Algorithms for solving Infinite Games on Graphs

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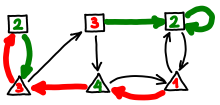

Why play games?

Set theory

Logic

Automata theory

Computational complexity

Game Theory

Verification

Controller synthesis

Algorithms

AI

Operations research

Economics

Why play games?

Vol B

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Operations research

Economics

Vol A

How to win infinite games?

Divide and conquer

Dynamic programming

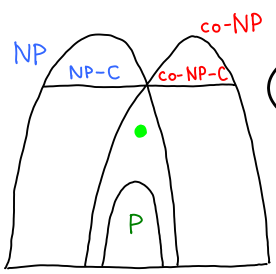
Algorithms

Mathematical programming

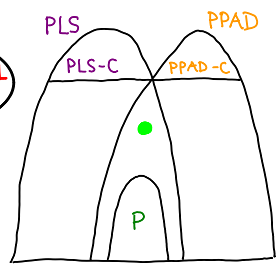
Optimization

Local search

How difficult is it to find a winning strategy?



Computational complexity



Computational complexity of finding equilibria

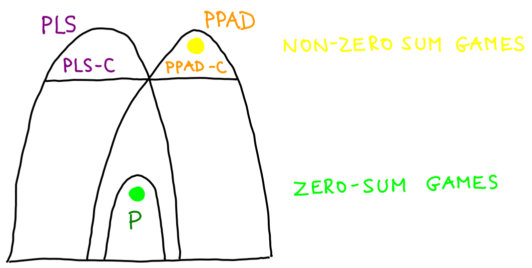
Thm [1950's]

Finding equilibria in zero-sum games (in s.f.) is poly-time equivalent to linear programming

Thm [DGP, CD 2006]

Finding equilibria in non-zero sum games (in s.f.) is poly-time equivalent to computing Brouwer fixed points

Computational complexity of finding equilibria



Plan

I Qualitative (ω-regular) games

1. Motivating example
2. Games on graphs
3. Reachability / safety games
4. Büchi / co-Büchi games
5. Parity games
6. ω-regular games
7. Two recent complexity improvements for parity games

Self-imposed limitations

- Finite graphs
- ω-Regular objectives
- Zero-sum
- Perfect-information
- Non-stochastic

Propositional formula evaluation

State s:

p	0
q	1
r	1

Property φ: $(r \vee p) \wedge (\neg p \wedge (q \vee \neg r))$

$$s \overset{?}{\models} \varphi$$

Propositional formula evaluation

State s:

p	0
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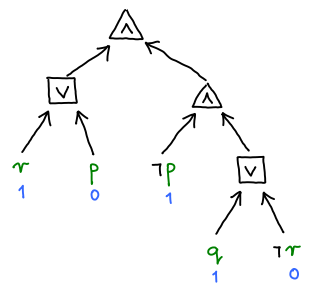
Property φ: $(r \vee p) \wedge (\neg p \wedge (q \vee \neg r))$

$C_{s,\varphi}$:



Propositional formula evaluation

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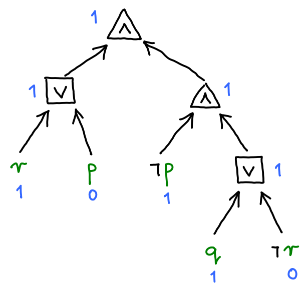


Fact

$s \models \varphi$

iff

the value of circuit $C_{s,\varphi}$ is 1

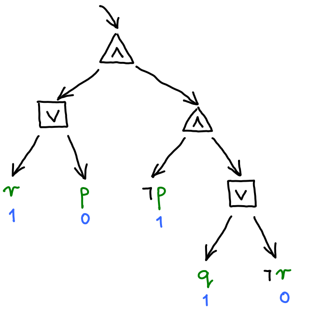
Propositional formula evaluation: Game

State s:

p	0
q	1
r	1

Property φ: $(r \vee p) \wedge (\neg p \wedge (q \vee \neg r))$

$\mathcal{G}_{s,\varphi}$:



Propositional formula evaluation: Game

State s:

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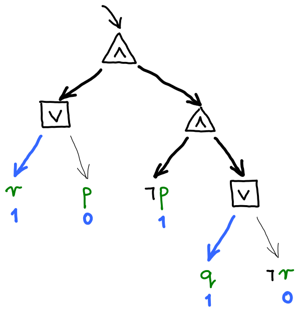
Property φ: $(r \vee p) \wedge (\neg p \wedge (q \vee \neg r))$

$\mathcal{G}_{s,\varphi}$:



Fact

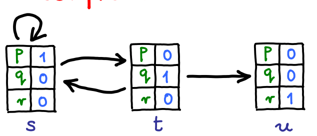
$s \models \varphi$

iff

$\boxed{\vee}$ has a strategy to reach 1 in $\mathcal{G}_{s,\varphi}$

Temporal formula evaluation

Kripke structure:



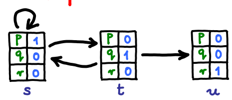
CTL property φ:

$E\big((p \vee q)\, U\, r\big)$

$s \overset{?}{\models} \varphi$

Temporal formula evaluation: Game

Kripke structure:



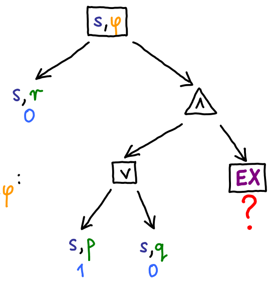
CTL property φ:

$\boxed{E\big((p \vee q)\, U\, r\big)}$

$\equiv$

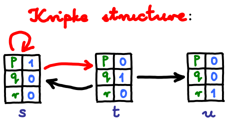
$r \vee \big((p \vee q) \wedge EX\,\varphi\big)$

$\mathcal{G}_{s,\varphi}$:



Temporal formula evaluation: Game

Kripke structure:



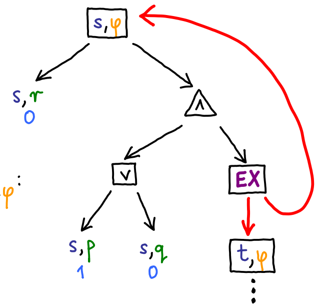
CTL property φ:

$\boxed{E\big((p \vee q)\, U\, r\big)}$

$\equiv$

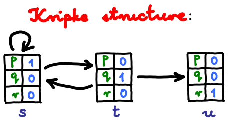
$r \vee \big((p \vee q) \wedge EX\,\varphi\big)$

