

# The Polynomial Hierarchy Collapses

Valentin Bura<sup>a</sup>

<sup>a</sup>*Department of Computer Science, University of Liverpool L69 3BX*

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## Abstract

This paper illustrates the power of Gaussian Elimination by adapting it to the problem of Exact Satisfiability. For 1-in-3 SAT instances with non-negated literals we are able to obtain considerably smaller equivalent instances of 0/1 Integer Programming restricted to equality only.

Thus we obtain an upper bound for the complexity of its counting version of  $\mathcal{O}(2\kappa r 2^{(1-\kappa)r})$  for number of variables  $r$  and clauses-to-variables ratio  $\kappa$ . Combining this method with previous results gives a time and space complexity for the counting problem of  $\mathcal{O}(4/3|V|2^{3|V|/8})$  and  $\mathcal{O}(4/3|V|2^{3|V|/16})$ .

Our method shows that Positive instances of 1-in-3 SAT may be reduced to significantly smaller instances of I.P. in the following sense: any such instance of  $|V|$  variables and  $|C|$  clauses can be polynomial-time reduced to an instance of 0/1 Integer Programming with equality only, of size at most  $2/3|V|$  variables and at most  $|C|$  clauses.

We then proceed to define formally the notion of a non-trivial kernel. For this, we define the problems considered as Constraint Satisfaction Problems. Considering recent advances in Computational Complexity relating to sparsification and existence of non-trivial kernels, we conclude by showing that the method presented here, giving a non-trivial kernel for positive 1-in-3 SAT, implies the existence of a non-trivial kernel for 1-in-3 SAT.

Our proof shows the structure known as the Polynomial Hierarchy collapses to the level above  $P = NP$ .

*Keywords:* Computational Complexity, Boolean Satisfiability, Kernelization

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## 1. Introduction

Recall that **SAT** and its restrictions **cnf-SAT**, **k-cnf-SAT** and **3-cnf-SAT** are **NP-complete** as shown in [1, 2, 3]. The **1-3-SAT** problem is that, given a collection of triples over some variables, to determine whether there exists a truth assignment to the variables so that each triple contains exactly one true literal and exactly two false literals.

Schaefer's reduction given in [4] transforms an instance of **3-cnf-SAT** into a **1-3-SAT** instance. A simple truth-table argument shows this reduction to be parsimonious, hence **1-3-SAT** is complete for the class  $\#P$  while a parsimonious reduction from **1-3-SAT** also shows **1-3-SAT**<sup>+</sup> to be complete for  $\#P$ .

The **1-K-SAT** problem, a generalization of **1-3-SAT**, is that, given a collection of tuples of size  $K$  over some variables, to determine whether there exists a truth assignment to the variables so that each  $K$ -tuple contains exactly one true and  $K - 1$  false literals.

The **1-K-SAT** problem has been studied before under the name of **XSAT**. In [5] very strong upper bounds are given for this problem, including the counting version. These bounds are  $\mathcal{O}(1.1907^{|V|})$  and  $\mathcal{O}(1.2190^{|V|})$  respectively, while in [6] the same bound of  $2^{|C|}|V|^{\mathcal{O}(1)}$  is given for both decision and counting, where  $|V|$  is the number of variables and  $|C|$  the number of clauses.

Gaussian Elimination was used before in the context of boolean satisfiability. In [7] the author uses this method for handling xor types of constraints. Other recent examples of Gaussian elimination used in exact algorithms or kernelization may be indeed found in the literature [8, 9].

Hence the idea that constraints of the type implying this type of exclusivity can be formulated in terms of equations, and therefore processed using Gaussian Elimination, is not new and the intuition behind it is very straightforward.

We mention the influential paper by Dell and Van Melkebeek [10] together with a continuation of their study by Jansen and Pieterse [11, 12]. It is shown in these papers that, under the assumption that  $\text{coNP} \not\subseteq \text{NP} \setminus \text{P}$ , there cannot exist a significantly small kernelization of various problems, of which exact satisfiability is one. We shall use these results directly in our current approach.

We begin our investigation by showing how a  $1\text{-}3\text{-SAT}^+$  instance can be turned into an integer programming version  $0\text{-}1\text{-IP}^=$  instance with fewer variables. The number of variables in the  $0\text{-}1\text{-IP}^=$  instance is at most two-thirds of the number of variables in the  $1\text{-}3\text{-SAT}^+$  instance. We achieve this by a straightforward preprocessing of the  $1\text{-}3\text{-SAT}^+$  instance using Gauss-Jordan elimination.

We are then able to count the solutions of the  $1\text{-}3\text{-SAT}^+$  instance by performing a brute-force search on the  $0\text{-}1\text{-IP}^=$  instance. This method gives interesting upper bounds on  $1\text{-}3\text{-SAT}^+$ , and the associated counting problem, though without a further analysis, the bounds thus obtained may not be the strongest upper bounds found in the literature for these problems.

Our method shows how instances become easier to solve with variation in clauses-to-variables ratio. For random  $k\text{-cnf-SAT}$  the ratio of clauses to variables has been studied intensively, for example [13] gives the proof that a formula with density below a certain threshold is with high probability satisfiable while above the threshold is unsatisfiable.

The ratio plays a similar role in our treatment of  $1\text{-}3\text{-SAT}$ . Another important observation is that in our case this ratio cannot go below  $1/3$  up to uniqueness of clauses, at the expense of polynomial time pre-processing of the problem instance. We note that, by reduction from  $3\text{-cnf-SAT}$  any instance of  $1\text{-}3\text{-SAT}$  in which the number of clauses does not exceed the number of variables is also **NP-complete**. Hence we restrict our attention to these instances.

Our preprocessing induces a certain type of “order” on the variables, such that some of the non-satisfying assignments can be omitted by our solution search. We therefore manage to dissect the **1-K-SAT** instance and obtain a “core” of variables on which the search can be performed. For a treatment of Parameterized Complexity the reader is directed to [14].

## 2. Outline

After a brief consideration of the notation used in Section 3, we define in Section 4 the problems **1-3-SAT**, **1-3-SAT<sup>+</sup>** and the associated counting problems **#1-3-SAT** and **#1-3-SAT<sup>+</sup>**.

