ELSEVIER Annals of Pure and Applied Logic 96 (1999) 187–207

ANNALS OF PURE AND APPLIED LOGIC

Dynamic linear time temporal logic

Jesper G. Henriksen[a,*], P.S. Thiagarajan[b,2]

[a] BRICS[1], Department of Computer Science, University of Aarhus, Ny Munkegade DK-8000 Aarhus C, Denmark

[b] SPIC Mathematical Institute, 92 G. N Chetty Road, T. Nagar, Chennai 600 017, India

Received 14 April 1997; received in revised form 13 August 1997; accepted 7 January 1998

Communicated by S.N. Artemov

Abstract

A simple extension of the propositional temporal logic of linear time is proposed. The extension consists of strengthening the until operator by indexing it with the regular programs of propositional dynamic logic. It is shown that DLTL, the resulting logic, is expressively equivalent to the monadic second-order theory of ω-sequences. In fact, a sublogic of DLTL which corresponds to propositional dynamic logic with a linear time semantics is already expressively complete. We show that DLTL has an exponential time decision procedure and admits a finitary axiomatization. We also point to a natural extension of the approach presented here to a distributed setting.

AMS classification: 03B45; 03B70; 68Q60; 68Q68

Keywords: Linear time temporal logic; Dynamic logic; ω-automata; Expressive completeness; Axiomatizations

1. Introduction

We present here a simple extension of the propositional temporal logic of linear time. The basic idea is to strengthen the until modality by indexing it with the regular programs of propositional dynamic logic. The resulting logic, called dynamic linear time temporal logic (DLTL), is easy to handle. It has the full expressive power of the monadic second-order theory of ω-sequences. Indeed a sublogic of DLTL is already

* Corresponding author.

[1] Basic Research in Computer Science, Centre of the Danish National Research Foundation.

[2] Part of this work was done while visiting BRICS. Part of this work has been supported by the Indo-French Centre for the Promotion of Advanced Research (IFCPAR) project 1502-1.

PII: S0168-0072(98)00039-6

expressively complete. A pleasant feature of this sublogic is that it is just propositional dynamic logic operating in a linear time framework.

In addition to our expressiveness results we show that DLTL has an exponential time decision procedure. We also extend the well-known axiomatization of propositional dynamic logic [11] to obtain an axiomatization of DLTL.

Our work may be viewed from two different perspectives. The first one is from the standpoint of process logics [6, 16, 18] which attempt a rapprochement between dynamic and temporal logics. However, the study of process logics is committed to viewing dynamic logic as a restricted kind of a *branching time* temporal logic. One then attempts to bring in some additional mechanisms for talking about computational paths. Our point of departure consists of merging, in a very simple way, dynamic logic and temporal logic in a *linear time* setting.

The second perspective has to do with attempts to augment the expressive power of linear time temporal logic. One route consists of permitting quantification over atomic propositions. The resulting logic called QPTL [20] is as expressive as S1S, the monadic second-order theory of sequences but its decision procedure has non-elementary time complexity. The second route consists of augmenting linear time temporal logic with the so-called automaton connectives. The resulting logic called ETL [26] is equal in expressive power to S1S while admitting an exponential time decision procedure.

Our logic is, in spirit, inspired by ETL and it can be easily translated into ETL. It may appear to be at first sight to be a mere reformulation of ETL with some cosmetic changes. This however has to do with the instinctive identification one makes between finite state automata and regular expressions. In fact, DLTL is quite different in terms of the mechanisms it offers for structuring formulas and we feel that it is more transparent and easier to work with. The results and the proofs we present here are designed to support this claim. Our approach also leads to smooth generalizations in non-sequential settings where similar extensions in terms of ETL will be hard to cope with.

In the next section, we start with an action-based version of of linear time temporal logic in order to fix terminology. In Section 3 we present DLTL and its semantics. This is then followed by a more detailed assessment of the similarities and the differences between ETL and DLTL.

In Section 4, we prove the decidability of DLTL by reducing it to the emptiness problem for Büchi automata. In Section 5, we show that DLTL^-, a sublogic of DLTL, has the same expressive power as S1S, the monadic second-order theory of sequences. We then establish similar results for the first-order fragment of S1S with the help of the "star-free" fragments of DLTL and DLTL^-.

In Section 6, we extend the axiomatization of PDL (propositional dynamic logic) and the completeness proof in [11] to obtain finitary axiomatizations of DLTL and DLTL^-. In the final section we point to a natural generalization in the setting of distributed systems. This generalization is eminently accessible and offers additional support to our belief that the synthesis of dynamic and temporal logics in a linear time framework as pursued here is a fruitful one.

2. Linear time temporal logic

One key feature of the syntax and semantics of our temporal logic is the treatment of *actions* as first class objects. To bring this out we formulate a version of LTL (linear time temporal logic) in which the next-state modality is indexed by actions taken from a fixed alphabet set.

Through the rest of the paper we fix a finite non-empty alphabet Σ. We let a, b range over Σ and refer to members of Σ as actions. Σ^* is the set of finite words and Σ^ω is the set of infinite words generated by Σ with $\omega = \{0, 1, 2, \ldots\}$. We set $\Sigma^\infty = \Sigma^* \cup \Sigma^\omega$ and denote the null word by ε. We let σ, σ' range over Σ^ω and τ, τ', τ'' range over Σ^*. Finally $\preccurlyeq$ is the usual prefix ordering defined over Σ^* and for $u \in \Sigma^\infty$, we let $\mathrm{prf}(u)$ be the set of finite prefixes of u.

Next we fix a countable set of atomic propositions $P = \{p_1, p_2, \ldots\}$ and let p, q range over P. The set of formulas of $\mathrm{LTL}(\Sigma)$ is then given by the syntax:

$$\mathrm{LTL}(\Sigma) ::= p \mid \sim\alpha \mid \alpha \vee \beta \mid \langle a \rangle \alpha \mid \alpha\, \mathscr{U}\, \beta.$$

Through the rest of this section α, β will range over $\mathrm{LTL}(\Sigma)$.

A model of $\mathrm{LTL}(\Sigma)$ is a pair $M = (\sigma, V)$ where $\sigma \in \Sigma^\omega$ and $V : \mathrm{prf}(\sigma) \longrightarrow 2^P$ is a valuation function. Let $M = (\sigma, V)$ be a model, $\tau \in \mathrm{prf}(\sigma)$ and α be a formula. Then $M, \tau \models \alpha$ will stand for α being satisfied at τ in M. This notion is defined inductively in the expected manner:

- $M, \tau \models p$ iff $p \in V(\tau)$.
- $M, \tau \models \sim\alpha$ iff $M, \tau \not\models \alpha$.
- $M, \tau \models \alpha \vee \beta$ iff $M, \tau \models \alpha$ or $M, \tau \models \beta$.
- $M, \tau \models \langle a \rangle \alpha$ iff $\tau a \in \mathrm{prf}(\sigma)$ and $M, \tau a \models \alpha$.
- $M, \tau \models \alpha\, \mathscr{U}\, \beta$ iff there exists τ' such that $\tau\tau' \in \mathrm{prf}(\sigma)$ and $M, \tau\tau' \models \beta$. Moreover for every τ'' such that $\varepsilon \preccurlyeq \tau'' \prec \tau'$, it is the case that $M, \tau\tau'' \models \alpha$.

We note that the next-state modality of LTL is definable via $O\alpha \stackrel{\Delta}{\Longleftrightarrow} \bigvee_{a \in \Sigma} \langle a \rangle \alpha$. It is well known [4, 10] that $\mathrm{LTL}(\Sigma)$ is expressively equivalent to the first-order theory of sequences. Hence, this temporal logic, relative to S1S, has limited expressive power. For instance, as pointed out by Wolper [25], the property "p holds at every even position" is not definable in this logic.

3. Dynamic linear time temporal logic

Our extension of $\mathrm{LTL}(\Sigma)$ basically consists of indexing the until operator with the programs of PDL (e.g. [3, 5]). We start by defining the set of programs (regular expressions) generated by Σ. This set is denoted by $\mathrm{Prg}(\Sigma)$ and is given by

$$\mathrm{Prg}(\Sigma) ::= a \mid \pi_0 + \pi_1 \mid \pi_0; \pi_1 \mid \pi^*.$$

Here and elsewhere, π, π' with or without subscripts will range over $\mathrm{Prg}(\Sigma)$. With each program we associate a set of finite words via the map $\|\cdot\| : \mathrm{Prg}(\Sigma) \longrightarrow 2^{\Sigma^*}$.

This map is defined in the standard fashion. As before, we fix a countable set of atomic propositions $P = \{p_1, p_2, \dots\}$ and let p, q range over P. The set of formulas of DLTL(Σ) is then given by the following syntax:

$$\mathrm{DLTL}(\Sigma) ::= p \mid \sim\alpha \mid \alpha \vee \beta \mid \alpha\, \mathscr{U}^{\pi} \beta.$$

Herein after, we let α, β range over DLTL(Σ). The notion of a model is as in the case of LTL(Σ). So let $M = (\sigma, V)$ be a model, $\tau \in \mathrm{prf}(\sigma)$ and $\alpha \in \mathrm{DLTL}(\Sigma)$. Then $M, \tau \models \alpha$ is defined inductively. The base case and the boolean connectives are handled as before. The semantics of the augmented until opeartor is given by

- $M, \tau \models \alpha\, \mathscr{U}^{\pi}\beta$ iff there exists $\tau' \in ||\pi||$ such that $\tau\tau' \in \mathrm{prf}(\sigma)$ and $M, \tau\tau' \models \beta$. Moreover, for every τ'' such that $\varepsilon \preccurlyeq \tau'' \prec \tau'$, it is the case that $M, \tau\tau'' \models \alpha$.

Thus DLTL(Σ) is obtained form LTL(Σ) by strengthening the until operator. To satisfy $\alpha\, \mathscr{U}^{\pi}\beta$, one must satisfy $\alpha \mathscr{U} \beta$ along some finite stretch of behaviour which is in the (linear time) behaviour of the program π.

As usual, $\alpha \in \mathrm{DLTL}(\Sigma)$ is *satisfiable* iff there exist a model $M = (\sigma, V)$ and $\tau \in \mathrm{prf}(\sigma)$ such that $M, \tau \models \alpha$.

Apart from the conventional derived propositional connectives such as $\wedge, \supset$ and $\equiv$ the derived modality $\langle\pi\rangle$ and its dual $[\pi]$ will play an important role in the sequel.

- $\top \overset{\Delta}{\Longleftrightarrow} p_1 \vee \sim p_1$. Recall that $P = \{p_1, p_2, \dots\}$.
- $\langle\pi\rangle\alpha \overset{\Delta}{\Longleftrightarrow} \top\, \mathscr{U}^{\pi}\alpha$.
- $[\pi]\alpha \overset{\Delta}{\Longleftrightarrow} \sim\langle\pi\rangle\sim\alpha$.

Suppose $M = (\sigma, V)$ is a model and $\tau \in \mathrm{prf}(\sigma)$. It is easy to see that $\sigma, \tau \models \langle\pi\rangle\alpha$ iff there exists $\tau' \in ||\pi||$ such that $\tau\tau' \in \mathrm{prf}(\sigma)$ and $\sigma, \tau\tau' \models \alpha$. It is also easy to see that $\sigma, \tau \models [\pi]\alpha$ iff for every $\tau' \in ||\pi||$, if $\tau\tau' \in \mathrm{prf}(\sigma)$ then $\sigma, \tau\tau' \models \alpha$. In this sense, the program modalities of PDL acquire a linear time semantics in the present setting.

