RATIONAL APPROXIMATIONS TO CERTAIN ALGEBRAIC NUMBERS

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1. Introduction

THE famous theorem of Roth (**3**) states that if α is an algebraic number, not rational, and $\kappa > 2$ then there are only finitely many rational numbers p/q $(q > 0)$ for which

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{\kappa}}. \tag{1}$$

This implies the existence of a positive constant c, depending only on α and κ, such that

$$\left|\alpha - \frac{p}{q}\right| > \frac{c}{q^{\kappa}} \tag{2}$$

for all rational numbers p/q. As is well known, Roth's method does not allow the constant c in (2) to be determined explicitly, and indeed this seems to be true of all other methods subsequent to that of Liouville (**2**), given in 1844. Liouville's comparatively crude result was that (2) holds for all rational p/q if κ is taken equal to the degree of α, and here c can easily be given a simple explicit value.

It is the main purpose of the present paper to obtain results of the type (2), with an explicit value of c, for a certain class of algebraic numbers α. These are essentially algebraic numbers of the form $(a/b)^{1/n}$, and the value of κ in the result depends upon conditions of magnitude satisfied by a and b. Under suitable conditions we are able to prove a result of the type (2) with κ only slightly greater than 2, that is, a result approaching Roth's in precision.

As is well known, there is an intimate connexion between results of the type (2) and results concerning the solutions of a class of Diophantine equations in two unknowns. Generally speaking, a result (2) without an explicit value of c corresponds to an estimate for the *number* of solutions of any Diophantine equation of the class; see, for example, Davenport and Roth (**1**). A result with an explicit value of c corresponds to an estimate for the *size* of any solution. We shall obtain such an estimate for certain equations of the form

$$ax^n - by^n = f(x, y), \tag{3}$$

where a, b, and n ($\geqslant 3$) are integers and $f(x,y)$ is a polynomial with rational coefficients of degree at most $n-3$. Some equations of this type were investigated by Thue (5) in 1918 and his results were subsequently sharpened by Siegel (4), who proved that, provided $|ab|$ is greater than an explicit function of the integer f, the inequality

$$|ax^n - by^n| > f \tag{4}$$

is satisfied by every pair of relatively prime integers x, y, with possibly one exception.

In the present paper we begin by obtaining inequalities of the form (4) with f replaced by a function of x, and holding without exception. We prove

THEOREM 1. *Suppose that $\kappa > 2$ and n, a, b are integers such that $n \geqslant 3$ and*

$$a > h^{\rho}(3n)^{2(\rho-1)}, \tag{5}$$

where $h = a - b > 0$ and

$$\rho = \tfrac{5}{2}(2\kappa - 1)(\kappa - 2)^{-1}. \tag{6}$$

Then

$$|ax^n - by^n| \geqslant C|x|^{n-\kappa} \tag{7}$$

for all integers x, y, where

$$C = (\tfrac{9}{2}an^2)^{-\sigma} \tag{8}$$

and

$$\sigma = (n-2)\{n + h + 1 + (2\rho - 5)(\log\{\tfrac{1}{2}(3n)^4\})^{-1}\log(\tfrac{9}{2}an^2)\}. \tag{9}$$

The results referred to above then follow as direct deductions. That on the solutions of equations of the form (3) is contained in

THEOREM 2. *Suppose that n, a, b are integers such that $n \geqslant 3$ and*

$$a > h^{25/2}(3n)^{23}, \tag{10}$$

where $h = a - b > 0$. Let

$$f(x,y) = \sum_{r+s \leqslant n-3} w_{rs} x^r y^s \tag{11}$$

be a polynomial of degree at most $n-3$ with rational coefficients which have absolute values less than W. If x, y are integers satisfying (3) *then $|x|, |y|$ are both less than M, where*

$$M = \{aW(bC)^{-1}(n-1)(n-2)\}^K, \tag{12}$$

$$K = 1 + 8(\log\{ah^{-25/2}(3n)^{-23}\})^{-1}\log(9hn^2), \tag{13}$$

and C is given by (8) *and* (9) *with $\kappa = 3 - K^{-1}$.*

The statement on an explicit estimate for the constant c in (2) for certain algebraic numbers α, depending on κ, is given precisely by

THEOREM 3. *Suppose the hypotheses of Theorem* 1 *hold. Let U, V, X, Y be integers and*

$$\alpha = \frac{Ua^{1/n} + Vb^{1/n}}{Xa^{1/n} + Yb^{1/n}} \tag{14}$$

be irrational. Then (2) *holds for all integers* p, q, $q > 0$, *with* c *given by*

$$c = 3^{1-n}(2a)^{-1} H^{-\kappa}(2+|\alpha|)^{1-\kappa} C, \tag{15}$$

where

$$H = \max(|U|, |V|, |X|, |Y|) \tag{16}$$

and C *is given by* (8) *and* (9).

