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ON SOME DECIDABILITY PROBLEMS FOR HD0L SYSTEMS WITH NONSINGULAR PARIKH MATRICES

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Abstract. The problem of decidability of ultimate equivalence, emptiness of intersection and finiteness of intersection for HD0L sequences the generating morphisms of which have nonsingular Parikh matrices is investigated. The first and the third of these decidability problems are shown to be closely connected with the equivalence problem for these sequences. In certain special cases the problems are proved to be solvable. A connection between the problem of effective findability of zeros in Z-rational sequences and the above problems is established (cf. [10]).

1. Introduction

Decidability problems connected with sequences of words generated by deterministic Lindenmayer systems are continuously a popular branch of L systems theory (for earlier results the reader is referred to the relevant chapters of [8]). Čulik II and Friš [4] solved the foremost of all these problems, the D0L equivalence problem. Among other recent positive results are the solvability of the ultimate equivalence problem for D0L systems proved by Čulik II [3] and the solvability of the equivalence problem as well as the ultimate equivalence problem for HD0L systems the eigenvalues of the Parikh matrices of which are of the form $r\varepsilon$ where r is a real number and ε is a root of unity (see [10]).

Here we concentrate ourselves on HD0L systems with nonsingular Parikh matrices. Using certain combinatorial word mappings introduced in [9] we are able to obtain the following results. The equivalence problem and the ultimate equivalence problem for the sequences generated by these systems are one and the same problem. Infinite intersections of these sequences must take place in an ultimately periodic fashion much in the same way as an infinite amount of zeros appear in a Z-rational sequence (cf. [1]). Finally we show that an algorithm for finding zeros of Z-rational sequences can be modified into an algorithm for deciding whether or not the intersection of two HD0L sequences the generating morphisms of which have nonsingular Parikh matrices is finite (or empty). (The existence of the first algorithm is one of the major open problems in the theory of Z-rational sequences.

In [10] it is shown to imply the solvability of the equivalence problem and the ultimate equivalence problem for *all* HD0L sequences.)

Further restriction to D0L sequences with generating morphisms having nonsingular Parikh matrices immediately implies (via the algorithm of Čulik II and Friš [4]) that the finiteness of intersection problem for these sequences is solvable.

It is also shown that the finiteness problem as well as the emptiness problem are solvable for intersections of HD0L sequences the eigenvalues of the Parikh matrices of the generating morphisms of which are of the form $r\varepsilon$ where $r>0$ and ε is a root of unity. Since also the equivalence problem is solvable for these sequences (see [10]), they form (essentially) the widest class of HD0L sequences for which we know all four decision problems (equivalence, ultimate equivalence, emptiness of intersection and finiteness of intersection) to be solvable.

2. Notation and preliminaries

An HD0L *system* is an ordered quintuple $G=(A, B, \delta, \sigma, \omega)$ where A and B are finite alphabets, δ is an endomorphism on A^*, $\sigma: A^* \to B^*$ is a morphism and $\omega \in A^*$. The *sequence generated by* G is

$$E(G)=\{(n, \sigma\delta^n(\omega)) \mid n \geq 0\}$$

which is also denoted by $(\sigma\delta^n(\omega))$. If $A=B$ and σ is identity endomorphism, then G is called a D0L *system* and we write $G=(A, \delta, \omega)$. Two (H)D0L sequences are called *equivalent* if they are the same and *ultimately equivalent* if they differ from each other only in a finite number of terms. For basic facts about (H)D0L systems and sequences the reader is referred to [8].

By $|P|$ we denote the length of the word P and by $[P]$ its Parikh vector. The *Parikh matrix* of a morphism μ on A^* is denoted by $[\mu]$, i.e.

$$[\mu]=\sum_{a \in A}[a]^{\mathrm{T}}[\mu(a)]$$

where T denotes transpose. The cardinality of a set S is denoted by card(S). The set of nonnegative integers is denoted by $\mathbf{N}$. Direct product of matrices is denoted by $\otimes$, i.e.

$$\Gamma \otimes \Delta=\left(\begin{array}{c|c}\Gamma & 0'' \\ \hline 0' & \Delta\end{array}\right)$$

where $0'$ and $0''$ are zero matrices of appropriate sizes.

3. Ultimate equivalence

For results in this section as well as in the following ones we recall some of our earlier results in [9].

