

Rudin-Shapiro Sums Via Automata Theory and Logic

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Abstract

We show how to obtain, via a unified framework provided by logic and automata theory, many classical results of Brillhart and Morton on Rudin-Shapiro sums. The techniques also facilitate easy proofs for new results.

1 Introduction

The Rudin-Shapiro coefficients $(a(n))_{n \geq 0}$ form an infinite sequence of ± 1 defined recursively by the identities

$$\begin{aligned} a(2n) &= a(n) \\ a(2n+1) &= (-1)^n a(n) \end{aligned}$$

and the initial condition $a(0) = 1$. It was apparently first discovered by Golay [8, 9], and later by Shapiro [18] and Rudin [15]. The map $a(n)$ can also be defined as $a(n) = (-1)^{r_n}$, where r_n counts the number of (possibly overlapping) occurrences of 11 in the binary representation of n [1].

The Rudin-Shapiro coefficients have many intriguing properties and have been studied by many authors; for example (just to name a few of the dozens of references), see [1, 4, 7, 12, 13].

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In a classic paper from 1978, Brillhart and Morton [3] studied sums of these coefficients, and defined the two sums¹

$$s(n) = \sum_{0 \leq i \leq n} a(i) \quad (1)$$

$$t(n) = \sum_{0 \leq i \leq n} (-1)^i a(i). \quad (2)$$

The first few values of the functions s and t are given in Table 1. They are sequences [A020986](#) and [A020990](#) respectively, in the *On-Line Encyclopedia of Integer Sequences* (OEIS) [19].

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$s(n)$	1	2	3	2	3	4	3	4	5	6	7	6	5	4	5	4	5	6	7	6	7
$t(n)$	1	0	1	2	3	2	1	0	1	0	1	2	1	2	3	4	5	4	5	6	7

Table 1: First few values of $s(n)$ and $t(n)$.

A priori it is not even clear that these sums are always non-negative, but Brillhart and Morton proved that they are, and also proved many other properties of them. Most of these properties can be proved by induction, sometimes rather tediously.

In this paper we show how to replace nearly all of these inductions with techniques from logic and automata theory. Ultimately, almost all the Brillhart-Morton results can be proved in a simple, unified manner, simply by stating them in first-order logic and applying the **Walnut** theorem-prover [14, 17]. In fact, there are basically only three simple inductions in our entire paper. One is the brief induction used to prove Lemma 18. The other two are the inductions used in Theorem 1 to prove the correctness of our constructed automata, and in these cases the induction step itself can be proved by **Walnut**! We are also able to easily derive and prove new results; see Section 9.

The paper is organized as follows: Section 2 gives basic notation used in describing base- b expansions. Section 3 gives the automata computing the functions $s(n)$ and $t(n)$ and justifies their correctness. In Section 4 we begin proving the results of Brillhart and Morton using our technique, and illustrate the basic ideas, and this continues in Section 5. In Section 6 we show how to compute various special values of s and t . Then in Section 7, we obtain the deepest results of Brillhart and Morton, on inequalities for the Rudin-Shapiro sums. In Section 8 we illustrate how our ideas can be used to enumerate some quantities connected with $s(n)$. In Section 9 we use our technique to prove various new results about the Rudin-Shapiro sums. In Section 10 we prove properties about a space-filling curve associated with s and t . Finally, in Section 11, we explain how the technique can be used to prove properties of two analogues of the Rudin-Shapiro sums.

¹One can make the case that these definitions are “wrong”, in the sense that many results become significantly simpler to state if the sums are taken over the range $0 \leq i < n$ instead. But the definitions of Brillhart-Morton are now very well-established, and using a different indexing would also make it harder to compare our results with theirs.

We assume the reader is familiar with the basics of automata and regular expressions, as discussed, for example, in [10].

2 Notation

Let $b \geq 2$ be an integer, and let Σ_b denote the alphabet $\{0, 1, \dots, b-1\}$.

For a string $x \in \Sigma_b^*$, we let $[x]_b$ denote the integer represented by x in base b , starting with the most significant digit. That is, if $x = c_1 c_2 \dots c_i$, then $[x]_b := \sum_{1 \leq j \leq i} c_j b^{j-i}$. For example, $[00101011]_2 = [223]_4 = 43$.

For an integer $n \geq 0$, we let $(n)_b$ denote the canonical base- b representation of n , starting with the most significant digit, with no leading zeros. For example, $(43)_2 = 101011$. We have $(0)_b = \epsilon$, the empty string.

3 Automata and logic

Walnut is free software, originally designed by Hamoon Mousavi, that can rigorously prove propositions about automatic sequences [14, 17]. If a proposition P has no free variables, one only has to state it in first-order logic and then the Walnut prover will prove or disprove it (subject to having enough time and space to complete the calculation). If the proposition P has free variables, the program computes a finite automaton accepting exactly the values of the free variables that make P evaluate to true.

Because of these two features, there is a philosophical choice in using Walnut. Either we can state the desired result as a first-order logical formula and verify it, or, if the theorem involves characterizing a set of numbers n with a certain property, we can simply create a formula with n as a free variable, and consider the resulting automaton as the desired characterization. In this case, if the automaton is simple enough, we can find a short regular expression specifying the accepted strings, and then interpret it as a function of n in terms of powers of the base b . The former is most appropriate when we already know the statement of the theorem we are trying to prove; the latter when we do not yet know the precise characterization that will become a theorem. In this paper we have used both approaches, to illustrate the ideas.

As is well-known, the Rudin-Shapiro sequence is 2-automatic and therefore, by a classic theorem of Cobham [6], also 4-automatic. This means there is a deterministic finite automaton with output (DFAO) that on input n , expressed in base 4, reaches a state with output $a(n)$. This Rudin-Shapiro automaton is illustrated below in Figure 1.

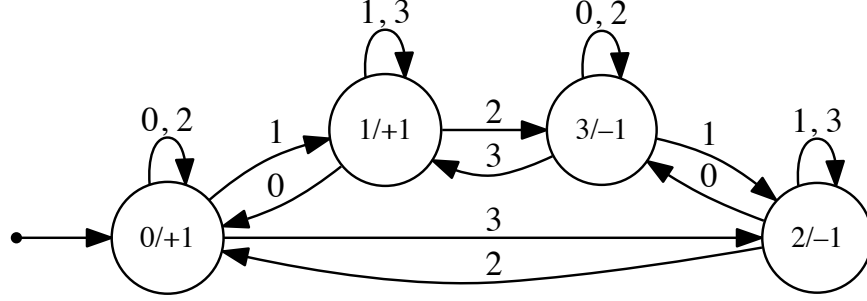


Figure 1: DFAO computing the Rudin-Shapiro function, in base 4.

Here states are labeled a/b , where a is the state number and b is the output. The initial state is state 0, and the automaton reads the digits of the base-4 representation of n , starting with the most significant digit. Leading zeros are allowed and do not affect the result.

Once the automaton in Figure 1 is saved as a file named `RS4.txt`, in `Walnut` we can refer to its value at a variable n simply by writing `RS4[n]`. We would like to do the same thing for the Rudin-Shapiro summatory functions $s(n)$ and $t(n)$ defined in Eqs. (1) and (2), but here we run into a fundamental limitation of `Walnut`: it can only directly deal with functions of finite range (like automatic sequences). Since $s(n)$ and $t(n)$ are unbounded, we must find another way to deal with them.

A common way to handle functions in first-order logic is to treat them as *relations*: instead of writing $f(n) = x$, we construct a relation $R_f(n, x)$ that is true iff $f(n) = x$. If the relation $R_f(n, x)$ is representable by a deterministic finite automaton (DFA) taking as input n in base b_1 and x in base b_2 , in parallel, and accepting iff $R_f(n, x)$ holds, then we say that f is (b_1, b_2) -synchronized. For more information about synchronized functions, see [16].

Our first step, then, is to show that the functions $s(n)$ and $t(n)$ are $(4, 2)$ -synchronized. These automata are illustrated in Figures 2 and 3. Here accepting states are labeled by double circles.

