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On the Valuedness of Finite Transducers

Andreas Weber
Fachbereich Informatik, J.W. Goethe-Universität, Postfach 111932,
D-6000 Frankfurt am Main 11, Federal Republic of Germany

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Summary. We investigate the valuedness of finite transducers in connection with their inner structure. We show: The valuedness of a finite-valued nondeterministic generalized sequential machine (NGSM) M with n states and output alphabet Δ is at most the maximum of $(1-\lfloor 1/\#\Delta \rfloor) \cdot (2^{k_1} \cdot k_3)^n \cdot n^n \cdot \#\Delta^{n^3 \cdot k_4/3}$ and $\lfloor 1/\#\Delta \rfloor \cdot (2^{k_2} \cdot k_3 \cdot (1+k_4))^n \cdot n^n$ where $k_1 \leqq 6.25$ and $k_2 \leqq 11.89$ are constants and $k_3 \geqq 1$ and $k_4 \geqq 0$ are local structural parameters of M. There are two simple criteria which characterize the infinite valuedness of an NGSM. By these criteria, it is decidable in polynomial time whether or not an NGSM is infinite-valued. In both cases, $\#\Delta > 1$ and $\#\Delta = 1$, the above upper bound for the valuedness is almost optimal. By reduction, all results can be easily generalized to normalized finite transducers.

Contents

0. Introduction

Let x be an input word of a normalized finite transducer (NFT) M (an NFT is a finite transducer with input length 0 or 1 on each transition; in this paper we consider finite transducers only in their normalized form). Every word which appears as output on some accepting path with input x in M is called a value for x. The valuedness of x in M is defined by the number of all different values for x. The valuedness of M is the maximal valuedness of an input word of M or is infinite, depending on whether or not a maximum exists. In the former (latter) case M is called finite-valued (infinite-valued). M is called single-valued, if its valuedness is at most 1.

The valuedness is a structural parameter of the relation realized by a finite transducer which received attention in connection with the equivalence problem: The equivalence of nondeterministic generalized sequential machines (NGSM's) is undecidable ([Gr68], [I78], [Li79], [Li83]; an NGSM is a finite transducer with input length 1 on each transition). The equivalence of finite-valued NFT's is decidable [CuKa86]. The equivalence problem for single-valued NFT's is PSPACE-complete ([Sch76], [BlHe77], [AnLi78], [GuI83]). A complete history of the topic is presented in excellent surveys by Karhumäki ([Ka86], [Ka87]) and Culik [Cu88]. For further background on finite transducers we refer to [Be79].

It is decidable whether or not an NFT is single-valued ([Sch76], [BlHe77], [AnLi78]). Moreover, given any fixed integer d, it can be tested in polynomial time whether or not the valuedness of an NFT is greater than d [GuI83]. It is decidable in DSPACE(2^{lin}) whether or not a nondeterministic sequential machine (NSM) is infinite-valued ([ChI83]; an NSM is an NGSM with output length 1 on each transition).

In this paper we clearly follow the improved contents of the Chap. 8 and 10–13 in [We87]. Our basic results are (see Sect. 2):

(1) The valuedness of a finite-valued NGSM M with n states and output alphabet Δ is at most the maximum of $(1-\lfloor 1/\#\Delta\rfloor)\cdot 2^{k_1\cdot n-2}\cdot n^n \cdot(k_3)^{n-1}\cdot \#\Delta^{(n^3-1)\cdot k_4/3}$ and $\lfloor 1/\#\Delta\rfloor\cdot 2^{k_2\cdot n-2}\cdot n^n\cdot(k_3)^{n-1}\cdot(1+k_4)^{n-1}$ where $k_1 \leqq 6.25$ and $k_2\leqq 11.89$ are constants and $k_3\geqq 1$ and $k_4\geqq 0$ are local structural parameters of M.

(2) There are two simple criteria (IV1) and (IV2) which characterize the infinite valuedness of an NGSM. These criteria can be also combined to a single criterion (IV).

According to (2), an NGSM is infinite-valued, if and only if it complies with (IV1) or (IV2). By a "machine oriented" approach due to Gurari and Ibarra [GuI81], the latter property is decidable in polynomial time. In Sect. 3 of this paper we present a "graph oriented" algorithm which leads to the same result in a more direct way. This shows that (2) implies (see Sect. 3):

(3) It is decidable in polynomial time whether or not an NGSM is infinite-valued.

Note that, in a preliminary form, (1) and (3) were already stated in [WeSe86]. An earlier attempt to establish a (nonefficient) decision procedure for (3) failed because of a fatal proof error [AnLi78].

In Sect. 4 we construct infinite series of NGSM's with large finite valuedness which demonstrate that the upper bound in (1) is almost optimal. In fact, it must be greater than $\#\Delta^{n^3\cdot k_4/343}$, if $\#\Delta>1$, and at least 2^{n-1}, if $\#\Delta=1$.

By reduction, the results (1)–(3) can be generalized to NFT's (see Sect. 5).

Our proofs clearly profit by the author's work together with Helmut Seidl on the degree of ambiguity of finite automata ([We87], [WeSe88.1], [WeSe88.2]).

Indeed, in order to prove (1) and (2), we introduce here new elementary methods for finite transducers which partly descend from ideas and techniques already presented in [We87] and in [WeSe88.1] (in the context of finite automata). First of all, we show that it is sufficient to consider chain NGSM's, which have a restricted structure. Then, for every input word x, we investigate a graph which describes all accepting paths with input x in the NGSM, and we use "pumping arguments" in these graphs. The criteria (IV 1) and (IV 2) describe simple reasons for an NGSM to be infinite-valued. In fact, we show that these are the only reasons. Note that, using the above mentioned work on finite automata, it is relatively easy to obtain (2) and (3) only for NSM's ([We87], Chap. 9).

Only recently, the ideas and techniques developed in this paper turned out to be fundamental for a proof of a decomposition theorem for finite-valued transducers: An NGSM M which does not comply with any of the criteria (IV 1), (IV 2) can be effectively decomposed into finitely many single-valued NGSM's $M_1, \ldots, M_N$ such that the relation realized by M is the union of the relations realized by $M_1, \ldots, M_N$ ([We88], [We89.2]). In fact, the above theorem directly implies (1) (with a slightly weaker upper bound) and (2). Moreover, it leads to a DTIME$(2^{2^{\text{poly}}})$-algorithm deciding the equivalence of finite-valued NFT's ([We88], [We89.2]).

In a finite transducer, instead of counting different values it is quite natural to consider only their lengths. This leads from the valuedness to another parameter called length-degree [We89.1]. Indeed, variants of (1)–(3) and of the above mentioned decomposition theorem and equivalence test hold true when the valuedness of an NGSM is replaced by its length-degree [We89.1]. In particular, the equivalence of NFT's with finite length-degree is recursively decidable.

The work presented in this paper and most of its sequel as described above was recently generalized by Helmut Seidl to bottom-up finite state tree transducers [Se89].

1. Definitions and Notations

A nondeterministic *normalized finite transducer* (short form: NFT) is a 6-tuple $M=(Q, \Sigma, \Delta, \delta, Q_I, Q_F)$ where Q, Σ and Δ denote nonempty, finite sets of states, input symbols and output symbols, respectively, $Q_I, Q_F \subseteq Q$ denote sets of initial resp. final (or accepting) states, and δ is a finite subset of $Q \times (\Sigma \cup \{\varepsilon\}) \times \Delta^* \times Q$. $\Sigma(\Delta)$ is called the input (output) alphabet of M, δ is called the transition relation of M. Each element of δ denotes a *transition* of M. M is called a *nondeterministic generalized sequential machine* (*nondeterministic sequential machine*), abbreviated NGSM (NSM), if δ is a finite subset of $Q \times \Sigma \times \Delta^* \times Q$ ($Q \times \Sigma \times \Delta \times Q$, resp.). M is called *ε-free*, if δ is a finite subset of $Q \times (\Sigma \cup \{\varepsilon\}) \times \Delta^+ \times Q$. In this paper we mainly deal with NGSM's.

The mode of operation of M is described by paths. A *path* π (of length m) for $x|z$ in M leading from p to q is a word $(q_1, x_1, z_1)\ldots(q_m, x_m, z_m)\, q_{m+1} \in (Q \times (\Sigma \cup \{\varepsilon\}) \times \Delta^*)^m \cdot Q$ so that $(q_1, x_1, z_1, q_2), \ldots, (q_m, x_m, z_m, q_{m+1})$ are transitions of M and the equalities $x = x_1 \ldots x_m$, $z = z_1 \ldots z_m$, $p = q_1$ and $q = q_{m+1}$ hold. π is said to realize $(x, z) \in \Sigma^* \times \Delta^*$ and to consume $x \in \Sigma^*$. π is called *accepting*, if $p \in Q_I$ and $q \in Q_F$. The *transduction* (or *relation*) *realized* by M, denoted by $T(M)$, is the set of pairs (in $\Sigma^* \times \Delta^*$) realized by all accepting paths in M. The *language recognized* by M, denoted by $L(M)$, is the domain of $T(M)$, i.e., the set of words (in Σ^*) consumed by all accepting paths in M.

We define $\hat{\delta} := \{(p, x, z, q) \in Q \times \Sigma^* \times \Delta^* \times Q \mid (x, z)$ is realized by some path in M leading from p to $q\}$. If M is an NGSM, then δ equals $\hat{\delta} \cap Q \times \Sigma \times \Delta^* \times Q$. In this case we rename $\hat{\delta}$ by δ. The reader may keep in mind that the Sects. 2–4 of this paper only deal with NGSM's. Thus, the notation $\hat{\delta}$ comes up again not before Sect. 5.

If $(x, z) \in \Sigma^* \times \Delta^*$ belongs to $T(M)$, then z is called a *value* for x in M. The *valuedness* of $x \in \Sigma^*$ in M (short form: $\mathrm{val}_M(x)$) is the number of all different values for x in M. The *valuedness* of M (short form: $\mathrm{val}(M)$) is the supremum of the set $\{\mathrm{val}_M(x) \mid x \in \Sigma^*\}$. Note that, for a given $x \in \Sigma^*$, $\mathrm{val}_M(x)$ may be infinite (see Sect. 5) whereas it is clearly finite, if M is an NGSM. M is called *infinite-valued* (*finite-valued*, *single-valued*), if its valuedness is infinite (finite, at most 1, respectively).

A state of M is called *useful*, if it appears on some accepting path in M; otherwise, this state is called useless. Useless states are irrelevant to the valuedness in M. If all states of M are useful, then M is called *trim*.

A state $p \in Q$ is said to be *connected* with a state $q \in Q$ (short form: $p \underset{M}{\longleftrightarrow} q$), if some paths in M lead from p to q and from q to p. An equivalence class w.r.t. the relation "$\underset{M}{\longleftrightarrow}$" is called a *strong component* of M. A transition (p, a, z, q) of M is called a *bridge*, if p is not connected with q.

We define:

$$\mathrm{val}(\delta) := \max(\{1\} \cup \{\#\{z \in \Delta^* \mid (p, a, z, q) \in \delta\} \mid p, q \in Q, a \in \Sigma \cup \{\varepsilon\}\}),$$

$$\mathrm{diff}(\delta) := \max(\{0\} \cup \{||z_1| - |z_2|| \mid \exists a \in \Sigma \cup \{\varepsilon\}: z_1, z_2 \in \{z \in \Delta^* \mid \delta \cap Q \times \{a\} \times \{z\} \times Q \neq \phi\}\}),$$

$$\mathrm{im}(\delta) := \{\varepsilon\} \cup \{z \in \Delta^* \mid \exists p, q \in Q \;\exists a \in \Sigma \cup \{\varepsilon\}: (p, a, z, q) \in \delta\},$$

$$\mathrm{length}(\delta) := \max\{|z| \mid z \in \mathrm{im}(\delta)\},$$

$$\mathrm{size}(\delta) := \sum_{(p, a, z, q) \in \delta} (1 + |z|).$$

Note that $\mathrm{val}(\delta) \leqq \#\mathrm{im}(\delta) \leqq \#(\Delta^{\leqq \mathrm{length}(\delta)})$[1] and $\mathrm{diff}(\delta) \leqq \mathrm{length}(\delta)$. If M is an NSM, then $\mathrm{val}(\delta) \leqq \#\Delta$, $\mathrm{diff}(\delta) = 0$ and $\mathrm{length}(\delta) \leqq 1$.

Let $M = (Q, \Sigma, \Delta, \delta, Q_I, Q_F)$ and $M' = (Q', \Sigma', \Delta', \delta', Q'_I, Q'_F)$ be two NFT's. M' is *included* in M (short form: $M' \subseteq M$), if inclusion holds in each component. Note: If M' is included in M, then $T(M') \subseteq T(M)$, $\mathrm{val}(\delta') \leqq \mathrm{val}(\delta)$, $\mathrm{diff}(\delta') \leqq \mathrm{diff}(\delta)$, $\mathrm{im}(\delta') \subseteq \mathrm{im}(\delta)$, $\mathrm{length}(\delta') \leqq \mathrm{length}(\delta)$ and $\mathrm{size}(\delta') \leqq \mathrm{size}(\delta)$.

In the rest of this section we consider an NGSM $M = (Q, \Sigma, \Delta, \delta, Q_I, Q_F)$.

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[1] $\Delta^{\leqq l}$ denotes the set $\bigcup_{i=0}^{l} \Delta^i$

M is said to be a *chain NGSM*, if, for some order $Q_1, \ldots, Q_k$ of the strong components of M, $(p_1, \ldots, p_k)$, $(q_1, \ldots, q_k) \in Q_1 \times \ldots \times Q_k$ exist such that, in M, $p_1(q_k)$ is the only possible initial (final) state and every bridge is of the form (q_i, a, z, p_{i+1}) where $i \in [k-1]$[2] and $(a, z) \in \Sigma \times \Delta^*$.

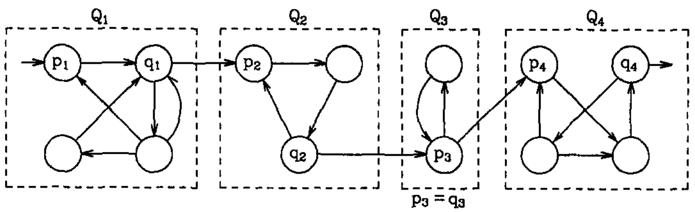

Fig. 1

Let M be a chain NGSM such that $L(M)$ is nonempty. Then, M is trim, $Q_I = \{p_1\}$ and $Q_F = \{q_k\}$.

Let $x = x_1 \ldots x_m \in \Sigma^*$ $(x_1, \ldots, x_m \in \Sigma)$. The *graph of accepting paths* in M consuming x (short form: $G_M(x)$) is the directed graph (V, E) where

$$\begin{aligned} V &:= \{(q,j) \in Q \times \{0, \ldots, m\} \mid \exists q_I \in Q_I \; \exists q_F \in Q_F \; \exists z_1, z_2 \in \Delta^*: \\ &\quad (q_I, x_1 \ldots x_j, z_1, q) \in \delta \;\&\; (q, x_{j+1} \ldots x_m, z_2, q_F) \in \delta\}, \\ E &:= \{((p, j-1), (q,j)) \in V^2 \mid j \in [m] \;\&\; \exists z \in \Delta^*: (p, x_j, z, q) \in \delta\}. \end{aligned}$$

Let $G_1 = (V_1, E_1)$ be a subgraph of $G_M(x)$. G_1 is said to be *LR-connected*, if, for each $(q,j) \in V_1$, some paths in G_1 lead from $Q_I \times \{0\}$ to (q,j) and from (q,j) to $Q_F \times \{m\}$. Let $e_0 = ((p, j_0 - 1), (q, j_0)) \in E_1$, and let π be a path in G_1 consisting of the edges $e_1, \ldots, e_l$. We define:

$$\begin{aligned} L(e_0) &:= \{z \in \Delta^* \mid (p, x_{j_0}, z, q) \in \delta\}, \\ L(\pi) &:= L(e_1) \cdot \ldots \cdot L(e_l), \\ L(G_1) &:= \bigcup_{\substack{\pi \text{ a path in } G_1 \text{ leading} \\ \text{from } Q_I \times \{0\} \text{ to } Q_F \times \{m\}}} L(\pi). \end{aligned}$$

Note: $\# L(e_0)$ is at most $\operatorname{val}(\delta)$, $L(G_M(x))$ equals the set of values for x in M, and $G_M(x)$ is LR-connected.

