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FINE HIERARCHY OF REGULAR APERIODIC ω-LANGUAGES

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We develop a theory of regular aperiodic ω-languages in parallel with the theory around the Wagner hierarchy. In particular, we characterize the Wadge degrees of regular aperiodic ω-languages, find an effective version of the Wadge reducibility adequate for this class of languages and prove "aperiodic analogs" of the Büchi-Landweber determinacy theorem and of the Landweber's characterization of regular open and regular G_δ sets.

Keywords: Aperiodic acceptor, aperiodic transducer, regular aperiodic ω-language, Wagner hierarchy, Wadge reducibility.

1. Introduction

This paper is devoted to the theory of infinite behavior of computing devices that is of primary importance for theoretical and practical computer science. More exactly, we consider topological aspects of this theory in the simplest case of finite automata.

A series of papers in this direction culminated with the paper [35] giving in a sense the finest possible topological classification of regular ω-languages (i.e., of the subsets of X^ω for a finite alphabet X recognized by finite automata) known as the Wagner hierarchy. In particular, K. Wagner completely described the (quotient structure of the) preorder $(\mathcal{R}; \leq_{CA})$ formed by the class $\mathcal{R}$ of regular subsets of X^ω and the reducibility by functions continuous in the Cantor topology on X^ω (note that in descriptive set theory the CA-reducibility is widely known as the Wadge reducibility denoted by $\leq_W$ but we follow the notation $\leq_{CA}$ from [35] for the sake of uniformity, because we will use also several versions of this notation).

In [18, 19, 22] the Wagner hierarchy of regular ω-languages was related to the Wadge hierarchy and to the author's fine hierarchy [20, 21]. This provided new

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proofs of results in [35] and yielded some new results on the Wagner hierarchy. See also an alternative approach [3, 4, 6].

Later some results from [22, 35] were extended to more complicated computing devices. In particular, the Wadge degrees of deterministic context-free ω-languages, of ω-languages recognized by deterministic Turing machines, and of infinite tree languages recognized by deterministic tree automata were determined in [5, 14, 24]. Note that in all these three cases some important properties of the Wagner hierarchy are either false or still open.

In this paper, we develop a complete analog of the theory from [22, 35] for the class $\mathcal{A}$ of regular aperiodic ω-languages. The class $\mathcal{A}$ is certainly the most important subclass of $\mathcal{R}$ which has several remarkable characterizations and plays a noticeable role in the field of specification and verification of finite-state systems. To explain our results, let us recall some results on the Wagner hierarchy in more details. In [35] the following results (among others) were established:

(1) The structure $(\mathcal{R}; \leq_{CA})$ is almost well-ordered with the order type ω^ω, i.e. there are $A_\alpha \in \mathcal{R}$, $\alpha < \omega^\omega$, such that $A_\alpha <_{CA} A_\alpha \oplus \overline{A}_\alpha <_{CA} A_\beta$ for $\alpha < \beta < \omega^\omega$ and any regular set is CA-equivalent to one of the sets $A_\alpha, \overline{A}_\alpha, A_\alpha \oplus \overline{A}_\alpha (\alpha < \omega^\omega)$.

(2) The CA-reducibility coincides on $\mathcal{R}$ with the DA-reducibility, i.e. the reducibility by functions computed by deterministic asynchronous finite transducers, and $\mathcal{R}$ is closed under the DA-reducibility.

(3) Any level $\mathcal{R}_\alpha = \{C \mid C \leq_{DA} A_\alpha\}$ of the Wagner hierarchy is decidable (even in polynomial time, as shown in [10, 37]).

In [22] the following additional facts (among others) about the Wagner hierarchy were established:

(4) Any class $\mathcal{R}_\alpha$ has a natural set-theoretic description in terms of the classes $\mathcal{L}_0$ of regular open sets and $\mathcal{L}_1$ of regular F_σ sets. In particular, there is a Boolean term $t(x_1, \ldots, x_n, y_1, \ldots, y_n)$ with $\mathcal{R}_\alpha = t(\mathcal{L}_0, \mathcal{L}_1)$ where $t(\mathcal{L}_0, \mathcal{L}_1)$ is the set of values of t when $x_1, \ldots, x_n$ range over $\mathcal{L}_0$ and $y_1, \ldots, y_n$ range over $\mathcal{L}_1$.

(5) For every term t as above, the set $t(\mathcal{L}_0, \mathcal{L}_1)$ coincides with one of the classes $\mathcal{R}_\alpha$ or their duals.

(6) If a regular set R is represented as $R = t(B_1, \ldots, B_n, C_1, \ldots, C_n)$ for some term t as above, open sets $B_1, \ldots, B_n$ and F_σ-sets $C_1, \ldots, C_n$, then there exist $B'_1, \ldots, B'_n \in \mathcal{L}_0$ and $C'_1, \ldots, C'_n \in \mathcal{L}_1$ with $R = t(B'_1, \ldots, B'_n, C'_1, \ldots, C'_n)$.

In this paper we will show that the sets A_α from 1) may be chosen from $\mathcal{A}$, and thus the Wadge degrees (as well as the DA-degrees) of sets in $\mathcal{A}$ are the same as those of sets in $\mathcal{R}$. We will find a reducibility $\leq_{AA}$ related to the class $\mathcal{A}$ in exactly the same way as the DA-reducibility is related to $\mathcal{R}$. Thus, we obtain the analogs of 1)—3). We also show that the levels $\mathcal{A}_\alpha = \{C \mid C \leq_{AA} A_\alpha\}$ of the fine hierarchy of regular aperiodic languages have the properties 4)—6), if we take the classes $\mathcal{K}_0$ of regular aperiodic open sets and $\mathcal{K}_1$ of regular aperiodic F_σ-sets in place of the classes $\mathcal{L}_0$ and $\mathcal{L}_1$, respectively. We obtain also some facts of independent interest, e.g. "aperiodic analogs" of the Büchi-Landweber determinacy theorem and of the

Landweber's characterization of regular open and regular G_δ sets (cf. the algebraic approach in [2]). Note that these results in fact strengthen the corresponding results about regular sets, hence this paper subsumes many results from [22, 35].

The rest of the paper is organized as follows. In Section 2 we collect notation and known facts we will rely upon. In Sections 3 and 4 we prove some necessary facts about the so called aperiodic automata (known also as the counter-free automata). In Section 5 we prove the "aperiodic analog" of the Büchi-Landweber determinacy theorem. In Section 6 we establish some facts about the regular aperiodic sets in the Borel hierarchy. Section 7 deals with the fine hierarchy of regular aperiodic sets, while Section 8 — with the reducibilities on such sets. We conclude in Section 9 with mentioning a couple of open questions.

2. Notation and Reminder

We use standard set-theoretic notation. For a set S, $P(S)$ is the class of subsets of S. For a class $\mathcal{C} \subseteq P(S)$, $\check{\mathcal{C}}$ is the dual class $\{\overline{C} \mid C \in \mathcal{C}\}$ and $BC(\mathcal{C})$ is the Boolean closure of $\mathcal{C}$.

We assume the reader to be acquainted with the notion of ordinal and with the main arithmetic operations on ordinals (see e.g. [9]). Actually, we consider only ordinals less than ω^ω which is the supremum of the ordinals ω^m, $m < \omega$, where $\omega = \{0, 1, \dots\}$ is the first infinite ordinal. Recall that ω^ω is the order type of the set of finite sequences $(k_0, \dots, k_n)$ of natural numbers $k_0 \geq \cdots \geq k_n$, ordered lexicographically. Every non-zero ordinal $\alpha < \omega^\omega$ is uniquely representable in the form $\alpha = \omega^{k_0} + \cdots + \omega^{k_n}$ with $\omega > k_0 \geq \ldots \geq k_n$.

Fix a finite alphabet X containing more than one symbol (for simplicity we may assume that $X = \{x \mid x < k\}$ for a natural number $k > 1$, so $0, 1 \in X$). Let X^* and X^ω denote respectively the sets of all words and of all ω-words (i.e. sequences $\xi : \omega \to X$) over X. The empty word is denoted by ε. Let $X^+ = X^* \setminus \{\varepsilon\}$ and $X^{\leq\omega} = X^* \cup X^\omega$. For $n < \omega$, let X^n be the set of words of length n. The sets of words $X^{\leq n}$ and $X^{>n}$ are defined in the same way. For $X = \{0, 1\}$ we write 2^* in place of X^*, 2^ω in place of X^ω and so on.

We use some almost standard notation concerning words and ω-words, so we are not too casual in reminding it here. For $w \in X^*$ and $\xi \in X^{\leq\omega}$, $w \sqsubseteq \xi$ means that w is a substring of ξ, $w \cdot \xi = w\xi$ denote the concatenation, $l = |w|$ is the length of $w = w(0) \cdots w(l-1) \in X^l$. For $w \in X^*, W \subseteq X^*$ and $A \subseteq X^{\leq\omega}$, let $w \cdot A = \{w\xi \mid \xi \in A\}$ and $W \cdot A = \{w\xi \mid w \in W, \xi \in A\}$. For $k, l < \omega$ and $\xi \in X^{\leq\omega}$, let $\xi[k, l) = \xi(k) \cdots \xi(l-1)$ and $\xi[k] = \xi[0, k)$. For $u \in X^*$ and $n < \omega$, u^n denote the concatenation of n copies of the word u. Our notation does not distinguish a word of length 1 and the corresponding letter.

Note that usually we work with the fixed alphabet X but sometimes we are forced to consider several alphabets simultaneously; in this case we denote the alphabets also by Y, Z, possibly with indices, and include the alphabets in the corresponding notation. The "fixed-alphabet mode" is the default one.

The set X^ω carries the Cantor topology with the open sets $W \cdot X^\omega$ where $W \subseteq X^*$. The continuous functions in this topology are called also *CA-functions*. A

CS-function is a function $f : X^\omega \to Y^\omega$ satisfying $f(\xi)(n) = \phi(\xi[n+1])$ for some $\phi : X^\omega \to X$. A *delayed CS-function* is a function $f : X^\omega \to Y^\omega$ satisfying $f(\xi)(n) = \phi(\xi[n])$ for some $\phi : X^\omega \to X$. Every delayed CS-function is a CS-function, and every CS-function is a CA-function. In descriptive set theory the CS-functions are known as Lipschitz functions. All three classes of functions are closed under composition.

For arbitrary alphabets $X_0, \ldots, X_{n-1}$, $n > 0$, there is a natural homeomorphism between the spaces $X_0^\omega \times \cdots \times X_{n-1}^\omega$ and $(X_0 \times \cdots \times X_{n-1})^\omega$ sending $(\xi_0, \ldots, \xi_{n-1})$ to the ω-word $\xi_0 \times \cdots \times \xi_{n-1}$ defined by

$$(\xi_0 \times \cdots \times \xi_{n-1})(i) = (\xi_0(i), \ldots, \xi_{n-1}(i)),\ i < \omega.$$

This reduces the consideration of continuous functions with many variables to the consideration of those with one variable. There is also a natural continuous function

$$f : X_0^\omega \times \cdots \times X_{n-1}^\omega \to (X_0 \cup \cdots \cup X_{n-1})^\omega$$

defined by

$$f(\xi_0, \ldots, \xi_{n-1})(i) = \xi_0(i) \cdots \xi_{n-1}(i).$$

In the particular case when $X_0, \ldots, X_{n-1}$ are one and the same alphabet X the function f is a homeomorphism between the n-th cartesian power of X^ω and X^ω. This gives rise to a bijective coding of the n-tuples $(\xi_0, \ldots, \xi_{n-1})$ of elements of X^ω by the elements

$$\langle \xi_0, \ldots, \xi_{n-1} \rangle = f(\xi_0, \ldots, \xi_{n-1})$$

of X^ω defined by

$$\langle \xi_0, \ldots, \xi_{n-1} \rangle (n \cdot i + r) = \xi_r(i),\ \ r < n.$$

The corresponding "projections" $p_r(\langle \xi_0, \ldots, \xi_{n-1} \rangle) = \xi_r$, $r < n$, are CA-functions on X^ω.

Let $\mathcal{B}$ denote the class of Borel subsets of X^ω, i.e. the smallest class containing the open sets and closed under the operations of complement and countable union. The Borel sets are organized in a hierarchy the lowest levels of which are as follows: G and F are the classes of open and closed sets, respectively; G_δ (F_σ) is the class of countable intersections (unions) of open (respectively, closed) sets; $G_{\delta\sigma}$ ($F_{\sigma\delta}$) is the class of countable unions (intersections) of G_δ- (respectively, of F_σ-) sets, and so on. In the modern notation of hierarchy theory,

$$\mathbf{\Sigma}^0_1 = G,\ \mathbf{\Sigma}^0_2 = F_\sigma,\ \mathbf{\Sigma}^0_3 = G_{\delta\sigma},\ \mathbf{\Sigma}^0_4 = F_{\sigma\delta\sigma}$$

and so on, $\mathbf{\Pi}^0_n$ is the dual class for $\mathbf{\Sigma}^0_n$, and $\mathbf{\Delta}^0_n = \mathbf{\Sigma}^0_n \cap \mathbf{\Pi}^0_n$. The sequence $\{\mathbf{\Sigma}^0_{n+1}\}_{n<\omega}$ is known as *the finite Borel hierarchy.* It may be in a natural way extended to the countable ordinals. The resulting sequence called *the Borel hierarchy* exhausts the class $\mathcal{B}$. For any $n > 0$, the class $\mathbf{\Sigma}^0_n$ contains $\emptyset, X^\omega$ and is closed under countable unions and finite intersections, while the class $\mathbf{\Delta}^0_n$ is closed under complementation and finite unions. For any $n > 0$, $\mathbf{\Sigma}^0_n \cup \mathbf{\Pi}^0_n \subseteq \mathbf{\Delta}^0_{n+1}$, and $\mathbf{\Sigma}^0_n \not\subseteq \mathbf{\Pi}^0_n$.

