Least and Greatest Fixpoints in Game Semantics

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Plan of the talk

I. (Co)Inductive Types

Induction / Coinduction

Languages with syntactic definitions for induction/coinduction:

- Dependent Type Theories (Paulin & Pfenning, 1990);
- λ-calculus (Abel & Altenkirch, 1991);
- MALL (Baelde & Miller, 2007)
- ...

Motivations:

- Model usual induction in mathematics;
- Increase the expressive power of a logic/total language while escaping:
  - Impredicative reasoning,
  - Exponential modalities

Inductive objects and initial algebras

- Consider an endofunctor T (often polynomial, like $T(X) = 1 + X$).
- An **algebra** of T is a pair (A, f) with $f : T(A) \to A$.
- An **initial algebra** is an algebra $(\mu T, i)$ such that :

$$\begin{tikzcd}
T(\mu T) \arrow[r, dotted, "T(f^\dagger)"] \arrow[d, "i"'] \arrow[dr, phantom, "="] & T(A) \arrow[d, "f"] \\
\mu T \arrow[r, dotted, "!"', "f^\dagger"'] & A
\end{tikzcd}$$

- Dually, coinductive objects relate to terminal coalgebras.

μLJ

- Formulas are built by the following grammar:

$$S, T \ ::= \ S \Rightarrow T \mid S + T \mid S \times T \mid \mu X.T \mid \nu X.T \mid X \mid 1 \mid 0$$

  We restrict to **closed** formulas.

- Derivation rules are those of LJ plus:

$$\frac{\Gamma \vdash T[\mu X.T/X]}{\Gamma \vdash \mu X.T}\,\mu_r \qquad \frac{T[A/X] \vdash A}{\mu X.T \vdash A}\,\mu_l$$

$$\frac{T[\nu X.T/X] \vdash B}{\nu X.T \vdash B}\,\nu_l \qquad \frac{A \vdash T[A/X]}{A \vdash \nu X.T}\,\nu_r$$

Cut "elimination"

$$
\dfrac{
  \dfrac{\dfrac{\pi_1}{\Gamma \vdash T[\mu X.T/X]}}{\Gamma \vdash \mu X.T}\,\mu_r
  \qquad
  \dfrac{\dfrac{\pi_2}{T[A/X] \vdash A}}{\mu X.T \vdash A}\,\mu_l
}{\Gamma \vdash A}\,Cut
\quad \rightsquigarrow \quad
\dfrac{
  \dfrac{
    \dfrac{\pi_1}{\Gamma \vdash T[\mu X.T/X]}
    \qquad
    \dfrac{\dfrac{\dfrac{\pi_2}{T[A/X] \vdash A}}{\mu X.T \vdash A}\,\mu_l}{T[\mu X.T/X] \vdash T[A/X]}\,[T]
  }{\Gamma \vdash T[A/X]}\,Cut
  \qquad
  \dfrac{\pi_2}{T[A/X] \vdash A}
}{\Gamma \vdash A}\,Cut
$$

- We add rules $[T]$ for functors and reductions for their unfoldings.
- Rules for ν are dual.
- This is just a 2-cell in the diagram of initial algebra !

Translation of Gödel's System T — easy part

$$\mathtt{nat}^\star = \mu X.1 + X \qquad 0^\star = \dfrac{\dfrac{\dfrac{}{\vdash 1}\, 1_r}{\vdash 1 + \mathtt{nat}}\, \overleftarrow{+}_r}{\vdash \mathtt{nat}}\, \mu_r$$

$$\mathtt{S}^\star = \dfrac{\dfrac{\dfrac{}{\mathtt{nat} \vdash \mathtt{nat}}\, ax}{\mathtt{nat} \vdash 1 + \mathtt{nat}}\, \overrightarrow{+}_r}{\mathtt{nat} \vdash \mathtt{nat}}\, \mu_r$$

Translation of Gödel's system T — less easy part

$$\mathtt{rec}^\star = \dfrac{\dfrac{\dfrac{}{T \vdash T}\,ax}{1, T \Rightarrow T, T \vdash T}\,W \qquad \dfrac{\dfrac{\dfrac{}{T \vdash T}\,ax \qquad \dfrac{}{T \vdash T}\,ax}{T, T \Rightarrow T \quad \vdash T}\Rightarrow_I}{T, T \Rightarrow T, T \vdash T}\,W}{\dfrac{1 + T, T \Rightarrow T, T \vdash T}{\mathtt{nat}, T \Rightarrow T, T \vdash T}\,\mu_I}\,+_I$$

This translation satisfies that if $M_1 \rightsquigarrow^T M_2$, then $M_1^\star \rightsquigarrow^* M_2^\star$.

