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Fundamental invariants of orbit closures



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ABSTRACT

For several objects of interest in geometric complexity theory, namely for the determinant, the permanent, the product of variables, the power sum, the unit tensor, and the matrix multiplication tensor, we introduce and study a fundamental SL-invariant function that relates the coordinate ring of the orbit with the coordinate ring of its closure. For the power sums we can write down this fundamental invariant explicitly in most cases. Our constructions generalize the two Aronhold invariants on ternary cubics. For the other objects we identify the invariant function conditional on intriguing combinatorial problems much like the well-known Alon–Tarsi conjecture on Latin squares. We provide computer calculations in small dimensions for these cases. As a main tool for our analysis, we determine the stabilizers, and we establish the polystability of all the mentioned forms and tensors (including the generic ones).

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1. Introduction

1.1. Motivation

In 1979 Valiant showed that for every polynomial p there exists a matrix A whose entries are affine linear forms in the variables of p such that $\det(A) = p$. The minimal size of such a matrix is called the *determinantal complexity* $\mathrm{dc}(p)$. The size $\mathrm{dc}(p)$ is closely related to the number of computation gates in arithmetic circuits computing p , see [53,52,32] for more information. Valiant’s flagship conjecture is concerned with the determinantal complexity of the permanent polynomial:

$$\mathrm{per}_m := \sum_{\pi \in S_m} X_{1,\pi(1)} X_{2,\pi(2)} \cdots X_{m,\pi(m)},$$

a homogeneous degree m polynomial in m^2 variables.

Conjecture 1.1. The sequence $\mathrm{dc}(\mathrm{per}_m)$ grows superpolynomially fast.

To study this conjecture, Mulmuley and Sohoni [40,41] proposed an approach through algebraic geometry and representation theory, for which they coined the term *geometric complexity theory*. The main idea is to study the GL_{n^2} -orbit closure $\overline{\mathrm{GL}_{n^2} \det_n}$ of the determinant polynomial

$$\det_n := \sum_{\pi \in \mathfrak{S}_n} \mathrm{sgn}(\pi) X_{1,\pi(1)} X_{2,\pi(2)} \cdots X_{n,\pi(n)}$$

in the space $\mathrm{Sym}^n \mathbb{C}^{n^2}$ of all homogeneous degree n polynomials in n^2 variables. For $n > m$ we compare this closure to the GL_{n^2} -orbit closure $\overline{\mathrm{GL}_{n^2} \mathrm{per}_{n,m}}$ of the *padded* permanent $\mathrm{per}_{n,m} := (X_{n,n})^{n-m} \mathrm{per}_m$. We consider the GL_{n^2} -action on the coordinate rings of these closures and note the following: If there exists an irreducible representation in $\mathcal{O}(\overline{\mathrm{GL}_{n^2} \mathrm{per}_{n,m}})$ that does not lie in $\mathcal{O}(\overline{\mathrm{GL}_{n^2} \det_n})$, then $\mathrm{dc}(\mathrm{per}_m) > n$.

A main approach towards understanding $\mathcal{O}(\overline{\mathrm{GL}_{n^2} \det_n})$ is by studying it as a subalgebra of the coordinate ring $\mathcal{O}(\mathrm{GL}_{n^2} \det_n)$ of the orbit $\mathrm{GL}_{n^2} \det_n$. The representation theoretic decomposition of $\mathcal{O}(\mathrm{GL}_{n^2} \det_n)$ can be deduced rather explicitly using the algebraic Peter–Weyl theorem [8, §4 and §5]. The main motivation of this paper is to improve our understanding of the connection between the coordinate ring of the orbit $\mathrm{GL}_{n^2} \det_n$ and its closure $\overline{\mathrm{GL}_{n^2} \det_n}$.

1.2. Results

We consider the space $\mathrm{Sym}^D \mathbb{C}^m$ of (homogeneous) forms of degree D in m variables with the natural action of the group GL_m . Let $w \in \mathrm{Sym}^D \mathbb{C}^m$ be a polystable form, which means that its SL_m -orbit is closed. We define the *stabilizer period* $a(w)$ as the

order of the subgroup of \mathbb{C}^\times generated by $\det(g)$, where g runs through the stabilizer group $\text{stab}(w)$ of w . In order to study in which degrees d there exist SL_m -invariants of $\mathcal{O}(\overline{Gw})$, we study the *degree monoid* of w :

$$\mathcal{E}(w) := \{d \in \mathbb{N} \mid \mathcal{O}(\overline{Gw})_d^{\text{SL}_m} \neq 0\}.$$

We note that $\mathcal{O}(\overline{Gw})_d^{\text{SL}_m}$ is at most one-dimensional for all $d \in \mathbb{Z}$. We prove that $\mathcal{E}(w)$ generates the group $b(w)\mathbb{Z}$ where $b(w) = \frac{m}{D}a(w)$, which we call the *degree period* of w (Theorem 3.4). A more geometric characterization of the degree monoid $\mathcal{E}(w)$ is as follows. Consider the well-defined map $\phi_w: Gw \rightarrow \mathbb{C}$, $g \mapsto \det(g)^{a(w)}$. We can show that the zero extension of ϕ_w to the boundary $\overline{Gw} \setminus Gw$ is continuous in the Euclidean topology. However, this extension might not be a regular function. In fact, for $k \in \mathbb{N}$, the k th power $(\phi_w)^k$ has a regular extension to \overline{Gw} iff $b(w)k \in \mathcal{E}(w)$.

Our primary object of interest is the *fundamental invariant* Φ_w of $w \in \text{Sym}^D \mathbb{C}^m$, which we define as the SL_m -invariant of minimal degree which occurs in $\mathcal{O}(\overline{Gw})$ (requiring $\Phi_w(w) = 1$ for uniqueness). Its degree $e(w)$ is the minimal positive element of $\mathcal{E}(w)$.

The boundary $\overline{Gw} \setminus Gw$ in \overline{Gw} is the zero set of Φ_w in $\mathcal{O}(\overline{Gw})$ and hence $\mathcal{O}(\overline{Gw})$ is the localization of $\mathcal{O}(Gw)$ with respect to Φ_w (Proposition 3.10). However, this holds only set-theoretically in most situations of interest. More specifically, we prove that if $b(w) < e(w)$, then the vanishing ideal of the boundary $\overline{Gw} \setminus Gw$ in \overline{Gw} is strictly larger than the principal ideal $\Phi_w \mathcal{O}(\overline{Gw})$. Moreover, \overline{Gw} is not a normal algebraic variety (see Theorem 3.11). The non-normality of $\overline{\text{GL}_n \det_n}$ (or even $\overline{\text{GL}_m X_1 \cdots X_m}$) is a considerable source of difficulty in implementing Mulmuley and Sohoni's strategy, see [5,26,4].

A result due to Howe [20] implies that for generic forms $w \in \text{Sym}^D \mathbb{C}^m$, with even D , the minimal degree of $\mathcal{E}(w)$ equals m . The corresponding fundamental invariant is obtained as the restriction of a beautiful SL_m -invariant $P_{D,m}: \text{Sym}^D \mathbb{C}^m \rightarrow \mathbb{C}$, for which we found an explicit expression; see (3.8). In the special case where D is even, this invariant apparently already had been discovered by Cayley in 1843 ("On the theory of determinants", Trans. Cambridge Phil Soc. VIII (1843): pp. 1–16). We call $P_{D,m}$ the *generic fundamental invariant* of $\text{Sym}^D \mathbb{C}^m$. It is an irreducible polynomial of degree m . If D is odd, the situation appears to be considerably more complicated. In the case where $D = m$ is odd, we show that for generic forms $w \in \text{Sym}^D \mathbb{C}^m$, the minimal degree of $\mathcal{E}(w)$ equals $m + 1$ and we explicitly construct this invariant in Section 3.2.1, cf. (3.10). This generalizes the degree 4 Aronhold invariant on ternary cubics (case $D = m = 3$); see also [20, eq. (80)].

In Section 3.3 we investigate the fundamental invariants for specific forms w that are of interest for geometric complexity theory. If $P_{D,m}(w) \neq 0$, then the fundamental invariant Φ_w is obtained from the generic fundamental invariant $P_{D,m}$ by restriction (and rescaling). It is therefore important to decide whether $P_{D,m}(w) \neq 0$. We prove that this is the case for the power sums $X_1^D + \cdots + X_m^D$ of even degree D . We note that this result is of crucial importance in a recent breakthrough in geometric complexity theory [7]. For odd D the minimal degree of the power sum equals $2m$ and we explicitly

construct its fundamental invariant, which generalizes Aronhold’s construction of the degree 6 invariant on ternary cubics (which we obtain as the special case $D = m = 3$). Quite interestingly, this construction only works as long as $2m \leq \binom{2D}{D}$, whereas for $2m > \binom{2D}{D}$, the invariant does not occur in any degree up to $2m$, see [Proposition 3.25](#). For other motivation from complexity theory, we note that the GL_m -orbit closure of the power sum $X_1^D + X_2^D + \cdots + X_m^D$ is closely related to $\Sigma\Lambda\Sigma$ circuit complexity [\[19\]](#) and to the polynomial Waring problem [\[28, §5.4\]](#).

The product of variables $X_1 \cdots X_m$ is of interest in geometric complexity theory because it serves as a simpler version of the determinant that is still interesting enough to deduce nontrivial facts about the determinant [\[26,4\]](#). However, already for $w = X_1 \cdots X_m$ (and m even), deciding $P_{m,m}(w) \neq 0$ turns out to be a difficult question: it is equivalent to the Alon–Tarsi conjecture [\[1\]](#) on Latin squares, which has so far only been verified in special cases. (The conjecture is known to be true if $n = p \pm 1$ where p is a prime, cf. [\[15,18\]](#).) For odd m we formulate an analogous combinatorial conjecture for deciding $P_{m,m}(w) \neq 0$.

For geometric complexity theory, it would be important to know whether P_{n,n^2} does not vanish on the determinant \det_n . We rephrase this as a combinatorial condition ([Proposition 3.29](#)) but only managed to verify it for $n = 2, 4$ by computer calculations.

The second part of the paper is devoted to analogous investigations for cubic tensors $w \in \otimes^3 \mathbb{C}^m$ with the action of the group $\mathrm{GL}_m \times \mathrm{GL}_m \times \mathrm{GL}_m$. These investigations are motivated by border rank and the complexity of matrix multiplication (see [\[3\]](#) for an overview), which can also be approached with geometric complexity theory [\[5,6\]](#).

The general theory is analogous to the case of forms. (It is obvious that everything could be developed in more generality, but we refrain from doing so.) We assign to a polystable tensor $w \in \otimes^3 \mathbb{C}^m$ its exponent monoid $\mathcal{E}(w)$ and show that $\mathcal{E}(w)$ generates the group $ma(w)$, where the stabilizer period $a(w)$ is the order of the subgroup of \mathbb{C}^\times generated by $\det(g_1)\det(g_2)\det(g_3)$, where (g_1, g_2, g_3) runs through the stabilizer group $\mathrm{stab}(w)$ of w ([Theorem 5.3](#)). We define the fundamental invariant Φ_w of w and prove that $\mathcal{O}(\overline{Gw}) = \mathcal{O}(Gw)_{\Phi_w}$.

Suppose now that $m = n^2$. In [Section 5.1](#) we show that for generic tensors $w \in \otimes^3 \mathbb{C}^m$, the minimal degree of $\mathcal{E}(w)$ equals n^3 . In fact, for generic w , the monoid $\mathcal{E}(w)$ is characterized in terms of the positivity of Kronecker coefficients of three rectangular shapes, see [\(5.2\)](#). The fundamental invariant of $w \in \otimes^3 \mathbb{C}^m$ is obtained as the restriction of a fascinating SL_m^3 -invariant $F_n: \otimes^3 \mathbb{C}^m \rightarrow \mathbb{C}$, for which we found an explicit expression; see [\(5.5\)](#). F_n is an irreducible polynomial of degree n^3 .

For understanding border rank, it would be important to verify that F_n does not vanish on the unit tensor $\langle n^2 \rangle$. We characterize this by a combinatorial condition, involving “Latin cubes”, which can be seen as a 3D version of the Alon–Tarsi conjecture ([Proposition 5.22](#)). However, so far we verified it only in the cases $n = 2, 4$ by computer calculations. We also express the nonvanishing of F_n on the matrix multiplication tensor $\langle n, n, n \rangle$ by a combinatorial condition ([Proposition 5.24](#)).

While developing the general theory, we determine the stabilizers of the forms and tensors listed above (including the generic ones) and use the Hilbert–Mumford criterion (as refined by Luna [31] and Kempf [24]) to prove the polystability of all objects under consideration. Some care has to be taken in the first few low-dimensional cases since irregular phenomena appear here. As a corollary of our observations we conclude the nonnormality of the orbit closures in most of the cases considered.

1.3. Organisation of the paper

Section 2 is devoted to determining the stabilizers and periods for certain forms and proving their polystability. In Section 3 we introduce the key concepts of the degree monoid and fundamental invariant for polystable forms, but postpone the proof of the main results to Section 6. Furthermore we study the generic fundamental invariant $P_{D,m}$ for forms and we investigate their vanishing at specific forms, arriving at intriguing combinatorial questions.

Section 4 provides the necessary background on stabilizers and polystability for tensors. Section 5 is devoted the analysis of the degree monoid and fundamental invariants for tensors. The remaining proofs are provided in Section 6. Finally, a short appendix collects information on the stabilizers of generic forms in special formats.

2. Stabilizer period and polystability of forms

2.1. Stabilizer and stabilizer period of forms

We consider $\text{Sym}^D \mathbb{C}^m$ with the natural action of GL_m . The *stabilizer* $\text{stab}(w)$ of a form $w \in \text{Sym}^D \mathbb{C}^m$ is defined as the following closed subgroup of GL_m :

$$\text{stab}(w) := \{g \in \text{GL}_m \mid gw = w\}. \quad (2.1)$$

It is clear that $\{\zeta \text{I}_m \mid \zeta^D = 1\}$ is contained in $\text{stab}(w)$. We say that w has a *trivial stabilizer* if equality holds.

The image H of $\text{stab}(w)$ under the determinant homomorphism $\det: \text{GL}_m \rightarrow \mathbb{C}^\times$ is a closed subgroup of \mathbb{C}^\times (cf. [21, §7.4, Prop. B]). Hence either $H = \mathbb{C}^\times$ or H is a finite subgroup of the group \mathbb{C}^\times and hence cyclic. We define the *stabilizer period* $a(w)$ as the order of H if H is finite and put $a(w) := \infty$ otherwise.

Lemma 2.1. *The stabilizer period $a(w)$ is a multiple of $D/\gcd(D, m)$ if $a(w) < \infty$.*

Proof. $H := \det(\text{stab}(w))$ contains the subgroup $H' := \{\zeta^m \mid \zeta^D = 1\}$ since $\{\zeta \text{I}_m \mid \zeta^D = 1\}$ is contained in $\text{stab}(w)$. But H' equals the group of D' th roots of unity, where $D' := D/\gcd(D, m)$. \square

Definition 2.2. The *reduced stabilizer period* $a'(w)$ of a form $w \in \text{Sym}^D \mathbb{C}^m$ is defined as $a'(w) := a(w) \gcd(D, m)/D$, where the *stabilizer period* $a(w)$ is defined as the order of the subgroup $\det(\text{stab}(w))$.

It follows from general principles that there exists $a(D, m)$ such that almost all $w \in \text{Sym}^D \mathbb{C}^m$ have the stabilizer period $a(D, m)$; see [47] or Popov and Vinberg [49, §7]. The *generic reduced stabilizer period* $a'(D, m)$, defined by $a'(D, m) := a(D, m) \gcd(D, m)/D$, can be explicitly determined. Matsumura and Monsky [38] showed that if $D > 2$ and $m > 3$, then almost all $w \in \text{Sym}^D \mathbb{C}^m$ have a trivial stabilizer, which implies $a'(D, m) = 1$. The following result is a consequence of this and a study of the stabilizers of generic binary and ternary forms. We indicate the proof in Appendix 7.

Theorem 2.3. Let $D > 2$ and $m \geq 1$. The set of $w \in \text{Sym}^D \mathbb{C}^m$ with finite stabilizer is a nonempty open subset of W . We have $a'(D, m) = 1$ except in the following cases: $a'(3, 2) = 2$, $a'(3, 3) = 2$, $a'(4, 3) = 2$.

We remark that in the exceptional case $D = 2$ of quadratic forms, the orbit of $X_1^2 + \cdots + X_m^2$ is dense in $\text{Sym}^2 \mathbb{C}^m$ and the stabilizer of $X_1^2 + \cdots + X_m^2$ equals the complex orthogonal group, which is not finite. We have $a(2, m) = 2$.

In the following we shall exclude the case $D = 1$, where the orbit of a nonzero element w equals $\text{Sym}^1 \mathbb{C}^m \setminus \{0\}$ and hence w has the stabilizer period ∞ .

The following result is well-known (compare [10, Chap. 2] for a proof of the second assertion).

Proposition 2.4.

1. The stabilizer of $X_1 \cdots X_m$ is generated by the permutation matrices and the diagonal matrices with determinant one. The stabilizer period of $X_1 \cdots X_m$ equals 2 if $m \geq 2$.
2. Suppose $D > 2$ and $m > 1$. The stabilizer of the power sum $X_1^D + \cdots + X_m^D$ is the subgroup generated by the permutation matrices and the diagonal matrices $\text{diag}(t_1, \dots, t_m)$, where $t_1^D = \cdots = t_m^D = 1$. The stabilizer period of $X_1^D + \cdots + X_m^D$ equals D if D is even and $2D$ if D is odd.

We study now the stabilizers and periods of the permanents and determinants. We note that the linear transformation $g_{A,B}: \mathbb{C}^{n \times n} \rightarrow \mathbb{C}^{n \times n}$, $X \mapsto AXB$, defined for $A, B \in \text{GL}_n$, satisfies $\det g_{A,B} = \det(A)^n \det(B)^n$.

Theorem 2.5 (Frobenius). The stabilizer of \det_n is generated by the transposition τ and the linear transformations $g_{A,B}$, where $A, B \in \text{SL}_n$. The stabilizer period of \det_n equals 1 if $n \bmod 4 \in \{0, 1\}$ and the stabilizer period of \det_n equals 2 if $n \bmod 4 \in \{2, 3\}$.

Proof. The stabilizer of \det_n has been determined by Frobenius [16] and, independently, by Dieudonné [12]. To determine the period, note that $\det g_{A,B} = \det(A)^n \det(B)^n = 1$

for $A, B \in \mathrm{SL}_n$. Moreover, $\det(\tau) = (-1)^{\frac{n(n-1)}{2}}$. Taking into account that $n(n-1)/2$ is even iff $n \bmod 4 \in \{0, 1\}$, the assertion about the period follows. \square

Theorem 2.6 (Marcus and May). *Let $n > 2$. The stabilizer of per_n is generated by the transposition τ , and the linear transformations $g_{A,B}$, where A and B are products of a permutation matrix with a diagonal matrix of determinant 1. The stabilizer period of per_n equals 2 unless $n \bmod 4 = 0$, in which case the stabilizer period equals 1.*

Proof. The stabilizer of per_n has been determined by Marcus and May [36]. If $A = P_\pi A_1$ and $B = P_\sigma B_1$ with permutation matrices P_π, P_σ and diagonal matrices $A_1, B_1 \in \mathrm{SL}_n$, then $\det g_{A,B} = \mathrm{sgn}(\pi)^n \mathrm{sgn}(\sigma)^n$. Moreover, recall that $\det(\tau) = (-1)^{\frac{n(n-1)}{2}}$. This implies that $\det(\mathrm{stab}(\mathrm{per}_n)) = \{-1, 1\}$ iff $n \bmod 4 \neq 0$. \square

2.2. Polystability of forms

The concept of polystability is crucial in geometric invariant theory. We review the basic facts we need.

