Using Determinancy of Games to Eliminate Quantifiers.

J. Richard Büchi, Purdue University

A. Combinatorial lemmas reduce quantifiers: To have recognized the fundamental role that quantifiers play, is one of Frege's contributions to mathematics. Elimination of quantifiers, however, was not invented by logicians. In fact, it is easily the most important thing that happens in any mathematical proof. Investigation would probably reveal a direct relation between the usefulness of a theorem, and its ability to simplify quantifications. (The same goes for notions, e.g., continuous everywhere versus uniformly continuous). In particular, (what some call) infinity lemmas, or (what others call) combinatorial lemmas, turn out to be simple instructions for replacing bad combinations $(\forall\exists)$ by more manageable ones $(\exists\forall)$. Here is a list of examples:

(1) Axiom of choice:

$$(\forall x)(\exists y)Rxy \;.\equiv.\; (\exists f)(\forall x)Rxfx$$

(2) Infinity lemma (compactness):

$$(\forall x)(\exists \overline{Z}x)(\forall t)^{x}M(\overline{X}t,\overline{Z}t) \;.\equiv.\; (\exists Z)(\forall t)M(\overline{X}t,\overline{Z}t)$$

(3) Ramsey's lemma:

$$(\forall Z)^{\inf}(\exists y)(\exists x)^{y}[Zx\wedge Zy\wedge\tilde{R}xy] \;.\supset.\; (\exists Z)^{\inf}(\forall y)(\forall x)^{y}[Zx\wedge Zy\supset Rxy]$$

Why, in the course of a proof, is the right side $\exists\forall$ more desirable? Having arrived at $(\exists x)(\forall y)Sxy$, I will simply say "let b be one of these x, and so $(\forall y)Sby$". Such an "existentiation" permanently eliminates a quantifier. In contrast, having arrived at $(\forall x)(\exists y)Sxy$, I might say "let b be any one of these x, and so $(\exists y)Sby$". This trick of "the generic element" eliminates the quantifier $(\forall x)$, but only temporarily. At any rate, in a later state I will have to recall that b was generic.

In the course of this lecture we will add to the list some combinatorial lemmas from automata theory. They all come with nice proofs, which are just as non-trivial as you may desire, and they all say very real things about rationals and their relation to reals. Here we add some remarks about the history of infinity lemmas.

The compactness lemma generalizes to Tychonoff's theorem, and in this form becomes another version of AC. In 1949 R. Rado proved a useful combinatorial lemma (and uses it where a straight application of AC would do). Actually, his lemma is just a way of stating Tychonoff's. In 1964 A. Robinson announced a lemma (which he uses in model theory). This turned out to be a special case of Rado's, and actually just a way of stating the infinity lemma. Ramsey also used his lemma where nothing that refined is needed. (So did I in 1960). Moral: If you discover a combinatorial lemma, be careful, it may be old hat, or you might not actually need it.

The axiom of choice does, of course, not actually deserve to be called a

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combinatorial lemma. For one thing, a c.l. comes with a nice proof. Better think of AC as a schematic form, which indicates what is wanted, and which is to be proved for special R's. The infinity lemma is an example of this, and so are other combinatorial lemmas.

B. Systematic elimination of quantifiers: The idea of methodically eliminating quantifiers is due to Löwenheim 1915. He thus obtained the now famous countable models for elementary axiomsystems, and showed that the monadic theory MT[S] of any set S is decidable (called Behmann's theorem). These were contributions to the axiomatic method, and the foundations of mathematics, of a new caliber. So, nobody seems to have paid much attention, except for Skolem. He extended Löwenheim's theorem, about the countable model, to infinite axiomsystems. He then found that Zermelo's "Aussonderungs Axiom", and the induction axiom could be more rigorously formalized, as axiom schemata. And now the method of quantifier elimination gave these exiting results: 1. There are (if any) countable models for set-theory. 2. There are non-standard models of the axioms of number theory. Skolem also showed how to use the infinity lemma, in place of the more controversial AC. In this form quantifier elimination was used by Gödel, to show that the basic notion of axiomatics "Σ is logical consequence of the axioms A" is semi-recursive.