Note that $a \in \Sigma$ is a member of Prg(Σ) and hence $\langle a\rangle\alpha$ is a derived modality. Letting $\Sigma = \{a_1, a_2, \dots, a_n\}$, it is also easy to see that the until operator of LTL(Σ) can be obtained via: $\alpha \mathscr{U} \beta \overset{\Delta}{\Longleftrightarrow} \alpha\, \mathscr{U}^{\Sigma^*} \beta$ with Σ as a shorthand for the program $a_1 + a_2 + \dots + a_n$. Thus LTL(Σ) is a fragment of DLTL(Σ) both in terms of syntax and semantics. To see that DLTL(Σ) is strictly more expressive than LTL(Σ), let $\pi_{ev} = (\Sigma; \Sigma)^*$. It is easy to see that $\alpha_{ev} = [\pi_{ev}]p$ is a specification of the property "p holds at every even position".

We shall close out the section by briefly discussing the key differences between DLTL(Σ) and ETL, the extension of LTL proposed by Wolper [25]. We shall present a simplified form of ETL so as to stay close to DLTL. First we fix an enumeration of $\Sigma = \{a_1, a_2, \dots, a_n\}$. The syntax of the logic that we shall name as ETL(Σ) is given by

$$\mathrm{ETL}(\Sigma) ::= p \mid \sim\phi \mid \phi \vee \phi' \mid \mathscr{A}(\phi_0, \phi_1, \dots, \phi_n).$$

Here $\mathscr{A}$ is a finite state automaton of the form $\mathscr{A} = (Q, \longrightarrow, Q_{\mathrm{in}}, F)$ with $\longrightarrow \subseteq Q \times \Sigma \times Q$ as the transition relation, $Q_{in} \subseteq Q$ as the initial states and $F \subseteq Q$ as the accepting states. Let $\mathscr{L}(\mathscr{A})$ be the language of finite words accepted by $\mathscr{A}$. We shall assume for the sake of convenience that $\varepsilon \notin \mathscr{L}(\mathscr{A})$ for each formula of the form $\mathscr{A}(\phi_0, \phi_1, \ldots, \phi_n)$.

A model for ETL(Σ) is, as before, a pair $M = (\sigma, V)$ with $V : \mathrm{prf}(\sigma) \longrightarrow 2^P$. Let $\tau \in \mathrm{prf}(\sigma)$. Then $M, \tau \models \phi$ is defined for the cases of atomic propositions and the boolean connectives in the expected manner. The automaton connective is interpreted as follows:

- $M, \tau \models \mathscr{A}(\phi_0, \phi_1, \ldots, \phi_n)$ iff there exists $a_{i_1} a_{i_2} \ldots a_{i_m} \in \mathscr{L}(\mathscr{A})$ such that the following conditions are satisfied:
  - $i_1, i_2, \ldots, i_m \in \{1, 2, \ldots, n\}$. (recall that $\Sigma = \{a_1, a_2, \ldots, a_n\}$).
  - $\tau a_{i_1} a_{i_2} \cdots a_{i_m} \in \mathrm{prf}(\sigma)$.
  - $M, \tau \models \phi_0$ and $M, \tau a_{i_1} \cdots a_{i_j} \models \phi_{i_j}$ for $1 \leqslant j \leqslant m$.

Though the technical details are somewhat different, ETL(Σ) captures the spirit of the logic presented in [24]. The key drawback of ETL(Σ), as we see it, lies in its lack of structuring principles for forming compound formulas. The only mechanism that ETL(Σ) has – apart from the boolean connectives – to form compound formulas is *by nesting* the automaton formulas. Thus, a typical compound formula would look like

$$\mathscr{A}^1(\phi_0^1, \mathscr{A}^2(\phi_0^2, \phi_1^2, \mathscr{A}^3(\phi_0^3, \ldots, \phi_n^3), \phi_3^2, \ldots, \phi_n^2), \phi_2^1, \ldots, \phi_n^1).$$

In contrast, DLTL(Σ) adds to the familiar mechanisms of LTL an *orthogonal* and well-understood component; namely, the language of regular expressions. Equally important, this orthogonal component is formulated purely in terms of Σ and not in terms of arbitrary formulas as is the case of ETL. In fact, ETL, as formulated in [24] has an uncontrolled amount of "external" elements in the sense that the states and the alphabets of the automata which are used to write down the automaton formulas have little to do with the logic under consideration.

It is an easy exercise to translate DLTL into ETL with only a linear blow-up in the size of the formulas. It will however be more productive and illuminating to give an independent treatment of DLTL as we shall do here.

4. A decision procedure for DLTL

The goal here is to show that the satisfiability problem for DLTL(Σ) can be solved in deterministic exponential time. This will be achieved by effectively constructing for each $\alpha \in$ DLTL(Σ), a Büchi automaton $\mathscr{B}_\alpha$ such that the language of ω-words accepted by $\mathscr{B}_\alpha$ is non-empty iff α is satisfiable.

A Büchi automaton over Σ is a tuple $\mathscr{B} = (Q, \longrightarrow, Q_{\mathrm{in}}, F)$ where

- Q is a finite non-empty set of states.
- $\longrightarrow \subseteq Q \times \Sigma \times Q$ is a transition relation.
- $Q_{\mathrm{in}} \subseteq Q$ is a set of initial states.
- $F \subseteq Q$ is a set of accepting states.

Let $\sigma \in \Sigma^{\omega}$. Then a run of $\mathscr{B}$ over σ is a map $\rho : \mathrm{prf}(\sigma) \longrightarrow Q$ such that
– $\rho(\varepsilon) \in Q_{\mathrm{in}}$.
– $\rho(\tau) \xrightarrow{a} \rho(\tau a)$ for each $\tau a \in \mathrm{prf}(\sigma)$.
The run ρ is accepting iff $\inf(\rho) \cap F \neq \emptyset$ where $\inf(\rho) \subseteq Q$ is given by $q \in \inf(\rho)$ iff $\rho(\tau) = q$ for infinitely many $\tau \in \mathrm{prf}(\sigma)$. Finally $\mathscr{L}(\mathscr{B})$, the language of ω-words accepted by $\mathscr{B}$, is

$$\mathscr{L}(\mathscr{B}) = \{\sigma \mid \exists \text{ an accepting run of } \mathscr{B} \text{ over } \sigma\}.$$

Through the rest of the section we fix a formula α_0. To construct $\mathscr{B}_{\alpha_0}$ we first define the (Fischer–Ladner) closure of α_0 as follows. $cl(\alpha_0)$ is the least set of formulas that satisfies
– $\alpha_0 \in cl(\alpha_0)$.
– If $\sim\beta \in cl(\alpha_0)$ then $\beta \in cl(\alpha_0)$.
– If $\alpha \vee \beta \in cl(\alpha_0)$ then $\alpha, \beta \in cl(\alpha_0)$.
– If $\alpha\, \mathscr{U}^{\pi} \beta \in cl(\alpha_0)$ then $\alpha, \beta \in cl(\alpha_0)$.
Now $CL(\alpha_0)$, the *closure* of α_0, is defined to be

$$CL(\alpha_0) = cl(\alpha_0) \cup \{\sim\beta \mid \beta \in cl(\alpha_0)\}.$$

In what follows, $\sim\sim\beta$ will be identified with β. Moreover, throughout the section, all the formulas that we encounter – unless stated otherwise – will be assumed to be members of $CL(\alpha_0)$. For convenience, we shall often write CL instead of $CL(\alpha_0)$.

$A \subseteq CL$ is called an *atom* iff it is a subset of CL satisfying
– $\beta \in A$ iff $\sim\beta \notin A$.
– $\alpha \vee \beta \in A$ iff $\alpha \in A$ or $\beta \in A$.
– If $\beta \in A$ and $\varepsilon \in ||\pi||$ then $\alpha\, \mathscr{U}^{\pi} \beta \in A$.

$AT(\alpha_0)$ is the set of atoms and again we shall often write AT instead of $AT(\alpha_0)$. Next we define $Req(\alpha_0)$, the set of until requirements of α_0, to be the subset of CL given by

$$Req(\alpha_0) = \{\alpha\, \mathscr{U}^{\pi} \beta \mid \alpha\, \mathscr{U}^{\pi} \beta \in CL\}.$$

We shall write Req instead $Req(\alpha_0)$ and take ξ, ξ' to range over Req. For each $\xi = \alpha\, \mathscr{U}^{\pi} \beta \in Req$ we fix a finite state automaton $\mathscr{A}_{\xi}$ such that $\mathscr{L}(\mathscr{A}_{\xi}) = ||\pi||$ where $\mathscr{L}(\mathscr{A}_{\xi})$ is the language of finite words accepted by $\mathscr{A}_{\xi}$. We shall assume each such $\mathscr{A}_{\xi}$ is of the form $\mathscr{A}_{\xi} = (Q_{\xi}, \longrightarrow_{\xi}, I_{\xi}, F_{\xi})$ where Q_{ξ} is the set of states, $\longrightarrow_{\xi} \subseteq Q_{\xi} \times \Sigma \times Q_{\xi}$ is the transition relation, $I_{\xi} \subseteq Q_{\xi}$ is the set of initial states and $F_{\xi} \subseteq Q_{\xi}$ is the set of final states. Without loss of generality, we shall assume that $\xi \neq \xi'$ implies $Q_{\xi} \cap Q_{\xi'} = \emptyset$ for every $\xi, \xi' \in Req$. We set $Q = \bigcup_{\xi \in Req} Q_{\xi}$ and $\widehat{Q} = Q \times \{0, 1\}$.

The Büchi automaton $\mathscr{B}_{\alpha_0}$ associated with α_0 (from now on denoted as $\mathscr{B}$) can now be defined as

$$\mathscr{B} = (S, \Longrightarrow, S_{\mathrm{in}}, F),$$

where the various components of $\mathscr{B}$ are specified as follows. We provide explanatory remarks immediately after the definition.

(1) $S \subseteq AT \times 2^{Q} \times 2^{\widehat{Q}} \times \{0,1\} \times \{\downarrow,\uparrow\}$ such that $(A, X, \widehat{X}, x, f) \in S$ iff the following conditions are satisfied for each $\xi = \alpha\, \mathscr{U}^{\pi} \beta$:
  (i) If $\beta \in A$ then $F_\xi \subseteq X$. (Recall that $\mathscr{A}_\xi = (Q_\xi, \longrightarrow_\xi, I_\xi, F_\xi)$).
  (ii) If $\alpha \in A$ and $q \in X$ for some $q \in I_\xi$ then $\alpha\, \mathscr{U}^{\pi} \beta \in A$.
  (iii) If $\alpha\, \mathscr{U}^{\pi} \beta \in A$ then either $\beta \in A$ and $\varepsilon \in \|\pi\|$ or $(q, 1-x) \in \widehat{X}$ for some $q \in I_\xi$. (Note that we are considering the candidate $(A, X, \widehat{X}, x, f)$ for membership in S).
  (iv) If $(q, z) \in \widehat{X}$ with $q \in Q_\xi$ and $q \notin F_\xi$ or $\beta \notin A$ then $\alpha \in A$.

