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The Similarity Reduction of Matrices over a Skew Field

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1. Introduction

Two $n \times n$ matrices A and B over a ring R are called *similar* if there is an invertible matrix P over R such that

$$P^{-1}AP=B.$$

The problem of similarity reduction consists in finding a particularly simple representative for each similarity class. When R is a commutative field, the solution is well known: it is the rational canonical form; in particular, over an algebraically closed field we have the Jordan normal form.

Our object in this note is to consider the corresponding problem over skew fields. Here it is necessary to single out the centre, or at least a central subfield k, and to distinguish the "algebraic" case, where the matrix satisfies an equation over k. In this case the reduction proceeds very much as in the commutative theory, and we show how to obtain the rational canonical form for such matrices in § 3. We also find that any matrix is similar to a diagonal sum of an algebraic part and a part containing no algebraic components (the "transcendental" component).

In the commutative case the most important tool is the notion of eigenvalue. This is not so in general, because we no longer have the Cayley-Hamilton theorem (as evidenced by the transcendental component), and the notion of algebraic closure for a skew field is less well developed than in the commutative case. Nevertheless, the eigenvalues of a matrix – or rather, their conjugacy classes – form an important similarity invariant, and in § 2 we shall see that they have properties entirely analogous to their commutative counterpart. The chief difference is that now there are left and right eigenvalues, which may be different in some cases, though over an existentially complete (skew) field (a particular notion of algebraic closure) they are the same. But this is a deeper result, which only appears at the end.

In § 4 we investigate the transcendental component of a matrix. Somewhat unexpectedly, the normal form for such a matrix is simpler even than in the commutative case. We show that two transcendental matrices are similar, over a suitable extension field, if and only if they have the same order. In particular, every transcendental matrix is similar to a scalar matrix (not merely diagonal), with a preassigned scalar.

The three sections of this paper are at rather different levels. § 2 is completely elementary, and uses almost only undergraduate algebra (though to my knowledge the skew field case has not been written down before). § 3 requires some results on bounded and invariant elements in principal ideal domains which goes back to the 1930's. Much of it is implicit in Ch. 3 of [5]; in the form needed here it is developed in [3]. Finally § 4 makes use of two recent results: the author's embedding theorem for free products of skew fields [2], and Bergman's theorem on coproducts of hereditary rings [1]. However, no acquaintance with these papers is necessary beyond the statements of the theorems, and these are given in § 4.

I should like to thank G. M. Bergman for the very elegant argument leading to the proof of the main theorem, Th. 4.2.

2. The Spectrum of a Matrix

Throughout, all rings have a unit-element which is preserved by homomorphisms and inherited by subrings; further, all modules are unital. By the term "field" we understand a not necessarily commutative division ring; occasionally the prefix "skew" is used for emphasis. If k is any commutative ring, a *k-algebra* is essentially a ring R with a homomorphism of k into the centre of R. When k is a field, a non-zero k-algebra is just a ring which has k as a subfield of the centre; in that case k is also called a *central subfield* of R. The ring of all $n \times n$ matrices over R is denoted by $\mathfrak{M}_n(R)$ or also R_n. The set of non-zero elements of R is denoted by R^*, and $\mathbf{GL}_n(R)$ is the group of invertible elements in the ring R_n.

Let n be a positive integer, K a field and let $A \in K_n$. An element $\alpha \in K$ is called a *right eigenvalue* of A if there is a non-zero column vector u over K, called an *eigenvector* for α, such that

$$Au = u\alpha.$$

The set of all right eigenvalues of A is called the *right spectrum* of A. Similarly, a *left eigenvalue* of A is an element $\beta \in K$ such that $vA = \beta v$ for some non-zero row v over K, itself called *eigenvector* for β, and the set of all such β is the *left spectrum* of A. By the *spectrum* of A, spec A, we understand the union of the left and right spectra.

