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ON NOTATION FOR ORDINAL NUMBERS

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Consider a system of formal notations for ordinal numbers in the first and second number classes, with the following properties. Given a notation for an ordinal, it can be decided effectively whether the ordinal is zero, or the successor of an ordinal, or the limit of an increasing sequence of ordinals. In the second case, a notation for the preceding ordinal can be determined effectively. In the third case, notations for the ordinals of an increasing sequence of type ω with the given ordinal as limit can be determined effectively.

Are there systems of this sort which extend farthest into the second number class? When the conditions for the systems have been made precise, the question will be answered in the affirmative. There is an ordinal ω_1 in the second number class such that there are systems of notations of the sort described which extend to all ordinals less than ω_1, but none in which ω_1 itself is assigned a notation.[1]

1. An effective or constructive operation on the objects of an enumerable class is one for which a fixed set of instructions can be chosen such that, for each of the infinitely many objects (or n-tuples of objects), the operation can be completed by a finite process in accordance with the instructions. This notion is made exact by specifying the nature of the process and set of instructions. It appears possible to do so without loss of generality.

A function of natural numbers, with natural numbers as values, is taken to be effective if it is Herbrand-Gödel recursive.[2] The set of instructions for a

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[1] A closely related result, and a discussion of the significance of these questions of notation, are given in Alonzo Church, *The constructive second number class*, ***Bulletin of the American Mathematical Society***, vol. 44 (1938), pp. 224–232. The ordinal ω_1 is the least ordinal not represented by formulas in the λ-notation, Alonzo Church and S. C. Kleene, *Formal definitions in the theory of ordinal numbers*, ***Fundamenta mathematicae***, vol. 28 (1936), pp. 11–21, Rules i–iv.

[2] Kurt Gödel, ***On undecidable propositions of formal mathematical systems***, mimeographed lecture notes, Princeton 1934, pp. 26–27; S. C. Kleene, *General recursive functions of natural numbers*, ***Mathematische Annalen***, vol. 112 (1936), pp. 727–742.

This notion of effectiveness appears, on the following evidence, to be general. A variety of particular effective functions and classes of effective functions (selected with the intention of exhausting known types) have been found to be recursive. Two other notions, with the same heuristic property, have been proved equivalent to the present one, viz., Church-Kleene λ-definability and Turing computability. Turing's formulation comprises the functions computable by machines. See S. C. Kleene, *λ-definability and recursiveness*, ***Duke mathematical journal***, vol. 2 (1936), pp. 340–353, and A. M. Turing, *On computable numbers, with an application to the Entscheidungsproblem*, ***Proceedings of the London Mathematical Society***, vol. 42 (1936–7), pp. 230–265, and *Computability and λ-definability*, (general) recursive function $\phi(x_1, \cdots, x_n)$ is given by means of a system of equations E. The process of computing the value of the function for a particular n-tuple of numbers $k_1, \cdots, k_n$ as arguments consists in performing certain operations of substitution on the equations. After a finite number of these operations, a unique equation of the form $\phi(k_1, \cdots, k_n)=k$, where k is a number, is obtained. The number k is the desired value.

If we omit the requirement that the computation process always terminate, we obtain a more general class of functions, each function of which is defined over a subset (possibly null or total) of the n-tuples of natural numbers, and possesses the property of effectiveness when defined. These functions we call partial recursive.

If $f(x_1, \cdots, x_n)$ is any function defined over a subset of the n-tuples of natural numbers, with natural numbers as values, a condition for effectiveness is that there exist a partial recursive function $\phi(x_1, \cdots, x_n)$ such that the equality $f(x_1, \cdots, x_n)=\phi(x_1, \cdots, x_n)$ holds on the range of definition of f (which may be less extensive than that of ϕ).[3]

Let $F(X_1, \cdots, X_n)$ be any operation on n-tuples of objects of certain categories, yielding objects of a certain category. If for each category, a natural number can be assigned to each object (distinct numbers to distinct objects) in a particular manner, which is acknowledged to be reciprocally effective, then the criterion that F be effective is reduced to the preceding case.

