# Bi-Arc Graphs and the Complexity of List Homomorphisms

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Received July 12, 2001; Revised June 12, 2002

DOI 10.1002/jgt.10073

**Abstract:** Given graphs $G, H$, and lists $L(v) \subseteq V(H), v \in V(G)$, a list homomorphism of $G$ to $H$ with respect to the lists $L$ is a mapping $f: V(G) \to V(H)$ such that $uv \in E(G)$ implies $f(u)f(v) \in E(H)$, and $f(v) \in L(v)$ for all $v \in V(G)$. The list homomorphism problem for a fixed graph $H$ asks whether or not an input graph $G$, together with lists

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$L(v) \subseteq V(H)$, $v \in V(G)$, admits a list homomorphism with respect to $L$. In two earlier papers, we classified the complexity of the list homomorphism problem in two important special cases: When $H$ is a reflexive graph (every vertex has a loop), the problem is polynomial time solvable if $H$ is an interval graph, and is $NP$-complete otherwise. When $H$ is an irreflexive graph (no vertex has a loop), the problem is polynomial time solvable if $H$ is bipartite and $\bar{H}$ is a circular arc graph, and is $NP$-complete otherwise. In this paper, we extend these classifications to arbitrary graphs $H$ (each vertex may or may not have a loop). We introduce a new class of graphs, called bi-arc graphs, which contains both reflexive interval graphs (and no other reflexive graphs), and bipartite graphs with circular arc complements (and no other irreflexive graphs). We show that the problem is polynomial time solvable when $H$ is a bi-arc graph, and is $NP$-complete otherwise. In the case when $H$ is a tree (with loops allowed), we give a simpler algorithm based on a structural characterization.  J Graph Theory 42: 61–80, 2003

Keywords: *list colorings; list homomorphisms; internal graphs, circular arc graphs, bi-arc graphs; edge asteroids; sheet width; polynomial-time algorithms; NP-completeness; complexity*

## 1. INTRODUCTION

This paper deals with graphs in which loops are allowed. On the other hand, we do not consider parallel edges. (They are irrelevant from our point of view.) A graph in which every vertex has a loop is called *reflexive*; a graph in which no vertex has a loop called *irreflexive*. In most situations, the definitions as applied to these graphs are the natural ones. For instance, a bipartite graph is, by definition, irreflexive. The only exception we shall make is the concept of a tree. We shall define, by convention, a tree in a way that allows loops. A *tree* is a connected graph without cycles of length greater than one. (Thus the standard trees shall be called irreflexive trees.)

A *homomorphism* $f$ of a graph $G$ to a graph $H$ is a mapping $f : V(G) \to V(H)$ such that $uv \in E(G)$ implies $f(u)f(v) \in E(H)$. Note the role of vertices with loops. Firstly, if a vertex $u$ has a loop in $G$, then $f(u)$ must also have a loop in $H$. Secondly, if $u$ and $v$ are adjacent in $G$ and $f(u) = f(v)$, then the vertex $f(u)$ of $G$ must have a loop. Homomorphisms to graphs in which loops are allowed are of interest in statistical physics, cf. [6,7].

For a fixed graph $H$, the *homomorphism problem* HOM$H$ asks whether or not an input graph $G$ admits a homomorphism to $H$. When $H$ is the complete irreflexive graph $K_n$, a homomorphism of $G$ to $H$ is simply an $n$-colouring of $G$, and hence HOM$H$ is precisely the $n$-colouring problem. It is well known that the $n$-colouring problem is $NP$-complete, when $n > 2$ and polynomial time solvable, when $n \leq 2$. The complexity of all homomorphism problems HOM$H$ has been classified in [12]. HOM$H$ is $NP$-complete if $H$ is an irreflexive nonbipartite graph, and is polynomial time solvable otherwise.

Given graphs $G, H$, and lists $L(v) \subseteq V(H), v \in V(G)$, a *list homomorphism* of $G$ to $H$ with respect to the lists $L$ is a homomorphism $f$ of $G$ to $H$, such that $f(v) \in L(v)$ for all $v \in V(G)$. For a fixed graph $H$, the *list homomorphism problem* LIST-HOM$H$ asks whether or not an input graph $G$ with lists $L$ admits a list homomorphism to $H$ with respect to $L$. As above, when $H = K_n$, a list homomorphism of $G$ to $H$ is simply a list colouring of $G$, and hence LIST-HOM$H$ is the list coloring problem. The complexity of list colouring problems has been studied by J. Kratochvil and Z. Tuza [15]; among other results, they have proved that the problem is polynomial time solvable when each list has at most two elements, but *NP*-complete even if each list is restricted to have at most three elements. List homomorphisms and list colourings also generalize problems of colouring graphs with some precolored vertices [1,14].

A homomorphism of $G$ to $H$ is sometimes referred to as an *$H$-colouring* of $G$, and HOM$H$ is referred to as the *$H$-colouring problem* [12]. In the same way, we could call a list homomorphism of $G$ to $H$ a *list $H$*-colouring (both with respect to some $L$) and also call LIST-HOM$H$ the *list $H$-colouring problem*.

We have the following classifications of the complexity of LIST-HOM$H$ in the reflexive and irreflexive cases.

**Theorem 1.1** [8]. *Let H be a reflexive graph. Then the problem* LIST-HOM*H is polynomial time solvable if H is an interval graph, and is NP-complete otherwise.*

**Theorem 1.2** [9]. *Let H be an irreflexive graph. Then the problem* LIST-HOM*H is polynomial time solvable if H is a bipartite graph whose complement* $\bar{H}$ *is a circular arc graph, and is NP-complete otherwise.*

$H$ is an *interval graph* if it admits an *interval representation*, i.e., a mapping that associates to each vertex $x$ of $H$ a real interval $I_x$ so that two vertices $x$ and $y$ are adjacent in $H$ if and only if $I_x$ and $I_y$ intersect. $H$ is a *circular arc graph* if it admits a *circular arc representation*, whereby each vertex is associated with an arc on a circle, with two vertices adjacent if and only if the associated arcs intersect. Note that $H$ is bipartite if and only if $\bar{H}$ has clique covering number two, thus the problem is polynomial time solvable for those graphs $H$ whose complement is a circular arc graph of clique covering number two.

In this paper, we complete the classification of the complexity of all list homomorphism problems LIST-HOM$H$—without restrictions on $H$.

## 2. BI-ARC REPRESENTATIONS

The polynomial algorithms for LIST-HOM$H$ in Theorems 1.1 and 1.2 take advantage of the geometric representation of $H$, i.e., the interval representation of a reflexive $H$, and the circular arc representation of the complement of an irreflexive $H$. With that in mind, we introduce a new kind of geometric representation.

Let $C$ be a circle with two specified points $p$ and $q$ on $C$. A *bi-arc* is an ordered pair of arcs $(N, S)$ on $C$ such that $N$ contains $p$ but not $q$, and $S$ contains $q$ but not $p$. A graph $H$ is a *bi-arc graph* if there is a family of bi-arcs $\{(N_x, S_x)$: $x \in V(H)\}$ such that, for any $x, y \in V(H)$, not necessarily distinct, the following hold:

- if $x$ and $y$ are adjacent, then neither $N_x$ intersects $S_y$ nor $N_y$ intersects $S_x$;
- if $x$ and $y$ are not adjacent, then both $N_x$ intersects $S_y$ and $N_y$ intersects $S_x$.

We shall refer to $\{(N_x, S_x) : x \in V(H)\}$ as a *bi-arc representation* of $H$. Note that a bi-arc representation can not contain bi-arcs $(N, S), (N', S')$ such that $N$ intersects $S'$, but $S$ does not intersect $N'$ or vice versa.

