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# A generalized permutahedron

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*Dedicated to the memory of Alan Day*

*Abstract.* With respect to a fixed $n$-element ordered set $P$, the *generalized permutahedron* **Perm**($P$) is the set of all ordered sets $P \cap L$, where $L$ is any permutation of the elements of the underlying $n$-element set. Considered as a subset of the extension lattice of an $n$-element set, **Perm**($P$) is cover-preserving. We apply this to deduce, for instance, that, in any finite ordered set $P$, there is a comparability whose removal will not increase the dimension, and there is a comparability whose addition to $P$ will not increase its dimension.

We establish further properties about the extension lattice which seem to be of independent interest, leading for example, to the characterization of those ordered sets $P$ for which this generalized permutahedron is itself a lattice.

The elements of the *extension lattice* $Ext(P)$ of an ordered set $P$ are the *extensions* of $P$, that is, all ordered sets on the same underlying set as $P$ in which $x < y$ whenever $x < y$ in $P$. Then $Ext(P)$ is itself ordered: for $Q, R \in Ext(P)$, $Q < R$ if $R$ itself is an extension of $Q$. It is convenient to adjoin a top element to $Ext(P)$, with which it now becomes a lattice ([1], [4], [10]) (see Figure 1).

An arbitrary $n$-element ordered set $P$ is an extension of the $n$-element antichain $\mathbf{A}_n$, so $P$ is an element of $Ext(\mathbf{A}_n)$. To $P$ we associate **Perm**($P$) the subset of all ordered sets in $Ext(\mathbf{A}_n)$ of the form $P \cap L$ where $L$ is any $n$-element linearly ordered set (equivalently, linear extension of $\mathbf{A}_n$ or, $n$-element permutation). **Perm**($P$) with the order induced from $Ext(\mathbf{A}_n)$ we call the *generalized permutahedron.* Our motivation to study it is twofold. In the first place it is a wonderfully concise setting for the study of the dimension of an ordered set obtained from a given one by adding or removing a comparability. In the second place, it extends the study of the so-called "weak Bruhat order" [2] which is the special case of the generalized permutahedron **Perm**($P$) in the case that $P$ itself is a linearly ordered set. The weak

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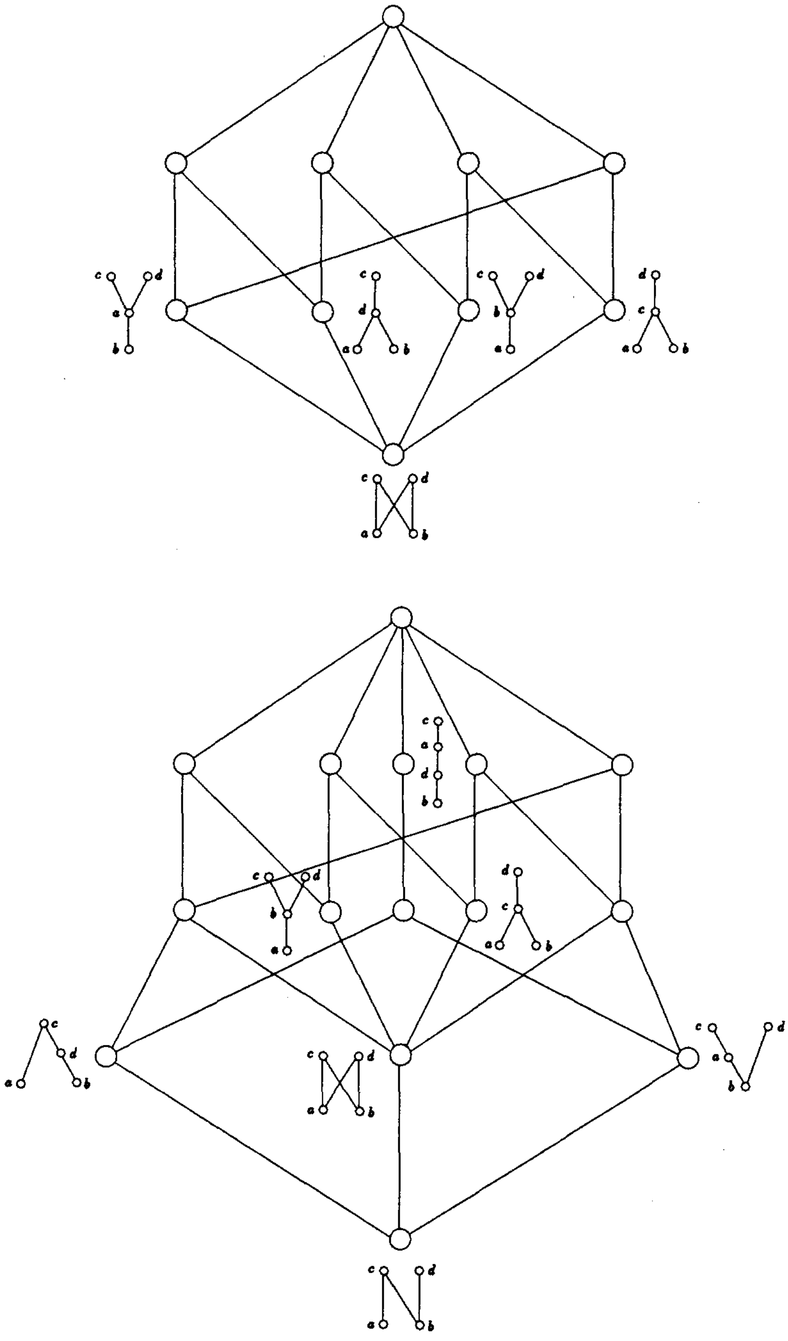


Figure 1. Examples of extension lattices.

