A Term Calculus for Intuitionistic Linear Logic

Nick Benton[1], Gavin Bierman[1], Valeria de Paiva[1] and Martin Hyland[2]

[1] Computer Laboratory, University of Cambridge, UK
[2] Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, UK

Abstract. In this paper we consider the problem of deriving a term assignment system for Girard's Intuitionistic Linear Logic for both the sequent calculus and natural deduction proof systems. Our system differs from previous calculi (e.g. that of Abramsky [1]) and has two important properties which they lack. These are the *substitution property* (the set of valid deductions is closed under substitution) and *subject reduction* (reduction on terms is well-typed). We also consider term reduction arising from cut-elimination in the sequent calculus and normalisation in natural deduction. We explore the relationship between these and consider their computational content.

1 Intuitionistic Linear Logic

Girard's Intuitionistic Linear Logic [3] is a refinement of Intuitionistic Logic where formulae must be used exactly once. Given this restriction the familiar logical connectives become divided into *multiplicative* and *additive* versions. Within this paper, we shall only consider the multiplicatives.

Intuitionistic Linear Logic can be most easily presented within the sequent calculus. The linearity constraint is achieved by removing the *Weakening* and *Contraction* rules. To regain the expressive power of Intuitionistic Logic, we introduce a new logical operator, !, which allows a formula to be used as many times as required (including zero). The fragment we shall consider is given in Fig. 1.

We use capital Greek letters Γ, Δ for sequences of formulae and A, B for single formulae. The system has multiplicative conjunction or tensor, $\otimes$, linear implication, $\multimap$, and a logical operator, !. The *Exchange* rule simply allows the permutation of assumptions. In what follows we shall consider this rule to be implicit, whence the convention that Γ, Δ denote multisets (and not sequences).

The '! rules' have been given names by other authors. $!_{\mathcal{L}-1}$ is called *Weakening*, $!_{\mathcal{L}-2}$ *Contraction*, $!_{\mathcal{L}-3}$ *Dereliction* and $(!_{\mathcal{R}})$ *Promotion*[3]. We shall use these terms throughout this paper. In the *Promotion* rule, $!\Gamma$ means that every formula in the set Γ is modal, in other words, if Γ is the set $\{A_1, \ldots, A_n\}$, then $!\Gamma$ denotes the set $\{!A_1, \ldots, !A_n\}$. We shall defer the question of a term assignment system until Section 3.

2 Linear Natural Deduction

In the natural deduction system, originally due to Gentzen [11], but expounded by Prawitz [9], a deduction is a derivation of a proposition from a finite set of assumption

[3] Girard, Scedrov and Scott [5] prefer to call this rule *Storage*.

$$\frac{}{A \vdash A}\ \textit{Identity}$$

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}\ \textit{Exchange}$$

$$\frac{\Gamma \vdash B \qquad B, \Delta \vdash C}{\Gamma, \Delta \vdash C}\ \textit{Cut}$$

$$\frac{\Gamma \vdash A}{\Gamma, I \vdash A}\ (I_{\mathcal{L}}) \qquad \frac{}{\vdash I}\ (I_{\mathcal{R}})$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\ (\otimes_{\mathcal{L}}) \qquad \frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\ (\otimes_{\mathcal{R}})$$

$$\frac{\Gamma \vdash A \qquad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}\ (\multimap_{\mathcal{L}}) \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\ (\multimap_{\mathcal{R}})$$

$$\frac{\Gamma \vdash B}{\Gamma, !A \vdash B}\ !_{\mathcal{L}-1} \qquad \frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\ !_{\mathcal{L}-2}$$

$$\frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}\ !_{\mathcal{L}-3} \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\ (!_{\mathcal{R}})$$

Fig. 1. (Multiplicative) Intuitionistic Linear Logic

packets, using some predefined set of inference rules. More specifically, these packets consist of a multiset of propositions, which may be empty. This flexibility is the equivalent of the Weakening and Contraction rules in the sequent calculus. Within a deduction, we may "discharge" any number of assumption packets. Assumption packets can be given natural number labels and applications of inference rules can be annotated with the labels of those packets which it discharges.

We might then ask what restrictions need we make to natural deduction to make it linear? Clearly, we need to withdraw the concept of packets of assumptions. A packet must contain exactly one proposition, i.e. a packet is now equivalent to a proposition. A rule which used to be able to discharge many packets of the same proposition, can now only discharge the one. Thus we can label every proposition with a *unique* natural number. We derive the inference rules given in Fig. 2.

The $(\multimap_{\mathcal{I}})$ rule says that we can discharge exactly one assumption from a deduction to form a linear implication. The $(\multimap_{\mathcal{E}})$ rule looks similar to the $(\supset_{\mathcal{E}})$ rule of Intuitionistic Logic. However it is implicit in the linear rule that the assumptions of the two upper deductions are disjoint, i.e. their set of labels are disjoint. This upholds the fundamental feature of linear natural deduction; that all assumptions must have *unique* labels.