The importance of eigenvalues is that they are invariant under similarity; we have

Proposition 2.1. *The (left, right) spectrum of a matrix A over a field K consists of complete conjugacy classes of K, and is a similarity invariant of A.*

Proof. Let α be a right eigenvalue of A, say $Au = u\alpha$, then for any $c \in K^*$ we have $Auc = u\alpha c = uc.c^{-1}\alpha c$, hence $c^{-1}\alpha c$ is again a right eigenvalue of A.

Secondly, let $P \in \mathbf{GL}_n(K)$, then $P^{-1}AP.P^{-1}u = P^{-1}Au = P^{-1}u.\alpha$, so α is also a right eigenvalue of $P^{-1}AP$. This proves the assertion for the right spectrum. The proof for the left spectrum is similar and by combining the results we get the proof for spec A.

The next result generalizes the well known fact that eigenvectors belonging to distinct eigenvalues (in the commutative case) are linearly independent; it also establishes a connexion between left and right eigenvalues.

Proposition 2.2. *The eigenvectors belonging to inconjugate right eigenvalues of a matrix A are linearly independent. If α is a right and β a left eigenvalue of A and α, β are not conjugate, then the eigenvectors belonging to them are orthogonal, i.e. if u is a column belonging to α and v is a row belonging to β, then $vu = 0$.*

Proof. Let $\alpha_1, \ldots, \alpha_r$ be right eigenvalues and $u_1, \ldots, u_r$ corresponding eigenvectors, and assume that the u's are linearly dependent. By taking a minimal linearly dependent set, we may assume that

$$u_1 = \sum_2^r u_i \lambda_i \qquad (\lambda_i \in K).$$

By the minimality, $\lambda_i \neq 0$, and $r > 1$, because $u_1 \neq 0$ by definition. Now $Au_1 = \sum Au_i\lambda_i = \sum u_i\alpha_i\lambda_i$, and $Au_1 = u_1\alpha_1 = \sum u_i\lambda_i\alpha_1$. By the minimality of r, $u_2, \ldots, u_r$ are linearly independent, hence $\alpha_i\lambda_i = \lambda_i\alpha_1$, i.e. $\alpha_i = \lambda_i\alpha_1\lambda_i^{-1}$, and so $\alpha_1, \ldots, \alpha_r$ are all conjugate.

Next let α be a right eigenvalue with eigenvector u and β a left eigenvalue with eigenvector v, then $Au = u\alpha$, $vA = \beta v$, hence $vAu = vu.\alpha = \beta.vu$, so if $vu \neq 0$, α and β are conjugate. This completes the proof.

This proposition shows in particular that spec A cannot consist of more than n conjugacy classes; this is the analogue of a theorem on equations by Gordon and Motzkin (cf. [3], Th. 8.5.1). But the impact of this result is lessened by the fact that K may well have only one conjugacy class outside its centre (cf. [2]). We can also write down sufficient conditions for reducibility to diagonal form, as in the commutative case; first a lemma which is of independent interest:

Lemma 2.3. *Let R and S be k-algebras and let M be an (R, S)-bimodule. Given $a \in R$, $b \in S$, suppose there is a polynomial f over k such that $f(a)$ is a unit, while $f(b) = 0$. Then for any $m \in M$, the equation*

$$ax - xb = m \tag{1}$$

has a unique solution $x \in M$.

Proof. In the endomorphism ring of M, as k-space, (1) may be written

$$x(L-R)=m, \tag{2}$$

where $L\colon x\mapsto ax$, $R\colon x\mapsto xb$. We note that $LR=RL$, and by hypothesis, $f(L)$ is a unit, while $f(R)=0$. Define the polynomial $\varphi(s,t)$ (in commuting indeterminates) by $\varphi(s,t)=[f(s)-f(t)](s-t)^{-1}$, then for any x satisfying (2) we have

$$m\varphi(L,R)=x(L-R)\varphi(L,R)=xf(L),$$

and this has the unique solution $x=m\varphi(L,R)f(L)^{-1}$. Inserting this value in (2), we obtain

$$m\varphi(L,R)f(L)^{-1}(L-R)=m\big(f(L)-f(R)\big)f(L)^{-1}=m,$$

and the proof is complete.

