

# Fixpoint operators for 2-categorical structures

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**Abstract**—Fixpoint operators are tools to reason on recursive programs and data types obtained by induction (e.g. lists, trees) or coinduction (e.g. streams). They were given a categorical treatment with the notion of categories with fixpoints. A theorem by Plotkin and Simpson characterizes existence and uniqueness of fixpoint operators for categories satisfying some conditions on bifree algebras and recovers the standard examples of the category  $\mathbf{Cppo}$  ( $\omega$ -complete pointed partial orders and continuous functions) in domain theory and the relational model in linear logic.

We present a categorification of this result and develop the theory of 2-categorical fixpoint operators where the 2-dimensional framework allows to model the execution steps for languages with (co)recursive principles. We recover the standard categorical constructions of initial algebras and final coalgebras for endofunctors as well as **fixpoints of generalized species and polynomial functors**.

## INTRODUCTION

Fixpoints operators play an important role to model infinite computation in a wide range of computer science disciplines: design of programming languages, verification, model checking, databases, concurrency theory, type theory, etc. A fixpoint for a program  $t$  is an input  $x$  such that there is a calculation sequence between  $t(x)$  and  $x$ . For example, a fundamental property of untyped  $\lambda$ -calculus is the existence of fixpoint combinators *i.e.* terms  $Y$  such that for any  $\lambda$ -term  $t$ , there is a reduction path connecting the terms  $t(Yt)$  and  $Yt$ .

From a set-theoretic viewpoint, the standard notion of fixpoint for an endomap  $f : A \rightarrow A$  is an element  $x \in A$  such that  $f(x) = x$  and it was axiomatized with the notion of categories with fixpoint operators [1]. There is however no notion of categorical fixpoint operator taking into account the computational reduction steps and not collapsing them into strict equalities.

The objective of this paper is to work in a 2-dimensional framework to model explicitly the reductions of languages with fixpoints and study their coherences *i.e.* the equations satisfied by the program computations steps. It fits into the line of research of *categorifying* models of computation by replacing semantics where types are sets or preorders with richer categorical structures to establish stronger mathematical invariants.

Categories in dimension one have objects and morphisms that can be composed. We can consider an additional dimension with the notion of 2-categories or bicategories which have objects, 1-morphisms that can be composed and 2-morphisms that can be composed in two different ways that verify compatibility conditions. These 2-morphisms are thought of as morphisms between 1-morphisms. When using bidimensional categorical structures to model computations, we can

study program execution steps as primitive objects as they become explicit 2-morphisms carrying information on program reductions. It has seen many applications in concurrency, game semantics, type theory, higher dimensional rewriting [2], [3], [4], [5], [6], [7], [8], [9], [10].

When generalizing preorder semantics to richer categorical semantics, least fixpoints become initial algebras and greatest fixpoints become final coalgebras. In both cases, strict equalities for fixpoints  $t(x) = x$  are now represented by explicit isomorphisms

$$t(x) \xrightarrow{\text{algebra}} x \quad x \xrightarrow{\text{coalgebra}} t(x)$$

capturing the dynamic aspect of fixpoint reductions. Initial algebras correspond to recursive definition with induction as a logical reasoning principle. They are typically used to model finite data types (such as finite lists and trees) with a constructor operation.

Dually, final coalgebras are the counterpart of corecursion and coinduction where structures are described with a destructor or observer operation. They are now widely used in computer science to model state-based systems with circular or non-terminating behavior (automata, transitions systems, network dynamics etc.). The categorical coalgebraic framework is also a standard tool to study notions such as bisimilarity and trace equivalence. It has been used in functional programming to model lazy datatypes in languages such as Haskell or reactive programs in proof assistants such as Coq and Agda.

Settings where initial algebra and final coalgebras coincide because of limit-colimit coincidence arguments have also been studied to solve equations with mixed variance variables (such as reflexive objects  $D \cong D \Rightarrow D$  for  $\lambda$ -calculus models [11]). These algebras are called *bifree* and they provide a framework where inductive and coinductive arguments are equivalent. An important result by Plotkin and Simpson in this area states that provided some conditions on bifree algebras are satisfied, we obtain the existence of a unique uniform fixpoint operator for 1-categories [1].

While using initial algebra or final coalgebra semantics to model infinite computations are now well-established traditions in computer science, there is no counterpart axiomatization of fixpoint operators for 2-categorical structures and the goal of this paper is develop their theory. In order to axiomatize the notion of 2-dimensional fixpoint operator, the main difficulty is to understand what axioms and coherences the rewriting 2-morphisms should satisfy. We both need to ensure that the equations are correct, *i.e.* that they hold in concrete models, and we also need completeness properties, *i.e.* we want to ensure that we have stated them all.

We proceed in two steps: we first categorify the restricted case of the Plotkin-Simpson theorem characterizing the existence of a unique uniform fixpoint operator for 1-categories to extract an axiomatization of pseudo-fixpoint operator for 2-categories. These operators form a category and uniqueness of the fixpoint operator is replaced by a contractibility property: there is a unique isomorphism between any two fixpoint operators. The second step is to generalize to models where fixpoints are not unique to verify that our axiomatization holds. Our motivation for choosing this approach is that the Plotkin-Simpson proof constructs explicitly a canonical fixpoint operator using the bifree algebras and then uses the fixpoint axioms to show that it is in fact unique. In the 2-dimensional case, the pseudo-bifree algebras allow us to construct a canonical pseudo-fixpoint operator and in order to obtain a contractibility property, we need to impose certain coherence equations on the structural reduction 2-morphisms which provides a guideline for the axiomatization of general pseudo-fixpoint operators.

#### Related works

Categorification of recursive domain theory has an established history with the work of Lambek, Freyd, Lehmann, Adamek, Taylor, Fiore, Winskel and Cattani [12], [13], [14], [15], [16], [17], [18]. In this paper, we mostly use results by Cattani, Fiore and Winskel generalizing the notion of algebraically compact categories for enriched categories to enriched bicategories and proving limit-colimit coincidence theorems in this setting with applications to presheaf models of concurrency [17], [18]. From a syntactic viewpoint, Pitts presented a candidate 2-dimensional type theory for fixpoints which can serve to prove coherence theorems for our notion of pseudo-fixpoint operators [19].

Ponto and Shulman have also studied a categorification of the notion of fixpoint and trace for bicategories in a different direction [20], [21]. They consider the trace of endo-2-cells  $\alpha : f \Rightarrow f$  for a 1-cell  $f : A \rightarrow B$ ; whereas in our case we still want to compute the trace or fixpoint of endo-1-cells but up to explicit rewriting 2-cells. We aim to investigate whether the two approaches can be compared for the cartesian closed setting in future work.

#### Plan of the paper

- We start in Section I by recalling the standard theory of 1-categorical fixpoint operators and give examples from domain theory and linear logic that can be recovered by the Plotkin-Simpson theorem.
- In Section II, we state the definitions of pseudo-fixpoint operators for 2-categories with uniformity and dinaturality axioms.
- We prove in Section III a generalization of the Plotkin-Simpson theorem for 2-categories where pseudo-fixpoint operators now form a category and uniqueness of the fixpoint operator is replaced by a contractibility property.

- We show in Section V how the notion of 2-categorical fixpoint we developed in Section II is verified in well-known 2-categorical models.

### I. FIXPOINT OPERATORS FOR 1-CATEGORIES

**Definition I.1.** Let  $\mathbb{D}$  be a category with a terminal object  $1$ , a *fixpoint operator* on  $\mathbb{D}$  is a family of functions  $(-)^* : \mathbb{D}(A, A) \rightarrow \mathbb{D}(1, A)$  indexed by the objects  $A$  of  $\mathbb{D}$  verifying that for all morphisms  $f : A \rightarrow A$ ,

$$f^* = f \circ f^*. \quad (\text{fix})$$

We can further require additional axioms such as uniformity and dinaturality which are usually satisfied in concrete models and can serve to characterize uniqueness properties of fixpoint operators [22]. We can also consider parametrized axioms giving the connection with traced monoidal categories [23] but we leave the parametrized case for an accompanying paper.

For a category with fixpoints, the uniformity principle (also called Plotkin's axiom) is relative to a subcategory of "strict maps" and is used to characterize the fixpoint operator uniquely without relying on order-theoretic arguments [24], [25]. For domain-like structures, strict maps are usually the ones which preserve bottom elements  $\perp$  whereas general maps are just assumed to be Scott-continuous. Another possibility is to consider linear maps for strict maps, *i.e.* maps which commute with all suprema not just directed ones and these are the typical examples in linear logic models with fixpoints. Freyd has also considered the case of a reflective subcategory for the subcategory of strict maps [26].

**Definition I.2.** Let  $\mathbb{C}, \mathbb{D}$  be categories with terminal objects and  $J : \mathbb{C} \rightarrow \mathbb{D}$  be an identity-on-objects functor preserving terminal objects.

- A fixpoint operator  $(-)^*$  on  $\mathbb{D}$  is said to be *uniform with respect to  $J$*  if for every  $s : A \rightarrow B$  in  $\mathbb{C}$  and  $f : A \rightarrow A, g : B \rightarrow B$  in  $\mathbb{D}$ , we have:

$$J(s) \circ f = g \circ J(s) \quad \text{implies} \quad J(s) \circ f^* = g^*. \quad (\text{unif})$$

- A fixpoint operator  $(-)^*$  on  $\mathbb{D}$  is *dinatural* if for every  $f : A \rightarrow B$  and  $g : B \rightarrow A$  in  $\mathbb{D}$ ,

$$(f \circ g)^* = f \circ (g \circ f)^*. \quad (\text{dinat})$$

*Remark 1.* We can in fact consider bijective-on-objects functors for  $J : \mathbb{C} \rightarrow \mathbb{D}$  but we restrict to identity-on-objects functors to make the notation less cumbersome.

As mentioned in the introduction, our approach is to use a categorification of the Plotkin-Simpson theorem which characterizes existence and uniqueness of fixpoint operators for 1-categories that are obtained as the Kleisli of a comonad satisfying some conditions on bifree algebras [1].

Recall that for an endofunctor  $T : \mathbb{D} \rightarrow \mathbb{D}$ , a *bifree  $T$ -algebra* (also called dialgebra, compact algebra or free bialgebra) is an initial  $T$ -algebra  $(A, a : TA \rightarrow A)$  such that the inverse of  $a$  is a final  $T$ -coalgebra  $(A, a^{-1} : A \rightarrow TA)$ .

**Theorem I.3** (Plotkin-Simpson [1]). *Let  $\mathbb{C}$  be a category equipped with a comonad  $(T, \delta, \varepsilon)$  and a terminal object. We denote by  $\mathbb{D}$  the co-Kleisli category  $\mathbb{C}_T$  and by  $J : \mathbb{C} \rightarrow \mathbb{D}$  the free functor induced by the comonadic adjunction.*

- 1) *If the endofunctor  $T$  has a bifree algebra, then  $\mathbb{D}$  has a unique uniform (with respect to  $J$ ) fixpoint operator.*
- 2) *If  $\mathbb{C}$  is cartesian and the endofunctor  $T \circ T$  has a bifree algebra, then  $\mathbb{D}$  has a unique uniform (with respect to  $J$ ) dinatural fixpoint operator.*

We proceed to give the general recipe to obtain bifree algebras for endofunctors in a suitable preorder-enriched setting in order to motivate the generalizations to dimension 2 in Section V.

We denote by  $\mathbf{Cpo}$  the category whose objects are  $\omega$ -complete partial orders and morphisms are Scott-continuous functions (monotone maps preserving colimits of  $\omega$ -chains). If we restrict the objects to *pointed* cpo's (with a bottom element  $\perp$ ), we denote this subcategory  $\mathbf{Cppo}$ . We can further restrict the morphisms to Scott-continuous functions that are *strict* (preserving bottom elements) and we denote by  $\mathbf{Cppo}_\perp$  the category of pointed cpo's and strict Scott-continuous functions. We are interested in  $\mathbf{Cppo}_\perp$ -enriched categories since they provide a well-behaved setting to compute bifree algebras as we will see below. Explicitly, a category  $\mathbb{C}$  is  $\mathbf{Cppo}_\perp$ -enriched if each homset  $\mathbb{C}(A, B)$  is a cpo with a bottom element  $\perp$  and composition  $\mathbb{C}(A, B) \times \mathbb{C}(B, C) \rightarrow \mathbb{C}(A, C)$  is Scott-continuous and strict in both components. Examples of  $\mathbf{Cppo}_\perp$ -enriched categories include the category  $\mathbf{Cppo}_\perp$  which is enriched over itself and the category  $\mathbf{Rel}$  whose objects are sets and morphisms are binary relations. Another example is the category  $\mathbf{Lin}$  of preorders and ideal relations (binary relations  $R \subseteq A \times B$  between preorders that are up-closed in  $A$  and down-closed in  $B$ ). The category  $\mathbf{Rel}$  can be viewed as the subcategory of  $\mathbf{Lin}$  containing discrete preorders.

The following theorem is a consequence of the limit-colimit coincidence theorem by Smyth and Plotkin [27] which was generalized by Fiore [16]:

**Theorem I.4.** *Let  $\mathbb{C}$  be a  $\mathbf{Cppo}_\perp$ -enriched category. If  $\mathbb{C}$  has an initial object and  $\omega$ -colimits of chains of embeddings, then  $\mathbb{C}$  is  $\mathbf{Cpo}$ -algebraically compact. It means that for every endofunctor  $T : \mathbb{C} \rightarrow \mathbb{C}$ , if  $T$  is  $\mathbf{Cpo}$ -enriched, i.e. for all  $A, B$ , the induced map*

$$\mathbb{C}(A, B) \longrightarrow \mathbb{C}(TA, TB)$$

*preserves colimits of  $\omega$ -chains, then  $T$  has a bifree algebra.*

We proceed to give examples of well-known fixpoint operators that can be recovered from the Plotkin-Simpson theorem. Consider the lifting comonad  $(-)_\perp$  on  $\mathbf{Cppo}_\perp$ : this comonad freely adjoins a bottom element and its co-Kleisli category is isomorphic to  $\mathbf{Cppo}$ . Since the endofunctors  $(-)_\perp$  and  $(-)_\perp \circ (-)_\perp$  are  $\mathbf{Cppo}$ -enriched, the category  $\mathbf{Cppo}$  has a unique dinatural fixpoint operator uniform with respect to the

free functor  $\mathbf{Cppo}_\perp \rightarrow \mathbf{Cppo}$  and it is given by the standard formula

$$f^* = \bigvee_{n \in \omega} f^n(\perp)$$

for  $f : A \rightarrow A$  in  $\mathbf{Cppo}$ . We can also consider the finite multiset comonad  $\mathcal{M}_{\text{fin}}$  on the category  $\mathbf{Rel}$ . It leads to a *quantitative* model of linear logic where multisets allow to count multiplicities of the inputs for a term [28]. As  $\mathcal{M}_{\text{fin}}$  and  $\mathcal{M}_{\text{fin}}\mathcal{M}_{\text{fin}}$  have bifree algebras, we recover the fixpoint operator in the relational model [29] and obtain that it is unique.

Lastly, the category  $\mathbf{Lin}$  (also called the linear Scott category **ScottL**) can be equipped with the  $\vee$ -semi-lattice comonad yielding a *qualitative* model of linear logic where substitution allows for duplication and erasure. This comonad also verifies the necessary enrichment conditions to obtain a unique uniform dinatural fixpoint operator.

## II. BIDIMENSIONAL FIXPOINTS OPERATORS

We state in this section the definition of fixpoint operators for 2-categories. We will show in the remaining sections how the axioms we present arise from the generalization of the Plotkin-Simpson construction in dimension 2 and how they are verified in concrete examples. When moving to a higher dimension, we have several possibilities depending on the degrees of strictness and direction of the 2-morphisms (strict, pseudo, (op)lax).

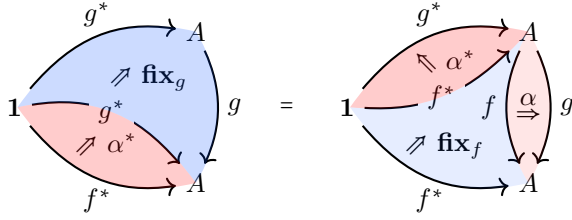
In this paper, for space considerations, we will focus on one case: *pseudo-fixpoint operators for 2-categories* and leave the remaining cases and the bicategorical weakening for the long version. Even if we only consider the pseudo case where the 2-cells are invertible, we will still provide a direction for the arrows to give a better understanding of where they come from and provide a guideline for the directed lax and oplax cases.

**Definition II.1.** Let  $\mathcal{D}$  be a 2-category with a terminal object  $1$ . A *pseudo-fixpoint operator* on  $\mathcal{D}$  consists of a family of functors indexed by the objects  $A$  of  $\mathcal{D}$ :

$$(-)_A^* : \mathcal{D}(A, A) \rightarrow \mathcal{D}(1, A)$$

together with a family of natural isomorphisms  $\text{fix}_A$  (we will omit the subscripts for objects to simplify the notation) with components:

for a 1-cell  $f : A \rightarrow A$  in  $\mathcal{D}$ . Naturality means that for an invertible 2-cell  $\alpha : f \Rightarrow g$  in  $\mathcal{D}$ , we have:



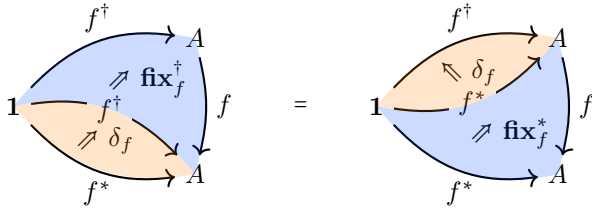
*Remark 2.* Strictly speaking, our pseudo-fixpoint operators act on the sub-2-category of  $\mathcal{D}$  where we only consider invertible 2-cells. Indeed, for a general 2-cell  $\alpha : f \Rightarrow g$ ,  $\alpha^*$  is not defined unless  $\alpha$  is invertible and we need to consider (op)lax-fixpoint operators to act on non-invertible 2-cells. In order to lighten the notation, for a 2-category  $\mathcal{D}$ , we implicitly use the same notation for  $\mathcal{D}$  and its sub-2-category containing only invertible 2-cells for the rest of the paper.

In the preordered case, we can compare fixpoint operators pointwise and we are usually interested in the least and greatest fixpoints. In the categorified setting, fixpoint operators form a category so there can be more than one way of comparing two fixpoint operators. Morphisms of pseudo-fixpoint operators are transformations that commute with the structural 2-cells  $\text{fix}$ . Initial objects in this category correspond to least fixpoints while terminal objects correspond to greatest fixpoints. The uniqueness property for fixpoint operators in the preorder setting now becomes a contractibility property for the category of fixpoint operators. A category is *contractible* when it is not empty and for any two objects, there is a unique isomorphism between them (in particular, it is a groupoid).

