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On well-quasi-ordering infinite trees

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Abstract. Let A be the set of all ascending finite sequences (with at least one term) of positive integers. Let $s, t \in A$. Write $s \lhd t$ if there exist $m, n, x_1, \ldots, x_n$ such that $m < n$ and $x_1 < \ldots < x_n$ and s is $x_1, \ldots, x_m$ and t is $x_2, x_3, \ldots, x_n$. Call a subset S of A a *P-block* if, for every infinite ascending sequence $x_1, x_2, \ldots$ of positive integers, there exists an m such that $x_1, \ldots, x_m$ belongs to S. A quasi-ordered set Q (i.e. a set on which a reflexive and transitive relation $\leqslant$ is defined) is *better-quasi-ordered* if, for every P-block S and every function $f: S \to Q$, there exist $s, t \in S$ such that $s \lhd t$ and $f(s) \leqslant f(t)$. It is proved that any set of (finite or infinite) trees is better-quasi-ordered if $T_1 \leqslant T_2$ means that the tree T_1 is homeomorphic to a subtree of the tree T_2. This establishes a conjecture of J. B. Kruskal that, if $T_1, T_2, \ldots$ is an infinite sequence of trees, then there exist i, j such that $i < j$ and $T_i \leqslant T_j$.

1. *Introduction.* A *graph* G consists of two disjoint (finite or infinite) sets $V(G)$, $E(G)$, together with a relationship whereby with each element λ of $E(G)$ we associate either two distinct elements of $V(G)$ which are said to be *joined* by λ or a single element of $V(G)$ which is said to be *joined* to itself by λ. The elements of $V(G)$ are called *vertices* of G and the elements of $E(G)$ are called *edges* of G. A vertex ξ and edge λ of G are *incident* with each other if λ joins ξ to itself or to another vertex. The *degree* of a vertex is the number of edges incident with it. We shall write

$$V(G) \cup E(G) = Z(G).$$

G is *finite* if the set $Z(G)$ is finite, and *infinite* otherwise. A finite sequence

$$\xi_0, \lambda_1, \xi_1, \lambda_2, \xi_2, \lambda_3, \ldots, \lambda_n, \xi_n \tag{1}$$

(where $n \geqslant 0$) is a *path* in G if $\xi_0, \ldots, \xi_n$ are vertices of G and $\lambda_1, \ldots, \lambda_n$ are *distinct* edges of G and λ_i joins ξ_{i-1} to ξ_i for $i = 1, \ldots, n$. Since n is allowed to be 0, a sequence whose sole term is a vertex is counted as a path. The path (1) is *strict* if $n > 0$ and $\xi_1, \ldots, \xi_n$ are distinct. A path *from* ξ *to* η is a path with first term ξ and last term η. The terms of a path other than its first and last terms will be called *mid-terms*. G is a *tree* if $V(G) \neq \varnothing$ and, for every element (ξ, η) of $V(G) \times V(G)$, there exists in G a unique path from ξ to η. The notations $\{x \mid \mathscr{S}\}$ and $\{x \in X \mid \mathscr{S}\}$ will be used for 'the set of all elements x such that $\mathscr{S}$' and 'the set of all elements x of the set X such that $\mathscr{S}$' respectively (where $\mathscr{S}$ is a statement specifying some restriction on x). If A, B are sets and $C \subset A \times B$ and $\xi \in A$, then $C \odot \xi$ will denote $\{\eta \in B \mid (\xi, \eta) \in C\}$. Such a set C may be thought of as a many-valued function from A into B, mapping each $\xi \in A$ into all the elements of $C \odot \xi$.

[However, the word 'function' is in this paper reserved for single-valued functions.] An *immersion* of a graph G in a graph H is a subset C of $Z(G) \times Z(H)$ satisfying the following conditions (i)–(iii).

(i) For every $\xi \in V(G)$, $C \odot \xi$ has just one element, which is a vertex of H.

(ii) If $\lambda \in E(G)$ and λ joins ξ to η, then $C \odot \lambda$ is the set of mid-terms of a strict path from the element of $C \odot \xi$ to the element of $C \odot \eta$.

(iii) If (α, γ) and (β, γ) both belong to C and $\alpha \neq \beta$, then $\alpha \in E(G)$, $\beta \in E(G)$ and $\gamma \in V(H)$. An *embedding* of G in H is an immersion C of G in H such that the sets $C \odot \alpha$ associated with the elements α of $Z(G)$ are disjoint (which, in view of (iii), is equivalent to saying that there are not two distinct edges λ, μ of G such that $C \odot \lambda$, $C \odot \mu$ have a vertex of H in common). The purpose of this paper is to prove the following conjecture of Kruskal ((2)):

Theorem 1. *If $T_1, T_2, \ldots$ is an infinite sequence of (finite or infinite) trees, there exist i and j such that $i < j$ and T_i can be embedded in T_j.*

The conjecture that Theorem 1 is true for finite trees was made by Vázsonyi. Higman ((1)) proved Theorem 1 for the case in which the T_i are finite and the degrees of their vertices are bounded. The theorem was proved for finite trees in general, and conjectured to be true without the restriction to finite trees, by Kruskal ((2)). A proof for finite trees was also, subsequently but independently, obtained by Tarkowski, and is briefly announced in (7). A shorter proof for finite trees was given by the author ((3)), and an extension of this result was obtained in (4).

Let us call a graph G *3-simple* if no vertex of G has degree greater than 3. Another conjecture of Vázsonyi was that, for every infinite sequence $G_1, G_2, \ldots$ of 3-simple graphs, there exist i and j such that $i < j$ and G_i can be embedded in G_j. In this connexion, it is of interest to make the further conjecture that, for every infinite sequence of graphs $G_1, G_2, \ldots$, there exist i and j such that $i < j$ and G_i can be immersed in G_j. For, if this conjecture is true, both Theorem 1 and Vázsonyi's conjecture concerning 3-simple graphs are consequences of it, since it is easily seen that an immersion of G_i in G_j is necessarily an embedding if G_j is 3-simple or if G_i and G_j are both trees.

In this paper, the word 'set' will (as in axiomatic set theory) mean a set with a definite cardinal number, and such entities as 'the collection of all trees' will not be referred to as 'sets'. A set of sets will sometimes be called a *class*. In the interests of clarity, we shall say that a set *includes* its elements and *contains* its subsets. We recall that a *function* is a set f of ordered pairs such that, if (x, y) and (x, z) belong to f, then $y = z$. The fact that a function is a set will play a significant role in the presentation of our arguments. The *domain* $\mathrm{D}f$ of f is the set of elements x such that $(x, y) \in f$ for some y and the *image* $\mathrm{Im} f$ of f is the set of elements y such that $(x, y) \in f$ for some x. If $(x, y) \in f$, we write $y = f(x)$ or $y = fx$ (whichever seems preferable). f is a function *from* the set A *into* the set B (in symbols, $f: A \to B$) if $\mathrm{D}f = A$ and $\mathrm{Im} f \subset B$. If α is an ordinal number, X_α will denote the set of non-zero ordinal numbers less than $1 + \alpha$. (Thus $X_0 = \varnothing$.) A *sequence* of *length* α on a set S may be defined to be a function from X_α into S. The length of a sequence s will be denoted by $l(s)$ (and will in this paper usually be finite or ω). The element $s(r)$, where $r \in \mathrm{D}s = X_{l(s)}$, is called the *rth term* of s. A sequence of finite length is a *finite* sequence, and a sequence of length ω is an

ω-*sequence*. A finite sequence s of length n will be denoted by $s(1), \ldots, s(n)$ or, in this paper, more commonly by $[s(1), \ldots, s(n)]$. In particular, $[x]$ is the sequence of length 1 with sole term x. Likewise, an ω-sequence s will be denoted by $s(1), s(2), \ldots$ or $[s(1), s(2), \ldots]$. A sequence of length 0 will be denoted by $\square$: it could, in fact, equally well be denoted by $\varnothing$, since it is according to our definitions an empty set of ordered pairs. For any sequence s and any $\alpha \leqslant l(s)$, the restriction of s to X_α (which is a sequence of length α) will be called a *left-segment* of s: if $\alpha < l(s)$, it is a *strict left-segment*. We write $s \prec t$ for 's is a left-segment of t' and $s \prec\prec t$ for 's is a strict left-segment of t'. If s is a finite sequence $[x_1, x_2, \ldots, x_n]$ and $m \leqslant n$, the sequence $[x_{m+1}, x_{m+2}, \ldots, x_n]$ will be denoted by ${}_{(m)}s$. We shall also write ${}_{(1)}s$ as ${}_*s$. If s is an ω-sequence $[x_1, x_2, \ldots]$, ${}_{(m)}s$ will denote the ω-sequence $[x_{m+1}, x_{m+2}, \ldots]$, and ${}_*s$ will denote ${}_{(1)}s$. The set of positive integers will be denoted by P. A sequence s on P will be called *ascending* if $l(s) \geqslant 1$† and $s(i) < s(j)$ whenever $i, j \in \mathrm{D}s$ and $i < j$. (The latter condition is vacuously satisfied if $l(s) = 1$.) The set of ascending finite sequences on a subset I of P will be denoted by $A(I)$, and the set of ascending sequences of length α on I by $A_\alpha(I)$. [The letter A, when not occurring in an expression of one of these forms, may be used also with meanings unconnected with this definition.] In introducing spaces of ascending sequences and the notation 'A' to describe them, we are using an idea due to Kruskal. If $B \subset A(P)$, $\bar{B}$ will denote the set of elements x of P such that x is a term of at least one element of B. In other words, $\bar{B}$ is the union of the images of the elements of B. B is a *block* if $\bar{B}$ is an infinite subset of P and every element of $A_\omega(\bar{B})$ has a left-segment belonging to B. If S is a set, an S-*function* is a function f such that $\mathrm{D}f \subset A(P)$ and $\mathrm{Im} f \subset S$. An S-*pattern* is an S-function whose domain is a block. If s, $t \in A(P)$, we write $s \lhd t$ iff there exists a $u \in A(P)$ such that $s \prec\prec u$ and $t = {}_*u$. ['Iff' stands for 'if and only if'.]

A *quasi-ordered* (*qo*) set is a set in which a reflexive and transitive relation $\leqslant$ is defined. The symbols Q and Q' will always denote qo sets. If $q_1, q_2 \in Q$ and $q_1 \leqslant q_2$, we shall say that q_1 *anticipates* q_2. $\mathfrak{S}Q$ will denote the class of all subsets of Q, and will be considered as a qo set in which $P_1 \leqslant P_2$ iff every element of P_1 anticipates an element of P_2 (where $P_1, P_2 \in \mathfrak{S}Q$). We shall write $Q = \mathfrak{S}^0 Q$, $\mathfrak{S}Q = \mathfrak{S}^1 Q$ and, in general, $\mathfrak{S}^n Q = \mathfrak{S}(\mathfrak{S}^{n-1}Q)$ for every $n \in P$. An ω-sequence σ on Q is *good* if there exist $i, j \in P$ such that $i < j$ and $\sigma(i) \leqslant \sigma(j)$, and is *bad* if not. A Q-function f is *good* if there exist $s, t \in \mathrm{D}f$ such that $s \lhd t$ and $f(s) \leqslant f(t)$, and is *bad* if not. Q is *well-quasi-ordered* (*wqo*) if every ω-sequence on Q is good. Q is *better-quasi-ordered* (*bqo*) if every Q-pattern is good.

LEMMA 1. *Every bqo set is wqo.*

Proof. Suppose that Q is bqo. Let σ be an ω-sequence on Q. Let f be the Q-pattern such that $\mathrm{D}f = A_1(P)$ and $f([i]) = \sigma(i)$ for every $i \in P$. Then f is good, and therefore σ is good.

Lemma 1 justifies the term 'better-quasi-ordered' in the sense of showing that it describes a stronger property than 'well-quasi-ordered'. It in fact describes a *strictly* stronger property, since examples of wqo sets that are not bqo have been constructed.

† This represents a slight departure from the convention of (5), where $\square$ was counted as ascending.

(See, for example, Theorem 2 of (6).) Such examples tend, however, to have a somewhat artificial character, and one is inclined to conjecture that most wqo sets which arise in a reasonably 'natural' manner are likely to be bqo.

It can be shown that $\mathfrak{S}^n Q$ being wqo is equivalent to the set $L^n Q$ of (4) being wqo, and that it is also equivalent to every Q-pattern with domain $A_n(P)$ being good. (Theorem 3 of (6), which constituted an important starting point for some of the ideas in the present paper, introduces something which approximates to this last condition in the case $n = 2$.) It also transpires that if, by a certain fairly natural extension of our definition of $\mathfrak{S}^n Q$, we define $\mathfrak{S}^\alpha Q$ for every ordinal α, then Q is bqo iff $\mathfrak{S}^\alpha Q$ is wqo for every ordinal α. To justify these statements would not be relevant here, but it was from this point of view that the author was first led to study bqo sets.

The main result of this paper will be

Theorem 2. *Any set of trees is bqo under the quasi-ordering in which $T_1 \leqslant T_2$ iff the tree T_1 can be embedded in the tree T_2.*

Theorem 1 is an immediate consequence of Theorem 2 and Lemma 1.

2. *An investigation leading to a lemma concerning the 'sharp' quasi-ordering on the class of subsets of a wqo set.*

Definitions. The cardinal number of a set A will be denoted by $|A|$. A function f from a subset A of Q into a subset B of Q is *weakly ascending* if $a \leqslant f(a)$ for every $a \in A$. If ν is a cardinal number, $\Lambda(\nu, Q)$ will denote the set of those elements of Q which anticipate at least ν elements of Q. Clearly

$$\Lambda(\nu, Q) \supset \Lambda(\nu', Q) \quad \text{if} \quad \nu \leqslant \nu'. \tag{2}$$

We call ν a *Q-number* if there is no $\nu' > \nu$ such that $\Lambda(\nu, Q) = \Lambda(\nu', Q)$. The set of those elements of Q which anticipate only finitely many elements of Q will be denoted by Q^*. If q_1, q_2 are elements of Q such that $q_1 \leqslant q_2$ and $q_2 \nleqslant q_1$, we write $q_1 < q_2$. If $\nu = \aleph_\alpha$, $\nu +$ will denote $\aleph_{\alpha+1}$.

Lemma 2. *Let ν be an infinite cardinal number and A, B, C be subsets of Q such that $|A| \leqslant \nu$, $|C| < \nu$ and each element of A anticipates at least ν elements of B. Then there exists a weakly ascending one-to-one function from A into $B - C$.*

Proof. Suppose, first, that A is an infinite set. Let $|A| = \aleph_\alpha$. Then we can write A in the form $\{a_\gamma | \gamma < \omega_\alpha\}$, where $a_\gamma \neq a_\delta$ if $\gamma \neq \delta$. Suppose that $\delta < \omega_\alpha$ and that, for each $\gamma < \delta$, we have selected an $f(a_\gamma) \in B - C$. Then, since $|C| < \nu$ and

$$|\{f(a_\gamma) | \gamma < \delta\}| < \aleph_\alpha \leqslant \nu$$

and a_δ anticipates at least ν elements of B, we can select an $f(a_\delta) \in B - (C \cup \{f(a_\gamma) | \gamma < \delta\})$ such that $a_\delta \leqslant f(a_\delta)$. In this way, we can define $f(a_\delta)$ for every $\delta < \omega_\alpha$ by transfinite induction so that f is a weakly ascending one-to-one function from A into $B - C$.

If A is finite, an obvious, purely notational modification of the above argument is needed.

