

REACHABILITY IN VECTOR ADDITION SYSTEMS IS PRIMITIVE-RECURSIVE IN FIXED DIMENSION

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ABSTRACT. The reachability problem in vector addition systems is a central question, not only for the static verification of these systems, but also for many inter-reducible decision problems occurring in various fields. The currently best known upper bound on this problem is not primitive-recursive, even when considering systems of fixed dimension. We provide significant refinements to the classical decomposition algorithm of Mayr, Kosaraju, and Lambert and to its termination proof, which yield an ACKERMANN upper bound in the general case, and primitive-recursive upper bounds in fixed dimension. While this does not match the currently best known TOWER lower bound for reachability, it is optimal for related problems.

KEYWORDS. Vector addition system, Petri net, reachability, fast-growing complexity

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1. INTRODUCTION

Vector addition systems with states (VASS) are basically finite state systems with vectors of integers as transition weights, as depicted in Fig. 1. Their semantics,

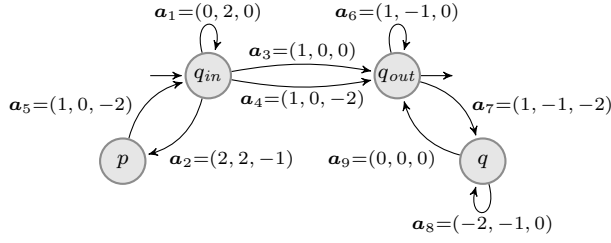


FIGURE 1. A vector addition system with states.

starting from an initial vector of natural numbers, simply adds component-wise the weights of the successive transitions, but the current values should remain non-negative at all times on every coordinate. For instance, in the three-dimensional system of Fig. 1,

$$\begin{aligned} q_{in}(0, 0, 2) &\xrightarrow{a_1} q_{in}(0, 2, 2) \xrightarrow{a_1} q_{in}(0, 4, 2) \xrightarrow{a_3} q_{out}(1, 4, 2) \\ &\xrightarrow{a_6} q_{out}(2, 3, 2) \xrightarrow{a_7} q(3, 2, 0) \xrightarrow{a_8} q(1, 1, 0) \xrightarrow{a_9} q_{out}(1, 1, 0) \end{aligned}$$

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is a path witnessing that $q_{out}(1, 1, 0)$ can be reached from $q_{in}(0, 0, 2)$, but for instance $q(1, 1, 0) \xrightarrow{as} q(-1, 0, 0)$ is not a valid execution step due to the negative value in the first coordinate.

Vector addition systems with states are equivalent to Petri nets, and well-suited whenever one needs to model discrete resources, for instance threads in concurrent computations, molecules in chemical reactions, organisms in biological processes, etc. They are also a crucial ingredient in many algorithms. In particular, the decidability of their *reachability problem* [21, 13, 14, 17] is the cornerstone of many decidability results—see for instance [28, Sec. 5] for a large sample of problems inter-reducible with VASS reachability in logic, formal languages, verification, etc.

In spite of its relevance to a wide range of problems, the complexity of the VASS reachability problem is still not well understood. Indeed, it turns out that this seemingly simple problem is both conceptually and computationally very complex.

On a conceptual level, the 1981 decidability proof by Mayr [21] was the culmination of more than a decade of research in the topic and is considered as one of the great achievements of theoretical computer science. Both Mayr’s *decomposition algorithm* and its proof are however quite intricate. Kosaraju [13] and Lambert [14] contributed several simplifications of Mayr [21]’s original arguments and Leroux and Schmitz [18] recast the decomposition algorithm in a more abstract framework based on well-quasi-order ideals, while Leroux [17] provides a very simple algorithm with a short but non constructive proof, but none of these developments can be called ‘easy’ and the problem seems inherently involved.

On a computational level, on the one hand, the best known lower bound—which was from 1976 until very recently **EXPSpace**-hardness [19]—is now **TOWER**-hardness [6]. This new lower bound puts the problem firmly in the realm of non-elementary complexity. In this realm, complexity is measured using the ‘fast-growing’ complexity classes $(F_\alpha)_\alpha$ from [27], which form a strict hierarchy indexed by ordinals. The already mentioned **TOWER** = F_3 corresponds to problems solvable in time bounded by a tower of exponentials; each F_k for a finite k is primitive recursive, and **ACKERMANN** = F_ω corresponds to problems solvable with Ackermannian resources (see Fig. 2). On the other hand, due to the intricacy of the decomposition algorithm, it eluded analysis for a long time until a ‘cubic Ackermann’ upper bound was obtained in [18] at level F_{ω^3} , with a slightly improved F_{ω^2} upper bound in [29].

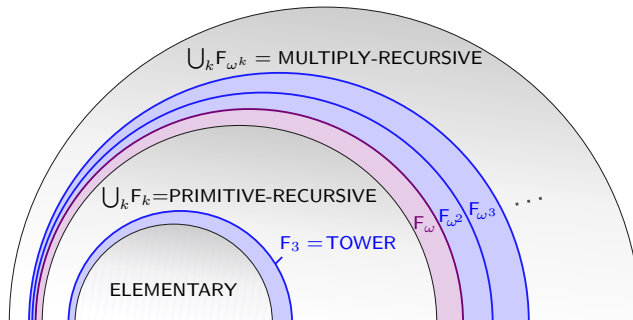


FIGURE 2. Pinpointing F_ω = **ACKERMANN** among the complexity classes beyond **ELEMENTARY** [27].

This leaves a gigantic gap between the known lower and upper bounds. This is however mitigated by the fact that the decomposition algorithm, on which the upper bounds were obtained, provably has a non primitive-recursive complexity. This was already observed by Müller [22], due to the algorithm’s reliance on Karp and Miller trees [12]. Moreover, the full decomposition produced by the algorithm contains more information than just the existence of a reachability witness (which exists if and only if the full decomposition is not empty). For instance, Lambert [14] exploits the full decomposition to derive a pumping lemma for labelled VASS languages, Habermehl et al. [10] further show that one can compute a finite-state automaton recognising the downward-closure of a labelled VASS language with respect to the scattered subword ordering, and Czerwinski et al. [5] show how to exploit the decomposition for deciding language boundedness properties. In particular, the result of Habermehl et al. means that one can decide, given two labelled VASS, whether an inclusion holds between the downward-closures of their languages, which is an ACKERMANN-hard problem [32]. Thus any algorithm that returns such a full decomposition must be non primitive-recursive.

is it
F_K - hard
in dimension
O(K)?
YES, q. below

Contributions. In this paper, we show that VASS reachability is in ACKERMANN = F_ω , and more precisely in F_{d+4} when the dimension d of the system is fixed. This improvement over the bound F_{ω^2} (resp. $F_{\omega \cdot (d+1)}$ in fixed dimension) shown in [29] is obtained by analysing a decomposition algorithm similar to those of Mayr [21], Kosaraju [13], and Lambert [14]. In a nutshell, a decomposition algorithm defines both

- a structure (resp. ‘regular constraint graphs’ for Mayr, ‘generalised VASSes’ for Kosaraju, and ‘marked graph-transition sequences’ for Lambert)—see Sec. 3—and
- a condition on this structure that ensures there is an execution witnessing reachability (resp. ‘consistent marking’, ‘property θ ’, and ‘perfectness’)—see Sec. 4.3.3.

The algorithms compute a decomposition by successive refinements of the structure until the condition is fulfilled, by which time the existence of an execution becomes guaranteed—see Sec. 4.

We work in this paper with a decomposition algorithm quite similar to that of Kosaraju [13], for which the reader will find good expositions for instance in [22, 25, 15]. We benefit however from two key insights (which in turn require significant adaptations throughout the algorithm).

The first key insight is a new termination argument for the decomposition process, based on the dimensions of the vector spaces spanned by the cycles of the structure (see Sec. 3.2). On its own, this new termination argument would already be enough to yield ACKERMANN upper bounds and primitive-recursive ones in fixed dimension.

The second key insight lies within the decomposition process itself, where we show using techniques inspired by Rackoff [24] that we can eschew the computation of Karp and Miller’s coverability trees, and therefore the worst-case Ackermannian blow-up that arises from their use [3]—see Sec. 4.2.1. In itself, this new decomposition algorithm would not bring the complexity below the previous bounds, but combined with the first insight, it yields rather tight upper bounds, at level F_{d+4} in fixed dimension d —see Sec. 5.

In fact, the new upper bounds apply to other decision problems. As we discuss in Sec. 6, Zetzsche's ACKERMANN lower bound [32] can be refined to prove that the inclusion problem between the downward-closures of two labelled VASS languages is F_d -hard in fixed dimension $d > 3$, thus close to matching the F_{d+4} upper bound one obtains by applying the results of Habermehl et al. [10] to our decomposition algorithm.

We start in Sec. 2 by recalling basic definitions and notations on vector addition systems. The full proofs for the decomposition algorithm are presented in Appendices A to C.

2. BACKGROUND

Notations. Let $\mathbb{N}_\omega \stackrel{\text{def}}{=} \mathbb{N} \uplus \{\omega\}$ extend the set of natural numbers with an infinite element ω with $n < \omega$ for all $n \in \mathbb{N}$. We also use the partial order \sqsubseteq over \mathbb{N}_ω defined by $x \sqsubseteq y$ if $y \in \{x, \omega\}$.

Let $d \in \mathbb{N}$ be a dimension. The relations \leq and \sqsubseteq are extended component-wise to vectors in \mathbb{N}_ω^d . The components of a vector that are equal to ω intuitively denote arbitrarily large values; we call a vector in \mathbb{N}_ω^d *finite*. Given a vector $\mathbf{x} \in \mathbb{N}_\omega^d$ and a subset $I \subseteq \{1, \dots, d\}$ of the components, we denote by $\mathbf{x}|_I$ the vector obtained from \mathbf{x} by replacing components *not* in I by ω . Note that $\mathbf{x} \sqsubseteq \mathbf{y}$ implies $\mathbf{x} \leq \mathbf{y}$ and that $\mathbf{x} \sqsubseteq \mathbf{x}|_I$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{N}_\omega^d$ and $I \subseteq \{1, \dots, d\}$. For instance, for $d = 3$, $(3, 2, 1) \leq (4, \omega, 1)$ but $(3, 2, 1) \not\sqsubseteq (4, \omega, 1)$; if $I = \{2, 3\}$, then $(3, 2, 1)|_I = (\omega, 2, 1)$ and $(4, \omega, 1)|_I = (\omega, \omega, 1)$, and then $(\omega, 2, 1) \sqsubseteq (\omega, \omega, 1)$. We let $\mathbf{0}$ denote the zero vector and $\boldsymbol{\omega}$ the vector with $\omega(i) \stackrel{\text{def}}{=} \omega$ for all $1 \leq i \leq d$. Observe that $\mathbf{x} \sqsubseteq \boldsymbol{\omega}$ for all $\mathbf{x} \in \mathbb{N}_\omega^d$.

For a vector $\mathbf{x} \in \mathbb{N}_\omega^d$, its norm $\|\mathbf{x}\|$ is defined over its finite components as $\sum_{1 \leq i \leq d | \mathbf{x}(i) < \omega} \mathbf{x}(i)$ (a sum over an empty set is zero); for a vector $\mathbf{x} \in \mathbb{Z}^d$, we let as usual $\|\mathbf{x}\| \stackrel{\text{def}}{=} \sum_{1 \leq i \leq d} |\mathbf{x}(i)|$. For instance, $\|(3, \omega, 1)\| = 4$ and $\|(-4, 2, 1)\| = 7$.

Vector Addition Systems. While we focus in this paper on reachability in vector addition systems with a finite set of control states, we also rely on notations for the simpler case of vector addition systems.

A *vector addition system* (VAS) [12] of dimension $d \in \mathbb{N}$ is a finite set $\mathbf{A} \subseteq \mathbb{Z}^d$ of vectors called *actions*. The semantics of a VAS is defined over *configurations* in \mathbb{N}_ω^d . We associate to an action $\mathbf{a} \in \mathbf{A}$ the binary relation $\xrightarrow{\mathbf{a}}$ over configurations by $\mathbf{x} \xrightarrow{\mathbf{a}} \mathbf{y}$ if $\mathbf{y} = \mathbf{x} + \mathbf{a}$, where addition is performed component-wise with the convention that $\omega + z = \omega$ for every $z \in \mathbb{Z}$. Given a finite word $\sigma = \mathbf{a}_1 \dots \mathbf{a}_k \in \mathbf{A}^*$ of actions we also define the binary relation $\xrightarrow{\sigma}$ over configurations by $\mathbf{x} \xrightarrow{\sigma} \mathbf{y}$ if there exists a sequence $\mathbf{c}_0, \dots, \mathbf{c}_k$ of configurations such that

$$\mathbf{x} = \mathbf{c}_0 \xrightarrow{\mathbf{a}_1} \mathbf{c}_1 \dots \xrightarrow{\mathbf{a}_k} \mathbf{c}_k = \mathbf{y}.$$

The *VAS reachability problem* consists in deciding given two finite configurations $\mathbf{c}_{in}, \mathbf{c}_{out}$ in \mathbb{N}^d and a VAS \mathbf{A} whether there exists a word $\sigma \in \mathbf{A}^*$ such that $\mathbf{c}_{in} \xrightarrow{\sigma} \mathbf{c}_{out}$.

Vector Addition Systems with States. A *vector addition system with states* (VASS) [11] of dimension $d \in \mathbb{N}$ is a triple $G = (Q, q_{in}, q_{out}, T)$ where Q is a non-empty finite set of states, $q_{in} \in Q$ is the input state, $q_{out} \in Q$ is the output state, and T is a finite set of transitions in $Q \times \mathbb{Z}^d \times Q$; $\mathbf{A} \stackrel{\text{def}}{=} \{\mathbf{a} \mid \exists p, q \in Q. (p, \mathbf{a}, q) \in T\}$ is the associated set of actions.

Example 2.1. Figure 1 depicts the VASS $G_{\text{ex}} = (Q_{\text{ex}}, q_{\text{in}}, q_{\text{out}}, T_{\text{ex}})$ of dimension 3 where $Q_{\text{ex}} = \{q_{\text{in}}, q_{\text{out}}, p, q\}$ and $T_{\text{ex}} = \{t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9\}$ with

$$\begin{aligned} t_1 &= (q_{\text{in}}, (0, 2, 0), q_{\text{in}}), & t_2 &= (q_{\text{in}}, (2, 2, -1), p), \\ t_3 &= (q_{\text{in}}, (1, 0, 0), q_{\text{out}}), & t_4 &= (q_{\text{in}}, (1, 0, -2), q_{\text{out}}), \\ t_5 &= (p, (1, 0, -2), q_{\text{in}}), & t_6 &= (q_{\text{out}}, (1, -1, 0), q_{\text{out}}), \\ t_7 &= (q_{\text{out}}, (1, -1, -2), q), & t_8 &= (q, (-2, -1, 0), q), \\ t_9 &= (q, (0, 0, 0), q_{\text{out}}). \end{aligned} \quad \square$$

We focus on VASSes in this paper rather than VASes, because we exploit the properties of their underlying directed graphs. A *path* π in a VASS G from a state p to a state q labelled by a word $\mathbf{a}_1 \dots \mathbf{a}_k$ of actions is a word of transitions of G of the form $(p_1, \mathbf{a}_1, q_1) \dots (p_k, \mathbf{a}_k, q_k)$ with $p_0 = p$, $q_k = q$, and $q_j = p_{j+1}$ for all $1 \leq j < k$. Such a path is *complete* if $p = q_{\text{in}}$ and $q = q_{\text{out}}$ are the input and output states of G . A *cycle* on a state q is a path from q to q .

Example 2.2. For instance, in Ex. 2.1, the execution presented in the introduction corresponds to the path $\pi_{\text{ex}} = t_1 t_1 t_3 t_6 t_7 t_8 t_9$ labelled by $\sigma_{\text{ex}} = \mathbf{a}_1 \mathbf{a}_1 \mathbf{a}_3 \mathbf{a}_6 \mathbf{a}_7 \mathbf{a}_8 \mathbf{a}_9$, and is complete. \square

We write $p \equiv_G q$ if there exists a path from p to q and a path from q to p ; this defines an equivalence relation whose equivalence classes are called the *strongly connected components* of G . In Ex. 2.1, the strongly connected components are $\{q_{\text{in}}, p\}$ and $\{q, q_{\text{out}}\}$. A VASS $G = (Q, q_{\text{in}}, q_{\text{out}}, T)$ is said to be *strongly connected* if Q is a strongly connected component of G .

The *Parikh image* of a path π is the function $\phi: T \rightarrow \mathbb{N}$ that maps each transition $t \in T$ to its number of occurrences in π . The *displacement* of a path π labelled by a word $\mathbf{a}_1 \dots \mathbf{a}_k$ of actions is the vector $\Delta(\pi) \stackrel{\text{def}}{=} \sum_{j=1}^k \mathbf{a}_j$; note that this is equal to $\Delta(\phi) \stackrel{\text{def}}{=} \sum_{t=(p, \mathbf{a}, q) \in T} \phi(t) \cdot \mathbf{a}$ if ϕ is the Parikh image of π .

Example 2.3. For the example path π_{ex} from Ex. 2.2, $\phi_{\text{ex}} = (2, 0, 1, 0, 0, 1, 1, 1, 1)$ and $\Delta(\pi_{\text{ex}}) = (1, 1, -2)$. \square

A *state-configuration* of a VASS $G = (Q, q_{\text{in}}, q_{\text{out}}, T)$ is a pair $(q, \mathbf{x}) \in Q \times \mathbb{N}_{\omega}^d$ denoted by $q(\mathbf{x})$ in the sequel. Given an action \mathbf{a} we define the step relation $\xrightarrow[\mathbf{G}]{\mathbf{a}}$ over state-configurations by $p(\mathbf{x}) \xrightarrow[\mathbf{G}]{\mathbf{a}} q(\mathbf{y})$ if $(p, \mathbf{a}, q) \in T$ and $\mathbf{x} \xrightarrow{\mathbf{a}} \mathbf{y}$. By extension, given a word σ of actions $\sigma = \mathbf{a}_1 \dots \mathbf{a}_k$, $p(\mathbf{x}) \xrightarrow[\mathbf{G}]{\sigma} q(\mathbf{y})$ if there exists a sequence $q_0(\mathbf{c}_0), \dots, q_k(\mathbf{c}_k)$ of state-configurations such that

$$p(\mathbf{x}) = q_0(\mathbf{c}_0) \xrightarrow[\mathbf{G}]{\mathbf{a}_1} q_1(\mathbf{c}_1) \dots \xrightarrow[\mathbf{G}]{\mathbf{a}_k} q_k(\mathbf{c}_k) = q(\mathbf{y}).$$

Notice that $p(\mathbf{x}) \xrightarrow[\mathbf{G}]{\sigma} q(\mathbf{y})$ if, and only if, there exists a path in G from p to q labelled by σ such that $\mathbf{x} \xrightarrow{\sigma} \mathbf{y}$. In Ex. 2.1, $q_{\text{in}}((0, 0, 2)) \xrightarrow[\mathbf{G}_{\text{ex}}]{\sigma_{\text{ex}}} q_{\text{out}}((1, 1, 0))$. Finally, we write $p(\mathbf{x}) \xrightarrow[\mathbf{G}]{*} q(\mathbf{y})$ if there exists $\sigma \in \mathbf{A}^*$ such that $p(\mathbf{x}) \xrightarrow[\mathbf{G}]{\sigma} q(\mathbf{y})$.

Reachability. We focus in this paper on the following decision problem.

