AUTOMORPHISM GROUPS OF INFINITE SEMILINEAR ORDERS (II)

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ABSTRACT

This paper and its predecessor examine certain infinite semilinear orders ('trees') and their automorphism groups. Here we classify weakly 2-transitive trees up to $L_{\infty\omega}$-equivalence, and countable weakly 2-transitive trees up to isomorphism. Various results are obtained about the automorphism groups, concerning torsion, divisibility, and subgroups of small index. The automorphism groups of some related treelike structures and their normal subgroup lattices are also examined.

1. *Introduction*

In this paper, which is a sequel to [**8**], we examine further a family of posets and their automorphism groups. Recall from [**8**] the definition of a tree, and in particular of a weakly 2-transitive tree. In [**8**], we examined the lattice of normal subgroups of the automorphism group of a weakly 2-transitive tree. Here we consider other properties of the automorphism group. We shall assume familiarity with the notation and definitions of [**8**].

In § 2 we give a classification by cardinal invariants of weakly 2-transitive trees, together with a construction of all countable such trees. It is shown that there are $2^{\aleph_0}$ countable weakly 2-transitive trees. We understand that S. Adeleke and P. M. Neumann (personal communication) have obtained related results in their work on Jordan groups.

In § 3 we discuss various structural questions about weakly 2-transitive automorphism groups of trees, particularly concerning divisibility and torsion. We also make some observations on the extent to which a weakly 2-transitive tree can be recovered from the abstract structure of its automorphism group. This topic has recently been treated in much more detail by M. Rubin (personal communication).

The final two sections do not require much understanding of the rest of this paper or of [**8**]. In § 4 we consider subgroups of small index of the automorphism groups of countable 2-homogeneous trees, proving that the 'small index property' holds. Some related model-theoretic structures, first described by Cameron, are discussed in § 5. In particular, we examine the lattice of normal subgroups of their automorphism groups, and show that some of the groups are simple.

2. *Countable trees and $L_{\infty\omega}$-equivalence*

Our first goal is to introduce the notions of a *normal* tree and an *almost normal* tree. A classification by triples of cardinal invariants will then be given for almost normal trees.

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For any poset $(P, \leqslant)$ and any $a,b \in P$, we say that b *covers* a if $a<b$ and there is no $x \in P$ with $a<x<b$. If $A,B \subseteq P$, we say that A is *B-dense in P* if for all $x,y \in B$ with $x<y$ there is $a \in A$ with $x<a<y$. Hence A is dense in P if and only if A is P-dense in P. Recall that if T is a tree then T^+ denotes the smallest tree which is a meet-semilattice and contains T. Recall also from § 6 of [8] the definition of a cone.

DEFINITION 2.1. Let T be a tree.

(a) If $a \in \mathrm{ram}(T)$, we let $C(a)$ be the set of all cones at a.

(b) If $a \in \mathrm{ram}(T)$, we say that a is a *special ramification point* of T if a has a cone which has a smallest element, that is, if a is covered in T^+ by some $b \in T$. Let $\mathrm{ram}_s(T)$ denote the set of all special ramification points of T. If $a \in \mathrm{ram}_s(T)$, we let $C_s(a)$ (respectively $C_n(a)$) denote the set of all cones at a with (respectively without) a smallest element.

(c) For each finite or infinite cardinal $k \geqslant 2$, let

$$\mathrm{ram}_k(T) = \{a \in \mathrm{ram}(T)\setminus \mathrm{ram}_s(T)\colon\ |C(a)| = k\}$$

and

$$\mathrm{ram}_\infty(T) = \{a \in \mathrm{ram}(T)\setminus \mathrm{ram}_s(T)\colon\ C(a) \text{ is infinite}\}.$$

DEFINITION 2.2. A tree $(T, \leqslant)$ is called *normal* if the following conditions are satisfied:

(1) T is dense and has no maximal or minimal elements;
(2) if $\mathrm{ram}_s(T) \neq \varnothing$ then for each $y \in T$ there is $x \in \mathrm{ram}_s(T)$ such that y covers x;
(3) if there is $a \in \mathrm{ram}_s(T)$ such that $C_n(a) \neq \varnothing$, then for all $x,y \in T$ with $x<y$ there are $z \in \mathrm{ram}_s(T)$ and a cone $Y \in C_n(z)$ such that $x<z<y$ and $y \in Y$;
(4) $|C_s(a)| = |C_s(b)|$ and $|C_n(a)| = |C_n(b)|$ for all $a,b \in \mathrm{ram}_s(T)$;
(5) either $T \cap \mathrm{ram}(T) = \varnothing$, or $T \subseteq \mathrm{ram}_k(T)$ for some finite or infinite cardinal $k \geqslant 2$;
(6) for each finite or infinite cardinal $k \geqslant 2$, $\mathrm{ram}_k(T)\setminus T$ is either empty or T-dense in T^+.

The proof of the following result is straightforward, and is omitted.

PROPOSITION 2.3. *Every weakly 2-transitive tree is normal.*

For the rest of this section, let $L = \{\leqslant\}$ be the first order language of posets. Recall that the infinitary language $L_{\infty\omega}$ contains all atomic formulas of L and is closed under negation, quantification over finitely many variables, and conjunction and disjunction of arbitrary sets of formulas which have altogether finitely many free variables.

Unfortunately, the class of all normal trees is not closed under $L_{\infty\omega}$-equivalence, as we shall see below. Therefore we now introduce 'almost normal' trees as follows. Let $\mathbb{N} = \{1,2,3,\dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$, and put $\infty + 1 = \infty$ and $i < \infty$ for each $i \in \mathbb{N}_0$.

DEFINITION 2.4. A tree $(T, \leqslant)$ is called *almost normal* if it satisfies Conditions

(1)–(3) of Definition 2.2 and also the following conditions:

(4) (i) either $|C_s(a)| = |C_s(b)| \in \mathbb{N}$ for all $a,b \in \mathrm{ram}_s(T)$, or $C_s(a)$ is infinite for all $a \in \mathrm{ram}_s(T)$;

(ii) either $|C_n(a)| = |C_n(b)| \in \mathbb{N}_0$ for all $a,b \in \mathrm{ram}_n(T)$, or $C_n(a)$ is infinite for all $a \in \mathrm{ram}_n(T)$;

(5) either $T \cap \mathrm{ram}(T) = \varnothing$ or $T \subseteq \mathrm{ram}_n(T)$ for some $2 \leqslant n \in \mathbb{N}_\infty$;

(6) for each $2 \leqslant n \in \mathbb{N}_\infty$, $\mathrm{ram}_n(T) \setminus T$ is either empty or T-dense in T^+.

Clearly any normal tree is almost normal, and for countable trees the converse is also true. If T is a tree such that either $T = \mathrm{ram}(T)$ or $T \cap \mathrm{ram}(T) = \varnothing$ and $\mathrm{ram}(T) = \mathrm{ram}_n(T)$ for some cardinal $n \geqslant 2$, then our present notions of normality and almost normality coincide with those of Droste [6, § 4]. Also, if T is 2-homogeneous, then $A(T)$ operates transitively on $\mathrm{ram}(T)$ and hence $\mathrm{ram}(T) = \mathrm{ram}_n(T)$ for some unique $n \in \mathbb{N}_\infty$. In Proposition 2.13 below, we will show that if T is just weakly 2-transitive, it is possible for any $A \subseteq \mathbb{N}_\infty \setminus \{1\}$ that $A := \{n \in \mathbb{N}_\infty \colon \mathrm{ram}_n(T) \neq \varnothing\}$. Hence there are just countably many countable 2-homogeneous trees, but $2^{\aleph_0}$ countable weakly 2-transitive trees. We now classify almost normal trees by triples of cardinal invariants.

