INVERSES OF MATRICES AND MATRIX-TRANSFORMATIONS

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Let $A = (a_{nk})$, $n, k = 1, 2, \cdots$, be a matrix of complex numbers. Let D be the set (linear sequence space) of sequences $x = \{x_n\}$ such that $y = Ax$ is defined; y being the sequence $\{y_n\}$, where $y_n = \sum_k a_{nk}x_k$ for each n. Let R be the set of all Ax, $x \in D$. We call D and R the domain and range of A. They are linear subspaces of (s), the space of all sequences.

To emphasize the distinction between inverse matrix and inverse transformation, we denote Ax by $T(x)$, thus defining $T: D \to R$, and investigate, under various hypotheses:

(a) the existence of right, left, and two-sided inverses for A, denoted by A', $'A$, A^{-1},

(b) the same for T, denoted by T', $'T$, T^{-1},

(c) connections between (a) and (b).

By A' we mean any matrix satisfying $AA' = I$, the identity matrix. By T' we mean any function $T': R \to D$ satisfying $T(T'(x)) = x$ for all $x \in R$. The other symbols are interpreted similarly. By "T' exists" we mean "there exists at least one T'." Similarly for the others.

Our main results concern *row-finite* matrices, i.e. such that almost all the elements in each row are zero; *column-finite* matrices, i.e. matrices whose transpose is row-finite; and *reversible* matrices, i.e. matrices A such that for each convergent sequence y, the equation $y = Ax$ has a unique solution (we shall see that if A is row-finite, reversibility is equivalent to the existence of a unique solution for all y). A discussion is given of the constants c_n of Banach [1, p. 50] which appear in the inverse transformation of a reversible matrix.

Let E be the (countably infinite-dimensional) set of sequences x such that $x_n = 0$ for almost all n, (c) the set of convergent sequences.

Clearly, $D \supset E$ for all A; A is row-finite if and only if $D = (s)$, column-finite if and only if $Ax \in E$ whenever $x \in E$, reversible if and only if $R \supset (c)$, and T is $1-1$ (i.e. to each $y \in R$ corresponds exactly one $x \in D$; A is 1-1 will mean that the associated T is 1-1).

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THEOREM 1 (INVERSE TRANSFORMATIONS).
(a) *A linear* T' *always exists,*
(b) $'T$ *exists (and is linear) if and only if* T *is* 1-1,
(c) *if* $'T$ *exists it is unique, and* $'T=T'=T^{-1}$,
(d) T' *is unique if and only if* $'T$ *exists.*

For example, (a) is proved by noting that T is 1-1 on the subspace of D complementary to the kernel of T.

THEOREM 2 (INVERSE MATRICES).
(a) A' *exists if and only if* $R \supset E$,
(b) *if* T *is* 1-1, *there exists at most one* A',
(c) *if* T *is not* 1-1, A' *is not unique, if it exists,*
(d) A^{-1} *may exist and not* $'T$, T^{-1} *may exist and not* $'A$. *In the former,* A *may be also row-finite and column-finite, in the latter,* A *may be also reversible.*
(e) *If a row-finite* $'A$ *exists,* $'T$ *must exist. More than one row-finite* $'A$ *may exist.*

For (a), if $R \supset E$, we take for the kth column of A' any solution of $\delta^k = Ax$, where $\delta^k = \{0, 0, \cdots, 0, 1, 0, \cdots\}$, 1 in the kth place.
For (d), consider the following examples.
EXAMPLE 1.

$$\left\| \begin{matrix} 1 & -1 & 0 & 0 & 0 \cdots \\ 0 & 1 & -1 & 0 & 0 \cdots \\ 0 & 0 & 1 & -1 & 0 \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \cdots \end{matrix} \right\|.$$

This matrix has the two-sided inverse given in Example 5, but it is not 1-1 since it transforms $\{1, 1, 1, \cdots\}$ to 0. Other examples of this type occur in Wilansky [**2**, p. 391] and (a particularly interesting one) in Agnew [**3**, p. 555].
EXAMPLE 2.

$$\left\| \begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 \cdots \\ 1 & 0 & 1 & 1 & 1 & 1 \cdots \\ 1 & 0 & 0 & 1 & 1 & 1 \cdots \\ 1 & 0 & 0 & 0 & 1 & 1 \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \cdots \end{matrix} \right\|.$$

