On the Expressiveness of Büchi Arithmetic

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Definition

The $\exists^*$ fragment (or $\forall^*, \exists^*\forall^*$...) of logical theory is *expressively complete* if and only if for every first-order logic formula ϕ of the logic there exists a formula ψ with quantifier prefix $\exists^*$ (or $\forall^*, \exists^*\forall^*$...) such that $\phi(\boldsymbol{x}) \iff \psi(\boldsymbol{x})$.

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Theorem

If L is a regular language of polynomial growth, then there exists an $\exists^*$ formula of Büchi arithmetic ϕ such that $\phi(x) \iff x \in L$.

Density function

Let $L \subset \mathbb{N}$. We introduce a function $d_L(k) := \#\{L \cap [p^{k-1}, p^k)\}$.

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Corollary

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Proof. For the language $L = (01|10)^*$ we have $d_L(k) \approx 2^{\frac{k}{2}}$. Thus, for any polynomial q and constant c for big enough k we have $q(k) \leq d_L(k) \leq c \cdot p^k$.

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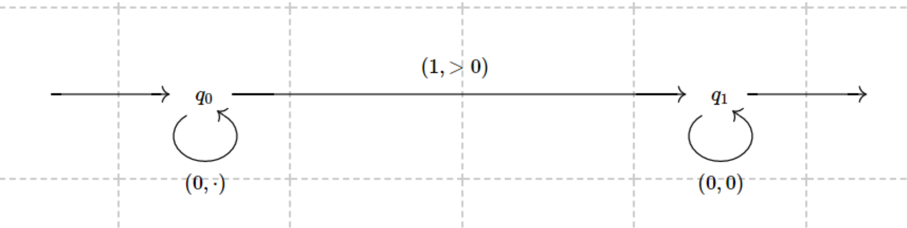

Automaton for linear equations

Fix $S : \boldsymbol{A} \cdot \boldsymbol{x} = \boldsymbol{c}$. Define $A(S) = (Q, \Sigma_p^d, \delta, \boldsymbol{q}_0, F)$ such that:

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- $Q = \mathbb{Z}^d$
- $\boldsymbol{q}_0 = 0$
- $\delta(\boldsymbol{q}, \boldsymbol{u}) = p \cdot \boldsymbol{q} + \boldsymbol{A} \cdot \boldsymbol{u}$
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Fact

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Now we define an automaton for a linear equation with valuation constraints as a product automaton of $A(S)$ and automata for V_p.

Counting loops

Definition

Let A be an automaton, $q \in Q(A)$ - a state and x - a variable. We define a counting function:

$$C_{q,x}(n) = \#\{\pi_x(w) : q \xrightarrow{w} q, w \in (\Sigma_p^d)^n\}$$

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Let $\phi_t(\boldsymbol{x})$ be a formula of Parametric Presburger Arithmetic. Then $\#\{\boldsymbol{x} : \phi_t(\boldsymbol{x})\}$ is an eventual quasi-polynomial.

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Lemma

Let S be a system of linear equations with valuation constraints with the associated DFA $A(S)$. Then either:

(i) $d_L(n) \geq c \cdot p^n$ for some fixed $c > 0$ and infinitely many $n \in \mathbb{N}$.

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- Can we classify all languages of existential Büchi arithmetic using the density function?
- Is it decidable to say whether a language is in existential Büchi arithmetic?
- Can we approach the generalized starheight problem using the languages that are in $\exists^*\forall^* \setminus \exists^*$ fragment of Büchi arithmetic?

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