We elaborate on the relationship between the number of clauses and the number of variables in **1-3-SAT<sup>+</sup>**. We give a proof sketch that **1-3-SAT** is **NP – complete** via a reduction from **3-cnf-SAT**, and a proof sketch that **1-3-SAT<sup>+</sup>** is **NP – complete** via reduction from **1-3-SAT**.

We conclude by remarking that due to this chain of reductions, the restriction of **1-3-SAT<sup>+</sup>** to instances with more variables than clauses is also **NP – complete**, since these kind of instances encode the **3-cnf-SAT** problem. We hence restrict our treatment of **1-3-SAT<sup>+</sup>** to these instances.

Section 5 presents our method of reducing a **1-3-SAT<sup>+</sup>** instance to an instance of 0/1 Integer Programming with Equality only. This results in a 0/1 I.P.E. instance with at most two thirds the number of variables found in the **1-3-SAT<sup>+</sup>** instance.

Essentially, our method describes the same method as the one presented by Jansen and Pieterse in an introductory paragraph of [12]. Jansen and Pieterse are not primarily interested in reduction of the number of variables, but reduction of number of constraints and they do not tackle the associated counting problem as such.

The method consists of encoding a **1-3-SAT<sup>+</sup>** instance into a system of linear equations and performing Gaussian Elimination on this system.

Linear Algebraic methods show the resulting matrix can be rearranged into an  $r \times r$  diagonal submatrix of “independent” columns, where  $r$  is the rank of the system, to which it is appended a submatrix containing the rest of the columns and the result column which correspond roughly to the 0/1 I.P.E. instance we have in mind. We further know the values in the independent submatrix can be scaled to 1.

The most pessimistic scenario complexity-wise is when the input clauses, or the rank of the resulting system, is a third the number of variables,  $|C| = 1/3|V|$ , from which we obtain our complexity upper bounds.

To this case, one may wish to contrast the case of the system matrix being full rank, for which Gaussian Elimination alone suffices to find a solution. Further to this, we explain how to solve the 0/1 I.P.E. problem in order to recover the number of solutions to the **1-3-SAT<sup>+</sup>** problem.

Section 6 outlines the complexity implications for **1-3-SAT<sup>+</sup>** by considering the 1981 results of Schroepel and Shamir [15]. Section 7 outlines the method of substitution, well-known to be equivalent to Gaussian Elimination.

Finally, Section 8 gives additional definitions, considers some of the very recent literature on sparsification, and gives an argument that the existence of the **1-3-SAT<sup>+</sup>** kernel found in previous sections implies the existence of a non-trivial kernel for the more general **1-3-SAT**.

### 3. Notation

We denote boolean variables by  $p_1, p_2, \dots, p_i, \dots$  and denote negation by  $\neg p_i$ . Whenever considered as binary variables over the set  $\{0, 1\}$  these will be written as  $\bar{p}_i$  in the positive case and  $-\bar{p}_i$  in the negative.

Denote the true and false constants by  $\top$  and  $\perp$  respectively. For any SAT formula  $\varphi$ , write  $\Sigma(\varphi)$  if  $\varphi$  is satisfiable and write  $\bar{\Sigma}(\varphi)$  otherwise. Reserve the notation  $a(p)$  for a truth assignment to the variable  $p$ .

We write  $\Phi(r, k)$  for the set of formulas in 3-CNF with  $r$  variables and  $k$  unique clauses. We also write  $\varphi(V, C)$  to specify concretely such a formula, where  $V, C$  shall denote the sets of variables and clauses of  $\varphi$ . For any formula  $\varphi \in \Phi(r, k)$  we let  $\kappa(\varphi) = \frac{k}{r}$ .

We will make use of the following properties of a given map  $f$ :

subadditivity:  $f(A + B) \leq f(A) + f(B)$

scalability:  $f(cA) = cf(A)$  for constant  $c$ .

For a given tuple  $s = (s_1, s_2, \dots, s_n)$  we let  $s(m)$  denote the element  $s_m$ . Finally, for given linear constraints  $\sum_{i \leq n} d_i x_i = R$  for some  $n$  and  $x_i \in \{0, 1\}$ , denote by  $\text{coef}(x_i)$  the value  $d_i$ .

#### 4. One-in-Three SAT

One-in-three satisfiability arose in late seventies as an elaboration relating to Schaefer's Dichotomy Theorem [4]. It is proved there using certain assumptions boolean satisfiability problems are either in **P** or they are **NP-complete**.

The counting versions of satisfiability problems were introduced in [16] and it is known in general that counting is in some sense strictly harder than the corresponding decision problem.

This is due to the fact that, for example, producing the number of satisfying assignments of a formula in 2-CNF is complete for  $\#\text{P}$ , while the corresponding decision problem is known to be in **P** [16]. We thus restrict our attention to **1-3-SAT** and more precisely **1-3-SAT<sup>+</sup>** formulas.

**Definition 4.1** (1-3-SAT). 1-3-SAT is defined as determining whether a formula  $\varphi \in \Phi(r, k)$  is satisfiable, where the formula comprises of a collection of triples

$$C = \{\{p_1^1, p_2^1, p_3^1\}, \{p_1^2, p_2^2, p_3^2\}, \dots, \{p_1^k, p_2^k, p_3^k\}\}$$

such that  $p_1^i, p_2^i, p_3^i \in V = \{p_1, \neg p_1, p_2, \neg p_2, \dots, p_r, \neg p_r\} \cup \{\perp\}$  and for any clause exactly one of the literals is allowed to be true in an assignment, and no clause may contain repeated literals or a literal and its negation, and such that every variable in  $V$  appears in at least one clause.

In the restricted case that  $p_1^i, p_2^i, p_3^i \in V^+ = \{p_1, p_2, \dots, p_r\} \cup \{\perp\}$  for  $1 \leq i \leq r$  we denote the problem as 1-3-SAT<sup>+</sup>.

In the extended case that we are required to produce the number of satisfying assignments, these problems will be denoted as #1-3-SAT and #1-3-SAT<sup>+</sup>.

**Example 4.1.** The 1-3-SAT<sup>+</sup> formula  $\varphi = \{\{p_1, p_2, p_3\}, \{p_2, p_3, p_4\}\}$  is satisfiable by the assignment  $a(p_2) = \top$  and  $a(p_j) = \perp$  for  $j = 1, 3, 4$ . The 1-3-SAT<sup>+</sup> formula  $\varphi = \{\{p_1, p_2, p_3\}, \{p_2, p_3, p_4\}, \{p_1, p_2, p_4\}, \{p_1, p_3, p_4\}\}$  is not satisfiable.

