INEVITABLE GRAPHS: A PROOF OF THE TYPE II CONJECTURE AND SOME RELATED DECISION PROCEDURES

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We verify the "Type II Conjecture" concerning the question of which elements of a finite monoid M are related to the identity in every relational morphism with a finite group. We confirm that these elements form the smallest submonoid, K, of M (containing 1 and) closed under "weak conjugation", that is, if $x \in K$, $y \in M$, $z \in M$ and $yzy = y$ then $yxz \in K$ and $zxy \in K$.

More generally, we establish a similar characterization of those directed graphs having edges are labelled with elements of M which have the property that for every such relational morphism there is a choice of related group elements making the corresponding labelled graph "commute". We call these "*inevitable M-graph*". We establish, using this characterization, an effective procedure for deciding from the multiplication table for M whether an "M-graph" is inevitable.

A significant stepping-stone towards this was Tilson's 1986 construction which established the Type II Conjecture for regular monoid elements, and this construction is used here in a slightly modified form. But substantial credit should also be given to Henckell, Margolis, Meakin and Rhodes, whose recent independent work follows lines very similar to our own.

Let M be a finite monoid. We consider all possible choices of a finite monoid R, a finite group G and homomorphisms $\mu : R \twoheadrightarrow M$, $\gamma : R \to G$. In [8] the question was proposed of determining which elements x of M have the property that for each such μ, γ there exists $r \in R$ with $\mu(r) = x$ and $\gamma(r) = 1$. Such elements were called "type II" elements in [8]. The set of all elements of type II is also called the "kernel" of M.

We answer this question and some related questions, for example the question of which elements $x_1, x_2, \ldots, x_k$ have the property that, for each such μ, γ, R, G there exist $r_1, r_2, \ldots, r_k \in R$ with each $\mu(r_i) = x_i$ and (i) $\gamma(r_1)\gamma(r_2)\ldots\gamma(r_k) = 1$, or alternatively, (ii) $\gamma(r_1) = \gamma(r_2) = \cdots = \gamma(r_k)$.

We define in Sec. 1 the notion of an *inevitable M-graph* and observe how the questions mentioned can be expressed in terms of this notion. In Sec. 2 we give the relevant definitions and state our principal result, Theorem 2.1, characterizing inevitable M-graphs.

In Sec. 3 we show how, in consequence of Theorem 2.1, we can effectively determine from the multiplication table of M whether a given finite M-graph is inevitable. From

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the comments of Sec. 1 it follows that our initial questions can also be effectively answered. In particular, from the details of the decision procedure, we confirm Conjecture 1.3 of [4], referred to in [6] and [8] as the "(Rhodes) Type II Conjecture" and suggested by the 1972 result of Rhodes and Tilson, [7]. This application was also discussed in [1].

The remaining sections, 4 to 10, of the paper are devoted to proving Theorem 2.1.

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The results of this paper were obtained by the author with the knowledge of [8], but without knowledge of the more recent [2], [3] or [5].

The definition of the relation S, introduced in Sec. 2, and its relevant properties were extrapolated by the author from [8], while the same relation was considered in [2] and [3] in which the two questions of the second paragraph of this introduction were also raised.

Likewise, the definition of the inverse semigroups $Q(G)$, introduced in Sec. 5, and their properties used here were developed by the author from the well-known constructions of McAlister and Munn for free inverse semigroups and independently of the more detailed treatment of [5]. (In [5], the phrase "E-unitary cover" is used where we have used the word "adequate".)

1. Inevitable M-graphs

Let X be any monoid. [We reserve the symbol M for *finite* monoids.] Define an *X-graph* to be a pair $\underset{\sim}{D} = (D, \{x_e\}_{e \in E})$ where D is a finite directed graph having E as its set of edges and, for each $e \in E$, x_e is an element of X.

If G is a group, we say that a G-graph $(D, \{g_e\}_{e \in E})$ *commutes* if, for each finite sequence $e_1, e_2, \ldots, e_k$ of edges of D forming an undirected circuit in D, the corresponding product $g_{e_1}^{\pm 1} g_{e_2}^{\pm 1} \ldots g_{e_k}^{\pm 1}$ in G is equal to 1, in which we take g_{e_i} if e_i is an edge in the forward direction of the circuit and $g_{e_i}^{-1}$ if e_i is in the backward direction.

For a finite monoid M, we define the M-graph $(D, \{x_e\}_{e \in E})$ to be *inevitable* (for finite groups) if, whenever R is a finite monoid, G a finite group, $\mu : R \twoheadrightarrow M$ and $\gamma : R \to G$ then there exists a choice for each $e \in E$ of $g_e \in \gamma(\mu^{-1}(x_e))$ such that the G-graph $(D, \{g_e\}_{e \in E})$ commutes.

From these definitions we see that an element $x \in M$ is of "type II" if and only if the M-graph consisting of a single vertex U and a single loop e at U with $x_e = x$ is inevitable.

The two conditions on $x_1, x_2, \ldots, x_k \in M$ mentioned above can similarly be re-expressed by the statements that the following M-graph are inevitable.

(i) The M-graph consisting only of a directed circuit having edges $e_1, e_2, \ldots, e_k$ in order where $x_{e_i} = x_i$.

(ii) The M-graph having two distinct vertices U and V and edges $e_1, e_2, \ldots, e_k$ each directed from U to V where $x_{e_i} = x_i$.

2. Principal Result

Let M be a finite monoid. Let us choose, and fix from now on, a finite set A and a homomorphism from A^* onto M, denoted by $w \mapsto [w]_M$. The arbitrary nature of these choices may, of course, be avoided by taking $A = M$.

(For any set Y, as usual Y^* denotes the free monoid on the set Y, regarded as the set of all finite sequences, or "words" from Y with concatenation written as juxtaposition.)

Let A^{-1} denote the set of symbols $\{a^{-1} : a \in A\}$, assumed to be disjoint from A. We give a necessary and sufficient condition that an M-graph is inevitable in terms of the relation $S \leq M \times (A \cup A^{-1})^*$ defined as follows. (This relation was also considered independently in [2] and [3].)

Definition. Let S be the submonoid of $M \times (A \cup A^{-1})^*$ generated by all the pairs $([a]_M, a)$ for $a \in A$ and the pairs (m, a^{-1}) for which $a \in A$ and $m[a]_M m = m$.

Thus, $(m, w) \in S$ iff m has a factorization $m_1 m_2 \dots m_k$ in M from which the word w arises on replacing each m_i either by some $a \in A$ for which $m_i = [a]_M$ or by some $a^{-1} \in A^{-1}$ for which $m_i[a]_M m_i = m_i$.

Let $FG(A)$ denote the free group on the set A and let $w \mapsto [w]_{FG(A)}$ denote the usual homomorphism from $(A \cup A^{-1})^*$ to $FG(A)$. For a sequence $\{w_e\}_{e\in E}$ from $(A \cup A^{-1})^*$, we refer more briefly to the $FG(A)$-graph $(D, \{[w_e]_{FG(A)}\}_{e\in E})$ as the $FG(A)$-graph $(D, \{w_e\}_{e\in E})$.

In Secs. 4–10 we establish the following:

Principal Theorem 2.1. *Let M be a finite monoid. Then an M-graph $(D, \{x_e\}_{e\in E})$ is inevitable if and only if there is a choice, for each $e \in E$ of $w_e \in (A \cup A^{-1})^*$ such that each $(x_e, w_e) \in S$ and the $FG(A)$-graph $(D, \{w_e\}_{e\in E})$ commutes.*

Comment. Note that here the choice of w_e depends on e only, not on x_e (as does the choice of g_e in the definition of an inevitable M-graph).

Theorem 2.1 is ultimately proved by the more technical Proposition 10.2 obtained using Proposition 9.2 from the results of Sec. 5 to Sec. 8 concerning inverse monoids and, in the easier direction, from Sec. 4. Before embarking on this proof, we show in Sec. 3 the consequences of this result for decidability.

3. Decidability

We show here how it follows from Theorem 2.1 that there is a decision procedure for determining, given the multiplication table of a finite monoid M, whether a given M-graph $(D, \{x_e\}_{e\in E})$ is inevitable.

If the undirected graph D has several connected components, then each component determines an M-subgraph and clearly the given M-graph is inevitable if and only if each such component is. It is therefore sufficient to consider only the case where the (undirected) graph D is connected.

Let us say that an M-graph $(D, \{x_e\}_{e\in E})$ is *M-simple* if for every two vertices U and V of D and each element x of M, there is *at most one* edge $e \in E$ directed from U to V for which $x_e = x$. (We allow the possibility that $U = V$, in which case e is a loop.)

For any M-graph $(D, \{x_e\}_{e \in E})$ we define its *M-simplification* to be the M-graph which results by identifying, for every ordered pair (U, V) of vertices of D and each $x \in M$, all edges e, if any, from U to V having $x_e = x$ to form a single edge. We note that, to within isomorphism, this M-simplification is also a subgraph of the original M-graph.

Clearly the given M-graph is inevitable if and only if its M-simplification is, and of course, if the first is connected then so is the second. It is therefore sufficient to determine whether a given connected, M-simple M-graph $(D, \{x_e\}_{e \in E})$ is inevitable.

The advantage of considering only M-simple M-graphs is that there are only finitely many such, for a given M and a given number of vertices of D. A further step towards the decision procedure is the following proposition, which we shall prove from Theorem 2.1.

Proposition 3.1. *For a finite monoid M, the connected M-simple, inevitable M-graphs form the smallest class $\mathscr{C}$ of M-graphs satisfying the following conditions:*

(A) $\circ$ *and* $\overset{\overset{1}{\frown}}{\circ}$ *are in* $\mathscr{C}$.