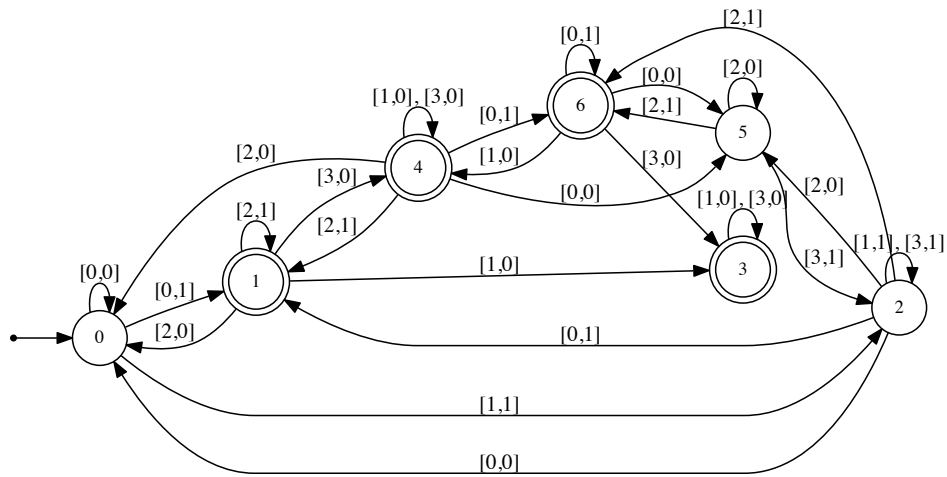


Figure 2: Synchronized automaton for $s(n)$.

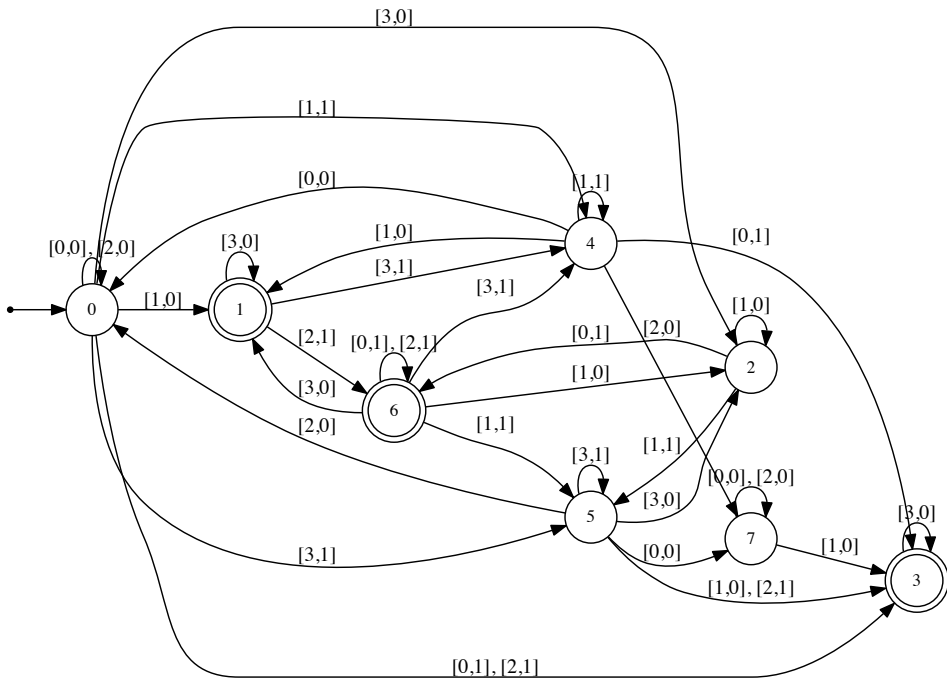


Figure 3: Synchronized automaton for $t(n)$.

We obtained these automata by “guessing” them from calculated initial values of the sequences s and t , using the Myhill-Nerode theorem [10, §3.4]. However, we will see below in Remark 6 that we could have also deduced them from Satz 3 of [3] (Lemma 1 of [5]).

Once we have guessed the automata, we need to verify they are correct.

Theorem 1. *The automata in Figs. (2) and (3) correctly compute $s(n)$ and $t(n)$.*

Proof. Let $s_1(n)$ (resp., $t_1(n)$) be the function computed by the automaton in Fig. (2) (resp., Fig. (3)). We prove that $s_1(n) = s(n)$ and $t_1(n) = t(n)$ by induction on n .

First we check that $s_1(0) = s(0) = 1$ and $t_1(0) = t(0) = 1$, which we can see simply by inspecting the automata.

Now assume that $n \geq 1$ and $s_1(n) = s(n)$ and $t_1(n) = t(n)$. We prove with Walnut that $s_1(n+1) = s_1(n) + a(n+1)$ and $t_1(n+1) = t_1(n) + (-1)^{n+1}a(n+1)$.

```
eval test1 "?msd_4 An,y ($rss(n,y) & RS4[n+1]=@1) => $rss(n+1,?msd_2 y+1)":
eval test2 "?msd_4 An,y ($rss(n,y) & RS4[n+1]=@-1) => $rss(n+1,?msd_2 y-1)":
# show that rss is correct
```

```
def even4 "?msd_4 Ek n=2*k":
def odd4 "?msd_4 Ek n=2*k+1":
eval test3 "?msd_4 An,y ($rst(n,y) & ((RS4[n+1]=@1 & $even4(n+1)) |
    (RS4[n+1]=@-1 & $odd4(n+1)))) => $rst(n+1,?msd_2 y+1)":
eval test4 "?msd_4 An,y ($rst(n,y) & ((RS4[n+1]=@-1 & $even4(n+1)) |
    (RS4[n+1]=@1 & $odd4(n+1)))) => $rst(n+1, ?msd_2 y-1)":
# show that rst is correct
```

and Walnut returns TRUE for all of these tests. Now the correctness of our automata follows immediately by induction. \square

Remark 2. Some Walnut syntax needs to be explained here. First, the capital A is Walnut’s abbreviation for \forall (for all); capital E is Walnut’s abbreviation for \exists (there exists); the jargon $?msd_b$ for a base b instructs that a parameter or expression is to be evaluated using base- b numbers, and an @ sign indicates the value of an automatic sequence (which is allowed to be negative). The symbol & is logical AND; the symbol | is logical OR; and the symbol => is logical implication.

Remark 3. There is a small technical wrinkle that we have glossed over, but it needs saying: the default domain for Walnut is \mathbb{N} , the natural numbers. But we do not know, a priori, that the functions s and t take only non-negative values. Therefore, it is conceivable that our verification might fail simply because negative numbers appear as intermediate values in a calculation. We can check that this does not happen simply by checking that rss and rst both are truly representations of functions:

```
eval test5 "?msd_4 (An Ey $rss(n,y)) &
    An ~Ex,y ($rss(n,x) & $rss(n,y) & (?msd_2 x!=y))":
eval test6 "?msd_4 (An Ey $rst(n,y)) &
    An ~Ex,y ($rst(n,x) & $rst(n,y) & (?msd_2 x!=y))":
```

These commands assert that for every n there is at least one value y such that $s(n) = y$, and there are not two different such y , and the same for t . Both evaluate to **TRUE**. This shows that the relations computed by **rss** and **rst** are well-defined functions, and take only non-negative values. As a result, we have already deduced Satz 11 of [3]: $t(n) \geq 0$ for all n .

The advantage of the representation of $s(n)$ and $t(n)$ as synchronized automata is that they essentially encapsulate all the needed knowledge about $s(n)$ and $t(n)$ to replace tedious inductions about them. The simple induction we used to verify them replaces, in effect, all the other needed inductions.

All the code necessary to verify the results in this paper can be found at

<https://cs.uwaterloo.ca/~shallit/papers.html> .

4 Proofs of results

We can now begin to *reprove*, and in some cases, *improve* some of the results of Brillhart and Morton. Let us start with their Satz 2 [3], reprised as Lemma 1 in [5]:

Theorem 4. *We have*

$$s(2n) = s(n) + t(n-1), \quad (n \geq 1); \quad (3)$$

$$s(2n+1) = s(n) + t(n), \quad (n \geq 0); \quad (4)$$

$$t(2n) = s(n) - t(n-1), \quad (n \geq 1); \quad (5)$$

$$t(2n+1) = s(n) - t(n), \quad (n \geq 0). \quad (6)$$

Proof. We use the following Walnut commands:

```
eval eq3 "?msd_4 An,x,y,z (n>=1 & $rss(2*n,x) & $rss(n,y) & $rst(n-1,z))
=> ?msd_2 x=y+z":
```

```
eval eq4 "?msd_4 An,x,y,z ($rss(2*n+1,x) & $rss(n,y) & $rst(n,z))
=> ?msd_2 x=y+z":
```

```
eval eq5 "?msd_4 An,x,y,z (n>=1 & $rst(2*n,x) & $rss(n,y) & $rst(n-1,z))
=> ?msd_2 x+z=y":
```

```
eval eq6 "?msd_4 An,x,y,z ($rst(2*n+1,x) & $rss(n,y) & $rst(n,z))
=> ?msd_2 x+z=y":
```

and Walnut returns **TRUE** for all of them.