(2) The transition relation $\Longrightarrow \subseteq S \times \Sigma \times S$ is defined as follows:

$$(A, X, \widehat{X}, x, f) \overset{a}{\Longrightarrow} (B, Y, \widehat{Y}, y, g)$$

iff the following conditions are satisfied for each $\xi = \alpha\, \mathscr{U}^{\pi} \beta$:
  (i) Suppose $q' \in Q_\xi \cap Y$ and $q \overset{a}{\longrightarrow}_\xi q'$ and $\alpha \in A$. Then $q \in X$.
  (ii) Suppose $(q, z) \in \widehat{X}$ with $q \in Q_\xi$. Suppose further that $q \notin F_\xi$ or $\beta \notin A$. Then $(q', z) \in \widehat{Y}$ for some q' with $q \overset{a}{\longrightarrow}_\xi q'$.
  (iii) If $f = \uparrow$ then $(y, g) = (1-x, \downarrow)$. If $f = \downarrow$ then,

$$(y, g) = \begin{cases} (x, \downarrow), & \text{if there exists } (q, x) \in \widehat{X} \text{ such that} \\ & q \notin F_\xi \text{ or } \beta \notin A. \\ (x, \uparrow), & \text{otherwise.} \end{cases}$$

(3) $S_{\text{in}} = \{(A, X, \widehat{X}, x, f) \mid \alpha_0 \in A \text{ and } (x, f) = (0, \uparrow)\}$.

(4) $F = \{(A, X, \widehat{X}, x, f) \mid f = \uparrow\}$.

To understand the functioning of the automaton $\mathscr{B}$, let (σ, V) be a model and ρ a run of $\mathscr{B}$ over σ. Assume further that $\tau \in \text{prf}(\sigma)$ and that $\rho(\tau) = (A, X, \widehat{X}, x, f)$. The role of the atom A, as usual, is to assert that the formulas in A will be satisfied at τ. To check this, the automaton should verify that all the until requirements are being satisfied. This work is divided into two phases; a 0-phase and a 1-phase. The value of the boolean variable x indicates the current phase of the automaton. The last component is used to signal the successful completion of one phase. The automaton will not toggle to the next phase until successful completion of the current phase. The component X corresponds to the so-called safety automaton in [23]. The point is that the automaton must assert $\alpha\, \mathscr{U}^{\pi} \beta$ at τ in case there is *some* possibility of satisfying this assertion in the unknown future. The component X, in combination with the transition relation, is designed to ensure this. The component $\widehat{X}$ is used to check the liveness requirements. The complication here is that while requirements of the form (q, x) are being checked, new requirements may come up. These will be tagged with the value $1-x$ but will have to be simultaneously checked. They cannot be ignored while working towards discharging the requirements in the current phase. The definition of the state set of the automaton as well as the transition relation have been guided by these considerations. It might be that this information could be maintained in a more compact form but it is a pointless optimization at this stage.

We wish to first prove that α_0 is satisfiable iff $\mathscr{L}(\mathscr{B}) \neq \emptyset$. Afterwards we will argue that the size of $\mathscr{B}$ can be chosen to be at most exponential in the size of α_0.

Lemma 1. *Suppose $\mathscr{L}(\mathscr{B}) \neq \emptyset$. Then α_0 is satisfiable.*

Proof. Let $\sigma \in \mathscr{L}(\mathscr{B})$ and $\rho : \mathrm{prf}(\sigma) \longrightarrow S$ be an accepting run. For each $\tau \in \mathrm{prf}(\sigma)$, let $\rho(\tau) = (A_\tau, X_\tau, \widehat{X}_\tau, x_\tau, f_\tau)$. Define the model $M = (\sigma, V)$ via:

$$V(\tau) = A_\tau \cap P \quad \text{for all } \tau \in \mathrm{prf}(\sigma).$$

Claim 2. *For all $\tau \in \mathrm{prf}(\sigma)$ and $\delta \in CL$,*

$$M, \tau \models \delta \quad \textit{iff} \quad \delta \in A_\tau.$$

First note that if the claim is true then Lemma 1 follows at once. This is so because ρ is a run of $\mathscr{B}$ and hence $\rho(\varepsilon) \in S_{in}$. But from (3), in the definition of $\mathscr{B}$, it follows that $\alpha_0 \in A_\varepsilon$.

In proving the claim we will repeatedly refer to various clauses in the definition of the Büchi automaton $\mathscr{B}$. We proceed by structural induction on δ. For the base case and the boolean connectives the claim is obvious. Hence assume that $\delta = \alpha \, \mathscr{U}^\pi \beta$.

Suppose that $M, \tau \models \alpha \, \mathscr{U}^\pi \beta$. Since $M, \tau \models \alpha \, \mathscr{U}^\pi \beta$ there exists $\tau' \in ||\pi||$ such that $\tau\tau' \in \mathrm{prf}(\sigma)$ and $M, \tau\tau' \models \beta$. Moreover, $M, \tau\tau'' \models \alpha$ for every $\tau'' \in \Sigma^*$ such that $\varepsilon \preccurlyeq \tau'' \prec \tau'$.

Suppose $\tau' = \varepsilon$. Then $\varepsilon \in ||\pi||$ and $M, \tau \models \beta$. By the induction hypothesis $\beta \in A_\tau$. From the definition of an atom it follows that $\alpha \, \mathscr{U}^\pi \beta \in A_\tau$.

So assume that $\tau' \neq \varepsilon$. Let $\xi = \alpha \, \mathscr{U}^\pi \beta$ and R be an accepting run of $\mathscr{A}_\xi$ over $\tau' = a_1 a_2 \ldots a_n$ with $R(\varepsilon) = q_0 \in I_\xi$ and $R(a_1 a_2 \ldots a_i) = q_i$ for $1 \leqslant i \leqslant n$ and $q_n \in F_\xi$. Since $M, \tau\tau' \models \beta$ we have from the induction hypothesis that $\beta \in A_{\tau\tau'}$. Hence by (1.i), $F_\xi \subseteq X_{\tau\tau'}$. Now by the definition of R we are assured that $q_{n-1} \xrightarrow{a_n}_\xi q_n$. On the other hand, the fact that $M, \tau \models \alpha \, \mathscr{U}^\pi \beta$ and the choice of τ' guarantee that $M, \tau a_1 \ldots a_{n-1} \models \alpha$ (with the convention that $\varepsilon = a_1 \ldots a_{n-1}$ in case $n = 1$). By the induction hypothesis $\alpha \in A_{\tau a_1 \ldots a_{n-1}}$, so by (2.i) and the fact that $q_n \in X_{\tau a_1 \ldots a_n}$, we have that $q_{n-1} \in X_{\tau a_1 \ldots a_{n-1}}$. In case $n \geqslant 2$ we repeat the above argument at q_{n-1} to conclude that $q_{n-2} \in X_{\tau a_1 \ldots a_{n-2}}$. Continuing this way we can finally arrive at $q_0 \in X_\tau$ and $\alpha \in A_\tau$. But $q_0 \in I_\xi$ and hence by (1.ii) we are assured that $\alpha \, \mathscr{U}^\pi \beta \in A_\tau$.

For the converse direction assume that $\alpha \, \mathscr{U}^\pi \beta \in A_\tau$. There are four cases to consider depending on the values of x_τ and f_τ. We will only prove one case. The remaining cases can be resolved by similar arguments.

So assume that $x_\tau = 0$ and $f_\tau = \downarrow$. Suppose first that $\beta \in A_\tau$ and $\varepsilon \in ||\pi||$. Then by the induction hypothesis $M, \tau \models \beta$ and hence we at once have $M, \tau \models \alpha \, \mathscr{U}^\pi \beta$. So assume that $\beta \notin A_\tau$ or $\varepsilon \notin ||\pi||$. Then by (1.iii), $(q_0, 1) \in \widehat{X}_\tau$ for some $q_0 \in I_\xi$. Suppose $q_0 \in F_\xi$. Then $\varepsilon \in ||\pi||$ and thus $\beta \notin A_\tau$. This implies, by (1.iv), that $\alpha \in A_\tau$, and by the induction hypothesis we have that $M, \tau \models \alpha$.

Now with ρ being an accepting run of $\mathscr{B}$ over σ there must exist τ_1 and τ_2 in Σ^* such that the following conditions are satisfied:

- $\tau_1 \neq \varepsilon$ and $\tau_2 \neq \varepsilon$ and $\tau\tau_1\tau_2 \in \mathrm{prf}(\sigma)$.
- $x_{\tau\tau_1} = 0$ and $x_{\tau\tau_1\tau_2} = 1$. (Recall the notational convention that $\rho(u) = (A_u, X_u, \widehat{X}_u, x_u, f_u)$ for each $u \in \mathrm{prf}(\sigma)$.)
- $f_{\tau\tau_1} = \uparrow$ and $f_{\tau\tau_1\tau_2} = \uparrow$.
- For each τ_1'' and τ_2'' in Σ^*, if $\varepsilon \preccurlyeq \tau_1'' \prec \tau_1$ then $f(\tau\tau_1'') \neq \uparrow$ and if $\varepsilon \prec \tau_2'' \prec \tau_2$ then $f(\tau\tau_1\tau_2'') \neq \uparrow$.

Let $\tau_1 = a_1a_2 \ldots a_n$ and $\tau_2 = b_1b_2 \ldots b_m$. Now $\rho(\tau) \overset{a_1}{\Longrightarrow} \rho(\tau a_1)$, $\alpha\, \mathscr{U}^\pi \beta \in A_\tau$ and $(q_0, 1) \in \widehat{X}_\tau$. Moreover, we have that $q_0 \notin F_\xi$ (if $\varepsilon \notin ||\pi||$) or $\beta \notin A_\tau$. Thus by (2.ii), there exists $q_1 \in Q_\xi$ such that $q_0 \xrightarrow{a_1}_\xi q_1$ and $(q_1, 1) \in \widehat{X}_{\tau a_1}$.

Now suppose $q_1 \in F_\xi$ and $\beta \in A_{\tau a_1}$. Then $a_1 \in ||\pi||$ and by the induction hypothesis $M, \tau a_1 \models \beta$. Since $M, \tau \models \alpha$ has already been deduced we have $M, \tau \models \alpha\, \mathscr{U}^\pi \beta$. So assume that $q_1 \notin F_\xi$ or $\beta \notin A_{\tau a_1}$. Then by repeating the arguments we had above for q_0 at q_1 we can arrive at $\alpha \in A_{\tau a_1}$, and hence by the induction hypothesis $M, \tau a_1 \models \alpha$. Moreover, we can conclude that there exists $q_2 \in Q_\xi$ such that $q_1 \xrightarrow{a_2}_\xi q_2$ and $(q_2, 1) \in \widehat{X}_{\tau a_1 a_2}$. Marching down τ_1 using this sequence of arguments we will either terminate with the conclusion $M, \tau \models \alpha\, \mathscr{U}^\pi \beta$ or we will exhaust all of τ_1 while being able to conclude that there must exist states $q_0, q_1, \ldots, q_n \in Q_\xi$ such that $q_0 \xrightarrow{a_1}_\xi q_1 \xrightarrow{a_2}_\xi q_2 \ldots q_{n-1} \xrightarrow{a_n}_\xi q_n$. Furthermore, we will be able to conclude that $M, \tau\tau_1'' \models \alpha$ for every τ_1'' such that $\varepsilon \preccurlyeq \tau_1'' \prec \tau_1$. Finally, we will also be assured that $(q_n, 1) \in \widehat{X}_{\tau\tau_1}$.