(B) *If $\underset{\sim}{D}_1$ and $\underset{\sim}{D}_2$ are in $\mathscr{C}$, have no edges in common and exactly one vertex in common, then each connected, M-simple M-subgraph of the union of $\underset{\sim}{D}_1$ and $\underset{\sim}{D}_2$ containing all their vertices is also in $\mathscr{C}$.*

(C)(i) *If $\underset{\sim}{D} = (D, \{x_e\}_{e \in E})$ is in $\mathscr{C}$, U is a vertex of D and $y \in M$, then any M-simple M-graph $\underset{\sim}{D}'$ is in $\mathscr{C}$* which arises from *$\underset{\sim}{D}$ by:*

(a) *changing x_e to $x'_e = yx_e$ for each edge e out of U which is not a loop,*

(b) *replacing each edge e into U, which is not a loop, by one or more edges e_i between the same vertices and taking $x'_{e_i} = x_e z_i$ for some $z_i \in M$ with $z_i y z_i = z_i$,*

(c) *replacing each loop e at U by one or more loops e_i at U and taking each $x'_{e_i} = ye_i z_i$ for some $z_i \in M$ with $z_i y z_i = z_i$.*

(C)(ii) *As for* C(i) *except that in* (a) *$x'_e = x_e y$ for each edge e into U, not a loop, in* (b) *each edge e out of U, not a loop, is replaced by one or more edges e_i between the same vertices with $x'_{e_i} = z_i x_e$ for some $z_i \in M$ with $z_i y z_i = z_i$, and in* (c) *each loop e at U is replaced by one or more loops e_i at U with $x'_{e_i} = z_i x_e y$ for some $z_i \in M$ with $z_i y z_i = z_i$.*

(D) *If $\underset{\sim}{D}$ is in $\mathscr{C}$ and has edges e_1, e_2 from vertices U_0 to U_1 and from U_1 to U_2 respectively, and if $\underset{\sim}{D}$ contains no edge e from U_0 to U_2 with $x_e = x_{e_1} x_{e_2}$, then the graph $\underset{\sim}{D}'$ obtained by adding such an edge e is also in $\mathscr{C}$.*

(E) *If $\underset{\sim}{D} = (D, \{x_e\}_{e \in E})$ is in $\mathscr{C}$ and U is any vertex of D, then so is the M-graph $\underset{\sim}{D}_1 = (D_1, \{x'_e\}_{e \in E_1})$ defined to be the M-graph having as vertices those of $\underset{\sim}{D}$ together with one new vertex U' and having as edges:*

(i) *the edges e of D with $x'_e = x_e$.*

(ii) *for each vertex V of D and each edge e of D from U to V, an edge f from U' to V with $x'_f = x_e$.*

(iii) *for each vertex V of D and each edge e of D from V to U, an edge g from V to U' with $x'_g = x_e$.*

(iv) *for each loop e at U, a loop h at U' with $x'_h = x_e$.*

(In (ii) *and* (iii) *above we do not exclude the case where $V = U$, so a loop at U in D gives rise to four edges of D_1 under each of the clauses.)*

Comment. Clearly, by (A) and (B), this smallest class $\mathscr{C}$ has the further property:

(B)′ If $\underset{\sim}{D}$ is in $\mathscr{C}$ then so is any connected M-subgraph of $\underset{\sim}{D}$ obtained by removing edges only. We wish, however, to exclude the removal of vertices, for the sake of Corollary 3.2.

Before proving Proposition 3.1, we complete the discussion of the resulting decision procedure. The implications (B), (C), (D), (E) were chosen to be of the form "If $\underset{\sim}{D}_1, \underset{\sim}{D}_2, \ldots, \underset{\sim}{D}_k \in \mathscr{C}$ then $\underset{\sim}{D}$ is in $\mathscr{C}$" where each $\underset{\sim}{D}_i$ has no more vertices than does $\underset{\sim}{D}$. The following is therefore immediate.

Corollary 3.2. *Let M be a finite monoid. Then the connected, M-simple, inevitable M-graphs having at most n vertices form the smallest class $\mathscr{C}_n$ of M-graphs satisfying condition* (A) *of Proposition* 3.1 *and the implications* (B), (C), (D), (E) *of Proposition* 3.1 *when applied only to M-simple graphs having at most n vertices.*

Decision procedure concluded

Given the multiplication table of a finite monoid M and a connected, M-simple M-graph $(D, \{x_e\}_{e\in E})$ having n vertices, by Corollary 3.2 we can obtain a complete list (up to isomorphism) of all the connected, M-simple, inevitable M-graphs having at most n vertices by beginning with the graphs given in (A) and repeatedly applying the implications (B), (C), (D), (E), restricted to yield only M-graphs of at most n vertices, until no new M-graphs appear. We may then verify whether the given graph appears in this list.

We proceed to prove Proposition 3.1, assuming Theorem 2.1.

Proof of Proposition 3.1. It is fairly easy to show directly, by induction on the number of steps (B), (C), (D), (E) needed, that any M-graph in $\mathscr{C}$ is inevitable. For steps C(i) and C(ii) we may appeal to Lemma 4.1, which is proved later but from first principles.

(From the argument in Cases 3(i), (ii) below it will be seen that (C)(i), (ii) could have been stated in the more restricted form where $y = [a]_M$, in which case the implication in this direction could proceed more simply by proving that every member of $\mathscr{C}$ has a suitable choice of $\{w_e\}_{e\in E}$ and appealing to Theorem 2.1.)

For the converse, let us define a *witnessing choice* for an M-graph $(D, \{x_e\}_{e\in E})$ to be a sequence $\{w_e\}_{e\in E}$ from $(A \cup A^{-1})^*$ for which the $FG(A)$-graph $(D, \{x_e\}_{e\in E})$ commutes. Then, by Theorem 2.1, every inevitable M-graph has a witnessing choice. For each witnessing choice $\{w_e\}_{e\in E}$ for $(D, \{x_e\}_{e\in E})$ and each $i = 0, 1, 2, \ldots$, let k_i be the number of edges e of D for which the word w_e has length i, let the *rank* of $\{w_e\}_{e\in E}$ be the ordinal number $\cdots + \omega^2\cdot k_2 + \omega\cdot k_1 + k_0$ (in which, of course, only finitely many terms are non-zero) and let the *rank* of an inevitable M-graph $\underset{\sim}{D} = (D, \{x_e\}_{e\in E})$ be $\omega^\omega\cdot n + \beta$ where n is the number of vertices of D and β is the smallest of the ranks of the witnessing choices for $\underset{\sim}{D}$. We show, by transfinite induction on the rank of a connected, M-simple, inevitable M-graph $\underset{\sim}{D}$ that $\underset{\sim}{D}$ is in $\mathscr{C}$.

Let U be an arbitrary vertex of D.

Case 1. The degree of U in the undirected graph D is 0. Then, since D is connected, it consists only of the vertex U and no edges, and so $\underset{\sim}{D}$ is in $\mathscr{C}$ by (A).

Case 2. $\underset{\sim}{D}$ is the union of two M-subgraphs $\underset{\sim}{D}_1$ and $\underset{\sim}{D}_2$ having no edges in common, having only U as their only common vertex and each having at least one edge. Then each $\underset{\sim}{D}_i$ is an M-subgraph of $\underset{\sim}{D}$, and therefore is inevitable, and each has strictly fewer edges and so has strictly smaller rank than that of $\underset{\sim}{D}$. So, by the induction hypothesis, each $\underset{\sim}{D}_i$ is in $\mathscr{C}$ and since $\underset{\sim}{D}$ arises from $\underset{\sim}{D}_1$ and $\underset{\sim}{D}_2$ by (B), we conclude that $\underset{\sim}{D}$ is in $\mathscr{C}$.

Now choose, and fix for the remainder of the argument, a witnessing choice $\{w_e\}_{e\in E}$ for $\underset{\sim}{D}$ having the least possible rank.

Case 3(i). There exists $a \in A$ such that, for every edge e out of U, w_e begins with a and, for every edge e into U, w_e ends with a^{-1}.

Then for each edge e out of U, not a loop, we have $x_e S w_e$ and $w = aw'_e$ for some $w'_e \in (A \cup A^{-1})^*$ where $x_e S a w'_e$ and so $x_e = [a]_M x'_e$ for some $x'_e \in M$ such that $x'_e S w'_e$. For every edge e into U, not a loop, $w_e = w'_e a^{-1}$ where $x_e S w'_e a^{-1}$ and so $x_e = x'_e z$ for some $x'_e, z \in M$ such that $x'_e S w'_e$ and $z[a]_M z = z$. For each loop e at U, since a, a^{-1} are assumed to be different symbols, $w_e = aw'_e a^{-1}$ where $x_e S a w'_e a^{-1}$, so $x_e = [a]_M x'_e z$ where $x'_e S w'_e$ and $z[a]_M z = z$.

In this case let $\underset{\sim}{D}_0$ be the result of replacing x_e by x'_e for the three kinds of edge e just considered. Then $\underset{\sim}{D}_0$ has a witnessing choice $\{w'_e\}$ obtained by replacing each w_e by w'_e for these same edges e, and so $\underset{\sim}{D}_0$ has smaller rank than that of $\underset{\sim}{D}$. As an M-subgraph of $\underset{\sim}{D}_0$, so therefore does the M-simplification $\underset{\sim}{D}_1$ of $\underset{\sim}{D}_0$, and so $\underset{\sim}{D}_1$ is in $\mathscr{C}$ by the induction hypothesis. Since $\underset{\sim}{D}$ arises from $\underset{\sim}{D}_1$ by C(i), we may conclude that also $\underset{\sim}{D}$ is in $\mathscr{C}$.

Case 3(ii). There exists $a \in A$ such that, for every edge *e into* U, w_e begins with a and for every edge *e out of* U, w_e ends with a^{-1}.

In this case, the argument is entirely analogous to that of Case 3(i), concluding that $\underset{\sim}{D}$ is in $\mathscr{C}$ by (C)(ii).

Now consider any undirected circuit $\gamma = (e_1, e_2, \ldots, e_k)$ in D, where $k \geq 1$. Define $\hat{w}_{e_i}$ to be w_{e_i} if e_i is in the direction of the circuit and to be the result of reversing the sequence w_{e_i} and replacing each $a \in A$ by a^{-1} and *vice versa* if e_i is in the opposite direction. Thus, in this second case, $\hat{w}_{e_i} = w_{e_i}^{-1}$ in $FG(A)$, and so by the assumption that $\{w_e\}_{e\in E}$ is a witnessing choice, the word $w_\gamma = \hat{w}_{e_1}\hat{w}_{e_2}\ldots\hat{w}_{e_k}$ has $w_\gamma = 1$ in $FG(A)$.