Lemma 3. *If Q is wqo, ν is an infinite cardinal number and S is the set of those elements of Q which anticipate less than ν elements of Q, then $|S| < \nu$.*

Proof. Suppose that $|S| \geqslant \nu$. For any $q \in Q$, let $M(q)$ be the set of elements of Q which q anticipates. Select a $q_1 \in S$. Then $S - M(q_1) \neq \varnothing$ since $|M(q_1)| < \nu \leqslant |S|$, and so we can select a $q_2 \in S - M(q_1)$. Since $|M(q_1) \cup M(q_2)| < \nu \leqslant |S|$, we can select a

$$q_3 \in S - (M(q_1) \cup M(q_2)),$$

and then likewise a $q_4 \in S - (M(q_1) \cup M(q_2) \cup M(q_3))$ and so on. Since $q_j \notin M(q_i)$ when $i < j$, the ω-sequence $[q_1, q_2, \ldots]$ is bad, which contradicts the hypothesis that Q is wqo.

LEMMA 4. *$\Lambda(\nu, Q) \neq \varnothing$ if and only if there is a Q-number greater than or equal to ν. Furthermore, if either of these equivalent conditions is satisfied, then*

$$\Lambda(\nu, Q) = \Lambda(\rho, Q),$$

where ρ is the smallest Q-number greater than or equal to ν.

Proof. If there is a Q-number $\tau \geqslant \nu$,

$$\Lambda(\nu, Q) \supset \Lambda(\tau, Q) \supset \Lambda(\tau+, Q) \neq \Lambda(\tau, Q)$$

by (2) and the fact that τ is a Q-number. Therefore $\Lambda(\nu, Q) \neq \varnothing$.

To complete the proof, we will now assume that $\Lambda(\nu, Q) \neq \varnothing$ and deduce that there is a Q-number greater than or equal to ν and that $\Lambda(\nu, Q) = \Lambda(\rho, Q)$ where ρ is the least such Q-number. Call a cardinal number ν' *ν-equivalent* if $\Lambda(\nu', Q) = \Lambda(\nu, Q)$. If ν' is ν-equivalent, our hypothesis implies that $\Lambda(\nu', Q) \neq \varnothing$, which implies that $\nu' \leqslant |Q|$ since no element of Q can anticipate more than $|Q|$ elements of Q. Therefore $|Q|$ is greater than or equal to all ν-equivalent cardinals. It follows, since any set of cardinal numbers is well-ordered, that there is a least cardinal ρ greater than or equal to all ν-equivalent cardinals. If $q \in \Lambda(\nu, Q)$ and σ is the number of elements of Q anticipated by q, then $q \in \Lambda(\nu', Q)$ for every ν-equivalent ν', and therefore $\sigma \geqslant \nu'$ for every such ν', and therefore $\sigma \geqslant \rho$ by the definition of ρ, and therefore $q \in \Lambda(\rho, Q)$. Hence $\Lambda(\nu, Q) \subset \Lambda(\rho, Q)$. But $\rho \geqslant \nu$ since ν is ν-equivalent, and therefore $\Lambda(\rho, Q) \subset \Lambda(\nu, Q)$ by (2). Therefore $\Lambda(\nu, Q) = \Lambda(\rho, Q)$. It follows that, if $\Lambda(\rho, Q) = \Lambda(\rho', Q)$, then ρ' is ν-equivalent and therefore less than or equal to ρ: hence ρ is a Q-number. We have seen that $\rho \geqslant \nu$, and ρ must in fact be the least Q-number greater than or equal to ν since, if ρ'' were a Q-number such that $\nu \leqslant \rho'' < \rho$, (2) and the fact that ρ'' is a Q-number would give

$$\Lambda(\rho'', Q) \neq \Lambda(\rho, Q) \subset \Lambda(\rho'', Q) \subset \Lambda(\nu, Q),$$

which is impossible since $\Lambda(\nu, Q) = \Lambda(\rho, Q)$.

LEMMA 5. *If ν, ν' are Q-numbers such that $\nu < \nu'$, then $\Lambda(\nu, Q) > \Lambda(\nu', Q)$ (in the quasi-ordering of $\mathfrak{S}Q$).*

Proof. By (2), $\Lambda(\nu', Q) \subset \Lambda(\nu, Q)$ and therefore $\Lambda(\nu', Q) \leqslant \Lambda(\nu, Q)$. Moreover, if $\Lambda(\nu, Q) \leqslant \Lambda(\nu', Q)$, it would follow that any element of $\Lambda(\nu, Q)$ anticipates an element of $\Lambda(\nu', Q)$ which anticipates at least ν' elements of Q, and therefore that

$$\Lambda(\nu, Q) \subset \Lambda(\nu', Q)$$

and therefore that $\Lambda(\nu, Q) = \Lambda(\nu', Q)$, which contradicts the hypothesis that ν is a Q-number. Therefore $\Lambda(\nu, Q) \nleqslant \Lambda(\nu', Q)$, and therefore $\Lambda(\nu, Q) > \Lambda(\nu', Q)$.

Lemma 6. *If Q is wqo, there are only finitely many Q-numbers.*

Proof. If our assertion is false, there exists an ascending infinite sequence $\nu_1 < \nu_2 < \ldots$ of Q-numbers. Let $\Lambda(\nu_i, Q) = R_i$. Then $R_1 > R_2 > \ldots$ by Lemma 5. Hence, for each positive integer i, we can select an element r_i of R_i which does not anticipate any element of R_{i+1}. If $i < j$, then $R_j \leqslant R_{i+1}$ and therefore r_j anticipates an element of R_{i+1}, and hence r_i would anticipate an element of R_{i+1} if $r_i \leqslant r_j$. Therefore, when $i < j$, r_i does not anticipate r_j, which contradicts the hypothesis that Q is wqo.

Lemma 7. *Let Q be wqo and not equal to Q^*. Then there exists at least one infinite Q-number. Futhermore, if $\nu_1, \ldots, \nu_n$ are the infinite Q-numbers in ascending order and $\Lambda(\nu_i, Q) = R_i$ and $R_j - R_{j+1} = N_j$ for $j < n$ and $R_n = N_n$, then $N_1, \ldots, N_n$ are disjoint, $Q - Q^* = N_1 \cup \ldots \cup N_n$ and $|N_i| \leqslant \nu_i$ for each N_i.*

Proof. Since $\varnothing \neq Q - Q^* = \Lambda(\aleph_0, Q)$, it follows from Lemma 4 that there is at least one infinite Q-number and that $\Lambda(\aleph_0, Q) = \Lambda(\nu_1, Q)$, i.e. $Q - Q^* = R_1$. But, since $R_1 \supset \ldots \supset R_n$ by (2), it follows that $N_1, \ldots, N_n$ are disjoint and that

$$N_1 \cup \ldots \cup N_n = R_1 = Q - Q^*.$$

Furthermore, if $i < n$, then $\Lambda(\nu_i +, Q) = R_{i+1}$ by Lemma 4 and therefore each element of N_i anticipates fewer than $\nu_i +$ elements of Q, whence $|N_i| \leqslant \nu_i$ by Lemma 3. Finally, since $\Lambda(\nu_n +, Q) = \varnothing$ by the first part of Lemma 4, every element of Q anticipates fewer than $\nu_n +$ elements of Q, whence $|N_n| \leqslant |Q| \leqslant \nu_n$ by Lemma 3.

Definitions. Let A, B be subsets of Q. Our definition of '$A \leqslant B$' amounts to saying that $A \leqslant B$ iff there exists a weakly ascending function from A into B. If there exists a weakly ascending *one-to-one* function from A into B, we shall write $A \leqslant {}^{\#}B$. Thus $\leqslant$ and $\leqslant {}^{\#}$ are two quasi-orderings on the class of subsets of Q. When this class of subsets is regarded as being quasi-ordered by $\leqslant {}^{\#}$, it will be denoted by $\mathfrak{S}^{\#}Q$. (It will as heretofore be denoted by $\mathfrak{S}Q$ when it is regarded as being quasi-ordered by $\leqslant$.) The class of finite subsets of Q, when regarded as quasi-ordered by $\leqslant {}^{\#}$, will be denoted by $\mathfrak{F}^{\#}Q$. The Cartesian product $Q \times Q'$ of two qo sets will be quasi-ordered by the rule that $(q_1, q_1') \leqslant (q_2, q_2')$ iff $q_1 \leqslant q_1'$ and $q_2 \leqslant q_2'$. The set of cardinal numbers less than or equal to $|Q|$ will be denoted by C_Q. The set of all ordered pairs (ν, R) such that ν is an infinite Q-number and $R = \Lambda(\nu, Q)$ will be denoted by $\Pi(Q)$. The set of all ordered pairs of the form $(1, \{q\})$, where $q \in Q^*$ and $\{q\}$ denotes the set whose sole element is q, will be denoted by $\Omega(Q)$; and $\Xi(Q)$ will denote $\Pi(Q) \cup \Omega(Q)$.

Lemma 8. *If Q is wqo, $\Xi(Q) \in \mathfrak{F}^{\#}(C_Q \times \mathfrak{S}Q)$.*

Proof. If $(\nu, R) \in \Xi(Q)$, it is clear from the definition of $\Xi(Q)$ that $R \in \mathfrak{S}Q$: furthermore, either $(\nu, R) \in \Omega(Q)$, in which case $\nu = 1 = |R| \leqslant |Q|$, or $(\nu, R) \in \Pi(Q)$, in which case ν is a Q-number and therefore $\Lambda(\nu, Q) \neq \varnothing$ by Lemma 4 and therefore $\nu \leqslant |Q|$. Hence $(\nu, R) \in C_Q \times \mathfrak{S}Q$. Thus $\Xi(Q)$ is a subset of $C_Q \times \mathfrak{S}Q$; and it is in fact a finite subset since Q^* and $\Pi(Q)$ are finite sets by Lemmas 3 and 6, respectively. Therefore

$$\Xi(Q) \in \mathfrak{F}^{\#}(C_Q \times \mathfrak{S}Q).$$

(Of course, $\Xi(Q)$ could equally well be regarded as an element of $\mathfrak{F}(C_Q \times \mathfrak{S}Q)$ or $\mathfrak{F}(C_Q \times \mathfrak{S}^{\#}Q)$, say: the choice is simply a matter of convenience.)

COROLLARY 8A. *If Q is wqo and $A \subset Q$, then* $\Xi(A) \in \mathfrak{F}^{\#}(C_Q \times \mathfrak{S}Q)$.

Explanation and Proof. A subset A of Q is of course itself quasi-ordered by the quasi-ordering in Q, or, more precisely, by the restriction of that quasi-ordering to A. If Q is wqo, so is A, and therefore $\Xi(A)$ belongs by Lemma 8 to $\mathfrak{F}^{\#}(C_A \times \mathfrak{S}A)$, which is a subset of $\mathfrak{F}^{\#}(C_Q \times \mathfrak{S}Q)$.

LEMMA 9. *If Q is wqo and A, $\bar{A}$ are subsets of Q such that $\Xi(A) \leqslant^{\#} \Xi(\bar{A})$ (in the quasi-ordering of $\mathfrak{F}^{\#}(C_Q \times \mathfrak{S}Q)$), then there exists a weakly ascending one-to-one function from A^* into $\bar{A}$.*

Proof. Since $\Xi(A) \leqslant^{\#} \Xi(\bar{A})$, there exists a weakly ascending one-to-one function f from $\Xi(A)$ into $\Xi(\bar{A})$. Write $A^* = A_1^* \cup A_2^*$, where $f((1,\{a\}))$ belongs to $\Omega(\bar{A})$ for $a \in A_1^*$ and to $\Pi(\bar{A})$ for $a \in A_2^*$. Since f is weakly ascending and one-to-one, a weakly ascending one-to-one function ψ from A_1^* into $(\bar{A})^*$ ($\subset \bar{A}$) is clearly defined by the relation $(1,\{\psi(a)\}) = f((1,\{a\}))$ (where $a \in A_1^*$). Moreover, if $a \in A_2^*$, then

$$f((1,\{a\})) = (\bar{\nu}_a, \Lambda(\bar{\nu}_a, \bar{A}))$$

for some infinite $\bar{A}$-number $\bar{\nu}_a$. Since f is weakly ascending, $\{a\} \leqslant \Lambda(\bar{\nu}_a, \bar{A})$ and therefore a anticipates an element of $\Lambda(\bar{\nu}_a, \bar{A})$ which anticipates at least $\bar{\nu}_a$ elements of $\bar{A}$. Therefore each element of A_2^* anticipates at least $\aleph_0$ elements of $\bar{A}$. Moreover, since A^* is finite by Lemma 3, A_2^* and $\operatorname{Im}\psi$ are both finite. Therefore there exists by Lemma 2 a weakly ascending one-to-one function χ from A_2^* into $\bar{A} - \operatorname{Im}\psi$. Clearly $\psi \cup \chi$ is a weakly ascending one-to-one function from A^* into $\bar{A}$.

LEMMA 10. *If Q is wqo and A, $\bar{A}$ are subsets of Q such that $\Xi(A) \leqslant^{\#} \Xi(\bar{A})$ (in the quasi-ordering of $\mathfrak{F}^{\#}(C_Q \times \mathfrak{S}Q)$), then $A \leqslant^{\#} \bar{A}$.*

Proof. Let f be a weakly ascending function from $\Xi(A)$ into $\Xi(\bar{A})$. By Lemma 9, there exists a weakly ascending one-to-one function ϕ_0 from A^* into $\bar{A}$. If $A = A^*$, it follows that $A \leqslant^{\#} \bar{A}$. If not, there exists by Lemma 7 at least one infinite A-number (and by Lemma 6 there are only finitely many). Let $\nu_1, \ldots, \nu_n$ be the infinite A-numbers in ascending order. Let $\Lambda(\nu_i, A) = R_i$ and $R_j - R_{j+1} = N_j$ for $j < n$ and $R_n = N_n$. Let $f((\nu_i, R_i)) = (\bar{\nu}_i, \bar{R}_i)$. Then, since f is weakly ascending, $\nu_i \leqslant \bar{\nu}_i$ (which implies that $\bar{\nu}_i$ is infinite) and $R_i \leqslant \bar{R}_i$. Since $\bar{\nu}_i$ is infinite, $(\bar{\nu}_i, \bar{R}_i) \in \Pi(\bar{A})$ and therefore $\bar{R}_i = \Lambda(\bar{\nu}_i, \bar{A})$. Since $R_i \leqslant \bar{R}_i = \Lambda(\bar{\nu}_i, \bar{A})$, each element of R_i anticipates an element of $\bar{R}_i$ which anticipates at least $\bar{\nu}_i$ ($\geqslant \nu_i$) elements of $\bar{A}$ and therefore in particular each element of N_i anticipates at least ν_i elements of $\bar{A}$. Furthermore, by Lemma 7, $|N_i| \leqslant \nu_i$ and by Lemma 3 A^* (and therefore also $\operatorname{Im}\phi_0$) is finite. Therefore, by n applications of Lemma 2, we can construct in succession weakly ascending one-to-one functions

$$\phi_1 : N_1 \to \bar{A} - \operatorname{Im}\phi_0, \quad \phi_2 : N_2 \to \bar{A} - (\operatorname{Im}\phi_0 \cup \operatorname{Im}\phi_1), \quad \ldots,$$
$$\phi_n : N_n \to \bar{A} - (\operatorname{Im}\phi_0 \cup \operatorname{Im}\phi_1 \cup, \ldots, \cup \operatorname{Im}\phi_{n-1}).$$

Since $N_1, \ldots, N_n$ are by Lemma 7 disjoint sets with union $A - A^*$, $\phi_0 \cup \phi_1 \cup \ldots \cup \phi_n$ is a function from A into $\bar{A}$. Since the ϕ_i are by construction weakly ascending one-to-one functions with disjoint images, $\phi_0 \cup \ldots \cup \phi_n$ is weakly ascending and one-to-one. Hence $A \leqslant^{\#} \bar{A}$.