Problem: VASS reachability.

input: a VASS $G = (Q, q_{\text{in}}, q_{\text{out}}, T)$ of dimension d and two finite configurations $\mathbf{c}_{\text{in}}, \mathbf{c}_{\text{out}} \in \mathbb{N}^d$

question: does $q_{\text{in}}(\mathbf{c}_{\text{in}}) \xrightarrow[\mathbf{G}]{*} q_{\text{out}}(\mathbf{c}_{\text{out}})$ hold?

The previously mentioned VAS reachability problem reduces to VASS reachability: given a VAS \mathbf{A} and two finite configurations $\mathbf{c}_{in}, \mathbf{c}_{out}$, it suffices to consider the VASS reachability problem with input $(\{q\}, q, q, \{q\} \times \mathbf{A} \times \{q\})$ and the same configurations $\mathbf{c}_{in}, \mathbf{c}_{out}$. A converse reduction is possible by encoding the states, at the expense of increasing the dimension by three [11].

3. DECOMPOSITION STRUCTURES

The version of the decomposition algorithm we present in Sec. 4 proceeds globally as the ones of Mayr, Kosaraju, and Lambert, and we call the underlying structures *KLM sequences* after them.

3.1. KLM Sequences. A *KLM sequence* ξ of dimension d is a sequence

$$\xi = (\mathbf{x}_0 G_0 \mathbf{y}_0) \mathbf{a}_1 (\mathbf{x}_1 G_1 \mathbf{y}_1) \dots \mathbf{a}_k (\mathbf{x}_k G_k \mathbf{y}_k) \quad (1)$$

where $\mathbf{x}_0, \mathbf{y}_0, \dots, \mathbf{x}_k, \mathbf{y}_k$ are configurations, G_0, \dots, G_k are VASSes of dimension d , and $\mathbf{a}_1, \dots, \mathbf{a}_k$ are actions. KLM sequences are essentially the same as Kosaraju's ‘generalised VASSes’ [13], except that we do not require G_0, \dots, G_k to be strongly connected.

The *action language* of a KLM sequence ξ is the set L_ξ of words of actions of the form $\sigma_0 \mathbf{a}_1 \sigma_1 \dots \mathbf{a}_k \sigma_k$ such that σ_j is the label of a complete path of G_j for every j , and such that there exists a sequence $\mathbf{m}_0, \mathbf{n}_0, \dots, \mathbf{m}_k, \mathbf{n}_k$ of configurations in \mathbb{N}^d such that

$$\mathbf{m}_0 \xrightarrow{\sigma_0} \mathbf{n}_0 \xrightarrow{\mathbf{a}_1} \dots \mathbf{m}_k \xrightarrow{\sigma_k} \mathbf{n}_k \quad (2)$$

where $\mathbf{m}_j \sqsubseteq \mathbf{x}_j$ and $\mathbf{n}_j \sqsubseteq \mathbf{y}_j$ for every $0 \leq j \leq k$.

Note that the reachability problem for a VASS G and two finite configurations $\mathbf{c}_{in}, \mathbf{c}_{out} \in \mathbb{N}^d$ reduces to the non-emptiness of the action language of the KLM sequence $(\mathbf{c}_{in} G \mathbf{c}_{out})$. In fact, in that case, the action language is the set of words $\sigma \in \mathbf{A}^*$ such that $q_{in}(\mathbf{c}_{in}) \xrightarrow{\sigma}_G q_{out}(\mathbf{c}_{out})$.

Example 3.1. In Ex. 2.1, $\xi_{ex} = ((0, 0, 2) G_{ex}(1, 1, 0))$ is a KLM sequence with action language

$$\begin{aligned} L_{\xi_{ex}} = & \{ \mathbf{a}_1^{2+3n} \mathbf{a}_3 \mathbf{a}_6^{1+4n} \mathbf{a}_7 \mathbf{a}_8^{1+2n} \mathbf{a}_9 \mid n \in \mathbb{N} \} \\ & \cup \{ \mathbf{a}_1^{2+3n} \mathbf{a}_3 \mathbf{a}_6^{4n} \mathbf{a}_7 \mathbf{a}_8^{1+2n} \mathbf{a}_9 \mathbf{a}_6 \mid n \in \mathbb{N} \} . \end{aligned} \quad \square$$

3.2. Ranks and Sizes.

Vector Spaces. We associate to a transition t of a VASS G the vector space $\mathbf{V}_G(t) \subseteq \mathbb{Q}^d$ spanned by the displacements of the cycles that contain t . The following lemma shows that this vector space only depends on the strongly connected components of G .

Lemma 3.2. *Let t be a transition of a strongly connected VASS G . Then the vector space $\mathbf{V}_G(t)$ is equal to the vector space spanned by the displacements of the cycles of G .*

Proof. Let \mathbf{V} be the vector space spanned by the displacements of the cycles of G . Naturally, we have $\mathbf{V}_G(t) \subseteq \mathbf{V}$. For the converse, let us consider a sequence $\theta_1, \dots, \theta_k$ of cycles such that θ_j is a cycle on a state q_j for every $1 \leq j \leq k$, and such that $\Delta(\theta_1), \dots, \Delta(\theta_k)$ span the vector space \mathbf{V} . Since G is strongly connected, there exists a path π_j from q_{j-1} to q_j for every $j \in \{1, \dots, k\}$ with $q_0 \stackrel{\text{def}}{=} q_k$. Moreover,

we can assume without loss of generality that t occurs in the cycle $\theta \stackrel{\text{def}}{=} \pi_1 \dots \pi_k$. Let θ'_j be the cycle obtained from θ by inserting θ_j in q_j and formally defined as $\theta'_j \stackrel{\text{def}}{=} \pi_1 \dots \pi_j \theta_j \pi_{j+1} \dots \pi_k$. Observe that $\Delta(\theta)$ and $\Delta(\theta'_j)$ are both in $\mathbf{V}_G(t)$ since t occurs in the cycles θ and θ'_j . As $\Delta(\theta_j) = \Delta(\theta'_j) - \Delta(\theta)$, it follows that $\Delta(\theta_j) \in \mathbf{V}_G(t)$. We derive that the vector space spanned by $\Delta(\theta_1), \dots, \Delta(\theta_k)$ is included in $\mathbf{V}_G(t)$. Hence $\mathbf{V} \subseteq \mathbf{V}_G(t)$. \square

As a corollary, if two transitions t and t' are induced by the same strongly connected component of a VASS G , then $\mathbf{V}_G(t) = \mathbf{V}_G(t')$.

Ranks. The *rank* of a VASS G is the tuple $\text{rank}(G) \stackrel{\text{def}}{=} (r_d, \dots, r_0) \in \mathbb{N}^{d+1}$ where r_i is the number of transitions $t \in T$ such that the dimension of $\mathbf{V}_G(t)$ is equal to i . The *rank* of a KLM sequence ξ defined as $(\mathbf{x}_0 G_0 \mathbf{y}_0) \mathbf{a}_1 (\mathbf{x}_1 G_1 \mathbf{y}_1) \dots \mathbf{a}_k (\mathbf{x}_k G_k \mathbf{y}_k)$ is the vector $\text{rank}(\xi) \stackrel{\text{def}}{=} \sum_{j=0}^k \text{rank}(G_j)$ where ranks are added component-wise.

Ranks are ordered lexicographically by the relation \leq_{lex} defined by $(r_d, \dots, r_0) \leq_{\text{lex}} (s_d, \dots, s_0)$ if they are equal or if the minimal i such that $r_i \neq s_i$ satisfies $r_i < s_i$. Note that the linear order $(\mathbb{N}^{d+1}, <_{\text{lex}})$ is well-founded, with order type ω^{d+1} . In Kosaraju's decomposition algorithm, the rank of a KLM sequence was defined as a multiset of triples $(n_{j,1}, n_{j,2}, n_{j,3})$ for all $0 \leq j \leq k$, with $n_{j,1} \leq d$, $n_{j,2} \stackrel{\text{def}}{=} |T_j|$, and $n_{j,3} \leq 2d$, where the triples are ordered lexicographically and the multisets using Dershowitz and Manna multiset ordering [7]. This ranking function ranged over an order type in ω^{ω^3} , and actually below $\omega^{\omega \cdot (d+1)}$ [29].

Example 3.3. In Ex. 2.1,

$$\mathbf{V}_G(t_3) = \mathbf{V}_G(t_4) = \{(0, 0, 0)\},$$

$$\mathbf{V}_G(t_1) = \mathbf{V}_G(t_2) = \mathbf{V}_G(t_5) = \text{span}((0, 2, 0), (3, 2, -3)),$$

$$\mathbf{V}_G(t_6) = \mathbf{V}_G(t_7) = \mathbf{V}_G(t_8) = \mathbf{V}_G(t_9) = \text{span}((-2, -1, 0), (1, -1, -2), (1, -1, 0)).$$

Thus $\text{rank}(G_{\text{ex}}) = (4, 3, 0, 2) = \text{rank}(\xi_{\text{ex}})$. \square

Sizes. The *size* of a VASS $G = (Q, q_{\text{in}}, q_{\text{out}}, T)$ is

$$|G| \stackrel{\text{def}}{=} |Q| + |T| + \sum_{t \in T} \|\Delta(t)\|. \quad (3)$$

The *size* of a KLM sequence ξ of the form $(\mathbf{x}_0 G_0 \mathbf{y}_0) \mathbf{a}_1 \dots \mathbf{a}_k (\mathbf{x}_k G_k \mathbf{y}_k)$ is the natural number

$$|\xi| \stackrel{\text{def}}{=} 2(d+1)^{d+1} \left(k + \sum_{j=1}^k \|\mathbf{a}_j\| + \sum_{j=0}^k (\|\mathbf{x}_j\| + |G_j| + \|\mathbf{y}_j\|) \right). \quad (4)$$

3.3. Characteristic System. The action language of a KLM sequence can be over-approximated thanks to a system of linear equations called its *characteristic system*, which we are about to define. Let us first associate to a VASS $G = (Q, q_{\text{in}}, q_{\text{out}}, T)$ a *Kirchhoff system* K_G of linear equations such that $\phi \in \mathbb{N}^T$ is a model of K_G if, and only if, the following constraint holds

$$\mathbf{1}_{q_{\text{out}}} - \mathbf{1}_{q_{\text{in}}} = \sum_{t=(p, \mathbf{a}, q) \in T} \phi(t) (\mathbf{1}_q - \mathbf{1}_p), \quad (5)$$

where $\mathbf{1}_q: Q \rightarrow \{0, 1\}$ is the *characteristic function* of $q \in Q$ defined by $\mathbf{1}_q(p) \stackrel{\text{def}}{=} 1$ if $p = q$ and $\mathbf{1}_q(p) \stackrel{\text{def}}{=} 0$ otherwise. Let us observe that the Parikh image of a path from q_{in} to q_{out} in G is a model of K_G .

A *characteristic sequence* of a KLM sequence of the form $\xi = (\mathbf{x}_0 G_0, \mathbf{y}_0) \mathbf{a}_1 \dots \mathbf{a}_k (\mathbf{x}_k, G_k, \mathbf{y}_k)$ where T_j is the set of transitions of G_j for each j is a sequence $\mathbf{h} = (\mathbf{m}_j, \phi_j, \mathbf{n}_j)_{0 \leq j \leq k}$ of triples $(\mathbf{m}_j, \phi_j, \mathbf{n}_j) \in \mathbb{N}^d \times \mathbb{N}^{T_j} \times \mathbb{N}^d$. We denote by $\|\mathbf{h}\|$ the value $\sum_{j=0}^k (\|\mathbf{m}_j\| + \sum_{t \in T_j} \phi_j(t) + \|\mathbf{n}_j\|)$. We also denote by $(\mathbf{m}_j^h, \phi_j^h, \mathbf{n}_j^h)$ the j th triple $(\mathbf{m}_j, \phi_j, \mathbf{n}_j)$ occurring in \mathbf{h} .

The *characteristic system* of ξ is the system E_ξ of linear equations such that a characteristic sequence $\mathbf{h} = (\mathbf{m}_j, \phi_j, \mathbf{n}_j)_{0 \leq j \leq k}$ is a model of E_ξ if, and only if, the following two conditions hold:

- (1) $\mathbf{m}_j^h \sqsubseteq \mathbf{x}_j, \phi_j^h \models K_{G_j}, \mathbf{n}_j^h = \mathbf{m}_j^h + \Delta(\phi_j^h)$, and $\mathbf{n}_j^h \sqsubseteq \mathbf{y}_j$ for every $0 \leq j \leq k$, and
- (2) $\mathbf{n}_{j-1}^h \xrightarrow{a_j} \mathbf{m}_j^h$ for every $1 \leq j \leq k$.

A KLM sequence ξ is said to be *satisfiable* if its characteristic system E_ξ is satisfiable. It is said to be *unsatisfiable* otherwise.

Example 3.4. Let us consider $\xi_{\text{ex}} = ((0, 0, 2)G_{\text{ex}}(1, 1, 0))$ from Ex. 2.1. Its characteristic system is

$$\begin{aligned}
& \mathbf{m} = (0, 0, 2) \wedge \mathbf{n} = (1, 1, 0) \\
\wedge \quad & \mathbf{n}(1) = \mathbf{m}(1) + 2\phi(t_2) + \phi(t_3) + \phi(t_4) + \phi(t_5) + \phi(t_6) + \phi(t_7) - 2\phi(t_8) \\
\wedge \quad & \mathbf{n}(2) = \mathbf{m}(2) + 2\phi(t_1) + 2\phi(t_2) - \phi(t_6) - \phi(t_7) - \phi(t_8) \\
\wedge \quad & \mathbf{n}(3) = \mathbf{m}(3) - \phi(t_2) - 2\phi(t_4) - 2\phi(t_5) - 2\phi(t_7) \\
\wedge \quad & -1 = -\phi(t_2) - \phi(t_3) - \phi(t_4) + \phi(t_5) \\
\wedge \quad & 0 = \phi(t_2) - \phi(t_3) \\
\wedge \quad & 0 = \phi(t_7) - \phi(t_9) \\
\wedge \quad & 1 = \phi(t_3) + \phi(t_4) - \phi(t_7) + \phi(t_9),
\end{aligned}$$

where the last four equations correspond to $K_{G_{\text{ex}}}$. One can check that the tuple $((0, 0, 2), \phi_{\text{ex}}, (1, 1, 0))$ is a model, where ϕ_{ex} was defined in Ex. 2.3. \square

Lemma 3.5. *The action language of an unsatisfiable KLM sequence is empty.*

Proof. Assume that L_ξ contains a word σ , and let us prove that E_ξ is satisfiable. As $\sigma \in L_\xi$, there exists a decomposition of σ into $\sigma_0 \mathbf{a}_1 \sigma_1 \dots \mathbf{a}_k \sigma_k$ such that σ_j is the label of a complete path π_j of G_j , and there exists a sequence $\mathbf{m}_0, \mathbf{n}_0, \dots, \mathbf{m}_k, \mathbf{n}_k$ of vectors in \mathbb{N}^d with $\mathbf{m}_j \sqsubseteq \mathbf{x}_j$ and $\mathbf{n}_j \sqsubseteq \mathbf{y}_j$ for every $0 \leq j \leq k$, and such that

$$\mathbf{m}_0 \xrightarrow{\sigma_0} \mathbf{n}_0 \xrightarrow{\mathbf{a}_1} \dots \xrightarrow{\mathbf{a}_k} \mathbf{m}_k \xrightarrow{\sigma_k} \mathbf{n}_k.$$

Let ϕ_j be the Parikh image of π_j ; then the characteristic sequence $(\mathbf{m}_j, \phi_j, \mathbf{n}_j)_{0 \leq j \leq k}$ is a model of E_ξ . \square

3.4. Homogeneous Characteristic System. In the sequel, variables whose values are bounded by the characteristic system will provide a way of decomposing KLM sequences. Since E_ξ is a system of linear equations, bounded variables are characterised thanks to the homogeneous form E_ξ^0 of E_ξ , called the *homogeneous characteristic system* of ξ that we are about to define.

First, we define the *homogeneous form* K_G^0 of the Kirchhoff system K_G as the system of linear equation such that $\phi \in \mathbb{N}^T$ is a model of K_G^0 if, and only if, the

following constraint holds

$$\sum_{t=(p,\mathbf{a},q) \in T} \phi(t)(\mathbf{1}_q - \mathbf{1}_p) = 0. \quad (6)$$

The *homogeneous characteristic system* E_ξ^0 is such that a sequence $(\mathbf{m}_0, \phi_0, \mathbf{n}_0), \dots, (\mathbf{m}_k, \phi_k, \mathbf{n}_k)$ of triples $(\mathbf{m}_j, \phi_j, \mathbf{n}_j) \in \mathbb{N}^d \times \mathbb{N}^{T_j} \times \mathbb{N}^d$ is a model of E_ξ^0 if, and only if, the following two conditions hold:

- (1) $\bigwedge_{i|\mathbf{x}_j(i) \neq \omega} \mathbf{m}_j(i) = 0$, $\phi_j \models K_{G_j}^0$, $\mathbf{n}_j = \mathbf{m}_j + \Delta(\phi_j)$, and $\bigwedge_{i|\mathbf{y}_j(i) \neq \omega} \mathbf{n}_j(i) = 0$ for every $0 \leq j \leq k$, and
- (2) $\mathbf{n}_{j-1} = \mathbf{m}_j$ for every $1 \leq j \leq k$.

Example 3.6. Let us consider $\xi_{\text{ex}} = ((0, 0, 2)G_{\text{ex}}(1, 1, 0))$ from Ex. 2.1. Its homogeneous characteristic system is

$$\begin{aligned} & \mathbf{m} = (0, 0, 0) \wedge \mathbf{n} = (0, 0, 0) \\ \wedge \quad & \mathbf{n}(1) = \mathbf{m}(1) + 2\phi(t_2) + \phi(t_3) + \phi(t_4) + \phi(t_5) + \phi(t_6) + \phi(t_7) - 2\phi(t_8) \\ \wedge \quad & \mathbf{n}(2) = \mathbf{m}(2) + 2\phi(t_1) + 2\phi(t_2) - \phi(t_6) - \phi(t_7) - \phi(t_8) \\ \wedge \quad & \mathbf{n}(3) = \mathbf{m}(3) - \phi(t_2) - 2\phi(t_4) - 2\phi(t_5) - 2\phi(t_7) \\ \wedge \quad & 0 = -\phi(t_2) - \phi(t_3) - \phi(t_4) + \phi(t_5) \\ \wedge \quad & 0 = \phi(t_2) - \phi(t_3) \\ \wedge \quad & 0 = \phi(t_7) - \phi(t_9) \\ \wedge \quad & 0 = \phi(t_3) + \phi(t_4) - \phi(t_7) + \phi(t_9), \end{aligned}$$

where the last four equations correspond to $K_{G_{\text{ex}}}^0$. \square

By using classical linear algebra results [e.g., 23, Thm. 1], in Appendix B we prove the following characterisation of the bounded variables of E_ξ .

Lemma 3.7. *Assume that $\xi = (\mathbf{x}_0 G_0 \mathbf{y}_0) \mathbf{a}_1 \dots (\mathbf{x}_k G_k \mathbf{y}_k)$ is satisfiable. Then for every $0 \leq j \leq k$ we have:*

- For every $1 \leq i \leq d$, the set of values $\mathbf{m}_j^{\mathbf{h}}(i)$ where \mathbf{h} is a model of E_ξ is unbounded if, and only if, there exists a model \mathbf{h}_0 of E_ξ^0 such that $\mathbf{m}_j^{\mathbf{h}_0}(i) > 0$.
- For every $t \in T_j$, the set of values $\phi_j^{\mathbf{h}}(t)$ where \mathbf{h} is a model of E_ξ is unbounded if, and only if, there exists a model \mathbf{h}_0 of E_ξ^0 such that $\phi_j^{\mathbf{h}_0}(t) > 0$.
- For every $1 \leq i \leq d$, the set of values $\mathbf{n}_j^{\mathbf{h}}(i)$ where \mathbf{h} is a model of E_ξ is unbounded if, and only if, there exists a model \mathbf{h}_0 of E_ξ^0 such that $\mathbf{n}_j^{\mathbf{h}_0}(i) > 0$.

