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THE REPRESENTATION OF RELATIONAL ALGEBRAS

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§1. Summary

A family of binary relations, or sets of ordered pairs, which is a boolean algebra, which is closed under the operations of forming converses and relative products, and which contains an identity relation, constitutes what Tarski has called a proper relational algebra. Tarski has given a set of axioms for abstract algebras of this type, and has raised the question whether every abstract algebraic system satisfying these axioms is isomorphic to a proper relational algebra.

It follows from the results established below that this is not the case.

In Part One we obtain, after some preliminaries, a set C of conditions which are necessary and sufficient for a *finite* algebra to be isomorphic to a proper relational algebra. A finite model is exhibited which satisfies all of Tarski's axioms, but which does not satisfy all the conditions C, and hence is not representable.

The conditions C, which are infinite in number and non-finitary in the sense that they contain bound variables, are necessary for any relational algebra, finite or infinite, that is complete as a boolean algebra, to be representable. In Part Two it is shown, by means of an infinite model, that no set of finitary axioms suffices to characterize the class of representable relational algebras.

The relation of these results to some unsolved problems is discussed in the last section (§15).

PART ONE

§2. Definitions and notation

For the operations of boolean algebra familiar notation will be used: $R \cup S$ for the *union*, $R \cap S$ for the *intersection*, and $R - S$ for the *difference* of two elements R and S. We shall write 0 for the *null element*, V for the *universal element*, and $R \subset S$ if R is included in S.

The union of a finite or infinite family of elements R_i will be written $\bigcup R_i$, and the intersection $\bigcap R_i$. An algebra which contains the union (and so also the intersection) of every family of its elements will be called *complete*. We shall assume throughout the distributive laws for infinite intersections and unions, whenever defined; under this assumption a complete boolean algebra is automatically atomistic. A *minimal element* is a non-null element that properly includes no other element.

A *proper relational algebra* (PRA), over a domain D, is a boolean algebra of sets R of ordered pairs of elements of D. The notation aRb will express, as usual, that the pair (a, b) belongs to R. The *identity relation* I is the set of all pairs

(a, a) for a in D, and the *universal relation* V is the set of all pairs (a, b) for a and b in D. The *converse* R^u of a relation R is defined in the usual way:

$$aR^ub \qquad \text{if and only if} \qquad bRa;$$

so also is the *product* (or 'composition') RS of two relations R and S:

$$aRSc \qquad \text{if and only if} \qquad aRb \text{ and } bSc \qquad \text{for some } b \text{ in } D.$$

A proper relational algebra is required to contain I and V, and to be closed under the operations converse and product.

It is easily verified that every proper relational algebra satisfies the following six axioms:

A1. $RI = R$

A2. $R^{uu} = R$

A3. $(RS)^u = S^uR^u$

A4. $RS \cap T^u = 0 \cdot \rightarrow \cdot ST \cap R^u = 0$

A5. $(RS)T = R(ST)$

A6. $R \neq 0 \cdot \rightarrow \cdot V(RV) = V$.

An (abstract) *relational algebra* (RA) is any abstract boolean algebra containing an 'identity', a unary 'converse' operation, and a binary 'product' operation, which satisfies A1 to A6.

A *complete relational algebra* is a RA that is complete as a boolean algebra.

Among the most important consequences of axioms A1 to A6 are the two *distributive laws for unions*:

$$(\bigcup R_i)^u = \bigcup (R_i^u)$$
$$(\bigcup R_i)S = \bigcup (R_iS),$$

which hold whenever the unions involved exist.

§3. The structure of relational algebras

(The informal remarks in this section are not strictly necessary to the proofs of the theorems of this paper, but are included to clarify the ideas involved.)

A comprehensive discussion of relational algebras has been given by Tarski;[1] the axioms A1 to A6 are essentially those given by him. The term relation(al) algebra was introduced for systems satisfying A1 to A5; among these generalized RA's those satisfying in addition axiom A6 are called *simple*.[2]

Jónsson and Tarski have proved the following theorem: Every (generalized) RA is isomorphic to a subalgebra of a complete (generalized) RA.[3]

Standard algebraic arguments suffice to prove also that: Every complete generalized RA is isomorphic to a direct union of simple RA's.[4] For PRA's this

[1] [11]; A1–5 correspond respectively to XI, VIII, IX, XIII and X of (11), while McKinsey and Tarski have shown that A6 is equivalent to XII (see [6], Result 4).

[2] [6]; also, axiom VII of [11]: $V \neq 0$, is omitted in [6] and in the present paper.

[3] [6]; Result 1.

[4] This exemplifies the type of general algebraic analysis discussed by Birkhoff in [1] and [2]. The components D_i are the domains of the minimal (principal) two-sided union-product ideals.

has the following interpretation: the universal element V is an equivalence over the domain D, and hence partitions D into disjoint orthogonal components D_i. The algebra obtained by contraction to each of these components D_i is simple, and the original algebra is the direct union of these simple subalgebras.[5]

This result reduces the representation problem for complete generalized RA's to that for simple RA's. For simple RA's, every homomorphic image is either an isomorphic image or a null-algebra. Further, if a simple RA has a representation in which the universal element V and the identity element I are perhaps not represented by the universal element and identity element over the domain D, it is always possible to derive a new representation in which V and I are properly represented.[6] Thus no reduction of the representation problem for simple RA's appears possible.

It is nonetheless possible to carry the algebraic analysis of simple RA's one step further. Following standard algebraic procedure, let the identity I be decomposed into its minimal elements, or units, I_i, which will then be orthogonal. For a PRA this amounts to decomposing the domain D into disjoint subdomains D_i, with the property that each minimal element of the algebra is a correspondence from all of some D_i onto all of some D_j.[7]

Those algebras in which I is already minimal satisfy the following stronger form of A6:

$$R \neq 0 \cdot \rightarrow \cdot RV = V,$$

which is equivalent to

$$R \neq 0 \cdot \; \& \; \cdot S \neq 0 \cdot \rightarrow \cdot RS \neq 0.$$

Such algebras, which are 'without 0-divisors,'[8] may be called *integral* RA's. If U is a unit in the RA A, then the subalgebra UAU is integral.

The minimal elements of an integral RA form the basis for what Wedderburn has termed a 'boolean linear associative algebra'.[9] They also constitute a 'regular multigroup' in the terminology of Dresher and Ore.[10] It may further be noted that every RA is a 'projective algebra' in the sense of Everett and Ulam.[11]

§4. The basis of a complete relational algebra

In a complete RA the set of all minimal elements constitutes a *basis* in the sense that every element is (uniquely) expressible as a union of minimal elements.

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[5] See in particular [2].

[6] [6]; Result 5 establishes the assertion concerning V, while that concerning I is obtained by an obvious identification.

[7] These minimal subdomains correspond to the minimal one-sided union-product ideals.

[8] [6]; problem 2.

[9] See [12]. Pursuing the analogy with classical linear algebra, we remark that every PRA is in a sense a boolean matrix algebra (see [3]); and some, but not all, non-representable RA's are matrix algebras in a natural and non-trivial sense.

[10] [4]; the representation mentioned in [4], which is not in general faithful (see the remark following the example cited), is faithful for those multigroups that are bases for integral RA's. However, distinct (minimal) elements do not always receive disjoint representations (see also [6], Result 1).

[11] [9]; also [7].

From the distributive laws for unions (§2) it is clear that the RA is completely determined by a knowledge of the *units*, or minimal elements contained in I, of the converse of each minimal element, and of the product of each two minimal elements.

The converse of a minimal element is clearly again a minimal element; however the product of two minimal elements will not always be a minimal element unless the basis constitutes a group. A minimal element X is a unit if and only if $XX^u = X$. Thus a complete RA may be characterized in terms of its basis by specifying:

(i) a mapping $X \to X^u$ of B onto itself, and

(ii) a complete set of 'incidence relations' $X \subset YZ$ for triples of elements of B.

It is useful to express axioms A2 to A6 in terms of the basis B. These axioms are equivalent to the following set:

B2. $X^{uu} = X$

B3. $X^u \subset YZ \cdot \to \cdot X \subset Z^u Y^u$

B4. $X^u \subset YZ \cdot \to \cdot Y^u \subset ZX$

B5. $T \subset XY \cdot \& \cdot U \subset TZ \cdot \to (\exists W)\, (W \subset YZ \cdot \& \cdot U \subset XW)$.

Axiom A1 does not assume a convenient form in terms of minimal elements.

It follows from B2, 3, and 4 that the six conditions

$$(1) \qquad \begin{matrix} X^u \subset YZ, & Y^u \subset ZX, & Z^u \subset XY, \\ X \subset Z^u Y^u, & Y \subset X^u Z^u, & Z \subset Y^u X^u, \end{matrix}$$

are all equivalent; a triple of relations satisfying these conditions will be said to constitute a *cycle*. Clearly, if an abstract set B is given, together with an involution $X \to X^u$, and a list of cycles, in order to show that B generates a RA it remains only to verify A1, A5, and A6.

§5. The Conditions C

An infinite family C of conditions will be defined, which will later be shown necessary for a complete RA to be representable, and also sufficient in the case of finite RA's. The simplest among these conditions will be examined in detail.