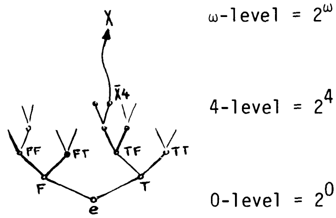
Skolem also worked on Löwenheim's decision method, and showed how to eliminate quantifiers from the theory of $\langle N, +\rangle$ (Presburger's decision method). Both these theories are included in MT[ω,o,'] (the monadic theory of one successor), and the early ideas can be recognized in the proofs we will discuss. Tarski's decision method for the ordered field of real numbers was the first to cover sentences which could (and I hear, did) occur as genuine mathematical problems. Of this same caliber is Rabin's method for the MT of two successors. We shall outline how, in this case, determinacy of games can be used, to systematically eliminate quantifiers.

Some have dreamed about implementing such decision methods on the beautiful modern machines. Some feel that the complexity boys have spoiled these dreams. But then, if one was to heed present complexity theory, how would he dare implement propositional calculus, and how else was he going to use the machine, if he was to believe it can't handle truth tables.

Others feel that the resulting decision method is just a nice way to summarize the work done. The crux of the matter are the ideas which go into the single step of quantifier elimination. I will take these ideas, and what they tell me about the nature of the theory under investigation; you can have the decision method. Apropo: such ideas do sometimes occur in papers which talk about model completeness. So you look up what this is all about; it's nice. But next time they will tell you about model companions. I am sure these are nice things too; but now you will have to work still more to find the place where the quantifier is being eliminated.

C. The two-splitting tree: Let $2 = \{0,1\} = \{F,T\}$ and let ω be the set of natural numbers. We use variables like t,x,y to range over ω, and X,Y,Z to range over 2^{ω}. So X may be viewed as a subset of ω, or alternatively, as an ω-sequence of members of 2. $\bar{X}t = Xo\ldots X(t-1)$ stands for the course of values of X, up to the place t. Thus $\bar{X}t$ is a member of 2^*, the set of all words over the symbols F,T. Variables like x, y, z are used to range over 2^*.

The e items provide a very natural and efficient set of notations, for discussing the oriented two splitting tree N_2. Namely, N_2 has the vertices 2^*, the empty word e is its root, and xF, xT are the (left,right) successors of the vertex x.



The elements of ω mark the levels of the tree N_2, and the members of 2^{ω} are just the infinite paths through N_2. Finally, $\bar{X}t$ is the vertex on the path X, which occurs at level t. Note that $N_2 = \langle 2^*, e, xF, xT \rangle \cong \langle \omega, 0, 2x+1, 2x+2 \rangle$ is the totally free algebra with one generator and two unary functions. It should be compared to the one-successor algebra $N_1 = \langle \omega, 0, x+1 \rangle$. In addition we use notations like M,Φ,Ψ for sets of vertices, i.e., functions of 2^* to 2. More generally M(or Z) may denote a function from 2^*(from ω) into any finite set n. Furthermore, Z may be an m-tuple $Z_1,\ldots,Z_m$, which we identify with the sequence of tuples $Z_1t,\ldots,Z_mt$. The function $M:2^* \to n$ puts a marker Mx onto every vertex x of N_2. So M may be visualized as a marked tree.

The brackets [,] are used so that you can spot propositional calculus from one mile off. So, $D[-]$ denotes a truth-function, or an n-vector of truth-functions, or function from a finite set to a finite set. Occasionally we will violate this convention. Namely, when we are dealing with ω^* and ω^{ω}, and we are doing the analog to a thing which, in the case of 2^* and 2^{ω}, requires truth functions.

