A Coinductive Confluence Proof for Infinitary Lambda-Calculus

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Abstract. We give a coinductive proof of confluence, up to equivalence of root-active subterms, of infinitary lambda-calculus. We also show confluence of Böhm reduction (with respect to root-active terms) in infinitary lambda-calculus. In contrast to previous proofs, our proof makes heavy use of coinduction and does not employ the notion of descendants.

1 Introduction

Infinitary lambda-calculus is a generalization of lambda-calculus that allows infinite lambda-terms and transfinite reductions. This enables the consideration of "limits" of terms under infinite reduction sequences. For instance, for a term $M \equiv (\lambda mx.mm)(\lambda mx.mm)$ we have

$$M \to_\beta \lambda x.M \to_\beta \lambda x.\lambda x.M \to_\beta \lambda x.\lambda x.\lambda x.M \to_\beta \ldots$$

Intuitively, the "value" of M is an infinite term L satisfying $L \equiv \lambda x.L$, where by $\equiv$ we denote identity of terms. In fact, L is the normal form of M in infinitary lambda-calculus.

In [6] it is shown that infinitary reductions may be defined coinductively. The standard non-coinductive definition makes explicit mention of ordinals and limits in a certain metric space [11, 14, 2]. Arguably, a coinductive approach is better suited to formalization in a proof-assistant.

We prove confluence of infinitary reduction up to equivalence of root-active subterms, and confluence of infinitary Böhm reduction w.r.t. root-active terms. These results have already been obtained in [11] with proofs involving the notion of descendants. However, our proof is coinductive. We show that the theory of infinitary lambda-calculus, to the extent studied here, may be entirely based on coinductive definitions and proofs, without even mentioning ordinals or metric convergence.

* Partly supported by Narodowe Centrum Nauki grant 2012/07/N/ST6/03398.

1.1 Related work

Infinitary lambda-calculus was introduced in [11, 10]. All results of this paper were already obtained in [11], by a different proof method. See also [14, 2, 5] for an overview of various results in infinitary lambda-calculus and infinitary rewriting. The coinductive definition of infinitary reductions was introduced in [6].

Our proof differs from the proof in [11]. Instead of using the notion of descendants, we rely on coinduction. Nonetheless, the overall structure of the whole proof and proofs of some lemmas are analogous to [11].

A method of coinductive confluence proofs somewhat similar to ours was given by Joachimski in [9]. However, Joachimski's notion of reduction does not correspond to strongly convergent reductions. Essentially, it allows for infinitely many parallel contractions in one step, but only finitely many reduction steps.

There are three well-known variants of infinitary lambda-calculus: the Λ^{111}, Λ^{001} and Λ^{101} calculi [2, 5, 11, 10]. The superscripts 111, 001, 101 indicate the depth measure used: *abc* means that we shall add $a/b/c$ to the depth when going down/left/right in the tree of the lambda-term [11, Definition 6]. We are concerned only with Λ^{111}. In this calculus, after addition of appropriate $\perp$-rules, every term has its Berarducci tree [10, 3] as the normal form. In Λ^{001} and Λ^{101}, the normal forms are, respectively, Böhm trees and Levy-Longo trees [11, 10]. With the addition of infinite eta reductions it is possible to also capture eta-Böhm trees as normal forms [13].

2 Preliminaries

2.1 Coinduction

In this section we give an introduction to coinduction, to the extent necessary to understand the proofs in this paper. Because of space limits and for the sake of readability we only present several examples from which we hope the general method should be clear. For more background on coinduction see e.g. [8, 12], or see [4] for a practical introduction to the usage of coinduction in the Coq proof assistant. Our use of coinduction does not correspond exactly to the coinduction principle of Coq [7] and some of our proofs cannot be directly formalized in Coq. They could probably be formalized in recent versions of Agda extended with copatterns and sized types [1]. Also, we do not directly employ the usual coinduction principle from [8]. The correctness criterion for our proofs is that they may all be interpreted in the way outlined below.

We do not attempt here to formulate a general coinduction principle or provide a formal system in which our proofs could be easily formalized. This is an interesting question by itself, but a seperate one from simply ensuring correctness of the proofs in each particular case, by indicating how to interpret them in ordinary set theory without using coinduction. In other words, our use of coinduction may be seen as a way of leaving implicit the tedious, annoying and purely technical details which would be necessary if the proofs were to be made inductive.

Consider the following definition by a grammar of a set $\mathbb{T}$ of terms, where V is an infinite set of variables.

$$\mathbb{T} ::= V \parallel A(\mathbb{T}) \parallel B(\mathbb{T},\mathbb{T})$$

Conventionally, this definition is interpreted inductively: the set $\mathbb{T}$ consists of all *finite* terms built up from variables and the constructors A and B. When interpreted coinductively, both *finite and infinite* terms are allowed. So then e.g. a term A^ω satisfying $A^\omega \equiv A(A^\omega)$ also belongs to $\mathbb{T}$. Formally, under the coinductive interpretation $\mathbb{T}$ is a final coalgebra of an appropriate endofunctor [8]. We will not get into details here. One may think of $\mathbb{T}$ as the set of all possibly infinite labelled trees with labels specified by the grammar.

A definition of a function f with codomain $\mathbb{T}$ is by guarded corecursion if each (co)recursive call of f occurs directly inside a constructor for $\mathbb{T}$. Such a definition determines a well-defined function and with the introduction of some technicalities it may be reformulated as an ordinary inductive definition (by induction on the length of positions in a term). An example of a function defined by guarded corecursion is substitution: a function taking two terms and a variable.

$$\begin{array}{lll} x[t/x] \equiv t & & A(s)[t/x] \equiv A(s[t/x]) \\ y[t/x] \equiv y & \text{if } y \neq x & B(s_1,s_2)[t/x] \equiv B(s_1[t/x], s_2[t/x]) \end{array}$$

Now consider a relation defined by the following derivation rules.

$$\frac{}{x \to_0 x} \qquad \frac{t \to_0 t'}{A(t) \to_0 B(t',t')} \qquad \frac{t \to_0 t'}{A(t) \to_0 A(t')} \qquad \frac{s \to_0 s' \quad t \to_0 t'}{B(s,t) \to_0 B(s',t')}$$

When interpreted coinductively, in addition to finite derivations we also allow infinite ones. More formally, one may interpret the relation $\to_0$ as the greatest fixpoint of a function $F : \mathcal{P}(\mathbb{T}\times\mathbb{T}) \to \mathcal{P}(\mathbb{T}\times\mathbb{T})$ defined as follows.

$$\begin{aligned} F(R) = \{\langle t_1,t_2\rangle \mid\ & (t_1 \equiv t_2 \equiv x) \vee \\ & \exists t,t' \, (t_1 \equiv A(t) \wedge t_2 \equiv B(t',t') \wedge R(t,t')) \vee \\ & \exists t,t' \, (t_1 \equiv A(t) \wedge t_2 \equiv A(t') \wedge R(t,t')) \vee \\ & \exists s,t,s',t' \, (t_1 \equiv B(s,t) \wedge t_2 \equiv B(s',t') \wedge R(s,s') \wedge R(t,t'))\} \end{aligned}$$

The function F is monotone, so by the Knaster-Tarski theorem its greatest fixpoint exists and may be obtained in the following way. By transfinite induction we define: $R_0 = \mathbb{T}\times\mathbb{T}$, $R_{\alpha+1} = F(R_\alpha)$, and $R_\lambda = \bigcap_{\alpha<\lambda} R_\alpha$ for λ a limit ordinal. Then there exists an ordinal ζ such that $R_\zeta = \to_0$ is the greatest fixpoint of F.

All coinductive proofs in this paper show statements of one of two forms: $\forall \overline{x}\,(\phi(\overline{x}) \to R(f(\overline{x})))$ or $\forall \overline{x}\,(\phi(\overline{x}) \to \exists y\,(R(\overline{x},y) \wedge S(\overline{x},y)))$, where R and S are relations defined coinductively by some derivation rules, $\overline{x}$ ranges over tuples of terms, y ranges over terms, and f is a function from tuples to tuples of terms.

