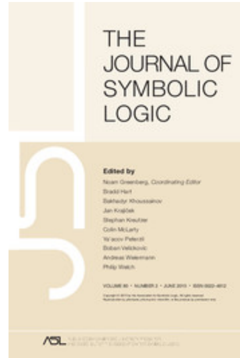

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Piergiorgio Odifreddi

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FORCING AND REDUCIBILITIES. II. FORCING IN FRAGMENTS OF ANALYSIS

PIERGIORGIO ODIFREDDI

In *Forcing and reducibilities*, I [20] we reviewed various forcing techniques in the context of arithmetic. This second part deals with the same topics in the context of analysis. The numbering of sections is the continuation of the numbering of the first part. For the reader's convenience, we collect in an independent bibliography the papers which are referred to in this part.

§5. Hyperarithmetical sets. The purpose of this section is to extend to every level of the hyperarithmetical hierarchy the structural properties proved for the finite levels of it in §3. The basic idea is simple: in the language of §1 we had a set constant X and, because of the presence of the quantifiers, we could define the finite jumps $X^{(n)}$. The obvious limit to the method there was in its ability to control arithmetical properties of the set X, but not more. In a sense, the natural object which is not definable in arithmetic is the ω-jump. We can then think to add a new constant for $X^{(\omega)}$ and this will allow us to take care of another segment of the arithmetical hierarchy. Then we add a constant for $X^{(\omega+\omega)}$ and so on.

Before we go on we recall some basic facts that will be needed in the future. A recursive ordinal is the ordinal of a recursive well ordering of ω. The rcursive ordinals form an initial segment of the ordinals, and the first nonrecursive ordinal is denoted by ω_1^{ck}, or simply ω_1 (the Church-Kleene ω_1). We do not get new ordinals if we consider Σ_1^1 well orderings of ω (Tanaka [32]), but there is a π_1^1 well ordering of ω whose ordinal is ω_1 (Feferman and Spector [3], Gandy [5]). For our purposes we fix a π_1^1 well ordering of ω with ordinal ω_1 and such that all the initial segments of it are recursive (Jockusch [10]). To simplify notation we identify it with the ordinal ω_1, so that $\alpha < \omega_1$ will mean that α is the unique natural number in its domain denoting the ordinal α and $\alpha < \beta$ will be interpreted similarly. A basic fact coming from Spector [30] is the boundedness property: any Σ_1^1 subset A of ω_1 is bounded, i.e. $A \subseteq \alpha$ for some $\alpha < \omega_1$.

We are ready now to give the basic definitions we need. We first define for $\alpha \leq \omega_1$, $\boldsymbol{O}^{(\alpha)}$ as the degree of $O^{(\alpha)}$, where $O^{(0)} = \emptyset$, $O^{(\beta+1)} =$ jump of $O^{(\beta)}$ and for λ limit, $\langle x, \beta\rangle \in O^{(\lambda)} \Leftrightarrow \beta < \lambda \wedge x \in O^{(\beta)}$. Recall that we are abusing notation, hence β and λ are actually integers and $\beta < \lambda$ is a recursive expression. As now, the definition of $O^{(\alpha)}$ seems to depend on the particular well ordering of ω with ordinal ω_1 we chose. Proposition 5.4 will however give a degree-theoretic definition of $O^{(\alpha)}$, thus proving its independence from the given well ordering.

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In analogy with the arithmetical hierarchy, where we had $\Sigma^0_{n+1}=$ sets Σ^0_1 in $O^{(n)}$, we now let, for any $\alpha<\omega_1$, $\Sigma^0_{\alpha+1}=$ sets Σ^0_1 in $O^{(\alpha)}$. We extend the definition to limit ordinals by letting $\Sigma^0_\lambda=\bigcup_{\alpha<\lambda}\Sigma^0_\alpha$ (sometimes in the literature Σ^0_λ is defined as the class of sets Σ^0_1 in $\bigcup_{\alpha<\lambda}\Sigma^0_\alpha$). Finally, $\Sigma^0_0=$ recursive sets. As usual we also let $\pi^0_\alpha=$ sets with complement in Σ^0_α and $\Delta^0_\alpha=\Sigma^0_\alpha\cap\pi^0_\alpha$. In particular, $\Delta^0_{\alpha+1}=$ sets recursive in $O^{(\alpha)}$. E.g. $\Sigma^0_\omega=\Delta^0_\omega=$ arithmetical sets. The collection $\bigcup_{\alpha<\omega_1}\Sigma^0_\alpha$ is particularly important and, by Kleene [12], consists exactly of the Δ^1_1 (or hyperarithmetical) sets.

All the definitions and results given above can be relativized to a given set A. E.g. ω_1^A is the first ordinal which is not the ordinal of a well ordering of ω recursive in A, and the sets Δ^1_1 in A are exactly the sets in $\bigcup_{\alpha<\omega_1^A}\Sigma^{0,A}_\alpha$.

In the following we give informal proofs. To make them precise, an appeal to the principle of recursive induction (see Rogers [22]) would have to be done. The language $\mathscr{L}_\alpha$ for $\alpha<\omega_1$, α limit consists of the language for first order arithmetic considered in §1, augmented by a set constant X and, for every $\beta<\omega_1$, β limit of a set constant $X^{(\beta)}$. Having quantifiers in the language, we may define $X^{(\beta)}$ for β successor ordinal in the standard way. Note that $\mathscr{L}_\omega$ is nothing else than $\mathscr{L}$ of §1.

Definition. Let φ be a sentence of $\mathscr{L}_\alpha$.

(a) $A\models\varphi$ means that φ is true when the set constants $X^{(\beta)}$ are interpreted as $A^{(\beta)}$.

(b) $A\Vdash\varphi$ means that for some $\sigma\subseteq A$, $\sigma\Vdash\varphi$ where the relation $\sigma\Vdash\varphi$ is defined as follows:

if φ is atomic and does not contain set constants, $\sigma\Vdash\varphi$ iff φ is true in arithmetic;

if $\varphi\equiv\bar n\in X$, $\sigma\Vdash\varphi$ iff $\sigma(n)=1$;

if $\varphi\equiv\overline{\langle x,\gamma\rangle}\in X^{(\beta)}$, $\sigma\Vdash\varphi$ iff $\gamma<\beta\wedge\sigma\Vdash\bar x\in X^{(\gamma)}$ (note that $X^{(\gamma)}$ may be a defined symbol of $\mathscr{L}_\alpha$);

if $\varphi\equiv\sim\psi$, $\sigma\Vdash\varphi$ iff $(\forall\tau\supseteq\sigma)(\tau\nVdash\psi)$;

if $\varphi\equiv\psi_0\vee\psi_1$, $\sigma\Vdash\varphi$ iff $\sigma\Vdash\psi_0$ or $\sigma\Vdash\psi_1$;

if $\varphi\equiv\exists x\,\psi(x)$, $\sigma\Vdash\varphi$ iff for some n, $\sigma\Vdash\psi(\bar n)$.

The only difference from the definitions of §1, therefore, is in the treatment of the new atomic formulas $z\in X^{(\beta)}$ for β limit and, in accordance with our previous discussion, forcing is set equal to truth for atomic formulas. As in §1 we get monotonicity, consistency and quasi-completeness for forcing.

Definition. Let $\alpha\le\omega_1$ be a limit.

(a) A is α-generic if for any sentence φ of $\mathscr{L}_\alpha$, $A\Vdash\varphi$ or $A\Vdash\sim\varphi$.

(b) If, moreover, $\alpha=\beta+\omega$ and β is limit, A is $(\beta+n)$-*generic* if for any sentence $\varphi\in\Sigma^0_{\beta+n}$ of $\mathscr{L}_\alpha$, $A\Vdash\varphi$ or $A\Vdash\sim\varphi$.

Forcing = truth for generic sets follows as usual. The main computation of this section is the following:

Proposition 5.1. Definability of forcing. *Let $\alpha<\omega_1$ be a limit.*

(a) *$\{(\sigma,\varphi)$: φ is a sentence of $\mathscr{L}_\alpha$ and $\sigma\Vdash\varphi\}$ is recursive in $O^{(\alpha)}$.*

(b) *if, moreover, $\alpha=\beta+\omega$, β limit and $n\ge1$, $\{(\sigma,\varphi)$: φ is a $\Sigma^0_{\beta+n}$ sentence of $\mathscr{L}_\alpha$ and $\sigma\Vdash\varphi\}$ is $\Sigma^0_{\beta+n}$, hence recursive in $O^{(\beta+n)}$. Similarly for $\pi^0_{\beta+n}$.*

Proof. By induction on φ. First we check if φ is a sentence of $\mathscr{L}_\alpha$: by our choice of the well ordering of ω_1, this is recursive. Then we find the biggest $\beta<\alpha$ such that $X^{(\beta)}$ occurs in φ (again, this is recursive). If there is no such β, we are reduced to the case of §1. Otherwise, we apply the induction hypothesis. Roughly, everything will be recursive in $O^{(\beta+\omega)}$ and hence in $O^{(\alpha)}$. E.g. suppose $\varphi \equiv \overline{\langle x, \gamma\rangle} \in X^{(\beta)}$. First we check if $\gamma < \beta$ (recursively), then if $\sigma \Vdash \bar{x} \in X^{(\gamma)}$: since $\gamma < \beta$, by induction hypothesis this is recursive in $O^{(\alpha)}$. The reader should note that this proof is one of the cases in which the principle of recursive induction must be used. □

PROPOSITION 5.2. *The hyperarithmetical sets are all hyperarithmetically implicitly definable. However for any $\alpha < \omega_1$, α limit there are sets recursive in $O^{(\alpha)}$ and not implicitly definable in $\mathscr{L}_\alpha$.*

PROPOSITION 5.3. *For any $\alpha < \omega_1$:*

(a) *The α-jump of Turing degrees has range $\{\boldsymbol{a}: \boldsymbol{a} \geq \boldsymbol{0}^{(\alpha)}\}$(MacIntyre* [18]) *and it is never one-one on its range.*

(b) *There are $\Sigma^0_{\alpha+1}$-incomparable $\Delta^0_{\alpha+2}$ sets.*

(c) *There are $\Delta^0_{\alpha+1}$-incomparable $\Sigma^0_{\alpha+1}$ sets (Hinman* [8]).

Nerode and Shore [19] have proved that if $\mathscr{D}_\alpha$ is the theory of Turing degrees with the usual ordering and the operation of α-jump, then for any α, $\beta \leq \omega_1$ such that $\alpha \neq \beta$, $\mathscr{D}_\alpha$ and $\mathscr{D}_\beta$ are not elementarily equivalent.

PROPOSITION 5.4. (a) *For any $\alpha < \omega_1$, α limit $\boldsymbol{O}^{(\alpha)}$ is the 2-least upper bound of $\{\boldsymbol{O}^{(\beta)}\}_{\beta<\alpha}$ (Sacks* [23]).