Let $x = x_1 \ldots x_m \in \Sigma^*$ $(x_1, \ldots, x_m \in \Sigma)$. Let $\pi = (q_1, x_1, z_1) \ldots (q_m, x_m, z_m)\, q_{m+1}$ and $\tilde{\pi} = (\tilde{q}_1, x_1, \tilde{z}_1) \ldots (\tilde{q}_m, x_m, \tilde{z}_m)\, \tilde{q}_{m+1}$ be paths in M consuming x. We define:

$$\operatorname{diff}(\pi, \tilde{\pi}) := \max\{ \big| |z_1 \ldots z_l| - |\tilde{z}_1 \ldots \tilde{z}_l| \big| \mid 0 \leq l \leq m\}.$$

Note that $\operatorname{diff}(\pi, \tilde{\pi})$ is at most $m \cdot \operatorname{diff}(\delta)$.

[2] $[m]$ denotes the set $\{1, \ldots, m\}$

2. Basic Results

Let $M=(Q, \Sigma, \Delta, \delta, Q_I, Q_F)$ be an NGSM with n states. We introduce the following criteria (IV 1) and (IV 2)[3] which characterize the infinite valuedness of M:

(IV 1) There are useful states $p, q_1, q_2, q_3, q \in Q$ such that, for some words $u, v, w \in \Sigma^*$ and $\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{w}_1, \tilde{w}_2, \tilde{w}_3 \in \Delta^*$, the following holds:

$\tilde{v}_1$ and $\tilde{v}_2$ have distinct lengths,

$(p, u, \tilde{u}_1, q_1), (q_1, v, \tilde{v}_1, q_1), (q_1, w, \tilde{w}_1, p) \in \delta$,

$(p, u, \tilde{u}_2, q_2), (q_2, v, \tilde{v}_2, q_2), (q_2, w, \tilde{w}_2, q) \in \delta$, and

$(q, u, \tilde{u}_3, q_3), (q_3, v, \tilde{v}_3, q_3), (q_3, w, \tilde{w}_3, q) \in \delta$.

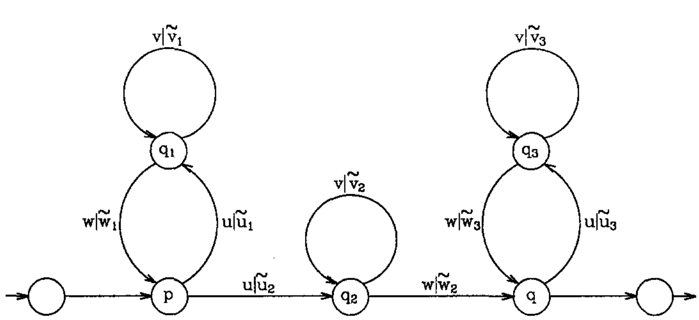

Fig. 2

In the next criterion we use the following notation: Let $z \in \Delta^*$ and $j \in [|z|]$, then $z(j) \in \Delta$ denotes the j-th letter of z. Let $z_1, z_2 \in \Delta^*$ and $j \in [\min\{|z_1|, |z_2|\}]$. We say that z_1 and z_2 *differ at position* j, if $z_1(j)$ and $z_2(j)$ are distinct.

(IV 2) There are useful states $p, q \in Q$ such that, for some words $v \in \Sigma^*$ and $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3 \in \Delta^*$, $\tilde{v}_1$ and $\tilde{v}_2$ differ at some position $j \in [\min\{|\tilde{v}_1|, |\tilde{v}_2|\}]$, and $(p, v, \tilde{v}_1, p), (p, v, \tilde{v}_2, q), (q, v, \tilde{v}_3, q) \in \delta$.

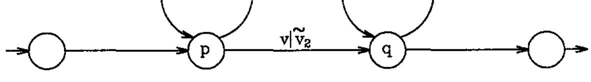

Fig. 3

(IV 1) and (IV 2) can be combined to the following criterion (IV):

[3] The notation does not correspond to those in [We87] and [We88]

(IV) There are useful states $q_1, q_2, q_3 \in Q$ such that, for some words $u, v \in \Sigma^*$ and $\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{u}_4, \tilde{v}_1, \tilde{v}_2, \tilde{v}_3 \in \Delta^*$, $\tilde{u}_1 \tilde{v}_1 \tilde{u}_2 \tilde{u}_3$ and $\tilde{u}_2 \tilde{v}_2 \tilde{u}_3 \tilde{u}_4$ are distinct, $(q_1, u, \tilde{u}_1, q_1), (q_1, u, \tilde{u}_2, q_2), (q_2, u, \tilde{u}_3, q_3), (q_3, u, \tilde{u}_4, q_3) \in \delta$, and $(q_1, v, \tilde{v}_1, q_1)$, $(q_2, v, \tilde{v}_2, q_2), (q_3, v, \tilde{v}_3, q_3) \in \delta$.

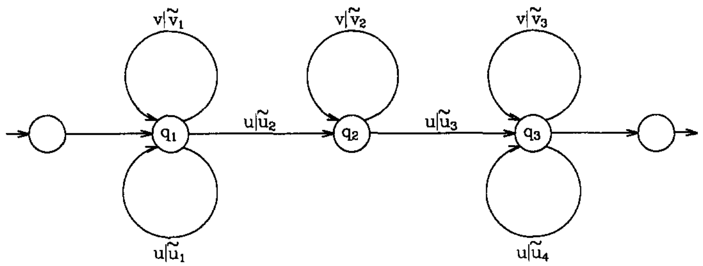

Fig. 4

In this Section we show:

(1) M complies with (IV), if and only if it complies with (IV 1) or (IV 2). In this assertion, the criteria (IV 1) and (IV 2) are both necessary.

(2) If M complies with (IV 1) or (IV 2), then it is infinite-valued.

(3) There are chain NGSM's $M_1, \ldots, M_N \subseteq M$ such that $N \leqq 5^{n/2}$ and M is finite-valued, if and only if $M_1, \ldots, M_N$ are all finite-valued. In the latter case, $\operatorname{val}(M)$ is at most $\sum_{i=1}^{N} \operatorname{val}(M_i)$.

(4) If M is a chain NGSM which does not comply with any of the criteria (IV 1), (IV 2), then the valuedness of M is at most

$$\begin{array}{ll} 3^n \cdot 2^{7n/2-2} \cdot n^n \cdot \operatorname{val}(\delta)^{n-1} \cdot \#\Delta^{(n^3-1)\cdot \operatorname{diff}(\delta)/3} & \text{if } \#\Delta > 1, \\ 3^{8n/3} \cdot 2^{13n/2-2} \cdot n^n \cdot \operatorname{val}(\delta)^{n-1} \cdot (1+\operatorname{diff}(\delta))^{n-1} & \text{if } \#\Delta = 1. \end{array}$$

The two basic results of this paper follow from (1)–(4):

Theorem 2.1. *Let $M = (Q, \Sigma, \Delta, \delta, Q_I, Q_F)$ be a finite-valued NGSM with n states. Then, the valuedness of M is at most*

$$\begin{array}{ll} (1/4) \cdot (5^{1/2} \cdot 3 \cdot 2^{7/2})^n \cdot n^n \cdot \operatorname{val}(\delta)^{n-1} \cdot \#\Delta^{(n^3-1)\cdot \operatorname{diff}(\delta)/3} & \textit{if } \#\Delta > 1, \\ (1/4) \cdot (5^{1/2} \cdot 3^{8/3} \cdot 2^{13/2})^n \cdot n^n \cdot \operatorname{val}(\delta)^{n-1} \cdot (1+\operatorname{diff}(\delta))^{n-1} & \textit{if } \#\Delta = 1. \end{array}$$

Theorem 2.2. *Let M be an NGSM. The assertions* (i)–(iii) *are equivalent:*
(i) *M is infinite-valued.*
(ii) *M complies with at least one of the criteria* (IV 1), (IV 2).
(iii) *M complies with* (IV).

Considering assertion (1) and Theorem 2.2, the reader may observe that the criterion (IV) is more succinct than the disjunction of (IV 1) and (IV 2) and also has some algebraic elegance. However, the latter two criteria meet the author's particular plain style of practical work in proofs and algorithms. Therefore, (IV 1) and (IV 2) are preferred to (IV) throughout the paper.

Let $M=(Q,\Sigma,\Delta,\delta,Q_I,Q_F)$ be a finite-valued NGSM with n states. Theorem 2.1 implies:

– If M is an NSM, then its valuedness is at most $(1/4)\cdot(5^{1/2}\cdot 3\cdot 2^{7/2})^n\cdot n^n\cdot \#\Delta^{n-1}$.

Specializing our proof of Theorem 2.1, we will obtain at the end of this Section (Corollary 2.12):

– If M is ε-free, then its valuedness is at most $(1/4)\cdot(5^{1/2}\cdot 3^{8/3}\cdot 2^{13/2})^n \cdot n^n\cdot \mathrm{val}(\delta)^{n-1}\cdot(1+\mathrm{diff}(\delta))^{n-1}$.

Using special methods, it is shown in [We87], Theorem 10.4:

– If M has only one input symbol, then its valuedness is at most $n\cdot 5^{n/2}\cdot \mathrm{val}(\delta)^{n-1}\cdot(1+\mathrm{diff}(\delta))$.

In Sect. 4 we state that each improvement of the upper bound in Theorem 2.1 has to stop above $\#\Delta^{n^3\cdot(\mathrm{diff}(\delta)/343}$, if $\#\Delta>1$ (Theorem 4.1), and at 2^{n-1}, if $\#\Delta=1$ (Theorem 4.2). By reduction, Theorem 2.1 can be generalized to NFT's (Theorem 5.1).

In Sect. 3 we state: It is decidable in polynomial time whether or not an NGSM complies with (IV 1) or (IV 2) (Lemmas 3.6 and 3.7). Thus, Theorem 2.2 implies: It is decidable in polynomial time whether or not an NGSM is infinite-valued (Theorem 3.1). By reduction, Theorem 2.2 and the last assertion can be generalized to NFT's (Theorems 5.2 and 5.3).

In the rest of this Section we consider an NGSM M in the notation $M=(Q,\Sigma,\Delta,\delta,Q_I,Q_F)$. We turn now to the proof of the assertions (1)–(4) stated above. For that purpose we first of all introduce two additional criteria (IV 2)′ and (IV 3) for M:

(IV 2)′ There are useful states $p,q\in Q$ such that, for some words $v\in\Sigma^*$, $\tilde v_1\in\Delta^+$ and $\tilde v_2,\tilde v_3,\tilde v_4\in\Delta^*$, $\tilde v_2$ and $\tilde v_3$ differ at some position $j\in[\min\{|\tilde v_2|,|\tilde v_3|\}]$, and $(p,v,\tilde v_1,p),(p,v,\tilde v_2,q),(p,v,\tilde v_3,q),(q,v,\tilde v_4,q)\in\delta$.

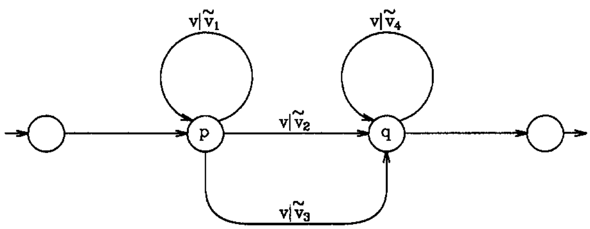

Fig. 5

(IV 3) There are useful states $p, q \in Q$ such that, for some words $v \in \Sigma^*$ and $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3 \in \Delta^*$, $\tilde{v}_1$ and $\tilde{v}_3$ have distinct lengths, and $(p, v, \tilde{v}_1, p), (p, v, \tilde{v}_2, q)$, $(q, v, \tilde{v}_3, q) \in \delta$.

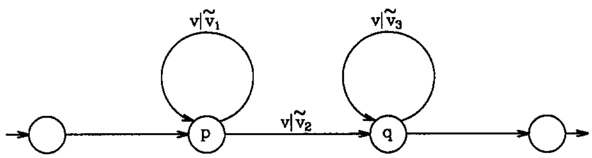

Fig. 6

Note that (IV 3) is a special case of (IV 1) and (IV). Our first two Lemmas show (1) and (2):

Lemma 2.3. *In an NGSM M, the criteria* (IV 1), (IV 2), (IV), (IV 2)′ *and* (IV 3) *are mutually related as follows:*

(i) *If M complies with* (IV 3), *then also with* (IV 1). *M complies with* (IV 2), *if and only if it complies with* (IV 2)′.

(ii) *M complies with* (IV), *if and only if it complies with* (IV 1) *or* (IV 2).

(iii) *NGSM's* $M_1, \ldots, M_5$ *exist such that* M_1 *complies with* (IV 1) *but not with any of* (IV 2), (IV 3), M_2 *complies with* (IV 2) *but not with* (IV 1), M_3 *complies with* (IV 1) *and* (IV 2) *but not with* (IV 3), M_4 *complies with* (IV 3) *but not with* (IV 2), *and* M_5 *complies with* (IV 2) *and* (IV 3).

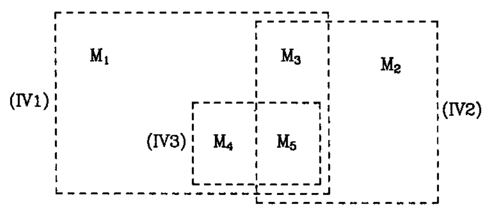

Fig. 7

Proof. (i) (IV 3) is a special case of (IV 1).

(IV 2) implies (IV 2)′: Let $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3 \in \Delta^*$ be selected according to the criterion (IV 2). Then, $\tilde{v}_1^2 \neq \varepsilon$, and $\tilde{v}_1 \tilde{v}_2$ and $\tilde{v}_2 \tilde{v}_3$ differ at some position $j \in [\min\{|\tilde{v}_1|, |\tilde{v}_2|\}]$.

(IV 2)′ implies (IV 2): Let $\tilde{v}_1 \in \Delta^+$ and $\tilde{v}_2, \tilde{v}_3, \tilde{v}_4 \in \Delta^*$ be selected according to the criterion (IV 2)′. Let $t \in \mathbb{N}$ so that $t \cdot |\tilde{v}_1| \geq \min\{|\tilde{v}_2|, |\tilde{v}_3|\}$. Then, for some $j \in [\min\{|\tilde{v}_2|, |\tilde{v}_3|\}]$ and some $\mu \in \{2, 3\}$, $\tilde{v}_1^t(j)$ and $\tilde{v}_\mu(j)$ are distinct. Thus, $\tilde{v}_1^t$ and $\tilde{v}_\mu \tilde{v}_4^{t-1}$ differ at position j.

(ii) (IV) implies (IV 1) or (IV 2)′: Let $\tilde{u}_1, \ldots, \tilde{v}_1, \ldots \in \Delta^*$ be selected according to the criterion (IV). If $|\tilde{u}_1| \neq |\tilde{u}_4|$ or $|\tilde{v}_1| \neq |\tilde{v}_2|$, then we have ((IV 3) or) (IV 1). In the other case, $\tilde{u}_1 \tilde{v}_1 \tilde{u}_2 \tilde{u}_3$ and $\tilde{u}_2 \tilde{v}_2 \tilde{u}_3 \tilde{u}_4$ are distinct words of equal length. Thus, since $\tilde{u}_1 \tilde{v}_1 \tilde{u}_1^2 \neq \varepsilon$, we have (IV 2)′.

(IV 1) implies (IV): Let $\tilde{u}_1, \ldots, \tilde{v}_1, \ldots, \tilde{w}_1, \ldots \in \Delta^*$ be selected according to the criterion (IV 1). Then, $|\tilde{u}_1 \tilde{w}_1| \neq |\tilde{u}_3 \tilde{w}_3|$ or $|(\tilde{w}_1 \tilde{u}_1)\tilde{v}_1(\tilde{w}_1 \tilde{u}_2)(\tilde{w}_2 \tilde{u}_3)| \neq |(\tilde{w}_1 \tilde{u}_2)\tilde{v}_2(\tilde{w}_2 \tilde{u}_3)(\tilde{w}_3 \tilde{u}_3)|$. Thus, we have ((IV 3) or) (IV). Note that (IV 3) is a special case of (IV).

(IV 2) implies (IV): Let $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3 \in \Delta^*$ be selected according to the criterion (IV 2). Then, $\tilde{v}_1^2 \tilde{v}_2 \tilde{v}_3$ and $\tilde{v}_2 \tilde{v}_3^3$ are distinct.

