Program Termination and Well Partial Orderings

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The following known observation is useful in establishing program termination: if a transitive relation R is covered by finitely many well-founded relations $U_1, \ldots, U_n$ then R is well-founded. A question arises how to bound the ordinal height $|R|$ of the relation R in terms of the ordinals $\alpha_i = |U_i|$. We introduce the notion of the *stature* $\|P\|$ of a well partial ordering P and show that $|R| \leq \|\alpha_1 \times \cdots \times \alpha_n\|$ and that this bound is tight. The notion of stature is of considerable independent interest. We define $\|P\|$ as the ordinal height of the forest of nonempty bad sequences of P, but it has many other natural and equivalent definitions. In particular, $\|P\|$ is the supremum, and in fact the maximum, of the lengths of linearizations of P. And $\|\alpha_1 \times \cdots \times \alpha_n\|$ is equal to the natural product $\alpha_1 \otimes \cdots \otimes \alpha_n$.

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1. INTRODUCTION

A program π, possibly nondeterministic, is *terminating* if every computation of π from an initial state is finite. If there is a computation of π from state

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x to state y, we say that y is *reachable* from x and write $y \prec x$. A state y is *reachable* if it is reachable from an initial state. In practice termination is often established by means of ranking functions. A *ranking function* for π is an ordinal-valued function f on the reachable states of π such that $f(y) < f(x)$ whenever $y \prec x$. Clearly π is terminating if and only if the reachability relation $\prec$ over reachable states is well-founded if and only if π admits a ranking function. If π is terminating then the smallest ordinal α such that π admits a ranking function with values $< \alpha$ is the *ranking height* of π. The following observation may be helpful in establishing termination.

LEMMA 1 (COVERING OBSERVATION). *Any transitive relation covered by finitely many well-founded relations is well-founded.*

In other words, if relations $U_1, \ldots, U_n$ are well-founded and $R \subseteq U_1 \cup \cdots \cup U_n$ is a transitive relation, then R is well-founded.

Apparently this observation was made independently a number of times, and each time it was related to the termination problem. As far as we know, the observation was made first by Alfons Geser in [Geser 1990, page 31]. A weaker form of the observation, in which the relations U_i are required to be transitive, had been proposed as a question on the Web by Geser, and he informed us that he received proofs of it from Jean-Pierre Jouannaud, Werner Nutt, Franz Baader, George McNulty, Thomas Streicher, and Dieter Hofbauer; see Lescanne [2003], discussion list, items 38–42, for all but the last two of these. Both of our two referees pointed out that the observation was made independently in Lee et al. [2001]. One of them wrote that “the covering observation lies at the heart of” Lee et al. [2001] where it “is used implicitly in Theorem 4.” The other referee pointed out that the covering observation was made independently in Dershowitz et al. [2001] and in Codish et al. [2003]; see Bruynooghe et al. [2006] in this connection. Recently the covering observation was rediscovered in Podelski and Rybalchenko [2004] and was used for proving termination in Podelski and Rybalchenko [2005], Cook et al. [2006], and Berdine et al. [2007]. A stronger version of the covering observation, using a hypothesis that is weaker (but more complicated) than transitivity of R, was given in Doornbos and von Karger [1998].

The covering observation is proved by a straightforward application of the infinite version of Ramsey's theorem. The transitivity of R is essential here. If a, b are distinct elements then the relation $\{(a, b), (b, a)\}$ is covered by the well-founded relations $\{(a, b)\}$ and $\{(b, a)\}$ but is not well-founded.

Example 2. Let π_1 be the program

```
while a ≤ 1000 ≤ b
   choose between
      a,b := a+2, b+1
      a,b := a-1, b-1
```

with integer variables a, b. Initially, a and b could be any integers. Since π_1's reachability relation $y \prec x$ is covered by well-founded relations $a_x < a_y \leq 1002$ and $999 \leq b_y < b_x$, π_1 terminates. Obviously the ranking height of π_1 isn't finite. In fact it is ω, the least infinite ordinal, because the function $|3b-2a|$ is a ranking function for π_1. (The absolute value is used to guarantee that all values of the function are natural numbers.) Let π_2 be the following modification of π_1:

```
while a ≤ 1000 ≤ b
   choose between
      a := a+1
      a,b := arbitrary integer, b-1
```

Again, the covering observation applies, with the well-founded covering relations $a_x < a_y \le 1001$ and $999 \le b_y < b_x$. It is easy to see that the ranking height of π_2 is ω^2. In particular the function $\omega b + |1000 - a|$ is a ranking function for π_2.

Example 3. Let π_3 be the program

```
while F ≠ ℕ²
   choose (a,b)∈ ℕ² − F
      F := F ∪ {(a′,b′)∈ ℕ²: a′ ≥ a and b′ ≥ b}
```

where $\mathbb{N}$ is the set of natural numbers, $F \subseteq \mathbb{N}^2$, and initially $F = \emptyset$. Think of F as the set of forbidden pairs. Initially, no pairs are forbidden, but once a pair becomes forbidden it remains so forever. As long as some pairs are not yet forbidden, the program nondeterministically chooses a nonforbidden pair (a, b) and forbids it and all pairs (a', b') such that $a' \ge a$ and $b' \ge b$. For every noninitial state x of π_3 let

$$A(x) = \min\{a : (a, b) \in F_x \text{ for some } b\},$$

where F_x is F at state x. If x is the initial state, define $A(x) = \infty$. Define the function $B(x)$ similarly. Define $C(x)$ to be the number of points $(a, b) \in \mathbb{N}^2 - F_x$ such that $a \ge A(x)$ and $b \ge B(x)$. It is not hard to see that $C(x)$ is always finite. π_3's reachability relation $y \prec x$ is covered by the three well-founded relations $A(y) < A(x)$, $B(y) < B(x)$, and $C(y) < C(x)$. By the covering observation, π_3 is terminating. But this time the ranking height of the program is less obvious. It is $\omega^2 + 1$; the initial state has rank ω^2, and the rank of a noninitial state x is $\omega \cdot (A(x) + B(x)) + C(x)$.

In Section 2.2, we recall the definition of the ordinal height $|R|$ of a well-founded relation R as well as the definition of the ordinal height $|x|_R$ of any element x in the domain of R. If a program π is terminating then the ordinal height $|x|_\prec$ is a ranking function, and $|x|_\prec \le f(x)$ for any ranking function f for π. It follows that the ranking height of π is the ordinal height $|\prec|$ of π's reachability relation $\prec$. Our colleague, Byron Cook, working on a program-termination prover [Cook et al. 2006], asked the following question [Cook 2005].

Question 4 (*Covering Question*). If a transitive relation R is covered by well-founded relations $U_1, \ldots, U_n$, what is the best bound on $|R|$ in terms of the ordinals $\alpha_i = |U_i|$?

The covering question led us to investigate well partially ordered sets (in short, wpo sets). Recall that a sequence $\langle x_0, x_1, \ldots\rangle$, finite or infinite, of elements of a partially ordered set P is *bad* if there are no indices $i < j$ with $x_i \le x_j$ and that a partially ordered set P is wpo if every bad sequence in P is finite. In Section 4, we introduce the key notion of this study, the *stature* $\|P\|$ of a wpo set P. We define $\|P\|$ as the ordinal height of the forest of nonempty bad sequences of P. We then give, in the same section, several alternative and equivalent definitions of $\|P\|$. In particular, $\|P\|$ is the height of the well-founded poset of proper ideals of P. In our view, the notion of stature is of central importance to the theory of wpo sets.

In Section 5 we prove the following theorem to reduce the covering question to a question about the stature of the direct product $\alpha_1 \times \cdots \times \alpha_n$ of ordinals α_i. (By Corollary 18, the direct product of finitely many wpo sets is wpo.)

THEOREM 5. *If $R, U_1, \ldots, U_n, \alpha_1, \ldots, \alpha_n$ are as in the covering question then*

$$|R| \le \|\alpha_1 \times \cdots \times \alpha_n\|$$

and this inequality is tight in the following sense. For any ordinals $\alpha_1, \ldots, \alpha_n$, there exist relations $R, U_1, \ldots, U_n$ such that R is transitive, $U_1, \ldots, U_n$ are well-founded, $R \subseteq U_1 \cup \cdots \cup U_n$, each $|U_i| = \alpha_i$, and $|R| = \|\alpha_1 \times \cdots \times \alpha_n\|$.

Applying this theorem to the reachability relation $\prec_{\pi_3}$ of program π_3 of Example 3, we get that $|\prec_{\pi_3}| \le \|\omega \times \omega \times \omega\|$. But what is $\|\alpha_1 \times \cdots \times \alpha_n\|$? This question is addressed in Section 6.

THEOREM 6. $\|\alpha_1 \times \cdots \times \alpha_n\| = \alpha_1 \otimes \cdots \otimes \alpha_n$.

Here $\alpha_1 \otimes \cdots \otimes \alpha_n$ is the natural product of the n ordinals. The natural sum and natural product of ordinals are recalled in Section 2.4. Combining Theorems 5 and 6, we obtain the answer to the covering question.

THEOREM 7 (MAIN THEOREM). *If $R, U_1, \ldots, U_n, \alpha_1, \ldots, \alpha_n$ are as in the covering question then*

$$|R| \le \alpha_1 \otimes \cdots \otimes \alpha_n$$

and this inequality is tight in the following sense. For any ordinals $\alpha_1, \ldots, \alpha_n$, there exist relations $R, U_1, \ldots, U_n$ such that R is transitive, $U_1, \ldots, U_n$ are well-founded, $R \subseteq U_1 \cup \cdots \cup U_n$, each $|U_i| = \alpha_i$, and $|R| = \alpha_1 \otimes \cdots \otimes \alpha_n$.

After this article was written, we were informed that this answer to the covering question had also been obtained by Christian Delhommé [2006].