By an *automaton* (over X) we mean a triple $\mathcal{M} = (Q, X, f)$ consisting of a finite non-empty set Q of states, an input alphabet X and a transition function $f : Q \times X \to Q$. The transition function is naturally extended to the function $f : Q \times X^* \to Q$ defined by induction $f(q, \varepsilon) = q$ and $f(q, u \cdot x) = f(f(q, u), x)$, where $u \in X^*$ and $x \in X$. Similarly, we may define the function $f : Q \times X^\omega \to Q^\omega$ by $f(q, \xi)(n) = f(q, \xi[n])$. Note that in this paper we consider only deterministic finite automata. A *partial automaton* is defined in the same way, only now $f : Q \times X \rightharpoonup Q$ is a partial function.

Automata equipped with appropriate additional structures are used as acceptors (devices accepting words or ω-words) and transducers (devices computing functions on words or ω-words). A *word acceptor* is a triple $(\mathcal{M}, i, F)$ consisting of an automaton $\mathcal{M}$, an initial state i of $\mathcal{M}$ and a set of final states $F \subseteq Q$. Such an acceptor recognizes the language

$$L(\mathcal{M}, i, F) = \{u \in X^* \mid f(i, u) \in F\}.$$

The languages recognized by such acceptors are called *regular*. The languages recognized by partial automata are defined in the same way and coincide with the regular languages.

For the case of ω-words, there are several notions of acceptors of which we will use only three. A *Büchi acceptor* has the form $(\mathcal{M}, i, F)$ as above and recognizes the set

$$L_\omega(\mathcal{M}, i, F) = \{\xi \in X^* \mid In(f(i, \xi)) \cap F \neq \emptyset\}$$

where $In(f(i, \xi))$ is the set of states which occur infinitely often in the sequence $f(i, \xi) \in Q^\omega$. A *Muller acceptor* has the form $(\mathcal{M}, i, \mathcal{F})$, where $\mathcal{M}, i$ are as above and $\mathcal{F} \subseteq P(Q)$; it recognizes the set

$$L_\omega(\mathcal{M}, i, \mathcal{F}) = \{\xi \in X^\omega \mid In(f(i, \xi)) \in \mathcal{F}\}.$$

A *Mostowski acceptor* (known also as the Rabin chain acceptor or parity acceptor) has the form $(\mathcal{M}, i, \Omega)$, where $\mathcal{M}, i$ are as above and $\Omega = (E_1, F_1, \ldots, E_n, F_n)$ for some $E_1 \subseteq F_1 \subseteq \cdots \subseteq E_n \subseteq F_n \subseteq Q$; it recognizes the set

$$L_\omega(\mathcal{M}, i, \Omega) = \{\xi \in X^\omega \mid \exists k(In(f(i, \xi)) \cap E_k = \emptyset \wedge In(f(i, \xi)) \cap F_k \neq \emptyset)\}.$$

It is well known that the Muller and Mostowski acceptors recognize the same ω-languages; these are called *regular ω-languages* or just regular sets. The class $\mathcal{R}$ of all regular ω-languages is a proper subclass of $BC(\mathbf{\Sigma}^0_2)$ that in turn is a proper subclass of $\mathbf{\Delta}^0_3$. The Büchi acceptors recognize a smaller class of sets, namely exactly the regular $\mathbf{\Pi}^0_2$-sets.

A *synchronous transducer* (over alphabets X, Y) is a tuple $\mathcal{T} = (Q, X, Y, f, g, i)$, also written as $\mathcal{T} = (\mathcal{M}, Y, g, i)$, consisting of an automaton $\mathcal{M}$ as above, an initial state i and an output function $g : Q \times X \to Y$. The output function is extended to the function $g : Q \times X^* \to Y^*$ defined by induction

$$g(q, \varepsilon) = \varepsilon, \quad g(q, u \cdot x) = g(q, u) \cdot g(f(q, u), x),$$

and to the function $g : Q \times X^\omega \to Y^\omega$ defined by

$$g(q, \xi) = g(q, \xi(0)) \cdot g(f(q, \xi(0)), \xi(1)) \cdot g(f(q, \xi[0, 2)), \xi(2)) \cdots . \tag{1}$$

In other notation, $g(q, \xi) = lim_n g(q, \xi[n])$. The transducer $\mathcal{T}$ *computes* the function $g_\mathcal{T} : X^\omega \to Y^\omega$ defined by $g_\mathcal{T}(\xi) = g(i, \xi)$. If the output function is of the form $g : Q \to Y$ (i.e., does not really depend on the second argument), then $\mathcal{T}$ is called a *delayed synchronous transducer.*

An *asynchronous transducer* (over alphabets X, Y) is defined as a synchronous transducer with only one exception: this time the output function g maps $Q \times X$ into Y^*. As a result, the value $g(q, \xi)$ defined as in (1) is now in $Y^{\leq\omega}$, and the function $g_\mathcal{T}$ maps X^ω into $Y^{\leq\omega}$. Nevertheless, we usually consider the case when $g_\mathcal{T}$ maps X^ω into Y^ω; this condition is easily characterized in terms of $\mathcal{T}$.

The functions computed by synchronous (delayed synchronous, asynchronous) transducers are called DS-functions (respectively delayed DS-functions and DA-functions). As is well known, all tree classes of functions are closed under composition, and every DS-function (delayed DS-function, DA-function) is a CS-function (delayed CS-function, CA-function).

We will study several reducibilities on subsets of X^ω. For $A, B \subseteq X^\omega$, A is said to be *CA-reducible* to B (in symbols $A \leq_{CA} B$), if $A = g^{-1}(B)$ for some CA-function $g : X^\omega \to X^\omega$. The relations $\leq_{DA}$, $\leq_{CS}$ and $\leq_{DS}$ on $P(X^\omega)$ are defined in the same way but using the other three classes of functions. The introduced relations on $P(X^\omega)$ are preorders. The CA-reducibility is widely known as Wadge reducibility, and the CS-reducibility as Lipschitz reducibility. The other two reducibilities are effective "automatic" versions of these. By $\equiv_{CA}$ we denote the induced equivalence relation which gives rise to the corresponding quotient partial order. Following a well established jargon, we call this partial order the structure of CA-degrees. The same applies to the other reducibilities (and to the reducibilities to be introduced later). In the "alphabet-dependent mode", we say that $A \subseteq X^\omega$ is CA-reducible to $B \subseteq Y^\omega$, if $A = g^{-1}(B)$ for some CA-function $g : X^\omega \to Y^\omega$. Sometimes such variations are also of use.

The operation

$$A \oplus B = \{0 \cdot \alpha, i \cdot \beta \mid 0 < i < k, \alpha \in A, \beta \in B\}$$

on subsets of X^ω, $X = \{0, \ldots, k-1\}$, induces the operation of least upper bound in the structures of degrees under all four reducibilities introduced above. Any level of the Borel hierarchy is closed under the CA-reducibility (and thus under all four reducibilities) in the sense that every set reducible to a set in the level is itself in that level. The class $\mathcal{R}$ is closed under the DA- and DS-reducibilities but is not closed under the CA- and CS-reducibilities. Every $\mathbf{\Sigma}$-level $\mathcal{C}$ (and also every $\mathbf{\Pi}$-level) of the Borel hierarchy has a CA-complete set C which means that $\mathcal{C} = \{A \mid A \leq_{CA} C\}$.

More detailed information related to the introduced notions may be found in many sources including [16, 27, 31, 33, 34].

3. Aperiodic Automata and Acceptors

Here we collect some facts on the regular aperiodic sets, the main object of this paper, and on the closely related aperiodic automata (known also as the counter-free automata).

The aperiodic languages were characterized in several ways, in particular as: the languages of words defined by generalized regular star-free expressions; the languages of words defined by first-order sentences of a natural signature; the languages of words satisfying a formula of linear time temporal logic; the languages recognized by aperiodic acceptors. Similar characterizations exist also for the regular aperiodic ω-languages (see e.g. [29, 31, 33] and references therein). It is well-known (and follows from the mentioned characterizations) that the classes of regular aperiodic languages and ω-languages are closed under the Boolean operations.

Let us illustrate the logical characterization by proving an easy fact which will be used later. First we recall some relevant terminology. Relate to any alphabet X the signature $\sigma_X = \{<, Q_x \mid x \in X\}$ where $<$ is a binary predicate symbol and Q_x are unary predicate symbols. Relate to any $\xi \in X^\omega$ the structure $\tilde{\xi} = (\omega; \sigma_X)$ of signature σ_X where $<$ is the usual order on ω and $Q_x(i) \leftrightarrow \xi(i) = x$ for all $i < \omega$ and $x \in X$. The logical characterization [11, 13, 15, 29, 30] states that a set $A \subseteq X^\omega$ is regular aperiodic iff $A = L_\phi$ for some first-order sentence ϕ of σ_X, where $L_\phi = \{\xi \in X^\omega \mid \tilde{\xi} \models \phi\}$.

Proposition 1 *Let X, Y be disjoint alphabets and $A \subseteq (X \times Y)^\omega$ be regular aperiodic. Then the language $A' = \{\langle \xi, \eta \rangle \mid \xi \in X^\omega, \eta \in Y^\omega, \xi \times \eta \in A\}$ is a regular aperiodic subset of $(X \cup Y)^\omega$. The same applies to any finite sequence of pairwise disjoint alphabets.*

Proof. Let θ be a sentence of $\sigma_{X \times Y}$ satisfying $A = L_\theta$. It suffices to find a sentence $\tilde{\theta}$ of $\sigma_{X \cup Y}$ satisfying $A' = L_{\tilde{\theta}}$. Since we use an induction on formulas, we are forced to employ also formulas with free variables, i.e. we want to relate to any formula $\phi(v_1, \dots v_n)$ of $\sigma_{X \times Y}$ with free variables among $v_1, \dots v_n$ a formula $\tilde{\phi}(v_1, \dots v_n)$ of $\sigma_{X \cup Y}$ such that

$$\widetilde{\xi \times \eta} \models \phi(i_1, \dots i_n) \text{ iff } \widetilde{\langle \xi, \eta \rangle} \models \tilde{\phi}(2i_1, \dots 2i_n) \tag{2}$$

for all $i_1, \dots i_n < \omega$.

Note that in defining the formula $\tilde{\phi}$ below we use not only the symbols from $\sigma_{X \cup Y}$ but also the symbols 0 and $+1$ with the usual interpretation; this is possible since these new symbols are easily definable in terms of the symbols in $\sigma_{X \cup Y}$. Relate first to any formula $\phi(v_1, \dots v_n)$ of $\sigma_{X \times Y}$ the formula $\phi'(v_1, \dots v_n)$ of $\sigma_{X \cup Y}$ by induction as follows:

if ϕ is $v_l < v_k$ or $v_l = v_k$ then ϕ' is ϕ;
if ϕ is $Q_{(x,y)}(v_l)$ then ϕ' is $Q_x(v_l) \wedge Q_y(v_l + 1)$;
if ϕ is $\phi_1 \wedge \phi_2$ then ϕ' is $\phi'_1 \wedge \phi'_2$ and similarly for the other Boolean connectives;
if ϕ is $\exists v \phi_1$ then ϕ' is $\exists v(Q_X(v) \wedge \phi'_1)$, where $Q_X(v)$ is $\bigvee_{x \in X} Q_x(v)$.

We did not consider $\forall$ since it may be expressed through $\exists$ and $\neg$. Finally, set

$$\tilde{\phi} = Q_X(0) \wedge \forall v(Q_X(v) \leftrightarrow \neg Q_X(v+1)) \wedge \phi'.$$

By induction on formulas it is easy to check the property (2). □

For our paper, the characterization in terms of aperiodic acceptors is the most relevant. Let us recall the corresponding definition from [13].

Definition 1 *An automaton $\mathcal{M} = (Q, X, f)$ is aperiodic if for all $q \in Q$, $u \in X^+$ and $n > 0$ the equality $f(q, u^n) = q$ implies $f(q, u) = q$. This is clearly equivalent to say that for all $q \in Q$ and $u \in X^+$ there is $m < \omega$ with $f(q, u^{m+1}) = f(q, u^m)$. An acceptor (or a transducer) is aperiodic if so is the corresponding automaton.*

The next basic fact is proved in [13] for the languages of finite words and follows from [13, 36] for the ω-languages. The reader not familiar with the aperiodic languages could take this fact as their definition.

