# **Dynamic Membership for Regular Languages**

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#### Abstract

We study the *dynamic membership problem* for regular languages: fix a language L, read a word w, build in time O(|w|) a data structure indicating if w is in L, and maintain this structure efficiently under substitution edits on w. We consider this problem on the unit cost RAM model with logarithmic world length, where the problem always has a solution in  $O(\log |w| / \log \log |w|)$ .

We show that the problem is in  $O(\log \log |w|)$  for languages in an algebraically-defined class **QSG**, and that it is in O(1) for another class **QLZG**. We show that languages not in **QSG** admit a reduction from the prefix problem for a cyclic group, so that they require  $\Omega(\log n/\log\log n)$  operations in the worst case; and that **QSG** languages not in **QLZG** admit a reduction from the prefix problem for the monoid  $U_1$ , which we conjecture cannot be maintained in O(1). This yields a conditional trichotomy. We also investigate intermediate cases between O(1) and  $O(\log\log n)$ .

Our results are shown via the dynamic word problem for monoids and semigroups, for which we also give a classification. We thus solve open problems of the paper of Skovbjerg Frandsen, Miltersen, and Skyum [25] on the dynamic word problem, and additionally cover regular languages.

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# 1 Introduction

This paper studies how to handle substitution updates on a word while maintaining its membership to a regular language. Specifically, we fix a regular language L and we are given a word w, which we preprocess in linear time – in particular testing if  $w \in L$ . Now, edits are applied to w, and we want to update w and keep track at each step of whether  $w \in L$ . We study this *dynamic membership* task from a theoretical angle, but it is also practically useful for maintaining a Boolean condition (expressed as a regular language) on a user-edited word.

Dynamic membership was studied for various update operations, e.g., append operations for streaming algorithms or the sliding window model [12, 11, 13], substitutions for the dynamic word problem for monoids [25], or concatenations and splits [18]. It was also studied in the case of pattern matching, where we check if the word contains some target pattern [8], which is also assumed to be editable. It is also connected to the incremental validation problem, which has been studied for strings and for XML documents [3].

Our focus in this work is to identify classes of fixed regular languages for which dynamic membership can be solved extremely efficiently, e.g., in constant time or sublogarithmic time. Thus, our update language only allows substitutions to the input word, as insertions and deletions already make it challenging to efficiently maintain the word itself (see Section 7). We work within the computational model of the unit-cost RAM, with logarithmic cell size.

**Dynamic word problem for monoids [25].** Our problem closely relates to the work by Skovbjerg Frandsen, Miltersen, and Skyum on the *dynamic word problem for monoids* [25]:

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fix a finite monoid, read a word which is a sequence of monoid elements, and maintain under substitution updates the image of these elements when composed according to the monoid law. Indeed, the dynamic membership problem for a language L reduces to the dynamic word problem for any monoid that recognizes L; but the converse is not true. Hence, studying the dynamic word problem for monoids is coarser than studying the dynamic membership problem for languages, although it is a natural first step and is already very challenging.

In the context of monoids, Skovbjerg Frandsen et al. [25] propose a general algorithm for the dynamic word problem in  $O(\log n/\log\log n)$ , for n the length of the word. This is a refinement of the elementary  $O(\log n)$  algorithm that decomposes the word as a balanced binary tree whose nodes are annotated with the monoid image of the corresponding infix. They show that this bound is tight for some monoids, namely noncommutative groups, and a generalization of them defined via an equation. This is obtained by a reduction from the so-called prefix- $\mathbb{Z}_d$  problem, for which an  $\Omega(\log n/\log\log n)$  lower bound [10] is known in the cell probe model [14]. We will reuse this lower bound in our work.

They also show that the problem is easier for some monoids. For instance, commutative monoids can be maintained in O(1), simply by maintaining the count of element occurrences. They also show a trickier  $O(\log \log n)$  upper bound for group-free monoids: this is based on a so-called Krohn-Rhodes decomposition [24] and uses a predecessor data structure implemented as a Van Emde Boas tree [30]. However, there are non-commutative monoids for which the problem is in O(1) (as we will show), and there is still a gap between group-free monoids (having an upper bound in  $O(\log \log n)$ ) and the monoids to which the  $O(\log n/\log \log n)$  lower bound applies. This was claimed as open in [25] and not addressed afterwards – there is a more recent study by Pătraşcu and Tarniță [20] but focusing on single-bit memory cells.

Our contributions. In this paper, we attack these problems using algebraic monoid theory. This unlocks new results: first on the dynamic word problem for monoids, where we extend the results of [25], and then on the dynamic membership problem for regular languages.

We start with our results on the dynamic word problem for monoids, which are summarized in Figure 1 in the appendix for common monoid classes. First, in Section 3, we show how a more elaborate  $O(\log\log n)$  algorithm can cover all monoids to which the  $\Omega(\log n/\log\log n)$  lower bound of [25] does not apply: we dub this class  $\mathbf{SG}$  and characterize it by the equation  $x^{\omega+1}yx^{\omega}=x^{\omega}yx^{\omega+1}$ , for any elements x and y, where  $\omega$  denotes the idempotent power. Our algorithm shares some ideas with the  $O(\log\log n)$  algorithm of [25], in particular it uses Van Emde Boas trees, but it faces significant new challenges. For instance, we can no longer use a Krohn-Rhodes decomposition, and proceed instead by a rather technical induction on the  $\mathcal{J}$ -classes of the monoid. Thus, we have an unconditional dichotomy between monoids in  $\mathbf{SG}$  which are in  $O(\log\log n)$ , and monoids outside  $\mathbf{SG}$  which are in  $O(\log\log n/\log\log n)$ .

Second, in Section 4, we generalize the O(1) result on commutative monoids to the monoid class  $\mathbf{ZG}$  [2]. This class is defined via the equation  $x^{\omega+1}y = yx^{\omega+1}$ , i.e., only the elements that are part of a group are required to commute with all other elements. We show that the dynamic word problem for these monoids is in O(1): we intuitively describe them algebraically as the combination of commutative monoids with monoids obtained from nilpotent semigroups, for which we design a simple but somewhat surprising algorithm. We also show a conditional lower bound: for any monoid M not in  $\mathbf{ZG}$ , we can reduce the prefix- $U_1$  problem to the dynamic word problem for M. This is the problem of maintaining a binary word under substitution updates while answering queries asking if a prefix contains a 0. It can be seen as a priority queue (slightly weakened), so we conjecture that no O(1) data structure for this problem exists in the RAM model. Under this conjecture,  $\mathbf{ZG}$  is

exactly the class of monoids having a dynamic word problem in O(1).

We then extend our results in Section 5 from monoids to the dynamic word problem for semigroups. Our results for  $\mathbf{SG}$  extend as-is, as the upper bound on  $\mathbf{SG}$  also applies to semigroups in  $\mathbf{SG}$ , and semigroups not in  $\mathbf{SG}$  must contain a submonoid not in  $\mathbf{SG}$  so covered by the lower bound. For  $\mathbf{ZG}$ , there are major complications, and we must study the class  $\mathbf{LZG}$  of semigroups where all submonoids are in  $\mathbf{ZG}$ . A semigroup not in  $\mathbf{LZG}$  is covered by our conditional lower bound, but it is very tricky to show the converse, i.e., that imposing the condition on  $\mathbf{LZG}$  suffices to ensure tractability. We do so by showing tractability for  $\mathbf{ZG} * \mathbf{D}$ , the semigroups generated by semidirect products of  $\mathbf{ZG}$  semigroups and so-called definite semigroups, and using a locality result establishing that  $\mathbf{ZG}*\mathbf{D} = \mathbf{LZG}$ .

Next, we extend our results in Section 6 from semigroups to languages. This is done using the notion of *stable semigroup* [4, 5], which we can use to lift our results and obtain:

- ▶ **Theorem 1.1.** Let L be a regular language, and consider the dynamic membership problem for L on the unit-cost RAM with logarithmic word length under substitution edits:
- If L is in the class  $\mathbf{QLZG}$ , then the problem is in O(1).
- If L is not in the class QLZG but is in the class QSG, then the dynamic membership problem is in  $O(\log \log n)$  with n the length of the word. Further, solving it in O(1) time gives an O(1) implementation of a prefix-U<sub>1</sub> structure.
- $\blacksquare$  If L is not in the class QSG, then the dynamic membership problem is in  $\Theta(\log n/\log\log n)$ .

We last present in Section 7 some extensions and questions for future work: preliminary observations on the precise complexity of languages in  $\mathbf{QSG} \setminus \mathbf{QLZG}$  (as the  $O(\log \log n)$  bound is not shown to be tight), the complexity of deciding which case of the theorem applies, the support for insertion and deletion updates, and the support for infix queries.

Note that this paper uses some new results about the class  $\mathbf{ZG}$ , namely its characterization via commutative monoids and nilpotent semigroups, and the locality result  $\mathbf{ZG} * \mathbf{D} = \mathbf{LZG}$ . As these results are of possible independent interest, and have a technical proof using unrelated tools (namely finite categories and Straubing's delay theorem [27]), we chose to defer them to a separate ICALP'21 submission [1].

#### 2 Preliminaries and Problem Statement

**Computation model.** We work in the RAM model with unit cost, i.e., each cell can store integers of value at most polynomial in O(|w|) where |w| is the length of the input, and arithmetic operations (addition, successor, modulo, etc.) on two cells takes unit time. As the integers have at most polynomial value, the memory usage is also polynomially bounded.

We consider *dynamic problems* where we are given a data structure, preprocess it in linear time, and must then handle *update operations* on the data structure as well as *query operations* on the current structure. The *complexity* of the problem is the worst-case complexity of handling an update or answering a query.

Like in [25], the lower bounds that we show will actually all hold in the coarser *cell probe* model, which only considers the number of memory cells accessed during a computation. Furthermore, they will all hold even without the assumption that the preprocessing is linear.

Given two dynamic problems A and B, we say that A has a constant-time reduction to B if we can implement a data structure for problem A having constant-time complexity when using as oracle a data structure for problem B (built during the preprocessing). In other words, queries and updates on the structure for A, in addition to constant-time computations on its own memory, can use the data structure for B as an oracle, i.e., perform a constant

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number of queries and updates on it, which are considered to run in O(1). We similarly talk of a dynamic problem having a reduction to multiple problems, meaning we can use all of them as oracle. If problem A reduces to problems  $B_1, \ldots, B_n$ , then any complexity upper bound that holds on all problems  $B_1, \ldots, B_n$  also holds for A, and any complexity lower bound on A extends to at least one of the  $B_i$ .

**Problem statement.** Our problems require some algebraic prerequisites. We refer the reader to the book of Pin [22] and his lecture notes [23] for basic concepts (monoids, semigroups, etc.), and only recall less common notions. All semigroups and monoids considered are finite.

A semigroup element  $s \in S$  is *idempotent* if ss = s. For  $x \in S$ , we denote by  $\omega$  the idempotent power of x, i.e., the smallest positive integer such that  $x^{\omega}$  is idempotent. A zero for S is an element  $0 \in S$  such that 0x = x0 = 0 for all  $x \in S$ : if it exists, it is unique. A variety of monoids (resp., semigroups) is a class of monoids (resp., semigroups) closed under direct product, quotient and submonoid (resp., subsemigroup). The variety of monoids (resp., semigroups) generated by a class V of monoids (resp., of semigroups) is the least variety closed under these operations and containing V.

We consider dynamic problems where we maintain a word w on an alphabet  $\Sigma$ , every letter being stored in a cell. We allow substitution updates (i,a) for  $1 \le i \le |w|$  and  $a \in \Sigma$  setting the i-th letter of w to be a. The size |w| of the word never changes, the preprocessing runs in O(|w|), and we measure the worst-case complexity of an update or query as a function of |w|. We do not limit the memory used, but all our upper complexity bounds will have memory usage in O(|w|), while all our lower bounds hold without this assumption.

We focus on three related dynamic problems, allowing different query operations. The first is the dynamic word problem for monoids: fix a monoid M, the alphabet  $\Sigma$  is M, and the query returns the evaluation of the current word w, i.e., the product of the elements of w (it is an element of M). This is the problem studied in [25]. The second is the dynamic word problem for semigroups, which is the same but with a semigroup, and assuming that |w| > 0. The third is the dynamic membership problem for regular languages: we fix a regular language L on the alphabet  $\Sigma$ , and the query checks if the current word belongs to L or not.

