∃ℝ-Completeness of Stationary Nash Equilibria in Perfect Information Stochastic Games

Kristoffer Arnsfelt Hansen and Steffan Christ Sølvsten

Aarhus University
{arnsfelt,soelvsten}@cs.au.dk

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Abstract

We show that the problem of deciding whether in a multi-player perfect information recursive game (i.e. a stochastic game with terminal rewards) there exists a stationary Nash equilibrium ensuring each player a certain payoff is ∃ℝ-complete. Our result holds for acyclic games, where a Nash equilibrium may be computed efficiently by backward induction, and even for deterministic acyclic games with non-negative terminal rewards. We further extend our results to the existence of Nash equilibria where a single player is surely winning. Combining our result with known gadget games without any stationary Nash equilibrium, we obtain that for cyclic games, just deciding existence of any stationary Nash equilibrium is ∃ℝ-complete. This holds for reach-a-set games, stay-in-a-set games, and for deterministic recursive games.

1 Introduction

The most common solution concept for noncooperative games is that of a Nash equilibrium (NE), which was shown by Nash [25] to be guaranteed to exist in finite games in strategic form. On the other hand, existence of a NE is not guaranteed in more general models of games, and one must therefore settle for weaker solutions. From a computational point of view this leads to the natural problem of deciding whether a given game admits a NE. Likewise, if a NE is guaranteed to exist this leads to the natural problem of computing a NE. In case a NE exists it will generally not be unique, and some NE may be more desirable than others. For instance, if comparing two different NE, *all* players may strictly prefer the first NE and we might consider the second NE undesirable. From a computational point of view this leads to the natural problem of deciding whether a given game admits a NE in which every player receives payoff meeting a given *payoff demand*. The computational complexity of these three basic problems naturally depends heavily on the model of games under consideration.

In the basic setting of finite games in strategic form, the computational complexity of these problems is now well understood. The problem of computing a NE was shown to be PPAD-complete for 2-player games by Daskalakis, Goldberg, and Papadimitriou [13] and Chen and Deng [11] and FIXP-complete for m-player games, when $m \geq 3$, by Etessami and Yannakakis [15]. The problem of deciding existence

1

of a NE meeting given payoff demands was shown to be NP-complete for 2-player games by Gilboa and Zemel [18] and $\exists\mathbb{R}$-complete for m-player games, when $m \geq 3$, by Garg et al. [17].

Littman et al. [24] studied the arguably much simpler case of two-player perfect information extensive form games, which we shall refer to simply as *tree games*. Here a NE is guaranteed to exist and may be computed efficiently by backward induction [33]. In this way one may in fact always find a pure NE. On the other hand, players are in general required to make probabilistic choices in order to ensure maximum possible payoff. While Littman et al. devise an efficient algorithm for computing the set of NE payoffs for *deterministic games*, they show that for two-player games with chance-nodes, it is NP-hard to decide existence of a NE meeting given payoff demands. One may for two-player games also prove NP-membership of this problem, thereby settling its complexity.

A more general setting where backward induction also show existence and efficient computation of NE is that of perfect information games that are given as a directed acyclic graph. We shall refer to these simply as *acyclic games*. Here the strategies of the players may in general depend on past history, but we shall here mainly be interested in the simple case when strategies just depend on the current node of the graph, i.e. stationary strategies.

Our main result is that for m-player perfect information acyclic games, $m \geq 7$, it is $\exists\mathbb{R}$-complete to decide existence of a stationary NE meeting given payoff demands. This problem is thus presumably significantly harder for acyclic games than for tree games. Recently several works have proved $\exists\mathbb{R}$-completeness for decision problems about NE in multiplayer games, but these all concerns games in strategic form [27, 17, 2, 3, 21, 1], or the even more general models of extensive-form games with perfect recall but imperfect information [21] and extensive form games with imperfect recall [20]. In contrast, our results are the first $\exists\mathbb{R}$-completeness results for *perfect information* games.

Acyclic games form a special case of perfect information recursive games, which again form a special case of perfect information stochastic games. The complexity of deciding existence of a NE meeting given payoff demands in multiplayer stochastic games was first studied systematically by Ummels and Wojtczak [32, 30]. Motivated by applications to verification and synthesis of reactive systems, they study the cases of games where players have ω-regular objectives and of mean-payoff games, in addition to the special case of recursive games. Ummels and Wojtczak show that the problem of existence of a NE meeting given payoff constraints[1] is *undecidable* for 10-player recursive games with non-negative terminal rewards or for deterministic 14-player recursive games. Since then, Das et al. [12] improved this, by showing undecidability of recursive games with non-negative terminal rewards with just 5 players. In the more general setting of *concurrent* games, Bouyer et al. [7] even showed undecidability of the problem of existence of a NE where a given player is surely winning for deterministic concurrent 3-players games with reachability objectives.

In order to obtain decidability, Ummels and Wojtczak considered positional and stationary NE. For existence of stationary NE meeting given payoff constraints, they prove NP-hardness for 2-player recursive games with non-negative terminal rewards and for n-player deterministic recursive games (with n being part of the input), and they prove SQRTSUM-hardness for 4-player recursive games with non-negative termi-

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[1]Ummels and Wojtczak consider having both lower bounds (i.e. demands) and upper bounds on payoffs. Their results however also holds with few changes assuming just payoff demands.

nal rewards and for 8-player deterministic recursive games. On the other hand, they show PSPACE-membership of existence of a NE meeting given payoff constraint for recursive games, games with common ω-regular objectives, and mean-payoff games. One may observe that their proofs in fact give $\exists\mathbb{R}$-membership (cf. Section 3.4).

From our initial $\exists\mathbb{R}$-completeness result we show that deciding existence of a stationary NE meeting given payoff demands is $\exists\mathbb{R}$-complete also for deterministic 13-player acyclic games with non-negative terminal rewards. To prove this we make use of a modified version of a gadget constructed by Ummels and Wojtczak [30] to simulate chance nodes. To use this modified gadget we rely on the fact, that we have proved $\exists\mathbb{R}$-hardness for acyclic games. In passing, we also observe that the chance node gadget can be combined with the NP-hardness result for tree games of Littman et al. [24] to give NP-hardness for deterministic tree games.

Combining our results for acyclic games with known gadget games without any stationary NE, we obtain that for cyclic games, just deciding existence of any stationary NE is $\exists\mathbb{R}$-complete. This holds for reach-a-set games, stay-in-a-set games, and for deterministic recursive games. Ummels previously proved NP-hardness and SQRT-SUM-hardness for deciding existence of any stationary NE in reach-a-set games [29, Corollary 4.9]. The gadgets used for the last two constructions were only constructed recently and to use them we again rely on the fact that we have proved $\exists\mathbb{R}$-hardness for acyclic games.

2 Preliminaries

For a finite set S, let $\Delta(S)$ denote the set of probability distributions on S. Denote by $\Delta^n \subseteq \mathbb{R}^{n+1}$ the standard n-simplex $\{x \in \mathbb{R}^{n+1} \mid x \geq 0 \land \sum_{i=1}^{n+1} x_i = 1\}$. We may then identify Δ^n and $\Delta(\{1,\ldots,n+1\})$ in the natural way. Denote by $\Delta_c^n \subseteq \mathbb{R}^n$ the standard corner n-simplex $\{x \in \mathbb{R}^n \mid x \geq 0 \land \sum_{i=1}^n x_i \leq 1\}$.