$\mathcal{G}_{s,\varphi}$:



Temporal formula evaluation: Game

Kripke structure:



CTL property φ:

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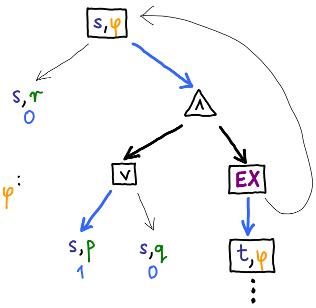
$\mathcal{G}_{s,\varphi}$:



Thm

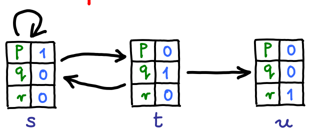
$s \models \varphi$

iff

$\boxed{\vee}$ has a strategy to reach 1 in $\mathcal{G}_{s,\varphi}$

FIXPOINT FORMULA EVALUATION

Kripke structure:



μ-calculus property φ:

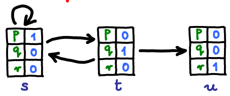
$$\boxed{\mu Z.\,(r \vee ((p \vee q) \wedge EX\, Z))}$$

$$\equiv\; r \vee ((p \vee q) \wedge EX\, \varphi)$$

$$\boxed{s \overset{?}{\models} \varphi}$$

FIXPOINT FORMULA EVALUATION

Kripke structure:

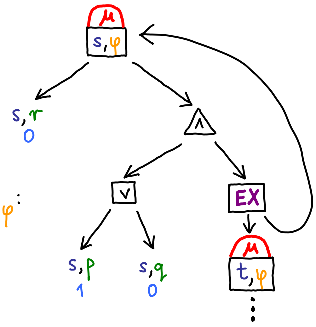


μ-calculus property φ:

$$\boxed{\mu Z.\,(r \vee ((p \vee q) \wedge EX\, Z))}$$

$$\equiv\; r \vee ((p \vee q) \wedge EX\, \varphi)$$

$\mathcal{G}_{s,\varphi}$:



THM

$s \models \varphi$

iff

$\boxed{\vee}$ has a **strategy** in $\mathcal{G}_{s,\varphi}$

- to reach 1, or
- outermost infinitely occurring fixpoint subformula is **not** a μ

PLAN

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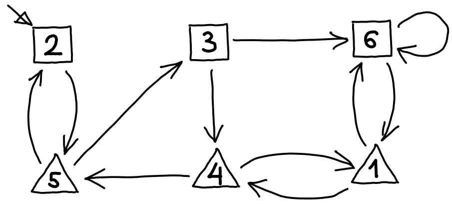
GAMES ON GRAPHS: GAME GRAPH

Players:

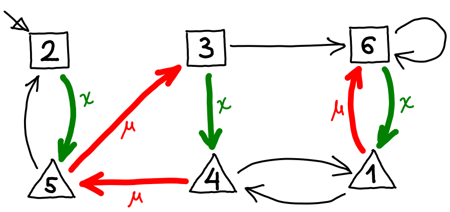
□

△

$n = |V|$ vertices, $m = |E|$ edges



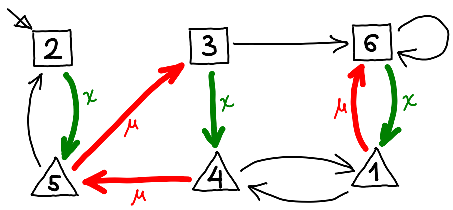
GAMES ON GRAPHS: POSITIONAL STRATEGIES



Positional **strategy** for □: $\boxed{\chi : V_\square \to V}$

Positional **strategy** for △: $\boxed{\mu : V_\triangle \to V}$

GAMES ON GRAPHS: PLAYS



Play $(2, \chi, \mu) = (2,5,3,4,5,3,4,5,\ldots) \in V^\omega$

Games on graphs: objectives

Objectives: $W_\square \subseteq V^\omega$

$W_\Delta = V^\omega \setminus W_\square$

Examples of objectives:

- **Reachability**: $W_\square = \{(v_0, v_1, v_2, \dots) : v_i \in T \text{ for some } i \in \mathbb{N}\}$
- **Safety**: $W_\square = \{(v_0, v_1, v_2, \dots) : v_i \in T \text{ for no } i \in \mathbb{N}\}$

where $T \subseteq V$

Games on graphs: winning strategies

$\chi \in \Sigma_\square$ is a **winning strategy** for $\square$ from $v \in V$,
if $\boxed{\mathrm{Play}(v, \chi, \mu) \in W_\square, \text{ for all } \mu \in \Sigma_\Delta}$

$\mu \in \Sigma_\Delta$ is a **winning strategy** for Δ from $v \in V$,
if $\boxed{\mathrm{Play}(v, \chi, \mu) \in W_\Delta, \text{ for all } \chi \in \Sigma_\square}$

Games on graphs: algorithmic problem

Given: game graph $(V = V_\square \uplus V_\Delta, E)$
objective $W_\square \subseteq V^\omega$
starting vertex $v \in V$

answer: if $\square$ has a winning strategy from v

Games on graphs: determinacy

A game $(V = V_\square \uplus V_\Delta, E; W_\square)$ is **determined**
if for every starting vertex $v \in V$

- $\square$ has a winning strategy from v, **or**
- Δ has a winning strategy from v

Martin's determinacy theorem

Thm [Martin 1975]
Every game $(V = V_\square \uplus V_\Delta, E; W_\square)$,
such that $W_\square \subseteq V^\omega$ is a Borel set,
is determined

Games on graphs: algorithmic problem

Given: game graph $(V = V_\square \uplus V_\Delta, E)$
objective $W_\square \subseteq V^\omega$

compute: $V = W_\square \uplus W_\Delta$ such that

- $\square$ has a winning strategy from $W_\square$, and
- Δ has a winning strategy from W_Δ

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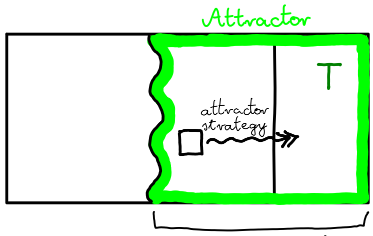
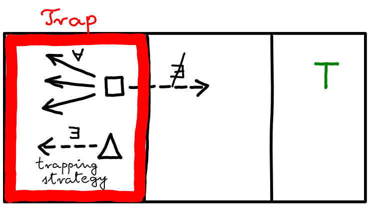
REACHABILITY GAMES