Let A be a finite alphabet. Then there is a sequence of mappings $\theta_i : A^* \to \mathbf{N}^{p_i}$ $(i = 0, 1, \ldots)$ such that

(i) $p_i = \text{card}(A) + \cdots + \text{card}(A^{i+1})$;

(ii) $P = Q$ whenever $\theta_i(P) = \theta_i(Q)$ and $2i \geq |P|, |Q|$;

(iii) $\theta_{i+1}(P) = (\theta_i(P) \mid \bar{\theta}_{i+1}(P))$ for some mapping $\bar{\theta}_{i+1} : A^* \to \mathbf{N}^{p_{i+1} - p_i}$;

(iv) $\theta_0(P) = [P]$.

Furthermore, for each morphism $\mu : A^* \to B^*$ where B is another finite alphabet there is a sequence of integer matrices (Γ_i) such that

(v) Γ_i is of size $p_i \times q_i$ where $q_i = \text{card}(B) + \cdots + \text{card}(B^{i+1})$;

(vi) $\tau_i \mu(P) = \theta_i(P)\Gamma_i$ where $\tau_i : B^* \to \mathbf{N}^{q_i}$ $(i = 0, 1, \ldots)$ is the sequence of mappings with domain B^* corresponding to (θ_i) on A^*;

(vii)

$$\Gamma_{i+1} = \left(\begin{array}{c|c} \Gamma_i & \bar{\Gamma}_{i+1} \\ \hline 0' & \Gamma_0^{(i+1)} \end{array} \right)$$

where $\Gamma_0^{(i+1)}$ is the $(i+1)$st Kronecker power (tensor power, Hadamard power) of Γ_0, $\bar{\Gamma}_{i+1}$ is an integer matrix and $0'$ a zero matrix of appropriate size;

(viii) $\Gamma_0 = [\mu]$.

The definitions of (θ_i) (resp. (τ_i)) and (Γ_i) are constructive but rather complicated (for this reason we will not give them here but instead refer to [9]); recently another sequence of mappings satisfying (i)–(viii) has been given in [5, 7].

Theorem 1. *Let* $G_1 = (A_1, B, \delta_1, \sigma_1, \omega_1)$ *and* $G_2 = (A_2, B, \delta_2, \sigma_2, \omega_2)$ *be* HD0L *systems such that* $[\delta_1]$ *and* $[\delta_2]$ *are nonsingular. If* $E(G_1)$ *and* $E(G_2)$ *differ only in a finite number of terms, then* $E(G_1) = E(G_2)$.

Proof. Let $E(G_1) = (\alpha_n)$ and $E(G_2) = (\beta_n)$. Further let (Γ_i), (Δ_i), (Λ_i) and (Ω_i) be the sequences of matrices associated with δ_1, δ_2, σ_1 and σ_2, respectively, as described above. By (vi) we see that

$$\gamma_n^{(i)} = \tau_i(\alpha_n) - \tau_i(\beta_n) = \theta_i^{(1)}(\omega_1)\Gamma_i^n\Lambda_i - \theta_i^{(2)}(\omega_2)\Delta_i^n\Omega_i \quad (n \geq 0)$$

where $(\theta_i^{(1)})$ on A_1^* and $(\theta_i^{(2)})$ on A_2^* correspond to (θ_i) on A^*. By the Cayley–Hamilton Theorem the sequence $(\gamma_n^{(i)})$ is governed by the linear recurrence equation

$$\rho_i(E)\gamma_n^{(i)} = \bar{0} \tag{1}$$

where ρ_i is the characteristic polynomial of $\Gamma_i \otimes \Delta_i$, E is the shift operator given by $E\gamma_n^{(i)} = \gamma_{n+1}^{(i)}$ and $\bar{0}$ is a zero row vector.

By assumption $\Gamma_0 = [\delta_1]$ and $\Delta_0 = [\delta_2]$ are nonsingular. Since raising a matrix to some Kronecker power does not remove its nonsingularity we see, by (vii), that also Γ_i and Δ_i and hence $\Gamma_i \otimes \Delta_i$ are nonsingular. So $\rho_i(0) \neq 0$.

Let now $E(G_1)$ and $E(G_2)$ differ only in a finite number of terms. Then $(\gamma_n^{(i)})$ is ultimately zero for each $i \geq 0$. Applying the recursion given by (1) in a 'backward' fashion and recalling that $\rho_i(0) \neq 0$ we see that $\gamma_n^{(i)} = \bar{0}$ for *all* $i, n \geq 0$. Assume now, contrary to the claim, that $\alpha_m \neq \beta_m$ for some $m \geq 0$. To derive a contradiction, let $2i \geq |\alpha_m|, |\beta_m|$. Since $\tau_i(\alpha_m) = \tau_i(\beta_m)$, it follows from (ii) that $\alpha_m = \beta_m$. So we conclude that $E(G_1) = E(G_2)$.