**Definition II.2.** Let  $((-)^*, \text{fix}^*)$  and  $((-)^{\dagger}, \text{fix}^{\dagger})$  be two pseudo-fixpoint operators on a 2-category  $\mathcal{D}$ . A pseudo-morphism of pseudo-fixpoint operators  $((-)^*, \text{fix}^*) \rightarrow ((-)^{\dagger}, \text{fix}^{\dagger})$  consists of a family of natural isomorphisms

$$\delta_A : (-)_A^* \Rightarrow (-)_A^{\dagger} : \mathcal{D}(A, A) \rightarrow \mathcal{D}(1, A)$$

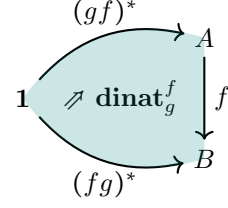
indexed by the objects  $A$  of  $\mathcal{D}$  that commutes with the structural 2-cells  $\text{fix}$ , i.e. it satisfies the following coherence for every  $f : A \rightarrow A$ :



We denote by  $\mathbf{Fix}(\mathcal{D})$  the category of pseudo-fixpoint operators on  $\mathcal{D}$ .

Dinatural transformations are used to model mixed variance operators using their argument both covariantly and contravariantly. The typical examples arising from semantics occur with (cartesian or monoidal) closed structure where the evaluation map is dinatural and with fixpoint operators. In our setting, we use the notion of pseudo-dinatural transformation for 2-functors (see Appendix A1).

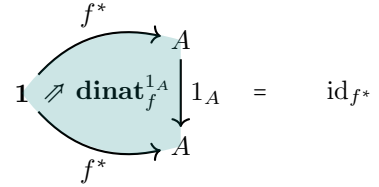
**Definition II.3.** A pseudo-dinatural fixpoint operator on  $\mathcal{D}$  consists a pseudo-fixpoint operator  $((-)^*, \text{fix})$  on  $\mathcal{D}$  together with a family of invertible 2-cells



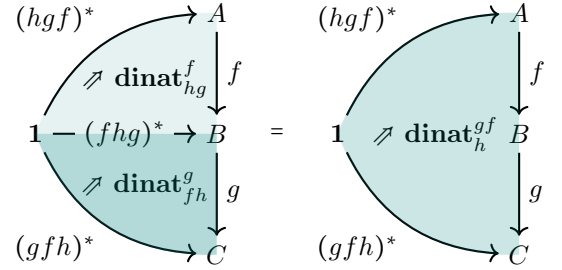
for 1-cells  $f : A \rightarrow B$  and  $g : B \rightarrow A$  in  $\mathcal{D}$  satisfying the following axioms:

1) **pseudo-dinaturality axioms:**

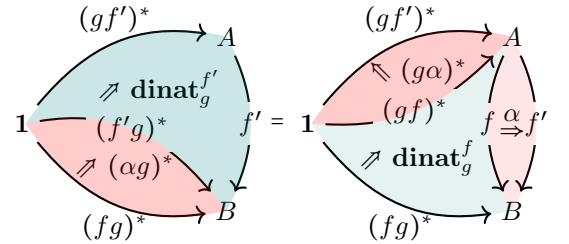
a) Unity axiom: for a 1-cell  $f : A \rightarrow A$ , we have



b) 1-naturality axiom: for all 1-cells  $f : A \rightarrow B$ ,  $g : B \rightarrow C$  and  $h : C \rightarrow A$  in  $\mathcal{D}$ :



c) 2-naturality axiom: for an invertible 2-cell  $\alpha : f \Rightarrow f' : A \rightarrow B$  in  $\mathcal{D}$  and a 1-cell  $g : B \rightarrow A$  in  $\mathcal{D}$ , we have



these three axioms induce a pseudo-dinatural transformation:

$$\text{dinat} : \mathcal{D}(-, -) \rightrightarrows \mathcal{D}(1, -) : \mathcal{D}^{\text{op}} \times \mathcal{D} \rightarrow \mathbf{CAT}$$

2) **coherence between dinat and fix:** for all 1-cells  $f : A \rightarrow B$  and  $g : B \rightarrow A$  in  $\mathcal{D}$ , we have:

$$\begin{array}{ccc}
\begin{array}{c} (f \circ g)^* \rightarrow B \\ \nearrow \text{dinat}_f^g \\ 1 \xrightarrow{(g \circ f)^*} A \\ \nwarrow \text{dinat}_g^f \\ (f \circ g)^* \rightarrow B \end{array} & = & \begin{array}{c} (f \circ g)^* \rightarrow B \\ \nearrow \text{fix}_{fg} \\ 1 \xrightarrow{(g \circ f)^*} A \\ \nwarrow \text{fix}_{fg} \\ (f \circ g)^* \rightarrow B \end{array}
\end{array}$$

**Remark 3.** Note that the coherence axiom between **dinat** and **fix** above implies that

$$\begin{array}{ccc}
\begin{array}{c} f^* \rightarrow A \\ \nearrow \text{dinat}_{1_A}^f \\ 1 \xrightarrow{(f \circ f)^*} A \\ \nwarrow \text{dinat}_{1_A}^f \\ f^* \rightarrow A \end{array} & = & \begin{array}{c} f^* \rightarrow A \\ \nearrow \text{fix}_f \\ 1 \xrightarrow{(f \circ f)^*} A \\ \nwarrow \text{fix}_f \\ f^* \rightarrow A \end{array}
\end{array}$$

since  $\text{dinat}_{1_A}^f = \text{id}_{f^*}$ . In the 1-categorical setting, if the axiom for dinaturality  $f \circ (g \circ f)^* = (f \circ g)^*$  holds, the fixpoint axiom  $f \circ f^* = f^*$  becomes redundant (it suffices to take  $g = \text{id}_A$ ). In the 2-dimensional case, the 2-cells **fix** are entirely determined by the 2-cells **dinat** and we therefore just write  $((-)^*, \text{dinat})$  for a pseudo-dinatural fixpoint operator instead of  $((-)^*, \text{fix}, \text{dinat})$ .

**Definition II.4.** Let  $((-)^*, \text{dinat}^*)$  and  $((-)^{\dagger}, \text{dinat}^{\dagger})$  be two pseudo-dinatural fixpoint operators on a 2-category  $\mathcal{D}$ . A pseudo-morphism of pseudo-dinatural fixpoint operators  $((-)^*, \text{dinat}^*) \rightarrow ((-)^{\dagger}, \text{dinat}^{\dagger})$  consists of a family of natural isomorphisms

$$\delta_A : (-)_A^* \Rightarrow (-)_A^{\dagger} : \mathcal{D}(A, A) \rightarrow \mathcal{D}(1, A)$$

indexed by the objects  $A$  of  $\mathcal{D}$  that commutes with the structural 2-cells **dinat**, i.e. it satisfies the following coherence for every  $f : A \rightarrow B$  and  $g : B \rightarrow A$ :

$$\begin{array}{ccc}
\begin{array}{c} (gf)^{\dagger} \rightarrow A \\ \nearrow \text{dinat}_g^{f^{\dagger}} \\ 1 \xrightarrow{(fg)^{\dagger}} B \\ \nwarrow \delta_{fg} \\ (fg)^* \rightarrow B \end{array} & = & \begin{array}{c} (gf)^{\dagger} \rightarrow A \\ \nearrow \delta_{gf} \\ 1 \xrightarrow{(gf)^*} A \\ \nwarrow \text{dinat}_g^{f^*} \\ (fg)^* \rightarrow B \end{array}
\end{array}$$

We denote by  $\mathbf{DinFix}(\mathcal{D})$  the category of pseudo-dinatural fixpoint operators on  $\mathcal{D}$ .

Dinatural transformations do not compose in general and therefore they do not form a category. In order to make the notion compositional, *strong dinatural transformations* were introduced. It was noted by Mulry that the uniformity axiom for fixpoints (Definition I.2) can be reformulated by requiring  $(-)^*$  to be part of a strong dinatural transformation with respect to strict maps [30] and we use this characterization

when moving to dimension 2. While generalizations of dinatural transformations for 2-categorical structures have been considered before [31], to our knowledge, there is no existing notion of strong dinatural transformations in dimension 2 (see Appendix A2).

**Definition II.5.** Let  $J : \mathcal{C} \rightarrow \mathcal{D}$  be an identity-on-objects 2-functor (strictly) preserving terminal objects. A pseudo-fixpoint operator on  $\mathcal{D}$  that is *uniform with respect to  $J$*  consists of a pseudo-fixpoint operator  $((-)^*, \text{fix})$  as in Definition II.1 together with a family of 2-cells

$$\begin{array}{c} f^* \rightarrow A \\ \nearrow \text{unif}_{\gamma} \\ 1 \xrightarrow{(g \circ f)^*} A \\ \nwarrow \text{unif}_{\gamma} \\ f^* \rightarrow B \end{array}$$

for every 1-cells  $s : A \rightarrow B$  in  $\mathcal{C}$ ,  $f : A \rightarrow A$  and  $g : B \rightarrow B$  in  $\mathcal{D}$  and invertible 2-cell  $\gamma$  as below:

$$\begin{array}{ccc} A & \xrightarrow{f} & A \\ Js \downarrow & \Downarrow \gamma & \downarrow Js \\ B & \xrightarrow{g} & B \end{array}$$

satisfying the following axioms:

1) **strong pseudo-dinaturality:**

a) Unity axiom: we have

$$\begin{array}{ccc} \begin{array}{c} f^* \rightarrow A \\ \nearrow \text{unif}_{\text{id}_f} \\ 1 \xrightarrow{(f \circ f)^*} A \\ \nwarrow \text{unif}_{\text{id}_f} \\ f^* \rightarrow A \end{array} & = & \text{id}_{f^*} \end{array}$$

b) 1-naturality: for two squares in  $\mathcal{D}$

$$\begin{array}{ccc} A & \xrightarrow{f} & A \\ Js \downarrow & \Downarrow \gamma & \downarrow Js \\ B & \xrightarrow{g} & B \end{array} \quad \begin{array}{ccc} B & \xrightarrow{g} & B \\ Jr \downarrow & \Downarrow \rho & \downarrow Jr \\ C & \xrightarrow{h} & C \end{array}$$

we have:

$$\begin{array}{ccc} \begin{array}{c} f^* \rightarrow A \\ \nearrow \text{unif}_{\gamma^* \nu \rho} \\ 1 \xrightarrow{(g \circ f)^*} B \\ \nwarrow \text{unif}_{\gamma^* \nu \rho} \\ h^* \rightarrow C \end{array} & = & \begin{array}{c} f^* \rightarrow A \\ \nearrow \text{unif}_{\gamma} \\ 1 \xrightarrow{g^*} B \\ \nwarrow \text{unif}_{\rho} \\ h^* \rightarrow C \end{array} \end{array}$$

where  $\gamma^* \nu \rho$  denotes the 2-cell corresponding to stacking the two squares vertically as follows:

$$\begin{array}{ccc}
\begin{array}{ccc} \Phi & \xrightarrow{f} & \Phi \\ Jrs \downarrow & \not\parallel \gamma * \rho & \downarrow Jrs \\ C & \xrightarrow{h} & C \end{array} & := & \begin{array}{ccc} A & \xrightarrow{f} & A \\ Js \downarrow & \not\parallel \gamma & \downarrow Js \\ B & \xrightarrow{g} & B \\ Jr \downarrow & \not\parallel \rho & \downarrow Jr \\ C & \xrightarrow{h} & C \end{array}
\end{array}$$

$$\begin{array}{ccc}
\begin{array}{ccc} & g^* & \\ 1 & \nearrow \text{fix}_g & B \\ f^* \nearrow & g^* & \searrow g \\ A & \circlearrowleft J_s & B \end{array} & = & \begin{array}{ccc} & g^* & \\ 1 & \nearrow \text{unif}_\gamma & B \\ f^* \nearrow & f^* \rightarrow A \circlearrowleft Js & \searrow g \\ A & \circlearrowleft Js & B \end{array}
\end{array}$$

c) 2-naturality: for every invertible 2-cell  $\theta : s \Rightarrow r$  in  $\mathcal{C}$  such that

$$\begin{array}{ccc}
\begin{array}{ccc} A & \xrightarrow{f} & A \\ Jr \downarrow & \not\parallel J\theta & \downarrow Js \\ B & \xrightarrow{g} & B \end{array} & = & \begin{array}{ccc} A & \xrightarrow{f} & A \\ Jr \downarrow & \not\parallel \rho & \downarrow Jr \\ B & \xrightarrow{g} & B \end{array}
\end{array}$$

we have

$$\begin{array}{ccc}
\begin{array}{ccc} f^* & & \\ 1 & \nearrow \text{unif}_\gamma & Js \\ g^* & & \end{array} & = & \begin{array}{ccc} f^* & & \\ 1 & \nearrow \text{unif}_\rho & Jr \circlearrowleft J\theta Js \\ g^* & & \end{array}
\end{array}$$

d) If the following equality holds,

$$\begin{array}{ccc}
\begin{array}{ccc} A & \xrightarrow{f} & A \\ Js \downarrow & \not\parallel \alpha & \downarrow Js \\ B & \xrightarrow{h} & B \\ & \not\parallel \rho & \\ & \xrightarrow{k} & \end{array} & = & \begin{array}{ccc} A & \xrightarrow{f} & A \\ Js \downarrow & \not\parallel \gamma & \downarrow Js \\ B & \xrightarrow{g} & B \\ & \not\parallel \beta & \\ & \xrightarrow{k} & \end{array}
\end{array}$$

then we have:

$$\begin{array}{ccc}
\begin{array}{ccc} f^* & & \\ 1 & \nearrow \alpha^* & A \\ h^* \nearrow & \text{unif}_\rho & \searrow \rho \\ k^* & & B \end{array} & = & \begin{array}{ccc} f^* & & \\ 1 & \nearrow g^* & A \\ g^* \nearrow & \text{unif}_\gamma & \searrow \beta^* \\ k^* & & B \end{array}
\end{array}$$

These four axioms induce a strong pseudo-dinatural transformation:

$$\text{unif} : \mathcal{D}(J(-), J(-)) \rightrightarrows \mathcal{D}(1, J(-)) : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathbf{CAT}$$

2) **coherence between fix and unif**: for an invertible 2-cell as below

$$\begin{array}{ccc}
A & \xrightarrow{f} & A \\ J(s) \downarrow & \not\parallel \gamma & \downarrow J(s) \\ B & \xrightarrow{g} & B
\end{array}$$

we have:

$$\rho * \gamma = \begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & A \\ Js \downarrow & \not\parallel \gamma & Jr \downarrow & \not\parallel \rho & \downarrow Js \\ C & \xrightarrow{h} & D & \xrightarrow{k} & C \end{array}$$

we have:

**Definition II.6.** Let  $((-)^*, \text{fix}^*, \text{unif}^*)$  and  $((-)^{\dagger}, \text{fix}^{\dagger}, \text{unif}^{\dagger})$  be two pseudo-fixpoint operators uniform with respect to  $J : \mathcal{C} \rightarrow \mathcal{D}$ . A pseudo-morphism of uniform fixpoint operators from  $((-)^*, \text{fix}^*, \text{unif}^*)$  to  $((-)^{\dagger}, \text{fix}^{\dagger}, \text{unif}^{\dagger})$  is a pseudo-morphism of fixpoint operators  $\delta : ((-)^*, \text{fix}^*) \rightarrow ((-)^{\dagger}, \text{fix}^{\dagger})$  as in Definition II.2 satisfying the additional coherence for every square in  $\mathcal{D}$ :

$$\begin{array}{ccc}
A & \xrightarrow{f} & A \\ J(s) \downarrow & \not\parallel \gamma & \downarrow J(s) \\ B & \xrightarrow{g} & B
\end{array}$$

we have:

$$\begin{array}{ccc}
\begin{array}{ccc} f^* & & \\ 1 & \nearrow \text{unif}_\gamma^* & Js \\ g^* \nearrow & \delta_g & \searrow g^{\dagger} \\ A & \circlearrowleft J_s & B \end{array} & = & \begin{array}{ccc} f^* & & \\ 1 & \nearrow \delta_f & A \\ f^{\dagger} \nearrow & \text{unif}_\gamma^{\dagger} & \searrow g^{\dagger} \\ A & \circlearrowleft J_s & B \end{array}
\end{array}$$

We denote by  $\mathbf{Fix}(\mathcal{D}, J)$  the category of pseudo-fixpoint operators on  $\mathcal{D}$  uniform with respect to  $J$ .

**Definition II.7.** Let  $J : \mathcal{C} \rightarrow \mathcal{D}$  be a identity-on-objects 2-functor (strictly) preserving terminal objects. A *pseudo-dinatural fixpoint operator uniform with respect to  $J$*  consists of a pseudo-dinatural fixpoint operator  $((-)^*, \text{dinat})$  on  $\mathcal{D}$  together with a strong dinatural transformation

$$\text{unif} : \mathcal{D}(J(-), J(-)) \rightrightarrows \mathcal{D}(1, J(-),) : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathbf{CAT}$$

satisfying the following additional coherence between **dinat** and **unif**: for two squares in  $\mathcal{D}$  of the form



where  $\gamma \star \rho$  corresponds to the following 2-cell:

Note that if we restrict to the case where  $A = B$ ,  $C = D$ ,  $g = 1_A$ ,  $k = 1_C$ ,  $r = s$  and  $\rho = \text{id}$ , we obtain the coherence axiom between **fix** and **unif** in Definition II.5. Pseudo-fixpoint operators uniform with respect to  $J$  are therefore a special case of pseudo-dinatural fixpoint operators uniform with respect to  $J$  as expected.

**Definition II.8.** Let  $((-)^*, \text{dinat}^*, \text{unif}^*)$  and  $((-)^{\dagger}, \text{dinat}^{\dagger}, \text{unif}^{\dagger})$  be two pseudo-dinatural fixpoint operators uniform with respect to  $J : \mathcal{C} \rightarrow \mathcal{D}$ . A pseudo-morphism of uniform dinatural fixpoint operators from  $((-)^*, \text{dinat}^*, \text{unif}^*)$  to  $((-)^{\dagger}, \text{dinat}^{\dagger}, \text{unif}^{\dagger})$  is a pseudo-morphism of dinatural fixpoint operators  $\delta : ((-)^*, \text{dinat}^*) \rightarrow ((-)^{\dagger}, \text{dinat}^{\dagger})$  as in Definition II.4 that is also a pseudo-morphism of uniform fixpoint operators as in Definition II.6, i.e.  $\delta$  commutes with both the dinaturality and uniformity structural 2-cells **dinat** and **unif**.

We denote by  $\mathbf{DinFix}(\mathcal{D}, J)$  the category of pseudo-dinatural fixpoint operators on  $\mathcal{D}$  uniform with respect to  $J$ .

Before stating the main theorem of the paper, we recall the notion of pseudo-bifree algebras for 2-functors.

