SOLUTIONS TO GENERAL NON-ZERO-SUM GAMES[1]

Donald B. Gillies

The main problem at present in n-person game theory is to determine whether each game ([1] pp. 504-533: General Non-Zero-Sum Games) has at least one solution in the von Neumann-Morgenstern sense. While a considerable amount of work has been done studying special classes of games, no general systematic theory has yet appeared. It is the purpose of this paper to give the beginnings of such a theory.

The principal results are these:

(1) A positive fraction of all games have solutions, and a positive fraction of all non-zero-sum games have unique solutions.

(2) In all but a lower-dimensional set of games, open regions about each vertex of the simplex of imputations are excluded from any solution, because some player would receive too much in such a region. Discriminatory solutions exist for every game in that lower dimensional set. Furthermore, the restricted set of imputations has the desirable property that any set stable for them is a solution. The excluded imputations just don't matter.

(3) In games for which the value is sufficiently small, a set of imputations, called the core, is in every solution. The core is closed and convex.

(4) A regular feature of games is that the relations, "dominates" and "does not dominate," are preserved if certain player sets are disregarded. If $v(S)$ is any set function, there exists a set function $v^*(S)$ which is superadditive and defines the same dominations. $v^*(S) \geq v(S)$ always, and if $v^*(S) > v(S)$ then S is certainly unnecessary.

(5) Games are considered equivalent if there exists a 1-1 domination-preserving mapping of their imputation spaces. If a game is defined by an arbitrary set-function, this game is equivalent to a game

[1] The preparation of this paper was sponsored (in part) by the Office of Naval Research.

defined by a superadditive set-function. Furthermore, two games can have different superadditive set-functions and still be equivalent.

(6) The definitions of a decomposable game and a dummy will be generalized by being treated from the point of view of solutions. A larger class of games will be shown to be decomposable (or, as a special case, to have dummies).

(7) An approach will be sketched, which might lead to a general existence theorem for solutions. An (n+1)-person "pyramid game" is obtained by adjoining one indispensible player. To obtain its old-game value in the pyramid game, a coalition must also contain the extra player; otherwise it is utterly defeated. In a pyramid game, domination is an acyclic relation, so it should be easier to show the existence of solutions. It seems to be true that every solution to the original game is contained in a solution to the pyramid game.

§1. STABLE SETS FOR GENERAL REGIONS

Let E^n be a real n-dimensional Euclidean space with fixed coordinates indexed by $1, 2, \dots, n$. Let $I_n = (1, 2, \dots, n)$ be the full index set. Subsets of I_n are written $S, S_1, \dots, T, T_1, \dots$.

DEFINITION. Subsets of I_n are called <u>coalitions</u>. The number of indices in a set S is written $|S|$. If $|S| = k$, S is called a <u>k-person set</u>, or <u>k-person coalition</u>. A one-person coalition is called a <u>player</u>.

Points in E^n are written $\alpha, \beta, \gamma, \dots$ etc. and are defined once and for all by

$$\begin{aligned}\alpha &= (a_1, a_2, \dots, a_n),\\ \beta &= (b_1, b_2, \dots, b_n),\\ \gamma &= (c_1, c_2, \dots, c_n), \quad \text{etc.}\end{aligned}$$

DEFINITION. For any given α in E^n, the coordinate a_i is interpreted as the <u>payoff to player (i)</u>. The payoff to a coalition is the sum of the payoffs to its several players. The payoff to coalition S will be written $\Sigma_S a_i$ meaning $\Sigma_{i \in S} a_i$.

Let v be a real-valued set-function defined on all sets $S \subseteq I_n$, satisfying $v(\emptyset) = 0$, where $\emptyset$ is the empty set.

DEFINITION. With respect to v, a set S is called <u>effective for</u> α if the payoff of S via α is not greater than $v(S)$, that is if $\Sigma_S a_i \leq v(S)$. S is called <u>strictly effective</u> (for α) if the inequality is strict, and <u>exactly effective</u> in the case of equality. If $\Sigma_S a_i > v(S)$,

S is called <u>ineffective</u> for α.

DEFINITION. The <u>S-dominion of</u> α, written $\mathrm{dom}_S\alpha$, is the empty set if S is ineffective, and if S is effective, $\mathrm{dom}_S\alpha = \{\beta \mid b_i < a_i$ for all i in $S\}$, which we will write $\{\beta \mid b_i < a_i$ for $S\}$. If $\beta \subseteq \mathrm{dom}_S\alpha$, α is said to <u>dominate</u> β <u>via</u> S, written $\alpha \underset{S}{\in} \beta$. The dominion of a point is the union of its S-dominions, and the dominion of a point-set is the union of the dominions of its various points.

THEOREM 1. dom_S and dom are monotone non-decreasing functions of point sets into open point-sets.

PROOF. $\mathrm{dom}_S\alpha$ is either the null set or $\bigcap_S \{\beta \mid b_i < a_i\}$, a finite intersection of open sets. Thus $\mathrm{dom}_S\alpha$ is an open set. If Q is any point set, $\mathrm{dom}_S Q = \bigcup_{\alpha \in Q} \mathrm{dom}_S \alpha$, which is a union of open sets and therefore open. Similarly $\mathrm{dom}\,(Q)$ is open. If R is any point set, $\mathrm{dom}_S(Q \cup R) = \mathrm{dom}_S Q \cup \mathrm{dom}_S R \supseteq \mathrm{dom}_S Q$. Hence dom_S is a monotone non-decreasing function. Similarly dom is a monotone non-decreasing function.

THEOREM 2. $\underset{S}{\in}$ is a partial ordering.

PROOF. (a) If $\alpha \underset{S}{\in} \beta$ then $a_i > b_i$ for S which contradicts $a_i = b_i$ or $b_i > a_i$ for S. Hence at most one of $\alpha \underset{S}{\in} \beta$, $\alpha = \beta$, $\beta \underset{S}{\in} \alpha$ can hold between α and β.

(b) If $\alpha \underset{S}{\in} \beta$, $\beta \underset{S}{\in} \gamma$, then

$$a_i > b_i \quad \text{for } S,$$
$$b_i > c_i \quad \text{for } S,$$

so

$$a_i > c_i \quad \text{for } S.$$

But since S is effective for α, $\alpha \underset{S}{\in} \gamma$.

DEFINITION. If K, P are point sets in E^n, and if $K = P - \mathrm{dom}\, K$, then K is called <u>P-stable</u>. Thus K consists of those points of P which are not in the dominion of K, so

(a) K is contained in P
(b) K and $\mathrm{dom}\, K$ are disjunct
(c) $K \cup \{P \cap \mathrm{dom}\, K\} = P$.

The pair (v, P) may be considered a generalized game, where v is a value without the restriction of superadditivity, P is a set of generalized extended imputations without necessarily the restriction

$a_i \geq v((i))$ for α in P, and K is a solution. Since various imputation spaces have been proposed for defining a solution, and since one deals with irregular subsets of imputations while studying solutions, it is preferable to keep P as general as possible, even though P-stable sets do not always exist.

DEFINITION. The set $P - \text{dom}\, P$ is called the <u>P-core</u> or <u>core</u>, which may be empty.

THEOREM 3. If K is P-stable and $K \subseteq Q \subseteq P$ then

(3.1) K is closed in the relative topology of P

(3.2) K contains the core

(3.3) K is Q-stable.

PROOF. (3.1) dom K is open, and $P \cap \text{dom}\, K$ is open in the relative topology of P. But K and $P \cap \text{dom}\, K$ are disjunct sets whose union is P, so K is closed in the relative topology.

(3.2) $P \supseteq K$ and since dom is a monotone non-decreasing mapping,

$$\text{dom}\, P \supseteq \text{dom}\, K,$$

hence

$$P - \text{dom}\, P \subseteq P - \text{dom}\, K = K\ ,$$

so

$$P - \text{dom}\, P \subseteq K\ .$$

Hence K contains the core.

(3.3) Since $P \supseteq Q \supseteq K$,

$$P - \text{dom}\, K \supseteq Q - \text{dom}\, K \supseteq K - \text{dom}\, K$$

but

$$P - \text{dom}\, K = K \quad \text{since } K \text{ is P-stable,}$$

and

$$K - \text{dom}\, K = K \quad \text{since } K \text{ and dom } K \text{ are disjunct,}$$

so

$$K \supseteq Q - \text{dom}\, K \supseteq K$$

hence

$$K = Q - \text{dom}\, K, \quad \text{and } K \text{ is Q-stable.}$$

We have found a set, possibly empty, called the core, which is in every stable set. The points in the core are those points undominated by any point in P, that is, the set of relative maxima in P with respect to the relation of domination. Hence all points of P which are not in the core have the property that they are dominated by points of P. For some points, such as those in the dominion of the core, the relation of being dominated is so strong that we can guarantee that such a point is in no stable set.

LEMMA 1. For a given α, if there exists a β in P such that $\beta \in\!\!- \alpha$ and if for every $\gamma \in\!\!- \beta$ it follows that $\gamma \in\!\!- \alpha$, then α cannot be in any stable set.

PROOF. If K is any stable set, then β is either in K or in dom K. If β is in K, α is in dom β, so α is in dom K. If β is in dom K, then K contains γ with $\gamma \in\!\!- \beta$. Hence $\gamma \in\!\!- \alpha$, and α is in dom K. Hence α is always in dom K, and never in K.

DEFINITION. If P contains α and β with $\beta \in\!\!- \alpha$, and if, for every γ in P for which $\gamma \in\!\!- \beta$ it follows that $\gamma \in\!\!- \alpha$, then β is said to <u>majorize</u> α, written $\beta \longrightarrow \alpha$.

Lemma 1 can now be re-stated: A sufficient condition for α to be in no P-stable set is the existence of β with $\beta \longrightarrow \alpha$. Note that no assumptions have been made regarding the sets effective for the dominations in question: if $\beta \longrightarrow \alpha$ and $\gamma \in\!\!- \beta$ we admit the possibility that $\gamma \in\!\!- \beta$ via S_1 and $\gamma \in\!\!- \alpha$ via S_2, where $S_1 \neq S_2$. Majorization will be shown to be very important in the study of stable sets. We show that it is a partial ordering with the property that every linearly ordered increasing subset has at least one limit point, and that the relation $\longrightarrow$ is preserved when taking limits. Then it will be shown that if P is closed, for any partial ordering of this type there exists a set of maxima. This permits the operation of simultaneously eliminating all majorized points -- any stable set is contained in the set of maxima, and any set stable over the maxima is stable over P.

LEMMA 2. $\longrightarrow$ is a partial ordering, that is,

(2.1) At most one of $\beta \longrightarrow \alpha$, $\beta = \alpha$, $\alpha \longrightarrow \beta$ can hold, and

(2.2) If $\gamma \longrightarrow \beta$ and $\beta \longrightarrow \alpha$ then $\gamma \longrightarrow \alpha$.

