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REACHABILITY PROBLEMS IN LOW-DIMENSIONAL ITERATIVE MAPS

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In this paper we analyze the dynamics of one-dimensional piecewise maps. We show that one-dimensional piecewise affine maps are equivalent to pseudo-billiard or so called "strange billiard" systems. We also show that use of more general classes of functions lead to undecidability of reachability problem for one-dimensional piecewise maps.

Keywords: Theory of computing; reachability problems; iterative piecewise maps; pseudo-billiard systems; universality.

1. Introduction

In the present work we investigate a class of hybrid systems defined by one-dimensional piecewise maps. We are mainly interested in analysis of iterative one-dimensional piecewise affine maps (PAMS) [2], piecewise rational maps and piecewise elementary maps. The analysis of piecewise-affine maps is one of the simplest model that generate complex behavior, see [2, 3, 4, 6, 7]. It is known that the reachability problem is undecidable for the two-dimensional case and it is open for dimension one [1, 2].

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It was recently shown that PAM is equivalent to hierarchical piecewise constant derivatives system (HPCD)[2]. In this paper we show that PAM is also equivalent to planar pseudo-billiard systems (PBSs) or so called "strange billiards" model that is a well known object in bifurcation and chaos theory [10, 11]. By the pseudo billiard we understand a system of borders in $\mathbb{R}^2$ with assigned to them vector fields. The computation in this system can be described by the dynamics of the particle, which initially moves with the constant velocity (in a particular direction) inside a given region and changes it instantaneously at the moment of a collision with the boundary to the velocity defined by a given vector field (not necessarily a constant one) on the boundary of the region.

Although the reachability for PAMs is known to be open we think that the shown equivalence between PBSs and PAMs can be useful and the results from chaos theory about "strange billiards" [10, 5, 11] could help understand the complexity in one-dimensional piecewise-affine maps.

Another question that we study in this paper is related to the computational power of other one-dimensional maps. Firstly, we consider a class of rational maps which generalizes a class of affine maps. We show that a system of one-dimensional piecewise rational maps (PRM) of degree 2 with a finite set of intervals can simulate a 2-counter Minsky machine. As a result we can state that there is a particular map, corresponding to universal Minsky machine, where the point-to-point reachability problem is undecidable.

In fact the above construction requires a finite set of intervals, but two of which are semi-infinite. We also show that there is another class of piecewise iterative maps, defined by a very restricted basis of elementary functions:

$$\{x^2, x^3, \sqrt[2]{x}, \sqrt[3]{x}, x \pm 1, 10 \cdot x\},$$

that can simulate a Minsky machine in dimension one with only a finite number of bounded intervals. As a result we can define a pseudo-billiard system with non-linear borders where the point-to-point reachability problem is undecidable.

2. Preliminaries

In what follows we use traditional denotations $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ for the sets of naturals, rationals and reals respectively.

A function from a set A to a set B, we will denote by $f : A \to B$. If f is an injection such that $dom(f) = A$, then it will be denoted by $f : A \hookrightarrow B$. In some cases we put $x \mapsto y$ under the definition of a function f to express that $y = f(x)$, for example:

$$\begin{array}{cccc} f: & \mathbb{R} & \to & \mathbb{R} \\ & x & \mapsto & ax + b \end{array}$$

is a way to say that $f(x) = ax + b$.

If we have a set A in a topological space (usually we will consider $\mathbb{R}$ or $\mathbb{R}^2$ with the euclidian topology), we will denote by $int(A)$ (the interior of A) the greatest open subset of A ($int(A) = \cup\{G : \ G \text{ open and } G \subseteq A\}$).

2.1. *Dynamical systems*

Definition 1. *A* dynamical transition system *is a triple* $S = (X, T, \Sigma)$*, where* X *is a set (the set of points of the system),* $T : X \to X$ *(the transition function that produces the evolution of the system), and* Σ *is a collection of subsets of* X *(this component is only considered in the case we are interested in the symbolic behavior of the system).*

Remark 2. *Usually, we will require* Σ *to be a partition of* X*, or at least to be a collection of pairwise disjoint subsets of* X *(in the case we are interested in the dynamical behavior of some parts of* X*, using the rest as* auxiliary computation*). Also, we will see* Σ *as an alphabet, and we will study the language generated by the system on this alphabet.*

Definition 3. *Let* $S = (X, T, \Sigma)$ *a dynamical system, and* $x \in X$*. The sequence* $\{x_n\}_{n \geq 0}$*, such that:*

- $x_0 = x$*,*
- *for every* $n \geq 0$*,* $x_{n+1} = T(x_n)$*.*

is called the orbit of x by the system S*, and it will be denoted as* $O_S(x)$*.*

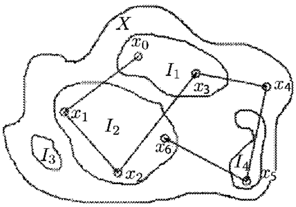


Fig. 1. Example of a dynamical system with an orbit in it.

