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ON THE MINIMAL COFINAL SUBSETS OF A DIRECTED QUASI-ORDERED SET

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It is shown that for any directed quasi-ordered set $(Q, \leqslant)$, there is a minimal ordinal number λ such that every cofinal subset of Q contains a cofinal subset which is the 0-th class original set of a pure λ-th class chain of Q. A special case of our results gives necessary and sufficient conditions for a directed set to contain a cofinal chain.

Introduction

It is well known that a chain (a totally ordered set) $(C, \leqslant)$ contains well-ordered cofinal subsets, and the minimal order type of such cofinal subsets, denoted by $\mathrm{cf}(C)$, is either 1 or a regular infinite cardinal depending upon whether or not C has a greatest element. The order type $\mathrm{cf}(C)$ is characteristic of the cofinal subsets of C in the sense that any cofinal subset of C contains a cofinal subset with order type $\mathrm{cf}(C)$. More recently several papers have been written (e.g. [1–7]) about the size and structure of the cofinal subsets of an arbitrary partially ordered set. In this case the situation is more complicated since it is not true in general that if $(\mathcal{P}, \leqslant)$ is a partially ordered set, then there is some partial sub-order $(K, \leqslant)$ such that every cofinal subset of $\mathcal{P}$ contains an isomorphic copy of K.

In this paper we solve this problem for directed sets (and we consider arbitrary quasi-orders rather than partial orders). We show that for any directed quasi-ordered set, $(Q, \leqslant)$, there is a minimal ordinal number λ such that every cofinal subset of Q contains a cofinal subset having a certain characteristic structure which we call the 0-th class original set of a pure λ-th class chain of Q. Very roughly this is a set whose elements form a chain of chains iterated λ times of Q. In order to give a precise description we need several definitions.

1. Basic definitions and preliminary results

The binary relation $\leqslant$ is a *quasi-order* on the non-empty set Q if it is reflexive and transitive. We write $a<b$ if $a\leqslant b$ and $b\nleqslant a$. If $a\leqslant b$ and $b\leqslant a$ both hold for $a, b \in Q$, we write $a\approx b$. Clearly $\approx$ is an equivalence relation on Q, and if Q' is a

subset containing exactly one element from each equivalence class, then $\leqslant$ is antisymmetric on Q', i.e. $(Q', \leqslant)$ is a partial order. A *chain* in the quasi-ordered set $(Q, \leqslant)$ is a set $C \subseteq Q$ such that for each pair of distinct elements $a, b \in C$ one of the relations $a < b$, $b < a$ holds. If α is an ordinal number and the elements a_ν $(\nu < \alpha)$ of a chain C are ordered so that $a_\nu < a_{\nu'}$ for $\nu < \nu' < \alpha$, then we indicate this briefly by writing $C = \{a_\nu : \nu < \alpha\}_<$.

The quasi-order $\leqslant$ on Q induces a natural quasi-order on $\mathcal{P}(Q)$, the set of subsets of Q. For $A, B \subseteq Q$ we write $A \leqslant B$ if and only if for each element $a \in A$, there is an element $b \in B$ such that $a \leqslant b$. This extends in an obvious way to $\mathcal{P}(\mathcal{P}(Q))$, $\mathcal{P}(\mathcal{P}(\mathcal{P}(Q)))$ etc. Although we use the same symbol $\leqslant$ to denote these different order relations, no confusion should arise since we only compare sets of the same complexity.

A subset $A \subseteq Q$ is *cofinal* in the quasi-ordered set $(Q, \leqslant)$ if and only if $Q \leqslant A$. We follow the notation of [4] by defining the *cofinality* of $(Q, \leqslant)$ to be $\mathrm{cf}(Q) = \min\{|A| : Q \leqslant A\}$, the size of the smallest cofinal subsets of Q. Here, as usual, $|X|$ denotes the cardinality of the set X. This notation agrees with the usual meaning of $\mathrm{cf}(\alpha)$ in the case when α is an ordinal number, where we identify α with the set of all smaller ordinals ordered by $\in$, the membership relation. The following lemma is an immediate consequence of the definitions.

Lemma 1.1. *Let Q' be a cofinal subset of the quasi-ordered set $(Q, \leqslant)$. Then*
(i) *Any cofinal subset of Q' is a cofinal subset of Q.*
(ii) $\mathrm{cf}(Q') = \mathrm{cf}(Q)$.
(iii) *If $X \subseteq Q$, $|X| < \mathrm{cf}(Q)$, then there is $a \in Q$ such that $\{a\} \nleqslant X$.*

It is well known that a chain $(C, \leqslant)$ contains a well-ordered cofinal subset (more generally, any quasi-ordered set $(Q, \leqslant)$ contains a well-founded cofinal subset), and in this case the cofinality, $\mathrm{cf}(C)$, is either 1, when $(C, \leqslant)$ has a greatest element, or an infinite regular cardinal, when there is no greatest element. We say that a chain C is *extendable* in $(Q, \leqslant)$ if there is an element $a \in Q$ such that $C < \{a\}$. Let $\mathcal{N}(Q)$ denote the set of all *non-extendable* chains in Q.

Lemma 1.2. *If, $A, B \in \mathcal{N}(Q)$ and $A \leqslant B$, then $\mathrm{cf}(A) = \mathrm{cf}(B)$.*

Proof. Let $A' = \{a_\alpha : \alpha < \mathrm{cf}(A)\}_<$ and $B' = \{b_\beta : \beta < \mathrm{cf}(B)\}_<$ be cofinal subchains of A and B respectively. Then $A' \leqslant B'$ and so, for each $\alpha < \mathrm{cf}(A)$, there is a smallest ordinal $\beta(\alpha) < \mathrm{cf}(B)$ such that $a_\alpha \leqslant b_{\beta(\alpha)}$. If $\mathrm{cf}(A) < \mathrm{cf}(B)$, then $B'' = \{b_{\beta(\alpha)} : \alpha < \mathrm{cf}(A)\}$ is not cofinal in B' and there is an element $b \in B'$ such that $a_\alpha \leqslant b_{\beta(\alpha)} < b$ holds for all $\alpha < \mathrm{cf}(A)$. This contradicts the fact that A is non-extendable in Q. Therefore, $\mathrm{cf}(B) \leqslant \mathrm{cf}(A)$. Now suppose that $\mathrm{cf}(B) < \mathrm{cf}(A)$. Then $\mathrm{cf}(A)$ is a regular infinite cardinal number and there is some element $b' \in B'$ such that the set $\{\alpha < \mathrm{cf}(A) : b_{\beta(\alpha)} = b'\}$ has cardinality $\mathrm{cf}(A)$ and hence is cofinal in cf(A). This implies that $a_\alpha < a_{\alpha+1} \leq b'$ for all $\alpha < \text{cf}(A)$ and again we obtain the contradiction that A is extendable. □

For the results presented in this paper the following definitions play a central role. For any quasi-ordered set $(Q, \leq)$ we define

$$\mu(Q) = \min\{|A| : A \in \mathcal{N}(Q)\},$$

and

$$\mathcal{D}(Q) = \{A \in \mathcal{N}(Q) : \text{tp}(A) = \mu(Q)\}.$$

Since Q is non-empty it follows from our earlier remarks that either $\mu(Q)$ is 1 (when Q has a maximal element) or an infinite regular cardinal number (if Q has no maximal elements). In the case when $\mu(Q) = 1$, we see that $\mathcal{D}(Q)$ is the set of all singletons of the form $\{a\}$, where a is a maximal element of Q. In the case when $\mu(Q) > 1$, the members of $\mathcal{D}(Q)$ are the well-ordered, non-extendable chains in Q having order type $\mu(Q)$.

Lemma 1.3. *If Q' is a cofinal subset of Q, then* (i) $\mu(Q) \leq \mu(Q')$ *and* (ii) *if* $\mu(Q) = \mu(Q')$, *then* $\mathcal{D}(Q') \subseteq \mathcal{D}(Q)$.

Proof. If a chain in Q' is extendable in Q, then it is also extendable in Q'. Thus $\mathcal{N}(Q') \subseteq \mathcal{N}(Q)$ and the result follows. □

Since $\mu(Q) \leq |Q|$, an immediate consequence of Lemma 1.3(i) is the following.

Corollary 1.4. $\mu(Q) \leq \text{cf}(Q)$.

It should be remarked that, in general, there may be strict inequality in (i) of Lemma 1.3 as the following example shows. Let $Q = \{a_{n\alpha} : n < \omega, \alpha < \omega_1\}$ and suppose that $a_{n\alpha} < a_{m\beta}$ holds if and only if either (i) $n = m$ and $\alpha < \beta$ or (ii) $n < m$ and $\alpha = 0$. The set $Q' = \{a_{n\alpha} : 0 < n < \omega, \alpha < \omega_1\}$ is cofinal in Q but $\mu(Q) = \omega < \mu(Q') = \omega_1$. However, for the case we shall be considering in this paper, when $(Q, \leq)$ is a directed set there is equality in (i) (see Lemma 2.3).