Moreover, the sum of the bounded values of E_ξ is bounded by $|\xi|^{|\xi|-1}$.

4. THE DECOMPOSITION ALGORITHM

Let us give an overview of the decomposition algorithm. Given an instance $(G, \mathbf{c}_{\text{in}}, \mathbf{c}_{\text{out}})$ of the VASS reachability problem, the algorithm takes as input the KLM sequence $\xi_0 \stackrel{\text{def}}{=} (\mathbf{c}_{\text{in}} G \mathbf{c}_{\text{out}})$. In an initialisation phase, the algorithm computes a finite set $\text{clean}(\xi_0)$ of so-called *clean* KLM sequences (see Lem. 4.17) such that $L_{\xi_0} = \bigcup_{\xi'_0 \in \text{clean}(\xi_0)} L_{\xi'_0}$. At each step of the algorithm, given a clean KLM sequence ξ ,

- either ξ is *normal*, which is a condition that ensures that the action language L_ξ is non-empty (see Lem. 4.19),
- or we can perform a *decomposition step* as per Lem. 4.18, which produces a *finite* (possibly empty) set $\text{dec}(\xi)$ of clean KLM sequences such that $\text{rank}(\xi') <_{\text{lex}} \text{rank}(\xi)$ for all $\xi' \in \text{dec}(\xi)$ and $L_\xi = \bigcup_{\xi' \in \text{dec}(\xi)} L_{\xi'}$.

Both the initialisation and the decomposition steps are the results of elementary steps presented in Sec. 4.1 and aiming to enforce various properties on KLM sequences.

By repeatedly applying decomposition steps, the decomposition algorithm explores a *decomposition forest* labelled with clean KLM sequences, where the roots are labelled by the elements $\xi'_0 \in \text{clean}(\xi_0)$, and where each node labelled by a non-normal KLM sequence ξ has a child labelled ξ' for each $\xi' \in \text{dec}(\xi)$. A decomposition forest has finitely many roots, finite branching degree, and, because the ranks decrease strictly along the branches and $(\mathbb{N}^d, <_{\text{lex}})$ is well-founded, it has finite branches. A decomposition forest is thus finite by König's Lemma, and the algorithm terminates.

Note that, in order to answer the VASS reachability problem, we only need to explore a decomposition forest nondeterministically in search of a leaf labelled by a normal KLM sequence. However, a *full decomposition* $\text{fdec}(\xi_0)$, which we define as the set of all the normal KLM sequences in a decomposition forest for ξ_0 , is computable, and such that

$$L_{\xi_0} = \bigcup_{\xi' \in \text{fdec}(\xi_0)} L_{\xi'} . \quad (7)$$

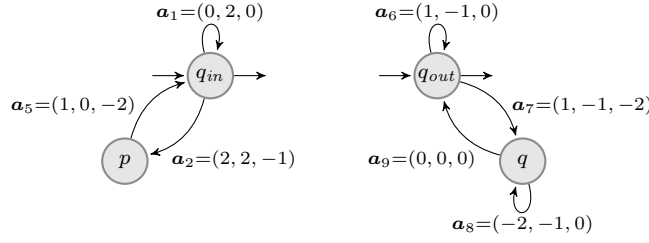
Remark 4.1. Note that decomposition steps are not deterministic, meaning that there might be several choices of sets $\text{dec}(\xi)$ for each ξ . Thus there might be several decomposition forests for a KLM sequence ξ_0 . This does not impact the correctness of the algorithm; in fact, we know from [18] that all the full decompositions one can obtain actually denote the same canonical *ideal decomposition*. \square

4.1. Elementary Decomposition Steps. As will be further explained in Sec. 4.3, clean KLM sequences are obtained in Lem. 4.17 by performing a decomposition into strongly connected components (Sec. 4.1.1), followed by a *saturation* step (Sec. 4.1.2), and keeping only the satisfiable KLM sequences according to their characteristic systems, which were defined in Sec. 3.3. A decomposition step according to Lem. 4.18 first unfolds *unpumpable* (Sec. 4.2.1) or *bounded* (Sec. 4.1.3) sequences, and then cleans up the resulting sequences thanks to Lem. 4.17.

4.1.1. Strongly Connected KLM Sequences. A KLM sequence $\xi = (\mathbf{x}_0 G_0 \mathbf{y}_0) \mathbf{a}_1 \dots \mathbf{a}_k (\mathbf{x}_k G_k \mathbf{y}_k)$ is said to be *strongly connected* if the VASSes G_0, \dots, G_k occurring in ξ are strongly connected.

Lemma 4.2. *For any KLM sequence ξ that is not strongly connected, we can compute in time $\exp(|\xi|)$ a finite set Ξ of strongly connected KLM sequences such that $L_\xi = \bigcup_{\xi' \in \Xi} L_{\xi'}$ and such that $\text{rank}(\xi') <_{\text{lex}} \text{rank}(\xi)$ and $|\xi'| \leq |\xi|$ for every $\xi' \in \Xi$.*

Proof. We just replace every triple $(\mathbf{x}G\mathbf{y})$ occurring in ξ where $G = (Q, q_{\text{in}}, q_{\text{out}}, T)$ is a non strongly connected VASS by all the possible sequences $(\mathbf{x}G_0\omega)\mathbf{a}_1 \dots (\omega G_n\mathbf{y})$

FIGURE 3. The strongly connected VASSes G_{ex}^1 (left) and G_{ex}^2 (right).

where $n \geq 1$, $G_j = (Q_j, r_j, s_j, T_j)$ is such that Q_0, \dots, Q_n are distinct strongly connected components of G , $T_j \stackrel{\text{def}}{=} T \cap (Q_j \times \mathbb{Z}^d \times Q_j)$ for every $0 \leq j \leq n$, $r_0 \stackrel{\text{def}}{=} q_{\text{in}}$, $s_n \stackrel{\text{def}}{=} q_{\text{out}}$, and $(s_{j-1}, \mathbf{a}_j, r_j)$ is a transition in T for every $1 \leq j \leq n$.

We obtain that way a finite set Ξ of strongly connected KLM sequences satisfying the lemma. In particular, regarding sizes, observe that $|(xG_0\omega)\mathbf{a}_1 \dots (\omega G_n\mathbf{y})| = 2(d+1)^{d+1}(\|\mathbf{x}\| + \|\mathbf{y}\| + (n + \sum_{j=1}^n \|\mathbf{a}_j\| + \sum_{j=0}^n |G_j|)) \leq 2(d+1)^{d+1}(\|\mathbf{x}\| + \|\mathbf{y}\| + |G|)$. \square

Example 4.3. Consider again the VASS G_{ex} of Ex. 2.1 and the KLM sequence $\xi_{\text{ex}} = ((0, 0, 2)G_{\text{ex}}(1, 1, 0))$. The decomposition into strongly connected KLM sequences yields a set $\{\xi_{\text{ex}}^1, \xi_{\text{ex}}^2\}$ where

$$\begin{aligned} \xi_{\text{ex}}^1 &\stackrel{\text{def}}{=} ((0, 0, 2)G_{\text{ex}}^1(\omega, \omega, \omega))\mathbf{a}_3((\omega, \omega, \omega)G_{\text{ex}}^2(1, 1, 0)) , \\ \xi_{\text{ex}}^2 &\stackrel{\text{def}}{=} ((0, 0, 2)G_{\text{ex}}^1(\omega, \omega, \omega))\mathbf{a}_4((\omega, \omega, \omega)G_{\text{ex}}^2(1, 1, 0)) , \end{aligned}$$

where G_{ex}^1 and G_{ex}^2 are displayed in Fig. 3. \square

4.1.2. Saturated KLM Sequences. A KLM sequence $\xi = (x_0G_0y_0)\mathbf{a}_1 \dots \mathbf{a}_k(x_kG_ky_k)$ is said to be *saturated* if for every $0 \leq j \leq k$ and for every $i \in \{1, \dots, d\}$ the following two conditions hold:

- (1) if $x_j(i) = \omega$, then the set of values $\mathbf{m}_j^{\mathbf{h}}(i)$ where \mathbf{h} is a model of E_ξ is unbounded, and
- (2) if $y_j(i) = \omega$, then the set of values $\mathbf{n}_j^{\mathbf{h}}(i)$ where \mathbf{h} is a model of E_ξ is unbounded.

Saturation corresponds essentially to Kosaraju's property $\theta 1(b)$.

Lemma 4.4. *From any strongly connected KLM sequence ξ , we can compute in time $\exp(|\xi|^{|\xi|})$ a finite set Ξ of saturated strongly connected KLM sequences such that $L_\xi = \bigcup_{\xi' \in \Xi} L_{\xi'}$, and such that $\text{rank}(\xi') \leq_{\text{lex}} \text{rank}(\xi)$ and $|\xi'| \leq |\xi|^{|\xi|}$ for every $\xi' \in \Xi$.*

Proof. Thanks to Lem. 3.7, we can saturate a KLM sequence. In fact, we just have to replace some ω components by all the possible bounded values $\leq |\xi|^{|\xi|-1}$ given by the characteristic system E_ξ for the variables $\mathbf{m}_j, \mathbf{n}_j$. \square

Example 4.5. Consider the KLM sequences ξ_{ex}^1 and ξ_{ex}^2 from Ex. 4.3. Lemma 4.4 yields respectively

$$\begin{aligned} \xi_{\text{ex}}^3 &\stackrel{\text{def}}{=} ((0, 0, 2)G_{\text{ex}}^1(0, \omega, 2))\mathbf{a}_3((1, \omega, 2)G_{\text{ex}}^2(1, 1, 0)) , \\ \xi_{\text{ex}}^4 &\stackrel{\text{def}}{=} ((0, 0, 2)G_{\text{ex}}^1(0, \omega, 2))\mathbf{a}_4((0, \omega, 0)G_{\text{ex}}^2(1, 1, 0)) . \end{aligned} \quad \square$$

4.1.3. Unbounded KLM Sequences. Consider a KLM sequence ξ of form $(\mathbf{x}_0 G_0 \mathbf{y}_0) \mathbf{a}_1 \dots \mathbf{a}_k (\mathbf{x}_k G_k \mathbf{y}_k)$, where T_j denotes the set of transitions of G_j . Observe that, if a transition t in T_j is such that the set of values $\phi_j^{\mathbf{h}}(t)$ where \mathbf{h} ranges over the models of the characteristic system E_ξ of ξ is bounded by some value B , then the number of times a word $\sigma \in L_\xi$ can use the transition t is bounded by B . It means that the VASS G_j can be replaced by at most B copies of itself without the transition t , joined using the action $\Delta(t)$ of t , while preserving the language L_ξ .

Formally, in such a situation, we define T'_j as the set of transitions $t \in T_j$ such that the set of values $\phi_j^{\mathbf{h}}(t)$ is unbounded. A KLM sequence ξ is said to be *unbounded* if $T'_j = T_j$ for every $0 \leq j \leq k$, and otherwise to be *bounded*. Unboundedness corresponds essentially to Kosaraju's property $\theta 1(a)$, but here we also need to show that the ranks decrease when performing this decomposition.

Lemma 4.6. *Whether a KLM sequence ξ is unbounded is in NP. Moreover, if ξ is strongly connected and bounded, we can compute in time $\exp(|\xi|^{|\xi|})$ a finite set Ξ of KLM sequences such that $L_\xi = \bigcup_{\xi' \in \Xi} L_{\xi'}$ and such that $\text{rank}(\xi') <_{\text{lex}} \text{rank}(\xi)$ and $|\xi'| \leq |\xi|^{|\xi|}$ for every $\xi' \in \Xi$.*

Proof. Let T'_j be the set of transitions $t \in T_j$ such that the set $\phi_j^{\mathbf{h}}(t)$ where \mathbf{h} is a model of E_ξ is unbounded. Let us introduce the VASS G'_j obtained from G_j by replacing T_j by T'_j . Let \mathbf{V}_j be the vector space spanned by the displacements of the cycles of G_j , and let \mathbf{V}'_j be the vector space generated by the displacements of the cycles of G'_j . Since $T'_j \subseteq T_j$, naturally $\mathbf{V}'_j \subseteq \mathbf{V}_j$. We are going to prove that if $\mathbf{V}'_j = \mathbf{V}_j$ then $T'_j = T_j$.

Claim 4.7. *Assume that E_ξ is satisfiable. For every j , if $\mathbf{V}'_j = \mathbf{V}_j$ then $T'_j = T_j$.*

Proof of Claim 4.7. Let us consider $j \in \{0, \dots, k\}$ such that $\mathbf{V}'_j = \mathbf{V}_j$ and let us prove that $T'_j = T_j$. By summing up a finite number of solutions of E_ξ^0 (one for each transition $t \in T'_j$), Lem. 3.7 shows that there exists a solution \mathbf{h}_0 of E_ξ^0 such that $\phi_j^{\mathbf{h}_0}(t) > 0$ for every $t \in T'_j$.

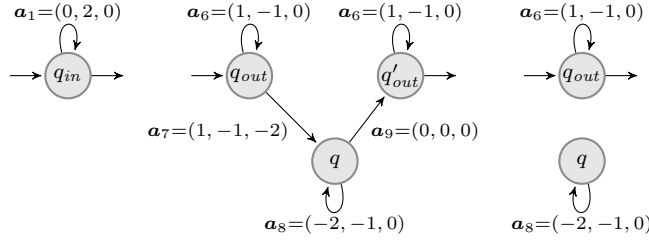
Let us consider a cycle of G_j that contains all the transitions of T_j ; such a cycle exists since G_j is strongly connected. We denote by ψ the Parikh image of that cycle. Notice that $\Delta(\psi) \in \mathbf{V}_j$; since $\mathbf{V}_j = \mathbf{V}'_j$, there exists a sequence $\theta_1, \dots, \theta_s$ of cycles of G'_j , and a sequence $\lambda_1, \dots, \lambda_s$ of rational numbers such that $\Delta(\psi) = \sum_{r=1}^s \lambda_r \Delta(\phi_r)$, where ϕ_r is the Parikh image of θ_r . Let $p > 0$ be a natural number such that $p\lambda_r \in \mathbb{Z}$ for every r . Since $\phi_j^{\mathbf{h}_0}(t) > 0$ for every $t \in T'_j$, there exists $q \in \mathbb{N}$ such that $p\lambda_r \phi_r \leq q\phi_j^{\mathbf{h}_0}$ for every r . It follows that $\phi'_r \stackrel{\text{def}}{=} q\phi_j^{\mathbf{h}_0} - p\lambda_r \phi_r$ maps every $t \in T_j \setminus T'_j$ to zero. Let ϕ' be the mapping $p\psi + \sum_{r=1}^s \phi'_r$. We deduce that

$$\Delta(\phi') = \Delta(qs\phi_j^{\mathbf{h}_0}) = qs\mathbf{n}_j^{\mathbf{h}_0} - qs\mathbf{m}_j^{\mathbf{h}_0} \quad (8)$$

since \mathbf{h}_0 is a model of E_ξ^0 .

It follows that the sequence \mathbf{h}'_0 obtained from $qs\mathbf{h}_0$ by replacing the j th tuple by $(qs\mathbf{m}_j^{\mathbf{h}_0}, \phi', qs\mathbf{n}_j^{\mathbf{h}_0})$ is a model of E_ξ^0 . Notice that $\phi_j^{\mathbf{h}'_0}(t) = \phi'(t) \geq p\psi(t) \geq 1$ for every $t \in T_j$. Lem. 3.7 shows that $T_j \subseteq T'_j$. Hence $T'_j = T_j$. \square

Let us return to the proof of Lem. 4.6. First observe that we can decide in nondeterministic polynomial time whether E_ξ is satisfiable. If it is not the case, then L_ξ is empty and we can return the empty set. Otherwise, Lem. 3.7 shows that

FIGURE 4. The VASSes G_{ex}^3 (left), G_{ex}^4 (middle), and G_{ex}^5 (right).FIGURE 5. The VASSes G_{ex}^6 (left) and G_{ex}^7 (right).

the sets T'_1, \dots, T'_j are computable in polynomial time. If $T'_j = T_j$ for every j , then ξ is unbounded. Otherwise, ξ is bounded, and there exists j such that T'_j is strictly included in T_j . Lemma 3.7 shows that a word $\sigma \in L_\xi$ cannot use a transition in $T_j \setminus T'_j$ more than $|\xi|^{|\xi|-1}$ times. It follows that we can replace the triple $\mathbf{x}_j G_j \mathbf{y}_j$ in ξ by a triple where the transitions in $T_j \setminus T'_j$ are taken at most $|\xi|^{|\xi|-1}$ times the VASS G'_j . Hence $|\xi'| \leq |\xi|^{|\xi|}$. Since G_j is strongly connected, Claim 4.7 shows that the KLM sequences ξ' obtained that way satisfy $\text{rank}(\xi') <_{\text{lex}} \text{rank}(\xi)$. \square

Example 4.8. Consider the KLM sequences ξ_{ex}^3 and ξ_{ex}^4 from Ex. 4.5. Lemma 4.6 yields respectively

$$\begin{aligned} \xi_{\text{ex}}^5 &\stackrel{\text{def}}{=} ((0, 0, 2)G_{\text{ex}}^3(0, \omega, 2))\mathbf{a}_3((1, \omega, 2)G_{\text{ex}}^4(1, 1, 0)) , \\ \xi_{\text{ex}}^6 &\stackrel{\text{def}}{=} ((0, 0, 2)G_{\text{ex}}^3(0, \omega, 2))\mathbf{a}_4((0, \omega, 0)G_{\text{ex}}^5(1, 1, 0)) , \end{aligned}$$

where G_{ex}^3 , G_{ex}^4 , and G_{ex}^5 are displayed in Fig. 4. Applying Lem. 4.2 and Lem. 4.4 to ξ_{ex}^5 yields

$$\begin{aligned} \xi_{\text{ex}}^7 &\stackrel{\text{def}}{=} ((0, 0, 2)G_{\text{ex}}^3(0, \omega, 2))\mathbf{a}_3((1, \omega, 2)G_{\text{ex}}^6(\omega, \omega, 2))\mathbf{a}_7 \\ &\quad ((\omega, \omega, 0)G_{\text{ex}}^7(\omega, \omega, 0))\mathbf{a}_9((\omega, \omega, 0)G_{\text{ex}}^6(1, 1, 0)) , \end{aligned}$$

where G_{ex}^6 and G_{ex}^7 are shown in Fig. 5. The KLM sequence ξ_{ex}^6 is unsatisfiable, thus by Lem. 3.5, it can be discarded. \square

4.2. Rigid KLM Sequences. A component i is said to be *fixed* by a VASS $G = (Q, q_{\text{in}}, q_{\text{out}}, T)$ if there exists a function $f_i: Q \rightarrow \mathbb{N}$ such that $f_i(q) = f_i(p) + \mathbf{a}(i)$ for every transition $(p, \mathbf{a}, q) \in T$. Notice that we can compute in polynomial time the set of fixed components of G , and given such a component i , we can compute in polynomial time a function $f_i: Q \rightarrow \mathbb{N}$ such that $f_i(q) = f_i(p) + \mathbf{a}(i)$ for every transition $(p, \mathbf{a}, q) \in T$.