In Figure 1 we have a dynamical system $S = (X, T, \Sigma = \{I_1, I_2, I_3, I_4\})$, where a partial orbit of a point, x_0, is shown.

Definition 4. *Let* $S = (X, T, \Sigma)$ *be a dynamical system, and* $x \in X$*. Let's associate the set* $X \setminus \cup_{w \in \Sigma} w$ *to the element* ε *(the empty word). The* symbolic dynamics of x in terms of Σ *is the set:*

$$\mathcal{S}_S(x) = \{w \in \Sigma' : \ \forall n \geq 0 \ (O_S(x)_n \in w_n)\}$$

Where $\Sigma' = \Sigma^* \cup \Sigma^\omega$ (all the words, finite ones and infinite ones, over Σ) and we use the notation $w = w_1 w_2 w_3 \ldots$

In example above, $\mathcal{S}_S(x_0) = I_1 I_2 I_2 I_1 \varepsilon I_4 I_2 = I_1 I_2 I_2 I_1 I_4 I_2$. Note that point x_4 in the orbit has no representation in its symbolic dynamics.

Remark 5. *If Σ is a collection of pairwise disjoint subsets of X, then for every point $x \in X$, $\mathcal{S}_S(x)$ has only one element.*

Definition 6. *Let $S_1 = (X_1, T_1, \Sigma_1)$ and $S_2 = (X_2, T_2, \Sigma_2)$ two dynamical systems. We will say that S_2 simulates S_1 if there exists an injection $\varphi : X_1 \hookrightarrow X_2$, and an injection $\sigma : \Sigma_1 \hookrightarrow \Sigma_2$ such that for every $x \in X_1$, we have:*

$$\mathcal{S}_{S_2}(\varphi(x)) = \widehat{\sigma}(\mathcal{S}_{S_1}(x))$$

where $\widehat{\sigma} : \Sigma_1' \hookrightarrow \Sigma_2'$ is the morphism generated by σ.

2.2. *Pseudo billiard systems*

Let us introduce the pseudo billiard model that already appeared in a different context and became an abstract framework for some practical problems. In this system we consider a number of segments with vector fields assigned to them. The computation in this system can be described by the dynamics of the particle, which initially moves with the constant velocity (in a particular direction) inside a given region (not necessarily a polyhedron) and changes it instantaneously at the moment of a collision with the boundary to the velocity defined by a given vector field on the boundary.

We start with a more general definition for PBS's, where we have no constraints on distributing the segments around the space. In this case, a particle can touch the segments by both faces, and therefore it may cross them by the action of their projection vectors.

Definition 7. *A Pseudo Billiard System (PBS) is a pair $(\mathcal{A}, \mathcal{V})$, where $\mathcal{A}$ is a set of pairwise disjoint segments in $\mathbb{R}^2$ (closed, open or semi-open), and $\mathcal{V} = \{\vec{v}_A\}_{A \in \mathcal{A}}$ is a set of vectors in $\mathbb{R}^2 - \{(0,0)\}$ ($\vec{v}_A$ is called the projection vector of A).*

The dynamics of a particle in PBS can be defined as follows. Let a particle P that is represented by a vector x and is located on a segment $A \in \mathcal{A}$, i.e. $x \in A$. The transition function that moves P from x to a position x' can be defined as follows: $x' = x + \lambda \vec{v}_A$, where $\lambda = \min\{\delta > 0 \ : \ x + \delta \vec{v}_A \in \bigcup_{A' \in \mathcal{A}} A'\}$. We will suppose that for every $x \in A$ there exists such a λ (the particle is trapped inside the system).

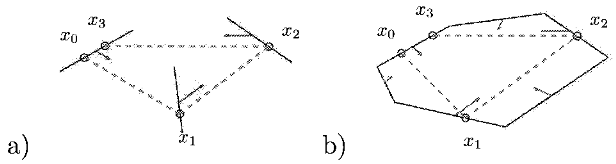


Fig. 2. An example of partial orbit: a) in a PBS, b) in a reflecting PBS.

The PBS can be seen as a dynamical transition system $S = (X, T, \Sigma)$ where:

- $X = \bigcup_{A \in \mathcal{A}} A$,
- $T(x) = x + \lambda \vec{v}_A$, where $x \in A$ and $\lambda = \min\{\delta > 0 \ : \ x + \delta \vec{v}_A \in \bigcup_{A' \in \mathcal{A}} A'\}$
- Σ is any collection of subsets of X (usually it will be a subset of $\mathcal{A}$).