$W_\square = attr_\square(T)$

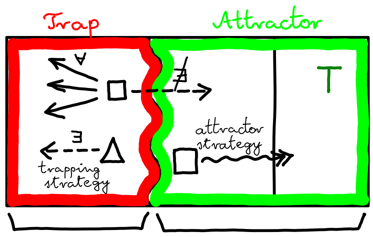
- **Reachability**: $W_\square = \{(v_0, v_1, v_2, \dots) : v_i \in T \text{ for some } i \in \mathbb{N}\}$

SAFETY GAMES



- **Reachability**: $W_\square = \{(v_0, v_1, v_2, \dots) : v_i \in T \text{ for some } i \in \mathbb{N}\}$
- **Safety**: $W_\Delta = \{(v_0, v_1, v_2, \dots) : v_i \in T \text{ for no } i \in \mathbb{N}\}$

REACHABILITY/SAFETY GAMES

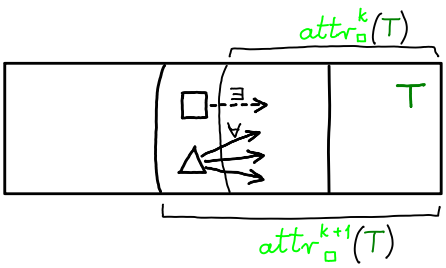


$W_\Delta = V \setminus attr_\square(T) \qquad W_\square = attr_\square(T)$

FACT

Reachability/safety games are **positionally determined**

SOLVING REACHABILITY/SAFETY GAMES: COMPUTING ATTRACTORS



FACT $attr_\square(T) = \bigcup_{k \geq 0} attr_\square^k(T)$

can be computed in $O(m)$ time

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Games on graphs: (co-)Büchi objectives

- **Büchi**:

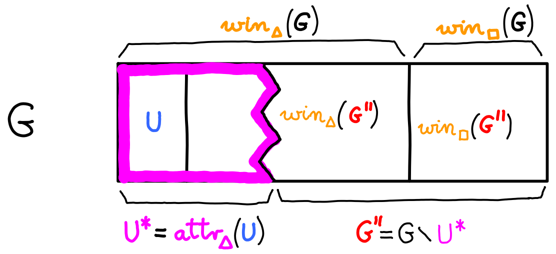
$$W_\square = \{(v_0, v_1, v_2, \ldots) : v_i \in T \text{ for } \textbf{infinitely many } i \in \mathbb{N}\}$$

- **co-Büchi**:

$$W_\Delta = \{(v_0, v_1, v_2, \ldots) : v_i \in T \text{ for } \textbf{finitely many } i \in \mathbb{N}\}$$

where $T \subseteq V$

Oracle Lemma

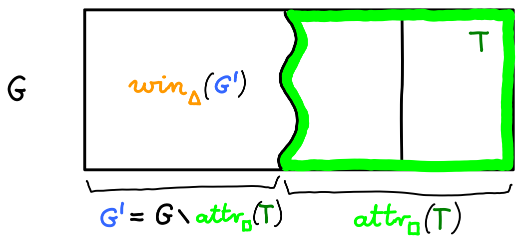


Lemma If $\boxed{U \subseteq win_\Delta(G)}$

then $win_\Delta(G) = U^* \cup win_\Delta(G'')$

$win_\square(G) = win_\square(G'')$

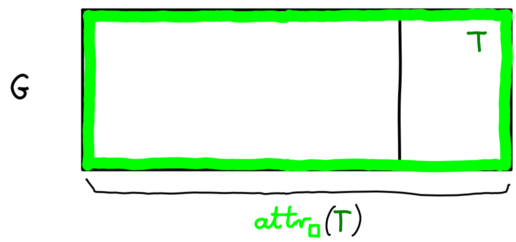
T-attractor Lemma



Lemma

1. $win_\Delta(G') = V \setminus attr_\square(T) \subseteq win_\Delta(G)$
2. If $G' = \emptyset$ then $win_\square(G) = V$

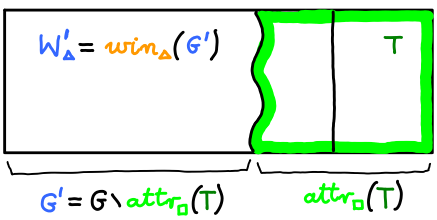
T-attractor Lemma



Lemma

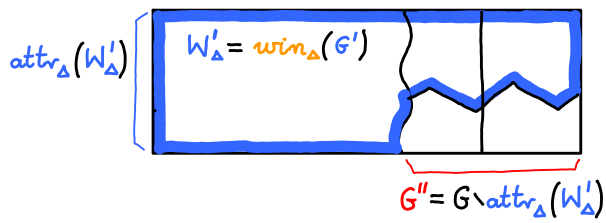
1. $win_\Delta(G') = V \setminus attr_\square(T) \subseteq win_\Delta(G)$
2. If $G' = \emptyset$ then $win_\square(G) = V$

A divide-and-conquer Algorithm



Win(G): if $W'_\Delta = \emptyset$ then $(W_\square, W_\Delta) := (V, \emptyset)$
else $(W''_\square, W''_\Delta) := \text{WIN}(G'')$
$(W_\square, W_\Delta) := (W''_\square, attr_\Delta(W'_\Delta) \cup W''_\Delta)$
return $(W_\square, W_\Delta)$

A divide-and-conquer Algorithm



Win(G): if $W'_\Delta = \emptyset$ then $(W_\square, W_\Delta) := (V, \emptyset)$
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$(W_\square, W_\Delta) := (W''_\square, attr_\Delta(W'_\Delta) \cup W''_\Delta)$
return $(W_\square, W_\Delta)$

Positional determinacy of (co-Büchi) games

Lemma

Büchi / co-Büchi games are **positionally determined**

(both on finite and **infinite graphs**)

Running time of the divide-and-conquer algorithm

$T(n)$

WIN(G):

if $W'_\triangle = \emptyset$ then $(W_\square, W_\triangle) := (V, \emptyset)$

else $(W''_\square, W''_\triangle) := \text{WIN}(G'')$

$(W_\square, W_\triangle) := \left(W''_\square, \text{attr}_\triangle(W'_\triangle) \cup W''_\triangle\right)$

return $(W_\square, W_\triangle)$

$T(n-1)$

Recurrence: $T(n) \leqslant T(n-1) + O(m)$

Solution: $T(n) = O(n \cdot m)$

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Games on graphs: parity objectives

$\text{Inf}(a_0, a_1, a_2, \dots) = \{ a : a_i = a \text{ for } \textbf{infinitely many } i \in \mathbb{N} \}$