3. *Some properties of barriers, arrays and bqo sets.*

Definitions. An element s of $A(P)$ is a *subsequence* of an element t of $A(P)$ if

$$\operatorname{Im} s \subset \operatorname{Im} t,$$

i.e. if every term of s is a term of t. If s is a subsequence of t and $s \neq t$, we call s a *strict subsequence* of t. If $s_1, \ldots, s_n \in A(P)$, $s_1 \circ s_2 \circ \ldots \circ s_n$ will denote the element of $A(P)$ whose terms are precisely those integers which are terms of one or more of $s_1, \ldots, s_n$. A *barrier* is a block which does not include two sequences s, t such that s is a strict subsequence of t. If B is a barrier, a finite sequence σ on B will be called *ascending* if $l(\sigma) \geqslant 1$ and $\sigma(i) \lhd \sigma(i+1)$ for $1 \leqslant i < l(\sigma)$. (We regard the latter condition as being vacuously satisfied if $l(\sigma) = 1$.) We shall denote the set of ascending sequences on B of length r by $A_r(B)$ and the set of all ascending finite sequences on B by $A(B)$. If $\sigma \in A(B)$, $a(\sigma)$ will denote the element $\sigma(1) \circ \sigma(2) \circ \ldots \circ \sigma(l(\sigma))$ of $A(P)$. B^r will denote the set of all sequences of the form $a(\sigma)$, where $\sigma \in A_r(B)$. $\bar{B}^r$ will mean $\bar{C}$, where $C = B^r$.

Lemma 11. *If B is a barrier, no element of $A(P) \cup A_\omega(P)$ can have more than one left-segment belonging to B.*

Proof. If two elements s, t of B are left-segments of the same sequence, one of s, t must be a subsequence of the other. But, since B is a barrier, neither of s, t can be a strict subsequence of the other, and therefore $s = t$.

Corollary 11A. *If B is a barrier, each element of $A_\omega(\bar{B})$ has a unique left-segment belonging to B.*

Lemma 12. *If B is a barrier and $s \in B$, then there exists a $t \in B$ such that $s \lhd t$.*

Proof. Let u be any element of $A_\omega(\bar{B})$ with s as a left-segment. Then, since B is a block, $_*u$ has a left-segment $t \in B$. Since $s, t \in B$, t cannot be a subsequence of $_*s$, and therefore $s \lhd t$.

Lemma 13. *Let D be a subset of $A(P)$ and I be an infinite subset of P such that $\bar{D} \subset I$ and every element of $A_\omega(I)$ has a left-segment which belongs to D. Then $\bar{D} = I$ and D is a block.*

Proof. Since every element of I is the first term of an element of $A_\omega(I)$, which has a left-segment in D, every element of I is the first term of an element of D. Therefore $I \subset \bar{D}$, and so, since $\bar{D} \subset I$ by hypothesis, $\bar{D} = I$. Since $\bar{D} = I$, the hypotheses of Lemma 13 imply that $\bar{D}$ is an infinite set and that every element of $A_\omega(\bar{D})$ has a left-segment in D.

Lemma 14. *If B is a barrier and I is an infinite subset of $\bar{B}$, then $B \cap A(I)$ is a barrier.*

Proof. Let $B \cap A(I) = D$. Since B is a block, each element of $A_\omega(I)$ has a left-segment which belongs to B and hence (being a left-segment of an element of $A_\omega(I)$) to $B \cap A(I) = D$. Moreover, it is obvious that $\bar{D} \subset I$. Therefore D is a block by Lemma 13. Moreover D, being a subset of B, cannot include two sequences such that one is a strict subsequence of the other, and must therefore be a barrier.

Lemma 15. *If B is a barrier and $r \in P$, then B^r is a barrier.*

Proof. Let $u \in A_\omega(\bar{B})$. For each non-negative integer i, $_{(i)}u$ has by Corollary 11A a unique left-segment $v_i \in B$. Since B is a barrier, v_{i+1} cannot be a strict subsequence of

v_i and therefore $v_i \lhd v_{i+1}$. Hence $[v_0, v_1, \ldots, v_{r-1}] \in A_r(B)$ and so u has a left-segment $v_0 \circ \ldots \circ v_{r-1}$ belonging to B^r. Thus every element of $A_\omega(\bar{B})$ has a left-segment in B^r. Since obviously $\bar{B^r} \subset \bar{B}$, it follows by Lemma 13 that B^r is a block. Suppose that $s, t \in B^r$ and that s is a subsequence of t. Write $s = a(\sigma)$, $t = a(\tau)$, where $\sigma, \tau \in A_r(B)$. Then ${}_{(r-1)}s = \sigma(r) \in B$, ${}_{(r-1)}t = \tau(r) \in B$, and, since s is a subsequence of t, ${}_{(r-1)}s$ is a subsequence of ${}_{(r-1)}t$. Hence, since B is a barrier, ${}_{(r-1)}s = {}_{(r-1)}t$. Since s is a subsequence of t and ${}_{(r-1)}s = {}_{(r-1)}t$, clearly $s = t$. Hence B^r does not include two sequences such that one is a strict subsequence of the other, and therefore B^r is a barrier.

LEMMA 16. *If B is a barrier and $\sigma, \tau \in A(B)$ and $a(\sigma) = a(\tau)$, then $\sigma = \tau$.*

Proof. Let $a(\sigma) = a(\tau) = u$, $l(\sigma) = m$, $l(\tau) = n$. Since $\sigma \in A(B)$ and $a(\sigma) = u$, clearly $\sigma(i) \prec {}_{(i-1)}u$ for $1 \leqslant i \leqslant m$ and $\sigma(m) = {}_{(m-1)}u$: similarly $\tau(j) \prec {}_{(j-1)}u$ for $1 \leqslant j \leqslant n$ and $\tau(n) = {}_{(n-1)}u$. Since ${}_{(m-1)}u = \sigma(m) \in B$, ${}_{(n-1)}u = \tau(n) \in B$, neither of these sequences can be a strict subsequence of the other: therefore $m = n$. Furthermore, $\sigma(i) = \tau(i)$ for $1 \leqslant i \leqslant m$ by Lemma 11 and the fact that $\sigma(i), \tau(i) \prec {}_{(i-1)}u$. Hence $\sigma = \tau$.

Definitions. If B is a barrier and $s \in B \cup B^2 \cup B^3 \cup \ldots$, the element σ of $A(B)$ such that $s = a(\sigma)$ will be denoted by $a^{-1}(s, B)$, and the first term of σ will be denoted by $s \wedge B$. These definitions are unambiguous by Lemma 16. In view of Lemma 11, $s \wedge B$ can also be characterized as the unique left-segment of s which belongs to B.

LEMMA 17. *If B is a barrier $(B^2)^2 = B^3$ and, for every $s \in B^3$, $(s \wedge B^2) \wedge B = s \wedge B$.*

Proof. If $s \in B^3$, then $s = t \circ u \circ v$ for some $[t, u, v] \in A_3(B)$. Since $t \lhd u \lhd v$, clearly $(t \circ u) \lhd (u \circ v)$ and hence $(t \circ u) \circ (u \circ v) \in (B^2)^2$, i.e. $s \in (B^2)^2$. Conversely, if $s \in (B^2)^2$, we can write $s = t \circ u$ where $[t, u] \in A_2(B^2)$ and $t = t_1 \circ t_2$, $u = u_1 \circ u_2$ where

$$[t_1, t_2], [u_1, u_2] \in A_2(B).$$

Then $t_2 = {}_*t \prec u$, $u_1 \prec u$ and therefore $t_2 = u_1$ by Lemma 11 since $t_2, u_1 \in B$. Therefore $[t_1, t_2, u_2] \in A_3(B)$ and $t_1 \circ t_2 \circ u_2 = t \circ u = s$, and so $s \in B^3$. Hence $(B^2)^2 = B^3$. Moreover, if $s \in B^3$, then $(s \wedge B^2) \wedge B$ and $s \wedge B$ are left-segments of s which belong to B and hence are equal by Lemma 11.

Definitions. A Q-function f is *perfect* if $f(s) \leqslant (t)$ for *every* pair s, t of elements of $\mathbf{D}f$ such that $s \lhd t$. A *Q-array* is a Q-function whose domain is a barrier. A subset T of $A(P)$ is *thin* if it does not include two sequences s, t such that $s \prec \prec t$. The second component y of an ordered pair (x, y) will be denoted by $c_2(x, y)$. The restriction of a function f to a subset A of $\mathbf{D}f$ will be denoted by $f|A$.

We refer the reader to section 2 of (5) for the proof of the following lemma:

LEMMA 18. *Let I be an infinite subset of P and $\{B, C\}$ be a partition of a thin subset T of $A(I)$ into two disjoint subsets. Then there exists an infinite subset K of I such that $T \cap A(K)$ is contained either in B or in C.*

LEMMA 19. *If f is a Q-array, $\mathbf{D}f$ contains a barrier B such that $f|B$ is either bad or perfect.*

Proof. Write $\mathbf{D}f = D$. For every $s \in D^2$, write $a^{-1}(s, D) = [s_1, s_2]$. Let X be the set of those $s \in D^2$ for which $f(s_1) \leqslant f(s_2)$ and Y be the set of those $s \in D^2$ for which $f(s_1) \nleqslant f(s_2)$. By Lemma 15, D^2 is thin; and obviously $D^2 \subset A(\bar{D})$. Therefore there exists by Lemma 18 an infinite subset K of $\bar{D}$ such that $D^2 \cap A(K)$ is contained in X or in Y. Write $D \cap A(K) = B$, which is a barrier by Lemma 14. For every pair s, t of elements of B such that $s \lhd t$, $s \circ t \in D^2 \cap A(K)$ and therefore

$$f(s) \leqslant f(t) \quad \text{if} \quad D^2 \cap A(K) \subset X \quad \text{and} \quad f(s) \nleqslant f(t) \quad \text{if} \quad D^2 \cap A(K) \subset Y.$$

Hence $f|B$ is bad or perfect.

Lemma 20. *Every block contains a barrier.*

Proof. Let B be a block. Call an element s of B *minimal* if no strict left-segment of s belongs to B. Let M be the set of minimal elements of B. Let X be the set of those elements of M which have a strict subsequence belonging to M. Since M is clearly thin, there exists by Lemma 18 an infinite subset K of $\bar{B}$ such that $M \cap A(K)$ is contained either in X or in $M - X$. Let t be an element of minimum length in $M \cap A(K)$. If $M \cap A(K) \subset X$, t has a strict subsequence $u \in M$ and, since $t \in A(K)$, u belongs to $A(K)$ and therefore to $M \cap A(K)$, which contradicts the minimality of $l(t)$. Therefore $M \cap A(K) \subset M - X$, and so no element of $M \cap A(K)$ has a strict subsequence belonging to $M \cap A(K)$. Since B is a block, every element of $A_\omega(\bar{B})$ has a left-segment belonging to B and therefore has a left-segment belonging to M. Therefore every element of $A_\omega(K)$ has a left-segment belonging to $M \cap A(K)$. Since obviously $\overline{M \cap A(K)} \subset K$, it follows by Lemma 13 that $M \cap A(K)$ is a block. Hence $M \cap A(K)$ is a barrier contained in B.

Lemma 21. *Q is bqo iff every Q-array is good.*

Proof. If Q is bqo, every Q-pattern, and therefore *a fortiori* every Q-array, is good. If every Q-array is good, then, for any Q-pattern f, $\mathbf{D}f$ contains by Lemma 20 a barrier B, and $f|B$, being a Q-array, must be good by hypothesis, and therefore f is good. Hence Q is bqo.

We shall henceforward take Lemma 21, in place of the definition given in section 1, as our characterization of bqo sets, since barriers will be more convenient to handle than blocks.

Lemma 22. *If Q is bqo and f is a bad $(Q \times Q')$-array, then there exists a bad Q'-array g such that $\mathbf{D}g \subset \mathbf{D}f$ and $g(s) = c_2 f(s)$ for every $s \in \mathbf{D}g$.*

Proof. Write $f(s) = (f_1(s), f_2(s))$ for every $s \in \mathbf{D}f$. By Lemma 19, there exists a barrier $B \subset \mathbf{D}f$ such that $f_2|B$ is either bad or perfect. However, since Q is bqo, $f_1|B$ is good, and therefore there exist $s, t \in B$ such that $s \lhd t$ and $f_1(s) \leqslant f_1(t)$. If $f_2|B$ were perfect, we should also have $f_2(s) \leqslant f_2(t)$ and therefore $f(s) \leqslant f(t)$, contrary to the hypothesis that f is bad. Therefore $f_2|B$ is bad, and is thus a g with the required properties.

Corollary 22A. *If Q and Q' are bqo, $Q \times Q'$ is bqo.*

Definitions. If s, t are sequences and $l(s)$ is finite, st will denote the sequence defined by the rule that $st(\alpha) = s(\alpha)$ for $1 \leqslant \alpha \leqslant l(s)$ and $st(l(s)+\beta) = t(\beta)$ for $1 \leqslant \beta < 1 + l(t)$. The last term of a sequence $s \in A(P)$ will be denoted by $\lambda(s)$. If S is a set and f is an S-function, $\bar{f}$ will denote $\overline{\mathbf{D}f}$. If f, g are $\mathfrak{F}^\# Q$-functions such that $\mathbf{D}f \subset \mathbf{D}g$ and $f(s) \subset g(s)$ for every $s \in \mathbf{D}f$, we shall write $f \angle g$. An $\mathfrak{F}^\# Q$-array f is *irreducibly bad* if f is bad and every bad $\mathfrak{F}^\# Q$-array g such that $g \angle f$ is a subset of f.

LEMMA 23. *Let f be a bad $\mathfrak{F}^{\#}Q$-array which is not irreducibly bad. Then there exists a pair e, s such that e is a bad $\mathfrak{F}^{\#}Q$-array and $e \angle f$ and $s \in \mathbf{D}f$ and $\mathbf{D}e$ includes every element of $\mathbf{D}f$ whose last term is less than or equal to $\lambda(s)$ and $e(s)$ is strictly contained in $f(s)$.*

Proof. Since f is not irreducibly bad, there exists a bad $\mathfrak{F}^{\#}Q$-array g such that $g \angle f$ and $g \not\subseteq f$. Since $\mathbf{D}g \subset \mathbf{D}f$ and $g \not\subseteq f$, $\{t \in \mathbf{D}g | g(t) \neq f(t)\}$ is non-empty. Select an element s of this set with $\lambda(s)$ as small as possible. Then $s \in \mathbf{D}f$. Let L be the set of elements of $\bar{f}$ less than or equal to $\lambda(s)$, and let $I = \bar{g} \cup L$. We note that

$$\mathbf{D}g \subset A(\bar{g}) \subset A(I)$$

and therefore $\mathbf{D}g \subset \mathbf{D}f \cap A(I)$. Define a function $e: \mathbf{D}f \cap A(I) \to \mathfrak{F}^{\#}Q$ by the rules:

$$e(t) = g(t) \quad \text{if} \quad t \in \mathbf{D}g,$$

$$e(t) = f(t) \quad \text{if} \quad t \in (\mathbf{D}f \cap A(I)) - \mathbf{D}g.$$

Then e is an $\mathfrak{F}^{\#}Q$-array by Lemma 14, and $e \angle f$ since $g \angle f$. If $t \in \mathbf{D}f$ and $\lambda(t) \leqslant \lambda(s)$, then $t \in A(L) \subset A(I)$ and therefore $t \in \mathbf{D}f \cap A(I) = \mathbf{D}e$. By the definition of s, $s \in \mathbf{D}g$ and $g(s) \neq f(s)$, and therefore, since $g \angle f$, $g(s)$ is strictly contained in $f(s)$, and therefore $e(s)$ is strictly contained in $f(s)$. It remains to be proved that e is bad.