$(\exists t)^x$ and $(\forall t)^x$ are bounded quantifiers, made famous by Skolem. $(\exists t)_x$, $(\forall t)_x$ stand for $(\exists t)[x \le t \ldots]$ and its dual. From these one defines $(\exists^{\omega} t):(\forall x)(\exists t)_x$ and $(\forall^{\omega} t):(\exists x)(\forall t)_x$. They mean "for infinitely many t" and "for all but finitely many t". For $Z:\omega \to n$ we let $\sup Z = \sup_t(Zt) = \{a:(\exists^{\omega} t)Zt=a\}$. So $\sup Z$ is the set of values which Z takes infinitely often. This provides for efficient and intuitively manageable handling of Boolean combinations of expressions, $(\exists^{\omega} t)L(t)$. Namely: Let $Z:\omega \to n$. Let $L(Z)$ be any Boolean combination of parts of form $(\exists^{\omega} t)C[Zt]$ and $(\forall^{\omega} t)C[Zt]$. Then $L(Z)$ is equivalent to $\sup Z \in \mathcal{U}$, whereby $\mathcal{U}$ is a set of subsets of n. Conversely, for any such $\mathcal{U}$, $\sup Z \in \mathcal{U}$ means $\bigvee_{D \in \mathcal{U}}[(\forall^{\omega} t)D[Zt] \wedge \bigwedge_{a \in D}(\exists^{\omega} t)Zt=a]$.

You might feel that all this makes a rather involved set of notations. I think that it is precisely fitted to talk about the very interesting structure N_2. This structure occurs in automata theory, and of course it also displays the basic (topological and algebraic) relationship between rational and real numbers.

t,x	nat. numbers	level of tree	pos. of approx.	time instance
x,y	rationals	vertices	approximations	input signals
X,Y	sets of n.n.	paths	reals	infinite signals
M,Φ	sets of rationals	marked trees	F,G,F_σ	strategies
$\bar{X}t$	vertex at level t on path X		t-th approximation of the real X	

Actually 2^ω more naturally carries the topology of the compact Cantor-set. The real topology is obtained by considering only the infinite subsets of ω (i.e., the infinitely right-turning path through the tree N_2). These are naturally identified with the members of ω^ω, for which we use notations like s,f. For ω^* we use the same notations as for 2^*; it will be clear that in the expression $M\bar{s}t$ the M means a function from ω^ω into 2 (or any finite set).

D. Descriptive set-theory is a beautiful collection of combinatorial lemmas: Cantor invented his set theory for the purpose of giving rigorous definitions to the basic concepts of analysis. For example, he defined reals to be Cauchy sequences of rationals. In particular he developed the fundamentals of topology, such as the theory of closed and open sets, and the set of limitpoints of a set. These investigations led to descriptive set-theory, a branch of analysis which flourished in the early 1900s, and has come back to life. Sets (and functions) of reals are here classified with regard to the form of defining expressions. The iterated occurrence of unions and intersections are recognized as the main ingredients in these expressions, and these are but another way of representing existential and universal quantifiers. A nice way of acquiring the ideas developed in the field, is to look at Hausdorff's book, and to do things in the special cases 2^ω and ω^ω. Here are the very simplest Borel sets.

$C(X) \equiv (\forall t)M\bar{X}t$	closed, or F-set
$C(X) \equiv (\exists t)M\bar{X}t$	open, or G-set
$C(X) \equiv (\forall^\omega t)M\bar{X}t$	F_σ-set
$C(X) \equiv (\exists^\omega t)M\bar{X}t$	G_δ-set
$C(X) \equiv \sup_t M\bar{X}t \in U$	$Bl(F_\sigma) = Bl(G_\delta)$ = Boolean over F_σ

You may prefer the notations $X \in C$ and $x \in M$; so do I at times. The class B of all Borel-sets is obtained from $F \cup G$, by closing up under the operation $C(X) \equiv (\exists t)C_t(X)$ and its dual. This process of "closing up" can be analyzed into steps, using Cantor's countable ordinals. Souslin sets are the simplest which can be obtained by using quantification over reals:

$$C(X) \equiv \begin{array}{l} (\exists Z)(\exists^{\omega} t)\, M(\bar{X}t,\bar{Z}t) \\ (\exists Z)\sup M(\bar{X}t,\bar{Z}t) \in U \\ (\exists f)(\forall t)\, M(\bar{X}t,\bar{f}t) \end{array} \qquad \begin{array}{l} \text{equivalent forms for} \\ \text{Souslin-, or } \mathcal{S}\text{-sets} \end{array}$$

Note that in the first line we can not put $(\forall t)$ in place of $(\exists^{\omega} t)$. Just look at our formulation of the infinity lemma! It says "what may look like a genuine Souslin set (right side of infinity lemma), actually is only a closed set (left side)". Consider this puzzle; the Souslin-like form is proof-technically more desirable than the closed form.