(b) *There is no β-least upper bound for $\{\boldsymbol{O}^{(\alpha)}\}_{\alpha<\omega_1}$, for any $\beta < \omega_1$ (Enderton and Putnam* [1]).

PROOF. Part (a) is obtained as in 3.14. To prove (b) we show that if B is any bound for $\{\boldsymbol{O}^{(\alpha)}\}_{\alpha<\omega_1}$, there is another upper bound A hyperarithmetically incomparable with B. This follows from the facts that B is not hyperarithmetical and the set of upper bounds for $\{\boldsymbol{O}^{(\alpha)}\}_{\alpha<\omega_1}$ is Σ^1_1, by the basis theorem of [6]. □

The last result gives a recursion-theoretic definition of the chain $\{\boldsymbol{O}^{(\alpha)}\}_{\alpha<\omega_1}$, in the theory of degrees with jump operator. Let $\mathscr{D}$ be the structure of degrees and $\mathscr{D}(\geq \boldsymbol{a})$ the structure of degrees above $\boldsymbol{a}$. The strong homogeneity conjecture says that $\mathscr{D}$ and $\mathscr{D}(\geq \boldsymbol{a})$ are jump-isomorphic for any $\boldsymbol{a}$ (i.e. there is an isomorphism preserving jumps).

PROPOSITION 5.5 *The strong homogeneity conjecture fails (Feiner* [4], *Jockusch).*

PROOF. If $\mathscr{D}$ and $\mathscr{D}(\geq \boldsymbol{a})$ are jump-isomorphic, the chain $\{O^{(\alpha)}\}_{\alpha<\omega_1}$ and $\{\boldsymbol{a}^{(\alpha)}\}_{\alpha<\omega_1^a}$ are isomorphic, in particular $\omega_1 = \omega_1^A$ for $A \in \boldsymbol{a}$. This is not always true. □

Actually the proof above (due to Jockusch) gives more: let $\mathcal{O}$ be a π^1_1-complete set and $\boldsymbol{a}$ be a degree such that $\boldsymbol{a} \geq$ degree of $\mathcal{O}$. Then $\mathscr{D}$ and $\mathscr{D}(\geq \boldsymbol{a})$ are not jump-isomorphic (because $\omega_1 < \omega_1^{\mathcal{O}}$: we have a π^1_1 well ordering of ω of ordinal ω_1). Shore has proved in [26] that for any such $\boldsymbol{a}$, $\mathscr{D}$ and $\mathscr{D}(\geq \boldsymbol{a})$ are not isomorphic and in [27] that they are not even elementarily equivalent.

Finally, we note that for any $\alpha < \omega_1$ closed under addition it is possible to consider a notion of α-degree obtained from the relation "A is Δ^0_α in B". A structure theory for such degrees can be obtained as in §4. In particular:

PROPOSITION 5.6. *For any $\alpha < \omega_1$ closed under addition every countable partial ordering is embeddable in the α-degrees of sets Turing reducible to $O^{(\alpha)}$.*

This says that, in a certain sense, the gap between $\{\boldsymbol{O}^{(\beta)}\}_{\beta<\alpha}$ and $\boldsymbol{O}^{(\alpha)}$ gets bigger and bigger as α grows toward ω_1 (α closed under addition, although the results hold in general for α limit — only we do not have a notion of degree). The chain

$\{O^{(\alpha)}\}_{\alpha<\omega_1}$ can be extended even beyond ω_1, as we will discuss in §9.

§6. Feferman forcing in analysis. The next major topic we want to study is the structure of hyperdegrees, i.e. the degrees obtained by the relation "A is Δ_1^1 in B". This will be done in §8. In this and the next section we introduce the notions of forcing we need. What we have to do is consider a second order language, with set quantifiers as well as number quantifiers. We could choose a language with set constants $X^{(\beta)}$ for any $\beta < \omega_1$, β limit and hence extend directly the various languages $\mathscr{L}_\alpha$ of §5. However, since we will have set variables and set quantifiers, we follow a slightly different approach that avoids the proliferation of set constants and simplifies the technicalities.

$\mathscr{L}_{\omega_1}$ is the language for first order arithmetic of §1, with individual constants for each integer, one set constant X and two types of set variables:

X_β, Y_β, Z_β ... for any $\beta \leq \omega_1$ (called ranked variables of rank β),

X, Y, Z ... (unranked variables).

$\mathscr{L}_{\omega_1}$ contains set quantifiers and the membership relation $\in$ as well. We say that a formula φ of $\mathscr{L}_{\omega_1}$ is ranked if it contains only ranked set variables. In this case we call rank of φ the least upper bound of the ranks of free variables and of the rank of quantified ranked variables plus one.

Since now a formula φ can have unranked set variables, the specification of the interpretation of X is not enough to determine the truth value of φ: we must also specify how we interpret those variables in relation to A. So far (in previous sections) we have interpreted the set constants in the recursion-theoretic way which was suggested by the applications we had in mind. The first thought that comes to mind here is: if A is the given interpretation of X, than interpret X_β as ranging over the βth level of the hyperarithmetical hierarchy relative to A, and X as ranging over the ω_1th level of it. This is the intuition behind the following definition:

DEFINITION. (a) For $\beta < \omega_1$, $\mathscr{M}_\beta(A)$ is defined by induction on β as the collection of sets of integers definable by formulas of rank $\leq \beta$, where the constant X is interpreted by A and the bounded set variables of rank γ are interpreted over $\mathscr{M}_\gamma(A)$.

(b) $\mathscr{M}(A) = \bigcup_{\beta<\omega_1}\mathscr{M}_\beta(A)$.

(c) If φ is a sentence of $\mathscr{L}_{\omega_1}$, we say $A \models \varphi$ if φ is true when the ranked variables of rank β are interpreted over $\mathscr{M}_\beta(A)$ and the unranked variables are interpreted over $\mathscr{M}(A)$ (we will also write $\mathscr{M}(A) \models \varphi$ for this).

The definition of $\mathscr{M}(A)$ goes back to Kleene [13] and gives the initial segment of length ω_1 of the ramified analytic hierarchy relative to A. Kleene proves there that the sets Δ_1^1 in A are exactly the sets in the initial segment of length ω_1^A of that hierarchy. Hence for every set A such that $\omega_1 = \omega_1^A$ we have $\mathscr{M}(A) = \Delta_1^{1,A}$.

DEFINITION. $A \Vdash \varphi$ means that for some $\sigma \subseteq A$, $\sigma \Vdash \varphi$ where the relation $\sigma \Vdash \varphi$ is defined by induction on φ as in §§1, 5 with the following additional clauses:

If $\varphi \equiv \exists X_\beta\,\psi(X_\beta)$, $\sigma \Vdash \varphi$ iff for some ranked formula $\psi_0(x)$ of rank $\leq \beta$ and with only x free, $\sigma \Vdash \psi(\hat{x}\cdot\psi_0(x))$ where $\psi(\hat{x}\cdot\psi_0(x))$ is the result of the replacement of every occurrence of $z \in X_\beta$ with $\psi_0(z)$ in ψ.

If $\varphi \equiv \exists X\psi(X)$,

$$\sigma \Vdash \varphi \text{ iff for some } \beta < \omega_1, \sigma \Vdash \exists X_\beta \psi(X_\beta).$$

The induction in the definition above is on the ordinal $\omega_1 \cdot a(\varphi) + \omega^2 \cdot b(\varphi) + \omega \cdot c(\varphi) + d(\varphi)$ where $a(\varphi)$ is the number of unranked quantifiers in φ, $b(\varphi)$ is the rank of φ (defined as for ranked formulas), $c(\varphi)$ is the number of ranked quantifiers in φ, and $d(\varphi)$ is the number of connectives of φ. As usual, the definition of forcing for the new clauses is in accordance with the definition of truth. Monotonicity, consistence and quasi-completeness are immediate.

We can define the usual hierarchy of second order formulas by letting Σ_0^1 be the set of formulas with no unranked set quantifiers, π_n^1 the set of negations of Σ_n^1 formulas, and Σ_{n+1}^1 the set of formulas obtained from π_n^1 formulas by adding one unranked existential set quantifier in front. We stress the fact that this is a hierarchy for formulas of $\mathscr{L}_{\omega_1}$ and must not be confused with the similar hierarchy for predicates of second order arithmetic, to which we refer when we say, e.g., that a set is π_1^1 or when we classify the complexity of the forcing relation in the next result. From now on $\mathcal{O}$ indicates a π_1^1-complete set.

PROPOSITION 6.1. DEFINABILITY OF FORCING. (a) $\{(\sigma, \varphi)$: *φ is a ranked sentence of $\mathscr{L}_{\omega_1}$ and $\sigma \Vdash \varphi\}$ is π_1^1.*

(b) $\{(\sigma, \varphi)$: *φ is a $\Sigma_1^1 \cup \pi_1^1$ sentence of $\mathscr{L}_{\omega_1}$ and $\sigma \Vdash \varphi\}$ is recursive in $\mathcal{O}$.*

(c) $\{(\sigma, \varphi)$: *φ is a sentence of $\mathscr{L}_{\omega_1}$ and $\sigma \Vdash \varphi\}$ is recursive in $\mathcal{O}^{(\omega)}$.*

(d) *For a fixed sentence φ of $\mathscr{L}_{\omega_1}$, $\{\sigma: \sigma \Vdash \varphi\}$ is arithmetical in $\mathcal{O}$.*

PROOF. (a) By induction on φ. The real difference with the cases of previous sections is that to check whether φ is a sentence of $\mathscr{L}_{\omega_1}$ is no longer recursive, because the language now uses ω_1. Because of our coding of ω_1, it is, however, π_1^1 and the forcing clauses do not lead outside π_1^1 (see 5.1).

(b) The clause relative to one unranked existential set quantifier again gives a π_1^1 predicate, and the one for negation then gives a Σ_1^1 one. In both cases the expression is recursive in $\mathcal{O}$. Basic examples:

$\sigma \Vdash \exists X \psi(X)$ iff $(\exists \beta)(\beta < \omega_1 \wedge \sigma \Vdash \exists X_\beta \psi(X_\beta))$. $\exists \beta$ is a number quantifier, $\beta < \omega_1$ is π_1^1 and so is $\sigma \Vdash \exists X_\beta \psi(X_\beta)$ by part (a). Hence the whole expression is π_1^1, since π_1^1 is closed under number quantifiers.

$\sigma \Vdash \sim \exists X \psi(X)$ iff $(\forall \tau \supseteq \sigma)(\tau \not\Vdash \exists X \psi(X))$.