(iii) We construct the NGSM's $M_1, \ldots, M_5$ (see Fig. 8):

- $M_1 := (Q_1, \{a_1, a_2\}, \{b\}, \delta_1, \{p_1\}, \{p_3\})$ where $Q_1 := \{p_1, p_2, p_3\}$ and
  $\delta_1 := \{(p_1, a_1, \varepsilon, p_1), (p_1, a_2, \varepsilon, p_1), (p_1, a_1, \varepsilon, p_2), (p_2, a_2, b, p_2), (p_2, a_1, \varepsilon, p_3), (p_3, a_1, \varepsilon, p_3), (p_3, a_2, \varepsilon, p_3)\}$;
- $M_2 := (Q_2, \{a_1\}, \{b_1, b_2\}, \delta_2, \{p_1\}, \{p_2\})$ where $Q_2 := \{p_1, p_2\}$ and
  $\delta_2 := \{(p_1, a_1, b_1, p_1), (p_1, a_1, b_2, p_2), (p_2, a_1, b_1, p_2)\}$;
- $M_3 := (Q_1, \{a_1, a_2\}, \{b_1, b_2\}, \delta_3, \{p_1\}, \{p_3\})$ where
  $\delta_3 := (\delta_1 \cap Q_1 \times \{a_1\} \times \{\varepsilon\} \times Q_1)$
  $\cup \{(p_1, a_2, b_1, p_1), (p_2, a_2, b_2 b_2, p_2), (p_3, a_2, b_1, p_3)\}$;
- $M_4 := (Q_2, \{a_1\}, \{b\}, \delta_4, \{p_1\}, \{p_2\})$ where
  $\delta_4 := \{(p_1, a_1, b, p_1), (p_1, a_1, \varepsilon, p_2), (p_2, a_1, \varepsilon, p_2)\}$;
- $M_5 := (Q_2, \{a_1\}, \{b_1, b_2\}, \delta_5, \{p_1\}, \{p_2\})$ where
  $\delta_5 := \{(p_1, a_1, b_1, p_1), (p_1, a_1, b_2, p_2), (p_2, a_1, \varepsilon, p_2)\}$.

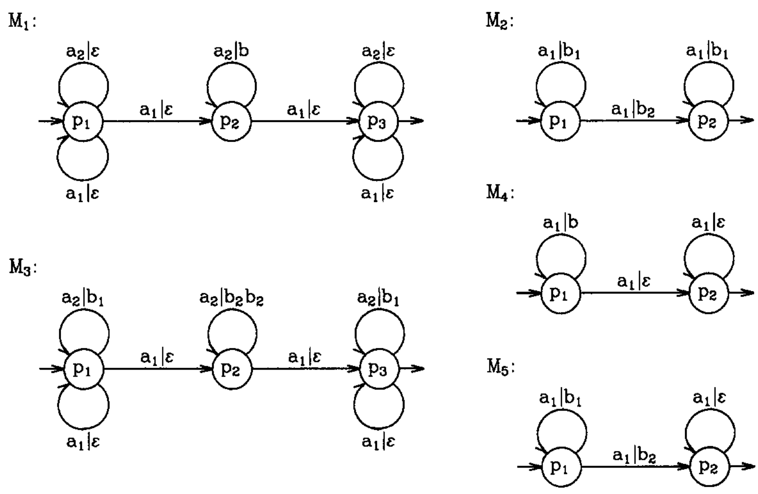

Fig. 8

By construction, M_1 complies with (IV 1), M_2 complies with (IV 2), M_3 complies with (IV 1) and (IV 2), M_4 complies with (IV 3), and M_5 complies with (IV 2) and (IV 3). Since M_1 and M_4 have only one output symbol, these NGSM's do not comply with (IV 2). Since M_2 is an NSM, it does not comply with

(IV 1). Assume that M_1 complies with (IV 3). Let $p, q \in Q_1$ and $v \in \{a_1, a_2\}^*$ be selected according to that criterion. Then, $p_2 \in \{p, q\}$ and $v \in \{a_2\}^*$. (Contradiction!) The same line of reasoning holds true, if M_1 is replaced by M_3. □

Lemma 2.4. *Let M be an NGSM. If M complies with at least one of the criteria* (IV 1), (IV 2), *then it is infinite-valued.*

Proof. Case 1. M complies with (IV 3).

Let $v \in \Sigma^*$ and $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3 \in \Delta^*$ be selected according to the criterion (IV 3). Then, $(u, \tilde{u}), (w, \tilde{w}) \in \Sigma^* \times \Delta^*$ exist such that, for all $i \in \mathbb{N}$, $\tilde{u}\tilde{v}_1^{j-1}\tilde{v}_2\tilde{v}_3^{i-j}\tilde{w}$ $(j=1, \ldots, i)$ are values for uv^iw in M. Since $|\tilde{v}_1| \neq |\tilde{v}_3|$, these values have pairwise distinct lengths. This implies for all $i \in \mathbb{N}$: $\mathrm{val}_M(uv^iw) \geqq i$. Thus, M is infinite-valued.

Case 2. M complies with (IV 1) but not with (IV 3).

Let $u, v, w \in \Sigma^*$ and $\tilde{u}_1, \ldots, \tilde{v}_1, \ldots, \tilde{w}_1, \ldots \in \Delta^*$ be selected according to the criterion (IV 1). Then, $|\tilde{u}_1\tilde{w}_1| = |\tilde{u}_3\tilde{w}_3|$ and $|\tilde{v}_1| = |\tilde{v}_3|$. By construction of (IV 1), $(u_0, \tilde{u}_0), (w_0, \tilde{w}_0) \in \Sigma^* \times \Delta^*$ exist such that, for all $i \in \mathbb{N}$,

$$\tilde{u}_0 \cdot \prod_{\lambda=1}^{j-1} (\tilde{u}_1\tilde{v}_1^\lambda\tilde{w}_1) \cdot \tilde{u}_2\tilde{v}_2^j\tilde{w}_2 \cdot \prod_{\lambda=j+1}^{i} (\tilde{u}_3\tilde{v}_3^\lambda\tilde{w}_3) \cdot \tilde{w}_0 \qquad (j=1, \ldots, i)$$

are values for $u_0 \cdot \prod_{\lambda=1}^{i} (uv^\lambda w) \cdot w_0$ in M. These values have lengths $|\tilde{u}_0\tilde{u}_2\tilde{w}_2\tilde{w}_0| + (i-1) \cdot |\tilde{u}_1\tilde{w}_1| + i \cdot (i+1) \cdot |\tilde{v}_1|/2 + j \cdot (|\tilde{v}_2| - |\tilde{v}_1|)$ $(j=1, \ldots, i)$. Since $|\tilde{v}_1| \neq |\tilde{v}_2|$, these lengths are pairwise distinct. This implies for all $i \in \mathbb{N}$: $\mathrm{val}_M(u_0 \cdot \prod_{\lambda=1}^{i} (uv^\lambda w) \cdot w_0) \geqq i$. Thus, M is infinite-valued.

Case 3. M complies with (IV 2).

Let $v \in \Sigma^*$ and $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3 \in \Delta^*$ be selected according to the criterion (IV 2). Then, $(u, \tilde{u}), (w, \tilde{w}) \in \Sigma^* \times \Delta^*$ exist such that, for all $i \in \mathbb{N}$, $\tilde{u}\tilde{v}_1^{j-1}\tilde{v}_2\tilde{v}_3^{i-j}\tilde{w}$ $(j=1, \ldots, i)$ are values for uv^iw in M. Since $\tilde{v}_1$ and $\tilde{v}_2$ differ at some position $\mu \in [\min\{|\tilde{v}_1|, |\tilde{v}_2|\}]$, these values are pairwise distinct. This implies for all $i \in \mathbb{N}$: $\mathrm{val}_M(uv^iw) \geqq i$. Thus, M is infinite-valued. □

The next Lemma will show (3). It is derived from Lemma 2.3 of [WeSe88.1]. In order to prove these Lemmas, we need the following Proposition:

Proposition 2.5. *Let $n = \sum_{i=1}^{k} n_i$ where $n_1, \ldots, n_k \in \mathbb{N}$*[4]*. Then,* $\prod_{i=1}^{k} (n_i^2+1) \leqq 5^{n/2}$.

Proof. It is easy to show by induction on j that for each $j \in \mathbb{N}$ $(j^2+1)^2 \leqq 5^j$. In turn, this implies: $\prod_{i=1}^{k} (n_i^2+1) \leqq \prod_{i=1}^{k} 5^{n_i/2} = 5^{n/2}$. □

Lemma 2.6. *Let M be an NGSM with n states and input alphabet Σ. There are chain NGSM's $M_1, \ldots, M_N \subseteq M$ such that $N \leqq 5^{n/2}$ and the following assertions are true:*

[4] $\mathbb{N}$ denotes the set of all nonnegative integers

(i) $\forall x \in \Sigma^*$: $\mathrm{val}_M(x) \leqq \sum_{i=1}^{N} \mathrm{val}_{M_i}(x)$.

(ii) *M is finite-valued, if and only if $M_1, \ldots, M_N$ are all finite-valued. In this case,* val(M) *is at most* $\sum_{i=1}^{N} \mathrm{val}(M_i)$.

Proof. Let $Q_1, \ldots, Q_k$ be an order of the strong components of M so that for all $i, j \in [k]$ the following holds:

$$\delta \cap Q_i \times \Sigma^* \times \Delta^* \times Q_j \neq \phi \Rightarrow i \leqq j.$$

Let K be a nonempty subset of $[k]$, let $1 \leqq i_1 < i_2 < \ldots < i_l \leqq k$ so that $K = \{i_1, \ldots, i_l\}$, and let $\underline{p} = (p_{i_1}, \ldots, p_{i_l})$, $\underline{q} = (q_{i_1}, \ldots, q_{i_l}) \in Q^{(K)} := Q_{i_1} \times \ldots \times Q_{i_l}$. We construct the NGSM $M^{(\underline{p}, \underline{q}, K)} = \left(\bigcup_{\lambda=1}^{l} Q_{i_\lambda}, \Sigma, \Delta, \delta^{(\underline{p}, \underline{q}, K)}, Q_I^{(\underline{p}, \underline{q})}, Q_F^{(\underline{p}, \underline{q})} \right)$:

$$Q_I^{(\underline{p}, \underline{q})} := Q_I \cap \{p_{i_1}\},$$
$$Q_F^{(\underline{p}, \underline{q})} := Q_F \cap \{q_{i_l}\},$$
$$\delta^{(\underline{p}, \underline{q}, K)} := \delta \cap \left(\bigcup_{\lambda=1}^{l} Q_{i_\lambda} \times \Sigma \times \Delta^* \times Q_{i_\lambda} \cup \bigcup_{\lambda=1}^{l-1} \{q_{i_\lambda}\} \times \Sigma \times \Delta^* \times \{p_{i_{\lambda+1}}\} \right).$$

$M^{(\underline{p}, \underline{q}, K)}$ is a chain NGSM, which is included in M. We observe:

$$T(M) = \bigcup_{\phi \neq K \subseteq [k]} \bigcup_{\underline{p}, \underline{q} \in Q^{(K)}} T(M^{(\underline{p}, \underline{q}, K)}).$$

From this follows for all $x \in \Sigma^*$:

$$\mathrm{val}_M(x) \leqq \sum_{\phi \neq K \subseteq [k]} \sum_{\underline{p}, \underline{q} \in Q^{(K)}} \mathrm{val}_{M^{(\underline{p}, \underline{q}, K)}}(x).$$

From Proposition 2.5 follows:

$$\sum_{\phi \neq K \subseteq [k]} \sum_{p, q \in Q^{(K)}} 1 = \sum_{(\sigma_1, \ldots, \sigma_k) \in \{0,1\}^k} \left(\prod_{i=1}^{k} (\# Q_i)^{2 \cdot \sigma_i} \right) - 1$$
$$= \prod_{i=1}^{k} ((\# Q_i)^2 + 1) - 1 \leqq 5^{n/2} - 1 < 5^{n/2}.$$

This completes the proof of (i). The assertion (ii) follows from (i). $\square$

The next Lemma shows (4):

Lemma 2.7. *Let $M=(Q,\Sigma,\Delta,\delta,Q_I,Q_F)$ be a chain NGSM with n states which does not comply with any of the criteria* (IV 1), (IV 2). *Then, the valuedness of M is at most*

$$3^n \cdot 2^{7n/2-2} \cdot n^n \cdot \operatorname{val}(\delta)^{n-1} \cdot \#\Delta^{(n^3-1)\cdot \operatorname{diff}(\delta)/3} \quad \text{if } \#\Delta>1,$$
$$3^{8n/3} \cdot 2^{13n/2-2} \cdot n^n \cdot \operatorname{val}(\delta)^{n-1} \cdot (1+\operatorname{diff}(\delta))^{n-1} \quad \text{if } \#\Delta=1.$$

In order to prove Lemma 2.7 we state three technical Lemmas (Lemmas 2.8–2.10). Lemma 2.8 will be used as an effective tool in the proof of Lemma 2.7, while the Lemmas 2.9 and 2.10 will guide the proof of Lemma 2.8. Therefore, we will successively prove all these Lemmas according to the order 2.9, 2.10, 2.8 and 2.7.

Lemma 2.8. *Let $M=(Q,\Sigma,\Delta,\delta,Q_I,Q_F)$ be a trim NGSM with n states which does not comply with any of the criteria* (IV 1), (IV 2). *Let $A\in 2^Q\setminus\{\phi\}$ and $y\in\Sigma^+$ such that the following holds:*

$$\forall r\in A\;\exists s\in A\;\exists z\in\Delta^*\colon (r,y,z,s)\in\delta \quad \& \quad \forall s\in A\;\exists r\in A\;\exists z\in\Delta^*\colon (r,y,z,s)\in\delta.$$

Let $U_1, U_2\in 2^Q$ so that $Q=U_1\,\dot\cup\, U_2$ and $\delta\cap U_2\times\Sigma^*\times\Delta^*\times U_1=\phi$; define $n_i:=\#U_i$ $(i=1,2)$. Let $p'\in A\cap U_1$ and $q'\in A\cap U_2$. Then, $\#\{z\in\Delta^*\,|\,(p',y,z,q')\in\delta\}$ is at most $t_0:=\max\{2\cdot(n_1 n_2 n-1)\cdot\operatorname{diff}(\delta)+1,\ 2\cdot\#\Delta^{(n_1n_2n-1)\cdot\operatorname{diff}(\delta)}-1\}$.

Lemma 2.9. *Let $M=(Q,\Sigma,\Delta,\delta,Q_I,Q_F)$ be an NGSM. Let $A\in 2^Q\setminus\{\phi\}$ and $y\in\Sigma^+$ such that the following holds:*

$$\forall r\in A\;\exists s\in A\;\exists z\in\Delta^*\colon (r,y,z,s)\in\delta \;\&\; \forall s\in A\;\exists r\in A\;\exists z\in\Delta^*\colon (r,y,z,s)\in\delta.$$

Let $p', q'\in A$. Then, the following assertion is true:

$$\begin{aligned}&\exists p,q\in A\;\exists p_1,p_2,p_3,p_4\in Q\;\exists l_1,l_2\in\mathbb{N}\;\exists z_1,\dots,z_4,z_6,\dots,z_9\in\Delta^*\colon\\&(p,y^{l_1},z_1,p_1),(p_1,y,z_2,p_2),(p_2,y^{l_2},z_3,p)\in\delta\;\&\\&(p,y^{l_1},z_4,p')\in\delta\;\&\;(q',y^{l_2},z_6,q)\in\delta\;\&\\&(q,y^{l_1},z_7,p_3),(p_3,y,z_8,p_4),(p_4,y^{l_2},z_9,q)\in\delta.\end{aligned}$$

Lemma 2.10. *Let $M=(Q,\Sigma,\Delta,\delta,Q_I,Q_F)$ be a trim NGSM which does not comply with the criterion* (IV 1). *Let $U_1', U_2', U_3', U_4'\in 2^Q$ so that $Q=U_1'\,\dot\cup\,U_2'\,\dot\cup\,U_3'\,\dot\cup\,U_4'$ and, for all $1\leqq i<j\leqq 4$, $\delta\cap U_j'\times\Sigma^*\times\Delta^*\times U_i'=\phi$; define $n_i':=\#U_i'$ $(i=1,2,3,4)$. Let $A\in 2^Q\setminus\{\phi\}$, $y\in\Sigma^+$, $p'\in A\cap U_2'$ and $q'\in A\cap U_3'$ such that the assertion of Lemma 2.9 is true. Let $p_1,p_2\in Q$ be taken from this assertion. If π and $\tilde\pi$ are paths in M consuming y and leading from p' to q' and from p_1 to p_2, respectively, then $\operatorname{diff}(\pi,\tilde\pi)$ is at most $((n_1'+n_2')\cdot(n_2'+n_3')\cdot(n_3'+n_4')-1)\cdot\operatorname{diff}(\delta)$.*