The class of bi-arc graphs contains all reflexive interval graphs and all complements of irreflexive circular arc graphs of clique covering number two. We first show how to obtain a bi-arc representation of a reflexive interval graph $H$ from its interval representation (see Fig. 1). Suppose first that $H$ is reflexive and is represented by a family of intervals. Let $C$ be a circle with two specified points $p$ and $q$ diametrically opposed on $C$. We transfer the family of intervals to a family of arcs by mapping each interval to an arc on the open segment extending from $p$ to $q$ clockwise. That is, each vertex $v$ of $H$ corresponds to an arc $I_v$ from a point $a_v$ to a point $b_v$ clockwise in such a way that two *distinct* vertices $v$ and $v'$ are adjacent in $H$ if and only if $I_v$ intersects $I_{v'}$. We obtain a second set of arcs $J_v$ in a similar way by transfering the family of intervals to a family of arcs on the open segment extending from $q$ to $p$. In the second set of arcs, each vertex $v$ of $H$ corresponds to an arc $J_v$, from a point $c_v$ to a point $d_v$ clockwise, in the open segment from $q$ to $p$. Now our bi-arc representation of $H$ will consist of pairs $(N_v, S_v)$, with $N_v$ being the arc from $d_v$ to $a_v$, and $S_v$ being the arc from $b_v$ to $c_v$. Clearly, each $N_v$ contains $p$ but not $q$ and each $S_v$ contains $q$ but not $p$. To see that this is a bi-arc representation of $H$, first note that for each vertex $v$ (which is adjacent to itself) $N_v$ is disjoint from $S_v$. Then consider two distinct vertices $v$ and $v'$ of $H$. If $v$ and $v'$ are adjacent, then $I_v$ intersects $I_{v'}$ and $J_v$ intersects $J_{v'}$. Since neither $N_v$ nor $S_{v'}$ contains any point in the intersection of $I_v$ and $I_{v'}$, $N_v$ does not intersect $S_{v'}$; similarly, $N_{v'}$ does not intersect $S_v$. If $v$ and $v'$ are not adjacent, then $I_v$ is disjoint from $I_{v'}$ and $J_v$ is disjoint from $J_{v'}$. We see that $N_v$ intersects $S_{v'}$ and $N_{v'}$ intersects $S_v$.

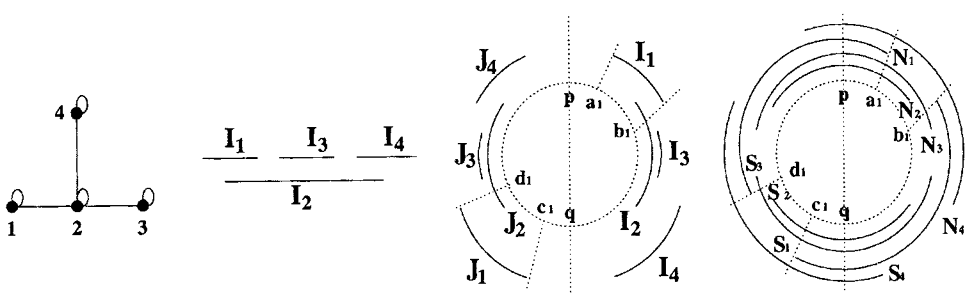


FIGURE 1

Suppose now that $H$ is a bipartite graph whose complement is a circular arc graph. Let $(A, B)$ be a bipartition of $H$. Let $C$ be a circle with the top point $p$, rightmost point $r$, bottom point $q$, and leftmost point $s$. According to [17] (see also [13]), $H$ can be represented by a family $\mathcal{F}_1 = \{N_v : v \in A\} \cup \{S_v : v \in B\}$ of arcs on the circle satisfying the following properties. Each arc $N_v$ contains both $p$ and $r$, but neither $q$ nor $s$, and each arc $S_v$ contains both $q$ and $s$, but neither $p$ nor $r$. Further, two vertices $v \in A$ and $v' \in B$ are adjacent in $H$ if and only if $N_v$ and $S_{v'}$ are disjoint. Let $\mathcal{F}_2 = \{S_v : v \in A\} \cup \{N_v : v \in B\}$ be a new family of arcs, where each $S_v$ with $v \in A$ (resp. $N_v$ with $v \in B$) is obtained by reflecting $N_v$ (resp. $S_v$) of $\mathcal{F}_1$ with respect to the line passing through $r$ and $s$. Thus, in $\mathcal{F}_2$, each arc $S_v$ contains both $q$ and $r$, but neither $p$ nor $s$, and each arc $N_v$ contains $p$ and $s$, but neither $q$ or $r$. Further, two vertices $v \in A$ and $v' \in B$ are adjacent in $H$ if and only if $S_v$ and $N_{v'}$ do not intersect. These two families $\mathcal{F}_1$ and $\mathcal{F}_2$ together form a bi-arc representation of $H$ (see Fig. 2).

As we shall show in the next section, the reflexive bi-arc graphs are precisely the reflexive interval graphs and the irreflexive bi-arc graphs are precisely the complements of irreflexive circular arc graphs of clique covering number two. Hence, when $H$ is reflexive or irreflexive, the list homomorphism problem LIST-HOM$H$ is polynomial time solvable if $H$ is a bi-arc graph and is *NP*-complete otherwise.

The main result of this paper is that this holds for *all* graphs.

**Theorem 2.1.** *The list homomorphism problem* LIST-HOM$H$ *is polynomial time solvable if H is a bi-arc graph, and is NP-complete otherwise.*

This will follow from Theorem 3.1 in the next section and Theorem 4.1 in Section 4.

## 3. *NP*-COMPLETENESS

Given a graph $H$, the *associated bipartite graph* $H^*$ is defined as follows. The vertex set of $H^*$ consists of sets $A_H = \{x' : x \in V(H)\}$ and $B_H = \{x'' : x \in V(H)\}$,

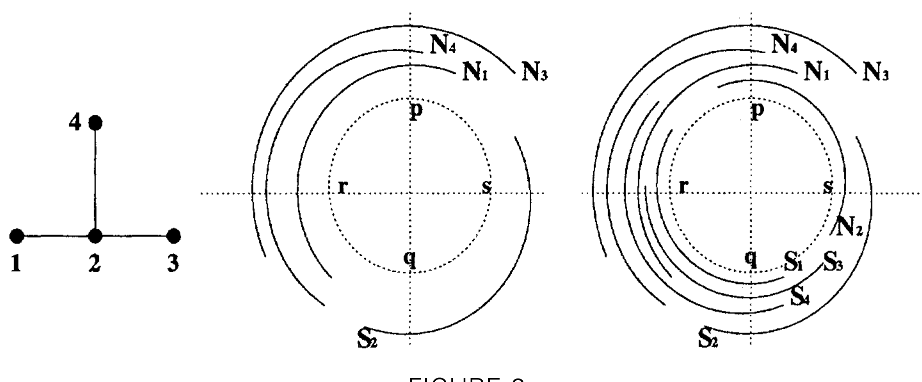


FIGURE 2

forming a bipartition of $H^*$. The edge set of $H^*$ consists of all edges $x'y''$ such that $xy$ is an edge of $H$. Note that there are two kinds of edges in $H^*$, namely edges $x'x''$ where $x$ has a loop in $H$, and edges $x'y''$ where $x \neq y$ and $xy \in E(H)$, which go in pairs, i.e., both $x'y''$ and $x''y'$ are present in $H^*$. (We observe that $H^*$ is the 'categorical product' of $H$ and $K_2$, cf. [4].)

A simple yet useful observation is that a bi-arc representation of $H$ corresponds precisely to a circular arc representation of the complement of $H^*$. Thus,

**Proposition 3.1.** *$H$ is a bi-arc graph if and only if the complement of $H^*$ is a circular arc graph.*

A *structure* $S = (V, E_1, E_2)$ consists of a finite set $V$, and two sets $E_1$ and $E_2$, each consisting of unordered pairs of elements from $V$. Each element of $V$ is called a *vertex* and each unordered pair of $E_1$ ($E_2$) is referred to as a *red* (*blue*) edge of the structure $S$. Although each of $(V, E_1)$ and $(V, E_2)$ is a graph, the structure $S = (V, E_1, E_2)$ is *not* a graph. However, a structure can be viewed as an edge-colored multigraph. If the vertex set $V$ can be partitioned into $A$ and $B$ such that no red or blue edge has its both ends in $A$ or in $B$, then the structure is called *bipartite* and $(A, B)$ is called a *bipartition* of the structure. Given a graph $H$, the *associated structure* $H^{**}$ has vertex set $V(H^{**}) = V(H^*)$, red edge set $E_1(H^{**}) = E(H^*)$, and blue edge set $E_2(H^{**}) = \{x'x'' : x \in V(H)\}$. Clearly, $H^{**}$ is bipartite with a bipartition $(A_H, B_H)$.