We will study the complexity of these problems in the rest of this paper. Let us first observe that, for monoids and more generally for semigroups, the usual algebraic operators of quotient, subsemigroup, and product, do not increase the complexity of the problem:

▶ Proposition 2.1. Let S and T be finite semigroups. The dynamic word problem of subsemigroups or quotients of S reduce in constant time to the problem for S, and the dynamic word problem of  $S \times T$  reduces to that of S and T.

**Hard problems.** All our lower bounds are by reducing from the problem prefix-M, for M a fixed monoid. In this problem, we maintain a word of  $M^*$  under substitution updates, and handle  $prefix\ queries$ : given a prefix length, return the evaluation of the prefix of that length.

In particular, for  $d \geq 2$ , we will consider the problem  $prefix\mathbb{Z}_d$  for  $\mathbb{Z}_d$  the cyclic group of order d, i.e.,  $\mathbb{Z}_d = \{0, \ldots, d-1\}$  with addition modulo d, where we must compute the sum of the prefix modulo d. The following lower bound is known already in the cell probe model:

▶ **Theorem 2.2** ([10, 25]). For any fixed  $d \ge 2$ , any structure for prefix- $\mathbb{Z}_d$  on a word of length n has complexity at least  $\Omega(\log n/\log\log n)$ .

We also consider the problem  $prefix-U_1$ , where  $U_1 = \{0,1\}$  with composition being the logical AND, i.e., prefix queries check if the prefix contains a 0. Equivalently, we must maintain a subset S of a universe  $\{1,\ldots,n\}$  (intuitively n is the length of the word) under

insertions and deletions, and support threshold queries that ask, given  $0 \le i \le n$ , whether S contains some element which is  $\le i$  (i.e., if some position  $\le i$  has a 0). The prefix- $U_1$  problem can be solved with a priority queue data structure in  $O(\log \log n)$  [29], or even in expected  $O(\sqrt{\log \log n})$  if we allow randomization [15]. Note that prefix- $U_1$  is slightly weaker than a priority queue as we can only *compare* the minimal value to a value given as input. We do not know of lower bounds on prefix- $U_1$ , but conjecture [16] that it cannot be solved in O(1):

▶ Conjecture 2.3. There is no structure for prefix- $U_1$  with complexity O(1).

Note that the best algorithm for prefix- $U_1$  is by sorting small sets of large integers. This takes linear time in the cell probe model, so does not rule out the existence of an O(1) priority queue [29]. Hence, a lower bound for prefix- $U_1$  would need to apply to the RAM model specifically, which would require new techniques.

Our last hard problem is prefix- $U_2$  where  $U_2$  is the monoid  $\{1, a, b\}$  with composition law xy = y for  $x, y \in \{a, b\}$ , i.e., queries check if the last non-neutral element is a or b (or nothing). By adapting known results on the *colored predecessor problem* [21], we have:

▶ **Theorem 2.4** (Adapted from [21]). Any structure for prefix- $U_2$  on a word of length n must be in  $\Omega(\log \log n)$ .

**General algorithms.** Of course, the "hard" prefix problems, and the three problems that we study, can all be solved in O(|w|) by re-reading the whole word at each update. We can improve this to  $O(\log |w|)$  by maintaining a balanced binary tree on the letters of the word, with each node of the tree carrying the evaluation in the monoid of the letters reachable from that node. Any edit to the word can be propagated up to the root in logarithmic time, and the annotation of the root is the evaluation of the word. This algorithm has been implemented in practice [18]. A finer bound is given in [25] using a foklore technique of working with  $(\log n)$ -ary trees rather than binary trees, and using the power of the RAM model. We recall it here for monoids (but it applies to all three problems):

▶ **Theorem 2.5** ([25]). For any monoid M, the dynamic word problem and prefix problem for M are in  $O(\log n/\log \log n)$ .

Our goal in this paper is to solve the dynamic word problem and dynamic membership problem more efficiently for specific classes of monoids, semigroups, and languages. We start our study with monoids in the next two sections, by studying the varieties **SG** and **ZG**.

## 3 Dynamic Word Problem for Monoids in SG

The class  $\mathbf{SG}$  of monoids is defined by the equation  $x^{\omega+1}yx^{\omega}=x^{\omega}yx^{\omega+1}$  for all x,y. It incidentally occurs in [9, Theorem 3.1], but to our knowledge was not otherwise studied. The name  $\mathbf{SG}$  means switching group, as we can "switch" group elements between occurrences of their idempotent. We first recall the lower bound from [25] on the dynamic word problem for monoids not in  $\mathbf{SG}$ , and then show an upper bound for monoids in  $\mathbf{SG}$ .

**Lower bound.** The monoids not in **SG** are in fact those covered by the lower bound of [25]. Namely, we have the following, implying the  $\Omega(\log n/\log\log n)$  lower bound by Theorem 2.2:

▶ Theorem 3.1 ([25], Theorem 2.5.1). For any monoid M not in  $\mathbf{SG}$ , there is some  $d \geq 2$  such that the prefix- $\mathbb{Z}_d$  problem reduces in constant time to the dynamic word problem for M.

**Upper bound.** The rest of this section presents our upper bound on monoids in **SG**. We will show a more general claim on the dynamic word problem for *semigroups* in **SG**, i.e., those satisfying the equation of **SG**: this covers in particular the monoids of **SG**. We show:

▶ **Theorem 3.2.** The dynamic word problem for any semigroup in  $\mathbf{SG}$  is in  $O(\log \log n)$ .

This result extends the result of [25] on group-free monoids, because **SG** captures both aperiodic monoids and commutative monoids. Indeed, in the case where a monoid is aperiodic, then it satisfies the equation  $x^{\omega+1} = x^{\omega}$ , and then  $x^{\omega+1}yx^{\omega} = x^{\omega}yx^{\omega} = x^{\omega}yx^{\omega+1}$ . Besides, commutative monoids clearly satisfy the equation. Of course, **SG** captures monoids not covered by [25], e.g., products of a commutative monoid and an aperiodic monoid.

The result of [25] is proved using van Emde Boas trees [30] (vEBs), which we will also use, and which we will extend to store values in an alphabet  $\Sigma$ . In this paper, fixing an alphabet  $\Sigma$ , a vEB tree is a data structure parametrized by an integer n called its span, which stores a set  $X \subseteq \{1, ..., n\}$  and a mapping  $\mu : X \to \Sigma$ , and supports the following operations: inserting an integer  $x \in \{1, ..., n\}$  with a label  $\mu(x) := a$ , removing an integer x and its label, retrieving the label of  $x \in \{1, ..., n\}$ , finding the next integer of X that follows an input  $x \in \{1, ..., n\}$  (or a special value if none exists), and finding the previous integer.

We can implement vEBs so that these five operations run in  $O(\log \log n)$  time in the worst case, and so that a vEB can be constructed in linear time from a ordered list.

We use vEBs in our inductive proof to represent words with "gaps": a vEB represents the word obtained by concatenating the labels of the elements of X in order. For a semigroup S and span  $n \in \mathbb{N}$ , the dynamic word problem on vEBs for S is to maintain a vEB T of span n on alphabet S under insertions and deletions while maintaining the evaluation in S of the word of T. The complexity is measured as a function of n (which never changes), and the preprocessing of T is in O(n). Note that when T is empty then its evaluation is not expressible in the semigroup S: we then return a special value.

It is then clear that Theorem 3.2 follows from its generalization to vEBs, as a word in the usual sense can be converted in linear time to a vEB where  $X = \{1, ..., n\}$ :

▶ **Theorem 3.3.** Let S be a semigroup in  $\mathbf{SG}$ . The dynamic word problem for S on a vEB of span n is in  $O(\log \log n)$ .

We show this result in the rest of the section. We assume without loss of generality that S has a zero, as otherwise we can simply add one. We first introduce some algebraic preliminaries, and then present the proof, which is an induction on  $\mathcal{J}$ -classes of the semigroup.

**Preliminaries and proof structure.** The  $\mathcal{J}$ -order of S is the preorder  $\leq_{\mathcal{J}}$  on S defined by  $s \leq_{\mathcal{J}} s'$  if  $SsS \subseteq Ss'S$ : its equivalence classes of its reflexive closure are called  $\mathcal{J}$ -classes. We lift the  $\mathcal{J}$ -order to  $\mathcal{J}$ -classes C, C's by writing  $C \leq_{\mathcal{J}} C'$  if  $u \leq_{\mathcal{J}} v$  for all  $u \in C$  and  $u' \in C'$ . A  $\mathcal{J}$ -class is maximal if it is maximal for this order.

We show Theorem 3.3 by induction on the number of  $\mathcal{J}$ -classes of the semigroup. More precisely, at every step, we consider a maximal  $\mathcal{J}$ -class C, and remove it by doing a constant-time reduction to the semigroup  $S \setminus C$  obtained by removing this class. Thus, as there are only constantly many classes, and as the constant number of operations on vEBs at each class each take time  $O(\log \log n)$ , the overall bound is indeed in  $O(\log \log n)$ .

The base case of the induction is that of a semigroup with a single  $\mathcal{J}$ -class; from our assumption that the semigroup has a zero, that  $\mathcal{J}$ -class must then consist of the zero, i.e., we have the trivial monoid  $\{0\}$ , and the image is always 0 (or undefined if the word is empty).

We now show the induction step of Theorem 3.3. Take a semigroup S with more than one  $\mathcal{J}$ -class, and fix a maximal  $\mathcal{J}$ -class C of S: we know that  $S \setminus C$  is not empty. What is more:

▶ Claim 3.4. For any x, y of S with  $xy \in C$ , then  $x \in C$  and  $y \in C$ .

Thus, whenever a combination of elements "falls" outside of the maximal class C, then it remains in  $S \setminus C$ ; and we can see  $S \setminus C$  as a semigroup, which still has a zero, and has strictly less  $\mathcal{J}$ -classes. So we will study how to reduce to  $S \setminus C$ . We now make a case disjunction on whether C is regular, i.e., it contains an idempotent element, or whether it does not.

**Non-regular maximal classes.** This case is easy, as the combination of any two elements in C always "falls" in  $S \setminus C$ . Formally, for a maximal  $\mathcal{J}$ -class C of S, we call a word w on  $S^*$  pair-collapsing for C if the product of any two adjacent letters of w is in  $S \setminus C$ . We show:

▶ **Lemma 3.5.** Let C be a maximal  $\mathcal{J}$ -class. If C is non-regular, then any word is pair-collapsing: for any  $x, y \in C$ , we have  $xy \in S \setminus C$ .

We can then show the following, which we will reuse for regular maximal classes:

▶ Lemma 3.6. Let S be a semigroup and let C be a maximal  $\mathcal{J}$ -class of S. Consider the dynamic word problem for S on vEBs of some span n where we assume that, at every step, the represented word is pair-collapsing for C. Then there is a constant-time reduction from that problem to the dynamic word problem same problem for  $S \setminus C$  on vEBs of span n.

**Proof sketch.** We group adjacent letters of the word into groups of  $\geq 2$  letters (into a new vEB of the same span), so that every group is in  $S \setminus C$ , and we can perform the evaluation with a structure for the dynamic word problem for  $S \setminus C$ . When handling edits, we maintain a constant bound on the size of the groups, without introducing singleton groups.

Thanks to Lemma 3.5, this allows us to settle the case of a non-regular maximal  $\mathcal{J}$ -class, using the induction hypothesis to maintain the problem on  $S \setminus C$ .

Case of a regular maximal class. We now consider a maximal  $\mathcal{J}$ -class C that is regular. Consider the semigroup  $C^0 := C \cup \{0\}$  for a fresh zero 0, i.e., the multiplication is that of C except that x0 = 0x = 0 for all  $x \in C$ . Note that 0 is unrelated to the zero which S was assumed to have; intuitively, the 0 of  $C^0$  stands for combinations of elements that "fall" outside of C. Another way to see  $C^0$  is as the quotient of S by the ideal  $S \setminus C$ , i.e., we identify all elements of  $S \setminus C$  to 0. By Prop. 4.35 of Chapter V of [23], we know that  $C^0$  is a so-called 0-simple semigroup. By the Rees-Sushkevich theorem (Theorem 3.33 of [23]), S is isomorphic to some Rees matrix semigroup with S. This is a semigroup S0 multiple of S1 with S2 and S3 multiple of S4 multiple of S5. The elements of the semigroup are S6 multiple of S7 and S8 multiple of S9 multiple of S1 multiple of S1 multiple of S2 multiple of S3 multiple of S4 multiple of S6 multiple of S6 multiple of S8 multiple of S9 multiple of S1 multiple of S1 multiple of S2 multiple of S3 multiple of S4 multiple of S5 multiple of S6 multiple of S6 multiple of S9 multip

With this representation, the idea is to collapse together the maximal runs of consecutive elements of  $C^0$  whose product is not 0, i.e., does not "fall" outside of C. Once this is done, the product of two elements always falls in  $S \setminus C$ , so we can conclude with Lemma 3.6.