Definition 2.5. (a) We call a triple $t = (m, (i, j), A)$ with $m \in \mathbb{N}_\infty$, i, $j \in \mathbb{N}_\infty \cup \{0\}$, and $A \subseteq \mathbb{N}_\infty \setminus \{1\}$, a *type* if

(i) either $i = j = 0$ or $i \neq 0$ and $i + j \geqslant 2$,

(ii) $m \geqslant 2$ or $i \neq 0$ or $A \neq \varnothing$.

(b) With each almost normal tree $(T, \leqslant)$ we associate a type $t(T) = (m, (i, j), A)$ in the following way:

(i) if $T \cap \mathrm{ram}(T) = \varnothing$ then $m := 1$; otherwise, m is such that $T \subseteq \mathrm{ram}_m(T)$;

(ii) if $\mathrm{ram}_s(T) = \varnothing$ then $i = j = 0$; otherwise, $i = |C_s(a)|$ if this is finite, and $i = \infty$ if it is infinite, and $j = |C_n(a)|$ if this is finite, and $j = \infty$ if it is infinite, where a is any element of $\mathrm{ram}_s(T)$;

(iii) $A = \{k \in \mathbb{N}_\infty \colon k \geqslant 2, \mathrm{ram}_k(T) \setminus T \neq \varnothing\}$.

We will call $t(T)$ the *type* of T, or the *triple of invariants* of T.

Proposition 2.6. *Let $(T_1, \leqslant)$ be an almost normal tree of type $t(T_1) = (m, (i, j), A)$, and let $(T_2, \leqslant)$ be an arbitrary poset. If either $(T_1, \leqslant)$ and $(T_2, \leqslant)$ are $L_{\infty\omega}$-equivalent, or $(T_1, \leqslant)$ and $(T_2, \leqslant)$ are elementarily equivalent and A is a finite set, then T_2 is an almost normal tree with the same type as T_1.*

Proof. This is lengthy but straightforward, and is therefore omitted.

We wish now to prove the converse of Proposition 2.6, namely that any two almost normal trees of the same type are $L_{\infty\omega}$-equivalent. The following remarks hold in a more general model-theoretic setting, but we formulate them only for posets.

If $(P_1, \leqslant)$ and $(P_2, \leqslant)$ are posets, $A \subseteq P_1$, $B \subseteq P_2$, and $\phi\colon A \to B$ is an isomorphism, then ϕ is called a *partial isomorphism* from P_1 into P_2. Suppose that there is a system S of partial isomorphisms from P_1 into P_2 such that whenever $\phi \in S$ and $a \in P_1$ ($b \in P_2$) there is $\psi \in S$ which extends ϕ such that the domain (range) of ψ contains a (b) respectively; then we say that $(P_1, \leqslant)$ and $(P_2, \leqslant)$ are *partially isomorphic* by S. An easy back-and-forth argument shows that any two countable partially isomorphic posets are isomorphic. In general, by a result of Karp [11], two posets are partially isomorphic if and only if they are $L_{\infty\omega}$-equivalent.

Next, let T_1, T_2 be two almost normal trees of the same type, $A \subseteq T_1^+$, $B \subseteq T_2^+$ be two sub meet-semilattices, and ϕ: $A \to B$ be an isomorphism. We say that ϕ is a *good* isomorphism if the following conditions hold:

(1) $(A \cap T_1)\phi = B \cap T_2$;

(2) $(A \cap \mathrm{ram}_s(T_1))\phi = B \cap \mathrm{ram}_s(T_2)$;

(3) $(A \cap \mathrm{ram}_i(T_1))\phi = B \cap \mathrm{ram}_i(T_2)$ for each $2 \leqslant i \in \mathbb{N}_\infty$;

(4) whenever $a_j \in \mathrm{ram}_s(T_j)$, $b_j \in T_j^+$, and $a_j < b_j$ $(j = 1, 2)$, with a_1, $b_1 \in A$, a_2, $b_2 \in B$, $a_1\phi = a_2$, $b_1\phi = b_2$, then b_1 covers a_1 if and only if b_2 covers a_2; furthermore, there is $x \in T$ such that $a_1 < x < b_1$ and x covers a_1 if and only if there is $y \in T$ such that $a_2 < y < b_2$ and y covers a_2.

PROPOSITION 2.7. *Let T_1, T_2 be two almost normal trees of the same type.*

(a) *$(T_1^+, \leqslant)$ and $(T_2^+, \leqslant)$ are partially isomorphic by the system S of all good isomorphisms ϕ: $A \to B$ between finite sub meet-semilattices $A \subseteq T_1^+$ and $B \subseteq T_2^+$.*

(b) *$(T_1, \leqslant)$ and $(T_2, \leqslant)$ are partially isomorphic by the system of all those isomorphisms ϕ: $A_1 \to A_2$ for which $A_i \subseteq T_i$ is finite and ϕ extends to a good isomorphism ψ: $B_1 \to B_2$ where B_i is the smallest sub meet-semilattice of T^+ containing A_i $(i = 1,2)$.*

Proof. (a) Let $A \subseteq T_1^+$, $B \subseteq T_2^+$ be two finite sub meet-semilattices, ϕ: $A \to B$ be a good isomorphism, and $a \in T_1^+ \setminus A$. We claim that there are two finite sub meet-semilattices $C \subseteq T_1^+$, $D \subseteq T_2^+$ and a good isomorphism ψ: $C \to D$ such that $A \cup \{a\} \subseteq C$, $B \subseteq D$, and ψ extends ϕ. This can be shown very similarly to the proof of Droste [6, Theorem 4.4], where the special case when the trees T_1, T_2 are of type $(n, (0, 0), \varnothing)$ or $(1, (0, 0), \{n\})$ for some $2 \leqslant n \in \mathbb{N}_\infty$ was considered. Therefore we consider only the case where $a < A$ in detail, leaving the other cases to the reader.

Thus suppose that $a < A$. Let $m = \inf A$. Then $m \in A$, whence $a < M$, and we put $C = A \cup \{a\}$. Choose $b \in T_2^+$ with $b < \inf B$ such that the following conditions are satisfied:

(i) $b \in T_2\,(\mathrm{ram}_s(T_2), \mathrm{ram}_i(T_2))$ if and only if $a \in T_1(\mathrm{ram}_s(T_1), \mathrm{ram}_i(T_1)$ respectively, where $2 \leqslant i \leqslant \mathbb{N}_\infty)$;

(ii) if $a \in \mathrm{ram}_s(T_1)$, then b is covered by $\inf B$ if and only if a is covered by m, and there is $y \in T_2$ such that $b < y < \inf B$ and y covers b if and only if there is $x \in T_1$ such that $a < x < m$ and x covers a.

Now let $D := B \cup \{b\}$, and define ψ: $C \to D$ by $\psi|_A = \phi$ and $\psi(a) = b$.

(b) This follows immediately from (a).

COROLLARY 2.8. *Let T_1, T_2 be two almost normal trees. Then T_1 and T_2 are $L_{\infty\omega}$-equivalent if and only if they have the same type.*

Proof. This is immediate from Propositions 2.6 and 2.7(b) and Karp's result.