Parity objective:

$W_\square = \{ (v_0, v_1, v_2, \dots) : \max\left(\text{Inf}(p(v_0), p(v_1), p(v_2), \dots)\right) \text{ is even} \}$

where $p : V \to \mathbb{N}$ is the priority function

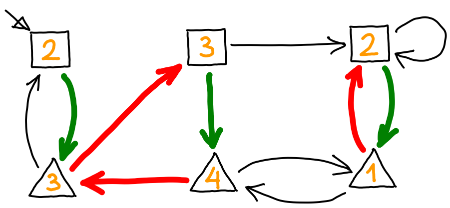
Parity Games

Players:

$\overset{6}{n}$ vertices, $\overset{11}{m}$ edges, $\overset{4}{d}$ priorities

Even = $\square$ = 0

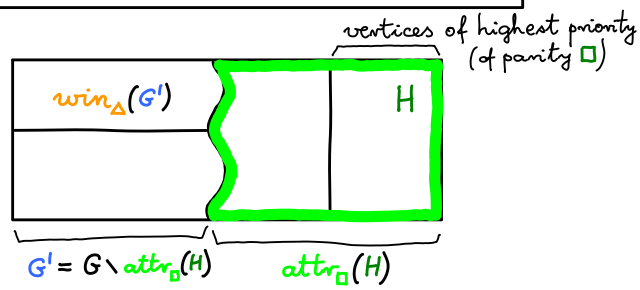
Odd = $\triangle$ = 1



Winner of an infinite play:

parity of the **highest priority** occurring **infinitely often**

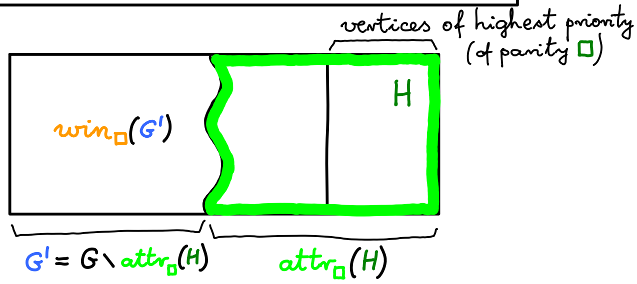
Highest-Priority Attractor Lemma



Lemma

1. If $\text{win}_\triangle(G') \neq \emptyset$ then $\text{win}_\triangle(G') \subseteq \text{win}_\triangle(G)$

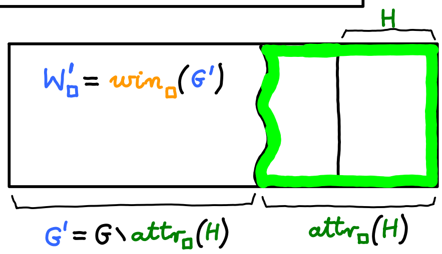
Highest-Priority Attractor Lemma



Lemma

1. If $win_\triangle(G') \neq \emptyset$ then $win_\triangle(G') \subseteq win_\triangle(G)$
2. If $win_\triangle(G') = \emptyset$ then $win_\square(G) = V$

A divide-and-conquer Algorithm



WIN(G): $(W'_\square, W'_\triangle) := \mathrm{WIN}(G')$
  if $W'_\triangle = \emptyset$ then $(W_\square, W_\triangle) := (V, \emptyset)$
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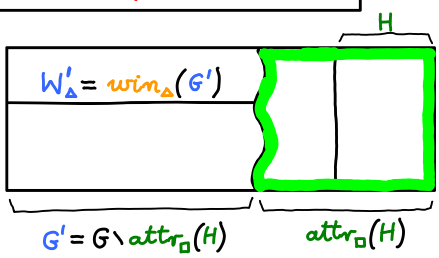
  return $(W_\square, W_\triangle)$

A divide-and-conquer Algorithm



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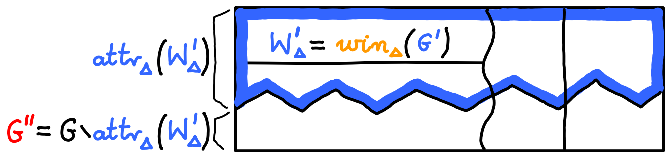
A divide-and-conquer Algorithm



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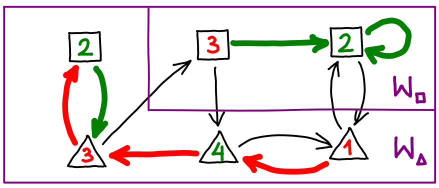
      $(W_\square, W_\triangle) := (W''_\square, attr_\triangle(W'_\triangle) \cup W''_\triangle)$
  return $(W_\square, W_\triangle)$

Positional Determinacy



Thm [Emerson, Jutla; Mostowski 1991]

Parity games are **positionally determined**
(also on **infinite graphs**)

Corollary

(Deciding the winner in parity games is in NP ∩ co-NP)

Running time of the divide-and-conquer algorithm

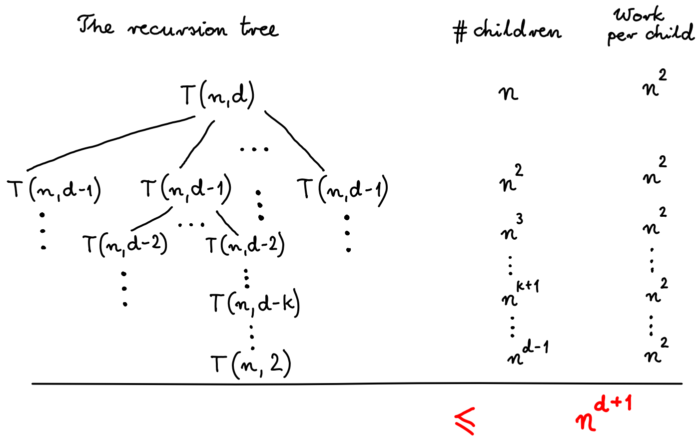
$T(n)$

WIN(G): $(W'_\square, W'_\triangle) := \mathrm{WIN}(G')$ $T(n-1)$
  if $W'_\triangle = \emptyset$ then $(W_\square, W_\triangle) := (V, \emptyset)$
    else $(W''_\square, W''_\triangle) := \mathrm{WIN}(G'')$ $T(n-1)$
      $(W_\square, W_\triangle) := (W''_\square, attr_\triangle(W'_\triangle) \cup W''_\triangle)$
  return $(W_\square, W_\triangle)$