$$\frac{\begin{array}{c}[A^x]\\ \vdots \\ B\end{array}}{A \multimap B}\,(\multimap_{\mathcal{I}})_x \qquad \frac{\begin{array}{cc}\vdots & \vdots\\ A\multimap B & A\end{array}}{B}\,(\multimap_{\mathcal{E}})$$

$$\frac{}{I}\,(I_{\mathcal{I}}) \qquad \frac{\begin{array}{cc}\vdots & \vdots\\ A & I\end{array}}{A}\,(I_{\mathcal{E}})$$

$$\frac{\begin{array}{cc}\vdots & \vdots\\ A & B\end{array}}{A\otimes B}\,(\otimes_{\mathcal{I}}) \qquad \frac{\begin{array}{cc} & [A^x][B^y]\\ \vdots & \vdots\\ A\otimes B & C\end{array}}{C}\,(\otimes_{\mathcal{E}})_{x,y}$$

$$\frac{\begin{array}{cc}\vdots & \vdots\\ !B & C\end{array}}{C}\,\textit{Weakening} \qquad \frac{\begin{array}{cc} & [!B^x][!B^y]\\ \vdots & \vdots\\ !B & C\end{array}}{C}\,\textit{Contraction}_{x,y}$$

$$\frac{\begin{array}{c}\vdots\\ !B\end{array}}{B}\,\textit{Dereliction} \qquad \frac{\begin{array}{cccc} & & & [!A_1^{x_1}\cdots !A_n^{x_n}]\\ \vdots & & \vdots & \vdots\\ !A_1 & \ldots & !A_n & B\end{array}}{!B}\,\textit{Promotion}_{x1,\ldots,xn}$$

Fig. 2. Inference Rules in Linear Natural Deduction

The $(\otimes_{\mathcal{I}})$ rule is similar to the $(\wedge_{\mathcal{I}})$ rule of Intuitionistic Logic. It has the same restriction of disjointness of upper deduction assumptions as $(\multimap_{\mathcal{E}})$. In Linear Logic this makes $\otimes$ a *multiplicative* connective. The $(\otimes_{\mathcal{E}})$ rule is slightly surprising. Traditionally in Intuitionistic Logic we provide two projection rules for $(\wedge_{\mathcal{E}})$, namely

$$\frac{A\wedge B}{A} \qquad \frac{A\wedge B}{B}$$

But Intuitionistic Linear Logic decrees that a multiplicative conjunction can *not* be projected over; but rather both components must be used[4]. In the $(\otimes_{\mathcal{E}})$ rule, both components of the pair $A\otimes B$ are used in the deduction of C.

Rules that are of a similar form to $(\otimes_{\mathcal{E}})$ have been considered in detail by Schroeder-Heister [10]. The astute reader will have noticed the similarity between our $(\otimes_{\mathcal{E}})$ rule and the $(\vee_{\mathcal{E}})$ rule of Intuitionistic Logic. This is interesting as we know that $(\vee_{\mathcal{E}})$ is not very well behaved as a logical rule [4, Chapter 10].

[4] Projections are only defined for *additive* conjunction.

Since we have defined a *linear* system, non-linear inference must be given explicitly. *Weakening* allows a deduction to be discarded provided that its conclusion is non-linear. *Contraction* allows a deduction to be duplicated provided that its conclusion is non-linear. *Dereliction* appears to offer two alternatives for formulation. We have given one in Fig. 2, but following the style advocated by Schroeder-Heister, we could give the alternative

$$\dfrac{\begin{array}{cc} & [B^x] \\ \vdots & \vdots \\ !B & C \end{array}}{C}\,\mathit{Dereliction}'_x$$

Most presentations we are aware of use this alternative rule (e.g. [12, 7, 6]); only O'Hearn [8] gives the same rule as ours (although for a variant of linear logic).

Promotion insists that all of the undischarged assumptions at the time of application are modal, i.e. they are all of the form $!A_i$. However, an additional fundamental feature of natural deduction is that it is *closed under substitution*. If we had taken *Promotion* as

$$\dfrac{\begin{array}{c} !A_1 \cdots !A_n \\ \vdots \\ B \end{array}}{!B}\,\mathit{Promotion}$$

(as in all other formulations we know of), then clearly this rule is *not* closed under substitution. For example, substituting for $!A_1$, the deduction

$$\dfrac{C \multimap !A_1 \qquad C}{!A_1}\,(\multimap_{\mathcal{E}})$$

we get the following deduction

$$\dfrac{\begin{array}{c} \dfrac{C \multimap !A_1 \qquad C}{!A_1}\,(\multimap_{\mathcal{E}}) \quad \cdots !A_n \\ \vdots \\ B \end{array}}{!B}\,\mathit{Promotion}$$

which is no longer a valid deduction (the assumptions are not all modal.) We conclude that *Promotion* must be formulated as in Fig. 2, where the substitutions are given explicitly[5].

It is possible to present natural deduction rules in a 'sequent-style', where given a sequent $\Gamma \vdash A$, the multiset Γ represents all the undischarged propositions so far in the deduction, and A represents conclusion of the deduction. We can still label the undischarged assumptions with a unique natural number, but we refrain from doing

[5] Prawitz [9, p.79] encountered similar problems when defining the rule for introduction of necessitation. He defined a notion of essentially modal formulae and needed to keep track of dependencies in the derivation.

so. This formulation should not be confused with the sequent calculus formulation, which differs by having operations which act on the left and right of the turnstile, rather than rules for the introduction and elimination of logical constants.

We now apply the Curry-Howard Correspondence to derive a term assignment system for this natural deduction formulation of Intuitionistic Linear Logic. The Curry-Howard Correspondence essentially annotates each stage of the deduction with a "term", which is an encoding of the construction of the deduction so far. This means that a logic can be viewed as a type system for a term assignment system. The Correspondence also links proof normalisation to term reduction. We shall use this feature in Section 4.

