ADVANCES IN MATHEMATICS **129**, 99–157 (1997)
ARTICLE NO. AI971649

Monoidal Bicategories and Hopf Algebroids

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Received January 2, 1997; accepted March 15, 1997

Why are groupoids such special categories? The obvious answer is because all arrows have inverses. Yet this is precisely what is needed mathematically to model symmetry in nature. The relation between the groupoid and the physical object is expressed by an action. The presence of inverses means that actions of a groupoid $\mathscr{G}$ behave much better than actions of an arbitrary category. The totality of actions of $\mathscr{G}$ on vector spaces forms a category $\mathrm{Mod}\,\mathscr{G}$ of modules. The feature of $\mathrm{Mod}\,\mathscr{G}$ which epitomises the fact that $\mathscr{G}$ is a groupoid is that the internal hom in $\mathrm{Mod}\,\mathscr{G}$ is calculated in a particularly simple way. More precisely, the functor out of $\mathrm{Mod}\,\mathscr{G}$ which forgets the actions preserves, not only the monoidal structure but also, the closed structure. With this as a guiding principle, we develop a general concept of "autonomous pseudomonoid" which includes ordinary Hopf algebras (indeed, Hopf algebroids) and autonomous (= compact = rigid) monoidal categories. This is intended to elucidate the interaction between Hopf algebras and autonomous monoidal categories in Tannaka duality as appearing in [JS2; D2], for example.

Given a topological monoid M, it is explained in [JS2, Section 8] why the monoidal category of finite-dimensional representations of M is equivalent to the monoidal category of finite-dimensional *comodules* over the bialgebra $R(M)$ of representative functions on M. This provides evidence that, when regarding a Hopf algebra as a quantum group, it is the finite-dimensional comodules (rather than modules) which should be regarded as the representations of the group. In dealing with comodules, we are using the coalgebra structure of the Hopf algebra H. A Hopf algebroid is an additive category (that is, "algebra with several objects"; that is, algebroid) with a comonoidal structure: modules over a Hopf algebroid make sense but comodules are not appropriate. What we need is a notion where the

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0001-8708/97 $25.00

"several object" aspect applies to the coalgebra structure rather than the algebra. We call these *Hopf opalgebroids*. A quantum groupoid[1] should be a Hopf opalgebroid with cobraiding and cotwist inducing a tortile structure on the monoidal category of finite-dimensional comodules over it.

We shall examine convolution product in a context general enough to include both that on $\mathrm{Hom}(C, A)$ for a coalgebra C and an algebra A (see [Swd]), and that on a functor category $[\mathscr{A}, \mathscr{V}]$ for $\mathscr{A}$ a promonoidal category with homs enriched in $\mathscr{V}$ (see [D1]).

That context is a monoidal bicategory $\mathscr{M}$ (by which we mean a one-object tricategory in the sense of [GPS]). We shall introduce versions of many concepts from category theory internal to our monoidal bicategory $\mathscr{M}$. It is an interesting fact[2] that, to internalize a given structure, the corresponding external higher-dimensional structure is required to be present on $\mathscr{M}$. This leads to a dichotomy of definitions into those for the external higher structures and those for the internal structures.

As a notational point, in any bicategory, we shall use $\circ$ for horizontal composition and juxtaposition for vertical composition. A monoidal bicategory $\mathscr{M}$ has another operation $\otimes : \mathscr{M} \times \mathscr{M} \to \mathscr{M}$ called *tensor product*. For simplicity in developing the theory, we shall suppose that $\mathscr{M}$ is a very special kind of monoidal bicategory: a Gray monoid. A Gray monoid is a 2-category $\mathscr{M}$ with an associative, unital multiplication $\otimes : \mathscr{M} \times \mathscr{M} \to \mathscr{M}$ which is not generally a 2-functor but merely a special kind of homomorphism of bicategories (called "cubical functor" in [GPS]). Precisely, a Gray monoid (also called "semistrict monoidal 2-category" [KV]) is a monoid in the monoidal category Gray of 2-categories and 2-functors with the strong Gray tensor product. In the first section, we shall spell out the definition of Gray monoid explicitly.

Examples naturally occur as monoidal bicategories rather than Gray monoids. However, the coherence theorem of [GPS] allows us to transfer our definitions and results. The main example we deal with is the monoidal bicategory of $\mathscr{V}$-enriched categories and $\mathscr{V}$-modules between them where $\mathscr{V}$ is a sufficiently complete braided monoidal category.

1. GRAY MONOIDS AND THEIR MORPHISMS

DEFINITION 1. *A Gray monoid $\mathscr{M}$* is a 2-category equipped with the following data:

[1] A different definition is given by [BM].

[2] John Baez and James Dolan have advised us that they call this phenomenon "the microcosm principle."

(a) an object I;

(b) for all objects A, two 2-functors $L_A, R_A : \mathscr{M} \to \mathscr{M}$ satisfying the conditions

$$L_A(B) = R_B(A) \qquad (\text{and define } A \otimes B = L_A(B)),$$

$$L_I = R_I = 1_{\mathscr{M}},$$

$$L_{A \otimes B} = L_A L_B, \qquad R_{A \otimes B} = R_B R_A, \qquad R_B L_A = L_A R_B,$$

for all objects A, B; and

(c) for all arrows $f : A \to A'$, $g : B \to B'$, an invertible 2-cell

$$\begin{tikzcd}
A \otimes B \arrow[r, "L_A(g)"] \arrow[d, "R_B(f)"'] \arrow[dr, phantom, "\overset{\Rightarrow}{c_{f,g}}"] & A \otimes B' \arrow[d, "R_{B'}(f)"] \\
A' \otimes B \arrow[r, "L_{A'}(g)"'] & A' \otimes B'
\end{tikzcd}$$

such that the following axioms hold:

(i) if both f and g are identity arrows then $c_{f,g}$ is an identity 2-cell,

(ii) for all arrows $f : A \to A'$, $g : B \to B'$, $h : C \to C'$, there are equalities

$$L_A(c_{g,h}) = c_{L_A(g),h}, \qquad c_{f,L_B(h)} = c_{R_B(f),h}, \qquad R_C(c_{f,g}) = c_{f,R_C(g)};$$

(iii) for all arrows $f, h : A \to A'$, $g, k : B \to B'$, and 2-cells $\alpha : f \Rightarrow h$, $\beta : g \Rightarrow k$,

$$\begin{tikzcd}
A \otimes B \arrow[r, bend left=50, "L_A(k)"] \arrow[r, bend right=50, "L_A(g)"'] \arrow[r, phantom, "\Uparrow L_A(\beta)"] \arrow[d, "R_B(f)"'] \arrow[dr, phantom, "\overset{\Rightarrow}{c_{f,g}}"] & A \otimes B' \arrow[d, bend right=50, "R_{B'}(f)"'] \arrow[d, bend left=50, "R_{B'}(h)"] \arrow[d, phantom, "\overset{\Rightarrow}{R_{B'}(\alpha)}"] \\
A' \otimes B \arrow[r, "L_{A'}(g)"'] & A' \otimes B'
\end{tikzcd}
=
\begin{tikzcd}
A \otimes B \arrow[r, "L_A(k)"] \arrow[d, bend right=50, "R_B(f)"'] \arrow[d, bend left=50, "R_B(h)"] \arrow[d, phantom, "\overset{\Rightarrow}{R_B(\alpha)}"] \arrow[dr, phantom, "\overset{\Rightarrow}{c_{h,k}}"] & A \otimes B' \arrow[d, "R_{B'}(h)"] \\
A' \otimes B \arrow[r, bend left=50, "L_{A'}(k)"] \arrow[r, bend right=50, "L_{A'}(g)"'] \arrow[r, phantom, "\Uparrow L_{A'}(\beta)"] & A' \otimes B'
\end{tikzcd}$$

(iv) for all arrows $f: A \to A'$, $g : B \to B'$, $f' : A' \to A''$, $g' : B' \to B''$,

$$
\begin{tikzcd}
A \otimes B \arrow[r, "L_A(g)"] \arrow[d, "R_B(f)"'] \arrow[dr, phantom, "\overset{\Rightarrow}{c_{f,g}}"] & A \otimes B' \arrow[r, "L_A(g')"] \arrow[d, "R_{B'}(f)"] \arrow[dr, phantom, "\overset{\Rightarrow}{c_{f,g'}}"] & A \otimes B'' \arrow[d, "R_{B''}(f)"] \\
A' \otimes B \arrow[r, "L_{A'}(g)"'] \arrow[d, "R_B(f')"'] \arrow[dr, phantom, "\overset{\Rightarrow}{c_{f',g}}"] & A' \otimes B' \arrow[r, "L_{A'}(g')"'] \arrow[d, "R_{B'}(f')"] \arrow[dr, phantom, "\overset{\Rightarrow}{c_{f',g'}}"] & A' \otimes B'' \arrow[d, "R_{B''}(f')"] \\
A'' \otimes B \arrow[r, "L_{A''}(g)"'] & A'' \otimes B' \arrow[r, "L_{A''}(g')"'] & A'' \otimes B''
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \otimes B \arrow[r, "L_A(g'g)"] \arrow[d, "R_B(f'f)"'] \arrow[dr, phantom, "\overset{\Rightarrow}{c_{f'f,g'g}}"] & A \otimes B' \arrow[d, "R_{B'}(f'f)"] \\
A' \otimes B \arrow[r, "L_{A'}(g'g)"'] & A' \otimes B'
\end{tikzcd}
$$

We put $L_A(g) = A \otimes g$, $L_A(\beta) = A \otimes \beta$, $R_B(f) = f \otimes B$, and $R_B(\alpha) = \alpha \otimes B$. However, notice that there are two isomorphic choices for what might be called $f \otimes g$; for definiteness we put

$$f \otimes g = R_{B'}(f) \circ L_A(g), \qquad \alpha \otimes \beta = R_{B'}(\alpha) \circ L_A(\beta),$$

and this naturally becomes a homomorphism of bicategories $\otimes : \mathscr{M} \times \mathscr{M} \to \mathscr{M}$. In this way, each Gray monoid becomes a monoidal bicategory.

Recall that we write $\mathscr{M}^{\mathrm{op}}$ for the bicategory obtained from $\mathscr{M}$ by reversing the direction of the arrows (so $\mathscr{M}^{\mathrm{op}}(A, B) = \mathscr{M}(B, A)$), and we write $\mathscr{M}^{\mathrm{co}}$ for the bicategory obtained from $\mathscr{M}$ by reversing the direction of the 2-cells (so $\mathscr{M}^{\mathrm{co}}(A, B) = \mathscr{M}(A, B)^{\mathrm{op}}$). If $\mathscr{M}$ is monoidal, we keep the same tensor product in $\mathscr{M}^{\mathrm{op}}$ and $\mathscr{M}^{\mathrm{co}}$. We write $\mathscr{M}^{\mathrm{rev}}$ for the monoidal bicategory $\mathscr{M}$ with the tensor product $A \otimes' B$ given by $A \otimes' B = B \otimes A$. Of course, if $\mathscr{M}$ is a Gray monoid then so are $\mathscr{M}^{\mathrm{op}}$, $\mathscr{M}^{\mathrm{co}}$, $\mathscr{M}^{\mathrm{rev}}$ in natural ways.

We shall distinguish a special kind of lax functor between tricategories [GPS, pp. 15–18].

Definition 2. A *weak monoidal homomorphism* $T : \mathscr{M} \to \mathscr{N}$ between Gray monoids $\mathscr{M}$, $\mathscr{N}$ is a pseudofunctor (= homomorphism of bicategories) from $\mathscr{M}$ to $\mathscr{N}$ equipped with arrows

$$\chi_{X, Y} : TX \otimes TY \to T(X \otimes Y), \qquad \iota : I \to TI$$

in $\mathscr{N}$ for all objects X, Y of $\mathscr{M}$, and invertible 2-cells,

$$
\begin{tikzcd}
TX \otimes TY \arrow[r, "\chi_{X,Y}"] \arrow[d, "Tf \otimes Tg"'] \arrow[dr, phantom, "\overset{\chi_{f,g}}{\Rightarrow}"] & T(X \otimes Y) \arrow[d, "T(f \otimes g)"] \\
TX' \otimes TY' \arrow[r, "\chi_{X',Y'}"'] & T(X' \otimes Y')
\end{tikzcd}
$$

$$
\begin{tikzcd}
TX \otimes TY \otimes TZ \arrow[r, "1 \otimes \chi_{Y,Z}"] \arrow[d, "\chi_{X,Y} \otimes 1"'] \arrow[dr, phantom, "\overset{\omega_{X,Y,Z}}{\Rightarrow}"] & TX \otimes T(Y \otimes Z) \arrow[d, "\chi_{X,Y \otimes Z}"] \\
T(X \otimes Y) \otimes TZ \arrow[r, "\chi_{X \otimes Y,Z}"'] & T(X \otimes Y \otimes Z)
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & TX \arrow[dl, "1 \otimes \iota"'] \arrow[d, "1"] \arrow[dr, "\iota \otimes 1"] & \\
TX \otimes TI \arrow[r, "\chi_{X,I}"'] \arrow[ur, phantom, "\underset{\zeta_X}{\Rightarrow}" near end] & TX & TI \otimes TX \arrow[l, "\chi_{I,X}"] \arrow[ul, phantom, "\underset{\kappa_X}{\Leftarrow}" near end]
\end{tikzcd}
$$

in $\mathscr{N}$ for all objects X, Y, Z and arrows f, g in $\mathscr{M}$, so that a pseudonatural transformation (= strong optransformation) χ and modifications ω, ζ, κ are determined satisfying the following conditions (where tensor symbols between objects have been omitted as a space-saving measure).

$$
\begin{tikzcd}[column sep=small]
 & TX\,TZ \arrow[dl, "1 \otimes \iota \otimes 1"'] \arrow[dr, "1"] \arrow[d, phantom, "\overset{1 \otimes \zeta}{\Rightarrow}"] & \\
TX\,TI\,TZ \arrow[rr, "1 \otimes \chi_{I,Z}"'] \arrow[d, "\chi_{X,I} \otimes 1"'] & {} & TX\,TZ \arrow[d, "\chi_{X,Z}"] \\
TX\,TZ \arrow[rr, "\chi_{X,Z}"'] \arrow[urr, phantom, "\overset{\omega_{X,I,Z}}{\Rightarrow}"] & & T(XZ)
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & TX\,TZ \arrow[dl, "1 \otimes \iota \otimes 1"'] \arrow[ddl, "1"] & \\
TX\,TI\,TZ \arrow[d, "\chi_{X,I} \otimes 1"'] \arrow[d, phantom, "\overset{\Rightarrow}{\kappa \otimes 1}", shift left=4ex] & & \\
TX\,TZ \arrow[rr, "\chi_{X,Z}"'] & & T(XZ)
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & T(WX)\,TY\,TZ \arrow[rr, "\chi\otimes 1"] \arrow[r, phantom, shift right=4ex, "\Downarrow \omega\otimes 1"] & & T(WXY)\,TZ \arrow[dr, "\chi"] & \\
TW\,TX\,TY\,TZ \arrow[ur, "\chi\otimes 1"] \arrow[rr, "1\otimes\chi\otimes 1"] \arrow[dr, "1\otimes\chi"'] \arrow[rr, phantom, shift right=6ex, "\Downarrow 1\otimes\omega"] & & TW\,T(XY)\,TZ \arrow[ur, "\chi\otimes 1"] \arrow[dr, "1\otimes\chi"] \arrow[r, phantom, "\Downarrow \omega"] & & T(WXYZ) \\
 & TW\,TX\,T(YZ) \arrow[rr, "1\otimes\chi"'] & & TW\,T(XYZ) \arrow[ur, "\chi"'] &
\end{tikzcd}
$$

$$\|$$

$$
\begin{tikzcd}
 & T(WX)\,TY\,TZ \arrow[rr, "\chi\otimes 1"] \arrow[dr, "1\otimes\chi"] & & T(WXY)\,TZ \arrow[dr, "\chi"] & \\
TX\,TY\,TZ \arrow[ur, "\chi\otimes 1"] \arrow[dr, "1\otimes\chi"'] \arrow[r, phantom, "\Downarrow c_{\chi,\chi}"] & & T(WX)\,T(YZ) \arrow[rr, "\chi"] \arrow[r, phantom, shift left=6ex, "\Downarrow \omega"] \arrow[r, phantom, shift right=6ex, "\Downarrow \omega"] & & T(WX) \\
 & TW\,TX\,T(YZ) \arrow[ur, "\chi\otimes 1"'] \arrow[rr, "1\otimes\chi"'] & & TW\,T(XYZ) \arrow[ur, "\chi"'] &
\end{tikzcd}
$$

Weak monoidal homomorphisms compose in a natural way. Weak monoidal structures can be transported across equivalences of homomorphisms. A *monoidal homomorphism* is a weak monoidal homomorphism for which $\chi_{X,\,Y}$ and ι are all equivalences.

EXAMPLE 1. If $\mathscr{M}$ is a Gray monoid, we have the representable 2-functor $\mathscr{M}(I, -) : \mathscr{M} \to \mathbf{Cat}$. In order to say that this provides an example of a weak monoidal homomorphism there is a small technical difficulty in that **Cat** is not a Gray monoid (the cartesian product of categories is not strictly associative). So we shall henceforth suppose that **Cat** denotes a Gray monoid which is equivalent to the monoidal bicategory usually intended by that name. Composing $\mathscr{M}(I, -)$ with the equivalence gives a weak monoidal homomorphism which we shall still denote by $\mathscr{M}(I, -) : \mathscr{M} \to \mathbf{Cat}$. Be aware, however, that this new $\mathscr{M}(I, -)$ is no longer a 2-functor, merely a homomorphism of bicategories.

DEFINITION 3 (Compare [GPS, Section 3.3]). Suppose $\mathscr{M}$, $\mathscr{N}$ are Gray monoids and $S, T : \mathscr{M} \to \mathscr{N}$ are weak monoidal homomorphisms. A *monoidal pseudonatural transformation* $\theta : S \to T$ consists of a pseudonatural transformation $\theta : S \to T$, together with an invertible modification,

$$\begin{tikzcd}
SX\otimes SY \arrow[r, "\chi_{X,Y}"] \arrow[d, "\theta_X\otimes\theta_Y"'] \arrow[dr, phantom, "\overset{\theta_{2;X,Y}}{\Rightarrow}"] & S(X\otimes Y) \arrow[d, "\theta_{X\otimes Y}"] \\
TX\otimes TY \arrow[r, "\chi_{X,Y}"'] & T(X\otimes Y)
\end{tikzcd}$$

and an invertible 2-cell,

$$\begin{tikzcd}
 & SI \arrow[d, "\theta_I"] \\
I \arrow[ur, "\iota"] \arrow[r, "\iota"'] \arrow[ur, phantom, shift right=3ex, "\Uparrow\theta_0"] & TI
\end{tikzcd}$$

subject to the following three equations:

$$\begin{tikzcd}
 & SX\otimes S(Y\otimes Z) \arrow[dr, "\chi_{X,Y\otimes Z}"] & \\
SX\otimes SY\otimes SZ \arrow[ur, "1\otimes\chi_{Y,Z}"] \arrow[r, "\chi_{X,Y}\otimes 1"] \arrow[d, "1\otimes 1\otimes\theta_Z"'] \arrow[dr, phantom, "\underset{c^{-1}}{\Rightarrow}"] & S(X\otimes Y)\otimes SZ \arrow[r, "\chi_{X\otimes Y,Z}"'] \arrow[d, "1\otimes\theta_Z"] \arrow[u, phantom, "\Uparrow\omega_{X,Y,Z}"] \arrow[ddr, phantom, "\overset{\theta_{2;X\otimes Y,Z}}{\Rightarrow}"] & S(X\otimes Y\otimes Z) \arrow[dd, "\theta_{X\otimes Y\otimes Z}"] \\
SX\otimes SY\otimes TZ \arrow[r, "\chi_{X,Y}\otimes 1"'] \arrow[d, "\theta_X\otimes\theta_Y\otimes 1"'] \arrow[dr, phantom, "\overset{\theta_{2;X,Y}\otimes 1}{\Rightarrow}"] & S(X\otimes Y)\otimes TZ \arrow[d, "\theta_{X\otimes Y}\otimes 1"] & \\
TX\otimes TY\otimes TZ \arrow[r, "\chi_{X,Y}\otimes 1"'] & T(X\otimes Y)\otimes TZ \arrow[r, "\chi_{X\otimes Y,Z}"'] & T(X\otimes Y\otimes Z)
\end{tikzcd}$$

$$\Vert$$

$$
\begin{tikzcd}
 & & SX\otimes S(Y\otimes Z) \arrow[drr, "\chi_{X,Y\otimes Z}"] \arrow[dd, "1\otimes\theta_{Y\otimes Z}"'] & & \\
SX\otimes SY\otimes SZ \arrow[urr, "1\otimes\chi_{Y,Z}"] \arrow[dd, "1\otimes\theta_Y\otimes\theta_Z"'] \arrow[drr, phantom, "\Rightarrow 1\otimes\theta_{2;Y,Z}"] & & & & S(X\otimes Y\otimes Z) \arrow[dddd, "\theta_{X\otimes Y\otimes Z}"] \\
 & & SX\otimes T(Y\otimes Z) \arrow[ddr, "\theta_X\otimes 1"] \arrow[urr, phantom, "\theta_{2;X,Y\otimes Z}\Rightarrow"] & & \\
SX\otimes TY\otimes TZ \arrow[urr, "1\otimes\chi_{Y,Z}"'] \arrow[dd, "\theta_X\otimes 1\otimes 1"'] \arrow[drr, phantom, "\overset{c}{\Rightarrow}"] & & & & \\
 & & & TX\otimes T(Y\otimes Z) \arrow[dr, "\chi_{X,Y\otimes Z}"] \arrow[d, phantom, "\Uparrow\omega_{X,Y,Z}"] & \\
TX\otimes TY\otimes TZ \arrow[urrr, "1\otimes\chi_{Y,Z}"'] \arrow[rr, "\chi_{X,Y}\otimes 1"'] & & T(X\otimes Y)\otimes TZ \arrow[rr, "\chi_{X\otimes Y,Z}"'] & & T(X\otimes Y\otimes Z)
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & SX \arrow[dl, "1\otimes\iota"'] \arrow[dr, "1"] \arrow[d, phantom, "\overset{\zeta_X}{\Rightarrow}"] & \\
SX\otimes SI \arrow[rr, "\chi_{X,I}"'] \arrow[d, "\theta_X\otimes\theta_I"'] \arrow[drr, phantom, "\overset{\theta_{2;X,I}}{\Rightarrow}"] & & SX \arrow[d, "\theta_X"] \\
TX\otimes TI \arrow[rr, "\chi_{X,I}"'] & & TX
\end{tikzcd}
=
\begin{tikzcd}
SX\otimes SI \arrow[d, "1\otimes\theta_I"'] \arrow[drr, phantom, "\overset{1\otimes\theta_0^{-1}}{\Rightarrow}"] & & SX \arrow[ll, "1\otimes\iota"'] \arrow[dll, "1\otimes\iota"] \arrow[dd, "\theta_X"] \\
SX\otimes TI \arrow[ddr, "\theta_X\otimes 1"'] \arrow[drr, phantom, "\underset{c^{-1}}{\Rightarrow}"] & & \\
 & & TX \arrow[dl, "1\otimes\iota"'] \arrow[d, "1"] \arrow[dl, phantom, "\overset{\zeta_X}{\Rightarrow}"] \\
 & TX\otimes TI \arrow[r, "\chi_{X,I}"'] & TX
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & SX \arrow[dl, "\iota\otimes 1"'] \arrow[dr, "1"] \arrow[d, phantom, "\overset{\kappa_X}{\Rightarrow}"] & \\
 & SI\otimes SX \arrow[rr, "\chi_{I,X}"] \arrow[dl, "\theta_I\otimes 1"'] \arrow[dd, "\theta_I\otimes\theta_X"] \arrow[ddrr, phantom, "\overset{\theta_{2;I,X}}{\Rightarrow}"] & & SX \arrow[dd, "\theta_X"] \\
TI\otimes SX \arrow[dr, "1\otimes\theta_X"'] \arrow[r, phantom, "\underset{c}{\Rightarrow}"] & & & \\
 & TI\otimes TX \arrow[rr, "\chi_{I,X}"'] & & TX
\end{tikzcd}
=
\begin{tikzcd}
SI\otimes SX \arrow[d, "\theta_I\otimes 1"'] \arrow[drr, phantom, "\overset{(\theta_0\otimes 1)^{-1}}{\Rightarrow}"] & & SX \arrow[ll, "\iota\otimes 1"'] \arrow[dll, "\iota\otimes 1"] \arrow[dd, "\theta_X"] \\
TI\otimes SX \arrow[ddr, "1\otimes\theta_X"'] \arrow[drr, phantom, "\underset{c}{\Rightarrow}"] & & \\
 & & TX \arrow[dl, "\iota\otimes 1"'] \arrow[d, "1"] \arrow[dl, phantom, "\overset{\kappa_X}{\Rightarrow}"] \\
 & TI\otimes TX \arrow[r, "\chi_{I,X}"'] & TX
\end{tikzcd}
$$

Suppose $S, T : \mathscr{M} \to \mathscr{N}$ are weak monoidal homomorphisms and $\theta, \phi : S \to T$ are monoidal pseudonatural transformations. A *monoidal modification* $s : \theta \to \phi$ is a modification $s : \theta \to \phi$ such that the following two equations hold:

$$
\begin{tikzcd}
SX\otimes SY \arrow[r, "\chi_{X,Y}"] \arrow[d, "\theta_X\otimes\theta_Y"'] \arrow[dr, phantom, "\overset{\theta_{2;X,Y}}{\Rightarrow}"] & S(X\otimes Y) \arrow[d, "\theta_{X\otimes Y}"'] \arrow[d, bend left=60, "\phi_{X\otimes Y}"] \arrow[d, phantom, shift left=3, "\overset{s_{X\otimes Y}}{\Rightarrow}"] \\
TX\otimes TY \arrow[r, "\chi_{X,Y}"'] & T(X\otimes Y)
\end{tikzcd}
=
\begin{tikzcd}
SX\otimes SY \arrow[r, "\chi_{X,Y}"] \arrow[d, "\theta_X\otimes\theta_Y"'] \arrow[d, bend left=60, "\phi_X\otimes\phi_Y"] \arrow[d, phantom, shift left=3, "\overset{s_X\otimes s_Y}{\Rightarrow}"] \arrow[dr, phantom, "\overset{\phi_{2;X,Y}}{\Rightarrow}"] & S(X\otimes Y) \arrow[d, "\phi_{X\otimes Y}"] \\
TX\otimes TY \arrow[r, "\chi_{X,Y}"'] & T(X\otimes Y)
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & SI \arrow[d, "\theta_I"'] \arrow[d, bend left=80, "\phi_I"] \arrow[d, phantom, shift left=4, "\overset{\Rightarrow}{s_I}"] \\
I \arrow[ur, "\iota"] \arrow[r, "\iota"'] \arrow[r, phantom, shift left=4, "\Uparrow\theta_0"] & TI
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & SI \arrow[d, "\phi_I"] \\
I \arrow[ur, "\iota"] \arrow[r, "\iota"'] \arrow[r, phantom, shift left=4, "\Uparrow\phi_0"] & TI
\end{tikzcd}
$$

There is a 2-category $\mathrm{WMonHom}(\mathscr{M}, \mathscr{N})$ whose objects are weak monoidal homomorphisms $S : \mathscr{M} \to \mathscr{N}$, whose arrows are monoidal pseudonatural transformations, and whose 2-cells are monoidal modifications; the compositions are defined in an obvious way, and we have a forgetful 2-functor $\mathrm{WMonHom}(\mathscr{M}, \mathscr{N}) \to \mathrm{Hom}(\mathscr{M}, \mathscr{N})$.