However, we cannot do this in a naive fashion. For instance, if we insert a letter in the vEB at the middle of such a maximal run, we cannot hope to split the run and know the exact group annotation of the two new runs – this could amount to solving a prefix- $\mathbb{Z}_d$  problem. Instead, we must now use the fact that S is in SG, and derive the consequences of the equation in terms of the Rees-Sushkevich representation. Intuitively, the equation ensures that the structuring group G is commutative, and that annotations in G can "move around" relative to other elements in S without changing the equation. Formally:

- $\triangleright$  Claim 3.7. The structuring group G is commutative.
- ▶ Claim 3.8. Let  $r, s, t \in S^*$  and  $(i, g, j), (i', g', j') \in I \times G \times J$ . Write w = r(i, g, j)s(i', g', j')tand w' = r(i, gg', j)s(i', e, j')t where e is the neutral element of G. Then eval(w) = eval(w').

This allows us to reduce from the dynamic word problem on S to the same problem while guaranteeing that the target word is pair-collapsing:

▶ Claim 3.9. There is a constant-time reduction from the incremental word problem for S on vEBs (of some span n) to the same problem on vEBs of span n where we additionally enforce that, at every step, the represented word is pair-collapsing.

**Proof sketch.** We maintain a mapping where all maximal runs of word elements evaluating to C are collapsed to a single element, which we can evaluate following the Rees-Sushkevich representation. The tricky case is whenever an edit breaks up a maximal run into two parts: we cannot recover the G-component of the annotation of each part, but we use Claim 3.8 to argue that we can simply put it on the left part without altering the evaluation in S.

This claim together with Lemma 3.6 implies that the dynamic word problem for S reduces to the same problem for  $S \setminus C$ , for which we use the induction hypothesis. This establishes the induction step and concludes the proof of Theorem 3.2.

## Dynamic Word Problem for Monoids in ZG

We pursue our study of the dynamic word problem for monoids with the class **ZG**, introduced in [2] and defined by the equation  $x^{\omega+1}y = yx^{\omega+1}$  for all x, y. This asserts that elements of the form  $x^{\omega+1}$ , which are the ones belonging to a subgroup of the monoid, are central, i.e., commute with all other elements. By the equations, and recalling that  $x^{\omega+1} = x^{\omega}x^{\omega+1}$ , clearly  $\mathbf{ZG} \subseteq \mathbf{SG}$ . In this section, we will show an upper bound on the dynamic word problem for monoids in **ZG**, and a conditional lower bound for any monoid not in **ZG**.

**Upper bound.** Recall the result on commutative monoids from [25]:

- ▶ Theorem 4.1 ([25]). The dynamic word problem for any commutative monoid is in O(1). Our goal is to generalize it to the following result:
- ▶ **Theorem 4.2.** The dynamic word problem for any monoid in  $\mathbf{ZG}$  is in O(1).

This generalizes Theorem 4.1 (as commutative monoids are clearly in  $\mathbb{ZG}$ ) and covers other monoids, e.g., the monoid  $M = \{1, a, b, ab, 0\}$  with  $a^2 = b^2 = ba = 0$ , where it intuitively suffices to track the position of a's and b's and compare them if there is only one of each.

We now prove Theorem 4.2. A semigroup S is *nilpotent* if it has a zero and it has some k>0 such that  $S^k=\{0\}$ , i.e., all products of k elements are equal to 0. Alternatively [23, Chapter X, Section 4], S is nilpotent iff it satisfies the equation  $x^{\omega}y = yx^{\omega} = x^{\omega}$ . We then define a monoid  $S^1$  by adding an *identity* to S, i.e., a fresh element 1 such that 1x = x1 = xfor all  $x \in S$ . Note that the monoid M given above is of this form. The variety generated by such monoids was studied by Straubing [26]. We can show:

**Proposition 4.3.** For any nilpotent S, the dynamic word problem for  $S^1$  is in O(1).

**Proof sketch.** We maintain a (non-sorted) doubly-linked list L of the positions with a nonneutral element. As S is nilpotent, the evaluation in 0 unless constantly many non-neutral letters remain, which we can then find in O(1) with L.

In [1] we show that **ZG** is generated by such monoids  $S^1$  and by commutative monoids:

▶ **Proposition 4.4** (Corollary 3.6 of [1]). The variety **ZG** is generated by commutative monoids and monoids of the shape  $S^1$  for S a nilpotent semigroup.

Theorem 4.1 and Proposition 4.3 allow us to conclude the proof of Theorem 4.2, thanks to the above result and to closure under the operations of a variety (Proposition 2.1).

**Lower bound.** We now show a conditional lower bound on the dynamic word problem for monoids outside of **ZG**. To do this, we will reduce from the prefix- $U_1$  problem:

▶ **Theorem 4.5.** For any monoid M in  $\mathbf{SG} \setminus \mathbf{ZG}$ , there is a constant-time reduction from the prefix- $U_1$  problem to the dynamic word problem for M.

**Proof sketch.** We show that  $\mathbf{ZG} = \mathbf{SG} \cap \mathbf{ZE}$ , and show for any monoid not in  $\mathbf{ZE}$  that we can encode the prefix- $U_1$  problem using carefully chosen elements.

By Conjecture 2.3, and together with Theorem 3.1 for the monoids not in SG, this implies a conditional super-constant lower bound for monoids outside ZG.

## 5 Dynamic Word Problem for Semigroups

We have classified the complexity of the dynamic word problem for monoids: it is in  $O(\log \log n)$  for monoids in  $\mathbf{SG}$ , in O(1) for monoids in  $\mathbf{ZG}$ , in  $O(\log n)$  for monoids not in  $\mathbf{SG}$ , and non-constant for monoids not in  $\mathbf{ZG}$  conditionally to Conjecture 2.3. In this section, we extend our results from monoids to *semigroups*.

**Sub-monoids and local monoids.** A *sub-monoid* of a semigroup S is a subset of the semigroup which is stable under its composition law and is a monoid. We first notice via Proposition 2.1 that a semigroup that contains a hard sub-monoid is also hard:

▶ Claim 5.1. The dynamic word problem for any sub-monoid of a semigroup S reduces in constant time to the same problem for S.

We will investigate if studying the sub-monoids of a semigroup suffices to understand the complexity of its dynamic word problem. To do so, we focus on a certain kind of sub-monoids: the local monoids. A sub-monoid N of S is a local monoid if there exists an idempotent element e of S such that N=eSe, i.e., N is the set of elements that can be written as ese for some  $s \in S$ . The point of local monoids is that every sub-monoid T of S is a sub-monoid of a local monoid: indeed, taking e the neutral element of T, all elements of T can be written as  $eTe \subseteq eSe$  and eSe is a local monoid. For V a variety of monoids, we denote by V the class of semigroups such that all local monoids are in V. As we explained, this is equivalent to imposing that all submonoids are in V (using the fact that V is closed by sub-monoid).

**Case of SG.** We now revisit our results on monoids to extend them to semigroups, starting with **SG**. We show that a semigroup where all local monoids are in **SG** must itself be in **SG**:

▶ Claim 5.2. We have LSG = SG as varieties of semigroups.

Semigroups in **SG** are already covered by our upper bound (Theorem 3.2), and semigroups not in **LSG** have a sub-monoid not in **SG**, so we can use Claim 5.1 and Theorem 3.1. Hence:

▶ Corollary 5.3. Let S be a semigroup. If S is in SG, then the dynamic word problem for S is in  $O(\log \log(n))$ . Otherwise, the dynamic word problem for S is in  $O(\log \log n)$ .

**Case of ZG.** The variety **ZG** is not equal to **LZG**. For instance, let S be the syntactic semigroup of  $a^*b^*$ , that is the semigroup  $\{a, b, ab, 0\}$  defined with  $a^2 = a$ ,  $b^2 = b$  and ba = 0. It is not in **ZG**, since a and b are idempotents that do not commute. However, its local monoids are either trivial or  $U_1$ , so they are all in **ZG**, showing that this semigroup is in **LZG**. Still, we can extend our characterization from monoids to semigroups up to studying **LZG**:

▶ **Theorem 5.4.** Let S be a semigroup. If S is in **LZG**, then the dynamic word problem for S is in  $O(\log \log(n))$ . Otherwise, unless prefix- $U_1$  is in O(1), the dynamic word problem for S is not in O(1).

The second part of the claim is by Claim 5.1 and Theorem 3.1, but the first part is much trickier. It uses an alternative definition of **LZG** via the *semidirect product* operator [27], a very technical *locality result* asserting that this is indeed equal to **LZG**, and an algorithm for the alternative definition. We prove Theorem 5.4 in the rest of this section.

Given two semigroups S and T, a semigroup action of S on T is defined by a map act :  $S \times T \to T$  such that  $\operatorname{act}(s_1,\operatorname{act}(s_2,t)) = \operatorname{act}(s_1s_2,t)$  and  $\operatorname{act}(s,t_1t_2) = \operatorname{act}(s,t_1)\operatorname{act}(s,t_2)$ . We then define the product  $\circ_{\operatorname{act}}$  on the set  $T \times S$  as follows: for all  $s_1, s_2$  in S and  $t_1, t_2$  in T, we have:  $(t_1,s_1)\circ_{\operatorname{act}}(t_2,s_2) := (t_1\operatorname{act}(s_1,t_2),s_1s_2)$ . The set  $T \times S$  equipped with the product  $\circ_{\operatorname{act}}$  is a semigroup called the semidirect product of S by T, denoted  $T \circ_{\operatorname{act}} S$ .

We say that a semigroup is definite if there exists an integer  $k \in \mathbb{N}$  such that for all  $y, x_1, \ldots, x_k$  in D, we have  $yx_1 \cdots x_k = x_1 \cdots x_k$ . Alternatively, a semigroup is definite iff it satisfies the equation  $yx^{\omega} = x^{\omega}$  [27, Proposition 2.2] for all x, y in T. In particular, every nilpotent semigroup is definite. We write  $\mathbf{D}$  for the variety of definite semigroups.

Our alternative definition of  $\mathbf{LZG}$  will be the variety of semigroups  $\mathbf{ZG} * \mathbf{D}$  generated by semigroups that are the semidirect product of a  $\mathbf{ZG}$  monoid by a definite semigroup.

The variety  $\mathbf{ZG} * \mathbf{D}$  of semigroups is not immediately related to the variety  $\mathbf{LZG}$  defined above. One can easily show that  $\mathbf{ZG} * \mathbf{D} \subseteq \mathbf{LZG}$ , but the other direction is much more challenging to establish. We show this as a so-called *locality theorem*, which we defer to a separate paper [1] because it uses different tools and is of possible independent interest:

▶ Theorem 5.5 ([1], Theorem 1.1). We have:  $\mathbf{ZG} * \mathbf{D} = \mathbf{LZG}$ .

To conclude the proof of Theorem 5.4, by the locality theorem above, it suffices to solve the dynamic word problem for semigroups in  $\mathbb{Z}G * \mathbb{D}$ . By Proposition 2.1, it suffices to consider the semigroups that generate the variety. We do this below, establishing Theorem 5.4:

▶ Proposition 5.6. Let S be a definite semigroup, let T be a semigroup of  $\mathbf{ZG}$ , and let act be an action of S on T. There is a constant-time reduction from the dynamic word problem for the semigroup  $T \circ_{\operatorname{act}} S$  to the same problem for T.

**Proof sketch.** We express the direct product of the word elements as a product involving elements of T and prefix sums of elements of S, which we can maintain in O(1).

#### 6 Dynamic Word Problem for Languages

We now turn to our results on the dynamic membership problem for regular languages, and show Theorem 1.1 using the three previous sections and some extra algebraic results.

Connection to the dynamic word problem. A regular language L is recognized by a finite monoid if there exists a morphism  $\eta: \Sigma^* \to M$  such that  $L = \eta^{-1}(\eta(L))$ . The syntactic

monoid of L is the least monoid M that recognizes L, and the morphism mapping  $\Sigma^*$  to M is called the *syntactic morphism*.