The term assignment system obtained is given in a 'sequent-style' in Fig. 3. We should point out that the unique natural number labels used above, are replaced by (the more familiar) unique variable names.

$$x : A \vdash x : A$$

$$\frac{\Gamma, x : A \vdash e : B}{\Gamma \vdash \lambda x^A.e : A \multimap B}\,(\multimap_{\mathcal{I}}) \qquad \frac{\Gamma \vdash e : A \multimap B \qquad \Delta \vdash f : A}{\Gamma, \Delta \vdash ef : B}\,(\multimap_{\mathcal{E}})$$

$$\vdash * : I \qquad \frac{\Gamma \vdash e : A \qquad \Delta \vdash f : I}{\Gamma, \Delta \vdash \mathsf{let}\ f\ \mathsf{be}\ *\ \mathsf{in}\ e : A}\,(I_{\mathcal{E}})$$

$$\frac{\Gamma \vdash e : A \qquad \Delta \vdash f : B}{\Gamma, \Delta \vdash e \otimes f : A \otimes B}\,(\otimes_{\mathcal{I}}) \qquad \frac{\Gamma \vdash e : A \otimes B \qquad \Delta, x : A, y : B \vdash f : C}{\Gamma, \Delta \vdash \mathsf{let}\ e\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ f : C}\,(\otimes_{\mathcal{E}})$$

$$\frac{\Delta_1 \vdash e_1 :!A_1 \quad \cdots \quad \Delta_n \vdash e_n :!A_n \qquad x_1 :!A_1, \ldots, x_n :!A_n \vdash f : B}{\Delta_1, \ldots, \Delta_n \vdash \mathsf{promote}\ e_1, \ldots, e_n\ \mathsf{for}\ x_1, \ldots, x_n\ \mathsf{in}\ f :!B}\,\textit{Promotion}$$

$$\frac{\Gamma \vdash e :!A \qquad \Delta \vdash f : B}{\Gamma, \Delta \vdash \mathsf{discard}\ e\ \mathsf{in}\ f : B}\,\textit{Weakening} \qquad \frac{\Gamma \vdash e :!A \qquad \Delta, x :!A, y :!A \vdash f : B}{\Gamma, \Delta \vdash \mathsf{copy}\ e\ \mathsf{as}\ x, y\ \mathsf{in}\ f : B}\,\textit{Contraction}$$

$$\frac{\Gamma \vdash e :!A}{\Gamma \vdash \mathsf{derelict}(e) : A}\,\textit{Dereliction}$$

Fig. 3. Term Assignment System for Linear Natural Deduction

We note at once a significant property of the term assignment system for linear natural deduction. Essentially the terms code the derivation trees so that any valid term assignment has a *unique* derivation.

Theorem 1 (Unique Derivation). *For any term t and proposition A, if there is a valid derivation of the form $\Gamma \vdash t : A$, then there is a unique derivation of $\Gamma \vdash t : A$.*

Proof. By induction on the structure of t. □

We are now in a position to consider the question of substitution. In previous work [12], it was shown that substitution does not hold for the term assignment systems considered hitherto. Some thought that this represented a mismatch between the semantics and syntax of linear logic. We can now see that this is not the case. For our system, the substitution property holds.

Theorem 2 Substitution. *If* $\Gamma \vdash a : A$ *and* $\Delta, x : A \vdash b : B$ *then* $\Gamma, \Delta \vdash b[a/x] : B$

Proof. By induction on the derivation $\Delta, x : A \vdash b : B$. □

Before we continue, a quick word concerning the *Promotion* rule. At first sight this seems to imply an ordering of the e_i and x_i subterms. However, the *Exchange* rule (which does not introduce any additional syntax) tells us that any such order is really just the effect of writing terms in a sequential manner on the page. This paper is not really the place to discuss such syntactical questions. Perhaps proof nets (or a variant of them) are the answer.

3 Sequent Calculus

Here we shall consider briefly term assignment for the sequent calculus. For the sequent calculus there is no Curry-Howard Correspondence because we can encode the same proof in many ways. We have to have some further insight into the logic before we can produce the terms. There are (at least) two ways of doing this: semantically or proof-theoretically. More specifically we can either use a model to suggest the term system or use the well-known relationship between sequent calculus and natural deduction directly. Of course, both methods should converge to a single solution! In our case both these methods lead to the term assignment system given in Fig. 4. Our term system is essentially the same as Abramsky's [1] except for the *Dereliction* and *Promotion* rules.

4 Proof Normalisation

Within natural deduction we can produce so-called "detours" in a deduction, which arise where we introduce a logical constant and then eliminate it immediately afterwards. We can define a procedure called *normalisation* which systematically eliminates such detours from a deduction. A deduction which has no such detours is said to be in *normal form*.

We can define the normalisation procedure by considering each pair of introduction and elimination rules in turn, of which there are six. Here we shall just give an example:

$$x : A \vdash x : A$$

$$\frac{\Gamma \vdash e : A \qquad \Delta, x : A \vdash f : B}{\Gamma, \Delta \vdash f[e/x] : B}\ \mathit{Cut}$$

$$\frac{\Gamma \vdash e : A \qquad \Delta, x : B \vdash f : C}{\Gamma, g : A \multimap B, \Delta \vdash f[(ge)/x] : C}\ (\multimap_{\mathcal{L}}) \qquad \frac{\Gamma, x : A \vdash e : B}{\Gamma \vdash \lambda x^A.e : A \multimap B}\ (\multimap_{\mathcal{R}})$$

$$\frac{\Gamma \vdash e : A}{\Gamma, x : I \vdash \mathsf{let}\ x\ \mathsf{be}\ *\ \mathsf{in}\ e : A}\ (I_{\mathcal{L}}) \qquad \frac{}{\vdash * : I}\ (I_{\mathcal{R}})$$