We next define the types of games, payoffs, and equilibria we consider in this paper. Striving for a uniform exposition we modify common definitions in slight and non-essential ways.

2.1 Perfect Information Stochastic Games

An m-player perfect information stochastic game G is given by a directed graph (digraph) $D = (V,A)$. For $u \in V$ denote by $N^+(u) = \{v \in V \mid (u,v) \in A\}$ the out-neighborhood of u. Let $T = \{u \in V \mid N^+(u) = \emptyset\}$ denote the set of sink nodes of D, also called the *terminals*. The non-terminal nodes are partitioned into disjoint sets $V \setminus T = V_0 \cup V_1 \cup \cdots \cup V_m$, where V_0 is the set of *chance nodes* and V_i is the set of *Player i nodes*, when $i \geq 1$. To each $v \in V_0$ is assigned a probability distribution $\pi_v \in \Delta(N^+(v))$. We say the game G is *deterministic* if $V_0 = \emptyset$.

We fix an initial node $u_0 \in V$ from which play proceeds in rounds. A history of play is an infinite sequence $(u_t)_{t \geq 0}$ such that $(u_t, u_{t+1}) \in A$ when $u_t \notin T$ and $u_{t+1} = u_t$ when $u_t \in T$. Let $\mathscr{H}_\infty$ denote the set of all such histories. A finite history is a prefix of a history of play. For $i \geq 0$ and $v \in V_i$, let $\mathscr{H}_{i,v}$ denote the set of finite histories $(u_t)_{t=0}^T$ ending in node $u_T = v$. For $i \geq 0$, let $\mathscr{H}_i = \cup_{v \in V_i} \mathscr{H}_{i,v}$ denote the finite histories ending in a node in V_i, and finally let $\mathscr{H} = \cup_{i \geq 0} \mathscr{H}_i$ denote the set of all finite histories. If some prefix of a play is contained in $\mathscr{H}_{i,v}$ for some i and $v \in V_i$ we say that the play *reaches v*. A finite history $h = (u_t)_{t=0}^T \in \mathscr{H}$ defines a *subgame G[h]* of G with u_T being the initial node of $G[h]$, play proceeding from u_T in rounds extending h.

2.1.1 Strategies and Equilibria

A strategy τ_i for Player i assigns to each $h \in \mathscr{H}_{i,v}$ a probability distribution $\tau_i(h) \in \Delta(\mathrm{N}^+(v))$. The strategy τ_i is *stationary* if $\tau_i(h) = \tau_i(h')$ for every $h, h' \in \mathscr{H}_{i,v}$ and every $v \in V_i$, i.e. when τ_i only depends on v. The strategy τ_i is *pure* if $\tau_i(h)$ is a single-point distribution for every $h \in \mathscr{H}_i$. A *positional* strategy is a strategy that is simultaneously pure and stationary.

A strategy profile $\tau = (\tau_1, \ldots, \tau_m)$ consists of a strategy for each player. The strategy profile is stationary, pure, or positional if all of its strategies are stationary, pure, or positional, respectively. The set of plays that extend a given finite history $h = (u_t)_{t=0}^T$ is called a cylinder set. The total probability of these plays is given by the product $\prod_{t=0}^{T-1} p_t \cdot (u_{t+1})$ where $p_t = \tau_i(u_0, \ldots, u_t)$ when $u_t \in V_i$ for some $i \geq 1$ and where $p_t = \pi_{u_t}$ when $u_t \in V_0$. By Carathéodory's extension theorem this defines a unique probability measure on the Borel σ-algebra generated by the cylinders sets. Assume now that each Player i is equipped with a bounded Borel measurable utility function $u_i : \mathscr{H}_\infty \to \mathbb{R}$. Let $u : \mathscr{H}_\infty \to \mathbb{R}^m$ denote the vector function of utilities $u(h) = (u_1(h), \ldots, u_m(h))$. Given a strategy profile τ, the expected payoff $U_i(x)$ for Player i is given by $U_i(\tau) = \mathrm{E}_\tau[u_i(h)]$. We let $U(\tau) = (U_1(\tau), \ldots, U_m(\tau))$ denote the payoff profile of τ.

Given a strategy profile τ we let $\tau_{-i} = (\tau_1, \ldots, \tau_{i-1}, \tau_{i+1}, \ldots, \tau_m)$ denote the strategy profile of all players except Player i. Given a strategy τ_i' for Player i, we let $(\tau_{-i}; \tau_i')$ denote the strategy profile $(\tau_1, \ldots, \tau_{i-1}, \tau_i', \tau_{i+1}, \ldots, \tau_m)$. We also denote $(\tau_{-i}; \tau_i')$ by $\tau \setminus \tau_i'$. We say that τ_i' is a *best reply* for Player i to τ if $u_i(\tau \setminus \tau_i') \geq u_i(\tau \setminus \tau_i'')$ for all strategies τ_i'' of Player i. We say that τ is a *Nash equilibrium* (NE) if τ_i is a best reply to τ for every Player i.

Any finite history $h \in \mathscr{H}$ induces a *conditional strategy* $\tau_i[h]$ in the subgame $G[h]$ from a strategy τ_i of Player i. We say that $\tau = (\tau_1, \ldots, \tau_m)$ is a *subgame perfect equilibrium* (SPE) if the conditional strategy profile $\tau[h] = (\tau_1[h], \ldots, \tau_m[m])$ is a NE in $G(h)$, for every $h \in \mathscr{H}$.

2.1.2 Utility Functions

We shall consider several different types of utility functions which in turn gives rise to different classes of games. In a *recursive game* [16] only plays that reach a terminal are assigned non-zero utility. We may thus view the utility functions as functions $u_i : T \to \mathbb{R}$, also known as *terminal rewards*. Recursive games where all terminal payoffs are non-negative or non-positive are respectively called non-negative recursive games and non-positive recursive games. If we normalize the utility functions to take values in the range $[-1, 1]$, every terminal reward vector $u(v)$, for $v \in T$, can be written as a convex combination $\sum_{i=1}^k \alpha_k p_k$ of vectors $p_k \in \{-1, 0, 1\}^m$. By replacing terminal nodes with payoff $u(v)$ with an additional chance node going to a terminal with payoff p_k with probability α_k, we transform a recursive game into an equivalent recursive game with terminal reward vectors from the set $\{-1, 0, 1\}^m$.

In a *mean-payoff game* [19, 14], Player i is given a reward function $r_i : V \to \mathbb{R}$ and the utility assigned to a play $h = (u_t)_{t \geq 0}$ is $u_i(h) = \liminf_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} r_i(u_t)$, for all i. Note that a recursive game is a special case of a mean-payoff game, where all non-terminal nodes are given reward 0.