Definition 8. *A PBS is* reflecting, *if for every $A \in \mathcal{A}$, the set $T^{-1}(A)$ and $T(A)$ are in the same half-plane determined by A.*

2.3. *Piecewise affine maps*

Definition 9. *We say that $f : \mathbb{R} \to \mathbb{R}$ is a piecewise affine map (PAM) if there exists a partition of dom(f) in a finite number of intervals of $\mathbb{R}$ (we allow the intervals to be closed, open or semi-open intervals), $\mathcal{I} = \{I_1, \ldots, I_k\}$, and for every interval $I_j \in \mathcal{I}$, there exists $a_j, b_j \in \mathbb{R}$ such that: $\forall x \in I_j,\ f(x) = a_j x + b_j$.*

Remark 10. *If we have $f(dom(f)) \subseteq dom(f)$, then we can consider a dynamical system associated to it, $S = (X, T, \Sigma)$ where:*

- *$X = dom(f)$,*
- *$T = f(x)$ and*
- *Σ is any collection of subsets of X (usually it will be a subset of $\mathcal{I}$).*

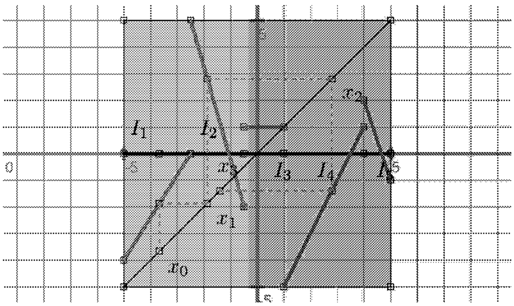


Fig. 3. An example of partial orbit in a PAM (represented on the diagonal).

Definition 11. *A rational function is a function defined as a ratio of polynomials. For a single variable x a typical rational function is therefore $f(x) = P(x)/Q(x)$, where P and Q are polynomials in x as indeterminate, and Q is not the zero polynomial.*

We also give the definition of a more general class of rational functions that we are going to study in the paper. We define it over $\mathbb{Q}$ to show that even in this case the predictability of its behavior is an undecidable problem.

Definition 12. *A Piecewise (one-dimensional) rational map (PRM) is a function that is defined on a finite sequence of disjoint intervals $I_- = (-\infty, r_-]$, $I_+ =$*

$[l_+, +\infty)$, $I_i = [l_i, r_i]$ with $r_-, l_+, l_i, r_i \in \mathbb{Q}$, $i = 1..k$ and uses rational functions for different parts of its domain.

The computation in the above system can be understood as a generation of sequence of points. One of the obvious problems that arises in such systems is a point-to-point reachability problem that can be formulated as follows:

Problem 1. *Given two points $x, y \in \mathbb{Q}$ and a one-dimensional piecewise map P. Decide whether y is reachable from x in P.*

3. Equivalence between Dynamical Systems

In this section we will study the equivalence between the models introduces above. We will say that two models are equivalent if for every system of one type there exists a system of another type that simulates it and vice versa. In particular we are giving geometrical constructions to show the equivalence of one-dimensional PAM, planar PBS and planar reflective PBS. Moreover using the result that model of hierarchical piecewise constant derivative systems (HPCDs) is equivalent to one-dimensional PAMs we can state that planar PBS is equivalent to two-dimensional HPCDs (see [2]). Hence the complexity that can be obtained with any of them is the same.

3.1. *PAM simulates PBS*

The first step through the equivalence will be devoted to prove that any PBS system (reflecting or not) can be simulated by a PAM system.

Theorem 13. *For every Pseudo Billiard System, $\{\mathcal{A}, \mathcal{V}\}$, there exists a Piecewise Affine Map that simulates it.*

Proof. Let us consider a PBS given by a set of segments $\mathcal{A} = \{A_i\}$ and a set of associated projection vectors $\{\vec{v}_i\}$. You can see an example on Figure 4 .a. □

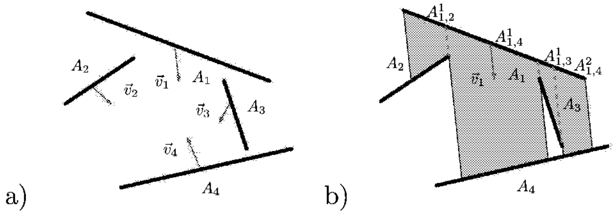


Fig. 4. a) Example of PBS to be simulated by a PAM; b) Projection of A_1 on other segments of the PBS, and the partition on A_1 that it generates.

The dynamics of the PBS is defined by projecting every point of A_i on some other segment by using the projection vector $\vec{v}_i$. It is clear, from the definition of

PBS, that we can make a partition of A_i in segments, $\{A^k_{i,j}\}$, in such a way that every point of $\{A^k_{i,j}\}$ is projected on a point of A_j (see Figure 4 .b). The next step is to associate for every segment of the system, A_i, an interval on the line, I_i, by using an affine bijection, $\mu_i : A_i \to I_i$. Also, we will require these intervals to be pairwise disjoint. For every i, j, k such that $A^k_{i,j} \neq \emptyset$, the projection $P^k_{i,j} : A^k_{i,j} \to A_j$ is an affine transformation. Also, all the functions μ_i are affine transformations.

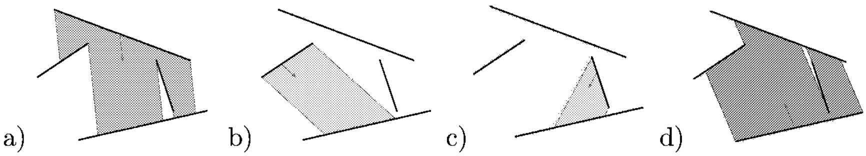


Fig. 5. Projections: a) from A_1; b) from A_2; c) from A_3; d) from A_4.