**Lemma 4.1.** Up to uniqueness of clauses and variable naming the set  $\Phi(r, r/3)$  determines one 1-3-SAT<sup>+</sup> formula and this formula is trivially satisfiable.

*Proof.* Consider the formula  $\varphi = \{\{p_{3i}, p_{3i+1}, p_{3i+2}\} \mid 1 \leq i \leq r/3\}$  which has  $r$  variables and  $r/3$  clauses, hence belongs to the set  $\Phi(r, r/3)$  and it is satisfiable, trivially, by any assignment that makes each clause evaluate to true.

Now take any clause  $\{a, b, c\} \in \varphi(V, C)$  with  $a, b, c \in V$ . We claim there is no other clause  $\{a', b', c'\} \in \varphi$  such that  $\{a, b, c\} \cap \{a', b', c'\} \neq \emptyset$ , for otherwise let  $a$  be in their intersection and we can see the number of variables used by the  $r/3$  clauses reduces by one variable, to be  $r - 1$ . Now, since the clauses of  $\varphi$  do not overlap in variables, we can see that our uniqueness claim must hold, since the elements of  $\varphi$  are partitions of the set of variables.  $\square$

**Remark 4.1.** For  $1\text{-3-SAT}^+$ , the sets  $\Phi(r, k)$  for  $k < r/3$  are empty.

Schaefer gives a polynomial time parsimonious reduction from  $3\text{-cnf-SAT}$  to  $1\text{-3-SAT}$  hence showing that  $1\text{-3-SAT}$  and its counting version  $\#1\text{-3-SAT}$  are **NP-complete** and respectively **#P-complete**.

**Theorem 4.1** (Schaefer, [4]).  $1\text{-3-SAT}$  is **NP-complete**.

*Proof sketch.* Proof by reduction from  $3\text{-cnf-SAT}$ . For any clause  $p \vee p' \vee p''$  create three  $1\text{-3-SAT}$  clauses  $\{\neg p, a, b\}$ ,  $\{p', b, c\}$ ,  $\{\neg p'', c, d\}$ . Hence, we obtain an instance with  $|V| + 4|C|$  variables and  $3|C|$  clauses, for the instance of  $3\text{-cnf-SAT}$  of  $|V|$  variables and  $|C|$  clauses.  $\square$

The following statement is given in [17]. For the sake of completeness, we provide a proof by a parsimonious reduction from  $1\text{-3-SAT}$ .

**Theorem 4.2** ([17]).  $1\text{-3-SAT}^+$  is **NP-complete**.

*Proof sketch.* Construct an instance of  $1\text{-3-SAT}^+$  from an instance of  $1\text{-3-SAT}$ . Add every clause in the  $1\text{-3-SAT}$  instance with no negation to the  $1\text{-3-SAT}^+$  instance. For every clause containing one negation  $\{\neg p, p', p''\}$ , add to the  $1\text{-3-SAT}^+$  two clauses  $\{\hat{p}, p', p''\}$  and  $\{\hat{p}, p, \perp\}$  where  $\hat{p}$  is a fresh variable. For a clause containing two negations  $\{\neg p, \neg p', p''\}$  we add two fresh variables  $\hat{p}, \hat{p}'$  and three clauses  $\{\hat{p}, \hat{p}', p''\}$ ,  $\{\hat{p}, p, \perp\}$  and  $\{\hat{p}', p', \perp\}$ . For a clause containing three negations  $\{\neg p, \neg p', \neg p''\}$  we add three fresh variables  $\hat{p}, \hat{p}', \hat{p}''$  and four clauses  $\{\hat{p}, \hat{p}', \hat{p}''\}$ ,  $\{\hat{p}, p, \perp\}$ ,  $\{\hat{p}', p', \perp\}$  and  $\{\hat{p}'', p'', \perp\}$ . We obtain a  $1\text{-3-SAT}^+$  formula with at most  $4|C|$  more clauses and at most  $|V| + 3|C|$  more variables, for initial number of clauses and variables  $|C|$  and  $|V|$  respectively. Our reduction is parsimonious, for it is verifiable by truth-table the number of satisfying assignments to the  $1\text{-3-SAT}$  clause  $\{\neg p, \neg p', \neg p''\}$  is the same as the number of satisfying assignments to the  $1\text{-3-SAT}^+$  collection of clauses  $\{\{\hat{p}, \hat{p}', \hat{p}''\}, \{\hat{p}, p, \perp\}, \{\hat{p}', p', \perp\}, \{\hat{p}'', p'', \perp\}\}$ .  $\square$



**Remark 4.2.** *In virtue of Theorem 4.1 and Theorem 4.2 we restrict ourselves to instances of  $1\text{-}3\text{-SAT}^+$   $\varphi \in \Phi(r, k)$  with  $r \geq k$ . For if an instance of  $3\text{-cnf-SAT}$   $\bar{\varphi} \in \Phi(r', k')$  is reduced to an instance of  $1\text{-}3\text{-SAT}$   $\hat{\varphi} \in \Phi(r'', k'')$  then our reduction entails  $r'' = r' + 4k'$  and  $k'' = 3k'$ .*

*We analyze the further reduction to the instance of  $1\text{-}3\text{-SAT}^+$   $\varphi \in \Phi(r, k)$ . Let  $C, C', C'', C'''$  be the collections of clauses in  $\hat{\varphi}$  containing, no negation, one negation, two negations and three negations respectively.*

*Our reduction implies  $r = r'' + |C'| + 2|C''| + 3|C'''|$  and  $k = |C| + 2|C'| + 3|C''| + 4|C'''| = k'' + |C'| + 2|C''| + 3|C'''|$ . Then,  $r - k = r'' + |C'| + 2|C''| + 3|C'''| - k'' - |C'| - 2|C''| - 3|C'''| = r'' - k'' = r' + 4k' - 3k' = r' + k' > 0$ .*

## 5. Gaussian elimination

A rank function is used as a measure of “independence” for members of a certain set. The dual notion of nullity is defined as the complement of the rank with respect to the size of the set.

**Definition 5.1.** *A rank function  $R$  obeys the following*

1.  $R(\emptyset) = 0$ ,
2.  $R(A \cup B) \leq R(A) + R(B)$ ,
3.  $R(A) \leq R(A \cup \{a\}) \leq R(A) + 1$ .

