

## Note

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# The equivalence problem of multitape finite automata

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### *Abstract*

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Using a result of B.H. Neumann we extend Eilenberg's Equality Theorem to a general result which implies that the multiplicity equivalence problem of two (nondeterministic) multitape finite automata is decidable. As a corollary we solve a long standing open problem in automata theory, namely, the equivalence problem for multitape deterministic finite automata. The main theorem states that there is a finite test set for the multiplicity equivalence of finite automata over conservative monoids embeddable in a fully ordered group.

## 1. Introduction

One of the oldest and most famous problems in automata theory is the equivalence problem for deterministic multitape finite automata. The notion of multitape finite automaton, or multitape automaton for short, was introduced by Rabin and Scott in their classic paper of 1959 [16]. They also showed that, unlike for ordinary (one-tape) finite automata, nondeterministic multitape automata are more powerful than the deterministic ones. This holds already in the case of two tapes.

As a central model of automata, multitape automata have gained plenty of attention. However, many important problems have remained open, including the equivalence problem in the deterministic case. For nondeterministic multitape

automata (even for two-tape automata, which are normally called finite transducers) the equivalence problem is a standard example of an undecidable problem (see [1]). This undecidability result was first proved by Griffiths in 1968 [8].

The equivalence problem of multitape deterministic automata has, as far as we know, been expected to be decidable. It seems that in this context “equivalence” implies “structural similarity”. Despite this, the equivalence problem has been solved only in a few special cases. The oldest result is that of Bird from 1973 [3] which solves the problem for two-tape deterministic automata. An alternative solution to the two-tape case was given in [19]. Numerous attempts (see [13, 12, 5]) to solve the general problem have led to only modest success so far. The difficulty of the equivalence problem is already manifested in the fact that the inclusion problem for multitape deterministic automata is easily seen to be undecidable. ~~XX~~ *tapes ?*

Our approach is as follows. Instead of deterministic multitape automata we consider nondeterministic multitape automata with multiplicities. Thus we ask whether two given multitape automata are multiplicity equivalent, that is, whether they accept the same  $n$ -tuples of words exactly the same number of times. The multiplicity equivalence clearly reduces to ordinary equivalence if the automata are deterministic, and even when they are unambiguous. The multiplicity equivalence problem for finite transducers has been considered an important open problem of its own (see [11]).

Consequently, we attack a nontrivial generalization of one of the famous open problems in automata theory. Hence a few explaining words are in order. First of all, intuitively, the deterministic behaviour of nondeterministic devices is in a sense captured by multiplicities. A nice example of this correspondence is Eilenberg's Equality Theorem [6]. It shows that in order to test the equivalence of two deterministic (one-tape) automata and to test the multiplicity equivalence of two nondeterministic (one-tape) automata it is enough to consider the computations of exactly the same lengths (in terms of the size of state sets). Secondly, the multiplicity considerations provide new tools to attack the original problem. Indeed, as shown by Eilenberg's proof of Equality Theorem, methods of classical algebra become applicable. This turns out to be decisive.

Eilenberg showed that the multiplicity equivalence of one-tape automata is decidable when the multiplicities are taken from a subsemiring  $R$  of a field. Eilenberg's proof extends immediately to one-tape automata, where the multiplicities are taken from a subsemiring  $R$  of a division ring. In particular, the semiring  $R$  need not be commutative. We shall use a restriction (to unary alphabet) of this Equality Theorem. However, a more general semiring  $R$  yields a proof of the decidability of the multiplicity equivalence problem for the multitape automata.

Neumann Here we need a result from [14], which states that the ring of all formal power series over a fully ordered group with well ordered supports is a division ring.

For the elementary results of division rings we refer to [10]. Formal power series in connection to automata theory are well treated in [18] and [2]. Fully ordered groups and the results needed for these, including the above mentioned Neumann's

result, can be found in [7] and in a more general setting in [4] and [15]. For the automata theoretic prerequisites we refer to [1] and [6].