PROOF. (2.1) If $\beta \longrightarrow \alpha$, then since $\beta \in\!\!- \alpha$, it follows $\beta \neq \alpha$. If $\beta \longrightarrow \alpha$ and $\alpha \longrightarrow \beta$ then: $\alpha \in\!\!- \beta$ from $\alpha \longrightarrow \beta$, but since $\alpha \in\!\!- \beta$ and $\beta \longrightarrow \alpha$, then $\alpha \in\!\!- \alpha$, a contradiction. Hence at most one of the three possibilities holds.

(2.2) Since $\gamma \longrightarrow \beta$, any point dominating γ must dominate β, and also γ dominates β. Since $\beta \longrightarrow \alpha$, any point dominating β must dominate α. Hence any point dominating γ, and γ itself, dominates α. Hence $\gamma \longrightarrow \alpha$ which was to be shown.

LEMMA 3. If $\{\alpha^i\}$ is an infinite sequence of points in E^n with at least one limit point, there exists a Cauchy subsequence in which each coordinate is monotone (constant, increasing, or decreasing), which converges to any given limit point of the original sequence.

PROOF. Let α be a given limit point of $\{\alpha^i\}$. Choose a subsequence $\{\beta^i\}$ of $\{\alpha^i\}$ as follows:

$$\beta^1 = \alpha^1$$

$$\beta^{i+1}$$

is a point α^j in the sequence $\{\alpha^i\}$ which occurs after the point β^i, and has the property

$$|b_k^{i+1} - a_k| < 2^{-i} \text{ for all } k, \quad i \geq 1.$$

Since α is a limit point of $\{\alpha^i\}$ this construction can always be made, and it can easily be shown that $\{\beta^i\}$ is a Cauchy subsequence with limit α.

For any coordinate x_j, the sequence of coordinates $\{b_j^i\}$ satisfies at least one of the following conditions:

(3.1) There is an infinity of points $\{\beta^i\}$ for which $b_j^i = a_j$.

(3.2) For every $\epsilon > 0$, there is an infinity of points $\{\beta^i\}$ for which $a_j - \epsilon < b_j^i < a_j$.

(3.3) For every $\epsilon > 0$, there is an infinity of points $\{\beta^i\}$ for which $a_j < b_j^i < a_j + \epsilon$.

Select one of these conditions that holds. If (3.1) is selected, choose a subsequence of $\{\beta^i\}$ satisfying $b_j^i = a_j$ for every point β^i of the subsequence. If (3.2) is selected, choose i_1 to be any index i for which $b_j^{i_1} < a_j$, i_2 any index greater than i_1 for which $b_j^{i_1} < b_j^{i_2} < a_j$. (The existence of such an index is shown by setting $\epsilon = a_j - b_j^{i_1} > 0$.) Continuing in this way an infinite subsequence can be found monotone increasing in the coordinate x_j. Similarly if (3.3) is selected a subsequence can be found monotone decreasing in the coordinate x_j. Since a subsequence of a Cauchy subsequence is itself a Cauchy subsequence with the same limit point, we have constructed a Cauchy subsequence monotone in the coordinate x_j.

Perform this construction first to $\{\beta^i\}$ with $j = 1$, then to the resulting subsequence with $j = 2$, then to the resulting subsequence with $j = 3$, etc. Continuing in this way, after n successive constructions, a Cauchy subsequence is obtained, monotone in all n coordinates, with limit α.

LEMMA 4. The relation $\longrightarrow$ is preserved in the limit, that is, if $\{\alpha^i\}$ is a sequence for which $\alpha^{i+1} \longrightarrow \alpha^i$ holds for every i, and if α is any limit point of this sequence, then $\alpha \longrightarrow \alpha^i$ for every i.

PROOF. From Lemma 3, select $\{\beta^i\}$, a Cauchy subsequence monotone in each coordinate, with limit α. To prove the lemma, it is sufficient to prove the following:

(4.1) If the lemma holds for $\{\beta^i\}$ it holds for $\{\alpha^i\}$.

(4.2) If $\gamma \in\!\!- \alpha$ then $\gamma \in\!\!- \beta^i$.

(4.3) $\alpha \in\!\!- \beta^i$.

(4.1): Since $\longrightarrow$ is a transitive relation, it can be shown by induction that $\alpha^j \longrightarrow \alpha^i$ for every $j > i$, and that $\beta^j \longrightarrow \beta^i$ for every $j > i$. Since $\{\beta^i\}$ is a subsequence, β^{i+1} coincides with some α^j for which $j \geq i + 1 > i$, so $\beta^{i+1} \longrightarrow \alpha^i$ always. But if $\alpha \longrightarrow \beta^{i+1} \longrightarrow \alpha^i$ then $\alpha \longrightarrow \alpha^i$, so it is sufficient to prove the lemma for $\{\beta^i\}$.

(4.2): If $\gamma \in\!\!- \alpha$ via S, then $c_j > a_j$ for S. Since S is a finite set, there exists a positive ϵ so small that

$$c_j > a_j + \epsilon \quad \text{for } S.$$

Since α is the Cauchy limit of $\{\beta^i\}$,

$$a_j + \epsilon > b_j^k \quad \text{for } j \text{ in } S,$$

holds for all sufficiently large k, and in particular for some $k > i$. Hence $c_j > b_j^k$ for S holds for this k, and

$$\gamma \in\!\!- \beta^k \longrightarrow \alpha^i ,$$

so

$$\gamma \in\!\!- \beta^i .$$

(4.3): Let S be the index set for which the coordinates of $\{\beta^i\}$ are monotone increasing functions of i. Since $\beta^{i+1} \longrightarrow \beta^i$,

$\beta^{i+1} \in\!\!- \beta^{i}$ via some set T_i defined for each i.

Since $b_j^{i+1} > b_j^i$ for j in T_i, $T_i \subseteq S$, and S is not empty. Now S contains a finite number of subsets, and each T_i coincides with one of these subsets. Since there is an infinity of T_i, then there is an infinity of T_i equal to some subset T of S. It will be shown that T is effective for α. For any β^i, there exists $j > i$ with $\beta^{j+1} \in\!\!- \beta^j$ via T, so T is strictly effective for β^j. But $b_k^j > b_k^i$ for k in $T \subseteq S$, since the coordinates are monotone increasing in S. Hence $\Sigma_T b_k^j > \Sigma_T b_k^i$, so T is strictly effective for β^i. Hence T is strictly effective for all β^i. Now $\Sigma_T a_k = \lim \Sigma_T b_k^i \leq v(T)$ since $\Sigma_T b_k^i < v(T)$ always. Hence T is effective for α. But $a_k > b_k^i$ for k in T so $\alpha \in\!\!- \beta^i$ via T. This proves (4.3) and completes the proof of the lemma.

LEMMA 5. If $P \supseteq Q \ni \alpha$ and if $\mathcal{R}$ is a partial ordering of P, define $Q(\alpha) = \{\beta | \alpha \mathcal{R} \beta, \beta \in Q\}$. Then Q has a maximum in P if and only if $Q - Q(\alpha)$ has a maximum in P.

PROOF. If γ is a maximum in P, it is sufficient to show that

(5.1) $\gamma \mathcal{R}(Q - \gamma)$ implies $\gamma \mathcal{R}(Q - Q(\alpha) - \gamma)$, and

(5.2) $\gamma \mathcal{R}(Q - Q(\alpha) - \gamma)$ implies $\gamma \mathcal{R}(Q - \gamma)$.

(5.1): By definition $Q \supseteq Q(\alpha)$, so $\gamma \mathcal{R}(Q - \gamma)$ implies a fortiori $\gamma \mathcal{R}(Q - Q(\alpha) - \gamma)$.

(5.2): Since $\alpha \notin Q(\alpha)$, and $\gamma \mathcal{R}(Q - Q(\alpha) - \gamma)$, either $\gamma = \alpha$ or $\gamma \mathcal{R} \alpha$. In either case, since $\mathcal{R}$ is transitive, $\gamma \mathcal{R} Q(\alpha)$, so $\gamma \mathcal{R}(Q - \gamma)$. This completes the proof of the lemma.

LEMMA 6. If every point α of P satisfies $\Sigma a_i \leq M$ for a fixed M, and if Q is a subset of P linearly ordered with respect to $\longrightarrow$, there exists Q^*, a bounded (linearly ordered) subset of Q, such that Q has a maximum in P if and only if Q^* has a maximum in P.

PROOF. Since $\Sigma a_i \leq M$, Q^* is bounded provided $a_i \geq k_i$ for $i = 1, 2, \ldots, n$, and k_i are constants. Suppose first that $a_i \geq v((i))$ for every i, and every α. Set $k_i = v((i))$ and $Q^* = Q$ and the lemma is trivially true. Otherwise choose α, i so $a_i < v((i))$, and define $Q_1 = Q - Q(\alpha)$. By Lemma 5, Q_1 is equivalent to Q as far as maxima are concerned. Suppose, per absurdum, that Q_1 contains β with $b_i < a_i$. Then $\beta \longrightarrow \alpha$ since β is in Q_1 and not in $Q(\alpha)$, and $\alpha \in\!\!- \beta$ via (i). Hence $\alpha \in\!\!- \alpha$, a contradiction. Hence Q_1 contains only points β for which $b_i \geq a_i$.

If Q_1, although bounded in x_i is not bounded in some coordinate x_j, a set Q_2 may be defined in a similar fashion, equivalent to Q_1 and to Q. Continuing in this fashion, a set Q_k is obtained, in at most n steps, bounded in every coordinate, and equivalent to Q. Set $Q^* = Q_k$. Then Q^* satisfies the lemma.

LEMMA 7. If

(7.1) $\mathcal{P}$ is a partial ordering of $P \subset E^n$.

(7.2) The limit α of a Cauchy sequence $\{\alpha^i\}$ satisfying $\alpha^{i+1} \mathcal{P} \alpha^i$ is maximum contained in P.

(7.3) Q is a bounded linearly ordered subset of P,

then Q has a maximum in P.

PROOF. If Q is finite, Q contains a maximum. Suppose then that Q contains an infinity of points but Q contains no maximum.

A point-set L is said to have property A, if, for any point β in Q there is an infinity of points $\{\alpha^i\}$ in $Q \cap L$ with $\alpha^i \mathcal{P} \beta$. Since Q has an infinity of points, is linearly ordered, and has no maximum, Q and any superset of Q have property A. Furthermore, if a set with property A be decomposed into sets V, W, not necessarily disjunct, then at least one of V and W has property A. For suppose not -- then there exist points β^1 and β^2 in Q such that (a) and (b) hold:

(a) Only a finite number of points $\{\alpha^i\}$ in $V \cap Q$ satisfy $\alpha^i \mathcal{P} \beta^1$.

(b) Only a finite number of points $\{\alpha^i\}$ in $W \cap Q$ satisfy $\alpha^i \mathcal{P} \beta^2$.

