On Infinite Transition Graphs Having a Decidable Monadic Theory

Didier Caucal

IRISA, Campus de Beaulieu, 35042 Rennes, France
E-mail: caucal@irisa.fr

Abstract. We define a family of graphs whose the monadic theory is linearly reducible to the monadic theory S2S of the complete deterministic binary tree. This family contains strictly the context-free graphs investigated by Muller and Schupp, and also the equational graphs defined by Courcelle. Using words for vertices, we give a complete set of representatives by prefix rewriting of rational languages. This subset is a boolean algebra preserved by transitive closure of arcs and by rational restriction on vertices.

1 Introduction

We consider the verification of properties by structures. The properties are the monadic second-order sentences, and the structures are the labelled directed graphs. Rabin has shown that the complete deterministic tree Λ on two labels has a decidable monadic theory [Ra 69] : we can decide whether a given property expressed by a monadic sentence, is satisfied by the tree Λ. Later Muller and Schupp have extended this decision result to the context-free graphs [MS 85] : a context-free graph is a rooted graph of finite degree which has a finite number of non isomorphic connected components by decomposition by distance from a (any) vertex. These context-free graphs are also the transition graphs of pushdown automata [MS 85]. Finally Courcelle has shown that the monadic theory remains decidable for the equational graphs [Co 90] : an equational graph is a graph generated by a deterministic graph grammar. For rooted graphs of finite degree, these equational graphs are the context-free graphs [Ca 90]. These decision results of [MS 85] and [Co 90] are extensions of the definability method used by Rabin.

Another approach is to find transformations on graphs which preserve the decision of the monadic theory, and to apply these transformations to graphs having a decidable monadic theory (see for instance [Th 91]). A first transformation has been given by Shelah [Sh 75] and proved by Stupp [St 75]: if a graph has a decidable monadic theory then its "tree-graph" has a decidable monadic theory.

A way to find a transformation f on graphs that preserves the decision of the monadic theory, is to translate f into an "equivalent" transformation f^* on monadic formulas: for any graph G, $f(G)$ satisfies a sentence φ if and only if G

satisfies $f^*(\varphi)$. This method has been applied for instance in [Co 94], [CW 95], and especially in [Sem 84], [W 96] for an extension of the tree-graph transformation. We give here two transformations on graphs which have direct equivalent transformations on monadic formulas: they are based on the fact that the existence of a path labelled by a rational language can be expressed by an equivalent monadic formula. By closure of Λ under these two operations, we get a family F of graphs which have a decidable monadic theory as a corollary of Rabin's theorem. We show that this family F is a strict extension of the equational graphs. By taking words as vertices, we extract a complete subset F_0 of representatives up to isomorphism, such that F_0 remains closed under the two operations defining F, and is a boolean algebra.

2 A family of graphs with a decidable monadic theory

We define two transformations on graphs which can be translated on formulas in such a way that the decision of the monadic theory is simply preserved. The first transformation is the rational restriction on labels and the second transformation is the inverse rational mapping on labels. We start with the complete and deterministic tree on two labels which has a decidable monadic theory [Ra 69]. By applying to this tree the second transformation followed by the first one, we obtain a family of graphs with a decidable monadic theory, and which is closed by these two transformations.

We take an alphabet (finite set of symbols) T of *terminals* containing at least two symbols a, b. Here a graph is a set of arcs labelled on T. This means that a *graph* G is a subset of $V{\times}T{\times}V$ where V is an arbitrary set. Any (s,a,t) of G is a *labelled arc* of *source* s, of *target* t, with *label* a, and is identified with the labelled transition $s \underset{G}{\overset{a}{\longrightarrow}} t$ or directly $s \overset{a}{\longrightarrow} t$ if G is understood. We denote by

$$V_G \ := \ \{\, s \mid \exists\, a \,\exists\, t,\ s \overset{a}{\longrightarrow} t \vee t \overset{a}{\longrightarrow} s \,\}$$

the set of *vertices* of G. A graph is *deterministic* if distinct arcs with the same source have distinct labels: if $r \overset{a}{\longrightarrow} s$ and $r \overset{a}{\longrightarrow} t$ then $s = t$. And a graph is *complete* if for every label a, every vertex is source of an arc labelled by a: $\forall\, a \in T,\ \forall\, s \in V_G,\ \exists\, t,\ s \overset{a}{\longrightarrow} t$.