$$\frac{\Delta, x : A, y : B \vdash f : C}{\Delta, z : A \otimes B \vdash \mathsf{let}\ z\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ f : C}\ (\otimes_{\mathcal{L}}) \qquad \frac{\Gamma \vdash e : A \qquad \Delta \vdash f : B}{\Gamma, \Delta \vdash e \otimes f : A \otimes B}\ (\otimes_{\mathcal{R}})$$

$$\frac{\Gamma \vdash e : B}{\Gamma, z : !A \vdash \mathsf{discard}\ z\ \mathsf{in}\ e : B}\ \mathit{Weakening} \qquad \frac{\Gamma, x : !A, y : !A \vdash e : B}{\Gamma, z : !A \vdash \mathsf{copy}\ z\ \mathsf{as}\ x, y\ \mathsf{in}\ e : B}\ \mathit{Contraction}$$

$$\frac{\Gamma, x : A \vdash e : B}{\Gamma, z : !A \vdash e[\mathsf{derelict}(z)/x] : B}\ \mathit{Dereliction} \qquad \frac{\overline{x} : !\Gamma \vdash e : A}{\overline{y} : !\Gamma \vdash \mathsf{promote}\ \overline{y}\ \mathsf{for}\ \overline{x}\ \mathsf{in}\ e : !A}\ \mathit{Promotion}$$

Fig. 4. Term Assignment System for Sequent Calculus

- *Promotion* with *Contraction*

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ !A_1\end{matrix}\ \dots\ \begin{matrix}\vdots\\ !A_n\end{matrix} \qquad \begin{matrix}[!A_1]\dots[!A_n]\\ \vdots\\ B\end{matrix}}{!B}\ \mathit{Prom.} \qquad \begin{matrix}[!B][!B]\\ \vdots\\ C\end{matrix}}{C}\ \mathit{Cont.}$$

normalises to

$$\dfrac{\begin{matrix}\dfrac{[!A_1]\dots[!A_n] \qquad \begin{matrix}[!A_1]\dots[!A_n]\\ \vdots\\ B\end{matrix}}{!B}\ \mathit{Prom.} \quad \dfrac{[!A_1]\dots[!A_n] \qquad \begin{matrix}[!A_1]\dots[!A_n]\\ \vdots\\ B\end{matrix}}{!B}\ \mathit{Prom.}\\ \vdots\\ C\end{matrix} \qquad \begin{matrix}\vdots\\ !A_1\end{matrix}\ \dots\ \begin{matrix}\vdots\\ !A_n\end{matrix}}{C}\ \mathit{Cont.}^*$$

As mentioned earlier, the Curry-Howard Correspondence tells us that we can relate proof normalisation to term reduction. Hence we can annotate the proof tree transformations to produce the (one-step) term reduction rules, which are given in full in Fig. 5. The astute reader will have noticed our use of shorthand in the last two rules. Hopefully, our notation is clear; for example, the term discard e_i in u represents the term discard e_1 in $\ldots$ discard e_n in u.

$(\lambda x^A.t)e$	$\rightarrow_\beta t[e/x]$
let $*$ be $*$ in e	$\rightarrow_\beta e$
let $e \otimes t$ be $x \otimes y$ in u	$\rightarrow_\beta u[e/x, t/y]$
derelict(promote e_i for x_i in t)	$\rightarrow_\beta t[e_i/x_i]$
discard (promote e_i for x_i in t) in u	$\rightarrow_\beta$ discard e_i in u
copy (promote e_i for x_i in t) as y, z in u	$\rightarrow_\beta$ copy e_i as x_i', x_i'' in u[promote x_i' for x_i in t/y, promote x_i'' for x_i in t/z]

Fig. 5. One-step β-reduction rules

Commuting Conversions

We follow a similar presentation to that of Girard [4, Chapter 10]. We use the shorthand notation

$$\frac{C \quad \vdots}{D}\, r$$

to denote an elimination of the premise C, where the conclusion is D and the ellipses represent possible other premises. This notation covers the six elimination rules: $(\multimap_\mathcal{E})$, $(I_\mathcal{E})$, $(\otimes_\mathcal{E})$, *Contraction*, *Weakening* and *Dereliction.* We shall follow Girard and commute the r rule upwards. Here we shall just give an example.

- Commutation of *Contraction*

$$\frac{\dfrac{\begin{array}{c}\vdots \\ !B\end{array} \quad \begin{array}{c}[!B][!B] \\ \vdots \\ C\end{array}}{C}\,\mathit{Contraction} \quad \vdots}{D}\, r$$

which commutes to

$$\dfrac{\begin{array}{c}\vdots\\ !B\end{array} \qquad \dfrac{\begin{array}{c}[!B][!B]\\ \vdots\\ C\end{array} \quad \vdots}{D}\,r}{D}\,\mathit{Contraction}$$

Again we can use the Curry-Howard Correspondence to get the term conversions. We give (all) the term conversions in Fig. 6. We use the symbol $\rightarrow_c$ to denote a commuting conversion.

These commuting conversions, although traditionally dismissed, appear to have some computational significance—they appear to reveal further β-redexes which exist in a term. Let us consider an example; the term

$$(\mathsf{copy}\ e\ \mathsf{as}\ x, y\ \mathsf{in}\ \lambda z^{!A}.\mathsf{discard}\ z\ \mathsf{in}\ x \otimes y)g$$

is in normal form. We can apply a commuting conversion to get the term

$$\mathsf{copy}\ e\ \mathsf{as}\ x, y\ \mathsf{in}\ (\lambda z^{!A}.\mathsf{discard}\ z\ \mathsf{in}\ x \otimes y)g$$

which has an (inner) β-redex. From an implementation perspective, such conversions would ideally be performed at compile-time (although almost certainly not at run-time). Again, as mentioned earlier, a better (i.e. less sequential) syntax might make such conversions unnecessary.