Since Q is linearly ordered, one of $\beta^1 \mathcal{P} \beta^2$, $\beta^2 \mathcal{P} \beta^1$ holds, say $\beta^1 \mathcal{P} \beta^2$. Since $V \cup W$ has property A, there is an infinity of points $\{\alpha^i\}$ in $(V \cup W) \cap Q$ for which $\alpha^i \mathcal{P} \beta^1$. Since only a finite number of these are in V (from (a)), W must contain an infinity of $\{\alpha^i\}$ with $\alpha^i \mathcal{P} \beta^1$. But $\beta^1 \mathcal{P} \beta^2$ so $\alpha^i \mathcal{P} \beta^2$ for each of these α^i, contradicting (b). Hence at least one of V, W has property A.

By repeated application of this argument, it can be shown that if a set having property A is decomposed into any finite number of subsets, not necessarily disjunct, then at least one of these subsets will have property A.

Since Q is bounded, it can be enclosed in a cube K_0, having property A since K_0 is a superset of Q. Bisect K_0 into 2^n closed subcubes by means of planes parallel to the coordinate axes. One of these, called K_1, will have property A. Similarly construct $K_2, K_3, \ldots$. By construction, K_i has property A for each i, and $\cap K_i$ is a point, called α. For any point β in Q, choose a point β^1 in $K_1 \cap Q$ with $\beta^1 \mathcal{R} \beta$, and in general, choose β^{i+1} in $K_{i+1} \cap Q$ with $\beta^{i+1} \mathcal{R} \beta^i$. Then $\{\beta^i\}$ is a Cauchy sequence with limit α, so $\alpha \mathcal{R} \beta$, for any point β in Q. Hence α is a maximum. This contradicts the assumption that Q has no maximum. This contradiction proves the lemma.

THEOREM 4. If

(4.1) P contains the limit of each Cauchy sequence $\{\alpha^i\}$ for which $\alpha^{i+1} \longrightarrow \alpha^i$ (for example if P is closed), and if

(4.2) there exists a constant M, such that $\Sigma a_i \leq M$ holds for every point α of P,

then the set P* of points of P maximal with respect to $\longrightarrow$, has the property that P*-stability is equivalent to P-stability.

PROOF. By Lemma 1, all stable sets are contained in P*, so <u>P-stability implies P*-stability</u>.

Let Q be any subset of P which is linearly ordered with respect to $\longrightarrow$. By Lemmata 3, 4, 5, 6 a bounded set Q* can be chosen such that Q has a maximum in P if and only if Q* has a maximum in P. By Lemma 7, Q* has a maximum in P. Hence, by Zorn's lemma there exists a set $P^* \subseteq P$ such that:

(a) If α, β are in P*, never $\alpha \longrightarrow \beta$ over P

(b) If α is in P - P*, there exists a β in P* with $\beta \longrightarrow \alpha$.

Let K be P*-stable. Then for any α in P - P*, there is a β in P* with $\beta \longrightarrow \alpha$. If β is in K, dom K contains α. If K contains γ such that $\gamma \in\!\!- \beta$, since $\beta \longrightarrow \alpha$ it follows that $\gamma \longrightarrow \alpha$. Hence P - P* is contained in dom K, and K is P-stable. Hence <u>P*-stability implies P-stability</u>.

COROLLARY 5. If P, P* are defined as in Theorem 4, and $P \supseteq Q \supseteq P^*$, then by (3.3) of Theorem 3, P-stability is equivalent to Q-stability. Hence any relation implying majorization but not necessarily implied by it, may be used to construct a set $Q \supseteq P^*$, to which the theorem applies.

COROLLARY 6. If P, P^* are defined as in Theorem 4, and C is the core, then P^* and dom C are disjunct, and a necessary and sufficient condition for the core to be the unique stable set is $P^* = C$.

PROOF. $C = P - \text{dom } P$, by definition. If α is a point in C, then α is not dominated by any point of P. Hence, if $\alpha \in \beta$, $\alpha \longrightarrow \beta$ too, so β must be in $P - P^*$. Hence P^* and dom C are disjunct. If $P^* \neq C$, there are points of P^* undominated by C, so $P^* = C$ is a necessary condition for C to be the unique stable set. It is also a sufficient condition, since any stable set K satisfies $P^* \supseteq K \supseteq C$.

Since P^* and dom C are disjunct, in passing from P to P^* we have eliminated the effect of the core: the two regions C, $P^* - C$ are independent in the sense that no point in one region can dominate a point in the other. The core is in every stable set, and stable sets differ only in which points of $P^* - C$ they contain.

COROLLARY 7. The relation of majorization may be redefined for points in P^* alone, (a weaker condition), and a set P^{**} constructed such that $P^{**} \subseteq P^*$, and P-stability is equivalent to P^{**}-stability.

PROOF. The existence of $P^{**} \subseteq P$ is guaranteed by Theorem 4, and it is only necessary to verify that P^{**} and $P - P^*$ are disjunct. Suppose not, and let α be a point in both sets. Since α is in $P - P^*$, P^* contains $\beta \longrightarrow \alpha$ where $\longrightarrow$ is defined in terms of P. A fortiori $\beta \longrightarrow \alpha$, when $\longrightarrow$ is defined in terms of P^*. Hence α is not a maximum of P^*, so α is not in P^{**}, the required contradiction.

It is therefore possible, in theory, to construct successively smaller sets P^{**}, P^{***}, ..., each of which contains all P-stable sets, and such that the stability of any set need only be verified for the smallest of the sequence. Examples have been found for which $P^{**} \neq P^*$, but it is not known whether this can happen when P is the usual simplex of imputations.

§2. SPECIAL REGIONS AND EQUIVALENCE OF SET-FUNCTIONS

For a fixed set-function $v(S)$ defined on the subsets of I_n, we have considered a general region $P \subset E^n$, restricted, at most, by the two conditions:

For α in P, $\Sigma a_i \leq M$
P contains the limits of
Cauchy majorization sequences.

(A sufficient condition would be that P is closed in any bounded region.)

To obtain sharper results, we now consider four regions of E^n:

$$\begin{aligned}
E &= \{\alpha \mid \Sigma a_i = M\} \\
\bar{E} &= \{\alpha \mid \Sigma a_i \leq M\} \\
A &= \{\alpha \mid \Sigma a_i = M \text{ and } a_i \geq v((i)) \text{ for all } i\} \\
\bar{A} &= \{\alpha \mid \Sigma a_i \leq M \text{ and } a_i \geq v((i)) \text{ for all } i\}
\end{aligned}$$

Two lemmata will be proved, leading to a theorem that all $\bar{E}$-stable sets ($\bar{A}$-stable sets) are obtained from the E-stable sets (A-stable sets) by adjoining the part of the $\bar{E}$-core ($\bar{A}$-core) lying outside E (or A). This reduces the problem to the study of E and A.

LEMMA 1. If $\bar{P} = \bar{E}$ or $\bar{A}$ has core $\bar{C}$ and $P = E$ or A respectively, and has core C; then $\bar{P}^* - \bar{C} \subseteq P$.

PROOF. It is sufficient to prove that if α is in $\bar{P} - P - \bar{C}$ ($\Sigma a_i < M$ and some point dominates α), then α is majorized.

Since some point dominates α, there is some set S strictly effective for α: $\Sigma_S a_i < v(S)$. Choose $\epsilon > 0$ such that

$$\epsilon \leq \frac{1}{|S|} \operatorname{Min} (M - \Sigma a_i,\ v(S) - \Sigma_S a_i)$$

and define

$$\begin{aligned}
\beta : b_i &= a_i + \epsilon \quad \text{for } i \text{ in } S \\
&= a_i \quad \text{otherwise.}
\end{aligned}$$

Then

$$\Sigma_S b_i = \Sigma_S a_i + |S|\epsilon \leq \Sigma_S a_i + v(S) - \Sigma_S a_i = v(S),$$

so S is effective for β, and $\Sigma b_i = \Sigma a_i + |S|\epsilon \leq \Sigma a_i + M - \Sigma a_i = M$, so β is in $\bar{P}$. Now $b_i > a_i$ for i in S so $\bar{P} \ni \beta \underset{S}{\leftarrow} \alpha$ and $b_i \geq a_i$ for all i, so any point which dominates β also dominates α. Hence $\bar{P} \ni \beta \longrightarrow \alpha$. This proves the lemma.

LEMMA 2. If α is a point of E^n, and β is a point of $\bar{P}$ such that $\beta \underset{S}{\leftarrow} \alpha$, $S \neq I_n$, then a point γ can be found in P, such that $\gamma \underset{S}{\leftarrow} \alpha$, and any point which dominates γ also dominates β. Therefore

$$\underset{S}{\operatorname{dom}} P = \underset{S}{\operatorname{dom}} \bar{P}$$

and

$$\underset{S}{\operatorname{dom}} C = \underset{S}{\operatorname{dom}} \bar{C}$$

and

$$C = \bar{C} \cap P \quad .$$

PROOF. Define γ by

$$\begin{aligned} c_i &= b_i \quad \text{for } i \text{ in } S \\ &= b_i + \epsilon \quad \text{otherwise,} \end{aligned}$$

where

$$\epsilon = \frac{1}{n - |S|}(M - \Sigma b_i) \geq 0 \quad .$$

Then $\Sigma c_i = \Sigma b_i + (n - |S|)\epsilon = M$, so γ is in P. Since $c_i = b_i$ for i in S and $\beta \underset{S}{\leftarrow} \alpha$, $\gamma \underset{S}{\leftarrow} \alpha$. Since $c_i \geq b_i$ always, any point that dominates γ, also dominates β.

(1) If β is a general point of $\bar{P}$ then

$$\underset{S}{\operatorname{dom}} \beta = \underset{S}{\operatorname{dom}} \gamma \quad ,$$

so

$$\underset{S}{\operatorname{dom}} P \supseteq \underset{S}{\operatorname{dom}} \bar{P} \quad .$$

Since $P \subseteq \bar{P}$

$$\underset{S}{\operatorname{dom}} P \subseteq \underset{S}{\operatorname{dom}} \bar{P} \quad .$$

Hence

$$\underset{S}{\text{dom}}\ P = \underset{S}{\text{dom}}\ \bar{P} \quad .$$

(2) Again, if β is a general point of $\bar{C}$, γ is in $\bar{C}$ also, so

$$\underset{S}{\text{dom}}\ \bar{C} = \underset{S}{\text{dom}}\ (\bar{C} \cap P).$$

Since $P \subseteq \bar{P}$,

$$\text{dom}\ P \subseteq \text{dom}\ \bar{P} \quad ,$$

$$P - \text{dom}\ P \supseteq P - \text{dom}\ \bar{P} \quad ,$$

$$C \supseteq \bar{C} \cap P \quad .$$

(3) Suppose α is a point in $C - \bar{C} \cap P$, which is therefore undominated by any point in P, but dominated by a point β in $\bar{P} - P$. Then $\Sigma a_i > \Sigma b_i$ so $\beta \in\!\!-\ \alpha$ via $S \neq I_n$. Then P contains $\gamma \in\!\!-\ \alpha$ via S, a contradiction. Hence

$$C - \bar{C} \cap P$$

is empty, and

$$C = \bar{C} \cap P$$

and

$$\underset{S}{\text{dom}}\ C = \underset{S}{\text{dom}}\ (\bar{C} \cap P) = \underset{S}{\text{dom}}\ \bar{C} \quad .$$

THEOREM 8. If $Q = P - C - \text{dom}\ C$, then K is P-stable if and only if $K - C$ is Q-stable, and K is $\bar{P}$-stable if and only if $K - \bar{C}$ is Q-stable.

