Well-Quasi-Ordering Hereditarily Finite Sets

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Abstract. Recently, strong immersion was shown to be a well-quasi-order on the class of all tournaments. Hereditarily finite sets can be viewed as digraphs, which are also acyclic and extensional. Although strong immersion between extensional acyclic digraphs is not a well-quasi-order, we introduce two conditions that guarantee this property. We prove that the class of extensional acyclic digraphs corresponding to *slim* sets (i.e. sets in which every memebership is necessary) of bounded *skewness* (i.e. sets whose $\in$-distance between their elements is bounded) is well-quasi-ordered by strong immersion.

Our results hold for sets of bounded cardinality and it remains open whether they hold in general.

Keywords: well-quasi-order, hereditarily finite set, strong immersion, extensional digraph.

1 Introduction

Sets are directed graphs (*digraphs*) when one interprets the membership relation $\in$ as the adjacency relation $\leftarrow$:

$$a \in b \Leftrightarrow a \leftarrow b.$$

Such a digraph is called the *membership digraph* of a set. According to this view, set-theoretic axioms isolate classes of digraphs whose study can be "assisted" by the underlying set-theoretic semantics. The most natural and initial of such classes of digraphs is the one defined using the axiom of *extensionality*: no two vertices of a digraph in such a class have the same *set* of out-neighbors. More intriguing (even though still very basic) axioms can be considered. In this paper we study the problem of existence of well-quasi-orders on subclasses of the class of digraphs, with such set-theoretic assistance. We start with finite extensional acyclic digraphs, that is, membership digraphs of *hereditarily finite sets*.

One of the earliest results on well-quasi-orderings of graphs belongs to Kruskal [8], who showed that the class of all finite trees is well-quasi-ordered by the topological minor relation. This study culminated with the celebrated theorem of Robertson and Seymour [13] stating that the minor relation is a well-quasi-order

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on the class of all finite graphs. Later [14], they showed that this is also the case for weak immersion between graphs, which was a conjecture of Nash-Williams [10]. In the case of *digraphs* not much is known. Immersion between *eulerian* digraphs was studied by Johnson (cf. [3, p. 517], [5]). Recently, Chudnovsky and Seymour [5] proved that *strong immersion* between digraphs is a well-quasi-order on the set of all tournaments (i.e., orientations of complete graphs). In view of this result, we will focus in this paper on strong immersion between digraphs.

Notice that, requiring acyclicity and extensionality, tournaments are forced to be isomorphic to the membership digraphs of the so-called von Neumann's numerals. Starting from this observation and given that the full family of hereditarily finite sets is not well-quasi-ordered by strong immersion (see below), it is natural to ask for other (sub-)collections of the hereditarily finite sets that could be well-quasi-ordered by strong immersion.

The result of this paper has been obtained by introducing two conditions: the first (*slimness*) guarantees that the number of sets at any given rank in the transitive closure of a set is bounded by its cardinality; the second (bounded *skewness*) guarantees that arcs (memberships) do not reach too freely into its membership digraph. Notice that both these facts are true in the membership digraphs of von Neumann's numerals.

Our result is based on the fact that slimness together with bounded cardinality and skewness are sufficient to ensure that the entire transitive closure of a hereditarily finite set can be described as a sequence of characters in a suitable finite alphabet. From this, the existence of a wqo follows by standard means.

It is not clear exactly to what extent slimness and bounded skewness are necessary, even though in the conclusion we observe that the bounds on cardinality and skewness cannot be both dropped on slim sets without losing wqo.

Wqo's proved to be a key ingredient in generalizing and unifying many results concerning the decidability of verification problems (e.g. coverability) on infinite-state transition systems (cf. [6,1] and the references therein). To be more precise, a transition system is said to be *well-structured* when its transition relation is monotonic w.r.t. a wqo of its states; the classical example is that of Petri nets: the states of the transition system is the set of all configurations of the net, while the wqo is the inclusion between their markings. In this light, our contribution can also be viewed as laying the set-theoretic groundwork for a class of well-structured transition systems having as states the hereditarily finite sets considered in this paper.

The outline of the paper is the following. In Sec. 2 we give the main definitions and argue that weak immersion is not a wqo on the class of all extensional acyclic digraphs. Sec. 3 gives some lemmas about the structure of slim sets, which are then used in Sec. 4 to provide an encoding for slim sets of bounded cardinality and skewness. All results are assembled in Sec. 5, by proving that strong immersion is a wqo on the class of membership digraphs corresponding to slim sets of bounded cardinality and skewness. Finally, some open questions are put forward in Sec. 6.

2 Basics and Notation

2.1 Sets and Digraphs

Our notation follows [9,3]. In this paper we consider hereditarily finite sets, that is, sets belonging to the first ω levels of the von Neumann's cumulative hierarchy. Their collection, which we denote by HF, is defined as $\mathsf{HF} = \bigcup_{i\in\omega} \mathcal{V}_i$, where

$$\mathcal{V}_0 = \emptyset, \quad \mathcal{V}_{i+1} = \mathscr{P}(\mathcal{V}_i),$$

and $\mathscr{P}(\cdot)$ stands for the power-set operator. We will use the following, standard, notion of *rank* of a set x:

$$\mathsf{rk}(x) =_{\mathrm{def}} \sup\,\{\mathsf{rk}(y) + 1 \mid y \in x\},$$

and we will denote the set of elements in x at a given rank r as $x^{=r} = \{y \in x \mid \mathsf{rk}(y) = r\}$. Analogously, $x^{\leqslant r} = \{y \in x \mid \mathsf{rk}(y) \leqslant r\}$. For any set x, we denote by $\mathsf{TrCl}(x)$ the *transitive closure* of x, defined through the recursion:

$$\mathsf{TrCl}(x) =_{\mathrm{def}} x \cup \bigcup_{y\in x} \mathsf{TrCl}(y),$$

where we denote by $\bigcup x$ the *union-set* of x, that is the set $\{z \mid z \in y \wedge y \in x\}$. We say that a set x is *transitive* if $\bigcup x \subseteq x$. Plainly, the transitive closure of a set x is a transitive set.

