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Magidor–Malitz reflection

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Abstract In this paper we investigate the consistency and consequences of the downward Löwenheim–Skolem–Tarski theorem for extension of the first order logic by the Magidor–Malitz quantifier. We derive some combinatorial results and improve the known upper bound for the consistency of Chang's conjecture at successor of singular cardinals.

Keywords Magidor–Malitz quantifiers · Chang's conjecture · Large cardinals

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1 Introduction

Let M be a model of size λ and let κ be an infinite cardinal below λ. The downward Löwenheim–Skolem–Tarski theorem says that there is an elementary submodel $N \prec M$ such that $|N| = \kappa$. This is one of the most basic results in model theory, and it can be viewed as a reflection principle. The metamathematical object which is reflected here is the first order logic. The theorem asserts that if M is a model of some first order sentence and $|M| > \aleph_0$, then a *strictly smaller* elementary submodel of M already satisfies this sentence.

The Löwenheim–Skolem–Tarski theorem is extremely useful in model theory, and it is quite natural that mathematicians investigate tentative generalizations of it. One way to do this is to strengthen the underlying logic, and an important case is second

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order logic, $\mathcal{L}^2$. In $\mathcal{L}^2$, one can quantify over subsets and predicates of the model and thus express much more of its behavior.

Moving to second order properties, we are catapulted into the realm of large cardinals:

Theorem 1 *(Magidor)* [9] *Assume that there is a cardinal κ such that for every model M with countable language, there is $N \prec_{\mathcal{L}^2} M$, $|N| < \kappa$, then there is a supercompact cardinal $\leq \kappa$.*

Indeed, even full Π_1^1-reflection implies the existence of a supercompact cardinal.

Magidor's theorem demonstrates an important difference between first order and second order logic. Focusing on first order logic, there are essentially no set-theoretical restrictions on our ability to reflect valid sentences. But if we wish to reflect all the second order properties, we need a supercompact cardinal.

In this paper we try to examine what happens at some intermediate logics between first order and second order logic. We shall focus mostly on first order logic extended by the following quantifiers:

Definition 2 *(Magidor–Malitz Quantifiers)* Let M be a model in the language $\mathcal{L}$. For a formula $\varphi(x_0, x_1, \ldots, x_{n-1}, p_0, \ldots, p_{m-1})$, we write

$$M \models Q^n x_0, \ldots, x_{n-1} \varphi(x_0, x_1, \ldots, x_{n-1}, p_0, \ldots, p_{m-1})$$

if there is a set $A \subseteq M$ with $|A| = |M|$ such that

$$\forall a_0, a_1, \ldots, a_{n-1} \in A, \ M \models \varphi(a_0, \ldots, a_{n-1}, p_0, \ldots, p_{m-1}).$$

We write $M \prec_{Q^n} N$ if M is an elementary submodel of N with respect to first order logic enriched with the quantifier Q^n. We Write $M \prec_{Q^{<\omega}} N$ if $M \prec_{Q^n} N$ for all $n < \omega$.

These quantifiers were defined by Menachem Magidor and Jerome Malitz in [10]. In this paper, Magidor and Malitz proved that if the set theoretical principle $\diamondsuit(\omega_1)$ holds, then this quantifier satisfies a certain compactness theorem for models of size $\aleph_1$. This was generalized by Shelah in [16] to arbitrary successor of regular cardinal, λ^+, under the assumption $\diamondsuit(\lambda) + \diamondsuit(\lambda^+)$. In [14], Shelah and Rubin showed that the the quantifiers Q^n form a strict hierarchy and discussed some cases in which compactness fails.

Basically, the Magidor–Malitz quantifiers express *some* of the second order properties of the model. This weakening enables us to get consistently some variants of the Löwenheim–Skolem–Tarski theorem at accessible cardinals. See for example Theorem 25 and Sect. 3.4. The hierarchy of languages between first order and second order logic reflects at the size of the large cardinals which are needed in order to get reflection principles for those logics. For example, one can get Q^1-reflection at successor cardinal, starting from subcompact cardinal, but in order to get a similar reflection principle relative to $Q^{<\omega}$ using the same approach, one needs Π_1^1-subcompact cardinal.

A more concrete approach is related to the celebrated Chang's conjecture. Recall:

Definition 3 Let $\kappa, \lambda, \mu, \nu$ be cardinals, $\kappa > \lambda, \mu > \nu$. We say that $(\kappa, \lambda) \twoheadrightarrow (\mu, \nu)$ is for every model M of the countable language $\mathcal{L}$ with distinct unary predicate A such that $|M| = \kappa$ and $|A| = \lambda$ there is an elementary submodel $N \prec M$ such that $|N| = \mu, |A \cap N| = \nu$.

Chang's conjecture is a natural strengthening of the model theoretical two cardinals theorems of Vaught and Chang. There is an extensive literature about Chang's conjecture for various parameters. See, for example, [1, section 7.3].

It turned out that some instances of Chang's conjecture are equivalent to reflection of sentences with the Chang's quantifier Q^1 (see below, Lemma 5). Some results in this paper point towards a similarity between Chang's conjecture and the reflection of $Q^{<\omega}$, while other demonstrate the difference between them. For example:

Theorem *It is consistent, relative to a $(+\omega + 1)$-subcompact cardinal, that*

$$(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0).$$

But in Question 2 we ask whether the same situation can occur at all for $Q^{<\omega}$.

Our notation is standard. We work in ZFC. We force downwards (namely, $p \leq q \implies p \Vdash q \in \dot{G}$, where $\dot{G}$ is the canonical name for the generic filter). For a formula $\varphi(x_0, \ldots, x_{n-1}, p)$ with free variables $x_0, \ldots, x_{n-1}$ and parameter p (for simplicity, we assume that there is only one parameter). A set $A \subseteq M$ is called φ-*cube* if for all $a_0, \ldots, a_{n-1} \in A, M \models \varphi(a_0, \ldots, a_{n-1}, p)$.

The paper is arranged in three sections. In Sect. 2 we define the $Q^{<\omega}$ analogue for Chang's conjecture and derive some reflection principles from it. In Sect. 3 we investigate the large cardinals which imply $Q^{<\omega}$ reflection and prove consistency results about some cases of $Q^{<\omega}$ reflection at small cardinals.

2 Combinatorial consequences of $Q^{<\omega}$ reflection

In this section we analyse the relationship between the reflection of the Magidor–Malitz quantifiers, $Q^{<\omega}$, and some square like principles.

The Magidor–Malitz quantifiers allow us to access some of the second order properties of the model. As we will see, the downward Löwenheim–Skolem–Tarski theorem for the quantifiers Q^n is a strong reflection principle, yet its consistently holds for some pairs of small cardinals (assuming the consistency of large cardinals).

Definition 4 Let λ, μ and η be cardinals. $\lambda \xrightarrow[Q^{<\omega}]{\eta} \mu$ iff for every model of cardinality λ, over a language of cardinality η, there is a $Q^{<\omega}$−elementary submodel of cardinality μ. When $\eta = \aleph_0$ we write $\lambda \xrightarrow[Q^{<\omega}]{} \mu$. Similarly, $\lambda \xrightarrow[Q^n]{} \mu$ iff for every model of cardinality λ there is a Q^n−elementary submodel of cardinality μ.

$\lambda \xrightarrow[Q^{<\omega}]{} < \mu$ abbreviates the assertion that for every model of cardinality λ there is a $Q^{<\omega}$−elementary submodel of cardinality less than μ.

Let us recall that a weak instance of the reflection principle $\lambda \xrightarrow[Q^{<\omega}]{} \mu$ is equivalent to Chang's conjecture:

Lemma 5 (Folklore) *Let $\mu < \lambda$ be cardinals. $\lambda^+ \xrightarrow[Q^1]{} \mu^+$ iff $(\lambda^+, \lambda) \twoheadrightarrow (\mu^+, \mu)$.*

Proof Let us assume that $\lambda^+ \xrightarrow[Q^1]{} \mu^+$. Let (M, A) be a model of type (λ^+, λ).

Assume, without loss of generality, that:

1. A is a predicate in the language
2. There is a definable well ordering $\leq^\star$ on M with order type λ^+
3. For every $a \in M$ there is a definable surjection from A onto the elements that are smaller than a in $\leq^\star$.

Let $N \prec_{Q^1} M$ be an elementary submodel of cardinality μ^+. We claim that $A^N = A \cap N$ has cardinality μ.

$M \models \neg Q^1 x \in A$ (since $|A| = \lambda < |M| = \lambda^+$) and therefore $N \models \neg Q^1 x \in A$, so $|A^N| \leq \mu$. On the other hand, by elementarity for every $a \in N$ there is a surjection from A^N onto $\{b \in N \mid b \leq^\star a\}$ so $|A|$ cannot be strictly smaller than μ.

On the other hand, assume that $(\lambda^+, \lambda) \twoheadrightarrow (\mu^+, \mu)$. By enriching the language, we may assume that for every formula $\phi(x, b)$ there is a function symbol f_ϕ such that $\{f_\phi(x, b) \mid x \in M\} = \{y \in M \mid M \models \phi(y, b)\} = \phi(M, b)$. Moreover, we may choose f_ϕ in a way that if $\phi(M, b)$ has cardinality $\leq \lambda$ then $x \to f_\phi(x, b)$ is onto already when restricted to A. Otherwise, we assume that $x \to f_\phi(x, b)$ is one to one.