The dynamic membership problem for a language clearly reduces to the dynamic word problem for its syntactic morphism. Yet, the converse is not true: the language  $L := (aa)^*ba^*$  has a syntactic monoid that can be shown to be outside of  $\mathbf{SG}$ , but we can solve dynamic membership in O(1) by counting the b's at even and odd positions. Intuitively, the syntactic monoid has a neutral element e so that its dynamic word problem has a reduction from prefix- $\mathbb{Z}_2$ , but e is not achieved by a letter of the alphabet so dynamic membership is easier.

We extend our results to languages using the *stable semigroup* [4, 5]. This allows us to remove the neutral element (as it is a semigroup not a monoid) and ensures that all semigroup elements can be achieved by subwords of some constant length (the *stability index*).

Formally, let L be a regular language and  $\eta: \Sigma^* \to M$  its syntactic morphism. The powerset of M is the monoid whose elements are subsets of M and for  $E, F \subseteq M$ , the product EF is  $\{xy \mid x \in E, y \in F\}$ . The stability index of L is the idempotent power s of  $\eta(\Sigma)$  in the powerset monoid. Intuitively, the choice of s ensures that, for any two words  $w_1, w_2 \in \Sigma^s$ , the effect  $\eta(w_1w_2)$  of their concatenation in the monoid can be achieved by another word of  $\Sigma^s$ , i.e.,  $\eta(w_1w_2) = \eta(w)$  for some  $w \in \Sigma^s$ . The stable semigroup of L is then  $S := \eta(\Sigma^s)$ , and it is a sub-semigroup of M, since  $(\eta(\Sigma^s))^2 = \eta(\Sigma^s)$ . For any class of semigroups V, we denote by  $\mathbf{Q}\mathbf{V}$  the class of languages whose stable semigroup is in  $\mathbf{V}$ .

**Upper bounds.** We first show that the dynamic membership problem for a regular language reduces more specifically to the dynamic word problem for its *stable semigroup*:

▶ Proposition 6.1. Let L be a regular language. The dynamic membership problem for L reduces in constant time to the dynamic word problem for the stable semigroup of L.

**Proof sketch.** We partition the word of L into chunks of size s (plus one of size s) for s the stability index, and feed them to the data structure for the stable semigroup of L.

Thanks to Corollary 5.3 and Theorem 5.4, this implies that languages in **QSG** (resp., in **QLZG**) have a dynamic membership problem in  $O(\log \log n)$  (resp., in O(1)).

**Lower bound.** We now show that languages whose stable semigroup is not in  $\mathbf{LV}$  admit a reduction from the dynamic word problem of a monoid of  $\mathbf{V}$ .

▶ Proposition 6.2. Let V be a variety of monoids and let L be a regular language not in  $\mathbf{QLV}$ . There is a monoid not in V whose dynamic word problem reduces in constant-time to the dynamic membership problem for L.

**Proof sketch.** If L is not in  $\mathbf{QLV}$ , then its stable semigroup contains a submonoid M not in  $\mathbf{V}$ , and all elements can be achieved by a block of s letters for s the stability index.

Again by Corollary 5.3 and Theorem 5.4, we deduce that languages outside of **QSG** have complexity at least  $\Omega(\log n/\log\log n)$ , and languages outside of **QLZG** do not have complexity O(1) assuming Conjecture 2.3.

#### 7 Extensions, Problem Variants, and Future Work

**Intermediate cases.** Our  $O(\log \log n)$  upper bound in Theorem 3.2 and its variants may not be tight. Still, we can identify a language  $L_{U_2}$  in  $\mathbf{QSG} \setminus \mathbf{QLZG}$  for which we show

that the dynamic membership problem is in  $\Theta(\log \log n)$  (even allowing randomization and allowing a probably correct answer), because the prefix- $U_2$  problem reduces to it.

We can also identify a language of  $\mathbf{QSG} \setminus \mathbf{QLZG}$  that reduces to prefix- $U_1$  and so can be solved in expected  $O(\sqrt{\log \log n})$ . This shows that languages in  $\mathbf{QSG} \setminus \mathbf{QLZG}$  have different complexity regimes, at least when allowing randomization.

▶ Proposition 7.1. There is a language  $L_{U_2}$  in QSG\QLZG which is equivalent to prefix- $U_2$  under constant-time reductions, and a language  $L_{U_1}$  in QSG\QLZG which is equivalent to prefix- $U_1$  under constant-time reductions.

**Deciding which case applies.** A natural question about our results is the question of efficiently identifying, given a regular language, which of the case of Theorem 1.1 applies, or in particular of determining, given an input monoid, if it is **SG**, or in **ZG**. This depends on how the input is represented. If we are given a monoid explicitly (as a table of its operations), then the equations of **ZG** and of **SG** can be checked in polynomial time. If the monoid is represented more concisely as a transition monoid of some automaton, then the verification can be performed in PSPACE. We do not know if the problems are PSPACE-hard, though this seems likely at least for **SG** because of its proximity to aperiodic monoids [6]. We leave open the precise complexities of this task, in particular for the **L** and **Q** operators.

Handling insertions and deletions. Another natural question is to handle insertion and deletion updates, i.e., inserting letter a at position k transforms the word  $w_1 \cdots w_{k-1} w_k \cdots w_n$  into  $w_1 \cdots w_{k-1} a w_k \cdots w_n$ , and deleting at position k does the opposite. Any regular language can be maintained under such updates in  $O(\log n)$  using a Fenwick tree, but it makes the problem much harder for most languages. For example, if the alphabet has two letters a and b, just testing if the word that we maintain contains an a requires  $\Omega(\log n/\log\log n)$  by [16]. This is why we do not study such updates in this work. Interestingly, notice that our algorithm in Theorem 3.3 supports insertions and deletions on words represented as vEBs, but the semantics are different (they use explicit positions in a fixed range).

Infix queries. A natural extension of dynamic membership for a regular language L is the dynamic infix membership problem, where we can query any infix of the word (identified by its endpoints) and know if it is in L. The  $O(\log n/\log\log n)$  algorithm of Theorem 2.5 supports this, and so can the  $O(\log\log n)$  algorithm of [25] for group-free monoids. However, the infix problem can be harder. Consider for instance the language  $L_2$  on  $\{a,b\}$  of words with an even number of a's. Dynamic membership has complexity O(1) because  $L_2$  is commutative, but infix queries (or even prefix queries) require  $\Omega(\log n/\log\log n)$  as prefix- $\mathbb{Z}_2$  reduces to it.

We leave open the study of the complexity of the infix problem. We note, however, that this problem for a language L can be studied as the dynamic membership problem for a regular language defined from L. So our results cover the infix problem via this transformation; we leave to future work a characterization of the resulting classes.

▶ Claim 7.2. For any fixed regular language L, the dynamic infix membership problem is equivalent up to constant-time reductions to the dynamic membership problem for the language  $\Sigma^* x L x \Sigma^*$  where x is a fresh letter.

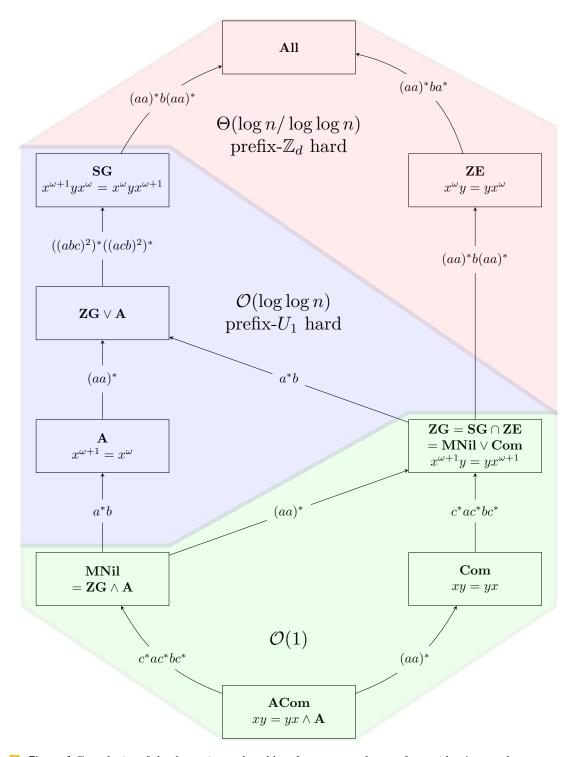
Other open questions. A natural question for future work would be to study the complexity of our problems in weaker models, e.g., pointer machines [28], or machines with counters. One could also extend our study to languages that are not regular: one result in this direction is a lower bound on maintaining the language of well-parenthesized strings [17, Proposition 1].

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**Figure 1** Complexity of the dynamic word problem for common classes of monoids. Arrows denote inclusion and are labeled with languages (with an implicit neutral letter e) whose syntactic monoids separate the classes. The classes  $\mathbf{ZG}$  and  $\mathbf{SG}$  are maximal for the O(1) region and  $O(\log\log n)$  region respectively.

# A Proofs for Section 2 (Preliminaries and Problem Statement)

#### A.1 Proof of Proposition 2.1

▶ Proposition 2.1. Let S and T be finite semigroups. The dynamic word problem of subsemigroups or quotients of S reduce in constant time to the problem for S, and the dynamic word problem of  $S \times T$  reduces to that of S and T.

**Proof.** In the case of submonoids, we can simply solve the dynamic word problem for the submonoid by performing the computations in S. In the case of quotient, we can represent each element of the quotient by some choice of representative element in S, perform the computation in S, and check in which equivalence class in the quotient does the result in S fall. For the product, we can simply maintain the structures for both S and T simultaneously.

#### A.2 Proof of Theorem 2.4

▶ **Theorem 2.4** (Adapted from [21]). Any structure for prefix- $U_2$  on a word of length n must be in  $\Omega(\log \log n)$ .

To prove this result, we first introduce prerequisites on the colored predecessor problem. We then explain how it relates to the prefix- $U_2$  problem. We then derive a lower bound for prefix- $U_2$ .

**Colored predecessor problem.** The colored predecessor problem is a problem parameterized by n. In this problem, we have a set  $S \subseteq \{1, ..., n\}$  and a color (black or white) for each element in S and we need to answer queries asking for the color  $c_y$  of the biggest element  $y \in S$  such that  $y \leq x$  for x a parameter of the query.

In the static version of the problem, the set S is given in advance and we are allowed to compute an index before receiving the query. The time complexity is then measured only in terms of the time required to answer the query.

In [21], it has been proved that, in the worst case, the static colored predecessor problem for S a subset of  $\{1, ..., n^2\}$  with |S| = n cannot be solved in less than  $\Omega(\log \log n)$  query time if the space allowed is bounded by  $n^{1+o(1)}$ . And this lower bound holds in the cell probe model, and still holds even if we allow randomization and the answer to be only probably correct.

Reduction from the static colored predecessor problem to prefix- $U_2$ . Now let us suppose that we have a maintenance scheme for prefix- $U_2$  with complexity d(n). Let  $S \subseteq \{1, ..., n^2\}$  be a instance of the static colored predecessor problem and let us show that we can build a data structure for S that takes  $O(n \times d(n^2))$  space and can answer predecessor queries with  $O(d(n^2))$  cell-probes. If we manage to do that then either  $d(n^2) = n^{o(1)}$  and thus we have  $d(n^2) = \Omega(\log \log n)$  using the static lower bound or  $d(n^2) \neq n^{o(1)}$ . In both cases we have  $d(n^2) = \Omega(\log \log n)$  and thus  $d(n) = \Omega(\log \log n)$ .

**Lower bound for prefix-** $U_2$ . Since  $S \subseteq \{1, ..., n^2\}$ , we create a word w of length  $n^2$  where all elements are the neutral element of  $U_2$ . Then we perform n substitution updates on w so that the i-th letter of w is a a when  $i \in S$  and i has color White, b when  $i \in S$  and i has color Black, and the neutral element of  $U_2$  when  $i \notin S$ . The index for prefix- $U_2$  on w now enables us to answer predecessor queries on S: if the prefix of length k gives out the color

of the predecessor query with parameter k (White when it is a, Black when it is b, and the neutral element when k has no predecessor in S).

Since queries on this index take  $d(n^2)$  time, we do have a scheme for the static predecessor problem in time  $d(n^2)$ , but, before concluding, we need to make sure that the space usage is bounded by  $O(n^{1+o(1)})$ . This is not obvious for two reasons: first because the initialization of the dynamic algorithm might modify  $O(n^2)$  memory cells during the initialization (or even more if we allow a superlinear preprocessing time) and second because during each update the dynamic algorithm might access a memory cell i with  $i \neq n^{1+o(1)}$  (e.g. it might use the memory cell  $n^2$ ). We now explain how to handle these two issues.