The proof of Theorem 1 depends on obtaining a sequence of rational approximations to $(a/b)^{1/n}$ almost as good as the convergents and in which the denominators increase comparatively slowly. For this purpose the hypergeometric function is used in a similar manner to that found in Siegel (**4**). Thus the proof of Theorem 3 does not depend on Roth's method or indeed on that of the earlier Thue–Siegel theorem.

I am indebted to Prof. Davenport for first introducing me to the problems discussed here and for his constant encouragement while I was working on them.

2. Lemmas

In the following lemmas, $F(\alpha, \beta, \gamma, x)$ denotes the hypergeometric function, given by

$$F(\alpha, \beta, \gamma, x) = \sum_{m=0}^{\infty} \left\{ \prod_{k=0}^{m-1} \frac{(\alpha+k)(\beta+k)}{(1+k)(\gamma+k)} \right\} x^m. \tag{17}$$

It satisfies the differential equation

$$x(x-1)\frac{d^2 F}{dx^2} + \{(1+\alpha+\beta)x - \gamma\}\frac{dF}{dx} + \alpha\beta F = 0. \tag{18}$$

LEMMA 1. *Suppose that* n *is an integer* $\geqslant 3$, *and that* $\nu = 1/n$. *For each positive integer* r, *let* $A_r(x)$, $B_r(x)$ *denote*

$$F(-\nu - r, -r, -2r, x), \quad F(\nu - r, -r, -2r, x),$$

and let

$$E_r(x) = \frac{F(-\nu + r + 1, r+1, 2r+2, x)}{F(-\nu + r + 1, r+1, 2r+2, 1)}. \tag{19}$$

Then, for every x *such that* $0 < x < 1$,

$$A_r(x) - (1-x)^{\nu} B_r(x) = x^{2r+1} A_r(1) E_r(x). \tag{20}$$

Proof. Let

$$f_1^{(r)}(x) = x^{2r+1} F(-\nu + r + 1, r+1, 2r+2, x), \tag{21}$$

$$f_2^{(r)}(x) = (1-x)^{\nu} B_r(x). \tag{22}$$

Then, by direct differentiation, it follows that $f_1^{(r)}(x)$, $f_2^{(r)}(x)$ both satisfy the hypergeometric differential equation for $A_r(x)$, and they are linearly independent since

$$f_1^{(r)}(0) = 0, \quad f_2^{(r)}(0) = 1. \tag{23}$$

Hence there are real numbers u_1, u_2 such that

$$A_r(x) = u_1 f_1^{(r)}(x) + u_2 f_2^{(r)}(x). \tag{24}$$

Since $A_r(0) = 1$, it follows from (23) that $u_2 = 1$, and from (21), (22), and (24) we obtain

$$A_r(1) = u_1 f_1^{(r)}(1) = u_1 F(-\nu+r+1, r+1, 2r+2, 1).$$

Then (20) results on substituting for u_1 and u_2 in (24).

LEMMA 2. *Suppose the hypotheses of Lemma* 1 *hold. Then, for every* x *such that* $0 < x < 1$,

$$A_r(x) = A_r(1) F(-\nu-r, -r, 1-\nu, 1-x), \tag{25}$$

$$A_r(1) = \frac{(r!)^2}{(2r)!} \prod_{m=1}^{r} \left(1 - \frac{\nu}{m}\right), \tag{26}$$

$$E_r(0) = \frac{(r!)^2}{(2r+1)!} \nu \prod_{m=1}^{r} \left(1 + \frac{\nu}{m}\right), \tag{27}$$

$$A_r(1) < A_r(x) < 1, \tag{28}$$

$$E_r(0) < E_r(x) < 1. \tag{29}$$

Proof. By definition, $A_r(x)$ and $B_r(x)$ are polynomials. The polynomial $F(-\nu-r, -r, 1-\nu, 1-x)$ also satisfies the hypergeometric differential equation for $A_r(x)$, and hence is linearly dependent on $A_r(x)$ and $(1-x)^\nu B_r(x)$. However, if it were not linearly dependent on $A_r(x)$ then $(1-x)^\nu$ could be expressed as a rational function of x. This contradiction proves (25).