In the case of the program π_3 of Example 3, we have $|\prec_{\pi_3}| \le \omega \otimes \omega \otimes \omega = \omega^3$. It is often convenient to generalize the notion of ranking function to allow such a function to have values in any well-ordered set. For each natural number ℓ, let $\mathbb{N}^\ell$ be the set of ℓ-tuples of natural numbers ordered lexicographically, so that the ordinal type of $\mathbb{N}^\ell$ is ω^ℓ.

COROLLARY 8. *Let $R, U_1, \ldots, U_n, \alpha_1, \ldots, \alpha_n$ be as in the covering question and assume that $\alpha_i \le \omega^{\ell_i}$, so that U_i admits a ranking function with values in $\mathbb{N}^{\ell_i}$. Here each ℓ_i is a positive integer. Let $\ell = \ell_1 + \cdots + \ell_n$. Then R admits a* *ranking function with values in N^ℓ but may not admit a ranking function with values in $\mathbb{N}^{\ell-1}$.*

At first sight, it seems that information like that in the main theorem is useless in the search for termination proofs. After all, it applies only to relations that are, by the covering observation, already known to be well-founded. Nevertheless, such information could be helpful. Here is an example. Suppose that you have an automated tool that, given a relation R, tries to find a ranking function for R. The power of a practical automated tool is necessarily restricted (and typically corresponds to a restricted system of arithmetic). There is an ordinal α such that the tool can only succeed for relations R of ordinal height $|R| < \alpha$. Let π be a program of ranking height $\beta \geq \alpha$. The tool cannot directly prove that π terminates. But if the reachability relation of π is covered with relations $U_1, \ldots, U_n$ whose ordinal heights $\alpha_1, \ldots, \alpha_n$ are below α, the tool might find ranking functions for them. For this approach to succeed, it is necessary, according to the main theorem, that $\alpha_1 \otimes \cdots \otimes \alpha_n \geq \beta \geq \alpha$. If we have some additional information about α, we can draw additional conclusions. For example, the approach does not work if $\alpha = \omega^\omega$ because in that case, $\alpha_1 \otimes \cdots \otimes \alpha_n < \alpha$.

As we mentioned earlier, the notion of stature is of independent interest. Section 7, the most involved section of this article, is devoted to a characterization of the stature of a wpo set P in terms of linearizations of P. Earlier, in Section 4, we notice that every linearization (that is, every linear extension with the same underlying set) of a wpo set P is well-founded and of length (that is ordinal height) $\leq \|P\|$. In the process of proving Theorem 6, we construct a linearization of $\alpha_1 \times \cdots \times \alpha_n$ of length $\|\alpha_1 \times \cdots \times \alpha_n\|$.

COROLLARY 9. *The supremum of the lengths of linearizations of $\alpha_1 \times \cdots \times \alpha_n$ is $\|\alpha_1 \times \cdots \times \alpha_n\|$ and the supremum is attained.*

It turns out that this corollary generalizes to all wpo sets, a result proved in de Jongh and Parikh [1977]. In Section 7, we shall obtain the same result simultaneously with Theorem 10 below, by a somewhat simpler argument. Because the supremum of the lengths of linearizations of a wpo set is attained, it is sometimes called its *maximal order type* [Schmidt 1979].

THEOREM 10. *The stature of any wpo set is its maximal order type.*

As one of the referees pointed out, it is fairly easy to prove Theorem 10 if one already knows that the supremum of the linearization lengths of a wpo set is attained. The idea is to show that this supremum satisfies the analog of Lemma 36, so that an induction on the stature of the wpo set shows that this supremum and the stature agree.

A part of our results on wpo sets was already known. As already mentioned, de Jongh and Parikh [1977] showed that the supremum of linearization lengths of a wpo set is attained. They also computed this supremum for disjoint unions and Cartesian products. We believe that our proofs are simpler than the proofs that either are available in the literature or could be obtained by combining those proofs. In Section 8 we touch upon the involved history of the theory of wpo sets and other related work.

We attempt to make this article self-contained. In Section 2 we give some preliminary information on partially ordered sets, well-founded partially ordered sets, wpo sets, ordinal arithmetic, and infinite combinatorics.

In Section 3 we introduce games that allow us to compare ordinal heights of well-founded sets. The game criterion for height inequalities proved to be very useful. It may be known, but we have not found an explicit statement of it in the literature.

Remark 11. The notations $|P|$ and $\|P\|$ have many different uses in the literature. But they are convenient for our purposes and so, with some apprehension, we use them.

2. PRELIMINARIES

We recall various definitions and facts and use this occasion to fix terminology and notation.

2.1 Partially Ordered Sets

A binary relation R can be viewed as a set of pairs of elements. A directed graph, in short digraph, is a pair (X, R), where X is a set and $R \subseteq X \times X$; the set X is the *domain* of the digraph. The smallest set X such that $R \subseteq X \times X$ will be called the *domain* of R.

A *poset* is a partially ordered set. In other words, a poset is a digraph where the relation is a partial order. Let P be a poset. The relation $<_P$ (respectively $\leq_P$) is the strict (respectively nonstrict) version of the partial order of P. If $x <_P y$, we say that x is lower than y in P and that y is higher than x in P. In this and other similar cases, the subscript may be omitted when it is clear.

A poset Q *extends* P if $\leq_P \subseteq \leq_Q$, so that the digraph $(\mathrm{Dom}(Q), Q)$ may have more elements as well as more edges than the digraph $(\mathrm{Dom}(P), P)$. A *linearization* of P is a linearly ordered set with the same domain that extends P.

An element $x \in \mathrm{Dom}(P)$ is the *top* of P if $x \geq_P y$ for every element y of P.

A subset A of (the domain of) P is an *antichain* of P if the elements of A are pairwise incomparable. A subset F of P is a *filter* of P if it is upward closed, so that $y \in F$ if $x \leq y$ for some $x \in F$. If X is a subset of P then $\mathrm{Min}(X)$ is the antichain of minimal elements of X and

$$\mathrm{Filter}_P(X) = \{y \ : \ y \geq_P x \text{ for some } x \in X\}.$$

$\mathrm{Filter}(X)$ is the smallest filter that includes X. If A is an antichain then $A = \mathrm{Min}(\mathrm{Filter}(A))$.

A subset D of P is an *ideal* if it is downward closed, so that $x \in D$ if $x \leq y$ for some $y \in D$. Ideals are the complements of filters in P, and the other way round. If $X \subseteq \mathrm{Dom}(P)$ then $\mathrm{Ideal}_P(X)$, or simply $\mathrm{Ideal}(X)$, is the ideal $\mathrm{Dom}(P) - \mathrm{Filter}(X)$. In other words,

$$\mathrm{Ideal}(X) = \{y \in \mathrm{Dom}(P) \ : \ (\forall x \in X)\, x \not\leq_P y\}.$$

An ideal D of P is *proper* if it does not contain all elements of P.

Warning 12. Ideal(X) is the largest ideal that avoids X, rather than—which is more usual—the smallest ideal that includes X. We will not use the latter notion while the first one will play a role in this article. Note also that many authors require filters to be not only upward closed but also downward directed (and dually for ideals). Such authors use terminology like *order-filter* or *up-set*, where we use *filter*, and they use *order-ideal* or *down-set* where we use *ideal*.

The sets $\mathrm{Filter}_P(X)$ and $\mathrm{Ideal}_P(X)$ inherit partial orderings from P and thus give rise to posets that are also called $\mathit{Filter}_P(X)$, and $\mathit{Ideal}_P(X)$, respectively. If $x \in \mathrm{Dom}(P)$ then

$$\mathrm{Filter}_P(x) = \mathrm{Filter}_P(\{x\}) = \{y \in \mathrm{Dom}(P) : x \leq_P y\},$$
$$\mathrm{Ideal}_P(x) = \mathrm{Ideal}_P(\{x\}) = \{y \in \mathrm{Dom}(P) : x \not\leq_P y\}.$$

Given finitely many posets $P_1, \ldots, P_n$, we can form the direct (or Cartesian) product $P_1 \times \cdots \times P_n$ with domain $\mathrm{Dom}(P_1) \times \cdots \times \mathrm{Dom}(P_n)$ where the n-tuples are ordered componentwise. The direct product operation generalizes to infinitely many components but we will not need the generalization.

2.2 Well-Founded Partially Ordered Sets

A poset is *well-founded* if it has no infinite descending sequence. A *well-ordered* set is a well-founded, linearly ordered set. Fix a well-founded poset P. If F is a filter of P then $F = \mathrm{Filter}(\mathrm{Min}(F))$.

Each element x of P has an ordinal *height* $|x|_P$ defined by the recursion

$$|x| = \min\{\text{ordinal } \alpha \ : \ \alpha > |y| \text{ for all } y <_P x\}.$$

The *height* $|P|$ of the poset P itself is the smallest ordinal $> |x|_P$ for all $x \in \mathrm{Dom}(P)$. If a poset Q is obtained from P by adding a new top element ∞ to P, then we have $|\infty|_Q = |P|$. The ordinal height of a well-ordered set is also called its *length*.

LEMMA 13. *For every $\alpha < |P|$, there is an element x with $|x|_P = \alpha$. If $\alpha < |y|_P$, then there is an element $x <_P y$ with $|x|_P = \alpha$.*

PROOF. The second claim follows from the first: consider the subposet given by the set $\{z : z < y\}$. To prove the first claim, notice that elements of height $\geq \alpha$ form a nonempty filter; any minimal element x of that filter is of height α. □

Definition 14. A binary relation R is *well-founded* if there is no infinite sequence $\langle x_0, x_1, \ldots\rangle$ such that $x_{n+1} R x_n$ holds for all n. In the obvious way, the definition of height generalizes to well-founded relations.

A well-founded relation does not have to be transitive. For example the successor relation on natural numbers is well-founded but not transitive. However, the transitive closure of a well-founded relation is well-founded as well.

2.3 Well Partially Ordered Sets

A good reference for this subsection is Kruskal [1972].

Definition 15. Let P be a poset. A sequence $\langle x_0, x_1, \ldots\rangle$ of elements of P, finite or infinite, is *bad* if there are no indices $i < j$ with $x_i \le x_j$. A poset P is *well partially ordered*, or *wpo*, if all bad sequences in P are finite.