Hence, we can define an affine map

$$\begin{array}{llll} f_{i,j,k}: & \mu_i(A^k_{i,j}) & \to & \mu_j(A_j) \\ & x & \mapsto & \mu_j(P^k_{i,j}(\mu_i^{-1}(x))) \end{array}$$

Since $\{A^k_{i,j}\}_{i,j,k}$ is a partition of the points of the PBS, $\{I^k_{i,j}\}_{i,j,k}$ is a partition of the set of intervals considered, hence we obtain that the map

$$\begin{array}{llll} f: & \bigcup_{i,j,k} I^k_{i,j} & \to & \bigcup_{i,j,k} I^k_{i,j} \\ & x & \mapsto & f_{i,j,k}(x), \text{ if } x \in I^k_{i,j} \text{ is a piecewise affine map.} \end{array}$$

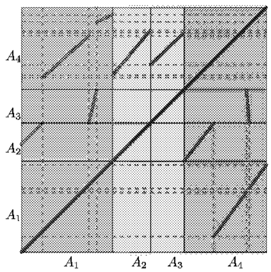


Fig. 6. PAM obtained after the process.

In order to prove the dynamical simulation, let us consider the following injection between the set of points of the systems:

$$\begin{array}{llll} \varphi: & \bigcup A_i & \hookrightarrow & \bigcup I_i \\ & x & \mapsto & \mu_i(x), \text{ if } x \in A_i. \end{array}$$

If Σ is the subset of A that produces the symbolic dynamics, then $\Sigma' = \{\varphi(I) : I \in \Sigma\}$ is the collection to be considered in the PAM, and $\sigma : \Sigma \to \Sigma'$ defined by $\sigma(I) = \varphi(I)$ is the injection for the dynamics. From the construction, it is obvious that f simulates the given PBS.

Lemma 14. *The number of affine functions we need in order to simulate a PBS is bounded*[a] *by* $|\mathcal{A}|(|\mathcal{A}|+2)$.

Proof. Idea: for every segment of the PBS, the partition we need to make all possible projections on the other segments is bounded in size by $|\mathcal{A}|+2$. □

3.2. *PBS simulates PAM*

Next, we will prove that for any PAM we can build a PBS simulating its dynamical behavior. Indeed, we will see that they can be simulated using only reflecting PBS's, hence there is no difference (regarding the dynamical complexity of the system) in using a general PBS or restrict ourselves to reflecting PBS's.

Nevertheless, we will prove in a first step that, for every PAM we can get a general PBS (usually not reflecting) that simulates the given PAM. The proof is based on the graphical idea about how to compute the orbit of a point directly on the graph generated by the PAM (where we consider the dynamics on the diagonal rather than on the X axis) using the iterated projections between the affine map and the diagonal.

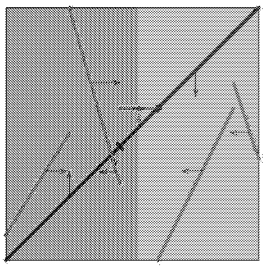


Fig. 7. PBS associated to a PAM.

In this example we can see that the PBS obtained by the dynamics of the PAM over the diagonal is, in general, a non reflecting one, because depending its definition, we will need to cross some of the affine map graphs to reach the diagonal. In any case, it is not a problem, because if we must cross one map from another one, it is because both of them are in the same half-plane from the diagonal, and then their associated vectors are parallel and in the same direction.

[a]In case of reflecting PBSs, the bound can be reduced to $|\mathcal{A}|+2$.

It is easy to see that, if we restrict the dynamics on the set of segments over the diagonal, rather than on the X axis, the system we obtain is equivalent to the original one.

Theorem 15. *For every Piecewise Affine Map there exists a Pseudo Billiard System that simulates it.*

Proof. Let $f = \bigcup_{i=1}^{n} f_i$ a PAM where every f_i is an affine map over an interval I_i. We consider the following segments on $\mathbb{R}^2$:

- For every $I_i = [a_i, b_i]$, we consider its projection on the diagonal, $x = y$, that we note as A_i.
- For every I_i we consider the segment given by $(a_i, f_i(a_i)) - (b_i, f_i(b_i))$, that we note as $f(A_i)$.

We can consider that there is no intersection between the interior of segments of the PAM (otherwise if any A_i intersects with some $f(A_j)$, we consider the intersection point, and subdivide both segments, leaving this point in the diagonal segment (in Figure 7 we have split the second affine function in order to have segments with no intersection in their interiors). Of course, the obtained PAM is equivalent to the original one.

The vectors associated with every segment is given by the following rule: for every i

- if $int(f(A_i))$ is inside the half-plane $x < y$ (the upper half) then the vector associated to A_i is $(0, 1)$, and the vector associated to $f(A_i)$ is $(1, 0)$.
- if $int(f(A_i))$ is inside the half-plane $x > y$ (the lower half) then the vector associated to A_i is $(0, -1)$, and the vector associated to $f(A_i)$ is $(-1, 0)$.

