On the Complexity of Equilibrium Computation in First-Price Auctions

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We consider the problem of computing a (pure) Bayes-Nash equilibrium in the first-price auction with continuous value distributions and discrete bidding space. We prove that when bidders have independent *subjective* prior beliefs about the value distributions of the other bidders, computing an ε-equilibrium of the auction is PPAD-complete, and computing an *exact* equilibrium is FIXP-complete.

CCS Concepts: • **Theory of computation** → **Problems, reductions and completeness**; **Exact and approximate computation of equilibria**; *Algorithmic mechanism design*; *Computational pricing and auctions*.

Additional Key Words and Phrases: first-price auctions; Bayes-Nash equilibria; approximate equilibria; subjective priors; PPAD; FIXP; generalized circuit

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1 INTRODUCTION

Auctions are prime examples of economic environments in which the element of strategic behavior is prevalent. The associated theory can be traced back to as early as the 1960s and the seminal work of Vickrey [67]. Over the years, auction theory and mechanism design have produced some of the most celebrated results in economics, as can be evidenced, e.g., by the relevant 1996, 2007 and 2020 Nobel Prizes.[1] Among the plethora of auction formats that this rich literature has proposed, some stand out, such as the second-price auction of Vickrey [67] or the revenue-maximizing auction of Myerson [54].

Arguably, though, the most fundamental auction format is that of the *first-price auction*, in which the highest bidder wins and is charged an amount equal to her bid. Compared to its counterparts mentioned above, the first-price auction does not enjoy the same desirable incentive properties: participants may have an incentive to misreport their true bids. At the same time, however, the first-price auction is very natural and simple to describe, implement and participate in, making it very suitable for a range of important applications. As a matter of fact, several online ad exchanges,

[1]For the official Nobel Prize announcements see here, here and here.

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including Google Ad Manager, have adopted this auction format for selling their ads, which has been coined "the first-price movement" (see, e.g., [20, 56]).

There has been a large body of work studying incentives and bidding behavior in first-price auctions, dating back to the original paper of Vickrey [67]. In particular, the literature has studied the equilibria of the auction in an incomplete information setting where the bidders have only probabilistic prior beliefs (or simply *priors*) about the values of other bidders, via the lens of Bayesian game theory [35] (see also [36, 55]). Several different scenarios of interest have been analyzed; see, e.g., [1, 5, 11, 18, 42, 43, 46, 48, 49, 58, 60, 61]. It is no exaggeration to say that understanding the Bayes-Nash equilibria of the first-price auction has historically been one of the most important questions of auction theory.

The aforementioned literature has been primarily concerned with identifying conditions under which (pure Bayes-Nash) equilibria are guaranteed to exist. Among those, the seminal paper of Athey [1] has been pivotal in establishing the existence of equilibria for fairly general settings with continuous priors. A natural follow-up question posed explicitly by Athey [1], which was also very much present in earlier works, is whether these equilibria can also be "found"; in the context of the related literature, this is usually interpreted as coming up with closed-form solutions that describe them.

One of the most significant contributions of computer science to the field of game theory is to formalize and systematically study this notion of "finding" or "computing" equilibria in games. Roughly speaking, an equilibrium can be efficiently computed if it can be found using a limited number of standard operations that can be performed by a computer, where "limited" here typically means a number which is a polynomial function of the size of the input parameters.[2] In perhaps the most important result in computational game theory, Daskalakis et al. [19] proved that in all likelihood, Nash equilibria of general games cannot always be computed efficiently. In particular, they proved that the problem of computing a Nash equilibrium is complete for the class PPAD [57], which is widely believed to include problems that are computationally hard to solve.

In this paper, we study the complexity of computing an equilibrium of the first-price auction, in settings with continuous priors and discrete bids. We offer the following main result.

Informal Theorem 1. *Computing a (pure, Bayes-Nash) equilibrium of a first-price auction with continuous subjective priors and discrete bids is* PPAD-*complete.*

This result can be interpreted intuitively as justification of why research in economics has only had limited success in providing closed forms or characterizations for the equilibria of the first-price auction. In addition, we consider it to be a quite valuable addition to the literature of total search problems [51], as it concerns the computation of equilibria of one of the most fundamental games in auction theory.

1.1 Discussion and Further Results

Below, we provide a more in-depth discussion of our main result and its assumptions, as well as some other related results that we obtain along the way.

Continuous Priors, Discrete Bids. Informal Theorem 1 applies to the case where the bidders' beliefs about the values of other bidders are continuous distributions, whereas the bidding space is a discrete set. The former assumption is standard in auction theory (see, e.g., [55, Sec. 3.11] or [41]). From a technical standpoint, this also guarantees the existence of equilibria [1].[3] The assumption

[2]We remark that contrary to earlier works in economics, Athey's interpretation of "finding" an equilibrium was very much of a computational nature.

[3]It is important to note here that in some versions of the problem, even *mixed* Bayes-Nash equilibria are not guaranteed to exist; see, e.g., [42].

of the discrete bidding space is clearly motivated by any real-world scenario, in which the bids will be increments of some minimum monetary amount, e.g., 1 dollar or 1 cent, depending on the application. This setting has in fact been studied in several works for first-price auctions in particular (see, e.g., [1, 8, 16, 21, 59]).

Subjective Priors. In Informal Theorem 1 we assume that the priors are subjective, meaning that two different bidders might have different beliefs about the values of some other bidder. In the auction theory literature, it is often assumed that a "universal" prior exists, which is common knowledge among all players; this is known as the *independent private values* model. Indeed, such common priors are quite convenient in settings where there is an aggregate objective that needs to be optimized in expectation (e.g., the social welfare or the seller's revenue), since it can be used by the designer to tune the parameters of the auction in a way that works best for the optimization goal at hand; this is the case, e.g., for Myerson's revenue-maximizing auction [54].

From our perspective however, where the goal is to study the players' incentives and compute an equilibrium, we believe it is natural to make the more general assumption that priors are still independent, but subjective: this is enough for the bidders to come up with their best responses. As a matter of fact, Harsanyi's original paper [35], as well as classic textbooks in economics (e.g., [37, 55]) introduce Bayesian games directly in the context of subjective beliefs.[4] Similar notions of subjective priors and "subjective equilibria" have also been studied rather extensively for general Bayesian games in economics [3, 4, 28, 34, 39, 40, 65] and computer science [27, 69].

The subjective priors assumption is necessary for our PPAD-hardness result, but we would of course be very interested in settling the complexity for the case of common priors as well. In fact, as we explain in Section 7, we consider this to be one of the most important open problems in computational game theory. Thus, besides being of standalone interest, one can also see our result for subjective priors as an important first step in the quest of answering this question. We remark that our PPAD-membership result obviously applies to common priors, as this is just a special case of subjective beliefs.

Approximate Equilibria. While Informal Theorem 1 states the PPAD-completeness of computing an equilibrium of the first-price auction, the formal statement is in fact about ε-equilibria, i.e., stable states in which bidders do not wish to unilaterally deviate unless they are better off by some small positive quantity ε. As we explain in Section 2, this is very much necessary: there are examples where the equilibrium is *irrational*, and therefore cannot be computed exactly in many standard models of computation. As a matter of fact, this is a common theme in most papers in equilibrium computation; see, e.g., [12, 19] or the survey of Goldberg [31] for a related discussion.

Of course, the focus on ε-equilibria is only relevant for the membership result in PPAD; the computational hardness result for approximate equilibria is clearly stronger. In fact, we show that under some standard assumptions (see Section 2), the problem is PPAD-hard even when ε is allowed to be a (sufficiently small) constant, independent of the input parameters. This is the strongest type of PPAD-hardness one could hope for. For the computation of *exact* equilibria, Etessami and Yannakakis [22] defined the computational class FIXP. At a high level, this class contains problems that can be stated as computations of (possibly irrational) fixed points of functions defined by means of arithmetic circuits (see [70]). We complement our main result about ε-equilibria with the following analogous result on exact ones:

INFORMAL THEOREM 2. *Computing an* exact *(pure, Bayes-Nash) equilibrium of a first-price auction with continuous subjective priors and discrete bids is* FIXP*-complete.*

[4] These works also usually provide discussions on "consistency" conditions, e.g., see [35] and [55, Sec. 2.8].

One way to interpret a FIXP-completeness result in the standard computational (Turing) model is in terms of *strong* vs *weak* approximations. A weak approximation is an ε-equilibrium as defined above and is captured by our PPAD-completeness result. A strong approximation is a set of strategies represented by rational numbers, which are "ε-close" to an exact equilibrium (in terms of the max norm), and is captured by our FIXP-completeness result. We remark that this is completely analogous to the computation of Nash equilibria in general games, see [22, 29] for a more in-depth discussion.

The Meaning of PPAD-completeness. As we mentioned earlier, a PPAD-hardness result is interpreted as an indication that the problem cannot be solved in polynomial time. In particular, it is as hard as finding Nash equilibria in general games [12, 19, 52, 64], market equilibria in Arrow-Debreu markets [13, 66] or solutions to fixed point theorems [32, 57]. Additionally, PPAD has been shown to be hard under various cryptographic assumptions (e.g., see [7, 14, 30, 62]), meaning that solving a PPAD-hard problem would "break" those assumptions as well. On the other hand, an "in PPAD" result can be interpreted as the existence of an (inefficient) algorithm that uses a path-following argument to reach a solution.

An Efficient Algorithm. Besides our main PPAD- and FIXP-completeness results, we identify a special case of the problem which can be solved efficiently, namely when the number of bidders and the size of the bidding space are constant, and the value distributions are "sufficiently smooth", in the sense that they are given by piecewise polynomial functions. To this end, we have the following theorem.

Informal Theorem 3. *A (pure, Bayes-Nash) equilibrium of the first-price auction can be computed in polynomial time when there is a constant number of bidders, a constant-size bidding space, and continuous (subjective) priors which are piecewise polynomial functions.*

Informal Theorem 3 complements our PPAD- and FIXP-hardness results rather tightly, as our reductions use a constant bidding space and very simple, piecewise constant distributions, but a large number of bidders.