PROOF. It is sufficient, by Corollary 6, to show

(8.1) $Q \supseteq P^* - C$

(8.2) $Q \supseteq \bar{P}^* - \bar{C}$.

(8.1) Since $P^* \supset C$ and $P^* \cap \text{dom}\ C = \emptyset$, and $P^* \subseteq P$, then

$$P^* - C = P^* - C - \text{dom}\ C \subseteq P - C - \text{dom}\ C = Q \quad .$$

Hence

$$Q \supseteq P^* - C \quad .$$

(8.2) By Lemma 1, $\bar{P}* - \bar{C} \subseteq P$. Now

$$\begin{aligned}\bar{P}* - \bar{C} &= \bar{P}* - \bar{C} - \mathrm{dom}\ \bar{C} \\ &\subseteq P - \bar{C} - \mathrm{dom}\ \bar{C} \\ &= P - \bar{C} \cap P - [\cup_{S \neq I_n} \mathrm{dom}\ \bar{C}] \cup \underset{I_n}{\mathrm{dom}}\ \bar{C} \\ &= P - C - \mathrm{dom}\ C \cup \underset{I_n}{\mathrm{dom}}\ \bar{C} \\ &= P - C - \mathrm{dom}\ C - \underset{I_n}{\mathrm{dom}}\ \bar{C} \quad .\end{aligned}$$

But if $\alpha \in\!\!- \beta$ via I_n, $\Sigma a_i > \Sigma b_i$, so $M > \Sigma b_i$, and β cannot be in P. Hence

$$P - \underset{I_n}{\mathrm{dom}}\ \bar{C} = P,$$

and

$$\bar{P}* - \bar{C} \subseteq P - C - \mathrm{dom}\ C = Q \quad .$$

We have shown that there is a 1-1 correspondence between A-stable sets and $\bar{A}$-stable sets, and between E-stable sets and $\bar{E}$-stable sets, namely to obtain an A-stable set from an $\bar{A}$-stable set, delete $\bar{C} - C$; to obtain an $\bar{A}$-stable set from an A-stable set, adjoin $\bar{C} - C$; etc. Stability for one implies stability for the other. We may therefore confine ourselves to the sets A, E.

THEOREM 9. The E-core is contained in the A-core, and is bounded.

PROOF. It will be shown first that the E-core is contained in A. Suppose not. Then there exists a point α in E - dom E and an index i such that $a_i < v((i))$. By hypothesis α is dominated by no point of E. Define β by

$$\begin{aligned} b_i &= a_i + (n - 1)\epsilon \\ b_j &= a_j - \epsilon \quad \text{for} \quad j \neq i \end{aligned}$$

where

$$0 < \epsilon \leq \frac{1}{n - 1} (v((i)) - a_i) \quad .$$

Then $\Sigma b_i = \Sigma a_i + (n - 1)\epsilon - (n - 1)\epsilon = \Sigma a_i = M$. Hence β is in E. But

$$a_i < b_i = a_i + (n - 1)\epsilon \leq v((i))$$

so $\beta \overset{\in\!\!-}{(i)} \alpha$, a contradiction. Hence the E-core is contained in A. Thus

$$E - \text{dom } E = A - \text{dom } E$$

but

$$\text{dom } E \supseteq \text{dom } A \quad \text{since} \quad E \supset A$$

so

$$A - \text{dom } E \subseteq A - \text{dom } A \quad .$$

Hence the E-core is contained in the A-core. Since the E-core is contained in A it is bounded.

THEOREM 10. For a given v, there exist n quantities $k_1, k_2, \cdots, k_n$, such that if α satisfies $a_i > k_i$ for some i, then either α is in the core or α is majorized.

PROOF. Define

$$k_i = \max_{S \supseteq i} (v(S) - v(S - (i))) \geq v((i))$$

since $v(\emptyset) = 0$, and suppose α satisfies $a_i > k_i$. Either α is in the core, or else α is dominated by some point, so that there is at least one set strictly effective for α. In the latter case we shall show that α is majorized. Choose $\epsilon > 0$ so small that β, defined by

$$b_i = a_i - (n - 1)\epsilon$$
$$b_j = a_j + \epsilon \quad \text{for} \quad j \neq i$$

has the same strictly effective sets, and $b_i \geq k_i$. Then $a_i > b_i$, and $a_j < b_j$ for $j \neq i$. To show $\beta \longrightarrow \alpha$ it is necessary to show

(10.1) $\beta \in\!\!-\!\!- \alpha$

(10.2) If $\gamma \in\!\!-\!\!- \beta$ then $\gamma \in\!\!-\!\!- \alpha$.

(10.1) Let S be a set strictly effective for β. If $S \not\ni i$, then $b_j > a_j$ for S so $\beta \overset{\in\!\!-}{S} \alpha$. If $S \ni i$, $v(S - (i)) \geq v(S) - k_i \geq \Sigma_S b_j - b_i = \Sigma_{S-(i)} b_j$ so $S - (i)$ is effective for β and $\beta \overset{\in\!\!-}{S-(i)} \alpha$.

(10.2) If $\gamma \in\!\!-\!\!- \beta$ via S then $c_j > b_j$. If $S \not\supset i$, then $b_j > a_j$ for S so $c_j > a_j$ for S and $\gamma \overset{\in\!\!-}{S} \alpha$. If $S \supset i$, $v(S - (i)) \geq v(S) - k_i \geq \Sigma_S c_j - c_i = \Sigma_{S-(i)} c_j$ since $c_i > b_i \geq k_i$, so $S - (i)$ is effective for γ. Now $c_j > b_j > a_j$ for $S - (i)$ so $\gamma \overset{\in\!\!-}{S-(i)} \alpha$.

COROLLARY 11. E-stable sets are bounded.

PROOF. If $R = \{\alpha \mid a_i \leq k_i \text{ for all } i\}$ where the quantities k_i are defined as in the proof of Theorem 10, then the core is contained in A, and any point outside of the core but in some stable set must be contained in R. Hence $R \cup A$ contains every stable set. Since both A and R are bounded, every stable set is bounded.

THEOREM 12.

(12.1). A necessary and sufficient condition for an A-stable set to be E-stable is that it touch each face of the simplex A, that is, for each j in I_n, the stable set contains a point $\beta^{(j)}$ such that $b_j^{(j)} = v((j))$.

(12.2). A necessary and sufficient condition for and E-stable set to be A-stable is that it be contained in A.

PROOF. (12.1). <u>Sufficiency</u>. Suppose K is A-stable and satisfies (12.1). Define

$$Q_j = \{\alpha \mid \Sigma a_i = M \text{ and } a_j < v((j))\}$$

$$\begin{aligned} E &= \{\alpha \mid \Sigma a_i = M\} \\ &= \{\alpha \mid \Sigma a_i = M \text{ and } a_j \geq v((j)) \text{ for all } j\} \cup \textstyle\bigcup_{j \in I_n} \{\alpha \mid \Sigma a_i = M \text{ and } a_j < v((j))\} \\ &= A \cup \textstyle\bigcup_{I_n} Q_j \quad . \end{aligned}$$

For every j, K contains $\beta^{(j)}$ such that

$$b_j^{(j)} = v((j)) \quad .$$

But if α satisfies

$$a_j < v((j))$$

then

$$v((j)) = b_j^{(j)} > a_j$$

so

$$\beta^{(j)} \underset{(j)}{\succ} \alpha \quad .$$

Hence

$$\underset{(j)}{\mathrm{dom}}\ \beta^{(j)} \supseteq Q_j$$

so

$$\mathrm{dom}\ K \supseteq \cup_{I_n} Q_j \quad .$$

Now

$$K = A - \mathrm{dom}\ K$$

so

$$K = A \cup_{I_n} Q_j - \mathrm{dom}\ k$$

$$= E - \mathrm{dom}\ K,$$

so K is E-stable.

Necessity. Suppose K is A-stable, but there exists some index j such that, for every β in K, $b_j > v((j))$. Since K is a bounded set, a quantity N can be chosen so large that, if $\alpha = \alpha(N)$ is defined by

$$a_i = \frac{M}{n} + N \quad \text{if} \quad i \neq j$$

$$a_j = \frac{M}{n} - (n - 1)N \quad ,$$

then for each β in K,

$$a_i > b_i \quad \text{for} \quad i \neq j$$

and

$$a_j < v((j)) < b_j \quad .$$

Suppose K is E-stable. Then K must contain some point β such that $\beta \in\!\!- A$ via some set S. Hence $b_i > a_i$ for S and S is effective for β. But $b_j > a_j$ only, so $S = (j)$, and, by hypothesis, (j) is not effective for β. Hence K is not E-stable. This proves that the condition was necessary.

(12.2). If $K = E - \mathrm{dom}\ K = A - \mathrm{dom}\ K$, then $K \subseteq A$, so the condition is necessary. If $K \subseteq A$ and $K = E - \mathrm{dom}\ K$, then $K \cap A = E \cap A - \mathrm{dom}\ K$ so $K = A - \mathrm{dom}\ K$. This proves the sufficiency of $K \subseteq A$.

THEOREM 13. A necessary condition for the core to be A-stable or E-stable is that it touches each face of A.

PROOF. By Theorem 12, it is sufficient to show that if $a_i > v((i))$ for some i and every α in C, then C is not A-stable. Suppose, therefore, that C is A-stable, so C is closed. Define

$$\bar{a}_i = \inf_{\alpha \in C} (a_i) > v((i)).$$

Then $\bar{a}_i$ is attained by some point $\bar{\alpha}$ in C. So $a_i \geq \bar{a}_i$ for every α in C, but for $\bar{\alpha}$ this is an equality. Define

$$\beta : b_i = v((i))$$

$$b_j = \bar{a}_j + \frac{\bar{a}_i - v((i))}{n - 1} \quad \text{for } j \neq i \quad .$$

Then β is in A, $b_j > \bar{a}_j$ for $j \neq i$ and $b_i < \bar{a}_i$. Since $b_i < \bar{a}_i$, β is not in C. Hence, by hypothesis, β must be in dom C. But if $\bar{\alpha} \in\!\!- \beta$, this domination must be via (i) which is impossible since $\bar{a}_i > v((i))$. Hence C contains $\gamma \neq \bar{\alpha}$ such that $\gamma \in\!\!- \beta$.