**Definition II.9.** For a 2-functor  $! : \mathcal{C} \rightarrow \mathcal{C}$ , a *pseudo-initial algebra* is a 1-cell  $R : !\Phi \rightarrow \Phi$  such that for every 1-cell  $f : !A \rightarrow A$ , there exists a pseudo-morphism of algebras  $(u_f, \mu_f) : R \rightarrow f$ , i.e. a 1-cell  $u_f : \Phi \rightarrow A$  and a 2-cell

verifying the following universal property: for any pseudo-algebra 1-cells  $(v, \nu), (w, \omega) : R \rightarrow f$ , there is a unique invertible algebra 2-cell  $\phi : (v, \nu) \Rightarrow (w, \omega)$ , i.e. a unique invertible 2-cell  $\phi : v \Rightarrow w$  in  $\mathcal{C}$  such that:

We can similarly define a dual notion of pseudo-final !-coalgebra. Lambek's theorem stating that an initial algebra or a final coalgebra is an invertible morphism is generalized to an adjoint equivalence:

**Lemma II.10** ([18]). *If  $R : !\Phi \rightarrow \Phi$  is a pseudo-initial !-algebra, then it is part of an adjoint equivalence  $(R : !\Phi \rightarrow \Phi, L : \Phi \rightarrow !\Phi, \eta : \text{id} \xRightarrow{\cong} RL, \varepsilon : LR \xRightarrow{\cong} \text{id})$ .*

**Definition II.11.** We say that  $R : !\Phi \rightarrow \Phi$  is a *pseudo-bifree algebra* if  $R$  is a pseudo-initial algebra and its (uniquely determined) left adjoint  $L$  is a pseudo-final coalgebra.

We can now state the main theorem of this paper which is categorification of Theorem I.3:

**Theorem II.12.** *Let  $\mathcal{C}$  be a 2-category equipped with a (strict) 2-comonad  $(!, \delta, \varepsilon)$  and a (strict) terminal object  $1$ . We denote by  $\mathcal{D}$  the co-Kleisli 2-category  $\mathcal{C}_!$  and by  $J : \mathcal{C} \rightarrow \mathcal{D}$  the free functor induced by the comonadic adjunction.*

- 1) *If the endofunctor  $!$  has a pseudo-bifree algebra, then the category  $\mathbf{Fix}(\mathcal{D}, J)$  of pseudo-fixpoint operators on  $\mathcal{D}$  uniform with respect to  $J$  is contractible.*
- 2) *If  $\mathcal{C}$  is cartesian and the endofunctor  $!!$  has a pseudo-bifree algebra, then the category  $\mathbf{DinFix}(\mathcal{D}, J)$  of pseudo-dinatural fixpoint operators on  $\mathcal{D}$  uniform with respect to  $J$  is contractible.*

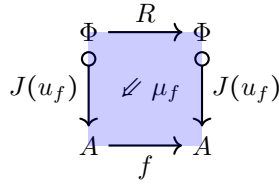
We proceed to prove this theorem in the next section by first constructing an explicit pseudo-fixpoint operator from the bifree algebras and showing that it verifies the required axioms and then proving the contractibility property i.e. for any other pseudo-fixpoint operator, there is a unique isomorphism between them.

### III. THE PLOTKIN-SIMPSON THEOREM FOR 2-CATEGORIES

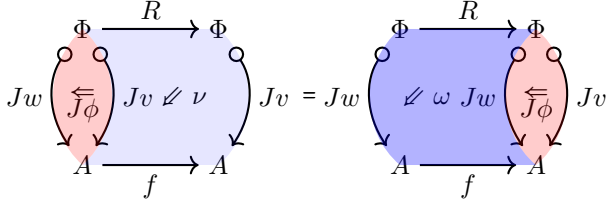
In this section, we fix a 2-category  $\mathcal{C}$  equipped with a (strict) 2-comonad  $(!, \delta, \varepsilon)$  and a (strict) terminal object  $1$ . We denote by  $\mathcal{D}$  the co-Kleisli 2-category  $\mathcal{C}_!$  and by  $J : \mathcal{C} \rightarrow \mathcal{D}$  the free functor induced by the comonadic adjunction. We assume further that the endofunctor  $!$  has a pseudo-bifree algebra  $R : !\Phi \rightarrow \Phi$ .

The following lemma is simply a reformulation of Definition II.9 from  $\mathcal{C}$  to the co-Kleisli  $\mathcal{D}$ :

**Lemma III.1.** *For any 1-cell  $f : A \rightarrow A$  in  $\mathcal{D}$ , there exists a 1-cell  $u_f : \Phi \rightarrow A$  in  $\mathcal{C}$  and a 2-cell  $\mu_f$*

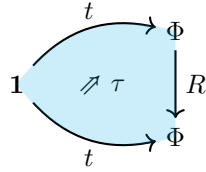


in  $\mathcal{D}$  verifying the following universal property: for any 1-cells  $v, w : \Phi \rightarrow A$  in  $\mathcal{C}$  and 2-cells  $\nu : J(v)R \Rightarrow fJ(v)$  and  $\omega : J(w)R \Rightarrow fJ(w)$  in  $\mathcal{D}$ , there exists a unique invertible 2-cell  $\phi : v \Rightarrow w$  in  $\mathcal{C}$  such that

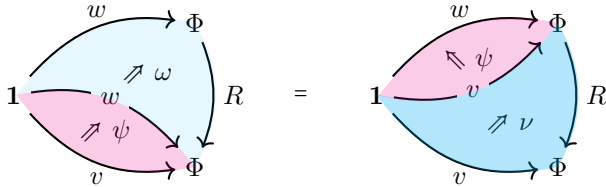


The next lemma uses the fact that  $R$  is pseudo-bifree and therefore part of an adjoint equivalence where the left adjoint  $L$  is pseudo-final.

**Lemma III.2.** *There exists a 1-cell  $t : \mathbf{1} \rightarrow \Phi$  and an invertible 2-cell  $\tau : t \Rightarrow Rt$  in  $\mathcal{D}$*



satisfying the following universal property: for any 1-cells  $v, w : \mathbf{1} \rightarrow \Phi$  in  $\mathcal{D}$  and invertible 2-cells  $\nu : v \Rightarrow Rv$  and  $\omega : w \Rightarrow Rw$ , there exists a unique invertible 2-cell  $\psi : v \Rightarrow w$  in  $\mathcal{D}$  such that



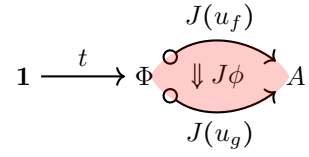
Using these two lemmas, we can now construct the pseudo-fixpoint operator on  $\mathcal{D}$ . We define a family of functors indexed by the objects  $A \in \mathcal{D}$

$$(-)_A^* : \mathcal{D}(A, A) \rightarrow \mathcal{D}(\mathbf{1}, A)$$

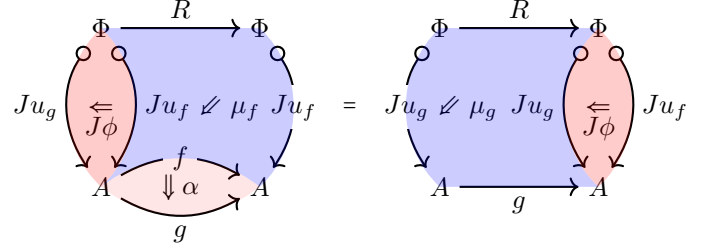
mapping a 1-cell  $f : A \rightarrow A$  to

$$f^* := \mathbf{1} \xrightarrow{t} \Phi \xrightarrow{J(u_f)} A$$

where  $u_f$  and  $t$  are obtained from Lemmas III.1 and III.2. For a 2-cell  $\alpha : f \Rightarrow g$  in  $\mathcal{D}(A, A)$ , define  $\alpha^* : f^* \Rightarrow g^*$  as

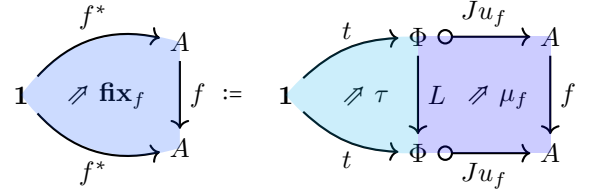


where  $\phi$  is the unique 2-cell  $u_f \Rightarrow u_g$  in  $\mathcal{C}$  such that



which exists by Lemma III.1.

We can now define the fixpoint 2-cell  $\text{fix}_f : f \circ f^* \Rightarrow f^*$  for a 1-cell  $f : A \rightarrow A$  in  $\mathcal{D}$  as:

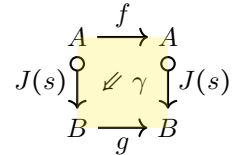


where  $\mu_f$  and  $\tau$  are obtained from Lemmas III.1 and III.2. To obtain that  $((-)^*, \text{fix})$  is a pseudo-fixpoint operator on  $\mathcal{D}$ , it only remains to show that  $\text{fix}$  is natural:

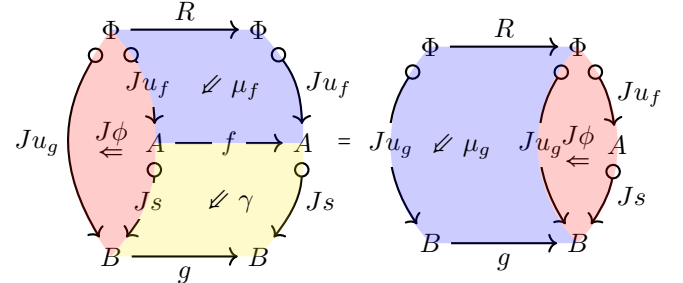
**Lemma III.3** (Naturality of  $\text{fix}$ ). *For a 2-cell  $\alpha : f \Rightarrow g$  in  $\mathcal{D}$ , we have  $\text{fix}_g \circ \alpha^* = (\alpha \cdot \alpha^*) \circ \text{fix}_f$ .*

Now that we have a constructed a pseudo-fixpoint operator on  $\mathcal{D}$ , we want to show that it is uniform with respect to the free functor  $J : \mathcal{C} \rightarrow \mathcal{D}$  from the base 2-category  $\mathcal{C}$  to the co-Kleisli  $\mathcal{D} = \mathcal{C}_1$ .

Assume that there exist a 1-cell  $s : A \rightarrow B$  in  $\mathcal{C}$  and  $f : A \rightarrow A$ ,  $g : B \rightarrow B$  in  $\mathcal{D}$  and a 2-cell



in  $\mathcal{D}$ . By Lemma III.1, there exists a unique 2-cell  $\phi : s \circ u_f \Rightarrow u_g$  in  $\mathcal{C}$  such that:



Define  $\text{unif}_\gamma$  as:



$$\begin{array}{c}
\begin{array}{ccc}
& f^* & \\
1 & \xrightarrow{\text{unif}_\gamma} & A \\
& g^* & \\
& B &
\end{array}
\end{array}
\begin{array}{c}
\downarrow J_S \\
:= \\
\begin{array}{ccc}
& Ju_f & \\
1 & \xrightarrow{t} & \Phi \\
& Ju_g & \\
& B &
\end{array}
\end{array}$$

**Proposition III.4.** *The 2-cells  $\text{unif}_\gamma$  we constructed yield a strong pseudo-dinatural transformation and they verify the coherence axiom between  $\text{fix}$  and  $\text{unif}$ .*

We have constructed a uniform pseudo-fixpoint operator showing that the category  $\mathbf{Fix}(\mathcal{D}, J)$  is inhabited, it remains to show that it is contractible.

Assume that there exists another pseudo-fixpoint operator  $((-)^{\dagger}, \text{fix}^{\dagger}, \text{unif}^{\dagger})$  on  $\mathcal{D}$  that is uniform with respect to  $J : \mathcal{C} \rightarrow \mathcal{D}$ . We want to show that there is a unique isomorphism of uniform pseudo-fixpoint operators  $\delta : ((-)^* \text{fix}^*, \text{unif}^*) \rightarrow ((-)^{\dagger}, \text{fix}^{\dagger}, \text{unif}^{\dagger})$ .

By Lemma III.2, there exists a unique invertible 2-cell  $\delta_0 : t \Rightarrow R^{\dagger}$  such that

$$\begin{array}{ccc}
\begin{array}{ccc}
& R^{\dagger} & \\
1 & \xrightarrow{\text{fix}_R^{\dagger}} & \Phi \\
& \delta_0 & \\
& t &
\end{array}
& = &
\begin{array}{ccc}
& R^{\dagger} & \\
1 & \xrightarrow{t} & \Phi \\
& \delta_0 & \\
& t &
\end{array}
\end{array}$$

Now, for every 1-cell  $f : A \rightarrow A$ , there exists by Lemma III.1 a 1-cell  $u_f : \Phi \rightarrow A$  in  $\mathcal{C}$  and a 2-cell

$$\begin{array}{ccc}
\Phi & \xrightarrow{R} & \Phi \\
J(u_f) \downarrow & \mu_f \lrcorner & \downarrow J(u_f) \\
A & \xrightarrow{f} & A
\end{array}$$

in  $\mathcal{D}$ . Since  $(-)^{\dagger}$  is uniform with respect to  $J$ , we have a 2-cell  $\text{unif}_{\mu_f}^{\dagger} : Ju_f \circ R^{\dagger} \Rightarrow f^{\dagger}$  and we define  $\delta_f : f^* \Rightarrow f^{\dagger}$  as:

$$\delta_f := \begin{array}{ccc}
& t & \\
1 & \xrightarrow{R^{\dagger}} & A \\
& \text{unif}_{\mu_f}^{\dagger} \lrcorner & \\
& f^{\dagger} & \\
& B &
\end{array}$$

**Proposition III.5.** *The category  $\mathbf{Fix}(\mathcal{D}, J)$  is contractible i.e.  $\delta$  is an isomorphism of uniform pseudo-fixpoint operators and it is unique.*

#### IV. DINATURALITY

To obtain that the pseudo-fixpoint operator is pseudo-dinatural, we want to construct an invertible dinaturality 2-cell for every 1-cell  $f : A \rightarrow B$  in  $\mathcal{D}$ ,

$$\begin{array}{ccccc}
\mathcal{D}(f, A) & \xrightarrow{\mathcal{D}(A, A)} & \mathcal{D}(1, A) & \xrightarrow{(-)_A^*} & \mathcal{D}(1, A) \\
\downarrow \mathcal{D}(B, A) & \nearrow \text{dinat}_g^f & \downarrow \mathcal{D}(1, f) & & \\
\mathcal{D}(B, f) & \xrightarrow{\mathcal{D}(B, B)} & \mathcal{D}(1, B) & \xrightarrow{(-)_B^*} & \mathcal{D}(1, B)
\end{array}$$

with components

$$\begin{array}{ccc}
& (gf)^* & \\
1 & \xrightarrow{\text{dinat}_g^f} & A \\
& (fg)^* & \\
& B &
\end{array}$$

for  $g$  in  $\mathcal{D}(B, A)$ . To do so, we need to assume further that the endofunctor  $!! : \mathcal{C} \rightarrow \mathcal{C}$  has a pseudo-bifree algebra and that 2-category  $\mathcal{C}$  is cartesian. Note that it implies that co-Kleisli  $\mathcal{D}$  is cartesian as well and that the free functor  $J : \mathcal{C} \rightarrow \mathcal{D}$  preserves the cartesian structure.

The following lemma uses the same argument as the strict case by Freyd [26].

**Lemma IV.1.** *Assume that  $!! : \mathcal{C} \rightarrow \mathcal{C}$  has a pseudo bifree algebra. Then, if  $R : !\Phi \rightarrow \Phi$  is a pseudo bifree  $!$ -algebra,*

$$!!\Phi \xrightarrow{!R} !\Phi \xrightarrow{R} \Phi$$

*is a pseudo bifree  $!!$ -algebra.*

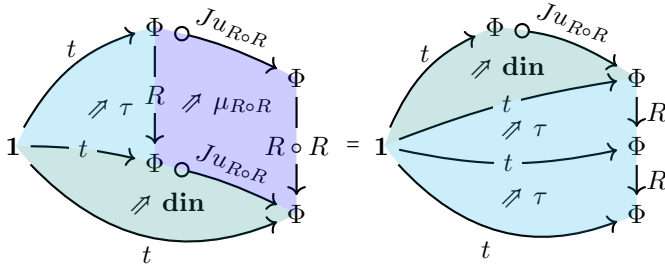
**Lemma IV.2.** *For 1-cells  $v, w : 1 \rightarrow \Phi$  in  $\mathcal{D}$  and 2-cells*

$$\begin{array}{ccc}
\begin{array}{ccc}
& v & \\
1 & \xrightarrow{\nu} & \Phi \\
& v &
\end{array}
& \text{and} &
\begin{array}{ccc}
& w & \\
1 & \xrightarrow{\omega} & \Phi \\
& w &
\end{array}
\end{array}$$

*in  $\mathcal{D}$ , there exists a unique invertible 2-cell  $\lambda : w \Rightarrow v$  in  $\mathcal{D}$  such that*

$$\begin{array}{ccc}
\begin{array}{ccc}
& v & \\
1 & \xrightarrow{\nu} & \Phi \\
& \lambda & \\
& w &
\end{array}
& \xrightarrow{R \circ R} &
\begin{array}{ccc}
& v & \\
1 & \xrightarrow{w} & \Phi \\
& \omega &
\end{array}
\end{array}$$

To construct the dinaturality 2-cells, we first start by constructing a 2-cell  $\text{din} : t \Rightarrow (R \circ R)^*$ . Let  $\text{din}$  be the unique invertible 2-cell from  $t$  to  $J(u_{R \circ R}) \circ t$  (obtained from Lemma IV.2) such that:



Once this isomorphism is obtained, we construct the dinaturality 2-cells  $\mathbf{dinat}_g^f$  using uniformity. Consider the following endo-1-cell:

$$A \times B \xrightarrow{f \times g} B \times A \xrightarrow{\sigma} A \times B$$

where  $\sigma$  is the symmetry obtained in the standard way as the pairing  $\langle \pi_1, \pi_2 \rangle$ . Using Lemma III.1, there is a 1-cell  $u_{\sigma(f \times g)} : \Phi \rightarrow A \times B$  in  $\mathcal{C}$  and a 2-cell  $\mu_{\sigma(f \times g)}$ :

$$\begin{array}{ccc} \Phi & \xrightarrow{R} & \Phi \\ J(u_{\sigma(f \times g)}) \downarrow & \mu_{\sigma(f \times g)} & \downarrow J(u_{\sigma(f \times g)}) \\ A & \xrightarrow{f} & A \end{array}$$

From the two squares

$$\begin{array}{ccccc} \Phi & \xrightarrow{R} & \Phi & \xrightarrow{R} & \Phi \\ J u_{\sigma(f \times g)} \downarrow & \mu_{\sigma(f \times g)} & J u_{\sigma(f \times g)} \downarrow & \mu_{\sigma(f \times g)} & J u_{\sigma(f \times g)} \downarrow \\ \Pi_{\sigma(f \times g)}^1 := A \times B & \xrightarrow{\sigma(f \times g)} & A \times B & \xrightarrow{\sigma(f \times g)} & A \times B \\ \pi_1 \downarrow & & \pi_2 \downarrow & & \pi_1 \downarrow \\ A & \xrightarrow{f} & B & \xrightarrow{g} & A \end{array}$$

$$\begin{array}{ccccc} \Phi & \xrightarrow{R} & \Phi & \xrightarrow{R} & \Phi \\ J u_{\sigma(f \times g)} \downarrow & \mu_{\sigma(f \times g)} & J u_{\sigma(f \times g)} \downarrow & \mu_{\sigma(f \times g)} & J u_{\sigma(f \times g)} \downarrow \\ \Pi_{\sigma(f \times g)}^2 := A \times B & \xrightarrow{\sigma(f \times g)} & A \times B & \xrightarrow{\sigma(f \times g)} & A \times B \\ \pi_2 \downarrow & & \pi_1 \downarrow & & \pi_2 \downarrow \\ B & \xrightarrow{g} & A & \xrightarrow{f} & B \end{array}$$

we obtain uniformity 2-cells

$$\begin{array}{ccc} (RR)^* & \rightarrow & \Phi \\ \downarrow J u_{\sigma(f \times g)} & & \downarrow J u_{\sigma(f \times g)} \\ 1 \not\Leftarrow \mathbf{unif}_{\Pi_{\sigma(f \times g)}^1} A \times B & \text{and} & 1 \not\Leftarrow \mathbf{unif}_{\Pi_{\sigma(f \times g)}^2} A \times B \\ \downarrow J \pi_1 & & \downarrow J \pi_2 \\ (gf)^* & \rightarrow & A \end{array}$$

We can now construct the general dinaturality 2-cell  $\mathbf{dinat}_g^f$  as in Figure 1.