Souslin's Criterion: If both C and $\tilde{C}$ are $\mathcal{S}$, then C is $\mathcal{B}$.

This is one of the quantifier lemmas, that make up descriptive set-theory. (Souslin's proof you will find in Hausdorff, p. 191). Others take the form of determinacy results for games. To discuss these we need operators.

E. Presentation of deterministic operators: Let $f:\omega \to \omega$ and let $\Psi: I^* \to J^*$, where I and J are finite sets. These items can be used to define an operator $Y = \Psi f X$ from 2^I to 2^J. Namely, $Yt = \Psi\bar{X}ft$. You may verify that this is a continuous map, and that every continuous map from 2^I to 2^J is of this form. We will be interested in the cases $ft = t$, and $ft = t'$:

$Yt = \Psi\bar{X}t$	the 1-delay operator $Y = (\Psi X)$
$Yt = \Psi\bar{X}t'$	the 0-delay operator $Y = [\Psi X]$

These are properly called deterministic operators, as the value Yt depends only on values Xv for times $v < t$ (1-delay) or $v \le t$ (0-delay). These same operators can be presented in a more general manner:

	det. operator $X \to Y$, presented by
$so = c,\ st' = F[st,Xt],\ Yt = D[st]$	an infinite-state induction
$Zo = c,\ Zt' = F[Zt,Xt],\ Yt = D[Zt]$	a finite-state recursion

As the type indicates, in the second line, Z takes values in a finite set K, F takes $K \times I$ to K (the transition operator), c is in K (the initial state), and D takes K to J (the output). In the first line s takes values in $K = \omega$. Clearly Yt depends on $Xo \ldots X(t-1)$ only. So these are 1-delay operators. The presentation of 0-delay operators are obtained by replacing the output by one of form $Yt = D[st,Xt]$. State inductions may be used to present Borel and Souslin sets in a more general form. For example:

$C(X) \equiv (\exists s)[so=c \wedge (\forall t)st'=F[st,Xt] \wedge \sup_t D[st] \in U]$
a $Bl(F_\sigma)$-set, presented by state-induction.

At first sight this may seem to be a Souslin set. However, the real quantification $(\exists s)$ is only apparent. Namely $s = gX$ is uniquely determined by the state-induction, and so $C(X)$ also has the form $(\forall s)[s=gX \supset \sup \in U]$. From this one sees (Souslin Criterium) that C is Borel. Actually C is a $Bl(F_\sigma)$. Namely, $C=g^{-1}(\mathcal{D})$ whereby $\mathcal{D}(s) \equiv \sup_t D[st] \in U$. So $\mathcal{D}$ is a $Bl(F_\sigma)$, and as the state-induction operator g is continuous, C is also $Bl(F_\sigma)$. Here it is essential that D takes its values in a finite set. Otherwise the sup-condition is not a Boolean expression (in fact it would contain real-number quantification).

F. Determinacy is a schema for quantifier elimination: Let I and J be finite sets, and let $C(X,Y)$ be a condition for $X:\omega \to I$, $Y:\omega \to J$.

(4) C is determinate: $\sim(\exists\Psi)(\forall X)C(X,(\Psi X)) \equiv (\exists\Phi)(\forall Y) \sim C([\Phi Y],Y)$
Ψ wins for J $\quad$ Φ wins for I

Consider a game between players I and J. At any time t, first J makes a move $Yt \in J$, and then I makes a move $Xt \in I$. The winner is picked on doomsday, when all the moves have been made. Namely, J wins if $C(X,Y)$, and I wins if $\sim C(X,Y)$. Both players have complete information on previous moves of the opponent. I.e., a strategy for J is a function $Yt = \Psi\bar{X}t$, which tells J what his next move Yt should be, given the previous moves $\bar{X}t$ of I. Thus, a strategy for J is a 1-delay operator $Y = (\Psi X)$. As I moves after J, his strategies are 0-delay operators $X = [\Phi Y]$. A winning strategy for J is one which, no matter how I picks his moves X, produces moves $Y = (\Psi X)$ for J, such that J wins, i.e., such that $C(X,Y)$. In turn, Φ wins for I if $C(X,Y)$, for every Y and $X = [\Phi Y]$. That C is determinate now takes this nice meaning: Either player J, or then player I has a winning strategy.