Remark 16. Admittedly, the terminology is not very good. But it is accepted. We discuss the issue in Section 8.

There are many equivalent characterizations of the wpo sets.

LEMMA 17. *Let P be a poset. The following are equivalent characterizations of the wpo property. In other words, each of the following claims is equivalent to the claim that P is wpo.*

(1) *Every infinite sequence $\langle x_0, x_1, \ldots\rangle$ of elements of P includes an infinite weakly increasing subsequence.*
(2) *P is well-founded, and all its antichains are finite.*
(3) *For every filter F of P, there is a finite antichain A such that $F = \mathit{Filter}(A)$.*
(4) *For every filter F of P, the antichain $\mathit{Min}(F)$ is finite and $F = \mathit{Filter}(\mathit{Min}(F))$.*
(5) *For every ideal D of P, there is a finite antichain A such that $D = \mathit{Ideal}(A)$.*
(6) *For every ideal D of P, the antichain $A = \mathit{Min}(\mathit{Dom}(P) - D)$ is finite and $D = \mathit{Ideal}(A)$.*

The proof is straightforward, using Ramsey's theorem, Theorem 21, for items 1 and 2.

COROLLARY 18. *The direct product of finitely many wpo set is wpo.*

PROOF. Use the first equivalent characterization of the wpo property in the preceding lemma. □

2.4 Ordinal Arithmetic

We recall some basic definitions of set theory on ordinals and cardinals. A good reference for this subsection is Zuckerman [1974], particularly Section 5.11.

In set theory, an ordinal α is defined as the set $\{\xi : \xi < \alpha\}$ of smaller ordinals. In particular, the first infinite ordinal ω is the set of natural numbers. Every well-ordered set P is isomorphic to a unique ordinal, namely, the length of P. The *cardinality* of a set X is the least ordinal α such that there is a bijection between X and α; ordinals that arise in this way are *cardinals*. If α and β are ordinals then $\alpha + \beta$ is the length of the set

$$\{(0, \mu) : \mu < \alpha\} \cup \{(1, \nu) : \nu < \beta\}$$

ordered lexicographically.

A *limit ordinal* is an ordinal $\alpha > 0$ not of the form $\beta + 1$ for any β. A set X of ordinals is *cofinal* in a limit ordinal α if α is the supremum of X and $\alpha \notin X$. Thus X is cofinal in α if and only if $X \subseteq \alpha$ and for every $\beta < \alpha$ there is an element of X that is $> \beta$. The *cofinality* of a limit ordinal α is the least cardinal κ such that α has a cofinal subset of cardinality κ. Alternatively and equivalently, the cofinality of a limit ordinal α can be defined as the least ordinal

κ such that there is a (strictly) increasing sequence $s = \langle \beta_\xi : \xi < \kappa \rangle$ of ordinals whose range is cofinal in α.

A cardinal is *regular* if it is equal to its own cofinality. It is easy to see that, for every limit ordinal α, the cofinality of α is a regular cardinal.

For any ordinal α, the ordinal ω^α is the length of the following well-ordered set P. Dom(P) is the set of functions $f : \alpha \to \omega$ such that the support $\{\xi < \alpha : f(\xi) > 0\}$ is finite. The order is reverse lexicographic. That is, $f_1 <_P f_2$ if and only if $f_1(\xi) < f_2(\xi)$ for the largest ξ with $f_1(\xi) \neq f_2(\xi)$. Such a largest ξ exists, whenever f_1 and f_2 are distinct, because the supports of f_1, f_2 are finite.

Any ordinal number α can be written in Cantor normal form (with base ω),

$$\alpha = \omega^{\alpha_1} + \omega^{\alpha_2} + \cdots + \omega^{\alpha_n},$$

for a unique finite sequence $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$. (There is an alternative version where the terms have the form $\omega^{\alpha_i} m_i$ with integer coefficients m_i and where the exponents are strictly decreasing. The two are obviously equivalent.)

The *natural sum* $\alpha \oplus \beta$ of two ordinals α and β is obtained by adding their Cantor normal forms as if they were polynomials (i.e., as if ω were an indeterminate), arranging the terms in nonincreasing order of exponents. It is well known and easy to check that the natural sum is strictly increasing in each of its arguments. (In fact, there is an equivalent definition of $\oplus$ by recursion: $\alpha \oplus \beta$ is the smallest ordinal strictly above $\alpha' \oplus \beta$ and $\alpha \oplus \beta'$ for all $\alpha' < \alpha$ and $\beta' < \beta$.)

The natural product, $\alpha \otimes \beta$, of two ordinals is defined by multiplying their Cantor normal forms as if they were polynomials in the indeterminate ω, using natural addition for the exponents, and arranging the resulting terms in nonincreasing order.

LEMMA 19. *Natural multiplication is commutative and associative, and it distributes over natural addition. It is a strictly increasing function of either argument as long as the other argument is not zero.*

Remark 20. The natural sum of two ordinals is at least as big as the ordinal sum in either order, but it may be strictly bigger than either of the two ordinal sums. For example, the natural sum of $\omega^2 + 1$ and ω is $\omega^2 + \omega + 1$, which is strictly greater than both $(\omega^2 + 1) + \omega = \omega^2 + \omega$ and $\omega + (\omega^2 + 1) = \omega^2 + 1$. Similarly, the natural product $\alpha \otimes \beta$ is always at least as big as either of the ordinal products $\alpha \cdot \beta$ and $\beta \cdot \alpha$, but it may be strictly bigger than both. An example of strict inequality is given by exponentiating the additive example above: take $\alpha = \omega^{\omega^2+1}$ and $\beta = \omega^\omega$.

2.5 Infinite Combinatorics

We recall the infinite version of Ramsey's theorem for pairs and one generalization of it. If S is a set then $[S]^2$ is the collection of two-element subsets of S.

THEOREM 21 (RAMSEY [1930]). *If S is an infinite set and $[S]^2$ is partitioned into finitely many pieces, $S_1, \ldots, S_m$, then there exists an infinite $T \subseteq S$ such that $[T]^2 \subseteq S_i$ for some i.*

THEOREM 22 (DUSHNIK AND MILLER [1941]). *If κ is a regular cardinal and $[\kappa]^2$ is partitioned into two pieces, S_1 and S_2, then either there exists a subset $T_1 \subseteq \kappa$ of cardinality κ with $[T_1]^2 \subseteq S_1$ or else there exists an infinite subset $T_2 \subseteq \kappa$ with $[T_2]^2 \subseteq S_2$.*

Theorem 21 easily follows from the special case where $m = 2$ and $S = \omega$, which is the special case $\kappa = \omega$ of Theorem 22. The proof of Theorem 22 is very similar to a standard argument for Ramsey's theorem. (Theorem 22 also holds for singular κ, but the proof, due to Erdős, is more complicated.)

3. GAMES

We give a useful way to compare the heights of well-founded posets.

Given two posets, P and Q, define a game $\Gamma(P, Q)$ between two players, called 1 and 2, played as follows. Player 1 (resp. 2) has a pebble which, at each noninitial stage of the game is at some element of P (respectively Q). Initially, the pebble is off the poset. Think about the initial position of the pebble being above all elements of the poset, at the *summit position* of P (respectively Q). This allows us to pretend that, at each stage, including the initial stage, the pebble occupies some position in the poset. Define the *height* of the summit position to be the height of the poset.

The players move alternately, with 1 moving first. A player's move shifts his pebble to a position lower than the current one. In particular, the first move puts the pebble at any element of the poset. If a player is unable to move, he loses the game, and his opponent wins. We say that a player *wins* the game if he has a winning strategy, that is, a strategy by which he wins no matter how the opponent plays.

PROPOSITION 23 (GAME CRITERION). *Let P and Q be posets.*

—*Suppose that P is well-founded. Then player 1 wins $\Gamma(P, Q)$ if and only if Q is well-founded and $|P| > |Q|$.*
—*Suppose that Q is well-founded. Then player 2 wins $\Gamma(P, Q)$ if and only if P is well-founded and $|P| \leq |Q|$.*

PROOF. Q is well-founded if P is well-founded and 1 wins the game. Indeed, if Q is not well-founded then it has an infinite descending sequence, and 1 cannot win a play where 2 moves his pebble along the descending sequence. Similarly, P is well-founded if Q is well-founded and 2 wins the game. In the rest of the proof, we may assume that both P and Q are well-founded. It remains to prove the following two claims:

(1) Player 1 wins $\Gamma(P, Q)$ if and only if $|P| > |Q|$.
(2) Player 2 wins $\Gamma(P, Q)$ if and only if $|P| \leq |Q|$.

The two right-to-left implications are easy because, in each case, the winning strategy is to move your pebble to a position whose height is at least as great as that of the other player's pebble. Once you have both right-to-left implications, the left-to-right ones follow, because at most one player can have a winning strategy and at least one of the ordinal inequalities must hold. □

A map f from a poset P to a poset Q is *monotone* if $f(x) <_Q f(y)$ whenever $x <_P y$.

COROLLARY 24. *If there is a monotone map from a poset P to a well-founded poset Q then P is well-founded and $|P| \leq |Q|$.*

PROOF. Player 2 has a winning strategy in $\Gamma(P, Q)$: whenever 1 moves to a position x, move to position $f(x)$. Now use Proposition 23. □

Remark 25. The game is even more natural for posets with a distinguished element (pointed posets). Then we don't need the summit position; initially the pebbles are at the distinguished elements. The proposition, appropriately adjusted, remains valid. The corollary, also properly adjusted, remains valid if we require that the monotone map takes the distinguished element to the distinguished element. We skip the details of adjustment as we will not use pointed posets.

Remark 26. The game, the proposition and the corollary generalize in a straightforward way to the case when P, Q are directed graphs.

4. STATURE OF A WPO SET

First we give a number of equivalent definitions of stature. Then we prove some useful facts related to statures and natural sums of ordinals.

4.1 Definition and Equivalent Characterizations

Fix a wpo set P.

