JOURNAL OF COMBINATORIAL THEORY, Series B 36, 49–64 (1984)

Graph Minors. III. Planar Tree-Width

NEIL ROBERTSON AND P. D. SEYMOUR*

Department of Mathematics, Ohio State University, 231 West 18th Avenue, Columbus, Ohio 43210

Communicated by the Editors

Received January 14, 1983

The "tree-width" of a graph is defined and it is proved that for any fixed planar graph H, every planar graph with sufficiently large tree-width has a minor isomorphic to H. This result has several applications which are described in other papers in this series.

1. INTRODUCTION

All graphs in this paper are finite, and may have loops or multiple edges. $V(G)$ and $E(G)$ denote the sets of vertices and edges of the graph G, respectively. A *tree-decomposition* of a graph G is a pair $(T, \mathscr{X})$, where T is a tree and $\mathscr{X} = (X_t : t \in V(T))$ is a family of subsets of $V(G)$, with the following properties:

(W1) $\bigcup(X_t : t \in V(T)) = V(G)$.

(W2) For every edge e of G there exists $t \in V(T)$ such that e has both ends in X_t.

(W3) For $t, t', t'' \in V(T)$, if t' is on the path of T between t and t'' then

$$X_t \cap X_{t''} \subseteq X_{t'}.$$

The *width* of the tree-decomposition is

$$\max_{t \in V(T)} (|X_t| - 1).$$

The graph G has *tree-width* w if w is minimum such that G has a tree-decomposition of width w.

Thus, for example, trees and forests have tree-width 1 (or 0 in degenerate

* Research partially supported by NSF Grant MCS 8103440.

0095-8956/84 $3.00

cases) and series-parallel graphs have tree-width $\leqslant 2$. For $n \geqslant 1$, the complete graph K_n has tree-width $n-1$, while for $n \geqslant 2$ the $n \times n$ grid (this has n^2 vertices—it is the adjacency graph of the chessboard with n^2 squares) can be shown to have tree-width n.

A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contraction. If $H \neq G$ it is a *proper* minor. Other papers in this series prove theorems concerning minors of graphs chosen from a set of graphs with bounded tree-width. For example,

(1.1) [3]. *If $\mathscr{F}$ is an infinite set of graphs all with tree-width $\leqslant w$, where w is some integer, then there exist $G, G' \in \mathscr{F}$ such that G is isomorphic to a minor of G'.*

(1.2) [2]. *If $\mathscr{F}$ is a (possibly infinite) set of graphs, all with tree-width $\leqslant w$, where w is some integer, then there is a polynomial algorithm which, given an arbitrary graph G, tests if there is a graph $G' \in \mathscr{F}$ such that G is isomorphic to a minor of G'.*

The purpose of this paper is to provide a useful source of sets of graphs with bounded tree-width. The set of all planar graphs is not such a set, because it contains all the grids. However, our main result may be expressed as follows.

(1.3) *Any proper subset of the set of all planar graphs which is closed under isomorphism and the taking of minors is a set of graphs with bounded tree-width.*

This may be reformulated in the following more convenient form.

(1.4) *For every planar graph H, there is a number w such that every planar graph with no minor isomorphic to H has tree-width $\leqslant w$.*

(1.1) and (1.4) together imply that planar graphs are "well-quasi-ordered" by the taking of minors; more precisely,

(1.5) *If $\mathscr{F}$ is an infinite set of planar graphs, then there exist distinct G, $G' \in \mathscr{F}$ such that G is isomorphic to a minor of G'.*

Proof. Choose $H \in \mathscr{F}$, and choose a number w as in (1.4). If there exists $G \in \mathscr{F} - \{H\}$ such that H is isomorphic to a minor of G then (1.5) is true. If not, then by our choice of w, each $G \in \mathscr{F} - \{H\}$ has tree-width $\leqslant w$, and the result follows from (1.1).

We remark that in [4] we shall prove that (1.4) is true with the second occurrence of "planar" deleted. We are publishing the weaker version of

(1.4) separately here because it is much easier to prove than the result of [4], and our proof here yields a very much better estimate of the value of w than that in [4].

The proof of (1.4) occupies the whole paper (except for some remarks at the end) and we begin by sketching it. It is enough to prove (1.4) when H is a "cylinder" (a grid with one pair of opposite sides identified) because every planar graph is isomorphic to a minor of some sufficiently large cylinder. We introduce the "sleeve union" of two graphs G_1, G_2, and observe that the tree-width of the resulting graph G is the maximum of the tree-widths of G_1, G_2, and that G_1, G_2 are isomorphic to minors of G. It follows that it is enough to prove that if a planar graph is not expressible as a sleeve union of two smaller planar graphs, and has no minor isomorphic to the $n \times n$ cylinder, then its tree-width is bounded by a function of n. We prove that in any plane drawing of such a graph there do not exist $n + \frac{1}{2}n^2$ vertex-disjoint circuits each inside the next, and deduce that it has tree-width $\leqslant 3(n + \frac{1}{2}n^2) - 2$. These steps are performed more or less in reverse order.

