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P-adic logarithmic forms and group varieties I

***Dedicated to Professor Alan Baker on the occasion of his forthcoming* 60*th birthday on* 19 *August* 1999**

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1. Introduction

There is an extensive literature on estimates for linear forms in p-adic logarithms; for an historical account see [24]. In fact the p-adic theory has followed closely its classical counterpart beginning with works of Mahler and Gelfond in the 1930's, and especially, since 1966, the celebrated work of Baker on linear forms in logarithms of algebraic numbers. In recent years, Kunrui Yu ([24], [25], [26]) has succeeded in establishing p-adic analogues of Baker [3], Theorem 2 and [2], Sharpening II. Nevertheless, the work predates the introduction of multiplicity estimates on group varieties (see Wüstholz [20], [21], [22]); this was used by Baker and Wüstholz in their fundamental memoir [5]. The purpose of the present paper is to bring the p-adic theory more in line with the complex theory as in [5].

Let $\alpha_1, \ldots, \alpha_n$ $(n \geqq 1)$ be non-zero algebraic numbers and K be a number field containing $\alpha_1, \ldots, \alpha_n$ with $d = [K:\mathbb{Q}]$. Denote by $\mathfrak{p}$ a prime ideal of the ring $\mathcal{O}_K$ of integers in K, lying above the prime number p, by $e_\mathfrak{p}$ the ramification index of $\mathfrak{p}$, and by $f_\mathfrak{p}$ the residue class degree of $\mathfrak{p}$. For $\alpha \in K$, $\alpha \neq 0$, write $\operatorname{ord}_\mathfrak{p} \alpha$ for the exponent to which $\mathfrak{p}$ divides the principal fractional ideal generated by α in K; define $\operatorname{ord}_\mathfrak{p} 0 = \infty$. We shall estimate $\operatorname{ord}_\mathfrak{p} \Xi$, where

$$\Xi = \alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1 \tag{1.1}$$

with $b_1, \ldots, b_n$ being rational integers and $\Xi \neq 0$. Let

$$h_j = \max\big(h_0(\alpha_j), \log p\big) \quad (1 \leqq j \leqq n),$$

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where $h_0(\alpha)$ for algebraic α is defined by (1.6) in the sequel, and let B satisfy (1.8) below. Then as a simple consequence of Theorem 1, we have

$$\operatorname{ord}_{\mathfrak{p}} \Xi < 12\left(\frac{6(n+1)d}{\sqrt{\log \mathfrak{p}}}\right)^{2(n+1)} (p^{f_{\mathfrak{p}}}-1)\log(e^5 nd)h_1\cdots h_n \log B\,.$$

Write $K_{\mathfrak{p}}$ for the completion of K with respect to the exponential valuation $\operatorname{ord}_{\mathfrak{p}}$; and the completion of $\operatorname{ord}_{\mathfrak{p}}$ will be denoted again by $\operatorname{ord}_{\mathfrak{p}}$. Now let Σ be an algebraic closure of $\mathbb{Q}_p$. Write $\mathbb{C}_p$ for the completion of Σ with respect to the valuation of Σ, which is the unique extension of the valuation $|\ |_p$ of $\mathbb{Q}_p$. Denote by $|\ |_p$ the valuation on $\mathbb{C}_p$, and by ord_p the exponential valuation on $\mathbb{C}_p$. Hence $|\gamma|_p = p^{-\operatorname{ord}_p \gamma}$ for all $\gamma \in \mathbb{C}_p$. According to Hasse [8], pp. 298–302, we can embed $K_{\mathfrak{p}}$ into $\mathbb{C}_p$: there exists a $\mathbb{Q}$-isomorphism ψ from K into Σ such that $K_{\mathfrak{p}}$ is value-isomorphic to $\mathbb{Q}_p(\psi(K))$, whence we can identify $K_{\mathfrak{p}}$ with $\mathbb{Q}_p(\psi(K))$. Thus

$$\operatorname{ord}_{\mathfrak{p}}\gamma = e_{\mathfrak{p}} \operatorname{ord}_p \gamma \quad \text{for all } \gamma \in K_{\mathfrak{p}}\,.$$

From now on to the end of the paper, except for § 2, § 3, § 4 and § 6, we assume that K satisfies the following condition:

$$\text{(1.2)} \qquad \begin{cases} \zeta_3 \in K, & \text{if } p = 2\,; \\ \text{either } p^{f_{\mathfrak{p}}} \equiv 3 \pmod 4 \text{ or } \zeta_4 \in K, & \text{if } p > 2\,, \end{cases}$$

where $\zeta_m = e^{2\pi i/m}$ $(m = 1, 2, \ldots)$. Set

$$\text{(1.3)} \qquad q = \begin{cases} 2, & \text{if } p > 2\,, \\ 3, & \text{if } p = 2\,. \end{cases}$$

Let $\mathbb{N}$ be the set of non-negative rational integers and set

$$\text{(1.4)} \qquad u = \max\{k \in \mathbb{N} \mid \zeta_{q^k} \in K\}, \quad \alpha_0 = \zeta_{q^u}\,.$$

Define

$$\text{(1.5)} \qquad h'(\alpha_j) = \max\left(h_0(\alpha_j), \frac{f_{\mathfrak{p}} \log p}{d}\right) \quad (1 \leqq j \leqq n)\,,$$

where $h_0(\alpha)$ denotes the absolute logarithmic Weil height of an algebraic number α, i.e.,

$$\text{(1.6)} \qquad h_0(\alpha) = \delta^{-1}\left(\log a_0 + \sum_{i=1}^{\delta} \log\max(1, |\alpha^{(i)}|)\right),$$

where the minimal polynomial for α is

$$a_0 x^\delta + \alpha_1 x^{\delta-1} + \cdots + a_\delta = a_0(x-\alpha^{(1)})\cdots(x-\alpha^{(\delta)}), \quad a_0 > 0\,.$$

Denote by

$$\text{(1.7)} \qquad \Phi(\mathfrak{p}) = p^{f_{\mathfrak{p}}} - 1$$

the Euler function of the prime ideal $\mathfrak{p}$. Let B be a real number satisfying

$$B \geqq \max(|b_1|, \ldots, |b_n|, 3). \tag{1.8}$$

Let $\omega_q(n)$ $(n = 1, 2, \ldots)$ be the two sequences (for $q = 2, 3$) of positive rational numbers, defined in § 5. We note that $\omega_q(n) \leqq n!/2^{n-1}$ for $n = 1, 2, \ldots$.

Theorem 1. *Suppose that*

$$\operatorname{ord}_{\mathfrak{p}} \alpha_j = 0 \quad (1 \leqq j \leqq n). \tag{1.9}$$

If $\Xi = \alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1 \neq 0$, *then*

$$\operatorname{ord}_{\mathfrak{p}} \Xi < C(n, d, \mathfrak{p})\, h'(\alpha_1) \cdots h'(\alpha_n) \log B,$$

where

$$C(n, d, \mathfrak{p}) = c a^n \cdot \frac{n^n (n+1)^{n+2}}{n!} \cdot \omega_q(n) \cdot \frac{\Phi(\mathfrak{p})}{q^u} \cdot \left(\frac{d}{f_{\mathfrak{p}} \log p}\right)^{n+2} \max\left(f_{\mathfrak{p}} \log p, \log\left(e^4 (n+1) d\right)\right), \tag{1.10}$$

with

$$a = 16, \qquad c = 1544, \qquad \text{if } p > 2,$$

$$a = 32, \qquad c = 81, \qquad \text{if } p = 2;$$

furthermore we can take $a = 8(p-1)/(p-2)$ *when* $p \geqq 5$ *with* $e_{\mathfrak{p}} = 1$.

Finally if $\alpha_1, \ldots, \alpha_n$ *satisfy*

$$[K(\alpha_0^{1/q}, \alpha_1^{1/q}, \ldots, \alpha_n^{1/q}) : K] = q^{n+1}, \tag{1.11}$$

then $C(n, d, \mathfrak{p})$ *can be replaced by* $C(n, d, \mathfrak{p})/\omega_q(n)$.

Let

$$C^*(n, d, \mathfrak{p}) = C(n, d, \mathfrak{p})/(n+1), \tag{1.12}$$

where $C(n, d, \mathfrak{p})$ is given by (1.10).

Theorem 2. *Suppose that* (1.9) *holds and*

$$\operatorname{ord}_p b_n = \min_{1 \leqq j \leqq n} \operatorname{ord}_p b_j. \tag{1.13}$$

Let B, B_n, Ψ *be such that*

$$\max_{1 \leqq j \leqq n} |b_j| \leqq B, \quad |b_n| \leqq B_n \leqq B, \quad \Psi = p^{f_{\mathfrak{p}}} \left(8 n^3 d \log(5d)\right)^n. \tag{1.14}$$

If $\Xi = \alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1 \neq 0$, *then for all real* δ *with*

$$0 < \delta \leqq d^{n-1} h'(\alpha_1) \cdots h'(\alpha_{n-1}) f_{\mathfrak{p}} \log p$$

we have

$$\operatorname{ord}_{\mathfrak{p}} \Xi < C^*(n, d, \mathfrak{p})\, d^{-n} \max\left(d^n h'(\alpha_1) \cdots h'(\alpha_n)\, \tilde{h}, \delta B / B_n\right),$$

where

$$\tilde{h} = 3 \log\left(\delta^{-1} \Psi\left(d^{n-1} h'(\alpha_1) \cdots h'(\alpha_{n-1})\right)^2 B_n\right).$$

If $\alpha_1, \ldots, \alpha_n$ *satisfy* (1.11), *then* $C^*(n, d, \mathfrak{p})$ *can be replaced by* $C^*(n, d, \mathfrak{p}) / \omega_q(n)$, *and* Ψ *in* (1.14) *can be replaced by* $\Psi = \max\left(p^{f_{\mathfrak{p}}}, (5n)^{2n} d\right)$.

It is straightforward to deduce, from Theorems 1 and 2, precise versions in terms of $K_0 = \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$ and $\operatorname{ord}_{\mathfrak{p}_0}$, and without assuming $\operatorname{ord}_{\mathfrak{p}_0} \alpha_j = 0$ $(1 \leqq j \leqq n)$, where $\mathfrak{p}_0$ is a prime ideal of the ring of integers in K_0, also versions for $\alpha_1, \ldots, \alpha_n$ being rational. See [26], §4.

The first improvement in our results upon [26] is the deletion of the factor $\log V$ from [26], Theorem 1 and the factor $\log V_{n-1}$ from [26], Theorem 2. This can be achieved by the multiplicity estimates ([21]); but the numerical values of the constants occurring in our expression for $C(n, d, \mathfrak{p})$ would be greater. In order to obtain the version given here we apply Kummer descent as in [5]. The extent of the descent is determined by the multiplicity estimates and depends here only on n, d and $\mathfrak{p}$; in [26] it depends also on the heights of the α's and this explains the occurrence of the extra factors $\log V$ and $\log V_{n-1}$ mentioned above.

To overcome the essential problem in applying Kummer descent to the p-adic case, we introduce the Vahlen-Capelli Theorem (see, for example, [25], p. 28) into our proof. Hypothesis (1.2) on the ground field K, which is equivalent to [26], $(0.3)^\circ$, is for this purpose.

We adopt the extrapolation and interpolation scheme of [5] with some modifications in the p-adic case (see §11). In this connexion, we have improved upon [24], Lemma 1.4 (based on Hermite interpolation) by our Lemma 2.1, proved by the use of Schnirelman integral [14] (see Adams [1] for a very useful summary), which provides an analogue of the Cauchy integral formula. In the same way, we prove Lemma 2.2 for the interpolation procedure in §11.

Another new feature of the present paper is that when we extrapolate on fractional points s/q (see the proof of Lemma 11.3), instead of considering the valuation of $K' = K(\alpha_0^{1/q}, \alpha_1'^{1/q}, \ldots, \alpha_r'^{1/q})$ at a single prime ideal of $\mathcal{O}_{K'}$ lying above $\mathfrak{p}$, we investigate the valuations at *all* prime ideals $\mathfrak{P}_1, \ldots, \mathfrak{P}_{q^{r_1}}$ of $\mathcal{O}_{K'}$ lying above $\mathfrak{p}$, and estimate

$$\sum_{j=1}^{q^{r_1}} \operatorname{ord}_p^{(j)} \varphi_b^{(I)}\left(\frac{s}{q}; t\right)$$

from below and above, where $\operatorname{ord}_p^{(j)} \beta = e_{\mathfrak{P}_j}^{-1} \operatorname{ord}_{\mathfrak{P}_j} \beta$ for non-zero β in $K'_{\mathfrak{P}_j}$.

All the above mentioned lead to the second improvement, which is on the values of a and c in [26], Theorem 2.1, where $a = 40$, $c = 199586$, if $p > 2$; $a = 48$, $c = 105204$, if $p = 2$.

We remark that $\omega_q(n)$ is a reduced cost for removing Kummer condition (1.11). In [25] and [26], it is basically $n!/2^{n-1}$. It is not difficult to deduce from the explicit formulae for $\omega_q(n)$ in § 5 that

$$\frac{\omega_2(n)}{n!/2^{n-1}} \leqq n^{-\sqrt{n}} \quad (n \geqq 83), \quad \frac{\omega_3(n)}{n!/2^{n-1}} \leqq (2/3)^n \cdot n^{-n^{2/3}} \quad (n \geqq 2833).$$

Note also that in our definition of modified height (1.5), we have deleted the complex logarithm $\log \alpha_j$ from [26], (0.7)* and (0.9)*. See § 4.

In a subsequent paper, *P*-adic logarithmic forms and group varieties II, we shall show that we can replace the term n^{n-1} that occurs in the expressions for $C(n, d, \mathfrak{p})$ and $C^*(n, d, \mathfrak{p})$ in Theorems 1 and 2 by $c_{\mathfrak{p}}^{n-1}$ where $c_{\mathfrak{p}} = 2e\theta e_{\mathfrak{p}} f_{\mathfrak{p}} \log p$ with θ given by (8.1) in § 8. Plainly this gives an improvement on our results when $n > c_{\mathfrak{p}}$ and this is significant in applications (see e.g. my subsequent joint paper with C. L. Stewart, On the *abc* conjecture II). The idea for the refinement came to the author during a lecture given by Matveev in Oberwolfach in 1996 (see [11]) in which he indicated that he could eliminate a term $n!$ from certain linear form estimates in the complex case. Though stimulated by Matveev's lecture, our work in the p-adic case involves a different approach and it is in substance quite independent.

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2. *P*-adic extrapolation and interpolation

The concepts of normal series and functions are due to Mahler [9]; see also [1] and [24]. For Schnirelman integral, we refer to [1]. For fixed γ, $\Gamma \in \mathbb{C}_p (\Gamma \neq 0)$ and for all $m \in \mathbb{Z}_{>0}$ with $|m|_p = 1$, the sequence

$$\gamma + \Gamma \xi_1^{(m)}, \ldots, \gamma + \Gamma \xi_m^{(m)},$$

where $\xi_1^{(m)}, \ldots, \xi_m^{(m)}$ denote the m-th roots of unity in $\mathbb{C}_p$, is called a discrete circle with centre γ and radius $|\Gamma|_p$. Let $f(z)$ be a $\mathbb{C}_p$-valued function defined for all points on the above discrete circle. Then define, if exists,

$$\int_{\gamma, \Gamma} f(z)\, dz = \lim_{\substack{m \to \infty \\ |m|_p = 1}} m^{-1} \sum_{k=1}^{m} f(\gamma + \Gamma \xi_k^{(m)}).$$

We shall use frequently the fact that

$$\left| \int_{\gamma, \Gamma} f(z)\, dz \right|_p \leqq \sup_{|z-\gamma|_p = |\Gamma|_p} |f(z)|_p. \tag{2.1}$$

Note that the most important property of the Schnirelman integral is the analogue of Cauchy's integral formula, that is, if $f(z)$ is analytic on $|z-\gamma|_p \leqq |\Gamma|_p$ and $x \in \mathbb{C}_p$ satisfies $|x-\gamma|_p < |\Gamma|_p$, then

$$f^{(k)}(x) = k! \int_{\gamma,\Gamma} \frac{f(z)(z-\gamma)}{(z-x)^{k+1}}\, dz, \quad k = 0,1,2,\ldots. \tag{2.2}$$

For $x \in \mathbb{R}_{>0}$, write $\log_p x$ for the usual logarithm of x with base p. In the following Lemmas 2.1 and 2.2, m and r denote rational integers.

Lemma 2.1. *Let $\theta \in \mathbb{Q}_{>0}$, $R \in \mathbb{Z}_{>0}$, $M \in \mathbb{Z}_{>0}$, q be a prime number with $q \neq p$. Suppose that $F(z)$ is a normal function and*

$$\min_{|r| \leqq R,\, 0 \leqq m \leqq M} \left\{ \operatorname{ord}_p \left(\frac{1}{m!} F^{(m)}(rp^\theta) \right) + m\big(\theta + \log_p(2R+1)\big) \right\} \tag{2.3}$$
$$\geqq (M+1)(2R+1)\,\theta + M \log_p(2R+1)\,.$$

Then

$$\operatorname{ord}_p F\left(\frac{l}{q} p^\theta\right) \geqq (M+1)(2R+1)\,\theta \quad \textit{for all } l \in \mathbb{Z}\,.$$

Proof. Obviously we may assume $l/q \neq r$ for all r with $|r| \leqq R$. We distinguish two cases:

(i) $\left|\frac{l}{q} - r_0\right|_p \leqq \frac{1}{2R+1}$ for some r_0 with $|r_0| \leqq R$;

(ii) $\left|\frac{l}{q} - r\right|_p > \frac{1}{2R+1}$ for all r with $|r| \leqq R$.

Note that for $-R \leqq r_1 < r_2 \leqq R$, we have $\operatorname{ord}_p(r_1 - r_2) \leqq \log_p(2R)$, whence

$$|r_1 - r_2|_p \geqq \frac{1}{2R}.$$

Case (i). Let $N \in \mathbb{Z}$ with $N \geqq 2$. Choose arbitrarily non-zero Γ, Γ', Γ_0, $\Gamma^* \in \mathbb{C}_p$ such that

$$|\Gamma|_p = p^{\left(1-\frac{1}{N}\right)\theta}, \quad |\Gamma'|_p < \frac{1}{4}\left|\frac{l}{q} - r_0\right|_p,$$

$$\frac{1}{3} < \frac{|\Gamma_0|_p}{\left|\frac{l}{q} - r_0\right|_p} := \varrho < \frac{1}{2}, \quad |\Gamma^*|_p < \frac{1}{2R+1}.$$

Set

$$f(z) = F(zp^\theta), \quad E(z) = \prod_{|r| \leqq R} (z-r), \quad G(z) = E(z)^{M+1}.$$

By (2.2) with $k = 0$,

$$\int_{l/q,\Gamma'} \frac{f(z)}{G(z)}\, dz = \frac{f(l/q)}{G(l/q)}.$$

Write

$$\mathscr{I} = \int_{0,\Gamma} \frac{f(z)\, z}{(z-l/q)\, G(z)}\, dz\,,$$

$$\mathscr{J} = \int_{r_0,\Gamma_0} \frac{f(z)(z-r_0)}{(z-l/q)\, G(z)}\, dz = \sum_{m=0}^{M} \frac{1}{m!} f^{(m)}(r_0) \int_{r_0,\Gamma_0} \frac{(z-r_0)^{m+1}}{(z-l/q)\, G(z)}\, dz\,,$$

$$\mathscr{K} = \sum_{|r| \leqq R,\, r \neq r_0} \int_{r,\Gamma^*} \frac{f(z)(z-r)}{(z-l/q)\, G(z)}\, dz$$

$$= \sum_{|r| \leqq R,\, r \neq r_0} \sum_{m=0}^{M} \frac{1}{m!} f^{(m)}(r) \int_{r,\Gamma^*} \frac{(z-r)^{m+1}}{(z-l/q)\, G(z)}\, dz\,.$$

By [1], Appendix, Theorem 13, we have

$$\frac{f(l/q)}{G(l/q)} = \mathscr{I} - \mathscr{J} - \mathscr{K}. \tag{2.4}$$

On $|z|_p = |\Gamma|_p$, we have

$$\left| \frac{z}{z-l/q} \right|_p = 1\,, \quad |G(z)|_p = p^{\left(1-\frac{1}{N}\right)(M+1)(2R+1)\theta}\,,$$

and

$$|f(z)|_p = |F(zp^{\theta})|_p \leqq 1\,,$$

since F is a normal function and $|zp^{\theta}|_p < 1$. By $|G(l/q)|_p \leqq 1$ and (2.1), we get

$$\operatorname{ord}_p\big(G(l/q)\mathscr{I}\big) \geqq \left(1-\frac{1}{N}\right)(M+1)(2R+1)\,\theta\,.$$

For every fixed r with $|r| \leqq R$, by $(p,q) = 1$ and a simple counting argument, we have

$$\operatorname{ord}_p \prod_{|s| \leqq R,\, s \neq r} (l/q - s) \geqq \operatorname{ord}_p\big((R+r)!\,(R-r)!\big)\,.$$

Hence

$$\prod_{|s| \leqq R,\, s \neq r} \left| \frac{l/q - s}{r - s} \right|_p \leqq 1\,. \tag{2.5}$$

Now on $|z-r_0|_p = |\Gamma_0|_p$,

$$|z-s|_p = |z-r_0+r_0-s|_p = |r_0-s|_p \quad \text{for all } s \text{ with } |s| \leqq R,\ s \neq r_0\,,$$

whence, by (2.5),

$$\left|\frac{E(l/q)}{E(z)}\right|_p = \frac{|l/q - r_0|_p}{|\Gamma_0|_p} \cdot \prod_{|s| \leqq R,\, s \neq r_0} \left|\frac{l/q - s}{r_0 - s}\right|_p \leqq \frac{1}{\varrho}$$

and

$$\left|\frac{G(l/q)}{G(z)}\right|_p \leqq \left(\frac{1}{\varrho}\right)^{M+1}.$$

Further on $|z - r_0|_p = |\Gamma_0|_p$ we have $|z - l/q|_p = |z - r_0 + r_0 - l/q|_p = |l/q - r_0|_p$. So

$$\left|\frac{(z - r_0)^{m+1}}{z - l/q}\right|_p = \varrho^{m+1} \left|\frac{l}{q} - r_0\right|_p^m \leqq \varrho^{m+1}.$$

Thus by (2.1)

$$\left|G(l/q) \int_{r_0, \Gamma_0} \frac{(z - r_0)^{m+1}}{(z - l/q)\, G(z)}\, dz\right|_p \leqq \varrho^{m-M}.$$

This together with (2.3), $f^{(m)}(r) = F^{(m)}(rp^\theta)\, p^{m\theta}$ and $\frac{1}{3} < \varrho < \frac{1}{2}$ implies

$$\operatorname{ord}_p\big(G(l/q)\mathscr{J}\big) \geqq (M+1)(2R+1)\,\theta.$$

On $|z - r|_p = |\Gamma^*|_p$ with $|r| \leqq R$, $r \neq r_0$, we have

$$|z - l/q|_p = |z - r + r - r_0 + r_0 - l/q|_p = |r - r_0|_p,$$
$$|z - s|_p = |z - r + r - s|_p = |r - s|_p \quad \text{for all } s \text{ with } |s| \leqq R,\ s \neq r,$$

whence, by (2.5) and the fact that $|l/q - r|_p = |r - r_0|_p$, we get

$$\left|\frac{E(l/q)}{E(z)}\right|_p = \left|\frac{l/q - r}{z - r}\right|_p \cdot \prod_{|s| \leqq R,\, s \neq r} \left|\frac{l/q - s}{r - s}\right|_p \leqq \frac{|r - r_0|_p}{|\Gamma^*|_p}$$

and

$$\left|\frac{G(l/q)}{G(z)}\right|_p \leqq \frac{|r - r_0|_p^{M+1}}{|\Gamma^*|_p^{M+1}}.$$

Thus, by (2.1),

$$\left|G(l/q) \int_{r, \Gamma^*} \frac{(z - r)^{m+1}}{(z - l/q)\, G(z)}\, dz\right|_p \leqq |\Gamma^*|_p^{m-M} |r - r_0|_p^M \leqq |\Gamma^*|_p^{m-M}.$$

This together with (2.3) and the fact that $\Gamma^* \in \mathbb{C}_p$ is arbitrarily chosen with $|\Gamma^*|_p < 1/(2R+1)$ implies

$$\operatorname{ord}_p\big(G(l/q)\mathscr{K}\big) \geqq (M+1)(2R+1)\,\theta.$$

Thus by (2.4), we obtain

$$\operatorname{ord}_p F\left(\frac{l}{q}p^{\theta}\right) = \operatorname{ord}_p f\left(\frac{l}{q}\right) \geqq \left(1-\frac{1}{N}\right)(M+1)(2R+1)\,\theta$$

for all $N \in \mathbb{Z}$ with $N \geqq 2$. This proves Lemma 2.1 in Case (i). The proof for Case (ii) is similar and simpler. We omit the details here.

Lemma 2.2. *Let $\theta \in \mathbb{Q}_{>0}$, $R \in \mathbb{Z}_{>0}$, $M \in \mathbb{Z}_{>0}$, q be a prime number with $q \neq p$ and $q|R$. Suppose that $F(z)$ is a normal function and*

$$\min_{\substack{|r| \leqq R,\,(r,q)=1 \\ 0 \leqq m \leqq M}} \left\{\operatorname{ord}_p\left(\frac{1}{m!}F^{(m)}(rp^{\theta})\right) + m\big(\theta + \log_p(2R)\big)\right\} \tag{2.6}$$
$$\geqq 2\left(1-\frac{1}{q}\right)R(M+1)\,\theta + (2M+1)\log_p(2R)\,.$$

Then

$$\operatorname{ord}_p F(lp^{\theta}) \geqq 2\left(1-\frac{1}{q}\right)R(M+1)\,\theta$$

for all $l \in \mathbb{Z}$ with $|l| \leqq R$ and $q|l$.