Definition 27. $\mathcal{B}(P)$ is the poset of nonempty bad sequences of P. The ordering is reverse extension: $s \leq_{\mathcal{B}} t$ if and only if t is an initial segment of s.

Thus $s \leq_{\mathcal{B}} t$ if and only if s is an end extension of t. It is easy to see that $\mathcal{B}$ is a forest. If $s <_{\mathcal{B}} t$ then we think of the longer sequence s being below the shorter t; in that sense the forest $\mathcal{B}$ grows downward. Every infinite downward path in $\mathcal{B}$ gives rise to an infinite bad sequence of P. But P is wpo and thus does not have infinite bad sequences. It follows that $\mathcal{B}$ is well-founded. Note that if P were not wpo then $\mathcal{B}$ would not be well-founded: an infinite bad sequence in P would give rise to an infinite downward path in $\mathcal{B}$.

Definition 28. The *stature* $\|P\|$ of P is the height $|\mathcal{B}(P)|$ of $\mathcal{B}(P)$.

It is easy to see that $\|P\| \geq |P|$. Indeed, it suffices to show that, if $x \in P$ and s is a bad sequence of P that ends with x then $|x|_P \leq |s|_{\mathcal{B}}$. This is easily proved by induction on $|x|_P$. And $\|P\|$ can exceed $|P|$. For a simple example consider the case when P is a finite antichain of $n > 1$ elements. In this case $|P| = 1$ and $\|P\| = |\mathcal{B}| = n$.

Remark 29. We could have defined $\mathcal{B}(P)$ to be the tree of all bad sequences of P including the empty sequence. The empty sequence would be the root and the top element of $\mathcal{B}$. Then $\|P\|$ could be defined as the height of the empty sequence in $\mathcal{B}$. One of our referees asked us why we didn't define $\mathcal{B}(P)$ as the tree of all bad sequences of P. This was partially a matter of taste, but it is convenient to deal with the height of $\mathcal{B}$ itself rather than the height of one of its elements.

We will give several equivalent characterizations of $\|P\|$. To this end, we define some useful posets.

Definition 30.

— $\mathcal{A}(P)$ is the set of nonempty antichains of P with partial order
$$A \leq_{\mathcal{A}} B \iff (\forall b \in B)(\exists a \in A)\, a \leq_P b.$$
— $\mathcal{I}(P)$ is the set of proper ideals of P ordered by inclusion.
— A *pointed ideal* is a pair (D, d) where D is an ideal and d is a maximal (not necessarily the greatest) element of D. $\mathcal{P}(P)$ is the set of pointed ideals of P with partial order
$$(D, d) <_{\mathcal{P}} (E, e) \iff D \subseteq E - \{e\}.$$

We take the liberty of omitting the argument of $\mathcal{A}, \mathcal{B}, \mathcal{I}, \mathcal{P}$ when it is clear from the context.

PROPOSITION 31. *$\mathcal{A}$, $\mathcal{I}$, and $\mathcal{P}$ are well-founded, and* $|\mathcal{A}| = |\mathcal{I}| = |\mathcal{P}| = |\mathcal{B}| = \|P\|$.

PROOF. We split the proposition into a number of claims and repeatedly use the game criterion of Section 3.

— *$\mathcal{P}$ is well-founded and* $|\mathcal{P}| \leq |\mathcal{B}|$. By the game criterion, it suffices to construct a winning strategy for player 2 in game $\Gamma(\mathcal{P}, \mathcal{B})$. When player 1 has just played (D, x), extend your previous bad sequence by appending x. This strategy always provides a legal move, so it wins.
— *$\mathcal{I}$ is well-founded and* $|\mathcal{I}| \leq |\mathcal{P}|$. We construct a winning strategy for player 2 in game $\Gamma(\mathcal{I}, \mathcal{P})$. Let $D_0 = \mathrm{Dom}(P)$. The ith move of 1 is some ideal $D_i \subsetneq D_{i-1}$. Choose any $y_i \in D_{i-1} - D_i$ and play (E_i, y_i) where $E_i = \mathrm{Ideal}(\{y_1, \ldots, y_i\}) \cup \{y_i\} = \{x \in \mathrm{Dom}(P) : x \not> y_i$ and, for all $j < i$, $x \not\geq y_j\}$. (The key is to play y_i in the second component; the first component E_i is chosen as big as possible subject to the requirement that y_i be maximal in it and that $E_i \subseteq E_j - \{y_j\}$ for all $j < i$.) This strategy always provides a legal move, so it wins.
— *$\mathcal{A}$ is well-founded and* $|\mathcal{A}| = |\mathcal{I}|$. The reason is simply that $\mathcal{A}$ and $\mathcal{I}$ are isomorphic. An isomorphism from $\mathcal{I}$ to $\mathcal{A}$ is given by $D \mapsto \mathrm{Min}(\mathrm{Dom}(P) - D)$, and its inverse is $X \mapsto \mathrm{Ideal}(X)$.
— $|\mathcal{B}| \leq |\mathcal{I}|$. We construct a winning strategy for player 2 in game $\Gamma(\mathcal{B}, \mathcal{I})$. When player 1 has just played a bad sequence $\langle x_1, \ldots, x_\ell \rangle$, reply with $\mathrm{Ideal}\{x_1, \ldots, x_\ell\}$. This strategy always provides a legal move, so it wins.

To summarize, we established that $\mathcal{A}, \mathcal{I}, \mathcal{P}$ are well-founded and that $|\mathcal{A}| = |\mathcal{I}| \leq |\mathcal{P}| \leq |\mathcal{B}| \leq |\mathcal{I}|$. It follows that $|\mathcal{A}| = |\mathcal{I}| = |\mathcal{P}| = |\mathcal{B}|$. It remains to recall that, by the definition of stature, $\|P\| = |\mathcal{B}|$. □

For future reference, we record the following fact about linearizations of P.

PROPOSITION 32. *Every linearization of a wpo P is well-founded and has length $\leq \|P\|$.*

PROOF. Let A be a linearization of P. By the definition of stature, $\|P\| = |\mathcal{B}|$. By the game criterion of Section 3, it suffices to construct a winning strategy for player 2 in $\Gamma(A, \mathcal{B})$. 1's moves up to stage ℓ form a decreasing sequence s_ℓ of length ℓ in A. Clearly s_ℓ is a bad sequence of P; use it as your reply at stage ℓ. (A direct proof that A is well-founded is to observe that every decreasing sequence in A is a bad sequence in P.) □

4.2 Statures and Natural Sums

Recall that any ordinal α has a unique Cantor normal form

$$\alpha = \omega^{\alpha_1} + \omega^{\alpha_2} + \cdots + \omega^{\alpha_n},$$

where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$, and that the natural sum $\alpha \oplus \beta$ of α and β is obtained by adding their Cantor normal forms as if they were polynomials.

LEMMA 33. *Let α have the Cantor normal form exhibited above, let $\beta < \alpha$, and let $\gamma < \omega^{\alpha_n}$. Then $\beta \oplus \gamma < \alpha$.*

PROOF. Increasing β if necessary, we may assume that it has the form

$$\beta = \omega^{\alpha_1} + \omega^{\alpha_2} + \cdots + \omega^{\alpha_{n-1}} + \delta,$$

where $\delta < \omega^{\alpha_n}$ (and where the exponents $\alpha_1, \ldots, \alpha_{n-1}$ are the same as in the normal form of α). As both γ and δ are $< \omega^{\alpha_n}$, their Cantor normal forms involve only exponents $< \alpha_n$. So the same is true of their natural sum. But

$$\beta \oplus \gamma = \omega^{\alpha_1} + \omega^{\alpha_2} + \cdots + \omega^{\alpha_{n-1}} + (\delta \oplus \gamma),$$

and so the required inequality follows. □

COROLLARY 34. *An ordinal of the form ω^β exceeds the natural sum of any two strictly smaller ordinals.*

The corollary is a bit stronger than the analogous assertion without the word *natural*, since $\xi + \eta \leq \xi \oplus \eta$ and the inequality can be strict. We will use the following obvious consequence of Corollary 34.

COROLLARY 35. *An ordinal of the form ω^β exceeds the natural sum of any finitely many strictly smaller ordinals.*

Our next goal is to relate the statures of a wpo set and its subsets. It is clear that $\|P\| \leq \|Q\|$ whenever $P \subseteq Q$; indeed the copycat strategy of player 2 wins $\Gamma(\mathcal{B}(P), \mathcal{B}(Q))$. On the other hand, we shall see that when two subsets of a wpo set P cover P their statures add (in the sense of $\oplus$) to at least the stature of P. For this, as well as for other purposes later, we need the following information about statures.

Recall that $\mathrm{Filter}_P(x)$ is the smallest filter that contains x, $\mathrm{Ideal}_P(x)$ is the largest ideal that avoids x, and that the corresponding posets (with partial orders inherited from P) are also denoted $\mathrm{Filter}_P(x)$ and $\mathrm{Ideal}_P(x)$, respectively.

LEMMA 36. *The stature $\|P\|$ of a wpo set P is the smallest ordinal strictly above $\|\mathrm{Ideal}_P(v)\|$ for all $v \in P$.*

PROOF. Since $\|P\|$ is the height of the forest $\mathcal{B}(P)$, it is the smallest ordinal strictly greater than the heights of all the roots $\langle v\rangle$ of the trees that constitute the forest $\mathcal{B}(P)$. The height of any $\langle v\rangle$ can be computed in the tree of which $\langle v\rangle$ is the root, and we shall complete the proof by checking that this height is exactly $\|\mathrm{Ideal}_P(v)\|$.

