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# Linear recurrence relations in the algebra of matrices and applications

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## Abstract

We study here some linear recurrence relations in the algebra of square matrices. With the aid of the Cayley–Hamilton Theorem, we derive some explicit formulas for  $A^n$  ( $n \geq r$ ) and  $e^{tA}$  for every  $r \times r$  matrix  $A$ , in terms of the coefficients of its characteristic polynomial and matrices  $A^j$ , where  $0 \leq j \leq r - 1$ . © 2001 Elsevier Science Inc. All rights reserved.

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## 1. Introduction

Let  $\{V_n\}_{n \geq 0}$  be a sequence of complex numbers defined by  $V_0 = \alpha_0$ ,  $V_1 = \alpha_1, \dots$ ,  $V_{r-1} = \alpha_{r-1}$  and the linear recurrence relation of order  $r$  ( $r \geq 2$ ),

$$V_{n+1} = a_0 V_n + a_1 V_{n-1} + \dots + a_{r-1} V_{n-r+1} \quad \text{for } n \geq r - 1, \quad (1)$$

where the two sequences of complex numbers  $a_0, a_1, \dots, a_{r-1}$  (with  $a_{r-1} \neq 0$ ) and  $\alpha_0, \alpha_1, \dots, \alpha_{r-1}$  are specified as the coefficients and initial values. The combinatorial form of *sequences as in* (1), known in the literature as  $r$ -generalized Fibon-

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acci sequences, has been studied by various methods (see [9,12–14] for example). Particularly, it was established in [12] that for any  $n \geq r$ ,

$$V_n = \rho(n, r)W_0 + \rho(n-1, r)W_1 + \cdots + \rho(n-r+1, r)W_{r-1}, \quad (2)$$

where  $W_s = a_{r-1}V_s + \cdots + a_sV_{r-1}$  ( $0 \leq s \leq r-1$ ) and

$$\rho(n, r) = \sum_{k_0+2k_1+\cdots+rk_{r-1}=n-r} \frac{(k_0+\cdots+k_{r-1})!}{k_0!k_1!\cdots k_{r-1}!} a_0^{k_0} a_1^{k_1} \cdots a_{r-1}^{k_{r-1}}, \quad (3)$$

with  $\rho(r, r) = 1$  and  $\rho(n, r) = 0$  if  $n \leq r-1$ .

For a given  $r \times r$  matrix  $A$ , the computation of  $A^n$  or more generally of  $e^{tA}$  is still an important problem in many fields of mathematics. In general, for such computations, properties of eigenvalues of  $A$  are considered (see [3,6,7,11,15] for example), or for some particular cases, other identities are considered (see [2,16] for example). In [3] Cheng and Yau study some explicit formulas for  $e^{tA}$ , where  $A \in \text{GL}(r; \mathbb{C})$  ( $r \geq 2$ ). They consider  $e^{tA} = \sum_{k=0}^{m-1} f_k(t)A^k$ , where  $m \leq r$  is the degree of the minimal polynomial  $M_A(\lambda)$  of  $A$  and  $f_k(t)$  ( $0 \leq k \leq r-1$ ) are some analytic functions, which satisfy a linear system of  $m$  equations (see [3, Eq. (9a)]). Hence, prior knowledge of the size of the largest Jordan blocks of eigenvalues is assumed. Thus the eigenvalues of  $A$  and the exponentials of the associated Jordan blocks play a central role in the method proposed in [3].

In this paper, we give a combinatorial expression for some sequences defined by linear recurrence relations in the algebra  $\text{GL}(d; \mathbb{C})$  of  $d \times d$  matrices, which extends the study in [12]. For  $A \in \text{GL}(r; \mathbb{C})$  the preceding result and the Cayley–Hamilton Theorem allow us to derive an explicit formula for  $A^n$  for  $n \geq r$ , in terms of the coefficients of the characteristic polynomial of  $A$  and the matrices  $I_r, A, \dots, A^{r-1}$ , where  $I_r$  is the  $r \times r$  identity matrix. More precisely, we express  $A^n$  ( $n \geq r$ ) in the form  $A^n = \sum_{k=0}^{r-1} \rho_k(n)A^k$ , where explicit formulas for  $\rho_k(n)$  ( $0 \leq k \leq r-1$ ) are derived from (2) and (3).