Proposition 2 *(i) A regular language $A \subseteq X^*$ is aperiodic iff it is recognized by an aperiodic acceptor.*

(ii) A regular ω-language $A \subseteq X^\omega$ is aperiodic iff it is recognized by an aperiodic Muller acceptor.

We call a partial automaton $\mathcal{M} = (Q, X, f)$ *aperiodic* if for all $q \in Q$ and $u \in X^+$ there is $m < \omega$ such that either $f(q, u^{m+1})$ is undefined or $f(q, u^{m+1}) = f(q, u^m)$.

Proposition 3 *A regular language is aperiodic iff it is recognized by a partial aperiodic acceptor.*

Proof. One direction is trivial. For the other direction, let $\mathcal{M} = (Q, X, f, i, F)$ be a partial aperiodic acceptor; it suffices to find an aperiodic acceptor $\mathcal{M}'$ recognizing the same language. This is obtained by taking a new sink state s and by extending f to the total transition function f' satisfying $f'(q, x) = s$ whenever $f(q, x)$ was undefined. □

Next we show that the class of aperiodic automata is closed under some operations. First we consider the well known product operation. Relate to any automata $\mathcal{M}_k = (Q_k, X_k, f_k)$, $k < 2$, the automaton $\mathcal{M} = (Q, X, f)$ called the *product* of $\mathcal{M}_0$ and $\mathcal{M}_1$ and defined by

$$Q = Q_0 \times Q_1,\ X = X_0 \times X_1,\ f((q_0, q_1), (x_0, x_1)) = (f_0(q_0, x_0), f_1(q_1, x_1)).$$

For the particular case $X_0 = X_1 = X$ we can define a version of the product automaton over the alphabet X by $f((q_0, q_1), x)) = (f_0(q_0, x), f_1(q_1, x))$. Of course, both versions of the product may be defined in the same way for any finite number k of automata, $2 < k < \omega$.

Proposition 4 *The class of aperiodic automata is closed under both kinds of product.*

Proof. Both assertions are proved similarly, so we consider only the second kind of product. For simplicity of notation we consider only the case $k = 2$. Let $q = (q_0, q_1) \in Q$ and $u \in X^+$. Since $\mathcal{M}_k$ is aperiodic, there are numbers m_k, $k < 2$, with $f_k(q_k, u^{m_k+1}) = f_k(q_k, u^{m_k})$. For the number $m = max(m_0, m_1)$ we then have

$$f_k(q_k, u^{m+1}) = f_k(q_k, u^{m_k}) = f_k(q_k, u^m),\ k < 2,$$

hence

$$f(q,u^{m+1}) = (f_0(q_0,u^{m+1}), f_1(q_1,u^{m+1}) = (f_0(q_0,u^m), f_1(q_1,u^m) = f(q,u^m).$$

□

Next we define an exotic operation (called the shrink operation for reference) which will be used only in Section 6 for establishing a useful property known as the reduction (or shrinking) property. Relate to any automaton $\mathcal{M} = (Q,X,f)$ and sets $T_0,T_1 \subseteq Q$ the automaton $\mathcal{M}' = (Q',X,f')$ defined by

$$Q' = (\{0,1\} \times (T_0 \cup T_1)) \cup (\{2\} \times \overline{(T_0 \cup T_1)})$$

and

$$f'((k,q),x) = \begin{cases} (2,f(q,x)) & \text{if } \ f(q,x) \notin T_0 \cup T_1 \\ (0,f(q,x)) & \text{if } \ f(q,x) \in T_0 \setminus T_1 \\ (1,f(q,x)) & \text{if } \ f(q,x) \in T_1 \setminus T_0 \\ (k,f(q,x)) & \text{if } \ f(q,x) \in T_0 \cap T_1 \wedge \ k < 2 \\ (0,f(q,x)) & \text{if } \ f(q,x) \in T_0 \cap T_1 \wedge \ k = 2. \end{cases}$$

Proposition 5 *The class of aperiodic automata is closed under the shrink operation.*

Proof. Let $(k,q) \in Q'$ and $u \in X^+$. Since $\mathcal{M}$ is aperiodic, $f(q,u^{m+1}) = f(q,u^m)$ for some m. W.l.o.g. we may assume that $m = 0$, i.e. $f(q,u) = q$ (otherwise, take $f(q,u^m)$ in place of q). Now consider the following four cases.

Case 1. $q \notin T_0 \cup T_1$.
By definition of f', $f'((j,q),u) = (2,q)$ for every $j < 3$. Therefore, $f'((k,q),u^2) = (2,q) = f'((k,q),u)$, as desired.

Case 2. $q \in T_0 \setminus T_1$.
By definition of f', $f'((j,q),u) = (0,q)$ for every $j < 3$. Therefore, $f'((k,q),u^2) = (0,q) = f'((k,q),u)$, as desired.

Case 3. $q \in T_1 \setminus T_0$.
This case is similar to the previous one.

Case 4. $q \in T_0 \cap T_1$.
Let $n < 3$ be the unique number satisfying $f'((k,q),u) = (n,q)$. Since $q \in T_0 \cap T_1$, by definition of f' we actually have $n < 2$. It suffices to show that $f'((n,q),u) = (n,q)$. Let $u = x_0 \cdots x_l$ where $x_0,\ldots,x_l \in X$. For any $i \leq l$, let $q_i = f(q,x_0 \cdots x_{i-1})$ and $k_i,n_i < 3$ be the unique numbers satisfying

$$(k_i,q_i) = f'((k,q),x_0 \cdots x_{i-1}) \text{ and } (n_i,q_i) = f'((n,q),x_0 \cdots x_{i-1}).$$

Then

$$f'((n,q),u) = f'((n_l,q_l),x_l) \text{ and } f'((k,q),u) = f'((k_l,q_l),x_l) = (n,q) = (n_0,q_0),$$

hence it suffices to show that $f'((n_l,q_l),x_l) = (n_0,q_0)$. If $q_1,\ldots,q_l \in T_0 \cap T_1$ then, by definition of f', $n_0 = \cdots = n_l < 2$, therefore $f'((n_l,q_l),x_l) = (n_0,q_0)$.

Now assume that $q_i \notin T_0 \cap T_1$ for some $i \in \{1, \dots, l\}$. By definition of f', k_i and n_i depend only on q_i (and does not depend on k_{i-1} or n_{i-1}), hence $k_i = n_i$. Therefore,

$$(k_j, q_j) = f'((k_i, q_i), x_i \cdots x_{j-1}) = f'((n_i, q_i), x_i \cdots x_{j-1}) = (n_j, q_j)$$

for all $j \in \{i, \dots, l\}$. Thus, $f'((n_l, q_l), x_l) = f'((k_l, q_l), x_l) = (n_0, q_0)$. □

Next we consider an operation (called the LAR operation for reference) used earlier by several authors (see e.g. remarks in [33], Theorem 6.5). Relate to any automaton $\mathcal{M} = (Q, X, f)$ the automaton $\mathcal{M}' = (Q', X, f')$ defined as follows. W.l.o.g. we may assume that $Q = \{1, \dots, n\}$. Let $S(Q)$ be the set of permutations of Q; the permutations will be written as "vectors" $v = (v(1), \dots, v(n))$. Let $Q' = Q \times S(Q)$ and for $(h, v) \in Q'$ and $x \in X$ let $f'((h, v), x) = (k, w)$ be the unique element of Q' satisfying

$$f(v(n), x) = v(k) \text{ and } w = (v(1), \dots, v(k-1), v(k+1), \dots, v(n), v(k)).$$

Proposition 6 *The class of aperiodic automata is closed under the LAR operation.*

Proof. Let $(h, v) \in Q'$ and $u \in X^+$. Since $\mathcal{M}$ is aperiodic, $f(v(n), u^{m+1}) = f(v(n), u^m)$ for some m. As in the previous proof, w.l.o.g. we may assume that $m = 0$, i.e. $f(v(n), u) = v(n)$. It suffices to show that $f'((h, v), u^2) = f'((h, v), u)$. Let $u = x_0 \cdots x_l$ where $x_0, \dots, x_l \in X$. For any $i \leq l + 1$, let (h_i, v_i) and (k_i, w_i) be the unique elements of Q' satisfying

$$(h_i, v_i) = f'((h, v), x_0 \cdots x_{i-1}) \text{ and } (k_i, w_i) = f'((h_{l+1}, v_{l+1}), x_0 \cdots x_{i-1}).$$

Since $(k_0, w_0) = f'(((h, v), u)$ and $(k_{l+1}, w_{l+1}) = f'((k_0, w_0), u)$, it remains to show that $(k_{l+1}, w_{l+1}) = (k_0, w_0)$.

By the definition of f',

$$w_0(n) = f(v_0(n), u) = f(v(n), u) = v(n),$$

$$w_{l+1}(n) = f(w_0(n), u) = f(v(n), u) = v(n)$$

and

$$f(v_i(n), x_i) = v_{i+1}(n), \ f(w_i(n), x_i) = w_{i+1}(n)$$

for all $i \leq l$. Thus, $w_{l+1}(n) = w_0(n)$ and

$$(v_0(n), \dots, v_l(n)) = (w_0(n), \dots, w_l(n)).$$

Omitting repetitions in the last sequence we obtain a sequence $(s_0, \dots, s_{m-1})$, where m is the number of elements in

$$\{v_0(n), \dots, v_l(n)\} = \{w_0(n), \dots, w_l(n)\}, \ 0 < m \leq n.$$

By the definition of f', the last m elements of the vectors v_l and w_l are exactly $s_0, \dots, s_{m-1}$, in this order. Since the first $n - m$ elements of the vectors v_l and w_l are not influenced by the function f' (when computing $w_0, \dots, w_l$) at all, we actually have $v_l = w_l$. Again by the definition of f', k_0 and k_{l+1} are positions of the same element $w_0(n) = w_{l+1}(n)$ in the same vector $v_l = w_l$, thus

$$(k_0, w_0) = f'((h_l, v_l), x_l) = f'((k_l, w_l), x_l) = (k_{l+1}, w_{l+1}).$$

□

We conclude this section by the "aperiodic analog" of a well known fact of automata theory.

Proposition 7 *A regular omega-language is aperiodic iff it is recognized by an aperiodic Mostowski acceptor.*

Proof. In the easy direction, we have to relate to any Mostowski aperiodic acceptor $(\mathcal{M}, i, \Omega)$, $\Omega = (E_1, F_1, \ldots, E_n, F_n)$, an aperiodic Muller acceptor recognizing the same language. It is obvious that the acceptor $(\mathcal{M}, i, \mathcal{F})$ where

$$\mathcal{F} = \{S \subseteq Q \mid \exists k \leq n(S \cap E_k = \emptyset \wedge S \cap F_k \neq \emptyset)\},$$

has the desired properties.

In the hard direction, we have to relate to any Muller aperiodic acceptor $(\mathcal{M}, i, \mathcal{F})$ an aperiodic Mostowski acceptor recognizing the same language A. Assuming w.l.o.g. that $Q = \{1, \ldots, n\}$ and $i = 1$, let $\mathcal{M}'$ be the automaton obtained from $\mathcal{M}$ by the LAR operation above, $i' = (1, (2, \ldots, n, 1))$ and $\Omega = (E_1, F_1, \ldots, E_n, F_n)$ where

$$E_1 = \{(n, v) \mid (\{v(n)\} \notin \mathcal{F})\},\ F_1 = E_1 \cup \{(n, v) \mid (\{v(n)\} \in \mathcal{F})\}$$

and, for every $i \in \{1, \ldots, n-1\}$,

$$E_{i+1} = F_i \cup \{(n-i+1, v) \mid (\{v(i+1), \ldots, v(n)\} \notin \mathcal{F})\}$$

and

$$F_{i+1} = E_{i+1} \cup \{(n-i+1, v) \mid (\{v(i+1), \ldots, v(n)\} \in \mathcal{F})\}.$$

By Proposition 6, $\mathcal{M}'$ is aperiodic. By the argument in the proof of Theorem 6.5 in [33] (that theorem is about the tree automata but the argument works also for the word automata), $(\mathcal{M}', i', \Omega)$ recognizes the language A. □

4. Aperiodic Transducers

Here we establish some results on the functions $g_{\mathcal{T}}$ computed by aperiodic transducers $\mathcal{T}$ (see Section 2). Though we are mostly interested in the functions from X^ω to Y^ω, sometimes we are forced to consider the more general case of functions from X^ω to $Y^{\leq\omega}$.

Definition 2 *A function $h : X^\omega \to Y^{\leq\omega}$ is called an AA-function (an AS-function) if it is computed by an aperiodic asynchronous (respectively, aperiodic synchronous) transducer $\mathcal{T}$ over X, Y, i.e. $h = g_{\mathcal{T}}$.*

We start with a couple of examples of AA-functions.