For example, the Walnut formula eq3 asserts that for all n, x, y, z if $n \geq 1$, $s(2n) = x$, $s(n) = y$, $t(n-1) = z$, then it must be the case that $x = y + z$. It is easily seen that this is equivalent to the statement of Eq. (3).

Note that in our Walnut proofs of Eqs. (4) and (5), we rearranged the statement to avoid subtractions. This is because Walnut's basic domain is \mathbb{N} , the natural numbers, and

subtractions that could potentially result in negative numbers might give anomalous results. \square

We can now verify Satz 3 of [3] (Lemma 2 of [5]):

Lemma 5. *For $n \geq 0$ we have*

$$s(4n) = 2s(n) - a(n) \tag{7}$$

$$s(4n + 1) = s(4n + 3) = 2s(n) \tag{8}$$

$$s(4n + 2) = 2s(n) + (-1)^n a(n). \tag{9}$$

Proof. We use the following Walnut commands:

```
eval eq7 "?msd_4 An,x,y ($rss(4*n,x) & $rss(n,y)) =>
  ((RS4[n]=@1 => ?msd_2 x+1=2*y) & (RS4[n]=@-1 => ?msd_2 x=2*y+1))":
```

```
eval eq8 "?msd_4 An,x,y,z ($rss(4*n+1,x) & $rss(4*n+3,y) & $rss(n,z))
  => ?msd_2 x=y & x=2*z":
```

```
eval eq9 "?msd_4 An,x,y ($rss(4*n+2,x) & $rss(n,y)) =>
  (((RS4[n]=@1 & $even4(n)) => ?msd_2 x=2*y+1) &
  ((RS4[n]=@-1 & $even4(n)) => ?msd_2 x+1=2*y) &
  ((RS4[n]=@1 & $odd4(n)) => ?msd_2 x+1=2*y) &
  ((RS4[n]=@-1 & $odd4(n)) => ?msd_2 x=2*y+1))":
```

and Walnut returns TRUE for all of them. \square

Remark 6. As it turns out, Lemma 5 is more or less equivalent to our synchronized automaton depicted in Figure 2. To see this, consider a synchronized automaton where the first component represents n in base 4, while the second component represents $s(n)$ in base 2, but using the nonstandard digit set $\{-1, 0, 1\}$ instead of $\{0, 1\}$. Reading a bit i in the first component is like changing the number n read so far into $4n + i$. Theorem 5 says that $s(4n + i)$ is twice $s(n)$, plus either $-1, 0$, or 1 , depending on the value of $a(n)$ and the parity of n , both of which are (implicitly) computed by the base-4 DFAO for $a(n)$. This gives us the automaton depicted in Figure 4.

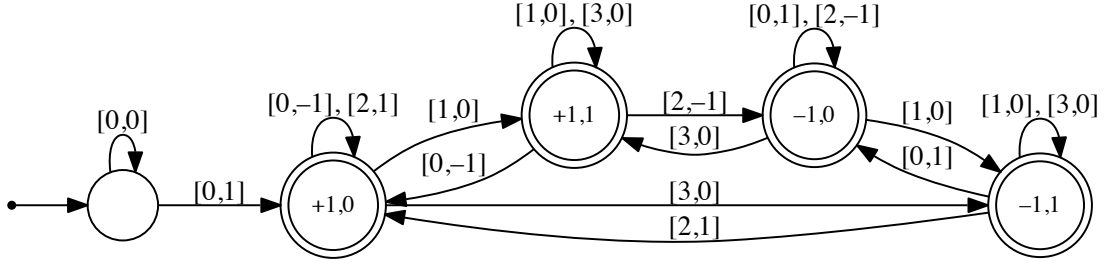


Figure 4: Synchronized automaton implementing Theorem 5. The state labels record $a(n)$ and $n \bmod 2$.

To get the automaton in Figure 2 from this one, we would need to combine it with a “normalizer” that can convert a nonstandard base-2 representation into a standard one.

The sequence $t(n)$ satisfies a similar set of recurrences, which are given as Satz 4 of [3].

Lemma 7. *We have*

$$t(4n) = 2t(n-1) + a(n), \quad (n \geq 1); \quad (10)$$

$$t(4n+1) = 2t(n-1), \quad (n \geq 1); \quad (11)$$

$$t(4n+2) = t(n) + t(n-1), \quad (n \geq 1); \quad (12)$$

$$t(4n+3) = 2t(n), \quad (n \geq 0). \quad (13)$$

Proof. We use the following Walnut commands:

```
eval eq10 "?msd_4 An,x,y (n>=1 & $rst(4*n,x) & $rst(n-1,y)) =>
  ((RS4[n]=@1 => ?msd_2 x=2*y+1) & (RS4[n]=@-1 => ?msd_2 x+1=2*y))":
```

```
eval eq11 "?msd_4 An,x,y (n>=1 & $rst(?msd_4 4*n+1,x) &
  $rst(?msd_4 n-1,y)) => ?msd_2 x=2*y":
```

```
eval eq12 "?msd_4 An,x,y,z (n>=1 & $rst(4*n+2,x) &
  $rst(n,y) & $rst(n-1,z)) => ?msd_2 x=y+z":
```

```
eval eq13 "?msd_4 An,x,y ($rst(4*n+3,x) & $rst(n,y)) => ?msd_2 x=2*y":
```

and Walnut returns TRUE for all of them. □

Next we give Theorem 1 of [5].

Theorem 8.

- (a) For $k \geq 1$, the minimum value of $s(n)$ for $n \in [4^k, 4^{k+1} - 1]$ is $2^k + 1$ and $s(n)$ attains this value only when $n = 4^k$ or $n = (5 \cdot 4^k - 2)/3$.
- (b) For $k \geq 0$, the maximum value of $s(n)$ for $n \in [4^k, 4^{k+1} - 1]$ is $2^{k+2} - 1$ and $s(n)$ attains this value only when $n = M_k := \frac{2}{3}(2^{2k+2} - 1)$.

Proof. We use the following Walnut commands:

```
reg rss_int msd_4 msd_4 "[0,0]*[1,3][0,3]*":
```

```
eval min_rss "?msd_4 n>=4 & $rss(n, x) & Ei,j $rss_int(i,j) &
  i<=n & n<=j & (Ay,m (i<=m & m<=j & $rss(m,y)) =>
  ?msd_2 y>=x)":
```

```
eval max_rss "?msd_4 $rss(n, x) & Ei,j $rss_int(i,j) &
  i<=n & n<=j & (Ay,m (i<=m & m<=j & $rss(m,y)) =>
  ?msd_2 y<=x)":
```

The output of these commands is the automata displayed in Figures 5 and 6. The automata accept pairs $((n)_4, (s(n))_2)$ where $s(n)$ is extremal for n in the specified interval. The first automaton accepts $[0, 0]^* [1, 1][0, 0]^* [0, 1]$ and $[0, 0]^* [1, 1][2, 0]^* [2, 1]$ and the second automaton accepts $[0, 0]^* [0, 1][2, 1][2, 1]^*$. From this one easily deduces the result. \square

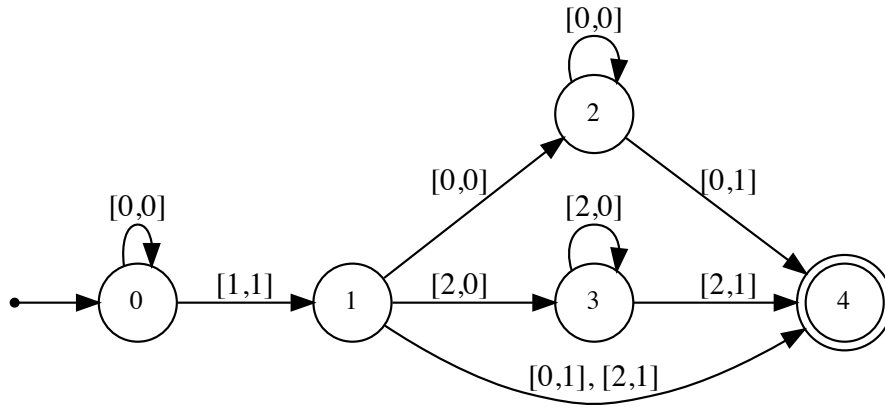


Figure 5: Automaton for the minimum value of $s(n)$, $n \in [4^k, 4^{k+1} - 1]$

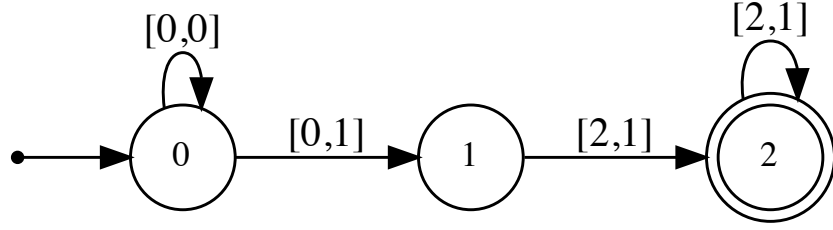


Figure 6: Automaton for the maximum value of $s(n)$, $n \in [4^k, 4^{k+1} - 1]$

4.1 The ω function

Brillhart and Morton defined a function, $\omega(k)$, as follows: $\omega(k)$ is the largest value of n for which $s(n) = k$. This is sequence [A020991](#) in the OEIS. We can create a $(2, 4)$ -synchronized automaton for ω as follows:

```
def omega "?msd_4 $rss(n,k) & At (t>n) => ~$rss(t,k)":
```