**Definition 5.2** (rank and nullity). *For a  $1\text{-}3\text{-SAT}^+$  formula  $\varphi(V, C)$  define the system of linear equations  $\mathbf{Sys}(\varphi)$  as follows:*

*for any clause  $\{p, p', p''\} \in C$  add to  $\mathbf{Sys}(\varphi)$  equation  $\bar{p} + \bar{p}' + \bar{p}'' = 1$ ;*

*Define the rank and nullity of  $\varphi$  as  $\eta(\varphi) = R(\mathbf{Sys}(\varphi))$  and  $\bar{\eta}(\varphi) = |V| - \eta(\varphi)$ .*

*If formula is clear from context, we also use the shorthand  $\eta$  and  $\bar{\eta}$ .*

**Remark 5.1.**  *$\eta$  is a rank function with respect to sets of  $1\text{-}3\text{-SAT}$  triples.*

**Lemma 5.1.** *For any 1-3-SAT<sup>+</sup> instance  $\varphi$  transformed into a linear system  $\text{Sys}(\varphi)$  we observe the following:*

$\bar{p} + \bar{p}' + \bar{p}'' = 1$  has a solution  $S \subset \{0, 1\}^3$  if and only if exactly one of  $\bar{p}, \bar{p}', \bar{p}''$  is equal to 1 and the other two are equal to 0.

**Proposition 5.1.** *For any formula  $\varphi \in \Phi(r, k)$  we have  $\Sigma(\varphi)$  if and only if  $\text{Sys}(\varphi)$  has at least one solution over  $\{0, 1\}^r$ .*

**Corollary 5.1.** *A formula  $\varphi \in \Phi(r, k)$  has as many satisfiability assignments as the number of solutions of  $\text{Sys}(\varphi)$  over  $\{0, 1\}^r$ .*

We define the binary integer programming problem with equality here and show briefly that 1-3-SAT<sup>+</sup> is reducible to a “smaller” instance of this problem.

**Definition 5.3** (0, 1-integer programming with equality). *The 0-1-IP<sup>=</sup> problem is defined as follows. Given a family of finite tuples  $s_1, s_2, \dots, s_k$  with each  $s_i \in \mathbb{Q}^S$  for some fixed  $S \in \mathbb{N}$ , and given a sequence  $q_1, q_2, \dots, q_k \in \mathbb{Q}$ , decide whether there exists a tuple  $T \in \{0, 1\}^S$  such that*

$$\sum_{j=1}^S s_i(j)T(j) = q_i \text{ for each } i \in \{1, 2, \dots, k\}$$

**Remark 5.2.** 0-1-IP<sup>=</sup>  $\in \mathcal{O}(k2^S)$  where  $k$  is the number of 0-1-IP<sup>=</sup> tuples and  $S$  is the size of the tuples.

*Proof sketch.* The bound is obtained through applying an exhaustive search. □

**Lemma 5.2.** *Let  $\varphi \in \Phi(r, k)$  be a 1-3-SAT<sup>+</sup> formula, then  $\eta(\varphi) \leq k$  and  $\bar{\eta}(\varphi) \geq r - k$ .*

*Proof.* Follows from the observation that  $\eta$  is a rank function. □

**Lemma 5.3.** Consider a 1-3-SAT<sup>+</sup> formula  $\varphi$  and suppose  $\eta(\varphi) = k$  and  $\bar{\eta}(\varphi) = r - k$ . The satisfiability of  $\varphi$  is decidable in  $\mathcal{O}(2k2^{r-k})$ .

*Proof.* The result of performing Gauss-Jordan Elimination on  $\text{Sys}(\varphi)$  yields, after a suitable re-arrangement of column vectors, the reduced echelon form

$$\text{GJE}(\text{Sys}(\varphi)) = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & x_{11} & x_{12} & \dots & x_{1d} & R_1 \\ 0 & 1 & 0 & \dots & 0 & x_{21} & x_{22} & \dots & x_{2d} & R_2 \\ 0 & 0 & 1 & \dots & 0 & x_{31} & x_{32} & \dots & x_{3d} & R_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & x_{k1} & x_{k2} & \dots & x_{kd} & R_k \end{bmatrix}$$

Now consider the following structure, obtained from the given dependencies above through ignoring the zero entries

$$\begin{bmatrix} 1 & x_{11} & x_{12} & \dots & x_{1d} & R_1 \\ 1 & x_{21} & x_{22} & \dots & x_{2d} & R_2 \\ 1 & x_{31} & x_{32} & \dots & x_{3d} & R_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & x_{k1} & x_{k2} & \dots & x_{kd} & R_k \end{bmatrix}$$

This induces an instance of 0-1-IP<sup>=</sup> which can be solved as follows

Initialize  $C := 0$   
Enumerate sequentially all sequences  $s \in S = \{0, 1\}^d$   
For each such sequence  $s$ :  
If  $\forall j \leq k [\sum_{i \leq d} s(i)x_{ji} = R_j \vee \sum_{i \leq d} s(i)x_{ji} = R_j - 1]$  then  $C \leftarrow C + 1$ .

We note the length of sequences  $s \in S$  is  $d = r - k$ , hence the brute force procedure has to enumerate  $2^{r-k}$  members of  $S$ . Furthermore, each such sequence  $s \in S$  is tested twice against all of the constraints  $x_{1i}, x_{2i}, \dots, x_{ki}$  for  $i \leq d$ , resulting in the claimed time complexity of  $\mathcal{O}(2k2^{r-k})$ .

To see the algorithm is correct, we give a proof that considers when the counter is incremented. Suppose for all  $j \leq k$  some  $s \in S$  is not a solution to either  $\sum_{i \leq d} s(i)x_{ji} = R_j$  or  $\sum_{i \leq d} s(i)x_{ji} = R_j - 1$ . In this case, the counter is not incremented and we claim  $s$  does not induce a solution to the 1-3-SAT<sup>+</sup> formula  $\varphi$ . For in this case  $s$  is not a 0/1 solution to the system  $\text{Sys}(\varphi)$  and hence by Corollary 5.1 cannot be a satisfying solution to  $\varphi$ . In effect, the counter is not incremented as we have not seen an additional satisfying solution.