Proposition 8 *Let $Y_0, \ldots, Y_n$ be pairwise disjoint alphabets, $Y = Y_0 \cup \cdots \cup Y_n$ and for every $k \leq n$ let $p_k : Y^\omega \to Y_k^{\leq\omega}$ be the function deleting all letters not in Y_k.*

(i) Any of $p_0, \ldots, p_n$ is an AA-function.

(ii) For any AA-functions $h_k : X^\omega \to Y_k^{\leq\omega}$, $k \leq n$, there is an AA-function $h : X^\omega \to Y^{\leq\omega}$ such that $h_k = p_k \circ h$ for all $k \leq n$.

Proof. (i) is easy, since any p_k is computed by a trivial transducer with only one state.

(ii) For every $k \leq n$ let $(\mathcal{M}_k, Y_k, g_k, i_k)$ be an aperiodic asynchronous transducer that computes h_k. Consider the transducer $\mathcal{T} = (\mathcal{M}, Y, g, i)$ where $\mathcal{M}$ is the product of the automata $\mathcal{M}_0, \ldots, \mathcal{M}_n$ with the common alphabet X, $i = (i_0, \ldots, i_n)$ and

$$g((q_0, \ldots, q_n), x) = g_0(q_0, x) \cdots g_n(q_n, x).$$

By Proposition 4, $\mathcal{M}$ is aperiodic, hence $h = g_\mathcal{T}$ is an AA-function. Clearly, $h_k = p_k \circ h$ for all $k \leq n$. □

Next we consider a simple encoding $e : Y^\omega \to X^\omega$ of ω-words in an arbitrary alphabet $Y = \{y_1, \ldots, y_n\}$ by ω-words in another (typically, small) alphabet X with $0, 1 \in X$ defined as follows: for every $k \in \{1, \ldots, n\}$ let $\tilde{y}_k = 01^k0$, and for every $\eta \in Y^\omega$ let $e(\eta) = \widetilde{\eta(0)}\widetilde{\eta(1)} \cdots$.

Proposition 9 *In notation of the previous paragraph, e is an AA-function and there is an AA-function $r : X^\omega \to Y^{\leq\omega}$ such that $r \circ e$ is the identity function on Y^ω.*

Proof. The function e is computed by a trivial transducer with only one state, hence it is an AA-function. Consider the transducer $\mathcal{T} = (Q, X, Y, f, g, i)$ where $Q = \{0, 1, \ldots, n+2\}$, $i = 0$ and f, g are defined as follows:

$f(0, 0) = 1$ and $g(0, 0) = \varepsilon$;
$f(1, 0) = n + 2$ and $g(1, 0) = \varepsilon$;
$f(k, x) = n + 2$ and $g(k, x) = \varepsilon$ for all $k < n + 2$ and $x \in X \setminus \{0, 1\}$;
$f(n + 2, x) = n + 2$ and $g(n + 2, x) = \varepsilon$ for all $x \in X$;
$f(k, 1) = k + 1$ and $g(k, 1) = \varepsilon$ for all $k \in \{1, \cdots, n\}$;
$f(k, 0) = 0$ and $g(k, 0) = y_{k-1}$ for all $k \in \{2, \cdots, n + 1\}$.

It is easy to check that the automaton (Q, X, f) is aperiodic and the AA-function $r = g_\mathcal{T}$ has the desired property. □

Next we state an important closure property of the introduced classes of functions.

Proposition 10 *The classes of AA-functions and of AS-functions are closed under composition.*

Proof. We consider only the asynchronous case but the proof applies also to the synchronous case. Let $h_k : X_k^\omega \to X_{k+1}^{\leq\omega}$, $k < 2$, be AA-functions computed by aperiodic transducers $\mathcal{T}_k = (Q_k, X_k, X_{k+1}, f_k, g_k, i_k)$. We have to show that the composition $h_2 = h_1 \circ h_0 : X_0^\omega \to X_2^{\leq\omega}$ is an AA-function. Define the transducer $\mathcal{T} = (Q, X_0, X_2, f, g, i)$ as follows: $Q = Q_0 \times Q_1$, $i = (i_0, i_1)$ and

$$f((q_0, q_1), x) = (f_0(q_0, x), f_1(q_1, g_0(q_0, x)), \; g((q_0, q_1), x) = g_1(q_1, g_0(q_0, x)).$$

As is well-known and easy to see, the transducer $\mathcal{T}$ computes h_2, hence it remains to show that the automaton $\mathcal{M} = (Q, X_0, f)$ is aperiodic.

Let $q = (q_0, q_1)$ and $u \in X_0^+$. Since (Q_0, X_0, f_0) is aperiodic, $f_0(q_0, u^{m+1}) = f_0(q_0, u^m)$ for some m. As above, w.l.o.g. we may assume that $m = 0$, i.e. $f_0(q_0, u) = q_0$. For any $n < \omega$ we have $f(q, u^n) = (q_0, f_1(q_1, g_0(q_0, u^n)))$. Set $v = g_0(q_0, u) \in X_1^*$ and consider two cases. If $v = \varepsilon$ then $f(q, u) = (q_0, f_1(q_1, v)) = q$, hence $\mathcal{M}$ is aperiodic. Now let $v \neq \varepsilon$. Since (Q_1, X_1, f_1) is aperiodic, $f_1(q_1, v^{m+1}) = f_1(q_1, v^m)$ for some m, and again we may assume that $m = 0$, i.e. $f_1(q_1, v) = q_1$. We then have $f(q, u) = (q_0, f_1(q_1, v)) = q$, hence $\mathcal{M}$ is aperiodic. $\square$

The last proposition implies some natural properties of the following reducibilities.

Definition 3 *A set $A \subseteq X^\omega$ is AA-reducible (AS-reducible) to a set $B \subseteq Y^\omega$, in symbols $A \leq_{AA} B$ ($A \leq_{AS} B$) if $A = g^{-1}(B)$ for some AA-function (respectively, AS-function) $g : X^\omega \to Y^\omega$. For $X = Y$ we obtain the relations $\leq_{AA}$ and $\leq_{AS}$ on $P(X^\omega)$ called the AA- and AS-reducibilities, respectively.*

Corollary 1 *The relations $\leq_{AA}$ and $\leq_{AS}$ on $P(X^\omega)$ are preorders. The corresponding quotient partial orders of the AA- and AS-degrees are upper semilattices under the supremum operation induced by the operations $\oplus$ from Section 2.*

Now we relate the AA-transducers to the regular aperiodic sets. We say that a set is *Büchi aperiodic* if it is recognized by an aperiodic Büchi acceptor.

Proposition 11 *Let $h : X^\omega \to Y^{\leq\omega}$ be an AA-function.*

(i) For every $F \subseteq Y$, the set $A_F = \{\xi \in X^\omega \mid In(h(\xi)) \cap F \neq \emptyset\}$ is Büchi aperiodic.

(ii) For every $\mathcal{F} \subseteq P(Y)$, the set $A_\mathcal{F} = \{\xi \in X^\omega \mid In(h(\xi)) \in \mathcal{F}\}$ is regular aperiodic.

Proof. (i) Let (Q, X, Y, f, g, i) be an AA-transducer that computes h. Let $V = g(Q, X) \subseteq Y^*$ and let v_0 be a fixed word from V. Define a function $p : (Q \times V) \times X \to Q \times V$ by $p((q, v), x) = (f(q, x), g(q, x))$. Then the automaton $\mathcal{M} = (Q \times V, X, p)$ is aperiodic. Indeed, let $(q, v) \in Q \times V$ and $u = x_0 \cdots x_l \in X^+$ where $x_0, \ldots, x_l \in X$. Since (Q, X, f) is aperiodic, $f(q, u^{m+1}) = f(q, u^m)$ for some m. As above, we may assume that $m = 0$, i.e. $f(q, u) = q$. From the definition of p it follows that $p((q, v), u) = (q, g(f(q, x_0 \cdots x_{l-1}), x_l))$, hence $\mathcal{M}$ is aperiodic. Let $G = \{(q, v) \in Q \times V \mid \exists k(v(k) \in F)\}$. Then the aperiodic Büchi acceptor $(\mathcal{M}, (i, v_0), G)$ recognizes A_F.

(ii) The set $A_\mathcal{F}$ is a Boolean combination of the regular aperiodic sets A_F, $F \subseteq Y$, hence it is regular aperiodic. $\square$

Proposition 12 *The classes of the Büchi aperiodic and of the regular aperiodic ω-languages are closed under the preimages of AA-functions.*

Proof. Let $h : X^\omega \to Y^{\leq\omega}$ be an AA-function computed by an AA-transducer $\mathcal{T}_0 = (Q_0, X, Y, f_0, g_0, i_0)$ and let $(\mathcal{M}_1, i_1, F)$ $((\mathcal{M}_1, i_1, \mathcal{F}))$ be an aperiodic Büchi (respectively, Muller) acceptor recognizing a set $A \subseteq Y^\omega$. We have to show that $h^{-1}(A)$ is Büchi (respectively, regular) aperiodic. Let $p : Y^\omega \to Q_1^\omega$ be the AS-function computed by the transducer $\mathcal{T}_1 = (Q_1, X, Q_1, f_1, f_1, i_1)$. Let $\mathcal{T}$ be the

AA-transducer constructed from $\mathcal{T}_0$ and $\mathcal{T}_1$ as in the proof of Proposition 10 (for $X_0 = X$, $X_1 = Y$, $X_2 = Q_1$ and $g_1 = f_1$). Then $g_{\mathcal{T}} = p \circ h$, hence $h^{-1}(A) = \{\xi \in X^\omega \mid In(g_{\mathcal{T}}(\xi)) \cap F \neq \emptyset\}$ (respectively, $h^{-1}(A) = \{\xi \in X^\omega \mid In(g_{\mathcal{T}}(\xi)) \in \mathcal{F}\}$). Therefore, $h^{-1}(A)$ is Büchi (respectively, regular) aperiodic. □

Corollary 2 *The classes of Büchi aperiodic and of regular aperiodic sets in X^ω are closed under the AA- and AS-reducibilities.*

For a function $h : X^\omega \to Y^\omega$, let $G_h = \{\xi \times h(\xi) : \xi \in X^\omega\}$ be the "graph" of h, $G_h \subseteq (X \times Y)^\omega$.

Proposition 13 *The graph of an AS-function is regular aperiodic.*

Proof. Let (Q, X, Y, f, g, i) be an AS-transducer that computes a function $h : X^\omega \to Y^\omega$. Define the automaton $\mathcal{M} = (Q', X \times Y, f')$ by $Q' = Q \cup \{s\}$ where $s \notin Q$ and

$$f'(r,(x,y)) = \begin{cases} f(r,x) & \text{if } \ r \in Q \wedge g(r,x) = y \\ s & \text{otherwise.} \end{cases}$$

It is easy to see that $\mathcal{M}$ is aperiodic and the Muller acceptor $(\mathcal{M}, i, P(Q))$ recognizes the graph G_h. □

5. Aperiodic Determinacy

Here we prove an "aperiodic version" of the Büchi-Landweber regular determinacy theorem. This may be of independent interest and is also an important technical tool to prove some results below.

We start with recalling some relevant information on the Gale-Stewart games. Relate to any set $A \subseteq (X \times Y)^\omega$ the Gale-Stewart game $G(A)$ played by two opponents 0 and 1 as follows. Player 0 chooses a letter $x_0 \in X$, then player 1 chooses a letter $y_0 \in Y$, then 0 chooses $x_1 \in X$, then 1 chooses $y_1 \in Y$ and so on. Each player knows all the previous moves. After ω moves, player 0 has constructed a word $\xi = x_0x_1 \cdots \in X^\omega$ while player 1 has constructed a word $\eta = y_0y_1 \cdots \in Y^\omega$. Player 1 wins this particular play if $\xi \times \eta \in A$, otherwise player 0 wins.

A *strategy for player* 1 (player 0) in the game $G(A)$ is a function $h : X^+ \to Y$ (respectively, $h : Y^* \to X$) that prompts the player 1's move (respectively, the player 0's move) for any finite string of the opponent's previous moves. It is clear that the strategies for player 1 (for 0) are in a bijective correspondence with the CS-functions $h : X^\omega \to Y^\omega$ (respectively, with the delayed CS-functions $h : Y^\omega \to X^\omega$); we identify strategies with the corresponding CS-functions.

A strategy h for player 1 (player 0) in the game $G(A)$ is *winning* if the player wins each play when following the strategy, i.e. if $\xi \times h(\xi) \in A$ for all $\xi \in X^\omega$ (respectively, $h(\eta) \times \eta \in \overline{A}$ for all $\eta \in Y^\omega$). A set $A \subseteq (X \times Y)^\omega$ is *determined* if one of the players has a winning strategy in $G(A)$. It is interesting and useful to know which sets are determined and, in case of determinacy, how complicated it is to find the winner and how complicated is his/her winning strategy.

One of the best results of descriptive set theory is the Martin determinacy theorem (see e.g. [7]) stating that any Borel set is determined. Note that, since any regular set is Borel, this implies the determinacy of regular sets. One of the best results of automata theory is the Büchi-Landweber regular determinacy theorem stating that for any regular set A the winner in $G(A)$ may be computed effectively, (s)he has a winning strategy which is a DS-function, and the strategy is also computed effectively.