Recurrence: $T(n) \leqslant 2 \cdot T(n-1) + O(n^2)$

Solution: $T(n) = O(2^n)$

Running time of the divide-and-conquer algorithm



Plan

I Qualitative (ω-regular) games

1. Motivating example
2. Games on graphs
3. Reachability / safety games
4. Büchi / co-Büchi games
5. Parity games
6. ω-regular games
7. Two recent complexity improvements for parity games

Games on graphs: ω-regular objectives

- **Muller** objectives:
  $W_\square = \{(v_0,v_1,v_2,\dots) : \mathrm{Inf}(\bar v) \in \mathcal{F}\}$ where $\mathcal{F} \subseteq 2^V$

- **Rabin** objectives:
  $W_\square = \{(v_0,v_1,v_2,\dots) :$ for **some** $j = 0,1,\dots,k-1$, we have $\mathrm{Inf}(\bar v) \cap G_j \neq \emptyset$ **and** $\mathrm{Inf}(\bar v) \cap R_j = \emptyset\}$

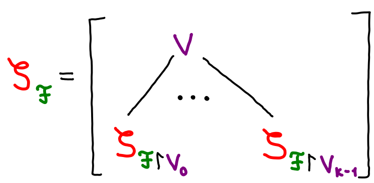
- **Streett** objectives:
  $W_\square = \{(v_0,v_1,v_2,\dots) :$ for **all** $j = 0,1,\dots,k-1$, we have $\mathrm{Inf}(\bar v) \cap G_j \neq \emptyset$ **implies** $\mathrm{Inf}(\bar v) \cap R_j \neq \emptyset\}$

where $G_0, R_0, G_1, R_1, \dots, G_{k-1}, R_{k-1} \subseteq V$

Split tree of a Muller objective

- If $\mathcal{F} = \emptyset$ or $\mathcal{F} = 2^V$ then $\mathcal{S}_\mathcal{F} = [V]$
- If $\emptyset \subsetneq \mathcal{F} \subsetneq 2^V$ and
  if $V_0, V_1, \dots, V_{k-1}$ are all **maximal** subsets of V, such that $V_j \in \mathcal{F}$ iff $V \notin \mathcal{F}$

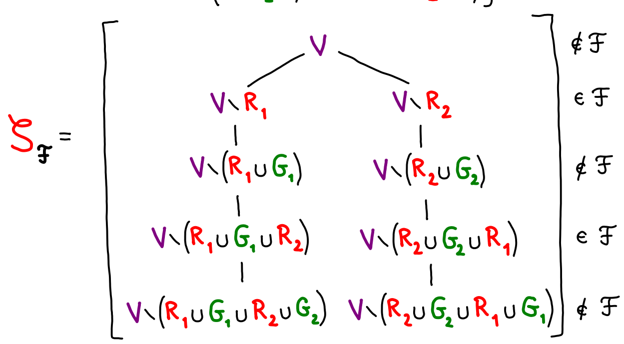
then



Split tree of a Rabin objective

Two **Rabin** pairs: $(G_1,R_1), (G_2,R_2)$

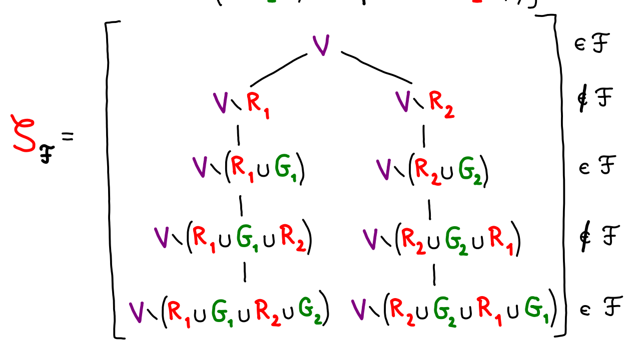
$\mathcal{F} = \{U \subseteq V : (U \cap G_1 \neq \emptyset$ **and** $U \cap R_1 = \emptyset)$ **or** $(U \cap G_2 \neq \emptyset$ **and** $U \cap R_2 = \emptyset)\}$



Split tree of a Streett objective

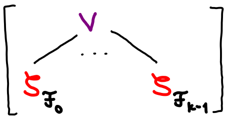
Two **Streett** pairs: $(G_1,R_1), (G_2,R_2)$

$\mathcal{F} = \{U \subseteq V : (U \cap G_1 \neq \emptyset$ **implies** $U \cap R_1 \neq \emptyset)$ **and** $(U \cap G_2 \neq \emptyset$ **implies** $U \cap R_2 \neq \emptyset)\}$



Memory number $m_{\mathcal{F}}$

For $\mathcal{Z}_{\mathcal{F}} =$



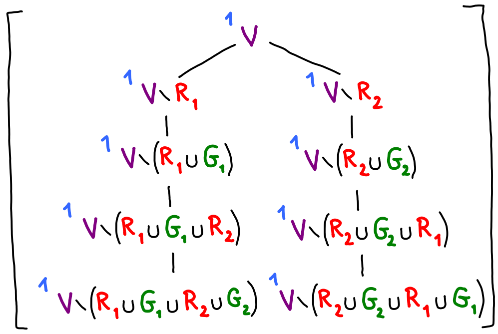
define:

$$m_{\mathcal{F}} = \begin{cases} 1 & \text{if } \mathcal{F} = \emptyset \text{ or } \mathcal{F} = 2^V \\ \max_{i=0}^{k-1} \{ m_{\mathcal{F}_i} \} & \text{if } V \notin \mathcal{F} \\ \sum_{i=0}^{k-1} m_{\mathcal{F}_i} & \text{if } V \in \mathcal{F} \end{cases}$$

Memory number of a Rabin objective

$\mathcal{Z}_{\mathcal{F}} =$

$m_{\mathcal{F}\restriction U}$



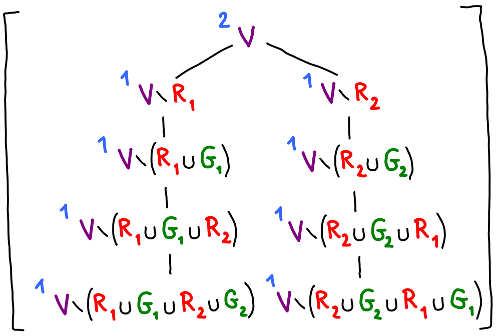
Proposition

If $\mathcal{F} \subseteq 2^V$ is Rabin then $\boxed{m_{\mathcal{F}} = 1}$

Memory number of a Streett objective

$\mathcal{Z}_{\mathcal{F}} =$

$m_{\mathcal{F}\restriction U}$



Proposition

If $\mathcal{F} \subseteq 2^V$ is Streett with k pairs then $\boxed{m_{\mathcal{F}} \leqslant k!}$

