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Linear systems over localizations of rings

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Abstract. We describe a method for solving linear systems over the localization of a commutative ring R at a multiplicatively closed subset S that works under the following hypotheses: the ring R is coherent, i.e., we can compute finite generating sets of row syzygies of matrices over R, and there is an algorithm that decides for any given finitely generated ideal $I \subseteq R$ the existence of an element r in $S \cap I$ and in the affirmative case computes r as a concrete linear combination of the generators of I.

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1. Introduction. The concept of rings equipped with algorithms for dealing with linear systems is fundamental in constructive algebra ([4]). A ring R is called coherent if we have an algorithm computing a finite generating set of the row syzygies of a given matrix over R. Moreover, R is called computable or coherent strongly discrete if we have an algorithm for finding a particular solution of an inhomogeneous linear system over R. Computable rings provide the basis for an effective categorical framework for homological algebra ([2,8,10]).

In this paper we will address the following problem: when is the localization $S^{-1}R$ of a coherent commutative ring R at a multiplicatively closed subset $S \subseteq R$ computable? With the investigation of this problem we wish to contribute to the powerful framework developed by Barakat and Lange-Hegermann ([2]) that renders the abelian category of finitely presented R-modules constructive whenever a ring R is known to be computable.

In [5, Section 2.8.8] it is shown how to solve linear equations over $S_>^{-1}k[x_1, \ldots, x_n]$, i.e., the polynomial ring over a computable field k in $n \in \mathbb{N}_0$

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indeterminates localized at the multiplicatively closed subset $S_>$ which consists of polynomials having leading monomial equal to 1 for a given monomial ordering $>$. In particular, this yields the computability of the localization $k[x_1,\dots,x_n]_{\langle x_{\ell+1},\dots,x_n\rangle} = S_>^{-1}k(x_1,\dots,x_\ell)[x_{\ell+1},\dots,x_n]$ by choosing a local ordering on $x_{\ell+1},\dots,x_n$ for $\ell \in \mathbb{N}_0$ (see [5, Example 1.5.3.4]).

The computability of the localization at a finitely generated maximal ideal $\mathfrak{m} \subset R$ of a computable ring R is established by Barakat and Lange-Hegermann in [2, Section 4]. Their algorithm avoids the computation of standard bases over a local ordering in the special case $R = k[x_1,\dots,x_n]$ and $\mathfrak{m} = \langle x_1,\dots,x_n\rangle$.

In this paper we describe a general method for solving linear systems over $S^{-1}R$ for those coherent rings R and multiplicatively closed subsets $S \subseteq R$ that can be equipped with the following extra data: an algorithm that decides for $\ell \in \mathbb{N}_0$ and any given finitely generated ideal $I = \langle f_1,\dots,f_\ell\rangle \subseteq R$ the existence of an element in $S \cap I$ and in the affirmative case computes $a_1,\dots,a_\ell \in R$ such that $\sum_{i=1}^{\ell} a_i f_i \in S \cap I$. From this general method[1], we can deduce the computability of $R_\mathfrak{p}$ for the localization of a computable ring R at a finitely generated prime ideal $\mathfrak{p}$ (Corollary 4.1).

The paper is structured as follows. After introducing the notion of coherent and computable rings in Section 2, we discuss linear systems over $S^{-1}R$ in Section 3. The idea of our method for solving such systems is a generalization of the following observation: given a matrix $A \in R^{m\times n}$ and a row $b \in R^{1\times n}$, the existence of a solution $x \in R^{1\times m}$ of the linear system $xA = b$ is equivalent to $1 \in \operatorname{ann}_R([b]_A)$, where $\operatorname{ann}_R([b]_A)$ denotes the annihilator of b regarded as an element in $\operatorname{coker}(R^{1\times m} \xrightarrow{A} R^{1\times n})$. We generalize this criterion to the localized case: if A and b are considered over $S^{-1}R$, then a solution x exists over $S^{-1}R$ if and only if $\operatorname{ann}_R([b]_A) \cap S$ is inhabited, i.e., there is an element r in this intersection (Lemma 3.5). In the affirmative case a solution over $S^{-1}R$ can be constructed from a concrete expression of r as a linear combination of a special set of generators of $\operatorname{ann}_R([b]_A)$. This will yield the computability of $S^{-1}R$ (Theorem 3.9).