Utility functions that are indicator functions of Borel sets of plays are called *objectives*. For convenience we simply identify the objective with its defining set of plays. For $S \subseteq V$, the *reachability* objective $\mathrm{Reach}(S)$ is the set of plays that reach a node in

S and the *safety* objective Safe(S) is the set of plays that only reach nodes in S. Games in which all players have reachability objectives are called *reach-a-set games* [10] and games in which all players have safety objectives are called *stay-in-a-set games* [28]. We say that Reach(S) is a terminal reachability objective if $S \subseteq T$ and similarly that Safe(S) is a terminal safety objective if $V \setminus T \subseteq S$. Note that a reach-a-set game with terminal reachability objectives is equivalent to a recursive game with terminal rewards from the set $\{0,1\}$. Likewise, a stay-in-a-set game with terminal safety objectives is equivalent to a recursive game with terminal rewards from the set $\{-1,0\}$. Other objectives of interest are the standard ω-regular objectives of Büchi, co-Büchi, Parity, Streett, Rabin, Muller objectives, see e.g. [32] for definitions. These objectives all generalize *terminal* reachability and safety objectives.

2.1.3 Games on Trees and DAGs

When the digraph D of a given perfect information stochastic game G is *acyclic* we refer to G as an *acyclic game*. Likewise, when D is a tree we refer to G as a *tree game*. A tree game is in particular an acyclic game.

In an acyclic game we have that every play reaches a terminal. For a general acyclic game there may be multiple plays reaching the same terminal, but for a tree game there is a unique play reaching each specific terminal. Thus for a tree game we may view the utility functions simply as terminal payoffs. This also means that tree games correspond exactly to perfect information extensive form games. The method of backward induction [33] shows existence of a (pure) SPE for any terminal payoff acyclic game, and by considering the unfolding of an acyclic game into a tree game, also a SPE for any acyclic game.

2.2 The Existential Theory of the Reals

The existential theory of the reals ETR is the set of all true sentences of the form $\exists x_1, \ldots, x_n \in \mathbb{R} : \varphi(x_1, \ldots, x_n)$, where φ is a quantifier-free Boolean formula of inequalities and equalities of polynomials with integer coefficients. Schaefer and Štefankovič [27] defined the complexity class $\exists \mathbb{R}$ as the closure of ETR under polynomial time many-one reductions. Alternatively, $\exists \mathbb{R}$ is equal to the constant-free Boolean part of the class $NP_{\mathbb{R}}$ [8], which is the analogue class to NP in the Blum-Shub-Smale model of computation [4]. Clearly $NP \subseteq \exists \mathbb{R}$ and from the decision procedure by Canny [9] we have that $\exists \mathbb{R} \subseteq PSPACE$.

A fundamental complete problem for $\exists \mathbb{R}$ is the problem QUAD of deciding whether a system $\mathscr{S}$ of quadratic polynomials in n variables with integer coefficients has a solution in $\mathbb{R}^n$ [4]. Schaefer [26] proved that the similar problem QUAD($B(\mathbf{0},\mathbf{1})$) of deciding whether the system $\mathscr{S}$ has a solution in the unit ball is also $\exists \mathbb{R}$-complete. Analogously one can prove (cf. [21]) that the problem QUAD(Δ_c) of deciding whether the system $\mathscr{S}$ has a solution in the corner simplex Δ_c^n is $\exists \mathbb{R}$-complete.

Define HOMQUAD(Δ) as the problem of deciding whether a system $\mathscr{S}'$ of homogeneous quadratic polynomials in n variables with integer coefficients has a solution in the unit simplex Δ^{n-1}. This problem will form the basis of our $\exists \mathbb{R}$-hardness results.

Proposition 1. HOMQUAD(Δ) *is $\exists \mathbb{R}$-complete.*

Proof. Membership of $\exists \mathbb{R}$ is straightforward. To obtain $\exists \mathbb{R}$-hardness we reduce from QUAD(Δ_c). Suppose $\mathscr{S}$ is a system of quadratic equations in $n-1$ variables $x_1, \ldots, x_{n-1}$.

Introduce the slack variable $x_n = 1 - \sum_{i=1}^{n-1} x_i$. We may then homogenize each polynomial of $\mathscr{S}$ forming the set of homogeneous quadratic polynomials $\mathscr{S}'$, replacing constant terms of the form a by $\sum_{i=1}^{n} \sum_{j=1}^{n} a x_i x_j$ and degree 1 terms of the form $a x_i$ by $\sum_{j=1}^{n} a x_i x_j$. Solutions of $\mathscr{S}$ in Δ_c^{n-1} then correspond exactly to solutions of $\mathscr{S}'$ in Δ^n, by either introducing or dropping the slack variable x_n. $\qquad\square$

3 $\exists\mathbb{R}$-Completeness of Stationary NE

Consider an m-player game G and let $L \in \mathbb{R}^m$ be a vector of *payoff demands*. We say that a strategy profile τ satisfies the payoff demands L if $U(\tau) \geq L$ (with componentwise comparison).

Our main result is a precise characterization of the complexity of deciding existence of stationary NE in perfect information recursive games satisfying given payoff demands.

Theorem 1. *It is $\exists\mathbb{R}$-complete to decide whether for a given m-player recursive game G and payoff demands $L \in \mathbb{R}^m$ there exists a stationary NE τ with $U(\tau) \geq L$. The problem is $\exists\mathbb{R}$-complete even for acyclic 7-player recursive games with non-negative rewards. The same result holds for the analogous problem for stationary SPE.*

Membership of $\exists\mathbb{R}$ follows by expressing that τ is a stationary NE (SPE) satisfying the given payoff demands by an existential first-order formula over the reals. This is done by expressing for all i that τ_i is an optimal solution of the Markov Decision Process (MDP) for Player i that results from fixing the strategies of the other players according to τ_{-i}. Ummels and Wojtczak give a detailed proof for the (more general) case of mean-payoff games [30, Theorem 7] (see the full version of the paper [31] for the actual proof). We return to this in Section 3.4.

Our proof of $\exists\mathbb{R}$-hardness is by reduction from the problem HOMQUAD(Δ) and involves several gadget games that we describe next. In the following let $\mathscr{S}$ be a system of homogeneous quadratic polynomials $q_1(x), \ldots, q_\ell(x)$ in variables $x = (x_1, \ldots, x_n)$. We write $q_k(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^k x_i x_j$ for $k = 1, \ldots, \ell$, and assume that coefficients are scaled to be rational numbers in the interval $[-1, 1]$. That is $a_{ij}^k \in \mathbb{Q}$ and $-1 \leq a_{ij}^k \leq 1$, for all i, j, k.

Remark 1. *For clarity, drawings of the many gadget games are provided in accompanying figures. Chance nodes $v \in V_0$ are diamond-shaped with out-going arcs labelled by the values of π_v. Nodes $v \in V_i$ controlled by Player i are circular nodes labelled with i above and unlabelled out-going arcs. Nodes themselves may also contain labels, though these labels are only used to refer to the specific nodes inside the proofs.*

The first gadget is the variable selection game $\mathscr{G}_{\mathrm{var}}$ shown in Figure 1. An initial chance node leads to Player 1 nodes $v_1, \ldots, v_n$, each chosen with probability $\frac{1}{n}$. In node v_i, Player 1 makes a binary choice between either giving payoff 1 to Player 2 and Player 4 or to Player 3 and Player 5 and all other players payoff 0. We let x_i denote the probability of the former choice, and let $x = (x_1, \ldots, x_n)$. Since $0 \leq x_i \leq 1$, it follows that $x \geq 0$ and $\|x\|_1 \leq n$.

The payoff analysis of $\mathscr{G}_{\mathrm{var}}$ is straightforward.