Case 4(i). For some such circuit γ at U and some e_i $(1 \leq i \leq k)$ in the direction of the circuit, there exist $u, v \in (A \cup A^{-1})^*$ of non-zero lengths for which $w_{e_i} = uv$ and, in $FG(A)$, $\hat{w}_{e_1}\ldots\hat{w}_{e_{i-1}}u = v\hat{w}_{e_{i+1}}\ldots\hat{w}_{e_k} = 1$.

Suppose that the arc $e_1, \ldots, e_{i-1}$ leads from U to the vertex V and that the arc $e_{i+1}, \ldots, e_k$ leads from W to U. Since $x_{e_i} Suv$ we have $x_{e_i} = rt$ where rSu and tSv. Let $\underset{\sim}{D}_0$ be the M-simple M-graph which arises from $\underset{\sim}{D}$ by deleting the edge e_i and inserting, if necessary, a new edge f from V to U with $x_f = r$ and, if necessary, a new edge g from U to W with $x_g = t$. (If $i = 1$ or $i = k$ then the new edge is a loop.) Then $\underset{\sim}{D}_0$ has a witnessing choice, taking $w_f = u$ and $w_g = v$, of smaller rank than that of $\underset{\sim}{D}$, so by the induction hypothesis $\underset{\sim}{D}_0$ is in $\mathscr{C}$ and thus, since $\underset{\sim}{D}$ arises from $\underset{\sim}{D}_0$ by condition (D) of Proposition 3.1, we conclude that also $\underset{\sim}{D}$ is in $\mathscr{C}$.

Case 4(ii). For some undirected circuit $\gamma = (e_1, \ldots, e_k)$ at U and some e_i $(1 \leq i \leq k)$ in the direction *opposite* to the circuit, there exist $u, v \in (A \cup A^{-1})^*$ of non-zero lengths for which $\hat{w}_{e_i} = uv$ and, in $FG(A)$, $\hat{w}_{e_1}\ldots\hat{w}_{e_{i-1}}u = v\hat{w}_{e_{i+1}}\ldots\hat{w}_{e_k} = 1$.

Then we may consider the same circuit γ in the *opposite* direction, showing that Case 4(i) applies and therefore that $\underset{\sim}{D}$ is in $\mathscr{C}$.

Case 5. For some undirected circuit $\gamma = (e_1, \ldots, e_k)$ at U, there exists $i\,(1 \leq i \leq k-1)$ for which, in $FG(A)$, $\hat{w}_{e_1} \ldots \hat{w}_{e_i} = \hat{w}_{e_{i+1}} \ldots \hat{w}_{e_k} = 1$. Suppose that the arc $e_1, \ldots, e_i$ leads from U to the vertex U'. Then $U' \neq U$. Let $\underset{\sim}{D}_0$ be the M-graph obtained by identifying U' with U, for example by deleting the vertex U' and replacing each edge e in $\underset{\sim}{D}$ from V to W by an edge e^* from V^* to W^* with $x_{e^*} = x_e$, where $V^* = V$ for $V \neq U'$ and $(U')^* = U$.

Then $\underset{\sim}{D}_0$ has a witnessing set $\{w_{e^*}\}$ where $w_{e^*} = w_e$, so $\underset{\sim}{D}_0$ is inevitable. Let $\underset{\sim}{D}_1$ be the M-simplification of $\underset{\sim}{D}_0$. Then $\underset{\sim}{D}_1$ is also inevitable and, since it has fewer vertices than $\underset{\sim}{D}$, by the induction hypothesis $\underset{\sim}{D}_1$ is in $\mathscr{C}$. Since $\underset{\sim}{D}$ arises from $\underset{\sim}{D}_1$ by (E), (A) and (B), $\underset{\sim}{D}$ is also in $\mathscr{C}$.

Case 6. There is an edge e at U, not a loop, with $x_e = 1$. Then, as in Case 5, $\underset{\sim}{D}$ arises by (E), (A) and (B), from the M-simplification $\underset{\sim}{D}_1$ of the result of identifying the endpoints of e, which is also inevitable and has fewer vertices and therefore smaller rank.

Case 7. None of Cases 1 to 6 applies. First suppose, for a contradiction, that there is no loop at U. Consider the equivalence relation defined on the set E of all edges of D by $e_1 \sim e_2$ if there is an undirected path (equivalently, an arc) in D containing at least one vertex of e_1, at least one vertex of e_2, but not containing U. (We include each trivial path consisting of a single vertex.)

For each equivalence class of edges, we may form the corresponding M-subgraph of $\underset{\sim}{D}$ consisting of these edges and their endpoints. Since D is connected, each such M-subgraph contains U (by considering an arc from U to any other vertex), while for any two different equivalence classes, the corresponding M-subgraphs can have no vertex in common other than U. Since Case 2 does not apply, there is at most one equivalence class of edges.

Thus, every two distinct edges at U form the first and last edges of an undirected circuit at U.

Now, since Case 1 does not apply and D is connected, there is at least one edge e at U. By symmetry (between the M-graph $\underset{\sim}{D}$ and that which arises by reversing the directions of all its edges) we may assume that e is an edge *out of* U. Since Case 6 does not apply, $w_e \neq 1$.

Let $w_e = cv$ where $c \in A \cup A^{-1}$. Then, since Cases 3(i) and 3(ii) do not apply, there is either an edge e' out of U with $w_{e'} = dv'$, $d \in A \cup A^{-1}$ and $d \neq c$ or an edge e' into U with $w_{e'} = v'd$, $d \in A \cup A^{-1}$ and, for each $a \in A$, both $(c, d) \neq (a, a^{-1})$ and $(c, d) \neq (a^{-1}, a)$.

In either case $e \neq e'$ so, by our previous conclusion, there is an undirected circuit $\gamma = (e_1, e_2, \ldots, e_k)$ at U (where $e_1 = e$ and $e_k = e'$) with $k \geq 2$ for which the product $w_\gamma = \hat{w}_{e_1}\hat{w}_{e_2} \ldots \hat{w}_{e_k}$ defined before Case 4 is, for each $a \in A$, not of either form aua^{-1} or $a^{-1}ua$.

By choice of $\{w_e\}_{e \in E}$, $w_\gamma = 1$ in $FG(A)$. The set of all $w \in (A \cup A^{-1})^*$ for which $w = 1$ in $FG(A)$ is the smallest subset $\mathscr{E}$, of $(A \cup A^{-1})^*$, for which (P) $1 \in \mathscr{E}$, (Q) if $u, v \in \mathscr{E}$ then

$uv \in \mathscr{E}$, (R) if $u \in \mathscr{E}$, $a \in A$ then aua^{-1}, $a^{-1}ua \in \mathscr{E}$. So, since (R) does not yield w_γ, and since $w_{e_1} \neq 1$ we also have $w_\gamma \neq 1$ (in $(A \cup A^{-1})^*$). Thus $w_\gamma = uv$, where u and v are strictly shorter words than w_γ, and so one of Cases 4(i), 4(ii) or 5 will apply. This completes the contradiction, showing that there must be a loop at U.

But now, since Case 2 does not apply, D consists of the single vertex U and a single loop e at U. Since Cases 3(i) and 3(ii) do not apply, w_e is, for each $a \in A$ not of either form aua^{-1} or $a^{-1}ua$. Since $w_e = 1$ in $FG(A)$, either $w_e = 1$ (in $(A \cup A^{-1})^*$) or $w_e = uv$ for $u, v = 1$ in $(A \cup A^{-1})^*$ shorter than w_e. In this second case Case 4(i) would apply, so we can only have $w_e = 1$. Since $x_e S w_e$, we therefore have $x_e = 1$.

Thus we conclude that the only remaining possibility for $\underset{\sim}{D}$ is a single vertex having a single loop e with $x_e = 1$, and then $\underset{\sim}{D} \in \mathscr{C}$ by (A).

This completes the transfinite induction. □

As remarked, an M-graph is inevitable if and only if the M-simplification of each of its connected components is inevitable. So it follows immediately from Proposition 3.1 that:

Proposition 3.3. *The class $\mathscr{C}'$ of all inevitable M-graphs is the smallest class of M-graphs satisfying conditions* (A), (B), (C), (D), (E) *of Proposition* 3.1 *stated for $\mathscr{C}'$ instead of $\mathscr{C}$ and also*:

(F) *The disjoint union of two M-graphs in $\mathscr{C}'$ is also in $\mathscr{C}'$.*

(G) *If $(D, \{x_e\}_{e \in E})$ is in $\mathscr{C}'$ and e is any edge of D then the result of adding a new edge e' between the same vertices as e and with $x_{e'} = x_e$ is also in $\mathscr{C}'$.*

Type II elements

As we have remarked, the element x of M is of type II if and only if the M-graph having one vertex and one loop at that vertex, labelled x, is inevitable. So, by Corollary 3.2, we need only consider M-graphs with one vertex and several loops. By Proposition 3.1(B) and (B)′, such multiple loops are inevitable if and only if each of the elements of M involved is of type II.

Clause (E) of Corollary 3.2. can thus be ignored, and it follows from Corollary 3.2 that the *subsets* of K_1 form the smallest class of subsets of M containing $\{1\}$, by (A), closed under the formation of conjugates as in C(i) and in C(ii), and closed under the addition of products, by (D).

It follows immediately that:

Proposition 3.4. *The class K_1 of type II elements of M is the smallest subset of M satisfying the conditions*

(1) $1 \in K_1$;

(2) *if* $x_1, x_2 \in K_1$ *then* $x_1 x_2 \in K_1$;

(3) *if* $x \in K_1$ *then* $yxz \in K_1$ *whenever* $y, z \in M$ *and either* $yzy = y$ *or* $zyz = z$.

This confirms the "type II conjecture" reported in [8].

Inevitable directed circuits

As an example of the way in which the argument of Proposition 3.1 can be varied we may consider the classes K_n of inevitable M-graphs which form directed circuits, having edges $e_1, e_2, \ldots, e_n$ in sequence and corresponding elements $x_1, x_2, \ldots, x_n$ of M. (Thus, the M-graph is specified by the sequence $(x_1, \ldots, x_n)$ from M.) These can be characterized by induction on n, appealing directly to Proposition 2.1 and taking a simpler notion of rank, namely the least possible sum of the lengths of $w_{e_1}, w_{e_2}, \ldots, w_{e_n}$ which provide a witnessing choice.