We then apply the above result to the computation of  $e^{tA}$  ( $t \in \mathbb{R}$ ), for any  $A \in \text{GL}(r; \mathbb{C})$ , in terms of  $I_r, A, \dots, A^{r-1}$  and a class of analytic functions, whose coefficients depend on  $\rho(n, r)$  given by (3). More precisely, our approach for computing  $e^{tA}$  in the form  $e^{tA} = \sum_{k=0}^{s-1} \Omega_k(t)A^k$  ( $s = r$  or  $m$ ), where  $\Omega_k(t)$  are some analytic functions, assuming the knowledge of the coefficients of the characteristic (or minimal) polynomial of  $A$ . Here, the  $\Omega_k(t)$  ( $0 \leq k \leq r-1$ ) are derived from (2) and (3). Also the relations between  $\rho(n, r)$  and the characteristic roots of (1) are derived, and the connection with [3] is discussed.

This paper is organized as follows. In Section 2, we give the combinatorial form of some sequences defined by linear recurrence relations in  $\text{GL}(d; \mathbb{C})$ . We also compute  $A^n$  for any  $A \in \text{GL}(d; \mathbb{C})$ . Section 3 is devoted to obtaining a new expression for  $e^{tA}$ . In Section 4 we give an expression for  $\rho(n, r)$ , with the aid of the characteristic roots of (1). In Section 5, we consider the connection of Sections 3 and 4 to those in [3]. Some other results are also given.

## 2. Linear recurrence relations in $GL(d; \mathbb{C})$

Let  $\{A_0, A_1, \dots, A_{r-1}\}$  ( $r \geq 2$ ) be in  $GL(d; \mathbb{C})$  such that  $A_{r-1} \neq \Theta_d$ , where  $\Theta_d$  is the zero matrix. Suppose that for  $0 \leq j \leq r-1$ ,

$$A_j = \text{diag}(a_0^j, a_1^j, \dots, a_{d-1}^j). \quad (4)$$

Consider also the sequence  $\{Y_n\}_{n \geq 0}$  of  $GL(d; \mathbb{C})$  defined as follows:

$$\begin{aligned} Y_0 &= V_0, \\ Y_1 &= V_1, \dots, Y_{r-1} = V_{r-1}, \\ Y_{n+1} &= A_0 Y_n + A_1 Y_{n-1} + \dots + A_{r-1} Y_{n-r+1}, \quad n \geq r-1, \end{aligned} \quad (5)$$

where  $\{V_0, V_1, \dots, V_{r-1}\}$  ( $r \geq 2$ ) is a given sequence of  $GL(d; \mathbb{C})$ . Set  $V_n = (v_{ij}^{(n)})_{0 \leq i, j \leq d-1}$  for  $0 \leq n \leq r-1$ , and  $Y_n = (y_{ij}^{(n)})_{0 \leq i, j \leq d-1}$  for  $n \geq r$ . Conditions (4) and (5) give rise to the family  $\{y_{ij}^{(n)}\}_{n \geq 0}$  of sequences of the form (1) defined as follows:  $y_{ij}^{(n)} = v_{ij}^{(n)}$  for  $n = 0, 1, \dots, r-1$  and  $y_{ij}^{(n+1)} = a_i^0 y_{ij}^{(n)} + a_i^1 y_{ij}^{(n-1)} + \dots + a_i^{r-1} y_{ij}^{(n-r+1)}$  for  $n \geq r-1$ . Hence,  $\{y_{ij}^{(n)}\}_{n \geq 0}$  is a sequence of the form (1), for every fixed  $i, j$ . From (2), (3) and (5) we derive the following result.