Lemma 1. *The payoff profile of the subgame of $\mathscr{G}_{\mathrm{var}}$ starting from node v_i is equal to $(0, x_i, 1 - x_i, x_i, 1 - x_i, 0, 0)$, for $i = 1, \ldots, n$. The payoff profile of the game $\mathscr{G}_{\mathrm{var}}$ itself is of the form*

$$\left(0, \tfrac{1}{n}\|x\|_1, 1 - \tfrac{1}{n}\|x\|_1, \tfrac{1}{n}\|x\|_1, 1 - \tfrac{1}{n}\|x\|_1, 0, 0 \right) \ .$$

(a) The nodes v_i of $\mathscr{G}_{\text{var}}$
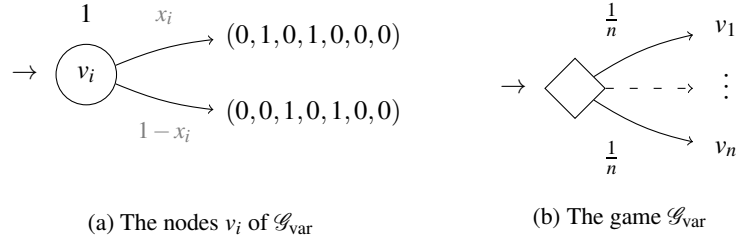
(b) The game $\mathscr{G}_{\text{var}}$

Figure 1: The variable selection game $\mathscr{G}_{\text{var}}$.

We eventually want to enforce that $x \in \Delta^{n-1}$ by payoff demands. Note that this can be obtained locally in $\mathscr{G}_{\text{var}}$ by payoff demands $\frac{1}{n}$ for Player 2 and $\frac{n-1}{n}$ for Player 3.

The second gadget is the multiplication game $\mathscr{G}_{\text{mul}}(i, j, \alpha)$, defined for $1 \le i, j \le n$ and $\alpha \in [0, 1]$ and shown in Figure 2. Note that it connects to nodes v_i and v_j of $\mathscr{G}_{\text{var}}$. By Lemma 1 these may be viewed as terminal nodes with reward vectors $(0, x_i, 1 - x_i, x_i, 1 - x_i, 0, 0)$ and $(0, x_j, 1 - x_j, x_j, 1 - x_j, 0, 0)$, and we shall do so in the analysis in order to be able to analyze $\mathscr{G}_{\text{mul}}(i, j, \alpha)$ separately.

First Player 2 and Player 3 are able to threat to leave to node v_i. Otherwise Player 1 is given a binary choice: either continue or give Player 1 and Player 3 reward 1. We denote by x_i' the probability of the former choice. If Player 1 continues, Player 4 and Player 5 are able to threat to leave to node v_j. Otherwise Player 1 is given a binary choice between terminal reward vectors $(1, 1, 0, 1, 0, \alpha, 1 - \alpha)$ and $(1, 1, 0, 0, 1, 0, 0)$. We denote by x_j' the probability the former choice.
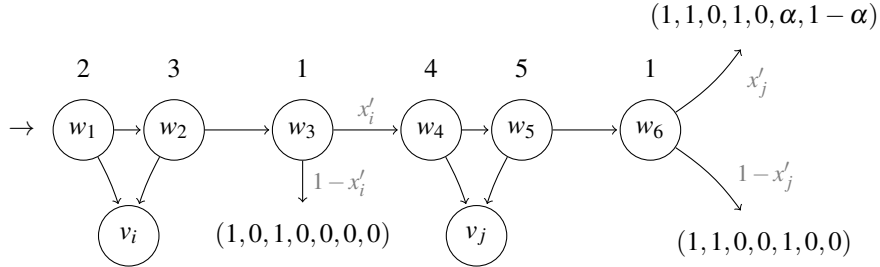

Figure 2: The multiplication game $\mathscr{G}_{\text{mul}}(i, j, \alpha)$.

Lemma 2. *Any NE payoff profile of $\mathscr{G}_{\text{mul}}(i, j, \alpha)$ in which Player 1 receives payoff 1 is of the form*

$$\left(1, x_i, 1 - x_i, x_i x_j, x_i (1 - x_j), \alpha x_i x_j, (1 - \alpha) x_i x_j\right) \ .$$

Proof. For Player 1 to receive payoff 1, neither of Player 2, 3, 4, or 5 execute their threats to leave to v_i or v_j with positive probability. Conditioned on play reaching node w_3, Player 2 and Player 3 receives payoff x_i' and $1 - x_i'$, respectively. Thus, unless $x_i' = x_i$, either Player 2 or Player 3 would gain by leaving to v_i in node w_1 or w_2. Similarly, conditioned on play reaching node w_6, Player 4 and Player 5 receive payoff x_j' and $1 - x_j'$, respectively. Thus, unless $x_j' = x_j$, either Player 4 or Player 5 would gain by leaving to v_j in node w_4 or w_5. It follows that the payoff profile is as claimed. $\square$

7

The third gadget is the polynomial evaluation game $\mathcal{G}_{\mathrm{poly}}(k)$ defined by the polynomial $q_k(x)$ and shown in Figure 3. First Player 6 and Player 7 are in turn able to threat to leave to a terminal giving payoff $1/(2n^2)$ (and all other players payoff 0). Otherwise a chance node leads to the game $\mathcal{G}_{\mathrm{mul}}(i, j, (1 + a_{ij}^k)/2)$, with probability $1/n^2$, for $i, j = 1, \ldots, n$.
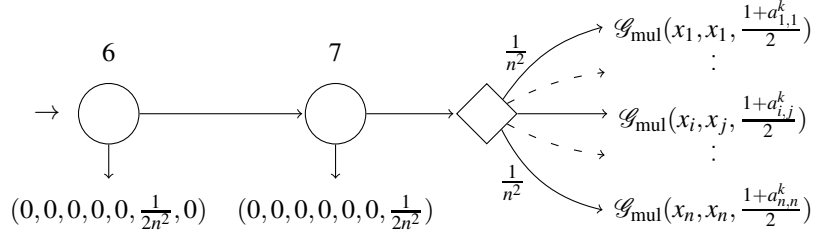

Figure 3: The polynomial evaluation game $\mathcal{G}_{\mathrm{poly}}(k)$

The analysis of $\mathcal{G}_{\mathrm{poly}}(k)$ follows by using Lemma 2.