Now suppose $q_n \in F_\xi$ and $\beta \in A_{\tau\tau_1}$. Then $\tau_1 \in ||\pi||$ and $M, \tau\tau_1 \models \beta$ by the induction hypothesis. Consequently $M, \tau \models \alpha\, \mathscr{U}^\pi \beta$. So assume that $q_n \notin F_\xi$ or $\beta \notin A_{\tau\tau_1}$. Then $\alpha \in A_{\tau\tau_1}$ (by (1.iv)) and hence $M, \tau\tau_1 \models \alpha$ by the induction hypothesis. Now by the choice of τ_1, we know that $(x_{\tau\tau_1}, f_{\tau\tau_1}) = (0, \uparrow)$ and hence $(x_{\tau\tau_1 b_1}, f_{\tau\tau_1 b_1}) = (1, \downarrow)$ by (2.iii). On the other hand, $\rho(\tau\tau_1) \overset{b_1}{\Longrightarrow} \rho(\tau\tau_1 b_1)$ implies that there exists $q_1' \in Q_\xi$ such that $q_n \xrightarrow{b_1}_\xi q_1'$ and $(q_1', 1) \in \widehat{X}_{\tau\tau_1 b_1}$. Again $q_1' \in F_\xi$ and $\beta \in A_{\tau\tau_1 b_1}$ will lead to the desired conclusion $M, \tau \models \alpha\, \mathscr{U}^\pi \beta$.

So suppose $q_1' \notin F_\xi$ or $\beta \notin A_{\tau\tau_1 b_1}$. Then as before, $\alpha \in A_{\tau\tau_1 b_1}$ and hence $M, \tau\tau_1 b_1 \models \alpha$ by induction hypothesis. By the choice of τ_2 we are assured that $m \geqslant 2$ because $f_{\tau\tau_1 b_1} = \downarrow$. So consider $\rho(\tau\tau_1 b_1) \overset{b_2}{\Longrightarrow} \rho(\tau\tau_1 b_1 b_2)$. Then again it follows easily that there must exist $q_2' \in Q_\xi$ such that $q_1' \xrightarrow{b_2}_\xi q_2'$ and $(q_2', 1) \in \widehat{X}_{\tau\tau_1 b_1 b_2}$. If $q_2' \in F_\xi$ and $\beta \in A_{\tau\tau_1 b_1 b_2}$ then we will at once obtain $M, \tau \models \alpha\, \mathscr{U}^\pi \beta$. If not, the facts that $(q_1', 1) \in \widehat{X}_{\tau\tau_1 b_1}$ and that $q_1' \notin F_\xi$ or $\beta \notin A_{\tau\tau_1 b_1}$ holds, guarantee us that $f_{\tau\tau_1 b_1 b_2} = \downarrow$ by (2.iii). Hence $m \geqslant 3$. Carrying on this way we will eventually exhaust all of τ_2 and while doing so, reach the desired conclusion $M, \tau \models \alpha\, \mathscr{U}^\pi \beta$. $\square$

Lemma 3. *Suppose* α_0 *is satisfiable. Then* $\mathscr{L}(\mathscr{B}) \neq \emptyset$.

Proof. Since our logic has no past modalities it is easy to see that if α_0 is satisfiable then there exists a model $M = (\sigma, V)$ such that $M, \varepsilon \models \alpha_0$. We shall show that

$\sigma \in \mathscr{L}(\mathscr{B})$ by constructing a map $\rho : \mathrm{prf}(\sigma) \longrightarrow S$ so that ρ is an accepting run of $\mathscr{B}$ over σ. For each $\tau \in \mathrm{prf}(\sigma)$ we set $\rho(\tau) = (A_\tau, X_\tau, \widehat{X}_\tau, x_\tau, f_\tau)$ and define ρ in a componentwise manner.

For each $\tau \in \mathrm{prf}(\sigma)$ define A_τ via

$$A_\tau = \{\alpha \mid M, \tau \models \alpha\}.$$

For each $\tau \in \mathrm{prf}(\sigma)$ define X_τ as follows. Suppose $\xi = \alpha\, \mathscr{U}^\pi \beta$ and $q \in Q_\xi$. Then $q \in X_\tau$ iff there exists a pair (τ', R') such that
- $\tau\tau' \in \mathrm{prf}(\sigma)$ and $M, \tau\tau' \models \beta$.
- For every τ'', if $\varepsilon \preccurlyeq \tau'' \prec \tau'$ then $M, \tau\tau'' \models \alpha$.
- $R' : \mathrm{prf}(\tau') \longrightarrow Q_\xi$ such that $R'(\varepsilon) = q$ and $R'(\tau') \in F_\xi$ and $R'(\tau'') \xrightarrow{a}_\xi R'(\tau'' a)$ for every $\tau'' a \in \mathrm{prf}(\tau')$.

To define the remaining three components we will first define the fourth and fifth components by mutual induction. To this end we shall make use of some terminology.

We shall call the pair (τ, ξ) an *obligation in M* if $\tau \in \mathrm{prf}(\sigma)$ and $\xi = \alpha\, \mathscr{U}^\pi \beta \in Req$ such that $M, \tau \models \alpha\, \mathscr{U}^\pi \beta$ but $M, \tau \not\models \beta$ or $\varepsilon \notin \|\pi\|$. Let (τ, ξ) be an obligation in M. We shall say that the pair (τ', R') is a *witness* for (τ, ξ) iff the following conditions are satisfied:
- $\tau\tau' \in \mathrm{prf}(\sigma)$ and $M, \tau\tau' \models \beta$ and for every τ'', $\varepsilon \preccurlyeq \tau'' \prec \tau'$ implies $M, \tau\tau'' \models \alpha$.
- $\tau' \in \|\pi\|$ and $R' : \mathrm{prf}(\tau') \longrightarrow Q_\xi$ such that $R'(\varepsilon) \in I_\xi$, $R'(\tau') \in F_\xi$ and $R'(\tau'') \xrightarrow{a}_\xi R'(\tau'' a)$ for every $\tau'' a \in \mathrm{prf}(\tau')$.

Note that if (τ', R') is a witness for the obligation (τ, ξ) then $\tau' \neq \varepsilon$. We shall fix a *chronicle set CH* for M. It is a set of quadruples which satisfies the following conditions:
- If $(\tau, \xi, \tau', R') \in CH$ then (τ, ξ) is an obligation in M and (τ', R') is witness for (τ, ξ).
- If (τ, ξ) is an obligation in M then $(\tau, \xi, \tau', R') \in CH$ for some witness (τ', R') for (τ, ξ).
- If (τ, ξ, τ', R'), $(\tau, \xi, \tau'', R'') \in CH$ then $(\tau', R') = (\tau'', R'')$.

It is easy to check that CH exists. (In fact, it can be chosen in a canonical manner by fixing a lexicographic order on Q_ξ for each $\xi \in Req$.)

With these definitions in place, we are now prepared to define the fourth and the fifth components of ρ by induction on τ. For the base case, we set $(x_\varepsilon, f_\varepsilon) = (0, \uparrow)$. Now consider the induction step where $\tau = \tau_0 a$ and assume that $(x_{\tau'}, f_{\tau'})$ is defined for every $\tau' \in \mathrm{prf}(\tau_0)$. If $f_{\tau_0} = \uparrow$ then $(x_\tau, f_\tau) = (1 - x_{\tau_0}, \downarrow)$. Suppose $f_{\tau_0} = \downarrow$. Then $(x_\tau, f_\tau) = (x_{\tau_0}, \downarrow)$ if there exists $(\tau_1, \xi_1, \tau_1', R_1') \in CH$ such that $\tau_1 \preccurlyeq \tau_0 \prec \tau_1\tau_1'$ and $x_{\tau_1} = 1 - x_{\tau_0}$. Otherwise, $f_\tau = \uparrow$ and $x_\tau = x_{\tau_0}$.

Finally, the third component of ρ can now be defined. For each $\tau \in \mathrm{prf}(\sigma)$, we define $\widehat{X}_\tau$ as follows. Suppose $\xi \in Req$ and $q \in Q_\xi$ and $z \in \{0,1\}$. Then $(q, z) \in \widehat{X}_\tau$ iff there exists $(\tau_1, \xi, \tau_1', R_1') \in CH$ such that for some $\tau_1'' \in \mathrm{prf}(\tau_1')$, $\tau_1 \preccurlyeq \tau = \tau_1\tau_1''$. Moreover, $R_1'(\tau_1'') = q$ and $x_{\tau_1} = 1 - z$.

We now wish to argue that $\rho : \mathrm{prf}(\sigma) \longrightarrow S$ is an accepting run of $\mathscr{B}$ over σ. First we shall show that ρ is well defined. Let $\tau \in \mathrm{prf}(\sigma)$ be given. We must show that $\rho(\tau) \in S$. It is easy to see that A_τ is an atom, $X_\tau \subseteq Q$, $\widehat{X}_\tau \subseteq \widehat{Q}$, $x_\tau \in \{0,1\}$ and $f_\tau \in \{\downarrow, \uparrow\}$. We will show that $\rho(\tau)$ satisfies all the clauses of the definition of $\mathscr{B}$.

So fix some $\alpha \mathscr{U}^\pi \beta = \xi$. Assume initially that $\beta \in A_\tau$ and $q \in F_\xi$. Then $M, \tau \models \beta$ by definition of A_τ. Now consider the pair (τ', R') where $\tau' = \varepsilon$ and $R'(\varepsilon) = q$. From the definition of X_τ it now follows that $q \in X_\tau$. Thus $F_\xi \subseteq X_\tau$ as required by (1.i).

Next assume that $\alpha \in A_\tau$ and $q \in X_\tau$ for some $q \in I_\xi$. From the definition of X_τ it follows that there exists a pair (τ', R') such that $\tau\tau' \in \mathrm{prf}(\sigma)$ and $M, \tau\tau' \models \beta$ and $M, \tau\tau'' \models \alpha$ for every τ'' such that $\varepsilon \preccurlyeq \tau'' \prec \tau'$. Furthermore, $R' : \mathrm{prf}(\tau') \longrightarrow Q_\xi$ such that $R'(\varepsilon) = q$ and $R'(\tau') \in F_\xi$ and $R'(\tau'') \xrightarrow{a}_\xi R'(\tau''a)$ for every $\tau''a \in \mathrm{prf}(\tau')$. But from the assumption that $q \in I_\xi$ we have that $\tau' \in ||\pi||$, because R' is an accepting run of $\mathscr{A}_\xi$ over τ'. Consequently $M, \tau \models \alpha \mathscr{U}^\pi \beta$ and this leads to the conclusion that $\alpha \mathscr{U}^\pi \beta \in A_\tau$ as required by (1.ii).

Next assume that $\alpha \mathscr{U}^\pi \beta \in A_\tau$ and $\beta \notin A_\tau$ or $\varepsilon \notin ||\pi||$. Then (τ, ξ) is an obligation in M since by the definition of A_τ, $M, \tau \models \alpha \mathscr{U}^\pi \beta$ but $M, \tau \not\models \beta$ or $\varepsilon \notin ||\pi||$. Hence there exists $(\tau, \xi, \tau', R') \in CH$. Let $R'(\varepsilon) = q$. From the fact that (τ', R') is a witness for (τ, ξ) we have that $q \in I_\xi$. Moreover, by the definition of $\widehat{X}_\tau$ and from $\tau \preccurlyeq \tau = \tau$ (i.e. $\tau_1 = \tau$ and $\tau_1'' = \varepsilon$), it follows that $(q, 1 - x_\tau) \in \widehat{X}_\tau$ as required by (1.iii).