Let $S = (V, E_1, E_2)$ be a fixed structure. The *structure list homomorphism problem* LIST-HOM$S$ asks whether or not an input structure $Z = (V(Z), E_1(Z), E_2(Z))$, together with lists $L(v) \subseteq V, v \in V(Z)$, admits a mapping $f : V(Z) \to V$ such that $uv \in E_1(Z)$ implies $f(u)f(v) \in E_1$ and $uv \in E_2(Z)$ implies $f(u)f(v) \in E_2$. That is, each red edge is mapped to a red edge and each blue edge is mapped to a blue edge. We shall refer to such a mapping as a *structure list homomorphism* from $Z = (V(Z), E_1(Z), E_2(Z))$ to $S$ with respect to the lists $L$. Viewing a structure as an edge-colored multigraph, a structure list homomorphism is simply a color preserving list homomorphism. Color preserving homomorphisms of edge-colored multigraphs have been studied for instance in [2,3].

**Lemma 3.1.** *For any graph $H$, the list homomorphism problem* LIST-HOM$H$ *and the structure list homomorphism problem* LIST-HOM$H^{**}$ *are polynomially equivalent.*

***Proof.*** There is a natural way to transform an instance of LIST-HOM$H$ to an instance of LIST-HOM$H^{**}$ as follows: for each input $G$ of LIST-HOM$H$ with lists $L(v) \subseteq V(H)$, $v \in V(G)$, we let $G^{**}$ be the associated structure of $G$ defined above. To define the lists for $G^{**}$, let $L(v') = \{x' : x \in L(v)\}$ and $L(v'') = \{x'' : x \in L(v)\}$. It is easy to see that $G$ admits a list homomorphism to $H$ if and only if $G^{**}$ admits a structure list homomorphism to $H^{**}$.

We now show how to reduce LIST-HOM$H^{**}$ to LIST-HOM$H$. Consider an instance (structure) $Z = (V(Z), E_1(Z), E_2(Z))$ of LIST-HOM$H^{**}$, together with lists $L(v) \subseteq V(H^{**})$, $v \in V(Z)$. Since $H^{**}$ is bipartite, we may assume that $Z$ is also bipartite with a bipartition $(C, D)$. Further, we may assume that $L(v) \subseteq A_H$ for each $v \in C$, and $L(v) \subseteq B_H$ for each $v \in D$. (Recall that $(A_H, B_H)$ is a bipartition of $H^{**}$. Thus each list in $L$ consists of either primed or double-primed vertices.) We construct an instance of LIST-HOM$H$. Let $G$ be the graph constructed from $Z$ as follows. For each (nontrivial) component $T$ induced by blue edges, contract $T$ into a single vertex $u$ and remove all blue loops resulting from the blue edges in $T$. The lists $L_G$ of vertices of $G$ are defined as follows: Firstly, we remove the primes and double primes from all the lists $L$ in $Z$, i.e., replace each $x'$ and $x''$ by $x$, in all lists in $Z$. Secondly, if $u$ is contracted from $T$, we let $L_G(u) = \cap_{v \in V(T)} L(v)$. (The lists of uncontracted vertices do not change further.) The construction of $G$ and the lists $L_G$ can be accomplished in polynomial time. It is easy to verify that $Z$ admits a structure list homomorphism to $H^{**}$ with respect to the lists $L$, if and only if $G$ admits a list homomorphism to $H$ with respect to the lists $L_G$. ■

**Lemma 3.2.** *For any graph H, the list homomorphism problem* LIST-HOM$H^*$ *can be reduced, in polynomial time, to the structure list homomorphism problem* LIST-HOM$H^{**}$.

***Proof.*** Let $G = (V(G), E(G))$ be an input graph of LIST-HOM$H^*$, together with lists $L(v) \subseteq V(H^*)$, $v \in V(G)$. Let $Z = (V(Z), E_1(Z), E_2(Z))$ be the structure with $V(Z) = V(G)$, $E_1(Z) = E(G)$, and $E_2(Z) = \emptyset$. It is obvious that $G$ admits a list homomorphism to $H^*$ with respect to the lists $L$, if and only if $Z$ admits a structure list homomorphism to $H^{**}$ with respect to the lists L. ■

**Theorem 3.1.** *The list homomorphism problem* LIST-HOM$H$ *is NP-complete, unless H is a bi-arc graph.*

***Proof.*** Suppose that $H$ is not a bi-arc graph, i.e., that the complement of $H^*$ is not a circular arc graph of clique covering number two. By Theorem 1.2, LIST-HOM$H^*$ is *NP*-complete, and hence, by Lemmas 3.1 and 3.2, LIST-HOM$H$ is also *NP*-complete. ■

An *edge-asteroid* in a bipartite graph, with the bipartition $(X, Y)$, is a set of $2k + 1$ edges $u_0v_0, u_1v_1, \ldots, u_{2k}v_{2k}$ ($k \geq 1$ and each $u_i \in X$ and $v_i \in Y$), and $2k + 1$ paths, $P_{0,1}, P_{1,2}, \ldots, P_{2k,0}$, where each $P_{i,i+1}$ joins $u_i$ to $u_{i+1}$, such that for each $i = 0, 1, \ldots, 2k$ there is no edge between $\{u_i, v_i\}$ and $\{v_{i+k}, v_{i+k+1}\} \cup V(P_{i+k,i+k+1})$. (Subscripts are modulo $2k + 1$.) We refer to the (odd) integer $2k + 1$ as the *order* of the edge-asteroid. We have the following characterization of circular arc graphs of clique covering number two.

**Theorem 3.2** [9]. *Let H be a bipartite graph. Then the complement of H is a circular arc graph if and only if H contains no induced cycles of length greater than four and no edge-asteroids.*

**Corollary 3.1.** *If $H^*$ contains a cycle of length greater than four, or an edge-asteroid, then* LIST-HOM$H$ *is NP-complete.*

***Proof.*** This is implied by Theorems 3.1 and 3.2 together with the fact that $H$ is a bi-arc graph if and only if the complement of $H^*$ is a circular arc graph. ■

**Theorem 3.3.** *Let H be a reflexive graph. Then H is a bi-arc graph if and only if H is an interval graph.*

***Proof.*** Sufficiency is shown in the previous section, so we only show necessity. Suppose that $H$ is not an interval graph. By a well-known theorem of Lekkerkerker and Boland [16], $H$ contains either an induced cycle of length greater than three or an asteroidal triple, i.e., a set of three vertices $v_0, v_1, v_2$, such that for each $i \in \{0, 1, 2\}$ there is a path joining $v_i$ and $v_{i+1}$ and avoiding the neighbors of $v_{i+2}$ (subscripts are modulo 3). Assume first that $x_1x_2 \cdots x_lx_1$ $(l > 3)$ is an induced cycle of $H$. Then $H^*$ contains an induced cycle of length greater than four (namely, $x'_1x''_1x'_2x''_3x'_4 \cdots x'_{l-2}x''_{l-1}x'_lx''_lx'_1$, when $l$ is even, and $x'_1x''_1x'_2x''_3x'_4 \cdots x''l - 2x'_{l-1}x''_lx'_1$, when $l$ is odd). By Theorem 3.2, the complement of $H^*$ is not a circular arc graph, and hence $H$ is not a bi-arc graph. Assume that $\{v_0, v_1, v_2\}$ is an asteroidal triple in $H$. Let $x_{i_1}x_{i_2} \cdots x_{i_t}$ be a path joining $v_i$ and $v_{i+1}$ and avoiding the neighbors of $v_{i+2}$. Then $\{v'_0v''_0, v'_1v''_1, v'_2v''_2\}$ forms an edge-asteroid in $H^*$. Indeed, each $x'_{i_1}x''_{i_1}x'_{i_2}x''_{i_2} \cdots x'_{i_t}x''_{i_t}$ is a path joining $v'_iv''_i$ and $v'_{i+1}v''_{i+1}$ and avoiding the neighbors of $v'_{i+2}v''_{i+2}$. Again, by Theorem 3.2, the complement of $H^*$ is not a circular arc graph, implying that $H$ is not a bi-arc graph. ■

**Theorem 3.4.** *Let H be an irreflexive graph. Then H is a bi-arc graph if and only if it is the complement of a circular arc graph of clique covering number two.*

***Proof.*** Again it only remains to show necessity. So assume that $H$ is an irreflexive bi-arc graph. Suppose that $H$ contains an induced odd cycle, say $x_1x_2 \cdots x_{2k+1}x_1$. Then $H^*$ contains an induced cycle of length greater than four (namely, $x'_1x''_2x'_3 \cdots x''_{2k}x'_{2k+1}x''_1x'_2 \cdots x''_{2k+1}x'_1$). By Theorem 3.2, the complement of $H^*$ is not a circular arc graph, and hence $H$ is not a bi-arc graph, contradicting the hypothesis. Hence $H$ contains no induced odd cycles, and therefore it must be a bipartite graph. It follows that $H^*$ is the union of two disjoint copies of $H$. Since $H$ is a bi-arc graph, $H^*$ is the complement of a circular arc graph of clique covering number two. Therefore, the induced subgraph $H$ of $H^*$ is the complement of a circular arc graph of clique covering number two. ■

## 4. POLYNOMIAL TIME ALGORITHMS

Combining Theorems 1.1, 1.2, 3.3, and 3.4, we obtain the following facts.