We can then create a $(2, 4)$ -synchronized automaton for $\omega(n+1) - \omega(n)$ as follows:

```
def omegadiff "?msd_4 Et,u $omega((?msd_2 n+1),t) &
  $omega(n,u) & t=x+u":
```

Let us now show that $\omega(s(n)) \leq (10n - 2)/3$ for $n \geq 2$.

```
def omegas "?msd_4 Ek $rss(n,k) & $omega(k,x)":
eval check_bounds "?msd_4 An,t (n>=2 & $omegas(n,t)) => 3*t+2<=10*n":
```

Also this bound is optimal, because it holds for $n = 2^{2i+1}$ and $i \geq 0$.

Now let us prove Lemma 5 of Brillhart and Morton [\[5\]](#).

Theorem 9. *We have*

$$\omega(2n) = 4\omega(n) + 3, \quad n \geq 1; \quad (14)$$

$$\omega(2n+1) = 4\omega(n+1) + 2, \quad n \geq 2, \quad n+1 \neq 2^r, \quad r \geq 2. \quad (15)$$

Proof. Let us prove Eq. (14).

```
eval eq14 "?msd_4 An,x,y ((?msd_2 n>=1) & $omega(n,x) &
  $omega((?msd_2 2*n), y)) => y=4*x+3":
```

Brillhart and Morton claim that the proof of the second equality is “much trickier”. However, with **Walnut** it is not really much more difficult than the previous one.

```
reg power2 msd_2 "0*10*":
eval eq15 "?msd_4 An,x,y (?msd_2 n>=2 & (~$power2(?msd_2 n+1)) &
    $omega((?msd_2 n+1),x) & $omega((?msd_2 2*n+1),y)) => y=4*x+2":
```

□

5 More lemmas

Let us now prove Lemma 3 of [5]:

Theorem 10. *We have*

$$s(n + 2^{2k}) = s(n) + 2^k, \quad 0 \leq n \leq 2^{2k-1} - 1, \quad k \geq 1; \quad (16)$$

$$s(n + 2^{2k}) = -s(n) + 3 \cdot 2^k, \quad 2^{2k-1} \leq n \leq 2^{2k} - 1, \quad k \geq 1; \quad (17)$$

$$s(n + 2^{2k+1}) = s(n) + 2^{k+1}, \quad 0 \leq n \leq 2^{2k} - 1, \quad k \geq 0; \quad (18)$$

$$s(n + 2^{2k+1}) = -s(n) + 2^{k+2}, \quad 2^{2k} \leq n \leq 2^{2k+1} - 1, \quad k \geq 0. \quad (19)$$

Proof. We can prove these identities with **Walnut**. One small technical difficulty is that the equation $x = 2^n$ is not possible to express in the particular first-order logic that **Walnut** is built on; it cannot even multiply arbitrary variables, or raise a number to a power. Instead, we assert that x is a power of 2 without exactly specifying *which* power of 2 it is. This brings up a further difficulty, which is that we need to simultaneously express 2^{2k} and 2^k . Normally this would also not be possible in **Walnut**. However, in this case the former is expressed in base 4 and the latter in base 2, we can achieve this using the **link42** automaton:

```
reg power4 msd_4 "0*10*":
reg link42 msd_4 msd_2 "([0,0] | [1,1])*":
```

Here **power4** asserts that its argument is a power of 4; specifically, that its base-4 representation looks like 1 followed by some number of 0's, and also allowing any number of leading zeros. If this is true for x , then $x = 4^k$ for some k , and **link42** applied to the pair (x, y) asserts that $y = 2^k$ (by asserting that the base-4 representation x is the same as the base-2 representation of y).

To verify Eqs. (16)–(19), we use the following **Walnut** code:

```
eval eq16 "?msd_4 An,x,y,z ($power4(x) & x>=4 & 2*n+2<=x &
    $rss(n,y) & $link42(x,z)) => $rss(n+x,?msd_2 y+z)":
eval eq17 "?msd_4 An,x,y,z ($power4(x) & x>=4 & 2*n>=x & n<x & $rss(n,y)
    & $link42(x,z)) => $rss(n+x,?msd_2 3*z-y)":
eval eq18 "?msd_4 An,x,y,z ($power4(x) & n<x & $rss(n,y) & $link42(x,z))
    => $rss(n+2*x,?msd_2 y+2*z)":
eval eq19 "?msd_4 An,x,y,z ($power4(x) & x<=n & n<2*x & $rss(n,y) &
```

`$link42(x,z)) => $rss(n+2*x,?msd_2 4*z-y)":`

□

We now turn to Lemma 4 in Brillhart and Morton [5]. It is as follows (where we have corrected a typographical error in the original statement).

Theorem 11. *Suppose $n \in [2^{2k}, 2^{2k+1})$. Then $s(n) \leq 2^{k+1}$, and furthermore, equality holds for n in this range iff $n = 2^{2k+1} - 1 - \sum_{0 \leq r < k} e_r 2^{2r+1}$, where the $e_r \in \{0, 1\}$.*

Proof. We can verify the first claim as follows:

```
eval lemma4 "?msd_4 An,x,y,z ($power4(x) & x<=n & n<2*x &
$link42(x,z) & $rss(n,y)) => ?msd_2 y<=2*z":
```

For the second, let us create a synchronized automaton accepting the base-4 representation of k and n for which $s(n) = 2^{k+1}$.

```
def lemma4a "?msd_4 Ez $power4(x) & x<=n & n<2*x &
$link42(x,z) & $rss(n,?msd_2 2*z)":
```

By examining the result, we see that the only accepted paths are labeled with $[1, 1]\{[1, 0], [3, 0]\}^*$. This is easily seen to be the same as the claim in the Brillhart-Morton result. □

6 Special values

Along the way, Brillhart and Morton proved a large number of results about special values of the functions s and t . These are very easily proved with Walnut.

Let us start with Examples (“Beispiel”) 5–10 of [3].

Theorem 12. *We have*

- (a) $s(2^k) = 2^{\lfloor (k+1)/2 \rfloor} + 1$ for $k \geq 0$;
- (b) $s(2^k - 1) = 2^{\lfloor (k+1)/2 \rfloor}$ for $k \geq 0$;
- (c) $s(2^k - 2) = 2^{\lfloor (k+1)/2 \rfloor} + (-1)^k$ for $k \geq 1$;
- (d) $s(3 \cdot 2^{2k} - 1) = 3 \cdot 2^k$ for $k \geq 0$;
- (e) $s(3 \cdot 2^{2k+1} - 1) = 2^{k+2}$ for $k \geq 0$;
- (f) $t(2^{2k}) = 2^k + 1$ for $k \geq 1$;
- (g) $t(2^{2k+1}) = 1$ for $k \geq 0$;

- (h) $t(2^{2k} - 1) = 2^k$ for $k \geq 0$;
- (i) $t(2^{2k+1} - 1) = 0$ for $k \geq 0$;
- (j) $t(2^{2k} - 2) = 2^k - 1$ for $k \geq 1$;
- (k) $t(2^{2k+1} - 2) = 1$ for $k \geq 0$;
- (l) $t(3 \cdot 2^k - 1) = 2^{\lfloor (k+1)/2 \rfloor}$ for $k \geq 0$.