Definition 1. *The* skewness *of a set x is*

$$\mathsf{skewness}(x) =_{\mathrm{def}} \sup\,\{\mathsf{rk}(y) - \mathsf{rk}(z) \mid y, z \in \mathsf{TrCl}(x) \wedge z \in y\}.$$

We will also consider Ackermann's order $\prec_{\mathsf{A}}$ between hereditarily finite sets [2], defined recursively as

$$x \prec_{\mathsf{A}} y \Leftrightarrow_{\mathrm{def}} \max_{\prec_{\mathsf{A}}}(x \setminus y) \prec_{\mathsf{A}} \max_{\prec_{\mathsf{A}}}(y \setminus x),$$

where we let, by convention, $\max_{\prec_{\mathsf{A}}} \emptyset \prec_{\mathsf{A}} x$, for any $x \in \mathsf{HF} \setminus \{\emptyset\}$.

Given a digraph $G = (V, E)$ we say that $V(G) =_{\mathrm{def}} V$ is its set of *vertices* and $E(G) =_{\mathrm{def}} E$ is its set of *arcs*. We will write uv as a shorthand for the arc $(u, v) \in E(G)$ (u and v are called its *endpoints*). Given $v \in V(G)$ we denote by $N^+(v)$ the set of its *out-neighbors* in G, namely the set $\{w \in V(G) \mid vw \in E(G)\}$. If $N^+(v) = \emptyset$, then we say that v is a *sink*, whereas v is a *source* if v is not an out-neighbor of any vertex of G. We say that H is a subdigraph of a digraph G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and every arc of G with both endpoints in $V(H)$ is also in $E(H)$. In this case, we say that H is a subdigraph of G induced by the vertices in $V(H)$.

Any set in HF can be seen as a digraph, whose vertices correspond to sets and whose arc relation corresponds to the inverse of the membership relation, $\ni$ (see Fig. 1). To be more precise, consider the following definition.

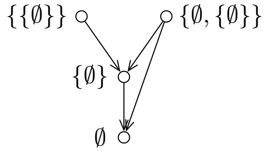

Fig. 1. The membership digraph of the set $x = \big\{\{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\big\}$; we have $\emptyset \prec_{\mathsf{A}} \{\emptyset\} \prec_{\mathsf{A}} \{\{\emptyset\}\} \prec_{\mathsf{A}} \{\emptyset, \{\emptyset\}\}$

Definition 2. *Given a set x, we denote by G_x the digraph (x, E_x), with*

$$E_x =_{\mathrm{def}} \{uv \mid u, v \in x \wedge v \in u\},$$

and call the membership digraph *of x the digraph $G_{\mathsf{TrCl}(x)}$.*

The well-foundedness of the membership relation among sets ensures that membership digraphs are *acyclic*, while the *extensionality* principle guarantees that they are also *extensional*, in the following sense:

Definition 3. *A digraph G is* extensional *if for any distinct vertices u and v in $V(G)$, it holds $N^+(v) \neq N^+(u)$.*

Moreover, the converse also holds: any finite extensional acyclic digraph is the membership digraph of a hereditarily finite set. The following notion is instrumental to our results and establishes a link between a set and its graph-theoretical representation.

Definition 4. *A set x is* slim *if the digraph obtained by removing any arc from $G_{\mathsf{TrCl}(x)}$ is not extensional.*

Def. 4 is equivalent to saying that x is slim if for any vertex y of $G_{\mathsf{TrCl}(x)}$ and for any out-neighbor of it z, there exists a vertex y' of $G_{\mathsf{TrCl}(x)}$ whose set of out-neighbors is precisely $N^+(y) \setminus \{z\}$. In set-theoretic terms, a set x is slim if $\forall y \in \mathsf{TrCl}(x)$ and $\forall z \in y$, it holds that $y \setminus \{z\} \in \mathsf{TrCl}(x)$. Observe that the transitive closure of a slim set x is closed under taking subsets for its elements, in the sense that for any $y \in \mathsf{TrCl}(x)$, $\mathscr{P}(y) \subset \mathsf{TrCl}(x)$. For example, the set x whose membership digraph is depicted in Fig. 1 is slim.

In equivalent graph-theoretic terms, the rank of a set is the length of the longest simple (i.e. without repeated vertices) directed path in its membership digraph. The notion of skewness of a set can thus be interpreted as the length of the longest simple directed path in its membership digraph, so that its two endpoints are also connected by an arc. See Fig. 2 for an example of a membership digraph of skewness 5. Finally, it can be also easily seen that the Ackermann's order among hereditarily finite sets can be analogously defined for the vertices of membership digraphs.

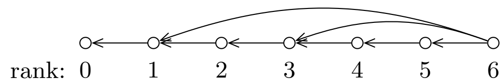

Fig. 2. A membership digraph of skewness 5

2.2 Well-Quasi-Orders and Digraph Immersion

A *quasi-order* is a pair $(Q, \preccurlyeq)$ where Q is a set and $\preccurlyeq$ is a transitive and reflexive binary relation on Q. We say that a quasi-order $(Q, \preccurlyeq)$ is a *well*-quasi-order, or *wqo*, if for every infinite sequence $(q_i)_{i=1,2,\ldots}$ of elements of Q, there exist $1 \leqslant i < j$ such that $q_i \preccurlyeq q_j$.