The tree with $\langle v\rangle$ as root consists of all the bad sequences of the form $\langle v\rangle^\frown s$. For such a sequence to be bad means that s is bad and that no term in s is $\geq_P v$. In other words, s must be a bad sequence in $\mathrm{Ideal}_P(v)$. Thus, the tree with root v is isomorphic to $\mathcal{B}(\mathrm{Ideal}_P(v))$ with a top element added, and so the proof is complete. □

LEMMA 37. *Let (the domain of) a wpo set Q be the union of (the domains of) finitely many subposets $Q_1, \ldots, Q_n$. The subposets may overlap. For any subposet P of Q, we have $\|P\| \leq \|Q_1\| \oplus \cdots \oplus \|Q_n\|$.*

PROOF. We proceed by induction on the stature $\|P\|$ of the focal poset P, so assume the lemma holds for all cases where the focal poset is of strictly smaller stature. In view of the preceding lemma, it suffices to prove that, for each $x \in P$, $\|\mathrm{Ideal}_P(x)\| < \|Q_1\| \oplus \cdots \oplus \|Q_n\|$. For this purpose, fix an arbitrary $x \in P$. If $x \in Q_i$ then $\mathrm{Ideal}_{Q_i}(x)$ is a well-defined proper ideal of Q_i; otherwise the notation $\mathrm{Ideal}_{Q_i}(x)$ has not yet been assigned a meaning but it is convenient to adopt the convention[1] that $\mathrm{Ideal}_{Q_i}(x) = Q_i$.

By the induction hypothesis, we have

$$\|\mathrm{Ideal}_P(x)\| \leq \|\mathrm{Ideal}_{Q_1}(x)\| \oplus \cdots \oplus \|\mathrm{Ideal}_{Q_n}(x)\|.$$

There is an index j such that $x \in Q_j$. By Lemma 36,

$$\|\mathrm{Ideal}_{Q_j}(x)\| < \|Q_j\|.$$

For Q_i with $i \neq j$ we cannot use the same argument, since we don't necessarily have $x \in Q_i$, but we still have

$$\|\mathrm{Ideal}_{Q_i}(x)\| \leq \|Q_i\| \text{ for all } i \neq j.$$

Combining the three displayed inequalities and the strict monotonicity of $\oplus$, we get $\|\mathrm{Ideal}_P(x)\| < \|Q_1\| \oplus \cdots \oplus \|Q_n\|$ as required. □

5. REDUCTION OF COVERING QUESTION

We return to the covering question discussed in the introduction and prove Theorem 5. Assume that $U_1, \ldots, U_n$ are well-founded relations and that R is a transitive relation included in $U_1 \cup \cdots \cup U_n$. Let $X = \mathrm{Dom}(R)$. Without loss of generality, each $\mathrm{Dom}(U_i) \subseteq X$. By an argument using Ramsey's theorem, Theorem 21, R is well-founded. We seek to bound its ordinal height in terms of

[1]We do not advocate adopting this convention in general, as it may have awkward consequences; for example $\mathrm{Ideal}_{Q_i}(x)$ is not always a monotone function of x. The convention is useful in the present proof, and that is why we introduce it here, but not in the general context of Section 2.

the ordinals $\alpha_i = |U_i|$. The direct product $\alpha_1 \times \cdots \times \alpha_n$ of the ordinals $\alpha_1, \ldots, \alpha_n$ can be seen as a poset where the n-tuples are ordered componentwise. That poset is wpo by Corollary 18.

Proposition 38. *Under the assumptions above,* $|R| \le \|\alpha_1 \times \cdots \times \alpha_n\|$.

Proof. According to Definition 28, we need to prove that $|R| \le |\mathcal{B}(\alpha_1 \times \cdots \times \alpha_n)|$. According to the game criterion of Section 3, it suffices to prove that player 2 has a winning strategy in game

$$\Gamma((X, R), \mathcal{B}(\alpha_1 \times \cdots \times \alpha_n)).$$

The desired strategy is simple. Whenever player 1 moves his pebble to a new point $x \in X$, extend the current bad sequence (or the empty sequence if this is the first move) by appending the element $(|x|_{U_1}, \ldots, |x|_{U_n})$ of $\alpha_1 \times \cdots \times \alpha_n$. Here $|x|_{U_i} = 0$ if $x \notin \mathrm{Dom}(U_i)$. We need to check only that this preserves badness of the sequence. Since the opponent's current move x is R-below all his previous moves y (thanks to transitivity of R) and since $R \subseteq U_1 \cup \cdots \cup U_n$, we have, for each earlier y, that $x\, U_i\, y$ and therefore $|x|_{U_i} < |y|_{U_i}$ hold for some i. Thus, the n-tuple $(|x|_{U_1}, \ldots, |x|_{U_n})$ cannot be $\ge$ the earlier n-tuple $(|y|_{U_1}, \ldots, |y|_{U_n})$, so badness persists. □

Proposition 39. *The bound in the previous proposition is tight. That is, given any ordinals $\alpha_1, \ldots, \alpha_n$, we can find well-founded relations $U_1, \ldots, U_n$ of at most these heights and we can find a transitive $R \subseteq U_1 \cup \cdots \cup U_n$ with $|R| = \|\alpha_1 \times \cdots \times \alpha_n\|$.*

Proof. Let (X, R) be the poset $\mathcal{B}(\alpha_1 \times \cdots \times \alpha_n)$ of nonempty bad sequences in $\alpha_1 \times \cdots \times \alpha_n$. If s, t are nonempty bad sequences with last members $(\xi_1, \ldots, \xi_n)$ and $(\eta_1, \ldots, \eta_n)$, respectively, set $tU_is \iff \eta_i < \xi_i$. Obviously, every U_i is well-founded and $|U_i| = \alpha_i$. If tRs holds, then t is an end-extension of s. By the definition of bad sequences, $(\xi_1, \ldots, \xi_n) \not\le (\eta_1, \ldots, \eta_n)$, and so tU_is holds for some i. Thus R is covered by the relations $U_1, \ldots, U_n$. □

Theorem 5 follows from the two propositions.

6. STATURE OF DIRECT PRODUCT OF SEVERAL ORDINALS

The goal of this section is to prove Theorem 6. The theorem asserts that $\|\alpha_1 \times \cdots \times \alpha_n\| = \alpha_1 \otimes \cdots \otimes \alpha_n$ for any natural number n and any ordinals $\alpha_1, \ldots, \alpha_n$. In Section 6.1 we will prove that $\|\alpha_1 \times \cdots \times \alpha_n\| \ge \alpha_1 \otimes \cdots \otimes \alpha_n$, and in Section 6.2 we will prove that $\|\alpha_1 \times \cdots \times \alpha_n\| \le \alpha_1 \otimes \cdots \otimes \alpha_n$. But first we state a characterization of natural sums for future reference.

Recall that the natural sum $\alpha \oplus \beta$ of two ordinals is defined by adding their Cantor normal forms as if they were polynomials (Section 2.4). It is clear from this definition that a well-ordered set of length $\alpha \oplus \beta$ can be partitioned into two subsets of lengths α and β.

Lemma 40. *$\alpha \oplus \beta$ is the largest ordinal that admits a partition into two subsets of lengths α and β.*

Though this is well-known, we point out that it follows also from Lemmas 37 and 46.

6.1 The Natural Product Is Small Enough

In this section, we prove that the stature of $\alpha_1 \times \cdots \times \alpha_n$ is at least as large as natural product of the n ordinals. We start with the case $n = 2$.

LEMMA 41. *Let α and β be arbitrary ordinals.*

(1) *There is a linearization of $\alpha \times \beta$ of length $\alpha \otimes \beta$.*
(2) $\|\alpha \times \beta\| \geq \alpha \otimes \beta$.

Remark 42. This lemma is part of de Jongh and Parikh [1977], Theorem 3.5. Indeed, if we presuppose Theorem 10, which relates stature to linearizations and is proved in Section 7 below, then Theorem 3.5 of de Jongh and Parikh [1977] implies all of Theorem 6. Our proof here uses only half of Theorem 10, namely, Proposition 32 proved above. By working directly with stature rather than linearizations in the next subsection, we shall obtain a simpler proof of that part of Theorem 6.

PROOF OF LEMMA 41. In virtue of Proposition 32, the first claim implies the second. So it suffices to prove claim 1. In the rest of the proof we construct the desired linearization. We do that by induction on $\alpha \otimes \beta$. The zero case is trivial. The induction step splits into two cases.

Case 1. At least one of α and β is not a power of ω. Without loss of generality, suppose it is α. Let the Cantor normal forms of α and β be

$$\begin{aligned} \alpha &= \omega^{\mu_1} + \cdots + \omega^{\mu_m}, \\ \beta &= \omega^{\nu_1} + \cdots + \omega^{\nu_n}, \end{aligned}$$

where $\mu_1 \geq \cdots \geq \mu_m$ and $\nu_1 \geq \cdots \geq \nu_n$ and $m > 1$. Concerning n, we know only that $n \geq 1$. The Cantor normal form of $\alpha \otimes \beta$ has the form

$$\alpha \otimes \beta = \omega^{\pi_1} + \cdots + \omega^{\pi_{mn}},$$

where $\pi_1 \geq \cdots \geq \pi_{mn}$ and every $\pi_p = \mu_i \oplus \nu_j$ for some $1 \leq i \leq m$ and $1 \leq j \leq n$, arranged in nonincreasing order. This gives rise to a bijection

$$f : \{1, \ldots, m\} \times \{1, \ldots, n\} \to \{1, \ldots, mn\}$$

such that $\pi_{f(i,j)} = \mu_i \oplus \nu_j$. Since the same ordinal can occur as $\mu_i \oplus \nu_j$ for several pairs (i, j), there is some freedom in the choice of f. It will be convenient to specify f so that $f(i, j) < f(k, l)$ if and only if one of the following three conditions is satisfied:

(1) $\mu_i \oplus \nu_j > \mu_k \oplus \nu_l$.
(2) $\mu_i \oplus \nu_j = \mu_k \oplus \nu_l$ and $i < k$.
(3) $\mu_i \oplus \nu_j = \mu_k \oplus \nu_l$ and $i = k$ and $j < l$.

There is a unique such function because the three clauses define a linear ordering of the pairs (i, j). The first condition suffices to ensure that the sequence

$\langle \pi_{f(i,j)} \rangle$ is nonincreasing, as required above. We record the following claim about f for future reference.