## 2. Reduction to one-tape automata

In this section we reduce the multiplicity equivalence problem for multitape automata to the corresponding problem for one-tape automata, which, in addition, has a unary input alphabet. In this reduction the behaviour of a multitape automaton is encoded to the multiplicities of the new one-tape automaton. *& many alphabet*

In order to make the result more general and at the same time to simplify the notations, we consider automata over (conservative) semigroups. A reader who wishes a less abstract treatment is advised to consult our first report of the subject [9].

Let  $S$  be a finitely generated monoid, that is, a semigroup with an identity element 1. We denote by  $S^+$  the subset  $S - \{1\}$ . For a subset  $B \subseteq S$  we let  $[B]$  denote the submonoid of  $S$  generated by  $B$ .

The monoid  $S$  is said to be finitely factored if each element  $s \in S$  has only finitely many factorizations  $s = s_1 \cdot s_2 \cdot \dots \cdot s_n$  with each  $s_i \in S^+$ . In a finitely factored monoid  $S$  from  $s_1 \cdot s_2 = 1$  it follows that  $s_1 = 1 = s_2$ , and thus  $S$  is torsion-free and  $S^+$  is a subsemigroup of  $S$ . Also, a finitely factored monoid cannot have idempotents other than the identity element 1.

A finitely factored  $S$  is said to be conservative if there is a generator set  $B$  for  $S$  such that  $s_1 \cdot s_2 \cdot \dots \cdot s_n = r_1 \cdot r_2 \cdot \dots \cdot r_m$ ,  $s_i, r_i \in B$ , implies that  $n = m$ .

As an example the free monoid  $\Sigma^*$  and the free commutative monoid  $c\Sigma^*$  over a (finite or infinite) alphabet  $\Sigma$  are conservative. Also, if  $S_1, \dots, S_k$  are conservative monoids then so is their direct product  $S_1 \times S_2 \times \dots \times S_k$  and their free product  $S_1 * S_2 * \dots * S_k$ .

Let now  $S$  be a finitely factored monoid and define

$$S^{(i)} = S - \bigcup_{j > i} (S^+)^j$$

for all  $i = 0, 1, \dots$ , where  $(S^+)^j = \{s_1 \cdot s_2 \cdot \dots \cdot s_j \mid s_k \in S^+ \text{ for } k = 1, 2, \dots, j\}$ . Thus  $S^0 = \{1\}$  because in a finitely factored monoid no product of nonidentity elements can be identity.

**Lemma 2.1.** *Let  $S$  be a finitely factored monoid. Then*

- (1)  $\bigcup_{i \geq 0} S^{(i)} = S$ ,
- (2)  $S^{(1)}$  generates  $S$ ,
- (3)  $S^{(i)} \subsetneq S^{(i+1)}$  for all  $i \geq 0$ .

*Moreover, if  $S$  is conservative then  $(S^+)^i = S^{(i)} - S^{(i-1)}$  for all  $i$ .*

**Proof.** Let  $s \in S$ . There are only finitely many factorizations of  $s$  and thus there exists a factorization of maximum length,  $s = s_1 \cdot s_2 \cdot \dots \cdot s_n$ , where  $s_i \in S^+$  for all

$i = 1, 2, \dots, n$ . This means that  $s \in S^{(n)}$  and hence the first claim follows. It follows also that  $s_i \notin (S^+)^j$  for  $j \geq 2$  and hence  $s_i \in S^{(1)}$  for all  $1 \leq i \leq n$ . Consequently,  $S^{(1)}$  generates  $S$ .

If  $S^{(i)} = S^{(i+1)}$  for some  $i \geq 0$ , then clearly  $S$  would be a finite semigroup and hence not finitely factored.