We define C by defining recursively a family E of expressions which, properly interpreted, express the conditions C. The variables entering in these expressions will be doubly indexed, X_{ij}, where the indices i and j are positive integers. The X_{ij} will be restricted to range over the minimal elements of the algebra. Following the occurrence of a variable X_{ij} in a quantifier, the sign X_{ji} is always to be understood as an abbreviation for X_{ij}^u. Further, every variable X_{ii} whose two indices are the same is to be understood as subject to the condition $X_{ii} \subset I$. (All these tacit conditions could be written explicitly into the expressions E, but would make them rather unwieldly.)

The recursive step will involve adjoining a further clause to the consequent of the last conditional occurring in a condition E, and this *within the scope of* *all quantifiers already present in E.* Thus, from an expression of the form "$\cdots \to C) \cdots$)" we obtain a new expression of the form "$\cdots \to C \,\&\, D) \cdots$)". The precise formulation of this construction is facilitated by the unambiguous convention of omitting all final right parentheses.

Recursive definition of the family of expressions E:

(i) The expression

$$E_0\colon (X_{12}, X_{23}, X_{31})\,(X_{23} \subset X_{21}X_{13}\cdot \to \cdot 0 = 0$$

belongs to the family.

(ii) If an expression E belongs to the family, and if the maximum index i or j occurring on a variable X_{ij} of E is $n-1$, then, for each pair of positive integers a_n and b_n not greater than $n-1$, the following expression E' belongs to the family:

$$\begin{aligned} E'\colon\ & E.\,\&\,(X_{a_n n}, X_{n b_n})\,(X_{a_n b_n} \subset X_{a_n n}X_{n b_n}\cdot \to :\\ & \to (\exists X_{1n})\,(X_{1n} \subset X_{1a_n}X_{a_n n} \cap X_{1b_n}X_{b_n n}\cdot \,\&\\ & \&\,(\exists X_{2n})\,(X_{2n} \subset X_{2a_n}X_{a_n n} \cap X_{2b_n}X_{b_n n} \cap X_{21}X_{1n}\cdot \,\&\, \cdot\\ & \cdot \qquad \cdot \qquad \cdot \qquad \cdot\\ & \&\,(\exists X_{jn})\,(X_{jn} \subset X_{ja_n}X_{a_n n} \cap X_{jb_n}X_{b_n n} \cap \bigcap_{i<j} X_{ji}X_{in}\cdot \,\&\, \cdot\\ & \cdot \qquad \cdot \qquad \cdot \qquad \cdot\\ & \&\,(\exists X_{n-1,n})\,(X_{n-1,n} \subset X_{n-1,a_n}X_{a_n n} \cap X_{n-1,b_n}X_{b_n n} \cap \bigcap_{i<n-1} X_{n-1,i}X_{in}\,. \end{aligned}$$

(In this expression the i and j range through the integers from 1 to $n-1$, omitting the values a_n and b_n.)

The family of conditions C is defined to be exactly the set of conditions expressed by the expressions E, under the interpretation prescribed above. In the study of RA's it is natural to regard as equivalent two of these conditions which are equivalent under the axioms A1 to A4 as hypotheses. (Axiom A5 will turn out to be one of the conditions C.) Thus E_0 expressed a tautologous condition, and it may happen that formally different expressions E may express equivalent conditions.

We shall now examine the first few propositions of the family C, that is, for $n \leqq 5$.

Case $n = 3$: Here we have only the proposition C0 expressed by E_0, which is clearly tautologous and so devoid of interest.

Case $n = 4$: The case $a_4 = b_4$ expresses a condition which is deducible from A1 $-$ 4 (notably from A1), and is in this sense trivial. If a_n and b_n are different, we may, on grounds of symmetry, take them to be 1 and 2. This yields an expression E of the following form

$$\begin{aligned} & (X_{12}, X_{13}, X_{32})\,(X_{12} \subset X_{13}X_{32}\cdot \to\\ E1 \qquad & \qquad \cdot 0 = 0\cdot \,\&\,(X_{14}, X_{42})\,(X_{12} \subset X_{14}X_{42}\cdot \to\\ & \qquad\qquad (\exists X_{34})\,(X_{34} \subset X_{31}X_{14} \cap X_{32}X_{24} \end{aligned}$$

which is clearly equivalent to the more concise condition

C1 $\quad X_{12} \subset X_{13}X_{32} \cap X_{14}X_{42} : \rightarrow (\exists X_{34})\ (X_{34} \subset X_{31}X_{14} \cap X_{32}X_{24}).$

But comparison with B5 shows that this is just a variant of the associative law, A5, expressed in terms of minimal elements.

It will be instructive to pause to give a graphical interpretation of $C1$. Let A be a PRA over domain D. If $X_{12} \subset X_{13}X_{32}$, then D must contain three elements, or 'points', p_1, p_2, p_3 (not necessarily distinct) such that the ordered pairs, or 'directed arcs', (p_ip_j) belong to the minimal elements X_{ij} : more briefly, $p_iX_{ij}p_j$, for $i, j \leqq 3$. Now, if further $X_{12} \subset X_{14}X_{42}$, then D must surely contain a further point p_4 such that $p_1X_{14}p_4$ and $p_4X_{42}p_2$. But then A must contain a minimal element X_{34} relating p_3 to p_4 ; thus the original complete 3-graph is extended to a complete 4-graph. This argument shows that the condition $C1$ must hold in any complete PRA.

Case $n = 5$: Again we ignore the trivial case $a = b$. On grounds of symmetry the number of distinct cases, for $a \neq b$, reduces to three. These correspond to the various ways in which a triangle $(a, b, 5)$ can be adjoined to the four-graph of case 4 having the side (a, b) in common with it. The resulting conditions may be interpreted as stating that the resulting 5-graph can always be completed, so that every arc (p_i, p_j) belongs to some X_{ij} and all the relations $X_{ik} \subset X_{ij}X_{jk}$ hold.

Subcase 5.1: $a, b = 1, 2$. This gives an expression

$$(X_{12}, X_{13}, X_{32})\ (X_{12} \subset X_{13}X_{32} \cdot \rightarrow$$
$$\rightarrow (X_{14}, X_{42})\ (X_{12} \subset X_{14}X_{42} \cdot \rightarrow (\exists X_{34})\ (X_{34} \subset X_{31}X_{14} \cap X_{32}X_{24} \cdot\ \&$$
$$\&\ (X_{15}, X_{52})\ (X_{12} \subset X_{15}X_{52} \cdot \rightarrow (\exists X_{35})\ (X_{35} \subset X_{31}X_{15} \cap X_{32}X_{25} \cdot\ \&$$
$$\&\ (\exists X_{45})\ (X_{45} \subset X_{41}X_{15} \cap X_{42}X_{25} \cap X_{43}X_{35} \cdot$$

Introducing the abbreviation $P_{hk}^{m\cdots n} = X_{hm}X_{mk} \cap \cdots \cap X_{hn}X_{nk}$, this can be written in an equivalent simpler form:

C2: $$P_{12}^{345} \neq 0 \cdot \rightarrow \cdot P_{45}^{12} \cap P_{43}^{12}P_{35}^{12} \neq 0.$$

Subcase 5.2: $a, b = 1, 3$, leads to a condition which reduces to

C3: $$P_{12}^{34} \neq 0 \cdot\ \&\ \cdot X_{13} \subset X_{15}X_{53} \cdot \rightarrow \cdot X_{51}X_{14} \cap P_{52}^{12}X_{24} \cap X_{53}P_{34}^{12} \neq 0.$$

Subcase 5.3: $a, b = 3, 4$, leads to a condition which may be written:

C4: $$X_{12} \subset P_{12}^{34} \cdot \rightarrow (\exists X_{34})(X_{34} \subset P_{34}^{12}$$
$$\&\ (X_{35}, X_{54})(X_{34} \subset X_{35}X_{54} \cdot \rightarrow \cdot X_{12} \subset P_{15}^{34}P_{52}^{34})).$$

It may be remarked that C1, C2, and C3 have been reduced to a form containing no bound variables; further, it may be verified that the force of these conditions is unaltered if the condition that the X_{ij} be minimal is removed. It will be shown in Part Two that C4, however, is not equivalent to any condition expressed without bound variables. In particular, although the condition which results from C4 by removing the restriction of the X_{ij} to minimal elements, which can be written

$$P_{34}^{12} \subset X_{35} X_{54} \cdot \rightarrow \cdot P_{12}^{34} \subset P_{15}^{34} P_{52}^{34},$$

is easily seen to be a consequence of C4, it follows that it must indeed be weaker than C4.

It is not hard to see that, with respect to implication, the family C constitutes a lattice. The tautologous expression C0 may be taken as the null element. The associative law C1 is the only element minimal over C0, while immediately over C1 are the three elements C2, C3, and C4. It appears that all the conditions C, except C0, 1, 2, and 3, are like C4 essentially non-finitary. It also seems intuitively certain that the lattice C has no maximal element, and hence is infinite.