2. Directed sets

The quasi-ordered set Q is *directed* (directed upwards) if for every pair of elements $a, b \in Q$ there is some $c \in Q$ such that $a \leq c$ and $b \leq c$. Note that for directed sets, a maximal element is also a greatest element. The main purpose of this paper is to describe the minimal cofinal subsets of a directed set, and from now on we always assume that $(Q, \leq)$ is a directed quasi-ordered set.

Lemma 2.1. *If Q' is a cofinal subset of Q, then $(Q', \leq)$ is also directed.*

Proof. If $a, b \in Q'$ then there is $c \in Q$ such that $a \leqslant c$, $b \leqslant c$, and there is $c' \in Q'$ so that $c \leqslant c'$. □

Lemma 2.2. *If $X \subseteq Q$ and $|X| < \mu(Q)$, then there is $a \in Q$ such that $X < \{a\}$.*

Proof. If $X = \emptyset$, the empty set, this is obvious. Suppose $X = \{x_\alpha : \alpha < \lambda\}$, where $\lambda = |X| < \mu(Q)$. Construct a chain $A = \{a_\alpha : \alpha < \lambda\}_<$ in Q as follows. Put $a_0 = x_0$. Let $\alpha < \lambda$ and suppose that a_β has been chosen for $\beta < \alpha$ so that $A_\alpha = \{a_\beta : \beta < \alpha\}_<$ in an increasing chain. Since A_α is a chain in Q of order type $\alpha < \mu(Q)$, it follows that A_α is extendable. Hence there is $a'_\alpha \in Q$ such that $a_\beta < a'_\alpha$ holds for all $\beta < \alpha$. Since Q is directed, there is $a_\alpha \in Q$ such that $a'_\alpha \leqslant a_\alpha$ and $x_\alpha \leqslant a_\alpha$. This defines a chain $A = \{a_\alpha : \alpha < \lambda\}$ in Q of type λ such that $x_\alpha \leqslant a_\alpha$ holds for all $\alpha < \lambda$. Again, since $\lambda < \mu(Q)$, A is extendable and there is $a \in Q$ such that $X \leqslant A < \{a\}$. □

For directed sets we can strengthen Lemma 1.3 as follows.

Lemma 2.3. *If Q' is cofinal in $(Q, \leqslant)$, then* (i) *$\mu(Q') = \mu(Q)$ and* (ii) *$\mathscr{D}(Q')$ is cofinal in $\mathscr{D}(Q)$.*

Proof. If $\mu(Q) = 1$, then Q has a greater element and the result is obvious. Suppose $\mu(Q) > 1$. Let $A = \{a_\alpha : \alpha < \mu(Q)\}_< \in \mathscr{D}(Q)$. We define elements $b_\alpha \in Q'$ for $\alpha < \mu(Q)$ by transfinite induction as follows. Let $\alpha < \mu(Q)$ and suppose we have already chosen b_β for $\beta < \alpha$. By Lemma 2.2 there is $a \in Q$ such that $a_\beta < a$ and $b_\beta < a$ holds for all $\beta < \alpha$. Since Q' is cofinal, there is $b_\alpha \in Q'$ such that $a \leqslant b_\alpha$. This defines a chain $B = \{b_\alpha : \alpha < \mu(Q)\}_<$ in Q' which is non-extendable since $A < B$ and A is non-extendable. This shows that $\mu(Q') \leqslant \mu(Q)$ and (i) now follows from Lemma 1.3(i). By Lemma 1.3(ii) we have $\mathscr{D}(Q') \subseteq \mathscr{D}(Q)$. Also, since A was an arbitrary element of $\mathscr{D}(Q)$ and the chain B constructed above belongs to $\mathscr{D}(Q')$, it follows that $\mathscr{D}(Q')$ is a cofinal subset of $\mathscr{D}(Q)$. □

Lemma 2.4. *For any element $a \in Q$, there is $A \in \mathscr{D}(Q)$ such that $\{a\} \leqslant A$.*

Proof. Since Q is directed, the set $Q' = \{x \in Q : a \leqslant x\}$ is cofinal in Q. Therefore, by Lemma 2.3(ii), $\mathscr{D}(Q')$ is cofinal in $\mathscr{D}(Q)$, and the result holds for any $A \in \mathscr{D}(Q')$. □

The following simple theorem is of basic importance for the results obtained in this paper.

Theorem 2.5. *If $(Q, \leqslant)$ is directed, then so also is $(\mathscr{D}(Q), \leqslant)$.*

Proof. We have to verify that if $A=\{a_\alpha:\alpha<\mu\}_<$ and $B=\{b_\alpha:\alpha<\mu\}_<$ ($\mu=\mu(Q)$) are members of $\mathscr{D}(Q)$, then there is $C\in\mathscr{D}(Q)$ such that $A\leqslant C$ and $B\leqslant C$.

The argument is similar to that used in the proof of Lemma 2.3. Let $\alpha<\mu$ and suppose that c_β has been defined for $\beta<\alpha$. By Lemma 2.2 there is $d_\alpha\in Q$ such that $c_\beta<d_\alpha$ holds for all $\beta<\alpha$. Since Q is directed, there is $c_\alpha\in Q$ such that $a_\alpha\leqslant c_\alpha$, $b_\alpha\leqslant c_\alpha$ and $d_\alpha\leqslant c_\alpha$. This defines the chain $c=\{c_\alpha:\alpha<\mu\}_<$ with the required properties. □

Theorem 2.5 shows that $\mathscr{D}$ may be regarded as an operator on directed sets and by iteration we define the directed sets $\mathscr{D}^n(Q)$ for $n<\omega$ so that

$$\mathscr{D}^0(Q)=Q,\qquad \mathscr{D}^{n+1}(Q)=\mathscr{D}(\mathscr{D}^n(Q)).$$

We also define the corresponding cardinal numbers

$$\mu_{n+1}(Q)=\mu(\mathscr{D}^n(Q)).$$

Thus $\mu_1(Q)=\mu(Q)$, $\mu_2(Q)=\mu(\mathscr{D}(Q))$ etc.

3. Conditions for a directed set to have a cofinal chain

We have already remarked that in a directed set $(Q,\leqslant)$ a maximal element is also a greatest element. Thus the statements:

(i) *Q has a greatest element,*
(ii) $\mathrm{cf}(Q)=1$,
(iii) $\mu_1(Q)=1$,

are all equivalent. The theorem proved in this section gives similar necessary and sufficient conditions for a directed set to have a cofinal chain.

Theorem 3.1. *For a directed set $(Q,\leqslant)$ the following statements are equivalent*:

(1) *Q has a cofinal chain.*
(2) $\mu_2(Q)=1$.
(3) *$\mathscr{D}(Q)$ has a maximal member.*
(4) $\mu_2(Q)\leqslant\mu_1(Q)$.
(5) $\mu_1(Q)=\mathrm{cf}(Q)$.

Proof. The equivalence of (2) and (3) is obvious. Suppose (1) holds and that B is a cofinal chain in Q. There is a subchain $B'\subseteq B$ which is cofinal in B and with order type $\mathrm{cf}(B)$. Clearly B' is non-extendable in Q and $Q\leqslant B'$. Thus, for any $A\in\mathscr{D}(Q)$, we have by Lemma 1.2, $\mu_1(Q)=\mathrm{cf}(A)=\mathrm{cf}(B')=\mathrm{tp}(B')$, and so $B'\in\mathscr{D}(Q)$. Clearly B' is a maximal member of $\mathscr{D}(Q)$ and (3) holds.

Conversely, if $\mathscr{D}(Q)$ has a maximal member, say X, then $A\leqslant X$ for every $A\in\mathscr{D}(Q)$ (since $\mathscr{D}(Q)$ is directed). By Lemma 2.4, for each $a\in Q$ there is

$A \in \mathscr{D}(Q)$ such that $\{a\} \leqslant A$ and hence $Q \leqslant X$. This proves the equivalence of (1), (2) and (3).

Clearly (2) implies (4). We now show that (4) implies (2).

Suppose (4) holds and that (2) is false. Then $\mu_2 = \mu_2(Q)$ is an infinite regular cardinal number, and since $\mu_2 \leqslant \mu_1$, μ_1 is also an infinite regular cardinal. Let $B = \{A_\nu : \nu < \mu_2\}_< \in \mathscr{D}_2(Q)$, where $A_\nu = \{a_{\rho\nu} : \rho < \mu_1\}_<$ $(\nu < \mu_2)$. By transfinite induction (using Lemma 2.2 and the assumption that $\mu_2 \leqslant \mu_1$) choose elements $a_\alpha \in Q (\alpha < \mu_1)$ so that

$$\{a_\beta : \beta < \alpha\} \cup \{a_{\rho\nu} : \rho < \alpha,\ \nu < \min\{\alpha, \mu_2\}\} < \{a_\alpha\}.$$

This defines a chain $C = \{a_\alpha : \alpha < \mu_1\}_<$ in Q so that $A_\nu \leqslant C$ holds for each $\nu < \mu_2$. Clearly C is non-extendable in Q (since each A_ν is non-extendable) and so $C \in \mathscr{D}_1(Q)$. We now have a contradiction since $A_\nu < A_{\nu+1} \leqslant C$ $(\nu < \mu_2)$ and so B is extendable in $\mathscr{D}_1(Q)$. Hence (1), (2), (3) and (4) are equivalent.

