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Prime and Prime Power Divisibility of Catalan Numbers

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For any prime p, the sequence of Catalan numbers

$$a_n = \frac{1}{n}\binom{2n-2}{n-1}$$

is divided by the a_n prime to p into blocks $B_k (k > 0)$ of a_n divisible by p. The lengths and positions of the B_k are determined. Additional results are obtained on prime power divisibility of Catalan numbers.

1. INTRODUCTION

The Catalan numbers, given by

$$a_n = \frac{1}{n}\binom{2n-2}{n-1}, \qquad n \geqslant 1 \tag{1}$$

arise in many combinatorial and geometric problems. For example, it can be shown that a_n is the number of non-commutative non-associative binary products of a single generator taken n times. Also, it was proved by Euler that a_n represents the number of ways that a planar, convex $(n + 1)$-gon may be divided into triangles by non-intersecting diagonals. A more detailed discussion of this can be found in Alter [1], who also gives a brief history of the subject. For an extensive bibliography see Alter [1], Brown [3], and Gould [4].

Defining $a_1 = 1$, the Catalan numbers can be represented recursively as

$$a_n = \frac{4n-6}{n} a_{n-1}, \qquad n \geqslant 2. \tag{2}$$

It is known that a_n is odd if and only if n is a power of 2. Several proofs of this can be found in Alter and Curtz [2]. Thus, the sequence of the a_n for n not a power of 2 is partitioned into a monotonically increasing sequence of blocks $\{B_k\}_{k\geqslant 1}$ of length $2^k - 1$, respectively. To be more precise, let the sequence of blocks $\{B_k\}_{k\geqslant 1}$ be such that each member, B_k, consists of the largest possible set of consecutive a_n, all of which are even. L_k, the length of the block B_k, represents the cardinality of B_k. Clearly

$$L_k = 2^k - 1, \qquad k \geqslant 1. \tag{3}$$

Also each individual block B_k consists of $2^k - 1$ consecutive Catalan numbers of the form

$$a_{2^k+1}, a_{2^k+2}, \ldots, a_{2^{k+1}-1}, \qquad k \geqslant 1. \tag{4}$$

The object of this paper is to generalize these results to all primes p and prime powers p^k. The result that $2 \nmid a_{2^k} (k \geqslant 0)$ is generalized in Theorem 1 by establishing for all prime numbers p that

$$p \nmid a_{p^k} \qquad (k \geqslant 0). \tag{5}$$

It is further established that the subsequence of Catalan numbers which are multiples of a given prime p and the subsequence of Catalan numbers which are not multiples of p also form into blocks. The length of the blocks in each of these cases is determined. It is proved that the set of Catalan numbers which are not divisible by an odd prime p occur in blocks $\bar{B}_k$ of a fixed length $\bar{L}_k$ where

$$\bar{L}_k = \frac{p + 3(1 + 2\delta)}{2}, \qquad \delta = \delta_{3p}, \tag{6}$$

and

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

is the Kronecker function. This generalizes the observation that the a_n which are not divisible by 2 occur in blocks of fixed length one. The Catalan numbers which are divisible by an odd prime p occur in blocks B_k of length L_k where

$$L_k = \frac{p^{m+1+\delta} - 3}{2} \tag{7}$$

and m is the highest power of $(p + 1)/2$ which divides k. Thus the length of the k-th block, L_k, is determined. This result generalizes the results stated in equation 3, since it establishes that the block lengths of the

Catalan numbers that are multiples of p are not fixed but depend on k for their size. However, for odd primes p, the sequence $\{B_k\}_{k\geqslant 1}$ of blocks consisting of Catalan numbers that are multiples of p is no longer monotone increasing but in fact possesses a somewhat different geometric property.