Notes on μLJ

- Rules for μ/ν are not relative to a context, but it is definable.
- Some close cousins: $\mu MALL$(Baelde & Miller), the λ^{μ}-calculus (Abel & Altenkirch),...
- Normalization/Consistency:
  - Translation in second-order LL for $\mu MALL$,
  - Predicative normalization proof for λ^{μ} (with reducibility candidates – for *strictly positive* inductive types),
  - No cut elimination in μLJ !
  - A variant (like $\mu MALL$) eliminates cuts, but loses subformula property...

Our game semantics will provide a proof of consistency (for what it is worth).

II. Game Semantics

Game semantics

Game semantics is the study of the interactive behaviour of a **program** (or proof) against its **environment** :

- A formula A is interpreted by a **game**
- A proof $\pi : A \Rightarrow B$ is interpreted as a **strategy**

Such that these data are compositional.

Interpretation of Logic with Games.

- An old idea (Lorenzen, 1960)
- A Semantics of Proofs and Refutation
- Lots of achievements: MLL, System F, LL, LK, LLP, ...
- Parity games for terms in the theory of μ-**lattices** (Santocanale, 2002) (data types, no functions)
- Still no such **interactive** model of languages with (co)inductive types.
- Our setting will be the Hyland-Ong-Nickau setting of Game Semantics.

Arenas

Arenas are the semantic counterparts of **formulas** or **data types**.

Definition (Arena)

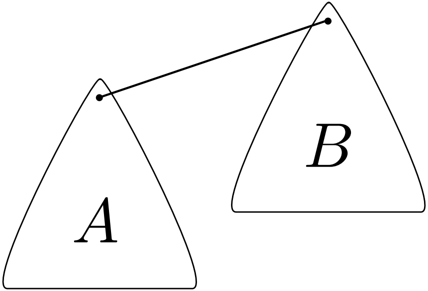
An arena A is a **tree***.*

- *The nodes of A are called* **moves***, denoted by M_A*
- *The existence of an edge $m \rightarrow n$ in A is called* **enabling** *denoted $m \vdash_A n$.*
- *A move is* **Player** *(P) if its depth is odd,* **Opponent** *(O) otherwise. This polarity function is denoted by λ_A.*

$$\mathbb{B} = \begin{tikzcd} & q \arrow[dl, no head, bend right] \arrow[dr, no head, bend left] & \\ \mathtt{tt} & & \mathtt{ff} \end{tikzcd}$$

Arrow construction.

$$A \Rightarrow B$$



Legal plays, strategies.

Definition (Legal plays)

Plays are sequences of **justified moves***:*

- *Each move, if not initial (i.e. the root), points to an earlier move*
- *These pointers are required to comply with* $\vdash_A$
- *Plays are* **alternating***.*

Legal plays on A *are denoted by* $\mathcal{L}_A$*.*

Definition (Strategy)

A **strategy** *on* A *is a subset of* $\mathcal{L}_A$*, which is:*

- **prefix-closed**,
- **deterministic** *(i.e. only Opponent branches)*

Composition

Definition (Composition)

If $\sigma : A \Rightarrow B$ and $\tau : B \Rightarrow C$, we define:

$$\sigma;\tau = \{u_{\restriction A,C} \mid u_{\restriction A,B} \in \sigma \wedge u_{\restriction B,C} \in \tau\}$$

We have $\sigma;\tau : A \Rightarrow C$.

Theorem

The following data defines a category:

- **objects** A**:** *arenas A,*
- **morphisms** $A \rightarrow B$**:** *strategies $\sigma : A \Rightarrow B$.*
- **identity** $A \rightarrow A$**:** *copycat strategy on $A \Rightarrow A$.*

The cartesian closed category of Innocent strategies.