Finally, if  $S$  is conservative then for each  $s \in S$  there is a unique integer  $k$  such that  $s = s_1 \cdot s_2 \cdot \dots \cdot s_n$  for  $s_i \in S^{(1)}$ ,  $i = 1, 2, \dots, n$ , implies that  $n = k$ . The last claim follows from this.  $\square$

The generator set  $S^{(1)}$  obtained for a finitely factored monoid  $S$  is unique as a minimal generator set (i.e., the base) for  $S$ . The existence of a base for a monoid is equivalent to its being finitely factored.

**Lemma 2.2.** *Monoid  $S$  is finitely factored if and only if  $S$  has a base.*

**Proof.** Suppose  $S$  has a base  $B$ . Then for every  $s \in S^{(1)}$  there are generators  $t_1, \dots, t_k$  in  $B$  such that  $s = t_1 \cdot t_2 \cdot \dots \cdot t_k$ . This implies immediately that  $k = 1$  and hence that  $B = S^{(1)}$ .  $\square$

For a conservative monoid  $S$  we let  $|s|$  denote the *length* of the element  $s \in S$ , that is,  $|s| = k$ , if  $s \in (S^{(1)})^k$ .

Clearly, for a finitely factored (resp. conservative) monoid  $S$  the subsemigroups  $[B]$  generated by subsets  $B \subseteq S^{(1)}$  are finitely factored (resp. conservative).

The following definition of an  $R$ - $S$ -automaton is in accordance with Eilenberg's definition of the corresponding one-tape automaton for free monoids [6]. We assume that a semiring  $R$  always contains an identity element.

Let  $S$  be a finitely factored semigroup and  $R$  a semiring. An  $R$ - $S$ -automaton is defined as a tuple  $A = (Q, E, \mu, I, T)$ , where  $Q$  is a finite set of nodes (or states),  $E \subseteq Q \times S^+ \times Q$  is a finite set of edges (or transitions),  $\mu: E \rightarrow R$  is a multiplicity (weight) function,  $I$  is a set of initial nodes and  $T$  is a set of final nodes.

A *path* of  $A$  is any sequence  $p = e_1 \cdot e_2 \cdot \dots \cdot e_k$  of edges such that  $e_i = (q_{i-1}, s_i, q_i) \in E$  ( $i = 1, 2, \dots, k$ ). The *label* of  $p$  is the element  $s = s_1 \cdot s_2 \cdot \dots \cdot s_k$ , and  $p$  is *successful* if  $q_0$  is an initial node and  $q_k$  is a final node. The *multiplicity* of the path  $p$  is the element  $p\mu = e_1\mu \cdot \dots \cdot e_k\mu$  of  $R$ . The *multiplicity* of an element  $s \in S$  is defined as the sum  $sA = \sum_p p\mu$  over all successful paths  $p$  with label  $s$ . Since  $S$  is supposed to be finitely factored,  $sA \in R$  for all  $s$ . Thus an  $R$ - $S$ -automaton  $A$  defines a mapping  $A: S \rightarrow R$ . Two  $R$ - $S$ -automata  $A_1$  and  $A_2$  are multiplicity equivalent if they define the same mappings.

An  $R$ - $S$ -automaton  $A$  is normalized if its edges are in  $Q \times S^{(1)} \times Q$ . Each  $R$ - $S$ -automaton  $A$  can be transformed to a multiplicity equivalent normalized  $R$ - $S$ -automaton by dissecting the edges  $e \in E$  of  $A$  to paths (with additional new nodes) corresponding to a chosen factorization of the label  $s \in S$  of  $e$ . The function  $\mu$  is modified so that it gives the value 1 for all but one of the added edges, and the

No tape emptiness tests

remaining edge obtains the value of the original edge  $e$  of  $A$ . We omit the details of this straightforward construction.