Note that for **fix** and **unif**, we wrote the direction of 2-cells without needing to take inverses whereas for dinaturality, a back-and-forth is needed to construct the 2-cells. It means that for the general directed case, we need both the lax and oplax directions in order to obtain dinaturality.

**Proposition IV.3.** *The 2-cells  $\mathbf{dinat}_g^f$  we constructed yield a pseudo-dinatural transformation and they verify the coherence axiom between **dinat** and **unif**.*

We have now shown that the category of uniform pseudo-dinatural fixpoint operators  $\mathbf{DinFix}(\mathcal{D}, J)$  is inhabited, it remains to show that it is contractible to complete the proof of Theorem II.12.

Assume that there is another uniform pseudo-dinatural fixpoint operators  $((-)^{\dagger}, \mathbf{dinat}^{\dagger}, \mathbf{unif}^{\dagger})$ . By Proposition III.5, we already have a constructed morphism of uniform pseudo-fixpoint operators  $\delta : ((-)^*, \mathbf{fix}^*, \mathbf{unif}^*) \rightarrow ((-)^{\dagger}, \mathbf{fix}^{\dagger}, \mathbf{unif}^{\dagger})$  where the 2-cells **fix** are induced from the 2-cells **dinat**. Therefore, it only remains to show:

**Proposition IV.4.** *The category  $\mathbf{DinFix}(\mathcal{D}, J)$  is contractible i.e.  $\delta$  commutes with the dinaturality 2-cells and it is unique.*

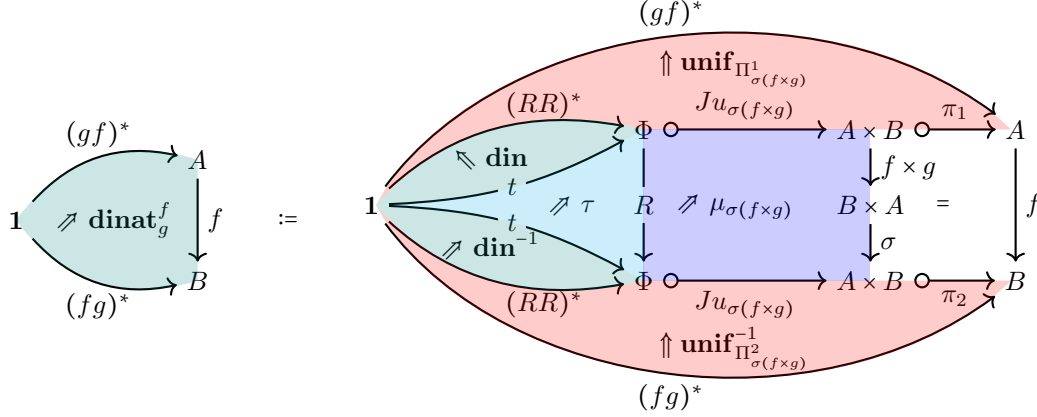
## V. EXAMPLES

We start by presenting examples where the pseudo-fixpoint operators are obtained as instances of Theorem II.12 and therefore for which the category of pseudo-fixpoint operators is contractible. In particular, the least and greatest fixpoint (i.e. the initial and final object of the category of pseudo-fixpoints) are isomorphic. We consider afterwards the example of polynomial functors which are not an instance of our theorem but where the axioms for pseudo-fixpoints presented in Section II are verified.

### A. Limit-colimit coincidence theorem for 2-categorical structures

We give a brief reminder of the general recipe to obtain pseudo-bifree algebras for endofunctors on 2-categories using the machinery developed by Cattani, Fiore and Winskel [16], [17], [18]. Instead of considering preorder-enriched categories, we move to 2-categories or bicategories whose hom-categories have colimits of  $\omega$ -chains and initial objects. The colimits of  $\omega$ -chains of embedding-projection pairs in the preorder-enriched setting are generalized to pseudo-colimits of  $\omega$ -

Figure 1. Construction of the dinaturality 2-cells



chains of co-reflections (adjunctions with invertible unit) in the categorified setting.

We say that a 2-category or bicategory  $\mathcal{C}$  is  $\mathbf{Cat}_\omega$ -enriched ( $\mathbf{Cat}_{\omega, \perp}$ -enriched) if for all objects  $A$  and  $B$ , its hom-category  $\mathcal{C}(A, B)$  have colimits of  $\omega$ -chains (and initial objects) and whose composition functors

$$\mathcal{C}(A, B) \times \mathcal{C}(B, C) \rightarrow \mathcal{C}(B, C)$$

preserve of colimits  $\omega$ -chains (and initial objects) in both variables. The analogue of the category  $\mathbf{Cppo}$  is the 2-category  $\mathbf{Cat}_\omega$  given by

- 0-cells: categories with colimits of  $\omega$ -chains and initial objects;
- 1-cells: functors preserving colimits of  $\omega$ -chains;
- 2-cells: natural transformations.

and its sub-2-category  $\mathbf{Cat}_{\omega, \perp}$  which restricts the 1-morphisms to those preserving initial objects (corresponding to the category  $\mathbf{Cppo}_\perp$  in the preorder setting).

In order to obtain pseudo-bifree algebras, we make use of the following theorem:

**Theorem V.1** ([18]). *Let  $\mathcal{C}$  be a  $\mathbf{Cat}_{\omega, \perp}$ -enriched 2-category. If  $\mathcal{C}$  has a pseudo-initial object and pseudo- $\omega$ -colimits of chains of coreflections (adjunctions with invertible units), then  $\mathcal{C}$  is  $\mathbf{Cat}_\omega$ -pseudo-algebraically compact. It means that for every pseudo-functor  $T : \mathcal{C} \rightarrow \mathcal{C}$ , if  $T$  is  $\mathbf{Cat}_\omega$ -enriched, i.e. for all  $A, B$ , the induced functor*

$$\mathcal{C}(A, B) \longrightarrow \mathcal{C}(TA, TB)$$

*preserves colimits of  $\omega$ -chains, then  $T$  has a pseudo-bifree algebra.*

### B. Categorical domain theory

The canonical example is the 2-category  $\mathbf{Cat}_{\omega, \perp}$  with the lifting 2-comonad  $(-)_{\perp} : \mathbf{Cat}_{\omega, \perp} \rightarrow \mathbf{Cat}_{\omega, \perp}$  which freely adjoins initial objects. Both  $(-)_{\perp}$  and  $(-)_{\perp} \circ (-)_{\perp}$  are  $\mathbf{Cat}_\omega$ -enriched and therefore the coKleisli 2-category (which is

equivalent to  $\mathbf{Cat}_\omega$ ) verifies that its category of pseudo-dinatural fixpoint operators uniform (with respect to the free functor  $\mathbf{Cat}_{\omega, \perp} \rightarrow \mathbf{Cat}_\omega$ ) is contractible. The standard Lambek construction for initial algebras for a finitary endofunctor  $f : \mathbb{A} \rightarrow \mathbb{A}$  on a category  $\mathbb{A}$  with an initial object  $\perp$  and  $\omega$ -colimits by calculating the colimit of the diagram

$$\perp \rightarrow f(\perp) \rightarrow f^2(\perp) \rightarrow \dots$$

provides both the initial and final pseudo-fixpoint operators [32]. As a consequence of Theorem II.12, Lambek's construction verifies the dinaturality axioms and is uniform with respect to sub-2-category of functors preserving initial objects up to isomorphism.

We can also recover Adamek's result for *Scott-complete categories* which can be viewed as a categorification of Scott domains [14]. He considers the 2-category  $\mathbf{SCC}$  given by

- 0-cells: Scott-complete categories (i.e finitely accessible category such that every diagram with a cocone has a colimit);
- 1-cells: functors preserving directed colimits;
- 2-cells: natural transformations.

and its sub-2-category  $\mathbf{SCC}_\perp$  whose 1-cells are restricted to functors preserving directed colimits and initial objects. The lifting 2-comonad on  $\mathbf{SCC}_\perp$  has the required bifree algebras so that the co-Kleisli  $\mathbf{SCC}$  has a contractible category of pseudo-dinatural fixpoint operators uniform with respect to the free 2-functor  $\mathbf{SCC}_\perp \rightarrow \mathbf{SCC}$ .

### C. Profunctors and linear logic models

Another example of  $\mathbf{Cat}_{\omega, \perp}$ -enriched bicategory is the bicategory of profunctors denoted by  $\mathbf{Prof}$  [33]. For small categories  $\mathbb{A}$  and  $\mathbb{B}$ , a profunctor  $P$  from  $\mathbb{A}$  to  $\mathbb{B}$  is a functor  $P : \mathbb{A} \times \mathbb{B}^{\text{op}} \rightarrow \mathbf{Set}$  or equivalently a functor from  $\mathbb{A}$  to the presheaf category  $\hat{\mathbb{B}}$ . Profunctors can be seen as a categorification of  $\mathbf{Rel}$  as a relation  $R \subseteq A \times B$  corresponds to a profunctor between discrete categories such that each component is either the empty set or a singleton.

Many pseudo-comonad structures were considered on **Prof** leading to models of linear logic with various notion of substitution [3], [4], [34]. In particular, the free symmetric monoidal completion comonad on **Prof** yields the model of *generalized species of structures* which encompasses Joyal’s combinatorial species and is also a categorification of the relational model linear logic [4]. The morphisms in the co-Kleisli bicategory correspond to the notion of *analytic functors* which are generalized power series with quotients [35]. We can also consider the pseudo-comonads freely adjoining finite coproducts or finite colimits to generalize the category **Lin** with the  $\vee$ -semi-lattice comonad in Section I or simply the comonad freely adjoining an initial object. We refer the reader to [36] for a general treatment of pseudo-(co)-monads on **Prof** and to [9] for other examples with applications to intersection typing systems.

The pseudo-comonads we consider verify the necessary conditions on bifree algebras as they are all  $\mathbf{Cat}_\omega$ -enriched and Theorem V.1 has been extended to bicategories and pseudo-functors [18]. For the colimit-completion cases, it is a straightforward consequence of the commutation of colimits and it only needs to be checked by hand for the free symmetric strict monoidal case.

Strictly speaking, we have only provided the notion of pseudo-fixpoint operators for 2-categories and not for bicategories. Even if we strictify the bicategory **Prof** to its biequivalent 2-category **Cocont** (concontinuous functors between presheaf categories), the comonads we consider are pseudo and their corresponding co-Kleisli are therefore bicategories and not 2-categories.

For space considerations, we do not give the proof of Theorem II.12 for bicategories in this paper but only state that we obtain as a corollary that species with the free symmetric strict monoidal completion, initial object, finite coproduct, finite colimit pseudo-comonads are all instances of this construction and therefore all have a contractible category of pseudo-dinatural fixpoint operators uniform with respect to the inclusion of profunctors into these generalized species.

#### D. Polynomial functors

Initial algebras (well-founded trees) and final coalgebras (non-well-founded trees) for polynomial functors have been extensively studied [37], [38], [39], [40] and are standard tools to model (co)inductive types in dependent type theories. If we fix a locally cartesian closed category  $\mathbb{C}$ , for objects  $I$  and  $J$  in  $\mathbb{C}$  a polynomial from  $I$  to  $J$  is a diagram of shape

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J$$

in the category  $\mathbb{C}$ . It induces a polynomial functor  $\mathbb{C}/I \rightarrow \mathbb{C}/J$  between the slice categories which form a 2-category with 2-cells given by cartesian transformations. We say that  $\mathbb{C}$  has **W**-types if all polynomial functors have initial algebras and that it has **M**-types if they have final coalgebras.

It is well-known that **W**-types and **M**-types do not coincide and they are therefore not an instance of the contractibility

property of Theorem II.12. This example is therefore an adequate test to verify that the axioms we stated in Section II are not just valid in the restricted contractible case but provide a general notion of pseudo-fixpoint operators.

The 2-naturality axiom of the operator computing initial algebras (or final coalgebras) is well known and is sometimes called the “functoriality property”. Dinaturality for the initial algebra or final coalgebra operators is also known in the literature [26]. The statement is that for functors  $F : \mathbb{A} \rightarrow \mathbb{B}$ ,  $G : \mathbb{B} \rightarrow \mathbb{A}$  such that  $GF$  and  $FG$  have initial algebras, if  $GFA \xrightarrow{a} A$  is  $GF$ -initial, then  $FGFA \xrightarrow{Fa} FA$  is  $FG$ -initial. To our knowledge, the axioms of a pseudo-dinatural transformation, while straightforward to check, have not been stated explicitly.

Uniformity on the other hand depends on how the initial algebras (or final coalgebras) are computed in the ambient 2-category. In general, if initial algebras are obtained as colimit constructions, then natural candidates for the 2-category of strict maps are functors preserving initial objects or cocontinuous functors whenever these notions are well-defined. Dually, if the pseudo-fixpoint operator is obtained by computing final coalgebras as certain limits then we can consider terminal object preserving functors or continuous functors as strict maps. In the case of polynomial functors over **Set**, **W**-types are uniform with respect to spans *i.e.* polynomial functors of shape

$$I \xleftarrow{s} E \xrightarrow{\cong} B \xrightarrow{t} J$$

and **M**-types are uniform with respect to monomials which are polynomial functors of shape

$$I \xleftarrow{s} E \xrightarrow{!} 1 \xrightarrow{t} J$$

where 1 is a singleton set.

#### CONCLUSION

We have presented the theory of fixpoint operators for 2-categories using a categorification of Plotkin-Simpson’s theorem as a guideline to derive the equations on the structural 2-cells in dimension 2. The concrete 2-categorical examples allow us to confirm that theses equations are verified.

In future work, we aim to extend our theory to *parametrized* and *guarded* fixpoint operators. Adding parameters allows to consider richer contexts for terms and guardedness restricts the possible infinite behavior of fixpoints to ensure properties such as solvability or productivity are satisfied. Since a parametrized fixpoint operator verifying the Conway axioms is equivalent to a *traced monoidal category* with the cartesian product as the chosen tensor, we aim to use our formalism to develop the theory of traced monoidal bicategories and establish new connections with cyclic  $\lambda$ -calculi [41], [42]. We also want to formulate the theory of 2-dimensional fixpoints in a cartesian closed framework where the fixpoint operator is expressed internally as a family of 1-cells of type  $(A \Rightarrow A) \rightarrow A$  bringing us closer to the intuition of fixpoint combinators for  $\lambda$ -calculus.