**Proposition 2.1.** *Let  $\{Y_n\}_{n \geq 0}$  be a sequence as in (5). Suppose that  $A_j$  satisfies (4) ( $j = 0, 1, \dots, r-1$ ). Then, for every  $n \geq r$ , we have*

$$Y_n = \rho(n, r) W_0 + \rho(n-1, r) W_1 + \dots + \rho(n-r+1, r) W_{r-1}, \quad (6)$$

where

$$W_s = A_{r-1} V_s + \dots + A_s V_{r-1} \quad (s = 0, 1, \dots, r-1),$$

$$\rho(r, r) = I_d, \quad \rho(p, r) = \Theta_d \quad \text{for } p < r,$$

and for all  $n \geq r$

$$\rho(n, r) = \sum_{k_0+2k_1+\dots+rk_{r-1}=n-r} \frac{(k_0+\dots+k_{r-1})!}{k_0!k_1!\dots k_{r-1}!} A_0^{k_0} A_1^{k_1} \dots A_{r-1}^{k_{r-1}}. \quad (7)$$

It is well known that if  $\{A_0, A_1, \dots, A_{r-1}\}$  is a family of commuting matrices such that one of them is diagonal, then there exists a nonsingular matrix  $B$  such that  $BA_jB^{-1}$  is diagonal for all  $j$  (see [5] for example). Hence Proposition 2.1 is still valid if we replace (4) by the condition:  $A_i A_j = A_j A_i$  for all  $i, j$  ( $0 \leq i, j \leq r-1$ ) and  $A_0$  diagonal.

If  $A_i = a_i I_d$  for any  $i$ , then expression (7) may be identified with (3). In particular, let  $A \in GL(r; \mathbb{C})$  and  $P_A(\lambda) = \lambda^r - a_0 \lambda^{r-1} - \dots - a_{r-1}$  with  $a_{r-1} \neq 0$  be its characteristic polynomial. Then, from the Cayley–Hamilton Theorem we derive that  $A^n = a_0 A^{n-1} + a_1 A^{n-2} + \dots + a_{r-1} A^{n-r}$  for any  $n \geq r$ . Thus  $\{A^n\}_{n \geq 0}$  is a sequence of the form (5), where  $A_i = a_i I_r$  ( $0 \leq i \leq r-1$ ) and  $V_n = A^n$  for  $0 \leq i \leq r-1$ . Hence Proposition 2.1 implies the following corollary.

**Corollary 2.1.** Let  $A \in \text{GL}(r; \mathbb{C})$  and  $P_A(\lambda) = \lambda^r - a_0\lambda^{r-1} - \dots - a_{r-1}$ , with  $a_{r-1} \neq 0$  be its characteristic polynomial. Then, for any  $n \geq r$ , we have

$$A^n = \rho(n, r)W_0 + \rho(n-1, r)W_1 + \dots + \rho(n-r+1, r)W_{r-1}, \quad (8)$$

where  $W_s = a_{r-1}A^s + \dots + a_sA^{r-1}$  ( $s = 0, 1, \dots, r-1$ ) and  $\rho(n, r)$  are given by (3) with  $\rho(r, r) = 1$  and  $\rho(p, r) = 0$  for  $p < r$ .

Expression (8) may be written in terms of  $I_r, A, \dots, A^{r-1}$  as follows:

$$A^n = \sum_{k=0}^{r-1} \left( \sum_{j=0}^k a_{r-k+j-1} \rho(n-j, r) \right) A^k \quad \text{for any } n \geq r. \quad (9)$$

The Cayley–Hamilton Theorem allows us to write  $A^n = \beta_0^{(n)}I_r + \beta_1^{(n)}A + \dots + \beta_{r-1}^{(n)}A^{r-1}$  ( $n \geq r$ ), but the coefficients  $\beta_j^{(n)}$  ( $0 \leq j \leq r-1, n \geq 0$ ) are not known explicitly in general (see [3,11,15] for example).