§6. Necessity of the conditions C

THEOREM I. *Every complete relational algebra that is isomorphic to a proper relational algebra satisfies all the conditions of the family C.*

PROOF. It is clearly sufficient to show that all the conditions C hold in every complete PRA over a domain D. The proof of this lies in formulating, inductively for each n, the argument in terms of graphs that was suggested in discussing condition C1.

To show that no condition C fails in A amounts to prescribing a rule for assigning values to the existentially quantified variables X_{jn} occurring in an expression E in such a way that by this choice E is always realized. This rule will give the X_{jn} ($j \neq a_n, b_n; j < n$) as functions of the earlier X_{hk} ($h, k < n$), of a_n and b_n, and of the values assigned to $X_{a_n n}$ and $X_{b_n n}$.

For $n - 1$ assume inductively that such a rule has been given, and indeed that it assigns values in such a way that there always exist $p_1, \cdots, p_{n-1}$ in D with $p_i X_{ij} p_j$ for all $i, j < n$. Let $a, b < n$, and let X_{an}, X_{nb} be arbitrary minimal elements of A, satisfying the condition $X_{ab} \subset X_{an} X_{nb}$ (if this condition fails, E' reduces to E, which is already realized). Then D must contain a point p_n such that $p_a X_{an} p_n$ and $p_n X_{nb} p_b$. But then A must contain minimal elements—X_{jn} ($j \neq a, b; j \leqq n$) such that $p_j X_{jn} p_n$; for these all the clauses $X_{ik} \subset X_{ij} X_{jk}$ ($i, j, k \leqq n$) are realized, and so E' is realized.

This construction extends the inductive hypothesis to the case n, and since the case $n = 3$ is trivial, establishes the existence of a rule for realizing the expressions E for all n. This completes the proof.

§7. Representability of finite relational algebras

For finite RA's, which are automatically complete, the converse of Theorem I holds.

THEOREM II. *Every finite relational algebra in which the conditions* C *all hold is isomorphic to a proper relational algebra.*

PROOF. By a *realization* of an expression E will be meant a function f which assigns to variables X_{ij} entering in E values fX_{ij} which are minimal elements, and this in such a way as to satisfy the conditions

$$\text{(i)} \qquad fX_{ji} = (fX_{ij})^u, \qquad fX_{ik} \subset fX_{ij}fX_{jk}$$

whenever all are defined. We shall prove the existence of a realization f, defined over X_{ij} for all positive integers i, j. This realization will be so constructed that the correspondence from a minimal element Y of A to the set of all couples of integers (i, j), such that $fX_{ij} = Y$, defines a representation of A over the positive integers.

We shall be concerned only with those realizations, f_{n-1}, whose domain is exactly the set of all X_{ij} for $i, j < n$, (for some n). Among these we define recursively a sequence of subsets F_n. For this we suppose the (finite) set of minimal elements of A indexed with positive integers: $Y^1, Y^2, \cdots$.

DEFINITION. Let f_{n-1} be in F_{n-1}; then f_n is an *immediate extension* of f_{n-1} if:
(ii) $f_{n-1}X_{ab} \subset Y^hY^k$, and yet for no $c < n$ is $f_{n-1}X_{ac} = Y^h$ and $f_{n-1}X_{cb} = Y^k$;
(iii) the quadruple $(a, b, h, k,)$ is lexicographically the first with this property;
(iv) f_n, defined for all X_{ij}, $i, j \leqq n$, coincides with f_{n-1} for $i, j < n$, while $f_nX_{an} = Y^h$ and $f_nX_{nb} = Y^k$.
The set F_3 is that of all realizations of the inclusion $X_{12} \subset X_{13}X_{32}$; inductively, F_n is defined as the set of all extensions f_n of elements f_{n-1} of the set F_{n-1}.

From the recursive definition of E' in terms of E, it can be seen that the hypothesis that all the propositions C hold in A ensures that none of the sets F_n is empty. (Unless possibly for some f_{n-1} no instance of (ii) is to be found; in this case the function f_{n-1} replaces the limit function f to be constructed below, and the remainder of the argument shows that f_{n-1} defines a representation of A over a finite set.)

Before continuing, it is advantageous to introduce one rather trivial modification of the foregoing definitions: we exclude from the list (Y^h) all *units* of A, that is, all minimal elements contained in the identity. Then, in view of (iv), no variable X_{an}, X_{nb} (i.e., no universally quantified variable in the corresponding expression E') can be assigned as value a unit. But neither can any (existentially quantified) variable X_{hn} $(h \neq a, b, n)$: for $X_{hn} \subset X_{ha}X_{an} \cap X_{hb}X_{bn}$, and $fX_{hn} \subset I$ implies $fX_{ah} = fX_{an}$ and $fX_{hb} = fX_{nb}$, whence $fX_{ab} \subset fX_{ah}\, fX_{hb}$, $a, b, h < n$, contrary to (ii) and (iv). The values fX_{nn} must of course always be units; but it is a trivial matter to see that the choice of these values is always possible, and is uniquely determined by the remaining fX_{ij}.

Since no F_m is empty, for all $m > n$ there must exist some $f_n^{(m)}$ in F_n possessing an extension f_m in F_m. But A is finite and so F_n is finite; hence there exists some particular f_n in F_n, such that for *all* $m > n$, f_n has an extension f_m in F_m. The same argument shows that f_n has an immediate extension f_{n+1} with the property that, for all $m > n + 1$, f_{n+1} has an extension f_m. This establishes by induction the existence of an infinite chain of extensions: $f_n, f_{n+1}, f_{n+2}, \cdots$. The limit (or union) of this sequence is a function f, with the properties: (i) and

(v) $fX_{ab} \subset Y^h Y^k$ implies that, for some n, $fX_{an} = Y^h$ and $fX_{nb} = Y^k$;

(vi) $fX_{ab} \subset I$ if and only if $a = b$.

With each minimal element Y of A, we associate a relation FY defined over the positive integers by setting

(viii) $a(FY)b$ if and only if $fX_{ab} = Y$.

Since the FY are clearly disjoint, they constitute a basis for a boolean algebra A' over the integers, and F is thus extended to an isomorphism of A, as a boolean algebra, onto A'. Further, it is immediately clear that F preserves converses and the identity relation.

Let $FY \cap (FZ)\ (FW) \neq 0$; then, for some i, j, k, we have $i(FY)k$, $i(FZ)j$, and $j(FW)k$. By (vii), $fX_{ik} = Y$, $fX_{ij} = Z$, and $fX_{jk} = W$, whence by (i) $Y \subset ZW$.

Conversely, let $Y \subset ZW$. If $i(FY)k$, then $fX_{ik} = Y \subset ZW$, and by (v) $fX_{ij} = Z$ and $fX_{jk} = W$ for some j. But then $i(FZ)j$ and $j(FW)k$, so that $i(FZ)\ (FW)k$. This proves that $Y \subset ZW$ implies $FY \subset (FZ)\ (FW)$.

From these two results it follows that $FY \cap (FZ)\ (FW) \neq 0$ implies $FY \subset (FZ)\ (FW)$, and hence that A' is closed under the operation of forming products. While also it follows that $FY \subset (FZ)\ (FW)$ if and only if $Y \subset ZW$, whence A' is a PRA isomorphic under F to the algebra A. This completes the proof of the theorem.

§8. A non-representable relational algebra

A finite relational algebra A (satisfying A1–6) will be defined in which the condition C2 fails, and which therefore is isomorphic to no proper relational algebra.[12]

Using the ideas developed in §4, we shall define A by listing its basis of minimal elements (56 in all), by defining their converses, and by listing all cycles.

The *basis* of A contains the following elements:

(i) x'_{ii}, x_{ii} for $i = 1, 2, 3, 4, 5$;

(ii) x_{12}, x'_{12}; x_{13}, x'_{13}; x_{34}, $\overset{1}{x}_{34}$, $\overset{2}{x}_{34}$; and all elements resulting from these by the permutation (1, 2) and any permutation of the indices (3, 4, 5);

(iii) all elements obtainable from these by reversing the order of subscripts.

The *converse* of any of these minimal elements r_{ij} is the element r_{ji} obtained by reversing the order of subscripts. The *cycles* of A are all the triples $(r_{ij}s_{jk}t_{ki})$ with 'matching subscripts,' *except the following*:

(i) $(r_{ij}s_{ji}x_{ii})$ unless $r_{ij} = s_{ij}$;

(ii) $(x_{34}\, x_{45}\, x_{53})$;

(iii) $(\overset{1}{x}_{34}x_{41}x_{13})$, and the analogous cycles obtained by permutation of 1 and 2, and of 3, 4, and 5.

The *universal element* V of A is, of course, the union of all 56 minimal elements. It is convenient to define V_{ij} for each pair i, j as the union of all elements r_{ij} bear-

[12] Other models, including one with only 52 minimal elements, have been constructed.

ing the subscripts i, j. (The indices i, j, etc. suggest subdomains; an element r_{ij} suggests a relation with range i and domain j.)