2. ADJOINT MONOIDAL HOMOMORPHISMS AND BIDUALS

Definition 4. Suppose $\mathscr{A}, \mathscr{B}, \mathscr{C}$ are bicategories. A *left parametrized biadjunction* consists of homomorphisms $T : \mathscr{A} \times \mathscr{C} \to \mathscr{B}$, $H : \mathscr{A}^{\mathrm{op}} \times \mathscr{B} \to \mathscr{C}$, together with equivalences

$$\pi_{A, B, C} : \mathscr{B}(T(A, C), B) \simeq \mathscr{C}(C, H(A, B))$$

which are pseudonatural in objects A, B, C of $\mathscr{A}, \mathscr{B}, \mathscr{C}$, respectively. The homomorphism H is determined by T up to pseudonatural equivalence. We call T the left biadjoint and H the right biadjoint. The pseudonatural parameterized counit ε with components

$$\varepsilon_{A, B} : T(A, H(A, B)) \to B$$

is determined up to isomorphism by $\pi_{A, B, H(A, B)}(\varepsilon_{A, B}) = 1_{H(A, B)}$. Similarly, there is a parameterized unit

$$\eta_{C, A} : C \to H(A, T(A, C)).$$

Proposition 1. *If $T : \mathscr{A} \times \mathscr{C} \to \mathscr{B}$ is a homomorphism and, for all objects A, B of $\mathscr{A}, \mathscr{B}$, there exists an object $H(A, B)$ of $\mathscr{C}$ and equivalences $\pi_{A, B, C} : \mathscr{B}(T(A, C), B) \simeq \mathscr{C}(C, H(A, B))$ pseudo-natural in $C \in \mathscr{C}$, then H can be extended to a homomorphism $H : \mathscr{A}^{\mathrm{op}} \times \mathscr{B} \to \mathscr{C}$ such that $\pi_{A, B, C}$ becomes pseudonatural in A, B, and so becomes a left parameterized biadjunction.*

Proof. This is a routine exercise in the bicategorical Yoneda lemma [St1]. Q.E.D.

Recall [K1] that a right adjoint to a monoidal functor is automatically weak monoidal. This generalizes to the parameterized case and to the next higher dimension.

PROPOSITION 2. *Suppose $\mathscr{A}, \mathscr{B}, \mathscr{C}$ are Gray monoids and T, H, π form a left parametrized biadjunction. If $T: \mathscr{A} \times \mathscr{C} \to \mathscr{B}$ is a monoidal homomorphism then $H: \mathscr{A}^{\mathrm{op}} \times \mathscr{B} \to \mathscr{C}$ becomes a weak monoidal homomorphism.*

Proof. The required arrow,

$$\chi_{(A, B), (A', B')}: H(A, B) \otimes H(A', B') \to H(A \otimes A', B \otimes B'),$$

corresponds under π to the composite

$$T(A \otimes A', H(A, B) \otimes H(A', B'))$$
$$\simeq T(A, H(A, B)) \otimes T(A', H(A', B')) \xrightarrow{\varepsilon_{A, B} \otimes \varepsilon_{A', B'}} B \otimes B'.$$

The extensive remaining details are left to the reader. Q.E.D.

DEFINITION 5. A monoidal bicategory $\mathscr{M}$ is called *right closed* when the homomorphism $\otimes: \mathscr{M} \times \mathscr{M} \to \mathscr{M}$ is the left biadjoint in a left parametrized biadjunction. The left biadjoint is denoted by $[\ ,\ ]: \mathscr{M}^{\mathrm{op}} \times \mathscr{M} \to \mathscr{M}$. A monoidal bicategory $\mathscr{M}$ is called *left closed* when $\mathscr{M}^{\mathrm{rev}}$ is right closed; it is *closed* when it is both left and right closed.

DEFINITION 6. Suppose $\mathscr{M}$ is a monoidal bicategory. An arrow $e: A \otimes B \to I$ is called an *exact pairing* when, for all objects C, D, the functor

$$e^{\#}: \mathscr{M}(C, B \otimes D) \to \mathscr{M}(A \otimes C, D),$$

given by $e^{\#}(f) = (e \otimes 1_D) \circ (1_A \otimes f)$, is an equivalence of categories. We call B (together with an exact pairing e) a *right bidual* for A; since right biduals are unique up to equivalence, we write A° for B. Taking $C = I$ and $D = A$, we obtain an *exact copairing* $n: I \to A^{\circ} \otimes A$ with $e^{\#}(n) = 1_A$. We call $\mathscr{M}$ *right autonomous* when each object has a right dual. Then the assignment $A \mapsto A^{\circ}$ extends to a biequivalence [St1] of bicategories

$$(\ )^{\circ}: \mathscr{M}^{\mathrm{oprev}} \to \mathscr{M}.$$

We call $\mathscr{M}$ *left autonomous* when each object has a left dual; a left and right autonomous monoidal bicategory is called *autonomous*.

PROPOSITION 3. *Each right autonomous monoidal bicategory is right closed. A right closed monoidal bicategory is right autonomous if and only if, for all objects A, B, the arrow*

$$[A, I] \otimes B \to [A, B],$$

corresponding to $\varepsilon_{A,I} \otimes 1_B : A \otimes [A, I] \otimes B \to B$, is an equivalence. In this case there are canonical equivalences

$$[A, I] \simeq A^\circ, \qquad A^\circ \otimes B \simeq [A, B].$$

3. PSEUDOMONOIDS

There is a well-known concept which, following the terminology scheme of [KS], we shall call "pseudomonoid"[3] A in $\mathscr{M}$. Explicitly, a *pseudomonoid* A in $\mathscr{M}$ consists of an object A, arrows $\mu : A \otimes A \to A$, $\eta : I \to A$ (called the *multiplication* and the *unit* for A), and invertible 2-cells a, l, r,

$$\begin{tikzcd}
A\otimes A\otimes A \arrow[r, "1\otimes\mu"] \arrow[d, "\mu\otimes 1"'] \arrow[dr, phantom, "\overset{a}{\Rightarrow}"] & A\otimes A \arrow[d, "\mu"] \\
A\otimes A \arrow[r, "\mu"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\eta\otimes 1"] \arrow[dr, "1"'] & A\otimes A \arrow[d, "\mu"] \arrow[dl, phantom, "\overset{\ell}{\Leftarrow}"] \arrow[dr, phantom, "\overset{r}{\Rightarrow}"] & A \arrow[l, "1\otimes\eta"'] \arrow[dl, "1"] \\
& A &
\end{tikzcd}$$

(called the *constraints of associativity*, *left unit*, *right unit*) such that the following two equations hold (where, for example, we have written 3 for $A \otimes A \otimes A$):

$$\begin{tikzcd}
4 \arrow[rr, "1\otimes1\otimes\mu"] \arrow[dr, "1\otimes\mu\otimes1"] \arrow[dd, "\mu\otimes1\otimes1"'] & & 3 \arrow[dr, "1\otimes\mu"] \arrow[dl, phantom, "\overset{1\otimes a}{\Rightarrow}"] & \\
& 3 \arrow[rr, "1\otimes\mu"] \arrow[dd, "\mu\otimes1"] \arrow[l, phantom, "\overset{a\otimes1}{\Rightarrow}"] \arrow[ddrr, phantom, "\overset{a}{\Rightarrow}"] & & 2 \arrow[dd, "\mu"] \\
3 \arrow[dr, "\mu\otimes1"'] & & & \\
& 2 \arrow[rr, "\mu"'] & & 1
\end{tikzcd}
=
\begin{tikzcd}
4 \arrow[rr, "1\otimes1\otimes\mu"] \arrow[dd, "\mu\otimes1\otimes1"'] \arrow[ddrr, phantom, "\overset{c_{\mu,\mu}}{\Rightarrow}"] & & 3 \arrow[dr, "1\otimes\mu"] \arrow[dd, "\mu\otimes1"] & \\
& & & 2 \arrow[dd, "\mu"] \arrow[dl, phantom, "\overset{a}{\Rightarrow}"] \\
3 \arrow[rr, "1\otimes\mu"] \arrow[dr, "\mu\otimes1"'] & & 2 \arrow[dr, "\mu"] \arrow[d, phantom, "\overset{a}{\Rightarrow}"] & \\
& 2 \arrow[rr, "\mu"'] & & 1
\end{tikzcd}$$

[3] Other words used at varying degrees of generality in the literature are "doctrine" [Z; St1] and "tensor object" [JS1, pp. 61–62]. Also, "pseudo-algebra" might be preferred by those who use "algebra" for "monoid" in a monoidal category.

$$\begin{tikzcd}
& 2 \arrow[dl, "1\otimes\eta\otimes 1"'] \arrow[dr, "1"] \arrow[d, phantom, "\overset{1\otimes \ell}{\Rightarrow}"] & \\
3 \arrow[rr, "1\otimes\mu"] \arrow[d, "\mu\otimes 1"'] \arrow[drr, phantom, "\underset{a}{\Rightarrow}"] & {} & 2 \arrow[d, "\mu"] \\
2 \arrow[rr, "\mu"'] & & 1
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& 2 \arrow[dl, "1\otimes\eta\otimes 1"'] \arrow[dr, "1"] \arrow[ddl, "1"] & \\
3 \arrow[d, "\mu\otimes 1"'] \arrow[d, phantom, shift left=3, "\overset{r\otimes 1}{\Rightarrow}"] & & 2 \arrow[d, "\mu"] \\
2 \arrow[rr, "\mu"'] & & 1
\end{tikzcd}$$

For example, a pseudomonoid in the cartesian closed 2-category **Cat** of categories, functors and natural transformations, is precisely a monoidal (or "tensor") category. A pseudomonoid in $\mathcal{M}^{\mathrm{op}}$ is called a *pseudocomonoid* in $\mathcal{M}$; the multiplication μ and unit η in $\mathcal{M}^{\mathrm{op}}$ are called and denoted *comultiplication* δ and *counit* ε in $\mathcal{M}$. However, notice that pseudomonoids in $\mathcal{M}^{\mathrm{co}}$ and $\mathcal{M}^{\mathrm{rev}}$ can be identified with pseudomonoids in $\mathcal{M}$.

Example 2. For any object X of a right closed Gray monoid, the internal endohom $[X, X]$ becomes a pseudomonoid under "internal composition." In fact, we have an obvious n-fold multiplication $\mu_n : [X, X] \otimes \cdots \otimes [X, X] \to [X, X]$ corresponding to the composite

$$\begin{aligned}
&X \otimes [X, X] \otimes \cdots \otimes [X, X] \\
&\xrightarrow{\varepsilon_{X,X} \otimes 1 \otimes \cdots \otimes 1} \cdots \xrightarrow{\varepsilon_{X,X} \otimes 1} X \otimes [X, X] \xrightarrow{\varepsilon_{X,X}} X.
\end{aligned}$$

There is a canonical constraint isomorphism between μ_n and any arrow $[X, X] \otimes \cdots \otimes [X, X] \to [X, X]$ obtained as a composite of arrows of the form $1 \otimes \cdots \otimes \mu_m \otimes \cdots \otimes 1$.

Example 3. If C is a pseudo-comonoid in a cartesian monoidal 2-category (that is, a 2-category with cartesian product as tensor product) then it easily seen that the left and right unit constraints form an isomorphism between the comultiplication δ and the diagonal $C \to C \times C$.

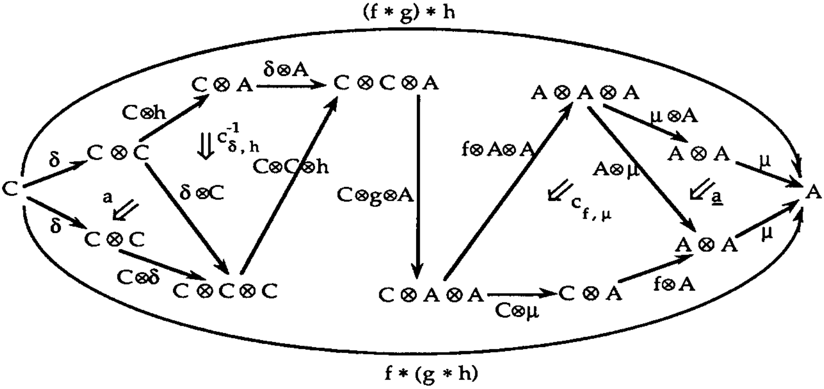
Proposition 4. *Suppose A is a pseudomonoid and C is a pseudocomonoid in $\mathcal{M}$. The category $\mathcal{M}(C, A)$ is equipped with a monoidal structure: the tensor product is* convolution *$f * g$ given by*

$$C \xrightarrow{\delta} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{\mu} A$$

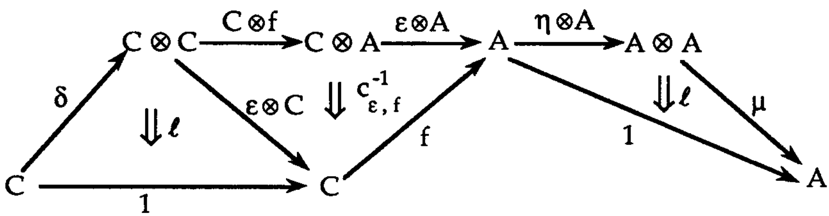
and the unit is given by

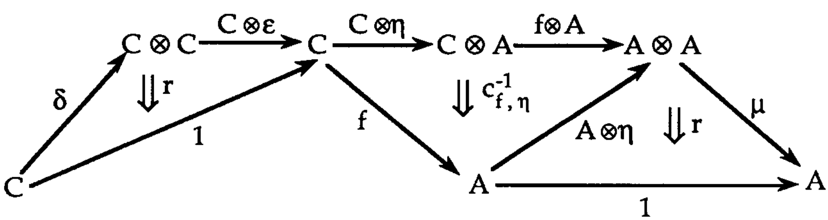
$$C \xrightarrow{\varepsilon} I \xrightarrow{\eta} A.$$

Proof. The associativity constraint is the following pasting composite:



The left and right unit constraints are the following pasting composites:





The associativity pentagon and triangle for unit constraints are easily checked, however, a more conceptual proof of this proposition will be given below. Q.E.D.

PROPOSITION 5. *Weak monoidal homomorphisms take pseudomonoids to pseudomonoids.*

Proof. Every homomorphism $H : \mathscr{A} \to \mathscr{B}$ is equivalent to a *normal* homomorphism (that is, one for which the constraints $1_{TA} \Rightarrow T(1_A)$ are identities for all objects A of $\mathscr{A}$). Pseudomonoids A in $\mathscr{M}$ can be identified with weak monoidal normal homomorphisms $A : 1 \to \mathscr{M}$. Thus, a weak monoidal homomorphism $T : \mathscr{M} \to \mathscr{N}$ yields a composite weak monoidal homomorphism $T \circ A$ in $\mathscr{N}$. The canonically equivalent normal homomorphism TA becomes weak monoidal by transport of structure. Q.E.D.

Recall from [SW] the versatility and expressive power of liftings and extensions in a bicategory $\mathscr{M}$. A diagram

$$\begin{tikzcd} & K \arrow[dl, "h"'] \arrow[dr, "f"] \arrow[d, phantom, "\lambda \Rightarrow"] & \\ A \arrow[rr, "j"'] & {} & X \end{tikzcd}$$

is said to *exhibit h as a right lifting of f through j* when, for all arrows $t : K \to A$, pasting with λ determines a bijection between 2-cells $t \Rightarrow h$ and 2-cells $j \circ t \Rightarrow f$. A *right extension* in $\mathscr{M}$ is a right lifting in $\mathscr{M}^{\mathrm{op}}$. Of course, not all right liftings exist in **Cat**; but they do in an interesting class of bicategories which we shall examine later in this paper.

PROPOSITION 6. *Suppose A is a pseudoalgebra in a Gray monoid $\mathscr{M}$ which admits all right liftings. Then the monoidal category $\mathscr{M}(I, A)$ is closed.*

Proof. For $f, h : I \to A$, let $[f, h] : I \to A$ be the right lifting of h along the composite

$$A \xrightarrow{f \otimes 1} A \otimes A \xrightarrow{\mu} A.$$

Then 2-cells $g \Rightarrow [f, h]$ are in natural bijection with 2-cells $f * g = \mu \circ (f \otimes 1) \circ g \Rightarrow h$. So $\mathscr{M}(I, A)$ is right closed. The right lifting of h through $\mu \circ (1 \otimes g)$ gives the left internal hom. So $\mathscr{M}(I, A)$ is closed as required. Q.E.D.

In a monoidal bicategory there is a generalized form of lifting to be considered.

DEFINITION 7. In a monoidal bicategory, a diagram

$$\begin{tikzcd} & U\otimes K \arrow[dl, "1_U\otimes h"'] \arrow[dr, "f"] \arrow[d, phantom, "\overset{\lambda}{\Rightarrow}"] & \\ U\otimes A \arrow[rr, "j"'] & & X \end{tikzcd}$$

is said to *exhibit* $h: K \to A$ *as a right lifting of* f *through* j *modulo* U when, for all arrows $t: K \to A$, pasting with λ determines a bijection between 2-cells $t \Rightarrow h$ and 2-cells $j \circ (1_U \otimes t) \Rightarrow f$. If the ambient bicategory is right closed then h can be obtained as a right lifting of the arrow $\pi(f): K \to [U, X]$ through the arrow $\pi(j): A \to [U, X]$. Hence, in a right closed bicategory in which all right liftings exist, all right liftings modulo objects also exist.

DEFINITION 8. Suppose $f: U\otimes A \to U\otimes B$ is an arrow in any monoidal bicategory $\mathscr{M}$. A *cotrace* $\int_U f$ *of* f *modulo* U is a right lifting of $f: U\otimes A \to U\otimes B$ through $1_{U\otimes B}: U\otimes B \to U\otimes B$ modulo U.

Recall [St2] that an arrow $f: X \to Y$ in a bicategory $\mathscr{M}$ is called a *map* when it has a right adjoint $f^r: Y \to X$. We write Map $\mathscr{M}$ for the sub-bicategory of $\mathscr{M}$ with the same objects as $\mathscr{M}$, with the maps as arrows, and with all 2-cells between these. Homomorphisms (= pseudofunctors) of bicategories preserve adjunctions, and hence, maps. It follows that, if $\mathscr{M}$ is a Gray monoid, then Map $\mathscr{M}$ is a Gray submonoid. However even when $\mathscr{M}$ and Map $\mathscr{M}$ are both left closed, the homs are generally quite different.

PROPOSITION 7. *If A is a pseudomonoid in* Map $\mathscr{M}$ *then A becomes a pseudocomonoid A^d in $\mathscr{M}$ by taking the comultiplication and counit to be the right adjoints of the multiplication and unit.*

Proof. This is a simple exercise in the calculus of "mates" [KS]. Q.E.D.

Let us call A a *map pseudomonoid* in $\mathscr{M}$ when it is a pseudomonoid in Map $\mathscr{M}$. Despite the last result, the reader should be warned that a map pseudomonoid is not necessarily a "pseudobimonoid" in $\mathscr{M}$. A pseudocomonoid in Map $\mathscr{M}$ will be called a *map pseudocomonoid.*

DEFINITION 9. Suppose $\mathscr{M}$ is a right autonomous monoidal bicategory. A *left parametrized right adjoint for an arrow* $t : A \otimes C \to B$ in $\mathscr{M}$ is an arrow $h : A^\circ \otimes B \to C$ which has the composite

$$C \xrightarrow{n \otimes 1} A^\circ \otimes A \otimes C \xrightarrow{1 \otimes t} A^\circ \otimes B$$

as a right adjoint. Of course, it follows that h is a map such that t is isomorphic to the following composite:

$$A \otimes C \xrightarrow{1 \otimes h^r} A \otimes A^\circ \otimes B \xrightarrow{e \otimes 1} B.$$

DEFINITION 10. Suppose A is a pseudomonoid in the right autonomous monoidal bicategory $\mathscr{M}$. A *right internal hom* for A is a left parametrized right adjoint $hom : A^\circ \otimes A \to A$ for $\mu : A \otimes A \to A$. We say A is *right closed* when a right internal hom exists.