**Corollary 4.1.** *When $H$ is a reflexive or an irreflexive bi-arc graph,* LIST-HOM$H$ *is polynomial time solvable.*

We proceed to prove that this remains so for all bi-arc graphs $H$.

**Theorem 4.1.** *When $H$ is any bi-arc graph,* LIST-HOM$H$ *is polynomial time solvable.*

We shall first consider the specialized algorithms for reflexive and irreflexive graphs. In [8,9] we gave algorithms which use the geometric representation of $H$ (by intervals or circular arcs) to encode LIST-HOM$H$ as an instance of 2-satisfiability. We now describe an alternate approach.

Let $H$ be an arbitrary graph. A function $g : V(H) \times V(H) \times V(H) \to V(H)$ is called a *majority choice function* if it satisfies the following properties:

(1) $g(x, y, z) \in \{x, y, z\}$,
(2) $g(x, w, w) = g(w, y, w) = g(w, w, z) = w$,
(3) $g(x, y, z)g(x', y', z') \in E(H)$ whenever $xx', yy', zz' \in E(H)$.

Feder and Vardi [11] proved that if $H$ admits a majority choice function, then LIST-HOM$H$ is polynomial time solvable. In fact, they consider the following inference procedure [11].

For each pair of distinct vertices $u, v \in V(G)$, let $S_{uv} = (L(u) \times L(v)) \cap E(H)$ if $uv \in E(G)$; otherwise let $S_{uv} = L(u) \times L(v)$. Here we assume that if $u \in V(G)$ has a loop, then every element of $L(u)$ also has a loop.

Next, repeatedly perform the following update operation for triples of distinct vertices $u, v, w \in V(G)$:

$$S_{uv} \leftarrow \{ab \in S_{uv} : \exists\, c \in V(H), ac \in S_{uw}, \text{ and } bc \in S_{vw}\}.$$

If no set $S_{uv}$ can be further reduced by an application of this operation, then the updating process terminates.

It is clear that if $f$ is a list homomorphism of $G$ to $H$ with respect to the lists $L$, then at all times we must have $f(u)f(v) \in S_{uv}$ for all distinct $u, v \in V(G)$. In particular, if the sets $S_{uv}$ become empty, then no list homomorphism exists.

On the other hand, in certain situations, we may be able to conclude conversely that if all sets $S_{uv}$ are nonempty, a list homomorphism of $G$ to $H$ with respect to the lists $L$ exists: We say that $H$ has *strict width two* if, for all graphs $G$ and lists $L$, whenever $f : U \to V(H)$, for some $U \subseteq V(G)$ with $|U| \geq 2$, is a mapping such that $f(u)f(v) \in S_{uv}$ for all distinct pairs of vertices $u, v \in U$, then there exists a list homomorphism $\hat{f}$ of $G$ to $H$ (with respect to the lists $L$) such that $\hat{f}(v) = f(v)$ for all $v \in U$. Feder and Vardi showed [11] that $H$ admits a majority choice function if and only if it has strict width two. (The terminology of [11] applies the term 'strict width two' to the *problem* LIST-HOM$H$, rather than to the graph $H$ as we do here. The results of [11] also apply in the more general context of constraint satisfaction problems.)

We now see that if $H$ has strict width two, then the fact that all sets $S_{uv}$ are nonempty, implies that a list homomorphism of $G$ to $H$ with respect to the lists $L$ exists (and can be efficiently found). Choose some $U \subseteq V(G)$ with $|U| = 2$, say $U = \{u, v\}$, and choose some $f(u)f(v) \in S_{uv}$. Then repeatedly choose a single vertex $w \in V(G) - U$ and a vertex $c \in V(H)$ such that $f(u)c \in S_{uw}$ for all $u \in U$, so that we can proceed to add $w$ to $U$ by setting $f(w) = c$. The process terminates with $U = V(G)$ and a list homomorphism $f$.

We now explain how to use these techniques to give new polynomial time algorithms for finding list homomorphisms to a reflexive interval graph $H$, and to an irreflexive bipartite graph $H$ whose complement is a circular arc graph.

**Lemma 4.1.** *If $H$ is a reflexive interval graph, or an irreflexive bipartite graph whose complement is a circular arc graph, then $H$ admits a majority choice function, i.e., has strict width two.*

***Proof.*** Suppose first that $H$ is a reflexive interval graph, and consider an interval representation of $H$. We shall define a majority choice function $g$ on $H$. Consider $x, y, z \in V(H)$, and the intervals $I_x, I_y, I_z$ representing them. Let $I$ be the interval whose left endpoint is the middle point of the three left endpoints of $I_x, I_y, I_z$, and whose right endpoint is the middle point of the three right endpoints of $I_x, I_y, I_z$. One of the three intervals $I_x, I_y, I_z$, say $I_t$, must contain $I$, since two of the three intervals have left endpoint not to the right of the left endpoint of $I$, and two of the three intervals have right endpoint not to the left of the right endpoint of $I$. We define $g(x, y, z) = t$. (If two of $x, y, z$ are the same vertex $a$, then we can make sure that $t$ was chosen equal to $a$, since $I_a$ will contain both the above middle points.)

It remains to show that if $xx', yy', zz' \in E(H)$, then $tt' \in E(H)$ for $t = g(x, y, z)$ and $t' = g(x', y', z')$. Suppose to the contrary that $I_t$ is disjoint from $I_{t'}$, say $I_t$ precedes $I_{t'}$. Then at least two of $I_x, I_y, I_z$ precede at least two of $I_{x'}, I_{y'}, I_{z'}$, so either $I_x$ is disjoint from $I_{x'}$, or $I_y$ from $I_{y'}$, or $I_z$ from $I_{z'}$, a contradiction.

If $H$ is an irreflexive bipartite graph (with parts $V_1, V_2$), whose complement is a circular arc graph, the construction is similar. If two arguments are in $V_1$ and one in $V_2$, we can let $g$ take the value of the first argument from $V_1$, and similarly if two arguments are in $V_2$ and one in $V_1$ let $g$ take the value of the first argument from $V_2$. Thus it remains to define $g$ when the three arguments are from the same part. Consider a circular arc representation of the complement of $H$. Vertices $x$ in $V_1$ are represented by arcs $N_x$ containing $p$ but not $q$, vertices $x'$ in $V_2$ by arcs $S_{x'}$ containing $q$ but not $p$, and $x \in V_1$ is adjacent in $H$ to $x' \in V_2$ if and only if $N_x$ is disjoint from $S_{x'}$. When $x, y, z \in V_1$, we treat the arcs $N_x, N_y, N_z$ as intervals in the line obtained by removing $q$, and choose $t$ from $x, y, z$ so that $N_t$ is *contained in* the arc whose left endpoint is the middle point of the three left endpoints of $N_x, N_y, N_z$, and whose right endpoint is the middle point of the three right endpoints of $N_x, N_y, N_z$. We define $g(x, y, z) = t$, making sure once again that if two of $x, y, z$ equal to $a$, then $t = a$. The definition of $g$ when all three arguments $x', y', z'$ are in $V_2$ is similar, by considering the corresponding intervals $S_{x'}, S_{y'}, S_{z'}$

containing $q$ but not $p$. The proof that $g$ is a majority choice function is analogous to the case of reflexive interval graphs. ■

Note that this lemma implies the existence of the polynomial algorithms in Theorems 1.1 and 1.2.