Let us clarify some terminology. A *walk* in a graph G is a sequence $v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_{k+1}$ of vertices and edges of G (where $k \geqslant 0$), such that for $1 \leqslant j \leqslant k$, the ends of e_j are v_j and v_{j+1}. The walk is *closed* if $v_1 = v_{k+1}$. A *path* is a walk such that $v_j \neq v_{j'}$ for $1 \leqslant j < j' \leqslant k+1$, and a *circuit* is a closed walk with $k \geqslant 1$ such that $v_j \neq v_{j'}$ for $1 \leqslant j < j' \leqslant k$. We often identify paths and circuits with the corresponding subgraphs of G when no confusion should arise. The *initial* and *terminal* vertices of the walk are v_1, v_{k+1}. When $u, v \in V(G)$, a path *between* u and v is a path with initial vertex u and terminal vertex v, and when $X, Y \subseteq V(G)$, a path *between* X and Y is a path with initial vertex in X and terminal vertex in Y. A path *avoids* $Z \subseteq V(G)$ if no vertex of it is in Z. When $Z \subseteq V(G)$, Z *separates* X and Y if no path of G between X and Y avoids Z. (Here X, Y may be vertices or sets of vertices.) If G_1, G_2 are subgraphs of G we denote by $G_1 \cup G_2$ the subgraph of G consisting of the vertices in $V(G_1) \cup V(G_2)$ and the edges in $E(G_1) \cup E(G_2)$. We define $G_1 \cap G_2$ similarly. If X is a vertex or an edge of G, or a set of vertices or edges, we denote by $G \backslash X$ the graph obtained by deleting X from G.

2. Radius

If G is a planar graph, it can be drawn without crossings in the plane, and by a "drawing" we shall always mean such a drawing. If M is a drawing of G and C is a circuit of G, then every region of M and every vertex and edge of G not in C is either *inside* or *outside* C in M, in the obvious sense. For every region R of a drawing M, we define $d(R)$ to be the minimum value of k such that there is a sequence $R_0, R_1, \ldots, R_k$ of regions of M, where R_0 is the infinite region, $R_k = R$, and for $1 \leqslant j \leqslant k$ there is a vertex v incident with both R_{j-1} and R_j. The *radius* $\rho(M)$ of M is the minimum value of d such that $d(R) \leqslant d$ for all regions R of M. The *radius* of a planar graph is the minimum of the radii of its drawings.

The object of this section is to prove that if a planar graph has radius d then its tree-width is at most $3d + 1$. The proof takes several steps.

(2.1) *Any planar graph of radius $\leqslant d$ is isomorphic to a minor of a simple planar graph with radius $\leqslant d$.*

Proof. We replace each edge of the graph (G, say) by three edges in series, forming a new graph G'. It is clear that G' is planar and has the same radius as G, and that G' is simple. Moreover G is isomorphic to a minor of G', as required.

Let M be a drawing of a planar graph G. By a *nested sequence of circuits of M* we mean a sequence $C_1, \ldots, C_d$ of circuits of G such that for $1 \leqslant j < j' \leqslant d$, C_j and $C_{j'}$ have no common vertices and $C_{j'}$ is inside C_j in M. By a *d-shell of M* (where $d \geqslant 0$ is an integer) we mean a nested sequence $C_1, \ldots, C_d$ of circuits of M such that

(S1) every vertex of G is either in one of $V(C_1), \ldots, V(C_d)$ or is inside C_d,

(S2) for $1 \leqslant j \leqslant d$, every edge of G which joins two vertices in $V(C_j)$ is in $E(C_j)$.

(2.2) *Let G be a simple planar graph of radius $\leqslant d$, where $d \geqslant 0$ is an integer. Then there exist a simple planar graph G' with a minor isomorphic to G and a drawing M' of G' such that $\rho(M') \leqslant d$ and M' has a d-shell.*

Proof. For $d' = 0, 1, \ldots, d$, we shall prove by induction on d' that there exist a simple planar graph G' with a minor isomorphic to G and a drawing M' of G' such that M' has a d'-shell and $\rho(M') \leqslant d$. This is clearly true when $d' = 0$. Now assume that $0 < d' \leqslant d$ and assume the inductive hypothesis that there is a planar graph G' with a drawing M' such that

(i) *G' is simple, G is isomorphic to a minor of G', and $\rho(M') \leqslant d$*

(ii) *M' has a $(d'-1)$-shell $C_1, \ldots, C_{d'-1}$.*

Let H be the graph obtained from G' by deleting the vertices in $V(C_1), \ldots, V(C_{d'-1})$ and the edges incident with them; and let N be the drawing of H obtained from M' by the same deletion.

We may assume that

(iii) *at least three vertices of H are on the infinite region of N.*

For if not, we may add isolated vertices to G' and to M' to make (iii) true without falsifying (i) or (ii).

We may also assume that

(iv) *every vertex of H is adjacent to at most one vertex of $C_{d'-1}$.*

For we may replace every edge of G' joining $V(H)$ and $V(C_{d'-1})$ by two edges in series, thereby making (iv) true without falsifying any of (i), (ii), or (iii).

Let $|V(H)| = k$, say. We may also assume that

(v) *subject to* (i)–(iv) *and to* $|V(H)| = k$, *G' is chosen with $|E(H)|$ maximum.*

(This is possible because G' is constrained to be simple and $|V(H)|$ is fixed.)

Let $C_{d'}$ be the subgraph of H consisting of the vertices and edges incident with the infinite region of N.

(1) *$C_{d'}$ is connected.*

For if not, it follows from (S2) applied to $C_{d'-1}$ that there are vertices v_1, v_2 of $C_{d'}$ in different components of $C_{d'}$ and hence of H but incident with the same region of M'. We add an edge joining them. The resulting graph still satisfies (i)–(iv), contrary to (v).