DEFINITION 11. Suppose A is a map pseudomonoid in a right-autonomous monoidal bicategory $\mathscr{M}$. A *right antipode* for A is a map $v : A^\circ \to A$ such that the composite

$$A^\circ \otimes A \xrightarrow{v \otimes 1} A \otimes A \xrightarrow{\mu} A$$

is a right internal hom for A. In other words, there should exist an isomorphism

$$\begin{tikzcd}
A \arrow[r, "\mu^r"] \arrow[d, "n \otimes 1"'] \arrow[dr, phantom, "\cong"] & A \otimes A \arrow[d, "v^r \otimes 1"] \\
A^\circ \otimes A \otimes A \arrow[r, "1 \otimes \mu"'] & A^\circ \otimes A
\end{tikzcd}$$

Equivalently, the following composite should be isomorphic to μ:

$$A \otimes A \xrightarrow{1 \otimes \mu^r} A \otimes A \otimes A \xrightarrow{1 \otimes v^r \otimes 1} A \otimes A^\circ \otimes A \xrightarrow{e \otimes 1} A.$$

A map pseudomonoid equipped with a right antipode will be called *right autonomous.*

PROPOSITION 8. *Suppose A is a right-autonomous map pseudomonoid in a right-autonomous monoidal bicategory $\mathscr{M}$. Suppose that the counit*

$v \circ v^r \to 1_A$ is invertible. If $f: I \to A$ is a map then it has a left dual $f^\vee$ in the convolution monoidal category $\mathscr{M}(I, A)$; indeed, $f^\vee$ can be taken to be the composite

$$I \xrightarrow{n} A^\circ \otimes A \xrightarrow{v \otimes 1} A \otimes A \xrightarrow{f^r \otimes 1} A.$$

Proof. If we take $f^\vee$ as suggested then $f^r \circ v \cong e \circ (f^\vee \otimes 1)$. Since the counit $v \circ v^r \to 1_A$ is invertible, we have an isomorphism $f^r \cong e \circ (f^\vee \otimes 1) \circ v^r$. Since right liftings along maps can be obtained by composition with the map's right adjoint, it follows from the proof of Proposition 6 that $(f^r \otimes 1) \circ \mu^r \circ h$ is a right internal hom of f, h in $\mathscr{M}(I, A)$; we need to prove that this is isomorphic to $f^\vee * h = \mu \circ (f^\vee \otimes 1) \circ h$. So it suffices to prove $(f^r \otimes 1) \circ \mu^r \cong \mu \circ (f^\vee \otimes 1)$. Using the defining property of antipode, the homomorphism property of tensor, and the formula for f^r in terms of $f^\vee$, we obtain isomorphisms

$$\begin{aligned} \mu \circ (f^\vee \otimes 1) &\cong (e \otimes 1) \circ (1 \otimes v^r \otimes 1) \circ (1 \otimes \mu^r) \circ (f^\vee \otimes 1) \\ &\cong (e \otimes 1) \circ (f^\vee \otimes 1 \otimes 1) \circ (v^r \otimes 1) \circ \mu^r \\ &\cong (f^r \otimes 1) \circ \mu^r. \end{aligned}$$

Q.E.D.

4. BRAIDED GRAY MONOIDS AND BRAIDED PSEUDOMONOIDS

DEFINITION 12. Suppose $\mathscr{M}$ is a Gray monoid. A *braiding* for $\mathscr{M}$ consists of arrows

$$\rho_{X,Y}: X \otimes Y \to Y \otimes X$$

which are equivalences for all objects X, Y with $\rho_{X,I} = \rho_{I,X} = 1_X$, and invertible 2-cells

$$\begin{tikzcd} X \otimes Y \arrow[r, "\rho_{X,Y}"] \arrow[d, "f \otimes g"'] \arrow[dr, phantom, "\overset{\rho_{f,g}}{\Rightarrow}"] & Y \otimes X \arrow[d, "g \otimes f"] \\ X' \otimes Y' \arrow[r, "\rho_{X',Y'}"'] & Y' \otimes X' \end{tikzcd}$$

$$\begin{tikzcd}
W\otimes X\otimes Y\otimes Z \arrow[r, "1\otimes\rho_{Y,Z}"] \arrow[d, "\rho_{W,X}\otimes 1"'] \arrow[dr, phantom, "\overset{\varpi_{W,X,Y,Z}}{\Rightarrow}"] & W\otimes X\otimes Z\otimes Y \arrow[d, "\rho_{W,X\otimes Z}\otimes 1"] \\
X\otimes W\otimes Y\otimes Z \arrow[r, "1\otimes\rho_{W\otimes Y,Z}"'] & X\otimes Z\otimes W\otimes Y
\end{tikzcd}$$

determining a pseudonatural transformation ρ and such that $\varpi_{W,I,I,Z}$ is the identity of $\rho_{W,Z}$ and the following two equalities hold:

$$\begin{tikzcd}
& XWUYVZ \arrow[rr, "1\otimes\rho_{WU,Y}\otimes 1"] & & XYWUVZ \arrow[dr, "1\otimes\rho_{WUV,Z}"] & \\
WXUYVZ \arrow[ur, "\rho_{W,X}\otimes 1"] \arrow[rr, "1\otimes\rho_{U,Y}\otimes 1"] \arrow[dr, "1\otimes\rho_{V,Z}"'] \arrow[urr, phantom, "\Downarrow\varpi_{W,X,U,Y}\otimes 1"] \arrow[drr, phantom, "\Downarrow 1\otimes\varpi_{U,Y,V,Z}"] & & WXYUVZ \arrow[ur, "\rho_{W,XY}\otimes 1"] \arrow[dr, "1\otimes\rho_{UV,Z}"] \arrow[rr, phantom, "\Downarrow\varpi_{W,XY,UV,Z}"] & & XYZWUV \\
& WXUYZV \arrow[rr, "1\otimes\rho_{U,YZ}\otimes 1"'] & & WXYZUV \arrow[ur, "\rho_{W,XYZ}\otimes 1"'] &
\end{tikzcd}$$

$$||$$

$$\begin{tikzcd}
& XWUYVZ \arrow[rr, "1\otimes\rho_{WU,Y}\otimes 1"] \arrow[dr, "1\otimes\rho_{V,Z}"] & & XYWUVZ \arrow[dr, "1\otimes\rho_{WUV,Z}"] \arrow[d, phantom, "\Downarrow 1\otimes\varpi_{WU,Y,V,Z}"] & \\
WXUYVZ \arrow[ur, "\rho_{W,X}\otimes 1"] \arrow[dr, "1\otimes\rho_{V,Z}"'] \arrow[rr, phantom, "\Downarrow c_{\rho,\rho}"] & & XWUYZV \arrow[rr, "1\otimes\rho_{WU,YZ}\otimes 1"'] & {} \arrow[d, phantom, "\Downarrow\varpi_{W,X,U,YZ}\otimes 1"] & XYZWUV \\
& WXUYZV \arrow[ur, "\rho_{W,X}\otimes 1"] \arrow[rr, "1\otimes\rho_{U,YZ}\otimes 1"'] & & WXYZUV \arrow[ur, "\rho_{W,XYZ}\otimes 1"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
A\otimes B\otimes C \arrow[rr, "\rho_{A,B}\otimes 1"] \arrow[dd, "1\otimes\rho_{B,C}"'] \arrow[dr, "\rho_{A\otimes B,C}"] \arrow[ddr, phantom, "\varpi_{A,I,B,C}\Downarrow"] & & B\otimes A\otimes C \arrow[dd, "\rho_{B\otimes A,C}"] \arrow[dl, phantom, "\Downarrow\rho_{\rho_{A,B},1}"] \\
& C\otimes A\otimes B \arrow[dr, "1\otimes\rho_{A,B}"] \arrow[d, phantom, "\Downarrow\varpi_{A,C,I,B}"] & \\
A\otimes C\otimes B \arrow[ur, "\rho_{A,C}\otimes 1"] \arrow[rr, "\rho_{A,C\otimes B}"'] & {} & C\otimes B\otimes A
\end{tikzcd}$$

$$||$$

$$\begin{tikzcd}
A\otimes B\otimes C \arrow[rr, "\rho_{A,B}\otimes 1"] \arrow[dd, "1\otimes\rho_{B,C}"'] \arrow[dr, "\rho_{A,B\otimes C}"'] & {} \arrow[d, phantom, "\Downarrow\varpi_{A,B,I,C}"] & B\otimes A\otimes C \arrow[dd, "\rho_{B\otimes A,C}"] \arrow[dl, "1\otimes\rho_{A,C}"'] \\
{} \arrow[r, phantom, "\Downarrow\rho^{-1}_{1,\rho_{B,C}}"] & B\otimes C\otimes A \arrow[dr, "\rho_{B,C}\otimes 1"'] \arrow[r, phantom, "\Leftarrow \varpi_{B,I,A,C}"] & {} \\
A\otimes C\otimes B \arrow[rr, "\rho_{A,C\otimes B}"'] & & C\otimes B\otimes A
\end{tikzcd}$$

The first equality means that the homomorphism $\otimes : \mathscr{M}\times\mathscr{M}\to\mathscr{M}$ becomes equipped with a monoidal structure by taking

$$\chi_{(W,X),(Y,Z)} = 1_W\otimes\rho_{X,Y}\otimes 1_Z : W\otimes X\otimes Y\otimes Z\to W\otimes Y\otimes X\otimes Z,$$

$$\imath = 1_I : I\to I\otimes I,$$

$$\omega_{(W,W'),(X,X'),(Y,Y')} = 1_W\otimes\varpi_{W',X,X',Y}\otimes 1_{Y'},$$

$$\zeta_{(X,X')} = \kappa_{(X,X')} = 1_{X\otimes X'}.$$

(Compare the one-dimensional case as in [JS1, Proposition 5.2].) The second equality means that the identity pseudonatural transformation in the commutative square

$$\begin{tikzcd}
\mathcal{M}\times\mathcal{M}\times\mathcal{M} \arrow[r, "1\times\otimes"] \arrow[d, "\otimes\times 1"'] & \mathcal{M}\times\mathcal{M} \arrow[d, "\otimes"] \\
\mathcal{M}\times\mathcal{M} \arrow[r, "\otimes"'] & \mathcal{M}
\end{tikzcd}$$

is enriched with a monoidal structure (Section 1, Definition 3) via the isomorphism

$$\begin{tikzcd}[column sep=huge]
A\otimes B\otimes C\otimes X\otimes Y\otimes Z \arrow[r, bend left=50, "\chi^{\otimes(\otimes\times 1)}"{name=U}] \arrow[r, bend right=50, "\chi^{\otimes(1\times\otimes)}"'{name=D}] & A\otimes X\otimes B\otimes Y\otimes C\otimes Z
\arrow[from=U, to=D, phantom, "\Downarrow\theta_{2;(A,B,C),(X,Y,Z)}"]
\end{tikzcd}$$

obtained as the pasted composite,

$$\begin{tikzcd}[column sep=large]
 & A\otimes X\otimes B\otimes C\otimes Y\otimes Z \arrow[dr, "1\otimes 1\otimes 1\otimes\rho_{C,Y}\otimes 1"] & \\
A\otimes B\otimes C\otimes X\otimes Y\otimes Z \arrow[ur, "1\otimes\rho_{B\otimes C,X}\otimes 1\otimes 1"] \arrow[r, "1\otimes 1\otimes\rho_{C,X}\otimes 1\otimes 1"'] \arrow[dr, "1\otimes 1\otimes\rho_{C,X\otimes Y}\otimes 1"'] \arrow[ur, phantom, shift right=6, "\Downarrow 1\otimes\varpi_{B,I,C,X}\otimes 1\otimes 1"] \arrow[dr, phantom, shift left=6, "\Downarrow 1\otimes 1\otimes\varpi_{C,X,I,Y}\otimes 1"] & A\otimes B\otimes X\otimes C\otimes Y\otimes Z \arrow[u, "1\otimes\rho_{B,X}\otimes 1\otimes 1\otimes 1"'] \arrow[d, "1\otimes 1\otimes 1\otimes\rho_{C,Y}\otimes 1"] \arrow[r, phantom, "\Downarrow 1\otimes c_{\rho,\rho}\otimes 1"] & A\otimes X\otimes B\otimes Y\otimes C\otimes Z \\
 & A\otimes B\otimes X\otimes Y\otimes C\otimes Z \arrow[ur, "1\otimes\rho_{B,X}\otimes 1\otimes 1\otimes 1"'] &
\end{tikzcd}$$

(In fact, this monoidal pseudonatural transformation is a blip in the technical sense of [GPS, (3.8)].)

A *braided Gray monoid* is a Gray monoid equipped with a braiding.

Notice that, if $\mathscr{M}$ is braided, the identity functor of $\mathscr{M}$ is equipped with a canonical structure of monoidal homomorphism $\mathscr{M}\to\mathscr{M}^{\mathrm{rev}}$.

Remark. This definition of two-dimensional braiding is equivalent to that of Kapranov and Voevodsky [KV1; KV2] as modified by Baez-Neuchl [BN][4]; in particular, the need for an axiom equivalent to our last axiom was observed by Larry Breen and Baez-Neuchl. The fact that these

[4] See the Appendix of [C] for many other comments on, and corrections, to [KV1].

definitions agree is a higher version of [JS1, Proposition 5.1]. To make the connection, put

$$r_{W,\,Y\,|\,Z} = \varpi_{W,\,I,\,Y,\,Z}, \qquad r_{W\,|\,X,\,Z} = \varpi_{W,\,X,\,I,\,Z}$$

which fit into triangles as shown below:

$$\begin{tikzcd}
W \otimes Y \otimes Z \arrow[r, "1 \otimes \rho_{Y,Z}"] \arrow[dr, "\rho_{W \otimes Y, Z}"'] & W \otimes Z \otimes Y \arrow[d, "\rho_{W,Z} \otimes 1"] \arrow[dl, phantom, "\overset{\Rrightarrow}{r}_{W,Y\,|Z}"] \\
 & Z \otimes W \otimes Y
\end{tikzcd}
\qquad
\begin{tikzcd}
W \otimes X \otimes Z \arrow[d, "\rho_{W,X} \otimes 1"'] \arrow[dr, "\rho_{W,X \otimes Z}"] & \\
X \otimes W \otimes Z \arrow[r, "1 \otimes \rho_{W,Z}"'] \arrow[ur, phantom, "\overset{\Rrightarrow}{r}_{W|X,Z}"] & X \otimes Z \otimes W
\end{tikzcd}$$

A right closed braided Gray monoid is closed with the left internal hom equal to the right. We have the following consequence of Proposition 5.

COROLLARY 9. *If $\mathscr{M}$ is a closed braided Gray monoid then* $[\ ,\ ] : \mathscr{M}^{\mathrm{op}} \times \mathscr{M} \to \mathscr{M}$ *is a weak monoidal homomorphism.*

Proof. Since $\mathscr{M}$ is braided, $\otimes : \mathscr{M} \times \mathscr{M} \to \mathscr{M}$ is a monoidal homomorphism. Q.E.D.

PROPOSITION 10. *If $\mathscr{M}$ is a closed braided Gray monoid, if C is a pseudocomonoid in $\mathscr{M}$ and A is a pseudomonoid in $\mathscr{M}$ then $[C, A]$ becomes a pseudomonoid with multiplication given by the composite*:

$$[C, A] \otimes [C, A] \xrightarrow{\ \chi\ } [C \otimes C, A \otimes A] \xrightarrow{[\delta, \mu]} [C, A].$$

Proof. Since (C, A) is a pseudomonoid in $\mathscr{M}^{\mathrm{op}} \times \mathscr{M}$, the result is a corollary of Propositions 5 and Corollary 9. Q.E.D.

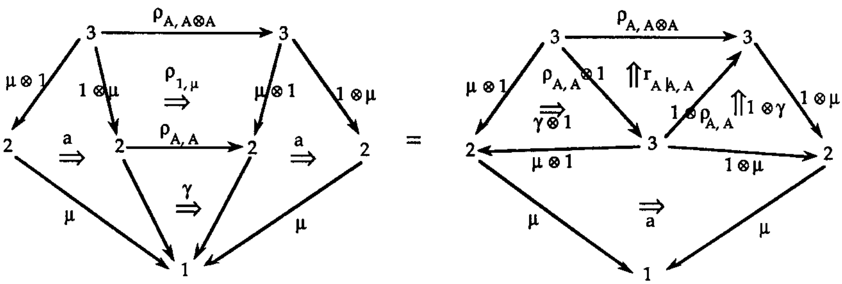
This provides a more conceptual proof of Proposition 4 since, applying the monoidal homomorphism $\mathscr{M}(I, -) : \mathscr{M} \to \mathbf{Cat}$ to the pseudomonoid $[C, A]$ in $\mathscr{M}$, we obtain a pseudomonoid $\mathscr{M}(I, [C, A])$ in $\mathbf{Cat}$; but this category is equivalent to $\mathscr{M}(C, A)$. So $\mathscr{M}(C, A)$ becomes a monoidal category via transport of structure across the equivalence; the tensor product is easily seen to be convolution.

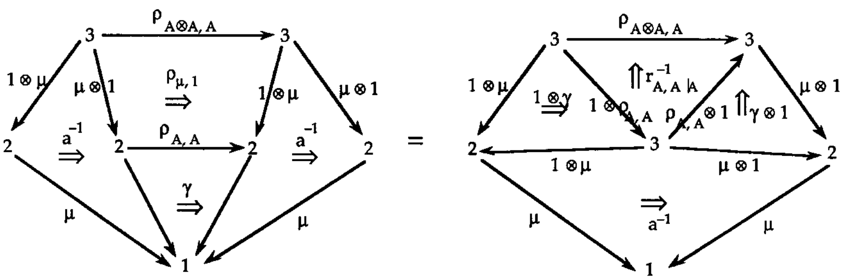
In fact, we use Proposition 10 to justify centring attention on the convolution structure of $\mathscr{M}(I, A)$, where A is a pseudomonoid, rather than the more general $\mathscr{M}(C, A)$, where C is any pseudocomonoid. For, $\mathscr{M}(C, A)$ is equivalent to $\mathscr{M}(I, [C, A])$ and $[C, A]$ is a pseudoalgebra. Our results for $\mathscr{M}(I, A)$ can, of course, be generalized to $\mathscr{M}(C, A)$ even when $\mathscr{M}$ is not closed or not braided, but we choose the simpler case for easier reading.

DEFINITION 13 (Compare [JS1, Definition 5.2]). Suppose $\mathscr{M}$ is a braided Gray monoid and A is a pseudomonoid in $\mathscr{M}$. A *braiding* for A is an invertible 2-cell γ as in the triangle

$$\begin{tikzcd} A \otimes A \arrow[rr, "\rho_{A,A}"] \arrow[dr, "\mu"'] & {} \arrow[d, phantom, "\overset{\gamma}{\Rightarrow}"] & A \otimes A \arrow[dl, "\mu"] \\ & A & \end{tikzcd}$$

such that the following two equations hold[5]:





Alternatively, the last two equations are equivalent to the following three equations:

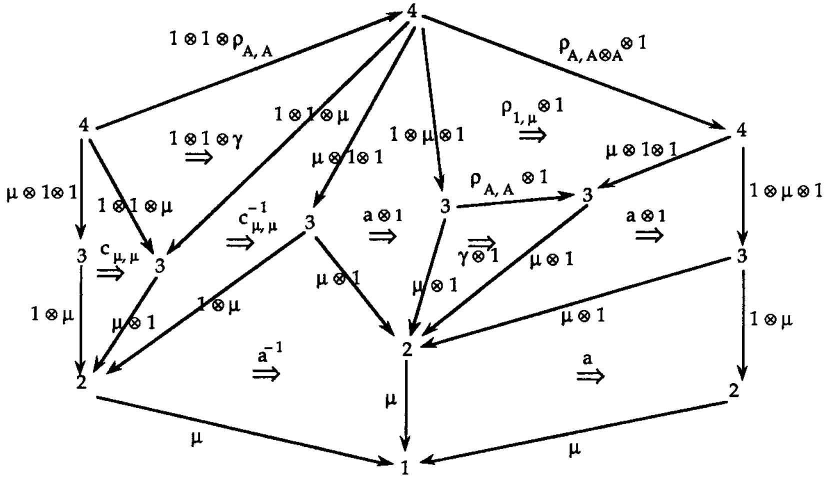
[5] As before, "3" is an abbreviation for "$A \otimes A \otimes A$."

$$
\begin{tikzcd}
A \arrow[rr, "\rho_{I,A}=1"] \arrow[d, "\eta\otimes 1"'] \arrow[drr, phantom, "\overset{\rho_{\eta,1}}{\Rightarrow}"] & & A \arrow[d, "1\otimes\eta"] \\
A\otimes A \arrow[rr, "\rho_{A,A}"] \arrow[dr, "\mu"'] & \overset{\gamma}{\Rightarrow} & A\otimes A \arrow[dl, "\mu"] \\
& A &
\end{tikzcd}
=
\begin{tikzcd}
& & A \arrow[dll, "\eta\otimes 1"'] \arrow[drr, "1\otimes\eta"] \arrow[dd, "1"] & & \\
A\otimes A \arrow[drr, "\mu"'] & \overset{\ell}{\Rightarrow} & & \overset{r^{-1}}{\Rightarrow} & A\otimes A \arrow[dll, "\mu"] \\
& & A & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[rr, "\rho_{A,I}=1"] \arrow[d, "1\otimes\eta"'] \arrow[drr, phantom, "\overset{\rho_{1,\eta}}{\Rightarrow}"] & & A \arrow[d, "\eta\otimes 1"] \\
A\otimes A \arrow[rr, "\rho_{A,A}"] \arrow[dr, "\mu"'] & \overset{\gamma}{\Rightarrow} & A\otimes A \arrow[dl, "\mu"] \\
& A &
\end{tikzcd}
=
\begin{tikzcd}
& & A \arrow[dll, "1\otimes\eta"'] \arrow[drr, "\eta\otimes 1"] \arrow[dd, "1"] & & \\
A\otimes A \arrow[drr, "\mu"'] & \overset{r}{\Rightarrow} & & \overset{\ell^{-1}}{\Rightarrow} & A\otimes A \arrow[dll, "\mu"] \\
& & A & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& & 4 \arrow[drr, "\rho_{A,A\otimes A}\otimes 1"] \arrow[d, phantom, "\Uparrow\varpi_{A,A,A,A}"] & & \\
4 \arrow[urr, "1\otimes 1\otimes\rho_{A,A}"] \arrow[rr, "\rho_{A,A}\otimes 1\otimes 1"] \arrow[d, "\mu\otimes 1\otimes 1"'] \arrow[dr, phantom, "\overset{\gamma\otimes 1\otimes 1}{\Rightarrow}"] & & 4 \arrow[rr, "1\otimes\rho_{A\otimes A,A}"] \arrow[dll, "\mu\otimes 1\otimes 1"] \arrow[dddl, "1\otimes 1\otimes\mu"] \arrow[dd, "1\otimes\mu\otimes 1"] \arrow[dr, phantom, "\overset{1\otimes\rho_{\mu,1}}{\Rightarrow}"] & & 4 \arrow[dl, "1\otimes 1\otimes\mu"'] \arrow[d, "1\otimes\mu\otimes 1"] \\
3 \arrow[ddd, "1\otimes\mu"'] \arrow[dr, phantom, "\overset{c_{\mu,\mu}}{\Rightarrow}"] & & & 3 \arrow[dddr, "1\otimes\mu"] \arrow[r, phantom, "\overset{1\otimes a^{-1}}{\Rightarrow}"] & 3 \arrow[ddd, "1\otimes\mu"] \\
& & 3 \arrow[ur, "1\otimes\rho_{A,A}"] \arrow[ddrr, "1\otimes\mu"'] \arrow[r, phantom, "\overset{1\otimes\gamma}{\Rightarrow}"] & & \\
& 3 \arrow[dl, "\mu\otimes 1"'] \arrow[drrr, "1\otimes\mu"] \arrow[ur, phantom, "\overset{1\otimes a^{-1}}{\Rightarrow}"] & & & \\
2 \arrow[drr, "\mu"'] \arrow[rrrr, phantom, "\overset{a}{\Rightarrow}"'] & & & & 2 \arrow[dll, "\mu"] \\
& & 1 & &
\end{tikzcd}
$$

$\parallel$

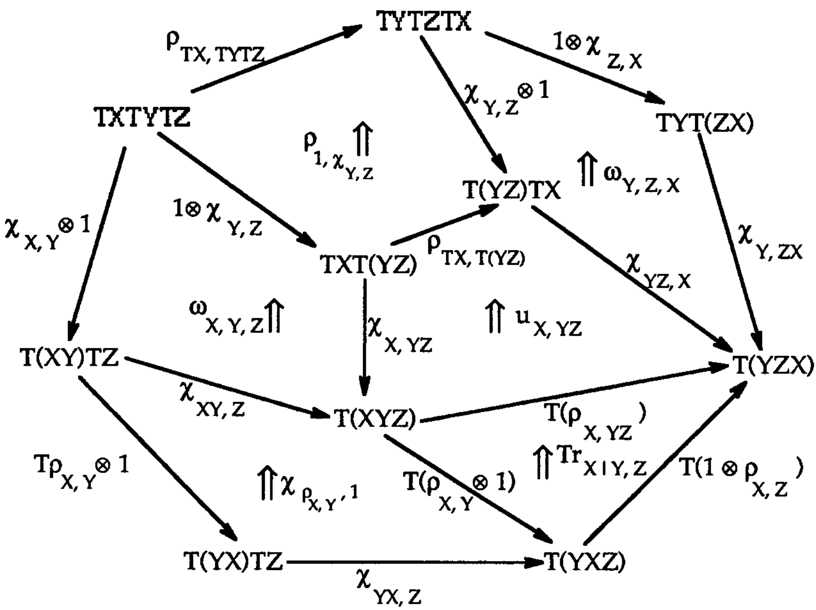


DEFINITION 14. Suppose $\mathscr{M}, \mathscr{N}$ are braided Gray monoids and $T : \mathscr{M} \to \mathscr{N}$ is a weak monoidal homomorphims. A *braiding* for T is an invertible modification,