Theorem 2.4. *Let K be a field and $A\in K_n$. Then* spec A *cannot contain more than n conjugacy classes, and when it consists of exactly n classes, all except at most one algebraic over the centre of K, then A is similar to a diagonal matrix.*

Proof. By Prop. 2.1, spec A consists of complete conjugacy classes. Let the right spectrum consist of r classes and let s be the number of conjugacy classes in the left spectrum which do not occur in the right spectrum. Then the space spanned by the columns corresponding to the right eigenvalues is at least r-dimensional, and the space of rows orthogonal to this is at least s-dimensional, by Prop. 2.2. Hence $r+s\leqq n$; but $r+s$ is just the number of conjugacy classes in spec A.

Suppose now that $r+s=n$; let $\alpha_1,\dots,\alpha_r$ be inconjugate right eigenvalues and $u_1,\dots,u_r$ the corresponding eigenvectors, while $\beta_1,\dots,\beta_s$ are the left eigenvalues not conjugate among themselves or to the α's, with corresponding eigenvectors $v_1,\dots,v_s$. By Prop. 2.2, the u's are right linearly independent, the v's are left linearly independent, and $v_ju_i=0$ ($i=1,\dots,r$, $j=1,\dots,s$). Let us write U_1 for the $n\times r$ matrix consisting of the columns $u_1,\dots,u_r$ and V_2 for the $s\times n$ matrix consisting of the rows $v_1,\dots,v_s$. Since the columns of U_1 are linearly independent, we can find an $r\times n$ matrix V_1 such that $V_1U_1=I$, and similarly there is an $n\times s$ matrix U_2 such that $V_2U_2=I$. Put $U=(U_1\,U_2)$, $V=\begin{pmatrix}V_1\\V_2\end{pmatrix}$, then

$$VU=\begin{pmatrix}V_1U_1 & V_1U_2\\ V_2U_1 & V_2U_2\end{pmatrix}=\begin{pmatrix}I & W\\ 0 & I\end{pmatrix}.$$

The matrix on the right is invertible, hence $U(VU)^{-1}=(U_1\ U_2-U_1 W)=V^{-1}$ (recall that in a matrix ring over a field, every one-sided inverse is two-sided, i.e. fields are *weakly finite* in the terminology of [3]). Thus we have

$$AV^{-1}=A(U_1\quad U_2-U_1 W)=(u_1\alpha_1,\dots,u_r\alpha_r,A(U_2-U_1 W)),$$

$$VA=\begin{pmatrix} V_1 A\\ \beta_1 v_1\\ \vdots\\ \beta_s v_s\end{pmatrix},$$

hence $VAV^{-1}=\begin{pmatrix}\alpha & T\\ 0 & \beta\end{pmatrix}$, where $\alpha=\operatorname{diag}(\alpha_1,\dots,\alpha_r)$, $\beta=\operatorname{diag}(\beta_1,\dots,\beta_s)$ and T is an $r\times s$ matrix. Now all the eigenvalues are inconjugate and all but at most one are algebraic over the centre, hence their minimal equations are distinct ([3], Prop. 8.5.2, p. 302). If only right or only left eigenvalues occur, we have diagonal form; otherwise let $\beta_1,\dots,\beta_s$ be algebraic, say. Taking f to be the product of their minimal polynomials, we have $f(\beta)=0$, while $f(\alpha)$ is a unit. By the lemma we can find an $r\times s$ matrix X over K such that $\alpha X-X\beta=T$, and transforming our matrix by $\begin{pmatrix}I & X\\ 0 & I\end{pmatrix}$, we reach diagonal form.

The restriction on the eigenvalues, that there is to be only one transcendental conjugacy class, is not as severe as appears at first sight, because as pointed out earlier, in many cases there is only one such class.