Reducing the memory footprint. To handle the first issue, we can notice that the whole preprocessing computation depends only on n and therefore if a cell has not been modified since the initialization of the dynamic algorithm then we can recover its value from n. For the second issue, we can notice that each update takes  $d(n^2)$  time, and therefore the total number of memory cells that have been modified during one of the updates is bounded by  $d(n^2) \times n$ . Therefore using a perfect hashing scheme [19] using  $O(d(n^2) \times n)$  memory cells, we can retrieve in O(1) for each address i whether i was modified during an update and if it has been modified, what is its current value.

All in all, we can modify the query function for our dynamic problem in the following way: whenever the dynamic algorithm wants to read the cell at address i we use the perfect hashing scheme to determine if i has been modified since the initialization, if it has been modified we retrieve its value and otherwise we recompute the value that can be deduced from n. Such an algorithm makes  $O(d(n^2))$  cell probes over a memory of size  $O(n \times d(n^2))$ . This is what we wanted to obtain, concluding the proof.

## A.3 Proof of Theorem 2.5

▶ **Theorem 2.5** ([25]). For any monoid M, the dynamic word problem and prefix problem for M are in  $O(\log n/\log \log n)$ .

We first show the claim for the dynamic word problem. Let  $(M, \odot)$  be a monoid and  $w = w_1 \dots w_n$  be a word. Let us show that we can maintain the value  $\nu(w) = w_1 \odot \cdots \odot w_n$  after substitution updates on w in  $O(\log n/\log \log n)$ . M will be considered fixed here and all complexities will thus only depend on n, the length of the word w to be maintained.

We denote by  $G_M$  the (infinite) directed graph whose nodes are the elements of  $M^*$  and where a node  $(m_1, \ldots, m_\ell)$  has  $\ell \times |M|$  outgoing edges each labeled with a different element of  $\{1 \cdots \ell\} \times M$ . For a node  $(m_1, \ldots, m_\ell)$ , the edge labeled (i, v) goes to  $(m_1, \ldots, m_{i-1}, v, m_{i+1}, \ldots, m_\ell)$ . Finally the value of the node  $(m_1, \ldots, m_\ell)$  is defined as  $m_1 \odot \cdots \odot m_\ell$ .

We denote by  $G_M^k$  the restriction of  $G_M$  to nodes  $(m_1, \ldots, m_\ell)$  where  $\ell \leq k$ .  $G_M^k$  has at most  $|M|^k \times k$  nodes and the degree of each node is bounded by  $|M| \times k$  so there exists  $\gamma$  (independent of n) such that for  $k = \lfloor \gamma \times \log(n) \rfloor$ , the graph  $G_M^k$  can be computed in  $O(n^{1/2})$  and stored with  $O(n^{1/2})$  space (in fact, for any  $\epsilon > 0$ , we could achieve  $n^{\epsilon}$  by changing  $\gamma$ ). We can also ensure that the value of each node is pre-computed (and takes constant time to access) and that retrieving the (i, v)-labeled neighbor of a given node takes constant time.

▶ Claim A.1. Once  $G_M^k$  is computed, we have a scheme  $S_k(n)$  with  $O(\log_k(n))$  update time that allows to maintain the value of  $\nu(w)$  after substitution update for w, a word composed of n letters.

**Proof.** This idea is based on a modification of the Fenwick tree data structure with a branching factor of k. The proof works by induction. Each scheme  $S_k(n)$  will maintain a

node in  $G_M^k$  such that the product of elements of the word is equal to the value of this node (which takes O(1) to retrieve).

- For  $0 \le n \le k$ , we will store the node  $(c_1, \ldots, c_n)$  where  $c_1 \ldots c_n$  is the word to maintain. We deal with substitutions using the transitions of the graph: to update the element at position j to the element v we simply replace the current node with its (j, v)-neighbor.
- For n > k, we note R the biggest power of k that is strictly less than n and we split the word into  $w_0, \ldots, w_\ell$  such that  $|w_0| = \cdots = |w_\ell| = R$  (note that  $\ell \le k$ ). At each step the current node will correspond to  $(\nu(w_0), \ldots, \nu(w_\ell))$ . For each of the words  $w_0, \ldots, w_\ell$  we use a scheme  $S_k(R)$  to maintain the value  $\nu(w_i)$ .

To update w at position j, we update the subword  $w_i$  with  $i = \left\lfloor \frac{j}{R} \right\rfloor$  at position  $j - i \times R$  which gives us the new value of  $\nu(w_i)$  then we update the current node by replacing it with its  $(\nu(w_i), i)$ -neighbor.

Notice that initialization time is linear, and each update will make  $\log_k(n)$  recursive calls each in constant time. To finish, let us recall that, for  $k = \gamma \times \log(n)$  then we have the computation of  $G_M^k$  that is  $O(n^{1/2})$  and each query that is in time  $O(\log_k(n)) = O\left(\frac{\log(n)}{\log(\log(n))}\right) = O\left(\frac{\log(n)}{\log(\log(n))}\right)$ . This concludes the proof of Theorem 2.5.

In terms of memory usage, the data structure uses O(n/k) that is  $O(n/\log n)$ . Note that this is sublinear thanks to the power of the RAM model. This is because nodes at the lowest level compress a factor of size k of the original word. In particular, this memory usage is no more than linear in the size of the original word.

We now turn to the prefix problem, and show that the structure can be used more generally to support prefix queries (see Section 7). By lowering the  $\gamma$  parameter, we can precompute for each node  $(m_1, \ldots, m_\ell)$  of  $G_M^k$  and for each  $1 \leq i \leq j \leq \ell$  the value  $m_i \odot \cdots \odot m_j$ . Then, in a scheme  $S_k(n)$  to compute the element of the monoid corresponding to the factor (i, j), either positions i and j belong to the same subword in which case we recurse on this subword or the factor (i, j) can be decomposed into a strict suffix of some subword  $w_{i'}$  followed by a possibly empty contiguous list of subwords followed by a strict prefix of a subword  $w_{j'}$ . If they are not empty, we recurse on  $w_{i'}$  and  $w_{j'}$  to get the value of the strict suffix and prefix parts and using our precomputation we get the value for the list of subwords. By composing the obtained values, we get the effect of the factor (i, j).

Note that a subword query on a scheme might trigger two recursive calls but this can only happen if the subword query is not a prefix nor a suffix query and the two recursive calls it makes are a prefix query and a suffix query. Therefore the overall algorithm does have the expected  $O\left(\frac{\log(n)}{\log(\log(n))}\right)$  complexity.

# B Proofs for Section 3 (Dynamic Word Problem for Monoids in SG)

#### **B.1** Details on Van Emde Boas Trees

In this appendix, we give some details about the vEB data structure. It supports the following operations:

- **insert**(x, a) that inserts the integer  $x \in \{1, ..., n\}$  in X (with  $x \notin X$ ) and sets  $\mu(x) := a$ ;
- **delete**(x) for  $x \in X$  that removes the integer x from X (and from the domain of  $\mu$ );
- **retrieve**(x) that returns the value  $\mu(x)$  if x is in X and a special value otherwise.

- **=** findPrev(y) for  $y \in \{1,...,n\}$  that returns the biggest  $x \in X$  such that  $x \leq y$ , and returns a special value if no such element exists, i.e., y is smaller than all elements in X;
- **findNext**(y) that works like findPrev but returns the smallest  $x \in X$  such that  $x \geq y$ .

The original definition [30] of vEB creates a tree with  $O(n \times \log \log n)$  cells and  $O(n \times \log \log n)$  initialization time. We present here an adaptation of the vEB tree that reduces the memory usage and the initialization to O(n), to guarantee that our preprocessing is linear.

**Linear initialization of vEB.** For this, we will reduce the predecessor problem over the range  $\{1,...,n\}$  to the predecessor problem over the range  $\{1,...,K(n)\}$  with  $K(x) = \lceil x/\log\log n \rceil$ . Note that the division here can be performed in O(1), for instance by filling sequentially a table for its results over  $\{1,...,n\}$ , as the arguments will always be in that range.

The idea is to create an array T of size n that will store the mapping  $\mu$  and a vEB V storing keys over the range  $\{1, ..., K(n)\}$ .

If we want to initialize our modified vEB with an empty set, we start with V empty and T filled with a special value  $\bot$  indicating that the domain of the associative array is empty. We then perform the vEB operations on this structure in the following way:

- for an operation  $\mathbf{insert}(x, a)$  we set T[x] := a and we insert into V the integer K(x) (if it is not already present);
- for an operation retrieve(x) we return T[x];
- for an operation  $\mathtt{delete}(x)$  we set  $T[x] := \bot$  and if there is no  $y \in K^{-1}(K(x))$  such that  $T[y] \neq \bot$  we call  $\mathtt{delete}(K(x))$  on V;
- for findPrev(x), we start by looking among the  $y \in K^{-1}(K(x))$  with  $y \le x$  and  $T[y] \ne \bot$ . If no such y exists, we set p to the result of findPrev(K(x) 1) on V. If  $p = \bot$ , it means that x has no predecessor and if p is set we find the predecessor of x among the elements of  $K^{-1}(p)$ ;
- **a** successor query can be done in a similar fashion.

To initialize the modified vEB with a set X in O(n) storing the function  $\mu$ , we do something similar to performing |X| insertions except that we first do all the modifications in T without inserting in V and afterwards we do all the insertions in V, making sure that each key is inserted once. The total initialization time is thus O(n) to modify in T followed by at most K(n) insertions in V each taking  $O(\log \log n)$ . Thus, the initialization time is O(n) overall.

Note that, by construction, for all x we have that  $|K^{-1}(x)| = \lceil \log \log n \rceil$  and thus all operations on our new vEB run in  $O(\log \log n)$ .

#### B.2 Proof of Theorem 3.2

▶ **Theorem 3.3.** Let S be a semigroup in  $\mathbf{SG}$ . The dynamic word problem for S on a vEB of  $span\ n$  is in  $O(\log \log n)$ .

We start by a preliminary remark pointing out that  $\mathbf{SG}$  is not equal to  $\mathbf{Com} \vee \mathbf{A}$ , as is illustrated in Figure 1:

▶ Remark B.1. Remark that **SG** contains both **Com** and **A**, the variety of aperiodic monoids. From [25], we know that both those varieties have a dynamic word problem in  $O(\log \log n)$ . Hence, their *join* **Com**  $\vee$  **A**, i.e., the variety that they generate (which is not illustrated in Figure 1), also has a word problem in  $O(\log \log n)$  since the complexity of this problem is preserved by the operations of a variety (Proposition 2.1).

However, **SG** is in fact not equal to this variety. First remark that both **Com** and **A** monoids satisfies the equation  $(xyz)^{\omega+1}(xzy)^{\omega} = (xyz)^{\omega}(xzy)^{\omega+1}$ . Therefore, the variety they generate also satisfies this equation. However, the language  $((abc)^2)^*((acb)^2)^*$  does not satisfy it by construction. Still, it is in **SG** and thus has a complexity in  $\log \log(n)$  by our results.

This remark will be refined in the next appendix section (see Remark C.1) to show that  $\mathbf{SG}$  is also larger than  $\mathbf{Com} \vee \mathbf{ZG}$ , for the class  $\mathbf{ZG}$  defined in Section 4.

In the rest of this appendix, we successively prove the results needed for the proof of Theorem 3.2.

▶ Claim 3.4. For any x, y of S with  $xy \in C$ , then  $x \in C$  and  $y \in C$ .

**Proof.** We first show the first claim. Note that we have  $SxyS \subseteq SxS$ : this is because any element of SxyS, say sxyt for  $s,t \in S$ , can be written as sx(ty) so it is in SxS also. Thus, we have  $xy \le x$  But as C is a maximal  $\mathcal{J}$ -class, we must have  $x \le xy$ . So x and xy are in the same  $\mathcal{J}$ -class and  $x \in C$ . The same reasoning shows that  $y \in C$ .

We now show the second claim. Let  $x \in S \setminus C$ , and  $y \in S$ . Let us show that  $xy \in S \setminus C$ . But we know that if  $xy \in C$ , then  $x \in C$  by the previous claim, a contradiction. The same reasoning shows that  $yx \in S \setminus C$ .