Proof. We can verify all of these by straightforward translation of the assertions into Walnut. First let's write a formula asserting $x = 2^k$ and $y = 2^{\lfloor (k+1)/2 \rfloor}$, where the former is expressed in base 4 and the latter in base 2.

```
reg sqrtpow2 msd_4 msd_2 "[0,0]*([1,1] | [0,1] [2,0])[0,0]*":
```

Then we can verify all the assertions as follows:

```
reg oddpow2 msd_4 "0*20*":
eval specval_a "?msd_4 Ax,y $sqrtpow2(x,y) => $rss(x,?msd_2 y+1)":
eval specval_b "?msd_4 Ax,y $sqrtpow2(x,y) => $rss(x-1,?msd_2 y)":
eval specval_c1 "?msd_4 Ax,y ($power4(x) & x>1 & $sqrtpow2(x,y))
=> $rss(x-2, ?msd_2 y+1)":
eval specval_c2 "?msd_4 Ax,y ($oddpow2(x) & $sqrtpow2(x,y))
=> $rss(x-2, ?msd_2 y-1)":
eval specval_d "?msd_4 Ax,y ($power4(x) & $link42(x,y))
=> $rss(3*x-1,?msd_2 3*y)":
eval specval_e "?msd_4 Ax,y ($oddpow2(x) & $sqrtpow2(x,y))
=> $rss(3*x-1,?msd_2 2*y)":
eval specval_f "?msd_4 Ax,y ($power4(x) & x>1 & $link42(x,y))
=> $rst(x,?msd_2 y+1)":
eval specval_g "?msd_4 Ax $oddpow2(x) => $rst(x,?msd_2 1)":
eval specval_h "?msd_4 Ax,y ($power4(x) & $link42(x,y))
=> $rst(x-1,?msd_2 y)":
eval specval_i "?msd_4 Ax $oddpow2(x) => $rst(x-1,?msd_2 0)":
eval specval_j "?msd_4 Ax,y ($power4(x) & x>1 & $link42(x,y))
=> $rst(x-2, ?msd_2 y-1)":
eval specval_k "?msd_4 Ax $oddpow2(x) => $rst(x-2,?msd_2 1)":
eval specval_l "?msd_4 Ax,y $sqrtpow2(x,y) => $rst(3*x-1,y)":
```

□

Satz 10 of [3] gives the values of n for which $t(n) = 0$. We can find these with the Walnut command

```
def satz10 "$rst(?msd_4 n,?msd_2 0)":
```

The resulting automaton appears in Figure 7.

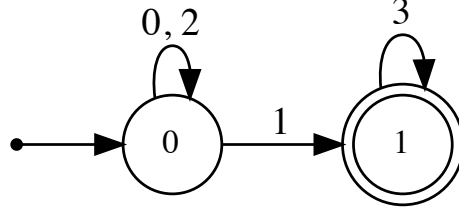


Figure 7: DFA accepting those n for which $t(n) = 0$, expressed in base 4.

Examining this automaton gives the following:

Theorem 13. *We have $t(n) = 0$ iff $(n)_4 \in 13^* \cup 2\{0, 2\}^*13^*$.*

Next let us determine those n for which $s(n) = t(n)$. We can define the base-4 representation of such n with the following Walnut command:

```
def same "Ex $rss(n,x) & $rst(n,x)":
```

The resulting automaton appears in Figure 8.

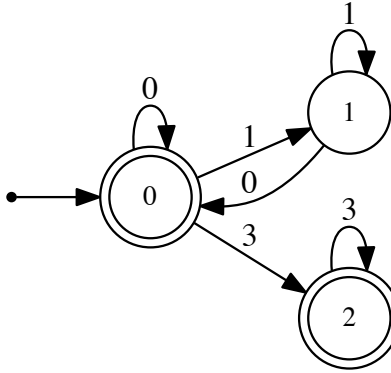


Figure 8: DFA accepting those n for which $s(n) = t(n)$, expressed in base 4.

We have therefore proved the following result, which is given as a Zusatz to Satz 10 of [3].

Theorem 14. *We have $s(n) = t(n)$ if and only if $n = 0$ or $(n)_4 \in 3^* \cup 1\{0, 1\}^* 03^*$.*

From Theorem 13 we see that the minimum value of $t(n)$ is 0 and $t(n)$ takes this value infinitely often. The next result, which is an analogue of Theorem 8 (Satz 12 of [3]), gives the maximum value of $t(n)$ on certain intervals.

Theorem 15.

- (a) *For $k \geq 1$, the maximum value of $t(n)$ for $n \in [4^k, 2 \cdot 4^k - 1]$ is $2^{k+1} - 1$ and $t(n)$ attains this value only when $n = 4(2^{2k} - 1)/3$.*
- (b) *For $k \geq 1$, the maximum value of $t(n)$ for $n \in [2 \cdot 4^k, 4^{k+1} - 1]$ is 2^{k+1} and $t(n)$ attains this value only when $n = 2^{2k+2} - 1$.*

Proof. We use the following Walnut commands:

```
reg rst_int1 msd_4 msd_4 "[0,0]*[1,1][0,3]*":
reg rst_int2 msd_4 msd_4 "[0,0]*[2,3][0,3]*":
```

```
eval max_rst1 "?msd_4 n>=2 & $rst(n, ?msd_2 x) & Ei,j $rst_int1(i,j) &
i<=n & n<=j & (Ay,m (i<=m & m<=j & $rst(m,?msd_2 y)) => ?msd_2 y<=x)":
```

```
eval max_rst2 "?msd_4 n>=4 & $rst(n, ?msd_2 x) & Ei,j $rst_int2(i,j) &
i<=n & n<=j & (Ay,m (i<=m & m<=j & $rst(m,?msd_2 y)) => ?msd_2 y<=x)":
```

The output of these commands is the automata displayed in Figures 9 and 10. The automata accept pairs $((n)_4, (t(n))_2)$ where $t(n)$ is extremal for n in the specified interval. The first automaton accepts $[0, 0]^*[1, 1][1, 1]^*[0, 1]$ and the second automaton accepts $[0, 0]^*[3, 0][3, 0][3, 0]^*$. From this one easily deduces the result. \square

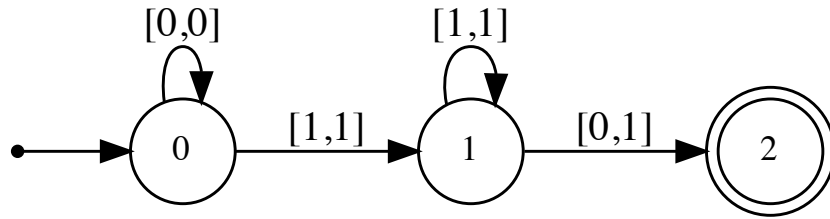


Figure 9: Automaton for the maximum value of $t(n)$, $n \in [4^k, 2 \cdot 4^k - 1]$

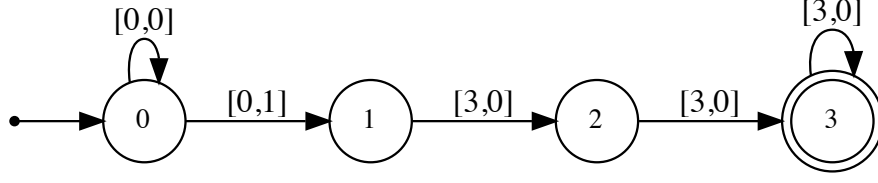


Figure 10: Automaton for the maximum value of $t(n)$, $n \in [2 \cdot 4^k, 4^{k+1} - 1]$.

Let us now prove Satz 14 of [3]:

Theorem 16. *We have*

- (a1) $s(4(2^{2k} - 1)/3) = 2^{k+1} - 1$ for $k \geq 0$;
- (a2) $s(2^{2k+2} - 1) = 2^{k+1}$ for $k \geq 0$;
- (b1) $t(2^{2k}) = 2^k + 1$ for $k \geq 1$;
- (b2) $t((5 \cdot 2^{2k} - 2)/3) = 2^k - 1$ for $k \geq 0$;
- (c) If $0 \leq s < 2^k$ then $t(2^{2k+1} - 1 - 2m(s)) = 2s$;
- (d) If $0 \leq s < 2^k$ then $t(2^{2k+2} - 1 - 2m(s)) = 2^{k+1} - 2s$;
- (e) $t(2(2^{2k+2} - 1)/3) = 1$ for $k \geq 0$.

