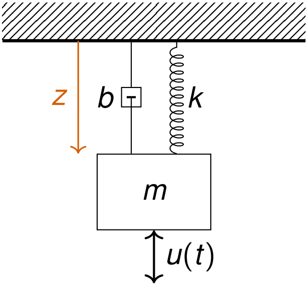
On the Decidability of Reachability in Linear Time-Invariant Systems

Nathanaël Fijalkow, Joël Ouaknine, Amaury Pouly, João Sousa-Pinto, James Worrell

CNRS, IRIF, Université Paris Diderot

16 april 2019

Example : mass-spring-damper system



Model with external input $u(t)$

$\rightarrow$ Linear time invariant system

$$X' = AX + Bu$$

with some constraints on u.

State : $X = z \in \mathbb{R}$

Equation of motion :

$$mz'' = -kz - bz' + mg + u$$

$\rightarrow$ Affine but not first order

State : $X = (z, z', 1) \in \mathbb{R}^3$

Equation of motion :

$$\begin{bmatrix} z \\ z' \\ 1 \end{bmatrix}' = \begin{bmatrix} z' \\ -\frac{k}{m}z - \frac{b}{m}z' + g + \frac{1}{m}u \\ 0 \end{bmatrix}$$

Linear dynamical systems

Discrete case

$$x(n+1) = Ax(n) + Bu(n)$$

- biology,
- software verification,
- probabilistic model checking,
- combinatorics,
- ....

Continuous case

$$x'(t) = Ax(t) + Bu(t)$$

- biology,
- physics,
- probabilistic model checking,
- electrical circuits,
- ....

Typical questions

- reachability
- safety
- controllability

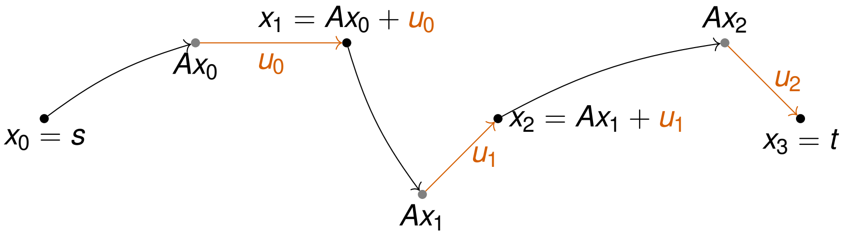
The problem

LTI-REACHABILITY

- a source $s \in \mathbb{Q}^d$,
- a target $t \in \mathbb{Q}^d$,
- a transition matrix $A \in \mathbb{Q}^{d \times d}$,
- a set of controls $U \subseteq \mathbb{R}^d$,

decide if $\exists T \in \mathbb{N}, u_0, \ldots, u_{T-1} \in U$ such that $x_T = t$ where

$$x_0 = s, \qquad x_{n+1} = Ax_n + u_n.$$



Existing work

LTI-REACHABILITY

- a source $s \in \mathbb{Q}^d$,
- a target $t \in \mathbb{Q}^d$,
- a transition matrix $A \in \mathbb{Q}^{d \times d}$,
- a set of controls $U \subseteq \mathbb{R}^d$,

decide if $\exists T \in \mathbb{N}, u_0, \ldots, u_{T-1} \in U$ such that $x_T = t$ where

$$x_0 = s, \qquad x_{n+1} = Ax_n + u_n.$$

Theorem (Lipton and Kannan, 1986)

LTI-REACHABILITY is decidable if U is an affine subspace of $\mathbb{R}^d$.

Almost no exact results for other classes of U in particular when U is bounded (which is the most natural case).

Our results : hardness

Study the impact of the control set on the hardness of reachability

Theorem

LTI−REACHABILITY is

- ***undecidable** if U is a finite union of affine subspaces.*
- ***Skolem-hard** if $U = \{0\} \cup V$ where V is an affine subspace*
- ***Positivity-hard** if U is a convex polytope*

Given $s \in \mathbb{Q}^d$ and $A \in \mathbb{Q}^{d \times d}$:

- Skolem problem : decide if $\exists T \in \mathbb{N}$ such that $(A^T s)_1 = 0$,
- Positivity problem : decide if $(A^T s)_1 \geqslant 0$ for all $T \in \mathbb{N}$.

Why is this a hardness result ?

Decidability of Skolen and Positivity has been open for 70 years !

Since we cannot solve Skolem/Positivity, we need some strong assumptions for decidability.

Our results : a positive result

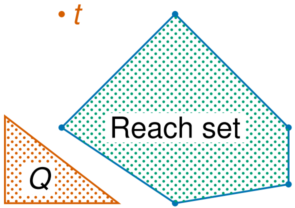
A LTI system (s, A, t, U) is simple if $s = 0$ and

- U is a bounded polytope that contains 0 in its (relative) interior,
- the spectral radius of A is less than 1 (stability),
- some positive power of A has exclusively real spectrum.