CLAIM 43. *If $f(i, j) < f(k,l)$ then $i < k$ or $j < l$ (or both).*

PROOF. The claim is obvious if the inequality $f(i, j) < f(k,l)$ holds by virtue of clause (2) or (3). If the inequality holds by virtue of clause (1), we argue by contradiction. If we had both $i \geq k$ and $j \geq l$, then, since the μ and ν sequences are non-increasing, we would have $\mu_i \leq \mu_k$ and $\nu_j \leq \nu_l$. But then $\mu_i \oplus \nu_j \leq \mu_k \oplus \nu_l$, contrary to clause (1). □

Now partition α into consecutive segments $A_1, \ldots, A_m$ of lengths $\omega^{\mu_1}, \ldots, \omega^{\mu_m}$, respectively, and partition β into consecutive segments $B_1, \ldots, B_n$ of lengths $\omega^{\nu_1}, \ldots, \omega^{\nu_n}$, respectively. Each $A_i \times B_j$ can be viewed as a partially ordered set where the order is componentwise. For each pair (i, j), fix a linearization $C_{f(i,j)}$ of $A_i \times B_j$ of height $\omega^{\mu_i \oplus \nu_j}$. Such linearizations exist by the induction hypothesis, since every $\omega^{\mu_i} < \alpha$ and every $\omega^{\nu_j} \neq 0$. By the definition of $\alpha \otimes \beta$, the concatenation

$$C = C_1^\frown \cdots {}^\frown C_{mn}$$

defined in the obvious way, is of length $\alpha \otimes \beta$. It remains to show that C extends the partially ordered set $\alpha \times \beta$. That will complete Case 1 of the proof of the proposition.

Within each block $A_i \times B_j$, there is no problem, since $C_{f(i,j)}$ extends $A_i \times B_j$ by the definition of $C_{f(i,j)}$. The only possible problem arises between elements of different blocks. Suppose, toward a contradiction, that something goes wrong, that is, we have

$$\begin{array}{rl} & f(i, j) \ < \ f(k,l), \\ \text{and} & (\gamma, \delta) \ \in \ C_{f(i,j)},\ (\varepsilon, \zeta) \in C_{f(k,l)}, \\ \text{but} & (\varepsilon, \zeta) \ \leq \ (\gamma, \delta) \text{ in } \alpha \times \beta. \end{array}$$

By Claim 43, either $i < k$ or $j < l$. If $i < k$, then the segment A_i of α, which contains γ, precedes the segment A_j, which contains ε; hence $\gamma < \varepsilon$. Similarly, $j < l$ implies $\delta < \zeta$. In either case, this contradicts that $(\varepsilon, \zeta) \leq (\gamma, \delta)$ in $\alpha \times \beta$.

Case 2. Both α and β are powers of ω, say $\alpha = \omega^\mu$ and $\beta = \omega^\nu$. We need to prove that there is a linearization of the poset $\omega^\mu \times \omega^\nu$ of length $\omega^{\mu \oplus \nu}$.

Recall that every ω^γ is the length of a linearly ordered set B_γ such that $\mathrm{Dom}(B_\gamma)$ is the set of finite-support functions $f : \gamma \to \omega$ and the order of B_γ is reverse lexicographic (Section 2.4). Since the posets $\omega^\mu \times \omega^\nu$ and $B_\mu \times B_\nu$ are isomorphic, it suffices to prove that there is a linearization of $B_\mu \times B_\nu$ of the length of $B_{\mu \oplus \nu}$. To this end, it suffices to produce a bijection C from $B_\mu \times B_\nu$ onto $B_{\mu \oplus \nu}$ that is monotone in both arguments. Indeed such a map gives a linearization

$$(f, g) < (f', g') \iff C(f, g) < C(f', g')$$

of $B_\mu \times B_\nu$ of the length of $B_{\mu \oplus \nu}$.

By the definition of natural sum (Section 2.4), $\mu \oplus \nu$ can be partitioned into two subsets, M of length μ and N of length ν. Let B_M (respectively B_N) be the poset of finite-support functions f from M (respectively N) to ω ordered in the reverse lexicographic way. Since posets $B_M \times B_N$ and $B_\mu \times B_\nu$ are isomorphic, it suffices to prove that there is a bijection D from $B_M \times B_N$ onto $B_{\mu\oplus\nu}$ that is monotone in both arguments.

The desired map D sends (f, g) to the disjoint union $f \cup g$ (where we view a function as a set of ordered pairs). Then D is clearly a one-to-one map from $B_M \times B_N$ to $B_{M\cup N} = B_{\mu\oplus\nu}$. D is monotone; that is, if $f \le f'$ and $g \le g'$ then $D(f, g) \le D(f', g')$. This is because the last place (in $\mu \oplus \nu$) where $D(f, g)$ and $D(f', g')$ differ either lies in M and is the last place where f and f' differ or lies in N and is the last place where g and g' differ.

It remains to check that every finite-support function $h : \mu \oplus \nu \to \omega$ has the form $f \cup g$ where $f : M \to \omega$ and $g : N \to \omega$. The desired f, g are the restrictions of h to M, N respectively. □

PROPOSITION 44. *For every natural number n and all ordinals $\alpha_1, \dots, \alpha_n$,*

(1) *there is a linearization of $\alpha_1 \times \cdots \times \alpha_n$ of length $\alpha_1 \otimes \cdots \otimes \alpha_n$;*
(2) $\|\alpha_1 \times \cdots \times \alpha_n\| \ge \alpha_1 \otimes \cdots \otimes \alpha_n$.

PROOF. Again, in virtue of Proposition 32, the first claim implies the second. So it suffices to prove claim 1. We do that by induction on n. Cases $n < 2$ are trivial, and case $n = 2$ is Lemma 41. We suppose that $n \ge 2$ and that claim 1 holds for all natural numbers $\le n$, and we prove the case $n + 1$ of claim 1. By the induction hypothesis, there is a linearization A of $\alpha_1 \times \cdots \times \alpha_n$ of length $\alpha_1 \otimes \cdots \otimes \alpha_n$. So the (componentwise) partial order of $\alpha_1 \times \cdots \times \alpha_n \times \alpha_{n+1}$ can be extended to the componentwise partial order $A \times \alpha_{n+1}$ isomorphic to $(\alpha_1 \otimes \cdots \otimes \alpha_n) \times \alpha_{n+1}$. By Lemma 41, this can in turn be extended to a linear order of length $\alpha_1 \otimes \cdots \otimes \alpha_n \otimes \alpha_{n+1}$. □

6.2 The Natural Product is Large Enough

In this section, we prove that the stature of $\alpha_1 \times \cdots \times \alpha_n$ is at most as large as the natural product of the n ordinals.

PROPOSITION 45. *For every natural number n and all ordinals $\alpha_1, \dots, \alpha_n$, we have $\|\alpha_1 \times \cdots \times \alpha_n\| \le \alpha_1 \otimes \cdots \otimes \alpha_n$.*

PROOF. Without loss of generality $n \ge 2$. We prove the proposition by induction on $\alpha_1 \otimes \cdots \otimes \alpha_n$. The base case $\alpha_1 \otimes \cdots \otimes \alpha_n = 0$ (so that one of the ordinals α_i is 0) is trivial. The induction step splits into two cases.

Case 1. At least one of ordinals α_i is not of the form ω^μ. Without loss of generality, the Cantor normal form of α_1 has at least two terms. Notice that any ordinal is the *natural* sum (as well as the ordinary sum) of the terms in its Cantor normal form. So we have $\alpha_1 = \alpha' \oplus \alpha'' = \alpha' + \alpha''$ with both summands $< \alpha_1$. (To be specific, take α'' to be the last term in the Cantor normal form of α_1 and α' to be the sum of all the earlier terms.) So α_1 is the disjoint union of its initial segment α' and a final segment F of length α''. Therefore

$$\alpha_1 \times \alpha_2 \times \cdots \times \alpha_n = (\alpha' \times \alpha_2 \times \cdots \times \alpha_n) \cup (F \times \alpha_2 \times \cdots \times \alpha_n).$$

By Lemma 37, we have

$$\begin{aligned}\|\alpha_1 \times \alpha_2 \times \cdots \times \alpha_n\| &\leq \|\alpha' \times \alpha_2 \times \cdots \times \alpha_n\| \oplus \|F \times \alpha_2 \times \cdots \times \alpha_n\| \\ &\leq (\alpha' \otimes \alpha_2 \otimes \cdots \otimes \alpha_n) \oplus (\alpha'' \otimes \alpha_2 \otimes \cdots \otimes \alpha_n) \\ &= \alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_n,\end{aligned}$$

where the second inequality comes from the induction hypothesis and the final equality is the distributivity of $\otimes$ over $\oplus$.

Case 2. $\alpha_i = \omega^{\mu_i}$ for $i = 1, \ldots, n$. Let $P = \alpha_1 \times \cdots \times \alpha_n$. By Lemma 36, $\|P\|$ is the smallest ordinal strictly above $\|\mathrm{Ideal}_P(\xi_1, \ldots, \xi_n)\|$ for any $\xi_i \in \alpha_i$. So it suffices to show that, for all such $\xi_1, \ldots, \xi_n$,

$$\|\mathrm{Ideal}_P(\xi_1, \ldots, \xi_n)\| < \alpha_1 \otimes \cdots \otimes \alpha_n.$$

Fix an arbitrary tuple $(\xi_1, \ldots, \xi_n) \in P$ and let $I = \mathrm{Ideal}_P(\xi_1, \ldots, \xi_n)$. We show that $\|I\| < \alpha_1 \otimes \cdots \otimes \alpha_n$.

The set I is covered by the subsets $P_1, \ldots, P_n$ where P_i is obtained from P by replacing the ith factor α_i with ξ_i. By Lemma 37, $\|I\| \leq \|P_1\| \oplus \cdots \oplus \|P_n\|$. By the induction hypothesis, each $\|P_i\|$ is bounded by the natural product of ξ_i and $n-1$ ordinals α_j with $j \neq i$. By the strict monotonicity of $\otimes$, we have

$$\|P_i\| < \alpha_1 \otimes \cdots \otimes \alpha_n = \omega^{\mu_1 \oplus \cdots \oplus \mu_n}.$$

for each i. By Corollary 35, the ordinal $\omega^{\mu_1 \oplus \cdots \oplus \mu_n}$ is strictly larger than the natural sum of any finite number of smaller ordinals. It follows that

$$\|I\| \leq \|P_1\| \oplus \cdots \oplus \|P_n\| < \omega^{\mu_1 \oplus \cdots \oplus \mu_n} = \alpha_1 \otimes \cdots \otimes \alpha_n. \quad \square$$

Propositions 44 and 45 imply Theorem 6.