Bruhat order is a lattice and we characterize all ordered sets $P$ such that **Perm**($P$) is a lattice.

Extension lattices are still largely unexplored although they have been occasionally studied without such explicit identification. It is known, however, that, for any linear extension $L$ of $P$, the down set of $L$ in $Ext(P)$ is *lower semimodular* (that is, if $a + b \succ b$ then $a \succ ab$) and even more, *locally distributive* (that is, for each $a$, the interval in $Ext(P)$ between $a$ and $b = \Pi_{c \prec a} c$ is isomorphic to some $\mathbf{Q}_m$, the $m$-cube ([1], [4]). Upper and lower covers of an element $Q \in Ext(P)$ also have familiar structural interpretations in terms of the ordered set $Q$ itself. Thus, $R \prec Q$ amounts precisely to removing from $Q$ the comparability corresponding to a covering relation in $Q$; on the other hand, an upper cover $R \succ Q$ amounts to adding the comparability $a < b$ corresponding to a *critical pair* $(a, b)$ of $Q$, that is, $a, b \in P$, $a$ noncomparable to $b$ in $Q$, and $x \geq a$ whenever $x \geq b$, $y \leq b$ whenever $y \leq a$.

Here is our starting point. Using (adjacent) transpositions, any permutation on a fixed set, can be transformed into any other permutation. It seems to be an item of order-theoretical folklore that any linear extension of an ordered set $P$ can be transformed into any other by a sequence of linear extensions of $P$, each obtained from the predecessor by a transposition. This idea has, in recent years, inspired several authors to study ordered sets whose linear extensions can be listed so that consecutive linear extensions differ by a transposition [6], [12], [13], [14], [11].

Our first theorem provides structural insight into this phenomenon. Let $\mathbf{m} = \{0 < 1 < 2 < \cdots < m-1\}$ the $m$-element chain, let $\mathbf{m} \times \mathbf{m}$ stand for the usual direct product of the $m$-element chain by itself, and let *halfgrid*$(\mathbf{m} \times \mathbf{m}) = \{(i, j \mid i + j \leq m\}$.

THEOREM 1. *Let $P$ be an $n$-element ordered set, let $L$, $L'$ be linear extensions of it, and let $m = |\{$noncomparabilities of $L \cap L'\}|$. Then there is a sequence $L = L_0, L_1, L_2, \ldots, L_m = L'$ of linear extensions of $P$ such that, for $0 \leq i \leq j \leq k \leq m$,*

(i) *$L_i$ is identical to $L_{i+1}$ – except for precisely one comparability – and, if $a < b$ in $L_j$ while $b < a$ in $L_i$, then $a < b$ in $L_k$,*

(ii) *the $L_i$'s generate, in $Ext(P)$, a cover-preserving meet semilattice with bottom $L \cap L'$ isomorphic to half$(\mathbf{m} \times \mathbf{m})$ and,*

(iii) *for any $n$-element ordered set $Q$, $Q \cap L_i \geq Q \cap L_k$ implies $Q \cap L_i \geq Q \cap L_j \geq Q \cap L_k$.*

*Proof.*

(i) Let $L \neq L' = \{a_1 < a_2 < \cdots a_n\}$ and let $i$ be the first index such that $a_i$ differs from the $i$th element in $L$. Thus, $a_i \not\succ a_{i-1}$ in $L = L_0$. The lower cover $b$, say, in $L_0$, of $a_i$, is noncomparable to $a_i$ in $P$ for, if $a_i > b$ in $P$, then in $L'$, too, that is $a_{i-1} \geq b$ in $L'$ which, according to the choice of $a_{i-1}$,

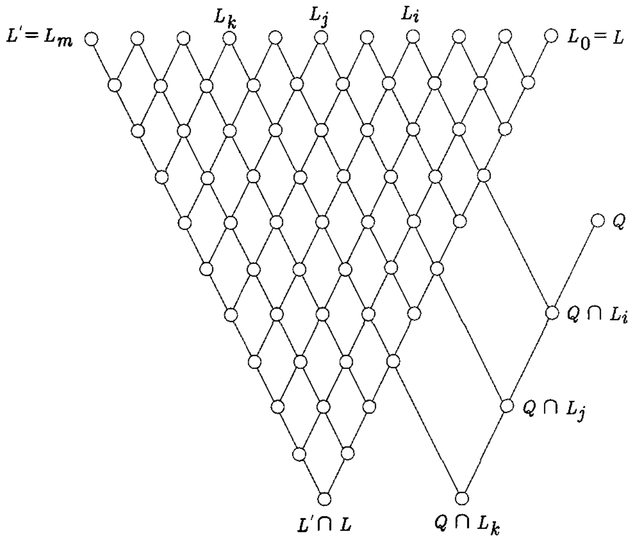



Figure 2. A schematic illustration of Theorem 1.

implies $a_i \geq b$ in $L_0$, a contradiction. Then $L_1$, constructed from $L_0$ by interchanging just the consecutive pair $a_i \succ b$ in $L_0$ to $b \succ a_i$, produces a linear extension of $P$ which is "closer" to $L'$, in the sense that $L_1$ and $L'$ now share the comparability $a_i < b$, as well. An induction completes this argument.