We next show that (1) implies (5). Suppose Q contains a cofinal chain C. Let C' be a cofinal sub-chain of C having order type $\mathrm{cf}(C)$. Then $Q \leqslant C'$ and so, by Lemma 1.2 $\mathrm{cf}(C) = |C'| = \mu_1(Q)$. This shows that $\mathrm{cf}(Q) \leqslant |C'| = \mu_1(Q)$. However, $\mu_1(Q) \leqslant \mathrm{cf}(Q)$ by Corollary 1.4 and therefore (5) holds.

Finally we show that (5) implies (1). Let $Q' = \{q_\alpha : \alpha < \mathrm{cf}(Q)\}$ be a cofinal subset of Q having cardinality $|Q'| = \mathrm{cf}(Q)$. We choose elements $a_\alpha \in Q$ for $\alpha < \mathrm{cf}(Q)$ so that

$$\bigcup_{\beta \leqslant \alpha} \{q_\beta\} \cup \bigcup_{\beta < \alpha} \{a_\beta\} < \{a_\alpha\}$$

This is possible by Lemma 2.2 and the assumption that $\mu_1 = \mathrm{cf}(Q)$. Thus $C = \{a_\alpha : \alpha < \mathrm{cf}(Q)\}_<$ is a chain in Q and $C \geqslant Q$ since, by the construction, $C \geqslant Q'$ and $Q' \geqslant Q$. $\square$

From the equivalence of (2) and (4) in Theorem 3.1, it follows that either $\mu_2(Q) = 1$ or $\mu_2(Q) > \mu_1(Q)$. Thus, writing $\mu_n = \mu_n(Q)$ $(n < \omega)$, it follows that EITHER (a) there is an integer k so that $\mu_1 < \mu_2 < \cdots < \mu_k$ and $1 = \mu_{k+1} = \mu_{k+2} = \cdots$, OR (b) $\mu_1 < \mu_2 < \mu_3 < \cdots$. We discuss case (a) in the next section and case (b) in Section 5.

4. n-th class chains

We have already defined the sets $\mathscr{D}^n = \mathscr{D}^n(Q)$ for $n < \omega$ obtained from the directed set Q by successive application of the directed operator $\mathscr{D}$. We call the elements of $\mathscr{D}^n$ the n-th *class chains* of Q. Thus, a 0-th class chain is simply an element of Q, a 1st-class chain is a non-extendable chain of length $\mu_1 = \mu(Q)$ in Q etc. In general, for $n < \omega$, an n-th class chain $A^n \in \mathscr{D}^n$ is a non-extendable chain of length $\mu_n = \mu_n(Q)$ in $\mathscr{D}^{n-1}$, and we write $A^n = \{A^{n-1}(\nu_n) : \nu_n < \mu_n\}_<$. If $n > 1$, each $A^{n-1}(\nu_n)$ for $\nu_n < \mu_n$ is a non-extendable chain $A^{n-1}(\nu_n) = \{A^{n-2}(\nu_{n-1}, \nu_n): \nu_{n-1} < \mu_{n-1}\}_<$ of length μ_{n-1} in $\mathcal{D}^{n-2}$. More generally, for $0 \leqslant i \leqslant n$ and fixed $\nu_j < \mu_j$ $(i < j \leqslant n)$, the corresponding i-th class chain derived from A^n is $A^i(\nu_{i+1}, \ldots, \nu_n)$. Thus $A^0(\nu_1, \ldots, \nu_n) \in Q$ and if $0 < i \leqslant n$, then

$$A^i(\nu_{i+1}, \ldots, \nu_n) = \{A^{i-1}(\nu_i, \nu_{i+1}, \ldots, \nu_n) : \nu_i < \mu_i\}_<$$

It is convenient to define a function U_i (more properly U_i^n, but we suppress the superfix) from $\mathcal{D}^n$ to the power set $\mathcal{P}(\mathcal{D}^i)$ for $i \leqslant n$ as follows. Let $A^n \in \mathcal{D}^n$. We define $U_n(A^n) = \{A^n\}$, and for $0 \leqslant i < n$,

$$U_i(A^n) = \bigcup U_{i+1}(A^n).$$

Thus

$$\begin{aligned} U_{n-1}(A^n) &= \bigcup \{A^n\} = A^n = \{A^{n-1}(\nu_n): \nu_n < \mu_n\}, \\ U_{n-2}(A^n) &= \bigcup \{A^{n-1}(\nu_n): \nu_n < \mu_n\} \\ &= \{A^{n-2}(\nu_{n-1}, \nu_n): \nu_{n-1} < \mu_{n-1}, \nu_{n-2} < \mu_n\} \quad \text{etc.} \end{aligned}$$

We call $U_i(A^n)$ the *i-th class original* set of A^n. In particular, $U_0(A^n) = \{A^0(\nu_1, \ldots, \nu_n): \nu_j < \mu_j\}$, the 0-th class original set of A^n, consists of all the elements of Q which appear in the chains of chains of chains etc. which make up A^n. The mapping

$$(\nu_1, \ldots, \nu_n) \mapsto A^0(\nu_1, \ldots, \nu_n) \tag{4.1}$$

is not injective in general since we may have $A^0(\nu_1, \ldots, \nu_n) = A^0(\nu_1', \ldots, \nu_n')$ for different sequences $(\nu_1, \ldots, \nu_n)$ and $(\nu_1', \ldots, \nu_n')$. In the case when (4.1) is an injective mapping we say that A^n is a *pure n-th class chain of Q*. In general we cannot specify precisely the order relations between different elements of the 0-*th* class original set $U_0(A^n)$ of A^n, but we do have the global relations that

$$A^i(\nu_{i+1}, \ldots, \nu_n) \leqslant A^i(\nu_{i+1}', \nu_{i+2}, \ldots, \nu_n)$$

holds in $\mathcal{D}^i(Q)$ if $\nu_{i+1} \leqslant \nu_{i+1}' < \mu_{i+1}$ and $\nu_j < \mu_j$ $(i+1 < j \leqslant n)$.

Lemma 4.1. *If $\mu_n \neq 1$ and $A^n \in \mathcal{D}^n(Q)$, then $|U_0(A^n)| \leqslant \mu_n$ and there is equality if A^n is pure.*

Proof. $|U_0(A^n)| \leqslant |\{(\nu_1, \ldots, \nu_n) : \nu_i \leqslant \mu_n\}| = \mu_1\mu_2 \cdots \mu_n = \mu_n$ and there is equality if A^n is pure. $\square$

Lemma 4.2. *If $A^n, B^n \in \mathcal{D}^n(Q)$, then $A^n \leqslant B^n$ holds in $\mathcal{D}^n(Q)$ if and only if $U_0(A^n) \leqslant U_0(B^n)$ holds in $\mathcal{P}(Q)$.*

Proof. We prove this by induction or n. For $n = 0$ the result is clear. Suppose $n > 0$.

If $\mu_n = 1$, then $\mathcal{D}^{n-1}$ has maximal elements, and any n-th class chain $A^n \in \mathcal{D}^n$ has the form $A^n = \{A^{n-1}\}$, where A^{n-1} is a maximal member of $\mathcal{D}^{n-1}$. Thus, if

$A^n = \{A^{n-1}\}$ and $B^n = \{B^{n-1}\}$ belong to $\mathscr{D}^n$, then $A^n \approx B^n$ and

$$U_0(A^n) = U_0(A^{n-1}) \approx U_0(B^{n-1}) = U_0(B^n)$$

since $A^{n-1} \approx B^{n-1}$ in $\mathscr{D}^{n-1}$. Therefore, we may assume $\mu_n \neq 1$.

If $A^n \leqslant B^n$ and $a \in U_0(A^n)$, then there is some chain $C^1 \in U_1(A^n)$ such that $a \in C^1$. By the induction hypothesis, since A^n, $B^n \in \mathscr{D}^{n-1}(\mathscr{D}(Q))$ and $A^n \leqslant B^n$, it follows that $U_1(A^n) \leqslant U_1(B^n)$ in $\mathscr{D}(Q)$. Hence there is some $D^1 \in U_1(B^n)$ such that $C^1 \leqslant D^1$ and so there is $b \in D^1 \subseteq U_0(B^n)$ such that $a \leqslant b$. This shows that $U_0(A^n) \leqslant U_0(B^n)$.