When the objects are formulas, consisting of finite sequences of symbols from a given list, the assignment of numbers can be effected by Gödel's method. Assuming this possibility for the systems of notations for ordinals, we simplify the discussion by treating the notations themselves as being natural numbers.

The conditions for a representation of ordinals by natural numbers, to be called here an *r-system*, are the following. (1) No number represents two distinct ordinals. (2) There is a partial recursive function K such that, if X is zero (X is the successor of an ordinal, X is the limit of an increasing sequence of ordinals), then for each number x which represents X, $K(x)=0$ ($K(x)=1$, $K(x)=2$). (3) There is a partial recursive function P such that, if X is the successor of the ordinal Y, then for each number x which represents X, $P(x)$ is a number which represents Y. (4) There is a partial recursive function Q such

this JOURNAL, vol. 2 (1937), pp. 153–163. Functions determined by algorithms and by the derivation in symbolic logics of equations giving their values (provided the individual steps have an effectiveness property which may be expressed in terms of recursiveness) are recursive. See Alonzo Church, *An unsolvable problem of elementary number theory*, *American journal of mathematics*, vol. 58 (1936), pp. 345–363, where it was first proposed to identify effectiveness with recursiveness. Church's remarks should be generalized in one particular, as will appear in Footnote 3.

[3] This condition is more general than potential recursiveness (*An unsolvable problem of elementary number theory*, p. 352). For there are functions $f(x_1, \cdots, x_n)$ not equal throughout their range of definition to any recursive function, e.g., $\mu y T_1(x, x, y)$, where μ is defined below, it being noted that the number f in *General recursive functions of natural numbers*, proof of Theorem XIV, belongs to the range of definition of $\mu y T_1(x, x, y)$. The condition is more general than partial recursiveness, since a partial recursive function has a range of definition of a special form, $(Ey)T_n(e, x_1, \cdots, x_n, y)$.

that, if X is the limit of an increasing sequence of ordinals, then for each number x which represents X, there is an increasing sequence of ordinals $\{Y_n\}$ of order type ω which has X as limit such that, for each natural number n, $Q(x, n)$ is a number which represents Y_n.

It follows from the conditions that for some ordinal ξ, at least one number x represents each ordinal $X < \xi$, and no number represents any ordinal $\geqq \xi$.

2. Partial recursive functions may be constructed by methods which have been considered in relation to general recursive functions, and other devices. We outline this theory.[4]

The equations of E are expressed in a particular formalism, with a particular functional variable ρ_{α_n} representing ϕ (as we now assume for definiteness). A given function $\phi(x_1, \cdots, x_n)$ is *partial recursive* if there is a system of equations E in ρ_{α_n} and possibly other functional variables such that for each n-tuple of numerals $k_1, \cdots, k_n$ there exists at most one numeral k for which $E \vdash_{1,3} \rho_{\alpha_n}(k_1, \cdots, k_n)=k$, specifically k exists when and only when ϕ is defined for the arguments represented by $k_1, \cdots, k_n$ and has then the value represented by k. We say that E *defines* ϕ *recursively.*

We abbreviate $x_1, \cdots, x_n$ to $\mathfrak{x}$. The equality of two functions $\phi(\mathfrak{x})$ and $\psi(\mathfrak{x})$ in range of definition and in value throughout that range is written $\phi(\mathfrak{x}) \simeq \psi(\mathfrak{x})$. A composite function written in the form $\phi(\psi(\mathfrak{x}))$ is to be interpreted as undefined for values of $\mathfrak{x}$ which make ψ undefined, unless the contrary is stated (likewise when ϕ has several arguments).