To prove the aperiodic version of the Büchi-Landweber theorem, we need some information about games played on bipartite graphs. Such games were considered by several authors (see e.g. [12, 32, 38] and the discussion in [33]). Recall that a *game graph* is a tuple $G = (V_0, V_1, E, c, C)$ where V_0, V_1 are (at most) countable disjoint non-empty sets of vertices, $E \subseteq (V_0 \times V_1) \cup (V_0 \times V_1)$ is a set of edges satisfying $\forall v \in V_i \exists w \in V_{1-i}((v, w) \in E)$ for $i < 2$, C is a finite set and $c : V \to C$, where $V = V_0 \cup V_1$. Games over G are again played by the opponents 0 and 1. The set V_i is intended as the set of game positions where it is the turn of player i to move. Winning conditions for such graph games may be specified in different ways.

E.g., by a *Mostowski game* we mean a triple (G, v, Ω), where G is a game graph as above, $v \in V_0$ and $\Omega = (E_1, F_1, \ldots, E_n, F_n)$ is a sequence of subsets of C with $E_1 \subseteq F_1 \subseteq \cdots \subseteq E_n \subseteq F_n$. A *play* in a Mostowski game (G, v, Ω) is a sequence $\gamma \in V^\omega$ such that $\gamma(0) = v$ and $(\gamma(k), \gamma(k+1)) \in E$ for all $k < \omega$. Player 1 wins the play γ iff

$$\exists k(In(c \circ \gamma) \cap E_k = \emptyset \wedge In(c \circ \gamma) \cap F_k \neq \emptyset).$$

A *memoryless strategy* for player i is a function $s : V_i \to V_{1-i}$ such that $(v, s(v)) \in E$ for all $v \in V_i$. A play γ *fits* a strategy s for i if $s(\gamma(2k+i)) = \gamma(2k+i+1)$ for all $k < \omega$. A strategy for player i is *winning* if player i, following this strategy, wins every play that fits the strategy. For a discussion and proof of the next important result known as the memoryless determinacy theorem, see e.g. [33].

Proposition 14 *Let (G, v, Ω) be a Mostowski game as above. Then one of the players has a memoryless winning strategy. If V is finite then the winner and some of his/her memoryless winning strategies may be found effectively.*

Now we are ready to prove the aperiodic version of the Büchi-Landweber theorem.

Theorem 1 *For any regular aperiodic set $A \subseteq (X \times Y)^\omega$ the winner of the game $G(A)$ may be computed effectively, (s)he has a winning strategy which is an AS-function, and the strategy is also computed effectively.*

Proof. W.l.o.g. we may assume X and Y to be disjoint. By Proposition 1, $A' = \{\langle \xi, \eta \rangle \mid \xi, \eta \in A\}$ is a regular aperiodic subset of $(X \cup Y)^\omega$. By Proposition 7, A' is recognized by an aperiodic Mostowski acceptor $(\mathcal{M}, i, \Omega)$, where $\mathcal{M} = (Q, X \cup Y, f)$ and $\Omega = (E_1, F_1, \ldots, E_n, F_n)$, $E_1 \subseteq F_1 \subseteq \cdots \subseteq E_n \subseteq F_n \subseteq Q$.

Define a finite game graph $G = (V_0, V_1, E, c, C)$ by

$$V_0 = Q \times \{0\},\ V_1 = Q \times \{1\},\ C = Q,\ c(q, k) = q,$$

and

$$E = \{((q, 0), (r, 1)) \mid \exists x \in X(f(q, x) = r)\} \cup \{((q, 1), (r, 0)) \mid \exists y \in Y(f(q, y) = r)\}$$

and consider the Mostowski game (G, v, Ω) where $v = (i, 0)$. By Proposition 14, some player i has a memoryless winning strategy $s : V_i \to V_{1-i}$, and the winner and a winning strategy are computed effectively.

Let first $i = 1$. Then

$$\forall q \in Q \exists y \in Y(s(q,1) = (f(q,y), 0)),$$

so there is a function $s' : Q \to Y$ with $s(q,1) = (f(q, s'(q)), 0)$. Define a function $g : Q \times X \to Y$ by $g(q,x) = s'(f(q,x))$. Let $h : X^\omega \to Y^\omega$ be the AS-function computed by the aperiodic transducer (Q, X, Y, f_X, g, i), where f_X is the restriction of f to $Q \times X$. Then h is a winning AS-strategy for player 1 in $G(A)$, i.e. $\xi \times h(\xi) \in A$ for all $\xi \in X^\omega$. Indeed, let $f(i, \xi \times h(\xi)) = q_0 q_1 \cdots$. Then the play $(q_0, 0), (q_1, 1), (q_2, 0), (q_3, 1), \ldots$ in (G, v, Ω) fits the strategy s, hence player 1 wins this play which means $\langle \xi, h(\xi) \rangle \in A'$, i.e. $\xi \times h(\xi) \in A$.

Now let $i = 0$. Then

$$\forall q \in Q \exists x \in X(s(q,0) = (f(q,x), 1)),$$

so there is a function $g : Q \to X$ with $s(q,0) = (f(q, g(q)), 1)$. Let $h : X^\omega \to Y^\omega$ be the delayed AS-function computed by the aperiodic transducer (Q, Y, X, f_Y, g, i) where f_Y is the restriction of f to $Q \times Y$. Then h is a winning AS-strategy for player 0 in $G(A)$, i.e. $h(\eta) \times \eta \in A$ for all $\eta \in Y^\omega$. Indeed, let $f(i, h(\eta) \times \eta) = q_0 q_1 \cdots$. Then the play $(q_0, 0), (q_1, 1), (q_2, 0), (q_3, 1), \ldots$ in (G, v, Ω) fits the strategy s, hence player 0 wins this play which means $\langle h(\eta, \eta \rangle \in A'$, i.e. $\in h(\eta) \times \eta \in A$. □

The last result and proof have several interesting corollaries. E.g. the proof works also for the case when A is regular (but not necessarily aperiodic) and yields the Büchi-Landweber theorem in the original form [1]. We conclude this section by a couple of other corollaries relevant to further results of this paper.

Corollary 3 *(i) Let $A \subseteq (X \times Y)^\omega$ be regular aperiodic and player i has a winning CS-strategy in the game $G(A)$. Then player i has also a winning AS-strategy.*

(ii) Let $B \subseteq X^\omega$ and $C \subseteq Y^\omega$ be regular aperiodic and let $h : X^\omega \to Y^\omega$ be a CS-function satisfying $B = h^{-1}(C)$. Then $B = g^{-1}(C)$ for some AS-function $g : X^\omega \to Y^\omega$.

(iii) Let $B, C \subseteq X^\omega$ be regular aperiodic. Then $B \leq_{AS} C$ or $\overline{C} \leq_{AS} B$.

Proof. (i) Suppose that player i has no winning AS-strategy in $G(A)$. By Theorem 1, player $1-i$ has a winning AS-strategy, hence also a winning CS-strategy in $G(A)$. Thus, both players have winning strategies in $G(A)$, a contradiction.

(ii) Let $A = \{\xi \times \eta \mid \xi \in B \leftrightarrow \eta \in C\}$. By Proposition 8, A is a regular aperiodic subset of $(X \times Y)^\omega$. Since $B = h^{-1}(C)$, h is a winning CS-strategy for player 1 in the game $G(A)$. By (i), player 1 has also a winning AS-strategy g in $G(A)$. Therefore, $B = g^{-1}(C)$.

(iii) Let A be as in the proof of (ii). If player 1 has a winning strategy in $G(A)$ then, by (ii), $B \leq_{AS} C$. Otherwise, player 0 has a winning strategy in $G(A)$, hence, by (ii), $\overline{C} \leq_{AS} B$. □

Define subsets K_0, K_1 of 2^ω which play a noticeable role in further considerations by

$$K_0 = \{\xi \mid \exists k(\xi(k) = 1)\} \text{ and } K_1 = \{\xi \mid \exists l \forall k > l(\xi(k) = 0)\}.$$

Corollary 4 *(i) The sets K_0 and $\overline{K}_1$ are Büchi aperiodic.*

(ii) $K_0 \in \mathbf{\Sigma}_1^0 \setminus \mathbf{\Pi}_1^0$ and $K_1 \in \mathbf{\Sigma}_2^0 \setminus \mathbf{\Pi}_2^0$.

(iii) Any $\mathbf{\Sigma}_1^0$-set ($\mathbf{\Sigma}_2^0$-set) $B \subseteq X^\omega$ is CS-reducible to K_0 (respectively, to K_1).

(iv) Any regular $\mathbf{\Sigma}_1^0$-set (regular $\mathbf{\Sigma}_2^0$-set) $B \subseteq X^\omega$ is DS-reducible to K_0 (respectively, to K_1).

(v) Any regular aperiodic $\mathbf{\Sigma}_1^0$-set (regular aperiodic $\mathbf{\Sigma}_2^0$-set) $B \subseteq X^\omega$ is AS-reducible to K_0 (respectively, to K_1).

Proof. The assertion (i) is obvious while (ii)—(iv) are well known. The assertion (v) follows from (iii) and Corollary 3(ii). □

Finally, we prove the following "aperiodic version" of a well known characterization of the DS-functions (see e.g. [34]).

Corollary 5 *A CS-function $h : X^\omega \to Y^\omega$ is an AS-function iff its graph G_h is a regular aperiodic subset of $(X \times Y)^\omega$.*

Proof. From left to right, this is Proposition 13. Conversely, let $A = G_h$ be regular aperiodic. Then h is a winning strategy for player 1 in the game $G(A)$. By Corollary 3(iii), there is a winning strategy g for player 1 in $G(A)$ which is an AS-function. Since $G_h = G_g$, we have $h = g$. □

6. Aperiodic $\mathbf{\Sigma}_n^0$-Sets

Here we establish some facts on the regular aperiodic sets in the Borel hierarchy.

First we recall some well known characterizations of the regular $\mathbf{\Sigma}_n^0$-sets. For a Muller acceptor $(\mathcal{M}, i, \mathcal{F})$, $\mathcal{M} = (Q, X, f)$, set $\mathcal{E} = \{In(f(i,\xi) \mid \xi \in X^\omega\}$. Thus, $\mathcal{E}$ is the class of sets of vertices of the "cycles" of the graph of $\mathcal{M} = (Q, X, f)$ reachable from i. Since $(\mathcal{M}, i, \mathcal{F} \cap \mathcal{E})$ recognizes the same ω-language as $(\mathcal{M}, i, \mathcal{F})$ does, we may w.l.o.g. think that $\mathcal{F} \subseteq \mathcal{E}$. Define a preorder $\leq_0$ and a partial order $\leq_1$ on $\mathcal{E}$ as follows: $U \leq_1 V$, if $U \supseteq V$, and $U \leq_0 V$, if for any $q \in U$ there exists a $w \in X^*$ with $f(q, w) \in V$. The next result is due to L. Landweber and K. Wagner, see e.g. [31, 35].

Proposition 15 *Let $(\mathcal{M}, i, \mathcal{F})$, $\mathcal{F} \subseteq \mathcal{E}$, be a Muller acceptor recognizing a set $A \subseteq X^\omega$. Then $A \in \mathbf{\Sigma}_1^0$ ($A \in \mathbf{\Sigma}_2^0$) iff $\mathcal{F}$ is closed upwards in $(\mathcal{F}; \leq_0)$ (respectively, in $(\mathcal{F}; \leq_1)$).*

Our first result is an "aperiodic version" of a slightly different Landweber's characterizations of the regular $\mathbf{\Sigma}_n^0$-sets (see e.g. [31]).

Theorem 2 *(i) A regular aperiodic set $A \subseteq X^\omega$ is $\mathbf{\Sigma}_1^0$ iff $A = W \cdot X^\omega$ for some regular aperiodic set $W \subseteq X^*$.*

(ii) A regular aperiodic set $A \subseteq X^\omega$ is $\mathbf{\Pi}_2^0$ iff A is Büchi aperiodic.

Proof. (i) Let A be $\mathbf{\Sigma}_1^0$ and $(\mathcal{M}, i, \mathcal{F})$ be an aperiodic Muller acceptor recognizing A. Let W be the regular aperiodic set recognized by the acceptor $(\mathcal{M}, i, F)$, where $F = \bigcup \mathcal{F}$. From Proposition 15(i) it easily follows that $A = W \cdot X^\omega$, as desired. The opposite implication is trivial (e.g., it follows immediately from the well known characterization of the regular aperiodic ω-languages in terms of regular expressions [31]).