Proof. The following straightforward translations of the assertions all evaluate to TRUE in Walnut:

```

eval eq24a1 "?msd_4 An,x,y,z ($power4(x) & $link42(x,y) & 3*n+4=4*x &
  $rss(n,z)) => ?msd_2 z+1=2*y":
eval eq24a2 "?msd_4 An,x,y,z ($power4(x) & $link42(x,y) & n+1=4*x &
  $rss(n,z)) => ?msd_2 z=2*y":
eval eq24b1 "?msd_4 Ax,y,z ($power4(x) & x>1 & $link42(x,y) & $rst(x,z))
  => ?msd_2 z=y+1":
eval eq24b2 "?msd_4 An,x,y,z ($power4(x) & $link42(x,y) & 3*n+2=5*x &
  $rst(n,z)) => ?msd_2 z+1=y":
eval eq24c "?msd_4 An,s,x,y,w,z ($power4(x) & $link42(x,w) & $link42(y,s)
  & (?msd_2 s<w) & n+2*y+1=2*x & $rst(n,z)) => ?msd_2 z=2*s":
eval eq24d "?msd_4 An,s,x,y,w,z ($power4(x) & $link42(x,w) & $link42(y,s)
  & (?msd_2 s<w) & n+2*y+1=4*x & $rst(n,z)) => ?msd_2 z+2*s=2*w":
eval eq24e "?msd_4 Ax,n ($power4(x) & 3*n+2=8*x) => $rst(n,?msd_2 1)":

```

□

7 Inequalities

Probably the most celebrated parts of the results of Brillhart and Morton on the Rudin-Shapiro sums are the inequalities they proved.

Theorem 17 (Brillhart & Morton). *For $n \geq 1$ we have*

$$\begin{aligned}\sqrt{3n/5} &\leq s(n) \leq \sqrt{6n} \\ 0 &\leq t(n) \leq \sqrt{3n}.\end{aligned}$$

Trying to prove these results by *directly* translating the claims into **Walnut** leads to two difficulties: first, automata cannot compute squares or square roots. Second, our synchronized automata work with n expressed in base 4, but $s(n)$ and $t(n)$ are expressed in base 2, and **Walnut** cannot directly compare arbitrary integers expressed in different bases.

However, there is a way around both of these difficulties. First, we define a kind of “pseudo-square” function as follows: $m(n) = [(n)_2]_4$. In other words, m sends n to the integer obtained by interpreting the base-2 expansion of n as a number in base 4. Luckily we have already defined an automaton for m called **link42**; to get an automaton for m , we only have to reverse the order of the arguments in **link42**!

Now we need to see how far away from a real squaring function our pseudo-square function $m(n)$ is.

Lemma 18. *We have $(n^2 + 2n)/3 \leq m(n) \leq n^2$.*

Proof. We can prove the bounds by induction on n . They are clearly true for $n = 0$. Assume $n \geq 1$ and the inequalities hold for all $n' < n$; we prove them for n .

Suppose n is even. Then $n = 2k$. Clearly $m(n) = 4m(k)$. By induction we have $(k^2 + 2k)/3 \leq m(k) \leq k^2$, and multiplying through by 4 gives

$$(n^2 + 2n)/3 = (4k^2 + 4k)/3 < 4(k^2 + 2k)/3 \leq 4m(k) \leq 4k^2 = n^2.$$

Suppose n is odd. Then $n = 2k + 1$. Clearly $m(n) = 4m(k) + 1$. By induction we have $(k^2 + 2k)/3 \leq m(k) \leq k^2$. Multiplying by 4 and adding 1 gives

$$(n^2 + 2n)/3 = ((2k+1)^2 + 2(2k+1))/3 = (4k^2 + 8k + 3)/3 \leq 4m(k) + 1 \leq 4k^2 + 1 \leq (2k+1)^2 = n^2,$$

as desired. □

We can now prove:

Lemma 19. *For $n \geq 1$ we have $\frac{3n+7}{5} \leq m(s(n)) \leq 3n + 1$, and the upper and lower bounds are tight.*

Proof. We use the **Walnut** code

```

def maps "?msd_4 Ex $rss(n,x) & $link42(y,x)":
eval ms_lowerbnd "?msd_4 An,y (n>=1 & $maps(n,y)) => y<=3*n+1":
eval ms_upperbnd "?msd_4 An,y (n>=1 & $maps(n,y)) => 3*n+7<=5*y":

```

To show they are tight, let us show there are infinitely many solutions to $m(s(n)) = 3n+1$ and $m(s(n)) = (3n+7)/5$:

```

eval lowerbnd_tight "?msd_4 Am En,y (n>m) & $maps(n,y) & y=3*n+1":
eval upperbnd_tight "?msd_4 Am En,y (n>m) & $maps(n,y) & 5*y=3*n+7":

```

□

As a consequence, we get one lower bound in Theorem 17.

Corollary 20. *For $n \geq 1$ we have*

$$s(n) \geq \sqrt{\frac{3n+7}{5}}.$$

Proof. From Theorem 18 we have $m(s(n)) \leq s(n)^2$ and from Lemma 19 we have $\frac{3n+7}{5} \leq m(s(n))$. Putting these two bounds together gives $\frac{3n+7}{5} \leq s(n)^2$. □

Note that our lower bound is actually slightly *stronger* than that of Brillhart-Morton!

To get the upper bound $s(n) \leq \sqrt{6n}$, as in Brillhart-Morton, we need to do more work, since the results we have proved so far only suffice to show that $s(n) \leq \sqrt{9n+3}$. To get their upper bound, Brillhart and Morton carved the various intervals for n up into three classes and proved the upper bound of $\sqrt{6n}$ for each class. We'll do the same thing, but use slightly different classes. By doing so we avoid their complicated induction entirely.

The first class is the easiest: those n for which $m(s(n)) \leq 2n$. For these n , Lemma 18 immediately gives us $s(n)^2 \leq 6n$, as desired. Furthermore, the “exceptional set” (that is, those n for which $m(s(n)) > 2n$) is calculatable with **Walnut**:

```

def exceptional_set "?msd_4 Em $maps(n,m) & m>2*n":

```

The resulting automaton is quite simple (2 states!) and recognizes the set of base-4 expansions $\{0,2\}^* \cup \{0,2\}^* 1 \{1,3\}^*$.

We readily see, then, that the exceptional set consists of

- (a) numbers whose base-4 expansion starts with a 1 and thereafter consists of 1's and 3's, and
- (b) the rest, which must start with a 2.

The numbers in group (a) are easiest to deal with, because they satisfy the inequality $M_k/2 \leq n < 2^{2k+1}$ for some $k \geq 0$. (Recall that $M_k = (2^{2k+3} - 2)/3$ was defined in Theorem 8.) Now for all n (not just those in the exceptional set) in the half-open interval $I_k := [M_k/2, 2^{2k+1})$ we can show with **Walnut** that $s(n) \leq 2^{k+1}$, as follows:

```
eval maxcheck "?msd_4 An,x,y,z ($power4(x) & 3*n+1>=4*x & n<2*x
& $rss(n,y) & $link42(x,z)) => ?msd_2 y<=2*z":
```

So for all $n \in I_k$ we have

$$\frac{s(n)^2}{n} \leq \frac{(\max_{n \in I_k} s(n))^2}{\min_{n \in I_k} n} = \frac{(2^{k+1})^2}{M_k/2} = 3 \frac{2^{2k+2}}{2^{2k+2} - 1} \leq 4.$$

This handles the numbers in group (a).

Finally, we turn to group (b), which are the hardest to deal with. These numbers lie in the interval $I'_k = [2^{2k+1}, M_k]$. We will split these numbers into the following intervals: $J_{k,i} := [M_k - M_i, M_k - M_{i-1})$ for $0 \leq i < k$. Since $M_k - M_{k-1} = 2^{2k+1}$, the union

$$J_{k,0} \cup J_{k,1} \cup \dots \cup J_{k,k-1} \cup \{M_k\}$$

forms a disjoint partition of the interval I'_k .

Now with Walnut we can prove that for $n \in J_{k,i}$ we have $s(n) \leq 2^{k+2} - 2^{i+1}$.

```
eval J_inequality "?msd_4 An,x,y,z,w,m ($rss(n,m) & $power4(x) & $power4(y)
& x>y & $link42(x,w) & $link42(y,z) & 8*x<=3*n+8*y & 3*n+2*y<8*x)
=> ?msd_2 m+2*z<=4*w":
```

It now follows that for $n \in J_{k,i}$, $k \geq 1$, and $0 \leq i < k$, we have

$$\frac{s(n)^2}{n} \leq \frac{(\max_{n \in J_{k,i}} s(n))^2}{\min_{n \in J_{k,i}} n} \leq \frac{(2^{k+2} - 2^{i+1})^2}{M_k - M_i}$$

and a routine manipulation² shows this is less than 6.

The only remaining case is M_k . But then $s(M_k) = 2^{k+2} - 1$, and then $s(M_k)^2 < 6M_k$ by another routine calculation.