If any $w_{e_i} = 1$ in $FG(A)$, then x_i is in K_1 and the result of removing the edge e_i and identifying its endpoints is in K_{n-1}. Otherwise, by choice of the w_{e_i}, $w = w_{e_1} w_{e_2} \ldots w_{e_n} \in (A \cup A^{-1})^*$ has $w = 1$ in $FG(A)$, so two possible situations remain.

The first is that $w = uv$ for shorter words u, v, in which case the arguments of Case 4(i) or Case 5 of the proof of Proposition 3.1 apply. But then the resulting simplified graph is obtained by (B) from two circuit graphs of smaller rank and no more vertices.

The remaining possibility is that one of Cases 3(i) or 3(ii) applies, in which case again the result has smaller rank. So the simplest statement we can find is:-

Proposition 3.5. *The classes K_n of inevitable circuit M-graphs may be obtained, by induction on n, as follows. The class K_1 is as in Propsition 3.4. For $n > 1$, K_n is the smallest class of circuit M-graphs $(x_1, x_2, \ldots, x_n)$ such that:*

(1) *(Because of this notation) every cyclic permutation of any sequence $(x_1, x_2, \ldots, x_n)$ from K_n is also in K_n.*
(2) *If $i + j = n$, $(x_1, \ldots, x_i)$ is in K_i and $(y_1, \ldots, y_j)$ is in K_j, then $(x_1, \ldots, x_i, y_1, \ldots, y_j)$ is in K_n.*
(3) *If $i + j = n + 1$, $(x_1, \ldots, x_i)$ is in K_i and $(y_1, \ldots, y_j)$ is in K_j, then $(x_1, \ldots, x_i y_1, \ldots, y_j)$ is in K_n.*
(4) *If $(x_1, x_2, \ldots, x_n)$ is in K_n then so is $(yx_1, x_2, \ldots, x_n z)$ whenever either $yzy = y$ or $zyz = z$.*

4. Sufficiency of the Condition

We now proceed with our proof of Theorem 2.1, that a necessary and sufficient condition for an M-graph $(D, \{x_e\}_{e \in E})$ to be inevitable is that there is a choice, $\{w_e\}_{e \in E}$ of elements of $(A \cup A^{-1})^*$ for which the $FG(A)$-graph $(D, \{w_e\}_{e \in E})$ commutes.

To show that this condition is sufficient, we use the following.

Lemma 4.1. *Let $\mu : R \twoheadrightarrow M$, $\gamma : R \to G$ be homomorphism, where R and M are finite monoids and G is a finite group. Suppose that $y, z \in M$, $g \in G$, $yzy = y$ and $g \in \gamma(\mu^{-1}(z))$. Then $g^{-1} \in \gamma(\mu^{-1}(y))$.*

Proof. Let $r \in R$ where $\gamma(r) = g$ and $\mu(r) = z$. Let s be arbitrary with $\mu(s) = y$ and put $\gamma(s) = h$. Let n be the exponent of G. Then $\mu(s(rs)^{n-1}) = y(zy)^{n-1} = y$ and $\gamma(s(rs)^{n-1}) = h(gh)^{-1} = g^{-1}$. □

Now, for each $a \in A$, we may arbitrarily choose $r_a \in R$ with $\mu(r_a) = [a]_M$ and put $\gamma(r_a) = [a]_G$. This choice of $[a]_G$ determines a homomorphism from $(A \cup A^{-1})^*$, through $FG(A)$, to G which we denote by $w \mapsto [w]_G$.

Let $R_1 = \{(\mu(r), \gamma(r)) : r \in R\}$. By choice of $[a]_G$, we have $([a]_M, [a]_G) \in R_1$, for each $a \in A$. By the Lemma, if $a \in A$ and $m[a]_M m = m$ then $(m, [a^{-1}]_G) \in R_1$. So, from the definition of S and since R_1 is clearly a submonoid of $M \times G$, we see that for all $w \in (A \cup A^{-1})^*$, if $(x, w) \in S$ then $(x, [w]_G) \in R_1$.

Thus, for each $e \in E$, we have $g_e \in \gamma(\mu^{-1}(x_e))$ where $g_e = [w_e]_G$, and certainly $(D, \{g_e\}_{e \in E})$ commutes, since $(D, \{[w_e]_{FG(A)}\}_{e \in E})$ does.

We may thus concentrate in the remaining sections on showing that this condition is also necessary. We see in Sec. 9 how we may obtain results for an arbitrary monoid from similar results for inverse monoids, and we first establish corresponding results for these. The result for inevitable M-loops for arbitrary monoids M depends on the result of Sec. 8 for inevitable (I, A)-circuit graphs for inverse monoids I.

5. A-Inverse Monoids

We write $\tilde{A}$ to denote the algebra $((A \cup A^{-1})^*, \cdot, (\ \ )^{-1})$, where $((A \cup A^{-1})^*, \cdot)$ remains the free monoid on $A \cup A^{-1}$ and where w^{-1} is obtained from w by reversing the order of the symbols from $A \cup A^{-1}$ and interchanging the symbols a and a^{-1} for $a \in A$. Where appropriate, we regard an inverse semigroup as being also of this type by adjoining the inverse operation, as for example in the following.

Let A be a finite set. We define an *A-inverse monoid* to consist of an inverse monoid I together with a surjective homomorphism $w \mapsto [w]_I$ from $\tilde{A}$ onto I. Between two A-inverse monoids, we consider only the homomorphism $[w]_{I_1} \mapsto [w]_{I_2}$, which we may call the *A-homomorphism*. If this is well-defined, we write $I_1 \twoheadrightarrow I_2$.

In Secs. 6 and 7, we prove the following:

Theorem 5.1. *Let I be a finite A-inverse monoid. Then an I-graph $(D, \{x_e\}_{e \in E})$ is inevitable if and only if there exists a choice for each $e \in E$ of $w_e \in \tilde{A}$ such that $[w_e]_I = x_e$ for each $e \in E$ and such that the $FG(A)$-graph $(D, \{w_e\}_{e \in E})$ commutes.*

Clearly, the condition is sufficient; if $x = [w]_I$ then, from the definition of S in Sec. 2, we have $(x, w) \in S$ and so the sufficiency follows from the argument of Sec. 3.

We prove the converse first in Sec. 6 for circuit graph D, and then in Sec. 7 for general graphs. If I is a finite A-inverse monoid then to show that an I-graph is not inevitable, it is clearly enough to exhibit a finite A-group which "spoils" it, according to the following.

Definition. For a finite A-inverse monoid I, we say that a finite A-group G *spoils* an I-graph $(D, \{x_e\}_{e \in E})$ if there is no choice, for each $e \in E$, of $u_e \in \tilde{A}$ such that, for each e, $[u_e]_I = x_e$ and the G-graph $(D, \{u_e\}_{e \in E})$ commutes.

Clearly, if a finite A-group G spoils an I-graph, then we may take $R = \{([u]_I, [u]_G) : u \in \tilde{A}\} \leq I \times G$ with μ, γ as the projections to show that the I-graph is not inevitable.

Conversely, if R, μ, γ, G show that an I-graph is not inevitable, then by choosing for each $a \in A$ some $[a]_G \in \gamma(\mu^{-1}([a]_I))$, we obtain a homomorphism from $\tilde{A}$ into G and thence an A-group which spoils the I-graph.

Definition. Let G be an A-group. Then G can be viewed as an *A-graph*, that is, a directed graph having edges labelled by elements of A, by taking one edge for each pair $(g, a) \in G \times A$ from g to $g[a]_G$. We use the same terminology as for automata and for $w_1, w_2 \in \tilde{A}$ define $[w_1]_{Q(G)} = [w_2]_{Q(G)}$ if $[w_1]_G = [w_2]_G$ and the two runs of w_1, w_2 in the A-graph G starting from (say) 1 use the same set of edges (in either direction).

The resulting quotient, $Q(G)$, of $\tilde{A}$ is then an A-inverse monoid.

For a finite A-inverse monoid, I, we may take a faithful representation of I by finite partial one-one functions and, extending the actions of elements of A to permutations, obtain an A-group G which is "adequate" for I according to the following:

Definition. We say that an A-group G is *adequate* for an A-inverse monoid I if for all $w \in \tilde{A}$, if $[w]_G = 1$ then $[w]_I$ is idempotent in I.

We have the following connection with the A-inverse monoids, $Q(G)$.

Proposition 5.2. *$Q(G) \twoheadrightarrow I$ if and only if G is adequate for I.*

Proof (Sketch). The condition is clearly necessary. Conversely, if G is adequate for I then we may prove for $u, v \in \tilde{A}$, by induction on the length of v, that if $u \leq v$ in $Q(G)$ then $u \leq v$ in I. □

Comment. The A-inverse monoids $Q(G)$, and the equivalence of Proposition 5.2 were considered independently in [5]. The phrase "an E-unitary cover" is used where we have used "adequate".

6. Circuit I-graphs

In this section, we prove Theorem 5.1 for I-graphs of the form $(D, \{x_e\}_{e \in E})$ in which the undirected graph D forms a circuit.

We note that, by Lemma 4.1, if an I-graph $(D, \{x_e\}_{e \in E})$ is inevitable, then so is the result of reversing the directions of some of the edges of D and replacing the corresponding x_e by x_e^{-1}. Since the first I-graph is also obtained thus from the second, one is inevitable if and only if the other is.

So, from the statement of Theorem 5.1, if the undirected graph D is a circuit, we need only consider the case where the edges are in the same direction round the circuit. labelled with elements $x_1, x_2, \ldots, x_n$, say, of I. We must show that if this I-graph is inevitable then there exists $w_i \in \tilde{A}$ with each $x_i = [w_i]_I$ such that $[w_1 w_2 \ldots w_n]_{FG(A)} = 1$. We proceed by induction on n.

