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Two non-holonomic lattice walks in the quarter plane

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ABSTRACT

We present two classes of random walks restricted to the quarter plane with non-holonomic generating functions. The non-holonomicity is established using the iterated kernel method, a variant of the kernel method. This adds evidence to a recent conjecture on combinatorial properties of walks with holonomic generating functions [M. Mishna, Classifying lattice walks in the quarter plane, J. Combin. Theory Ser. A 116 (2009) 460–477]. The method also yields an asymptotic expression for the number of walks of length n.

1. Introduction

Previous studies of random walks on regular lattices had led some to conjecture that random walks in the quarter plane are likely to have holonomic generating functions. This is not unreasonable, given that random walks confined to the half plane have algebraic generating functions ([1] proves this for directed paths, but their results extend to more general random walks). This was, however, countered by Bousquet-Mélou and Petkovšek with their result that knight's walks confined to the quarter plane are not holonomic [5].

In this paper, we give two new examples of random walks whose generating functions are not holonomic. We use a technique developed in a recent study of self-avoiding walks in wedges [11], which is also at the root of the work Bousquet-Mélou and Petkovšek [5], called the iterated kernel method. This is an adaptation of the kernel method [3,15], in which we express a generating function in terms of iterated compositions of a kernel solution. This approach can also be traced back to the study of they symmetries of the kernel in [7]. We show that our generating functions have an infinite number of poles; this is a property which is incompatible with holonomy.

The aims of this work are two-fold. We illustrate a potentially general technique to prove that a generating function satisfying a certain kind of functional equation is not holonomic. Further, this work was completed in the context of a generating function classification (by the first author) of all nearest neighbour lattice walks in the quarter plane [14], and lends evidence to a general conjecture on the connection between different symmetries of the step sets and the analytic nature of their generating functions.

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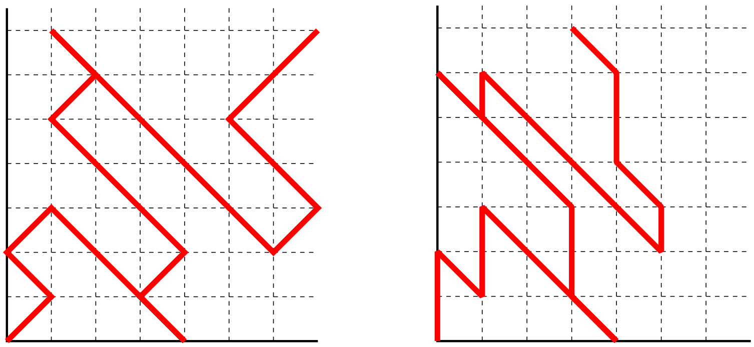


Fig. 1. Sample walks with steps from $\mathcal{S} = \{\mathsf{NW}, \mathsf{NE}, \mathsf{SE}\}$ (left) and $\mathcal{T} = \{\mathsf{NW}, \mathsf{N}, \mathsf{SE}\}$ (right).

1.1. Walks and their generating functions

The objects under consideration are walks in $\mathbb{N} \times \mathbb{N}$, the first quadrant of the integer lattice, with steps taken from $\mathcal{S} = \{(-1, 1), (1, 1), (1, -1)\}$ in the first case, and $\mathcal{T} = \{(-1, 1), (0, 1), (1, -1)\}$ in the second case. We also label these steps using compass directions: $\mathcal{S} = \{\mathsf{NW}, \mathsf{NE}, \mathsf{SE}\}$ and $\mathcal{T} = \{\mathsf{NW}, \mathsf{N}, \mathsf{SE}\}$. Two sample walks are given in Fig. 1.

To each step set we associate two formal power series: $W(t)$ a (univariate) counting generating function and $Q(x, y; t)$, a (multivariate) generating function which refines $W(t)$. The series, $W(t)$, is the ordinary generating function for the number of walks, that is, the coefficient of t^n is the number of walks of length n. The complete generating function, $Q(x, y; t)$, encodes more information. The coefficient of $x^iy^jt^n$ in $Q(x, y; t)$ is the number of walks of length n ending at the point (i, j). Note that the specialisation $x = y = 1$ in the complete generating function is precisely the counting series, i.e. $Q(1, 1; t) = W(t)$. If the choice of step set is not clear by context, we add a subscript.

In part, our interest in the complete generating function stems from the fact it satisfies a very useful functional equation which we derive using the recursive definition of a walk: a walk of length n is a walk of length $n-1$ plus a step. The quarter plane condition asserts itself by restricting our choice of step should the walk of length $n-1$ end on a boundary (i.e. an axis).

The step set $\mathcal{S}$ leads to the following equation:

$$Q(x, y; t) = 1 + t\left(xy + \frac{x}{y} + \frac{y}{x}\right) Q(x, y; t) - t\frac{x}{y}Q(x, 0; t) - t\frac{y}{x}Q(0, y; t), \tag{1}$$

and set $\mathcal{T}$ defines a comparable equation:

$$Q(x, y; t) = 1 + t\left(y + \frac{x}{y} + \frac{y}{x}\right) Q(x, y; t) - t\frac{x}{y}Q(x, 0; t) - t\frac{y}{x}Q(0, y; t). \tag{2}$$

These two equations are very similar with the only difference arising from the coefficient of $Q(x, y; t)$. Also note that the first equation is $x \leftrightarrow y$ symmetric, while the second is not.

In the text that follows we will frequently use a bar over a variable or function to denote its reciprocal, for example: $\bar{x} \equiv \frac{1}{x}$.

1.2. Properties of holonomic functions

We are interested in understanding the analytic nature of the generating functions. This gives a basic first classification of structures and also some general properties, for example, about the asymptotic growth of the coefficients. See, for example, Bousquet-Mélou's recent summary classifying combinatorial families with rational and algebraic generating functions [4]. We are interested in generating functions which are holonomic, also known as *D-finite*. Let $\mathbf{x} = x_1, x_2, \ldots, x_n$.

A multivariate function $G(\mathbf{x})$ is *holonomic* if the vector space generated by the partial derivatives of G (and their iterates), over rational functions of $\mathbf{x}$ is finite dimensional. This is equivalent to the existence of n partial differential equations of the form

$$p_{0,i}f(\mathbf{x}) + p_{1,i}\frac{\partial f(\mathbf{x})}{\partial x_i} + \cdots + p_{d_i,i}\frac{\partial^{d_i} f(\mathbf{x})}{(\partial x_i)^{d_i}} = 0,$$

where $1 \leq i \leq n$ and the $p_{j,i}$ are all polynomials in $\mathbf{x}$.

In the univariate case, this implies that there is at most a finite number of singularities, which can be recovered as zeros of the leading coefficient, $p_{d,1}(x)$.

1.3. Main results

Holonomic functions are closed under algebraic substitution [13,17], thus to establish that $Q(x, y; t)$ is not holonomic, it is sufficient to show that $W(t) = Q(1, 1; t)$ is not holonomic. We prove, for both step sets, that $W(t)$ has an infinite number of poles and thus is not holonomic [17]. These are summarized in our two main theorems.

Theorem 1. *Let $Q_{\mathcal{S}}(x, y; t)$ be the complete generating function for random walks on the first quadrant of the integer lattice with steps taken from* {NW, SE, NE} *and let $W_{\mathcal{S}}(t) = Q_{\mathcal{S}}(1, 1; t)$ be the corresponding counting generating function. Neither of these functions are holonomic.*

Theorem 2. *Let $Q_{\mathcal{T}}(x, y; t)$ be the complete generating function for random walks on the first quadrant of the integer lattice with steps taken from* {NW, SE, N} *and let $W_{\mathcal{T}}(t) = Q_{\mathcal{T}}(1, 1; t)$ be the corresponding counting generating function. Neither of these functions are holonomic.*

We prove these results by studying closed form expressions for the generating function. We find these using the same variation of the kernel method used in [11], and which is similar to that used in [5].