Lemma 3. *Any NE payoff profile of $\mathcal{G}_{\mathrm{poly}}(k)$ in which Player 1 receive payoff 1 is of the form*

$$\left(1, \tfrac{1}{n}\|x\|_1, 1 - \tfrac{1}{n}\|x\|_1, (\tfrac{1}{n}\|x\|_1)^2, \tfrac{1}{n}\|x\|_1(1 - \tfrac{1}{n}\|x\|_1), \tfrac{1}{2n^2}(\|x\|_1^2 + q_k(x)), \tfrac{1}{2n^2}(\|x\|_1^2 - q_k(x))\right) \ .$$

Proof. For Player 1 to receive payoff 1, neither Player 6 nor Player 7 execute their threats to leave directly to the terminal nodes. Likewise, Player 1 must receive payoff 1 in each of the games $\mathcal{G}_{\mathrm{mul}}(i, j, (1 + a_{ij}^k)/2)$, each of which by Lemma 2 then has the payoff profile $(1, x_i, 1 - x_i, x_i x_j, x_i(1 - x_j), (1 + a_{ij}^k)x_i x_j/2, (1 - a_{ij}^k)x_i x_j/2)$. Taking the average of this over all pairs $i, j \in \{1, \ldots, n\}$ is easily seen to yield the claimed payoff vector. For instance, the payoff of Player 6 is equal to

$$\frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{1 + a_{ij}^k}{2}\right) x_i x_j = \frac{1}{2n^2} \left(\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{j=1}^{n} x_j\right) + \left(\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^k x_i x_j\right)\right)$$

$$= \frac{1}{2n^2}(\|x\|_1^2 + q_k(x)) \ .$$

$\square$

Corollary 1. *In a NE of $\mathcal{G}_{\mathrm{poly}}$ where $\|x\|_1 = 1$ and Player 1 receives payoff 1 we must have that $q_k(x) = 0$.*

Proof. Again, for Player 1 to receive payoff 1, neither of Player 6 and Player 7 execute their threats to leave directly to the the terminal nodes. For this to happen it is required that $\frac{1}{2n^2}(\|x\|_1^2 + q_k(x)) \geq \frac{1}{2n^2}$ and $\frac{1}{2n^2}(\|x\|_1^2 - q_k(x)) \geq \frac{1}{2n^2}$. When $\|x\|_1 = 1$ this implies that $\frac{1}{2n^2}q_k(x) \geq 0$ and $-\frac{1}{2n^2}q_k(x) \geq 0$, and thus $q_k(x) = 0$. $\square$

We now have all the ingredients needed for our $\exists\mathbb{R}$-hardness proof.

Proof of Theorem 1. We already discussed the proof of $\exists\mathbb{R}$ membership. For proving $\exists\mathbb{R}$ hardness we reduce from HOMQUAD(Δ). As above, let $\mathcal{S}$ be a system of homogeneous quadratic polynomials $q_1(x), \ldots, q_\ell(x)$ in variables $x = (x_1, \ldots, x_n)$. We construct
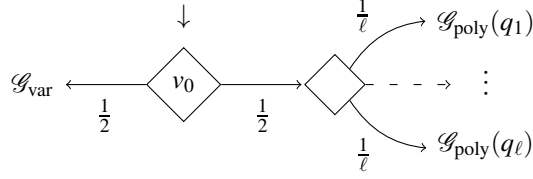
8

Figure 4: The game $\mathscr{G}(\mathscr{S})$.

the game $\mathscr{G}(\mathscr{S})$ as shown in Figure 4. Using initial chance nodes, play proceeds to $\mathscr{G}_{\mathrm{var}}$ with probability $\frac{1}{2}$ and to $\mathscr{G}_{\mathrm{poly}}(k)$ with probability $\frac{1}{2\ell}$, for $k = 1, \ldots, \ell$.

We shall prove that $\mathscr{G}(\mathscr{S})$ has a stationary NE satisfying the payoff demands

$$L = \left(\frac{1}{2}, \frac{1}{n}, 1 - \frac{1}{n}, \frac{1+n}{2n^2}, \frac{2n-1}{2n^2}, \frac{1}{4n^2}, \frac{1}{4n^2} \right) \ ,$$

if and only if there exists $x \in \Delta^{n-1}$ such that $q_k(x) = 0$, for all k.

Suppose first that $\mathscr{G}(\mathscr{S})$ has a NE satisfying the payoff demands L. Since Player 1 receives payoff 0 in $\mathscr{G}_{\mathrm{var}}$, Player 1 must receive payoff 1 in every game $\mathscr{G}_{\mathrm{poly}}(k)$. Thus by Lemma 3 Player 2 and Player 3 receive payoff $\frac{1}{n}\|x\|_1$ and $1 - \frac{1}{n}\|x\|_1$, respectively, which by Lemma 1 also is their payoff in $\mathscr{G}_{\mathrm{var}}$. We conclude that $\frac{1}{n}\|x\|_1$ and $1 - \frac{1}{n}\|x\|_1$ is also the payoff of Player 2 and Player 3 in $\mathscr{G}(\mathscr{S})$. The payoff demands L gives that $\frac{1}{n}\|x\|_1 \geq \frac{1}{n}$ and $1 - \frac{1}{n}\|x\|_1 \geq 1 - \frac{1}{n}$, which implies $\|x\|_1 = 1$. By Corollary 1 this implies $q_k(x) = 0$ for all k.

Suppose now that $x \in \Delta^{n-1}$ is such that $q_k(x) = 0$ for all k. We let Player 1 play according to x in $\mathscr{G}_{\mathrm{var}}$ and consistent to that (i.e. also according to x) in $\mathscr{G}_{\mathrm{mul}}(i, j, (1 + a_{ij}^k)/2)$, for all i, j, k. We let all other players *not* execute any of their threats. It remains to be shown that this strategy profile τ is a NE. No strategy profile yields payoff larger than $\frac{1}{2}$ to Player 1, so Player 1 has no incentive to change strategy. What remains to prove is that no player gains from executing a threat. In $\mathscr{G}_{\mathrm{mul}}(i, j, (1 + a_{ij}^k)/2)$, if either Player 2 or 3 execute their threat to v_i in $\mathscr{G}_{\mathrm{var}}$ then their payoff stays unchanged, since Player 1 is playing according to x_i in both v_i and w_3. Likewise, the payoffs for Player 4 and Player 5 are neither improved by executing their threat to v_j. In $\mathscr{G}_{\mathrm{poly}}(k)$, since $\|x\|_1 = 1$ and $q_k(x) = 0$, Player 6 and Player 7 are both receiving payoff $\frac{1}{2n^2}$ which is also exactly what they would receive by executing their threat. This concludes the proof that x defines a NE. Let us finally note that the payoff profile of $\mathscr{G}_{\mathrm{var}}$ is $(0, \frac{1}{n}, 1 - \frac{1}{n}, \frac{1}{n}, 1 - \frac{1}{n}, 0, 0)$ and the (average of) the payoff profiles of $\mathscr{G}_{\mathrm{poly}}(k)$ is $(1, \frac{1}{n}, 1 - \frac{1}{n}, \frac{1}{n^2}, \frac{1}{n}(1 - \frac{1}{n}), \frac{1}{2n^2}, \frac{1}{2n^2})$, and the average of these is exactly L. Let us finally note that τ is easily seen to in fact be a SPE. $\qquad\square$

Remark 2. *We only used the first* 3 *entries of the payoff demands L to argue a NE satisfying the payoff demand implies the system $\mathscr{S}$ is satisfied. We could therefore equivalently have used the demands $L = (\frac{1}{2}, \frac{1}{n}, \frac{n-1}{n}, 0, 0, 0, 0)$.*

3.1 Deterministic Games

Ummels and Wojtczak [30] constructed a gadget that allows for simulation of a chance node by a deterministic game under certain conditions. Ummels and Wojtczak used this to prove that deciding existence of a stationary NE is SQRTSUM-hard for 8-player

recursive games. Their proof constructs games with both positive and negative terminal rewards. Terminals with negative rewards are used to make a player prefer infinite play away from terminals to such a terminal. We describe their gadget below, modified to have non-negative terminal rewards (and thus not applicable in the reduction of Ummels and Wojtczak). In acyclic games, as we have constructed, any play reaches a terminal, and in turn makes non-negative rewards sufficient.