Definition 2.7. A form $w \in \mathrm{Sym}^D \mathbb{C}^m$ is called *polystable* iff the SL_m -orbit of w is closed.

We present a convenient test for polystability that is based on a refinement of the Hilbert–Mumford criterion due to Luna [31] and Kempf [24]. The *support* of the form $w \in \mathrm{Sym}^D \mathbb{C}^m$ is defined as $\mathrm{supp}(w) := \{\alpha \in \mathbb{N}^m \mid w_\alpha \neq 0\}$, where $w = \sum_\alpha w_\alpha X^\alpha$ and $X^\alpha := X_1^{\alpha_1} \cdots X_m^{\alpha_m}$. (Clearly, this is a coordinate dependent notion.)

Proposition 2.8. *Let the form $w \in \mathrm{Sym}^D \mathbb{C}^m$ satisfy the following two properties:*

1. *There is a reductive subgroup R of $\mathrm{SL}_m \cap \mathrm{stab}(w)$ such that the centralizer of R in SL_m is contained in the group of diagonal matrices.*
2. *The convex cone generated by $\mathrm{supp}(w)$ contains $(1, \dots, 1)$.*

Then w is polystable.

Proof. Suppose that w is not polystable. Let Y be an SL_m -orbit in $\overline{\mathrm{SL}_m w} \setminus \mathrm{SL}_m w$ of minimal dimension. Then Y must be closed. The Hilbert–Mumford criterion states that there exists a one-parameter subgroup $\sigma: \mathbb{C}^\times \rightarrow \mathrm{SL}_m$ such that $y := \lim_{t \rightarrow 0} \sigma(t)w \in Y$. Recall that R is a reductive subgroup of $\mathrm{SL}_m \cap \mathrm{stab}(w)$. A result due to Luna [31, Cor. 1] and Kempf [24, Cor. 4.5] states that σ may be chosen such that the image of σ is contained in the centralizer of R in SL_m . Hence, by our first assumption, σ maps \mathbb{C}^\times to the diagonal matrices, say $\sigma(t) = \mathrm{diag}(t^{\mu_1}, \dots, t^{\mu_m})$ for $t \in \mathbb{C}^\times$. We have $\sum_i \mu_i = 0$ since $\det \sigma(t) = 1$. Note that

$$\sigma(t)w = \sum_{\alpha} w_{\alpha} (t^{\mu_1} X_1)^{\alpha_1} \cdots (t^{\mu_m} X_m)^{\alpha_m} = \sum_{\alpha} t^{\langle \alpha, \mu \rangle} w_{\alpha} X^{\alpha}, \quad (2.2)$$

where $\langle \alpha, \mu \rangle := \sum_i \alpha_i \mu_i$. The existence of $\lim_{t \rightarrow 0} \sigma(t)w$ implies that $\langle \alpha, \mu \rangle \geq 0$ for all $\alpha \in \text{supp}(w)$. Our second assumption states that there are $c_\alpha \geq 0$ such that $(1, \dots, 1) = \sum_\alpha c_\alpha \alpha$. Hence $\sum_i \mu_i = \langle (1, \dots, 1), \mu \rangle \geq 0$. On the other hand, $\sum_i \mu_i = 0$. This implies $\mu_i = 0$ for all i and hence $s(t) = 1$ for all t . Therefore $y = w$, which contradicts the fact the y lies in the boundary of the SL_m -orbit of w . \square

Corollary 2.9. *The forms $X_1 \cdots X_m$, $X_1^D + \cdots + X_m^D$ if $D > 1$, \det_n , and per_n are all polystable.*

Proof. We use [Proposition 2.8](#). For $w = X_1 \cdots X_m$ we take for R the group of diagonal matrices with determinant 1.

For $X_1^D + \cdots + X_m^D$ we take for R the group of diagonal matrices whose entries are D th roots of unity. Let g be in the centralizer of R . If $D > 1$, then for all $i < j$ there is $r \in R$ such that $r_i \neq r_j$. Hence $rg = gr$, which implies $r_i g_{ij} = g_{ij} r_j$, hence $g_{ij} = 0$. So g must be diagonal and the first property is satisfied.

For \det_n and per_n we take for R the group of diagonal matrices in SL_{n^2} . The support of both \det_n and per_n consists of the permutation matrices. Summing up the n permutation matrices of the cyclic shifts we get the matrix with only 1s, which provides the second property in [Proposition 2.8](#). \square

Proposition 2.10. *If $D > 1$, then almost all $w \in \text{Sym}^D \mathbb{C}^m$ are polystable.*

Proof. If $D = 2$, we can interpret $\text{Sym}^2 \mathbb{C}^m$ as the space of symmetric matrices of size m . If A is an invertible symmetric matrix, then $\text{SL}_m A = \{B \in \text{Sym}^2 \mathbb{C}^m \mid \det B = \det A\}$, which is closed.

Now assume $D > 2$. [Theorem 2.3](#) tells us that $\text{stab}(w)$ is finite for almost all $w \in \text{Sym}^D \mathbb{C}^m$. A general result due to Luna [\[30\]](#), see also [\[25, II 4.3D, Folgerung, p. 142\]](#), implies that $\text{SL}_m w$ is closed for almost all w if $\text{SL}_m \cap \text{stab}(w)$ is finite for almost all w . (This general result only requires that SL_m is a semisimple group.) Hence, almost all $w \in \text{Sym}^D \mathbb{C}^m$ are polystable. \square

Proposition 2.11. *A polystable nonzero form has a finite stabilizer period.*

Proof. Suppose that $w \in \text{Sym}^D \mathbb{C}^m$ has infinite stabilizer period, that is, $\det(\text{stab}(w)) = \mathbb{C}^\times$. Then there exists a sequence $g_n \in \text{stab}(w)$ such that $\lim_{n \rightarrow \infty} |\det(g_n)| = \infty$. We decompose $g_n = \iota(t_n)h_n$, where $h_n \in \text{SL}_m$ and $\iota(t) := t\text{I}_m$. Then $\det(g_n) = t_n^m$. We have $w = g_n w = \iota(t_n)h_n w = \overline{t_n^m} h_n w$, hence $t_n^{-m} w = h_n w \in \text{SL}_m w$. Taking the limit $n \rightarrow \infty$, we conclude that $0 \in \overline{\text{SL}_m w}$, hence $\text{SL}_m w$ is not closed or $w = 0$. \square

2.3. Numerical semigroups

We provide some useful background, see Alfonsín [\[45\]](#). A submonoid S of \mathbb{N} is a subset that is closed under addition and contains 0. A *numerical semigroup* S is defined

as a submonoid of \mathbb{N} that generates the group \mathbb{Z} . The submonoid of \mathbb{N} generated by the positive integers a_1, \dots, a_n consists of the nonnegative integer linear combinations $x_1 a_1 + \dots + x_n a_n$, where $x_i \in \mathbb{N}$. Clearly, S is a numerical semigroup iff $\gcd(a_1, \dots, a_n) = 1$. It is well-known that any submonoid S of \mathbb{N} is finitely generated.

Assume that S is a numerical semigroup. It is a well known fact that $\mathbb{N} \setminus S$ is a finite set. The elements of $\mathbb{N} \setminus S$ are called the *gaps* (or *holes*) of S . For example, the monoid S generated by 2 and 5 is a numerical semigroup with the gaps 1, 3. The largest gap of S is called the *Frobenius number* $g(S)$ of S .

Remark 2.12. If S is generated by a_1, a_2 , then Sylvester showed that $g(a_1, a_2) = a_1 a_2 - a_1 - a_2$. For $n = 3$, the formula for $g(a_1, a_2, a_3)$ is quite involved. In fact, the structure of numerical semigroups is intricate. It is known that the computation of $g(a_1, \dots, a_n)$ is an NP-hard problem.

3. Fundamental invariant of forms

Throughout this section, we assume that $w \in \text{Sym}^D \mathbb{C}^m$ is a polystable form. It will be convenient to write $G := \text{GL}_m$ in the following.

We denote by $\mathcal{O}(Gw)^{\text{SL}_m}$ the ring of SL_m -invariants of $\mathcal{O}(Gw)$ and, for $d \in \mathbb{Z}$, by $\mathcal{O}(Gw)_d$ its degree d part.

Definition 3.1. The *degree period* $b(w)$ of $w \in \text{Sym}^D \mathbb{C}^m$ is defined as $b(w) := \frac{m}{D} a(w)$.

The reason for the naming of $b(w)$ will be evident from next lemma. Note that $b(w)$ is an integer since $b(w) = \frac{m}{D} \frac{D}{\gcd(D, m)} a'(w) = \frac{m}{\gcd(D, m)} a'(w)$, where $a'(w)$ denotes the reduced stabilizer period of w .

Consider the map $\phi_w: Gw \rightarrow \mathbb{C}$ defined by $\phi_w(w) := \det(g)^{a(w)}$, which is well defined by the definition of the stabilizer period $a(w)$.

Lemma 3.2.

1. We have $\phi_w(tv) = t^{b(w)} \phi_w(v)$ for $t \in \mathbb{C}^\times$ and $v \in Gw$.
2. $\mathcal{O}(Gw)_d^{\text{SL}_m}$ is one-dimensional if d is a multiple of the degree period $b(w)$, and zero otherwise. More specifically, if $d = b(w)k$ with $k \in \mathbb{Z}$, we have $\mathcal{O}(Gw)_d^{\text{SL}_m} = \mathbb{C} \cdot (\phi_w)^k$.
3. For $k \in \mathbb{N}$, we have $\mathcal{O}(\overline{Gw})_{b(w)k}^{\text{SL}_m} = \mathbb{C} \cdot (\phi_w)^k$ iff $(\phi_w)^k$ has a regular extension to \overline{Gw} , otherwise $\mathcal{O}(\overline{Gw})_{b(w)k}^{\text{SL}_m} = 0$.

Proof. 1. For $g = tI_m$ we have $\det(g) = t^m$ and $gw = t^D w$. Hence,

$$\phi_w(t^D w) = \phi_w(gw) = \det(g)^{a(w)} = (t^m)^{a(w)} = (t^D)^{b(w)}.$$

2. Generally, let $V_G(\lambda)$ denote an irreducible G -module of highest weight λ and $V_G(\lambda)^{\text{stab}(w)}$ denote its subspace of $\text{stab}(w)$ -invariants. By the algebraic Peter–Weyl

theorem [8, §4.1], $V_G(\lambda)^*$ occurs in $\mathcal{O}(Gw)$ with the multiplicity $\dim V_G(\lambda)^{\text{stab}(w)}$. Suppose now that $V_G(\lambda) = \mathbb{C}$ corresponds to the character $\gamma \mapsto \det(g)^\ell$, where $\ell \in \mathbb{Z}$. Since $\dim V_G(\lambda) = 1$, $\dim V_G(\lambda)^{\text{stab}(w)} \leq 1$. Moreover, $\dim V_G(\lambda)^{\text{stab}(w)}$ is nonzero iff $\det(g)^\ell = 1$ for all $g \in \text{stab}(w)$. This means that ℓ is a multiple of $a(w)$, say $\ell = a(w)k$. Hence $F = (\phi_w)^k$.

3. The third assertion is immediate from the second. \square

We note the following useful relation between the degree period and the stabilizer period:

$$D b(w) = m a(w). \quad (3.1)$$

We turn now to the ring $\mathcal{O}(\overline{Gw})^{\text{SL}_m}$ of SL_m -invariants of the orbit closure \overline{Gw} and study in which degrees d there is a nonzero contribution.

Definition 3.3. The *degree monoid* of $w \in \text{Sym}^D \mathbb{C}^m$ is defined as

$$\mathcal{E}(w) := \{d \in \mathbb{N} \mid \mathcal{O}(\overline{Gw})_d^{\text{SL}_m} \neq 0\}.$$

We call the minimal positive element of $\mathcal{E}(w)$ the *minimal degree of w* and denote it by $e(w)$.

The minimal degree is well-defined due to the following result whose proof is postponed to Section 6.2.

Theorem 3.4. The degree monoid $\mathcal{E}(w)$ generates the group $b(w)\mathbb{Z}$.

We can give the numerical semigroup $\frac{1}{b(w)}\mathcal{E}(w)$ a somewhat different interpretation. Recall the map $\phi_w: Gw \rightarrow \mathbb{C}$, $g \mapsto \det(g)^{a(w)}$.

Definition 3.5. The *exponent monoid* $\mathcal{E}'(w)$ of $w \in \text{Sym}^D \mathbb{C}^m$ is defined as

$$\mathcal{E}'(w) := \{k \in \mathbb{N} \mid (\phi_w)^k \text{ has a regular extension to } \overline{Gw}\}.$$

The *minimal exponent* $e'(w)$ is defined as the minimal positive element of $\mathcal{E}'(w)$.

Lemma 3.2(3) immediately implies that $\mathcal{E}(w) = b(w)\mathcal{E}'(w)$ and $e(w) = b(w)e'(w)$. We illustrate now these notions for generic forms.

Definition 3.6. The *generic degree monoid of forms* of format (D, m) is defined as

$$\mathcal{E}(D, m) := \{d \in \mathbb{N} \mid \mathcal{O}(\text{Sym}^D \mathbb{C}^m)_d^{\text{SL}_m} \neq 0\}.$$

We call the minimal positive element of $\mathcal{E}(w)$ the *generic minimal degree* of format (D, m) and denote it by $e(D, m)$.

The namings of these objects are justified by the following insight.

Proposition 3.7. *We have $\mathcal{E}(w) = \mathcal{E}(D, m)$, and hence $e(w) = e(D, m)$, for almost all $w \in \text{Sym}^D \mathbb{C}^m$.*

Proof. Consider the surjective, graded ring homomorphism $\mathcal{O}(\text{Sym}^D \mathbb{C}^m)^{\text{SL}_m} \rightarrow \mathcal{O}(\overline{Gw})^{\text{SL}_m}$ defined by the restriction to \overline{Gw} . This implies $\mathcal{E}(w) \subseteq \mathcal{E}(D, m)$. Recall that every submonoid of \mathbb{N} is finitely generated. Assume that d_1, \dots, d_s generate the monoid $\mathcal{E}(D, m)$. Then there are nonzero $f_i \in \mathcal{O}(\text{Sym}^D \mathbb{C}^m)_{d_i}^{\text{SL}_m}$ for $i = 1, \dots, s$. If $f_1(w) \cdots f_s(w) \neq 0$, then $d_1, \dots, d_s \in \mathcal{E}(w)$, and hence $\mathcal{E}(w) = \mathcal{E}(D, m)$. \square

Proposition 3.7 combined with Theorem 3.4 implies that $\mathcal{E}(D, m)$ generates the group $b(D, m)\mathbb{Z}$, where the *generic degree period* $b(D, m)$ is given by

$$b(D, m) := \frac{m}{D} a(D, m) = a'(D, m) \frac{m}{\gcd(D, m)}.$$

Recall that $a'(D, m)$ is determined in Theorem 2.3.

Example 3.8. We have $b(2, 2) = 2$ and $\mathcal{E}(2, 2) = 2\mathbb{N}$, cf. [17, p. 153]. We have $b(3, 3) = 2$. Using the SCHUR software package (<http://schur.sourceforge.net>) we calculated that $\mathcal{E}(3, 3) = 2(\mathbb{N} \setminus \{1\})$ and hence $e(3, 3) = 2 \cdot 2 = 4$. Moreover, $b(4, 4) = 1$ and $\mathcal{E}(4, 4) = \{0, 4, 6, 7, 8, 10, 11, \dots\}$, so the gaps are 1, 2, 3, 5, 9 and we have $e(4, 4) = 4$.

In the sequel, we assume that $0 \neq w \in \text{Sym}^D \mathbb{C}^m$ is polystable. We can now define the key objects of study of this paper.

Definition 3.9. Let $w \in \text{Sym}^D \mathbb{C}^m$ be polystable. By the *fundamental invariant* Φ_w of w we understand the SL_m -invariant $\Phi_w \in \mathcal{O}(\overline{Gw})$ of the (minimal) degree $e(w)$ satisfying $\Phi_w(w) = 1$.

The explicit descriptions of the fundamental invariant Φ_w for certain w (cf. Theorem 3.16 and Theorem 3.23) are the main results of this paper concerning forms.

We note that $\Phi_w = (\phi_w)^{e'(w)}$ on Gw by Lemma 3.2. In particular, Φ_w is nonzero on the orbit Gw .

Proposition 3.10.

1. The zero set of Φ_w in \overline{Gw} equals the boundary $\overline{Gw} \setminus Gw$.
2. The ring $\mathcal{O}(Gw)$ of regular functions on Gw equals the localization of $\mathcal{O}(\overline{Gw})$ with respect to fundamental invariant Φ_w , i.e.,

$$\mathcal{O}(Gw) = \mathcal{O}(\overline{Gw})_{\Phi_w} = \left\{ \frac{f}{(\Phi_w)^s} \mid f \in \mathcal{O}(\overline{Gw}), s \in \mathbb{N} \right\}.$$

Proof. 1. It is sufficient to prove that the fundamental invariant Φ_w vanishes on the boundary $\overline{Gw} \setminus Gw$. For this, let $u \in \overline{Gw} \setminus Gw$ and assume that $g_n \in G$ is such that $\lim_{n \rightarrow \infty} g_n w = u$ (with respect to the Euclidean topology). Since $\Phi_w(g_n) = (\phi_w(g_n))^{e'(w)}$, it is sufficient to show that $\lim_{n \rightarrow \infty} \det(g_n) = 0$.

Let $\|\cdot\|$ be a norm on $\mathbb{C}^{m \times m}$ and write $G_s := \mathrm{SL}_m$. Since $G_s w$ is closed and $0 \notin G_s w$, there is $\varepsilon > 0$ such that $\|\tilde{g}w\| \geq \varepsilon$ for all $\tilde{g} \in G_s$. For each n there are $\tilde{g}_n \in G_s$ and $t_n \in \mathbb{C}^\times$ such that $g_n = t_n \tilde{g}_n$. Hence $\|g_n w\| = |t_n| \|\tilde{g}_n w\|$. Since $\lim_{n \rightarrow \infty} \|g_n w\| = \|u\|$ and $\|\tilde{g}_n w\| \geq \varepsilon$ we conclude that $|t_n| \leq \|g_n w\|/\varepsilon$ is bounded.

If $\lim_{n \rightarrow \infty} t_n = 0$ were false, there would be some nonzero limit point δ of the sequence (t_n) . After going over to a subsequence, we have $\lim_{n \rightarrow \infty} t_n = \delta$. From $g_n w = t_n \tilde{g}_n w$ we get $\lim_{n \rightarrow \infty} \tilde{g}_n w = \delta^{-1}u$. Hence $\delta^{-1}u \in \overline{G_s u} = G_s u$, which implies the contradiction $u \in Gw$.