We can now prove subject reduction; namely that (β and commuting) reduction ($\rightarrow_{\beta,c}$) is well-typed. Again this property was thought not to hold [6, 8].

Theorem 3 (Subject Reduction). *If $\Gamma \vdash e : A$ and $e \rightarrow_{\beta,c} f$ then $\Gamma \vdash f : A$.*

Proof. By induction on the derivation of $e \rightarrow_{\beta,c} f$. □

It is evident that the above theorem also holds for $\rightarrow^*_{\beta,c}$ the reflexive and transitive closure of $\rightarrow_{\beta,c}$.

5 Cut Elimination for Sequent Calculus

In this section we consider cut elimination for the sequent calculus formulation of Intuitionistic Linear Logic. Suppose that a derivation in the term assignment system of Fig. 4 contains a cut:

$$\dfrac{\dfrac{}{\Gamma \vdash e : A}\,D_1 \qquad \dfrac{}{\Delta, x : A \vdash f : B}\,D_2}{\Gamma, \Delta \vdash f[e/x] : B}\,\mathit{Cut}$$

If $\Gamma \vdash e : A$ is the direct result of a rule D_1 and $\Delta, x : A \vdash f : B$ the result of a rule D_2, we say that the cut is a (D_1, D_2)-cut. A step in the process of eliminating cuts in the derivation tree will replace the subtree with root $\Gamma, \Delta \vdash f[e/x] : B$ with a tree with root of the form

(let e be $x\otimes y$ in f)g	$\rightarrow_c$ let e be $x\otimes y$ in (fg)
let (let e be $x\otimes y$ in f) be $p\otimes q$ in g	$\rightarrow_c$ let e be $x\otimes y$ in (let f be $p\otimes q$ in g)
discard (let e be $x\otimes y$ in f) in g	$\rightarrow_c$ let e be $x\otimes y$ in (discard f in g)
copy (let e be $x\otimes y$ in f) as p, q in g	$\rightarrow_c$ let e be $x\otimes y$ in (copy f as p, q in g)
let (let e be $x\otimes y$ in f) be $*$ in g	$\rightarrow_c$ let e be $x\otimes y$ in (let f be $*$ in g)
derelict(let e be $x\otimes y$ in f)	$\rightarrow_c$ let e be $x\otimes y$ in (derelict(f))
(let e be $*$ in f)g	$\rightarrow_c$ let e be $*$ in (fg)
let (let e be $*$ in f) be $p\otimes q$ in g	$\rightarrow_c$ let e be $*$ in (let f be $p\otimes q$ in g)
discard (let e be $*$ in f) in g	$\rightarrow_c$ let e be $*$ in (discard f in g)
copy (let e be $*$ in f) as p, q in g	$\rightarrow_c$ let e be $*$ in (copy f as p, q in g)
let (let e be $*$ in f) be $*$ in g	$\rightarrow_c$ let e be $*$ in (let f be $*$ in g)
derelict(let e be $*$ in f)	$\rightarrow_c$ let e be $*$ in (derelict(f))
(discard e in f)g	$\rightarrow_c$ discard e in (fg)
let (discard e in f) be $p\otimes q$ in g	$\rightarrow_c$ discard e in (let f be $p\otimes q$ in g)
discard (discard e in f) in g	$\rightarrow_c$ discard e in (discard f in g)
copy (discard e in f) as p, q in g	$\rightarrow_c$ discard e in (copy f as p, q in g)
let (discard e in f be $*$ in g	$\rightarrow_c$ discard e in (let f be $*$ in g)
derelict(discard e in f)	$\rightarrow_c$ discard e in (derelict(f))
(copy e as x, y in f)g	$\rightarrow_c$ copy e as x, y in (fg)
let (copy e as x, y in f) be $p\otimes q$ in g	$\rightarrow_c$ copy e as x, y in (let f be $p\otimes q$ in g)
discard (copy e as x, y in f) in g	$\rightarrow_c$ copy e as x, y in (discard f in g)
copy (copy e as x, y in f) as p, q in g	$\rightarrow_c$ copy e as x, y in (copy f as p, q in g)
let (copy e as x, y in f) be $*$ in g	$\rightarrow_c$ copy e as x, y in (let f be $*$ in g)
derelict(copy e as x, y in f)	$\rightarrow_c$ copy e as x, y in (derelict(f))

Fig. 6. Commuting Conversions

$$\Gamma, \Delta \vdash t : B$$

The terms in the remainder of the tree may be affected as a result.

Thus to ensure that the cut elimination process extends to derivations in the term assignment system, we must insist on an equality $f[e/x] = t$, which we can read from left to right as a term reduction. In fact we must insist on arbitrary substitution instances of the equality, as the formulae in Γ and Δ may be subject to cuts in the derivation tree below the cut in question.

In this section we are mainly concerned to describe the equalities/reductions which result from the considerations just described. Note however that we cannot be entirely blithe about the process of eliminating cuts at the level of the propositional logic. As we shall see, not every apparent possibility for eliminating cuts should be realized in practice. This is already implicit in our discussion of natural deduction.

As things stand there are 11 rules of the sequent calculus aside from *Cut* (and *Exchange*) and hence 121 a priori possibilities for (D_1, D_2)-cuts. Fortunately most of these possibilities are not computationally meaningful in the sense that they have no effect on the terms. We say that a cut is ***insignificant*** if the equality $f[e/x] = t$ we derive from it as above is actually an identity (up to α-equivalence) on terms (so in executing the cut the term at the root of the tree does not change). Let us begin by considering the insignificant cuts.