Proof of Lemma 2.9. We proceed like in the proof of Lemma 3.3 of [WeSe88.1]. Let $p',q'\in A$. We construct states $r_i\in A$ $(i\geqq 1)$ as follows: Define $r_1:=p'$. Choose $r_i\in A$ and $z_i^{(1)}\in\Delta^*$ so that $(r_i,y,z_i^{(1)},r_{i-1})\in\delta$ $(i=2,3,\dots)$. There are $i_1,i_2\in\mathbb{N}$ such that $r_{i_1}=r_{i_1+i_2}=:p\in A$ (see Fig. 9). We construct states $s_i\in A$ $(i\geqq 0)$ as follows: Define $s_0:=q'$. Choose $s_i\in A$ and $z_i^{(2)}\in\Delta^*$ so that $(s_{i-1},y,z_i^{(2)},s_i)\in\delta$ and, for all $j\in[i-1]$, $s_{i-1}=s_{j-1}$ implies $s_i=s_j$ $(i=1,2,\dots)$. There are $i_3\in\mathbb{N}$ and $i_4\in\mathbb{N}$ such that $s_{i_3}=s_{i_3+i_4}=:q\in A$ and $i_1+i_3=0 \bmod i_2\cdot i_4$ (see Fig. 9). In conclusion, we know (see Fig. 10):

$$(p, y^{i_2}, z^{(1)}_{i_1+i_2} \dots z^{(1)}_{i_1+1}, p)\in\delta,\ (p, y^{i_1-1}, z^{(1)}_{i_1} \dots z^{(1)}_{2}, p')\in\delta,$$
$$(q', y^{i_3}, z^{(2)}_{1} \dots z^{(2)}_{i_3}, q)\in\delta,\ (q, y^{i_4}, z^{(2)}_{i_3+1} \dots z^{(2)}_{i_3+i_4}, q)\in\delta.$$

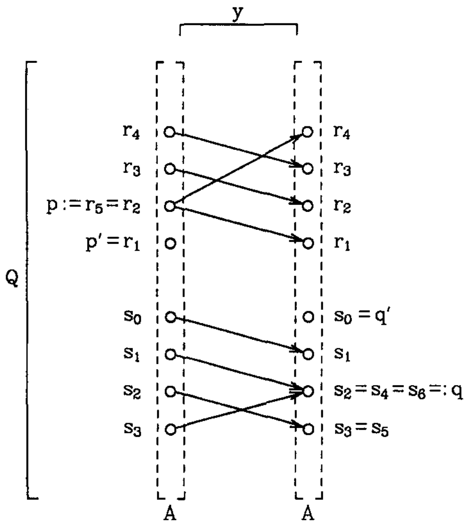

Fig. 9

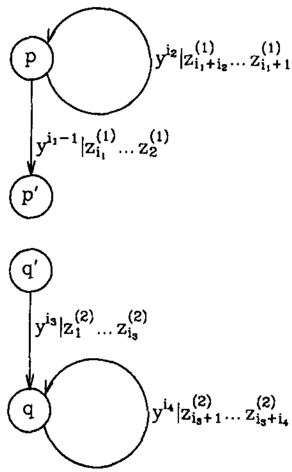

Fig. 10

Let $j_1\in\mathbb{N}$ so that $i_1+i_3=j_1\cdot i_2\cdot i_4$. Then, we have:

$$(p, y^{j_1 i_2 i_4}, (z^{(1)}_{i_1+i_2} \dots z^{(1)}_{i_1+1})^{j_1 i_4}, p)\in\delta,$$
$$(p, y^{i_1-1}, z^{(1)}_{i_1} \dots z^{(1)}_{2}, p')\in\delta \ \ \&\ \ (q', y^{i_3}, z^{(2)}_{1} \dots z^{(2)}_{i_3}, q)\in\delta,$$
$$(q, y^{j_1 i_2 i_4}, (z^{(2)}_{i_3+1} \dots z^{(2)}_{i_3+i_4})^{j_1 i_2}, q)\in\delta.$$

This implies the assertion of the Lemma. □

Proof of Lemma 2.10. From the assumption of the Lemma we know (see Fig. 11):

$$(*)\quad (p, y^{l_1}, z_1, p_1), (p_1, y, z_2, p_2), (p_2, y^{l_2}, z_3, p)\in\delta,$$
$$(p, y^{l_1}, z_4, p')\in\delta, \qquad (q', y^{l_2}, z_6, q)\in\delta, p'\in U'_2, q'\in U'_3,$$
$$(q, y^{l_1}, z_7, p_3), (p_3, y, z_8, p_4), (p_4, y^{l_2}, z_9, q)\in\delta.$$

We show by induction on the length of x (see Fig. 11):

$$(\#)\quad \forall x, x'\in\Sigma^*\ \forall r_1, r_2, r_3\in Q\ \forall z^{(1)}_1, \dots, z^{(1)}_6\in\Delta^*:$$
$$(p_1, x, z^{(1)}_1, r_1), (r_1, x', z^{(1)}_2, p_2)\in\delta\ \&$$
$$(p', x, z^{(1)}_3, r_2), (r_2, x', z^{(1)}_4, q')\in\delta\ \&$$
$$(p_3, x, z^{(1)}_5, r_3), (r_3, x', z^{(1)}_6, p_4)\in\delta \Rightarrow$$
$$||z^{(1)}_3|-|z^{(1)}_1|| \leqq ((n'_1+n'_2)\cdot(n'_2+n'_3)\cdot(n'_3+n'_4)-1)\cdot \mathrm{diff}(\delta).$$

By definition of the diff-operator for paths, the Lemma follows from (#).

Proof of (#). If $|x| \leqq (n'_1+n'_2)\cdot(n'_2+n'_3)\cdot(n'_3+n'_4)-1$, then (#) is correct by the definition of diff(δ). Therefore, let $|x| \geqq (n'_1+n'_2)\cdot(n'_2+n'_3)\cdot(n'_3+n'_4)$. From (*) we know: $p_1, p_2 \in U'_1 \cup U'_2$, $p', q' \in U'_2 \cup U'_3$, $p_3, p_4 \in U'_3 \cup U'_4$. Thus, it follows from the assumption of (#) (see Fig. 11):

$$\begin{aligned}
&\exists q_1 \in U'_1 \cup U'_2\ \exists q_2 \in U'_2 \cup U'_3\ \exists q_3 \in U'_3 \cup U'_4\ \exists y_1, y_2, y_3 \in \Sigma^*\ \exists z_1^{(2)}, \ldots, z_9^{(2)} \in \Delta^*:\\
&x = y_1\, y_2\, y_3,\ y_2 \neq \varepsilon,\ z_1^{(1)} = z_1^{(2)}\, z_2^{(2)}\, z_3^{(2)},\ z_3^{(1)} = z_4^{(2)}\, z_5^{(2)}\, z_6^{(2)},\ z_5^{(1)} = z_7^{(2)}\, z_8^{(2)}\, z_9^{(2)}\ \&\\
&(p_1, y_1, z_1^{(2)}, q_1), (q_1, y_2, z_2^{(2)}, q_1), (q_1, y_3, z_3^{(2)}, r_1) \in \delta\ \&\\
&(p', y_1, z_4^{(2)}, q_2), (q_2, y_2, z_5^{(2)}, q_2), (q_2, y_3, z_6^{(2)}, r_2) \in \delta\ \&\\
&(p_3, y_1, z_7^{(2)}, q_3), (q_3, y_2, z_8^{(2)}, q_3), (q_3, y_3, z_9^{(2)}, r_3) \in \delta.
\end{aligned}$$

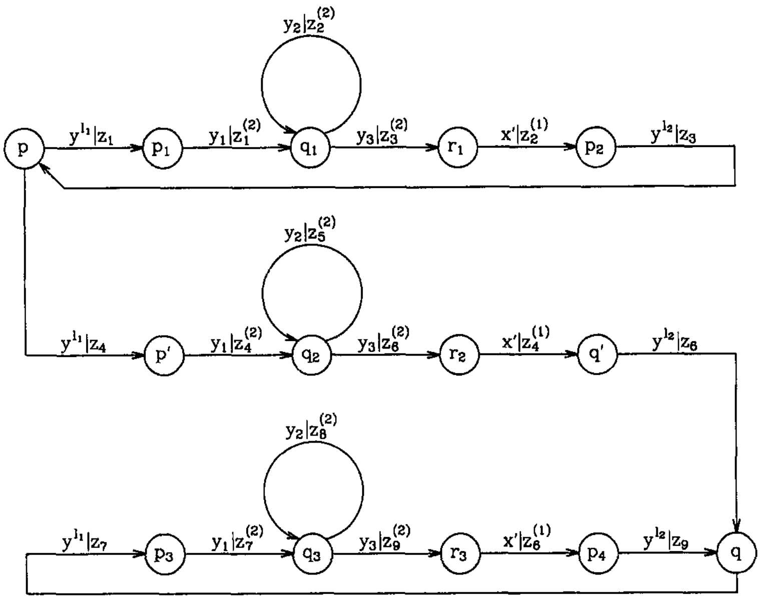

Fig. 11

From this follows that $|z_2^{(2)}| = |z_5^{(2)}|$ (because otherwise, since M is trim, (*) and the assumption of (#) would imply that M complies with (IV 1), see Fig. 11). With the induction hypothesis follows:

$$\begin{aligned}
||z_3^{(1)}| - |z_1^{(1)}|| &= ||z_4^{(2)}\, z_5^{(2)}\, z_6^{(2)}| - |z_1^{(2)}\, z_2^{(2)}\, z_3^{(2)}|| = ||z_4^{(2)}\, z_6^{(2)}| - |z_1^{(2)}\, z_3^{(2)}||\\
&\leqq ((n'_1+n'_2)\cdot(n'_2+n'_3)\cdot(n'_3+n'_4)-1)\cdot \mathrm{diff}(\delta). \quad \square
\end{aligned}$$

Proof of Lemma 2.8. Defining $U'_1 := \phi$, $U'_2 := U_1$, $U'_3 := U_2$ and $U'_4 := \phi$, we apply the Lemmas 2.9 and 2.10. Let $z_2 \in \Delta^*$ be taken from the assertion of Lemma 2.9.

Case 1. $z_2 \neq \varepsilon$.

M is trim and, by Lemma 2.3, does not comply with (IV2)'. Thus, we observe: $\#\{z \in \Delta^* | (p', y, z, q') \in \delta\} = \#\{|z| | (p', y, z, q') \in \delta\} \leqq 2 \cdot (n_1 n_2 n - 1) \cdot \mathrm{diff}(\delta) + 1 \leqq t_0$.

Case 2. $z_2 = \varepsilon$.

Then: $\#\{z \in \Delta^* | (p', y, z, q') \in \delta\} \leqq \#(\Delta^{\leqq (n_1 n_2 n - 1) \cdot \mathrm{diff}(\delta)}) \leqq \max \{(n_1 n_2 n - 1) \cdot \mathrm{diff}(\delta) + 1, 2 \cdot \#\Delta^{(n_1 n_2 n - 1) \cdot \mathrm{diff}(\delta)} - 1\} \leqq t_0$. □

Note that the Lemmas 2.9 and 2.10 can be also found in [We88] and [We89.2] where they turn out to be fundamental for the proof of a decomposition theorem for finite-valued NGSM's. From the point of view only of this paper, U_1' and U_4' in Lemma 2.10 could be omitted, and both Lemmas could be combined into one. Note that Lemma 2.10 remains true, if the partition of Q is missing. In this case, the upper bound for $\mathrm{diff}(\pi, \tilde{\pi})$ is $(n^3 - 1) \cdot \mathrm{diff}(\delta)$.

In order to prove Lemma 2.7, we still need the following Proposition:

Proposition 2.11. *Let $n, k \in \mathbb{N}$. If $n \geqq k \geqq 3$, then $n^k \leqq k^n$.*

Proof. We prove the Proposition by induction on $n - k$. Indeed, it is correct for $n = k$. Let $n + 1 > k \geqq 3$. Then, we conclude with the induction hypothesis:

$$(n+1)^k = \sum_{i=0}^{k-2} \binom{k}{i} \cdot n^{k-i} + (k \cdot n + 1) < (k-1) \cdot n^k + n^k = k \cdot n^k \leqq k^{n+1}. \quad \square$$

Proof of Lemma 2.7. Let $Q_1, \ldots, Q_k \subseteq Q$ and $(p_1, \ldots, p_k), (q_1, \ldots, q_k) \in Q_1 \times \ldots \times Q_k$ be given in correspondence with the definition of a chain NGSM. Let w.l.o.g.[5] $\mathrm{val}(M) > 0$. Then, M is trim, $Q_I = \{p_1\}$ and $Q_F = \{q_k\}$. We will show by induction on κ:

(*) Let $\kappa \in \{-1\} \cup \mathbb{N}$. If $2^\kappa \leqq k \leqq 2^{\kappa+1}$, then $\mathrm{val}(M)$ is at most

$$\mathrm{val}(\delta)^{k-1} \cdot 2^{n \cdot \lceil \log_2 k \rceil - k + 1} \cdot 2^{2^{\lceil \log_2 k \rceil} - 1} \cdot \begin{cases} f_1(n, \kappa) \cdot \#\Delta^{(n^3 - 1) \cdot \mathrm{diff}(\delta)/3} & \text{if } \#\Delta > 1 \\ f_2(n, \kappa) \cdot (1 + \mathrm{diff}(\delta))^{k-1} & \text{if } \#\Delta = 1 \end{cases}$$

$$\text{where } f_1(n, \kappa) := \begin{cases} 3^{n \cdot (\kappa+1)/3} & \text{if } \kappa \in \{-1, 0, 1\} \\ 2^{n \cdot (3/2 - (\kappa+1)/2^{\kappa-1})} \cdot 3^n & \text{else} \end{cases}$$

$$\text{and } f_2(n, \kappa) := \begin{cases} 3^{n \cdot (\kappa+1)} & \text{if } \kappa \in \{-1, 0, 1\} \\ 2^{n \cdot (9/2 - (5\kappa+6)/2^\kappa)} \cdot 3^{8n/3} & \text{else} \end{cases}$$

The Lemma follows from (*): It is shown in the proof of Lemma 2.5 of [WeSe88.1]: $n \cdot \left\lceil \log_2 k \right\rceil - k + 1 \leqq n \cdot \log_2 n$. Moreover, $2^{2^{\lceil \log_2 k \rceil} - 1} \leqq 2^{2n-2}$, $f_1(n, \kappa) \leqq 2^{3n/2} \cdot 3^n$, and $f_2(n, \kappa) \leqq 2^{9n/2} \cdot 3^{8n/3}$.

[5] Without loss of generality

Proof of (*).

Base of induction: $\kappa = -1$. Since $k=1$, some path in M leads from q_k to p_1. Let $x \in \Sigma^*$, and let $z_1, z_2 \in \Delta^*$ be values for x in M, i.e., (p_1, x, z_1, q_k), $(p_1, x, z_2, q_k) \in \delta$. Then, since M does not comply with any of the criteria (IV 1), (IV 2), z_1 and z_2 coincide (see Fig. 12). From this follows: M is single-valued, i.e., $\mathrm{val}(M) \leqq 1$.

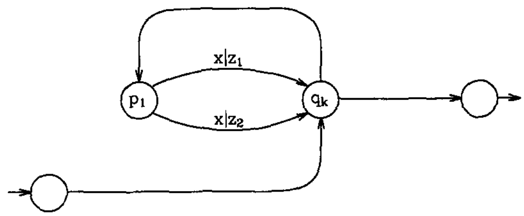

Fig. 12

Induction step: Let $\kappa \in \mathbb{N}$ so that $2^\kappa \leqq k \leqq 2^{\kappa+1}$. Since $k=1$ implies that M is single-valued (as shown in the base of induction), we may assume that $k \geqq 2$. Define $l := \lceil k/2 \rceil$. Like in the proof of Lemma 2.5 of [WeSe88.1], we (uniquely) divide M into the NGSM's $M_1 = \left(\bigcup_{i=1}^{l} Q_i, \Sigma, \Delta, \delta_1, \{p_1\}, \{q_l\} \right)$ and $M_2 = \left(\bigcup_{i=l+1}^{k} Q_i, \Sigma, \Delta, \delta_2, \{p_{l+1}\}, \{q_k\} \right)$ so that $\delta = \delta_1 \cup \delta_2 \cup (\delta \cap \{q_l\} \times \Sigma \times \Delta^* \times \{p_{l+1}\})$. M_1 and M_2 are chain NGSM's with $n_1 := \sum_{i=1}^{l} \# Q_i$ and $n_2 := \sum_{i=l+1}^{k} \# Q_i$ states, respectively, which do not comply with any of (IV 1), (IV 2). Moreover, $\mathrm{val}(M_1) > 0$ and $\mathrm{val}(M_2) > 0$.

Let $x = x_1 \ldots x_m \in L(M)$ $(x_1, \ldots, x_m \in \Sigma)$. Let $G_1 = (V_1, E_1)$ be an LR-connected subgraph of $G_M(x) = (V, E)$. Define $D(G_1) := \{((q_l, j-1), (p_{l+1}, j)) \in E_1 \mid j \in [m]\}$. $D(G_1)$ is the set of all edges in G_1 "leading from Q_l to Q_{l+1}". We are able to choose G_1 so that $L(G_1)$ equals $L(G_M(x))$ and $\# D(G_1)$ is minimal.