(ii) Notice that consecutive $L_i$'s intersect to produce an extension $K_i = L_i \cap L_{i+1}$ with precisely one noncomparability and that, according to the construction scheme, for $i \neq j$, $K_i \neq K_j$. Moreover, for $i \leq j \leq k$, $L_i \cap L_j \cap L_k = L_i \cap L_k$ and $depth(L_i \cap L_k) = k - i$, where $depth(Q) = |\{\text{non-comparabilities of } Q\}|$.

(iii) As $Q \cap L_k \leq L_i$ and $Q \cap L_k \leq L_k$ it follows from (ii) that $Q \cap L_k \leq L_j$ so $Q \cap L_k \leq Q \cap L_j$. Suppose that $Q \cap L_j \nleq Q \cap L_i$. Then there is $a < b$ in $Q \cap L_j$ while $a \nless b$ in $Q \cap L_i$. Then $a < b$ in $L_j$ while $b < a$ in $L_i$. From (i), $a < b$ in $L_k$. Then $a < b$ in $Q \cap L_k$ whence $a < b$ in $Q \cap L_i$, a contradiction. This completes the proof.

Our next result provides structural insight about the extension lattice. Actually it generalizes a result about the permutahedron **Perm**($P$) (cf. [7]) in the case that $P$ is itself a linear extension.

THEREOM 2. *For any ordered set $P$,* **Perm**($P$) *is cover-preserving in $Ext(\mathbf{A}_n)$.*

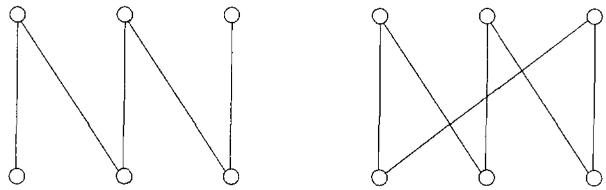


Figure 3. An ordered set whose dimension (two) can increase by adding a comparability.

*Proof.* Let $P \cap L = R > R' = P \cap L'$, $L$, $L'$ linear extensions of $\mathbf{A}_n$. According to Theorem 1(i) there is an $m$ and a sequence $L = L_1, L_2, \ldots, L_m = L'$ of linear extensions of $\mathbf{A}_n$ which, with its consecutive intersections, forms a cover-preserving zigzag joining $L$ to $L'$. Let $i$ be the least index such that $L_i \not\geq R$. From Theorem 1(iii), for each $j = 1, 2, \ldots, m$, $R = P \cap L_1 \geq P \cap L_j \geq P \cap L_m = R'$. As $L_i \cap L_{i-1} \prec L_{i-1}$ and $P \cap L_{i-1} \nleq P \cap L_i \cap L_{i-1}$, it follows from semimodularity, that $R = P \cap L_{i-1} \succ P \cap (L_{i-1} \cap L_i) = P \cap L_i \geq R'$.

It is possible, too, to give a direct proof of Theorem 2, independent of Theorem 1.

It is a longstanding and well known problem in ordered sets whether any ordered set (with at least three elements) contains a pair of elements whose removal decreases the (order) dimension by at most one? ([8]). There are also numerous "removal" theorems according to which distinguished isomorphic types are removed without lowering the dimension (too much) (cf. [9]).

Recall too, that dimension is monotone with respect to removing elements. On the contrary, this is not the case for the removal of comparabilities; the study of such effects is quite recent. It is known, for instance, that adding or removing a single comparability changes the dimension by at most one [10], and, indeed, there are examples in which the dimension increases, by the addition or removal of a single comparability (see Figure 3 and Figure 4). Nonetheless, there is always, as we shall see, a single comparability whose addition or removal does not increase the dimension. Although it is this result that has inspired this paper, our original proof establishes it in terms of the extension lattice.

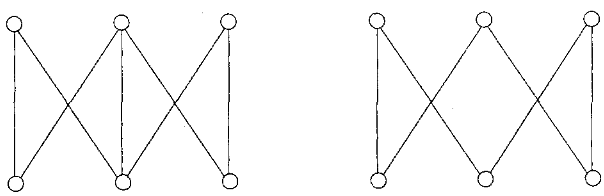


Figure 4. An ordered set whose dimension (two) can increase by removing a comparability.

The result which actually motivated this note is now a straightforward deduction.

COROLLARY 3.
(i) *Any finite ordered set (not an antichain or a chain), has a comparability whose removal does not increase its dimension and,*
(ii) *to any finite ordered set (not a chain) it is possible to add a comparability without increasing its dimension.*

*Proof.* Let $Q$ be an $n$-element ordered set and $Q = \bigcap_{i\in I} L_{Q,i}$ be an irredundant representation of $P$ by linear extensions $L_{Q,i}$ of $Q$, and let $P = \bigcap_{i\in I, i\neq 1} L_{Q,i}$.