First note that any cut involving an axiom rule

$$\frac{}{x : A \vdash x : A}\ \textit{Identity}$$

is insignificant; and the cut just disappears (hence instead of 121 we must now account for 100 cases). These 100 cases of cuts we will consider as follows: 40 cases of cuts the form (R, D) as we have 4 right rules and 10 others; 24 cases of cuts of the form (L, R) as we have 6 left-rules and 4 right ones and finally 36 cases of cuts of the form (L, L). Let us consider these three groups in turn.

Firstly we observe that there is a large class of insignificant cuts of the form (R, D) where R is a right rule: $(\otimes_{\mathcal{R}})$, $(I_{\mathcal{R}})$, $(\multimap_{\mathcal{R}})$, *Promotion*. Indeed all such cuts are insignificant with the following exceptions:

- *Principal cuts.* These are the cuts of the form $((\otimes_{\mathcal{R}}), (\otimes_{\mathcal{L}}))$, $((I_{\mathcal{R}}), (I_{\mathcal{L}}))$, $((\multimap_{\mathcal{R}}), (\multimap_{\mathcal{L}}))$, (*Promotion*, *Dereliction*), (*Promotion*, *Weakening*), (*Promotion*, *Contraction*) where the cut formula is introduced on the right and left of the two rules.
- Cases of the form (R, *Promotion*) where R is a right rule. Here we note that cuts of the form $((\otimes_{\mathcal{R}}))$, *Promotion*), $((I_{\mathcal{R}}))$, *Promotion*) and $((\multimap_{\mathcal{R}})$, *Promotion*) cannot occur; so the only possibility is (*Promotion*, *Promotion*).

Next any cut of the form (L, R) where L is one of the left rules $(\otimes_{\mathcal{L}})$, $(I_{\mathcal{L}})$, $(\multimap_{\mathcal{L}})$, *Weakening*, *Contraction*, *Dereliction* and R is one of the simple right rules $(\otimes_{\mathcal{R}})$, $(I_{\mathcal{R}})$, $(\multimap_{\mathcal{R}})$ is insignificant (18 cases). Also cuts of the form $((\multimap_{\mathcal{L}})$, *Promotion*) and (*Dereliction*, *Promotion*) are insignificant (2 cases). This is one of the things we gain by having actual substitutions in the $(\multimap_{\mathcal{L}})$ and *Dereliction* rules. Thus there remains four further cases of cuts of the form (L, *Promotion*) where L is a left rule.

Lastly the 36 cuts of the form (L_1, L_2), where the L_i are both left rules. Again we derive some benefit from our rules for $(\multimap_{\mathcal{L}})$ and *Dereliction*: cuts of the form $((\multimap_{\mathcal{L}}), L)$ and (*Dereliction*, L) are insignificant. There are hence 24 remaining cuts of interest.

We now summarize the cuts of which we need to take some note. They are:

- Principal cuts. There are six of these.
- Secondary Cuts. The single (strange) form of cut: (*Promotion*, *Promotion*) and the four remaining cuts of form (L, *Promotion*) where L is a left rule other than $(\multimap_{\mathcal{L}})$ or *Dereliction*.
- Commutative Cuts. The twenty-four remaining cuts of the form (L_1, L_2) just described.

5.1 Principal Cuts

The first three cases are entirely familiar and we simply state the resulting β-rules.

$$\mathsf{let}\ f \otimes g\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ h\ \rhd\ h[f/x, g/y] \tag{1}$$

$$\mathsf{let} * \mathsf{be} * \mathsf{in}\ h \rhd h \tag{2}$$

$$h[(\lambda x^A.f)g/y] \rhd h[f[g/x]/y] \tag{3}$$

We shall consider in detail the principal cuts involving the *Promotion* rule.

- (*Promotion*, *Dereliction*)-cut. The derivation

$$\dfrac{\dfrac{!\Gamma \vdash B}{!\Gamma \vdash !B}\,Promotion \qquad \dfrac{B, \Delta \vdash C}{!B, \Delta \vdash C}\,Dereliction}{!\Gamma, \Delta \vdash C}\,Cut$$

is reduced to

$$\dfrac{!\Gamma \vdash B \qquad B, \Delta \vdash C}{!\Gamma, \Delta \vdash C}\,Cut$$

This reduction yields the following term reduction.

$$(f[\mathsf{derelict}(q)/p])[\mathsf{promote}\ y_i\ \mathsf{for}\ x_i\ \mathsf{in}\ e/q] \rhd f[e/p] \tag{4}$$

- (*Promotion*, *Weakening*)-cut. The derivation

$$\dfrac{\dfrac{!\Gamma \vdash B}{!\Gamma \vdash !B}\,Promotion \qquad \dfrac{\Delta \vdash C}{!B, \Delta \vdash C}\,Weakening}{!\Gamma, \Delta \vdash C}\,Cut$$

is reduced to

$$\dfrac{\Delta \vdash C}{!\Gamma, \Delta \vdash C}\,Weakening^*$$

where *Weakening** corresponds to many applications of the *Weakening* rule.