LEMMA 23A.† *If $t \in \mathbf{D}g$ and $u \in \mathbf{D}e$ and $t \lhd u$ and $e(t) \leqslant^{\#} e(u)$, then $u \in \mathbf{D}g$.*

Proof. Since f is bad, $f(t) \nleqslant^{\#} f(u)$. Since $e \angle f$, $e(u) \subset f(u)$ and therefore $e(u) \leqslant^{\#} f(u)$. Hence $f(t) \nleqslant^{\#} e(u)$. But $e(t) \leqslant^{\#} e(u)$ by hypothesis. Therefore $f(t) \neq e(t) = g(t)$, and therefore, by the definition of s, $\lambda(t) \geqslant \lambda(s)$. But, since $t \lhd u$, we can write $u = {}_{*}tv$, where all terms of v are greater than $\lambda(t)$. Hence all terms of v are greater than $\lambda(s)$, i.e. none of them belong to L. But $v \in A(I)$ since $u \in \mathbf{D}e \subset A(I)$. Therefore $v \in A(\bar{g})$, and hence, since $t \in \mathbf{D}g$, it follows that $u = {}_{*}tv \in A(\bar{g})$. We can therefore select a $w \in A_{\omega}(\bar{g})$ such that $u \prec w$. Since $\mathbf{D}g$ is a barrier, w has a left-segment $u' \in \mathbf{D}g$. Since u, u' are left-segments of w and $u \in \mathbf{D}e \subset \mathbf{D}f$ and $u' \in \mathbf{D}g \subset \mathbf{D}f$, it follows by Lemma 11 that $u = u'$. Therefore $u \in \mathbf{D}g$.

To prove that e is bad, suppose now that t, u are any two elements of $\mathbf{D}e$ such that $t \lhd u$. Then either (i) $t \in \mathbf{D}e - \mathbf{D}g$ or (ii) $t \in \mathbf{D}g$. In Case (i), $e(t) = f(t)$ by the definition of e and $f(t) \nleqslant^{\#} e(u)$ by the argument in the first three sentences of the proof of Lemma 23A, and therefore $e(t) \nleqslant^{\#} e(u)$. In Case (ii), if $e(t) \leqslant^{\#} e(u)$, it would follow from Lemma 23A that $u \in \mathbf{D}g$ and hence (since g is bad) that $e(t) = g(t) \nleqslant^{\#} g(u) = e(u)$, which is a contradiction; hence we may again infer that $e(t) \nleqslant^{\#} e(u)$. Therefore e is bad.

LEMMA 24. *If f_0 is a bad $\mathfrak{F}^{\#}Q$-array, there exists an irreducibly bad $\mathfrak{F}^{\#}Q$-array f such that $f \angle f_0$.*

Proof. Select an $s_1 \in \mathbf{D}f_0$ with $\lambda(s_1)$ as small as possible. Then select a bad $\mathfrak{F}^{\#}Q$-array f_1 such that $f_1 \angle f_0$ and $s_1 \in \mathbf{D}f_1$ and, subject to these conditions, $|f_1(s_1)|$ is as small as possible. (This selection is certainly possible since f_0 is a bad $\mathfrak{F}^{\#}Q$-array and $f_0 \angle f_0$ and $s_1 \in \mathbf{D}f_0$.) Then select an $s_2 \in \mathbf{D}f_1 - \{s_1\}$ with $\lambda(s_2)$ as small as possible. Then select a bad $\mathfrak{F}^{\#}Q$-array f_2 such that $f_2 \angle f_1$ and $s_1, s_2 \in \mathbf{D}f_2$ and, subject to these requirements,

† In this paper, the designations 'Lemma nA', 'Lemma nB', etc., denote propositions in which the hypotheses of Lemma n are assumed and the notation of the proof of Lemma n is employed, and whose statement and proof constitute a part of the proof of Lemma n.

$|f_2(s_2)|$ is as small as possible (which can be done since f_1 is a bad $\mathfrak{F}^{\#}Q$-array and $f_1 \angle f_1$ and $s_1, s_2 \in \mathbf{D}f_1$). Then select an $s_3 \in \mathbf{D}f_2 - \{s_1, s_2\}$ with $\lambda(s_3)$ as small as possible. Then select a bad $\mathfrak{F}^{\#}Q$-array f_3 such that $f_3 \angle f_2$ and $s_1, s_2, s_3 \in \mathbf{D}f_3$ and, subject to these requirements, $|f_3(s_3)|$ is as small as possible. Then select an $s_4 \in \mathbf{D}f_3 - \{s_1, s_2, s_3\}$ with $\lambda(s_4)$ as small as possible; and so on. Let $f = \{(s_i, f_i(s_i)) \mid i \in P\}$.

Lemma 24A. *If* $0 \leqslant i \leqslant j$, *then* $f_j \angle f_i$.

Proof. $f_j \angle f_i$ since $f_{r+1} \angle f_r$ for each r and $\angle$ is clearly reflexive and transitive.

Lemma 24B. *If* $1 \leqslant i \leqslant j$, *then* $\lambda(s_i) \leqslant \lambda(s_j)$.

Proof. $\mathbf{D}f_{j-1} \subset \mathbf{D}f_{i-1}$ by Lemma 24A and therefore

$$s_j \in \mathbf{D}f_{j-1} - \{s_1, \ldots, s_{j-1}\} \subset \mathbf{D}f_{i-1} - \{s_1, \ldots, s_{i-1}\}.$$

Therefore $\lambda(s_j) \geqslant \lambda(s_i)$ by the manner of choice of s_i.

Lemma 24C. $s_j \in \mathbf{D}f_i$ *for every* $j \geqslant 1, i \geqslant 0$.

Proof. $s_j \in \mathbf{D}f_j, \mathbf{D}f_{j+1}, \mathbf{D}f_{j+2}, \ldots$ by the definitions of $f_j, f_{j+1}, \ldots$. Moreover, if $i < j$, then $s_j \in \mathbf{D}f_j \subset \mathbf{D}f_i$ by Lemma 24A.

Lemma 24D. *f is an $\mathfrak{F}^{\#}Q$-array.*

Proof. Since $s_1, s_2, \ldots$ are all distinct by hypothesis, the definition of f implies that it is an $\mathfrak{F}^{\#}Q$-function with domain $\{s_1, s_2, \ldots\}$. Since $\mathbf{D}f$ is an infinite subset of $A(P)$, $\bar{f}$ is an infinite subset of P. Let $u \in A_\omega(\bar{f})$. Since $\mathbf{D}f = \{s_1, s_2, \ldots\} \subset \mathbf{D}f_i$ by Lemma 24C, $\bar{f} \subset \bar{f}_i$ and therefore $u \in A_\omega(\bar{f}_i)$ for every i. Hence, since $\mathbf{D}f_i$ is a barrier, u has a left-segment $u_i \in \mathbf{D}f_i$ $(i = 0, 1, \ldots)$. If $i < j$, then, since $\mathbf{D}f_j \subset \mathbf{D}f_i$ by Lemma 24A, u_i and u_j both belong to $\mathbf{D}f_i$, which is a barrier. Therefore $u_i = u_j$ by Lemma 11. Hence

$$u_0 = u_1 = u_2 = \ldots.$$

Since $s_1, s_2, \ldots$ are distinct ascending sequences, there exists a j such that $\lambda(s_j) > \lambda(u_0)$, which, by the manner of choice of s_j, implies that $u_0 \notin \mathbf{D}f_{j-1} - \{s_1, \ldots, s_{j-1}\}$. But

$$u_0 = u_{j-1} \in \mathbf{D}f_{j-1}.$$

Therefore $u_0 = s_i$ for some $i < j$, and therefore $u_0 \in \mathbf{D}f$. Hence any $u \in A_\omega(\bar{f})$ has a left-segment in $\mathbf{D}f$. Finally, since by Lemma 24C $s_1, s_2, \ldots$ all belong to $\mathbf{D}f_0$, which is a barrier, no element of $\{s_1, s_2, \ldots\} = \mathbf{D}f$ can be a strict subsequence of another. Hence $\mathbf{D}f$ is a barrier and so f is an $\mathfrak{F}^{\#}Q$-array.

Lemma 24E. *If* $1 \leqslant i \leqslant j$, *then* $f_j(s_i) = f(s_i)$.

Proof. f_j is a bad $\mathfrak{F}^{\#}Q$-array by definition and $f_j \angle f_{i-1}$ and $s_1, \ldots, s_i \in \mathbf{D}f_j$ by Lemmas 24A and 24C. But f_i is by definition a bad $\mathfrak{F}^{\#}Q$-array such that $f_i \angle f_{i-1}$ and

$$s_1, \ldots, s_i \in \mathbf{D}f_i$$

and, subject to these requirements, $|f_i(s_i)|$ is as small as possible. Therefore

$$|f_j(s_i)| \geqslant |f_i(s_i)|.$$

But $f_j(s_i) \subset f_i(s_i)$ by Lemma 24A. Therefore $f_j(s_i) = f_i(s_i) = f(s_i)$.

Lemma 24F. $f \angle f_i$ *for every* $i \geqslant 0$.

Proof. $\mathbf{D}f = \{s_1, s_2, \ldots\} \subset \mathbf{D}f_i$ by Lemma 24C. Furthermore, $f(s_j) = f_j(s_j) \subset f_i(s_j)$ if $j \geqslant i$ by Lemma 24A and $f(s_j) = f_i(s_j) \subset f_i(s_j)$ if $j < i$ by Lemma 24E; hence $f(s) \subset f_i(s)$ for every $s \in \mathbf{D}f$.

By Lemma 24D, f is an $\mathfrak{F}^{\#}Q$-array; and $f \angle f_0$ by Lemma 24F. Moreover, $\mathbf{D}f = \{s_1, s_2, \ldots\}$ by the definition of f, and, if $s_i \lhd s_j$, then $i < j$ by Lemma 24B and

$$f(s_i) = f_j(s_i) \nleqslant {}^{\#} f_j(s_j) = f(s_j)$$

by Lemma 24E and the badness of f_j. Hence f is bad. Therefore, if f is not irreducibly bad, there exists by Lemma 23 a pair e, s such that e is a bad $\mathfrak{F}^{\#}Q$-array and $e \angle f$ and $s \in \mathbf{D}f$ and $\mathbf{D}e$ includes every element of $\mathbf{D}f$ whose last term is less than or equal to $\lambda(s)$ and $e(s)$ is strictly contained in $f(s)$. Since $s \in \mathbf{D}f$, $s = s_r$ for some r. Since $\mathbf{D}e$ includes every element of $\mathbf{D}f$ whose last term is less than or equal to $\lambda(s)$, it includes $s_1, \ldots, s_r$ by Lemma 24B. Since $e \angle f$ and $f \angle f_{r-1}$ by Lemma 24F, $e \angle f_{r-1}$. Hence e is a bad $\mathfrak{F}^{\#}Q$-array such that $e \angle f_{r-1}$ and $s_1, \ldots, s_r \in \mathbf{D}e$ and

$$|e(s_r)| < |f(s_r)| = |f_r(s_r)|.$$

Since this contradicts the definition of f_r, we infer that f must be irreducibly bad; and Lemma 24 is proved.

Lemma 25. *If f is a bad $\mathfrak{F}^{\#}Q$-array, there exists a bad Q-array ϕ such that $\mathbf{D}\phi \subset \mathbf{D}f$ and $\phi(s) \in f(s)$ for every $s \in \mathbf{D}\phi$.*

Proof. By Lemma 24, there exists an irreducibly bad $\mathfrak{F}^{\#}Q$-array g such that $g \angle f$. Since g is bad, there is no $s \in \mathbf{D}g$ for which $g(s) = \varnothing$ (since, if there were, we could by Lemma 12 select a $t \in \mathbf{D}g$ such that $s \lhd t$ and we should have $g(s) = \varnothing \leqslant {}^{\#} g(t)$). Therefore we can select a $\psi(s) \in g(s)$ for every $s \in \mathbf{D}g$. Write $h(s) = g(s) - \{\psi(s)\}$ for every $s \in \mathbf{D}g$: this defines an $\mathfrak{F}^{\#}Q$-array h with the same domain as g. By Lemma 19, $\mathbf{D}g$ contains a barrier B such that $h|B$ is either bad or perfect. However, if $h|B$ were bad, then, since $h|B \angle g$ and g is irreducibly bad, it would follow that $h|B \subset g$ and hence that $h(s) = g(s)$ for every $s \in B$, which by the definition of h is not so. Therefore $h|B$ is perfect. It follows that, if $s, t \in B$ and $s \lhd t$, then $h(s) \leqslant {}^{\#} h(t)$; and therefore $\psi(s) \nleqslant \psi(t)$ since the relations $h(s) \leqslant {}^{\#} h(t)$, $\psi(s) \leqslant \psi(t)$ would together imply that $g(s) \leqslant {}^{\#} g(t)$, whereas g is bad. It follows that $\psi|B$ is bad. Hence, if we write $\psi|B = \phi$, then ϕ is a bad Q-array, and (since $g \angle f$) $\mathbf{D}\phi = B \subset \mathbf{D}g \subset \mathbf{D}f$ and $\phi(s) = \psi(s) \in g(s) \subset f(s)$ for every $s \in \mathbf{D}\phi$.

Lemma 26. *If f is a bad $\mathfrak{S}Q$-array, there exists a bad Q-array ϕ such that $\mathbf{D}\phi = (\mathbf{D}f)^2$ and $\phi(s) \in f(s \wedge \mathbf{D}f)$ for every $s \in \mathbf{D}\phi$.*

Proof. If $s \in (\mathbf{D}f)^2$, then $s = u \circ v$ for some $u, v \in \mathbf{D}f$ such that $u \lhd v$, and this clearly implies that $u = s \wedge \mathbf{D}f$ and $v = {}_*s$. Since $u \lhd v$ and f is bad, $f(u) \nleqslant f(v)$, i.e.

$$f(s \wedge \mathbf{D}f) \nleqslant f({}_*s).$$

Hence, for every $s \in (\mathbf{D}f)^2$, we can select an element $\phi(s)$ of $f(s \wedge \mathbf{D}f)$ which does not anticipate any element of $f({}_*s)$, thus defining a function $\phi\colon (\mathbf{D}f)^2 \to Q$, which, by Lemma 15, is a Q-array. Suppose that $s, t \in (\mathbf{D}f)^2$ and $s \lhd t$. Then ${}_*s \prec t$ and $t \wedge \mathbf{D}f \prec t$ and $t \wedge \mathbf{D}f$ and (in view of what was said in the first sentence of this proof) ${}_*s$ belong to $\mathbf{D}f$: therefore ${}_*s = t \wedge \mathbf{D}f$ by Lemma 11. But $\phi(t) \in f(t \wedge \mathbf{D}f) = f({}_*s)$ and $\phi(s)$ does not anticipate any element of $f({}_*s)$ and therefore $\phi(s) \nleqslant \phi(t)$. Hence ϕ is bad.