The idea behind the iterated kernel method is similar to other variants of the kernel method; we write our functional equation in the so-called *kernel form*, determine particular values of x and y which annihilate the kernel, and then use these to build new equations. For these walks, the kernel iterates form an infinite group whose elements each introduce new poles into the generating function. The main difficulty in proving that the generating functions are not holonomic lies in demonstrating that the poles introduced by the kernel iterates are indeed present in the sum.

The step set $\mathcal{T}$ is not $x \leftrightarrow y$ symmetric and the equations obtained are more complex. As a result the details of the proof of Theorem 2 are more cumbersome, though the argument is essentially the same.

2. $Q_{\mathcal{S}}(x, y; t)$ is not holonomic

We use the iterated kernel method to find an explicit expression for the generating function $W_{\mathcal{S}}(t)$.

2.1. Defining the iterates of the kernel solutions

To begin, consider Eq. (1), written in what we refer to as its kernel form:

$$\left(xy - tx^2y^2 - tx^2 - ty^2\right) Q(x, y) = xy - tx^2 Q(x, 0) - ty^2 Q(y, 0). \tag{3}$$

Here, for brevity we write $Q(x, y; t)$ as $Q(x, y)$, and have used the $x \leftrightarrow y$ symmetry to rewrite $Q(0, y)$ as $Q(y, 0)$. We will similarly suppress t as an argument of functions in the text below — for example $Y_{\pm 1}(x) \equiv Y_{\pm 1}(x; t)$.

The coefficient of $Q(x, y)$ in Eq. (3) is known as the *kernel* of the equation, and is denoted $K(x, y)$. The kernel $K(x, y) = xy - tx^2y^2 - tx^2 - ty^2$ is a quadratic polynomial in y, and hence it has two solutions:

$$Y_{\pm 1}(x; t) \equiv Y_{\pm 1}(x) = \frac{x}{2t(1 + x^2)} \left(1 \mp \sqrt{1 - 4t^2(1 + x^2)}\right). \tag{4}$$

Note that as formal power series in t, these roots are:

$$Y_1(x) = xt + O(t^3)$$
$$Y_{-1}(x) = \frac{x}{(1 + x^2)} \frac{1}{t} - xt + O(t^3).$$

The kernel may be rewritten as $y = tx + ty^2(x + 1/x)$. From this it follows that $Y_1(x)$ is a power series in t with positive polynomial coefficients in x. We use iterated compositions of Y_1, denoted $Y_n(x) = (Y_1 \circ)^n (x)$.

Lemma 3. *The kernel roots obey the relation:*

$$Y_1(Y_{-1}(x)) = x \quad \text{and} \quad Y_{-1}(Y_1(x)) = x \tag{5}$$

and the set $\{Y_n \mid n \in \mathbb{Z}\}$ forms a group, under the operation $Y_n(Y_m(x)) = Y_n \circ Y_m = Y_{n+m}$, with identity $Y_0 = x$.

Furthermore, they obey the relation:

$$\frac{1}{Y_1(x)} + \frac{1}{Y_{-1}(x)} = \frac{1}{tx}, \tag{6}$$

which extends (by substituting $x = Y_{n-1}(x)$) to:

$$\frac{1}{Y_n(x)} = \frac{1}{tY_{n-1}(x)} - \frac{1}{Y_{n-2}(x)}. \tag{7}$$

Proof. To prove Eq. (5), consider the product involving the four possible compositions of the kernel roots and a formal parameter z:

$$p(z) = \prod_{k,\ell=\pm 1} (z - Y_k(Y_\ell(x))) .$$

One may verify (by expanding the expression using a computer algebra system) that

$$(x^2 + t^2)p(z) = (z - x)^2 \left((t^2 + x^2)z - x(1 - 2t^2)z + t^2x^2\right) .$$

Remark that as $(z - x)$ is a factor with multiplicity 2, two of the compositions must be equal to x. It is not the composition $Y_+(Y_+(x)) = xt^2 + O(t^4)$, nor $Y_-(Y_-(x)) = 1/x + O(t^2)$, hence it must be that $Y_+(Y_-(x)) = Y_-(Y_+(x)) = x$. Note that this also shows the remaining compositions are, in fact solutions of a quadratic and not a quartic as one might expect.

Eq. (6) follows since $Y_{\pm 1}$ are roots of a quadratic — in particular $Y_{+1}Y_{-1}$ and $Y_{+1}+Y_{-1}$ are rational functions of x and t. □

2.2. An expression for $Q(x, 0)$ in terms of the kernel iterates

We are now able to write an explicit expression for the generating function.

Theorem 4. *The generating function $Q(x, 0)$ satisfies*

$$Q(x, 0) = \frac{1}{x^2 t} \sum_{n\geq 0} (-1)^n Y_n(x)Y_{n+1}(x). \tag{8}$$

Consequently, $W(t)$ satisfies

$$W(t) = \frac{1 - 2tQ(1, 0)}{1 - 3t} = \frac{1 - 2\sum_{n\geq 0}(-1)^n Y_n(1)Y_{n+1}(1)}{1 - 3t}. \tag{9}$$

Proof. By construction we have $K(x, Y_1(x)) = 0$, thus substituting $y = Y_1(x)$ into Eq. (3) gives (after a little tidying) the equation:

$$\frac{Y_0(x)Y_1(x)}{t} = Y_0(x)^2 Q(Y_0(x), 0) + Y_1(x)^2 Q(Y_1(x), 0),$$

where we have made use of the fact that $Y_0(x) = x$. Now we substitute $x = Y_n(x)$ into this equation to obtain

$$\frac{1}{t} Y_n(x)Y_{n+1}(x) = Y_n(x)^2 Q(Y_n(x), 0) + Y_{n+1}(x)^2 Q(Y_{n+1}(x), 0). \tag{10}$$

Since this expression gives $Q(Y_n, 0)$ in terms of $Q(Y_{n+1}, 0)$, we can successively eliminate $Q(Y_1, 0)$, $Q(Y_2, 0)$ and so forth. In particular, the alternating sum of Eq. (10) for n from 0 to $N - 1$ leads to the following telescoped expression for $Q(x, 0)$ in terms of Y_n, $n = 0, 1, 2, \ldots, N$:

$$\frac{1}{t} \sum_{n=0}^{N-1} (-1)^n Y_n(x)Y_{n+1}(x) = x^2 Q(x, 0) + (-1)^N Y_N(x)^2 Q(Y_N(x), 0). \tag{11}$$

Since $Y_1(x) = xt + O(t^3)$ is a power series in t with positive polynomial coefficients in x, it follows that $Y_n(x) = xt^n + o(xt^n)$. Hence we have that $\lim_{N\to\infty} Y_N(x) = 0$ as a formal power series in t. The theorem follows. □

2.3. The abundant singularities of $W(t)$

Given Eq. (9), there are 3 potential sources of singularities in $W(t)$:

(1) the simple pole at $t = 1/3$;
(2) singularities from the $Y_n(1; t)$;
(3) other singularities caused by the infinite sum.

Next we prove that the singularity at $t = 1/3$ is the dominant singularity (Lemma 5), and there is an infinite collection of singularities given by (2) (Lemma 8), and that there are no singularities given by (3) inside the unit circle (shown in Section 2.5). This allows us to conclude that W (and thus Q) are not holonomic.

Lemma 5. *The generating function $W(t)$ has a simple pole at $t = 1/3$.*

Proof. To show that $t = 1/3$ is a simple pole, we show that the numerator of Eq. (9), $1-2tQ(1, 0)$, is absolutely convergent and non-zero at this point.

In the proof of Proposition 9, we show that radius of convergence of $Q(1, 0)$ is bounded below by $8^{-1/2} \approx 0.335$, and thus $1 - 2tQ(t, 0)$ converges absolutely at $|t| = 1/3$.