This matrix has no left inverse since the first row $\{b_1, b_2, \cdots\}$ of such a matrix would have to satisfy $\sum b_i = 1$, $0 = b_1 = b_1 + b_2 = b_1 + b_2$

$+b_3 = \cdots$. On the other hand A is reversible. It is 1-1 since $Ax=0$ implies $0 = \sum x_i = \sum x_i - x_2 = \sum x_i - x_2 - x_3 = \cdots$, hence $x=0$; also $R \supset (c)$, for let $y \in (c)$ and $x_1 = \lim y_n$, $x_n = y_{n-1} - y_n$ for $n = 2, 3, \cdots$, then $y = Ax$.

For the second part of (e) we consider:

EXAMPLE 3.

$$\begin{Vmatrix} 0 & 0 & 0 \cdots \\ 1 & 0 & 0 \cdots \\ 0 & 1 & 0 \cdots \\ 0 & 0 & 1 \cdots \\ \cdot & \cdot & \cdot \cdots \end{Vmatrix}.$$

This has

$$\begin{Vmatrix} a & 1 & 0 & 0 \cdots \\ 0 & 0 & 1 & 0 \cdots \\ 0 & 0 & 0 & 1 \cdots \end{Vmatrix}$$

as left inverse, for arbitrary a.

THEOREM 3 (ROW-FINITE MATRICES). *Let A be row-finite. Then*

(a) *A' exists if and only if $R = (s)$,*

(b) *A is reversible if and only if it is* 1-1 *and $R = (s)$,*

(c) *if A' exists, a row-finite A' exists.*

However,

(d) *there exists a row-finite, column-finite matrix B such that B^{-1} exists, but no row-finite $'B$.*

The interest of (d) lies in the fact that the row-finite B' given by (c) is different from B^{-1}.

Assume that A' exists. Then $R \supset E$. Let (s) be given the linear metric $|x| = \sum 2^{-n} |x_n|/(1+|x_n|)$. A theorem of Toeplitz asserts that R is closed in (s) when A is row-finite. We give a proof of this theorem in an appendix at the end of this paper. Since E is dense in (s), we conclude that $R = (s)$.

We shall show that a row-finite A' exists, assuming $R = (s)$; this will complete the proof of (a) and (c). Assume first that T is 1-1. Then T is a linear homeomorphism of (s) onto itself, hence (Banach [1, Theorem 5, p. 41]) so also is T^{-1}, and so it is given by a row-finite matrix. This matrix is A^{-1}, as a computation shows.

If T is not necessarily 1-1; since A' exists, by Theorem 2(a) the rows of A are linearly independent elements of E. There exists a basis, necessarily countable, for E, which includes the rows of A. We form a matrix B whose rows are the elements of this basis and whose odd-numbered rows are the rows of A. Then B is 1-1 and B' exists, by Toeplitz's theorem (see the appendix). Hence, by the above argument B' is row-finite. Finally, by omitting the even-numbered columns of B' we have a row-finite A'.

Part (b) is now clear. Part (d) is given by Example 1, the two-sided inverse being unique as a left inverse.

The next result shows that the hypothesis "row-finite" cannot be dropped.

THEOREM 4 (COMPLEMENT TO THEOREM 3).

(a) *There exists a matrix A for which A^{-1} exists, but no row-finite A'.*

(b) *There exists a matrix A (for which a row-finite, column-finite A^{-1} exists) such that $E \subset R \neq (s)$, and R is not closed in (s).*

For (a), consider

EXAMPLE 4.

$$\begin{Vmatrix} 1 & 1 & 1 & 1 & 1 \cdots \\ 0 & 1 & 0 & 0 & 0 \cdots \\ 0 & 0 & 1 & 0 & 0 \cdots \\ 0 & 0 & 0 & 1 & 0 \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \quad \cdot \end{Vmatrix}.$$

This matrix is easily seen to be 1-1, hence has at most one A'. However, A' can be explicitly calculated and seen to be A^{-1} and not row-finite.

For (b), Example 4 would do, except for the part in parentheses.