In the case where $n = 1$, the I-graph is a loop labelled by x_1, as previously noted, there is an A-group G for which $[w]_G = 1$ only if $[w]_I$ is idempotent in I. Thus if the loop is inevitable, then x_1 is idempotent in I and so, taking any u with $x_1 = [u]_I$ we also have $x_1 = [uu^{-1}]_I$ and we may take $w_1 = uu^{-1}$.

In the case where $n = 2$, we may begin with the same G. For each pair $u_1, u_2 \in \tilde{A}$ with $x_i = [u_i]_I$, we show that either there is a group automaton of less than $2|G|$ states in which the action of $u_1 u_2$ is not the identity, or that there exist suitable $w_1, w_2 \in \tilde{A}$ for x_1, x_2. If the first option holds for all such u_1, u_2 then the direct product of all the non-isomorphic A-groups of this size shows that the I-graph is not inevitable. Thus, if it is inevitable, the second option must arise, giving the desired conclusion.

The first option may arise as follows. If $[u_1 u_2]_G \neq 1$ then it is already achieved in G, so we may assume that $[u_1 u_2]_G = 1$. Let G_1 and G_2 be two copies of the A-graph given by G except that G_1 contains only the edges used (in either direction) in the run of u_1 from 1 to $[u_1]_G$ and G_2 only those used in the run of u_2 from $[u_1]_G$ to 1. Having first made G_1 and G_2 disjoint, we identify $[u_1]_G$ in G_1 with $[u_1]_G$ in G_2, and whichever other elements then need to be identified to make the actions of the elements of A one-one and single-valued.

Clearly, the effect of this is that those pairs $g^{(1)}$ and $g^{(2)}$ of copies in G_1, G_2 of the same $g \in G$ are identified for which there is an undirected path from $[u_1]_G$ to g in the A-graph G consisting of edges used in both of the mentioned runs of u_1 and u_2. If 1_G is *not* such a g then the run in the composite automaton of $u_1 u_2$ from $1^{(1)}$ terminates at $1^{(2)} \neq 1^{(1)}$ and (on extending the actions of elements of A to permutations) we have the first option.

Otherwise, there is such an undirected path in G from $[u_1]_G$ to 1_G. Let $p \in \tilde{A}$ be such that the run of p from $[u_1]_G$ to 1_G follows such a path. Then we have $u_1 = u_1 u_1^{-1} p^{-1}$ and $u_2 = u_2 u_2^{-1} p$ in $Q(G)$. By choice of G, we have an A-homomorphism from $Q(G)$ to I, so the same equations are true in I. So we may take $w_1 = u_1 u_1^{-1} p^{-1}$ and $w_2 = u_2 u_2^{-1} p$, and we have the second option.

For $n \geq 3$ we proceed similarly. First, let G be chosen such that G spoils every circuit I-graph having $k < n$ edges which is not inevitable. This is possible, by taking direct products, since there are only finitely many sequences $y_1, \ldots, y_k$ from I, where $k < n$.

Now, given $x_1, \ldots, x_n \in I$, we show that for $u_1, \ldots, u_n \in \tilde{A}$ with $x_i = [u_i]_I$, either there is a group automaton having less than $n|G|$ states in which the action of $u_1 u_2 \ldots u_n$ is not the identity or that there exist $w_1, w_2, \ldots, w_n \in \tilde{A}$ for which each $x_i = [w_i]_I$ and $w_1 w_2 \ldots w_n = 1$ in $FG(A)$.

The first option may arise as follows. Possibly $u_1 u_2 \ldots u_n \neq 1$ in G, already. Otherwise we have $u_1 u_2 \ldots u_n = 1$ in G.

Consider the run of $u_1 u_2 \ldots u_n$ from 1 in G. Suppose that the run of u_i in this is from g_{i-1} to g_i, where $g_0 = g_n = 1$. Let $G_1, \ldots, G_n$ be n copies of the A-graph G, with $G_i = \{g^{(i)} : g \in G\}$ except that G_i has only the edges of G used in this run of u_i. We wish to identify $g_i^{(i)}$ with $g_i^{(i+1)}$ for $i = 1, \ldots, (n-1)$ and whatever else is necessary.

Let X_i, for $i = 1, \ldots, (n-1)$ be the set of those $g \in G$ for which there is an undirected path in G from g_i to g of edges used in both of the mentioned runs of u_i and u_{i+1}. If no two consecutive sets in the sequence $X_1, X_2, \ldots, X_{n-1}$ intersect, then we need only identify $g^{(i)}$ with $g^{(i+1)}$ for $g \in X_i$, and so we do not identify $1^{(1)}$ with $1^{(n)}$ and the action of $u_1 u_2 \ldots u_n$ in the composite automaton is not the identity.

If this supposition fails, then for some k, the sets X_k and X_{k+1} intersect. Then, by the definitions of the X_i, we may find p, $q \in \tilde{A}$ with $u_{k+1} = pq$ in G and $u_k = u_k p p^{-1}$, $u_{k+1} = p q u_{k+1}^{-1} u_{k+1}$, $u_{k+2} = q^{-1} q u_{k+2}$ in $Q(G)$ and therefore in I. Now $u_1 u_2 \ldots (u_k p)(q u_{k+2}) \ldots u_n = 1$ in G, so by choice of G the corresponding circuit graph of $(n-1)$ edges is inevitable. So, by induction hypothesis, we have $w_1, \ldots, w_{k-1}, r, s, w_{k+3}, \ldots, w_n \in \tilde{A}$ such that $[u_i]_I = [w_i]_I$ for $i = 1, \ldots, k-1, k+3, \ldots, n$, $[u_k p]_I = [r]_I$, $[q u_{k+2}]_I = [s]_I$ and $w_1 \ldots w_{k-1} r s w_{k+3} \ldots w_n = 1$ in $FG(A)$.

But then, in I, $u_k = u_k p p^{-1} = r p^{-1}$ and $u_{k+2} = q^{-1} q u_{k+2} = q^{-1} s$. So we may take

$w_k = rp^{-1}$, $w_{k+2} = q^{-1}s$ and $w_{k+1} = pqu_{k+1}^{-1}u_{k+1}$ to obtain $x_i = [w_i]_I$, and then, in $FG(A)$,

$$\begin{aligned} w_1 w_2 \dots w_n &= w_1 \dots w_{k-1} rp^{-1} pqu_{k+1}^{-1} u_{k+1} q^{-1} s w_{k+3} \dots w_n \\ &= w_1 \dots w_{k-1} rs w_{k+3} \dots w_n \\ &= 1. \end{aligned}$$

Thus in this case we have the second option, as desired.

7. General I-graphs

We now prove Theorem 5.1 for all I-graphs in a slightly different form. An $\tilde{A}$-tree means an $\tilde{A}$-graph of which the undirected graph forms a tree.

Definition. We say that f is an *association* from an I-graph $\underset{\sim}{D} = (D, \{x_e\}_{e\in E})$ to an $\tilde{A}$-tree $\underset{\sim}{T} = (T, \{w_{\bar{e}}\}_{\bar{e}\in \bar{E}})$, and write $f : \underset{\sim}{D} \to \underset{\sim}{T}$, if f is a function from the vertices of D to those of T such that, for each $e \in E$ from vertex U to vertex V, say, $x_e \le [w_e]_I$ where w_e is the product obtained from the unique arc $\bar{e}_1, \bar{e}_2, \dots, \bar{e}_k$ in T from $f(U)$ to $f(V)$ by letting $w_e = \hat{w}_{\bar{e}_1}\hat{w}_{\bar{e}_2}\dots\hat{w}_{\bar{e}_k}$ where $\hat{w}_{\bar{e}_i}$ is $w_{\bar{e}_i}$ if $\bar{e}_i$ is in the forward direction of the arc and is $w_{\bar{e}_i}^{-1}$ otherwise.

In this case, we say that $\underset{\sim}{T}$ is an $\tilde{A}$-tree associated with $\underset{\sim}{D}$.

Comments. (1) In this case, if $e_1, e_2, \dots, e_k$ form an undirected closed path in $\underset{\sim}{D}$, then the corresponding product $w_{e_1}^{\pm 1} w_{e_2}^{\pm 1} \dots w_{e_k}^{\pm 1} = 1$ in $FG(A)$, since T is a tree. Thus the $FG(A)$-graph $(D, \{w_e\}_{e\in E})$ commutes. In this case, a suitable witnessing choice for Theorem 5.1 is obtained by taking $w'_e = u_e u_e^{-1} w_e$ where $[u_e]_I = x_e$.

(2) For any association $f : \underset{\sim}{D} \to \underset{\sim}{T}$, one can modify $\underset{\sim}{T}$ by removing all the vertices of degree 1 or 2 which are not in the range of f, replacing the two edges incident on such a vertex of degree 2 by a single edge labelled by a corresponding product. This gives an association $f : \underset{\sim}{D} \to \underset{\sim}{T}'$ where $\underset{\sim}{T}'$ is an $\tilde{A}$-tree having, by a simple counting argument, no more than $2m - 2$ vertices, where m is the number of vertices of D.

(3) If desired, we may ensure that an association $f : \underset{\sim}{D} \to \underset{\sim}{T}$ is one-one, by inserting new vertices into $\underset{\sim}{T}$ and new edges labelled with $1 \in \tilde{A}$.

Theorem 7.1. *For each inevitable I-graph there is an associated $\tilde{A}$-tree.*

Proof. We need consider only connected I-graphs, since otherwise each connected component is clearly inevitable and we may join up associated trees for these components arbitrarily. (Alternatively, we could make the analogous definition of an associated *forest*.)

If the I-graph consists of a single circuit which, as remarked, we may take to be a *directed* circuit, then the result follows from Theorem 5.1 for circuit I-graphs, established in Sec. 6, because we may use the tree obtained from the run of the product $w_1 w_2 \dots w_n$ in $FG(A)$.

We prove the result for connected inevitable I-graphs $(D, \{x_e\}_{e\in E})$ in general by induction on $|E|$, simultaneously for all A-inverse monoids I.

Let D_0 result from D by removing an edge e_0 of D, from a vertex U_0 to a vertex U_1. Let $E_0 = E - \{e_0\}$ denote the set of edges of D_0. Then clearly $(D, \{x_e\}_{e \in E})$ is also inevitable.