Let $N \prec M$ be an elementary submodel with $|A^N| = \mu$. Let us look at the formula $Q^1 x \phi(x, b)$. If it holds in M, then $f_\phi(x, b)$ enumerates the set of witnesses and when restricting this function to N, we get a one to one function from N such that $N \models \forall x \phi(f_\phi(x, b), b)$. Thus $|\{x \in N \mid \phi(x, b)\}| = |N|$. On the other hand, if $\neg Q^1 x \phi(x, b)$ then

$$M \models \forall x \phi(x, b) \to \exists a \in A, \ x = f_\phi(a, b).$$

Therefore, N satisfies the same formula and $|\{x \in N \mid \phi(x, b)\}| \leq |A^N|$. $\square$

The previous lemma shows that the reflection principle $\mu^+ \xrightarrow[Q^{<\omega}]{} \kappa$ is at least as strong as Chang's Conjecture. For example, since $(\aleph_{\omega+1}, \aleph_\omega) \nrightarrow (\aleph_n, \aleph_{n-1})$ for all $n \geq 4$, we conclude that $\aleph_{\omega+1} \xcancel{\xrightarrow[Q^{<\omega}]{}} \aleph_n$ for all $n \geq 4$.

The proof of the lemma shows that if $\lambda^+ \xrightarrow[Q^1]{} \mu$ then μ must be a successor cardinal and in particular regular. Similarly, if $\lambda^+ \xrightarrow[Q^1]{} < \kappa$ then we may assume always that the cardinalities of the elementary submodels are successor cardinals.

Let us start with the following useful observation which shows that models that are obtained from Chang's conjecture can be assumed to have a specific order type.

Lemma 6 *Assume $\lambda \xrightarrow[Q^1]{} \mu$. Then for every model $\mathcal{A}$ on set of ordinals of order type λ there is an elementary submodel, $\mathcal{B}$, such that $\mathrm{otp}\,\mathcal{B} = \mu$.*

Proof Assume that the language of $\mathcal{A}$ has a predicate $<$, interpreted as the order of the ordinals. Let us reflect the statement:

$$\forall \alpha \neg Q^1 \beta, \ \beta < \alpha$$

from $\mathcal{A}$ into $\mathcal{B}$. Observe that for every $\alpha \in \mathcal{B}$, $\text{otp}(\mathcal{B} \cap \alpha) < \mu$. Therefore $\mathcal{B}$ is an increasing union of chain of models, ordered by end-extension, where the order type of the chain is μ and all the substructures in the chain has order type strictly smaller than μ. $\qquad\square$

Let us recall the definition of $\square(\kappa)$. Let A be a set of ordinals, we denote:

$$\text{acc } A = \{\beta \mid \sup A \cap \beta = \beta\}.$$

Definition 7 Let $\mathcal{C} = \langle C_\alpha \mid \alpha < \kappa \rangle$ be a sequence of closed sets such that:

1. $\sup C_\alpha = \alpha$ for all limit ordinal α.
2. If $\beta \in \text{acc } C_\alpha$ then $C_\alpha \cap \beta = C_\beta$.
3. There is no club D such that $\forall \alpha \in (\text{acc } D) \cap \kappa$, $D \cap \alpha = C_\alpha$.

Then $\mathcal{C}$ is called a $\square(\kappa)$ sequence. We say that $\square(\kappa)$ holds if there is a $\square(\kappa)$ sequence.

This definition, due to Todorčević, is pivotal in the research of reflection properties, in particular when dealing with Π_1^1-statements. See [13] for extensive review.

Theorem 8 *Assume that $\kappa \xrightarrow[Q^2]{} \mu$ where μ is regular. Then $\square(\kappa)$ fails.*

Proof Let $\mathcal{C} = \langle C_\alpha \mid \alpha < \kappa \rangle$ be a coherent C-sequence, i.e., a sequence that satisfies the first two conditions in Definition 7. We will show that there is a *thread*, namely a club D such that for every $\alpha \in (\text{acc } D) \cap \kappa$, $D \cap \alpha = C_\alpha$.

Let $\mathcal{A} \prec H(\chi)$, for some large enough regular χ, with $\kappa + 1 \subseteq \mathcal{A}$, $\mathcal{C} \in \mathcal{A}$ and $|\mathcal{A}| = \kappa$.

Let $\mathcal{B} \prec_{Q^2} \mathcal{A}$ with $|\mathcal{B}| = \mu$ and assume that $\kappa, \mathcal{C} \in \mathcal{B}$. Since μ is a regular cardinal, by Lemma 6, we can take $\mathcal{B}$ so that $\sup(\mathcal{B} \cap \kappa) = \rho$, $\text{cf } \rho = \mu$.

Let us look at $\delta \in \text{acc } C_\rho \cap \text{acc}(\mathcal{B} \cap \kappa)$ below ρ. Let $\beta = \min(\mathcal{B} \cap \kappa \setminus \delta)$. β is well defined, since $\delta < \rho = \sup(\mathcal{B} \cap \kappa)$. Let us show that $\delta \in \text{acc } C_\beta$. If $\delta = \beta$, then this is clearly true, so let us assume that $\delta \neq \beta$.

Let $\alpha \in \mathcal{B} \cap \beta$. By the minimality of β, $\alpha < \delta$. Let γ be $\min C_\beta \setminus \alpha$. This ordinal is definable from $\alpha, \beta, \mathcal{C}$ and therefore $\gamma \in \mathcal{B}$. Since C_β is cofinal at β, $\gamma < \beta$. By using the minimality of β again, we conclude that $\gamma < \delta$. Therefore, δ is an accumulation point of ordinals in $\mathcal{B}$. We conclude that the set of ordinals in C_β below δ is unbounded, as required.

Since $\mathcal{C}$ is coherent, $C_\delta \trianglelefteq C_\beta$ (i.e. $C_\beta \cap \delta = C_\delta$).

Now, let $\delta < \delta'$ be in $\text{acc } C_\rho \cap \text{acc}(\mathcal{B} \cap \kappa)$. Let $\beta = \min(\mathcal{B} \cap \kappa \setminus \delta)$, $\beta' = \min(\mathcal{B} \cap \kappa \setminus \delta')$. We claim that $C_\beta \trianglelefteq C_{\beta'}$, since otherwise there is some $\gamma < \beta$ such that $\gamma \in C_\beta \triangle C_{\beta'}$. Such γ must appear in $\mathcal{B}$ (by elementarity), so it is smaller than δ. But $C_\delta \trianglelefteq C_{\delta'} \trianglelefteq C_{\beta'}$ - a contradiction.

We conclude that

$$\mathcal{B} \models Q^2 \alpha, \beta < \kappa, \; \alpha \geq \beta \bigvee C_\alpha \trianglelefteq C_\beta$$

Therefore, $\mathcal{A}$ contains a set of cardinality κ, I, of elements which are compatible in $\mathcal{C}$. $D = \bigcup_{\alpha \in I} C_\alpha$ is a thread. $\qquad\square$

2.1 The tree property at successor of singular

In this section we will show that reflection of the Q^2-quantifier can behave, in some ways, similarly to the existence of strongly compact cardinals. In particular, we will show that in the successor of singular limit of cardinals in which some $Q^{<\omega}$-reflection holds, the tree property holds.

Let us recall the following definition:

Definition 9 [11] A triplet $\mathcal{S} = \langle I, \kappa, \mathcal{R} \rangle$ is called a *system* if:

1. $\mathcal{R}$ is a set of partial orders on $I \times \kappa$.
2. For every $\alpha < \beta$ in I there are $\zeta < \xi < \kappa$ and $\leq_i \in \mathcal{R}$ such that $\langle \alpha, \xi \rangle \leq_i \langle \beta, \zeta \rangle$.
3. For every $\leq_i \in \mathcal{R}$, $\langle \alpha, \xi \rangle \leq_i \langle \beta, \zeta \rangle$ implies that either $\alpha < \beta$ or that $\alpha = \beta$ and $\xi = \zeta$.
4. For every $\leq_i \in \mathcal{R}$, if $\langle \alpha, \xi \rangle \leq_i \langle \gamma, \rho \rangle$ and $\langle \beta, \zeta \rangle \leq_i \langle \gamma, \rho \rangle$ then $\langle \alpha, \xi \rangle \leq_i \langle \beta, \zeta \rangle$.

The system $\mathcal{S}$ is *narrow* if $\kappa^+, |\mathcal{R}|^+ < \sup I$.

A branch in the system $\mathcal{S}$ is a partial function $b \subseteq I \times \kappa$ such that there is $\leq_i \in \mathcal{R}$ such that for every $\alpha < \beta$ in dom b, $\langle \alpha, b(\alpha) \rangle \leq_i \langle \beta, b(\beta) \rangle$. A branch b is *cofinal* if $\sup \operatorname{dom} b = \sup I$.

The *Narrow System Property* holds at a cardinal ν if every narrow system $\mathcal{S} = \langle I, \kappa, \mathcal{R} \rangle$ with $\sup I = \nu$ has a cofinal branch.

For full discussion about narrow systems and the Narrow System Property, see [8].

Theorem 10 *Let μ be a singular cardinal and assume that $\langle \kappa_i \mid i < \operatorname{cf} \mu \rangle$ is cofinal in μ, $\kappa_0 \geq \operatorname{cf} \mu$, $\operatorname{cf} \kappa_i = \kappa_i$ for all i. If for every $i < \operatorname{cf} \mu$, $\mu^+ \xrightarrow[Q^{<\omega}]{\kappa_i} \kappa_{i+1}$ then the tree property holds at μ^+.*

Proof We prove the theorem in two steps. First we apply $Q^{<\omega}$-reflection in order to find for a given μ^+-tree T a narrow subsystem. At this step we will use $\mu^+ \xrightarrow[Q^{<\omega}]{\operatorname{cf} \mu} \lambda$ for some regular $\lambda < \mu^+$. Then we pick i large enough so that κ_i is larger than the width of the system and use $\mu^+ \xrightarrow[Q^2]{\kappa_i} \kappa_{i+1}$ in order to get a branch through the narrow system.

Let T be a μ^+-tree and assume, without loss of generality, that $T = \langle \mu^+ \times \mu, \leq_T \rangle$, i.e. that the α-th level of T is given by $\{\alpha\} \times \mu$. Let $\mathcal{A}_0$ be an elementary substructure of $H(\chi)$ for some large enough χ, such that $|\mathcal{A}_0| = \mu^+$, $\mu^+ \subseteq \mathcal{A}_0$ and $T \in \mathcal{A}_0$.