In order to prove the dynamical simulation, let us consider the following injection between the set of points of the systems:

$$\begin{array}{cccc} \varphi : & \bigcup I_i & \hookrightarrow & \bigcup A_i \\ & x & \mapsto & (x, x) \end{array}$$

and, following the same procedure as in theorem 1, the same injection between the dynamics of the systems.

From the construction, it is obvious that the resulting PBS simulates the given PAM (note that we use only a part of the dynamics of the PBS, considering only the dynamics on the diagonal, and not the evolution of the points through the other segments, necessary for the correct computing of the evolution, but not for the dynamics itself). □

3.3. *Reflecting PBS simulates PAM*

Theorem 16. *For every Piecewise Affine Map there exists a Reflecting Pseudo Billiard System that simulates it.*

Proof. Let $f : I \to I$ be a PAM expressed in such a way that $I = \bigcup_{i=1}^{n} I_i$ is union of pairwise disjoint intervals, and for every i, $f_{|I_i} = f_i$, where $f_i(x) = a_i x + b_i$ is an affine function.

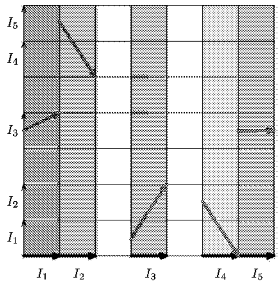


Fig. 8. PAM to be simulated.

The first step of the proof consists in assigning to every interval of the PAM a segment in $\mathbb{R}^2$ where we simulate the dynamics of the system. Since $f_i : I_i \to I$ is affine, and I_i is an interval, $f_i(I_i)$ must be an interval too. Hence, the image of every interval of our partition must be inside an union of intervals of our partition that constitutes a larger interval. To make more direct the proof, we will maintain the continuity among intervals of f by considering for every interval, $I_i \subseteq \mathbb{R}$, of f, the segment $A_i = I_i \times \{0\} \subseteq \mathbb{R}^2$.

Now, we will simulate the dynamics of each affine map separately. Because the segments A_i are in the same line, we can't go directly from one to another by using projections, therefore we will make use of auxiliary reflection segments to produce the same result as f produces.

Depending on the coefficients of the affine map, there are three different cases:

(1) **Case 1:** $a_i > 0$. In this case there is no flip from A_i to $f_i(A_i)$, so we will need only one reflecting auxiliary segment to simulate the application of f, B_i (see figure 9 .a).

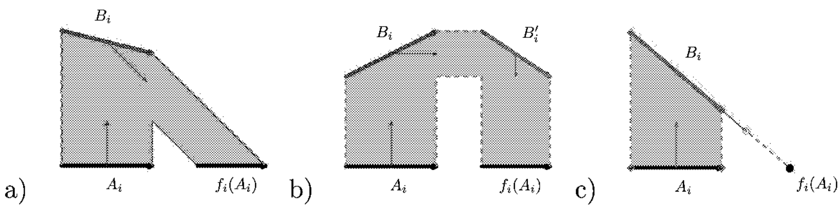
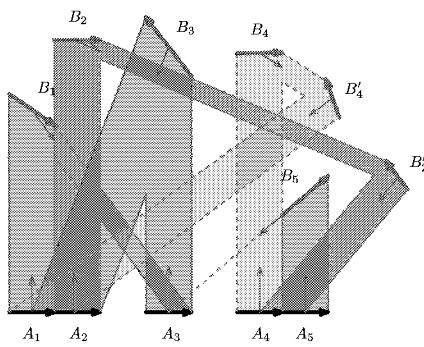


Fig. 9. a) Case 1: $a_i > 0$ b) Case 2: $a_i < 0$ c) Case 3: $a_i = 0$.

(2) **Case 2:** $a_i < 0$. In this case there is a flip from A_i to $f_i(A_i)$, so we will need two reflecting auxiliary segments, B_i and B_i', to simulate the application of f (see figure 9 .b).
(3) **Case 3:** $a_i = 0$. In this case $f(A_i)$ is a point, and we will make use of only one reflecting auxiliary segment, B_i, to project to this point (see Figure 9 .c). Indeed, it can be seen as a extremal subcase of case 1.

We can construct simultaneously all these segments with projection vectors on $\mathbb{R}^2$ without disturbing one to each other, obtaining a reflecting PBS (see Figure 10 for a complete construction for PAM in Figure 8).

a) 

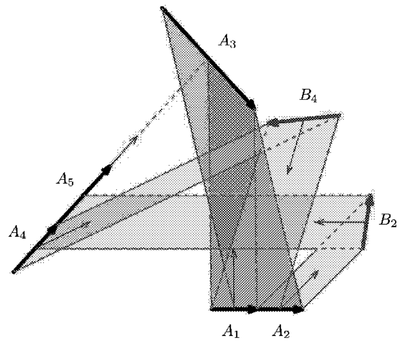
b) 

Fig. 10. a) Reflecting PBS simulating PAM b) PBS with reduced number of segments.