- **Innocent strategies** (not defined here) are blind to the Opponent's duplications.
- **Products** are handled by generalizing tree-arenas to **forests**.
- The former category is thus refined into the CCC of arenas and innocent strategies.

Interpretation of LJ.

- A sequent $A_1, \ldots, A_n \vdash B$ is interpreted by an arena $[\![A_1]\!] \times \cdots \times [\![A_n]\!] \Rightarrow [\![B]\!]$;
- Identity group:

$$\left[\!\!\left[\frac{}{A \vdash A}\, ax\right]\!\!\right] = id_A \qquad \left[\!\!\left[\frac{\dfrac{\pi_1}{\Gamma \vdash A} \qquad \dfrac{\pi_2}{\Delta, A \vdash B}}{\Delta, \Gamma \vdash B}\, Cut\right]\!\!\right] = id_\Delta \times [\![\pi_1]\!]; [\![\pi_2]\!]$$

- Structural group is interpreted by copycat strategies,
- Logical group is interpreted with the cartesian closed structure, and a weak sum.

III. Recursive Types and Loops

Recursive Types à la McCusker

We define an ordering on arenas $\trianglelefteq$ by $A \trianglelefteq B$ iff

$$\begin{array}{rcl} M_A & \subseteq & M_B \\ \lambda_A & = & \lambda_{B\restriction_{M_A}} \\ \vdash_A & = & \vdash_B \cap (M_A \times M_A) \end{array}$$

defining a dcpo of arenas.

Recursive Types à la McCusker

If $F : \mathcal{I} \to \mathcal{I}$ is an endofunctor which is continous for $\trianglelefteq$, we define:

$$D = \bigsqcup_{n=0}^{\infty} F^n(I)$$

to get a solution $D = F(D)$.

This generalizes to functors of mixed variance.

Recursive Types à la McCusker

A functor F is **closed** if its action can be internalized as a map:

$$(A \Rightarrow B) \to (FA \Rightarrow FB)$$

Theorem (McCusker, 1995)

If F is closed and preserves injection and projection morphisms (corresponding to $\trianglelefteq$), then $D = \bigsqcup_{n=0}^{\infty} F^n(I)$ defines a minimal invariant (Freyd, 1990) for F.

We will now revisit McCusker's work in terms of **loops** in **open arenas**.

Open arenas.

Let T be a countable set of **names**.

Definition

An **open arena** *is an arena A with distinguished moves called* **holes**.

- *Each hole is labelled with an element of T;*
- *A hole with name X will be denoted by $\Box_X$; (or $\Box_X^P$ or $\Box_X^O$, depending on its polarity).*
- *There can be distinct holes in A sharing the same name;*
- *An arena A with holes labelled by $X_1, \ldots, X_n$ will be denoted by $A[X_1, \ldots, X_n]$.*

From $A[X_1, \ldots, X_n]$, we will define the **open functor**

$$A[X_1, \ldots, X_n] : (\mathcal{I}^{op} \times \mathcal{I})^n \to \mathcal{I}$$

Open arenas.

$$\square_X \Rightarrow \square_X \qquad\qquad 1 + \square_X$$

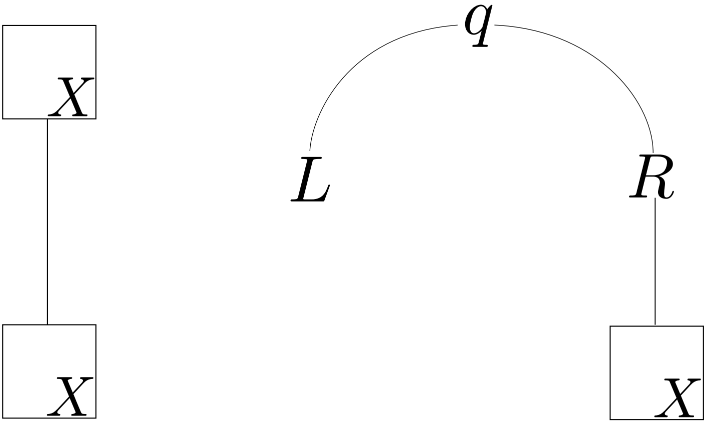


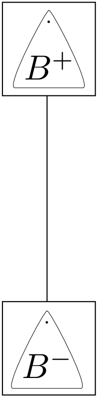
Image of arenas.