In coinductive proofs we appeal to the "coinductive hypothesis", which at first sight may seem like assuming what we are supposed to prove. The trick is that we are allowed to use the result of an application of the coinductive hypothesis only in certain ways: we *have to* use it *directly* as a premise of some derivation rule, and we *must not* manipulate the resulting derivation in any other way. In contrast, in an inductive proof, the result of an application of the inductive hypothesis may be used in an arbitrary way, but there is a restriction on the parameters of the hypothesis – they should be smaller in an appropriate sense. We shall now give an example of a coinductive proof.

Example 1. We show by coinduction: if $t \in \mathbb{T}$ then $t \to_0 t$. If $t \equiv x$ then $t \to_0 t$ holds by the first rule. If $t \equiv A(s)$ then $s \to_0 s$ by the coinductive hypothesis. So $t \to_0 t$ by the third rule. If $t \equiv B(t_1, t_2)$ then $t_1 \to_0 t_1$ and $t_2 \to_0 t_2$ by the coinductive hypothesis. Hence $t \to_0 t$ by the fourth rule.

Formally, a coinductive proof of $\forall \overline{x} \, (\phi(\overline{x}) \to R(f(\overline{x})))$ may be interpreted as a proof by transfinite induction on an ordinal α of $\forall \overline{x} \, (\phi(\overline{x}) \to R_\alpha(f(\overline{x})))$ for $\alpha \leq \zeta$. The cases $\alpha = 0$ and α a limit ordinal are trivial and left implicit. A coinductive proof may be read as a proof of the inductive step for α a successor ordinal, where "coinductive hypothesis" means "inductive hypothesis" and the ordinal indices are left implicit.

In a few proofs we actually violate the above interpretation slightly by applying derivation rules to the result of an application of the coinductive hypothesis more than once. However, this is easily seen to be correct because $R_\beta \subseteq R_\alpha$ for $\alpha \leq \beta$.

The formal interpretation of a coinductive proof of a statement of the form $\forall \overline{x} \, (\phi(\overline{x}) \to \exists y \, (R(\overline{x}, y) \wedge S(\overline{x}, y)))$ is slightly more involved. Instead we show a statement $\forall \overline{x} \, (\phi(\overline{x}) \to (R(\overline{x}, f(\overline{x})) \wedge S(\overline{x}, f(\overline{x}))))$ with an appropriately chosen function f, whose corecursive definition is implicit in the proof. We indicate this with the following example.

Example 2. We show: if $t \to_0 t_1$ and $t \to_0 t_2$ then there exists t_3 such that $t_1 \to_0 t_3$ and $t_2 \to_0 t_3$. Coinduction with case analysis on $t \to_0 t_1$. For instance, assume $t \equiv A(t')$ and $t_1 \equiv B(t_1', t_1')$ with $t' \to_0 t_1'$. There are two cases.

1. $t_2 \equiv B(t_2', t_2')$ with $t' \to_0 t_2'$. By the coinductive hypothesis there is t_3' such that $t_1' \to_0 t_3'$ and $t_2' \to_0 t_3'$. Thus $t_1 \equiv B(t_1', t_1') \to_0 B(t_3', t_3')$ and $t_2 \equiv B(t_2', t_2') \to_0 B(t_3', t_3')$, by the last rule. Hence, we may take $t_3 \equiv B(t_3', t_3')$.
2. $t_2 \equiv A(t_2')$ with $t' \to_0 t_2'$. By the coinductive hypothesis there is t_3' such that $t_1' \to_0 t_3'$ and $t_2' \to_0 t_3'$. Hence $t_1 \equiv B(t_1', t_1') \to_0 B(t_3', t_3')$ by the last rule, and $t_2 \equiv A(t_2') \to_0 B(t_3', t_3')$ by the second rule. Thus we may take $t_3 \equiv B(t_3', t_3')$.

Formally, the above proof may be interpreted as actually showing: if $t \to_0 t_1$ and $t \to_0 t_2$ then $t_1 \to_0 f(t_1, t_2)$ and $t_2 \to_0 f(t_1, t_2)$ where f is defined corecursively:

$$
\begin{aligned}
f(x, x) &= x \\
f(A(t), B(s, s)) &= B(f(t, s), f(t, s)) \\
f(B(t, t), A(s)) &= B(f(t, s), f(t, s)) \\
f(A(t), A(s)) &= A(f(t, s)) \\
f(B(s, t), B(s', t')) &= B(f(s, s'), f(t, t'))
\end{aligned}
$$

The corecursive definition of f is implicit in the proof. When we obtain an element from an existential quantifier in the coinductive hypothesis, this corresponds to a corecursive invocation of f. Later when we exhibit an element for a proof of an existential statement, this implicitly defines f in the case specified by the assumptions currently active in the proof.

2.2 Infinitary lambda-calculus

In this section we define the syntax of infinitary lambda-calculus. We also provide definitions of various notions of reduction and other relations in infinitary lambda-calculus. Our definitions are coinductive. For standard introduction to infinitary lambda-calculus see e.g. [14, 11].

Definition 1. The set of Λ^∞-terms is defined coinductively:

$$\Lambda^\infty ::= V \parallel \lambda V.\Lambda^\infty \parallel \Lambda^\infty \Lambda^\infty$$

where V is an infinite set of variables.

Capture-avoiding substitution is defined by guarded corecursion.

$$\begin{array}{llll} x[t/x] \equiv t & & (s_1 s_2)[t/x] \equiv s_1[t/x] s_2[t/x] & \\ y[t/x] \equiv y & \text{when } x \neq y & (\lambda y.s)[t/x] \equiv \lambda y.s[t/x] & \text{when } y \notin FV(t) \end{array}$$

The relation $\to_\beta$ of *β-contraction* is defined inductively by the following rules.

$$\frac{}{(\lambda x.s)t \to_\beta s[t/x]} \qquad \frac{s \to_\beta s'}{st \to_\beta s't} \qquad \frac{t \to_\beta t'}{st \to_\beta st'} \qquad \frac{s \to_\beta s'}{\lambda x.s \to_\beta \lambda x.s'}$$

The relation $\to_\beta^*$ of *β-reduction* is the transitive-reflexive closure of $\to_\beta$.

The relation $\to_\beta^\infty$ of *infinitary β-reduction* is defined coinductively.

$$\frac{s \to_\beta^* x}{\overline{s \to_\beta^\infty x}} \qquad \frac{s \to_\beta^* t_1 t_2 \quad t_1 \to_\beta^\infty t_1' \quad t_2 \to_\beta^\infty t_2'}{\overline{s \to_\beta^\infty t_1' t_2'}} \qquad \frac{s \to_\beta^* \lambda x.r \quad r \to_\beta^\infty r'}{\overline{s \to_\beta^\infty \lambda x.r'}}$$

We will disregard the usual problems with α-conversion. In the infinitary setting it presents some additional, but purely technical difficulties [11, 6].

The idea with the definition of $\to_\beta^\infty$ is that the depth at which a redex is contracted should tend to infinity. This is achieved by defining $\to_\beta^\infty$ in such a way that always after finitely many reduction steps the subsequent contractions may be performed only under a constructor. So the depth of the contracted redex always ultimately increases. In [6] it is shown that the above definition of $\to_\beta^\infty$ coincides with the standard definition based on strongly convergent reductions.

Definition 2. Let $\bot$ be a constant, i.e. a variable which is assumed to never occur bound. A Λ^∞-term t is *root-stable* if either $t \equiv x$ with $x \not\equiv \bot$, or $t \equiv \lambda x.t'$, or $t \equiv t_1 t_2$ and there does not exist s such that $t_1 \to_\beta^\infty \lambda x.s$. A Λ^∞-term t is *root-active* if there does not exist a root-stable s such that $t \to_\beta^\infty s$.

Given $t, s \in \Lambda^\infty$, the relation $t \sim s$ is defined by coinduction.

$$\frac{t, s \text{ are root-active}}{\overline{t \sim s}} \qquad \frac{}{\overline{x \sim x}} \qquad \frac{t \sim s}{\overline{\lambda x.t \sim \lambda x.s}} \qquad \frac{t_1 \sim s_1 \quad t_2 \sim s_2}{\overline{t_1 t_2 \sim s_1 s_2}}$$

We finish this section with several lemmas concerning the introduced notions. The first three lemmas have essentially been shown in [6, Lemma 4.3-4.5].