Thus HD0L sequences the generating endomorphisms of which have nonsingular Parikh matrices cannot differ only in a finite but nonzero number of terms. If they differ from each other this must take place in infinitely many terms. We discuss the case where the sequences, however, have infinitely many common terms in the next section.

Concerning D0L sequences we have the following corollary to Theorem 1.

Corollary 1. *It is decidable of any two given* D0L *systems G_1 and G_2 with nonsingular Parikh matrices whether or not $E(G_1)$ and $E(G_2)$ are ultimately equivalent.*

Proof. By Theorem 1, if $E(G_1)$ and $E(G_2)$ differ only in a finite number of terms they must be equivalent. Thus it suffices to decide whether or not $E(G_1) = E(G_2)$ which can be done by the algorithm of Čulik II and Friš [4].

In order to somewhat sharpen Theorem 1 we note that although some HD0L system $G = (A, B, \delta, \sigma, \omega)$ may not satisfy the conditions of the theorem, i.e. $[\delta]$ is singular, it can be the case that, for some $m \geq 1$, $\delta^m = \varepsilon\kappa\nu$ where $[\nu\varepsilon\kappa]$ is nonsingular. Instead of G we may then consider the HD0L systems

$$G_k = (A', B', \nu\varepsilon\kappa, \sigma\delta^{k-1}\varepsilon\kappa, \nu(\omega)) \quad (k = 1, \ldots, m)$$

where A' and B' are properly chosen. Let us take the following example:

Example. Let $G = (\{a, b, c\}, \delta, b)$ where δ is given by

$$\delta(a) = \lambda, \qquad \delta(b) = ab^2, \qquad \delta(c) = ac$$

(λ is the empty word) whence $[\delta]$ is singular. Let us write $\delta^2 = \varepsilon\kappa\nu$ where ε, κ and ν are given by

$$\varepsilon(b) = \delta(b), \qquad \varepsilon(c) = \delta(c),$$

$$\nu(a) = \lambda, \qquad \nu(b) = b, \qquad \nu(c) = c,$$

$$\kappa = \nu\varepsilon.$$

Then $[\nu\varepsilon\kappa] = [\kappa^2]$ is nonsingular and instead of investigating the D0L system G we investigate the HD0L systems

$$G_1 = (\{b, c\}, \{a, b, c\}, \nu\varepsilon\kappa, \varepsilon\kappa, b)$$

and

$$G_2 = (\{b, c\}, \{a, b, c\}, \nu\varepsilon\kappa, \delta\varepsilon\kappa, b).$$

Starting from two equivalent (H)D0L sequences we can always obtain two ultimately equivalent ones simply by adding some finite amount of 'initial mess' in the beginning of the sequences. Adding initial mess necessarily makes the Parikh matrices of the generating endomorphisms singular since some symbols appear in the sequences only in a finite but nonzero number of terms. All ultimately equivalent pairs of D0L sequences, of course, are not obtained in this fashion, for instance the pair $a, ab, (ab)^2, \ldots$ and $b, ab, (ab)^2, \ldots$.

4. Infinite intersection of HD0L sequences

Let us now see in what way the intersection of two HD0L sequences the generating endomorphisms of which have nonsingular Parikh matrices can be infinite.

Theorem 2. *Let* $G_1 = (A_1, B, \delta_1, \sigma_1, \omega_1)$ *and* $G_2 = (A_2, B, \delta_2, \sigma_2, \omega_2)$ *be* HD0L *systems such that* $[\delta_1]$ *and* $[\delta_2]$ *are nonsingular and let* $E(G_1) = (\alpha_n)$ *and* $E(G_2) = (\beta_n)$ *have an infinite intersection. Then there exists a computable integer* $a \geq 1$ *and numbers* $u \leq a$ *and* $0 \leq b_1, \ldots, b_u < a$ *such that*

$$T = \{n \mid \alpha_n = \beta_n\} = \left(\bigcup_{j=1}^{u} \{as + b_j \mid s \geq 0\}\right) \cup F$$

where F is a finite set.