2. The second assertion follows from general principles; cf. [48, §4.2, p. 50]. \square

According to Proposition 3.10, extending $\phi_w: Gw \rightarrow \mathbb{C}$ to the boundary $\overline{Gw} \setminus Gw$ by the value zero yields a continuous function $\overline{Gw} \rightarrow \mathbb{C}$ with respect to the \mathbb{C} -topology on \overline{Gw} . However, if $1 < e'(w)$, that is, $b(w) < e(w)$, this extension cannot be regular on \overline{Gw} . This implies that \overline{Gw} is not normal.

Here is a more detailed result, telling us that \overline{Gw} is a complicated object if $b(w) < e(w)$. We postpone its proof to Section 6.3, since it heavily relies on the arguments used for the proof of Theorem 3.4.

Theorem 3.11. *If we assume that $b(w) < e(w)$, that is, $1 < e'(w)$, then the vanishing ideal of the boundary $\overline{Gw} \setminus Gw$ in \overline{Gw} is strictly larger than the principal ideal $\Phi_w \mathcal{O}(\overline{Gw})$. Moreover, \overline{Gw} is not a normal algebraic variety. In particular, $\mathcal{O}(\overline{Gw})$ is not a valuation ring.*

3.1. Constructing invariants of forms

We will construct the fundamental invariant in several cases. This section explains the recipe underlying these constructions. While the construction process always returns an element P in $\mathcal{O}(\mathrm{Sym}^D \mathbb{C}^m)_d^{\mathrm{SL}_m}$, it does not guarantee the nonzeroness of P . Nonzeroness has to be verified separately, for example by evaluation of P at a carefully chosen point.

For $\mathcal{O}(\mathrm{Sym}^D \mathbb{C}^m)_d^{\mathrm{SL}_m}$ to be nonzero we need that $s := Dd/m$ is an integer. We start with an $m \times s$ array T of numbers whose entries are $\{1, 2, \dots, d\}$, each number appearing exactly D times with the restriction that no number occurs in a column more than once. This is a special case of a Young tableau. For example, if $D = 6$, $d = 4$, $m = 4$ we could have

$$T = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 & 2 & 2 \\ \hline 3 & 3 & 3 & 3 & 3 & 3 \\ \hline 4 & 4 & 4 & 4 & 4 & 4 \\ \hline \end{array}. \quad (3.2)$$

Tableaux of this type will be discussed in Section 3.2. Or, if for example $D = 3$, $d = 4$, $m = 2$ we could have

$$T = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 & 2 \\ \hline 3 & 3 & 4 & 3 & 4 & 4 \\ \hline \end{array}, \quad (3.3)$$

which we will treat in Section 3.3. An example for the case that we treat Section 3.2.1 looks like

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 4 & 1 & 2 & 3 \\ \hline 3 & 4 & 1 & 2 \\ \hline \end{array}, \quad (3.4)$$

where $m = D = 3$, $d = 4$.

To a fixed tableau T as above we assign now a bijection

$$[D] \times [d] \rightarrow [m] \times [s], \quad (\iota, i) \mapsto (T^{\iota, i}, T_{\iota, i}) = (\kappa, j) \quad (3.5)$$

as follows. We read the tableau T columnwise from left to right. For $(\iota, i) \in [D] \times [d]$ we denote by (κ, j) the position of T with the ι th appearance of i (recall that i can appear in a column at most once). For example, in (3.4), the second 4 from left to right appears at position $(3, 2)$, so $(T^{2,4}, T_{2,4}) = (3, 2)$. The inverse of the bijection (3.5) is given by

$$[m] \times [s] \rightarrow [D] \times [d], \quad (\kappa, j) \mapsto (\tilde{T}(\kappa, j), T(\kappa, j)) = (\iota, i), \quad (3.6)$$

where i is the entry in T at the position (κ, j) and ι equals the number of entries in T in the first j columns that contain i .

A map $v: [m]^D \rightarrow \mathbb{C}$ provides a coordinate description of the tensor

$$v := \sum_{1 \leq \nu_1, \dots, \nu_D \leq m} v(\nu_1, \dots, \nu_D) |\nu_1 \cdots \nu_D\rangle \in \bigotimes^D \mathbb{C}^m,$$

that we denote by the same symbol. We used here the notation $|\nu_1 \cdots \nu_D\rangle = |\nu_1\rangle \otimes \cdots \otimes |\nu_D\rangle$, where $|\nu\rangle$ denotes the ν th standard basis vector of \mathbb{C}^m . To a tableau T as above we assign a homogeneous degree d polynomial $P_T: \bigotimes^D \mathbb{C}^m \rightarrow \mathbb{C}$ by setting:

$$P_T(v) := \sum_{\sigma_1, \dots, \sigma_s} \left[\prod_{j=1}^s \text{sgn}(\sigma_j) \right] \prod_{i=1}^d v(\sigma_{T_{1,i}}(T^{1,i}), \dots, \sigma_{T_{D,i}}(T^{D,i})), \quad (3.7)$$

where the sum is over all s -tuples $(\sigma_1, \dots, \sigma_s)$ of permutations of $[m]$.

Before proving that P_T is an SL_m -invariant, let us look at the special case where $d = m$ (thus $s = D$) and T equals the $m \times D$ tableau that has all i s in row i , see the tableau (3.2). In this case, the expression (3.7) takes the considerably simpler form

$$P_{D,m}(v) := P_T(v) = \sum_{\sigma_1, \dots, \sigma_D} \left[\prod_{j=1}^D \operatorname{sgn}(\sigma_j) \right] \cdot \prod_{i=1}^m v(\sigma_1(i), \dots, \sigma_D(i)), \quad (3.8)$$

where the sum is over all D -tuples $(\sigma_1, \dots, \sigma_D)$ of permutations of $[m]$. Remarkably, the invariant $P_{D,m}$ was already studied by Cayley in 1843, see [9]. We shall investigate $P_{D,m}$ in Section 3.2 in more detail, but already remark here that for $D = 2$, it specializes to the usual determinant of a matrix: $P_{2,m}(w) = m! \det[w(\nu_1, \nu_2)]$, where $w = \sum_{\nu_1, \nu_2} w(\nu_1, \nu_2) |\nu_1 \nu_2\rangle$.

Theorem 3.12. *We have $P_T(gv) = (\det g)^s P_T(v)$ for all $g \in \operatorname{GL}_m$ and $v \in \otimes^D \mathbb{C}^m$.*

Proof. We recall that $gv := g^{\otimes D}(v)$ defines the action of GL_m on $\otimes^D \mathbb{C}^m$. In coordinates, $w = gv$ reads as follows:

$$w(\mu_1, \dots, \mu_D) = \sum_{r: [D] \rightarrow [m]} v(r) \prod_{\iota=1}^D g(\mu_\iota; r(\iota)), \quad (3.9)$$

where the sum is over all maps r from $[D]$ to $[m]$. Let $w = gv$ and $\sigma_1, \dots, \sigma_s \in S_m$. We use (3.9) to write

$$w(\sigma_{T_{1,i}}(T^{1,i}), \dots, \sigma_{T_{D,i}}(T^{D,i})) = \sum_{r: [D] \rightarrow [m]} v(r) \prod_{\iota=1}^D g(\sigma_{T_{\iota,i}}(T^{\iota,i}); r(\iota)).$$

Plugging this into the defining formula (3.7) for $P_T(w)$ we obtain

$$\begin{aligned} P_T(w) &= \sum_{\sigma_1, \dots, \sigma_s} \left[\prod_{j=1}^s \operatorname{sgn}(\sigma_j) \right] \cdot \sum_{r_1, \dots, r_d} \prod_{i=1}^d \left[v(r_i) \prod_{\iota=1}^D g(\sigma_{T_{\iota,i}}(T^{\iota,i}); r_i(\iota)) \right] \\ &= \sum_{\sigma_1, \dots, \sigma_s} \sum_{r_1, \dots, r_d} \left[\prod_{j=1}^s \operatorname{sgn}(\sigma_j) \right] \cdot \left[\prod_{i=1}^d v(r_i) \right] \prod_{i=1}^d \prod_{\iota=1}^D g(\sigma_{T_{\iota,i}}(T^{\iota,i}); r_i(\iota)) \\ &= \sum_{\sigma_1, \dots, \sigma_s} \sum_{r_1, \dots, r_d} \left[\prod_{i=1}^d v(r_i) \right] \left[\prod_{j=1}^s \operatorname{sgn}(\sigma_j) \right] \cdot \prod_{\iota=1}^D \prod_{i=1}^d g(\sigma_{T_{\iota,i}}(T^{\iota,i}); r_i(\iota)) \\ &= \sum_{r_1, \dots, r_d} \left[\prod_{i=1}^d v(r_i) \right] \sum_{\sigma_1, \dots, \sigma_s} \left[\prod_{j=1}^s \operatorname{sgn}(\sigma_j) \right] \cdot \prod_{\iota=1}^D \prod_{i=1}^d g(\sigma_{T_{\iota,i}}(T^{\iota,i}); r_i(\iota)) \\ &\stackrel{(3.6)}{=} \sum_{r_1, \dots, r_d} \left[\prod_{i=1}^d v(r_i) \right] \sum_{\sigma_1, \dots, \sigma_s} \left[\prod_{j=1}^s \operatorname{sgn}(\sigma_j) \right] \cdot \prod_{j=1}^s \prod_{\kappa=1}^m g(\sigma_j(\kappa); r_{T(\kappa,j)}(\tilde{T}(\kappa, j))) \\ &= \sum_{r_1, \dots, r_d} \left[\prod_{i=1}^d v(r_i) \right] \sum_{\sigma_1, \dots, \sigma_s} \prod_{j=1}^s \left[\operatorname{sgn}(\sigma_j) \prod_{\kappa=1}^m g(\sigma_j(\kappa); r_{T(\kappa,j)}(\tilde{T}(\kappa, j))) \right] \end{aligned}$$

$$= \sum_{r_1, \dots, r_d} \left[\prod_{i=1}^d v(r_i) \right] \underbrace{\prod_{j=1}^s \left[\sum_{\sigma \in S_m} \operatorname{sgn}(\sigma) \prod_{\kappa=1}^m g(\sigma(\kappa); r_{T(\kappa, j)}(\tilde{T}(\kappa, j))) \right]}_{=:\Delta(r_1, \dots, r_d)}.$$

For a tuple $r = (r_1, \dots, r_d)$ of maps r_i , each mapping from $[D]$ to $[m]$, let us write $\Delta(r) := \Delta(r_1, \dots, r_d)$. We interpret the tuple r as a map $r : [D] \times [d] \rightarrow [m]$. Using the bijection (3.6) we can reinterpret r as a map $r : [m] \times [s] \rightarrow [m]$, i.e., r assigns a number from $[m]$ to each position in T . This assignments results in a tableau that we denote by $\Gamma(r)$. The tableau $\Gamma(r)$ has the same shape as T , but its entries are taken from $[m]$. Lemma 3.13 below tells us that $\Delta(r) = 0$ if at least one column in $\Gamma(r)$ is not a permutation and otherwise $\Delta(r) = \det(g)^s \prod_{j=1}^s \operatorname{sgn}(\gamma(r, j))$, where $\gamma(r, j) \in S_m$ denotes the j th column of $\Gamma(r)$, that is, $\gamma(r, j)(\kappa) = r_{T(\kappa, j)}(\tilde{T}(\kappa, j))$. So it suffices to sum over all s -tuples of permutations $\gamma_1, \dots, \gamma_s$ of $[m]$ and we get

$$P_T(w) = \det(g)^s \sum_{\gamma_1, \dots, \gamma_s} \left[\prod_{j=1}^s \operatorname{sgn}(\gamma_j) \right] \cdot \prod_{i=1}^d v(\gamma_{T_{1,i}}(T^{1,i}), \dots, \gamma_{T_{D,i}}(T^{D,i})) = \det(g)^s P_T(v),$$

which completes the proof. \square

Lemma 3.13. *Let $g \in \operatorname{GL}_m$ and $\gamma : [m] \rightarrow [m]$ be a map. Then the sum*

$$\sum_{\sigma \in S_m} \operatorname{sgn}(\sigma) \prod_{\kappa=1}^m g(\sigma(\kappa); \gamma(\kappa))$$

equals $\operatorname{sgn}(\gamma) \det(g)$ if γ is a permutation and zero otherwise.

Proof. Writing $\tilde{g}(k; \kappa) := g(k; \gamma(\kappa))$, we get

$$\sum_{\sigma \in S_m} \operatorname{sgn}(\sigma) \prod_{\kappa=1}^m g(\sigma(\kappa); \gamma(\kappa)) = \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma) \prod_{\kappa=1}^m \tilde{g}(\sigma(\kappa); \kappa) = \det(\tilde{g}).$$

If γ is not a permutation, then \tilde{g} has two repeated columns and hence $\det(\tilde{g}) = 0$. Otherwise, clearly $\det(\tilde{g}) = \operatorname{sgn}(\gamma) \det(g)$. \square

Remark 3.14. A Young tableau T is called *semistandard* if the entries in each column of T increase from top to bottom and in each row they never decrease from left to right. The example (3.2) is semistandard, while (3.3) and (3.4) are not. The degree d invariant space $\mathcal{O}(\operatorname{Sym}^{D\mathbb{C}^m}_d)^{\operatorname{SL}_m}$ is generated by the set of all invariants P_T , where T runs over all semistandard tableaux of shape $m \times s$ with exactly D entries of each number $1, \dots, d$. The proof works similarly to the argument in Theorem 5.13. See for example [22, Sec. 3(A)].

3.2. Generic fundamental invariant of forms

The following result is due to Howe [20].

Theorem 3.15 (Howe). *We have $\mathcal{O}(\mathrm{Sym}^D \mathbb{C}^m)_d^{\mathrm{SL}_m} = 0$ if $d < m$. Moreover, $\mathcal{O}(\mathrm{Sym}^D \mathbb{C}^m)_m^{\mathrm{SL}_m}$ is one-dimensional if D is even and zero-dimensional if D is odd.*

If D is even, then Theorem 3.15 states that, up to a scaling factor, there is exactly one homogeneous SL_m -invariant $P_{D,m}: \mathrm{Sym}^D \mathbb{C}^m \rightarrow \mathbb{C}$ of degree m (and no nonzero invariant of smaller degree). We call $P_{D,m}$ the *generic fundamental invariant* of $\mathrm{Sym}^D \mathbb{C}^m$.

Before discussing the explicit formula for $P_{D,m}$, we observe the following.

Theorem 3.16. *Assume that $w \in \mathrm{Sym}^D \mathbb{C}^m$ is polystable. Then $m \leq e(w)$. If D is odd, then this inequality is strict. Moreover, if D is even, then we have $m = e(w)$ iff $P_{D,m}(w) \neq 0$. In this case, we have*

$$\Phi_w = P_{D,m}(w)^{-1} P_{D,m} \quad \text{on } \overline{Gw}.$$

Proof. Note that $\mathcal{O}(\overline{Gw})_d^{\mathrm{SL}_m} \neq 0$ implies $\mathcal{O}(\mathrm{Sym}^D \mathbb{C}^m)_d^{\mathrm{SL}_m} \neq 0$ and hence $m \leq d$ due to Theorem 3.15. (If D is odd we even have $m < d$.) If D is even, then Theorem 3.15 implies that $\mathcal{O}(\overline{Gw})_{e(w)}^{\mathrm{SL}_m}$ is generated by the restriction of $P_{D,n}$. So $\mathcal{O}(\overline{Gw})_{e(w)}^{\mathrm{SL}_m}$ does not vanish iff $P_{D,n}(w) \neq 0$, and in this case we have $\Phi_w = P_{D,m}(w)^{-1} P_{D,m}$. \square

The following is an immediate consequence. (It is instructive to test this in Example 3.8.)

Corollary 3.17. *If D is even, then the generic minimal degree is given by $e(D, m) = m$.*

We can now conclude that orbit closures of forms are rarely normal. In fact, a small stabilizer period already implies nonnormality of the orbit closure. (Recall that $a'(D, m) = 1$ except in a few cases by Theorem 2.3.)

Corollary 3.18. *1. If $w \in \mathrm{Sym}^D \mathbb{C}^m$ satisfies $a(w) < D$, then \overline{Gw} is not normal.*

2. Suppose that $a'(D, m) = 1$ and let $w \in \mathrm{Sym}^D \mathbb{C}^m$ be generic. Then \overline{Gw} is not normal if D is odd, or if D is even and $\gcd(D, m) > 1$.

Proof. 1. $a(w) < D$ implies $b(w) < m \leq e(w)$, where the last inequality is due to Theorem 3.16. Now apply Theorem 3.11.

2. We have $a(D, m) = a'(D, m) \frac{D}{\gcd(D, m)} < D$ by assumption. Now apply part one to a generic w . \square

In the following we assume that $D \geq 2$ is even. We take $d = m \geq 1$ and let T be the $m \times D$ tableau that has all is in row i , cf. (3.2). According to Section 3.1, we have a corresponding invariant P_T , that specializes to the form $P_{D,m}$ defined in (3.8).

Theorem 3.19. 1. We have $P_{D,m}(gv) = (\det g)^D P_{D,m}(v)$ for all $g \in \mathrm{GL}_m$ and $v \in \otimes^D \mathbb{C}^m$.
 2. Let $m \geq 2$. Then the polynomial $P_{D,m}$ is nonzero iff D is even. More specifically, $P_{D,m}(X_1^D + \cdots + X_m^D) = m! \neq 0$.

Proof. The first assertion follows by Theorem 3.12.

For the second assertion, assume $m \geq 2$. For a permutation σ_j of S_m we set $\tilde{\sigma}_j := \sigma_j \circ (12)$. By the defining equation (3.8),

$$\begin{aligned} P_{D,m}(v) &= \sum_{\tilde{\sigma}_1, \dots, \tilde{\sigma}_D} \left[\prod_{j=1}^D \mathrm{sgn}(\tilde{\sigma}_j) \right] \cdot \prod_{i=1}^m v(\tilde{\sigma}_1(i), \dots, \tilde{\sigma}_D(i)) \\ &= \sum_{\sigma_1, \dots, \sigma_D} (-1)^D \left[\prod_{j=1}^D \mathrm{sgn}(\sigma_j) \right] \cdot \prod_{i=1}^m v(\sigma_1(i), \dots, \sigma_D(i)) = -P_{D,m}(v), \end{aligned}$$

if D is odd. Therefore, $P_{D,m}(w) = 0$ for any $w \in \otimes^D \mathbb{C}^m$.

Suppose now that D is even and consider the symmetric tensor $w = \sum_{i=1}^m |i \cdots i\rangle \in \otimes^D \mathbb{C}^m$ corresponding to the power sum $X_1^D + \cdots + X_m^D$. Thus $w(\mu_1, \dots, \mu_D)$ is nonzero iff $\mu_1 = \dots = \mu_D$. Hence in (3.8), we only have contributions if $\sigma_1 = \dots = \sigma_D$ and therefore $P_{D,m}(w) = \sum_{\sigma \in S_m} (\mathrm{sgn}(\sigma))^D = m! \neq 0$. \square

Lemma 3.20. $P_{D,m}$ is an irreducible polynomial of degree m if D is even.