1.2 Related Work

As we mentioned earlier, there is a significant amount of work in economic theory on the equilibria of the first-price auction [1, 5, 18, 42, 43, 46, 48, 49, 58, 60, 61]. Among those, the most relevant work to us is that of Athey [1], who established the existence of pure Bayes-Nash equilibria in games with discontinuous payoffs which satisfy the *single crossing property* of Milgrom and Shannon [53], of which the first-price auction is a special case. Athey's proof applies to both discrete and continuous bidding spaces, and in fact the latter is established through the former, via a limit argument similar in spirit to [42, 49].

To the best of our knowledge, there are only a few prior works on the computational complexity of equilibria in first-price auctions. Escamocher et al. [21] study the problem of computing equilibria when *both* the priors and the bidding space are discrete. In that case, it is not hard to construct counter-examples that show that pure equilibria may not exist, and therefore they are concerned with the question of *deciding* their existence. Their results do not provide a conclusive answer (i.e., neither NP-hardness nor polynomial-time solvability is proven), except for the very special case of two bidders with bi-valued distributions. Wang et al. [68] very recently studied the equilibrium computation problem in settings with *discrete priors* and *continuous bids* (in a sense, the opposite of what we do here), and under the *Vickrey tie-breaking rule* for deciding the winner of the auction in case of a tie. According to this rule, ties are resolved by running an auxiliary second-price (Vickrey) auction among the potential winners of the first-price auction; effectively this allocates the item to the bidder with highest true valuation. This tie-breaking rule was introduced by Maskin and Riley [49] primarily as a technical tool in proving their existence results for the *uniform tie-breaking rule*, where ties are broken uniformly at random among the bidders with the highest bid. Our results are proven for the uniform tie-breaking rule, which is the standard rule in the literature of the problem [1, 41, 42, 49].

Finally, we remark that while we consider an equilibrium computation setting, our results are markedly different from other works on such problems, e.g., [19]. This is because it concerns a much more specific and structured game, and crucially, a game which is *Bayesian*, which is not the case for most prior work. Conceptually closer to our work is the paper by Cai and Papadimitriou [9] who study the complexity of Bayesian *combinatorial* auctions, a more complicated auction format which typically involves multiple items for sale and more complex agents' valuations over subsets of items. The complexity of *general* Bayesian games (beyond auctions) has been studied in the literature, primarily resulting in NP-hardness results for several cases of interest, e.g., see [17, 33].

2 MODEL AND NOTATION

In a (Bayesian) *first-price auction (FPA)*, there is a set $N = \{1, 2, \dots, n\}$ of *bidders* (or *players*) and one item for sale. Each player i submits a *bid* $b_i \in B$, where the *bidding space* $B \subseteq [0, 1]$ is a finite set. We will also make the standard assumption (often referred to as the "null bid" in the literature) that $0 \in B$, which can be interpreted as the option of the bidders to not participate in the auction (see, e.g., [1, 49]).

The item is allocated to the player with the highest bid, who is charged a payment equal to her bid. If there are multiple players submitting the same highest bid, the winner is determined based on the *uniform tie-breaking* rule. Formally, for a *bid profile* $\boldsymbol{b} = (b_1, \dots, b_n)$, the *ex-post utility* of player i with true value v_i is given by

$$\tilde{u}_i(\boldsymbol{b}; v_i) \equiv \begin{cases} \frac{1}{|W(\boldsymbol{b})|}(v_i - b_i), & \text{if } i \in W(\boldsymbol{b}), \\ 0, & \text{otherwise,} \end{cases} \qquad \text{where } W(\boldsymbol{b}) = \operatorname*{argmax}_{j \in N} b_j \tag{1}$$

For each pair of players $i, j \in N$, $i \neq j$, there is a continuous value distribution $F_{i,j}$ over $[0, 1]$; we call this distribution the *prior* of bidder i over the values of bidder j. The *subjective belief* of player i for the values $\boldsymbol{v}_{-i} = (v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$ of the other bidders is then given by the product distribution $\boldsymbol{F}_{-i} \equiv \times_{j \neq i} F_{i,j}$. In other words, from the perspective of bidder i, the values v_j for $j \neq i$ are drawn *independently* from distributions $F_{i,j}$. Notice that the special case where $F_{i,j} = F_{i',j}$ for all $j \in N$ and $i, i' \in N \setminus \{j\}$ corresponds to the classic *independent private values* model of auction theory, where the value of each bidder is drawn (independently of the others) from a single distribution. More formally, simplifying the notation by using F_j instead of $F_{i,j}$, $\boldsymbol{v}$ is drawn from the *common prior* distribution $\boldsymbol{F} = \times_{i \in N} F_i$. Obviously, while our hardness results rely on the fact that priors are subjective, all of our positive results trivially extend to the case of common priors as well.

The FPA described above naturally induces a game in which each bidder i selects her bid based on her own (true) value v_i, and her beliefs $\boldsymbol{F}_{-i}$. A *strategy* of bidder i is a function $\beta_i : [0, 1] \longrightarrow B$ mapping values to bids. Given a strategy profile $\boldsymbol{\beta}_{-i}$ of the other players, the (ex-interim) *utility* of player i with true value v_i when bidding $b \in B$ is

$$u_i(b, \boldsymbol{\beta}_{-i}; v_i) \equiv \mathbb{E}_{\boldsymbol{v}_{-i} \sim \boldsymbol{F}_{-i}} \left[\tilde{u}_i(b, \boldsymbol{\beta}_{-i}(\boldsymbol{v}_{-i}); v_i) \right],$$

where $\boldsymbol{\beta}_{-i}(\boldsymbol{v}_{-i})$ is a shorthand for $(\beta_1(v_1), \dots, \beta_{i-1}(v_{i-1}), \beta_{i+1}(v_{i+1}), \cdots, \beta_n(v_n))$. Intuitively, the player calculates her (expected) utility by drawing a value v_j for each bidder $j \neq i$ from her corresponding subjective prior distribution $F_{i,j}$, and then using the strategy "rules" $\boldsymbol{\beta}_{-i}$ of the others to map their values to actual bids in B.

We are interested in "stable" states of the FPA, i.e., strategy profiles from which no bidder would like to unilaterally deviate to a different strategy. Formally, we have the following definition.

Definition 1 (ε-Bayes-Nash equilibrium of the FPA). Let $\varepsilon > 0$. A strategy profile $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n)$ is a (pure, ex-interim) ε-Bayes-Nash equilibrium (ε-BNE) of the FPA if for any bidder $i \in N$ and any value $v_i \in [0, 1]$,

$$u_i(\beta_i(v_i), \boldsymbol{\beta}_{-i}; v_i) \geq u_i(b, \boldsymbol{\beta}_{-i}; v_i) - \varepsilon \qquad \text{for all } \ b \in B.$$

Given a fixed strategy profile $\boldsymbol{\beta}_{-i}$ of the other bidders, we will denote the set of *ε-best responses* of player i by

$$BR_i^{\varepsilon}(\boldsymbol{\beta}_{-i}) = \left\{ \beta_i \;\middle|\; u_i(\beta_i(v_i), \boldsymbol{\beta}_{-i}; v_i) \geq \max_{b \in B} u_i(b, \boldsymbol{\beta}_{-i}; v_i) - \varepsilon \quad \text{for all } \ v_i \in [0, 1] \right\}$$

Using this, the condition in Definition 1 can be equivalently written as $\beta_i \in BR_i^{\varepsilon}(\boldsymbol{\beta}_{-i})$ for all players i. For the special case of $\varepsilon = 0$, i.e. *exact* best-responses, we will drop the ε superscript.

Notice that, in Definition 1 we define a relaxed equilibrium concept, in which the bidder does not want to change to a different strategy unless it increases her utility by an additive factor larger than ε; obviously, when $\varepsilon = 0$ we recover the standard definition of the (exact) Bayes-Nash equilibrium.

No Overbidding. As part of our model, we will make the assumption that bidders will never submit a bid b_i which is higher than their valuation v_i. This is a standard assumption in the literature of the first-price auction [21, 44, 49, 50, 68] and auctions in general [6, 10, 15, 25, 45, 47]. The rationale behind it stems from the fact that, given the format of the utilities in the FPA (see (1)), it is arguably unreasonable to overbid, as bidding 0 will *always* result in at least the same utility. In game-theoretic terms, the overbidding strategy is *weakly dominated* by bidding 0, which can be interpreted as abstaining from the auction. These strategies are typically excluded from consideration to rule out unnatural equilibria (see [25] for a discussion).

We are now ready to formally define our computational problem of finding an equilibrium of the FPA:

ε-BAYES-NASH EQUILIBRIUM IN THE FIRST-PRICE AUCTION (ε-BNE-FPA)

INPUT:

- a set of bidders $N = \{1, 2, \dots, n\}$;
- a finite bidding space $B \subseteq [0, 1]$;
- for each pair of bidders $i, j \in N$, a continuous value distribution $F_{i,j}$ over $[0, 1]$.

OUTPUT: An ε-Bayes-Nash equilibrium $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n)$.

We will use the term EXACT-BNE-FPA instead of 0-BNE-FPA to denote the computational problem of finding an exact Bayes-Nash equilibrium of the auction. Some remarks related to the definition above are in order.

The Input Model for the Distributions. We have intentionally vaguely stated that the distributions $F_{i,j}$ should be provided as input to the problem, but we have not specified exactly how. Our positive results hold even when the functions $F_{i,j}$ are fairly general, and can be concisely and efficiently represented in a form that is appropriate for computation. In the interest of clarity, we omit the technical details here, and we refer the reader to the full version [26] where we provide all the details of the input model. For the negative results on the other hand, we use fairly simple distributions $F_{i,j}$ – this only makes our results stronger. In particular, we use *piecewise-constant* density functions, which can be represented by the endpoints and the value for each interval.

Explicit Bidding Space. We assume that the bidding space is explicitly given as part of the input. This assumption is required in Section 3 in order to show that we can compute best-responses efficiently. Even in the mildest of settings where the bidding space is given implicitly, computing best-responses turns out to be computationally and information-theoretically hard. We show this in the full version [26].