(ii) Let $A \subseteq X^\omega$ be regular aperiodic and $\mathbf{\Pi}^0_2$. By Corollary 4(v), $\overline{A} = g^{-1}(K_1)$ for some AS-function $h : X^\omega \to 2^\omega$, so $A = g^{-1}(\overline{K_1})$. By Corollary 4(i), $\overline{K_1}$ is Büchi aperiodic. By Proposition 12, so is also the set A, as desired. The opposite implication is trivial. □

Recall that a class $\mathcal{C}$ has the *reduction property*, if for all $C_0, C_1 \in \mathcal{C}$ there are disjoint $C'_0, C'_1 \in \mathcal{C}$ such that $C'_i \subseteq C_i$ for both $i < 2$ and $C_0 \cup C_1 = C'_0 \cup C'_1$; such a pair (C'_0, C'_1) is called a *reduct* of (C_0, C_1). A class $\mathcal{C}$ has the *separation property*, if all disjoint sets $C_0, C_1 \in \mathcal{C}$ are separable by a set $B \in \mathcal{C} \cap \check{\mathcal{C}}$ (i.e. $C_0 \subseteq B \subseteq \overline{C}_1$). It is well known that if a class $\mathcal{C}$ has the reduction property then the dual class $\check{\mathcal{C}}$ has the separation property.

As is well known [7], any level $\mathbf{\Sigma}^0_n$ of the Borel hierarchy has the reduction property. In [22] it was shown that the classes $\mathcal{L}_n = \mathcal{R} \cap \mathbf{\Sigma}^0_{n+1}$, $n < 2$, have the reduction property (this trivially holds also for $n \geq 2$, because in this case we have $\mathcal{L}_n = \mathcal{R}$ and this class is closed under the Boolean operations). The next result is an "aperiodic version" of the last fact. Set $\mathcal{K}_n = \mathcal{A} \cap \mathbf{\Sigma}^0_{n+1}$ where $\mathcal{A}$ is the class of regular aperiodic ω-languages. Since again $\mathcal{K}_n = \mathcal{A}$ is closed under the Boolean operations for $n \geq 2$, these classes trivially have the reduction property.

Theorem 3 *The classes $\mathcal{K}_0$ and $\mathcal{K}_1$ have the reduction property.*

Proof. Let $C_0, C_1 \in \mathcal{K}_0$. By Theorem 2(i), $C_k = W_k \cdot X^\omega$, $k < 2$, for some regular aperiodic sets $W_0, W_1 \subseteq X^*$. Let $C'_k = W'_k \cdot X^\omega$, $k < 2$, where $W'_0, W'_1 \subseteq X^*$ are defined by

$$W'_0 = \{u \in W_0 \mid \neg\exists v \sqsubset u(v \in W_0 \cup W_1)\}$$

and

$$W'_1 = \{u \in W_1 \setminus W_0 \mid \neg\exists v \sqsubseteq u(v \in W_0 \cup W_1)\}.$$

It is easy to check that (C'_0, C'_1) is a reduct of (C_0, C_1), hence, by Theorem 2(i), it remains to show that W'_0 and W'_0 are regular aperiodic.

For any $k < 2$, let $(\mathcal{M}_k, i_k, F_k)$, $\mathcal{M}_k = (Q_k, X, f_k)$, be an aperiodic acceptor recognizing W_k. Let $\mathcal{M} = (Q, X, f)$ be the product of automata $\mathcal{M}_0$ and $\mathcal{M}_1$. By Proposition 4, the automaton $\mathcal{M}$ is aperiodic. It is clear that W_0 (W_1) is recognized by $(\mathcal{M}, i, F_0 \times Q_1)$ (respectively, by $(\mathcal{M}, i, Q_0 \times F_1)$) where $i = (i_0, i_1)$. Let $\mathcal{M}'$ be the partial automaton obtained from (the graph of) $\mathcal{M}$ by removing all the edges with origins (q_0, q_1) such that $q_0 \in F_0$ or $q_1 \in F_1$. It is easy to see that W'_0 (W'_1) is recognized by the partial aperiodic acceptor $(\mathcal{M}', i, F_0 \times Q_1)$ (respectively, $(\mathcal{M}', i, (Q_0 \setminus F_0) \times F_1)$. By Proposition 3, W'_0 and W'_1 are regular aperiodic.

It remains to prove the reduction property for $\mathcal{K}_1$. Let $C_0, C_1 \in \mathcal{K}_1$. By Theorem 2(ii), $\overline{C}_k$, $k < 2$, is recognized by an aperiodic Büchi acceptor $(\mathcal{M}_k, i_k, F_k)$, where $\mathcal{M}_k = (Q_k, X, f_k)$. Let $\mathcal{M} = (Q, X, f)$ be the product of the automata $\mathcal{M}_0$ and $\mathcal{M}_1$. By Proposition 4, the automaton $\mathcal{M}$ is aperiodic. It is clear that

$$C_k = \{\xi \in X^\omega \mid In(f(i, \xi)) \subseteq T'_k\}$$

where $k < 2$, $i = (i_0, i_1)$, $T_0 = (Q_0 \setminus F_0) \times Q_1$ and $T_1 = Q_0 \times (Q_1 \setminus F_1)$. Let $\mathcal{M}' = (Q', X, f')$ be the shrink automaton constructed from $\mathcal{M}$ and the sets T_0, T_1 as in Proposition 5. By that proposition, the automaton $\mathcal{M}'$ is aperiodic. Set

$$C'_k = \{\xi \in X^\omega \mid In(f'(i', \xi)) \subseteq T'_k\}$$

where $k < 2$, $i' = (i, 0)$ and $T'_k = \{k\} \times T_k$. By Proposition 2(ii), $C'_0, C'_1 \in \mathcal{K}_1$, so it remains to check that (C'_0, C'_1) is a reduct of (C_0, C_1).

Since T'_0 and T'_1 are disjoint, so are also C'_0 and C'_1. The inclusion $C'_k \subseteq C_k$, $k < 2$, follows easily from the definition of f' (see Proposition 5). It remains to show the inclusion $C_0 \cup C_1 \subseteq C'_0 \cup C'_1$. Let $\xi \in C_0 \cup C_1$, then $\xi \in C_0 \setminus C_1$ or $\xi \in C_1 \setminus C_0$ or $\xi \in C_0 \cap C_1$. In the first case we have $In(f(i, \xi)) \subseteq T_0$ and $In(f(i, \xi)) \not\subseteq T_1$. In other words, $r_l \in T_0$ for almost all l and $r_l \in T_0 \setminus T_1$ for infinitely many l, where $r_0 r_1 \cdots = In(f(i, \xi))$. By the definition of f', $In(f'(i', \xi)) \subseteq T'_0$, hence $\xi \in C'_0 \subseteq C'_0 \cup C'_1$. The second case is considered similarly. Finally, let $\xi \in C_0 \cap C_1$. Then there is an l_0 such that $r_l \in T_0 \cap T_1$ for all $l \geq l_0$. By the definition of f', there is a $k < 2$ such that $r_l = (k, q_l)$ for all $l \geq l_0$. Therefore, $\xi \in C'_k \subseteq C'_0 \cup C'_1$. $\square$

Corollary 6 *The classes $\check{\mathcal{K}}_0$ and $\check{\mathcal{K}}_1$ have the separation property.*

Next we want to establish the "aperiodic version" of a theorem due to L. Staiger and K. Wagner [28] but first we recall some facts on the difference hierarchy. Let S be a set and $\mathcal{C} \subseteq P(S)$ a class of sets closed under finite unions and intersections. For any $n < \omega$, let $D_n(\mathcal{C})$ be the n-th level of the difference hierarchy over $\mathcal{C}$ consisting of the sets $\bigcup_i (A_{2i} \setminus A_{2i+1})$ where $A_i \in \mathcal{C}$, $A_0 \supseteq A_1 \supseteq \cdots$ and $A_n = \emptyset$. The next fact is well known.

Proposition 16 *In notation of the previous paragraph, $D_n(\mathcal{C}) \cup \check{D}_n(\mathcal{C}) \subseteq D_{n+1}(\mathcal{C})$ for every $n < \omega$, and $\bigcup_n D_n(\mathcal{C}) = BC(\mathcal{C})$.*

For $\xi \in X^\omega$ and $x \in X$, let $\#_x(\xi)$ denote the number of occurrences of x in ξ, thus $\#_x(\xi) \leq \omega$. For any $n < \omega$, set

$$B_n = \{\xi \in X^\omega : \#_1(\xi) \geq n\},\ C_{2n} = \bigcup_{i<n} (B_{2i+1} \setminus B_{2i+2}),\ C_{2n+1} = C_{2n} \cup B_{2n+1}$$

(recall that $0, 1 \in X$).

Proposition 17 *For any $n < \omega$, C_n is regular aperiodic and CS-complete (DS-complete, AS-complete) in $D_n(\mathbf{\Sigma}^0_1)$ (respectively in $D_n(\mathcal{L}_0)$, $D_n(\mathcal{K}_0)$).*

Proof. The regular aperiodicity of C_n is obvious. Its CS-completeness in $D_n(\mathbf{\Sigma}^0_1)$ is well known and easy to check. Its DS-completeness in $D_n(\mathcal{L}_0)$ follows from the results in [35]. Its AS-completeness in $D_n(\mathcal{K}_0)$ follows from Corollary 3(ii). $\square$

Now we are ready to establish the "aperiodic version" of the Staiger-Wagner theorem.

Theorem 4 *Every regular aperiodic $\mathbf{\Delta}^0_2$-set is a Boolean combination of open regular aperiodic sets. In symbols, $\mathcal{K}_1 \cap \check{\mathcal{K}}_1 = BC(\mathcal{K}_0)$.*

Proof. The inclusion $BC(\mathcal{K}_0) \subseteq \mathcal{K}_1 \cap \check{\mathcal{K}}_1$ is obvious. Conversely, let $A \in \mathcal{K}_1 \cap \check{\mathcal{K}}_1$, then $A \in \mathcal{L}_1 \cap \check{\mathcal{L}}_1$. By the Staiger-Wagner theorem, $A \in BC(\mathcal{L}_0)$. By Proposition 16, $A \in D_n(\mathcal{L}_0)$ for some $n < \omega$. By Proposition 17, $A \leq_{DS} C_n$. Since A is regular aperiodic, by Proposition 17 we have $A \leq_{DS} C_n$. Therefore, $A \in D_n(\mathcal{K}_0) \subseteq BC(\mathcal{K}_0)$. □

We conclude this section by a corollary which is crucial for the subsequent sections. Recall that *a base* (in a set S) is a sequence $\{L_n\}_{n<\omega}$ of sublattices of $(P(S); \cup, \cap, 0, 1)$ satisfying $L_n \cup \check{L}_n \subseteq L_{n+1}$.

Definition 4 *(i) A base L is reducible, if every L_n has the reduction property.*

(ii) A base L is interpolable, if for all $n < \omega$ any two disjoint elements $A, B \in \check{L}_{n+1}$ are separable by a Boolean combination of elements of L_n.

In [22] we have shown that the base $\{\mathcal{L}_n\}_{n<\omega}$ is reducible and interpolable. From the results of this section we obtain

Corollary 7 *The base $\{\mathcal{K}_n\}_{n<\omega}$ is reducible and interpolable.*

Proof. The reducibility was established earlier. Since $\mathcal{A} = BC(\mathcal{K}_1) = \mathcal{K}_n$ for every $n > 1$, the interpolability property for the levels $\mathcal{K}_n$, $n > 1$, is obvious. It remains to show that any two disjoint sets $A, B \in \check{\mathcal{K}}_1$ are separable by a Boolean combination of elements of $\mathcal{K}_0$. Since, by Corollary 6, the class $\check{\mathcal{K}}_1$ has the separation property, A is separated from B by a set in $\mathcal{K}_1 \cap \check{\mathcal{K}}_1$. By Theorem 4, A is separated from B by a set in $BC(\mathcal{K}_0)$. □

7. Fine Hierarchy

Here we describe some basic properties of the fine hierarchy of regular aperiodic ω-languages which is just the fine hierarchy over the base $\mathcal{K} = \{\mathcal{K}_n\}_{n<\omega}$. For a background on the fine hierarchy see e.g. [20, 21, 22]. The results of this section are particular cases of the corresponding general facts about the fine hierarchy [17, 19], so the proofs are omitted.

In our definition of the fine hierarchy we use the difference hierarchy from the previous section and the following operation *bisep* on classes of ω-languages: let $bisep(\mathcal{C}, \mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2)$ be the class

$$\{(C_0 \cap D_0) \cup (C_1 \cap D_1) \cup (\overline{C}_0 \cap \overline{C}_1 \cap D_2) \mid C_i \in \mathcal{C}, D_j \in \mathcal{D}_j, C_0 \cap C_1 = 0\}.$$

Definition 5 *The fine hierarchy over $\mathcal{K}$ is the sequence $\{\mathcal{A}_\alpha\}_{\alpha<\omega^\omega}$ defined as follows:*

$\mathcal{A}_n = D_n(\mathcal{K}_0)$ *for* $n < \omega$*;*

$\mathcal{A}_{\omega^n} = D_n(\mathcal{K}_1)$ *for* $0 < n < \omega$*;*

$\mathcal{A}_{\beta+\omega^n} = bisep(\mathcal{K}_0, \mathcal{A}_\beta, \check{\mathcal{A}}_\beta, \mathcal{A}_{\omega^n})$ *for* $0 < n < \omega$ *and* β *of the form* $\beta = \omega^n \cdot \beta_1$ *for some* β_1, $0 < \beta_1 < \omega^\omega$*;*

$\mathcal{A}_{\beta+1} = bisep(\mathcal{K}_0, \mathcal{A}_\beta, \check{\mathcal{A}}_\beta, \mathcal{A}_0)$ *for* $\omega \leq \beta < \omega^\omega$.