Proof. First we note that for r_1, r_2 with $-R \leqq r_1 < r_2 \leqq R$ and $(r_i, q) = 1$ $(i = 1, 2)$, we have $\operatorname{ord}_p(r_1 - r_2) \leqq \log_p\big(2(R-1)\big)$, whence

$$|r_1 - r_2|_p \geqq \frac{1}{2(R-1)}\,.$$

Similarly, for l, r with $|l| \leqq R$, $|r| \leqq R$ and $q|l$, $(r, q) = 1$, we have

$$|l - r|_p \geqq \frac{1}{2R-1}\,.$$

Let $N \in \mathbb{Z}$ with $N \geqq 2$. Choose arbitrarily non-zero $\Gamma, \Gamma^* \in \mathbb{C}_p$ such that

$$|\Gamma|_p = p^{\left(1-\frac{1}{N}\right)\theta}, \quad |\Gamma^*|_p < \frac{1}{2R}\,.$$

Set

$$f(z) = F(zp^{\theta}), \quad \tilde{E}(z) = \prod_{|r| \leqq R,\,(r,q)=1} (z - r), \quad \tilde{G}(z) = \tilde{E}(z)^{M+1}\,.$$

By (2.2) with $k = 0$

$$\int_{l,\Gamma^*} \frac{f(z)}{\tilde{G}(z)}\,dz = \frac{f(l)}{\tilde{G}(l)}\,.$$

Write

$$\mathscr{I}' = \int_{0,\Gamma} \frac{f(z)\,z}{(z-l)\,\tilde{G}(z)}\,dz\,,$$

$$\begin{aligned}\mathscr{J}' &= \sum_{|r| \leqq R,\,(r,q)=1} \int_{r,\Gamma^*} \frac{f(z)(z-r)}{(z-l)\,\tilde{G}(z)}\,dz \\ &= \sum_{|r| \leqq R,\,(r,q)=1} \sum_{m=0}^{M} \frac{1}{m!} f^{(m)}(r) \int_{r,\Gamma^*} \frac{(z-r)^{m+1}}{(z-l)\,\tilde{G}(z)}\,dz\,.\end{aligned}$$

By [1], Appendix, Theorem 13, we have

$$\frac{f(l)}{\tilde{G}(l)} = \mathscr{I}' - \mathscr{J}'\,. \tag{2.7}$$

By the fact that $F(z)$ is a normal function and that

$$\sum_{|r| \leqq R,\,(r,q)=1} 1 = 2\left(1 - \frac{1}{q}\right) R\,,$$

we get (similarly to the proof of Lemma 2.1)

$$\operatorname{ord}_p\big(\tilde{G}(l)\mathscr{I}'\big) \geqq \left(1 - \frac{1}{N}\right) \cdot 2\left(1 - \frac{1}{q}\right) R(M+1)\,\theta\,.$$

On $|z-r|_p = |\Gamma^*|_p$ with $|r| \leqq R$, $(r,q) = 1$, we have

$$|z-l|_p = |z-r+r-l|_p = |l-r|_p\,,$$

$$|z-s|_p = |z-r+r-s|_p = |r-s|_p \quad \text{for all } s \text{ with } |s| \leqq R \text{ and } s \neq r\,.$$

Hence

$$\begin{aligned}\left|\frac{\tilde{E}(l)}{\tilde{E}(z)}\right|_p &= \frac{|l-r|_p}{|\Gamma^*|_p} \cdot \prod_{\substack{|s| \leqq R \\ (s,q)=1,\, s \neq r}} \left|\frac{l-s}{r-s}\right|_p \\ &= \frac{|l-r|_p}{|\Gamma^*|_p} \cdot \prod_{\substack{|s| \leqq R \\ s \neq l,\, s \neq r}} \left|\frac{l-s}{r-s}\right|_p \cdot \prod_{\substack{|s| \leqq R \\ q \mid s,\, s \neq l}} \left|\frac{r-s}{l-s}\right|_p\,.\end{aligned}$$

Now by (2.5) with R replaced by R/q, we have

$$\prod_{\substack{|s| \leqq R \\ q \mid s,\, s \neq l}} \left|\frac{r-s}{l-s}\right|_p = \prod_{|s| \leqq \frac{R}{q},\, s \neq \frac{l}{q}} \left|\frac{s-r/q}{s-l/q}\right|_p \leqq 1\,.$$

Further

$$\operatorname{ord}_p \prod_{\substack{|s| \leqq R \\ s \neq l,\, s \neq r}} \frac{s-l}{s-r} = \operatorname{ord}_p \frac{(R+l)!\,(R-l)!}{(R+r)!\,(R-r)!} \geqq -\log_p(2R)\,.$$

Thus on $|z-r|_p = |\Gamma^*|_p$,

$$\left|\frac{\tilde{E}(l)}{\tilde{E}(z)}\right|_p \leqq \frac{|l-r|_p}{|\Gamma^*|_p} \cdot 2R, \quad \left|\frac{\tilde{G}(l)}{\tilde{G}(z)}\right|_p \leqq \frac{|l-r|_p^{M+1}}{|\Gamma^*|_p^{M+1}} \cdot (2R)^{M+1}.$$

By (2.1) we obtain

$$\left|\tilde{G}(l) \int_{r,\Gamma^*} \frac{(z-r)^{m+1}}{(z-l)\,\tilde{G}(z)}\, dz\right|_p \leqq |\Gamma^*|_p^{m-M} (2R)^{M+1}.$$

This together with (2.6), the fact that $f^{(m)}(r) = F^{(m)}(rp^\theta)\, p^{m\theta}$ and that Γ^* is arbitrarily chosen with $|\Gamma^*|_p < 1/(2R)$, implies

$$\operatorname{ord}_p\big(\tilde{G}(l)\mathscr{I}'\big) \geqq 2\left(1-\frac{1}{q}\right) R(M+1)\theta.$$

Finally, from (2.7) we see that

$$\operatorname{ord}_p F(lp^\theta) = \operatorname{ord}_p f(l) \geqq \left(1-\frac{1}{N}\right) \cdot 2\left(1-\frac{1}{q}\right) R(M+1)\,\theta$$

for all $N \in \mathbb{Z}$ with $N \geqq 2$, and the lemma follows at once.

3. Discriminant

We need to estimate the discriminant that always appears in the Bombieri-Vaaler [6] version of Siegel's lemma (see, for example, [5], Lemma 1).

Recall that the absolute height $H(\alpha)$ of an algebraic number α is equal to $e^{h_0(\alpha)}$. With his gracious permission, we include here a lemma (and its proof) of D.W. Masser [10].

Lemma 3.1. *If a number field of degree d is generated over $\mathbb{Q}$ by algebraic numbers of absolute heights at most H, then its discriminant is at most $d^d H^{2d(d-1)}$ in absolute value.*

Proof. For any algebraic number α of degree D we define the numbers

$$\begin{aligned}
\alpha(0) &= 1, \\
\alpha(1) &= a_0\alpha, \\
\alpha(2) &= a_1\alpha + a_0\alpha^2, \\
&\cdots \\
\alpha(D-1) &= a_{D-2}\alpha + \cdots + a_0\alpha^{D-1}
\end{aligned} \tag{3.1}$$

following a talk of W. M. Schmidt, where $a_0x^D + \cdots + a_D$ is the minimal polynomial of α over $\mathbb{Z}$. Since

$$\alpha(\delta) = -(a_\delta + a_{\delta+1}\alpha^{-1} + \cdots + a_D \alpha^{-(D-\delta)}) \quad (1 \leqq \delta < D)$$

and locally either α or α^{-1} is integral, we see that $\alpha(0), \ldots, \alpha(D-1)$ are algebraic integers.

Suppose now $k = \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$ (with $[k:\mathbb{Q}] = d$) for $H(\alpha_i) \leqq H$; and we define

$$[\mathbb{Q}(\alpha_1, \ldots, \alpha_i) : \mathbb{Q}(\alpha_1, \ldots, \alpha_{i-1})] = d_i \quad (1 \leqq i \leqq n)$$

so that $d_1 \cdots d_n = d$. Then the degree D_i of α_i satisfies $d_i \leqq D_i \leqq d_1 \cdots d_i$ $(1 \leqq i \leqq n)$.

For integers $\delta_1, \ldots, \delta_n$ with $0 \leqq \delta_i < d_i$ write

$$\alpha(\delta_1, \ldots, \delta_n) = \alpha_1(\delta_1) \cdots \alpha_n(\delta_n).$$

These are algebraic integers in k, and the matrix M expressing them in terms of the $\alpha_1^{\delta_1} \cdots \alpha_n^{\delta_n}$ is the tensor product (see Yokonuma [23], pp. 16–17) of $M_1, \ldots, M_n$, where M_i expresses $\alpha_i(0), \ldots, \alpha_i(d_i - 1)$ in terms of $1, \alpha_i, \ldots, \alpha_i^{d_i - 1}$. So

$$\det M = \{(\det M_1)^{1/d_1} \cdots (\det M_n)^{1/d_n}\}^d.$$

Also from (3.1) we see that $\det M_i = A_i^{d_i - 1}$, where A_i is the leading coefficient of the minimal polynomial of α_i over $\mathbb{Z}$. It follows that the discriminant $\mathscr{D}$ of the $\alpha(\delta_1, \ldots, \delta_n)$ satisfies

$$(3.2) \qquad |\mathscr{D}| = \{A_1^{(d_1-1)/d_1} \cdots A_n^{(d_n-1)/d_n}\}^{2d} |\mathscr{D}'|,$$

where $\mathscr{D}'$ is the discriminant of the $\alpha_1^{\delta_1} \cdots \alpha_n^{\delta_n}$. Since these latter are linearly independent over $\mathbb{Q}$, we see that $\mathscr{D} \neq 0$, and so $|\mathscr{D}(k)| \leqq |\mathscr{D}|$.

To estimate $\mathscr{D}'$, we note that $\sqrt{|\mathscr{D}'|}$ is the absolute value of a determinant of order d whose entries are the $(\alpha_1^{\delta_1} \cdots \alpha_n^{\delta_n})^\sigma$ as σ runs over all embeddings of k into $\mathbb{C}$. Thus, denoting by $(\delta_1, \ldots, \delta_n)$ the index for row, by σ the index for column, the Euclidean norm of the σ-th column of the determinant is at most $d^{1/2} \max(1, |\alpha_1^\sigma|)^{d_1 - 1} \cdots \max(1, |\alpha_n^\sigma|)^{d_n - 1}$; and by Hadamard inequality

$$\sqrt{|\mathscr{D}'|} \leqq d^{d/2} A_1'^{d_1 - 1} \cdots A_n'^{d_n - 1},$$

where $A_i' = \prod_\sigma \max(1, |\alpha_i^\sigma|)$. But $A_i' = B_i^{d/D_i}$, where B_i is the product of the absolute values of the distinct conjugates of α_i outside the unit circle. Combining with (3.2) and noting $d_i \leqq D_i$, we find

$$|\mathscr{D}| \leqq d^d \{(A_1 B_1)^{(d_1-1)/d_1} \cdots (A_n B_n)^{(d_n-1)/d_n}\}^{2d}.$$

Finally $A_i B_i = H(\alpha_i)^{D_i} \leqq H^{D_i}$ and $D_i \leqq d_1 \cdots d_i$, so we end up with

$$|\mathscr{D}| \leqq d^d \{(H^{d_1})^{(d_1-1)/d_1} (H^{d_1 d_2})^{(d_2-1)/d_2} \cdots (H^{d_1 \cdots d_n})^{(d_n-1)/d_n}\}^{2d} = d^d H^{2d(d-1)}.$$

This completes the proof.

We remark that Lemma 3.1 improves Silverman [15], Theorem 2 (when $F=\mathbb{Q}$ and $S_{\mathbb{Q}}$ contains only the archimedean valuation), in which H is replaced with the (probably larger) "affine" absolute height $H(1, \alpha_1, \ldots, \alpha_n)$. Note also that Lemma 3.1 improves [5], Lemma 2.

4. A problem of Lehmer and linear dependence of logarithms

For Lehmer's problem, we refer to [5], § 8. Recently Voutier [17] proved sharp estimates on this problem. In particular, [17], Corollary 1 states that if α is a non-zero algebraic number of degree $\delta \geqq 2$ which is not a root of unity then

$$h_0(\alpha) > \frac{2}{\delta\big(\log(3\delta)\big)^3}.$$

This together with the fact that if $\beta \in \mathbb{Q}$ and $\beta \neq 0, 1, -1$, then $h_0(\beta) \geqq \log 2$, yields

$$h_0(\alpha) > \frac{2}{\delta\big(\log(5\delta)\big)^3} \tag{4.1}$$

for any non-zero algebraic number α of degree δ which is not a root of unity.

Let K be an algebraic number field and $d = [K:\mathbb{Q}]$.

Lemma 4.1. *Suppose that $l_1, \ldots, l_m \in \mathbb{C}$ are linearly dependent over $\mathbb{Q}$ and $e^{l_j} \in K$ $(1 \leqq j \leqq m)$. Then there exist $t_1, \ldots, t_m \in \mathbb{Z}$, not all zero, such that*

$$t_1 l_1 + \cdots + t_m l_m = 0$$

and

$$|t_k| \leqq \big(4(m-1)\, d^2 \log\log(6d)\big)^{m-1} V_1 \cdots V_m / V_k \quad (1 \leqq k \leqq m),$$

where

$$V_j = V(l_j) := \max\left(h_0(e^{l_j}), \frac{|l_j|}{2\pi d}\right) \quad (1 \leqq j \leqq m). \tag{4.2}$$

Proof. This is a slight improvement upon Waldschmidt [18], Lemma 4.1. By virtue of (4.1) we may replace the definition of $C_0(D)$ and c_j^{-1} in [18] by

$$C_0(d) = 0.5\, d\big(\log(5d)\big)^3, \quad c_j^{-1} = (m-1) \max\left(C_0(d)\, h_0(e^{l_j}), \varphi_{-1}(d) \frac{|l_j|}{2\pi}\right),$$

and the lemma follows at once, observing

$$0.5\big(\log(5x)\big)^3 < 4x \log\log(6x) \quad \text{for } x \geqq 1.$$

For $z \in \mathbb{C}$, $z \neq 0$, write $\operatorname{Log} z = \log|z| + i \arg z$ with $\arg z \in (-\pi, \pi]$.

Lemma 4.2. *For any non-zero $\alpha \in K$, we have*

$$\frac{|\mathrm{Log}\,\alpha|}{2\pi d} \leqq \max\left(h_0(\alpha), \frac{\log 2}{d}\right).$$

Proof. We may assume $|\alpha| \geqq 1$, since $|\mathrm{Log}(\alpha^{-1})| = |\mathrm{Log}\,\alpha|$ and $h_0(\alpha^{-1}) = h_0(\alpha)$. Thus, $0 \leqq \log|\alpha| \leqq d h_0(\alpha)$ by (1.6), and

$$|\mathrm{Log}\,\alpha|^2 \leqq \big(d h_0(\alpha)\big)^2 + \pi^2 \leqq 4\pi^2 \max\big((d h_0(\alpha))^2, (\log 2)^2\big).$$

Remark. Recalling (1.5) and (4.2), Lemma 4.2 implies

$$V(\mathrm{Log}\,\alpha) \leqq h'(\alpha) \tag{4.3}$$

for any non-zero $\alpha \in K$.

5. Successive minima and Kummer descent

For $q = 2, 3$ we define $\omega_q(1) = \omega_q(2) = 1$ and for $n > 2$

$$\omega_2(n) = 4^{s-n} \cdot (s+n+1)!/(2s+1)!\,, \tag{5.1}$$

$$\omega_3(n) = 6^{t-n} \cdot (2t+n+1)!/(3t+1)!\,, \tag{5.2}$$

where

$$s = [1/4 + \sqrt{n + (17/16)}] \tag{5.3}$$

and t is the unique rational integer such that

$$g(t) := 9t^3 - 8t^2 - (8n+5)t - 2n(n+1) \leqq 0 \quad \text{and} \quad g(t+1) > 0\,. \tag{5.4}$$

Hence $t = [x_n]$, where x_n is the unique real zero of $g(x)$, which can be determined explicitly by the Cardano's formula.

Here we record a result due to Weyl [19]. Let Ω be a lattice in $\mathbb{R}^n$ and let f be a distance function determined by a bounded convex body symmetric about the origin. We denote by $\lambda_1, \ldots, \lambda_n$ the corresponding successive minima. Let M_n be a sequence given by $M_n = \omega_2(n)$ $(n = 1, 2, \ldots)$. Then [19], Theorem VI states that there exists a basis $w_1, \ldots, w_n$ for Ω such that

$$f(w_1) \cdots f(w_n) \leqq M_n \lambda_1 \cdots \lambda_n\,. \tag{5.5}$$

Let

$$\mathscr{L}_{K,\mathfrak{p}} = \{l \in \mathbb{C} \,|\, \alpha := e^l \in K \text{ and } \mathrm{ord}_{\mathfrak{p}}\,\alpha = 0\} \tag{5.6}$$

and

(5.7) $$l_j = \operatorname{Log} \alpha_j \quad (0 \leqq j \leqq n),$$

where Log is determined in §4 and α_0 is defined by (1.4). Suppose that $\alpha_1, \ldots, \alpha_n$ satisfy (1.9) and are multiplicatively independent. Thus

(5.8) $l_j \in \mathscr{L}_{K,\mathfrak{p}}$ $(0 \leqq j \leqq n)$ and $l_0, l_1, \ldots, l_n$ are linearly independent over $\mathbb{Q}$.

Now we establish inductively that for each $v = 0, 1, \ldots$ there exist $l_0^{(v)}, \ldots, l_n^{(v)} \in \mathscr{L}_{K,\mathfrak{p}}$ such that for some non-negative integers $v_0, \ldots, v_n$ with $v_0 + \cdots + v_n \leqq v$ we have

(5.9) $$q^v l_j^{(v)} = \sum_{i=0}^{j} l_i m_{ij} \quad (0 \leqq j \leqq n),$$

where the m_{ij} $(0 \leqq j \leqq n, 0 \leqq i \leqq j)$ are integers with

$$m_{jj} = q^{v - v_j}, \quad |m_{ij}| \leqq \frac{1}{2} q^{v - v_j} \quad \text{and} \quad m_{ij} = 0 \quad \text{whenever } v_j = 0.$$

Let $\alpha_j^{(v)} = e^{l_j^{(v)}}$ $(0 \leqq j \leqq n)$. We choose for each $v = 0, 1, \ldots$

(5.10) $$v_0 = 0, \quad l_0^{(v)} = l_0, \quad \alpha_0^{(v)} = \alpha_0;$$

and note, by (1.4), that this is the only possible choice for v_0, $l_0^{(v)}$, $\alpha_0^{(v)}$. By construction, $\alpha_0^{(v)}, \ldots, \alpha_n^{(v)}$ will satisfy the Kummer condition

(5.11) $$[K(\alpha_0^{(v)1/q}, \ldots, \alpha_n^{(v)1/q}) : K] = q^{n+1}$$

as soon as the equation $v_0 + \cdots + v_n = v$ fails to hold for $v + 1$, and we shall prove shortly that the latter must occur for some v.

For $v = 0$, we simply take $l_j^{(v)} = l_j$ and $v_j = 0$ $(0 \leqq j \leqq n)$. Suppose that $l_0^{(v)}, \ldots, l_n^{(v)}$ have been constructed to satisfy the inductive hypothesis with $v_0 + \cdots + v_n = v$. Then either (5.11) holds or it does not. In the former case we put $l_j^{(v+1)} = l_j^{(v)}$ $(0 \leqq j \leqq n)$ and take the same values of $v_0, \ldots, v_n$. In the latter case, let k be the least positive integer such that

$$[K(\alpha_0^{(v)1/q}, \ldots, \alpha_k^{(v)1/q}) : K] < q^{k+1}.$$

We take the same values of $v_0, \ldots, v_{k-1}, v_{k+1}, \ldots, v_n$ and replace v_k by $v_k + 1$. (Thus the equality $v_0 + \cdots + v_n = v$ holds for $v + 1$.) Put $l_j^{(v+1)} = l_j^{(v)}$ $(0 \leqq j < k)$, and replace m_{ij} $(0 \leqq i \leqq j < k)$ by $q m_{ij}$. On noting that $\zeta_q \in K$ by (1.2) and (1.3), Baker and Stark [4], Lemma 3 gives

(5.12) $$l_k^{(v)} = m_0 l_0^{(v)} + \cdots + m_{k-1} l_{k-1}^{(v)} + q l$$

for some $l \in \mathscr{L}_{K,\mathfrak{p}}$ with $m_i \in \{0, 1, \ldots, q-1\}$ $(0 \leqq i < k)$. From (5.9) and (5.12) we see that

$$q^{v+1} l = \sum_{i=0}^{k} l_i m'_{ik}$$

for some integers m'_{ik} with $m'_{kk} = m_{kk}$. Now write $m'_{k-1,k} = r_{k-1} - s_{k-1} q^{v+1-v_{k-1}}$, where r_{k-1}, s_{k-1} are integers with $|r_{k-1}| \leqq \frac{1}{2} q^{v+1-v_{k-1}}$. So

$$q^{v+1}(l + s_{k-1} l_{k-1}^{(v+1)}) = \sum_{i=0}^{k-2} l_i m''_{ik} + l_{k-1} r_{k-1} + l_k m'_{kk}.$$

Repeat this process, we can find integers r_i, s_i $(i = k-2, \ldots, 0)$ with $|r_i| \leqq \frac{1}{2} q^{v+1-v_i}$ such that

$$q^{v+1}\left(l + \sum_{i=0}^{k-1} s_i l_i^{(v+1)}\right) = l_0 r_0 + \cdots + l_{k-1} r_{k-1} + l_k m'_{kk}.$$

On taking

$$l_k^{(v+1)} = l + \sum_{i=0}^{k-1} s_i l_i^{(v+1)}, \quad m_{ik} = r_i \quad (0 \leqq i < k), \quad m_{kk} = m'_{kk},$$

we see that (5.9) with $j = k$ holds for $v+1$. Similarly, we can prove that (5.9) with $k < j \leqq n$ holds for $v+1$. Note that the equalities $m_{ij} = 0$ $(0 \leqq i < j)$ (whenever $v_j = 0$) follow from the construction, on noting the definition of k.

For each $j = 1, \ldots, n$, let

$$(5.13) \quad \delta_j = \begin{cases} 1, & \text{if } v_j = 0, \\ q^{-1}, & \text{if } v_j > 0, \end{cases} \quad \tau_j = \begin{cases} \delta_j, & \text{if } \delta_j = 1, \\ \max\left(1, \delta_j + \frac{1}{2}(\delta_1 + \cdots + \delta_{j-1})\right), & \text{if } \delta_j = q^{-1}. \end{cases}$$

Thus $\tau_1 = \tau_2 = 1$. Suppose now

$$(5.14) \qquad h'(\alpha_1) \leqq h'(\alpha_2) \leqq \cdots \leqq h'(\alpha_n).$$

Then by (5.9) and the basic properties of the absolute logarithmic Weil height, noting $h_0(\alpha_0) = 0$, we obtain

$$(5.15) \qquad h'(\alpha_j^{(v)}) \leqq \tau_j h'(\alpha_j) \quad (1 \leqq j \leqq n).$$

Let

$$(5.16) \qquad \eta = \eta(\delta_1, \ldots, \delta_n) = \tau_1 \cdots \tau_n.$$

We now prove that

$$(5.17) \qquad \max_{\delta_1, \ldots, \delta_n} \eta(\delta_1, \ldots, \delta_n) = \omega_q(n), \quad n = 1, 2, \ldots,$$

where each $\delta_j = 1$ or q^{-1}, $\omega_q(n)$ is given at the beginning of this section. It is easy to see (5.17) holds for $1 \leqq n \leqq 4$ and we may assume $n \geqq 5$. For each $j = 1, \ldots, n-1$ and any fixed choice of δ_i $(i \neq j, j+1)$, the value of $\eta(\delta_1, \ldots, \delta_n)$ with $\delta_j = 1$, $\delta_{j+1} = q^{-1}$ is not less than that with $\delta_j = q^{-1}$, $\delta_{j+1} = 1$. Further let $\eta_q(i)$ $(0 \leqq i \leqq n)$ be the value of $\eta(\delta_1, \ldots, \delta_n)$ with

$$\delta_k = 1 \quad (1 \leqq k \leqq i), \quad \delta_k = q^{-1} \quad (i < k \leqq n).$$

We have

$$\eta_q(0) \leqq \eta_q(1) \leqq \eta_q(2) \quad \text{and} \quad \eta_q(n) = 1 . \tag{5.18}$$

Thus

$$\max_{\delta_1, \ldots, \delta_n} \eta(\delta_1, \ldots, \delta_n) = \max_{2 \leqq j < n} \eta_q(j) . \tag{5.19}$$

For $2 \leqq j < n$

$$\begin{aligned} \eta_q(j) &= \prod_{k=j+1}^{n} \left\{ \frac{1}{q} + \frac{1}{2}\left(j + \frac{k-j-1}{q} \right) \right\} \\ &= (2q)^{j-n} \prod_{k=j+1}^{n} \big(k + (q-1)j + 1\big) . \end{aligned} \tag{5.20}$$

Thus for $3 \leqq j < n$

$$\frac{\eta_2(j)}{\eta_2(j-1)} = \frac{2(j+n+1)}{j(2j+1)} \quad \text{and} \quad \frac{\eta_3(j)}{\eta_3(j-1)} = \frac{2(2j+n)(2j+n+1)}{(3j-1)j(3j+1)} .$$

This together with (5.18) gives

$$\eta_2(j)/\eta_2(j-1) \geqq 1 \quad (1 \leqq j \leqq s) \quad \text{and} \quad \eta_2(j)/\eta_2(j-1) < 1 \quad (s < j < n) ,$$

where s is given by (5.3); and

$$\eta_3(j)/\eta_3(j-1) \geqq 1 \quad (1 \leqq j \leqq t) \quad \text{and} \quad \eta_3(j)/\eta_3(j-1) < 1 \quad (t < j < n) ,$$

where t is given by (5.4). Thus (5.17) follows from the fact that for $n > 2$ (5.1) is just $\omega_2(n) = \eta_2(s)$ and (5.2) is exactly $\omega_3(n) = \eta_3(t)$.

By (5.17) we have

$$\omega_q(n) \leqq \omega_q(n+1), \quad n = 1, 2, \ldots .$$

For $n > 2$, we get, from (5.20) and $t \geqq s$,

$$\eta_3(t) \leqq \eta_2(t) \leqq \eta_2(s) \quad \text{and} \quad \frac{\eta_2(s)}{\eta_3(t)} \leqq \frac{\eta_2(s)}{\eta_3(s)} \leqq \left(\frac{3}{2}\right)^{n-s} .$$

On recalling the definition of M_n, these yield

$$1 \leqq \frac{M_n}{\omega_q(n)} \leqq \left(\frac{q}{2}\right)^{n-s}, \quad n = 1, 2, \ldots . \tag{5.21}$$

We now apply the Geometry of Numbers. Consider the distance function

$$F(x) = |x_1| h'(\alpha_1) + \cdots + |x_n| h'(\alpha_n)$$

and the convex body $F(x) < 1$ with volume

$$V(F) = (2^n/n!)\bigl(h'(\alpha_1)\cdots h'(\alpha_n)\bigr)^{-1}.$$

Let Ω be the lattice in $\mathbb{R}^n$ generated by the vectors $v_1, \ldots, v_n$, where

$$v_j = q^{-\nu}(m_{1j}, \ldots, m_{jj}, 0, \ldots, 0) \quad (1 \leqq j \leqq n)$$

with m_{ij} given in (5.9). Ω has determinant $\Delta = q^{-(\nu_1 + \cdots + \nu_n)}$. By Minkowski's theorem, the successive minima $\lambda_1, \ldots, \lambda_n$ of $F(x) < 1$ satisfy

$$\lambda_1 \cdots \lambda_n \leqq 2^n \Delta / V(F).$$

Further, by (5.5), there exists a basis $w_1, \ldots, w_n$ for Ω such that

$$F(w_1) \cdots F(w_n) \leqq M_n \lambda_1 \cdots \lambda_n.$$

We can express w_j uniquely as

$$w_j = v_1 u_{1j} + \cdots + v_n u_{nj} = q^{-\nu}(w_{1j}, \ldots, w_{nj}) \quad (1 \leqq j \leqq n),$$

where u_{ij}, w_{ij} are integers and $\det(u_{ij}) = \pm 1$. Let

$$w_{00} = m_{00}, \quad w_{i0} = 0 \quad (1 \leqq i \leqq n), \quad w_{0j} = \sum_{i=1}^{n} m_{0i} u_{ij} \quad (1 \leqq j \leqq n).$$

Thus

$$l_0'' := q^{-\nu} \sum_{i=0}^{n} l_i w_{i0} = l_0 = l_0^{(\nu)}$$

by (5.10), and

$$(5.22) \qquad l_j'' := q^{-\nu} \sum_{i=0}^{n} l_i w_{ij} = \sum_{i=1}^{n} l_i^{(\nu)} u_{ij} \in \mathscr{L}_{K,\mathfrak{p}} \quad (1 \leqq j \leqq n).$$

By the equality $\det(u_{ij}) = \pm 1$, we see that $l_0'' = l_0, l_1'', \ldots, l_n''$ are linearly independent over $\mathbb{Q}$, $\alpha_j'' = e^{l_j''}$ is not a root of unity $(1 \leqq j \leqq n)$, and

$$[K(\alpha_0''^{1/q}, \ldots, \alpha_n''^{1/q}) : K] = q^{n+1}$$

whenever (5.11) holds. From (4.1), (5.22) and $h_0(\alpha_0) = 0$ we get

$$\frac{2}{d \log^3(5d)} < h_0(\alpha_j'') \leqq F(w_j) \quad (1 \leqq j \leqq n).$$

Hence

$$\left(\frac{2}{d \log^3(5d)}\right)^n < F(w_1) \cdots F(w_n) \leqq q^{-(\nu_1 + \cdots + \nu_n)} n!\, M_n h'(\alpha_1) \cdots h'(\alpha_n).$$

Thus for $\nu = 0, 1, \ldots,$ the value $\nu_0 + \nu_1 + \cdots + \nu_n = \nu_1 + \cdots + \nu_n$ (see (5.10)) is bounded above. Henceforth we denote by ν the least value of ν such that $\nu_0 + \nu_1 + \cdots + \nu_n = \nu$ and the equation fails to hold for $\nu + 1$. Then, as remarked earlier, (5.11) holds.