A KLM sequence $\mathbf{x}G\mathbf{y}$ where $G = (Q, q_{\text{in}}, q_{\text{out}}, T)$ is a VASS is said to be *rigid* if for every component i that is fixed by G there exists a function $g_i: Q \rightarrow \mathbb{N}$ such that $g_i(q) = g_i(p) + \mathbf{a}(i)$ for every transition $(p, \mathbf{a}, q) \in T$, and such

that $g_i(q_{in}) \sqsubseteq \mathbf{x}(i)$ and $g_i(q_{out}) \sqsubseteq \mathbf{y}(i)$. More generally, a KLM sequence $\xi = (\mathbf{x}_0 G_0 \mathbf{y}_0) \mathbf{a}_1 \dots \mathbf{a}_k (\mathbf{x}_k G_k \mathbf{y}_k)$ is said to be *rigid* if $\mathbf{x}_j G_j \mathbf{y}_j$ is rigid for every $0 \leq j \leq k$. Rigidity corresponds essentially to the rigid components introduced by Kosaraju.

Lemma 4.9. *From any strongly connected KLM sequence ξ , we can decide in time $\text{poly}(|\xi|)$ whether ξ is not rigid. Moreover, in that case we can compute in time $\text{poly}(|\xi|)$ a KLM sequence ξ' such that $L_\xi = L_{\xi'}$, $\text{rank}(\xi') <_{lex} \text{rank}(\xi)$, and $|\xi'| \leq |\xi|$.*

Proof. Let us assume that ξ is the KLM sequence $\mathbf{x}G\mathbf{y}$ where $G = (Q, q_{in}, q_{out}, T)$ is strongly connected (the general case can be obtained the same way). We can compute in polynomial time by a straightforward constant propagation algorithm the set I of components that are fixed by G and for every $i \in I$ a function $f_i: Q \rightarrow \mathbb{N}$ such that $f_i(q) = f_i(p) + \mathbf{a}(i)$ for every transition $(p, \mathbf{a}, q) \in T$.

Claim 4.10. *ξ is rigid if and only if the following three conditions hold for every $i \in I$ and for every $q \in Q$:*

- (i) $\mathbf{y}(i) - f_i(q_{out}) = \mathbf{x}(i) - f_i(q_{in})$ if $\mathbf{x}(i), \mathbf{y}(i) \in \mathbb{N}$,
- (ii) $\mathbf{x}(i) - f_i(q_{in}) + f_i(q) \geq 0$ if $\mathbf{x}(i) \in \mathbb{N}$, and
- (iii) $\mathbf{y}(i) - f_i(q_{out}) + f_i(q) \geq 0$ if $\mathbf{y}(i) \in \mathbb{N}$.

Proof of Claim 4.10. Assume first that ξ is rigid. In that case, for every $i \in I$ there exists a function $g_i: Q \rightarrow \mathbb{N}$ such that $g_i(q) = g_i(p) + \mathbf{a}(i)$ for every transition $(p, \mathbf{a}, q) \in T$ and such that $g_i(q_{in}) \sqsubseteq \mathbf{x}(i)$ and $g_i(q_{out}) \sqsubseteq \mathbf{y}(i)$. Since G is strongly connected, it follows that there exists an integer $z_i \in \mathbb{Z}$ such that $g_i(q) = z_i + f_i(q)$ for every $q \in Q$. This equality in q_{in} and q_{out} provides $z_i = \mathbf{x}(i) - f_i(q_{in})$ if $\mathbf{x}(i) \in \mathbb{N}$ and $z_i = \mathbf{y}(i) - f_i(q_{out})$ if $\mathbf{y}(i) \in \mathbb{N}$. We deduce that conditions (i), (ii), and (iii) hold.

Conversely, assume that these conditions hold and let us prove that ξ is rigid. Let $i \in I$ and let us prove that there exists a function $g_i: Q \rightarrow \mathbb{N}$ such that $g_i(q) = g_i(p) + \mathbf{a}(i)$ for every transition $(p, \mathbf{a}, q) \in T$ and such that $g_i(q_{in}) \sqsubseteq \mathbf{x}(i)$ and $g_i(q_{out}) \sqsubseteq \mathbf{y}(i)$. If $\mathbf{x}(i) = \omega$ and $\mathbf{y}(i) = \omega$, notice that $g_i \stackrel{\text{def}}{=} f_i$ fullfills the required conditions. If $\mathbf{x}(i) \in \mathbb{N}$, then condition (i) shows that we define $g_i: Q \rightarrow \mathbb{N}$ by $g_i(q) \stackrel{\text{def}}{=} \mathbf{x}(i) - f_i(q_{in}) + f_i(q)$. Notice that for every transition $(p, \mathbf{a}, q) \in T$, we have $g_i(q) = g_i(p) + \mathbf{a}(i)$. Observe that $g_i(q_{in}) = \mathbf{x}(i)$. Let us show that $g_i(q_{out}) \sqsubseteq \mathbf{y}(i)$. If $\mathbf{y}(i) = \omega$, the relation is immediate. Otherwise, by condition (i), we get $g_i(q_{out}) = \mathbf{y}(i)$. We have proved that g_i fullfills the required conditions. Symmetrically, we obtain the case $\mathbf{y}(i) \in \mathbb{N}$ and $\mathbf{x}(i) \in \mathbb{N}_\omega$. We have shown that ξ is rigid. \square

By Claim 4.10, we can decide in polynomial time whether ξ is rigid. Moreover, if ξ is not rigid, we can compute in polynomial time both $i \in I$ and $q \in Q$ such that one of the three conditions (i), (ii), and (iii) does not hold. If condition (i) does not hold, then ξ cannot be satisfiable, and in particular $L_\xi = \emptyset$. Thus we can consider for ξ' the KLM sequence obtained from ξ by removing all the transitions and all the states except q_{in} and q_{out} .

Otherwise, if condition (i) holds, then either (ii) or (iii) does not hold. Since (i) holds, it follows that $q \notin \{q_{in}, q_{out}\}$. Let us show that $L_\xi = L_{\xi'}$ where $\xi' \stackrel{\text{def}}{=} \mathbf{x}G'\mathbf{y}$ and $G' \stackrel{\text{def}}{=} (Q', q_{in}, q_{out}, T')$, $Q' \stackrel{\text{def}}{=} Q \setminus \{q\}$, $T' \stackrel{\text{def}}{=} T \cap (Q' \times \mathbb{Z}^d \times Q')$. To prove this inclusion, let us consider any $\sigma \in L_\xi$. There exists two configurations \mathbf{m} and \mathbf{n} and a word $\sigma = \mathbf{a}_1 \dots \mathbf{a}_k$ of actions such that $q_{in}(\mathbf{m}) \xrightarrow[G]{\sigma} q_{out}(\mathbf{n})$. Thus there exists a

sequence $q_0(\mathbf{c}_0), \dots, q_k(\mathbf{c}_k)$ of state-configurations such that

$$q_{in}(\mathbf{m}) = q_0(\mathbf{c}_0) \xrightarrow[G]{\mathbf{a}_1} \dots \xrightarrow[G]{\mathbf{a}_k} q_k(\mathbf{c}_k) = q_{out}(\mathbf{n}). \quad (9)$$

Observe that if $\mathbf{x}(i) \in \mathbb{N}$, then $\mathbf{c}_0(i) = \mathbf{x}(i)$ and by induction we get $\mathbf{x}(i) - f_i(q_{in}) + f_i(q_j) = \mathbf{c}_j(i) \geq 0$ for every $0 \leq j \leq k$. Symmetrically, if $\mathbf{y}(i) \in \mathbb{N}$, then $\mathbf{y}(i) - f_i(q_{out}) + f_i(q_j) = \mathbf{c}_j(i) \geq 0$ for every $0 \leq j \leq k$. Thus $q \notin \{q_0, \dots, q_k\}$ and in particular $\sigma \in L_{\xi'}$. \square

4.2.1. Pumpable KLM Sequences. Given a VASS G and two configurations \mathbf{x}, \mathbf{y} , the *forward and backward accelerations* are the vectors $\text{Facc}_G(\mathbf{x})$ and $\text{Bacc}_G(\mathbf{y})$ in \mathbb{N}_ω^d defined respectively for every $i \in \{1, \dots, d\}$ as follows:

$$\begin{aligned} \text{Facc}_G(\mathbf{x})(i) &\stackrel{\text{def}}{=} \begin{cases} \omega & \text{if } \exists \mathbf{x}' \geq \mathbf{x} \text{ with } \mathbf{x}'(i) > \mathbf{x}(i) \text{ s.t.} \\ & q_{in}(\mathbf{x}) \xrightarrow[G]{*} q_{in}(\mathbf{x}') \\ \mathbf{x}(i) & \text{otherwise} \end{cases} \\ \text{Bacc}_G(\mathbf{y})(i) &\stackrel{\text{def}}{=} \begin{cases} \omega & \text{if } \exists \mathbf{y}' \geq \mathbf{y} \text{ with } \mathbf{y}'(i) > \mathbf{y}(i) \text{ s.t.} \\ & q_{out}(\mathbf{y}') \xrightarrow[G]{*} q_{out}(\mathbf{y}) \\ \mathbf{y}(i) & \text{otherwise} \end{cases} \end{aligned}$$

Observe that $\text{Facc}_G(\mathbf{x})(i) = \mathbf{x}(i)$ and $\text{Bacc}_G(\mathbf{y})(i) = \mathbf{y}(i)$ for every component i fixed by G . A triple $(\mathbf{x}G\mathbf{y})$ is said to be *pumpable* if $\text{Facc}_G(\mathbf{x})(i) = \omega$ and $\text{Bacc}_G(\mathbf{y})(i) = \omega$ for every component i not fixed by G . More generally, a KLM sequence $\xi = (\mathbf{x}_0G_0\mathbf{y}_0)\mathbf{a}_1(\mathbf{x}_1G_1\mathbf{y}_1) \dots \mathbf{a}_k(\mathbf{x}_kG_k\mathbf{y}_k)$ is said to be *pumpable* if $(\mathbf{x}_jG_j\mathbf{y}_j)$ is pumpable for every $0 \leq j \leq k$, and otherwise to be *unpumpable*.

Remark 4.11. Pumpability, rigidity, and saturation together correspond essentially to Kosaraju's property θ_2 . In fact, we show in Appendix C.2 that if a KLM sequence $\mathbf{x}G\mathbf{y}$ is pumpable, rigid, and saturated, then there exists a function $f: Q \rightarrow \mathbb{N}_\omega^d$ such that $f(q) = f(p) + \mathbf{a}$ for every $(p, \mathbf{a}, q) \in T$, and such that $f(q_{in}) = \text{Facc}_G(\mathbf{x})$ and $f(q_{out}) = \text{Bacc}_G(\mathbf{y})$. \square

Example 4.12. The KLM sequence ξ_{ex}^7 from Ex. 4.8 is unpumpable: indeed, in the triple $((\omega, \omega, 0)G_{\text{ex}}^6(1, 1, 0))$, the components 1 and 2 are not fixed, but we find $\text{Bacc}_{G_{\text{ex}}^6}((1, 1, 0))(1) = \text{Bacc}_{G_{\text{ex}}^6}((1, 1, 0))(2) = 1$. \square

Deciding Pumpability. Observe that $\text{Facc}_G(\mathbf{x})$ and $\text{Bacc}_G(\mathbf{y})$ are computable by performing $2d$ calls to an oracle for the *coverability problem* [see, e.g., 16, Lem. 3.3]. By the results of Rackoff [24], we can therefore decide in exponential space whether a KLM sequence ξ is pumpable.

Unfolding. When a KLM sequence ξ is unpumpable, there is a triple $(\mathbf{x}G\mathbf{y})$ and a component i not fixed by G such that $\text{Facc}_G(\mathbf{x})(i) < \omega$ or $\text{Bacc}_G(\mathbf{y})(i) < \omega$. Assume that we are in the former case. If ξ is strongly connected, then there exists a finite $B \in \mathbb{N}$ such that $\text{Facc}_G(\mathbf{x})(i) = B$, and the idea is then to *unfold* G by tracking the value of the i th component in the control state. Classically, such a bound B is computed by constructing a Karp and Miller coverability tree, but this has a worst-case Ackermannian complexity [3]. Thus the decomposition algorithms of Mayr, Kosaraju, and Lambert might use an Ackermannian time in their very first decomposition step.

Here, we refine this decomposition step using insights from Rackoff's results in [24]. We show that, if there is a component i not fixed by G such that $\text{Facc}_G(\mathbf{x})(i) < \omega$, then there exists a component i' not fixed by G and such that a double exponential B suffices. Formally, let $\mathbb{N}_B \stackrel{\text{def}}{=} \{0, \dots, B-1, \omega\}$. Consider any $i \in \{1, \dots, d\}$, $r \in \mathbb{N}_B$, and $\mathbf{x}(i) \in \mathbb{N}_B$; the *forward* (i, B, r) -*unfolding* of a KLM triple $\mathbf{x}G\mathbf{y}$ is the KLM triple $\mathbf{x}G'\mathbf{y}$ where $G' \stackrel{\text{def}}{=} (Q \times \mathbb{N}_B, (q_{in}, \mathbf{x}(i)), (q_{out}, r), T')$ and T' is the set of transitions $((p, m), \mathbf{a}, (q, n))$ where $(p, \mathbf{a}, q) \in T$ and $m, n \in \mathbb{N}_B$ satisfy $n = m + \mathbf{a}(i)$ or $(n = \omega \wedge m + \mathbf{a}(i) \geq B)$, and such that $m = \omega$ implies $q \neq q_{in}$. (The *backward* (i, B, r) -*unfolding* is defined symmetrically.) We show the following in Appendix A.

Lemma 4.13. *Let $\xi = \mathbf{x}G\mathbf{y}$ be a KLM sequence and let I be the set of components $i \in \{1, \dots, d\}$ that are not fixed by G and such that $\text{Facc}_G(\mathbf{x})(i) < \omega$. If I is not empty, then there exists $i \in I$ such that $L_\xi = \bigcup_{r \in \mathbb{N}_B} L_{\xi_r}$ where ξ_r is the forward (i, B, r) -unfolding of ξ and $B \stackrel{\text{def}}{=} (\|\mathbf{x}\| + 2|G|)^{1+d^d}$.*

Of course, we also require that unfolding $\mathbf{x}G\mathbf{y}$ decreases the rank. The condition that $m = \omega$ must imply $q \neq q_{in}$ in the unfolding is central for the proof of the following lemma.

Lemma 4.14. *Let $\xi = \mathbf{x}G\mathbf{y}$ be a strongly connected KLM sequence and let i be a component not fixed by G and such that $\mathbf{x}(i) \in \mathbb{N}_B$ for some $B \in \mathbb{N}$. Then the (i, B, r) -unfolding ξ' of ξ satisfies $\text{rank}(\xi') <_{lex} \text{rank}(\xi)$ for all $r \in \mathbb{N}_B$.*

Proof. Assume that $G = (Q, q_{in}, q_{out}, T)$ and let $\xi' = \mathbf{x}G'\mathbf{y}$ be the (i, B, r) -unfolding of G where $G' = (Q \times \mathbb{N}_B, (q_{in}, \mathbf{x}(i)), (q_{out}, r), T')$. Let \mathbf{V} be the vector space generated by the displacements of the cycles of G . As G is strongly connected, Lem. 3.2 shows that $\mathbf{V}_G(t) = \mathbf{V}$ for every transition t in T .

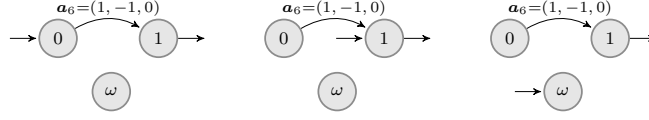
Observe that since i is not fixed by G , it means that there exists a vector $\mathbf{v} \in \mathbf{V}$ such that $\mathbf{v}(i) \neq 0$. In particular the dimension of \mathbf{V} is larger than or equal to one.

Let us observe that every cycle of G' labelled by a word σ corresponds (by projecting on the first component of its control states) to a cycle of G also labelled by σ . It follows that the displacement of every cycle of G' is in \mathbf{V} , therefore $\mathbf{V}_{G'}(t') \subseteq \mathbf{V}$ for every transition t' in T' . Let us consider such a transition $t' = ((p, m), \mathbf{a}, (q, n))$ from T' , such that $(p, \mathbf{a}, q) \in T$ and $m, n \in \mathbb{N}_B$. For the transitions $t' \in T'$ such that $m = \omega$, then $n = \omega$ and $q \neq q_{in}$, thus there are at most $|T| - 1$ such transitions. For the other transitions in T' , i.e., such that $m \neq \omega$, let us prove that $\mathbf{V}_{G'}(t')$ is strictly included in \mathbf{V} . If there is no cycle using t' , then $\mathbf{V}_{G'}(t') = \{\mathbf{0}\}$ and we are done. Otherwise, notice that this cycle keep tracks in G' of the precise displacement on the component i since there is no way to move from a state in $Q \times \{\omega\}$ to a state in $Q \times \{0, \dots, B-1\}$. It follows that the displacement of such a cycle is zero on component i . Hence the vector \mathbf{v} we singled out earlier is not in $\mathbf{V}_{G'}(t')$ and we have proven that $\mathbf{V}_{G'}(t')$ is strictly included in \mathbf{V} .

This shows that $\text{rank}(G') <_{lex} \text{rank}(G)$. □

Together, the previous two lemmas allow to show the following.

Lemma 4.15. *Whether a KLM sequence ξ is pumpable is in EXPSPACE. Moreover, if ξ is strongly connected and unpumpable, we can compute in time $\exp(|\xi|^{2+d^d})$ a finite set Ξ of KLM sequences such that $L_\xi = \bigcup_{\xi' \in \Xi} L_{\xi'}$ and such that $\text{rank}(\xi') <_{lex} \text{rank}(\xi)$ and $|\xi'| \leq |\xi|^{2+d^d}$ for every $\xi' \in \Xi$.*

FIGURE 6. The VASSes G_{ex}^8 (left), G_{ex}^9 (middle), and G_{ex}^{10} (right).

Proof. We have already argued that pumpability is decidable in exponential space. Assume that ξ is strongly connected and unpumpable. Then there is a triple $\mathbf{x}G\mathbf{y}$ in ξ and a component i not fixed by G such that $\text{Facc}_G(\mathbf{x})(i) < \omega$ or $\text{Bacc}_G(\mathbf{y})(i) < \omega$. Let us consider the former case and define $B \stackrel{\text{def}}{=} (\|\mathbf{x}\| + 2|G|)^{1+d^d}$.