The definition uses some ordinal arithmetic as described e.g. in [KM67]. It is correct since every non-zero ordinal $\alpha < \omega^\omega$ is uniquely representable in the form $\alpha = \omega^{n_0} + \cdots + \omega^{n_k}$ for a finite sequence $n_0 \geq \cdots \geq n_k$ of ordinals $< \omega$. Applying the definition we subsequently get $\mathcal{A}_{\omega^{n_0}}, \mathcal{A}_{\omega^{n_0}+\omega^{n_1}}, \ldots, \mathcal{A}_\alpha$. Note that the definition applies to any base in place of the base $\mathcal{K}$ above. For this paper, the fine hierarchy $\{\mathcal{S}_\alpha\}$ over the base $\{\mathbf{\Sigma}^0_{n+1}\}$ and the fine hierarchy $\{\mathcal{R}_\alpha\}$ over the base $\mathcal{L} = \{\mathcal{L}_n\}_{n<\omega}$ are also relevant. The hierarchy $\{\mathcal{R}_\alpha\}$ coincides with the Wagner hierarchy [22].

Note that in general the levels of the fine hierarchy are numbered by the ordinals less than the well known ordinal $\varepsilon_0 = sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots\}$. Since in our case the base $\mathcal{K}$ has the property $\mathcal{K}_{n+2} = \mathcal{A}$, we have $\mathcal{A}_\alpha = \mathcal{A}$ for $\alpha \geq \omega^\omega$, hence only the levels $< \omega^\omega$ are interesting (the same applies to the hierarchy $\{\mathcal{R}_\alpha\}$). Note also that our definition here slightly differs from (but is equivalent to) the definition of the fine hierarchy in [22].

Proof of the next result uses heavily the fact from the previous section that the base $\mathcal{K}$ is reducible and interpolable.

Theorem 5 *(i) For all $\alpha < \beta < \omega^\omega$, $\mathcal{A}_\alpha \cup \check{\mathcal{A}}_\alpha \subseteq \mathcal{A}_\beta$.*

(ii) $\bigcup_{\alpha<\omega^\omega} \mathcal{A}_\alpha = \mathcal{A}$.

(iii) For any limit ordinal $\lambda < \omega^\omega$, $\mathcal{A}_\lambda \cap \check{\mathcal{A}}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{A}_\alpha$.

(iv) For any $\alpha < \omega^\omega$, $\mathcal{A}_{\alpha+1} \cap \check{\mathcal{A}}_{\alpha+1}$ coincides with the class of sets $(A \cap C) \cup (B \cap \overline{C})$ where $A \in \mathcal{A}_\alpha$, $B \in \check{\mathcal{A}}_\alpha$ and $C \in \mathbf{\Delta}^0_1 = \mathcal{K}_0 \cap \check{\mathcal{K}}_0$.

(v) For any $\alpha < \omega^\omega$, $\check{\mathcal{A}}_\alpha$ has the separation property.

Next we give an alternative characterization of the fine hierarchy $\{\mathcal{A}_\alpha\}$.

Definition 6 *(i) A typed Boolean term is a term of signature $\{\cup, \cap, \bar{\ }, 0, 1\}$ with variables $v^0_n, v^1_n (n < \omega)$. Variables v^0_n are of type 0 while variables v^1_n are of type 1.*

(ii) For a typed Boolean term t and a base L, let $t(L_0, L_1)$ be the set of values of t when variables $v^i_n (n < \omega)$ of type i range over L_i, $i < 2$.

The next result shows the close relation of the classes $t(\mathcal{K}_0, \mathcal{K}_1)$ to the fine hierarchy $\{\mathcal{A}_\alpha\}$. The result follows from the reducibility of the base $\mathcal{K}$. Similar facts for the bases $\{\mathbf{\Sigma}^0_{n+1}\}$ and $\mathcal{L}$ were established in [20, 22].

Theorem 6 *For every $\alpha < \omega^\omega$ one can effectively find a typed Boolean term $t = t_\alpha$ such that $\mathcal{A}_\alpha = t(\mathcal{K}_0, \mathcal{K}_1)$. Conversely, for every typed Boolean term t the class $t(\mathcal{K}_0, \mathcal{K}_1)$ coincides with one of the classes $\mathcal{A}_\alpha, \check{\mathcal{A}}_\alpha$ for some $\alpha < \omega^\omega$, and the class is effectively computable from t.*

We give a couple of examples. If $t = v^0_0 \cap \overline{v^0_1}$, then $t(\mathcal{K}_0, \mathcal{K}_1) = D_2(\mathcal{K}_0)$. Let $u \triangle v = (u \cap \overline{v}) \cup (v \cap \overline{u})$ be the symmetric difference operation and let $t = v^0_0 \triangle \cdots \triangle v^0_n$. Then $t(\mathcal{K}_0, \mathcal{K}_1) = D_{n+1}(\mathcal{K}_0)$ for any $n < \omega$ (this follows from a result in [8]).

We conclude this section by a generalization of the last example. This is the "aperiodic version" of Theorem 8.6 in [22] that gives a set-theoretic description of a subsequence of $\{\mathcal{R}_\alpha\}_{\alpha<\omega^\omega}$ called in [35] the coarse structure. For classes of ω-languages $\mathcal{C}$ and $\mathcal{D}$, set $\mathcal{C} + \mathcal{D} = \{C \triangle D \mid C \in \mathcal{C}, D \in \mathcal{D}\}$. The operation $+$ is clearly associative and commutative. For any $m < \omega$, set $m \cdot \mathcal{C} = \mathcal{C} + \cdots + \mathcal{C}$ (m summands, assuming $0 \cdot \mathcal{C} = \{\emptyset\}$). By the mentioned result in [8], $m \cdot \mathcal{K}_i = D_m(\mathcal{K}_i)$ for all $m < \omega$ and $i < 2$.

Theorem 7 *For all $m, n < \omega$, $\mathcal{A}_{\omega^{n+1} \cdot (m+1)} = \mathcal{A}_{\omega^{n+1}} + \mathcal{A}_m = (n+1) \cdot \mathcal{K}_1 + m \cdot \mathcal{K}_0$.*

A principal question about any hierarchy is whether it collapses or not (for the case of the hierarchy $\{\mathcal{A}_\alpha\}$ the non-collapse property means that $\mathcal{A}_\alpha \not\subseteq \check{\mathcal{A}}_\alpha$ for all $\alpha < \omega^\omega$). It is well-known (see e.g. [22, 35]) that the hierarchies $\{\mathcal{S}_\alpha\}$ and $\{\mathcal{R}_\alpha\}$ do not collapse. The hierarchy $\{\mathcal{A}_\alpha\}$ does not collapse too, as we shall see in the next section.

8. Reducibilities on $\mathcal{A}$

Here we establish the non-collapse property of the fine hierarchy $\{\mathcal{A}_\alpha\}_{\alpha<\omega^\omega}$ from the previous section and describe the structures of AA- and AS-degrees of regular aperiodic sets.

First we will show that for every $\alpha < \omega^\omega$ the class $\mathcal{A}_\alpha$ has an AA-complete set. This result is parallel to the result from [35] that any $\mathcal{R}_\alpha$ has a DA-complete set. Since our proof will be a modification of the corresponding proof of the last fact in [22], we start with reproducing a version of the mentioned proof. This version uses an idea from [17] and is a bit shorter than the proof in [22] since it does not use the induction on α.

By Corollary 6.8 in [22], for every $\alpha < \omega^\omega$ there is a typed Boolean term $t = t_\alpha$ such that $\mathcal{R}_\alpha = t(\mathcal{L}_0, \mathcal{L}_1)$. Thus, it suffices to construct for any given typed Boolean term $t = t(x_1, \ldots, x_n, y_1, \ldots, y_n)$ (where x_i are variables of type 0 and y_i are variables of type 1) a set R_t which is DA-complete in $\mathcal{R}_t = t(\mathcal{L}_0, \mathcal{L}_1)$. Recall that we work with a fixed alphabet X containing 0 and 1.

Let $p_1, \ldots, p_{2n} : X^\omega \to X^\omega$ be the "projections" from Section 2 satisfying $p_i(\langle \xi_1, \ldots, \xi_{2n} \rangle) = \xi_i$. Since p_i are DA-functions, $p_i^{-1}(K_0) \in \mathcal{L}_0$ for $i \in \{1, \ldots, n\}$ and $p_i^{-1}(K_1) \in \mathcal{L}_1$ for $i \in \{n+1, \ldots, 2n\}$ where K_0, K_1 are the sets from Corollary 4(iv). Therefore, the set

$$R_t = t(p_1^{-1}(K_0), \ldots, p_n^{-1}(K_0), p_{n+1}^{-1}(K_1), \ldots, p_{2n}^{-1}(K_1))$$

is in $\mathcal{R}_t$. It remains to show that $S \leq_{DA} R_t$ for every $S \in \mathcal{R}_t$. Let $B_1, \ldots, B_n \in \mathcal{L}_0$ and $B_{n+1}, \ldots, B_{2n} \in \mathcal{L}_1$ satisfy $S = t(B_1, \ldots, B_n, B_{n+1}, \ldots, B_{2n})$. Since K_0 and K_1 are DA-complete respectively in $\mathcal{L}_0$ and $\mathcal{L}_1$ by Corollary 4(iv), there are DA-functions $f_1, \ldots, f_{2n} : X^\omega \to X^\omega$ such that $B_i = f_i^{-1}(K_0)$ for $i \in \{1, \ldots, n\}$ and $B_i = f_i^{-1}(K_1)$ for $i \in \{n+1, \ldots, 2n\}$. For the DA-function $f(\xi) = \langle f_1(\xi), \ldots, f_{2n}(\xi) \rangle$ we have $p_i \circ f = f_i$ for $i \in \{1, \ldots, 2n\}$ and

$$f^{-1}(R_t) = t(f^{-1}p_1^{-1}(K_0), \ldots, f^{-1}p_{2n}^{-1}(K_1)) = t(B_1, \ldots, B_{2n}) = S.$$

Therefore, $S \leq_{DA} R_t$.

Unfortunately, the proof in the previous paragraph does not work in the aperiodic case because the projections p_i are not AA-functions and so the set R_t need not be aperiodic. In the proof of the next result we use the more subtle coding from Section 4 which requires to work with different alphabets simultaneously, so in the proof of the next result we switch to the alphabet-dependent notation.

Theorem 8 *For every $\alpha < \omega^\omega$ the class $\mathcal{A}_\alpha$ has an AA-complete set.*

Proof. As above, by Proposition 6 it suffices to construct for any given typed Boolean term $t = t(x_1, \ldots, x_n, y_1, \ldots, y_n)$ (where x_i are variables of type 0 and y_i are variables of type 1) a set A_t which is AA-complete in $\mathcal{A}_t = t(\mathcal{K}_0, \mathcal{K}_1)$; we have here in mind the fixed alphabet X containing 0 and 1.

For any alphabet Y, let $\mathcal{A}_t(Y)$ be the class of sets of the form $t(B_1, \ldots, B_{2n})$ where $B_1, \ldots, B_n \in \mathcal{K}_0(Y)$ and $B_{n+1}, \ldots, B_{2n} \in \mathcal{K}_1(Y)$. In this proof, by an AA-complete set in $\mathcal{A}_t(Y)$ we mean a set $C \in \mathcal{A}_t(Y)$ such that for any alphabet Z and any $D \in \mathcal{A}_t(Z)$ there is an AA-function $f : Z^\omega \to Y^\omega$ with $D = f^{-1}(C)$.