In particular, any two countable almost normal trees of the same type are isomorphic. We also obtain:

COROLLARY 2.9. *Let T be a countable tree. Then the following are equivalent:*
(i) *T is normal;*
(ii) *T is almost normal;*
(iii) *T is weakly 2-transitive.*

Proof. (i)$\Rightarrow$(ii). This is trivial.
(ii)$\Rightarrow$(iii). Let $A_i \subseteq T$ be a chain with $|A_i| = 2$ $(i = 1,2)$ and let ϕ: $A_1 \rightarrow A_2$ be the unique isomorphism. Then ϕ belongs to the system of partial isomorphisms described in Proposition 2.7(b). Hence, as T is countable, ϕ extends to an automorphism of T.
(iii)$\Rightarrow$(i). This is immediate from Proposition 2.3.

Let $(P, \leqslant)$ be a poset. Then $\mathrm{Th}(P)$, the complete first order theory of P, is called *$\aleph_0$-categorical* if any two countable posets which are elementarily equivalent to P are isomorphic. By a theorem of Engeler, Ryll-Nardzewski and Svenonius (cf. Sacks [13]) $\mathrm{Th}(P)$ is $\aleph_0$-categorical if and only if $\mathrm{Aut}(P)$ has finitely many orbits an unordered k-subsets of P for all $k \in \mathbb{N}$.

COROLLARY 2.10. *Let T be a weakly 2-transitive tree of type $(m, (i, j), A)$. Then $\mathrm{Th}(T)$ is $\aleph_0$-categorical if and only if A is finite.*

Proof. If A is finite, the $\aleph_0$-categoricity follows from Propositions 2.6 and 2.7(b). So suppose A is infinite. For each $n \in A$ choose $a_n \in \mathrm{ram}_n(T)\setminus T$ and then b_n, $c_n \in T$ with $b_n \parallel c_n$ and $a_n = \inf\{b_n, c_n\}$. Then the pairs $\{b_n, c_n\}$ $(n \in A)$ all belong to different orbits of $A(T)$, so $\mathrm{Th}(T)$ is not $\aleph_0$-categorical.

We now wish to construct for each type t a countable normal tree T realising this type. In this generality, the construction is quite complicated; for motivation and some simpler cases (types $(n, (0, 0), \varnothing)$ and $(1, (0, 0), \{n\})$ where $2 \leqslant n \in \mathbb{N}_\infty$) we refer to Droste [5, pp. 56–58]. Before describing the construction, we give an example to indicate how to visualise these trees.

EXAMPLE 2.11. *A normal tree $(T, <)$ of type* $(1, (\infty, 0), \varnothing)$. Let C be a dense unbounded chain. Fix $c \in C$, and put $C' = \{x \in C\colon c \leqslant x\}$. By induction, we define a sequence of posets $(A_i, \leqslant)$ $(i \in \mathbb{N})$. Let $(A_1, \leqslant)$ be $(C, \leqslant)$. Now let $i \in \mathbb{N}$, and assume $(A_i, \leqslant)$ is already defined. For each $a \in A_i$, let $C_{a,i}$ be a copy of C'. We put $A_{i+1} = A_i \dot\cup \dot\bigcup(C_{a,i}\colon a \in A_i)$, and let the order of A_{i+1} extend the order of A_i and of each chain $C_{a,i}$ $(a \in A_i)$ such that $z < C_{a,i}$ for each $z \in A_i$ with $z < a$ (and no other relations hold; thus, $a \parallel C_{a,i}$ and $C_{a,i} \parallel C_{b,i}$ for all $a,b \in A_i$ with $a \neq b$). Let $(T, \leqslant) := \bigcup_{i \in \mathbb{N}}(A_i, \leqslant)$. Then $(T, \leqslant)$ is a normal tree with type as prescribed.

CONSTRUCTION 2.12. Let $t = (m, (i, j), A)$ be a type. We put $A^* = A$ if $j = 0$, and $A^* = A \dot\cup \{1\}$ otherwise. Let C be a dense unbounded chain with Dedekind completion $\bar{C}$ such that there are $C_n \subseteq \bar{C}\setminus C$ (for all $n \in A^*$) which are dense in $\bar{C}$. If $i = j = 0$, let $C^+ := C \dot\cup \dot\bigcup(C_n\colon n \in A) \subseteq \bar{C}$, and put $\tilde{C} := \bar{C}$. If $i \neq 0$, let $C^* := \{c^*\colon c \in C\}$ be a copy of $(C, \leqslant)$, put $C^+ := C \dot\cup C^* \dot\cup \dot\bigcup(C_n\colon n \in A^*)$ and

$\tilde{C} := \bar{C} \cup C^*$; also define a linear order $\leqslant$ on $\tilde{C}$ in the natural way so that it extends the orders of $\bar{C}$ and of C^* and $\{z \in \bar{C}\colon z < c\} < c^* < c$ in $\tilde{C}$ for each $c \in C$ (so, c covers c^* in $(\tilde{C}, \leqslant)$). Finally, fix $x \in C$ and $x_n \in C_n$ for each $n \in A^*$.

For each subset $Z \subseteq \tilde{C}$, let $T(Z) = T(C, x, (C_n, x_n)_{n \in A^*}, t, Z)$ be the set of all elements y of the form either $y = (z)$ for some $z \in Z$, or $y = ((a_1, n_1), \ldots, (a_p, n_p); a_{p+1})$ where $p \in \mathbb{N}$, $a_{p+1} \in Z$, and the following conditions hold for each $1 \leqslant k \leqslant p$:

(1) $a_k \in C^+$, and if $m = 1$ then $a_k \notin C$ (note that always $m \geqslant 2$ or $i \neq 0$ or $A^* \neq \varnothing$);
(2) if $a_k \in C$, then $2 \leqslant n_k \leqslant m$ and $x < a_{k+1}$;
(3) if $a_k \in C_n$, then $2 \leqslant n_k \leqslant n$ and $x_n < a_{k+1}$ $(n \in A)$;
(4) if $a_k \in C^*$, then either $2 \leqslant n_k \leqslant i$ and $x \leqslant a_{k+1}$, or $n_k = z^*$ for some $z \in \mathbb{N}$ with $1 \leqslant z \leqslant j$ and $x_1 < a'_{k+1}$;
(5) if $a_k \in C_1$, then either $1 \leqslant n_k \leqslant i$ and $x \leqslant a_{k+1}$, or $n_k = z^*$ for some $z \in \mathbb{N}$ with $2 \leqslant z \leqslant j$ and $x_1 < a_{k+1}$.

For $p = 0$ and $z \in Z$ we also formally write $(z) = ((a_1, n_1), \ldots, (a_p, n_p); z)$.

Next, we define a partial ordering lexicographically on $T(Z)$ as follows. Let $a' = ((a_1, n_1), \ldots, (a_p, n_p); a)$, $b' = ((b_1, m_1), \ldots, (b_q, m_q); b) \in T(Z)$. We put $a' \leqslant b'$ if and only if both the following hold:

(i) $p \leqslant q$ and $a_k = b_k$, $n_k = m_k$ for all $k = 1, \ldots, p$;
(ii) $a \leqslant b$ if $p = q$, and $a \leqslant b_{p+1}$ if $p < q$.

PROPOSITION 2.13. *Under the assumptions of Construction* 2.12, $(T(C), \leqslant)$ *is a normal tree of type t.*

Proof. Let T denote $T(C)$. Clearly T is a dense tree without maximal or minimal elements. As in [**5**, Proposition 6.3] or [**6**, Proposition 3.5], the tree $(T(C^+), \leqslant)$ is the smallest meet-semilattice contained in $T(\tilde{C}, \leqslant)$ and containing T. Hence $T^+ = T(C^+)$.