We refer here to the notions of *weak* and *strong immersion*, as considered in [5]. A *weak immersion* of a digraph H into G is a map η such that:

- for every $v \in V(H)$, $\eta(v) \in V(G)$;
- for every $u, v \in V(H)$ with $u \neq v$, it holds that $\eta(u) \neq \eta(v)$;
- for each arc $uv \in E(H)$, $\eta(uv)$ is a directed path in G from $\eta(u)$ to $\eta(v)$ (all paths considered are *simple*, i.e., do not have repeated vertices);
- if $e, f \in E(H)$ are distinct, then $\eta(e)$ and $\eta(f)$ have no arcs in common, although they may share vertices.

The map η is called a *strong immersion* when it also holds that if $v \in V(H)$, $e \in E(H)$, and e is not incident with v in H, then $\eta(v)$ is not a vertex on the directed path $\eta(e)$. We say that a digraph H is *weakly (strongly) immersed* into a digraph G, and write $H \preccurlyeq_i^w G$ ($H \preccurlyeq_i^s G$), if there exists a weak (strong) immersion of H into G.

As already observed in [5], weak immersion is not a wqo on the set of all digraphs. Just consider the acyclic digraphs D_n formed by orienting the edges of a cycle of length $2n$ alternately clockwise and counterclockwise (see Fig. 3). That being so, the collection $\{D_n \mid n \geqslant 2\}$ has the property that none of its elements can be weakly immersed into another one.

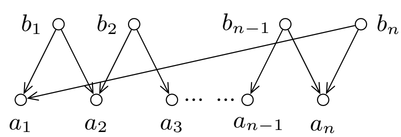

Fig. 3. Digraphs D_n, $n \geqslant 2$

Since the collection of transitive closures of sets is a collection of digraphs, it is natural to ask whether $\preccurlyeq_i^w$ or $\preccurlyeq_i^s$ form a wqo on such a collection. The answer is *no*, since the collection of membership digraphs F_n, $n \geqslant 3$, constructed as in Fig. 4 has, analogously to the corresponding D_n's, the property that no digraph can be weakly immersed into another one.

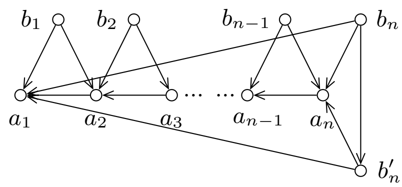

Fig. 4. Membership digraphs F_n, $n \geqslant 3$; notice that F_n is acyclic and extensional

Indeed, given F_n and F_m, with $n < m$, and supposing that η is a weak immersion of F_n into F_m, observe that $\eta(b_n)$ must be b_m, since b_n has 3 out-neighbors in F_n and b_m is the only vertex of F_m having more than 2 out-neighbors. Then, $\eta(a_1) = a_1$ and $\eta(b'_n) = b'_m$. Next, $\eta(b_1) = b_1$, since otherwise the directed paths $\eta(b_1a_1)$ and $\eta(a_2a_1)$ would have to share the arc a_2a_1, against the fact that η is an immersion. This implies that $\eta(a_2) = a_2$. By a similar argument, inductively, $\eta(b_i) = b_i$ and $\eta(a_{i+1}) = a_{i+1}$ hold for all $1 \leqslant i < n$. Moreover, for all $1 \leqslant i < n$, the image of the arc b_ia_i is the 2-vertex directed path (b_i, a_i), and the image of $a_{i+1}a_i$ is the directed path (a_{i+1}, a_i). At this point, the arc b_na_n must be mapped to the directed path $(b_m, a_m, a_{m-1}, \ldots, a_n)$, and b'_na_n is mapped to $(b'_m, a_m, a_{m-1}, \ldots, a_n)$, contradicting the arc-disjointness of η.

3 Slim Sets and Discrimination

Given a slim set x and a subset $\mathcal{F} \subseteq \mathsf{TrCl}(x)$, we will characterize in this section the elements of $\bigcup \mathcal{F}$. In particular, Lemma 4 tackles the case when $\mathcal{F} = \mathsf{TrCl}(x)^{=r}$, for some rank r. To begin with, consider a slightly weaker version of slimness.

Definition 5. *Let $\mathcal{F}$ be a finite family of sets. A $\mathbf{z} \in \bigcup \mathcal{F}$ is said to be redundant if $|\mathcal{F}| = |\{\mathbf{v} \setminus \{\mathbf{z}\} \mid \mathbf{v} \in \mathcal{F}\}|$. We say that $\mathcal{F}$ is irredundant if no element of $\bigcup \mathcal{F}$ is redundant.*

Plainly, a slim set $\mathcal{F}$ whose elements have the same rank is irredundant, since if there would exist a redundant $\mathbf{z} \in \bigcup \mathcal{F}$ so that $\mathbf{z} \in \mathbf{v} \in \mathcal{F}$, then the removal of the arc $\mathbf{vz}$ from $G_{\mathsf{TrCl}(\mathcal{F})}$ would maintain the resulting digraph extensional.

If $\mathcal{F}$ is irredundant then $\bigcup \mathcal{F}$ is also called a *minimal differentiating set for* $\mathcal{F}$. The following lemma puts a bound on the size of minimal differentiating sets.

Lemma 1 ([12,4]). *Given an n-element irredundant family $\mathcal{F}$,*

$$\left|\bigcup \mathcal{F}\right| \leqslant n - 1.$$

Lemma 2 (Discrimination lemma [11]). *Given an n-element nonempty family $\mathcal{F} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, we can determine $\mathbf{z}_1, \ldots, \mathbf{z}_k \in \bigcup \mathcal{F}$, with $k \leqslant n - 1$, so that the family*

$$\{\mathbf{v}_i \cap \{\mathbf{z}_1, \ldots, \mathbf{z}_k\} \mid \mathbf{v}_i \in \mathcal{F}\}$$

is irredundant and has cardinality n.