Let  $S$  be a monoid and consider the set of all formal power series,  $R\langle\langle S \rangle\rangle$ , over  $S$  with coefficients in the semiring  $R$ . Define the sum and product in  $R\langle\langle S \rangle\rangle$  in a natural way:

$$\sum_s n_s s + \sum_s m_s s = \sum_s (n_s + m_s) s,$$

$$\left( \sum_s n_s s \right) \left( \sum_s m_s s \right) = \sum_s \left( \sum_{uv=s} n_u m_v \right) s.$$

We note that the product in  $R\langle\langle S \rangle\rangle$  is well defined for finitely factored monoids  $S$ . In fact, the product is needed here only in the polynomial semiring  $R\langle S \rangle$  which consists of all finite formal sums  $\sum_{s \in S} n_s s$  and which is a subsemiring of  $R\langle\langle S \rangle\rangle$ . However, in Section 3 formal power series over (fully ordered) groups are used and then the product is not automatically well defined any more.

If  $S$  is a finitely factored monoid then in an  $R$ - $S$ -automaton  $A$  the multiplicity  $sA \in R$  for each  $s \in S$ .

**We reduce the equivalence problem of an  $R$ - $S$ -automata with  $S$  conservative to the equivalence problem of  $R\langle S \rangle$ - $\Sigma^*$ -automata**, where  $\Sigma$  is a finite alphabet. In fact,  $\Sigma$  will be a one-letter alphabet,  $\Sigma = \{a\}$ .

Let  $S$  be a conservative monoid and let  $A = (Q, E, \mu, I, T)$  be a normalized  $R$ - $S$ -automaton. Define an  $R\langle S \rangle$ - $\Sigma^*$ -automaton  $A^{(a)} = (Q, E^{(a)}, \mu^{(a)}, I, T)$  with

$$E^{(a)} = \{(q, a, p) \mid (q, s, p) \in E\},$$

$$(q, a, p) \mu^{(a)} = \sum_{s \in S^{(1)}} (q, s, p) \mu \cdot s.$$

$\approx \mathbb{N} \rightarrow R\langle S \rangle$  linear recursive sequences over a ring

$$f(a \cdot w) = \alpha f(w) + \beta p(w)$$

$$f(b \cdot w) = \gamma f(w) + \delta q(w)$$

$$\leadsto f(m+2) = (\alpha \cdot a + \gamma \cdot b) f(m) + (\beta a + \delta b) g(m).$$

**Theorem 2.3.** Let  $A_1$  and  $A_2$  be two normalized  $R$ - $S$ -automata for a conservative monoid  $S$ . Then  $sA_1 = sA_2$  holds for all  $s \in S$  if and only if  $uA_1^{(a)} = uA_2^{(a)}$  holds for all  $u \in \Sigma^*$ .

**Proof.** Every successful path for  $s$  in  $A_i$ ,  $i = 1, 2$ , has length  $|s|$  and thus  $sA_i = sA_2$  for all  $s \in S$  if and only if for all  $k \geq 0$ ,  $sA_i = sA_2$  for all  $s \in S^{(k)}$ . This is equivalent to  $\sum_{|s|=k} (sA_1)s = \sum_{|s|=k} (sA_2)s$  for all  $k$ . Here  $\sum_{|s|=k} (sA_i)s = a^k A_i^{(a)}$  for  $i = 1, 2$ , and hence the claim follows.  $\square$

Above we assumed that an  $R$ - $S$ -automaton reads a nonidentity element in its edges, that is, we do not allow edges of the form  $(q, 1, p)$ . In fact, the multiplicity equivalence problem for finite transducers is undecidable if we allow the transitions which "read and write" the empty word, that is,  $(q, (1, 1), p) \in E$ . To see this, consider two finite transducers  $T_1$  and  $T_2$  and define two new transducers by adding the loops  $(q, (1, 1), q)$  to each state  $q$  of  $T_1$  and  $T_2$ , respectively. The new transducers have infinitely many computations for each input-output pair and, consequently the original transducers are equivalent (without multiplicity) just in case the modified transducers are "multiplicatively" equivalent. Hence the undecidability follows.

$$f(a \cdot u, v) = \alpha \cdot f(u, v) + \dots; f(u, a \cdot v) = \beta \cdot f(u, v) + \dots$$

$$f(b \cdot u, v) = \gamma \cdot f(u, v) + \dots; f(u, b \cdot v) = \delta \cdot f(u, v) + \dots$$

$$\leadsto f(m+1) = (\alpha \cdot \langle a, \epsilon \rangle + \beta \cdot \langle \epsilon, a \rangle + \gamma \cdot \langle b, \epsilon \rangle + \delta \cdot \langle \epsilon, b \rangle) \cdot f(m)$$

Free abelian groups are fully ordered  $\approx$  lex order

on  $\mathbb{Z}^k$

(when finitely generated)

### 3. Decidability results

We firstly restate Eilenberg's Equality Theorem [6, pp. 143–145] for division rings.