We may now derive from the list of cycles the

Multiplication table for A:

I. Every $r_{ij}s_{hk} = 0$ for $j \neq h$;

II. Every $r_{ij}s_{jk} = V_{ik}$ *except the following*:

(1) $x_{ii}r_{ij} = r_{ij}$;

(2) $r_{ij}s_{ji} = x'_{ii}$ unless $r_{ij} = s_{ij}$;

(3) $x_{34}x_{45} = \overset{1}{x}_{35} \cup \overset{2}{x}_{35}$;

(4) $x_{13}\overset{1}{x}_{34} = x'_{14}$;

(5) $x_{31}x_{14} = x_{34} \cup \overset{2}{x}_{34}$;

(6) all products obtainable from these by permutation of 1 and 2, and of 3, 4, and 5;

(7) all products obtainable from these in accordance with axiom A3.

LEMMA. *Axioms* A1–4 *hold in A*.

PROOF. A2, 3, and 4 follow automatically from the manner in which A was defined. If we note that $I = x_{11} \cup x_{22} \cup \cdots \cup x_{55}$, that A1 holds follows directly from the multiplication table.

LEMMA. *For all* r_{ij}, $r_{ij}V_{jk} = V_{ik}$, *and* $V_{ki}r_{ij} = V_{kj}$.

PROOF. In the list of cycles, for no r_{ij} and t_{ik} is every cycle $(r_{ij}s_{jk}t_{ki})$ excluded.

COROLLARY. *Axiom* A6 *holds in A*.

PROOF. It is enough to show that $V(r_{ij}V) = V$ for each r_{ij}. But $V_{hk} \subset V_{hi}V_{ik} = V_{hi}(r_{ij}V_{jk}) \subset V(r_{ij}V)$.

LEMMA. *The assosiative law*, A5, *holds in A*.

PROOF. It will clearly be enough to prove associativity for minimal elements with 'matching subscripts': $r_{ij}(s_{jk}t_{kh}) = (r_{ij}s_{jk})t_{kh}$. This can fail only in case at least one member is less than V_{ih}, and by symmetry (using A2, 3) we may assume that $(r_{ij}s_{jk})t_{kh} \neq V_{ih}$. But by the preceding lemma this is possible only in case $r_{ij}s_{jk} \neq V_{ik}$.

Thus it will suffice to consider each of the products rs of types (1–5) in the multiplication table, together with the product corresponding to it under A3. For each of these products rs, we need verify associativity for those t only such that $(rs)t \neq V_{ih}$.

CASE 1. If r, or indeed s or t, is a unit, associativity is immediate; henceforth we tacitly exclude this possibility.

CASE 2. $r = r_{ij}$, $s = s_{ji}$, and $r_{ij} \neq s_{ij}$. Then, by (1), $rs = x'_{ii}$, and inspection of the multiplication table shows that in every case $(rs)t_{ik} = V_{ik}$, contrary to assumption.

CASE 2′ obtained by applying A3 to (2), takes exactly the same form as Case 2.

CASE 3. $rs = x_{34}x_{45} = \overset{1}{x}_{35} \cup \overset{2}{x}_{35}$. Inspection of the multiplication table shows that either $\overset{1}{x}_{35}t_{5k}$ or $\overset{2}{x}_{35}t_{5k}$, and so their union, will be V_{3k} except in the case $t = x_{53}$. For this value of t, $(rs)t = \overset{1}{x}_{35}x_{53} \cup \overset{2}{x}_{35}x_{53} = x'_{33} \cup x'_{33} = x'_{33}$. But $r(st) = x_{34}(x_{45}x_{53}) = x_{34}(\overset{1}{x}_{43} \cup \overset{2}{x}_{43}) = x'_{33} \cup x'_{33} = x'_{33}$, and associativity holds.

CASE 3′ has the same form as Case 3, with 3, 4, and 5 permuted.

CASE 4. $rs = x_{13}x_{34}^{1} = x_{14}'$; then $(rs)t_{4k} = V_{1k}$ for all t_{4k}.

CASE 4′. The same argument applies.

CASE 5. $rs = x_{31}x_{14} = x_{34} \cup x_{34}^{2}$. Here $(rs)t = V_{3k}$ only in case $t = x_{43}^{1}$, when $(rs)t = x_{33}'$. But then $r(st) = x_{31}(x_{14}x_{43}^{1}) = x_{31}x_{13}' = x_{33}'$, and associativity holds.

CASE 5′ has the same form as Case 5, with 3 and 4 exchanged.

This disposes of all possible cases under which A5 might be violated, and so completes the proof of the lemma.

COROLLARY. *A is a relational algebra.*

LEMMA. *The condition* C2 *fails in A.*

PROOF. Condition C2 requires, in particular, that

$$x_{13}x_{32} \cap x_{14}x_{42} \cap x_{15}x_{52} \neq 0.$$
$$\rightarrow \cdot (x_{41}x_{15} \cap x_{42}x_{25}) \cap (x_{41}x_{13} \cap x_{42}x_{23})(x_{31}x_{15} \cap x_{32}x_{25}) \neq 0.$$

Since x_{12} is included in the left member of the first inequality, the antecedent holds. However, $(x_{41}x_{15} \cap x_{42}x_{25}) = (x_{45} \cup x_{45}^{2}) \cap (x_{45} \cup x_{45}^{1}) = x_{45}$, and similarly for the other terms in parentheses, whence the left member of the second inequality becomes $x_{45} \cap x_{43}x_{35} = x_{45} \cap (x_{45}^{1} \cup x_{45}^{2}) = 0$, and the consequent fails.

From this, and Theorem I, it follows that A is not representable.

THEOREM *III. There exists a finite relational algebra* (*satisfying* A1 *to* A6) *which is not isomorphic to any proper relational algebra.*

PART TWO

§9. The model M

The proof that the class of all representable relational algebras cannot be characterized by algebraic axioms depends upon the comparison of two infinite algebras M and M'. We begin by defining the algebra M.

Let P be the set of all rational numbers p, with $0 < p < 1$, to which has been adjoined a single irrational number Y lying in the same interval. The model M will be defined in terms of its basis in the same way as was the model A of §8. M will have a denumerably infinite basis, with certain of its minimal elements indexed by the set P.

Basis: $m_{ii}, m_{12}, m_{13}, m_{15}^{3p}, m_{34}^{p}, m_{35}^{p}$; $i = 1, 2, \cdots, 5$

$m_{ii}', m_{12}', m_{13}', m_{15}^{4p}, m_{34}', m_{35}'$; p in P

and the same under permutation of 1 and 2, of 3 and 4, and under reversal of the order of subscripts.

Converses: $(r_{ij})^{u} = r_{ji}$.

Cycles: all cycles $(r_{ij}s_{jk}t_{ki})$ with matching subscripts, *except*

$r_{ij}s_{ji}m_{ii}$	for $r_{ij} \neq s_{ij}$;
$m_{12}m_{52}^{hp}m_{51}^{kp}$	$h, k = 3, 4; p \neq Y$;
$m_{34}^{p_1}m_{45}^{p_2}m_{53}^{p_3}$	$p_i \neq p_j = p_k \neq Y$;

(i.e., if exactly two of the superscripts are equal, but are different from Y)

$$m_{13}m_{35}^{p}m_{51}^{3q} \quad \text{for } p < q;$$
$$m_{13}m_{35}^{p}m_{51}^{4q} \quad \text{for } q < p.$$

The identity of M is $I = m_{11} \cup m_{22} \cup m_{33} \cup m_{44} \cup m_{55}$; from the manner of definition it follows directly that M satisfies the axioms A1–4.

From the list of cycles we obtain the following *Multiplication table*: Products $r_{ij}s_{jk} \neq V_{ik}$, where neither r nor $s \subset I$; of each pair rs and $s^u r^u$ only one is listed.

1) $m_{12}m_{25}^{hp} = \bigcup_{p \neq r} m_{15}^{kr}$ $\qquad h, k = 3, 4; p \neq Y$

2) $m_{15}^{hp}m_{52}^{kp} = m_{12}'$ $\qquad h, k = 3, 4; p \neq Y$

3) $m_{34}^{p}m_{45}^{p} = m_{35}^{p} \cup m_{35}'$ $\qquad p \neq Y$

4) $m_{34}^{p}m_{45}^{q} = \bigcup_{r \neq p} m_{35}^{r} \cup m_{35}^{Y} \cup m_{35}'$ $\qquad p, q$ in P

(3) and 4): and the same for any permutation of (3, 4, 5))

5) $m_{13}m_{35}^{p} = \bigcup_{p \leq q} m_{15}^{3q} \cup \bigcup_{q \leq p} m_{15}^{4q}$ $\qquad p, q$ in P

6) $m_{35}^{p}m_{51}^{3q} = m_{31}'$ $\qquad p < q$, in P

$m_{35}^{p}m_{51}^{4q} = m_{31}'$ $\qquad q < p$, in P

7) $m_{31}m_{15}^{3p} = \bigcup_{q \leq p} m_{35}^{q} \cup m_{35}'$ $\qquad p, q$ in P

$m_{31}m_{15}^{4p} = \bigcup_{p \leq q} m_{35}^{q} \cup m_{35}'$

(5)–7): and the same under permutation of (3, 4))

§10. Representability of M

PROPOSITION. *M is a representable relational algebra.*

The *proof* will closely parallel that of Theorem II. However, since M has a denumerably infinite number of elements, the argument for the existence of an infinite chain of f_n must be modified. We shall define the sets F_n more narrowly, so that *every* f in F_n is extendible to an infinite chain.