Theorem

$\mathtt{LTI\text{-}REACHABILITY}$ is decidable for simple systems.

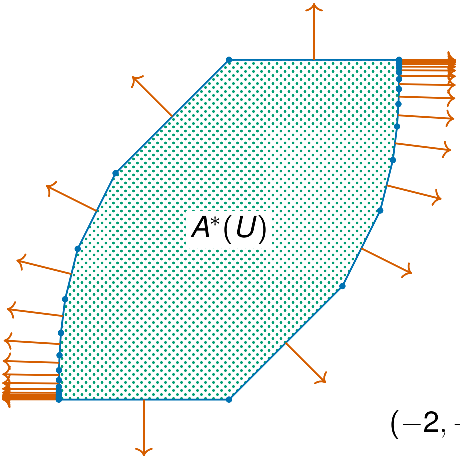
Remark : in fact we can decide reachability to a convex polytope Q.



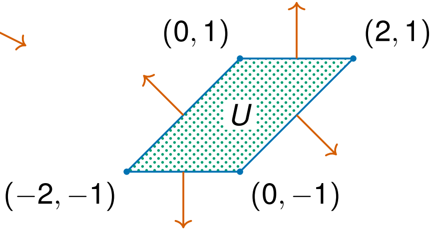
Assumptions imply that the reachable set is an open convex bounded set, but not always a polytope !

Why is this problem hard

The reachable set $A^*(U)$ can have **infinitely** many faces.

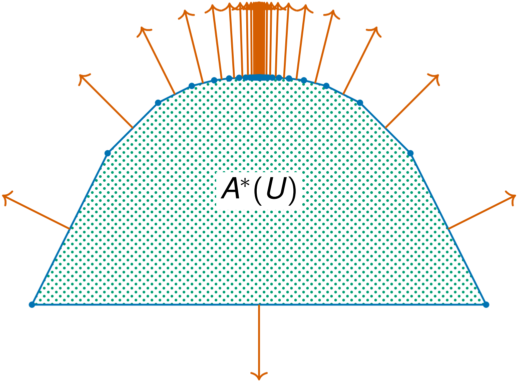


$$A = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{bmatrix}$$

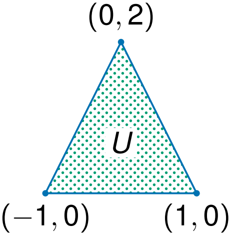


Why is this problem hard

The reachable set $A^*(U)$ can have **faces of lower dimension** : the "top" extreme point does not belong to any facet.



$$A = \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$



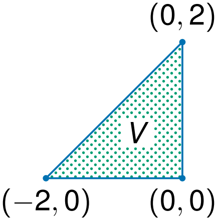
Why is this problem hard

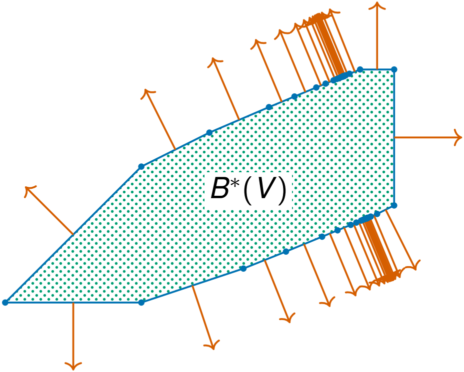
Approach : two semi-decision procedures

- reachability : under-approximations of the reachable set
- non-reachability : **separating hyperplanes**

Why is this problem hard

$$B = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 \end{bmatrix}$$





Even more difficulty : $B^*(V)$ has two extreme points that do not belong to any facet and have rational coordinates, but whose (unique) separating hyperplane requires the use of algebraic irrationals

Theorem (Non-reachable instances)

There is a separating hyperplane with algebraic coefficients.

Conclusion and future work

Exact reachability of $x_{n+1} = Ax_n + u_n$:

- decidability crucially depends on the shape of the control set
- even with convex bounded inputs, the problem is very hard (Skolem/Positivity, open for 70 years)
- we can recover decidability using strong spectral assumptions

Open questions :

- for convex bounded inputs, is it Positivity-easy ?
- weaken spectral assumptions ? Minimal difficult example :

$$A = \frac{1}{2}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad U = [0,1] \times \{0\}.$$

Decidability of $t \leqslant \sum_{n=0}^{\infty} \max(0, 2^{-n}\cos(n\theta))$ unknown.

Future work : continuous case $x'(t) = Ax(t) + u(t)$