(2) *$C_{d'}$ is a circuit.*

For $C_{d'}$ has at least three vertices by (iii). If it is not a circuit, it has a vertex v such that $C_{d'}\backslash v$ is disconnected, and hence such that $H\backslash v$ is disconnected. Let D be the vertex set of one component of $C_{d'}\backslash v$, and let

$$D' = V(C_{d'}) - (D \cup \{v\}).$$

Let $v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_{k+1}(=v_1)$ be the closed walk in H which is the perimeter of the infinite region of N. There exist distinct i, j with $1 \leqslant i, j \leqslant k$, such that $v_i = v_j = v$, and $v_{i-1}, v_{j+1} \in D$, and $v_{i+1}, v_{j-1} \in D'$ (where v_0 means v_k; similarly, later, e_{k+1} will mean e_1, and e_0 will mean e_k). Now by (iv), there is at most one edge of G' incident with v which is not in $E(H)$; and so without loss of generality we assume that there is a region R of M' for which the path $v_{i-1}, e_{i-1}, v, e_i, v_{i+1}$ forms part of the perimeter, and for which (a) if $d' = 1$, R is the infinite region of M', and (b) if $d' > 1$, R is incident with a vertex in $C_{d'-1}$. We add a new edge e to G' joining v_{i-1} and v_{i+1}, and make the corresponding extension of M'. R is thus divided into two regions, one incident with v_{i-1}, e, v_{i+1} and (a) being the infinite region if $d' = 1$, and (b) being incident with a vertex of $C_{d'-1}$ if $d' > 1$, and the other a triangle bounded by $v_{i-1}, e_{i-1}, v, e_i, v_{i+1}, e$. We claim that this enlarged graph satisfies (i)–(iv), contrary to (v). Verifying this is straightforward and is left to the reader. (The only part which is not quite

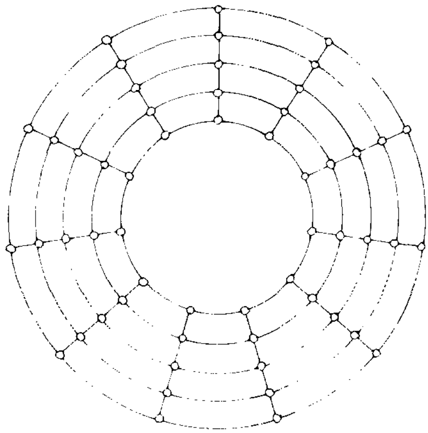

FIGURE 1

trivial is checking the radius, using the condition $d' - 1 < d$, and this too is easy.)

We replace every edge of G' which joins two vertices of $V(C_{d'})$ but which is not in $E(C_{d'})$ by two edges in series. The graph thus obtained is a simple planar graph with a minor isomorphic to G, and the drawing we obtain of it has a d'-shell and radius $\leqslant d$. This completes our inductive argument and hence proves (2.2).

When $r \geqslant 0$, $s > 0$ are integers, the $r \times s$ *cylinder* is the graph shown in Fig. 1, with r radial lines and s circles.

The "circular" circuits are called the *circles* of the cylinder, and the innermost one is called the *central circle*. When $\lambda \geqslant 1$ is an integer, the graph Y_λ is a tree as shown in Fig. 2.

Thus Y_λ has $3.2^{\lambda-1}$ end-vertices. For $d, \lambda \geqslant 1$, take the $3.2^{\lambda-1} \times d$ cylinder, and let $C_1, \ldots, C_d$ be its circles, in order, where C_d is the central circle. Identify the vertices of the central circle with the end-vertices of Y_λ, in the cyclic order provided by the drawing of Fig. 2. The resulting graph we call $N(d, \lambda)$. (See Fig. 3.) We also set $N(0, \lambda) = Y_\lambda$.

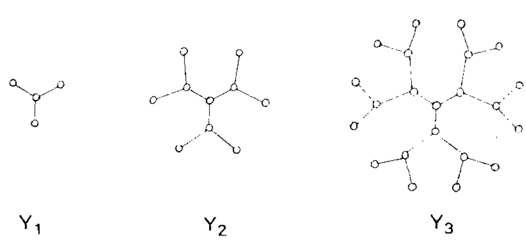

FIGURE 2

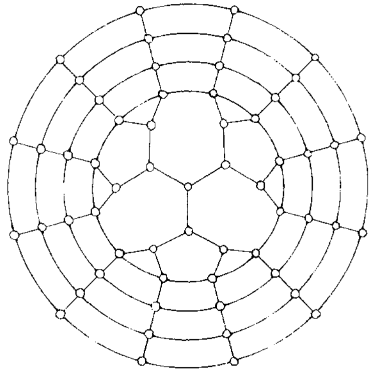

FIGURE 3

(2.3) *Let G be a simple planar graph, with a drawing M such that $\rho(M) \leqslant d$ and M has a d-shell $C_1, \ldots, C_d$. Then G is isomorphic to a minor of $N(d, \lambda)$ for some sufficiently large λ.*

Proof. We observe that the graph obtained from G by deleting $V(C_1), \ldots, V(C_d)$ is a forest, since if the corresponding drawing had a finite region R, we would have $d(R) > d$ in M which is impossible. But any forest is isomorphic to a minor of one of Y_1, Y_2, $Y_3, \ldots$. The rest of the proof is obvious, although tedious to write out, and we omit it.

(2.4) *If G is a planar graph with radius $\leqslant d$ then G is isomorphic to a minor of $N(d, \lambda)$ for some sufficiently large λ.*

This follows from (2.1), (2.2), and (2.3).