If $A[X]$ is an open arena, and B^-, B^+ are arenas:

$$\begin{array}{rcl}
M_{A(B^-,B^+)} & = & (M_A \setminus \{\Box_X\}) + M_{B^-} + M_{B^+} \\
\lambda_{A(B^-,B^+)} & = & [\lambda_A, \overline{\lambda_{B^-}}, \lambda_{B^+}] \\
m \vdash_{A(B^-,B^+)} n & \Leftrightarrow & \left\{ \begin{array}{l}
m \vdash_A \Box_X^P \wedge n \in I_{B^+} \\
m \vdash_A \Box_X^O \wedge n \in I_{B^-} \\
m \in I_{B^+} \wedge \Box_X^P \vdash_A n \\
m \in I_{B^-} \wedge \Box_X^O \vdash_A n \\
m \vdash_{B^+} n \\
m \vdash_{B^-} n \\
m \vdash_A n
\end{array} \right.
\end{array}$$

Image of arenas.

$(\square_X \Rightarrow \square_X)(B^-, B^+)$



$(1 + \square_B)(B)$

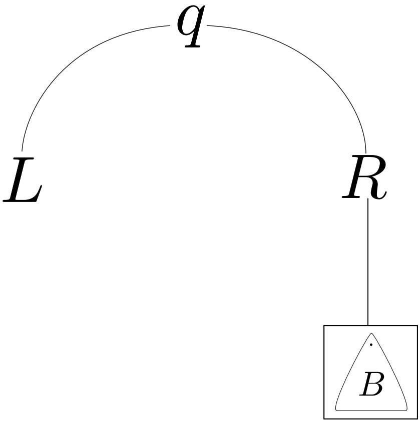
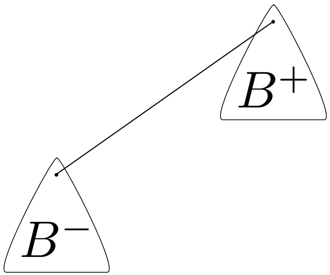


Image of arenas.

$B^- \Rightarrow B^+$



$1 + B$

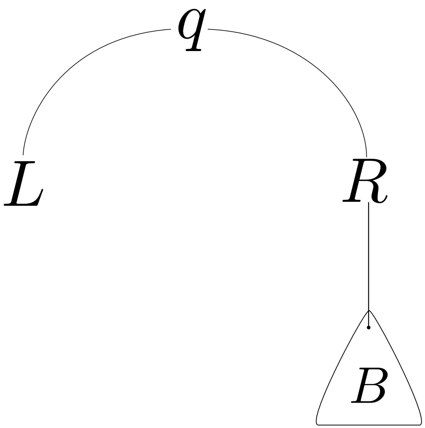


Image of strategies.

$A[X]$ is an open arena. Let $\sigma^+ : B^+ \to C^+$ and $\sigma^- : C^- \to B^-$. We will define:

$$A(\sigma^-, \sigma^+) : A(B^-, B^+) \to A(C^-, C^+)$$

$s \in A(\sigma^-, \sigma^+)$?

- $s_{\restriction B^+, C^+} \in \sigma^+$,
- $s_{\restriction C^-, B^-} \in \sigma^-$,
- s is copycat on $A[X] \Rightarrow A[X]$:
  - Replace each initial move of the Bs and Cs by $\Box_X$,
  - Delete all remaining inner moves of the Bs and Cs,
  - Is the resulting play in $id_{A[X]}$?

Example

$$(\Box_X \Rightarrow \Box_X)(\sigma^-, \sigma^+) = \sigma^- \Rightarrow \sigma^+$$

Open functors.

Theorem

Let $A[X_1, \ldots, X_n]$ be an open arena. Then we have a functor:

$$A[X_1, \ldots, X_n] : (\mathcal{I}^{op} \times \mathcal{I})^n \to \mathcal{I}$$

which is monotone and continuous with respect to $\trianglelefteq$.

Theorem

Open functors have the following additional properties:

- *They are closed (as introduced previously),*
- *They preserve injections and projections.*

Thus McCusker's theorem applies.

Loops.

We generalize arenas to **rooted directed bipartite graphs**. Thus:

- Possibility of loops in arena constructions,
- Changes almost nothing in the construction of $\mathcal{I}$.