Lemma 3 (Slim discrimination lemma). *Given an n-element slim and nonempty family $\mathcal{F} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ such that $\emptyset \notin \mathcal{F}$, we can determine $\mathbf{z}_1, \dots, \mathbf{z}_k \in \bigcup \mathcal{F}$, with $k \leqslant n$, so that:*

(1) *the family $\{\mathbf{v}_i \cap \{\mathbf{z}_1, \dots, \mathbf{z}_k\} \mid \mathbf{v}_i \in \mathcal{F}\} \cup \{\emptyset\}$ is irredundant, of cardinality $n+1$ and*
(2) *if $\mathsf{rk}(\mathcal{F}) = \varrho + 2$ and there is a $1 \leqslant j \leqslant n$ so that $|\mathbf{v}_j \cap \{\mathbf{z}_1, \dots, \mathbf{z}_k\}| > 1$, then $|\{\mathbf{z}_i \mid \mathsf{rk}(\mathbf{z}_i) = \varrho\}| < n$.*

Proof. We reason by induction on n. If $n = 1$ the claim is clear, since $\mathbf{v_1} \neq \emptyset$ and we can thus find $\mathbf{z_1} \in \mathbf{v_1}$ to satisfy (1) and (2).

If $n > 1$, apply the discrimination lemma to $\mathcal{F} \cup \{\emptyset\}$ and let $\mathbf{z}_1, \dots, \mathbf{z}_k \in \bigcup \mathcal{F}$, with $k \leqslant n$, so that (1) holds for $\mathbf{z}_1, \dots, \mathbf{z}_k$. If there is a $\mathbf{v}_j$ with $\mathsf{rk}(\mathbf{v}_j) \leqslant \varrho$, then, since $\mathbf{v}_j \cap \{\mathbf{z}_1, \dots, \mathbf{z}_k\} \neq \emptyset$, one of $\mathbf{z}_1, \dots, \mathbf{z}_k$ is of rank strictly smaller than ϱ, and we are done. Therefore, we can assume that $k = n$, and that for all $1 \leqslant j \leqslant n$, $\mathsf{rk}(\mathbf{v}_j) = \varrho + 1$ and $\mathsf{rk}(\mathbf{z}_j) = \varrho$.

Accordingly, suppose for a contradiction that there exists $1 \leqslant j \leqslant n$ so that

$$\mathbf{v}_j \cap \{\mathbf{z}_1, \dots, \mathbf{z}_n\} = \{\mathbf{z}'_1, \dots, \mathbf{z}'_l\}, \text{ where } l \geqslant 2.$$

Denote by $P(\mathbf{v}_j)$ the set

$$P(\mathbf{v}_j) =_{\text{def}} \{\mathbf{v} \subseteq \mathbf{v}_j \mid \mathbf{v} \cap \{\mathbf{z}'_1, \dots, \mathbf{z}'_l\} \neq \emptyset\}$$

and observe that any element of $P(\mathbf{v}_j)$ belongs to $\mathcal{F}$, since $\mathsf{rk}(\mathbf{z}'_1) = \cdots = \mathsf{rk}(\mathbf{z}'_l) = \varrho$ and $\mathcal{F}$ is slim. Consider now $\mathcal{F}' =_{\text{def}} \mathcal{F} \setminus P(\mathbf{v}_j)$ and $\mathcal{Z}' =_{\text{def}} \{\mathbf{z}_1, \dots, \mathbf{z}_n\} \setminus \{\mathbf{z}'_1, \dots, \mathbf{z}'_l\}$, which entails $\mathcal{Z}' \subseteq \bigcup \mathcal{F}'$. Since $l \geqslant 2$, $|P(\mathbf{v}_j)| \geqslant 2^l - 1 > l$, and thus $|\mathcal{F}'| < |\mathcal{Z}'| < n$. This implies that there exists $\mathbf{v}' \in \mathcal{F}'$ with $|\mathbf{v}' \cap \mathcal{Z}'| > 1$. At this point, we can repeatedly apply the above argument to the strictly smaller finite sets $\mathcal{F}'$ and $\mathcal{Z}'$, which brings the desired contradiction. $\square$

Lemma 4 (Slim structure lemma). *For any slim set x and for every $0 < r < \mathsf{rk}(x)$, the following properties hold:*

(1) $|\mathsf{TrCl}(x)^{=r-1} \setminus x| \leqslant |\mathsf{TrCl}(x)^{=r}|$*;*
(2) $|\mathsf{TrCl}(x)^{=r}| \leqslant |x^{\geqslant r}|$.

Proof. To prove (1), assume for a contradiction that $\mathbf{r}$ is a rank such that $|\mathsf{TrCl}(x)^{=\mathbf{r}-1} \setminus x| > |\mathsf{TrCl}(x)^{=\mathbf{r}}|$; this entails $\mathbf{r} < \mathsf{rk}(x) - 1$, since $\mathsf{TrCl}(x)^{=\mathsf{rk}(x)-1} \setminus x = \emptyset$.