(i) $Q$ has the form $P \cap L$, $L$ a linear extension of $\mathbf{A}_n$ and $\mathbf{A}_n$, too, has the form $P \cap L^d$, where $L^d$ is the dual of $L$. As $Q > \mathbf{A}_n$, Theorem 2 implies that there is a lower cover $Q' = P \cap L'$ of $Q$, where $L'$ is some linear extension of $\mathbf{A}_n$. It follows that $dim(Q') \leq dim(Q)$.

(ii) Theorem 2 implies that there is an upper cover $Q' \leq P$ of $Q$ of the form $P \cap L'$.

Let $L = \{a_1 < a_2 < \cdots < a_n\}$ be a linear extension of an ordered set $P$. A consecutive pair $a_i < a_{i+1}$ of $L$ is a *jump* if $a_i \not< a_{i+1}$ in $P$. In Theorem 1(i), any pair of consecutive linear extensions differs precisely by a jump; this jump is "interchanged" to produce the companion linear extension. As a matter of fact, the comparability added to produce an ordered set whose dimension does not increase, is found by locating a jump in a linear extension $L_j$, chosen from a minimal family $(L_j)$ of linear extensions with $P = \bigcap_j L_j$. On the other hand, a consecutive pair $a_i < a_{i+1}$ in $L$ such that $a_i < a_{i+1}$ in $P$, too, is called a *climb*; the comparability removed to produce an ordered set whose dimension does not increase is found, loosely speaking, by "interchanging" a climb in a linear extension $L_j$ chosen from among the $L_j$'s with $P = \bigcap_j L_j$. In fact, as W. T. Trotter has pointed out to us, it is enough to choose, in any $L_j$, any comparability of $P$, whose distance in $L_j$ is minimum.

All of these remarks can be computationally cast to efficiently construct the sought after extensions.

Although it seems possible, then, to provide direct proofs to this Corollary, without an apparently explicit reference to the extension lattice, our results about the extension lattice seem to have further consequences and, therefore, they seem to be of independent interest. W. Kern, for instance, has suggested to us another argument which opens with an order embedding of an ordered set $P$ into a direct product of $dim(P)$ chains such that no two elements of $P$ have any identical projections. Then "shifting" a prescribed element until it acquires just one new comparability and without altering any other comparabilities, produces the required extension.

An immediate consequence of Corollary 3 is this.

COROLLARY 4. *For a finite ordered set $P$, the number of upper covers of $P$, in its extension lattice, each with dimension at most $dim(P)$, is at least $dim(P)$.*

The next result answers positively, a conjecture posed in [10].

COROLLARY 5. *For any $n$-element ordered set $P$ (not a chain) there is, in $Ext(\mathbf{A}_n)$, a maximal chain $\mathbf{A}_n = P_{-r} \prec P_{-r+1} \prec \cdots \prec P_0 = P \prec P_1 \prec P_2 \prec \cdots \prec P_s \prec top$ such that $dim(P_i) \leq dim(P_{i+1})$ for $i = -r, -r+1, \ldots, -1, 0$ and $dim(P_{j+1}) \leq dim(P_j)$ for $j = 0, 1, 2, \ldots, s$.*

A result similar to Corollary 5 can be established for the companion parameter *jump number* $jump(P)$ of $P$, that is, $jump(P) = min\{jump(P, L) \mid L$ a linear extension of $P\}$, where $jump(P, L)$ stands for the number of jumps of $L$ with respect to $P$. Our next result answers positively a conjecture posed in [3].

PROPOSITION 6. *In the extension lattice of any finite ordered set $P$ there is a maximal chain $P = P_0 \prec P_1 \prec P_2 \prec \cdots \prec P_s \prec$ top such that $jump(P_i) \leq jump(P_{i-1})$, for each $i = 1, 2, \ldots, s$.*

*Proof.* Let $L$ be a "jump optimal" linear extension of $P$, that is, $jump(P) = jump(P, L)$. Choose a critical pair $(a, b)$ of $P$ such that $a < b$ in $L$ (for instance, a pair $(a, b)$ such that $a$ noncomparable to $b$ in $P$, $a < b$ in $L$, and which maximizes the size of the segment $a \leq x \leq b$ in $L$). Let $P_1$ be the extension of $P$ constructed by adding the comparability $a < b$. Then $P_1 \succ P$ and, as $L$ is still a linear extension of $P_1$, we conclude that $jump(P) = jump(P, L) \geq jump(P_1, L) \geq jump(P_1)$.

It is straightforward to check that **Perm**($P$) is not a lattice if $P$ is isomorphic either to a four-element crown or to $\mathbf{2}^2$ the direct product of the 2-element chain by itself (cf. Figure 1). Our next result establishes just when **Perm**($P$) is a lattice: namely, the covering graph of $P$ is a tree and, both the down set and the up set of any element of $P$ must be a chain. The result is an extension of the much older result in the case that $P$ is a linearly ordered set [7].

THEOREM 7. *For an ordered set $P$,* **Perm**($P$) *is a lattice if and only if $P$ contains no subset isomorphic to a crown and to* $\mathbf{2}^2$.

Before we turn to its proof it is convenient and interesting to set down some of the basic properties of the generalized permutahedron.