(c) With many unranked set quantifiers, we can filter through numerical quantifiers expressions like $\beta < \omega_1$, and we get a matrix recursive in $\mathcal{O}$ and a string of numerical quantifiers, depending uniformly on the complexity of the formula. Hence for φ a sentence, $\sigma \Vdash \varphi$ is recursive in $\mathcal{O}^{(\omega)}$. □

We now have many options for genericity, depending on the sentences we decide to force:

DEFINITION. (a) A is *weakly ω_1-generic* if for every ranked sentence φ of $\mathscr{L}_{\omega_1}$, $A \Vdash \varphi$ or $A \Vdash \sim\varphi$.

(b) A is *ω_1-generic* if the same happens for $\varphi \in \Sigma_1^1$ or φ obtained from a Σ_1^1 formula by adding finitely many number quantifiers in front of it.

(c) A is *strongly ω_1-generic* if the same happens for every sentence of $\mathscr{L}_{\omega_1}$.

The condition in part (b) of the definition is perhaps not very elegant, but it turns out that it is not enough to restrict the consideration to Σ_1^1 formulas. Actually, the formulas of interest in the following are of the form $(\forall x)\psi(x)$ with $\psi \in \Sigma_1^1$.

Note however that in the classification of predicates of second order arithmetic the distinction disappears, because Σ_1^1 and π_1^1 are then closed under numerical quantifiers (see [22, p. 375]), For this reason, if φ is a sentence of $\mathscr{L}_{\omega_1}$ of the kind considered in part (b) above, $\sigma \Vdash \varphi$ still has complexity π_1^1.

We do not know yet which kind of genericity will be useful for the applications. What we lack now is a connection between formulas of $\mathscr{L}_{\omega_1}$ (to which forcing applies) and formulas of analysis (which are used in, say, the definition of hyperdegree). The next result is instrumental in this respect.

PROPOSITION 6.2. *If A is ω_1-generic then $\mathscr{M}(A) = \Delta_1^{1,A}$ (Feferman [2]).*

PROOF. The inclusion $\mathscr{M}(A) \subseteq \Delta_1^{1,A}$ is true in general and does not use genericity. The basic fact here is that, similarly to 6.1(a), the satisfaction relation $\mathscr{M}(A) \models \varphi$ for ranked formulas is $\pi_1^{1,A}$. If then $B \in \mathscr{M}(A)$, for some ranked formula φ,

$$x \in B \Leftrightarrow \mathscr{M}(A) \models \varphi(x) \Leftrightarrow \sim(\mathscr{M}(A) \models \sim\varphi(x))$$

and, hence, $B \in \Delta_1^{1,A}$.

For the inclusion $\Delta_1^{1,A} \subseteq \mathscr{M}(A)$ we make use of the fact, proved in Kreisel [14], that $\Delta_1^{1,A}$ is the smallest model of $\Delta_1^{1,A}$-comprehension: it is then enough to prove that $\mathscr{M}(A)$ satisfies this. A typical instance of the $\Delta_1^{1,A}$-comprehension is

$$\begin{aligned}(\forall x)[(\exists Y)\varphi(x, Y) \Leftrightarrow (\forall Z)\psi(x, Z)]& \\ \Rightarrow (\exists X)(\forall x)[x \in X \Leftrightarrow (\exists Y)\varphi(X, Y)]&\end{aligned}$$

where φ, ψ are arithmetical in A. The antecedent says that $(\exists Y)\varphi(x, Y)$ is a $\Delta_1^{1,A}$ formula, and the conclusion says that it defines a set. Suppose

$$\mathscr{M}(A) \models (\forall x)[(\exists Y)\varphi(x, Y) \Leftrightarrow (\forall Z)\psi(x, Z)],$$

where $\varphi(x, Y)$ has only Y as (free) unranked variable and similarly for ψ. Then

$$\mathscr{M}(A) \models (\forall x)(\forall Y)(\forall Z)[\varphi(x, Y) \Rightarrow \psi(x, Z)]$$

and

$$\mathscr{M}(A) \models (\forall x)(\exists Y)(\exists Z)[\psi(x, Z) \Rightarrow (x, Y)].$$

Suppose we can find $\beta < \omega_1$ such that

$$\mathscr{M}(A) \models (\forall x)(\exists Y_\beta)(\exists Z_\beta)[\psi(x, Z_\beta) \Rightarrow \varphi(x, Y_\beta)].$$

Then also

$$\mathscr{M}(A) \models (\forall x)(\forall Y_\beta)(\forall Z_\beta)[\varphi(x, Y_\beta) \Rightarrow \psi(x, Z_\beta)]$$

and hence

$$\mathscr{M}(A) \models (\forall x)[(\exists Y_\beta)\varphi(x, Y_\beta) \Leftrightarrow (\forall Z_\beta)\psi(x, Z_\beta)].$$

But now the right-hand side is a ranked formula that can be realized by a set in $\mathscr{M}(A)$ by definition. By contraction of variables it is then enough to prove that, if $\chi(x, Y)$ is an arithmetic formula with only Y as (free) unranked set variable and

$$\mathscr{M}(A) \models (\forall x)(\exists Y)\chi(x, Y),$$

then for some $\beta < \omega_1$,

$$\mathscr{M}(A) \models (\forall x)(\exists Y_\beta)\chi(x, Y_\beta).$$

Suppose the hypothesis holds. By ω_1-genericity of A and forcing = truth there is $\sigma \subseteq A$ such that $\sigma \Vdash (\forall x)(\exists Y)\chi(x, Y)$. By definition of forcing,

$$(\forall \tau \supseteq \sigma)(\forall x)(\exists \tau' \supseteq \tau)(\exists \beta)\,[\beta < \omega_1 \wedge \tau' \Vdash \exists Y\, \chi(x, Y_\beta)].$$

By definability the matrix is π_1^1, and by Kreisel [15] there is a Δ_1^1 function giving τ', β in terms of τ, x. By boundedness (Spector [30]) the ordinals β are actually bounded below ω_1, so for some $\beta < \omega_1$,

$$(\forall \tau \supseteq \sigma)(\forall x)(\exists \tau' \supseteq \tau)(\tau' \Vdash \exists Y_\beta\, \chi(x, Y_\beta)).$$

Hence $\sigma \Vdash (\forall x)(\exists Y_\beta)\chi(x, Y_\beta)$ and, by forcing = truth, $\mathscr{M}(A) \models (\forall x)(\exists Y_\beta)\chi(x, Y_\beta)$. □

For future reference, note that ω_1-genericity was used only in the last part of the proof, to get a kind of boundedness result. It seems that weak ω_1-genericity would not be enough for this: Sacks [24] claims the existence of a weakly ω_1-generic A such that $\mathcal{O}$ is $\Delta_1^{1,A}$, hence such that $\omega_1 < \omega_1^A$. It follows instead from $\mathscr{M}(A) = \Delta_1^{1,A}$ (and is actually equivalent to this, by Kleene [13]) that $\omega_1 = \omega_1^A$.

We are now ready to tie up definability in $\mathscr{L}_{\omega_1}$ and definability in analysis. Let us call a set Σ_1^1 (definable) over $\mathscr{M}(A)$ if there is a Σ_1^1 formula of $\mathscr{L}_{\omega_1}$ that defines it, when X is interpreted as A and the set variables are interpreted over $\mathscr{M}(A)$ in the way explained at the beginning of this section. Let now A be ω_1-generic: then Σ_1^1 over $\mathscr{M}(A)$ means Σ_1^1 over $\Delta_1^{1,A}$ in the usual sense, and by Kleene [13], Gandy [5] and Spector [31] this means $\pi_1^{1,A}$. Hence, except for this switching type of quantifiers, the language of analysis and the language $\mathscr{L}_{\omega_1}$ have the same power of expression relative to an ω_1-generic set, as far as formulas with *one* (unranked) set quantifier are concerned. Since these are the formulas used in the study of hyperdegrees, the applications will follow easily. We stress however that the correspondence does not hold for formulas with more than one set quantifier: a direct computation shows that every set definable over $\mathscr{M}(A)$ is actually $\Delta_2^{1,A}$. We will return to this in a moment.

PROPOSITION 6.3. GENERIC EXISTENCE THEOREM. (a) *Weakly ω_1-generic sets are not π_1^1, but there are ω_1-generic sets recursive in $\mathcal{O}$.*

(b) *There are strongly ω_1-generic sets recursive in $\mathcal{O}$.*

PROOF. The positive parts are standard. For the negative one, note that a weakly ω_1-generic set is infinite. If it were π_1^1, it would contain an infinite Δ_1^1 set B. There is a ranked formula of $\mathscr{L}_{\omega_1}$ expressing the fact that $B \subseteq X$ (this does not use 6.2: every Δ_1^1 set is in the ramified analytic hierarchy relative to A at level ω_1, for any A, hence it is in $\mathscr{M}(A)$ for every A). This gives the usual contradiction, being forced by a finite string. □

Although we do not use them in the following, we give forcing-free characterizations of genericity. First we need some computations.

PROPOSITION 6.4. *For $n \geq 1$, B is Σ_0^n in $\mathcal{O}$ iff B is Σ_{n+1}^1 over Δ_1^1 (Jockusch and Simpson* [11]).

PROOF. The basic fact, already noted, is that $\mathcal{O}$ (being π_1^1) is Σ_1^1 over Δ_1^1. A matrix recursive in $\mathcal{O}$ is hence Δ_2^1 over Δ_1^1. Any numerical quantifier can be transformed into a function quantifier restricted to Δ_1^1, so e.g., Σ_1^0 in $\mathcal{O}$ becomes Σ_2^1 over Δ_1^1, etc.

If B is Σ_1^1 over Δ_1^1 then it is π_1^1 and hence recursive in $\mathcal{O}$. Since there is a π_1^1 enumeration (on ω) of the Δ_1^1 sets (Kleene [13]), all the other quantifiers over Δ_1^1 can be transformed into number quantifiers restricted to $\mathcal{O}$ and Σ_{n+1}^1 over Δ_1^1 becomes Σ_n^0 in $\mathcal{O}$. $\square$

In particular, arithmetical in $\mathcal{O}$ means second order definable over Δ_1^1, and recursive in $\mathcal{O}$ means Δ_2^1 over Δ_1^1. Both these facts will be useful in the following.

PROPOSITION 6.5. (a) *B is π_1^1 iff B is Σ_1^1 invariantly definable over $\mathcal{M}(A)$ for A ω_1-generic* (*Gandy and Sacks* [7]).

(b) *B is arithmetical in $\mathcal{O}$ iff B is invariantly definable over $\mathcal{M}(A)$ for A strongly ω_1-generic* (*Sacks* [23]).