If $\gamma \in\!\!- \beta$ via S then S contains i and $c_i = \bar{a}_i$, for if

$$S \not\ni i,\ c_j > b_j > \bar{a}_j \quad \text{for } S \text{ so } \gamma \in\!\!- \bar{\alpha}$$

and if

$$c_i > \bar{a}_i,\ c_j > \bar{a}_j \quad \text{for } S \text{ so } \gamma \in\!\!- \bar{\alpha}$$

since it is not true that $c_i > b_i > \bar{a}_i$. Now

$$v(S) \geq \Sigma_S c_j = \bar{a}_i + \Sigma_{S-(i)} c_j > \bar{a}_i + \Sigma_{S-(i)} b_j > \bar{a}_i + \Sigma_{S-(i)} \bar{a}_j$$

so

$$v(S) \geq \Sigma_S c_j > \Sigma_S \bar{a}_j \quad .$$

Since γ, $\bar{\alpha}$ are in A, $S \neq I_n$ so $I_n - S$ is not empty and

$$\Sigma_{I_n - S} \bar{a}_j > \Sigma_{I_n - S} c_j \quad .$$

We can therefore choose a point δ such that

$$d_j = c_j \quad \text{for } j \text{ in } I_n - S$$

$$d_j = \frac{\bar{a}_j + c_j}{2} \text{ for } j \text{ in } S - (i)$$

and

$$d_i = \bar{a}_i + \Sigma_{S-(i)} \frac{c_j - \bar{a}_j}{2} .$$

Now $\Sigma d_j = \Sigma c_j$ so δ is in E. Since

$$c_j > \bar{a}_j \quad \text{for} \quad j \text{ in } S - (i) ,$$

$$d_j > \bar{a}_j \quad \text{for} \quad j \text{ in } S - (i)$$

and

$$d_i > \bar{a}_i .$$

Hence δ is in A and $d_j > \bar{a}_j$ for S. But

$$\Sigma_S d_i = \Sigma_S c_j \leq v(S)$$

so S is effective for δ. Hence $\delta \succ \bar{\alpha}$, contradicting the original assumption that $\bar{\alpha}$ was in C. Hence C is not A-stable if it does not touch each face of A.

§3. EQUIVALENCE OF GAMES

In Section 1, an arbitrary real set-function v, satisfying only $v(\emptyset) = 0$, was used to define domination and stability. On the basis of this, two sets, the core C and the set of unmajorized points P* were defined, having the properties that if K is P-stable, then

$$C \subseteq K \subseteq P^* \quad \text{and also}$$

P*-stability implies P-stability.

In Section 2, the region P was limited to one of four possible regions, and the stability problem was reduced to the stability problem for two of these regions: A and E. General theorems were obtained without the usual restrictions: $M = v(I_n)$, or v is superadditive. These restrictions appear to have originated in the economic interpretation of games, and imposing them does not seem to simplify the search for stable sets. Indeed, the most convenient way to approach the solutions to decomposable games as well as game equivalence is to reject these conditions.

For simplicity we shall now confine ourselves to A-stability, although, a parallel theory could be constructed for E-stability.

DEFINITION. A game is a real valued set function v, satisfying

$v(\emptyset) = 0$, defined on the subsets of I_n, together with a constant M from which A is defined. A solution to (M, v) is an A-stable set. A point in A is called an imputation.

DEFINITION. Two games (M, v) and (M', v') are called equivalent if there exists a $1 - 1$ dominion-preserving transformation between their imputation-spaces A and A'.

If T is such a transformation, it is sufficient that T preserve dominions of single imputations, since the dominion of a set is the union of the dominions of its members. Since T carries complements of dominions into complements of dominions, it carries solutions into solutions. The relation of game equivalence is reflexive, symmetric and transitive, so it is an equivalence relation, and if $A = A'$, the class of dominion-preserving transformations forms a group. T need not carry S-dominions into S-dominions, and there are important cases when it does not.

THEOREM 14. (M, v) is equivalent to a game in which

(14.1) Players are re-numbered (a player permutation) or

(14.2) $v'(S) = \lambda v(S)$ for λ some positive constant, and $a_i' = \lambda a_i$ for $\alpha \sim \alpha'$, $M' = \lambda M$ or

(14.3) $v'(S) = v(S) + \Sigma_S \ell_i$ where $\ell_1, \ell_2, \ldots, \ell_n$ are fixed constants, $a_i' = a_i + \ell_i$ for $\alpha \sim \alpha'$, $M' = M + \Sigma \ell_i$

or a combination of these transformations.

Since this theorem is a trivial extension of the results in Theory of Games [1], the proof will be omitted.

DEFINITION. A game (M, v) is said to be in $0 - 1$ reduced form if $v((i)) = 0$ for $i = 1, 2, \ldots, n$ and $M =$ either $1, 0$ or -1.

A reduced form (M', v') of any game (M, v) is obtained by setting

$$M' = \frac{M - \Sigma_{I_n} v((i))}{|M - \Sigma_{I_n} v((i))|} \quad \text{if this is not } \frac{0}{0}$$

and

$$M' = 0 \quad \text{if} \quad M - \Sigma_{I_n} v((i)) = 0 \ ,$$

and setting

$$v'(S) = v(S) - \Sigma_S v((i)) \quad \text{if} \quad M' = 0,$$

$$= \frac{v(S) - \Sigma_S v((i))}{|M - \Sigma_{I_n} v((i))|} \quad \text{if} \quad M' \neq 0 .$$

It may be readily verified that (M', v') is equivalent to (M, v), that $v'((i)) = 0$ for $i = 1, 2, \ldots, n$, and that $M' = 1, 0$ or -1. If $M' = 0$, A consists of the point $(0, 0, \ldots, 0)$ which is the unique solution. In this case, (M, v) is called an inessential game. If $M' = -1$, A is empty so there is no problem. There remains only the case $M' = 1$.

THEOREM 15. Any game is equivalent to a game with superadditive set function.

PROOF. It is sufficient to consider (M, v) already in $0 - 1$ reduced form. If $M = -1$ or 0, there are, respectively, no imputations or one imputation in A. In neither case does A contain a point dominated by another point, so (M, v) is equivalent to (M, v') for any set function v' satisfying $v'((i)) = 0$ for $i = 1, 2, \ldots, n$. If we define v' by $v'(S) = 0$ for every S, then v' is superadditive, and (M, v') is equivalent to (M, v).

There remains only the case $M = 1$. For a set $S \subseteq I_n$, let $\mathcal{S}$ be the class of partitions of S into non-empty subsets:

$$\mathcal{S} = \left\{(S_1, S_2, \ldots,) \mid S_i \neq \emptyset, \quad S_i \cap S_j = \emptyset, \quad \cup S_i = S\right\} .$$

Define

$$v'(S) = \operatorname*{Max}_{\mathcal{S}} \Sigma v(S_i).$$

Since $\mathcal{S}$ contains the partition $S_1 = S$, $v'(S) \geq v(S)$. We show first that v' is superadditive, that is, if S and T are disjunct,

$$v'(S) + v'(T) \leq v'(S \cup T) .$$

Let $\mathcal{T}$ be the class of partitions of T, and $\mathcal{U}$ the class of partitions of $S + T$. Denote by $\mathcal{S} \times \mathcal{T}$ the topological product of $\mathcal{S}$ and $\mathcal{T}$, that is, the class of partitions of $\mathcal{U}$ comprising every combination of one partition of S and one partition of T. Then $\mathcal{U} \supseteq \mathcal{S} \times \mathcal{T}$, and

$$\begin{aligned} v'(S \cup T) &= \operatorname*{Max}_{\mathcal{U}} \Sigma v(S_i) \\ &\geq \operatorname*{Max}_{\mathcal{S}\times\mathcal{T}} \Sigma v(S_i) \\ &= \operatorname*{Max}_{\mathcal{S}} \Sigma v(S_i) + \operatorname*{max}_{\mathcal{T}} \Sigma v(S_i) \\ &= v'(S) + v'(T) . \end{aligned}$$

Hence v' is superadditive.

To show that (M, v) is equivalent to (M, v'), given that $v'(S) \geq v(S)$, it is sufficient to show that $\alpha \in\!\!- \beta$ via S for v' implies $\alpha \in\!\!- \beta$ for v. Now $a_i > b_i$ for S, and

$$\begin{aligned} \Sigma_S a_i &\leq v'(S) \\ &= \operatorname*{Max}_{\mathcal{S}} \Sigma v(S_i) . \end{aligned}$$

Suppose that this maximum is attained by the partition

$$(S_1, S_2, \ldots, S_P), \quad P \geq 1 .$$

Then

$$v'(S) = v(S_1) + v(S_2) + \ldots + v(S_P) .$$

Then one of $S_1, S_2, \ldots, S_P$ is effective for α, for v. Suppose not. Then

$$\begin{aligned} &\Sigma_{S_1} a_i > v(S_1) \\ &\vdots \\ &\Sigma_{S_P} a_i > v(S_P) \end{aligned}$$

so

$$\Sigma_{S_1} a_i + \ldots + \Sigma_{S_P} a_i > v(S_1) + \ldots + v(S_P) = v'(S)$$

so

$$\Sigma_S a_i > v'(S),$$

a contradiction. Hence some S_j is effective for α, for v, and $a_i > b_i$ for S_j so $\alpha \in\!\!- \beta$ via S_j, for v. This completes the proof of the theorem.

The preceding proof illustrates the fact that if a subset of S is effective, it does not matter whether S is effective or not - any domination via S can be achieved via the effective subset instead. For such sets, the value of the set-function can be altered, provided it is not made too large, for S is never a minimal winning set. For A, the value $v(I_n)$ plays no role, since domination in A is never via I_n.

The quantity $v(I_n) - M$ is analogous to the excess ([1], p. 364), and, as will be shown, if M is allowed to vary, the critical question is when A-stability and $\bar{A}$-stability coincide. This happens provided $\bar{C} = C$ which means that either C is empty or $v(I_n) \geq M$ so $\bar{C} - C$ is empty.

DEFINITION. A set S is called <u>vital</u> for (M, v) if there exists imputations α, β in A for which $\alpha \underset{S}{\in\!\!-} \beta$, and this domination is not achieved via any subset of S.

EXAMPLE. In the $0 - 1$ reduced form, if $v(12) = v(23) = v(31) = \frac{1}{2}$ and $v(123) \leq \frac{3}{4}$ then (123) is not vital, since α in A implies $a_1 \geq 0$, $a_2 \geq 0$, $a_3 \geq 0$, and therefore if $a_1 + a_2 + a_3 \leq 3/4$, one of: $a_1 + a_2 \leq \frac{1}{2}$, $a_2 + a_3 \leq \frac{1}{2}$, $a_3 + a_1 \leq \frac{1}{2}$ must hold. It is evident that for a set to be vital, it is not sufficient that its value merely exceed the sum of the values of the sets in any non-trivial partition.

THEOREM 16. Domination can always be achieved via a vital set.

PROOF. Suppose $\alpha \in\!\!- \beta$ via S, and S is not vital. Then S properly contains a subset S_1 such that $\alpha \in\!\!- \beta$ via S_1. If S_1 is vital, we are done. Otherwise S_1 properly contains a subset S_2 such that $\alpha \in\!\!- \beta$ via S_2. Continuing in this way, after at most a finite number of steps, a set S_k is obtained, such that $|S_k| \geq 2$ (since <u>players</u> are not effective for any A-domination, $|S_k| \neq 1$), $\alpha \underset{S_k}{\in\!\!-} \beta$, and via no subset of S_k does $\alpha \in\!\!- \beta$. Hence S_k is vital.