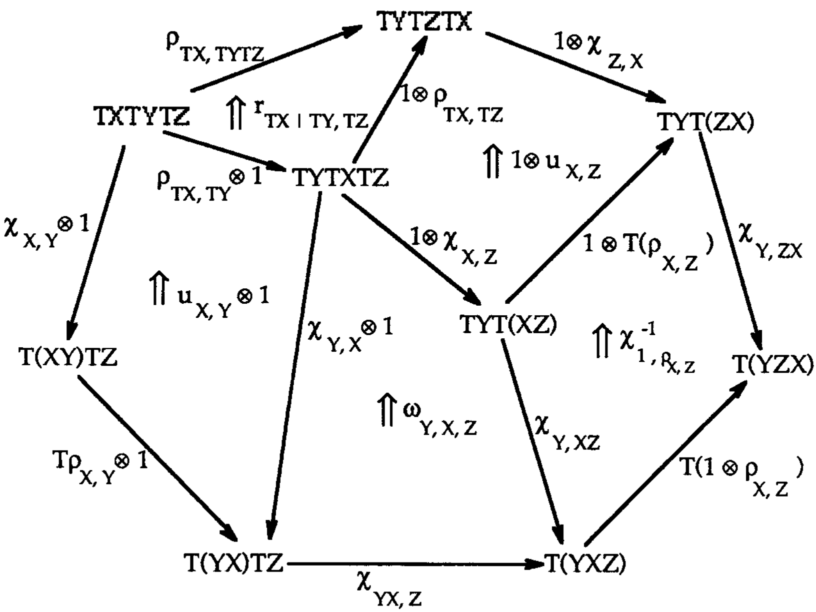
$$\begin{tikzcd}
TX \otimes TY \arrow[r, "\rho_{TX, TY}"] \arrow[d, "\chi_{X,Y}"'] \arrow[dr, phantom, "\overset{u_{X,Y}}{\Rightarrow}"] & TY \otimes TX \arrow[d, "\chi_{Y,X}"] \\
T(X \otimes Y) \arrow[r, "T(\rho_{X,Y})"'] & T(Y \otimes X)
\end{tikzcd}$$

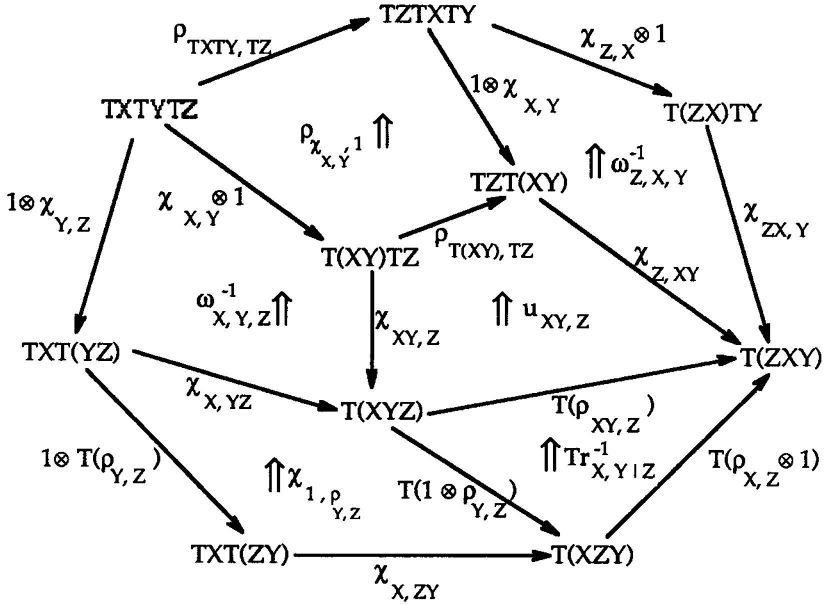
such that the following two equations (expressing $u_{X,\, Y \otimes Z}$ in terms of $u_{X,\, Y}$ and $u_{X,\, Z}$, and $u_{X \otimes Y,\, Z}$ in terms of $u_{X,\, Y}$ and $u_{X,\, Z}$) hold[6]:

[6] For $\alpha : gf \Rightarrow h$, we abusively write $T\alpha : Tg \circ Tf \Rightarrow Th$ for the composite of the real $T\alpha : T(gf) \Rightarrow Th$ and the canonical isomorphism $Tg \circ Tf \Rightarrow T(gf)$.

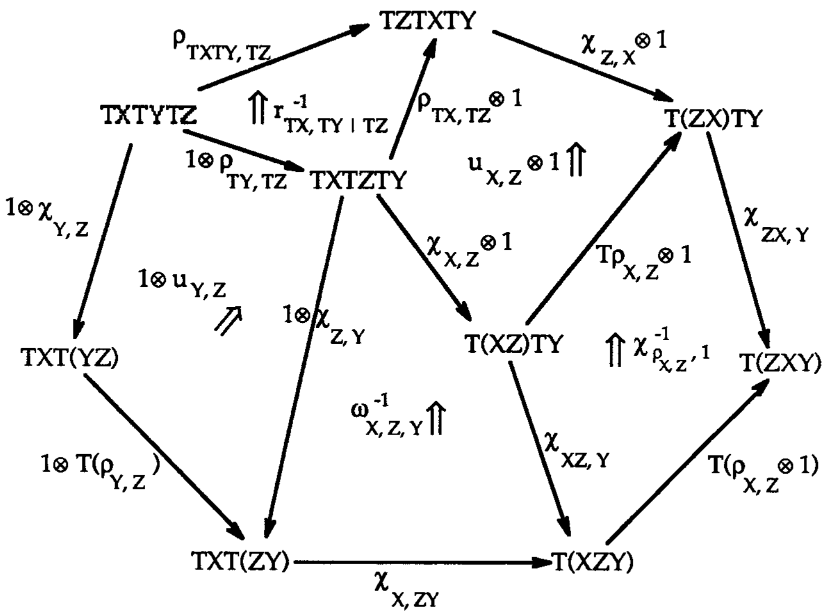

TYTZTX
TXTYTZ
TYT(ZX)
T(YZ)TX
TXT(YZ)
T(XY)TZ
T(YZX)
T(XYZ)
T(YX)TZ
T(YXZ)
$\rho_{TX,TYTZ}$
$1\otimes\chi_{Z,X}$
$\chi_{Y,Z}\otimes 1$
$\rho_{1,\chi_{Y,Z}}$
$\omega_{Y,Z,X}$
$\chi_{X,Y}\otimes 1$
$1\otimes\chi_{Y,Z}$
$\rho_{TX,T(YZ)}$
$\chi_{YZ,X}$
$\chi_{Y,ZX}$
$\omega_{X,Y,Z}$
$\chi_{X,YZ}$
$u_{X,YZ}$
$\chi_{XY,Z}$
$T(\rho_{X,YZ})$
$Tr_{X|Y,Z}$
$T\rho_{X,Y}\otimes 1$
$\chi_{\rho_{X,Y},1}$
$T(\rho_{X,Y}\otimes 1)$
$T(1\otimes\rho_{X,Z})$
$\chi_{YX,Z}$

=


TYTZTX
TXTYTZ
TYT(ZX)
TYTXTZ
T(XY)TZ
TYT(XZ)
T(YZX)
T(YX)TZ
T(YXZ)
$\rho_{TX,TYTZ}$
$1\otimes\chi_{Z,X}$
$r_{TX|TY,TZ}$
$1\otimes\rho_{TX,TZ}$
$1\otimes u_{X,Z}$
$\rho_{TX,TY}\otimes 1$
$\chi_{X,Y}\otimes 1$
$1\otimes\chi_{X,Z}$
$1\otimes T(\rho_{X,Z})$
$\chi_{Y,ZX}$
$u_{X,Y}\otimes 1$
$\chi_{Y,X}\otimes 1$
$\chi^{-1}_{1,\rho_{X,Z}}$
$\omega_{Y,X,Z}$
$\chi_{Y,XZ}$
$T\rho_{X,Y}\otimes 1$
$T(1\otimes\rho_{X,Z})$
$\chi_{YX,Z}$

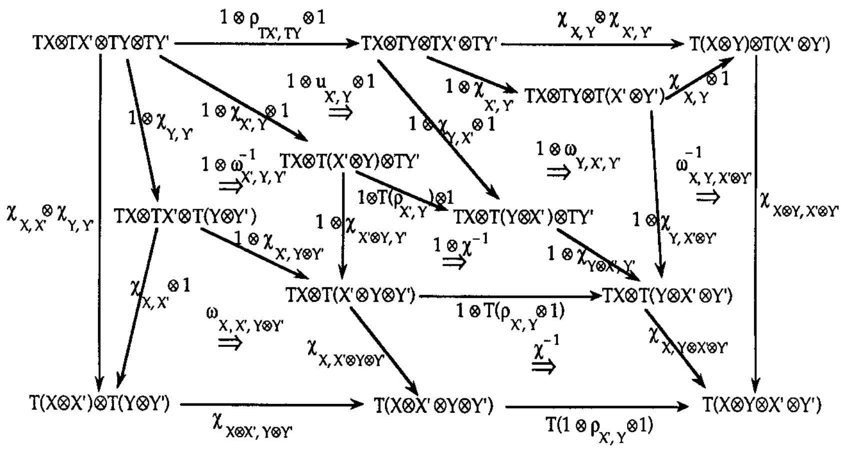

$TZTXTY$
$\rho_{TXTY,TZ}$
$\chi_{Z,X}\otimes 1$
$TXTYTZ$
$1\otimes\chi_{X,Y}$
$T(ZX)TY$
$\rho_{\chi_{X,Y},1}\Uparrow$
$\Uparrow\omega^{-1}_{Z,X,Y}$
$TZT(XY)$
$1\otimes\chi_{Y,Z}$
$\chi_{X,Y}\otimes 1$
$\chi_{ZX,Y}$
$\rho_{T(XY),TZ}$
$T(XY)TZ$
$\chi_{Z,XY}$
$\omega^{-1}_{X,Y,Z}\Uparrow$
$\Uparrow u_{XY,Z}$
$\chi_{XY,Z}$
$TXT(YZ)$
$T(ZXY)$
$\chi_{X,YZ}$
$T(XYZ)$
$T(\rho_{XY,Z})$
$1\otimes T(\rho_{Y,Z})$
$\Uparrow Tr^{-1}_{X,Y\mid Z}$
$T(\rho_{X,Z}\otimes 1)$
$\Uparrow\chi_{1,\rho_{Y,Z}}$
$T(1\otimes\rho_{Y,Z})$
$TXT(ZY)$
$T(XZY)$
$\chi_{X,ZY}$

$$=$$


$TZTXTY$
$\rho_{TXTY,TZ}$
$\chi_{Z,X}\otimes 1$
$TXTYTZ$
$\Uparrow r^{-1}_{TX,TY\mid TZ}$
$\rho_{TX,TZ}\otimes 1$
$T(ZX)TY$
$1\otimes\rho_{TY,TZ}$
$TXTZTY$
$u_{X,Z}\otimes 1\Uparrow$
$1\otimes\chi_{Y,Z}$
$\chi_{X,Z}\otimes 1$
$T\rho_{X,Z}\otimes 1$
$\chi_{ZX,Y}$
$1\otimes u_{Y,Z}$
$1\otimes\chi_{Z,Y}$
$TXT(YZ)$
$T(XZ)TY$
$\Uparrow\chi^{-1}_{\rho_{X,Z},1}$
$T(ZXY)$
$\omega^{-1}_{X,Z,Y}\Uparrow$
$\chi_{XZ,Y}$
$1\otimes T(\rho_{Y,Z})$
$T(\rho_{X,Z}\otimes 1)$
$TXT(ZY)$
$T(XZY)$
$\chi_{X,ZY}$

$$
\begin{tikzcd}
\mathcal{M}\times\mathcal{M} \arrow[r, "\otimes"] \arrow[d, "T\times T"'] \arrow[dr, phantom, "\overset{\chi}{\Rightarrow}"] & \mathcal{M} \arrow[d, "T"] \\
\mathcal{N}\times\mathcal{N} \arrow[r, "\otimes"'] & \mathcal{N}
\end{tikzcd}
$$

These conditions are equivalent to the requirement that the pseudo-natural transformation χ (above) should become monoidal when equipped with the structure $\chi_{2;(X,X'),(Y,Y')}$ and χ_0 given, respectively, by the following two pasting composites:



$$
\begin{tikzcd}
 & & TI\otimes TI \arrow[dd, "\chi_{I,I}"] \\
I \arrow[urr, "\iota\otimes\iota"] \arrow[r, "\iota"] \arrow[drr, "\iota"'] & TI \arrow[ur, "\iota\otimes 1"'] \arrow[dr, "1"] \arrow[r, phantom, "\overset{\kappa_I}{\Rightarrow}"] & \\
 & & TI
\end{tikzcd}
$$

A weak monoidal homomorphism is called *braided* when it is equipped with a braiding.

Suppose $S, T : \mathcal{M} \to \mathcal{N}$ are braided weak monoidal homomorphisms. A monoidal pseudo-natural transformation $\theta : S \to T$ is called *braided* when it satisfies the following condition:

$$
\begin{tikzcd}[column sep=small]
 & SY\otimes SX \arrow[rr, "\chi_{Y,X}"] & & S(Y\otimes X) \arrow[dr, "\theta_{Y\otimes X}"] & \\
SX\otimes SY \arrow[ur, "\rho_{SX,SY}"] \arrow[rr, "\chi_{X,Y}"'] \arrow[dr, "\theta_X\otimes\theta_Y"'] \arrow[urr, phantom, "\Uparrow u_{X,Y}"] & & S(X\otimes Y) \arrow[ur, "S(\rho_{X,Y})"] \arrow[dr, "\theta_{X\otimes Y}"] \arrow[rr, phantom, "\Uparrow\theta^{-1}_{\rho_{X,Y}}"] & & T(Y\otimes X) \\
 & TX\otimes TY \arrow[rr, "\chi_{X,Y}"'] \arrow[ur, phantom, "\Uparrow\theta_{2;X,Y}"] & & T(X\otimes Y) \arrow[ur, "T(\rho_{X,Y})"'] &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
 & SY\otimes SX \arrow[rr, "\chi_{Y,X}"] \arrow[dr, "\theta_Y\otimes\theta_X"] & & S(Y\otimes X) \arrow[dr, "\theta_{Y\otimes X}"] \arrow[d, phantom, "\Uparrow\theta_{2;Y,X}"] & \\
SX\otimes SY \arrow[ur, "\rho_{SX,SY}"] \arrow[dr, "\theta_X\otimes\theta_Y"'] \arrow[rr, phantom, "\Uparrow\rho_{\theta_X,\theta_Y}"] & & TY\otimes TX \arrow[rr, "\chi_{Y,X}"'] & & T(Y\otimes X) \\
 & TX\otimes TY \arrow[rr, "\chi_{X,Y}"'] \arrow[ur, "\rho_{TX,TY}"] & & T(X\otimes Y) \arrow[ur, "T(\rho_{X,Y})"'] \arrow[u, phantom, "\Uparrow u_{X,Y}"] &
\end{tikzcd}
$$

This is equivalent to the requirement that the following modification should be monoidal:

$$
\begin{tikzcd}
\mathcal{M}\times\mathcal{M} \arrow[rr, "\otimes"] \arrow[d, "S\times S"'] \arrow[d, bend left=60, "T\times T"] \arrow[d, phantom, shift left=3ex, "\overset{\theta\times\theta}{\Rightarrow}"] & \arrow[d, phantom, "\overset{\chi}{\Rightarrow}"] & \mathcal{M} \arrow[d, "T"] \\
\mathcal{N}\times\mathcal{N} \arrow[rr, "\otimes"'] & \, & \mathcal{N}
\end{tikzcd}
\overset{\theta_2}{\Rrightarrow}
\begin{tikzcd}
\mathcal{M}\times\mathcal{M} \arrow[rr, "\otimes"] \arrow[d, "S\times S"'] & \arrow[d, phantom, "\overset{\chi}{\Rightarrow}"] & \mathcal{M} \arrow[d, "T"] \arrow[d, bend right=60, "S"'] \arrow[d, phantom, shift right=3ex, "\overset{\theta}{\Rightarrow}"] \\
\mathcal{N}\times\mathcal{N} \arrow[rr, "\otimes"'] & \, & \mathcal{N}
\end{tikzcd}
$$

EXAMPLE 4. If $\mathscr{M}$ is a braided Gray monoid then $\mathscr{M}(I, -) : \mathscr{M} \to \mathbf{Cat}$ is a braided weak monoidal homomorphism.

PROPOSITION 11. *Braided weak monoidal homomorphisms preserve braided pseudomonoids.*

Proof. Braided pseudomonoids can be identified with braided weak monoidal (normal) homomorphisms $1 \to \mathscr{M}$, and a composite of braided weak monoidal homomorphisms is canonically braided. Q.E.D.

EXAMPLE 5. If $\mathscr{M}$ is a braided Gray monoid and A is a braided pseudomonoid in $\mathscr{M}$ then $\mathscr{M}(I, A)$ is a braided monoidal category. The braiding is given by the following pasting composite:

$$
\begin{tikzcd}
 & A\otimes A \arrow[dr, "\mu"] \arrow[dd, "\rho_{A,A}"] & \\
I \arrow[ur, "f\otimes g"] \arrow[dr, "g\otimes f"'] \arrow[r, phantom, "\Downarrow\rho_{f,g}"] & \, \arrow[r, phantom, "\Downarrow\gamma"] & A \\
 & A\otimes A \arrow[ur, "\mu"'] &
\end{tikzcd}
$$

PROPOSITION 12. *In the situation of Proposition* 2, *if T is braided then so is H.*

Proof. Since $T : \mathscr{A} \times \mathscr{C} \to \mathscr{B}$ is braided, we have data as follows in which the vertical arrows are equivalences:

$$\begin{tikzcd}
T(A,C)\otimes T(A',C') \arrow[r, "\rho_{T(A,C),T(A',C')}"] \arrow[d, "\chi_{(A,C),(A',C')}"'] \arrow[dr, phantom, "\overset{u_{(A,C),(A',C')}}{\Rightarrow}"] & T(A',C')\otimes T(A,C) \arrow[d, "\chi_{(A',C'),(A,C)}"] \\
T(A\otimes A',C\otimes C') \arrow[r, "T(\rho_{A,A'},\rho_{C,C'})"'] & T(A'\otimes A,C'\otimes C)
\end{tikzcd}$$

What we require are data as displayed in the following square:

$$\begin{tikzcd}
H(A,B)\otimes H(A',B') \arrow[r, "\rho_{H(A,B),H(A',B')}"] \arrow[d, "\chi_{(A,B),(A',B')}"'] \arrow[dr, phantom, "\overset{u_{(A,B),(A',B')}}{\Rightarrow}"] & H(A',B')\otimes H(A,B) \arrow[d, "\chi_{(A',B'),(A,B)}"] \\
H(A\otimes A',B\otimes B') \arrow[r, "H(\rho_{A,A'},\rho_{B,B'})"'] & H(A'\otimes A,B'\otimes B)
\end{tikzcd}$$

We saw in the proof of Proposition 2 what the vertical arrows in the last square correspond to under the pseudo-natural equivalence π of the parameterized biadjunction. So we obtain the required data $u_{(A,B),(A',B')}$ corresponding to the following pasting composite (in which we write χ^{-1} for an inverse adjoint equivalence for χ).

$$\begin{tikzcd}
& T(A'\otimes A, H(A,B)\otimes H(A',B')) \arrow[dl, "T(\rho_{A,A'},1)"'] \arrow[dr, "T(1,\rho_{H(A,B),H(A',B')})"] \arrow[d, phantom, "\cong"] & \\
T(A\otimes A', H(A,B)\otimes H(A',B')) \arrow[rr, "T(\rho_{A,A'},\rho_{H(A,B),H(A',B')})"] \arrow[d, "\chi^{-1}_{(A,H(A,B)),(A',H(A',B'))}"'] \arrow[drr, phantom, "\overset{\chi^{-1}u\,\chi^{-1}}{\cong}"] & {} & T(A'\otimes A, H(A',B')\otimes H(A,B)) \arrow[d, "\chi^{-1}_{(A',H(A',B')),(A,H(A,B))}"] \\
T(A,H(A,B))\otimes T(A',H(A',B')) \arrow[rr, "\rho_{T(A,H(A,B)),T(A',H(A',B'))}"'] \arrow[d, "\varepsilon_{A,B}\otimes\varepsilon_{A',B'}"'] \arrow[drr, phantom, "\cong"] & & T(A',H(A',B'))\otimes T(A,H(A,B)) \arrow[d, "\varepsilon_{A',B'}\otimes\varepsilon_{A,B}"] \\
B\otimes B' \arrow[rr, "\rho_{B,B'}"'] & & B'\otimes B
\end{tikzcd}$$

The axioms required on these data for H to become braided can be verified. Q.E.D.

5. SYLLEPTIC AND SYMMETRIC GRAY MONOIDS, AND SYMMETRIC PSEUDOMONOIDS

Recall from [JS1, Proposition 5.4] that a braiding on a monoidal category is a symmetry if and only if the tensor-product functor is braided monoidal. We shall consider the corresponding situation for monoidal bicategories.