Corollary 26A. *If Q is bqo, $\mathfrak{S}Q$ is bqc*

Lemma 27. *Every well-ordered set is bqo.*

Proof. Let W be a well-ordered set, and f be a W-array. By repeated application of Lemma 12, we can find an infinite sequence $s_1, s_2, \ldots$ of elements of $\mathbf{D}f$ such that $s_1 \lhd s_2 \lhd \ldots$. Since W is well-ordered, it cannot be the case that $f(s_1) > f(s_2) > \ldots$, and therefore f is good.

Lemma 28. *If Q is wqo and f is a bad $\mathfrak{S}^{\#}Q$-array, there exists a bad Q-array ϕ such that $\mathbf{D}\phi \subset (\mathbf{D}f)^2$ and $\phi(s) \in f(s \wedge \mathbf{D}f)$ for every $s \in \mathbf{D}\phi$.*

Proof. If $u, v \in \mathbf{D}f$ and $u \lhd v$, then $f(u) \nleq {}^{\#} f(v)$ since f is bad, and therefore $\Xi f(u) \nleq {}^{\#} \Xi f(v)$ by Lemma 10. By this remark and Corollary 8A, the composition Ξf is a bad $\mathfrak{F}^{\#}(C_Q \times \mathfrak{S}Q)$-array with domain $\mathbf{D}f$. Therefore, by Lemma 25, there exists a bad $(C_Q \times \mathfrak{S}Q)$-array ψ such that $\mathbf{D}\psi \subset \mathbf{D}f$ and $\psi(t) \in \Xi f(t)$ for every $t \in \mathbf{D}\psi$. By Lemma 27, C_Q is bqo, and therefore by Lemma 22 there exists a bad $\mathfrak{S}Q$-array g such that $\mathbf{D}g \subset \mathbf{D}\psi \subset \mathbf{D}f$ and $g(t) = c_2\psi(t)$ for every $t \in \mathbf{D}g$. By Lemma 26, there exists a bad Q-array ϕ such that $\mathbf{D}\phi = (\mathbf{D}g)^2$ and $\phi(s) \in g(s \wedge \mathbf{D}g)$ for every $s \in \mathbf{D}\phi$. Since $\mathbf{D}\phi = (\mathbf{D}g)^2$ and $\mathbf{D}g \subset \mathbf{D}f$, it follows that $\mathbf{D}\phi \subset (\mathbf{D}f)^2$ and that $s \wedge \mathbf{D}g = s \wedge \mathbf{D}f$ for every $s \in \mathbf{D}\phi$. Thus, if $s \in \mathbf{D}\phi$, $\phi(s) \in g(s \wedge \mathbf{D}f) = c_2\psi(s \wedge \mathbf{D}f)$. But, since

$$\psi(s \wedge \mathbf{D}f) \in \Xi f(s \wedge \mathbf{D}f)$$

and since, by the definition of Ξ, the second component of any element of $\Xi(A)$ is a subset of A, it follows that $c_2\psi(s \wedge \mathbf{D}f) \subset f(s \wedge \mathbf{D}f)$ and hence that $\phi(s) \in f(s \wedge \mathbf{D}f)$.

Corollary 28A. *If Q is bqo, $\mathfrak{S}^{\#}Q$ is bqo.*

Proof. If Q is bqo, it is also wqo by Lemma 1. Therefore there is no bad $\mathfrak{S}^{\#}Q$-array since the existence of a bad $\mathfrak{S}^{\#}Q$-array would by Lemma 28 entail the existence of a bad Q-array.

Note. An alternative method of obtaining Corollary 28A might be to attempt to prove, by an argument on the lines of that of section 4 of (5), that any set of (transfinite) sequences on a bqo set is bqo; but I have not yet fully explored this possibility. An argument of this type would probably be complicated, but it might yield a variant of Lemma 28 in which Q was not required to be wqo, thus avoiding certain complications which arise towards the end of section 4 (below) from the fact that certain lemmas in that section are proved for 'well-branched' sets of trees only.

4. *Proof of the bqo property for rooted trees.* A *rooted tree* is a tree T in which a vertex ρ_T is distinguished. We call ρ_T the *root* of T. Let $\mathscr{Z}$ be a set, which we shall think of as being fixed throughout this section, and let $\mathfrak{R}$ be the set of all rooted trees T such that $Z(T) \subset \mathscr{Z}$. $C_{\mathfrak{R}}$ will for simplicity be denoted by C. If T is a rooted tree and

$$\eta \in V(T) - \{\rho_T\}$$

and $$\xi_0, \lambda_1, \xi_1, \lambda_2, \ldots, \lambda_n, \xi_n$$

(where $\xi_0 = \rho_T$, $\xi_n = \eta$) is the unique path from ρ_T to η in T, ξ_{n-1} will be called the *predecessor* of η and denoted by pre η. We call η a *successor* of ξ if ξ is the predecessor of η. The set of successors of a vertex ξ in a rooted tree T will be denoted by ξT: we further write $\{\xi\} = \xi T^0$, $\xi T = \xi T^1$ and in general $\xi T^n = \bigcup\{\eta T : \eta \in \xi T^{n-1}\}$ for every

$n \in P$. [The notation $\bigcup\{X : \mathscr{S}\}$ means 'the union of all sets X such that $\mathscr{S}$'; and a similar notation will be used for intersections.] Thus ξT^n is the set of vertices ζ of T such that ξ is a term of the path from ρ_T to ζ and exactly n edges intervene between ξ, ζ in this path. The set $\xi T^0 \cup \xi T^1 \cup \xi T^2 \cup \ldots$ will be denoted by ξT^ω. A vertex ζ will be said to be *above* ξ if $\zeta \in \xi T^\omega$. (Note that ξ is above itself.) The *branch of* T *at* ξ (denoted by $\mathrm{br}\,\xi$) is the rooted tree U such that $\rho_U = \xi$, $V(U) = \xi T^\omega$, $E(U)$ is the set of edges of T joining pairs of elements of ξT^ω, and each element of $E(U)$ joins the same vertices in U as in T. If $S \subset V(T)$, $\mathrm{BR}\,S$ will denote $\{\mathrm{br}\,\xi | \xi \in S\}$, i.e. a set of branches of T. We shall write $\mathrm{BR}\,V(T) = \mathbf{B}T$, i.e. $\mathbf{B}T$ is the set of all branches of T, including T itself.

If $T, U \in \mathfrak{R}$, $\xi \in V(T)$ and $\eta \in V(U)$, a function $f: \xi T \to V(U)$ is *η-based* if there exists a *one-to-one* function g from ξT into ηU such that $f(\xi')$ is above $g(\xi')$ for every $\xi' \in \xi T$. A *conveyance* of T into U is a function $c: V(T) \to V(U)$ such that $c|\xi T$ is $c\xi$-based for every $\xi \in V(T)$. If $T, U \in \mathfrak{R}$, the statement that $T \leqslant U$ will, for the purposes of this section, mean that there exists a conveyance of T into U, and $\mathfrak{R}$ (or any subset of $\mathfrak{R}$) will be considered as being quasi-ordered by this definition. (Obviously every branch of T anticipates T in this quasi-ordering.) Two elements of a qo set are *equivalent* if each anticipates the other. A *strict* branch of T is a branch of T which is inequivalent to T, i.e. a branch U of T such that $T \nleqslant U$. The set of strict branches of T will be denoted by $\mathbf{S}T$. The set of vertices ξ of T such that $\mathrm{br}\,\xi$ is a strict branch of T (i.e. such that $T \nleqslant \mathrm{br}\,\xi$) will be denoted by $S(T)$, and the set of vertices ξ of T such that $\mathrm{br}\,\xi$ is equivalent to T (i.e. such that $T \leqslant \mathrm{br}\,\xi$) will be denoted by $R(T)$: thus $R(T)$, $S(T)$ are disjoint sets with union $V(T)$. If $T \in \mathfrak{R}$ and $\xi \in V(T)$, $\Gamma_T(\xi)$ will denote the ordered pair $(|\xi T \cap R(T)|, \mathrm{BR}\,(\xi T \cap S(T)))$, and will be considered as an element of $C \times \mathfrak{S}^{\#}\mathfrak{R}$. $\Theta(T)$ will denote $\{\Gamma_T(\xi) \,|\, \xi \in V(T)\}$, and will be considered as an element of $\mathfrak{S}(C \times \mathfrak{S}^{\#}\mathfrak{R})$, so that, for instance, '$\Theta(T) \leqslant \Theta(U)$' means that $\Theta(T)$ anticipates $\Theta(U)$ in $\mathfrak{S}(C \times \mathfrak{S}^{\#}\mathfrak{R})$.

LEMMA 29. *If $T, U \in \mathfrak{R}$ and $\Theta(T) \leqslant \Theta(U)$, then $T \leqslant U$.*

Proof. Let W be the set of all elements (ξ, η) of $V(T) \times V(U)$ such that either $\mathrm{br}\,\xi \leqslant \mathrm{br}\,\eta$ or $(\xi, \eta) \in R(T) \times R(U)$. Let $\overline{W}$ be the set of all elements (ξ, η) of $V(T) \times V(U)$ such that W contains an η-based function from ξT into $V(U)$.

LEMMA 29A. *If $(\xi, \eta) \in W$, then there exists a ζ above η in U such that $(\xi, \zeta) \in \overline{W}$.*

Proof. Since $(\xi, \eta) \in W$, either (i) $\mathrm{br}\,\xi \leqslant \mathrm{br}\,\eta$, or (ii) $(\xi, \eta) \in R(T) \times R(U)$.

In case (i), there exists a conveyance c of $\mathrm{br}\,\xi$ into $\mathrm{br}\,\eta$. If $f = c|\xi T$, then f is $c\xi$-based. Moreover, if $\alpha \in \xi T$, then $\mathrm{br}\,\alpha \leqslant \mathrm{br} f\alpha$ since $c|\alpha T^\omega$ is a conveyance of $\mathrm{br}\,\alpha$ into

$$\mathrm{br}\, c\alpha = \mathrm{br} f\alpha,$$

and therefore $(\alpha, f\alpha) \in W$. Hence $f \subset W$, and so $(\xi, c\xi) \in \overline{W}$. Since $c\xi$ is above η by the definition of c, Lemma 29A is proved in Case (i).

In case (ii), since $\eta \in R(U)$, there exists a conveyance d of U into $\mathrm{br}\,\eta$. Since

$$\Theta(T) \leqslant \Theta(U),$$

there exists a $\sigma \in V(U)$ such that $\Gamma_T(\xi) \leqslant \Gamma_U(\sigma)$, which implies that

$$|\xi T \cap R(T)| \leqslant |\sigma U \cap R(U)|$$

and $\text{BR}(\xi T \cap S(T)) \leqslant {}^{\#}\text{BR}(\sigma U \cap S(U))$. Hence there exists a one-to-one function $g: \xi T \to \sigma U$ such that $g\alpha \in \sigma U \cap R(U)$ when $\alpha \in \xi T \cap R(T)$ and $\text{br}\,\alpha \leqslant \text{br}\,g\alpha$ when $\alpha \in \xi T \cap S(T)$. Let $f = \{(\alpha, dg\alpha) | \alpha \in \xi T\}$, which is a function from ξT into $V(U)$. Since g is one-to-one and d is a conveyance, f is clearly $d\sigma$-based. Let $\alpha \in \xi T$. Then $d|\,\text{br}\,g\alpha$ is a conveyance of $\text{br}\,g\alpha$ into $\text{br}\,dg\alpha = \text{br}\,f\alpha$, and therefore $\text{br}\,g\alpha \leqslant \text{br}\,f\alpha$. Thus, if $\alpha \in R(T)$, we have $g\alpha \in R(U)$ and therefore $U \leqslant \text{br}\,g\alpha \leqslant \text{br}\,f\alpha$, which implies that $f\alpha \in R(U)$ and hence that $(\alpha, f\alpha) \in R(T) \times R(U) \subset W$, whilst if $\alpha \in S(T)$, then $\text{br}\,\alpha \leqslant \text{br}\,g\alpha \leqslant \text{br}\,f\alpha$ and so once again $(\alpha, f\alpha) \in W$. Hence $f \subset W$, and so $(\xi, d\sigma) \in \overline{W}$. Since $d\sigma$ is above η by the definition of d, Lemma 29A is proved in Case (ii) also.

Lemma 29B. *If $(\xi, \eta) \in \overline{W}$, then $\overline{W}$ contains an η-based function from ξT into $V(U)$.*

Proof. By the definition of $\overline{W}$, W contains an η-based function f from ξT into $V(U)$. For each $\alpha \in \xi T$, we can select by Lemma 29A a $g\alpha$ above $f\alpha$ such that $(\alpha, g\alpha) \in \overline{W}$. This clearly defines an η-based function $g: \xi T \to V(U)$ contained in $\overline{W}$.

To complete the proof of Lemma 29, we construct a conveyance ϕ of T into U as follows. Since $(\rho_T, \rho_U) \in R(T) \times R(U) \subset W$, we can by Lemma 29A select a $\phi\rho_T \in V(U)$ such that $(\rho_T, \phi\rho_T) \in \overline{W}$. We may say that this defines $\phi\xi$ for every $\xi \in \rho_T T^0$ in such a way that $(\xi, \phi\xi) \in \overline{W}$ for every such ξ. Moreover, if $\phi\xi$ has been defined and $(\xi, \phi\xi) \in \overline{W}$ for every $\xi \in \rho_T T^n$, then by Lemma 29B there exists for each $\xi \in \rho_T T^n$ a $\phi\xi$-based function ψ_ξ from ξT into $V(U)$ such that $\psi_\xi \subset \overline{W}$. If, for each $\xi' \in \rho_T T^{n+1}$, we write $\phi\xi' = \psi_{\text{pre}\,\xi'}\xi'$, then this defines $\phi\xi'$ in such a way that $(\xi', \phi\xi') \in \overline{W}$ for every $\xi' \in \rho_T T^{n+1}$. It is clear that the function $\phi : V(T) \to V(U)$ constructed in this manner is a conveyance.

Definitions. Let $T \in \mathfrak{R}$. If U is a branch of T, we write $U \leqslant^* T$ or $T \geqslant^* U$. If U is a strict branch of T, we write $U <^* T$ or $T >^* U$. Thus $U \leqslant T$ if $U \leqslant^* T$ and $U < T$ if $U <^* T$. T is *descensionally infinite* if there exists an ω-sequence u on $\mathbf{B}T$ such that $u(1) >^* u(2) >^* \ldots$. If no such ω-sequence exists, T is *descensionally finite.* The set of descensionally finite branches of T will be denoted by $\mathbf{F}T$. $I(T)$ will denote the set of vertices ξ of T such that $\text{br}\,\xi$ is descensionally infinite and $F(T)$ will denote the set of vertices ξ of T such that $\text{br}\,\xi$ is descensionally finite. For any $\xi \in V(T)$, $\Delta_T(\xi)$ will denote the element $(|\xi T \cap I(T)|, \text{BR}(\xi T \cap F(T)))$ of $C \times \mathfrak{S}^{\#}\mathfrak{R}$, and $\Phi(T, \xi)$ will denote the element $\{\Delta_T(\eta) | \eta \in \xi T^\omega\}$ of $\mathfrak{S}(C \times \mathfrak{S}^{\#}\mathfrak{R})$.