**Theorem 3.1.** Let  $R$  be a subsemiring of a division ring  $P$  and let  $\Sigma$  be an alphabet. Then the normalized  $R$ - $\Sigma^*$ -automata  $A_1$  and  $A_2$  are multiplicatively equivalent if and only if  $sA_1 = sA_2$  for all  $s \in \Sigma^*$  with  $|s| < \text{card}(Q_1) + \text{card}(Q_2)$ , where  $Q_i$  is the state set of  $A_i$  for  $i = 1, 2$ .

The proof in [6] for a subsemiring of a (commutative) field  $K$  is based on the following properties of fields. Let  $V$  be a finite dimensional vector space over the field  $K$ . If  $U$  is a subspace of  $V$  then the dimension of  $U$  is at most that of  $V$ , and if  $U$  has the same dimension as  $V$  then  $U = V$ . These dimension results are equally valid for division rings, see [10], and hence the proof of Theorem 8.1 of [6] generalizes as such to subsemirings of division rings.

Our next step is to use a result of Neumann [14]. We refer also to [4], [7] and [15] for this result. We begin by introducing the formal power series needed in the theorem.

A group  $G$  is fully ordered if there exists a linear ordering  $\leq$  of  $G$ , which respects right and left multiplication: for all  $g, h, t$ , if  $g < h$  then  $gt < ht$  and  $tg < th$ . Let  $P$  be a division ring,  $G$  a fully ordered group and let  $B = \sum_g n_g g \in P\langle\langle G \rangle\rangle$ . The support of  $B$  is the set  $\{g \in G : n_g \neq 0\}$ . Let  $P_{\text{wo}}\langle\langle G \rangle\rangle$  denote the family of all series from  $P\langle\langle G \rangle\rangle$  with well ordered support, i.e., the series for which every subset of the support has a least element with respect to the ordering of  $G$ . The well ordered power series are used instead of the general power series in order to obtain a ring structure. Indeed, the product of two (general) series from  $P\langle\langle G \rangle\rangle$  is not necessarily well defined (see e.g. [4]).

need to avoid infinite sums...

**Theorem 3.2.** Let  $G$  be a fully ordered group and  $P$  a division ring. Then  $P_{\text{wo}}\langle\langle G \rangle\rangle$  is a division ring. For instance, Laurent series form a field.

If  $S$  is a submonoid of a fully ordered group  $G$  then  $P(S)$  is a subsemiring of  $P(G)$  and the latter is contained in the division ring  $P_{\text{wo}}\langle\langle G \rangle\rangle$ . Hence

**Theorem 3.3.** Let  $S$  be a submonoid of a fully ordered group and let  $R$  be a subsemiring of a division ring. Then  $R(S)$  is a subsemiring of a division ring.

Combining the above two theorems with Theorem 2.1 we obtain the following general result.

**Theorem 3.4.** Let  $R$  be a subsemiring of a division ring  $P$  and let  $S$  be a conservative monoid that can be embedded in a fully ordered group  $G$ . Then the normalized  $R$ - $S$ -automata  $A_1$  and  $A_2$  are multiplicatively equivalent if and only if  $sA_1 = sA_2$  for all  $s \in S$  with  $|s| < \text{card}(Q_1) + \text{card}(Q_2)$ , where  $Q_i$  is the state set of  $A_i$  for  $i = 1, 2$ .