7. STATURE IS MAXIMAL LINEARIZATION LENGTH

In this section, we prove Theorem 10: the stature of a wpo set P is the largest among the lengths of linearizations of P. In Section 4.1, we showed that every linearization of P is well-founded and has length $\leq \|P\|$ (Proposition 32). It remains to prove that the supremum of linearization lengths of P is attainable and equal to $\|P\|$. This is easy if P is linearly ordered.

LEMMA 46. *If P is a well-ordered set then the supremum of linearization lengths of P is attainable and equal to $\|P\|$.*

PROOF. The first claim is trivial as there is only one linearization and so the supremum is $|P|$. To prove the second claim, recall that, by Proposition 31, $\|P\| = |\mathcal{P}(P)|$. Since P is linear, $\mathcal{P}(P)$ is isomorphic to P via the map $(D, d) \mapsto d$. $\square$

In the rest of this section, we prove that, for any wpo set P, the supremum of linearization heights of P is attainable and equal to $\|P\|$.

7.1 Long Consistent Sequence Suffices

Definition 47. Two posets P and Q are *consistent* if there is no pair $\{x, y\}$ such that $x <_P y <_Q x$.

A sequence $s = \langle x_\beta : \beta < \alpha \rangle$ of distinct elements of a poset P can be viewed as a linearly ordered set where $x_\beta \leq_s x_\gamma$ if and only if $\beta \leq \gamma$.

Definition 48. A sequence $s = \langle x_\beta : \beta < \alpha \rangle$ of distinct elements of P is *consistent* with P if the posets P and s are consistent.

In this subsection, we prove that a wpo set P has a linearization of length $\|P\|$ if it has a consistent sequence of elements of length $\|P\|$. We start with an auxiliary result.

LEMMA 49. *Let P be a poset (not necessarily wpo), and let A be a linearly ordered set with* $\mathrm{Dom}(A) \subseteq \mathrm{Dom}(P)$. *If A and P are consistent then there is a linearization of P that extends A.*

PROOF. Let R be the binary relation $\leq_P \cup \leq_A$. It suffices to prove that the digraph $G = (\mathrm{Dom}(P), R)$ is acyclic. Indeed, if G is acyclic then the transitive closure R^* of R is a partial order. Extend R^* to a linear order (by Zorn's lemma, any poset can be extended to a linearly ordered set) and get the desired linearization of P.

So suppose, toward a contradiction, that G has a cycle C. Since $\leq_P$ and $\leq_A$ are transitive, we can combine consecutive "steps" in the same ordering. Thus, without loss of generality, C has the form $x_0, x_1, \ldots, x_{n-1}$ where $x_i <_P x_{i+1}$ for even i and $x_i <_A x_{i+1}$ for odd i. Here we take the subscripts modulo n, so that when $i = n-1$ we interpret $i+1$ as 0. Also, n is even, because otherwise the steps from x_{n-2} to x_{n-1} and then to x_0 would be in the same one of A and P and could be combined. Note that each x_i is in A because it is related by $<_A$ to either x_{i-1} or x_{i+1}. Since $\leq_A$ linearly orders A, let j be the index for which x_j is largest, with respect to $\leq_A$.

In particular, $x_j >_A x_{j+1}$. But if j is odd then $x_j <_A x_{j+1}$, a contradiction. So j is even and $x_j <_P x_{j+1}$. But then A and P are inconsistent contrary to the hypothesis of the lemma. □

Remark 50. The linearity of A is essential for the proof of the lemma. It is not true that, if two posets P, Q with the same domain are consistent, then there is a partial order extending both of them. For a counterexample, take the common domain to be a four-element set $\{a, b, c, d\}$, take $<_P = \{(a, b), (c, d)\}$, and take $<_Q = \{(b, c), (d, a)\}$. Then P and Q are consistent, yet the union of the order relations contains a cycle.

LEMMA 51. *Let P be a wpo set, and suppose that there is a sequence $s = \langle x_\alpha : \alpha < \|P\| \rangle$ of elements of P consistent with P. Then there is a linearization of P of length $\|P\|$.*

PROOF. By Lemma 49, there is a linearization A of P that extends s. By Proposition 32, A is well-founded and $|A| \leq \|P\|$. Since A extends s, we have $|A| \geq |s|$; one easy way to see that $|A| \geq |s|$ is to use a game as in Section 3. But $|s| = \|P\|$. So $|A| = \|P\|$. □

7.2 Producing a Long Consistent Sequence

PROPOSITION 52. *For every wpo set P, there is a linearization of P of length* $\|P\|$.

PROOF. Fix a wpo set P. According to Lemma 51, it suffices to prove that there is a sequence $s = \langle x_\alpha : \alpha < \|P\|\rangle$ of elements of P consistent with P. We do that by induction on $\|P\|$.

Case 1. $\|P\| = 0$. Trivial.

Case 2. $\|P\|$ is a successor ordinal $\alpha + 1$. By Lemma 36, there is an element $x \in P$ such that $\|\mathrm{Ideal}_P(x)\| = \alpha$. Let I be $\mathrm{Ideal}_P(x)$ (viewed as a poset). By induction hypothesis, we have an α-sequence s of elements of I consistent with I. Appending x to s, we get the desired $\|P\|$-sequence of elements of P consistent with P.

Case 3. $\|P\|$ is a limit ordinal but not of the form ω^α. So the Cantor normal form of $\|P\|$ has at least two summands; let ω^ε be the last summand, and let δ be the sum of all the other terms in the Cantor normal form. So

$$\|P\| = \delta + \omega^\varepsilon = \delta \oplus \omega^\varepsilon.$$

By Lemma 36, there is $x \in P$ such that $\delta < \|\mathrm{Ideal}_P(x)\| < \|P\|$. Applying Lemma 37 with the posets $\mathrm{Ideal}_P(x)$ and $\mathrm{Filter}_P(x)$ in the roles of Q_1 and Q_2, we obtain

$$\|P\| \le \|\mathrm{Ideal}_P(x)\| \oplus \|\mathrm{Filter}_P(x)\|.$$

It follows, by Lemma 33, that we cannot have $\|\mathrm{Filter}_P(x)\| < \omega^\varepsilon$.

By induction hypothesis, $\mathrm{Ideal}_P(x)$ contains a consistent sequence s of length $\|\mathrm{Ideal}_P(x)\| > \delta$. And $\mathrm{Filter}_P(x)$ contains a consistent sequence t of length $\ge \omega^\varepsilon$. Indeed, let Q be the poset $\mathrm{Filter}_P(x)$. If $\|Q\| < \|P\|$ then the induction hypothesis gives t of length $\|\mathrm{Filter}_P(x)\| \ge \omega^\varepsilon$. Otherwise $\|Q\| = \|P\|$. Use Lemma 36 to find $w \in Q$ such that $\omega^\varepsilon \le \|\mathrm{Ideal}_Q(w)\| < \|P\|$. By applying the induction hypothesis to $\mathrm{Ideal}_Q(w)$, we again get a consistent sequence t of length $\ge \omega^\varepsilon$.

The concatenation $s^\frown t$ has length at least $\delta + \omega^\varepsilon = \|P\|$. It is consistent with P because s and t are consistent with P and because all elements of t and no elements of s are $\ge_P x$.

Case 4. $\|P\| = \omega^\alpha$ for some nonzero ordinal α. Let κ be the cofinality of ω^α, and let $\langle \beta_\xi : \xi < \kappa\rangle$ be a strictly increasing sequence of ordinals cofinal with ω^α. Recall that κ, being the cofinality of something, must be a regular cardinal.

LEMMA 53. *There is an increasing sequence $\langle x_\xi : \xi < \kappa\rangle$ of elements of P such that $\|\mathrm{Ideal}_P(x_\xi)\| > \beta_\xi$ for all $\xi < \kappa$.*

PROOF. For each $\xi < \kappa$, use Lemma 36 to obtain some $y_\xi \in P$ with $\|\mathrm{Ideal}_P(y_\xi)\| > \beta_\xi$. Although there may be repetitions in the sequence $\langle y_\xi\rangle$, no single element y can be y_ξ for κ different ordinals ξ. The reason is that, if there were such a y, then $\|\mathrm{Ideal}_P(y)\|$ would be greater than the corresponding ordinals β_ξ. As any κ of these ordinals have supremum ω^α, we would have $\|\mathrm{Ideal}_P(y_\xi)\| \ge \omega^\alpha = \|P\|$, which contradicts Lemma 36.

Because no element occurs κ times in the sequence $\langle y_\xi\rangle$ and because κ is regular, the set $S = \{\xi : y_\xi \ne y_\eta \text{ for all } \eta < \xi\}$ is of cardinality κ. We can therefore extract a subsequence of $\langle y_\xi\rangle$ in which there are no repetitions. Specifically, let $f(\xi)$ be the ξth ordinal in S, and let $y'_\xi = y_{f(\xi)}$. Then all the y'_ξ are distinct and, since $f(\xi) \geq \xi$, we have $\|\mathrm{Ideal}_P(y'_\xi)\| > \beta_{f(\xi)} \geq \beta_\xi$. (That $f(\xi) \geq \xi$ is probably intuitively evident; for a proof see Zuckerman [1974], Theorem 5.1.1). From now on, we work with the y'_ξ and we omit the primes.