Lemma 30. *Let $T \in \mathfrak{R}$ and σ be an element of $I(T)$ such that $\Phi(T, \sigma) \leqslant \Phi(T, \tau)$ for every $\tau \in \sigma T^\omega \cap I(T)$. Then* $\text{br}\,\sigma \leqslant \text{br}\,\tau$ *for every $\tau \in \sigma T^\omega \cap I(T)$.*

Proof. Let W be the set of all elements (ξ, η) of $\sigma T^\omega \times \sigma T^\omega$ such that either $\text{br}\,\xi \leqslant \text{br}\,\eta$ or $(\xi, \eta) \in I(T) \times I(T)$. Let $\overline{W}$ be the set of all elements (ξ, η) of $\sigma T^\omega \times \sigma T^\omega$ such that W contains an η-based function from ξT into σT^ω.

Lemma 30A. *If $(\xi, \eta) \in W$, then there exists a ζ above η such that $(\xi, \zeta) \in \overline{W}$.*

Proof. Since $(\xi, \eta) \in W$, either (i) $\text{br}\,\xi \leqslant \text{br}\,\eta$ or (ii) $(\xi, \eta) \in I(T) \times I(T)$.

In Case (i), there exists a conveyance c of $\text{br}\,\xi$ into $\text{br}\,\eta$. If $f = c|\xi T$, then f is $c\xi$-based. Moreover, if $\alpha \in \xi T$, then $\text{br}\,\alpha \leqslant \text{br}\,f\alpha$ since $c|\alpha T^\omega$ is a conveyance of $\text{br}\,\alpha$ into

$$\text{br}\,c\alpha = \text{br}\,f\alpha,$$

and therefore $(\alpha, f\alpha) \in W$. Hence $f \subset W$, and so $(\xi, c\xi) \in \overline{W}$. Since $c\xi$ is above η by the definition of c, Lemma 30A is proved in Case (i).

In Case (ii), since $\eta \in \sigma T^{\omega} \cap I(T)$, $\Phi(T,\sigma) \leqslant \Phi(T,\eta)$ by the hypothesis of Lemma 30. Therefore the element $\Delta_T(\xi)$ of $\Phi(T,\sigma)$ anticipates an element of $\Phi(T,\eta)$, i.e. there exists a $\zeta \in \eta T^{\omega}$ such that $\Delta_T(\xi) \leqslant \Delta_T(\zeta)$, which implies that

$$|\xi T \cap I(T)| \leqslant |\zeta T \cap I(T)|$$

and $\mathrm{BR}(\xi T \cap F(T)) \leqslant {}^{\#}\,\mathrm{BR}(\zeta T \cap F(T))$. Hence there exists a one-to-one function $g: \xi T \to \zeta T$ such that $g\alpha \in \zeta T \cap I(T)$ when $\alpha \in \xi T \cap I(T)$ and $\mathrm{br}\,\alpha \leqslant \mathrm{br}\,g\alpha$ when $\alpha \in \xi T \cap F(T)$. These properties of g imply that $(\alpha, g\alpha) \in W$ whenever $\alpha \in \xi T$, i.e. that $g \subset W$. Since g is clearly ζ-based, we have $(\xi,\zeta) \in \overline{W}$ and Lemma 30A is proved in Case (ii) also.

LEMMA 30B. *If $(\xi,\eta) \in \overline{W}$, then $\overline{W}$ contains an η-based function from ξT into σT^{ω}.*

Proof. By the definition of $\overline{W}$, W contains an η-based function f from ξT into σT^{ω}. For each $\alpha \in \xi T$, we can select by Lemma 30A a $g\alpha$ above $f\alpha$ such that $(\alpha, g\alpha) \in \overline{W}$. This clearly defines an η-based function $g: \xi T \to \sigma T^{\omega}$ contained in $\overline{W}$.

Let $\tau \in \sigma T^{\omega} \cap I(T)$. We shall construct a conveyance ϕ of $\mathrm{br}\,\sigma$ into $\mathrm{br}\,\tau$ as follows. Since

$$(\sigma,\tau) \in (\sigma T^{\omega} \cap I(T)) \times (\sigma T^{\omega} \cap I(T)) \subset W,$$

we can by Lemma 30A select a $\phi\sigma$ above τ such that $(\sigma, \phi\sigma) \in \overline{W}$. We may say that this defines $\phi\xi$ for every $\xi \in \sigma T^0$ in such a way that $(\xi, \phi\xi) \in \overline{W}$ for every such ξ. Moreover, if $\phi\xi$ has been defined and $(\xi,\phi\xi) \in \overline{W}$ for every $\xi \in \sigma T^n$, then by Lemma 30B there exists for each $\xi \in \sigma T^n$ a $\phi\xi$-based function ψ_{ξ} from ξT into σT^{ω} such that $\psi_{\xi} \subset \overline{W}$. If, for each $\xi' \in \sigma T^{n+1}$, we write $\phi\xi' = \psi_{\mathrm{pre}\,\xi'}\xi'$, then this defines $\phi\xi'$ in such a way that $(\xi', \phi\xi') \in \overline{W}$ for every $\xi' \in \sigma T^{n+1}$. It is clear that the function $\phi: \sigma T^{\omega} \to \tau T^{\omega}$ constructed in this manner is a conveyance of $\mathrm{br}\,\sigma$ into $\mathrm{br}\,\tau$; and hence $\mathrm{br}\,\sigma \leqslant \mathrm{br}\,\tau$.

LEMMA 31. *If $T \in \mathfrak{R}$ and $\sigma \in I(T)$, then there exists a $\tau \in \sigma T^{\omega} \cap I(T)$ such that*

$$\Phi(T,\tau) < \Phi(T,\sigma).$$

Proof. Since $\sigma \in I(T)$, $\mathrm{br}\,\sigma$ is descensionally infinite. Therefore there exists an infinite sequence $\sigma_1, \sigma_2, \ldots$ of vertices of $\mathrm{br}\,\sigma$ such that $\mathrm{br}\,\sigma_1 >{}^{*}\,\mathrm{br}\,\sigma_2 >{}^{*} \ldots$. Since $\mathrm{br}\,\sigma_2 >{}^{*}\,\mathrm{br}\,\sigma_3 >{}^{*} \ldots$, it follows that $\sigma_2 \in I(T)$ and therefore that $\sigma_2 \in \sigma T^{\omega} \cap I(T)$. But $\mathrm{br}\,\sigma \nleqslant \mathrm{br}\,\sigma_2$ since $\mathrm{br}\,\sigma_2 < \mathrm{br}\,\sigma_1 \leqslant \mathrm{br}\,\sigma$. Therefore, by Lemma 30, there is a $\tau \in \sigma T^{\omega} \cap I(T)$ such that $\Phi(T,\sigma) \nleqslant \Phi(T,\tau)$. But $\Phi(T,\tau) \leqslant \Phi(T,\sigma)$ since

$$\Phi(T,\tau) \subset \Phi(T,\sigma).$$

Therefore $\Phi(T,\tau) < \Phi(T,\sigma)$.

LEMMA 32. *If $T \in \mathfrak{R}$ and $\mathbf{F}T$ is bqo, then T is descensionally finite.*

Proof. Write $\rho_T = \sigma_1$. Suppose that T is descensionally infinite. Then $\sigma_1 \in I(T)$, and therefore by Lemma 31 there is a $\sigma_2 \in \sigma_1 T^{\omega} \cap I(T)$ such that $\Phi(T,\sigma_2) < \Phi(T,\sigma_1)$. By Lemma 31 again, there is a $\sigma_3 \in \sigma_2 T^{\omega} \cap I(T)$ such that $\Phi(T,\sigma_3) < \Phi(T,\sigma_2)$. Continuing in this manner, we construct an infinite descending chain

$$\Phi(T,\sigma_1) > \Phi(T,\sigma_2) > \ldots \tag{3}$$

of elements of $\mathfrak{S}(C \times \mathfrak{S}^{\#}\mathbf{F}T)$. But, since $\mathbf{F}T$ is bqo, $\mathfrak{S}(C \times \mathfrak{S}^{\#}\mathbf{F}T)$ is bqo by Lemma 27 and Corollaries 22A, 26A and 28A, and hence is wqo by Lemma 1. Therefore $\Phi(T,\sigma_i) \leqslant \Phi(T,\sigma_j)$ for some i,j such that $i<j$. Since this contradicts (3), T must be descensionally finite.

Definitions. The set of descensionally finite elements of $\mathfrak{R}$ will be denoted by $\mathfrak{R}_0$. If $\mathfrak{U} \subset \mathfrak{R}$, $\mathbf{S}\mathfrak{U}$ will denote $\bigcup\{\mathbf{S}T : T \in \mathfrak{U}\}$. $\mathfrak{U}$ is *well-branched* if $\mathbf{S}\mathfrak{U}$ is wqo. $\mathfrak{U}$ is *closed* if $\mathbf{B}T \subset \mathfrak{U}$ for every $T \in \mathfrak{U}$. An $\mathfrak{R}$-function f will be said to *forerun* an $\mathfrak{R}$-function g if $\bar{g} \subset \bar{f}$ and, for every $t \in \mathbf{D}g$, either $t \in \mathbf{D}f$ and $g(t) = f(t)$ or t has a strict left-segment s such that $s \in \mathbf{D}f$ and $g(t) < {}^*f(s)$. We shall say that f *warily foreruns* g if f foreruns g and there is an $s \in \mathbf{D}f - \mathbf{D}g$ such that $\bar{g}$ includes all elements of $\bar{f}$ less than or equal to $\lambda(s)$.

Let δ be an ordinal number. If there exists an ordinal γ such that $\delta = \gamma + 1$, we shall write $\gamma = \delta - 1$. If γ does not exist and $\delta \neq 0$, δ is a *limit ordinal.* If δ is a limit ordinal and S_α is a set for every $\alpha < \delta$, $\liminf_{\alpha \to \delta} S_\alpha$ will denote $\bigcup_{\alpha<\delta} \bigcap_{\alpha \leqslant \beta < \delta} S_\beta$, i.e. the set to which an element x belongs iff there is an $\alpha < \delta$ such that $x \in S_\beta$ whenever $\alpha \leqslant \beta < \delta$.

Lemma 33. *Let κ be a limit ordinal, and let an $\mathfrak{R}$-function f_β be given for every $\beta \leqslant \kappa$ in such a way that*

(i) *f_β foreruns $f_{\beta+1}$ for every $\beta < \kappa$,*
(ii) *$f_\beta = \liminf_{\alpha \to \beta} f_\alpha$ for every limit ordinal $\beta \leqslant \kappa$.*

Then f_γ foreruns f_ϵ whenever $\gamma \leqslant \epsilon \leqslant \kappa$.

Proof. We shall consider a fixed γ ($\leqslant \kappa$) and prove the required result by transfinite induction on ϵ. Since f_γ clearly foreruns itself, the result is true for $\epsilon = \gamma$. Suppose that $\gamma < \epsilon \leqslant \kappa$ and that f_γ foreruns f_δ whenever $\gamma \leqslant \delta < \epsilon$. We will deduce that f_γ foreruns f_ϵ by considering two cases.

(*a*) Suppose that ϵ is not a limit ordinal. Then $f_{\epsilon-1}$ foreruns f_ϵ by (i) and f_γ foreruns $f_{\epsilon-1}$ by the inductive hypothesis. Therefore $\bar{f}_\epsilon \subset \bar{f}_{\epsilon-1} \subset \bar{f}_\gamma$. Let $u \in \mathbf{D}f_\epsilon$. Then, since $f_{\epsilon-1}$ foreruns f_ϵ, u has a left-segment $t \in \mathbf{D}f_{\epsilon-1}$ such that either $t = u$ and $f_{\epsilon-1}(t) = f_\epsilon(u)$ or $t \prec\prec u$ and $f_{\epsilon-1}(t) > {}^*f_\epsilon(u)$; and, since f_γ foreruns $f_{\epsilon-1}$, t has a left-segment $s \in \mathbf{D}f_\gamma$ such that either $s = t$ and $f_\gamma(s) = f_{\epsilon-1}(t)$ or $s \prec\prec t$ and $f_\gamma(s) > {}^*f_{\epsilon-1}(t)$. It follows from these properties of t and s that s is a left-segment of u belonging to $\mathbf{D}f_\gamma$ such that either $s = u$ and $f_\gamma(s) = f_\epsilon(u)$ or $s \prec\prec u$ and $f_\gamma(s) > {}^*f_\epsilon(u)$. Hence f_γ foreruns f_ϵ.

(*b*) Suppose that ϵ is a limit ordinal. Let $n \in \bar{f}_\epsilon$. Then n is a term of some $u \in \mathbf{D}f_\epsilon$. Since $(u, f_\epsilon(u)) \in f_\epsilon = \liminf_{\alpha \to \epsilon} f_\alpha$, there is a δ such that $\gamma \leqslant \delta < \epsilon$ and $(u, f_\epsilon(u)) \in f_\delta$. Therefore $u \in \mathbf{D}f_\delta$ and therefore $n \in \bar{f}_\delta$, which by the inductive hypothesis is contained in $\bar{f}_\gamma$. Hence $\bar{f}_\epsilon \subset \bar{f}_\gamma$. Furthermore, if $u \in \mathbf{D}f_\epsilon$, there exists as above a δ such that $\gamma \leqslant \delta < \epsilon$ and $(u, f_\epsilon(u)) \in f_\delta$. Hence $u \in \mathbf{D}f_\delta$ and $f_\delta(u) = f_\epsilon(u)$. Therefore, by the inductive hypothesis, either $u \in \mathbf{D}f_\gamma$ and $f_\gamma(u) = f_\delta(u) = f_\epsilon(u)$ or u has a strict left-segment $s \in \mathbf{D}f_\gamma$ such that $f_\gamma(s) > {}^*f_\delta(u) = f_\epsilon(u)$. Hence f_γ foreruns f_ϵ, and the proof of Lemma 33 is completed.

Lemma 34. *Let κ be a limit ordinal, and let a bad $\mathfrak{R}_0$-array f_β be given for every $\beta < \kappa$ in such a way that*

(i) *f_β warily foreruns $f_{\beta+1}$ for every $\beta < \kappa$,*
(ii) *$f_\beta = \liminf_{\alpha \to \beta} f_\alpha$ for every limit ordinal $\beta < \kappa$.*

Let f_κ denote $\liminf_{\alpha \to \kappa} f_\alpha$. Then f_κ is a bad $\mathfrak{R}_0$-array. Furthermore, the sets $\mathbf{D}f_\alpha$ ($\alpha < \kappa$) are all distinct.

Proof. If $(s, T) \in f_\kappa$, then, by the definition of f_κ, $(s, T) \in f_\epsilon \subset A(P) \times \mathfrak{R}_0$ for some $\epsilon < \kappa$. Hence $f_\kappa \subset A(P) \times \mathfrak{R}_0$. If (s, T), $(s, T') \in f_\kappa$, then, by the definition of f_κ, there exists an $\epsilon < \kappa$ such that (s, T), (s, T') both belong to f_ϵ, and therefore, since f_ϵ is a function, $T = T'$. Hence f_κ is an $\mathfrak{R}_0$-function, and therefore, by Lemma 33, f_γ foreruns f_ϵ whenever $\gamma \leqslant \epsilon \leqslant \kappa$. Therefore

$$\bar{f}_\gamma \supset \bar{f}_\epsilon \quad (\gamma \leqslant \epsilon \leqslant \kappa). \tag{4}$$

Moreover, if $s \in \mathbf{D}f_\kappa$, then $(s, f_\kappa(s)) \in f_\kappa = \liminf_{\alpha \to \kappa} f_\alpha$ and therefore there is a $\zeta(s) < \kappa$ such that $(s, f_\kappa(s)) \in f_\theta$ for $\zeta(s) \leqslant \theta < \kappa$, i.e.

$$s \in \mathbf{D}f_\theta, \quad f_\kappa(s) = f_\theta(s) \quad \text{for} \quad \zeta(s) \leqslant \theta < \kappa. \tag{5}$$

Lemma 34A. *If $\gamma \leqslant \epsilon < \kappa$ and $u \in \mathbf{D}f_\epsilon$, then u has a unique left-segment in $\mathbf{D}f_\gamma$.*

Proof. Since f_γ foreruns f_ϵ (by Lemma 33), u has a left-segment in $\mathbf{D}f_\gamma$, which is unique by Lemma 11.