Let $\mathcal{B}_0$ be a $Q^{<\omega}$-elementary substructure of $\mathcal{A}_0$, containing $\operatorname{cf} \mu$ such that the order type of $\mathcal{B}_0 \cap \mu^+$ has cofinality above $\operatorname{cf} \mu$. We may assume, without loss of generality, that $\{\kappa_i \mid i < \operatorname{cf} \mu\} \subseteq \mathcal{B}_0$. Let $\Delta = \mathcal{B}_0 \cap \mu^+$ - the set of levels that appear in $\mathcal{B}_0$. Let $\delta = \sup \Delta$ and let us consider the branch below $\langle \delta, 0 \rangle$. Since T is a tree, for every $\alpha \in \Delta$ there is $\zeta < \mu$ such that $\langle \alpha, \zeta \rangle \leq_T \langle \delta, 0 \rangle$.

Since μ is singular and $\operatorname{cf} \mu \subseteq \mathcal{B}_0$, there is some $i \in \mathcal{B}_0$ such that $\zeta < \kappa_i$. If α, β are both in Δ, and $\langle \alpha, \zeta \rangle, \langle \beta, \xi \rangle \leq_T \langle \delta, 0 \rangle$ then $\langle \alpha, \zeta \rangle \leq_T \langle \beta, \xi \rangle$. If we assume that $\zeta, \xi < \kappa_i$ then by elementarity there are $\tilde{\zeta}, \tilde{\xi} \in \mathcal{B}_0 \cap \kappa_i$ such that $\langle \alpha, \tilde{\zeta} \rangle \leq_T \langle \beta, \tilde{\xi} \rangle$.

Since the cofinality of δ is larger than cf μ, there is $i < \text{cf } \mu$ such that $\mathcal{B}_0$ satisfies the $Q^{<\omega}$-formula:

$$Q^2\alpha, \beta \exists \zeta, \xi < \kappa_i, \langle \alpha, \zeta \rangle \leq_T \langle \beta, \xi \rangle$$

By $Q^{<\omega}$-elementarity there is some subset $I \subseteq \mathcal{A}_0$ with cardinality μ^+ such that every element of I is an ordinal and the elements of I satisfy the same compatibility relation. Therefore, we can define a narrow system on $I \times \kappa_i$ (with only single relation), by the restriction of the tree T to this set.

Let us show that the Narrow System Property follows from the reflection assumption $\mu^+ \xrightarrow[Q^2]{\kappa_i} \kappa_{i+1}$ for cofinal set of regular $\kappa_i < \mu$.

Lemma 11 *Let μ be a singular cardinal and assume that for cofinal set of regular cardinals $\kappa < \mu$, there is a regular cardinal λ, such that $\kappa < \lambda < \mu$ and $\mu^+ \xrightarrow[Q^{<\omega}]{\kappa} \lambda$.*

Then the narrow system property holds at μ^+.

Proof Let $\mathcal{S}$ be a narrow system with height μ^+ and less than κ many order relations and elements in each level.

Let $\mathcal{A}_1$ be an elementary substructure of $H(\chi)$ containing all ordinals in μ^+ and $\mathcal{S}$. Let us pick a $Q^{<\omega}$−elementary substructure $\mathcal{B}_1$ of $\mathcal{A}_1$, of cardinality strictly larger than κ, containing all ordinals below κ. Let $\delta = \sup(\mathcal{B}_1 \cap \mu^+)$ and let us pick some element $\epsilon \in I \setminus \delta$. Since $\mathcal{S}$ is a narrow system, for every $\alpha \in \mathcal{B}_1$ there are $\zeta, \xi < \kappa$ and index $i < \kappa$ such that $\langle \alpha, \zeta \rangle \leq_i \langle \epsilon, \xi \rangle$. By [Theorem 6](#), we may assume that otp $\mathcal{B}_1$ is regular and therefore, for unbounded many ordinals below ϵ in $\mathcal{B}_1$ the tuple (ζ, ξ, i) is constant. Therefore, for some $\zeta_\star, i_\star < \kappa$, $\mathcal{B}_1$ satisfies:

$$Q^2\alpha, \beta, \langle \alpha, \zeta_\star \rangle \leq_{i_\star} \langle \beta, \zeta_\star \rangle$$

The same holds in $\mathcal{A}_1$, and therefore there is a branch in $\mathcal{S}$. $\qquad\square$

Applying Lemma 11 on $T \upharpoonright I$, we obtain a cofinal branch through $T \upharpoonright I$, b'. The set $\{s \in T \mid \exists s \in b, \ t \leq s\}$ is a cofinal branch through T. $\qquad\square$

The assumptions of Theorem 10 and Lemma 11 can be weakened to the assumption that for every model $\mathcal{A}$ of cardinality μ^+ over a language of cardinality $\eta < \mu$ there is some regular cardinal $\kappa < \mu$ and a Q^2-elementary submodel $\mathcal{B}$ of cardinality κ (note that in this case, $\eta < \kappa$). This is true, since the proof does not use the fact that the values of the cardinals λ_n are pre-determined. The proof only uses the fact that for every $\eta < \mu$ (which is the width of the narrow system), we can find a Q^2-elementary submodel of some fragment of the universe, $\mathcal{B}$, such that $\eta \subseteq \mathcal{B}$ and otp $\mathcal{B}$ is regular and large enough.

This type of reflection principle follows from large cardinals at the level of supercompact (see Theorem 17). It is unclear whether one can derive this kind of reflection from strongly compact cardinals. In fact, it is unclear even if one can derive some instances of Chang's conjecture from strongly compact cardinals.

The assumption that $\mu^+ \xrightarrow[Q^{<\omega}]{} \kappa$ for cofinally many $\kappa < \mu$ seems to be stronger than the narrow system property. For example, it cannot hold for $\mu = \aleph_\omega$, since

$(\aleph_{\omega+1}, \aleph_\omega) \not\twoheadrightarrow (\aleph_{n+1}, \aleph_n)$ for every $n \geq 3$. This fact is a combination of PCF related results of Cummings, Foreman, Magidor and Shelah. For a proof, see [15, Section 4].

3 Consistency results

This section is dedicated to the derivation of some consistency results regarding the reflection principles that were defined above.

3.1 Chang's conjecture at $\aleph_{\omega+1}$

We begin this section with two theorems about the consistency of Chang's conjecture at successor of singular cardinals.

Definition 12 (*Jensen*) [12] A cardinal κ is $(+\alpha)$ *subcompact* if for every $A \subseteq H(\kappa^{+\alpha})$ there are $\rho < \kappa$, $B \subseteq H(\rho^{+\alpha})$ and an elementary embedding

$$j \colon \langle H(\rho^{+\alpha}), \in, \rho, B \rangle \to \langle H(\kappa^{+\alpha}), \in, \kappa, A \rangle$$

where ρ is the critical point of j. A cardinal κ is *subcompact* if it is $(+1)$-subcompact.

In order to get a general feeling about the place of this type of cardinals in the large cardinal hierarchy, let us remark that if κ is $\kappa^{+\omega+1}$ supercompact and $\kappa^{+\omega}$ is strong limit then κ is $(+\omega + 1)$-subcompact and it has a normal measure concentrating on $(+\omega + 1)$-subcompact cardinals below it. On the other hand, if a cardinal κ is $(+\omega + 1)$-subcompact then it is κ^{+n}-supercompact for every $n < \omega$ and it has a normal measure concentrating on cardinals ρ which are ρ^{+n} supercomapct.

Theorem 13 *Let κ be $(+\omega + 1)$-subcompact cardinal, $\kappa^{+\omega}$ strong limit. There is $\rho < \kappa$ such that $(\kappa^{+\omega+1}, \kappa^{+\omega}) \twoheadrightarrow (\rho^{+\omega+1}, \rho^{+\omega})$.*

Proof Let $\mu = \kappa^{+\omega}$. Assume otherwise, and let us pick for every $\rho < \kappa$ a function $f_\rho \colon (\mu^+)^{<\omega} \to \mu^+$ such that for all $R \subseteq \mu^+$ of cardinality $\rho^{+\omega+1}$, $|f_\rho''[R]^{<\omega} \cap \mu| \neq \rho^{+\omega}$.

Let us code this sequence of functions as a subset of $H(\kappa^{+\omega+1})$, A.

Let $j \colon \langle H(\rho^{+\omega+1}), B \rangle \to \langle H(\kappa^{+\omega+1}), A \rangle$ be an elementary embedding as in the definition of $(+\omega + 1)$-subcompactness. B codes a sequence of functions from $(\rho^{+\omega+1})^{<\omega}$ to $\rho^{+\omega+1}$, $\langle g_\eta \mid \eta < \rho \rangle$, witnessing the failure of Chang's conjecture. Note that $\rho^{+\omega}$ is strong limit.

Let us look at f_ρ. Let $R = j''\rho^{+\omega+1} \in H(\kappa^{+\omega+1})$. By our assumption, $|f_\rho''[R]^{<\omega} \cap \mu| > \rho^{+\omega}$. Let n be the first ordinal such that $|f''[R]^{<\omega} \cap \kappa^{+n}| = \rho^{+\omega+1}$.

Since $\operatorname{cf} \rho^{+\omega} = \omega$ and $\rho^{+\omega+1}$ is regular, it is impossible that $n = \omega$, so n is a natural number.

Let $\langle \vec{\alpha}_\xi \mid \xi < \rho^{+\omega+1} \rangle$ be a sequence of elements in $(\rho^{+\omega+1})^{<\omega}$, and assume that $\langle f_\rho(j(\vec{\alpha}_\xi)) \mid \xi < \rho^{+\omega+1} \rangle$ is strictly increasing sequence of ordinals below κ^{+n}.