In general there will be no other connexion between the left and right eigenvalues of a matrix; in particular, a matrix may well have an element α as a right but not left eigenvalue. E.g. let B be a 2×2 matrix with no left eigenvalues; such a matrix is obtained by adjoining 4 non-commuting indeterminates $b_{ij}\,(i,j=1,2)$ to a commutative field k and forming $B=(b_{ij})$ over the universal field of fractions. Let c_1, c_2, α be further indeterminates over k, then the matrix

$$A=\begin{pmatrix}\alpha & c_1 & c_2\\ 0 & & \\ 0 & \multicolumn{2}{c}{B} \end{pmatrix}$$

has α as right eigenvalue, but has no left eigenvalue within the universal field of fractions, as is easily verified (because B has no left eigenvalues).

Besides the left and right eigenvalues considered here there is a third kind. Let us call $\alpha\in K$ a *singular eigenvalue* of $A\in K_n$ if the matrix $A-\alpha$ is singular. The singular eigenvalues are not related in any obvious way to the left and right eigenvalues, except that central eigenvalues of all three kinds coincide:

Proposition 2.5. *Let K be a field with central subfield k. If $A\in K_n$ and $\alpha\in k$, then the following three assertions are equivalent:*

(a) *α is a right eigenvalue of A,*

(b) *α is a left eigenvalue of A,*

(c) *α is a singular eigenvalue of A.*

Proof. Since α centralizes K, it is a right eigenvalue if and only if the equation $Au-\alpha u=0$ has a non-zero solution vector u, and this is so precisely when $A-\alpha$ is singular. Thus (a)$\Leftrightarrow$(c), and by symmetry, (b)$\Leftrightarrow$(c).

The singular eigenvalues do not share such properties as invariance under similarity, in fact they arise in a rather different context (the study of equations) and will be discussed elsewhere.

3. The Rational Canonical Form

The importance of the similarity reduction arises from the way matrices are used to describe linear transformations in vector spaces. Given a field K, let V be a right K-space with basis $e_1, \dots, e_n$, then an endomorphism θ of V is completely determined by its effect on a basis: if

$$\theta e_j=\sum e_i a_{ij}, \tag{1}$$

then the correspondence $\theta\mapsto A$, where $A=(a_{ij})$, defines an endomorphism between $\mathrm{End}_K(V)$ and K_n. Moreover, the matrices of the endomorphism θ in different bases just constitute the matrices similar to A. We note the significance of the eigenvalues in this interpretation, omitting the (easy) proofs. The right eigenvalues of A correspond to scalars α for which a non-zero vector $u\in V$ exists such that $\theta u=u\alpha$. To interpret the left eigenvalues we need the dual space $V^*=\mathrm{Hom}_K(V, K)$; this is a left vector space of dimension n, and θ has an adjoint θ^* acting in V^*, with the same matrix A relative to the dual basis of V^*. Now a left eigenvalue β of A corresponds to a non-zero element $v\in V^*$ such that $v\theta^*=\beta v$, and Prop. 2.1–2.2 have an immediate interpretation. By contrast, the singular eigenvalues do not have a meaning for θ, because they are not similarity invariants of the matrix.

The above construction of V as a $(k[\theta], K)$-bimodule has the effect of separating the action of θ from that of K, but for some purposes it is better not to make this separation. Thus we again take a right K-space V, but this time write θ on the same side as the scalars. Writing $R=K[t]$, where t is a commuting indeterminate, we consider V as right R-module by letting $\sum t^i c_i$ correspond to $\sum \theta^i c_i$. We shall call V the R-module associated to A; it is clear that two matrices are similar precisely when the associated R-modules are isomorphic.

By the diagonal reduction in R ([3], Ch. 8) there exist P, $Q \in \mathbf{GL}_n(R)$ such that

$$P(t-A)\,Q = \operatorname{diag}(\lambda_1, \ldots, \lambda_n), \tag{2}$$

where λ_{i-1} is a total divisor of λ_i $(i=2, \ldots, n)$. The λ_i are just the invariant factors of $t-A$, and as right R-module V is isomorphic to the direct sum

$$R/\lambda_1 R \oplus \cdots \oplus R/\lambda_n R. \tag{3}$$

In order to describe the canonical form we need another definition. Given a square matrix A over a field K with central subfield k, we shall call A *algebraic over* k, if it satisfies a polynomial equation over k. In the commutative case, where one normally takes $K=k$, every matrix is algebraic, by the Cayley-Hamilton theorem, but when $k \neq K$, this need no longer be the case. If $f(A)$ is non-singular for every non-zero polynomial f over k, we shall call A *transcendental over* k. Whether a matrix A is algebraic or transcendental (or neither) depends clearly only on its similarity class; we can therefore also define these concepts for endomorphisms of a vector space over K.