Strategies with memory

- Positional strategies:

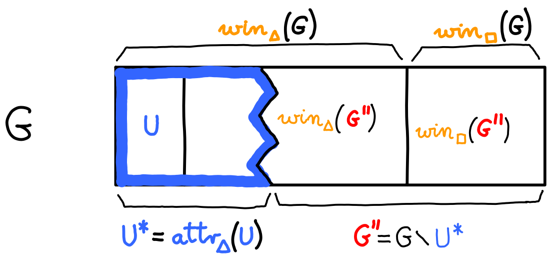
$$\chi : V_\square \longrightarrow V$$

- Strategies with **memory**:

$$\chi : M \times V_\square \longrightarrow M \times V$$

automaton states alphabet output

Oracle Lemma

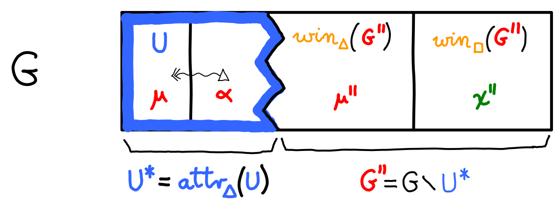


Lemma If $\boxed{U \subseteq win_\Delta(G)}$

then $win_\Delta(G) = U^* \cup win_\Delta(G'')$

$win_\square(G) = win_\square(G'')$

Oracle lemma with memory



Lemma

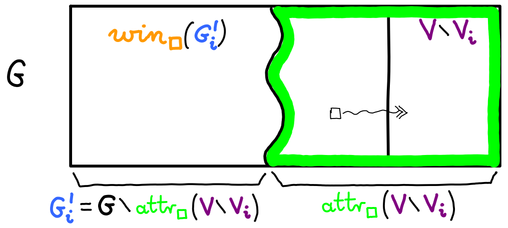
If μ is winning for Δ from U

μ'' is winning for Δ from $win_\Delta(G'')$

then $(\mu \cup \alpha \cup \mu'')$ is winning for Δ from $(U^* \cup win_\Delta(G''))$

$$|Memory(\mu \cup \alpha \cup \mu'')| \leqslant \max\{Memory(\mu), Memory(\mu'')\}$$

Muller games: $V\setminus V_i$ attractors lemma

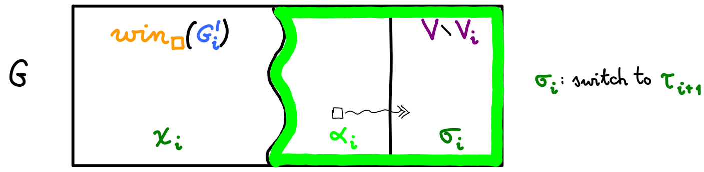


Lemma

If $win_\triangle(G_i^!) = \emptyset$ for all $i = 0, 1, \dots, k-1$

then $win_\square(G) = V$

Muller games: $V\setminus V_i$ attractors lemma



Lemma

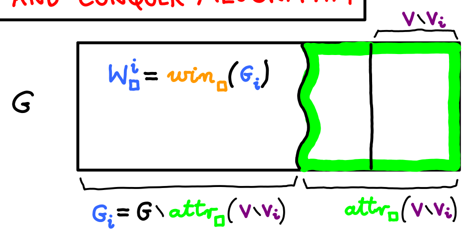
If χ_i is winning for $\square$ in $G_i^!$ for all $i = 0, 1, \dots, k-1$

$\tau_i = \chi_i \cup \alpha_i \cup \sigma_i$

then $\boxed{\tau = \tau_0 \oplus \tau_1 \oplus \cdots \oplus \tau_{k-1}}$ is winning for $\square$ from V

$$|Memory(\tau)| = \sum_{i=0}^{k-1} |Memory(\tau_i)| \leq \sum_{i=0}^{k-1} m_{\mathcal{F}_i} = m_{\mathcal{F}}$$

A divide-and-conquer Algorithm



$\underline{\text{Win}}(G)$: $(W_\square^i, W_\triangle^i) := \underline{\text{Win}}(G_i)$ for all $i = 0, 1, \dots, k-1$

if $W_\triangle^i = \emptyset$ for all $i = 0, 1, \dots, k-1$

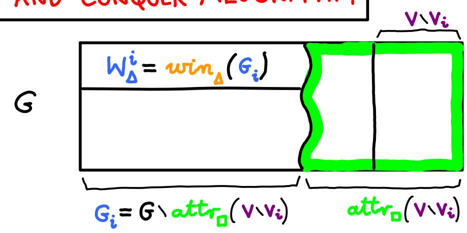
then $(W_\square, W_\triangle) := (V, \emptyset)$

else $(W_\square^{"}, W_\triangle^{"}) := \text{Win}(G_i^{"})$ s.t. $W_\triangle^i \neq \emptyset$

$(W_\square, W_\triangle) := (W_\square^{"}, attr_\triangle(W_\triangle^i) \cup W_\triangle^{"})$

return $(W_\square, W_\triangle)$

A divide-and-conquer Algorithm



$\underline{\text{Win}}(G)$: $(W_\square^i, W_\triangle^i) := \underline{\text{Win}}(G_i)$ for all $i = 0, 1, \dots, k-1$

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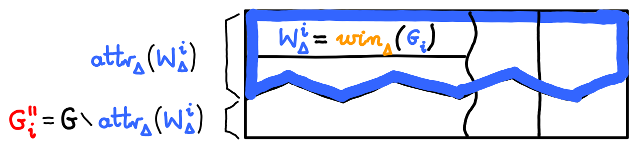
A divide-and-conquer Algorithm



$\underline{\text{Win}}(G)$: $(W_\square^i, W_\triangle^i) := \underline{\text{Win}}(G_i)$ for all $i = 0, 1, \dots, k-1$

if $W_\triangle^i = \emptyset$ for all $i = 0, 1, \dots, k-1$

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$(W_\square, W_\triangle) := (W_\square^{"}, attr_\triangle(W_\triangle^i) \cup W_\triangle^{"})$

return $(W_\square, W_\triangle)$

Forgetful determinacy of ω-regular games

Thm [Gurevich, Harrington 1982; Zielonka 1998]