The existence of a *path* in G from vertex s to vertex t and labelled by a word $w \in T^*$ is denoted by $s \underset{G}{\overset{w}{\Longrightarrow}} t$ or directly by $s \overset{w}{\Longrightarrow} t$ if G is understood: $s \overset{\epsilon}{\Longrightarrow} s$ and $s \overset{aw}{\Longrightarrow} t$ if there is some vertex r such that $s \overset{a}{\longrightarrow} r$ and $r \overset{w}{\Longrightarrow} t$.

We say that a vertex r is a *root* of G if every vertex is accessible from r: $\forall\, s \in V_G,\ \exists\, w \in T^*,\ r \overset{w}{\Longrightarrow} s$. And a graph is a *tree* if it has a root r which is target of no arc, and every vertex $s \neq r$ is target of a unique arc.

To construct monadic second-order formulas, we take two disjoint denumerable sets: a set of *vertex variables* and a set of *vertex set variables*. *Atomic formulas* have one of the following two forms:

$$x \in X \quad \text{or} \quad x \xrightarrow{a} y$$

where X is a vertex set variable, x and y are vertex variables, and $a \in T$.
From the atomic formulas, we construct as usual the *monadic second-order formulas* with the propositional connectives $\neg, \wedge$ and the existential quantifier $\exists$ acting on these two kind of variables. A *sentence* is a formula without free variable. The set of monadic second-order sentences $MTh(G)$ satisfied by a graph G forms the *monadic theory* of G.
Note that two isomorphic graphs satisfy the same sentences. Instead of renaming vertices, we consider the *restriction* $G_{|U}$ of G to an arbitrary set U as follows:

$$G_{|U} \;:=\; G \cap (U{\times}T{\times}U) \;=\; \{\, s \xrightarrow[G]{a} t \mid s,t \in U \,\}.$$

Analogously we consider the restriction φ_U (resp. $\varphi_{|U}$) of any sentence φ to a set U by imposing that vertex variables (resp. and vertex set variables) are interpreted only by vertices in U (resp. by subsets of vertices in U) i.e. by induction on the structure of any formula:

$$\begin{array}{rlcrl}
(x \in X)_U &= x \in X & \qquad & (x \in X)_{|U} &= x \in X \\
(x \xrightarrow{a} y)_U &= x \xrightarrow{a} y & & (x \xrightarrow{a} y)_{|U} &= x \xrightarrow{a} y \\
(\neg\, \varphi)_U &= \neg(\varphi_U) & & (\neg\, \varphi)_{|U} &= \neg(\varphi_{|U}) \\
(\varphi \wedge \psi)_U &= \varphi_U \wedge \psi_U & & (\varphi \wedge \psi)_{|U} &= \varphi_{|U} \wedge \psi_{|U} \\
(\exists\, x\ \ \varphi)_U &= \exists\, x\ (x \in U \wedge \varphi_U) & & (\exists\, x\ \ \varphi)_{|U} &= \exists\, x\ (x \in U \wedge \varphi_{|U}) \\
(\exists\, X\ \ \varphi)_U &= \exists\, X\ \ \varphi_U & & (\exists\, X\ \ \varphi)_{|U} &= \exists\, X\ (X \subseteq U \wedge \varphi_{|U})
\end{array}$$

These restrictions of graphs and sentences are dual.

Lemma 2.1 *Given a graph G, a set U and a monadic sentence φ, we have*

$$G_{|U} \models \varphi \iff G \models \varphi_U \iff G \models \varphi_{|U}\,.$$

Note that φ_U and $\varphi_{|U}$ are not monadic formulas. But for any monadic formula φ, we give a general condition on U to transform φ_U (or $\varphi_{|U}$) into an equivalent monadic formula i.e. to transform $x \in U$ into a monadic formula.
Basic transformations are well-known to express for instance the propositional connectives $\vee$, $\Longrightarrow$ and the universal quantifier $\forall$:

$\varphi \vee \psi : \neg\,(\neg\,\varphi \wedge \neg\,\psi) \qquad \varphi \Rightarrow \psi : \neg\,(\varphi \wedge \neg\,\psi) \qquad \forall X\ \ \varphi : \neg\,(\exists\, X\ \neg\,\varphi)$

Similarly the existence of a path $s \overset{L}{\Longrightarrow} t$ from s to t and labelled by a word in $L \subseteq T^*$ can be expressed by a monadic formula when $L \in Rat(T^*)$ is a rational language; this is done by induction on the rational structure of L:

$x \overset{\emptyset}{\Longrightarrow} y : \exists\, X\ (x \in X \wedge \neg\,(x \in X))$ i.e. a false formula

$x \overset{\{a\}}{\Longrightarrow} y : x \xrightarrow{a} y$

$x \overset{L+M}{\Longrightarrow} y : x \overset{L}{\Longrightarrow} y \vee x \overset{M}{\Longrightarrow} y$

$x \overset{L.M}{\Longrightarrow} y : \exists\, z\ (x \overset{L}{\Longrightarrow} z \wedge z \overset{M}{\Longrightarrow} y)$

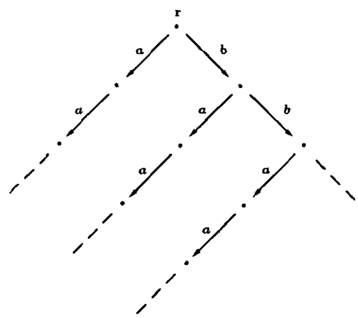
$x \overset{L^*}{\Longrightarrow} y : \forall X\ ((x \in X \wedge \forall p \forall q((p \in X \wedge p \overset{L}{\Longrightarrow} q) \Rightarrow q \in X)) \Rightarrow y \in X)$

where the transformation for $\stackrel{L^*}{\Longrightarrow}$ is the reflexive and transitive closure $(\stackrel{L}{\Longrightarrow})^*$ of $\stackrel{L}{\Longrightarrow}$: $x \stackrel{L^*}{\Longrightarrow} y$ if and only if every vertex set X containing x and closed by $\stackrel{L}{\Longrightarrow}$ contains y.

So we consider the restriction $G_{\|r,L}$ of a graph G to the vertices accessible from a vertex r by a path labelled in $L \subseteq T^*$:

$$G_{\|r,L} \ := \ G_{|\{s \ | \ r \stackrel{L}{\Longrightarrow} s\}} \ = \ \{ \ s \underset{G}{\stackrel{a}{\longrightarrow}} t \ | \ r \stackrel{L}{\Longrightarrow} s \ \wedge \ r \stackrel{L}{\Longrightarrow} t \ \} \ .$$

For instance taking the following deterministic complete tree Λ on $\{a, b\}$ and its root r i.e. its vertex satisfying the formula $\neg \ (\exists \ y \ \ y \stackrel{a}{\longrightarrow} x)$, then its restriction $\Lambda_{\|r,b^*a^*}$ is the following graph:



By Lemma 2.1, the restriction of a graph preserves the decision of the monadic theory if we restrict to the vertices accessible from a given vertex by a path labelled in a given rational language.

Proposition 2.2 *Given a graph G, a rational language L over T, and a monadic formula $\varphi(x)$ satisfied by a unique vertex r, we have*

$$MTh(G) \ \textit{decidable} \ \Longrightarrow \ MTh(G_{\|r,L}) \ \textit{decidable}.$$

To move by inverse arcs, we introduce a new alphabet $\overline{T} \ := \ \{ \ \overline{a} \ | \ a \in T \ \}$ in bijection with T. Any transition $u \stackrel{\overline{a}}{\longrightarrow} v$ means that $v \stackrel{a}{\longrightarrow} u$ is an arc of G.

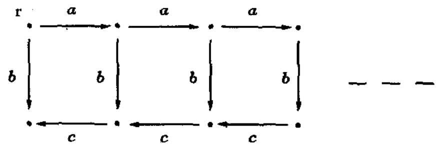
We extend by composition the existence of a path $\stackrel{w}{\Longrightarrow}$ labelled by a word w in $(T \cup \overline{T})^*$.

A *label-mapping* is a mapping from T into the family $2^{(T \cup \overline{T})^*}$ of languages over $T \cup \overline{T}$.