Finally, suppose that $(q,z) \in \widehat{X}_\tau$ with $q \in Q_\xi$ such that $q \notin F_\xi$ or $\beta \notin A$. Now $(q,z) \in \widehat{X}_\tau$ implies, by the definition of $\widehat{X}_\tau$, that there exists $(\tau_1, \xi, \tau_1', R_1') \in CH$ such that for some $\tau_1'' \in \mathrm{prf}(\tau_1')$, $\tau_1 \preccurlyeq \tau = \tau_1\tau_1''$ and $R_1'(\tau_1'') = q$ and $x_{\tau_1} = 1 - z$. But (τ_1', R_1') is a witness for the obligation (τ_1, ξ) and hence $R_1'(\tau_1') \in F_\xi$ and $M, \tau_1\tau_1' \models \beta$. Since $\beta \notin A_\tau$ or $q \notin F_\xi$ it must be the case that $\tau_1'' \prec \tau_1'$ and hence $M, \tau_1\tau_1'' \models \alpha$. But then $\tau = \tau_1\tau_1''$ now leads to $\alpha \in A_\tau$ as required by (1.iv).

We have now shown that ρ is well defined. Next we wish to show that ρ is a run of $\mathscr{B}$ over σ. Since $M, \varepsilon \models \alpha_0$ we have $\alpha_0 \in A_\varepsilon$. By definition, $(x_\varepsilon, f_\varepsilon) = (0, \uparrow)$. Hence $\rho(\varepsilon) \in S_{in}$.

Now suppose $\tau a \in \mathrm{prf}(\sigma)$. We must show that $\rho(\tau) \overset{a}{\Longrightarrow} \rho(\tau a)$. For this purpose we fix $\alpha \mathscr{U}^\pi \beta = \xi \in Req$. Suppose $q, q' \in Q_\xi$ with $q' \in X_{\tau a}$ such that $q \xrightarrow{a}_\xi q'$. Further suppose $\alpha \in A_\tau$. Now $q' \in X_{\tau a}$ implies that there exists a pair (τ', R') such that $R'(\varepsilon) = q'$ and $R'(\tau') \in F_\xi$ and $R'(\tau'') \xrightarrow{b}_\xi R'(\tau''b)$ for every $\tau''b \in \mathrm{prf}(\tau')$. Furthermore, $M, \tau a\tau' \models \beta$ and $M, \tau a\tau'' \models \alpha$ for every τ'' such that $\varepsilon \preccurlyeq \tau'' \prec \tau'$. Now consider the pair $(a\tau', R_a')$ where $R_a' : \mathrm{prf}(a\tau') \longrightarrow Q_\xi$ is given as $R_a'(\varepsilon) = q$ and for every $\tau'' \in \mathrm{prf}(\tau')$, $R_a'(a\tau'') = R'(\tau'')$. From $M, \tau \models \alpha$ (as $\alpha \in A_\tau$ by assumption) it now follows at once that $q \in X_\tau$ as required by (2.i).

Suppose now that $q \in Q_\xi$ and $(q,z) \in \widehat{X}_\tau$ but $q \notin F_\xi$ or $\beta \notin A_\tau$. Since $(q,z) \in \widehat{X}_\tau$ there must exist $(\tau_1, \xi, \tau_1', R_1') \in CH$ and $\tau_1'' \in \mathrm{prf}(\tau_1')$ such that $\tau_1 \preccurlyeq \tau = \tau_1\tau_1''$ and $x_{\tau_1} = 1 - z$ and $R_1'(\tau_1'') = q$. But (τ_1', R_1') is a witness for (τ_1, ξ) and hence $R_1'(\tau_1') \in F_\xi$ and $M, \tau_1\tau_1' \models \beta$. Consequently $\tau_1'' \prec \tau_1'$ and thus $\tau_1''a \in \mathrm{prf}(\tau_1')$ for the unique a. This implies that $R_1'(\tau_1'') \xrightarrow{a}_\xi R_1'(\tau_1''a)$. Let $R_1'(\tau_1''a) = q'$. Then $q \xrightarrow{a}_\xi q'$. But then it follows directly from the definition of $X_{\tau a}$, that $(q', 1 - z) \in \widehat{X}_{\tau a}$ as required by (2.ii).

Next suppose that $f_\tau = \uparrow$. Then clearly $(x_{\tau a}, f_{\tau a}) = (1 - x_\tau, \downarrow)$ by the definition of ρ. So assume that $f_\tau = \downarrow$. Supposing there exists $\alpha\, \mathscr{U}^\pi \beta = \xi$ in Req and there exists $q \in Q_\xi$ such that $(q, z) \in \widehat{X}_\tau$ where $z = x_\tau$. Further suppose $q \notin F_\xi$ or $\beta \notin A_\tau$. Now $(q, z) \in \widehat{X}_\tau$ implies that there exists $(\tau_1, \xi, \tau_1', R_1') \in CH$ such that $\tau_1 \preccurlyeq \tau = \tau_1\tau_1''$ for some $\tau_1'' \in \mathrm{prf}(\tau_1')$ with the further property that $x_{\tau_1} = 1 - z$. From the definitions and the fact that $q \notin F_\xi$ or $\beta \notin A_\tau$ it follows that $\tau_1 \preccurlyeq \tau \prec \tau_1\tau_1'$. Hence, by the definition of ρ it follows that $(x_{\tau a}, f_{\tau a}) = (x_\tau, \downarrow)$ as required by (2.iii). On the other hand, if such a $(q, z) \in \widehat{X}_\tau$ does not exist, then it follows directly from the definition that $(x_{\tau a}, f_{\tau a}) = (x_\tau, \uparrow)$ as required by (2.iii).

We have now verified that ρ is a run of $\mathscr{B}$ over σ. To show that ρ is accepting it suffices to prove that for any $\tau \in \mathrm{prf}(\sigma)$ there exists τ' such that $\tau\tau' \in \mathrm{prf}(\sigma)$ and $f_{\tau\tau'} = \uparrow$.

Case 1: $(x_\tau, f_\tau) = (0, \uparrow)$.
By picking $\tau' = \varepsilon$ the desired conclusion follows trivially.

Case 2: $(x_\tau, f_\tau) = (0, \downarrow)$. Define the set $\Gamma_\tau \subseteq CH$ as follows. Let (τ, ξ, τ', R') be a member of the chronicle set CH. Then $(\tau_1, \xi_1, \tau_1', R_1') \in \Gamma_\tau$ iff $\tau_1 \preccurlyeq \tau \prec \tau_1\tau_1'$ and $x_{\tau_1} = 1$. Now if $\Gamma_\tau = \emptyset$ then it is easy to see that with $\tau' = a$ where $\tau a \in \mathrm{prf}(\sigma)$ we must have $f_{\tau\tau'} = \uparrow$ as required.

So suppose $\Gamma_\tau \neq \emptyset$. Define, for each $ch = (\tau_1, \xi_1, \tau_1', R_1') \in \Gamma_\tau$, $k_{ch} = |\tau_1\tau_1'| - |\tau|$ and set $k_\tau = \max(\{k_{ch}\}_{ch \in \Gamma_\tau})$. Let $\tau a \in \mathrm{prf}(\sigma)$. Then it is easy to see that $(x_{\tau a}, f_{\tau a}) = (0, \downarrow)$. But it is also easy to verify $\Gamma_{\tau a} = \emptyset$ or $k_{\tau a} < k_\tau$. Proceeding in this way the required conclusion can be drawn eventually.

The two other cases can be resolved by similar arguments. □

It is now straightforward to establish the main result of this section. To start with we define the size of a formula α, denoted $|\alpha|$, via

- $|p| = 1$, $|{\sim}\alpha| = 1 + |\alpha|$ and $|\alpha \vee \beta| = 1 + |\alpha| + |\beta|$.
- $|\alpha\, \mathscr{U}^\pi \beta| = 1 + |\alpha| + |\pi| + |\beta|$,

where $|\pi|$ is given by $|a| = 1$, $|\pi + \pi'| = |\pi; \pi'| = 1 + |\pi| + |\pi'|$ and $|\pi^*| = 1 + |\pi|$.

Theorem 4. *For each $\alpha \in \mathrm{DLTL}(\Sigma)$ the question whether or not α is satisfiable can be decided in time $2^{\mathrm{O}(|\alpha|)}$.*

Proof. Let $\alpha_0 \in \mathrm{DLTL}(\Sigma)$. Then α_0 is satisfiable iff $\mathscr{L}(\mathscr{B}_{\alpha_0}) \neq \emptyset$ where α_0 is the Büchi automaton constructed above. The emptiness problem for $\mathscr{B}_{\alpha_0}$ can be settled in time $\mathrm{O}(|S|)$ where S is the set of states of $\mathscr{B}$ [22].

Clearly, $CL(\alpha_0)$ is linear in the size of α_0 and hence $|AT| = 2^{\mathrm{O}(|\alpha_0|)}$. Let $\alpha\, \mathscr{U}^\pi \beta \in Req$. It is known that for $\pi \in \mathrm{Prg}(\Sigma)$, we can construct in polynomial time a non-deterministic finite state automaton $\mathscr{A}_\xi$ with $\mathscr{L}(\mathscr{A}) = ||\pi||$ such that $|Q_\xi|$ is linear in the size of π (see [9] for a recent account on converting regular expression to small finite state automata).

Let $Req = \{\alpha_1 \mathscr{U}^{\pi_1}\beta_1, \ldots, \alpha_m \mathscr{U}^{\pi_m}\beta_m\}$. Then $|\pi_1| + |\pi_2| + \cdots + |\pi_m| \leqslant |\alpha_0|$. Consequently, both Q and $\widehat{Q}$ are linear in the size of α_0. It is now easy to see that $|S| = 2^{O(|\alpha_0|)}$. $\square$

As usual, the decision procedure can be applied to solve the associated model checking problem but we will not enter into details here.

5. Some expressiveness results

Our main goal here is to show that DLTL(Σ) has the same expressive power as the monadic second-order theory of infinite sequences over Σ. Towards the end of the section we will also establish that a natural sublogic of DLTL(Σ) captures the first-order theory of infinite sequences over Σ.

In order to obtain clean formulations of the expressiveness results, we shall banish atomic propositions through the rest of the paper. Instead, we will just work with the constant $\top$ and its negation $\sim\top \stackrel{\Delta}{\iff} \bot$. To be precise, the syntax of DLTL(Σ) will be from now on assumed to be

$$\mathrm{DLTL}(\Sigma) ::= \top \mid \sim\alpha \mid \alpha \vee \beta \mid \alpha\, \mathscr{U}^{\pi}\beta,$$

where $\pi \in \mathrm{Prg}(\Sigma)$ with $\mathrm{Prg}(\Sigma)$ defined as before.

A model is now just a ω-sequence $\sigma \in \Sigma^{\omega}$. For $\tau \in \mathrm{prf}(\sigma)$ we define $\sigma, \tau \models \alpha$ via:
- $\sigma, \tau \models \top$.
- All the other clauses are filled in exactly as in Section 3 while replacing M by σ in the appropriate places.

Each formula α now defines a ω-language $L_\alpha \subseteq \Sigma^{\omega}$ given by

$$L_\alpha = \{\sigma \mid \sigma, \varepsilon \models \alpha\}.$$

We say that $L \subseteq \Sigma^{\omega}$ is DLTL(Σ)-*definable* iff there exists some $\alpha \in \mathrm{DLTL}(\Sigma)$ such that $L = L_\alpha$.