Lemma 1. *If* $s \to_\beta^\infty s'$ *and* $t \to_\beta^\infty t'$ *then* $s[t/x] \to_\beta^\infty s'[t'/x]$.

Lemma 2. *If* $t_1 \to_\beta^\infty t_2 \to_\beta t_3$ *then* $t_1 \to_\beta^\infty t_3$.

Lemma 3. *If* $t_1 \to_\beta^\infty t_2 \to_\beta^\infty t_3$ *then* $t_1 \to_\beta^\infty t_3$.

Lemma 4. *If* s *is root-active and* $s \to_\beta^\infty t$*, then* t *is root-active.*

Proof. If $t \to_\beta^\infty t'$ for some root-stable t', then also $s \to_\beta^\infty t'$ by Lemma 3.

The following lemma was first shown in [11, Lemma 43] by a different proof.

Lemma 5. *If* $t_1, t_2 \in \Lambda^\infty$ *and* t_1 *is root-active, then so is* $t_1[t_2/x]$.

Proof. We write $s \succ_x s'$ if x is not bound in s', i.e. s' does not contain subterms of the form $\lambda x.u$, and s' may be obtained from s by changing some arbitrary subterms in s into some terms having the form $xu_1 \ldots u_n$. It is easy to show by induction that

(a) if $t \to_\beta^* s$ and $t \succ_x t'$, then there exists s' such that $t' \to_\beta^* s'$ and $s \succ_x s'$,
(b) if $t' \to_\beta^* s'$ and $t \succ_x t'$, then there exists s such that $t \to_\beta^* s$ and $s \succ_x s'$.

By coinduction we show

(c) if $t \to_\beta^\infty s$ and $t \succ_x t'$, then there exists s' such that $t' \to_\beta^\infty s'$ and $s \succ_x s'$,
(d) if $t' \to_\beta^\infty s'$ and $t \succ_x t'$, then there exists s such that $t \to_\beta^\infty s$ and $s \succ_x s'$.

We only give the proof for (c), since the proof for (d) is analogous. There are three cases.

- $t \to_\beta^* s \equiv y$. Then the claim follows directly from (a).
- $t \to_\beta^* t_1 t_2$, $s \equiv s_1 s_2$ and $t_i \to_\beta^\infty s_i$. By (a) there is u such that $t' \to_\beta^* u$ and $t_1 t_2 \succ_x u$. If u has the form $xu_1 \ldots u_n$ then we may take $s' \equiv u$. Otherwise $u \equiv u_1 u_2$ with $t_i \succ_x u_i$. By the coinductive hypothesis there are s_1', s_2' such that $u_i \to_\beta^\infty s_i'$ and $s_i \succ_x s_i'$. Thus we may take $s' \equiv s_1' s_2'$.
- $t \to_\beta^* \lambda y.u$, $s \equiv \lambda y.u'$ and $u \to_\beta^\infty u'$. By (a) there is w such that $t' \to_\beta^* w$ and $\lambda y.u \succ_x w$. If w has the form $xu_1 \ldots u_n$ then we may take $s' \equiv w$. Otherwise $w \equiv \lambda y.w_0$ with $u \succ_x w_0$. By the coinductive hypothesis there is w_1 such that $w_0 \to_\beta^\infty w_1$ and $u' \succ_x w_1$. So we may take $s' \equiv \lambda y.w_1$.

Now we show

(e) if $s \succ_x s'$ and s is root-stable, then so is s'.

So suppose $s \succ_x s'$ and s is root-stable. If s' has the form $xu_1 \ldots u_n$ then it is obviously root-stable. Otherwise, $s' \equiv y$, $s' \equiv \lambda y.s''$ or $s' \equiv s_1' s_2'$ with $s \equiv s_1 s_2$ and $s_i \succ_x s_i'$. In the first two cases s' is root-stable. So assume $s' \equiv s_1' s_2'$, $s \equiv s_1 s_2$ and $s_i \succ_x s_i'$. If s' is not root-stable, then $s_1' \to_\beta^\infty \lambda z.u'$. But then by ($d$) there is w such that $s_1 \to_\beta^\infty w$ and $w \succ_x \lambda z.u'$. So w must have the form $\lambda z.u$. Contradiction.

Finally, suppose $t_1[t_2/x]$ is not root-active. Then $t_1[t_2/x] \to_\beta^\infty s$ with s root-stable. Without loss of generality $t_1[t_2/x] \succ_x t_1$. Hence by (c) there is s' such that $t_1 \to_\beta^\infty s'$ and $s \succ_x s'$. By (e) we conclude that s' is root-stable. This means that t_1 is not root-active.

3 Confluence up to equivalence of root-active subterms

In this section we show that the relation $\to_\beta^\infty$ is confluent up to $\sim$. More precisely, we prove the following theorem.

Theorem 1. *If $t \sim t'$, $t \to_\beta^\infty s$ and $t' \to_\beta^\infty s'$, then there exist r, r' such that $s \to_\beta^\infty r$, $s' \to_\beta^\infty r'$ and $r \sim r'$.*

The general proof strategy is similar to that in [11] and is illustrated in Fig. 1. We introduce an ϵ-calculus – a modified infinitary lambda-calculus. We show confluence of infinitary reduction in the ϵ-calculus and then translate this result into confluence of $\to_\beta^\infty$ up to $\sim$.

$$\begin{tikzcd}
& t \arrow[dl, "\beta"'] & \prec \quad t_0 \arrow[dl, "\epsilon"'] \arrow[dr, "\epsilon"] \quad \succ & t' \arrow[dr, "\beta"] & \\
s \quad \prec \arrow[dr, "\beta"'] & s_0 \arrow[dr, "\epsilon"'] & & s_0' \arrow[dl, "\epsilon"] & \succ \quad s' \arrow[dl, "\beta"] \\
& r & \prec \quad r_0 \quad \succ & r' &
\end{tikzcd}$$

Fig. 1. Confluence proof for $\to_\beta^\infty$ up to $\sim$.

Definition 3. The set of Λ^ϵ-terms is defined by coinduction.

$$\Lambda^\epsilon ::= V \parallel \lambda V.\Lambda^\epsilon \parallel \Lambda^\epsilon \Lambda^\epsilon \parallel \epsilon(\Lambda^\epsilon)$$

We say that $t \in \Lambda^\epsilon$ *starts with* ϵ if $t \equiv \epsilon(t')$ for some t'. The relation $\to_\epsilon$ of ϵ-*contraction* is defined as the compatible closure of the reduction rules

$$\epsilon^n(\lambda x.s)t \to_\epsilon \epsilon(s[t/x])$$

for $n \in \mathbb{N}$. The relation $\to_\epsilon^*$ of ϵ-*reduction* is the transitive-reflexive closure of $\to_\epsilon$.

The relation $\to_1$ is defined coinductively.

$$\frac{}{x \to_1 x} \qquad \frac{s \to_1 s'}{\lambda x.s \to_1 \lambda x.s'} \qquad \frac{s \to_1 s' \quad t \to_1 t'}{st \to_1 s't'} \qquad \frac{t \to_1 t'}{\epsilon(t) \to_1 \epsilon(t')}$$

$$\frac{t_1[t_2/x] \to_1 t'}{\epsilon^n(\lambda x.t_1)t_2 \to_1 \epsilon(t')}$$

The relation $\to_\epsilon^\infty$ of *infinitary ϵ-reduction* is defined coinductively.

$$\frac{}{x \to_\epsilon^\infty x} \qquad \frac{s \to_\epsilon^\infty s'}{\lambda x.s \to_\epsilon^\infty \lambda x.s'} \qquad \frac{s \to_\epsilon^\infty s' \quad t \to_\epsilon^\infty t'}{st \to_\epsilon^\infty s't'} \qquad \frac{s \to_1^* \epsilon(t) \quad t \to_\epsilon^\infty t'}{s \to_\epsilon^\infty \epsilon(t')}$$

For $s \in \Lambda^\epsilon$ and $s' \in \Lambda^\infty$, the relation $s \succ s'$ is defined coinductively

$$\frac{}{\epsilon^n(x) \succ x} \qquad \frac{s' \text{ is root-active}}{\epsilon^\omega \succ s'} \qquad \frac{s \succ s'}{\epsilon^n(\lambda x.s) \succ \lambda x.s'} \qquad \frac{s \succ s' \quad t \succ t'}{\epsilon^n(st) \succ s't'}$$

where $n \in \mathbb{N}$. If $s \succ s'$ then s' is an *erasure* of s.