**Remark 2.1.** Suppose that there exists a polynomial  $R(\lambda) = \lambda^s - b_0\lambda^{s-1} - \dots - b_{s-1}$  ( $2 \leq s \leq r$ ) such that  $R(A) = \Theta_r$  (in particular the minimal polynomial  $M_A(\lambda)$  of  $A$ ). Then, the general process (6) and (7) allows us to derive that  $A^n = \sum_{k=0}^{s-1} \phi_k(n) A^k$  for all  $n \geq r$ . More precisely, (8) and (9) are still valid when  $r, a_0, \dots, a_{r-1}$  are replaced by  $s, b_0, \dots, b_{s-1}$  (respectively). Our choice of the characteristic polynomial of  $A$  is due to the fact that its expression is defined by the determinant  $P_A(\lambda) = \det(\lambda I_r - A)$ , which is more practical for computation.

**Remark 2.2.** In [10] Liu studies the combinatorial form of solutions of some nonhomogeneous recurrence relations of order  $r$ . By using some combinatorial techniques and applying the Cayley–Hamilton Theorem to the companion matrix of  $P(\lambda) = \lambda^r - a_0\lambda^{r-1} - \dots - a_{r-1}$  (the characteristic polynomial of sequences as in (1)), Liu derives that these solutions depend on  $\rho(n, r)$  given by (3).

### 3. Computation of $e^{tA}$

Let  $A$  be in  $\text{GL}(r; \mathbb{C})$  ( $r \geq 2$ ) and  $P_A(\lambda) = \lambda^r - a_0\lambda^{r-1} - \dots - a_{r-1}$  be its characteristic polynomial. In this section we are interested in the computation of  $e^{tA}$  ( $t \in \mathbb{R}$ ), with the aid of expressions (8) and (9).

**Proposition 3.1.** Let  $A$  be in  $\text{GL}(r; \mathbb{C})$  ( $r \geq 2$ ) and  $P_A(\lambda) = \lambda^r - a_0\lambda^{r-1} - \dots - a_{r-1}$  its characteristic polynomial, with  $a_{r-1} \neq 0$ . Then,  $e^{tA} = \sum_{k=0}^{r-1} \Omega_k(t) A^k$ , where

$$\Omega_k(t) = \frac{t^k}{k!} + \sum_{n=r}^{+\infty} \frac{t^n}{n!} \rho_k(n), \quad (10)$$

$\rho_k(n) = \sum_{j=0}^k a_{r-k+j-1} \rho(n-j, r)$ , and  $\rho(n, r)$  are given by (3).

**Proof.** Set  $A^n = \sum_{k=0}^{r-1} \rho_k(n) A^k$  ( $n \geq r$ ), where  $\rho_k(n) = \sum_{j=0}^k a_{r-k+j-1} \rho(n-j, r)$ . Thus we have

$$e^{tA} = \sum_{k=0}^{r-1} \frac{t^k}{k!} A^k + \sum_{n=r}^{+\infty} \frac{t^n}{n!} \left( \sum_{k=0}^{r-1} \rho_k(n) A^k \right). \quad \square$$

**Remark 3.1.** Consider a polynomial  $R(\lambda) = \lambda^s - b_0 \lambda^{s-1} - \dots - b_{s-1}$  ( $2 \leq s \leq r$ ) such that  $R(A) = \Theta_r$  (in particular the minimal polynomial  $M_A(\lambda)$  of  $A$  (see Section 5)). Then, the general relations (6) and (7), and Remark 2.1 show that  $e^{tA}$  may be also computed as follows:  $e^{tA} = \sum_{k=0}^{s-1} f_k(t) A^k$ . More precisely, the preceding process shows that (10) is still valid when  $r, a_0, \dots, a_{r-1}$  are replaced by  $s, b_0, \dots, b_{s-1}$  (respectively). As for Remark 2.1, our choice of the characteristic polynomial of  $A$  is due to its practical computation from the determinant.