It has been suggested that determinacy should be an axiom, replacing AC. But then, setting up an axiom can be a very destructive affair. In the present case it would eliminate several interesting proofs of descriptive set theory. They use AC, but to me they still show clearly that determinacy is not evident. Furthermore, there is a proof of determinacy which does not use AC, and is very much non-trivial (see Buchi and Landweber 1969).

A lopsided game is one in which player J may make "long moves", picking a word $Yt \in J^*$ in place of just an element of J. The original Cantor-Bendixson proof, as it stands, shows that lopsided closed games are determinate. Countable ordinals are used here, and I think Cantor invented them for the purpose. W. H. Young extended the result to G_δ. Hausdorff (see p. 179) gives a proof which is easily fixed up to show: every S is determinate in the lopsided sense. Another way of stating "C is lopsided determinate" is "C is countable or contains a perfect set" (if the moves of both players are in 2). So these results say that closed, G_δ, and even Souslin sets satsify the continuums hypothesis.

That S games are determinate was shown by Martin 1970, using large cardinals. He now can handle $\mathcal{B}$'s without the super cardinals. Hausdorff's proof and the modern results do not use the Cantor-Bendixson-type argument. Therefore (so it seems to me) they do not provide any presentation of the winner's strategy. But this is what is needed, if determinacy is to be used for systematic quantifier elimination.

G. Regularity: Let M be a subset of 2*, or more generally a map from 2* into a finite set n. Think of M as being a marked tree, and consider the marked subtree M_x, which stands over the root x. Define,

$$x \approx y(M) \equiv (\forall u)M(xu) = M(yu) \qquad \text{The congruence induced by } M$$

This is actually a congruence relation on the free algebra N_2, and it simply means $M_x = M_y$. The index, rkM, of this congruence is called the rank of M. It tells how many of the marked trees M_x are different from each other. The simplest possibility is that rkM is finite; we then say that M is regular.

Regular M's are exactly those which may be presented by a finite-state recursion. That is, in the form $M\overline{X}t \equiv (\exists Z)[Zo = c \wedge (\forall x)^t Zx' = F[Zx,Xx] \wedge D[Zt]]$. It now is clear what a regular closed (Borel, Souslin) set is. Here are examples:

$C(X) \equiv (\exists Z)[Zo=c, Zt'=F[Zt,Xt], \sup Z \in U]$	$Bl(F_\sigma)^{reg}$ given by f.s.r.
$C(X) \equiv (\exists Z)(\exists^\omega t)M(\overline{X}t,\overline{Z}t)$	S^{reg}, if M is regular

In these days of super-set-theory one finds those who sneer at sets of rationals, particularly at simple sets of rationals. Others like simple concepts, especially if they are finiteness notions, and very non-trivial things can be said about them. The fact is that regularity theory has produced a series of combinatorial lemmas, with the kind of proof which can be seen in any company. These ideas must be useful in other places.

H. Monadic second order theories: The formulas of these contain two sorts of variables, individual variables which are intended to range over a basic domain D, and set-variables which range over some or all subsets of D. The monadic theory, MT[S], of a structure S consists of all monadic sentences which are true in S, whereby the range of set-variables is taken to be all subsets of S. If S is just a set, MT[S] is decidable (Löwenheim). We will discuss $MT_1 = MT[\omega,o,']$ = the monadic theory of one successor, and $MT_2 = MT[2^*,e,xF,xT]$ = the monadic theory of two successors. The range of the variables, in both cases is of cardinality 2^ω. We must (by Skolem-Löwenheim) have equivalent countable structures. In fact these are obtained by restricting the set-variables to range over regular sets only. (Note that $\omega = 1^*$, and so its regular subsets are just the ultimately periodic sets). In other words, $MT_i = MT_i^{reg}$. Of course this is not obvious, but requires elimination of quantifiers, like every other proof of elementary equivalence of two structures.