Let us return to the expression (3) for the R-module associated with a matrix A. It shows that A is algebraic if and only if λ_n divides a polynomial with coefficients in k. In what follows we shall take k to be the precise centre of K; then a polynomial over K is invariant if and only if it is associated to a polynomial over k ([3], p. 297), hence A is algebraic if and only if λ_n is bounded. To find when A is transcendental, assume that some λ_i has a bounded factor, p say. Then the R-module V has an element annihilated by $p(t)$ and hence by $p^*(t)$, where p^* is the bound of p. Now p^* is invariant, and $p^*(A)$ is singular, so that A cannot be transcendental. Conversely, if A is not transcendental, then the R-module associated with A has an element annihilated by an invariant polynomial, and hence some invariant factor of A must have a factor which is bounded. Hence A is transcendental over the centre of K precisely when λ_n has no bounded non-unit factor, i.e. λ_n is *totally unbounded*. When this holds, then $\lambda_1 = \cdots = \lambda_{n-1} = 1$, for if there were two λ's different from 1, one would be a total divisor of the other, and so λ_n would have an invariant element as divisor. We sum up these results as

Theorem 3.1. *Let K be a field with centre k, and let $A \in K_n$. Then A is algebraic over k if and only if its last invariant factor is bounded, and A is transcendental over k if and only if its last invariant factor is totally unbounded.*

If the last invariant factor of $t-A$ is totally unbounded, all the others must be 1, as we have seen, and this means that the associated R-module

(3) is cyclic. In that case the matrix A is also called *cyclic*, and so we obtain the

Corollary. *A matrix over a field K, which is transcendental over the centre of K is cyclic.*

To obtain a reduction of A we recall Th. 6.5.4 of [3], p. 229. For a principal ideal domain (the case needed here) this states that every cyclic module R/aR has a direct decomposition

$$R/aR \cong R/q_1 R \oplus \cdots \oplus R/q_v R \oplus R/uR,$$

where each q_i is a product of **GL**-related bounded atoms, while atoms in different q's are not **GL**-related, and u is totally unbounded. Here two elements a, b are called **GL**-*related*[1] when $R/aR \cong R/bR$. Applying this result to (3), and observing that each λ_i for $i<n$ is necessarily bounded, we obtain a direct decomposition

$$R/\alpha_1 R \oplus \cdots \oplus R/\alpha_r R \oplus R/uR, \tag{4}$$

where each α_i is a product of **GL**-related bounded atoms (but now atoms in different α's may be **GL**-related), and u is totally unbounded. The term R/uR corresponds to the transcendental part of θ and is left unchanged. The $R/\alpha_i R$ correspond to the algebraic parts of θ; they are not necessarily indecomposable, as in the commutative case, but by decomposing any $R/\alpha_i R$, where possible, we may assume each α_i indecomposable in (4). The decomposition is then unique up to isomorphism, by the Krull-Schmidt theorem (because the algebraic part is itself unique). The resulting polynomials $\alpha_1, \ldots, \alpha_r$ are the *elementary divisors* of A.

Relative to a basis of V adapted to the decomposition (4), θ has a matrix which is a diagonal sum

$$A_1 \dotplus \cdots \dotplus A_r \dotplus U, \tag{5}$$

where A_i is an algebraic matrix with a single elementary divisor α_i, and U is transcendental. We shall leave U aside for the moment (it will be taken up again in § 4), and show that A_i has a normal form much as in the commutative case.