Next, we prove that if $i \neq 0$, $j \neq 0$, and $a \in T(C^* \cup C_1)$, then $a \in \mathrm{ram}_s(T)$, $|C_s(a)| = i$, and $|C_n(a)| = j$. Suppose first that $a = (s, c^*) \in T(C^*)$ where $c \in C$ and s is a p-tuple of the prescribed form $(p \in \mathbb{N}_0)$. Then

$$M_1 := \{(s, (c^*, n); x)\colon 2 \leqslant n \leqslant i\} \cup \{(s; c)\}$$

is the set of all elements of T^+ covering a. Hence $a \in \mathrm{ram}_s(T)$ and $|C_s(a)| = |M_1| = i$. Choose $b \in C$ with $x_1 < b$. Then the set $M_2 := \{(s, (c^*, z^*); b)\colon 1 \leqslant z \leqslant j\}$ is maximal in the class of all antichains $M' \subseteq T$ such that

(i) $a = \inf S$ for all $S \subseteq M'$ with $|S| = 2$,
(ii) for all $y \in M'$ there is no $w \in T$ with $a < w \leqslant y$ which covers a.

Hence $|C_n| = |M_2| = j$. If $a = (s; c) \in T(C_1)$ with $c \in C_1$, choose $b \in C$ with $\{x_1, c\} < b$. Let

$$M_1 := \{(s, (c, n); x)\colon 1 \leqslant n \leqslant i\}$$

and

$$M_2 := \{(s, (c, z^*); b)\colon 2 \leqslant z \leqslant j\} \cup \{(s, b)\},$$

and argue analogously as before.

In particular, $T(C^* \cup C_1) \subseteq \mathrm{ram}_s(T_1)$ if $i \neq 0$ and $j \neq 0$. A similar argument shows that if $i \neq 0$ and $j = 0$, then $T(C^*) \subseteq \mathrm{ram}_s(T)$ and $|C_s(a)| = i$, $C_n(a) = \varnothing$ for each $a \in T(C^*)$. Furthermore, $T(C_n) \subseteq \mathrm{ram}_n(T) \backslash T$ and $T(C_n)$ is T-dense in T^+ if $n \in A$, and $T \subseteq \mathrm{ram}_m(T)$ if $m \geqslant 2$. Also, $T \cap \mathrm{ram}(T) = \varnothing$ if $m = 1$. Thus $\mathrm{ram}(T) = T^+ \backslash T = T(C^+ \backslash C)$ if $m = 1$, and $\mathrm{ram}(T) = T^+ = T(C^+)$ if $m \geqslant 2$. It follows that the above four inclusions are, in fact, equalities, and also that $\mathrm{ram}_s(T) = \varnothing$ if $i = 0$, and $\mathrm{ram}_n(T) \backslash T = \varnothing$ if $2 \leqslant n \in \mathbb{N}_\infty \backslash A$.

It remains only to check conditions 2.2(2) and 2.2(3). For 2.2(2), let $i \neq 0$ and $y = (s\,; c) \in T$ where $c \in C$ and s is a p-tuple ($p \in \mathbb{N}_0$). Then $x = (s, c^*) \in T(C^*) \subseteq \mathrm{ram}_s(T)$ and y covers x. For 2.2(3), let $i \neq 0$, $j \neq 0$ and $x = (s\,; a)$, $y \in T$ with $x < y$. Then $a \in C$ and y is of the form either $y = (s\,; b)$ or $y = (s, (b, n), \ldots)$ for some $b \in C^+$ with $a < b$, or else $y = (s, (a, n)\,; b)$ or $y = (s, (a, n), (b, n'), \ldots)$ with $x < b \in C^+$ (in this case $m \geqslant 2$). In the first case, choose $c \in C_1$ with $a < c < b$ and put $z = (s\,; c)$. In the second case, choose $c \in C_1$ with $x < c < b$ and let $z = (s, (a, n)\,; c)$. Then, in either case, $z \in T(C_1) \subseteq \mathrm{ram}_s(T)$, $x < z < y$, and any $w \in T$ which covers z is of the form $w = (s, (c, n_1)\,; x)$ or $w = (s, (a, n), (c, n_1)\,; x)$ for some $1 \leqslant n_1 \leqslant i$, and hence satisfies $w \parallel y$. The result follows.

We note that, using Proposition 2.13 and Theorem 3.3 of [**8**], much as in Step I of the argument for [**6**, Theorem 3.8], we can prove the following result which yields weakly 2-transitive trees of arbitrary types and cardinalities. Recall from the end of § 2 of [**8**] the definition of an $\mathcal{S}$-isomorphism between two chains, where $\mathcal{S} = (S_i\colon\ i \in I)$ is a family of sets.

THEOREM 2.14. *Under the assumption of Construction* 2.12, *suppose also that whenever* $a, b, c, d \in C^+$ *all belong to the same one of the chains* C, C_n $(n \in A^*)$ *with* $a < b$ *and* $c < d$, *then the intervals* $[a, b]_{C^+}$ *and* $[c, d]_{C^+}$ *are* $(C, C_n\colon n \in A^*)$-*isomorphic. Then* $T = T(C)$ *is a weakly* 2-*transitive normal tree of type* t *and cardinality* $|T| = |C| + \sum_{n \in A^*} |C_n|$.

We can now prove the main result of this section.

THEOREM 2.15. *Up to isomorphism, there is for each type* t *a unique countable weakly* 2-*transitive tree of type* t.

Proof. The uniqueness comes from Propositions 2.3 and 2.7(b). The existence follows from Proposition 2.13 and Corollary 2.9 by taking, in Construction 2.12, $C = \mathbb{Q}$ and countable subsets $C_n \subseteq \mathbb{R} \backslash \mathbb{Q}$ $(n \in A^*)$ with the prescribed properties.

A result similar to the following has also been proved by Adeleke and Neumann (personal communication).

COROLLARY 2.16. *There are* $2^{\aleph_0}$ *pairwise non-isomorphic countable weakly* 2-*transitive trees.*

REMARK. Corollary 2.16 contrasts with Theorem 6.21 and Corollary 6.23 of [**5**], which show that there are $\aleph_0$ countable 2-homogeneous trees. The countable 2-homogeneous trees are precisely the trees of type $(n, (0, 0), \varnothing)$ or $(1, (0, 0), \{n\})$ for some $2 \leqslant n \in \mathbb{N}_\infty$.

3. *Further algebraic properties of $A(T)$*

Let T be a tree. We consider here questions concerning the divisibility of the group $A(T)$ and the presence of torsion elements.

We first consider divisibility. A subgroup U of a group G will be called *divisible in G* if, whenever $u \in U$ and $n \in \mathbb{N}$, there is $g \in G$ with $u = g^n$.

THEOREM 3.1. *Let T be a weakly 2-transitive tree. Then the following are equivalent:*

(i) *$S(T)$ is divisible in $A(T)$;*
(ii) *$S(T)$ is torsion-free;*
(iii) *$A(T)$ is torsion-free;*
(iv) *each normal subgroup of $A(T)$ is divisible;*
(v) *each ramification point $z \in \mathrm{ram}(T)$ has only pairwise non-isomorphic cones.*

In any of these cases, if $N \trianglelefteq A(T)$, then each element of N is a commutator in N.

Proof. (i)$\Rightarrow$(v), (ii)$\Rightarrow$(v). Assume that some $z \in \mathrm{ram}(T)$ has at least two isomorphic cones, say A and B. There is $f \in A(T)$ interchanging A and B and fixing $T \setminus (A \,\dot{\cup}\, B)$, with $f^2 = 1$. As $f \in S(T)$, this shows that $S(T)$ is not torsion-free, so we must show that $S(T)$ is not divisible in $A(T)$.