COROLLARY 17. If (M, v) is in the $0 - 1$ reduced form, and v' is defined by

$$v'(S) = 0 \text{ if } S \text{ is not vital,}$$
$$v'(S) = \text{Min } (v(S), M) \text{ if } S \text{ is vital,}$$

then (M, v) is equivalent to (M, v').

PROOF. If, for v, $\alpha \in\!\!- \beta$ then this domination can be achieved via some vital set S. Hence $\Sigma_S a_i \leq v(S)$. But since $a_i \geq v((i)) = 0$ for all i,

$$\Sigma_{I_n - S} a_i \geq 0$$

Now

$$M = \Sigma a_i = \Sigma_{I_n - S} a_i + \Sigma_S a_i \geq \Sigma_S a_i .$$

Hence

$$\Sigma_S a_i \leq M$$

and

$$\Sigma_S a_i \leq v(S)$$

so

$$\Sigma_S a_i \leq \min\ (v(S),\ M) = v'(S) ,$$

so S is effective for α when v' is the set function. Hence $\alpha \in\!\!- \beta$ via S for v'. Similarly, if $\alpha \in\!\!- \beta$ via S for v',

$$\Sigma_S a_i \leq \min\ (v(S),\ M)$$

so, a fortiori,

$$\Sigma_S a_i \leq v(S)$$

and $\alpha \in\!\!- \beta$ via S for v. This proves the corollary.

(M, v') as defined in the corollary will be called the <u>normal form</u>. The set function v is adjusted so that non-vital sets have value 0 and are effective nowhere, and vital sets have their original value, provided that this value does not exceed M. If the value exceeds M, it is replaced by M, since such sets are still effective everywhere.

A set S is vital if and only if the equalities and inequalities

$$a_i \geq 0, \ \Sigma a_i = 1, \ \Sigma_S a_i \leq v(S), \ \Sigma_T a_i > v(T) \quad \text{for each} \quad T \subset S$$

have at least one solution, so the enumeration of vital sets is a linear programming problem.

THEOREM 18. The core is the intersection with A of closed half-spaces of the form

$$\{\alpha \mid \Sigma_S a_i \geq v(S) \quad \text{where} \quad S \quad \text{is vital}\},$$

where v is the normalized value.

PROOF. Define

$$H_S = \{\alpha \mid \Sigma_S a_i \geq v(S)\} .$$

It is required to prove

$$C = A \bigcap_{\substack{\text{vital} \\ \text{sets} \\ S}} H_S .$$

Suppose first that α is a point in C and that there exists a vital set S such that α is not in H_S. Then

$$M = 1 \geq v(S) > \Sigma_S a_i$$

$$1 = \Sigma_{I_n} a_i$$

so

$$\Sigma_{I_n - S} a_i > 0 .$$

Choose $\epsilon : 1 > \epsilon > 0$ so small that

$$\Sigma_S a_i + \epsilon \Sigma_{I_n - S} a_i \leq v(S)$$

and define β by

$$b_i = (1 - \epsilon) a_i \quad \text{for } i \text{ in } I_n - S$$

$$= a_i + \frac{\epsilon}{|S|} \Sigma_{I_n - S} a_j \quad \text{for } i \text{ in } S.$$

Now β is in A, since

$$\Sigma b_i = \Sigma_S a_i + \epsilon \Sigma_{I_n - S} a_i + (1 - \epsilon) \Sigma_{I_n - S} a_i = \Sigma a_i = 1 ,$$

and $a_i \geq 0$ for i in $I_n - S$ implies $b_i \geq 0$ for i in $I_n - S$, and $b_i > a_i$ for i in S. But

$$\Sigma_S b_i = \Sigma_S a_i + \epsilon \Sigma_{I_n - S} a_i \leq v(S),$$

so S is effective for β. Hence $\beta \succ \alpha$, a contradiction, since α was supposed to be contained in the core. Hence $C \subseteq A \cap H_S$, where the intersection is taken over vital sets.

Suppose next that α is a point in $A \cap H_S - C$. Hence A contains $\beta \succ \alpha$, and this domination can be achieved via some vital set S. Now $\Sigma_S b_i \leq v(S)$, and $a_i < b_i$ for S, so $\Sigma_S a_i < \Sigma_S b_i \leq v(S)$, and α is not in H_S. This contradicts the assumption that α is in $A \cap H_S$.

This contradiction proves the theorem.

COROLLARY 19. The core is closed and convex; its boundary consists of points having at least one exactly effective vital set or player; and its interior consists of points having no vital set effective.

PROOF. H_S and A are closed and convex, so C is a finite intersection of closed convex sets. Therefore C is closed and convex. The boundary of $\cap H_S$ consists of points

$$\{\alpha \mid \Sigma_S a_i \geq v(S) \text{ for every vital set, } \Sigma_S a_i = v(S) \text{ for some vital set}\},$$

so the boundary of $\cap H_S$ consists of points having at least one exactly effective vital set. The boundary of A consists of points

$$\{\alpha \mid \Sigma a_i = 1,\ a_i \geq 0, \text{ and } a_j = 0 \text{ for some } j\}$$

so it consists of points having at least one player exactly effective. The boundary of $A \cap H_S$ consists of boundary points either of A or of $\cap H_S$, so it consists of points having at least one exactly effective vital set or player. Similarly it may be shown that the interior of C consists of points having no vital set effective.

DEFINITION. A game (M, v) in normalized form is called <u>semi-simple</u> if there exists some vital set S, having no vital subsets, satisfying $v(S) = 1 = M$.

THEOREM 20. Every semi-simple game has a solution.

PROOF. Let (M, v) be semi-simple and let S be a minimal vital set satisfying $v(S) = 1$. Since S is vital, S is neither I_n nor a player. Define

$$V = \{\alpha \mid \Sigma_S a_i = 1, \text{ and } \alpha \text{ is in } A\}.$$

It will be shown that V is a solution, that is:

(20.1) $$\operatorname{dom} V \supseteq A - V$$

(20.2) $$V \cap \operatorname{dom} V = \emptyset\ .$$

(20.1): For β in A, $\Sigma b_i = 1$, so

$$A - V = \{\beta \mid b_j > 0 \text{ for some } j \text{ in } I_n - S,\ \beta \text{ in } A\}.$$

For any β in $A - V$, define

$$a_i = 0 \text{ for } i \text{ in } I_n - S$$

$$a_i = b_i + \frac{1}{|S|} \Sigma_{I_n - S} b_j \text{ for } i \text{ in } S.$$

It may be verified that

$$a_i \geq 0 \text{ always}$$

$$\Sigma a_i = 1$$

$$\Sigma_S a_i = 1 = v(S)$$

so α is in V, and S is effective for α. But since

$$b_i \geq 0 \text{ always}$$

and

$$b_j > 0 \text{ for some } j \text{ in } I_n - S$$

therefore

$$\frac{1}{|S|} \Sigma_{I_n - S} b_i > 0$$

so

$$a_i > b_i \text{ for } S .$$

Hence $\alpha \underset{S}{\in\!\!-} \beta$ and $\operatorname{dom} V \supseteq A - V$.

(20.2): Suppose $V \cap \operatorname{dom} V \neq \emptyset$, that is, there exist α, β in V such that $\alpha \in\!\!- \beta$ via some set T. So

$$a_i > b_i \text{ for } T$$

but

$$a_i = b_i = 0 \text{ for } I_n - S$$

so

$$T \subseteq S .$$

Since T may be chosen to be a vital set, and S has no proper subsets, $T = S$. Hence $\Sigma_S a_i > \Sigma_S b_i$ or $1 > 1$, a contradiction. Hence $V \cap \operatorname{dom} V = \emptyset$, and V is a solution.

For any point α in A, define two classes $\mathcal{S}$, $\mathcal{S}_1$, of vital sets effective for α as follows:

If S is in $\mathcal{S}$, and $T \subset S$, never $v(T) - \Sigma_T a_i \geq v(S) - \Sigma_S a_i$

If S is in $\mathcal{S}_1$, and $T \subset S$, never $v(T) - \Sigma_T a_i > v(S) - \Sigma_S a_i$.

Then $\mathcal{S}_1 \supseteq \mathcal{S}$. Define

$$R = \cup_{S\in\mathcal{S}} S, \quad R_1 = \cup_{S\in\mathcal{S}_1} S, \quad \text{so} \quad R_1 \supseteq R.$$

THEOREM 21.

(21.1) If R is not empty, and β is such that $b_i < a_i$ for R, then $\alpha \longrightarrow \beta$.

(21.2) If R_1 is not empty, and there exists an i in $I_n - R_1$ such that $a_i > 0$, then α is majorized unless α is in the core.

PROOF. (21.1): Since R is not empty, $\mathcal{S}$ contains some effective vital set S, and $a_i > b_i$ for $S \subseteq R$. Hence $\alpha \leftarrow\!\!\!\in \beta$ via S. Suppose next that $\gamma \leftarrow\!\!\!\in \alpha$ via some set S not in $\mathcal{S}$. Then, by the definition of $\mathcal{S}$, there is a $T \subset S$ such that

$$(21.1a) \qquad \begin{aligned} v(T) - \Sigma_T a_i &\geq v(S) - \Sigma_S a_i \\ v(T) &\geq v(S) - \Sigma_S a_i + \Sigma_T a_i. \end{aligned}$$

We show that this set T can be chosen to be an element of $\mathcal{S}$. First, this set T is effective for γ (shown in the present proof) so $\gamma \underset{T}{\leftarrow\!\!\!\in} \alpha$. Hence, T can be chosen to be vital. Now if $T \notin \mathcal{S}$, we can again choose a subset, by the definition of $\mathcal{S}$. This process will eventually terminate, yielding a $T \in \mathcal{S}$, since each time a genuinely smaller subset is chosen, and since $|T| = 2$ trivially belongs to $\mathcal{S}$, since its only subsets are players. Since $\gamma \leftarrow\!\!\!\in \alpha$ via S, $c_i > a_i$ for S and

$$\Sigma_S c_i \leq v(S) .$$

Substituting this in (21.1a), we have

$$\begin{aligned} v(T) &\geq \Sigma_S c_i - \Sigma_S a_i + \Sigma_T a_i \\ &= \Sigma_T c_i + \Sigma_{S-T}(c_i - a_i) \\ &> \Sigma_T c_i \quad \text{since} \quad c_i > a_i \quad \text{for} \quad S - T . \end{aligned}$$

Hence T is effective for γ and $\gamma \leftarrow\!\!\!\in \alpha$ via $T \subset R$. Hence $c_i > a_i > b_i$ for T and $\gamma \leftarrow\!\!\!\in \beta$ via T. Hence $\alpha \longrightarrow \beta$.