Lemma 4.13 shows that $L_\xi = \bigcup_{r \in \mathbb{N}_B} L_{\xi_r}$, where ξ_r is the KLM sequence obtained from ξ by replacing the KLM triple $\mathbf{x}G\mathbf{y}$ by its (i', B, r) -unfolding for a suitable i' . Lemma 4.14 shows that $\text{rank}(\xi_r) <_{\text{lex}} \text{rank}(\xi)$. Finally, $B < |\xi|^{1+d^d}$ and thus $|\xi_r| \leq (1+B)|\xi| \leq |\xi|^{2+d^d}$. \square

Example 4.16. Consider again Ex. 4.12 and in particular component 1. Then $B = 1$ suffices, and we can unfold along the first component, yielding three new KLM triples

$$\begin{aligned} \xi_{\text{ex}}^8 &\stackrel{\text{def}}{=} ((0, 0, 2)G_{\text{ex}}^3(0, \omega, 2))\mathbf{a}_3((1, \omega, 2)G_{\text{ex}}^6(\omega, \omega, 2))\mathbf{a}_7 \\ &\quad ((\omega, \omega, 0)G_{\text{ex}}^7(\omega, \omega, 0))\mathbf{a}_9((\omega, \omega, 0)G_{\text{ex}}^8(1, 1, 0)) , \\ \xi_{\text{ex}}^9 &\stackrel{\text{def}}{=} ((0, 0, 2)G_{\text{ex}}^3(0, \omega, 2))\mathbf{a}_3((1, \omega, 2)G_{\text{ex}}^6(\omega, \omega, 2))\mathbf{a}_7 \\ &\quad ((\omega, \omega, 0)G_{\text{ex}}^7(\omega, \omega, 0))\mathbf{a}_9((\omega, \omega, 0)G_{\text{ex}}^9(1, 1, 0)) , \\ \xi_{\text{ex}}^{10} &\stackrel{\text{def}}{=} ((0, 0, 2)G_{\text{ex}}^3(0, \omega, 2))\mathbf{a}_3((1, \omega, 2)G_{\text{ex}}^6(\omega, \omega, 2))\mathbf{a}_7 \\ &\quad ((\omega, \omega, 0)G_{\text{ex}}^7(\omega, \omega, 0))\mathbf{a}_9((\omega, \omega, 0)G_{\text{ex}}^{10}(1, 1, 0)) , \end{aligned}$$

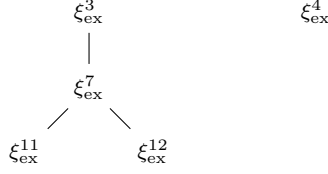
where G_{ex}^8 , G_{ex}^9 , and G_{ex}^{10} are shown in Fig. 6. When applying lemmata 4.2 and 4.4, ξ_{ex}^8 and ξ_{ex}^9 are respectively decomposed into

$$\begin{aligned} \xi_{\text{ex}}^{11} &\stackrel{\text{def}}{=} ((0, 0, 2)G_{\text{ex}}^3(0, \omega, 2))\mathbf{a}_3((1, \omega, 2)G_{\text{ex}}^6(\omega, \omega, 2))\mathbf{a}_7 \\ &\quad ((\omega, \omega, 0)G_{\text{ex}}^7(0, 2, 0))\mathbf{a}_9((0, 2, 0)G_{\text{ex}}^{11}(0, 2, 0))\mathbf{a}_6 \\ &\quad ((1, 1, 0)G_{\text{ex}}^{11}(1, 1, 0)) , \\ \xi_{\text{ex}}^{12} &\stackrel{\text{def}}{=} ((0, 0, 2)G_{\text{ex}}^3(0, \omega, 2))\mathbf{a}_3((1, \omega, 2)G_{\text{ex}}^6(\omega, \omega, 2))\mathbf{a}_7 \\ &\quad ((\omega, \omega, 0)G_{\text{ex}}^7(1, 1, 0))\mathbf{a}_9((1, 1, 0)G_{\text{ex}}^{11}(1, 1, 0)) , \end{aligned}$$

where $G_{\text{ex}}^{11} = (\{q\}, \{q\}, \{q\}, \emptyset)$ is the trivial VASS with no transitions, while G_{ex}^{10} is discarded. \square

4.3. Normal KLM Sequences. A KLM sequence is said to be *clean* if it is satisfiable (see Sec. 3.3), strongly connected (see Sec. 4.1.1), and saturated (see Sec. 4.1.2). It is *normal* if it is clean, rigid (see Sec. 4.2), pumpable (see Sec. 4.2.1), and unbounded (see Sec. 4.1.3).

4.3.1. Cleaning Lemma. We can transform any KLM sequence into a finite set of clean KLM sequences thanks to the following lemma.

FIGURE 7. A decomposition forest for ξ_{ex} .

Lemma 4.17 (Cleaning). *From any KLM sequence ξ , we can compute in time $\exp(g(|\xi|))$ a finite set $\text{clean}(\xi)$ of clean KLM sequences such that $L_\xi = \bigcup_{\xi' \in \text{clean}(\xi)} L_{\xi'}$ and such that $\text{rank}(\xi') \leq_{\text{lex}} \text{rank}(\xi)$ and $|\xi'| \leq g(|\xi|)$ for every $\xi' \in \text{clean}(\xi)$, where $g(x) \stackrel{\text{def}}{=} x^x$.*

Proof. By Lem. 4.2, we can compute a finite set Ξ of strongly connected KLM sequences such that $L_\xi = \bigcup_{\xi' \in \Xi} L_{\xi'}$ and such that $\text{rank}(\xi') \leq_{\text{lex}} \text{rank}(\xi)$ and $|\xi'| \leq |\xi|$ for every $\xi' \in \Xi$. By applying Lem. 4.4 to each KLM sequence in Ξ , we compute in exponential time a finite set Ξ' of saturated strongly connected KLM sequences such that $\bigcup_{\xi' \in \Xi} L_{\xi'} = \bigcup_{\xi'' \in \Xi'} L_{\xi''}$ and such that $\text{rank}(\xi'') \leq_{\text{lex}} \text{rank}(\xi)$ and $|\xi''| \leq |\xi|^{|\xi|}$ for every $\xi'' \in \Xi'$. By Lem. 3.5, we can safely remove the unsatisfiable KLM sequences from Ξ' —which can be performed in nondeterministic time polynomial in $\sum_{\xi'' \in \Xi'} |\xi''|$ since each $E_{\xi''}$ is of size polynomial in $|\xi''|$ —and we obtain a set $\text{clean}(\xi)$ satisfying the lemma. \square

4.3.2. Decomposition Lemma. In order to decompose a KLM sequence into a finite set of normal KLM sequences, the decomposition algorithm applies as many times as possible the *decomposition step* defined by the following lemma.

Lemma 4.18 (Decomposition). *Let ξ be a clean KLM sequence. If ξ is not normal, we can compute in time $\exp(h(|\xi|))$ a finite set $\text{dec}(\xi)$ of clean KLM sequences such that $L_\xi = \bigcup_{\xi' \in \text{dec}(\xi)} L_{\xi'}$ and such that $\text{rank}(\xi') <_{\text{lex}} \text{rank}(\xi)$ and $|\xi'| \leq h(|\xi|)$ for every $\xi' \in \text{dec}(\xi)$, where $h(x) \stackrel{\text{def}}{=} x^{x^{1+x}}$.*

Proof. Lemmata 4.6, 4.9 and 4.15 show that we can compute in double exponential time a finite set Ξ of KLM sequences such that $L_\xi = \bigcup_{\xi' \in \Xi} L_{\xi'}$ and such that $\text{rank}(\xi') <_{\text{lex}} \text{rank}(\xi)$ and $|\xi'| \leq |\xi|^{|\xi|}$ for every $\xi' \in \Xi$ by observing that $2 + d^d \leq |\xi|$. For each KLM sequence $\xi' \in \Xi$, by applying Lem. 4.17 we compute in time exponential in $g(|\xi'|)$ a finite set $\text{clean}(\xi')$ of clean KLM sequences such that $L_{\xi'} = \bigcup_{\xi'' \in \text{clean}(\xi')} L_{\xi''}$ and such that $\text{rank}(\xi'') \leq_{\text{lex}} \text{rank}(\xi')$ and $|\xi''| \leq g(|\xi'|)$ for each $\xi'' \in \text{clean}(\xi')$. We deduce the statement by letting $\text{dec}(\xi) \stackrel{\text{def}}{=} \bigcup_{\xi' \in \Xi} \text{clean}(\xi')$. \square

4.3.3. Bounded Witness Lemma. Thanks to the following lemma, we can stop the decomposition once we obtain a normal KLM sequence. The proof given in Appendix C follows the same lines as Kosaraju's, with the added twist that we extract a bound on the length of minimal words in L_ξ .

Lemma 4.19 (Bounded Witness). *From any normal KLM sequence ξ , we can compute in space $O(\ell(|\xi|))$ a word $\sigma \in L_\xi$ such that $|\sigma| \leq \ell(|\xi|)$ where $\ell(x) \stackrel{\text{def}}{=} x^{3x}$.*

Example 4.20. Let us consider examples 4.3 to 4.16. We have $\text{clean}(\xi_{\text{ex}}) = \{\xi_{\text{ex}}^3, \xi_{\text{ex}}^4\}$, which are both bounded, and then $\text{dec}(\xi_{\text{ex}}^3) = \{\xi_{\text{ex}}^7\}$ and $\text{dec}(\xi_{\text{ex}}^4) = \emptyset$. Then, ξ_{ex}^7

is unpumpable and $\text{dec}(\xi_{\text{ex}}^7) = \{\xi_{\text{ex}}^{11}, \xi_{\text{ex}}^{12}\} = \text{fdec}(\xi_{\text{ex}})$, since those last two KLM sequences are normal. The corresponding decomposition forest is depicted in Fig. 7. Observe that the union provided in Ex. 3.1 for $L_{\xi_{\text{ex}}}$ corresponds exactly to the union of $L_{\xi_{\text{ex}}}^{12}$ and $L_{\xi_{\text{ex}}}^{11}$. \square

5. COMPLEXITY UPPER BOUNDS

In this section, we derive upper bounds on the lengths of the branches in a decomposition forest of a KLM sequence ξ_0 , from which we can in turn provide upper bounds on the size of normal KLM sequences, the length of small witnesses, the running time of the decomposition algorithm, and the size of the full decomposition. The idea is to exploit the ranking function defined in Sec. 3.2 in order to bound how many decomposition steps can be performed along a branch of a decomposition forest. We rely for this on a so-called ‘length function theorem’ from [26] to bound the length of descending sequences of ordinals. Finally, we classify the running time complexity using the ‘fast-growing’ complexity classes defined in [28]. A general introduction to these techniques can be found in [29].

5.1. Controlled Sequences of Ranks. For the purposes of this section, it is more convenient to recast the ranking function $\text{rank}()$ on KLM sequences from Sec. 3.2 in terms of ordinals. If $\text{rank}(\xi) = (r_d, \dots, r_0)$, then we associate to ξ the *ordinal rank* in ω^{d+1} defined by

$$\alpha_\xi \stackrel{\text{def}}{=} \omega^d \cdot r_d + \omega^{d-1} \cdot r_{d-1} + \dots + \omega^0 \cdot r_0. \quad (10)$$

This is just a reformulation, because $\text{rank}(\xi) <_{\text{lex}} \text{rank}(\xi')$ if and only if $\alpha_\xi < \alpha_{\xi'}$. Along a branch $\xi'_0, \xi_1, \xi_2, \dots$ of a decomposition forest for a KLM sequence ξ_0 , we see therefore a descending sequence of ordinal ranks

$$\alpha_{\xi'_0} > \alpha_{\xi_1} > \alpha_{\xi_2} > \dots \quad (11)$$

Though all descending sequences of ordinals are finite, we cannot bound their lengths in general; e.g., $K+1 > K > K-1 > \dots > 0$ and $\omega > K > K-1 > \dots > 0$ are descending sequences of length $K+2$ for all K in \mathbb{N} . Nevertheless, a descending sequence of ordinal ranks like (11), found along a branch of a decomposition forest, is not arbitrary, because the successive KLM sequences are either $\xi'_0 \in \text{clean}(\xi_0)$ or the result of some decomposition step, hence one cannot use an arbitrary K as in these examples.

5.1.1. Controlled Sequences of Ordinals. The previous intuition is captured by the notion of *controlled sequences*. In general, for an ordinal $\alpha < \omega^\omega$ (like the ordinal ranks defined by (10)), let us write α in Cantor normal form as $\alpha = \omega^n \cdot c_n + \dots + \omega^0 \cdot c_0$ with c_0, \dots, c_n and n in \mathbb{N} , and define its *size* as $N\alpha \stackrel{\text{def}}{=} \max\{n, \max_{0 \leq i \leq n} c_i\}$. Thus, for the ordinal rank α_ξ defined in (10) for a KLM sequence ξ with $\text{rank}(\xi) = (r_d, \dots, r_0)$,

$$N\alpha_\xi = \max\{d, \max_{0 \leq i \leq d} r_i\}. \quad (12)$$

Let n_0 be a natural number in \mathbb{N} and $h: \mathbb{N} \rightarrow \mathbb{N}$ a monotone inflationary function, i.e., $x \leq h(x)$ and $x \leq y$ implies $h(x) \leq h(y)$. A sequence $\alpha_0, \alpha_1, \dots$ of ordinals below ω^ω is (n_0, h) -controlled if, for all j in \mathbb{N} ,

$$N\alpha_j \leq h^j(n_0), \quad (13)$$

i.e., the size of the j th ordinal α_j is bounded by the j th iterate of h applied to n_0 ; in particular, $N\alpha_0 \leq n_0$ for the first element of the sequence. Because for each $n \in \mathbb{N}$, there are only finitely many ordinals below ω^ω of size at most n , the length of controlled descending sequences is bounded [see, e.g., 26]. One can actually give a precise bound on this length in terms of *subrecursive functions*, whose definition we are about to recall.

5.1.2. Subrecursive Functions. Algorithms shown to terminate via an ordinal ranking function can have a very high worst-case complexity. In order to express such large bounds, a convenient tool is found in subrecursive hierarchies, which employ recursion over ordinal indices to define faster and faster growing functions. We define here two such hierarchies.

Fundamental Sequences. A *fundamental sequence* for a limit ordinal λ is a strictly ascending sequence $(\lambda(x))_{x < \omega}$ of ordinals $\lambda(x) < \lambda$ with supremum λ . We use the standard assignment of fundamental sequences to limit ordinals $\lambda < \varepsilon_0$, where ε_0 denotes the least solution of $x = \omega^x$. For the purposes of this paper, it actually suffices to consider the case $\lambda \leq \omega^\omega$, defined inductively by

$$\omega^\omega(x) \stackrel{\text{def}}{=} \omega^{x+1}, \quad (\beta + \omega^{k+1})(x) \stackrel{\text{def}}{=} \beta + \omega^k \cdot (x+1),$$

where $\beta + \omega^{k+1}$ is in Cantor normal form. This particular assignment satisfies, e.g., $0 < \lambda(x) < \lambda(y)$ for all $x < y$. For instance, $\omega(x) = x+1$ and $(\omega^3 + \omega^3 + \omega)(x) = \omega^3 + \omega^3 + x+1$.

Hardy and Cichoń Hierarchies. In the context of controlled sequences, the hierarchies of Hardy and Cichoń turn out to be especially well-suited [4]. Let $h: \mathbb{N} \rightarrow \mathbb{N}$ be a function. For each such h , the *Hardy hierarchy* $(h^\alpha)_{\alpha \leq \omega^\omega}$ and the *Cichoń hierarchy* $(h_\alpha)_{\alpha \leq \omega^\omega}$ relative to h are two families of functions $h^\alpha, h_\alpha: \mathbb{N} \rightarrow \mathbb{N}$ defined by induction over α by

$$\begin{aligned} h^0(x) &\stackrel{\text{def}}{=} x, & h_0(x) &\stackrel{\text{def}}{=} 0, \\ h^{\alpha+1}(x) &\stackrel{\text{def}}{=} h^\alpha(h(x)), & h_{\alpha+1}(x) &\stackrel{\text{def}}{=} 1 + h_\alpha(h(x)), \\ h^\lambda(x) &\stackrel{\text{def}}{=} h^{\lambda(x)}(x), & h_\lambda(x) &\stackrel{\text{def}}{=} h_{\lambda(x)}(x). \end{aligned}$$

The Hardy functions are well-suited for expressing a large number of iterations of the provided function h . For instance, h^k for some finite k is simply the k th iterate of h . This intuition carries over: h^α is a ‘transfinite’ iteration of the function h , using a kind of diagonalisation in the fundamental sequences to handle limit ordinals. For instance, if we use the successor function $H(x) = x+1$ as our function h , we see that a first diagonalisation yields $H^\omega(x) = H^{x+1}(x) = 2x+1$. The next diagonalisation occurs at $H^{\omega \cdot 2}(x) = H^{\omega+x+1}(x) = H^\omega(2x+1) = 4x+3$. Fast-forwarding a bit, we get for instance a function of exponential growth $H^{\omega^2}(x) = 2^{x+1}(x+1) - 1$, and later a non-elementary function H^{ω^3} akin to a tower of exponentials, and a non primitive-recursive function H^{ω^ω} of Ackermannian growth.

In the following, we will use the fact that, if h is monotone inflationary, then so is h^α : if $x \leq y$, then $x \leq h^\alpha(x) \leq h^\alpha(y)$. Regarding the Cichoń functions, if h is monotone inflationary, then by induction on α ,

$$h^\alpha(x) \geq h_\alpha(x) + x. \quad (14)$$

But the main interest of Cichón functions is that they capture how many iterations are performed by Hardy functions [4]:

$$h^{h_\alpha(x)}(x) = h^\alpha(x) . \quad (15)$$

5.1.3. Length Function Theorem. We can now state a ‘length function theorem’ for controlled descending sequences of ordinals.

Theorem 5.1 ([26, Thm. 3.3]). *Let $n_0 \geq d + 1$. The maximal length of (n_0, h) -controlled descending sequences of ordinals in ω^{d+1} is $h_{\omega^{d+1}}(n_0)$.*

Let us apply Thm. 5.1 to the descending sequences of ordinal ranks from (11) found along a branch of a decomposition forest of ξ_0 . Observe that by (4) and (12), $N\alpha_\xi \leq |\xi|$ for any KLM sequence ξ . Thus, by monotonicity, a sequence like (11) is $(g(|\xi_0|), h)$ -controlled, where g was defined in Lem. 4.17 and h in Lem. 4.18. By Thm. 5.1 and because $g(|\xi_0|) \geq d + 1$, the branches of a decomposition forest for ξ_0 are of length at most

$$L \stackrel{\text{def}}{=} h_{\omega^{d+1}}(g(|\xi_0|)) . \quad (16)$$

In turn, by (14) and (16),

$$L \leq h^{\omega^{d+1}}(g(|\xi_0|)) , \quad (17)$$

and if ξ is any KLM sequence labelling a node of a decomposition forest for ξ_0 , then by (15) and (16),

$$|\xi| \leq h^L(g(|\xi_0|)) = h^{\omega^{d+1}}(g(|\xi_0|)) . \quad (18)$$

Consider now a VASS G of dimension d and two finite configurations \mathbf{c}_{in} and \mathbf{c}_{out} . Then according to (3) and (4),

$$|\mathbf{c}_{in}G\mathbf{c}_{out}| = 2(d+1)^{d+1}(|G| + \|\mathbf{c}_{in}\| + \|\mathbf{c}_{out}\|) . \quad (19)$$

Thus, by combining (18) with Lem. 4.19, we obtain the following small witness property.

Property 5.2 (Small Witness). *Let $G = (Q, q_{in}, q_{out}, T)$ be a VASS of dimension d , \mathbf{c}_{in} and \mathbf{c}_{out} be two finite configurations in \mathbb{N}^d , and $n \stackrel{\text{def}}{=} 2(d+1)^{d+1}(|G| + \|\mathbf{c}_{in}\| + \|\mathbf{c}_{out}\|)$. If $q_{in}(\mathbf{c}_{in}) \xrightarrow[G]{\sigma} q_{out}(\mathbf{c}_{out})$ for some σ , then there exists a word $\sigma' \in \mathbf{A}^*$ such that $q_{in}(\mathbf{c}_{in}) \xrightarrow[G]{\sigma'} q_{out}(\mathbf{c}_{out})$ and*

$$|\sigma'| \leq \ell(h^{\omega^{d+1}}(g(n))) ,$$

where g , h , and ℓ are defined in lemmata 4.17 to 4.19.

5.2. Fast-Growing Complexity. We wish now to exploit the upper bounds from (16–18) and Pty. 5.2 to provide complexity upper bounds for the decomposition algorithm and the reachability problem. We will employ for this the *fast-growing* complexity classes defined in [28]. This is an ordinal-indexed hierarchy of complexity classes $(F_\alpha)_{\alpha < \varepsilon_0}$, that uses the Hardy functions $(H^\alpha)_\alpha$ relative to $H(x) \stackrel{\text{def}}{=} x + 1$ as a standard against which we can measure high complexities.