From Gauss's formula

$$F(\alpha, \beta, \gamma, 1) = \frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)} \tag{30}$$

we obtain, on putting $x = 0$ in (25),

$$\begin{aligned} A_r(1) &= \frac{\Gamma(r+1)\,\Gamma(r+1-\nu)}{\Gamma(1-\nu)\,\Gamma(2r+1)} \\ &= \frac{(r!)\,(r-\nu)\,(r-\nu-1)\ldots(1-\nu)\,\Gamma(1-\nu)}{\Gamma(1-\nu)\,(2r)!}, \end{aligned}$$

and (26) follows. Similarly, from (19) and (30),

$$\begin{aligned} E_r(0) &= \{F(-\nu+r+1, r+1, 2r+2, 1)\}^{-1} \\ &= \frac{\Gamma(r+1+\nu)\,\Gamma(r+1)}{\Gamma(2r+2)\,\Gamma(\nu)} \\ &= \frac{(r+\nu)\,(r+\nu-1)\ldots\nu\Gamma(\nu)\,(r!)}{(2r+1)!\,\Gamma(\nu)}, \end{aligned}$$

and (27) follows.

Finally, since all the coefficients in the series for $F(-\nu-r, -r, 1-\nu, x)$ and the series for $F(-\nu+r+1, r+1, 2r+2, x)$ are positive, we obtain from (25) and (19), if $0<x<1$,

$$A_r(1) < A_r(x) < A_r(0),$$
$$E_r(0) < E_r(x) < E_r(1),$$

where $A_r(0) = 1$ and $E_r(1) = 1$. This completes the proof of Lemma 2.

LEMMA 3. *Suppose the hypotheses of Lemma 1 hold. Let a, h be positive integers, $a > 12h$, and let s_r be the product of all the prime factors of $r!$ which divide n, counted with multiplicities. For each positive integer r, let*

$$q_r = \binom{2r}{r} s_r (an)^r A_r\left(\frac{h}{a}\right), \tag{31}$$

and let p_r be given by (31) with $A_r(h/a)$ replaced by $B_r(h/a)$. Then p_r, q_r are integers such that $p_r > 0$,

$$(\tfrac{3}{2}an)^r < q_r < (\tfrac{9}{2}an^2)^r, \tag{32}$$

and

$$q_{r+1} > q_r. \tag{33}$$

Proof. By definition,

$$\begin{aligned} A_r(x) &= \sum_{m=0}^{r} \left\{ \prod_{k=0}^{m-1} \frac{r+\nu-k}{2r-k} \right\} \binom{r}{m} (-x)^m \\ &= \sum_{m=0}^{r} l_r^{(m)} n^{-m} \frac{(r!)^2}{(2r)!\, m!} \binom{2r-m}{r} (-x)^m, \end{aligned}$$

where

$$l_r^{(m)} = \prod_{k=r-m+1}^{r} (kn+1).$$

If p is a prime which does not divide n then, for every positive integer j, exactly $[mp^{-j}]$ of the numbers $1, 2, \ldots, m$, and at least $[mp^{-j}]$ of the m factors in $l_r^{(m)}$, are divisible by p^j. Hence the exponent to which p divides $m!$ is

$$\sum_{j=1}^{\infty} [mp^{-j}],$$

and this is not greater than the exponent to which p divides $l_r^{(m)}$. Thus, for each r, the coefficients of

$$\binom{2r}{r} s_r n^r A_r(x) \tag{34}$$

are integers relative to all p which do not divide n. As for primes p which divide n, the power of p in $m!$ is cancelled by that in s_r. Hence the coefficients are integers. Similarly the coefficients of (34) with $A_r(x)$ replaced by $B_r(x)$ are integers, and since $A_r(x), B_r(x)$ are polynomials of degree r it follows that p_r and q_r are integers.

To prove (32) we first deduce bounds for s_r. We have

$$\begin{aligned}\log s_r &= \sum_{\substack{p\mid n\\ p\text{ prime}}} \log p\left(\sum_{j=1}^{\infty}[rp^{-j}]\right)\\ &\leqslant r\sum_{p\mid n}(p-1)^{-1}\log p\\ &< r\sum_{p\mid n}\log p\\ &\leqslant r\log n,\end{aligned}$$

so that

$$1 \leqslant s_r < n^r.$$

Then

$$\begin{aligned}q_r &= s_r\sum_{m=0}^{r} l_r^{(m)}(m!)^{-1}\binom{2r-m}{r}(an)^{r-m}(-h)^m\\ &\leqslant n^r\,2^{2r}\sum_{m=0}^{r} l_r^{(m)}(m!)^{-1}(an)^{r-m}h^m\\ &\leqslant (4n)^r\sum_{m=0}^{r}\left(\prod_{k=r-m+1}^{r}\tfrac{4}{3}kn\right)(m!)^{-1}(an)^{r-m}h^m\\ &= (2n)^{2r}\sum_{m=0}^{r}\binom{r}{m}a^{r-m}(\tfrac{4}{3}h)^m\\ &= (2n)^{2r}(a+\tfrac{4}{3}h)^r\end{aligned}$$

and, since $a > 12h$, we obtain the right-hand inequality of (32).