EXAMPLE 5.

$$\begin{Vmatrix} 1 & 1 & 1 & 1 & 1 \cdots \\ 0 & 1 & 1 & 1 & 1 \cdots \\ 0 & 0 & 1 & 1 & 1 \cdots \\ 0 & 0 & 0 & 1 & 1 \cdots \end{Vmatrix}.$$

If $y = Ax$, we have $x_n = y_n - y_{n+1}$, $n = 1, 2, \cdots$. Thus $\{n\} \notin R$, since $y_n = n$ would imply that $x_n = -1$, for which Ax does not exist. But consideration of sequences of the type $\{0, 0, \cdots, 0, -1, 1, 0, 0, \cdots\}$ in D shows that $R \supset E$. Thus R is not closed in (s).

THEOREM 5 (COLUMN-FINITE MATRICES). *Let A be column-finite. Then if T is* 1-1 *and $R \supset E$, A^{-1} exists.*

Theorem 2, part (d) shows that the hypothesis "column-finite" cannot be dropped, even if A is assumed reversible. Example 1 shows that the converse is false.

To prove the theorem, we have, by Theorem 1, a linear T^{-1}. Since $R\supset E$, there exists a matrix B such that $Bx=T^{-1}(x)$ for $x\in E$. Since A is column-finite, $Ax\in E$ if $x\in E$, hence we have, for $x\in E$, $B(Ax)=x$, and so $BA=I$. Also, for $x\in E$, $A(Bx)=T(T^{-1}(x))=x$, and so $AB=I$. This concludes the proof.

For the remainder of this note, let A be reversible. Then, as shown in Banach [1, p. 50], $y=Ax$ has, for $y\in(c)$, the solution

$$x_n = c_n \lim y + \sum_k b_{nk}y_k, \tag{1}$$

with $\sum_k |b_{nk}| < \infty$.

Setting $y=\delta^k$, $k=1, 2, \cdots$, yields the fact that $B=A'$. Here of course A' is unique since T is 1-1. Example 2 shows that $'A$ need not exist. MacPhail [5] has shown that the sequence $\{c_n\}$ may be unbounded, even if A is conservative, i.e. if $Ax\in(c)$ whenever $x\in(c)$. Wilansky [2] has shown that $c_n=0$ for all n if A is row-finite.

Suppose that A has convergent columns, i.e. let $a_k=\lim_n a_{nk}$ be assumed to exist for each k. For example, a conservative matrix has this property, while a regular matrix has $a_k=0$ for all k. For all x such that $Ax\in(c)$ we have, from (1),

$$x_n = c_n \lim_m \sum_{k=1}^{\infty} a_{mk}x_k + \sum_{k=1}^{\infty} b_{nk} \sum_{r=1}^{\infty} a_{kr}x_r.$$

Our assumption that A has convergent columns implies that this identity holds, in particular, for $x=\delta^k$, $k=1, 2, \cdots$, and it then reads $I=D+(A')A$, where $D=(c_na_k)$. We have proved:

THEOREM 6 (REVERSIBLE MATRICES). *Let A be reversible, and with column limits a_k. Then A' satisfies $(A')A=I-D$, where $D=(c_na_k)$, the c_n being defined by* (1).

If A is regular and reversible, this theorem shows that A^{-1} exists; we shall prove more than this, however, namely, that $c_n=0$ for all n, and under wider hypotheses.

Let us call A *co-regular* if the number $\rho(A)=\lim_n \sum_k a_{nk}-\sum a_k$ exists and is not 0. The role of this number in the theory of summability has been shown elsewhere by the authors. A regular matrix is co-regular.

THEOREM 7. *Let A be reversible, co-regular. Then $c_n=0$ for all n, A^{-1} exists and is the matrix of T^{-1}.*