PROOF. B being Σ_1^1 invariantly definable over $\mathcal{M}(A)$ for A ω_1-generic means: for some Σ_1^1 formula φ of $\mathscr{L}_{\omega_1}$ and every ω_1-generic set A, $x \in B \Leftrightarrow \mathcal{M}(A) \models \varphi(x)$. Then $x \in B \Leftrightarrow (\exists\sigma)(\sigma \Vdash \varphi(\bar{x}))$ because if $x \in B$ then fix A: $(\exists\sigma \subseteq A)(\sigma \Vdash \varphi(\bar{x}))$; if $\sigma \Vdash \varphi(\bar{x})$ then choose $A \supseteq \sigma$ ω_1-generic. By 6.1 $B \in \pi_1^1$. Let now $B \in \pi_1^1$. Then for some φ arithmetical, $x \in B \Leftrightarrow (\exists X \in \Delta_1^1)(\varphi(x, X))$. We prove that for A ω_1-generic, $x \in B \Leftrightarrow \mathcal{M}(A) \models (\exists X)\varphi(x, X)$. One direction is immediate because $\Delta_1^1 \subseteq \mathcal{M}(A)$. If $\mathcal{M}(A) \models (\exists X)\varphi(x, X)$ by genericity and definition of forcing, for some $\sigma \subseteq A$ and $\beta \le \omega_1$, $\sigma \Vdash \exists X_\beta \varphi(x, X_\beta)$. It is then enough to take $C \supseteq \sigma$ hyperarithmetic and sufficiently generic (as in §5) to have forcing $=$ truth for it and $\mathcal{M}(C) = \Delta_1^1$. Hence $(\exists X \in \Delta_1^1)\varphi(x, X)$ and $x \in B$.

(b) Similar to (a). $\square$

PROPOSITION 6.6. (a) *A is weakly ω_1-generic iff $\{\sigma: \sigma \subseteq A\}$ meets every dense Δ_1^1 set of strings.*

(b) *A is ω_1-generic iff for every π_1^1 set of strings S, there is $\sigma \subseteq A$ such that either $\sigma \in S$ or $(\forall\tau \supseteq \sigma)(\tau \notin S)$.*

(c) *A is strongly ω_1-generic iff $\{\sigma: \sigma \subseteq A\}$ meets every dense set of strings arithmetical in $\mathcal{O}$.*

PROOF. We prove, e.g., (b). Let A be ω_1-generic and $S \in \pi_1^1$. Then by 6.5(a) there is a sentence φ of $\mathscr{L}_{\omega_1}$ (obtained by adding an existential number quantifier in front of a Σ_1^1 formula) such that $\mathcal{M}(B) \models \varphi$ iff $(\exists\tau \subseteq B)(\tau \in S)$, whenever B is ω_1-generic. We can then proceed as in 1.9. The converse is immediate, by considering the π_1^1 set $\{\sigma: \sigma \Vdash \varphi\}$ for any fixed $\varphi \in \Sigma_1^1$ or obtained from a Σ_1^1 formula by adding finitely many number quantifiers in front of it. $\square$

§7. **Sacks-forcing in analysis.** In this section we force with hyperarithmetical perfect closed sets. P, Q, R ..., will denote such sets. Note that there is a π_1^1 enumeration (on ω) of them. As in the case of forcing with arithmetical perfect closed sets, we have some trouble with the condition $P \Vdash \sim\varphi$ iff $(\forall Q \subseteq P)(Q \nVdash \varphi)$ because this involves a quantification over a π_1^1 set. If we want to retain some of the definability results of §6, we will have to adopt a nonstandard approach for our definition of forcing, and our choice is the following:

DEFINITION. $A \Vdash \varphi$ means that for some P such that $A \in P$, $P \Vdash \varphi$ where $P \Vdash \varphi$ is defined by induction on the sentences φ of $\mathscr{L}_{\omega_1}$ as follows:

If φ is ranked, $P \Vdash \varphi$ iff $(\forall X \in P)(\mathcal{M}(X) \models \varphi)$.

If φ is not ranked, the definition of $P \Vdash \varphi$ is parallel to the one of §6.

This allows the usual definability results. We just state:

PROPOSITION 7.1. *$\{(P, \varphi)$: φ is a Σ_1^1 sentence of $\mathscr{L}_{\omega_1}$ and $P \Vdash \varphi\}$ is π_1^1 (Gandy and Sacks* [7]).

PROOF. First note that not only is the set of sentences of $\mathscr{L}_{\omega_1}$ π_1^1, but so is the set of the forcing conditions. For φ ranked, we already know from §6 that $\mathscr{M}(X) \models \varphi$ is $\pi_1^{1,X}$. Then if P is Δ_1^1,

$$P \Vdash \varphi \text{ iff } (\forall X)(X \in P \Rightarrow \mathscr{M}(X) \models \varphi)$$

is π_1^1. Finally, if $\varphi \equiv \exists X\psi(X)$ then

$$P \Vdash \varphi \text{ iff } (\exists\beta)(\beta < \omega_1 \wedge P \Vdash \exists X_\beta \psi(X_\beta))$$

and if ψ does not contain unranked set variables other than X, then $\exists X_\beta\, \psi(X_\beta)$ is ranked and the whole expression is π_1^1. □

We can introduce at this point local forcing, and prove as in §2 quasi-completeness for ranked formulas (for which we defined forcing as truth). The appropriate definability results are obtained from §5. Note that quasi-completeness for unranked formulas is immediate by the definition of forcing for negation in this case.

The distinction (made in §6) between ω_1-genericity and strong ω_1-genericity was justified by the fact that the forcing relation for formulas obtained from a Σ_1^1 formula by adding number quantifiers in front of it was considerably simpler than the one for formulas of $\mathscr{L}_{\omega_1}$ (π_1^1 versus recursive in $\mathscr{O}^{(\omega)}$). This is not true anymore for Sacks forcing, due to the clause $P \Vdash \sim\varphi(\forall Q \subseteq P)(Q \nVdash \varphi)$, which introduces a quantification on the π_1^1 set of forcing conditions. Thus there is no need to keep the distinction between ω_1-genericity and strong ω_1-genericity, and we define:

DEFINITION. (a) A is *weakly Sacks ω_1-generic* if for every ranked sentence φ of $\mathscr{L}_{\omega_1}$, $A \Vdash \varphi$ or $A \Vdash \sim\varphi$.

(b) A is *Sacks ω_1-generic* if the same happens for every sentence φ of $\mathscr{L}_{\omega_1}$.

Having the quasi-completeness property we can derive by standard methods:

PROPOSITION 7.2. GENERIC EXISTENCE THEOREM (a). *There is a perfect tree of Sacks ω_1-generic (contained in any given condition).*

(b) *Weakly Sacks ω_1-generic sets are not Δ_1^1 and there are Sacks ω_1-generic recursive in $\mathscr{O}^{(\omega)}$.*

In §6 we had the stronger fact that weakly ω_1-generic sets are not π_1^1. We will prove in §9 that there are instead weakly Sacks ω_1-generic π_1^1 sets. From 7.6 it will follow that *Sacks ω_1-generic sets are not π_1^1* (because if A is such, $A \notin \Delta_1^1$ but $\omega_1 = \omega_1^A$).

Our goal now is to obtain the basic result: if A is Sacks ω_1-generic then $\mathscr{M}(A) = \Delta_1^{1,A}$. A little more work than was necessary in §6 for the analogue result is, however, needed in this context. The following fact will be crucial. The definition of $P \Vdash^w \varphi$ is, as usual, $(\forall Q \subseteq P)(\exists R \subseteq Q)(R \Vdash \varphi)$.

PROPOSITION 7.3. FUSION LEMMA (GANDY AND SACKS [7]). *If $\{\varphi_n\}_{n\in\omega}$ is a hyperarithmetical set of Σ_1^1 sentences of $\mathscr{L}_{\omega_1}$ and $(\forall n)(P \Vdash^w \varphi_n)$, then $(\exists Q \subseteq P)(\forall n)$ $(Q \Vdash \varphi_n)$.*

PROOF. By hypothesis $(\forall n)(\forall Q \subseteq P)(\exists R \subseteq Q)(R \Vdash \varphi_n)$. Since φ_n is Σ_1^1, $R \Vdash \varphi_n$

is π_1^1. We can therefore build as usual a subtree Q of P by successively forcing each of the φ_n and splitting. The procedure is π_1^1 and total, hence Q is Δ_1^1. $\square$

Note that if the φ_n's were only ranked sentences, then by boundedness (since they are hyperarithmetic sets) their rank would be bounded below ω_1 and a local forcing argument as in 2.5 would suffice. But local forcing is no longer enough when $\varphi_n \in \Sigma_1^1$. On the other hand the simple argument above does not work in 2.5 because we cannot prove that the Q it produces is arithmetic.

If we wanted, we could prove the fusion lemma at the beginning and then use it, without introducing local forcing at all, to prove quasi-completeness by induction on the complexity of ranked sentences, as indicated in §2.

We now turn to the proof of $\mathscr{M}(A) = \Delta_1^{1,A}$ when A is Sacks ω_1-generic. The argument of 6.2 indicates that the only thing we have to prove is:

If $\mathscr{M}(A) \models (\forall x)(\exists Y)\varphi(x, Y)$ then for some $\beta < \omega_1$, $\mathscr{M}(A) \models (\forall x)(\exists Y_\beta)\varphi(x, Y_\beta)$, where φ is arithmetical with Y as the only unranked variable. By forcing = truth the premise gives $P \Vdash (\forall x)(\exists Y)\varphi(x, Y)$ for some P, $A \in P$.

PROPOSITION 7.4. *If* $P \Vdash (\forall x)(\exists Y)\varphi(x, Y)$ *then for some* $\beta < \omega_1$ *and* $Q \subseteq P$, $Q \Vdash (\forall x)(\exists Y_\beta)\varphi(x, Y_\beta)$.

PROOF. By definition of forcing with unranked sentences we have $(\forall x)(\forall Q \subseteq P)(\exists R \subseteq Q)(R \Vdash \exists Y\, \varphi(x, Y))$. Hence $(\forall x)(P \Vdash^w \exists Y\, \varphi(x, Y))$ and by the fusion lemma we get $Q \subseteq P$ such that $(\forall x)(Q \Vdash \exists Y\, \varphi(x, Y))$. By definition of forcing, $(\forall x)(\exists \beta < \omega_1)(Q \Vdash \exists Y_\beta\, \varphi(x, Y_\beta))$ and by boundedness $(\exists \beta < \omega_1)(\forall x)(Q \Vdash \exists Y_\beta\, \varphi(x, Y_\beta))$. $\square$

This, however, is not enough: to conclude $\mathscr{M}(A) \models (\forall x)(\exists Y_\beta)\varphi(x, Y_\beta)$ we should know that $A \in Q$. The lemma above, however, tells that a certain π_1^1 set of conditions is dense, hence we need to know that if A is Sacks ω_1-generic, $\{P: A \in P\}$ meets every π_1^1 dense set of conditions.