Suppose there is $g \in A(T)$ with $g^2 = f$. Since f acts as an odd permutation on the set of cones at z, we have $zg \neq z$, so, as $zg^2 = z$, we have $z \parallel zg$. Let $C := Ag$ and $D := Bg$. Then $Cg = B$ and $Dg = A$; hence $Cf = Cg^2 = D \neq C$. However, as $zg < C$ and $z < \mathrm{Supp}(f)$, f fixes C pointwise, a contradiction. Hence $S(T)$ is not divisible in $A(T)$.

(v)$\Rightarrow$(iv), (v)$\Rightarrow$(iii). Let $N \trianglelefteq A(T)$ and $f \in N$. We show that for each $n \in \mathbb{N}$ there is $g \in \langle f \rangle^{A(T)}$ with $f^n = f^g$. Then (iv), (iii), and the final remark follow, for, in particular, $f = [f, g]$ for some $g \in N$.

We first show that any $x \in \mathrm{Supp}(f)$ satisfies $x \in S^{\mathrm{cl}}$ for some non-trivial orbital S of f. We can assume that $x \parallel xf$. Let $y = \inf\{x, xf\} \in \bar{T}$ (the Dedekind-completion of T). Since $\{y, yf\} < xf$, we have either $y \leqslant yf$ or $yf \leqslant y$. If $y = yf$, then the points x and xf belong to different cones at y, contradicting (iv). Hence $y < yf$ or $yf < y$. Let S_1 be the convexification of $\{yf^i\colon i \in \mathbb{Z}\}$ in $\bar{T}$. Then our claim follows for $S := S_1 \cap T$.

Now, for each non-trivial orbital S of f, there are (by Lemma 3.5 of [**8**]) automorphisms g_S, $k_{S,1}$, $k_{S,2} \in A(T)$ which map S^{cl} onto itself and fix $T \setminus S^{\mathrm{cl}}$ pointwise, such that $f^n = g_S^{-1} f g_S$ on S^{cl} and $g_S = f^{k_{S,1}}(f^{-1})^{k_{S,2}}$ on T. Now define automorphisms g, k_1, $k_2 \in A(T)$ so that they coincide on S^{cl} (for each non-trivial orbital S of f) with g_S, $k_{S,1}$, $k_{S,2}$ respectively, and fix the rest of T pointwise. Then $f^n = f^g$ and $g = f^{k_1}(f^{-1})^{k_2} \in \langle f \rangle^{A(T)}$, as claimed.

(iv)$\Rightarrow$(i), (iii)$\Rightarrow$(ii). These implications are trivial.

We say that a cone A in T without a smallest element has *countable coinitiality* if it contains a countable chain which is unbounded below in A.

PROPOSITION 3.2. *Let T be a weakly 2-transitive tree.*

(a) *Let $a, b \in \mathrm{ram}(T)$, and A, B be cones at a, b respectively such that A and B have no smallest elements and countable coinitiality. Then $A \cong B$.*

(b) *Let $z \in \mathrm{ram}(T)$. Then the following are equivalent:*
   (i) *z has only pairwise non-isomorphic cones;*
   (ii) *z has at most one cone with a smallest element, the remaining cones at z are pairwise non-isomorphic, and $z \notin T$.*

Proof. (a) This is an easy adaptation of the proof of [**5**, Corollary 5.36].

(b) (i)$\Rightarrow$(ii). Let A, B be distinct cones at z with smallest elements $a \in A$, $b \in B$. Then there is $f \in A(T)$ with $af = b$, so $f|_A$ is an isomorphism from A to B, a contradiction. If $z \in T$, choose distinct cones A, B at z with $a \in A$, $b \in B$. There is $g \in A(T)$ with $zg = z$, $ag = b$. Then $Ag = B$, again a contradiction.

(ii)$\Rightarrow$(i). This is trivial.

Corollary 3.3. *Up to isomorphism, there is a unique countable weakly 2-transitive tree T whose automorphism group is torsion-free (or, equivalently, divisible).*

Proof. By Theorem 3.1 and Proposition 3.2(b), any such tree has type $(1, (1, 1), \varnothing)$. By Theorem 2.15, there is a unique countable weakly 2-transitive tree of this type.

We now use our analysis of torsion to show that there are many non-isomorphic, and indeed elementarily inequivalent, groups $A(T)$ arising from countable 2-homogeneous trees T.

Proposition 3.4. *Let T be a weakly 2-transitive tree, and let $2 \leqslant n \in \mathbb{N}$. Then $S(T)$ contains an element of order n if and only if, for each prime divisor p of n, T has a ramification point with at least p pairwise isomorphic cones.*

Proof. First, assume that $f \in A(T)$ has order p, where p is prime. Let $a \in \mathrm{Supp}(f)$. Then $\{af^i\colon 1 \leqslant i \leqslant p\}$ is an antichain in T and $z = \inf\{a, af\} \in \mathrm{ram}(T)$ satisfies $zf = z$ and $z = \inf\{af^i, af^j\}$ for each $1 \leqslant i < j \leqslant p$. Hence the elements af^i $(1 \leqslant i \leqslant p)$ belong to p different pairwise isomorphic cones at z.

Conversely, let p be a prime and let $z \in \mathrm{ram}(T)$ have distinct isomorphic cones $A_1, \ldots, A_p$. Then there is $f \in S(T)$ such that $z < \mathrm{Supp}(f)$, f has order p, and f permutes $\{A_1, \ldots, A_p\}$ in a single cycle. Next, choose $x \in A_1$ and $y \in \mathrm{ram}(T)$ with $x < y$ such that y again has at least p isomorphic cones, say $B_1, \ldots, B_p$. Define $g \in S(T)$ as before so that $y < \mathrm{Supp}(g)$, $g^p = 1$, and g acts transitively on $\{B_1, \ldots, B_p\}$. Then $h := fg \in S(T)$ has order p^2. An inductive argument shows that $S(T)$ contains elements of order p^m for all $m \in \mathbb{N}$. Finally, note that if $f, g \in S(T)$ have disjoint supports and relatively prime orders q and r respectively, then $fg \in S(T)$ has order qr.

Corollary 3.5. *There are infinitely many countable 2-homogeneous trees T such that*
   (i) *the groups $S(T)$ are elementarily inequivalent in the language $L = (1, \cdot, {}^{-1})$ of groups,*
   (ii) *the groups $A(T)$ are elementarily inequivalent in L.*

Proof. (i) This follows directly from Proposition 3.4.

(ii) By (i), it suffices to show that $S(T)$ is definable in $A(T)$ (over L). By Theorem 1.1 of **[8]**, $f \in S(T)$ if and only if, for each non-trivial $g \in A(T)$, there are $h_1, \ldots, h_4 \in A(T)$ with $f = (g^{-1})^{h_1} g^{h_2} (g^{-1})^{h_3} g^{h_4}$. The latter property is first order.

QUESTION 3.6. Suppose T_1, T_2 are countable 2-homogeneous trees whose automorphism groups are elementarily equivalent as abstract groups. Can it be that just one of T_1, T_2 is a meet-semilattice?

QUESTION 3.7. How many groups $S(T)$ (where T is weakly 2-transitive) are there up to isomorphism, or up to elementary equivalence?

M. Rubin (personal communication) now has detailed results on a number of questions along these lines, and can handle more general classes of trees.