The *inverse* $h^{-1}(G)$ of a graph G according to a label-mapping h is the following graph:

$$h^{-1}(G) \ := \ \{ \ s \stackrel{a}{\longrightarrow} t \ | \ \exists \ w \in h(a), \ s \underset{G}{\stackrel{w}{\Longrightarrow}} t \ \} \ .$$

For instance for $h(a) = \{b\}$, $h(b) = \{b\overline{b}a\}$, $h(c) = \{\overline{a}\overline{b}a\}$, $h(d) = \emptyset$ for every other d in T, the inverse $h^{-1}(\Lambda_{\|r,b^*a^*})$ of the previous graph by h is the following graph:



Similarly to the restriction, we define the transformation φ_{h} by h of any formula φ by replacing each atomic formula $x \xrightarrow{a} y$ by the existence of a path $x \overset{h(a)}{\Longrightarrow} y$ labelled in $h(a)$ i.e. by induction on the structure of any formula:

$$\begin{array}{ll} (x \in X)_{h} = x \in X & (x \xrightarrow{a} y)_{h} = x \overset{h(a)}{\Longrightarrow} y \\ (\neg\, \varphi)_{h} = \neg(\varphi_{h}) & (\varphi \wedge \psi)_{h} = \varphi_{h} \wedge \psi_{h} \\ (\exists\, x \;\; \varphi)_{h} = \exists\, x \;\; \varphi_{h} & (\exists\, X \;\; \varphi)_{h} = \exists\, X \;\; \varphi_{h} \end{array}$$

Note that $(\varphi_{U})_{h} = (\varphi_{h})_{U}$ which is denoted by $\varphi_{U,\,h}$ and $(\varphi_{|U})_{h} = (\varphi_{h})_{|U}$ which is denoted by $\varphi_{|U,\,h}$.

Lemma 2.3 *Given a graph G, a label-mapping h and a monadic sentence φ, we have*

$$h^{-1}(G) \models \varphi \iff G \models \varphi_{U,\,h} \iff G \models \varphi_{|U,\,h}$$

where $U = V_{h^{-1}(G)}$ is the set of vertices of $h^{-1}(G)$.

However $\varphi_{U,\,h}$ and $\varphi_{|U,\,h}$ are not monadic formulas. But we can transform

$$x \in V_{h^{-1}(G)} \quad \text{by} \quad \exists\, y\; (\, x \overset{L}{\Longrightarrow} y \;\vee\; y \overset{L}{\Longrightarrow} x\,) \;\; \text{for} \;\; L \;=\; \textstyle\bigcup_{a \in T}\; h(a)\;,$$

and we have seen that $x \overset{h(a)}{\Longrightarrow} y$ can be transformed into a monadic formula if $h(a)$ is rational (and with $x \overset{\{\overline{a}\}}{\Longrightarrow} y$ transformed into $y \xrightarrow{a} x$).

So the inverse according to a label-mapping h preserves the decision of the monadic theory when h is rational: $h(a) \in Rat((T \cup \overline{T})^{*})$ for any $a \in T$.

Proposition 2.4 *Given a graph G and a rational label-mapping h, we have*

$$MTh(G) \text{ decidable} \implies MTh(h^{-1}(G)) \text{ decidable.}$$

Let us compose Proposition 2.2 with Proposition 2.4.

Proposition 2.5 *Given a graph G with a unique root r, a rational label-mapping h, and a rational label language $L \in Rat((T \cup \overline{T})^{*})$, we have*

$$MTh(G) \text{ decidable} \implies MTh(h^{-1}(G)_{\,|L_G}) \text{ decidable}$$

with $L_G \;:=\; \{\; s \mid r \overset{L}{\underset{G}{\Longrightarrow}} s \;\}$ the set of vertices accessible in G from r by a path in L.

Note that Proposition 2.5 is a corollary of Proposition 3.1 in [Co 94] because the transformation of G to $h^{-1}(G)_{|L_G}$ is a noncopying monadic second-order definable transduction. Furthermore this transformation is a linear reduction for the monadic theory.
We will study the family REC_{Rat} of graphs obtained by applying to the tree Λ an inverse rational label-mapping followed by a rational restriction:

$$REC_{Rat} := \left\{ h^{-1}(\Lambda)_{|L_\Lambda} \;\middle|\; \begin{array}{l} \Lambda \text{ is a complete deterministic tree on } \{a,b\} \\ h : T \longrightarrow Rat(\{a,b,\overline{a},\overline{b}\}^*) \\ L \in Rat(\{a,b\}^*) \end{array} \right\}$$

We consider also the sub-family REC_{Fin} by using only finite label-mappings:

$$REC_{Fin} := \left\{ h^{-1}(\Lambda)_{|L_\Lambda} \;\middle|\; \begin{array}{l} \Lambda \text{ is a complete deterministic tree on } \{a,b\} \\ h : T \longrightarrow Fin(\{a,b,\overline{a},\overline{b}\}^*) \\ L \in Rat(\{a,b\}^*) \end{array} \right\}$$

where $Fin(V)$ is the set of finite subsets of a set V.
We start with any complete and deterministic tree on two labels because Rabin has shown that we can decide any monadic sentence on it.