MT_2 contains a significant fragment of the topology of reals. Namely: 1. In MT_2 one can define $Pth(\Phi) \equiv \Phi$ is a path through 2^*. So the variables $X,Y,\ldots$ can be introduced in MT_2, by restricting its set-variables $\Phi,\Psi,\ldots$ to Pth. 2. We can use the set-variable Φ to simulate a variable over F, as $(\forall t)\Phi\bar{X}t$ is easily expressed as a formula $C(\Phi,X)$ of MT_2. The same goes for F_σ, and similarly one simulates a variable over larger and larger parts of $Bl(F_\sigma)$. Thus, MT_2 contains a good part of the elementary theory ET of the structure $\langle Bl(F_\sigma),\subseteq,F_\sigma,F,B_0\rangle$, where B_0 is the set of all clopen sets.

Problem: Does MT_2 contain all of ET? Are the two theories equivalent? If not, is ET decidable?

I. The basic facts about regularity are quantifier-eliminators: We are now ready to continue the list of infinity lemmas, started in section A.

(5) Büchi 1960: For every regular M one can construct a regular $\bar{M}$ such that

$$\sim(\exists Z)(\exists^\omega t)M(\bar{X}t,\bar{Z}t) .\equiv. (\exists V)(\exists^\omega t)\bar{M}(\bar{X}t,\bar{V}t).$$

More carefully stated, the construction does of course apply to a given finite-state recursion for M, and yields one for $\bar{M}$. Lemmas of this type, where the new form is precisely the dual to the starting form, are called complementation lemmas. Such complementation lemmas are particularly suited for systematic elimination of quantifiers! One needs but iterate complementations to reduce a long prefix $\exists Z_1\forall Z_2 \ldots \ldots \forall Z_{n-1}\exists Z_n$ to the final $\exists Z$. Our lemma thus yields a decision method for MT_1. The same complementation also holds in MT_1^{reg}, and so $MT_1 = MT_1^{reg}$. Just as interesting is this interpretation: The lemma says that every regular Souslin set C has a complement which is still regular Souslin. So, by Souslin's criterion: Every regular Souslin set is Borel; $S^{reg} \subseteq B$. A much stronger result is this:

(6) McNaughton 1966: For every regular M one can construct a regular $\bar{M}$ and a U such that

$$(\exists Z)(\exists^\omega t)M(\bar{X}t,\bar{Z}t) .\equiv. \sup_t \bar{M}(\bar{X}t) \in U.$$

In short, every regular Souslin set is actually in the Boolean algebra over F_σ; $S^{reg} \subseteq Bl(F_\sigma)^{reg}$. McNaughton's is a quite ingenius proof. It shows how a bounded memory (finite automaton) can be made to keep track of essential information, accruing in an ever growing past. You will find a presentation of the matter in Büchi 1973, which also contains a version of the complementation lemma which works at ω_1. We will use McNaughton's lemma in this form.

$$(6')\quad (\forall Y)(\exists Z)[Zo=c, Zt'=F[Zt,Xt,Yt], \sup Z \in \mathcal{U}\ .\equiv.$$
$$(\exists S)[So=d, St'=L[St,Xt], \sup S \in \mathcal{V}]$$

What is meant here is that from $c,F,\mathcal{U}$ one can construct $a,G,\mathcal{V}$ such that the equivalence holds. McNaughton also conjectured this result:

(7) Landweber 1969: Regular $\mathcal{B}l(F_\sigma)$ games are determinate. One can tell who has the win, and the winner has a winning strategy which is finite-state (i.e., can be presented by a finite state recursion).