Invoking again the regularity of κ, we can apply the Dushnik-Miller theorem, Theorem 22, to the partition where $S_1 = \{\{\xi < \eta\} : y_\xi \leq_P y_\eta\}$ and $S_2 = [\kappa]^2 - S_1$. If an infinite subset T of κ had $[T]^2 \subseteq S_2$, then the first ω elements of T would constitute an infinite bad sequence, contrary to the assumption that P is wpo. So, by the Dushnik-Miller theorem, there must be a κ-element subset $T \subseteq \kappa$ such that $[T]^2 \subseteq S_1$. Letting $g(\xi)$ denote the ξth ordinal in T and letting $x_\xi = y_{g(\xi)}$, we obtain the conclusion of the lemma. Indeed, the homogeneity of T ensures that the sequence $\langle x_\xi : \xi < \kappa\rangle$ is increasing, and because $g(\xi) \geq \xi$ we have $\|\mathrm{Ideal}_P(x_\xi)\| = \|\mathrm{Ideal}_P(y_{g(\xi)})\| > \beta_{g(\xi)} \geq \beta_\xi$. $\square$

LEMMA 54. *There is an increasing sequence $\langle x_\xi : \xi < \kappa\rangle$ of elements of P such that*

$$\|\mathrm{Ideal}(x_{\xi+1})\| > \|\mathrm{Ideal}(x_\xi)\| \oplus \beta_\xi \text{ for all } \xi < \kappa.$$

PROOF. By Lemma 53, there is an increasing sequence $s = \langle y_\xi : \xi < \kappa\rangle$ such that every $\|\mathrm{Ideal}_P(y_\xi)\| > \beta_\xi$. The desired $\langle x_\xi : \xi < \kappa\rangle$ is a subsequence of s built by recursion. Start with $x_0 = y_0$ and, at limit stages of the recursion, simply take the next y_η after all those previously taken. The nontrivial case is the successor step, where we already have x_ξ and must find an appropriate $x_{\xi+1}$. Since the statures of the sets $\mathrm{Ideal}(y_\eta)$ approach $\|P\| = \omega^\alpha$, it suffices to check that $\|\mathrm{Ideal}(x_\xi)\| \oplus \beta_\xi < \omega^\alpha$. Fortunately, this follows immediately from Corollary 34. This completes the proof of the lemma. $\square$

Let $\langle x_\xi : \xi < \kappa\rangle$ be as in Lemma 54. Temporarily fix some $\xi < \kappa$. Since $\mathrm{Ideal}(x_{\xi+1})$ is obviously the union of $\mathrm{Ideal}(x_\xi)$ and $\mathrm{Ideal}(x_{\xi+1}) \cap \mathrm{Filter}(x_\xi)$, Lemma 37 gives us that

$$\|\mathrm{Ideal}(x_{\xi+1})\| \leq \|\mathrm{Ideal}(x_\xi)\| \oplus \|\mathrm{Ideal}(x_{\xi+1}) \cap \mathrm{Filter}(x_\xi)\|.$$

Comparing this with Lemma 54, we find that

$$\|\mathrm{Ideal}(x_{\xi+1}) \cap \mathrm{Filter}(x_\xi)\| > \beta_\xi.$$

Applying the induction hypothesis to $\mathrm{Ideal}(x_{\xi+1}) \cap \mathrm{Filter}(x_\xi)$ (which is a subset of $\mathrm{Ideal}(x_{\xi+1})$ and so, by Lemma 36, has lower stature than P), we obtain, in $\mathrm{Ideal}(x_{\xi+1}) \cap \mathrm{Filter}(x_\xi)$, a sequence s_ξ of length at least β_ξ which is consistent with $\mathrm{Ideal}(x_{\xi+1}) \cap \mathrm{Filter}(x_\xi)$ and therefore is consistent with P.

Now unfix ξ. Let t be the concatenation of all the sequences s_ξ, in order of increasing ξ. The length of t is, for each ξ, at least β_ξ, since s_ξ is a segment of t. So the length of t is at least the supremum of the β_ξ's, which is ω^α.

To complete the proof, it remains only to check that t is consistent with P. Since each s_ξ has this property, the only thing that can go wrong is that there are $\xi < \eta$ with some y in s_η being $\leq_P$ some $x \in s_\xi$. To see that this cannot happen, suppose it did, and recall where these sequences s_ξ and s_η came from. The former was chosen from $\mathrm{Ideal}(x_{\xi+1}) \cap \mathrm{Filter}(x_\xi)$, so $x \not\geq x_{\xi+1}$, while the latter was chosen from $\mathrm{Ideal}(x_{\eta+1}) \cap \mathrm{Filter}(x_\eta)$, so $y \geq x_\eta$. Since the sequence

$\langle x_\xi : \xi < \kappa \rangle$ is increasing, and since $\xi < \eta$, we have

$$x \geq y \geq x_\eta \geq x_{\xi+1},$$

a contradiction. Proposition 52 is proved.

Propositions 32 and 52 imply Theorem 10.

8. RELATED WORK

We describe in this section earlier work on two concepts central to this article, namely, well partially ordered sets and natural products of ordinals.

8.1 Natural Products

Natural sums and natural products of ordinals are defined in Hausdorff [1927], pp. 68–70. Hausdorff credits these concepts to Hessenberg [1906], Section 75, but the cited section contains only natural sums, not products, nor have we found natural products elsewhere in Hessenberg [1906].

Carruth [1942] proved that every linearization of the componentwise partial order on $\alpha \times \beta$ has length at most $\alpha \otimes \beta$. In our presentation, this fact is a consequence of Propositions 32 and 45. Carruth's argument is fairly complex, using neither any notion of stature nor indeed any notion of well partially ordered set. Carruth's motivation came from the theory of ordered Abelian groups; he showed how to bound, in terms of the length of a well-ordered set X of positive elements in such a group, the length of the (necessarily also well-ordered) subsemigroup generated by X.

8.2 Well Partially Ordered Sets

Maurice Janet [1920] published a proof that a sequence of monomials in a fixed number n of variables must be finite if no monomial in the list divides a later one. This amounts to the statement that $\mathbb{N}^n$ is well partially ordered. See Lescanne [1989] for the relevant passages from Janet's article along with an English translation.

The general notion of well partially ordered sets was introduced by Higman [1952]. He called them *partially ordered sets with the finite basis property*. This terminology refers to the characterization given by item 3 in our Lemma 17, which Higman used as the definition. He proved several equivalent characterizations, including the main points of Lemma 17 and the well-foundedness of $\mathcal{I}$. His main result is that when P is wpo then so is the set of finite sequences from P, ordered by "componentwise majorized by a subsequence of."

The second author [Gurevich 1969] independently discovered the notion of wpo, introduced the terminology *tight partial order*, and proved some cases of Higman's [1952] result that he needed for investigations about decidability in predicate logic. The word *tight* was meant to refer to a boot, where one cannot move downward or sideways but only upward.

Kruskal [1960] developed the theory of wpo sets further, proving a celebrated result about certain posets of trees being wpo. He seems to have been the first to use the terminology *well-quasi-ordering*. (*Quasi* in place of *partial* means that

$\leq$ is not required to be antisymmetric. Many authors write *preorder* instead of *quasi-order*, but *prewellorder* means something different from well-quasi-order. A prewellorder is a preorder whose partially ordered quotient, obtained by identifying x and y whenever $x \leq y \leq x$, is a well-order.) Kruskal mentioned that previous authors had used the terms *well-partial-ordering* and *partial well-ordering*. Even at this early stage of the development of wpo theory, the terminology had become so chaotic that Kruskal gave, at the end of his 1960 article, a glossary for matching his terminology with Higman's [1952].

Kruskal [1972] described much of the early history of the wpo concept (though he was unaware of Gurevich [1969]). He mentioned yet another name for the concept, *fairly well-ordered*, used by Michael [1960].

De Jongh and Parikh [1977] gave several equivalent characterizations of wpo, adding to Higman's [1952] list the property that all linearizations are well-ordered. Furthermore, they showed that among the ordinal lengths of these linear orderings there is a largest one. In the case of a Cartesian product $\alpha \times \beta$ of ordinals, they showed that this largest length of a linearization is $\alpha \otimes \beta$. Recall that, by Theorem 10, the stature of a wpo set P equals the largest length of a linearization, which de Jongh and Parikh called $o(P)$. In this sense, their 1977 article can be regarded as introducing the notion of stature, though without a name and without other equivalent descriptions (such as our definition in terms of the forest of nonempty bad sequences).

The definition of stature that we use, the height of the forest of nonempty bad sequences, was studied by Kříž and Thomas [1990], who called it the *type* of P and used the notation $c(P)$ for it. They asserted (in their Theorem 4.7) that this equals the largest length of a linearization, but there seems to be a problem with the proof. Their Theorem 4.6 uses in an essential way that the disjoint union of two posets is defined with the two parts incomparable, but then this theorem is applied in a situation where the incomparability requirement is violated. Nevertheless, their Lemma 4.5 motivated our use of the Dushnik-Miller theorem in the proof of Lemma 53; in fact, their Lemma 4.5 essentially reproves the relevant case of the Dushnik-Miller theorem.

Remark 55. As already indicated, the notion of wpo set has acquired many names as a result of being discovered many times. (Yet another name, *Noetherian*, was used in Aschenbrenner and Pong [2004, p 33]; other authors, however, have used *Noetherian* to mean that the reverse ordering is well-founded.) If we could choose between the many names, we would prefer *tight*, and not just because one of us introduced it. It's short and (with the boot metaphor) descriptive, and it doesn't use *well* as an adjective (as in *well partial order*). A second choice would probably be *finite basis property*. Although longer, it summarizes nicely one of the equivalent characterizations of the notion. It also has the advantage of being the name used by Higman [1952], who introduced the concept.

Unfortunately, the terminology *well partially ordered* and its close relative *well quasi-ordered* are used so commonly, and the alternatives so rarely, that it seems hopeless to advocate a change of terminology now. We have therefore resigned ourselves to *wpo*.

An imaginative interpretation of *well partial order* is to invoke the other meaning of *well*, namely, a source of water from underground. Like a boot, a well (at least an old-fashioned one) is closed at the bottom and sides but open at the top. Think of a well partial order as a partial order where, as in a well, the only direction for unrestricted motion is upward.

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