(21.2): define

$$\begin{aligned} \beta : b_j &= a_j + \epsilon \quad \text{for } R_1 \\ &= a_j \quad \text{for } I_n - R_1 - (i) \\ b_j &= a_1 - |R_1| \epsilon \end{aligned}$$

where ϵ is chosen so small that

(1) $\Sigma_S a_j < v(S)$ implies $\Sigma_S b_j < v(S)$
(2) $\Sigma_S a_j > v(S)$ implies $\Sigma_S b_j > v(S)$
(3) for $S \supset T$ and T in $\mathcal{S}$
$v(T) - \Sigma_T a_j > v(S) - \Sigma_S a_j$ implies
$v(T) - \Sigma_T b_j > v(S) - \Sigma_S b_j$.

There is a finite number of inequalities, each satisfied by any sufficiently small positive ϵ, so there exists $\epsilon > 0$ satisfying them all. For β, define $\mathcal{S}(\beta)$ by analogy with $\mathcal{S}$.

Since α is not in the core, there is some set strictly effective for α in R_1 and therefore strictly effective for β, so $\mathcal{S}(\beta)$ is not empty. If $\mathcal{S}(\beta) \subseteq \mathcal{S}_1$ then $R(\beta) \subseteq R_1$ and $b_i > a_i$ for R_1 so $\beta \longrightarrow \alpha$. It is therefore sufficient to show that $\mathcal{S}(\beta) \subseteq \mathcal{S}_1$. Suppose S is a set in $\mathcal{S}(\beta)$ but not in $\mathcal{S}_1$. Then since $\Sigma_S a_j > v(S)$ implies $\Sigma_S b_j > v(S)$, S must be effective for α. Hence it may be shown, by reasoning similar to that of the proof to (21.1), that there exists $T \subset R_1$ such that

$$v(T) - \Sigma_T a_j > v(S) - \Sigma_S a_j$$

so

$$v(T) - \Sigma_T b_j > v(S) - \Sigma_S b_j \quad .$$

Hence S is not in $\mathcal{S}(\beta)$, a contradiction which proves that $\mathcal{S}(\beta) \subseteq \mathcal{S}_1$. Hence $\beta \longrightarrow \alpha$.

Theorem 21 is a generalization of Theorem 10. Suppose the core is empty. Theorem 10 states that there is a definite maximum amount which a player can hope to receive in any imputation of a solution, namely the amount by which he increases the value of some coalition by joining it. Theorem 16 states that the strategic possibilities of an imputation depend only on the vital coalitions whose payoff does not exceed the value of the game. Theorem 21 states that certain imputations are excluded because the vital coalitions are penalized in an inequitable way. A vital set S, instead of receiving $v(S)$, is offered $\Sigma_S a_i$ and is asked to accept a deficit $v(S) - \Sigma_S a_i$. The most injured coalitions are those for which no subset is offered as great a deficit. If any player is not a member of a most injured coalition, but would receive more than the absolute minimum (zero), then the imputation is untenable.

Theorem 21 is best possible for the 3-person non-zero sum games - the union of all solutions to each game just fills the region not eliminated by Theorem 21.

DEFINITION. A player in no vital set is a <u>dummy</u>.

DEFINITION. A game for which the core is empty is called a strong game.

DEFINITION. The game (M, v) is said to be <u>decomposable</u> if there exists a partition of I_n into two or more disjunct sets such that any vital set is contained in some one of the sets of the partition.

Let $S_1, S_2, \ldots, S_p$ be a decomposition of I_n minimal in the sense that if any set S_i is split into two or more sets, or if one or more players are removed from S_i, then the resulting partition is not a decomposition partition. Since each player in $\cup S_i$ is in at least one vital set, $I_n - \cup S_i$ consists of the dummies, if any. Therefore, "minimal decomposition" is used in this paper, not in the usual sense, since single element sets (dummies) are excluded from the minimal decomposition.

THEOREM 22. If $S_1, S_2, \ldots, S_p$ is a minimal decomposition of a strong game (M, v), and α is a point in a solution,

(22.1) For each S_i there is at least one vital set contained in S_i effective for α,

(22.2) If β is a point in the same solution, then $\Sigma_{S_i} a_j = \Sigma_{S_i} b_j$ holds for each S_i of the partition.

PROOF. (22.1): A useful, but weaker, form of Theorem 21 is this: If an imputation is in $A^* - C$, the effective vital sets cover each non-zero component. Since α is in a solution, α is in A^*, and C is empty since (M, v) is strong. If $a_j > 0$ for some j in S_i, there is a vital set containing j effective for α. Since (M, v) is decomposable, this set is contained in S_i. But if $a_j = 0$ for every j in S_i, since S_i does not consist of dummies, there exists a set $S \subseteq S_i$ such that $v(S) > 0$. Hence S is effective for α.

(22.2): Suppose $\Sigma_{S_i} a_j \neq \Sigma_{S_i} b_j$ for some S_i of the partition. Define γ by

$$c_j = a_j \text{ if } j \text{ is in a set } S_k \text{ of the partition for which } \Sigma_{S_k} b_j \geq \Sigma_{S_k} a_j \,.$$

$c_j = b_j$ if j is in a set S_k of the partition for which $\Sigma_{S_k} b_j < \Sigma_{S_k} a_j$.

Then $c_j \geq \min(a_j, b_j)$ and $\Sigma c_j < M$, so γ is in $\bar{A} - A$. Since the core is empty, $\bar{A} - A$ is in the dominion of any A-stable set, so the solution contains $\delta \succ \gamma$ via S. But S may be chosen to be a vital set contained in some S_k, so either

$$c_i = a_i \quad \text{for} \quad S \subseteq S_k$$

or

$$c_i = b_i \quad \text{for} \quad S \subseteq S_k.$$

Hence either $\delta \underset{S}{\succ} \alpha$ or $\delta \underset{S}{\succ} \beta$, a contradiction. Hence $\Sigma_{S_i} a_j = \Sigma_{S_i} b_j$.

THEOREM 23. If V is a solution to the strong decomposable game (M, v), and α is any point in V, and $M_i = \Sigma_{S_i} a_j$, then

(23.1) $\Sigma M_i = M$ since each dummy receives 0,

(23.2) For β in V, $\Sigma_{S_i} b_j = M_i$

(23.3) The point

$$\gamma : c_j = a_j \quad \text{for } j \text{ in } S_i$$
$$c_j = b_j \quad \text{otherwise,}$$

is in V.

(23.4) If γ is a point such that $\Sigma_{S_i} c_j \leq M_i$, and for no point α of V does $c_j = a_j$ for S_i, then γ is dominated via a subset of S_i by some point of V.

PROOF. (23.1): Suppose i belongs to no vital set, and there exists an $\alpha \in V$ with $a_i > 0$. Construct β such that

$$b_i = a_i - (n - 1)\epsilon$$
$$b_j = a_j + \epsilon \qquad\qquad j \neq i \ .$$

Either $\beta \succ \alpha$ via some set $S \subseteq S_i$, or else all $S \subseteq S_i$ are either ineffective or exactly effective for α so that α is in the core. In either case we have a contradiction.

(23.2): This is a re-statement of (22.2).

(23.3): γ is either in V or dom V. If γ is in dom V, there exists δ in V with $\delta \succ \gamma$ via S, and S may be taken to be a vital set contained in some S_k. If $k = i$, $\delta \succ \alpha$; if $k \neq i$, $\delta \succ \beta$, a contradiction in either case. Hence γ is not in dom V, but in V.

(23.4): γ satisfies $\Sigma_{S_i} c_j \leq M_i$, and for no α of V does $c_j = a_j$ for S_i. Let α be any point of V, and define

$$\beta : b_j = a_j \text{ for } j \text{ not in } S_i ,$$

$$b_j = c_j + \frac{1}{|S_i|} (M_i - \Sigma_{S_i} c_j) \text{ for } j \text{ in } S_i .$$

If β is in V, then $M_i - \Sigma_{S_i} c_j > 0$ since $\beta \neq \gamma$ so $b_j > c_j$ for S_i. Since β must have some effective vital set S contained in S_i, $\beta \succ \gamma$ via S. If β is not in V, then V contains some point $\delta \succ \beta$ via some set S. If S is contained in $S_k \neq S_i$ then $\delta \succ \alpha$, a contradiction. Hence S is contained in S_i and $d_i > b_i \geq c_i$ so $\delta \succ \gamma$ via $S \subseteq S_i$, which was to be shown.

A subgame or component game (M_i, v_i) is a game defined on the $|S_i|$-dimensional Euclidean space E^{S_i} whose coordinates are indexed by the integers in S_i, and

$$A_i = \left\{\alpha \mid \Sigma a_i + M_i,\ a_i \geq 0 \text{ and } \alpha \text{ is in } E^{S_i}\right\}$$

$$\bar{A}_i = \left\{\alpha \mid \Sigma a_i \leq M_i,\ a_i \geq 0 \text{ and } \alpha \text{ is in } E^{S_i}\right\} .$$

v_i is the restriction of v to S_i and its subsets. Let V_i be the projection of V on A_i, and call C_i, $\bar{C}_i$ the A_i-core and $\bar{A}_i$-core, respectively. The results obtained so far may be re-stated:

1. Any point α of A which is the composition of one point from each of $V_1, V_2, \ldots, V_p$, is in V.
2. V_i is $\bar{A}_i$-stable, which means that V_i is A-stable and $\bar{C}_i = C_i$.

These conditions have been shown to be necessary.

THEOREM 24. Necessary and sufficient conditions that V be a solution to the strong decomposable game (M, v) are

(24.1) There exist p constants $M_1, M_2, \ldots, M_p$ such that $\Sigma M_i = M$, and for any α in V, $\Sigma_{S_i} a_i = M_i$

(24.2) For each (M_i, v_i), $\bar{C}_i = C_i$ and V_i is A_i-stable.

(24.3) Any point composed of a point each from $V_1, V_2, \ldots, V_p$ is in V.

PROOF. Necessity has been shown. For sufficiency, suppose β is a point in $A - V$ but not in dom V. Since $\Sigma b_j = 1$, it is not possible that $\Sigma_{S_i} b_j > M_i$ for all i. Hence either

$$\Sigma_{S_i} b_j < M_i \quad \text{for some } i,$$

or

$$\Sigma_{S_i} b_j = M_i \quad \text{for } i = 1, 2, \ldots, p.$$

If $\Sigma_{S_i} b_j < M_i$ for some i, the point $\beta^{(i)}$ formed by projecting β on E^{S_i}, is in $\bar{A}_i - A_i$, so V_i contains some point $\alpha^{(i)} \succ \beta^{(i)}$. Hence V contains some point $\alpha \succ \beta$. Similarly, if $\Sigma_{S_i} b_j = M_i$ for $i = 1, 2, \ldots, p$ define $\beta^{(1)}, \beta^{(2)}, \ldots, \beta^{(p)}$ to be the projections of β on $E^{S_1}, E^{S_2}, \ldots, E^{S_p}$. If $\beta^{(i)} \in V_i$ for $i = 1, 2, \ldots, p$ then β is in V. Otherwise there exists an i for which $\beta^{(i)} \notin V_i$. Hence $\beta^{(i)}$ is in $A_i - V_i$ so there exists $\alpha^{(i)}$ in V_i with $\alpha^{(i)} \succ \beta^{(i)}$. Hence there exists α in V with $\alpha \succ \beta$. This proves the sufficiency of the conditions.