(2.5) *For all $d \geqslant 0$, $\lambda \geqslant 1$, the graph $N(d, \lambda)$ has tree-width $\leqslant 3d + 1$.*

Proof. If $d = 0$ the result is clear, and we assume $d > 0$. We construct a tree-decomposition of $N(d, \lambda)$ as follows. Let T be the tree obtained from $N(d, \lambda)$ by deleting the $3.2^{\lambda-1} \times d$ cylinder, and let C be the central circle of the cylinder. For each vertex v of C, let P_v be the set of all vertices of the cylinder on the same radial line as v. Thus $|P_v| = d$. Let t be a vertex of T. Now t is on precisely three regions R_1, R_2, R_3 say (taking a drawing M of $N(d, \lambda)$ as in Fig. 3). For $j = 1, 2, 3$, there are two vertices of C on R_j and they are adjacent. Let them be u_j, v_j, where v_j follows u_j in the anticlockwise orientation of C. Define

$$X_t = P_{v_1} \cup P_{v_2} \cup P_{v_3} \cup \{t, t'\},$$

where $t' = t$ if t is the "center" of T, and t' is the neighbour of t closer to the center if t' is not the center of T. It is easy to verify that $(T, \mathscr{X})$ is a tree-decomposition of $N(d, \lambda)$ of width $\leqslant 3d+1$, where $\mathscr{X} = (X_t : t \in V(T))$. This completes the proof.

Finally, we need

(2.6) *If G' is isomorphic to a minor of G then the tree-width of G' is not greater than the tree-width of G.*

The proof is clear. From (2.4), (2.5), and (2.6) we deduce

(2.7) *If G is planar and has radius $\leqslant d$ then its tree-width is at most $3d+1$.*

This completes the proof of the main result of this section. We shall also need the following lemma, which is easy and is left to the reader.

(2.8) *Let G be a planar graph with radius $\geqslant d$, and let M be a drawing of G. Then there is a nested sequence $C_1, C_2, \ldots, C_d$ of circuits of M.*

3. Sleeve Unions

We shall require some results about tree-decompositions which are proved in [2], but which we prove again here for the reader's convenience.

Let $(T, \mathscr{X})$ be a tree-decomposition of G, where $\mathscr{X} = (X_t : t \in V(T))$. For each $t \in V(T)$, the connected components of $T \backslash t$ are called the *branches* of T at t.

(3.1) *For $t \in V(T)$ and $v \in V(G)$, either $v \in X_t$, or there is a branch of T at t which contains all $t' \in V(T)$ with $v \in X_{t'}$.*

This follows immediately from (W1), (W3). If $v \notin X_t$, let $T_t(v)$ denote this branch. (It is unique.)

(3.2) *If $v, v' \notin X_t$ and v, v' are adjacent in G then $T_t(v) = T_t(v')$.*

Proof. By (W2), there exists $t' \in V(T)$ such that $v, v' \in X_{t'}$. Then $t' \neq t$, and so t' is in some branch B of T at t. But then $B = T_t(v)$, because $v \in X_{t'}$, and similarly $B = T_t(v')$.

(3.3) *If $v, v' \notin X_t$ and v, v' are not separated in G by X_t, then $T_t(v) = T_t(v')$.*

Proof. Let $v_1, v_2, \ldots, v_k$ be a sequence of vertices of G not in X_t, each adjacent to the next, with $v_1 = v$ and $v_k = v'$. Then by (3.2),

$$T_t(v_1) = T_t(v_2) = \cdots = T_t(v_k)$$

and so $T_t(v) = T_t(v')$.

(3.4) *Let e be an edge of T with ends t, t' say, and let N, N' be the vertex sets of the two components of $T\backslash e$. Then $X_t \cap X_{t'}$ separates $\bigcup(X_n : n \in N)$ and $\bigcup(X_n : n \in N')$.*

Proof. Suppose not. Then from (W1), there exist

$$v, v' \in V(G) - (X_t \cap X_{t'})$$

with $v \in X_n$ and $v' \in X_{n'}$ say, where $n \in N$ and $n' \in N'$, such that either $v = v'$ or v, v' are adjacent in G. From (W1), (W2) there exists $t'' \in V(T)$ such that $v, v' \in X_{t''}$. We assume without loss of generality that $t'' \in N'$. Then t, t' are both on the path of T between n and t'', and so $X_n \cap X_{t''} \subseteq X_t$, $X_{t'}$ by (W3). Hence $v \in X_t \cap X_{t'}$, a contradiction, as required.

(3.5) *Let $|V(T)| \geqslant 2$, and for each $t \in V(T)$ let G_t be a connected subgraph of G with $V(G_t) \cap X_t = \varnothing$. Then there exist t, $t' \in V(T)$, adjacent in T, such that $X_t \cap X_{t'}$ separates $V(G_t)$ and $V(G_{t'})$ in G.*

Proof. For each $t \in V(T)$, let B_t be a branch of T at t such that $T_t(v) = B_t$ for all $v \in V(G_t)$. This is possible by (3.3). (We use the fact that $|V(T)| \geqslant 2$ to ensure that T has a branch at t.) Let e_t be the edge of T joining t to a vertex of B_t. Now T has fewer edges than vertices, and so there exist distinct t, $t' \in V(T)$ with $e_t = e_{t'}$. Then t, t' are adjacent in T. Let N, N' be the vertex sets of the two components of $T\backslash e_t$, with $t \in N$, $t' \in N'$. Then $N = V(B_{t'})$, $N' = V(B_t)$, and

$$V(G_t) \subseteq \bigcup (X_n : n \in N'), \qquad V(G_{t'}) \subseteq \bigcup (X_n : n \in N)$$

and the result follows from (3.4).