For instance, Laurent monomial terms  $x^a y^b$ ,  $a, b \in \mathbb{Z}$  form a fully ordered (abelian) group.

In particular,  $\Sigma^*$  is a submonoid of a ordered group

**Proof.** Let  $r = \text{card}(Q_1) + \text{card}(Q_2)$  and let  $A_i^{(a)}$ ,  $i = 1, 2$ , be as in Theorem 2.1. Now by Neumann's result  $P_{\text{wo}}\langle\langle G \rangle\rangle$  is a division ring. The polynomial semiring  $R\langle S \rangle$  is a subsemiring of this division ring and thus we can consider  $A_1^{(a)}$  and  $A_2^{(a)}$  as  $P_{\text{wo}}\langle\langle G \rangle\rangle$ - $\Sigma^*$ -automata. By Equality Theorem  $A_1^{(a)} = A_2^{(a)}$  if and only if  $a^k A_1^{(a)} = a^k A_2^{(a)}$  for all  $k < r$ . But, by the proof of Theorem 2.1 this is equivalent to  $\sum_{|s|=k} (sA_1)s = \sum_{|s|=k} (sA_2)s$  for all  $k < r$ . This proves the theorem.  $\square$

Thus the multiplicity equivalence problem for  $R$ - $S$ -automata satisfying the demands of the previous theorem reduces to testing the equality for a finite set of elements. The effectiveness of the testing depends on the semiring  $R$  and the conservative monoid  $S$ .

For our original goals we need the result that every direct product of free groups is fully ordered. It is by no means an easy task to fully order a free group. This can be done using a general result of Neumann stating that a group  $G$  is fully ordered if the factor groups in the lower central series of  $G$  are torsion-free and the series terminates at the trivial group. The Magnus-Witt theorem says that the free groups possess this lower central series property. Another proof makes use of the following Vinogradov's result (see [15]). The free product of fully ordered groups is again fully ordered. Now a free group is the free product of cyclic groups and since the cyclic groups are easily fully ordered we obtain the result for free groups.

**Theorem 3.5.** Every free group is fully ordered.

→ all elements except 1 have infinite order  
Also every torsion-free Abelian group is fully ordered and hence the free Abelian groups are fully ordered (see [7]).

Now, if the groups  $G_1, \dots, G_k$  are fully ordered then their direct product  $G_1 \times G_2 \times \dots \times G_k$  can also be fully ordered. To see this let  $\leq_i$  be the full ordering for  $G_i$ ,  $i = 1, 2, \dots, k$ , and define the ordering  $\leq$  for  $G_1 \times G_2 \times \dots \times G_k$  by:  $(g_1, \dots, g_k) < (h_1, \dots, h_k)$  if and only if there is a  $j$  such that  $g_j < h_j$  and for all  $i < j$ ,  $g_i = h_i$ . Clearly, the ordering  $\leq$  is a full ordering in the direct product.

lex  
order

**Theorem 3.6.** The direct product of fully ordered groups is fully ordered.

The direct product  $S = \Sigma_1^* \times \Sigma_2^* \times \dots \times \Sigma_k^*$  of the free monoids  $\Sigma_i^*$  is a submonoid of the fully ordered group  $F_1 \times F_2 \times \dots \times F_k$ , where  $F_i$  is the free group generated by  $\Sigma_i$  for  $i = 1, 2, \dots, k$ . Thus we have the following special case of Theorem 3.4.

**Theorem 3.7.** Let  $S$  be a direct product of free monoids and let  $R$  be a subsemiring of a division ring. For two normalized  $R$ - $S$ -automata  $A_1$  and  $A_2$ ,  $sA_1 = sA_2$  holds for all  $s \in S$  if and only if  $sA_1 = sA_2$  holds for all  $s$  with  $|s| < \text{card}(Q_1) + \text{card}(Q_2)$ , where  $Q_i$  is the state set for  $A_i$ ,  $i = 1, 2$ .