Proof. This can be directly derived from the fact that, by Theorem 3.15, $P_{D,m}$ is a nonzero SL_m -invariant on $\mathrm{Sym}^D \mathbb{C}^m$ of smallest degree. For an analog argument see for example [43, p. 105]. \square

Problem 3.21. What is the computational complexity to evaluate $P_{D,m}$?

3.2.1. The case where $D = m$ is odd

We now treat the case where D is odd, but only in the special case $D = m$.

Theorem 3.22 (Howe). If D is odd, $\dim \mathcal{O}(\mathrm{Sym}^D \mathbb{C}^D)^{\mathrm{SL}_D} = 1$, otherwise $\mathcal{O}(\mathrm{Sym}^D \mathbb{C}^D)^{\mathrm{SL}_D} = 0$.

The tableau T that we base our construction of the generic fundamental invariant on is the $D \times (D+1)$ tableau T that has the entry $((j-i+1) \bmod (D+1))$ in row i and column j . Here we used the convention that $k \bmod (D+1) \in \{1, \dots, D+1\}$. An example is provided in (3.4). We write $P_D := P_T$. Interpreting (3.7) we see that

$$P_D := \sum_{\sigma_1, \dots, \sigma_{D+1}} \left[\prod_{j=1}^{D+1} \mathrm{sgn}(\sigma_j) \right] \prod_{i=1}^{D+1} v(\sigma_i(1), \dots, \sigma_{i+D-1}(D)), \quad (3.10)$$

where the indices of σ have to be taken modulo $D+1$.

Theorem 3.23. 1. We have $P_D(gv) = (\det g)^{D+1} P_D(v)$ for all $g \in \mathrm{GL}_D$ and $v \in \otimes^D \mathbb{C}^D$.
 2. The polynomial P_D is nonzero iff D is odd.

Proof. The first assertion follows by Theorem 3.12. Suppose now that $D = m$ is odd and for $\alpha \in \mathbb{C}$ consider the symmetric tensor $w = \alpha|(1 + 2 + \cdots + D) \cdots (1 + 2 + \cdots + D)\rangle + \sum_{i=1}^D |i \cdots i\rangle \in \otimes^D \mathbb{C}^D$ corresponding to the polynomial $\alpha(X_1 + X_2 + \cdots + X_D)^D + X_1^D + X_2^D + \cdots + X_D^D$. Thus $w(\mu_1, \dots, \mu_D) = 1 + \alpha$ iff $\mu_1 = \dots = \mu_D$ and $w(\mu_1, \dots, \mu_D) = \alpha$ otherwise. The evaluation of (3.10) at w is a polynomial in α of degree at most $D + 1$. (In fact, it is homogeneous of degree D , but we will not need that fact.) We analyze the coefficient of α^D and show that it is nonzero.

Let us study the summands in the evaluation of (3.10) that contribute to the coefficient of α^D . For such a summand there exists a cardinality D subset $I \subset \{1, \dots, D + 1\}$ such that $\sigma_i(1) = \dots = \sigma_{i+D-1}(D)$ for all $i \in I$ (all indices of the σ are to be taken modulo $D + 1$). Set $\bar{i} \in \{1, \dots, D + 1\}$ to be the unique number such that $(\bar{i} + 1) \bmod (D + 1) \notin I$, so we have $\sigma_i(1) = \dots = \sigma_{\bar{i}}(\bar{i} - i + 1) = \dots = \sigma_{i+D-1}(D)$ for all $i \in I$. Intuitively \bar{i} is the “rightmost” column index of I , modulo $D + 1$. We see that all σ_i , $1 \leq i \leq D + 1$, are completely determined by $\sigma_{\bar{i}}$, because in order to ensure that all σ_i are permutations we must have $\sigma_{\bar{i}+1}(1) = \sigma_{\bar{i}}(D)$, $\sigma_{\bar{i}+2}(2) = \sigma_{\bar{i}}(D - 1)$, \dots , $\sigma_{\bar{i}+D}(D) = \sigma_{\bar{i}}(1)$. In particular $\sigma_{\bar{i}+1}(1), \dots, \sigma_{\bar{i}+D}(D)$ are pairwise distinct, so $v(\sigma_{\bar{i}+1}(1), \dots, \sigma_{\bar{i}+D}(D)) = 1$. It is crucial to observe that for all i the permutation σ_i differs from σ_{i+1} by a cyclic permutation of order D , i.e., by a shift by one. Since D is odd, this implies that all $D + 1$ permutations σ_i have the same sign. Therefore the product $\prod_{i=1}^{D+1} \mathrm{sgn}(\sigma_i)$ is either $1^{D+1} = 1$ or $(-1)^{D+1} = 1$, since $D + 1$ is even. We conclude that each such summand contributes 1 to the coefficient of α^D , which results in the coefficient being $D!$, which is nonzero. \square

Problem 3.24. In which degree does the generic fundamental invariant appear for odd D and arbitrary m ?

3.3. Minimal degree for specific forms

We can determine, or at least combinatorially characterize, the minimal degree of a few interesting forms.

Proposition 3.25. Let $w = X_1^D + \cdots + X_m^D$.

1. If D is even, then w has the degree period $b(w) = m$ and minimal degree $e(w) = m$.
2. If D is odd, then w has the degree period $b(w) = 2m$. If additionally $2m \leq \binom{2D}{D}$, then we have $e(w) = 2m$.
3. If D is odd and $2m > \binom{2D}{D}$, we have $e(w) > 2m$.

In the cases 1 and 2, we have $e'(w) = 1$. In case 3, we have $e'(w) > 1$.

Proof. 1. The first statement follows from [Theorem 3.19](#), [Theorem 3.23](#), and [Proposition 2.4](#).

2. For the construction of the invariant we choose a tableau T with m rows, $2D$ columns, and entries $1, \dots, 2m$, each of which appears D times. See [\(3.3\)](#) for an example. For all $i \in \{1, 2, \dots, 2m\}$ all occurrences of i will be located in row $\lceil \frac{i}{2} \rceil$. In particular for odd i the entries of i and $i+1$ are in the same row. Let $J_i \subset \{1, \dots, 2D\}$ denote the cardinality D subsets that specifies the columns of T in which i appears. The arrangement of numbers in T should be done in a way that for distinct i, j we have $J_i \neq J_j$. To construct such an arrangement, we proceed row by row in a greedy fashion, i.e., we first construct J_1 and J_2 , then construct J_3 and J_4 , and so on. We always choose J_i such that $J_i \neq J_j$ for all $j < i$. When we chose J_i for an odd i , this defines J_{i+1} since we assume that J_i and J_{i+1} are disjoint and hence J_{i+1} is the complement of J_i in $\{1, 2, \dots, 2D\}$. But since $J_i \neq J_j$ for all $j < i$, it follows by construction that also $J_{i+1} \neq J_j$ for all $j < i+1$. We can proceed in this way until we run out of cardinality D subsets of $\{1, 2, \dots, 2D\}$, so as long as $2m \leq \binom{2D}{D}$ we can finish this construction and end up with a tableau T .

By [Theorem 3.12](#), P_T is SL_m -invariant. By construction, for each i and j that are in different rows in T , we have

$$J_i \cap J_j \neq \emptyset. \quad (*)$$

We now evaluate P_T at the power sum $w = X_1^D + \dots + X_m^D$. We analyze the nonzero summands in the evaluation $P_T(w)$, see [\(3.7\)](#). Let $r_i = \lceil \frac{i}{2} \rceil$ denote the row in which i appears in T and let $J_i(1), \dots, J_i(D)$ denote the columns in which i appears. We remark that in the notation of [Section 3.1](#) we have $T^{j,i} = r_i$ and $T_{j,i} = J_i(j)$. Clearly $w(\mu_1, \dots, \mu_D) = 1$ iff $\mu_1 = \dots = \mu_D$ and $w(\mu_1, \dots, \mu_D) = 0$ otherwise. Therefore we only have to consider the cases where

$$\sigma_{J_i(1)}(r_i) = \sigma_{J_i(2)}(r_i) = \dots = \sigma_{J_i(D)}(r_i). \quad (**)$$

Assuming [\(*\)](#) and [\(**\)](#) we show now that

$$\begin{aligned} \sigma_{J_i(1)}(r_i) &= \sigma_{J_i(2)}(r_i) = \dots = \sigma_{J_i(D)}(r_i) \\ &= \sigma_{J_j(1)}(r_j) = \sigma_{J_j(2)}(r_j) = \dots = \sigma_{J_j(D)}(r_j) \end{aligned} \quad (\dagger)$$

in all cases where $r_i = r_j$. First we see that among the $2Dm$ numbers $\sigma_{J_i(\kappa)}(r_i)$, $1 \leq i \leq 2m$, $1 \leq \kappa \leq D$, each number from $\{1, \dots, m\}$ appears exactly $2D$ times: This is clear because the σ_j are $2D$ permutations of $[m]$. By [\(**\)](#) this implies that for each number $1 \leq \kappa \leq m$ there exist exactly two distinct indices i and j such that $\sigma_{J_i(1)}(r_i) = \sigma_{J_j(1)}(r_j)$. The fact that all σ_j are permutations rules out all pairs (i, j) for which $J_i \cap J_j \neq \emptyset$. These are all pairs that are not in the same row, as stated in property [\(*\)](#). The remaining pairs are $(i, i+1)$ with i odd. This implies [\(\dagger\)](#).

Hence $\sigma_1 = \sigma_2 = \dots = \sigma_{2D}$. Since $2D$ is even, it follows that $\prod_{j=1}^{2D} \text{sgn}(\sigma_j) = \text{sgn}(\sigma_1)^{2D} = 1$. This results in $m!$ summands, all of which contribute 1 to the evaluation. Therefore $P_T(w) = m! \neq 0$.

3. The reason is a vanishing plethysm coefficient: There is no nonzero SL_m -invariant in degree $2m$ if $\binom{2D}{D} < 2m$. For the proof of the upper bound we use the upper bound from Appendix 7, Corollary 7.2. It suffices to show that there are less than $2m$ distinct cardinality D subsets of $\{1, \dots, 2D\}$. But there are only $\binom{2D}{D}$ many cardinality D subsets of $\{1, \dots, 2D\}$ and $\binom{2D}{D} < 2m$. \square

A *Latin square of size n* is map $T: [n]^2 \rightarrow [n]$, viewed as a $n \times n$ matrix with entries in $[n]$, such that in each row and each column each entry in $[n]$ appears exactly once. So in each column and row we get a permutation of $[n]$. The column sign of T is defined as the product of the signs of column permutations. The Latin square T is called *column-even* if this sign equals one, otherwise T is called *column-odd*.

The Alon–Tarsi conjecture [1] states that the number of column-even Latin squares of size n is different from the number of column-odd Latin squares of size n if n is odd. This conjecture is known to be true if $n = p \pm 1$ where p is a prime, cf. [15,18]. The following observation is due to Kumar [26] and Kumar and Landsberg [27].

Proposition 3.26. *Let m be even. Then $e(X_1 \dots X_m) \geq m$ and equality holds iff the Alon–Tarsi conjecture is true for m .*

Proof. By Theorem 3.16 it is sufficient to show that $P_{m,m}$ evaluated at $X_1 \dots X_m$ is nonzero iff the Alon–Tarsi conjecture holds for m .

In order to show this, consider the symmetric tensor

$$w_m := \frac{1}{m!} \sum_{\pi \in S_m} |\pi(1) \dots \pi(m)\rangle$$

corresponding to the polynomial $X_1 \dots X_m$. Hence $w(\mu_1, \dots, \mu_m)$ is nonzero iff μ is a permutation of $[m]$. We evaluate now $P_{m,m}(w_m)$ according to (3.8). If we interpret $\sigma_1, \dots, \sigma_m$ as the columns of the matrix $T: [m] \times [m] \rightarrow [m]$, $(i, j) \mapsto \sigma_i(j)$, then we get a nonzero contribution iff T is a latin square of size m , that is, in each row and in each column, each symbol in $[m]$ appears exactly once. Therefore, $P_{m,m}(w_m)$ equals the difference of the column-even and the column-odd latin squares of size m . The Alon–Tarsi conjecture [1] states that this difference is nonzero if m is even. \square

Remark 3.27. Proposition 3.26 can be generalized to odd m by reinterpreting the Alon–Tarsi conjecture as follows. An $m \times d$ *Latin Annulus* is defined to be an $m \times d$ matrix where in each column and in each diagonal $((1, i), (2, i+1), (3, i+2), \dots, (m, i+m-1))$ each number from $1, \dots, m$ occurs exactly once, where the column indices are taken modulo d . The column-sign of a Latin Annulus is the product of the signs of the permutations in its columns. The Alon–Tarsi conjecture is equivalent to saying that for even m

the number of even $m \times m$ Latin Annuli is different from the number of odd $m \times m$ Latin Annuli. Analogously to the proof of [Proposition 3.26](#) we see that for odd m $P_m(w_m) \neq 0$ iff the number of even $m \times (m+1)$ Latin Annuli is different from the number of odd $m \times (m+1)$ Latin Annuli. Therefore for odd m we have $e(X_1 \dots X_m) = m+1$ iff the number of even $m \times (m+1)$ Latin Annuli is different from the number of odd $m \times (m+1)$ Latin Annuli. We verified that the number of even $m \times (m+1)$ Latin Annuli differs from the number of odd $m \times (m+1)$ Latin Annuli for $m = 1, 3, 5$, and 7 .

The period of \det_n and per_n was characterized in [Theorem 2.5](#). Combined with [Theorem 3.16](#), we obtain that $e(\det_n) \geq n^2$, with equality holding iff $P_{n,n^2}(\det_n) \neq 0$. (An analogous statement holds for per_n .)

Remark 3.28. \det_2 and per_2 are quadratic forms of full rank, they are equivalent to $X_1^2 + \dots + X_4^2$. Hence $P_{2,4}(\det_2)$ and $P_{2,4}(\text{per}_2)$ are nonzero due to [Theorem 3.19](#).

For geometric complexity, it would be of great interest to know whether $e(\det_n) = n^2$. If equality holds, then we have an explicit description of the fundamental invariant of \det_n , which provides a link between the coordinate ring of $\text{GL}_{n^2} \det_n$ and its closure.

We now provide a combinatorial description of $P_{n,n^2}(\det_n)$ in the spirit of the Alon–Tarsi conjecture if n is even.

We consider maps $S: [n^2] \times [n] \rightarrow [n]$, viewed as matrices of format $n^2 \times n$, with entries in $[n]$, in which each row $S(i, \cdot)$ is a permutation of $[n]$. Let us call the product of the signs of the rows the row-sign of S . We call a pair (S, T) of maps $[n^2] \times [n] \rightarrow [n]$ an *admissible n -table* iff each column of the table (S, T) enumerates $[n] \times [n]$, that is, for all j , $[n^2] \rightarrow [n] \times [n]$, $i \mapsto (S(i, j), T(i, j))$ is bijective. The sign of this permutation is well-defined after fixing an ordering of $[n] \times [n]$. We denote by $\text{colsgn}(S, T)$ the product of these signs and call it the column-sign of the admissible table (S, T) . The table is called *column-even* iff $\text{colsgn}(S, T) = 1$ and *column-odd* otherwise.

We define the row-sign of an admissible table (S, T) as the product of the row-sign of S and the row-sign of T . The sign of (S, T) is defined as the product of its row and column sign. Accordingly, we call an admissible table *even* iff it has sign 1 and *odd* otherwise.

Proposition 3.29. $P_{n,n^2}(\det_n)$ is the difference between the number of even and odd admissible n -tables. $P_{n,n^2}(\text{per}_n)$ is the difference between the number of column-even and column odd admissible n -tables.

Proof. The permanent $\text{per}_n = \sum_{\pi \in S_n} X_{\pi(1),1} \dots X_{\pi(n),n}$ corresponds to the following symmetric tensor in $\text{Sym}^n \mathbb{C}^{n \times n}$ (denoted by the same symbol)

$$\begin{aligned} \text{per}_n &:= \frac{1}{n!} \sum_{\pi \in S_n} \sum_{\sigma \in S_n} |(\sigma\pi(1), \sigma(1)) \cdots (\sigma\pi(n), \sigma(n))\rangle \\ &= \frac{1}{n!} \sum_{\rho \in S_n} \sum_{\sigma \in S_n} |(\rho(1), \sigma(1)) \cdots (\rho(n), \sigma(n))\rangle. \end{aligned}$$

Thus an n -tuple $((\mu_1, \nu_1), \dots, (\mu_n, \nu_n))$ in $[n]^2$ lies in the support of per_n iff μ and ν are permutations of $[n]$. Similarly, the determinant $\det_n = \sum_{\pi \in S_n} \text{sgn}(\pi) X_{\pi(1),1} \cdots X_{\pi(n),n}$ corresponds to

$$\det_n := \frac{1}{n!} \sum_{\rho \in S_n} \sum_{\sigma \in S_n} \text{sgn}(\rho) \text{sgn}(\sigma) \cdot |(\rho(1), \sigma(1)) \cdots (\rho(n), \sigma(n))\rangle.$$

The assertion now follows from the explicit description (3.8) of fundamental invariants. \square

Based on Proposition 3.29 we have verified that $P_{4,16}(\text{per}_4)$ and $P_{4,16}(\det_4)$ are nonzero, using computer calculations.

Corollary 3.30. 1. The orbit closure of $X_1 \cdots X_m$ is not normal if $m > 2$.

2. The orbit closures of \det_n and per_n are not normal if $n > 2$.

Proof. Combine Theorem 3.11 with Theorem 3.16 and the determination of the periods in Proposition 2.4, Theorem 2.5, and Theorem 2.6. \square

We note that the orbit closures of $X_1 X_2$ equals $\text{Sym}^2 \mathbb{C}^2$ and hence is normal. Similarly, the orbit closures of \det_2 and per_2 equal $\text{Sym}^2 \mathbb{C}^4$ and hence are normal.

Let us point out again that the non-normality of orbit closures is one of the main sources of difficulty in implementing Mulmuley and Sohoni's strategy, see [26,4].

4. Stabilizer period and polystability of tensors

Our investigations here are analogous to Section 2.1 and Section 2.2 for forms. We confine ourselves to cubic tensors. Recall that the group GL_m^3 acts on $W := \otimes^3 \mathbb{C}^m$ via the tensor product. Some of the results appeared in [5].

4.1. Stabilizer and stabilizer period of tensors

The subgroup $K := \{(\zeta_1 I_m, \zeta_2 I_m, \zeta_3 I_m) \mid \zeta_i \in \mathbb{C}^\times, \zeta_1 \zeta_2 \zeta_3 = 1\}$ acts trivially on $W = \otimes^3 \mathbb{C}^m$, so we have an induced action of $G := \text{GL}_m^3 / K$ on W . The stabilizer $\text{stab}(w)$ of a tensor $w \in \otimes^3 \mathbb{C}^m$ is defined as the following closed subgroup of GL_m^3 :

$$\text{stab}(w) := \{(g_1, g_2, g_3) \in \text{GL}_m^3 \mid (g_1 \otimes g_2 \otimes g_3)w = w\}. \quad (4.1)$$

Clearly, $K \subseteq \text{stab}(w)$. All the information about $\text{stab}(w)$ is contained in the *reduced stabilizer* $\text{stab}'(w) := \text{stab}(w)/K$ of w , which is defined as the image of $\text{stab}(w)$ under the canonical morphism $\text{GL}_m^3 \rightarrow G$. We say that w has a *trivial stabilizer* if $\text{stab}'(w)$ consists of the unit element only.