Let (A) be the set of sequences x such that $Ax \in (c)$, let $\|x\| = \sup_n \left| \sum_k a_{nk}x_k \right|$ for $x \in (A)$. Then, clearly, (A) is a Banach space, since the mapping T from (A) to (c) is an equivalence, (c) having its usual norm, $\sup_n |x_n|$. From (1), we have $|x_n| \leqq (|c_n| + \sum_k |b_{nk}|) \cdot \|x\|$, hence x_n is, for each n, a continuous linear functional on (A). Using the known form of the most general continuous linear functional on (c) we have, for such a functional f on (A), $f(x) = tA(x) + \sum g_r A_r(x)$, where $A_r(x) = \sum_k a_{rk}x_k$, $A(x) = \lim_n A_n(x)$, $\sum |g_r| < \infty$. Let $\dot{1}$ denote the sequence $\{1, 1, 1, \cdots\}$. A straightforward computation yields $\rho(f) \equiv f(\dot{1}) - \sum f(\delta^k) = t\rho(A)$. Now, for any n, the functional f given by $f(x) = x_n$ has $\rho(f) = 0$. By hypothesis, $\rho(A) \neq 0$, hence $t = 0$. We have proved that $x_n = \sum_{r=1}^{\infty} g_{rn}A_r(x)$ for all $x \in (A)$. Comparing this with (1) yields $c_n \lim y + \sum_k b_{nk}y_k = \sum_k g_{nk}y_k$ for all $y \in (c)$, for each n. Hence $c_n = 0$ for each n.

REMARKS. 1. By introducing a weaker linear topology for (A), more information about the c_n is available.

2. That $B = 'A$, it is not sufficient that $B(Ax) = x$ for all $x \in (A)$. For example if $Ax = \{x_1, 0, x_1, x_2, 0, x_1, x_2, x_3, 0, \cdots\}$ one sees that (A) contains only the zero sequence.

Appendix. Toeplitz's theorem. The following result due to Toeplitz [**6**] is quoted (incorrectly) in Banach [**1**, p. 51, Theorem 12]. (See Zeller [**4**, p. 47].) We give a short proof.

THEOREM. *Let A be row-finite. Then $y \in R$ if and only if $\sum_{i=1}^{r} h_i y_i = 0$ whenever $h_1, h_2, \cdots, h_r$ is a set of numbers such that $\sum_{i=1}^{r} h_i a_{ik} = 0$ for $k = 1, 2, \cdots$.*

Necessity is trivial. Now assume that y satisfies the stated conditions.

Let a^i denote the ith row of A, so that $a^i \in E$. By hypothesis, $\sum h_i y_i = 0$ whenever $h_1, h_2, \cdots, h_r$ satisfy $\sum h_i a^i = 0$. Let $f(a^i) = y_i$ for $i = 1, 2, \cdots$. Then f can be extended so as to be a linear functional defined on all of E; for it can first be extended to the linear extension of $\{a^i\}$ by $f(\sum t_i a^i) = \sum t_i y_i$, and thence to all of E, for example, by means of a Hamel basis. Let $x_i = f(\delta^i)$ define x. Then clearly $y = Ax$, hence $y \in R$.

COROLLARY. *Let A be row-finite. Then R is closed in (s).*

Let H be the set of continuous linear functionals f on (s) such that $f(a^i) = 0$, $i = 1, 2, \cdots$. Then, by the theorem, R is the set of points at which every $f \in H$ vanishes, i.e. R is the intersection of a collection of closed sets.

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DIFFERENTIAL IDEALS[1]

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0. **Introduction.** In this paper we investigate the membership of power products in certain differential ideals. The questions examined were motivated by results by Levi,[2] which we use extensively. Levi has obtained for $[y^p]$ and $[uv]$ sufficiency conditions for membership of a pp. in the ideal, which tests membership, in certain cases, by a calculation using only the weight and degree of the pp. In Theorem IV we show that a more refined criteria is required for the determination of membership of a pp. in $[y^p]$. Whether a necessity criteria for membership of a pp. in $[uv]$ will require more information than the weight and degree of the pp. is not known.

Levi has also shown that the totality of pp. in u and v are divided by a single calculation into three nonempty sets: the α-terms, which are outside the ideal $[uv]$, another set all of whose members are in the ideal, and a third set concerning whose elements membership in the ideal is undecided. The number of elements known to be outside the ideal is increased by Theorem II, and a dependence of the set of elements whose membership in $[uv]$ is undecided upon one of its proper subsets is demonstrated in Theorem III. Carrying out the reduction

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[1] The nomenclature and notation are as in J. F. Ritt's *Differential algebra*, Amer. Math. Soc. Colloquium Publications, vol. 33, New York, 1950.

[2] H. Levi, *On the structure of differential polynomials and on their theory of ideals*, Trans. Amer. Math. Soc. vol. 51 (1942) pp. 532–568.