Thus let A be a matrix which is algebraic with a single elementary divisor α, then α is a product of **GL**-related atoms, say $\alpha = p_1 p_2 \ldots p_s$. Each p_j has the same degree d say, and $sd = n$ is the order of A. Let V be the R-module associated with A; since A is cyclic, we can find a vector v in V which generates V as R-module, and a familiar argument shows that

[1] This is one of several equivalent terms, the usual name being similarity; we shall not use that term here to avoid confusion.

$v, v\theta, \ldots, v\theta^{n-1}$ is a K-basis for V. We still have a basis if we take v, $v\theta, \ldots, v\theta^{d-1}, vp_s, v\theta p_s, \ldots, vp_{s-1}p_s, \ldots, v\theta^{d-1}p_2 \ldots p_s$. Relative to this basis, θ has the matrix

$$\begin{pmatrix} P_s & N & 0 & 0 & \ldots & 0 \\ 0 & P_{s-1} & N & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & P_2 & N \\ 0 & 0 & 0 & \ldots & 0 & P_1 \end{pmatrix}$$

where P_i is the companion matrix of p_i and $N = e_{s1}$ (the $s \times s$ matrix with 1 in the SW-corner and 0's elsewhere) (cf. [4], Ch. 11).

This describes completely the algebraic part of any matrix. We see that it has a normal form very much like the classical form. A linear elementary divisor corresponds to an eigenvalue in K and algebraic over the centre, and as in the commutative case, an algebraic matrix is diagonalizable if and only if each elementary divisor is linear. It remains to find a criterion for two matrices to be similar. We recall that A, B are similar if and only if their associated R-modules are isomorphic. Further, in a principal ideal domain, two bounded indecomposable elements are **GL**-related if and only if they have the same bound ([3], p. 231). Hence we obtain

Theorem 3.2. *Every square matrix over a field K is similar to the diagonal sum of an algebraic and a transcendental part, themselves unique up to similarity. Two algebraic matrices of the same order are similar if and only if their elementary divisors can be paired off so that corresponding ones have the same bound.*

The question when two transcendental matrices are similar will be taken up in § 4.

4. The Reduction of Transcendental Matrices

In [2] it was shown that for any field K with a central subfield k, if $a, b \in K$ are transcendental over k, then K can be embedded in a field L in which a and b are conjugate. Our objective is to prove a corresponding result, involving matrices rather than elements. Here and in what follows it is understood that all extensions are k-algebras and all homomorphisms are k-linear. All we shall need from [2] is the following result:

Theorem A. *Let K be a field with central subfield k, and let F_1, F_2 be subfields of K, isomorphic under a mapping $\varphi\colon F_1 \to F_2$. Then K can be embedded in a field L such that L is a k-algebra and φ is realized by conjugation by an element x of L, i.e. $a\varphi = x^{-1}ax$ for all $a \in F_1$.*

Let R be any ring; then R is a full $n\times n$ matrix ring (over some ring) if and only if it contains a set of n^2 matrix units e_{ij} satisfying the familiar identities. It follows in particular that any ring containing a full matrix ring is itself a full matrix ring. For any ring R we have denoted the full $n\times n$ matrix ring over R by R_n, but we shall also write $\mathfrak{M}_n(R)$ for this ring, especially when we wish to fix a particular set of matrix units; thus $\mathfrak{M}_n(R)$ is to be thought of not just as a ring, but as a ring with n^2 constant operators e_{ij} satisfying $e_{ij}e_{kl}=\delta_{jk}e_{il}$, $\sum e_{ii}=1$.