Theorem 2.6 **[Ra 69]** *Any complete and deterministic tree on two labels has a decidable monadic theory.*

Let us apply Proposition 2.5 to this result of Rabin.

Corollary 2.7 *Any graph in REC_{Rat} has a decidable monadic theory.*

In fact REC_{Rat} is the closure of the complete and deterministic trees on $\{a,b\}$ by the operations of Proposition 2.2 and Proposition 2.4.

Proposition 2.8 *The family REC_{Rat} is closed by rational restriction and by inverse rational label-mapping.*

Proposition 2.8 is proved using a complete set of representatives. We will obtain other closure properties and particularly we will deduce that this family contains strictly the graphs generated by deterministic graph grammars.

3 Complete sets of representatives

We show that the rational restrictions on vertices of prefix transition graphs of labelled word rewriting systems is a complete set of representatives of REC_{Fin}

(Corollary 3.4). This set of representatives contains the context-free graphs of [MS 85]. In fact REC_{Fin} is exactly the class of regular graphs of finite degree (Theorem 3.9) where a regular graph (or equational graph) is a graph generated by a deterministic graph grammar.
We show that the rational restrictions on vertices of prefix transition graphs of labelled recognizable relations is a complete set of representatives of REC_{Rat} (Corollary 3.4). It follows that REC_{Rat} contains strictly the class of regular graphs (Proposition 3.10).
Finally we extend these sets of representatives to the rationally controlled prefix transition graphs of labelled recognizable relations. This set is also a complete set of representatives of REC_{Rat} (Proposition 3.13). But it is a boolean algebra preserved by inverse rational label-mapping and by rational restriction on vertices (Theorems 3.14).

We take an alphabet $N \subseteq T$ containing the symbols a, b. A representant of the complete deterministic trees labelled on N is the tree Δ_N in $N^* \times N \times N^*$ defined by

$$\Delta_N := \{ u \xrightarrow{a} au \mid a \in N \ \wedge \ u \in N^* \} .$$

The *right closure* $G.N^*$ of a graph G in $N^* \times T \times N^*$ is

$$G.N^* := \{ uw \xrightarrow{a} vw \mid u \xrightarrow{a} v \ \in G \ \wedge \ w \in N^* \}$$

the set of prefix transitions of G. For instance

$$\Delta_N = \{ \epsilon \xrightarrow{a} a \mid a \in N \}.N^* .$$

Note that a finite graph G in $N^* \times T \times N^*$ is a labelled rewriting system i.e. a finite set of rules over N and labelled in T; and the right closure of G is the prefix rewriting according to G. For instance the *identity graph* $\{ u \xrightarrow{a} u \mid a \in T \ \wedge \ u \in N^* \}$ is the right closure of the finite graph $\{ \epsilon \xrightarrow{a} \epsilon \mid a \in T \}$.
We denote by $U \xrightarrow{a} V := \{ u \xrightarrow{a} v \mid u \in U \ \wedge \ v \in V \}$ the graph of the transitions from $U \subseteq N^*$ to $V \subseteq N^*$ and labelled by $a \in T$. A *recognizable graph* is a finite union of such graphs $U \xrightarrow{a} V$ where $U, V \in Rat(N^*)$. We denote by $Rec(N^* \times T \times N^*)$ the family of recognizable graphs.
Note that the unlabelled recognizable graphs $\{ (u,v) \mid \exists\, a \in T,\ u \xrightarrow{a} v \ \in G \}$ are the *recognizable relations* in $N^* \times N^*$ (by Mezei's theorem) [Be 79]. For instance, the *full graph* $\{ u \xrightarrow{a} v \mid a \in T \ \wedge \ u, v \in N^* \}$ is the recognizable graph $\bigcup \{ N^* \xrightarrow{a} N^* \mid a \in T \}$ and is equal to its right closure.
The right closures of recognizable graphs in $N^* \times T \times N^*$ are exactly the inverse rational mappings $h^{-1}(\Delta_N)$ of Δ_N i.e. for $h : T \longrightarrow Rat((N \cup \overline{N})^*)$.