The notions are extended to propositional functions of natural numbers, called relations, under the correspondence between a relation and the function which takes the value 0 or 1, or is undefined, according as the relation is true or false, or is undefined. If $R(\mathfrak{x}, y)$ is a relation, then $\mu y R(\mathfrak{x}, y)$ denotes the function of $\mathfrak{x}$ which, for each fixed $\mathfrak{x}$, takes as value the least y such that $R(\mathfrak{x}, y)$ is true, provided such a y exists and $R(\mathfrak{x}, y)$ is defined for the preceding values of y, and is undefined otherwise.

A *primitive recursive* function is one which can be generated from the functions $C(x)=0$, $S(x)=x+1$ and $U_i^n(x_1, \cdots, x_n)=x_i$ by zero or more applications of the schemata $\phi(\mathfrak{x}) \simeq \theta(\chi_1(\mathfrak{x}), \cdots, \chi_m(\mathfrak{x}))$ and $\phi(0, \mathfrak{x}) \simeq \psi(\mathfrak{x})$, $\phi(y+1, \mathfrak{x}) \simeq \chi(y, \phi(y, \mathfrak{x}), \mathfrak{x})$. Let applications of a third schema $\phi(\mathfrak{x}) \simeq \mu y R(\mathfrak{x}, y)$ be admitted, together with applications of the other two. The functions obtained are partial recursive. [For the proof that a system E defines one of these functions recursively, it is required that E yield formally only the desired values. When E is chosen suitably, this is inferred from a verifiability property of the equations, relative to certain interpretations of the functional variables, which is preserved under the operations R_1 and R_3.]

Employing a particular primitive recursive function $S(z, y)$ [$=\mathrm{Val}(H(z, y))$] and for each n a particular primitive recursive relation $T_n(z, \mathfrak{x}, y)$, we set $\Phi_n(z, \mathfrak{x}) \simeq S(z, \mu y T_n(z, \mathfrak{x}, y))$. Then Φ_n is a partial recursive function of $n+1$ variables,

[4] For details, see *General recursive functions of natural numbers.* Dr. Barkley Rosser has called my attention to an error in No. 17, p. 733, which is corrected by reading

$$\text{“}\binom{k+1+l(\mathrm{Sb}(k, x, v, y)) \dot{-} l(x)}{y}\text{”} \text{ instead of } \text{“}\binom{k+1}{y}\text{”} \text{ at two places.}$$

with the range of definition $(Ey)T_n(z, \mathfrak{x}, y)$, i.e., the range consists of the $n+1$-tuples $(z, \mathfrak{x})$ satisfying this relation. Every partial recursive function $\phi(\mathfrak{x})$ of n variables is expressible thus, $\phi(\mathfrak{x}) \simeq \Phi_n(e, \mathfrak{x})$, where e is a number, which is said to *define* ϕ *recursively*. Among the numbers which define ϕ recursively are the Gödel numbers of the systems of equations which define ϕ recursively. The range of definition of ϕ is $(Ey)T_n(e, \mathfrak{x}, y)$.

For any partial recursive functions $\phi(x)$ and $\psi(\mathfrak{x})$ and partial recursive relation $R(x)$, the function $\phi(\psi(\mathfrak{x}))$ and the relation $R(\psi(\mathfrak{x}))$ are partial recursive (likewise when ϕ or R has several arguments). Thus, $\phi(\mathfrak{x}) = \psi(\mathfrak{x})$ and $\phi(\mathfrak{x}) < \psi(\mathfrak{x})$ are partial recursive relations, $\phi(\mathfrak{x}) + \psi(\mathfrak{x})$, $\phi(\mathfrak{x}) \cdot \psi(\mathfrak{x})$ and $\phi(\mathfrak{x})^{\psi(\mathfrak{x})}$ are partial recursive functions, etc. For any partial recursive relation $R(\mathfrak{x}, y)$, the function $\mu y R(\mathfrak{x}, y)$ is partial recursive.[5] [The preceding result ensures that $\phi(x)$, $\psi(\mathfrak{x})$ and the functions corresponding to $R(\mathfrak{x})$ and $R(\mathfrak{x}, y)$ can be generated using the three schemata.]