LEMMA 8. *For ordered sets $P$ and $Q$, $P$ an extension of $Q$, these conditions are equivalent:*

(i) $Q \in \mathbf{Perm}(P)$;
(ii) $P \backslash Q \in \mathbf{Perm}(P)$;
(iii) *the transitive closure of* $Q \cup (P \backslash Q)^d$ *is an order.*

*Proof.* If $Q \in \mathbf{Perm}(P)$ then $Q = L \cap P$ for some linear extension $L$ of $\mathbf{A}_n$. Since $P \backslash Q \subseteq L^d$ then $P \backslash Q = L^d \cap P$.

If $P \backslash Q \in \mathbf{Perm}(P)$ then $P \backslash Q = L \cap P$ for some linear extension $L$ of $\mathbf{A}_n$. Since $Q$ and $(P \backslash Q)^d$ are included in $L^d$, then the transitive closure of the union is an order.

If the transitive closure of $Q \cup (P \backslash Q)^d$ is an order then, for any linear extension $L$ of it, we have $Q = L \cap P$.

LEMMA 9. *For any ordered set* $P$, $\mathbf{Perm}(P)$ *is self dual.*

*Proof.* For each suborder $Q$ of $P$ define $\varphi(Q) = P \backslash Q$. Evidently, $\varphi$ is an order-reversing self-map of $\mathbf{Perm}(P)$ and $\varphi \circ \varphi$ is the identity.

There are cases in which $Q$ and $\varphi(Q)$ are ordered sets while $Q \notin \mathbf{Perm}(P)$. For instance, neither $Q$ nor $\varphi(Q)$ has the form $P \cap L$, for the four-cycle $P$, and the disjoint union $Q$ of two, two-element chains, as illustrated in Figure 5. Indeed, if $x_1 < x_2$ in $L$ then $x_1 < y_2$ in $P \cap L$, and if $x_2 < x_1$ in $L$ then $x_2 < y_1$ in $P \cap L$. Lemma 10 (below) shows that $Q$ and $\varphi(Q)$ are ordered sets if and only if, for every $x$, $y$, $z$, if $x < y < z$ in $P$ and $x < z$ in $Q$, then $x < y$ in $Q$ or $y < z$ in $Q$. In case $P$ and $Q$ satisfy this condition we say that $P$ is a *nonseparating extension* of $Q$, otherwise, $Q$ *separates* $x < y < z$ *in* $P$. The case in which $P$ is a linear extension of $\mathbf{A}_n$ was introduced in [5] to characterize those subsets of linear extensions of ordered sets of dimension two which can be extended to a realizer. Let $\mathbf{N}(P)$ stand for the set of all ordered sets $Q$ such that $P$ is a nonseparating extension of $Q$. Clearly $\mathbf{Perm}(P) \subseteq \mathbf{N}(P)$, for $x < y < z$ in $P$ and $x < z$ in $P \cap L$ imply $x < y$ or $y < z$ in $P \cap L$.

LEMMA 10. *For any ordered set* $P$, $Q \in \mathbf{N}(P)$ *if and only if* $P \backslash Q$ *is an ordered set.*

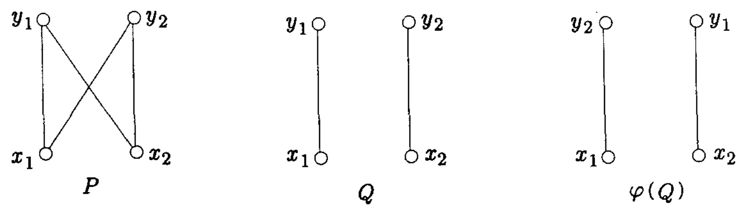



Figure 5.

*Proof.* Let $Q \in \mathbf{N}(P)$. It suffices to show that $P \backslash Q$ is transitive. Let $x < y$ and $y < z$ in $P \backslash Q$. Suppose $x < z$ not in $P \backslash Q$. Since $P$ is an ordered set, $x < z$ in $Q$. Since $P$ is a nonseparating extension of $Q$ then either $x < y$ or $y < z$ in $Q$, a contradiction. Let $Q \notin \mathbf{N}(P)$ and suppose that $Q$ separates $x < y < z$ in $P$. Clearly $x < y$ and $y < z$ in $P \backslash Q$ but $x \nless z$ in $P \backslash Q$.

LEMMA 11. *The supremum of $Q_1$ and $Q_2$ in* $\mathbf{N}(P)$, *respectively in* $\mathbf{Perm}(P)$, *provided it exists, is the transitive closure of* $Q_1 \cup Q_2$.

*Proof.* Let $R$ be one of the sets $\mathbf{N}(P)$, $\mathbf{Perm}(P)$. Let $Q_1, Q_2 \in R$ and $Q$ be the join of $Q_1, Q_2$ in $R$. Suppose that $Q'$, the transitive closure of $Q_1 \cup Q_2$, is distinct from $Q$. Select $a < b$ in $Q \backslash Q'$. Since $a$ and $b$ are noncomparable in $Q'$, there is some linear extension $L$ containing $Q'$ and $b < a$. Since $P \cap L \in \mathbf{Perm}(P) \subseteq \mathbf{N}(P)$, it follows that $P \cap L \in R$. Since $P$ and $L$ are upper bounds of $Q_1, Q_2$, it follows that $Q \subseteq P \cap L$, hence $a < b$ in $L$, a contradiction.