Let the minimal elements of M, other than units, be arranged in sequence: $Y^1, Y^2, Y^3, \cdots$. We define for each integer $n \geqq 3$ inductively a family F_n of partial representations.

I. Let g be the first integer such that $Y^g \subset Y^1Y^2$; then F_3 contains a single element f_3, where f_3 is a function of the six symbols X_{ij} for $i, j \leqq 3$ and $i \neq j$ (or more exactly, of the six ordered pairs (i, j)), which assumes as values minimal elements of M. Specifically, $fX_{12} = Y^g$, $fX_{13} = Y^1$, $fX_{32} = Y^2$, and $fX_{ji} = (fX_{ij})^u$.

II. Let f be an element of F_{n-1}, such that fX_{ij} is defined for all $i, j < n$. Let the 5-uple (max (a, b, h, k), a, b, h, k) be lexicographically the first such that

(i) $fX_{ab} \subset Y^hY^k$

(ii) for no $c < n$ is $fX_{ac} = Y^h$ and $fX_{cb} = Y^k$.

(From the fact that the number of triples $Y^g \subset Y^hY^k$ is infinite, it is not hard to see that a, b, c, h, and k satisfying (i) and (ii) always exist.)

(iii) $f'X_{ij} = fX_{ij}$ for $i, j < n$, and $i \neq j$.

(iv) $f'X_{an} = Y^h$ and $f'X_{nb} = Y^k$

(v) $f'X_{hn}$ is some Y^g for all $h < n$, $h \neq a, b$

(vi) $f'X_{ji} = (f'X_{ij})^u$ for all $i, j \leqq n$, $i \neq j$

(vii) $f'X_{ik} \subset (f'X_{ij})(f'X_{jk})$ for all $i, j, k \leqq n, i \neq j \neq k \neq i$.
(viii) no $f'X_{hn}$ is $m_{12}, m_{13}, m_{14}, m_{23}, m_{24}$, or their converses, for $h \neq a, b$
(ix) $f'X_{hn} = m'_{34}, m^p_{34}$ $(p \neq Y)$, $h \neq a, b$, or their converses only in case

$(f'X_{hj})(f'X_{jn})$ does not contain m^Y_{34} or m^Y_{43} for some $j < n, j \neq h$.

If f' is related to f in this fashion, we shall call f' an *immediate extension* of f.

The meaning of this construction is roughly as follows: f in F_{n-1} is a function assigning each ordered couple (i, j), for $i, j \leqq n - 1$, to a minimal element fX_{ij} in such a way that every triple i, j, k is related in a manner compatible with the multiplication table for M: iY^gk and iY^hj, jY^sk implies $Y^g \subset Y^hY^s$.

The construction (i, ii, iv) is designed to insure that conversely, if $Y^g \subset Y^hY^s$ and iY^gk, then for some j, iY^hj and jY^sk. We suppose that for some i, k, h, s this condition is not realized; then f' is constructed to remedy this by adjoining the point $j = n$, and setting $f'X_{in} = Y^h$ and $f'X_{nj} = Y^s$ (the condition on the 5-uple serves to insure that, by a diagonal enumeration, every such condition will be realized for n sufficiently large.)

The crux of the problem lies in showing that f' can be extended over all pairs of indices less than n, in the same way as f was. Now the elements m'_{ij} and m^Y_{34} give the largest products; in order to make the extension of f' as easy as possible, we choose the $f'X_{in}$ $(i \neq a, b)$ to have these values whenever possible (viii, ix). We defer proof of the crucial

LEMMA. *Every f in F_{n-1} has an immediate extension f' in F_n.*

Supposing this to be true, there exists an infinite chain of extensions of f_3: $f_3 \subset f_4 \subset \cdots \subset f_{n-1} \subset f_n \subset \cdots$. Together, these define a limit(or union) function f. Now f is defined for all integers $i, j, i \neq j$, and the same reasoning as in the proof of Theorem II shows that f determines a representation of M.

PROOF OF THE LEMMA. This proceeds by contradiction. We shall assume that f is in F_{n-1}, that for some $a, b < n$ $fX_{ab} \subset Y^hY^k$, and that there is no way of choosing fX_{in} for all $i < n$ in accordance with (iii).

For brevity we shall occasionally write simply X_{ij} for fX_{ij}. Then the conditions (iv), (vi), and (vii) on the X_{in}, over and above those that are assured by the hypotheses, are:
(vi′) $X_{ni} = X^u_{in}$
(iv′) $X_{an} = Y^h$ and $X_{nb} = Y^k$.
(vii′) $X_{jn} \subset X_{ji}X_{in}$ for all $i, j < n, i \neq j$.

In view of (1) of §4, half the conditions (vii′) are redundant, and we can reformulate (vii′) in the more economical form

$$X_{h_1n} \subset X_{h_1a}X_{an} \cap X_{h_1b}X_{bn}. \tag{1}$$

$$\cdots$$

$$X_{h_jn} \subset X_{h_ja}X_{an} \cap X_{h_jb}X_{bn} \cap \bigcap_{i<j} X_{h_jh_i}X_{h_in}$$

$$\cdots$$

$$X_{h_sn} \subset X_{h_sa}X_{an} \cap X_{h_sb}X_{bn} \cap \bigcap_{i<s} X_{h_sh_i}X_{h_in}$$

$$X_{tn} \subset X_{ta}X_{an} \cap X_{tb}X_{bn} \cap \bigcap_{i\leqq s} X_{th_i}X_{h_in}.$$

Here the indices $(h_1, \cdots . h_s, t)$ are the set $(1, \cdots, n-1) - (a, b)$ arranged in some order. In the future we shall write $W_{h_1}, \cdots, W_{h_s}, W_t$ for the expressions appearing on the right in (1).

The hypothesis from which we must derive a contradiction is now, that for the values assigned to the X_{ij}, $i\ j < n$, and the values (iv′) of X_{an}, X_{nb}, it is impossible to assign values to $X_{h_1n}, \cdots, X_{h_sn}, X_{tn}$ in accordance with (vii) and (ix). But it can be seen that any product which contains m_{12} also contains m_{12}', and that any product with a factor m_{12} is contained in the corresponding product with m_{12} replaced by m_{12}'; and analogously for the other m_{ij} of (viii). Hence if any assignment of values satisfies (1), we can obtain from it one that satisfies also (viii) by replacing m_{12} by m_{12}'. Further, the condition (ix) merely states that a variable X_{hn} will be given the value m_{34}^Y or m_{43}^Y whenever this is compatible with the conditions (1), and so imposes no real limitation.

Therefore we may replace our hypothesis by the following: that no values whatsoever can be assigned to the $X_{h_1n}, \cdots, X_{h_sn}, X_{tn}$ so as to satisfy (1). But this amounts to the condition

$$X_{h_1n} \subset W_{h_1} \cdot\& \cdots \&\cdot X_{h_sn} \subset W_{h_s} :$$
$$\rightarrow : \sim (\exists X_{tn})(X_{tn} \subset W_t),$$

or equivalently

$$X_{h_1n} \subset W_{h_1} \cdot\&\cdot \cdots \cdot\&\cdot X_{h_sn} \subset W_{h_j} : \rightarrow : W_t = 0. \tag{2}$$

We make one more modification of our hypothesis, namely that the set of indices $(h_1, \cdots, h_s, t)$ are perhaps not all the indices $1, \cdots, n-1$ except a, b; but rather they are the smallest subset for which a condition of form (2) holds. In other words we assume that in (2) the index s is minimal.

Details of proof. Since the X_{h_ja}, X_{ta}, X_{an}, are given, each W_{h_j} (and W_t) is contained in some definite V_{gk}; in other words, the choice of each X_{hn} is confined to the set of r_{gk} bearing some fixed set of subscripts gk.

Now, unless $(g, k) = (1, 5), (5, 1), (5, 2)$, any product $rs \subset V_{gk}$ will contain m_{gk}'. (This uses the fact that no factor may be a unit.) If W_t were an intersection of such products in V_{gk}, W_t would contain m_{gk}' and could not be void, as required by (2).

Hence, by the symmetry of indices 1 and 2, we may suppose that $W_t \subset V_{15}$ or $W_t \subset V_{51}$.

Suppose first that $W_t \subset V_{51}$. Then X_{an} is some r_{g1}, and every $W_{h_i} \subset V_{k1}$ for some k. As before, if $k \neq 5$, $m_{k1}' \subset W_{h_i}$, so that we may choose $X_{h_in} = m_{k1}'$. Then the products $X_{h_jh_i}X_{h_in}$ and $X_{th_i}X_{h_in}$ in the later W_{h_j} and W_t are of the form $r_{gk}m_{k1}' = V_{g1}$ and are without effect. But this means that the variable X_{h_in} is superfluous in condition (2), contrary to the hypothesis that s was minimal. On the other hand, if $k = 5$, then (since units are excluded) $X_{th_i} = m_{55}'$ and again the variable X_{h_in} is superfluous. This leads to the conclusion that $s = 0$, and $W_t = X_{ta}X_{an} \cap X_{tb}X_{bn} \subset V_{51}$. Now W_t can be made to contain only two minimal elements (as the intersection of the two products of type 5 in the multiplication table), but $W_t = 0$ is impossible. Thus the assumption $W_t \subset V_{51}$ contradicts (2).