Assume that $\# D(G_1) > 2^{n-1} \cdot (1 + n_1 n_2 t_0)$ where $t_0 := \max\{2 \cdot (n_1 n_2 n - 1) \cdot \mathrm{diff}(\delta) + 1,\ 2 \cdot \# \Delta^{(n_1 n_2 n - 1) \cdot \mathrm{diff}(\delta)} - 1\}$. Let $J \subseteq [m]$ so that $D(G_1) = \{((q_l, j-1), (p_{l+1}, j)) \mid j \in J\}$. Let $j \in J$: Define $A_j := \{q \in Q \mid (q, j) \in V_1\}$. Clearly, $p_{l+1} \in A_j \subseteq Q$. Since $\# D(G_1) > 2^{n-1} \cdot (1 + n_1 n_2 t_0)$, $j_1, j_2 \in J$ exist such that $j_1 < j_2$, $A_{j_1} = A_{j_2}$ and $\#\{j \in J \mid j_1 < j \leqq j_2\} > n_1 \cdot n_2 \cdot t_0$. Since G_1 is LR-connected, we can apply Lemma 2.8 to $A := A_{j_1} = A_{j_2}$, $y := x_{j_1+1} \ldots x_{j_2}$, $U_1 := \bigcup_{i=1}^{l} Q_i$ and $U_2 := \bigcup_{i=l+1}^{k} Q_i$ (see Fig. 13) which yields:

$$\forall p' \in A \cap U_1\ \forall q' \in A \cap U_2:$$
$$\#\Big(\bigcup_{\substack{\pi \text{ a path in } G_1 \text{ leading} \\ \text{from } (p', j_1) \text{ to } (q', j_2)}} L(\pi) \Big) \leqq \#\{z \in \Delta^* \mid (p', y, z, q') \in \delta\} \leqq t_0.$$

Therefore, $n_1 \cdot n_2 \cdot t_0$ paths in G_1 leading from $(A \cap U_1) \times \{j_1\}$ to $(A \cap U_2) \times \{j_2\}$ and $n_1 \cdot n_2 \cdot t_0$ edges in $\{((q_l, j-1), (p_{l+1}, j)) | j \in J, j_1 < j \leqq j_2\} \subseteq D(G_1)$ suffice in order to obtain that $L(G_1)$ equals $L(G_M(x))$ (see Fig. 13). Hence, $\# D(G_1)$ is not minimal. (Contradiction!)

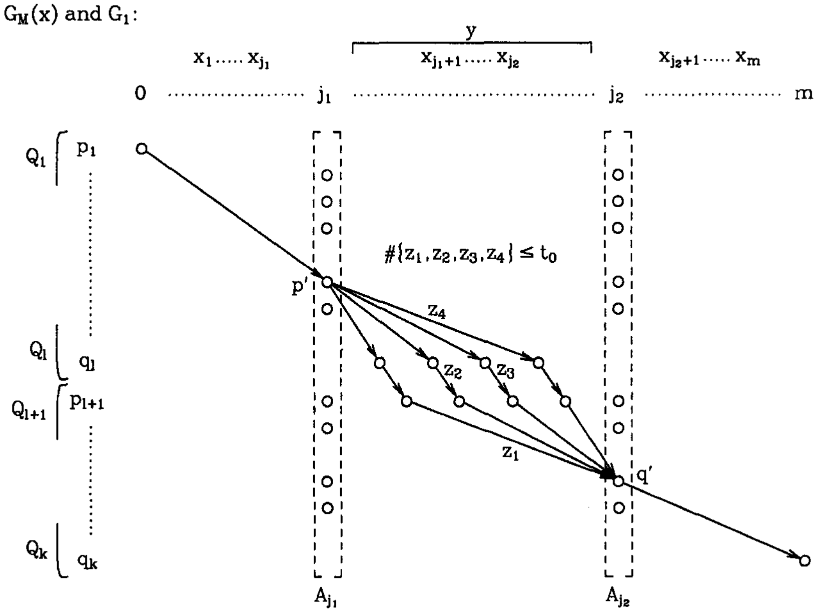

Fig. 13

Thus, we know:

$$\begin{aligned}\# D(G_1) &\leqq 2^{n-1} \cdot (1 + n_1\, n_2\, t_0) \leqq 2^{n-1} \cdot n_1\, n_2 \cdot (1 + t_0) \\ &= 2^{n-1} \cdot 2 n_1\, n_2 \cdot \max\{(n_1\, n_2\, n - 1) \cdot \operatorname{diff}(\delta) + 1,\ \# \Delta^{(n_1 n_2 n - 1) \cdot \operatorname{diff}(\delta)}\}.\end{aligned}$$

The above constructions were made in order to observe:

$$\begin{aligned}\operatorname{val}_M(x) &= \# L(G_M(x)) = \# L(G_1) \\ &\leqq \sum_{j \in J} \operatorname{val}_{M_1}(x_1 \ldots x_{j-1}) \cdot \# L(((q_l, j-1), (p_{l+1}, j))) \cdot \operatorname{val}_{M_2}(x_{j+1} \ldots x_m) \\ &\leqq \operatorname{val}(\delta) \cdot \# D(G_1) \cdot \operatorname{val}(M_1) \cdot \operatorname{val}(M_2).\end{aligned}$$

In the following estimations we will use Proposition 2.11 and the fact that, for all $j \in \mathbb{N}$, $j \leqq 3^{j/3}$ and, for all $j \in \mathbb{N} \setminus \{3\}$, $j \leqq 2^{j/2}$. Note that $2^{\kappa-1} \leqq \lfloor k/2 \rfloor \leqq \lceil k/2 \rceil \leqq 2^{\kappa}$. With the induction hypothesis follows:

If $\# \Delta > 1$:

$$\begin{aligned}\operatorname{val}_M(x) &\leqq \operatorname{val}(\delta) \cdot 2^{n-1} \cdot 2 n_1\, n_2 \cdot \# \Delta^{(n_1 n_2 n - 1) \cdot \operatorname{diff}(\delta)} \cdot \operatorname{val}(\delta_1)^{\lceil k/2 \rceil - 1} \cdot \operatorname{val}(\delta_2)^{\lfloor k/2 \rfloor - 1} \\ &\quad \cdot 2^{n_1 \cdot \lceil \log_2 \lceil k/2 \rceil \rceil - \lceil k/2 \rceil + 1} \cdot 2^{n_2 \cdot \lceil \log_2 \lfloor k/2 \rfloor \rceil - \lfloor k/2 \rfloor + 1} \cdot 2^{2^{\lceil \log_2 \lceil k/2 \rceil \rceil} - 1} \cdot 2^{2^{\lceil \log_2 \lfloor k/2 \rfloor \rceil} - 1} \\ &\quad \cdot f_1(n_1, \kappa - 1) \cdot f_1(n_2, \kappa - 1) \cdot \# \Delta^{(n_1^3 - 1) \cdot \operatorname{diff}(\delta_1)/3} \cdot \# \Delta^{(n_2^3 - 1) \cdot \operatorname{diff}(\delta_2)/3} \\ &\leqq \operatorname{val}(\delta)^{k-1} \cdot 2^{n \cdot \lceil \log_2 k \rceil - k + 1} \cdot 2^{2^{\lceil \log_2 k \rceil} - 1} \cdot f_1(n, \kappa) \cdot \# \Delta^{(n^3 - 1) \cdot \operatorname{diff}(\delta)/3}.\end{aligned}$$

Note: $\lceil \log_2 \lceil k/2 \rceil \rceil = \lceil \log_2 (k/2) \rceil$; if $\kappa \leqq 2$, then: $n_1 n_2 \cdot f_1(n_1, \kappa-1) \cdot f_1(n_2, \kappa-1) \leqq 3^{n/3} \cdot 3^{n \cdot \kappa/3} = 3^{n \cdot (\kappa+1)/3} = f_1(n, \kappa)$; if $\kappa \geqq 3$, then $n_i \geqq 2^{\kappa-1} \geqq 4$ $(i=1, 2)$ implies: $n_1 n_2 \cdot f_1(n_1, \kappa-1) \cdot f_1(n_2, \kappa-1) \leqq 2^{(\kappa-1) \cdot n/2^{\kappa-1}} \cdot 2^{n \cdot (3/2 - \kappa/2^{\kappa-2})} \cdot 3^n = 2^{n \cdot (3/2 - (\kappa+1)/2^{\kappa-1})} \cdot 3^n = f_1(n, \kappa)$; $(n_1^3 - 1)/3 + (n_2^3 - 1)/3 + n_1 n_2 n - 1 = n^3/3 - 5/3 < (n^3 - 1)/3$.

If $\# \Delta = 1$:

$$\begin{aligned}
\mathrm{val}_M(x) \leqq{} & \mathrm{val}(\delta) \cdot 2^{n-1} \cdot 2 n_1 n_2 \cdot ((n_1 n_2 n - 1) \cdot \mathrm{diff}(\delta) + 1) \\
& \cdot \mathrm{val}(\delta_1)^{\lceil k/2 \rceil - 1} \cdot \mathrm{val}(\delta_2)^{\lfloor k/2 \rfloor - 1} \\
& \cdot 2^{n_1 \cdot \lceil \log_2 \lceil k/2 \rceil \rceil - \lceil k/2 \rceil + 1} \cdot 2^{n_2 \cdot \lceil \log_2 \lfloor k/2 \rfloor \rceil - \lfloor k/2 \rfloor + 1} \cdot 2^{2^{\lceil \log_2 \lceil k/2 \rceil \rceil} - 1} \cdot 2^{2^{\lceil \log_2 \lfloor k/2 \rfloor \rceil} - 1} \\
& \cdot f_2(n_1, \kappa - 1) \cdot f_2(n_2, \kappa - 1) \cdot (1 + \mathrm{diff}(\delta_1))^{\lceil k/2 \rceil - 1} \cdot (1 + \mathrm{diff}(\delta_2))^{\lfloor k/2 \rfloor - 1} \\
\leqq{} & \mathrm{val}(\delta)^{k-1} \cdot 2^{n \cdot \lceil \log_2 k \rceil - k + 1} \cdot 2^{2^{\lceil \log_2 k \rceil} - 1} \cdot f_2(n, \kappa) \cdot (1 + \mathrm{diff}(\delta))^{k-1}.
\end{aligned}$$

Note: $(n_1 n_2 n - 1) \cdot \mathrm{diff}(\delta) + 1 \leqq n_1 n_2 n \cdot (1 + \mathrm{diff}(\delta))$; if $\kappa \in \{0, 1\}$, then: $n_1^2 n_2^2 n \cdot f_2(n_1, \kappa-1) \cdot f_2(n_2, \kappa-1) \leqq 3^n \cdot 3^{n \cdot \kappa} = 3^{n(\kappa+1)} = f_2(n, \kappa)$; if $\kappa = 2$, then $n \geqq 2^\kappa = 4$ implies: $n_1^2 n_2^2 n \cdot f_2(n_1, \kappa-1) \cdot f_2(n_2, \kappa-1) \leqq 3^{2n/3} \cdot 2^{n/2} \cdot 3^{2n} = 2^{n/2} \cdot 3^{8n/3} = f_2(n, \kappa)$; if $\kappa \geqq 3$, then $n \geqq 2^\kappa \geqq 8$ and $n_i \geqq 2^{\kappa-1} \geqq 4$ $(i=1, 2)$ implies: $n_1^2 n_2^2 n \cdot f_2(n_1, \kappa-1) \cdot f_2(n_2, \kappa-1) \leqq 2^{2(\kappa-1) \cdot n/2^{\kappa-1}} \cdot 2^{\kappa \cdot n/2^\kappa} \cdot 2^{n \cdot (9/2 - (5\kappa+1)/2^{\kappa-1})} \cdot 3^{8n/3} = 2^{n \cdot (9/2 - (5\kappa+6)/2^\kappa)} \cdot 3^{8n/3} = f_2(n, \kappa)$.

This completes the proof of Lemma 2.7. □

By this, we have established the Theorems 2.1 and 2.2. The dominating factor $\# \Delta^{(n^3-1) \cdot \mathrm{diff}(\delta)/3}$ in the upper bound for the valuedness of a finite-valued NGSM stated in Theorem 2.1 is caused by case 2 in the proof of Lemma 2.8. In order to avoid this case, let us assume in the Lemmas 2.6–2.8 that M is ε-free. Then, we have in Lemma 2.8: $\#\{z \in \Delta^* | (p', y, z, q') \in \delta\} \leqq 2 \cdot (n_1 n_2 n - 1) \cdot \mathrm{diff}(\delta) + 1$. This implies for Lemma 2.7: $\mathrm{val}(M) \leqq 3^{8n/3} \cdot 2^{13n/2 - 2} \cdot n^n \cdot \mathrm{val}(\delta)^{n-1} \cdot (1 + \mathrm{diff}(\delta))^{n-1}$. In Lemma 2.6, the chain NGSM's $M_1, \ldots, M_N \subseteq M$ are ε-free, just like M. In conclusion, we have shown:

Corollary 2.12. *Let $M = (Q, \Sigma, \Delta, \delta, Q_I, Q_F)$ be a finite-valued ε-free NGSM with n states. Then, the valuedness of M is at most* $(1/4) \cdot (5^{1/2} \cdot 3^{8/3} \cdot 2^{13/2})^n \cdot n^n \cdot \mathrm{val}(\delta)^{n-1} \cdot (1 + \mathrm{diff}(\delta))^{n-1}$.

3. Deciding Infinite Valuedness

In Sect. 2 we have shown that an NGSM is infinite-valued, if and only if it complies with at least one of the criteria (IV 1), (IV 2). In this Section we use the above characterization in order to show: It is decidable in polynomial time whether or not an NGSM is infinite-valued.

Let $M = (Q, \Sigma, \Delta, \delta, Q_I, Q_F)$ be an NGSM with n states. Using a result of Gurari and Ibarra [GuI81], it is easy to sketch an algorithm which decides in polynomial time whether or not M complies with (IV 1) or (IV 2):

– Remove all useless states from M. Assume that M is trim.

- For all $p, q_1, q_2, q_3, q \in Q$ do:
  - Decide whether or not $L_1 \cdot L_3$ is empty where $L_1 := \{u \in \Sigma^* \mid \exists z_1, z_2, z_3 \in \Delta^*: (p, u, z_1, q_1), (p, u, z_2, q_2), (q, u, z_3, q_3) \in \delta\}$ and $L_3 := \{w \in \Sigma^* \mid \exists z_1, z_2, z_3 \in \Delta^*: (q_1, w, z_1, p), (q_2, w, z_2, q), (q_3, w, z_3, q) \in \delta\}$.
  - Construct a nondeterministic finite 1-turn 2-counter machine (see [GuI81]) of polynomial size accepting the language $L_2 := \{v \in \Sigma^* \mid \exists z_1, z_2, z_3 \in \Delta^*: |z_1| \neq |z_2| \,\&\, (q_1, v, z_1, q_1), (q_2, v, z_2, q_2), (q_3, v, z_3, q_3) \in \delta\}$.
  - Decide whether or not L_2 is empty [GuI81].
  - If $L_1 \cdot L_2 \cdot L_3 \neq \phi$, then M complies with (IV 1).
- For all $p, q \in Q$ do:
  - Construct a nondeterministic finite 1-turn 2-counter machine (see [GuI81]) of polynomial size accepting the language $L_0 := \{v \in \Sigma^* \mid \exists z_1, z_2, z_3 \in \Delta^*: \exists j \in [\min\{|z_1|, |z_2|\}]: z_1(j) \neq z_2(j) \,\&\, (p, v, z_1, p), (p, v, z_2, q), (q, v, z_3, q) \in \delta\}$.
  - Decide whether or not L_0 is empty [GuI81].
  - If $L_0 \neq \phi$, then M complies with (IV 2).

According to [GuI81], the above algorithm requires polynomial time. However, the time bounds coming from [GuI81] are rather "machine oriented". It seems lengthy to analyse them exactly.

Using an elementary, "graph oriented" approach, we will show in the Lemmas 3.6 and 3.7:

(1) It is decidable in time $O((n^9 + n^6 \cdot \#\Sigma) \cdot (1 + \mathrm{diff}(\delta))^2 + \mathrm{size}(\delta))$ whether or not M complies with (IV 1).

(2) Assume that M does not comply with (IV 1). Then, it is decidable in time $O(n^9 \cdot (1 + \mathrm{diff}(\delta)) \cdot (1 + \mathrm{length}(\delta)) \cdot \#\Delta + n^6 \cdot \#\Sigma \cdot (1 + \mathrm{length}(\delta))^2 \cdot \#\Delta + \mathrm{size}(\delta))$ whether or not M complies with (IV 2).