Now suppose that $U_0(A^n) \leqslant U_0(B^n)$. Consider any term $A^{n-1}(\nu_n) \in A^n$, where $\nu_n < \mu_n$. For any element $x \in U_0(A^{n-1}(\nu_n)) \subseteq U_0(A^n)$, there is some $y(x) \in U_0(B^n)$ such that $x \leqslant y(x)$, and there is $\rho_n(x) < \mu_n$ such that $y(x) \in U_0(B^{n-1}(\rho_n(x)))$. By Lemma 4.1, $|U_0(A^{n-1}(\nu_n))| \leqslant \mu_{n-1} < u_n$. Therefore, since μ_n is a regular cardinal, there is $\lambda_n < \mu_n$ such that $\rho_n(x) < \lambda_n$ holds for every $x \in U_0(A^{n-1}(\nu_n))$. Since $B^{n-1}(\rho_n(x)) \leqslant B^{n-1}(\lambda_n)$ $(x \in U_0(A^{n-1}(\nu_n)))$ it follows from the induction hypothesis that $U_0(B^{n-1}(\rho_n(x))) \leqslant U_0(B^{n-1}(\lambda_n))$ and since $x \leqslant y(x) \in U_0(B^{n-1}(\rho_n(x)))$, it follows that $U_0(A^{n-1}(\nu_n)) \leqslant U_0(B^{n-1}(\lambda_n))$. By the induction hypothesis again, this implies that $A^{n-1}(\nu_n) \leqslant B^{n-1}(\lambda_n)$ in $\mathscr{D}^{n-1}(Q)$. Since ν_n was an arbitrary ordinal less than μ_n, it follows that $A^n \leqslant B^n$ in $\mathscr{D}^n(Q)$. □

Lemma 4.3. *If $A^n \in \mathscr{D}^n(Q)$, then $U_0(A^n)$ is cofinal in Q if and only if A^n is a maximal element of $\mathscr{D}^n(Q)$.*

Proof. Suppose A^n is maximal in $\mathscr{D}^n$. Let $B^0 \in Q$ be an arbitrary element. By Lemma 2.4 there are $B^i \in \mathscr{D}^i(Q)$ such that $\{B^{i-1}\} \leqslant B^i$ for $1 \leqslant i \leqslant n$. Since $A^n \geqslant B^n$ it follows from Lemma 4.2 that $U_0(A^n) \geqslant U_0(B^n) \geqslant \{B^0\}$. Therefore, $U_0(A^n)$ is cofinal in Q.

Conversely, suppose $U_0(A^n)$ is cofinal in Q. Thus $U_0(B^n) \leqslant U_0(A^n)$ for any $B^n \in \mathscr{D}^n(Q)$ and therefore, by Lemma 4.2, $B^n \leqslant A^n$, i.e. A^n is maximal in $\mathscr{D}^n(Q)$. □

Lemma 4.4. *If $1 \leqslant n < \omega$, $\mu_n \neq 1$, $B^n \in \mathscr{D}^n(Q)$ and A^{n-1} is a pure $(n-1)$-class chain, then there is a pure $A^n \in \mathscr{D}^n(Q)$ such that $A^n(0) = A^{n-1}$ and $A^n \geqslant B^n$.*

Proof. Suppose first that $n = 1$. Put $A^0(0) = A^0$. Now let $1 \leqslant \alpha < \mu_1$ and suppose that $A^0(\nu)$ has been chosen for $\nu < \alpha$. By Lemma 2.2 there is an element $a \in Q$ such that $A^0(\nu) < a$ for all $\nu < \alpha$. Since Q is directed there is $A^0(\alpha) \in Q$ such that $A^0(\alpha) \geqslant B^0(\alpha)$ and $A^0(\alpha) \geqslant a$. This defines $A^1 = \{A^0(\alpha) : \alpha < \mu_1\}_< \geqslant B^1$ and so $A^1 \in \mathscr{D}^1$. Of course, any first-class chain is pure.

We now assume that $n > 1$ and use induction on n. Put $A^{n-1}(0) = A^{n-1}$. Let $1 \leqslant \alpha < \mu_n$ and suppose that we have already constructed pure $(n-1)$-th class chains $A^{n-1}(\nu)$ for $\nu < \alpha$ so that (i) $A^{n-1}(\nu) < A^{n-1}(\nu')$ for $\nu < \nu' < \alpha$, (ii) $B^{n-1}(\nu) \leqslant A^{n-1}(\nu)$ for $1 \leqslant \nu < \alpha$ and (iii) the sets $U_0(A^{n-1}(\nu))(\nu < \alpha)$ are pairwise disjoint. By Lemma 2.2 there is $C^{n-1} \in \mathscr{D}^{n-1}$ such that $A^{n-1}(\nu) < C^{n-1}$ for $\nu < \alpha$. Since $\mu_n \neq 1$, $\mathscr{D}^{n-1}$ has no maximal element and so, by Lemma 4.3, $U_0(C^{n-1})$ is not cofinal in Q and there is an element $a \in Q$ such that $\{a\} \not\leq U_0(C^{n-1})$. The set $\check{a} = \{x \in Q : x \geq a\}$ is cofinal in Q and therefore $\mathscr{D}^{n-1}(\check{a})$ is cofinal in $\mathscr{D}^{n-1}(Q)$ by Lemma 2.3. Hence there is $C_1^{n-1} \in \mathscr{D}^{n-1}(\check{a})$ such that $C_1^{n-1} \geq C^{n-1}$ and also so that $C_1^{n-1} \geq B^{n-1}(\alpha)$. By the induction hypothesis, there is a pure $(n-1)$-th class chain $A^{n-1}(\alpha) \in \mathscr{D}^{n-1}(\check{a})$ such that $A^{n-1}(\alpha) \geq C_1^{n-1}$. Clearly (i) and (ii) now hold for $\alpha + 1$ and so also does (iii) since all elements of $U_0(A^{n-1}(\alpha))$ are above a. The n-th class chain $A^n = \{A^{n-1}(\nu) : \nu < \mu_n\}_<$ constructed in this way has the desired properties. (Note that the hypothesis $\mu_n \neq 1$ is used since, in this cse, $B^{n-1}(0) < B^{n-1}(1)$ and (ii) implies that $A^n \geq B^n$.)

Lemma 4.5. $\mu_n \leq \mathrm{cf}(Q, \leq)$.

Proof. If $\mu_n = 1$ this is clear. Suppose $\mu_n > 1$. Let Q' be any cofinal subset of $(Q, \leq)$ having cardinality $|Q'| = \mathrm{cf}(Q, \leq)$. By Lemma 2.3 we have that $\mathscr{D}^n(Q') \approx \mathscr{D}^n(Q)$. By Lemma 4.4 there is a pure n-th class chain $A^n \in \mathscr{D}^n(Q')$. Therefore, by Lemma 4.1, $\mu_n = |A^n| \leq |Q'| = \mathrm{cf}(Q, \leq)$. □

We now prove the following generalization of Theorem 3.1.

Theorem 4.6. *Let $(Q, \leq)$ be a directed set, n a positive integer. If $\mu_n \neq 1$, then the following statements are equivalent:*

(1) *There is a (pure) n-th class chain A^n whose 0-th class original set $U_0(A^n)$ is cofinal in Q.*

(2) $\mu_{n+1} = 1$.

(3) *$\mathscr{D}^n(Q)$ has a maximal element.*

(4) $\mu_{n+1} \leq \mu_n$.

(5) $\mu_n = \mathrm{cf}(Q)$.

Proof. If there is an n-th class chain A^n such that $U_0(A^n)$ is cofinal in Q, then there is also a pure n-th class chain with the same property by Lemmas 4.2 and 4.4. Thus the truth of (1) does not depend under whether the word 'pure' is included or not.

The equivalence of (2) and (3) is clear, (1) implies (3) by Lemma 4.3, and (3) implies (1) by Lemmas 4.2, 4.3 and 4.4. The equivalence of (2) and (4) follows from Theorem 3.1. (1) implies (5) by Lemmas 4.1 and 4.5, and (5) implies (4) by Lemma 4.5. □

5. ω-th and (ω+1)-th class chains

Suppose now that $\mu_n = \mu_n(Q) \neq 1$ for all $n \geqslant 1$. Then the sequence of cardinals $\mu_1, \mu_2, \ldots$ is strictly increasing and we define

$$\mu_\omega = \mu_\omega(Q) = \lim_{n<\omega} \mu_n(Q).$$

Also we define $\mathscr{D}^\omega(Q)$ to be the set of all sets the form $A^\omega = \{A^1, A^2, \ldots, A^n, \ldots\}$, where $A^n \in \mathscr{D}^n(Q)$ and, moreover, $A^n = A^{n+1}(0)$, i.e. the n-th class chain A^n is the first term of the $(n+1)$-th class chain A^{n+1} (which is a chain of length μ_n in $\mathscr{D}^n(Q)$). We call the members of $\mathscr{D}^\omega(Q)$, the ω-th *class chains* of Q. The 0-th class original sets of the terms in the sequence A^ω,

$$U_0(A^1), \quad U_0(A^2), \ldots$$

form an increasing sequence of subsets of Q and we define the 0-th *class original set* of A^ω to be

$$U_0(A^\omega) = \bigcup_{1 \leqslant n < \omega} U_0(A^n).$$

Also, for $0 < i < \omega$, we define the *i-th class original set* of A^ω to be

$$U_i(A^\omega) = \bigcup_{i \leqslant n < \omega} U_i(A^n)$$

which is a subset of $\mathscr{D}^i(Q)$.