Our object in this section is to introduce the "sleeve union" of two graphs, and to prove that the tree-width of the union is the maximum of the tree-widths of the two pieces. We need some further lemmas.

Let S be the $r \times s$ cylinder, where $r \leqslant 2s$, and let $C_1, \ldots, C_s$ be its circles in order, where C_s is the central circle.

(3.6) *Suppose $X \subseteq V(S)$ and $|X \cap V(C_j)| \geqslant 2$ $(1 \leqslant j \leqslant s)$. Then there are r vertex-disjoint paths of S, each between X and $V(C_s)$.*

Proof. By Menger's theorem, it is enough to show that if $Y \subseteq V(S)$ and $|Y| < r$, there is a path of S between X and $V(C_s)$ which avoids Y. Thus, let $Y \subseteq V(S)$, with $|Y| < r$. Now since $r \leqslant 2s$, there exists j with $1 \leqslant j \leqslant s$ such that $|V(C_j) \cap Y| \leqslant 1$. Choose $v \in V(C_j)$ such that $V(C_j) \cap Y \subseteq \{v\}$. Then $V(C_j) - \{v\}$ contains no vertex of Y. However, it does contain a vertex of X, since $|V(C_j) \cap X| \geqslant 2$. Let L be the set of vertices in a radial line of S such that $L \cap Y = \varnothing$. (This is possible since S has r radial lines and $|Y| < r$.)

Then $L \cup (V(C_j) - \{v\})$ is a subset of $V(S) - Y$, and induces a subgraph of S which contains a path of S between X and C_s avoiding Y, as required.

(3.7) *Let* $(T, \mathscr{X})$ *be a tree-decomposition of* S, *where* $\mathscr{X} = (X_t : t \in V(T))$. *Then there exist* $t_0 \in V(T)$ *and* r *vertex-disjoint paths of* S, *each between* X_{t_0} *and* C_s.

Proof. Let the vertex sets of the radial lines of S be $L_1, ..., L_r$. If some $t \in V(T)$ has $X_t \cap L_i \neq \emptyset$ for all i $(1 \leqslant i \leqslant r)$, the theorem is obviously true. We assume then that for each $t \in V(T)$ there exists i with $1 \leqslant i \leqslant r$ such that $X_t \cap L_i = \emptyset$. From (3.5) there exist t, $t' \in V(T)$, adjacent in T, such that $X_t \cap X_{t'}$ separates L_i and $L_{i'}$ for some i, i' with $1 \leqslant i$, $i' \leqslant r$, and $X_t \cap X_{t'}$ intersects neither of L_i, $L_{i'}$. Clearly $i \neq i'$, and for $1 \leqslant j \leqslant k$ the two paths in C_j from L_i to $L_{i'}$ both meet $X_t \cap X_{t'}$, and so $|X_t \cap V(C_j)| \geqslant 2$ $(1 \leqslant j \leqslant s)$. The result follows from (3.6).

(3.8) *With* S, $C_1, ..., C_s$ *as before, let* $V(C_s) = \{u_1, ..., u_r\}$. *Let* H *be another graph, vertex-disjoint from* S, *and let* $v_1, ..., v_r$ *be distinct vertices of* H. *Construct* G *by making the identifications* $u_1 = v_1, ..., u_r = v_r$. *If* G *has tree-width* $\leqslant w$, *then* H *has a tree-decomposition* $(T, \mathscr{X})$ *where* $\mathscr{X} = (X_t : t \in V(T))$ *say, of width* $\leqslant w$, *such that* $\{v_1, ..., v_r\} \subseteq X_{t_0}$ *for some* $t_0 \in V(T)$.

Proof. Let $(T, \mathscr{X}_1)$ be a tree-decomposition of G of width $\leqslant w$, where $\mathscr{X}_1 = (Y_t : t \in V(T))$ say. Define

$$\mathscr{X}_2 = (Y_t \cap V(S) : t \in V(T)).$$

Clearly $(T, \mathscr{X}_2)$ is a tree-decomposition of S, and so by (3.7) there exists $t_0 \in V(T)$ and vertex-disjoint paths $P_1, ..., P_r$ of S, each between $Y_{t_0} \cap V(S)$ and $\{u_1, ..., u_r\}$. Order these so that for $1 \leqslant i \leqslant r$, u_i is the terminal vertex of P_i. For $t \in V(T)$, define

$$X_t = (Y_t \cap V(H)) \cup \{v_i : 1 \leqslant i \leqslant r, Y_t \cap V(P_i) \neq \emptyset\}.$$

Put $\mathscr{X} = (X_t : t \in V(T))$. We claim that $(T, \mathscr{X})$ is a tree-decomposition of H. For suppose that t, t', $t'' \in V(T)$, and t' lies on the path of T between t and t'', and suppose that $v \in X_t$, $X_{t''}$. We must show that $v \in X_{t'}$. Now if $v \neq v_1, ..., v_r$ then $v \in Y_t \cap Y_{t''} \subseteq Y_{t'}$, and so $v \in X_{t'}$. If $v = v_i$ say, then $Y_t \cap V(P_i) \neq \emptyset$ and $Y_{t''} \cap V(P_i) \neq \emptyset$. But $Y_{t'}$ separates Y_t and $Y_{t''}$ in G by (3.3), and so $Y_{t'} \cap V(P_i) \neq \emptyset$ and again $v \in X_{t'}$. Thus $(T, \mathscr{X})$ is a tree-decomposition of H. Clearly it has width $\leqslant w$, and

$$\{v_1, ..., v_r\} \subseteq X_{t_0},$$

as required.

