On the locality of arb-invariant first-order logic with modulo counting quantifiers

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Highlights 2013

Structures & Logic

Our aim: understanding the expressive power of first-order logic (FO) extended with

modulo m counting (FO+MOD_m)

numerical relations $(<, +, \times, \dots)$

over structures that are finite and relational.

in this talk:
we consider only colored finite directed graphs $G = (V, E, C_1, \dots, C_\ell)$ viewed as structures with signature $\sigma := \{\mathrm{E}, C_1, \dots, C_\ell\}$.

Modulo counting

FO+MOD_m: FO + modulo counting quantifiers $\exists^{j \bmod m}$

where

$$G \models \exists^{j \bmod m} x\, \varphi(x)$$

if there are $j \pmod m$ vertices v such that $G \models \varphi(v)$.

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Numerical relations

Allowing arithmetic in formulas

k-ary numerical relation: subset of $\mathbb{N}^k$.

$\mathcal{ARB}$: set of all ("arbitrary") numerical relations.

Examples : $<$, $+$, $\times$, HALT etc., where e.g.

- $+ := \{(x, y, z) \in \mathbb{N}^3 : x + y = z\}$,
- $\text{HALT} = \{i \in \mathbb{N} : \text{TM } i \text{ halts on } \epsilon\}$.

embedding (G, f) of G: f is a bijection of V and $\{1, \dots, |V|\}$.

Consider a set $\mathcal{N} \subseteq \mathcal{ARB}$ of numerical relations.

Interpret FO[$\sigma, \mathcal{N}$]-formulas φ in embeddings of graphs.

Question: How to define $G \models \varphi$?

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Making arithmetic well-behaved

Definition ($\mathcal{N}$-invariance):

A formula $\varphi \in \mathrm{FO}[\sigma, \mathcal{N}]$ is $\mathcal{N}$-invariant if for all finite graphs G and all embeddings E_1, E_2 of G:

$$E_1 \models \varphi \iff E_2 \models \varphi.$$

$\mathcal{N}$-inv-FO$[\sigma]$: set of all $\mathcal{N}$-invariant $\varphi \in \mathrm{FO}[\sigma, \mathcal{N}]$.

For $\varphi \in \mathcal{N}$-inv-FO$[\sigma]$, define $G \models \varphi$ as

$$E \models \varphi, \text{ for some embedding } E = (G, f) ,$$

Example: An $+$-invariant $\mathrm{FO}[\sigma, <, +]$-sentence φ defining the class of all finite graphs with an even number of vertices:

$$\varphi \;:=\; \exists x \, \exists z \Big(x + x = z \,\wedge\, \forall y \, (y < z \,\vee\, y = z) \Big)$$

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Gaifman local queries

k-ary query q : mapping of graphs G to k-ary relations $q(G)$, which is closed under isomorphism.

r-Ball at $\vec{a}$ in G : vertices at distance $\leq r$ to some a_i

r-Neighborhood at $\vec{a}$: subgraph induced by the r-ball at $\vec{a}$, with distinguished vertices $\vec{a}$

Definition (Gaifman locality):

Let $f : \mathbb{N} \to \mathbb{N}$.
A k-ary query q is Gaifman $f(n)$-local if

for sufficiently large numbers n, and
all graphs G on n vertices,
and all k-ary tuples $\vec{a}, \vec{b}$ of vertices,

if $\vec{a}$ and $\vec{b}$ have isomorphic $f(n)$-neighborhoods,
then $\vec{a} \in q(G) \iff \vec{b} \in q(G)$.

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Definition (Weak Gaifman locality):

Let $f : \mathbb{N} \to \mathbb{N}$.
A k-ary query q is weakly Gaifman $f(n)$-local if

for sufficiently large numbers n, and
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and all k-ary tuples $\vec{a}, \vec{b}$ of vertices with disjoint $f(n)$-balls,

if $\vec{a}$ and $\vec{b}$ have isomorphic $f(n)$-neighborhoods,
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Gaifman locality of FO[σ]

Each query q that is FO[σ]-definable is
Gaifman c-local for some constant $c = c(q)$.
(Hella, Libkin, Nurmonen 1990s; Gaifman '82)

Each query q that is $<$-inv-FO[σ]-definable is
Gaifman c-local for some constant $c = c(q)$.
(Grohe, Schwentick '98)

Each query q that is $\mathcal{ARB}$-inv-FO[σ]-definable is
Gaifman $(\log n)^c$-local for some constant $c = c(q)$.
(Anderson, van Melkebeek, Schweikardt, Segoufin '11)

Question: Are all FO+MOD$_m$-definable queries Gaifman local?

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> For each $m \geq 2$, there exists an $<$-inv-FO+MOD$_m[\sigma]$-definable unary query that is not $o(n)$-local.
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Question: Are all FO+MOD$_m$-definable queries Gaifman local? Weakly Gaifman local?

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For even m there exists an $<$-inv-FO+MOD$_m[\sigma]$-definable unary query that is not weakly $o(n)$-local.

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Results

Negative results:

$<$-inv-FO$+$MOD$_p[\sigma]$ <u>is not</u> Gaifman $o(n)$-local.
(not even for unary queries).

$<$-inv-FO$+$MOD$_2[\sigma]$ <u>is not</u> weakly Gaifman $o(n)$-local
(not even for unary queries on strings).

We introduce a new notion of locality, called **shift locality**.

Positive results:

$\mathcal{ARB}$-inv-FO$+$MOD$_p[\sigma]$ <u>is</u> shift local, for **prime powers** p.

The proof uses lower bounds from circuit complexity.

Shift locality is easily applied to derive non-expressibility results.

$\mathcal{ARB}$-inv-FO$+$MOD$_p[\sigma]$ <u>is</u> weakly Gaifman polylog-local, if p is an **odd prime power**.

Follows easily from shift locality.

Extending our results beyond prime powers p should be very hard, because

this would imply NLOGSPACE $\subseteq$ ACC^0, and the
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$<$-inv-FO+MOD$_2[\sigma]$ is more expressive than FO+MOD$_2[\sigma]$.

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