DEFINITION 15. A *syllepsis* for a braided Gray monoid $\mathscr{M}$ consists of an invertible modification,

$$\begin{tikzcd}
X\otimes Y \arrow[rr, "1"] \arrow[dr, "\rho_{X,Y}"'] & \arrow[d, phantom, "\Downarrow v_{X,Y}"] & X\otimes Y \\
& Y\otimes X \arrow[ur, "\rho_{Y,X}"'] &
\end{tikzcd}$$

such that the following two equalities hold:

$$\begin{tikzcd}
X\otimes Y\otimes Z \arrow[rr, "1"] \arrow[d, "1\otimes\rho_{Y,Z}"'] \arrow[drr, "\rho_{X\otimes Y,Z}"] \arrow[drr, phantom, shift left=4ex, "\Downarrow v_{X\otimes Y,Z}"] \arrow[drr, phantom, shift right=4ex, "r_{X,Y|Z}\Downarrow"'] & & X\otimes Y\otimes Z \\
X\otimes Z\otimes Y \arrow[rr, "\rho_{X,Z}\otimes 1"'] & & Z\otimes X\otimes Y \arrow[u, "\rho_{Z,X\otimes Y}"']
\end{tikzcd}
=
\begin{tikzcd}
X\otimes Y\otimes Z \arrow[rr, "1"] \arrow[dd, "1\otimes\rho_{Y,Z}"'] \arrow[dr, phantom, "\Downarrow 1\otimes v_{Y,Z}"] & & X\otimes Y\otimes Z \\
& X\otimes Z\otimes Y \arrow[ur, "1\otimes\rho_{Z,Y}"] \arrow[r, phantom, "\overset{\Rightarrow}{r_{Z|X,Y}}"] \arrow[d, phantom, "\underset{\Rightarrow}{v_{X,Z}\otimes 1}"] & \\
X\otimes Z\otimes Y \arrow[ur, "1"] \arrow[rr, "\rho_{X,Z}\otimes 1"'] & \ & Z\otimes X\otimes Y \arrow[ul, "\rho_{Z,X}\otimes 1"'] \arrow[uu, "\rho_{Z,X\otimes Y}"']
\end{tikzcd}$$

$$\begin{tikzcd}
X\otimes Y\otimes Z \arrow[rr, "1"] \arrow[d, "\rho_{X,Y\otimes Z}"'] \arrow[rr, phantom, shift right=5ex, "v_{X,Y\otimes Z}\Downarrow"] & & X\otimes Y\otimes Z \\
Y\otimes Z\otimes X \arrow[rr, "1\otimes\rho_{Z,X}"'] \arrow[urr, "\rho_{Y\otimes Z,X}"] \arrow[rr, phantom, shift left=5ex, "\Downarrow r_{Y,Z|X}"] & & Y\otimes X\otimes Z \arrow[u, "\rho_{Y,X}\otimes 1"']
\end{tikzcd}
=
\begin{tikzcd}
X\otimes Y\otimes Z \arrow[rr, "1"] \arrow[dd, "\rho_{X,Y\otimes Z}"'] \arrow[dr, "\rho_{X,Y}\otimes 1"] & \arrow[d, phantom, "\Downarrow v_{X,Y}\otimes 1"] & X\otimes Y\otimes Z \\
\arrow[r, phantom, "\overset{\Leftarrow}{r_{X|Y,Z}}"] & Y\otimes X\otimes Z \arrow[dl, "1\otimes\rho_{X,Z}"'] \arrow[dr, "1"] \arrow[d, phantom, "1\otimes v_{X,Z}\Leftarrow"] & \\
Y\otimes Z\otimes X \arrow[rr, "1\otimes\rho_{Z,X}"'] & \ & Y\otimes X\otimes Z \arrow[uu, "\rho_{Y,X}\otimes 1"']
\end{tikzcd}$$

It follows that $v_{I,X}=v_{X,I}=1$ (the proof is similar to that in [JS1, Proposition 2.1]). Syllepses are in bijection with braidings for the monoidal homomorphism $\otimes : \mathscr{M} \times \mathscr{M} \to \mathscr{M}$: given a braiding w for $\otimes : \mathscr{M} \times \mathscr{M} \to \mathscr{M}$ constituted by the data

$$\begin{tikzcd}
W\otimes X\otimes Y\otimes Z \arrow[r, "\rho_{W\otimes X,\, Y\otimes Z}"] \arrow[d, "1\otimes\rho_{X,Y}\otimes 1"'] \arrow[dr, phantom, "\overset{w_{W,X,Y,Z}}{\Rightarrow}"] & Y\otimes Z\otimes W\otimes X \arrow[d, "1\otimes\rho_{Z,W}\otimes 1"] \\
W\otimes Y\otimes X\otimes Z \arrow[r, "\rho_{W,Y}\otimes\rho_{X,Z}"'] & Y\otimes W\otimes Z\otimes X
\end{tikzcd}$$

the corresponding syllepsis v is given by the formula $v_{X,Y} = w_{X,I,I,Y}$; given a syllepsis v, the corresponding braiding w has $w_{W,X,Y,Z}$ equal to the following pasting composite:

$$\begin{tikzcd}
W\otimes X\otimes Y\otimes Z \arrow[rr, "\rho_{W\otimes X,\, Y\otimes Z}"] \arrow[dd, "1\otimes\rho_{X,Y}\otimes 1"'] \arrow[dr, "1\otimes\rho_{X,Y\otimes Z}"] & \Uparrow r^{-1}_{W,X\,|\,Y\otimes Z} & Y\otimes Z\otimes W\otimes X \arrow[ddrr, "1\otimes\rho_{Z,W}\otimes 1"] & & \\
\Rightarrow 1\otimes r_{X\,|\,Y,Z} & W\otimes Y\otimes Z\otimes X \arrow[ur, "\rho_{W,Y\otimes Z}\otimes 1"] \arrow[dr, "\rho_{W,Y}\otimes 1\otimes 1"'] \arrow[r, phantom, "\Leftarrow r_{W\,|\,Y,Z}\otimes 1"] & {} & \Uparrow 1\otimes v_{W,Z}\otimes 1 & \\
W\otimes Y\otimes X\otimes Z \arrow[ur, "1\otimes 1\otimes\rho_{X,Z}"] \arrow[rr, "\rho_{W,Y}\otimes\rho_{X,Z}"'] & & Y\otimes W\otimes Z\otimes X \arrow[uu, "1\otimes\rho_{W,Z}\otimes 1"'] \arrow[rr, "1"'] & & Y\otimes W\otimes Z\otimes X
\end{tikzcd}$$

A Gray monoid is called *sylleptic*[7] when it is equipped with a braiding and a syllepsis.

COROLLARY 13. *If $\mathscr{M}$ is a closed sylleptic Gray monoid then* $[\ ,\ ] : \mathscr{M}^{\mathrm{op}} \times \mathscr{M} \to \mathscr{M}$ *is a braided weak monoidal homomorphism.*

PROPOSITION 14. *If $\mathscr{M}$ is a closed sylleptic Gray monoid, if C is a braided pseudocomonoid in $\mathscr{M}$ and A is a braided pseudomonoid in $\mathscr{M}$ then the convolution pseudomonoid structure on $[C, A]$ acquires a braiding given by the following pasting composite*:

[7] The term "weakly involutory" has been used by Baez-Neuchl [BN].

$$
\begin{tikzcd}
{[C,A]\otimes[C,A]} \arrow[rr, "\rho_{[C,A],[C,A]}"] \arrow[d, "\chi_{[C,A],[C,A]}"'] \arrow[drr, phantom, "\overset{u_{[C,A],[C,A]}}{\Rightarrow}"] & & {[C,A]\otimes[C,A]} \arrow[d, "\chi_{[C,A],[C,A]}"] \\
{[C\otimes C, A\otimes A]} \arrow[rr, "{[\rho_{C,C}, \rho_{A,A}]}"'] \arrow[dr] & {} \arrow[d, phantom, "\overset{[\gamma,\gamma]}{\Rightarrow}"] & {[C\otimes C, A\otimes A]} \arrow[dl] \\
& {[C,A]} &
\end{tikzcd}
$$

Definition 16. Suppose $\mathscr{M}, \mathscr{N}$ are sylleptic Gray monoids. A weak monoidal homomorphism $T : \mathscr{M} \to \mathscr{N}$ is called *sylleptic* when it is braided and the following equation holds:

$$
\begin{tikzcd}
TX\otimes TY \arrow[r, "\rho_{TX,TY}"] \arrow[d, "\chi_{X,Y}"'] \arrow[rr, bend right, "1_{TX,TY}"'] & TY\otimes TX \arrow[r, "\rho_{TY,TX}"] \arrow[d, phantom, "\Uparrow v_{TX,TY}"] & TX\otimes TY \arrow[d, "\chi_{X,Y}"] \\
T(X\otimes Y) \arrow[rr, "1_{T(X\otimes Y)}"] \arrow[rr, bend right, "T(1_{X\otimes Y})"'] & {} \arrow[d, phantom, "\wr\|"] & T(X\otimes Y) \\
& {} &
\end{tikzcd}
$$

$$\|$$

$$
\begin{tikzcd}
TX\otimes TY \arrow[r, "\rho_{TX,TY}"] \arrow[d, "\chi_{X,Y}"'] \arrow[dr, phantom, "\overset{u_{X,Y}}{\Rightarrow}"] & TY\otimes TX \arrow[r, "\rho_{TY,TX}"] \arrow[d, "\chi_{Y,X}"] \arrow[dr, phantom, "\overset{u_{Y,X}}{\Rightarrow}"] & TX\otimes TY \arrow[d, "\chi_{X,Y}"] \\
T(X\otimes Y) \arrow[r, "T(\rho_{X,Y})"'] \arrow[rr, bend right=20, "T(\rho_{Y,X}\circ\rho_{X,Y})"] \arrow[rr, bend right=70, "T(1_{X\otimes Y})"'] & T(Y\otimes X) \arrow[r, "T(\rho_{Y,X})"'] \arrow[d, phantom, "\wr\|"] & T(X\otimes Y) \\
& {} \arrow[d, phantom, "\Uparrow T(v_{X,Y})"] & \\
& {} &
\end{tikzcd}
$$

The composite of two sylleptic weak homomorphisms is sylleptic.

Example 6. If $\mathcal{M}$ is a sylleptic Gray monoid then $\mathcal{M}(I, -) : \mathcal{M} \to \mathbf{Cat}$ is a sylleptic weak monoidal homomorphism.

Proposition 15. *In the situation of Propositions* 2 *and* 12, *if T is sylleptic then so is H.*

Definition 17. Suppose $\mathcal{M}$ is a sylleptic Gray monoid. A *symmetry* for a pseudomonoid A in $\mathcal{M}$ is a braiding γ satisfying the following condition:

$$
\begin{tikzcd}
A \otimes A \arrow[r, "\rho_{A,A}"] \arrow[dr, "\mu"'] \arrow[r, phantom, shift right=6ex, "\overset{\gamma}{\Rightarrow}"] & A \otimes A \arrow[r, "\rho_{A,A}"] \arrow[d, "\mu"] \arrow[r, phantom, shift right=6ex, "\overset{\gamma}{\Rightarrow}"] & A \otimes A \arrow[dl, "\mu"] \\
& A &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \otimes A \arrow[r, "\rho_{A,A}"] \arrow[dr, "1"'] & A \otimes A \arrow[d, "\rho_{A,A}"] \arrow[dl, phantom, shift left=2ex, "\overset{v_{A,A}}{\Rightarrow}"] & \\
& A \otimes A \arrow[r, "\mu"'] & A
\end{tikzcd}
$$

Clearly a symmetric pseudomonoid corresponds to a sylleptic weak homomorphism $1 \to \mathcal{M}$.

Proposition 16. *Sylleptic weak monoidal homomorphisms preserve symmetric pseudomonoids.*

Example 7. If $\mathcal{M}$ is a sylleptic Gray monoid and A is a symmetric pseudomonoid then $\mathcal{M}(I, A)$ is a symmetric monoidal category.

Definition 18. A Gray monoid $\mathcal{M}$ is called *symmetric* when it is sylleptic and the following equation holds:

$$
\begin{tikzcd}
X \otimes Y \arrow[rr, "1"] \arrow[dr, "\rho_{X,Y}"'] & \arrow[d, phantom, "\Downarrow v_{X,Y}"] & X \otimes Y \arrow[r, "\rho_{X,Y}"] & Y \otimes X \\
& Y \otimes X \arrow[ur, "\rho_{Y,X}"'] & &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
X \otimes Y \arrow[r, "\rho_{X,Y}"] & Y \otimes X \arrow[rr, "1"] \arrow[dr, "\rho_{Y,X}"'] & \arrow[d, phantom, "\Downarrow v_{Y,X}"] & Y \otimes X \\
& & X \otimes Y \arrow[ur, "\rho_{X,Y}"'] &
\end{tikzcd}
$$

This is equivalent to the requirement that the braided homomorphism $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ should be sylleptic.

Corollary 17. *If $\mathcal{M}$ is a closed symmetric Gray monoid then $[\ ,\ ] : \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{M}$ is a sylleptic weak monoidal homomorphism.*

Proposition 18. *If $\mathcal{M}$ is a closed symmetric Gray monoid, if C is a symmetric pseudocomonoid in M and A is a symmetric pseudomonoid in $\mathcal{M}$ then the braiding of Proposition* 14 *is a symmetry.*

6. BALANCED GRAY MONOIDS, AND BALANCED PSEUDOMONOIDS

There is a bicategorical generalization of balanced monoidal category [JS1].

DEFINITION 19. A *twist* for a braided Gray monoid $\mathscr{M}$ is a monoidal pseudo-natural equivalence from S to T, where $S : \mathscr{M} \to \mathscr{M}^{\mathrm{rev}}$ is the identity homomorphism equipped with the canonical monoidal structure determined by the given braiding, while $T : \mathscr{M} \to \mathscr{M}^{\mathrm{rev}}$ is the identity homomorphism equipped with the monoidal structure from a quasi-inverse braiding. The data for a twist (t, v) are displayed as the following squares in which the 2-cells are isomorphisms and the horizontal arrows are equivalences (we take the isomorphism $t_I \cong 1_I$ to be an identity for simplicity):

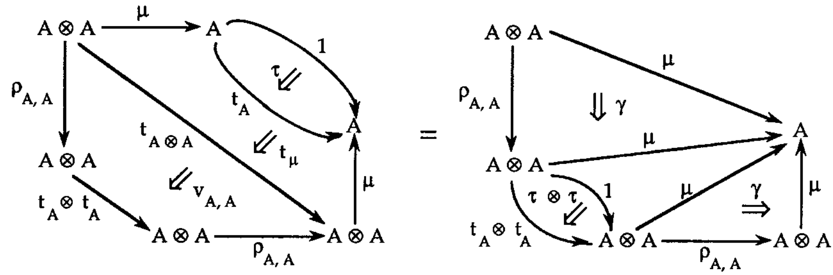
$$\begin{tikzcd}
X \arrow[r, "t_X"] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{t_f}{\Rightarrow}"] & X \arrow[d, "f"] \\
Y \arrow[r, "t_Y"'] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
X \otimes Y \arrow[r, "t_{X \otimes Y}"] \arrow[d, "\rho_{X,Y}"'] \arrow[dr, phantom, "\Downarrow v_{X,Y}"] & X \otimes Y \\
Y \otimes X \arrow[r, "t_Y \otimes t_X"'] & Y \otimes X \arrow[u, "\rho_{Y,X}"']
\end{tikzcd}$$

A *balanced* Gray monoid is a braided Gray monoid equipped with a twist. Notice that a syllepsis is a twist with $t = 1$.

DEFINITION 20. Suppose A is a braided pseudomonoid in a balanced Gray monoid $\mathscr{M}$. A *twist* for A is an invertible 2-cell $\tau : 1_A \Rightarrow t$ such that the following two equations hold:

$$\begin{tikzcd}
I \arrow[r, "\eta"] & A \arrow[r, bend left=60, "1"] \arrow[r, "t_A"'] \arrow[r, phantom, bend left=30, "\Downarrow\tau"] & A
\end{tikzcd}
=
\begin{tikzcd}
I \arrow[r, "\eta"'] \arrow[rr, bend left=50, "\eta"] & A \arrow[r, "t_A"'] \arrow[u, phantom, "\Downarrow t_\eta"] & A
\end{tikzcd}$$



7. ENRICHED CATEGORIES

It is now an appropriate point to turn to more explicit examples of the concepts developed so far. In this section, we merely provide a class of examples of monoidal bicategories. We would like to approach these examples by specialization. For any monoidal bicategory $\mathscr{M}$, we have the braided monoidal category

$$\mathscr{V} = \mathscr{M}(I, I)$$

(it is the hom category in the monoidal full subbicategory $\Sigma\mathscr{V}$ of $\mathscr{M}$ consisting of the single object I; that is, we have a one object, one arrow tricategory, and so, a braided monoidal category [GPS]). Our question now is: *when can we recapture* $\mathscr{M}$ *itself from* $\mathscr{V}$? An answer is provided by [St3]. If $\mathscr{M}$ admits collages, is locally cocomplete, and the single object I is a cauchy generator for $\mathscr{M}$, then $\mathscr{M}$ is biequivalent (as a bicategory) to $\mathscr{V}$-**Mod**. Some of these terms could be unfamiliar. "Collages" are lax colimits. A bicategory is "locally cocomplete" when each hom category $\mathscr{M}(X, Y)$ is cocomplete and horizontal composition with arrows, on either side, preserves the colimits. The "cauchy generator" condition here means that the homomorphism $\mathscr{M}(I, -) : \mathscr{M} \to \mathbf{Cat}$ should reflect equivalences. It follows that every object X of $\mathscr{M}$ is a collage

$$\begin{tikzcd} I \arrow[rr, "y^r \circ x"] \arrow[dr, "x"'] & {} \arrow[d, phantom, "\overset{\varepsilon \circ x}{\Leftarrow}"] & I \arrow[dl, "y"] \\ & X & \end{tikzcd}$$

where x, y run over all maps $I \to X$ and ε is the counit for the adjunction $y \dashv y^r$. We shall soon remind the reader what we mean by the bicategory

$\mathscr{V}$-**Mod** of $\mathscr{V}$-categories and $\mathscr{V}$-modules, although [St2; BCSW] are suitable references. The tensor product of $\mathscr{V}$-categories, for $\mathscr{V}$ braided, was described in [JS1]. Of course, in order for the biequivalence between $\mathscr{M}$ and $\mathscr{V}$-**Mod** to be monoidal, we need the tensor product on $\mathscr{M}$ to preserve collages in each variable. Now, although $\mathscr{V}$ is braided as a monoidal category, this does not mean $\Sigma\mathscr{V}$ is braided as a monoidal bicategory; for this we require $\mathscr{V}$ to be symmetric. Yet, if $\mathscr{V}$ is symmetric, we shall see that $\mathscr{V}$-**Mod** is a symmetric monoidal bicategory. On the other hand, if $\mathscr{M}$ is a braided monoidal bicategory then $\mathscr{V}$ is a symmetric monoidal category; so, whenever $\mathscr{V}$-**Mod** is braided, it is symmetric. This implies that, if we are interested in examples of braided or sylleptic monoidal bicategories, we must look beyond $\mathscr{V}$-**Mod**. Yet this example will suffice for the present.

Continuing for the moment with this kind of monoidal bicategory $\mathscr{M}$ and the braided monoidal category $\mathscr{V} = \mathscr{M}(I, I)$, we should also point out that each hom category $\mathscr{M}(I, A)$ becomes a $\mathscr{V}$-category by taking the internal hom of $f, g : I \to A$ to be the right lifting of g through f. (There is a generally different way to do this using right extension, but the two ways are isomorphic when $\mathscr{M}$ is braided.) So, in fact, we can improve on Proposition 6 and Examples 5 and 6. The convolution structure on $\mathscr{M}(I, A)$ actually makes it a closed monoidal $\mathscr{V}$-category. If the object A is braided as in Example 5, then $\mathscr{M}(I, A)$ becomes a braided $\mathscr{V}$-category; moreover, there is now a converse: a $\mathscr{V}$-enriched braiding on $\mathscr{M}(I, A)$ gives a braiding on A. Symmetries behave similarly.

Remark. If $\mathscr{M}$ is right closed, notice that each hom category $\mathscr{M}(A, B)$ becomes $\mathscr{V}$-enriched because $\mathscr{M}(I, [A, B])$ does. This shows that the impulse to deal with bicategories $\mathscr{M}$ enriched in another monoidal bicategory (such as $\mathscr{V}$-**Cat**) can be avoided in the present paper. For our purposes that structure is implicit in the mere monoidal bicategory itself.

Suppose $\mathscr{V}$ is any monoidal category. We shall consider $\mathscr{V}$-*categories* in the sense of [EK]; also see [L; K2]. A $\mathscr{V}$-*module*[8] $\Phi : \mathscr{A} \to \mathscr{B}$ consists of objects $\Phi(A, B)$ of $\mathscr{V}$ for all objects A, B of $\mathscr{A}, \mathscr{B}$, respectively, together with an associative, unital right action $m_r : \Phi(A, B) \otimes \mathscr{A}(A', A) \to \Phi(A', B)$ of $\mathscr{A}$ and an associative, unital left action $m_1 : \mathscr{B}(B, B') \otimes \Phi(A, B) \to \Phi(A, B')$ which satisfy the obvious compatibility axiom for a bimodule. Suppose $\Phi_1 : \mathscr{A}_0 \to \mathscr{A}_1$, $\Phi_2 : \mathscr{A}_1 \to \mathscr{A}_2, \ldots, \Phi_n : \mathscr{A}_{n-1} \to \mathscr{A}_n$, $\Psi : \mathscr{A}_0 \to \mathscr{A}_n$ are $\mathscr{V}$-modules. A $\mathscr{V}$-*form* $\alpha : (\Phi_1, \Phi_2, \ldots, \Phi_n) \Rightarrow \Psi$ is a family of arrows

$$\alpha_{A_0, \ldots, A_n} : \Phi_n(A_{n-1}, A_n) \otimes \cdots \otimes \Phi_1(A_0, A_1) \to \Psi(A_0, A_n)$$

such that the following diagrams commute:

[8] Modules have been called "bimodules" [L], "distributors" and "profunctors."

$$\begin{tikzcd}[column sep=tiny]
& \Phi_n(A_{n-1}, A_n) \otimes \dots \otimes \Phi_{k+1}(L, A_{k+1}) \otimes \mathscr{A}_k(K, L) \otimes \Phi_k(A_{k-1}, K) \otimes \dots \otimes \Phi(A_0, A_1) \arrow[dl, "1 \otimes \dots \otimes m_r \otimes 1 \otimes \dots \otimes 1"'] \arrow[dr, "1 \otimes \dots \otimes 1 \otimes m_\ell \otimes \dots \otimes 1"] & \\
\Phi_n(A_{n-1}, A_n) \otimes \dots \otimes \Phi_{k+1}(K, A_{k+1}) \otimes \Phi_k(A_{k-1}, K) \otimes \dots \otimes \Phi(A_0, A_1) \arrow[dr, "\alpha_{A_0, \dots, A_{k-1}, K, A_{k+1}, \dots, A_n}"'] & & \Phi_n(A_{n-1}, A_n) \otimes \dots \otimes \Phi_{k+1}(L, A_{k+1}) \otimes \Phi_k(A_{k-1}, L) \otimes \dots \otimes \Phi(A_0, A_1) \arrow[dl, "\alpha_{A_0, \dots, A_{k-1}, L, A_{k+1}, \dots, A_n}"] \\
& \Psi(A_0, A_n) &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathscr{A}_n(K, L) \otimes \Phi_n(A_{n-1}, K) \otimes \dots \otimes \Phi_1(A_0, A_1) \arrow[r, "1 \otimes \alpha_{A_0, \dots, A_{n-1}, K}"] \arrow[d, "m_\ell \otimes \dots \otimes 1"'] & \mathscr{A}_n(K, L) \otimes \Psi(A_0, K) \arrow[d, "m_\ell"] \\
\Phi_n(A_{n-1}, L) \otimes \dots \otimes \Phi_1(A_0, A_1) \arrow[r, "\alpha_{A_0, \dots, A_{n-1}, L}"'] & \Psi(A_0, L)
\end{tikzcd}$$

$$\begin{tikzcd}
\Phi_n(A_{n-1}, A_n) \otimes \dots \otimes \Phi_1(L, A_1) \otimes \mathscr{A}_0(K, L) \arrow[r, "\alpha_{L, A_1, \dots, A_n} \otimes 1"] \arrow[d, "1 \otimes \dots \otimes m_r"'] & \Psi(L, A_n) \otimes \mathscr{A}_0(K, L) \arrow[d, "m_r"] \\
\Phi_n(A_{n-1}, A_n) \otimes \dots \otimes \Phi_1(K, A_1) \arrow[r, "\alpha_{K, A_1, \dots, A_n}"'] & \Psi(K, A_n)
\end{tikzcd}$$

In particular, if $\Phi, \Psi : \mathscr{A} \to \mathscr{B}$ are modules,[9] a form $\alpha : \Phi \Rightarrow \Psi$ is called a *module morphism*, and these can be composed in the obvious way. More generally, there is an operation of *substitution* of forms; for example, if $\alpha_1 : (\Phi_{11}, \Phi_{12}) \Rightarrow \Psi_1$, $\alpha_2 : (\Phi_{21}, \Phi_{22}) \Rightarrow \Psi_2$, $\beta : (\Psi_1, \Psi_2) \Rightarrow \Theta$ are forms, we obtain a form $\beta(\alpha_1, \alpha_2) : (\Phi_{11}, \Phi_{12}, \Phi_{21}, \Phi_{22}) \Rightarrow \Theta$ in the obvious way.

A *composite* for modules $\Theta : \mathscr{A} \to \mathscr{B}$, $\Phi : \mathscr{B} \to \mathscr{C}$ is a module $\Phi \circ \Theta : \mathscr{A} \to \mathscr{C}$, together with a form $v : (\Theta, \Phi) \Rightarrow \Phi \circ \Theta$ such that, for all modules $\Psi : \mathscr{A} \to \mathscr{C}$, substitution with v determines a bijection between module morphisms $\alpha : \Phi \circ \Theta \Rightarrow \Psi$ and forms $\beta : (\Theta, \Phi) \Rightarrow \Psi$.

[9] We drop the prefix "$\mathscr{V}$-" when the context makes it clear.

A *right lifting* (or "right hom") of a module $\Psi : \mathscr{A} \to \mathscr{C}$ through $\Phi : \mathscr{B} \to \mathscr{C}$ is a module $\Phi/\Psi : \mathscr{A} \to \mathscr{B}$, together with a form $\varepsilon : (\Phi/\Psi, \Phi) \Rightarrow \Psi$ such that, for all modules $\Theta : \mathscr{A} \to \mathscr{B}$, substitution with ε determines a bijection between module morphisms $\alpha : \Theta \Rightarrow \Phi/\Psi$ and forms $\beta : (\Theta, \Phi) \Rightarrow \Psi$. A *right extension* (or "left hom") of a module $\Psi : \mathscr{A} \to \mathscr{C}$ along $\Theta : \mathscr{A} \to \mathscr{B}$ is a module $\Theta\backslash\Psi : \mathscr{B} \to \mathscr{C}$, together with a form $\varepsilon' : (\Theta, \Theta\backslash\Psi) \Rightarrow \Psi$ such that, for all $\mathscr{V}$-modules $\Phi : \mathscr{B} \to \mathscr{C}$, substitution with ε' determines a bijection between module morphisms $\alpha : \Phi \Rightarrow \Theta\backslash\Psi$ and forms $\beta : (\Theta, \Phi) \Rightarrow \Psi$.