We claim that there exists a $\mathbf{z} \in \mathsf{TrCl}(x)^{=\mathbf{r}-1} \setminus x$ which is not an element of any $y \in \mathsf{TrCl}(x)^{=\mathbf{r}}$. If this were not to hold, then by the discrimination lemma applied to the family $\mathsf{TrCl}(x)^{=\mathbf{r}} \cup \{\emptyset\}$ we obtain $\mathbf{z}_1, \dots, \mathbf{z}_k$, $k \leqslant |\mathsf{TrCl}(x)^{=\mathbf{r}}|$, so that the family $\{v \cap \{\mathbf{z}_1, \dots, \mathbf{z}_k\} \mid v \in \mathsf{TrCl}(x)^{=\mathbf{r}}\} \cup \{\emptyset\}$ is irredundant, of cardinality $|\mathsf{TrCl}(x)^{=\mathbf{r}}| + 1$. Therefore, there exists $\mathbf{t} \in \left(\mathsf{TrCl}(x)^{=\mathbf{r}-1} \setminus x\right) \setminus \{\mathbf{z}_1, \dots, \mathbf{z}_k\}$ so that any arc from any element $y \in \mathsf{TrCl}(x)^{=\mathbf{r}}$ to $\mathbf{t}$ can be removed from $G_{\mathsf{TrCl}(x)}$ without interfering with its extensionality. This contradicts the slimness of x.

Accordingly, taking $\mathbf{z} \in \mathsf{TrCl}(x)^{=\mathbf{r}-1} \setminus x$ which is not an element of any $y \in \mathsf{TrCl}(x)^{=\mathbf{r}}$, there must be a $w \in \mathsf{TrCl}(x)^{>\mathbf{r}}$ so that $\mathbf{z} \in w$; take such a $\mathbf{w}$ to be $\subseteq$-minimal in $\mathsf{TrCl}(x)^{>\mathbf{r}}$. Since $\mathsf{rk}(\mathbf{w}) > \mathbf{r}$, there must exist $\mathbf{u} \in \mathbf{w}$ of rank greater than or equal to $\mathbf{r}$. From the minimality of $\mathbf{w}$, $\mathsf{rk}(\mathbf{w} \setminus \{\mathbf{u}\}) = \mathbf{r}$. Since no $y \in \mathsf{TrCl}(x)^{=\mathbf{r}}$ contains $\mathbf{z}$ as element, we have that $\mathbf{w} \setminus \{\mathbf{u}\} \notin \mathsf{TrCl}(x)$. Thus, removing the arc $\mathbf{wu}$ from $G_{\mathsf{TrCl}(x)}$ maintains its extensionality, which is against the slimness of x.

To see that (2) holds as well, proceed again by contradiction and take $\mathbf{r}$ maximum such that $|\mathsf{TrCl}(x)^{=\mathbf{r}}| > |x^{\geqslant \mathbf{r}}|$. Observe that $\mathbf{r} < \mathsf{rk}(x) - 1$, since otherwise $\mathsf{TrCl}(x)^{=\mathsf{rk}(x)-1} = x^{\geqslant \mathsf{rk}(x)-1}$. The choice of $\mathbf{r}$ implies that $|\mathsf{TrCl}(x)^{=\mathbf{r}+1}| \leqslant |x^{\geqslant \mathbf{r}+1}|$. From (1) we thus have

$$|\mathsf{TrCl}(x)^{=\mathbf{r}} \setminus x| \leqslant |\mathsf{TrCl}(x)^{=\mathbf{r}+1}| \leqslant |x^{\geqslant \mathbf{r}+1}|,$$

which brings the desired contradiction, since

$$|\mathsf{TrCl}(x)^{=\mathbf{r}}| \leqslant |\mathsf{TrCl}(x)^{=\mathbf{r}} \setminus x| + |x^{=\mathbf{r}}| \leqslant |x^{\geqslant \mathbf{r}+1}| + |x^{=\mathbf{r}}| = |x^{\geqslant \mathbf{r}}|. \qquad \square$$

4 Encoding

Let us begin by considering the set of $\in$-digraphs of slim sets of bounded cardinality and skewness:

$$\mathsf{M}_h^s = \{G_{\mathsf{TrCl}(x)} \mid x \text{ is slim} \wedge |x| \leqslant s \wedge \mathsf{skewness}(x) \leqslant h\}.$$

Point (2) of the slim structure lemma implies that at any rank of $G_{\mathsf{TrCl}(x)} \in \mathsf{M}_h^s$ there are at most s vertices. Point (1) implies that the number of vertices at any given rank r of $G_{\mathsf{TrCl}(x)}$ is non-increasing (for decreasing r), save for at most $|x| \leqslant s$ times, when a bounded number of sources (which are elements of x) appear in $G_{\mathsf{TrCl}(x)}$. Additionally, from the slim discrimination lemma it follows that whenever $x \cap \mathsf{TrCl}(x)^{=r} = \emptyset$ and $|\mathsf{TrCl}(x)^{=r+1}| = |\mathsf{TrCl}(x)^{=r}|$, then each element of $\mathsf{TrCl}(x)^{=r+1}$ has cardinality 1, and thus has exactly one out-going arc towards an element of $\mathsf{TrCl}(x)^{=r}$, and conversely, each element of $\mathsf{TrCl}(x)^{=r}$ is the out-neighbor of exactly one element of $\mathsf{TrCl}(x)^{=r+1}$.

Let us say that a rank r is a *juncture rank* of a slim set x if there is a source at rank r in $G_{\mathsf{TrCl}(x)}$, or there is a change in cardinality when passing from $\mathsf{TrCl}(x)^{=r}$ to $\mathsf{TrCl}(x)^{=r-1}$. For uniformity, we also consider 0 to be a juncture rank. The previous two observations imply that two factors need to be taken into account in order to show that strong immersion is a wqo on M_h^s.