For the left-hand inequality of (32) we use Lemma 2. From (25),

$$\begin{aligned}A_r(x) &= A_r(1)\sum_{m=0}^{r}\left\{\prod_{k=0}^{m-1}\frac{(r+\nu-k)(r-k)}{(1+k)(1-\nu+k)}\right\}(1-x)^m\\ &= A_r(1)\sum_{m=0}^{r}\left\{\prod_{k=1}^{m}(kn-1)\right\}^{-1} l_r^{(m)}\binom{r}{m}(1-x)^m.\end{aligned}$$

Hence, from (26),

$$\begin{aligned}q_r &= s_r(r!)^{-1}\sum_{m=0}^{r}\left\{\prod_{k=m+1}^{r}(kn-1)\right\}l_r^{(m)}\binom{r}{m}a^{r-m}(a-h)^m \qquad (35)\\ &\geqslant (r!)^{-1}\sum_{m=0}^{r}\left\{\prod_{k_1=r-m+1}^{r}k_1 n\right\}\left\{\prod_{k_2=m+1}^{r}\tfrac{2}{3}k_2 n\right\}\binom{r}{m}a^{r-m}(a-h)^m\\ &\geqslant (r!)^{-1}n^r\sum_{m=0}^{r}\left\{\prod_{k_1=r-m+1}^{r}k_1\right\}\left\{\prod_{k_2=1}^{r-m}k_2\right\}\binom{r}{m}(\tfrac{2}{3}a)^{r-m}(a-h)^m\\ &= n^r\sum_{m=0}^{r}\binom{r}{m}(\tfrac{2}{3}a)^{r-m}(a-h)^m\\ &= n^r(\tfrac{5}{3}a-h)^r,\end{aligned}$$

and (32) follows, since $a > 12h$. Similarly we see that $p_r > 0$.

Finally, since

$$s_r \leqslant s_{r+1}, \quad l_r^{(m)} \leqslant l_{r+1}^{(m)}, \quad \binom{r}{m} \leqslant \binom{r+1}{m},$$

and

$$r+1 < (r+1)n-1,$$

it follows from (35) that

$$q_r < s_{r+1}\{(r+1)!\}^{-1} \sum_{m=0}^{r} \left\{ \prod_{k=m+1}^{r+1} (kn-1) \right\} l_{r+1}^{(m)} \binom{r+1}{m} a^{r+1-m}(a-h)^m.$$

Now q_{r+1} exceeds the right-hand side by a term corresponding to $m = r+1$. Hence we obtain (33), and Lemma 3 is proved.

LEMMA 4. *Suppose the hypotheses of Lemma* 3 *hold. Then*

$$(2n)^{-1}\, 48^{-r} < \left(\frac{a}{h}\right)^{2r+1} \left\{ \frac{a}{a-h} - \left(\frac{p_r}{q_r}\right)^n \right\} < 2^{n+1}. \tag{36}$$

Proof. Put $x = h/a$ in (20). Then

$$\frac{a}{a-h}\left\{1 - t_r\left(\frac{h}{a}\right)^{2r+1}\right\}^n = \left(\frac{p_r}{q_r}\right)^n, \tag{37}$$

where

$$t_r = A_r(1)\, E_r\!\left(\frac{h}{a}\right) \left\{ A_r\!\left(\frac{h}{a}\right) \right\}^{-1}.$$

From (28) and (26),

$$1 > A_r\!\left(\frac{h}{a}\right) > A_r(1) > 2^{-2r}(\tfrac{2}{3})^r,$$

and, from (29) and (27),

$$1 > E_r\!\left(\frac{h}{a}\right) > E_r(0) > (r!)^2\{(2r+1)!\,n\}^{-1} \geqslant n^{-1}\, 2^{-3r-1}.$$