By elementarity, for every $\xi \neq \xi'$,

$$\langle g_\eta(\vec{\alpha}_\xi) \cap \rho^{+n} \mid \eta < \rho \rangle \neq \langle g_\eta(\vec{\alpha}_{\xi'}) \cap \rho^{+n} \mid \eta < \rho \rangle$$

Otherwise, for every $\tilde{\rho} < \kappa$ we would get that

$$f_{\tilde{\rho}}(j(\vec{\alpha}_\xi)) \cap \kappa^{+n} = f_{\tilde{\rho}}(j(\vec{\alpha}_{\xi'})) \cap \kappa^{+n}$$

and evaluating at $\tilde{\rho} = \rho$ we get a contradiction.

But the number of possible sequences of this form is $(\rho^{+n})^\rho < \rho^{+\omega}$ - a contradiction. $\qquad \square$

Theorem 14 *Let κ be a $(+\omega + 1)$-subcompact cardinal, $\kappa^{+\omega}$ strong limit. There is $\rho < \kappa$ such that forcing with $\mathrm{Col}(\omega, \rho^{+\omega}) \times \mathrm{Col}(\rho^{+\omega+2}, < \kappa)$ forces the instance of Chang's conjecture $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$.*

Proof Let $\mu = \kappa^{+\omega}$.

Assume otherwise, and let us pick for all $\rho < \kappa$ a $\mathrm{Col}(\omega, \rho^{+\omega}) \times \mathrm{Col}(\rho^{+\omega+2}, < \kappa)$-name for a function $f_\rho \colon (\mu^+)^{<\omega} \to \mu^+$, witnessing the failure of Chang's conjecture in the generic extension. Note that we can still code this set of names as a subset of $H(\kappa^{+\omega+1})$, A.

Let $j \colon \langle H(\rho^{+\omega+1}), \in, B \rangle \to \langle H(\kappa^{+\omega+1}), \in, A \rangle$ be the subcompact embedding. As in the previous proof, we denote by $\dot{g}_\eta$ the names of the functions coded by B. Let us look at $\dot{f}_\rho$ and the forcing $\mathrm{Col}(\omega, \rho^{+\omega}) \times \mathrm{Col}(\rho^{+\omega+2}, < \kappa)$.

Let $R = j''\rho^{+\omega+1}$.

By the assumption, $\Vdash |f''_\rho[R]^{<\omega} \cap \mu| = \rho^{+\omega+1}$ and by the same argument as before, there is some condition (p_0, p_1) and a minimal ordinal n such that

$$(p_0, p_1) \Vdash |f''_\rho[R]^{<\omega} \cap \kappa^{+n}| = \rho^{+\omega+1}.$$

Since $\rho^{+\omega+1}$ is still regular after the forcing, n must be finite.

Let $\{\dot{a}_\xi \mid \xi < \rho^{+\omega+1}\}$ be a sequence of names of finite sequences of ordinals below $\rho^{+\omega+1}$ such that it is forced by the empty condition that $f_\rho(j(\dot{a}_\xi)) < f_\rho(j(\dot{a}_{\xi'})) < \kappa^{+n}$ for all $\xi < \xi'$.

Since the $\mathrm{Col}(\rho^{+\omega+2}, < \kappa)$ is $\rho^{+\omega+2}$-closed, we can find a condition that below it the value of $\dot{a}_\xi$ is determined only by the first coordinate for all $\xi < \rho^{+\omega+1}$.

There are $\rho^{+\omega}$ many conditions in $\mathrm{Col}(\omega, \rho^{+\omega})$. Therefore, there is a single condition p and a set of size $\rho^{+\omega+1}$ of finite sequences such that p decides the value of all of them. Let $\{b_\xi \mid \xi < \rho^{+\omega+1}\}$ be an enumeration of this set.

Back in $H(\rho^{+\omega+1})$, for every pair of ordinals $\xi < \xi'$ there is $\eta < \rho$ and a condition $q \in \mathrm{Col}(\omega, \eta^{+\omega}) \times \mathrm{Col}(\eta^{+\omega+2}, < \rho)$ that forces $g_\eta(b_\xi) < g_\eta(b_{\xi'}) < \rho^{+n}$.

This defines a coloring $[\rho^{+\omega+1}]^2 \to \rho \times V_\rho$. Let us restrict the coloring to the first $(2^{\rho^{+n}})^+$ elements. By the Erdős–Rado theorem, there is a homogeneous set of cardinality ρ^{+n+1}, H. Let (η, q) be the color of all pairs in H. So for every $\xi < \xi'$ in H,

$$q \Vdash_{\mathrm{Col}(\omega, \eta^{+\omega}) \times \mathrm{Col}(\eta^{+\omega+2}, <\rho)} \dot{g}_\eta(b_\xi) < \dot{g}_\eta(b_{\xi'}) < \rho^{+n}$$

and in particular, after forcing with $\mathrm{Col}(\omega, \eta^{+\omega}) \times \mathrm{Col}(\eta^{+\omega+2}, < \rho)$ below the condition q, there is a set of order type ρ^{+n+1} below ρ^{+n}, which is impossible. $\qquad \square$

Assuming the consistency of a $(+\omega+1)$-subcompact cardinal, it is consistent that κ is $(+\omega+1)$-subcompact and $\square(\kappa^{+\omega+1})$ holds (yet $\square_{\kappa^{+\omega}}^{\star}$ and even the approachability property must fail, by Theorem 13). Therefore in the model of Theorem 14 we may have $\square(\aleph_{\omega+1})$ and therefore $\aleph_{\omega+1} \xnrightarrow{Q^{<\omega}} \aleph_1$.

The assumption in both Theorems 13 and 14 is slightly below the assumption of κ being $\kappa^{+\omega+1}$-supercompact.

Question 1 *Is Chang's conjecture $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ consistent assuming the consistency of a strongly compact cardinal?*

3.2 Magidor–Malitz submodels

In this subsection we investigate which large cardinal assumptions imply the $Q^{<\omega}$-reflection. We first deal with the case $\lambda \xrightarrow{Q^{<\omega}} \kappa$ for λ successor cardinal.

3.2.1 Π_1^1 Subcompact cardinals

The following large cardinal notion was defined by Neeman and Steel in [12]. We will use a slightly different notation than the one used in [12].

Definition 15 A cardinal κ is Π_1^1-$(+\alpha)$-subcompact, if for every $A \subseteq H(\kappa^{+\alpha})$ and Π_1^1-statement ϕ such that $\langle H(\kappa^{+\alpha}), \kappa, \in, A \rangle \models \phi$ there is $\rho < \kappa$ and $B \subseteq H(\rho^{+\alpha})$ such that $\langle H(\rho^{+\alpha}), \in, \rho, B \rangle \models \phi$.[1]

In order to get a general feeling above the place of Π_1^1-subcompact cardinals in the large cardinal hierarchy, we remark that a Π_1^1-$(+0)$-subcompact is weakly compact, while $(+0)$-subcompact cardinal is inaccessible cardinal[2].

Lemma 16 *Let κ be a Π_1^1-$(+\alpha)$-subcompact cardinal, $\alpha < \kappa$. Then there is a stationary subset of κ of $(+\alpha)$-subcompact cardinals.*

Proof Let us note that being a $(+\alpha)$-subcompact cardinal is equivalent to a seemly weaker assumption in which we assume only that j is Σ_1 elementary relative to additional predicate (by coding the full elementary diagram). Using this interpretation, the statement "κ is $(+\alpha)$-subcompact" is Π_1^1-statement and therefore reflects downwards. By adding a predicate C for a club, we obtain the desired result. $\square$

Theorem 17 *Let κ be Π_1^1-$(+\alpha)$ subcompact, $\alpha < \kappa$ successor ordinal and assume that $|H(\kappa^{+\alpha})| = \kappa^{+\alpha}$. Then $\kappa^{+\alpha} \xrightarrow{Q^{<\omega}} < \kappa$.*

---

[1] In [12], Neeman and Steel denoted by Π_1^2-subcompact the large cardinal notion that is denoted here by Π_1^1-$(+1)$-subcompact.

[2] This depends on the precise definition of $H(\chi)$ for singular χ. If we define $H(\chi)$ to be the collection of sets of cardinality $< \chi$ such that every member of them belongs to $H(\chi)$ we conclude that $(+0)$-subcompact cardinal is Mahlo.

Proof Let $\lambda = \kappa^{+\alpha}$.

Let $\mathcal{A}$ be an algebra on λ. $\mathcal{A}$ can be coded by a single predicate on $H(\lambda)$, A. Moreover, we assume that A codes also the truth predicate of $\mathcal{A}$ and that the language of $\mathcal{A}$ contains some bijection between $H(\lambda)$ and λ.

For every formula φ in the language of the model $\mathcal{A}$ with the $Q^{<\omega}$ quantifiers we enrich the language of $\mathcal{A}$ by adding one function symbol. For formula φ of the form $Q^n x_0, \ldots, x_{n-1} \psi(x_0, \ldots, x_{n-1}, b)$, let us add the function symbol $F_\varphi \colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and interpret it such that whenever $\mathcal{A} \models \varphi(b)$ (where $b \in \mathcal{A}$), the function $x \to F_\varphi(x, b)$ is one to one and its image, I, witness φ. Namely, $\forall x_0, \ldots, x_{n-1} \in \mathcal{A}$, $\mathcal{A} \models \psi(F_\varphi(x_0, b), \ldots, F_\varphi(x_{n-1}, b), b)$. We will assume that the truth predicate A contains also the truth value of all formulas in the enriched language.