In the last Section 4 we give some examples for our method. In particular, we can describe how to find particular solutions of inhomogeneous linear systems over $(k[x_1,\dots,x_n]/I)_\mathfrak{p}$ *without* the usage of Mora's tangent cone algorithm, where I is an ideal of $k[x_1,\dots,x_n]$ and $\mathfrak{p}$ is a prime ideal of $k[x_1,\dots,x_n]/I$.

2. Computable rings. In this paper R will always denote a commutative unital ring. Recall that R is called **coherent** if it comes equipped with an algorithm for computing syzygies:

1. Given a matrix $A \in R^{m\times n}$ for $m, n \in \mathbb{N}_0$, we can find an $o \in \mathbb{N}_0$ and a matrix $L \in R^{o\times m}$ such that $LA = 0$. Furthermore, L is universal with this property in the sense that for every other $p \in \mathbb{N}_0$ and matrix $T \in R^{p\times m}$ such that $TA = 0$, there exists a $U \in R^{p\times o}$ such that $UL = T$.

[1] For Ore localizations of non-commutative domains finding an element in $S \cap I$ is even the key problem for performing basic arithmetics like addition and multiplication [7].

Remark 2.1. Following a fully constructive reading of this definition, the quantifier claiming the existence of U for given A and T also has to be realized algorithmically. We will come back to this very important point in Remarks 2.2 and 3.2.

Following [2] we call a coherent ring R **computable** if it additionally comes equipped with an algorithm for computing lifts:

2. Given two matrices $A \in R^{m \times n}$ and $B \in R^{q \times n}$, we can decide whether there exists a matrix $X \in R^{q \times m}$ such that $XA = B$, and in the affirmative case construct such an X. We call X a **lift of** B **along** A.

In constructive algebra computable rings are also known as **coherent strongly discrete rings** [4].

We will refer to statement (1) as the **syzygy problem for** R. Statement (2) is called the **lifting problem for** R, since it can be nicely rephrased as follows: if we interpret the matrices A and B as R-module homomorphisms between free modules $R^{1\times m} \to R^{1\times n}$ and $R^{1\times q} \to R^{1\times n}$, respectively, then we ask whether the cospan $(R^{1\times q} \xrightarrow{B} R^{1\times n} \xleftarrow{A} R^{1\times m})$ admits a lift, i.e., if there exists a module homomorphism X making the diagram

$$\begin{tikzcd}
R^{1\times q} \arrow[d, "B"'] \arrow[drr, dashed, "X"] & & \\
R^{1\times n} & & R^{1\times m} \arrow[ll, "A"']
\end{tikzcd}$$

commutative, and in the affirmative case we ask for a specific instance of such an X.

Remark 2.2. Turning the existential quantifier in the definition of a coherent ring algorithmic as proposed in Remark 2.1 can be seen as a special instance of the lifting problem, namely finding a solution of $XL = T$. In actual implementations of computable rings it is advisable to have separate algorithms for the special and the general lifting problem, since in the special case we can benefit from additional knowledge. For example, if we deal with matrices over the polynomial ring, the syzygy matrix L could have been already computed as a Gröbner basis with respect to the induced ordering, and this knowledge turns the special lifting problem into a simple reduction (see Remark 3.2 for another example). Note that at the moment, there is no interface in the `homalg`-project [8] for the special lifting problem in computable rings.[2]

From a categorical point of view, being coherent for a ring means the existence of so-called weak kernels in the category of row modules (see [10]). In this light having a complete set of algorithms dealing with weak kernels, including their weak kernel lifts, appears very natural.

3. Solving linear systems over localizations of rings. Let R be a unital commutative ring equipped with a multiplicatively closed subset $S \subseteq R$, i.e., $1 \in S$

[2]However, for a fixed ordering $<$ in the current session, `homalg` matrices over polynomial rings that are created as Gröbner bases w.r.t. $<$ store this knowledge, and utilize it whenever appropriate.

and $r, s \in S$ implies $rs \in S$. In this section we investigate the computability of the localization[3] $S^{-1}R$ of R at S.