4. *Subgroups of small index*

In **[12]** it was conjectured that if M is an $\aleph_0$-categorical structure, $G := \operatorname{Aut} M$, and $H \leqslant G$ with $|G : H| \leqslant \aleph_0$, then there is a finite subset $S \subseteq M$ with $G_{(S)} \leqslant H$. Hrushovski has since shown the conjecture to be false in general, but it does hold for a number of structures. In **[4]** it was shown that if G is the full symmetric group on a countable set X, and $H \leqslant G$ with $|G : H| < 2^{\aleph_0}$, then there is a finite set $S \subseteq X$ with $G_{(S)} \leqslant H \leqslant G_S$. Evans **[9]** and Truss **[14]** have proved the corresponding results for $G = \mathrm{GL}(\aleph_0, R)$ (R any division ring) and for $G = \operatorname{Aut}(\mathbb{Q}, <)$, respectively. Here we prove the following result. It is likely that the result generalizes to other trees, and to some of the structures discussed in the next section, but we have not checked this.

THEOREM 4.1. *Let T be a countable 2-homogeneous tree which is a meet-semilattice, and let $H \leqslant A(T)$ with $|A(T) : H| < 2^{\aleph_0}$. Then there is a finite subset $S \subseteq T$ with $A(T)_{(S)} \leqslant H$.*

Note that we assumed T is a meet-semilattice in order to ensure that if C is a maximal chain in T then $A(T)^C_C = A(C)$. This enables us to use the result of Truss quoted above. However, it is easy to adapt our proof to obtain the same result for countable 2-homogeneous trees which are not meet-semilattices, by passing to T^+. One then uses the corresponding small index theorem, also proved by Truss in **[14]**, for countable chains with points coloured densely with two colours.

LEMMA 4.2. *Let T and H satisfy the assumptions of Theorem 4.1, and suppose there is a maximal chain $C \subseteq T$ such that H^C_C is transitive. Then $H = A(T)$.*

Proof. Since $|A(T)^C_C : H^C_C| < 2^{\aleph_0}$, the result of Truss quoted above shows that there is a finite set $S \subseteq C$ such that $A(C)_{(S)} \leqslant H^C_C \leqslant A(C)_S$. Hence, as H^C_C is transitive, $H^C_C = A(C)$.

Now for each $c \in C$ let T_c be the union of the cones at c which do not meet C. Then $A(T)^{T \setminus C}_{(C)} = \prod_{c \in C} \operatorname{Aut}(T_c)$ (the unrestricted direct product) in the natural actions. Since $|A(T)_{(C)} : H_{(C)}| < 2^{\aleph_0}$, and the projections of $H_{(C)}$ into $\operatorname{Aut}(T_c)$ are all isomorphic (for $c \in C$), it follows that $H_{(C)}^{T \setminus C} = \prod_{c \in C} \mathrm{Aut}(T_C)$. From this and the transitivity of H_C^C it follows that for any two maximal chains D_1, D_2 of T with $D_1 \neq C$ and $D_2 \neq C$, there is $h \in H_C$ with $D_1 h = D_2$. Clearly H does not fix C (for C has $2^{\aleph_0}$ conjugates under $A(T)$), so H is transitive on the maximal chains of T, and the lemma follows.

LEMMA 4.3. *Let T and H satisfy the assumptions of Theorem* 4.1, *and let Ω be an infinite orbit of H on T. Then Ω contains an infinite chain.*

Proof. It suffices to find $a, b \in \Omega$ with $a < b$. So suppose Ω is an antichain. Then it can be checked that one of the following happens.

(i) There is $x \in T$ and $\{a_i\colon i < \omega\} \subseteq \Omega$ with $a_i > x$ and $\inf(a_i, a_j) = x$ for all $i, j < \omega$ (see Fig. 1).

(ii) There are a chain $\{b_i\colon i < \omega\} \subseteq T$ and a set $\{a_i\colon i < \omega\} \subseteq \Omega$ such that, for all $i < \omega$, we have $b_i < b_{i+1}$, $b_i < a_i$, and $b_{i+1} \parallel a_i$ (see Fig. 2).

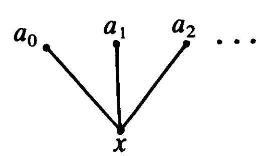

FIG. 1

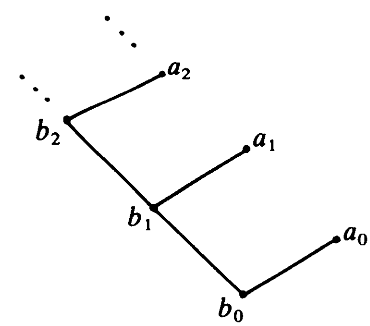

FIG. 2

In Case (i) it follows as in the proof of Lemma 4.2, that $|A(T)_x : H_x| = 2^{\aleph_0}$, and in Case (ii) that if B is a maximal chain containing all the b_i $(i < \omega)$ then $|A(T)_{(B)} : H_{(B)}| = 2^{\aleph_0}$. Either case gives a contradiction.

We introduce some notation. If K is a permutation group on a set Y, then let $\psi(K) := \bigcup \{S \subseteq Y\colon S \text{ is a finite orbit of } K\}$, and put $n(K) := |\psi(K)|$. The proof of Theorem 4.1 will be by induction on $n(H)$, and the following lemma starts the induction.

LEMMA 4.4. *Theorem* 4.1 *holds in the case when* $n(H) = 0$.

Proof. Suppose $n(H) = 0$, and let C be a maximal chain in T. By the result of Truss mentioned above, there is a finite set $S \subseteq C$ with $H_C^C = A(C)_{(S)}$, and by Lemma 4.2 we may suppose S is non-empty. Let $a := \mathrm{Max}\, S$. Also, for each $c \in C$ let R_c denote the cone at c which meets C.

Claim 1. For all $b \in R_a \cap C$, if $E := \{x \in T\colon b \not\leq x\}$, then $H_{(E)}^{R_b} = A(R_b)$.

Proof of Claim. By the result of Truss, $H_{(E),C}^{C \setminus E} = A(C \setminus E)$. The claim now follows from Lemma 4.2.

Claim 2. For every $b \in R_a$, a has infinitely many translates under H_b.

Proof of Claim. It suffices to show that a is not fixed by H_b. So suppose there is $b \in R_a$ such that H_b fixes a, and let Ω be the H-orbit of b. By Lemma 4.3, Ω contains an infinite chain $\{b_i: i<\omega\}$ with $b_i<b_{i+1}$ for all $i<\omega$. Let B be any maximal chain containing $\{b_i: i<\omega\}$. For each $i<\omega$, there is a unique a_i such that (b_i, a_i) and (b, a) are in the same H-orbit; clearly $b_i>a_i$ $(i<\omega)$. By Truss's result, there is a finite $Q \subseteq B$ such that $H^B_B = A(B)_{(Q)}$. For some $i<\omega$, $\{a_i, b_i\} \cap Q = \varnothing$, giving a contradiction.

Claim 3. There is $b \in R_a$ and a maximal chain B through b which meets R_b such that a has infinitely many translates under $H_{B,b}$.

Proof of Claim. Choose $b, c \in R_a \cap C$, with $c \in R_b$. For every maximal chain D through c, H^D_D has finitely many orbits. Hence, as a has infinitely many translates under H_c (by Claim 2), there is a maximal chain B through c such that a has infinitely many translates under H^B_B. Now (again by the theorem of Truss) b and B satisfy the claim.

To complete the proof of the lemma, let b, B be as in Claim 3, and choose $\{h_i: i<\omega\} \subseteq H_{B,b}$ such that the ah_i $(i<\omega)$ are all distinct. By Claim 1, for each $i<\omega$ there is g_i such that $ah_ig_i = ah_i$ and $Ch_ig_i = C$. This contradicts the assumption that H_C fixes a.