One may try to define a majority choice function for any bi-arc graph $H$; this however does not appear to be an easy task. We show instead that $H$ inherits strict width two from $H^*$ (the associated bipartite graph of $H$).

**Lemma 4.2.** *If $H^*$ has strict width two, then $H$ also has strict width two.*

***Proof.*** Let $G$ be an instance of LIST-HOM$H$. Consider the final sets $S_{uv}$ obtained with the above inference procedure, and suppose they are nonempty. Suppose we have a mapping $f : U \to V(H)$ for some $U \subseteq V(G)$ with $|U| \geq 2$ such that $f(u)f(v) \in S_{uv}$ for all pairs of distinct $u, v \in U$. To establish the strict width 2 property, we must show that there exists a list homomorphism $\hat{f}$ of $G$ to $H$ such that $\hat{f}(u) = f(u)$ for $u \in U$.

For distinct vertices $u, v \in V(G)$, let $S_u = \{a \in V(H) : \exists b \in V(H), ab \in S_{uv}\}$; note that $S_u$ does not depend on the choice of $v$. Let $G^*$ be the associated bipartite graph for $G$. If $u \in V(G)$, define lists for the corresponding vertices in $G^*$ by $L^*(u') = \{a' : a \in S_u\}$, and $L^*(u'') = \{a'' : a \in S_u\}$. Treat $G^*$ with these lists as an instance of LIST-HOM$H^*$, which has strict width 2, and obtain final sets $S^*_{uv}$. Note by comparison of the two inference procedures, that the sets $S^*_{uv}$ are no more restrictive than the sets $S_{uv}$. That is, if $a \in S_u$, then $a'a'' \in S^*_{u'u''}$; if $ab \in S_{uv}$, then $a'b' \in S^*_{u'v'}$, $a'b'' \in S^*_{u'v''}$, $a''b' \in S^*_{u''v'}$, and $a''b'' \in S^*_{u''v''}$. This follows by induction on the successive updates of the $S^*_{uv}$ when compared with the final $S_{uv}$ (which satisfy their update rule with equality).

Consider the mapping $f^*$ that sets $f^*(u') = a'$ and $f^*(u'') = a''$, whenever $f(u) = a$. By the above observation, this mapping satisfies the $S^*_{uv}$ conditions, and since LIST-HOM$H^*$ has strict width 2, there must exist a list homomorphism $\hat{f}^*_1$ of $G^*$ to $H^*$ that coincides with $f^*$ where $f^*$ is defined.

Say that a pair of vertices $u', u'' \in V(G^*)$ corresponding to $u \in V(G)$ is *matched* in $\hat{f}^*_1$ if whenever $\hat{f}^*_1(u') = a'$, we also have $\hat{f}^*_1(u'') = a''$. If all such pairs are matched in $\hat{f}^*_1$, then we can let $\hat{f}$ be the corresponding list homomorphism of $G$ to $H$, i.e., $\hat{f}(u) = a$ if $\hat{f}^*_1(u') = a'$ and $\hat{f}^*_1(u'') = a''$; this list homomorphism extends $f$ as desired.

Suppose that not all pairs are matched in $\hat{f}^*_1$. We show how to select any one unmatched pair $u', u''$, and obtain a different list homomorphism $\hat{f}^*$ that has all the matched pairs from $\hat{f}^*_1$, with the same corresponding values as in $\hat{f}^*_1$, and in addition the pair $u', u''$ is also matched in $\hat{f}^*$. Repeating this process, we eventually get all pairs matched and this gives as mentioned above the desired list homomorphism of $G$ to $H$ that extends $f$.

First, we define the list homomorphism $\hat{f}^*_2$ of $G^*$ to $H^*$ obtained from $\hat{f}^*_1$ by exchanging the two sides of the bipartite graphs $G^*, H^*$, by taking advantage of the symmetry of these two graphs and of the lists $L^*$. That is, if $\hat{f}_1^*(v') = b'$ and $\hat{f}_1^*(v'') = c''$, we let $\hat{f}_2^*(v') = c'$ and $\hat{f}_2^*(v'') = b''$. Note that $\hat{f}_1^*$ and $\hat{f}_2^*$ have the same matched pairs with the same corresponding values.

Consider now the partial mapping that sets $f^*(u') = \hat{f}_1^*(u')$ and $f^*(u'') = \hat{f}_1^*(u'')$ for all matched pairs $u', u''$ in $\hat{f}_1^*$, and for some pair $v', v''$ as above that is not matched, sets $f^*(v') = \hat{f}_1^*(v') = b'$ and $f^*(v'') = \hat{f}_2^*(v'') = b''$. The mapping $f^*$ satisfies all the $S^*_{uv}$ conditions, since it satisfies those not involving $v''$ because of $\hat{f}_1^*$, those not involving $v'$ because of $\hat{f}_2^*$, and the single extra condition $b'b'' \in S^*_{v'v''}$, as mentioned before, because $b \in S_v$.

Therefore, since LIST-HOM$H^*$ has strict width 2, $f^*$ can be extended to a list homomorphism $\hat{f}^*$ of $G^*$ to $H^*$ with at least one more matched pair than in $\hat{f}_1^*$, while the pairs that were already matched in $\hat{f}_1^*$ have the same values in $\hat{f}^*$. By induction, we eventually make all pairs be matched, and then $\hat{f}^*$ yields the desired list homomorphism $\hat{f}$ of $G$ to $H$. This completes the proof of the lemma. ■

If $H$ is a bi-arc graph, then the associated bipartite graph $H^*$ is the complement of a circular arc graph with clique covering number two. Thus $H^*$ has strict width two, and therefore $H$ has also strict width two, and hence LIST-HOM$H$ is polynomial time solvable. This completes the proof of Theorem 4.1 and hence also of our main theorem. ■

**Corollary 4.2.** *Every bi-arc graph admits a majority choice function.*

We observe that if $P \neq NP$, then the converse also holds, as a graph $H$ which is not a bi-arc graph yields an $NP$-complete problem LIST-HOM$H$, and hence $H$ cannot have a majority choice function. In another paper [4] we prove this converse without assuming $P \neq NP$.

**Theorem 4.2.** [4]. *A graph admits a majority choice function if and only if it is a bi-arc graph.*

## 5. BI-ARC TREES

Since each bi-arc graph $H$ has strict width two, it must have a majority choice function. We do not know a direct construction of a majority choice function for an arbitrary bi-arc graph $H$. It is always possible to find such a function $g$ for a given bi-arc graph $H$ in polynomial time, if we so desire, by expressing the conditions on $g$ as an instance of LIST-HOM$H$ of size cubic in the size of $H$, and solving the problem by the above inference procedure. In this section, we explicitly construct these majority choice functions for bi-arc *trees*. The construction relies on the structure of these trees, described in Theorem 5.1, and proved in a companion paper [10].

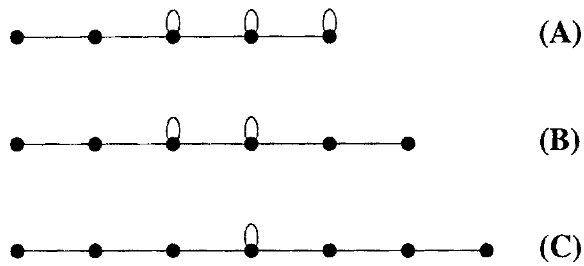

FIGURE 3

For a graph $H$, we use $H^o$ to denote the subgraph of $H$ induced by vertices which have loops. We say that $H^o$ is *convex*, if every induced path $P$ of $H$ joining two vertices of $H^o$ satisfies $V(P) \subseteq V(H^o)$.