In order to prove the dynamical simulation, let us consider the following injection between the set of points of the systems:

$$\begin{array}{cccc} \varphi: & \bigcup I_i & \hookrightarrow & \bigcup A_i \\ & x & \mapsto & (x, 0) \end{array}$$

and the same injection between the dynamics as in the previous theorems.

From the construction, it is obvious that the resulting PBS simulates the given PAM (note that, again, we use only a part of the dynamics of the PBS, considering only the dynamics on the segments A_i, and not the evolution of the points through the other segments, necessary for the correct computing of the evolution, but not for the dynamics itself). □

From above construction, we obtain an upper bound to the number of segments we need in a reflecting PBS to simulate a PAM.

Corollary 17. *Let f be a PAM with N affine functions. Let R be the number of affine maps, f_i, with $a_i < 0$. Then, there is a reflecting PBS simulating f using, at most, $2N + R$ reflecting segments.*

Remark 18. *The method of construction presented previously is not efficient in general, but it works for any possible PAM. In a number of PAM's, it is possible to reduce the number of elements of the PBS simulating the PAM. For example, in Figure 10 .a, we can identify segment A_3 with B_3 (of course, taking A_3 out of the X-axis), making unnecessary the use of B_1, B_3 and B_5. Also, in this example, if we change the orientation of A_4 and A_5 we can avoid the use of some auxiliary segments, B_2' and B_4'. We have reduced the construction from 12 segments to only 7 (it is easy to check that in this example we need, at least, 5 segments in order to simulate the dynamics), see Figure 10 .b.*

Since $\mathbb{Q}$ is closed under linear rational transformations, if we restrict segments and vectors in $\mathbb{Q}^2$ for the PBS, and intervals and coefficients in $\mathbb{Q}$ for the PAMs, everything can be proved in the same way and the equivalence remains true.

4. Unpredictability in Rational Piecewise Maps

In this section we show that the reachability problem in one dimensional rational piecewise maps is undecidable since for every Minsky machine [9] we can define a PRM that simulates its computation. Actually we need to show how the states, transition function and updates of integer counters can be simulated by a piecewise rational map P. The main idea of simulating two dimensional computation using one dimensional function is based on abilities of some function to create a copy of information within the same number.

Let A be a 2-counter machine with a set of states $S = \{1, 2, \ldots, n\}$. The configuration of A is a triple $[k, l, s]$ where k and l are values of two counters and s is a current state of A. Let us define the mapping $\phi : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$ that is an isomorphism between a configuration $[k, l, s]$ of A and a rational number $s + \frac{1}{2^{k+1}3^{l+1}}$ that is shifted to the interval (0,1)

$$\phi([k, l, s]) \to \frac{1}{10^H}(s + \frac{1}{2^{k+1}3^{l+1}}), H = \lceil \lg(|S|) \rceil$$

Instead of classical Minsky machine from now on we will consider a well-known equivalent model of two counter machine, where one of the counters is used as a scratchpad and another counter holds an integer whose prime factorization is $2^c \cdot 3^d$. The exponents c, d can be thought of as two virtual counters that are being simulated. If the real counter is set to zero then incremented once, that is equivalent to setting all the virtual counters to zero. If the real counter is doubled, that is equivalent to incrementing c, and if it is halved, that is equivalent to decrementing c. By a similar procedure, it can be multiplied or divided by 3, which is equivalent to incrementing or decrementing d.

To check if a virtual counter such as c (d) is equal to zero, just divide the real counter by 2 (3), see what the remainder is, then multiply by 2 (3) and add back the remainder. That leaves the real counter unchanged. The remainder will have been nonzero if and only if c (d) was zero.

Let A be in configuration $[k, l, s]$ and it is represented by a number

$$x = \frac{1}{10^H}(s + \frac{1}{2^{k+1}3^{l+1}}).$$

Let us show that we can perform the operations of multiplication and division by 2 and 3 in a piecewise rational map P. To multiply/divide virtual counter by 2 or/and 3 we can use the following expression for x, where a,b are integers:

$$\frac{(10^H x - s)2^a 3^b + s}{10^H}$$

Now, we construct a system of intervals with rational functions, associated to them, that allows us to check divisibility of the value of the virtual counter by 2 and 3 or in other words to perform a zero testing on counters of original Minsky machine. For each state s of a counter machine we define the following intervals and functions:

Let us assume that the current configuration $[k, l, s]$ of a machine M is represented by a rational number x. If M is in a state s then x belongs to the interval $[\frac{s}{10^H}, \frac{s+1}{10^H}]$. Assuming that we know the current state we can add to x an integer $2^{k+1}3^{l+1}$ by expression $\frac{1}{(10^H x - s)} + x$. In fact for further simulation of checking the emptiness of the one Minsky machine counter we would need to add an integer $2^k 3^{l+1}$ using the expression $\frac{1}{2(10^H x - s)} + x$. Such operation gives us an extra information about the counter values in integer part of the number. It is important that we can use it now for some temporal computation and keep another copy of the current state and counter values in the decimal part of the number.