Hence

$$(2n)^{-1}\, 48^{-r} < t_r < 1. \tag{38}$$

From (37) we obtain first

$$\begin{aligned} \frac{a}{a-h} - \left(\frac{p_r}{q_r}\right)^n &= \frac{a}{a-h} \sum_{m=1}^{n} \binom{n}{m} \left(\frac{h}{a}\right)^{m(2r+1)} (-1)^{m-1} t_r^m \\ &< 2\left(\frac{h}{a}\right)^{2r+1} \sum_{m=1}^{n} \binom{n}{m} \left(\frac{h}{a}\right)^{(m-1)(2r+1)} t_r^m \\ &< 2\left(\frac{h}{a}\right)^{2r+1} \sum_{m=1}^{n} \binom{n}{m} \\ &< 2^{n+1}\left(\frac{h}{a}\right)^{2r+1}, \end{aligned}$$

and secondly, using (38),

$$\frac{a}{a-h}-\left(\frac{p_r}{q_r}\right)^n=\frac{a}{a-h}\left(1-\left\{1-t_r\left(\frac{h}{a}\right)^{2r+1}\right\}^n\right)$$
$$>1-\left\{1-t_r\left(\frac{h}{a}\right)^{2r+1}\right\}$$
$$>(2n)^{-1}48^{-r}\left(\frac{h}{a}\right)^{2r+1}.$$

Hence Lemma 4 is proved.

LEMMA 5. *Suppose that δ, ζ, η are positive and that n, a, b are integers such that $n \geqslant 3$ and $a > b > 0$. Suppose also that there is a sequence of pairs of integers u_r, v_r $(r = 1, 2, \ldots)$ such that, for each r, $u_r > 0$,*

$$0 < v_r < v_{r+1} < v_r^{1+\eta}, \tag{39}$$

and

$$|av_r^n - bu_r^n| < v_r^{n-2+\zeta}. \tag{40}$$

Let u, v be integers such that $u > 0$, $v \geqslant v_1$, and

$$|av^n - bu^n| < v^{n-2-\delta}. \tag{41}$$

If

$$1+\eta \leqslant (1-\zeta)(1+\delta) \tag{42}$$

then there is an integer j such that $u/v = u_j/v_j$ and $v_j < v^{1+\delta}$.

Proof. From (42) we see that $\zeta < 1$, and it follows from (40) that

$$bu_r^n > av_r^n - v_r^{n-2+\zeta}$$
$$> av_r^n - v_r^{n-1}$$
$$\geqslant \tfrac{1}{2}av_r^n,$$

that is

$$u_r > \left(\frac{a}{2b}\right)^{1/n} v_r.$$

Similarly, from (41) we obtain

$$u > \left(\frac{a}{2b}\right)^{1/n} v.$$

Let j be the integer such that

$$v_{j+1}^{(1+\delta)^{-1}} \geqslant v > v_j^{(1+\delta)^{-1}}. \tag{43}$$

Then $v_j < v^{1+\delta}$ and from (39), (42), and (43) we obtain

$$v \leqslant v_{j+1}^{(1+\delta)^{-1}}$$
$$< v_j^{(1+\eta)(1+\delta)^{-1}}$$
$$\leqslant v_j^{1-\zeta}.$$

Hence

$$\begin{aligned} v^n|av_j^n - bu_j^n| + v_j^n|av^n - bu^n| & \\ &< v^n v_j^{n-2+\zeta} + v_j^n v^{n-2-\delta} \\ &< v^{n-1} v_j^{n-1} + v_j^{n-1} v^{n-1} \\ &= 2(vv_j)^{n-1}. \end{aligned}$$

However, if $u/v \neq u_j/v_j$ then

$$\begin{aligned} & v^n|av_j^n - bu_j^n| + v_j^n|av^n - bu^n| \\ &\geq b|(vu_j)^n - (uv_j)^n| \\ &= b|vu_j - uv_j|\{(vu_j)^{n-1} + (vu_j)^{n-2}(uv_j) + \ldots + (uv_j)^{n-1}\} \\ &\geq bn(a/2b)^{(n-1)/n} v^{n-1} v_j^{n-1} \\ &\geq n(vv_j)^{n-1} \\ &\geq 3(vv_j)^{n-1}. \end{aligned}$$

This contradiction proves Lemma 5.

It may be remarked here that if each pair u_r, v_r is relatively prime then there are no integers u, v satisfying the hypotheses of Lemma 5. For no multiple of v_j occurs in the range (43).

3. Proof of Theorem 1

If x^n, y^n have opposite signs then (7) holds. Hence, without loss of generality, we consider only pairs of positive integers x, y.