The purpose of our ϵ-calculus is similar to the ϵ-calculus in [11]. To make a direct coinductive confluence proof feasible, each contraction must produce at least one new constructor. In [11] the ϵ-contraction rule is $\epsilon^n(\lambda x.s)t \to \epsilon^{n+2}(s[t/x])$, so it additionally does not decrease the depth of other redexes. This is not necessary for a coinductive proof. The relation $\to_1$ is analogous to a development, but it also allows contracting some redexes which were not present in the original term. The difference is for purely technical reasons. If we used a development, then in the proof of Lemma 8 we would have to apply Lemma 7 to derivations obtained from the coinductive hypothesis. Hence, the proof would not conform to the interpretation from Sect. 2.1 which would make its correctness non-obvious.

Our first aim is to show that $\to_\epsilon^\infty$ has the Church-Rosser property. For this we need several lemmas.

Lemma 6. *Let $t_1, t_2, t_3 \in \Lambda^\epsilon$ and $y \notin FV(t_3)$. Then:*

$$t_1[t_2/y][t_3/x] \equiv t_1[t_3/x][(t_2[t_3/x])/y].$$

Proof. By coinduction with case analysis on t_1. See also [14, Chapter 12].

Lemma 7. *If $s \to_1 s'$ and $t \to_1 t'$ then $s[t/x] \to_1 s'[t'/x]$.*

Proof. Coinduction with case analysis on $s \to_1 s'$, using Lemma 6.

Lemma 8. *If $t \to_1 t_1$ and $t \to_1 t_2$ then there exists t_3 such that $t_1 \to_1 t_3$ and $t_2 \to_1 t_3$.*

Proof. By coinduction. We have the following cases.

1. $t \equiv t_1 \equiv x$. Then we must also have $t_2 \equiv x$ and we may take $t_3 \equiv x$.
2. $t \equiv \lambda x.t'$ and $t_1 \equiv \lambda x.t_1'$ with $t' \to_1 t_1'$. Then $t_2 \equiv \lambda x.t_2'$ with $t' \to_1 t_2'$. By the coinductive hypothesis, there is t_3' with $t_1' \to_1 t_3'$ and $t_2' \to_1 t_3'$. Thus take $t_3 \equiv \lambda x.t_3'$.
3. $t \equiv s_1 s_2$ and $t_1 \equiv u_1 u_2$ with $s_1 \to_1 u_1$ and $s_2 \to u_2$. Then one of the following holds.
   (a) $t_2 \equiv r_1 r_2$ with $s_1 \to_1 r_1$ and $s_2 \to_1 r_2$. By the coinductive hypothesis there are v_1, v_2 with $r_1 \to_1 v_1$, $u_1 \to_1 v_1$, $r_2 \to_1 v_2$ and $u_2 \to_1 v_2$. So $t_1 \equiv u_1 u_2 \to_1 v_1 v_2$ and $t_2 \equiv r_1 r_2 \to_1 v_1 v_2$. Thus take $t_3 \equiv v_1 v_2$.
   (b) $t_2 \equiv \epsilon(t_2')$ and $s_1 \equiv \epsilon^n(\lambda x.s_1')$ with $s_1'[s_2/x] \to_1 t_2'$. It follows directly from the definition of $\to_1$ that $u_1 \equiv \epsilon^n(\lambda x.u_1')$ with $s_1' \to_1 u_1'$. By Lemma 7, $s_1'[s_2/x] \to_1 u_1'[u_2/x]$. By the coinductive hypothesis there exists t_3' such that $u_1'[u_2/x] \to_1 t_3'$ and $t_2' \to_1 t_3'$. Hence $t_1 \equiv u_1 u_2 \equiv \epsilon^n(\lambda x.u_1')u_2 \to_1 \epsilon(t_3')$ and $t_2 \equiv \epsilon(t_2') \to_1 \epsilon(t_3')$. Thus take $t_3 \equiv \epsilon(t_3')$.

4. $t \equiv \epsilon(t')$ and $t_1 \equiv \epsilon(t_1')$ with $t' \to_1 t_1'$. Then $t_2 \equiv \epsilon(t_2')$ with $t' \to_1 t_2'$. By the coinductive hypothesis there exists t_3' such that $t_1' \to_1 t_3'$ and $t_2' \to_1 t_3'$. Hence we may take $t_3 \equiv \epsilon(t_3')$.
5. $t \equiv \epsilon^n(\lambda x.s)r$ and $t_1 \equiv \epsilon(t_1')$ with $s[r/x] \to_1 t_1'$. There are two possibilities.
   (a) $t_2 \equiv u_1 u_2$ with $\epsilon^n(\lambda x.s) \to_1 u_1$ and $r \to_1 u_2$. Then the proof is analogous to case 3(b).
   (b) $t_2 \equiv \epsilon(t_2')$ with $s[r/x] \to_1 t_2'$. By the coinductive hypothesis there exists t_3' such that $t_1' \to_1 t_3'$ and $t_2' \to_1 t_3'$. Hence we may take $t_3 \equiv \epsilon(t_3')$.

Lemma 9. *If $t_1 \to_1 t_2 \to_\epsilon^\infty t_3$ then $t_1 \to_\epsilon^\infty t_3$.*

Proof. Coinduction with case analysis on $t_2 \to_\epsilon^\infty t_3$.

Lemma 10. *If $\epsilon(t) \to_\epsilon^\infty s$ then $s \equiv \epsilon(s')$ with $t \to_\epsilon^\infty s'$.*

Proof. It follows directly from the definition of $\to_\epsilon^\infty$ that $s \equiv \epsilon(s')$ with $\epsilon(t) \to_1^* \epsilon(t')$ and $t' \to_\epsilon^\infty s'$. From the definition of $\to_1$ it follows that $t \to_1^* t'$. Thus $t \to_\epsilon^\infty s'$ by repeated application of Lemma 9.

Lemma 11. *If $s \to_\epsilon^\infty s'$ and $t \to_\epsilon^\infty t'$ then $s[t/x] \to_\epsilon^\infty s'[t'/x]$.*

Proof. Coinduction with case analysis on $s \to_\epsilon^\infty s'$, using that $s \to_1^* t$ implies $s[u/x] \to_1^* t[u/x]$, which follows from Lemma 7.

Lemma 12. *If $t \to_\epsilon^\infty t_1$ and $t \to_1 t_2$ then there exists t_3 such that $t_1 \to_1 t_3$ and $t_2 \to_\epsilon^\infty t_3$.*

Proof. Coinduction, analysing $t \to_\epsilon^\infty t_1$. There are two interesting cases.

1. $t \equiv \epsilon^n(\lambda x.s)r$, $t_1 \equiv u_1 u_2$, $t_2 \equiv \epsilon(t_2')$ with $\epsilon^n(\lambda x.s) \to_\epsilon^\infty u_1$, $r \to_\epsilon^\infty u_2$ and $s[r/x] \to_1 t_2'$. Because $\epsilon^n(\lambda x.s) \to_\epsilon^\infty u_1$, it follows from Lemma 10 and the definition of $\to_\epsilon^\infty$ that $u_1 \equiv \epsilon^n(\lambda x.u_1')$ with $s \to_\epsilon^\infty u_1'$. Hence $s[r/x] \to_\epsilon^\infty u_1'[u_2/x]$ by Lemma 11. By the coinductive hypothesis there exists t_3' such that $u_1'[u_2/x] \to_1 t_3'$ and $t_2' \to_\epsilon^\infty t_3'$. Hence $t_2 \equiv \epsilon(t_2') \to_\epsilon^\infty \epsilon(t_3')$ and $t_1 \equiv u_1 u_2 \equiv \epsilon^n(\lambda x.u_1')u_2 \to_1 \epsilon(t_3')$. Thus we may take $t_3 \equiv \epsilon(t_3')$.
2. $t_1 \equiv \epsilon(t_1')$ with $t \to_1^* \epsilon(s)$ and $s \to_\epsilon^\infty t_1'$. Since $t \to_1^* \epsilon(s)$ and $t \to_1 t_2$, it follows from Lemma 8 by an easy diagram chase that there exists s' such that $t_2 \to_1^* \epsilon(s')$ and $\epsilon(s) \to_1 \epsilon(s')$. From the definition of $\to_1$ we obtain $s \to_1 s'$. By the coinductive hypothesis there exists t' such that $s' \to_\epsilon^\infty t'$ and $t_1' \to_1 t'$. Hence $t_2 \to_\epsilon^\infty \epsilon(t')$, because $t_2 \to_1^* \epsilon(s')$ and $s' \to_\epsilon^\infty t'$. Also $t_1 \equiv \epsilon(t_1') \to_1 \epsilon(t')$. We may thus take $t_3 \equiv \epsilon(t')$.