5.2.1. *Fast-Growing Complexity Classes.* Let us first define

$$\mathcal{F}_\alpha \stackrel{\text{def}}{=} \bigcup_{\beta < \omega^\alpha} \text{FDTIME}(H^\beta(n)) \quad (20)$$

as the class of functions computed by deterministic Turing machines in time $O(H^\beta(n))$ for some $\beta < \omega^\alpha$. This captures for instance the class of Kalmar elementary functions as \mathcal{F}_3 and the class of primitive-recursive functions as \mathcal{F}_ω [20, 30]. Then we let

$$\mathbf{F}_\alpha \stackrel{\text{def}}{=} \bigcup_{p \in \mathcal{F}_\alpha} \text{DTIME}(H^{\omega^\alpha}(p(n))) \quad (21)$$

denote the class of decision problems solved by deterministic Turing machines in time $O(H^{\omega^\alpha}(p(n)))$ for some function $p \in \mathcal{F}_\alpha$. The intuition behind this quantification over p is that, just like e.g. $\text{EXP} = \bigcup_{p \in \text{poly}} \text{DTIME}(2^{p(n)})$ quantifies over polynomial functions to provide enough ‘wigggle room’ to account for polynomial reductions, \mathbf{F}_α is closed under \mathcal{F}_α reductions [28, Thms. 4.7 and 4.8].

For instance, $\text{TOWER} \stackrel{\text{def}}{=} \mathbf{F}_3$ defines the class of problems that can be solved using computational resources bounded by a tower of exponentials of elementary height in the size of the input, $\bigcup_{k \in \mathbb{N}} \mathbf{F}_k$ is the class of primitive-recursive decision problems, and $\text{ACKERMANN} \stackrel{\text{def}}{=} \mathbf{F}_\omega$ is the class of problems that can be solved using computational resources bounded by the Ackermann function applied to some primitive-recursive function of the input size—here it does not matter for $\alpha > 2$ whether we are considering deterministic, nondeterministic, alternating, time, or space bounds [28, Sec. 4.2.1]. See Fig. 2 for a depiction.

5.2.2. *Complexity Upper Bounds.* Let us first observe that, by Lem. 4.18, the branching degree $|\text{dec}(\xi)|$ of a node labelled by ξ in a decomposition forest for ξ_0 is exponential in $h(|\xi|)$, thus elementary in $|\xi|$. Furthermore, by Lem. 4.17, the number $|\text{clean}(\xi_0)|$ of initial clean KLM sequences is exponential in $g(|\xi_0|)$, thus elementary in $|\xi_0|$. Thus, by (18), the size of the entire forest—i.e., the number of decomposition steps performed by the decomposition algorithm—is also elementary in $h^{\omega^{d+1}}(g(|\xi_0|))$. Finally, still by Lem. 4.18, each decomposition step on a KLM sequence ξ can be performed in time elementary in $|\xi|$, hence the entire decomposition forest can be computed in time elementary in $h^{\omega^{d+1}}(g(|\xi_0|))$.

Lemma 5.3. *Given a KLM sequence ξ_0 of dimension d , we can compute $\text{fdec}(\xi_0)$ in time $e(h^{\omega^{d+1}}(g(|\xi_0|)))$ for some fixed elementary function e and where g and h are defined in lemmata 4.17 and 4.18.*

Consider an instance of the VASS reachability problem, namely a VASS G of dimension d and two finite configurations \mathbf{c}_{in} and \mathbf{c}_{out} , and let $\xi_0 \stackrel{\text{def}}{=} (\mathbf{c}_{in} G \mathbf{c}_{out})$. Then $\text{fdec}(\xi_0) = \emptyset$ if and only if $q_{in}(\mathbf{c}_{in}) \xrightarrow{*}_G q_{out}(\mathbf{c}_{out})$, where by (19), $|\xi_0|$ is elementary in the size of the instance. Let us examine the bound $e(h^{\omega^{d+1}}(g(|\xi_0|)))$ from Lem. 5.3 and express it in the form of (21). The innermost g function composed with the blow-up incurred by (19) is a fixed elementary function in $\mathcal{F}_{<3}$, thus is captured by the quantification over $p \in \mathcal{F}_{<\alpha}$ for all $\alpha \geq 3$. The inner h function is also fixed and in $\mathcal{F}_{<3}$, and [27, Thm. 4.2] allows to over-approximate $h^{\omega^{d+1}}$ in terms of $H^{\omega^{d+4}}$. Finally, the outermost function e is also fixed and in $\mathcal{F}_{<3}$, and [27, Lem. 4.6] shows how to ‘shift’ it into the innermost position.

Theorem 5.4 (Upper Bound). *VASS reachability is in ACKERMANN, and in F_{d+4} if the dimension d is fixed.*

5.2.3. Combinatorial Algorithm. An alternative proof of Thm. 5.4 could also exploit the following *combinatorial algorithm*. By Pty. 5.2, if $q_{in}(\mathbf{c}_{in}) \xrightarrow{\sigma}_G q_{out}(\mathbf{c}_{out})$ for some σ , then there is a small witness σ' of length at most $\ell(h^{\omega^{d+1}}(g(n)))$. It suffices therefore to compute this upper bound—which can be performed in time elementary in the bound [27, Thm. 5.1]—, and to enumerate the paths in G of length up to that bound until we find a witness or exhaust the search space.

6. APPLICATION: DOWNWARD LANGUAGE INCLUSION

The $\text{ACKERMANN} = F_\omega$ upper bound provided by Thm. 5.4 for the VASS reachability problem is still quite far from the currently best known lower bound, which is $\text{TOWER} = F_3$ hardness [6]. As mentioned in the introduction, this upper bound is nevertheless rather tight as far as the decomposition algorithm is concerned. In this section, we illustrate the usefulness of our new upper bound for another decision problem.

Labelled VASSes. A *labelled VASS* (G, Σ, λ) is a VASS $G = (Q, q_{in}, q_{out}, T)$ of dimension d together with a finite alphabet Σ and labelling function $\lambda: T \rightarrow \Sigma \cup \{\varepsilon\}$, which is lifted homomorphically to a function $T^* \rightarrow \Sigma^*$. We overload the notations for step relations by writing $p(\mathbf{x}) \xrightarrow{w}_G q(\mathbf{y})$ if there exists a path $\pi \in T^*$ from p to q labelled by σ such that $p(\mathbf{x}) \xrightarrow{\sigma}_G q(\mathbf{y})$ and $\lambda(\pi) = w \in \Sigma^*$. Given two finite configurations \mathbf{c}_{in} and \mathbf{c}_{out} in \mathbb{N}^d , its *labelled language* is

$$L_\lambda(\mathbf{c}_{in}, G, \mathbf{c}_{out}) \stackrel{\text{def}}{=} \{w \in \Sigma^* \mid q_{in}(\mathbf{c}_{in}) \xrightarrow{w}_G q_{out}(\mathbf{c}_{out})\}.$$

Downward-Closures. For two finite words u and v in Σ^* , we say that u *embeds* into v , denoted $u \leq_* v$, if $u = a_1 \cdots a_k$ and $v = v_0 a_1 v_1 a_2 \cdots a_k v_k$ for some $a_1, \dots, a_k \in \Sigma$ and $v_0, \dots, v_k \in \Sigma^*$. In other words, u embeds into v if we can obtain u from v by ‘dropping’ some letters from v ; for instance, $bca \leq_* aabacba$. For a language $L \subseteq \Sigma^*$, its *downward-closure* is $\downarrow L \stackrel{\text{def}}{=} \{u \in \Sigma^* \mid \exists v \in L. u \leq_* v\}$. A consequence of Higman’s Lemma also known as Haine’s Theorem is that, for any $L \subseteq \Sigma^*$, $\downarrow L$ is a regular language.

Example 6.1. Let us consider again the VASS G_{ex} from Ex. 2.1, along with the alphabet $\Sigma \stackrel{\text{def}}{=} \{a_j \mid 1 \leq j \leq 9\}$ and the labelling function defined by $\lambda(t_j) \stackrel{\text{def}}{=} a_j$ for all $1 \leq j \leq 9$. Then $\downarrow L_\lambda((0, 0, 2), G_{\text{ex}}, (1, 1, 0))$ is the language denoted by the regular expression

$$a_1^*(a_3 + \varepsilon)a_6^*(a_7 + \varepsilon)a_8^*(a_9 + \varepsilon)(a_6 + \varepsilon). \quad \square$$

Downward Language Inclusion. We are interested in this section in the following decision problem.

Problem: VASS downward language inclusion.

input: Two labelled VASSes (G, Σ, λ) and (G', Σ, λ') and four finite configurations \mathbf{c}_{in} and \mathbf{c}_{out} of G and \mathbf{c}'_{in} and \mathbf{c}'_{out} of G' .

question: Is $\downarrow L_\lambda(\mathbf{c}_{in}, G, \mathbf{c}_{out}) \subseteq \downarrow L_{\lambda'}(\mathbf{c}'_{in}, G', \mathbf{c}'_{out})$?

Now, by Haine’s Theorem, $\downarrow L_\lambda(\mathbf{c}_{in}, G, \mathbf{c}_{out})$ is regular for any labelled VASS. However, that does not necessarily mean that one can actually compute a finite automaton \mathcal{A} such that $L(\mathcal{A}) = \downarrow L_\lambda(\mathbf{c}_{in}, G, \mathbf{c}_{out})$ from the labelled VASS and

configurations. Nevertheless, Habermehl, Meyer, and Wimmel [10, Prop. 1] show that, given a full decomposition $\text{fdec}(\xi_0)$ of the KLM sequence $\xi_0 = (\mathbf{c}_{in}G\mathbf{c}_{out})$, one can construct such a finite automaton in logarithmic space¹—as a hint, the reader might see some resemblance between the regular expression of Ex. 6.1 and the full decomposition $\text{fdec}(\xi_{ex}) = \{\xi_{ex}^{11}, \xi_{ex}^{12}\}$ from Ex. 4.16. Since the inclusion problem for two regular languages represented by finite automata is in PSPACE, Lem. 5.3 entails the following.

Corollary 6.2 (of [10, Prop. 1]). *The VASS downward language inclusion problem is in ACKERMANN, and in F_{d+4} if the dimension of the labelled VASSes is fixed to d .*

Lower Bounds. The computational and the descriptive complexity of computing downward-closures of languages is rather well studied [e.g., 31]. In the case of labelled VASS languages, Atig et al. [1, Thm. 10] show that there exists a family of labelled VASSes such that any finite automaton \mathcal{A} such that $L(\mathcal{A}) = \downarrow L_\lambda(\mathbf{c}_{in}, G, \mathbf{c}_{out})$ requires a number of states Ackermannian in the size of G . A stronger result was shown by Zetsche [32, Cor. 17], which bars any alternative algorithm for the VASS downward language inclusion problem from performing significantly better than Cor. 6.2.

Theorem 6.3 ([32, Cor. 17]). *The VASS downward language inclusion problem is ACKERMANN-hard, and F_d -hard if the dimension of the labelled VASSes is fixed to $d \geq 3$.*

Proof. The lower bound in the general case is stated in [32, Cor. 17]. Regarding the case in fixed dimension $d \geq 1$, [32, Thm. 15] shows how to derive F_d -hardness provided we can ‘weakly implement’ the Ackermann function $A_d(x)$ with a VASS of dimension d and size polynomial in d , where $A_d(x)$ is defined inductively by $A_1(x) \stackrel{\text{def}}{=} 2x$ and $A_{d+1}(x) \stackrel{\text{def}}{=} A_d^x(1)$. The existence of such weak implementations is well-known; see for instance [29, Sec. 4.2.2]. \square

7. CONCLUDING REMARKS

We have proven that a refinement of the decomposition algorithms of Mayr [21], Kosaraju [13], and Lambert [14] runs in Ackermannian time, and in primitive-recursive time for VASSes of fixed dimension. In turn, this provides respectively ACKERMANN and F_{d+4} upper bounds for both the VASS reachability and the VASS downward language inclusion problems. While the former only needs to find *some* normal KLM sequence in a decomposition forest and is only known to be TOWER-hard [6], the latter essentially requires to construct a full decomposition and was already known to be ACKERMANN-hard [32], and therefore ACKERMANN-complete by our results. Thus it is unclear at the moment whether a better algorithm for VASS reachability might exist.

Another line for future research is the complexity of VASS reachability in fixed dimension. With a binary encoding, the problem is NP-complete in dimension one [9] and PSPACE-complete in dimension two [2]; with a unary encoding, both are NL-complete [8]. In dimension three and above, our F_{d+4} bound is currently the best known upper bound, but we expect that this could be refined further.

¹The result of Habermehl et al. is stated in terms of a full decomposition constructed by Lambert’s algorithm, but the adaptation to our full decomposition is mostly straightforward.

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APPENDIX A. PUMPABILITY

In this section we provide the details of the proof of Lem. 4.15.

A.1. Rackoff Extraction. We start by proving the following result inspired by Rackoff [24].

Lemma A.1. *Let us assume that*

$$q_0(\mathbf{c}_0) \xrightarrow{G} q_1(\mathbf{c}_1) \cdots \xrightarrow{G} q_k(\mathbf{c}_k).$$

Let $C \geq |G|$ and assume that for every $i \in \{1, \dots, d\}$ there exists $j \in \{0, \dots, k\}$ such that

$$\mathbf{c}_j(i) \geq C^{1+d^d}.$$

In that case there exists a configuration $\mathbf{c} \in \mathbb{N}_\omega^d$ such that $\mathbf{c}(i) \geq C - |G|$ for every $i \in \{1, \dots, d\}$, and a word σ such that $|\sigma| < C^{(d+1)^{d+1}}$ and $q_0(\mathbf{c}_0) \xrightarrow{\sigma} q_k(\mathbf{c}_k)$.

The previous lemma is a direct consequence of the following statement.

Lemma A.2. *Let us assume that*

$$q_0(\mathbf{c}_0) \xrightarrow{G} q_1(\mathbf{c}_1) \cdots \xrightarrow{G} q_k(\mathbf{c}_k).$$

Let $C \geq |G|$ be such that for every $i \in \{1, \dots, d\}$ there exists $j \in \{0, \dots, k\}$ such that

$$\mathbf{c}_j(i) \geq C^{1+n^n}$$

where $n \stackrel{\text{def}}{=} |\{i \mid \mathbf{c}_0(i) \in \mathbb{N}\}|$. Then, there exists a configuration \mathbf{c} and a word σ such that $q_0(\mathbf{c}_0) \xrightarrow{\sigma} q_k(\mathbf{c}_k)$, $\mathbf{c}(i) \geq C - |G|$ for every i , and

$$|\sigma| < C^{(n+1)^{n+1}}.$$

Proof. Let T be the set of transitions of G . Notice that if T is empty, the lemma is immediate. So, we can assume that $T \neq \emptyset$, and in particular that $|G| \geq 2$.

We prove the lemma by induction on n . Naturally, for the base case $n = 0$, then $\mathbf{c}_0 = \omega$ and the proof is immediate by selecting for σ the label of a simple path from q_0 to q_k ; notice that $|\sigma| < |Q| \leq C$. For the induction step, let us assume that the lemma holds as soon as the cardinal of $H \stackrel{\text{def}}{=} \{i \in \{1, \dots, d\} \mid \mathbf{c}_0(i) \neq \omega\}$ is strictly bounded by $n \geq 1$, and let us consider an instance of the lemma such that $|H| = n$. We have by assumption

$$q_0(\mathbf{c}_0) \xrightarrow{a_1} q_1(\mathbf{c}_1) \cdots \xrightarrow{a_k} q_k(\mathbf{c}_k). \quad (22)$$

Assume that for every $i \in \{1, \dots, d\}$ there exists $j \in \{0, \dots, k\}$ such that $\mathbf{c}_j(i) \geq C^{1+n^n}$.

Notice that for every $i \in H$, there exists a minimal $k_i \in \{0, \dots, k\}$ such that $\mathbf{c}_{k_i}(i) \geq C^{1+n^n}$. Let $\hat{k} \stackrel{\text{def}}{=} \min\{k_i \mid i \in H\}$. Observe that for every $j \in \{0, \dots, \hat{k} - 1\}$ and for every $i \in H$, by minimality of k_i we deduce that $\mathbf{c}_j(i) < C^{1+n^n}$. It follows that the cardinal of the set $\{q_j(\mathbf{c}_j) \mid 0 \leq j < \hat{k}\}$ is bounded by $|Q| \cdot (C^{1+n^n})^n \leq C^{1+n+n^{n+1}}$. By removing cycles that occur on the execution from q_0 to $q_{\hat{k}}$, we can assume without loss of generality that $\hat{k} \leq C^{1+n+n^{n+1}}$.

Let $I \stackrel{\text{def}}{=} \{i \in H \mid k_i > \hat{k}\}$. Observe that $\hat{n} \stackrel{\text{def}}{=} |I|$ satisfies $\hat{n} < n$. Let us define $\mathbf{x}_j \stackrel{\text{def}}{=} \mathbf{c}_j|_I$. Then

$$q_{\hat{k}}(\mathbf{x}_{\hat{k}}) \xrightarrow{a_{\hat{k}+1}} q_{\hat{k}+1}(\mathbf{x}_{\hat{k}+1}) \cdots \xrightarrow{a_k} q_k(\mathbf{x}_k). \quad (23)$$

Moreover, for every $i \in \{1, \dots, d\}$ there exists $j \in \{\hat{k}, \dots, k\}$ such that $\mathbf{x}_j(i) \geq C^{1+\hat{n}^{\hat{n}}}$. In fact, if $i \notin I$ then $\mathbf{x}_{\hat{k}}(i) = \omega \geq C^{1+\hat{n}^{\hat{n}}}$, and if $i \in I$, and since $k_i > \hat{k}$, there exists $j \in \{\hat{k} + 1, \dots, k\}$ such that $\mathbf{x}_j(i) \geq C^{1+n^n} \geq C^{1+\hat{n}^{\hat{n}}}$. By induction hypothesis, there exists a configuration $\hat{\mathbf{c}}$ and a word $\hat{\sigma}$ such that $q_{\hat{k}}(\mathbf{x}_{\hat{k}}) \xrightarrow{\hat{\sigma}} q_k(\hat{\mathbf{c}})$, $\hat{\mathbf{c}}(i) \geq C - |G|$ for every i , and $|\hat{\sigma}| < C^{(\hat{n}+1)^{\hat{n}+1}} \leq C^{m^n}$.

Let $\sigma \stackrel{\text{def}}{=} a_1 \dots a_{\hat{k}} \hat{\sigma}$. Observe that $|\sigma| < C^{1+n+n^{n+1}} + C^{m^n}$. Let us prove that $C^{1+n+n^{n+1}} + C^{m^n} \leq C^{(n+1)^{n+1}}$. Since $C \geq 2$, it is sufficient to prove that $(1 + n + n^{n+1}) + n^n \leq (n+1)^{n+1}$. If $n = 1$ the inequality is trivial. Otherwise, the first two elements of the binomial decomposition of $(n+1)^{n+1}$ provide $(n+1)^{n+1} \geq n^{n+1} + (n+1)n^n$. Moreover $(n+1)n^n \geq n^n + n^{n+1} \geq n^n + 2n \geq n^n + n + 1$. We have proven the inequality.