When we choose  $R$  to be  $\mathbb{N}$ ,  $\mathbb{Z}$  or  $\mathbb{Q}$  in the above theorem we can decide whether or not two given polynomials from  $R\langle S \rangle$  are equal.

**Theorem 3.8.** *Let  $S$  be a direct product of finitely many free monoids. Then it is decidable whether or not two  $\mathbf{Q}$ - $S$ -automata are multiplicatively equivalent.*

The same reasoning is valid for direct products of free commutative monoids.

*Then  $S$  is finitely factored*  
**Theorem 3.9.** *Let  $S$  be a direct product of finitely many free commutative monoids and free monoids. Then it is decidable whether or not two  $\mathbf{Q}$ - $S$ -automata are multiplicatively equivalent.*

$$S = (\Sigma^*)^m$$

We obtain the  $n$ -tape finite automata as  $\mathbf{Q}$ - $S$ -automata by setting the multiplicity  $\mu$  equal to 1 for each edge  $e = (p, s, q)$  of the automaton.

**Theorem 3.10.** *The multiplicity equivalence problem for  $n$ -tape finite automata is decidable.*

An  $n$ -tape finite automaton  $A$  is unambiguous if for each  $s \in S$  there is at most one computation of  $A$  which accepts  $s$ . Hence  $A$  is unambiguous if and only if  $s\mu = 1$  when  $A$  accepts  $s$  and  $s\mu = 0$  otherwise. In this case the semiring  $R$  does not play any role and by the above theorem we can decide whether two given unambiguous  $n$ -tape finite automata are equivalent. We note here that it is undecidable to determine whether a multitape automaton is unambiguous.

No matter how an  $n$ -tape deterministic finite automaton is defined in details (for example, with or without endmarkers) it is natural to require the unambiguity. Hence we have a solution to the equivalence problem in this case.

**Theorem 3.11.** *The equivalence problem for the  $n$ -tape deterministic finite automata is decidable.*

#### 4. Discussion

In solving the equivalence problem of deterministic multitape automata we generalized the problem considerably to the multiplicity equivalence problem for nondeterministic multitape automata. The generalization to conservative monoids is not essential for this problem. However, we have tried to be general because the approach of this paper may turn out to be useful in solving other equivalence problems. In particular, we had in mind another famous open problem of formal languages: the equivalence problem of deterministic pushdown automata.

As in our considerations above the fact that the ordinary equivalence problem (of nondeterministic devices) is undecidable does not mean that the multiplicity equivalence should be undecidable as well. Indeed, we have the following example of a subfamily of context-free languages supporting this view.

Let us call a linear context-free grammar  $G = (N, \Sigma, S, P)$  marked, if either

(i) the sets  $\bigcup_{p \in P} \{a \in \Sigma \mid \exists (X, \alpha) \in P: \alpha \in \Sigma^* \underline{a} \Sigma^* N \Sigma^*\}$  and  $\bigcup_{p \in P} \{a \in \Sigma \mid \exists (X, \alpha) \in P: \alpha \in \Sigma^* N \Sigma^* \underline{a} \Sigma^*\}$  are disjoint, or

*Algebraic approach to multiplicity equivalence  
of context-free grammars / pushdown automata?*



(ii) for each  $(X, \alpha) \in P$  with  $\alpha \in \Sigma^*$ , there exists a letter in  $\text{alph}(\alpha) - \{a \in \Sigma \mid \exists (Y, \beta) \in P: \beta \in \Sigma^* a \Sigma^* N \Sigma^* \cup \Sigma^* N \Sigma^* a \Sigma^*\}$ , where  $\text{alph}(\alpha)$  denotes the letters occurring in  $\alpha$ .

Consequently, a linear grammar is marked if the "middle" of each generated word can be locally identified.

Now, since marked linear context-free grammars can be simulated by two-tape automata (respecting the multiplicities), it follows from Theorem 3.10 that the multiplicity equivalence of marked linear context-free grammars is decidable. On the other hand, it follows from [17] that their ordinary equivalence is undecidable.

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