First we prove the auxiliary assertion that there is an alphabet Y such that $\mathcal{A}_t(Y)$ has an AA-complete set. Let $Y = Y_1 \cup \cdots \cup Y_{2n}$ where $Y_1, \ldots, Y_{2n}$ are some pairwise disjoint binary alphabets, $Y_i = \{y_0^i, y_1^i\}$. Let $p_i : Y^\omega \to Y_i^{\leq\omega}$, $i \in \{1, \ldots, 2n\}$, be the AA-function that erases all letters not in Y_i. Let

$$A_t(Y) = t(p_1^{-1}(K_0^1), \ldots, p_n^{-1}(K_0^n), p_{n+1}^{-1}(K_1^{n+1}), \ldots, p_{2n}^{-1}(K_1^{2n}))$$

where

$$K_0^i = \{\xi \in Y_i^\omega \mid \exists k(\xi(k) = y_1^i)\} \text{ for } i \in \{1, \ldots, n\}$$

and

$$K_1^i = \{\xi \in Y_i^\omega \mid \exists k \forall l \geq k(\xi(l) = y_0^i)\} \text{ for } i \in \{n+1, \ldots, 2n\}.$$

By Corollary 4(v), the sets K_0 and K_1 are AA-complete respectively in $\mathcal{K}_0(\{0,1\})$ and $\mathcal{K}_1(\{0,1\})$. Thus, $A_t(Y) \in \mathcal{A}_t(Y)$. It remains to AA-reduce any set $S \in \mathcal{A}_t(Z)$ to $A_t(Y)$. Let $B_1, \ldots, B_n \in \mathcal{K}_0(Z)$ and $B_{n+1}, \ldots, B_{2n} \in \mathcal{K}_1(Z)$ satisfy $S = t(B_1, \ldots, B_n, B_{n+1}, \ldots, B_{2n})$. From Corollary 4(v) it follows that there are AA-functions $f_i : Z^\omega \to Y_i^\omega$, $i \in \{1, \ldots, 2n\}$ such that $B_i = f_i^{-1}(K_0^i)$ for $i \in \{1, \ldots, n\}$ and $B_i = f_i^{-1}(K_0^i)$ for $i \in \{n+1, \ldots, 2n\}$. For the AA-function $f : Z^\omega \to Y^\omega$ defined by $f(\zeta) = \langle f_1(\zeta), \ldots, f_{2n}(\zeta)\rangle$ we have $p_i \circ f = f_i$ for $i \in \{n+1, \ldots, 2n\}$. Therefore, $f^{-1}(A_t(Y)) = S$ completing the proof of the auxiliary assertion.

Now it suffices to show that for any alphabet X with $0, 1 \in X$ there are a set $A_t(X) \in \mathcal{A}_t(X)$ and an AA-function $e : Y^\omega \to X^\omega$ such that $A_t(Y) = e^{-1}(A_t(X))$. Let e, r be the functions from Proposition 9 and let $A_t(X) = r^{-1}(A_t(Y))$. From Proposition 9 it follows that $A_t(X)$ and e have the desired property. □

Let A_α be the AA-complete set in $\mathcal{A}_\alpha$ constructed in the last proof. It has also some additional nice properties related to the hierarchies $\{\mathcal{S}_\alpha\}$ and $\{\mathcal{R}_\alpha\}$ from the previous section.

Corollary 8 *Let $\alpha < \omega^\omega$.*

(i) The set A_α is DA-complete in $\mathcal{R}_\alpha$ and CA-complete in $\mathcal{S}_\alpha$.

(ii) The set $A_\alpha \oplus \bar{A}_\alpha$ is AA-complete in $(\mathcal{A}_{\alpha+1} \cap \check{\mathcal{A}}_{\alpha+1}) \setminus (\mathcal{A}_\alpha \cup \check{\mathcal{A}}_\alpha)$ and DA-complete in $(\mathcal{R}_{\alpha+1} \cap \check{\mathcal{R}}_{\alpha+1}) \setminus (\mathcal{R}_\alpha \cup \check{\mathcal{R}}_\alpha)$.

Proof. (i) Since for the hierarchies $\{\mathcal{S}_\alpha\}$ and $\{\mathcal{R}_\alpha\}$ the analogs of Theorem 6 hold [22], the proof of Theorem 8 applies uniformly to the both reducibilities.

(ii) The assertion follows from (i), Theorem 8, Theorem 5(iv) and the similar fact on the Wagner hierarchy. □

The non-collapse of the fine hierarchy from the previous section is also an easy corollary.

Corollary 9 *The hierarchy $\{\mathcal{A}_\alpha\}$ does not collapse, i.e. $\mathcal{A}_\alpha \not\subseteq \check{\mathcal{A}}_\alpha$ for all $\alpha < \omega^\omega$.*

Proof. Suppose the contrary: $\mathcal{A}_\alpha \subseteq \check{\mathcal{A}}_\alpha$ for some $\alpha < \omega^\omega$. Then $A_\alpha \in \check{\mathcal{A}}_\alpha \subseteq \check{\mathcal{S}}_\alpha$. Since $\check{\mathcal{S}}_\alpha$ is closed under CA-reducibility, $\mathcal{S}_\alpha \subseteq \check{\mathcal{S}}_\alpha$. This contradicts to a well-known property of the Wadge hierarchy (see e.g. [22]). □

The relation of the hierarchy $\{\mathcal{A}_\alpha\}$ to the AA-reducibility is even tighter than Theorem 8 and Corollary 8 suggest.

Theorem 9 *For every $\alpha < \omega^\omega$, $(\mathcal{A}_{\alpha+1} \cap \check{\mathcal{A}}_{\alpha+1}) \setminus (\mathcal{A}_\alpha \cup \check{\mathcal{A}}_\alpha) = \{C \mid C \equiv_{AA} A_\alpha \oplus \bar{A}_\alpha\}$ and $\mathcal{A}_\alpha \setminus \check{\mathcal{A}}_\alpha = \{C \mid C \equiv_{AA} A_\alpha\}$.*

Proof. We have seen that $A_\alpha \in \mathcal{A}_\alpha \setminus \check{\mathcal{A}}_\alpha$, so for proving the first equality it remains to show that $C \equiv_{AA} A_\alpha$ for any $C \in \mathcal{A}_\alpha \setminus \check{\mathcal{A}}_\alpha$. The reduction $C \leq_{AA} A_\alpha$ follows from Theorem 8. It remains to show that $A_\alpha \leq_{AA} C$. Suppose the contrary, then $A_\alpha \not\leq_{AS} C$. By Corollary 3(iii), $C \leq_{AS} \overline{A}_\alpha$, hence $C \leq_{AA} \overline{A}_\alpha$ and $C \in \check{\mathcal{A}}_\alpha$. A contradiction.

The second equality is checked in the same way. □

Let us summarize some facts on the AA-degrees of regular aperiodic sets established above. The last assertion (v) in the next corollary refines Theorem 32 in [10].

Corollary 10 *(i) For any $\alpha < \omega^\omega$, $A_\alpha \not\leq_{AA} \overline{A}_\alpha$.*

(ii) For all $\alpha < \beta < \omega^\omega$, $A_\alpha \oplus \overline{A}_\alpha <_{AA} A_\beta$.

(iii) Any regular aperiodic ω-language is AA-equivalent to exactly one of the sets A_α, $\overline{A}_\alpha$, $A_\alpha \oplus \overline{A}_\alpha$ $(\alpha < \omega^\omega)$.

(iv) The relations $\leq_{AA}$, $\leq_{DA}$ and $\leq_{CA}$ coincide on $\mathcal{A}$.

(v) For every $R \in \mathcal{R}$ there is $A \in \mathcal{A}$ with $R \equiv_{DA} A$ (and hence $R \equiv_{CA} A$). In particular, the Wadge degrees of regular aperiodic sets coincide with the Wadge degrees of regular sets.

The next corollary states interesting relationships between the hierarchies $\{\mathcal{A}_\alpha\}$ and $\{\mathcal{S}_\alpha\}$ parallel to the relationships between hierarchies $\{\mathcal{R}_\alpha\}$ and $\{\mathcal{S}_\alpha\}$ established in [22].

Corollary 11 *(i) For any $\alpha < \omega^\omega$, $\mathcal{A}_\alpha = \mathcal{A} \cap \mathcal{S}_\alpha = \mathcal{A} \cap \mathcal{R}_\alpha$.*

(ii) Let $t = t(x_1, \ldots, x_n, y_1, \ldots, y_n)$ be a typed Boolean term where x_i are variables of type 0 and y_i are variables of type 1. Let A be a regular aperiodic set such that $A = t(B_1, \ldots, B_n, C_1, \ldots, C_n)$ for some $B_1, \ldots, B_n \in \mathbf{\Sigma}^0_1$ and $C_1, \ldots, C_n \in \mathbf{\Sigma}^0_2$. Then $A = t(B'_1, \ldots, B'_n, C'_1, \ldots, C'_n)$ for some $B'_1, \ldots, B'_n \in \mathcal{K}_0$ and $C'_1, \ldots, C'_n \in \mathcal{K}_1$.

From the last result and the well known facts that the classes $\mathcal{A}$ and $\mathcal{R}_\alpha$ (for every $\alpha < \omega^\omega$) are decidable (i.e. from a given Muller automaton one can effectively determine whether the corresponding language belongs to the class) we immediately obtain

Corollary 12 *For any $\alpha < \omega^\omega$, the class $\mathcal{A}_\alpha$ is decidable.*

We conclude this section by a characterization of the structure $(\mathcal{A}; \leq_{AS})$ similar to the characterization of the structure $(\mathcal{R}; \leq_{DS})$ in [22, 35]. To this end, we define for all $\alpha < \omega^\omega$ and $n < \omega$ the regular aperiodic set

$$A^n_\alpha = 0^{n+1} \cdot A_\alpha \cup (\bigcup \{u \cdot \overline{A}_\alpha \mid u \in X^{k+1}, u \neq 0^{n+1}\}).$$

Theorem 10 *Let $\alpha < \omega^\omega$, $n < \omega$ and $C \in \mathcal{A}$.*

(i) $C \leq_{AA} A_\alpha$ iff $C \leq_{AS} A_\alpha$.

(ii) $\mathcal{A}_\alpha \setminus \check{\mathcal{A}}_\alpha = \{C : C \equiv_{AS} A_\alpha\}$.

(iii) $A^n_\alpha \equiv_{AA} A_\alpha \oplus \bar{A}_\alpha$, $A^n_\alpha <_{AS} A^{n+1}_\alpha$ and $A^n_\alpha \equiv_{AS} \overline{A}^n_\alpha$.

(iv) $C_\alpha \equiv_{AA} A_\alpha \oplus \overline{A}_\alpha$ iff $C_\alpha \equiv_{AS} A^k_\alpha$ for a unique $k < \omega$.

(v) The analogs of (i)—(iv) hold for $\mathcal{R}$ in place of $\mathcal{A}$ and DS-reducibility in place of AS-reducibility.

Proof. (i) follows from Corollary 10, Corollary 3(ii) and the well known property of the Wadge and Lipschitz reducibilities that $C \leq_{CA} A_\alpha$ iff $C \leq_{CS} A_\alpha$.

(ii) follows from (i) and Theorem 9.

(iii) is proved as the corresponding assertion in Theorem 9.1 of [22] for the DS-reducibility.

(iv) follows from (iii).

(v) follows from Corollary 8(i) and Theorem 9.1 of [22]. □

Let us summarize some facts on the AS-degrees of regular aperiodic sets established above.

Corollary 13 *(i) For all $\alpha < \beta < \omega^\omega$ and $n < \omega$, $A_\alpha, \overline{A}_\alpha <_{AS} A^n_\alpha <_{AS} A^{n+1}_\alpha <_{AS} A_\beta, \overline{A}_\beta$.*

(ii) Any regular aperiodic ω-language is AS-equivalent to exactly one of the sets A_α, $\overline{A}_\alpha$, A^n_α $(\alpha < \omega^\omega, n < \omega)$.

(iii) The relations $\leq_{AS}$, $\leq_{DS}$ and $\leq_{CS}$ coincide on $\mathcal{A}$.

(iv) For every $R \in \mathcal{R}$ there is $A \in \mathcal{A}$ with $R \equiv_{DS} A$ (and hence $R \equiv_{CS} A$). In particular, the Lipschitz degrees of regular aperiodic sets coincide with the Lipschitz degrees of regular sets.

9. Conclusion

In this paper we answered most natural questions on the *topological* fine hierarchy of regular aperiodic ω-languages. Along with the topological classification, there are some alternative classifications of regular aperiodic ω-languages. E.g., one could consider the *logical* fine hierarchy of regular aperiodic ω-languages which is the fine hierarchy over the base $\{L_n\}$ where L_n is the class of regular aperiodic ω-languages defined by the first-order Σ^0_{n+1}-sentences of the signature σ_X as explained in Section 2. Some related classifications of the regular aperiodic languages of finite words were discussed in [23]. We plan to consider relations of the topological fine hierarchy of regular aperiodic ω-languages to some alternative classifications in a separate paper.

There are some other natural open questions related to the results of this paper. E.g., consider the structures

$$(B(\mathbf{\Sigma}^0_2); \mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_1, \cup, \cap, ^-),\ (\mathcal{R}; \mathcal{L}_1, \mathcal{L}_0, \cup, \cap, ^-) \text{ and } (\mathcal{A}; \mathcal{K}_1, \mathcal{K}_0, \cup, \cap, ^-)$$

where $\mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_1, \mathcal{L}_1, \ldots$ are treated as unary predicates on the corresponding universes. The results of this paper and of [22] (in particular, Corollary 11) show some striking similarities between the three structures which suggest that all the structures may turn out to be elementary equivalent. It may even turn out that each of the two last structures is an elementary substructure of the previous one. The author does not currently know whether this is really the case.

Another natural open question is to understand the structures $(\mathcal{R}; \leq_{AA})$ and $(\mathcal{R}; \leq_{AS})$. These structures may turn out to be more complicated than the structures discussed in the previous section. In particular, we do not know whether these structures are almost well ordered, or whether some of them is computable.

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