Elements in R give rise to elements in the localization via the not necessarily injective natural map $R \to S^{-1}R : r \mapsto \frac{r}{1}$. When we write $\frac{A}{d} \in S^{-1}R^{m\times n}$, we mean that A is a matrix in $R^{m\times n}$, $d \in S$, and $\frac{A}{d} = \left(\frac{A_{ij}}{d}\right)_{ij}$. Note that every matrix with entries in $S^{-1}R$ has such a representation by choosing a common denominator and representatives in R.

The solvability of the syzygy problem for $S^{-1}R$ is easy.

Lemma 3.1. *If we can solve the syzygy problem for R, then the same is true for $S^{-1}R$.*

Proof. [4] Let $\frac{A}{d} \in S^{-1}R^{m\times n}$ and let $L \in R^{o\times m}$ be a solution of the syzygy problem of $A \in R^{m\times n}$. This is equivalent to $R^{1\times o} \xrightarrow{L} R^{1\times m} \xrightarrow{A} R^{1\times n}$ being an exact sequence. Applying the exact localization functor $(\frac{-}{1})$ yields an exact sequence which proves that $\frac{L}{1}$ is a solution of the syzygy problem of $\frac{A}{1}$ and thus, by the invertibility of d, also of $\frac{A}{d}$. $\square$

Remark 3.2. The proof of Lemma 3.1 actually hides how we can find for given $\frac{T}{d'} \in S^{-1}R^{p\times m}$ with $\frac{T}{d'} \cdot \frac{A}{1} = 0$ a matrix $\frac{U}{d''} \in S^{-1}R^{p\times o}$ such that $\frac{U}{d''} \cdot \frac{L}{1} = \frac{T}{d'}$, so, we will briefly explain how it can be done. First, since $\frac{T}{d'} \cdot \frac{A}{1} = 0$ we can find an $s \in S$ such that $sTA = 0$. Next, since L consists of row syzygies of A, we have a $U \in R^{p\times o}$ such that $UL = sT$. It follows that

$$\left(\frac{U}{d's}\right) \cdot \frac{L}{1} = \frac{T}{d'}.$$

Thus, we could quite easily establish a matrix $\frac{U}{d's}$ that solves a special instance of the lifting problem.

We turn to the general lifting problem for $S^{-1}R$. Let $\frac{A}{d_A} \in S^{-1}R^{m\times n}$, $\frac{B}{d_B} \in S^{-1}R^{q\times n}$, and $\frac{X}{d} \in S^{-1}R^{q\times m}$ for $m, n, q \in \mathbb{N}_0$. Since

$$\frac{X}{d} \cdot \frac{A}{d_A} = \frac{B}{d_B} \iff \frac{d_B X}{d d_A} \cdot \frac{A}{1} = \frac{B}{1},$$

finding a lift of $\frac{B}{d_B}$ along $\frac{A}{d_A}$ is equivalent to finding a lift of $\frac{B}{1}$ along $\frac{A}{1}$, which in turn is equivalent to finding lifts $\frac{X_i}{d_i} \in S^{-1}R^{1\times m}$ for all rows of $\frac{B}{1}$ along $\frac{A}{1}$, i.e., $\frac{X_i}{d_i} \cdot \frac{A}{1} = \frac{B_{i,-}}{1}$ for $i = 1, \ldots, q$. Thus, it suffices to deal with the case $q = 1$ in which B is a row vector (from now on we will call it b) and the diagram of the simplified lifting problem is given by

$$\begin{tikzcd}
S^{-1}R^{1\times 1} \arrow[d, "\frac{b}{1}"'] & \\
S^{-1}R^{1\times n} & S^{-1}R^{1\times m}. \arrow[l, "\frac{A}{1}"']
\end{tikzcd}$$

[3]If S contains a zero divisor, then $S^{-1}R$ is the zero ring. Since we will never use $1 \neq 0$ in $S^{-1}R$, all statements that we make in this paper are also true in this special case.