Lemma 13. *If $t \to_\epsilon^\infty t_1$ and $t \to_\epsilon^\infty t_2$ then there exists t_3 such that $t_1 \to_\epsilon^\infty t_3$ and $t_2 \to_\epsilon^\infty t_3$.*

Proof. By coinduction. There is one non-trivial case, when e.g. $t_2 \equiv \epsilon(t_2')$ with $t \to_1^* \epsilon(s)$ and $s \to_\epsilon^\infty t_2'$. By repeated application of Lemma 12 we obtain u with $t_1 \to_1^* u$ and $\epsilon(s) \to_\epsilon^\infty u$. By Lemma 10, $u \equiv \epsilon(s')$ with $s \to_\epsilon^\infty s'$. By the coinductive hypothesis there is t_3' with $t_2' \to_\epsilon^\infty t_3'$ and $s' \to_\epsilon^\infty t_3'$. So $t_1 \to^\infty \epsilon(t_3')$, since $t_1 \to_1^* \epsilon(s')$ and $s' \to_\epsilon^\infty t_3'$. Also $t_2 \equiv \epsilon(t_2') \to_\epsilon^\infty \epsilon(t_3')$. So take $t_3 \equiv \epsilon(t_3')$.

The above lemma states the Church-Rosser property of $\to_\epsilon^\infty$. Now we need to translate this result into confluence of $\to_\beta^\infty$ up $\sim$. For this purpose we need to be able to transform infinitary ϵ-reductions on Λ^ϵ-terms into infinitary β-reductions on their erasures, and vice versa. This is achieved in Lemmas 16, 21.

Lemma 14. *If $t_1 \succ u_1$ and $t_2 \succ u_2$ then $t_1[t_2/x] \succ u_1[u_2/x]$.*

Proof. Coinduction with case analysis on $t_1 \succ u_1$, using Lemma 5.

Lemma 15. *If $t_1 \succ u_1 \to_\beta u_2$ then there exists t_2 such that $t_1 \to_\epsilon t_2 \succ u_2$. Moreover, if the contraction $u_1 \to_\beta u_2$ occurs at the root, then t_2 starts with ϵ.*

Proof. Induction on $u_1 \to_\beta u_2$. If $t_1 \equiv \epsilon^\omega$ then u_1 is root-active. By Lemma 4 so is u_2, since $u_1 \to_\beta u_2$. Thus take $t_2 \equiv \epsilon^\omega$. If $t_1 \not\equiv \epsilon^\omega$ then it suffices to consider the case when $t_1 \equiv \epsilon^m(\epsilon^n(\lambda x.s_1)s_2)$, $u_1 \equiv (\lambda x.v_1)v_2$ and $u_2 \equiv v_1[v_2/x]$ with $s_1 \succ v_1$ and $s_2 \succ v_2$. But then $t_1 \to_\epsilon \epsilon^{m+1}(s_1[s_2/x])$. Also $s_1[s_2/x] \succ v_1[v_2/x] \equiv u_2$ by Lemma 14. If $s_1[s_2/x] \equiv \epsilon^\omega$ then $\epsilon^{m+1}(s_1[s_2/x]) \equiv \epsilon^\omega$, and thus $\epsilon^{m+1}(s_1[s_2/x]) \succ u_2$. Otherwise $\epsilon^{m+1}(s_1[s_2/x]) \succ u_2$ from the definition of $\succ$. So take $t_2 \equiv \epsilon^{m+1}(s_1[s_2/x])$.

Lemma 16. *If $t_1 \succ u_1 \to_\beta^\infty u_2$ then there exists t_2 such that $t_1 \to_\epsilon^\infty t_2 \succ u_2$.*

Proof. Coinduction with case analysis on $u_1 \to_\beta^\infty u_2$. There are three cases.

1. $u_2 \equiv x$ and $u_1 \to_\beta^* x$. By repeated application of Lemma 15 we obtain t_2 such that $t_1 \to_\epsilon^* t_2 \succ x$. Then also $t_1 \to_\epsilon^\infty t_2$ and we are done.
2. $u_2 \equiv v_1'v_2'$ with $u_1 \to_\beta^* v_1v_2$ and $v_i \to_\beta^\infty v_i'$. By Lemma 15 there is s such that $t_1 \to_\epsilon^* s \succ v_1v_2$. There are two possibilities.
   (a) $s \equiv \epsilon^\omega$ and v_1v_2 is root-active. Since $v_1v_2 \to_\beta^\infty v_1'v_2' \equiv u_2$, by Lemma 4, u_2 is root-active. Thus $t_1 \to_\epsilon^\infty \epsilon^\omega \succ u_2$ and we take $t_2 \equiv \epsilon^\omega$.
   (b) $s \equiv \epsilon^n(s_1s_2)$ with $s_i \succ v_i$. If $n = 0$ then $t_1 \equiv r_1r_2$ and $u_1 \equiv w_1w_2$ with $r_i \succ w_i$. Moreover, by the second part of Lemma 15, no contraction in $u_1 \to_\beta^* v_1v_2$ occurs at the root. Thus $w_i \to_\beta^* v_i \to_\beta^\infty v_i'$. Hence $w_i \to_\beta^\infty v_i'$. By the coinductive hypothesis there are s_1', s_2' with $r_i \to_\epsilon^\infty s_i' \succ v_i'$. Hence $t_1 \equiv r_1r_2 \to_\epsilon^\infty s_1's_2' \succ v_1'v_2' \equiv u_2$ and we take $t_2 \equiv s_1's_2'$. Now assume $n > 0$. Since $s_i \succ v_i \to_\beta^\infty v_i'$, by the coinductive hypothesis there are s_1', s_2' with $s_i \to_\epsilon^\infty s_i' \succ v_i'$. Hence $\epsilon^n(s_1's_2') \succ v_1'v_2' \equiv u_2$. Also $t_1 \to_\epsilon^\infty \epsilon^n(s_1s_2)$, because $t_1 \to_1^* \epsilon(\epsilon^{n-1}(s_1s_2))$ and $s_i \to_\epsilon^\infty s_i'$ (to obtain the derivation apply once the penultimate rule and then n times the the last rule). Thus take $t_2 \equiv \epsilon^n(s_1s_2)$.
3. $u_2 \equiv \lambda x.u_2'$ with $u_1 \to_\beta^* \lambda x.u_1'$ and $u_1' \to_\beta^\infty u_2'$. By Lemma 15 there is s with $t_1 \to_\epsilon^* s \succ \lambda x.u_1'$. Obviously, $\lambda x.u_1'$ is not root-active, so the only possibility for $s \succ \lambda x.u_1'$ to hold is when $s \equiv \epsilon^n(\lambda x.s')$ with $s' \succ u_1'$. If $n = 0$ then $t_1 \equiv \lambda x.t_1'$ and $u_1 \equiv \lambda x.u_0$ with $t_1' \succ u_0 \to_\beta^* u_1' \to_\beta^\infty u_2'$. Thus $t_1' \succ u_0 \to_\beta^\infty u_2'$. By the coinductive hypothesis there is r with $t_1' \to_\epsilon^\infty r \succ u_2'$. So $t_1 \equiv \lambda x.t_1' \to_\epsilon^\infty \lambda x.r \succ \lambda x.u_2' \equiv u_2$. So take $t_2 \equiv \lambda x.r$. Now assume $n > 0$. Since $s' \succ u_1' \to_\beta^\infty u_2'$, by the coinductive hypothesis there is r with $s' \to_\epsilon^\infty r \succ u_2'$. Thus $\epsilon^n(\lambda x.r) \succ u_2'$. Because $t_1 \to_1^* s \equiv \epsilon(\epsilon^{n-1}(\lambda x.s'))$ and $s' \to_\epsilon^\infty r$, also $t_1 \to_\epsilon^\infty \epsilon^n(\lambda x.r)$. So take $t_2 \equiv \epsilon^n(\lambda x.r)$.