Given relations $A(\mathfrak{x})$ and $B(\mathfrak{x})$, which may be undefined for some values of $\mathfrak{x}$, we shall interpret $\bar{A}(\mathfrak{x})$, $A(\mathfrak{x}) \vee B(\mathfrak{x})$, $A(\mathfrak{x}) \& B(\mathfrak{x})$, $A(\mathfrak{x}) \rightarrow B(\mathfrak{x})$ and $A(\mathfrak{x}) \equiv B(\mathfrak{x})$ as follows.

$\bar{A}$

A	t	f	u
	f	t	u

$A \vee B$

B \ A	t	f	u
t	t	t	t
f	t	f	u
u	t	u	u

$A \& B$

B \ A	t	f	u
t	t	f	u
f	f	f	f
u	u	f	u

$A \rightarrow B$

B \ A	t	f	u
t	t	t	t
f	f	t	u
u	u	t	u

$A \equiv B$

B \ A	t	f	u
t	t	f	u
f	f	t	u
u	u	u	u

u: undefined
t: true
f: false

(Not all equivalences of the classical calculus of propositions hold.) When $A(\mathfrak{x})$ and $B(\mathfrak{x})$ are partial recursive, the composite relations are partial recursive. [Let $\alpha(\mathfrak{x})$, $\beta(\mathfrak{x})$ and $\gamma(\mathfrak{x})$ correspond to $A(\mathfrak{x})$, $B(\mathfrak{x})$ and $A(\mathfrak{x}) \vee B(\mathfrak{x})$, respectively. Let a and b define α and β recursively. Then $\gamma(\mathfrak{x}) \simeq 1 \text{ Gl } \mu w \{[T_n(a, \mathfrak{x}, 2 \text{ Gl } w) \ \& \ S(a, 2 \text{ Gl } w) = 0 \ \& \ 1 \text{ Gl } w = 0] \vee [T_n(b, \mathfrak{x}, 3 \text{ Gl } w) \ \& \ S(b, 3 \text{ Gl } w) = 0 \ \& \ 1 \text{ Gl } w = 0] \vee [T_n(a, \mathfrak{x}, 2 \text{ Gl } w) \ \& \ S(a, 2 \text{ Gl } w) = 1 \ \& \ T_n(b, \mathfrak{x}, 3 \text{ Gl } w) \ \& \ S(b, 3 \text{ Gl } w) = 1 \ \& \ 1 \text{ Gl } w = 1]\}$.]

There is a primitive recursive function $S_n^m(z, y_1, \cdots, y_m)$ such that, if e defines recursively $\phi(y_1, \cdots, y_m, \mathfrak{x})$ as a function of $m+n$ variables, and $k_1, \cdots, k_m$ are fixed numbers, then $S_n^m(e, k_1, \cdots, k_m)$ defines recursively $\phi(k_1, \cdots, k_m, \mathfrak{x})$ as a function of the n remaining variables. [The construction of S_n^m is based on the relation $\phi(k_1, \cdots, k_m, \mathfrak{x}) \simeq \Phi_{m+n}(e, k_1, \cdots, k_m, \mathfrak{x})$.] If $\psi(z, \mathfrak{x})$ is any partial recursive function, there is a number f which defines $\psi(f, \mathfrak{x})$ recursively. [Let e define $\psi(S_n^1(y, y), \mathfrak{x})$ recursively, and set $f = S_n^1(e, e)$.]