By (4.1) and (1.5) we have

$$(5.23) \qquad h'(\alpha_j'') \leqq c' h_0(\alpha_j'') \leqq c' F(w_j)\,,$$

where $c' = 0.5\log^3(5d)\cdot f_{\mathfrak{p}}\log p$. We put

$$(5.24) \qquad c^* = c'^n n!\, M_n/\omega_q(n)$$

and define

$$l_j' = l_j^{(v)},\quad \alpha_j' = \alpha_j^{(v)} \quad (0 \leqq j \leqq n),\quad \text{if } q^v \leqq c^*;$$

$$l_j' = l_j'',\quad \alpha_j' = \alpha_j'' \quad (0 \leqq j \leqq n),\quad \text{if } q^v > c^*.$$

Then, in either case, $\alpha_1', \ldots, \alpha_n'$ are $\mathfrak{p}$-adic units and satisfy

$$[K(\alpha_0^{1/q}, \alpha_1'^{1/q}, \ldots, \alpha_n'^{1/q}) : K] = q^{n+1}\,,$$

noting that $\alpha_0' = \alpha_0$. Moreover we have

$$(5.25) \qquad l_j' = q^{-v} L_j(l_0, \ldots, l_n) \quad (0 \leqq j \leqq n)\,,$$

where $L_0, \ldots, L_n$ are linearly independent linear forms in $z_0, \ldots, z_n$ with integer coefficients; indeed L_j is given by

$$\sum_{i=0}^{j} z_i m_{ij} = \sum_{i=0}^{n} z_i m_{ij} \quad \text{or} \quad \sum_{i=0}^{n} z_i w_{ij}\,,$$

where $m_{ij} = 0$ whenever $i > j$. Now we have

$$(5.26) \qquad h'(\alpha_j') \leqq \sigma_j \quad (1 \leqq j \leqq n)\,,$$

where $\sigma_j = \tau_j h'(\alpha_j)$ ((5.15)) if $q^v \leqq c^*$ (when $v = 0$, i.e., (1.11) holds, we simply have $\alpha_j' = \alpha_j$, $\sigma_j = h'(\alpha_j)$) and $\sigma_j = c' F(w_j)$ if $q^v > c^*$ (see (5.23)). Thus

$$(5.27) \qquad \sigma_1 \cdots \sigma_n \leqq \min(1, c^*/q^v)\, \omega_q(n)\, h'(\alpha_1) \cdots h'(\alpha_n)\,,$$

and if $v = 0$ then $\omega_q(n)$ can be deleted from (5.27). Further, we see that, in either case,

$$(5.28) \qquad \sum_{i=1}^{n} |\partial L_j/\partial z_i|\, h'(\alpha_i) \leqq q^v \sigma_j \quad (1 \leqq j \leqq n)\,.$$

Remark. For any permutation of $\alpha_1, \ldots, \alpha_n$ we can maintain (5.25)–(5.28) by applying the same permutation to

$$\{l_1, \ldots, l_n\},\ \{x_1, \ldots, x_n\},\ \{z_1, \ldots, z_n\},\ \{m_{1j}, \ldots, m_{nj}\},\ \{w_{1j}, \ldots, w_{nj}\} \quad (1 \leqq j \leqq n)$$

and modifying accordingly the definitions of the distance function $F(x)$, of $L_1, \ldots, L_n$ and of $\sigma_1, \ldots, \sigma_n$. Thus we may omit (5.14) and still have (5.25)–(5.28).

6. Multiplicity estimates

Here we follow [5] and summarize the multiplicity estimates proved in [21]; note that the proposition given here is a slightly modified version of that in [5].

Let $\mathscr{L}$ be a linear form $B_1 Z_1 + \cdots + B_r Z_r$ with rationals $B_1, \ldots, B_r$ and assume that $B_r \neq 0$. We consider the polynomial ring $\mathbb{C}[Y_0, \ldots, Y_r]$ and we introduce the differential operators

$$\partial_0 = \frac{\partial}{\partial Y_0}, \quad \partial_j = B_r Y_j \frac{\partial}{\partial Y_j} - B_j Y_r \frac{\partial}{\partial Y_r} \quad (1 \leqq j < r).$$

Our proposition concerns a non-zero polynomial $\mathscr{P}(Y_0, \ldots, Y_r)$ defined over $\mathbb{C}$ with degree at most $\mathscr{D}_j$ in Y_j for $j = 0, 1, \ldots, r$.

Let $\mathscr{S}_0 \geqq \cdots \geqq \mathscr{S}_r \geqq 0$, let $\mathscr{T}_0 \geqq \cdots \geqq \mathscr{T}_r \geqq 0$ be integers and suppose that $\mathscr{S}_0 + \cdots + \mathscr{S}_r \leqq \mathscr{S}$ and $\mathscr{T}_0 + \cdots + \mathscr{T}_r \leqq \mathscr{T}$. Suppose further that

$$(6.1) \qquad \mathscr{S}_m \binom{\mathscr{T}_m + m + \delta_{m,r}}{m + \delta_{m,r}} \geqq (m+1)!\, \mathscr{D}_0^{m_0} \cdots \mathscr{D}_r^{m_r}$$

for $m = 0, \ldots, r$ and for all $m_0, \ldots, m_r$, 0 or 1, where $m_0 + \cdots + m_r = m + 1$ and $\delta_{m,r} = 0$ for $m = r$, $\delta_{m,r} = 1$ otherwise. (Here, by convention, $x^0 = 1$ for all x including $x = 0$.)

Let $\theta_1, \ldots, \theta_r$ be algebraic numbers, which are multiplicatively independent. Let N be a positive integer. We assume that

$$(6.2) \qquad \mathscr{T}_r + r < r(r+1)\mathscr{D}_0 .$$

Then the following holds.

Proposition 6.1. *Suppose that for all non-negative integers $t_0, \ldots, t_{r-1}$ with*

$$t_0 + \cdots + t_{r-1} \leqq \mathscr{T}$$

we have

$$\partial_0^{t_0} \cdots \partial_{r-1}^{t_{r-1}} \mathscr{P}(Ns, \theta_1^{Ns}, \ldots, \theta_r^{Ns}) = 0$$

for integers s with $0 \leqq s \leqq \mathscr{S}$. Then there exists an integer ϱ with $1 \leqq \varrho < r$ having the following property. For any positive real numbers $a_1, \ldots, a_r$ there is a set of primitive linear forms $\mathscr{L}_1, \ldots, \mathscr{L}_\varrho$ in $Z_1, \ldots, Z_r$ with integer coefficients such that $\mathscr{L}$ is in the module generated by $\mathscr{L}_1, \ldots, \mathscr{L}_\varrho$ over $\mathbb{Q}$ and, on defining

$$\mathscr{R}_j = \sum_{i=1}^{r} |\partial \mathscr{L}_j / \partial Z_i| a_i \quad (1 \leqq j \leqq \varrho)$$

and $\mathscr{C}(\varrho) = 2^{1-\varrho}(\varrho!)^2 (r\varrho)^\varrho$, we have at least one of the following inequalities:

$$(6.3) \quad \mathscr{R}_1 \cdots \mathscr{R}_\varrho \mathscr{S}_{\varrho-1} \binom{\mathscr{T}_{\varrho-1} + \varrho - 1}{\varrho - 1} \leqq \mathscr{C}(\varrho) \max\left((a_1 \mathscr{D}_1)^{m_1} \cdots (a_r \mathscr{D}_r)^{m_r}\right)$$

and

$$(6.4)\qquad \mathscr{R}_1\cdots\mathscr{R}_\varrho\mathscr{S}_\varrho\binom{\mathscr{T}_\varrho+\varrho}{\varrho}\leqq(\varrho+1)\mathscr{C}(\varrho)\mathscr{D}_0\max\left((a_1\mathscr{D}_1)^{m_1}\cdots(a_r\mathscr{D}_r)^{m_r}\right),$$

where both of the maxima are over all $m_1,\ldots,m_r$, 0 *or* 1, *with* $m_1+\cdots+m_r=\varrho$.

Here a set of linear forms $\mathscr{L}_1,\ldots,\mathscr{L}_\varrho$ in $Z_1,\ldots,Z_r$ with integer coefficients is said to be primitive if the minors of order ϱ of the matrix of coefficients are relatively prime.

Note that $\theta_1'=\theta_1^N,\ldots,\theta_r'=\theta_r^N$ are multiplicatively independent. By considering $\mathscr{P}_1(Y_0,Y_1,\ldots,Y_r)=\mathscr{P}(NY_0,Y_1,\ldots,Y_r)$ instead of $\mathscr{P}(Y_0,Y_1,\ldots,Y_r)$, considering $\theta_1',\ldots,\theta_r'$ instead of $\theta_1,\ldots,\theta_r$, we may, without loss of generality, assume $N=1$ in the deduction of our proposition.

We indicate how to derive our proposition from the basic [21], I, Theorem 1.1. Since $\mathscr{P}$ is a non-zero polynomial, Theorem 1.1 implies that there exists a connected algebraic subgroup H of $G=\mathbb{G}_a\times\mathbb{G}_m^r$ such that the property (i) or (ii) stated there holds. As shown in [5], §10, H is non-trivial and proper, and H must have the form $\mathbb{G}_a\times\mathbb{H}$ or $\mathbb{O}\times\mathbb{H}$ for some connected subgroup $\mathbb{H}$ of $\mathbb{G}_m^r$, where $\mathbb{O}$ is trivial. If $H=\mathbb{G}_a\times\mathbb{H}$ then $\mathbb{H}$ is proper and if $H=\mathbb{O}\times\mathbb{H}$ then $\mathbb{H}$ is non-trivial. Since $\theta_1,\ldots,\theta_r$ are multiplicatively independent, the property (ii), that is, $(s,\theta_1^s,\ldots,\theta_r^s)\in H$ for some positive integer s, can be excluded. Thus we see that $\mathbb{H}$ is always non-trivial and proper, by the same argument as given in [5], §10, on noting that the condition $\mathscr{T}_r<\mathscr{D}_0$ there can be replaced by our (6.2).

We note now that $\mathbb{H}$ can be defined in the Lie algebra of $\mathbb{G}_m^r$ by a primitive set of ϱ linear forms in $Z_1,\ldots,Z_r$ with integer coefficients. The set of these forms defines a free module $\mathscr{Z}$ over the integers and ϱ is equal to the codimension of $\mathbb{H}$ in $\mathbb{G}_m^r$. Hence, since $\mathbb{H}$ is non-trivial and proper, we have $1\leqq\varrho<r$. We identify $\mathscr{Z}$ with a submodule $\mathscr{Z}'$ of $\mathbb{Z}^r$ by attaching to a linear form its vector of coefficients. In [21], II, §2, the volume $\mathscr{V}$ of $\mathscr{Z}'$ is defined in terms of $a_1,\ldots,a_r$ and the minors of order ϱ of the matrix formed by the vectors of coefficients of the above ϱ primitive linear forms; the absolute values of the minors are in fact the partial degrees of $\mathbb{H}$ and bounds for these can be calculated by [21], I, Theorem 1.1, (i). Thus if $H=\mathbb{G}_a\times\mathbb{H}$ then the codimension of H in G is ϱ and

$$(6.5)\qquad \mathscr{V}\mathscr{S}_{\varrho-1}\binom{\mathscr{T}_{\varrho-1}+\varrho-1}{\varrho-1}\leqq\varrho!\max\left((a_1\mathscr{D}_1)^{m_1}\cdots(a_r\mathscr{D}_r)^{m_r}\right),$$

if $H=\mathbb{O}\times\mathbb{H}$ then the codimension of H in G is $\varrho+1$ and

$$(6.6)\qquad \mathscr{V}\mathscr{S}_\varrho\binom{\mathscr{T}_\varrho+\varrho}{\varrho}\leqq(\varrho+1)!\,\mathscr{D}_0\max\left((a_1\mathscr{D}_1)^{m_1}\cdots(a_r\mathscr{Z}_r)^{m_r}\right),$$

where both of the maxima are over $m_1,\ldots,m_r$, 0 or 1, with $m_1+\cdots+m_r=\varrho$. Now, by an argument similar to that of [5], §10, we see that (6.5) implies (6.3) and (6.6) yields (6.4). The assertion that $\mathscr{L}$ is in the module generated by $\mathscr{L}_1,\ldots,\mathscr{L}_\varrho$ over $\mathbb{Q}$ is equivalent to the property of the Lie algebra in (i) by duality.

7. A central result

We state now a central result, which implies Theorem 1 and Theorem 2 of §1. We maintain the notation introduced in §1.

Theorem 7.1. *Suppose that* (1.9) *and* (1.13) *hold*, $\alpha_1, \ldots, \alpha_n$ *are multiplicatively independent, and* $b_1, \ldots, b_n$ *are not all zero. Then*

$$\operatorname{ord}_{\mathfrak{p}} \Xi < C^*(n, d, \mathfrak{p})\, h'(\alpha_1) \cdots h'(\alpha_n)(h^* + \log c^*)\,, \tag{7.1}$$

where $C^*(n, d, \mathfrak{p})$ *is given by* (1.12), c^* *is given by* (5.24), *and*

$$h^* = \max\left\{\log\left(\frac{f_{\mathfrak{p}} \log p}{2d} \max_{1 \leqq j < n}\left(\frac{|b_n|}{h'(\alpha_j)} + \frac{|b_j|}{h'(\alpha_n)}\right)\right),\right. \tag{7.2}$$
$$\left. \log B^\circ,\ 6n\log(5n) + 1.2\log d,\ 2 f_{\mathfrak{p}} \log p\right\}$$

with

$$B^\circ = \min_{1 \leqq j \leqq n,\, b_j \neq 0} |b_j|\,. \tag{7.3}$$

Furthermore if $\alpha_1, \ldots, \alpha_n$ *satisfy* (1.11), *then* $C^*(n, d, \mathfrak{p})$ *and* $h^* + \log c^*$ *can be replaced by* $C^*(n, d, \mathfrak{p})/\omega_q(n)$ *and* h^*, *respectively.*

The following §8–§13 will be devoted to prove Theorem 7.1.

8. Basic hypothesis

We record [26], Lemma 1.1 as

Lemma 8.1. *Let* $\kappa \geqq 0$ *be the rational integer satisfying*

$$\phi(p^\kappa) \leqq 2e_{\mathfrak{p}} < \phi(p^{\kappa+1})\,,$$

where ϕ *is the Euler's* ϕ*-function. If* $\beta \in \mathbb{C}_p$ *satisfies*

$$\operatorname{ord}_p(\beta - 1) \geqq \frac{1}{e_{\mathfrak{p}}}\,,$$

then

$$\operatorname{ord}_p(\beta^{p^\kappa} - 1) \geqq \frac{p^\kappa}{2e_{\mathfrak{p}}} + \frac{1}{p-1}\,.$$

Set θ by

$$\left(1 + \frac{1}{2n}\cdot 10^{-100}\right)\theta = \begin{cases} (p-2)/(p-1), & \text{if } p \geqq 5 \text{ with } e_{\mathfrak{p}} = 1\,, \\ p^\kappa/(2e_{\mathfrak{p}}), & \text{otherwise.} \end{cases} \tag{8.1}$$

Let

$$(8.2) \qquad c_2 = \begin{cases} 2, & \text{if } p > 2, \\ 16/9, & \text{if } p = 2. \end{cases}$$

Let $b_1, \ldots, b_n$ be the rational integers in Theorem 7.1, satisfying (1.13), and set

$$(8.3) \qquad L = b_1 z_1 + \cdots + b_n z_n .$$

Let v be defined as in § 5, $l_0, l_1, \ldots, l_n$ be defined by (5.7).

Our *basic hypothesis* is that *there exists a set of linear forms* $L_0, L_1, \ldots, L_r$ *in* $z_0, z_1, \ldots, z_n$ *with rational integer coefficients having the following properties*:

(i) $L_0 = q^v z_0$; $L_0, L_1, \ldots, L_r$ *are linearly independent*; *and*

$$(8.4) \qquad L = B_0 L_0 + B_1 L_1 + \cdots + B_r L_r$$

for some rationals $B_0, B_1, \ldots, B_r$, *with* $B_r \neq 0$.

(ii) *On writing*

$$l_i' = q^{-v} L_i(l_0, l_1, \ldots, l_n) \quad (1 \leqq i \leqq r),$$

the numbers $\alpha_i' = e^{l_i'}$ $(1 \leqq i \leqq r)$ *are in* K, *and satisfy* $\operatorname{ord}_{\mathfrak{p}} \alpha_i' = 0$ $(1 \leqq i \leqq r)$ *and*

$$(8.5) \qquad [K(\alpha_0^{1/q}, \alpha_1'^{1/q}, \ldots, \alpha_r'^{1/q}) : K] = q^{r+1} .$$

(iii) *We have*

$$(8.6) \qquad h'(\alpha_i') \leqq \sigma_i \quad (1 \leqq i \leqq r)$$

and

$$(8.7) \qquad \sum_{j=1}^{n} |\partial L_i / \partial z_j| h'(\alpha_j) \leqq q^v \sigma_i \quad (1 \leqq i \leqq r)$$

for some positive real numbers $\sigma_1, \ldots, \sigma_r$ *satisfying*

$$(8.8) \qquad \sigma_1 \cdots \sigma_r \leqq \psi(r) h'(\alpha_1) \cdots h'(\alpha_n),$$

where

$$(8.9) \qquad \psi(r) = \left(e c_2 q \frac{p^{\kappa}}{e_{\mathfrak{p}} \theta} (n+1) \frac{d}{f_{\mathfrak{p}} \log p} \right)^{n-r} \omega_q(n) \min\left(\frac{c^*}{q^v}, 1 \right)$$

with c^* *given by* (5.24). *Furthermore, if* $v = 0$ *then* $\psi(r)$ *in* (8.8) *is replaced by* $\psi(r)/\omega_q(n)$.

The construction of § 5 establishes the existence of linear forms as above for $r = n$. *We now take* r *as the least integer for which such a set of linear forms exists. Hence* $B_i \neq 0$ $(1 \leqq i \leqq r)$ *and we may assume*

(8.10)
$$\sigma_r = \max_{1 \leqq i \leqq r} \sigma_i$$

in our basic hypothesis.

Lemma 8.2. *If* $r = 1$, *then Theorem* 7.1 *holds.*

Proof. Note that $B_1 \neq 0$. Write $B_1 = p_1/q_1$ with $p_1, q_1 \in \mathbb{Z}$, $(p_1, q_1) = 1$, $q_1 > 0$. By (8.4) we have

(8.11)
$$q_1 L = q_1 B_0 q^v z_0 + p_1 L_1 .$$

Thus
$$q_1 B_0 q^v = -p_1 \partial L_1/\partial z_0 \in \mathbb{Z} \quad \text{and} \quad p_1 | b_j \qquad (1 \leqq j \leqq n),$$

whence $|p_1| \leqq B^\circ$, where B° is defined by (7.3). On noting (1.9) and that α_1' is not a root of unity, (8.11) and [25], Lemma 1.4 give

(8.12)
$$\begin{aligned}\operatorname{ord}_\mathfrak{p} \Xi &\leqq \operatorname{ord}_\mathfrak{p}\big((\alpha_1^{b_1} \cdots \alpha_n^{b_n})^{q_1 q^u} - 1\big) = \operatorname{ord}_\mathfrak{p}(\alpha_1'^{q^{v+u} p_1} - 1) \\ &\leqq \frac{d}{f_\mathfrak{p} \log p} \{\log(2 q^{v+u} B^\circ) + 2(p^{f_\mathfrak{p}} - 1) e_\mathfrak{p} h'(\alpha_1')\} .\end{aligned}$$

Now it is readily to deduce Lemma 8.2 from (8.12), (1.4), (7.2), (8.6), (8.8), (8.9) and

(8.13)
$$v \log q \leqq \log c^* \cdot \max\left(\frac{q^v}{c^*}, 1\right).$$

We omit the details here.

By Lemma 8.2, *we may assume* $r \geqq 2$ *in our basic hypothesis.*

Proposition 6.1 will be applied to a polynomial $\mathscr{P}(Y_0, \ldots, Y_r)$ with the differential operators $\partial_1, \ldots, \partial_{r-1}$ replaced by a new set as follows. We write

(8.14)
$$\partial_j^* = (1/B_r) \sum_{i=1}^{r-1} (b_n \partial L_i/\partial z_j - b_j \partial L_i/\partial z_n) \partial_i \quad (1 \leqq j < n) .$$

Here the matrix of coefficients of $\partial_1, \ldots, \partial_{r-1}$ has rank $r-1$; for if the expressions on the right vanish when rationals B_i' are substituted for ∂_i $(1 \leqq i < r)$ then

$$b_n \sum_{i=1}^{r-1} B_i'(L_i - z_0 \partial L_i/\partial z_0) = b_n' L, \quad \text{where} \quad b_n' = \sum_{i=1}^{r-1} B_i' \partial L_i/\partial z_n ,$$

whence the linear independence of $L_0 = q^v z_0$, $L_1, \ldots, L_r$ implies that all the B_i' are 0. It follows that this matrix has a non-zero minor of order $r-1$. We may, without loss of generality (see the remark in § 5), assume that

(8.15)
$$\det(b_n \partial L_i/\partial z_j - b_j \partial L_i/\partial z_n)_{1 \leqq i < r, 1 \leqq j < r} \neq 0 \quad \text{and has the minimal}$$
p-adic order among the minors of order $r-1$ of this matrix.

Then $\partial_1^*, \ldots, \partial_{r-1}^*$ are linearly independent over $\mathbb{Q}$, and Proposition 6.1 holds with $\partial_1^*, \ldots, \partial_{r-1}^*$ in place of $\partial_1, \ldots, \partial_{r-1}$. Furthermore, ∂_j^* $(r \leqq j < n)$ are linear combinations of $\partial_1^*, \ldots, \partial_{r-1}^*$ with coefficients in $\mathbb{Q} \cap \mathbb{Z}_p$, where $\mathbb{Z}_p$ is the ring of p-adic integers. Note that the asterisked operators can be written in the form

$$\partial_j^* = \sum_{i=1}^{r} (b_n \partial L_i / \partial z_j - b_j \partial L_i / \partial z_n) Y_i \partial / \partial Y_i . \tag{8.16}$$

9. Choices of parameters and Proposition 9.1

We define $g_0, \ldots, g_{12}, \varepsilon_1, \varepsilon_2, f_6$ by the following formulae, which are quoted as

(9.1)
$$g_0 = 6r\log(5r) + 1.2\log d,$$

$$g_1 = \log\big(e^4(r+1)\,d\big),$$

$$g_2 = 2c_3 q(r+1)\,d,$$

$$g_3 = 2c_0c_4\left(c_2 q\frac{p^\kappa}{e_\mathfrak{p}\theta}\right)^r \frac{r^r(r+1)^{r+1}}{r!}(q-1)\frac{g_1}{\theta}, \quad 1+\varepsilon_1 = \left(1+\frac{r}{g_3}\right)^r,$$

$$g_4 = 2c_0c_4\left(c_2\frac{p^\kappa}{e_\mathfrak{p}\theta}\right)^{r-1} q^r\cdot\frac{r^{r-1}(r+1)^r}{r!}\frac{f_\mathfrak{p}}{\theta}, \quad 1+\varepsilon_2 = \left(1-\frac{1}{g_4}\right)^{-r},$$

$$g_5 = 2c_0c_4\left(c_2 q\frac{p^\kappa}{e_\mathfrak{p}\theta}\right)^{r-1}\frac{r^{r-1}(r+1)^r}{r!}\cdot\frac{p^{f_\mathfrak{p}}-1}{f_\mathfrak{p}\log p}\cdot(q-1)\frac{g_1}{\theta},$$

$$g_6 = 2c_0c_3\left(c_2 q\frac{p^\kappa}{e_\mathfrak{p}\theta}\right)^r (q-1)\cdot\frac{r^r(r+1)^{r+1}}{r!}\cdot\frac{p^{f_\mathfrak{p}}-1}{f_\mathfrak{p}\log p}\cdot e_\mathfrak{p},$$

$$g_7 = 2c_0c_1c_4\left(c_2 q\frac{p^\kappa}{e_\mathfrak{p}\theta}\right)^r\left(1-\frac{1}{q}\right)\cdot\frac{r^r(r+1)^r}{r!}\cdot\frac{p^{f_\mathfrak{p}}-1}{f_\mathfrak{p}\log p}\cdot g_1 e_\mathfrak{p},$$

$$g_8 = g_7 f_\mathfrak{p}\log p,$$

$$g_9 = \frac{d}{g_2 g_8}\left(\log\left(\frac{g_2}{d}\right) + (r+1)\log g_8\right),$$

$$g_{10} = \frac{2}{g_0}\exp(-1+e^{-g_0-0.9})\frac{1}{c_2 q p^\kappa}\cdot\frac{r-1}{r+1},$$

$$g_{11} = \frac{1}{c_2c_5qr}\cdot\frac{\theta e_\mathfrak{p}}{p^\kappa}\cdot\left(q\left(1-\frac{c_5}{r+1}\right)^{r+1}\right)^{-(\log g_5)/\log q},$$

$$g_{12} = \frac{1}{g_2}\left(\frac{d-1}{g_7} + \frac{d\log d}{2g_8}\right),$$

$$f_6 = (1+\varepsilon_1)(1+\varepsilon_2)(2+1/g_2)\, c_0 c_1 c_3 c_4 q^2 (1+10^{-100})\,.$$

In (9.1), θ is given by (8.1); c_0, c_1, c_2, c_3, c_4, c_5 are given in the following table, denoted by (9.2). The upper bounds for f_6 can be obtained from the above formulae by direct calculation. Blocks I, II, III and IV are for cases (I) $p=3$; or $p=5$ with $e_\mathfrak{p} \geqq 2$, (II) $p \geqq 5$ with $e_\mathfrak{p} = 1$, (III) $p \geqq 7$ with $e_\mathfrak{p} \geqq 2$ and (IV) $p=2$, respectively.