First, the subdigraphs of $G_{\mathsf{TrCl}(x)}$ induced by the vertices at a juncture rank and by their out-neighbors can be encoded by characters drawn from an alphabet $\Sigma_{s,h}$ consisting of all such membership digraphs. On the other hand, in order to manage the directed paths linking these subdigraphs, it suffices to encode their lengths and their endpoints. We will use natural numbers for the first item, while for the second, we will require that the membership digraphs from the alphabet $\Sigma_{s,h}$ have their sinks labeled. Entering into details, $\Sigma_{s,h}$ consists of all pairs (D, λ) satisfying the following properties:

(i) D is a digraph, which is acyclic and *weakly extensional*, in the sense that for any distinct vertices u and v in $V(D)$, if u is not a sink, then $N^+(u) \neq N^+(v)$;

(ii) removing any arc from D either increases the number of sinks of D or makes it no longer weakly extensional;
(iii) D has at most s sources;
(iv) for any $u, v \in V(D)$ with $v \in N^+(u)$ it holds that $\mathsf{rk}(u) - \mathsf{rk}(v) \leqslant h$;
(v) if all vertices of D at some rank r have exactly one out-neighbor, then either
  (a) there is a source of D of rank r, or
  (b) there exist $u \in V(D)$ and $v \in N^+(u)$ such that $\mathsf{rk}(v) < r < \mathsf{rk}(u)$;
(vi) denoting by T the set of sinks of D, λ is a label-assigning bijection $\lambda : T \to \{1, \ldots, |T|\}$.

Given a pair $(\mathcal{D}, \lambda) \in \Sigma_{s,h}$, we can define an Ackermann-line order on the vertices of $\mathcal{D}$ in the following recursive way:

- for any sinks u, v of $\mathcal{D}$, let $u \prec_{(\mathcal{D},\lambda)} v \Leftrightarrow_{\text{def}} \lambda(u) < \lambda(v)$;
- for any sink u of $\mathcal{D}$ any any other non-sink vertex v of $\mathcal{D}$, set $u \prec_{(\mathcal{D},\lambda)} v$;
- for any non-sink vertices u, v of $\mathcal{D}$, let $u \prec_{(\mathcal{D},\lambda)} v \Leftrightarrow_{\text{def}} \max_{\prec_{(\mathcal{D},\lambda)}}(u \setminus v) \prec_{(\mathcal{D},\lambda)} \max_{\prec_{(\mathcal{D},\lambda)}}(v \setminus u)$.

Given a slim set x having $|x| \leqslant s$ and $\mathsf{skewness}(x) \leqslant h$, we give two encodings for it: $\sigma(x)$, a string over the alphabet $\Sigma_{s,h}$, and $\delta(x)$, a string of positive integers. They are obtained in the following algorithmic way. First, compute Ackermann's order $\prec_{\mathsf{A}}$ on the elements of $\mathsf{TrCl}(x)$. Put also $\varrho = 1 + \max\{\mathsf{rk}(y) \mid y \in \mathsf{TrCl}(x)\}$.

The ith ($i = 1, 2, ...$) characters of $\sigma(x)$ and $\delta(x)$ are obtained as follows. Let r be the greatest juncture rank of x satisfying $r < \varrho$, and let r' be the smallest rank such that $r' \leqslant r$, and the subdigraph induced by the vertices in $\mathsf{TrCl}(x)^{=r} \cup \mathsf{TrCl}(x)^{=r-1} \cup \cdots \cup \mathsf{TrCl}(x)^{=r'}$, which we denote by D_i, is isomorphic to a digraph appearing in a pair of the alphabet $\Sigma_{s,h}$. Then, the ith character of $\sigma(x)$ is that pair $(\mathcal{D}, \lambda)$ such that there exists an isomorphism $f : D_i \to \mathcal{D}$ with the additional property that for any sinks $u, v \in V(D_i)$ it holds $u \prec_{\mathsf{A}} v \Leftrightarrow \lambda(f(u)) < \lambda(f(v))$. It thus holds that for any u, v sources of D_i, we have $u \prec_{\mathsf{A}} v \Leftrightarrow f(u) \prec_{(\mathcal{D},\lambda)} f(v)$.

Moreover, set the ith character of $\delta(x)$ to be $\varrho - r$. Next, update ϱ to be r' and go on to computing the $(i+1)$th characters of $\sigma(x)$ and $\delta(x)$. Observe that the strings $\sigma(x)$ and $\delta(x)$ have the same lengths, and that, for uniformity, we have chosen to always set the first character of $\delta(x)$ as 1.

An example of a membership digraph of a slim set x is given in Fig. 5, where the digraphs appearing in the characters of $\sigma(x)$ are marked in gray.

5 Main Result

The proof that strong immersion if a wqo on M_h^s proceeds as follows. First, we argue that for any such slim set x of bounded cardinality and skewness, the length of the string $\sigma(x)$ is bounded. Thus, given an infinite sequence $(x_i)_{i=1,2,...}$ of such sets, we can find an infinite subsequence $(x_{i_j})_{j=1,2,...}$ such that $\sigma(x_{i_1}) = \sigma(x_{i_2}) = \ldots$. This implies that between the membership digraphs corresponding to two consecutive characters of $\sigma(x_{i_j})$, for any j, we have the same number of directed paths. Observe also that $|\delta(x_{i_1})| = |\delta(x_{i_2})| = \ldots$.