The classical Cantor-Bendixson process will give this result for F^{reg} in place of $\mathcal{B}l(F_\sigma)^{reg}$. In fact, the process will terminate before ω (instead of before ω_1). Landweber rediscovered a rather sophisticated version of this process. In the next section we will show how his result, and McNaughton's remove quantifiers in MT_2^{reg}.

$$(8)\quad \text{Rabin 1969: } \sim(\exists Z)\cdot Ze=c, A[Zx, ZxF, ZxT, \mathfrak{X}x],\ (\forall X)\sup(ZX) \in \mathcal{U}\ .\equiv.$$
$$(\exists H)\cdot He=d, B[Hx, HxF, HxT, \mathfrak{X}x],\ (\forall X)\sup(HX) \in \mathcal{V}$$

This is the complementation lemma which eliminates quantifiers in MT_2. The same formula also holds in MT_2^{reg}, and so $MT_2=MT_2^{reg}$. Rabin's proof also contains a Cantor-Bendixson process (now extending up to ω_1), and it contains ideas from McNaughton (without actually using his lemma). We will show how a stronger version of Landweber yields Rabin.

K. Strong determinacy of Boolean F_σ-games: An important determinacy proof was Morton Davis (1964) for $F_{\sigma\delta}$. This implies the non-constructive part of Landweber's result, but does not seem to tell anything about the form of the winner's strategy. I think that our 1969 proof should be a first step in a very interesting and fruitful investigation of the forms of strategies. Here is the second step:

(9) Büchi 1973: $\mathcal{B}l(F_\sigma)$ games are determinate. The winner has a strategy which, up to finite-state, is no more complex as a given presentation of the game.

I have tried to fix up our 1969 proof; can you do it? A reasonably short proof only shows the second part of (9) and leaves the existence part to Davis. A much longer proof does both. Here is a precise statement of the result:

Given a presentation $so=c, st'=F[st,Xt,Yt], \sup_t D[st] \in \mathcal{U}$ of a game (D takes values in a finite set!) One can make up a finite state recursion $Vo=v_0, Vt'= G[Vt,D[st]]$ (in fact Vt is the order-vector of the values $D[st]$), such that the winner has a winning strategy presented by the joint inductions for s and V, and an output $Yt=B[st,Vt]$ (if J has the win) or $Xt=A[st,Vt,Yt]$ (if I has the win).

In the remaining sections we will show how this, and McNaughton's lemma yield Rabin's complementation lemma, and hence the decidability of MT_2. Actually this requires but a special case of (9). Can you tell me what more the full result will do? Here is what is needed for MT_2:

(9') Given the game $Zo=c_0, Zt'=F[Zt,Xt,Yt,\mathfrak{X}\overline{X}t], \sup Z \in \mathcal{U}$, whereby $I = 2$.
J has no winning strategy, if and only if, I has a winning strategy presented by the recursion for Z, the ordervector recursion $Vo=v_0$, $Vt'=G[Vt,Zt]$ and an output $Xt=\Phi_{Zt,Vt,Yt}\overline{X}t$.

Note that $I=2$, J any finite set. As Z,V,Y all take values in finite sets, the strategy for I, in essence, consists of finitely many components $\Phi_{c,v,b} \subseteq 2^*$. To recognize this as a special case of (9), let $st=\langle Zt,\overline{X}t\rangle$, $D[st]=Zt$, and note that the game in (9'), is presented by a state-induction for s. In the sequel we will call these games <u>the special</u> $Bl(F_\sigma)$ <u>games</u>.

<u>L. The monadic theory MT_2 is just right to handle special Boolean F_σ games:</u>

Every sentence of MT_2, if put into prenex form and squeezed just a little more, will take the form $(\exists\mathfrak{X}_1)(\forall\mathfrak{X}_2)\ldots(\forall\mathfrak{X}_n)$ [J has a winning strategy in G]. Here G is a special $Bl(F_\sigma)$-game into which the $\mathfrak{X}_i$ enter. This will be easily established by one who knows monadic theories. Here is how the existential formula $C(\mathfrak{X})$ in Rabin's lemma is transformed.