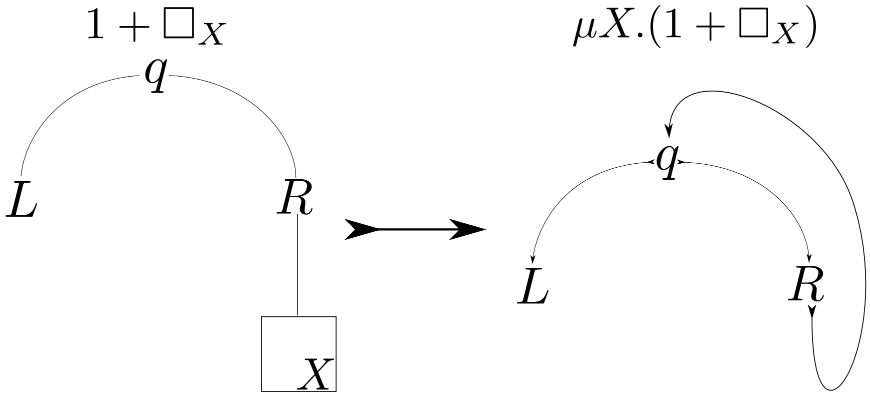
Definition

Suppose $A[X_1, \ldots, X_n]$ is an open arena, such that $\square_{X_i}$ appears only in non-initial, positive position in A. Then we define $\mu X_i.A$:

$$\begin{array}{rcl}
M_{\mu X_i.A} & = & (M_A \setminus \square_{X_i}) \\
\lambda_{\mu X_i.A} & = & \lambda_{A \upharpoonright M_{\mu X_i.A}} \\
m \vdash_{\mu X_i.A} n & \Leftrightarrow & \left\{ \begin{array}{l} m \vdash_A n \\ m \vdash_A \square_{X_i} \wedge n \in I_A \end{array} \right.
\end{array}$$

This generalizes to holes in arbitrary polarities, but the arena is no longer bipartite. . .

Loops.



Loops and recursive types.

Definition

A **path** *isomomorphism between two graphs is a bijection ϕ between the paths in A and the paths in B such that $\phi(ip(s)) = ip(\phi(s))$.*

Theorem (Laurent, 2003)

Two arenas A and B are game-isomorphic if and only if they are path-isomorphic.

Loops and recursive types.

Corollary

If $A[X]$ is an open arena, $\mu X.A[X] \simeq \bigsqcup_{n=0}^{\infty} A^n(I)$.

Corollary

If $A[X]$ is an open arena, $\mu X.A[X]$ is a minimal invariant for $A[X]$.

IV. Least and Greastest Fixed Points

Winning and Totality

Winning functions:

- Method to ensure compositionality of totality,
- First used in Game Semantics by Abramsky&Jagadeesan (1992)
- Strongly related to realizability

Win-games.

Let $\overline{\mathcal{L}_A}$ denote the set of possibly infinite legal plays on A.

Definition

A **win-game** *is a pair* $(A, \mathcal{G}_A)$ *where* A *is an arena and* $\mathcal{G}_A : \overline{\mathcal{L}_A} \to \{W, L\}$ *is a winning function such that:*

- $\mathcal{G}_A(s) = W \Leftrightarrow$ *for all thread* t *of* s, $\mathcal{G}_A(t) = W$,
- *If* s *is finite,* $\mathcal{G}_A(s) = W \Leftrightarrow |s|$ *is even-length.*

Constructions $\times, +, \Rightarrow$ extends to win-games as follows:

$$\begin{aligned}
\mathcal{G}_{A\times B}(s) = W &\Leftrightarrow \mathcal{G}_A(s_{\restriction A}) = W \wedge \mathcal{G}_B(s_{\restriction B}) = W \\
\mathcal{G}_{A+B}(s) = W &\Leftrightarrow \mathcal{G}_A(s_{\restriction A}) = W \wedge \mathcal{G}_B(s_{\restriction B}) = W \\
\mathcal{G}_{A\Rightarrow B}(s) = W &\Leftrightarrow \mathcal{G}_A(s_{\restriction A}) = W \implies \mathcal{G}_B(s_{\restriction B}) = W
\end{aligned}$$

Win-games.