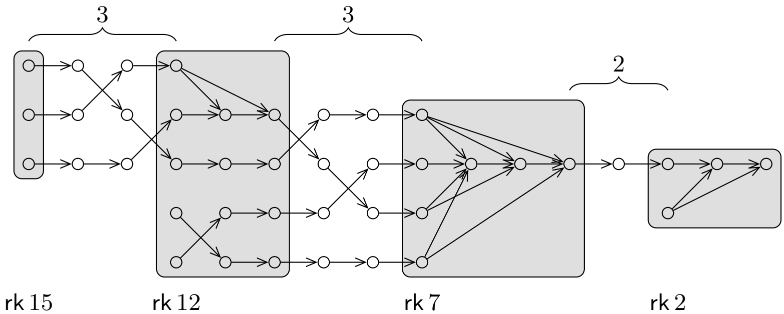

Fig. 5. The membership digraph of a slim set x of cardinality 6 and skewness 3; we have $\delta(x) = 1\ 3\ 3\ 2$

However, this not yet sufficient, since the directed paths between these digraphs can have arbitrary lengths. For this reason, we use a basic fact from the theory of wqo's (see e.g. [7]). If $(Q, \preccurlyeq)$ is a wqo, then so is the set of fixed length sequences over Q, componentwise ordered by $\preccurlyeq$. That is, the pair $(Q^\ell, \preccurlyeq^\ell)$ is a wqo, where for any $(x_1, \ldots, x_\ell), (y_1, \ldots, y_\ell) \in Q^\ell$ we have $(x_1, \ldots, x_\ell) \preccurlyeq^\ell (y_1, \ldots, y_\ell) \Leftrightarrow_{\text{def}} x_i \preccurlyeq y_i$, for all $1 \leqslant i \leqslant \ell$. Since $(\mathbb{N}, \leqslant)$ is a wqo, this implies that we can find x_{i_j} and x_{i_k} in the aforementioned infinite sequence such that taking $\ell = |\delta(x_{i_1})|$ we have $\delta(x_{i_j}) \leqslant^\ell \delta(x_{i_k})$. To ensure that these directed paths also have corresponding endpoints, we will finally make use of the labeling of the sinks, obtained by Ackermann's order.

Lemma 5. *For any $s, h \geqslant 1$, there exists a computable function $g(s,h)$ such that for any slim set x, with $|x| \leqslant s$ and $\mathsf{skewness}(x) \leqslant h$, it holds $|\sigma(x)| \leqslant g(s,h)$.*

Proof. We first argue that the cardinality of $\Sigma_{s,h}$ is finite. Given $(\mathcal{D}, \lambda) \in \Sigma_{s,h}$, observe that point (2) of the slim structure lemma entails that the number of vertices of $\mathcal{D}$ at a given rank is bounded by s. Then, for any t, $1 \leqslant t \leqslant s$, there are at most sh ranks r with the property that $\mathcal{D}$ has exactly t vertices of rank r. Indeed, (ii), (iv) and the slim structure lemma imply that there can be at most h consecutive ranks at which there are exactly t vertices of $\mathcal{D}$. Moreover, this cardinality t of vertices at the same rank can repeat itself for at most s non-consecutive ranks, the culprits being the sources of $\mathcal{D}$, which are at most s. To conclude, $\mathcal{D}$ can have at most $sh \sum_{t=1}^{s} t$ vertices, and hence the cardinality of $\Sigma_{s,h}$ is bounded by $s!3^{\binom{\frac{1}{2}s^2(s+1)h}{2}}$, since there can be at most s sinks in a character of $\Sigma_{s,h}$, and $s!$ distinct ways to label them.

Observe now that a character inside the encoding $\sigma(x)$ of a slim set x can appear more than once, but only because of a source of $G_{\mathsf{TrCl}(x)}$ (that is, one of the elements of x). Since the cardinality of x is bounded by s, then $g(s,h)$ can be taken to be $s|\Sigma_{s,h}|$. $\square$

Theorem 1. *The pair $(\mathsf{M}^s_h, \preccurlyeq^s_i)$ is a wqo.*

Proof. Let $(G_{\mathsf{TrCl}(x_i)})_{i=1,2,...}$ be an infinite sequence of membership digraphs belonging to M_h^s. By Lemma 5, there exists an infinite subsequence of it, $(G_{\mathsf{TrCl}(x_{i_j})})_{j=1,2,...}$, such that $\sigma(x_{i_1}) = \sigma(x_{i_2}) = \ldots$. As noted before, $|\sigma(x_{i_j})| = |\delta(x_{i_j})|$ holds for any j, and if we denote their length by ℓ, we have that there exists $1 \leqslant j < k$ such that for $\mathbf{y} =_{\text{def}} x_{i_j}$ and $\mathbf{z} =_{\text{def}} x_{i_k}$, we have $\delta(\mathbf{y}) \leqslant^{\ell} \delta(\mathbf{z})$.

Given a string σ, denote by $\sigma[m]$ its mth character, $1 \leqslant m \leqslant |\sigma|$. Let $(\mathcal{D}_m, \lambda_m) =_{\text{def}} \sigma(\mathbf{y})[m]$, for any $1 \leqslant m \leqslant \ell$. Let also $r(m)$ be the juncture rank of $\mathbf{y}$ responsible for the introduction of the mth character of $\sigma(\mathbf{y})$. Recall that $r(m) > r(m+1)$, for any $1 \leqslant m < \ell$. We will show by induction on t, $0 \leqslant t < \ell$, that we have a strong immersion of the subdigraph of $G_{\mathsf{TrCl}(\mathbf{y})}$ induced by the elements of $\mathsf{TrCl}(\mathbf{y})^{\leqslant r(\ell-t)}$ into $G_{\mathsf{TrCl}(\mathbf{z})}$.

The subdigraph induced by $\mathsf{TrCl}(\mathbf{y})^{\leqslant r(\ell)}$ is isomorphic to $\mathcal{D}_\ell$, and hence isomorphic to $\mathsf{TrCl}(\mathbf{z})^{\leqslant r(\ell)}$, which proves our claim for $t = 0$.