If H is a minor of G then each vertex u of H is formed by taking a nonempty subset $Z_u \subseteq V(G)$ and identifying the vertices in Z_u by contraction. If $v \in Z_u$ we write $v \to u$.

A *separation* of a graph G is a pair (H_1, H_2), where H_1, H_2 are subgraphs of G such that

$$H_1 \cup H_2 = G, \qquad E(H_1) \cap E(H_2) = \emptyset.$$

Let (H_1, H_2) be a separation of G. Let

$$V(H_1) \cap V(H_2) = \{v_1, ..., v_r\},$$

say. Let s be the least integer such that $2s \geqslant r$. Let S_1, S_2 be graphs, both isomorphic to the $r \times s$ cylinder, and let the vertex set of the central circles of S_1, S_2 be $\{u_1^1, ..., u_r^1\}$ and $\{u_1^2, ..., u_r^2\}$, respectively, in order. Suppose that S_1 is a proper minor of H_2, where $v_1 \to u_1^1, ..., v_r \to u_r^1$, and S_2 is a proper minor of H_1 where $v_1 \to u_1^2, ..., v_r \to u_r^2$.

Let G_k be the graph obtained from H_k, S_k by making the identifications $v_i = u_i^k$ $(1 \leqslant i \leqslant r)$ $(k = 1, 2)$. We say that G is the *sleeve union* of G_1, G_2.

(3.9) *If G is the sleeve union of G_1, G_2 then G_1, G_2 are both isomorphic to proper minors of G.*

This is clear from the construction.

(3.10) *If G is the sleeve union of G_1, G_2 then the tree-width of G is the maximum of the tree-widths of G_1, G_2.*

Proof. Let w be the maximum of the tree-widths of G_1, G_2. By (3.9) and (2.6), the tree-width of G is at least w. We must prove the reverse inequality. G_1 has tree-width $\leqslant w$, and so by (3.8), H_1 has a tree-decomposition $(T_1, \mathscr{X}_1)$ of width $\leqslant w$, where $\mathscr{X}_1 = (X_t : t \in V(T_1))$ say, such that $\{v_1, ..., v_r\} \subseteq X_{t_1}$ for some $t_1 \in V(T_1)$. Define T_2, $\mathscr{X}_2$, t_2 similarly for H_2, arranged so that T_1, T_2 are disjoint. Let T be the tree constructed from T_1, T_2 by adding an edge joining t_1, t_2. Put $\mathscr{X} = (X_t : t \in V(T))$. It is easy to see that $(T, \mathscr{X})$ is a tree-decomposition for G of width $\leqslant w$, as required.

4. The Production of Cylinders

Let $r \geqslant 0$, $s \geqslant 1$ be integers, and let M be a drawing of a planar graph G. We say that M *majors* the $r \times s$ cylinder if there is a nested sequence $C_1, ..., C_s$ of circuits of M and vertex-disjoint paths $P_1, ..., P_r$ of G, each between $V(C_1)$ and $V(C_s)$, such that for $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant s$, $P_i \cap C_j$ is a path.

(4.1) *Let $r \geqslant 0$, $s \geqslant 1$ be integers, and let M be a drawing of the planar graph G. Suppose that there is a nested sequence $C_1,..., C_s$ of circuits of M such that there are r vertex-disjoint paths of G, each between $V(C_1)$ and $V(C_s)$. Then M majors the $r \times s$ cylinder.*

Proof. Let $C_1,..., C_s$ be a nested sequence of circuits of M, and let $P_1,..., P_r$ be vertex-disjoint paths of G, each between $V(C_1)$ and $V(C_s)$. Choose $C_1,..., C_s, P_1,..., P_r$ with

$$|E(C_1 \cup \cdots \cup C_s \cup P_1 \cup \cdots \cup P_r)| \text{ minimum.} \tag{1}$$

We shall show that for $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant s$, $P_i \cap C_j$ is a path.

Suppose that $C_1 \cap P_i$ is not a path for some i, say $i = 1$. Let e be the first edge of P_1 not in C_1. Then there is a subpath of P_1 (P_1' say) between C_1 and C_s not using e; and then e lies in none of $C_1,..., C_s, P_1', P_2,..., P_r$, contrary to (1). Thus $C_1 \cap P_i$ is a path, and similarly $C_s \cap P_i$ is a path, for each i ($1 \leqslant i \leqslant r$).

Suppose for a contradiction that there is some value of j with $1 \leqslant j \leqslant s$ such that $P_i \cap C_j$ is not a path for some i ($1 \leqslant i \leqslant r$). Choose such a value of j, as small as possible. Then by the foregoing, $2 \leqslant j \leqslant s-1$.