Section 2 contains some preliminary results in which certain congruence properties of the Catalan numbers are established and the theorems that deal with the a_n that are not multiples of a given odd prime p are proved. Some results on the Catalan numbers which are multiples of a given prime p are presented along with a statement and discussion of the Main Theorem. In Section 3 the Main Theorem is proved. Section 4 contains remarks and observations regarding the divisibility of Catalan numbers by prime powers; in this section some of the earlier results for primes are generalized to prime powers. Section 5 discusses the calculation of the lengths of some of the p^k-blocks that arise in Section 4. Also, in this final section, more general recursive number sequences and some of their divisibility properties are discussed.

2. Preliminary Results

THEOREM 1. *For all primes p and $k \geqslant 1$ it is true that*

$$p \nmid a_{p^k}. \tag{5}$$

Proof. As already stated, Theorem 1 is true for $p = 2$. Let p be an odd prime. Then

$$a_{p^k} = \frac{1}{p^k}\binom{2p^k - 2}{p^k - 1} = \frac{1}{p^k}\frac{(2p^k - 2)!}{((p^k - 1)!)^2}.$$

Choose M_1, M_2, and s to be the highest power of p that divides

$$(2p^k - 2)!,\ (p^k - 1)!,\ \text{and } a_{p^k},\ \text{respectively.}$$

It is well known that $p \mid n!$ exactly

$$\sum_{m=1}^{\infty}\left(\frac{n}{p^m}\right)$$

times, where $[x]$ is the greatest integer $\leqslant x$; thus

$$M_1 = \sum_{m=1}^{\infty}\left[\frac{2p^k - 2}{p^m}\right] = 2\sum_{i=0}^{k-1} p^i - k = \frac{2(p^k - 1)}{p - 1} - k$$

and

$$M_2 = \sum_{m=1}^{\infty} \left[\frac{p^k - 1}{p^m}\right] = \sum_{i=1}^{k-1} p^i - (k-1) = \frac{p^k - p}{p-1} - (k-1).$$

Clearly,

$$s = M_1 - 2M_2 - k = 0,$$

and this completes the proof of Theorem 1.

Continuing along the lines of Theorem 1, the next two theorems are also about Catalan numbers which are not multiples of a given prime p. There are two theorems because the situation for the prime number 3 is a little different from that of arbitrary primes $p > 3$.

THEOREM 2. *If n is chosen so that $3 \mid a_{n-1}$ and $3 \nmid a_n$, then*

(a) $n \equiv 0$ (modulo 9),

(b) $3 \nmid \prod_{i=0}^{5} a_{n+i}$,

(c) $3 \mid a_{n+6}$.

Proof. The congruence relation (a) is immediate from the recurrence relation (2). Similarly (b) follows by applying (2) five times. Finally, (c) is immediate from (b) and the fact that

$$a_{n+6} = \frac{4n+18}{n+6} a_{n+5}.$$

THEOREM 3. *Let $p > 3$ be prime. If n is chosen so that $p \mid a_{n-1}$* and *$p \nmid a_n$*, then

(a) $n \equiv 0$ (modulo p),

(b) $p \nmid \prod_{i=0}^{(p+1)/2} a_{n+i}$,

(c) $p \mid a_{n+(p+3)/2}$.

Proof. The proof is similar to the proof of Theorem 2 in that (a) is immediate from (2), (b) follows from (a) and (2), and (c) follows from (b) and (2). However (b) and (c) require the additional observation that, if $n \equiv 0$ (modulo p), then $4(n+k) - 6 \equiv 0$ (modulo p) if and only if

$$k \equiv \frac{p+3}{2} \qquad \text{(modulo } p\text{)}.$$

Theorems 2 and 3 establish that the set of Catalan numbers which are not divisible by an odd prime p occur in blocks of a fixed length 6 for $p = 3$ and a fixed length $(p+3)/2$ for $p > 3$. The next theorem deals with the set of Catalan numbers that are multiples of a given prime p.

THEOREM 4. *If n is chosen so that $p \nmid a_{n-1}$ and $p \mid a_n$, then $n \equiv (p+3)/2$ (modulo p).*

Proof. Immediate from recurrence relation (2).