This gives the term reduction

$$\mathsf{discard}\ (\mathsf{promote}\ e_i\ \mathsf{for}\ x_i\ \mathsf{in}\ f)\ \mathsf{in}\ g \rhd \mathsf{discard}\ e_i\ \mathsf{in}\ g \tag{5}$$

- (*Promotion*, *Contraction*)-cut. The derivation

$$\dfrac{\dfrac{!\Gamma \vdash B}{!\Gamma \vdash !B}\,Promotion \qquad \dfrac{!B, !B, \Delta \vdash C}{!B, \Delta \vdash C}\,Contraction}{!\Gamma, \Delta \vdash C}\,Cut$$

is reduced to

$$\dfrac{\dfrac{\dfrac{!\Gamma \vdash B}{!\Gamma \vdash !B}\,Promotion \qquad \dfrac{\dfrac{!\Gamma \vdash B}{!\Gamma \vdash !B}\,Promotion \qquad !B, !B, \Delta \vdash C}{!\Gamma, !B, \Delta \vdash C}\,Cut}{!\Gamma, !\Gamma, \Delta \vdash C}\,Cut}{!\Gamma, \Delta \vdash C}\,Contraction^*$$

or to the symmetric one where we cut against the other B first. This gives the term reduction

$$\begin{array}{r} \mathsf{copy}\ (\mathsf{promote}\ e_i\ \mathsf{for}\ x_i\ \mathsf{in}\ f)\ \mathsf{as}\ y, y'\ \mathsf{in}\ g\ \rhd \\ \mathsf{copy}\ e_i\ \mathsf{as}\ z_i, z_i'\ \mathsf{in}\ g[\mathsf{promote}\ z_i\ \mathsf{for}\ x_i\ \mathsf{in}\ f/y, \mathsf{promote}\ z_i'\ \mathsf{for}\ x_i\ \mathsf{in}\ f/y'] \end{array} \tag{6}$$

Note that the three cases of cut elimination above involving *Promotion* are only considered by Girard, Scedrov and Scott [5] when the context ($!\Gamma$) is empty. If the context is non-empty these are called ***irreducible cuts.***

The principal cuts correspond to the β-reductions in natural deduction. Hence the reductions that we have just given are almost the same as those given in Fig. 5. The differences arise because in the sequent calculus some 'reductions in context' are effected directly by the process of moving cuts upwards. Hence some of the rules just given appear more general.

5.2 Secondary Cuts

We now consider the cases where the *Promotion* rule is on the right of a cut rule. The first case is the 'strange' case of cutting *Promotion* against *Promotion*, then we have the four cases $(\otimes_{\mathcal{L}})$, $(I_{\mathcal{L}})$, *Weakening* and *Contraction* against the rule *Promotion*.

• (*Promotion*, *Promotion*)-cut. The derivation

$$\dfrac{\dfrac{!\Gamma \vdash B}{!\Gamma \vdash !B}\ \mathit{Promotion} \qquad \dfrac{!B, !\Delta \vdash C}{!B, !\Delta \vdash !C}\ \mathit{Promotion}}{!\Gamma, !\Delta \vdash !C}\ \mathit{Cut}$$

reduces to

$$\dfrac{\dfrac{\dfrac{!\Gamma \vdash B}{!\Gamma \vdash !B}\ \mathit{Promotion} \qquad !B, !\Delta \vdash C}{!\Gamma, !\Delta \vdash C}\ \mathit{Cut}}{!\Gamma, !\Delta \vdash !C}\ \mathit{Promotion}$$

Note that it is always possible to permute the cut upwards, as all the formulae in the antecedent are nonlinear.

This gives the term reduction

$$\begin{array}{r} \mathsf{promote}\ (\mathsf{promote}\ z\ \mathsf{for}\ x\ \mathsf{in}\ f)\ \mathsf{for}\ y\ \mathsf{in}\ g\ \rhd \\ \mathsf{promote}\ w\ \mathsf{for}\ z\ \mathsf{in}\ (g[\mathsf{promote}\ z\ \mathsf{for}\ x\ \mathsf{in}\ f/y]) \end{array} \tag{7}$$

• $((\otimes_{\mathcal{L}})$, *Promotion*)-cut. The derivation

$$\dfrac{\dfrac{A, E, \Gamma \vdash !B}{A\otimes E, \Gamma \vdash !B}\ (\otimes_{\mathcal{L}}) \qquad \dfrac{!\Delta, !B \vdash C}{!B, !\Delta \vdash !C}\ \mathit{Promotion}}{A\otimes E, \Gamma, !\Delta \vdash !C}\ \mathit{Cut}$$

reduces to

$$\dfrac{A, E, \Gamma \vdash !B \qquad \dfrac{!B, !\Delta \vdash C}{!B, !\Delta \vdash !C}\,\textit{Promotion}}{\dfrac{A, E, \Gamma, !\Delta \vdash !C}{A \otimes E, \Gamma, !\Delta \vdash !C}\,(\otimes_{\mathcal{L}})}\,\textit{Cut}$$

This gives the term reduction

$$\textsf{promote}\ (\textsf{let}\ z\ \textsf{be}\ x, y\ \textsf{in}\ f)\ \textsf{for}\ w\ \textsf{in}\ g\ \rhd\ \textsf{let}\ z\ \textsf{be}\ x, y\ \textsf{in}\ (\textsf{promote}\ f\ \textsf{for}\ w\ \textsf{in}\ g) \tag{8}$$

- $((I_{\mathcal{L}}), \textit{Promotion})$-**cut**. The derivation

$$\dfrac{\dfrac{\Gamma \vdash !B}{I, \Gamma \vdash !B}\,(I_{\mathcal{L}}) \qquad \dfrac{!\Delta, !B \vdash C}{!B, !\Delta \vdash !C}\,\textit{Promotion}}{I, \Gamma, !\Delta \vdash !C}\,\textit{Cut}$$

reduces to

$$\dfrac{\Gamma \vdash !B \qquad \dfrac{!B, !\Delta \vdash C}{!B, !\Delta \vdash !C}\,\textit{Promotion}}{\dfrac{\Gamma, !\Delta \vdash !C}{I, \Gamma, !\Delta \vdash !C}\,(I_{\mathcal{L}})}\,\textit{Cut}$$