The group homomorphism

$$\chi: \text{GL}_m^3 \rightarrow \mathbb{C}^\times, (g_1, g_2, g_3) \mapsto \det(g_1) \det(g_2) \det(g_3) \quad (4.2)$$

factors through K ; we denote the resulting homomorphism $G \rightarrow \mathbb{C}^\times$ by χ as well. The image $H := \chi(\text{stab}(w))$ of the stabilizer under χ is a closed subgroup of \mathbb{C}^\times (cf. [21, §7.4, Prop. B]). Either $H = \mathbb{C}^\times$ or H equals the group μ_a of a th roots of unity, where $a = |H|$.

Definition 4.1. The *stabilizer period* $a(w)$ of a tensor $w \in \otimes^3 \mathbb{C}^m$ is defined as the order of $\chi(\text{stab}(w))$.

It is clear that a tensor with trivial stabilizer has the stabilizer period 1. It follows from general principles that there exists $a(m)$ such that almost all $w \in \otimes^3 \mathbb{C}^m$ have the stabilizer period $a(m)$; see [47] or Popov and Vinberg in [49, §7].

Theorem 4.2. We have $a(m) = 1$ for $m \geq 3$ and $a(2) = 2$.

Proof. A.M. Popov [44, Thm. 2] proved that if $m > 3$, then almost all $w \in \otimes^3 \mathbb{C}^m$ have a trivial stabilizer group, hence $a(m) = 1$. (His result is more general and covers also noncubic dimension formats.)

The isomorphism classes of tensors in $\otimes^3 \mathbb{C}^3$ have been classified by Thrall and Chanler [51], see also [42]. This reveals that $a(w) = 1$ for a generic $w \in \otimes^3 \mathbb{C}^3$.

Almost all $w \in \otimes^3 \mathbb{C}^2$ are in the same GL_2^3 -orbit as the unit tensor $\langle 2 \rangle := |1\rangle \otimes |1\rangle \otimes |1\rangle + |2\rangle \otimes |2\rangle \otimes |2\rangle$, which has a nontrivial stabilizer and stabilizer period two, cf. Theorem 4.3. \square

We shall next determine the stabilizer and the stabilizer period of some interesting tensors. We first proceed with a general useful reasoning (compare [10, Chap. 2]) that will also be used in Appendix 7. The *Hessian* $H(f)$ of a polynomial f in the variables X_1, \dots, X_n is defined as the symmetric matrix given by the second order partial derivatives of f :

$$H(f) := [\partial_{X_i X_j}^2 f]_{i,j \leq n}.$$

Let $A \in \text{GL}_n$ and put $\tilde{f}(X) := f(AX)$, where $X = (X_1, \dots, X_n)^T$. The chain rule easily implies that

$$H(\tilde{f})(X) = A^T \cdot H(f)(AX) \cdot A,$$

which implies $\det H(\tilde{f})(X) = (\det A)^2 \cdot \det H(f)(AX)$. In particular, if $f(AX) = f(X)$, we get

$$\det H(f)(X) = (\det A)^2 \cdot \det H(f)(AX). \quad (4.3)$$

It is useful to interpret a tensor $w \in \otimes^3 \mathbb{C}^m$ as a trilinear form $\otimes^3 (\mathbb{C}^m)^* \rightarrow \mathbb{C}$. If $X_1, \dots, X_m, Y_1, \dots, Y_m$, and Z_1, \dots, Z_m denote the standard basis of $(\mathbb{C}^m)^*$ in the three factors, respectively, we can write w as the multilinear polynomial $w = \sum_{ijk} w_{ijk} X_i Y_j Z_k$.

The m th unit tensor is defined as $\langle m \rangle := \sum_{i=1}^m |i\rangle \otimes |i\rangle \otimes |i\rangle$. It corresponds to the trilinear form $\sum_{i=1}^m X_i Y_i Z_i$.

The following result is known, but our proof is more elementary than the one given in [11]; see also [5].

Theorem 4.3. *The stabilizer H_m of the unit tensor $\langle m \rangle$ in GL_m^3 is the semidirect product of the subgroup*

$$\mathcal{T}_m := \{(\mathrm{diag}(a), \mathrm{diag}(b), \mathrm{diag}(c)) \mid \forall i \ a_i b_i c_i = 1\},$$

and the symmetric group S_m diagonally embedded in G_m via $\pi \mapsto (P_\pi, P_\pi, P_\pi)$. (Here P_π denotes the permutation matrix of π .) The stabilizer period of $\langle m \rangle$ equals 2 if $m > 1$.

Proof. A straightforward calculation shows that the Hessian $H(f)$ of the trilinear form $f = \sum_{i=1}^m X_i Y_i Z_i$ is the block direct sum of the matrices $\begin{bmatrix} 0 & Z_i & Y_i \\ Z_i & 0 & X_i \\ Y_i & X_i & 0 \end{bmatrix}$ for $i = 1, \dots, m$. We have

$$\det H(f) = \prod_{i=1}^m X_i Y_i Z_i. \quad (4.4)$$

Suppose the linear transformation $X_i \mapsto X'_i, Y_i \mapsto Y'_i, Z_i \mapsto Z'_i$ fixes f . By (4.3), the determinant of the Hessian of f is preserved up to a scalar. Hence (4.4) implies that $\prod_{i=1}^m X_i Y_i Z_i = c \prod_{i=1}^m X'_i Y'_i Z'_i$ for some $c \in \mathbb{C}^\times$. The uniqueness of factorization of polynomials implies that $X'_i = a_i X_{\rho(i)}, Y'_i = b_i Y_{\sigma(i)}, Z'_i = c_i Z_{\tau(i)}$ for $\rho, \sigma, \tau \in S_m$ and $a_i, b_i, c_i \in \mathbb{C}^\times$. Since the transformation fixes f , we must have $\rho = \sigma = \tau$ and $a_i b_i c_i = 1$ for all i . \square

It is remarkable that $\langle m \rangle$ is uniquely determined by its stabilizer up to a scalar.

Proposition 4.4. *If the stabilizer of some tensor $w \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ contains the stabilizer H_m of the unit tensor $\langle m \rangle$, then $w = c \langle m \rangle$ for some $c \in \mathbb{C}$.*

Proof. Assume the stabilizer of $w = \sum w_{ijk} |ijk\rangle$ contains H_m . By contradiction, we suppose that $w_{ijk} \neq 0$ for some i, j, k with $i \neq k$. For any $(\mathrm{diag}(a), \mathrm{diag}(b), \mathrm{diag}(c)) \in \mathcal{T}_m$ we have $a_i b_j c_k w_{ijk} = w_{ijk}$ and hence $a_i b_j c_k = 1 = a_k b_k c_k$, which implies $a_i = a_k b_k / b_j$. However, defining $\tilde{a}_i = 2a_i, \tilde{c}_i = \frac{1}{2}c_i, \tilde{a}_\ell = a_\ell, \tilde{c}_\ell = c_\ell$ for $\ell \neq i$ we

get $(\text{diag}(\tilde{a}), \text{diag}(b), \text{diag}(\tilde{c})) \in \mathcal{T}_m$. This yields the contradiction $\tilde{a}_i = \tilde{a}_k b_k / b_j = a_k b_k / b_j = a_i$. We have thus shown that $w_{ijk} \neq 0$ implies $i = k$. By symmetry, we conclude that $w_{ijk} = 0$ unless $i = j = k$. Finally, from the invariance of w under S_m , we get $w_{iii} = w_{111}$ for all i . Hence $w = w_{111} \langle m \rangle$. \square

We turn now to the matrix multiplication tensor $\langle n, n, n \rangle$, which can be interpreted as the trilinear form $\text{trace}(XYZ)$, where X, Y, Z are $n \times n$ matrices of variables.

The following result is due to de Groote [11] and relies on the Skolem–Noether theorem; see also [5, Prop. 5.1].

Theorem 4.5. *The stabilizer of $\text{trace}(XYZ)$ in GL_n^3 is given by the transformations $X \mapsto AXB^{-1}, Y \mapsto BXC^{-1}, Z \mapsto CXA^{-1}$, where $A, B, C \in \text{GL}_n$.*

Corollary 4.6. *The tensor $\langle n, n, n \rangle$ of matrix multiplication has the stabilizer period 1.*

Proof. We recall that the linear transformation $g_{A,B}: \mathbb{C}^{n \times n} \rightarrow \mathbb{C}^{n \times n}, X \mapsto AXB^{-1}$, defined for $A, B \in \text{GL}_n$, satisfies $\det(g_{A,B}) = \det(A)^n \det(B)^{-n}$. If $g \in \text{GL}_n^3$ is given by the transformations $X \mapsto AXB^{-1}, Y \mapsto BXC^{-1}, Z \mapsto CXA^{-1}$, then

$$\begin{aligned} \chi(g) &= \det(g_{A,B}) \det(g_{B,C}) \det(g_{C,A}) \\ &= \det(A)^n \det(B)^{-n} \det(B)^n \det(C)^{-n} \det(C)^n \det(A)^{-n} = 1. \end{aligned}$$

Hence the period equals 1. \square

It is remarkable that $\langle n, n, n \rangle$ is uniquely determined by its stabilizer up to a scalar.

Proposition 4.7. *If the stabilizer of some tensor $w \in \otimes^3 \mathbb{C}^{n \times n}$ contains the stabilizer of $\langle n, n, n \rangle$, then $w = c \langle n, n, n \rangle$ for some $c \in \mathbb{C}$.*

Proof. It is helpful have a coordinate-free description of $\langle n, n, n \rangle$. Let U be a \mathbb{C} -vector space of dimension n . Recall the canonical isomorphism $U^* \otimes U \simeq \text{End}(U)$, $\ell \otimes v \mapsto (u \mapsto \ell(u)v)$, which is $\text{GL}(U) \times \text{GL}(U)$ -equivariant. Choosing a basis of U and its dual basis of U^* , identifying $\text{End}(U)$ with $\mathbb{C}^{m \times m}$, and $\text{GL}(U)$ with GL_m , the resulting action of $\text{GL}_m \times \text{GL}_m$ on $\mathbb{C}^{m \times m}$ turns out to be $(A, B)X = AX(B^T)^{-1}$. (The transpose comes from going over to the dual; this is sometimes confusing.) We note that the space of $\text{GL}(U) \times \text{GL}(U)$ -invariants in $\text{End}(U) = U^* \otimes U$ is one-dimensional and equals Cid_U .

Let now U_1, U_2, U_3 be three \mathbb{C} -vector space of dimension n . We have a canonical isomorphism

$$\text{End}(U_1) \otimes \text{End}(U_2) \otimes \text{End}(U_3) \simeq (U_1^* \otimes U_2) \otimes (U_2^* \otimes U_3) \otimes (U_3^* \otimes U_1), \quad (4.5)$$

which results from permuting the factors in a cyclic way and rearranging parentheses. It is easy to check that the image of $\text{Cid}_{U_1} \otimes \text{Cid}_{U_2} \otimes \text{Cid}_{U_3}$ on the right-hand side corresponds

to the composition $\text{Hom}(U_2, U_1) \times \text{Hom}(U_3, U_2) \rightarrow \text{Hom}(U_3, U_1)$, $(\varphi, \psi) \mapsto \varphi \circ \psi$ of linear maps. Therefore, the image of $\text{id}_{U_1} \otimes \text{id}_{U_2} \otimes \text{id}_{U_3}$ is the coordinate-free description of the matrix multiplication tensor $\langle n, n, n \rangle$.

According to [Theorem 4.5](#), the action of the stabilizer of $\langle n, n, n \rangle$, realized on the left-hand side of [\(4.5\)](#), is the usual action of $\text{GL}(U_1) \times \text{GL}(U_2) \times \text{GL}(U_3)$. On the other hand, the space of $\text{GL}(U_1) \times \text{GL}(U_2) \times \text{GL}(U_3)$ -invariants of $\text{End}(U_1) \otimes \text{End}(U_2) \otimes \text{End}(U_3)$ is one-dimensional and given by $\mathbb{C} \text{id}_{U_1} \otimes \text{id}_{U_2} \otimes \text{id}_{U_3}$. \square

4.2. Polystability of tensors

We call a tensor $w \in \otimes^3 \mathbb{C}^m$ *polystable* iff the SL_m^3 -orbit of w is closed. Our goal is a criterion for polystability, analogous to [Proposition 2.8](#). We define the *support* of a tensor $w \in \otimes^3 \mathbb{C}^m$ as

$$\text{supp}(w) := \{(i, j, k) \in [m]^3 \mid w_{ijk} \neq 0\}, \quad (4.6)$$

where $w = \sum_{ijk} w_{ijk} |ijk\rangle$. Suppose we have a map $\alpha: \text{supp}(w) \rightarrow \mathbb{R}$. Its *first marginal* is defined by $\alpha^1: [m] \rightarrow \mathbb{R}$, $i \mapsto \sum_{j,k} \alpha(i, j, k)$, and its second and third marginals α^2, α^3 are defined similarly. Note that if α is a probability distribution on $\text{supp}(w)$, then its marginals are probability distributions on $[m]$.

Let T_m denote the subgroup of GL_m consisting of the diagonal matrices and consider the subgroup $T_m^3 := T_m \times T_m \times T_m$ of GL_m^3 . We have the following analogue of [Proposition 2.8](#).

Proposition 4.8. *Let the tensor $w \in \otimes^3 \mathbb{C}^m$ satisfy the following two properties:*

1. *There is a reductive subgroup R of $\text{SL}_m^3 \cap \text{stab}(w)$ such that the centralizer of R in SL_m^3 is contained in T_m^3 .*
2. *There is a probability distribution α on $\text{supp}(w)$ such that its marginals $\alpha^1, \alpha^2, \alpha^3$ are the uniform distributions on $[m]$.*

Then w is polystable.

Proof. We proceed as for [Proposition 2.8](#). Suppose that w is not polystable and let Y be a nonempty closed SL_m^3 -orbit in $\text{SL}_m^3 w \setminus \text{SL}_m^3 w$. By the Hilbert–Mumford criterion, refined by Luna [\[31, Cor. 1\]](#) and Kempf [\[24, Cor. 4.5\]](#), there exists a one-parameter subgroup $\sigma: \mathbb{C}^\times \rightarrow \text{SL}_m^3$ with $y := \lim_{t \rightarrow 0} \sigma(t)w \in Y$, such that the image of σ is contained in the centralizer of R in SL_m^3 . By our first assumption, σ maps \mathbb{C}^\times to T_m^3 . So there exist $\mu, \nu, \pi \in \mathbb{Z}^m$ such that $\sigma(t) = (\text{diag}(t^{\mu_1}, \dots, t^{\mu_m}), \text{diag}(t^{\nu_1}, \dots, t^{\nu_m}), \text{diag}(t^{\pi_1}, \dots, t^{\pi_m}))$ for $t \in \mathbb{C}^\times$. Note that $\sum_i \mu_i = \sum_i \nu_i = \sum_i \pi_i = 0$ since the image of σ is in SL_m^3 .

We have

$$\sigma(t)w = \sum_{i,j,k} t^{\mu_i + \nu_j + \pi_k} w_{ijk}.$$

The existence of $\lim_{t \rightarrow 0} \sigma(t)w$ implies that $\mu_i + \nu_j + \pi_k \geq 0$ for all $(i, j, k) \in \text{supp}(w)$. Taking convex combinations, this condition implies that $\langle \alpha^1, \mu \rangle + \langle \alpha^2, \nu \rangle + \langle \alpha^3, \pi \rangle \geq 0$ for all probability distributions α on $\text{supp}(w)$.

Our second assumption states that there exists a probability distribution α on $\text{supp}(w)$ whose marginals are the uniform distributions on $[m]$. But then $\langle \alpha^1, \mu \rangle = \frac{1}{m} \sum_i \mu_i = 0$ and similarly, $\langle \alpha^2, \nu \rangle = \langle \alpha^3, \pi \rangle = 0$. This implies $\mu = \nu = \pi = 0$ and hence $\sigma(t) = 1$ for all t . Therefore $y = w$, which contradicts the fact the y lies in the boundary of the SL_m -orbit of w . \square

The following result was shown by Meyer [39].

Corollary 4.9. *The unit tensors $\langle m \rangle$ and the matrix multiplication tensors $\langle n, n, n \rangle$ are polystable.*

Proof. We use Proposition 4.8. For $\langle m \rangle$ we take $R := \mathcal{T}_m$ as defined in Corollary 4.3. Note that \mathcal{T}_m is contained in $\text{SL}_m^3 \cap \text{stab}(\langle m \rangle)$. It is easy to check that the centralizer of \mathcal{T}_m in SL_m^3 is contained in T_m^3 . For the second condition, we note that the marginals of the uniform distribution on the support of $\langle m \rangle$ are the uniform distributions on $[m]$. Hence $\langle m \rangle$ is polystable.

For $\langle n, n, n \rangle$ we take for R the group consisting of the transformations $X \mapsto AXB^{-1}, Y \mapsto BXC^{-1}, Z \mapsto CXA^{-1}$, defined by diagonal matrices $A, B, C \in T_n$. In other words, R consists of the triples of diagonal matrices

$$(\text{diag}(a_i b_j^{-1}), \text{diag}(b_i c_j^{-1}), \text{diag}(c_i a_j^{-1})),$$

where $a_1, \dots, c_n \in \mathbb{C}^\times$. Note that R is indeed contained in $\text{SL}_{n^2}^3 \cap \text{stab}(\langle n, n, n \rangle)$. If $g \in \text{SL}_{n^2}$ commutes with all $\text{diag}(a_i b_j^{-1})$, then g must be diagonal, since we may choose the a_i, b_j such that $a_i b_j^{-1}$ are pairwise distinct for $i, j = 1, \dots, n$. Hence the centralizer of R is contained in T_m^3 . Finally, it is easy to check that the marginals of the uniform distribution on the support of $\langle n, n, n \rangle$ are uniform as well. \square

Proposition 4.10. *Almost all $w \in \otimes^3 \mathbb{C}^m$ are polystable.*

Proof. Recall that the GL_m^3 -action induces an action of $G = \text{GL}_m^3/K$ on $\otimes^3 \mathbb{C}^m$. Theorem 4.2 states that the reduced stabilizer $\text{stab}'(w) = \text{stab}(w)/K$ is finite for almost all $w \in \otimes^3 \mathbb{C}^m$. The group $G_s := \text{SL}_m^3/K$ is a homomorphic image of SL_m^3 and hence semisimple. Clearly, $\text{SL}_m^3 w = G_s w$. According to [30], or [25, II 4.3D, Folgerung, p. 142], $G_s w$ is closed for almost all w since $\text{stab}'(w) = \text{stab}(w)/K$ is finite for almost all w . \square

Any polystable tensor has a finite stabilizer period (the proof is as for [Proposition 2.11](#)).

5. Fundamental invariant of tensors

We proceed similarly as in [Section 3](#). Throughout this section, we assume that $w \in \otimes^3 \mathbb{C}^m$ is polystable.

Consider the map $\phi_w: Gw \rightarrow \mathbb{C}$ defined by $\phi_w(w) := \chi(g)^{a(w)}$, which is well defined by the definition of the stabilizer period $a(w)$. The following is analogous to [Lemma 3.2](#).

Lemma 5.1.