**Proposition 5.1.** *If $H$ is a bi-arc graph, then $H^o$ is convex.*

***Proof.*** Suppose that $H^o$ is not convex. Then there is an induced path $x_1x_2\cdots x_k$ ($k \geq 3$), such that $x_i \in V(H^o)$ if and only if $i = 1$ or $k$. Thus in $H^*$, the vertices $x_1', x_1'', x_2', x_2'', \ldots, x_k', x_k''$ form an induced cycle of length greater than four. By Theorem 3.2, the complement of $H^*$ is not a circular arc graph and hence $H$ is not a bi-arc graph. ■

In [10], we verify that if $H$ is one of graphs in Figure 3 or 4, then $H^*$ contains an edge-asteroid. Hence we have the following.

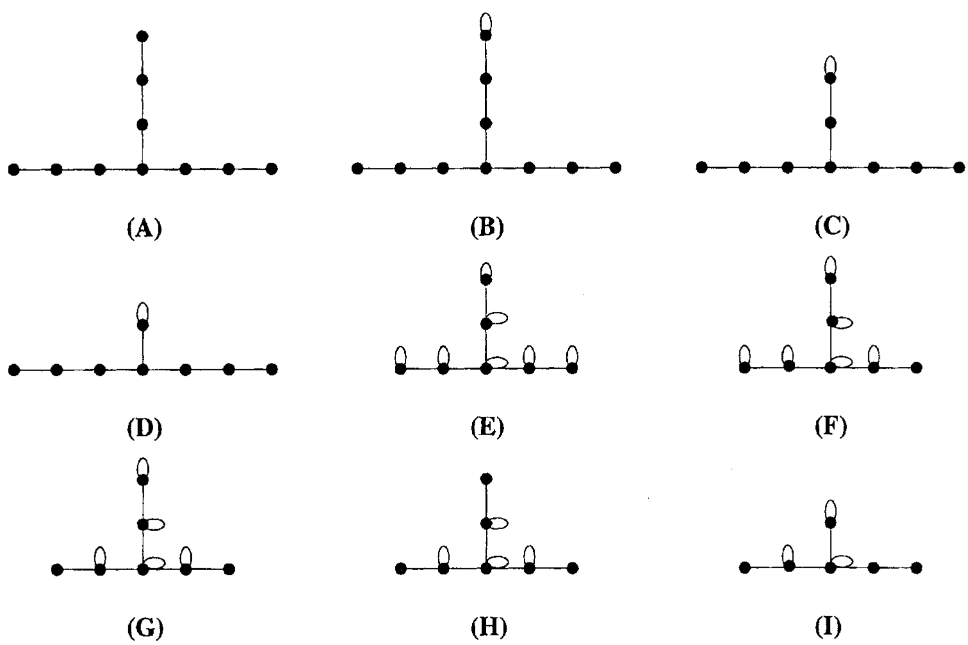

FIGURE 4

**Proposition 5.2** [10]. *A bi-arc graph does not contain any graph from Figure* 3 *or from Figure* 4 *as an induced subgraph.*

When $H$ is a reflexive tree, $H$ is an interval graph if and only if it does not contain the graph in Figure 4E [16]. Such a tree is also known as a *reflexive caterpillar.* Thus it follows from Theorem 3.3 and Corollary 4.1 that in the case when $H$ is reflexive, LIST-HOM$H$ is polynomial time solvable if $H$ is a caterpillar, and is *NP*-complete otherwise. We shall refer to a reflexive caterpillar also as a *reflexive polynomial tree*. When $H$ is an irreflexive tree, it follows from Theorem 3.2 that the complement of $H$ is a circular arc graph if and only if $H$ does not contain the graph in Figure 4A (since $H$ does not contain any cycle and if it contains an edge-asteroid then it must contain the graph in Fig. 4A). Hence, by Theorem 3.4 and Corollary 4.1, when $H$ is irreflexive, LIST-HOM$H$ is polynomial time solvable if $H$ does not contain the graph in Figure 4A, and is *NP*-complete otherwise. We shall refer to an irreflexive tree which does not contain the graph in Figure 4A as an *irreflexive polynomial tree*.

We define a vertex $v$ in a tree to be a *good vertex* if there does not exist a path $P$ of length 6, with the middle vertex $u$, such that $v$ is joined to $u$ by a path (possibly of length 0) which is internally vertex-disjoint from $P$. For instance, every vertex of the graph in Figure 3A is a good vertex and so is every vertex of the graph in Figure 3B. All vertices except the center vertex of the graph in Figure 3C are good vertices. The graph in Figure 4A has no good vertices.

In [10], we prove that the above figures list all forbidden subtrees for polynomial trees. In fact, we have the following structure theorem.

**Theorem 5.1** [10]. *Let $H$ be a tree such that $H^o$ is a subtree of $H$. Then the following three statements are equivalent:*

(i) *$H$ is a bi-arc graph;*
(ii) *$H$ does not contain as an induced subgraph any graph from Figure 3 or from Figure 4;*
(iii) *$H$ is obtained either*
*from a reflexive polynomial tree by deleting the loops at a subset (possibly empty) of leaves, or*
*from an irreflexive polynomial tree by performing one of the following three operations:*

- *adding no loops, or adding a loop on a good vertex $v$,*
- *adding a loop on a good vertex $v$ and on a neighbor $w$ of $v$, such that each neighbor of $w$ other than $v$ is a leaf,*
- *adding a loop on a good vertex $v$, and loops on any number of neighbors of $v$ which are leaves.*

The cases are illustrated in Figures 5 and 6.

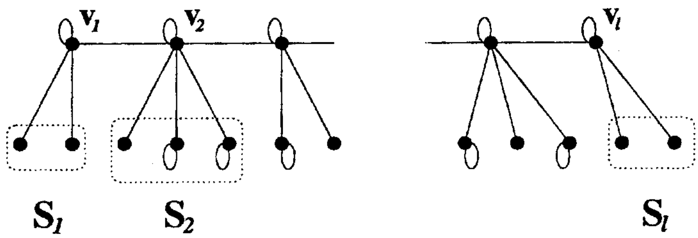


FIGURE 5

Recall that we have no explicit construction of the majority choice function for an arbitrary bi-arc graph (even though Corollary 4.2 guarantees that one exists). We shall now explicitly construct these majority choice functions for bi-arc trees, using the structure theorem, Theorem 5.1.

## A. *H* Is Obtained From a Reflexive Polynomial Tree

Suppose $H$ is a bi-arc tree obtained from a reflexive polynomial tree by deleting the loops at a subset of leaves. We may assume that $V(H^o) \neq \emptyset$ because of Lemma 4.1. Let $P : v_1 v_2 \cdots v_l$ be a longest path in $H^o$. Since $H$ is a caterpillar, so is $H^o$. Each vertex of $H^o - V(P)$ and hence of $H - V(P)$ is a leaf of $H$. Let $S_i$ be the set of vertices of $H - V(P)$ adjacent to $v_i \in V(P)$ for each $i = 1, 2, \ldots, l$. Thus for each $x \in V(H)$, there exists a unique $\pi(x)$ such that $x \in S_{\pi(x)} \cup \{v_{\pi(x)}\}$. For each $i$, $i = 1, 2, \ldots, l$, we let $O_i$ be an ordering of the vertices of $S_i \cup \{v_i\}$, in which $v_i$ is the first vertex.

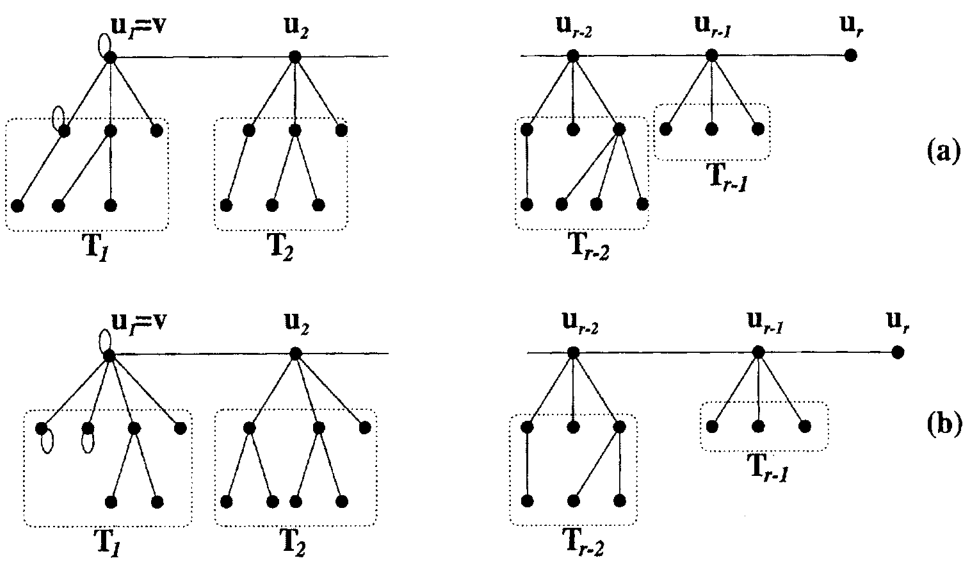


FIGURE 6

We define a function $f : V(H) \times V(H) \times V(H) \to V(H)$ as follows. Let $x, y, z$ be three vertices of $H$. If any two of $x, y, z$ are the same vertex, say $w$, then $f(x, y, z) = w$; otherwise, let $m$ be the median of $\pi(x), \pi(y), \pi(z)$ and set $f(x, y, z)$ to be the first vertex of $\{x, y, z\}$ in the ordering $O_m$.