Muller games are **forgetfully determined**:

- $\square$ has a winning strategy with **memory** $m_{\mathcal{F}}$
- $\triangle$ has a winning strategy with **memory** $m_{\overline{\mathcal{F}}}$

Corollary [Klarlund 1991]

Player $\square$ has a **positional** winning strategy in **Rabin games**

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TWO RECENT IMPROVEMENTS

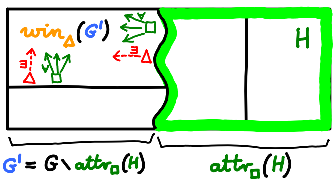
Many priorities $d = \Omega(n^{\frac{1}{2}+\varepsilon})$		Few priorities $d = O(n^{1/2})$	
Randomized	Deterministic		
Expected $n^{O(\sqrt{n})}$	$n^{O(\sqrt{n})}$	$O(n^{d/2})$	$O(n^{d/3})$
old	new	old	new

TWO RECENT IMPROVEMENTS

Many priorities $d = \Omega(n^{\frac{1}{2}+\varepsilon})$		Few priorities $d = O(n^{1/2})$	
Randomized	Deterministic		
Expected $n^{O(\sqrt{n})}$	$n^{O(\sqrt{n})}$	$O(n^{d/2})$	$O(n^{d/3})$
old	new	old	new

DOMINIONS

DEF $D \subseteq V$ is an **dominion** for Δ if Δ has a **trapping** strategy in D that is **winning** for her



FACT $win_\Delta(G')$ is a dominion for Δ in G

FACT A dominion of size $\leq \ell$ can be found in $n^{O(\ell)}$ time

A SUBEXPONENTIAL ALGORITHM

$T(n)$ | WIN(G): If there is a dominion $D \neq \emptyset$ of size $\leq \ell$ — $n^{O(\ell)}$
then return WIN$(G \setminus D) \oplus D$ — $T(n-1)$
else
$(W'_\square, W'_\Delta) :=$ WIN(G') — $T(n-1)$
if $W'_\Delta = \emptyset$ then $(W_\square, W_\Delta) := (V, \emptyset)$
else $(W''_\square, W''_\Delta) :=$ WIN(G'') — $T(n-\ell)$
$(W_\square, W_\Delta) := (W''_\square, attr_\Delta(W'_\Delta) \cup W''_\Delta)$
return $(W_\square, W_\Delta)$

Recurrence: $T(n) \leq n^{O(\ell)} + T(n-1) + T(n-\ell)$

Solution: $\boxed{T(n) = n^{O(\sqrt{n})}}$ if $\ell = \sqrt{n}$

TWO RECENT IMPROVEMENTS

Many priorities $d = \Omega(n^{\frac{1}{2}+\varepsilon})$		Few priorities $d = O(n^{1/2})$	
Randomized	Deterministic		
Expected $n^{O(\sqrt{n})}$	$n^{O(\sqrt{n})}$	$O(n^{d/2})$	$O(n^{d/3})$
old	new	old	new

A SUFFICIENT CONDITION FOR WINNING PLAYS

$$\overset{v_1}{\boxed{1}} \to \overset{v_2}{\boxed{2}} \to \overset{v_3}{\boxed{3}} \to \overset{v_4}{\boxed{2}} \to \overset{v_5}{\boxed{1}} \to \overset{v_6}{\boxed{1}} \to \overset{v_7}{\boxed{2}} \to \overset{v_8}{\boxed{3}} \to \overset{v_9}{\boxed{4}} \cdots$$

μ_3: $2 \geq 2 \geq 2 > 1 \geq 1 \geq 1 \geq 1 \geq 1 > 0$

μ_1: $1 > 0$ $\quad\quad\quad 2 > 1 > 0$

FACT

$v_1, v_2, v_3, \dots$ is winning for $\square$

if

there are $\mu_1, \mu_3, \dots, \mu_{d-1} : V \to \mathbb{N}$, s.t.

- $\mu_k(v_i) \geq \mu_k(v_{i+1})$ for $k \in [d-1, p(v_i))$
- $\mu_k(v_i) > \mu_k(v_{i+1})$ for $k = p(v_i)$

A SUFFICIENT CONDITION FOR WINNING PLAYS

$$\overset{v_1}{\boxed{1}} \to \overset{v_2}{\boxed{2}} \to \overset{v_3}{\boxed{3}} \to \overset{v_4}{\boxed{2}} \to \overset{v_5}{\boxed{1}} \to \overset{v_6}{\boxed{1}} \to \overset{v_7}{\boxed{2}} \to \overset{v_8}{\boxed{3}} \to \overset{v_9}{\boxed{4}} \cdots$$

$$\begin{pmatrix}\mu_3\\ \mu_1\end{pmatrix}: \begin{pmatrix}2\\1\end{pmatrix} >_{lex} \begin{pmatrix}2\\0\end{pmatrix} \geq_{lex} (2) >_{lex} (1) \geq_{lex} \begin{pmatrix}1\\2\end{pmatrix} >_{lex} \begin{pmatrix}1\\1\end{pmatrix} >_{lex} \begin{pmatrix}1\\0\end{pmatrix} \geq_{lex} (1) >_{lex} (0)$$

FACT

$v_1, v_2, v_3, \dots$ is winning for $\square$

if

there are $\mu_1, \mu_3, \dots, \mu_{d-1} : V \to \mathbb{N}$, s.t.

$$\boxed{\begin{pmatrix}\mu_{d-1}\\ \vdots \\ \mu_{p(v_i)}\end{pmatrix}(v_i) \geq^{lex} \begin{pmatrix}\mu_{d-1}\\ \vdots \\ \mu_{p(v_i)}\end{pmatrix}(v_{i+1})} \quad \boxed{>^{lex} \text{ if } p(v_i) \text{ is odd!}}$$

CHARACTERIZING EXISTENCE OF WINNING PLAYS

LEMMA

There **is** a winning play (from every vertex) for $\square$

if

there are $\mu_1, \mu_3, \dots, \mu_{d-1} : V \to \mathbb{N}$, s.t.

$$\boxed{\begin{pmatrix}\mu_{d-1}\\ \vdots \\ \mu_{p(v_i)}\end{pmatrix}(v) \geq^{lex} \min_{(v,w)\in E} \begin{pmatrix}\mu_{d-1}\\ \vdots \\ \mu_{p(v_i)}\end{pmatrix}(w)} \quad \boxed{>^{lex} \text{ if } p(v) \text{ is odd!}}$$

CHARACTERIZING UNIVERSALITY OF WINNING PLAYS

LEMMA

ALL plays (from every vertex) are winning for $\square$

if

there are $\mu_1, \mu_3, \dots, \mu_{d-1} : V \to \mathbb{N}$, s.t.

$$\boxed{\begin{pmatrix}\mu_{d-1}\\ \vdots \\ \mu_{p(v_i)}\end{pmatrix}(v) \geq^{lex} \max_{(v,w)\in E} \begin{pmatrix}\mu_{d-1}\\ \vdots \\ \mu_{p(v_i)}\end{pmatrix}(w)} \quad \boxed{>^{lex} \text{ if } p(v_i) \text{ is odd!}}$$

CHARACTERIZING EXISTENCE OF WINNING STRATEGIES

THEOREM

There is a winning strategy for $\square$ (from every vertex)

iff

there are $\mu_1, \mu_3, \dots, \mu_{d-1} : V \to \mathbb{N}$, s.t.