1. We have $\phi_w(tv) = t^{ma(w)}\phi_w(v)$ for $t \in \mathbb{C}^\times$ and $v \in Gw$.
2. $\mathcal{O}(Gw)_d^{\text{SL}_m}$ is one-dimensional if d is a multiple of $ma(w)$, and zero otherwise. More specifically, if $d = ma(w)k$ with $k \in \mathbb{Z}$, then we have $\mathcal{O}(Gw)_d^{\text{SL}_m} = \mathbb{C} \cdot (\phi_w)^k$.
3. For $k \in \mathbb{N}$, we have $\mathcal{O}(\overline{Gw})_{ma(w)k}^{\text{SL}_m} \neq 0$ iff $(\phi_w)^k$ has a regular extension to \overline{Gw} .

Proof. Let $\iota(t) := (tI_m, I_m, I_m) \bmod K \in G$ for $t \in \mathbb{C}^\times$. We have $\chi(\iota(t)) = t^m$ and $\iota(t)w = tw$. We thus get $\phi_w(tw) = \phi_w(\iota(t)w) = \chi(\iota(t))^{a(w)} = t^{ma(w)}$, hence $ma(w)$ equals the degree of F . This shows the first assertion. The remaining assertions are shown as for [Lemma 3.2](#). \square

Definition 5.2. The degree monoid $\mathcal{E}(w)$ of $w \in \otimes^3 \mathbb{C}^m$ is defined as

$$\mathcal{E}(w) := \{d \in \mathbb{N} \mid \mathcal{O}(\overline{Gw})_d^{\text{SL}_m^3} \neq 0\},$$

and we call the minimal positive element $e(w)$ of $\mathcal{E}(w)$ the *minimal degree* of w .

The following is analogous to [Theorem 3.4](#), cf. [Section 6.4](#) for the proof.

Theorem 5.3. The degree monoid $\mathcal{E}(w)$ generates the group $ma(w)\mathbb{Z}$.

Again we have $\mathcal{E}(w) = ma(w)\mathcal{E}'(w)$ with the *exponent monoid* characterized by

$$\mathcal{E}'(w) := \{k \in \mathbb{N} \mid (\phi_w)^k \text{ has a regular extension to } \overline{Gw}\}.$$

The *minimal exponent* $e'(w)$ of $\mathcal{E}'(w)$ is defined as the minimal positive element of $\mathcal{E}'(w)$.

To illustrate the notions introduced, recall that $a(m)$ denotes the stabilizer period of a generic tensor $w \in \otimes^3 \mathbb{C}^m$. From [Theorem 4.2](#) we know that $a(m) = 1$ if $m > 2$ and $a(2) = 2$. We define the *generic degree monoid of tensors* (of the cubic format m) as

$$\mathcal{E}(m) := \{d \in \mathbb{N} \mid \mathcal{O}(\otimes^3 \mathbb{C}^m)_d^{\text{SL}_m^3} \neq 0\}, \quad (5.1)$$

and we call its minimal positive value $e(m)$ the *generic minimal degree*. It is easy to see that $\mathcal{E}(w) = \mathcal{E}(m)$, and hence $e(w) = e(m)$, for almost all $w \in \otimes^3 \mathbb{C}^m$. Theorem 5.3 implies that $\mathcal{E}(m)$ generates the group $ma(m)\mathbb{Z}$. We call $\mathcal{E}'(m) := \frac{1}{ma(m)}\mathcal{E}(m)$ the *generic exponent monoid of tensors* (of the cubic format m) and its minimal positive element $e'(m) := \frac{1}{ma(m)}\varepsilon(m)$ the *generic minimal exponent of tensors*.

It is a well-known fact, e.g., see [29], that

$$\dim \mathcal{O}(\otimes^3 \mathbb{C}^m)_{m\delta}^{\text{SL}_m^3} = k_m(\delta),$$

where $k_m(\delta) := k(m \times \delta, m \times \delta, m \times \delta)$ is the *Kronecker coefficient* assigned to three partitions of the same rectangular shape $m \times \delta := (\delta, \dots, \delta)$ (m times). We therefore obtain that

$$\mathcal{E}'(m) = \{\delta \in \mathbb{N} \mid k_m(\delta) > 0\} \quad \text{if } m > 2. \quad (5.2)$$

Remark 5.4. The case $m = 2$ is special since $a(2) = 2$. Here we have $\{\delta \in \mathbb{N} \mid k_2(\delta) > 0\} = 2\mathcal{E}'(2) = 2\mathbb{N}$. In fact, it is known that $k_2(\delta)$ equals 1 if δ is even and $k_2(\delta) = 0$ otherwise; cf. [46].

Example 5.5. The generic exponent monoid $\mathcal{E}'(3) = \{0, 2, 3, 4, \dots\}$ is generated by 2, 3. Thus 1 is the only gap. The first function values of k_3 are given by

δ	0	1	2	3	4	5	6	7	8	9	10	11	12
$k_3(\delta)$	1	0	1	1	2	1	3	2	4	3	5	4	7

The above table is consistent with the fact that $\mathcal{O}(\otimes^3 \mathbb{C}^3)_{\text{SL}_3^3}$ is generated by three homogeneous generators having the degrees 6, 9, 12, see [54, 2]. We also remark that $k_3(3) = 1$ states the existence and uniqueness of Strassen's invariant [50]. It is instructive to verify here the monotonicity relation $k_m(\delta) \leq k_m(\delta + \ell)$, holding if $\ell \in \mathcal{E}'(m)$. (This property follows from the general observation $k(\lambda, \mu, \nu) \leq k(\lambda + \alpha, \mu + \beta, \nu + \gamma)$ if $k(\alpha, \beta, \gamma) > 0$, resulting from interpreting the Kronecker coefficients as the dimensions of highest weight vector spaces; cf. [34].)

The following computations were obtained with the DERKSEN program for computing Kronecker coefficients. (This is an adaption by J. Hüttenhain of a code originally written by H. Derksen.)

Example 5.6. We have $\mathcal{E}'(4) = \mathcal{E}'(3) = \{0, 2, 3, 4, \dots\}$ generated by 2, 3, with the only gap 1. Further, we have $\mathcal{E}'(5) = \mathcal{E}'(6) = \mathcal{E}'(8) = \mathcal{E}'(9) = \{0, 3, 4, 5, 6, \dots\}$ generated by 3, 4, 5, with the gaps 1, 2. But we have $\mathcal{E}'(7) = \{0, 4, 5, 6, 7, 8, \dots\}$ generated by 4, 5, 6, 7, with the gaps 1, 2, 3. This shows that $e'(3) = e'(4) = 2$, $e'(5) = e'(6) = e'(8) = e'(9) = 3$, $e'(7) = 4$. Computing a little further, we obtain

$\mathcal{E}'(10) = \mathcal{E}'(11) = \mathcal{E}'(12) = \{0, 4, 5, 6, 7, \dots\}$ and hence $e'(10) = e'(11) = e'(12) = 4$. Further, $e'(13) = e'(14) = e'(15) = e'(16) = 4$.

We can now define the key objects of study for tensors, as in Section 3.2.

Definition 5.7. By the *fundamental invariant* Φ_w of $w \in \otimes^3 \mathbb{C}^m$ we understand the SL_m^3 -invariant Φ_w in $\mathcal{O}(\overline{Gw})$ of the (minimal) degree $e(w)$ satisfying $\Phi_w(w) = 1$.

We note that $\Phi_w = (\phi_w)^{e'(w)}$ on Gw by Lemma 5.1.

Theorem 5.8.

1. The zero set of Φ_w in \overline{Gw} equals the boundary $\overline{Gw} \setminus Gw$.
2. The coordinate ring of the orbit Gw equals the localization of the coordinate of the orbit closure \overline{Gw} with respect to the fundamental invariant Φ_w .
3. If $ma(w) < e(w)$, that is, $1 < e'(w)$, then the vanishing ideal of the boundary $\overline{Gw} \setminus Gw$ in \overline{Gw} is strictly larger than the principal ideal $\Phi_w \mathcal{O}(\overline{Gw})$. Moreover, \overline{Gw} is not a normal algebraic variety. In particular, it is not a valuation ring.

The proof is completely analogous to the one of Proposition 3.10 and Theorem 3.11 and therefore omitted.

5.1. Generic fundamental invariant of tensors

We investigate the generic minimal exponent $e'(m)$.

Theorem 5.9.

1. We have $\lceil \sqrt{m} \rceil \leq e'(m)$ if $m > 2$.
2. We have $e'(m) \leq m$ if the Alon–Tarsi conjecture holds for m .
3. If $m = n^2$, then $k_{n^2}(n) = 1$. In particular, $e'(n^2) = n$.

Proof. 1. According to (5.2), $e'(m)$ is the minimum δ such that $k_m(\delta) > 0$ (recall $m > 2$). The symmetry property of Kronecker coefficients allows to transpose two partitions without changing the value: $k(m \times \delta, m \times \delta, m \times \delta) = k(m \times \delta, \delta \times m, \delta \times m)$. Moreover, it is known that $k(\lambda, \mu, \nu) = 0$ if $\ell(\lambda) > \ell(\mu)\ell(\nu)$. Hence $k_m(\delta) = 0$ if $m > \delta^2$. This implies $e'(m) \geq \lceil \sqrt{m} \rceil$.

2. We construct an SL_m^3 -invariant P in $\mathcal{O}(\otimes^3 \mathbb{C}^m)_{m^2}$, we evaluate it at the unit tensor $\langle m \rangle$, and we show that $P(\langle m \rangle)$ is the difference between the number of column-even and the number of column-odd Latin squares of order m .

Let $\{e_i\}_{1 \leq i \leq m}$ be the standard basis of \mathbb{C}^m . We observe that $\zeta := (\bigwedge_{i=1}^m e_i)^{\otimes m}$ is an SL_m -invariant in $\otimes^m(\otimes^m(\mathbb{C}^m))$. We define ζ^t to be the image under the canonical

automorphism of $\otimes^m(\otimes^m(\mathbb{C}^m))$ that sends $v_{1,1} \otimes v_{1,2} \otimes \cdots \otimes v_{m,m}$ to $v_{1,1} \otimes v_{2,1} \otimes \cdots \otimes v_{m,m}$ (“transposition”). We interpret both ζ and ζ^t as elements in $\otimes^{m^2} \mathbb{C}^m$. Now $\zeta \otimes \zeta^t \otimes \zeta^t$ is an SL_m^3 -invariant in $\otimes^3(\otimes^{m^2}(\mathbb{C}^m))$. We use the canonical isomorphism $\otimes^3(\otimes^{m^2}(\mathbb{C}^m)) \simeq \otimes^{m^2}(\otimes^3(\mathbb{C}^m))$ to interpret $\zeta \otimes \zeta^t \otimes \zeta^t$ as an element in $\otimes^{m^2}(\otimes^3(\mathbb{C}^m))$. Applying the symmetrization map $\otimes^{m^2}(\otimes^3(\mathbb{C}^m)) \rightarrow \mathcal{O}(\otimes^3(\mathbb{C}^m))_{m^2}$ to $\zeta \otimes \zeta^t \otimes \zeta^t$ (identifying \mathbb{C}^m with its dual), we obtain a function P in $\mathcal{O}(\otimes^3(\mathbb{C}^m))_{m^2}$. The principle of polarization and restitution says that $P(w)$ equals the value of the scalar product

$$\langle \zeta \otimes \zeta^t \otimes \zeta^t, w^{\otimes m^2} \rangle.$$

Setting $w = \langle m \rangle$ to be the unit tensor we obtain

$$w^{\otimes m^2} = \sum_{i_{1,1}, i_{1,2}, \dots, i_{m,m}=1}^m e_{i_{1,1}}^{\otimes 3} \otimes e_{i_{1,2}}^{\otimes 3} \otimes \cdots \otimes e_{i_{m,m}}^{\otimes 3}$$

We write

$$P(w) = \sum_{i_{1,1}, i_{1,2}, \dots, i_{m,m}=1}^m \langle \zeta \otimes \zeta^t \otimes \zeta^t, e_{i_{1,1}}^{\otimes 3} \otimes e_{i_{1,2}}^{\otimes 3} \otimes \cdots \otimes e_{i_{m,m}}^{\otimes 3} \rangle. \quad (5.3)$$

Using the special structure of ζ we see that each contraction of $\zeta \otimes \zeta^t \otimes \zeta^t$ with a tensor of the form $e_{i_{1,1}}^{\otimes 3} \otimes e_{i_{1,2}}^{\otimes 3} \otimes \cdots \otimes e_{i_{m,m}}^{\otimes 3}$ is a product of determinants of $3m$ matrices $A_1, A_2, \dots, A_m, A_1, A_2, \dots, A_m, B_1, B_2, \dots, B_m \in \mathbb{C}^{m \times m}$. For $1 \leq a \leq m$ the matrix A_a is the concatenation of the column vectors $e_{i_{b,a}} \in \mathbb{C}^m$ for $1 \leq b \leq m$. For $1 \leq a \leq m$ the matrix B_a is the concatenation of the column vectors $e_{i_{a,b}} \in \mathbb{C}^m$ for $1 \leq b \leq m$. Note that all matrices A_a and B_a only have entries from $\{0, 1\}$. In order for a summand in (5.3) to be nonzero there can be no double columns in any of the matrices A_a or B_a (so all A_a and B_a must be permutation matrices), which can be rephrased as: the numbers $1 \leq i_{1,1}, i_{1,2}, \dots, i_{m,m} \leq m$ must form a Latin square T . Now $(\prod_{a=1}^m \det(A_a))(\prod_{a=1}^m \det(A_a))(\prod_{a=1}^m \det(B_a)) = \prod_{a=1}^m \det(B_a)$, which is the product of the signs of the permutations in the columns of T . Therefore $P(w)$ is the difference between the number of column-even and the number of column-odd Latin squares of order m .

3. For the third statement, we need to show that $k(n^2 \times n, n^2 \times n, n^2 \times n) = 1$. We use the following symmetry relation from [22, Cor. 4.4.15]:

$$k(\lambda, \mu, \nu) = k(n^2 \times n - \lambda, n^2 \times n - \mu, n^2 \times n - \nu), \quad (5.4)$$

holding for any partitions λ, μ, ν contained in $n^2 \times n$. (Here, $n^2 \times n - \lambda$ denotes the partition corresponding to the complement of the Young diagram of λ in the rectangle $n^2 \times n$.) In particular, setting $\lambda = \mu = \nu = n^2 \times n$, we get $k(\lambda, \mu, \nu) = k(0, 0, 0) = 1$. \square

Remark 5.10. The fact $k_{n^2}(n) = 1$ also immediately follows combinatorially from the upper and lower bounds on the Kronecker coefficient in [33] and [6], see also [23, Lemma 2.3].

If $m = n^2$ is a square, then Theorem 5.9(3) states that, up to a scaling factor, there is exactly one homogeneous SL_m^3 -invariant $F_n: \otimes^3 \mathbb{C}^m \rightarrow \mathbb{C}$ of degree $e(m) = ma(m)e'(m) = n^3$ (and no nonzero invariant of smaller degree). We call F_n the *generic fundamental invariant* of $\otimes^3 \mathbb{C}^{n^2}$.

Before providing an explicit formula for F_n , we state our main general result concerning tensors. The proof is analogous to Theorem 3.16 and therefore omitted.

Theorem 5.11. Assume $w \in \otimes^3 \mathbb{C}^m$ is polystable. Then $\sqrt{m}/a(w) \leq e'(w)$ if $m > 2$. Moreover, if $m = n^2$, we have equality iff $F_n(w) \neq 0$. In this case, we have $\Phi_w = F_n(w)^{-1}F_n$ on \overline{Gw} .

The following is immediate. Note that the orbit closure of almost all $w \in \otimes^3 \mathbb{C}^2$ equals $\otimes^3 \mathbb{C}^2$ and hence is normal.

Corollary 5.12. 1. \overline{Gw} is not normal if $a(w) < \sqrt{m}$.

2. Let $m \geq 3$. Then \overline{Gw} is not normal for almost all $w \in \otimes^3 \mathbb{C}^m$.

In the following, we assume $m = n^2$, and provide the promised explicit description of the invariant F_n .

Consider the combinatorial cube $[n]^3$. We shall call the elements $p \in [n]^3$ *points*. For $\ell \in [n]$ we call $\{(i, j, k) \in [n]^3 \mid i = \ell\}$ its ℓ th slice in x -direction. (In other words, we fix the first coordinate to ℓ .) Similarly, we define its y -slices by fixing the second coordinate, and its z -slices by fixing the third coordinate.

We fix a total ordering of $[n]^3$ once and for all. This defines an ordering of each of the slices of $[n]^3$. By a *labeling* we shall understand a map $\alpha: [n]^3 \rightarrow [n^2]$. Suppose that a labeling defines a bijection on each x -slice. Then the resulting permutation of each x -slice has a well-defined sign. We define $\mathrm{sgn}_x(\alpha) \in \{-1, 1\}$ of the labeling α to be the product of the signs of the permutations of all x -slices. If the labeling α fails to be a bijection on an x -slice, we write $\mathrm{sgn}_x(\alpha) := 0$. Similarly, we define $\mathrm{sgn}_y(\alpha)$ and $\mathrm{sgn}_z(\alpha)$.

We express a tensor $w = \sum_{abc} w_{abc} |abc\rangle \in \otimes^3 \mathbb{C}^{n^2}$ in the standard basis $|abc\rangle$ with coordinates $w_{abc} \in \mathbb{C}$. Consider the following homogeneous polynomial $F_n: \otimes^3 \mathbb{C}^{n^2} \rightarrow \mathbb{C}$ of degree n^3 defined by

$$F_n(w) := \sum_{\alpha, \beta, \gamma: [n]^3 \rightarrow [n^2]} \mathrm{sgn}_x(\alpha) \mathrm{sgn}_y(\beta) \mathrm{sgn}_z(\gamma) \prod_{p \in [n]^3} w_{\alpha(p)\beta(p)\gamma(p)}, \quad (5.5)$$

where the sum is over all triples α, β, γ of labelings $[n]^3 \rightarrow [n^2]$.

The following result shows that F_n is indeed the invariant we are looking for; see also [6, Example 4.12].

Theorem 5.13. *We have $F_n((g_1 \otimes g_2 \otimes g_3)w) = (\det(g_1) \det(g_2) \det(g_3))^n F_n(w)$ for all $(g_1, g_2, g_3) \in \mathrm{GL}_{n^2}$ and $w \in \otimes^3 \mathbb{C}^{n^2}$. Moreover, $F_n \neq 0$.*

Proof. The proof is rather indirect. Let $d = n^3$ and $m = n^2$. We will explicitly construct a generating set for the space of SL_m^3 -invariants in $\mathcal{O}(\otimes^3 \mathbb{C}^m)_d$ and then see that this vector space is *at most* 1-dimensional with F_n being the only element up to scale. From Proposition 5.9(3) we know that $k_{n^2}(n) = 1$, therefore $F_n \neq 0$.

The following construction arises by specializing the description of “obstruction designs” in [22] and [6] from highest weight vectors to invariants. We refer to these papers for more references.