Proof. Let us use the notation of the proof of Theorem 1. Then we know that, for each $i \geq 0$, the sequence $(\gamma_n^{(i)})$ has infinitely many zero terms. Also each of the sequences $(\gamma_n^{(i)}\eta_r)$, where η_r is a column vector the only nonzero entry of which is the rth one which has the value 1, have infinitely many zero terms. By the Skolem–Mahler–Lech Theorem (see e.g. [6 or 1]), then

$$T_{ir} = \{n \mid \gamma_n^{(i)}\eta_r = 0\} = \bigcup_{j=1}^{u_{ir}} \{a_{jir}s + b_{jir} \mid s \geq 0\}$$

where l.c.m.$\{a_{jir} \neq 0 \mid j = 1, \ldots, u_{ir}\}$ divides v_i, the l.c.m. of the degrees of those primitive roots of unity which can be expressed as quotients of eigenvalues of $\Gamma_i \otimes \Delta_i$. So we may write

$$T_{ir} = \left(\bigcup_{j=1}^{w_{ir}} \{v_i s + c_{jir} \mid s \geq 0\}\right) \cup F_{ir}$$

where F_{ir} is a finite set and further

$$T_i = \bigcap_r T_{ir} = \left(\bigcup_{j=1}^{u_i} \{v_i s + b_{ji} \mid s \geq 0\}\right) \cup F_i$$

where F_i is a finite set.

To get an upper bound for the numbers $v_0, v_1, \ldots$ we take advantage of the theory of field extensions (see e.g. [2]). Let K be the field which we get by adjoining the eigenvalues of $\Gamma_0 \otimes \Delta_0$ into Q, the field of rationals. Then the degree $[K:Q]$ of this extension does not exceed $(\mathrm{card}(A_1)+\mathrm{card}(A_2))!$. By (vii) in Section 3, we see that the eigenvalues of Γ_i (resp. Δ_i) consist of all products $\zeta_1 \ldots \zeta_y$ where $1 \le y \le i+1$ and ζ_j are eigenvalues of Γ_0 (resp. Δ_0). So, all eigenvalues of $\Gamma_i \otimes \Delta_i$ as well as their quotients are in K. It follows that $\varphi(v_i) \le (\mathrm{card}(A_1)+\mathrm{card}(A_2))!$ for each $i \ge 0$ where φ denotes Euler's totient function. This gives us the desired effective bound c on v_i: c can be taken to be the largest number such that $\varphi(c) \le (\mathrm{card}(A_1)+\mathrm{card}(A_2))!$.

We may write further

$$T_i = \left(\bigcup_{j=1}^{w_i} \{c!s + c_{ji} \mid s \ge 0\}\right) \cup F_i.$$

We show then that we may assume that $c_{ji} < c!$ for all j and i. Let $c_{ji} = c!c''_{ji} + c'_{ji}$ where $0 \le c'_{ji} < c!$. The sequence

$$\xi_n^{(i)} = \tau_i(\alpha_{c'_{ji}+nc!}) - \tau_i(\beta_{c'_{ji}+nc!}) \quad (n = 0, 1, \ldots)$$

has only finitely many nonzero terms and is governed by the linear recurrence equation $\pi_i(E)\xi_n^{(i)} = \bar{0}$ where π_i is the characteristic polynomial of $\Gamma_i^{c!} \otimes \Delta_i^{c!}$. Since $[\delta_1] = \Gamma_0$ and $[\delta_2] = \Delta_0$ are nonsingular it follows that also $\Gamma_i^{c!} \otimes \Delta_i^{c!}$ is nonsingular whence $\pi_i(0) \ne 0$ and $\xi_n^{(i)} = \bar{0}$ for all $i, n \ge 0$. We replace c_{ji} by c'_{ji}.

Consider now $T = T_0 \cap T_1 \cap \ldots$. Denote

$$C_j = \{c!s + c_{j0} \mid s \ge 0\} \quad (1 \le j \le w_0).$$

If, for some j, $C_j \not\subseteq T$, then $C_j \not\subseteq T_i$ for some i and hence $C_j \cap T \subseteq F_i$. Thus

$$T = \left(\bigcup_{j=1}^{u} \{c!s + b_j \mid s \ge 0\}\right) \cup F$$

for some $v_j < c!$ and some finite set F.

The theorem follows when we write $a = c!$. Naturally we may take $u \le a$ because otherwise the union is not disjoint and u can be reduced.