Lemma 17. *If $t \to_1^* \epsilon(t')$ then there exists s such that $t \to_\epsilon^* \epsilon(s)$ and $s \to_1^* t'$, where the length of $s \to_1^* t'$ is not larger than the length of $t \to_1^* \epsilon(t')$.*

Proof. Induction on the length l of the reduction $t \to_1^* \epsilon(t')$. If $t \equiv \epsilon(u)$ then $u \to_1^* t'$ follows from the definition of $\to_1$. Hence, take $s \equiv u$. Otherwise $t \equiv u_1 u_2$ and we may decompose $t \to_1^* \epsilon(t')$ into reductions: $u_1 \to_1^* \epsilon^m(\lambda x.v)$ of length l_1, $u_2 \to_1^* w$ of length l_1, $\epsilon^m(\lambda x.v)w \to_1 \epsilon(r)$ and $r \to_1^* t'$ of length l_2, where $l_1+l_2 < l$. By applying the inductive hypothesis m times, we conclude there is u_1' with $u_1 \to_\epsilon^* \epsilon^m(u_1')$ and $u_1' \to_1^* \lambda x.v$, where the length of $u_1' \to_1^* \lambda x.v$ is at most l_1. By the definition of $\to_1$, we have $u_1' \equiv \lambda x.v_0$ with $v_0 \to_1^* v$ of length at most l_1. By repeated application of Lemma 7, there is a reduction $v_0[u_2/x] \to_1^* v[w/x]$ of length at most l_1. Since $\epsilon^m(\lambda x.v)w \to_1 \epsilon(r)$, $v[w/x] \to_1 r$. Hence, there is a reduction $v_0[u_2/x] \to_1^* v[w/x] \to_1 r \to_1^* t'$ of length at most $l_1 + l_2 + 1 \leq l$. Also $t \equiv u_1 u_2 \to_\epsilon^* \epsilon^m(\lambda x.v_0)u_2 \to_\epsilon v_0[u_2/x]$. Thus take $s \equiv v_0[u_2/x]$.

Lemma 18. *If $s \to_\epsilon^\infty t$ and $t \not\equiv \epsilon^\omega$, then there exists s' such that $s \to_\epsilon^* s'$ and one of the following holds:*

- *$s' \equiv \epsilon^n(x) \equiv t$, or*
- *$s' \equiv \epsilon^n(\lambda x.r)$ and $t \equiv \epsilon^n(\lambda x.r')$ with $r \to_\epsilon^\infty r'$, or*
- *$s' \equiv \epsilon^n(r_1 r_2)$ and $t \equiv \epsilon^n(r_1' r_2')$ with $r_1 \to_\epsilon^\infty r_1'$ and $r_2 \to_\epsilon^\infty r_2'$.*

Proof. By the definition of $\to_\epsilon^\infty$ there is s' such that $s \to_1^* s'$ and one of the three conditions hold. For instance, suppose $s' \equiv \epsilon^n(r_1 r_2)$ and $t \equiv \epsilon^n(r_1' r_2')$ with $r_1 \to_\epsilon^\infty r_1'$ and $r_2 \to_\epsilon^\infty r_2'$. By applying Lemma 17 n times we obtain s_1, s_2 with $s \to_\epsilon^* \epsilon^n(s_1 s_2)$ and $s_1 s_2 \to_1^* r_1 r_2$. By the definition of $\to_1$ we have $s_1 \to_1^* r_1$ and $s_2 \to_1^* r_2$. By Lemma 9, $s_1 \to_\epsilon^\infty r_1'$ and $s_2 \to_\epsilon^\infty r_2'$. This finishes the proof.

Lemma 19. *If $t_1 \to_\epsilon t_2$ and $t_1 \succ u_1$, then there exists u_2 such that $u_1 \to_\beta u_2$ and $t_2 \succ u_2$. Moreover, if t_1 does not start with ϵ but t_2 does, then the contraction $u_1 \to_\beta u_2$ occurs at the root.*

Proof. We proceed by induction on $t_1 \to_\epsilon t_2$. All cases are trivial except when $t_1 \equiv \epsilon^m(\epsilon^n(\lambda x.s)t) \to_\epsilon \epsilon^{m+1}(s[t/x]) \equiv t_2$. Then $u_1 \equiv (\lambda x.v_1)v_2$ with $s \succ v_1$ and $t \succ v_2$. Thus $u_1 \to_\beta v_1[v_2/x]$. By Lemma 14, $s[t/x] \succ v_1[v_2/x]$. Let $u_2 \equiv v_1[v_2/x]$. If $s[t/x] \equiv \epsilon^\omega$ then $\epsilon^{m+1}(s[t/x]) \equiv \epsilon^\omega$, and thus $\epsilon^{m+1}(s[t/x]) \succ u_2$. Otherwise $\epsilon^{m+1}(s[t/x]) \succ u_2$ also holds by the definition of $\succ$.

Lemma 20. *If $t \to_\epsilon^\infty \epsilon^\omega$ and $t \succ u$, then u is root-active.*

Proof. Suppose $u \to_\beta^\infty u'$ with u' root-stable. Then by Lemma 16 there is t' with $t \to_\epsilon^\infty t' \succ u'$. Since ϵ^ω is in normal form w.r.t. infinitary ϵ-reduction, by Lemma 13, $t' \to_\epsilon^\infty \epsilon^\omega$. Since $t' \succ u'$ and u' is not root-active, $t' \equiv \epsilon^n(s)$ where s does not start with ϵ. From the definition of $\to_\epsilon^\infty$ we have $s \to_1^* \epsilon(s')$ for some s'. By Lemma 17 there is r with $s \to_\epsilon^* \epsilon(r)$. Since $t' \succ u'$ we have $s \succ u'$. By Lemma 19 there is w such that $u' \to_\beta^* w$ with at least one contraction occuring at the root. But this contradicts the fact that u' is root-stable.

Lemma 21. *If $t_1 \to_\epsilon^\infty t_2$ and $t_1 \succ u_1$, then there exists u_2 such that $t_2 \succ u_2$ and $u_1 \to_\beta^\infty u_2$.*

Proof. We proceed by coinduction. First, assume that $t_2 \equiv \epsilon^\omega$. By Lemma 20, u_1 is root-active. Take $u_2 \equiv u_1$. So assume $t_2 \not\equiv \epsilon^\omega$. Then by Lemmas 18, 19 there are s, u_1' with $t_1 \to_\epsilon^* s \succ u_1'$, $u_1 \to_\beta^* u_1'$ and we have one of three possibilities.

- $s \equiv \epsilon^n(x) \equiv t_2$. Then $u_1' \equiv x$ and $u_1 \to_\beta^\infty x$. So we may take $u_2 \equiv u_1'$.
- $s \equiv \epsilon^n(\lambda x.r)$ and $t_2 \equiv \epsilon^n(\lambda x.r')$ with $r \to_\epsilon^\infty r'$. Then $u_1' \equiv \lambda x.w$ with $r \succ w$. By the coinductive hypothesis there is w' with $w \to_\beta^\infty w'$ and $r' \succ w'$. So $t_2 \equiv \epsilon^n(\lambda x.r') \succ \lambda x.w'$. Also $u_1 \to_\beta^\infty \lambda x.w'$, since $u_1 \to_\beta^* \lambda x.w$ and $w \to_\beta^\infty w'$. Thus take $u_2 \equiv w'$.
- $s \equiv \epsilon^n(r_1 r_2)$ and $t_2 \equiv \epsilon^n(r_1' r_2')$ with $r_1 \to_\epsilon^\infty r_1'$ and $r_2 \to_\epsilon^\infty r_2'$. Analogous to the previous case.