Lemma 34B. *If $\gamma < \epsilon < \kappa$ and $u \in \mathbf{D}f_\gamma - \mathbf{D}f_{\gamma+1}$, then $u \notin \mathbf{D}f_\epsilon$.*

Proof. If u belonged to $\mathbf{D}f_\epsilon$, then by Lemma 34A u would have a left-segment $t \in \mathbf{D}f_{\gamma+1}$ and t would have a left-segment $s \in \mathbf{D}f_\gamma$. Since s would then be a subsequence of u and both u and s would belong to $\mathbf{D}f_\gamma$, it would follow (since $\mathbf{D}f_\gamma$ is a barrier) that $u = s$ and hence that $u = t \in \mathbf{D}f_{\gamma+1}$, contrary to hypothesis.

We can now prove the last sentence in the statement of Lemma 34. For, if $\gamma < \epsilon < \kappa$, there exists by (i) an $s \in \mathbf{D}f_\gamma - \mathbf{D}f_{\gamma+1}$. By Lemma 34B, $s \notin \mathbf{D}f_\epsilon$, and hence $\mathbf{D}f_\gamma \neq \mathbf{D}f_\epsilon$.

Lemma 34C. *If $\gamma \leqslant \epsilon < \kappa$, $s \in \mathbf{D}f_\gamma$, $t \in \mathbf{D}f_\epsilon$ and $s \prec t$, then either $s = t$ and $f_\gamma(s) = f_\epsilon(t)$ or $s \prec\prec t$ and $f_\gamma(s) >^* f_\epsilon(t)$.*

Proof. Since f_γ foreruns f_ϵ (by Lemma 33), t has a left-segment $s' \in \mathbf{D}f_\gamma$ such that either $s' = t$ and $f_\gamma(s') = f_\epsilon(t)$ or $s' \prec\prec t$ and $f_\gamma(s') >^* f_\epsilon(t)$. By Lemma 34A, $s = s'$.

Lemma 34D. *If θ is a limit ordinal and $\theta \leqslant \kappa$, every element of $A_\omega(\bigcap\{\bar{f}_\alpha : \alpha < \theta\})$ has a left-segment in $\mathbf{D}f_\theta$.*

Proof. Let $u \in A_\omega(\bigcap\{\bar{f}_\alpha : \alpha < \theta\})$. Then, for any $\alpha < \theta$, $u \in A_\omega(\bar{f}_\alpha)$ and so has, by Corollary 11A, a unique left-segment $u_\alpha \in \mathbf{D}f_\alpha$. If $\gamma \leqslant \epsilon < \theta$, u_ϵ has by Lemma 34A a left-segment $u_{\epsilon,\gamma} \in \mathbf{D}f_\gamma$, and, since $u_{\epsilon,\gamma}$ is a left-segment of u which belongs to $\mathbf{D}f_\gamma$, $u_\gamma = u_{\epsilon,\gamma} \prec u_\epsilon$. Hence $u_\gamma \prec u_\epsilon$ whenever $\gamma \leqslant \epsilon < \theta$. It follows that either there is a δ such that $u_\delta = u_\epsilon$ whenever $\delta \leqslant \epsilon < \theta$ or there is an infinite ascending sequence $\phi < \psi < \chi < \ldots$ of ordinals less than θ such that $u_\phi \prec\prec u_\psi \prec\prec u_\chi \prec\prec \ldots$. But the latter alternative would by Lemma 34C imply that $f_\phi(u_\phi) >^* f_\psi(u_\psi) >^* \ldots$, which is impossible since $f_\phi(u_\phi)$ belongs to $\mathfrak{R}_0$ and so is descensionally finite. Therefore δ exists, and, since $u_\delta = u_\epsilon$ for $\delta \leqslant \epsilon < \theta$, it follows from Lemma 34C that $f_\delta(u_\delta) = f_\epsilon(u_\epsilon)$ for $\delta \leqslant \epsilon < \theta$, so that $(u_\delta, f_\delta(u_\delta)) = (u_\epsilon, f_\epsilon(u_\epsilon)) \in f_\epsilon$ for $\delta \leqslant \epsilon < \theta$, and therefore

$$(u_\delta, f_\delta(u_\delta)) \in \liminf_{\alpha \to \theta} f_\alpha = f_\theta,$$

and therefore $u_\delta \in \mathbf{D}f_\theta$. Hence u has a left-segment in $\mathbf{D}f_\theta$, and Lemma 34D is proved.

Corollary 34DA. *If θ is a limit ordinal and $\theta \leqslant \kappa$ and either $\bar{f}_\theta$ or $\bigcap\{\bar{f}_\alpha : \alpha < \theta\}$ is an infinite set, then $\bar{f}_\theta = \bigcap\{\bar{f}_\alpha : \alpha < \theta\}$ and $\mathbf{D}f_\theta$ is a block.*

Proof. By (4), $\bar{f}_\theta \subset \bigcap\{\bar{f}_\alpha : \alpha < \theta\}$. From this fact and the hypothesis that one of these two sets is infinite, it follows that $\bigcap\{\bar{f}_\alpha : \alpha < \theta\}$ is infinite. By Lemma 34D, every element of $A_\omega(\bigcap\{\bar{f}_\alpha : \alpha < \theta\})$ has a left-segment in $\mathbf{D}f_\theta$. Therefore, by Lemma 13,

$$\bar{f}_\theta = \bigcap\{\bar{f}_\alpha : \alpha < \theta\}$$

and $\mathbf{D}f_\theta$ is a block.

COROLLARY 34DB. *If θ is a limit ordinal and $\theta < \kappa$, then $\bar{f}_\theta = \bigcap\{\bar{f}_\alpha : \alpha < \theta\}$.*

Proof. In this case f_θ is by the hypotheses of Lemma 34 an $\mathfrak{R}_0$-array: therefore $\bar{f}_\theta$ is infinite and Corollary 34DA applies.

LEMMA 34E. *$\bigcap\{\bar{f}_\alpha : \alpha < \kappa\}$ is an infinite set.*

Proof. Suppose that $\bigcap\{\bar{f}_\alpha : \alpha < \kappa\}$ is finite. Let L be the set of those positive integers which are less than or equal to at least one element of this intersection. Then L is also a finite set. Therefore, since $\bar{f}_\gamma$ is an infinite set, $\bar{f}_\gamma - L$ is non-empty for every $\gamma < \kappa$. Write $\gamma(0) = 0$. Let m_0 be the smallest element of $\bar{f}_{\gamma(0)} - L$. Let $\gamma(1)$ be the smallest γ such that $m_0 \notin \bar{f}_\gamma$: $\gamma(1)$ exists and is less than κ since $m_0 \notin L$ and therefore $m_0 \notin \bigcap\{\bar{f}_\alpha : \alpha < \kappa\}$. Let m_1 be the smallest element of $\bar{f}_{\gamma(1)} - L$. Let $\gamma(2)$ be the smallest γ such that $m_1 \notin \bar{f}_\gamma$ (which exists and is less than κ as before), and let m_2 be the smallest element of $\bar{f}_{\gamma(2)} - L$. Let $\gamma(3)\,(< \kappa)$ be the smallest γ such that $m_2 \notin \bar{f}_\gamma$, m_3 be the smallest element of $\bar{f}_{\gamma(3)} - L$, and so on. Let $i \in P$. Then, since m_{i-1} belongs to $\bar{f}_{\gamma(i-1)}$ but not to $\bar{f}_{\gamma(i)}$,

$$\gamma(i-1) < \gamma(i) \tag{6}$$

by (4). By the definition of $\gamma(i)$ and Corollary 34DB, $\gamma(i)$ is not a limit ordinal, and by (6) $\gamma(i) > 0$. Therefore $\gamma(i) - 1$ exists and hence, by (i), there is an $s_i \in \mathbf{D}f_{\gamma(i)-1} - \mathbf{D}f_{\gamma(i)}$ such that $\bar{f}_{\gamma(i)}$ includes all elements of $\bar{f}_{\gamma(i)-1}$ less than or equal to $\lambda(s_i)$. But, by the definition of $\gamma(i)$, m_{i-1} is an element of $\bar{f}_{\gamma(i)-1}$ which is not in $\bar{f}_{\gamma(i)}$. Therefore $m_{i-1} > \lambda(s_i)$. But $s_i \in A(\bar{f}_{\gamma(i)-1}) \subset A(\bar{f}_{\gamma(i-1)})$ by (4) and (6), and m_{i-1} is the smallest element of $\bar{f}_{\gamma(i-1)}$ not in L. Therefore $s_i \in A(L)$. Moreover, if $1 \leqslant j < k$, then $\gamma(j) < \gamma(k)$ by (6) and therefore by Lemma 34B $s_j \notin \mathbf{D}f_{\gamma(k)-1}$ and therefore $s_j \neq s_k$. Hence $s_1, s_2, \ldots$ are distinct elements of $A(L)$, which is impossible since L is finite. Hence our assumption that $\bigcap\{\bar{f}_\alpha : \alpha < \kappa\}$ was finite must have been false, and Lemma 34E is proved.

To complete the proof of Lemma 34, we observe that, by Lemma 34E and Corollary 34DA, $\mathbf{D}f_\kappa$ is a block. Moreover, if $s, t \in \mathbf{D}f_\kappa$ and $\zeta = \max(\zeta(s), \zeta(t))$, then by (5) s, t both belong to $\mathbf{D}f_\zeta$, which is a barrier, and hence neither of s, t can be a strict subsequence of the other. Therefore $\mathbf{D}f_\kappa$ is a barrier, and hence f_κ (which we have already proved to be an $\mathfrak{R}_0$-function) is an $\mathfrak{R}_0$-array. Finally, if $s, t \in \mathbf{D}f_\kappa$ and $s \lhd t$, and if $\zeta = \max(\zeta(s), \zeta(t))$, then by (5) s, t both belong to $\mathbf{D}f_\zeta$ and $f_\kappa(s) = f_\zeta(s)$, $f_\kappa(t) = f_\zeta(t)$ and hence $f_\kappa(s) \nleqslant f_\kappa(t)$ since f_ζ is bad. Hence f_κ is bad.

LEMMA 35. *Let B, D be barriers such that $D \subset B^3$. Then there exists a barrier $F \subset B \cup D$ and a $w \in B - F$ such that $\bar{F}$ includes all elements of $\bar{B}$ less than or equal to $\lambda(w)$.*

Proof. Since $D \subset B^3$, $\bar{D} \subset \bar{B}$ and therefore each element of $A_\omega(\bar{D})$ has a left-segment in B. Hence $B \cap A(\bar{D}) \neq \varnothing$. Select a $w \in B \cap A(\bar{D})$ with $\lambda(w)$ as small as possible. Let W be the set of elements of $\bar{B}$ less than or equal to $\lambda(w)$, and let $W \cup \bar{D} = \bar{\bar{D}}$. Let $B_0 = B \cap (A(\bar{\bar{D}}) - A(\bar{D}))$, and let $F = B_0 \cup D$. Then $F \subset B \cup D$.

Let $u \in A_\omega(\bar{\bar{D}})$. We will show that u has a left-segment in F. In the first place, $u \in A_\omega(\bar{B})$ since $\bar{\bar{D}} \subset \bar{B} \cup \bar{D} \subset \bar{B} \cup \bar{B}^3 = \bar{B}$. Therefore, since B is a barrier, u has a left-segment $s \in B$. Since $u \in A_\omega(\bar{\bar{D}})$, it follows that $s \in B \cap A(\bar{\bar{D}})$ and therefore either $s \in B_0 \subset F$ or $s \in B \cap A(\bar{D})$. If $s \in F$, u has a left-segment in F, and so we may suppose that $s \in B \cap A(\bar{D})$. Then $\lambda(s) \geqslant \lambda(w)$ by the definition of w. Write $u = sv$. Since $u \in A_\omega(\bar{\bar{D}})$ and $\lambda(s) \geqslant \lambda(w)$, the terms of v belong to $\bar{\bar{D}}$ and are greater than $\lambda(w)$ and therefore belong to $\bar{D}$. Since we are assuming that $s \in A(\bar{D})$, this implies that $u = sv$ belongs to $A_\omega(\bar{D})$ and so (since D is a barrier) has a left-segment in D and hence in F. Thus every element of $A_\omega(\bar{\bar{D}})$ has a left-segment in F. Moreover $\bar{\bar{D}}$ (since it contains $\bar{D}$) is an infinite set; and $\bar{F} \subset \bar{\bar{D}}$ since $B_0 \subset A(\bar{\bar{D}})$ and $D \subset A(\bar{D}) \subset A(\bar{\bar{D}})$. Therefore by Lemma 13 $\bar{F} = \bar{\bar{D}}$ and F is a block.

Let $s, t \in F = B_0 \cup D$: then we will show that s cannot be a strict subsequence of t. If $s, t \in B_0 (\subset B)$, then s cannot be a strict subsequence of t since B is a barrier, and the same conclusion follows if $s, t \in D$ since D is a barrier. If $s \in D$ and $t \in B_0$, then, since $D \subset B^3$, s has a subsequence $s' \in B$. Since s', $t \in B$, s' cannot be a strict subsequence of t and therefore nor can s. Finally, if $s \in B_0$ and $t \in D$, then, by the definition of B_0, $s \notin A(\bar{D})$ and so (since $t \in D$) s cannot be a subsequence of t. We have thus established that F is a barrier.

By Lemma 16, an element of B cannot belong to B^3 and therefore cannot belong to D: hence $w \notin D$. Furthermore, $w \notin B_0$ since $w \in B \cap A(\bar{D})$, and hence $w \notin F$. Therefore $w \in B - F$. Finally, since we have shown that $\bar{F} = \bar{\bar{D}}$, $\bar{F}$ contains W, i.e. includes all elements of $\bar{B}$ less than or equal to $\lambda(w)$.

LEMMA 36. *Let $\mathfrak{U}$ be a subset of $\mathfrak{R}$ and f be a bad $\mathfrak{U}$-array. If there exists a bad $\mathfrak{U}$-array g such that $\mathbf{D}g \subset (\mathbf{D}f)^3$ and $g(s) < {}^*f(s \wedge \mathbf{D}f)$ for every $s \in \mathbf{D}g$, then there exists a bad $\mathfrak{U}$-array h such that f warily foreruns h.*

Proof. By Lemma 35, there exists a barrier $F \subset \mathbf{D}f \cup \mathbf{D}g$ and a $w \in \mathbf{D}f - F$ such that $\bar{F}$ includes all elements of $\bar{f}$ less than or equal to $\lambda(w)$. We note that, since $\mathbf{D}g \subset (\mathbf{D}f)^3$, $\mathbf{D}f$ and $\mathbf{D}g$ are by Lemma 16 disjoint, so that we can unambiguously define a function $h: F \to \mathfrak{U}$ by writing $h(s) = f(s)$ if $s \in F \cap \mathbf{D}f$ and $h(s) = g(s)$ if $s \in F \cap \mathbf{D}g$. If $s \in \mathbf{D}h$, then either $s \in F \cap \mathbf{D}f$ or $s \in F \cap \mathbf{D}g$. In the former case, $s \in \mathbf{D}f$ and $h(s) = f(s)$, and in the latter case, since $\mathbf{D}g \subset (\mathbf{D}f)^3$, s has a strict left-segment $s \wedge \mathbf{D}f$ in $\mathbf{D}f$ and

$$h(s) = g(s) < {}^*f(s \wedge \mathbf{D}f).$$

Furthermore, $\mathbf{D}h = F \subset \mathbf{D}f \cup \mathbf{D}g \subset \mathbf{D}f \cup (\mathbf{D}f)^3$ and therefore $\bar{h} \subset \bar{f}$. Hence f foreruns h. But $w \in \mathbf{D}f - F = \mathbf{D}f - \mathbf{D}h$ and $\bar{F} (= \bar{h})$ includes all elements of $\bar{f}$ less than or equal to $\lambda(w)$. Therefore f warily foreruns h. It remains to be shown that h is bad.