Equilibrium Representation. Besides the representation of the input, the output of our computational problem, i.e., the equilibrium of the FPA, should also be represented in some concise and efficient way. Following the standard literature of the problem, we will consider equilibria for which the strategy $\beta_i(v_i)$ of each bidder is a non-decreasing function of her value v_i (e.g., see [1, 49, 60] and [41, Appendix G]) for which the existence of an equilibrium is always guaranteed [1]. These equilibria are in a sense the only "natural" ones, as, similar to the case of overbidding (see earlier discussion), any bidder's strategy is weakly dominated by a non-decreasing strategy.

Based on this, there is a straightforward and computationally efficient way of representing the best response of each player, as a step function with a finite set of "jump points", corresponding to the values at which the bidder "jumps" from one bid to the next [1]. Formally, we define

$$\alpha_i(b) = \sup \{ v \mid \beta_i(v) \le b \} . \tag{2}$$

Intuitively, $\alpha_i(b)$ is the largest value for which player i would bid b or lower. With a slight abuse of notation, we can write $\alpha_i = \beta_i^{-1}$, that is, α_i can be interpreted as an *inverse bidding* strategy. In that way, we can also rework β_i from α_i, as $\beta_i(v) = b$, where $v \in (\alpha_i(b^-), \alpha_i(b)]$ for any $b \in B$. Here we let b^- denote the previous bid, i.e., the largest $b' \in B$ with $b' < b$. Finally, to be able to handle the corner cases in a unified way, we set $a_i(b^-) = 0$ when $b = 0$ and $\alpha_i(b) = 1$ when $b = \max B$.

In particular, this implies that bidding strategies are left-continuous (which is without loss of generality given our value distributions), as shown in Figure 1.

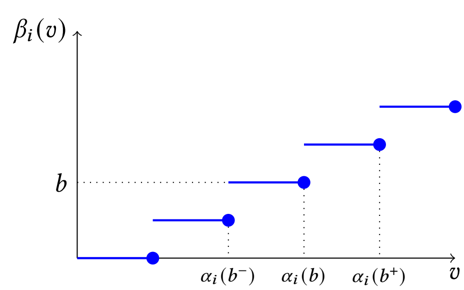

Fig. 1. A monotone bidding strategy $\beta_i(\cdot)$ can be succinctly represented by its jump points, $\alpha_i(b)$ for $b \in B$.

Irrational Equilibria. As discussed in our introduction, for our PPAD-completeness result, we will be looking for an ε-approximate equilibrium, rather than an exact one. Of course, this only makes our hardness results even stronger; but besides that, it is actually very much necessary for our membership result in PPAD as well. In particular, as demonstrated by the example below, the FPA can have *only irrational* equilibria, even when all input parameters are rational numbers.

Example 1. Consider a FPA with $n = 3$ bidders and common priors, whose values are independently and identically distributed according to the uniform distribution on $[0, 1]$; that is, $F_i(x) = x$

for $i = 1, 2, 3$. Let the bidding space be $B = \{0, 1/2\}$. Clearly, this auction can be represented with piecewise-constant density functions (with a single piece) and with a finite number of rational quantities. It can be verified that the auction has a unique equilibrium, where a bidder bids 0 iff her valuation is below $\frac{-1+\sqrt{5}}{2} \approx 0.618$; therefore, the unique equilibrium is irrational. We provide the detailed derivation in the full version [26].

The appropriate setting for studying the computation of *exact* equilibria is the class FIXP of Etessami and Yannakakis [22]. In Sections 4.2 and 5 we show that the problem of exact equilibrium computation of the FPA is FIXP-complete.

Further Notation. We conclude the section with the following terminology which will be useful in multiple sections of our paper. For $t_1 < t_2$, we will let $\mathrm{T}_{[t_1,t_2]}$ denote the *truncation* of a value x to $[t_1, t_2]$, i.e., $\mathrm{T}_{[t_1,t_2]}(x) = \max\{t_1, \min\{t_2, x\}\}$. Furthermore, for $k \in \mathbb{N}$ we sometimes use $[k]$ to denote $\{1, 2, \ldots, k\}$.

2.1 Outline

In Section 3 we provide a useful characterization of BNE and then show how to compute the best responses in polynomial time. In Section 4, first we provide a new existence proof via Brouwer's fixed point theorem, and then proceed to prove the membership of the equilibrium computation problems in PPAD and FIXP. In Section 5 we show the computational hardness for these classes. In Section 6 we present an efficient algorithm for a natural special case. We conclude with some interesting future directions in Section 7.

3 EQUILIBRIUM CHARACTERIZATION AND BEST RESPONSE COMPUTATION

In this section we begin by presenting a useful characterization of ε-BNE that is crucial for many parts of the paper. Then, we show how best-responses of bidders can be checked and computed in polynomial time. We remark that the reductions that we will construct in Section 4 to show the PPAD-membership and the FIXP-membership of the problem do not technically require the computation of the whole best-response function, but rather only the probabilities of winning the item given the bidder's bid and the bidding strategies of the other bidders. However, the best-response computation is interesting in its own right, and that is why we present this here.

Characterization. The following lemma essentially states that an ε-BNE is characterized by the behavior of the bidding function at the jump points. Recall that for any bid b, we let b^- denote the previous bid, and we use the conventions $\alpha_i(b^-) = 0$ when $b = 0$, and $\alpha_i(b) = 1$ when $b = \max B$.

Lemma 3.1 (Characterization of ε-BNE). *Fix an $\varepsilon \geq 0$. A strategy profile $\boldsymbol{\beta}$ is an ε-BNE of the FPA, if and only if, for every bidder i and every bid b with $\alpha_i(b^-) < \alpha_i(b)$,*

$$u_i(b, \boldsymbol{\beta}_{-i}; \alpha_i(b^-)) \geq u_i(b', \boldsymbol{\beta}_{-i}; \alpha_i(b^-)) - \varepsilon \qquad \text{for all } b' < b \tag{3}$$

and

$$u_i(b, \boldsymbol{\beta}_{-i}; \alpha_i(b)) \geq u_i(b', \boldsymbol{\beta}_{-i}; \alpha_i(b)) - \varepsilon \qquad \text{for all } b' > b. \tag{4}$$

The H-functions. Before proving this characterization, we introduce some useful notation. We use the term $H_i(b, \boldsymbol{\beta}_{-i})$ to denote the (perceived) probability that bidder i wins the item with bid b, when the other bidders use bids according to the bidding strategy $\boldsymbol{\beta}_{-i}$, i.e.,

$$H_i(b, \boldsymbol{\beta}_{-i}) = \Pr\left[\text{bidder } i \text{ wins} | b, \boldsymbol{\beta}_{-i}\right]$$

The utility can easily be expressed in terms of this function, namely $u_i(b, \boldsymbol{\beta}_{-i}; v_i) = (v_i - b) \cdot H_i(b, \boldsymbol{\beta}_{-i})$.

Proof of Lemma 3.1. ($\Rightarrow$): Fix a bidder i and a bid b with $\alpha_i(b^-) < \alpha_i(b)$. Since bidder i bids b inside the non-empty interval $(\alpha_i(b^-), \alpha_i(b)]$, and $\boldsymbol{\beta}$ is an ε-BNE, we get that $u_i(b, \boldsymbol{\beta}_{-i}; v_i) \geq u_i(b', \boldsymbol{\beta}_{-i}; v_i) - \varepsilon$ for every $v_i \in (\alpha_i(b^-), \alpha_i(b)]$ and $b' \neq b$. Since the utilities are continuous functions on v_i, the inequalities must also hold at the interval endpoints.

($\Leftarrow$): Suppose (3, 4) hold. Take any bidder i and any valuation v_i, and let $(\alpha_i(b^-), \alpha_i(b)]$ be the interval containing v_i. Notice that the utilities $u_i(b, \boldsymbol{\beta}_{-i}; v_i)$, $u_i(b', \boldsymbol{\beta}_{-i}; v_i)$ are linear functions on v_i, with slopes given by $H_i(b, \boldsymbol{\beta}_{-i})$, $H_i(b', \boldsymbol{\beta}_{-i})$ respectively. For $b' < b$, we know that $H_i(b', \boldsymbol{\beta}_{-i}) \leq H_i(b, \boldsymbol{\beta}_{-i})$ and $u_i(b, \boldsymbol{\beta}_{-i}; v) \geq u_i(b', \boldsymbol{\beta}_{-i}; v) - \varepsilon$ holds at $v = \alpha_i(b^-)$; therefore it must hold also at $v = v_i$. Similarly for $b' > b$, we know that $H_i(b', \boldsymbol{\beta}_{-i}) \geq H_i(b, \boldsymbol{\beta}_{-i})$ and $u_i(b, \boldsymbol{\beta}_{-i}; v) \geq u_i(b', \boldsymbol{\beta}_{-i}; v) - \varepsilon$ holds at $v = \alpha_i(b)$; therefore it must hold also at $v = v_i$. We thus conclude that $\boldsymbol{\beta}$ is an ε-BNE. □

We now consider the basic computational problems of checking and computing best-responses of bidders. We assume throughout that bidding strategies provided in the input are given via rational quantities corresponding to the jump points $\alpha_j(b)$, as defined in Section 2. The first step to be able to check or compute best-responses is the efficient computation of the H-functions defined above.

Computation of the H-functions. Recall that $H_i(b, \boldsymbol{\beta}_{-i}) = \Pr[\text{bidder } i \text{ wins} | b, \boldsymbol{\beta}_{-i}]$. This probability clearly depends on bidder i's prior on the other bidders' distributions, as well as on whether b is the highest bid, and if it is, how many other highest bids there are in the auction, in case of a tie. While the form of the functions H_i can be be devised analytically, the expression involves exponentially many terms in the number of bidders n; therefore it is not obvious that it can be computed efficiently. The following lemma states that this is in fact possible.