Let Φ be the Π_1^1 sentence: For every $X \subseteq H(\lambda)$ one of the following cases hold:

1. There is $y \in X$ which is not of the form $\langle \phi, p, x \rangle$ where ϕ is a (Gödel number of a) $Q^{<\omega}$-formula, $x, p \in H(\lambda)$.
2. There is a pair of elements $\langle \phi, p, x \rangle$, $\langle \phi', p', x' \rangle \in X$ with $\langle \phi, p \rangle \neq \langle \phi', p' \rangle$
3. $\mathcal{A} \models \phi(p)$
4. $\phi = Q^n x_0, \ldots, x_{n-1} \varphi(x_0, \ldots, x_{n-1}, p)$ and there are $a_0, \ldots, a_{n-1}$ such that $\forall i < n$, $\langle \phi, p, a_i \rangle \in X$ and $\mathcal{A} \models \neg \varphi(a_0, \ldots, a_{n-1})$.

Where all the truth values are evaluated using the truth predicate and the variation of arity in the last case is solved using standard coding.

Let ρ, B and $j \colon \langle H(\rho^{+\alpha}), \in, B \rangle \to \langle H(\kappa^{+\alpha}), \in, A \rangle$ witness the assumption that κ is Π_1^1-$(+\alpha)$-subcompact relative to the Π_1^1 formula Φ.

Let us claim that $\mathcal{B} = j'' H(\rho^{+\alpha})$ is a $Q^{<\omega}$ elementary substructure of $\mathcal{A}$.

We need to show that for a $Q^{<\omega}$-formula φ, and $b \in \mathcal{B}$, $\mathcal{B} \models \varphi(b)$ if and only if $\mathcal{A} \models \varphi(b)$.

We prove the claim by induction on the complexity of φ. Elementarity for first order quantifiers and connectives follows from the elementarity of j. Let us assume that φ has the form $Q^n x_0, \ldots, x_{n-1} \psi(x_0, \ldots, x_{n-1}, y)$, and that the induction hypothesis holds for all formulas in the complexity level of ψ.

If $\mathcal{A} \models \varphi(b)$, then $g(x) = F_\varphi(x, b)$ enumerates some set I such that for every $a_0, \ldots, a_{n-1} \in I$, $\mathcal{A} \models \psi(a_0, \ldots, a_{n-1})$. By elementarity of j, when restricting g to elements of $\mathcal{B}$ its range will be a subset of $\mathcal{B}$ which is a ψ-block of cardinality $|\mathcal{B}|$.

Let us assume that $\mathcal{B} \models \varphi$. Then, by the elementarity of j,

$$\langle H(\rho^{+\alpha}), \in, B \rangle \models \exists I \text{ unbounded} \, \forall a_0, \ldots, a_{n-1} \in I, \, \psi(a_0, \ldots, a_{n-1}, j^{-1}(b))$$

Let X to be $\{\langle \varphi, b, x \rangle \mid x \in I\}$. So $\langle H(\rho^{+\alpha}), \in, B \rangle$ is a model for the Σ_1^1-statement - there is X witnessing the truth of $Q^n x_0, \ldots, x_{n-1} \psi(x_0, \ldots, x_{n-1}, j^{-1}(b))$.

By Π_1^1-elementarity, the same holds at $\langle H(\kappa^{+\alpha}), \in, A \rangle$, and therefore $\mathcal{A} \models \varphi$. $\square$

Theorem 17 is parallel to Theorem 13. Unfortunately, we do not know how to generalize the stronger result of Theorem 14. For successors of regulars and target $\aleph_1$, Sect. 3.4 gives some partial results.

3.2.2 Inaccessible cardinals

For inaccessible cardinals, the consistency strength seems to be lower.

Theorem 18 *Let κ be Ramsey cardinal. Then for every regular cardinal $\omega < \mu < \kappa$,*
$$\kappa \xrightarrow[Q^{<\omega}]{} \mu.$$

Proof Let $\mathcal{A}$ be an algebra on κ. Let I be a set of indiscernibles for $\mathcal{A}$ and let $\mathcal{B}$ be the substructure of $\mathcal{A}$ generated by the first μ indiscernibles.

As in the previous proof, in order to show that $\mathcal{B} \prec_{Q^{<\omega}} \mathcal{A}$, we may enrich the language of $\mathcal{A}$ by functions that produce witnesses for all $Q^{<\omega}$-formulas that hold in $\mathcal{A}$ and show only that for every $Q^{<\omega}$-formula $\varphi = Q^n x_0, \dots, x_n \psi$, if $\mathcal{B} \models \varphi$ then $\mathcal{A} \models \varphi$.

Let $J \subseteq \mathcal{B}$ be any set of cardinality μ. Every element $a \in J$ can be represented as $f(\alpha_0, \dots, \alpha_{m-1})$ where f is one of the Skolem functions of $\mathcal{A}$ and $\alpha_i \in I$.

Since there are only countably many Skolem functions, f, there is some fixed $f_\star$ and uncountable subset of J, K, such that for every $a \in K$ there are indiscernibles $\alpha_0, \dots, \alpha_{m-1}$ such that $a = f_\star(\alpha_0, \dots, \alpha_{m-1})$. Moreover, if γ is the maximal indiscernible that appears in the description of the parameters of the formula ψ, we may assume that for all $a \in K$, $a = f_\star(\alpha_0, \dots, \alpha_{m-1})$ and the set $\{\alpha_i \mid i < m, \alpha_i \leq \gamma\}$ is fixed (since μ is a regular cardinal, and there are less than μ finite sequences of indiscernibles below γ).

By Δ-system arguments, there is some finite set $r \in I^k$ and a set $\tilde{J} \subseteq I^{m-k}$ such that $|\tilde{J}| = \mu$ and $f_\star(r^\frown s) \in I$ for all $s \in \tilde{J}$. Moreover, we may assume that for all $s \neq s'$, $f_\star(r^\frown s) \neq f_\star(r^\frown s')$ and $\max s < \min s'$ or $\min s > \max s'$. Otherwise, by indiscerniblity, every two members of K were equal.

Since the members of I are indiscernible, and by our assumption on K, we have that for every $\beta_0 < \beta_1 < \cdots < \beta_{(m-k)\cdot n - 1}$ in I if we let

$$b_i = f_\star \left(r^\frown \langle \beta_{(m-k)i}, \beta_{(m-k)i+1}, \dots, \beta_{(m-k)(i+1)-1} \rangle \right)$$

then $\psi(b_0, \dots, b_{n-1})$. This provides a set of cardinality κ in $\mathcal{A}$ which is a ψ-cube. $\square$

3.3 $Q^{<\omega}$-reflection at successor cardinals

In this subsection we will discuss some cases in which the one can force the $Q^{<\omega}$-reflection at successor of regular cardinals, starting from large cardinals at the level of huge cardinals.

We will start with the following technical definition:

Definition 19 Let $\mathbb{P}$ be a forcing notion. We say that $\mathbb{P}$ is κ-$Q^{<\omega}$ preserving if for every algebra of cardinality κ, $\mathcal{A}$, $Q^{<\omega}$-formula φ, and $G \subseteq \mathbb{P}$ a generic filter

$$V \models \mathcal{A} \models \varphi \iff V[G] \models \mathcal{A} \models \varphi.$$

The class of κ-$Q^{<\omega}$ preserving forcings is closed under finite iterations. Note that in order to show that a forcing is κ-$Q^{<\omega}$-preserving, it is enough to show that for every formula of the right signature φ, if $V[G] \models \mathcal{A} \models Q^n\varphi$ then also $V \models \mathcal{A} \models Q^n\varphi$.

Lemma 20 κ-*closed forcing notion is* κ-$Q^{<\omega}$ *preserving.*

Proof Let $\mathbb{P}$ be a κ-closed forcing notion and let $\mathcal{A}$ be a model of cardinality κ. Let φ be a formula and assume that

$$V[G] \models \mathcal{A} \models Q^n x_0, \ldots, x_{n-1}\, \varphi(x_0, \ldots, x_{n-1}, \vec{p}).$$

By induction on the complexity of the formula φ we may assume that the satisfaction of φ is absolute between V and $V[G]$. Let $\dot{I}$ be a $\mathbb{P}$-name for a subset of $\mathcal{A}$, and let $p_0 \in \mathbb{P}$ be a condition that forces $|\dot{I}| = \kappa$ and that $\dot{I}$ is a φ-block. Let us construct, by induction, a decreasing sequence of conditions $p_i \in \mathbb{P}$, of length κ, such that $p_{i+1} \Vdash \check{a}_i \in \dot{I}$ for some $a_i \in \mathcal{A}$, and for every $i \neq j, a_i \neq a_j$.

By the κ-closure - this is possible. Let $J = \{a_i \mid i < \kappa\}$. J is a φ-block since for every $\vec{a} \in J^n$, if we take ξ to be above all indices of the elements of $\vec{a}$,

$$p_\xi \Vdash \mathcal{A} \models \varphi(\vec{a}, \vec{p})$$

and therefore $\mathcal{A} \models \varphi(\vec{a}, \vec{p})$. Thus, in V, J is a φ block of size κ. $\qquad\square$

A forcing $\mathbb{P}$ has the $(\lambda, \kappa, < \zeta)$-c.c. if every set of cardinality λ of conditions, A, has a subset $B \subseteq A$ of cardinality κ such that for every $C \subseteq B, |C| < \zeta$, there is a lower bound for C. We will be interested in the case of $(\kappa, \kappa, < \omega)$-c.c. which is a minor strengthening of κ-Knaster property. We use the following terminology: For a forcing notion $\mathbb{P}$, we say that $\mathbb{P}$ has precaliber-κ if it is $(\kappa, \kappa, < \omega)$-c.c.

Lemma 21 *Assume that* κ *is regular. Every forcing notion that has precaliber-*κ *is* κ-$Q^{<\omega}$ *preserving.*

Proof Let $\mathbb{P}, \mathcal{A}, \dot{I}$ and φ be as in the proof of 20. Let us construct a sequence of conditions $p_i \in \mathbb{P}$ such that $p_i \Vdash \check{a}_i \in \dot{I}$ and for every $i \neq j, a_i \neq a_j$. Note that we cannot assume that p_i is compatible with p_j for every i, j.