Now we can easily check whether a virtual counter is divisible by 2 iteratively applying $x - 2$ while the point x is in the interval $[3, +\infty)$. Finally a point x should reach either the interval $[2, 3]$, which corresponds to $k \neq 0$, or the interval $[1, 2]$, which corresponds to $k = 0$.

In a similar way we can check divisibility by 3 from a state s using negative numbers. If $x \in [\frac{s}{10^H}, \frac{s+1}{10^H}]$ we apply $-(\frac{1}{3(10^H x - s)} + x)$ and then $x + 3$ for any point in the interval $(-\infty, -4]$. Next the number x should appear in the interval $[-4, -3]$, which corresponds to $l \neq 0$ or in the interval $[-3, -1]$, which corresponds to $l = 0$.

Now we define a piecewise rational map to simulate all operations of Minsky machine such as state transitions, update of counters and testing them for zero. Initially let us define two intervals for intermediate computation related to the zero testing in counters:

If $x \in [3, +\infty)$ then apply $x - 2$, If $x \in (-\infty, -4]$ then apply $x + 3$

Next for every command of the Minsky machine

State s: IF $k \neq 0$ THEN k=k+a, l=l+b GOTO State t ELSE GOTO State p

we define a set of intervals with assigned rational functions:

If $x \in [\frac{s}{10^H}, \frac{s+1}{10^H}]$ then apply $\frac{1}{2(10^H x - s)} + x$

If $x \in [2 + \frac{s}{10^H}, 2 + \frac{s+1}{10^H}]$ then apply $\frac{(10^H(x-2)-s)\cdot 2^a 3^b + t}{10^H}$

If $x \in [1 + \frac{s}{10^H}, 1 + \frac{s+1}{10^H}]$ then apply $\frac{(10^H(x-1)-s)+p}{10^H}$

where $a \in \mathbb{Z}$ stands for increasing (decreasing) of the first counter by an integer a, and $b \in \mathbb{Z}$ stands for increasing (decreasing) of the second counter by an integer b.

Next for every command of the Minsky machine with testing of the second counter for zero

State s: IF $l \neq 0$ THEN k=k+a, l=l+b GOTO State t ELSE GOTO State p

We define a set of intervals in a similar way:

If $x \in [\frac{s}{10^H}, \frac{s+1}{10^H}]$ then apply $-(\frac{1}{3(10^H x - s)} + x)$

If $x \in [-(3 + \frac{s}{10^H}), -(3 + \frac{s+1}{10^H})]$ then apply $\frac{(10^H(x+4)-s)\cdot 2^a 3^b + t}{10^H}$

If $x \in [-(2 + \frac{s}{10^H}), -(2 + \frac{s+1}{10^H})]$ then apply $\frac{(10^H(x+3)-s)+p}{10^H}$

If $x \in [-(1 + \frac{s}{10^H}), -(1 + \frac{s+1}{10^H})]$ then apply $\frac{(10^H(x+2)-s)+p}{10^H}$

Since the computation of a Minsky machine can be simulated by a specially designed PRM the following theorem holds:

Theorem 19. *One-dimensional piecewise rational map with a finite number of intervals is a universal model of computation.*

Corollary 20. *The reachability problem (Problem 1) for one-dimensional PRM is undecidable.*

Corollary 21. *There exists a particular one-dimensional PRM , that corresponds to the universal Minsky machine, for which the point-to-point reachability problem is undecidable.*

5. Elementary Maps with Bounded Intervals

In the case of rational maps with undecidable reachability problem two of the intervals were semi-infinite. In this section we construct another map with a finite set of bounded intervals using a richer set of one-dimensional functions.

Let us consider a piecewise elementary map (PEM) P which is defined by composition of x^2, x^3, $\sqrt[2]{x}$, $\sqrt[3]{x}$, $x \pm 1$ and $10 \cdot x$ for different parts of a piecewise function

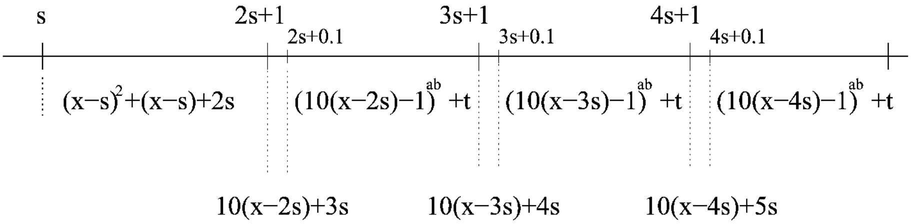


Fig. 11. A part of universal one-dimensional piecewise elementary map.

domain. In order to prove the universality of one-dimensional PEM we show how to simulate states and transitions of a 2-counter machine (Minsky machine) A, in a similar way to our construction from the section 4.

Let us define a new mapping $\psi : \mathbb{N}\times\mathbb{N}\times\mathbb{N} \to \mathbb{Q}$ that is an isomorphism between a configuration $[k, l, s]$ of A and a rational number $s.0^{2^k 3^l}1$:

$$\psi([k,l,s]) \to s + \frac{1}{10^{2^k \cdot 3^l + 1}},$$

where k and l are values of two counters and s is a current state of A.