(9.2)

case	r	c_0	c_1	c_2	c_3	c_4	c_5	$f_6 \leqq$
I	$2 \leqq r \leqq 6$	2.7	0.7322	2	1.5527	23.71	0.5	598
	$7 \leqq r \leqq 15$	2.6	0.6636	2	1.3794	23.94	0.53	461
	$r \geqq 16$	2.55	0.6479	2	1.346	24.005	0.54	430
II	$2 \leqq r \leqq 6$	2.8	0.7096	2	3.265	20.378	0.48	1072
	$7 \leqq r \leqq 15$	2.6	0.6485	2	2.945	21.14	0.512	845
	$r \geqq 16$	2.5	0.65	2	2.576	24.9	0.545	837
III	$2 \leqq r \leqq 6$	2.85	0.6825	2	5.1869	19.046	0.47	1544
	$7 \leqq r \leqq 15$	2.625	0.6404	2	4.6135	20.083	0.504	1249
	$r \geqq 16$	2.5	0.6332	2	4.495	21.01	0.52	1197
IV	$2 \leqq r \leqq 6$	2.5	0.8672	16/9	0.5115	3.942	0.72	81
	$7 \leqq r \leqq 15$	2.2	0.7705	16/9	0.4795	4	0.786	60
	$r \geqq 16$	2.16	0.746	16/9	0.4611	4.04	0.81	55

Let

(9.3)
$$\eta = 1 - \frac{c_5}{r+1}\,.$$

It can be verified that c_0, c_1, c_2, c_3, c_4, c_5 satisfy (9.4), (9.5), (9.6), (9.7) and (9.9) below.

(9.4)

(i) $$2c_5\eta^{r-1}\left(1-\frac{1}{2g_2}\right) - \frac{\log q}{\log(q\eta)}\left(\frac{1}{c_2} + \left(1+\frac{1}{g_6}\right)\frac{\tau(p)}{c_4 g_1}\cdot\frac{1}{q}\right) \geqq 0, \quad 2 \leqq r \leqq 15,$$

(ii) $$2c_5 e^{-c_5}\left(1-\frac{1}{2g_2}\right) - \frac{\log q}{\log(q\eta)}\left(\frac{1}{c_2} + \left(1+\frac{1}{g_6}\right)\frac{\tau(p)}{c_4 g_1}\cdot\frac{1}{q}\right) \geqq 0, \quad r \geqq 16,$$

where $\tau(p) = 1$ or $\frac{3}{2}$, according to $p > 2$ or $p = 2$,

$$(9.5)\qquad 2c_5 q\left(1 - \frac{1}{2g_2}\right)$$
$$\geqq c_1\left\{g_{12} + \left(1 + \frac{1}{2(c_0-1)}\right)g_9\right\} + \left\{q + \frac{1}{2(c_0-1)}\left(1 + \frac{1}{2g_2+1}\right)\right\}\frac{1}{c_2}$$
$$+ \left\{\frac{107}{103}\,\frac{1}{e_{\mathfrak{p}}\theta_0}\left(1 - \frac{c_5}{r+1} + \frac{1}{c_0-1}\right) + \left(1 + \frac{1}{c_0-1}\right)g_{10}\right\}\frac{1}{c_3}$$
$$+ \left(1 + \frac{1}{g_6}\right)\left\{1 + \frac{1}{c_0-1} + \left(\theta + \frac{1}{p-1}\right)\frac{1}{f_{\mathfrak{p}}}\right\}\frac{1}{c_4},$$

where and in the sequel $(1 + 10^{-100})\,e_{\mathfrak{p}}\theta_0 = \frac{3}{2}, \frac{3}{4}, \frac{1}{2}, 2$, according to cases (I), (II), (III), (IV), whence $e_{\mathfrak{p}}\theta \geqq e_{\mathfrak{p}}\theta_0$,

$$(9.6)\qquad c_1 \geqq c_5(\eta^r + g_{11})\left\{2 + \frac{1}{g_2} + \frac{1}{(r+1)\,q^{r+1}}\cdot\frac{1}{e_{\mathfrak{p}}\theta_0}\cdot\frac{1}{c_3}\right\},$$

$$(9.7)\quad \text{(i)}\quad \left(1 + \frac{1}{g_6}\right)\frac{c_2}{c_4}\cdot\frac{\log q}{q}\cdot\frac{I}{\max(f_{\mathfrak{p}}\log p, g_1)} + \frac{1}{(q\eta^{r+1})^I} \leqq 1, \quad \text{if } p > 2,$$

$$\text{(ii)}\quad \left(1 + \frac{1}{g_6}\right)\frac{c_2}{c_4}\cdot\frac{\log q}{q^2}\cdot\frac{I}{\max(f_{\mathfrak{p}}\log p, g_1)} + \frac{1}{(q\eta^{r+1})^I} \leqq 1, \quad \text{if } p = 2,$$

for $1 \leqq I < I^*$, where

$$(9.8)\qquad I^* = [5\max(f_{\mathfrak{p}}\log p, g_1)/\log(q\eta^{r+1})] + 1,$$

$$(9.9)\qquad \eta^{-(r+1)} + g_2^{-1} \leqq q.$$

Note that in verifying (9.4), (9.5), (9.6), (9.7) and (9.9), we have used the fact that $\eta^{r+1} = \left(1 - c_5/(r+1)\right)^{r+1}$ is increasing and $\eta^r = \left(1 - c_5/(r+1)\right)^r$ is decreasing in each of the three ranges $2 \leqq r \leqq 6$, $7 \leqq r \leqq 15$ and $r \geqq 16$. For the details of the verification, see the paragraph next to the last paragraph of this section.

Let

$$(9.10)\quad h = \max\left\{\log\left(\frac{f_{\mathfrak{p}}\log p}{2d}\max_{1\leqq j<n}\left(\frac{|b_n|}{h'(\alpha_j)} + \frac{|b_j|}{h'(\alpha_n)}\right)\right), \log B^\circ, g_0, 2f_{\mathfrak{p}}\log p\right\},$$

where B° is given by (7.3), and

$$(9.11)\qquad G_0 = (p^{f_{\mathfrak{p}}} - 1)/q^u,$$

which is a positive integer by Hasse [8], p. 220 and (1.3), (1.4). Set

$$S = \frac{c_3 q(r+1) d(h + v\log q)}{f_{\mathfrak{p}} \log p}, \tag{9.12}$$

$$D = (1+\varepsilon_1)(1+\varepsilon_2)\left(2 + \frac{1}{g_2}\right) c_0 c_1 c_4 \left(c_2 q \frac{p^\kappa}{e_{\mathfrak{p}} \theta}\right)^r \frac{r^r (r+1)^r}{r!} \cdot G_0 \frac{d^{r+1}\sigma_1 \cdots \sigma_r}{(f_{\mathfrak{p}} \log p)^r} \max(f_{\mathfrak{p}} \log p, g_1), \tag{9.13}$$

$$T = \frac{q(r+1) D}{c_1 \theta e_{\mathfrak{p}} f_{\mathfrak{p}} \log p}, \tag{9.14}$$

$$\tilde{D}_{-1} = h + v\log q - 1, \quad D_{-1} = [\tilde{D}_{-1}], \tag{9.15}$$

$$\tilde{D}_0 = \frac{SDd^{-1}}{c_1 c_4 (D_{-1}+1) \max(f_{\mathfrak{p}} \log p, g_1)}, \quad D_0 = [\tilde{D}_0], \tag{9.16}$$

$$\tilde{D}_i = \frac{D}{c_1 c_2 r p^\kappa d\sigma_i}, \quad D_i = [\tilde{D}_i], \quad 1 \leqq i \leqq r. \tag{9.17}$$

In the sequel we need

$$d/e_{\mathfrak{p}} \geqq q^{u-1}(q-1), \tag{9.18}$$

which follows from the fact that p is unramified in $\mathbb{Q}(\zeta_{q^u})$. It is readily seen that the following inequalities hold. We give proof for part of them, when it is necessary.

$$S \geqq g_2. \tag{9.19}$$

$$T \geqq g_3, \quad (1 + r/T)^r \leqq (1 + r/g_3)^r = 1 + \varepsilon_1. \tag{9.20}$$

$$\frac{\tilde{D}_i}{G_0} \geqq g_4 \ (1 \leqq i \leqq r), \quad \prod_{i=1}^r \left(1 - \frac{G_0}{\tilde{D}_i}\right) \geqq \left(1 - \frac{1}{g_4}\right)^r = \frac{1}{1+\varepsilon_2}. \tag{9.21}$$

$$\tilde{D}_i \geqq g_5 \quad (1 \leqq i \leqq r). \tag{9.22}$$

$$(D_{-1}+1)(D_0+1) G_0^{-1} \prod_{i=1}^r (D_i + 1 - G_0) \geqq c_0 (2S+1) \binom{[T]+r}{r}. \tag{9.23}$$

$$p^\kappa S \sum_{i=1}^r D_i \sigma_i \leqq \frac{1}{c_1 c_2} \frac{SD}{d}. \tag{9.24}$$

$$T(\tilde{D}_{-1}+1) \leqq \frac{1}{c_1 c_3 e_{\mathfrak{p}} \theta} \frac{SD}{d}. \tag{9.25}$$

$$\tilde{D}_0 \geqq g_6, \quad (D_{-1}+1)(D_0+1)\max(f_{\mathfrak{p}} \log p, g_1) \leqq \left(1 + \frac{1}{g_6}\right) \frac{1}{c_1 c_4} \frac{SD}{d}. \tag{9.26}$$

$$D \geqq g_7 d\sigma_i \quad (1 \leqq i \leqq r), \quad D \geqq g_8. \tag{9.27}$$

(9.28) $$\log\left(\prod_{i=-1}^{r}(D_i+1)\right) \leqq g_9\,\frac{SD}{d}\,.$$

Proof. By (9.22) and (9.26) we have

$$\prod_{i=-1}^{r}(D_i+1) \leqq \frac{1}{g_1}\left(1+\frac{1}{g_6}\right)\frac{1}{c_1c_4}\left(1+\frac{1}{g_5}\right)^r\frac{SD}{d}\,\tilde{D}_1\cdots\tilde{D}_r \leqq \frac{SD^{r+1}}{d}\,,$$

whence (9.28) follows by (9.19) and (9.27).

(9.29) $$\log\left(e\left(2+\frac{qS}{D_{-1}+1}\right)\right) \leqq g_1\,.$$

Proof. By (9.12) and (9.15), the left side is at most

$$\log\left(e\left(2+\frac{c_3q^2(r+1)\,d}{f_{\mathfrak{p}}\log p}\cdot\frac{g_0}{g_0-1}\right)\right).$$

Now (9.29) follows from the inequality

(9.30) $$\frac{2}{(r+1)\,d}+\frac{c_3q^2}{f_{\mathfrak{p}}\log p}\cdot\frac{g_0}{g_0-1} \leqq e^3\,,$$

which can be verified, using (9.2) and the fact that $f_{\mathfrak{p}} \geqq 2$ when $p=2$ (see [25], Appendix).

(9.31) $$x\log\left(\frac{1}{e^h}+\frac{2(r-1)\,D}{c_1c_2p^{\kappa}f_{\mathfrak{p}}\log p}\cdot\frac{1}{x}\right) \leqq g_{10}\,\frac{1}{c_1c_3}\,\frac{SD}{d} \quad \text{for } x \geqq 1\,.$$

Proof. Let $A=2(r-1)D/(c_1c_2p^{\kappa}f_{\mathfrak{p}}\log p)$ and $f(x)=x\log(1/e^h+A/x)$. By (9.27), we have $A>e$. Obviously, we may assume $1 \leqq x \leqq A/(1-e^{-h})$. Putting $y=1+e^hA/x$, it is readily seen that

$$f(x)=x(\log y-h),\quad f'(x)=g(y)=\log y+1/y-h-1\,,$$
$$f''(x)=(1/y-1/y^2)\cdot(-e^hA/x^2)<0 \quad \text{for } 1 \leqq x \leqq A/(1-e^{-h})\,.$$

Now the facts that $dg/dy>0$ for $y>1$, $g(e^h)<0$ and $g(e^{h+1})>0$ imply that $g(y)$ has only one zero in $(1,\infty): y=y_0=e^{h+1-\delta}$ for some $\delta \in (0,1)$. Thus

(9.32) $$\delta=e^{-h-1+\delta}=1/y_0\,.$$

Set $x_0=e^hA/(y_0-1)$, we see that $1<x_0<A/(1-e^{-h})$ and

$$f(x) \leqq f(x_0)=x_0(\log y_0-h)=x_0(1-1/y_0)=e^{-1+\delta}A \quad \text{for all } x \geqq 1\,.$$

Note that $\delta<0.1$ by (9.32), $0<\delta<1$ and $h \geqq g_0>27$. So (9.32) gives further

$$\delta<\exp(-g_0-0.9)\,.$$

Now (9.31) follows from (9.12) at once.

Set

$$U = \frac{q^{r+1}}{e_{\mathfrak{p}} f_{\mathfrak{p}} \log p} SD. \tag{9.33}$$

Proposition 9.1. *Under the hypotheses of Theorem* 7.1, *we have*

$$\operatorname{ord}_p \Xi < U.$$

Note that Proposition 9.1 implies Theorem 7.1. We verify this for the case $v > 0$. By (9.12), (9.13), (8.8), (8.9), (8.13), (7.2), (9.10), (8.1), (8.2), (1.3), (9.2) and the inequality $n^n/n! \geqq 2^{n-r} \cdot r^r/r!$, we have

$$\frac{e_{\mathfrak{p}} U}{C^*(n, d, \mathfrak{p})\, h'(\alpha_1) \cdots h'(\alpha_n)(h^* + \log c^*)}$$
$$\leqq \frac{(1+\varepsilon_1)(1+\varepsilon_2)(2+1/g_2)\, c_0 c_1 c_3 c_4 q^2 \big(c_2 q^2 p^\kappa/(e_{\mathfrak{p}}\theta)\big)^n}{c a^n} \leqq \frac{f_6}{c} \leqq 1.$$

The verification in the case $v = 0$, i.e., when $\alpha_1, \ldots, \alpha_n$ satisfy (1.11), is similar.

In the following §10–§13, we shall prove Proposition 9.1.

Now we indicate how we verify (9.4), (9.5), (9.6), (9.7) and (9.9). We divide the verification into the four cases listed in table (9.2) and note that we have

$$\begin{array}{llllllll} \text{(9.34)}\ \text{(I)} & q=2, & d \geqq 1, & p^\kappa/(e_{\mathfrak{p}}\theta) \geqq 2, & \theta \leqq 3/2, & f_{\mathfrak{p}} \geqq 1, & e_{\mathfrak{p}} \geqq 1, & p^\kappa \geqq 3; \\ \text{(II)} & q=2, & d \geqq 1, & p^\kappa/(e_{\mathfrak{p}}\theta) \geqq 1, & \theta \leqq 1, & f_{\mathfrak{p}} \geqq 1, & e_{\mathfrak{p}} = 1, & p^\kappa = 1; \\ \text{(III)} & q=2, & d \geqq 2, & p^\kappa/(e_{\mathfrak{p}}\theta) \geqq 2, & \theta \leqq 7/6, & f_{\mathfrak{p}} \geqq 1, & e_{\mathfrak{p}} \geqq 2, & p^\kappa \geqq 1; \\ \text{(IV)} & q=3, & d \geqq 2, & p^\kappa/(e_{\mathfrak{p}}\theta) \geqq 2, & \theta \leqq 2, & f_{\mathfrak{p}} \geqq 2, & e_{\mathfrak{p}} \geqq 1, & p^\kappa \geqq 4. \end{array}$$

We can prove (9.4), (9.5), (9.6), (9.7) and (9.9) for $r = 2, 3, 4, \ldots, 15$ by direct computation, using (9.1)–(9.3) and (9.34). It remains to verify them for $r \geqq 16$. By direct computation, we see that (9.4) (ii) is true for $r = 16$, whence it holds for $r \geqq 16$, since its left side is an increasing function of r. Further we have

$$\begin{aligned} &\left\{\frac{107}{103}\,\frac{1}{e_{\mathfrak{p}}\theta_0} \cdot \frac{c_5}{r+1} - \left(1 + \frac{1}{c_0 - 1}\right) g_{10}\right\} \frac{1}{c_3} - \frac{c_5 q}{g_2} \\ &- c_1 \left\{g_{12} + \left(1 + \frac{1}{2(c_0-1)}\right) g_9\right\} - \frac{1}{2(c_0-1)(2g_2+1)c_2} \\ &- \frac{1}{g_6}\left\{1 + \frac{1}{c_0-1} + \left(\theta + \frac{1}{p-1}\right)\frac{1}{f_{\mathfrak{p}}}\right\} \frac{1}{c_4} \geqq 0, \quad \text{for } r \geqq 16, \end{aligned} \tag{9.35}$$

since its left side is positive for $r = 16$, decreasing in r and tends to 0 as $r \to \infty$. Also it is readily verified, using (9.1)–(9.3) and (9.34), that for $r \geqq 16$

$$(9.36) \qquad 2c_5 q \geqq \left\{ q + \frac{1}{2(c_0 - 1)} \right\} \frac{1}{c_2} + \frac{107}{103} \frac{1}{e_{\mathfrak{p}} \theta_0} \left(1 + \frac{1}{c_0 - 1} \right) \frac{1}{c_3} + \left\{ 1 + \frac{1}{c_0 - 1} + \left(\theta + \frac{1}{p-1} \right) \frac{1}{f_{\mathfrak{p}}} \right\} \frac{1}{c_4}.$$

Now (9.5) for $r \geqq 16$ follows from (9.35) and (9.36). Finally (9.6), (9.7) and (9.9) for $r = 16$ can be verified by direct computation, using (9.1)–(9.3) and (9.34), whence they hold for $r \geqq 16$ by monotonicity in r. Our computation is carried out on a SUN SPARCstation 10 with PARI GP 1.39.

In order to prove Lemma 11.2 of §11 in the sequel, we show the following inequality. Let $I \in \mathbb{Z}$ satisfy $0 \leqq I < I^*$ with I^* given by (9.8) and $\delta_I = 0$ or 1 according to $I = 0$ or $I > 0$. Then for $k = 0, \ldots, r-1$ when $I = 0$ and for $k = 1, \ldots, r-1$ when $I > 0$ we have

$$(9.37) \qquad 2c_5 q^{k+1} \eta^k \left(1 - \frac{1}{2g_2} \right) \geqq c_1 \left\{ g_{12} + \left(1 + \frac{1}{2(c_0 - 1)} \right) g_9 \right\} + \left\{ \frac{q^{k+1}}{(q\eta^{r+1})^I} + \frac{1}{2(c_0-1)} \left(1 + \frac{1}{2g_2 + 1} \right) \right\} \frac{1}{c_2}$$
$$+ \left\{ \frac{107}{103} \frac{1}{e_{\mathfrak{p}} \theta_0} \left(\eta^{k+1} + \frac{1}{c_0 - 1} \right) + \left(1 + \frac{1}{c_0 - 1} \right) g_{10} \right\} \frac{1}{c_3}$$
$$+ \left(1 + \frac{1}{g_6} \right) \left\{ 1 + \frac{1}{c_0 - 1} + \frac{[k + \delta_I(I + 1/(q-1))] \log q}{\max(f_{\mathfrak{p}} \log p, g_1)} + \left(\theta + \frac{1}{p-1} \right) \frac{1}{f_{\mathfrak{p}}} \right\} \frac{1}{c_4}.$$

Proof. By (9.7), we see that the right side of (9.37) is bounded above by $\mathfrak{R}(k)$ which is obtained from the right side of (9.37) by replacing $q^{k+1}/(q\eta^{r+1})^I$ with q^{k+1}, replacing $k + \delta_I(I + 1/(q-1))$ with $\tau(p)k$, and replacing $\max(f_{\mathfrak{p}} \log p, g_1)$ with g_1. Write $\mathfrak{L}(k)$ for the left side of (9.37). Now (9.4) implies $(\mathfrak{L}(x) - \mathfrak{R}(x))' > 0$ for $0 \leqq x \leqq r-1$ and (9.5) implies $\mathfrak{L}(0) - \mathfrak{R}(0) \geqq 0$. Hence $\mathfrak{L}(k) \geqq \mathfrak{R}(k)$ for $k = 0, \ldots, r-1$, which yields (9.37).

10. The auxiliary polynomials

Let

$$(10.1) \qquad G = p^{f_{\mathfrak{p}}} - 1, \quad G_0 = G/q^u, \quad G_1 = G/q^\mu \quad \text{with} \quad \mu = \mathrm{ord}_q G.$$

Denote by ζ a fixed G-th primitive root of 1 in $K_{\mathfrak{p}}$ such that

$$(10.2) \qquad \zeta^{G_0} = \zeta_{q^u} (= \alpha_0),$$

by ξ a fixed (qG)-th root of 1 in $\mathbb{C}_p$, and by $\alpha_0^{1/q}$ a q-th root of α_0 in $\mathbb{C}_p$, satisfying

(10.3) $$\xi^q = \zeta \quad \text{and} \quad \xi^{G_0} = \alpha_0^{1/q}.$$

By (1.9), there exist $\tilde{a}_1, \ldots, \tilde{a}_n \in \mathbb{N}$ such that $\alpha_j \zeta^{\tilde{a}_j} \equiv 1 \pmod{\mathfrak{p}}$ $(1 \leqq j \leqq n)$. Now Lemma 8.1 yields

(10.4) $$\operatorname{ord}_p(\alpha_j^{p^\kappa} \zeta^{a_j} - 1) > \theta + \frac{1}{p-1}, \quad 1 \leqq j \leqq n,$$

where $a_j = p^\kappa \tilde{a}_j$ and θ is given by (8.1). Note also, by (10.2),

(10.5) $$\alpha_0^{p^\kappa} \zeta^{a_0} = 1, \quad \text{where} \quad a_0 = p^\kappa (G - G_0).$$

Thus the p-adic logarithms of $\alpha_j^{p^\kappa} \zeta^{a_j}$ satisfy

(10.6) $$\log(\alpha_0^{p^\kappa} \zeta^{a_0}) = 0, \quad \operatorname{ord}_p \log(\alpha_j^{p^\kappa} \zeta^{a_j}) > \theta + \frac{1}{p-1}, \quad 1 \leqq j \leqq n.$$

We shall freely use the fundamental properties of the p-adic exponential and logarithmic functions (see, for example, [24], §1.1).

Let $L_i(z_0, \ldots, z_n)$ and α_i' $(1 \leqq i \leqq r)$ be specified in the basic hypothesis in §8. We assert that there exist $a_1', \ldots, a_r' \in \mathbb{N}$ such that

(10.7) $$\exp\left(\frac{1}{q^\nu} L_i\big(0, \log(\alpha_1^{p^\kappa} \zeta^{a_1}), \ldots, \log(\alpha_n^{p^\kappa} \zeta^{a_n})\big)\right) = \alpha_i'^{p^\kappa} \zeta^{a_i'}, \quad 1 \leqq i \leqq r.$$

Proof. Note that the left-hand side of (10.7) is equal to

$$\exp\left(\frac{1}{q^\nu} \log \prod_{j=0}^{n} (\alpha_j^{p^\kappa} \zeta^{a_j})^{\partial L_i / \partial z_j}\right) = \exp\left(\frac{1}{q^\nu} \log(\alpha_i'^{q^\nu p^\kappa} \zeta^{L_i(a_0, \ldots, a_n)})\right).$$

Denote by x_i the quotient of the left-hand side of (10.7) divided by $\alpha_i'^{p^\kappa}$. By the fact that $\operatorname{ord}_p(q^\nu) = 0$, we have

$$x_i^{q^\nu G} = \{\exp\big(\log(\alpha_i'^{q^\nu p^\kappa} \zeta^{L_i(a_0, \ldots, a_n)})\big)\}^G / \alpha_i'^{q^\nu p^\kappa G} = 1.$$

Thus x_i is a root of 1 in $K_{\mathfrak{p}}$, and we denote by d_i the order of x_i. Now d_i, being a divisor of $q^\nu G$, is prime to p, whence $d_i \mid G$ (by [8], p. 220) and $x_i = \zeta^{a_i'}$ for some $a_i' \in \mathbb{N}$. This establishes (10.7).

By (10.6) and (10.7) we have

(10.8) $$\operatorname{ord}_p(\alpha_i'^{p^\kappa} \zeta^{a_i'} - 1) > \theta + \frac{1}{p-1}, \quad 1 \leqq i \leqq r.$$

We define $(\alpha_i'^{p^\kappa} \zeta^{a_i'})^{1/q}$ by the p-adic exponential and logarithmic functions:

(10.9) $$(\alpha_i'^{p^\kappa} \zeta^{a_i'})^{1/q} = \exp\left(\frac{1}{q} \log(\alpha_i'^{p^\kappa} \zeta^{a_i'})\right), \quad 1 \leqq i \leqq r;$$

and we fix a choice of q-th roots of $\alpha'_1, \ldots, \alpha'_r$ in $\mathbb{C}_p$, denoted by $\alpha_1'^{1/q}, \ldots, \alpha_r'^{1/q}$, such that

$$(10.10) \qquad (\alpha_i'^{p^\kappa} \zeta^{a'_i})^{1/q} = (\alpha_i'^{1/q})^{p^\kappa} \xi^{a'_i}, \quad 1 \leqq i \leqq r,$$

where ξ is given by (10.3).

We shall use the notation introduced in [5], §12:

$$\Delta(z;k) = (z+1)\cdots(z+k)/k! \quad \text{for} \quad k \in \mathbb{Z}_{>0} \quad \text{and} \quad \Delta(z;0) = 1,$$

$$\Pi(z_1, \ldots, z_{r-1}; t_1, \ldots, t_{r-1}) = \prod_{i=1}^{r-1} \Delta(z_i; t_i) \quad (t_1, \ldots, t_{r-1} \in \mathbb{N}),$$

$$\Theta(z;k,l,m) = \frac{1}{m!}\left(\frac{d}{dz}\right)^m (\Delta(z;k))^l \quad (l, m \in \mathbb{N}).$$

By the argument in Tijdeman [16], p. 200, we see that [26], Lemma 1.3 and the first assertion of [16], Lemma T1 remain valid for $x \leqq 0$.

Recalling (10.1) and writing $\lambda = (\lambda_{-1}, \ldots, \lambda_r)$, we define

$$(10.11) \qquad \mathscr{B} = \{b \in \mathbb{N} \mid b < q^{\mu - u}, \gcd(a'_1, \ldots, a'_r, G_0) \mid bG_1\}.$$

Let

$$(10.12) \quad \Lambda = \left\{\lambda \in \mathbb{N}^{r+2} \mid \lambda_i \leqq D_i, -1 \leqq i \leqq r, \sum_{i=1}^r a'_i \lambda_i \equiv 0 \pmod{G_1}\right\}.$$

For $b \in \mathscr{B}$, set

$$(10.13) \qquad \Lambda_b = \left\{\lambda \in \Lambda \mid \sum_{i=1}^r a'_i \lambda_i \equiv bG_1 \pmod{G_0}\right\}.$$

Obviously Λ has a partition

$$(10.14) \qquad \Lambda = \bigcup_{b \in \mathscr{B}} \Lambda_b.$$

We shall construct a polynomial $P = P(Y_0, \ldots, Y_r)$ of the form

$$(10.15) \qquad P = \sum_{\lambda \in \Lambda} \varrho(\lambda) \big(\Delta(Y_0 + \lambda_{-1}; D_{-1} + 1)\big)^{\lambda_0 + 1} Y_1^{\lambda_1} \cdots Y_r^{\lambda_r}$$

with coefficients $\varrho(\lambda) = \varrho(\lambda_{-1}, \ldots, \lambda_r)$ in $\mathcal{O}_K$. We write

$$P = \sum_{b \in \mathscr{B}} P_b,$$

where P_b is given by the right side of (10.15) with Λ replaced by Λ_b.