Let $p \in \Delta_c^n$ with $\|p\|_1 < 1$. We construct a gadget game $\mathscr{G}_{\text{chance}}(p)$ with designated nodes $u_1 \dots, u_n$ in order to simulate a single chance node that for each $i = 1, \dots, n$ continues play in nodes u_i with probability p_i and with the remaining probability $1 - \|p\|_1 > 0$ leads to a terminal $\perp$.

Define $q_1, \dots, q_n$ by

$$q_i = \frac{1 - \sum_{j=i}^n p_j}{1 - \sum_{j=i+1}^n p_j} \quad .$$

Note that $\prod_{j=i}^n q_j = 1 - \sum_{j=i}^n p_j$ for all $i = 1, \dots, n$. The chance node described above can be simulated by the following stochastic process in steps $k = 0, \dots, n$. When $k < n$, we select node u_{n-k} as the outcome with probability $1 - q_{n-k}$, and otherwise proceed to the next step $k + 1$ with probability q_{n-k}. When $k = n$, we end with outcome $\perp$. Then the probability of outcome u_i is equal to

$$(1 - q_i) \prod_{j=i+1}^n q_j = \left(\prod_{j=i+1}^n q_j \right) - \left(\prod_{j=i}^n q_j \right) = \left(1 - \sum_{j=i+1}^n p_j\right) - \left(1 - \sum_{j=i}^n p_j\right) = p_i$$
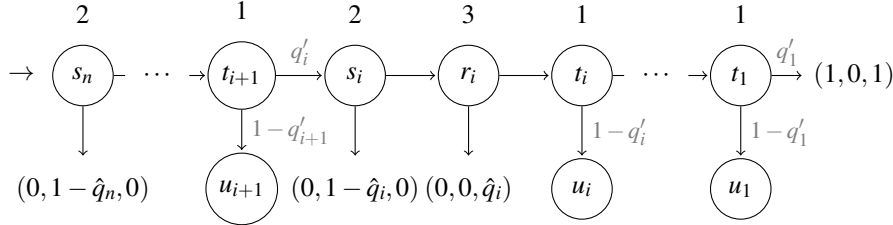
as required.

Figure 5: The game $\mathscr{G}_{\text{chance}}(p)$.

The game $\mathscr{G}_{\text{chance}}(p)$ shown in Figure 5 consists of non-terminal nodes s_i, t_i, r_i and u_i, for $i = 1, \dots, n$, with the initial node being s_n. Player 1 has the role of implementing the chance node, whereas Player 2 and Player 3 incentivize Player 1 to play using the probabilities $q_1, \dots, q_n$ by means of threats. In nodes t_i Player 1 has the choice between node u_i, or when $i > 1$ continuing in node s_{i-1} and when $i = 1$ end in a terminal with rewards $(1, 0, 1)$, corresponding to $\perp$. Before each node t_i, Player 2 and Player 3 are able to threat to end in terminals with rewards $(0, 1 - \hat{q}_i, 0)$ and $(0, 0, \hat{q}_i)$ from nodes s_i and t_i, respectively, where we define $\hat{q}_i$ by

$$\hat{q}_i = \prod_{j=1}^i q_j = \frac{1 - \sum_{j=1}^n p_j}{1 - \sum_{j=i+1}^n p_j} \quad .$$

Lemma 4. *Consider the game derived from $\mathscr{G}_{\text{chance}}(p)$ where each node u_i is changed to be a terminal node with rewards $(1, 1, 0)$. Then, play according to any stationary NE in which Player 1 receives payoff 1 reaches terminal u_i with probability p_i, for all i.*

10

Proof. For Player 1 to receive payoff 1, play must reach either one of the terminals u_i or $\perp$ with probability 1, so no threat is executed by Player 2 and Player 3. Suppose Player 1 chooses node u_i with probability $1 - q_i'$, for every i. Since Player 3 only receives a positive reward in $\perp$, play must reach $\perp$ with positive probability which means $q_i' > 0$ for all i. For a given i and conditioned on play reaching s_i, Player 2 receives payoff $1 - \prod_{j=1}^{i} q_j'$ and Player 3 receives payoff $\prod_{j=1}^{i} q_j'$. For Player 2 and Player 3 to not execute their threats in s_i and r_i it is required that $1 - \prod_{j=1}^{i} q_j' \geq 1 - \prod_{j=1}^{i} q_j$ and $\prod_{j=1}^{i} q_j' \geq \prod_{j=1}^{i} q_j$, which implies $\prod_{j=1}^{i} q_j' = \prod_{j=1}^{i} q_j$. Since this must hold for all i, we have $q_i' = q_i$ for all i, and thus play reaches terminal u_i with probability p_i for all i. □

Using the construction above, we are able to replace the chance nodes in $\mathscr{G}(\mathscr{S})$ used to prove Theorem 1. The chance node v_0 and its two immediate chance nodes can be combined into a single one with outgoing arcs to $v_1, v_2, \ldots, v_n$ in $\mathscr{G}_{\mathrm{var}}$ with probability $\frac{1}{4n}$ and arcs to $\mathscr{G}_{\mathrm{poly}}(q_1), \ldots, \mathscr{G}_{\mathrm{poly}}(q_\ell)$ with probability $\frac{1}{4\ell}$. With the remaining probability of $\frac{1}{2}$ the chance node leads to a new terminal $\perp_0$ where all 7 original players of $\mathscr{G}(\mathscr{S})$ receive payoff 0. This modified chance node can be replaced with the gadget of Lemma 4, which adds three new players to the construction. In the terminals of all subgames (including the terminals added next), the first two newly added players receive payoff 1 while the third receives 0. Similarly, the chance node within $\mathscr{G}_{\mathrm{poly}}(q_k)$ can be replaced with a chance node, that leads with probability $\frac{1}{2}$ to a terminal $\perp_k$ and with probability $\frac{1}{2n^2}$ to $\mathscr{G}_{\mathrm{mul}}(x_i, x_j, (1 + a_{i,j}^k)/2)$, for all i, j, k. To compensate for this new terminal, the payoff in the threats by the original sixth and seventh player is decreased to $\frac{1}{4n^2}$. Since each $\mathscr{G}_{\mathrm{poly}}(q_k)$ is independent of another, these chance nodes can be replaced by only adding another 3 players, rather than 3ℓ.

We therefore obtain the following result for deterministic recursive games.

Theorem 2. *It is $\exists\mathbb{R}$-complete to decide whether for a given m-player deterministic recursive game G and payoff demands $L \in \mathbb{R}^m$ there exists a stationary NE τ with $U(\tau) \geq L$. The problem is $\exists\mathbb{R}$-complete even for 13-player acyclic deterministic games with non-negative rewards. The same result holds for the analogous problem for stationary SPE.*

Proof. The result follows by similar argumentation as in the proof of Theorem 1 on the payoff vector $L = (\frac{1}{8}, \frac{3}{8n}, \frac{3}{8}(1 - \frac{1}{n}), 0, 0, 0, 0, 1, 0, 0, \frac{1}{4}, 0, 0)$ together with Lemma 4. □

3.2 Stationary NE where a Player Wins Almost Surely

Theorem 1 and 2 are both concerned with existence of a Stationary NE given a payoff demand. Both were proven using a payoff demand L that is non-zero for more than one player. In applications of verification and synthesis it is of interest to discern whether there exists a Nash equilibria, where a single player can expect the maximal payoff 1; that is, where a player is *almost surely winning*.