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## REFERENCES

- [1] A. Simpson and G. Plotkin, “Complete axioms for categorical fixed-point operators,” in *Proceedings Fifteenth Annual IEEE Symposium on Logic in Computer Science (Cat. No. 99CB36332)*. IEEE, 2000, pp. 30–41.
- [2] M. P. Fiore, *Axiomatic domain theory in categories of partial maps*. Cambridge University Press, 2004, vol. 14.
- [3] G. L. Cattani and G. Winskel, “Profunctors, open maps and bisimulation,” *Mathematical Structures in Computer Science*, vol. 15, no. 3, pp. 553–614, 2005.
- [4] M. Fiore, N. Gambino, M. Hyland, and G. Winskel, “The cartesian closed bicategory of generalised species of structures,” *J. Lond. Math. Soc. (2)*, vol. 77, no. 1, pp. 203–220, 2008. [Online]. Available: <http://dx.doi.org/10.1112/jlms/jdm096>
- [5] D. Mazza, L. Pellissier, and P. Vial, “Polyadic approximations, fibrations and intersection types,” *Proceedings of the ACM on Programming Languages*, vol. 2, no. POPL:6, 2018.
- [6] M. Fiore and P. Saville, “A type theory for cartesian closed bicategories,” in *2019 34th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*. IEEE, 2019, pp. 1–13.
- [7] P.-A. Melliès, “Template games and differential linear logic,” in *2019 34th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, 2019, pp. 1–13.
- [8] M. Fiore and P. Saville, “Coherence and normalisation-by-evaluation for bicategorical cartesian closed structure,” in *Proceedings of the 35th Annual ACM/IEEE Symposium on Logic in Computer Science*, 2020, pp. 425–439.
- [9] F. Olimpieri, “Intersection type distributors,” in *36th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2021, Rome, Italy, June 29 - July 2, 2021*. IEEE, 2021, pp. 1–15. [Online]. Available: <https://doi.org/10.1109/LICS52264.2021.9470617>
- [10] A. Kerinec, G. Manzonetto, and F. Olimpieri, “Why are proofs relevant in proof-relevant models?” *PACMPL*, vol. 7, no. POPL, pp. 8:1–8:31, 2023. [Online]. Available: <https://doi.org/10.1145/3571201>
- [11] D. Scott and C. Strachey, *Toward a mathematical semantics for computer languages*. Oxford University Computing Laboratory, Programming Research Group Oxford, 1971, vol. 1.
- [12] J. Lambek, “A fixpoint theorem for complete categories,” *Mathematische Zeitschrift*, vol. 103, pp. 151–161, 1968. [Online]. Available: <http://eudml.org/doc/170906>
- [13] D. J. Lehmann and M. B. Smyth, “Igebraic specification of data types: A synthetic approach,” *Mathematical systems theory*, vol. 14, no. 1, pp. 97–139, 1981.
- [14] J. Adámek, “A categorical generalization of scott domains,” *Mathematical Structures in Computer Science*, vol. 7, no. 5, p. 419–443, 1997.
- [15] P. Taylor, “The limit-colimit coincidence for categories,” *unpublished preprint, Imperial College, London*, 1988.
- [16] M. P. Fiore, *Axiomatic domain theory in categories of partial maps*. Cambridge University Press, 2004, vol. 14.
- [17] G. L. Cattani, M. Fiore, and G. Winskel, “A theory of recursive domains with applications to concurrency,” in *Proceedings of the 13th Annual IEEE Symposium on Logic in Computer Science*, ser. LICS ’98. USA: IEEE Computer Society, 1998, p. 214.
- [18] G. L. Cattani and M. P. Fiore, “The bicategory-theoretic solution of recursive domain equations,” *Electronic Notes in Theoretical Computer Science*, vol. 172, pp. 203–222, 2007.
- [19] A. Pitts, “An elementary calculus of approximations,” 1987.
- [20] K. A. Ponto, *Fixed point theory and trace for bicategories*. The University of Chicago, 2007.
- [21] K. Ponto and M. Shulman, “Shadows and traces in bicategories,” *Journal of Homotopy and Related Structures*, vol. 8, no. 2, pp. 151–200, 2013.
- [22] A. K. Simpson, “A characterisation of the least-fixed-point operator by dinaturality,” *Theoretical Computer Science*, vol. 118, no. 2, pp. 301–314, 1993.
- [23] A. Joyal, R. Street, and D. Verity, “Traced monoidal categories,” in *Mathematical proceedings of the cambridge philosophical society*, vol. 119, no. 3. Cambridge University Press, 1996, pp. 447–468.
- [24] S. Eilenberg, “The category  $C$ ,”
- [25] G. Plotkin, “Domains. pisa notes, 1983,” *University of Edinburgh*.
- [26] P. Freyd, “Algebraically complete categories,” in *Category Theory*. Springer, 1991, pp. 95–104.
- [27] M. B. Smyth and G. D. Plotkin, “The category-theoretic solution of recursive domain equations,” *SIAM Journal on Computing*, vol. 11, no. 4, pp. 761–783, 1982.
- [28] J. Girard, “Linear logic,” *Theor. Comput. Sci.*, vol. 50, pp. 1–102, 1987. [Online]. Available: [https://doi.org/10.1016/0304-3975\(87\)90045-4](https://doi.org/10.1016/0304-3975(87)90045-4)
- [29] C. Grellois and P.-A. Melliès, “An infinitary model of linear logic,” in *International Conference on Foundations of Software Science and Computation Structures*. Springer, 2015, pp. 41–55.
- [30] P. S. Mulry, “Strong monads, algebras and fixed points,” *Applications of Categories in Computer Science*, vol. 177, pp. 202–216, 1992.
- [31] J. Climent Vidal and J. Soliveres Tur, “A 2-categorical generalization of the concept of institution,” *Studia Logica*, vol. 95, no. 3, pp. 301–344, 2010.
- [32] J. Lambek, “A fixpoint theorem for complete categories,” *Mathematische Zeitschrift*, vol. 103, no. 2, pp. 151–161, 1968.
- [33] J. Bénabou, “Distributors at work,” 2000, lecture notes written by Thomas Streicher. [Online]. Available: <https://www2.mathematik.tu-darmstadt.de/~streicher/FIBR/DiWo.pdf>
- [34] Z. Galal, “A profunctorial Scott semantics,” in *5th International Conference on Formal Structures for Computation and Deduction (FSCD 2020)*. Schloss Dagstuhl-Leibniz-Zentrum für Informatik, 2020.
- [35] M. Fiore, “Analytic functors between presheaf categories over groupoids,” *Theoretical Computer Science*, vol. 546, pp. 120–131, 2014.
- [36] M. Fiore, N. Gambino, M. Hyland, and G. Winskel, “Relative pseudomonads, kleisli bicategories, and substitution monoidal structures,” *Selecta Mathematica*, vol. 24, no. 3, pp. 2791–2830, 2018.
- [37] I. Moerdijk and E. Palmgren, “Wellfounded trees in categories,” *Annals of Pure and Applied Logic*, vol. 104, no. 1-3, pp. 189–218, 2000.
- [38] P. Aczel, J. Adámek, and J. Velebil, “A coalgebraic view of infinite trees and iteration,” *Electronic Notes in Theoretical Computer Science*, vol. 44, no. 1, pp. 1–26, 2001.
- [39] N. Gambino and M. Hyland, “Wellfounded trees and dependent polynomial functors,” in *International Workshop on Types for Proofs and Programs*. Springer, 2004, pp. 210–225.
- [40] B. van den Berg and F. De Marchi, “Non-well-founded trees in categories,” *Annals of Pure and Applied Logic*, vol. 146, no. 1, pp. 40–59, 2007.
- [41] M. Hasegawa, “Recursion from cyclic sharing: traced monoidal categories and models of cyclic lambda calculi,” in *International Conference on Typed Lambda Calculi and Applications*. Springer, 1997, pp. 196–213.
- [42] N. Benton and M. Hyland, “Traced premonoidal categories,” *RAIRO-Theoretical Informatics and Applications*, vol. 37, no. 4, pp. 273–299, 2003.

## A. Dinatural and strong dinatural transformations for 2-categories

We start by recalling the 1-categorical notions of dinatural and strong dinatural transformations and proceed with the 2-categorical generalizations.

## 1) Dinatural transformations:

**Definition A.1.** For categories  $\mathbb{C}, \mathbb{D}$  and functors  $F, G : \mathbb{C}^{\text{op}} \times \mathbb{C} \rightarrow \mathbb{D}$ , a *dinatural transformation*  $\theta : F \Rrightarrow G$  consists of a family of 1-cells  $\{\theta_c : F(c, c) \rightarrow G(c, c)\}_{c \in \mathbb{C}}$  indexed by the objects  $c$  in  $\mathbb{C}$  such that for every morphism  $f : c \rightarrow d$  in  $\mathbb{C}$ , the following hexagon commutes:

$$\begin{array}{ccccc}
 & & \theta_c & & \\
 & \nearrow F(f, c) & F(c, c) & \xrightarrow{\theta_c} & G(c, c) & \searrow G(c, f) \\
 & F(d, c) & & & G(c, d) \\
 & \searrow F(d, f) & F(d, d) & \xrightarrow{\theta_d} & G(d, d) & \nearrow G(f, d)
 \end{array}$$

**Definition A.2.** For 2-categories  $\mathcal{C}, \mathcal{D}$  and 2-functors  $F, G : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{D}$ , a *lax dinatural transformation*  $\theta : F \Rrightarrow G$  consists of:

- a family of 1-cells  $\{\theta_c : F(c, c) \rightarrow G(c, c)\}_{c \in \mathcal{C}}$  indexed by the objects  $c$  in  $\mathcal{C}$ ;
- for every 1-cell  $f : c \rightarrow d$  in  $\mathcal{C}$ , a 2-cell:

$$\begin{array}{ccccc}
 & \nearrow F(f, c) & F(c, c) & \xrightarrow{\theta_c} & G(c, c) & \searrow G(c, f) \\
 & F(d, c) & & \Downarrow \theta_f & G(c, d) \\
 & \searrow F(d, f) & F(d, d) & \xrightarrow{\theta_d} & G(d, d) & \nearrow G(f, d)
 \end{array}$$

satisfying the following axioms:

- 1) *unity*: for every object  $c \in \mathcal{C}$ ,  $\theta_{1_c} = \text{id}_{\theta_c}$ ,
- 2) *1-naturality*: for every 1-cells  $f : c \rightarrow d$  and  $g : d \rightarrow e$  in  $\mathcal{C}$ ,

$$\begin{array}{ccccc}
 & \nearrow F(f, c) & F(c, c) & \xrightarrow{\theta_c} & G(c, c) & \searrow G(c, f) \\
 & F(d, c) & & \Downarrow \theta_f & G(c, d) \\
 & \searrow F(d, f) & F(d, d) & \xrightarrow{\theta_d} & G(d, d) & \nearrow G(f, d) \\
 \theta_{gf} = & F(e, c) & = & F(d, d) & \xrightarrow{\theta_d} & G(d, d) & = & G(c, d) & \xrightarrow{G(c, g)} & G(c, e) \\
 & \nearrow F(g, c) & F(d, c) & \xrightarrow{F(d, f)} & F(d, d) & \searrow F(d, g) & & G(f, d) & \searrow G(c, g) & \\
 & F(e, c) & & & F(d, d) & & & G(d, d) & \searrow G(d, g) & G(d, e) & \nearrow G(f, e) \\
 & \searrow F(e, f) & F(e, d) & \xrightarrow{F(g, d)} & F(d, d) & \searrow F(e, g) & & G(d, d) & \searrow G(d, g) & G(d, e) & \nearrow G(f, e) \\
 & & & & F(e, e) & \xrightarrow{\theta_e} & G(e, e) & \xrightarrow{G(g, e)} & G(e, e) & & 
 \end{array}$$

- 3) *2-naturality*: for every 2-cell  $\alpha : f \Rightarrow f'$  in  $\mathcal{C}$ :



$$\begin{array}{ccc}
\begin{array}{c}
\begin{array}{ccccc}
& F(f,c) & \xrightarrow{\quad} & F(c,c) & \xrightarrow{\theta_c} & G(c,c) & \xrightarrow{G(c,f)} & \\
& \downarrow F(\alpha,c) & \searrow & & & \searrow G(\alpha,f) & & \\
F(d,c) & \xrightarrow{F(f',c)} & & & & & G(c,d) & \\
& \downarrow F(d,\alpha) & \searrow & & & \searrow G(\alpha,d) & & \\
& F(d,f') & \xrightarrow{\quad} & F(d,d) & \xrightarrow{\theta_d} & G(d,d) & \xrightarrow{G(f',d)} & \\
& & & & & & & 
\end{array} \\
\Downarrow \theta_{f'} \\
\begin{array}{ccccc}
& F(d,c) & \xrightarrow{F(d,f)} & F(d,d) & \xrightarrow{\theta_d} & G(d,d) & \xrightarrow{G(f',d)} & \\
& \downarrow F(d,\alpha) & \searrow & & & \searrow G(\alpha,d) & & \\
& F(d,f') & \xrightarrow{\quad} & F(d,d) & \xrightarrow{\theta_d} & G(d,d) & \xrightarrow{G(f',d)} & 
\end{array}
\end{array}
= 
\begin{array}{ccc}
\begin{array}{ccccc}
& F(f,c) & \xrightarrow{\quad} & F(c,c) & \xrightarrow{\theta_c} & G(c,c) & \xrightarrow{G(c,f)} & \\
& \downarrow F(\alpha,c) & \searrow & & & \searrow G(\alpha,f) & & \\
F(d,c) & \xrightarrow{F(d,f)} & & & & & G(c,d) & \\
& \downarrow F(d,\alpha) & \searrow & & & \searrow G(\alpha,d) & & \\
& F(d,f') & \xrightarrow{\quad} & F(d,d) & \xrightarrow{\theta_d} & G(d,d) & \xrightarrow{G(f',d)} & \\
& & & & & & & 
\end{array} \\
\Downarrow \theta_f \\
\begin{array}{ccccc}
& F(d,c) & \xrightarrow{F(d,f)} & F(d,d) & \xrightarrow{\theta_d} & G(d,d) & \xrightarrow{G(f',d)} & \\
& \downarrow F(d,\alpha) & \searrow & & & \searrow G(\alpha,d) & & \\
& F(d,f') & \xrightarrow{\quad} & F(d,d) & \xrightarrow{\theta_d} & G(d,d) & \xrightarrow{G(f',d)} & 
\end{array}
\end{array}
\end{array}$$

For an *oplax dinatural transformation*, the 2-cells  $\theta_f$  go in the opposite direction. When the 2-cells  $\theta_f$  are invertible, we obtain the notion of *pseudo dinatural transformation* and when they are strict identities, we obtain *strict dinatural transformations*.

**Definition A.3.** A *modification between lax dinatural transformation*  $\Phi : \theta \Rightarrow \delta : F \Rrightarrow G : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{D}$  consists of a family of 2-cells  $\{\Phi_c : \theta_c \Rightarrow \delta_c\}_{c \in \mathcal{C}}$  such that for every 1-cell  $f : c \rightarrow d$  in  $\mathcal{C}$  the following equality holds:

$$\begin{array}{ccc}
\begin{array}{c}
\begin{array}{ccccc}
& F(f,c) & \xrightarrow{\quad} & F(c,c) & \xrightarrow{\theta_c} & G(c,c) & \xrightarrow{G(c,f)} & \\
& \downarrow F(\alpha,c) & \searrow & & & \searrow G(\alpha,f) & & \\
F(d,c) & \xrightarrow{F(f',c)} & & & & & G(c,d) & \\
& \downarrow F(d,\alpha) & \searrow & & & \searrow G(\alpha,d) & & \\
& F(d,f') & \xrightarrow{\quad} & F(d,d) & \xrightarrow{\theta_d} & G(d,d) & \xrightarrow{G(f',d)} & \\
& & & & & & & 
\end{array} \\
\Downarrow \theta_f \\
\begin{array}{ccccc}
& F(d,c) & \xrightarrow{F(d,f)} & F(d,d) & \xrightarrow{\theta_d} & G(d,d) & \xrightarrow{G(f',d)} & \\
& \downarrow F(d,\alpha) & \searrow & & & \searrow G(\alpha,d) & & \\
& F(d,f') & \xrightarrow{\quad} & F(d,d) & \xrightarrow{\theta_d} & G(d,d) & \xrightarrow{G(f',d)} & 
\end{array}
\end{array}
= 
\begin{array}{ccc}
\begin{array}{ccccc}
& F(f,c) & \xrightarrow{\quad} & F(c,c) & \xrightarrow{\theta_c} & G(c,c) & \xrightarrow{G(c,f)} & \\
& \downarrow F(\alpha,c) & \searrow & & & \searrow G(\alpha,f) & & \\
F(d,c) & \xrightarrow{F(d,f)} & & & & & G(c,d) & \\
& \downarrow F(d,\alpha) & \searrow & & & \searrow G(\alpha,d) & & \\
& F(d,f') & \xrightarrow{\quad} & F(d,d) & \xrightarrow{\theta_d} & G(d,d) & \xrightarrow{G(f',d)} & \\
& & & & & & & 
\end{array} \\
\Downarrow \delta_f \\
\begin{array}{ccccc}
& F(d,c) & \xrightarrow{F(d,f)} & F(d,d) & \xrightarrow{\delta_d} & G(d,d) & \xrightarrow{G(f',d)} & \\
& \downarrow F(d,\alpha) & \searrow & & & \searrow G(\alpha,d) & & \\
& F(d,f') & \xrightarrow{\quad} & F(d,d) & \xrightarrow{\delta_d} & G(d,d) & \xrightarrow{G(f',d)} & 
\end{array}
\end{array}$$

Similarly to the 1-categorical case, we cannot compose lax dinatural transformations horizontally and therefore the following data

- 0-cells: mixed variance 2-functors  $F, G : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{D}$ ;
- 1-cells: lax dinatural transformations  $\theta : F \Rrightarrow G$ ;
- 2-cells: modifications  $\Phi : \theta \Rightarrow \delta$  between them.

does not constitute a 2-category and we need the notion of strong lax dinatural transformation to make it compositional.

## 2) Strong dinatural transformations:

**Definition A.4.** For categories  $\mathbb{C}, \mathbb{D}$  and functors  $F, G : \mathbb{C}^{\text{op}} \times \mathbb{C} \rightarrow \mathbb{D}$ , a *strong dinatural transformation*  $\gamma : F \Rrightarrow G$  consists of a family of 1-cells  $\{\gamma_c : F(c, c) \rightarrow G(c, c)\}_{c \in \mathbb{C}}$  indexed by the objects  $c$  in  $\mathbb{C}$  such that for every morphism  $f : c \rightarrow d$  in  $\mathbb{C}$  and for every span  $F(c, c) \xleftarrow{q_c} Q \xrightarrow{q_d} F(d, d)$  in  $\mathbb{D}$ , if the square

$$\begin{array}{ccc}
& F(c,c) & \xrightarrow{F(c,f)} & F(c,d) \\
q_c \nearrow & & & \\
Q & & & \\
q_d \searrow & & & \\
& F(d,d) & \xrightarrow{F(f,d)} & 
\end{array}$$

commutes, then so does the hexagon:

$$\begin{array}{ccccc}
& & F(c,c) & \xrightarrow{\gamma_c} & G(c,c) & \xrightarrow{G(c,f)} & \\
& & \downarrow q_c & & \downarrow \gamma_c & & \\
& & Q & & & & \\
& & \downarrow q_d & & \downarrow \gamma_d & & \\
& & F(d,d) & \xrightarrow{\gamma_d} & G(d,d) & \xrightarrow{G(f,d)} & 
\end{array}$$

For functors  $F, G, H : \mathbb{C}^{\text{op}} \times \mathbb{C} \rightarrow \mathbb{D}$  and strong dinatural transformations  $\gamma : F \rightrightarrows G$  and  $\delta : G \rightrightarrows H$ , the transformation

$$\{\delta_c \circ \gamma_c : F(c, c) \rightarrow H(c, c)\}_{c \in \mathbb{C}}$$

is also strongly dinatural so that strong dinatural transformations form a category. To obtain the 2-dimensional analogue, we first need to consider the notion of mixed variance cones:

**Definition A.5** (mixed variance cones). Let  $\mathcal{C}, \mathcal{D}$  be 2-categories and  $F : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{D}$  be a 2-functor. For an object  $Q$  of  $\mathcal{D}$ , a *lax cone* of  $Q$  over  $F$  consists of:

- a family of 1-cells  $\{q_c : Q \rightarrow F(c, c)\}_{c \in \mathcal{C}}$  indexed by the objects  $c$  in  $\mathcal{C}$ ;
- for every 1-cell  $f : c \rightarrow d$  in  $\mathcal{C}$ , a 2-cell:

$$\begin{array}{ccc} & q_c & \rightarrow F(c, c) \xrightarrow{F(c, f)} \\ & \searrow & \downarrow q_f \quad \nearrow \\ Q & & F(c, d) \\ & \nearrow & \downarrow q_d \quad \searrow \\ & q_d & \rightarrow F(d, d) \xrightarrow{F(f, d)} \end{array}$$

satisfying the following axioms:

- 1) unity:  $q_{1_c} = \text{id}_{q_c}$
- 2) 1-naturality: for 1-cells  $f : c \rightarrow d$  and  $g : d \rightarrow e$ ,

$$\begin{array}{ccc} & q_c & \rightarrow F(c, c) \xrightarrow{F(c, gf)} \\ & \searrow & \downarrow q_{gf} \quad \nearrow \\ Q & & F(c, e) \\ & \nearrow & \downarrow q_e \quad \searrow \\ & q_e & \rightarrow F(e, e) \xrightarrow{F(gf, d)} \end{array} = \begin{array}{ccccc} & & F(c, f) & \rightarrow & F(c, d) \\ & q_c & \nearrow & & \nearrow \\ & \searrow & \downarrow q_f & & \downarrow q_g \\ Q & \xrightarrow{q_d} & F(d, d) & \xrightarrow{F(f, d)} & F(c, e) \\ & \nearrow & \downarrow q_g & & \nearrow \\ & q_e & \rightarrow & F(e, e) & \xrightarrow{F(d, g)} \\ & & \searrow & & \searrow \\ & & F(g, e) & \rightarrow & F(d, e) \end{array}$$