COROLLARY. *If n is chosen so that $3 \nmid a_{n-1}$, then $3 \mid a_n$ if and only if $n \equiv 6$ (modulo 9).*

From theorems 2, 3, and 4 it is clear that the Catalan numbers which are divisible by a given prime p occur in blocks. It now remains to determine the length of these blocks. This is done in Theorem 5, the proof of which is given in the Section 3.

THEOREM 5. *Let the sequence $\{B_k\}_{k \geqslant 1}$ represent the sequence of blocks of Catalan numbers which are divisible by an odd prime p, and L_k be the corresponding length of the block B_k. If m is chosen so that*

$$\left(\frac{p+1}{2}\right)^m \mid k \qquad \textit{and} \qquad \left(\frac{p+1}{2}\right)^{m+1} \nmid k,$$

then

$$L_k = \frac{p^{m+1+\delta} - 3}{2}. \tag{7}$$

Theorem 5 is established constructively by first showing that the block configuration of the Catalan numbers, a_n, which are multiples of p in the range $[0, p^{m+1})$ is really $(p-1)/2$ copies (or linear translations, to be more precise) of those numbers in the range $[0, p^m)$ followed by a copy of the block configuration for those in the range $[0, (p^m+3)/2)$, followed by a single block of a_n which are all multiples of p.

3. PROOF OF THE MAIN THEOREM

Theorem 5 has been stated and discussed in Section 2. In order to simplify the proof it is desirable to define a new sequence a_n by

$$a_0 = -1, \qquad a_n = \frac{2n-3}{n} a_{n-1} \qquad (n \geqslant 1). \tag{8}$$

Iterating the above sequence r times yields for all r $(0 \leqslant r \leqslant n)$ that

$$a_n = \frac{(2n-3)(2n-5)\cdots(2n-2r-1)}{n(n-1)\cdots(n-r+1)} a_{n-r}. \tag{9}$$

Let p be an odd prime and let $a, b \in Q$ be rational numbers. Then a and b are said to be associates, $a \sim b$, if and only if

$$v_p(a) = v_p(b),$$

where v_p is the p-adic valuation of Q.

Since the new a_n's are associates of the old a_n's, it suffices to prove Theorem 5 for the new a_n's. In order to prove Theorem 5, it is convenient to prove first the following two lemmas. The proof of Theorem 5 will immediately follow the proof of Lemma 2.

LEMMA 1. *For p an odd prime let s, m, n be integers satisfying*

$$sp^m \leqslant n < (s+1)\,p^m \qquad (m \geqslant 0, 0 < s < p).$$

Then

(a) $a_n \sim a_{n-sp^m}$, *if* $n < (p^{m+1}+3)/2$,

(b) $a_n \sim pa_{n-sp^m}$, if $n \geqslant (p^{m+1}+3)/2$.

Proof. For $m = 0$ it is clear that $n = s$ and by choosing $n = r$ in (9) the desired result follows because the denominator is prime to p and the numerator is a multiple of p if and only if $n \geqslant (p+3)/2$. For $m > 0$, consider (9) with $r = sp^m$. The factors of the denominator consist of s complete sets of representatives of the integers modulo p^m. The largest factor being $n < p^{m+1}$ and any two that are congruent modulo p^m are associates. Since p is odd, the numerator also consists of s complete sets of a and b representatives (mod p^m). If a and b are two factors of the numerator and $a \equiv b \not\equiv 0 \pmod{p^m}$, then $a \sim b$. Because $2n - 3 < 2p^{m+1}$ there is at most one factor $a \equiv 0 \pmod{p^{m+1}}$, in this case $a = p^{m+1}$. It follows that

$$a_n \sim pa_{n-sp^m} \qquad \text{or} \qquad a_n \sim a_{n-sp^m},$$

depending on whether or not p^{m+1} occurs as a factor of the numerator. If p^{m+1} is a factor of the numerator, then there is a t such that

$$2n - 2t - 3 = p^{m+1} \qquad \text{and} \quad 0 \leqslant t < sp^m,$$

or

$$n - t = \frac{p^{m+1}+3}{2}.$$

Hence a necessary condition is that

$$n \geqslant \frac{p^{m+1}+3}{2}. \tag{10}$$

Conversely, if (10) holds, then let t be such that

$$t = n - \frac{p^{m+1} + 3}{2} < (s + 1)\, p^m - \frac{p^{m+1} + 3}{2} < sp^m.$$

Thus (10) is a sufficient condition and the proof is complete.