Suppose now that $\mathscr{V}$ is (right and left) closed, complete, and cocomplete. Then, for all modules $\Theta : \mathscr{A} \to \mathscr{B}$, $\Psi : \mathscr{A} \to \mathscr{C}$, $\Phi : \mathscr{B} \to \mathscr{C}$ a composite $\Phi \circ \Theta$ exists if $\mathscr{B}$ is small, a right lifting Φ/Ψ exists if $\mathscr{C}$ is small, and a right extension $\Theta\backslash\Psi$ exists if $\mathscr{A}$ is small. We now have the bicategory $\mathscr{V}$-**Mod** in which all right liftings and right extensions exist: the objects are small $\mathscr{V}$-categories, the arrows are $\mathscr{V}$-modules, and the 2-cells are $\mathscr{V}$-module morphisms. Moreover, the bicategory $\mathscr{V}$-**Mod** is locally cocomplete.

There is also a 2-category $\mathscr{V}$-**Cat** whose objects are small $\mathscr{V}$-categories, whose arrows are $\mathscr{V}$-functors, and whose 2-cells are $\mathscr{V}$-natural transformations. There is a locally fully faithful inclusion homomorphism from $\mathscr{V}$-**Cat** to $\mathscr{V}$-**Mod**; it is obtained by identifying each $\mathscr{V}$-functor $F : \mathscr{A} \to \mathscr{B}$ with the $\mathscr{V}$-module $F : \mathscr{A} \to \mathscr{B}$ given by $F(A, B) = \mathscr{B}(FA, B)$. In fact, $F : \mathscr{A} \to \mathscr{B}$ is a map in $\mathscr{V}$-**Mod**; the right adjoint $\mathscr{V}$-module $F^r : \mathscr{B} \to \mathscr{A}$ is given by $F^r(A, B) = \mathscr{B}(B, FA)$.

In order for $\mathscr{V}$-**Mod** to be a monoidal bicategory, we have seen that it is necessary for $\mathscr{V}$ to be braided. This is also sufficient. The *tensor product* $\mathscr{A} \otimes \mathscr{B}$ of $\mathscr{V}$-categories $\mathscr{A}$, $\mathscr{B}$ has objects pairs (A, B) of objects A, B of $\mathscr{A}$, $\mathscr{B}$, respectively; the homs are given by

$$(\mathscr{A} \otimes \mathscr{B})((A, B), (C, D)) = \mathscr{A}(A, C) \otimes \mathscr{B}(B, D);$$

the composition $(\mathscr{A} \otimes \mathscr{B})((C, D), (E, F)) \otimes (\mathscr{A} \otimes \mathscr{B})((A, B), (C, D)) \to (\mathscr{A} \otimes \mathscr{B})((A, B), (E, F))$ is the composite (and note the use of the braiding)

$$\begin{aligned} &\mathscr{A}(C, E) \otimes \mathscr{B}(D, F) \otimes \mathscr{A}(A, C) \otimes \mathscr{B}(B, D) \\ &\qquad \xrightarrow{1 \otimes c \otimes 1} \mathscr{A}(C, E) \otimes \mathscr{A}(A, C) \otimes \mathscr{B}(D, F) \otimes \mathscr{B}(B, D) \\ &\qquad \xrightarrow{\text{comp} \otimes \text{comp}} \mathscr{A}(A, E) \otimes \mathscr{B}(B, F); \end{aligned}$$

and the unit $I \to (\mathscr{A} \otimes \mathscr{B})((A, B), (A, B))$ is just given by the tensor product of the units $I \to \mathscr{A}(A, A)$, $I \to \mathscr{B}(B, B)$. We write I for the $\mathscr{V}$-category with one object O and $I(O, O) = I$. The reader will have no problem in providing the definition of the *tensor product* $\Phi \otimes \Psi : \mathscr{A} \otimes \mathscr{B} \to \mathscr{A}' \otimes \mathscr{B}'$ of modules $\Phi : \mathscr{A} \to \mathscr{A}'$, $\Psi : \mathscr{B} \to \mathscr{B}'$; it is very similar to the definition of $\mathscr{A} \otimes \mathscr{B}$. For forms such as $\alpha : (\Theta, \Phi) \Rightarrow \Lambda$, $\beta : (\Pi, \Psi) \Rightarrow \Omega$, we easily define a *tensor product* $\alpha \otimes \beta : (\Theta \otimes \Pi, \Phi \otimes \Psi) \Rightarrow \Lambda \otimes \Omega$ of forms. In particular, this shows how to make $\otimes : \mathscr{V}\text{-}\mathbf{Mod} \times \mathscr{V}\text{-}\mathbf{Mod} \to \mathscr{V}\text{-}\mathbf{Mod}$ into a homomorphism of bicategories. There are associativity and unit constraints which are actually invertible $\mathscr{V}$-functors. (If we take our category **Set** of small sets to be skeletal, take $\mathscr{V}$ to be braided strict monoidal, and restrict attention to $\mathscr{V}$-categories whose sets of objects lie in **Set**, then these constraints can be taken to be identities; so the axioms for a monoidal bicategory are trivial.) The tensor product on $\mathscr{V}$-**Mod** restricts to $\mathscr{V}$-**Cat** and the latter becomes a monoidal 2-category.

The (*left*) *opposite* $\mathscr{A}^{\mathrm{op}}$ of a $\mathscr{V}$-category $\mathscr{A}$ has the same objects as $\mathscr{A}$, homs given by $\mathscr{A}^{\mathrm{op}}(A, B) = \mathscr{A}(B, A)$, composition $\mathscr{A}^{\mathrm{op}}(B, C) \otimes \mathscr{A}^{\mathrm{op}}(A, B) \to \mathscr{A}^{\mathrm{op}}(A, C)$ equal to the composite,

$$\mathscr{A}(C, B) \otimes \mathscr{A}(B, A) \xrightarrow{\ c\ } \mathscr{A}(B, A) \otimes \mathscr{A}(C, B) \xrightarrow{\mathrm{comp}} \mathscr{A}(C, A),$$

and units those of $\mathscr{A}$. There is a $\mathscr{V}$-module $\Xi : \mathscr{A}^{\mathrm{op}} \otimes \mathscr{A} \to I$ given by $\Xi(B, A, O) = \mathscr{A}(A, B)$ and a $\mathscr{V}$-module $N : I \to \mathscr{A} \otimes \mathscr{A}^{\mathrm{op}}$ given by $N(O, A, B) = \mathscr{A}(B, A)$. These modules induce the equivalence of categories,

$$\mathscr{V}\text{-}\mathbf{Mod}(\mathscr{B}, \mathscr{A} \otimes \mathscr{C}) \simeq \mathscr{V}\text{-}\mathbf{Mod}(\mathscr{A}^{\mathrm{op}} \otimes \mathscr{B}, \mathscr{C}),$$

under which $\Phi : \mathscr{B} \to \mathscr{A} \otimes \mathscr{C}$ corresponds to $\Psi : \mathscr{A}^{\mathrm{op}} \otimes \mathscr{B} \to \mathscr{C}$ via $\Phi(B, A, C) = \Psi(A, B, C)$. So $\mathscr{V}$-**Mod** is a left autonomous monoidal bicategory. In fact, $\mathscr{V}$-**Mod** is autonomous since we can define a *right opposite* $\mathscr{A}^{\mathrm{op}}$ using the inverse braiding to define composition. It follows that $\mathscr{V}$-**Mod** is a closed monoidal bicategory.

We can regard $\mathscr{V}$ itself as a $\mathscr{V}$-category with the left internal hom as the $\mathscr{V}$-valued hom. Then $\mathscr{V}$-functors $\Phi : \mathscr{A} \to \mathscr{V}$ can be identified with $\mathscr{V}$-modules $\Phi : I \to \mathscr{A}$. This implies that $\mathscr{V}$-modules $\Phi : \mathscr{A} \to \mathscr{B}$ can be identified with $\mathscr{V}$-functors $\Phi : \mathscr{A}^{\mathrm{op}} \otimes \mathscr{B} \to \mathscr{V}$.

We should also point out that $\mathscr{V}$-**Cat** is a closed 2-category. For any endomodule $\Phi : \mathscr{A} \to \mathscr{A}$ with $\mathscr{A}$ small, there is a (*left*) *cotrace* (or "end") $\int_{\mathscr{A}} \Phi$ which is the object of $\mathscr{V}$ obtained as the right extension of the corresponding module $\Phi : \mathscr{A}^{\mathrm{op}} \otimes \mathscr{A} \to I$ along the module $\Xi : \mathscr{A}^{\mathrm{op}} \otimes \mathscr{A} \to I$ (compare with Definition 8),

$$\begin{tikzcd}
\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A} \arrow[r, "\Xi"] \arrow[d, "1 \otimes \Phi"'] \arrow[dr, phantom, "\Leftarrow"] & I \arrow[d, "\int_{\mathscr{A}} \Phi"] \\
\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A} \arrow[r, "\Xi"'] & I
\end{tikzcd}$$

For any $\mathscr{V}$-category $\mathscr{X}$, we can define a *$\mathscr{V}$-functor $\mathscr{V}$-category* $[\mathscr{A}, \mathscr{X}]$ whose objects are $\mathscr{V}$-functors $F : \mathscr{A} \to \mathscr{X}$ and whose homs are given by

$$[\mathscr{A}, \mathscr{X}](F, G) = \int_{\mathscr{A}} G^r \circ F.$$

Then $[\mathscr{A}, \mathscr{X}]$ is a left internal hom of $\mathscr{V}$-categories; there is a natural *isomorphism* between the category of $\mathscr{V}$-functors $\mathscr{C} \otimes \mathscr{A} \to \mathscr{X}$ and the category of $\mathscr{V}$-functors $\mathscr{C} \to [\mathscr{A}, \mathscr{X}]$. A *right cotrace* of $\Phi : \mathscr{A} \to \mathscr{A}$ is given by taking the right lifting of the corresponding $\Phi : I \to \mathscr{A} \otimes \mathscr{A}^{\mathrm{op}}$ through $N : I \to \mathscr{A} \otimes \mathscr{A}^{\mathrm{op}}$. This leads to a right internal hom $[\mathscr{A}, \mathscr{X}]_r$ of $\mathscr{V}$-categories.

The objects of $[\mathscr{A}, \mathscr{V}]$ are $\mathscr{V}$-functors $S : \mathscr{A} \to \mathscr{V}$ and the $\mathscr{V}$-valued hom $[\mathscr{A}, \mathscr{V}](S, T)$ is determined up to isomorphism by the existence of a natural bijection between arrows $X \to [\mathscr{A}, \mathscr{V}](S, T)$ in $\mathscr{V}$ and $\mathscr{V}$-natural transformations $X \otimes S \to T$. There is a $\mathscr{V}$-functor $Y : \mathscr{A}^{\mathrm{op}} \to [\mathscr{A}, \mathscr{V}]$, called the *Yoneda embedding*, which takes A to the representable $\mathscr{V}$-functor $\mathscr{A}(A, -) : \mathscr{A} \to \mathscr{V}$. Restriction along the Yoneda embedding produces an equivalence between the category of $\mathscr{V}$-functors

$$F : [\mathscr{A}, \mathscr{V}] \to [\mathscr{B}, \mathscr{V}],$$

which admit right $\mathscr{V}$-adjoints, and the category of $\mathscr{V}$-functors $\mathscr{A}^{\mathrm{op}} \to [\mathscr{B}, \mathscr{V}]$; and this last category is isomorphic to $\mathscr{V}$-**Mod**$(\mathscr{A}, \mathscr{B})$. Indeed, the $\mathscr{V}$-module $\Phi : \mathscr{A} \to \mathscr{B}$ corresponding to F is determined up to isomorphism by

$$\Phi(A, B) = F(\mathscr{A}(A, -))(B),$$

$$F(S)(B) = \int^{A} S(A) \otimes \Phi(A, B).$$

By [GPS] there exists a Gray monoid equivalent to the monoidal bicategory $\mathscr{V}$-**Mod**. Yet, when $\mathscr{V}$ is symmetric, there is a fairly canonical choice $\mathscr{V}$-**Mog** for such a Gray monoid. First, we choose a skeletal category **Set** of small sets so that cartesian product makes it a *strict* monoidal category. Next we replace $\mathscr{V}$ by a monoidally equivalent strict monoidal category; this is possible by Mac Lane's coherence theorem, but see [JS1] for another choice.

Then we need to replace the bicategory $\mathscr{V}$-**Mod** by a biequivalent 2-category $\mathscr{V}$-**Mog** described as follows. The objects of $\mathscr{V}$-**Mog** are $\mathscr{V}$-categories whose sets of objects lie in the skeletal category **Set** and whose homs lie in the strict monoidal category $\mathscr{V}$. The arrows $\mathscr{A} \to \mathscr{B}$ are $\mathscr{V}$-functors $[\mathscr{A}, \mathscr{V}] \to [\mathscr{B}, \mathscr{V}]$ with right $\mathscr{V}$-adjoints. The 2-cells are $\mathscr{V}$-natural transformations between the $\mathscr{V}$-functors. The ordinary compositions of $\mathscr{V}$-functors and $\mathscr{V}$-natural transformations make this a 2-category.

From our previous discussion, it is clear that $\mathscr{V}$-**Mog** is biequivalent to $\mathscr{V}$-**Mod**.

For each $\mathscr{A}, \mathscr{B}$ in $\mathscr{V}$-**Mog** we define 2-functors $L_{\mathscr{A}}$, $R_{\mathscr{B}} : \mathscr{V}\text{-}\mathbf{Mog} \to \mathscr{V}\text{-}\mathbf{Mog}$ on objects by $L_{\mathscr{A}}(\mathscr{C}) = \mathscr{A} \otimes \mathscr{C}$, $R_{\mathscr{B}}(\mathscr{C}) = \mathscr{C} \otimes \mathscr{B}$. For a left-adjoint $\mathscr{V}$-functor $G : [\mathscr{C}, \mathscr{V}] \to [\mathscr{D}, \mathscr{V}]$, define $L_{\mathscr{A}}(G)$, $R_{\mathscr{B}}(G)$ by commutativity in the squares

$$\begin{tikzcd}
{[\mathscr{A}\otimes\mathscr{C},\mathscr{V}]} \arrow[r, "L_{\mathscr{A}}(G)"] \arrow[d, "\cong"'] & {[\mathscr{A}\otimes\mathscr{D},\mathscr{V}]} \arrow[d, "\cong"] \\
{[\mathscr{A},[\mathscr{C},\mathscr{V}]]} \arrow[r, "{[1,G]}"'] & {[\mathscr{A},[\mathscr{D},\mathscr{V}]]}
\end{tikzcd}
\qquad
\begin{tikzcd}
{[\mathscr{C}\otimes\mathscr{B},\mathscr{V}]} \arrow[r, "R_{\mathscr{B}}(G)"] \arrow[d, "\cong"'] & {[\mathscr{D}\otimes\mathscr{B},\mathscr{V}]} \arrow[d, "\cong"] \\
{[\mathscr{B},[\mathscr{C},\mathscr{V}]]} \arrow[r, "{[1,G]}"'] & {[\mathscr{B},[\mathscr{D},\mathscr{V}]]}
\end{tikzcd}$$

where the vertical arrows are the canonical isomorphisms (those for the second square use the symmetry to ensure the identification $[\mathscr{A}, \mathscr{X}]_r = [\mathscr{A}, \mathscr{X}]$). If G corresponds to the $\mathscr{V}$-module $\Psi : \mathscr{C} \to \mathscr{D}$, then, of course, $L_{\mathscr{A}}(G)$, $R_{\mathscr{B}}(G)$ correspond to the $\mathscr{V}$-modules $\mathscr{A} \otimes \Psi$, $\Psi \otimes \mathscr{B}$ whose components are $\mathscr{A}(A, A') \otimes \Psi(C, D)$, $\Psi(C, D) \otimes \mathscr{B}(B, B')$. Further, if F corresponds to the $\mathscr{V}$-module $\Phi : \mathscr{A} \to \mathscr{B}$, there is a canonical isomorphism $c_{f,g} : L_{\mathscr{B}}(G) \circ R_{\mathscr{C}}(F) \cong R_{\mathscr{D}}(F) \circ L_{\mathscr{A}}(G)$ since both composites correspond, up to isomorphism, to the $\mathscr{V}$-module $\Phi \otimes \Psi$. The axioms for a Gray monoid can be verified.

A pseudomonoid $\mathscr{A}$ in $\mathscr{V}$-**Mod** is precisely a *promonoidal $\mathscr{V}$-category* as defined by [D1] (except that, in that first paper, the word "premonoidal" was used). The $\mathscr{V}$-modules $\mu : \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$, $\eta : I \to \mathscr{A}$ amount to $\mathscr{V}$-functors $P : \mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}^{\mathrm{op}} \otimes \mathscr{A} \to \mathscr{V}$, $J : \mathscr{A} \to \mathscr{V}$. The convolution monoidal structure on $\mathscr{V}\text{-}\mathbf{Mod}(I, \mathscr{A}) = [\mathscr{A}, \mathscr{V}]$ is given by the formula

$$(F * G)(A) \cong \int^{X,\,Y} P(X, Y, A) \otimes F(X) \otimes G(Y)$$

with unit J. By Proposition 6, $[\mathscr{A}, \mathscr{V}]$ is closed. From Sections 4, 5, 6 we see that $[\mathscr{A}, \mathscr{V}]$ is braided, symmetric, balanced if $\mathscr{A}$ is (and conversely, using comments at the beginning of this section).

Example 8. There is a promonoidal $\mathscr{V}$-category associated with any small $\mathscr{V}$-category $\mathscr{A}$ [D1; pp. 36–37]. The underlying $\mathscr{V}$-category is $\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}$ while the $\mathscr{V}$-functors

$$P : \mathscr{A} \otimes \mathscr{A}^{\mathrm{op}} \otimes \mathscr{A} \otimes \mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}^{\mathrm{op}} \otimes \mathscr{A} \to \mathscr{V}, \qquad J : \mathscr{A}^{\mathrm{op}} \otimes \mathscr{A} \to \mathscr{V}$$

are given by

$$P((A_1, A_2), (B_1, B_2), (C_1, C_2)) = \mathscr{A}(C_1, A_1) \otimes \mathscr{A}(A_2, B_1) \otimes \mathscr{A}(B_2, C_2),$$

$$J(A_1, A_2) = \mathscr{A}(A_1, A_2).$$

The convolution tensor product induced on $[\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}, \mathscr{V}]$ by this promonoidal structure is precisely the transportation of composition of modules across the equivalence

$$[\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}, \mathscr{V}] \simeq \mathscr{V}\text{-}\mathbf{Mod}(\mathscr{A}, \mathscr{A}).$$

EXAMPLE 9. There is a braided promonoidal category[10] associated with any groupoid $\mathscr{G}$. Let $\mathscr{G}^{\mathrm{Z}}$ denote the category of automorphisms in $\mathscr{G}$; that is, the objects are pairs (A, a), where $a : A \to A$ is an arrow in $\mathscr{G}$, and the arrows $f : (A, a) \to (B, b)$ are arrows $f : A \to B$ in $\mathscr{G}$ such that $f \circ a = b \circ f$. It is convenient to write this last equation as $b = {}^f a$, where ${}^f a = f \circ a \circ f^{-1}$ is conjugation by f. A promonoidal structure on $\mathscr{G}^{\mathrm{Z}}$ is defined by

$$P((A, a), (B, b), (C, c)) = \{(u : A \to C, v : B \to C) \mid {}^u a \circ {}^v b = c\},$$

$$P(f, g, h)(u, v) = (h \circ u \circ f, h \circ v \circ g).$$

The braiding is the natural isomorphism $\gamma : P \to P$ whose component at $(A, a), (B, b), (C, c)$ takes (u, v) to $({}^u a \circ v, u)$.

This construction is related to the Drinfeld double of a group [Kl], Yetter's crossed bimodules over a Hopf algebra [Kl], and the Freyd–Yetter braiding on crossed G-sets [JS1; Example 5.1]. To see this, consider the functor category $[\mathscr{G}, \mathbf{Set}]$ as monoidal under cartesian product, and consider the functor category $[\mathscr{G}^{\mathrm{Z}}, \mathbf{Set}]$ as a braided monoidal category under convolution of the above braided promonoidal structure. Notice that there is an equivalence of categories

$$[\mathscr{G}^{\mathrm{Z}}, \mathbf{Set}] \simeq [\mathscr{G}, \mathbf{Set}]/\mathrm{Aut}$$

where $\mathrm{Aut} : \mathscr{G} \to \mathbf{Set}$ is the functor given by

$$\mathrm{Aut}(A) = \mathscr{G}(A, A), \qquad \mathrm{Aut}(f)(a) = {}^f a,$$

so the convolution structure transports to a braided monoidal structure on $[\mathscr{G}, \mathbf{Set}]/\mathrm{Aut}$; when $\mathscr{G} = G$ has one object this agrees with the Freyd–Yetter braided monoidal structure on crossed G-sets.

[10] Here the base category $\mathscr{V}$ is of course the category of small sets.

Furthermore, there is an equivalence of braided monoidal categories

$$[\mathscr{G}^{Z}, \mathbf{Set}] \simeq \mathscr{Z}[\mathscr{G}, \mathbf{Set}],$$

where $\mathscr{Z}\mathscr{V}$ denotes the *center* [JS1; Example 2.3] of a monoidal category $\mathscr{V}$. The base can be changed to the category of $\mathbf{k}$-vector spaces by taking, for any category $\mathscr{C}$, the $\mathbf{k}$-linear category $\mathbf{k}_*\mathscr{C}$ with the same objects as $\mathscr{C}$ and with homs the free vector spaces on the homs $\mathscr{C}(A, B)$ of $\mathscr{C}$. Then we obtain an equivalence of $\mathbf{k}$-linear braided monoidal categories

$$[\mathbf{k}_*\mathscr{G}^{Z}, \mathbf{Vect}_{\mathbf{k}}] \simeq \mathscr{Z}[\mathbf{k}_*\mathscr{G}, \mathbf{Vect}_{\mathbf{k}}].$$

It follows that, when $\mathscr{G} = G$, the Drinfield double $D(\mathbf{k}_*G)$ of the group algebra $\mathbf{k}_*G$ is Morita equivalent to the algebroid $\mathbf{k}_*G^{Z}$.

8. ENRICHED DUALS AND HOPF ALGEBROIDS

Suppose $\mathscr{V}$ is a complete cocomplete closed braided monoidal category, and suppose $\mathscr{A}$ is a monoidal $\mathscr{V}$-category (that is, a pseudomonoid in $\mathscr{V}$-**Cat**). Then $\mathscr{A}$ is a map pseudomonoid in $\mathscr{V}$-**Mod**. The multiplication $\mu : \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$ is a $\mathscr{V}$-functor whose value at (A, B) will be denoted by $A \blacklozenge B$.

According to Definition 10, the condition for a $\mathscr{V}$-functor $hom : \mathscr{A}^{\mathrm{op}} \otimes \mathscr{A} \to \mathscr{A}$ to be a right internal hom for $\mathscr{A}$ is that the $\mathscr{V}$-module $\mathscr{A} \to \mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}$ corresponding to $\mu : \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$ (under the equivalence arising from the right dual $\mathscr{A}^{\mathrm{op}}$ of $\mathscr{A}$ in $\mathscr{V}$-**Mod**) is a right adjoint of *hom*. In other words, there should be a $\mathscr{V}$-module isomorphism

$$\mathscr{A}(A \blacklozenge B, C) \cong \mathscr{A}(B, hom(A, C)).$$

According to Definition 11, the condition for a $\mathscr{V}$-functor $v : \mathscr{A}^{\mathrm{op}} \to \mathscr{A}$ to be a right antipode for $\mathscr{A}$ is that there should be a $\mathscr{V}$-natural isomorphism

$$hom(A, C) \cong v(A) \blacklozenge C.$$

That is, there should be a $\mathscr{V}$-module isomorphism

$$\mathscr{A}(A \blacklozenge B, C) \cong \mathscr{A}(B, v(A) \blacklozenge C).$$

Suppose we regard $\mathscr{A}^{\mathrm{op}}$ as a monoidal $\mathscr{V}$-category with the same tensor product $A \blacklozenge B$ as $\mathscr{A}$ (not the reverse). We claim that an antipode for $\mathscr{A}^{\mathrm{op}}$ amounts to an inverse equivalence for v. To see this, suppose v' is an antipode for $\mathscr{A}^{\mathrm{op}}$. Then we have a $\mathscr{V}$-module isomorphism

$$\begin{aligned}\mathscr{A}(v'(v(A)), B) &\cong \mathscr{A}^{\mathrm{op}}(B, v'(v(A)))\\ &\cong \mathscr{A}^{\mathrm{op}}(v(A) \blacklozenge B, I)\\ &\cong \mathscr{A}(I, v(A) \blacklozenge B)\\ &\cong \mathscr{A}(A, B),\end{aligned}$$

and so, by the enriched Yoneda Lemma, a $\mathscr{V}$-natural isomorphism $v'(v(A)) \cong A$. Similarly, $v(v'(B)) \cong B$. So v' is an inverse equivalence for v. The converse is easy.