Corresponding to each sequence ending in a string of zeros of the form $(\nu_1, \nu_2, \ldots, \nu_n, 0, 0, \ldots)$, where $n < \omega$, $\nu_i < \mu_i$ $(i \leqslant n)$, there is a unique element

$$A^0(\nu_1, \ldots, \nu_n, 0, 0, \ldots) = A^0(\nu_1, \ldots, \nu_n)$$

of the 0-th class original set of A^ω. This notation is unambiguous in view of the fact, for example, that

$$A^0(\nu_1, \ldots, \nu_n) = A^0(\nu_1, \ldots, \nu_n, 0)$$

since A^n is the same as $A^{n+1}(0)$. In general, different sequences $(\nu_1, \ldots, \nu_n, 0, 0, \ldots)$, $(\nu'_1, \nu'_2, \ldots, \nu'_n, 0, 0, \ldots)$ may give rise to the same element $A^0(\nu_1, \ldots, \nu_n, 0, 0, \ldots) = A^0(\nu'_1, \ldots, \nu'_n, 0, 0, \ldots)$ in $U_0(A^\omega)$. We say that $A^\omega = \{A^1, A^2, \ldots\} \in \mathscr{D}^\omega(Q)$ is *pure* if each A^n is pure, and in this case the above correspondence is a bijection. By analogy, we can introduce a similar coordination for the i-th class chains in the i-th class original set $U_i(A^\omega)$ of A^ω. Thus, if $\nu_j < \mu_j$ for $i < j < \omega$ and if only finitely many of the ν_j are non-zero, then

$$A^i(\nu_{i+1}, \nu_{i+2}, \ldots) = A^i(\nu_{i+1}, \nu_{i+2}, \ldots, \nu_l),$$

where $\nu_j = 0$ for $j > l$.

In general, we cannot describe the order relations in Q between the different elements of $U_0(A^\omega)$, but since $A^{i+1}(\nu_{i+2}, \ldots, \nu_l)$ is an increasing chain of length

μ_{i+1} in $\mathscr{D}^i$, it follows that the inequality

$$A^i(\nu_{i+1}, \nu_{i+2}, \dots, \nu_l) \leq A^i(\nu'_{i+1}, \nu_{i+2}, \dots, \nu_l)$$

holds in $\mathscr{D}^i$ if $\nu_{i+1} \leq \nu'_{i+1} < \mu_{i+1}$.

We define an ordering on $\mathscr{D}^\omega(Q)$ by writing $A^\omega \leq B^\omega$ if and only if $U_0(A^\omega) \leq U_0(B^\omega)$ holds in $\mathscr{P}(Q)$.

Theorem 5.1. *$(\mathscr{D}^\omega(Q), \leq)$ is a directed set.*

Proof. Clearly $\leq$ is a quasi-ordering of $\mathscr{D}^\omega(Q)$. Let $A^\omega = \{A^1, A^2, \dots, A^n, \dots\}$, $B^\omega = \{B^1, B^2, \dots, B^n, \dots\} \in \mathscr{D}^\omega(Q)$. Since $\mathscr{D}^i$ is directed, there is $C^i \in \mathscr{D}^i$ such that $A^i \leq C^i$ and $B^i \leq C^i$ in $\mathscr{D}^i$. Moreover, if $i > 0$ and C^{i-1} has already been chosen, then by Lemma 4.4, we can choose the i-th class chain C^i so that $C^i(0) = C^{i-1}$. Therefore, $C^\omega = \{C^1, C^2, \dots, C^n, \dots\} \in \mathscr{D}^\omega(Q)$. Moreover, by Lemma 4.2, $A^\omega \leq C^\omega$ and $B^\omega \leq C^\omega$. □

The cardinal number $\mu_\omega(Q) = \lim_{n<\omega} \mu_n$ is singular and has cofinality $\mathrm{cf}(\mu_\omega) = \omega$. Since each $\mu_n < \mathrm{cf}(Q)$ we have

Lemma 5.2. $\mu_\omega(Q) \leq \mathrm{cf}(Q)$.

Corresponding to Lemma 4.1 we also have

Lemma 5.3. *If $A^\omega \in \mathscr{D}^\omega(Q)$, then $|U_0(A^\omega)| \leq \mu_\omega$ and there is equality if A^ω is pure.*

Proof. This is obvious from the fact that the set $\bigcup_{1 \leq n < \omega} \{(\nu_1, \dots, \nu_n, 0, 0, \dots) : \nu_i < \mu_i\}$ has cardinality $\mu_1 + \mu_2 + \cdots = \mu_\omega$. □

From Lemma 2.3 and Lemma 4.4 we easily obtain the following.

Lemma 5.4. *If Q' is a cofinal subset of Q, then $\mu_\omega(Q') = \mu_\omega(Q)$ and $\mathscr{D}^\omega(Q')$ is a cofinal subset of $\mathscr{D}^\omega(Q)$.*

Also by Lemma 4.4, we have

Lemma 5.5. *If $A^\omega \in \mathscr{D}^\omega(Q)$, then there is a pure ω-th class chain B^ω such that $A^\omega \leq B^\omega$.*

Since $\mathscr{D}^\omega(Q)$ is directed by Theorem 5.1, we may define $\mu_{\omega+1} = \mu(\mathscr{D}^\omega(Q))$ and $\mathscr{D}^{\omega+1}(Q) = \mathscr{D}(\mathscr{D}^\omega(Q))$. An element $A^{\omega+1} = \{A^\omega(\nu) : \nu < \mu_{\omega+1}\}_<$ is *pure* if each $A^\omega(\nu)$ is pure and if the 0th-class original sets $U_0(A^\omega(\nu))$ $(\nu < \mu_{\omega+1})$ are pairwise disjoint. Clearly $\mu_{\omega+1}$ is either 1, when $\mathscr{D}^\omega(Q)$ has a maximal element, or an infinite regular cardinal if there is no maximal element.

We now prove the following analogue of Theorem 4.6.

Theorem 5.6. *If $(Q, \leqslant)$ is a directed set, and $\mu_\omega \neq 1$, then the following statements are equivalent.*

(1) *There is a (pure) ω-th class chain $A^\omega \in \mathscr{D}^\omega(Q)$ such that $U_0(A^\omega)$ is cofinal in Q.*

(2) $\mu_{\omega+1} = 1$.

(3) *$\mathscr{D}^\omega(Q)$ has a maximal element.*

(4) $\mu_{\omega+1} \leqslant \mu_\omega$.

(5) $\mu_\omega = \mathrm{cf}(Q)$.

Proof. As in the proof of Theorem 4.6, it does not matter if the word 'pure' in (1) is included or not (use Lemma 5.5). The equivalence of (2) and (3) is clear. Suppose (3) holds. Let $B^0 \in Q$. Then by 5.4 we can construct $B^\omega = \{B^1, B^2, \ldots, B_n, \ldots\} \in \mathscr{D}^\omega(Q)$, so that B^0 is the first element of B^1. If A^ω is a maximal element of $\mathscr{D}^\omega(Q)$, then $U_0(B^\omega) \leqslant U_0(A^\omega)$ so that $U_0(A^\omega)$ is cofinal in Q and (1) holds. Obviously (1) implies (3) for if $U_0(A^\omega)$ is cofinal in Q, then $B^\omega \leqslant A^\omega$ for any $B^\omega \in \mathscr{D}^\omega(Q)$. Therefore (1), (2) and (3) are equivalent.

Obviously (2) implies (4). We now prove the converse. Suppose (4) holds. Let $A^{\omega+1} = \{A^\omega(\nu) : \nu < \mu_{\omega+1}\}_<$ be a non-extendable chain in $\mathscr{D}^\omega(Q)$. We will show that there is a pure $C^\omega \in \mathscr{D}^\omega(Q)$ such that $A^\omega(\nu) \leqslant C^\omega$ holds for all $\nu < \mu_{\omega+1}$. Since the terms of A^ω are strictly increasing, this implies that either $A^{\omega+1}$ is extendable in $\mathscr{D}^\omega(Q)$ (a contradiction), or that $\mu_{\omega+1} = 1$ (as required). Since $\mu_n < \mu_{n+1}$ for $0 < n < \omega$, it follows from Lemma 2.2 that there is $B^n \in \mathscr{D}^n(Q)$ such that[1] $A^n(\nu) < B^n$ holds for all $\nu < \min\{\mu_n, \mu_{\omega+1}\}$ choose $C^0 \in Q$ arbitrarily. Let $n > 0$ and suppose that the pure $(n-1)$-th class chain $C^{n-1} \in \mathscr{D}^{n-1}$ has already been chosen. Then by Lemma 4.4 there is a pure n-th class chain $C^n \geqslant B^n$ such that $C^n(0) = C^{n-1}$. This defines $C^\omega = \{C^1, C^2, \ldots\} \in \mathscr{D}^\omega(Q)$. By Lemma 4.2, we have that $U_0(C^n) \geqslant U_0(B^n) \geqslant U_0(A^n(\nu))$ for $\nu < \min\{\mu_n, \mu_{\omega+1}\}$. Therefore, $U_0(C^\omega) \geqslant U_0(A^\omega(\nu))$ for each $\nu < \mu_{\omega+1}$ ($\leqslant \mu_\omega$). This proves the equivalence of (1), (2), (3), and (4).

By Lemma 5.2 we know that $\mu_\omega \leqslant \mathrm{cf}(Q)$, and if (1) holds then, by Lemma 5.3, $\mathrm{cf}(Q) \leqslant \mu_\omega$. Thus (1) implies (5). We conclude the proof by showing that (5) implies (4).