[4]This proof can also be found in [2, Lemma 4.3]. It had a typo in the exact sequence that had been communicated to the authors and now is fixed (v5).

The key to the lifting problem lies in the following definition.

Definition 3.3. For a matrix $A \in R^{m\times n}$ and a row $b \in R^{1\times n}$, we set

$$\operatorname{ann}_R([b]_A) := \{r \in R \mid \exists x \in R^{1\times m} : xA = rb\} \subseteq R.$$

Note that $\operatorname{ann}_R([b]_A)$ can also be described as the annihilator of b regarded as an element in $\operatorname{coker}(R^{1\times m} \xrightarrow{A} R^{1\times n})$. In particular, $\operatorname{ann}_R([b]_A)$ is an ideal of R.

Remark 3.4. Whether a lift of b along A exists can be read off from $\operatorname{ann}_R([b]_A)$:

$$\big(\exists x \in R^{1\times m} : xA = b\big) \iff 1 \in \operatorname{ann}_R([b]_A).$$

The last remark generalizes to the localized case.

Lemma 3.5. *Given $A \in R^{m\times n}$ and $b \in R^{1\times n}$. Then there exists a lift $\frac{x}{d} \in S^{-1}R^{1\times m}$ such that $\frac{x}{d} \cdot \frac{A}{1} = \frac{b}{1}$ if and only if there exists an element $r \in \operatorname{ann}_R([b]_A) \cap S$.*

Proof. A lift of $\frac{b}{1}$ along $\frac{A}{1}$ exists if and only if $1 \in \operatorname{ann}_{S^{-1}R}\left([\frac{b}{1}]_{\frac{A}{1}}\right)$ (Remark 3.4). From the exactness of the localization functor, we get $\langle \frac{a}{1} \mid a \in \operatorname{ann}_R([b]_A)\rangle_{S^{-1}R} = \operatorname{ann}_{S^{-1}R}\left([\frac{b}{1}]_{\frac{A}{1}}\right)$. Now, the claim follows from the fact that for any ideal $I \subseteq R$, we have $1 \in \langle \frac{a}{1} \mid a \in I\rangle_{S^{-1}R}$ if and only if there exists an element $r \in I \cap S$. $\square$

We turn to the case where R is a coherent ring.

Construction 3.6. If R is a coherent ring, then $\operatorname{ann}_R([b]_A)$ can be constructed as follows. First, we find an $o \in \mathbb{N}_0$ and a solution $L \in R^{o\times(m+1)}$ of the syzygy problem

$$X \cdot \begin{pmatrix} b \\ A \end{pmatrix} = 0.$$

Next, we decompose this solution L as follows:

$$L = \left(\begin{array}{c|c} r_1 & L_1 \\ \vdots & \vdots \\ r_o & L_o \end{array}\right),$$

where $r_i \in R$ and $L_i \in R^{1\times m}$ for $i = 1, \dots, o$. Then it easily follows that

$$\operatorname{ann}_R([b]_A) = \langle r_1, \dots, r_o\rangle_R.$$

In particular, $\operatorname{ann}_R([b]_A)$ is a finitely generated ideal. We should not discard the L_i after this computation, since their true value lies in the construction of lifts:

Lemma 3.7. *A ring R is computable if and only if*

1. *R is coherent*
2. *we can effectively decide $1 \in I$ for any finitely generated ideal $I = \langle f_1, \dots, f_\ell\rangle_R \subseteq R$, i.e., construct a linear combination $a_1, \dots, a_\ell \in R$: $\sum_{i=1}^{\ell} a_i f_i = 1$ or disprove its existence.*

Proof. The "only if" direction is trivial, so we prove the "if" direction. Using the notation of Construction 3.6, a solution of $x \cdot A = b$ is simply given by

$$\left(- \sum_{i=1}^{o} a_i L_i \right) \cdot A = b,$$

where we use a linear combination $\sum_{i=1}^{o} a_i r_i = 1$. □

Our strategy for proving computability of $S^{-1}R$ is to generalize Lemma 3.7. Instead of finding a linear combination of 1, we need to be able to find a linear combination of an element of S:

Definition 3.8. Deciding whether for a given $\ell \in \mathbb{N}_0$ and finitely generated ideal $I = \langle f_1, \ldots, f_\ell \rangle_R \subseteq R$, there exists an element in $I \cap S$, and in the affirmative case constructing elements $a_1, \ldots, a_\ell \in R$ such that $\sum_{i=1}^{\ell} a_i f_i \in I \cap S$ is what we call the **localization problem for R at S**.