We recall the following basic result of Bergman's ([1], Cor. 2.5–6), which will also be needed:

Theorem B. *Let E be a semisimple ring and (R_λ) a family of E-rings, and denote by R the coproduct of the R_λ over E (in the category of rings). Then*

(i) *the right global dimension of R is given by the formula:*

$$\text{r. gl. dim. } R=\begin{cases}\sup_\lambda \{\text{r. gl. dim. } R_\lambda\}, & \textit{if this is positive,}\\ 0 \textit{ or } 1, & \textit{if each } R_\lambda \textit{ has } \text{r. gl. dim. } 0;\end{cases}$$

(ii) *every projective right R-module has the form of a direct sum $\bigoplus M_\lambda\otimes R$, where M_λ is a projective R_λ-module, for each λ, and the tensor product is taken over R_λ.*

Here an E-ring means a ring R with a homomorphism $E\to R$. We shall want to use this theorem in the following situation: Let A and B be firs (=free ideal rings, cf. [3], Ch. 1) over a common subfield E, and consider $R=\mathfrak{M}_n(A)*_E B$. By (i) this is hereditary and by (ii) every projective R-module is a direct sum of copies of $P\otimes R$, where P is a minimal projective for $\mathfrak{M}_n(A)$. Since $P^n\otimes R=\mathfrak{M}_n(A)\otimes R\cong R$, it follows that R is projective trivial, and by Th. 1.4.2 of [3] it is a full matrix ring over a fir. From the proof of that theorem it is clear that $R\cong\mathfrak{M}_n(C)$, where C is a fir.

The next lemma, the remark following it and its application to Th. 4.2 are due to G.M. Bergman.

Lemma 4.1. *Let K be a field and $n\geqq 1$. Suppose that $\mathfrak{M}_n(K)$ contains a subfield E which in turn contains two subfields F_1, F_2, isomorphic under a mapping φ: $F_1\to F_2$. Then there is an extension field L of K such that φ is realized by conjugation by a unit x in $\mathfrak{M}_n(L)$.*

Proof. By Theorem A, E has an extension field E' with an element x inducing φ. Consider $R=\mathfrak{M}_n(K)*_E E'$; by the remarks preceding the lemma, $R=\mathfrak{M}_n(G)$, where G is a fir containing K. Let L be the universal field of fractions of G ([3], Ch. 7), then L contains K and $\mathfrak{M}_n(L)$ contains the element x inducing the isomorphism φ. This establishes the lemma.

Now let K be a field and suppose that $\mathfrak{M}_n(K)$ contains isomorphic subfields F_1, F_2, F_3, with isomorphisms f: $F_1\to F_2$, g: $F_2\to F_3$ say, such that F_1, F_2 lie in a common subfield of $\mathfrak{M}_n(K)$ and so do F_2 and F_3. Then by the lemma we can enlarge K to a field L and obtain a unit x such that conjugation by x induces f; F_2, F_3 still lie within a common subfield of $\mathfrak{M}_n(L)$, and enlarging L further we obtain a unit y inducing the isomorphism g between F_2 and F_3. Now xy induces the isomorphism fg: $F_1 \to F_3$. In this way the scope of the lemma can be extended.

We now come to the main result of the paper.

Theorem 4.2. *Let K be a field with a central subfield k. Then there is an extension field L of k (still with k as central subfield), such that any two square matrices over L of the same order and both transcendental over k are similar.*

Proof. Let A, B be square matrices of the same order n over K, both transcendental over k; we must find an extension field of K over which A and B are similar.

Consider the field $K((t))$ of all formal Laurent series, i.e. series $\sum a_i t^i$ with $a_i \in K$ $(i \in \mathbb{Z})$ and $a_i = 0$ for $i < i_0$ (where i_0 depends on the series). Clearly $K((t))$ is again a field and (as L. Small has pointed out) we have a natural isomorphism

$$\mathfrak{M}_n(K((t))) \cong \mathfrak{M}_n(K)((t)). \tag{6}$$

Now the subfield $k(A)$ generated by the matrix A over k is clearly a purely transcendental extension of k. Write $F_1 = k(A)$, $F_2 = k(t)$, $F_3 = k(B)$, then F_1, F_2, F_3 are isomorphic subfields of the matrix ring (6). Moreover, F_1 and F_2 are contained in the subfield $k(A)((t))$, while F_2 and F_3 are contained in $k(B)((t))$. Hence we can apply the remark following the lemma and obtain an extension field H of $K((t))$ such that $\mathfrak{M}_n(H)$ contains a unit z inducing the k-isomorphism between $k(A)$ and $k(B)$ in which $A \mapsto B$. We now repeat the process until we have a field $K_1 \supseteq K$ such that any two matrices of the same order over K and transcendental over k are similar over K_1. If we apply the same construction to K_1 we get a chain of fields, all with k as central subfield:

$$K \subseteq K_1 \subseteq K_2 \subseteq \cdots.$$

Their union L is clearly a field with the required properties.