COROLLARY. *For p an odd prime and all $m \geqslant 0$ it is true that*

(a) $a_{p^m} \sim 1$,

(b) $a_{(p^{m+1}+1)/2} \sim 1$,

(c) $a_n \equiv 0$ (modulo p) for $(p^{m+1} + 3)/2 \leqslant n < p^{m+1}$.

Proof. (a) $a_{p^m} \sim a_{p^m - p^m} = a_0 = -1$.

(c) Immediate from Lemma 1 since $v_p(a_n) \geqslant 0$ for all $n \geqslant 0$.

(b) Choose $n = (p^m + 1)/2$ and $s = (p - 1)/2$, then from part (a) of the lemma it follows that

$$a_{(p^{m+1}+1)/2} \sim a_{(p^m+1)/2}\,.$$

Since (b) holds for $m = 0$, the proof is complete.

LEMMA 2. *Let p be an odd prime and $m \geqslant 1$ be an integer. (If $p = 3$, suppose $m \geqslant 2$.) Then for any natural number $s \leqslant (p - 1)/2$, there are precisely $s((p + 1)/2))^{m+1-\delta}$ blocks in the interval* $[0, sp^m)$.

Proof. Because of Lemma 1 it suffices to consider the case in which $s = 1$. For $m = 1$ and $p > 3$ this has already been proved and it is easy to see that the case $m = 2, p = 3$ also holds. The proof is completed by induction on m. There are

$$\frac{p-1}{2}\left(\frac{p+1}{2}\right)^{m-1-\delta} \text{ blocks in the interval } \left[0, \frac{p-1}{2}\, p^m\right).$$

Noting that

$$\frac{p-1}{2}\, p^m + \frac{p^m + 3}{2} = \frac{p^{m+1} + 3}{2},$$

it is easy to see that there are

$$\left(\frac{p+1}{2}\right)^{m-1-\delta} - 1 \text{ blocks in the interval } \left[\frac{p-1}{2}\, p^m, \frac{p^{m+1} + 3}{2}\right)$$

and

$$\text{one block in the interval } \left[\frac{p^{m+1} + 3}{2}, p^{m+1}\right).$$

This gives a total of

$$\left(\frac{p+1}{2}\right)^{m-\delta}$$

blocks in the interval $[0, p^{m+1})$. The Main Theorem now follows.

Proof of Theorem 5. If k is a power of $(p+1)/2$, then (7) follows from Lemma 2 and the corollary to Lemma 1. Suppose

$$k = \sum_{i=m}^{t} c_i \left(\frac{p+1}{2}\right)^i, \qquad 0 \leqslant c_i < \frac{p+1}{2}, \qquad c_m, c_t > 0. \quad (11)$$

Since there are

$$c_n \left(\frac{p+1}{2}\right)^n$$

blocks in the interval $0 \leqslant n \leqslant c_n p^{n+1+\delta}$, it follows, using also Lemma 1, that there are exactly n blocks in the interval

$$0 \leqslant n < \sum_{i=m}^{t} c_i p^{i+1+\delta}.$$

It is easy to see that

$$\begin{aligned}\sum_{i=m}^{t} c_i p^{i+1+\delta} &\leqslant \sum_{j=0}^{t} \frac{p-1}{2} p^{j+1+\delta} \\ &= \frac{p^{1+\delta}(p-1)}{2} \sum_{j=0}^{t} p^j \\ &= \tfrac{1}{2}(p^{t+2+\delta} - p^{1+\delta}) < \tfrac{1}{2}(p^{t+2+\delta} + 3).\end{aligned}$$

Using Lemmas 1 and 2, it now follows that this B_k has the same length as the last block in the interval

$$0 \leqslant n < \left(\sum_{i=m}^{t} c_i p^{i+1+\delta}\right) - p^{t+1+\delta}.$$

Thus B_k has the same length as Block $B_{k'}$, where $k' = k - ((p+1)/2)^t$. The proof of the theorem is completed by induction.