We are ready to state and prove our main theorem.

Theorem 3.9. *Let R be a coherent ring and $S \subseteq R$ a multiplicatively closed subset. Then $S^{-1}R$ is a computable ring if we can algorithmically solve the localization problem for R at S.*

Proof. Assume we have an algorithm solving the localization problem for R at S. Then we may use it to decide whether there exists an element in $\mathrm{ann}_R([b]_A) \cap S$ which by Lemma 3.5 is the case if and only if a lift of $\frac{b}{1}$ along $\frac{A}{1}$ exists. In the affirmative case our algorithm gives us $a_1, \ldots, a_o \in R$ such that $\sum_{i=1}^{o} a_i r_i \in \mathrm{ann}_R([b]_A) \cap S$, where the r_i are the generators described in Construction 3.6. Now, we can benefit from the already computed L_i in Construction 3.6:

$$\sum_{i=1}^{o} a_i \left(r_i \; L_i \right) \cdot \begin{pmatrix} b \\ A \end{pmatrix} = 0 \quad \Longleftrightarrow \quad \left(\sum_{i=1}^{o} a_i r_i \right) b + \left(\sum_{i=1}^{o} a_i L_i \right) A = 0$$

which in turn gives us a concrete formula for our desired lift:[5]

$$\left(\frac{\sum_{i=1}^{o} a_i L_i}{- \sum_{i=1}^{o} a_i r_i} \right) \cdot \frac{A}{1} = \frac{b}{1}.$$

□

4. Examples.

Of special importance in algebraic geometry are localizations of rings at prime ideals.

Corollary 4.1. *Let R be a computable ring with a finitely generated prime ideal $\mathfrak{p} = \langle p_1, \ldots, p_m \rangle \subseteq R$ for $m \in \mathbb{N}_0$. Then $R_{\mathfrak{p}} = S^{-1}R$ is a computable ring, where $S := R \setminus \mathfrak{p}$.*

[5] If R is a computable ring, then for any given $r \in \mathrm{ann}_R([b]_A) \cap S$ we can solve the lifting problem $xA = rb$ in R. It follows that $\frac{x}{r}$ is a lift of $\frac{b}{1}$ along $\frac{A}{1}$. However, this strategy does not benefit from the already computed L_i and thus can lead to slower computations.

Proof. By Theorem 3.9 we need to show how to solve the localization problem of R at S. Given $\ell \in \mathbb{N}_0$ and a finitely generated ideal $I = \langle f_1, \dots, f_\ell \rangle \subseteq R$, then

$$\exists r \in I \cap S \iff \exists r \in I - \mathfrak{p} \iff \exists i \in \{1, \dots, \ell\} : f_i \notin \mathfrak{p}.$$

So, all we have to do is to test whether $f_i \in \mathfrak{p}$, which is equivalent to solving

$$X \cdot \begin{pmatrix} p_1 \dots p_m \end{pmatrix}^{\mathrm{tr}} = f_i.$$

□

Remark 4.2. Let k be a computable field, $n \in \mathbb{N}_0$, $I \subseteq k[x_1, \dots, x_n]$ an ideal. We set $R := k[x_1, \dots, x_n]/I$. Then Corollary 4.1 gives us an algorithm to solve linear systems over $(k[x_1, \dots, x_n]/I)_{\mathfrak{p}}$ for prime ideals $\mathfrak{p} \subseteq R$, since R is a computable ring by means of Gröbner bases. In particular, we do not need the computation of a standard basis over a *local* monomial ordering by means of the tangent cone algorithm ([9]).

Since the localization of a polynomial ring at a prime ideal is a very interesting special case for computer algebra, we discuss it at length in the following construction.