We start by constructing a generating set for the SL_m^3 -invariants in $\otimes^d(\otimes^3 \mathbb{C}^m)$. Let $(\otimes^d(\otimes^3 \mathbb{C}^m))^{\mathrm{SL}_m^3} \subseteq \otimes^d(\otimes^3 \mathbb{C}^m)$ denote the SL_m^3 -invariant subspace. Clearly the actions of S_d^3 and SL_m^3 on $\otimes^d(\otimes^3 \mathbb{C}^m)$ commute. By Schur–Weyl duality, the S_d^3 -representation $(\otimes^d(\otimes^3 \mathbb{C}^m))^{\mathrm{SL}_m^3}$ is irreducible. Therefore, if we have a nonzero element $P \in (\otimes^d(\otimes^3 \mathbb{C}^m))^{\mathrm{SL}_m^3}$, we know that $\{\pi P \mid \pi \in S_d^3\}$ is a generating set of $(\otimes^d(\otimes^3 \mathbb{C}^m))^{\mathrm{SL}_m^3}$. Let $\{e_1, \dots, e_m\}$ be the standard basis of \mathbb{C}^m . We consider the SL_m -invariant $\zeta := (\bigwedge_{i=1}^m e_i)^{\otimes n} \in \otimes^d \mathbb{C}^m$ and put $P := \zeta \otimes \zeta \otimes \zeta \in (\otimes^3(\otimes^d \mathbb{C}^m))^{\mathrm{SL}_m^3}$. Using the canonical isomorphism $\otimes^3(\otimes^d \mathbb{C}^m) \simeq \otimes^d(\otimes^3 \mathbb{C}^m)$ we interpret $P \in (\otimes^d(\otimes^3 \mathbb{C}^m))^{\mathrm{SL}_m^3}$. Therefore $(\otimes^d(\otimes^3 \mathbb{C}^m))^{\mathrm{SL}_m^3}$ is generated by $\{\pi P \mid \pi \in S_d^3\}$. The vector ζ wedges together basis vectors in tuples of cardinality m . We want to describe this combinatorially by saying that ζ has the following wedge list:

$$(1, 2, \dots, m), (m+1, m+2, \dots, 2m), \dots, ((n-1)m+1, \dots, d).$$

We call *blocks* the subsets in which the parentheses partition the set $\{1, \dots, d\}$. The combinatorial description of P is given by a triple t of wedge lists and the action of S_d^3 permutes the entries in these lists. We denote by $P_t \in (\otimes^d(\otimes^3 \mathbb{C}^m))^{\mathrm{SL}_m^3}$ the invariant corresponding to t . The structure of ζ implies that if we permute whole blocks in t while preserving the order inside the blocks, P_t does not change. If only entries inside a block are changed, then, up to sign, P_t does not change. So we see that P_t can be combinatorially defined (up to a sign) by a triple of set partitions of the set $\{1, \dots, d\}$ into n sets of cardinality m . A crucial observation is the following: If there exist two numbers i and j whose block coincides in the first wedge list of t and whose block also coincides in the second and third wedge list, then P_t vanishes under the projection $\Psi : \otimes^d(\otimes^3 \mathbb{C}^m) \rightarrow \mathcal{O}(\otimes^3 \mathbb{C}^m)_d$ given by $\Psi := \frac{1}{d!} \sum_{\sigma \in S_d} \sigma$. The reason is that if S_2 is the symmetric group on $\{i, j\}$, then P_t vanishes already under the projection $\frac{1}{2} \sum_{\sigma \in S_2} \sigma$, because $\frac{1}{2} \sum_{\sigma \in S_2} \sigma P_t = \frac{1}{2} (P_t - P_t) = 0$. We call this the *zero pattern*. The main observation which we prove next is that there exist only very few triples of set partitions of the set $\{1, \dots, d\}$ into n sets of cardinality m that avoid the zero pattern. Indeed, these triples all coincide up to the action of S_d , which renames the numbers in all three wedge lists simultaneously. Therefore the corresponding P_t all get mapped to the same element under Ψ .

The argument in [22, Claim 7.2.17] or [6, Exa. 4.12] is as follows. From a triple of set partitions that avoid the zero pattern we can construct a cardinality d subset of \mathbb{N}^3 as follows: In each of the three wedge lists we order the blocks from 1 to n in an arbitrary way. Now for each number $i \in \{1, \dots, d\}$ we let the x -coordinate of the i th point be the number of its block in the first wedge list. The y coordinate shall be the number of the block in the second wedge list and the z coordinate shall be the number of the block in the third wedge list. No point is repeated because the wedge list triple avoids the zero pattern. This construction can be reversed in the obvious way, but the reverse process is only unique up to the action of S_d . Since there are only n blocks in each wedge list this construction results in a subset of the $n \times n \times n$ cube. But since we constructed $d = n^3$ points, the point set is *precisely* the $n \times n \times n$ cube. Let t be a wedge list triples that avoids the zero pattern. We have seen that $\mathcal{O}(\otimes^3 \mathbb{C}^m)_d^{\text{SL}_m^3}$ is at most 1-dimensional and generated by $\Psi(P_t)$. If we evaluate $\Psi(P_t)$ at a point w , i.e., if we contract P_t with a tensor $w^{\otimes d}$, we get precisely the sum in equation (5.5), hence $(\Psi(P_t))(w) = F_n(w)$. With Proposition 5.9(3) it follows that $F_n \neq 0$. \square

As for Lemma 3.20, we can show that the generic fundamental invariant F_n is an irreducible polynomial.

Remark 5.14. The above has a natural generalization to noncubic formats. Let $n_1, n_2, n_3 \geq 1$. We know from [22, Cor. 4.4.15] that $k(n_2 n_3 \times n_1, n_1 n_3 \times n_2, n_1 n_2 \times n_3) = 1$. The corresponding invariant $F_{n_1 n_2 n_3}$ of degree $n_1 n_2 n_3$ evaluated at a tensor $w = \sum_{abc} w_{abc} |abc\rangle \in \mathbb{C}^{n_2 n_3} \otimes \mathbb{C}^{n_1 n_3} \otimes \mathbb{C}^{n_1 n_2}$ is given as in (5.5), where the sum is over the labelings $\alpha: P \rightarrow [n_2 n_3], \beta: P \rightarrow [n_1 n_3], \gamma: P \rightarrow [n_1 n_2]$ with $P := [n_1] \times [n_2] \times [n_3]$. In the special case $n_2 = n_3 = 1$ we get $F_{n11}(w) = n! \det(w)$, when we interpret $w \in \mathbb{C}^1 \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ as a matrix.

We finish this section with a couple of interesting problems related with $\mathcal{E}(m)$.

Problem 5.15. Give a direct proof of $F_n \neq 0$ by evaluating F_n at a (generic) $w \in \otimes^3 \mathbb{C}^m$.

Problem 5.16. What is the computational complexity to evaluate F_n ?

Problem 5.17. How close is $e'(m)$ to \sqrt{m} if m is not a square?

Remark 5.18. We note that $e'(m) = \lceil \sqrt{m} \rceil$ is false for $m = 7$, which is the only counterexample for $m \leq 16$. We have $e'(7) = 4$. We further note that a computation showed that $k_m(e'(m)) = 1$ for $m = 2, 3, 4, 5, 6, 8, 9, 14, 15, 16$, but $k_7(e'(7)) = k_7(4) = 14$, and $k_{10}(e'(10)) = k_{10}(4) = 13$, $k_{11}(e'(11)) = k_{11}(4) = 6$, $k_{12}(e'(12)) = 5$, $k_{13}(e'(13)) = 2$. So in general, we do not have $k_m(e'(m)) = 1$.

Problem 5.19. Do we have $\mathcal{E}'(m) = \{0\} \cup (e(m) + \mathbb{N})$ if $m > 2$? In other words, are $1, 2, \dots, e(m) - 1$ the only possible gaps? (We have verified this for $m \leq 12$.)

Problem 5.20. Determine the asymptotic growth of k_m . More specifically, find c_m and μ_m such that $k_m(\delta) \sim c_m \delta^{\mu_m}$ for $\delta \rightarrow \infty$.

In [Example 5.5](#) $k_3(\delta)$ is a quasipolynomial of period 12 and $k_3(\delta) \sim \frac{1}{48} \delta^2$ for $\delta \rightarrow \infty$, see the OEIS sequence A005044, where all 12 polynomials are specified: $\delta^2/48$, $(\delta^2 + 6\delta - 7)/48$, $(\delta^2 - 4)/48$, $(\delta^2 + 6\delta + 21)/48$, $(\delta^2 - 16)/48$, $(\delta^2 + 6\delta - 7)/48$, $(\delta^2 + 12)/48$, $(\delta^2 + 6\delta + 5)/48$, $(\delta^2 - 16)/48$, $(\delta^2 + 6\delta + 9)/48$, $(\delta^2 - 4)/48$, $(\delta^2 + 6\delta + 5)/48$.

5.2. Minimal degree for specific tensors

We can combinatorially characterize the minimal exponent in a few interesting cases.

To evaluate F_n at the unit tensor $\langle n^2 \rangle = \sum_a |aaa\rangle \in \otimes^3 \mathbb{C}^{n^2}$, we introduce the following 3D generalization of latin squares. Recall that we fixed a total ordering of $[n]^3$.

Definition 5.21. A *Latin cube* of size n is a map $\alpha: [n]^3 \rightarrow [n^2]$ that is a bijection on each of the x -slices, y -slices, and z -slices of the combinatorial cube $[n]^3$. The latin cube is called *even* if the product of the signs of the resulting permutations of all x -slices, y -slices, and z -slices equals one. Otherwise, the latin cube is called *odd*.

Proposition 5.22. 1. $F_n(\langle n^2 \rangle)$ equals the difference between the number of even Latin cubes of size n and the number of odd Latin cubes of size n .

2. We have $F_n(\langle n^2 \rangle) = 0$ if n is odd.

Proof. The first assertion is obvious from [\(5.5\)](#).

For the second assertion, we consider $[n^2]$ as the set of symbols. Exchanging two fixed symbols (e.g., 1 with 2) defines an involution $\alpha \mapsto \alpha'$ of the set of Latin cubes of size n . We have $\text{sgn}(\alpha') = (-1)^{3n} \text{sgn}(\alpha) = -\text{sgn}(\alpha)$, since the involution creates in each slice a transposition, there is a total of $3n$ slices, and n is odd. \square

The following question, which is analogous to the Alon–Tarsi conjecture, is important for understanding tensor border rank. We have verified this in the cases $n = 2$ and $n = 4$.

Problem 5.23. Let n be even. Is the number of even Latin cubes of size n different from the number of odd Latin cubes of size n ?

The matrix multiplication tensor $\langle n, n, n \rangle = \sum_{ijk} |(ij)(jk)(ki)\rangle \in \otimes^3 \mathbb{C}^{n \times n}$ has the dimension format (n^2, n^2, n^2) . We are going to describe the evaluation of F_n at the matrix multiplication tensor. A labeling $\alpha: [n]^3 \rightarrow [n]^2$ is given by its two coordinate maps $\mu, \nu: [n]^3 \rightarrow [n]$, i.e., $\alpha(p) = (\mu(p), \nu(p))$. The following characterization is obvious from [\(5.5\)](#) and the structure of the matrix multiplication tensor.

Proposition 5.24. *We have*

$$F_n(\langle n, n, n \rangle) = \sum_{\mu, \nu, \pi: [n]^3 \rightarrow [n]} \operatorname{sgn}_x(\mu, \nu) \operatorname{sgn}_y(\nu, \pi) \operatorname{sgn}_z(\pi, \mu).$$

Corollary 5.25.

1. We have $e'(\langle m \rangle) \geq \frac{1}{2} \lceil \sqrt{m} \rceil$ if $m > 1$.
2. Let $m = n^2$ be even. Then $e'(\langle n^2 \rangle) = n/2$ iff the number of even Latin cubes of size n is different from the number of odd Latin cubes of size n .
3. We have $e'(\langle n, n, n \rangle) \geq n$, with equality holding iff $F_n(\langle n, n, n \rangle) \neq 0$. (See Lemma 5.24 for a combinatorial expression for this condition.)

Proof. This follows from Theorem 5.11 and the determination of the periods in Corollary 4.3 and Corollary 4.6. \square

Theorem 5.11 immediately implies the following. This result already appeared in [5].

Corollary 5.26. *The orbit closure of the unit tensor $\langle m \rangle$ is not normal if $m \geq 5$. The orbit closure of the matrix multiplication tensor $\langle n, n, n \rangle$ is not normal if $n \geq 2$.*

6. Proofs for Section 3 and Section 5

6.1. Algebraic curves with dense \mathbb{C}^\times -orbit

We present here a general observation. Let Z be an affine algebraic curve over \mathbb{C} and assume the group \mathbb{C}^\times acts rationally on Z . We further assume that there exists $p \in Z$ such that the orbit morphism $\sigma: \mathbb{C}^\times \rightarrow Z, t \mapsto tp$ has a regular extension to a surjective morphism $\sigma: \mathbb{C} \rightarrow Z$. In particular, Z is irreducible. Note that \mathbb{C}^\times acts nontrivially on Z (otherwise, Z would be a point). We consider now $\sigma^*: \mathcal{O}(Z) \hookrightarrow \mathcal{O}(\mathbb{C}) = \mathbb{C}[T]$ and write

$$\sigma^*(\mathcal{O}(Z)) = \bigoplus_{k \in S} \mathbb{C}T^k \tag{6.1}$$

with a submonoid $S \subseteq \mathbb{N}$. The stabilizer $\{t \in \mathbb{C}^\times \mid tp = p\}$ of p is a finite subgroup of \mathbb{C}^\times and thus equals μ_a for some $a \geq 1$.

Lemma 6.1. *The monoid S generates the subgroup $a\mathbb{Z}$ of \mathbb{Z} .*

Proof. Suppose that S generates the subgroup $\ell\mathbb{Z}$ of \mathbb{Z} , where $\ell \geq 1$. Then $\sigma^*(\mathcal{O}(Z)) \subseteq \mathbb{C}[T^\ell]$ and hence σ^* factors into the chain of ring homomorphisms $\mathcal{O}(Z) \rightarrow \mathbb{C}[T^\ell] \hookrightarrow \mathbb{C}[T]$. This yields the factorization of σ into the chain of regular maps $\mathbb{C} \xrightarrow{\phi} \mathbb{C} \xrightarrow{\tilde{\sigma}} Z$, where $\phi(t) = t^\ell$. This implies $\mu_\ell \subseteq \{t \in \mathbb{C}^\times \mid tp = p\} = \mu_a$, hence ℓ divides a .

For the converse, we observe that σ factors via a map $\tilde{\sigma}: \mathbb{C} \rightarrow Z$ as $\sigma = \tilde{\sigma} \circ \phi$, where $\phi: \mathbb{C} \rightarrow \mathbb{C}^\times$ is given by $\phi(t) = t^a$. The map $\tilde{\sigma}: \mathbb{C}^\times \rightarrow Z$ is regular by the universal property of the quotient $\mathbb{C}^\times \xrightarrow{\phi} \mathbb{C}^\times/\mu_a$ of algebraic groups provided by ϕ . Moreover, $\tilde{\sigma}$ is continuous at 0 because σ is so. It follows that $\tilde{\sigma}$ is regular on \mathbb{C} (e.g., because \mathbb{C} is normal). We deduce that $\sigma^* = \phi^* \circ \tilde{\sigma}^*$, and hence $\sigma^*(\mathcal{O}(Z)) \subseteq \mathbb{C}[T^a]$. Since $\ell\mathbb{Z}$ is the subgroup generated by S , we get $\ell\mathbb{Z} \subseteq a\mathbb{Z}$, hence a divides ℓ . We have thus shown that $\ell = a$. \square

6.2. Proof of Theorem 3.4

Let us recall the situation. We have a group embedding $\iota: \mathbb{C}^\times \rightarrow G = \mathrm{GL}_m$, via $\iota(t) = \mathrm{tid}_m$. Moreover, G is the product of the normal subgroup $\mathbb{C}^\times \mathrm{id}_m$ and the subgroup $G_s := \mathrm{SL}_m$. Recall the action of G on $W := \mathrm{Sym}^D \mathbb{C}^m$ and note that

$$\iota(t)w = t^D w \quad \text{for } t \in \mathbb{C}^\times, w \in \mathrm{Sym}^D \mathbb{C}^m. \quad (6.2)$$

This means that $t \in \mathbb{C}^\times$, when seen as embedded in G via ι , acts by the D th power of t . From this it follows that the orbit closure $X := \overline{Gw}$ is invariant under scalar multiplication and $0 \in X$. The scalar multiplication on W defines a \mathbb{C}^\times -action on X , which commutes with the G_s -action. Hence we get an induced \mathbb{C}^\times -action on $\mathcal{O}(X)$, which commutes with the G_s -action as well. Therefore, the subring $\mathcal{O}(X)^{G_s}$ of G_s -invariants is \mathbb{C}^\times -invariant.

For the following facts from geometric invariant theory, see [25]. Since G_s is reductive, the ring $\mathcal{O}(X)^{G_s}$ is a finitely generated \mathbb{C} -algebra and therefore it can be interpreted as the coordinate ring of an affine variety Z . The inclusion $\mathcal{O}(Z) \hookrightarrow \mathcal{O}(X)$ defines a morphism $\pi: X \rightarrow Z$ of algebraic varieties, which is called the *GIT-quotient* of X by the action of the group G_s . We have a \mathbb{C}^\times -action on Z , defined by the \mathbb{C}^\times -action on $\mathcal{O}(Z)$, and π is \mathbb{C}^\times -equivariant.

By definition, π is constant on G_s -orbits. It is an important property of GIT-quotients that disjoint orbit closures are mapped to distinct points under π . It is also known that π is surjective, cf. [25, II. 3.2]. Consider the orbit map of $p := \pi(w)$

$$\sigma: \mathbb{C}^\times \rightarrow Z, \quad t \mapsto tp = \pi(tw), \quad (6.3)$$

which has a regular extension $\sigma: \mathbb{C} \rightarrow Z$ given by $\sigma(0) := \pi(0)$.

It will be convenient to use the notation $X^\circ := Gw$ and $\partial X := X \setminus X^\circ$. Note that $0 \in \partial X$.

Lemma 6.2. *Z is an irreducible affine curve and the map $\sigma: \mathbb{C} \rightarrow Z$ is surjective. Moreover, $\pi(X^\circ) = \sigma(\mathbb{C}^\times)$, $\pi(\partial X) = \{\pi(0)\}$, and $p \notin \sigma(\mathbb{C}^\times)$.*

Proof. (1) Any $g \in G$ can be written as $g = \iota(t)g_s$, where $t \in \mathbb{C}^\times$ and $g_s \in G_s$. By (6.2), we have $gw = t^D g_s w$, and hence

$$\pi(gw) = t^D \pi(g_s w) = t^D \pi(w) = \sigma(t^D),$$

hence $\pi(X^\circ) \subseteq \sigma(\mathbb{C}^\times)$. The other inclusion is obvious.

(2) Let $u \in \partial X$, say $u = \lim_{n \rightarrow \infty} g_n w$ for some sequence g_n in G . Write $g_n = \iota(t_n) \tilde{g}_n$, where $\tilde{g}_n \in G_s$. Then, $\det(g_n) = \det(\iota(t_n)) = t_n^m$. We have $\lim_{n \rightarrow \infty} t_n = 0$ since $\lim_{n \rightarrow \infty} \det(g_n) = 0$ by Proposition 3.10. We obtain

$$\pi(g_n w) = \pi(t_n^D \tilde{g}_n w) = t_n^D \pi(\tilde{g}_n w) = t_n^D \pi(w) = \pi(t_n^D w).$$

Hence, $\lim_{n \rightarrow \infty} \pi(g_n w) = \pi(0)$. On the other hand, $\lim_{n \rightarrow \infty} \pi(g_n w) = \pi(u)$ by the continuity of π . It follows that $\pi(u) = \pi(0)$. This shows that $\pi(\partial X) = \{\pi(0)\}$.