If $s \in \overline{\mathcal{L}_A}$, we say that $s \in \overline{\sigma}$ if all finite prefixes of s are in σ.

Definition

A strategy $\sigma : A$ is **winning** *if for all even-length or infinite $s \in \overline{\sigma}$, $\mathcal{G}_A(s) = W$.*

Definition

A strategy $\sigma : A$ is **total** *if for all odd-length $sa \in \sigma$, there is b such that $sab \in \sigma$.*

Proposition

There is a CCC $\mathcal{WG}$ of win-games and total winning strategies.

We can now interpret (co)inductive types in $\mathcal{WG}$.

Open win-functor.

Definition

An **open win-functor** *is a functor*

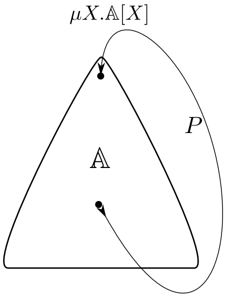
$$\mathbb{A}[X] : \mathcal{WG}^{op} \times \mathcal{WG} \to \mathcal{WG}$$

such that there is an open arena $A[X]$ such that for all win-game $\mathbb{B} = (B, \mathcal{G}_B)$, the base arena of $\mathbb{A}(\mathbb{B})$ has base arena $A(B)$.

And similarly for n-ary open win-functors

$$\mathbb{A}[X_1, \ldots, X_n] : (\mathcal{WG}^{op} \times \mathcal{WG})^n \to \mathcal{WG}$$

Least fixed point.



$\mathcal{G}_{\mu X.\mathbb{A}[X]}(s) = W$ if and only if both these conditions are satisfied :

- There is $N \in \mathbb{N}$ such that no path of s crosses the external more than N times, **and**
- $\mathcal{G}_{\mathbb{A}[X]}(s_{\upharpoonright \mathbb{A}}) = W$

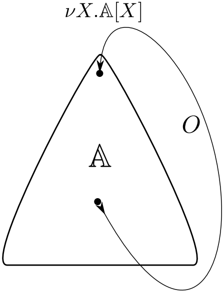
$(\mu X.\mathbb{A}[X], id_{\mu X.\mathbb{A}[X]})$ defines an **initial algebra** for $\mathbb{A}[X]$.

Least fixed point.

$$
\begin{tikzcd}
\mathbb{T}(\mu X.\mathbb{T}) \arrow[r, dotted, "\mathbb{T}(\sigma^\dagger)"] \arrow[d, "i"'] \arrow[dr, phantom, "="] & \mathbb{T}(\mathbb{A}) \arrow[d, "\sigma"] \\
\mu\mathbb{T} \arrow[r, dotted, "\sigma^\dagger"'] & \mathbb{A}
\end{tikzcd}
$$

$$
\begin{aligned}
\sigma^\dagger &= \cdots \xrightarrow{\mathbb{T}^3(\sigma)} \mathbb{T}^3(\mathbb{A}) \xrightarrow{\mathbb{T}^2(\sigma)} \mathbb{T}^2(\mathbb{A}) \xrightarrow{\mathbb{T}(\sigma)} \mathbb{T}(\mathbb{A}) \xrightarrow{\sigma} \mathbb{A} \\
\sigma^{(1)} &= \sigma \\
\sigma^{(n+1)} &= \mathbb{T}^n(\sigma); \sigma^{(n)} \\
\sigma^\dagger &= \{s \in \mathcal{L}_{\mu X.\mathbb{T} \Rightarrow \mathbb{A}} \mid \exists n \in \mathbb{N}^*,\ s \in \sigma^{(n)}\}
\end{aligned}
$$

Greatest fixed point.



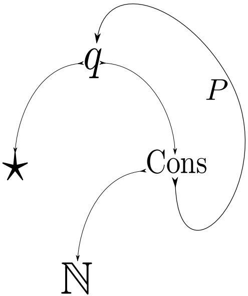
$\mathcal{G}_{\nu X.\mathbb{A}[X]}(s) = W$ if and only if one of these conditions are satisfied :

- For any bound $N \in \mathbb{N}$, there is a path of s crossing the external loop more than N times, **or**
- $\mathcal{G}_{\mathbb{A}[X]}(s_{\upharpoonright \mathbb{A}}) = W$

$(\nu X.\mathbb{A}[X], id_{\nu X.\mathbb{A}[X]})$ defines an **terminal coalgebra** for $\mathbb{A}[X]$.