Construction 4.3. Let k be a computable field and $R := k[x_1, \dots, x_n]$ the polynomial ring in $n \in \mathbb{N}_0$ indeterminates. Let $\mathfrak{p} \subset R$ be a prime ideal with generators $p_1, \dots, p_m$ for $m \in \mathbb{N}_0$. Given a linear system

$$X \cdot \frac{A}{1} = \frac{b}{1}$$

over $R_{\mathfrak{p}}$, where $\frac{A}{1} \in R_{\mathfrak{p}}^{m \times n}$ and $\frac{b}{1} \in R_{\mathfrak{p}}^{1 \times n}$ for $m, n \in \mathbb{N}_0$, we can find a solution (or disprove its existence) as follows:

1. Find a solution $L \in R^{o \times (m+1)}$ of the syzygy problem

$$X \cdot \begin{pmatrix} b \\ A \end{pmatrix} = 0.$$

   This can be done with Gröbner basis techniques ([5, Algorithm 2.5.4]), e.g., by computing a Gröbner basis of the rows of

$$\left(\begin{array}{c|c} b & \multirow{2}{*}{I_{m+1}} \\ A & \end{array} \right)$$

   with a monomial ordering giving priority to the components in the left block.
2. For $i = 1, \dots, o$, let $\begin{pmatrix} r_i & L_i \end{pmatrix}$ denote the i-th row of L, where $r_i \in R$, $L_i \in R^{1 \times m}$. We check if $r_i \in \mathfrak{p}$ with an algorithm[6] for ideal membership [5, Section 1.8.1]. The first i such that $r_i \notin \mathfrak{p}$ gives us the desired solution

$$\left(-\frac{L_i}{r_i} \right) \cdot \frac{A}{1} = \frac{b}{1}.$$

[6] Such an algorithm needs a Gröbner basis of $\mathfrak{p}$. So, if we need to solve many different linear systems over the same ring $R_{\mathfrak{p}}$, then the determination of such a Gröbner basis can be seen as a preprocessing step.

If there is no such i, then we successfully disproved the existence of a solution.

The author believes that it is worth to implement Construction 4.3 for two reasons: first, it covers localizations at all prime ideals (with given finite set of generators) and therefore generalizes the work of Barakat and Lange-Hegermann [2], in which computability of a ring is sufficient to provide a whole framework for effective homological algebra. Second, it is not a priori clear how Construction 4.3 performs in practice compared to the algorithm described in [2, Proposition 4.5]:

Proposition 4.4. (Barakat, Lange-Hegermann) *Let R be a computable ring with a maximal ideal $\mathfrak{m} = \langle m_1, \dots, m_\ell \rangle$ for $\ell \in \mathbb{N}_0$. A linear system*

$$X \cdot \frac{A}{1} + \frac{b}{1} = 0$$

over $R_{\mathfrak{m}}$, where $\frac{A}{1} \in R_{\mathfrak{m}}^{m \times n}$ and $\frac{b}{1} \in R_{\mathfrak{m}}^{1 \times n}$ for $m, n \in \mathbb{N}_0$, has a solution if and only if the following linear system over R has a solution:

$$X \cdot \begin{pmatrix} A \\ (m_1, \dots, m_\ell)^{\mathrm{tr}} \cdot b \end{pmatrix} + b = 0.$$

In the case $R = k[x_1, \dots, x_n]$ the costly part in solving this linear system involves the computation of a Gröbner basis of the given stacked matrix, whereas the costly part in Construction 4.3 is the computation of syzygies. Testing Construction 4.3 for the applications described in [2] would be an interesting project.

We end this section with two more examples.

Example 4.5. Let R be a computable commutative ring, $m \in \mathbb{N}_0$, and $L = \langle h_1, \dots, h_m \rangle_R \subseteq R$ an ideal. Then $S := 1 + L$ is a multiplicatively closed set, and $S^{-1}R$ the Zariskification (see Section 5 in [9]) of R at L. For any finitely generated ideal $I = \langle f_1, \dots, f_\ell \rangle_R \subseteq R$ with $\ell \in \mathbb{N}_0$, we have

$$\exists r \in S \cap I \Longleftrightarrow \exists r_1, \dots, r_m, r'_1, \dots, r'_\ell \in R : 1 = \sum_{i=1}^{m} r_i h_i + \sum_{i=1}^{\ell} r'_i f_i.$$

Since R is computable, we can effectively solve this equation. Thus, Theorem 3.9 implies computability of the Zariskification.