Remark. Using the Main Theorem in conjunction with Lemma 1 it follows that the blocks of Catalan numbers that are not multiples of a given odd prime p all have the same length. In fact, if $p \neq 3$, then the length is $(p+3)/2$ and the length is 6 for $p = 3$. However, this result has already been discussed in Section 2 since it followed immediately from Theorems 2 and 3, and was thus proved independently of Theorem 5.

By using Theorem 5 the first and last value of the block B_k can be obtained. To be more precise:

COROLLARY 1. *If k is a positive integer, say*

$$k = \sum_{i=m}^{t} c_i \left(\frac{p+1}{2}\right)^i \qquad \left(\text{with } 0 \leqslant c_i < \frac{p+1}{2}, c_m, c_t > 0\right),$$

then B_k begins at

$$\frac{p^{m+1+\delta}+3}{2} + \sum_{i=m}^{t} (c_i - \delta_{mi})\, p^{i+1+\delta}$$

and ends at

$$\sum_{i=m}^{t} c_i p^{i+1+\delta} - 1.$$

Proof. By the proof of Theorem 5, there are precisely k blocks in the interval $0 \leqslant n < \sum_{i=m}^{t} c_i p^{i+1+\delta}$; the k-th block ends at $\sum_{i=m}^{t} c_i p^{i+1+\delta} - 1$, and has length

$$L_k = \frac{p^{m+1+\delta}-3}{2}.$$

So the k-th block begins at

$$\sum_{i=m}^{t} c_i p^{i+1+\delta} - L_k = \frac{p^{m+1+\delta}+3}{2} + \sum_{i=m}^{t} (c_i - \delta_{mi})\, p^{i+1+\delta}.$$

COROLLARY 2. *The $((p+1)/2)^m$-th block begins at*

$$\frac{p^{m+1+\delta}+3}{2}.$$

4. PRIME POWER DIVISIBILITY

In this section, prime power divisibility of the Catalan numbers is studied. For simplicity only the general case of powers of a prime $p > 3$ will be considered. Because the proofs are rather similar to those of previous sections the proofs will often only be outlined.

Several kinds of blocks will be discussed and relationships between the different types will be found. In general, if P is a property of the non-negative integers N, then a P-block is an interval $[n, m]$, a subset of N, such that P holds for all i such that $n \leqslant i \leqslant m$ and does not hold for $n - 1$ or $m + 1$. The P-blocks are numbered according to the usual order on N. In particular, for $k \geqslant 0$, let $A_k = \{A_{ki}\}_{i \geqslant 1}$, $D_k = \{D_{ki}\}_{i \geqslant 1}$ (large p^k-blocks), and $E_k = \{E_{ki}\}_{i \geqslant 1}$ (small p^k-blocks) be the sets of blocks defined by the properties $v_p(a_n) = k$, $v_p(a_n) \geqslant k$, and $v_p(a_n) \leqslant k$, respectively. The union $\bigcup_{k \geqslant 0} A_k$ partitions the set of non-negative integers into blocks which will be denoted by C_i, $i > 0$.

Note that Theorems 2 and 3 of Section 2 are results about the E_{0i} and that the blocks B_i of Sections 2 and 3 are the same as the D_{1i}. The blocks A_{ki} have been introduced because of their simple structure. The next theorem illustrates this:

THEOREM 6. *The beginning points of the blocks C_i are the non-negative integers* $\equiv 0, (p + 3)/2/(\text{mod}\, p)$. *$C_i$ is of length* $(p + 3)/2$ (*respectively,* $(p - 3)/2$) *if and only if it begins with an* $n \equiv 0\ (\text{mod}\, p)$(*respectively,* $n \equiv (p + 3)/2\ (\text{mod}\, p)$). *If C_i is a p^k-block, then*

(i) *If C_i is of length $(p + 3)/2$, one of C_{i-1}, C_{i+1} is a p^{k+1}-block and the other is a p^{k+r}-block for some $r \geqslant 1$.*

(ii) *If C_i is of length $(p - 3)/2$, one of C_{i-1}, C_{i+1} is a p^{k-1}-block and the other is a p^{k-r}-block for some $r \geqslant 1$.*

Proof. It is easy to show this follows from the recursive definition of the a_n's.