The possible range of each individual M_i may be computed by using the condition $\bar{C}_i = C_i$. Let μ_i be the lower bound of the numbers $\{\mu\}$ for which the equations and inequalities

$$a_j \geq 0 \quad \text{for } S_i$$

$$\Sigma_S a_j > v(S) \quad \text{for every } S \subseteq S_i$$

$$\Sigma_{S_i} a_j = \mu,$$

have a solution. Then μ_i is not attained, and the range of M_i is

$$0 \leq M_i \leq \mu_i \, .$$

THEOREM 25. A positive fraction of all n-person games have a unique solution consisting of the core.

PROOF. The value v is defined by the $2^n - n - 2$ numbers $\{v(S)\}$ for $S \neq \emptyset$, I_n or a player. Imposing the restriction $v(S) < \frac{1}{n}$ for every S selects a full-dimensional class of values $\{v\} \cdot \{v\}$

contains, incidentally, a full dimensional class of values for which every $S \neq \emptyset$, I_n or a player, is a vital set. But $\max_i \{v(S \cup \{i\}) - v(S)\} < \frac{1}{n}$, and for any point α in A, since $\Sigma a_i = 1$, there is some component $a_j \geq \frac{1}{n}$. Hence by Theorem 10, α is majorized unless it is in the core. Since the core contains the point $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$ it is not empty, and $A^* = C$ since A is closed and bounded. Since C is C-stable (A^*-stable) it is A-stable. Hence the core is the unique solution.

COROLLARY 26. A positive fraction of all (n+1)-person constant-sum games have discriminatory solutions.

PROOF. If v is super-additive and satisfies $v(S) < \frac{1}{n}$ for $S \neq I_n$ and $v(I_n) = 1$, an (n+1)-person constant-sum game can be defined over $I_n \cup (n + 1)$ by setting

$$v^*(S) = v(S) \quad \text{for} \quad S \subseteq I_n$$

$$v^*(S) = 1 - v(I_n - S) \quad \text{otherwise.}$$

It can be readily verified that the set of points consisting of points in the core of the original game with an (n+1)st coordinate, identically 0, adjoined, is a solution to the (n+1)-person constant-sum game. Since the value of an (n+1)-person constant-sum game is given by $2^n - n - 2$ numbers, the class of constant-sum (n+1)-person games having discriminatory solutions of this form is full-dimensional.

§4. AN APPROACH TO A GENERAL EXISTENCE THEOREM FOR SOLUTIONS

The results obtained so far - that every semi-simple game has a solution, and that every game for which the value is sufficiently small (and hence the core sufficiently large) has a unique solution consisting of the core alone - have depended on somehow setting up a partial ordering which implies domination but need not be implied by it. Another approach is to define, for the game (M, v) a related game called a pyramid game, for which domination is acyclic. It is believed that the task of finding solutions to the pyramid game is easier than for the original game, and it is also believed that a solution to the one problem would also solve the other.

DEFINITION. An <u>essential player</u> is a player contained in every vital set. A <u>pyramid game</u> is a game with one or more essential players.

Given a game (M, w), where w is defined for subsets of I_n, a game (M, v) can be defined, where v is defined on the subsets of I_{n+1} as follows:

$$v(S) = 0 \quad \text{if} \quad S \subseteq I_n$$
$$v(S + (n + 1)) = w(S).$$

If the original simplex A_0 for (M, w) is embedded in the new simplex A for (M, v), it consists of the face $x_{n+1} = 0$. If this face is visualized as the base of the simplex A, then S-dominions or "domination cones" open downward toward this base because of the necessary condition

$$\alpha \in\!\!-\, \beta \quad \text{implies} \quad a_{n+1} > b_{n+1} \ .$$

Thus one can visualize the passage from (M, w) to (M, v) as erecting a "pyramid" over the original simplex. The point $(0, 0, \ldots, 0, 1)$ has no strictly effective set and therefore is contained in the core. For any number m in $0 \leq m \leq M$, the simplex

$$A_m = \{\alpha \mid a_{n+1} \geq m, \ \alpha \ \text{in} \ A\}$$

may be defined. A_m consists of those points of A for which $x_{n+1} \geq m$. Note that if the essential player is discarded and the original game value restored for I_n, the base of A_m which is $\{\alpha \mid \Sigma_{I_n} a_i = M - m\}$ corresponds to the situation in decomposable games in which a set is given a constant payoff. In this sense we are stacking or pyramiding all of these various games together.

THEOREM 26. If V is a solution to the pyramid game (M, v) then $V \cap A_m$ is a solution to (M, v) defined over A_m.

PROOF. If α is a point in V but not in A_m, then $a_{n+1} < m$. If $\alpha \in\!\!-\, \beta$, then

$$a_{n+1} > b_{n+1} \ .$$

Hence

$$\text{dom } \alpha \cap A_m = \emptyset \ .$$

Hence

$$A_m - \text{dom } V = A_m - A_m \cap \text{dom } V = A_m \cap V \ .$$

THEOREM 27. If Q is a set in A composed of sets $\{P_i\}$ such that $P_i \subseteq \{\alpha \mid a_{n+1} = k_i\}$, a plane parallel to the base of the pyramid, where $\{k_i\}$

are constants, and either

(27.1) The planes are finite in number,
or

(27.2) The planes are denumerably infinite in number, and such that each limit plane (which need not contain points of Q) is a finite distance from the next lower plane.

Then there exists a unique Q-stable set.

(This is an instance of a "regressively finite" relation, for which solutions have been shown to exist by Richardson [4].)

PROOF. (27.1): By a permutation of indices, $\{k_i\}$ satisfy

$$1 \geq k_1 \geq k_2 \geq \cdots \geq k_r \geq 0 .$$

Define

$$Q_1 = P_1$$
$$Q_{i+1} = P_{i+1} - \text{dom} \bigcup_{j \leq i} Q_j \qquad 1 \leq i \leq r - 1 .$$

Any Q-stable set must contain Q_1, and hence Q_2, ..., and hence Q_r. Since $Q \cap \text{dom}\, Q_r = \emptyset$

$$\cup Q_i = \cup P_i - \text{dom} \cup Q_i$$

so

$$\cup Q_i \text{ is Q-stable} .$$

(27.2): This follows from (27.1) by transfinite induction since solutions are closed in the relative topology of Q.

In particular, if Q is any finite set of imputations, planes can be found containing every point of Q and satisfying (27.1), so any finite set Q has a unique Q-stable set. There is the possibility of approximating to an A-stable set by inserting more and more planes $\{P_i\}$. But since the values $\{k_i\}$ are nowhere dense in $[0, 1]$ this task seems formidable.

THEOREM 28. If m is sufficiently close to 1, an A_m-stable set exists.

PROOF. Let

$$\epsilon = \max_{v(S)<1} (v(S)) \quad .$$

Let $1 > m > \epsilon$, and let α be any point of A_m. Then for S to be effective for α,

$$\Sigma_S a_i \leq v(S)$$

$$a_{n+1} + \sum_{S-(n+1)} a_i \leq v(S) \; .$$

But

$$a_{n+1} \geq m > \epsilon$$

so

$$v(S) - \epsilon > \sum_{S-(n+1)} a_i \quad \text{and} \quad v(S) > \epsilon \; .$$

Hence $v(S) = 1$, and S is effective throughout A_m. If no set S satisfies $v(S) = 1$, then the core coincides with A_m and is the unique stable-set. Otherwise the game is semi-simple in A_m and a stable set exists.

DEFINITION. If Q satisfies $Q \cap \text{dom}\, Q = \emptyset$, and no point in $A - \text{dom}\, Q$ is distant more than ϵ from Q, Q is called an ϵ-solution.

Intuitively this means that no change greater than ϵ in the payoff can be proposed because of the ϵ-stability of Q.

THEOREM 29. For any $\epsilon > 0$, a pyramid game has a finite ϵ-solution.

PROOF. A rule for adding points, one at a time, on a descending sequence of planes parallel to the base will be given. Assume inductively that the process has been continued down to the plane $x_{n+1} = t$ $(1 \geq t \geq 0)$, that is, a sequence $\alpha^1, \alpha^2, \ldots, \alpha^k$ of points has been found such that

$$1 \geq a^1_{n+1} \geq a^2_{n+1} \geq a^k_{n+1} = t \quad ,$$

and never

$$\alpha^i \in\!\!- \alpha^j \; ,$$

and if β is not in $\text{dom}(\alpha^1, \alpha^2, \ldots, \alpha^k)$, either $b_{n+1} \leq t$ or there exists some j such that α^j is distant not more than ϵ from β.

Let $\mathfrak{A}^k$ be the point-set in A distant at least ϵ from every point of $(\alpha^1, \alpha^2, \ldots, \alpha^k)$ and undominated by any of these. $\mathfrak{A}^k$ is closed since it is the intersection of two closed sets. Define

$$\bar{t} = \sup_{\beta \in \mathcal{U}^k} (b_{n+1}) \quad .$$

Then $\bar{t}$ is attained for some β called α^{k+1} and $\bar{t} \leq t$. α^{k+1} is the next point added, and satisfies the inductive assumptions. Since A is compact, this process terminates after a finite number of steps since no two points are nearer than ϵ from each other.

If ϵ tends to zero, it can be shown that limit sets exist, however they may not be A-stable because

(1) the limit of points having a set S ineffective may have S exactly effective

(2) the limit of points, each dominating a fixed point, need not dominate the fixed point, since one or more components may become equal, in the limit, to components of the fixed point.

The importance of pyramid games lies in the possibility that a solution, under suitable conditions (for example, connectedness) when intersected with the base, always solves the subgame. In the case of a three-person non-zero-sum game, a pyramid game can be constructed over it, and the connected solutions, intersected with any plane parallel to the base, solve the corresponding games. Thus solutions to games with excess are unified if suitable properties can be established generally.

Also, for all but a lower dimensional set of games, $v(S) < 1$ for all S, and the core is the unique solution to the pyramid game in a neighbourhood of $(0, 0, \ldots, 0, 1)$. We can pass from any solution of the game on the base to any other solution by going up the pyramid, possibly as high as the core, and taking another continuation down again. This would unify all solutions to a particular game -- a perplexing problem at present.

BIBLIOGRAPHY

[1] von NEUMANN, J. and MORGENSTERN, O., Theory of Games and Economic Behavior, Princeton 1944, 2nd edition, 1947.

[2] SHAPLEY, L. S., "Notes on the n-person game III: some variants of the von Neumann-Morgenstern definition of solution", RAND Corporation report RM-817 (unpublished).

[3] McKINSEY, J. C. C., "Isomorphism of games and strategic equivalence", Contributions to the Theory of Games, Vol. 1. Annals of Mathematics Study No. 24 (Princeton, 1950), pp. 117-130.

[4] RICHARDSON, Moses, "Solutions of irreflexive relations", Annals of Mathematics 58 (1953), pp. 573-590.

Donald B. Gillies

University of Illinois