Theorem 3.1 *The inverse rational (resp. finite) label-mappings $h^{-1}(\Delta_N)$ of Δ_N are effectively the right closures $G.N^*$ of the recognizable graphs (resp. finite graphs) G :*

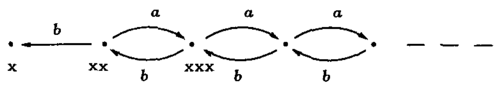
$$h^{-1}(\Delta_N) = G.N^*$$

with h rational (resp. finite) $\iff$ G recognizable (resp. finite).

This theorem implies some direct generalizations of known results. A first consequence follows from the closure by composition of (extended) rational label-mappings.

Corollary 3.2 *The right closures of recognizable graphs is a class closed effectively by inverse rational label-mapping.*

For instance take the right closure $G.N^*$ of $G = \{x \xrightarrow{a} \epsilon\ ,\ x^2 \xrightarrow{b} x^3\}$ with $N = \{x\}$. Its inverse $h^{-1}(G.N^*)$ by h defined by $h(a) = \{b\}$ and $h(b) = \{baa\}$, is the following graph:



which is the right closure of $\{x^2 \xrightarrow{a} x^3\ ,\ x^2 \xrightarrow{b} x\}$.
A consequence of Corollary 3.2 is that the unlabelled right closures of recognizable graphs are preserved by reflexive and transitive closure. More precisely, the *prefix rewriting* $\underset{R}{\longmapsto}$ of any binary relation R on N^* is the unlabelled graph $R.N^*$, i.e.

$$\underset{R}{\longmapsto}\ :=\ \{\ (uw, vw)\ |\ u\ R\ v\ \wedge\ w \in N^*\ \}$$

and its reflexive and transitive closure $\overset{*}{\underset{R}{\longmapsto}}$ is the *prefix derivation* of R.

Corollary 3.3 *The prefix derivation of any recognizable relation is effectively the prefix rewriting of a recognizable relation.*

In particular for any finite relation, its prefix derivation is a rational transduction [BN 84], and this remains true for any recognizable relation. Thus for the right closure of any recognizable graph, the set of vertices accessible from any rational set is rational; this extend the rationality of words accessible from a given word by prefix derivation of a finite relation [Bü 64].
Another consequence of Theorem 3.1 is that the following family:

$$REC_{|Rat}\ :=\ \{\ (G.N^*)_{|U}\ |\ G \in Rec(N^* \times T \times N^*)\ \wedge\ U \in Rat(N^*)\ \}$$

is a complete set of representatives of REC_{Rat} .

Corollary 3.4 *The set $REC_{|Rat}$ of the rational restrictions on vertices of the right closures of recognizable graphs (resp. finite graphs) is a complete set of representatives of REC_{Rat} (resp. REC_{Fin}).*

Note that Corollary 3.4 is true in particular for $N = \{a, b\}$. By Corollary 2.7, any rational restriction on vertices of the right closure of any recognizable graph has a decidable monadic theory.

Corollary 3.5 *$MTh((G.N^*)_{|U})$ is decidable for any $G \in Rec(N^* \times T \times N^*)$ and for any $U \in Rat(N^*)$.*

A particular case are the pushdown transition graphs (called also context-free graphs) considered in [MS 85]. A *pushdown transition graph* is the graph $R.N^*_{|U}$ of the right closure of a pushdown automaton R in $Q.P \times T \times Q.P^*$ with $N = P \cup Q$, and restricted to the set $U = \{ s \mid r \underset{R}{\overset{*}{\longmapsto}} s \}$ of vertices accessible from a given axiom $r \in Q.P^*$. By Corollary 3.3, we deduce the well-known fact that U is rational, and it remains to apply Corollary 3.5 to get Theorem 4.4 of [MS 85].

Corollary 3.6 [MS 85] *Any pushdown transition graph has a decidable monadic theory.*

But the pushdown automata define up to isomorphism the same accessible prefix transition graphs than the labelled rewriting systems.

Proposition 3.7 [Ca 90] *The pushdown transition graphs form effectively a complete set of representatives of the rooted right closures of finite graphs.*

Instead of labelled (word) rewriting systems, we can also use a subclass of term context-free grammars [Ca 92]. We will now show that REC_{Rat} contains also the graphs generated by deterministic graph grammars. These graphs are the equational graphs of [Co 90], and are called here *regular graphs*. Several basic properties of regular graphs are given in [Ca 95].
The regular graphs generalize the pushdown transition graphs. In fact the pushdown transition graphs are the rooted graphs of finite degree which can be decomposed by distance from any vertex [MS 85]. As the decomposition is dual to the generation, this implies that any pushdown transition graph is a rooted regular graph of finite degree, and this is a result due to Muller and Schupp. Furthermore the inverse inclusion remains true and this correspondance is effective.