Next, we show that the singularity is not removable by proving that $1-2tQ(1,0)$ is non-zero at $t=1/3$. Note that $Y_0(1;1/3)=1$ and $Y_1(1;1/3)=1/2$. Using the recurrence for Y_n in Eq. (7) we have

$$\frac{1}{Y_n(1;1/3)}=\frac{3}{Y_{n-1}(1;1/3)}-\frac{1}{Y_{n-2}(1;1/3)}$$

which leads to $Y_n(1;1/3)=1/F_{2n+1}$, *i.e.* the reciprocal of the $2n+1^{\text{st}}$ Fibonacci number. Hence $Y_nY_{n+1}\to 0$ as $n\to\infty$ and hence the alternating (real) sum $Q(1,1;1/3)$ is convergent. Further, we may write

$$\begin{aligned}\sum_{n\geq 0}\frac{(-1)^n}{F_{2n+1}F_{2n+3}}&=\frac{1}{2}-\sum_{k\geq 1}\frac{F_{4k+3}-F_{4k-1}}{F_{4k-1}F_{4k+1}F_{4k+3}}\\&=\frac{1}{2}-\frac{1}{10}+\sum_{k\geq 1}\frac{F_{4k+5}-F_{4k+1}}{F_{4k+1}F_{4k+3}F_{4k+5}}.\end{aligned}$$

Since the summands are strictly positive, it follows that the sum is bounded between $2/5$ and $1/2$. This implies that $1-\frac{2}{3}Q(1,0;1/3)$ is non-zero, and so $t=1/3$ is indeed a simple pole of $Q(1,1)$. □

2.4. An infinite set of singularities

Next, we show that $W(t)$ possesses an infinite number of singularities coming from simple poles of the Y_n. In order to do this we make the substitution $t\mapsto\frac{q}{1+q^2}$ which allows us to write Y_n in a nice closed form. This substitution does not affect whether or not the generating function is holonomic. Recall that we use $\overline{q}$ to denote $1/q$.

Lemma 6. *Define* $\overline{Y}_n(q):=\left(Y_n\left(1;\frac{q}{1+q^2}\right)\right)^{-1}$*. Then*

$$\overline{Y_1}(q)=\frac{q+\overline{q}}{2}+\frac{1}{2}\sqrt{\overline{q}^2+q^2-6},\tag{12}$$

$$\begin{aligned}\textit{and}\quad \overline{Y_n}(q)&=\left(\frac{\overline{Y_1}(q)-q}{\overline{q}-q}\right)\cdot\overline{q}^n+\left(\frac{\overline{q}-\overline{Y_1}(q)}{\overline{q}-q}\right)\cdot q^n\\&=\left(\frac{1-q^{2n}}{1-q^2}\right)\cdot q^{1-n}\cdot\overline{Y}_1(q)-\left(\frac{1-q^{2n-2}}{1-q^2}\right)\cdot q^{2-n}.\end{aligned}\tag{13}$$

Proof. To prove this lemma, substitute $t=\frac{q}{1+q^2}$ into $Y_1(x;t)$ and the recurrence in Eq. (7). The roots of the corresponding characteristic equation are simply q and $\overline{q}$ and the recurrence can be solved using standard methods. We remark that care must be taken when simplifying; we use a branch of the square root that preserves invariance under $q\mapsto\overline{q}$. □

We observe from these equations that $\overline{Y_n}(q)=\overline{Y_n}(\overline{q})$. This is a particularly useful property for our purposes. We can deduce the location of the zeros of $\overline{Y_n}(q)$, and hence the poles of $Y_n\left(1;\frac{q}{1+q^2}\right)$ and so those of $Y_n(1;t)$.

Lemma 7. *Suppose q_c is a zero of* $\overline{Y}_n(q):=Y_n\left(1;\frac{q}{1+q^2}\right)^{-1}$*, and that $q_c\neq 0$. Then*

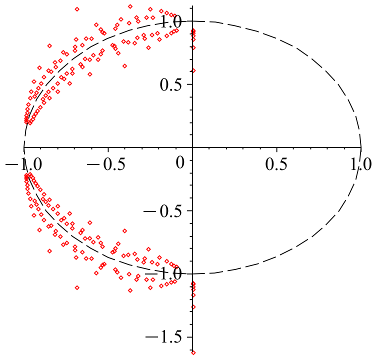
(1) *$q_c\neq\pm 1$;*
(2) *q_c is a solution of $q^{2n}+q^{-2n}+q^2+q^{-2}=4$;*
(3) *q_c is not a kth root of unity for any k; and*
(4) *for all $k\neq n$, $\overline{Y}_k(q_c)\neq 0$.*

Note that not all solutions of $q^{2n}+q^{-2n}+q^2+q^{-2}=4$ are zeros of $\overline{Y}_n(q)$ and in Section 2.6 we show that the solutions of this polynomial are partitioned into the zeros of $\overline{Y}_n(q)$ and $\overline{Y}_{-n}(q)$. We have found (numerically) that the zeros of $\overline{Y}_n(q)$ are those solutions with $\Re(q_c)<0$ or $q=ir$ with $r\in(-\infty,-1)\cup(0,1)$. Fig. 2 illustrates the zeros of $\overline{Y_n}(q)$ for $n=2\ldots 15$, and the zeros of $\overline{Y_{15}}(q)$.

Proof. Substitute $q=\pm 1$ into Eq. (12) to deduce that $Y_1(\pm 1)=\pm(1-\mathrm{i})/2$. Next, by first simplifying the rational expressions in Eq. (13) we compute the following non-zero values for $\overline{Y}_n$ evaluated at $q=\pm 1$:

$$\begin{aligned}\overline{Y}_n(1)&=\frac{n}{2}(1-\mathrm{i})-(n-1);\\\overline{Y}_n(-1)&=(-1)^n\left(\frac{1-\mathrm{i}}{2}+(n-1)\right).\end{aligned}$$

Thus $q=\pm 1$ are not zeros of $\overline{Y}_n(q)$, for any n.



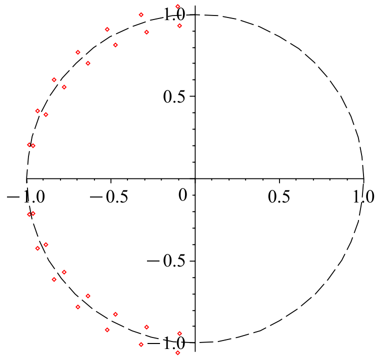


Fig. 2. The location of the zeros of $\overline{Y_n}(q)$ in the complex q-plane for $n = 2 \dots 15$ (left) and of $\overline{Y}_{15}(q)$ (right). The unit circle is indicated with a dashed line.

If $\overline{Y_n}(q_c) = 0$ we can rewrite Eq. (13) as (since $q \neq \pm 1$):

$$q_c(1 - q_c^{2n})\overline{Y}_1(q_c) = (q_c^2 - q_c^{2n}).$$

Using the expression for $\overline{Y}_1(q)$ in Eq. (6) and assuming that $q_c \neq 0$ we can rearrange the above equation as follows:

$$\begin{aligned} (1 - q_c^{2n}) \left(1 + q_c^2 - q_c\sqrt{\overline{q_c}^2 + q_c^2 - 6}\right) &= 2(q_c^2 - q_c^{2n}) \\ \Longrightarrow \qquad (1 - q_c^{2n})q_c\sqrt{\overline{q_c}^2 + q_c^2 - 6} &= 2(q_c^2 - q_c^{2n}) - (1 - q_c^{2n})(q_c^2 + 1) \\ &= (q_c^2 - 1)(q_c^{2n} + 1). \end{aligned} \tag{14}$$

By squaring both sides, expanding and collecting terms we arrive at the equation

$$q_c^{4n} + q_c^{2n}\left(\frac{1 - 4q_c^2 + q_c^4}{q_c^2}\right) + 1 = 0,$$

which can in turn be reduced to

$$q_c^{2n} + q_c^{-2n} + q_c^2 + q_c^{-2} = 4.$$

It is important to note that by squaring both sides of the above equation we introduce extraneous solutions that are not solutions of the original expression. These extra solutions are those q which satisfy

$$-(1 - q^{2n})q\sqrt{\overline{q}^2 + q^2 - 6} = 2(q^2 - q^{2n}) - (1 - q^{2n})(q + \overline{q}). \tag{15}$$

Effectively we have multiplied the left-hand side of Eq. (14) by -1. Remark that we obtain exactly this expression if we substitute $\overline{Y}_{-1}(q)$ for $\overline{Y}_1(q)$ in Eq. (13) and repeat the arguments leading to Eq. (14). That is, we solve the recurrence in Eq. (7) but with a different initial condition. We will return to this expression in Section 2.6, and use it to show that $\overline{Y}_{-n}(-q) = (-1)^n\overline{Y}_n(q)$, and so show how the solutions of $q_c^{2n} + q_c^{-2n} + q_c^2 + q_c^{-2} - 4 = 0$ partition themselves into poles of $Y_n(1; \frac{q}{1+q^2})$ and $Y_n(1; \frac{-q}{1+q^2})$.