Using the Skolem–Mahler–Lech Theorem we have obtained an analogy of the theorem for sequences of words. We have however restricted ourselves to HD0L sequences of a special kind. It would be interesting to know whether the result holds true for HD0L sequences in general. Taking into account also the refinements of the result given by the proof of Theorem 2 we know at least that these cannot be incorporated into the general case. To see this consider the D0L systems $G_1 = (\{a, b\}, \delta_1, a)$ and $G_2 = (\{a, b\}, \delta_2, a)$ where

$$\delta_1(a) = \delta_2(b) = ab, \qquad \delta_1(b) = \delta_2(a) = ba.$$

Thus $[\delta_1]=[\delta_2]$ is singular. Since δ_1 and δ_2 are monomorphisms, $\delta_1^2=\delta_2^2$ and $\delta_1(a)\neq\delta_2(a)$ we see that the result of Theorem 2 holds with $a=2$ but not with $a=1$. However, the eigenvalues of $[\delta_1]=[\delta_2]$ are nonnegative numbers and thus $v_i=1$ in the proof of the theorem.

To be able to compute u and $b_1,\ldots,b_u$ in Theorem 2 one should be able to decide whether or not the sequences $(\sigma_1\delta_1^{j+an}(\omega_1))$ and $(\sigma_2\delta_2^{j+an}(\omega_2))$ are equivalent ($j=0,\ldots,a-1$). This can be done if G_1 and G_2 are D0L systems (by the result of Čulik II and Friš [4]) and also if the eigenvalues of $[\delta_1]\otimes[\delta_2]$ are of the form $r\varepsilon$ where r is a real number and ε is a root of unity (see [10]). So we have the following corollaries.

Corollary 2. *It is decidable of any two given* D0L *systems G_1 and G_2 with nonsingular Parikh matrices whether or not $E(G_1)\cap E(G_2)$ is infinite.*

Corollary 3. *It is decidable of any two given* HD0L *systems G_1 and G_2, such that the eigenvalues of the Parikh matrices of G_1 and G_2 are of the form $r\varepsilon$ where $r>0$ and ε is a root of unity, whether or not $E(G_1)\cap E(G_2)$ is infinite.*

Moreover, results in [10] imply

Corollary 4. *An algorithm for finding the zeros of Z-rational sequences can be modified into an algorithm which of any two given* HD0L *systems G_1 and G_2 with nonsingular Parikh matrices decides whether or not $E(G_1)\cap E(G_2)$ is infinite.*

5. Empty intersection of HD0L sequences

Let $G_1=(A_1,B,\delta_1,\sigma_1,\omega_1)$ and $G_2=(A_2,B,\delta_2,\sigma_2,\omega_2)$ be HD0L systems such that $[\delta_1]$ and $[\delta_2]$ are nonsingular. Using Theorem 2 we may assume that card $(E(G_1)\cap E(G_2))<\infty$ because instead of G_1 and G_2 we may investigate the pairs

$$\begin{aligned} G_1^{(k)} &= (A_1,B,\delta_1^a,\sigma_1,\delta_1^{k-1}(\omega_1)),\\ G_2^{(k)} &= (A_2,B,\delta_2^a,\sigma_2,\delta_2^{k-1}(\omega_2)) \end{aligned}\qquad (k=1,\ldots,a)$$

of HD0L systems. So the following theorem holds.

Theorem 3. *To obtain an algorithm for deciding of any two given* HD0L *systems G_1 and G_2 with nonsingular Parikh matrices whether or not $E(G_1)\cap E(G_2)$ is empty it suffices to do this in the case where we know in advance that $E(G_1)\cap E(G_2)$ is finite.*

Here we have again an analogy to the situation with Z-rational sequences (see [1]).

Recalling the argumentation of the proof of Theorem 2 we see that each of the sequences $(\gamma_n^{(i)})$ either has finitely many zero terms or then is a zero sequence (we assume that the original G_1 and G_2 are replaced by the pairs of HD0L systems given above). If the latter alternative holds true for large i, then by (ii) in Section 3 $E(G_1) \cap E(G_2)$ is nonempty (it suffices to take $2i \geq |\sigma_1(\omega_1)|, |\sigma_2(\omega_2)|$). We may assume that the former alternative holds true for some i and we have

Theorem 4. *An algorithm for finding the zeros of Z-rational sequences gives an algorithm for deciding of any two given* HD0L *systems G_1 and G_2 with nonsingular Parikh matrices whether or not $E(G_1) \cap E(G_2)$ is empty.*

Such an algorithm is known to exist in certain special cases and so we have the following corollary.

Corollary 5. *It is decidable of any two given* HD0L *systems G_1 and G_2, such that the eigenvalues of the Parikh matrices of G_1 and G_2 are of the form $r\varepsilon$, where $r > 0$ and ε is a root of unity, whether or not $E(G_1) \cap E(G_2)$ is empty.*

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