The monadic second-order theory of infinite sequences over Σ is denoted S1S(Σ). Its vocabulary consists of a family of unary predicates $\{R_a\}_{a\in\Sigma}$, one for each $a \in \Sigma$; a binary predicate $\leqslant$; a binary predicate $\in$; a countable supply of individual variables $Var = \{x, y, z, \ldots\}$; a countable supply of set variables (i.e. monadic predicate variables) $SVar = \{X, Y, Z, \ldots\}$. The formulas of S1S($\Sigma$) are then built up by:
- $R_a(x)$, $x \leqslant y$ and $x \in X$ are atomic formulas.
- If ϕ and ϕ' are formulas then so are $\sim\phi$, $\phi \vee \phi'$, $(\exists x)\phi$ and $(\exists X)\phi$.

A structure for S1S(Σ) is a ω-sequence $\sigma \in \Sigma^{\omega}$. Let $\mathscr{I}$ be an interpretation of the variables with $\mathscr{I} : Var \longrightarrow \omega$ and $\mathscr{I} : SVar \longrightarrow 2^{\omega}$. Then the notion of σ being a model of ϕ under the interpretation $\mathscr{I}$, denoted $\sigma \models_{\mathscr{I}} \phi$, is defined in the expected manner. In particular, $\sigma \models_{\mathscr{I}} R_a(x)$ iff $\sigma(\mathscr{I}(x)) = a$ (note that $\sigma \in \Sigma^{\omega}$ is viewed as

$\sigma : \omega \longrightarrow \Sigma$); $\sigma \models_{\mathcal{I}} x \leqslant y$ iff $\mathcal{I}(x) \leqslant \mathcal{I}(y)$ (here $\leqslant$ is the usual ordering over ω); $\sigma \models_{\mathcal{I}} x \in X$ iff $\mathcal{I}(x) \in \mathcal{I}(X)$.

As usual, a sentence is a formula with no free variables. Each sentence ϕ defines a ω-language denoted L_ϕ where

$$L_\phi = \{\sigma \mid \sigma \models \phi\}.$$

We say that $L \subseteq \Sigma^\omega$ is S1S(Σ)-*definable* iff there exists a sentence $\phi \in$ S1S(Σ) such that $L = L_\phi$.

Lemma 5. *Let $L \subseteq \Sigma^\omega$. If L is* DLTL(Σ)*-definable then L is* S1S(Σ)*-definable.*

Proof. Consider the construction from the previous section which associates a Büchi automaton $\mathcal{B}_{\alpha_0}$ with each formula $\alpha_0 \in$ DLTL(Σ). Suppose we apply this construction to formulas arising from the restricted syntax assumed in the present section. Then it is easy to see that, in the absence of atomic propositions, $L_{\alpha_0} = \mathcal{L}(\mathcal{B}_{\alpha_0})$. But then the classic result of Büchi [1] asserts that $L \subseteq \Sigma^\omega$ is S1S(Σ)-definable iff there exists a Büchi automaton $\mathcal{B}$ operating over Σ such that $L = \mathcal{L}(\mathcal{B})$. $\square$

Next we wish to show that if $L \subseteq \Sigma^\omega$ is S1S(Σ)-definable then L is DLTL(Σ)-definable. In fact, it turns out that it suffices to consider a natural fragment of DLTL(Σ) denoted DLTL$^-$(Σ) whose syntax is given by

$$\text{DLTL}^-(\Sigma) ::= \top \mid {\sim}\alpha \mid \alpha \vee \beta \mid \langle \pi \rangle \alpha,$$

where $\pi \in$ Prg(Σ).

Here $\langle \pi \rangle \alpha$ is interpreted as $\top\, \mathcal{U}^\pi \alpha$ with the resulting semantics. Thus DLTL$^-$ is PDL equipped with a linear time semantics. As before $L \subseteq \Sigma^\omega$ is said to be DLTL$^-$(Σ)-*definable* iff there exists $\alpha \in$ DLTL$^-$(Σ) such that $L = L_\alpha$, where L_α is defined as for DLTL(Σ). To get at the result we are after we need to work with Muller automata operating over Σ of the form $\mathcal{A} = (Q, \longrightarrow, Q_{\text{in}}, \mathcal{F})$ where:
- $Q, \longrightarrow$ and Q_{in} are as in the case of a Büchi automaton.
- $\mathcal{F} \subseteq 2^Q$ is a family of accepting sets of states.

Let $\sigma \in \Sigma^\omega$. Then the notion of a run $\rho : \text{prf}(\sigma) \longrightarrow Q$ of $\mathcal{A}$ over σ is as in the case of a Büchi automaton. The definition of inf(ρ) is also as before. The run ρ is said to be accepting iff $\text{inf}(\rho) \in \mathcal{F}$. Naturally $\mathcal{L}(\mathcal{A})$, the ω-language accepted by $\mathcal{A}$, is given by: $\sigma \in \mathcal{L}(\mathcal{A})$ iff there exists an accepting run of $\mathcal{A}$ over σ.

The Muller automaton $\mathcal{A} = (Q, \longrightarrow, Q_{\text{in}}, \mathcal{F})$ is deterministic iff $|Q_{in}| = 1$ and whenever $q \xrightarrow{a} q'$ and $q \xrightarrow{a} q''$, we have $q' = q''$. The well-known theorem of McNaughton [14] guarantees that $L \subseteq \Sigma^\omega$ is S1S(Σ)-definable iff there exists a deterministic Muller automaton operating over Σ such that $L = \mathcal{L}(\mathcal{A})$. This fact will be the basis for the proof of the next result.

Lemma 6. *Let $L \subseteq \Sigma^\omega$. If L is* S1S(Σ)*-definable then L is* DLTL$^-$(Σ)*-definable.*

Proof. As remarked above, L is $\mathrm{S1S}(\Sigma)$-definable implies that there exists a *deterministic* Muller automaton $\mathscr{A} = (Q, \longrightarrow, \{q_{\mathrm{in}}\}, \mathscr{F})$ operating over Σ such that $L = \mathscr{L}(\mathscr{A})$. We will exhibit a formula $\alpha_{\mathscr{A}} \in \mathrm{DLTL}^{-}(\Sigma)$ such that $L_{\alpha_{\mathscr{A}}} = \mathscr{L}(\mathscr{A})$.

An easy argument shows that it involves no loss of generality to assume that $\mathscr{A}$ – apart from determinacy – has two additional properties:

(i) $\emptyset \notin \mathscr{F}$.

(ii) $\forall q \in Q\ \forall a \in \Sigma.\quad \exists q'.\ q \xrightarrow{a} q'$.

Determinacy and (ii) ensure that for *every* $\sigma \in \Sigma^{\omega}$ the Muller automaton $\mathscr{A}$ has a *unique* run over σ. This fact will be crucial in what follows.

If $\mathscr{F} = \emptyset$ we have that $L = \emptyset$, so we set $\alpha_{\mathscr{A}} = \bot$. So suppose that $\mathscr{F} \neq \emptyset$. For each $F \in \mathscr{F}$ we shall construct a formula α_F expressing acceptance by F. The required formula $\alpha_{\mathscr{A}}$ defining L will then be the disjunction of all such α_F.

First we extend $\longrightarrow\ \subseteq Q \times \Sigma \times Q$ to $\longrightarrow_*$, where $\longrightarrow_*$ is the least subset of $Q \times \Sigma^* \times Q$ satisfying

- $q \xrightarrow{\varepsilon}_* q$ for every $q \in Q$.
- If $q \xrightarrow{\tau}_* q'$ and $q' \xrightarrow{a} q''$ then $q \xrightarrow{\tau a}_* q''$.

Next define, for each $q, q' \in Q$, the language of finite words $L_{q,q'} \subseteq \Sigma^*$ by

$$L_{q,q'} = \{\tau \mid q \xrightarrow{\tau}_* q'\}.$$

It is easy to see that each $L_{q,q'}$ is a regular subset of Σ^*. Hence we can fix a regular expression $\pi_{q,q'} \in \mathrm{Prg}(\Sigma)$ such that $L_{q,q'} = ||\pi_{q,q'}||$. Due to the determinacy of $\mathscr{A}$ it follows at once that if $q, q', q'' \in Q$ such that $L_{q,q'} \cap L_{q,q''} \neq \emptyset$ then $q' = q''$.

Now let $F = \{q_0, q_1, \ldots, q_{n-1}\}$ with $n \geqslant 1$. Then the formula α_F is given by

$$\alpha_F = \bigvee_{q \in F} \langle \pi_{q_{in},q} \rangle \left(\bigwedge_{q' \notin F} [\pi_{q,q'}]\bot \ \wedge \ \bigwedge_{j=0}^{n-1} [\pi_{q,q_j}] \langle \pi_{q_j, q_{j \oplus 1}} \rangle \top \right),$$

where $\oplus$ denotes addition modulo n.

The required formula $\alpha_{\mathscr{A}}$ describing $\mathscr{L}(\mathscr{A})$ is then defined as

$$\alpha_{\mathscr{A}} = \bigvee_{F \in \mathscr{F}} \alpha_F.$$

Clearly, $\alpha_{\mathscr{A}} \in \mathrm{DLTL}^{-}(\Sigma)$. It is easy to check that $L_{\alpha_{\mathscr{A}}} = \mathscr{L}(\mathscr{A})$. □

Theorem 7. *Let $L \subseteq \Sigma^{\omega}$. Then the following statements are equivalent:*

(i) *L is $\mathrm{S1S}(\Sigma)$-definable.*

(ii) *L is $\mathrm{DLTL}(\Sigma)$-definable.*

(iii) *L is $\mathrm{DLTL}^{-}(\Sigma)$-definable.*

Proof. Follows immediately from Lemmas 5, 6 and the fact that $\mathrm{DLTL}^{-}(\Sigma)$ is a sublogic of $\mathrm{DLTL}(\Sigma)$. □

At present we do not know of a direct translation of DLTL(Σ)-formulas into DLTL$^-$(Σ)-formulas. Although these two logics have the same expressive power in the sense of Theorem 7, it appears that DLTL(Σ) will admit more natural specifications. In addition, it is a conservative extension of LTL(Σ) even from a syntactic standpoint and hence conventional LTL specifications can be brought in with no overhead translation costs.

We shall conclude this section by pointing out that star-free programs can be used to capture the first-order definable subsets of Σ^ω. Admittedly this is not a big surprise, but it illustrates once more that our method of augmenting the expressive power of LTL is a natural one.

FO(Σ) will denote the first-order theory of ω-sequences generated by Σ. It is the fragment of S1S(Σ) obtained by eliminating set variables from the syntax. We shall say that $L \subseteq \Sigma^\omega$ is FO(Σ)-*definable* iff there exists a sentence ϕ in FO(Σ) such that $L = L_\phi$.

The set of *star-free regular programs* over Σ is denoted $\mathrm{Prg}_{\mathrm{SF}}(\Sigma)$ and its syntax is given by

$$\mathrm{Prg}_{\mathrm{SF}}(\Sigma) ::= 0 \mid a \mid \pi + \pi' \mid \pi;\pi' \mid \overline{\pi}.$$

The set of finite words denoted by each star-free program is obtained via the map $||\cdot|| : \mathrm{Prg}_{\mathrm{SF}}(\Sigma) \longrightarrow 2^{\Sigma^*}$ which is defined as follows: $||\overline{\pi}|| = \Sigma^* - ||\pi||$ and $||0|| = \emptyset$. The remaining cases are handled as before.