Now if $q_c = e^{i\theta}$, it follows that $\cos(2n\theta) + \cos(2\theta) = 2$. Hence that the only possible zeros on the unit circle are at $q_c = \pm 1$. We have already excluded these, so all the zeros lie off the unit circle.

Finally, we prove part (4) of the lemma. It suffices to show that the following equations do share common zeros except at $q = \pm 1$ unless $n = k$:

$$\begin{aligned} q^{2n} + q^{-2n} &= 4 - (q^2 + q^{-2}), \\ q^{2k} + q^{-2k} &= 4 - (q^2 + q^{-2}). \end{aligned}$$

Let $d = n - k \neq 0$ and subtracting the second equation from the first:

$$\begin{aligned} 0 &= q^{2n} - q^{2k} - q^{-2k} + q^{-2n} \\ &= q^{2k}(q^{2d} - 1) - q^{-2n}(q^{2d} - 1) = (q^{2d} - 1)(q^{2k} - q^{-2n}). \end{aligned} \tag{16}$$

Thus any common solution to both $\overline{Y_n}(q) = 0$ and $\overline{Y_k}(q) = 0$ lies on the unit circle. We have already excluded this possibility and so (4) follows. $\square$

Next we show that for any $n \geq 0$, all zeros of $\overline{Y_n}(q)$ are poles of $Q\left(1, 0; \frac{q}{1+q^2}\right)$, and hence of $W\left(\frac{q}{1+q^2}\right)$, by Eq. (9).

Lemma 8. *Let $q = q_c \neq 0$ be a zero of $\overline{Y}_n(q)$. The function $Q\left(1, 0; \frac{q}{1+q^2}\right)$ has a pole at the $q = q_c$.*

Proof. In order to verify that q_c is a pole of $Q\left(1, 0; \frac{q}{1+q^2}\right)$, we examine the series defined in Theorem 4. We show that the terms arising when $k = n, n-1$ contribute a pole, while the remaining terms are convergent at $q = q_c$.

By Lemma 7 (part 4), $\overline{Y}_n(q_c) \neq 0$ if $n \neq k$. Hence the sum of the first $n-1$ summands is a finite sum of functions that are analytic at q_c, and so is itself analytic at q_c. Thus we consider the tail of the sum, given by $\sum_{k>n}(-1)^k Y_k Y_{k+1}$.

The term $Y_k Y_{k+1}$ can be simplified using Eq. (13):

$$Y_k Y_{k+1} = \frac{q^{2k+2}-1}{q^{2k+1}(q\overline{Y}_1 - 1) - \overline{Y}_1 + q} - \frac{q^{2k}-1}{q^{2k-1}(q\overline{Y}_1 - 1) - \overline{Y}_1 + q}. \tag{17}$$

We know that $|q_c| \neq 1$, and that Y_n is invariant under $q \mapsto 1/q$. So we only need to consider the case $|q_c| < 1$. By manipulating the the explicit expression for $Y_k Y_{k+1}$ one can show that when $0 < |q| < 1$ (for *any* q, not just the singularities),

$$\lim_{k\to\infty} \left| \frac{Y_k Y_{k+1}}{Y_k Y_{k-1}} \right| = |q^2|.$$

Since $|q| < 1$ this limit is less than 1 and so, by the ratio test, the series converges.

Finally, we consider the only remaining term, $Y_n(Y_{n+1} - Y_{n-1})$ (up to a sign). By Lemma 7(4), both Y_{n+1}, Y_{n-1} are analytic at $q = q_c$. Since $\overline{Y_n}(q_c) = 0$, Eq. (7) implies that $\overline{Y}_{n+1}(q_c) = -\overline{Y}_{n-1}(q_c)$, and so $Y_{n+1} - Y_{n-1} = 2Y_{n+1}$ is also analytic at q_c.

Furthermore, using the expression for $\overline{Y}_n(q)$ in Eq. (13), we see that the potential singularities of $\overline{Y}_{n+1}(q)$ are ± 1, 0, and the square root singularity arising from $\overline{Y}_1(q)$. Thus it does not have a pole at q_c, and so Y_{n+1} is not zero at q_c. We conclude that $Q(1, 0)$ has a pole at q_c. □

2.5. Possible singularities at other points?

The argument given in the above proof can be recycled to show that for any q_0 not on the unit circle, the series is analytic: Either it is a pole of the above type, or every Y_n is analytic at q_0, and we can show absolute convergence at the point by the same ratio test argument. The same argument also holds when q_0 is one of the four square-root singularities of $\overline{Y}_1(1)$.

This argument does not apply for points on the unit circle. The solutions of $q^n + q^{-n} + q^2 + q^{-2} - 4 = 0$ approach the $|q| = 1$ as n tends to infinity. In this case, we have not determined what happens; for the purposes of our proof we only need to demonstrate that there are an infinite number of singularities.

2.6. The zeros of $q^{2n} + q^{-2n} + q^2 + q^{-2} - 4 = 0$ which contribute poles

The points in Fig. 2 were selected based on numerical computation. How does one decide if a solution to $q^{2n} + q^{-2n} + q^2 + q^{-2} - 4 = 0$ is a pole, and does it matter? By tracing the origin of this equation, we can express the additional zeros in a very straightforward manner, and so complete the picture in a satisfying way.

Since $Y_{+1}(Y_{-1}(x)) = x$ we were able to define $Y_{-n}(x)$ (for $n > 0$) as $Y_{-n} = (Y_{-1}\circ)^n(x)$. Substituting $x = \overline{Y}_{-n}$ and $t = \frac{q}{1+q^2}$ into Eq. (6) gives

$$\overline{Y}_{-n} = \frac{1+q^2}{q}\overline{Y}_{-(n-1)} - \overline{Y}_{-(n-2)}.$$

We can solve this above recurrence using standard methods:

$$\overline{Y}_{-n}(q) = \left(\frac{1-q^{2n}}{1-q^2}\right) \cdot q^{1-n} \cdot \overline{Y}_{-1}(q) - \left(\frac{1-q^{2n-2}}{1-q^2}\right) \cdot q^{2-n} \tag{18}$$

where $\overline{Y}_{-1}(q) = \frac{q+\overline{q}}{2} - \frac{1}{2}\sqrt{\overline{q}^2 + q^2 - 6}$.

The solutions to the equation $Y_{-n}(q) = 0$ satisfy Eq. (15), which is precisely Eq. (14) with the left-hand side multiplied by -1; this describes exactly the set of extraneous solutions which were added in the proof of Lemma 7.

Since $\overline{Y}_{-1}(-q) = -\overline{Y}_1(q)$, one can verify that $\overline{Y}_n(q) = (-1)^n \overline{Y}_{-n}(-q)$. Consequently, if $q = q_c$ is a solution of $q^{2n} + q^{-2n} + q^2 + q^{-2} - 4 = 0$, then one of q_c or $-q_c$ is a zero of $\overline{Y}_n(q)$ (the other being a zero of $\overline{Y}_{-n}(-q)$). Since $q^{2n} + q^{-2n} + q^2 + q^{-2} - 4$ is invariant under $q \mapsto -q$, we see that precisely half of the solutions to this equation (excluding $q = \pm 1$) are poles of $Y_n\left(\frac{q}{1+q^2}\right)$, and of $W\left(\frac{q}{1+q^2}\right)$.