PROPOSITION 7.5. *A is Sacks ω_1-generic iff $\{P: A \in P\}$ meet every dense set of conditions arithmetical in $\mathcal{O}$ (Sacks [23]).*

PROOF. As in 6.6, using the analogue of 6.5. E.g. we prove that $B \in \pi_1^1$ iff B is Σ_1^1 invariantly definable over $\mathscr{M}(A)$ for A Sacks ω_1-generic. If B is Σ_1^1 invariantly defined by φ, then $x \in B \Leftrightarrow (\exists P)(P \Vdash \varphi(\bar{x}))$, hence $B \in \pi_1^1$. Conversely, if $B \in \pi_1^1$ then $x \in B \Leftrightarrow (\exists X \in \Delta_1^1)\, \varphi(x, X)$ and for A Sacks ω_1-generic, $x \in B \Leftrightarrow \mathscr{M}(A) \models \exists X \varphi(x, X)$. In the only interesting direction, if $\mathscr{M}(A) \models \exists X\, \varphi(x, X)$ then $P \Vdash \exists X_\beta\, \varphi(x, X_\beta)$ for some $\beta < \omega_1$, $A \in P$. By definition of forcing for ranked formulas, $\mathscr{M}(C) \models \exists X_\beta\, \varphi(x, X_\beta)$ for every $C \in P$; taking $C \in \Delta_1^1$ (since $P \in \Delta_1^1$) we get such an $X_\beta \in \mathscr{M}(C) = \Delta_1^1$. Hence $x \in B$.

There is still one point that prevents us from translating the proof of 6.5 into a proof of the present result: we have to consider sets of conditions (i.e. sets of trees) but the result above applies only to sets of numbers. We can however make use of the fact that there is a π_1^1 enumeration of the Δ_1^1 sets over ω, and eliminate the direct occurrences of conditions in favour of their indices. E.g. above S will be a π_1^1 set of indices of conditions, and we can immediately obtain φ such that $\mathscr{M}(A) \models \varphi$ iff $(\exists P \in S)(A \in P)$ whenever A is Sacks ω_1-generic and then proceed as in 6.5. $\square$

Note that we do not have any such characterization for weak Sacks ω_1-genericity because any natural set of conditions must express the fact that every element of it is a Δ_1^1 perfect closed set and this is already π_1^1. For a similar reason we did not have similar characterizations in §2, because being an arithmetical perfect closed set is not an arithmetical condition.

PROPOSITION 7.6. *If A is Sacks ω_1-generic then $\mathscr{M}(A) = \Delta_1^{1,A}$ (Gandy and Sacks [7]).*

PROOF. Let $P \Vdash (\forall x)(\exists Y)\varphi(x, Y)$ and $A \in P$. Consider the π_1^1 set of conditions S so defined:

$$\{Q\colon (\forall X \in Q)(X \notin P) \text{ or } Q \subseteq P \wedge (\exists \beta < \omega_1)(Q \Vdash (\forall x)(\exists Y_\beta)\varphi(x, Y_\beta))\}.$$

To show S is dense, let φ be ranked such that $\mathscr{M}(X) \models \varphi \Leftrightarrow X \in P$. By quasi-completeness, given R there is $R' \subseteq R$ such that $R' \Vdash \varphi$ or $R' \Vdash \sim\varphi$. If $R' \Vdash \varphi$ then $(\forall X \in R')(\mathscr{M}(X) \models \varphi)$ so $R' \subseteq P$ and, by definition of forcing, $R' \Vdash (\forall x)(\exists Y)\varphi(x, \cdot Y)$, so by 7.4 there is $Q \subseteq R' \subseteq R$, $Q \in S$. If $R' \Vdash \sim\varphi$ then $(\forall X \in R')(X \notin P)$ and $R' \in S$. In both cases, given R there is $Q \subseteq R$, $Q \in S$ and S is dense. By 7.5 there is $Q \in S$, $A \in Q$. Since $A \in P$ the second case must occur, i.e. $Q \subseteq P$ and for some $\beta < \omega_1$, $Q \Vdash (\forall x)(\exists Y_\beta)\varphi(x, Y_\beta)$. Hence $\mathscr{M}(A) \models (\forall x)(\exists Y_\beta)\varphi(x, Y_\beta)$. $\square$

§8. **Hyperdegrees.** In this section we derive the consequences of the work of §§6, 7 for the structure of *hyperdegrees*, obtained in the usual way from the relation $A \leq_h B$ iff A is Δ_1^1 in B. The structure of hyperdegrees admits a natural partial order $\leq$ induced by $\leq_h$, and a natural jump operation (hyperjump) induced by the operation of taking, given A, the complete π_1^1 set in A (we denote this by $\mathcal{O}^A$). Moreover there is a natural ordinal assignment induced by ω_1^A (that this is well defined on hyperdegrees comes from Spector [30], since $A \leq_h B \Rightarrow \omega_1^A \leq \omega_1^B$). We use the usual notations: $\boldsymbol{a}$ is a hyperdegree, $\boldsymbol{O}$ is the smallest hyperdegree (containing exactly the hyperarithmetical sets), $\boldsymbol{a}'$ is the hyperdegree of the hyperjump of any set in $\boldsymbol{a}$, $\omega_1^{\boldsymbol{a}}$ is the ordinal ω_1^A for any $A \in \boldsymbol{a}$, $\boldsymbol{a} \cup \boldsymbol{b}$ is the l.u.b. of $\boldsymbol{a}$ and $\boldsymbol{b}$ (the hyperdegree of $A \oplus B$ for $A \in \boldsymbol{a}$, $B \in \boldsymbol{b}$), and $\boldsymbol{a} \cap \boldsymbol{b}$ is the g.l.b. of $\boldsymbol{a}$ and $\boldsymbol{b}$ (when it exists). A useful fact to remember is the Spector criterion for ordinals: $\omega_1 < \omega_1^{\boldsymbol{a}}$ iff $\boldsymbol{O}' \leq \boldsymbol{a}$ or, in relativized form, $\boldsymbol{a} \leq \boldsymbol{b} \Rightarrow (\omega_1^{\boldsymbol{a}} < \omega_1^{\boldsymbol{b}}$ iff $\boldsymbol{a}' \leq \boldsymbol{b})$. We have already noted that if A is ω_1-generic then $\omega_1 = \omega_1^A$.

PROPOSITION 8.1. *If A is ω_1-generic then:*

(a) $\mathcal{O}^A \equiv_T A \oplus \mathcal{O}$ (*Thomason* [33]).

(b) *The components of A are hyperarithmetically independent* (*Feferman* [2]).

PROOF. (a) $A \oplus \mathcal{O} \leq_T \mathcal{O}^A$ is true in general. Since $\mathcal{O}^A$ is $\pi_1^{1,A}$ then for some $\varphi \in \Sigma_1^1$ of $\mathscr{L}_{\omega_1}$ and by genericity, $x \in \mathcal{O}^A \Leftrightarrow (\exists \sigma \subseteq A)(\sigma \Vdash \varphi(\bar{x}))$ and $\mathcal{O}^A$ is r.e. in $A \oplus \mathcal{O}$ (because $\mathcal{O}$ is π_1^1-complete). Similarly $\mathcal{O}^A$ is co-r.e. in $A \oplus \mathcal{O}$. So $\mathcal{O}^A \leq A \oplus \mathcal{O}$.

(b) As in 3.4. $\square$

PROPOSITION 8.2. *If $\mathcal{O} \leq_T A$ then for some ω_1-generic B, $\mathcal{O}^B \equiv_T A$ (MacIntyre [18]).*

We only quote the results that can be obtained, using genericity, as in Part I.

PROPOSITION 8.3. (a) *Every countable partial ordering is embeddable in the hyperdegrees below* 0′ (*Feferman* [2]).

(b) *The hyperdegees below* $\mathbf{0}'$ *are not a lattice.*

The next result collects the facts about the hyperjump.

PROPOSITION 8.4. (a) *The hyperjump has range* $\{\boldsymbol{a}: \boldsymbol{a} > \mathbf{0}'\}$ *and is never one-one on its range* (*Thomason* [33]).

(b) *If* $\boldsymbol{a} < \mathbf{0}'$ *then* $\boldsymbol{a}' = \mathbf{0}'$ (*Spector* [30]).

PROOF. (a) E.g. the fact that $\boldsymbol{a} \geq \mathbf{0}'$ is the hyperjump of some $\boldsymbol{b}$ is obtained from 8.2 as follows: given $A \in \boldsymbol{a}$, by hypothesis $\mathcal{O} \leq_h A$. By Kleene [12] there is some $\alpha < \omega_1^A$ such that $\mathcal{O} \leq_T A^{(\alpha)}$ (the α-jump of A in the sense of §5). By 8.2 there is B such that $\mathcal{O}^B \equiv_T A^{(\alpha)}$. Then $\mathcal{O}^{(B)} \equiv_h A$.

(b) From Spector criterion for ordinals we have $\omega_1 < \omega_1^{\boldsymbol{a}}$ iff $\mathbf{0}' \leq \boldsymbol{a}$. Since $\boldsymbol{a} < \mathbf{0}'$ then $\omega_1 = \omega_1^{\boldsymbol{a}}$ and hence $\omega_1^{\boldsymbol{a}} < \omega_1^{\mathbf{0}'}$. From Spector criterion relativized (since $\boldsymbol{a} \leq \mathbf{0}'$), it follows that $\boldsymbol{a}' \leq \mathbf{0}'$.

Thomason [33] has proved that there are two hyperdegrees below $\mathbf{0}'$ whose hyperjumps are arithmetically incomparable. His technique consists of enlarging $\mathcal{L}_\omega$ by adding a constant for the hyperjump of the generic set. This has some advantages, namely it does not require the introduction of the structure $\mathcal{M}(A)$ for the definition of truth and the treatment is more in the spirit of Part I. It also yields stronger results, like the one quoted above. There are, however, some technical complications and the treatment is not apt for the generalizations we have in mind for Part III. However, this method may be applied to the languages of §5 and gives, for any $\alpha < \omega_1$, α limit *two hyperdegrees below* $\mathbf{0}'$ *whose hyperjumps are* α*-incomparable.*

PROPOSITION 8.5. *For any* $\boldsymbol{a} > \mathbf{0}$ *there is a hyperdegree* $\boldsymbol{b}$ *incomparable with it. If, moreover,* $\mathbf{0} < \boldsymbol{a} < \mathbf{0}'$, *then* $\boldsymbol{b}$ *can be chosen such that* $\mathbf{0} < \boldsymbol{b} < \mathbf{0}'$ (*Thomason* [33]).