Let $s, t \in \mathbf{D}h$ $(= F)$ and let $s \lhd t$. If $s, t \in \mathbf{D}f$, then $h(s) = f(s) \nleq f(t) = h(t)$ since f is bad, and a similar argument shows that $h(s) \nleq h(t)$ if $s, t \in \mathbf{D}g$. If $s \in \mathbf{D}f$ and $t \in \mathbf{D}g$, then, since $s \lhd t$, either $s \lhd t \wedge \mathbf{D}f$ or $t \wedge \mathbf{D}f$ is a strict subsequence of s. The latter alternative is, however, excluded since s, $t \wedge \mathbf{D}f \in \mathbf{D}f$, and hence $s \lhd t \wedge \mathbf{D}f$. Therefore $f(s) \nleq f(t \wedge \mathbf{D}f)$ since f is bad, and therefore, since $g(t) < {}^*f(t \wedge \mathbf{D}f)$, it follows that $f(s) \nleq g(t)$, i.e. $h(s) \nleq h(t)$. The case $s \in \mathbf{D}g$, $t \in \mathbf{D}f$ cannot arise, since, if $s \in \mathbf{D}g$, then $s \in (\mathbf{D}f)^3$ and therefore ${}_*s$ has a left-segment $u \in \mathbf{D}f$, so that, since $s \lhd t$, $u \prec {}_*s \prec \prec t$

and therefore t cannot belong to $\mathbf{D}f$ since its strict subsequence u is in $\mathbf{D}f$. We have thus shown that $h(s) \nleq h(t)$ in all possible cases. Hence h is bad.

Lemma 37. *If $\mathfrak{W}$ is a closed well-branched subset of $\mathfrak{R}_0$ and f is a bad $\mathfrak{W}$-array, there exists a bad $\mathfrak{W}$-array h such that f warily foreruns h.*

Proof. Since (for any $T \in \mathfrak{R}$) the second component of an element of $\Theta(T)$ is a set of strict branches of T, the composition Θf is an $\mathfrak{S}(C \times \mathfrak{S}^{\#}\mathbf{S}\mathfrak{W})$-array with domain $\mathbf{D}f$. Furthermore, Θf is bad since, if $u, v \in \mathbf{D}f$ and $u \lhd v$, then $f(u) \nleq f(v)$ since f is bad and therefore $\Theta f(u) \nleq \Theta f(v)$ by Lemma 29. Therefore, by Lemma 26, there exists a bad $(C \times \mathfrak{S}^{\#}\mathbf{S}\mathfrak{W})$-array ϕ such that $\mathbf{D}\phi = (\mathbf{D}(\Theta f))^2 = (\mathbf{D}f)^2$ and $\phi(s) \in \Theta f(s_f)$ for every $s \in \mathbf{D}\phi$, where for brevity we use notation such as s_f in place of $s \wedge \mathbf{D}f$. By Lemmas 22 and 27, there exists a bad $\mathfrak{S}^{\#}\mathbf{S}\mathfrak{W}$-array g such that $\mathbf{D}g \subset \mathbf{D}\phi$ and $g(s) = c_2\phi(s)$ for every $s \in \mathbf{D}g$. Since $\mathfrak{W}$ is well-branched, there exists by Lemma 28 a bad $\mathbf{S}\mathfrak{W}$-array ψ such that $\mathbf{D}\psi \subset (\mathbf{D}g)^2$ and $\psi(s) \in g(s_g)$ for every $s \in \mathbf{D}\psi$. We observe that

$$\mathbf{D}\psi \subset (\mathbf{D}g)^2 \subset (\mathbf{D}\phi)^2 = ((\mathbf{D}f)^2)^2 = (\mathbf{D}f)^3 \tag{7}$$

by Lemma 17. Let $s \in \mathbf{D}\psi$. Then $s_g = s_\phi = s \wedge (\mathbf{D}f)^2$ since $\mathbf{D}g \subset \mathbf{D}\phi = (\mathbf{D}f)^2$, and therefore $s_g \wedge \mathbf{D}f = s_f$ by Lemma 17. Therefore $\phi(s_g) \in \Theta f(s_f)$. Therefore $c_2\phi(s_g)$ is the second component of an element of $\Theta f(s_f)$ and hence is (by the definition of Θ) a set of strict branches of $f(s_f)$. Hence, since $\psi(s) \in g(s_g) = c_2\phi(s_g)$, it follows that

$$\psi(s) <^{*} f(s_f). \tag{8}$$

Moreover, since $\mathfrak{W}$ is closed and ψ is a bad $\mathbf{S}\mathfrak{W}$-array, ψ is a bad $\mathfrak{W}$-array. In view of this fact and (7) and (8), there exists by Lemma 36 a bad $\mathfrak{W}$-array h such that f warily foreruns h.

Lemma 38. *Every closed well-branched subset of $\mathfrak{R}_0$ is bqo.*

Proof. Let $\mathfrak{W}$ be a closed well-branched subset of $\mathfrak{R}_0$. Let μ be a limit ordinal such that the set of ordinals less than μ has a greater cardinal number than the set of all barriers. Suppose that $\mathfrak{W}$ is not bqo. Then there exists a bad $\mathfrak{W}$-array f_0. We now define a set f_β for every β such that $0 < \beta < \mu$ by transfinite induction on β as follows. Suppose that f_α has been defined for $0 \leqslant \alpha < \beta$. If β is a limit ordinal, write $f_\beta = \liminf_{\alpha \to \beta} f_\alpha$. If β is not a limit ordinal and $f_{\beta-1}$ is a bad $\mathfrak{W}$-array, take f_β to be a bad $\mathfrak{W}$-array such that $f_{\beta-1}$ warily foreruns f_β. (This is possible by Lemma 37.) If β is not a limit ordinal and $f_{\beta-1}$ is not a bad $\mathfrak{W}$-array, write $f_\beta = \varnothing$. Then in fact f_β is a bad $\mathfrak{W}$-array for every $\beta < \mu$. For otherwise there would be a smallest κ such that f_κ is not a bad $\mathfrak{W}$-array; but κ cannot be 0 by the definition of f_0, and κ cannot be a non-limit ordinal greater than 0 since then $f_{\kappa-1}$ would be a bad $\mathfrak{W}$-array and therefore so would f_κ, and κ cannot be a limit ordinal by Lemma 34. But it now follows from the last part of Lemma 34 that the $\mathbf{D}f_\beta$ $(\beta < \mu)$ are distinct barriers, which is impossible by our choice of μ. Hence $\mathfrak{W}$ must be bqo.

Lemma 39. *If s is a strict left-segment of a bad ω-sequence on $\mathfrak{R}_0$, then there exists a $T \in \mathfrak{R}_0$ such that*

(i) *$s[T]$ is a left-segment of a bad ω-sequence on $\mathfrak{R}_0$,*

(ii) *there is no $T' \in \mathbf{S}T$ such that $s[T']$ is a left-segment of a bad ω-sequence on $\mathfrak{R}_0$.*

Proof. Let $\mathfrak{U}$ be the set of those $T \in \mathfrak{R}_0$ which have the property that $s[T]$ is a left-segment of a bad ω-sequence on $\mathfrak{R}_0$. Then $\mathfrak{U} \neq \varnothing$ since s is a strict left-segment of such an ω-sequence. Therefore we can select a $T_1 \in \mathfrak{U}$. If $\mathfrak{U} \cap \mathbf{S}T_1 \neq \varnothing$, select a $T_2 \in \mathfrak{U} \cap \mathbf{S}T_1$. If $\mathfrak{U} \cap \mathbf{S}T_2 \neq \varnothing$, select a $T_3 \in \mathfrak{U} \cap \mathbf{S}T_2$, and so on. If this process did not terminate, we should have constructed an infinite sequence $T_1 >^* T_2 >^* \dots$, which is impossible since $T_1 \in \mathfrak{R}_0$. Therefore there is an i such that $T_i \in \mathfrak{U}$ and $\mathfrak{U} \cap \mathbf{S}T_i = \varnothing$, which amounts to saying that T_i is a T satisfying (i) and (ii).

LEMMA 40. *Let $[T_1, T_2, \dots]$ be a bad ω-sequence on $\mathfrak{R}_0$. If there is no bad ω-sequence on $\mathfrak{R}_0$ which has a left-segment of the form $[T_1, T_2, \dots, T_{j-1}, T]$ where $j \in P$ and $T \in \mathbf{S}T_j$, then $\mathbf{S}T_1 \cup \mathbf{S}T_2 \cup \dots$ is wqo.*

Proof. If $\mathbf{S}T_1 \cup \mathbf{S}T_2 \cup \dots$ is not wqo, there exists a bad ω-sequence $[R_1, R_2, \dots]$ on this set. Let $R_i \in \mathbf{S}T_{f(i)}$, and let k be a positive integer such that $f(k) \leqslant f(l)$ for every l. Since clearly any branch of an element of $\mathfrak{R}_0$ belongs to $\mathfrak{R}_0$,

$$[T_1, T_2, \dots, T_{f(k)-1}, R_k, R_{k+1}, R_{k+2}, \dots]$$

is an ω-sequence on $\mathfrak{R}_0$. Since $R_k \in \mathbf{S}T_{f(k)}$, this ω-sequence must, by the hypothesis of our lemma, be good. Therefore either $T_i \leqslant T_j$ for some i, j such that $i < j < f(k)$ or $R_i \leqslant R_j$ for some i, j such that $k \leqslant i < j$ or $T_i \leqslant R_j$ for some i, j such that $i < f(k)$, $j \geqslant k$. But the first two alternatives contradict the hypotheses that $[T_1, T_2, \dots]$ and $[R_1, R_2, \dots]$ are bad, and the third would imply that $T_i \leqslant R_j <^* T_{f(j)}$, which, since $i < f(k) \leqslant f(j)$, again contradicts the badness of $[T_1, T_2, \dots]$. Hence $\mathbf{S}T_1 \cup \mathbf{S}T_2 \cup \dots$ must be wqo.

LEMMA 41. *$\mathfrak{R}_0$ is wqo.*

Proof. If $\mathfrak{R}_0$ is not wqo, there exists a bad ω-sequence on $\mathfrak{R}_0$, and $\square$ is a left-segment of this sequence. Therefore by Lemma 39 we can select a $T_1 \in \mathfrak{R}_0$ such that

(i) $[T_1]$ is a left-segment of a bad ω-sequence on $\mathfrak{R}_0$,

(ii) there is no $T \in \mathbf{S}T_1$ such that $[T]$ is a left-segment of a bad ω-sequence on $\mathfrak{R}_0$.

By Lemma 39 again, we can select a $T_2 \in \mathfrak{R}_0$ such that

(i) $[T_1, T_2]$ is a left-segment of a bad ω-sequence on $\mathfrak{R}_0$,

(ii) there is no $T \in \mathbf{S}T_2$ such that $[T_1, T]$ is a left-segment of a bad ω-sequence on $\mathfrak{R}_0$.

By Lemma 39 again, we can select a $T_3 \in \mathfrak{R}_0$ such that

(i) $[T_1, T_2, T_3]$ is a left-segment of a bad ω-sequence on $\mathfrak{R}_0$,

(ii) there is no $T \in \mathbf{S}T_3$ such that $[T_1, T_2, T]$ is a left-segment of a bad ω-sequence on $\mathfrak{R}_0$;

and so on. In this way, we can construct an ω-sequence $[T_1, T_2, \dots]$ on $\mathfrak{R}_0$. If $i < j$, then $T_i \nleqslant T_j$ since $[T_1, \dots, T_j]$ is a left-segment of a bad ω-sequence. Therefore $[T_1, T_2, \dots]$ is bad, and hence $\mathbf{S}T_1 \cup \mathbf{S}T_2 \cup \dots$ is wqo by Lemma 40. Since it is easily seen that

$$\mathbf{S}(\mathbf{B}T_1 \cup \mathbf{B}T_2 \cup \dots) = \mathbf{S}T_1 \cup \mathbf{S}T_2 \cup \dots,$$

it follows that $\mathbf{B}T_1 \cup \mathbf{B}T_2 \cup \dots$ is a well-branched (closed) subset of $\mathfrak{R}_0$, and is therefore wqo by Lemmas 38 and 1. This, however, contradicts the fact that $[T_1, T_2, \dots]$, which is an ω-sequence on $\mathbf{B}T_1 \cup \mathbf{B}T_2 \cup \dots$, is bad. Hence our original assumption that $\mathfrak{R}_0$ was not wqo must have been false.

Lemma 42. $\mathfrak{R}$ *is bqo.*

Proof. Since obviously $\mathbf{S}\mathfrak{R}_0 \subset \mathfrak{R}_0$, $\mathfrak{R}_0$ is by Lemma 41 a well-branched subset of itself, and is therefore by Lemma 38 bqo. Therefore any subset of $\mathfrak{R}_0$ is bqo. But, if $T \in \mathfrak{R}$, then $\mathbf{F}T \subset \mathfrak{R}_0$: therefore $\mathbf{F}T$ is bqo and so by Lemma 32 $T \in \mathfrak{R}_0$. Hence $\mathfrak{R} = \mathfrak{R}_0$, which we have just shown to be bqo.

5. *Proof of Theorem* 2. Given any set $\mathfrak{T}$ of trees, we may take the set $\mathscr{Z}$ of section 4 to be $\bigcup\{Z(T): T \in \mathfrak{T}\}$ and define $\mathfrak{R}$ (in relation to $\mathscr{Z}$) as before. Let f be any $\mathfrak{T}$-array. Then we form an $\mathfrak{R}$-array g with the same domain as f by taking $g(s)$ to be (for each $s \in \mathbf{D}f$) a rooted tree obtained from $f(s)$ by selecting an arbitrary vertex as root. Then g is good by Lemma 42. Therefore there exist $s, t \in \mathbf{D}f$ such that $s \lhd t$ and $g(s) \leqslant g(t)$ (where $\leqslant$ has the meaning defined in section 4). From this we easily infer that $f(s) \leqslant f(t)$ in the sense defined in the statement of Theorem 2. In fact, if c is a conveyance of $g(s)$ into $g(t)$, an embedding I of $f(s)$ in $f(t)$ is defined by the rules that

$$I \odot \xi = \{c(\xi)\}$$

for every $\xi \in V(f(s))$ and that, if an edge λ joins the vertices ξ and η in $f(s)$, then $I \odot \lambda$ is the set of mid-terms of the path from $c(\xi)$ to $c(\eta)$ in $f(t)$. Hence the $\mathfrak{T}$-array f is good. This argument proves that $\mathfrak{T}$ is bqo.

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