2.7. Proof of Theorem 1

We now have all the components in place to prove the main result.

Proof (*Theorem 1*). By Lemma 8, the function $W_{\mathcal{S}}(\frac{q}{1+q^2})$ has a set of poles given by the zeros of the $\overline{Y}_n$. By Lemma 7(4), this is an infinite set. Thus, $W_{\mathcal{S}}(\frac{q}{1+q^2})$ is not holonomic. If a multivariate series is holonomic, all of its algebraic specialisations must be holonomic. As $W_{\mathcal{S}}(\frac{q}{1+q^2}) = Q_{\mathcal{S}}\left(1, 1, \frac{q}{1+q^2}\right)$ is an algebraic specialisation of both $Q_{\mathcal{S}}(x, y; t)$ and $Q_{\mathcal{S}}(1, 1; t)$, it follows that neither of these two functions are holonomic. □

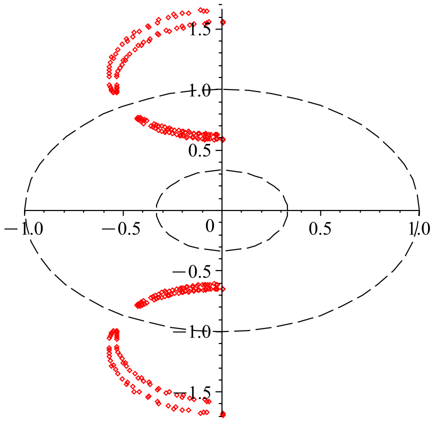


Fig. 3. A sample of singularities of $W(t)$ in the complex t-plane. The curves $|t| = 1/3, 1$ are indicated with dashed lines.

It is natural to ask to where the singularities are mapped in the t variable. There are two possible transformations to map $W(q/(1+q^2))$ to $W(t)$; if they are called τ_1, and τ_2, then we have that $\tau_1(q) = \tau_2(\overline{q})$. Thus, as our function is invariant under the transformation $q \mapsto \overline{q}$, either is applicable. Fig. 3 shows the the location of the singularities arsing from $Y_n(1)$, $n = 10 \dots 15$ in the t variable.

2.8. Asymptotics of the number of walks

We conclude our study of these walks by deriving their asymptotics.

Proposition 9. *The number of walks of length n with steps from step set* {NE, SE, NW}, *confined to the quarter plane is asymptotic to*

$$c_n \sim \alpha_{\mathcal{S}} 3^n + O(8^{n/2}) \tag{19}$$

where $\alpha_{\mathcal{S}}$ is a constant given by

$$\alpha_{\mathcal{S}} = 1 - 2\sum_{n\geq 0} \frac{(-1)^n}{F_{2n+1}F_{2n+3}} = 0.1731788835\cdots.$$

Proof. We need to show that the dominant singularity is the simple pole at $t = 1/3$. Let us rewrite Eq. (3) at $x = y = 1$:

$$(1-3t)Q(1,1) = 1 - 2tQ(1,0) \quad \text{or} \quad Q(1,1) = \frac{1-2Q(1,0)}{1-3t}. \tag{20}$$

Hence there are 2 sources of singularities for $Q(1,1)$ — the simple pole at $t = 1/3$ and the singularities of $Q(1,0)$. The residue at the simple pole may be computed quite directly (see the proof of Lemma 5).

We now bound the singularities of $Q(1,0)$ away from $|t| = 1/3$. The series $Q(1,0)$ enumerates random walks confined to the quarter plane which end on the line $y = 0$. Consider a family of random walks with the same step-set, ending on the line $y = 0$, but no longer confined to the quarter plane; rather they are confined to the half-plane $y \geq 0$. The generating function of these walks obeys the functional equation

$$P(y) = 1 + t\left(2y + \frac{1}{y}\right)P(y) - t\frac{1}{y}P(0). \tag{21}$$

The generating function of these walks can be computed using the (standard) kernel method to give:

$$P(0) = \frac{1-\sqrt{1-8t^2}}{4t^2}. \tag{22}$$

Hence the number of these walks grows as $O(8^{n/2})$. Consequently the walks enumerated by $Q(1,0)$ cannot grow faster than $O(8^{n/2})$, and so the radius of convergence of $Q(1,0)$ is bounded below by $8^{-1/2}$. The result follows. □

Note that we used last part of this proof to prove Lemma 5.

3. $Q_{\mathcal{T}}(x, y; t)$ is not holonomic

The step set $\mathcal{T} = \{\text{N, SE, NW}\}$ is not symmetric across the line $x = y$, thus we do not have $x \leftrightarrow y$ symmetry in the complete generating function, and we should expect a more complicated scenario. The general argument is the same: we use the iterated kernel method to determine an expression for the univariate generating function as an infinite sum of terms that are rational functions in $\mathbb{Q}(t, \sqrt{1-4t})$. Each term contributes a distinct, finite collection of poles. The other terms converge at these poles, and so the generating function has an infinite collection of singularities. In several places below we have omitted the details of the proof since they are essentially the same as those of the symmetric case considered above.

3.1. Defining two sets of iterated compositions

We begin by rewriting the functional equation in kernel form:

$$K(x,y)Q(x,y) = xy - txQ(x,0) - ty^2Q(0,y), \tag{23}$$

with kernel $K(x,y) = xy - t(xy^2 + x^2 + y^2)$. Because of the asymmetry in the variables, we require zeros of the kernel for both fixed x and fixed y, which we denote respectively with X and Y. This gives

$$\begin{aligned} X_{\pm1}(y;t) &\equiv X_{\pm1}(y) = \frac{y}{2t}\left(ty - 1 \mp \sqrt{(1-ty)^2 - 4t^2}\right), \quad \text{and} \\ Y_{\pm1}(x;t) &\equiv Y_{\pm1}(x) = \frac{x}{2t(1+x)}\left(1 \mp \sqrt{1-4t^2(x+1)}\right). \end{aligned} \tag{24}$$

Analogously to the previous case:

$$X_{\pm1}\left(Y_{\mp1}(y)\right) = y \qquad Y_{\pm1}\left(X_{\mp1}(x)\right) = x. \tag{25}$$

Furthermore,

$$\begin{aligned} \frac{1}{X_{+1}(y)} + \frac{1}{X_{-1}(y)} &= \frac{1}{ty} - 1 \\ \frac{1}{Y_1(x)} + \frac{1}{Y_{-1}(x)} &= \frac{1}{tx}. \end{aligned} \tag{26}$$

Setting $y = Y_1(x)$ and $x = X_1(y)$ in Eq. (23) yields respectively,

$$0 = xY_1(x) - L(x) - R(Y_1(x)) \tag{27}$$

$$0 = X_1(y)y - L(X_1(y)) - R(y), \tag{28}$$

where $L(x;t) \equiv L(x) = tx^2Q(x,0)$ and $R(y;t) \equiv R(y) = ty^2Q(0,y)$. Setting $y = Y_1(x)$ in Eq. (28) results in

$$R(Y_1(x)) = X_1(Y_1(x))Y_1(x) - L(X_1(Y_1(x))). \tag{29}$$

Combining Eqs. (27) and (29) produces an expression suitable for iteration:

$$L(x) = Y_1(x)\,(x - X_1(Y_1(x))) + L(X_1(Y_1(x))).$$

In a similar way we also obtain

$$R(y) = X_1(y)\,(y - Y_1(X_1(y))) + R(Y_1(X_1(y))).$$

Iterating these equations gives

$$L(x) = Y_1(x)\big(x - X_1(Y_1(x))\big) + Y_1(X_1(Y_1(x)))\big(X_1(Y_1(x)) - X_1(Y_1(X_1(Y_1(x))))\big) + \cdots \tag{30}$$

$$R(y) = X_1(y)\big(y - Y_1(X_1(y))\big) + X_1(Y_1(X_1(y)))\big(Y_1(X_1(y)) - Y_1(X_1(Y_1(X_1(y))))\big) + \cdots, \tag{31}$$

where we have assumed the convergence of these compositions. Below we do indeed prove the convergence of these expressions as formal power series.