Now suppose for all  $j \leq k$  some  $s \in S$  is a solution to either  $\sum_{i \leq d} s(i)x_{ji} = R_j$  or  $\sum_{i \leq d} s(i)x_{ji} = R_j - 1$ . In this case, the counter is incremented and we claim  $s$  is indeed a solution to the 1-3-SAT<sup>+</sup> formula  $\varphi$ .

For if  $s$  is a solution to all  $j$ th rows constraint  $\sum_{i \leq d} s(i)x_{ji} = R_j$  then  $s$  satisfies the constraint  $x_{j1} + x_{j2} + \dots + x_{jd} = R_j$  giving the satisfying assignment  $a(p) = \perp$  for all variables  $p$  corresponding to variables in the diagonal matrix, and  $a(p) = \perp$  for variables corresponding to column  $i$  for which  $s(i) = 0$ , and  $a(p) = \top$  for variables corresponding to column  $i$  for which  $s(i) = 1$ .

Similarly, if  $s$  is a solution to all  $j$ th rows constraint  $\sum_{i \leq d} s(i)x_{ji} = R_j - 1$  then  $s$  satisfies the constraint  $1 + x_{j1} + x_{j2} + \dots + x_{jd} = R_j$  giving the satisfying assignment  $a(p) = \top$  if  $p$  corresponds to the diagonal variable  $(j, j)$ ,  $a(p) = \perp$  for all variables  $p$  corresponding to all other variables in the diagonal matrix, and  $a(p) = \perp$  for variables corresponding to column  $i$  for which  $s(i) = 0$ , and  $a(p) = \top$  for variables corresponding to column  $i$  for which  $s(i) = 1$ .  $\square$

**Corollary 5.2.**  $1\text{-}3\text{-SAT}^+ \leq_{\text{poly}} 0\text{-}1\text{-IP}^=$ .

*Proof.* By the pre-processing of the problem instance using Gaussian Elimination, shown above, 1-3-SAT<sup>+</sup> is reduced in polynomial time to 0-1-IP<sup>=</sup>.  $\square$

**Theorem 5.1.**  $\#1\text{-}3\text{-SAT}^+ \in \mathcal{O}(\eta 2^{\bar{\eta}+1})$  for formula rank  $\eta$  and nullity  $\bar{\eta}$ .

*Proof.* There are  $\eta$ -many equations to satisfy by any assignment, and there are  $\bar{\eta}$ -many variables to search through exhaustively in order to solve the  $0\text{-}1\text{-IP}^=$  problem, which in turn solves the  $1\text{-}3\text{-SAT}^+$  problem.  $\square$

**Corollary 5.3.**  $\#1\text{-}3\text{-SAT}^+ \in \mathcal{O}(2\kappa r 2^{(1-\kappa)r})$  for any instance  $\varphi \in \Phi(r, k)$  and  $\kappa = k/r$ .

## 6. Implications for positive 1-in-3 SAT

In [15] it is shown that  $1\text{-}3\text{-SAT}$  can be solved in time  $\mathcal{O}(2^{|V|/2})$  and space  $\mathcal{O}(2^{|V|/4})$  through the 4-table method. The same result entails the same upper bounds for the  $0\text{-}1\text{-IP}^=$  problem.

**Theorem 6.1** ([15]).  $\#1\text{-}3\text{-SAT}$  can be solved in time  $\mathcal{O}(2^{|V|/2})$  and space  $\mathcal{O}(2^{|V|/4})$ .

*Proof sketch:.* Split the problem instance in two partitions of roughly  $|V|/2$  variables, solve each partition by exhaustive search and combine the two sets of solutions to obtain the solutions to the original problem.  $\square$

**Theorem 6.2** ([15]).  $\#0\text{-}1\text{-IP}^=$  can be solved in time  $\mathcal{O}(2|C|2^{|V|/2})$  and space  $\mathcal{O}(2^{|V|/4})$ .

*Proof sketch:.* Each constraint of  $0\text{-}1\text{-IP}^=$  can be seen as a subset-sum problem, hence solvable by the 4-table method.  $\square$

**Corollary 6.1.**  $\#1\text{-}3\text{-SAT}^+$  can be solved in time  $\mathcal{O}(4/3|V|2^{3|V|/8})$  and space  $\mathcal{O}(4/3|V|2^{3|V|/16})$ .

The proof idea is to split the problem instance into quarters as per the method of [15] and instead of solving each quarter using as resources  $\mathcal{O}(2^{|V|/2})$  time and  $\mathcal{O}(2^{|V|/4})$  space, solve the associated  $0\text{-}1\text{-IP}^=$  instance of at most  $3|V|/8$  variables in  $\mathcal{O}(4/3|V|2^{3|V|/8})$  time and in  $\mathcal{O}(4/3|V|2^{3|V|/16})$  space.

In other words, we split the  $\#1\text{-}3\text{-SAT}^+$  instance in two sub-formulas with roughly  $|V|/2$  variables each and we may apply our kernel method by splitting each  $\#0\text{-}1\text{-IP}^=$  constraint in half again and performing an exhaustive search on the halved instances, as per the 4-table method of [15].

It is essential to note that the  $\#0\text{-}1\text{-IP}^=$  instance can be viewed as multiple subset-sum instances that can be solved in a divide-and-conquer fashion.

## 7. The method of Substitution

For ease of analysis we shall suppose Gauss-Jordan Elimination above is replaced by the method of substitution. The algorithm is depicted below in Fig. 1. Let  $n(c), m(c), s(c)$  be the lowest, middle and highest labeled variable in clause  $c$ . By definition, these values must be distinct. Assume the formula  $\varphi$  is sorted in ascending order of  $n(c)$ . Represent clause  $c$  in equation form as  $n(c) = 1 - m(c) - s(c)$ .

```

i ← k
while i ≥ 1 do
  j ← k − 1
  while j ≥ 1 do
    if  $c(j) = A_j - (\sum_r C(r)B(r)) - c(i)B(i)$ , and  $c(i) = A_i - (\sum_t C(t)B(t))$ 
    then
       $c(j) = (A_j - A_i B(i)) - (\sum_r C(r)B(r) + \sum_t C(t)B(t))$ 
    end if
    j ← j − 1
  end while
  i ← i − 1
end while

```

Figure 1: Substitution algorithm

After the substitution process is finished, each of the clauses is expressed in terms of independent variables, variables which cannot be expressed in terms of other variables. We denote by  $|n(i)|$  the number of variables in constraint  $c(i)$  induced by the substitution method, excluding the variable  $n(i)$ .