**Proposition 5.3.** *The function $f$ is a majority choice function.*

***Proof.*** Clearly, $f$ satisfies Properties 1 and 2 in the definition of a majority choice function. To verify Property 3, let $xx', yy', zz'$ be three edges (each possibly a loop) of $H$. Let $m$ be the median of $\pi(x), \pi(y), \pi(z)$, and $m'$ the median of $\pi(x'), \pi(y'), \pi(z')$. Then we must have $|m - m'| \leq 1$. If, say $m \leq m' - 2$, then at least two of $x, y, z$ have $\pi$ values less than or equal to $m$ and at least two of $x', y', z'$ have $\pi$ values greater than or equal to $m + 2$, which contradicts the fact that $xx', yy', zz'$ are all edges of $H$.

Assume first that $x, y, z$ are distinct, and $x', y', z'$ are also distinct. If $v_m \in \{x, y, z\}$ and $v_{m'} \in \{x', y', z'\}$, then $f(x, y, z) = v_m$ is adjacent to $f(x', y', z') = v_{m'}$. If $v_m \notin \{x, y, z\}$ or $v_{m'} \notin \{x', y', z'\}$, then we must have $m = m'$. In this case, if $v_m \in \{x, y, z, x', y', z'\}$, then one of $f(x, y, z)$ and $f(x', y', z')$ is $v_m$ and the other is in $S_m$. Hence $f(x, y, z)$ and $f(x', y', z')$ are adjacent. So assume that $v_m \notin \{x, y, z, x', y', z'\}$. It is then easy to see that $S_m \cap \{x, y, z\} = S_m \cap \{x', y', z'\}$ and each vertex of $S_m \cap \{x, y, z\}$ has a loop. This implies that $f(x, y, z)$ and $f(x', y', z')$ are adjacent.

Assume without loss of generality that two of $x, y, z$ are the same vertex $w$. By definition, $f(x, y, z) = w$. If any two of $x', y', z'$ are the same vertex $w'$, then clearly $ww' \in E(H)$, i.e., $f(x, y, z)f(x', y', z') \in E(H)$. So assume that $x', y', z'$ are all distinct. Let $v$ be the first vertex in $O_{m'}$ from $\{x', y', z'\}$. By definition, $f(x', y', z') = v$. Since $w$ is adjacent to at least two of $x', y', z'$, $w$ is adjacent to $v$, i.e., $f(x, y, z)f(x', y', z') \in E(H)$. ■

## B. *H* Is Obtained From an Irreflexive Polynomial Tree

Suppose that $H$ is obtained from an irreflexive polynomial tree by performing one of the three operations listed in Theorem 5.1. We may again assume that $H$ contains at least one loop. Let $Q : u_1 u_2 \cdots u_r$ be a longest path in $H$ such that $u_1$ is the vertex $v$ in $V^o(H)$ described in Theorem 5.1, and it is the only vertex in $Q$ that has a loop. Note that the vertices of $H - V(Q)$ can again be partitioned into sets $T_1, T_2, \ldots, T_r$ such that $T_i \cup \{u_i\}$ induces a subtree for each $i = 1, 2, \ldots, r$. (Note that the maximality of $Q$ assures that $T_r = \emptyset$.) Thus for each $x \in V(H)$, there is again a unique $\sigma(x)$ such that $x \in T_{\sigma(x)} \cup \{u_{\sigma(x)}\}$. Note that $\sigma(x) = 1$ for all $x \in V(H^o)$. Since $v$ is a good vertex and $Q$ is a longest path, the distance between $u_i$ and any other vertex in $T_i \cup \{u_i\}$ is either one or two.

We order the vertices of $H$ as follows. First for each $i = 1, 2, \ldots, r$ we obtain an ordering $O_i$ of the vertices of $T_i \cup \{u_i\}$, using the depth-first search starting at $u_i$. When $i = 1$, the search is in favor of vertices of $H^o$ (note that

$V(H^o) \subseteq T_1 \cup \{u_1\}$). Then we combine these orderings into a vertex ordering $O$ of $H$ by concatenating them in the order $O_1, O_2, \ldots, O_r$. Suppose that $O$ is the ordering $v_1, v_2, \ldots, v_n$. Next, we color the vertices of $H$ with two colors so that two distinct adjacent vertices receive different colors. Call each of the two sets of vertices of the same color a *color class*.

We now define a function $g : V(H) \times V(H) \times V(H) \to V(H)$ as follows: Let $x, y, z$ be three vertices of $H$.

(a) If any two of $x, y, z$ are the same vertex, say $w$, then $g(x, y, z) = w$.
(b) Suppose that $x, y, z$ are distinct and in the same color class. Let $m$ be the median of $\sigma(x)$, $\sigma(y)$, $\sigma(z)$. We define $g(x, y, z)$ to be the *first* vertex among $x, y, z$ in the ordering $O_m$, except in the following four exceptional situations:

- all three vertices $x, y, z$ lie in $T_m$ with $m \geq 2$;
- all three vertices $x, y, z$ lie in $T_1$ and at most one of them has a loop;
- exactly two of $x, y, z$ lie in $T_1$ and exactly one of them has a loop;
- exactly two of $x, y, z$ lie in $T_1$, neither has a loop, exactly one of them is adjacent to the unique neighbor of $u_1$ with a loop, and the third vertex of $x, y, z$ is not $u_1$. (This last case only applies in the situation in Fig. 6A.)

In these situations, we define $g(x, y, z)$ be the *second* vertex among $x, y, z$ in the ordering $O_m$.

(c) Suppose that $x, y, z$ are distinct but not all in the same color class. We define $g(x, y, z)$ to be the first vertex in $O$ of the two vertices in the same color class, except when $\{x, y, z\}$ contains $u_1$ and at least one of its leaf neighbors with a loop, in which case we define $g(x, y, z) = u_1$. (This exceptional case only applies in the situation in Fig. 6B.)

**Proposition 5.4.** *The function g is a majority choice function.*

***Proof.*** To verify that $g$ is a majority choice function, we need only show $g(x, y, z)g(x', y', z') \in E(H)$ when $xx', yy', zz' \in E(H)$.