PROOF. The second fact comes from the existence of two hyperdegrees with l.u.b. $\mathbf{0}'$ and g.l.b. $\mathbf{0}$, proved as in 4.3. The first one comes from the fact that given $A \notin \Delta_1^1$, there are comeager many sets hyperarithemtically incomparable with it. But the following sets are comeager: the set of B's such that $B \not\leq_h A$ (since there are only countably many B's such that $B \leq_h A$); the set of B's such that $\omega_1 = \omega_1^B$ (since the generic sets are such); the set of B's such that B is Σ_α^0-incomparable with A for some fixed $\alpha < \omega_1$ (by 3.7 and §5). Hence the intersection of all these sets (for all $\alpha < \omega_1$) is comeager (because they are countably many). We claim that if B is in the intersection, B is hyperarithmetically incomparable with A: by definition $B \not\leq_h A$ and $A \notin \bigcup_{\alpha<\omega} \Sigma_\alpha^{0,B}$. But by Kleene [12] if $A \in \Delta_1^{1,B}$ then $A \in \bigcup_{\alpha<\omega_1^B} \Sigma_\alpha^{0,B}$ and $\omega_1^1 = \omega_1^B$. □

We quote here an unpublished result of Steel: *the hyperdegrees below* 0′ *are complemented*, i.e. for every $\boldsymbol{a}$ such that $\mathbf{0} < \boldsymbol{a} < \mathbf{0}'$ there is $\boldsymbol{b}$ such that $\boldsymbol{a} \cap \boldsymbol{b} = \mathbf{0}$ and $\boldsymbol{a} \cup \boldsymbol{b} = \mathbf{0}'$. The idea is to mix the technique to get $\boldsymbol{b}$ such that $\boldsymbol{a} \cup \boldsymbol{b} = \mathbf{0}'$ (see [21] for the analogue result for Turing degrees) with the procedure to get $\boldsymbol{b}$ ω_1-generic over $\boldsymbol{a}$ (this insures $\boldsymbol{a} \cap \boldsymbol{b} = \mathbf{0}$ automatically, see 4.3). The reason why this is simpler than the analogous result for Turing degrees (due to Posner) is because every hyperdegree strictly below $\mathbf{0}'$ has hyperjump $\mathbf{0}'$ and the difficult case for Turing degrees is when $\boldsymbol{a}'' > \mathbf{0}''$.

We turn now to arguments involving trees. The basic lemma is the following:

PROPOSITION 8.6. *Let* φ *be a ranked formula of* $\mathcal{L}_{\omega_1}$ *with only one free variable* *and let $x \in A_\varphi \Leftrightarrow A \models \varphi(x)$. Then for every P there is $Q \subseteq P$ such that one of the following holds*:

(a) $(\forall A \in Q)$ (A_φ *is hyperarithmetical*).

(b) $(\forall A \in Q)$ $(A \leq A_\varphi)$ (*Gandy and Sacks* [7]).

PROOF. Similar to 2.2(c). There are two cases:

(a) $(\exists R \subseteq P)(\forall R_1 \subseteq R)(\forall R_2 \subseteq R)(\forall x) \sim (R_1 \Vdash \varphi(\bar{x}) \wedge R_2 \Vdash \sim\varphi(\bar{x}))$. Take $Q = R$.

(b) Case (a) does not hold. Take $Q = \bigcap T_\sigma$ where T_σ is defined by induction as follows. $T_\varnothing = P$. Given T_σ, $T_{\sigma*0}$ and $T_{\sigma*1}$ are disjoint hyperarithmetical subtrees of T_σ such that for some x, one of them forces $\varphi(x)$. and the other forces $\sim\varphi(x)$. $\square$

We could think at this point to reproduce the arguments of §4 and build directly, e.g., a minimal hyperdegree by iterating 8.6. There is, however, an obvious trouble in proving that a set so obtained has really minimal hyperdegrees: we lack the usual connection between definability in analysis and definability in $\mathscr{L}_{\omega_1}$. This is easily overridden by simultaneously building our set Sacks ω_1-generic. Hence now forcing plays a crucial role, contrary to the case of §§3, 4. Once we see that genericity is needed, we may actually avoid the construction outlined above because the result is automatic:

PROPOSITION 8.7. *If A is Sacks ω_1-generic, A has minimal hyperdegree* (*Gandy and Sacks* [7]).

PROOF. A is not Δ_1^1 by 7.2. Let $B \leq_h A$: we want to show that $B \in \Delta_1^1$ or $A \leq_h B$. Since $B \in \Delta_1^{1,A}$, $B \in \mathscr{M}(A)$ by 7.6 and hence for some ranked φ, $x \in B \Leftrightarrow A \models \varphi(x)$. Consider the set of Q's such that either $(\forall X \in Q)(X_\varphi$ is hyperarithmetical) or $(\forall X \in Q)(X \leq_h X_\varphi)$: this is a π_1^1 set of conditions and by 8.6 is dense. By 7.5 there is such a Q such that $A \in Q$: since $B = A_\varphi$ we have the result. $\square$

We then get immediately from 7.2:

PROPOSITION 8.8. (a) *There are uncountably many minimal hyperdegrees.*

(b) *There are minimal hyperdegrees below* $\mathbf{0}'$ (*Gandy and Sacks* [7]).

The behaviour of the hyperjump operator restricted to minimal hyperdegrees is the following:

PROPOSITION 8.9. (a) *If $\boldsymbol{a}$ is minimal then $\boldsymbol{a}' = \boldsymbol{a} \cup \mathbf{0}'$.*

(b) *If $\boldsymbol{a} \geq \mathbf{0}'$ then for some minimal $\boldsymbol{b}$, $\boldsymbol{a} = \boldsymbol{b}'$*. (*Simpson* [28].)

PROOF. (a) Since $\mathbf{0}' \not\leq \boldsymbol{a}$ because $\mathbf{0}'$ is not minimal, from the Spector criterion we have $\omega_1 = \omega_1^{\boldsymbol{a}} < \omega_1^{\boldsymbol{a} \cup \mathbf{0}'}$. By the relativization of Spector criterion, $\boldsymbol{a}' \leq \boldsymbol{a} \cup \mathbf{0}'$ (since $\boldsymbol{a} \leq \boldsymbol{a} \cup \mathbf{0}'$).

(b) Let T be a tree recursive in $\mathcal{O}$ of Sacks ω_1-generic sets (as in 7.2). We may view T as a function from strings to strings. If we let $B = \bigcup_{\sigma \subseteq A} T(\sigma)$ where $A \in \boldsymbol{a}$, then:

$B \oplus \mathcal{O} \leq_h A$ by construction (since $\mathcal{O} \leq_h A$).

$A \leq_h B \oplus \mathcal{O}$ since A may be recovered from T and B, and $T \leq_h \mathcal{O}$.

Since B has minimal hyperdegree, by part (a) $\mathcal{O}^B \equiv_h B \oplus \mathcal{O}$. $\square$

Similarly, Simpson [28] has proved that if $\boldsymbol{a} \geq \mathbf{0}'$ then there are $\boldsymbol{b}$, $\boldsymbol{c}$ minimal hyperdegrees such that $\boldsymbol{b}' = \boldsymbol{c}' = \boldsymbol{b} \cup \boldsymbol{c} = \boldsymbol{a}$.

We will continue our investigation of the structure of hyperdegrees in §11. For now we just note that we cannot immediately prove the existence of minimal upper bounds for countable sets of hyperdegrees: Proposition 8.6 is easily relativized, but we need an appropriate notion of genericity. By 8.7, Sacks ω_1-genericity gives minimal hyperdegrees and hence is not appropriate. By the way, it is still an open problem whether every countable set of hyperdegrees has a minimal upper bound.

Working in another direction, Thomason [34] has proved that every finite distributive lattice is embeddable as an initial segment of the hyperdegrees, and this implies that *the theory of hyperdegrees is undecidable*. Simpson [29] has announced that this theory is actually recursively equivalent to second order arithmetic, but the proof of this does not follow under the general framework of Nerode and Shore [19] because not every countable ideal of hyperdegrees has an exact pair (see §11).

§9. π_1^1 sets. We proved by §6 that weakly ω_1-generic sets are not π_1^1 and in §7 that Sacks ω_1-generic sets are not π_1^1. We also promised to build Sacks weakly ω_1-generic π_1^1 sets. Before we get to this, we want to understand why this result is interesting.

π_1^1 sets are in many ways analogous to Σ_1^0 sets (see Rogers [22, Chapter 16]). It is then natural to expect them to have a structure theory similar to the one given by the Turing degrees of Σ_1^0 sets. Since Turing degrees are obtained from the relation "Δ_1^0-in", the naive approach is to consider the hyperdegrees of π_1^1 sets, since they are obtained from the relation "Δ_1^1-in". Spector [30] proved, however, that the analogy breaks down dramatically: there are only two hyperdegrees containing π_1^1 sets, namely $\mathbf{0}$ and $\mathbf{0}'$. We will prove this in a minute. So either we are satisfied with this and hence accept the fact that the analogy Σ_1^0 sets-π_1^1 sets is only superficial, or we look for some more sophisticated notions. Since it is the relationship between π_1^1 sets on one side and hyperdegrees on the other that is not satisfactory, we may look for an improvement of just one of them or of both.

In [17] Kripke introduced the following formalism for computations on ordinals less than ω_1. We have symbols $f, g \ldots$ for functions; $x, y \ldots$ for variables; constants $\bar{\alpha}$ for each $\alpha < \omega_1$; successor; equality; a bounded quantifier $(\exists x <)$. The terms are defined by induction: variables and constants are terms; if t is a term, so is t' (the successor of t); if $t_1, \ldots, t_n$ are terms and f is an n-ary function symbol, $f(t_1, \ldots, t_n)$ is a term; if t_1, t_2 are terms, so is $(\exists x < t_1)(t_2(x))$. An equation is an expression of the form $t_1 = t_2$, where t_1, t_2 are terms. The following are the rules of calculation:

(R1) Substitution of all occurrences of a variable in an equation with a constant.

(R2) Substitution of one side of an equation with the other side.

(R3) If for some $\alpha < \beta$, $t(\bar{\alpha}) = \bar{0}$ then $(\exists x < \bar{\beta})(t(x)) = \bar{0}$.
If for all $\alpha < \beta$, $t(\bar{\alpha}) = \bar{1}$ then $(\exists x < \bar{\beta})(t(x)) = \bar{1}$.

The third rules mean that the term $(\exists x < t_1)(t_2(x))$ is interpreted as the characteristic function of the bounded existential quantifier. If E is a finite system of equations, we define: $S_0^E = E$, $S_{\alpha+1}^E =$ the set of immediate consequences (via R1–R3) of equations in S_α^E, $S_\lambda^E = \bigcup_{\alpha<\lambda} S_\alpha^E$ if λ is limit, $S^E = \bigcup_\alpha S_\alpha^E$. It is easy to see that $S^E = \bigcup_{\alpha<\omega_1} S_\alpha^E$. We can relativize this to any $B \subseteq \omega_1$ by $S_0^{E,B} = E \cup \{f(\bar{\alpha}) = \bar{0}\colon \alpha \in B\} \cup \{f(\bar{\alpha}) = \bar{1}\colon \alpha \notin B\}$ for some symbol f not occurring in E. We get similarly $S^{E,B} = \bigcup_{\alpha<\omega_1^B} S_\alpha^{E,B}$.