Lemma 22. *If $u_1, u_2 \in \Lambda^\infty$ then: $u_1 \sim u_2$ iff there exists $t \in \Lambda^\epsilon$ such that $t \succ u_1$ and $t \succ u_2$.*

Proof. By coinduction.

Theorem 1. *If $t \sim t'$, $t \to_\beta^\infty s$ and $t' \to_\beta^\infty s'$, then there exist r, r' such that $s \to_\beta^\infty r$, $s' \to_\beta^\infty r'$ and $r \sim r'$.*

Proof. By Lemma 22 there is $t_0 \in \Lambda^\epsilon$ such that $t_0 \succ t$ and $t_0 \succ t'$. By Lemma 16 there are s_0, s_0' such that $s_0 \succ s$, $s_0' \succ s'$, $t_0 \to_\epsilon^\infty s_0$ and $t_0 \to_\epsilon^\infty s_0'$. By Lemma 13 there is r_0 such that $s_0 \to_\epsilon^\infty r_0$ and $s_0' \to_\epsilon^\infty r_0$. By Lemma 21 there are r, r' such that $r_0 \succ r$, $r_0 \succ r'$, $s \to_\beta^\infty r$ and $s' \to_\beta^\infty r'$. By Lemma 22, $r \sim r'$. See Fig. 1.

4 Confluence of Böhm reduction

In this section we show that the relation $\to_{\beta\bot}^\infty$ of infinitary Böhm reduction is confluent. The high-level strategy of the proof resembles that of the corresponding proof in [11]. At the end of this section we also indicate how to show that our definition of infinitary Böhm reduction corresponds to the definition in [11].

Definition 4. Given the $\bot$-rules

$$t \to \bot \qquad \text{if } t \text{ is root-active and } t \not\equiv \bot$$

we define the relation $\to_{\beta\bot}$ of *$\beta\bot$-contraction* as the compatible closure of the β-rule and the $\bot$-rules. The relation $\to_{\beta\bot}^*$ of *$\beta\bot$-reduction* is the transitive-reflexive closure of $\to_{\beta\bot}$.

The relation $\to_{\beta\bot}^\infty$ of *infinitary Böhm reduction* is defined coinductively.

$$\frac{s \to_{\beta\bot}^* x}{s \to_{\beta\bot}^\infty x} \qquad \frac{s \to_{\beta\bot}^* t_1 t_2 \quad t_1 \to_{\beta\bot}^\infty t_1' \quad t_2 \to_{\beta\bot}^\infty t_2'}{s \to_{\beta\bot}^\infty t_1' t_2'} \qquad \frac{s \to_{\beta\bot}^* \lambda x.r \quad r \to_{\beta\bot}^\infty r'}{s \to_{\beta\bot}^\infty \lambda x.r'}$$

The relation $\to_\bot$ is defined coinductively.

$$\frac{s \text{ is root-active and } s \not\equiv \bot}{s \to_\bot \bot} \qquad \frac{}{x \to_\bot x} \qquad \frac{s_1 \to_\bot t_1 \quad s_2 \to_\bot t_2}{s_1 s_2 \to_\bot t_1 t_2} \qquad \frac{s \to_\bot s'}{\lambda x.s \to_\bot \lambda x.s'}$$

Lemma 23. *If $s \to_\perp s'$ and $t \to_\perp t'$ then $s[t/x] \to_\perp s'[t'/x]$.*

Proof. Coinduction with case analysis on $s \to_\perp s'$, using Lemma 5.

Lemma 24. *If $t_1 \to_\perp t_2 \to_\beta t_3$ then there exists t_1' such that $t_1 \to_\beta t_1' \to_\perp t_3$.*

Proof. Induction on $t_2 \to_\beta t_3$. The only interesting case is when $t_2 \equiv (\lambda x.s_1)s_2$ and $t_3 \equiv s_1[s_2/x]$. We exclude $t_2 \equiv \perp$, because then $t_3 \equiv \perp$. So $t_1 \equiv (\lambda x.u_1)u_2$ with $u_i \to_\perp s_i$. By Lemma 23, $u_1[u_2/x] \to_\perp s_1[s_2/x]$. Thus take $t_1' \equiv u_1[u_2/x]$.

Lemma 25. *If $t \succ u$ and u is root-active, then $t \to_\epsilon^\infty \epsilon^\omega$.*

Proof. By coinduction. If $t \equiv \epsilon^\omega$ then the claim is obvious, so suppose otherwise. Since u is root-active, $u \equiv u_1 u_2$ with $u_1 \to_\beta^\infty \lambda x.s$. Then $u_1 \to_\beta^* \lambda x.s'$ for some s'. Since $t \not\equiv \epsilon^\omega$, we have $t \equiv \epsilon^n(t_1 t_2)$ with $t_1 \succ u_1$ and $t_2 \succ u_2$. By Lemma 15 there is t_1' such that $t_1 \to_\epsilon^* t_1' \succ \lambda x.s'$. We have $t_1' \equiv \epsilon^m(\lambda x.r)$ with $r \succ s'$. Hence $t \equiv t_1 t_2 \to_1^* \epsilon^m(\lambda x.r)t_2 \to_1 \epsilon(r[t_2/x])$. By Lemma 14, $r[t_2/x] \succ s'[u_2/x]$. Since u is root-active and $u \equiv u_1 u_2 \to_\beta^\infty s'[u_2/x]$, by Lemma 4 we conclude that $s'[u_2/x]$ is root-active. By the coinductive hypothesis $r[t_2/x] \to_\epsilon^\infty \epsilon^\omega$. Therefore $t \to_\epsilon^\infty \epsilon^\omega$, by applying the last rule in the definition of $\to_\epsilon^\infty$.

Lemma 26. *If $s \to_\perp t$ then $s \sim t$.*

Proof. By coinduction.

Lemma 27. *If t is root-active and $s \to_\perp t$ or $t \to_\perp s$, then s is root-active.*

Proof. By Lemma 26, $s \sim t$. By Lemma 22 there is r with $r \succ s$ and $r \succ t$. Since t is root-active, by Lemma 25, $r \to_\epsilon^\infty \epsilon^\omega$. But $r \succ s$, so s is root-active by Lemma 20.

Lemma 28. *If $t_1 \to_\perp t_2 \to_\perp t_3$ then $t_1 \to_\perp t_3$.*

Proof. Coinduction with case analysis on $t_2 \to_\perp t_3$, using Lemma 27.

Lemma 29. *If $s \to_{\beta\perp}^* t$ then there exists r such that $s \to_\beta^* r \to_\perp t$.*

Proof. Induction on the length of $s \to_{\beta\perp}^* t$, using Lemma 24 and Lemma 28.

Lemma 30. *If $t_1 \to_\perp t_2 \to_{\beta\perp}^\infty t_3$ then $t_1 \to_{\beta\perp}^\infty t_3$.*

Proof. Coinduction with case analysis on $t_2 \to_\beta^\infty t_3$, using Lemmas 29, 24, 28.

Lemma 31. *If $s \to_{\beta\perp}^\infty t$ then there exists r such that $s \to_\beta^\infty r \to_\perp t$.*

Proof. By coinduction with case analysis on $s \to_{\beta\perp}^\infty t$, using Lemmas 29, 30.

Lemma 32. *If $t \to_\perp t_1$ and $t \to_\perp t_2$ then there exists t_3 such that $t_1 \to_\perp t_3$ and $t_2 \to_\perp t_3$.*

Proof. Coinduction with case analysis on $t \to_\perp t_1$. The only non-trivial case is when $t \equiv s_1 s_2$ is root-active and e.g. $t_1 \equiv \perp$. But since $t \to_\perp t_2$, by Lemma 27, t_2 is also root-active. Thus $t_2 \to_\perp \perp \equiv t_1$ and we take $t_3 \equiv t_1$.