From (1), (2) and Theorem 2.2 follows:

Theorem 3.1. *Let $M = (Q, \Sigma, \Delta, \delta, Q_I, Q_F)$ be an NGSM with n states. It is decidable in time $O(n^9 \cdot (1 + \mathrm{diff}(\delta)) \cdot (1 + \mathrm{length}(\delta)) \cdot \#\Delta + n^6 \cdot \#\Sigma \cdot (1 + \mathrm{length}(\delta))^2 \cdot \#\Delta + \mathrm{size}(\delta))$ whether or not M is infinite-valued.*

In order to show (1) and (2) we need some preliminaries:

In a finite, directed graph $G = (V, E)$ we use the following notations: Let $p, q \in V$. We write $p \xrightarrow[G]{} q$, if some path in G leads from p to q. We write $p \xleftrightarrow[G]{} q$, if some paths in G lead from p to q and from q to p. $p \xleftrightarrow[G]{} q$ means: p is strongly connected with q. An equivalence class w.r.t. the relation "$\xleftrightarrow[G]{}$" is called a *strong component* of G.

Let $M = (Q, \Sigma, \Delta, \delta, Q_I, Q_F)$ be an NGSM with n states. The criteria (IV 1) and (IV 2) for M will be transformed into assertions on graphs. For this, we define the directed graphs $G_3 = (Q^3, E_3)$, $G_3' = (Q^3, E_3 \cup E_3')$, $G_4 = (V_4, E_4)$, $G_4' = (V_4, E_4 \cup E_4')$, $G_5 = (V_5, E_5)$ and $G_5' = (V_5, E_5 \cup E_5')$ and the set $V_4' \subseteq Q^3$:

$$E_3 := \{((p_1, p_2, p_3), (q_1, q_2, q_3)) \in Q^3 \times Q^3 \mid \exists a \in \Sigma\ \exists z_1, z_2, z_3 \in \Delta^*\ \forall i \in \{1, 2, 3\}: (p_i, a, z_i, q_i) \in \delta\},$$

$$E_3' := \{((p, q, q), (p, p, q)) \mid p, q \in Q\},$$

$$V_4 := Q^3 \times Z_4 \text{ where } Z_4 := \{t \in \mathbb{Z} \mid |t| \leqq (n^3-1)\cdot \operatorname{diff}(\delta)\} \cup \{\infty\},$$
$$E_4 := \{((p_1,p_2,p_3,t),(q_1,q_2,q_3,t')) \in (V_4)^2 \mid \exists a \in \Sigma\ \exists z_1,z_2,z_3 \in \Delta^*:$$
$$[\forall i \in \{1,2,3\}: (p_i,a,z_i,q_i) \in \delta] \ \&\ t' = \min(\{t+|z_1|-|z_2|,\infty\} \cap Z_4)\},$$
$$E'_4 := \{((q_1,q_2,q_3,\infty),(q_1,q_2,q_3,0)) \mid q_1,q_2,q_3 \in Q\},$$
$$V'_4 := \{(q_1,q_2,q_3) \in Q^3 \mid (q_1,q_2,q_3,0) \xleftrightarrow[G'_4]{} (q_1,q_2,q_3,\infty)\},$$
$$V_5 := Q^3 \times (\Delta \,\dot\cup\, \{1,2\}) \times Z_5 \text{ where } Z_5 := \{t \in \mathbb{Z} \mid |t| \leqq (n^3-1)\cdot \operatorname{diff}(\delta)\},$$
$$E_5 := \{((p_1,p_2,p_3,b,t),(q_1,q_2,q_3,b',t')) \in (V_5)^2 \mid \exists a \in \Sigma\ \exists z_1,z_2,z_3 \in \Delta^*:$$
$$[\forall i \in \{1,2,3\}: (p_i,a,z_i,q_i) \in \delta] \ \&\ [(b=b'=1 \ \&\ t'=t+|z_1|-|z_2|)$$
$$\vee (b=1 \ \&\ \exists j \in [|z_1|]: b'=z_1(j) \in \Delta \ \&\ t'=t+j-|z_2|>0)$$
$$\vee (b=1 \ \&\ \exists j \in [|z_2|]: b'=z_2(j) \in \Delta \ \&\ t'=t+|z_1|-j<0)$$
$$\vee (b=b' \in \Delta \ \&\ t'=t-|z_2|>0) \vee (b=b' \in \Delta \ \&\ t'=t+|z_1|<0)$$
$$\vee ((b,t) \in \Delta \times [|z_2|] \ \&\ b \neq z_2(t) \ \&\ (b',t')=(2,0))$$
$$\vee ((b,-t) \in \Delta \times [|z_1|] \ \&\ b \neq z_1(-t) \ \&\ (b',t')=(2,0))$$
$$\vee (b=1 \ \&\ \exists j \in [|z_1|]: t+j \in [|z_2|] \ \&\ z_1(j) \neq z_2(t+j) \ \&\ (b',t')=(2,0))$$
$$\vee (b=b'=2 \ \&\ t=t'=0)]\},$$
$$E'_5 := \{((p,q,q,2,0),(p,p,q,1,0)) \mid p,q \in Q\}.$$

The construction of G_4 and G_5 requires some explanation:

Let π_1,π_2,π_3 be paths in M all consuming the same word in Σ^*. Assume that π_i realizes $(v,\tilde v_i) \in \Sigma^* \times \Delta^*$ and leads from $p_i \in Q$ to $q_i \in Q$ $(i=1,2,3)$. Then, there is a path π in G_4 which simultaneously "simulates" π_1,π_2,π_3, and which "decides" whether or not $\operatorname{diff}(\pi_1,\pi_2)$ exceeds $(n^3-1)\cdot\operatorname{diff}(\delta)$. This path π leads from $(p_1,p_2,p_3,0)$ to (q_1,q_2,q_3,t) where $t=|\tilde v_1|-|\tilde v_2|$ or $t=\infty$, depending on whether or not $\operatorname{diff}(\pi_1,\pi_2)$ is at most $(n^3-1)\cdot\operatorname{diff}(\delta)$.

Assume that $\operatorname{diff}(\pi_1,\pi_2)$ is at most $(n^3-1)\cdot\operatorname{diff}(\delta)$. Then, there is a path π' in G_5 which simultaneously "simulates" π_1,π_2,π_3, and which is able to "verify" that $\tilde v_1$ and $\tilde v_2$ differ at some position $j \in [\min\{|\tilde v_1|,|\tilde v_2|\}]$. This path π' leads from $(p_1,p_2,p_3,1,0)$ to (q_1,q_2,q_3,b,t) where either $(b,t)=(1,|\tilde v_1|-|\tilde v_2|)$ or, for some "guessed" $j \in \{|\tilde v_2|+1,\ldots,|\tilde v_1|\}$, $(b,t)=(\tilde v_1(j), j-|\tilde v_2|)$ (and $t>0$) or, for some "guessed" $j \in \{|\tilde v_1|+1,\ldots,|\tilde v_2|\}$, $(b,t)=(\tilde v_2(j),|\tilde v_1|-j)$ (and $t<0$) or, for some "guessed" $j \in [\min\{|\tilde v_1|,|\tilde v_2|\}]$, $\tilde v_1(j) \neq \tilde v_2(j)$ and $(b,t)=(2,0)$.

Let M be trim. The graphs G_4 and G'_4 were constructed in order to establish the following Fact:

Fact 3.2. *Let $q_1,q_2,q_3 \in Q$. Then, the following assertions are equivalent:*

(i) $\exists v \in \Sigma^*\ \exists \tilde v_1,\tilde v_2,\tilde v_3 \in \Delta^*: |\tilde v_1| \neq |\tilde v_2| \ \&\ \forall i \in \{1,2,3\}: (q_i,v,\tilde v_i,q_i) \in \delta$.

(ii) *There are paths π_1,π_2,π_3 in M all consuming the same word in Σ^* such that* $\operatorname{diff}(\pi_1,\pi_2)$ *exceeds* $(n^3-1)\cdot\operatorname{diff}(\delta)$ *and, for all $i \in \{1,2,3\}$, π_i leads from q_i to q_i.*

(iii) $(q_1,q_2,q_3,0) \xrightarrow[G_4]{} (q_1,q_2,q_3,\infty)$.

(iv) $(q_1,q_2,q_3,0) \xleftrightarrow[G'_4]{} (q_1,q_2,q_3,\infty)$.

Proof. By construction of G_4, (ii) and (iii) are equivalent.

(i) implies (ii): Let $v \in \Sigma^*$ be as in (i). Then, (ii) is ensured for some paths consuming $v^{1+(n^3-1)\cdot \mathrm{diff}(\delta)}$.

(ii) implies (i): By (ii), $r_1, r_2, r_3 \in Q$ exist such that, for some words $y_1, y_2, y_3 \in \Sigma^*$ and $z_1, \ldots, z_9 \in \Delta^*$, the following holds:

$$\begin{aligned}
&z_2 \text{ and } z_5 \text{ have distinct lengths,}\\
&(q_1, y_1, z_1, r_1), (r_1, y_2, z_2, r_1), (r_1, y_3, z_3, q_1) \in \delta,\\
&(q_2, y_1, z_4, r_2), (r_2, y_2, z_5, r_2), (r_2, y_3, z_6, q_2) \in \delta, \text{ and}\\
&(q_3, y_1, z_7, r_3), (r_3, y_2, z_8, r_3), (r_3, y_3, z_9, q_3) \in \delta.
\end{aligned}$$

Defining $v := y_1 y_3$ or $v := y_1 y_2 y_3$, this implies (i).

Obviously, (iii) implies (iv). (iv) implies (iii): By (iv), $r_1, r_2, r_3 \in Q$ exist so that $(q_1, q_2, q_3, 0) \xrightarrow[G_4]{} (r_1, r_2, r_3, \infty) \xrightarrow[G_4']{} (q_1, q_2, q_3, \infty)$. Under the last arrow, G_4' may be replaced by G_4. $\square$

Using Fact 3.2, G_3 and V_4' allow to rewrite (IV 1) as follows:

(*) There are states $p, q_1, q_2, q_3, q \in Q$ such that $(p, p, q) \xrightarrow[G_3]{} (q_1, q_2, q_3) \xrightarrow[G_3]{} (p, q, q)$ and $(q_1, q_2, q_3) \in V_4'$.

It is easy to verify that (*) is equivalent to $\overline{(\mathrm{IV}\,1)}$:

$\overline{(\mathrm{IV}\,1)}$ There is a strong component U of G_3' so that $U \cap V_4' \neq \phi$ and $U^2 \cap E_3' \neq \phi$.

Assume that M is trim and does not comply with (IV 1). The graphs G_5 and G_5' were constructed in order to establish the following Fact:

Fact 3.3. *Let $p, q \in Q$. Then, the following assertions are equivalent:*

(i) $\exists v \in \Sigma^* \, \exists \tilde{v}_1, \tilde{v}_2, \tilde{v}_3 \in \Delta^*$: $\tilde{v}_1$ and $\tilde{v}_2$ differ at some position $j \in [\min\{|\tilde{v}_1|, |\tilde{v}_2|\}]$ & $(p, v, \tilde{v}_1, p), (p, v, \tilde{v}_2, q), (q, v, \tilde{v}_3, q) \in \delta$.

(ii) $(p, p, q, 1, 0) \xrightarrow[G_5]{} (p, q, q, 2, 0)$.

(iii) $(p, p, q, 1, 0) \xleftrightarrow[G_5']{} (p, q, q, 2, 0)$.

Proof. Consider the following assertion:

(*) There are paths π_1, π_2, π_3 in M realizing some $(v, \tilde{v}_1), (v, \tilde{v}_2), (v, \tilde{v}_3) \in \Sigma^* \times \Delta^*$ and leading from p to p, from p to q and from q to q, respectively, such that $\mathrm{diff}(\pi_1, \pi_2)$ is at most $(n^3-1)\cdot \mathrm{diff}(\delta)$ and $\tilde{v}_1$ and $\tilde{v}_2$ differ at some position $j \in [\min\{|\tilde{v}_1|, |\tilde{v}_2|\}]$.

By construction of G_5, (*) implies (ii) and (ii) implies (i). (i) implies (*): Let π_1 and π_2 be paths in M realizing $(v, \tilde{v}_1)$ and $(v, \tilde{v}_2)$ and leading from p to p and from p to q, respectively. Since M is trim and does not comply with (IV 1), we can apply a slight variation of Lemma 2.10 to $A := Q$, $y := v$, $p' := p_1 := p_2 := p$, $q' := p_3 := p_4 := q$, $l_1 := l_2 := 0$, $z_2 := \tilde{v}_1$ and $z_8 := \tilde{v}_3$ (note that the partition of Q is missing) which yields: $\mathrm{diff}(\pi_1, \pi_2) \leq (n^3-1)\cdot \mathrm{diff}(\delta)$.

Obviously, (ii) implies (iii). (iii) implies (ii): By (iii), $r_1, r_2 \in Q$ exist so that $(p, p, q, 1, 0) \xrightarrow[G_5]{} (r_1, r_2, r_2, 2, 0) \xrightarrow[G_5']{} (p, q, q, 2, 0)$. Under the last arrow, G_5' may be replaced by G_5. $\square$

Using Fact 3.3, it is easy to verify that (IV 2) is equivalent to $\overline{\text{(IV 2)}}$:

$\overline{\text{(IV 2)}}$ There is a strong component U of G_5' so that $U^2 \cap E_5' \neq \phi$.

Let M be now without any restriction. In order to prove (1) and (2), we still need the two following Facts:

Fact 3.4. $\#(E_4 \cup E_4') = O(n^9 \cdot (1 + \mathrm{diff}(\delta))^2)$ *and* $\#(E_5 \cup E_5') = O(n^9 \cdot (1 + \mathrm{diff}(\delta)) \cdot (1 + \mathrm{length}(\delta)) \cdot \#\Delta)$.

Proof. Let $(p_1, p_2, p_3), (q_1, q_2, q_3) \in Q^3$. By the definition of diff(δ) and length(δ) we observe:

$$\begin{aligned}
\forall t \in Z_4:\ & \#\{t' \in Z_4 | ((p_1, p_2, p_3, t), (q_1, q_2, q_3, t')) \in E_4\} \leqq 1 + 2 \cdot \mathrm{diff}(\delta)\ \& \\
\forall t \in Z_5:\ & \#\{(b, b', t') \in (\Delta \cup \{1, 2\})^2 \times Z_5 | ((p_1, p_2, p_3, b, t), (q_1, q_2, q_3, b', t')) \in E_5\} \\
& \leqq (1 + 2 \cdot \mathrm{diff}(\delta)) + 2 \cdot (2 \cdot \mathrm{length}(\delta) + 1) \cdot \#\Delta + \#\Delta + 2 \\
& \leqq 6 \cdot (1 + \mathrm{length}(\delta)) \cdot \#\Delta.
\end{aligned}$$

From this follows:

$$\begin{aligned}
&\#(E_4 \cup E_4') = \#E_4 + \#E_4' \leqq n^6 \cdot \#Z_4 \cdot (1 + 2 \cdot \mathrm{diff}(\delta)) + n^3 \leqq 4 \cdot n^9 \cdot (1 + \mathrm{diff}(\delta))^2\ \& \\
&\#(E_5 \cup E_5') = \#E_5 + \#E_5' \leqq n^6 \cdot \#Z_5 \cdot 6 \cdot (1 + \mathrm{length}(\delta)) \cdot \#\Delta + n^2 \\
&\qquad \leqq 12 \cdot n^9 \cdot (1 + \mathrm{diff}(\delta)) \cdot (1 + \mathrm{length}(\delta)) \cdot \#\Delta. \quad \square
\end{aligned}$$

Fact 3.5.

(i) G_4' *can be constructed in time* $O(\mathrm{size}(\delta) + (n^6 \cdot \#\Sigma + n^9) \cdot (1 + \mathrm{diff}(\delta))^2)$.

(ii) G_5' *can be constructed in time* $O(\mathrm{size}(\delta) + n^6 \cdot \#\Sigma \cdot (1 + \mathrm{length}(\delta))^2 \cdot \#\Delta + n^9 \cdot (1 + \mathrm{diff}(\delta)) \cdot (1 + \mathrm{length}(\delta)) \cdot \#\Delta)$.

We omit the proof of Fact 3.5.

The following Lemma shows (1):

Lemma 3.6. *Let* $M = (Q, \Sigma, \Delta, \delta, Q_I, Q_F)$ *be an NGSM with n states. It is decidable in worst case time* $O((n^9 + n^6 \cdot \#\Sigma) \cdot (1 + \mathrm{diff}(\delta))^2 + \mathrm{size}(\delta))$ *(on a RAM without multiplications and divisions using the uniform cost criterion) whether or not M complies with* (IV 1).