Since $\mathbb{P}$ has precaliber-κ, there is a subset $J \subseteq \kappa$ such that for every finitely many elements from $J, \xi_0, \ldots, \xi_{m-1}$, the conditions $p_{\xi_0}, \ldots, p_{xi_{m-1}}$ have a common lower bound.

In particular, for every n elements from $J, \xi_0, \ldots, \xi_{n-1}$, there is a condition $q \in \mathbb{P}$ stronger than $p_{\xi_0}, \ldots, p_{xi_{n-1}}$ and q forces $\varphi(\vec{a}, \vec{p})$ where $\vec{a} = \langle a_{\xi_0}, \ldots, a_{\xi_{n-1}} \rangle$. Therefore, $\{a_\xi \mid \xi \in J\}$ is a φ-block in the ground model. $\qquad\square$

We remark that κ-c.c. forcing notions may not be κ-$Q^{<\omega}$ preserving (e.g. a forcing that adds a branch to a κ-Suslin tree does not preserve the $Q^{<\omega}$ sentence "there is no set of cardinality κ of incompatible elements").

Lemma 22 *If there is a projection from* $\mathbb{P}$ *onto* $\mathbb{Q}$ *and* $\mathbb{P}$ *is* κ-$Q^{<\omega}$ *preserving then* $\mathbb{Q}$ *is also* κ-$Q^{<\omega}$ *preserving.*

A κ-$Q^{<\omega}$ preserving forcing notion does not collapse κ. Otherwise, if $|\kappa| = \mu$ in the generic extension, for some $\mu < \kappa$, then the truth value of the $Q^{<\omega}$ formula $Q^1 x$, $x < \mu$ in the model $\langle \kappa, \leq \rangle$ was changed.

For the next theorem we would like to have a forcing notion that collapses cardinals below a Mahlo cardinal and behaves nicely under iterations and elementary embeddings. There are several such forcing notions in the literature (see [2,5,7,17] and others).

For our results, in order to keep this paper as self-contained as possible, we will use a simple variation of the forcing notion that was defined in [5].

Definition 23 Let $\mu < \kappa$ be regular cardinals. We denote by $\mathbb{EC}(\mu, < \kappa)$ the Easton support product $\prod_{\mu \leq \alpha < \kappa} \mathrm{Col}(\alpha, < \kappa)$, where the product ranges over regular cardinals.

Let $\mathbb{P} = \mathbb{EC}(\mu, < \kappa)$. $\mathbb{P}$ is the set of all partial functions such that:

1. $\mathrm{dom}(f) \subseteq \{\langle \alpha, \beta, \gamma \rangle \mid \mu \leq \alpha < \kappa, \beta \in [\alpha, \kappa) \text{ regular cardinals}, \gamma < \alpha\}$.
2. $f(\alpha, \beta, \gamma) \in \beta$.
3. $|\{\alpha \mid \exists \langle \alpha, \beta, \gamma \rangle \in \mathrm{dom}(f)\} \cap \rho| < \rho$ for all inaccessible ρ.
4. For all $\alpha < \kappa$, $|\{\langle \beta, \gamma \rangle \mid \langle \alpha, \beta, \gamma \rangle \in \mathrm{dom}(f)\}| < \alpha$.

Lemma 24 Let $\mu < \kappa$ be regular cardinals and assume that κ is Mahlo.

1. $\mathbb{P}$ has precaliber-κ and it is μ-closed.
2. $\mathbb{P}$ collapses every cardinal between μ and κ.

Proof Let $\{p_i \mid i < \kappa\}$ be a sequence of conditions in $\mathbb{P}$. Let

$$g(i) = \sup\{\alpha, \beta, \gamma, p_i(\alpha, \beta, \gamma) \mid \langle \alpha, \beta, \gamma \rangle \in \mathrm{dom}(p_i)\}.$$

For all $i < \kappa$, $g(i) < \kappa$ and therefore there is a club $C \subseteq \kappa$ of cardinals such that $\forall \rho \in C$, $\sup g''(\rho) \leq \rho$. Let $\{\rho_i \mid i < \kappa\}$ be an increasing enumeration of all the strongly inaccessible cardinals in C. Note that this is a stationary subset of κ.

Let $q_i = p_i \upharpoonright [\rho_i, \kappa) \times [\rho_i, \kappa)$ namely the function p_i restricted to inputs of the form $\langle \alpha, \beta, \gamma \rangle$ where $\rho_i \leq \alpha, \beta$. By the definition of C, for every $i < j$, q_i and q_j are compatible, since their domain are disjoint - the domain of q_i is a subset of $\rho_{i+1} \times \rho_{i+1} \times \rho_{i+1}$ while the domain of q_i does not contain any triplet of the form $\langle \alpha, \beta, \gamma \rangle$ with $\alpha, \beta < \rho_j$.

Let us look at $r_i = p_i \upharpoonright \rho_i \times \rho_i$. Since ρ_i is strongly inaccessible, $r_i \in V_{\rho_i}$ (its domain is bounded below ρ_i). By fixing some enumeration of V_κ that maps elements of V_ρ to ordinals below ρ for every inaccessible ρ, the function $\rho_i \to r_i$ is equivalent to a regressive function on a stationary set. Therefore, by Fodor's lemma, there is a stationary subset S, such that for all $\rho_i, \rho_j \in S$, $r_i = r_j$. We conclude that for every $\rho_i, \rho_j \in S$, p_i is compatible with p_j. □

Theorem 25 Let $\mu < \kappa \leq \lambda < \delta$ be regular cardinals and assume that there is an elementary embedding $j : V \to M$ with $j(\kappa) = \delta$, $M^{j(\lambda)} \subseteq M$.

Let $\mathbb{P} = \mathbb{EC}(\mu, < \kappa)$ with Easton support and let $\dot{\mathbb{Q}}$ be the $\mathbb{P}$-name for the forcing notion $\mathbb{EC}(\lambda, < \delta)$ as defined in $V^{\mathbb{P}}$.

Then $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models j(\lambda) \xrightarrow[Q^{<\omega}]{} \lambda$.

Proof By Lemma 24, $\mathbb{P}$ has precaliber-κ and $\dot{\mathbb{Q}}$ is forced to have precaliber-δ in $V^{\mathbb{P}}$.

Let $\mathbb{N}$ be the termspace forcing for $\dot{\mathbb{Q}}$ and let $\mathbb{R}$ be $\mathbb{EC}(\lambda, < \delta)^V$. □

Lemma 26 *There is a projection from $\mathbb{R}$ onto $\mathbb{N}$.*

Proof Using the fact that κ is a huge cardinal and in particular δ-supercompact, for every regular cardinal, $\rho \geq \kappa$, $\rho^{<\kappa} = \rho$. In particular, we can construct a bijection between all the nice $\mathbb{P}$-names of ordinals below a regular ρ and ρ (using the fact that $\mathbb{P}$ is κ-c.c.). Using this bijection we can identify a condition in $\mathbb{R}$ as a partial function to names of ordinals in the appropriate domain and get a projection. □

By elementarity, $j(\mathbb{P} * \dot{\mathbb{Q}}) = j(\mathbb{P}) * j(\dot{\mathbb{Q}})$. We want to show that one can find a weak master condition for this forcing.

$j(\mathbb{P}) = \prod_{\mu \leq \alpha < \delta} \mathrm{Col}(\alpha, < \delta)$. Clearly, $\mathbb{P}$ embeds into $j(\mathbb{P})$. Moreover, $\mathbb{R}$ also embeds into $j(\mathbb{P})$ (and into a disjoint set of coordinates) and thus after forcing with $j(\mathbb{P})$ we have a generic filter $G \subseteq \mathbb{P}$ and a generic filter $H \subseteq \dot{\mathbb{Q}}$. Note that $\tilde{q} = \bigcup_{q \in H} j(q)$ is a condition - its domain is bounded by $\sup j " \delta < j(\delta)$ and it is clearly Easton. For every inaccessible $\alpha < \delta$, and every $\beta < j(\alpha)$, $\tilde{q}(\beta)$ is the union of at most α many functions. Therefore, by the closure of the forcing, $\tilde{q}$ is a condition.

Using Silver criteria for extending elementary embeddings to generic extension, one can extend j by forcing with $j(\mathbb{P}) * j(\dot{\mathbb{Q}})/(\mathbb{P} * \dot{\mathbb{Q}})$. $j(\dot{\mathbb{Q}})$ is a $j(\mathbb{P})$-name for a highly directed closed (at least δ-closed). We claim that $j(\mathbb{P}) * j(\dot{\mathbb{Q}})/(\mathbb{P} * \dot{\mathbb{Q}})$ is $j(\lambda)$-$Q^{<\omega}$-preserving.

By the structure of the projection - we can split the discussion into two parts. First, note that $j(\dot{\mathbb{Q}})$ is forced to be $j(\lambda)$-closed in $V^{j(\mathbb{P})}$, and therefore $j(\lambda)$-$Q^{<\omega}$-preserving. $j(\mathbb{P})/(\mathbb{P} * \dot{\mathbb{Q}})$ is the quotient of two forcing notions of preclibre-δ. If $j(\lambda) > \delta$, then since the forcing notion $j(\mathbb{P})/(\mathbb{P} * \dot{\mathbb{Q}})$ has cardinality $< j(\lambda)$, it automatically has precaliber $j(\lambda)$. Otherwise, we use the following claim:

Lemma 27 *The quotient of two precaliber-δ forcing notions by a projection that respects maximal lower bounds of finite sets has preclibre δ.*

Proof Let $\pi: \mathbb{R} \to \mathbb{S}$ be a projection. Let $\dot{K}$ be the canonical name for the generic filter for $\mathbb{S}$. Let $\dot{I}$ be a name for a subset of conditions in $\mathbb{R}/\dot{K}$ of size δ that has no subset of cardinality δ in which every finitely many conditions are compatible.