Consider the row vector  $\Omega(t) = (\Omega_0(t), \Omega_1(t), \dots, \Omega_{r-1}(t))$  and the column vector of matrices  $V(A) = {}^t(I_r, A, \dots, A^{r-1})$ . Then, (10) may be written as the matrix product

$$e^{tA} = \Omega(t) V(A). \quad (11)$$

In (10) the computation of  $e^{tA}$  depends only on the coefficients  $a_0, a_1, \dots, a_{r-1}$  of the characteristic polynomial of  $A$ .

Note that for any  $k$  ( $0 \leq k \leq r-1$ ) the series  $\Omega_k(t)$  converges in  $\mathbb{R}$ , because  $e^{tA}$  is defined for any  $t \in \mathbb{R}$ .

We can also verify that even if  $a_{r-1} = 0$ , Proposition 3.1 is still valid. As an example, let us consider the following classical case (see [2]).

**Example 3.1.** Let

$$A = \begin{bmatrix} 0 & -a & b \\ a & 0 & -c \\ -b & c & 0 \end{bmatrix},$$

where  $a, b$  and  $c$  are real numbers. The characteristic polynomial of  $A$  is  $P(\lambda) = \lambda^3 + \omega^2 \lambda$ , where  $\omega^2 = a^2 + b^2 + c^2$ . Hence  $a_0 = 0, a_1 = -\omega^2$  and  $a_2 = 0$ , which implies that  $\rho(n, 3) = (-1)^{k+1} \omega^{n-3}$  if  $n = 2k + 1 \geq 3$  and  $\rho(n, 3) = 0$  if not. A straight computation obtains

$$\begin{aligned} \Omega_0(t) &= 1, \\ \Omega_1(t) &= \frac{1}{\omega} \sum_{n=0}^{+\infty} (-1)^n \frac{(\omega t)^{2n+1}}{(2n+1)!} \\ \Omega_2(t) &= \frac{1}{\omega^2} \left( 1 - \sum_{n=0}^{+\infty} (-1)^n \frac{(\omega t)^{2n}}{(2n)!} \right). \end{aligned}$$

Then, we have

$$e^{tA} = I_3 + \frac{1}{\omega} \sin(\omega t)A + \frac{1 - \cos(\omega t)}{\omega^2} A^2.$$

This expression can also be obtained using the roots of the characteristic polynomial of  $A$ .

**Remark 3.2.** We can show that in the computation of  $\Omega_k(t)$  ( $0 \leq k \leq r-1$ ), the following class of power series appears:  $\omega_j(t) = \sum_{n=r}^{+\infty} (t^n/n!) \rho(n-j, r)$ , where  $0 \leq j \leq r-1$ . For every  $j$  we can verify directly that  $(d^j \omega_j / dt^j)(t) = \omega_0(t)$ .

**Remark 3.3.** For classical differential systems  $(dX/dt)(t) = AX(t)$ , where  $A \in GL(r; \mathbb{R})$  and  $X(t) : \mathbb{R} \rightarrow \mathbb{R}^r$ ,  $t \in \mathbb{R}$ , expression (10) may be used to obtain the solutions expressed in series form, depending only on the coefficients of the characteristic polynomial of  $A$ .

#### 4. Relation between $\rho(n, r)$ and characteristic roots of (1)

This section is devoted to a study of the connection between  $\rho(n, r)$  and the characteristic roots of the sequence (1), through some properties of  $e^{tA}$ . Here the knowledge of the eigenvalues of  $A$  (or characteristic roots) is assumed.