As was the case above, we make the substitution $t \mapsto \frac{q}{1+q^2}$ to simplify the subsequent expressions. To (somewhat) lighten the notation, we also define six sequences of functions:

$$\left.\begin{aligned} A_n(x;q) &\equiv A_n(x) = ((X_1 \circ Y_1)\circ)^n(x) & A_0(x) &= x \\ B_n(x;q) &\equiv B_n(x) = Y_1(A_n(x)) & & \\ C_n(y;q) &\equiv C_n(y) = ((Y_1 \circ X_1)\circ)^n(y) & C_0(y) &= y \\ D_n(y;q) &\equiv D_n(y) = X_1(C_n(y)) & & \\ \Delta_{B,n}(x) &= B_n(x) - B_{n-1}(x) & \Delta_{B,0} &= B_0(x) \\ \Delta_{C,n}(y) &= C_n(y) - C_{n+1}(y) & & \end{aligned}\right\} \tag{32}$$

These expression allow us to rewrite Eqs. (30) and (31) (with some rearranging) as

$$\begin{aligned} L\left(x; \frac{q}{1+q^2}\right) &= \sum_{n\geq 0} B_n(x)\left(A_n(x) - A_{n+1}(x)\right) \\ &= A_0B_0 + \sum_{n\geq 1} A_n(x)\left(B_n(x) - B_{n-1}(x)\right), \\ &= \sum_{n\geq 0} A_n(x)\Delta_{B,n}(x) \\ R\left(y; \frac{q}{1+q^2}\right) &= \sum_{n\geq 0} D_n(y)\left(C_n(y) - C_{n+1}(y)\right) = \sum_{n\geq 0} D_n(y)\Delta_{C,n}(y). \end{aligned} \tag{33}$$

We proceed by showing that $L(1; \frac{q}{1+q^2})$ has an infinite number of singularities arising from the poles of the $A_n(1)$. We then show that $R(1; \frac{q}{1+q^2})$ is analytic at an infinite subset of these poles. From this we conclude in Section 3.6 that $Q(1, 1; t) = (1-3t)^{-1}(1 + tQ(1, 0) + tQ(0, 1))$ has an infinite number of singularities, and is thus not holonomic.

Theorem 10. *Let $\mathcal{P}_n$ be the set of poles of $A_n(1; q)$. Then the following statements are true:*

(1) *$\rho \in \mathcal{P}_n$ satisfies $\rho^{2n} + \rho^{-2n} - 4 + \rho^2 + \rho^{-2} = 0$;*
(2) *if $n \neq k$, then $\mathcal{P}_n \cap \mathcal{P}_k \subseteq \{+1, -1\}$.*

Furthermore, for every even n, there is some $\rho = r\mathrm{i} \in \mathcal{P}_n$, with $r \in \mathbb{R}$ for which the following statements are true:

(3) *$E_n = A_n(B_n - B_{n-1}) - (A_{n+1}B_n + A_{n-1}B_{n-1})$ has a pole at ρ;*
(4) *$L(1; \frac{q}{1+q^2}) - E_n$ is analytic at ρ;*
(5) *$R(1; \frac{q}{1+q^2})$ is analytic at ρ.*

From this, we extract the following corollary, essential to the proof of Theorem 2:

Corollary 11. *The function $L(1; \frac{q}{1+q^2})$ has an infinite set of distinct poles along the imaginary axis and hence it is not holonomic.*

The proof of Theorem 10 is distributed across the next few sections.

3.2. Explicit expressions for $\overline{A}_n$, $\overline{B}_n$, $\overline{C}_n$, and $\overline{D}_n$

To find equations satisfied by the poles of A_n, B_n, C_n, and D_n, we use the recurrences in Eq. (28) and the identities from Eq. (25), to find a pair of coupled recurrences for $\overline{A}_n$ and $\overline{B}_n$ and another pair for $\overline{C}_n$ and $\overline{D}_n$. When $x = y = 1$ these functions satisfy the following recurrences:

$$\begin{aligned} &\overline{A}_n(1) = (q+\overline{q})\,\overline{B}_{n-1}(1) - \overline{A}_{n-1}(1) - 1 \\ &\overline{B}_n(1) = (q+\overline{q})\,\overline{A}_n(1) - \overline{B}_{n-1}(1) \\ &\overline{C}_n(1) = (q+\overline{q})\,\overline{D}_{n-1}(1) - \overline{C}_{n-1}(1) \\ &\overline{D}_n(1) = (q+\overline{q})\,\overline{C}_n(1) - \overline{D}_{n-1}(1) - 1 \end{aligned} \tag{34}$$

with initial conditions:

$$\begin{aligned} &\overline{A}_0 = 1 \qquad \overline{B}_0 = \tfrac{q+\overline{q}}{2} + \frac{1}{2}\sqrt{q^2+\overline{q}^2-6} \\ &\overline{C}_0 = 1 \qquad \overline{D}_0 = \frac{q+\overline{q}-1}{2} + \frac{1}{2}\sqrt{q^2+\overline{q}^2-2(q+\overline{q})-1}. \end{aligned} \tag{35}$$

These are constant coefficient recurrences and so may be solved using standard techniques. We can express these functions in a closed form using the following two functions:

$$\alpha(q) = \frac{(\overline{q}^2-2) + (q-\overline{q})\overline{Y}_1(q)}{(\overline{q}-q)^2}, \qquad \gamma(q) = \frac{(q-\overline{q})\overline{X}_1(q) - (1+\overline{q}-\overline{q}^2)}{(\overline{q}-q)^2}. \tag{36}$$

These expressions are

$$\begin{aligned} &\overline{A}_n(1;q) = \alpha(q)\,q^{2n} + \alpha(\overline{q})\,\overline{q}^{2n} + \frac{2}{(\overline{q}-q)^2} \\ &\overline{B}_n(1;q) = \alpha(q)\,q^{2n+1} + \alpha(\overline{q})\,\overline{q}^{2n+1} + \frac{q+\overline{q}}{(\overline{q}-q)^2} \\ &\overline{C}_n(1;q) = \gamma(q)\,q^{2n} + \gamma(\overline{q})\,\overline{q}^{2n} + \frac{q+\overline{q}}{(\overline{q}-q)^2} \\ &\overline{D}_n(1;q) = \gamma(q)\,q^{2n+1} + \gamma(\overline{q})\,\overline{q}^{2n+1} + \frac{2}{(\overline{q}-q)^2}. \end{aligned} \tag{37}$$

First, from this form, it is straightforward to verify that each of these functions is symmetric under the transformation $q \mapsto \overline{q}$.

Table 1
Equations satisfied by the poles of A_n, B_n, C_n, D_n, and the zeros of $\Delta_{B,n}$ and $\Delta_{C,n}$.