3. We now exhibit an r-system S_1, for which ξ has a greatest possible value. S_1 is the representation of ordinals by natural numbers in which a number represents an ordinal as required by the following rules and only then. (1) 1 represents 0. (2) If y represents Y, then 2^y represents $Y+1$. (3) If $\{Y_n\}$

[5] There is also a partial recursive function $\nu y_{(r)} R(\mathfrak{x}, y)$ of $\mathfrak{x}$ which, for each fixed $\mathfrak{x}$, takes as value a y such that $R(\mathfrak{x}, y)$ is true, provided such a y exists (whether or not $R(\mathfrak{x}, y)$ is defined for all the preceding values of y), and is undefined otherwise. For let r be a number defining recursively the function corresponding to $R(\mathfrak{x}, y)$, and set $\nu y_{(r)} R(\mathfrak{x}, y) \simeq 1 \text{ Gl } \mu w \{T_n(r, \mathfrak{x}, 1 \text{ Gl } w, 2 \text{ Gl } w) \ \& \ S(r, 2 \text{ Gl } w) = 0\}$.

is an increasing sequence of ordinals of order type ω, if for each n y_n represents Y_n, and if y defines recursively y_n as a function of n_o, where n_o represents n as a finite ordinal, then $3 \cdot 5^y$ represents $\lim_n \{Y_n\}$.

S_1 is an r-system. [$K(x) = \epsilon y\{y \leqq 2 \ \&\ y \text{ Gl } x \neq 0\}$. $P(x) = 1 \text{ Gl } x$. $Q(x, n) \simeq \Phi_1(3 \text{ Gl } x, n_o)$, where $0_o = 1$, $(n+1)_o = 2^{n_o}$.] Given any r-system S, there is a partial recursive function F such that, for any number x which represents an ordinal in S, $F(x)$ represents the same ordinal in S_1. The function F is given the following properties. If $K(x)=0$, then $F(x)=1$. If $K(x)=1$, then $F(x)=2^{F(P(x))}$. If $K(x)=2$, then $F(x)=3 \cdot 5^{a_{F,x}}$, where $a_{F,x}$ defines recursively $F(Q(x, n))$ as a function of n_o. [To obtain F as a partial recursive function, set $\phi(x, z, y) \simeq \Phi_1(z, Q(x, \mathfrak{J}(y)))$, where $\mathfrak{J}(y) = \epsilon x\{x \leqq y \ \&\ y = x_o\}$. Let p define ϕ recursively. Set $\psi(z, x) \simeq \mu y\{(K(x) = 0 \ \&\ y = 1) \vee (K(x) = 1 \ \&\ y = 2^{\Phi_1(z, P(x))}) \vee (K(x) = 2 \ \&\ y = 3 \cdot 5^{S_1^2(p, x, z)})\}$. There is a number f defining $\psi(f, x)$ recursively. Set $F(x) \simeq \psi(f, x)$.] Hence S_1 is an r-system with ξ a maximum.

The system S_1 is modeled after the system of formulas in the λ-notation assigned to represent ordinals by Church and Kleene.[6] The treatment of the two systems is analogous. Moreover, if the formulas in principal normal form which represent ordinals are replaced by their Gödel numbers, the resulting system S_2 is an r-system. Given any r-system S, there is a partial recursive function G such that, if x represents an ordinal in S, $G(x)$ represents the same ordinal in S_2. Hence S_2 is also an r-system with the maximum possible value of ξ.

The ordinal obtained (non-constructively) as the maximum value of ξ was designated as ω_1 by Church and Kleene. Arguing non-constructively, the notations are enumerable. The second number class is not enumerable. Therefore ω_1 is in the second number class.

Church and Kleene have considered functions of ordinals which are λ-definable, i.e. expressible in the λ-notation as operations on the formulas representing the ordinals. This class of functions is identical with the class of functions which correspond to partial recursive functions of the notations in S_1 (likewise for functions with some variables in the domain of ordinals and others in the domain of natural numbers).[7] Partial recursive functions applying to the notations in S_1, such as $x +_o y$ and $\sum_{m=0}^{n}{}_o\, \phi(m)$ where ϕ is recursive and $\phi(n)$ for each natural number n represents an ordinal, are distinguished here from like functions applying to the natural numbers as such by the subscript o.

4. One might expect to obtain systems extending to greater ordinals by employing conditions like those for r-systems but with $\{Y_n\}$ allowed to be a

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[6] Loc. cit. Proof of some remarks which follow may be based on *λ-definability and recursiveness*.