LEMMA 12. *An ordered set $P$ contains no induced suborder isomorphic to* $\mathbf{2}^2$ *if and only if every pair of elements of* $\mathbf{N}(P)$ *has a supremum.*

*Proof.* Suppose that $P$ contains no induced suborder isomorphic to $\mathbf{2}^2$. Let $Q_1$, $Q_2 \in \mathbf{N}(P)$ and let $Q$ be the transitive closure of $Q_1 \cup Q_2$. Let $x < y < z$ in $P$ such that $x < z$ in $Q$. Let $x_0, x_1, x_2, \ldots, x_n = z$ such that $x_i < x_{i+1}$ in $Q_1$ or in $Q_2$ for $i = 0, 1, 2, \ldots, n-1$. Since $P$ contains no induced suborder isomorphic to $\mathbf{2}^2$ each $x_i$ is comparable to $y$ with respect to $P$. Let $i_0$ be chosen such that $x_{i_0} \leq y \leq x_{i_0+1}$ in $P$. Since $x_{i_0} \leq x_{i_0+1}$ in $Q_j$ (for $j = 1$ or 2) and $P$ is a nonseparating extension of $Q_j$ we have either $x_{i_0} \leq y$ in $Q_j$ (hence $x < y$ in $Q$), or $y \leq x_{i_0+1}$ in $Q_j$ (hence $y < z$ in $Q$). This proves that $P$ is a nonseparating extension of $Q$.

Conversely, suppose that $P$ contains a suborder $\{x, y, u, z\} \cong \mathbf{2}^2$, where $x < y < z$ and $x < u < z$ are the only comparabilities among these elements in $P$. Let $Q_1$ be the ordered set consisting of all comparabilities in $P$ of the form $a < y$ where $x \leq a \leq y$, and let $Q_2$ be the ordered set consisting of all comparabilities in $P$ of the form $a < z$ where $y \leq a \leq z$. Clearly the transitive closure $Q$ of $Q_1 \cup Q_2$ is not in $\mathbf{N}(P)$ (it separates $x < u < z$ in $P$) although $Q_1$ and $Q_2$ do belong to $\mathbf{N}(P)$.

PROPOSITION 13. *For any ordered set $P$,* $\mathbf{Perm}(P) = \mathbf{N}(P)$ *if and only if $P$ contains no crowns.*

*Proof.* Suppose that $P$ contains a crown $C = \{x_1 < y_1 > x_2 < y_2 > \cdots > x_n < y_n > x_1\}$, $n \geq 2$. Let $Q$ be the ordered set consisting of all comparabilities in $P$ of the form $a \leq y_i$, where $x_i \leq a \leq y_i$ in $P$, for some $i = 1, 2, \ldots, n$. Clearly $Q$ is an ordered set and $Q \in \mathbf{N}(P)$. However, $x_i \leq y_i(Q)$ for $i = 1, 2, \ldots, n$ and $y_i < x_{i+1}$ in

$(P\backslash Q)^d$ for $i = 1, 2, \ldots, n-1$, and since $x_1 < y_n$ in $P$ this shows that the transitive closure of $Q \cup (P\backslash Q)^d$ is not an order. Therefore $Q \notin \mathbf{Perm}(P)$.

To prove the converse, suppose that $P$ contains no crown. Let $Q \in \mathbf{N}(P)$ and let $R$ be the transitive closure of $Q \cup (P\backslash Q)^d$. Suppose that $R$ is not an ordered set. Then there is a shortest sequence $x_1 < x_2 < \cdots x_m < x_1$ in $R$. We may assume that there are no further comparabilities in $R$ among the $x_i$s (for otherwise we could shorten the sequence). As $Q \in \mathbf{N}(P)$ then $(P\backslash Q)^d$ is an ordered set (cf. Lemma 10), hence we may assume that the comparabilities in this sequence are alternating between $Q$ and $(P\backslash Q)^d$. We omit the details of the straightforward, but lengthy, verification that $m$ is even and $\{x_1, x_2, \ldots, x_m\}$ forms a crown in $P$, which is a contradiction.

Finally, we are ready to turn to the proof of Theorem 7.

*Proof of Theorem* 7. If $P$ contains no crown and no induced suborders isomorphic to $\mathbf{2}^2$ then, according to Lemma 12 and Proposition 13, $\mathbf{Perm}(P)$ is a supremum semilattice. By Lemma 9, $\mathbf{Perm}(P)$ is self dual, whence it is a lattice.

For the converse, suppose that $P$ contains a cycle $\{x_1, y_1, x_2, y_2, \ldots, x_n, y_n\}$, $n \geq 2$. Let $Q_1$ be the suborder of $P$ consisting of all comparabilities in $P$ of the form $a \leq y_i$, where $x_i \leq a \leq y_i$ in $P$ for some $i = 1, 2, \ldots, n-1$. Let $Q_2$ be the suborder of $P$ consisting of all comparabilities in $P$ of the form $a \leq y_n$ where $x_n \leq a \leq y_n$ in $P$. The transitive closure $Q$ of $Q_1 \cup Q_2$ is not in $\mathbf{Perm}(P)$ (cf. the proof of Proposition 13), contradicting the fact that $\mathbf{Perm}(P)$ is a lattice. Since $P$ contains no cycle, then, from Proposition 13, $\mathbf{Perm}(P) = \mathbf{N}(P)$, hence from our hyposthesis, $\mathbf{N}(P)$ is a lattice. From Lemma 12 $P$ has no induced suborder isomorphic to $\mathbf{2}^2$. This completes the proof.