Proof of Theorem 4.1. We first note that $n(H)$ is finite. This can be checked directly, but follows from the following unpublished result of D. M. Evans: if $\mathcal{M}$ is an $\aleph_0$-categorical structure and $H \leqslant \operatorname{Aut}\mathcal{M}$ with $|\operatorname{Aut}\mathcal{M} : H| < 2^{\aleph_0}$, then H has finitely many orbits on unordered k-subsets of $\mathcal{M}$ for all $k \in \mathbb{N}$. We prove Theorem 4.1 by induction on $n(H)$. By Lemma 4.4, the theorem holds for $n(H)=0$. Assume that it holds for all T_1, H_1 satisfying the assumptions of the theorem with $n(H_1)<k$, and suppose $n(H)=k$. Let a be the unique smallest element of $\psi(H)$. Let $\{C_i: i \in I\}$ be the set of cones at a (with I finite or infinite), and let $E := \{x \in T: x<a\}^{\mathrm{cl}}$. Arguments as in Lemma 4.2 show that $H^E_{(T\setminus E)} = A(T)^E_E$. Furthermore, by the theorem mentioned earlier from [4], there is finite $P \subseteq I$ such that $H_{a,\{C_i:\ i\in P\}}$ induces $\operatorname{Sym}(I\setminus P)$ on $\{C_i: i \in I\setminus P\}$; indeed if $F := \bigcup(C_i: i \in I\setminus P)$ then $H^F_{(T\setminus F)} \cong A(T) \operatorname{Wr} \operatorname{Sym}(I\setminus P)$ in the natural action. For each $i \in P$, $n(H^{C_i}_{C_i})<k$, so as $H^{C_i}_{(T\setminus C_i)} \trianglelefteq H^{C_i}_{C_i}$, $n(H^{C_i}_{(T\setminus C_i)})<k$. Hence by the inductive hypothesis there is finite $S_i \subseteq C_i$ $(i \in P)$ such that $A(C_i)_{(S_i)} \leqslant H^{C_i}_{(T\setminus C_i)}$. It follows that if $S = \{a\} \cup \bigcup\{S_i: i \in P\}$, then S is finite and $A(T)_{(S)} \leqslant H$. This proves the theorem.

Remark. Unlike the cases of [4] and [9] and the result of Truss mentioned earlier, we could not hope to prove that if T, H satisfy the assumptions of Theorem 4.1 then there is finite $S \subseteq T$ with $A(T)_{(S)} \leqslant H \leqslant A(T)_S$. For let T be a countable 2-homogeneous tree which is a meet-semilattice with two cones at each point, let $x \in T$ and let C_1, C_2 be the two cones at x. Then $H := A(T)_{x,C_1}$ has index $\aleph_0$ in $A(T)$, but there is no finite S with $A(T)_{(S)} \leqslant H \leqslant A(T)_S$.

5. *Some related groups*

In this section we discuss certain treelike objects related to countable trees, and investigate the normal subgroup structure of their automorphism groups. The structures are discussed in detail by Cameron in [**2**, **3**], and in forthcoming work of Adeleke and Neumann, but for completeness we sketch their constructions. In this section Γ_n, respectively Δ_n $(n = 2, 3, \ldots, \omega)$ will denote the countable 2-homogeneous tree with precisely n cones at each point which is, respectively is not, a meet-semilattice. Recall from § 1 of [**8**] the definition of a homogeneous structure.

For each $n = 3, 4, \ldots, \omega$, let U_n be the homogeneous structure defined at the end of [**2**] and in § 9 of [**3**]. One description of U_n is as follows. Let L be a language with a ternary relation $x : yz$ and a quaternary relation $xy \mid zw$. Let T be a finite unrooted tree (in the graph-theoretic sense) whose vertices all have degree at most n. If x, y, z, w are vertices of T, write $x : yz$ if the unique path from y to z passes through x, and $xy \mid zw$ if the paths joining x, y, z, w form a tree homeomorphic to that in Fig. 3.

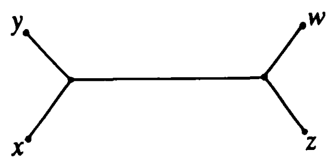

FIG. 3

Let C_n be the class of all finite structures arising from trees in this way. Then (as shown in [**3**, § 9]) C_n has the hereditary and amalgamation properties, so by a theorem of Fraissé [**10**, Chapter 11, 1.6], there is a unique countable homogeneous L-structure U_n whose finite substructures are precisely the members of C_n. The structure U_n may loosely be regarded as an unrooted dense tree of ramification order n.

Since U_n is homogeneous and the relations in L have arity greater than 2, $\mathrm{Aut}(U_n)$ is doubly transitive as a permutation group. Furthermore, if $x \in U_n$ then the point stabiliser $\mathrm{Aut}(U_n)_x$ acts imprimitively with a system of n blocks, each block carrying a structure isomorphic to Γ_{n-1}. Thus, $\mathrm{Aut}(U_n)_x$ acts like $A(\Gamma_{n-1}) \operatorname{wr} S_n$ on $U_n \setminus \{x\}$. (If $n = \omega$, then $n - 1 := \omega$, and also S_n is the full symmetric group of countable degree; here, as always in this section, the wreath product is unrestricted.) We shall often refer to the blocks of $\mathrm{Aut}(U_n)_x$ as the *trees at x*.

THEOREM 5.1. *For $n = 3, 4, \ldots, \omega$ the group* $\mathrm{Aut}(U_n)$ *is simple.*

We first prove the following lemma.

LEMMA 5.2. *Let $m \in \{2, 3, \ldots, \omega\}$, let T_1 and T_2 be trees isomorphic to Γ_m, and suppose $G := A(T_1) \times A(T_2)$ acts naturally on $T_1 \cup T_2$. If N is a normal subgroup of G inducing $A(T_i)$ on T_i $(i = 1,2)$, then $N = G$.*

Proof. Since $N_{(T_1)}$ induces a normal subgroup of $A(T_2)$ on T_2, it suffices by Theorem 1.3 of [8] to find some $k \in N$ fixing T_1 pointwise and inducing on T_2 an automorphism not in $R(T_2)$. Let $l \in N$ induce on T_2 an element of $A(T_2) \backslash R(T_2)$. Clearly there is $g \in G_{(T_1)}$ such that if $k := g^{-1}l^{-1}gl$ then k induces an element of $A(T_2) \backslash R(T_2)$. Now $k \in N_{(T_1)}$, as required.

Proof of Theorem 5.1. Put $U = U_n$ and $G = \mathrm{Aut}(U_n)$ for some fixed $n \in \{3, 4, \ldots, \omega\}$. Let N be a non-trivial normal subgroup of G. Note that since G is 2-transitive, N is transitive on U. Choose $x \in U$ and let $\{T_i\colon i \in I\}$ be the set of trees at x (here we take $I = \{I, \ldots, n\}$ or $I = \omega$).

Claim 1. N_x induces $\mathrm{Sym}(I)$ on $\{T_i\colon i \in I\}$.

Proof of Claim. Let $g \in N$ with $xg = y \neq x$. Suppose that $T_i g = R_i$ for $i \in I$, where the R_i are the trees at y. We may assume that $x \in R_1$. Let $\pi \in \mathrm{Sym}(I)_1$. There is $h \in G_{(R_1)}$ fixing setwise each T_i such that $R_i h = R_{i\pi}$ for all $i \in I$. Put $k = ghg^{-1}h^{-1}$. Then $T_i k = T_{i\pi}$ for all $i \in I$. Thus, since $k \in N_x$, N_x induces at least $\mathrm{Sym}(I)_1$ on $\{T_i\colon i \in I\}$. Since $n > 2$ and N_x induces a normal subgroup of $\mathrm{Sym}\,(I)$, the claim follows.

Claim 2. N_x induces $A(\Gamma_{n-1}) \operatorname{wr} \mathrm{Sym}\,(I)$ on $U \backslash \{x\}$.