Define C_i to be of type $(k, \pm 1)$ if C_i is a p^k-block of length $(p \pm 3)/2$, respectively. The C_k is made up of the large and small p^k-blocks of minimal length. More precisely, Theorem 6 has the following corollary:

COROLLARY. *The p^k-blocks of type $(k, -1)$ are the large p^k-blocks of length $(p - 3)/2$, and the p^k-blocks of type $(k, +1)$ are the small p^k*-blocks *of length $(p + 3)/2$.*

Before continuing, observe that Lemma 1 of Section 3 may be expressed in the following useful form:

THEOREM 7. *Let $n \geqslant 0$ be any non-negative integer and write* $n = \sum_{i=0}^{m} r_i p^i$, *where* $0 \leqslant r_i < p$. *Set*

$$n_i = \sum_{j=0}^{i} r_j p^j \quad \textit{and} \quad \epsilon_i = \begin{cases} 1, & \textit{if } n_i \geqslant (p^{i+1} + 1)/2, \\ 0, & \textit{otherwise.} \end{cases}$$

Then the number of times p divides a_n is precisely $\sum_{i=0}^{m} \epsilon_i$ times.

This will be used to prove the following reflection property of the p^k-blocks.

THEOREM 8. *Suppose that there are k_m blocks C_i in the interval $[0, p^m)$. Then, for $i \leqslant k_m$, C_i is of type $(k, \pm 1)$ if and only if C_{k_m+1-i} is of type $(k, \mp 1)$.*

Proof. For $0 \leqslant n < p^m$, let

$$f(n) = p^m - \frac{p-3}{2} - n.$$

Then f maps the beginning of C_i onto that of C_{k_m+1-i}. Further, the beginning points of C_i and C_{k_m+1-i} differ by $(p-3)/2$ modulo p. Hence the theorem follows from Theorem 6 and the following lemma:

LEMMA. *For $0 \leqslant n < p^m$ let*

$$f(n) = p^m - \frac{p-3}{2} - n.$$

If

$$n \equiv 0, \frac{p+3}{2} \pmod{p},$$

then

$$v_p(a_n) = n - v_p(a_{f(n)}).$$

Proof. Let $n = \sum_{i=0}^{m-1} r_i p^i$ where $r_0 = 0$ or $(p+3)/2$ and $0 \leqslant r_i < p$ for $i = 1, 2, \ldots, m-1$. Then

$$\begin{aligned} f(n) &= p^m - \frac{p-3}{2} - n \\ &= \left(p - r_0 - \frac{p-3}{2}\right) + (p - r_1 - 1)p + \cdots + (p - r_{m-1} - 1)p^{m-1}. \end{aligned}$$

In order to apply Theorem 7 let $n_i = \sum_{j=0}^{i} r_j p^j$, and define

$$(f(n))_i = \left(p - r_0 - \frac{p-3}{2}\right) + \sum_{j=1}^{i} (p - r_j - 1)\, p^j.$$

Then

$$(f(n))_i = p^{i+1} - \frac{p-3}{2} - n_i\,.$$

Recall that, for n, $\epsilon_i = 0$ if and only if $n_i < (p^{i+1} + 3)/2$. But $n_i < (p^{i+1} + 3)/2$ if and only if

$$p^{i+1} - \frac{p-3}{2} - n_i = (f(n))_i \geqslant \frac{p^{i+1} - p + 2}{2}.$$

Also

$$\frac{p^{i+1} - p + 2}{2} \equiv 1 \pmod{p},$$

whereas $(f(n))_i \equiv 0$ or $(p + 3)/2 \pmod{p}$. Hence the last inequality is equivalent to $(f(n))_i \geqslant (p^{i+1} + 3)/2$, which says that ϵ_i for $f(n)$ is $+1$. The sum of the ϵ_i's for n plus the sum of the ϵ_i's for $f(n)$ is therefore m, which completes the proof.