Assume now that η is a strong immersion of $\mathsf{TrCl}(\mathbf{y})^{\leqslant r(\ell-t)}$ into $G_{\mathsf{TrCl}(\mathbf{z})}$, for $t < \ell - 1$. Construct the strong immersion η' of $\mathsf{TrCl}(\mathbf{y})^{\leqslant r(\ell-(t+1))}$ into $G_{\mathsf{TrCl}(\mathbf{z})}$ as follows. For any vertex in $\mathsf{TrCl}(\mathbf{y})^{\leqslant r(\ell-t)}$ and any arc between two such vertices, let η' coincide with η. The elements of $\mathsf{TrCl}(\mathbf{y})^{\leqslant r(\ell-(t+1))} \setminus \mathsf{TrCl}(\mathbf{y})^{\leqslant r(\ell-t)}$ can be divided into those vertices forming the subdigraph $D^{\mathbf{y}}$ of $G_{\mathsf{TrCl}(\mathbf{y})}$ isomorphic to $\mathcal{D}_{\ell-(t+1)}$, and those vertices forming the subdigraph $P^{\mathbf{y}}$ of $G_{\mathsf{TrCl}(\mathbf{y})}$ consisting of vertex-disjoint directed paths linking the sinks of $D^{\mathbf{y}}$ to vertices at juncture rank $r(\ell - t)$.

Map the vertices of $D^{\mathbf{y}}$ to those vertices of $G_{\mathsf{TrCl}(\mathbf{z})}$ to which they are mapped according to the isomorphisms which gave $\sigma(\mathbf{y})[\ell - (t+1)] = \sigma(\mathbf{z})[\ell - (t+1)]$. To be more precise, denote by $D^{\mathbf{z}}$ the subdigraph of $G_{\mathsf{TrCl}(\mathbf{z})}$ responsible for the choice of $(\mathcal{D}_{\ell-(t+1)}, \lambda_{\ell-(t+1)})$ as the $(\ell - (t+1))$th character of $\sigma(\mathbf{z})$. Denote also by $f_{\mathbf{y}} : D^{\mathbf{y}} \to \mathcal{D}_{\ell-(t+1)}$ and by $f_{\mathbf{z}} : D^{\mathbf{z}} \to \mathcal{D}_{\ell-(t+1)}$ the corresponding label-preserving isomorphisms. Then, let $\eta'(v) = f_{\mathbf{z}}^{-1}(f_{\mathbf{y}}(v))$ and $\eta'(uv) = (f_{\mathbf{z}}^{-1}(f_{\mathbf{y}}(u)), f_{\mathbf{z}}^{-1}(f_{\mathbf{y}}(v)))$ for any $v, u \in V(D^{\mathbf{y}})$.

Let now $P^{\mathbf{z}}$ be the induced subdigraph of $G_{\mathsf{TrCl}(\mathbf{z})}$ consisting of those vertex-disjoint paths between the sinks of $D^{\mathbf{z}}$ to the vertices of $G_{\mathsf{TrCl}(\mathbf{z})}$ at that juncture rank responsible for the introduction of the $(\ell - (t+1))$th character of $\sigma(\mathbf{z})$. Clearly, $P^{\mathbf{y}}$ and $P^{\mathbf{z}}$ consist of the same number of vertex disjoint paths, since $\sigma(\mathbf{y}) = \sigma(\mathbf{z})$. Moreover, all paths of $P^{\mathbf{y}}$ and $P^{\mathbf{z}}$, have the same lengths, respectively. Additionally, since $\delta(\mathbf{y}) \leqslant^{\ell} \delta(\mathbf{z})$, we have that all paths of $P^{\mathbf{y}}$ have length less than, or equal, to all paths of $P^{\mathbf{z}}$. It remains to show that their endpoints correspond.

From the way $\sigma(\mathbf{y})$ and $\sigma(\mathbf{z})$ were constructed, for any sinks u, v of $D^{\mathbf{y}}$ it holds that $u \prec_{\mathsf{A}} v \Leftrightarrow \eta'(u) \prec_{\mathsf{A}} \eta'(v)$. Similarly, for any vertices u, v of $G_{\mathsf{TrCl}(\mathbf{y})}$ at juncture rank $r(\ell - t)$, $u \prec_{\mathsf{A}} v \Leftrightarrow \eta'(u) \prec_{\mathsf{A}} \eta'(v)$ also holds. Consider now a directed path $p^{\mathbf{y}}$ of $P^{\mathbf{y}}$ with endpoints v, a sink of $D_{\mathbf{y}}$, and w, a vertex of rank $r(\ell - t)$. From the above two observations and the definition of Ackermann's order, it follows that in $P^{\mathbf{z}}$ there is a directed path $p^{\mathbf{z}}$ with endpoints $\eta'(v)$ and $\eta'(w)$. Since $|p^{\mathbf{y}}| \leqslant |p^{\mathbf{z}}|$, $p^{\mathbf{y}}$ can thus be strongly immersed into $p^{\mathbf{z}}$ by an extension of η'. $\square$

6 Conclusion

We consider the results of this paper a first step in studying digraph immersion and well-quasi-orders for various classes of sets. It is thus of particular interest to what extent the constraints considered here can be weakened. Observe, however, that dropping the bounds on cardinality and on skewness at the same time no longer ensures the wqo property for slim sets, since the membership digraphs F_n in Fig. 4, which have unbounded cardinality and skewness, can be easily rendered slim (it suffices to add 4 vertices with sets of out-neighbors $\{a_n\}$, $\{b'_n\}$, $\{a_n, b'_n\}$, and $\{a_1, b'_n\}$, respectively).

It would be interesting to know if the same strategy employed in this paper could be extended to prove that the collection of hereditarily finite slim sets of bounded skewness (with no bound on the cardinality) admits a wqo. In order to do this a generalization of the results in Sec. 5 dealing with alphabets $\Sigma_{s,h}$ for various values of s would be sufficient.

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