Let A be in a state s and its current configuration is represented by a number $s.0^m 1$. The multiplication and division of a counters k and l in a piecewise elementary map P will be performed as follows. First, we construct a system of intervals with elementary functions, associated to them, that allow us to check divisibility of m by 2 and 3 or in other words to perform a zero testing on counters of original Minsky machine. For each state s of a counter machine we define the following intervals and functions:

$$\begin{array}{c}
\text{If } x \in (s, s+1) \text{ then apply } (x-s)^2 + (x-s) + 2s \\
\text{If } x \in (2s, 2s+0.1) \text{ then apply } 10(x-2s) + 3s \\
\text{If } x \in (3s, 3s+0.1) \text{ then apply } 10(x-3s) + 4s \\
\text{If } x \in (4s, 4s+0.1) \text{ then apply } 10(x-4s) + 5s \\
\text{If } x \in (5s, 5s+0.1) \text{ then apply } 10(x-5s) + 6s \\
\text{If } x \in (6s, 6s+0.1) \text{ then apply } 10(x-6s) + 7s \\
\text{If } x \in (7s, 7s+0.1) \text{ then apply } 10(x-7s) + 2s
\end{array}$$

It is easy to see that in this system of intervals any point of the form $s.0^m 1$ will be mapped to $2s + 0.0^m 10^m 1$ and then after m iterations to the point $i \cdot s + 0.10^m 1$. Note that $(i-1)$ is divided by 2 (3), if and only if m is divided by 2 (3).

Now we extend our piecewise function to simulate state transitions and update of counters by the following intervals and functions:

$$\text{If } x \in (i \cdot s + 0.1, i \cdot s + 1),\ i \in \{2,3,4,5,6,7\} \text{ then apply } (10(x - i \cdot s) - 1)^{a \cdot b} + t$$

where $a = 2$ ($a = \frac{1}{2}$) stands for increasing (decreasing) of the first counter by 1, and $b = 3$ ($a = \frac{1}{3}$) stands for increasing (decreasing) of the second counter by 1 (see Figure 11). Thus, this part of piecewise function express a transition of machine A

from state s to state t and counters updates assuming that their values satisfy to divisibility by 2 and 3.

In order to finish the construction of 1-dimensional PEM that models machine A we redefine a set of states S by changing it to a set $S' = \{8 \cdot s | s \in S\}$ that gives us $7 \cdot |S|$ disjoint intervals in piecewise elementary function. Since the computation of a Minsky machine can be simulated by a specially designed PEM the following theorem holds:

Theorem 22. *One-dimensional piecewise elementary map is the universal model of computation.*

Corollary 23. *The reachability problem (Problem 1) for 1-dimensional PEM is undecidable.*

Corollary 24. *There exists a particular one-dimensional PEM, that corresponds to the universal Minsky machine, for which the point-to-point reachability problem is undecidable.*

6. Discussions

In this paper we show that the model of one-dimensional PAMs is equivalent to a known model of strange billiards from bifurcation and chaos theory. On the other hand we show that predictability in more general one-dimensional class of functions is not possible since we can encode a universal model of computation such as Minsky Machine.

Although the point-to-point reachability for pseudo billiard system with linear borders is an open problem it is easy to see that we can design pseudo billiard system with non-linear borders that can simulate computation in one-dimensional PEM. Let one-dimensional PEM be defined on the domain $[l, r] \subset \mathbb{R}$. One of the solution would be to start computation from the border $\{(x, y) | l \leq x \leq r, y = 0\}$ with assigned vector $(1, 0)$ (vertical up) and to place each function f_i of PEM on a different horizontal level in $\mathbb{R}^2$, i,e, $y = f_i(x) + i \cdot h_i$, where $i \cdot h$ will be a vertical shift and h is a large enough constant. Then we assign vectors $(1, 0)$ (the right horizontal direction) for increasing parts of PEM and $(-1, 0)$ (the left horizontal direction) for decreasing parts of PEM. Finally we need to redirect all computed values to the original segment $\{(x, y) | l \leq x \leq r, y = 0\}$ via two extra reflections. Thus, the reachability problem for pseudo billiard system with non-linear borders is undecidable.

In terms of the future work it would be interesting to investigate a natural class of one-dimensional piecewise linear rational maps that is in between affine and rational maps. The main motivation for this class of systems is based on the fact that the reachability in one-dimensional piecewise linear rational maps can be seen as parameterized reachability in two dimensional linear[b] maps. Another interesting

[b]A two dimensional linear function f is a function of the following type $f(x, y) = ax + by$.

question is nondeterministic maps where transformations can be applied in any order. In this case reachability problems for nondeterministic linear rational maps corresponds to parameterized membership in 2×2 matrix semigroups. According to undecidability in PRM we think that it is also very likely that a nondeterministic version of one-dimensional rational maps has undecidable reachability problem as well.

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