- 3) 2-naturality: for every 2-cell  $\alpha : f \Rightarrow f'$ ,

$$\begin{array}{ccc} & q_c & \rightarrow F(c, c) \xrightarrow{F(c, f)} \\ & \searrow & \downarrow q_f \quad \nearrow \\ Q & & F(c, d) \\ & \nearrow & \downarrow q_{f'} \quad \searrow \\ & q_d & \rightarrow F(d, d) \xrightarrow{F(f', d)} \end{array} = \begin{array}{ccc} & q_c & \rightarrow F(c, c) \xrightarrow{F(c, f)} \\ & \searrow & \downarrow q_f \quad \nearrow \\ Q & & F(c, d) \\ & \nearrow & \downarrow q_d \quad \searrow \\ & q_d & \rightarrow F(d, d) \xrightarrow{F(f', d)} \end{array}$$

**Definition A.6** (morphism of mixed variance cones). For two lax cones  $(Q, q)$  and  $(P, p)$  over  $F : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{D}$ , a lax morphism from  $(P, p)$  to  $(Q, q)$  consists of

- a 1-cell  $u : P \rightarrow Q$  in  $\mathcal{D}$ ,
- a family of 2-cells  $\{\Gamma_c : p_c \rightarrow q_c u\}_{c \in \mathcal{C}}$  indexed by the objects  $c$  in  $\mathcal{C}$

such that for all  $f : c \rightarrow d$  in  $\mathcal{C}$ , the following two diagrams are equal:

**Definition A.7** (strong lax dinatural transformation). For 2-categories  $\mathcal{C}$ ,  $\mathcal{D}$  and 2-functors  $F, G : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{D}$ , a *strong lax dinatural transformation*  $\gamma : F \rightrightarrows G$  consists of:

- a family of 1-cells  $\{\gamma_c : F(c, c) \rightarrow G(c, c)\}_{c \in \mathcal{C}}$  indexed by the objects  $c$  in  $\mathcal{C}$ ;
- for every lax cone  $(Q, q)$  over  $F$  and for every 1-cell  $f : c \rightarrow d$  in  $\mathcal{C}$ , a 2-cell:

satisfying the following axioms:

- 1) unity: for every object  $c \in \mathcal{C}$ ,  $\gamma_{q, 1_c} = \text{id}_{\gamma_c \circ q_c}$ ,
- 2) 1-naturality: for every 1-cells  $f : c \rightarrow d$  and  $g : d \rightarrow e$ ,

- 3) 2-naturality: for every 2-cell  $\alpha : f \Rightarrow f'$  in  $\mathcal{C}$ :

- 4) for every morphism of lax cones  $(u, \Gamma) : (P, p) \rightarrow (Q, q)$  over  $F$  and for every  $f : c \rightarrow d$ ,

$$\begin{array}{ccc}
\begin{array}{ccccc}
& p_c & & \gamma_c & \\
& \curvearrowright & & \downarrow \gamma_{q,f} & \\
P & \xrightarrow{u} & Q & & G(c,d) \\
& \searrow q_d & \nearrow q_c & & \nearrow G(f,d) \\
& F(d,d) & \xrightarrow{\gamma_d} & G(d,d) & \\
& & & & \nearrow G(f,d)
\end{array}
& = &
\begin{array}{ccccc}
& p_c & & \gamma_c & \\
& \curvearrowright & & \downarrow \gamma_{p,f} & \\
P & \xrightarrow{u} & Q & & G(c,d) \\
& \searrow q_d & \nearrow q_c & & \nearrow G(f,d) \\
& F(d,d) & \xrightarrow{\gamma_d} & G(d,d) & \\
& & & & \nearrow G(f,d)
\end{array}
\end{array}$$

**Definition A.8.** A modification between strong lax dinatural transformation  $\Phi : \gamma \Rightarrow \delta : F \rightrightarrows G$  consists of a family of 2-cells  $\{\Phi_c : \gamma_c \Rightarrow \delta_c\}_{c \in \mathcal{C}}$  such that for every lax cone  $(Q, q)$  over  $F$  and 1-cell  $f : c \rightarrow d$  in  $\mathcal{C}$  the following equality holds:

$$\begin{array}{ccc}
\begin{array}{ccccc}
& q_c & & \gamma_c & \\
& \curvearrowright & & \downarrow \gamma_{q,f} & \\
Q & \xrightarrow{q_c} & F(c,c) & \xrightarrow{\gamma_c} & G(c,c) \\
& \searrow q_d & \nearrow q_c & & \nearrow G(f,f) \\
& F(d,d) & \xrightarrow{\gamma_d} & G(d,d) & \\
& & & & \nearrow G(f,d)
\end{array}
& = &
\begin{array}{ccccc}
& q_c & & \gamma_c & \\
& \curvearrowright & & \downarrow \delta_{q,f} & \\
Q & \xrightarrow{q_c} & F(c,c) & \xrightarrow{\delta_c} & G(c,c) \\
& \searrow q_d & \nearrow q_c & & \nearrow G(f,f) \\
& F(d,d) & \xrightarrow{\gamma_d} & G(d,d) & \\
& & & & \nearrow G(f,d)
\end{array}
\end{array}$$

**Definition A.9.** For 2-categories  $\mathcal{C}$  and  $\mathcal{D}$ , we denote by  $\mathbf{StDin}(\mathcal{C}, \mathcal{D})$  the following data:

- 0-cells: mixed variance 2-functors  $F, G : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{D}$ ;
- 1-cells: strong lax dinatural transformations  $\gamma : F \rightrightarrows G$ ;
- 2-cells: modifications  $\Phi : \gamma \Rightarrow \delta$  between them.

**Proposition A.10.** For 2-categories  $\mathcal{C}$  and  $\mathcal{D}$ ,  $\mathbf{StDin}(\mathcal{C}, \mathcal{D})$  defined above forms a 2-category.

B. Proofs of Section III

**Lemma III.2.** There exists a 1-cell  $t : \mathbf{1} \rightarrow \Phi$  and an invertible 2-cell  $\tau : t \Rightarrow Rt$  in  $\mathcal{D}$

$$\begin{array}{ccc}
& t & \\
& \curvearrowright & \\
\mathbf{1} & \xrightarrow{t} & \Phi \\
& \nearrow \tau & \downarrow R \\
& \xrightarrow{t} & \Phi
\end{array}$$

satisfying the following universal property: for any 1-cells  $v, w : \mathbf{1} \rightarrow \Phi$  in  $\mathcal{D}$  and invertible 2-cells  $\nu : v \Rightarrow Rv$  and  $\omega : w \Rightarrow Rw$ , there exists a unique invertible 2-cell  $\psi : v \Rightarrow w$  in  $\mathcal{D}$  such that

$$\begin{array}{ccc}
\begin{array}{ccc}
& w & \\
& \curvearrowright & \\
\mathbf{1} & \xrightarrow{w} & \Phi \\
& \nearrow \omega & \downarrow R \\
& \nearrow \psi & \downarrow R \\
& v & \xrightarrow{v} & \Phi
\end{array}
& = &
\begin{array}{ccc}
& w & \\
& \curvearrowright & \\
\mathbf{1} & \xrightarrow{w} & \Phi \\
& \nearrow \psi & \downarrow R \\
& v & \xrightarrow{v} & \Phi
\end{array}
\end{array}$$

*Proof.* By Lemma II.10,  $R$  is part of an adjoint equivalence  $(R : \mathbf{!}\Phi \rightarrow \Phi, L : \Phi \rightarrow \mathbf{!}\Phi, \eta : \text{id} \xRightarrow{\cong} RL, \varepsilon : LR \xRightarrow{\cong} \text{id})$  where  $L : \Phi \rightarrow \mathbf{!}\Phi$  is pseudo-final. Therefore, there exists a 1-cell  $t : \mathbf{!}\mathbf{1} \rightarrow \Phi$  and a 2-cell  $\xi$  in  $\mathcal{C}$  as below:

$$\begin{array}{ccc}
\mathbf{!}\mathbf{1} & \xrightarrow{\delta_1} & \mathbf{!}\mathbf{1} \\
t \downarrow & \nearrow \xi & \downarrow !t \\
\Phi & \xrightarrow{L} & \mathbf{!}\Phi
\end{array}$$

and we define  $\tau$  as the following 2-cell in  $\mathcal{C}$

$$\tau := \begin{array}{ccc} !1 & \xrightarrow{\delta_1} & !!1 \\ t \downarrow & \nearrow \xi & \downarrow !t \\ \Phi & \xrightarrow{L} & !\Phi \xrightarrow{R} \Phi \\ & \nwarrow \eta & \uparrow \\ & 1 & \end{array}$$

which corresponds to a 2-cell with boundaries  $t \Rightarrow Rt$  in  $\mathcal{D}$  as desired.

Assume now that we have 2-cells  $\nu : v \Rightarrow Rv$  and  $\omega : w \Rightarrow Rw$  in  $\mathcal{D}$ . By the universal property of  $L$ , there exists a unique 2-cell  $\psi : v \Rightarrow w$  such that

$$\begin{array}{ccc} !1 & \xrightarrow{\delta_1} & !!1 \\ v \downarrow & \nearrow \omega & \downarrow !w \\ \Phi & \xrightarrow{L} & !\Phi \xrightarrow{R} \Phi \\ & \nwarrow \varepsilon & \uparrow 1 \end{array} \quad = \quad \begin{array}{ccc} !1 & \xrightarrow{\delta_1} & !!1 \\ v \downarrow & \nearrow \nu & \downarrow !v \\ \Phi & \xrightarrow{L} & !\Phi \xrightarrow{R} \Phi \\ & \nwarrow \varepsilon & \uparrow 1 \end{array}$$

in  $\mathcal{C}$ . Cancelling the 2-cells  $\varepsilon$  on both sides using the adjunction identities

$$\begin{array}{ccc} !1 & \xrightarrow{\delta_1} & !!1 \\ v \downarrow & \nearrow \omega & \downarrow !w \\ \Phi & \xrightarrow{L} & !\Phi \xrightarrow{R} \Phi \\ & \nwarrow \varepsilon & \uparrow 1 \\ & \eta \Rightarrow & \\ & 1 & \end{array} \quad = \quad \begin{array}{ccc} !1 & \xrightarrow{\delta_1} & !!1 \\ v \downarrow & \nearrow \nu & \downarrow !v \\ \Phi & \xrightarrow{L} & !\Phi \xrightarrow{R} \Phi \\ & \nwarrow \varepsilon & \uparrow 1 \\ & \eta \Rightarrow & \\ & 1 & \end{array}$$

and rewriting the equality above in  $\mathcal{D}$ , we obtain that there exists a unique invertible 2-cell  $\psi : v \Rightarrow w$  in  $\mathcal{D}$  such that:

$$\begin{array}{ccc} w & \xrightarrow{\quad} & \Phi \\ 1 & \searrow \omega & \nearrow R \\ & \psi & \\ & v & \end{array} \quad = \quad \begin{array}{ccc} w & \xrightarrow{\quad} & \Phi \\ 1 & \searrow \psi & \nearrow R \\ & v & \end{array}$$

□

**Lemma III.3** (Naturality of  $\mathbf{fix}$ ). For a 2-cell  $\alpha : f \Rightarrow g$  in  $\mathcal{D}$ , we have  $\mathbf{fix}_g \circ \alpha^* = (\alpha \cdot \alpha^*) \circ \mathbf{fix}_f$ .

*Proof.* Immediate unfolding of the corresponding 2-cells. □

**Proposition III.4.** The 2-cells  $\mathbf{unif}_\gamma$  we constructed yield a strong pseudo-dinatural transformation and they verify the coherence axiom between  $\mathbf{fix}$  and  $\mathbf{unif}$ .

*Proof.* For the strong pseudo-dinatural axioms, we will only prove the 2-naturality axiom, as the other cases work similarly. Assume that we have the following equality in  $\mathcal{D}$ :

Let  $\phi : r \circ u_f \Rightarrow u_g$  be the unique 2-cell in  $\mathcal{C}$  such that:

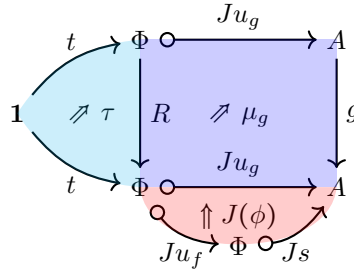
so that  $\mathbf{unif}_\rho = J\phi \cdot t : J(r) \circ J(u_f) \circ t \Rightarrow J(u_g) \circ t$ . We want to show that  $\mathbf{unif}_\gamma = \mathbf{unif}_\rho \circ (J\theta \cdot (J(u_f) \circ t))$  which is immediate from the equality below:

To prove that the operator  $((-)^*, \mathbf{fix}, \mathbf{unif})$  we constructed is a pseudo-fixpoint operator on  $\mathcal{D}$  uniform with respect to  $J$ , it only remains to show the coherence axiom between  $\mathbf{fix}$  and  $\mathbf{unif}$  i.e. that for a 2-cell

in  $\mathcal{D}$ , we have:

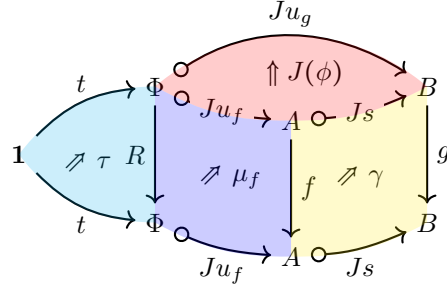
The left-hand side is equal to:





where  $\phi : s \circ u_f \Rightarrow u_g$  is the unique invertible 2-cell in  $\mathcal{C}$  such that:

so that the left-hand side is equal to the diagram below



which corresponds to the right-hand side of the desired equality.  $\square$

**Proposition III.5.** *The category  $\mathbf{Fix}(\mathcal{D}, J)$  is contractible i.e.  $\delta$  is an isomorphism of uniform pseudo-fixpoint operators and it is unique.*

*Proof.* We first show that  $\delta$  is a morphism of pseudo-fixpoint operators by proving that it commutes with the **fix** 2-cells:

The left hand diagram is equal to:

where the equality above follows from the coherence between  $\mathbf{fix}^\dagger$  and  $\mathbf{unif}^\dagger$ . By definition of  $\delta_0$ , the right-hand diagram is equal to:

and we obtain the desired equality. We now need to show that  $\delta$  is a morphism of uniform pseudo-fixpoint operators, *i.e.* for every square in  $\mathcal{D}$ :

we have:

By definition of  $\mathbf{unif}_\gamma^*$  and the 2-naturality axiom for  $\mathbf{unif}^\dagger$ , the left hand diagram is equal to:

where  $\phi$  is the unique 2-cell  $s \circ u_f \Rightarrow u_g$  such that:

The right hand diagram is equal to

where the equality above follows from the 1-naturality axiom for  $\text{unif}^\dagger$ .

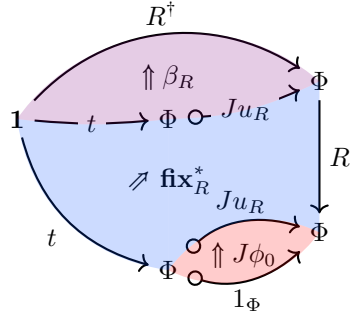
For uniqueness, assume that there exists another morphism of uniform pseudo-fixpoint operators  $\beta : (-)^* \rightarrow (-)^\dagger$ . We proceed to show that for every  $f : A \rightarrow A$ ,  $\delta_f = \beta_f$ . Let  $\phi_0$  be the unique invertible 2-cell  $1_\Phi \Rightarrow u_R$  in  $\mathcal{C}$  such that:

which we obtain from Lemma III.1. We show that

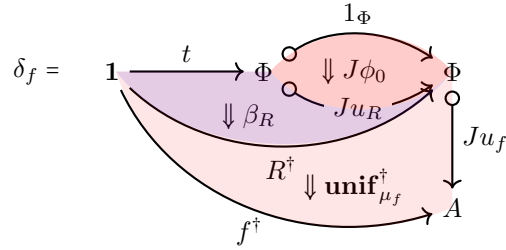
by proving the following equality:

and using the universal property of  $\delta_0$ . By definition of  $\phi_0$ , the left-hand side is equal to:

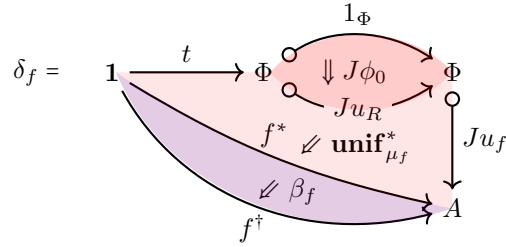
Since  $\beta$  is a morphism of pseudo fixpoint operators, the right-hand side is equal to:



so both sides are equal as desired by definition of  $\text{fix}_R^*$ . We therefore obtain that

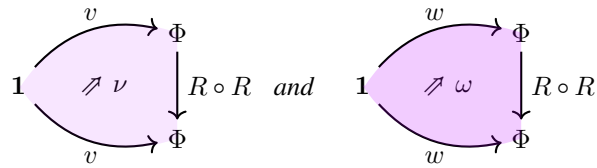


Since  $\beta$  is a morphism of uniform pseudo-fixpoint operators,  $\delta_f$  is equal to:

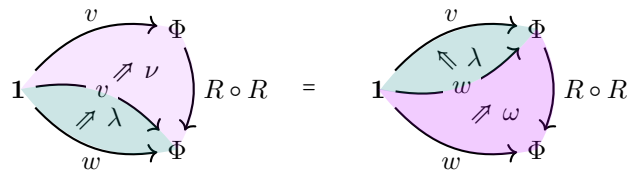


Using the universal property of  $\text{unif}_{\mu_f}^*$ , we obtain that  $\text{unif}_{\mu_f}^* \circ (J\phi_0 \cdot t) = \text{id}$  giving us the desired conclusion.  $\square$

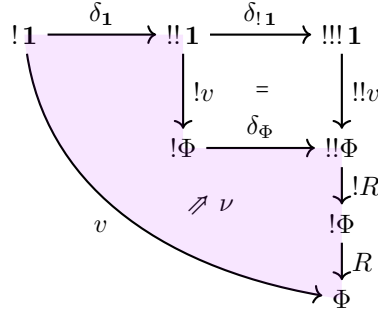
**Lemma IV.2.** For 1-cells  $v, w : 1 \rightarrow \Phi$  in  $\mathcal{D}$  and 2-cells



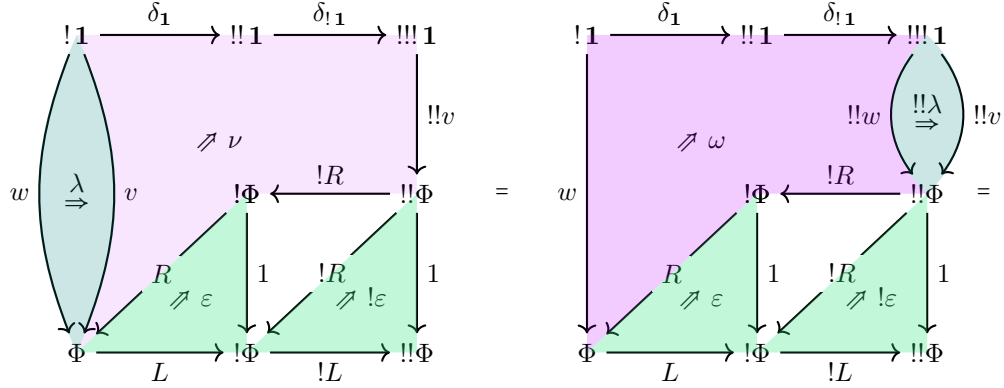
in  $\mathcal{D}$ , there exists a unique invertible 2-cell  $\lambda : w \Rightarrow v$  in  $\mathcal{D}$  such that



*Proof.* The 2-cell  $\nu$  in  $\mathcal{D}$  has the following boundary in  $\mathcal{C}$ :



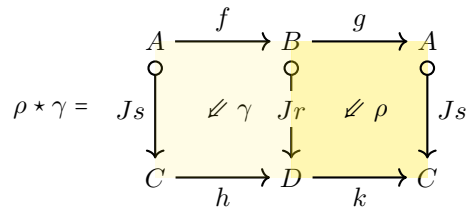
Since  $!L \circ L$  is a pseudo-final  $!!$ -coalgebra, there is a unique isomorphism  $\lambda: w \Rightarrow v$  such that:



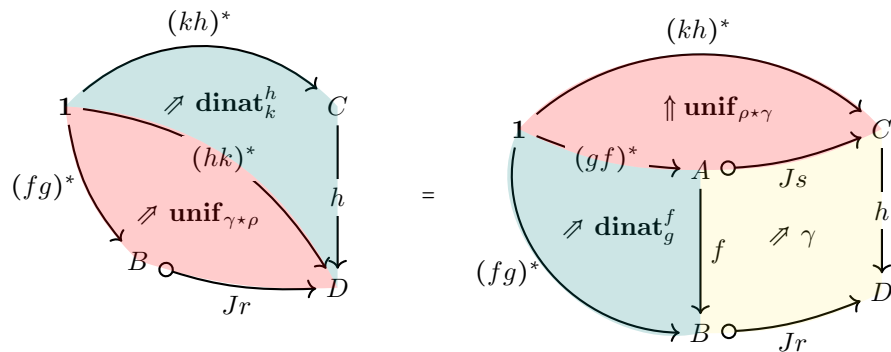
Cancelling the 2-cells  $\varepsilon$  and  $!\varepsilon$  on both sides using the adjunction identities and rewriting the equality above in  $\mathcal{D}$ , we obtain the desired equality in  $\mathcal{D}$ .  $\square$

**Proposition IV.3.** The 2-cells  $\text{dinat}_g^f$  we constructed yield a pseudo-dinatural transformation and they verify the coherence axiom between  $\text{dinat}$  and  $\text{unif}$ .