It follows from Proposition 8 that, in the situation of the last paragraph, if $F : \mathscr{A} \to \mathscr{V}$ is in the Cauchy completion of $\mathscr{A}^{\mathrm{op}}$ then F has a right dual $F^{\vee}$ in the convolution monoidal category $[\mathscr{A}, \mathscr{V}]$.

DEFINITION 21. A *Hopf* $\mathscr{V}$*-algebroid* is a $\mathscr{V}$-category $\mathscr{H}$, together with $\mathscr{V}$-functors

$$D : \mathscr{H} \to \mathscr{H} \otimes \mathscr{H}, \qquad E : \mathscr{H} \to I, \qquad S : \mathscr{H} \to \mathscr{H}^{\mathrm{op}}$$

whose right adjoints in $\mathscr{V}$-**Mod** are equipped with the structure of right autonomous map pseudomonoid in $(\mathscr{V}\text{-}\mathbf{Mod})^{\mathrm{co}}$.

A more explicit description of Hopf $\mathscr{V}$-algebroids is possible. This will help in the recognition of examples. It will also show that, to define this concept, far less structure is required of $\mathscr{V}$ than required to have $\mathscr{V}$-**Mod**.

First notice that the $\mathscr{V}$-functors $D : \mathscr{H} \to \mathscr{H} \otimes \mathscr{H}$, $E : \mathscr{H} \to I$ are equipped with the structure of comonoidal $\mathscr{V}$-category; that is, pseudo-comonoid in $\mathscr{V}$-**Cat** (we call the Hopf algebroid *strict* if it is actually a comonoid). Then notice that the $\mathscr{V}$-functor $D : \mathscr{H} \to \mathscr{H} \otimes \mathscr{H}$ is isomorphic to one given on objects by the diagonal; that is, we can assume $D(X) = (X, X)$ for all objects X of $\mathscr{H}$. This follows from the fact that the left and right unit constraints give isomorphisms $D(X) \cong (X, X)$ and the fact that if $F : \mathscr{A} \to \mathscr{B}$ is a $\mathscr{V}$-functor, $G : \operatorname{obj} \mathscr{A} \to \operatorname{obj} \mathscr{B}$ is a function, and $FA \cong GA$, $A \in \mathscr{A}$, is a family of isomorphisms, then G is the object function for a $\mathscr{V}$-functor $G : \mathscr{A} \to \mathscr{B}$ unique such that the family of isomorphisms provides a $\mathscr{V}$-natural isomorphism $F \cong G$.

The antipode $S : \mathscr{H} \to \mathscr{H}^{\mathrm{op}}$ is a $\mathscr{V}$-functor satifying the condition that there should be a $\mathscr{V}$-natural isomorphism

$$\mathscr{H}(A, S(B)) \otimes \mathscr{H}(B, C) \cong \mathscr{H}(A, C) \otimes \mathscr{H}(B, C).$$

Of course, the repetition of the object B on the left-hand side and the object C on the right is made $\mathscr{V}$-functorial by means of the $\mathscr{V}$-functor D which is suppressed from the notation. As we shall now see, it is this property of antipode which implies that the internal hom for the convolution monoidal structure on $[\mathscr{H}, \mathscr{V}]$ is formed "pointwise."

PROPOSITION 19. *For any Hopf $\mathscr{V}$-algebroid $\mathscr{H}$ and $\mathscr{V}$-functor $M : \mathscr{H} \to \mathscr{V}$, there is a $\mathscr{V}$-natural isomorphism*

$$MSB \otimes \mathscr{H}(B, C) \cong MC \otimes \mathscr{H}(B, C).$$

Furthermore, the convolution monoidal structure on $[\mathscr{H}, \mathscr{V}]$ has tensor product $M \otimes N$ and right internal hom $[M, L]$ given by

$$(M \otimes N)\, C = MC \otimes NC, \qquad [M, L]\, B = [MSB, LB],$$

where the $\mathscr{V}$-functoriality of the right-hand sides in C, B uses $D : \mathscr{H} \to \mathscr{H} \otimes \mathscr{H}$.

Proof. Recall that the Yoneda lemma can be expressed as a $\mathscr{V}$-natural isomorphism

$$MC \cong \int^{A} \mathscr{H}(A, C) \otimes MA.$$

We have the following sequence of isomorphisms:

$$\begin{aligned} MC \otimes \mathscr{H}(B, C) &\cong \int^{A} \mathscr{H}(A, C) \otimes \mathscr{H}(B, C) \otimes MA \\ &\cong \int^{A} \mathscr{H}(A, SB) \otimes \mathscr{H}(B, C) \otimes MA \\ &\cong MSB \otimes \mathscr{H}(B, C). \end{aligned}$$

The formula for $M \otimes N$ comes directly from the definition of convolution tensor. That the right internal hom is as asserted follows from

$$\begin{aligned} &[\mathscr{H}, \mathscr{V}](M \otimes N, L) \\ &\quad \cong \int_{C} [MC \otimes NC, LC] \cong \int_{B, C} [MC \otimes \mathscr{H}(B, C) \otimes NB, LC] \\ &\quad \cong \int_{B, C} [MSB \otimes \mathscr{H}(B, C) \otimes NB, LC] \\ &\quad \cong \int_{B, C} [\mathscr{H}(B, C), [MSB \otimes NB, LC]] \cong \int_{B} [MSB \otimes NB, LB] \\ &\quad \cong \int_{B} [NB, [MSB, LB]]. \end{aligned}$$

Q.E.D.

EXAMPLE 10. Any groupoid $\mathscr{G}$ is a strict Hopf algebroid for the base category $\mathscr{V} = \mathbf{Set}$. The functor $D : \mathscr{G} \to \mathscr{G} \times \mathscr{G}$ is the diagonal, the functor $E : \mathscr{G} \to 1$ is the only one possible, and the antipode $S : \mathscr{G} \to \mathscr{G}^{\mathrm{op}}$ is the identity on objects and is given on arrows by $S(f) = f^{-1}$. The natural isomorphism

$$\mathscr{G}(A, S(B)) \times \mathscr{G}(B, C) \cong \mathscr{G}(A, C) \times \mathscr{G}(B, C)$$

takes (f, g) to $(g \circ f, g)$. Changing the base from $\mathbf{Set}$ to $\mathbf{Vect_k}$ as in Example 9, we linearly induce a Hopf $\mathbf{Vect_k}$-algebroid structure on $\mathbf{k}_* \mathscr{G}$ from that on $\mathscr{G}$.

EXAMPLE 11. Let H be a Hopf algebra over the field $\mathbf{k}$. Let $\mathscr{H}$ denote the $\mathbf{Vect_k}$-category with one object A and $\mathscr{H}(A, A) = H$; the unit and multiplication of H provide the identity arrow and composition for $\mathscr{H}$. The comultiplication and counit for H provide the linear functors $D : \mathscr{H} \to \mathscr{H} \otimes \mathscr{H}$, $E : \mathscr{H} \to I$. The antipode $S : \mathscr{H} \to \mathscr{H}^{\mathrm{op}}$ is the identity on objects with $S(x) = v(x)$, where v is the antipode of H and $x \in \mathscr{H}(A, A) = H$. The isomorphism

$$\mathscr{H}(A, S(A)) \otimes \mathscr{H}(A, A) \cong \mathscr{H}(A, A) \otimes \mathscr{H}(A, A)$$

is the ("fusion operator" [St6]) isomorphism

$$H \otimes H \xrightarrow{\delta \otimes 1} H \otimes H \otimes H \xrightarrow{1 \otimes \mu} H \otimes H$$

which has inverse

$$H \otimes H \xrightarrow{\delta \otimes 1} H \otimes H \otimes H \xrightarrow{1 \otimes v \otimes 1} H \otimes H \otimes H \xrightarrow{1 \otimes \mu} H \otimes H.$$

In this way, Hopf $\mathbf{k}$-algebras can be identified with strict one-object Hopf $\mathbf{Vect_k}$-algebroids.

9. OPCATEGORIES AND HOPF OPALGEBROIDS

For any monoidal category $\mathscr{V}$, the opposite category $\mathscr{V}^{\mathrm{op}}$ is also monoidal via the same tensor product.[11] A $\mathscr{V}$*-opcategory* is defined to be

[11] The reader should keep in mind, however, that $\mathscr{V}^{\mathrm{op}}$ will not generally be closed if $\mathscr{V}$ is; indeed, if $X \otimes -$ preserves colimits in $\mathscr{V}$, it will preserve limits in $\mathscr{V}^{\mathrm{op}}$ and not generally colimits (unless, of course, X has a dual).

a category $\mathscr{A}$ with homs enriched in $\mathscr{V}^{\mathrm{op}}$; the point is that there are families of *cocomposition* and *counit* arrows

$$\delta : \mathscr{A}(A, C) \to \mathscr{A}(B, C) \otimes \mathscr{A}(A, B), \qquad \varepsilon : \mathscr{A}(A, A) \to I$$

in $\mathscr{V}$ indexed by objects A, B, C of $\mathscr{A}$.

Enriched categories always have underlying categories. The underlying category $\mathscr{A}_0$ of a $\mathscr{V}$-opcategory has the same objects as $\mathscr{A}$ and has, as arrows $f : A \to B$, the arrows $f : \mathscr{A}(A, B) \to I$ in $\mathscr{V}$. The composite $g \circ f : \mathscr{A}(A, C) \to I$ of $f : \mathscr{A}(A, B) \to I$ and $g : \mathscr{A}(B, C) \to I$ as arrows in $\mathscr{A}_0$ is the composite of the arrows

$$\delta : \mathscr{A}(A, C) \to \mathscr{A}(B, C) \otimes \mathscr{A}(A, B), \qquad g \otimes f : \mathscr{A}(B, C) \otimes \mathscr{A}(A, B) \to I$$

in $\mathscr{V}$.

Also, *$\mathscr{V}$-opfunctor*, *$\mathscr{V}$-opnatural transformation* just mean $\mathscr{V}^{\mathrm{op}}$-functor and $\mathscr{V}^{\mathrm{op}}$-natural transformation. So a $\mathscr{V}$-opfunctor $S : \mathscr{A} \to \mathscr{D}$ assigns an object $SA \in \mathscr{D}$ to each object $A \in \mathscr{A}$ and arrows $S_{A,B} : \mathscr{D}(SA, SB) \to \mathscr{A}(A, B)$ in $\mathscr{V}$ respecting the counits and cocompositions. A $\mathscr{V}$-opnatural transformation $\alpha : S \to T$ consists of arrows $\alpha_A : \mathscr{D}(SA, TA) \to I$ in $\mathscr{V}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathscr{D}(SA, TB) \arrow[r, "\delta"] \arrow[d, "\delta"'] & \mathscr{D}(TA, TB) \otimes \mathscr{D}(SA, TA) \arrow[r, "1 \otimes \alpha_A"] & \mathscr{D}(TA, TB) \arrow[d, "T_{A,B}"] \\
\mathscr{D}(SB, TB) \otimes \mathscr{D}(SA, SB) \arrow[r, "\alpha_B \otimes 1"'] & \mathscr{D}(SA, SB) \arrow[r, "S_{A,B}"'] & \mathscr{A}(A, B)
\end{tikzcd}$$

With the obvious compositions, we have a 2-category $\mathscr{V}$-**opCat** of $\mathscr{V}$-opcategories.

If $\mathscr{V}$ is left closed with left internal hom $[X, Y]$, we have the weak monoidal functor $[-, I] : \mathscr{V}^{\mathrm{op}} \to \mathscr{V}^{\mathrm{rev}}$. Hence [EK] we get a 2-functor $(-)^{\clubsuit} = [-, I]_* : \mathscr{V}$-**opCat** $\to \mathscr{V}^{\mathrm{rev}}$-**Cat** which takes each $\mathscr{V}$-opcategory $\mathscr{A}$ to a $\mathscr{V}^{\mathrm{rev}}$-category $\mathscr{A}^{\clubsuit}$ with the same objects as $\mathscr{A}$ and with $\mathscr{A}^{\clubsuit}(A, B) = [\mathscr{A}(A, B), I]$. The underlying category of $\mathscr{A}^{\clubsuit}$ is isomorphic to the underlying category of $\mathscr{A}$ using the natural bijection between arrows $I \to [\mathscr{A}(A, B), I]$ and arrows $\mathscr{A}(A, B) \to I$. If $\mathscr{V}$ is symmetric and, for each object $X \in \mathscr{V}$, the canonical arrow $X \to [[X, I], I]$ is a monomorphism, it can be see that $(-)^{\clubsuit}$ is locally fully faithful; that is, for all $\mathscr{V}$-opfunctors $S, T : \mathscr{A} \to \mathscr{A}'$, each $\mathscr{V}$-natural transformation $S^{\clubsuit} \to T^{\clubsuit}$ is of the form $\alpha^{\clubsuit}$ for a unique $\mathscr{V}$-opnatural transformation $\alpha : S \to T$.

Let $\mathscr{A}$ be a $\mathscr{V}$-opcategory. A (*right*) *$\mathscr{A}$-comodule* is defined to be a $\mathscr{V}^{\mathrm{op}}$-module $M : \mathscr{A} \to I$ as defined in Section 7; there is a family of *coaction* arrows

$$\delta : M(A) \to M(B) \otimes \mathscr{A}(A, B)$$

in $\mathscr{V}$ indexed by objects A, B of $\mathscr{A}$. An *$\mathscr{A}$-comodule morphism $f : M \to N$* is a family of arrows $f_A : M(A) \to N(A)$ in $\mathscr{V}$ which commute in the obvious way with the coactions. This defines a category *Comod*($\mathscr{A}$) of $\mathscr{A}$-comodules. If $\mathscr{V}$ is complete and left closed, and $\mathscr{A}$ is small, it is possible to make *Comod*($\mathscr{A}$) into a $\mathscr{V}$-category; the hom object *Comod*($\mathscr{A}$) (M, N) is defined to be the equalizer of the obvious pair of arrows in $\mathscr{V}$ from the product of all the objects $[M(A), N(A)]$, for $A \in \mathscr{A}$, to the product of all the objects $[M(A), N(B) \otimes \mathscr{A}(A, B)]$, for $A, B \in \mathscr{A}$. In other words, we have the following limit diagram in $\mathscr{V}$:

$$\begin{tikzcd}
\mathit{Comod}(\mathscr{A})(M,N) \arrow[rr, dashed] \arrow[d, dashed] & & {[M(A),N(A)]} \arrow[d, "{[1,\delta]}"] \\
{[M(B),N(B)]} \arrow[r, "- \otimes \mathscr{A}(A,B)"'] & {[M(B) \otimes \mathscr{A}(A,B), N(B) \otimes \mathscr{A}(A,B)]} \arrow[r, "{[\delta,1]}"'] & {[M(A),N(B) \otimes \mathscr{A}(A,B)]}
\end{tikzcd}$$

For each object $A \in \mathscr{A}$, there is an evaluation $\mathscr{V}$-functor $E_A : \mathit{Comod}(\mathscr{A}) \to \mathscr{V}$ which takes M to $M(A)$. We write $\mathit{Comod}_f(\mathscr{A})$ for full the sub-$\mathscr{V}$-category of *Comod*($\mathscr{A}$) consisting of those $\mathscr{A}$-comodules M for which each $M(A)$ has a left dual $M(A)^*$.

EXAMPLE 12. Suppose $\mathscr{B}$ is a $\mathscr{V}$-category for which each hom $\mathscr{B}(B, A)$ has a left dual $\mathscr{B}(B, A)^*$ in $\mathscr{V}$. Then we obtain a $\mathscr{V}$-opcategory $\mathscr{B}^*$ with $\mathscr{B}^*(A, B) = \mathscr{B}(B, A)^*$. Right $\mathscr{B}^*$-comodules M are in bijection with $\mathscr{V}$-modules $M : \mathscr{B} \to I$ (we use the same letter M since, under the bijection, the family of objects $M(A)$ remains the same while the coaction is the mate of the action). Furthermore, we have an isomorphism of $\mathscr{V}$-categories $\mathit{Comod}(\mathscr{B}^*) \cong \mathscr{V}\text{-}\mathbf{Mod}(\mathscr{B}, I)$ $(= [\mathscr{B}^{\mathrm{op}}, \mathscr{V}]$ when the terms make sense). So, for a general $\mathscr{V}$-opcategory $\mathscr{A}$, we might think of *Comod*($\mathscr{A}$) as the presheaf $\mathscr{V}$-category of the perhaps-non-existent $\mathscr{V}$-category $\mathscr{B}$ with $\mathscr{A} = \mathscr{B}^*$. Of course, if $\mathscr{V}$ is symmetric and each $\mathscr{A}(A, B)$ has a dual then $\mathscr{B} = \mathscr{A}^{\clubsuit}$.

For comodules, there is an analogue of the Yoneda lemma.

PROPOSITION 20. *Suppose $\mathscr{A}$ is a $\mathscr{V}$-opcategory and M is a right $\mathscr{A}$-comodule. Then the following diagram exhibits M as a limit in the $\mathscr{V}$-category Comod$(\mathscr{A})$.*

$$\begin{tikzcd}
M \arrow[rr, dashed, "\lambda_A"] \arrow[d, dashed, "\lambda_B"'] & & M(A) \otimes \mathscr{A}(-, A) \arrow[d, "\delta \otimes 1"] \\
M(B) \otimes \mathscr{A}(-, B) \arrow[rr, "1 \otimes \delta"'] & & M(B) \otimes \mathscr{A}(A, B) \otimes \mathscr{A}(-, A)
\end{tikzcd}$$

Furthermore, the evaluation functor E_A takes this limit to an absolute[12] *limit in $\mathscr{V}$.*

Proof. This is an application of an aspect of the generalised Beck monadicity theorem of [St0; p. 40]. Q.E.D.

Now suppose $\mathscr{C}$ is a small $\mathscr{V}$-category. We shall produce a $\mathscr{V}$-opcategory $\mathscr{C}^{\vee}$ whose objects are the $\mathscr{V}$-functors $X : \mathscr{C} \to \mathscr{V}$ whose values have left duals; that is, for all objects $U \in \mathscr{C}$, the object $X(U) \in \mathscr{V}$ has a left dual $\chi(U)^*$. The hom object $\mathscr{C}^{\vee}(X, Y)$ for $\mathscr{C}^{\vee}$ is given by the coend

$$\int^{U} Y(U)^* \otimes X(U)$$

in $\mathscr{V}$. This hom object has a universal property; there is a natural bijection between arrows $\mathscr{C}^{\vee}(X, Y) \to K$ in $\mathscr{V}$ and $\mathscr{V}$-natural transformations $X \to Y \otimes K$, where $Y \otimes K : \mathscr{C} \to \mathscr{V}$ is the $\mathscr{V}$-functor given by $(Y \otimes K)(U) = Y(U) \otimes K$ on objects and in the obvious way on homs. There is a *coevaluation* $\mathscr{V}$-natural transformation

$$d : X \to Y \otimes \mathscr{C}^{\vee}(X, Y)$$

corresponding to the identity of $\mathscr{C}^{\vee}(X, Y)$. The composite of $d : X \to Y \otimes \mathscr{C}^{\vee}(X, Y)$ and $d \otimes 1 : Y \otimes \mathscr{C}^{\wedge}(X, Y) \to Z \otimes \mathscr{C}^{\vee}(Y, Z) \otimes \mathscr{C}^{\vee}(X, Y)$ then corresponds to our required cocomposition arrow

$$\delta : \mathscr{C}^{\vee}(X, Z) \to \mathscr{C}^{\vee}(Y, Z) \otimes \mathscr{C}^{\vee}(X, Y).$$

[12] A limit in a $\mathscr{V}$-category X is *absolute* when it is preserved by all functors out of X.

The counit $\varepsilon : \mathscr{C}^\vee(X, X) \to I$ corresponds to the identity $\mathscr{V}$-natural transformation $X \to X$.

We write $\mathscr{V}_f$ for the full sub-$\mathscr{V}$-category of $\mathscr{V}$ consisting of those objects which have a left dual. In terms of the philosophy at the end of Example 12, we now show that the $\mathscr{V}$-category $[\mathscr{C}, \mathscr{V}_f]^{\mathrm{op}}$ is trying to be the perhaps-non-existent $\mathscr{V}$-category $\mathscr{B}$ with $\mathscr{C}^\vee = \mathscr{B}^*$.

PROPOSITION 21. *Suppose $\mathscr{V}$ is symmetric closed monoidal, complete and cocomplete. For any $\mathscr{V}$-category $\mathscr{C}$, there is a canonical isomorphism $\mathscr{C}^{\vee\clubsuit} \cong [\mathscr{C}, \mathscr{V}_f]$.*

Proof. If $X, Y : \mathscr{C} \to \mathscr{V}$ are $\mathscr{V}$-functors whose values have duals then we shall show there is a canonical isomorphism

$$[\mathscr{C}, \mathscr{V}](X, Y) \cong [\mathscr{C}^\vee(X, Y), I]$$

by showing that both sides of the isomorphism have the same universal property. By the universal property of the hom in $\mathscr{V}$ and by symmetry, the arrows $K \to [\mathscr{C}^\vee(X, Y), I]$ are in natural bijection with arrows $K \otimes \mathscr{C}^\vee(X, Y) \to I$, and these are in natural bijection with arrows $\mathscr{C}^\vee(X, Y) \to [K, I]$. By the universal property of $\mathscr{C}^\vee(X, Y)$, these arrows are in natural bijection with $\mathscr{V}$-natural transformations $X \to Y \otimes [K, I]$. Since the values of Y have duals, we have the string of isomorphisms:

$$\begin{aligned} Y \otimes [K, I] &\cong [Y(-)^*, [K, I]] \cong [Y(-)^* \otimes K, I] \cong [K, [Y(-)^*, I]] \\ &\cong [K, Y(-)^{**}] \cong [K, Y(-)] = [K, Y]. \end{aligned}$$

But $[K, Y]$ is the cotensor of K with Y in the $\mathscr{V}$-category $[\mathscr{C}, \mathscr{V}]$; so arrows $X \to [K, Y]$ are in natural bijection with arrows $K \to [\mathscr{C}, \mathscr{V}](X, Y)$. Q.E.D.

The following adjunction underlies the "several object" generalization of Tannaka duality. With the philosophy of Example 12, the adjunction is a variant of that between the functors indicated by the assignments $\mathscr{B} \mapsto [\mathscr{B}^{\mathrm{op}}, \mathscr{V}]$ and $\mathscr{C} \mapsto [\mathscr{C}, \mathscr{V}]^{\mathrm{op}}$.

PROPOSITION 22. *The constructions $\mathscr{C} \mapsto \mathscr{C}^\vee$ and $\mathscr{A} \mapsto Comod_f(\mathscr{A})$ are adjoint*[13] *to each other as*

[13] In stating Proposition 22 in this concise way, we are taking the liberty of ignoring some size questions. The heart of the proposition is an equivalence of hom categories which does not involve these size worries. Moreover, in the examples of current interest, $\mathscr{V}_f$ is essentially a small category so the codomains of the functors in the adjunction are indeed as shown.

$$\begin{tikzcd} \mathcal{V}\text{-}\mathbf{opCat}^{\mathrm{op}} \arrow[r, bend right, "Comod_f"'] \arrow[r, phantom, "\perp"] & \mathcal{V}\text{-}\mathbf{Cat} \arrow[l, bend right, "(-)^{\vee}"'] \end{tikzcd}$$

Proof. We begin by examining what is involved in giving an arrow $G : \mathscr{C}^{\vee} \to \mathscr{A}$ in $\mathscr{V}$-$\mathbf{opCat}^{\mathrm{op}}$; that is, in giving a $\mathscr{V}$-opfunctor $G : \mathscr{A} \to \mathscr{C}^{\vee}$. We must have

(a) for each object A of $\mathscr{A}$, a $\mathscr{V}$-functor $G(A, -) : \mathscr{C} \to \mathscr{V}$ with pointwise left duals, and

(b) for each pair of objects A, B of $\mathscr{A}$, an arrow $G_{A,B} : \mathscr{C}^{\vee}(G(A, -), G(B, -)) \to \mathscr{A}(A, B)$ in $\mathscr{V}$ which respects cocomposition and counit.

