On properties of logical sentences with arbitrary monadic predicates [1]

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Highlights of logic, Game and Automata
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Introduction

A nice theorem [Barrington, Compton, Straubing and Thérien, 92]:

$$\mathbf{AC}^0 \cap \mathbf{REG} = \text{"almost aperiodic regular languages"}$$

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Aim

Investigate the sub-case of *monadic numerical predicates* over **MSO**.

Monadic Second Order Logic

$$\mathbf{MSO}[P] = \exists x \phi(x) \mid \exists X \phi(X) \mid \neg\phi \mid \psi \wedge \phi \mid x \in X \mid a(x) \mid P(x_1, \ldots, x_r)$$

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<u>Notation:</u>
Class of numerical predicates of arity at most r: $\mathcal{N}_r$.
Class of uniform numerical predicates of arity at most r: $\mathcal{UN}_r$.

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Notation:
Class of all numerical predicates : $\mathcal{N}$.
Class of all uniform numerical predicates : $\mathcal{UN}$.

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Examples :
Arity 2 non uniform predicate: $2x + y = max$.
Arity 3 uniform predicate: $x + y = z$.

Regular Predicates

A predicate P is regular if, and only if, its a boolean combination of $x \leq y, x \equiv r \bmod q, x = y + k$ with k fixed $, min, max$.

Notation :
The class of regular predicates : $\mathcal{REG}$.

Examples :
Arity 1 non uniform predicate: $x = max - 3$.
Arity 2 uniform predicate: $x < y + 3$.

Straubing and Crane-Beach Properties

$L \in \mathcal{N}_e\mathcal{L}$ if there is e such that for all words u, v:

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Let $\mathcal{F}[<]$ be your favourite fragment of logic and $\mathcal{P}$ a class of numerical predicates.

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Straubing Property	Crane-Beach Property
$\mathcal{F}[<, \mathcal{P}] \cap \mathbf{REG} = \mathcal{F}[<, \mathcal{P} \cap \mathcal{REG}]$	$\mathcal{F}[<, \mathcal{P}] \cap \mathcal{N}_e\mathcal{L} = \mathcal{F}[<] \cap \mathcal{N}_e\mathcal{L}$

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- with: $\textbf{FO}[\mathcal{N}]$

Crane-Beach Property

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- with: $\mathbf{FO}[\mathcal{N}]$
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Theorem
The languages definable in $\mathbf{MSO}[<, \mathcal{N}_1]$ are exactly those recognized by *non-uniform advice automata*.

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Corollary

$\mathbf{MSO}[<, \mathcal{N}_1]$ has the Crane-Beach Property:

$$\mathbf{MSO}[<, \mathcal{N}_1] \cap \mathcal{N}_e\mathcal{L} = \mathbf{MSO}[<] \cap \mathcal{N}_e\mathcal{L}$$

Substitution Property

Definition

A fragment $\mathcal{F}[<, \mathcal{P}]$ has the Substitution Property if any regular language define by $\varphi(P_1, \ldots, P_k) \in \mathcal{F}[<, \mathcal{P}]$ can be defined by $\varphi(R_1, \ldots, R_k)$ such that $R_i \in \mathcal{P} \cap \mathcal{REG}$ with the **same** formula φ.

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$\mathbf{MSO}[<, \mathcal{UN}_1]$ and $\mathbf{MSO}[<, \mathcal{N}_1]$ have the Substitution Property.

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Corollary

1. $\mathcal{F}[<,\mathcal{N}_1]$ and $\mathcal{F}[<,\mathcal{UN}_1]$ have the Straubing Property.
2. $\mathcal{F}[<,\mathcal{N}_1]$ has *almost* the Crane-Beach property:

$$\mathcal{F}[<,\mathcal{N}_1] \cap \mathcal{N}_e\mathcal{L} = \mathcal{F}[<,\mathcal{REG}_1] \cap \mathcal{N}_e\mathcal{L}$$