(3) Suppose by way of contradiction that $\pi(0) \in \sigma(\mathbb{C}^\times)$, say $\pi(0) = \pi(tw)$ for some $t \in \mathbb{C}^\times$. Then the closures of the orbits $G_s 0$ and $G_s tw$ must intersect. But $G_s 0 = \{0\}$ and $G_s tw = tG_s w$ is closed since $G_s w$ is closed by assumption. Therefore, $0 \in G_s tw$, which implies the contradiction $w = 0$.

(4) We obtain $Z = \pi(X) = \pi(X^\circ) \cup \pi(\partial X) = \sigma(\mathbb{C}^\times) \cup \{\pi(0)\}$. Hence $\sigma: \mathbb{C} \rightarrow Z$ is surjective. \square

We now determine the stabilizer of p under the \mathbb{C}^\times -action on Z . Recall the degree period $b(w)$ from Definition 3.1.

Lemma 6.3. *We have $\{t \in \mathbb{C}^\times \mid tp = p\} = \mu_{b(w)}$.*

Proof. Suppose that $\pi(tw) = \pi(w)$. Then, since π is a GIT quotient, the orbit closures of tw and w intersect; cf. [25, II.3.2]. Since $G_s w$ is closed, this means that $tw \in G_s w$. Therefore, there exists $g_s \in G_s$ such that $tw = g_s^{-1} w$, hence $g_s tw = w$. If we choose $t_1 \in \mathbb{C}^\times$ such that $t = t_1^D$, then $g_s tw = g_s \iota(t_1) w$, hence $g_s \iota(t_1) \in \text{stab}(w)$. By the definition of the period $a(w)$, using (3.1), we obtain

$$1 = \det(g_s \iota(t_1))^{a(w)} = \det(\iota(t_1))^{a(w)} = t_1^{ma(w)} = t_1^{Db(w)} = t^{b(w)}.$$

We have shown that $\{t \in \mathbb{C}^\times \mid tp = p\} \subseteq \mu_{b(w)}$.

For the converse, assume that $t \in \mu_{b(w)}$. Choose t_1 with $t = t_1^D$. Then, similarly as before, $1 = t^{b(w)} = t_1^{Db(w)} = t_1^{ma(w)}$. Since $\det(\text{stab}(w)) = \mu_{a(w)}$, there exists $g \in \text{stab}(w)$ such that $\det(g^{-1}) = t_1^m = \det(\iota(t_1))$. Hence $g_s := \iota(t_1)g \in G_s$. We obtain

$$g_s w = \iota(t_1)gw = \iota(t_1)w = t_1^D w,$$

and hence $\pi(w) = \pi(g_s w) = t_1^D \pi(w) = t\pi(w)$, that is, $p = tp$. \square

Remark 6.4. The map $\mathbb{C}/\mu_{b(w)} \rightarrow Z$, resulting from σ by factoring out the stabilizer, is the normalization of the curve Z .

Proof of Theorem 3.4. Lemma 6.2 implies that Z is a curve and the orbit map $\sigma: \mathbb{C} \rightarrow Z$, $t \mapsto tp$ is surjective. Moreover, Lemma 6.3 tells us that the stabilizer of p with respect to the \mathbb{C}^\times -action equals $\mu_{b(w)}$. We apply now Lemma 6.1 to this setting. Accordingly, we have

$$\sigma^*(\mathcal{O}(Z)) = \bigoplus_{k \in S} \mathbb{C} T^{kb(w)}, \quad (6.4)$$

where the submonoid $S \subseteq \mathbb{N}$ generates the group \mathbb{Z} . It is sufficient to prove that $S = \mathcal{E}'(w)$.

Suppose $k \in S$. Then there is $h \in \mathcal{O}(Z)$ such that $\sigma^*(h) = T^{kb(w)}$. For $t \in \mathbb{C}^\times$ and $g_s \in G_s$ we have

$$h(\pi(tg_s w)) = h(\pi(tw)) = h(\sigma(t)) = t^{kb(w)}.$$

On the other hand, writing $t = t_1^D$, we get (recall $\phi_w(g) = \det(g)^{a(w)}$)

$$\phi_w(g_s tw) = \phi_w(g_s \iota(t_1)w) = \det(g_s \iota(t_1))^{a(w)} = t_1^{ma(w)} = t_1^{Db(w)} = t^{b(w)}. \quad (6.5)$$

Therefore, $(\phi_w(g_s tw))^k = h(\pi(tg_s w))$ and we see that $h \circ \pi$ is a regular extension of $(\phi_w)^k$. Thus $k \in \mathcal{E}'(w)$ and we have shown that $S \subseteq \mathcal{E}'(w)$.

For the converse, assume $k \in \mathcal{E}'(w)$. Let $f \in \mathcal{O}(X)$ be a regular extension of $(\phi_w)^k: Gw \rightarrow \mathbb{C}$. Then $f \in \mathcal{O}(X)^{G_s}$, hence there exists $h \in \mathcal{O}(Z)$ such that $h \circ \pi = f$. Since f is homogeneous, h is homogeneous as well, say $h(t\pi(w)) = ct^{sb(w)}$ for $c \in \mathbb{C}$ and $s \in S$. In particular, using (6.5),

$$t^{kb(w)} = (\phi_w(tw))^k = f(tw) = h(\pi(tw)) = ct^{sb(w)},$$

for all $t \in \mathbb{C}^\times$. Hence $k = s$, which shows $k \in S$ and proves the reverse inclusion $\mathcal{E}'(w) \subseteq S$. \square

6.3. Proof of Theorem 3.11

We go back to the setting of the proof of Theorem 3.4. Put $X := \overline{Gw}$ and $\partial X := X \setminus Gw$. Recall the GIT-quotient $\pi: X \rightarrow Z$, where $\mathcal{O}(Z) \simeq \mathcal{O}(X)^{G_s}$. Lemma 6.2 implies that $\pi^{-1}(\pi(0)) = \partial X$, hence we obtain $I(\pi(0)) = (\pi^*)^{-1}(I(\partial X))$ for the vanishing ideals. Moreover, the map $\sigma: \mathbb{C} \rightarrow Z$ defined in (6.3) satisfies $\sigma^{-1}(\pi(0)) = \{0\}$, hence $I(\pi(0)) = (\sigma^*)^{-1}(T\mathbb{C}[T])$. Using (6.4), we obtain

$$\sigma^*(I(\pi(0))) = \bigoplus_{e \in \mathcal{E}(w), e \neq 0} \mathbb{C} T^e. \quad (6.6)$$

Let $\varphi \in \mathcal{O}(Z)$ be such that $\Phi_w = \varphi \circ \pi$. Equation (6.5) implies that $\varphi \circ \sigma = T^{e(w)}$.

For the first assertion, suppose that Φ_w generates the vanishing ideal $I(\partial X)$ in $\mathcal{O}(X)$. Then Φ_w generates the ideal $I(\partial X) \cap \mathcal{O}(X)^{G_s}$. (Indeed, if $h \in \mathcal{O}(X)$ is such that $\Phi_w h$ is G_s -invariant, then h is G_s -invariant.) That is, φ generates the ideal $I(\pi(0)) = (\pi^*)^{-1}(I(\partial X))$ in $\mathcal{O}(Z)$. Hence $T^{e(w)}$ generates the ideal $\bigoplus_{e \in \mathcal{E}(w), e \neq 0} \mathbb{C} T^e$. This implies $\mathcal{E}(w) \subseteq e(w)\mathbb{N}$ and hence $e(w) = b(w)$, since $\mathcal{E}(w)$ generates the group $b(w)\mathbb{Z}$.

For the second assertion, note that it is a known fact that if $\mathcal{O}(X)$ is normal, then its ring of invariants $\mathcal{O}(X)^{G_s}$ is normal as well, e.g., see [13, Prop. 3.1]. Hence, in the situation from before, Z is normal if X is normal. Thus it suffices to prove that Z is not normal if $e'(w) > 1$.

Recall from (6.4) in the proof of Theorem 3.4 that $A := \sigma^*(\mathcal{O}(Z)) = \bigoplus_{s \in \mathcal{E}'(w)} \mathbb{C} T^{sb(w)}$. Suppose that $e'(w) > 1$. We have $e := e'(w) - 1 \notin \mathcal{E}'(w)$ since $e'(w)$ is defined as the minimal positive element of $\mathcal{E}'(w)$. Theorem 3.4 states that $\mathcal{E}'(w)$ generates the group \mathbb{Z} . So we have $ke \in \mathcal{E}'(w)$ for a sufficiently large $k \in \mathbb{N}$ (cf. Section 2.3) and hence $T^{keb(w)} \in A$. The element $T^{eb(w)}$ satisfies the monic equation $x^k - T^{keb(w)} = 0$ with coefficients in A , hence it lies in the integral closure of A .

On the other hand, there are $s_1, s_2 \in \mathcal{E}'(w)$ such that $e = s_1 - s_2$, since $\mathcal{E}'(w)$ generates the group \mathbb{Z} . It follows that $T^{eb(w)} = T^{s_1 b(w)} / T^{s_2 b(w)}$ is in the quotient field of A , but not in A . Therefore, A is not integrally closed.

Finally, $\mathcal{O}(X)$ is not a valuation ring since it is known that valuation rings are integrally closed [37, Thm. 10.3]. \square

6.4. Proof of Theorem 5.3

Recall the character $\chi: G \rightarrow \mathbb{C}^\times$ from (4.2), where $G = \mathrm{GL}_m^3/K$ and define the subgroup $G_s := \ker \chi = \mathrm{SL}_m^3/K$. We have a group embedding

$$\iota: \mathbb{C}^\times \rightarrow G, t \mapsto (tI_m, I_m, I_m) \bmod K.$$

(Note that $(tI_m, I_m, I_m) \equiv (I_m, tI_m, I_m) \equiv (I_m, I_m, tI_m) \bmod K$.) The image $\iota(\mathbb{C}^\times)$ is a normal subgroup of G and G is the product of $\iota(\mathbb{C}^\times)$ and the subgroup G_s . The group G acts on $W := \otimes^3 \mathbb{C}^m$ and we have $\iota(t)w = tw$ for $t \in \mathbb{C}^\times$ and $w \in W$ (compare (6.2)). We form the GIT-quotient of $X := \overline{Gw}$ by the action of G_s , obtaining a \mathbb{C}^\times -equivariant morphism $\pi: X \rightarrow Z$. Lemma 6.2 holds with essentially the same proof. The proof of Theorem 5.3 proceeds along the same lines as for Theorem 3.4, using the following result, which is similar to Lemma 6.3.

Lemma 6.5. *The stabilizer of $p := \pi(w)$ satisfies $\{t \in \mathbb{C}^\times \mid tp = p\} = \mu_{ma(w)}$.*

Proof. Suppose that $\pi(tw) = \pi(w)$. Then, since π is a GIT quotient, the orbit closures of tw and w intersect; cf. [25, II. §3.2]. Since $G_s w$ is closed, this means that $tw \in G_s w$. Therefore, there exists $g_s \in G_s$ such that $tw = g_s^{-1}w$, hence $g_s tw = w$. As before, we

have $g_s t w = g_s \iota(t) w$, hence $g_s \iota(t) \in \text{stab}(w)$. By the definition of the period $a = a(w)$, we obtain

$$1 = \chi(g_s \iota(t))^a = \chi(\iota(t))^a = (t^m)^a = t^{ma}.$$

Therefore, $\{t \in \mathbb{C}^\times \mid tp = p\} \subseteq \mu_{ma}$. The reverse inclusion is proven analogously. \square

7. Appendix

We believe the results here should be known, but they are hard to locate in the literature. We are therefore brief.

7.1. Plethysms

Proposition 7.1. *Let D be odd and let λ be a partition of Dd into at most m parts. Let $\text{mult}_\lambda(\mathcal{O}(\text{Sym}^D \mathbb{C}^m)_d)$ denote the multiplicity of the irreducible GL_m -representation of type λ^* in $\mathcal{O}(\text{Sym}^D \mathbb{C}^m)_d$. A set S of subsets of $\{1, \dots, \lambda_1\}$ is called of type λ in degree d if*

- The cardinality $|S|$ of S is d ,
- each element $s \in S$ has cardinality D ,
- each number i occurs in exactly λ_i^t of the subsets s of S , where λ_i^t is the length of the i th column of λ .

Let $q_\lambda(d)$ denote the number of sets S of type λ in degree d . Then

$$\text{mult}_\lambda(\mathcal{O}(\text{Sym}^D \mathbb{C}^m)_d) \leq q_\lambda(d).$$

Proof. Using [35, Fact 6.1] we know that the plethysm coefficient $\text{mult}_\lambda(\mathcal{O}(\text{Sym}^D \mathbb{C}^m)_d)$ equals $\text{mult}_{\lambda^t}(\bigwedge^d \bigwedge^D \mathbb{C}^m)$ if D is odd, where λ^t is the partition to the transposed Young diagram of λ . This multiplicity is bounded from above by the dimension of the λ^t -weight space $(\bigwedge^d \bigwedge^D \mathbb{C}^m)^{\lambda^t} \subseteq \bigwedge^d \bigwedge^D \mathbb{C}^m$:

$$\text{mult}_{\lambda^t}(\bigwedge^d \bigwedge^D \mathbb{C}^m) \leq \dim(\bigwedge^d \bigwedge^D \mathbb{C}^m)^{\lambda^t}.$$

The right hand side is precisely the number of cardinality d sets of cardinality D subsets of $\{1, \dots, D\}$ such that each number i occurs in exactly λ_i^t subsets. \square

For SL_m -invariants this specializes as follows.

Corollary 7.2. *Let D be odd and let Dd be divisible by m . The dimension of the SL_m -invariant space in $\mathcal{O}(\text{Sym}^D \mathbb{C}^m)_d$ is bounded from above by the number of cardinality d sets of cardinality D subsets of $\{1, \dots, dD/m\}$ such that each number occurs in precisely m of the subsets.*

We state without proof that [Proposition 7.1](#) can be strengthened as follows by factoring out the relations among the weight vectors:

Proposition 7.3. *Let λ be a partition of Dd into at most m parts. For D odd, the plethysm coefficient $\text{mult}_\lambda(\mathcal{O}(\text{Sym}^D \mathbb{C}^m)_d)$ is bounded from above by the number of semistandard tableaux*

- of shape λ
- in which each number from the set $\{1, \dots, D\}$ appears exactly d times
- and in which there do not exist two distinct numbers i and j with the property that they occur in precisely the same d columns.

7.2. Stabilizers

Let $\mu_d \subseteq \mathbb{C}^\times$ denote the group of d th roots of unity. We first consider binary forms.

Proposition 7.4.

1. For a generic $w \in \text{Sym}^3 \mathbb{C}^2$ we have $\text{stab}(w) \simeq \mu_3 \rtimes S_2$. Hence $a(3, 2) = 6$ and $a'(3, 2) = 2$.
2. For a generic $w \in \text{Sym}^4 \mathbb{C}^2$ we have $\text{stab}(w) \simeq C_2 \times C_2$. Hence $a(4, 2) = 2$ and $a'(4, 2) = 1$.
3. Let $D \geq 5$. The stabilizer of a generic $w \in \text{Sym}^D \mathbb{C}^2$ is trivial, hence $a'(D, 2) = 1$.

Proof. Suppose $D = 3$. It is known that all $w \in \text{Sym}^3 \mathbb{C}^2$ with three distinct zeros are equivalent to $v = X^3 + Y^3$. Using [Proposition 2.4](#) we get $\text{stab}(v) \simeq \mu_3 \rtimes S_2$ and the first assertion follows.

Suppose $D = 4$. Almost all $w \in \text{Sym}^4 \mathbb{C}^2$ are equivalent to a form $F_\mu := X^4 + \mu X^2 Y^2 + Y^4$. (In order to show this, check that the derivative of the orbit map $\text{GL}_2 \times \mathbb{C} \rightarrow \text{Sym}^4 \mathbb{C}^2, (g, \mu) \mapsto gF_\mu$ is surjective if $\mu \neq \pm 2$.) The Hessian of F_μ equals

$$H(F_\mu) = 24\mu(X^4 + Y^4) + 12(12 - \mu^2)X^2Y^2.$$

It is easy to check that $\text{span}\{F_\mu, H(F_\mu)\} = \text{span}\{X^4 + Y^4, X^2Y^2\}$ if $\mu \neq \pm 2$. Suppose that $g \in \text{stab}(F_\mu)$ satisfies $\det g = 1$. By [\(4.3\)](#), g stabilizes $H(F_\mu)$ as well, hence g stabilizes $X^4 + Y^4$. [Proposition 2.4](#) implies that $\text{stab}(F_\mu)$ is generated by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and the matrices $\begin{bmatrix} t_1 & 0 \\ 0 & \pm t_1^{-1} \end{bmatrix}$, where $t_1 \in \mu_4$. This implies $\text{stab}(F_\mu) \simeq C_2 \times C_2$, $a(F_\mu) = 2$ and the second assertion follows.

We omit the proof in the case $D \geq 5$. \square

Now we consider ternary forms.

Proposition 7.5.

1. A generic $w \in \text{Sym}^3 \mathbb{C}^3$ satisfies $\text{stab}(w) \simeq \mu_3 \rtimes S_3$. Hence $a'(3, 3) = 2$.
2. For a generic $w \in \text{Sym}^4 \mathbb{C}^3$ we have $\text{stab}(w) \simeq C_2$. Hence $a'(4, 3) = 2$.
3. Let $D \geq 5$. The stabilizer of a generic $w \in \text{Sym}^D \mathbb{C}^3$ is trivial, hence $a'(D, 3) = 1$.

Proof. Case $D = 3$: A generic ternary form is in the same orbit as a smooth cubic in Hesse normal form $v := X^3 + Y^3 + Z^3 + \lambda XYZ$ where $\lambda^3 \neq -27$, cf. [14, §3.1.2]. In order to determine the stabilizer, we compute the Hessian $H(v) = (216 + 2\lambda^3)XYZ - 6\lambda^2(X^3 + Y^3 + Z^3)$ and check that $\text{span}\{v, H(v)\} = \text{span}\{X^3 + Y^3 + Z^3, XYZ\}$ due to $\lambda^3 \neq -27$. If $g \in \text{stab}(v)$ such that $\det(g) = 1$, then $g \in \text{stab} H(v)$ by (4.3), hence $g \in \text{stab} H(XYZ)$. The assertion follows with Proposition 2.4.

In the case $D = 4$ we can use a normal form in [14, §6.5.2, Table 6.1, p. 266].

In the case $D \geq 5$ we can argue similarly as in Matsumura and Monsky [38]. However, some care has to be taken since their proof actually contains an error, which fortunately does not affect its statement under the hypothesis $D > 2$ and $m > 3$. Applying the same technique in the case $D \geq 5$ and $m = 3$, and correcting the error, yields the assertion. (We note that Matsumura and Monsky's [38] uncorrected proof technique yields the wrong answer in the case $D = 4$ and $m = 3$.) \square

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