The hypotheses of Theorem 1 on a, h, and n imply those of Lemmas 3 and 4, and hence there is a sequence of pairs of positive integers p_r, q_r $(r = 1, 2, \ldots)$ such that (32), (33), and (36) hold. Let $\kappa' = 3\kappa(\kappa+1)^{-1}$,

$$\zeta = \tfrac{4}{5}(\kappa'-2)(\kappa'-1)^{-1}, \tag{44}$$

and

$$\eta = \tfrac{1}{5}(\kappa'-2), \tag{45}$$

so that equality holds in (42) with $\delta = \kappa' - 2$. We now prove that if

$$r > \sigma(n-2)^{-1} - 1 \tag{46}$$

then

$$q_r^{n-\kappa'} < aq_r^n - bp_r^n < q_r^{n-2+\zeta} \tag{47}$$

and

$$q_r < q_{r+1} < q_r^{1+\eta}. \tag{48}$$

(i) We first prove the right-hand inequality of (47). Since $\kappa > 2$, it follows that $\rho > 5/2$ and hence, from (5),

$$\begin{aligned} a &> 2^{1-\frac{1}{2}\rho} h^\rho (3n)^{2(\rho-1)} \\ &= (\tfrac{9}{2}n^2)^{\rho-1}(2h^2)^{\frac{1}{2}\rho}. \end{aligned}$$

Using $\rho = 2\zeta^{-1}$ we obtain

$$a^{\zeta} > (\tfrac{9}{2}n^2)^{2-\zeta}\, 2h^2.$$

From (46), $r \geqslant n+h$ so that

$$\begin{aligned} a^{1+r\zeta} &> a(\tfrac{9}{2}n^2)^{r(2-\zeta)}\,(2h^2)^r \\ &> b(\tfrac{9}{2}n^2)^{r(2-\zeta)}\, 2^{n+h}\, h^{2r} \\ &\geqslant b2^{n+1}\, h^{2r+1}(\tfrac{9}{2}n^2)^{r(2-\zeta)}. \end{aligned} \tag{49}$$

From (32), (36), and (49) it follows that

$$\begin{aligned} \frac{a}{b}-\left(\frac{p_r}{q_r}\right)^n &< 2^{n+1}\left(\frac{h}{a}\right)^{2r+1} \\ &= 2^{n+1}\, h^{2r+1}\, a^{r(\zeta-2)}\, a^{-(1+r\zeta)} \\ &< b^{-1}(\tfrac{9}{2}an^2)^{r(\zeta-2)} \\ &< b^{-1}\, q_r^{\zeta-2}. \end{aligned}$$

(ii) Secondly, we deduce the left-hand inequality of (47). From (5) and (46),

$$r > 5(\{\kappa'-2\}\log\{\tfrac{1}{2}(3n)^4\})^{-1}\log(\tfrac{9}{2}an^2) \tag{50}$$

$$\begin{aligned} &> (\eta\log\{\tfrac{1}{2}(3n)^4\})^{-1}\log\{\tfrac{1}{2}(3n)^5\} \\ &> \eta^{-1} \end{aligned}$$

and from (5) again,

$$a \geqslant (3n)^{2(\rho-1)} > 9^{5(\kappa'-2)^{-1}} > 48^{\frac{1}{4}\eta^{-1}}.$$

Hence

$$\begin{aligned} a^{5r\eta-1} &> a^{4r\eta} \\ &> 48^r \\ &> 48^r(\tfrac{3}{2}n)^{-r\kappa'}\, 2nb^{-1}. \end{aligned} \tag{51}$$

Then using (32), (36), and (51) we obtain

$$\begin{aligned} \frac{a}{b}-\left(\frac{p_r}{q_r}\right)^n &> \left(\frac{h}{a}\right)^{2r+1} 48^{-r}(2n)^{-1} \\ &= h^{2r+1}\, 48^{-r}(2n)^{-1}\, a^{-r\kappa'}\, a^{5r\eta-1} \\ &> b^{-1}(\tfrac{3}{2}na)^{-r\kappa'} \\ &> b^{-1}\, q_r^{-\kappa'}. \end{aligned}$$

