Lyndon interpolation for modal mu-calculus

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Direct interpolation for modal mu-calculus

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The modal μ-calculus is an extension of modal logic by two quantifiers, μ and ν, that bind propositional variables. The formulæ $\mu x A$ and $\nu x A$ are interpreted over Kripke frames (labelled transition systems) respectively as the least and greatest fixed points of the function $x \mapsto A(x)$. Modal μ-calculus can thus be thought of as a logic that allows for restricted second-order quantification while still maintaining computationally attractive properties such as decidability of validity and the finite model property. Moreover, many program logics (LTL, PDL, CTL, etc) can be embedded into μ-calculus making it an important metatheory [3, 4].

Modal logics are known to widely enjoy interpolation (see e.g. [8, 19]) and μ-calculus does so in a very strong sense: given a formula A and a finite set of propositions and modality operators L, there exists a formula B (an interpolant) in the language L such that $A \to B$ is valid and for *every* formula C whose common language with A lies within L, $A \to C$ is valid if and only if $B \to C$ is valid. This property, called *uniform interpolation*, easily implies *Craig interpolation*: if $A \to C$ is valid then there is a formula B in the language common to A and C such that both $A \to B$ and $B \to C$ are valid.

Uniform interpolation for modal μ-calculus was established by D'Agostino and Hollenberg in [6]. The proof is very interesting as it involves both semantic and syntactic arguments. The authors utilise (disjunctive) modal automata from [10] to show that a form of propositional quantifier, called bisimulation quantifiers ([17, 15]), is definable in modal μ-calculus and can readily be used to define the interpolants.

Aside from the method of propositional quantification, there are a number of other ways to approach interpolation in non-classical and modal logics (see e.g. [5, 12]) among which is the syntactical approach via sequent calculus. If one has at hand a (complete) sequent calculus that admits elimination of cuts then Craig interpolation is often (but not always) provable via induction over the cut-free derivations. Indeed, there is an intimate connection between interpolation and the existence of various forms of sequent calculi, as demonstrated in [9, 13].

Following the proof-theoretic approach to interpolation, in this paper we show how to directly extract interpolants for modal μ-calculus formulæ via a sequent calculus.

The first proof system to consider is Kozen's original axiomatisation [11] which expands the standard axioms of the modal system K by regeneration and induction rules for the least (μ) and greatest (ν) fixed point quantifiers:

μ-regeneration:	$A(\mu x A(x)) \to \mu x A(x)$	μ-induction:	$A(B) \to B \vdash \mu x A(x) \to B$
ν-regeneration:	$\nu x A(x) \to A(\nu x A(x))$	ν-induction:	$B \to A(B) \vdash B \to \nu x A(x)$

Completeness for Kozen's system was established in [18]. The proof, imitated for the natural sequent formulation of the system, makes essential use of the cut rule and it remains a significant open problem whether Kozen's system without cut is also complete. But aside from cut, the induction rules themselves do not preserve interpolants,[1] so an alternate proof calculus is still needed.

[1] For example, B is an obvious interpolant for $A(B) \to B$ but it need not be an interpolant of $\mu x A \to B$.

To pursue our goal we utilise instead the finitary and complete 'circular' proof systems that were introduced in [1]. One of these systems discards the induction rules in favour of generalising the regeneration inferences. The new rules have the form

$$\dfrac{\begin{matrix}[\Gamma \Rightarrow \Delta, \nu x A]\\ \vdots \\ \Gamma \Rightarrow \Delta, A(\nu x A)\end{matrix}}{\Gamma \Rightarrow \Delta, \nu x A}\,\nu\mathrm{R} \qquad\qquad \dfrac{\begin{matrix}[\Gamma, \mu x A \Rightarrow \Delta]\\ \vdots \\ \Gamma, A(\mu x A) \Rightarrow \Delta\end{matrix}}{\Gamma, \mu x A \Rightarrow \Delta}\,\mu\mathrm{L}$$

where the sequent within the brackets is understood as an assumption of the proof which is discharged. Applications of the rules are subject to the condition that there is a thread from the formula $A(\sigma x A)$ in the premise to the formula $\sigma x A$ in the discharged sequent that does not regenerate fixed point variables subsuming x. This restriction is formalised by annotating formulæ: each formula in the proof is labelled by a word from a finite set of *names* in such a way as to record the regenerations of formulæ induced by the μ and ν inferences; the condition on applications of the inference above is then represented by the local requirement that the premise and discharged assumptions of the rule have identical annotations (for details, see [1]).

Our construction of interpolants from valid implications rests on utilising circular proofs which are both cut-free and analytic. The impredicativity inherent in the sequent presentation of Kozen's axiomatisation still remains, but instead of existing at the level of inference rules, where it naturally blocks a step-wise construction of interpolants, it manifests at the level of proofs in the choice of the assumption and discharge rules utilised. Indeed, it is precisely the two rules νR and μL that introduce quantifiers into interpolants. A similar approach using circular proofs for Gödel-Löb logic GL is [16].

The interpolants obtained in this way are structurally identical to the proof witnessing the interpolated implication and from this observation one can immediately infer results on the logical form of the interpolant. For example, given a valid formula $A \to C$ there exists a conjunction-free interpolant if there is a proof of $A \to C$ avoiding the rules of conjunction introduction on the right or disjunction introduction on the left. As it is the quantifiers that are the source of expressibility and complexity in the μ-calculus, it is of interest to examine the quantifier structure of interpolants and their dependence on the structure of proofs. We classify sufficient conditions for ensuring interpolants from complexity classes Π_1, Σ_1 and the alternation-free fragment of μ-calculus.

This work can be placed within a broader programme of adapting and applying proof-theoretic methods to coalgebraic logics in general. Modal μ-calculus is a special case of the coalgebraic fixed point logic which is known to have uniform interpolation [7, 14]. Potential avenues of future research include applying the above methods and systems to other modal and temporal logics with fixed points and a proof-theoretic treatment of uniform interpolation for μ-calculus, such as is carried out for modal logic in [2]. The former direction depends on sound and complete cut-free sequent calculi for different modal logics, a question which warrants further study independently.

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