Suppose on the contrary that $\mu_\omega = \mathrm{cf}(Q) < \mu_{\omega+1}$. Let Q' be a cofinal subset of Q having cardinality $|Q'| = \mu_\omega$. We shall obtain a contradiction by constructing a chain $\{A^\omega(\nu) : \nu < \mu_{\omega+1}\}_<$ of ω-class chains such that the corresponding 0-class original sets $U_0(A^\omega(\nu))$ are pairwise disjoint subsets of Q'. Let $\alpha < \mu_{\omega+1}$ and suppose that we have already chosen $A^\omega(\nu)$ for $\nu < \alpha$. By the definition of $\mu_{\omega+1} = \mu(\mathscr{D}^\omega(Q))$, there is an ω-class chain $B^\omega \in \mathscr{D}^\omega(Q)$ such that $A^\omega(\nu) \leqslant B^\omega$ holds for all $\nu < \alpha$. By the equivalence of (1) and (4) we can assume that (1) is

[1] There is a slight notational problem here. $A^n(\nu)$ refers to the n-th class chain in the set $A^\omega(\nu) = \{A^n(\nu) : 1 \leqslant n < \omega\}$, and not to the ν-th term of some $(n+1)$-th class chain A^{n+1}. However, we believe that the context makes this clear.

false, and so $U_0(B^\omega)$ is not cofinal in Q. Therefore, there is an element $a \in Q$ such that $\{a\} \not\leqslant U_0(B^\omega)$. Since $Q'' = \{x \in Q' : x \geqslant a\}$ is also cofinal in Q, using Lemma 4.4, we may construct an ω-class chain $A^\omega(\alpha) \in \mathscr{D}^\omega(Q'')$ such that $A^\omega(\alpha) \geqslant B^\omega$. Since $U_0(B^\omega) \geqslant U_0(A^\omega(\nu))$ for $\nu < \alpha$ and $U_0(B^\omega) \not\geqslant \{a\}$, and since $U_0(A^\omega(\alpha)) \subseteq Q''$, it follows that $U_0(A^\omega(\alpha))$ is disjoint from $\bigcup_{\nu<\alpha} U_0(A^\omega(\nu))$. In this way we construct $\mu_{\omega+1}$ non-empty, pairwise disjoint subsets $U_0(A^\omega(\alpha))$ $(\alpha < \mu_{\omega+1})$ of Q'. This is a contradiction since $|Q'| = \mu_\omega$. □

6. The general case

We now define by transfinite induction for any ordinal α the directed sets $\mathscr{D}^\alpha = \mathscr{D}^\alpha(Q)$ and the cardinal number $\mu_{1+\alpha} = \mu_{1+\alpha}(Q)$. Put $\mathscr{D}^0 = Q$, $\mathscr{D}^{\beta+1} = \mathscr{D}(\mathscr{D}^\beta)$, and $\mu_{\beta+1} = \mu(\mathscr{D}^\beta)$. If α is a limit ordinal, then we define $\mathscr{D}^\alpha$ to be the set of all sets of the form

$$A^\alpha = \{A^{\nu+1} : \nu < \alpha\},$$

where $A^{\nu+1} = \{A^\nu(\rho) : \rho < \mu_{\nu+1}\}_< \in \mathscr{D}^{\nu+1}$ is a $(\nu+1)$-th class chain $(\nu < \alpha)$, and the elements of A^α satisfy the conditions that

$$A^\nu = A^{\nu+1}(0) \qquad \text{for } \nu < \alpha,$$

and

$$A^\sigma = \{A^{\nu+1} : \nu < \sigma\} \in \mathscr{D}^\sigma \quad \text{for every limit ordinal } \sigma < \alpha.$$

Also, in the case of limit ordinal α, we define

$$\mu_\alpha = \limsup_{\beta<\alpha} \mu_\beta.$$

These definitions are not quite complete since we must also define the order relation on $\mathscr{D}^\alpha$ in the case of limit α in a suitable way to ensure that $\mathscr{D}^\alpha$ is indeed a directed set.

In order to make the induction work we shall also define by transfinite induction on α certain operators $U_\gamma^\alpha : \mathscr{D}^\alpha \to \mathscr{P}(\mathscr{D}^\gamma)$ for $\gamma \leqslant \alpha$ as follows. Put $U_\alpha^\alpha(A^\alpha) = \{A^\alpha\}$. If $\alpha = \beta + 1$ is a successor and $\gamma < \alpha$ we define

$$U_\gamma^\alpha(A^\alpha) = \bigcup \{U_\gamma^\beta(A^\beta(\rho)) : \rho < \mu_{\beta+1}\},$$

where $A^\alpha = \{A^\beta(\rho) : \rho < \mu_{\beta+1}\}_<$. If α is a limit ordinal and $\gamma < \alpha$, put

$$U_\gamma^\alpha(A^\alpha) = \bigcup \{U_\gamma^{\nu+1}(A^{\nu+1}) : \gamma \leqslant \nu + 1 < \alpha\},$$

where $A^\alpha = \{A^{\nu+1} : \nu < \alpha\}$.

For $A^\alpha \in \mathscr{D}^\alpha$ and $\gamma \leqslant \alpha$ we call $U_\gamma^\alpha(A^\alpha)$ the *γ-class original set* of A^α. For a limit ordinal α we define the order on $\mathscr{D}^\alpha$ by setting $A^\alpha \leqslant B^\alpha$ if and only if the corresponding relation

$$U_0^\alpha(A^\alpha) \leqslant U_0^\alpha(B^\alpha)$$

between their 0-class original sets holds in $\mathcal{P}(Q)$.

As a notational convenience we shall, for any ordinals γ, α with $\gamma \leqslant \alpha$, denote by L_γ^α the set of all sequences of the form $\boldsymbol{\nu} = \langle \nu_{\rho+1} : \gamma \leqslant \rho < \alpha \rangle$, where $\nu_{\rho+1} < \mu_{\rho+1}$ and only finitely many of the terms $\nu_{\rho+1}$ are different from zero. Note that, in particular $L_\alpha^\alpha = \{\emptyset\}$, and if $\alpha = \beta + 1$ is a successor and $\gamma < \alpha$, then each sequence in L_γ^α has a last element.

For the purposes of our induction we now assume that $\alpha > 1$ and that the following statements $(1)_\beta - (10)_\beta$ hold for $1 \leqslant \beta < \alpha$ and then we shall verify that these statements also hold when $\beta = \alpha$.

$(1)_\beta$ *$\mathcal{D}^\beta$ is directed.*

$(2)_\beta$ *If $\mu_\beta \neq 1$ and $\gamma < \beta$, then $\mu_\gamma < \mu_\beta$.*

$(3)_\beta$ *If $\gamma \leqslant \beta$ and $A^\beta \in \mathcal{D}^\beta$, then the γ-class original set $U_\gamma^\beta(A^\beta)$ may be coordinated by L_γ^β, i.e.*

$$(3.1)_{\beta,\gamma} \quad U_\gamma^\beta(A^\beta) = \{A^\gamma(\boldsymbol{\nu}) : \boldsymbol{\nu} \in L_\gamma^\beta\},$$

and, moreover, for fixed $\langle \nu_{\gamma+2}, \nu_{\gamma+3}, \ldots \rangle \in L_{\gamma+1}^\beta$, we have

$$(3.2)_{\beta,\gamma} \quad A^{\gamma+1}(\nu_{\gamma+2}, \nu_{\gamma+3}, \ldots) = \{A^\gamma(\nu, \nu_{\gamma+2}, \nu_{\gamma+3}, \ldots) : \nu < \mu_{\gamma+1}\}_<.$$

Note that, in general, the mapping $\boldsymbol{\nu} \mapsto A^0(\boldsymbol{\nu})$ from L_0^β to $U_0^\beta(A^\beta)$ is not one-to-one. If it is one-to-one, then we call A^β a *pure* β-class chain.

$(4)_\beta$ *If $\mu_\beta \neq 1$ and $A^\beta \in \mathcal{D}^\beta$, then $|U_0^\beta(A^\beta)| \leqslant \mu_\beta$ and there is equality if A^β is pure.*

$(5)_\beta$ *If $A^\beta, B^\beta \in \mathcal{D}^\beta$, then $A^\beta \leqslant B^\beta$ holds if and only if $U_0^\beta(A^\beta) \leqslant U_0^\beta(B^\beta)$ holds in $\mathcal{P}(Q)$.*

$(6)_\beta$ *A^β is a maximal element of $\mathcal{D}^\beta$ if and only if $U_0^\beta(A^\beta)$ is a cofinal subset of Q.*

$(7)_\beta$ *If $\gamma < \beta$, $A^\beta \in \mathcal{D}^\beta$, $B^\gamma \in \mathcal{D}^\gamma$ and if B^γ is pure, then there is a pure β-class chain $C^\beta \in \mathcal{D}^\beta$ such that $A^\beta \leqslant C^\beta$ and $B^\beta = C^\beta(0, 0, \ldots)$.*

$(8)_\beta$ *If Q' is a cofinal subset of Q, then $\mu_\beta(Q') = \mu_\beta(Q)$ and $\mathcal{D}^\beta(Q')$ is a cofinal subset of $\mathcal{D}^\beta(Q)$.*

$(9)_\beta$ *$\mu_\beta \leqslant \mathrm{cf}(Q)$.*

$(10)_\beta$ *If $\beta = \gamma + 1$ is a successor and $\mu_\gamma \neq 1$, then the following statements are equivalent:*

(i) $\mu_\beta = 1$.