Our hypotheses have now led to the conclusion that $W_t \subset V_{15}$. Arguments analogous to those just used show that every W_h must be contained in V_{11} or V_{12}, and hence that each X_{th} must be one of m'_{11}, m'_{12}, or m_{12}. Since products with a factor m'_{ij} have no effect on W_t, we find that, for any choice of the X_{hn}, W_t will reduce to a product of the form

$$(3) \quad W_t = X_{ta}X_{an} \cap X_{tb}X_{bn} \cap \bigcap m_{12}m_{25}^{kq_i} = X_{ta}X_{an} \cap X_{tb}X_{bn} \cap \bigcup_{r \neq q_i} m_{15}^{kr}$$

for a certain *finite* set of q_i, which (by 1 of the multiplication table) we may suppose are all different from Y.

Now since the (variable) union $\bigcup_{r \neq q_i} m_{25}^{kr}$ contains (in any case) all but a finite set of the minimal elements r_{25} in V_{25}, $W_t = 0$ only in case $X_{ta}X_{an} \cap X_{tb}X_{bn}$ contains only a finite number of minimal elements. Inspection of the multiplication table (entry 5) shows that this will be so only if

$$(4) \qquad X_{ta}X_{an} \cap X_{tb}X_{bn} = m_{13}m_{35}^{p} \cap m_{14}m_{45}^{p} = m_{15}^{3p} \cup m_{15}^{4p},$$

where we must always have p among the q_i, and so $p \neq Y$.

A similar argument shows that if some X_{h_in} takes a value m_{15}^{hq} (or m_{25}^{hq}), the effect of the products $X_{h_jh_i}X_{h_in}$ and $X_{th_i}X_{h_in}$, which are at least $m_{gk}m_{k5}^{hq} = \bigcup_{r \neq q} m_{g5}^{hr}$ is to exclude at most two minimal elements from the subsequent W_{h_j} and W_t. If W_{h_i} contains an infinite number of minimal elements, with different q, from which X_{h_in} may be chosen, it follows that the products with factor X_{h_in} impose no restriction at all on the choice of the X_{h_jn} and X_{tn}. Then the variable X_{h_in} is superfluous, contrary to the hypothesis that s was minimal.

Therefore all the W_h must be finite, and in order that there actually occur q_i in (3), some W_h must be contained in V_{25} with $X_{th} = m_{12}$. Then, analogously to (4)

$$(5) \qquad X_{ha}X_{an} \cap X_{hb}X_{bn} = m_{23}m_{35}^{p} \cap m_{24}m_{45}^{p} = m_{25}^{3p} \cup m_{25}^{4p}$$

where, since $X_{an} = m_{35}^{p}$ (or symmetrically, m_{45}^{p}) occurs in both (4) and (5), the subscript p is the same as in (4). But (5) requires that $X_{hn} = m_{25}^{kp}$ ($k = 3, 4$), so that $W_t \subset X_{th}X_{hn} = m_{12}m_{25}^{kp} \bigcup_{r \neq p} m_{15}^{gr}$. Combined with (4) this gives $W_t = 0$. Thus all the other X_{h_in} are superfluous, $h = h_1$, and $s = 1$.

Conclusion of proof. The hypothesis that f cannot be extended has led us in view of (4) and (5), to the conclusion that

$$(6) \qquad \begin{aligned} &fX_{th} = m_{12}, fX_{ta} = m_{13}, fX_{tb} = m_{14}, fX_{ha} = m_{23} \\ &fX_{hb} = m_{25}, fX_{an} = m_{35}^{p}, f'X_{bn} = m_{45}^{p}, \text{ for } p \neq Y. \end{aligned}$$

It will be shown that this contradicts the hypotheses that f was in F_{n-1} and that $fX_{ab} \subset (f'X_{an})(f'X_{nb})$.

Let $g < i < j < k < n$; then from the recursive definition of f, it is clear that at most two of X_{gk}, X_{ik}, X_{jk} can correspond to the X_{ak}, X_{bk} of (iv), and hence that at least one of them must have been chosen in accordance with the rules (viii), (ix). Hence, since a, b, h, and t are four different integers less than n, the values of one of the six variables X_{ab}, X_{ah}, X_{at}, X_{bh}, X_{bt}, X_{ht} must have been chosen in accordance with (ix) and (viii). But (6) shows this was not the case for any except X_{ab}. Hence fX_{ab} was chosen in accordance with (ix).

Now $fX_{ab} \subset (f'X_{an})(f'X_{nb}) = m_{35}^{p}m_{54}^{p} = m_{34}^{p} \cup m_{34}'$ $(p \neq Y)$, and therefore by (ix) there is some c, certainly less than n, such that $(fX_{ac})(fX_{cb})$ does not contain m_{34}^{Y}. Inspection of the multiplication table shows that this requires

$$(7) \qquad fX_{ac} = m_{35}^{p} \text{ and } fX_{cb} = m_{54}^{p},$$

Since f is in F_{n-1} (vi) and (vii) are satisfied, and in particular (abbreviating X_{ij} for fX_{ij})

$$(8) \qquad X_{tc} \subset X_{ta}X_{ac} \cap X_{tb}X_{bc}$$

$$(9) \qquad X_{hc} \subset X_{ha}X_{ac} \cap X_{hb}X_{bc} \cap X_{ht}X_{tc}.$$

Substituting values according to (6) and (7), condition (8) becomes

$$X_{tc} \subset m_{13}m_{35}^{p} \cap m_{14}m_{45}^{p]} = m_{15}^{3p} \cup m_{15}^{4p}$$

whence X_{tc} is m_{15}^{kp} $(k = 3, 4)$. Substituting this value, together with values from (6) and (7), into (9) gives

$$X_{hc} \subset m_{23}m_{35}^{p} \cap m_{24}m_{45}^{p} \cap m_{21}m_{15}^{kp} = (m_{25}^{3p} \cup m_{25}^{4p}) \cap \textstyle\bigcup_{r \neq p} m_{25}^{gr} = 0,$$

a contradiction.

This completes the proof of the lemma, and hence that M is representable.

§11. The model M'

The model M' is defined by exactly the same formal conditions as M, but with the range of upper indices p now restricted to the set P' of all rationals p, $0 < p < 1$, where *no* irrational Y is adjoined. M' is thus in a weak sense a 'subalgebra' of M; however, M' has a different universal element from that of M, and hence in M' complementation has a different meaning. That M' is a relational algebra can be verified directly, as for the example A of Part One, or it can be seen to follow from the results of §13, below. Only A5 presents any difficulty, and it is perhaps instructive to note that a slight modification of the argument of the last section suffices to prove A5, the first of the non-vacuous conditions C. One has only to note that for A5, $n = 4$, and the situation (6), which is the only possibility of a failure of A5, cannot arise since there are not four distinct indices a, b, h and t all less than $n = 4$.

However, the inductive argument of §10 cannot be carried beyond this initial stage, for

LEMMA. *The condition C5:*

$$X_{12} \subset X_{13}X_{32} \cap X_{14}X_{42}\cdot \rightarrow (\exists X_{34})(X_{34} \subset X_{31}X_{14} \cap X_{32}X_{24}\cdot \;\&$$
$$\& \; (X_{35}, X_{54})(X_{34} \subset X_{35}X_{54}\cdot \rightarrow \cdot X_{12} \subset (X_{13}X_{35} \cap X_{14}X_{45})$$
$$(X_{53}X_{32} \cap X_{54}X_{42})))$$

fails in M′.

PROOF. For the antecedent we verify that $m_{12} \subset m_{13}m_{32} \cap m_{14}m_{42} = V_{12}$. The consequent requires that there exist $X_{34} \subset m_{31}m_{14} \cap m_{32}m_{24} = V_{34}$, satisfying further conditions. Such an X_{34} must be m'_{34} or some m^p_{34}. In either case $X_{34} \subset m^p_{35}m^p_{54}$ for some p, and the remaining clauses of the consequent require that $X_{12} = m_{12}$ be contained in

$$(m_{13}m^p_{35} \cap m_{14}m^p_{45})(m^p_{53}m_{32} \cap m^p_{54}m_{42}) = (m^{3p}_{15} \cup m^{4p}_{15})(m^{3p}_{52} \cup m^{4p}_{52}) = m'_{12},$$

which is not the case.

COROLLARY. *The relational algebra M′ is not representable.*

§12. Finitely generated subalgebras of M

For any subset A of P, define

(1) $m^{3A}_{15} = \bigcup_{p \text{ in } A} m^{3p}_{15}$, and m^{4A}_{15}, m^{A}_{34}, etc. analogously.

