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McNaughton's Theorem

In the logic and automata community, Robert McNaughton is perhaps best known for his determinisation result for automata on infinite words, as mentioned in the obituary. Below, I try to explain the significance of this result.

Automata can also be run on infinite words, i.e. words where positions are indexed by natural numbers. When automata are used as a tool for logics, such as temporal logics used in verification, infinite words are at least as important as finite words.

For infinite words, it is not immediately clear what the acceptance condition should be, since the automaton runs forever. In a seminal paper, J. R. Büchi proposed what is now called the Büchi condition: a run of an automaton is considered accepting if accepting states are visited infinitely often. He proved that nondeterministic automata with this acceptance condition have the same expressive power as monadic second-order logic (MSO). The key lemma in Büchi's proof says that the class of languages recognised by nondeterministic Büchi automata is closed under complementation. To prove this lemma, Büchi did not use determinisation – instead, he showed how to directly convert a nondeterministic automaton into another nondeterministic automaton recognising the complement language. In fact, determinisation fails for Büchi automata, e.g. the language "infinite words over the alphabet $\{a,b\}$ with finitely many occurrences of the letter b" is recognised by a nondeterministic Büchi automaton, but not by any deterministic Büchi automaton.

After Büchi's paper, it was known that, over infinite words, nondeterministic automata (with the Büchi acceptance condition) are equivalent to MSO, and both are equivalent to ω-regular expressions (a natural notion of regular expressions for infinite words). This left open the problem: is there also a deterministic model of automata for infinite words?

This problem was solved by McNaughton in 1966, in a paper called "Testing and Generating Infinite Sequences by a Finite Automaton". This paper shows that every ω-regular expression can be converted into a deterministic automaton, which easily implies that nondeterministic Büchi automata can be converted into deterministic automata. As mentioned before, the Büchi condition is not sufficient for deterministic automata, so McNaughton used deterministic automata with a more general accepting condition, which is currently called the Muller condition. To specify the Muller condition, one gives a family of subsets of states (as opposed to the single set of states used in the Büchi condition). A run is considered accepting if the set of states that it visits infinitely often belongs to the family. A typical Muller condition would be: "either states p and q appear infinitely often but not state r, or states p and r appear infinitely often but not state q". The condition was proposed by D. E. Muller in 1963, in a paper on determinisation which unfortunately contained a flaw.

McNaughton's result was a true breakthrough. Apart from answering a very natural and deep problem, the determinisation result became a crucial element of later fundamental results on automata and logic, like Rabin's theorem MSO on infinite trees, or Büchi and Landweber's solution to Church's synthesis problem. There is now a rich bibliography on determinisation, with many alternative approaches, of which probably the best known is the Safra construction. After almost fifty years, the topic is still a lively research area, and new papers on determinisation appear every year (e.g. at least four papers in the last two years). Nevertheless, even with the simplest modern proofs, determinisation remains difficult, and there are no proofs which are substantially shorter than McNaughton's original 4-page proof.