From this it follows that the sequence of large p^k-blocks in $[0, p^m)$ is just like the sequence of small p^k-blocks in the same interval but with the order reversed. More precisely,

COROLLARY. *Let t_m be the number of large p^k-blocks in the interval $[0, p^m)$. For $1 \leqslant i \leqslant t_m$, D_{ki} is of length $sp + (p + 3)/2 (s \geqslant 0)$ if and only if A_{k,t_m+1-i} is of length $sp + (p - 3)/2$.*

From the above corollary it is clear that the lengths of D_{ki} and A_{k,t_m+1-i} differ by 3.

It will be established, in Theorem 9, that the p^k-block structure determines and is determined by the large p^k-block structure. In fact, it has already been seen that the p^k-blocks of length $(p - 3)/2$ are the same as the large p^k-blocks of the same length.

THEOREM 9. *Every large p^k-blocks contains a p^k-block and every p^k-block is contained in a large p^k-block. If D_{ki} is a large p^k-block of length $L_{ki} > (p - 3)/2$, then*

(a) $D_{k,i-1}$ and $D_{k,i+1}$ are length $(p - 3)/2$,

(b) $L_{ki} = (p^{m+1} - 3)/2$ for some $m \geqslant 1$,

(c) D_{ki} contains

$$\left(\frac{p-1}{2}\right)\left(\frac{p+1}{2}\right)^{m-1} p^k\text{-blocks of length } \frac{p+3}{2},$$

(d) D_{ki} contains no p^k-blocks of length $(p - 3)/2$.

Proof. By Lemma 1 of Section 3, one reduces first to the case of $k = 1$, and then to the case in which D_{li} is of the form

$$\frac{p^{m+1}+3}{2} \leqslant n < p^{m+1}.$$

In this case, the same lemma together with an easy counting argument allows one to finish the proof.

Remark. Theorem 5 of Section 3 says roughly that the large p-block lengths are like representations of integers to the base $(p+1)/2$. By the last theorem, the "blocks of p-blocks of length $(p+3)/2$" behave in the same way.

5. Remarks

Using counting arguments for decompositions (i.e., partitions of natural numbers in which the order of the summands counts), one can derive formulae for the number of p^k-blocks in the interval $[0, p^m)$. The formulae, however, are rather complicated and it is not obvious how to calculate the lengths of the "blocks of p^k-blocks of length $(p+3)/2$" from them.

Nevertheless these can be obtained from the block lengths of other sequences. Indeed, most of the results in this paper can, with suitable modifications, be extended to sequences defined by a recurrence like

$$b_n = \frac{an+b}{n}\, b_{n-1}.$$

For example, consider the sequence $b_n = \binom{2n}{n}$. This sequence satisfies $b_n = (2(2n-1))/(n-1)\, b_{n-1}$, and Lemma 1 of Section 3 takes on the form:

LEMMA 1′. *If p is an odd prime and s, m, n are integers satisfying*

$$sp^m < n \leqslant (s+1)\, p^m \qquad (m \geqslant 0, 0 \leqslant s < p),$$

then

(a) $a_n \sim a_{n-sp}$, if $n < \frac{1}{2}(p^{m+1}+1)$,

(b) $a_n \sim pa_{n-sp}$, if $n \geqslant \frac{1}{2}(p^{m+1}+1)$.

The rest of the results follow through. It is easily seen that there is a natural one-to-one correspondence between the blocks A_{ki}, D_{ki}, and E_{ki} for $\binom{2n-2}{n-1}/n$ and those for $\binom{2n}{n}$.

References

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