Suppose that for some i ($1 \leqslant i \leqslant r$) there are two distinct vertices v, v' say of $P_i \cap C_j$, such that all edges and all interior vertices of the subpath of P_i between v and v' are outside C_j. Let this subpath be P, say. Now no interior vertex of P is in $V(C_{j'})$ for any j' with $1 \leqslant j' \leqslant s$; for all are outside C_j, and none are on C_{j-1}, by the minimality of j, as is easily seen. Let the subpaths of C_j between v and v' be Q_1, Q_2. Then $P \cup Q_1$, $P \cup Q_2$ are circuits, vertex-disjoint from every $C_{j'}$ ($1 \leqslant j' \leqslant s, j' \neq j$). Now C_{j+1} is inside one of them, say $P \cup Q_1$. But then

$$C_1,..., C_{j-1}, P \cup Q_1, C_{j+1},..., C_s$$

is a nested sequence of circuits, contrary to (1), for some edge of Q_2 is not in any of $P_1,..., P_r$. Thus there is no such value of i.

It follows that

$$\text{for every vertex } v \text{ of } C_j \text{, if } v \text{ is on } P_i \text{ where } 1 \leqslant i \leqslant r \text{ then there is a subpath of } P_i \text{ between } v \text{ and } C_s \text{ which uses no edges outside } C_j. \tag{2}$$

By the choice of j there exists some i ($1 \leqslant i \leqslant r$) such that $P_i \cap C_j$ is not a path; and then there are distinct vertices v, v' of $P_i \cap C_j$ such that all edges and all interior vertices of the subpath of P_i between v and v' are inside C_j. Let this subpath be P, and let Q_1, Q_2 be the paths of C_j between v and v'. Now $P_i \cap C_s$ is a path, and so P does not neet C_s. Clearly there is no path of G between $V(Q_1)$ and $V(Q_2)$ which avoids $V(P)$ and which has no edges outside C_j. From these two facts it follows that for one of Q_1, Q_2, say Q_k, there is no path of G from $V(Q_k)$ to C_s which avoids $V(P)$ and has no edges outside C_j. But then by (2), no vertex of Q_k is on any $P_{i'}$ with $1 \leqslant i' \leqslant r$, $i' \neq i$. Let P' be the subgraph of G obtained from $P_i \cup Q_k$ by deleting the edges and interior vertices of P. Then P' contains a path between C_1 and C_s, P_i' say, and

$$P_1, ..., P_{i-1}, P_i', P_{i+1}, ..., P_r$$

are vertex-disjoint. But this contradicts (1), since the first edge of P is not in any of $C_1, ..., C_s$.

Thus the choice of j is impossible, and the theorem is proved.

(4.2) *Let $r \geqslant 0$, $s \geqslant 1$ be integers, and let M be a drawing of the planar graph G. Let $C_1, ..., C_s$ be a nested sequence of circuits of M. Then one of the following holds:*

(i) *M majors the $r \times s$ cylinder,*

(ii) *there exists $X \subseteq V(G)$ with $|X| < r$, such that X separates $V(C_1)$ and $V(C_s)$.*

Proof. If (ii) is false, then by Menger's theorem there are r vertex-disjoint paths of G between $V(C_1)$ and $V(C_s)$, and so (i) holds, from (4.1).

(4.3) *Let $r \geqslant 0$ be an integer, and let $r' = r + 1$ if r is odd, and $r' = r$ if r is even. Let $s > r'$ be an integer. Let M be a drawing of the planar graph G, and suppose that M majors the $r \times s$ cylinder. Then either M majors the $(r + 1) \times (s - r')$ cylinder, or G is expressible as the sleeve union of two planar graphs.*

Proof. Let $C_1, ..., C_s$ be a nested sequence of circuits of M, and let $P_1, ..., P_r$ be vertex-disjoint paths of G, each between $V(C_1)$ and $V(C_s)$, such that for $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant s$, $P_i \cap C_j$ is a path. Put $r' = 2d$. Now C_{d+1}, $C_{d+2}, ..., C_{s-d}$ is a nested sequence of $s - r'$ circuits of M. By (4.2) either M majors the $(r + 1) \times (s - r')$ cylinder or there exists $X \subseteq V(G)$ with $|X| \leqslant r$ which separates $V(C_{d+1})$ and $V(C_{s-d})$. We assume the latter. For $1 \leqslant i \leqslant r$, let P_i' be the subpath of P_i between $V(C_{d+1})$ and $V(C_{s-d})$, with no internal vertex in $V(C_{d+1})$ or $V(C_{s-d})$. Then $X \cap V(P_i') \neq \emptyset$; choose $v_i \in X \cap V(P_i')$ $(1 \leqslant i \leqslant r)$. We have $\{v_1, ..., v_r\} \subseteq X$ and $|X| \leqslant r$, and so $X = \{v_1, ..., v_r\}$. Let (H_1, H_2) be a separation of G with $V(H_1) \cap V(H_2) = X$, where $C_1, ..., C_{d+1}$ are circuits of H_1 and $C_{s-d}, ..., C_s$ are circuits of H_2. Now H_1 has a proper minor isomorphic to the $r \times d$ cylinder in the required way for sleeve unions, and so has H_2; thus G is expressible as a sleeve union of two planar graphs. This completes the proof.

(4.4) *Let $r \geqslant 0$, $s \geqslant 1$ be integers, and let M be a drawing of the planar graph G. Suppose that G is not expressible as a sleeve union of two planar* *graphs, and M does not major the $r \times s$ cylinder. Then G has radius $<k$, where*

$$k = s + \tfrac{1}{2}r^2 \qquad (r \text{ even})$$
$$= s + \tfrac{1}{2}(r^2 - 1) \qquad (r \text{ odd}).$$

Proof. We proceed by induction on r. If $r = 0$ the result is true by (2.8). We assume then that $r > 0$. If r is even, then by (4.3) M does not major the $(r-1) \times (s+r)$ cylinder, and hence by induction the radius of G is

$$<s + r + \tfrac{1}{2}((r-1)^2 - 1) = s + \tfrac{1}{2}r^2$$

as required. If r is odd we argue similarly.