Finally, we should verify that we have really covered all the possible n :

```
def left_endpoint "?msd_4 3*z+8*y=8*x":
def right_endpoint "?msd_4 3*z+2*y=8*x":
eval check_all "?msd_4 An (n>=1) => ((~$exceptional_set(n)) |
(Ex $power4(x) & 4*x<=3*n+1 & n<2*x) |
(Ex,y,z,w $power4(x) & $power4(y) & x>y &
$left_endpoint(x,y,z) & $right_endpoint(x,y,w) & n>=z & n<w) |
(Ex $power4(x) & 3*n+2=8*x))":
```

which evaluates to TRUE.

Thus we have proved one upper bound from Theorem 17:

²Here are the details. Since $k \geq 1$ and $0 \leq i < k$ we clearly have $2^{k+2} > 5 \cdot 2^i = 2^{i+2} + 2^i$. Multiplying by 2^{i+2} gives us $2^{k+i+4} > 2^{2i+4} + 2^{2i+2}$. Adding 2^{2k+4} to both sides, and rearranging gives $2^{2k+4} - 2^{k+i+4} + 2^{2i+2} < 2^{2k+4} - 2^{2i+4}$. In other words, $(2^{k+2} - 2^{i+1})^2 < 2(2^{2k+3} - 2^{2i+3})$. Hence $(2^{k+2} - 2^{i+1})^2 / (2^{2k+3} - 2^{2i+3}) < 2$, and so $(2^{k+2} - 2^{i+1})^2 / (M_k - M_i) < 6$.

Theorem 21. $s(n) \leq \sqrt{6n}$ for $n \geq 1$.

Using exactly the same techniques we can prove

Lemma 22. For all $n \geq 0$ we have $m(t(n)) \leq n + 1$, with equality iff $(n)_4 \in (0 \cup 11^*0)^*3^*$.

Proof. We use the following Walnut code:

```
def mapt "?msd_4 Ex $rst(n,x) & $link42(y,x)":
eval bnd "?msd_4 An,z $mapt(n,z) => z<=n+1":
def except2 "?msd_4 Ez $mapt(n,z) & z=n+1":
```

The command `bnd` returns TRUE, and the command `except2` computes a simple automaton of 3 states accepting the regular expression $(0 \cup 11^*0)^*3^*$. \square

From the first claim of Lemma 22 we see that $m(t(n)) \leq n + 1$, and by Lemma 18 we have $t(n)^2/3 \leq m(t(n))$. Putting these bounds together gives us $t(n) \leq \sqrt{3(n+1)}$, which is very close to the Brillhart-Morton upper bound for $t(n)$.

To get the other Brillhart-Morton upper bound of Theorem 17, we just use Eqs. (3) and (4), just as Brillhart and Morton did. This gives us

$$t(n) = s(n/2) - t(n/2 - 1) \leq s(n/2) \leq \sqrt{3n}$$

for $n \geq 2$ even and

$$t(n) = s((n-1)/2) - t((n-1)/2) \leq s((n-1)/2) \leq \sqrt{3(n-1)}$$

for n odd. Thus we have proved

Theorem 23. We have $t(n) \leq \sqrt{3n}$ for $n \geq 1$.

We can also reprove, in an extremely simple fashion, an inequality of Brillhart and Morton on the ω function introduced previously. Let us start by showing

Lemma 24. For $k \geq 0$ we have $\omega(k) \leq \frac{5}{3}m(k)$.

Proof. We verify this with the following Walnut command.

```
eval omegabound "?msd_4 Ak,x,y ($omega(k,x) & $link42(y,k)) => 3*x<=5*y":
and it returns TRUE.  $\square$ 
```

This, combined with Lemma 18 gives a proof of Theorem 5 in Brillhart and Morton [5]:

Corollary 25. We have $\omega(k) \leq \frac{5}{3}k^2$ for $k \geq 0$.

8 Counting the k for which $s(k) = n$

One of the most fun properties of the Rudin-Shapiro summation function $s(n)$ is Satz 22 of [3]:

Theorem 26. *There are exactly n values of k for which $s(k) = n$.*

Proof. We can prove this theorem “purely mechanically” by using another capability of **Walnut**: the fact that it can create base- b linear representations for values of synchronized sequences. By a *base- b linear representation* for a function $f(n)$ we mean vectors v, w , and a matrix-valued morphism γ such that $f(n) = v\gamma(x)w$ for all strings x representing n in base b . So let us find a base-2 linear representation for the number of such k for which $s(k) = n$:

```
eval satz22 n "$rss(?msd_4 k,n)":
```

This gives us a base-2 linear representation of rank 7 computing some function $f(n)$. Next we use **Walnut** to compute a base-2 linear representation for the function $g(n) = n$:

```
eval gfunc n "i<n":
```

From this, we can easily compute a base-2 linear representation for $f(n) - g(n)$, and minimize it using the algorithm³ of Schützenberger [2]. When we do so, we get the representation for the 0 function, so $f(n) = n$. \square

9 New results

One big advantage to the synchronized representation of the Rudin-Shapiro sum functions is that it becomes almost trivial to explore and rigorously prove new properties. As a new result, let’s consider the analogue of Theorem 26, but for the function t . Here we run into the problem that every natural number k appears as a value of $t(n)$ infinitely often:

```
eval tvalues "?msd_4 An,k Em (m>n) & $rst(m,k)":
```

and **Walnut** returns **TRUE**.

So it makes sense to count the number of times k appears as a value of $t(n)$ in some initial segment, say the first $0, 1, \dots, 2^r - 1$.

Some empirical calculations suggest the following conjecture:

Theorem 27.

- (a) For $n \in [0, 4^m/2)$, 0 appears as a value of $t(n)$ exactly 2^{m-1} times, and k appears exactly $2^m - k$ times for $1 \leq k < 2^m$.
- (b) For $n \in [0, 4^m)$, 0 appears as a value of $t(n)$ exactly $2^m - 1$ times, 2^m appears exactly once, and k appears $2(2^m - k)$ times for $1 \leq k < 2^m$.

³**Maple** code implementing this algorithm is available from the second author.

Proof. We use the following Walnut commands.

```
def counta1 k x "?msd_4 $rst(n,k) & $power4(x) & n<x":

def counta2 k x "?msd_4 Ey $power4(x) & $link42(x,y) &
  (?msd_2 (k=0 & n+1<y)|(k=y & n=0)|(1<=k & k<y & n+2*k<2*y))":

def countb1 k x "?msd_4 $rst(n,k) & $power4(x) & x>1 & 2*n<x ":

def countb2 k x "?msd_4 Ey $power4(x) & x>1 & $link42(x,y) &
  (?msd_2 (k=0 & 2*n<y)|(1<=k & k<y & n+k<y))":
```

The first two statements are used for part (a). The code `counta1` asserts that $x = 4^m$ for some m , and that $t(n) = k$ for some $n < x$. It returns a linear representation for the number of n for which this holds, as a function of k and x . The code `counta2` creates a formula that says that the number of n fulfills the conclusion of the theorem. From these linear representations we can create a linear representation for their difference. When we minimize it, we get the linear representation for the 0 function, so they compute the same function.

The same approach is used for (b). □

9.1 The α function

We now introduce an analogue of Brillhart and Morton's ω function, but for the *first* occurrence of each distinct value of $s(n)$; i.e., we define $\alpha(k)$ to be the smallest value of n for which $s(n) = k$. We can create a $(2, 4)$ -synchronized automaton for α as follows:

```
def alpha "?msd_4 $rss(n,k) & At (t<n) => ~$rss(t,k)":
```

This automaton is given in Figure 11.

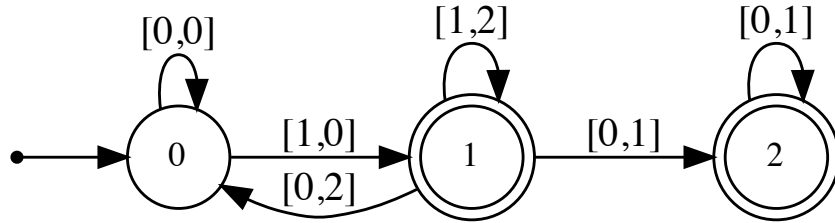


Figure 11: $(2, 4)$ -synchronized automaton for α .