Example 4.6. Let R be a computable commutative ring. For a polynomial $p = \sum_{i=0}^{d} a_i t^i \in R[t]$ where $d \in \mathbb{N}_0$, $a_i \in R$ for $i = 0, \dots d$, and $a_d \neq 0$ we define the leading term as $\mathrm{LT}(p) = a_d t^d$. Such a polynomial is called monic if $a_d = 1$. Now, consider the multiplicatively closed subset

$$S := \{p \in R[t] \mid p \text{ is monic}\} \subseteq R[t]$$

and the localization $R(t) := S^{-1}R[t]$. Let $I \subseteq R[t]$ be a finitely generated ideal with standard basis $G = \{g_1, \dots, g_n\} \subseteq I$ for $n \in \mathbb{N}_0$, i.e.,

$$\langle \mathrm{LT}(g) \mid g \in G \rangle_{R[t]} = \langle \mathrm{LT}(f) \mid f \in I \rangle_{R[t]} =: \mathrm{LT}(I).$$

We write $\mathrm{LT}(g_i) = c_i \cdot t^{d_i}$ for $c_i \in R$ and $d_i \in \mathbb{N}_0$. Then

$$\begin{aligned}\exists r \in I \cap S &\iff \exists m \in \mathbb{N}_0 : t^m \in \mathrm{LT}(I)\\ &\iff \exists m \in \mathbb{N}_0\ \exists a_i \in R : \sum_{i \in \{1 \dots n \mid m-d_i \geq 0\}} (a_i t^{m-d_i})(c_i t^{d_i}) = t^m\\ &\iff \exists a_i \in R : \sum_{i=1}^{n} a_i c_i = 1.\end{aligned}$$

Since R is computable, we can solve this last equation and if $(a_1, \dots, a_n)$ is such a solution, then $\left(\sum_{i=1}^n a_i t^{M-d_i} g_i\right) \in S \cap I$ where $M \geq \max\{d_1, \dots, d_n\}$. Thus, whenever we can compute a standard basis of finitely generated ideals in $R[t]$ (e.g., if $R = \mathbb{Z}$), then $R(t)$ is a computable ring. See [1, Chapter 4] for details on Gröbner bases over polynomial rings with coefficients in a commutative noetherian ring.

5. Outlook. Computations within the algorithmic model of the abelian category of finitely presented modules $S^{-1}R$-fpmod over a localized ring $S^{-1}R$ as implemented in the `homalg`-project are capable of outperforming equivalent methods based on Mora's algorithm (see [2, Section 6]). However, the implementation in `homalg` is limited to the case of localizations of computable rings R at maximal ideals $\mathfrak{m}$. The methods described in this paper make it possible to model the categories $S^{-1}R$-fpmod on the computer for rings beyond $R_\mathfrak{m}$. Their implementation is planned within Cap ([6]), a software project facilitating the implementation of category theory-based constructions. For example, $S^{-1}R$-fpmod can be categorically constructed as the so-called Freyd category of row modules over $S^{-1}R$ (see [10]).

Yet another drawback of the implementation of $R_\mathfrak{m}$-fpmod in `homalg` is the dependency on Mora's algorithm for the computation of Hilbert series (see [2, Remark 4.8]). However, at least for modules of finite length, as they appear for example in computations of intersection multiplicities, we can get rid of this dependency by using a purely categorical description of the filtration of a module induced by $\mathfrak{m}$.

The idea of a category theory-based alternative approach to localization in computer algebra can even be taken one step further: instead of localizing the ring R, we can localize the whole category R-fpmod in the sense of Serre quotients. This localization is again algorithmic ([3]) and provides a framework for the category of coherent sheaves on quasi-affine schemes, a proper generalization of $S^{-1}R$-fpmod.

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