(ii) *$\mathcal{D}^\gamma$ has a maximal element.*

(iii) $\mu_\beta \leqslant \mu_\gamma$.

(iv) *There is a (pure) γ-class chain $A^\gamma \in \mathcal{D}^\gamma$ such that the* 0-th *class original set of A^γ, $U_0^\gamma(A^\gamma)$, is cofinal in Q.*

(v) $\mu_\gamma = \mathrm{cf}(Q)$.

We now verify that above ten statements hold for $\beta = \alpha$.

Proof of $(1)_\alpha$. If $\alpha=\beta+1$ is a successor, then $(1)_\alpha$ follows from $(1)_\beta$ and Theorem 2.5. Suppose α is a limit ordinal. Let $A^\alpha=\{A^{\rho+1}:\rho<\alpha\}$, $B^\alpha=\{B^{\rho+1}:\rho<\alpha\}$ be any two elements of $\mathscr{D}^\alpha$. By $(1)_1$ there is $C^1\in\mathscr{D}^1$ such that $A^1\leqslant C^1$ and $B^1\leqslant C^1$. Now let $1<\gamma<\alpha$ and suppose that the pure δ-class chain $C^\delta\in\mathscr{D}^\delta$ has already been chosen for $1\leqslant\delta<\gamma$ so that $A^\delta\leqslant C^\delta$ and $B^\delta\leqslant C^\delta$ holds in $\mathscr{D}^\delta$ and, further, so that

$$C^\delta=C^{\delta+1}(0)\quad\text{if } \delta+1<\gamma$$

and

$$C^\delta=\{C^{\nu+1}:\nu<\delta\}\in\mathscr{D}^\delta\quad\text{when } \delta \text{ is a limit ordinal.}$$

If γ is a limit ordinal, then we simply put $C^\gamma=\{C^{\nu+1}:\nu<\gamma\}$. Clearly $C^\gamma\in\mathscr{D}^\gamma$ and the relations $A^\gamma\leqslant C^\gamma$, $B^\gamma\leqslant C^\gamma$ hold because of $(5)_\gamma$. Now suppose $\gamma=\delta+1$ is a successor. If $\mu_\delta=1$, then $A^{\delta+1}=\{A^\delta\}$, $B^{\delta+1}=\{B^\delta\}$ where A^δ, B^δ are maximal elements of $\mathscr{D}^\delta$, and since $A^\delta\leqslant C^\delta$, $B^\delta\leqslant C^\delta$ we can put $C^{\delta+1}=\{C^\delta\}$. So we can assume that $\mu_\delta\neq 1$. Since $\mathscr{D}^\delta$ is directed, the set $\{X^\delta\in\mathscr{D}^\delta:X^\delta\geqslant C^\delta\}$ is cofinal in $\mathscr{D}^\delta$ and so by $(8)_\delta$ and $(8)_{\delta+1}$, there is $C^{\delta+1}=\{C^\delta(\nu):\nu<\mu_\delta\}_<\in\mathscr{D}^{\delta+1}$ so that $A^{\delta+1}\leqslant C^{\delta+1}$, $B^{\delta+1}\leqslant C^{\delta+1}$ and $C^\delta=C^{\delta+1}(0)$. In fact, by $(7)_{\delta+1}$ we can also ensure that $C^{\delta+1}$ is a pure $(\delta+1)$-class chain. This defines $C^\gamma\in\mathscr{D}^\gamma$ for all $\gamma<\alpha$ and it is clear from the construction that $C^\alpha=\{C^{\nu+1}:\nu<\alpha\}\in\mathscr{D}^\alpha$ and $A^\alpha\leqslant C^\alpha$ and $B^\alpha\leqslant C^\alpha$.

Proof of $(2)_\alpha$. In the case when α is a limit ordinal this is obvious. For if $\mu_\alpha\neq 1$, then $\mu_\beta\neq 1$ for all $\beta<\alpha$ and so the cardinals μ_β $(\beta<\alpha)$ are strictly increasing by $(1)_\beta$ and $\mu_\alpha=\sup_{\beta<\alpha}\mu_\beta$. Also, if $\alpha=\gamma+2$, the result is an immediate consequence of Theorem 3.1 since $\mu_{\gamma+1}=\mu_1(\mathscr{D}^\gamma)$ and $\mu_{\gamma+2}=\mu_2(\mathscr{D}^\gamma)$. We may therefore assume that $\alpha=\gamma+1$ and that γ is a limit ordinal. We shall imitate the proof of Theorem 5.6.

Suppose on the contrary that $1\neq\mu_{\gamma+1}\leqslant\mu_\gamma$. Consider any non-extendable chain $A^{\gamma+1}=\{A^\gamma(\nu):\nu<\mu_{\gamma+1}\}_<$ in $\mathscr{D}^\gamma$ of length $\mu_{\gamma+1}$. As in the proof of Theorem 5.6, it will be enough to show that there is some $B^\gamma\in\mathscr{D}^\gamma$ such that $A^\gamma(\nu)\leqslant B^\gamma$ for all $\nu<\mu_{\gamma+1}$ since this will imply the contradiction $\mu_{\gamma+1}=1$. Suppose $A^\gamma(\nu)=\{A^{\rho+1}(\nu):\rho<\gamma\}$ for $\nu<\mu_{\gamma+1}$. Put $B^1=A^1(0)$. Now let $\rho<\gamma$ and suppose that we have already chosen pure $B^\sigma\in\mathscr{D}^\sigma$ for $\sigma<\rho$ so that $B^\sigma=B^{\sigma+1}(0)$ for $\sigma+1<\rho$ and so that $B^\sigma=\{B^{\tau+1}:\tau<\sigma\}$ for limit $\sigma<\rho$. If ρ is a limit we put $B^\rho=\{B^{\sigma+1}:\sigma<\rho\}$. Suppose $\rho=\sigma+1$ is a successor. Since $\mu_{\sigma+1}>\mu_\sigma$, it follows from Lemma 3.2 that there is $C^{\sigma+1}\in\mathscr{D}^{\sigma+1}$ such that

$$A^{\sigma+1}(\nu)\leqslant C^{\sigma+1}\quad\text{for all } \nu<\min\{\mu_\sigma,\mu_{\gamma+1}\}.$$

Also by $(7)_{\sigma+1}$ there is a pure $(\sigma+1)$-class chain $B^{\sigma+1}$ such that $C^{\sigma+1}\leqslant B^{\sigma+1}$ and $B^\sigma=B^{\sigma+1}(0)$. This defines $B^\gamma=\{B^{\sigma+1}:\sigma<\gamma\}\in\mathscr{D}^\gamma$. It is clear from $(5)_{\sigma+1}$ and the construction that

$$U_0(A^{\sigma+1}(\nu))\leqslant U_0(B^{\sigma+1})\leqslant U_0(B^\gamma)$$

holds for $\nu<\min\{\mu_\sigma, \mu_{\gamma+1}\}$. However, since the right side of this inequality does not depend upon σ, we have that $U_0(A^{\sigma+1}(\nu))\leq U_0(B^\gamma)$ holds for all $\nu<\mu_{\gamma+1}$ and so $A^\gamma(\nu)\leq B^\gamma$ for all $\nu<\mu_{\gamma+1}$ and the result follows.

Proof of (3)$_\alpha$. If $\gamma\leq\beta\leq\alpha$ and $\boldsymbol{\nu}=\langle\nu_{\rho+1}:\gamma\leq\rho<\alpha\rangle\in L^\alpha_\gamma$, then $\boldsymbol{\nu}=\boldsymbol{\nu}_1\boldsymbol{\nu}_2$ is the concatenation of $\boldsymbol{\nu}_1=\langle\nu_{\rho+1}:\gamma\leq\rho<\beta\rangle\in L^\beta_\gamma$ and $\boldsymbol{\nu}_2=\langle\nu_{\rho+1}:\beta\leq\rho<\alpha\rangle\in L^\alpha_\beta$.

Let $A^\alpha\in\mathscr{D}^\alpha$. Since $U^\alpha_\alpha(A^\alpha)=\{A^\alpha\}$ and $L^\alpha_\alpha=\{\emptyset\}$ are both singletons, it is clear that $(3.1)_{\alpha\alpha}$ holds if we set $A^\alpha(\emptyset)=A^\alpha$. Assume now that $\gamma<\alpha$.