$(\exists Z)\cdot Ze=c, A[Zx,ZxF,ZxT,\mathfrak{X}x], (\forall X)\sup(ZX) \in \mathcal{U}$

let $\Psi x= ZxF,ZxT$

$(\exists\Psi)(\exists Z)\cdot Ze=c, ZxF=\Psi_1 x, ZxT=\Psi_2 x, A[Zx,\Psi x,\mathfrak{X}x], (\forall X)\sup(ZX) \in \mathcal{U}$

let $F[Zt,F,\Psi Xt]=\Psi_1\overline{X}t$, $F[Zt,T,\Psi\overline{X}t]=\Psi_2\overline{X}t$

$(\exists\Psi)(\forall X)(\exists Z)\cdot Zo=c, Zt'=F[Zt,Xt,\Psi Xt], A[Zt,\Psi\overline{X}t,\mathfrak{X}\overline{X}t], \sup Z \in \mathcal{U}$

let $Ut\equiv(\forall x)^t A[Zt,\Psi\overline{X}t,\mathfrak{X}\overline{X}t]$

$(\exists\Psi)(\forall X)(\exists ZU)\cdot Zo=c, Zt'=F[Zt,Xt,\Psi\overline{X}t], \sup Z \in \mathcal{U}$

$Uo\equiv T, Ut'\equiv Ut \wedge A[Zt,\Psi\overline{X}t,X\overline{X}t], \sup U = \{T\}$

The last formula $C'(X)$ is the statement "J has winning strategy in the special game....".

M. <u>The complementation lemma for MT_2 proved by determinacy:</u> As seen in section L, Rabin's (8) comes to put the negation of "J has win in the special game G" back into the same form. Here it goes, using the strong determinacy of $Bl(F_\sigma)$-games and McNaughton's lemma:

$$\sim(\exists\Psi)(\forall X)(\exists Z)\cdot Zo=c_0, Zt'=F[Zt,Xt,\Psi\bar{X}t,\chi\bar{X}t],\ \sup Z \in U$$

use (9'), Φ stands for all $\Phi_{c,v,b}$, c,v,b, the states of Z,V,Y

$$(\exists\Phi)(\forall XY)(\exists Z)\cdot \begin{array}{l} Zo=c_0, Zt'=F[Zt,Xt,Yt,\chi\bar{X}t] \\ Vo=v_0, Vt'=G[Vt,Zt] \end{array},\ \forall t[X_t=\Phi_{Zt,Vt,Yt}\bar{X}t] \supset \sup Z \notin U$$

let $Wt\equiv(\forall x)^t Xx=\Phi_{Zt,Vt,Yt}\bar{X}t$

$$(\exists\Phi)(\forall XY)(\exists ZW)\cdot \begin{array}{l} Zo=c_0, Zt'=F[Zt,Xt,Yt,\chi\bar{X}t] \\ Vo=v_0, Vt'=G[Vt,Zt] \\ Wo\equiv T, Wt\equiv \bigwedge_{c,v,b}[Zt=c\wedge Vt=v\wedge yt=b \supset X_t=\Phi_{c,v,b}\bar{X}t] \end{array} \quad (\exists^\omega t)Wt \vee \sup Z \in U$$

abbreviated, using new Z for Z,W

$$(\exists\Phi)(\forall X)(\forall Y)(\exists Z)\cdot Zo=c, Zt'=H[Zt,Xt,Yt,\Phi\bar{X}t,\chi\bar{X}t], \sup Z \in V$$

use (6'), to eliminate Y

$$(\exists\Phi)(\forall X)(\exists S).\ So=d,\ St'=L[St,Xt,\Phi\bar{X}t,\chi\bar{X}t],\ \sup S \in W$$

As the last formula is again in the form "J has win in the special game...", this proves our complementation lemma for MT_2. Precisely the same transformations will do, if we work in MT_2^{reg} (only now (7) will be used to show correctness of the step where (9') was used). Hence we get $MT_2=MT_2^{reg}$. Having stepwise reduced the quantifiers of a sentence C it eventually takes the form $(\exists\Phi)[\Phi$ wins for J in $G]$, where G is finite state. So now the truth of C can be decided using (7).

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