If e_0 is not part of a circuit in D, then D_0 has two connected components. The induction hypothesis gives associations f_0 and f_1 of these with trees T_0 and T_1 and we may take a disjoint union of T_0 and T_1 with an additional edge from $f_0(U_0)$ to $f_1(U_1)$ labelled with any $w_e \in \tilde{A}$ for which $x_e \leq [w_e]_I$.

So we may assume that e_0 is part of at least one circuit in D, and let $e_0, e_1, e_2, \ldots, e_r$ be such a circuit, having vertices $U_0, U_1, \ldots, U_r$, where, as before, we may assume for simplicity that each e_i is in the direction of the circuit.

First, choose an A-group G_0 which spoils each non-inevitable I-circuit graph having fewer than $2m$ edges, where m is the number of vertices of D. Now choose an A-group $G_1 \twoheadrightarrow G_0$ which spoils every non-inevitable $Q(G_0)$-graph having fewer than $|E|$ edges.

Since $\underset{\sim}{D} = (D, \{x_e\}_{e \in E})$ is assumed to be inevitable, it is not spoilt in G_1, and we deduce from this that there is an associated tree.

Since $\underset{\sim}{D} = (D, \{x_e\}_{e \in E})$ is not spoilt in G_1, there exist $u_e \in \tilde{A}$ with each $x_e = [u_e]_I$ for which the G_1-graph $\underset{\sim}{D}' = (D, \{[u_e]_{G_1}\}_{e \in E})$ commutes. Then the G_1-graph, $\underset{\sim}{D}'_0 = (D_0, \{[x_e]_{G_1}\}_{e \in E_0})$ also commutes, and so, by choice of G_1, the $Q(G_0)$-graph $\underset{\sim}{D}_0 = (D_0, \{[u_e]_{Q(G_0)}\}_{e \in E_0})$ is inevitable. Thus, by the induction hypothesis, (applied to $Q(G_0)$, rather than I) there is an association f from the $Q(G_0)$-graph D_0 to some $\tilde{A}$-tree, say $\underset{\sim}{T}_0 = (T_0, \{v_{\bar{e}}\}_{\bar{e} \in \bar{E}})$. As remarked, $\underset{\sim}{T}_0$ may be chosen to have at most $2m - 2$ vertices, and f to be one-one.

Now let $\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_s$ be the arc in T_0 from $f(U_1)$ to $f(U_0)$, having vertices, say $V_0, V_1, \ldots, V_s$, where $V_0 = f(U_1)$, $V_s = f(U_0)$. For simplicity, we may assume that $\underset{\sim}{T}_0$ is modified so that each edge $\bar{e}_i$ is from V_{i-1} to V_i.

Now $e_1, \ldots, e_r$ is an arc in D_0 having vertices $U_1, U_2, \ldots, U_r, U_0$. Since $f : \underset{\sim}{D}_0 \to \underset{\sim}{T}_0$ is an association, we have $u_{e_1} u_{e_2} \ldots u_{e_r} \leq w$ in $Q(G_0)$ where $w \in \tilde{A}$ is the product $p_1 p_2 \ldots p_r$ and $p_1, p_2, \ldots, p_r$ are in turn the products of the appropriate $v_{\bar{e}}$ or $v_{\bar{e}}^{-1}$ along the arcs in T_0 from $f(U_1)$ to $f(U_2)$, $f(U_2)$ to $f(U_3)$, $\ldots$, $f(U_r)$ to $f(U_0)$. Thus w is the product of the appropriate $v_{\bar{e}}$ or $v_{\bar{e}}^{-1}$ along *some* path in T_0 from $f(U_1)$ to $f(U_0)$ and so, since $\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_s$ is an arc from $f(U_1)$ to $f(U_0)$ in T_0 and T_0 is a tree, we have $w \leq v_{\bar{e}_1} v_{\bar{e}_2} \ldots v_{\bar{e}_s}$ in $FIM(A)$ and thus also in $Q(G_0)$. So $u_{e_1} u_{e_2} \ldots u_{e_r} \leq v_{\bar{e}_1} v_{\bar{e}_2} \ldots v_{\bar{e}_s}$ in $Q(G_0)$ and hence $u_{e_1} u_{e_2} \ldots u_{e_r} = v_{\bar{e}_1} v_{\bar{e}_2} \ldots v_{\bar{e}_s}$ in G_0. But also, since the G_1-graph $\underset{\sim}{D}'$ commutes, and $e_0, e_1, \ldots, e_r$ is a circuit in D, we have $u_{e_0} u_{e_1} \ldots u_{e_r} = 1$ in G_1, and hence in G_0. So $u_{e_0} v_{\bar{e}_1} v_{\bar{e}_2} \ldots v_{\bar{e}_s} = 1$ in G_0.

Now consider the circuit I-graph, $\underset{\sim}{C}$, consisting of the vertices $f(U_1) = V_0, V_1, \ldots, V_s = f(U_0)$ of T_0, the edges $\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_s$ between these, labelled with the corresponding $[v_{\bar{e}_i}]_I$, together with a further edge e'_0 from V_s to V_0 labelled with $[u_{e_0}]_I$. Since $u_{e_0} v_{\bar{e}_1} v_{\bar{e}_2} \ldots v_{\bar{e}_s} = 1$ in G_0, $\underset{\sim}{C}$ is not spoilt in G_0. But $\underset{\sim}{C}$ is a circuit I-graph having $s + 1$ edges, where $s \leq 2m - 2$ by choice of T_0. Hence, by choice of G_0, $\underset{\sim}{C}$ is an inevitable I-graph. From the result of Sec. 6 for circuit I-graphs, it follows that there is a one-one association, h, say, from $\underset{\sim}{C}$ to some $\tilde{A}$-tree $\underset{\sim}{T}' = (T', \{v_{e'}\}_{e' \in E'})$.

Let the vertices and edges of T' be re-named to be distinct from those of T_0, except that each $h(V_i)$ in T' is identified with the corresponding V_i in T_0. Let the tree T be the union of the resulting T_0 and T' except that each of the edges $\bar{e}_1, \ldots, \bar{e}_s$ of T_0 is removed, so that T is connected but circuit-free. Let E^* be the set of edges of T and, for each $e^* \in E^*$ let w_{e^*} be $v_{e'}$ or $v_{\bar{e}}$ according to whether e^* is an edge e' of T' or an edge $\bar{e}$ of T_0, other than $\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_s$. We now show that $f: \underset{\sim}{D} \to \underset{\sim}{T}$ is an association from the I-graph $\underset{\sim}{D}$ to the $\tilde{A}$-tree $\underset{\sim}{T} = (T, \{w_{e^*}\}_{e^* \in E^*})$.

Recall that each $x_e = [u_e]_I$. If $e = e_0$, then the corresponding arc in T is that from $f(U_0) = V_s = h(V_s)$ to $f(U_1) = V_0 = h(V_0)$, which lies entirely within T'. By choice of $\underset{\sim}{C}$ and $\underset{\sim}{T}'$, we then have $[u_e]_I \le [p]_I$, where p is the corresponding product of the $w_{e'}^{\pm 1} = v_{e'}^{\pm 1}$ from $\underset{\sim}{T}'$, as required.

If $e \in E$, $e \ne e_0$, then, by choice of f and T_0, we have $u_e \le p_0$ in $Q(G_0)$, where p_0 is the product of the $v_{\bar{e}}^{\pm 1}$ along the corresponding arc in T_0. In the product, p, of the $v_{e^*}^{\pm 1}$ along the corresponding arc in T, each factor $v_{\bar{e}_i}^{\pm 1}$ of p_0 obtained from some $\bar{e}_i$ for $i = 1, 2, \ldots, s$ is replaced by the product q_i of the $v_{e'}^{\pm 1}$ along the arc in T' from $h(V_{i-1})$ to $h(V_i)$. But, by choice of h and $\underset{\sim}{T}'$, we have $v_{\bar{e}_i} \le q_i$ in I. Thus $p_0 \le p$ in I, and since $u_e \le p_0$ in $Q(G_0)$ and so also in I, we have $u_e \le p$ in I, as required.

In the case where $U_0 = U_1$, so e_0 is a loop, this argument with the appropriate conventions reduces to saying that the loop alone is inevitable, hence x_{e_0} is idempotent. Then we may take $\underset{\sim}{T}$ to be $\underset{\sim}{T}_0$, since the corresponding arc in T for e_0 is a trivial arc with one vertex, for which the product is, of course, taken to be $1 \in \tilde{A}$. $\square$

Theorem 5.1 is therefore now proved for all I-graphs, in view of Comment (1) of this section.

8. (I, A)-graphs

Let I be a finite A-inverse monoid. We define an *(I, A)-graph* to be a structure $\underset{\sim}{D} = (D, \{y_e\}_{e \in E})$ where D is a finite directed graph having E as its set of edges and each y_e is *either* an element of I *or* an element of A (asumed to be treated as a set disjoint from I).

We extend the previous definition of an inevitable I-graph and define an (I, A)-graph $\underset{\sim}{D} = (D, \{y_e\}_{e \in E})$ to be inevitable if for every finite A-group G there exist $u_e \in \tilde{A}$ such that for each $e \in E$, if $y_e \in I$ then $[u_e]_I = y_e$, while if $y_e \in A$ then $u_e = y_e$, and such that the G-graph $(D, \{[u_e]_G\})$ commutes.

Proposition 8.1. *An (I, A)-graph $(D, \{y_e\}_{e \in E})$ is inevitable if and only if there exists a choice, for each $e \in E$ of $w_e \in \tilde{A}$ such that if $y_e \in I$ then $[w_e]_I = y_e$, while if $y_e \in A$ then $w_e = y_e$ and such that the $FG(A)$-graph $(D, \{w_e\}_{e \in E})$ commutes.*

Proof. As usual, the condition is clearly sufficient. We may obtain the converse from Proposition 5.1.

Suppose that the (I, A)-graph $(D, \{y_e\}_{e \in E})$ is inevitable. Let G be adequate for I and such that, for each $a \in A$, $a \ne 1$ in G and let $I' = Q(G)$. Then, since T' is finite and $I' \twoheadrightarrow I$, there must be a choice of $y'_e \in I' \cup A$ such that $y'_e \mapsto y_e$ if $y_e \in I$ and $y'_e = y_e$ if $y_e \in A$, and the (I', A)-graph $(D, \{y'_e\}_{e \in E})$ is inevitable. If this were not so, then the direct product of the finitely many groups needed to spoil $(D, \{y'_e\}_{e \in E})$ for each of the finitely many such choices of the $\{y'_e\}$ would also spoil $(D, \{y_e\})$.