This gives a natural notion of relative computability for subsets of ω_1. For a fixed ordinal α we say *A is α-computable from B* ($A \leq_\alpha B$) if only deductions from B of length less than α are used, i.e. if for some function symbol g:

$$x \in A \text{ iff } (g(\bar{x}) = \bar{0}) \in \bigcup_{\beta<\alpha} S_\beta^{E,B},$$

$$x \notin A \text{ iff } (g(\bar{x}) = \bar{1}) \in \bigcup_{\beta<\alpha} S_\beta^{E,B}.$$

We also say *A is ω_1-calculable from B* ($A \leq_{c\omega_1} B$) if for some α, $A \leq_\alpha B$, i.e. if any length of computation is allowed.

The connection with hyperdegrees is the following: if $A, B \subseteq \omega$ then

$$A \leq_h B \text{ iff } A \leq_{c\omega_1} B \text{ iff } A \leq_{\omega_1^B} B.$$

This is easily seen by arithmetization since the two formalisms of calculability and of the relativized ramified analytic hierarchy are equivalent. For our purposes we note that:

(a) If E is a finite system of equations, e is an equation and α is an ordinal greater than those occurring in E, e then we can find a ranked sentence φ of $\mathscr{L}_{\omega_1}$ such that for all B, $e \in S_\alpha^{E,B}$ iff $\mathscr{M}(B) \models \varphi$.

(b) Given φ ranked we can find E, e, α such that for all B, $\mathscr{M}(B) \models \varphi$ iff $e \in S_\alpha^{E,B}$.

PROPOSITION 9.1. (a) *If $A \in \pi_1^1$ then $A \leq_\omega \mathcal{O}$.*

(b) *If $A \in \pi_1^1 - \Delta_1^1$ then $\mathcal{O} \leq_{\omega_1+1} A$ (Spector* [30]).

PROOF. (a) Since $A \in \pi_1^1$ and $\mathcal{O}$ is π_1^1 complete, for some recursive f, $x \in A \Leftrightarrow f(x) \in \mathcal{O}$. A finite number of steps (using only the rules R1–R2) gives the value $f(x)$ from an appropriate system of equations, and one step more gives the value of the characteristic function of $\mathcal{O}$ on $f(x)$. So every computation of this kind is finite.

(b) Let f be as in part (a). For our purposes we can think of $\mathcal{O}$ as the well ordering of ω with ordinal ω_1 of §5. Hence for $x \in A$, $f(x)$ is (the notation of) an ordinal less than ω_1 and these ordinals are not bounded below ω_1 (otherwise $A \in \Delta_1^1$). Hence $z \in \mathcal{O} \Leftrightarrow (\exists x \in A)(z < f(x))$. If $z \in \mathcal{O}$ then we can find this with a computation of length less than ω_1; if $z \notin \mathcal{O}$ then we can find this one step after we checked for all $x \in A$ that $z \not< f(x)$, hence in $\omega_1 + 1$ steps. □

Since $\omega_1 < \omega_1^A$ when $A \in \pi_1^1 - \Delta_1^1$, we have in particular that $\mathcal{O} \leq_h A$ and every $\pi_1^1 - \Delta_1^1$ set is in the hyperdegree $\mathbf{0}'$. We also see where the analogy between Turing degrees and hyperdegrees breaks down: in relative recursive computations the number of steps is always finite and hence independent of the sets we are computing from, but it is not so for relative Δ_1^1 computations. By combining the two parts of 9.1 we get that any two $\pi_1^1 - \Delta_1^1$ sets are actually mutually $\leq_{\omega_1+1}$-reducible. Hence the best possible result for such sets would be to have two of them $\leq_{\omega_1}$-incomparable. In terms of degrees this would amount to a solution of Post's problem for "ω_1-degrees" of π_1^1 sets. We do not state the result this way because $\leq_{\omega_1}$ does not induce a notion of degree: $\leq_{\omega_1}$ *is not transitive* (intuitively the difficulty is this: suppose $A \leq_{\omega_1} B$ and $B \leq_{\omega_1} C$. To reduce A to C we test e.g. $x \in A$, find the questions about B needed to answer this—using the first reduction—and then answer each of them—using the second reduction—in terms of questions about C. If we do not require any bound on the lengths of the computations, this works and gives the transitivity of $\leq_{c\omega_1}$. But otherwise we run into trouble, because each individual computation is bounded below ω_1, but we might use complicated parts of B and we can only get boundedness for all of them if these parts are Σ_1^1). Good notions of degrees are discussed in Kreisel and Sacks [16]. Since all of them have the feature that computations are bounded below ω_1, the solution of Post's problem for these degrees of π_1^1 sets comes from 9.5.

Given a tree P, let P^+ be the set of elements which are in every branch of P, and P^- be the set of elements which are in no branch of P. The conditions we use in the first part of this section are hyperarithmetical perfect trees P of a particular kind: they are completely determined by P^+ and P^-, in the sense that P consists of all strings which are 1 on elements of P^+ and 0 on elements of P^-. These trees are also called *hyperarithmetical coinfinite conditions*, since the complement of $P^+ \cup P^-$ is infinite (P being perfect). To say $A \in P$ now means that $P^+ \subseteq A$, $P^- \subseteq \bar{A}$, It is not difficult to see that the general theory of forcing developed in §7 goes through for forcing with hyperarithmetical coinfinite conditions (the set of the forcing conditions is again π_1^1). When we speak of Sacks weakly ω_1-generic sets, we refer to forcing with coinfinite conditions. The next two theorems illustrate the connections between genericity and ω_1-computability.

PROPOSITION 9.2. *A is Sacks weakly ω_1-generic iff $A \notin \Delta_1^1$ and every computation from it of length less than ω_1 only uses Δ_1^1 parts of A* (*Sacks* [24]).

PROOF. Let A be Sacks weakly ω_1-generic. $A \notin \Delta_1^1$ as in 7.2(b). Let e be an equation and E a finite system of equations. Then by the observations made before there are $\alpha < \omega_1$, φ ranked such that

$$e \in S_\alpha^{E,A} \text{ iff } \mathcal{M}(A) \models \varphi \text{ iff } (\exists P)(A \in P \wedge P \Vdash \varphi).$$

Since φ is ranked, forcing = truth on all the branches of P. Moreover P^+, P^- are Δ_1^1 and $P^+ \subseteq A$, $P^- \subseteq \bar{A}$. So every computation from A of length less than ω_1 only uses Δ_1^1 parts of A.

Let now φ be ranked. By definition of forcing it is enough to prove that $\mathcal{M}(A) \models \varphi \Rightarrow (\exists P)(A \in P \wedge P \Vdash \varphi)$ to have A weakly Sacks ω_1-generic. Let e, E, α be such that $\mathcal{M}(A) \models \varphi$ iff $e \in S_\alpha^{E,A}$. Let A_0, A_1 be the Δ_1^1 parts of A, $\bar{A}$ used in the right-hand side computation. Build P such that $P^+ = A_0$, $P^- = A_1$: P is Δ_1^1 and is perfect because $\overline{A_0 \cup A_1}$ is infinite (since $A \in P$ and $A \notin \Delta_1^1$). Moreover $P \Vdash \varphi$ by definition. □

Sets with the property that every computation from them of length less than ω_1 only uses Δ_1^1 parts of them are called *ω_1-subgeneric*.

It follows from 7.6 that if A is Sacks ω_1-generic then

$$B \text{ is } \Delta_1^1 \text{ over } \mathcal{M}(A) \text{ iff } B \leq_h A$$

(because $\mathcal{M}(A) = \Delta_1^{1,A}$ and Σ_1^1 over $\Delta_1^{1,A} = \pi_1^{1,A}$). Similarly we have:

PROPOSITION 9.3. *If A is Sacks weakly ω_1-generic then for any B, B is Δ_1^1 over $\mathcal{M}(A)$ iff $B \leq_{\omega_1} A$* (*Sacks* [24]).

Note that in 7.6 we needed to control computations from A of any given length (i.e. of length bounded by ω_1^A) and we used ω_1-genericity to insure $\omega_1^A = \omega_1$. Here we decided to restrict our attention to computations bounded by ω_1, and these are controlled by ranked formulas (see the observations before 9.1). Herce weak ω_1-genericity is enough.

PROPOSITION 9.4. *There is a π_1 Sacks weakly ω_1-generic set* (*Sacks* [24]).

PROOF. We enumerate $A \subseteq \omega$ in ω_1 many stages, and at each stage $\alpha < \omega_1$ we will determine a Δ_1^1 part A_α of it, uniformly in α. This makes A π_1^1 because $x \in A \Leftrightarrow (\exists\alpha < \omega_1)(x \in A_\alpha)$. We have a (partial) π_1^1 enumeration $\{\varphi_n\}_{n\in\omega}$ of the ranked sentences of $\mathscr{L}_{\omega_1}$. We can approximate it in a Δ_1^1 way along our construction, and we say that φ_n is defined at stage $\alpha < \omega_1$ if φ_n has appeared in the approximation at this stage. We use priority in the usual way: if $n < m$ then φ_n has higher priority than φ_m. The condition R_n requires that A force φ_n or $\sim \varphi_n$.

Suppose at stage $\alpha < \omega_1$ we have A_α and try to satisfy R_n, where n is the least integer x such that φ_x is defined at stage α, R_x has not yet been satisfied and not injured afterwards, and ($A_\alpha \cup$ the set of elements restrained out of A for the sake of conditions with higher priority) is coinfinite. Let P be the tree such that: $P^+ = A_\alpha$, $P^- =$ elements restrained out of A by conditions of higher priority. Since by hypothesis $P^+ \cup P^-$ is coinfinite, P is perfect. It is Δ_1^1 by induction hypothesis. So by quasi-completeness (§7) there is $Q \subseteq P$ such that $Q \Vdash \varphi_n$ or $Q \Vdash \sim\varphi_n$. Take such a Q and let $A_{\alpha+1} = Q^+$. Moreover restrain out of A the elements of Q^-. Again $Q^+ \cup Q^-$ is coinfinite (Q is perfect).