Lemma 3.2. *Given a bidder i, a bid b and bidding strategies $\boldsymbol{\beta}_{-i}$ of the other bidders, the probability $H_i(b, \boldsymbol{\beta}_{-i})$ of bidder i winning the item can be computed in polynomial time.*

Proof. For ease of notation, we present the proof for bidder $i = n$. The cases for the other bidders are analogous and can be handled, e.g., via an appropriate relabeling. The probability that bidder n wins (given her bid and the bidding strategies of the other bidders) can be written as

$$H_n(b, \boldsymbol{\beta}_{-n}) = \sum_{k=0}^{n-1} \frac{1}{k+1} T(b, n-1, k), \tag{5}$$

where, for $0 \leq k \leq \ell \leq n-1$, we use $T(b, \ell, k)$ to denote the probability that *exactly* k out of the first ℓ bidders bid exactly b, and the remaining $\ell - k$ bidders all bid below b; in other words, for the special case where $\ell = n-1$ in the above expression, $T(b, n-1, k)$ is the probability of the highest bid being b, with $k+1$ bidders (including bidder n) being tied for the highest bid. Next, for a given bidder j, let

$$G_{j,b^-} = F_{n,j}(\alpha_j(b^-)) = \Pr\left[\beta_j(v_j) < b\right], \quad g_{j,b} = F_{n,j}(\alpha_j(b)) - G_{jb^-} = \Pr\left[\beta_j(v_j) = b\right]$$

denote the (perceived from the perspective of bidder n) probabilities that bidder j bids below b, and exactly b, respectively. Note that G_{j,b^-} and $g_{j,b}$ can be efficiently computed with access to $F_{n,j}$ and $\boldsymbol{\alpha}_{-n}$. Moreover, one could write

$$T(b, n-1, k) = \sum_{\substack{S \subseteq [n-1] \\ |S| = k}} \prod_{j \in S} g_{j,b} \cdot \prod_{j \notin S} G_{j,b^-}. \tag{6}$$

Notice that (6) does not yield an efficient way of computing the probabilities, as the number of summands can be exponential in n. To bypass this obstacle, we observe that, more generally, the probabilities $T(b, \ell, k)$ can be computed from G_{ℓ,b^-} and $g_{\ell,b}$ via dynamic programming, by conditioning on bidder ℓ's bid, in the following way:

$$\begin{aligned}
T(b, 0, 0) &= 1; \\
T(b, \ell, k) &= 0, && \text{for } k > \ell; \\
T(b, \ell + 1, 0) &= T(b, \ell, 0)G_{\ell+1,b^-}; \\
T(b, \ell + 1, k + 1) &= T(b, \ell, k)g_{\ell+1,b} + T(b, \ell, k + 1)G_{\ell+1,b^-}; && \text{for } k \leq \ell.
\end{aligned}$$

Thus, all values of $T(b, n - 1, k)$, for $k = 0, \ldots, n - 1$, can be computed with a total number of $O(n^2)$ recursive calls, so that $H_n(b, \boldsymbol{\beta}_{-n})$ can be computed in polynomial time. □

Lemma 3.2 implies that the utilities in (3, 4) of the characterization (Lemma 3.1) can be computed in polynomial time. Since there are $O(n|B|^2)$ inequalities to check in Lemma 3.1, we immediately conclude the following.

Corollary 3.3. *Given $\varepsilon \geq 0$, and a strategy profile $\boldsymbol{\beta}$ in a first-price auction with subjective priors, one can determine in polynomial time if $\boldsymbol{\beta}$ constitutes an ε-BNE.*

Using Lemma 3.2, we can now also efficiently compute best-responses, and, in fact, even *exact* best-responses (i.e., ε-best-responses for $\varepsilon = 0$).

Theorem 3.4. *In a first-price auction with subjective priors, the bidders' best-responses can be computed in polynomial time.*

Proof. Given a bidder i and the vector of bidding strategies $\boldsymbol{\beta}_{-i}$, one can compute in polynomial time the probabilities $H_i(b, \boldsymbol{\beta}_{-i})$ for each bid $b \in B$ using Lemma 3.2. Now recall that the utility of bidder i, when having a valuation of v_i and bidding b, is given by $u_i(b, \boldsymbol{\beta}_{-i}; v_i) = (v_i - b) \cdot H_i(b, \boldsymbol{\beta}_{-i})$, which is a linear function on v_i having slope $H_i(b, \boldsymbol{\beta}_{-i})$. Thus, maximizing the utility amounts to taking the maximum (or *upper envelope*) of $|B|$ linear functions; the result is a piecewise linear function whose jump points can be efficiently computed by solving linear equations. In particular, given bids $b < b'$, we can compute $\alpha = \tilde{\alpha}_i(b, b')$ as the solution of $u_i(b, \boldsymbol{\beta}_{-i}; \alpha) = u_i(b', \boldsymbol{\beta}_{-i}; \alpha)$, that is,

$$\tilde{\alpha}_i(b, b') = \begin{cases} \frac{b' H_i(b', \boldsymbol{\beta}_{-i}) - b H_i(b, \boldsymbol{\beta}_{-i})}{H_i(b', \boldsymbol{\beta}_{-i}) - H_i(b, \boldsymbol{\beta}_{-i})} & \text{if } \ H_i(b', \boldsymbol{\beta}_{-i}) \neq H_i(b, \boldsymbol{\beta}_{-i}), \\ +\infty & \text{otherwise.} \end{cases}$$

Intuitively, $\tilde{\alpha}_i(b, b')$ is the jump point corresponding to bidding b versus bidding b': bidder i achieves higher utility by bidding b iff $v_i < \tilde{\alpha}_i(b, b')$. Now the highest value for which bidder i (weakly) prefers bidding b versus any other higher bid is $\min_{b'>b} \tilde{\alpha}_i(b, b')$; if at this valuation, bidding b also achieves higher utility than bidding any other lower bid, then $\min_{b'>b} \tilde{\alpha}_i(b, b')$ is indeed one of the desired jump points. Otherwise, b is a degenerate bid, in the sense that there is no valuation for which b is an optimal response. Therefore, the jump points introduced in (2) are given by $\alpha_i(b) = \max_{b' \leq b} \min_{b''>b'} \tilde{\alpha}_i(b', b'')$.[5] Clearly then, the $\alpha_i(b)$ can be found in polynomial time. □

4 EXISTENCE AND MEMBERSHIP IN PPAD AND FIXP

The existence of equilibria in our setting can essentially be established by adapting a proof by Athey [1], which relies on Kakutani's fixed point theorem. Unfortunately, proofs that are based on this fixed point theorem cannot easily be turned into membership results for computational classes such as PPAD and FIXP. This is especially true for FIXP which is essentially defined as the class of all problems that can be solved by finding a Brouwer fixed point. In order to circumvent this

[5]The maximization over $b' \leq b$ serves to exclude degenerate cases, e.g. if $b' < b < b''$ but $\tilde{\alpha}_i(b, b'') < \tilde{\alpha}_i(b', b'') < \tilde{\alpha}_i(b, b')$.

obstacle we present a new proof that uses Brouwer's fixed point theorem. In this section, we first present this proof, and then utilize it to prove membership of our problems of interest in PPAD and FIXP.

4.1 Existence of Equilibria via Brouwer's Fixed Point Theorem

Theorem 4.1. *Every first-price auction with continuous subjective priors and finite bidding space admits a monotone non-decreasing and non-overbidding pure Bayes-Nash equilibrium.*

Proof. Let $N = \{1, 2, \ldots, n\}$ be the set of bidders, $F_{i,j}$ the continuous subjective priors, and $0 = b_0, b_1, \ldots, b_m$ be the ordered list of bids, i.e., the elements of $B \subseteq [0, 1]$. Recall that a monotone non-decreasing strategy $\beta_i : [0, 1] \to B$ can be represented by its jump points $\alpha_i(b)$. Let

$$\mathcal{D} = \{\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in ([0, 1]^m)^n \mid \forall i \in N, j \in [m] : \alpha_i(b_{j-2}) \leq \alpha_i(b_{j-1}) \wedge b_j \leq \alpha_i(b_{j-1})\}$$

where we use the convention $\alpha_i(b_{-1}) := 0$ to keep the notation simple. The domain $\mathcal{D}$ is the set of all monotone non-decreasing non-overbidding strategy profiles, represented by their jump points. Note that $\mathcal{D}$ is compact and convex.

In what follows we slightly abuse notation by replacing the strategy profile $\boldsymbol{\beta}$ by its representation $\boldsymbol{\alpha}$ in some terms. Recall the functions $H_i(b, \boldsymbol{\alpha}_{-i})$ defined in Section 3, which represent the probability that bidder i wins the auction, if they bid b. By inspecting the proof of Lemma 3.2, it is easy to see that the quantities G_{jb^-} and g_{jb} are continuous with respect to $\boldsymbol{\alpha}_{-i}$, since the distributions are continuous. As a result, the terms $T(b, n-1, j)$ are also continuous in $\boldsymbol{\alpha}_{-i}$ (by (6)), which implies that $H_i(b, \boldsymbol{\beta}_{-i})$ is also continuous in $\boldsymbol{\alpha}_{-i}$. Since the utility functions can be written as $u_i(b, \boldsymbol{\alpha}_{-i}; v_i) = (v_i - b) \cdot H_i(b, \boldsymbol{\alpha}_{-i})$, it follows that the functions $(\boldsymbol{\alpha}_{-i}, v_i) \mapsto u_i(b, \boldsymbol{\alpha}_{-i}; v_i)$ are continuous.

We now construct a function $G : \mathcal{D} \to \mathcal{D}$. For any bidder $i \in N$ and any $j \in [m]$, define the continuous function $\Delta_j^i : \mathcal{D} \to \mathbb{R}$ by

$$\Delta_j^i(\boldsymbol{\alpha}) = u_i(b_{j-1}, \boldsymbol{\alpha}_{-i}; \alpha_i(b_{j-1})) - \max_{\ell \geq j} u_i(b_\ell, \boldsymbol{\alpha}_{-i}; \alpha_i(b_{j-1})).$$

Now, for any $\boldsymbol{\alpha} \in \mathcal{D}$, let $G(\boldsymbol{\alpha}) = \boldsymbol{\alpha}'$, where for all $i \in N$ and $j = 1, 2, \ldots, m$ (consecutively and in that order)

$$\alpha_i'(b_{j-1}) = \mathrm{T}_{[\max\{b_j, \alpha_i'(b_{j-2})\}, 1]}(\alpha_i(b_{j-1}) + \Delta_j^i(\boldsymbol{\alpha})). \tag{7}$$

Note in particular that this is well-defined, since $\alpha_i'(b_{j-2})$ is defined before $\alpha_i'(b_{j-1})$. The truncation operator immediately ensures that $\boldsymbol{\alpha}' \in \mathcal{D}$. Since G is also clearly continuous, and $\mathcal{D}$ is compact and convex, it follows by Brouwer's fixed point theorem that there exists a $\boldsymbol{\alpha} \in \mathcal{D}$ with $G(\boldsymbol{\alpha}) = \boldsymbol{\alpha}$. It remains to prove that $\boldsymbol{\alpha}$ corresponds to an equilibrium of the auction.