#	Quantity	Satisfying equation
1	Double pole of A_n	$q^{2n}+q^{-2n}+q^2+q^{-2}-4=0$
2	Simple pole of B_n	$q^{2n}+q^{-2n}+q^2+q^{-2}-4=0$
3		$q^{2n+2}+q^{-2n-2}+q^2+q^{-2}-4=0$
5	Simple pole of C_n	$q^{2n+1}+q^{-2n-1}+q^2+q^{-2}-q-q^{-1}-2=0$
6		$q^{2n-1}+q^{-2n+1}+q^2+q^{-2}-q-q^{-1}-2=0$
4	Double pole of D_n	$q^{2n+1}+q^{-2n-1}+q^2+q^{-2}-q-q^{-1}-2=0$
7	Simple zeros of $\Delta_{B,n}$	$q^{2n+1}+q^{-2n-1}+q^2+q^{-2}-4=0$
8		$q^{2n+1}+q^{-2n-1}-q^2-q^{-2}+4=0$
9	Simple zeros of $\Delta_{C,n}$	$q^{2n+1}+q^{-2n-1}+q^2+q^{-2}-q-q^{-1}-2=0$
10		$q^{2n+1}+q^{-2n-1}-q^2-q^{-2}+q+q^{-1}+2=0$

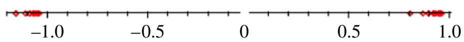


Fig. 4. A plot of r such that ri is a pole of $A_{2n}(x)(B_{2n}(x)-B_{2n-2}(x))$ for $n=1\dots10$.

We can then use these (and the expressions for X_1 and Y_1) to find the poles of these functions (A_n, B_n, C_n, D_n), or equivalently the zeros of their reciprocals $(\overline{A}_n, \overline{B}_n, \overline{C}_n, \overline{D}_n)$ in the same manner as in Lemma 7. Note that the poles of A_n and B_n are the same as those of the Y_n considered in symmetric case. The equations satisfied by the locations of the zeros and poles of the various components of $Q(1,0)$ and $Q(0,1)$ are summarized in Table 1.

3.3. An infinite set of singularities

As we have remarked, the singularities of A_n satisfy the same relation as the Y_n in the symmetric case. Thus, the following two lemmas, which together prove Theorem 10 Part (2), follow directly from Lemma 7. First, however, we remark that in this case A_n is not finite at $q=1$.

Lemma 12. *The poles of $A_n(1;q)$ do not lie on the unit circle, excepting possibly at $q=\pm1$.*

Lemma 13. *The functions A_n and A_k share common poles if, and only if $q=\pm1$, or $n=k$.*

3.4. The nature of the poles

Theorem 10(3) contends that there is an infinite collection of poles along the imaginary axis.

Lemma 14. *The equation $q^{2n}+q^{-2n}+q^2+q^{-2}=4$ has purely imaginary zeros for even n. Further, there are at least two zeros $q=\pm ir$ with $0<r<1$ and at least another two zeros $q=\pm ir$ with $r>1$.*

Proof. Substitute $q=\pm ri$ and $n=2k$ into the equation satisfied by the zeros:

$$r^{4k}+r^{-4k}-r^2-r^{-2}=4. \tag{38}$$

When $r=1$, the left-hand side is strictly less than 4. For $r>2$ and $0<r<1/2$ (and $k\geq1$) the left-hand side is strictly greater than 4. Hence there is at least one value of $r>1$ at least one value of $0<r<1$ such that the above equation is satisfied. □

To translate this result into a proof that $\overline{A}_{2n}$ has imaginary zeros, we must determine which set of zeros of the equation form come from solving $A_n(q)=0$. This situation is again extremely similar to the symmetric case. We can define A_{-n} and B_{-n} for $n>0$ and similarly show that $A_{-n}(-q)=A_n(q)$ and $B_{-n}(-q)=-B_n(q)$. Hence the zeros of $A_n(-q)$ and $A_{-n}(q)$ account for all of the zeros of $q^{2n}+q^{-2n}+q^2+q^{-2}=4$. Thus, if q_c is a solution to this equation, precisely one of q_c or $-q_c$ is a pole of A_n. Since both ri and $-ri$ are on the imaginary axis, A_{2n} does indeed have purely imaginary singularities. These are illustrated for $n=1\dots10$ in Fig. 4.

Next, we show that $E_n=A_n(B_n-B_{n-1})-(A_{n+1}B_n+A_{n-1}B_{n-1})$ has a pole at ρ.

As noted in Table 1, A_n has a double pole at q_c and (B_n-B_{n-1}) is not zero at that point. Thus $A_n(B_n-B_{n-1})$ has a pole of order at least 2 at that point. In fact, the simple poles that happen respectively at B_n and B_{n-1} imply that this is a pole of order 3.

The two other terms potentially have simple poles arising from the B terms. A simple pole cannot cancel a pole of higher order. Thus, it remains to show that A_{n-1} and A_{n+1} are analytic at q_c. This is true by Theorem 10(4).

Finally, to conclude that it the rest of the series is convergent at this point we apply a ratio test argument. When $|q| < 1$

$$\lim_{k\to\infty}\left|\frac{A_k\Delta_{B,k}}{A_{k-1}\Delta_{B,k-1}}\right| = |q^4|.$$

This value is less than one for $|q| < 1$, and hence the series is convergent. (When $|q| > 1$, this limit is $|\frac{1}{q}|^4$, and the series is equally convergent.)

3.5. There is no cancellation

Finally, we prove that there is no cancellation of the poles of $L(1)$ with those of $R(1)$ along the imaginary axis. It suffices to show that along the imaginary axis, $\overline{C}_n$ and $\overline{D}_n$ are non-zero and $R(q)$ is convergent.

Lemma 15. *The functions $\overline{C}_n$ and $\overline{D}_n$ do not have purely imaginary zeros.*

Proof. Substitute $q = ir$ into the equation satisfied by the zeros of $\overline{C}_n$ and $\overline{D}_n$:

$$(ir)^{2n+1} + (ir)^{-2n-1} - r^2 - r^{-2} = 2 + (ir) + (ir)^{-1}. \tag{39}$$

The real part of this equation is $r^2 + r^{-2} = -2$. This has no solution for real valued r. Hence the equation has no purely imaginary solutions. □

Finally, if ρ is a pole of A_n on the imaginary axis, as before, we use the fact that for $|q| < 1$,

$$\lim_{k\to\infty}\left|\frac{D_k\Delta_{C,k}}{D_{k-1}\Delta_{C,k-1}}\right| = |q^4|,$$

hence $R(1)$ is analytic at those singularities on the imaginary line inside the unit circle. Of course, $R(1)$ is also invariant under the transformation $q \mapsto \overline{q}$, hence this is also true for the remaining singularities.

3.6. Proof of Theorem 2

We summarize our arguments in the following proof of Theorem 2.

Proof (*Theorem 2*). The function $W_{\mathcal{T}}(\frac{q}{1+q^2})$ has a set of poles given by the zeros of the $\overline{A}_n$ along the imaginary axis. By Theorem 10, this is an infinite set. Thus, $W_{\mathcal{T}}(\frac{q}{1+q^2})$ is not holonomic. If a multivariate series is holonomic, all of its algebraic specialisations must be holonomic. As $W_{\mathcal{T}}(\frac{q}{1+q^2}) = Q_{\mathcal{T}}\left(1, 1; \frac{q}{1+q^2}\right)$ is an algebraic specialisation of both $Q_{\mathcal{T}}(x, y; t)$ and $Q_{\mathcal{T}}(1, 1; t)$, it follows that neither of these two functions are holonomic. □

Remark. We believe that one could make very similar arguments to prove that $Q(0, 1; t)$ is also non-holonomic by considering the part of the unit circle where $\overline{D}_n$ has no zeros, but we have not pursued this.

3.7. Asymptotics of the number of walks

We now turn to the asymptotics of the number of walks. Rewriting Eq. (23) at $x = y = 1$

$$Q(1, 1) = \frac{1}{1-3t}\left(1 - tQ(1, 0) - tQ(0, 1)\right).$$

One expects that the dominant singularity is a simple pole at $t = 1/3$, however one can explicitly show that residue at $t = 1/3$ is zero. At $t = 1/3$, the functions A_n, B_n, C_n and D_n can be expressed simply in terms of Fibonacci numbers and $tQ(1, 0) + tQ(0, 1)$ becomes a simple telescoping sum that equals one.

Because of this we have not been able to prove as much about the asymptotics of the number of these walks as we were able to for the symmetric case.