(iii) To prove the right-hand inequality of (48) we again use (50). This implies that

$$(3n)^{\frac{3}{8}(\kappa'-2)} > (\tfrac{9}{2}an^2)^{r^{-1}}(\tfrac{3}{2}n)^{-\frac{1}{8}(\kappa'-2)},$$

that is

$$(3n)^{3\eta} > (\tfrac{9}{2}an^2)^{r^{-1}}(\tfrac{3}{2}n)^{-\eta}.$$

Then, since $2\rho = 5+\eta^{-1}$, it follows from (5) that

$$\begin{aligned} a^{\eta} &\geqslant (3n)^{2\eta(\rho-1)} \\ &= (3n)^{1+3\eta} \\ &> (3n)\,(\tfrac{9}{2}an^2)^{r^{-1}}\,(\tfrac{3}{2}n)^{-\eta} \end{aligned}$$

and hence

$$a^{r\eta-1} > (\tfrac{9}{2}n^2)^{r+1}\,(\tfrac{3}{2}n)^{-r(1+\eta)}. \tag{52}$$

From (32) and (52) we obtain

$$\begin{aligned} q_{r+1} &< (\tfrac{9}{2}an^2)^{r+1} \\ &= (\tfrac{9}{2}n^2)^{r+1}\,a^{1-r\eta}\,a^{r(1+\eta)} \\ &< (\tfrac{3}{2}an)^{r(1+\eta)} \\ &< q_r^{1+\eta}. \end{aligned}$$

Hence (47) and (48) are proved.

It follows from Lemma 5 that if x and y are positive integers such that

$$x \geqslant q_R,$$

where

$$R = [\sigma(n-2)^{-1}], \tag{53}$$

and if

$$|ax^n - by^n| < x^{n-\kappa'},$$

then there is an integer j such that $y/x = p_j/q_j$ and $q_j < x^{\kappa'-1}$. However $\kappa > \kappa'(\kappa'-1) > \kappa'$, and for each $r \geqslant R$ we have

$$|aq_r^n - bp_r^n| > q_r^{n-\kappa'}.$$

Hence

$$|ax^n - by^n| > x^{n-\kappa}$$

for all positive integers x, y such that $x \geqslant q_R$. The condition (5) implies that if x, y are integers not both zero then $ax^n \neq by^n$. For otherwise there are integers w, x_1, y_1, such that $a = wx_1^n$, $b = wy_1^n$, $x_1 \neq y_1$, and, by application of the mean-value theorem, it would follow that

$$1 \leqslant |x_1 - y_1| = |(a^{1/n} - b^{1/n})\,w^{-1/n}| \leqslant hn^{-1}\,a^{1/n-1} < h^{-1/2},$$

a contradiction. Hence if x, y are integers such that

$$0 < x < q_R$$

then, using (32) and (53), we see that

$$|ax^n - by^n| \geqslant 1 > (\tfrac{9}{2}an^2)^{-R(n-2)}\,x^{n-\kappa} \geqslant Cx^{n-\kappa}.$$

This completes the proof of Theorem 1.

4. Proof of Theorem 2

Let $\kappa = 3 - K^{-1}$ and let ρ be given by (6). Then, from (13),

$$\begin{aligned} a &= h^{25/2}(3n)^{23}\,(9hn^2)^{8(K-1)^{-1}} \\ &> h^{25/2}(3n)^{23}\,(9hn^2)^{15(K-1)^{-1}/2} \\ &= h^{\rho}(3n)^{2(\rho-1)}. \end{aligned}$$

Hence the hypotheses of Theorem 1 are satisfied and it follows that

$$|ax^n - by^n| \geqslant C|x|^{n-\kappa} \tag{54}$$

for all integers x, y.

Suppose that x, y are integers, not both zero, which satisfy (3). We now distinguish three cases.

(i) If $|y| \leqslant |x|$ then

$$\begin{aligned} |f(x,y)| &\leqslant W\{1 + 2|x| + \ldots + (n-2)|x|^{n-3}\} \\ &\leqslant \tfrac{1}{2}W(n-1)(n-2)|x|^{n-3}. \end{aligned} \tag{55}$$

From (3), (54), and (55) it follows that

$$C|x|^{n-\kappa} \leqslant \tfrac{1}{2}W(n-1)(n-2)|x|^{n-3}$$

and hence

$$|x| \leqslant \{\tfrac{1}{2}C^{-1}W(n-1)(n-2)\}^K < M.$$

(ii) If $|y| > |x|$ and

$$|ax^n - by^n| \leqslant \tfrac{1}{2}b|y|^n$$

then

$$\tfrac{1}{2}b|y|^n \leqslant a|x|^n,$$

so that

$$|y| \leqslant \left(\frac{2a}{b}\right)^{1/n}|x|.$$

From $|y| > |x|$ alone, we obtain

$$\begin{aligned} |f(x,y)| &\leqslant W\{1 + 2|y| + \ldots + (n-2)|y|^{n-3}\} \\ &\leqslant \tfrac{1}{2}W(n-1)(n-2)|y|^{n-3}. \end{aligned} \tag{56}$$