The star-free version of DLTL(Σ) will be denoted – for want of a better notation – by $\mathrm{DLTL}_{\mathrm{SF}}(\Sigma)$ and its syntax is given by

$$\mathrm{DLTL}_{\mathrm{SF}}(\Sigma) ::= \top \mid \sim\alpha \mid \alpha \vee \beta \mid \alpha\, \mathcal{U}^\pi \beta \quad (\pi \in \mathrm{Prg}_{\mathrm{SF}}(\Sigma)).$$

Thus, the only difference is that the programs that are used to build up the until-formulas are required to be star-free programs. The fragment of $\mathrm{DLTL}_{\mathrm{SF}}(\Sigma)$ which corresponds to DLTL$^-$(Σ) has the syntax

$$\mathrm{DLTL}^-_{\mathrm{SF}}(\Sigma) ::= \top \mid \sim\alpha \mid \alpha \vee \beta \mid \langle\pi\rangle\alpha \quad (\pi \in \mathrm{Prg}_{\mathrm{SF}}(\Sigma)).$$

Theorem 8. *Let $L \subseteq \Sigma^\omega$. Then the following statements are equivalent:*
(i) *L is* FO(Σ)*-definable.*
(ii) *L is* $\mathrm{DLTL}_{\mathrm{SF}}(\Sigma)$*-definable.*
(iii) *L is* $\mathrm{DLTL}^-_{\mathrm{SF}}(\Sigma)$*-definable.*

Proof. Trivially (iii) implies (ii). The proof that (ii) implies (i) utilizes the well-known fact [15] that FO(Σ)-definable languages over finite strings and the languages described by star-free regular expressions coincide. It is then straightforward to exhibit a syntactic translation of formulas of $\mathrm{DLTL}_{\mathrm{SF}}(\Sigma)$ to FO(Σ) essentially re-expressing the semantics by relativizing the formulas arising from the star-free expressions. The details can be found in [7].

That (i) implies (iii) is a consequence of the fact that the abovementioned characterization of FO(Σ) and star-free regular expressions can be extended to languages of ω-sequences [22]. A linear translation from the star-free ω-regular expressions to $\mathrm{DLTL}^{-}_{\mathrm{SF}}(\Sigma)$ is then obtained by inductively translating the boolean operations to their logical counterparts, while left concatenation with a star-free language of finite strings is handled by the $\langle\pi\rangle$-modality. Once again, the details can be found in [7]. □

6. Axiomatizations

Our axiomatization of the set of valid formulas of DLTL is an extension of Segerberg's axiomatization of PDL [19]. Moreover, our completeness argument is based on the elegant proof of completeness of Segerberg's axioms due to Kozen and Parikh [11]. It will be convenient to first axiomatize DLTL^{-}.

We begin by augmenting the set of regular programs with the atomic program 1. We set $||1|| = \{\varepsilon\}$. By abuse of notation this augmented set of programs will also be denoted as $\mathrm{Prg}(\Sigma)$. Next we define the transition relation $\longrightarrow_{\mathrm{Prg}(\Sigma)}$ (from now on written as just $\longrightarrow$) to be the least subset of $\mathrm{Prg}(\Sigma) \times \Sigma \times \mathrm{Prg}(\Sigma)$ yielded by the following rules:

$$\frac{}{a \xrightarrow{a} 1}$$

$$\frac{\pi \xrightarrow{a} \pi_1}{\pi + \pi' \xrightarrow{a} \pi_1} \qquad \frac{\pi \xrightarrow{a} \pi_1}{\pi' + \pi \xrightarrow{a} \pi_1}$$

$$\frac{\pi \xrightarrow{a} \pi_1}{\pi;\pi' \xrightarrow{a} \pi_1;\pi'} \quad \text{if} \quad \pi_1 \neq 1$$

$$\frac{\pi \xrightarrow{a} 1}{\pi;\pi' \xrightarrow{a} \pi'}$$

$$\frac{\pi' \xrightarrow{a} \pi''}{\pi;\pi' \xrightarrow{a} \pi''} \quad \text{if} \quad \varepsilon \in ||\pi||$$

$$\frac{\pi \xrightarrow{a} \pi'}{\pi^* \xrightarrow{a} \pi';\pi^*}.$$

This transition relation is extended to the relation $\longrightarrow_* \subseteq \mathrm{Prg}(\Sigma) \times \Sigma^* \times \mathrm{Prg}(\Sigma)$ via

$\pi \xrightarrow{\varepsilon}_* \pi$
If $\pi \xrightarrow{\tau}_* \pi'$ and $\pi' \xrightarrow{a} \pi''$ then $\pi \xrightarrow{\tau a}_* \pi''$.

Finally the sets of programs $\delta_a(\pi)$ and $\delta_*(\pi)$ for each π and each a are defined as follows:

$$\delta_a(\pi) = \{\pi' \mid \pi \xrightarrow{a} \pi'\},$$
$$\delta_*(\pi) = \{\pi' \mid \exists\tau.\ \pi \xrightarrow{\tau}_* \pi'\}.$$

Proposition 9. *For each π and each a, both $\delta_a(\pi)$ and $\delta_*(\pi)$ are finite sets.*

Proof. The proof follows easily by structural induction on π. □

We are now ready to present an axiomatization of DLTL$^-$ (Recall that $O\alpha \overset{\Delta}{\Longleftrightarrow} \bigvee_{a\in\Sigma}\langle a\rangle\alpha$). The logical system $\mathscr{DLTL}^-$ is given as follows.

Axiom schemes:

(A0) All the tautologies of propositional calculus.

(A1) $[\pi]\ (\alpha \supset \beta) \supset ([\pi]\alpha \supset [\pi]\beta)$.

(A2) $\langle \pi + \pi' \rangle\alpha \equiv \langle\pi\rangle\alpha \vee \langle\pi'\rangle\alpha$.

(A3) $\langle \pi; \pi' \rangle\alpha \equiv \langle\pi\rangle\langle\pi'\rangle\alpha$.

(A4) $\langle \pi^* \rangle\alpha \equiv \alpha \vee \langle\pi\rangle\langle\pi^*\rangle\alpha$.

(A5) $[\pi^*](\alpha \supset [\pi]\alpha) \supset (\alpha \supset [\pi^*]\alpha)$.

(A6) $\alpha \equiv \langle 1 \rangle\alpha$.

(A7) $O\top$.

(A8) $\langle a\rangle\top \supset \bigwedge_{b\neq a}[b]\bot$.

(A9) $\langle a\rangle\alpha \supset [a]\alpha$.

(A10) $\langle\pi\rangle\alpha \equiv \alpha \vee \left(\bigvee_{a\in\Sigma}\langle a\rangle \bigvee_{\pi'\in\delta_a(\pi)}\langle\pi'\rangle\alpha\right)$, $(\varepsilon \in ||\pi||)$.

(A11) $\langle\pi\rangle\alpha \equiv \bigvee_{a\in\Sigma}\langle a\rangle \bigvee_{\pi'\in\delta_a(\pi)}\langle\pi'\rangle\alpha$, $(\varepsilon \notin ||\pi||)$.

Inference rules:

$$\text{(MP)} \quad \frac{\alpha \quad \alpha \supset \beta}{\beta},$$

$$\text{(TG)} \quad \frac{\alpha}{[\pi]\alpha}.$$

(A0)–(A5) and the inference rules together constitute an axiomatization of PDL. The behaviour of 1 is captured by (A6). The remaining axiom schemes describe the linear time semantics provided for regular programs in the setting of DLTL$^-$. Due to Proposition 9, both (A10) and (A11) are well-defined. It is easy to see that the axioms are valid and that the inference rules preserve validity.

We shall say, as usual, that a formula α is ($\mathscr{DLTL}^-$ –) consistent in case $\sim\alpha$ is not a thesis derivable from the system $\mathscr{DLTL}^-$. We shall prove that every consistent formula is satisfiable. To this end, fix a consistent formula α_0. Define $\widehat{cl}(\alpha_0)$ just as we defined $cl(\alpha_0)$ in Section 4. In addition, the following conditions are required to be satisfied:

- If $\langle\pi\rangle\alpha \in \widehat{cl}(\alpha_0)$ and $\pi' \in \delta_a(\pi)$ then $\langle\pi'\rangle\alpha, \langle a\rangle\langle\pi'\rangle\alpha \in \widehat{cl}(\alpha_0)$.
- If $\langle 1\rangle\alpha \in \widehat{cl}(\alpha_0)$ then $\alpha \in \widehat{cl}(\alpha_0)$.
- $\langle a\rangle\top \in \widehat{cl}(\alpha_0)$ for every $a \in \Sigma$.

Next define $\widehat{CL}(\alpha_0)$ as $\widehat{CL}(\alpha_0) = \widehat{cl}(\alpha_0) \cup \{\sim\beta \mid \beta \in \widehat{cl}(\alpha_0)\}$. As usual, we will identify $\sim\sim\beta$ with β in what follows.

Proposition 10. *$\widehat{CL}(\alpha_0)$ is a finite set.*

Proof. Follows at once from Proposition 9. □

In this section, an atom is a maximal consistent subset of $\widehat{CL}(\alpha_0)$. If A is an atom then $\widehat{A}$ will be the conjunction of all the formulas in A. Let AT_0 be the set of all atoms.

We now define the transition system $TS_0 = (AT_0, \Longrightarrow)$ where $\Longrightarrow \subseteq AT_0 \times \Sigma \times AT_0$ is given by $A \stackrel{a}{\Longrightarrow} B$ iff $\widehat{A} \wedge \langle a \rangle \widehat{B}$ is consistent. As before, the transition relation $\Longrightarrow$ is extended to $\Longrightarrow_* \subseteq AT_0 \times \Sigma^* \times AT_0$ in the obvious way.

Lemma 11. (i) *Suppose $A, B \in AT_0$ and $\pi \in \mathrm{Prg}(\Sigma)$ such that $\widehat{A} \wedge \langle \pi \rangle \widehat{B}$ is consistent. Then there exists $\tau \in ||\pi||$ such that $A \stackrel{\tau}{\Longrightarrow}_* B$.*

(ii) *Suppose $\langle \pi \rangle \alpha \in A \in AT_0$. Then there exists $B \in AT_0$ and $\tau \in ||\pi||$ such that $\alpha \in B$ and $A \stackrel{\tau}{\Longrightarrow}_* B$.*

Proof. Part (i) can be established by just repeating the proof of [11, Lemma 1]. Now part (ii) follows easily from part (i) with the help of a few tautologies of propositional calculus. □

We are now ready to extract a model of α_0 from TS_0. We shall do so by inductively defining a map $\widehat{\rho} : \omega \longrightarrow AT_0$ and an ascending chain of sequences $\tau_0 \prec \tau_1 \prec \cdots$ where each τ_i is in Σ^*. In what follows we will denote $\widehat{\rho}(i)$ by A_i for each $i \in \omega$. We shall also assume that we have fixed an enumeration of the countable set $\widehat{CL}(\alpha_0) \times \Sigma^*$.