Proof. For background information on RAM's we refer to [P78] and [AHoU74]. We present an informal algorithm deciding whether or not M complies with $\overline{\text{(IV 1)}}$. Note that this algorithm uses well-known graph algorithms (see [AHoU74]) as subroutines:

1. Remove all useless states from M. Assume that M is trim.
2. Construct E_3', G_3' and G_4' (see Fact 3.5).

3. Compute the strong components of G'_3 and G'_4. Construct V'_4.

4. Decide whether or not M complies with $\overline{(\text{IV}\,1)}$, i.e., check whether or not there is a strong component U of G'_3 so that $U \cap V'_4 \neq \phi$ and $U^2 \cap E'_3 \neq \phi$.

It can be easily seen that the above algorithm has worst case time complexity $O((n^9 + n^6 \cdot \#\Sigma) \cdot (1 + \text{diff}(\delta))^2 + \text{size}(\delta))$. $\square$

The following Lemma shows (2):

Lemma 3.7. *Let* $M = (Q, \Sigma, \Delta, \delta, Q_I, Q_F)$ *be an NGSM with n states which does not comply with the criterion* (IV 1). *It is decidable in worst case time* $O(n^9 \cdot (1 + \text{diff}(\delta)) \cdot (1 + \text{length}(\delta)) \cdot \#\Delta + n^6 \cdot \#\Sigma \cdot (1 + \text{length}(\delta))^2 \cdot \#\Delta + \text{size}(\delta))$ *(on a RAM without multiplications and divisions using the uniform cost criterion) whether or not M complies with* (IV 2).

Proof. We present an informal algorithm deciding whether or not M complies with $\overline{(\text{IV}\,2)}$ (for further comments see Lemma 3.6):

1. Remove all useless states from M. Assume that M is trim.
2. Construct E'_5 and G'_5 (see Fact 3.5).
3. Compute the strong components of G'_5.
4. Decide whether or not M complies with $\overline{(\text{IV}\,2)}$, i.e., check whether or not there is a strong component U of G'_5 so that $U^2 \cap E'_5 \neq \phi$.

It can be easily seen that the above algorithm has worst case time complexity $O(n^9 \cdot (1 + \text{diff}(\delta)) \cdot (1 + \text{length}(\delta)) \cdot \#\Delta + n^6 \cdot \#\Sigma \cdot (1 + \text{length}(\delta))^2 \cdot \#\Delta + \text{size}(\delta))$. $\square$

4. Large Finite Valuedness

In this Section we construct two infinite series of NGSM's with large finite valuedness. These constructions lead to the two following Theorems:

Theorem 4.1. *Let* $d \in \mathbb{N}$, *and let* Δ *be a nonempty, finite set of output symbols. For all* $N \in \mathbb{N}$ *there is, for some* $n \in \{N+6, \ldots, 5 \cdot N + 9\}$, *an NGSM* $M_n = (Q, \Sigma, \Delta, \delta, Q_I, Q_F)$ *with n states such that the following assertions are true:*

(i) $\#\Sigma = 5 \cdot \#(\Delta^{\leq d})$, $\#\delta = 3 \cdot (n+1) \cdot \#(\Delta^{\leq d})$, $\text{val}(\delta) = 1$, $\text{diff}(\delta) = d$, $\text{im}(\delta) = \Delta^{\leq d}$.

(ii) $\#\Delta^{n^3 \cdot \text{diff}(\delta)/343} < \text{val}(M_n) < \infty$.

Theorem 4.2. *Let* Δ *be a set of one output symbol. For all* $n \in \mathbb{N} \setminus \{1\}$ *there is an NGSM* $M_n = (Q, \Sigma, \Delta, \delta, Q_I, Q_F)$ *with n states such that the following assertions are true:*

(i) $\#\Sigma = n$, $\text{val}(\delta) = 2$, $\text{diff}(\delta) = 1$, $\text{im}(\delta) = \{\varepsilon\} \cup \Delta$.

(ii) $2^{n-1} \leq \text{val}(M_n) < \infty$.

Theorem 4.1 states that, if $\#\Delta > 1$ and $\text{diff}(\delta) > 0$, the upper bound for the valuedness of a finite-valued NGSM presented in Theorem 2.1 is asymptotically optimal apart from a constant factor in the exponent. Theorem 4.2 states that in the case $\#\Delta = 1$ the bound of Theorem 2.1 is "almost" optimal.

In addition to the Theorems 4.1 and 4.2, it is shown in [We87], Theorems 12.2 and 12.4:

- Let Δ be a nonempty, finite set of output symbols. For all $n \in \mathbb{N}$ there is a chain NSM $M_n = (Q, \Sigma, \Delta, \delta, Q_I, Q_F)$ with n states such that $\#\Sigma = 3$, $\mathrm{val}(\delta) = \#\Delta$ and $(2 \cdot \#\Delta - 1)^{n-1} \leqq \mathrm{val}(M_n) \leqq n^n \cdot \#\Delta^{n-1}$.
- Let $d \in \mathbb{N}$, let Σ be a set of one input symbol, and let Δ be a nonempty, finite set of output symbols. For all $n \in \mathbb{N} \setminus \{1\}$ there is a chain NGSM $M_n = (Q, \Sigma, \Delta, \delta, Q_I, Q_F)$ with n states such that $\mathrm{val}(\delta) = \#\Delta^d$, $\mathrm{diff}(\delta) = d$ and $\mathrm{val}(M_n) = \#\Delta^{d \cdot (n-1)}$.

We turn now to the proof of the Theorems 4.1 and 4.2. Our proof of Theorem 4.1 is based on the following Lemma:

Lemma 4.3. *Let $n_1, n_2, n_3 \in \mathbb{N} \setminus \{1\}$ so that $gcd(n_1, n_2, n_3) = 1$, let $d \in \mathbb{N}$, and let Δ be a nonempty, finite set of output symbols. Then, there is a trim NGSM $M = (Q, \Sigma, \Delta, \delta, Q_I, Q_F)$ with $n_1 + n_2 + n_3$ states such that the following assertions are true:*

(i) $\#\Sigma = 5 \cdot \#(\Delta^{\leqq d})$, $\#\delta = 3 \cdot (n_1 + n_2 + n_3 + 1) \cdot \#(\Delta^{\leqq d})$, $\mathrm{val}(\delta) = 1$, $\mathrm{diff}(\delta) = d$, $\mathrm{im}(\delta) = \Delta^{\leqq d}$.

(ii) $\#(\Delta^{\leqq d \cdot n_1 \cdot n_2 \cdot n_3}) \leqq \mathrm{val}(M) < \infty$.

Proof. We construct $M = (Q, \Sigma, \Delta, \delta, Q_I, Q_F)$ (see Fig. 14):

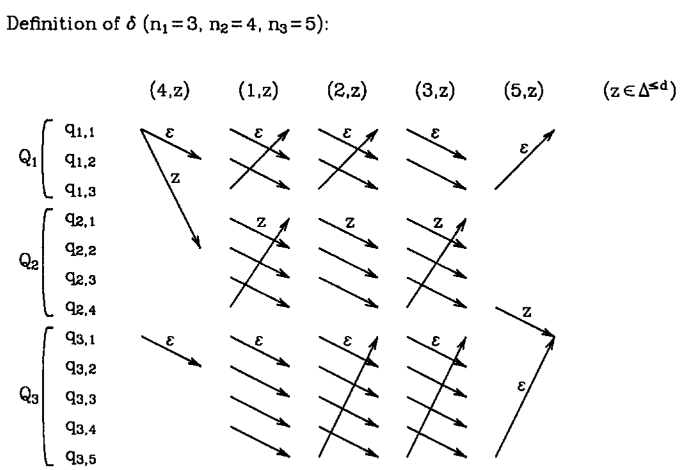

Fig. 14

$Q := Q_1 \dot\cup Q_2 \dot\cup Q_3$ where $Q_i := \{q_{i,1}, \ldots, q_{i,n_i}\}$ $(i = 1, 2, 3)$,

$\Sigma := [5] \times \Delta^{\leqq d}$, $Q_I := \{q_{1,1}\}$, $Q_F := \{q_{3,1}\}$,

$\delta := \bigcup_{a \in \Sigma} \delta(a)$ where for all $z \in \Delta^{\leqq d}$:

$$\delta((\mu, z)) := \{(q_{1,j}, (\mu, z), \varepsilon, q_{1,j+1}) \mid 1 \leqq j \leqq n_1 - 1\}$$
$$\cup \{(q_{2,j}, (\mu, z), z, q_{2,j+1}) \mid 1 \leqq j \leqq n_2 - 1\}$$
$$\cup \{(q_{3,j}, (\mu, z), \varepsilon, q_{3,j+1}) \mid 1 \leqq j \leqq n_3 - 1\}$$
$$\cup \begin{cases} \{(q_{1,n_1}, (\mu, z), \varepsilon, q_{1,1}), (q_{2,n_2}, (\mu, z), z, q_{2,1})\} & \text{if } \mu = 1 \\ \{(q_{1,n_1}, (\mu, z), \varepsilon, q_{1,1}), (q_{3,n_3}, (\mu, z), \varepsilon, q_{3,1})\} & \text{if } \mu = 2 \\ \{(q_{2,n_2}, (\mu, z), z, q_{2,1}), (q_{3,n_3}, (\mu, z), \varepsilon, q_{3,1})\} & \text{if } \mu = 3 \end{cases} \quad (\mu = 1, 2, 3),$$
$$\delta((4, z)) := \{(q_{1,1}, (4, z), \varepsilon, q_{1,2}), (q_{1,1}, (4, z), z, q_{2,2}), (q_{3,1}, (4, z), \varepsilon, q_{3,2})\},$$
$$\delta((5, z)) := \{(q_{1,n_1}, (5, z), \varepsilon, q_{1,1}), (q_{2,n_2}, (5, z), z, q_{3,1}), (q_{3,n_3}, (5, z), \varepsilon, q_{3,1})\}.$$

Obviously, M is a trim NGSM which meets the assertion (i). By construction, Q_1, Q_2, Q_3 are the strong components of M. M is planned to recognize a suitable input word by "counting" through all elements of $Q_1 \times Q_2 \times Q_3$ except $(q_{1,1}, q_{2,1}, q_{3,1})$.

Claim 1. M is finite-valued.

According to Theorem 2.2, it suffices to show that M does not comply with any of the criteria (IV 1), (IV 2). First of all, we observe for all $a \in \Sigma$:

$$(*) \quad \forall p \in Q_2: \quad \# \{(q, z) \in Q \times \Delta^* \mid (p, a, z, q) \in \delta\} \leqq 1,$$
$$\forall q \in Q_2: \quad \# \{(p, z) \in Q \times \Delta^* \mid (p, a, z, q) \in \delta\} \leqq 1.$$

Assume that M complies with (IV1). Let $p, q_1, q_2, q_3, q \in Q$, $u, v, w \in \Sigma^*$ and $\tilde{u}_1, \ldots, \tilde{v}_1, \ldots, \tilde{w}_1, \ldots \in \Delta^*$ be selected according to that criterion. From (*) follows: $p, q \in Q_1 \cup Q_3$. Therefore, $\tilde{v}_1 = \varepsilon$ and $\tilde{v}_2 \neq \varepsilon$. By construction of M, this implies: p, $q_1 \in Q_1$, $q_2 \in Q_2$, q_3, $q \in Q_3$, $v \in ([3] \times \Delta^{\leqq d})^*$. Let $j_1 \in [n_1]$, $j_2 \in [n_2]$, $j_3 \in [n_3]$ so that $(q_1, q_2, q_3) = (q_{1,j_1}, q_{2,j_2}, q_{3,j_3})$. Since $gcd(n_1, n_2, n_3) = 1$, the triples $(q_{1,(j_1+j) \bmod n_1}, q_{2,(j_2+j) \bmod n_2}, q_{3,(j_3+j) \bmod n_3}) \in Q_1 \times Q_2 \times Q_3$ $(j = 0, \ldots, n_1 n_2 n_3 - 1)$ are pairwise distinct. Therefore, since $(q_{1,j_1}, v, \tilde{v}_1, q_{1,j_1})$, $(q_{2,j_2}, v, \tilde{v}_2, q_{2,j_2})$, $(q_{3,j_3}, v, \tilde{v}_3, q_{3,j_3}) \in \delta$, there is a letter (μ, z) in v such that $(q_{1,n_1}, (\mu, z), \varepsilon, q_{1,1})$, $(q_{2,n_2}, (\mu, z), z, q_{2,1})$, $(q_{3,n_3}, (\mu, z), \varepsilon, q_{3,1}) \in \delta$. (Contradiction!)

Assume that M complies with (IV2). Let p, $q \in Q$, $v \in \Sigma^*$ and $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3 \in \Delta^*$ be selected according to that criterion. From (*) follows: p, $q \in Q_1 \cup Q_3$. By construction of M, this implies that $\tilde{v}_1 = \tilde{v}_3 = \varepsilon$. (Contradiction!)

Claim 2. $\mathrm{val}(M) \geqq \#(\Delta^{\leqq d \cdot n_1 \cdot n_2 \cdot n_3})$.

Let $z \in \Delta^{\leqq d \cdot n_1 \cdot n_2 \cdot n_3}$, and let $z_1, \ldots, z_{n_1 n_2 n_3} \in \Delta^{\leqq d}$ so that $z = z_1 \ldots z_{n_1 n_2 n_3}$. We construct $x := x(z) := x_1 \ldots x_{n_1 n_2 n_3} \in \Sigma^*$:

$$x_1 := (4, z_1), x_{n_1 n_2 n_3} := (5, z_{n_1 n_2 n_3}),$$

$$x_j := \begin{cases} (1, z_j) & \text{if } n_3 \nmid j \\ (2, z_j) & \text{if } n_3 \mid j \;\&\; n_2 \nmid j \\ (3, z_j) & \text{if } n_3 \mid j \;\&\; n_2 \mid j \;\&\; n_1 \nmid j \end{cases} \qquad (j = 2, \ldots, n_1 n_2 n_3 - 1).$$

It is easy to see that, in M, there are paths for $x|\varepsilon$ leading from $q_{1,1}$ to $q_{1,1}$, for $x|z$ leading from $q_{1,1}$ to $q_{3,1}$, and for $x|\varepsilon$ leading from $q_{3,1}$ to $q_{3,1}$ (see Fig. 15). Define $y := \prod_{z \in \Delta^{\leqq d \cdot n_1 \cdot n_2 \cdot n_3}} x(z)$. By the above, each $z \in \Delta^{\leqq d \cdot n_1 \cdot n_2 \cdot n_3}$ is a value for y in M. This implies: $\#(\Delta^{\leqq d \cdot n_1 \cdot n_2 \cdot n_3}) \leqq \mathrm{val}_M(y) \leqq \mathrm{val}(M)$.