Our final result asserts that $\mathbf{N}(P)$ is a cover-preserving subset of $Ext(\mathbf{A}_n)$.

THEOREM 14. *For any ordered set $P$, $\mathbf{N}(P)$ is a subdiagram of $Ext(\mathbf{A}_n)$.*

*Proof.* Let $Q_2$ be an upper cover of $Q_1$ in $\mathbf{N}(P)$. Choose a comparability $x < y$ in $Q_2\backslash Q_1$ such that the largest chain in the interval $[x, y]$ in $P$ is maximal. Let $Q = Q_2\backslash\{x < y\}$ and suppose that $Q \notin \mathbf{N}(P)$. Thus $Q$ separates a three-element chain $u < v < w$ in $P$. Since $Q_2 \in \mathbf{N}(P)$ then $u = x$ and $v = y$ or, $v = x$ and $w = y$. Suppose that $u = x$ and $v = y$, then clearly the largest chain in the interval $[x, w]$ in $P$ is bigger than the largest chain in $[x, y]$ and therefore the comparability $x < w$ is in $Q_1$. Then $Q_1$ separates $x < y < w$ in $P$ contradicting $Q_1 \in \mathbf{N}(P)$. The same argument applies in the case $v = x$ and $w = y$, completing the proof.

Theorem 7 holds for infinite ordered sets, too. In fact, $\mathbf{Perm}(P)$ is a complete lattice whenever it is a lattice. (Note that the supremum of an arbitrary family of

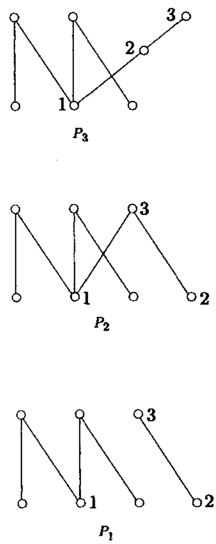



Figure 6. A three-element interval $\{P_1 \prec P_2 \prec P_3\}$ in $Ext(\mathbf{A}_7)$ whose endpoints have dimension two, and whose unique interior element has dimension three.

elements $Q_i$ of $\mathbf{N}(P)$ (or $\mathbf{PERM}(P)$), provided it exists, is the transitive closure of their union.)

## Examples

The results presented in this note were originally inspired by the conjecture that, in the extension lattice, between any two ordered sets of the same dimension, there is always a covering pair of ordered sets with the same dimension. The example illustrated in Figure 6 shows, however, that this need not be the case.

This example concerns a three-element interval within an extension lattice. It is straightforward to verify that any longer interval in an extension lattice cannot be just a chain. Let $P_0 \prec P_1 \prec P_2 \prec P_3$ in an extension lattice. Then there are critical pairs $(a_i, b_i)$ in $P_{i-1}$ such that $a_i < b_i$ in $P_i$. In this case, $a_1 \leq a_2 \leq a_3$ and $b_3 \leq b_2 \leq b_1$. As not all of the elements in each of these three-element chains can be the same, say $a_1 < a_2$ and $b_3 < b_2$ then, adding the comparability $a_2 < b_3$, produces an extension $P'$ distinct from the $P_i$'s and $P_0 < P' < P_3$.

As a matter of fact, the example illustrated in Figure 6 of a three-element interval in an extension lattice, can be generalized to an interval $\{P_1 \leq Q \leq P_k\}$ of arbitrary length $k$ in which any upper cover of $P_1$, in this interval, has dimension three and $dim(P_1) = dim(P_k) = 2$ (see Figure 7).

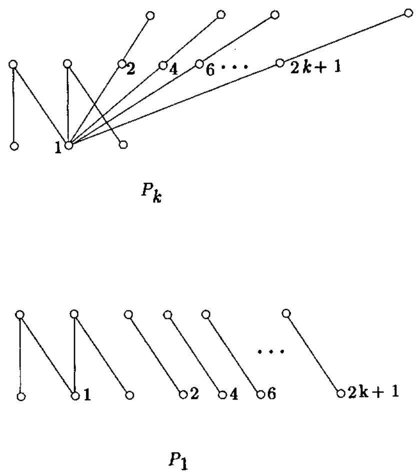



Figure 7. An interval $\{P_1 \leq Q \leq P_k\}$ in $Ext(\mathbf{A}_{2k+5})$ whose endpoints have dimension two, and whose interior elements all have dimension three.

According to Corollary 3, every ordered set has an upper cover whose dimension does not increase. It may be, however, that every upper cover has dimension which does not decrease. It is easy to construct ordered sets $P$ each of whose upper covers and each of whose lower covers has the same dimension as $P$ – for instance, the four-element cycle. Notice that any extension constructed by adding any single comparability to a cycle with at least eight elements, contains a cycle with at least six elements, whence it retains the dimension three.