Consider the game $\mathscr{G}_{\mathrm{sure}}(\mathscr{S})$ in Figure 6, where Player 1, Player 2, and Player 3 can choose to not continue into the game $\mathscr{G}(\mathscr{S})$ used in the proof of Theorem 1, but instead end the game early at a terminal with payoff L of Remark 2. An eighth player is added, who always gains payoff 1 in $\mathscr{G}(\mathscr{S})$ but only payoff 0 at the newly added terminal. We then obtain the following extensions of Theorem 1.
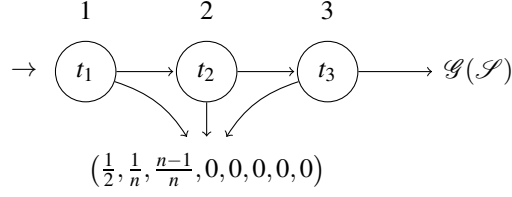
11

Figure 6: The game $\mathscr{G}_{\text{sure}}(\mathscr{S})$

Theorem 3. *It is $\exists\mathbb{R}$-complete to decide whether for a given m-player recursive game G and a given k, there exists a stationary NE in which Player k is almost surely winning. The problem is $\exists\mathbb{R}$-complete even for acyclic 8-player recursive games with non-negative rewards. The same result holds for the analogous problem for stationary SPE.*

Proof. In a stationary NE of $\mathscr{G}_{\text{sure}}(\mathscr{S})$, Player 8 is almost surely winning if and only if neither Player 1, Player 2, nor Player 3 execute their threat to end the game early. The payoff for executing the threat is respectively $\frac{1}{2}$, $\frac{1}{n}$, and $\frac{n-1}{n}$, which also is their payoff in the proof of Theorem 1. That is, the three players effectively enforce the payoff demand L of Remark 2 to $\mathscr{G}(\mathscr{S})$. □

To extend Theorem 2, we add two more threats t_8 and t_{10} to $\mathscr{G}_{\text{sure}}(\mathscr{S})$ before going into $\mathscr{G}(\mathscr{S})$. Here, Player 8 and Player 10, who implement the chance nodes, enforce their part of the payoff. The payoff on all threats is then the payoff demand mentioned in the proof of Theorem 2.

Theorem 4. *It is $\exists\mathbb{R}$-complete to decide whether for a given m-player deterministic recursive game G and a given k, there exists a stationary NE in which Player k is almost surely winning. The problem is $\exists\mathbb{R}$-complete even for 14-player acyclic deterministic games with non-negative rewards. The same result holds for the analogous problem for stationary SPE.*

3.3 Stationary NE without Payoff Demands

Theorem 1 settles the complexity of deciding existence of stationary NE satisfying payoff demands. While deciding the mere existence of any NE may seem to be an easier problem, we show that is exactly as hard.

Suppose that we have an *m*-player gadget game $\mathscr{G}_{\text{noNE}}$ which does not have a stationary NE and Player 1, 2, and 3 receive payoff 0 for *all* strategy profiles of the players; examples of such gadgets will be elaborated below. Let L of Remark 2 to $\mathscr{G}(\mathscr{S})$. Construct now the game $\mathscr{G}_{\exists\text{NE}}(\mathscr{S})$ shown in Figure 7, where similar to $\mathscr{G}_{\text{sure}}(\mathscr{S})$ the first three players can choose not to go into $\mathscr{G}(\mathscr{S})$ but into a different subgame. On this alternative path, a chance node t_4 leads with probability $\frac{1}{2}$ to a terminal, which has twice the payoff demand L. In the other case, the chance node leads to $\mathscr{G}_{\text{noNE}}$.

Lemma 5. *The game $\mathscr{G}_{\exists\text{NE}}(\mathscr{S})$ has a stationary NE if and only if there is a stationary NE of $\mathscr{G}(\mathscr{S})$ satisfying the payoff demands L.*

Proof. Since $\mathscr{G}_{\text{noNE}}$ does not permit a stationary NE, the game $\mathscr{G}_{\exists\text{NE}}(\mathscr{S})$ has a NE if and only if none of Player 1, Player 2, and Player 3 execute their threat to go to t_4.
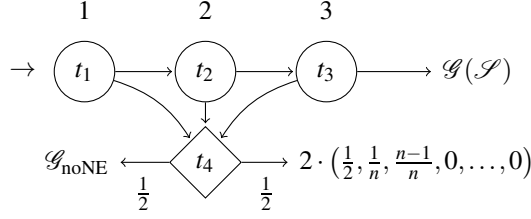
Figure 7: The game $\mathscr{G}_{\exists NE}(\mathscr{S})$

Similarly to the proof of Theorem 3, the three players enforce the payoff demand L of Remark 2 to $\mathscr{G}(\mathscr{S})$. □

Boros and Gurvich [5] and Kuipers et al. [23] (cf. [29, Proposition 3.3]) construct a (cyclic) 3-player recursive game with non-negative rewards which has no stationary NE. We may let Player 4, 5, and 6 take the role of playing in this game, letting Player 1, 2, and 3 receive reward 0 in all terminals. Together with Theorem 1 we obtain the following result.

Theorem 5. *It is $\exists\mathbb{R}$-complete to decide whether a given m-player recursive game has a stationary NE, even for 7-player recursive games with non-negative rewards.*
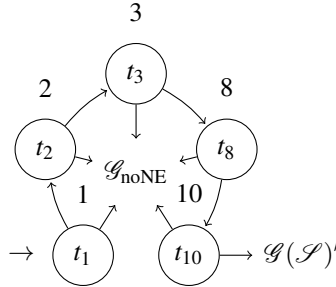

Figure 8: The game $\mathscr{G}'_{\exists NE}(\mathscr{S})$

In continuation of Section 3.1 we would like to dispense with the chance node t_4 to thereby combine Theorem 2 with Theorem 5. We thus consider the game in Figure 8, where t_4 has been removed in favor of going directly to $\mathscr{G}_{noNE}$ and a threat is added for the chance node implemented by Player 8 and 10. Unlike above, since play never reaching a terminal results in payoff 0, then it is not possible to guarantee positive payoffs in the $\mathscr{G}_{noNE}$. Instead, let $\mathscr{G}(\mathscr{S})'$ be the game obtained from $\mathscr{G}(\mathscr{S})$ of Theorem 2 where all terminal rewards of Player 1, 2, 3, 8, and 10 have been decreased by $\frac{1}{8}, \frac{3}{8n}, \frac{3n-3}{8n}, 1$, and $\frac{1}{4}$, respectively. Since the game is acyclic, and hence reaches a terminal with probability 1, this does not change the NE of the game, but just subtracts $\frac{1}{8}, \frac{3}{8n}, \frac{3n-3}{8n}, 1$, and $\frac{1}{4}$ from the NE payoffs of Player 1, 2, 3, 8, and 10, respectively.

Boros et al. [6] recently constructed a *deterministic* 3-player recursive game without a stationary NE. As above, we can let Player 4, 5, and 6 take the role of playing in this game, letting Player 1, 2, and 3 receive reward 0 in all terminals. Repeating the arguments in the proof of Lemma 5 and Theorem 2 we obtain the following result.