Consider some bidder $i \in N$. We will show that α_i is a best-response to $\boldsymbol{\alpha}_{-i}$ using the characterization of Lemma 3.1. Consider any non-empty interval of non-empty interior $[\alpha_i(b_{j-1}), \alpha_i(b_j)]$, for some $j \in \{0, 1, \ldots, m\}$, where we use the convention that $\alpha_i(b_{-1}) = 0$ and $\alpha_i(b_m) = 1$.

- First, we show that $u_i(b_j, \boldsymbol{\alpha}_{-i}; \alpha_i(b_j)) \geq \max_{\ell > j} u_i(b_\ell, \boldsymbol{\alpha}_{-i}; \alpha_i(b_j))$. Clearly, for $j = m$ this holds trivially. For $j < m$, this can immediately be rephrased as showing $\Delta_{j+1}^i(\boldsymbol{\alpha}) \geq 0$. Now, note that by assumption we have $\alpha_i(b_j) > \alpha_i(b_{j-1})$. Thus, since $\alpha_i(b_j)$ remains fixed under G, it must be that $\alpha_i(b_j) = b_{j+1}$ or $\Delta_{j+1}^i(\boldsymbol{\alpha}) \geq 0$. However, if $\alpha_i(b_j) = b_{j+1}$, then it also trivially holds that $\Delta_{j+1}^i(\boldsymbol{\alpha}) \geq 0$.
- Next, we show that $u_i(b_j, \boldsymbol{\alpha}_{-i}; \alpha_i(b_{j-1})) \geq \max_{\ell < j} u_i(b_\ell, \boldsymbol{\alpha}_{-i}; \alpha_i(b_{j-1}))$. Again, this holds trivially for $j = 0$, so we now consider $j > 0$. By the first bullet above, it holds that

$$u_i(b_j, \boldsymbol{\alpha}_{-i}; \alpha_i(b_j)) = \max_{\ell \geq j} u_i(b_\ell, \boldsymbol{\alpha}_{-i}; \alpha_i(b_j)).$$

As a result, by the monotonicity of the H-functions (see the proof of Lemma 3.1), this continues to hold if we replace $\alpha_i(b_j)$ by $\alpha_i(b_{j-1})$, i.e.,

$$u_i(b_j, \boldsymbol{\alpha}_{-i}; \alpha_i(b_{j-1})) = \max_{\ell \geq j} u_i(b_\ell, \boldsymbol{\alpha}_{-i}; \alpha_i(b_{j-1})).$$

On the other hand, since $\alpha_i(b_{j-1}) < \alpha_i(b_j)$, it follows in particular that $\alpha_i(b_k) < 1$ for all $k < j$. As a result, since $\alpha_i(b_k)$ remains fixed under G, it must be that $\Delta^i_{k+1}(\boldsymbol{\alpha}) \leq 0$ for all $k < j$, i.e.,

$$u_i(b_k, \boldsymbol{\alpha}_{-i}; \alpha_i(b_k)) \leq \max_{\ell \geq k+1} u_i(b_\ell, \boldsymbol{\alpha}_{-i}; \alpha_i(b_k))$$

which by monotonicity of the H-functions (see the proof of Lemma 3.1), continues to hold if we replace $\alpha_i(b_k)$ by $\alpha_i(b_{j-1})$, i.e., for all $k < j$ we have

$$u_i(b_k, \boldsymbol{\alpha}_{-i}; \alpha_i(b_{j-1})) \leq \max_{\ell \geq k+1} u_i(b_\ell, \boldsymbol{\alpha}_{-i}; \alpha_i(b_{j-1})).$$

As a result it follows by induction that for all $k < j$

$$u_i(b_k, \boldsymbol{\alpha}_{-i}; \alpha_i(b_{j-1})) \leq \max_{\ell \geq j} u_i(b_\ell, \boldsymbol{\alpha}_{-i}; \alpha_i(b_{j-1})) = u_i(b_j, \boldsymbol{\alpha}_{-i}; \alpha_i(b_{j-1})).$$

By Lemma 3.1, it immediately follows that α_i is a best-response to $\boldsymbol{\alpha}_{-i}$. Since this holds for all bidders $i \in N$, $\boldsymbol{\alpha}$ is an equilibrium. □

4.2 FIXP Membership

In order to study the exact equilibrium problem for the first-price auction in the context of FIXP, we consider the model where the distributions $F_{i,j}$ are given by algebraic circuits using the operations $\{+, -, \times, /, \max, \min, \sqrt[k]{\cdot}\}$ and rational constants, as is usual in this setting [22]. We show that the proof of existence in the previous section can be turned into a reduction.

Theorem 4.2. *The problem* EXACT-BNE-FPA *lies in* FIXP.

Proof. Clearly, the domain $\mathcal{D}$ of the function $G : \mathcal{D} \to \mathcal{D}$ from the proof of Theorem 4.1 can be represented by a set of linear inequalities that can be constructed in polynomial time in n, m and the representation length of B. Thus, it remains to show that we can construct in polynomial time an algebraic circuit that computes G.

We now describe how to construct a circuit for G that only uses operations $\{+, -, \times, /, \max, \min, \sqrt[k]{\cdot}\}$ and rational constants. First of all, note that probabilities of the form $\Pr_{v_j \sim F_{i,j}}[\beta_j(v_j) \leq b] = F_{i,j}(\alpha_j(b))$ can easily be computed by the circuit, since the (cumulative) distribution functions $F_{i,j}$ are provided as algebraic circuits, and $\boldsymbol{\alpha}$ is the input to the circuit. It follows that the quantities G_{jb^-} and g_{jb} defined in the proof of Lemma 3.2 can also be computed by the circuit. As a result, we can use the dynamic programming procedure described in the proof of Lemma 3.2, to compute the terms $T(b, n-1, j)$ by only using a polynomial number of operations. Note in particular, that the dynamic programming assignment rules can all be implemented using the available set of operations. With the terms $T(b, n-1, j)$ in hand, we can then easily compute the terms $H_i(b, \boldsymbol{\alpha}_{-i})$ for all $b \in B$, and thus evaluate the utility function $u_i(b, \boldsymbol{\alpha}_{-i}; v_i) = (v_i - b) \cdot H_i(b, \boldsymbol{\alpha}_{-i})$ at any given $v_i \in [0, 1]$. Finally, using the utility functions and the max operation we can now compute the terms $\Delta^i_j(\boldsymbol{\alpha})$ from the proof of Theorem 4.1, and then using +, max, min and the constant 1 we can output $\boldsymbol{\alpha}' = G(\boldsymbol{\alpha})$ by noting that

$$\alpha'_i(b_{j-1}) = \max\{\max\{b_j, \alpha'_i(b_{j-2})\}, \min\{1, \alpha_i(b_{j-1}) + g^i_j(\boldsymbol{\alpha})\}\}.$$ □

4.3 PPAD Membership

In order to study the approximate equilibrium problem for the first-price auction in the context of PPAD, we consider a model where the distributions $F_{i,j}$ are *polynomially-computable*, i.e., can be evaluated in polynomial time by a Turing machine.[6] In order to guarantee that an approximate equilibrium with polynomial bit complexity exists, we also assume that the distributions are *polynomially continuous*. For a formal definition of these two standard properties in the context of PPAD, see the full version [26]. In this section we show that in this model, the problem of computing an ε-BNE lies in the class PPAD. We begin by observing that the polynomial-continuity of the distribution functions $F_{i,j}$ implies that the utility functions are also polynomially-continuous. A proof is included in the full version [26].

Lemma 4.3. *If the distributions $F_{i,j}$ are polynomially-continuous, then so are the utility functions $\boldsymbol{\alpha} \mapsto u_i(b, \boldsymbol{\alpha}_{-i}; v_i)$. In more detail, given $\varepsilon > 0$, we can in polynomial time compute $\delta > 0$ such that for all $i \in N$, $b \in B$ and $v_i \in [0, 1]$*

$$\|\boldsymbol{\alpha} - \boldsymbol{\alpha}'\|_\infty \leq \delta \implies |u_i(b, \boldsymbol{\alpha}_{-i}; v_i) - u_i(b, \boldsymbol{\alpha}'_{-i}; v_i)| \leq \varepsilon.$$

In particular, δ can be represented using a polynomial number of bits.

We are now ready to state the main result of this section.

Theorem 4.4. *The problem* ε-BNE-FPA *lies in* PPAD.

Proof Sketch. We show that the existence proof of Theorem 4.1 can be turned into a polynomial-time many-one reduction to the problem of computing an approximate Brouwer fixed point of a polynomially-computable and polynomially-continuous function over a bounded polytope given by linear inequalities, known to lie in PPAD [22, Proposition 2].

Since the distributions $F_{i,j}$ are polynomially-computable, and by the arguments provided in the proof of Theorem 4.2 (including the dynamic programming procedure from Lemma 3.2), it immediately follows that G is polynomially-computable. The polynomial-continuity of G also immediately follows from the polynomial-continuity of the utility functions (Lemma 4.3). Thus, the problem of computing an approximate fixed point of G lies in PPAD. Finally, we show that an approximate fixed point of G yields an approximate equilibrium of the auction. The proof can be found in the full version [26]. □

5 COMPUTATIONAL HARDNESS

In this section we prove computational hardness results for the problem of computing an equilibrium of a first-price auction with subjective priors. Namely, we show that computing an ε-BNE is PPAD-hard, while computing an exact BNE is FIXP-hard. Our computational hardness results are particularly robust, because they hold even if we apply all of the following restrictions:

- the bidding space is $B = \{0, 1/5, 2/5, 3/5, 4/5\}$,
- the value distributions $F_{i,j}$ are given by very simple piecewise constant density functions,
- ε is some sufficiently small *constant*. *(only relevant for ε-BNE)*

In particular, by a simple rescaling argument, the hardness results also hold when the bidding space consists of all monetary amounts that are increments of some fixed denomination (e.g., one cent) up

[6] Note that a function represented as an algebraic circuit (as in the previous section on FIXP) is not necessarily polynomially-computable, e.g., because the circuit can use "repeated squaring" to construct numbers with exponential bit complexity. Conversely, a function that is polynomially-computable cannot necessarily be represented as an algebraic circuit, because a Turing machine is not restricted to using arithmetic gates. We note that these two different models for representing functions are standard for FIXP and PPAD respectively.

to some number m.[7] For example, there exists a sufficiently small constant ε such that it is PPAD-hard to compute an ε-BNE when the bidding space is $B = \{0, 1/100, 2/100, \dots, 99/100, 1, 101/100, \dots, m - 1/100, m\}$.