Denote by $\partial_1^*, \ldots, \partial_{n-1}^*$ the differential operators specified in §8 and put $\partial_0^* = \partial / \partial Y_0$. Then we have

$$\partial_j^* Y_1^{\lambda_1} \cdots Y_r^{\lambda_r} = \gamma_j Y_1^{\lambda_1} \cdots Y_r^{\lambda_r} \quad (1 \leqq j < n),$$

where

$$\gamma_j = \sum_{i=1}^{r} (b_n \partial L_i / \partial z_j - b_j \partial L_i / \partial z_n) \lambda_i . \tag{10.16}$$

For any $t = (t_0, \dots, t_{r-1}) \in \mathbb{N}^r$, we write $|t| = t_0 + \cdots + t_{r-1}$ and put

$$\Pi(t) = \Pi(\gamma_1, \dots, \gamma_{r-1}; t_1, \dots, t_{r-1}),$$

$$\Theta(Y_0; t) = \big(v(D_{-1}+1)\big)^{t_0} \Theta(Y_0 + \lambda_{-1}; D_{-1}+1, \lambda_0+1, t_0),$$

where

$$v(k) = \mathrm{lcm}(1, 2, \dots, k) \quad \text{for} \quad k \in \mathbb{Z}_{>0},$$

and we record

$$\log v(k) < 1.03883 k < \frac{107}{103} k \tag{10.17}$$

by [12], p. 71, (3.35). We introduce further polynomials $Q(t) = Q(Y_0, \dots, Y_r; t)$ by

$$Q(t) = \sum_{\lambda \in \Lambda} \varrho(\lambda) \Pi(t) \Theta(Y_0; t) Y_1^{\lambda_1} \cdots Y_r^{\lambda_r}, \tag{10.18}$$

and write

$$Q(t) = \sum_{b \in \mathscr{B}} Q_b(t),$$

where $Q_b(t)$ is given by the right side of (10.18) with Λ replaced by Λ_b.

We shall use the notation of heights introduced in [5], § 2. Now we apply [5], Lemma 1, which is a consequence of Bombieri and Vaaler [6], Theorem 9, to prove the following lemma, where

$$\varrho = \big(\varrho(\lambda) : \lambda \in \Lambda\big) \in \mathbb{P}^N \quad \text{with } N = \text{the number of elements of } \Lambda .$$

Lemma 10.1. *There exist*

$$\varrho(\lambda) \in \mathcal{O}_K, \quad \lambda \in \Lambda,$$

not all zero, with

$$h_0(\varrho) \leqq \frac{SD}{d} \Bigg\{ g_{12} + \frac{1}{c_0 - 1} \Bigg[\frac{1}{2} g_9 + \frac{1}{2} \left(1 + \frac{1}{2g_2 + 1}\right) \frac{1}{c_1 c_2} + \left(\frac{107}{103} \frac{1}{e_{\mathfrak{p}} \theta} + g_{10}\right) \frac{1}{c_1 c_3} + \left(1 + \frac{1}{g_6}\right) \frac{1}{c_1 c_4} \Bigg] \Bigg\}, \tag{10.19}$$

such that for all $b \in \mathscr{B}$ we have

$$Q_b\big(s, (\alpha_1'^{p^\kappa} \zeta^{a_1'})^s, \dots, (\alpha_r'^{p^\kappa} \zeta^{a_r'})^s; t\big) = 0 \tag{10.20}$$

for $s \in \mathbb{Z}$ with $|s| \leqq S$ and $t \in \mathbb{N}^r$ with $|t| \leqq T$.

Remark. In the sequel s always denotes a rational integer and t always denotes an r-tuple $(t_0, \ldots, t_{r-1}) \in \mathbb{N}^r$. The expressions $s \in \mathbb{Z}$ and $t \in \mathbb{N}^r$ will be omitted.

Proof. For each $\lambda \in \Lambda_b$, by (10.13), there exists $w(\lambda) \in \mathbb{Z}$ such that

$$a_1' \lambda_1 + \cdots + a_r' \lambda_r = b G_1 + w(\lambda) G_0 ,$$

whence, by (10.2),

$$\prod_{i=1}^{r} (\alpha_i'^{p^\kappa} \zeta^{a_i'})^{s\lambda_i} = \zeta^{sbG_1} \alpha_0^{sw(\lambda)} \prod_{i=1}^{r} \alpha_i'^{p^\kappa s \lambda_i} .$$

Thus it suffices to construct $\varrho(\lambda) \in \mathcal{O}_K$, $\lambda \in \Lambda$, not all zero, such that

$$\sum_{\lambda \in \Lambda_b} \varrho(\lambda) \Pi(t) \Theta(s; t) \alpha_0^{sw(\lambda)} \prod_{i=1}^{r} \alpha_i'^{p^\kappa s \lambda_i} = 0 \tag{10.21}$$

$$\text{for } b \in \mathscr{B},\ |s| \leqq S,\ |t| \leqq T .$$

(10.21) is a system of

$$M \leqq \tilde{M} = q^{\mu - u}(2S+1)\binom{[T]+r}{r}$$

homogeneous linear equations in N unknowns $\varrho(\lambda)$, $\lambda \in \Lambda$, with coefficients in

$$E = \mathbb{Q}(\alpha_0, \alpha_1', \ldots, \alpha_r') \subseteq K .$$

We now determine $\varrho(\lambda) \in \mathcal{O}_E$ by [5], Lemma 1. Let $d' = [E : \mathbb{Q}]$ and $\mathscr{D}'$ be the discriminant of E. Then Lemma 3.1, (9.19) and (9.27) yield

$$\frac{1}{2d'} \log|\mathscr{D}'| \leqq \frac{1}{2} \log d' + (d'-1) \max_{1 \leqq i \leqq r} \sigma_i \leqq g_{12} \frac{SD}{d} . \tag{10.22}$$

From

$$N \geqq (D_{-1}+1)(D_0+1) G_1^{r-1} \prod_{i=1}^{r} \left[\frac{D_i + 1}{G_1} \right]$$

we see that N is at least $q^{\mu - u}$ times the left side of (9.23), whence (9.23) gives

$$N \geqq c_0 \tilde{M} \geqq c_0 M .$$

Further by (9.28) we have

$$\frac{M}{N-M} \cdot \frac{1}{2} \log N \leqq \frac{1}{c_0 - 1} \cdot \frac{1}{2} g_9 \frac{SD}{d} . \tag{10.23}$$

By (8.7), $|\partial L_i / \partial z_j| \leqq q^\nu \sigma_i / h'(\alpha_j)$ for $1 \leqq i \leqq r$, $1 \leqq j \leqq n$. So by (10.16), (9.10) and (9.17)

$$\sum_{j=1}^{r-1} |\gamma_j| \leqq q^v e^h \cdot \frac{2(r-1)D}{c_1 c_2 p^\kappa f_{\mathfrak{p}} \log p}.$$

Thus for $t \in \mathbb{N}^r$ with $1 \leqq T' = t_1 + \cdots + t_{r-1}$ and $\lambda \in \Lambda$ we have, by (9.15), (9.31) and (9.10),

$$\begin{aligned}(10.24) \qquad \log|\Pi(t)| &= \log|\Pi(\gamma_1, \ldots, \gamma_{r-1}; t)| \\ &\leqq T' \log\bigl(e\bigl(1 + (|\gamma_1| + \cdots + |\gamma_{r-1}|)/T'\bigr)\bigr) \\ &\leqq T'(\tilde{D}_{-1} + 2) + T' \log\left(\frac{1}{e^h} + \frac{2(r-1)D}{c_1 c_2 p^\kappa f_{\mathfrak{p}} \log p} \cdot \frac{1}{T'}\right) \\ &\leqq \left(1 + \frac{1}{g_0}\right) T'(\tilde{D}_{-1} + 1) + g_{10} \frac{1}{c_1 c_3} \frac{SD}{d}.\end{aligned}$$

Note that (10.24) is trivially true for $T' = 0$, since in this case $\Pi(t) = 1$.

By Lemma 1.6 of [25], (10.17), (9.29) and (9.26) we have, for $\lambda \in \Lambda$, $|s| \leqq S$, $|t| \leqq T$

$$\begin{aligned}(10.25) \quad \log|\Theta(s; t)| &= \log|\bigl(v(D_{-1} + 1)\bigr)^{t_0} \Theta(s + \lambda_{-1}; D_{-1} + 1, \lambda_0 + 1, t_0)| \\ &\leqq \frac{107}{103} t_0 (D_{-1} + 1) + (D_{-1} + 1)(D_0 + 1) \log\left(e\left(1 + \frac{S + D_{-1}}{D_{-1} + 1}\right)\right) \\ &\leqq \frac{107}{103} t_0 (D_{-1} + 1) + \left(1 + \frac{1}{g_6}\right) \frac{1}{c_1 c_4} \frac{SD}{d}.\end{aligned}$$

On combining (10.24) with (10.25), and by (9.25) and $g_0 > 27$, we obtain

$$(10.26) \quad \log|\Pi(t)\,\Theta(s; t)| \leqq \left\{\left(\frac{107}{103} \frac{1}{e_{\mathfrak{p}} \theta} + g_{10}\right) \frac{1}{c_1 c_3} + \left(1 + \frac{1}{g_6}\right) \frac{1}{c_1 c_4}\right\} \frac{SD}{d}$$

$$\text{for } \lambda \in \Lambda,\ |s| \leqq S,\ |t| \leqq T.$$

Now let $|\ |_v$ be an absolute value on E normalized as in [5], §2. On noting that $\Pi(t)\,\Theta(s; t) \in \mathbb{Z}$, we have for $v \mid \infty$,

$$\log\left|\Pi(t)\,\Theta(s; t)\,\alpha_0^{sw(\lambda)} \prod_{i=1}^{r} \alpha_i'^{\,p^\kappa s \lambda_i}\right|_v \leqq \log|\Pi(t)\,\Theta(s; t)| + \sum_{i=1}^{r} D_i \log\max(1, |\alpha_i'^{\,p^\kappa s}|_v)$$

$$\text{for } \lambda \in \Lambda,\ |s| \leqq S,\ |t| \leqq T;$$

and for $v \nmid \infty$,

$$\log\left|\Pi(t)\,\Theta(s; t)\,\alpha_0^{sw(\lambda)} \prod_{i=1}^{r} \alpha_i'^{\,p^\kappa s \lambda_i}\right|_v \leqq \sum_{i=1}^{r} D_i \log\max(1, |\alpha_i'^{\,p^\kappa s}|_v)$$

$$\text{for } \lambda \in \Lambda,\ |s| \leqq S,\ |t| \leqq T.$$

Thus, by (10.26), (8.6) and (9.24),

$$\text{(10.27)} \qquad \frac{1}{d'}\sum_v \log \max_{\lambda\in\Lambda_b}\left|\Pi(t)\,\Theta(s;t)\,\alpha_0^{sw(\lambda)}\prod_{i=1}^{r}\alpha_i'^{p^\kappa s\lambda_i}\right|_v$$

$$\leqq \frac{1}{c_1c_2}\frac{|s|D}{d}+\left\{\left(\frac{107}{103}\frac{1}{e_{\mathfrak{p}}\theta}+g_{10}\right)\frac{1}{c_1c_3}+\left(1+\frac{1}{g_6}\right)\frac{1}{c_1c_4}\right\}\frac{SD}{d}$$

$$\text{for } b\in\mathscr{B},\ |s|\leqq S,\ |t|\leqq T.$$

Note that, by (9.19), $[S]([S]+1)\leqq S(2S+1)\cdot\frac{1}{2}\big(1+1/(2g_2+1)\big)$. Hence

$$\text{(10.28)} \quad \frac{1}{N-M}\sum_{\substack{b\in\mathscr{B}\\ |s|\leqq S,\,|t|\leqq T}}\frac{1}{d'}\sum_v \log\max_{\lambda\in\Lambda_b}\left|\Pi(t)\,\Theta(s;t)\,\alpha_0^{sw(\lambda)}\prod_{i=1}^{r}\alpha_i'^{p^\kappa s\lambda_i}\right|_v$$

$$\leqq \frac{1}{c_0-1}\left\{\frac{1}{2}\left(1+\frac{1}{2g_2+1}\right)\frac{1}{c_1c_2}+\left(\frac{107}{103}\frac{1}{e_{\mathfrak{p}}\theta}+g_{10}\right)\frac{1}{c_1c_3}\right.$$

$$\left.+\left(1+\frac{1}{g_6}\right)\frac{1}{c_1c_4}\right\}\frac{SD}{d}.$$

Now by Lemma 1 of [5], Lemma 10.1 follows from (10.22), (10.23) and (10.28).

11. Double inductive procedure

For $\varepsilon^{(I)}\in\mathbb{N}$, $D_i^{(I)}\in\mathbb{N}$ $(-1\leqq i\leqq r)$ and $\varrho^{(I)}(\lambda)=\varrho^{(I)}(\lambda_{-1},\dots,\lambda_r)\in\mathcal{O}_K$, which will be constructed in the following main inductive argument, we set

$$\text{(11.1)} \qquad \mathscr{B}^{(I)}=\{b\in\mathbb{N}\,|\,b<q^{\mu-u},\ \gcd(a_1',\dots,a_r',G_0)\,|\,(\varepsilon^{(I)}+bG_1)\}\,,$$

$$\text{(11.2)}\quad \Lambda^{(I)}=\left\{\lambda\in\mathbb{N}^{r+2}\,\middle|\,\lambda_i\leqq D_i^{(I)},\ -1\leqq i\leqq r,\ \sum_{i=1}^{r}a_i'\lambda_i\equiv\varepsilon^{(I)}\,(\mathrm{mod}\,G_1)\right\}.$$

For $b\in\mathscr{B}^{(I)}$, set

$$\text{(11.3)} \qquad \Lambda_b^{(I)}=\left\{\lambda\in\Lambda^{(I)}\,\middle|\,\sum_{i=1}^{r}a_i'\lambda_i\equiv\varepsilon^{(I)}+bG_1\,(\mathrm{mod}\,G_0)\right\}.$$

We put

$$\text{(11.4)}\quad Q^{(I)}(t)=Q^{(I)}(Y_0,\dots,Y_r;t)=\sum_{\lambda\in\Lambda^{(I)}}\varrho^{(I)}(\lambda)\,\Pi(t)\,\Theta(q^{-I}Y_0;t)\,Y_1^{\lambda_1}\cdots Y_r^{\lambda},$$

and write

$$Q^{(I)}(t)=\sum_{b\in\mathscr{B}^{(I)}}Q_b^{(I)}(t)\,,$$

where $Q_b^{(I)}(t)$ is given by the right side of (11.4) with $\Lambda^{(I)}$ replaced by $\Lambda_b^{(I)}$.

We now define the linear forms

$$M_i = L_i - \frac{1}{b_n}\frac{\partial L_i}{\partial z_n} L \quad (1 \leqq i \leqq r)\,, \tag{11.5}$$

where $L = b_1 z_1 + \cdots + b_n z_n$. Then

$$b_n M_i = b_n(\partial L_i/\partial z_0)\, z_0 + \sum_{j=1}^{n-1} (b_n \partial L_i/\partial z_j - b_j \partial L_i/\partial z_n)\, z_j \quad (1 \leqq i \leqq r)\,.$$

For $z_0, z_1, \ldots, z_n$ with $\operatorname{ord}_p z_0 \geqq 0$ and $\operatorname{ord}_p z_j > 1/(p-1)$ $(1 \leqq j \leqq n)$, we define the p-adic functions

$$\varphi^{(I)}(z_0, \ldots, z_n; t) = Q^{(I)}(z_0, e^{L_1(0, z_1, \ldots, z_n)}, \ldots, e^{L_r(0, z_1, \ldots, z_n)}; t)\,, \tag{11.6}$$

$$f^{(I)}(z_0, \ldots, z_{n-1}; t) = Q^{(I)}(z_0, e^{M_1(0, z_1, \ldots, z_{n-1})}, \ldots, e^{M_r(0, z_1, \ldots, z_{n-1})}; t)\,. \tag{11.7}$$

Let v be defined as in § 5 (see the paragraph above (5.23)). We put for $z \in \mathbb{Z}_p$

$$\varphi^{(I)}(z; t) = \varphi^{(I)}\big(z, zq^{-v}\log(\alpha_1^{p^\kappa}\zeta^{a_1}), \ldots, zq^{-v}\log(\alpha_n^{p^\kappa}\zeta^{a_n}); t\big)\,, \tag{11.8}$$

$$f^{(I)}(z; t) = f^{(I)}\big(z, zq^{-v}\log(\alpha_1^{p^\kappa}\zeta^{a_1}), \ldots, zq^{-v}\log(\alpha_{n-1}^{p^\kappa}\zeta^{a_{n-1}}); t\big)\,. \tag{11.9}$$

For $b \in \mathscr{B}^{(I)}$, let $\varphi_b^{(I)}(z_0, \ldots, z_n; t)$ and $f_b^{(I)}(z_0, \ldots, z_{n-1}; t)$ be given by the right side of (11.6) and (11.7), respectively, with $Q^{(I)}$ replaced by $Q_b^{(I)}$; let $\varphi_b^{(I)}(z; t)$ and $f_b^{(I)}(z; t)$ be given by the right side of (11.8) and (11.9), with $\varphi^{(I)}$ and $f^{(I)}$ replaced by $\varphi_b^{(I)}$ and $f_b^{(I)}$, respectively. Note that by (10.7), we have

$$\varphi^{(I)}(z; t) = Q^{(I)}\big(z, (\alpha_1'^{p^\kappa}\zeta^{a_1'})^z, \ldots, (\alpha_r'^{p^\kappa}\zeta^{a_r'})^z; t\big) \quad \text{for any } z \in \mathbb{Z}_p\,, \tag{11.10}$$

$$\varphi_b^{(I)}(z; t) = Q_b^{(I)}\big(z, (\alpha_1'^{p^\kappa}\zeta^{a_1'})^z, \ldots, (\alpha_r'^{p^\kappa}\zeta^{a_r'})^z; t\big) \quad \text{for any } z \in \mathbb{Z}_p\,. \tag{11.11}$$

Recall η, I^*, S, T given by (9.3), (9.8), (9.12), (9.14). Let

$$S^{(I)} = \eta^{-(r+1)I} S, \quad T^{(I)} = \eta^{(r+1)I} T. \tag{11.12}$$

We write $\varrho^{(I)} = \big(\varrho^{(I)}(\lambda) : \lambda \in \Lambda^{(I)}\big)$.

The main inductive argument. *Suppose that Proposition* 9.1 *is false, that is,*

$$\operatorname{ord}_p \Xi \geqq U \tag{11.13}$$

for some $\alpha_1, \ldots, \alpha_n \in K$ *and* $b_1, \ldots, b_n \in \mathbb{Z}$ *satisfying* (1.9) *and* (1.13) *with* $\alpha_1, \ldots, \alpha_n$ *multiplicatively independent and* $b_1, \ldots, b_n$ *not all zero. Then for every* $I \in \mathbb{Z}$ *with*

$$0 \leqq I \leqq \min([\log D_r/\log q] + 1, I^*)$$

there exist $\varepsilon^{(I)} \in \mathbb{N}$, $D_i^{(I)} \in \mathbb{N}$ $(-1 \leqq i \leqq r)$ *with*

$$\gcd(a_1', \ldots, a_r', G_1) \mid \varepsilon^{(I)}\,,$$

$$D_i^{(I)} = D_i \quad (i = -1, 0), \quad D_i^{(I)} \leqq q^{-I} D_i \quad (1 \leqq i \leqq r)\,,$$

and $\varrho^{(I)}(\lambda) = \varrho^{(I)}(\lambda_{-1}, \ldots, \lambda_r) \in \mathcal{O}_K$ $(\lambda \in \Lambda^{(I)})$, *not all zero, with*

$$(11.14)\quad h_0(\varrho^{(I)}) \leqq \frac{SD}{d}\left\{g_{12} + \frac{1}{c_0 - 1}\left[\frac{1}{2}g_9 + \frac{1}{2}\left(1 + \frac{1}{2g_2+1}\right)\frac{1}{c_1c_2}\right.\right.$$
$$\left.\left. + \left(\frac{107}{103}\frac{1}{e_{\mathfrak{p}}\theta} + g_{10}\right)\frac{1}{c_1c_3} + \left(1 + \frac{1}{g_6}\right)\frac{1}{c_1c_4}\right]\right\},$$

such that

$$(11.15)\qquad \varphi_b^{(I)}(s;t) = 0 \quad \text{for all } b \in \mathscr{B}^{(I)},\ |s| \leqq qS^{(I)},\ |t| \leqq \eta T^{(I)}.$$

In the remaining of this section, we always keep (11.13).

Lemma 11.1. *Suppose* $b \in \mathscr{B}^{(I)}$ *and* $\varrho^{(I)}(\lambda) \in \mathcal{O}_K$ $(\lambda \in \Lambda_b^{(I)})$ *are not all zero. Then for* $y \in \mathbb{Q} \cap \mathbb{Z}_p$, $|t| \leqq T$ *we have*

$$\operatorname{ord}_p\left(f_b^{(I)}(y;t) - \varphi_b^{(I)}(y;t)\right) \geqq U - \operatorname{ord}_p b_n + \min_{\lambda \in \Lambda_b^{(I)}} \operatorname{ord}_p \varrho^{(I)}(\lambda).$$

Proof. By [5], §12, we have

$$\Pi(t) \in \mathbb{Z}, \quad \operatorname{ord}_p \Theta(q^{-I}y;t) \geqq 0.$$

Further, similarly to the proof of [24], Lemma 3.2, we have $a_1b_1 + \cdots + a_nb_n \equiv 0 \pmod{G}$, whence by (11.13) we get

$$\begin{aligned}\operatorname{ord}_p L\left(\log(\alpha_1^{p^\kappa}\zeta^{a_1}), \ldots, \log(\alpha_n^{p^\kappa}\zeta^{a_n})\right) &= \operatorname{ord}_p\left(p^\kappa \log(\alpha_1^{b_1}\cdots\alpha_n^{b_n})\right)\\ &\geqq \operatorname{ord}_p \Xi \geqq U.\end{aligned}$$

Thus (11.5) gives

$$\begin{aligned}&\operatorname{ord}_p\Bigg(\sum_{i=1}^r \lambda_i M_i\left(0, \log(\alpha_1^{p^\kappa}\zeta^{a_1}), \ldots, \log(\alpha_{n-1}^{p^\kappa}\zeta^{a_{n-1}})\right)\\ &\qquad - \sum_{i=1}^r \lambda_i L_i\left(0, \log(\alpha_1^{p^\kappa}\zeta^{a_1}) \ldots, \log(\alpha_n^{p^\kappa}\zeta^{a_n})\right)\Bigg)\\ &= \operatorname{ord}_p\left(\left(-\frac{1}{b_n}\sum_{i=1}^r \lambda_i \frac{\partial L_i}{\partial z_n}\right) L\left(\log(\alpha_1^{p^\kappa}\zeta^{a_1}), \ldots, \log(\alpha_n^{p^\kappa}\zeta^{a_n})\right)\right)\\ &\geqq U - \operatorname{ord}_p b_n.\end{aligned}$$

Using (10.7), the remaining part of the proof is similar to that of [24], Lemma 3.2. We omit the details here.

Let $\varepsilon^{(0)} = 0$, $D_i^{(0)} = D_i$ $(-1 \leqq i \leqq r)$. Then $\mathscr{B}^{(0)} = \mathscr{B}$, $\Lambda^{(0)} = \Lambda$, $\Lambda_b^{(0)} = \Lambda_b$. Let

$$\varrho^{(0)}(\lambda) = \varrho(\lambda) \quad (\lambda \in \Lambda^{(0)}),$$

which are determined by Lemma 10.1. Thus $Q^{(0)}(t) = Q(t)$, $Q_b^{(0)}(t) = Q_b(t)$, and by Lemma 10.1, (11.11) and (11.12), we have

$$\varphi_b^{(0)}(s;t) = 0 \quad \text{for all } b \in \mathscr{B}^{(0)},\ |s| \leqq S^{(0)},\ |t| \leqq T^{(0)}. \tag{11.16}$$

Lemma 11.2. *Suppose $I = 0$ or I is a positive integer with*

$$I \leqq \min([\log D_r/\log q], I^* - 1) \tag{11.17}$$

for which the main inductive argument holds. Then for $J = 1, 2, \ldots, r$, we have

$$\varphi_b^{(I)}(s;t) = 0 \quad \text{for all } b \in \mathscr{B}^{(I)},\ |s| \leqq q^J S^{(I)},\ |t| \leqq \eta^J T^{(I)}. \tag{11.18}$$

Proof. We abbreviate $(\tau_0, \ldots, \tau_{r-1}) \in \mathbb{N}^r$ to τ, $(\mu_0, \ldots, \mu_{n-1}) \in \mathbb{N}^n$ to μ and write $|\tau| = \tau_0 + \cdots + \tau_{r-1}$, $|\mu| = \mu_0 + \cdots + \mu_{n-1}$. (There should be no confusion with that μ defined by (10.1).) We now prove that for every $m \in \mathbb{N}$ we have

$$\frac{1}{m!}\left(\frac{d}{dz}\right)^m f_b^{(I)}(z;t) = \sum_{|\mu| = m} \binom{\tau_0}{\mu_0} \frac{q^{-I\mu_0}(b_n q^v)^{-(m-\mu_0)}}{(v(D_{-1}+1))^{\mu_0}} \cdot \prod_{j=1}^{n-1} \frac{(\log(\alpha_j^{p^\kappa}\zeta^{a_j}))^{\mu_j}}{\mu_j!} \sum_{\tau_1, \ldots, \tau_{r-1}} C(\mu,\tau) f_b^{(I)}(z;\tau), \tag{11.19}$$

where $\tau = (\tau_0, \tau_1, \ldots, \tau_{r-1})$ with $\tau_0 = t_0 + \mu_0$, the second sum is over $\tau_1, \ldots, \tau_{r-1}$ with $|\tau| \leqq |t| + m$, and $C(\mu,\tau) \in \mathbb{Q} \cap \mathbb{Z}_p$. To see this, observe that by (11.5) and (10.16) we have

$$f_b^{(I)}(z;t) = \sum_{\lambda \in \Lambda_b^{(I)}} \varrho^{(I)}(\lambda)\, \Pi(t)\, \Theta(q^{-I}z;t) \prod_{j=1}^{n-1} (\alpha_j^{p^\kappa}\zeta^{a_j})^{\frac{\gamma_j}{b_n}q^{-v}z}. \tag{11.20}$$

Note that

$$\frac{1}{\mu_0!}\left(\frac{d}{dz}\right)^{\mu_0} \Theta(q^{-I}z;t) = \binom{\tau_0}{\mu_0} \frac{q^{-I\mu_0}}{(v(D_{-1}+1))^{\mu_0}}\, \Theta(q^{-I}z;\tau) \tag{11.21}$$

for all $\tau \in \mathbb{N}^r$ with $\tau_0 = t_0 + \mu_0$. Further for any $\mu \in \mathbb{N}^n$ with $|\mu| = m$ we have

$$\prod_{j=1}^{n-1} \frac{1}{\mu_j!}\left(\frac{d}{dz}\right)^{\mu_j} \left\{(\alpha_j^{p^\kappa}\zeta^{a_j})^{\frac{\gamma_j}{b_n}q^{-v}z}\right\} = (b_n q^v)^{-(m-\mu_0)} \prod_{j=1}^{n-1} \frac{(\log(\alpha_j^{p^\kappa}\zeta^{a_j}))^{\mu_j}}{\mu_j!} \cdot \gamma_1^{\mu_1} \cdots \gamma_{n-1}^{\mu_{n-1}} \prod_{j=1}^{n-1} (\alpha_j^{p^\kappa}\zeta^{a_j})^{\frac{\gamma_j}{b_n}q^{-v}z}. \tag{11.22}$$

In virtue of (8.15) and (10.16), each of $\gamma_r, \ldots, \gamma_{n-1}$ is a linear combination of $\gamma_1, \ldots, \gamma_{r-1}$ with coefficients in $\mathbb{Q} \cap \mathbb{Z}_p$. Now by [24], (3.72), which is based on Lemma 2.6 there, we obtain

$$\Pi(t)\gamma_1^{\mu_1} \cdots \gamma_{n-1}^{\mu_{n-1}} = \sum_{\tau_1, \ldots, \tau_{r-1}} C(\mu,\tau)\, \Pi(\tau) \tag{11.23}$$

where $\tau = (\tau_0, \tau_1, \ldots, \tau_{r-1})$ with $\tau_0 = t_0 + \mu_0$, the sum is over $\tau_1, \ldots, \tau_{r-1}$ with $|\tau| \leqq |t| + m$, and $C(\mu, \tau) \in \mathbb{Q} \cap \mathbb{Z}_p$. Now (11.19) follows readily from (11.20)–(11.23).