Proposition 16. *Let c_n be the number of walks of length n with steps from step set* {N, SE, NW}*, confined to the quarter plane. The following holds*

$$\lim_{n\to\infty} c_n^{1/n} = 3. \tag{40}$$

More precisely, c_n is asymptotic to

$$c_n \sim \alpha_{\mathcal{T}}\frac{3^n}{\sqrt{n}}\left(1 + o(1)\right) \tag{41}$$

where $0 \le \alpha_{\mathcal{T}} \le \sqrt{\frac{3}{\pi}}$.

Proof. One can show that the above limit exists and is equal to 3 using the arguments from Section 2 of [11].

The number of walks in the quarter plane is bounded above by the number of walks with the same step-set in the half plane $\mathbb{N} \times \mathbb{Z}$. These walks are in bijection with left-factors of Motzkin paths and their generating function, $P(x)$ of these walks satisfy the following functional equation

$$P(x) = 1 + t\left(x + 1 + \frac{1}{x}\right)P(x) - t\frac{1}{x}P(0). \tag{42}$$

This can be solved using the kernel method to give

$$P(1) = \frac{1}{2t}\left(-1 + \sqrt{\frac{1+t}{1-3t}}\right). \tag{43}$$

Singularity analysis then shows that these walks grow as

$$[t^n]P(1) \sim \sqrt{\frac{3}{\pi n}}3^n\left(1 + o(1)\right), \tag{44}$$

which gives the upper bound on $\alpha_{\mathcal{T}}$. The lower bound is trivial. □

Note that a similar analysis shows that $Q(0, 1)$ has a radius of convergence of at least $1/\sqrt{8}$ and so does not further contribute to the asymptotics of c_n. On the other hand, $X(1; t)$ has a square root singularity at $t = 1/3$. Thus one expects that the functions D_n and C_n and $Q(1, 0)$ also have square-root singularities at $t = 1/3$. Numerical analysis of the first few hundred series terms of $Q(1, 0)$ strongly suggests that this singularity is also a square root singularity.

We have not shown rigorously that $Q(1, 0)$ has a square root singularity, which would in turn prove that $\alpha_{\mathcal{T}} > 0$. However, since we can write $Q(1, 0)/(1 - 3t)$ as the sum $\sum(D_k(C_k - C_{k+1})/(1 - 3t))$, we have studied the asymptotics of each summand. Adding these asymptotic forms together (and ignoring the contributions from the simple pole) gives

$$\alpha_{\mathcal{T}} = 0.097559712851970777240\cdots.$$

The estimate converges quite quickly with increasing k. However, since we have not proved that the asymptotics of the summands converge uniformly in k, the above is not rigorous. That being said, simple linear fitting of $c_n 3^{-n}\sqrt{n}$ against $1/n$ (for $n \le 400$) gives the rough estimate of $\alpha_{\mathcal{T}} \approx 0.0977$ and that more careful analysis of the first four hundred series terms gives $\alpha_{\mathcal{T}} \approx 0.097559$.

4. Conclusions

4.1. How robust is this method at detecting non-holonomy?

This is a natural question, and it speaks to the ability of this approach to be applied to other problems. In all the cases of walks in the quarter plane known to give a holonomic generating function, the group of kernel iterates is finite (of order 8 or less) [14]. If one then applies the iterated kernel method to find an explicit form for the generating functions for the iterates, one obtains a finite set equations, and thus the iteration process is finite. It is not clear that solution of these equations has singularities appearing in the same way as occurred here.

On the other hand, there is a growing list of instances where either the group of kernel iterates, or the Galois group of the kernel is infinite, and the corresponding generating function is not holonomic [5,6].

4.2. Combinatorial intuitions of non-holonomy

More generally, we are interested in developing an intuition for when a combinatorial object will have a non-holonomic generating function. If we decompose the walks considered above according to their NE or N steps, a source of singularities becomes apparent. Unfortunately, since we are unable to prove that the singularities do not cancel, the details we present in this section do not constitute a proof of non-holonomy, however they are nonetheless instructive for understanding a potential source of the singularities.

Any walk from either of these two families can be decomposed into simpler walks by cutting them after each NE or N step. The components are then directed paths in strips of finite height. Further the height of the confining strip increases by one in each successive component. See Fig. 5 for an example of such a decomposition for step set $\mathcal{S}$. Thus, a potential strategy groups lattice walks that end on the line $x + y = k$, and decomposes any such walk into a triple: A walk that ends on the line $x + y = k - 2$, a NE step, and a directed path in a strip of height k.

To describe the generating function, we use Example 11 of [8], a generating function for paths of length n in strip of height k of walks that begin at a given height, and end at a given height.

Define $D_k(y; t)$ as the generating function for the subset of walks ending on $x + y = k$ where y marks the final height of the walk. We can easily translate the above decomposition into a functional equation for the generating function.

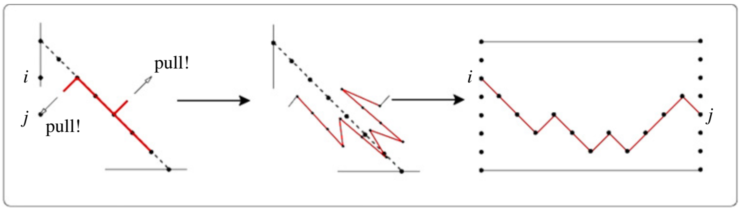


Fig. 5. Stretching the walk to find a directed path in a strip.

If we then make the substitution $t \mapsto \frac{q}{1+q^2}$, the expression simplifies remarkably into the following recurrence for $D_k(y) = D_k(y, \frac{q}{1+q^2})$:

$$D_k(y) = \frac{q^3 D_{k-2}(q)(y^{k+2}+1) - qy^2 D_{k-2}(y)(q^{k+2}+1)}{(q^{k+2}+1)(yq-1)(y-q)}. \tag{45}$$

In fact, for our purposes it suffices to consider:

$$D_k(1) = \frac{q(q^{k+2}+1)D_{k-2}(1) - 2q^3 D_{k-2}(q)}{(q^{k+2}+1)(q-1)^2}. \tag{46}$$

From this formula, and from computations for various values of k, $D_k(1)$ is a rational function in q, and it seems clear that the set (taken over all k) of poles of $D_k(1)$ is dense in the unit circle. Were this so, we would apply the following theorem (from [2]) to the generating function $Q_{\mathcal{S}}(s, s; \frac{q}{1+q^2}) = \sum D_k(1; \frac{q}{1+q^2}) s^k$, and thus conclude the non-holonomy of $Q_{\mathcal{S}}(x, y; t)$.

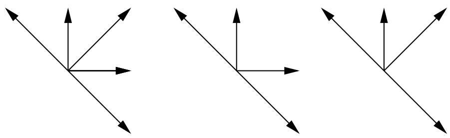
Theorem 17. *Let $f(x; t) = \sum_n c_n(x) t^n$ be a holonomic power series in $\mathbb{C}(x)[[t]]$. with rational coefficients in x. For $n \geq 0$ let S_n be the set of poles of $c_n(y)$, and let $S = \bigcup S_n$. Then S has only a finite number of accumulation points.*

Again, the principal difficulty is showing that the singularities do not cancel; that solutions to $q^{k+2}+1$ are indeed poles of D_k.

This approach was pioneered by Guttmann and Enting [10], and has been fruitful for several different models [9,16]. Unfortunately it is not clear how to apply their arguments successfully to this problem.

4.3. Related walks

We expect walks with steps from the following sets to also have non-holonomic generating functions because the groups of their kernel iterates are infinite. It is seems likely this can be proved in a manner similar to Theorems 1 and 2.



A second family that appears to be non-holonomic is the set of walks restricted to the interior of the wedge in the left half plane bounded by $y = \pm mx$, for rational m, with steps from $\{\mathtt{N}, \mathtt{E}, \mathtt{S}\}$. Remark that when $m = 1$, these are in bijection with the walks in our first case, using Step set $\mathcal{S}$. These satisfy a parametrized recurrence similar to Eq. (45) (see [12]).

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