Consider  $A \in GL(r; \mathbb{C})$ , whose characteristic polynomial is  $P_A(\lambda) = \lambda^r - a_0 \lambda^{r-1} - \dots - a_{r-1}$ . For reasons of simplicity, suppose that  $P_A(\lambda)$  has  $r$  distinct roots  $\lambda_0, \lambda_1, \dots, \lambda_{r-1}$ . Then,  $e^{tA}$  may be written in the following form:

$$e^{tA} = e^{\lambda_0 t} U_0 + e^{\lambda_1 t} U_1 + \dots + e^{\lambda_{r-1} t} U_{r-1}, \quad (12)$$

where  $U_0, \dots, U_{r-1}$  are elements of  $GL(r; \mathbb{C})$ . From (12) we derive that the sequence of matrices  $U_0, \dots, U_{r-1}$  satisfies the following system of equations:

$$\lambda_0^k U_0 + \dots + \lambda_{r-1}^k U_{r-1} = A^k \quad \text{for any } k \geq 0.$$

Hence, from Proposition 3.1, (11) and (12) we have

$$(\Omega_0(t), \Omega_1(t), \dots, \Omega_{r-1}(t))M = (e^{t\lambda_0}, \dots, e^{t\lambda_{r-1}}), \quad (13)$$

where  $M$  is the matrix whose  $j$ th row  $L_j$  ( $0 \leq j \leq r-1$ ) is given by  $L_j = (\lambda_0^j, \dots, \lambda_{r-1}^j)$ . Thus, we derive from (13) that  $\Omega_0(t), \Omega_1(t), \dots, \Omega_{r-1}(t)$  satisfy the following Vandermonde system of  $r$  equations:

$$\Omega_0(t) + \lambda_j \Omega_1(t) + \dots + \lambda_j^{r-1} \Omega_{r-1}(t) = e^{t\lambda_j}, \quad 0 \leq j \leq r-1.$$

Thus, we have

$$\Omega_0(t) = (-1)^{r-1} \sum_{j=0}^{r-1} \frac{(\pi_{i=0, i \neq j}^{r-1} \lambda_i)}{P'(\lambda_j)} e^{t\lambda_j}. \quad (14)$$

From (9) and (10), for any  $n \geq r$ , the  $n$ th derivative of  $\Omega_0(t)$  at  $t = 0$  is given by  $\Omega_0^{(n)}(0) = \rho_0(n) = a_{r-1}\rho(n, r)$ . Thus, expression (14) allows to state the following result.

**Proposition 4.1.** *Let  $A$  be in  $GL(r; \mathbb{C})$  such that its eigenvalues  $\lambda_0, \dots, \lambda_{r-1}$  are simple. Then, for every  $n \geq r$ , we have*

$$\rho(n, r) = \frac{\lambda_0^{n-1}}{P'(\lambda_0)} + \frac{\lambda_1^{n-1}}{P'(\lambda_1)} + \dots + \frac{\lambda_{r-1}^{n-1}}{P'(\lambda_{r-1})}. \quad (15)$$

For  $V_n = \rho(n, r)$  ( $n \geq 0$ ), we can see that  $\{V_n\}_{n \geq 0}$  is a sequence of the form (1), where  $V_0 = \dots = V_{r-2} = 0$  and  $V_{r-1} = 1$  (see [9,12] for example). Then, expression (15), written as

$$\rho(n, r) = \frac{1}{\lambda_0 P'(\lambda_0)} \lambda_0^n + \frac{1}{\lambda_1 P'(\lambda_1)} \lambda_1^n + \dots + \frac{1}{\lambda_{r-1} P'(\lambda_{r-1})} \lambda_{r-1}^n,$$

represents the Binet formula for the sequence  $\{\rho(n, r)\}_{n \geq 0}$  (see [4,8] for example).

Let us illustrate the above method in the case of  $r = 3$ .

**Example 4.1.** For  $r = 3$ , suppose that the characteristic polynomial of  $A \in GL(3; \mathbb{C})$  had three distinct roots  $\alpha, \beta, \mu$  in  $\mathbb{C}$ . Then, by the above we obtain

$$\Omega_0(t) = \frac{\beta\mu(\mu - \beta)e^{t\alpha} - \alpha\mu(\mu - \alpha)e^{t\beta} + \alpha\beta(\beta - \alpha)e^{t\mu}}{(\mu - \alpha)(\beta - \alpha)(\mu - \beta)}.$$