Note that (11.18) holds for $J = 0$ when $I = 0$ by (11.16), and for $J = 1$ when $I > 0$ by (11.15). Assume (11.18) holds for $J = k$ with $0 \leqq k \leqq r$ when $I = 0$, and with $1 \leqq k \leqq r$ when $I > 0$. We shall prove (11.18) for $J = k + 1$ with $k < r$ and include the case $k = r$ for later use.

Clearly, for any fixed $b \in \mathscr{B}^{(I)}$, we may assume $\varrho^{(I)}(\lambda)$, $\lambda \in \Lambda_b^{(I)}$, are not all zero, and we write

$$\varrho_b^{(I)} = \left(\varrho^{(I)}(\lambda) : \lambda \in \Lambda_b^{(I)}\right).$$

From (10.4), (10.16) and (1.13) we see that

$$\prod_{j=1}^{n-1} (\alpha_j^{p^\kappa} \zeta^{a_j})^{\frac{\gamma_j}{b_n} q^{-v} p^{-\theta} z}$$

is a p-adic normal function. Further

$$p^{(D_{-1}+1)(D_0+1)\theta} \left((D_{-1}+1)!\right)^{D_0+1} \Theta(q^{-I} p^{-\theta} z; t)$$

is a p-adic normal function. Hence, by (11.20), for $|t| \leqq \eta^{k+1} T^{(I)}$

$$F_b^{(I)}(z; t) := p^{(D_{-1}+1)(D_0+1)\left(\theta + \frac{1}{p-1}\right) - \Delta_b^{(I)}} f_b^{(I)}(p^{-\theta} z; t) \tag{11.24}$$

are p-adic normal functions, where

$$\Delta_b^{(I)} = \min_{\lambda \in \Lambda_b^{(I)}} \operatorname{ord}_p \varrho^{(I)}(\lambda).$$

Obviously

$$\frac{1}{m!}\left(\frac{d}{dz}\right)^m F_b^{(I)}(sp^\theta; t) = p^{(D_{-1}+1)(D_0+1)\left(\theta + \frac{1}{p-1}\right) - \Delta_b^{(I)} - m\theta} \frac{1}{m!}\left(\frac{d}{dz}\right)^m f_b^{(I)}(s; t). \tag{11.25}$$

We now apply Lemma 2.1 to each function in (11.24), taking

$$R = [q^k S^{(I)}], \quad M = \left[\frac{c_5}{r+1} \eta^k T^{(I)}\right]. \tag{11.26}$$

Note that by (9.10) and (9.15), we have

$$\operatorname{ord}_p b_n \leqq \log_p B^\circ \leqq (h + v \log q)/\log p,$$
$$\operatorname{ord}_p v(D_{-1}+1) \leqq (h + v \log q)/\log p.$$

By (11.25), (11.19), (11.18) with $J = k$ and Lemma 11.1 we obtain for $|t| \leqq \eta^{k+1} T^{(I)}$

$$\min_{|s| \leqq R,\, 0 \leqq m \leqq M} \left\{ \operatorname{ord}_p\left(\frac{1}{m!}\left(\frac{d}{dz}\right)^m F_b^{(I)}(sp^\theta; t)\right) + m\left(\theta + \log_p(2R+1)\right)\right\}$$

$$\geqq U + (D_{-1}+1)(D_0+1)\left(\theta + \frac{1}{p-1}\right)$$
$$+ \min_{0 \leqq m \leqq M} \{m \log_p (2R+1) - (m+1)(h + v \log q)/\log p\}\,.$$

Thus condition (2.3) holds for each $F_b^{(I)}(z;t)$ with $|t| \leqq \eta^{k+1} T^{(I)}$ whenever

$$U + (D_{-1}+1)(D_0+1)\left(\theta + \frac{1}{p-1}\right) \tag{11.27}$$
$$\geqq (M+1)(2R+1)\theta + (M+1)\frac{\max\left(h + v\log q, \log(2R+1)\right)}{\log p}\,.$$

We now verify (11.27). By (11.17), $q\eta^{r+1} > 1$, (9.22), (8.6) and (9.1), we have

$$\left(\frac{c_5}{r+1}\eta^{(r+1)I}T\right)^{-1} \leqq \frac{r+1}{c_5 T}\left(\frac{q}{q\eta^{r+1}}\right)^{\log \tilde{D}_r/\log q}$$
$$\leqq \frac{(r+1)\tilde{D}_r}{c_5 T}(q\eta^{r+1})^{-(\log g_5)/\log q} \leqq g_{11}\,.$$

Hence

$$\frac{c_5}{r+1}\eta^{k+(r+1)I}T < M+1 \leqq \frac{c_5}{r+1}\eta^{(r+1)I}T(\eta^k + g_{11})\,. \tag{11.28}$$

Further (11.12), (9.19), (9.1) and (9.2) yield

$$q^k \eta^{-(r+1)I} S\left(2 - \frac{1}{g_2}\right) < 2R+1 \leqq q^k \eta^{-(r+1)I} S\left(2 + \frac{1}{g_2}\right). \tag{11.29}$$

Thus by (9.14) and (9.33), we have

$$\frac{2c_5}{c_1} q^{k-r}\eta^k\left(1 - \frac{1}{2g_2}\right)U < (M+1)(2R+1)\theta \tag{11.30}$$
$$\leqq \frac{2c_5}{c_1} q^{k-r}(\eta^k + g_{11})\left(1 + \frac{1}{2g_2}\right)U\,.$$

From (11.29), (9.12), (9.10), (9.1) and (9.2), we see that

$$\eta^{(r+1)I}\log(2R+1) \leqq h + v\log q\,.$$

This together with (11.28), (9.12), (9.14) and (9.33) gives

$$(M+1)\frac{\max\left(h + v\log q, \log(2R+1)\right)}{\log p} \leqq (\eta^k + g_{11})\frac{1}{(r+1)q^{r+1}} \cdot \frac{1}{e_{\mathfrak{p}}\theta} \cdot \frac{c_5}{c_1 c_3} U. \tag{11.31}$$

It is readily seen that the sum of the (extreme) right sides of (11.30) and (11.31) is at most its value at $k = r$:

$$\frac{U}{c_1} \cdot c_5 (\eta^r + g_{11}) \left\{ 2 + \frac{1}{g_2} + \frac{1}{(r+1)\, q^{r+1}} \cdot \frac{1}{e_{\mathfrak{p}}\theta} \cdot \frac{1}{c_3} \right\}.$$

Thus (11.27) follows from (9.6). By Lemma 2.1 and (11.24), we get

$$\operatorname{ord}_p f_b^{(I)}\left(\frac{s}{q}; t\right) \geqq (M+1)(2R+1)\theta - (D_{-1}+1)(D_0+1)\left(\theta + \frac{1}{p-1}\right) + \Delta_b^{(I)}$$
$$\text{for} \quad s \in \mathbb{Z} \quad \text{and} \quad |t| \leqq \eta^{k+1} T^{(I)}.$$

Further Lemma 11.1 and (11.27) give for $s \in \mathbb{Z}$ and $|t| \leqq \eta^{k+1} T^{(I)}$

$$\operatorname{ord}_p \varphi_b^{(I)}\left(\frac{s}{q}; t\right) \geqq (M+1)(2R+1)\theta - (D_{-1}+1)(D_0+1)\left(\theta + \frac{1}{p-1}\right) + \Delta_b^{(I)}.$$

So we obtain, by (11.30),

$$(11.32) \qquad \operatorname{ord}_p \varphi_b^{(I)}\left(\frac{s}{q}; t\right) + (D_{-1}+1)(D_0+1)\left(\theta + \frac{1}{p-1}\right) - \Delta_b^{(I)}$$
$$> \frac{U}{c_1} \cdot 2 c_5 q^{k-r} \eta^k \left(1 - \frac{1}{2g_2}\right)$$
$$\text{for} \quad s \in \mathbb{Z} \quad \text{and} \quad |t| \leqq \eta^{k+1} T^{(I)}.$$

Now we assume $k < r$ and prove (11.18) for $J = k+1$. Suppose it were false, i.e., $\varphi_b^{(I)}(s; t) \neq 0$ for some s, t with $|s| \leqq q^{k+1} S^{(I)}$, $|t| \leqq \eta^{k+1} T^{(I)}$. We proceed to deduce a contradiction. In the remaining part of the proof we fix this set of s, t.

For $\lambda \in \Lambda_b^{(I)}$, by (11.3), there exists $w^{(I)}(\lambda) \in \mathbb{Z}$ such that

$$a_1' \lambda_1 + \cdots + a_r' \lambda_r = \varepsilon^{(I)} + b G_1 + w^{(I)}(\lambda) G_0,$$

whence by (10.2), (10.3) and (10.10), we have the following two equalities, quoted as

$$(11.33) \qquad \prod_{i=1}^{r} (\alpha_i'^{p^\kappa} \zeta^{a_i'})^{s\lambda_i} = \zeta^{s(\varepsilon^{(I)} + bG_1)} \alpha_0^{s w^{(I)}(\lambda)} \prod_{i=1}^{r} \alpha_i'^{p^\kappa s \lambda_i},$$
$$\prod_{i=1}^{r} (\alpha_i'^{p^\kappa} \zeta^{a_i'})^{\frac{s}{q}\lambda_i} = \xi^{s(\varepsilon^{(I)} + bG_1)} (\alpha_0^{1/q})^{s w^{(I)}(\lambda)} \prod_{i=1}^{r} (\alpha_i'^{1/q})^{p^\kappa s \lambda_i}.$$

Let

$$\delta_I = 0 \quad \text{if } I = 0, \quad \delta_I = 1 \quad \text{if } I > 0.$$

Then, by [26], Lemma 1.3 (see the remark before (10.11)),

$$(11.34) \qquad q^{\delta_I \{I(D_{-1}+1)(D_0+1) + (D_0+1)\operatorname{ord}_q((D_{-1}+1)!)\}} \Theta(q^{-I}s; t)\, \Pi(t) \in \mathbb{Z}.$$

By the first equality in (11.33), we have

$$\operatorname{ord}_p \varphi_b^{(I)}(s;t) = \operatorname{ord}_p \varphi',$$

where

$$\varphi' = \sum_{\lambda \in \Lambda_b^{(I)}} \varrho^{(I)}(\lambda)\big(q^{\delta_I(D_0+1)\{(D_{-1}+1)I + \operatorname{ord}_q((D_{-1}+1)!)\}} \cdot \Theta(q^{-I}s;t)\Pi(t)\big)\alpha_0^{sw^{(I)}(\lambda)} \prod_{i=1}^{r} \alpha_i'^{p^\kappa s\lambda_i}$$

is in K and non-zero. Now let $|\ |_v$ be an absolute value on K normalized as in [5], § 2, and let $|\ |_{v_0}$ be the one corresponding to $\mathfrak{p}$, whence

$$(11.35)\quad \operatorname{ord}_p \varphi' = \frac{1}{e_\mathfrak{p} f_\mathfrak{p} \log p}(-\log|\varphi'|_{v_0}) = \frac{1}{e_\mathfrak{p} f_\mathfrak{p} \log p} \sum_{v \neq v_0} \log|\varphi'|_v,$$

by the product formula on K. Note that

$$(11.36)\quad \log \sum_{\lambda \in \Lambda_b^{(I)}} 1 \leqq g_9 \frac{SD}{d}$$

by (9.28); and by $h_0(\varrho_b^{(I)}) \leqq h_0(\varrho^{(I)})$, we have

$$(11.37)\quad \sum_{v \neq v_0} \log \max_{\lambda \in \Lambda_b^{(I)}} |\varrho^{(I)}(\lambda)|_v \leqq d h_0(\varrho^{(I)}) - \log \max_{\lambda \in \Lambda_b^{(I)}} |\varrho^{(I)}(\lambda)|_{v_0}.$$

Now by (9.26)

$$(11.38)\quad \delta_I(D_0+1)\{(D_{-1}+1)I + \operatorname{ord}_q((D_{-1}+1)!)\}\log q \leqq \left(1+\frac{1}{g_6}\right)\frac{\delta_I(I+1/(q-1))\log q}{\max(f_\mathfrak{p}\log p, g_1)} \cdot \frac{1}{c_1 c_4} \cdot \frac{SD}{d}.$$

It is readily seen that (10.24) holds for the fixed t and any $\lambda \in \Lambda_b^{(I)}$. Further, by (10.17), [25], Lemma 1.6 and (11.12), $q\eta^{r+1} > 1$, (9.29), we have

$$\begin{aligned}
&\log|\Theta(q^{-I}s;t)| \\
&\leqq \frac{107}{103} t_0(D_{-1}+1) + (D_{-1}+1)(D_0+1)\left(1+\log\left(1+\frac{q^{-I}|s|+D_{-1}}{D_{-1}+1}\right)\right) \\
&\leqq \frac{107}{103} t_0(D_{-1}+1) + (D_{-1}+1)(D_0+1)\max(f_\mathfrak{p}\log p, g_1)\left(1+\frac{k\log q}{\max(f_\mathfrak{p}\log p, g_1)}\right).
\end{aligned}$$

This together with (10.24), (9.25), (9.26) and $g_0 > 27$ implies for all $\lambda \in \Lambda_b^{(I)}$

$$(11.39)\quad \log|\Theta(q^{-I}s;t)\Pi(t)| \leqq \left\{\left(\frac{107}{103}\frac{1}{e_\mathfrak{p}\theta}\eta^{k+1} + g_{10}\right)\frac{1}{c_1 c_3} + \left(1+\frac{k\log q}{\max(f_\mathfrak{p}\log p, g_1)}\right)\left(1+\frac{1}{g_6}\right)\frac{1}{c_1 c_4}\right\}\frac{SD}{d}.$$

Observe for $\lambda \in \Lambda_b^{(I)}$

$$(11.40)\qquad \log\left|\alpha_0^{sw^{(I)}(\lambda)} \prod_{i=1}^{r} \alpha_i'^{p^\kappa s \lambda_i}\right|_v \leqq p^\kappa \sum_{i=1}^{r} D_i^{(I)} \log\max(1, |\alpha_i'^s|_v)$$

$$\leqq p^\kappa q^{-I} \sum_{i=1}^{r} D_i \log\max(1, |\alpha_i'^s|_v)\,.$$

Also, by (8.6), for $1 \leqq i \leqq r$,

$$(11.41)\qquad \sum_{v \neq v_0} \log\max(1, |\alpha_i'^s|_v) \leqq d h_0(\alpha_i'^s) \leqq d|s|\sigma_i \leqq d q^{k+1} S^{(I)} \sigma_i$$

$$= q^{k+1} \eta^{-(r+1)I} S d \sigma_i\,.$$

Now using (11.34)–(11.41), (11.14), (9.24), (9.26), (9.33), we obtain

$$(11.42)\qquad \operatorname{ord}_p \varphi_b^{(I)}(s;t) + (D_{-1}+1)(D_0+1)\left(\theta + \frac{1}{p-1}\right) - \Delta_b^{(I)}$$

$$\leqq \frac{U}{c_1 q^{r+1}} \times \text{the right side of (9.37)}\,.$$

Hence by (9.37), (11.42) contradicts (11.32). This contradiction proves (11.18) for $J = k+1$. Thus the induction on J is complete and Lemma 11.2 follows at once.

Lemma 11.3. *For every I as in Lemma* 11.2 *we have*

$$(11.43)\qquad \varphi_b^{(I)}\left(\frac{s}{q}; t\right) = 0 \quad \textit{for all } b \in \mathscr{B}^{(I)},\ |s| \leqq q([S^{(I+1)}]+1),\ |t| \leqq T^{(I+1)}\,.$$

Proof. Note that $T^{(I+1)} = \eta^{r+1} T^{(I)}$ and by (9.9) we have

$$(11.44)\qquad |s/q| \leqq [S^{(I+1)}] + 1 \leqq \eta^{-(r+1)I} S(\eta^{-(r+1)} + g_2^{-1}) \leqq q S^{(I)}\,,$$

whence (11.43) for s with $q|s$ follows from Lemma 11.2 with $J=1$. Now we consider s with $(s,q)=1$. For any fixed $b \in \mathscr{B}^{(I)}$, we may assume that $\varrho^{(I)}(\lambda)$, $\lambda \in \Lambda_b^{(I)}$, are not all zero. Now, by (11.32) with $k = r$ we have for $s \in \mathbb{Z}$, $|t| \leqq T^{(I+1)}$

$$(11.45)\qquad \operatorname{ord}_p \varphi_b^{(I)}\left(\frac{s}{q}; t\right) + (D_{-1}+1)(D_0+1)\left(\theta + \frac{1}{p-1}\right) - \Delta_b^{(I)}$$

$$> \frac{U}{c_1} \cdot 2 c_5 \eta^r \left(1 - \frac{1}{2g_2}\right).$$

Let $K' = K(\alpha_0^{1/q}, \alpha_1'^{1/q}, \dots, \alpha_r'^{1/q})$. By consecutively applying Fröhlich and Taylor [7], III. 2, (**2.28**) (c) $r+1$ times, we see that

$$\mathfrak{p}\,\mathcal{O}_{K'} = \mathfrak{P}_1 \mathfrak{P}_2 \cdots \mathfrak{P}_{q^{r_1}}$$

for some r_1 with $0 \leqq r_1 \leqq r+1$, where $\mathfrak{P}_j$ are distinct prime ideals of $\mathcal{O}_{K'}$ with ramification index and residue class degree (over $\mathbb{Q}$)

$$e_{\mathfrak{P}_j} = e_{\mathfrak{p}}, \quad f_{\mathfrak{P}_j} = q^{r+1-r_1} f_{\mathfrak{p}}, \quad j = 1, \ldots, q^{r_1}.$$

Denote by $|\ |_{v'}$ an absolute value on K' normalized as in [5], §2, and let $|\ |_{v_j'}$ be the one corresponding to $\mathfrak{P}_j$, and $K'_{\mathfrak{P}_j}$ be the completion of K' with respect to $|\ |_{v_j'}$. The embedding of $K_{\mathfrak{p}}$ into $\mathbb{C}_p$ (see §1) can be extended to an embedding of $K'_{\mathfrak{P}_j}$ into $\mathbb{C}_p$ and we define for $\beta \in K'_{\mathfrak{P}_j}$ with $\beta \neq 0$

$$\operatorname{ord}_p^{(j)} \beta := \frac{1}{e_{\mathfrak{P}_j} f_{\mathfrak{P}_j} \log p} (-\log |\beta|_{v_j'}) = \frac{1}{q^{r+1-r_1} e_{\mathfrak{p}} f_{\mathfrak{p}} \log p} (-\log |\beta|_{v_j'}).$$

On noting that $\varphi_b^{(I)}(s/q; t) \in K_{\mathfrak{p}}$ $(\subset K'_{\mathfrak{P}_j})$, we have

$$\operatorname{ord}_p^{(j)} \varphi_b^{(I)}(s/q; t) = \operatorname{ord}_p \varphi_b^{(I)}(s/q; t) \quad (j = 1, \ldots, q^{r_1}),$$

whence (11.45) yields

$$(11.45)' \qquad \sum_{j=1}^{q^{r_1}} \operatorname{ord}_p^{(j)} \varphi_b^{(I)}\left(\frac{s}{q}; t\right) + q^{r_1}(D_{-1}+1)(D_0+1)\left(\theta + \frac{1}{p-1}\right) - q^{r_1} \Delta_b^{(I)} > \frac{U}{c_1} \cdot 2 c_5 q^{r_1} \eta^r \left(1 - \frac{1}{2 g_2}\right).$$

Suppose that (11.43) were false, that is,

$$\varphi_b^{(I)}\left(\frac{s}{q}; t\right) \neq 0$$

for some s, t with

$$(s, q) = 1, \quad |s| \leqq q([S^{(I+1)}]+1), \quad |t| \leqq T^{(I+1)}.$$

We proceed to deduce a contradiction. In the sequel we fix this set of s, t.

Now by [26], Lemma 1.3, we have for $\lambda \in \Lambda_b^{(I)}$

$$(11.46) \qquad q^{(D_0+1)\{(D_{-1}+1)(I+1) + \operatorname{ord}_q((D_{-1}+1)!)\}} \Theta(q^{-(I+1)} s; t) \Pi(t) \in \mathbb{Z}.$$

Hence, by the second equality in (11.33), we have for $j = 1, \ldots, q^{r_1}$

$$(11.47) \qquad \operatorname{ord}_p^{(j)} \varphi_b^{(I)}\left(\frac{s}{q}; t\right) = \operatorname{ord}_p^{(j)} \varphi'',$$

where

$$(11.48) \qquad \varphi'' = \sum_{\lambda \in \Lambda_b^{(I)}} \varrho^{(I)}(\lambda) \big(q^{(D_0+1)\{(D_{-1}+1)(I+1) + \operatorname{ord}_q((D_{-1}+1)!)\}} \cdot \Theta(q^{-(I+1)} s; t) \Pi(t)\big) (\alpha_0^{1/q})^{s w^{(I)}(\lambda)} \prod_{i=1}^{r} (\alpha_i'^{1/q})^{p^\kappa s \lambda_i} \neq 0$$

is in $K' = K(\alpha_0^{1/q}, \alpha_1'^{1/q}, \ldots, \alpha_r'^{1/q})$. Then, by the product formula on K',

$$(11.49) \qquad \sum_{j=1}^{q^{r_1}} \operatorname{ord}_p^{(j)} \varphi'' = \frac{1}{q^{r+1-r_1} e_{\mathfrak{p}} f_{\mathfrak{p}} \log p} \left(- \sum_{j=1}^{q^{r_1}} \log |\varphi''|_{v_j'} \right) = \frac{1}{q^{r+1-r_1} e_{\mathfrak{p}} f_{\mathfrak{p}} \log p} \sum{}' \log |\varphi''|_{v'},$$

where $\sum'$ signifies the summation over all $v' \neq v_1', \ldots, v_{q^{r_1}}'$. Note that, by $h_0(\varrho_b^{(I)}) \leqq h_0(\varrho^{(I)})$, we have

$$(11.50) \quad \sum{}' \log \max_{\lambda \in \Lambda_b^{(I)}} |\varrho^{(I)}(\lambda)|_{v'} \leqq [K' : \mathbb{Q}]\, h_0(\varrho^{(I)}) - \sum_{j=1}^{q^{r_1}} \log \max_{\lambda \in \Lambda_b^{(I)}} |\varrho^{(I)}(\lambda)|_{v_j'} = [K' : \mathbb{Q}]\, h_0(\varrho^{(I)}) + q^{r+1} e_{\mathfrak{p}} f_{\mathfrak{p}} \log p \cdot \Delta_b^{(I)}.$$

By (9.26),

$$(11.51) \qquad (D_0 + 1)\{(D_{-1} + 1)(I + 1) + \operatorname{ord}_q((D_{-1} + 1)!)\} \log q \leqq \left(1 + \frac{1}{g_6}\right) \frac{(I + q/(q-1)) \log q}{\max(f_{\mathfrak{p}} \log p, g_1)} \cdot \frac{1}{c_1 c_4} \cdot \frac{SD}{d}.$$

By [25], Lemma 1.6 and (11.44), $q\eta^{r+1} > 1$, (9.29), we have

$$\log |\Theta(q^{-(I+1)} s; t)| \leqq \frac{107}{103} t_0 (D_{-1} + 1) + (D_{-1} + 1)(D_0 + 1) g_1.$$

This together with (10.24), (9.25), (9.26), $g_0 > 27$ and $t_0 + T' = |t| \leqq T^{(I+1)} \leqq \eta^{r+1} T$ (see (11.43), (11.12)) implies that for all $\lambda \in \Lambda_b^{(I)}$

$$(11.52) \quad \log |\Theta(q^{-(I+1)} s; t)\, \Pi(t)| \leqq \left(\frac{107}{103} \frac{1}{e_{\mathfrak{p}} \theta} \eta^{r+1} + g_{10} \right) \cdot \frac{1}{c_1 c_3} \cdot \frac{SD}{d} + \left(1 + \frac{1}{g_6}\right) \cdot \frac{1}{c_1 c_4} \cdot \frac{SD}{d}.$$

Evidently

$$(11.53) \quad \log \left| \prod_{i=1}^{r} (\alpha_i'^{1/q})^{p^\kappa s \lambda_i} \right|_{v'} \leqq p^\kappa q^{-I} \sum_{i=1}^{r} D_i \cdot \log \max(1, |(\alpha_i'^{1/q})^s|_{v'}).$$

Also, by (8.6) and (11.44), we have for $1 \leqq i \leqq r$

$$(11.54) \qquad \sum{}' \log \max(1, |(\alpha_i'^{1/q})^s|_{v'}) \leqq q\eta^{-(r+1)I} S [K' : \mathbb{Q}] \sigma_i.$$

Observe that by (8.5) we have $[K' : \mathbb{Q}] = q^{r+1} d$. Utilizing (11.36), (11.46)–(11.54), (11.14), (9.24), (9.26) and (9.33), we see that

$$(11.55)\quad \sum_{j=1}^{q^{r_1}} \operatorname{ord}_p^{(j)} \varphi_b^{(I)}\left(\frac{s}{q};t\right) + q^{r_1}(D_{-1}+1)(D_0+1)\left(\theta+\frac{1}{p-1}\right) - q^{r_1}\Delta_b^{(I)}$$

$$\leqq \frac{U}{q^{r+1-r_1}c_1}\left\{c_1\left[g_{12}+\left(1+\frac{1}{2(c_0-1)}\right)g_9\right]\right.$$

$$+\left[\frac{q}{(q\eta^{r+1})^I}+\frac{1}{2(c_0-1)}\left(1+\frac{1}{2g_2+1}\right)\right]\frac{1}{c_2}$$

$$+\left[\frac{107}{103}\frac{1}{e_{\mathfrak{p}}\theta}\left(\eta^{r+1}+\frac{1}{c_0-1}\right)+\left(1+\frac{1}{c_0-1}\right)g_{10}\right]\frac{1}{c_3}$$

$$\left.+\left(1+\frac{1}{g_6}\right)\left[1+\frac{1}{c_0-1}+\frac{(I+q/(q-1))\log q}{\max(f_{\mathfrak{p}}\log p, g_1)}+\left(\theta+\frac{1}{p-1}\right)\frac{1}{f_{\mathfrak{p}}}\right]\frac{1}{c_4}\right\}.$$

Now we prove

$$(11.56)\quad \frac{q^{r+1-r_1}c_1}{U}\times \text{the right side of (11.55)} \leqq (q\eta)^r \times \text{the right side of (9.5)}.$$

If $p>2$, then by (11.17) and (9.7) (i), the left side of (11.56) as a function of I attains its maximum at $I=0$, whence (11.56) follows from the following inequality (which is a consequence of (9.1)–(9.3))

$$(q\eta)^r - 1 \geqq 2^r e^{-c_5} - 1 > \frac{2\log 2}{\log(e^4\cdot 3)} \geqq \frac{q\log q}{(q-1)g_1}.$$

If $p=2$, then by (11.17), (9.7) (ii) and (9.8), the value of the left side of (11.56) increases when $q/(q\eta^{r+1})^I$ is replaced by q and $I\log q/\max(f_{\mathfrak{p}}\log p, g_1)$ is replaced by

$$5(1-1/q)\log q/\log(q\eta^{r+1}),$$

whence (11.56) follows from

$$((q\eta)^r-1)\frac{q}{c_2} \geqq \left(1+\frac{1}{g_6}\right)\frac{1}{c_4}\left\{\frac{q\log q}{(q-1)g_1}+\left(1-\frac{1}{q}\right)\frac{5\log q}{\log(q\eta^{r+1})}\right\},$$

which can be verified by direct computation, using (9.1)–(9.3) and (9.34). Thus (11.56) is proved. By (11.56) and (9.5), (11.55) contradicts (11.45)′. This contradiction proves (11.43), and the proof of Lemma 11.3 is thus complete.