Proof of Claim. Let g, y and the R_i be as in Claim 1. By Claim 1, we may suppose that $y \in T_1$. Let $h \in G_x$ fix each T_i setwise. Then there is $k \in G_{xy}$ fixing $\bigcup_{i \neq 1} T_i$ pointwise, such that if $l := k^{-1}gkg^{-1}$ then lh^{-1} fixes $\bigcup_{i \neq 1} T_i$ pointwise (here k is chosen to agree with $g^{-1}hg$ on $\bigcup_{i \neq 1} R_i$). Since $l \in N_x$ and l acts like h on $U \backslash T_1$, it follows that N_x induces $A(\Gamma_{n-1}) \operatorname{wr} \mathrm{Sym}(I \backslash \{1\})$ on $U \backslash (\{x\} \cup T_1)$. Applying Lemma 5.2 to the subgroup of G_x fixing $U \backslash (T_1 \cup T_2)$ pointwise, we see that $N_x \geqslant \prod_{i \in I} A(\Gamma_{n-1})$ in the natural action. Claim 2 now follows from Claim 1.

We complete the proof of Theorem 5.1. Let $g \in G$. Since N is transitive on U, there is $h \in N$ such that gh^{-1} fixes $x \in U$. Then $gh^{-1} \in N$ (by Claim 2), so $g \in N$.

REMARK. Theorem 5.1 would fail if $n = 2$, since U_2 is simply the betweenness relation induced from a countable dense linear order, and $\mathrm{Aut}(U_2)$ has precisely two proper non-trivial normal subgroups, namely the order-preserving members of $\mathrm{Aut}(U_2)$, and the bounded order-preserving members of $\mathrm{Aut}(U_2)$.

The structures U_n above correspond to meet-semilattices in that the trees at each point are meet-semilattices. It is also possible to define a homogeneous structure $\mathcal{M}_n$ (for each $n = 3, 4, \ldots, \omega$) which is treelike but does not branch at points of the structure. It will have the same language with a ternary and a quaternary relation, but the corresponding class C_n of finite structures arises as follows. Let T be an unrooted finite tree with vertices of valency at most n, and take as the corresponding C_n-structure the set of vertices of T of valency 1 or 2, with the two relations defined as before. Then, as in [3, § 9], it ean be checked that C_n has the hereditary and amalgamation properties. Let $\mathcal{M}_n$ be the corresponding countable homogeneous object. Then $\mathcal{M}_n$ may be regarded as an unrooted dense tree in which the branching points are not points of the structure. Let $\mathcal{M}_n^+$ be obtained by adjoining to $\mathcal{M}_n$ the branching points of $\mathcal{M}_n$, that is, points such as x in Fig. 4 (to formalize this notion, we would have to introduce Dedekind-completions for unrooted trees).

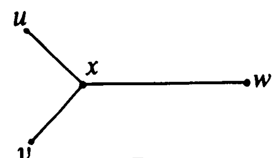

FIG. 4

We could construct $\mathcal{M}_n^+$ as a homogeneous structure in a language $L \cup \{P\}$, where L is the language for $\mathcal{M}_n$ and P is a unary predicate picking out the branching points.

It can be checked that $\mathrm{Aut}(\mathcal{M}_n)$ is a doubly transitive permutation group, and that for each $x \in \mathcal{M}_n$ we have $(\mathrm{Aut}(\mathcal{M}_n))_x \cong A(\Delta_{n-1}) \operatorname{wr} S_n$. Furthermore, $\mathrm{Aut}(\mathcal{M}_n^+)|_{\mathcal{M}_n} = \mathrm{Aut}(\mathcal{M}_n)$.

PROPOSITION 5.3. *For each $n \geqslant 3$, the group* $\mathrm{Aut}(\mathcal{M}_n)$ *is simple.*

Proof. Let $N \trianglelefteq \mathrm{Aut}(\mathcal{M}_n)$ with $N \neq 1$. The group $\mathrm{Aut}(\mathcal{M}_n)$ is doubly transitive on $\mathcal{M}_n^+ \setminus \mathcal{M}_n$, so N is transitive on $\mathcal{M}_n^+ \setminus \mathcal{M}_n$. The argument in the proof of Theorem 4.1 shows that if $x \in \mathcal{M}_n^+ \setminus \mathcal{M}_n$, then N_x induces $A(\Delta_{n-1}) \operatorname{wr} S_n$ on $\mathcal{M}_n$. The result follows immediately.

A further class of permutation groups J_n (for $n = 3, 4, \ldots, \omega$) is defined in [**2, 3**]. As in [**3**], we shall denote the corresponding homogeneous object by T_n. An uncountable model P_n of $\mathrm{Th}(T_n)$ (over a quaternary language) can be defined as follows: let U_n be the structure defined at the beginning of the section, adjoin a leaf at each end of each maximal chain in U_n, and let the domain of P_n be the set of leaves (so $|P_n| = 2^\omega$). There is a quaternary relation $xy \mid zw$ defined on P_n by the rule: $xy \mid zw$ holds if the set of paths in U_n joining x, y, z, w forms a tree homeomorphic to that in Fig. 3. The corresponding countable homogeneous object T_n is formed by choosing a countable dense set X of maximal chains in U_n (that is, a set such that each point of U_n lies on a chain in X), adjoining leaves to each end of each chain in X, and proceeding as before. There is also a rooted structure ∂T_n corresponding to T_n (as shown in [**3**]). It can be constructed either as a homogeneous structure in a language with one ternary relation, or by taking a countable dense set Y of maximal chains in Γ_{n-1} and giving it the natural ternary relation (here Y is dense if every point of Γ_{n-1} lies on a chain in Y). The case where $n = 3$ is examined in detail in [**2**]. Note that if $x \in T_n$, then $\mathrm{Aut}(T_n)_x^{T_n \setminus \{x\}}$ and $\mathrm{Aut}(\partial T_n)$ are isomorphic as permutation groups. Note too that if ∂T_n has domain Y as above, then $A(\Gamma_{n-1})_Y^Y$ and $\mathrm{Aut}(\partial T_n)$ are isomorphic as permutation groups, so we may identify $\mathrm{Aut}(\partial T_n)$ with a subgroup of $A(\Gamma_{n-1})$. Recall the definitions given in § 7 of [**8**].

THEOREM 5.4. $\mathrm{Aut}(\partial T_n)$, *in its action on* Γ_{n-1} *arising as above, is weakly 2-transitive and closed under piecewise and disjoint patching.*

Proof. This is straightforward, and omitted. In fact, $\mathrm{Aut}(\partial T_n)$ and $A(\Gamma_{n-1})$ have the same orbits on finite ordered subsets of Γ_{n-1}.

Now put $A(\partial T_n) := \mathrm{Aut}(\partial T_n)$, $R(\partial T_n) := A(\partial T_n) \cap R(\Gamma_{n-1})$, $S(\partial T_n) := A(\partial T_n) \cap S(\Gamma_{n-1})$. The remark at the end of § 7 of [**8**] shows that many of the theorems of [**8**] hold with $A(\partial T_n)$ in place of $A(\Gamma_{n-1})$, and with other replacements as above. Furthermore, an argument as in the proof of Theorem 5.1 above gives:

THEOREM 5.5. *For $n = 3, 4, \ldots, \omega$, the group* $\mathrm{Aut}(T_n)$ *is simple.*

Clearly, many of the questions of this paper, concerning torsion, divisibility, subgroups of small index, and uncountable analogues of the structures, can also be raised about these groups. These are also questions about whether some of the groups in this section are isomorphic as abstract groups.

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