This gives the term reduction

$$\textsf{promote}\ (\textsf{let}\ z\ \textsf{be}\ *\ \textsf{in}\ f)\ \textsf{for}\ w\ \textsf{in}\ g\ \rhd\ \textsf{let}\ z\ \textsf{be}\ *\ \textsf{in}\ (\textsf{promote}\ f\ \textsf{for}\ w\ \textsf{in}\ g) \tag{9}$$

- $(\textit{Weakening}, \textit{Promotion})$-**cut**. The derivation

$$\dfrac{\dfrac{\Gamma \vdash !B}{!A, \Gamma \vdash !B}\,\textit{Weakening} \qquad \dfrac{!\Delta, !B \vdash C}{!B, !\Delta \vdash !C}\,\textit{Promotion}}{!A, \Gamma, !\Delta \vdash !C}\,\textit{Cut}$$

reduces to

$$\dfrac{\Gamma \vdash !B \qquad \dfrac{!B, !\Delta \vdash C}{!B, !\Delta \vdash !C}\,\textit{Promotion}}{\dfrac{\Gamma, !\Delta \vdash !C}{!A, \Gamma, !\Delta \vdash !C}\,\textit{Weakening}}\,\textit{Cut}$$

This gives the term reduction

$$\textsf{promote}\ (\textsf{discard}\ x\ \textsf{in}\ f)\ \textsf{for}\ y\ \textsf{in}\ g\ \rhd\ \textsf{discard}\ x\ \textsf{in}\ (\textsf{promote}\ f\ \textsf{for}\ y\ \textsf{in}\ g) \tag{10}$$

• (*Contraction*, *Promotion*)-**cut**. The derivation

$$\dfrac{\dfrac{!A, !A, \Gamma \vdash !B}{!A, \Gamma \vdash !B}\,\textit{Contraction} \qquad \dfrac{!\Delta, !B \vdash C}{!B, !\Delta \vdash !C}\,\textit{Promotion}}{!A, \Gamma, !\Delta \vdash !C}\,\textit{Cut}$$

reduces to

$$\dfrac{\dfrac{!A, !A, \Gamma \vdash !B \qquad \dfrac{!B, !\Delta \vdash C}{!B, !\Delta \vdash !C}\,\textit{Promotion}}{!A, !A, \Gamma, !\Delta \vdash !C}\,\textit{Cut}}{!A, \Gamma, !\Delta \vdash !C}\,\textit{Contraction}$$

This gives the term reduction

$$\mathsf{promote}\ (\mathsf{copy}\ x\ \mathsf{as}\ y, z\ \mathsf{in}\ f)\ \mathsf{for}\ y\ \mathsf{in}\ g\ \rhd\ \mathsf{copy}\ x\ \mathsf{as}\ y, z\ \mathsf{in}\ (\mathsf{promote}\ f\ \mathsf{for}\ y\ \mathsf{in}\ g) \quad (11)$$

One is tempted to suggest that perhaps the reason why the rule *Promotion* gives us reductions with some sort of computational meaning is because this rule is not clearly either a left or a right rule. It introduces the connective on the right (so it is mainly a right rule), but it imposes conditions on the context on the left. Indeed there does not appear to be any analogous reductions in natural deduction.

5.3 Commutative Cuts

Next we consider briefly the 24 significant cuts of the form (L_1, L_2) where the L_i are both left rules. These correspond case by case to the commutative conversions for natural deduction considered in Section 4. For the most part the reduction rules we obtain from cut elimination are identical with those in Fig. 6. The exceptions are the cases where $(\multimap_{\mathcal{L}})$ is the (second) rule above the cut. In these cases we obtain in place of the first rules in the four groups of six in Fig. 6, the following stronger rules:

$$v[(\mathsf{let}\ z\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ t)u/w] \longrightarrow \mathsf{let}\ z\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ v[tu/w]$$

$$v[(\mathsf{let}\ z\ \mathsf{be}\ *\ \mathsf{in}\ t)u/w] \longrightarrow \mathsf{let}\ z\ \mathsf{be}\ *\ \mathsf{in}\ v[tu/w]$$

$$v[(\mathsf{discard}\ z\ \mathsf{in}\ t)u/w] \longrightarrow \mathsf{discard}\ z\ \mathsf{in}\ v[tu/z]$$

$$v[(\mathsf{copy}\ z\ \mathsf{as}\ x, y\ \mathsf{in}\ t)u/w] \longrightarrow \mathsf{copy}\ z\ \mathsf{as}\ x, y\ \mathsf{in}\ v[tu/w]$$

6 Conclusions and Further Work

In this paper we have considered the question of a term assignment system for both the natural deduction and sequent calculus system. We have also considered the question of reduction in both systems; in natural deduction resulting from normalisation of proofs and in the sequent calculus from the process of eliminating cuts. Our systems have important properties which previous proposals lack, namely closure under substitution and subject reduction.

In the full version of this paper [2] we give a general categorical model for our calculus. We also show how the term assignment system can be alternatively suggested by our semantics.

This paper represents preliminary work; much more remains. Given our term calculus it is our hope that this refined setting should shed new light on various properties of the λ-calculus such as Church-Rosser, strong normalisation and optimal reductions.

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