Lemma 33. *If $t_1 \sim t_2$ then there is s with $t_1 \to_\perp s$ and $t_2 \to_\perp s$.*

Lemma 34. *If $t_1 \sim t_2 \to_\beta^\infty t_3$ then there is t_2' such that $t_1 \to_\beta^\infty t_2' \sim t_3$.*

Proof. By Lemma 22 there is s with $s \succ t_1$ and $s \succ t_2$. Since $s \succ t_2 \to_\beta^\infty t_3$, by Lemma 16 there is s' with $s \to_\epsilon^\infty s' \succ t_3$. Since $s \succ t_1$ and $s \to_\epsilon^\infty s'$, by Lemma 21 there is t_2' with $t_1 \to_\beta^\infty t_2'$ and $s' \succ t_2'$. Since $s \succ t_2'$ and $s' \succ t_3$, by Lemma 22 we obtain $t_2' \sim t_3$.

Lemma 35. *If $t \to_\beta^\infty s$ and s is root-active, then so is t.*

Proof. Suppose $t \to_\beta^\infty t'$ for a root-stable t'. By Theorem 1 there are s_1, s_2 with $t' \to_\beta^\infty s_2$, $s \to_\beta^\infty s_1$ and $s_1 \sim s_2$. First, we show s_2 is root-stable, from which we obtain that s_1 is root-stable – a contradiction. Without loss of generality, assume $t' \equiv t_1 t_2$, $s_2 \equiv r_1 r_2$ with $t_i \to_\beta^\infty r_i$. If $r_1 \to_\beta^\infty \lambda x.r_0$ for some r_0, then $t_1 \to_\beta^\infty \lambda x.r_0$ by Lemma 3, which would contradict the fact that t' is root-stable. Since s_2 is root-stable, $s_1 \equiv u_1 u_2$ with $u_i \sim r_i$. Because s_1 is root-active, there is w with $u_1 \to_\beta^\infty \lambda x.w$. Then by Lemma 34 there is w' with $r_1 \to_\beta^\infty \lambda x.w' \sim \lambda x.w$. This contradicts the fact that s_2 is root-stable.

Lemma 36. *If $t \to_{\beta\perp}^\infty s$ and s is root-active, then so is t.*

Proof. Follows from Lemmas 31, 27, 35.

Lemma 37. *If $t_1 \to_\beta^\infty t_2 \to_\perp t_3$ then $t_1 \to_{\beta\perp}^\infty t_3$.*

Proof. Coinduction with case analysis on $t_1 \to_\beta^\infty t_2$, using Lemma 36.

Theorem 2. *If $t \to_{\beta\perp}^\infty t_1$ and $t \to_{\beta\perp}^\infty t_2$ then there exists t_3 such that $t_1 \to_{\beta\perp}^\infty t_3$ and $t_2 \to_{\beta\perp}^\infty t_3$.*

Proof. The proof is illustrated by the following diagram.

$$
\begin{tikzcd}
t \arrow[rrr, "\infty", "\beta"'] \arrow[ddd, "\infty", "\beta"'] & & & t_1' \arrow[r, "\perp"'] \arrow[dd, "\infty", "\beta"'] & t_1 \arrow[d, "\infty", "\beta"'] \\
& & & & r_1 \arrow[d, "\perp"'] \\
& & & s_1 \arrow[r, "\perp"'] \arrow[d, "\perp"'] & w_1 \arrow[d, "\perp"'] \\
t_2' \arrow[rr, "\infty", "\beta"'] \arrow[d, "\perp"'] & & s_2 \arrow[r, "\perp"'] \arrow[d] & s \arrow[r, "\perp"'] \arrow[d, "\perp"'] & \bullet \arrow[d, "\perp"'] \\
t_2 \arrow[r, "\infty", "\beta"'] & r_2 \arrow[r, "\perp"'] & w_2 \arrow[r, "\perp"'] & \bullet \arrow[r, "\perp"'] & \bullet
\end{tikzcd}
$$

By Lemma 31 there are t_1', t_2' with $t \to_\beta^\infty t_i' \to_\perp t_i$. By Theorem 1 there are s_1, s_2 with $t_i' \to_\beta^\infty s_i$ and $s_1 \sim s_2$. By Lemma 33 there is s with $s_i \to_\perp s$. By Lemma 26, $t_i' \sim t_i$. By Lemma 34 there are r_1, r_2 with $t_i \to_\beta^\infty r_i \sim s_i$. By Lemma 33 there are w_1, w_2 with $r_i \to_\perp w_i$ and $s_i \to_\perp w_i$. The remaining squares follow from Lemma 32. The claim then follows from Lemmas 28, 37.

4.1 Equivalence with the standard definition

Let $\to_{\mathcal{B}}$, $\to_\beta^i$ and $\to_\perp^i$ denote infinitary Böhm reduction w.r.t. root-active terms, infinitary β-reduction and parallel $\perp$-reduction, all as defined in [11] by means of strong convergence. From [6] we have: (a) $t \to_\beta^i s$ iff $t \to_\beta^\infty s$. We show: (b) $t \to_{\mathcal{B}} s$ iff $t \to_{\beta\perp}^\infty s$. This could probably be shown by easy modification of the proof of (a) from [6]. We derive (b) from (a) using some results from [11]. Suppose $t \to_{\mathcal{B}} s$. In [11] it is shown that then there is r with $t \to_\beta^i r \to_\perp^i s$. By (a), $t \to_\beta^\infty r$. From definitions $r \to_\perp^i s$ implies $r \to_\perp s$. Then $t \to_{\beta\perp}^\infty s$ by Lemma 37. Now suppose $t \to_{\beta\perp}^\infty s$. Then by Lemma 31 there is r with $t \to_\beta^\infty r \to_\perp s$. By (a), $t \to_\beta^i r$. From definitions $r \to_\perp^i s$. Then $t \to_{\mathcal{B}} s$.

References

1. Andreas Abel and Brigitte Pientka. Wellfounded recursion with copatterns: a unified approach to termination and productivity. In Greg Morrisett and Tarmo Uustalu, editors, *ICFP*, pages 185–196. ACM, 2013.
2. Henk Barendregt and Jan Willem Klop. Applications of infinitary lambda calculus. *Information and Computation*, 207(5):559–582, 2009.
3. Alessandro Berarducci. Infinite lambda-calculus and non-sensible models. In Aldo Ursini and Paolo Aglianò, editors, *Logic and Algebra, Lecture Notes in Pure and Applied Mathematics Series*, volume 180, pages 339–378. 1996.
4. Adam Chlipala. *Certified Programming with Dependent Types*, chapter 5. The MIT Press, 2013.
5. Jörg Endrullis, Dimitri Hendriks, and Jan Willem Klop. Highlights in infinitary rewriting and lambda calculus. *Theoretical Computer Science*, 464:48–71, 2012.
6. Jörg Endrullis and Andrew Polonsky. Infinitary rewriting coinductively. In Nils Anders Danielsson and Bengt Nordström, editors, *TYPES*, volume 19 of *LIPIcs*, pages 16–27. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2011.
7. Eduardo Giménez. Codifying guarded definitions with recursive schemes. In Peter Dybjer, Bengt Nordström, and Jan M. Smith, editors, *TYPES*, volume 996 of *Lecture Notes in Computer Science*, pages 39–59. Springer, 1994.
8. Bart Jacobs and Jan J.M.M. Rutten. An introduction to (co)algebras and (co)induction. In *Advanced Topics in Bisimulation and Coinduction*, pages 38–99. Cambridge University Press, 2011.
9. Felix Joachimski. Confluence of the coinductive [lambda]-calculus. *Theoretical Computer Science*, 311(1-3):105–119, 2004.
10. Richard Kennaway, Jan Willem Klop, M. Ronan Sleep, and Fer-Jan de Vries. Infinitary lambda calculi and Böhm models. In Jieh Hsiang, editor, *RTA*, volume 914 of *Lecture Notes in Computer Science*, pages 257–270. Springer, 1995.
11. Richard Kennaway, Jan Willem Klop, M. Ronan Sleep, and Fer-Jan de Vries. Infinitary lambda calculus. *Theoretical Computer Science*, 175(1):93–125, 1997.
12. Jan J. M. M. Rutten. Universal coalgebra: a theory of systems. *Theoretical Computer Science*, 249(1):3–80, 2000.
13. Paula Severi and Fer-Jan de Vries. An extensional Böhm model. In Sophie Tison, editor, *RTA*, volume 2378 of *Lecture Notes in Computer Science*, pages 159–173. Springer, 2002.
14. Terese. *Term Rewriting Systems*, volume 55 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 2003.