Thus from (15) we derive that for any  $n \geq 3$ ,

$$\rho(n, 3) = \frac{1}{a_2} \frac{\beta\mu(\mu - \beta)\alpha^n - \alpha\mu(\mu - \alpha)\beta^n + \alpha\beta(\beta - \alpha)\mu^n}{(\mu - \alpha)(\beta - \alpha)(\mu - \beta)}.$$

We now suppose that  $\alpha = \beta \neq \mu$ . Then  $e^{tA}$  is of the following form:

$$e^{tA} = \begin{cases} e^{t\alpha}U_0 + te^{t\alpha}U_1 + e^{t\mu}U_2 & \text{if } A \text{ is not diagonalizable,} \\ e^{t\alpha}V_0 + e^{t\alpha}V_1 + e^{t\mu}V_2 & \text{if } A \text{ is diagonalizable,} \end{cases}$$

where  $U_0, U_1, U_2, V_0, V_1, V_2$  are elements of  $GL(3, \mathbb{C})$ . Hence, from Proposition 3.1 we obtain that for any  $n \geq 0$ ,

$$\begin{aligned} \alpha^k U_0 + k\alpha^{k-1}U_1 + \mu^k U_2 &= A^k & \text{if } A \text{ is not diagonalizable,} \\ \alpha^k V_0 + \alpha^k V_1 + \mu^k V_2 &= A^k & \text{if } A \text{ is diagonalizable.} \end{aligned}$$

Thus, in the two cases, by  $A^n = \sum_{k=0}^2 \rho_k(n)A^k$  we have that for any  $n \geq 0$ ,

$$\rho_0(n) + \alpha\rho_1(n) + \alpha^2\rho_2(n) = \alpha^n.$$

Then as before we obtain

$$\rho(n+1, 3) = a_0\rho(n, 3) + a_1\rho(n-1, 3) + a_2\rho(n-2, 3).$$

This result is also given in [12], using the Fibonacci sequences properties. From the Binet formula we then get

$$\rho(n, 3) = c_0\alpha^n + c_1n\alpha^n + c_2\mu^n \quad (c_i \in \mathbb{C}).$$

Thus, for  $n \geq 3$ , we have

$$\rho(n, 3) = \frac{(n-2)\alpha^{n-1} - (n-1)\alpha^{n-2}\mu^1 + \mu^{n-1}}{(\mu - \alpha)^2}.$$

**Remark 4.1.** Note that  $\rho(n, r)$  ( $n \geq r$ ) is related to the sequence of multivariate Fibonacci polynomials  $\{H_n^{(r)}(a_0, \dots, a_{r-1})\}_{n \geq 0}$  of the order  $r$  of Philippou (see [1]) as follows:  $\rho(n, r) = H_{n-r+1}^{(r)}(a_0, \dots, a_{r-1})$ . Thus (15) allows another expression for the multivariate Fibonacci polynomials of Philippou.

We conclude this section by giving some comments on the applicability of (10) and (15). The substitution of  $\rho(n-j, r)$  given by (15) in  $\rho_k(n) = \sum_{j=0}^k a_{r+j-k-1} \rho(n-j, r)$  allows us to write expression (9) of  $A^n$  as follows:  $A^n = \sum_{i=0}^{r-1} \phi_i(n; \lambda_0, \dots, \lambda_{r-1}) A^i$  for every  $n \geq r$ . The substitution of the above expression in (10) implies that

$$\Omega_k(t) = \frac{t^k}{k!} + \sum_{n=r}^{\infty} \phi_k(n; \lambda_0, \dots, \lambda_{r-1}) \frac{t^n}{n!}.$$

Hence, we can show that  $\Omega_k(t)$  ( $0 \leq k \leq r-1$ ) may be expressed in terms of eigenvalues of  $A$ , relating our results to the results of Ref. [3].