But then, *a fortiori*, the I'-graph $(D, \{x_e\}_{e\in E})$ is inevitable, where $x_e = y'_e$ if $y'_e \in I'$ and $x_e = [a]_{I'}$ if $y'_e = y_e = a \in A$. So, by Proposition 5.1, there exist $w_e \in \tilde{A}$ such that each $[w_e]_{I'} = x_e$ and the $FG(A)$-graph $(D, \{w_e\}_{e\in E})$ commutes.

Then, for $y_e \in I$, we have $[w_e]_{I'} = y'_e$ and since $y'_e \mapsto y_e$, $[w_e]_I = y_e$. For $y_e = a \in A$ we have $y'_e = a$ and so $[w_e]_{I'} = [a]_{I'}$ but since $I' = Q(G)$ this gives that $w_e = a(a^{-1}a)^n$ for some $n = 0, 1, 2, \ldots$ so $w_e = a$ in $FG(A)$. So putting $w'_e = w_e$ if $y_e \in I$ and $w'_e = y_e$ if $y_e \in A$ gives $[w'_e]_I = y_e$ if $y_e \in I$, $w'_e = y_e$ if $y_e \in A$ and the $FG(A)$-graph $(D, \{w'_e\}_{e\in E})$ also commutes, since in either case, $w'_e = w_e$ in $FG(A)$. □

9. A-monoids

Analogously, we define an *A-monoid* to be a monoid M together with a homomorphism from A^* onto M, denoted by $w \mapsto [w]_M$.

We show in Proposition 9.2 how to construct a related A-inverse monoid from the regular $\mathscr{R}$-classes of M. Our construction and definitions for this are virtually reformulations of those of [8]. First we establish a significant Lemma.

Lemma 9.1. *Suppose that $w \in A^*$, $y \in M$ and $y[w]_M y = y$. Then there exists $w' \in \tilde{A}$ for which ySw' and $w' \le w^{-1}$ in $FIM(A)$.*

Proof. In this argument, $[w]$ abbreviates $[w]_M$ for $w \in A^*$ and, for $w_1, w_2 \in \tilde{A}$, $w_1 \le w_2$ abbreviates $w_1 \le w_2$ in $FIM(A)$.

First suppose that $w \ne 1$. Then we show, by induction on the length, $|w|$, of w that in fact ySw^{-1}. For $|w| = 1$ we have $w = a$ for some $a \in A$. Then $y[w]y$ gives $y[a]y$ and so ySa^{-1} from the definition of S.

For $|w| > 1$, let $w = w_1 a$ where $a \in A$ and $1 \le |w_1| < |w|$. Then $y[w]y = y$ gives $y[w_1][a]y = y$ and so $(y[w_1])[a](y[w_1]) = y[w_1]$ and also $([a]y)[w_1]([a]y) = [ay]$.

These give $y[w_1]Sa^{-1}$ by definition of S and also $[a]ySw_1^{-1}$ by the induction hypothesis. So $(y[w_1])([a]y)Sa^{-1}w_1^{-1}$, that is, $y[w]ySw^{-1}$, completing the induction for $w \ne 1$.

Now suppose that $w = 1$, so the supposition is that $y^2 = y$. If $y = 1$ then certainly ySw^{-1} since $1S1$ and $1^{-1} = 1$. So now suppose that $y^2 = y$ and $y \ne 1$. Let $w_1 \in A^*$ with $[w_1] = y$. Then $w_1 \ne 1$ (since $y \ne 1$) and $y^2 = y$ gives $y^3 = y$ and thus $y[w_1]y = y$, so by the previous case we have ySw_1^{-1}. But also, since $w_1 \in A^*$ and $y = [w_1]$, we have ySw_1. So $y = y^2Sw_1w_1^{-1} \le 1 = w^{-1}$, as required. □

Proposition 9.2. *For each finite A-monoid M there is a finite A-inverse monoid I such that, if $u \in A^*$, $v \in \tilde{A}$, $[u]_M$ is regular in M and $[u]_I \le [v]_I$, then there exists $w \in \tilde{A}$ for which $[u]_M Sw$ and $w \le v$ in $FIM(A)$.*

Proof. We consider any regular $\mathscr{R}$-class X of M and obtain a suitable I having this property for all $[u]_M \in X$. The result follows on taking the direct product of all such I.

First consider X as an A-simple A-graph with $x \xrightarrow{a} y$ iff $y = x[a]_M$ for $x, y \in X$. Then X can be regarded as a (non-deterministic) $\tilde{A}$-graph in the sense that each $a^{-1} \in A$ allows the backward transition from y to any x for which $x \xrightarrow{a} y$.

Now define an equivalence relation $\approx$ on X by $x \approx y$ if there exists $w \in \tilde{A}$ with $w = 1$ in $FG(A)$ and there is a run of w in the $\tilde{A}$-graph X from x to y. This is clearly an equivalence relation, since the set $D = \{w \in \tilde{A} : w = 1 \text{ in } FG(A)\}$ has $1 \in D$; $w \in D \Rightarrow w^{-1} \in D$ and $w_1, w_2 \in D \Rightarrow w_1 w_2 \in D$.

Denote the equivalence class of $x \in X$ under $\approx$ by x^0. Then we may define an A-simple A-graph with vertices $X^0 = X/\approx$ by $x^0 \xrightarrow{a} y^0$ if there exist $x_1 \approx x$ and $y_1 \approx y$ for which $x_1 \xrightarrow{a} y_1$ in the A-graph X. Now, since the set D, above, also has $w \in D$, $a \in A \Rightarrow a^{-1}wa$, $awa^{-1} \in D$, the actions of $a \in A$ on X^0 are quickly seen to be partial one-one functions, so X^0 can be considered as a (deterministic) $\tilde{A}$-graph.

Now let us define the congruence $\equiv$ on $\tilde{A}$ by $w_1 \equiv w_2$ if for all $x^0 \in X^0$ *either* there are no runs of w_1 or of w_2 in X^0 from x^0, *or* the (only possible) runs of w_1 and w_2 from x^0 both exist, lead to the same $y^0 \in X^0$ and use the same edges of X^0, in one or other direction. One easily verifies that $\equiv$ respects $\cdot$ and $(\ )^{-1}$ in $\tilde{A}$ and, writing I for $\tilde{A}/\equiv$, that I is finite. Clearly, for $w \in \tilde{A}$, we also have $ww^{-1}w \equiv w$.

To show that the $\tilde{A}$-monoid I is in fact an A-inverse monoid, we need further to observe that, for w_1, $w_2 \in \tilde{A}$ we have $w_1 w_1^{-1} w_2 w_2^{-1} \equiv w_2 w_2^{-1} w_1 w_1^{-1}$, which follows quickly on observing that the actions of $w_i w_i^{-1}$ on X^0 can only be partial identity functions. Thus, also $[w]_I$ is idempotent in I iff the action of w on X^0 is a partial identity function.

Now suppose that $u \in A^*$, $v \in \tilde{A}$, $[u]_M \in X$ and that $[u]_I \leq [v]_I$. Then $[u]_I = [vr]_I$ for some idempotent $[r]_I$ of I. So, for each $x^0 \in X^0$, if the run of u in X^0 exists and finishes at y^0, then also the run of v exists, from x^0 to y^0, and this run uses only those edges used in this run of u.

In particular, we may choose x^0 to be e^0 where $e \in X$ is such that $e[u]_M = [u]_M$, in which case the run of u in the A-graph X from e to $[u]_M$ certainly exists and therefore so does the run of u in X^0 from e^0 to $[u]_M^0$. By the previous remark concerning the run of $v \in \tilde{A}$ from e^0 to $[u]_M^0$ in the case where $[u]_I \leq [v]_I$, it is sufficient for us to show the following.

(P) Let X be a regular $\mathscr{R}$-class of M, $u \in A^*$, let $e \in X$ be an idempotent of M such that $e[u]_M = [u]_M$ and let $v \in \tilde{A}$ be such that the run of v in X^0 from e^0 exists, finishes at $[u]_M^0$ and uses, in either direction, only the edges used by the run of u in X^0 from e^0 to $[u]_M^0$. Then there exists $w \in \tilde{A}$ with $[u]_M S w$ and $w \leq v$ in $FIM(A)$.

This follows easily by induction on the length of v once we make the following observations. Again, from now on, $[w]$ means $[w]_M$ and $w_1 \leq w_2$ means $w_1 \leq w_2$ in $FIM(A)$.

(1) If $x \xrightarrow{a} y$ in X then there exists $z \in M$ with $yz = x$ and $z[a]z = z$. This follows from the fact that $x \mathscr{R} y$ by choosing $t \in M$ with $yt = x$, choosing n such that $(t[a])^n$ is idempotent in M and putting $z = (t[a])^{2n-1}t$.

Since M is an A-monoid, we may equally say that there exists $p \in A^*$ with $y[p] = x$ and $[pap] = [p]$.

(2) If $x \approx y$ then there exist $w^+ \in A^*$ and $w \in \tilde{A}$ for which $x[w^+] = y$, $[w^+] S w$ and $w = 1$ in $FG(A)$, that is, $w \leq 1$. This follows from (1) since, by definition of $\approx$, there exist $w \in \tilde{A}$ and a run in X of w from x to y. But then replacing each occurrence of any a^{-1} in w by a corresponding $p \in A^*$ gives w^+ as desired.

Now we prove (P), by induction on $|v|$.

Case 1. Here $v = 1$. Then the trivial run of v from e^0 finishes at e^0, so $[u]^0 = e^0$, that is, $[u] \approx e$. So, by observation (2), there exist $w^+ \in A^*$ and $w_1 \in \tilde{A}$ with $[u] = e[w^+]$, $[w^+]Sw_1$ and $w_1 = 1$ in $FG(A)$, so $w_1 \leq 1$. But also $e1e = e$ and so, by Lemma 9.1, we have eSw' for some $w' \in \tilde{A}$ with $w' \leq 1^{-1} = 1$. So $[u] = e[w^+]Sw'w_1 \leq 1 = v$. Thus $w = w'w_1$ is as required.