**Remark 7.1.** *The largest number of expansions determined by running substitution on the collection of clauses, is  $|n(k)| = 2$ ,  $|n(k-1)| = 3$ ,  $|n(k-2)| = 5, \dots$ ,  $|n(k-i)| = \text{Fib}(i+3)$ .*

*Proof.* We need to maximize the number of substitutions performed at each step. Hence, at first step we encounter two substitutions, at the second we encounter three substitutions, while at every subsequent step we must assume there exist two variables for which we can substitute in terms of previously found variables, which indicates that the formula for the Fibonacci expansion describes our process.  $\square$

**Definition 7.1** (Representation). *The size of a representation for a given instance of  $1\text{-}3\text{-SAT}^+ \in \Phi(r, k)$  expressed by substitution as  $n(1), n(2), \dots, n(k)$  is given by the formula*

$$r \times \log\left(\sum_{i \leq k} |n(i)|\right)$$

**Remark 7.2.** *The size of the resulting representation associated to formulas treated by Remark 7.1 converges asymptotically to  $r^2 \times \log(1.62)$ .*

*Proof.* The bound is given by an analysis of the growth of the Fibonacci sequence. It is well known the rate of growth of the sequence converges approximately to  $1.62^n$ .  $\square$

**Remark 7.3.** *Contrast the scenario in Remark 7.1, to the case in which there are no substitutions induced, i.e.  $\varphi = \{p_{3i+1}, p_{3i+2}, p_{3i+3}\}, i \leq 1/3k\}$ .*

**Remark 7.4.** *The size of the resulting representation associated to formulas treated by Remark 7.3 is  $r \times \log(2/3r)$ .*

*Proof.* In this case we have  $2k$  independent variables, for a value of  $k$  of  $1/3r$ .  $\square$

**Theorem 7.1.** *Any 1-3-SAT<sup>+</sup> formula admits a representation with size  $S$  for*

$$r \times \log(2/3r) \leq S \leq r^2 \times \log(1.62)$$

**Remark 7.5.** *The size of any representation is bounded above by  $r^{2-\epsilon}$  for*

$$\epsilon = \frac{0.52}{\log(r)}$$

*Proof.*  $r^{2-\epsilon} = r^2 \times \log(1.62)$  implies  $2 - \epsilon = 2 + \frac{\log \log(1.62)}{\log(r)}$  and therefore

$$\epsilon = \frac{0.52}{\log(r)}$$

$\square$

## 8. Implications for Computational Complexity

Dell and Melkebeek [10] give a rigorous treatment of the concept of “sparsification”. In their framework, an oracle communication protocol for a language  $L$  is a communication protocol between two players.

The first player is given the input  $x$  and is only allowed to run in time polynomial in the length of  $x$ . The second player is computationally unbounded, without initial access to  $x$ . At the end of communication, the first player should be able to decide membership in  $L$ . The cost of the protocol is the length in bits of the communication from the first player to the second.

Therefore, if the first player is able to reduce, in polynomial time, the problem instance significantly, the cost of communicating the “kernel” to the second player would also decrease, hence providing us with a very natural formal account for the notion of sparsification.



Jansen and Pieterse in [11] state and give a procedure for any instance of Exact Satisfiability with unbounded clause length to be reduced to an equivalent instance of the same problem with only  $|V| + 1$  clauses, for number of variables  $|V|$ . Their argument uses Gaussian Elimination and in essence their method corresponds to the method presented above. The concern regarding the number of clauses in  $1\text{-}3\text{-SAT}^+$  can be addressed, as we have done above, by observing that for any instance  $C$  of  $3\text{-cnf-SAT}$ , the chain of polynomial-time parsimonious reductions  $C \rightarrow \hat{C} \rightarrow \bar{C}$ , for  $\hat{C}$  and  $\bar{C}$  instances of  $1\text{-}3\text{-SAT}$  and  $1\text{-}3\text{-SAT}^+$  respectively, implies that the variables of  $\hat{C}$  and  $\bar{C}$  outnumber the clauses.

What is also claimed in [11] is that, assuming  $\text{coNP} \not\subseteq \text{NP} \setminus \text{P}$ , no polynomial time algorithm can in general transform an instance of Exact Satisfiability of  $|V|$ -many variables to a significantly smaller equivalent instance, i.e. an instance encoded using  $\mathcal{O}(|V|^{2-\epsilon})$  for any  $\epsilon > 0$ .

We believe it is already transparent that, in fact, we have obtained a significantly smaller kernel for  $1\text{-}3\text{-SAT}^+$  above, i.e. transforming parsimoniously an instance of  $|V|$  variables to a “compressed” instance of  $0\text{-}1\text{-IP}^=$  of at most  $2/3|V|$  variables.

**Definition 8.1** (Constraint Satisfaction Problem). *A csp is a triple  $(S, D, T)$  where*

- *$S$  is a set of variables,*
- *$D$  is the discrete domain the variables may range over, and*
- *$T$  is a set of constraints.*

*Every constraint  $c \in T$  is of the form  $(t, R)$  where  $t$  is a subset of  $S$  and  $R$  is a relation on  $D$ . An evaluation of the variables is a function  $v: S \rightarrow D$ . An evaluation  $v$  satisfies a constraint  $(t, R)$  if the values assigned to elements of  $t$  by  $v$  satisfies relation  $R$ .*

**Remark 8.1.** *3-cnf-SAT is a csp, rewritten in csp form as*

$$S = V, D = \{\top, \perp\},$$

*members of  $T$  are defined as the 3-cnf-SAT clauses together with the standard boolean function that evaluates disjunctions, i.e.  $R(a, b, c) = \top$  iff at least one of  $a, b, c$  is true.*

**Remark 8.2.** *1-3-SAT is a csp, rewritten in csp form as*

$$S = V, D = \{\top, \perp\},$$

*members of  $T$  are defined as the 1-3-SAT clauses together with the function  $R(a, b, c) = \top$  iff exactly one of  $a, b, c$  is true.*

**Remark 8.3.** *1-3-SAT<sup>+</sup> is a csp, rewritten in csp form as*

$$S = V^+, D = \{\top, \perp\},$$

*members of  $T$  are defined as the 1-3-SAT clauses together with the function  $R(a, b, c) = \top$  iff exactly one of  $a, b, c$  is true.*