## 5. Relation to the results in [3]

Let  $A$  be an  $r \times r$  matrix and  $P_A(\lambda)$ ,  $M_A(\lambda)$  be its characteristic and minimal polynomial (respectively). It is clear that  $P_A(\lambda)$  is of degree  $r$  and the degree  $m$  of  $M_A(\lambda)$  is  $\leq r$ . Let  $\lambda_1, \dots, \lambda_s$  be the eigenvalues of  $A$ , with multiplicities  $m_1, \dots, m_s$  (respectively). It was established in [3] (see Lemma 4 and Theorem 2) that

$$e^{tA} = \sum_{k=0}^{m-1} f_k(t) A^k,$$

where  $f_k(t)$  ( $0 \leq k \leq r-1$ ) are some analytic functions which satisfy the following system of  $m$  equations:

$$\sum_{k=i}^{m-1} \binom{k}{i} \lambda_j^{k-i} f_k(t) = \frac{t^i}{i!} e^{\lambda_j t}, \quad 1 \leq j \leq s, \quad 0 \leq i \leq m_j - 1. \quad (16)$$

From Proposition 2 of [3] and Proposition 3.1 we can derive the solutions of the system of  $m = r$  equations (16) as follows.



**Proposition 5.1.** *Let  $A \in \text{GL}(r; \mathbb{C})$  such that  $P_A(\lambda) = M_A(\lambda)$ . Then the solutions  $f_k(t)$  ( $0 \leq k \leq r-1$ ) of system (16) are given by*

$$f_k(t) = \Omega_k(t) = \frac{t^k}{k!} + \sum_{n=r}^{+\infty} \frac{t^n}{n!} \rho_k(n), \quad 0 \leq k \leq r-1,$$

where  $\rho_k(n) = \sum_{j=0}^k a_{m-k+j-1} \rho(n-j, m)$  and  $\rho(n, m)$  are given by (3).

The analytic functions  $f_k(t)$  ( $0 \leq k \leq r-1$ ) above are expressed in terms of the coefficients of the characteristic polynomial  $P_A(\lambda)$ , which are implicit functions of the eigenvalues  $\lambda_1, \dots, \lambda_s$  of  $A$ .

Suppose now that  $m \leq r-1$  and consider  $M_A(\lambda) = X^m - b_0 X^{m-1} - \dots - b_{m-1}$  ( $2 \leq m \leq r$ ). Then, as was shown in Remark 4.2, the following result can be derived from (6) and (7).

**Proposition 5.2.** *Let  $A$  be in  $\text{GL}(r; \mathbb{C})$  ( $r \geq 2$ ) and  $M_A(\lambda) = \lambda^m - b_0 \lambda^{m-1} - \dots - b_{m-1}$  be its minimal polynomial, with  $b_{m-1} \neq 0$ . Then,  $e^{tA} = \sum_{k=0}^{m-1} f_k(t) A^k$ , where*

$$f_k(t) = \frac{t^k}{k!} + \sum_{n=m}^{+\infty} \frac{t^n}{n!} \rho_k(n), \quad 0 \leq k \leq m-1, \quad (17)$$

with  $\rho_k(n) = \sum_{j=0}^k b_{m-k+j-1} \rho(n-j, m)$  and  $\rho(n, m)$  given by (3).

Here  $f_k(t)$  ( $0 \leq k \leq m-1$ ) are expressed in terms of the coefficients  $b_j$  ( $0 \leq j \leq m-1$ ) of the minimal polynomial. Thus the eigenvalues  $\lambda_0, \dots, \lambda_{s-1}$  appear implicitly in these functions too.

From Propositions 3.1 and 5.2 we also obtain that matrices with the same characteristic (respectively, minimal) polynomial have the same explicit formula (10) (respectively, (17)). In particular, if  $B$  is a matrix similar to  $A$  we have

$$e^{tB} = \sum_{k=0}^{s-1} \Omega_k(t) B^k,$$

where  $s = r$  or  $m$  and  $\Omega_k(t)$  ( $0 \leq k \leq s-1$ ) are given by (10) (or (17)). This is in essence Proposition 1 and Corollary 2 of [3].

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