▶ **Lemma 3.5.** Let C be a maximal  $\mathcal{J}$ -class. If C is non-regular, then any word is pair-collapsing: for any  $x, y \in C$ , we have  $xy \in S \setminus C$ .

**Proof.** We proceed by contraposition and show that if there exist  $x, y \in C$  such that  $xy \in C$ , then C is regular.

Assume that we have  $x, y \in C$  such that  $xy \in C$ . We then know that x and xy are in the same  $\mathcal{J}$ -class, so SxS = SxyS. Thus, as  $x \in SxS$ , there exists  $s, t \in S$  such that sxyt = x. By reinjecting the left-hand side in itself, this equation implies that  $s^ix(yt)^i = x$  for all  $i \in \mathbb{N}$ . Let  $\omega$  be the idempotent power of S. We have  $s^\omega x(yt)^\omega = x$ . As  $x \in C$ , Claim 3.4 implies that  $s^\omega \in C$ . Hence, C contains an idempotent element, so it is regular.

▶ Lemma 3.6. Let S be a semigroup and let C be a maximal  $\mathcal{J}$ -class of S. Consider the dynamic word problem for S on vEBs of some span n where we assume that, at every step, the represented word is pair-collapsing for C. Then there is a constant-time reduction from that problem to the dynamic word problem same problem for  $S \setminus C$  on vEBs of span n.

**Proof.** We will always refer to w to mean the word on S (represented as a vEB with some span), and w' the word on  $S \setminus C$  (represented as a vEB with the same span). Recall that the case where w is empty is special (as the image may not be representable in the semigroup S). The case where w contains only a single element is also special, as the result may not be in  $S \setminus C$ . Up to maintaining a count of the number of letters of w, we can handle this special case easily: when the number of letters of w becomes equal to 1, we locate the remaining letter using the vEB, and this gives us immediately the evaluation of w. So we assume in the sequel that w contains at least 2 elements.

We will maintain a function  $\psi$ , called a *position mapping*, from the positions of w to those of w', with the following requirements:

- $\psi$  is surjective: every position of w' is reached by  $\psi$
- $\psi$  is nondecreasing
- $\psi$  is bounded injective: there are at most 3 positions of w mapping to any position of w'
- $\psi$  is pair-grouping: there are at least 2 positions of w mapping to any position of w'

- $\psi$  preserves evaluations: for every position (i, a) of w', the letter a in S' is exactly the composition of the letters  $a_1, \ldots, a_n$  of the pairs  $(i_1, a_1), \ldots, (i_n, a_n)$  of the vEB w that are mapped to (i, a) by  $\psi$ : because  $\psi$  is nondecreasing, this is a segment of successive letters in w (and  $2 \le n \le 3$ )
- For any letter (i, a) in the image of  $\psi$ , i is the index of the last letter of w that is mapped to i by  $\psi$

Thanks to the fourth condition, as the word is pair-collapsing, we know that the fifth condition indeed yields an evaluation in  $S \setminus C$ . Also note that the position mapping (specifically conditions 1, 2 and 5) guarantees that w and w' indeed evaluate to the same monoid element, ensuring the correctness of the reduction: the answer to the dynamic word problem for S on w is the same as for S' on w'. So it only remains to explain how we can construct  $\psi$  and w' from w (preprocessing), and how we can maintain  $\psi$  and w' under updates to w.

For the preprocessing, to initialize  $\psi$ , we traverse w sequentially (this can be done in linear time if we simply assume that vEBs on span  $\{1, \ldots, n\}$  always store their contents in an array maintained in parallel to the structure), and construct w' and  $\psi$  sequentially, by grouping the letters of w in groups of 2 or possibly 3 for the last group (to avoid leaving one letter alone). This can clearly be done in time O(n) with n the span of w.

To maintain  $\psi$  under insertions, we find a predecessor or successor of the new letter in w using the vEB, and we set the image of the new letter by  $\psi$  to be that of that predecessor or successor, and update the image letter in w' to reflect this, in constant time because  $\psi$  is bounded injective. The only problem is that now the new  $\psi$  may not be bounded injective anymore, because we could have 4 letters of w (including the new one) mapping to a position of w'. If this happens, we split the group by inserting a letter in w' for the two first letters of w that mapped there (put at the position of the second letter to satisfy condition 6) and updating the letter of w' at the position of the fourth letter (by condition 6) to be that of the two last letters. All of this can be performed with constantly many updates on w' and constantly many operations on the vEB w.

To maintain  $\psi$  under deletions, we look at the image of the deleted letter by  $\psi$ , we find the other letters which are in the same group (i.e., constantly many neighbors, which we can find using the vEB w), we delete all letters of the group along with the group and modify  $\psi$ (preserving the invariant), and then we insert back the letters of the group that we did not intend to delete, as explained in the previous paragraph. This concludes the proof.

### ightharpoonup Claim 3.7. The structuring group G is commutative.

**Proof.** As C is regular, let (i, g, j) be an idempotent of C. We have  $(i, g, j) = (i, g, j)^2$ , which implies that  $p_{j,i} \neq 0$ , and in fact that the idempotent is of the form  $(i, p_{j,i}^{-1}, j)$ .

Let us write  $H_{i,j} = \{(i,g,j) \mid g \in G\}$ : this subset is closed under the semigroup operation because  $p_{j,i} \neq 0$ , so it is a subsemigroup, and in fact it is a group, with neutral element  $(i,p_{j,i}^{-1},j)$ , and with  $(i,p_{j,i}^{-1}g^{-1}p_{j,i}^{-1},j)$  the inverse of (i,g,j). Indeed, for any (i,g,j) we have  $(i,g,j)(i,p_{j,i}^{-1},j) = (i,p_{j,i}^{-1},j)(i,g,j) = (i,g,j)$  so indeed  $(i,p_{j,i}^{-1},j)$  is neutral, and we have  $(i,g,j)(i,p_{j,i}^{-1}g^{-1}p_{j,i}^{-1},j) = (i,p_{j,i}^{-1}g^{-1}p_{j,i}^{-1},j)(i,g,j) = (i,p_{j,i}^{-1},j)$  so indeed the inverses are as described.

Let us show that the group  $H_{i,j}$  is commutative. To see why, take any  $g, g' \in G$ , and take x = (i, g, j) and y = (i, g', j). Let us show that xy = yx. Applying the equation to S, we know that  $x^{\omega+1}yx^{\omega} = x^{\omega}yx^{\omega+1}$ , where  $\omega$  is the idempotent power of x. Now,  $x^{\omega} = x^{2\omega}$  by definition, and in the group  $H_{i,j}$  this implies that  $x^{\omega} = e$ , with e the neutral element

of the group, by composing the equation with  $(x^{\omega})^{-1}$ . Thus, injecting this in the equation yields xy = yx, as claimed. Thus,  $H_{i,j}$  is commutative.

We now show this implies that G is a commutative group. Let us first note that, for any  $g \in G$ , as (i,g,j)(i,e,j) = (i,e,j)(i,g,j) because  $H_{i,j}$  is commutative, we have  $gp_{j,i} = p_{j,i}g$ . Now, take any  $g,g' \in G$ , and let us show that gg' = g'g. We have  $(i,g,j)(i,g',j) = (i,gp_{j,i}g',j) = (i,p_{j,i}gg',j)$  by what we just argued. As  $H_{i,j}$  is commutative, the latter is also equal to (i,g',j)(i,g,j), which is equal to  $(i,p_{j,i}g'g,j)$ . Identifying the two and composing by  $p_{j,i}^{-1}$ , which is possible because we argued that  $p_{j,i} \neq 0$ , we obtain gg' = g'g. So indeed G is commutative.

▶ Claim 3.8. Let  $r, s, t \in S^*$  and  $(i, g, j), (i', g', j') \in I \times G \times J$ . Write w = r(i, g, j)s(i', g', j')t and w' = r(i, gg', j)s(i', e, j')t where e is the neutral element of G. Then eval(w) = eval(w').

**Proof.** As D is regular, let us consider an idempotent in D. By the same reasoning as the first paragraph of the proof of Claim 3.7, we know that the idempotent is of the form  $(a, p_{b,a}^{-1}, b)$  with  $p_{b,a}$  being nonzero. What is more, like in the proof of Claim 3.7, we know that  $H_{a,b} = \{(a, g, b) \mid g \in G\}$  is a group.

In all equations that follow, we abuse notation and write equalities between words of S to mean that they evaluate to the same element (not that the words are the same).

Let us now observe that:

$$(i', g', j') = (i', e, b)(a, p_{b,a}^{-1}g', b)(a, p_{b,a}^{-1}, b)(a, p_{b,a}^{-1}, j').$$

Indeed, this is immediate by the law of the Rees semigroup with zero.

Let us now take  $x = (a, p_{b,a}^{-1} g', b)$ , and take  $\omega$  its idempotent power. We have  $x^{\omega} = x^{2\omega}$ , and as this operation happens in the group  $H_{a,b}$  we know that  $x^{\omega}$  is the neutral element of this group, which is  $(a, p_{b,a}^{-1}, b)$ .

So we can rewrite the above equation as:

$$(i', g', j') = (i', e, b)xx^{\omega}(a, p_{h,q}^{-1}, j').$$

Similarly to the first equation, we have

$$(i,g,j) = (i,p_{b,a}^{-1},b)(a,g,b)x^{\omega}(a,p_{b,a}^{-1},j).$$

So let us take:

$$y = (a, p_{b,a}^{-1}, j)s(i', e, b)$$

The equation defining **SG**, applied to x and y, now tells us that  $x^{\omega+1}yx^{\omega}=x^{\omega}yx^{\omega+1}$ . So let us now write w, i.e.:

$$w = r(i, p_{b,a}^{-1}, b)(a, g, b) x^{\omega}(a, p_{b,a}^{-1}, j) s(i', e, b) x x^{\omega}(a, p_{b,a}^{-1}, j') t$$

We recognise the definition of y, so we have:

$$w = r(i, p_{b,a}^{-1}, b)(a, g, b)x^{\omega}yx^{\omega+1}(a, p_{b,a}^{-1}, j')t$$

Using the equation, we have:

$$w = r(i, p_{b,a}^{-1}, b)(a, g, b) x^{\omega + 1} y x^{\omega}(a, p_{b,a}^{-1}, j') t$$

Injecting back the definition of y:

$$w = r(i, p_{b,a}^{-1}, b)(a, g, b)x^{\omega+1}(a, p_{b,a}^{-1}, j)s(i', e, b)x^{\omega}(a, p_{b,a}^{-1}, j')t$$

The part between r and s is:

$$(i,p_{b,a}^{-1},b)(a,g,b)(a,p_{b,a}^{-1}g',b)(a,p_{b,a}^{-1},b)(a,p_{b,a}^{-1},j) \\$$

which evaluates by the Reese law to (i, gg', j); and the part between s and t evaluates according to the same to (i', e, j'), so we have obtained:

$$w = r(i, qq', j)s(i', e, j')t$$

Which establishes that eval(w) = eval(w') and concludes the proof.

▶ Claim 3.9. There is a constant-time reduction from the incremental word problem for S on vEBs (of some span n) to the same problem on vEBs of span n where we additionally enforce that, at every step, the represented word is pair-collapsing.

**Proof.** A maximal run of a word of  $S^*$  is a non-empty maximal contiguous subsequence of elements of the word whose image is in C. We will maintain the target word w' as a vEB along with a function  $\psi$  from the positions of w to that of w', with the following requirements:

- $\psi$  is surjective: every position of w' is reached by  $\psi$ ;
- $\psi$  is nondecreasing;
- For any letter of w not part of a maximal run, i.e., an element of  $S \setminus C$ , then the image of this letter by  $\psi$  has the same letter and it has has exactly this element as preimage;
- For any maximal run of w, all its letters have the same image by  $\psi$ , the maximal run is precisely the preimage of that element, and the label of this maximal run is of the form (i,g,j) for some g, where (i,g',j) for some g' is the actual evaluation of this maximal run in w;
- For any letter (i, a) in the image of  $\psi$ , i is the index of the last letter of w that is mapped to i by  $\psi$

We additionally require that the evaluation of w' and of w in S is the same. It is crucial that this condition is enforced globally, not locally, as we intend to use Claim 3.8.

For the preprocessing, given the word w, we process it sequentially (again assuming that vEBs also store their contents as an array), and we create the mapping sequentially, using the Rees-Sushkevich representation to determine when a maximal run ends. This takes time O(n).