Proposition 3.8 [Ca 90] *The pushdown transition graphs form effectively a complete set of representatives of the rooted regular graphs of finite degree.*

For instance every deterministic graph grammar generating a rooted graph G of finite degree, is mapped effectively into a pushdown automaton with an axiom such that its accessible prefix transition graph is isomorphic to G, and the reverse transformation is also effective.
To generalize Proposition 3.8 to all the regular graphs of finite degree, it suffices to take the class of graphs of all the prefix transitions of pushdown automata

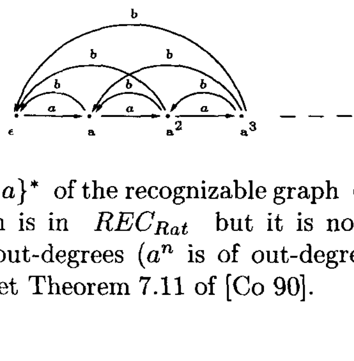
(or labelled rewriting systems) and to extend this class by restriction to rational vertex sets instead to the rational set of vertices accessible from an axiom.

Theorem 3.9 [Ca 95] *REC_{Fin} is effectively the family of regular graphs of finite degree.*

We get the regular graphs of infinite degree with inverse rational mappings.

Proposition 3.10 *REC_{Rat} $\supset$ effectively the regular graphs.*

For the strict containment, we consider the rational label-mapping h defined by $h(a) = a$ and $h(b) = \overline{a}^+$, and the rational language $L = a^*$.
Then $h^{-1}(\Delta_{\{a,b\}})_{|L} = h^{-1}(\Delta_{\{a\}})_{|L}$ is the following graph:



i.e. the right closure $G.\{a\}^*$ of the recognizable graph $G = \{\epsilon \xrightarrow{a} a \ , \ a^+ \xrightarrow{b} \epsilon\}$. By definition this graph is in REC_{Rat} but it is not regular because it has infinitely many vertex out-degrees (a^n is of out-degree $n+1$). It remains to apply Corollary 2.7 to get Theorem 7.11 of [Co 90].

Corollary 3.11 [Co 90] *Any regular graph has a decidable monadic theory.*

Thus Corollary 3.6 and Corollary 3.11 have been obtained by using the following complete set $REC_{|Rat}$ of representatives of REC_{Rat} (see Corollary 3.4). Although $REC_{|Rat}$ is obviously closed by rational restriction on vertices, it is not closed for instance by inverse finite label-mapping, nor by union. A simple extension is the family $\bigcup_f REC_{|Rat}$ of finite unions of graphs in $REC_{|Rat}$. But $\bigcup_f REC_{|Rat}$ is not closed by complement. So we introduce another complete set of representatives of REC_{Rat}. Following [Ch 82], we extend the right closures of recognizable graphs to rational right closures.

Definition 3.12 A *rational right closure of a recognizable graph* is a finite union of graphs

$$(U \xrightarrow{a} V).W \ := \ \{ \ uw \xrightarrow{a} vw \mid u \in U \ \wedge \ v \in V \ \wedge \ w \in W \ \}$$

where $U, V, W \in Rat(N^*)$.

This extension remains in REC_{Rat} .

Proposition 3.13 *The rational right closures of recognizable graphs form a complete set of representatives of* REC_{Rat} .

Contrary to $REC_{|Rat}$ the rational right closures of recognizable graphs are preserved by boolean operations, by composition and by transitive closure, hence by inverse rational label-mapping.

Theorem 3.14 *The rational right closures of recognizable graphs form an effective boolean algebra effectively preserved by inverse rational label-mapping and by rational restriction on vertices.*

Theorem 3.14 with Proposition 3.13 give Proposition 2.8.