- $\widehat{\rho}(0) = A_0$ where $A_0 \in AT_0$ such that $\alpha_0 \in A_0$. Further, $\tau_0 = \varepsilon$.
- Assume $\widehat{\rho}(i)$ and τ_i are defined. We say that the pair $(\langle \pi \rangle \alpha, \tau)$ is a requirement at stage i provided the following conditions are satisfied:
  - $\tau \preccurlyeq \tau_i$ and $\langle \pi \rangle \alpha \in A_j$ where $|\tau| = j$.
  - For every $\tau' \in \Sigma^*$, if $\tau\tau' \preccurlyeq \tau_i$ then $\tau' \notin ||\pi||$ or $\alpha \notin A_k$ where $|\tau\tau'| = k$.

Let RQ_i be the set of requirements at stage i. Suppose that $RQ_i = \emptyset$. Let $a \in \Sigma$ such that $\langle a \rangle \top \in A_i$. The fact that such an a exists and is unique is guaranteed by (A7) and (A8). Since $\bigvee_{A \in AT_0} \widehat{A}$ is a thesis, it follows from simple propositional reasoning that $\widehat{A} \wedge \langle a \rangle \widehat{B}$ is consistent for some $B \in AT_0$. Consequently $A \stackrel{a}{\Longrightarrow} B$. Now let $\widehat{\rho}(i+1) = B$ and $\tau_{i+1} = \tau_i a$. The construction now proceeds from stage $i+1$.

Suppose now that $RQ_i \neq \emptyset$. Let $(\langle \pi \rangle \alpha, \tau)$ be the least member of RQ_i in the enumeration we have fixed for $\widehat{CL}(\alpha_0) \times \Sigma^*$. Let $j = |\tau|$ and $\tau\tau' = \tau_i$. Then using (A10) and (A11) it is easy to show that there exists π' such that $\pi \stackrel{\tau'}{\longrightarrow}_* \pi'$ and $\langle \pi' \rangle \alpha \in A_i$. Moreover $\alpha \notin A_i$ or $\varepsilon \notin ||\pi'||$. By part (ii) of Lemma 11, there exists $B \in AT_0$ and $\tau'' \in ||\pi'||$ such that $A_i \stackrel{\tau''}{\Longrightarrow}_* B$ and $\alpha \in B$. Let $\tau'' = b_1 b_2 \ldots b_m$. Then we can find $B_0, B_1, \cdots, B_m \in AT_0$ such that $A_i = B_0$ and $B_m = B$ and $B_k \stackrel{b_k}{\Longrightarrow} B_{k+1}$ for $0 \leqslant k < m$. We now extend $\widehat{\rho}$ by:

$$\widehat{\rho}(i+k) = B_k \quad \text{for} \quad 1 \leqslant k \leqslant m.$$

Further we define $\tau_{i+k} = \tau_i b_1 b_2 \ldots b_k$ for $1 \leqslant k \leqslant m$. The construction now proceeds from stage $i+m$.

Now consider the model $M_0 = (\sigma, V_0)$ where $\sigma \in \Sigma^\omega$ is the sequence satisfying that $\tau_i \preccurlyeq \sigma$ for every $i \in \omega$. Further, $V_0(\tau) = A_{|\tau|} \cap P$ for each $\tau \in \mathrm{prf}(\sigma)$. It is a routine exercise to establish that for all $\tau \in \mathrm{prf}(\sigma)$ and $\alpha \in \widehat{CL}(\alpha_0)$, $M_0, \tau \models \alpha$ iff $\alpha \in A_{|\tau|}$. Hence $M_0, \varepsilon \models \alpha_0$ as required.

The system $\mathscr{DLTL}$ is obtained by *replacing* (A10) and (A11) with the following axiom schemes:

(A12) $\alpha\,\mathscr{U}^{\pi}\beta \supset \langle\pi\rangle\beta$.

(A13) $\alpha\,\mathscr{U}^{\pi}\beta \equiv \beta \vee \left(\alpha \wedge \bigvee_{a\in\Sigma}\langle a\rangle \bigvee_{\pi'\in\delta_a(\pi)} \alpha\,\mathscr{U}^{\pi'}\beta\right)$, $(\varepsilon \in ||\pi||)$.

(A14) $\alpha\,\mathscr{U}^{\pi}\beta \equiv \alpha \wedge \bigvee_{a\in\Sigma}\langle a\rangle \bigvee_{\pi'\in\delta_a(\pi)} \alpha\,\mathscr{U}^{\pi'}\beta$, $(\varepsilon \notin ||\pi||)$.

It is an easy exercise to extend the completeness argument for $\mathscr{DLTL}^-$ to $\mathscr{DLTL}$. Thus we have:

Theorem 12. (i) *$\mathscr{DLTL}^-$ is a sound and complete axiomatization of the set of valid formulas of* DLTL$^-(\Sigma)$.

(ii) *$\mathscr{DLTL}$ is a sound and complete axiomatization of the set of valid formulas of* DLTL(Σ).

7. Conclusion

We have presented here an enriched version of LTL called DLTL. The extension is obtained by indexing the until operator of LTL with regular programs. We have shown that in terms of the complexity of the decision procedure and expressiveness, DLTL compares very favourably with ETL. It is worth pointing out here that the decision procedure for DLTL is carried out directly in terms of Büchi automata whereas for ETL it is carried out in terms of the so-called set-subword automata, which are then translated to Büchi automata [24]. Two additional results that are available for DLTL are: A characterization of the first-order fragment of S1S in terms of the sublogics DLTL$^-_{\mathrm{SF}}$ and DLTL$_{\mathrm{SF}}$; and a relatively clean axiomatization of DLTL$^-$ and DLTL. All these results demonstrate that our means of bringing together propositional dynamic and temporal logics in a linear time setting is natural.

It turns out that our idea extends smoothly to richer domains. In particular, we can obtain similar results concerning the so-called ω-regular product languages [21] in terms of the product version of DLTL [8]. Roughly speaking, a ω-regular product language is a ω-regular language $L \subseteq \Sigma^{\omega}$ generated by a distributed alphabet $\{\Sigma_i\}_{i=1}^{K}$ with $\Sigma = \bigcup_{i=1}^{K} \Sigma_i$. The language L is a product language in the sense it is a finite union languages of the form $L_1 \otimes L_2 \otimes \cdots \otimes L_K$ with each L_i a regular subset of finite and infinite strings over Σ_i and $\otimes$ standing for the synchronized product operation. In other words $\sigma \in \Sigma^{\omega}$ is in $L_1 \otimes L_2 \otimes \cdots \otimes L_K$ iff $\sigma \upharpoonright \Sigma_i$ (i.e. the sequence obtained by erasing all symbols from σ that are not in Σ_i) is in L_i for each i. The interesting distributed alphabets are of course those in which the component alphabets are *not* pairwise disjoint. The ω-regular product languages can be used to capture the linear time behaviour of a widely used model of distributed programs. These programs consist of a fixed set of finite state sequential programs that coordinate their behaviours by performing common actions together. Our logical characterization of ω-regular product languages is obtained by taking boolean combinations of formulas in $\bigcup_{i=1}^{K}$ DLTL(Σ_i). More details can be found in [8]. It seems likely that one can find a nice generalization of this distributed version of DLTL to capture the full class of ω-regular trace languages.

References

[1] J.R. Büchi, On a decision method in restricted second order arithmetic, Proc. Internat. Congress on Logic, Methodology and Philosophy of Science, Stanford University Press, Stanford, 1960, pp. 1–11.
[2] V. Diekert, G. Rozenberg (Eds.), The Book of Traces, World Scientific, Singapore, 1995.
[3] M.J. Fischer, R.E. Ladner, Propositional dynamic logic of regular programs, J. Comput. System Sci. 18(2) (1979) 194–211.
[4] A. Gabbay, A. Pnueli, S. Shelah, J. Stavi, On the temporal analysis of fairness, Proc. 7th Ann. Symp. on Principles of Programming Languages, ACM, New York, 1980, pp. 163–173.
[5] D. Harel, Dynamic logic, in: D. Gabbay, F. Guenthner (Eds.), Handbook of Philosophical Logic, vol. II, Reidel, Dordrecht, 1984, pp. 497–604.
[6] D. Harel, D. Kozen, R. Parikh, Process logic: expressiveness, decidability, completeness, J. Comput. System Sci. 25 (1982) 144–170.
[7] J.G. Henriksen, P.S. Thiagarajan, Dynamic linear time temporal logic, Report RS-97-8, BRICS, Department of Computer Science, University of Aarhus, 1997.
[8] J.G. Henriksen, P.S. Thiagarajan, A product version of dynamic linear time temporal logic, Proc. 8th Internat. Conf. on Concurrency Theory, Lecture Notes in Computer Science, vol. 1243, Springer, Berlin, 1997, pp. 45–58.
[9] J. Hromkovič, S. Seibert, T. Wilke, Translating regular expressions into small ε-free nondeterministic automata, Proc. 12th Ann. Symp. on Theoretical Aspects of Computer Science, Lecture Notes in Computer Science, vol. 1200, Springer, Berlin, 1997, pp. 55–66.
[10] H.R. Kamp, Tense logic and the theory of linear order, Ph.D. Thesis, University of California, 1968.
[11] D. Kozen, R. Parikh, An elementary proof of the completeness of PDL, Theoret. Comput. Sci. 14 (1981) 113–118.
[12] Z. Manna, A. Pnueli, The temporal logic of reactive and concurrent systems (specification), Springer, Berlin, 1992.
[13] A. Mazurkiewicz, Concurrent program schemes and their interpretations, Tech. Report DAIMI PB-78, Computer Science Department, University of Aarhus, 1977.
[14] R. McNaughton, Testing and generating infinite sequences by a finite automaton, Inform. Control 9 (1966) 521–530.
[15] R. McNaughton, S. Papert, Counter-Free Automata, MIT Press, Cambridge, MA 1971.
[16] H. Nishimura, Descriptively complete process logic, Acta Inform. 14(4) (1980) 359–369.
[17] A. Pnueli, The temporal logic of programs, Proc. 18th Ann. Symp. on Foundations of Computer Science, IEEE, New York, 1977, pp. 46–57.
[18] V.R. Pratt, Process logic, Proc. 6th Symp. on Principles of Programming Languages, ACM, New York, 1979, pp. 93–100.
[19] K. Segerberg, A completeness theorem in the modal logic of programs, Notices AMS 24(6) (1977) A-522.
[20] A.P. Sistla, Theoretical issues in the design and verification of distributed systems, Ph.D. Thesis, Harvard University, 1983.
[21] P.S. Thiagarajan, PTL over product state spaces, Report TCS-95-4, School of Mathematics, SPIC Science Foundation, Madras, 1995.
[22] W. Thomas, Automata over infinite objects, in: J. van Leeuwen, (Ed.), Handbook of Theoretical Computer Science, vol. B: Formal Models and Semantics, Elsevier/MIT Press, Amsterdam/Cambridge, MA 1990, pp. 133–191.
[23] M. Vardi, P. Wolper, An automata-theoretic approach to automatic program verification, Proc. 1st Symposium on Logic in Computer Science, IEEE, New York, 1986, pp. 332–344.
[24] M. Vardi, P. Wolper, Reasoning about infinite computations, Inform. Comput. 115(1) (1994) 1–35.
[25] P. Wolper, Temporal logic can be more expressive, Proc. 22nd Ann. Symp. on Foundations of Computer Science, IEEE, New York, 1981, pp. 340–348.
[26] P. Wolper, M.Y. Vardi, A.P. Sistla, Reasoning about infinite computation paths, Proc. 24th Ann. Symp. on Foundations of Computer Science, IEEE, New York, 1983, pp. 185–194.