13

Theorem 6. *It is $\exists\mathbb{R}$-complete to decide whether a given m-player deterministic recursive game has a stationary NE, even for $m = 13$.*

3.4 ω-Regular Objectives and Mean-Payoff Games

Ummels and Wojtczak proved membership of PSPACE by giving reductions to ETR for the problem of deciding existence of a stationary NE meeting given payoff constraints in several classes of games. For perfect information stochastic games where all players have Streett or Rabin objectives (called Streett-Rabin games), or where all players have Muller objectives a non-determistic polynomial time many-one reduction to ETR is constructed [32]. For perfect information mean-payoff games a (deterministic) polynomial time many-one reduction to ETR is constructed [30]. Using the characterization of $\exists\mathbb{R}$ in terms of nondeterministic Blum-Shub-Smale machines it is straightforward that the many-one reductions may be combined with decision of ETR, thereby proving $\exists\mathbb{R}$-membership for the problems of deciding existence of NE meeting given payoff constraints.

Street-Rabin games generalize reach-a-set games and stay-in-a-set games where all objectives are terminal. One may prove $\exists\mathbb{R}$-membership for general reach-a-set and stay-in-a-set games in a similar way as Ummels and Wojtczak did.

Theorem 7. *It is $\exists\mathbb{R}$-complete to decide whether a given m-player perfect information reach-a-set game has a stationary NE, even for $m = 7$.*

Proof. Recursive games with non-negative rewards may, after normalizing rewards to $[0, 1]$, be viewed as a special case of reach-a-set games. The result then follows from Theorem 5. □

Theorem 8. *It is $\exists\mathbb{R}$-complete to decide whether a given m-player perfect information stay-in-a-set game has a stationary NE, even for $m = 7$.*

Proof. Hansen and Raskin [22] constructed a 2-player perfect information stay-in-a-set game without any stationary NE. We may use this game in place of $\mathscr{G}_{noNE}$ in the proof of Theorem 5. Namely, consider transforming the game $\mathscr{G}_{\exists NE}(\mathscr{S})$ by first dividing all rewards by 2 and then subtracting 1 from all rewards. This does not alter the set of NE of the game, but maps all rewards to the interval $[-1, 0]$, which may then be viewed as a stay-in-a-set game with terminal safety objectives. We may then replace $\mathscr{G}_{noNE}$ by the 2-player stay-in-a-set game of Hansen and Raskin, where we let Player 4 and Player 5 take the role of the 2 players and all nodes of this game are excluded from the safe sets of Player 1, 2, and 3. □

Let us finally consider mean-payoff games. Ummels and Wojtczak [30] note that in mean-payoff games, non-negative fractional terminal rewards may be simulated with a simple cycle where all rewards are chosen from the set $\{0, 1\}$. From Theorem 2 and the above $\exists\mathbb{R}$-membership result we thus obtain the following.

Theorem 9. *It is $\exists\mathbb{R}$-complete to decide whether a given m-player deterministic perfect information mean-payoff game where all rewards are 0 or 1 has a stationary NE that satisfies a given payoff demand, even for $m = 13$.*

4 Equilibria in Tree Games

Littman et al. [24] proved it NP-hard to decide existence of a NE satisfying given payoff demands for 2-player game trees. The proof applies both to positional and stationary NE, as well as positional and stationary SPE. We describe a variation of their proof below for completeness.

Recall that PARTITION is the NP-complete problem of deciding whether for given $a \in \mathbb{Z}_+^n$ there exist $S \subseteq \{1, \ldots, n\}$ such that $\sum_{i \in S} a_i = K/2$, where $K = \sum_{i=1}^n a_i$. For a rational valued vector $a \in \mathbb{Q}_+^n$, which is possible in our constructions, the problem is even strongly NP-complete [34]. We define $\mathscr{G}_{\text{PARTITION}}(a)$ to be the game depicted in Figure 9, where an initial chance node selects an item i uniformly at random, Player 1 can then choose to give the item to Player 2 (i.e. choose rewards $(0, a_i)$), or pass the turn to Player 2, who may either give the item to Player 1 (i.e. choose rewards $(a_i, 0)$ or discard it (i.e. choose rewards $(0, 0)$). Consider the subgame at node u_i, let p_i be
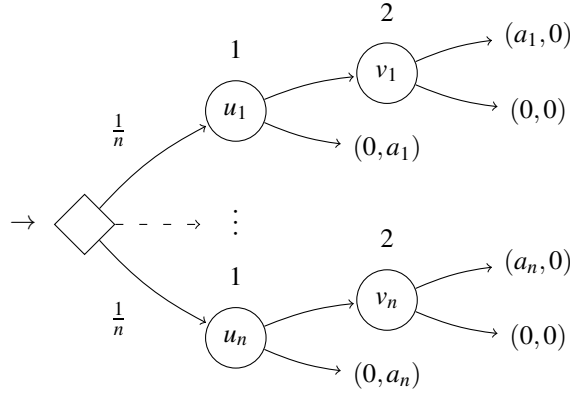

Figure 9: The partition game $\mathscr{G}_{\text{PARTITION}}(a)$.

the probability that Player i gives the item to the other player, $i = 1, 2$. In a NE we have either $(p_1, p_2) = (1, 0)$ or $p_1 = 0$ (and p_2 arbitrary). Thus the only NE (which are also positional and subgame perfect) where the total payoff of the players is a_i are $(p_1, p_2) = (1, 0)$ and $(p_1, p_2) = (0, 1)$. It follows that NE in the game $\mathscr{G}_{\text{PARTITION}}(a)$ in which both players receive payoff $K/2$ are positional and SPE and they correspond exactly to equal partitions of the integers $a_1, \ldots, a_n$. This gives a reduction showing NP-hardness of deciding existence of a NE satisfying payoff demands, even for 2-player games.

Theorem 10 (Littman et al. [24]). *It is* NP-*hard to decide whether for a given 2-player tree game there exist a NE satisfying given payoff demands. This hold for both positional NE, stationary NE, positional SPE and stationary SPE.*

It is not difficult to prove NP-membership, and with it strongly NP-completeness due to [34], for existence of NE (SPE) for 2-player tree games or for existence of *positional* NE (SPE) for *m*-player tree games, that satisfies given payoff demands. Proving NP-membership for existence of stationary NE (SPE) in *m*-player tree games satisfying given payoff demands is to our best knowledge yet an open problem.

Using the chance node gadget of Ummels and Wojtczak of Lemma 4 gives strongly NP-hardness for deterministic tree games.

Corollary 2. *It is strongly* NP-*hard to decide whether for a given 5-player deterministic tree game there exists a stationary NE satisfying given payoff demands. This holds also for positional SPE.*

5 Conclusion

In this paper we have focused on the complexity of decision problems concerning stationary NE in perfect information stochastic games, and we have obtained the first ∃ℝ-completeness result for such games, even for *acyclic* games. While existence of NE with payoff constraints is undecidable for general games, as shown by Ummels and Wojtczak [32], it is decidable for acyclic games. Indeed, for acyclic games, (general) NE correspond to stationary NE in the unfolding of the game into a tree game. We consider it an interesting problem to determine the precise complexity of existence of (general) NE meeting given payoff demands for acyclic games.

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