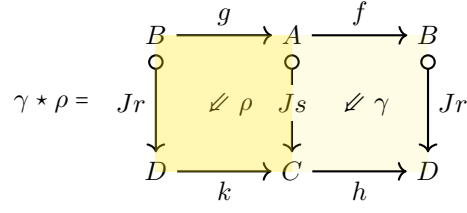
*Proof.* **Coherence between  $\text{dinat}$  and  $\text{unif}$ :** for two squares in  $\mathcal{D}$  of the form



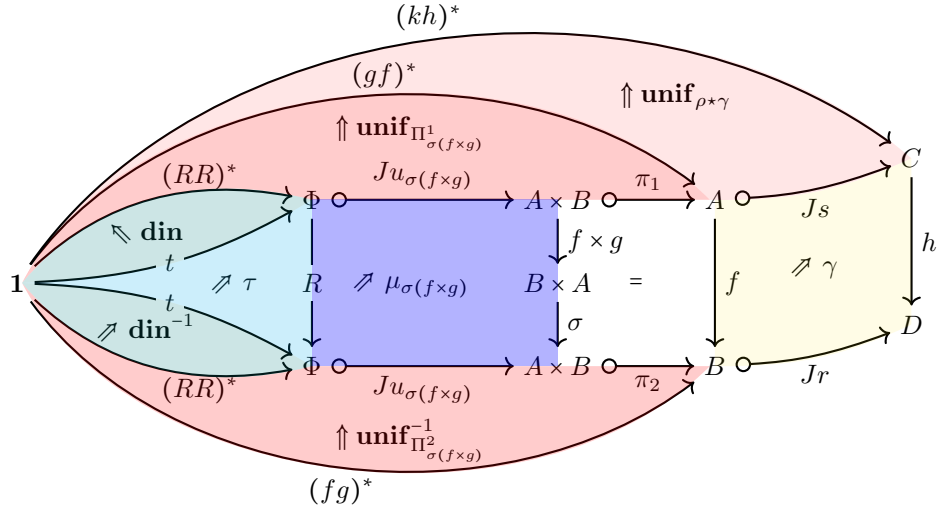
we have:



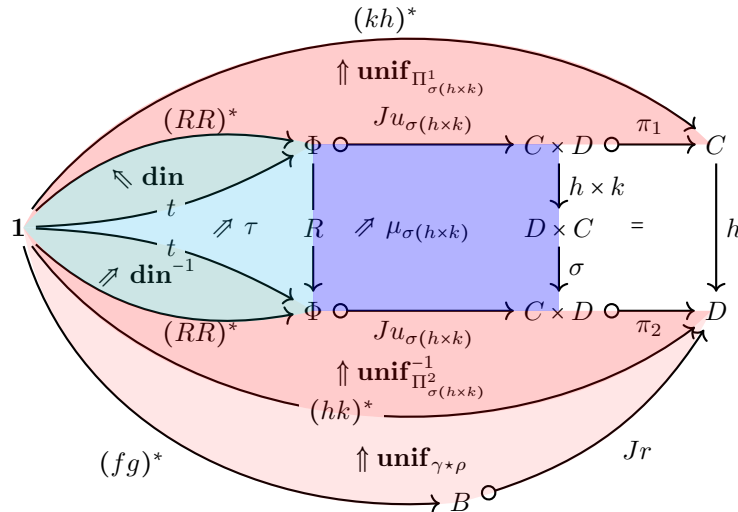
where  $\gamma \star \rho$  corresponds to the following 2-cell:



The right-hand diagram is equal to:



and the left-hand diagram is equal to:





Let  $\phi : (s \times r) \circ u_{\sigma(f \times g)} \Rightarrow u_{\sigma(h \times k)}$  be the unique invertible 2-cell in  $\mathcal{C}$  such that

$$\begin{array}{ccc}
 \Phi & \xrightarrow{R} & \Phi \\
 \downarrow Ju_{\sigma(f \times g)} & \swarrow \mu_{\sigma(f \times g)} & \downarrow Ju_{\sigma(f \times g)} \\
 Ju_{\sigma(h \times k)} \leftarrow J\phi \leftarrow A \times B & \xrightarrow{f \times g} & B \times A \xrightarrow{\sigma} A \times B \\
 \downarrow J(s \times r) & \swarrow \gamma \times \rho & \downarrow J(r \times s) \\
 C \times D & \xrightarrow{h \times k} & D \times C \xrightarrow{\sigma} C \times D
 \end{array}
 =
 \begin{array}{ccc}
 \Phi & \xrightarrow{R} & \Phi \\
 \downarrow Ju_{\sigma(h \times k)} & \swarrow \mu_{\sigma(h \times k)} & \downarrow Ju_{\sigma(h \times k)} \\
 Ju_{\sigma(h \times k)} \leftarrow J\phi \leftarrow A \times B & \xrightarrow{f \times g} & B \times A \xrightarrow{\sigma} A \times B \\
 \downarrow J(s \times r) & \swarrow \gamma \times \rho & \downarrow J(r \times s) \\
 C \times D & \xrightarrow{h \times k} & D \times C \xrightarrow{\sigma} C \times D
 \end{array}$$

The left-hand diagram of the desired equality is then equal to

$$\begin{array}{c}
 (kh)^* \\
 \uparrow \\
 (RR)^* \quad \Phi \quad \xrightarrow{Ju_{\sigma(f \times g)}} \quad C \times D \quad \xrightarrow{J(s \times r)} \quad C \times D \quad \xrightarrow{\pi_1} \quad C \\
 \uparrow \text{din} \quad \uparrow t \quad \uparrow \tau \quad \uparrow R \quad \uparrow \mu_{\sigma(f \times g)} \quad \uparrow D \times C \quad \uparrow J(r \times s) \quad \uparrow \sigma \quad \uparrow h \times k \\
 1 \quad \xrightarrow{t} \quad \Phi \quad \xrightarrow{Ju_{\sigma(f \times g)}} \quad C \times D \quad \xrightarrow{J(s \times r)} \quad C \times D \quad \xrightarrow{\pi_2} \quad D \\
 \uparrow \text{din}^{-1} \quad \uparrow t \quad \uparrow \tau \quad \uparrow R \quad \uparrow \mu_{\sigma(f \times g)} \quad \uparrow D \times C \quad \uparrow J(r \times s) \quad \uparrow \sigma \quad \uparrow h \times k \\
 (RR)^* \quad \Phi \quad \xrightarrow{Ju_{\sigma(f \times g)}} \quad C \times D \quad \xrightarrow{J(s \times r)} \quad C \times D \quad \xrightarrow{\pi_2} \quad D \\
 \uparrow \text{unif}_{\Pi^2_{\sigma(h \times k)}}^{-1} \quad \uparrow J\phi^{-1} \quad \uparrow Ju_{\sigma(h \times k)} \quad \uparrow (hk)^* \quad \uparrow \text{unif}_{\gamma \star \rho} \quad \uparrow Jr \\
 (fg)^* \quad \xrightarrow{\quad} \quad B
 \end{array}$$

From the universal property of **unif**, we can now verify that

$$\text{unif}_{\Pi^1_{\sigma(h \times k)}} \circ (\pi_1 \cdot J\phi \cdot (RR)^*) = \text{unif}_{\rho \star \gamma} \circ (Js \cdot \text{unif}_{\Pi^1_{\sigma(f \times g)}}) \quad \text{and}$$

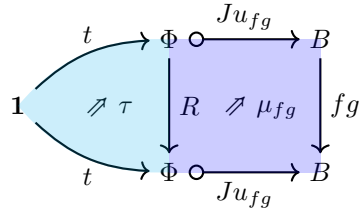
$$(\pi_2 \cdot J\phi^{-1} \cdot (RR)^*) \circ \text{unif}_{\Pi^2_{\sigma(h \times k)}}^{-1} \circ \text{unif}_{\gamma \star \rho} = Jr \cdot \text{unif}_{\Pi^2_{\sigma(f \times g)}}^{-1}$$

to obtain the desired equality.

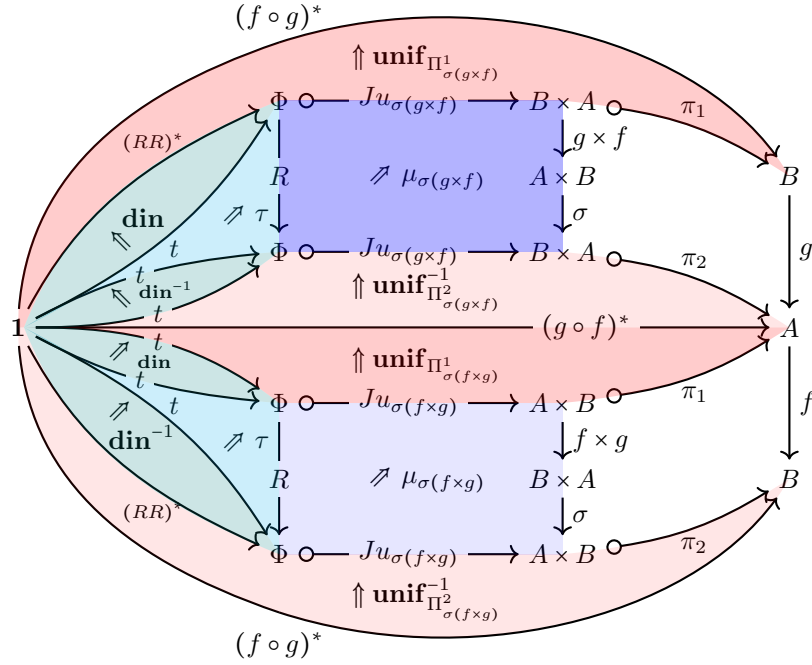
**Coherence between dinat and fix:** for 1-cells  $f : A \rightarrow B$  and  $g : B \rightarrow A$  in  $\mathcal{D}$ , we want to show:

$$\begin{array}{ccc}
 (f \circ g)^* & \xrightarrow{\quad} & B \\
 \uparrow \text{dinat}_f^g & \uparrow g & \\
 1 - (g \circ f)^* & \xrightarrow{\quad} & A \\
 \uparrow \text{dinat}_g^f & \uparrow f & \\
 (f \circ g)^* & \xrightarrow{\quad} & B
 \end{array}
 =
 \begin{array}{ccc}
 (f \circ g)^* & \xrightarrow{\quad} & B \\
 \uparrow \text{fix}_{fg} & \uparrow & \\
 1 & \xrightarrow{\quad} & A \\
 \uparrow & \uparrow f & \\
 (f \circ g)^* & \xrightarrow{\quad} & B
 \end{array}$$

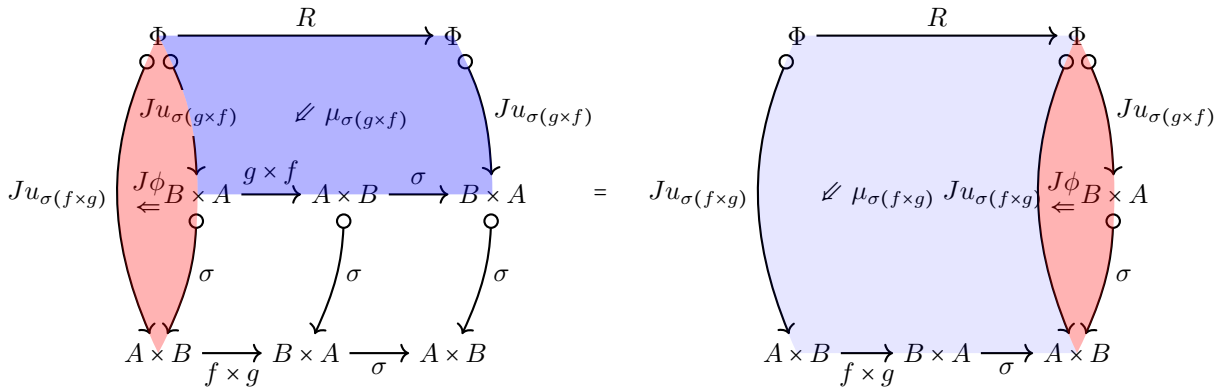
The right-hand diagram is equal to:



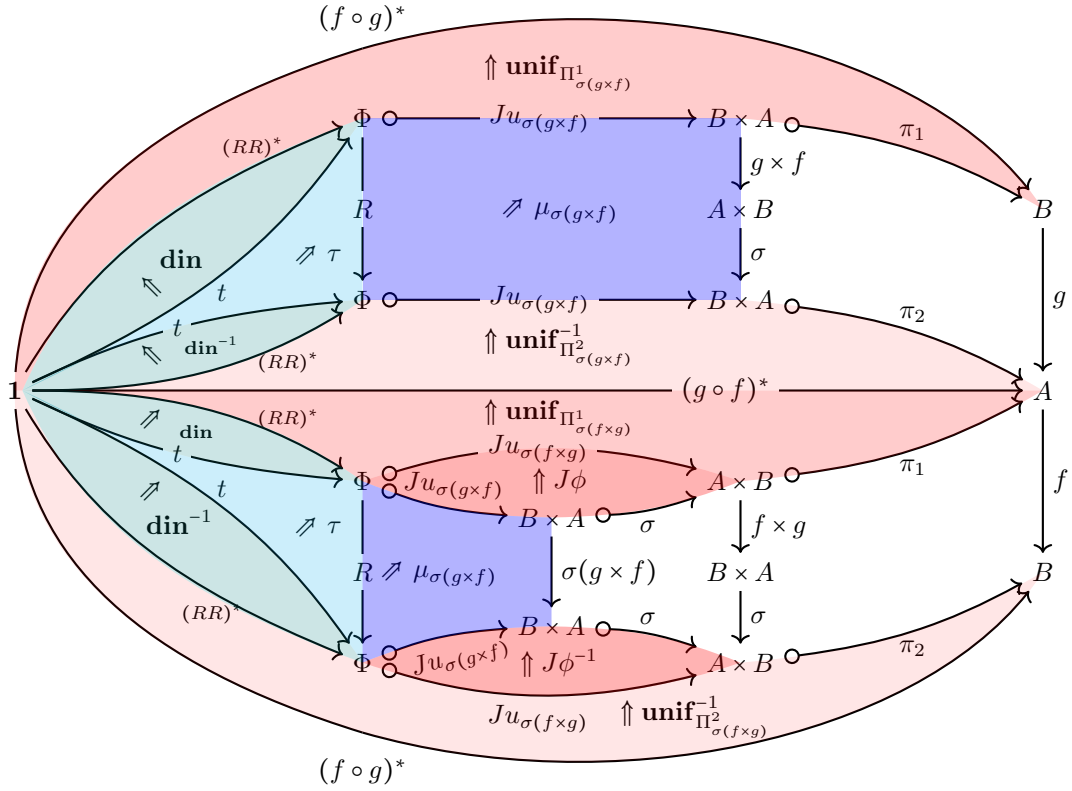
and the left-hand diagram is equal to:



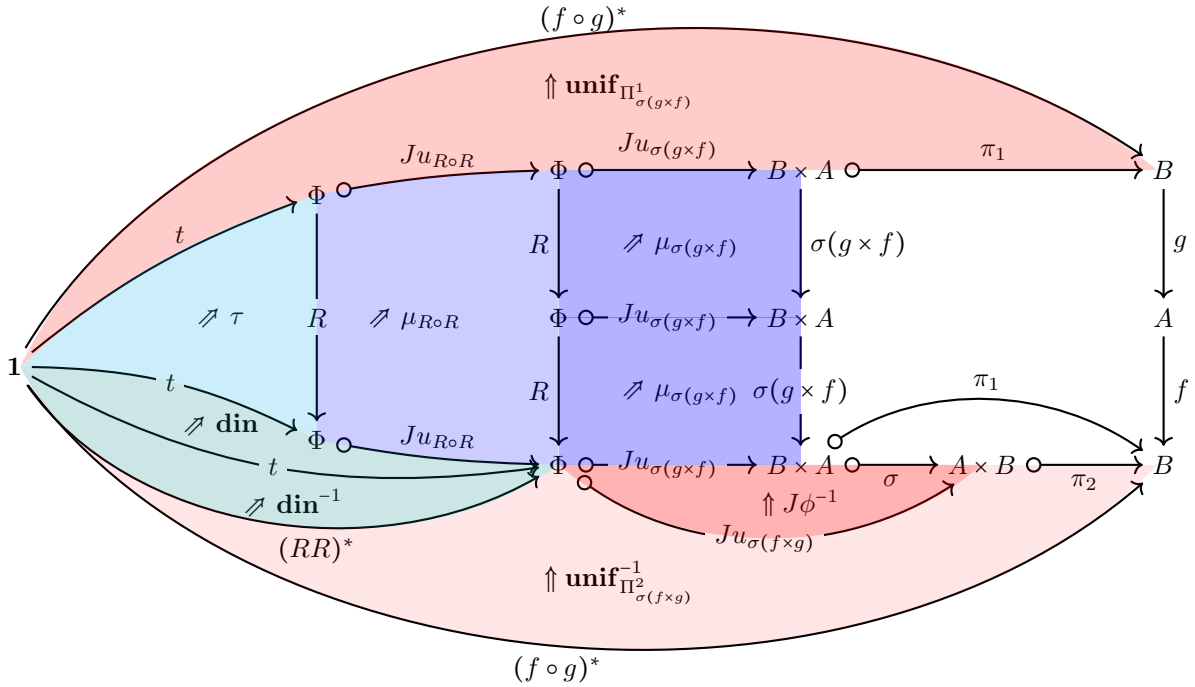
We use a similar reasoning as before and let  $\phi : \sigma \circ u_{\sigma(g \times f)} \Rightarrow u_{\sigma(f \times g)}$  be the unique invertible 2-cell in  $\mathcal{C}$  such that