Suppose that two of $x, y, z$ are the same vertex $w$ (and thus $g(x, y, z) = w$). If any two of $x', y', z'$ are the same, then clearly $g(x, y, z)g(x', y', z') \in E(H)$. So assume $x', y', z'$ are distinct. We first consider the case when $x', y', z'$ are in the same color class. Let $m'$ be the median of $\sigma(x')$, $\sigma(y')$, $\sigma(z')$. Suppose that $m' \geq 2$ and $\{x', y', z'\} \subseteq T_{m'}$. By definition, $g(x', y', z')$ is the second vertex of $\{x', y', z'\}$ in $O_{m'}$. Since the vertex $w$ is adjacent to at least two vertices of $\{x', y', z'\}$, it must be in $T_{m'} \cup \{u_{m'}\}$, and the depth-first search ordering $O_{m'}$ ensures that $w$ is adjacent to $g(x', y', z')$. A similar discussion applies when $\{x', y', z'\} \subseteq T_1$ but at most one of $x', y', z'$ has a loop. Suppose that exactly two of $x', y', z'$ are in $T_1$. If exactly one of them has a loop, then both of them are neighbors of $u_1$ and we must have $w = u_1$; since $w$ is adjacent to at least two vertices of $x', y', z'$ it can not be the case that neither has a loop, exactly one is adjacent to the neighbor of $u_1$ with a loop, and $u_1 \notin \{x', y', z'\}$. Now in all other cases $g(x', y', z')$ is defined to be the first vertex of $x', y', z'$ in $O_{m'}$. If $u_{m'} \in \{x', y', z'\}$, then $g(x', y', z') = u_{m'}$ and it is easy to see that $w$ is adjacent to $u_{m'}$. So assume that $u_{m'} \notin \{x', y', z'\}$. If $m' \geq 2$ and $T_{m'}$ contains one or two vertices from $\{x', y', z'\}$, then $w \in T_{m'} \cup \{u_{m'}\}$ and $w$ must be adjacent to the first vertex of $\{x', y', z'\}$ in the ordering $O_{m'}$, which is $g(x', y', z')$. If at least two of $x', y', z'$ are in $T_1$ and both of them have loops, then they are both neighbors of $u_1$ and we must have that $w = u_1$. If exactly two of $x', y', z'$ are in $T_1$, and neither has a loop or is adjacent to a neighbor of $u_1$ with a loop, then it is also easy to see that $w$ is adjacent to the first vertex of $x', y', z'$, which is $g(x', y', z')$. We now consider the case that $x', y', z'$ are not all in the same color class. Without loss of generality, assume that $y', z'$ are in the same color class and that $y'$ precedes $z'$ in $O$. Then by definition $g(x', y', z') = x'$ or $g(x', y', z') = y'$. When $g(x', y', z') = x'$, it must be the case that $x' = u_1$ and $y'$ is a leaf neighbor of $u_1$ with a loop. It is easy to see that $w$ must be adjacent to $u_1 = g(x', y', z')$. On the other hand, when $g(x', y', z') = y'$, $wy' \in E(H)$, as otherwise at least one of $x'$ and $z'$ would have to have a loop, which is easy to see can not be the case.

Suppose that $x, y, z$ are distinct and in the same color class, and that $x', y', z'$ are also distinct and in the same color class. Observe that either $x = x'$, $y = y'$, and $z = z'$ or $\{x, y, z\}$ and $\{x', y', z'\}$ are in different color classes. In the former case, each of $x, y, z$ has a loop and clearly $g(x, y, z)g(x', y', z') \in E(H)$. For the latter case, assume first that $\sigma(x) < \sigma(y) < \sigma(z)$. It is easy to see that $\sigma(x') \leq \sigma(y') \leq \sigma(z')$. Since $x', y', z'$ are distinct, either $\sigma(x') < \sigma(y')$ or $\sigma(y') < \sigma(z')$. Thus, there are three possibilities. If $\sigma(x') < \sigma(y') < \sigma(z')$, then it is easy to see that $g(x, y, z)g(x', y', z') = yy' \in E(H)$. If $\sigma(x') < \sigma(y') = \sigma(z')$, then $\sigma(y) = \sigma(z) - 1$ and we have either $g(x, y, z)g(x', y', z') = yy' = u_{\sigma(y)}u_{\sigma(z)} \in E(H)$ or $g(x, y, z)$ $g(x', y', z') = yz' = yu_{\sigma(y)} \in E(H)$. Finally, if $\sigma(x') = \sigma(y') < \sigma(z')$, then $\sigma(y) = \sigma(x) + 1$, and we have either $g(x, y, z)g(x', y', z') = yy' = u_{\sigma(y)}u_{\sigma(x)} \in E(H)$ or $g(x, y, z)g(x', y', z') = yx' = yu_{\sigma(y)} \in E(H)$. Assume then that $\sigma(x) < \sigma(y) = \sigma(z)$. Also assume that $g(x, y, z) = y$ (i.e., that $y$ precedes $z$ in $O_{\sigma(y)}$). If $y = u_{\sigma(y)}$, then $y$ is adjacent to both $u_{\sigma(y)-1}$ and $u_{\sigma(y)+1}$, and also to all vertices of $T_{\sigma(y)}$ in the different color class. It is easy to see that $g(x', y', z')$ must be one of these vertices and hence $g(x, y, z)g(x', y', z') \in E(H)$. If $y \neq u_{\sigma(y)}$, then $\sigma(y) = \sigma(y') = \sigma(z')$. If $u_{\sigma(y)} \in \{x', y', z'\}$, then $u_{\sigma(y)} = g(x', y', z')$ which must be adjacent to $y$; otherwise if $u_{\sigma(y)} \notin \{x', y', z'\}$, the depth-first ordering $O_{\sigma(y)}$ ensures that $y$ is adjacent to the first vertex of $\{x', y', z'\}$ in $O_{\sigma(y)}$, which is $g(x', y', z')$. The case when $\sigma(x) = \sigma(y) < \sigma(z)$ can be treated similarly as for the case when $\sigma(x) < \sigma(y) = \sigma(z)$. So we assume that $\sigma(x) = \sigma(y) = \sigma(z)$. If $u_{\sigma(x)} \in \{x, y, z\}$, then $g(x, y, z) = u_{\sigma(x)}$ and $g(x', y', z')$ is one of vertices in $T_{\sigma(x)}$ adjacent to $u_{\sigma(x)}$. If $u_{\sigma(x)} \in \{x', y', z'\}$, then $g(x', y', z') = u_{\sigma(x)}$ is adjacent to all of $x, y, z$. If $u_{\sigma(x)} \notin \{x, y, z, x', y', z'\}$, then the middle vertex, say $y$, of $\{x, y, z\}$ is adjacent to the middle vertex $y'$ of $\{x', y', z'\}$ in the ordering $O_{\sigma(x)}$.

Finally we consider the case when $x, y, z$ are distinct but are not all in the same color class. Suppose that $x = u_1$ and that $z$ is a leaf neighbor of $u_1$ with a loop. Then $g(x, y, z) = x$. Then by analyzing all possible cases, we see that

$g(x,y,z)g(x',y',z') \in E(H)$. So we assume that $x,y$ are in one color class and $z$ is in the other color class. Assume further that $g(x,y,z) = x$ which precedes $y$ in $O$. When $x',y',z'$ are in the same color class, either $x = x'$, $y = y'$, and $z \neq z'$ or $x \neq x'$, $y \neq y'$ and $z = z'$. In the former case, we have $g(x,y,z)g(x',y',z') = xx' \in E(H)$. If the latter case occurs, then either $z = z' = u_1$ and $g(x,y,z)$ $g(x',y',z') = xz' \in E(H)$ or $z = z' \neq u_1$ and $g(x,y,z)g(x',y',z') = xx' \in E(H)$ (note that $x'$ appears in the middle of $\{x',y',z'\}$ in $O_1$). Assume that $x',y',z'$ are not all in the same color class. There are three possibilities. Suppose that $x \neq x'$, $y \neq y'$, and $z \neq z'$. If $x'$ appears before $y'$, then $g(x,y,z)g\ (x',y',z') = xx' \in E(H)$; otherwise we must have that $\sigma(x) = \sigma(y)$ and $x = u_{\sigma(x)}$ and hence $g(x,y,z)$ $g(x',y',z') = xy' \in E(H)$. Suppose that $x = x'$, $y \neq y'$, and $z \neq z'$. Then $g(x',y',z') = x'$ or $g(x',y',z') = u_1$ and we have $g(x,y,z)g\ (x',y',z') \in E(H)$. The case when $x \neq x'$, $y = y'$, and $z = z'$ do not occur because this means that $y$ and $z$ are adjacent vertices with loops and hence one of them is $u_1$, which contradicts the assumption that $g(x,y,z) = x$ (note also that $x,y,z$ are distinct). Suppose that $x \neq x'$, $y = y'$, and $z \neq z'$. Then $y$ has a loop and hence, by the assumption $g(x,y,z) = x$, $x$ also has a loop. So both $x$ and $y$ are leaf neighbors of $u_1$. Therefore, $g(x',y',z') = x' = u_1$ and $g(x,y,z)g(x',y',z') \in E(H)$. This completes the proof. ∎

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