- $\boxed{\begin{pmatrix}\mu_{d-1}\\ \vdots \\ \mu_{p(v_i)}\end{pmatrix}(v) \geq^{lex} \min_{(v,w)\in E} \begin{pmatrix}\mu_{d-1}\\ \vdots \\ \mu_{p(v_i)}\end{pmatrix}(w)} \quad \boxed{>^{lex} \text{ if } p(v) \text{ is odd!}}$ for $v \in V_\square$
- $\boxed{\begin{pmatrix}\mu_{d-1}\\ \vdots \\ \mu_{p(v_i)}\end{pmatrix}(v) \geq^{lex} \max_{(v,w)\in E} \begin{pmatrix}\mu_{d-1}\\ \vdots \\ \mu_{p(v_i)}\end{pmatrix}(w)} \quad \boxed{>^{lex} \text{ if } p(v) \text{ is odd!}}$ for $v \in V_\triangle$

SMALL PROGRESS MEASURES

THEOREM

There is a winning strategy for $\square$ (from every vertex)

iff

there are $\mu_1, \mu_3, \dots, \mu_{d-1} : V \to \mathbb{N}$, s.t.

- $\boxed{\begin{pmatrix}\mu_{d-1}\\ \vdots \\ \mu_{p(v_i)}\end{pmatrix}(v) \geq^{lex} \min_{(v,w)\in E} \begin{pmatrix}\mu_{d-1}\\ \vdots \\ \mu_{p(v_i)}\end{pmatrix}(w)} \quad \boxed{>^{lex} \text{ if } p(v) \text{ is odd!}}$ for $v \in V_\square$
- $\boxed{\begin{pmatrix}\mu_{d-1}\\ \vdots \\ \mu_{p(v_i)}\end{pmatrix}(v) \geq^{lex} \max_{(v,w)\in E} \begin{pmatrix}\mu_{d-1}\\ \vdots \\ \mu_{p(v_i)}\end{pmatrix}(w)} \quad \boxed{>^{lex} \text{ if } p(v) \text{ is odd!}}$ for $v \in V_\triangle$

and $\boxed{\mu_k : V \to \{0, 1, 2, \dots, n_k\}}$ where $n_k = |p^{-1}(k)|$

Small progress measures

Lemma

The number of tuples

$$\begin{pmatrix} \mu_{d-1} \\ \vdots \\ \mu_1 \end{pmatrix} \in \mathbb{N}^{d/2}$$

s.t. $\mu_k \in \{0,1,2,\ldots,n_k\}$ where $n_k = |p^{-1}(k)|$

is $\boxed{\leqslant \left(\frac{n}{d/2}\right)^{d/2}}$

Smaller Progress Measures for Dominions

Thm If $D \subseteq V$ is a dominion of size $\leqslant \ell$, then there is a progress measure $\mu : V \to M_G^\infty(\ell)$, s.t. $\mu(D) \subseteq M_G(\ell)$,

where $\boxed{M_G(\ell) = \left\{ \mu \in \mathbb{N}^{d/2} : \sum_{k=1}^{d} \mu_k \leqslant \ell \right\}}$

Cor

1. All dominions of size $\leqslant \ell$ can be found in time $\binom{\ell + \frac{d}{2}}{\frac{d}{2}}$

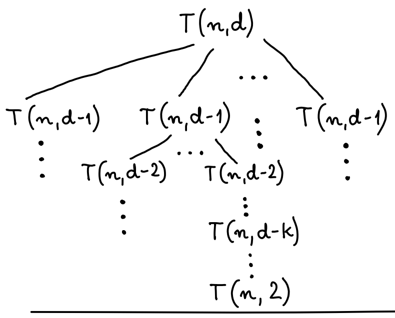
2. All dominions of size $\leqslant n^{2/3}$ can be found in time $n^{d/3}$

Progress Measure Lifting Algorithm

1. Start with $\mu : V \ni v \mapsto (0,0,\ldots,0)$
2. While μ violates the progress inequality at some $v \in V$, lift $\mu(v)$ (minimally) so that it doesn't

Running time: $\leqslant n \cdot \left(\frac{n}{d/2}\right)^{d/2}$

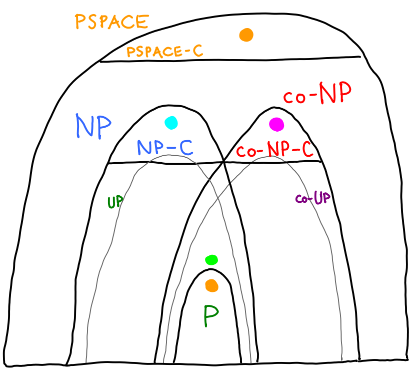
The Recursion Tree for Schewe's Algorithm



	# children	Dominion preprocessing
$T(n,d)$	$\leqslant n^{1/3}$	$n^{d/3}$
$T(n,d-1)$	$\leqslant n^{2/3}$	$n^{(d-1)/3}$
$T(n,d-2)$	$\leqslant n^{3/3}$	$n^{(d-2)/3}$
$\vdots$	$\vdots$	$\vdots$
$T(n,d-k)$	$\leqslant n^{(k+1)/3}$	$n^{(d-k)/3}$
$\vdots$	$\vdots$	$\vdots$
$T(n,2)$	$\leqslant n^{(d-1)/3}$	$n^{2/3}$

$\leqslant n \cdot n^{(d+1)/3}$

Computational complexity of solving infinite games



MULLER GAMES

STREETT GAMES

RABIN GAMES

PARITY GAMES

EXPLICIT MULLER GAMES