Hence

$$|f(x,y)| \leqslant \tfrac{1}{2}W(n-1)(n-2)\left(\frac{2a}{b}\right)^{(n-3)/n}|x|^{n-3}. \tag{57}$$

From (3), (54), and (57) it follows that

$$C|x|^{n-\kappa} \leqslant \tfrac{1}{2}W(n-1)(n-2)\left(\frac{2a}{b}\right)^{(n-3)/n}|x|^{n-3}.$$

Hence

$$|x| \leqslant \left\{\tfrac{1}{2}C^{-1}W(n-1)(n-2)\left(\frac{2a}{b}\right)^{(n-3)/n}\right\}^K$$

and

$$\begin{aligned} |y| &\leqslant \left(\frac{2a}{b}\right)^{1/n}|x| \\ &< \{aW(bC)^{-1}(n-1)(n-2)\}^K \\ &= M. \end{aligned}$$

(iii) Finally we suppose that $|y| > |x|$ and

$$|ax^n - by^n| > \tfrac{1}{2}b|y|^n.$$

Then the condition $|y|>|x|$ implies that (56) holds, and from this and (3) we obtain

$$\tfrac{1}{2}b\,|y|^n < \tfrac{1}{2}W(n-1)(n-2)\,|y|^{n-3},$$

that is

$$|y| < \{b^{-1}\,W(n-1)(n-2)\}^{\frac{1}{3}} < M.$$

Hence Theorem 2 is proved.

5. Proof of Theorem 3

Let $\nu = 1/n$ and $\varphi = e^{2\pi i\nu}$. It follows from (7) of Theorem 1 that

$$\prod_{k=0}^{n-1} |a^\nu x\varphi^k - b^\nu y| \geqslant C\,|x|^{n-\kappa} \tag{58}$$

for all integers x, y. Suppose that $x \neq 0$ and

$$|a^\nu x - b^\nu y| < 1. \tag{59}$$

Then

$$|b^\nu y| < 1 + |a^\nu x|,$$

and using $a < \tfrac{3}{2}b$ we see that $|y| \leqslant 2|x|$. Hence

$$\prod_{k=1}^{n-1} |a^\nu x\varphi^k - b^\nu y| \leqslant \prod_{k=1}^{n-1} (|a^\nu x\varphi^k| + |b^\nu y|)$$
$$\leqslant (3a^\nu |x|)^{n-1},$$

and from (58) it follows that

$$|a^\nu x - b^\nu y| \geqslant C(3a^\nu)^{1-n}\,|x|^{1-\kappa}. \tag{60}$$

We note that (60) remains valid if (59) does not hold.

Now suppose that p, q are integers such that $q > 0$ and

$$\left|\alpha - \frac{p}{q}\right| < 1,$$

so that

$$|p| < (1+|\alpha|)\,q.$$

We suppose also that $p \neq 0$, for otherwise (2) clearly holds. Put

$$x = Xp - Uq,$$
$$y = Vq - Yp.$$

Then

$$|x| < H(2+|\alpha|)\,q,$$

and hence from (60), if $x \neq 0$,

$$|(Xa^\nu + Yb^\nu)p - (Ua^\nu + Vb^\nu)q|$$
$$> C(3a^\nu)^{1-n}\{H(2+|\alpha|)\,q\}^{1-\kappa}. \tag{61}$$

If $x = 0$ then (61) continues to hold, for α is irrational so that $UY \neq VX$ and hence

$$|U\{(Xa^\nu + Yb^\nu)p - (Ua^\nu + Vb^\nu)q\}|$$
$$= b^\nu |(UY - VX)p| \geqslant 1 > |U| H^{1-\kappa}.$$

Finally, since

$$|Xa^\nu + Yb^\nu| < 2Ha^\nu,$$

(2) follows and Theorem 3 is proved.

Note added in proof. The proof of Theorem 2 can be modified to give the following result. If $n \geqslant 3$ and $f(x,y)$ satisfies the conditions of Theorem 2 then all integral solutions $x, y, z, x^n \neq y^n$, of

$$z(x^n - y^n) = x^n + y^n + f(x,y)$$

satisfy

$$|z| < \{2^{n+5}\sqrt{(W+1)}\}^5.$$

We show that it suffices to treat the case $x > y > \frac{1}{4}z > 0$ and the proof then follows on the same lines as that of Theorem 2 with $a = z+1$ and $h = 2$.

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