$G_M(x)$ $(n_1=2,\ n_2=3,\ n_3=5)$:

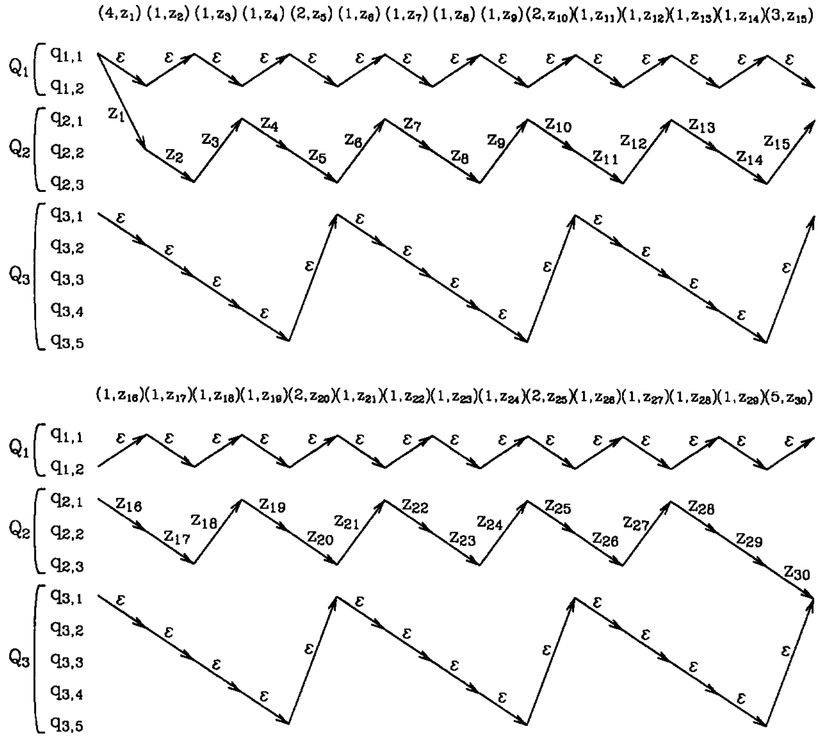

Fig. 15

From Claims 1 and 2 follows that M meets the assertion (ii). □

Proof of Theorem 4.1. According to Bertrand's postulate (see [HaWr60], Theorem 418), there are prime numbers n_1, n_2, n_3 so that $\lceil N/3\rceil < n_1 \leqq 2\cdot\lceil N/3\rceil$, $n_1 < n_2 \leqq 2\cdot n_1$ and $n_2 < n_3 \leqq 2\cdot n_2$. Set $n := n_1 + n_2 + n_3$, then: $N+6 \leqq 3\cdot\lceil N/3\rceil + 6 \leqq 3\cdot n_1 + 3 \leqq n = n_1 + n_2 + n_3 \leqq 7\cdot n_1 \leqq 14\cdot N/3 + 28/3 < 5\cdot N + 10$. Since $n \leqq 7\cdot n_1$, we know: $n_1\, n_2\, n_3 > n_1^3 \geqq n^3/343$. Therefore, we can select M_n according to Lemma 4.3. □

For the proof of Theorem 4.2 we take advantage of a nondeterministic finite automaton constructed in [WeSe88.1]:

Proof of Theorem 4.2. Let $\Delta = \{b_1\}$, and let $n \in \mathbb{N}\setminus\{1\}$. Analogously to Lemma 5.2 of [WeSe88.1], we construct the NGSM $M := M_n = (Q, \Sigma, \Delta, \delta, Q_I, Q_F)$ with n states (see Fig. 16):

$$Q := \{p_1, \dots, p_n\}, \quad \Sigma := \{a_1, \dots, a_n\}, \quad Q_I := \{p_1\}, \quad Q_F := \{p_n\},$$

$$\delta := \{p_1\} \times \{a_1\} \times \{b_1\} \times \{p_1, \dots, p_{n-1}\} \cup \bigcup_{i=2}^{n} [\{(p_j, a_i, b_1, p_j) \mid 1 \leqq j \leqq i-2\}$$
$$\cup \{(p_{i-1}, a_i, b_1, p_j) \mid i \leqq j \leqq n-1\} \cup \{(p_n, a_i, \varepsilon, p_n)\}]$$
$$\cup \{(p_{n-1}, a_n, b_1, p_n), (p_{n-1}, a_n, \varepsilon, p_n)\}.$$

Definition of δ (n=6):

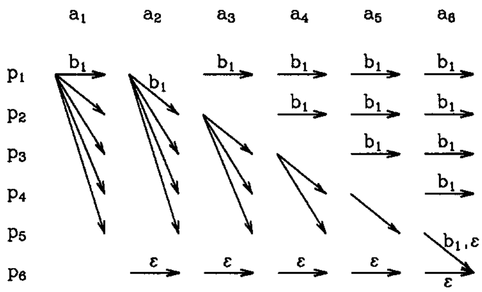

Fig. 16

Obviously, M meets the assertion (i) of the Theorem. Note that every transition of M is of the form (p_i, a, z, p_j) where $1 \leqq i \leqq j \leqq n$ and $(a, z) \in \Sigma \times \Delta^*$. In particular, M has only unitary strong components, i.e., "$\xleftrightarrow[M]{}$" is the equality relation. M is planned to recognize a suitable input word by "counting" through almost all sets of its states.

Claim 1. M is finite-valued.

According to Theorem 2.2, it suffices to show that M does not comply with any of the criteria (IV 1), (IV 2). In fact, since $\#\Delta = 1$, M does not comply with (IV 2). Assume that M complies with (IV 1). Let $p, q \in Q$ be selected according to that criterion. Then, $v \in \Sigma^*$ and $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3 \in \Delta^*$ exist such that $|\tilde{v}_1| \neq |\tilde{v}_2|$ and $(p, v, \tilde{v}_1, p)$, $(p, v, \tilde{v}_2, q)$, $(q, v, \tilde{v}_3, q) \in \delta$. This implies that p and q are distinct. (Contradiction!)

Claim 2. $\mathrm{val}(M) \geqq 2^{n-1}$.

We construct the words $y_0 := \varepsilon$, $y_i := y_{i-1}\, a_n\, a_{n-i}\, y_{i-1}$ $(i = 1, \ldots, n-2)$, and $y := a_1\, y_{n-2}\, a_n$. Considering selected accepting paths in M it is easy to show that $b_1, b_1^2, \ldots, b_1^{|y|}$ are values for y in M (see Fig. 17). Since $|y| = 2^{n-1}$, this implies: $2^{n-1} \leqq \mathrm{val}_M(y) \leqq \mathrm{val}(M)$.

$G_M(y)$ (n=6):

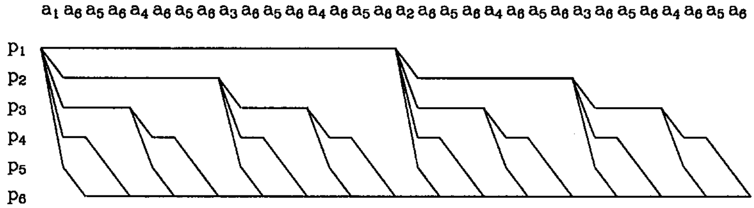

Fig. 17

From Claims 1 and 2 follows that M meets the assertion (ii) of the Theorem. $\square$

5. Normalized Finite Transducers

Let $M=(Q, \Sigma, \Delta, \delta, Q_I, Q_F)$ be an NFT. We introduce the criterion (ε-IV):

(ε-IV) There is a useful state $q \in Q$ such that, for some word $\tilde{v} \in \Delta^+$, $(q, \varepsilon, \tilde{v}, q) \in \hat{\delta}$.

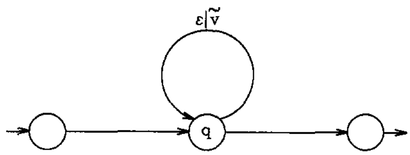

Fig. 18

Moreover, we introduce the criteria (IV 1), (IV 2), (IV), (IV 2)′ and (IV 3) for M by declaring that they are identical with the corresponding criteria for an NGSM (see Sect. 2) except that δ is replaced by $\hat{\delta}$. Note that (ε-IV) is a special case of (IV 3) and that (IV 3) is a special case of (IV 1) and (IV).

By reduction and using the above criteria, we can generalize the main results of this paper (Theorems 2.1, 2.2 and 3.1) to NFT's:

Theorem 5.1. *Let $M=(Q, \Sigma, \Delta, \delta, Q_I, Q_F)$ be a finite-valued NFT with n states. Then, the valuedness of M is at most*

$$(1/4)\cdot(5^{1/2}\cdot 3\cdot 2^{7/2})^n\cdot n^n\cdot \mathrm{val}(\delta)^{n-1}\cdot \#\Delta^{(n^3-1)\cdot \mathrm{length}(\delta)/3} \quad \text{if } \#\Delta>1,$$
$$(1/4)\cdot(5^{1/2}\cdot 3^{8/3}\cdot 2^{13/2})^n\cdot n^n\cdot \mathrm{val}(\delta)^{n-1}\cdot(1+\mathrm{length}(\delta))^{n-1} \quad \text{if } \#\Delta=1.$$

Theorem 5.2. *Let M be an NFT. The assertions* (i)–(iii) *are equivalent:*
- (i) *M is infinite-valued.*
- (ii) *M complies with at least one of the criteria* (IV 1), (IV 2).
- (iii) *M complies with* (IV).

Theorem 5.3. *Let $M=(Q, \Sigma, \Delta, \delta, Q_I, Q_F)$ be an NFT with n states. It is decidable in time $O((n^9+n^6\cdot \#\Sigma)\cdot(1+\mathrm{length}(\delta))^2\cdot \#\Delta+\mathrm{size}(\delta))$ whether or not M is infinite-valued.*

We prove the Theorems 5.1–5.3 by reduction to the Theorems 2.1, 2.2 and 3.1. The reduction is based on the two following Lemmas:

Lemma 5.4. *Let M be an NFT. Then, the following assertions are true:*
- (i) *M complies with* (IV), *if and only if it complies with* (IV 1) *or* (IV 2).
- (ii) *If M complies with* (IV 1) *or* (IV 2), *then it is infinite-valued.*

Proof. Using the criteria (IV2)′ and (IV3), the assertion (i) can be shown just like in the proof of Lemma 2.3 (except that δ is replaced by $\hat{\delta}$). Using the criterion (IV3), the assertion (ii) can be shown just like in the proof of Lemma 2.4 (except that δ is replaced by $\hat{\delta}$). □

We add to the proof of Lemma 5.4: If M complies with (ε-IV), then $(u, \tilde{u})$, $(w, \tilde{w}) \in \Sigma^* \times \Delta^*$ and $\tilde{v} \in \Delta^+$ exist such that, for all $i \in \mathbb{N}$, $\tilde{u}\tilde{v}^i\tilde{w}$ is a value for uw in M. This implies that the valuedness of the word uw in M (and hence also the valuedness of M) is infinite.

Lemma 5.5 *Let $M = (Q, \Sigma, \Delta, \delta, Q_I, Q_F)$ be a trim NFT with n states which does not comply with (ε-IV). Then, an NGSM $M' = (Q, \Sigma', \Delta, \delta', Q_I, Q_F)$ effectively exists such that the following assertions are true:*

(i) $\#\Sigma' = \#\Sigma + 1$, $\mathrm{val}(\delta') = \mathrm{val}(\delta)$, $\mathrm{im}(\delta') = \mathrm{im}(\delta)$, $\mathrm{diff}(\delta') \leqq \mathrm{length}(\delta') = \mathrm{length}(\delta)$, $\mathrm{size}(\delta) \leqq \mathrm{size}(\delta') \leqq \mathrm{size}(\delta) + n$.

(ii) *M' has the same valuedness as M.*

(iii) *If M' complies with* (IV1) (*with* (IV2)), *then M also complies with* (IV1) (*with* (IV2)).

Proof. We construct the NGSM $M' = (Q, \Sigma', \Delta, \delta', Q_I, Q_F)$:

$$\Sigma' := \Sigma \dot{\cup} \{a_0\},$$
$$\delta' := (\delta \cap Q \times \Sigma \times \Delta^* \times Q) \cup \{(p, a_0, z, q) | (p, \varepsilon, z, q) \in \delta\} \cup \{(q, a_0, \varepsilon, q) | q \in Q\}.$$

Obviously, $\mathrm{im}(\delta') = \mathrm{im}(\delta)$, $\mathrm{diff}(\delta') \leqq \mathrm{length}(\delta') = \mathrm{length}(\delta)$, and $\mathrm{size}(\delta') = \mathrm{size}(\delta) + \#\{q \in Q | (q, \varepsilon, \varepsilon, q) \notin \delta\}$. Since M is trim and does not comply with (ε-IV), $\mathrm{val}(\delta')$ equals $\mathrm{val}(\delta)$. Thus, M' meets the assertion (i). The proof of the assertions (ii) and (iii) is based on the two following Claims:

Claim 1. $\forall x' \in (\Sigma')^* \; \exists x \in \Sigma^* \; \forall p, q \in Q \; \forall z \in \Delta^*$: $(p, x', z, q) \in \delta' \Rightarrow (p, x, z, q) \in \hat{\delta}$.

Proof. Let $x' = a_0^{i_1} x_1 \ldots a_0^{i_m} x_m a_0^{i_{m+1}} \in (\Sigma')^*$ ($m \in \mathbb{N}$, $x_1, \ldots, x_m \in \Sigma$, $i_1, \ldots, i_{m+1} \in \mathbb{N}$). Select $x := x_1 \ldots x_m \in \Sigma^*$.

Claim 2. $\forall x \in \Sigma^* \; \exists x' \in (\Sigma')^* \; \forall p, q \in Q \; \forall z \in \Delta^*$: $(p, x, z, q) \in \hat{\delta} \Rightarrow (p, x', z, q) \in \delta'$.

Proof. Let $x = x_1 \ldots x_m \in \Sigma^*$ ($x_1, \ldots, x_m \in \Sigma$). Define $x' := a_0^{n-1} x_1 \ldots a_0^{n-1} x_m a_0^{n-1} \in (\Sigma')^*$. Let $p, q \in Q$ and $z \in \Delta^*$. Let

$$\pi = \prod_{i=1}^{m} ((q_{i,1}, \varepsilon, z_{i,1}) \ldots (q_{i,l_i}, \varepsilon, z_{i,l_i})(q_{i,l_i+1}, x_i, z_i))$$
$$\cdot (q_{m+1,1}, \varepsilon, z_{m+1,1}) \ldots (q_{m+1,l_{m+1}}, \varepsilon, z_{m+1,l_{m+1}}) \, q_{m+1,l_{m+1}+1}$$

be a path for $x|z$ in M leading from p to q. Thus, $l_1, \ldots, l_{m+1} \in \mathbb{N}$, and the equalities $z = \prod_{i=1}^{m} (z_{i,1} \ldots z_{i,l_i} z_i) \cdot z_{m+1,1} \ldots z_{m+1,l_{m+1}}$, $p = q_{1,1}$ and $q = q_{m+1,l_{m+1}+1}$ hold. Since M is trim and does not comply with (ε-IV), we may assume that $l_1, \ldots, l_{m+1} \leqq n-1$. Therefore,

$$\pi' = \prod_{i=1}^{m} ((q_{i,1}, a_0, \varepsilon)^{n-1-l_i}(q_{i,1}, a_0, z_{i,1}) \ldots (q_{i,l_i}, a_0, z_{i,l_i})(q_{i,l_i+1}, x_i, z_i))$$
$$\cdot (q_{m+1,1}, a_0, \varepsilon)^{n-1-l_{m+1}}(q_{m+1,1}, a_0, z_{m+1,1}) \ldots$$
$$(q_{m+1,l_{m+1}}, a_0, z_{m+1,l_{m+1}}) \, q_{m+1,l_{m+1}+1}$$

is a path for $x'|z$ in M' leading from p to q.

From Claims 1 and 2 follows:

$$\forall x' \in (\Sigma')^* \, \exists x \in \Sigma^*: \ \mathrm{val}_{M'}(x') \leqq \mathrm{val}_M(x) \ \& \ \forall x \in \Sigma^* \, \exists x' \in (\Sigma')^*: \ \mathrm{val}_M(x) \leqq \mathrm{val}_{M'}(x').$$

This implies (ii). The assertion (iii) follows immediately from Claim 1. □

Proof of Theorem 5.1. Let M be a finite-valued NFT which is w.l.o.g. trim. Since (ε-IV) is a special case of (IV 1), Lemma 5.4 (ii) shows that M does not comply with (ε-IV). Let M' be the finite-valued NGSM which is attached to M in Lemma 5.5. Then, Theorem 2.1 applied to M' yields the announced upper bound for the valuedness of M. □

Proof of Theorem 5.2. According to Lemma 5.4, the assertions (ii) and (iii) of the Theorem are equivalent, and (ii) implies the assertion (i). Thus, it remains to show that (i) implies (ii): Let M be an infinite-valued NFT which is w.l.o.g. trim. Since (ε-IV) is a special case of (IV 1), we may assume that M does not comply with (ε-IV). Consider the infinite-valued NGSM M' which is attached to M in Lemma 5.5. By Theorem 2.2, M' complies with (IV 1) or (IV 2). Thus, according to Lemma 5.5, M complies with (IV 1) or (IV 2). □

Proof of Theorem 5.3. Let $M=(Q, \Sigma, \Delta, \delta, Q_I, Q_F)$ be an NFT with n states. We sketch an algorithm which decides in polynomial time whether or not M is infinite-valued:

- Remove all useless states from M. Assume that M is trim.
- Construct the directed graph $G_1=(Q, E_1)$ where $E_1 := \{(p,q) \in Q^2 \mid \exists z \in \Delta^*: (p, \varepsilon, z, q) \in \delta\}$ and the subset $E_1' := \{(p,q) \in Q^2 \mid \exists z \in \Delta^+: (p, \varepsilon, z, q) \in \delta\}$ of E_1.
- Compute the strong components of G_1.
- Decide whether or not M complies with (ε-IV), i.e., check whether or not there is a strong component U of G_1 so that $U^2 \cap E_1' \neq \phi$.
- If M complies with (ε-IV), then it is infinite-valued.
- Construct the NGSM M' attached to M in Lemma 5.5 (recall: $\mathrm{val}(M') = \mathrm{val}(M)$).
- Decide according to Theorem 3.1 whether or not M' is infinite-valued.

By Theorem 3.1, the above algorithm requires time $O((n^9 + n^6 \cdot \#\Sigma) \cdot (1 + \mathrm{length}(\delta))^2 \cdot \#\Delta + \mathrm{size}(\delta))$. □

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