Let us conclude with a remark. We have seen that the decision of the monadic theory is preserved by inverse rational label-mapping. But this is false by inverse context-free label-mapping. For instance with $N = \{x,y\}$, the following (linear) context-free mapping:

$h(a) = \{\overline{x}xx\ ,\ \overline{y}yx\}$ and $h(b) = \{\ (\overline{x}\overline{x}x{+}\overline{x}\overline{y}y)^n\overline{y}yy(\overline{x}xx + \overline{y}yx)^n \mid n \geq 0\ \}$ defines by inverse of Δ_N followed by the restriction to the rational language $L = x^*y^+$ a graph $h^{-1}(\Delta_N)_{\,|L}$ which is the usual grid having an undecidable monadic theory (see among others [Se 91]).

Acknowledgements

Let me thank Wolfgang Thomas for a remark at the origin of Theorem 3.1, hence of this paper. This paper has been partly supported by ESPRIT BRA 6317 (ASMICS). Due to a lack of space, this paper has been seriously reduced. The full version (with proofs) can be accessed by ftp irisa (anonymous)
in the file: (/site/irisa/ftp) local/caucal/monadic.ps.gz
or by Web ftp://ftp.irisa.fr/local/caucal/monadic.ps.gz

References

[Be 79] J. BERSTEL *Transductions and context-free languages*, Ed. Teubner, pp. 1–278, 1979.

[BN 84] L. BOASSON and M. NIVAT *Centers of context-free languages*, Internal Report LITP 84–44, 1984.

[Bü 64] R. BÜCHI *Regular canonical systems*, Archiv für Mathematische Logik und Grundlagenforschung 6, pp. 91–111, 1964 or in *The collected works of J. Richard Büchi*, edited by S. Mac Lane and D. Siefkes, Springer-Verlag, New York, pp. 317–337, 1990.

[Ca 90] D. CAUCAL *On the regular structure of prefix rewriting*, CAAP 90, LNCS 431, pp. 87–102, 1990, extended in TCS 106, pp. 61–86, 1992.

[Ca 92] D. CAUCAL *Monadic theory of term rewritings*, 7^{th} IEEE Symp., LICS 92, pp. 266–273, 1992.

[Ca 95] D. CAUCAL *Bisimulation of context-free grammars and of pushdown automata*, CSLI volume 53 "Modal logic and process algebra", pp. 85–106, Stanford, 1995. The full version is accessible by ftp (see acknowledgements) in the file bisimulation.ps.gz

[Ch 82] L. CHOTTIN *Langages algébriques et systèmes de réécriture rationnels*, RAIRO-TIA 16-2, pp. 93–112, 1982.

[Co 90] B. COURCELLE *Graph rewriting: an algebraic and logic approach*, Handbook of TCS, Vol. B, Elsevier, pp. 193–242, 1990.

[Co 94] B. COURCELLE *Monadic second-order definable graph transductions: a survey*, TCS 126, pp. 53–75, 1994.

[CW 95] B. COURCELLE and I. WALUKIEWICZ *Monadic second-order logic, graphs and unfolding of transition systems*, CSL 95, 1995.

[MS 85] D. MULLER and P. SCHUPP *The theory of ends, pushdown automata, and second-order logic*, TCS 37, pp. 51–75, 1985.

[Ra 69] M. RABIN *Decidability of second-order theories and automata on infinite trees*, Trans. Amer. Math. Soc. 141, pp. 1–35, 1969.

[Sa 79] J. SAKAROVITCH *Syntaxe des langages de Chomsky; essai sur le déterminisme*, Thèse de doctorat d'état Paris VII, pp. 1–175, 1979.

[Se 91] D. SEESE *The structure of the models of decidable monadic theories of graphs*, Annals of Pure and Applied Logic 53, pp. 169–195, 1991.

[Sem 84] A. SEMENOV *Decidability of monadic theories*, MFCS 84, LNCS 176, pp. 162–175, 1984.

[Sé 93] G. SÉNIZERGUES *Formal languages and word-rewriting*, Ecole de Printemps 93.

[Sh 75] S. SHELAH *The monadic theory of order*, Ann. Math. 102, pp. 379–419, 1975.

[St 75] J. STUPP *The lattice-model is recursive in the original model*, Manuscript, The Hebrew University, 1975.

[Th 90] W. THOMAS *Automata on infinite objects*, Handbook of TCS, Vol. B, Elsevier, pp. 135–191, 1990.

[Th 91] W. THOMAS *On logics, tilings, and automata*, ICALP 91, LNCS 510, pp. 191–231, 1991.

[W 96] I. WALUKIEWICZ *Monadic second order logic on tree-like structures*, STACS 96, LNCS 1046, pp. 401–413, 1996.