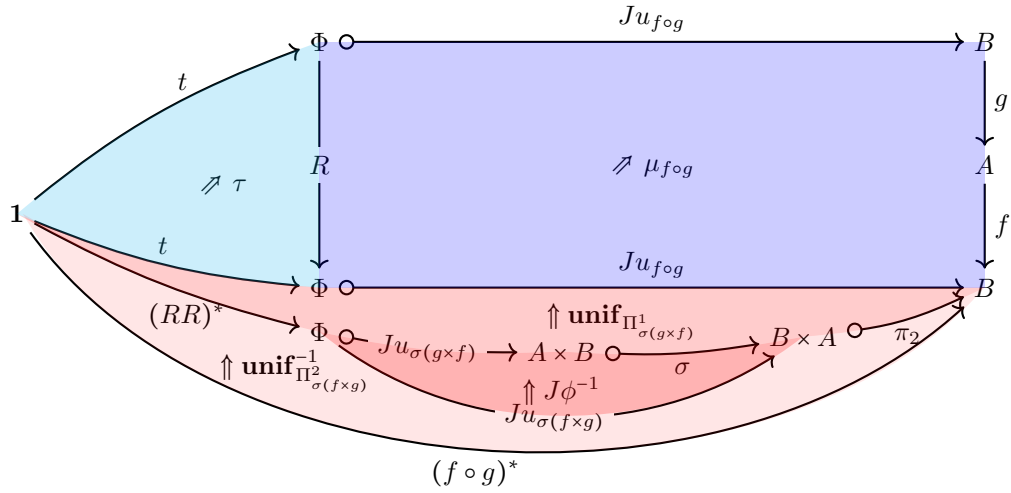
so that we can rewrite the previous diagram as:



Using the universal property of  $\mathbf{unif}$ , we can show that  $\mathbf{unif}_{\Pi^1_{\sigma(g \times f)}}^{-1} \circ \mathbf{unif}_{\Pi^1_{\sigma(f \times g)}} \circ (\pi_1 \cdot J\phi \cdot t) = \text{id}$  so that we can simplify the diagram to

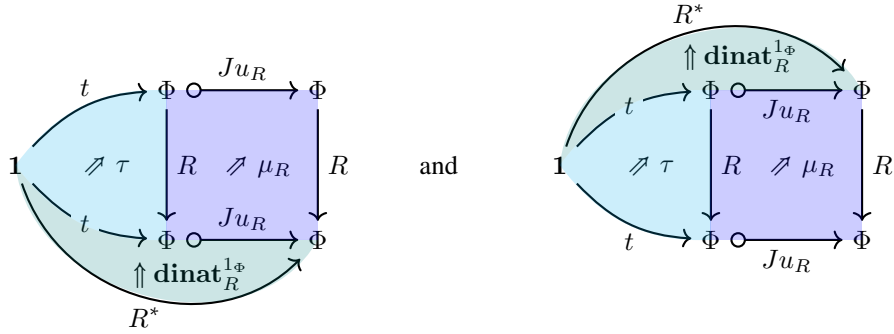


Using the definition of  $\mathbf{unif}_{\Pi^1_{\sigma(g \times f)}}$  and the coherence between  $\mathbf{unif}$  and  $\mathbf{fix}$  we rewrite the diagram above as:

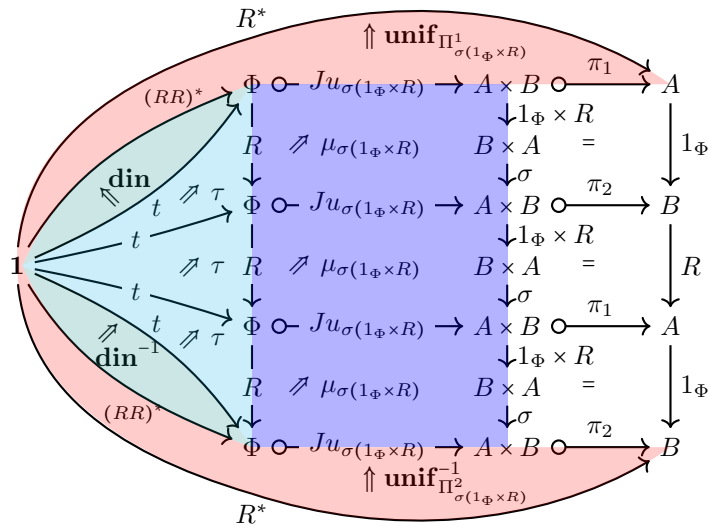


It remains to verify that  $\text{unif}_{\Pi_{\sigma(g \times f)}^1} \circ (\pi_2 \cdot J\phi^{-1} \cdot (RR)^*) \circ \text{unif}_{\Pi_{\sigma(f \times g)}^{-1}}^1 = \text{id}$  and we obtain the desired equality.

**Pseudo-dinaturality axioms for dinat:** We only show the unity axiom as the others work similarly. We want to show that for all  $f : A \rightarrow A$ ,  $\text{dinat}_f^{1_A} = \text{id}_{f^*}$ . We start by proving it for  $f = R$  and obtain the general statement by uniformity. We first note that the two diagrams below



are both equal to:



Using Lemma III.2, we conclude that  $\text{dinat}_R^{1_\Phi} = \text{id}_{R^*}$ .

For a general  $f : A \rightarrow A$  in  $\mathcal{D}$ , to show that  $\mathbf{dinat}_f^{1_A} = \text{id}_{f^*}$ , we make use of the coherence axiom between **unif** and **dinat** we proved previously. Consider the two squares:

$$\begin{array}{ccccc}
 \Phi & \xrightarrow{1_\Phi} & \Phi & \xrightarrow{R} & \Phi \\
 \downarrow Ju_f & & \downarrow Ju_f & \searrow \mu_f & \downarrow Ju_f \\
 A & \xrightarrow{1_A} & A & \xrightarrow{f} & A
 \end{array}$$

we obtain:

$$\begin{array}{c}
 \begin{array}{ccc}
 & f^* & \\
 \nearrow & & \searrow \\
 1 & \xrightarrow{\mathbf{dinat}_f^{1_A}} & A \\
 \nearrow & & \searrow \\
 R^* & \xrightarrow{\text{unif}_{\mu_f}} & \Phi \\
 \searrow & & \nearrow \\
 & Ju_f & \\
 & A & 
 \end{array}
 \quad = \quad
 \begin{array}{ccc}
 & f^* & \\
 \nearrow & & \searrow \\
 1 & \xrightarrow{\mathbf{dinat}_R^{1_\Phi}} & \Phi \\
 \nearrow & & \searrow \\
 R^* & \xrightarrow{R^*} & \Phi \\
 \searrow & & \nearrow \\
 & Ju_f & \\
 & A & 
 \end{array}
 \end{array}$$

since we have shown that  $\mathbf{dinat}_R^{1_\Phi} = \text{id}_{R^*}$ , we obtain that  $\mathbf{dinat}_f^{1_A} = \text{id}_{f^*}$  from the equality above.  $\square$

**Lemma A.11.** For the pseudo-bifree algebra  $R$ , the 2-cell  $\mathbf{dinat}_R^R$  is equal to:

$$\mathbf{dinat}_R^R = 1 \begin{array}{c} \nearrow \\ \nearrow \text{din} \\ \nearrow t \\ \nearrow \tau \\ \nearrow \text{din}^{-1} \\ \searrow \\ \searrow (RR)^* \end{array}$$

*Proof.* Using Lemma III.2, there exists a unique invertible 2-cell  $\psi : t \Rightarrow (RR)^*$  such that:

$$\begin{array}{ccc}
 \begin{array}{ccc}
 & (RR)^* & \\
 \nearrow & & \searrow \\
 1 & \xrightarrow{\mathbf{dinat}_R^R} & R \\
 \nearrow & & \searrow \\
 & (RR)^* & \\
 & \psi & 
 \end{array}
 \quad = \quad
 \begin{array}{ccc}
 & (RR)^* & \\
 \nearrow & & \searrow \\
 1 & \xrightarrow{t} & \Phi \\
 \nearrow & & \searrow \\
 & \psi & \\
 & \tau & 
 \end{array}
 \end{array}$$

It suffices to show that  $\psi = \mathbf{din}$  by proving that the equality below holds:

$$\begin{array}{ccc}
 \begin{array}{ccc}
 & \Phi & \\
 \nearrow & & \searrow \\
 1 & \xrightarrow{t} & \Phi \\
 \nearrow & & \searrow \\
 & \psi & \\
 & \tau & 
 \end{array}
 \quad = \quad
 \begin{array}{ccc}
 & \Phi & \\
 \nearrow & & \searrow \\
 1 & \xrightarrow{t} & \Phi \\
 \nearrow & & \searrow \\
 & \psi & \\
 & \tau & 
 \end{array}
 \end{array}$$

By definition of  $\psi$  and the coherence axiom between **dinat** and **fix**, the left-hand diagram is equal to

and we obtain the desired equality.  $\square$

**Proposition IV.4.** *The category  $\mathbf{DinFix}(\mathcal{D}, J)$  is contractible i.e.  $\delta$  commutes with the dinaturality 2-cells and it is unique.*

*Proof.* Let  $((-)^{\dagger}, \mathbf{dinat}^{\dagger}, \mathbf{unif}^{\dagger})$  be another pseudo-dinatural fixpoint operators uniform with respect to  $J$  on  $\mathcal{D}$ . We show that the morphism  $\delta : ((-)^*, \mathbf{unif}^*) \rightarrow ((-)^{\dagger}, \mathbf{unif}^{\dagger})$  we defined in Section III also commutes with the structural 2-cells **dinat**, i.e. it satisfies the following coherence for every  $f : A \rightarrow B$  and  $g : B \rightarrow A$ :

As before, we first show it for  $f = g = R$

and obtain the general result using the coherence between **unif** and **dinat**. Using Lemma A.11, the right-hand diagram is equal to:

where  $\phi_0 : 1_{\Phi} \Rightarrow J(u_R)$  is the unique invertible 2-cell such that  $R \cdot J\phi_0 = \mu_R \circ (J\phi_0 \cdot R)$  (see proof of Proposition III.5).



Let  $\zeta : R^\dagger \Rightarrow (RR)^\dagger$  be the unique invertible 2-cell (obtained from Lemma IV.2) such that

$$\begin{array}{c} (RR)^\dagger \\ \nearrow \zeta \\ 1 \xrightarrow{R^\dagger} \Phi \\ \nearrow \text{fix}_R^\dagger \\ R^\dagger \end{array} = \begin{array}{c} (RR)^\dagger \\ \nearrow \text{fix}_{RR}^\dagger \\ 1 \xrightarrow{(RR)^\dagger} \Phi \\ \nearrow \zeta \\ R^\dagger \end{array}$$

It plays a similar role as  $\mathbf{din} : t \Rightarrow (RR)^*$  for the operator  $(-)^{\dagger}$  as we can show that:

$$\mathbf{dinat}_R^{R^\dagger} = 1 \xrightarrow{(RR)^\dagger} \Phi$$

We can also prove that:

$$\begin{array}{c} (RR)^\dagger \\ \uparrow \delta_{RR} \\ 1 \xrightarrow{(RR)^*} \Phi \\ \uparrow \mathbf{din} \\ t \end{array} = \begin{array}{c} (RR)^\dagger \\ \uparrow \zeta \\ 1 \xrightarrow{R^\dagger} \Phi \\ \uparrow \delta_R \\ t \end{array}$$

$$\begin{array}{c} R^\dagger \\ \uparrow \delta_R \\ 1 \xrightarrow{R^\dagger} \Phi \\ \uparrow \mathbf{din}^{-1} \\ (RR)^* \end{array} = \begin{array}{c} R^\dagger \\ \uparrow \zeta^{-1} \\ 1 \xrightarrow{(RR)^\dagger} \Phi \\ \uparrow \delta_{RR} \\ (RR)^* \end{array}$$

We can now conclude:

$$\begin{array}{c} (RR)^\dagger \\ \uparrow \delta_{RR} \\ 1 \xrightarrow{(RR)^*} \Phi \\ \uparrow \mathbf{din} \\ t \end{array} = \begin{array}{c} (RR)^\dagger \\ \uparrow \zeta \\ 1 \xrightarrow{R^\dagger} \Phi \\ \uparrow \delta_R \\ t \end{array} = \begin{array}{c} (RR)^\dagger \\ \nearrow \mathbf{dinat}_R^{R^\dagger} \\ 1 \xrightarrow{(RR)^\dagger} \Phi \\ \nearrow \delta_{RR} \\ (RR)^* \end{array}$$

For general morphisms  $f : A \rightarrow B$  and  $g : B \rightarrow A$  in  $\mathcal{D}$ , we consider the squares:

$$\begin{array}{ccccc}
 \Phi & \xrightarrow{R} & \Phi & \xrightarrow{R} & \Phi \\
 \downarrow Ju_{\sigma(f \times g)} & \nearrow \mu_{\sigma(f \times g)} & \downarrow Ju_{\sigma(f \times g)} & \nearrow \mu_{\sigma(f \times g)} & \downarrow Ju_{\sigma(f \times g)} \\
 A \times B & \xrightarrow{\sigma(f \times g)} & A \times B & \xrightarrow{\sigma(f \times g)} & A \times B \\
 \downarrow \pi_1 & & \downarrow \pi_2 & & \downarrow \pi_1 \\
 A & \xrightarrow{f} & B & \xrightarrow{g} & A
 \end{array}$$

and obtain from the coherence between **unif** and **dinat**:

$$\begin{array}{c}
 \begin{array}{c}
 (gf)^\dagger \\
 \nearrow \text{dinat}_g^{f\dagger} \\
 1 \quad \nearrow \text{unif}_{\Pi_{\sigma(f \times g)}^2}^\dagger \\
 \searrow (RR)^\dagger \\
 \Phi \quad \searrow J\pi_2(u_{\sigma(f \times g)}) \\
 B
 \end{array}
 \quad = \quad
 \begin{array}{c}
 (gf)^\dagger \\
 \nearrow \text{unif}_{\Pi_{\sigma(f \times g)}^1}^\dagger \\
 1 \quad \nearrow (RR)^\dagger \\
 \searrow \text{dinat}_R^{R\dagger} \\
 \Phi \quad \searrow J\pi_2(u_{\sigma(f \times g)}) \\
 B
 \end{array}
 \end{array}$$

Therefore, we have:

$$\begin{array}{c}
 \begin{array}{c}
 (gf)^\dagger \\
 \nearrow \text{dinat}_g^{f\dagger} \\
 1 \quad \nearrow \delta_{fg} \\
 \searrow (fg)^* \\
 B
 \end{array}
 \quad = \quad
 \begin{array}{c}
 (gf)^\dagger \\
 \nearrow \text{unif}_{\Pi_{\sigma(f \times g)}^1}^\dagger \\
 1 \quad \nearrow (RR)^\dagger \\
 \searrow \text{dinat}_R^{R\dagger} \\
 \Phi \quad \searrow J\pi_2(u_{\sigma(f \times g)}) \\
 B
 \end{array}
 \end{array}$$

$$\begin{array}{ccc}
\begin{array}{c}
(gf)^\dagger \\
\uparrow \text{unif}_{\Pi^1_{\sigma(f \times g)}}^\dagger \\
(RR)^\dagger \nearrow \Phi \circ Ju_{\sigma(f \times g)} \rightarrow A \times B \circ \pi_1 \rightarrow A \\
\text{dinat}_R^{R^\dagger} \nearrow R \nearrow \mu_{\sigma(f \times g)} \quad B \times A \downarrow f \times g \\
(RR)^\dagger \nearrow \Phi \circ Ju_{\sigma(f \times g)} \rightarrow A \times B \circ \pi_2 \rightarrow B \\
\delta_{RR} \nearrow \\
(RR)^* \nearrow \Phi \circ Ju_{\sigma(f \times g)} \rightarrow A \times B \circ \pi_2 \rightarrow B \\
\uparrow \text{unif}_{\Pi^2_{\sigma(f \times g)}}^{*-1} \\
(fg)^*
\end{array}
& = &
\begin{array}{c}
(gf)^\dagger \\
\uparrow \text{unif}_{\Pi^1_{\sigma(f \times g)}}^\dagger \\
(RR)^\dagger \nearrow \Phi \circ Ju_{\sigma(f \times g)} \rightarrow A \times B \circ \pi_1 \rightarrow A \\
\delta_{RR} \nearrow (RR)^* \nearrow \Phi \circ Ju_{\sigma(f \times g)} \rightarrow A \times B \circ \pi_2 \rightarrow B \\
\text{dinat}_R^{R^*} \nearrow R \nearrow \mu_{\sigma(f \times g)} \quad B \times A \downarrow f \times g \\
(RR)^* \nearrow \Phi \circ Ju_{\sigma(f \times g)} \rightarrow A \times B \circ \pi_2 \rightarrow B \\
\uparrow \text{unif}_{\Pi^2_{\sigma(f \times g)}}^{*-1} \\
(fg)^*
\end{array}
\\[20pt]
\begin{array}{c}
(gf)^* \\
\uparrow \delta_{gf} \\
(gf)^\dagger \nearrow \Phi \circ Ju_{\sigma(f \times g)} \rightarrow A \times B \circ \pi_1 \rightarrow A \\
\text{dinat}_R^{R^*} \nearrow R \nearrow \mu_{\sigma(f \times g)} \quad B \times A \downarrow f \times g \\
(RR)^* \nearrow \Phi \circ Ju_{\sigma(f \times g)} \rightarrow A \times B \circ \pi_2 \rightarrow B \\
\uparrow \text{unif}_{\Pi^2_{\sigma(f \times g)}}^{*-1} \\
(fg)^\dagger
\end{array}
& = &
\begin{array}{c}
(gf)^\dagger \nearrow A \\
\delta_{gf} \nearrow (gf)^* \nearrow A \\
\text{dinat}_g^{f^*} \nearrow B \\
(fg)^* \nearrow B
\end{array}
\end{array}$$

□