**Remark 8.4.** *In what follows we switch between notations and write a csp in a more general form, with a problem  $(S, D, T)$  written as  $L \subseteq \mathbb{N} \times \Sigma^*$ , with instances  $(k, x)$  such that  $k = |S|$  and  $x$  a string representation of  $D$  and  $T$ .*

**Definition 8.2** (Kernelization). *Let  $L, M$  be two parameterized decision problems, i.e.  $L, M \subseteq \mathbb{N} \times \Sigma^*$  for some finite alphabet  $\Sigma$ .*

*A kernelization for the problem  $L$  parameterized by  $k$  is a polynomial time reduction of an instance  $(k, x)$  to an instance  $(k', x')$  such that:*

- $(k, x) \in L$  if and only if  $(k', x') \in M$ ,
- $k' \in \mathcal{O}(k)$ , and
- $|x'| \in \mathcal{O}(|x|)$ .

*In the extended case of referring to the counting versions  $\#L, \#M$  we additionally require the kernelization to be parsimonious and we refer to it as a parsimonious kernelization.*

**Definition 8.3** (Encoding). *An encoding of a problem  $L \subseteq \mathbb{N} \times \Sigma^*$  is a bijection  $h: L \rightarrow \mathbb{N}$  such that for any  $(k, x) \in \mathbb{N} \times \Sigma^*$  we have  $h(k, x) \in \mathcal{O}(|x|)$ .*

**Definition 8.4.** *A non-trivial kernel for 3-cnf-SAT is a kernelization of this problem transforming any instance  $\varphi \in \Phi(r, k)$  to an instance  $(f(r), g(k))$  of an arbitrary NP-complete csp  $M$ , such that  $f(r) \in \mathcal{O}(r)$  and  $g(k) \leq h(k, r)$  with  $h(k, r) \in \mathcal{O}(r^{3-\epsilon})$  for an encoding  $h$  of  $\varphi$  and some  $\epsilon > 0$ .*

**Remark 8.5** (Dell and Melkebeek [10]). *3-cnf-SAT admits a trivial kernel  $(f(r), g(k))$  with  $g(k) \leq h(k, r)$  and  $h(k, r) \in \mathcal{O}(r^3)$ .*

*Proof sketch:* Represent the problem instance as a  $r \times k$  matrix  $M$  with the value  $M(i, j) = 1$  iff variable  $i$  is in constraint  $j$ , otherwise  $M(i, j) = 0$ . There are in total  $r^3$  possible constraints for a general problem. The hypothesis is then that the number of constraints in a general instance cannot be reduced significantly, i.e. as to reduce a general instance to  $g(k, r) \in \mathcal{O}(r^{3-\epsilon})$  for some positive  $\epsilon$ . It is shown in [10] that if this hypothesis fails, then  $\text{coNP} \subseteq \text{NP} \setminus \text{P}$  and the Polynomial Hierarchy collapses to its third level.  $\square$

**Lemma 8.1** (Dell and Melkebeek [10]). *If 3-cnf-SAT admits a non-trivial kernel, then  $\text{coNP} \subseteq \text{NP} \setminus \text{P}$ .*

**Definition 8.5.** *A non-trivial kernel for 1-3-SAT is a kernelization of this problem transforming any instance  $\varphi \in \Phi(r, k)$  to an equivalent instance  $(f(r), g(k))$  of an arbitrary NP-complete csp  $M$ , such that  $f(r) \in \mathcal{O}(r)$  and  $g(k) \leq h(k, r)$  with  $h(k, r) \in \mathcal{O}(r^{2-\epsilon})$  for an encoding  $h$  of  $\varphi$  and some  $\epsilon > 0$ .*

**Remark 8.6** (Jansen and Pieterse [11]). *1-3-SAT admits a kernel  $(f(r), g(k))$  with  $g(k) \leq h(k, r)$  and  $h(k, r) \in \mathcal{O}(r^2)$ .*

*Proof sketch.* Gaussian Elimination may be used to reduce a general instance  $\varphi(r, k)$  to an instance of  $g(k) = r + 1$  clauses. By a similar matrix representation as in Remark 8.5 we obtain that  $g(k) \in \mathcal{O}(r^2)$ .  $\square$

The following statement is given in [11]. The authors elaborate on the results of [10] to analyze combinatorial problems from the perspective of sparsification, and give several arguments that non-trivial kernels for such problems would entail a collapse of the Polynomial Hierarchy to the level above  $\mathbf{P} = \mathbf{NP}$ .

It is essential to note here that this line of reasoning was used by researchers studying sparsification with the intention of proving lower bounds on the existence of kernels, while the results presented by us are slightly more optimistic.

**Lemma 8.2** (Jansen and Pieterse [11]). *If 1-3-SAT admits a non-trivial kernel, then  $\text{coNP} \subseteq \mathbf{NP} \setminus \mathbf{P}$ .*

**Lemma 8.3.** *If 1-3-SAT<sup>+</sup> admits a non-trivial kernel, then 1-3-SAT admits a non-trivial kernel.*

*Proof.* Let  $\varphi \in \Phi(r, k)$  be an instance of 1-3-SAT. By Schaeffer's results it follows  $\varphi$  can be parsimoniously polynomial time reduced to a 1-3-SAT<sup>+</sup> formula  $\bar{\varphi} \in \Phi(r', k')$  with  $r' = r + 4k$  and  $k' = 3k$ .

Assuming 1-3-SAT<sup>+</sup> admits a non-trivial kernel, this implies 1-3-SAT admits a non-trivial kernel, and therefore through Lemma 8.1  $\text{coNP} \subseteq \mathbf{NP} \setminus \mathbf{P}$ .

To spell this out, suppose we have non-trivial kernel  $(f(r'), g(k'))$  for the problem 1-3-SAT, with  $g(k') \leq h(k', r')$  and  $h(k', r') \in \mathcal{O}(r'^{2-\epsilon})$ . We observe using the reduction from 1-3-SAT,  $f(r + 4k) \leq f(r) + 4f(k) \leq 5f(r)$  and therefore  $f(r') \in \mathcal{O}(r)$  and, we obtain via the reduction the existence of a non-trivial kernel for 1-3-SAT, that is  $g(3k) \leq 3g(k) \leq 3h(k, r)$  with  $h(k, r) \in \mathcal{O}(r^{2-\epsilon})$ .  $\square$

Essentially the following result is a restatement of Corollary 5.3.