(4.5) *Let $r \geqslant 0$, $s \geqslant 1$ be integers. If G is a planar graph which is not expressible as the sleeve union of two planar graphs, then either G has a minor isomorphic to the $r \times s$ cylinder, or it has tree-width at most*

$$3(s + \tfrac{1}{2}r^2) - 2 \qquad (r \text{ even})$$
$$3(s + \tfrac{1}{2}(r^2 - 1)) - 2 \qquad (r \text{ odd}).$$

Proof. Take a drawing M of G. If M majors the $r \times s$ cylinder then G has a minor isomorphic to the $r \times s$ cylinder. If not, then G has radius $<k$, where

$$k = s + \tfrac{1}{2}r^2 \qquad (r \text{ even})$$
$$= s + \tfrac{1}{2}(r^2 - 1) \qquad (r \text{ odd})$$

by (4.4), and the result follows from (2.7).

5. The Main Result

Now we prove (1.4). We need two more lemmas.

(5.1) *For every planar graph H there is a number $r \geqslant 1$ such that the $r \times r$ cylinder has a minor isomorphic to H.*

The reader will probably have no trouble convincing himself of this. A proof of (5.1) and of a stronger result will appear in [5].

(5.2) *Let $r \geqslant 1$ be an integer. If G is planar and has no minor isomorphic to the $r \times r$ cylinder then G has tree-width $\leqslant \tfrac{3}{2}(r^2 + 2r) - 2$.*

Proof. We proceed by induction on the size of G (that is, $|V(G)| +$

$|E(G)|$). Suppose that G is expressible as the sleeve union of two planar graphs G_1, G_2. Then G_1, G_2 are isomorphic to proper minors of G, by (3.9), and so each has no minor isomorphic to the $r \times r$ cylinder. Moreover, the sizes of G_1, G_2 are less than the size of G, and so by induction G_1, G_2 both have tree-width $\leqslant \frac{3}{2}(r^2 + 2r) - 2$. Hence by (3.10), G has tree-width $\leqslant \frac{3}{2}(r^2 + 2r) - 2$. We may therefore assume that G is not expressible as a sleeve union of two planar graphs; but then the result follows from (4.5). This completes the proof.

We deduce (1.4), which we restate.

(5.3) *For every planar graph H there is a number w such that every planar graph with no minor isomorphic to H has tree-width $\leqslant w$.*

Proof. From (5.1) there is a number $r \geqslant 1$ such that the $r \times r$ cylinder has a minor isomorphic to H. Let G be any planar graph with no minor isomorphic to H. Then G certainly has no minor isomorphic to the $r \times r$ cylinder, and so by (5.2) G has tree-width $\leqslant \frac{3}{2}(r^2 + 2r) - 2$. Thus (5.3) is true with any number $w \geqslant \frac{3}{2}(r^2 + 2r) - 2$.

6. Remarks

One natural question concerning the tree-width of planar graphs is, what is the tree-width of the geometric dual? It seems that the tree-width of a planar graph and the tree-width of its geometric dual are approximately equal—indeed, we have convinced ourselves that they differ by at most one. However, they need not be equal. The two graphs of Fig. 4 are duals; but the first has tree-width 4 while the second has tree-width 5.

Concerning our main theorem (1.4), one method of generalization is the

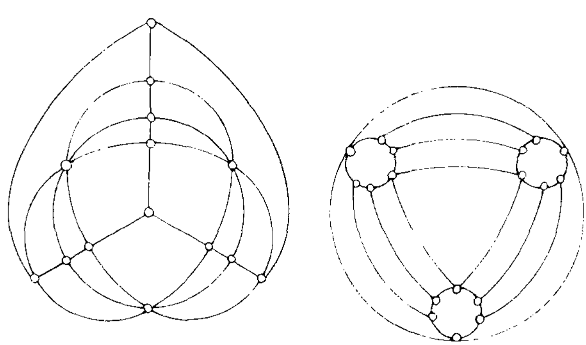

FIGURE 4

result of [4] which was described after the proof of (1.5). There is another, however, concerning surfaces of higher genus, which rather surprisingly is simpler than (1.4) itself. A proof of the following will appear in [6].

(6.1) *Let S be a surface of genus $g > 0$, and let H be a graph which may be embedded in S. Then there is a number k with the following property. For every graph G which has no minor isomorphic to H but which can be embedded in S, there exists $X \subseteq V(G)$ with $|X| \leqslant k$ such that $G \backslash X$ can be embedded on a surface of genus $<g$.*

References

1. N. Robertson and P. D. Seymour, Graph minors. I. Excluding a forest, *J. Combin. Theory Ser. B* **35** (1983), 39–61.
2. N. Robertson and P. D. Seymour, Graph minors. II. Algorithmic aspects of tree-width, submitted.
3. N. Robertson and P. D. Seymour, Graph minors. IV. Tree-width and well-quasi-ordering, submitted.
4. N. Robertson and P. D. Seymour, Graph minors. V. Excluding a planar graph, submitted.
5. N. Robertson and P. D. Seymour, Graph minors. VI. Disjoint paths on a surface, in preparation.
6. N. Robertson and P. D. Seymour, Graph minors. VII. Disjoint paths on a surface, in preparation.