If $\alpha=\beta+1$, then $A^\alpha=\{A^\beta(\xi):\xi<\mu_\alpha\}_<$ and

$$\begin{aligned}U^\alpha_\gamma(A^\alpha)&=\bigcup\{U^\beta_\gamma(A^\beta(\xi)):\xi<\mu_\alpha\}\\&=\{A^\gamma(\xi)(\boldsymbol{\nu}):\boldsymbol{\nu}\in L^\beta_\gamma,\ \xi<\mu_\alpha\}=\{A^\gamma(\boldsymbol{\nu}'):\boldsymbol{\nu}'\in L^\alpha_\gamma\},\end{aligned}$$

where, for $\boldsymbol{\nu}'=\langle\nu_{\gamma+1},\ldots,\nu_{\beta+1}\rangle\in L^\alpha_\gamma$ we define

$$A^\gamma(\boldsymbol{\nu}')=A^\gamma(\nu_{\beta+1})(\boldsymbol{\nu})\quad\text{and}\quad\boldsymbol{\nu}=\langle\nu_{\rho+1}:\gamma\leq\rho<\beta\rangle\in L^\alpha_\gamma.$$

Thus $(3.1)_{\alpha\gamma}$ holds. Likewise, $(3.2)_{\alpha\gamma}$ holds since, for $\langle\nu_{\gamma+2},\ldots,\nu_{\beta+1}\rangle\in L^\alpha_{\gamma+1}$ we have

$$\begin{aligned}A^{\gamma+1}(\nu_{\gamma+2},\ldots,\nu_{\beta+1})&=A^{\gamma+1}(\nu_{\beta+1})(\nu_{\gamma+2},\ldots)\\&=\{A^\gamma(\nu_{\beta+1})(\nu,\nu_{\gamma+2},\ldots):\nu<\mu_{\gamma+1}\}_<\\&=\{A^\gamma(\nu,\nu_{\gamma+2},\ldots,\nu_{\beta+1}):\nu<\mu_{\gamma+1}\}_<.\end{aligned}$$

If α is a limit, then $A^\alpha=\{A^{\xi+1}:\xi<\alpha\}$, where $A^\xi=A^{\xi+1}(0)$ for $\xi<\alpha$. In this case

$$\begin{aligned}U^\alpha_\gamma(A^\alpha)&=\bigcup\{U^{\xi+1}_\gamma(A^{\xi+1}):\gamma\leq\xi+1<\alpha\}\\&=\{A^\gamma(\boldsymbol{\nu}):\boldsymbol{\nu}\in\bigcup\{L^{\xi+1}_\gamma:\gamma\leq\xi+1<\alpha\}\}.\end{aligned}$$

Now, for any $\boldsymbol{\nu}=\langle\nu_{\gamma+1},\nu_{\gamma+2},\ldots\rangle\in L^\alpha_\gamma$ there is an ordinal $\rho=\rho(\boldsymbol{\nu})$ such that $\nu_{\rho+1}\neq0$ and $\nu_\sigma=0$ for $\rho+1<\sigma<\alpha$. Since $A^{\rho+1}=A^{\rho+2}(0)=A^{\rho+3}(0,0)=\cdots$, it follows that $A^\gamma(\boldsymbol{\nu}\restriction\rho+1)=A^\gamma(\boldsymbol{\nu}\restriction\rho+2)=\cdots$, where $\boldsymbol{\nu}\restriction\delta=\langle\nu_{\sigma+1}:\gamma\leq\sigma<\delta\rangle$. If we identify $A^\gamma(\boldsymbol{\nu})$ with $A(\boldsymbol{\nu}\restriction\rho+1)$, then we see that

$$U^\alpha_\gamma(A^\alpha)=\{A^\alpha(\boldsymbol{\nu}):\boldsymbol{\nu}\in L^\alpha_\gamma\},$$

so that $(3.1)_{\alpha\gamma}$ holds. It is clear that $(3.2)_{\alpha\gamma}$ holds in this case also since, if $\langle\nu_{\gamma+2},\ \nu_{\gamma+3},\ldots\rangle\in L^\alpha_{\gamma+1}$, there is $\beta<\alpha$ such that $A^{\gamma+1}(\nu_{\gamma+2},\ \nu_{\gamma+3},\ldots)=A^{\gamma+1}(\nu_{\gamma+2},\ldots,\nu_{\beta+1})$.

Proof of (4)$_\alpha$. This is an immediate consequence of $(3.1)_{\alpha0}$.

Proof of (5)$_\alpha$. If α is a limit ordinal the result is obvious from the definition of $A^\alpha\leq B^\alpha$. If $\alpha=\beta+1$ is a successor, the proof is almost identical with the proof of Lemma 4.2 and we omit the details.

Proof of (6)$_\alpha$. Suppose α is a limit ordinal. Let A^α be a maximal element of $\mathscr{D}^\alpha$ and $B^0 \in Q$. Then by $(7)_\beta$ we can construct pure β-class chains B^β for $\beta<\alpha$ so that $B^\beta = B^{\beta+1}(0)$. Then $B^\alpha = \{B^{\rho+1} : \rho<\alpha\} \in \mathscr{D}^\alpha$ and, since $A^\alpha \geqslant B^\alpha$, we have $U_0(A^\alpha) \geqslant U_0^\alpha(B^\alpha) \geqslant \{B^0\}$. This shows that $U_0^\alpha(A^\alpha)$ is cofinal in Q. Conversely, if $U_0^\alpha(A^\alpha)$ is cofinal in Q, then $U_0^\alpha(A^\alpha) \geqslant U_0^\alpha(B^\alpha)$ for any $B^\alpha \in \mathscr{D}^\alpha$ and so A^α is maximal in $\mathscr{D}^\alpha$.

If $\alpha=\beta+1$ is a successor, then the proof is similar to the proof of Lemma 4.3.

Proof of (7)$_\alpha$. For $\alpha=\beta+1$ a successor, the proof is similar to the proof of Lemma 4.4. Suppose α is a limit ordinal. Let $A^\alpha \in \mathscr{D}^\alpha$, $B^\gamma \in \mathscr{D}^\gamma$, where $\gamma<\alpha$ and B^γ is pure. For $\varepsilon \leqslant \gamma$, put $C^\varepsilon = B^\varepsilon(\boldsymbol{\nu}(\varepsilon))$ where $\boldsymbol{\nu}(\varepsilon) \in L_\varepsilon^\gamma$ is the vector having all components equal to zero. Now let $\gamma<\delta<\alpha$ and suppose that we have already chosen pure ε-class chains B^ε for $\varepsilon<\delta$ so that $B^\varepsilon = B^{\varepsilon+1}(0)$ for $\varepsilon+1<\delta$. If δ is a limit, put $B^\delta = \langle B^{\rho+1} : \rho<\delta\rangle$. If $\delta=\sigma+1$ is a successor, then by $(7)_\sigma$ there is a pure δ-class chain B^δ such that $B^\delta \geqslant A^\delta$ and $B^\delta(0)=B^\sigma$. This defines by transfinite induction a pure α-class chain $B^\alpha = \langle B^{\rho+1} : \rho<\alpha\rangle$ with the required properties.

Proof of (8)$_\alpha$. Let Q' be a cofinal subset of Q. If $\alpha=\beta+1$ is a successor, then $(8)_\alpha$ follows from $(8)_\beta$ and $(8)_1$. Suppose α is a limit. Then

$$\mu_\alpha(Q') = \limsup_{\beta<\alpha} \mu_\beta(Q') = \limsup_{\beta<\alpha} \mu_\beta(Q) = \mu_\alpha(Q).$$

Also, if $A^\alpha \in \mathscr{D}^\alpha$, then by $(7)_\beta$ and $(8)_\beta$ we can construct pure $C^\beta \in \mathscr{D}^\beta(Q')$ so that $C^\beta = C^{\beta+1}(0)$ and $C^\beta \geqslant A^\beta$ $(\beta<\alpha)$ and then $C^\alpha = \{C^{\rho+1} : \rho<\alpha\} \in \mathscr{D}^\alpha(Q')$ and $C^\alpha \geqslant A^\alpha$.

Proof of (9)$_\alpha$. This is an immediate consequence of $(4)_\alpha$ and $(8)_\alpha$.

Proof of (10)$_\alpha$. Suppose $\alpha=\gamma+1$ is a successor and $\mu_\gamma \neq 1$. The equivalence of (i) and (ii) for $\beta=\alpha$ is obvious and the equivalence of (i) and (iii) and follows from $(2)_\alpha$. Also the equivalence of (ii) and (iv) follows from $(6)_\alpha$. Therefore (i)–(iv) are all equivalent.

If (iv) holds for $\beta=\alpha$, then there is $A^\gamma \in \mathscr{D}^\gamma$ such that $U_0^\gamma(A^\gamma)$ is cofinal in Q and hence, by $(9)_\gamma$ and $(4)_\gamma$,

$$\mu_\gamma \leqslant \mathrm{cf}(Q) \leqslant |U_0^\gamma(A^\gamma)| \leqslant \mu_\gamma$$

and so (v) holds. Conversely, if (v) holds for $\beta=\alpha$, then by $(9)_\alpha$, we have $\mu_\alpha \leqslant \mathrm{cf}(Q) = \mu_\gamma$ and so (iii) holds. This completes the proof of $(10)_\alpha$ and therefore, by induction, $(1)_\beta$–$(10)_\beta$ hold for all β.

To conclude we summarize the results of this section in the following Theorem.

Theorem 6.1. *If $(Q, \leqslant)$ is a directed quasi-ordered set, then there is an ordinal $\lambda=\lambda(Q)$ such that $\mu_{\lambda+1}=1$ and $\mu_1<\mu_2<\cdots<\mu_\lambda=\mathrm{cf}(Q)$. Also, every cofinal* *subset of Q contains a cofinal subset of the form $U_0^\lambda(A^\lambda)$, where A^λ is a pure λ-class chain of Q.*

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