**Theorem 8.1.** *1-3-SAT<sup>+</sup> admits a non-trivial kernel.*

*Proof.* Follows from Lemma 5.3 by observing the following strategy: the first player preprocesses the input in polynomial time using Gaussian Elimination and passes the input to the second player which makes use of its unbounded resources to provide a solution to this kernel.

It remains to be inferred that the cost of this computation is bounded non-trivially, i.e.  $h(k, r) \in \mathcal{O}(r^{2-\epsilon})$  for  $\epsilon > 0$ . Follows from Lemma 5.3, for the instance of 0-1-IP<sup>=</sup> to which we reduce has at most  $f(r') \leq 2/3r$  variables and at most  $g(k') \leq r$  clauses. By the same argument as in Remark 8.5, we are able to store the resulting instance of 0-1-IP<sup>=</sup> in a  $(2/3r + 1) \times r$  matrix  $M$  with polynomial-bounded entries, such that  $M(i, j) = d$  iff  $d$  is the coefficient of variable  $i$  in constraint  $j$ , to which we add the result column.

From Remark 7.5 we obtain indeed that the bit representation of this kernel is indeed  $r^{2-\epsilon}$  for some non-negative  $\epsilon$ . Since Gaussian Elimination and Substitution are equivalent methods, this statement is correct.  $\square$

**Corollary 8.1.** *Every 1-3-SAT<sup>+</sup> instance  $\varphi \in \Phi(r, k)$  admits a non-trivial kernel  $(f(r), g(k))$ , such that  $f(r) \in \mathcal{O}(r)$ ,  $g(k) \leq h(k, r)$  and  $h(k, r) \in \mathcal{O}(r^{2-\epsilon})$  with  $\epsilon = \log(2/3)/\log(r)$ .*

*Proof.* We have established in Theorem 8.1 the kernel found in Lemma 5.3 may be stored in a matrix  $M$  with polynomial-bounded entries, such that  $M(i, j) = d$  iff  $d$  is the coefficient of variable  $i$  in constraint  $j$ , and we have established further that  $i \leq 2/3|V| + 1$ ,  $j \leq |V|$  and  $d \in \mathcal{O}(|V|^3)$ . It follows that the kernel size is at most  $2/3\bar{C} \times |V|^5$  for some constant  $\bar{C}$ , which gives a size in bits for the kernel of  $\log(C) + 5 \log(|V|)$  for  $C = 2/3\bar{C}$ .  $\square$

**Corollary 8.2.**  $\text{coNP} \subseteq \text{NP} \setminus \text{P}$

*Proof.* Follows from Lemma 8.3, Theorem 8.1 and Lemma 8.2.  $\square$

## 9. Conclusion

Our contribution is in pointing out the relevance of Gaussian Elimination in dealing with certain types of constraint satisfaction problems, such as the ones that presuppose a type of exclusivity between constraints.

These problems may be encoded into linear systems of equations which can be solved using the method of Gaussian Elimination, or the equivalent method of Substitution.

The challenge we faced was a rigorous analysis of the fully defined linear systems, together with a type of brute-force approach in solving these systems.

Another important detail we had to analyze rigorously was the issue of growth in size of representation of the matrix in question.

The most important question in Theoretical Computer Science remains open.

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## References

- [1] S. A. Cook, The complexity of theorem-proving procedures, in: Proceedings of the third annual ACM symposium on Theory of computing, ACM, 1971, pp. 151–158.
- [2] R. M. Karp, Reducibility among combinatorial problems, in: Complexity of computer computations, Springer, 1972, pp. 85–103.
- [3] L. A. Levin, Universal sequential search problems, Problemy Peredachi Informatsii 9 (3) (1973) 115–116.
- [4] T. J. Schaefer, The complexity of satisfiability problems, in: Proceedings of the tenth annual ACM symposium on Theory of computing, ACM, 1978, pp. 216–226.
- [5] V. Dahllöf, P. Jonsson, R. Beigel, Algorithms for four variants of the exact satisfiability problem, Theoretical Computer Science 320 (2-3) (2004) 373–394.
- [6] A. Björklund, T. Husfeldt, Exact algorithms for exact satisfiability and number of perfect matchings, Algorithmica 52 (2) (2008) 226–249.
- [7] M. Soos, Enhanced gaussian elimination in dpll-based sat solvers., in: POS@ SAT, 2010, pp. 2–14.
- [8] M. Wahlström, Abusing the tutte matrix: An algebraic instance compression for the k-set-cycle problem, arXiv preprint arXiv:1301.1517.
- [9] A. C. Giannopoulou, D. Lokshtanov, S. Saurabh, O. Suchy, Tree deletion set has a polynomial kernel but no  $\text{opt}^{o(1)}$  approximation, SIAM Journal on Discrete Mathematics 30 (3) (2016) 1371–1384.
- [10] H. Dell, D. Van Melkebeek, Satisfiability allows no nontrivial sparsification unless the polynomial-time hierarchy collapses, Journal of the ACM (JACM) 61 (4) (2014) 23.

- [11] B. M. Jansen, A. Pieterse, Optimal sparsification for some binary csps using low-degree polynomials, arXiv preprint arXiv:1606.03233.
- [12] B. M. Jansen, A. Pieterse, Sparsification upper and lower bounds for graph problems and not-all-equal sat, *Algorithmica* 79 (1) (2017) 3–28.
- [13] J. Ding, A. Sly, N. Sun, Proof of the satisfiability conjecture for large  $k$ , in: Proceedings of the forty-seventh annual ACM symposium on Theory of computing, ACM, 2015, pp. 59–68.
- [14] R. G. Downey, M. R. Fellows, Fundamentals of parameterized complexity, Vol. 201, Springer, 2016.
- [15] R. Schroeppel, A. Shamir, A  $t=o(2^{n/2})$ ,  $s=o(2^{n/4})$  algorithm for certain np-complete problems, *SIAM journal on Computing* 10 (3) (1981) 456–464.
- [16] L. G. Valiant, The complexity of computing the permanent, *Theoretical computer science* 8 (2) (1979) 189–201.
- [17] M. R. Garey, D. S. Johnson, Computers and intractability, W.H. Freeman, New York, 1979.