Lemma 11.4. *For every I as in Lemma* 11.2, *there exist*

$$\varepsilon^{(I+1)} \in \mathbb{N}, \quad D_i^{(I+1)} \in \mathbb{N} \quad (-1 \leqq i \leqq r)$$

with

$$(11.57)\quad \gcd(a_1', \ldots, a_r', G_1) \mid \varepsilon^{(I+1)},$$

$$(11.58)\quad D_i^{(I+1)} = D_i \quad (i=-1,0), \quad D_i^{(I+1)} \leqq q^{-(I+1)} D_i \quad (1 \leqq i \leqq r),$$

and $\varrho^{(I+1)}(\lambda) = \varrho^{(I+1)}(\lambda_{-1}, \ldots, \lambda_r) \in \mathcal{O}_K \, (\lambda \in \Lambda^{(I+1)})$, *not all zero, satisfying* (11.14) *with* I *replaced by* $I+1$, *such that*

$$(11.59) \quad \varphi_b^{(I+1)}(s;t) = 0 \quad \text{for all } b \in \mathscr{B}^{(I+1)}, |s| \leqq q([S^{(I+1)}]+1), |t| \leqq \eta T^{(I+1)},$$

where $\mathscr{B}^{(I+1)}$, $\Lambda^{(I+1)}$, $\Lambda_b^{(I+1)}$, $\varphi_b^{(I+1)}(z;t)$ *are obtained from* (11.1)–(11.3) *and* (11.11) *with* I *replaced by* $I+1$.

Proof. From Lemma 11.3, we have

$$\varphi^{(I)}\left(\frac{s}{q}; t\right) = \sum_{b \in \mathscr{B}^{(I)}} \varphi_b^{(I)}\left(\frac{s}{q}; t\right) = 0 \quad \text{for } |s| \leqq q([S^{(I+1)}]+1), |t| \leqq T^{(I+1)}.$$

By (1.2), (8.5), (10.3) and an argument similar to that in the proof of [26], Lemma 2.5, it is readily seen that

$$[K(\xi^{G_1})(\alpha_1'^{1/q}, \ldots, \alpha_r'^{1/q}) : K(\xi^{G_1})] = q^r.$$

Now, we can establish, similarly to the proof of [25], Lemma 2.5, the existence of $\varepsilon^{(I+1)}$, $D_i^{(I+1)}$ $(-1 \leqq i \leqq r)$ satisfying (11.57), (11.58) and $\varrho^{(I+1)}(\lambda) \in \mathcal{O}_K$ $(\lambda \in \Lambda^{(I+1)})$, not all zero, satisfying (11.14) with I replaced by $I+1$, such that

$$(11.60) \qquad \varphi_b^{(I+1)}(s;t) = 0$$
$$\text{for all } b \in \mathscr{B}^{(I+1)}, |s| \leqq q([S^{(I+1)}]+1) \text{ with } (s,q) = 1, |t| \leqq T^{(I+1)}.$$

It remains to verify (11.59) for s with $q|s$. In order to prove (11.59) with any fixed $b \in \mathscr{B}^{(I+1)}$, we may assume $\varrho^{(I+1)}(\lambda)$, $\lambda \in \Lambda_b^{(I+1)}$, are not all zero, and set

$$\Delta_b^{(I+1)} = \min_{\lambda \in \Lambda_b^{(I+1)}} \operatorname{ord}_p \varrho^{(I+1)}(\lambda).$$

We now apply Lemma 2.2 to each function in (11.24) with I replaced by $I+1$ and with $|t| \leqq \eta T^{(I+1)}$, taking

$$(11.61) \qquad R = q([S^{(I+1)}]+1), \quad M = \left[\frac{c_5}{r+1} T^{(I+1)}\right].$$

Similarly to the deduction of (11.27), by utilizing (11.25), (11.19), Lemma 11.1 (with I replaced by $I+1$) and (11.60), we see that the condition (2.6) holds for each $F_b^{(I+1)}(z;t)$ with $|t| \leqq \eta T^{(I+1)}$ whenever

$$(11.62) \qquad U + (D_{-1}+1)(D_0+1)\left(\theta + \frac{1}{p-1}\right)$$
$$\geqq 2\left(1 - \frac{1}{q}\right) R(M+1)\theta + (2M+2)\frac{\max\left(h + v\log q, \log(2R)\right)}{\log p}.$$

We now verify (11.62). Similarly to the proof of (11.28), we have

$$(11.63)\qquad \frac{c_5}{r+1}\,\eta^{(r+1)(I+1)}T < M+1 \leqq \frac{c_5}{r+1}\,\eta^{(r+1)I}\,T(\eta^{r+1}+g_{11})\,.$$

From (11.44) and (11.61), we get

$$(11.64)\qquad q\eta^{-(r+1)(I+1)}S < R \leqq q^2\eta^{-(r+1)I}S\,,$$

whence by (9.12), (9.10), (9.1) and (9.2), we see that

$$\eta^{(r+1)I}\log(2R) \leqq h + v\log q\,.$$

This together with (11.63), (9.12), (9.14) and (9.33) gives

$$(11.65)\qquad (2M+2)\,\frac{\max\bigl(h+v\log q, \log(2R)\bigr)}{\log p} \leqq (\eta^{r+1}+g_{11})\,\frac{1}{(r+1)\,q^{r+1}}\cdot\frac{1}{e_{\mathfrak{p}}\theta}\cdot\frac{2c_5}{c_1c_3}\,U\,.$$

By (11.63), (11.64), (9.14) and (9.33) we have

$$(11.66)\qquad \frac{2c_5}{c_1}\cdot\frac{q-1}{q^r}\,U < 2\left(1-\frac{1}{q}\right)R(M+1)\,\theta \leqq \frac{2c_5}{c_1}\cdot\frac{q-1}{q^{r-1}}\cdot(\eta^{r+1}+g_{11})\,U\,.$$

From (8.1) and (9.2), we see that

$$\frac{2(q-1)}{q^{r-1}} + \frac{1}{(r+1)\,q^{r+1}e_{\mathfrak{p}}\theta c_3} \leqq 2\,.$$

Hence

$\frac{c_1}{U}\times$ the sum of the (extreme) right sides of (11.65)

and (11.66) $\leqq$ the right side of (9.6).

Thus (11.62) follows from (9.6), whence (2.6) holds for each $F_b^{(I+1)}(z;t)$ with $|t| \leqq \eta T^{(I+1)}$. By applying Lemma 2.2 to $F_b^{(I+1)}(z;t)$, and utilizing (11.24), Lemma 11.1 with I replaced by $I+1$, (11.62) and (11.66), we obtain

$$(11.67)\qquad \operatorname{ord}_p \varphi_b^{(I+1)}(s;t) + (D_{-1}+1)(D_0+1)\left(\theta+\frac{1}{p-1}\right) - \Delta_b^{(I+1)} > \frac{2c_5}{c_1}\cdot\frac{q-1}{q^r}\,U$$
$$\text{for } |s| \leqq q([S^{(I+1)}]+1) \text{ with } q|s,\ |t| \leqq \eta T^{(I+1)}\,.$$

We now assume $\varphi_b^{(I+1)}(s;t) \neq 0$ for some s, t in the range stated in (11.67) and proceed to deduce a contradiction. In the sequel we fix this set of s, t.

For $\lambda \in \Lambda_b^{(I+1)}$, by [26], Lemma 1.3 and the fact that $q|s$, we see that (recalling $\delta_I = 0$ if $I = 0$ and $\delta_I = 1$ if $I > 0$)

$$q^{\delta_I(D_0+1)\{(D_{-1}+1)I+\operatorname{ord}_q((D_{-1}+1)!)\}}\,\Theta(q^{-(I+1)}s;t)\,\Pi(t)\in\mathbb{Z}\,. \tag{11.68}$$

Now by the first equality in (11.33) with I replaced by $I+1$, we have

$$\operatorname{ord}_p \varphi_b^{(I+1)}(s;t) = \operatorname{ord}_p \varphi''', \tag{11.69}$$

where

$$\varphi''' = \sum_{\lambda\in\Lambda_b^{(I+1)}} \varrho^{(I+1)}(\lambda)\big(q^{\delta_I(D_0+1)\{(D_{-1}+1)I+\operatorname{ord}_q((D_{-1}+1)!)\}} \cdot \Theta(q^{-(I+1)}s;t)\,\Pi(t)\big)\,\alpha_0^{sw^{(I+1)}(\lambda)}\prod_{i=1}^{r}\alpha_i'^{\,p^\kappa s\lambda_i} \tag{11.70}$$

is in K and non-zero. Let $|\;|_v$ and $|\;|_{v_0}$ be as in the proof of Lemma 11.2. Thus

$$\operatorname{ord}_p \varphi''' = \frac{1}{e_\mathfrak{p} f_\mathfrak{p}\log p}(-\log|\varphi'''|_{v_0}) = \frac{1}{e_\mathfrak{p} f_\mathfrak{p}\log p}\sum_{v\neq v_0}\log|\varphi'''|_v\,, \tag{11.71}$$

by the product formula on K. Note that (11.36) and (11.37) with I replaced by $I+1$ remain valid. On noting $|t|\leqq \eta T^{(I+1)}\leqq \eta^{r+2}T$ and (9.29), (10.24), we have

$$\log|\Theta(q^{-(I+1)}s;t)\,\Pi(t)| \leqq \left(\frac{107}{103}\frac{1}{e_\mathfrak{p}\theta}\eta^{r+2}+g_{10}\right)\cdot\frac{1}{c_1c_3}\cdot\frac{SD}{d} + \left(1+\frac{1}{g_6}\right)\cdot\frac{1}{c_1c_4}\cdot\frac{SD}{d}\,. \tag{11.72}$$

Further, (11.40) with I replaced by $I+1$ holds for every $\lambda\in\Lambda_b^{(I+1)}$. Also, by (11.44), for $1\leqq i\leqq r$

$$\sum_{v\neq v_0}\log\max(1,|\alpha_i'^{\,s}|_v)\leqq q^2\eta^{-(r+1)I}Sd\sigma_i\,. \tag{11.73}$$

Summing up, and using (11.38), we obtain

$$\begin{aligned} &\operatorname{ord}_p \varphi_b^{(I+1)}(s;t) + (D_{-1}+1)(D_0+1)\left(\theta+\frac{1}{p-1}\right) - \Delta_b^{(I+1)} \qquad (11.74)\\ &\leqq \frac{U}{c_1q^{r+1}}\left\{c_1\left[g_{12}+\left(1+\frac{1}{2(c_0-1)}\right)g_9\right]\right.\\ &\quad+\left[\frac{q}{(q\eta^{r+1})^I}+\frac{1}{2(c_0-1)}\left(1+\frac{1}{2g_2+1}\right)\right]\frac{1}{c_2}\\ &\quad+\left[\frac{107}{103}\frac{1}{e_\mathfrak{p}\theta}\left(\eta^{r+2}+\frac{1}{c_0-1}\right)+\left(1+\frac{1}{c_0-1}\right)g_{10}\right]\frac{1}{c_3}\\ &\quad\left.+\left(1+\frac{1}{g_6}\right)\left[1+\frac{1}{c_0-1}+\frac{\delta_I(I+1/(q-1))\log q}{\max(f_\mathfrak{p}\log p,g_1)}+\left(\theta+\frac{1}{p-1}\right)\frac{1}{f_\mathfrak{p}}\right]\frac{1}{c_4}\right\}. \end{aligned}$$

If $I = 0$, then

$$\frac{c_1 q^{r+1}}{U} \times \text{the right side of (11.74)} \leqq \text{the right side of (9.5)},$$

whence, by (9.5), (11.74) contradicts (11.67). If $I > 0$ and $p > 2$ (thus $q = 2$), then, by (11.17) and (9.7) (i), we have

$$\begin{aligned}
&\frac{c_1 q^{r+1}}{U} \times \text{the right side of (11.74)} \\
&\leqq \text{the right side of (9.5)} + \frac{107}{103}\frac{1}{e_{\mathfrak{p}}\theta_0}\frac{1}{c_3}(\eta^{r+2} - \eta) + \left(1 + \frac{1}{g_6}\right)\frac{1}{c_4}\frac{\log q}{(q-1)g_1} \\
&\leqq \text{the right side of (9.5)},
\end{aligned}$$

since

$$\frac{107}{103}\frac{1}{e_{\mathfrak{p}}\theta_0}\frac{1}{c_3}(\eta - \eta^{r+2}) \geqq \frac{107}{103}\frac{1}{e_{\mathfrak{p}}\theta_0}\frac{1}{c_3}\eta(1 - e^{-c_5}) \geqq \left(1 + \frac{1}{g_6}\right)\frac{1}{c_4}\frac{\log q}{(q-1)g_1}.$$

Now by (9.5), (11.74) contradicts (11.67). Finally, if $I > 0$ and $p = 2$ (thus $q = 3$), then

$$\text{(11.75)} \quad \frac{c_1 q^{r+1}}{U} \times \text{the right side of (11.74)} \leqq 2 \times \text{the right side of (9.5)},$$

since by (11.17), (9.7) (ii) and (9.8), the value of the left side of (11.75) increases when $q/(q\eta^{r+1})^I$ is replaced by q and $I \log q/\max(f_{\mathfrak{p}} \log p, g_1)$ is replaced by

$$5(1 - 1/q)\log q/\log(q\eta^{r+1}),$$

whence (11.75) follows from

$$\begin{aligned}
&\left(q + \frac{1}{2(c_0 - 1)}\right)\frac{1}{c_2} + \frac{107}{103}\frac{1}{e_{\mathfrak{p}}\theta_0}\left(\eta + \frac{1}{c_0 - 1}\right)\frac{1}{c_3} + \left(1 + \frac{1}{c_0 - 1}\right)\frac{1}{c_4} \\
&\geqq \left(1 + \frac{1}{g_6}\right)\frac{1}{c_4}\left\{\frac{\log q}{(q-1)g_1} + \left(1 - \frac{1}{q}\right)\frac{5\log q}{\log(q\eta^{r+1})}\right\},
\end{aligned}$$

which can be verified by direct computation, using (9.1)–(9.3) and (9.34). By (11.75) and (9.5), (11.74) contradicts (11.67). The fact that (11.74) contradicts (11.67) for all I as in the hypothesis of the lemma proves

$$\varphi_b^{(I+1)}(s;t) = 0 \quad \text{for } |s| \leqq q([S^{(I+1)}] + 1) \text{ with } q|s, |t| \leqq \eta T^{(I+1)}.$$

Since $b \in \mathscr{B}^{(I+1)}$ is arbitrarily chosen, this together with (11.60) establishes Lemma 11.4.

By applying Lemma 11.2 to $I = 0$ and taking $J = 1$, and by applying Lemma 11.4 to $I = 0$, we see that the main inductive argument is true for $I = 0, 1$. Now the main inductive argument follows by the induction on I, utilizing Lemma 11.4.

12. Simple reduction

We first deal with the case when $[\log D_r/\log q]+1 \leqq I^*$. Let

$$I=[\log D_r/\log q]+1\,.$$

Thus $D_r^{(I)}=0$. On applying the main inductive argument and defining

$$\varrho^{(I)}(\lambda_{-1},\ldots,\lambda_r)=0$$

for $0\leqq \lambda_i \leqq D_i^{(I)}$ $(-1\leqq i\leqq r)$ with λ not in $\Lambda^{(I)}$, we have

$$\begin{aligned}(12.1)\qquad &\sum_{\substack{0\leqq\lambda_i\leqq D_i^{(I)}\\ -1\leqq i<r}} \varrho^{(I)}(\lambda_{-1},\ldots,\lambda_{r-1},0)\,\Theta(q^{-I}s;t)\\ &\cdot\Pi(\gamma_1,\ldots,\gamma_{r-1};t_1,\ldots,t_{r-1})\cdot\prod_{i=1}^{r-1}(\alpha_i'^{p^\kappa}\zeta^{a_i'})^{s\lambda_i}=0\\ &\text{for}\quad |s|\leqq qS^{(I)},\quad |t|\leqq \eta T^{(I)},\end{aligned}$$

where

$$(12.2)\qquad \gamma_j=\sum_{i=1}^{r-1}(b_n\partial L_i/\partial z_j-b_j\partial L_i/\partial z_n)\,\lambda_i\quad(1\leqq j<r)\,,$$

since $\lambda_r=0$ by $D_r^{(I)}=0$. In virtue of (8.15) and (12.2), each of $\lambda_1,\ldots,\lambda_{r-1}$ is a linear combination of $\gamma_1,\ldots,\gamma_{r-1}$. Thus $\prod_{i=1}^{r-1}\Delta(\lambda_i;t_i)$ $(t_i\in\mathbb{N},\ 1\leqq i<r)$ is a linear combination of $\gamma_1^{\tau_1}\cdots\gamma_{r-1}^{\tau_{r-1}}$, whence, by [24], Lemma 2.6, a linear combination of

$$\Pi(\gamma_1,\ldots,\gamma_{r-1};\tau_1,\ldots,\tau_{r-1})$$

with $(\tau_1,\ldots,\tau_{r-1})\in\mathbb{N}^{r-1}$ and $\tau_1+\cdots+\tau_{r-1}\leqq t_1+\cdots+t_{r-1}$. So (12.1) gives

$$\begin{aligned}(12.3)\qquad &\sum_{\substack{0\leqq\lambda_i\leqq D_i^{(I)}\\ -1\leqq i<r}} \varrho^{(I)}(\lambda_{-1},\ldots,\lambda_{r-1},0)\,\Theta(q^{-I}s;t)\prod_{i=1}^{r-1}\Delta(\lambda_i;t_i)\cdot\prod_{i=1}^{r-1}(\alpha_i'^{p^\kappa}\zeta^{a_i'})^{s\lambda_i}=0\\ &\text{for}\quad |s|\leqq qS^{(I)},\quad |t|\leqq\eta T^{(I)}\,.\end{aligned}$$

Note that

$$D_1^{(I)}+\cdots+D_{r-1}^{(I)}\leqq\frac{1}{2}\eta T^{(I)}$$

by (11.12), $D_i^{(I)}\leqq q^{-I}D_i$ $(1\leqq i\leqq r)$, (9.14), (9.17), (9.2), (8.1), $q\eta^{r+1}>1$, $r\geqq 2$ and (9.3). Thus (12.3) holds for $|s|\leqq qS^{(I)}$ and t with

$$0\leqq t_0\leqq\frac{1}{2}\eta T^{(I)},\quad 0\leqq t_i\leqq D_i^{(I)}\quad(1\leqq i<r)\,.$$

This yields, by an argument similar to that in [24], § 3.5, that for any fixed $\lambda_1,\ldots,\lambda_{r-1}$ with $0\leqq\lambda_i\leqq D_i^{(I)}$ $(1\leqq i<r)$, the polynomial (recalling $D_i^{(I)}=D_i$, $i=-1,0$)

$$\sum_{\lambda_{-1}=0}^{D_{-1}} \sum_{\lambda_0=0}^{D_0} \varrho^{(I)}(\lambda_{-1},\ldots,\lambda_{r-1},0)\big(\Delta(x+\lambda_{-1};D_{-1}+1)\big)^{\lambda_0+1}, \tag{12.4}$$

whose degree is at most $(D_{-1}+1)(D_0+1)$, has at least

$$M := \big(2[qS^{(I)}]+1\big)\left(\left[\frac{1}{2}\eta T^{(I)}\right]+1\right)$$

zeros. Now, by $\theta \leqq 2$ (see (8.1)), $g_2 > 1$, $g_6 > 10^4$, $c_4 > 3.9$, $c_5 < 1$ (see (9.1), (9.2)), (9.3), (9.14), (9.19), (9.26) and (11.12), we have

$$M > \left(2q - \frac{1}{g_2}\right)\cdot \frac{1}{2}\eta S^{(I)} T^{(I)} \geqq (D_{-1}+1)(D_0+1)\,.$$

Thus the polynomial in (12.4) is identically zero. Further,

$$\big(\Delta(x+\lambda_{-1};D_{-1}+1)\big)^{\lambda_0+1} (0 \leqq \lambda_i \leqq D_i,\ i=-1,0)$$

are linearly independent over $\mathbb{C}$ (see [5], §12). Hence (recalling $D_r^{(I)}=0$),

$$\varrho^{(I)}(\lambda_{-1},\ldots,\lambda_r) = 0\,, \quad 0 \leqq \lambda_i \leqq D_i^{(I)} \quad (-1 \leqq i \leqq r)\,,$$

contradicting the construction in the main inductive argument that $\varrho^{(I)}(\lambda)$, $\lambda \in \Lambda^{(I)}$, are not all zero. This contradiction proves Proposition 9.1 in the case when $[\log D_r/\log q]+1 \leqq I^*$.

13. Group variety reduction

It remains to prove Proposition 9.1 in the case

$$I^* < [\log D_r/\log q]+1\,, \tag{13.1}$$

where I^* is given by (9.8). Take

$$I = I^* \tag{13.2}$$

in the main inductive argument. There exists $b \in \mathscr{B}^{(I)}$ such that $\varrho^{(I)}(\lambda)$, $\lambda \in \Lambda_b^{(I)}$, are not all zero, since $\varrho^{(I)}(\lambda)$, $\lambda \in \Lambda^{(I)}$, are not all zero and $\Lambda^{(I)} = \bigcup_{b \in \mathscr{B}^{(I)}} \Lambda_b^{(I)}$. For every s with $q^u | s$ and $\lambda \in \Lambda_b^{(I)}$, we have, by (11.33) and (1.4),

$$\prod_{i=1}^{r} (\alpha_i'^{p^\kappa}\zeta^{a_i'})^{s\lambda_i} = \zeta^{s(\varepsilon^{(I)}+bG_1)} \prod_{i=1}^{r} \alpha_i'^{p^\kappa s\lambda_i}\,.$$

Hence we get from (11.15)

$$\sum_{\lambda \in \Lambda_b^{(I)}} \varrho^{(I)}(\lambda)\,\Pi(t)\,\Theta(q^{-I}q^u s;t) \prod_{i=1}^{r} \alpha_i'^{p^\kappa q^u s\lambda_i} = 0, \tag{13.3}$$

$$|s| \leqq q^{1-u} S^{(I)}, \quad |t| \leqq \eta T^{(I)}\,.$$

Now we take

$$(13.4)\qquad \mathscr{P}(Y_0, \ldots, Y_r) = \sum_{\lambda \in \Lambda_b^{(I)}} \varrho^{(I)}(\lambda)\big(\Delta(q^{-I}p^{-\kappa}Y_0; D_{-1}+1)\big)^{\lambda_0+1} Y_1^{\lambda_1}\cdots Y_r^{\lambda_r},$$

$$(13.5)\qquad N = p^{\kappa}q^{u}, \quad \mathscr{S} = q^{1-u}S^{(I)}, \quad \mathscr{T} = \eta T^{(I)}, \quad \theta_i = \alpha_i' \quad (1 \leqq i \leqq r).$$

Recall that $\partial_0^* = \partial_0 = \partial/\partial Y_0$ and $\partial_1^*, \ldots, \partial_{r-1}^*$ are the differential operators specified in §8, and that

$$\partial_j^* Y_1^{\lambda_1}\cdots Y_r^{\lambda_r} = \gamma_j Y_1^{\lambda_1}\cdots Y_r^{\lambda_r} \quad (1 \leqq j < r),$$

with γ_j given in (10.16). By [24], Lemma 2.6, we obtain from (13.3) –(13.5) that

$$(13.6)\qquad \partial_0^{*t_0}\partial_1^{*t_1}\cdots\partial_{r-1}^{*t_{r-1}}\mathscr{P}(Ns, \theta_1^{Ns}, \ldots, \theta_r^{Ns}) = 0$$
$$\text{for}\quad 0 \leqq s \leqq \mathscr{S}, \quad t_0 + \cdots + t_{r-1} \leqq \mathscr{T}.$$

As remarked in §8, Proposition 6.1 holds with $\partial_1^*, \ldots, \partial_{r-1}^*$ in place of $\partial_1, \ldots, \partial_{r-1}$.

Let

$$(13.7)\qquad \mathscr{D}_0 = (D_{-1}+1)(D_0+1), \quad \mathscr{D}_i = q^{-I}\tilde{D}_i \quad (1 \leqq i \leqq r).$$

Put

$$(13.8)\qquad \mathscr{S}_0 = \left[\frac{1}{3}\mathscr{S}\right], \quad \mathscr{S}_i = \left[\frac{1}{r}\cdot\frac{2}{3}\mathscr{S}\right] (1 \leqq i \leqq r), \quad \mathscr{T}_i = \left[\frac{1}{r+1}\mathscr{T}\right] (0 \leqq i \leqq r).$$

Then $\mathscr{S}_0 \geqq \mathscr{S}_1 = \cdots = \mathscr{S}_r$ since $r \geqq 2$, $\mathscr{T}_0 = \cdots = \mathscr{T}_r$ and

$$\mathscr{S}_0 + \cdots + \mathscr{S}_r \leqq \mathscr{S}, \quad \mathscr{T}_0 + \cdots + \mathscr{T}_r \leqq \mathscr{T}.$$

For later convenience, we list the following inequalities derived from (8.1), (8.2) and §9. We shall use them frequently in the remaining of this section.

$$c_3/c_4 > 1/18, \quad c_5 \geqq 0.47,$$
$$r \geqq 2, \quad g_3/r > 2\cdot 10^5, \quad g_6 > 10^4, \quad (1+\varepsilon_1)(1+\varepsilon_2) < 1.001,$$
$$h \geqq \max(2f_{\mathfrak{p}}\log p, g_0) \geqq \max(f_{\mathfrak{p}}\log p, g_1), \quad \theta < p/(p-1) \leqq 2,$$
$$e_{\mathfrak{p}}\theta > 0.49, \quad c_2 q p^{\kappa}/(e_{\mathfrak{p}}\theta) > 4, \quad q\eta^{r+1} > 1, \quad \eta^{r+1} < e^{-c_5} \leqq e^{-0.47},$$
$$I = I^* \geqq [5g_1/\log(q\eta^{r+1})] + 1 \geqq 106, \quad \eta^{(r+1)I} < 10^{-21}.$$

By (13.5), (13.8) and (9.20) we have for $\varrho \in \mathbb{Z}$ with $1 \leqq \varrho \leqq r$

$$(13.9)\qquad \mathscr{T}_{\varrho} + \varrho \leqq \left(\frac{\eta^{(r+1)I}}{r+1} + \frac{\varrho}{g_3}\right)T < 10^{-5}T.$$

Evidently

$$10^{-5}/(e_{\mathfrak{p}}\theta) < 10^{-3} c_3/c_4 . \tag{13.10}$$

Now (13.9), (13.10), (13.7), (9.12), (9.14) and (9.16) yield

$$\mathscr{T}_\varrho + \varrho < 0.001 \mathscr{D}_0 \quad (1 \leqq \varrho \leqq r), \tag{13.11}$$

whence (6.2) follows. Further by (13.7), (9.17), (9.16), (9.12), (8.6), we get

$$\mathscr{D}_i < \mathscr{D}_0 \quad (1 \leqq i \leqq r). \tag{13.12}$$

Now we verify (6.1).