Theorem 28. *Let $k \geq 1$ and write $k = k'2^\ell$ where k' is odd. Then*

$$\alpha(k) = m(k)/2 - (2 \cdot 4^{\ell-1} + 1)/3.$$

Proof. Let (k, n) be a pair accepted by the automaton in Figure 11. Suppose $\ell \geq 2$. Note that $(k)_2$ ends with ℓ 0's and $(n)_4$ ends with ℓ 1's. Let $r = [2^{\ell-2}3]_4$. We observe that $(4(n+r))_4$ consists only of 0's and 2's and has 2's exactly where $(k)_2$ has 1's. It follows that $4(n+r) = 2m(k)$; i.e.,

$$4 \left(n + 2 \sum_{i=0}^{\ell-2} 4^i + 1 \right) = 2m(k),$$

which gives $\alpha(k) = n = m(k)/2 - (2 \cdot 4^{\ell-1} + 1)/3$, as required. The cases $\ell = 0, 1$ are similar. \square

Similarly, we can consider the analogue of the α function for t instead of s ; let us call it α' . Remarkably, α' has a very simple expression in terms of known functions:

Theorem 29. *Define $\alpha'(k) = \min\{n : t(n) = k\}$. Then $\alpha'(k) = m(k) - 1$ for all $k \geq 1$.*

Proof. We use the following Walnut code:

```
def alphap "?msd_4 $rst(n,k) & At (t<n) => ~$rst(t,k)":
eval verify_alphap "?msd_4 Ak,t ((?msd_2 k>=1) & $alphap(k,t)) =>
$link42(t+1,k)":
```

and Walnut returns TRUE. \square

10 Plane-filling curves

In this section we show how to use our automata to prove results about the space-filling curve generated by connecting the lattice points P_n in the plane defined by $P_n = (x(n), y(n))$ for $n \geq 0$. This curve was previously explored in the papers [13, 12, 7]. The first 1024 points of this curve are illustrated in Figure 12, where rounded edges are used to make the curve clear.

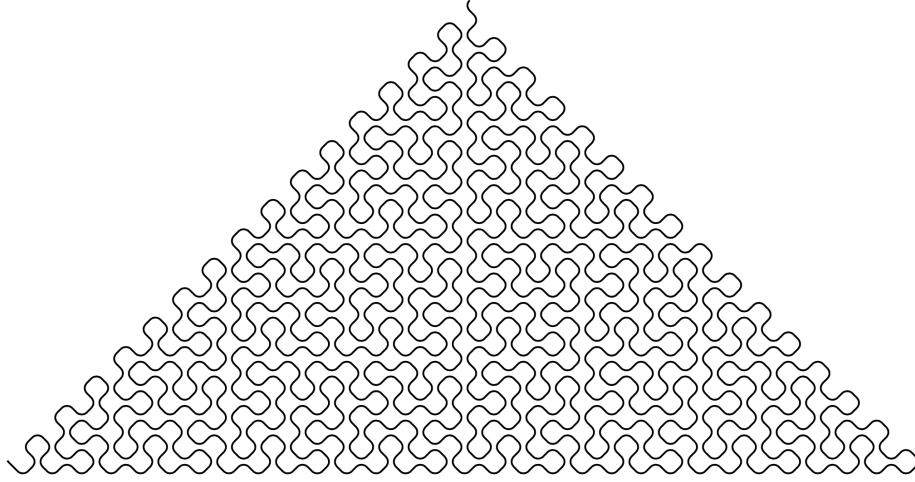


Figure 12: The curve fills one-eighth of the plane.

First let us determine exactly which lattice points are hit.

Theorem 30. *We have $(x, y) = (s(n), t(n))$ for $(x, y) \in \mathbb{N} \times \mathbb{N}$ if and only if $x \geq y$ and $(x, y) \neq (0, 0)$ and $x \equiv y \pmod{2}$. Furthermore, for each such pair (x, y) , there are at most two such n .*

Proof. We use the following Walnut code:

```
def even2 "Ek n=2*k":
def curve "?msd_4 $rss(n,x) & $rst(n,y)":
eval curvecheck "?msd_4 Ax,y (?msd_2 x>=y & x+y>0 & $even2(?msd_2 x-y)) <=>
  En $curve(n,x,y)":
eval curvecheck3 "?msd_4 Ex,y,n1,n2,n3 n1<n2 & n2<n3 &
  $curve(n1,x,y) & $curve(n2,x,y) & $curve(n3,x,y)":
```

The first check returns TRUE and the second, asserting a point that is hit three times, returns FALSE. □

Theorem 31. *The curve defined by $(P_n)_{n \geq 0}$ is not self-intersecting.*

Proof. Because of the parity condition on x, y in the pairs visited it suffices to show that we never traverse the same segment (P_n, P_{n+1}) twice for different n , either in the same direction, or the reverse direction. We do this as follows: we assert the existence of these traversals.

```
eval selfint1 "?msd_4 Em,x1,y1,x2,y2,n $curve(m,x1,y1) &
  $curve(m+1,x2,y2) & $curve(n,x1,y1) & $curve(n+1,x2,y2) & m!=n":

eval selfint2 "?msd_4 Em,x1,y1,x2,y2,n $curve(m,x1,y1) &
  $curve(m+1,x2,y2) & $curve(n+1,x1,y1) & $curve(n,x2,y2) & m!=n":
```

And Walnut returns FALSE for both. □

11 Going further

Since, as mentioned in the introduction, $a(n)$ is $+1$ or -1 , according to whether the number of 11's occurring in $(n)_2$ are even or odd, this suggests considering the analogous function $a'(n)$, where we instead count the number of 00's occurring in $(n)_2$. Then it is easy to see that $a'(n)$ obeys the recursion

$$\begin{aligned} a'(2n) &= (-1)^{n+1} a'(n) & (n \geq 1); \\ a'(2n+1) &= a'(n) & (n \geq 0), \end{aligned}$$

with initial condition $a'(0) = 1$. Then, in analogy with $s(n)$ and $t(n)$, one can consider the sums

$$\begin{aligned} s'(n) &= \sum_{0 \leq i \leq n} a'(i) \\ t'(n) &= \sum_{0 \leq i \leq n} (-1)^i a'(i). \end{aligned}$$

The first few values of these sequences are given in Table 2.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$a'(n)$	1	1	1	1	-1	1	1	1	1	-1	1	1	-1	1	1	1
$s'(n)$	1	2	3	4	3	4	5	6	7	6	7	8	7	8	9	10
$t'(n)$	1	-1	2	-2	1	-3	2	-4	3	-3	4	-4	3	-5	4	-6

Table 2: First few values of $a'(n)$, $s'(n)$, and $t'(n)$.

It turns out that both $s'(n)$ and $t'(n)$ are synchronized functions, which makes it possible to carry out the same kinds of analysis that we did for the Rudin-Shapiro sequence. However, since $t'(n)$ takes negative values, it's easier to work with $1 - t'(n)$ instead. Then it is possible to prove that both $s'(n)$ and $1 - t'(n)$ are $(4, 2)$ -synchronized.

We just mention a few results without proof, leaving proofs and further explorations to the reader.

Theorem 32.

(a) *We have*

$$\begin{aligned} s'(2n) &= s'(n-1) - t'(n) + 2, & (n \geq 1) \\ s'(2n+1) &= s'(n) - t'(n) + 2, & (n \geq 0) \\ t'(2n) &= -t'(n) - s'(n-1) + 2, & (n \geq 0) \\ t'(2n+1) &= -t'(n) - s'(n) + 2 & (n \geq 0). \end{aligned}$$

(b) We have

$$\begin{aligned}
s'(4n) &= 2s'(n) - (2 - (-1)^n)r'_n + 2, & (n \geq 1) \\
s'(4n+1) &= 2s'(n) - 2r'_n + 2, & (n \geq 0) \\
s'(4n+2) &= 2s'(n) - r'_n + 2, & (n \geq 0) \\
s'(4n+3) &= 2s'(n) + 2, & (n \geq 0).
\end{aligned}$$

(c) For $k \geq 1$, the minimum value of $s'(n)$ for $n \in [4^k, 4^{k+1})$ is $2^{k+1} - 1$ and $s'(n)$ attains this value only when $n = (4^{k+1} - 4)/3$.

For $k \geq 0$, the maximum value of $s'(n)$ for $n \in [4^k, 4^{k+1})$ is $3 \cdot 4^{k-1} - 2$ and $s'(n)$ attains this value only when $n = 4^{k+1} - 1$.

(d) For $n \geq 1$ we have $3\sqrt{n}/2 \leq s'(n) \leq \sqrt{75n}/7$.

(e) $\liminf s'(n)/\sqrt{n} = \sqrt{3}$.

(f) For $n \geq 0$ we have $-\sqrt{24n/7} \leq t'(n) \leq 0$.

Similarly, many of the results in [11] can be rederived using a $(4, 2)$ -synchronized automaton for their summation function $S(N)$.

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