Let us pick conditions $s_i \in \mathbb{S}$ such that $s_i \Vdash \check{r}_i \in \dot{I}$, r_i are all distinct and $i < \delta$. Note that in particular, $s_i \leq \pi(r_i)$. Since $\mathbb{S}$ has precaliber-δ, there is a subset of δ of cardinality δ, J, such that any finitely many conditions $s_{\xi_0}, \ldots, s_{\xi_{n-1}}, \xi_0, \ldots, \xi_{n-1} \in J$, are compatible. Since $\mathbb{R}$ has precaliber-δ, there is a set $J' \subseteq J$ of cardinality δ such that for every $\xi_0, \ldots, \xi_{n-1} \in J'$, $r_{\xi_0}, \ldots, r_{\xi_{n-1}}$ are compatible. Note that is s is the maximal lower bound of $s_{\xi_0}, \ldots, s_{\xi_{n-1}}$ and r is the maximal lower bound of $r_{\xi_0}, \ldots, r_{\xi_{n-1}}$ then $s \leq \pi(r)$ and therefore $s \Vdash \check{r} \in \mathbb{R}/\dot{K}$.

Let us show that there is a condition $s \in \mathbb{S}$ that forces $\{\xi \in J' \mid s_\xi \in \dot{K}\}$ is unbounded. Otherwise, by the chain condition of $\mathbb{S}$, there was a bound that was forced by the weakest condition of $\mathbb{S}$, β. But for $\xi > \beta, \xi \in J'$, s_ξ provides the contradiction. □

Let $\mathcal{A}$ be an algebra on $j(\lambda)$. Let us extend j to an embedding $\tilde{j}$ by forcing with $j(\mathbb{P} * \dot{\mathbb{Q}})/(\mathbb{P} * \dot{\mathbb{Q}})$. In $M[j(G * H)]$, $\tilde{j}''\mathcal{A}$ is an elementary substructure of $j(\mathcal{A})$ of cardinality $j(\lambda)$. We want to show that it is $Q^{<\omega}$-elementary.

Since the forcing $j(\mathbb{P} * \dot{\mathbb{Q}})/(\mathbb{P} * \dot{\mathbb{Q}})$ is $j(\lambda)\text{-}Q^{<\omega}$ preserving, every $Q^{<\omega}$ formula that holds in $j''\mathcal{A}$ (and hence in $\mathcal{A}$) in $M[j(G)][j(H)]$, holds in $\mathcal{A}$ in $V[G][H]$ as well. Therefore, it holds in $j(\mathcal{A})$ - as wanted. □

Taking $\lambda = \kappa$ and μ regular we obtain $\mu^{++} \xrightarrow[Q^{<\omega}]{} \mu^+$. The proof shows that the result holds also for languages of cardinality $< \kappa$, so we can write $\mu^{++} \xrightarrow[Q^{<\omega}]{\mu} \mu^+$. Assuming that GCH holds in the ground model, it also holds in the generic extension.

Corollary 28 *It is consistent, relative to a huge cardinal, that* $\aleph_3 \xrightarrow[Q^{<\omega}]{} \aleph_2$ *and* GCH *holds.*

Using $\lambda = \kappa^{+\omega+1}$ in order to obtain a gap we can get:

Corollary 29 *It is consistent, relative to a 2-huge cardinal, that* $\aleph_{\omega\cdot2+1} \xrightarrow[Q^{<\omega}]{} \aleph_{\omega+1}$.

3.4 Magidor–Malitz reflection to $\aleph_1$

In the subsection we will show how to derive instances of $Q^{<\omega}$-reflection from some cardinal to $\aleph_1$ using a sufficiently good ideal. For a survey about ideals and their connection to large cardinals, see [4]. In particular we will obtain the consistency of $\aleph_2 \xrightarrow[Q^{<\omega}]{} \aleph_1$ from a measurable cardinal.

In an unpublished work from 1978, Shelah showed that the instance of Magidor–Malitz reflection, $\aleph_2 \xrightarrow[Q^{<\omega}]{} \aleph_1$, is consistent relative to a Ramsey cardinal. Our proof gives weaker consistency result, but it shows an implication between the existence of sufficiently good generically large cardinal and $Q^{<\omega}$-reflection.

Let $\mathcal{I}$ be an ideal on κ such that:

(1) $\{\alpha\} \in \mathcal{I}$ for all $\alpha < \kappa$.
(2) $\mathcal{I}$ is κ-complete.
(3) The forcing that adds a generic ultrafilter to $\mathcal{P}(\kappa)/\mathcal{I}$ is $\omega + 1$-strategically closed.

Theorem 30 *Assume that there is ideal $\mathcal{I}$ as above and $\diamondsuit(\omega_1)$. Then* $\kappa \xrightarrow[Q^{<\omega}]{} \omega_1$.

Proof The proof follows closely the proof of the completeness theorem for the logic $\mathcal{L}(Q^{<\omega})$ - the first order logic extended by the Magidor–Malitz quantifier. This result requires $\diamondsuit(\omega_1)$ as well. See [6, Section 7.3]. The proof also resemble the proofs in [3]. In this paper, similar methods are used in order to construct models of size $\aleph_1$ that witness the completeness of extensions of $\mathcal{L}(Q^{<\omega})$ (some of them under large cardinal assumptions).

We start with a countable model M_0, and repeatedly add new elements to it. At each step we essentially enlarge M_α by adding some ordinal $\zeta < \kappa$ which is generic over M_α for the forcing $\mathcal{P}(\kappa)/\mathcal{I}$ (i.e. $\{A \in M_\alpha \mid \zeta \in A\}$ is an M_α-generic ultrafilter). Eventually, we will show that we can arrange the limit model M_{ω_1} to be a $Q^{<\omega}$ elementary submodel of some elementary substructure of $H(\chi)$ (χ large enough regular)

of cardinality κ that contains all ordinals below κ. Note that this is the general case, as for any algebra $\mathcal{A}$ on κ, we may assume that $\mathcal{A} \in M_0$.

Throughout the rest of the proof, χ is a regular cardinal above $2^{2^{\kappa}}$.

Lemma 31 *Let M be a countable elementary substructure of $H(\chi)$. There is $\zeta < \kappa$ such that $\{A \in M \mid \zeta \in A\}$ is M-generic for the forcing $(\mathcal{P}(\kappa)/\mathcal{I})^M$. Moreover, $M^{\star} := \{f(\zeta) \mid f : \kappa \to V, \ f \in M\}$ is a proper extension of M and $M^{\star} \prec H(\chi)$.*

Proof Let $\{I_n \mid n < \omega\}$ list all the maximal antichains of the forcing $(\mathcal{P}(\kappa)/\mathcal{I})^M$ in M. Since the forcing, consists of the $\mathcal{I}$-positive sets is σ-strategically closed, there is a sequence of $\mathcal{I}$-positive sets $\langle A_n \mid n < \omega \rangle$ such that $A_{n+1} \subseteq A_n$, $A_n \in I_n$ and $\bigcap_{n < \omega} A_n \notin \mathcal{I}$. Any $\zeta \in \bigcap A_n$ will generate a M-generic filter.

Since for every $x \in M$ the constant function $c_x(\alpha) = x$ is in M, $M \subseteq M^{\star}$. Since the identity function $id(\alpha) = \alpha$ is in M, $\zeta \in M^{\star}$ so $M^{\star}$ is strictly larger than M.

In order to show that $M^{\star} \prec H(\chi)$ we use Tarski-Vaught criterion. Let $\varphi(x, b)$ be a formula with $b \in M^{\star}$, and assume $H(\chi) \models \exists x \varphi(x, b)$. We need to show that $M^{\star} \models \exists x \varphi(x, b)$. $b = g(\zeta)$ for some $g \in M$. Let

$$B = \{\alpha < \kappa \mid H(\chi) \models \exists x \varphi(x, g(\alpha))\}.$$

B is definable from parameters in M and therefore it is a member of M. $B \notin \mathcal{I}$, since otherwise, we would have that $\zeta \notin B$. Thus, applying the axiom of choice inside of M, there is a function $f \in M$ that assign to every element $\alpha \in B$ a witness $f(\alpha)$ such that $\varphi(f(\alpha), g(\alpha))$. In M,

$$B = \{\alpha < \kappa \mid \varphi(f(\alpha), g(\alpha))\}$$

and by elmentarity, the same holds in V. Since $\zeta \in B$, $\varphi(f(\zeta), g(\zeta))$, so $f(\zeta)$ witnesses $M^{\star} \models \exists x \varphi(x, b)$. $\square$

Let us define a sequence of models. $M_0 = M$, $M_{\alpha+1} = M_{\alpha}^{\star}$ (we will define the M_{α}-generic filters more explicitly in the course of the proof). For limit ordinal $\beta \leq \omega_1$, $M_{\beta} = \bigcup_{\alpha < \beta} M_{\alpha}$.

We would like to get that M_{ω_1} witnesses an instance of $\kappa \xrightarrow[Q^{<\omega}]{} \omega_1$. In order to achieve this, during the iteration we will pick the generic elements in a way that will handle any potential counterexample for the reflection $M_{\omega_1} \cap \kappa \prec_{Q^{<\omega}} H(\chi) \cap \kappa$.

Let $\phi(x)$ be a formula (with parameters from M) and let A be an $\mathcal{I}$-positive set. we define the following formula in the language of set theory:

$$\partial_A \phi(w) := \text{``}w \text{ is a function from } \kappa \text{ and } \{\alpha \in A \mid \neg\phi(w(\alpha))\} \in \mathcal{I}\text{''}.$$

For a type $\Phi(x)$ with parameters in M, (not necessary in M), we define

$$\partial_A \Phi(w) = \{\partial_A \phi \mid \phi(x) \in \Phi(x)\}.$$

These types control which types will be omitted in the next step of the construction. If w realizes $\partial_A \Phi$ in M, then for every choice of $\zeta \in A$, except an $\mathcal{I}$-null set, Φ is

realized in $M^\star$. On the other hand, if for every $\zeta \in A$, Φ is realized in $M^\star$ (where it is defined using ζ) then there is some positive set $B \subseteq A$ such that $\partial_B \Phi$ is realized. Otherwise, we could remove, outside of M, an $\mathcal{I}$-null set and verify that this is not the case. This process cannot be done inside of M, since in general $\Phi \notin M$.