To handle updates on w, there are several cases:

- When we insert an element of  $S \setminus C$ , we check the target vEB to know if this insertion happens within a maximal run or not:
  - If the insert is not within a maximal run, then we simply reflect the same change in w' and  $\psi$ .
  - When we insert an element x of  $S \setminus C$  within a maximal run, then the maximal run is broken. We use the vEB w to find the preceding element  $(i_-, g_-, j_-)$  and  $(i_+, g_+, j_+)$  of w relative to the insertion: they are necessarily in C because we are within a maximal run.

We now replace the element (i, g, j) of w' which was the image of the maximal run by  $\psi$  by three elements: one of the form  $(i, g_1, j_-)$  corresponding to the remaining prefix of the maximal run, one corresponding to the inserted element, and one of the form  $(i_+, g_2, j)$  corresponding to the remaining suffix of the maximal run. We set  $g_1 := gp_{j_-, i_+}^{-1}$  and  $g_2 := e$  for e the neutral element of G.

We must argue that the evaluation of w' and of w is still the same. To do this, write the new w' as  $r(i, g_1, j_-)x(i_+, g_2, j)t$ . Consider now  $(i, g'_1, j_-)$  and  $(i_+, g'_2, j)$ ,

the respective actual evaluations of the prefix and suffix after the update. We know thanks to the invariant that w and w' had the same evaluation before the update, i.e., w evaluates to the same in S as r(i,g,j)t, and after the update w evaluates to the same as  $r(i,g_1',j_-)x(i_+,g_2',j)t$ , and by Rees-Sushkevich we have  $g=g_1'p_{j_-,i_+}g_2'$ . Instead, our definition of w' evaluates to  $r(i,g_1,j_-)x(i_+,e,j)t$ . The values  $g_1'$  and  $g_2'$  are intuitively the ones that we cannot retrieve from our data structure. However, thanks to Claim 3.8, as  $g_1=gp_{j_-,i_+}^{-1}=g_1'g_2'$  (using the commutativity of G, Claim 3.7), we know that w' as we defined it evaluates to the correct value. Thus, the invariant is preserved.

- When we delete an element of  $S \setminus C$ :
  - If the deletion does not connect together two maximal runs (i.e., it is preceded and succeeded in w' by elements that are not both in C, or that are both in C but whose composition in C yields 0), then we simply reflect the change in w';
  - When we delete an element of  $S \setminus C$  and connect together two maximal runs, then we update w' to delete the element and then delete the two elements (i, g, j) and (i', g', j') corresponding to the two runs by an element  $(i, gp_{j,i'}g', j')$  corresponding to the new run.
- When we insert an element (i, g, j) of C, we check w' to distinguish many possible cases:
  - If this happens within a maximal run (as ascertained using the vEB structure on w'), then let  $(i_-, g_-, j_-)$  and  $(i_+, g_+, j_+)$  be respectively the preceding and succeeding elements in w, obtained from the vEB of w: they are both in C. We update the G-annotation of the image by  $\psi$  of this maximal run to add  $p_{j_-,i_+}^{-1}$ , i.e., the inverse of the Rees matrix element obtained between the preceding and succeeding element: call this step (\*). Now:
    - \* If both  $p_{j_-,i}$  and  $p_{j,i_+}$  are nonzero then we simply update the G-annotation again to add  $p_{j_-,i}$  and  $p_{j,i_+}$ .
    - \* If both  $p_{j_-,i}$  and  $p_{j,i_+}$  are zero, then we break the maximal run in three parts: the part before the insertion, the insertion which is a maximal run of its own, and the part after the insertion. We replace the element (i',g',j') in w' (with g' already modified by step (\*)) that corresponded to the whole run by three elements  $(i',g',j_-)$ , (i,g,j), and  $(i_+,e,j')$ . The preservation of the global invariant is again by Claim 3.8.
    - \* The two cases where exactly one of  $p_{j_-,i}$  and  $p_{j,i_+}$  is zero and the other is nonzero are analogous to the above.
  - If this does not happen within a maximal run:
    - \* If the preceding element is in  $S \setminus C$  or is an element of C which combined with (i, g, j) gives a zero according to the Rees matrix, and the same is true of the succeeding element, then (i, g, j) is a new maximal run of its own. We insert it as-is in w'
    - \* If the preceding element is in  $S \setminus C$  or an element of C which combined with (i, g, j) gives a zero, but the next element  $(i_+, g_+, j_+)$  is in C and combining (i, g, j) with it does not give a zero (i.e.,  $p_{j,i_+}$  is nonzero), then we are extending the maximal run that follows. We modify its G-annotation to add  $p_{j,i_+}g$ , we remove the element in w' corresponding to that consecutive run and add it back to the new end position of the extended maximal run, but changed so that its I-index becomes I.
    - \* The case of an insertion extending the maximal run that precedes is analogous except that we can simply update the element in w' without having to move it, and change its J-index and not I-index.

- \* If both the preceding element  $(i_-, g_-, j_-)$  and succeeding elements  $(i_+, g_+, j_+)$  are in C and neither gives a zero together with the newly inserted element, then by our assumption that we are not within a maximal run, it must be that  $p_{j_-,i_+} = 0$  and we are merging together the two maximal runs of these elements. We reflect this in the additional structure, in the G-annotation (adding the two Rees matrix terms  $p_{j_-,i}$  and  $p_{j,i_+}$  which are nonzero by assumption, in addition to g), removing in g0 the element for the first run, and copying its g1-index and group information to the element for the original run (now the element for the second run)
- When we remove an element (i, g, j) of C, we distinguish between several cases:
  - Removal within a maximal run which does not change the run endpoints, i.e., the preceding and succeeding elements do not combine to a zero in the Rees matrix semigroup. Then we simply update the G-annotation to add the inverse of the two Rees matrix entries that are no longer realized and add  $g^{-1}$ , and add the new nonzero Rees matrix entry which is realized.
  - Removal within a maximal run which breaks up the run into two nonempty runs at the deletion point. We insert the information for these two runs, putting the group information of the split run on the first run (along with  $g^{-1}$  and the inverse p-term for the two elements that are no longer adjacent) and again use Claim 3.8 to argue for correctness.
  - Removal of the first letter of a maximal run. We update the G-annotation with the inverse of the Rees matrix entry which is no longer realized, and update the I-index of the end of the run.
  - Removal of the last letter of a maximal run. This is like the previous case except we change the J-index instead of the I-index, and we must move the element in w' corresponding to the run to sit at the new ending position of the run.
  - Removal that eliminates a singleton maximal run. We simply delete it from w'.

# C Proofs for Section 4 (Dynamic Word Problem for Monoids in ZG)

We start by another preliminary remark on the relationship between  $\mathbf{SG}$ ,  $\mathbf{ZG}$ , and  $\mathbf{A}$  (see Figure 1):

▶ Remark C.1. Following Remark B.1, we could be tempted to claim that  $\mathbf{SG}$  is equal to the variety generated by  $\mathbf{ZG}$  and  $\mathbf{A}$ . However, as monoids in  $\mathbf{ZG}$  also satisfy the equation  $(xyz)^{\omega+1}(xzy)^{\omega} = (xyz)^{\omega}(xzy)^{\omega+1}$ , the same argument shows that it is not the case.

We then provide the omitted proofs for the results in the main text.

#### C.1 Proof of Upper Bound Results

▶ **Theorem 4.1** ([25]). The dynamic word problem for any commutative monoid is in O(1).

**Proof.** We give a proof sketch for completeness. We easily maintain the vector of the number of occurrences of each letter. The problem boils down to testing its membership to a subset  $\mathbb{N}^{|\Sigma|}$  defined following the monoid. It is known by [7, Proposition 9] that such *recognizable subsets* are defined by a finite number of congruence and threshold conditions, so this can be checked in O(1).

▶ **Proposition 4.3.** For any nilpotent S, the dynamic word problem for  $S^1$  is in O(1).

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**Proof.** Let k > 0 be the constant integer such that  $S^k = 0$ . Given a word  $w \in S^*$ , we prepare in linear time from w a doubly-linked list L containing the positions of w having an element which is not the identity of  $S^1$ , along with a table T of size |w| where the i-th cell contains a pointer to the list element in L representing the i-th element if some exists, and a dummy value otherwise. We can construct L in linear time.

We will maintain the invariant that L contains all positions of w containing a non-neutral element (note that we do not assume that L is in sorted order), and that T contains one pointer per element of L leading to the cell corresponding to that element in L.

We can easily use L to determine the image of the current word in  $S^1$ . We first check if L contains  $\geq k$  elements, which can be done in time O(k), hence O(1), by navigating the list. If this is the case, then as  $S^k = 0$ , we know that the word evaluates to 0. Otherwise, we know the < k non-neutral elements of w, and we can evaluate their product in O(1) to know the answer.

We now explain how to maintain L in constant time per edit. When a substitution replaces 1 by 1, or a non-neutral element by a non-neutral element, then we do nothing. When a substitution replaces a neutral element by a non-neutral element at position i, then we add i to L, and let T[i] be a pointer to the new list item, in time O(1). When a substitution replaces a non-neutral element by a neutral element at position i, we use T[i] to find the element for i in L, remove it in time O(1), and erase T[i], all in O(1). This concludes the proof.

#### C.2 Proof of Lower Bound Results: Theorem 4.5

We prove Theorem 4.5 in the rest of this section. To do this, let us introduce **ZE** as the variety of monoids whose idempotents are central, i.e., the variety defined by the equation  $x^{\omega}y = yx^{\omega}$ . Note that **ZG**  $\subseteq$  **ZE**, as is clear from the equations. We show a useful claim:

▶ Claim C.2. We have:  $\mathbf{ZG} = \mathbf{SG} \cap \mathbf{ZE}$ .

**Proof.** Let M be in **ZG**. We first show that M is in **ZE**. Consider arbitrary elements x and y. We have  $x^{\omega} = (x^{\omega})^{\omega+1}$  by definition of  $x^{\omega}$ . Thus,  $x^{\omega}y = (x^{\omega})^{\omega+1}y = y(x^{\omega})^{\omega+1} = yx^{\omega}$ . Thus, M is in **ZG**. Furthermore, as **ZG**  $\subseteq$  **SG**, clearly M is in **SG**.

Let M be in  $\mathbf{SG} \cap \mathbf{ZE}$ . Then

$$x^{\omega+1}y = x^{\omega+1}x^{\omega}y$$
 (By definition of  $x^{\omega}$ )  

$$= x^{\omega+1}yx^{\omega}$$
 (By  $M \in \mathbf{ZE}$ )  

$$= x^{\omega}yx^{\omega+1}$$
 (By  $M \in \mathbf{SG}$ )  

$$= yx^{\omega}x^{\omega+1}$$
 (By  $M \in \mathbf{ZE}$ )  

$$= yx^{\omega+1}$$

This concludes the proof of the claim.

We can now conclude the proof of Theorem 4.5:

**Proof.** Let M be a monoid in  $SG \setminus ZG$ . By Claim C.2, we know that M is not in ZE.

Let us now assume that  $M \notin \mathbf{ZE}$ . By definition, there exist  $x, y \in M$ , such that  $x^{\omega}y \neq yx^{\omega}$ . Notice that we cannot have both  $x^{\omega}y = x^{\omega}yx^{\omega}$  and  $yx^{\omega} = x^{\omega}yx^{\omega}$ , so one of these equations must be false. Without loss of generality, we can assume that  $yx^{\omega} \neq x^{\omega}yx^{\omega}$ . Indeed, if we have  $x^{\omega}y \neq x^{\omega}yx^{\omega}$  instead, then we can show instead the lower bound for the reversal  $M^t$  defined by  $x \cdot_{M^t} y = y \cdot x$  with  $\cdot_{M^t}$  and  $\cdot_{M}$  the internal laws of  $M^t$  and

M respectively. Now, we can reduce from the dynamic word problem for  $M^t$  to the same problem for M by an obvious constant-time reduction where we reverse the input word and perform the updates at the mirror position. Thus, it suffices to consider the case where  $yx^{\omega} \neq x^{\omega}yx^{\omega}$ 

We now show that that any solution to the dynamic membership problem of M can be used to solve the prefix- $U_1$  problem. To do so, we consider a word w on  $\{0,1\}$  of length n, and encode it as a word w' of length 2n + 2. All letters of w' the neutral element e of M except that  $w_{2n+2} = x^{\omega}$  and  $w_{2i} = x^{\omega}$  whenever  $w_i = 0$ . This can be done in linear time during the preprocessing. Now, any edit that writes 0 or 1 in  $w_i$  is done by writing respectively  $x^{\omega}$  or e to  $w'_{2i}$ .