If, for any subset A of P, we define

(2) $$A^- = A \cup (0, \sup A) \quad\text{and}\quad A^+ = A \cup (\inf A, 1),$$

we observe that q is in A^- if and only if $q \leqq p$ for some p in A, and that q is in A^+ if and only if $p \leqq q$ for some p in A. We shall write $\bar{A}$ for the complement, $P - A$, of A. We may now calculate the

Multiplication table for elements of type (1)

(3) $m_{12}m^{kA}_{25} = m^{3\bar{A}}_{15} \cup m^{4\bar{A}}_{15}$ if $A = (p)$, a single element, and $p \neq Y$ (and, as usual, otherwise the product is V_{15});

(4) $m^{A}_{34}m^{B}_{35} = m^{\bar{A}\cap\bar{B}}_{35} \cup m'_{35}$ if $A = (p), B = (q), p \neq q, p, q \neq Y$

$= m^{A}_{35} \cup m'_{35}$ if $A = (p), B = (p), p \neq Y$

$= m^{\bar{A}}_{35} \cup m'_{35}$ if $A = (p), B \neq (q)$ for any $q \neq Y$, and $p \neq Y$;

(5) $m_{13}m^{A}_{35} = m^{3A^+}_{15} \cup m^{4A^-}_{15}$;

(6) $m^{A}_{35}m^{3B}_{51} = m'_{31}$ if $p < q$ for all p in A, q in B;

$m^{A}_{35}m^{4B}_{51} = m'_{31}$ if $q < p$ for all p in A, q in B;

(7) $m_{31}m^{3A}_{15} = m^{A^-}_{35} \cup m'_{35}$

$m_{31}m^{4A}_{15} = m^{A^+}_{35} \cup m'_{35}$.

It is possible to express any element R of M as a union of elements m_{ij} and m'_{ij}, together with elements of type (1); and for each R, at most ten sets A will be required for upper indices. If $R_1, \cdots, R_n$ is a finite set of elements of M, the set of A_i required to express them in this way will be finite. Let D_o denote the boolean algebra over P generated by these A_i and their complements.

LEMMA. *Let F be the subalgebra of M generated by the elements $R_1, \cdots, R_n$, I, and V. Let D be the smallest complete boolean algebra over P which contains D_0 and is closed under the operations of forming A^- and A^+. Then every element of F can be expressed as a union of elements m_{ij} and m'_{ij} together with elements of type (1) with indices from D.*

PROOF. The element I requires no upper indices. From the fact that D is a boolean algebra it follows that any boolean combination of elements with indices in D will in turn have indices in D. The operation of forming converses requires no new indices. Finally, the operation of forming products requires, in addition to boolean combinations of indices, only indices of the form A^+ and A^-, where A is already in D. But since D is closed under these further operations, no indices beyond those available in D are needed.

LEMMA. *There exists an open interval J in P such that for no element A of D does* $\inf A$ *or* $\sup A$ *lie in J.*

PROOF. The number of minimal elements in D_o is finite, and hence also the number having non-void intersection with any given interval J. Let $J = (a, b)$ (where a, b may of course be irrational) be such that the number, h, of minimal elements A_i of D_0 for which $A_i \cap J \neq 0$ is minimal. Suppose now that, for some A_i, $a < \inf(A_i \cap J) = c < b$ (or, symmetrically, the same for $\sup(A_i \cap J)$). Then the interval $J' = (a, c)$ would not meet A_i, nor any A_j that did not meet J, and hence would meet at most $h - 1$ minimal elements of D_o, contrary to the assumption that h was minimal.

This proves, for $D' = D_o$, the inductive hypotheses:

(i) $A_1, \cdots, A_h$ are the only minimal elements of D' such that $A_i \cap J \neq 0$;
(ii) for no minimal element A of D' is $\inf(A \cap J)$ or $\sup(A \cap J)$ in J.

Now D can be obtained from D_o in the limit by alternating the process of passing from an algebra D' to its closure under the operations A^+ and A^-, with the process of passing from D' to its closure under the operation of forming infinite intersections. But (ii) ensures that both properties (i) and (ii) are preserved under the first process, and (i) that they are preserved under the second. Hence the limit D of this construction will have these properties, and in particular (ii), which proves the lemma.

It is not difficult to see that, in fact, the minimal elements of D are all either single points, $A = (p)$, or sets contained in the closure of such intervals J, and having the same upper and lower limits. A natural ordering relation can be introduced among the minimal elements of D by defining:

$$\begin{aligned} &A \equiv B &&\text{if} \quad \inf A = \inf B &&\text{and} \quad \sup A = \sup B, \\ &A < B &&\text{if} \quad \inf A < \inf B &&\text{or} \quad \sup A < \sup B, \\ &A \leqq B &&\text{if} \quad A < B &&\text{or} \quad A \equiv B. \end{aligned} \tag{8}$$

LEMMA. *If A and B are minimal elements of D, then one and only one of the relations $A \equiv B$, $A < B$, and $B < A$ holds; moreover, $A < B$ if and only if* sup $A \leqq \inf B$, *and* $A \neq B$.

PROOF. We prove the second assertion first. If $\sup A < \sup B$, then $B \not\subset A^-$, and since B is minimal $B \cap A^- = 0$, whence $\sup A \leqq \inf B$. If $\inf A < \inf B$, an analogous argument applies. Conversely, if $\sup A \leqq \inf B$, then $\inf A \leqq \sup A \leqq B \leqq \sup B$, and all cannot be equal unless $A = B$, a single point; but otherwise $A < B$. Now one of $A \equiv B$, $A < B$, and $B < A$ surely holds, and the first is incompatible with the other two; but, finally, if $A < B$ and $B < A$, then $\sup A \leqq \inf B \leqq \sup B \leqq \inf A \leqq \sup A$, and all are equal, contrary to the hypothesis that $A < B$.

LEMMA. *$A \leqq B$ if and only if $A \subset B^-$, and symmetrically, $B \leqq A$ if and only if $A \subset B^+$.*

PROOF. Let $A \subset B^- = B \cup (0, \sup B)$; then $\sup A \leqq \sup B$. Now if $B < A$, then $\sup A \leqq \sup B \leqq \inf A$, whence A must reduce to the single point $(\sup B)$. But $A = (\sup B) \subset B^- = B \cup (0, \sup \mathrm{B})$ implies $A = (\sup B) \subset B$, and since both are minimal, that $A = B$, contrary to hypothesis.

Conversely, if $A \leqq B$, then $B < A$ fails, and $\inf A < \sup B$; but then $A \cap B^- \neq 0$, whence $A \subset B^-$.

In terms of this ordering relation we can rewrite the last three entries of the *multiplication table*

$$(5') \qquad m_{13}m_{35}^{A} = \bigcup_{A \leqq B} m_{15}^{3B} \cup \bigcup_{B \leqq A} m_{15}^{4B};$$

$$(6') \qquad m_{35}^{A}m_{51}^{3B} = m_{31}' \text{ if } A < B;$$

$$m_{35}^{A}m_{51}^{4B} = m_{31}' \text{ if } B < A;$$

$$(7') \qquad m_{31}m_{15}^{3A} = \bigcup_{B \leqq A} m_{35}^{B} \cup m_{35}'; \; m_{31}m_{15}^{4A} = \bigcup_{A \leqq B} m_{35}^{B} \cup m_{35}'.$$

§13. Finitely generated subalgebras of M'

Everything that has been said about finitely generated subalgebras F of M applies equally, with P replaced by P', to finitely generated subalgebras of M'.

LEMMA. *Every finitely generated subalgebra of M' is isomorphic to a finitely generated subalgebra of M.*

PROOF. Let the elements $I, V, R_1, \cdots, R_n$ of M' determine a boolean algebra D over P' in the manner described in the preceding section, and let F be the subalgebra of those elements of M' expressible by means of indices from the set D. It will clearly suffice to show that F is isomorphic to some subalgebra F'' of M.

We begin by constructing an isomorphism K of the boolean algebra D over P onto a boolean algebra D'' over P. Since P' is contained in P, and differs from P only in lacking the element Y, we may, for each minimal element $A \subset P'$, let KA be the same set regarded as a subset of P, except that we must assign the extra element Y to some KA. Let J be an open interval in P' such that $A_1, \cdots, A_h$ are the only minimal elements of D for which $A \cap J \neq 0$, and such

that no inf A or sup A is in J. There is clearly no loss in generality in assuming that Y lies in the image of J in P. We now define
(i) $KA = A$ for $A \neq A_1$, a minimal element of D;
(ii) $KA_1 = A_1 \cup (Y)$.

Since the elements KA are disjoint and cover P, they generate as minimal elements a boolean algebra D'' over P which is isomorphic under an extension of K to D. Now D'' in turn determines a subalgebra F'' of M, and K induces a one-to-one mapping of F onto F''. This correspondence clearly preserves boolean operations and converses, and to show that it preserves products it will be enough, in view of the multiplication table with (5, 6, 7) amended to (5′, 6′, 7′), to show that the isomorphism K of D onto D'' preserves the ordering relation $A < B$.

By virtue of definition (8), it will suffice to show that K preserves upper and lower limits. For $A \neq A_1$ this is immediate, while for A_1 it follows from the fact that Y is in J, while neither inf A_1 nor sup A_1 is in J, and hence that inf $A_1 < Y <$ sup A_1. This completes the proof that F is isomorphic to F''.