One can repeat this process countably many times (using the strategically closure of the forcing) and verify that for a countable set of types $\{\Phi_n(x) \mid n < \omega\}$ if $\partial_A \Phi_n$ is omitted in M for all $n < \omega$ and $A \in M$ then $\Phi_n(x)$ is omitted in $M^\star$.

In M, there are names for a positive sets in $M^\star$. Those are essentially the functions $f : \kappa \to \mathcal{I}^+$ that appear in M. One can define, for a given formula φ, a positive set A and a name of a positive set $\dot{B}$ the formula $\partial_A \partial_{\dot{B}} \varphi$, in the natural way. We define $\partial_A \partial_{\dot{B}} \Phi$ for a type Φ accordingly. We can continue this way and define the derivative of a type relative to any finite sequence of names of positive sets in the iterated forcing (in the narrow sense: the m-th set $\dot{B}$ is a function from κ^m to the positive sets).

Let us enrich the language of set theory by all the members of M (as constants). For simplicity of notations, we will use the fact that M is closed under pairs and we will not distinguish between formulas which are provably equivalent.

Lemma 32 *Let φ be a formula with k free variables. Let $Z \subseteq M$ be a maximal φ-cube. Let Φ be the type*

$$\{\psi(x, p) \mid p \in M, \forall a \in Z, \ \psi(a, p)\} \bigcup \{x \neq a \mid a \in Z\}.$$

If $V \models \neg Q^k \varphi$ then M omits all the derivatives of Φ.

Proof Φ contains the formulas $\varphi(a_0, \dots, a_{k-2}, x)$ for all $a_i \in Z$ and therefore M does not realize Φ, by the maximality of Z.

Let us denote by $\forall^\star \alpha \varphi(\alpha)$ the assertion that $\{\alpha \mid \neg \varphi(\alpha)\} \in \mathcal{I}$.

Assume that M realizes $\partial_{A_0} \partial_{\dot{A}_1} \cdots \partial_{\dot{A}_{m-1}} \Phi$ for some $A_0, \dots, \dot{A}_{m-1} \in M$. So there is some $b \in M$ such that:

$$\forall^\star \alpha_0 \in A_0 \forall^\star \alpha_1 \in \dot{A}_1(\alpha_0) \cdots \forall^\star \alpha_{m-1} \in \dot{A}_{m-1}(\alpha_0, \dots, \alpha_{m-2}) \psi(b(\alpha_0, \dots, \alpha_{m-1}))$$

for every $\psi \in \Phi$.

We may assume that for all $x \in M$, $\forall^\star \alpha_0, \dots \forall^\star \alpha_{m-1} b(\alpha_0, \dots, \alpha_{m-1}) \neq x$. For all relevant ordinal (one which escape all the $\mathcal{I}$-null sets in the quantifiers), this is true by the maximality of Z and the fact that b is "forced" to be different than all members of Z in M. We can complete the rest of the values (which are essentially elements outside the sets in range A_i, $i < m$ and A_0) with dummy values.

Taking $\psi(x)$ to be $\varphi(a_0, \dots, a_{k-2}, x)$ (and omitting the evaluations in the $\dot{A}_i$) we get:

$$\forall^\star \alpha_0 \in A_0 \forall^\star \alpha_1 \in \dot{A}_1 \cdots \forall^\star \alpha_{m-1} \in \dot{A}_{m-1} \varphi(a_0, \dots, a_{k-2}, b(\alpha_0, \dots, \alpha_{m-1}))$$

By the definition of Φ, replacing a_{k-2} by the variable x, we obtain a formula in Φ. So we conclude that:

$$\forall^\star \alpha_0 \in A_0 \forall^\star \alpha_1 \in \dot{A}_1 \cdots \forall^\star \alpha_{m-1} \in \dot{A}_{m-1}$$
$$\forall^\star \beta_0 \in A_0 \forall^\star \beta_1 \in \dot{A}_1 \cdots \forall^\star \beta_{m-1} \in \dot{A}_{m-1}$$
$$\varphi(a_0, \ldots, a_{k-3}, b(\beta_0, \ldots, \beta_{m-1}), b(\alpha_0, \ldots, \alpha_{m-1}))$$

Repeating this process and relabelling:

$$\forall^\star \alpha_0^0 \in A_0 \forall^\star \alpha_1^0 \in \dot{A}_1 \cdots \forall^\star \alpha_{m-1}^0 \in \dot{A}_{m-1}$$
$$\forall^\star \alpha_0^1 \in A_0 \forall^\star \alpha_1^1 \in \dot{A}_1 \cdots \forall^\star \alpha_{m-1}^1 \in \dot{A}_{m-1}$$
$$\vdots$$
$$\forall^\star \alpha_0^{k-1} \in A_0 \forall^\star \alpha_1^{k-1} \in \dot{A}_1 \cdots \forall^\star \alpha_{m-1}^{k-1} \in \dot{A}_{m-1}$$
$$\varphi(b(\alpha_0^0, \ldots, \alpha_{m-1}^0), b(\alpha_0^1, \ldots, \alpha_{m-1}^1), \ldots, b(\alpha_0^{k-1}, \ldots, \alpha_{m-1}^{k-1}))$$

Let us look on this last formula (which is true in M) and let us say that a set D is *solid* iff for all $a_0, \ldots, a_{r-1} \in D$ ($0 \le r \le k$),

$$\forall^\star \alpha_0^0 \in A_0 \forall^\star \alpha_1^0 \in \dot{A}_1 \cdots \forall^\star \alpha_{m-1}^0 \in \dot{A}_{m-1}$$
$$\forall^\star \alpha_0^1 \in A_0 \forall^\star \alpha_1^1 \in \dot{A}_1 \cdots \forall^\star \alpha_{m-1}^1 \in \dot{A}_{m-1}$$
$$\vdots$$
$$\forall^\star \alpha_0^{r-k-1} \in A_0 \forall^\star \alpha_1^{r-k-1} \in \dot{A}_1 \cdots \forall^\star \alpha_{m-1}^{r-k-1} \in \dot{A}_{m-1}$$
$$\varphi(a_0, \ldots, a_{r-1}, b(\alpha_0^0, \ldots, \alpha_{m-1}^0), b(\alpha_0^1, \ldots, \alpha_{m-1}^1), \ldots, b(\alpha_0^{k-r-1}, \ldots, \alpha_{m-1}^{k-r-1}))$$

The empty set is solid. Using Zorn's lemma in M, we can find a maximal solid set, $D \in M$.

Lemma 33 $M \models |D| = \kappa$.

Proof Assume otherwise. We will find $c \in M$ and outside D such that $\{c\} \cup D$ is solid. If $b^{-1}(D)$ is $\mathcal{I}$-positive then there must be some $d \in D$ such that $b^{-1}(\{d\})$ is $\mathcal{I}$-positive, and we assumed that this is not the case.

Let us iteratively narrow down, in V, the positive sets $A_0, \dot{A}_1, \ldots$ and replace the quantifier $\forall^\star$ by $\forall$. We would still remain with a positive sets. Moreover, we may assume that all of them are disjoint from $b^{-1}(D)$. Pick any $\alpha_0 \in A_0, \alpha_1 \in \dot{A}_1(\alpha_0), \ldots, \alpha_{m-1} \in \dot{A}_{m-1}(\alpha_0, \ldots, \alpha_{m-2})$. Let $c = b(\alpha_0, \ldots, \alpha_{m-1})$. $c \notin D$ and for every $a_i \in D$, $\varphi(a_0, \ldots, a_{k-2}, c)$, as wanted. $\square$

We conclude that D is a φ-cube of cardinality κ. But by elementarity, D is a φ-cube in V as well. $\square$

In order to finish the proof we need to explain how to choose the sets Z. Here the diamond comes into the picture. Let $\langle S_\alpha \mid \alpha < \omega_1 \rangle$ a $\diamondsuit(\omega_1)$ sequence. For convenience, we will assume that S_α is a pair (A_α, ϕ_α) where $A_\alpha \subseteq \alpha$ and ϕ_α is a $Q^{<\omega}$-formula $Q^n \varphi$ with parameters in α.

For every $i < \omega_1$, let us choose a bijection between $M_{i+1} \setminus M_i$ and $\omega \cdot (i+1) \setminus \omega \cdot i$. Connecting those bijections we obtain a continuous bijection between M_{ω_1} and ω_1.

For every α, if A_α is a maximal φ_α-cube in M_α, we define Φ_α to be the type which was defined in Lemma 32. Otherwise, we do nothing.

When enlarging M_i to be M_{i+1} we omit the types $\{\Phi_j \mid j \leq i\}$. Let $\psi = Q^n\varphi$ be a $Q^{<\omega}$-formula. Assume that M_{ω_1} satisfies ψ and that Z is a maximal φ-cube. Then on club many points, $Z \cap \alpha$ is a maximal φ-cube. Therefore, there is a point $\alpha < \omega_1$ such that $Z \cap \alpha = A_\alpha$, $\varphi_\alpha = \varphi$. But the corresponding type was not omitted, since it was enlarged, so $V \models \psi$, as needed.

We remark that for successor of a regular cardinal κ, the existence of such an ideal $\mathcal{I}$ is equiconsistent with the existence of a measurable cardinal. Unfortunately, for successor of singular cardinals of countable cofinality, such ideal cannot exist.

Question 2 *Is* $\aleph_{\omega+1} \xrightarrow[Q^{<\omega}]{} \aleph_1$ *consistent?*

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