Now, to perform at prefix- $U_1$  query with argument j, we write  $w_{2j+1} := y$ , use the incremental word problem data structure to get the evaluation of the word, then write back  $w_{2j+1} := e$ . The evaluation result, after removing neutral elements, is  $x^{k\omega}yx^{k'\omega}$  where k is the number of 0's in the prefix of length j in w, and k' is the number of 0's in the rest of w, which is  $\geq 1$ . Because  $x^{\omega} = x^{2\omega}$  this is equivalent to  $x^{\omega}yx^{\omega}$  when the prefix contained a 0, or to  $yx^{\omega}$  when it did not. We have shown that these two elements are different, so we can indeed recover the answer to the prefix query, concluding the proof.

## D Proofs for Section 5 (Dynamic Word Problem for Semigroups)

▶ Claim 5.1. The dynamic word problem for any sub-monoid of a semigroup S reduces in constant time to the same problem for S.

**Proof.** This is immediate by Proposition 2.1 (and also intuitively): we simply solve the problem with a structure for S but where we only use elements of the sub-monoid.

▶ Claim 5.2. We have LSG = SG as varieties of semigroups.

**Proof.** Clearly  $\mathbf{SG} \subseteq \mathbf{LSG}$ . Let S be a semigroup of  $\mathbf{LSG}$ . For all elements x,y of S, letting  $e := x^{\omega}$ , we have that the local monoid N = eSe is in  $\mathbf{SG}$ . Now, since x' = exe and y' = eye are in N, we have  $(x')^{\omega+1}y'(x')^{\omega} = (x')^{\omega}y'x^{\omega+1}$ . Furthermore,  $(x')^{\omega+1} = x^{\omega+1}$ . Thus  $x^{\omega+1}yx^{\omega} = x^{\omega} = yx^{\omega+1}$ .

▶ Proposition 5.6. Let S be a definite semigroup, let T be a semigroup of  $\mathbf{ZG}$ , and let act be an action of S on T. There is a constant-time reduction from the dynamic word problem for the semigroup  $T \circ_{\operatorname{act}} S$  to the same problem for T.

**Proof.** Given a word  $w = (t_1, s_1), \dots, (t_n, s_n)$  of  $(T \circ_{\text{act}} S)^*$ , we note that the second component of its evaluation is  $s_1 \cdots s_n$ . As S is definite, we can compute this and maintain it in constant time, simply by looking at the k last elements, for k the integer witnessing that S is definite.

As for the first component, it can be shown to evaluate to the following word of  $T^*$ :

$$t_1 \cdot \operatorname{act}(s_1, t_2) \cdot \operatorname{act}(s_1 s_2, t_3) \cdot \cdot \cdot \operatorname{act}(s_1 s_2 \cdot \cdot \cdot s_n, t_n)$$

We initialize a structure for the dynamic word problem on T with this word w' to obtain the first component. Now, when the word w is updated at position i, we perform the updates on this word w' by changing  $t_i$  in cell i (a single edit), and by changing the k cumulative sums of s that have changed, i.e., the products in the first component of act in the up to k cells starting at cell i: this amounts to k edits, and for each of them the new cumulative sum can be computed by looking at the k last elements, so this is a constant time computation and a constant number of operations.

# **E** Proofs for Section 6 (Dynamic Word Problem for Languages)

▶ Proposition 6.1. Let L be a regular language. The dynamic membership problem for L reduces in constant time to the dynamic word problem for the stable semigroup of L.

**Proof.** Let L be a regular language with S its stable semigroup. We reduce the dynamic membership problem for L to the dynamic word problem for S as follows. For any word  $u \in \Sigma^*$ , we decompose u into  $u = u_1 u_2 \cdots u_n v$  with each  $u_i$  of length s and v having length s, where s denotes the stability index. The image by s of this decomposition gives a word of s. We use a maintenance scheme for the word of s, and propagate the updates on s to updates of this word in s dividing the position of the update by a constant, we look a constant number of neighboring positions to find the entire s in the decomposition of s and evaluate s to compute the new image. Note that the case of updates to s is immediate as s has constant size. We can then use the structure for the dynamic word problem on s to obtain the image of s in the stable semigroup, which we can compute together with s.

We now prove Proposition 6.2 in the rest of the appendix. We first establish a general claim formalizing the connection between the syntactic monoid and the dynamic word problem:

▶ Proposition E.1. Let L be a regular language and  $\eta$  its syntactic morphism. The dynamic membership problem for L is equivalent under constant-time reductions to the dynamic word problem for its syntactic monoid where we require that we only use elements of  $\eta(\Sigma)$ .

**Proof.** Clearly, a maintenance scheme for the dynamic word problem provides a maintenance scheme for L as it is sufficient to check that the resulting element belongs to  $\eta(L)$ , and we will indeed only use elements of  $\eta(\Sigma)$ .

In the other direction, we simply use that for all m, the language  $L_m := \eta^{-1}(m)$  is in the Boolean algebra closed under the quotient operator generated by L. In other words, any language  $L_m$  can be expressed using L, Boolean operations, and the quotient operator; none of these operations changes the alphabet. Thus, we can reduce the dynamic problem to the problem of checking if the image is in  $L_m$  for every possible choice of  $m \in M$ . Now, each of the  $L_m$  reduces to L by Proposition 2.1 without changing the alphabet. So we can solve the dynamic word problem for the syntactic monoid under the assumption that we gave, by building data structures for these  $L_m$ : this uses the assumption that we are only using letters from  $\eta(\Sigma)$ .

We can now prove Proposition 6.2:

**Proof of Proposition 6.2.** Assume L is not in  $\mathbf{QLV}$ . Let  $\eta$  be its syntactic morphism, s be its stability index and S its stable semigroup. Then, by definition of L, there exists a submonoid N of S which is not in  $\mathbf{V}$ . By definition of the stable semigroup, there exists a mapping  $\psi$  from N to  $A^s$  such that for all  $n \in N$ , we have  $\eta(\psi(n)) = n$ , for s the stability index. We simulate the dynamic word problem for N by inserting for each update at position i, the corresponding word at position  $s \times i$ . We can perform the evaluation in N by evaluating the image by  $\eta$  of the resulting word. We know that evaluating the image by  $\eta$  reduce in constant time to L by Proposition E.1, where we use the fact that the resulting word (formed of the blocks of size s) only consists of letters from the original alphabet.

# F Proofs for Section 7 (Extensions, Problem Variants, and Future Work)

▶ Proposition 7.1. There is a language  $L_{U_2}$  in QSG\QLZG which is equivalent to prefix- $U_2$  under constant-time reductions, and a language  $L_{U_1}$  in QSG\QLZG which is equivalent to prefix- $U_1$  under constant-time reductions.

**Proof.** We show each claim of the statement separately.

Claim on  $L_{U_2}$ . Let  $L_{U_2}$  be the regular language over the alphabet  $\Sigma = \{a, b, c, x\}$  that contains all words where there is only one x and the closest preceding non-c letter exists and is a b, i.e.,  $L_{U_2}$  can be defined through the regular expression  $(a + b + c)^*bc^*x(a + b + c)^*$ . This language is in  $\mathbf{SG}$  and hence can be maintained in  $O(\log \log n)$ .

We first show a constant-time reduction from the prefix- $U_2$  problem to the dynamic membership problem for  $L_{U_2}$ , which implies that there is an  $\Omega(\log \log n)$  lower bound on that problem.

Let w be the word over the alphabet 1, a, b to maintain for prefix- $U_2$ . The idea of the reduction consists in encoding  $w = w_1 \cdots w_n$  into  $h(w) = h(w_1) \cdots h(w_n)$  where h(1) = c, h(a) = a and h(b) = b. Clearly  $h(w) \notin L_{U_2}$  as there is no x in the word.

Now, to handle a prefix- $U_2$  query with parameter k, we look at the k-th letter of w. If it is a or b we can answer immediately with the value of  $w_k$ . In the remaining case of  $w_k = 1$  we set  $w'_k$  to x. If we now have  $w' \in L_{U_2}$  it means that the answer of the prefix query is b, otherwise it means that the answer is either a or 1. To distinguish the two cases we first look at  $w_1$ , if  $w_1 \neq 1$  then the answer is a otherwise we set  $w'_1$  to b, if w' belongs to  $L_{U_2}$  then the answer is 1 otherwise it is a. In all these cases after answering the query we restore w' to its previous state.

We then show a constant-time reduction from the dynamic membership problem for  $L_{U_2}$  to the prefix- $U_2$  problem. We simply encode a word of  $\{a,b,c,x\}$  by writing a as a, b as b, c as 1, and x as 1. We also maintain a doubly-linked list as in the proof of Proposition 4.3 to store all occurrences of a. Now, whenever the number of x's is different from 1 then the word does not belong to the language. Otherwise, we use L to find the position of the one x. The word is then in the language iff the closest preceding non-c letter exists and is a b, which we can determine with the prefix- $U_2$  data structure by testing for the corresponding prefix and checking if the answer is b.

Claim on  $L_{U_1}$ . Let  $L_{U_1}$  be the regular language over the alphabet  $\Sigma = \{a, b, c\}$  defined by the expression  $c^*x(a+c)^*$ , i.e., there is exactly one x and it precedes all a's.

We first show a constant-time reduction from the dynamic membership problem for  $L_{U_1}$  to the prefix- $U_1$  problem. Indeed, we translate an input word w on  $\{a, c, x\}$  to a word on  $\{0, 1\}$  by writing 0 for a and 1 for a and x, and maintain this under updates. We also maintain a doubly-linked list as in the proof of Proposition 4.3 to store all occurrences of x. Now, whenever the number of x's is different from 1 then the word does not belong to the language. When the number becomes equal to 1, the list L allows us to know its position, and a prefix query on the prefix- $U_1$  structure allows us to know if there is an a before this x (in which case the word is not in the language) or if there is none (in which case it is).

We then show a constant-time reduction from the prefix- $U_1$  problem to the dynamic membership problem for  $L_{U_1}$ . Indeed, we simply encode 1 as c and 0 as a. When a prefix query arrives for a prefix of length i, we check if the i-th letter is a 0, in which case we

return 0, and otherwise we edit the word on  $\{a, b, x\}$  to insert an x at that position. Now, the resulting word belongs to the language iff there is no preceding 0.

▶ Claim 7.2. For any fixed regular language L, the dynamic infix membership problem is equivalent up to constant-time reductions to the dynamic membership problem for the language  $\Sigma^* x L x \Sigma^*$  where x is a fresh letter.

**Proof.** Let  $L' := \Sigma^* x L x \Sigma^*$ , where x is a fresh letter not in  $\Sigma$ . We first explain how to use a dynamic membership data structure for L' to solve the dynamic infix membership problem for L. Given a word over  $\Sigma^*$ , we initialize the structure for L' on this word where we add one letter to the beginning of the word and one letter to the end of the word, say a. When substitutions are performed on the word for L, we perform them in the data structure by offsetting them by 1. Now, whenever we receive an infix query for a subword, we perform two substitution updates on L' to replace the characters immediately before and after the subword by x, check if the resulting word belongs to L' using the data structure, and undo these two operations to put back the correct characters. It is clear that the modified word is in L' iff the infix is in L. Note that the addition of the two fixed letters at the beginning and end of the word guarantee that the characters immediately before and after the subword are indeed defined.

Second, we explain how to use a dynamic infix membership data structure for L to solve the dynamic membership problem for L'. Given a word over  $(\Sigma \cup \{x\})^*$ , we initialize the structure for L by replacing every occurrence of x by some arbitrary character of  $\Sigma$ . We also prepare a doubly-linked list, like in the proof of Proposition 4.3, storing all occurrences of the letter x, as well as a pointer from positions containing x to the element of the doubly-linked list storing this copy of x. All of this can be performed as part of the preprocessing.

Given updates to the word on L', we maintain the doubly-linked list and pointers, and we replicate these updates on the word on L, except that occurrences of x are replaced by some arbitrary character of  $\Sigma$ .

Now, to know if the current word belongs to L' or not, first observe that this is never the case if the number of occurrences of x is different from 2, which we can check in constant time using the doubly-linked list. If there are exactly two x's, we know their position, and can perform an infix query on the data structure for L for this infix. The current word is in L' iff this query returns true. Thus, by performing this infix query after every update to the word on L' (whenever the current word contains exactly two x's), we obtain the desired information. This concludes the proof.