Together with the corresponding membership results proved in the previous section (Theorems 4.2 and 4.4), we thus obtain the following two theorems, which are the main results of this paper.

Theorem 5.1. *There exists a constant $\varepsilon > 0$ such that the problem ε-BNE-FPA is PPAD-complete.*

Theorem 5.2. *The problem* exact-BNE-FPA *is* FIXP-*complete.*

In the rest of this section, we present the proof of our hardness results. A nice feature of our proof is that we provide a *single* reduction to prove both PPAD- and FIXP-hardness. In more detail, we reduce from the so-called *Generalized Circuit problem*, which has been instrumental for proving PPAD-hardness results for Nash equilibrium computation problems [12, 19, 64]. In fact, we show that it suffices to consider significantly restricted versions of the Generalized Circuit problem when proving hardness results, and that an exact version of the problem can also be used to prove FIXP-hardness. Since we believe that these points may be of independent interest for future works, they are presented separately in Section 5.1. Our reduction from this problem to equilibrium computation in first-price auctions is then presented in Section 5.2.

5.1 The Generalized Circuit Problem

Generalized circuits, defined by Chen et al. [12], can be viewed as a generalization of arithmetic circuits where we also allow *cycles*. This means that instead of representing a function, a generalized circuit represents a certain kind of constraint satisfaction problem. Indeed, the goal in the Generalized Circuit problem is to assign a value to each gate of the circuit such that all the gates are (approximately) satisfied. Importantly, gates are only allowed to take values in $[0, 1]$ and arithmetic operations are truncated accordingly. As a result, it can be shown that by Brouwer's fixed point theorem, there always exists an assignment of values that satisfies all the gates. However, computing even an approximate assignment is already PPAD-hard, i.e., essentially as hard as any Brouwer fixed point computation. We now provide some formal definitions.

Definition 2. A *generalized circuit*[8] with gate-types $\mathcal{G}$ is a list of gates $g_1, g_2, \dots, g_{\mathfrak{m}}$. Every gate g_i is a 3-tuple $g_i = (G, j, k)$, where $G \in \mathcal{G}$ is the type of the gate, and $j, k \in [\mathfrak{m}] = \{1, \dots, \mathfrak{m}\}$ are the indices of the input gates g_j, g_k (i, j, k distinct).

Before describing possible types of gates, we introduce some notation. Let $\mathrm{T} = \mathrm{T}_{[0,1]}$. Furthermore, we use the notation $x = y \pm \varepsilon$ to denote that $|x - y| \le \varepsilon$.

Consider a generalized circuit $g_1, g_2, \dots, g_{\mathfrak{m}}$ and an assignment $\mathbf{v} : [\mathfrak{m}] \to [0, 1]$ of values to its gates. We say that a gate is ε-satisfied by the assignment, if the constraint imposed by this gate is satisfied with error at most ε. The constraint that a gate $g_i = (G, j, k)$ must satisfy depends on its gate-type $G \in \mathcal{G}$, e.g.,

- if $G = G_1$, then $\mathbf{v}[g_i] = 1 \pm \varepsilon$ *(constant 1)*
- if $G = G_+$, then $\mathbf{v}[g_i] = \mathrm{T}(\mathbf{v}[g_j] + \mathbf{v}[g_k]) \pm \varepsilon$ *(addition)*

[7]Note that m should be provided in the input in *unary* representation. This is necessary to ensure that the bidding space has polynomial size, thus allowing efficient computation of best-responses. See the discussion in Section 2 regarding our assumption of an explicit bidding space.

[8]Note that in the usual definition of generalized circuits, every gate also contains a rational parameter $\zeta \in [0, 1]$, which is used by some gate-types, e.g., a gate performing multiplication by the constant ζ. In our definition, gates do not contain this rational parameter, because, as we show in Propositions 5.3 and 5.4, these gate-types are actually not needed for the problems to be hard.

- if $G = G_-$, then $\mathbf{v}[g_i] = \mathrm{T}(\mathbf{v}[g_j] - \mathbf{v}[g_k]) \pm \varepsilon$ *(subtraction)*
- if $G = G_{1-}$, then $\mathbf{v}[g_i] = 1 - \mathbf{v}[g_j] \pm \varepsilon$ *(complement)*
- if $G = G_{\times 2}$, then $\mathbf{v}[g_i] = \mathrm{T}(2 \cdot \mathbf{v}[g_j]) \pm \varepsilon$ *(multiplication by 2)*
- if $G = G_\times$, then $\mathbf{v}[g_i] = \mathbf{v}[g_j] \cdot \mathbf{v}[g_k] \pm \varepsilon$ *(multiplication)*
- if $G = G_{(\cdot)^2}$, then $\mathbf{v}[g_i] = (\mathbf{v}[g_j])^2 \pm \varepsilon$ *(square)*

We are now ready to define the associated computational problem.

Definition 3. Let $\varepsilon > 0$. The problem ε-Gcircuit with gate-types $\mathcal{G}$ is defined as follows: given a generalized circuit $g_1, g_2, \ldots, g_\mathfrak{m}$ with gate-types $\mathcal{G}$, find an assignment $\mathbf{v} : [\mathfrak{m}] \to [0, 1]$ to the gates such that they are all ε-satisfied.

Rubinstein [64] proved that this problem is PPAD-complete for some sufficiently small constant $\varepsilon > 0$ and a relatively large set of gate-types $\mathcal{G}$. In the full version [26], we prove that the problem remains hard, even with a very restricted set of gate-types.

Proposition 5.3. *There exists a constant $\varepsilon > 0$ such that the problem ε-*Gcircuit *with gate-types $\mathcal{G} = \{G_+, G_{1-}\}$ is* PPAD*-complete. This continues to hold if we instead take $\mathcal{G} = \{G_1, G_-\}$.*

We can also define a problem exact-Gcircuit, where the goal is to find an assignment that *exactly* satisfies all constraints (i.e., with $\varepsilon = 0$). In the full version [26], we prove the following result.

Proposition 5.4. *The problem* exact-Gcircuit *with gate-types $\mathcal{G} = \{G_{1-}, G_+, G_{(\cdot)^2}\}$ is* FIXP*-complete. This continues to hold if we instead take $\mathcal{G} = \{G_{1-}, G_{\times 2}, G_\times\}$.*

5.2 Reduction to BNE-FPA

In this section, we present a reduction that achieves the following: given a generalized circuit, it constructs (in polynomial time) an instance of the first-price auction problem, such that for all $\varepsilon \in [0, 1/10^5]$, from any ε-BNE we can extract an 500ε-satisfying assignment for the generalized circuit. Furthermore, this "extraction" of the assignment from an ε-BNE can be done efficiently and, in fact, using a simple so-called separable linear transformation. This ensures that in the case $\varepsilon = 0$, we obtain a so-called SL-reduction from exact-Gcircuit, which yields the FIXP-hardness result [22]. If we let $\tilde{\varepsilon} > 0$ be a constant such that $\tilde{\varepsilon}$-Gcircuit is PPAD-hard, then for $\varepsilon = \min\{1/10^5, \tilde{\varepsilon}/500\}$ the reduction is a valid polynomial-time many-one reduction, which yields the PPAD-hardness result.

An obstacle to obtaining the desired reduction is that it is unclear how to simulate a G_+-gate or a $G_\times$-gate. As a result, we reduce from the Gcircuit problem with gate-types $\mathcal{G} = \{G_{\times 2}, G_{1-}, G_\phi\}$, where $\phi : [0, 1]^2 \to [0, 1]$, $(x, y) \mapsto \frac{1}{4}(x + 1)(y + 1)$. This means that a gate $g_i = (G_\phi, j, k)$ enforces the constraint $\mathbf{v}[g_i] = \phi(\mathbf{v}[g_j], \mathbf{v}[g_k]) \pm \varepsilon$. In the full version [26] we prove that this set of gate-types is sufficient for our desired hardness results.

Lemma 5.5. *Let $\mathcal{G} = \{G_{\times 2}, G_{1-}, G_\phi\}$. There exists a constant $\tilde{\varepsilon} > 0$ such that the problem $\tilde{\varepsilon}$-*Gcircuit *with gate-types $\mathcal{G}$ is* PPAD*-complete. Furthermore,* exact-Gcircuit *with gate-types $\mathcal{G}$ is* FIXP*-complete.*

The reduction. We begin with a high-level description of the reduction. Consider a generalized circuit $g_1, g_2, \ldots, g_\mathfrak{m}$ with gate-types $\mathcal{G} = \{G_{\times 2}, G_{1-}, G_\phi\}$. We construct a first-price auction with bidding space $B = \{0, 1/5, 2/5, 3/5, 4/5\}$ and a set of bidders $N = \{1, 2, \ldots, n\}$ where $n = 10\mathfrak{m}$. For every $i \in [\mathfrak{m}]$, bidder i will "correspond" to gate g_i, in the sense that, in any ε-BNE $\boldsymbol{\beta}$, the position of the second jump point of β_i, i.e., $\alpha_i(1/5)$ will encode the value $\mathbf{v}[g_i]$ that we will assign to gate g_i. Thus, we will refer to the bidders $1, 2, \ldots, \mathfrak{m}$ as *gate-bidders*. The rest of the bidders will be used as intermediate steps to enforce the desired constraints on the strategies of the gate-bidders.