Let $\mathbb{Z}_\omega \stackrel{\text{def}}{=} \mathbb{Z} \uplus \{\omega\}$, and let us introduce for every prefix u of $\hat{\sigma}$ the vector \mathbf{z}_u in \mathbb{Z}_ω^d defined by $\mathbf{z}_u \stackrel{\text{def}}{=} \mathbf{c}_{\hat{k}} + \Delta(u)$. Observe that $\mathbf{z}_u|_I = \mathbf{x}_{\hat{k}} + \Delta(u)$. Since $q_{\hat{k}}(\mathbf{x}_{\hat{k}}) \xrightarrow{\hat{\sigma}} q_k(\hat{\mathbf{c}})$, there exists a configuration $q_u(\hat{\mathbf{z}}_u)$ such that $q_{\hat{k}}(\mathbf{x}_{\hat{k}}) \xrightarrow{u} q_u(\hat{\mathbf{z}}_u)$. In particular $\hat{\mathbf{z}}_u = \mathbf{x}_{\hat{k}} + \Delta(u)$, and we have proven that $\mathbf{z}_u|_I = \hat{\mathbf{z}}_u$. It follows that $\mathbf{z}_u(i) \geq 0$ for every $i \in I$. Notice that for every $i \notin H$, we have $\mathbf{z}_u(i) = \omega \geq 0$. Moreover, for every $i \in H \setminus I$, we have $\mathbf{z}_u(i) = \mathbf{c}_{\hat{k}}(i) + \Delta(u)(i) \geq C^{1+n^n} - |G| \cdot |u| \geq C^{1+n^n} - |G| \cdot C^{m^n} = C^{m^n} \cdot (C - |G|) \geq C - |G|$. Hence $\mathbf{z}_u(i) \geq C - |G|$. We have proven that $\mathbf{z}_u \in \mathbb{N}_\omega^d$. In particular, it follows that $q_{\hat{k}}(\mathbf{c}_{\hat{k}}) \xrightarrow{\hat{\sigma}} q_k(\mathbf{c})$ where $\mathbf{c} = \mathbf{z}_{\hat{\sigma}}$. Notice that we have $\mathbf{c}(i) \geq C - |G|$ for every i . In fact, if $i \in H \setminus I$ then $\mathbf{c}(i) = \mathbf{z}_{\hat{\sigma}}(i)$, if $i \in I$ then $\mathbf{c}(i) = \hat{\mathbf{c}}(i)$, and if $i \notin H$ then $\mathbf{c}(i) = \omega$. Finally, as $q_0(\mathbf{c}_0) \xrightarrow{\sigma_0} q_{\hat{k}}(\mathbf{c}_{\hat{k}})$ we deduce that $q_0(\mathbf{c}_0) \xrightarrow{\sigma} q_k(\mathbf{c})$ and we have proven the induction. \square

A.2. Unfoldings. Recall that $\mathbb{N}_B \stackrel{\text{def}}{=} \{0, \dots, B-1, \omega\}$ for any $B \in \mathbb{N}$, and that if $i \in \{1, \dots, d\}$, $r \in \mathbb{N}_B$, and $\mathbf{x}(i) \in \mathbb{N}_B$, then the *forward* (i, B, r) -*unfolding* of a KLM triple $\mathbf{x}G\mathbf{y}$ is the KLM triple $\mathbf{x}G'\mathbf{y}$ where $G' \stackrel{\text{def}}{=} (Q \times \mathbb{N}_B, (q_{in}, \mathbf{x}(i)), (q_{out}, r), T')$ and T' is the set of transitions $((p, m), \mathbf{a}, (q, n))$ where $(p, \mathbf{a}, q) \in T$ and $m, n \in \mathbb{N}_B$ satisfy $n = m + \mathbf{a}(i)$ or $(n = \omega \wedge m + \mathbf{a}(i) \geq B)$, and such that $m = \omega$ implies $q \neq q_{in}$. The *backward* (i, B, r) -*unfolding* is defined symmetrically.

Lemma 4.13. *Let $\xi = \mathbf{x}G\mathbf{y}$ be a KLM sequence and let I be the set of components $i \in \{1, \dots, d\}$ that are not fixed by G and such that $\text{Facc}_G(\mathbf{x})(i) < \omega$. If I is not empty, then there exists $i \in I$ such that $L_\xi = \bigcup_{r \in \mathbb{N}_B} L_{\xi_r}$ where ξ_r is the forward (i, B, r) -unfolding of ξ and $B \stackrel{\text{def}}{=} (\|\mathbf{x}\| + 2|G|)^{1+d^d}$.*

Proof. The inclusion of the right hand side to the left hand side is immediate. Let us prove the other inclusion.

Let $\sigma \stackrel{\text{def}}{=} \mathbf{a}_1 \dots \mathbf{a}_k$ be a word in L_ξ . Then there exists a sequence $q_0(\mathbf{m}_0), \dots, q_k(\mathbf{m}_k)$ of state-configurations such that $q_0 = q_{in}$, $q_k = q_{out}$, $\mathbf{m}_0, \mathbf{m}_k \in \mathbb{N}^d$, $\mathbf{m}_0 \sqsubseteq \mathbf{x}$, $\mathbf{m}_k \sqsubseteq \mathbf{y}$, and such that

$$q_0(\mathbf{m}_0) \xrightarrow[G]{\mathbf{a}_1} q_1(\mathbf{m}_1) \cdots \xrightarrow[G]{\mathbf{a}_k} q_k(\mathbf{m}_k).$$

Without loss of generality, by restricting the set of states of G to $\{q_0, \dots, q_k\}$, we can assume that every state of G is visited. Let us prove that for every component i fixed by G such that $\text{Facc}_G(\mathbf{x})(i) \neq \omega$, there exists a function $f_i: Q \rightarrow \mathbb{N}$ such that $f(q_{in}) = \text{Facc}_G(\mathbf{x})(i)$ and such that $f_i(q) = f_i(p) + \mathbf{a}(i)$ for every $(p, \mathbf{a}, q) \in T$. Since i is fixed by G , there exists a function $f_i: Q \rightarrow \mathbb{Z}$ such that $f_i(q) = f_i(p) + \mathbf{a}(i)$ for every $(p, \mathbf{a}, q) \in T$. As $\text{Facc}_G(\mathbf{x})(i) \in \mathbb{N}$, by translating f_i we can assume that $f_i(q_{in}) = \text{Facc}_G(\mathbf{x})(i)$. Notice that $\mathbf{m}_0 \sqsubseteq \mathbf{x} \sqsubseteq \text{Facc}_G(\mathbf{x})$. It follows that $\mathbf{m}_0(i) = \text{Facc}_G(\mathbf{x})(i)$ and in particular that $f_i(q_0) = \mathbf{m}_0(i)$. Because $q_{j-1}(\mathbf{m}_{j-1}) \xrightarrow[G]{\mathbf{a}_j} q_j(\mathbf{m}_j)$, we deduce by induction on j that $f_i(q_j) = \mathbf{m}_j(i)$ for every $0 \leq j \leq k$. As $Q = \{q_0, \dots, q_k\}$, we deduce that $f_i(q) \in \mathbb{N}$ for every $q \in Q$.

Observe that, if there exists $i \in I$ such that for every $j \in \{0, \dots, k\}$ we have $\mathbf{m}_j(i) < B$, then $\sigma \in L_{\xi_r}$ where $r \stackrel{\text{def}}{=} \mathbf{m}_k(i)$. Thus, we can assume that for every $i \in I$, there exists $j \in \{0, \dots, k\}$ such that $\mathbf{m}_j(i) \geq B$. Let k' be the minimal natural number such that, for every $i \in I$, there exists $j \in \{0, \dots, k'\}$ such that $\mathbf{m}_j(i) \geq B$. Since $\mathbf{m}_0(i) < B$ for every $i \in I$, it follows that $k' \geq 1$. By minimality of k' , there exists $i \in I$ such that for every $j \in \{0, \dots, k' - 1\}$ we have $\mathbf{m}_j(i) < B$. Observe that if $q_{in} \notin \{q_{k'}, \dots, q_k\}$, we deduce that $\sigma \in L_{\xi_\omega}$. So, it just remains to prove that $q_{in} \notin \{q_{k'}, \dots, q_k\}$.

Assume by contradiction that $q_{in} \in \{q_{k'}, \dots, q_k\}$. Since $\mathbf{m}_0 \sqsubseteq \mathbf{x} \sqsubseteq \text{Facc}_G(\mathbf{x})$ and $I \subseteq \{i \mid \text{Facc}_G(\mathbf{x})(i) \in \mathbb{N}\}$, it follows that $\mathbf{m}_0|_I = \text{Facc}_G(\mathbf{x})|_I$. Let $\mathbf{c}_j \stackrel{\text{def}}{=} \mathbf{m}_j|_I$ for every $0 \leq j \leq k$. Notice that we have

$$q_{in}(\text{Facc}_G(\mathbf{x})|_I) = q_0(\mathbf{c}_0) \xrightarrow[G]{\mathbf{a}_1} q_1(\mathbf{c}_1) \cdots \xrightarrow[G]{\mathbf{a}_{k'}} q_{k'}(\mathbf{c}_{k'}). \quad (24)$$

For every $i \in \{1, \dots, d\}$, there exists $j \in \{0, \dots, k'\}$ such that $\mathbf{c}_j(i) \geq C^{1+d^d}$, where $C \stackrel{\text{def}}{=} \|\mathbf{x}\| + 2|G|$. In fact, if $i \notin I$ then $\mathbf{c}_0(i) = \omega$, and if $i \in I$ then there exists $j \in \{0, \dots, k'\}$ such that $\mathbf{m}_j(i) \geq B$; From $\mathbf{c}_j(i) = \mathbf{m}_j(i)$ we are done. Lemma A.1 shows that there exists a configuration $\bar{\mathbf{x}}$ and a word u' such that $q_0(\mathbf{c}_0) \xrightarrow[G]{u'} q_{k'}(\bar{\mathbf{x}})$ and such that $\bar{\mathbf{x}}(i) \geq C - |G|$ for every $1 \leq i \leq d$.

Since $q_{in} \in \{q_{k'}, \dots, q_k\}$, we deduce that there exists a path in G from $q_{k'}$ to q_{in} . We can consider a simple path of that form. Let u'' be the label of that path. Because $|u''| < |Q|$, we know that for every prefix v of u'' we have $\Delta(v)(i) > -|G|$. It follows that $\bar{\mathbf{x}} + \Delta(v) \geq \mathbf{0}$. We have proven that $q_{k'}(\bar{\mathbf{x}}) \xrightarrow{u''}_G q_{in}(\mathbf{z})$ where \mathbf{z} satisfies $\mathbf{z}(i) > C - 2|G|$ for every $1 \leq i \leq d$. As $q_0(\mathbf{c}_0) = q_{in}(\mathbf{x}|_I)$, we also know that $q_{in}(\text{Facc}_G(\mathbf{x})|_I) \xrightarrow{u}_G q_{in}(\mathbf{z})$ where $u = u'u''$. Since $\mathbf{z}(i) > \mathbf{x}(i) = \text{Facc}_G(\mathbf{x})(i)$ for every $i \in I$, we deduce that $\Delta(u)(i) > 0$ for every $i \in I$.

Finally, let us prove that there exists a configuration $\mathbf{x}' \geq \text{Facc}_G(\mathbf{x})$ such that $q_{in}(\text{Facc}_G(\mathbf{x})) \xrightarrow{u}_G q_{in}(\mathbf{x}')$, and such that $\mathbf{x}'(i) > \text{Facc}_G(\mathbf{x})(i)$ for every $i \in I$. Let v be a prefix of u and let us first prove that $\text{Facc}_G(\mathbf{x})(i) + \Delta(v)(i) \geq 0$ for every $1 \leq i \leq d$. Note that if $i \in I$, then because $q_{in}(\text{Facc}_G(\mathbf{x})|_I) \xrightarrow{u}_G q_{in}(\mathbf{z})$, we get the property. If $\text{Facc}_G(\mathbf{x})(i) = \omega$ then $\text{Facc}_G(\mathbf{x})(i) + \Delta(v)(i) \geq 0$ is immediate. Therefore we can assume that $\text{Facc}_G(\mathbf{x})(i) \neq \omega$ and $i \notin I$. In that case i is fixed by G . We have seen in that case that there exists a function $f_i: Q \rightarrow \mathbb{N}$ such that $f_i(q_{in}) = \text{Facc}_G(\mathbf{x})(i)$ and $f_i(q) = f_i(p) + \mathbf{a}(i)$ for every $(p, \mathbf{a}, q) \in T$. We deduce that $\text{Facc}_G(\mathbf{x})(i) + \Delta(u)(i) = f_i(q) \geq 0$ where q is any state reachable from q_{in} by a path labelled by v . It follows that $q_{in}(\text{Facc}_G(\mathbf{x})) \xrightarrow{u}_G q_{in}(\mathbf{x}')$ for $\mathbf{x}' \stackrel{\text{def}}{=} \text{Facc}_G(\mathbf{x}) + \Delta(u)$. Notice that for every $i \notin I$ this shows that $\mathbf{x}'(i) = \text{Facc}_G(\mathbf{x})(i)$, and that for every $i \in I$ and because $\Delta(u)(i) > 0$, we have $\mathbf{x}'(i) > \text{Facc}_G(\mathbf{x})(i)$.

By monotony, notice that there exist a word σ and a configuration $\mathbf{x}'' \geq \mathbf{x}$ such that $q_{in}(\mathbf{x}) \xrightarrow{\sigma}_G q_{in}(\mathbf{x}'')$ and such that $\mathbf{x}''(i) > \mathbf{x}(i)$ for every i such that $\mathbf{x}(i) \in \mathbb{N}$ and $\text{Facc}_G(\mathbf{x})(i) = \omega$. It follows that, for every $n \in \mathbb{N}$ large enough, there exists a configuration $\mathbf{x}_n \geq \mathbf{x}$ such that $q_{in}(\mathbf{x}) \xrightarrow{\sigma^n u}_G q_{in}(\mathbf{x}_n)$. Notice that for n large enough, we have $\mathbf{x}_n \geq \mathbf{x}$. Moreover, we have $\mathbf{x}_n(i) > \mathbf{x}(i)$ for every $i \in I$. Hence $\text{Facc}_G(\mathbf{x})(i) = \omega$ for every $i \in I$ and we get a contradiction. \square

APPENDIX B. UNBOUNDED EQUATIONS

We recall some elements of linear algebra adapted from [23].

Lemma B.1 (corollary of [23, Thm. 1]). *Let $(a_{i,j})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$ be a sequence of integers and let $\mathbf{c} \in \mathbb{Z}^m$ and let us define two sets \mathbf{X} and \mathbf{X}_0 by*

$$\mathbf{X} \stackrel{\text{def}}{=} \left\{ \mathbf{x} \in \mathbb{N}^n \mid \bigwedge_{i=1}^m \sum_{j=1}^n a_{i,j} \mathbf{x}(j) = \mathbf{c}(i) \right\}, \quad \mathbf{X}_0 \stackrel{\text{def}}{=} \left\{ \mathbf{x} \in \mathbb{N}^n \mid \bigwedge_{i=1}^m \sum_{j=1}^n a_{i,j} \mathbf{x}(j) = 0 \right\}.$$

Then every vector in \mathbf{X} can be decomposed as the sum of a vector \mathbf{x} in \mathbf{X} and a finite sum of vectors \mathbf{x}_0 in \mathbf{X}_0 such that:

$$\|\mathbf{x}\| \leq \|\mathbf{c}\| \cdot (2 + \max_{1 \leq i \leq m} \sum_{j=1}^n |a_{i,j}|)^m, \quad \|\mathbf{x}_0\| \leq (2 + \max_{1 \leq i \leq m} \sum_{j=1}^n |a_{i,j}|)^m.$$

Proof. Observe that if $\mathbf{c}(i) < 0$ for some $1 \leq i \leq m$, by replacing $\mathbf{c}(i)$ by $-\mathbf{c}(i)$ and $a_{i,j}$ by $-a_{i,j}$ for every $1 \leq j \leq n$, we do not modify the sets \mathbf{X} and \mathbf{X}_0 . So, without loss of generality, we can assume that $\mathbf{c} \in \mathbb{N}^m$.

We call P the set of pairs $(\mathbf{u}, \mathbf{v}) \in \mathbb{N}^n \times \mathbb{N}^m$ such that $\bigwedge_{i=1}^m \sum_{j=1}^n a_{i,j} \mathbf{v}(j) = \mathbf{u}(i)$. We denote by P_0 the set of minimal (for \leq) non-zero pairs in P . In [23, Theorem 1]

it is shown that every pair in P is a finite sum of pairs in P_0 , and moreover, every pair (\mathbf{u}, \mathbf{v}) in P_0 satisfies $\|\mathbf{u}\| + \|\mathbf{v}\| \leq C$ where

$$C \stackrel{\text{def}}{=} (1 + \max_{1 \leq i \leq m} (\sum_{j=1}^n |a_{i,j}| + 1))^m.$$

Let $\mathbf{x}' \in \mathbf{X}$. Since $(\mathbf{c}, \mathbf{x}')$ is in P , it can be decomposed as a finite sum of pairs in P_0 . On the one hand, notice that the pairs (\mathbf{u}, \mathbf{v}) with $\mathbf{u} = \mathbf{0}$ provides us with vectors $\mathbf{v} \in \mathbf{X}_0$ satisfying $\|\mathbf{v}\| \leq C$. On the other hand, notice that the decomposition of (\mathbf{v}, \mathbf{x}) cannot contains more that $\|\mathbf{c}\|$ pairs (\mathbf{u}, \mathbf{v}) with $\mathbf{u} \neq \mathbf{0}$. Notice that such a pair satisfies $\|\mathbf{v}\| \leq C - 1$. It follows that those pairs sum up to a pair (\mathbf{c}, \mathbf{x}) in P such that $\|\mathbf{x}\| \leq \|\mathbf{c}\| \cdot (C - 1)$. As $(\mathbf{c}, \mathbf{x}) \in P$, it follows that $\mathbf{x} \in \mathbf{X}$. This concludes the proof. \square

Corollary B.2. *Every model \mathbf{h} of E_ξ can be decomposed as the sum of a model \mathbf{h}' of E_ξ and a finite sum of models \mathbf{h}_0 of E_ξ^0 such that*

$$\|\mathbf{h}'\| \leq |\xi|^{|\xi|-1}, \quad \|\mathbf{h}_0\| \leq |\xi|^{|\xi|-3}.$$

Proof. Just apply Lem. B.1 where $(a_{i,j})_{i,j}$ corresponds to the coefficients occurring in the characteristic system of ξ in front of variables, and \mathbf{c} corresponds to the constant terms. Observe that $2 + \max_{1 \leq i \leq m} \sum_{j=1}^n |a_{i,j}| \leq |\xi|$, $m \leq |\xi| - 3$, and $\|\mathbf{c}\| \leq |\xi|^2$. \square

Lemma 3.7. *Assume that $\xi = (\mathbf{x}_0 G_0 \mathbf{y}_0) \mathbf{a}_1 \dots (\mathbf{x}_k G_k \mathbf{y}_k)$ is satisfiable. Then for every $0 \leq j \leq k$ we have:*

- *For every $1 \leq i \leq d$, the set of values $\mathbf{m}_j^{\mathbf{h}}(i)$ where \mathbf{h} is a model of E_ξ is unbounded if, and only if, there exists a model \mathbf{h}_0 of E_ξ^0 such that $\mathbf{m}_j^{\mathbf{h}_0}(i) > 0$.*
- *For every $t \in T_j$, the set of values $\phi_j^{\mathbf{h}}(t)$ where \mathbf{h} is a model of E_ξ is unbounded if, and only if, there exists a model \mathbf{h}_0 of E_ξ^0 such that $\phi_j^{\mathbf{h}_0}(t) > 0$.*
- *For every $1 \leq i \leq d$, the set of values $\mathbf{n}_j^{\mathbf{h}}(i)$ where \mathbf{h} is a model of E_ξ is unbounded if, and only if, there exists a model \mathbf{h}_0 of E_ξ^0 such that $\mathbf{n}_j^{\mathbf{h}_0}(i) > 0$.*

Moreover, the sum of the bounded values of E_ξ is bounded by $|\xi|^{|\xi|-1}$.