Expanding (a), we have, for all $A \in \mathscr{A}$, $U \in \mathscr{C}$, objects $G(A, U) \in \mathscr{V}$ with left duals, and further, for $V \in \mathscr{C}$, arrows $G(A, -)_{U,V} : \mathscr{C}(U, V) \to \mathscr{V}(G(A, U), G(A, V))$ in $\mathscr{V}$ respecting composition and unit in $\mathscr{C}$. We note that the arrows $G(A, -)_{U,V}$ can be replaced by arrows

$$\mu : \mathscr{C}(U, V) \otimes G(A, U) \to G(A, V)$$

which are associative and preserve unit as needed for an action by $\mathscr{C}$. Turning to (b), we see that the arrows $G_{A,B}$ can be replaced by arrows

$$\delta : G(A, U) \to G(B, U) \otimes \mathscr{A}(A, B)$$

which respect cocomposition and counit of $\mathscr{A}$ and are $\mathscr{V}$-natural in U in the sense that the following *compatibility square* should commute:

$$\begin{tikzcd} \mathcal{C}(U, V) \otimes G(A, U) \arrow[r, "1 \otimes \delta"] \arrow[d, "\mu"'] & \mathcal{C}(U, V) \otimes G(B, U) \otimes \mathcal{A}(A, B) \arrow[d, "\mu \otimes 1"] \\ G(A, V) \arrow[r, "\delta"'] & G(B, V) \otimes \mathcal{A}(A, B) \end{tikzcd}$$

Now we examine what is involved in giving a $\mathscr{V}$-functor $F : \mathscr{C} \to Comod_f(\mathscr{A})$. We must have

(i) for each object U of $\mathscr{C}$, an $\mathscr{A}$-comodule $F(U, -)$ for which $F(U, A)$ has a left dual for each $A \in \mathscr{A}$, and

(ii) for $U, V \in \mathscr{C}$, an arrow $F_{U,V} : \mathscr{C}(U, V) \to Comod(\mathscr{A})(F(U, -), F(V, -))$ in $\mathscr{V}$ which respects composition and unit in $\mathscr{C}$.

Expanding (i), we have, for all $A \in \mathscr{A}$, $U \in \mathscr{C}$, objects $F(U, A) \in \mathscr{V}$ with left duals, and further, for $B \in \mathscr{A}$, arrows

$$\delta : F(U, A) \to F(U, B) \otimes \mathscr{A}(A, B)$$

which respect cocomposition and counit of $\mathscr{A}$. Turning to (ii), we see that the arrows $F_{U,V}$ can be replaced by arrows

$$\mu : \mathscr{C}(U, V) \otimes F(U, A) \to F(V, A)$$

which respect composition and unit in $\mathscr{C}$, and are opnatural in $\mathscr{A}$ in the sense that the above compatibility square should commute with $G(A, U)$ replaced by $F(U, A)$.

The required natural bijection between the arrows $G : \mathscr{C}^{\vee} \to \mathscr{A}$ in $\mathscr{V}$-**opCat**$^{\text{op}}$ and the $\mathscr{V}$-functors $F : \mathscr{C} \to Comod_f(\mathscr{A})$ is therefore given by $F(U, A) = G(A, U)$. It is easy to see that this is in fact an isomorphism of categories $\mathscr{V}$-**opCat**$^{\text{op}}(C^{\vee}, \mathscr{A}) \cong \mathscr{V}$-**Cat**$(\mathscr{C}, Comod_f(\mathscr{A}))$ so that we have a 2-adjunction. Q.E.D.

As a consequence of this adjunction, there are a "unit" and a "counit" which both look more like "units" because the opposite of the 2-category $\mathscr{V}$-**opCat** is involved. For each $\mathscr{V}$-category $\mathscr{C}$, we have the $\mathscr{V}$-functor $N : \mathscr{C} \to Comod_f(\mathscr{C}^{\vee})$ whose value at the object $U \in \mathscr{C}$ is the $\mathscr{C}^{\vee}$-comodule $N(U)$ with $N(U)(X) = X(U)$ for all $X \in \mathscr{C}^{\vee}$: the following triangle commutes:

$$\begin{tikzcd} C \arrow[rr, "N"] \arrow[dr, "X"'] & & Comod_f(C^{\vee}) \arrow[dl, "E_X"] \\ & \mathcal{V} & \end{tikzcd}$$

For each $\mathscr{V}$-opcategory $\mathscr{A}$, we have the $\mathscr{V}$-opfunctor $L : \mathscr{A} \to Comod_f(\mathscr{A})^{\vee}$ whose value at the object $A \in \mathscr{A}$ is the $\mathscr{V}$-functor $L(A) : Comod_f(\mathscr{A}) \to \mathscr{V}$ given by $L(A)(M) = M(A)$.

If $\mathscr{V}$ is a braided monoidal category then so is $\mathscr{V}^{\text{op}}$, so there is a well-defined tensor product of $\mathscr{V}$-opcategories as well as tensor product of $\mathscr{V}$-categories. For $\mathscr{V}$-opcategories $\mathscr{A}$, $\mathscr{A}'$, we have a canonical $\mathscr{V}$-functor

$$\Phi : Comod_f(\mathscr{A}) \otimes Comod_f(\mathscr{A}') \to Comod_f(\mathscr{A} \otimes \mathscr{A}').$$

For an $\mathscr{A}$-comodule M and $\mathscr{A}'$-comodule M', the $\mathscr{A} \otimes \mathscr{A}'$-comodule $\Phi(M, M')$ has

$$\Phi(M, M')(A, A') = M(A) \otimes M'(A'),$$

and the coaction

$$\delta : \Phi(M, M')(A, A') \to \Phi(M, M')(B, B') \otimes \mathscr{A}(A, B) \otimes \mathscr{A}'(A', B')$$

is the composite

$$\begin{aligned} M(A) \otimes M'(A') &\xrightarrow{\delta \otimes \delta} M(B) \otimes \mathscr{A}(A, B) \otimes M'(B') \otimes \mathscr{A}(A', B') \\ &\xrightarrow{1 \otimes c \otimes 1} M(B) \otimes M'(B') \otimes \mathscr{A}(A, B) \otimes \mathscr{A}'(A', B'). \end{aligned}$$

The effect of Φ on homs is induced by the effects

$$[M(A), N(A)] \otimes [M'(A'), N'(A')] \to [M(A) \otimes M'(A'), N(A) \otimes N'(A')]$$

of the tensor product of $\mathscr{V}$ on internal homs. There is also the canonical $\mathscr{V}$-functor

$$\Phi_0 : I \to Comod_f(I) = \mathscr{V}_f$$

which picks out the object $I \in \mathscr{V}_f$. This enriches the 2-functor $Comod_f$: $\mathscr{V}$-$\mathbf{opCat}^{\mathrm{op}} \to \mathscr{V}$-$\mathbf{Cat}$ with a weak monoidal structure. Consequently (Proposition 5), $Comod_f$ will preserve pseudo-monoidal structures, which brings us to the next definition.

DEFINITION 22. A *comonoidal $\mathscr{V}$-opcategory* is a pseudomonoid in $\mathscr{V}$-$\mathbf{opCat}^{\mathrm{op}}$.

To be more explicit, a comonoidal structure on a $\mathscr{V}$-opcategory $\mathscr{A}$ consists of $\mathscr{V}$-opfunctors $D : \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}$ and $E : \mathscr{A} \to I$, together with invertible coherent coassociativity and counit constraints. As with Hopf algebroids, it is possible to replace D up to isomorphism by a D which is given on objects by $D(A) = (A, A)$. The effects of D and E on homs then amount to arrows,

$$\mu : \mathscr{A}(A, B) \otimes \mathscr{A}(A, B) \to \mathscr{A}(A, B), \qquad \eta : I \to \mathscr{A}(A, A)$$

in $\mathscr{V}$. The associativity constraint has component $a_A : \mathscr{A}(A, A) \otimes \mathscr{A}(A, A) \otimes \mathscr{A}(A, A) \to I$ at $A \in \mathscr{A}$ satisfying invertibility, opnaturality, and coherency conditions; the unity constraints can be taken equal to the counit $\varepsilon : \mathscr{A}(A, A) \to I$.

If $\mathscr{A}$ is a comonoidal $\mathscr{V}$-opcategory then $Comod_f(\mathscr{A})$ becomes a monoidal category since $Comod_f$ is weak monoidal. We shall describe this structure in more detail. The tensor product $M \otimes N$ of $\mathscr{A}$-comodules M, N is given by $(M \otimes N)(A) = M(A) \otimes N(A)$ with coaction given by commutativity of the following square:

$$\begin{tikzcd}
M(A) \otimes N(A) \arrow[r, "\delta"] \arrow[d, "\delta \otimes \delta"'] & M(B) \otimes N(B) \otimes \mathscr{A}(A,B) \\
M(B) \otimes \mathscr{A}(A,B) \otimes N(B) \otimes \mathscr{A}(A,B) \arrow[r, "1 \otimes c \otimes 1"'] & M(B) \otimes N(B) \otimes \mathscr{A}(A,B) \otimes \mathscr{A}(A,B) \arrow[u, "1 \otimes 1 \otimes \mu"']
\end{tikzcd}$$

The associativity constraint $a : (M \otimes N) \otimes L \to M \otimes (N \otimes L)$ is the following composite:

$$\begin{aligned}
&(M(A) \otimes N(A)) \otimes L(A) \xrightarrow{\delta \otimes \delta \otimes \delta} M(A) \otimes \mathscr{A}(A,A) \otimes N(A) \otimes \mathscr{A}(A,A) \otimes L(A) \otimes \mathscr{A}(A,A) \\
&\xrightarrow{1 \otimes c \otimes c \otimes 1} M(A) \otimes N(A) \otimes \mathscr{A}(A,A) \otimes L(A) \otimes \mathscr{A}(A,A) \otimes \mathscr{A}(A,A) \\
&\xrightarrow{1 \otimes 1 \otimes c \otimes 1 \otimes 1} M(A) \otimes N(A) \otimes L(A) \otimes \mathscr{A}(A,A) \otimes \mathscr{A}(A,A) \otimes \mathscr{A}(A,A) \\
&\xrightarrow{1 \otimes 1 \otimes 1 \otimes a} M(A) \otimes (N(A) \otimes L(A))
\end{aligned}$$

For each $A \in \mathscr{A}$, the evaluation $\mathscr{V}$-functor $E_A : Comod_f(\mathscr{A}) \to \mathscr{V}$ becomes what Majid [Md] calls a "multiplicative" functor: it preserves the tensor product up to $\mathscr{V}$-natural isomorphisms, but these isomorphisms are not required to respect the associativity constraints as would be required for a strong monoidal functor.

The left adjoint of a weak monoidal 2-functor is weak comonoidal. We shall identify the weak comonoidal structure on the 2-functor $(-)^\vee$: $\mathscr{V}$-**Cat** $\to \mathscr{V}$-**opCat**$^{\mathrm{op}}$ which is the same as a weak monoidal structure on $(-)^\vee : \mathscr{V}$-**Cat**$^{\mathrm{op}} \to \mathscr{V}$-**opCat**. For $\mathscr{V}$-categories $\mathscr{C}$, $\mathscr{D}$, we have a $\mathscr{V}$-opfunctor $\Psi : \mathscr{C}^\vee \otimes \mathscr{D}^\vee \to (\mathscr{C} \otimes \mathscr{D})^\vee$ given on objects by $\Psi(X, Y) = X \underline{\otimes} Y$ which denotes the composite $\mathscr{V}$-functor

$$\mathscr{C} \otimes \mathscr{D} \xrightarrow{X \otimes Y} \mathscr{V} \otimes \mathscr{V} \xrightarrow{- \otimes -} \mathscr{V}.$$

Moreover, the effect of Ψ on homs is invertible:

$$\int^{U, V} (X' \underline{\otimes} Y')(U, V)^* \otimes (X \underline{\otimes} Y)(U, V)$$

$$\cong \int^{U} X'(U)^* \otimes X(U) \otimes \int^{V} Y'(V)^* \otimes Y(V).$$

Now suppose $\mathscr{C}$ is a monoidal $\mathscr{V}$-category. We write $\mathscr{C}^{\vee\otimes}$ for the $\mathscr{V}$-opcategory whose objects are the *multiplicative* $\mathscr{V}$-functors $X:\mathscr{C}\to\mathscr{V}$; that is, there are isomorphisms

$$X(I)\cong I,\qquad X(U\otimes V)\cong X(U)\otimes X(V),$$

where the latter are $\mathscr{V}$-natural in $U, V\in\mathscr{C}$. Also put $\mathscr{C}^{\vee\otimes}(X, Y)=\mathscr{C}^{\vee}(X, Y)$. A monoidal functor is a multiplicative functor for which the above isomorphisms are compatible with the associativity and unit constraints.

PROPOSITION 23. *For monoidal $\mathscr{V}$-categories $\mathscr{C}, \mathscr{D}$, the $\mathscr{V}$-opfunctor Ψ induces an equivalence of $\mathscr{V}$-opcategories*

$$\mathscr{C}^{\vee\otimes}\otimes\mathscr{D}^{\vee\otimes}\simeq(\mathscr{C}\otimes\mathscr{D})^{\vee\otimes}.$$

Proof. Since the effect of Ψ on homs is invertible, it remains to prove that each multiplicative $Z:\mathscr{C}\otimes\mathscr{D}\to\mathscr{V}$ is isomorphic to a $\mathscr{V}$-functor in the image of Ψ. We define $X, Y:\mathscr{C}\to\mathscr{V}$ by

$$X(U)=Z(U, I),\qquad Y(V)=Z(I, V).$$

Then $(X\underline{\otimes}Y)(U, V)=X(U)\otimes Y(V)=Z(U, I)\otimes Z(I, V)\cong Z((U, I)\otimes(I, V))\cong Z(U, V)$. Checking a few straightforward details, we obtain $Z\cong X\underline{\otimes}Y$. Q.E.D.

When $\mathscr{C}$ is a monoidal $\mathscr{V}$-category, there is a natural comonoidal structure on the $\mathscr{V}$-opcategory $\mathscr{C}^{\vee\otimes}$. To give the structure arrows $\mu:\mathscr{C}^{\vee\otimes}(X, Y)\otimes\mathscr{C}^{\vee\otimes}(X, Y)\to\mathscr{C}^{\vee\otimes}(X, Y)$, $\eta: I\to\mathscr{C}^{\vee\otimes}(X, X)$ for any multiplicative $\mathscr{V}$-functors $X, Y:\mathscr{C}\to\mathscr{V}$, we must provide arrows

$$\int^{U, V} Y(U)^*\otimes X(U)\otimes Y(V)^*\otimes X(V)\to\int^{W} Y(W)^*\otimes X(W),$$

$$I\to\int^{W} X(W)^*\otimes X(W).$$

For the first we take the arrow whose composite with the (U, V)-coprojection into the source coend is the composite of the isomorphism

$$\int^{U, V} Y(U)^*\otimes X(U)\otimes Y(V)^*\otimes X(V)\cong\int^{U, V} Y(U\otimes V)^*\otimes X(U\otimes V)$$

with the W-coprojection into the target coend with $W = U \otimes V$. For the second we take the composite of the isomorphism

$$I \cong X(I)^* \otimes X(I)$$

and the W-coprojection with $W = I$.

PROPOSITION 24. *Suppose $\mathscr{C}$ is a monoidal $\mathscr{V}$-category and $\mathscr{A}$ is a comonoidal $\mathscr{V}$-opcategory. Suppose $F : \mathscr{C} \to Comod_f(\mathscr{A})$ is a $\mathscr{V}$-functor and $G : \mathscr{A} \to \mathscr{C}^{\vee}$ is the $\mathscr{V}$-opfunctor corresponding to F under the adjunction of Proposition 22. Structures of multiplicative $\mathscr{V}$-functor on F are in natural bijection with strict comonoidal liftings $G' : \mathscr{A} \to \mathscr{C}^{\vee \otimes}$ of G through the forgetful $\mathscr{V}$-opfunctor $\mathscr{C}^{\vee \otimes} \to \mathscr{C}^{\vee}$. A multiplicative $\mathscr{V}$-functor structure on F is monoidal if and only if the corresponding G' lands in the full sub-$\mathscr{V}$-opcategory of $\mathscr{C}^{\vee \otimes}$ consisting of the monoidal functors.*

DEFINITION 23. A *Hopf $\mathscr{V}$-opalgebroid* $\mathscr{L}$ is a Hopf $\mathscr{V}^{\mathrm{op}}$-algebroid; that is, it is a comonoidal $\mathscr{V}$-opcategory $\mathscr{L}$, together with a $\mathscr{V}$-opfunctor $S : \mathscr{L}^{\mathrm{op}} \to \mathscr{L}$ (called the *antipode*) and a $\mathscr{V}$-opnatural isomorphism

$$\mathscr{L}(S(A), C) \otimes \mathscr{L}(A, B) \cong \mathscr{L}(B, C) \otimes \mathscr{L}(A, B).$$

The next thing to see is that $Comod_f(\mathscr{L})$ is right autonomous for each Hopf $\mathscr{V}$-opalgebroid $\mathscr{L}$. The approach is similar to that for Hopf algebroids (Proposition 19); however, for comodules we do not have the calculus of ends at our disposal and tensor product in $\mathscr{V}$ does not preserve all limits.

PROPOSITION 25. *If $\mathscr{L}$ is a Hopf $\mathscr{V}$-opalgebroid and M is any right $\mathscr{L}$-comodule then there is a canonical $\mathscr{V}$-opnatural isomorphism*

$$M(S(A)) \otimes \mathscr{L}(A, B) \cong M(B) \otimes \mathscr{L}(A, B).$$

Proof. Evaluate the diagram of Proposition 20 at $C \in \mathscr{L}$ and at $S(D) \in \mathscr{L}$. The two resultant diagrams are absolute limits in $\mathscr{V}$. Apply the functor $- \otimes \mathscr{L}(D, C)$ to the diagrams to obtain two diagrams which express $M(C) \otimes \mathscr{L}(D, C)$ and $M(S(D)) \otimes \mathscr{L}(D, C)$ as limits. Using the defining isomorphisms for antipode (Definition 23), we see that the two diagrams are isomorphic, so their limits must be isomorphic. Q.E.D.

Suppose $M \in Comod_f(\mathscr{L})$. Define an $\mathscr{L}$-comodule M^* by putting

$$M^*(A) = M(S(A))^*$$

and letting the coaction $\delta : M(S(A))^* \to M(S(B))^* \otimes \mathscr{L}(A, B)$ be the mate of the composite

$$M(S(B)) \xrightarrow{\delta} M(S(A)) \otimes \mathscr{L}(S(B), S(A)) \xrightarrow{1 \otimes S} M(S(A)) \otimes \mathscr{L}(A, B)$$
$$\xrightarrow{c} \mathscr{L}(A, B) \otimes M(S(A)).$$

PROPOSITION 26. *If $\mathscr{L}$ is a Hopf $\mathscr{V}$-opalgebroid and $\mathscr{M} \in Comod_f(\mathscr{L})$ then M^* is a right dual for M in $Comod(\mathscr{L})$. Consequently, $Comod_f(\mathscr{L})$ is right autonomous.*

Proof. We must prove that there is an isomorphism

$$Comod(\mathscr{L})(L \otimes M^*, N) \cong Comod(\mathscr{L})(L, N \otimes M)$$

$\mathscr{V}$-natural in L, N. This is determined by a canonical bijection between comodule morphisms $f: K \otimes (L \otimes M^*) \to N$ and comodule morphisms $g: K \otimes L \to N \otimes M$ for all $K \in \mathscr{V}$. To describe the bijection, notice that the component $f_A: K \otimes L(A) \otimes M(S(A))^* \to N(A)$ has a mate $K \otimes L(A) \to N(A) \otimes M(S(A))$ which, by using Proposition 20 to express $N(A)$ as an absolute limit, we see corresponds to a family of arrows $K \otimes L(A) \to N(B) \otimes M(S(A)) \otimes \mathscr{L}(A, B)$ compatible with the coactions. By Proposition 25, this amounts to a compatible family of arrows $K \otimes L(A) \to N(B) \otimes M(B) \otimes \mathscr{L}(A, B) = (N \otimes M)(B) \otimes \mathscr{L}(A, B)$. By Proposition 20 applied to the comodule $N \otimes M$, these induce a family of arrows $g_A: K \otimes L(A) \to (N \otimes M)(A)$ which can be seen to be a comodule morphism $g: K \otimes L \to N \otimes M$. The remaining details are left to the reader. Q.E.D.

For each monoidal $\mathscr{V}$-category $\mathscr{C}$, we write $\mathscr{C}^{\vee a}$ for the full sub-$\mathscr{V}$-opcategory of $\mathscr{C}^{\vee \otimes}$ consisting of the monoidal $\mathscr{V}$-functors $X: \mathscr{C} \to \mathscr{V}$. The comonoidal structure on $\mathscr{C}^{\vee \otimes}$ restricts to $\mathscr{C}^{\vee a}$. Suppose $\mathscr{C}$ is right autonomous. We shall define an antipode S for $\mathscr{C}^{\vee a}$. The $\mathscr{V}$-opfunctor $S: \mathscr{C}^{\vee a\, \mathrm{op}} \to \mathscr{C}^{\vee a}$ is given by the identity $S(X) = X$ on objects $X \in \mathscr{C}^{\vee a}$. The effect of S on the hom of $X, Y \in \mathscr{C}^{\vee a}$ is the arrow $S_{X,Y}: \mathscr{C}^{\vee}(X, Y) \to \mathscr{C}^{\vee}(Y, X)$ uniquely determined by commutativity of the following diagram in which the top-left isomorphism arises from the fact that (strong) monoidal $\mathscr{V}$-functors preserve duals.

$$\begin{tikzcd}
Y(U)^* \otimes X(U) \arrow[r, "\cong"] \arrow[d, "\mathrm{copr}_U"'] & Y(U^*) \otimes X(U^*)^* \arrow[r, "c"] & X(U^*)^* \otimes Y(U^*) \arrow[d, "\mathrm{copr}_{U^*}"] \\
C^{\vee}(X, Y) \arrow[rr, "S_{X,Y}"'] & & C^{\vee}(Y, X)
\end{tikzcd}$$

PROPOSITION 27. *If $\mathscr{C}$ is a right autonomous monoidal $\mathscr{V}$-category then $\mathscr{C}^{\vee a}$ is a Hopf $\mathscr{V}$-opalgebroid.*

Proof. For all (strong) monoidal $\mathscr{V}$-functors $X, Y, Z : \mathscr{C} \to \mathscr{V}$, the following calculation, invoking the Yoneda lemma, provides the required isomorphism:

$$\begin{aligned}
&\mathscr{C}^{\vee}(S(X), Z) \otimes \mathscr{C}^{\vee}(X, Y) \\
&\quad \cong \int^{U,V} Z(U)^* \otimes X(U) \otimes Y(V)^* \otimes X(V) \\
&\quad \cong \int^{U,V} Z(U)^* \otimes Y(V)^* \otimes X(V \otimes U) \\
&\quad \cong \int^{U,V,W} \mathscr{C}(W, V \otimes U) \otimes Z(U)^* \otimes Y(V)^* \otimes X(W) \\
&\quad \cong \int^{U,V,W} \mathscr{C}(W \otimes U^*, V) \otimes Z(U)^* \otimes Y(V)^* \otimes X(W) \\
&\quad \cong \int^{U,W} Z(U)^* \otimes Y(W \otimes U^*)^* \otimes X(W) \\
&\quad \cong \int^{U,W} Z(U)^* \otimes Y(U) \otimes Y(W)^* \otimes X(W) \\
&\quad \cong \mathscr{C}^{\vee}(Y, Z) \otimes \mathscr{C}^{\vee}(X, Y).
\end{aligned}$$

Q.E.D.

ACKNOWLEDGMENTS

We are deeply grateful to Martin Neuchl and the referee for noticing that, in our original preprint, our unpackaging of the definitions of Gray monoid and braided Gray monoid had some omissions. We are also grateful to Margaret McIntyre and the referee for other astute corrections. This work was supported in part by grants from the Australian Research Council.

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Printed in Belgium