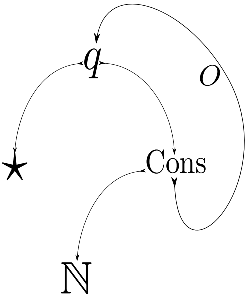
Example.

$$List(A) = \mu X.1 + A \times X$$



$$[1; 2; 3; 4; \ldots; n]$$

$$Stream(A) = \nu X.1 + A \times X$$



$$[1; 2; 3; 4; \ldots]$$

Open fixed points.

These definitions generalizes to *n*-ary open win-functors:

Proposition

If $\mathbb{A}[X_1, \ldots, X_n]$ *is an open win-functor, then*

$$\mu X_i.\mathbb{A}[X_1 \ldots X_{i-1} X_{i+1} \ldots X_n] : (\mathcal{WG}^{op} \times \mathcal{WG})^{n-1} \to \mathcal{WG}$$

is an open win-functor.

Proposition

If $\mathbb{A}[X_1, \ldots, X_n]$ *is an open win-functor, then*

$$\nu X_i.\mathbb{A}[X_1 \ldots X_{i-1} X_{i+1} \ldots X_n] : (\mathcal{WG}^{op} \times \mathcal{WG})^{n-1} \to \mathcal{WG}$$

is an open win-functor.

Interpretation of Formulas.

$$
\begin{aligned}
[\![1]\!] &= \mathbb{I} \\
[\![0]\!] &= \bot \\
[\![A + B]\!] &= [\![A]\!] + [\![B]\!] \\
[\![A \times B]\!] &= [\![A]\!] \times [\![B]\!]
\end{aligned}
\qquad
\begin{aligned}
[\![A \Rightarrow B]\!] &= [\![A]\!] \Rightarrow [\![B]\!] \\
[\![X]\!] &= \square_X \\
[\![\mu X.T]\!] &= \mu X.[\![T]\!] \\
[\![\nu X.T]\!] &= \nu X.[\![T]\!]
\end{aligned}
$$

Interpretation of Proofs.

$$\left[\!\!\left[\dfrac{\dfrac{\pi}{\Gamma \vdash T[\mu X.T/X]}}{\Gamma \vdash \mu X.T}\,\mu_r \right]\!\!\right] = [\![\pi]\!]; i_{[\![T]\!]}$$

$$\left[\!\!\left[\dfrac{\dfrac{\pi}{T[A/X] \vdash A}}{\mu X.T \vdash A}\,\mu_l \right]\!\!\right] = [\![\pi]\!]^{\dagger}$$

$$\left[\!\!\left[\dfrac{\dfrac{\pi}{T[\nu X.T/X] \vdash B}}{\nu X.T \vdash B}\,\nu_l \right]\!\!\right] = i^{-1}_{[\![T]\!]}; [\![\pi]\!]$$

$$\left[\!\!\left[\dfrac{\dfrac{\pi}{A \vdash T[A/X]}}{A \vdash \nu X.T}\,\nu_r \right]\!\!\right] = [\![\pi]\!]^{\ddagger}$$

$$\left[\!\!\left[\dfrac{\dfrac{\pi}{A \vdash B}}{T(A) \vdash T(B)}\,[T] \right]\!\!\right] = [\![T]\!]([\![\pi]\!])$$

A model of μLJ.

Theorem

This gives a sound interpretation of μLJ in $\mathcal{WG}$.

Corollary

μLJ is consistent: there is no proof of 0.

Completeness.

- Infinite winning strategies: defininability does not terminate,
- Find the adequate subclass of recursive winning strategies,
- Intensional Completeness vs. Universality

V. Conclusion

Contributions.

- Open arenas/functors
- Loops for recursive types,
- Winning conditions for least and greatest fixpoints,
- Sound model of μLJ.

Future work.

- Investigate completeness of the system,
- Investigate connections with parity games,
- Model induction/coinduction in dependent type systems,
- Dybjer's Inductive/Recursive definitions ?

Thanks.

QUESTIONS ?