The strategy for limit stages λ should be to let $A_\lambda = \bigcup_{\alpha<\lambda} A_\alpha$, and to retain as active only those requirements that have not been satisfied infinitely often before λ. There is, however, a problem: how do we insure that even at limit stages the induction hypothesis is preserved, in particular that A_λ is still coinfinite (otherwise there would be no room to satisfy other requirements after stage λ)? We have simply to modify the construction above and add some more restraint. Namely, when we consider R_n and have ($A_\alpha \cup$ the set of elements restrained out of A for the sake of R_m's, $m < n$) coinfinite, we also restrain out of A the first n elements of its complement. So if we satisfy infinitely many conditions below λ then (except for injuries) we preserve increasingly long segments of the complement above, and at the limit A_λ is coinfinite. □

PROPOSITION 9.5. *There are two π_1^1 sets ω_1-incomparable* (*Sacks* [24]).

PROOF. Similar to 3.6. We build two Sacks weakly ω_1-generic sets A, B such that A is not Δ_1^1 over $\mathscr{M}(B)$ and B is not Δ_1^1 over $\mathscr{M}(A)$. This is enough by 9.3 and can be achieved by a routine argument mixing forcing and priority, on the line of 9.4. As in 3.6, conflicts may arise both for the sake of making the sets incomparable and for the sake of making them generic. □

We only quote here that 9.5 actually provides a solution to Post's problem in the context of recursion in E (see Sacks [25] for an expository paper on this subject).

We turn now to a different matter, namely the problem of extending in some natural way the chain of Turing degrees $\{\mathbf{0}^{(\alpha)}\}_{\alpha<\omega_1}$. Since the natural nonhyperarithmetical object is $\mathcal{O}$, we simply define $\mathbf{0}^{(\omega_1)}$ as the Turing degree of $\mathcal{O}$. We cannot, however, expect to find a useful characterization of it from $\{\mathbf{0}^{(\alpha)}\}_{\alpha<\omega_1}$ in terms of β-least upper bounds because of 5.4. The start, however, comes from 6.4: $\mathbf{0}^{(\omega_1)}$ is the largest Turing degree among the degrees of sets Δ_2^1 over Δ_1^1. We have to learn how to deal with these sets.

The analogue of the Enderton-Putnam [1] computation used in §5 is now the following:

PROPOSITION 9.6. *$\mathcal{O}$ is Δ^0_4 in any minimal upper bound for* $\{0^{(\alpha)}\}_{\alpha<\omega_1}$ (*Jockusch and Simpson* [11]).

PROOF. By the Gandy [5]-Spector [31] theorem for some $P \in \pi^0_1$, $x \in \mathcal{O} \Leftrightarrow (\exists f \in \Delta^1_1)P(x, f)$. Let A be minimal upper bound for $\{0^{(\alpha)}\}_{\alpha<\omega_1}$. It is immediate that $x \in \mathcal{O} \Rightarrow (\exists f \leq_T A)P(x, f)$. Moreover, P can be chosen so to have the property that if $P(x, f)$ but $x \notin \mathcal{O}$, then the degree of f is not minimal over Δ^1_1 [31]. Hence $x \in \Leftrightarrow (\exists f \leq {}_T A)P(x, f)$. Since $\leq_T$ is expressible in Σ^0_3 form, $\mathcal{O}$ in Σ^0_3 in A (hence Δ^0_4 in A). □

It follows, in particular, that if $\boldsymbol{a}$ is a minimal upper bound for $\{\mathbf{0}^{(\alpha)}\}_{\alpha<\omega_1}$ then $\mathbf{0}^{(\omega_1)} \leq \boldsymbol{a}^{(3)}$. We will get a characterization of $\mathbf{0}^{(\omega_1)}$ if we can find such an $\boldsymbol{a}$ s.t. $\mathbf{0}^{(\omega_1)} = \boldsymbol{a}^{(3)}$. We build A by using recursively pointed Δ^1_1 trees (as in §3) to get a minimal upper bound of Δ^1_1. The whole problem is to insure that $A^{(3)} \leq_T \mathcal{O}$: for this it will be enough to get $A^{(3)}$ Δ^1_2 over Δ^1_1. Since the 3-jump is controlled by Σ^0_3 sentences, our condition is:

$$X\ \Sigma^0_3 \text{ in } A \Rightarrow X\ \Sigma^1_1 \text{ over } \Delta^1_1.$$

(We request this by symmetry, since it is easier to implement than the weaker condition $X\ \Sigma^0_3$ in $A \Rightarrow X\ \Delta^2_1$ over Δ^1_1, although this is really what we want.) Suppose we make A pointedly generic with respect to Σ^0_3 sentences. Then if X is Σ^0_3 in A via $\varphi \in \Sigma^0_3$,

$$x \in X \Leftrightarrow A \models \varphi(x) \Leftrightarrow (\exists P)(A \in P \wedge P \Vdash \varphi).$$

If our forcing conditions are Δ^1_1 (recursively pointed) trees then the quantifier $(\exists P)$ is a quantifier over Δ^1_1, hence we would need $P \Vdash \varphi(\bar{x})$ to be Σ^1_1 over Δ^1_1 (i.e. π^1_1) to have right-hand side Σ^1_1 over Δ^1_1. There are two problems to override to get this.

The restriction to Δ^1_1 trees is not a problem because we can quantify over Δ^1_1, but the use of recursively pointed trees is: P is recursively pointed iff $(\forall a \in P)$ $(P \leq_T A)$ iff $(\forall A \in P)(\exists e)(P \simeq \{e\}^A)$, hence this is a π^1_1 condition (it might be Δ^1_1 or even arithmetical, but this is not known). This leads to quantifications over π^1_1 instead of Δ^1_1.

The solution consists in using uniformly pointed trees, i.e. trees P such that $(\exists e)(\forall A \in P)(P \simeq \{e\}^A)$. The use of such trees is not more complicated than the use of pointed trees (Lemma 2.6 goes through with a similar proof) but now we have an arithmetical condition, since P is uniformly recursively pointed iff $(\exists e)$ $(\forall A)(A \in P \Rightarrow P \simeq \{e\}^A)$, the condition $P \simeq \{e\}^A$ is π^0_2 and π^0_2 is closed under universal set quantification because Σ^0_1 is (by König's lemma).

The second problem is that if we define forcing similarly to finite forcing, then the natural complexity of $P \Vdash \varphi$ for $\varphi \in \Sigma^0_n$ would be Σ^1_n over Δ^1_1 but we want, instead, to save two quantifiers and have complexity Σ^1_n over Δ^1_1 when $\varphi \in \Sigma^0_{n+2}$. On the other hand we cannot naively define forcing as truth for all branches, because we would not have generic sets (see §2). We have, however, room for exactly what we want:

DEFINITION. The relation $P \Vdash \varphi$ for P uniformly pointed tree and φ sentence of $\mathcal{L}_\omega$ is defined by induction on φ:

If $\varphi \in \Sigma_2^0 \cup \pi_2^0$, $P \Vdash \varphi$ iff $(\forall A \in P)(A \models \varphi)$.

If $\varphi \notin \Sigma_2^0 \cup \pi_2^0$ then $P \Vdash \varphi$ is defined similarly to finite forcing.

PROPOSITION 9.7. DEFINABILITY OF FORCING (JOCKUSCH AND SIMPSON [11]).

(a) *$\{(P, \varphi): \varphi \in \pi_2^0$ and $P \Vdash \varphi\}$ is Σ_3^0 over Δ_1^1.*

(b) *$\{(P, \varphi): \varphi \in \Sigma_{n+2}^0$ and $P \Vdash \varphi\}$ is Σ_n^1 over Δ_1^1 and similarly for $\varphi \in \pi_{n+2}^0$.*

PROOF. (a) Let $\varphi \in \pi_2^0$. Then $P \Vdash \varphi$ iff P is uniformly pointed and $(\forall A \in P)$ $(A \models \varphi)$. The first condition is Σ_3^0 (see above): $A \models \varphi$ is π_2^0 and π_2^0 is closed under universal set quantifier. So everything is Σ_3^0.

(b) Immediate by induction. □

As usual when the definition of forcing is not standard, quasi-completeness may become a problem. Since, however, we applied the nonstandard definition only to sentences with two quantifiers, Lemma 2.7 helps us to prove it in this case. We therefore get.

PROPOSITION 9.8. *$\mathbf{0}^{(\omega_1)}$ is the least element of $\{\boldsymbol{a}^{(3)} : \mathbf{a}$ is a minimal upper bound for $\{0^{(\alpha)}\}_{\alpha<\omega_1}\}$ (Jockusch and Simpson [11]).*

PROOF. One part comes from 9.6. By the discussion above, it is enough to build A minimal upper bound for $\{0^{(\alpha)}\}_{\alpha<\omega_1}$ and uniformly pointedly generic for Σ_3^0 sentences. This can be done recursively in $\mathcal{O}$. □

This characterization is similar to the one obtained in §§3, 5 for $\mathbf{0}^{(\alpha)}$, α limit: $\mathbf{0}^{(\alpha)}$ is the least element of $\{a^{(2)} : a$ is a minimal upper bound for $\{\mathbf{0}^{(\beta)}\}_{\beta<\alpha}\}$.

Similarly to 9.8 another characterization can be obtained: $\mathbf{0}^{(\omega_1)}$ is the least element of $\{(\boldsymbol{a} \cup \boldsymbol{b})^{(3)} : (\boldsymbol{a}, \boldsymbol{b})$ is an exact pair for the ideal generated by $\{\mathbf{0}^{(\alpha)}\}_{\alpha<\omega_1}\}$. In one direction (similar to 9.6) we just note that if (A, B) is an exact pair for Δ_1^1 then $x \in \mathcal{O} \Leftrightarrow (\exists f \leq_T A \oplus B)P(x, f)$, even without using special properties of P. In the other direction we use product forcing similarly to what we did (the conditions are pairs of uniformly pointed Δ_1^1 trees of the same Turing degree). The advantage of this characterization is that it extends far beyond the ω_1th iteration: it is possible to build a chain $\{\mathbf{0}^{(\alpha)}\}_{\alpha<\omega_1^L}$ of degrees cofinal in the degrees of the constructible sets. At successor stages jumps are taken, and at limit ordinals α, $\mathbf{0}^{(\alpha)}$ is the least element of $\{(\boldsymbol{a} \cup \boldsymbol{b})^{(\gamma)} : (\boldsymbol{a}, \boldsymbol{b})$ is an exact pair for the ideal generated by $\{\mathbf{0}^{(\beta)}\}_{\beta<\alpha}\}$ for the least $\gamma < \omega_1^L$ for which it exists. All the $\gamma < \omega_1^L$ are eventually considered. See Jockusch and Simpson [11] and Hodes [9]. It is not known if the stronger characterization of $\mathbf{0}^{(\omega_1)}$ given by 9.8 can be extended in a similar way through ω_1^L. We will consider other chains for the constructible sets in §11 (hyperdegrees) and §12 (Δ_2^1-degrees).

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UNIVERSITY OF TURIN
TURIN, ITALY