Corollary. *Every finitely generated subalgebra of the unrepresentable relational algebra M' is representable.*

§14. Independence of C5 of algebraic axioms

An *algebraic formula* in a relational algebra is any formula containing only the following primitive symbols:
(i) the constants 0, I, and V;
(ii) variables $X_1, X_2, \cdots$;
(iii) the equality sign, and signs for the operations of union, intersection, complementation, converse, and product;
(iv) the sentential connectives 'and', 'or', 'not', and 'if . . . then'.

It will be noted that an algebraic formula can contain only free variables, and these only in finite number.

An *algebraic axiom* for the class of relational algebras is any algebraic formula which is true in every proper relational algebra.

Lemma. *Every algebraic axiom holds in the unrepresentable relational algebra M'.*

Proof. The algebra M is isomorphic to a PRA (§10), whence every subalgebra F'' of M is likewise representable; thus every algebraic axiom holds in M and in all its subalgebras F''. Now let the algebraic axiom $Q(X_1, \cdots, X_n)$ fail in M'. Then for some $R_1, \cdots, R_n$ in M', $Q(R_1, \cdots, R_n)$ must fail, and hence Q must fail in the finitely generated subalgebra F containing $R_1, \cdots, R_n$. But this contradicts the fact that the algebraic axiom Q must hold in the subalgebra F'' of M, which is isomorphic to F.

Corollary. *The condition* C5, *which is necessary for a relational algebra to be representable, is not a consequence of any set of algebraic axioms.*

Theorem *IV. The class of all relational algebras that are isomorphic to proper relational algebras cannot be characterized by any set of algebraic axioms.*

§15. Discussion of results

We shall try to indicate briefly the scope of the results obtained above, and to point out some of the closely connected problems which remain unsolved.

Finite Relational Algebras. The conditions C have been shown necessary and sufficient for a finite RA to be represented by a PRA. To improve this result one would naturally seek to replace the non-algebraic conditions C by finitary conditions, which preferably made no reference to minimal elements. There seems little doubt that each condition C could indeed be replaced by an infinite set of conditions of this type, but it is doubtful if this artificial procedure could be regarded as an improvement on the result already obtained.

It would also be desirable to replace the infinite set C by some finite set of conditions of the same nature; but it seems doubtful if this is possible. For the sequence of conditions:

$$C^n: X_{12} \subset X_{13}X_{32} \cap \cdots \cap X_{1n}X_{n2} \,.\rightarrow .$$
$$(\exists X_{ij}\,, \text{all } i, j, 2 < i, j \leqq n)(X_{ik} \subset X_{ij}X_{jk}\,, \text{all } i, j, k \leqq n)$$

appears to be strictly increasing in logical force, and it is difficult to imagine a condition of the same nature from which all the C^n would be deducible.

Infinite complete RA's. It has been shown that the conditions C are necessary for representability, and that no set of algebraic conditions is sufficient. It seems doubtful if the conditions C can be enlarged to a set of conditions of the same nature which would be at the same time sufficient. For the conditions C^n have an 'infinite' analogue C^ω, which is surely necessary; but any formulation of C^ω would appear to require variables of higher logical type. Since the property of being representable can be expressed directly if no restriction is imposed on the logical type of the variables employed, there seems little room for improvement in this direction.

Incomplete RA's. It has been shown that no set of algebraic conditions is sufficient for representability. Indeed, the situation here is more difficult than that for infinite complete RA's, since the conditions C cannot in general be expressed without reference to minimal elements. It is hard to see how these conditions could even be reformulated for incomplete RA's, for a given RA will have various completions which are not isomorphic as boolean algebras, while even those that are isomorphic as boolean algebras need not be isomorphic as RA's.

Broader or narrower classes of algebras. For any enlarged class of algebras, involving the same operations, but satisfying weaker axioms than RA's, the representation problem should, in general, be at least as difficult as for RA's. However, for special classes of RA's, characterized by further axioms (which are not true for all PRA's), a positive solution to the representation problem may hold. An extreme example is the class of complete RA's whose minimal elements constitute a group. A non-trivial example is that of the class of RA's which McKinsey has shown isomorphic to PRA's whose minimal elements each contain a single ordered couple.[13] A third class of RA's, for which the representation problem remains unsolved (and apparently at least as difficult as that studied above) is the class of all integral RA's;[14] adjoined to this question is that of whether every integral PRA is isomorphic to an algebra of subsets (or, 'complexes') of elements from some group.[15]

Stronger or weaker structures. By reducing the operations admitted in a RA one imposes fewer conditions on a representation. Thus RA's, regarded merely as boolean algebras,[16] as projective algebras,[11] or (if integral) as multigroups,[10] are representable. A more difficult problem in this line would seem to be that of characterizing 'relational rings', lacking the operation of complementation.

The possibilities of strengthening RA's by introducing new operations appear to be almost limitless. All new operations should be invariant under permutation of the domain of a PRA. It can be seen that 0, V, I, and $V - I$ are the only invariant constants, or 0-ary operations. Let a 5-ary operation be defined by setting

$a\, F(A, B, C, D, E)\, b$ if and only if

$$(\exists c, \exists d)(aAc\cdot \ \& \ \cdot aBd\cdot \ \& \ \cdot cCd\cdot \ \& \ \cdot cDb\cdot \ \& \ dEb);$$

from the graph naturally associated with this operation it can be seen that it is not analyzable in terms of intersections, products, etc. A new unary operation may be derived from this by setting $F(A) = F(A, A, A, A, A)$.[17] Although the introduction of additional operations demands more of a representation, it enables one to formulate stronger axioms. However, it is hard to see how any finite set of new operations would make it possible to alter essentially the character of an infinite family of conditions such as C.

Many-termed relations. The best approach to a theory of n-ary relations seems to be through the study of higher dimensional projective algebras (readily generalized from the 2-dimensional projective algebras defined by Everett and Ulam[11]). Two-dimensional projective algebras, which have a weaker structure than RA's, are representable.[18] Every RA determines a three dimensional projective algebra, and conversely, every three dimensional projective algebra that contains an element with the properties of the line $x = y = z$ is determined (within some minor modifications) by a RA. Thus the structures of RA's and of 3-PA's are essentially equivalent, and the greater part of the present analysis carries over. However, the operations of 3-PA are, in a seemingly minor point, slightly stronger than those of RA, with the result that every denumerable 3-PA is finitely gen-

[13] See [8]; a stronger result in this direction is announced in [6], Result 2.

[14] Cf. §4 above.

[15] [6]; problem 2.

[16] See [10].

[17] A more familiar unary operation which is not definable in abstract RA is the ***ancestral***; for a study of algebras with this operation, see [9].

[18] See [7].

erated; thus the analogue of the proof of Theorem IV cannot be carried out for projective algebras.[19]

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BIBLIOGRAPHY

1. G. BIRKHOFF. *On the structure of abstract algebras*, Proc. Cambridge. Philos. Soc. 31 (1935), 433–454.
2. G. BIRKHOFF. *Subdirect unions in universal algebra*, Bull. Amer. Math. Soc. 50 (1944), 764–768.
3. I. M. COPILOWISH. *Matrix development of the calculus of relations*, J. Symbolic Logic 13 (1948), 193–203.
4. M. DRESHER and O. ORE. *The theory of multigroups*, Amer. J. Math. 60 (1938), 705–733.
5. C. J. EVERETT and S. ULAM, *Projective algebra, I*, Amer. J. Math. 68 (1946), 77–88.
6. B. JÓNSSON and A. TARSKI, *Representation problems for relation algebras* (Abstract 89t), Bull. Amer. Math. Soc. 54 (1948), 80.
7. J. C. C. MCKINSEY. *On the representation of projective algebras*, Amer. J. Math. 70 (1948), 375–384.
8. J. C. C. MCKINSEY. *Postulates for the calculus of binary relations*, J. Symbolic Logic 5 (1940), 85–97.
9. O. ORE. *Theory of equivalence relations*, Duke Math. J. 9 (1942), 573–627.
10. M. H. STONE. *The theory of representations for Boolean algebras*, Trans. Amer. Math. Soc. 40 (1936), 37–111.
11. A. TARSKI. *On the calculus of relations*, J. Symbolic Logic 6 (1941), 73–89.
12. J. H. M. WEDDERBURN. *Boolean linear associative algebras*, Ann. of Math. 35, (1934), 185–194.

[19] The close connection between relational algebra and projective algebra was pointed out to me by Tarski. Both Professor Tarski and Professor A. Mostowski have suggested to me a number of interesting connections between the present theory and the subject of mathematical logic. I shall limit myself here to pointing out the contrast between the results of the present investigation, interpreted as a study of the theory of two-termed predicates, and the Stone representation theory for boolean algebras [10], as interpreted in the theory of one-termed predicates.

I should like to cite at this point a remark of Mr. F. B. Thompson: From Theorem IV, together with Theorem 10 of Birkhoff [1], it follows that not every homomorphic image of a proper relational algebra is representable. This situation is exemplified in the standard direct limit construction, whereby the unrepresentable RA M' is obtained as the homomorphic image of a (representable) subalgebra of the (representable) direct product of all its finitely generated subalgebras. Indeed (as with boolean algebras), if an ideal is not principal, a representation of an algebra does not induce naturally a representation for the quotient algebra.