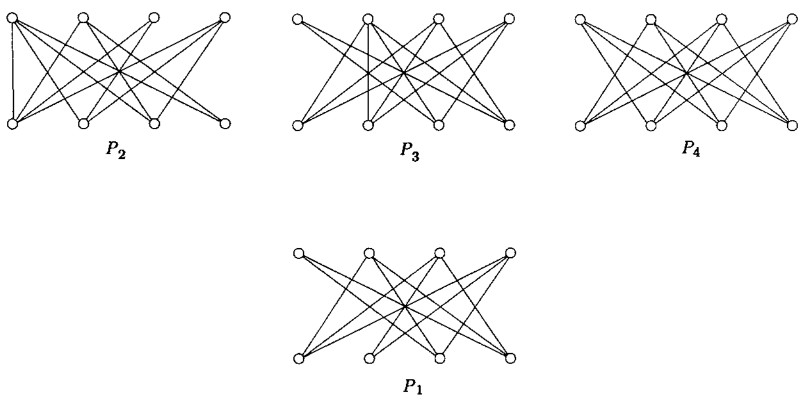
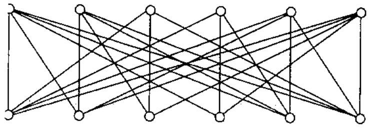
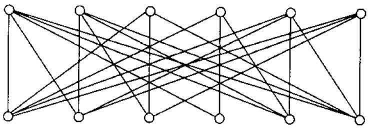


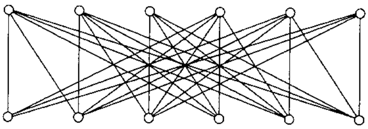

Figure 8. An ordered set $P_1$ with $dim(P_1) = 3$ with upper covers $P_2$, $P_3$, $P_4$ such that $dim(P_i) = i$ for $i = 2, 3, 4$.



$P_3$



$P_2$



$P_1$

Figure 9. A three-element covering chain $P_1 \prec P_2 \prec P_3$ in $Ext(\mathbf{A}_{12})$ with strictly increasing dimension.

Can the dimension continue decreasing as comparabilities are successively removed? Can the dimension continue increasing as comparabilities are successively removed? The answer to both questions is yes, for the $2n$-element ordered set consisting of the singletons and $n-1$-element subsets of an $n$-element set has dimension $n$ yet successively adding comparabilities between a minimal and a maximal successively decreases the dimension by one. An example for the second question is given in Figure 9.

For any finite ordered set $P$, may we expect that the subset of $Ext(\mathbf{A}_n)$, consisting of the ordered sets comparable to $P$ and of dimension at most $dim(P)$, has more (cover-preserving) "grid" structure than yet illuminated either in Theorem 1(ii), Corollary 5 (or Proposition 6)? The naive conjecture inspired by Theorem 1(ii) that $Ext(P)$ contains a "halfgrid" of $\mathbf{m}^{dim(P)}$ is, however, false, as would be illustrated by the extension lattice of the six-element cycle, were the margins of this paper wider [sic].

REFERENCES

[1] AVANN, S. P., *The lattice of natural partial orders*, Aequationes Math. *8* (1972), 95–102.
[2] BJÖRNER, *A., Orderings of Coxeter groups*, Amer. Math. Soc. Contemp. Math. *34* (1984), 175–195.
[3] BRUALDI, R. A. and JUNG, H. C., *On the poset of all posets on n elements*, 1991, preprint.

[4] DEAN, R. A. and KELLER, G., *Natural partial orders*, Canad. J. Math. *20* (1968), 535–554.
[5] DUSHNIK, B. and MILLER, E. W., *Partially ordered sets*, Amer. J. Math. *63* (1941), 600–610.
[6] EADES, P., HICKEY, M. and READ, R. C., *Some Hamilton paths and a minimal change algorithm*, J. Assoc. Comp. Mach. *31* (1984), 19–29.
[7] GUILBAUD, G. T. and ROSENSTIEHL, P., *Analyse algébrique d'un scrutin*, Math. et Science Humaines *4* (1963), 9–33. [Reprinted in B. Monjardet, *Treillis d'ordre*, in *Ordres Totaux Finis*, Gauthier-Villars, Paris, 1971, pp. 29–45.
[8] KELLY, D., *Removable pairs in dimension theory*, in *Unsolved Problems*, Order *1* (1984), 217–218.
[9] KELLY, D. and TROTTER, W. T., *Dimension theory for ordered sets*, in *Ordered Sets* (I. Rival, ed.), Reidel, 1982, pp. 171–211.
[10] POUZET, M. and RIVAL, I., *Structural arithmetic of the extension lattice of an order*, Technical Report TR-91-11, University of Ottawa.
[11] PRUESSE, G. and RUSKEY, F., *Generating the linear extensions of certain posets by transpositions*, SIAM J. Discrete Math. *4* (1991), 413–422.
[12] RUSKEY, F., *Adjacent interchange generation of combinations*, J. Algorithms *4* (1988), 162–180.
[13] RUSKEY, F., *Research problem 90*, Discrete Math. *70* (1988), 111–112.
[14] RUSKEY, F., *Transposition generation of alternating permutations*, Order *6* (1989), 227–233.

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