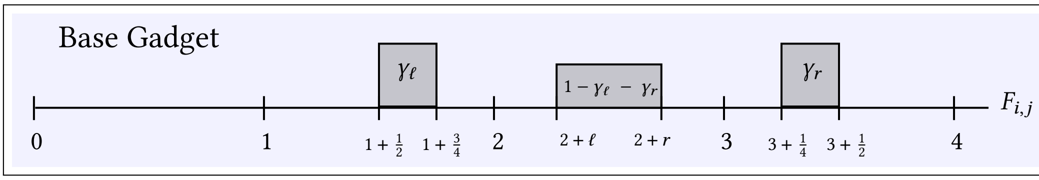

Fig. 2. An illustration of the base gadget. The density of $F_{i,j}$ is depicted. When $\gamma_\ell = \gamma_r = 1/3$, $\ell = 1/3$ and $r = 2/3$ we obtain a *standard* base gadget, which essentially (approximately) "copies" the value $\mathbf{v}[i]$ of the input bidder i to the value $\mathbf{v}[j]$ of the output bidder j.

Accordingly, we will refer to them as *auxiliary-bidders*. Note that for every gate-bidder, there are 9 auxiliary-bidders available (if needed). For convenience, we describe the construction with the value space $[0, 5]$ instead of $[0, 1]$. This is without consequence, since this re-scaling of the instance simply means that we have to replace ε by 5ε at the end. Note that as a result of the re-scaling, the bidding space is now simply $B = \{0, 1, 2, 3, 4\}$.

Valid strategies and encoded value. Let $\boldsymbol{\beta}$ be any ε-BNE of the auction. A bidder $i \in N$ is said to be *valid*, if $\alpha_i(0) \in [1, 1+1/2]$, $\alpha_i(1) \in [2+1/3-2\varepsilon, 2+2/3+2\varepsilon]$, $\alpha_i(2) \in [3+1/2, 5]$ and $\alpha_i(3) = 5$. The bidder i is *almost-valid*, if the condition on $\alpha_i(1)$ is relaxed to $\alpha_i(1) \in [2, 3]$. For every bidder $i \in N$, we define the value encoded by bidder i according to $\boldsymbol{\beta}$, as

$$\mathbf{v}_{\boldsymbol{\beta}}[i] = \begin{cases} \mathrm{T}_{[0,1]}(3(\alpha_i(1) - 2 - 1/3)) & \text{if } i \text{ is valid,} \\ \text{null} & \text{otherwise.} \end{cases}$$

Note that we always have $\mathbf{v}_{\boldsymbol{\beta}}[i] \in [0, 1] \cup \{\text{null}\}$. In the rest of the proof, we drop the subscript $\boldsymbol{\beta}$, since it is understood from the context. Our construction will ensure that for all $i \in [\mathfrak{m}]$, bidder i is valid and as a result $\mathbf{v}[i] \in [0, 1]$. Furthermore, letting $\mathbf{v}[g_i] := \mathbf{v}[i]$ will yield an 100ε-satisfying assignment of the generalized circuit.

Gadgets. The rest of the proof describes the construction of the distribution functions $F_{i,j}$. We begin by constructing some *unary* gadgets. A unary gadget has a single "input" bidder $j \in N$ and an output bidder $i \in N \setminus \{j\}$. The goal of such a gadget is to establish a constraint on β_i that depends on β_j, but not on the strategy of any other bidder. This is achieved by setting $F_{i,k}$ for all $k \in N \setminus \{i, j\}$, such that its (piecewise constant) density function has a single piece of volume 1 lying in $[0, 1]$. As a result, because of the no-overbidding assumption, bidder i will believe that all bidders $k \in N \setminus \{i, j\}$ bid 0 with probability 1. The behavior of the gadget is then determined by the precise construction of $F_{i,j}$.

Base Gadget. The base gadget with input bidder j and output bidder i has four parameters $\gamma_\ell, \gamma_r, \ell, r \in [0, 1]$ with $\gamma_\ell + \gamma_r < 1$ and $r - \ell > 0$. The piecewise constant density function of $F_{i,j}$ is defined as follows. There is a piece of volume γ_ℓ in the interval $[1+1/2, 1+3/4]$, a piece of volume $1 - \gamma_\ell - \gamma_r$ in $[2+\ell, 2+r]$, and finally a piece of volume γ_r in $[3+1/4, 3+1/2]$. See Figure 2 for an illustration.

When the parameters are $(\gamma_\ell, \gamma_r, \ell, r) = (1/3, 1/3, 1/3, 2/3)$, we call this the *standard* base gadget. It will immediately follow from Claim 1 below that if the input bidder j of the standard base gadget is valid, then so is the output bidder i, and furthermore $\mathbf{v}[i] = \mathbf{v}[j] \pm 6\varepsilon$. In other words, this gadget can be used to copy the value encoded by one bidder onto some other bidder. The proof of the claim can be found in the full version [26].

Claim 1. *Let $\gamma_\ell, \gamma_r, \ell, r \in [0, 1]$ with $\gamma_\ell, \gamma_r \geq 1/20$, $\gamma_\ell + \gamma_r < 1$ and $\ell < r$. Consider a base gadget with input bidder j and output bidder i, and parameters $(\gamma_\ell, \gamma_r, \ell, r)$. It holds that:*

- *If the input bidder j is almost-valid, then the output bidder i is also almost-valid.*
- *If $\gamma_\ell, \gamma_r \geq 1/3$ and j is almost-valid, then i is valid and*

$$\mathbf{v}[i] = (3\gamma_\ell - 1) + 3(1 - \gamma_\ell - \gamma_r)\frac{T_{[2+\ell,2+r]}(\alpha_j(1)) - (2+\ell)}{r-\ell} \pm 6\varepsilon.$$

Projection Gadget. The projection gadget with input bidder j and output bidder i, uses two additional auxiliary-bidders k and k', and consists of three uses of the standard base gadget. Concretely, the first standard base gadget has input j and output k, the second such gadget has input k and output k', and the third has input k' and output i. As stated in the claim below, the projection gadget has the notable property that the output bidder i is *always* valid. This gadget will be used to ultimately ensure that all the gate-bidders are valid. The proof of the claim, as well as an illustration of the gadget, can be found in the full version [26].

Claim 2. *The projection gadget with input bidder j and output bidder i ensures that:*
- *the output bidder i is valid, and*
- *if the input bidder j is valid, then* $\mathbf{v}[i] = \mathbf{v}[j] \pm 18\varepsilon$.

$G_{\times 2}$ Gadget. The $G_{\times 2}$ gadget with input bidder j and output bidder i, uses an additional auxiliary-bidder k, and consists of one use of the base gadget and one use of the projection gadget. In more detail, the base gadget has input j, output k and parameters $(\gamma_\ell, \gamma_r, \ell, r) = (1/3, 1/3, 1/3, 1/2)$, while the projection gate has input k and output i. The proof of the following claim, as well as an illustration of the gadget, can be found in the full version [26].

Claim 3. *The $G_{\times 2}$ gadget with input bidder j and output bidder i ensures that:*
- *the output bidder i is valid, and*
- *if the input bidder j is valid, then* $\mathbf{v}[i] = \mathrm{T}(2 \cdot \mathbf{v}[j]) \pm 24\varepsilon$.

G_{1-} Gadget. The G_{1-} gadget with input bidder j and output bidder i uses three additional auxiliary-bidders k_1, k_2, k_3. First, a base gadget is used with input j, output k_1 and parameters $(\gamma_\ell, \gamma_r, \ell, r) = (1/6, 2/3, 1/3, 2/3)$. Next, the density function of F_{k_2,k_1} has a block of volume $2/3$ in $[1+1/2, 1+3/4]$, and a block of volume $1/3$ in $[4, 5]$. Then, we use a base gadget with input k_2, output k_3 and parameters $(\gamma_\ell, \gamma_r, \ell, r) = (1/3, 1/3, 2/3, 5/6)$. Finally, we use a projection gadget with input k_3 and output i.

The crucial idea behind the construction of this gadget is that the third jump point (instead of the second one) is used to encode information in some intermediate step. This allows us to simulate the non-monotone operation $x \mapsto 1 - x$. The proof of the following claim, as well as an illustration of the gadget, can be found in the full version [26].

Claim 4. *The G_{1-} gadget with input bidder j and output bidder i ensures that:*
- *the output bidder i is valid, and*
- *if the input bidder j is valid, then* $\mathbf{v}[i] = 1 - \mathbf{v}[j] \pm 60\varepsilon$.

G_ϕ Gadget. The G_ϕ gadget with input bidders j_1 and j_2 and output bidder i is a binary gadget with additional auxiliary-bidders k_1, k_2, k_3. First of all, for all $t \in N \setminus \{j_1, j_2, k_1\}$, we set $F_{k_1,t}$ to have density function with a single block of volume 1 in $[0, 1]$. We set *both* F_{k_1,j_1} and F_{k_1,j_2} to be distributions as in our construction of the base gadget with parameters $(\gamma_\ell, \gamma_r, \ell, r) = (1/20, 8/20, 1/3, 2/3)$. The density function of F_{k_2,k_1} has a block of volume $1/2$ in $[1 + 1/2, 1 + 3/4]$, and a block of volume $1/2$ in $[3 + 1/2, 5]$. Next, we use a base gadget with input k_2, output k_3 and parameters $(\gamma_\ell, \gamma_r, \ell, r) = (1/3, 1/3(1 + 1/4), 104/200, 779/800)$. Finally, we use a G_{1-} gadget with input k_3 and output i. See Figure 3 for an illustration. We have the following claim, proved in the full version [26].