(i) $m = 0$. By (13.8), (13.5), (11.12), (9.18), (9.19), we have

$$\mathscr{S}_0(\mathscr{T}_0 + 1) > \left(\frac{(q-1)\,e_{\mathfrak{p}}}{3\,d} - \frac{10^{-21}}{g_2}\right)\frac{\eta}{r+1}\,ST. \tag{13.13}$$

From (13.13), (13.7), (9.3), (9.26), (9.14), we obtain

$$\mathscr{S}_0(\mathscr{T}_0 + 1) > \mathscr{D}_0 .$$

This and (13.12) establish (6.1) with $m = 0$.

(ii) $1 \leqq m < r$. By (13.8), (13.5), (11.12), (9.18), (9.19), we have

$$\mathscr{S}_m \binom{\mathscr{T}_m + m + \delta_{m,r}}{m + \delta_{m,r}} > \left(\frac{1}{r}\cdot\frac{2}{3}\cdot(q-1)\frac{e_{\mathfrak{p}}}{d} - \frac{10^{-21}}{g_2}\right)\frac{\eta^{m+1+(r+1)Im}}{(r+1)^{m+1}(m+1)!}\,ST^{m+1}. \tag{13.14}$$

By (13.7), (13.12), (9.26), (9.17) and (8.6) we get

$$(m+1)!\,\mathscr{D}_0^{m_0}\cdots\mathscr{D}_r^{m_r} \leqq \left(1 + \frac{1}{g_6}\right)\frac{1}{c_4}\cdot\frac{1}{c_1^{m+1}(c_2 p^\kappa)^m} \cdot\frac{(m+1)!}{r^m q^{Im}}\cdot\frac{SD^{m+1}}{d(f_{\mathfrak{p}}\log p)^{m+1}}, \tag{13.15}$$

where $m_i \in \{0,1\}$ with $m_0 + \cdots + m_r = m+1$. Now, by (13.14), (13.15), (9.14) and

$$(q\eta^{r+1})^{Im} > \big(e^4(r+1)\,d\big)^{5m} \quad \text{(see (13.2), (9.8), (9.1))},$$

we obtain (6.1) for $1 \leqq m < r$.

(iii) $m = r$. We have, similarly to (13.14),

$$(13.16) \quad \mathscr{S}_r\binom{\mathscr{T}_r + r}{r} > \left(\frac{1}{r}\cdot\frac{2}{3}\cdot(q-1)\frac{e_{\mathfrak{p}}}{d} - \frac{10^{-21}}{g_2}\right)\frac{\eta^{r+(r+1)I(r-1)}}{(r+1)^r r!} S T^r .$$

By (13.7), (9.26), (9.17), we have

$$(13.17) \quad (r+1)!\,\mathscr{D}_0\mathscr{D}_1\cdots\mathscr{D}_r$$
$$\leqq \frac{(r+1)!}{q^{Ir}r^r}\left(1+\frac{1}{g_6}\right)\frac{SD^{r+1}}{c_1^{r+1}(c_2 p^{\kappa})^r c_4 d^{r+1}\sigma_1\cdots\sigma_r \max(f_{\mathfrak{p}}\log p, g_1)} .$$

In virtue of (13.16), (13.17), (9.13), (9.14), in order to prove (6.1) with $m = r$, it suffices to show

$$(13.18) \quad \left(\frac{1}{r}\cdot\frac{2}{3}(q-1)\frac{e_{\mathfrak{p}}}{d} - \frac{10^{-21}}{g_2}\right)(q\eta^{r+1})^{Ir}\eta^{r-(r+1)I}$$
$$\geqq \left(1+\frac{1}{g_6}\right)(1+\varepsilon_1)(1+\varepsilon_2)\left(2+\frac{1}{g_2}\right)c_0(r+1)!\,(r+1)^r\cdot\frac{p^{f_{\mathfrak{p}}}-1}{q^u} .$$

Now, by (13.2), (9.8) and (9.1),

$$(q\eta^{r+1})^I > \exp\big(5\max(f_{\mathfrak{p}}\log p, g_1)\big) \geqq p^{f_{\mathfrak{p}}}\big(e^4(r+1)\,d\big)^4 .$$

This implies (13.18) at once. Hence (6.1) with $m = r$ is valid.

Note that in (13.6) $\mathscr{P}(Y_0, \ldots, Y_r) \neq 0$ and $\theta_i = \alpha_i'$ $(1 \leqq i \leqq r)$ are multiplicatively independent, since $l_0', \ldots, l_r'$ are linearly independent over $\mathbb{Q}$ by (5.8) and § 8, (i), (ii). Having verified (6.1) and (6.2), we can apply Proposition 6.1 with $a_i = \sigma_i$ $(1 \leqq i \leqq r)$. Thus there exists $\varrho \in \mathbb{Z}$ with $1 \leqq \varrho < r$ and there is a set of primitive linear forms $\mathscr{L}_1, \ldots, \mathscr{L}_\varrho$ in $Z_1, \ldots, Z_r$ with coefficients in $\mathbb{Z}$ such that $B_1 Z_1 + \cdots + B_r Z_r$ is in the module generated by $\mathscr{L}_1, \ldots, \mathscr{L}_\varrho$ over $\mathbb{Q}$ and, on defining

$$(13.19) \quad \mathscr{R}_i = \sum_{j=1}^{r} |\partial\mathscr{L}_i/\partial Z_j|\,\sigma_j \quad (1 \leqq i \leqq \varrho) ,$$

we have at least one of (6.3) and (6.4), whence (6.4) holds always, since (6.3) implies (6.4) by (13.5), (13.7), (13.8) and (13.11). Now

$$(13.20) \quad \mathscr{L}_i' := \mathscr{L}_i(L_1, \ldots, L_r) \quad (1 \leqq i \leqq \varrho)$$

are linear forms in $z_0, z_1, \ldots, z_n$ with coefficients in $\mathbb{Z}$ having the following properties:

(i) The $\varrho + 1$ linear forms $L_0 = q^v z_0$, $\mathscr{L}_1', \ldots, \mathscr{L}_\varrho'$ are linearly independent and

$$(13.21) \quad L = B_0 L_0 + B_1'\mathscr{L}_1' + \cdots + B_\varrho'\mathscr{L}_\varrho'$$

for some rationals $B_1', \ldots, B_\varrho'$, not all zero, since $\{\mathscr{L}_1, \ldots, \mathscr{L}_\varrho\}$ is a primitive set of linear forms with coefficients in $\mathbb{Z}$ and $B_1 Z_1 + \cdots + B_r Z_r$ is in the module generated by $\mathscr{L}_1, \ldots, \mathscr{L}_\varrho$ over $\mathbb{Q}$.

(ii) On writing

$$l_i'' = q^{-\nu} \mathscr{L}_i'(l_0, l_1, \ldots, l_n) \quad (1 \leqq i \leqq \varrho),$$

the numbers $\theta_i' = e^{l_i''}$ $(1 \leqq i \leqq \varrho)$ are in K, and satisfy $\mathrm{ord}_{\mathfrak{p}} \theta_i' = 0$ $(1 \leqq i \leqq \varrho)$ and

$$[K(\alpha_0^{1/q}, \theta_1'^{1/q}, \ldots, \theta_\varrho'^{1/q}) : K] = q^{\varrho+1}.$$

To see this, we note that, by (13.20) and § 8, (ii),

$$l_i'' = \mathscr{L}_i(l_1', \ldots, l_r') \quad (1 \leqq i \leqq \varrho). \tag{13.22}$$

Now the above statement follows from § 8, (ii) and the fact that $\{\mathscr{L}_1, \ldots, \mathscr{L}_\varrho\}$ is a primitive set of linear forms with coefficients in $\mathbb{Z}$.

(iii) We have $h'(\theta_i') \leqq \mathscr{R}_i$ $(1 \leqq i \leqq \varrho)$, since by (13.19) and (13.22), we have

$$(f_{\mathfrak{p}} \log p)/d \leqq \mathscr{R}_i$$

and

$$h_0(\theta_i') \leqq \sum_{j=1}^{r} |\partial \mathscr{L}_i / \partial Z_j| h_0(\alpha_j') \leqq \mathscr{R}_i.$$

Further (8.7) and (13.19) give

$$\begin{aligned} \sum_{j=1}^{n} |\partial \mathscr{L}_i' / \partial z_j| h'(\alpha_j) &\leqq \sum_{j=1}^{n} \sum_{k=1}^{r} |\partial \mathscr{L}_i / \partial Z_k| |\partial L_k / \partial z_j| h'(\alpha_j) \\ &\leqq q^\nu \sum_{k=1}^{r} |\partial \mathscr{L}_i / \partial Z_k| \sigma_k = q^\nu \mathscr{R}_i \quad (1 \leqq i \leqq \varrho). \end{aligned}$$

We shall prove shortly that (6.4) implies

$$\mathscr{R}_1 \cdots \mathscr{R}_\varrho \leqq \psi(\varrho) h'(\alpha_1) \cdots h'(\alpha_n), \tag{13.23}$$

where $\psi(\varrho)$ is given by (8.9) with r replaced by ϱ; thus the basic hypothesis in § 8 holds with ϱ in place of r. By the minimal choice of r, we have a contradiction and this establishes Proposition 9.1.

Now, by (6.4), (13.5), (13.7), (13.8), (9.13), (9.14), (9.17), (9.18), (9.19), (9.26), (8.8), (8.9) and $e^r \geqq r^r/r!$, in order to prove (13.23), it suffices to show

$$\frac{e^\varrho(\varrho+1)(\varrho!)^3 \varrho^\varrho (r+1)^\varrho p^{f_{\mathfrak{p}}}}{\dfrac{1}{r} \cdot \dfrac{2}{3}(q-1)\dfrac{e_{\mathfrak{p}}}{d} - \dfrac{10^{-21}}{g_2}} \leqq (q\eta^{r+1})^{I\varrho}. \tag{13.24}$$

Note that $(\varrho+1)(\varrho!)^3 \varrho^\varrho (r+1)^\varrho \leqq (r+1)^{5\varrho-2}$ and that by $I = I^*$ and (9.8) we have

$$(q\eta^{r+1})^{I\varrho} > p^{f_{\mathfrak{p}}} \big(e^4 (r+1) d\big)^{5\varrho-1}.$$

Hence (13.24) follows. The proof of Proposition 9.1 is thus complete, whence Theorem 7.1 is established.

14. Proof of Theorem 1

Lemma 14.1. *It suffices to prove Theorem* 1 *on a further assumption that* $\alpha_1, \ldots, \alpha_n$ *are multiplicatively independent.*

Proof. Without loss of generality, we may assume

$$h'(\alpha_1) \leqq \cdots \leqq h'(\alpha_n) \tag{14.1}$$

in Theorem 1. Then by [24], (2.6), we have

$$\operatorname{ord}_{\mathfrak{p}} \Xi \leqq \frac{d}{f_{\mathfrak{p}} \log p} \big(nBh'(\alpha_n) + \log 2\big), \tag{14.2}$$

whence the theorem follows in the case when $B \leqq 4ndp^{f_{\mathfrak{p}}/2}$, here we have used a consequence of (9.18):

$$q^u \leqq 2d/e_{\mathfrak{p}} \leqq 2d. \tag{14.3}$$

Thus we may assume in the sequel

$$B > 4ndp^{f_{\mathfrak{p}}/2}. \tag{14.4}$$

We prove the lemma by induction on n. We assert that the theorem is true for $n = 1$, for by (14.4) with $n = 1$ and [25], Lemma 1.4, we have

$$\operatorname{ord}_{\mathfrak{p}} \Xi \leqq \frac{2d}{f_{\mathfrak{p}} \log p} \{\log B + (p^{f_{\mathfrak{p}}} - 1) e_{\mathfrak{p}} h'(\alpha_1)\} < C(1, d, \mathfrak{p}) h'(\alpha_1) \log B.$$

Suppose now $n > 1$ and the theorem holds for $n - 1$. Assume that $\alpha_1, \ldots, \alpha_n$ are multiplicatively dependent, i.e., $l_0 = 2\pi i/q^u$, $l_1, \ldots, l_n$ are linearly dependent. Let m be the least integer such that $l_0, \ldots, l_{m-1}$ are linearly independent and $l_0, \ldots, l_{m-1}, l_m$ are linearly dependent. Then $1 \leqq m \leqq n$. By Lemma 4.1, (14.1), and the remark in §4, there exist $t_0, \ldots, t_m \in \mathbb{Z}$ with $t_m > 0$, such that $t_0 l_0 + \cdots + t_m l_m = 0$ and for $1 \leqq k \leqq m$

$$|t_k| \leqq \big(4md^2 \log\log(6d)\big)^m (q^u d)^{-1} \big(h'(\alpha_m)\big)^{m-1}. \tag{14.5}$$

Then

$$t_m(b_1 l_1 + \cdots + b_n l_n) = -b_m t_0 l_0 + \sum_{1 \leqq i \leqq n,\, i \neq m} b_i'' l_i,$$

where $b_i'' = t_m b_i - t_i b_m$, with $t_i = 0$ $(m < i \leqq n)$. Let $b_i' = q^u b_i''$ $(1 \leqq i \leqq n, i \neq m)$. Then, by (14.5), for $1 \leqq i \leqq n$ $(i \neq m)$

(14.6) $$|b_i'| \leqq 2B\big(4nd\log\log(6d)\big)^n\big(dh'(\alpha_m)\big)^{m-1} =: B'.$$

We may assume $(\alpha_1^{b_1}\cdots\alpha_n^{b_n})^{q^u t_m} \neq 1$, since otherwise we have, by [24], (2.6),

$$\operatorname{ord}_{\mathfrak{p}} \Xi \leqq \log 2 \cdot d/(f_{\mathfrak{p}}\log p),$$

whence the theorem holds trivially. Now we have, by (1.9) and the inductive hypothesis,

$$\begin{aligned}\operatorname{ord}_{\mathfrak{p}} \Xi &\leqq \operatorname{ord}_{\mathfrak{p}}\big((\alpha_1^{b_1}\cdots\alpha_n^{b_n})^{q^u t_m}-1\big)\\ &= \operatorname{ord}_{\mathfrak{p}}(\alpha_1^{b_1'}\cdots\alpha_{m-1}^{b_{m-1}'}\alpha_{m+1}^{b_{m+1}'}\cdots\alpha_n^{b_n'}-1)\\ &< C(n-1,d,\mathfrak{p})\,h'(\alpha_1)\cdots h'(\alpha_n)\big(h'(\alpha_m)\big)^{-1}\log B'.\end{aligned}$$

From (1.10),

$$\frac{C(n,d,\mathfrak{p})}{C(n-1,d,\mathfrak{p})} > 8(n+1)\frac{d}{f_{\mathfrak{p}}\log p}.$$

Hence it suffices to show

(14.7) $$\log B' \leqq 8(n+1)\frac{dh'(\alpha_m)}{f_{\mathfrak{p}}\log p}\log B.$$

Now (14.7) follows from (14.6), (14.4) and the fact that $dh'(\alpha_m) \geqq f_{\mathfrak{p}}\log p > 1$ (recalling that $f_{\mathfrak{p}} \geqq 2$ when $p=2$, see [25], Appendix). The proof of the lemma is thus complete.

Proof of Theorem 1. As indicated in the proof of Lemma 14.1, Theorem 1 is true for $n=1$. Thus we may assume that $n \geqq 2$ and $\alpha_1, \ldots, \alpha_n$ are multiplicatively independent by Lemma 14.1. Note that

$$\frac{d}{f_{\mathfrak{p}}\log p}\log 2 < 0.001\,C(n,d,\mathfrak{p})\,h'(\alpha_1)\cdots h'(\alpha_n)\log B.$$

Hence by (14.2) we may assume

$$\frac{B}{\log B} > 0.999 \times 1000 \cdot 8^n(n+1)^{n+2}\frac{d^2}{f_{\mathfrak{p}}\log p}\cdot\frac{p^{f_{\mathfrak{p}}}-1}{q^u} > e^{15.3}$$

(by $n \geqq 2$ and (14.3)). So, by (14.3), we obtain

(14.8) $$B > 5000 \cdot 8^n(n+1)^{n+2}dp^{f_{\mathfrak{p}}}/(f_{\mathfrak{p}}\log p).$$

We may further assume, without loss of generality, that (1.13) is satisfied, since the main inequality in Theorem 1 is symmetric in $\alpha_1, \ldots, \alpha_n$. On appealing to Theorem 7.1 and observing that (14.8) implies

$$(n+1)\log B \geqq \log\max\{\tilde{c}B, \tilde{c}(5n)^{6n}d^{1.2}, \tilde{c}p^{2f_{\mathfrak{p}}}\} \geqq h^* + \log c^*,$$

where h^* is given by (7.2) and

$$\tilde{c} = \left(\frac{3}{4}\log^3(5d)\cdot f_{\mathfrak{p}}\log p\right)^n \cdot n! \geqq c^* \tag{14.9}$$

by (5.21) and (5.24), Theorem 1 follows at once.

15. Proof of Theorem 2

Lemma 15.1. *It suffices to prove Theorem* 2 *on a further assumption that* $\alpha_1, \ldots, \alpha_n$ *are multiplicatively independent.*

Proof. We prove the lemma by induction on n. It is readily verified, by Theorem 1 with $n=1$, that the theorem holds for $n=1$. Suppose $n>1$ and the theorem holds for $n-1$, and $\alpha_1, \ldots, \alpha_n$ are multiplicatively dependent. We may assume that

$$h'(\alpha_1) \leqq \cdots \leqq h'(\alpha_{n-1})\,. \tag{15.1}$$

We define m as in the proof of Lemma 14.1 and divide the proof into two cases.

(i) $m=n$. By Lemma 4.1, there exist $t_0, \ldots, t_n \in \mathbb{Z}$ with $t_n > 0$ such that $t_0 l_0 + \cdots + t_n l_n = 0$ and

$$|t_j| \leqq \big(4nd^2 \log\log(6d)\big)^n (q^u d)^{-1} \max\big(h'(\alpha_1), h'(\alpha_n)\big) \prod_{j=2}^{n-1} h'(\alpha_j) \tag{15.2}$$

for $1 \leqq j \leqq n$. We may assume $(\alpha_1^{b_1} \cdots \alpha_n^{b_n})^{q^u t_n} \neq 1$, since otherwise the theorem holds trivially. Let $b'_j = q^u(t_n b_j - t_j b_n)$ $(1 \leqq j < n)$. Then we have for $1 \leqq j < n$

$$|b'_j| \leqq 2B\big(4nd\log\log(6d)\big)^n d^{n-1} \prod_{j=1}^{n-1} h'(\alpha_j) \cdot \max\left(1, \frac{h'(\alpha_n)}{h'(\alpha_1)}\right) =: B'\,. \tag{15.3}$$

On applying Theorem 1 to $\mathrm{ord}_{\mathfrak{p}}(\alpha_1^{b'_1} \cdots \alpha_{n-1}^{b'_{n-1}} - 1)$ and using (1.12), we obtain

$$\begin{aligned} \mathrm{ord}_{\mathfrak{p}}\, \Xi &\leqq \mathrm{ord}_{\mathfrak{p}}\big((\alpha_1^{b_1} \cdots \alpha_n^{b_n})^{q^u t_n} - 1\big) = \mathrm{ord}_{\mathfrak{p}}(\alpha_1^{b'_1} \cdots \alpha_{n-1}^{b'_{n-1}} - 1) \\ &< nC^*(n-1, d, \mathfrak{p})\, h'(\alpha_1) \cdots h'(\alpha_{n-1}) \log B'\,. \end{aligned} \tag{15.4}$$

It is easily seen that (15.3) yields

$$B' \leqq e^{\tilde{h}/3} \cdot \frac{\delta B/B_n}{d^{n-1} h'(\alpha_1) \cdots h'(\alpha_{n-1})\, f_{\mathfrak{p}} \log p} \cdot \max\left(1, \frac{h'(\alpha_n)}{h'(\alpha_1)}\right). \tag{15.5}$$

Note that by (1.12) and (1.10) we have

$$\frac{C^*(n, d, \mathfrak{p})}{C^*(n-1, d, \mathfrak{p})} > 8(n+1)\frac{d}{f_{\mathfrak{p}} \log p}\,. \tag{15.6}$$

Now the theorem follows from (15.4)–(15.6).

(ii) $m<n$. By Lemma 4.1, there exist $t_0, \dots, t_m \in \mathbb{Z}$ with $t_m \neq 0$ such that $t_0 l_0 + \cdots + t_m l_m = 0$. Let k with $1 \leqq k \leqq m$ satisfy

$$\operatorname{ord}_p t_k = \min_{1 \leqq j \leqq m} \operatorname{ord}_p t_j . \tag{15.7}$$

We may assume that $t_k > 0$ and $(\alpha_1^{b_1} \cdots \alpha_n^{b_n})^{q^u t_k} \neq 1$. Let

$$b_j' = q^u (t_k b_j - t_j b_k) \quad (1 \leqq j \leqq n, j \neq k) \quad \text{with} \quad t_j = 0 \quad (m < j \leqq n) .$$

Then we have, by Lemma 4.1,

$$\max_{1 \leqq j \leqq n, j \neq k} |b_j'| \leqq 2B\big(4md \log\log(6d)\big)^m d^{m-1} h'(\alpha_2) \cdots h'(\alpha_m) =: B' , \tag{15.8}$$

$$|b_n'| \leqq 2B_n\big(4md \log\log(6d)\big)^m d^{m-1} h'(\alpha_2) \cdots h'(\alpha_m) =: B_n' , \tag{15.9}$$

$$\operatorname{ord}_p b_n' = \min_{1 \leqq j \leqq n, j \neq k} \operatorname{ord}_p b_j' \quad \text{(by (15.7) and (1.13))} .$$

Let

$$\Psi' = p^{f_\mathfrak{p}}\big(8(n-1)^3 d \log(5d)\big)^{n-1}, \quad \delta' = \frac{\delta}{dh'(\alpha_k)} ,$$

$$\tilde{h}' = 3 \log\Big((\delta')^{-1} \Psi' \big(d^{n-2} h'(\alpha_1) \cdots h'(\alpha_{k-1}) h'(\alpha_{k+1}) \cdots h'(\alpha_{n-1})\big)^2 B_n'\Big) .$$

Now we can apply the inductive hypothesis and obtain

$$\operatorname{ord}_\mathfrak{p} \Xi \leqq \operatorname{ord}_\mathfrak{p}\big((\alpha_1^{b_1} \cdots \alpha_n^{b_n})^{q^u t_k} - 1\big) = \operatorname{ord}_\mathfrak{p}\Bigg(\prod_{1 \leqq j \leqq n, j \neq k} \alpha_j^{b_j'} - 1\Bigg) \tag{15.10}$$

$$< C^*(n-1, d, \mathfrak{p})\, d^{-(n-1)} \max\Bigg(d^{n-1} \tilde{h}' \prod_{1 \leqq j \leqq n, j \neq k} h'(\alpha_j), \delta' B'/B_n'\Bigg) .$$

From (15.9) we get

$$\tilde{h}' \leqq 2\tilde{h} . \tag{15.11}$$

Now the theorem follows from (15.8)–(15.11) and (15.6). This completes the proof of the lemma.

Proof of Theorem 2. As indicated in the proof of Lemma 15.1, Theorem 2 is true for $n=1$. Thus we may assume that $n \geqq 2$ and $\alpha_1, \dots, \alpha_n$ are multiplicatively independent by Lemma 15.1. We observe first that $\tilde{h} \geqq 3 \log \Psi > 30$, whence

$$e^{\tilde{h}/3} \geqq \tilde{h} . \tag{15.12}$$

Define

$$h_n = \max\left(h'(\alpha_n), \frac{\delta B}{B_n d^n h'(\alpha_1) \cdots h'(\alpha_{n-1}) \tilde{h}}\right) . \tag{15.13}$$

The desired conclusion of the theorem can be rewritten as

$$\operatorname{ord}_{\mathfrak{p}} \Xi < C^*(n,d,\mathfrak{p})\, h'(\alpha_1)\cdots h'(\alpha_{n-1})\, h_n \tilde{h}\,. \tag{15.14}$$

Note that Theorem 7.1 remains true if we replace $h'(\alpha_n)$ in (7.1) and (7.2) by any number $\geqq h'(\alpha_n)$, since we can replace $h'(\alpha_n)$ by this number in § 5, § 8 and § 9. Now we apply Theorem 7.1 with h_n in place of $h'(\alpha_n)$. Note that

$$\frac{f_{\mathfrak{p}}\log p}{2d} \max_{1\leqq j<n}\left(\frac{|b_n|}{h'(\alpha_j)}+\frac{|b_j|}{h_n}\right) \leqq \delta^{-1} d^{n-1} h'(\alpha_1)\cdots h'(\alpha_{n-1})(f_{\mathfrak{p}}\log p)\, B_n \tilde{h}\,.$$

So Theorem 7.1 gives

$$\operatorname{ord}_{\mathfrak{p}} \Xi < C^*(n,d,\mathfrak{p})\, h'(\alpha_1)\cdots h'(\alpha_{n-1})\, h_n(\bar{h}+\log c^*)\,, \tag{15.15}$$

where

$$\bar{h} = \max\{\log\left(\delta^{-1} d^{n-1} h'(\alpha_1)\cdots h'(\alpha_{n-1})(f_{\mathfrak{p}}\log p)\, B_n \tilde{h}\right), \\ \log B^\circ, 6n\log(5n)+1.2\log d, 2f_{\mathfrak{p}}\log p\}\,.$$

Now by direct calculation we see that

$$e^{\tilde{h}} \geqq \max\{\tilde{c}\delta^{-1} d^{n-1} h'(\alpha_1)\cdots h'(\alpha_{n-1})(f_{\mathfrak{p}}\log p)\, B_n \tilde{h}, \tilde{c}B^\circ, \tilde{c}(5n)^{6n} d^{1.2}, \tilde{c}p^{2f_{\mathfrak{p}}}\}\,,$$

where $\tilde{c}$ ($\geqq c^*$) is given by (14.9) (in dealing with the first term on the right, we need (15.12), and for the third term on the right, we use the Stirling's formula). Hence $\tilde{h} \geqq \bar{h} + \log c^*$, and (15.14) follows from (15.15) at once. Further, if $\alpha_1, \ldots, \alpha_n$ satisfy (1.11), then, by Theorem 7.1, we can replace $C^*(n,d,\mathfrak{p})$ by $C^*(n,d,\mathfrak{p})/\omega_q(n)$ in (15.15), whence we can do the same in (15.14); we can also replace Ψ in (1.14) by $\Psi = \max\left(p^{f_{\mathfrak{p}}}, (5n)^{2n} d\right)$, so that $\tilde{h} \geqq \bar{h}$. The proof of the theorem is thus complete.

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