Proof. This is a direct consequence of Cor. B.2 by observing that if \mathbf{h} is a model of E_ξ and \mathbf{h}_0 is a model of E_ξ^0 then $\mathbf{h} + n\mathbf{h}_0$ is a model of E_ξ for every $n \in \mathbb{N}$. \square

APPENDIX C. NORMAL KLM SEQUENCES

In this section, we prove Lem. 4.19. Throughout this appendix, we assume that ξ denotes a *normal* KLM sequence of the form $(\mathbf{x}_0 G_0 \mathbf{x}_1) \mathbf{a}_1 \dots (\mathbf{x}_k, G_k, \mathbf{y}_k)$, where $G_j = (Q_j, q_{in,j}, q_{out,j}, T_j)$ for each $0 \leq j \leq k$.

C.1. Models of Normal KLM Sequences.

Claim C.1. *There exists a model \mathbf{h} of E_ξ such that $\phi_j^{\mathbf{h}}(t) > 0$ for every $t \in T_j$, and such that $\|\mathbf{h}\| \leq 2|\xi|^{|\xi|-1}$.*

Proof. As ξ is satisfiable, there exists a model \mathbf{h} of E_ξ . By decomposing \mathbf{h} thanks to Cor. B.2, we can assume that $\|\mathbf{h}\| \leq |\xi|^{|\xi|-1}$. Moreover, since ξ is unbounded, Cor. B.2 shows that for every $0 \leq j \leq k$ and for every $t \in T_j$, there exists a model \mathbf{h}_0 of E_ξ^0 such that $\|\mathbf{h}_0\| \leq |\xi|^{|\xi|-3}$ and such that $\phi_j^{\mathbf{h}_0}(t) > 0$. By adding to \mathbf{h} , at most $\sum_{j=0}^k |T_j|$ models of E_ξ^0 , we get a model of E_ξ satisfying the claim. \square

Claim C.2. *There exists a model \mathbf{h}_0 of E_ξ^0 such that for every $0 \leq j \leq k$, for every $1 \leq i \leq d$, and for every $t \in T_j$,*

- if $\mathbf{x}_j(i) = \omega$ then $\mathbf{m}_j^{\mathbf{h}_0}(i) > 0$,
- if $\mathbf{y}_j(i) = \omega$ then $\mathbf{n}_j^{\mathbf{h}_0}(i) > 0$, and
- $\phi_j^{\mathbf{h}_0}(t) > 0$,

and moreover,

$$\|\mathbf{h}_0\| \leq |\xi|^{|\xi|-2}.$$

Proof. Since ξ is saturated, Cor. B.2 shows that for every $i \in \{1, \dots, d\}$ and $j \in \{0, \dots, k\}$:

- if $\mathbf{x}_j(i) = \omega$ then there exists a model \mathbf{h}_0 of E_ξ^0 such that $\|\mathbf{h}_0\| \leq |\xi|^{|\xi|-3}$ and such that $\mathbf{m}_j^{\mathbf{h}_0}(i) > 0$, and
- if $\mathbf{y}_j(i) = \omega$ then there exists a model \mathbf{h}_0 of E_ξ^0 such that $\|\mathbf{h}_0\| \leq |\xi|^{|\xi|-3}$ and such that $\mathbf{n}_j^{\mathbf{h}_0}(i) > 0$.

Moreover, since ξ is unbounded, Cor. B.2 shows that for every $j \in \{0, \dots, k\}$ and for every $t \in T_j$ there exists a model \mathbf{h}_0 of E_ξ^0 such that $\|\mathbf{h}_0\| \leq |\xi|^{|\xi|-3}$ and such that $\phi_j^{\mathbf{h}_0}(t) > 0$.

It follows that by summing up at most $2d(k+1) + \sum_{0 \leq j \leq k} |T_j|$ models of E_ξ^0 , we get a model \mathbf{h}_0 of E_ξ^0 satisfying the claim. \square

C.2. Flow Functions.

Claim C.3. *For all $0 \leq j \leq k$, there exists a function $F_j: Q_j \rightarrow \mathbb{N}_\omega^d$ such that $F_j(q) = F_j(p) + \mathbf{a}$ for every transition $(p, \mathbf{a}, q) \in T$, and such that $F_j(q_{in,j}) = \text{Facc}_{G_j}(\mathbf{x}_j)$ and $F_j(q_{out,j}) = \text{Bacc}_{G_j}(\mathbf{y}_j)$.*

Proof. It suffices to prove the claim for some KLM triple $\mathbf{x}G\mathbf{y}$ which is pumpable, rigid, and saturated. Let us prove that for every $i \in \{1, \dots, d\}$, there exists a function $f_i: Q \rightarrow \mathbb{N}_\omega$ such that $f_i(q) = f_i(p) + \mathbf{a}(i)$ for every transition $(p, \mathbf{a}, q) \in T$ and such that $f_i(q_{in}) = \text{Facc}_G(\mathbf{x})(i)$ and $f_i(q_{out}) = \text{Bacc}_G(\mathbf{y})(i)$. Let $i \in \{1, \dots, d\}$. Observe that if i is not fixed by G , then $\text{Facc}_G(\mathbf{x})(i) = \omega = \text{Bacc}_G(\mathbf{y})(i)$, and we can let f_i be the constant function mapping to ω . Otherwise, if i is not fixed, then $\text{Facc}_G(\mathbf{x})(i) = \mathbf{x}(i)$ and $\text{Bacc}_G(\mathbf{y})(i) = \mathbf{y}(i)$. Since $\mathbf{x}G\mathbf{y}$ is saturated, we deduce that $\mathbf{x}(i) \in \mathbb{N} \iff \mathbf{y}(i) \in \mathbb{N}$. Note that, if $\mathbf{x}(i) = \omega = \mathbf{y}(i)$, then we can let f_i be the constant function mapping to ω . So, we can assume that $\mathbf{x}(i), \mathbf{y}(i) \in \mathbb{N}$. Since $\mathbf{x}G\mathbf{y}$ is rigid, we deduce that there exists a function $g_i: Q \rightarrow \mathbb{N}$ such that $g_i(q) = g_i(p) + \mathbf{a}(i)$ for every transition $(p, \mathbf{a}, q) \in T$, and such that $g_i(q_{in}) \sqsubseteq \mathbf{x}(i)$ and $g_i(q_{out}) \sqsubseteq \mathbf{y}(i)$. As $\mathbf{x}(i), \mathbf{y}(i) \in \mathbb{N}$, we deduce that $g_i(q_{in}) = \mathbf{x}(i)$ and $g_i(q_{out}) = \mathbf{y}(i)$. It follows that we can let $f_i \stackrel{\text{def}}{=} g_i$ in that case. \square

C.3. Pumping in Normal KLM Sequences. Let us introduce the *acceleration operator* ∇ that maps a pair of configurations $(\mathbf{x}, \mathbf{x}')$ such that $\mathbf{x} \leq \mathbf{x}'$ into the configuration $\mathbf{x}\nabla\mathbf{x}'$ defined for every $1 \leq i \leq d$ by:

$$(\mathbf{x}\nabla\mathbf{x}')(i) \stackrel{\text{def}}{=} \begin{cases} \omega & \text{if } \mathbf{x}(i) < \mathbf{x}'(i) \\ \mathbf{x}(i) & \text{otherwise} \end{cases}$$

Claim C.4. *There exists a sequence $(u_j, v_j)_{0 \leq j \leq k}$ of pairs of words such that $|u_j|, |v_j| < |\xi|^{(d+1)^{d+1}}$, and a sequence $(\mathbf{x}'_j, \mathbf{y}'_j)_{0 \leq j \leq k}$ of pairs of configurations $\mathbf{x}'_j \geq \mathbf{x}_j$ and $\mathbf{y}'_j \geq \mathbf{y}_j$ such that for every $0 \leq j \leq k$:*

- $q_{in,j}(\mathbf{x}_j) \xrightarrow{u_j}_{G_j} q_{in,j}(\mathbf{x}'_j)$ and $\text{Facc}_{G_j}(\mathbf{x}_j) = \mathbf{x}_j \nabla \mathbf{x}'_j$,
- $q_{out,j}(\mathbf{y}'_j) \xrightarrow{v_j}_{G_j} q_{out,j}(\mathbf{y}_j)$ and $\text{Bacc}_{G_j}(\mathbf{y}_j) = \mathbf{y}_j \nabla \mathbf{y}'_j$.

Proof. Consider some triple $\mathbf{x}G\mathbf{y} \stackrel{\text{def}}{=} \mathbf{x}_jG_j\mathbf{y}_j$ for some $0 \leq j \leq k$; since ξ is normal, this triple is pumpable. We just provide a proof for u and \mathbf{x}' since v and \mathbf{y}' can be obtained by symmetry. Let I be the set of components $i \in \{1, \dots, d\}$ such that $\mathbf{x}(i) \in \mathbb{N}$ and $\text{Facc}_G(\mathbf{x})(i) = \omega$. Fix some $i \in I$. Notice that there exists a cycle θ_i on q_{in} labelled by a word σ_i , and a configuration $\mathbf{x}_i \geq \mathbf{x}$ such that $\mathbf{x} \xrightarrow{\sigma_i} \mathbf{x}_i$ and $\mathbf{x}_i(i) > \mathbf{x}(i)$.

Let us prove that for every $n \in \mathbb{N}$, there exists a configuration $\mathbf{c} \geq (n, \dots, n)$ such that $q_{in}(\mathbf{x}|_I) \xrightarrow{*}_G q_{in}(\mathbf{c})$. Notice that $q_{in}(\mathbf{x}|_I) \xrightarrow{\sigma_i}_G q_{in}(\mathbf{x}_i|_I)$. From $\mathbf{x}|_I \leq \mathbf{x}_i|_I$, we deduce that there exists a configuration $\mathbf{c}_i \in \mathbb{N}^d$ such that $\mathbf{x}_i|_I = \mathbf{x}|_I + \mathbf{c}_i$. As $\mathbf{x}|_I(i) < \mathbf{x}_i|_I(i)$, we get $\mathbf{c}_i(i) > 0$. By monotony, notice that we have $q_{in}(\mathbf{x}|_I) \xrightarrow{*}_G q_{in}(\mathbf{c})$ where $\mathbf{c} \stackrel{\text{def}}{=} \mathbf{x}|_I + \sum_{i \in I} n\mathbf{c}_i$. Notice that $\mathbf{c} \geq (n, \dots, n)$.

By selecting n large enough, and letting $C \stackrel{\text{def}}{=} |\xi|$, Lem. A.1 shows that there exists another configuration \mathbf{c} and a word u such that $q_{in}(\mathbf{x}|_I) \xrightarrow{u}_G q_{in}(\mathbf{c})$ and such that $\mathbf{c}(i) \geq C - |G|$ for every i , and such that $|u| < C^{(d+1)^{d+1}}$.

Let us prove that there exists a configuration \mathbf{x}' such that $q_{in}(\mathbf{x}) \xrightarrow{u}_G q_{in}(\mathbf{x}')$. Let v be a prefix of u and let us prove that $\mathbf{x}(i) + \Delta(u)(i) \geq 0$ for every i . If $i \in I$, the previous paragraph provides the bound. If $\mathbf{x}(i) = \omega$, the proof is immediate. If $i \notin I$ and $\mathbf{x}(i) \neq \omega$, then the function F_j introduced in Claim C.3 shows that $\mathbf{x}(i) + \Delta(u)(i) = F_j(q)(i) \geq 0$ where q is the state reached after reading v from $q_{in,j}$. Hence, we have proven the existence of \mathbf{x}' . Notice that $\mathbf{x}'(i) = \mathbf{x}(i)$ if $i \notin I$ and $\mathbf{x}'(i) \geq C - |G| > \mathbf{x}(i)$ for every $i \in I$. We have proven the claim. \square

C.4. Proof of Lemma 4.19.

Lemma 4.19 (Bounded Witness). *From any normal KLM sequence ξ , we can compute in space $O(\ell(|\xi|))$ a word $\sigma \in L_\xi$ such that $|\sigma| \leq \ell(|\xi|)$ where $\ell(x) \stackrel{\text{def}}{=} x^{3x}$.*

Proof. We use the models \mathbf{h} and \mathbf{h}_0 of E_ξ and E_ξ^0 defined in claims C.1 and C.2, and the sequence $(u_j, v_j)_{0 \leq j \leq k}$ and $(\mathbf{x}'_j, \mathbf{y}'_j)_{0 \leq j \leq k}$ defined in Claim C.4.

Now, let ψ_{u_j} be the Parikh image of a cycle in G_j on $q_{in,j}$ labelled by u_j , and let ψ_{v_j} be the Parikh image of a cycle in G_j on $q_{out,j}$ labelled by v_j . We define $\phi_j \stackrel{\text{def}}{=} r\phi_j^{\mathbf{h}_0} - (\psi_{u_j} + \psi_{v_j})$ where $r \stackrel{\text{def}}{=} 2|\xi|^{1+(d+1)^{d+1}}$. Observe that $\phi_j(t) > 0$ for every $t \in T_j$. Moreover, as ϕ_j satisfies the homogeneous Kirchhoff system $K_{G_j}^0$ and G_j is strongly connected, Euler's Lemma shows that there exists a cycle on $q_{in,j}$ labelled by some word w_j with a Parikh image equals to ϕ_j . Notice that $|w_j| =$

$\sum_{t \in T_j} \phi_j(t) \leq r|\xi|^{|\xi|-3}$. Let $s \stackrel{\text{def}}{=} r|\xi|^{|\xi|-2}$, thus such that $|\xi||w_j| \leq s$. In particular $\Delta(w_j) = \Delta(\phi_j) = r\Delta(\phi_j^{h_0}) - (\Delta(u_j) + \Delta(v_j))$. From $\Delta(\phi_j^{h_0}) = \mathbf{n}_j^{h_0} - \mathbf{m}_j^{h_0}$ we deduce

$$\mathbf{n}_j^0 = \mathbf{m}_j^0 + \Delta(w_j) \quad (25)$$

where $\mathbf{m}_j^0 \stackrel{\text{def}}{=} r\mathbf{m}_j^{h_0} + \Delta(u_j)$ and $\mathbf{n}_j^0 \stackrel{\text{def}}{=} r\mathbf{n}_j^{h_0} - \Delta(v_j)$.

Let I_j be the set of components fixed by G_j . Let us prove that $\mathbf{m}_j^0, \mathbf{n}_j^0 \in \mathbb{N}^d$ and $\mathbf{m}_j^0(i), \mathbf{n}_j^0(i) > 0$ for every $i \notin I_j$. Observe that if $i \in I_j$ then $\Delta(u_j)(i) = 0$ since u_j is the label of a cycle and in particular $\mathbf{m}_j^0(i) = r\mathbf{m}_j^{h_0}(i) \geq 0$. If $i \notin I_j$ and $\mathbf{x}_j(i) \in \mathbb{N}$, because $\text{Facc}_{G_j}(\mathbf{x}_j)(i) = \omega$ we know that $\Delta(u_j)(i) > 0$. If $i \notin I_j$ and $\mathbf{x}_j(i) = \omega$, then $\mathbf{m}_j^{h_0}(i) > 0$ and in particular $\mathbf{m}_j^0(i) \geq r + \Delta(u_j)(i) \geq r + \Delta(u_j)(i) \geq 1$ by definition of r and since $|u_j| < |\xi|^{(d+1)^{d+1}}$. We have proven that $\mathbf{m}_j^0 \geq \mathbf{0}$ and $\mathbf{m}_j^0(i) > 0$ for every $i \notin I_j$. Symmetrically, we see that $\mathbf{n}_j^0 \in \mathbb{N}^d$ and $\mathbf{n}_j^0(i) > 0$ for every $i \notin I_j$.

Notice that $\mathbf{x}_j = \mathbf{m}_j^h + \omega\mathbf{m}_j^{h_0}$. Since u_j is fireable from \mathbf{x}_j and $|\xi||u_j| \leq r$, we deduce that u_j is fireable from $\mathbf{m}_j^h + r\mathbf{m}_j^{h_0}$, thus

$$q_{in,j}(\mathbf{m}_j^h + r\mathbf{m}_j^{h_0}) \xrightarrow{G_j} q_{in,j}(\mathbf{m}_j^h + \mathbf{m}_j^0). \quad (26)$$

By monotony, this means that

$$q_{in,j}(\mathbf{m}_j^h + s\mathbf{m}_j^{h_0}) \xrightarrow{G_j} q_{in,j}(\mathbf{m}_j^h + s\mathbf{m}_j^0), \quad (27)$$

and by symmetry, we also have

$$q_{out,j}(\mathbf{n}_j^h + s\mathbf{n}_j^0) \xrightarrow{G_j} q_{out,j}(\mathbf{n}_j^h + s\mathbf{n}_j^{h_0}). \quad (28)$$

Moreover, since $|\xi||w_j| \leq s$ and $\mathbf{m}_j^0(i), \mathbf{n}_j^0(i) > 0$ for every $i \notin I_j$, we deduce

$$q_{in,j}(\mathbf{m}_j^h + s\mathbf{m}_j^0) \xrightarrow{G_j} q_{in,j}(\mathbf{m}_j^h + s\mathbf{n}_j^0). \quad (29)$$

Observe that ϕ_j^h satisfies the Kirchhoff system K_{G_j} and $\phi_j^h(t) > 0$ for every $t \in T_j$. As G_j is strongly connected, Euler's Lemma shows that ϕ_j^h is the Parikh image of a path from $q_{in,j}$ to $q_{out,j}$ labelled by a word σ_j . The function F_j introduced in Claim C.3 shows that we have:

$$q_{in,j}(\text{Facc}_{G_j}(\mathbf{x}_j)) \xrightarrow{G_j} q_{out,j}(\text{Bacc}_{G_j}(\mathbf{y}_j))$$

Notice that $\text{Facc}_{G_j}(\mathbf{x}_j) = \mathbf{m}_j^{veh} + \omega\mathbf{n}_j^0$ and $\text{Bacc}_{G_j}(\mathbf{y}_j) = \mathbf{n}_j^h + \omega\mathbf{n}_j^0$. Since $|\sigma_j| \leq \sum_{t \in T_j} \phi_j^h(t) \leq \|\mathbf{h}\| \leq 2|\xi|^{2+|\xi|}$. It follows that $|\xi||\sigma_j| \leq s$. We deduce:

$$q_{in,j}(\mathbf{m}_j^h + s\mathbf{n}_j^0) \xrightarrow{G_j} q_{out,j}(\mathbf{n}_j^h + s\mathbf{n}_j^0). \quad (30)$$

Thus, for every $0 \leq j \leq k$,

$$q_{in,j}(\mathbf{m}_j^h + s\mathbf{m}_j^{h_0}) \xrightarrow{G_j} q_{out,j}(\mathbf{n}_j^h + s\mathbf{n}_j^{h_0}). \quad (31)$$

This entails that $\sigma \stackrel{\text{def}}{=} (u_0^s w_0^s \sigma_0 v_0^s) \mathbf{a}_1 \dots (u_k^s w_k^s \sigma_k v_k^s)$ is in L_ξ . Notice that $|\sigma| \leq k + (k+1) \cdot s \cdot (2|\xi|^{(d+1)^{d+1}} + 2|\xi|^{|\xi|-1} + r|\xi|^{|\xi|-3}) \leq 7|\xi|^{2(d+1)^{d+1}+2|\xi|-1}$. Observe that $2(d+1)^{d+1} \leq |\xi|$ and $7 \leq |\xi|$. Hence $|\sigma| \leq |\xi|^{3|\xi|}$. \square