This result has some remarkable consequences. Thus by taking B to be a scalar we obtain

Corollary 1. *Any matrix A over a field K which is transcendental over a central subfield k, is similar (over a suitable extension field of K) to a scalar α. Moreover, α may be taken to be any preassigned element of K transcendental over k.*

Secondly, let f be a polynomial in $K[t]$; we can always find a matrix over K with f as its only invariant factor, namely the companion matrix of f. By Theorem 3.1 this matrix is transcendental if and only if f is totally unbounded, hence we obtain

Corollary 2. *Let K be a field, then any two totally unbounded polynomials in $K[t]$ of the same degree are* **GL**-*related over a suitable extension field.*

By combining the results of §§ 3 and 4 we obtain the Jordan normal form for a general matrix. To state the result it is convenient to operate in a field with some algebraic closure property, but in contrast to the commutative case, there are several different notions for the algebraic closure of a skew field (cf. [6]). We shall only need a fairly weak concept, the existential closure. An *existentially closed field*, EC-*field* for short, is a field K such that any existential sentence which holds in some extension of K already holds in K. E.g., if two matrices A, B are similar over an extension of an EC-field K, then they are similar over K itself. For the similarity of $A=(a_{ij})$ and $B=(b_{ij})$ is expressed by the solubility of the equations

$$\sum a_{ij} x_{jk} = \sum x_{ij} b_{jk}, \qquad \sum x_{ij} y_{jk} = \sum y_{ij} x_{jk} = \delta_{ik}.$$

It is proved in [6], p. 17 that the centre of an EC-field is necessarily the prime subfield, but this more precise information on the centre will not be needed here. Let k be any commutative subfield of an EC-field K, then K contains an algebraic closure of k. For if $\bar{k}$ is an algebraic closure of k, we can form the free product $K *_k \bar{k}$; its universal field of fractions L is an extension field of K in which every polynomial over k splits into linear factors, hence it already splits in K.

Theorem 4.3. *Let K be an* EC-*field with centre k. Then any square matrix A over K is similar to a diagonal sum*

$$A_1 \dotplus \cdots \dotplus A_r \dotplus u, \tag{7}$$

where A_i is a Jordan block, consisting of conjugate scalars c_{ij} along the main diagonal, 1's *above it and* 0's *elsewhere. Moreover, each c_{ij} is algebraic over k, while u is transcendental over k.*

Proof. In § 3 we obtained the reduction (5), where A_i is the matrix corresponding to the elementary divisor α_i, while U is a transcendental matrix. The transcendental part is similar to a scalar matrix u by Th. 4.2, Cor. 1. To find the form taken by the algebraic part, let A be a matrix with a single elementary divisor α. We know that α is a product of **GL**-related bounded atoms. Let p be a bounded atom occurring in α, and p^* its bound, then p^* is a polynomial with coefficients in k, and by the remark preceding the theorem, this splits into linear factors. Thus α has the form

$$\alpha=(t-c_1)\dots(t-c_s).$$

Moreover, $t-c_i$ and $t-c_j$ are **GL**-related, which means that c_i and c_j are conjugate. Thus A is similar to a matrix

$$\begin{pmatrix} c_1 & 1 & 0 & \dots & 0 & 0 \\ 0 & c_2 & 1 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & c_{s-1} & 1 \\ 0 & 0 & 0 & \dots & 0 & c_s \end{pmatrix}. \tag{8}$$

This completes the proof.

It is easily seen that the first diagonal element in (8), c_1, is a right eigenvalue, while the last is a left eigenvalue. Since c_1 and c_s are conjugate, we obtain the

Corollary. *For every square matrix the left and right spectrum over an* EC-*field extension coincide.*

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