Case 2. Here v is of the form $v_1 a$ where $a \in A$. By the assumptions of (P), the last edge used in the run of v from e^0 in X^0 is used in the run of u from e^0 and finishes with $(e[u])^0 = [u]^0$. So we have $u_1, u_2 \in A^*$ with $u = u_1 a u_2$, a run of v_1 in X^0 from e^0 to $[u_1]^0$ and $[u]^0 = [u_1 a]^0$, i.e., $[u] \approx [u_1 a]$.

By observation (1), there exists $p \in A^*$ with $[u_1 ap] = [u_1]$, and by observation (2) since $[u] \approx [u_1 a]$, we have $r^+, s^+ \in A^*$, $r, s \in \tilde{A}$ with $[us^+] = [u_1 a]$, $[u_1 ar^+] = [u]$, $r^+ Sr \leq 1$ and $s^+ Ss \leq 1$. Now $us^+ p$ and v_1 satisfy the assumptions of (P) and $|v_1| < |v|$, so by the induction hypothesis there exists $w_1 \in \tilde{A}$ with $[us^+ p]Sw_1 \leq v_1$. Then $[u] = [us^+ par^+] = [us^+ p][a][r^+]Sw_1 ar \leq v_1 a1 = v$, so $w = w_1 ar$ is as required.

Case 3. Here v is of the form $v_1 a^{-1}$ where $a \in A$. Now the assumptions of (P) give $u = u_1 a u_2$ where v_1 runs in X^0 from e^0 to $[u_1 a]^0$ and $[u]^0 = [u_1]^0$, i.e., $[u] \approx [u_1]$. By observations (1) and (2), we have $p \in A^*$ with $[pap] = [p]$ and $u_1 ap = u_1$ and now $q^+, t^+ \in A^*$, $q, t \in \tilde{A}$ with $[u_1 q^+] = [u]$, $[ut^+] = [u_1]$, $q^+ Sq \leq 1$ and $t^+ St \leq 1$.

Now $ut^+ a$ and v_1 satisfy the assumptions of (P) and so by the induction hypothesis there exists $w_1 \in \tilde{A}$ with $[ut^+ a]Sw_1 \leq v_1$. Then $[u] = [ut^+ apq^+]Sw_1 a^{-1} q \leq v_1 a^{-1} 1 = v$, so $w = w_1 a^{-1} q$ is as required. □

10. (M, A)-graphs

We should note that there is no confusion whether a finite A-group is considered as an A-monoid or as an A-inverse monoid. As in Sec. 5, if M is a finite A-monoid then an M-graph $(D, \{x_e\}_{e \in E})$ is inevitable if and only if, for every finite A-group G there exists $\{u_e\}_{e \in E}$ where each $u_e \in A^*$ such that $[u_e]_M = x_e$ for each $e \in E$ and such that the G-graph $(D, \{u_e\}_{e \in E})$ commutes.

We generalize this notion, in the style of Sec. 8. We define an (M, A)-graph to be a structure $(D, \{y_e\}_{e \in E})$ for which D is a directed graph having E as its set of edges and for each $e \in E$, *either* $y_e \in M$ *or* $y_e \in A$ (treated as disjoint from M).

Let us define such an (M, A)-graph to be inevitable if, for every finite A-group G there exist $\{u_e\}_{e \in E}$ where each $u_e \in A^*$ such that $[u_e]_M = y_e$ if $y_e \in M$, $u_e = y_e$ if $y_e \in A$ and such that the G-graph $(D, \{u_e\}_{e \in E})$ commutes.

We prove, in Proposition 10.2, the appropriate analogue of Proposition 8.1 but involving the relation S, which clearly has Theorem 2.1 as a special case, concluding our work. We obtain Proposition 10.2 by Ramsey's Theorem from Proposition 10.1 for *regular* (M, A)-graphs, that is, those in which the $y_e \in M$ are all regular elements of M.

Proposition 10.1. *Let M be a finite A-monoid and let $(D, \{y_e\}_{e \in E})$ be an inevitable regular (M, A)-graph. Then there exists $w_e \in \tilde{A}$ for which $y_e S w_e$ if $y_e \in M$, $y_e = w_e$ if $y_e \in A$ and such that the $FG(A)$-graph $(D, \{w_e\}_{e \in E})$ commutes.*

Proof. Let I be an A-inverse monoid chosen for M as in Proposition 9.2. Let G be a finite A-group in which every (I, A)-graph $(D, \{z_e\}_{e\in E})$, for this particular D, is either inevitable or is spoilt in G. This is possible, by taking a finite direct product of A-groups, since there are only finitely many such (I, A)-graphs.

Since the (M, A)-graph $(D, \{y_e\}_{e\in E})$ is assumed to be inevitable, there exist $\{u_e\}_{e\in E}$ such that each $u_e \in A^*$, $[u_e]_M = y_e$ if $y_e \in M$, $u_e = y_e$ if $y_e \in A$ and such that the G-graph $(D, \{u_e\}_{e\in E})$ commutes. This means that the (I, A)-graph $(D, \{z_e\}_{e\in E})$ is not spoilt in G, where $z_e = [u_e]_I$ if $y_e \in M$ and $z_e = u_e = y_e$ if $y_e \in A$. Thus, by choice of G, this (I, A)-graph is inevitable.

Now, by Proposition 8.1 applied to $(D, \{z_e\}_{e\in E})$, there exist $\{v_e\}_{e\in E}$ from $\tilde{A}$ such that $[v_e]_I = z_e$ if $y_e \in M$, $v_e = z_e = y_e$ if $y_e \in A$ and such that the $FG(A)$-graph $(D, \{v_e\}_{e\in E})$ commutes. But then, if $y_e \in M$ we have $[v_e]_I = z_e = [u_e]_I$ and $[u_e]_M$ is regular, so by Proposition 9.2, there exists $w_e \in \tilde{A}$ for which $u_e S w_e$ and $w_e \leq v_e$ in $FIM(A)$, and thus $w_e = v_e$ in $FG(A)$.

So if we also define $w_e = v_e = y_e$ when $y_e \in A$, then the $FG(A)$-graph $(D, \{w_e\}_{e\in E})$ still commutes and the $\{w_e\}_{e\in E}$ are as required. □

We proceed from regular (M, A)-graphs to arbitrary ones using Ramsey's Theorem.

Proposition 10.2. *Let M be a finite A-monoid. Then an (M, A)-graph $(D, \{y_e\}_{e\in E})$ is inevitable if and only if there exist $w_e \in \tilde{A}$ for which $y_e S w_e$ if $y_e \in M$, $y_e = w_e$ if $y_e \in A$ and the $FG(A)$-graph $(D, \{w_e\}_{e\in E})$ commutes.*

Proof. As in Sec. 4, the condition is clearly necessary. For the converse, we use the following consequence of Ramsey's Theorem that, since M is finite, there exists a K such that, for each $u \in A^*$, u has a factorization $u_1 u_2 \ldots u_k$ in A^*, where $k \leq K$ such that, for each i, either $u_i = a \in A$ or $[u_i]_M$ is regular. Thus, there are finitely many corresponding factorizations $y = y_1 y_2 \ldots y_k$ of each $y \in M$, obtained from all the $u \in A^*$ with $[u]_M = y$. For each $e \in E$ with $y_e \in M$ and each choice of one of these factorizations $y_e = y_1 y_2 \ldots y_k$ and obtain a *regular* (M, A)-graph by replacing each such $e \in E$ by a new arc $e'_1, e'_2, \ldots, e'_k$ and letting $y_{e'_i} = y_i$ if y_i is regular, and otherwise letting $y_{e'_i} = a$ for some $a \in A$ with $y_i = [a]_M$.

Suppose now that $\underset{\sim}{D}$ is inevitable. Then, of the finitely many possible regular (M, A)-graphs $\underset{\sim}{D}_j$ obtained from $\underset{\sim}{D}$ in this way, we claim that at least one is inevitable. Assuming otherwise, for a contradiction, for each j there is a finite A-group G_j which spoils $\underset{\sim}{D}_j$. Let G be the A-group obtained from the direct product of these G_j. Then, since $\underset{\sim}{D}$ is inevitable, there exist $u_e \in A^*$ with $[u_e]_M = y_e$ if $y_e \in M$, $u_e = y_e$ if $y_e \in A$ such that the G-graph $(D, \{u_e\}_{e\in E})$ commutes. Then, taking the corresponding short factorizations of those u_e for which $y_e \in M$ gives a $\underset{\sim}{D}_j$ which is not spoilt by G, and hence not by G_j, contradicting the choice of G_j.

So one of these $\underset{\sim}{D}_j = (D', \{y'_{e'}\}_{e'\in E})$ is an inevitable regular (M, A)-graph. Then by Proposition 10.1, there exist $w_{e'} \in \tilde{A}$ for which $y_{e'} S w_{e'}$ if $y_{e'} \in M$, $w_{e'} = y_{e'}$ if $y_{e'} \in A$ and the $FG(A)$-graph $(D', \{w_{e'}\}_{e'\in E})$ commutes.

From this choice of $w_{e'}$ for $e' \in E'$ we obtain a suitable choice of w_e for $e \in E$. If $y_e \in A$, then e is unchanged on forming D_j, that is, $e' = e$, and we let $w_e = w_{e'} = y_e$. If $y_e \in M$, then e is replaced in D_j with the new arc $e'_1, e'_2, \ldots, e'_k$ and we take $w_e = w_{e'_1} w_{e'_2} \ldots w_{e'_k}$.

Then, since $y_e = y_1 y_2 \dots y_k$ and by choice of the $y_{e_i'}$ and of the $w_{e_i'}$, we have $y_e S w_e$, and the $FG(A)$-graph $(D, \{w_e\}_{e \in E})$ commutes because the $FG(A)$-graph $(D', \{x_{e'}\}_{e' \in E'})$ does. □

As previously noted, Theorem 2.1 is a special case of Proposition 10.2, so our Principal Theorem is now proved.

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