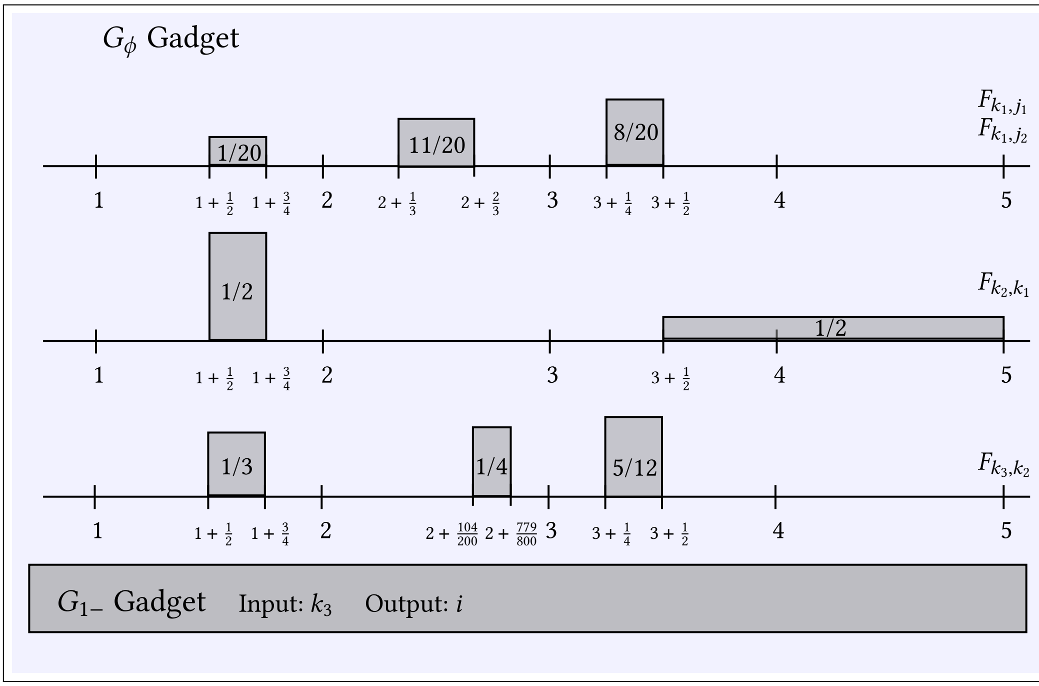

Fig. 3. The G_ϕ gadget. The probability density functions of the corresponding subjective priors are shown.

Claim 5. *The G_ϕ gadget with input bidders j_1, j_2 and output bidder i ensures that:*

- *the output bidder i is valid, and*
- *if the input bidders j_1 and j_2 are valid, then*

$$\mathbf{v}[i] = \phi(\mathbf{v}[j_1], \mathbf{v}[j_2]) \pm 86\varepsilon = \frac{1}{4}(\mathbf{v}[j_1] + 1)(\mathbf{v}[j_2] + 1) \pm 86\varepsilon.$$

Finishing the proof. Using the gadgets we have described above we can now enforce the constraints of the Gcircuit instance. Indeed, for each gate $g_i = (G, j, k)$ where $G \in \mathcal{G} = \{G_{\times 2}, G_{1-}, G_\phi\}$, it suffices to use the gadget corresponding to the gate-type G, with output bidder i and input bidder j (as well as k, in the case $G = G_\phi$). Since the distributions are subjective, we can re-use a bidder j as an input to multiple different gadgets, without any interference. By Claims 3 to 5 it immediately follows that the gate-bidders $1, 2, \ldots, \mathfrak{m}$ must all be valid, since each of them is the output of some gadget. But this means that for any gate $g_i = (G, j, k)$, the input bidder j (and k, if applicable) will be valid, because she is also a gate-bidder. As a result, again by Claims 3 to 5, it follows that the gadgets will correctly enforce their constraints on all values $\mathbf{v}[i]$.

To obtain a solution, it suffices to set $\mathbf{v}[g_i] := \mathbf{v}[i]$ for all $i \in [\mathfrak{m}]$. For the case $\varepsilon = 0$, note that since every gate-bidder i is valid, we have that $\alpha_i(1) \in [2 + 1/3, 2 + 2/3]$ and as a result $\mathbf{v}[i] = \mathrm{T}_{[0,1]}(3(\alpha_i(1) - 2 - 1/3)) = 3(\alpha_i(1) - 2 - 1/3)$, which indeed yields an SL-reduction [22]. By scaling back to the original value space $[0, 1]$, the proof yields that for all $\varepsilon \in [0, 1/10^5]$, from any ε-BNE of the auction we can extract an 500ε-satisfying assignment for the generalized circuit. As discussed at the beginning of the section, this yields both PPAD- and FIXP-hardness.

6 AN EFFICIENT ALGORITHM FOR A CONSTANT NUMBER OF BIDDERS AND BIDS

In this section, we design an algorithm which computes an ε-Bayes-Nash equilibrium of the FPA when (a) the number of bidders n is constant, (b) the size of the bidding space $|B|$ is constant, and (c) the value distributions $F_{i,j}$ of the bidders are *piecewise polynomial.*

To be more precise, our input comprises of:

- a set of bids[9] $B = \{b_0, b_1, \dots, b_{|B|-1}\} \subset [0, 1]$
- a partition[10] of $[0, 1]$ into K intervals $[x_{\ell-1}, x_\ell]$, $\ell = \{1, 2, \dots, K\}$, with rational endpoints
- for each distribution $F_{i,j}$ and each subinterval $[x_{\ell-1}, x_\ell]$, a vector of rationals $(a_0^{i,j,\ell}, \dots, a_d^{i,j,\ell})$.

Then, (the cumulative distribution function of) $F_{i,j}$ is defined as

$$F_{i,j}(z) = F_{i,j}^{\ell}(z), \qquad \text{for} \;\; z \in [x_{\ell-1}, x_\ell],$$

where

$$F_{i,j}^{\ell}(z) = \sum_{\kappa=0}^{d} a_\kappa^{i,j,\ell} z^\kappa \tag{8}$$

is the polynomial representation of $F_{i,j}$ in the ℓ-th interval. Of course, the input should respect the conditions

$$F_{i,j}^1(0) \geq 0, \quad F_{i,j}^K(1) = 1, \quad F_{i,j}^{\ell}(x_\ell) = F_{i,j}^{\ell+1}(x_\ell) \;\; \text{for} \;\; \ell = 1, 2, \dots, K-1,$$

and that each $F_{i,j}^{\ell}$ is nondecreasing on $[x_{\ell-1}, x_\ell]$.

Finally, when we say that n and $|B|$ are fixed, we mean that they are constant functions of the other parameters of the input.

We have the following theorem.

Theorem 6.1. *For a fixed number of bidders, a fixed bidding space, and piecewise polynomial value distributions, an ε-BNE of the first-price auction can be computed in polynomial time, even for subjective priors and even when ε is inversely-exponential in the input size.*

At a high level, the algorithm will perform the following four steps:

(1) It "guesses", for each bidder, an assignment of the jump points of her best-response strategy to the K sub-intervals $[x_{\ell-1}, x_\ell]$ above; intervals may be allocated zero or multiple jump points. Since the number of bidders and the size of the bidding space are constant, there is a total constant number of jump points for all bidders. Therefore, this "guessing" step is an enumeration of all such possible assignments; the subsequent steps of the algorithm are run for any such assignment.

(2) It "guesses" a set of *effective* jump points and bids. This is a technical corner case, to eliminate degenerate cases in which multiple jump points coincide. Again, this can be done via enumeration given that the number of jump points is constant.

(3) It formulates the problem of finding the *exact positions* of the effective jump points (within the intervals corresponding to the guessed allocation above) as a system of polynomial inequalities of polynomially-large degree. A δ-approximate solution to this system can be found using standard methods, in time polynomial in $\log(1/\delta)$ and the input parameters.

[9]Recall that here $|B|$ is *fixed*, i.e., not part of the input.

[10]Our assumption here of a *common* interval partition for the piecewise polynomial representation of all subjective priors $F_{i,j}$ is for the sake of simplicity, and it is not critical for the positive results of this section. In particular, it is not difficult to see that our model can handle different partitions $[x_{\ell-1}^{i,j}, x_\ell^{i,j}]$ with just a polynomial blow-up in the size of the representation; essentially one needs to take the interval partition induced by all points $\{x_\ell^{i,j}\}$.

(4) It "projects" the approximate solution to the "equilibrium space", as defined by the constraints of the aforementioned system, ensuring that the resulting object is indeed an ε-BNE, for some ε that can be made as small as needed, by making δ as small as needed.

The details of the algorithm and the proof of Theorem 6.1 can be found in the full version [26].

7 CONCLUSION AND FUTURE DIRECTIONS

In this paper, we have classified the complexity of computing a Bayes-Nash equilibrium of the first-price auction with subjective priors, by proving that it is PPAD-complete. As we explained in the introduction, our result contributes fundamentally to our understanding of this celebrated auction format, as well as the literature on total search problems and TFNP. The challenging next step is to move towards the special case of the common priors assumption, where the value distribution of each bidder is common knowledge ($F_{i,j} = F_{i',j}$ for all i, i'). Our PPAD-membership result obviously already extends to this case, as it is a special case of the subjective priors setting. The really intriguing question is to extend our PPAD-hardness result to this case as well. To this end, we state the following open problem, which we consider to be one of the most important problems both in computational game theory and in the literature of total search problems.

Open Problem. *What is the complexity of computing an ε-Bayes-Nash equilibrium of the first-price auction with common priors? Is it* PPAD*-complete? Is it polynomial-time solvable? Or could it be complete for some other (smaller) sub-class of* PPAD*?*

A potential candidate for such a smaller class could be the class PPAD ∩ PLS, which was recently shown by Fearnley et al. [24] and Babichenko and Rubinstein [2] to capture the complexity of interesting problems related to optimization via gradient descent, and computing mixed Nash equilibria in congestion games [63] respectively. The class PLS was introduced by Johnson et al. [38] and captures the computation of local minima of some objective function, and notably characterizes the complexity of finding *pure* Nash equilibria in congestion games [23].

Another very meaningful question is to study the case where both the value distributions and the bidding space are discrete. A special case of this setting was studied by Escamocher et al. [21], but they only obtained conclusive results for the case of two bidders with bi-valued distributions. We believe that some of our technical contributions (e.g., the computation of the best response functions or the gadgets used in the PPAD-hardness proof) can be adapted to show similar results for that case as well; we leave the details for future work. Finally, it would be very interesting to identify further tractable special cases for our problem; for example, can we obtain a positive result similar to Theorem 6.1 for more general value distributions?

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