[7] At this point, the analogy between the two systems is improved by writing $\{z\}(x_1, \cdots, x_n)$ or $z(x_1, \cdots, x_n)$ as an abbreviation for $\Phi_n(z, x_1, \cdots, x_n)$. Then $\phi(x_1, \cdots, x_n)$ is expressible as $e(x_1, \cdots, x_n)$, where e is a number defining ϕ recursively. The use of the numbers e rather than the functions ϕ conforms to the finitary standpoint. For only the numbers e (or the systems of equations E) are given directly, and there is no effective decision in general whether two e's (or two E's) define the same function.

sequence of ordinals of order type ω(not necessarily increasing) which contains no greatest and has X as least upper bound.

However it is seen non-constructively that the ordinal ω_1 is still a maximum ξ. For given any system S of this sort, there is a partial recursive function H such that, for any number x which represents an ordinal in S, $H(x)$ represents an equal or greater ordinal in S_1. The function H is given the following properties. If $K(x)=0$, then $H(x)=1$. If $K(x)=1$, then $H(x)=2^{H(P(x))}$. If $K(x)=2$, then $H(x)=3\cdot 5^{b_{H,x}}$, where $b_{H,x}$ defines recursively $\sum_{m=0}^{n}{}_o\, 1_o +_o H(Q(x, m))$ as a function of n_o.

5. Following Church, a modification is now made in the system S_1, which is regarded from the finitary viewpoint as a correction, in that it eliminates the presupposition of the classical (non-constructive) second number class.[8] The modification consists in replacing the ordering relation $<$ between ordinals by a partially ordering relation $<_o$ between the notations. The modified system is designated as S_3.

Let numbers belong to a class O, and stand in a relation $<_o$, as required by the following rules and only then. (1) $1 \in O$. (2) If $y \in O$, then $2^y \in O$ and $y <_o 2^y$. (3) Using n_o as above, if the sequence of numbers $\{y_n\}$ of order type ω has the property that, for each n, $y_n \in O$ and $y_n <_o y_{n+1}$, and if y defines recursively y_n as a function of n_o, then $3\cdot 5^y \in O$ and, for each n, $y_n <_o 3\cdot 5^y$. (4) If $x, y, z \in O$ and $x <_o y$ and $y <_o z$, then $x <_o z$.

S_3 is the system determined by the first two conditions for S_1 and the following condition. If $\{y_n\}$ is an increasing sequence, in terms of the relation $<_o$, of numbers which $\in O$, of order type ω, and if, for each n, y_n represents an ordinal Y_n, and if y defines recursively y_n as a function of n_o, then $3\cdot 5^y$ represents $\lim_n \{Y_n\}$.

S_3 is a subsystem of S_1, and is an r-system. We see non-constructively that the least ordinal not represented in S_3 is ω_1. For the function H, when constructed with S_1 itself as S, has the property that, for any number x which represents an ordinal in S_1, $H(x)$ represents an equal or greater ordinal in S_3.

Following Church, if x is a number representing an ordinal X in S_3, the subsystem of S_3 consisting of x and the numbers which precede x in terms of the relation $<_o$ is an r-system in which a unique number is assigned to each ordinal $< X+1$. Given any univalent r-system S, the numbers $F(x)$, for the values of x which represent ordinals in S, constitute an r-subsystem of S_3 simply ordered by the relation $<_o$.

Thus ω_1 is the least ordinal not representable in any univalent r-system.[9]

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[8] *The constructive second number class.* Stated by Church for the system of formulas in the λ-notation.

[9] The writer does not know whether any one univalent r-system contains notations for all the ordinals $< \omega_1$, as is the case for multivalent r-systems. It is possible to describe non-constructively an r-subsystem of S_3 simply ordered by the relation $<_o$, which assigns a number to each ordinal $< \omega^2$, and which does not admit of extension to ω^2 preserving the simple ordering.