The Independence of Inherent Ambiguity From Complementedness Among Context-Free Languages

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Abstract. Call a (context-free) language unambiguous if it is not inherently ambiguous. In the absence of evidence to the contrary, the suspicion has arisen that the unambiguous languages might be precisely those languages with context-free complements. The two theorems presented in this paper lay the suspicion to rest by providing (1) an inherently ambiguous language with context-free complement and (2) an unambiguous language without context-free complement. This establishes the independence of inherent ambiguity from complementedness among the context-free languages.

Definitions and notation used in this paper are as in [2].[1] We understand the claim that a set X has context-free complement as asserting that there is a finite alphabet Σ such that $X \subseteq \theta(\Sigma)$ and $\theta(\Sigma) - X$ is a language. Note that if $X \subseteq \theta(\Sigma)$, $\Sigma \subseteq \Sigma'$, Σ' finite, then $\theta(\Sigma) - X$ is a language if and only if $\theta(\Sigma') - X$ is.

Call a language *unambiguous* if it is not inherently ambiguous. In the absence of evidence to the contrary, the suspicion has arisen that the unambiguous languages might be precisely those languages with context-free complements. It is well known that there are unambiguous languages with context-free complements; each regular set is such. On the other hand, the inherently ambiguous languages given in [4] and [2] all fail to have context-free complement. The two theorems presented here lay the suspicion to rest by providing (1) an inherently ambiguous language with context-free complement and (2) an unambiguous language without context-free complement. This establishes the independence of inherent ambiguity from complementedness among the context-free languages.

THEOREM 1. *Let* $L = \{a^p b^q c^r d^s e^t \mid (p = q \wedge r = s) \vee (q = r \wedge s = t)\}$, $\Sigma = \{a, b, c, d, e\}$. *$L$ is an inherently ambiguous language and $\theta(\Sigma) - L$ is a language.*

PROOF. (1) *L is an inherently ambiguous language.*

Let τ be defined on N^5 by $\tau(p, q, r, s, t) = a^p b^q c^r d^s e^t$. Then $\tau^{-1}(L) = \{(p, q, r, s, t) \mid (p = q \wedge r = s) \vee (q = r \wedge s = t)\}$.

By [1, Th. 2.1], L is a language if and only if $\tau^{-1}(L)$ is *stratified semilinear*, that is, is a finite union of linear sets, each with a stratified set of periods. Set $Q_1 = L((0, 0, 0, 0, 0); (1, 1, 0, 0, 0), (0, 0, 1, 1, 0), (0, 0, 0, 0, 1))$ and $Q_2 = L((0, 0, 0, 0, 0); (1, 0, 0, 0, 0), (0, 1, 1, 0, 0), (0, 0, 0, 1, 1))$. Then $Q_1 \cup Q_2$ is stratified semilinear. Since $Q_1 \cup Q_2 = \tau^{-1}(L)$, L is a language.

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[1] In particular, "language" means context-free language and, for alphabet Δ, $\theta(\Delta)$ is the set of words over Δ.

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As in [2], a FUDLIP is a finite union of disjoint linear sets, each with independent periods. By a *stratified* FUDLIP we will mean a FUDLIP in which each set of periods is stratified. By [2, Th. 5.1] L is inherently ambiguous if and only if $\tau^{-1}(L)$ is not a stratified FUDLIP.

Suppose $\tau^{-1}(L) = \bigcup_{i=1}^{k} M_i$, where the $M_i = L(c_i ; P_i)$[2] are pairwise disjoint subsets of N^5 and each P_i is stratified. For $1 \leq i \leq k$ let $\Pi_i = L((0, 0, 0, 0, 0); P_i)$. Then Π_i is closed under addition and $M_i = \{c_i + g \mid g \text{ in } \Pi_i\}$. Let h be in Π_i for some i, let j be in $\{2, 3, 4\}$ and suppose $h(j - 1) \neq h(j)$.[3]

Since nh is in Π_i for every n, $c_i + nh$ is always in $M_i \subseteq \tau^{-1}(L)$. Therefore, for every n, either $(c_i + nh)(j - 1) = (c_i + nh)(j)$ or $(c_i + nh)(j) = (c_i + nh)\cdot(j + 1)$. Now $(c_i + nh)(\delta) = c_i(\delta) + nh(\delta)$ for $1 \leq \delta \leq 5$. So for every n, the quantity $c_i(j) + nh(j)$ is equal either to $c_i(j - 1) + nh(j - 1)$ or to $c_i(j + 1) + nh(j + 1)$. For those n for which the former equality holds we have $h(j) = h(j - 1) + [c_i(j - 1) - c_i(j)]/n$; since $h(j - 1) \neq h(j)$ there can be at most one such n. So for infinitely many n, $h(j) = h(j + 1) + [c_i(j + 1) - c_i(j)]/n$. It follows that $h(j) = h(j + 1)$, showing that:

(i) No Π_i contains h with $h(j-1) \neq h(j)$ and $h(j) \neq h(j + 1)$, j in $\{2, 3, 4\}$.

Suppose some Π_i contains h with $h(1) > 0$, $h(5) > 0$, $h(2) = h(3) = h(4) = 0$. Since $c_i + nh$ is always in $M_i \subseteq \tau^{-1}(L)$, either $(c_i + nh)(1) = (c_i + nh)(2)$ or $(c_i + nh)(4) = (c_i + nh)(5)$ for any given n. But if the former holds we have $c_i(1) + nh(1) = c_i(2)$; if the latter, $c_i(4) = c_i(5) + nh(5)$. Neither equation can hold for more than a single value of n. This establishes:

(ii) No Π_i contains h with $h(1) > 0$, $h(5) > 0$ and $h(2) = h(3) = h(4) = 0$.

Since $P_i \subseteq \Pi_i$ for every i, we may supplant "Π_i" by "P_i" in (i) and (ii) and so obtain consequences called (i′) and (ii′), respectively. Now let h be in some P_i, and suppose $h(1) > 0$. By (i′) either $h(1) = h(2)$ or $h(2) = h(3)$. Since P_i is stratified, it follows that if $h(1) = h(2)$ then $h(3) = h(4) = h(5) = 0$, while if $h(2) = h(3)$ then $h(2) = h(3) = 0$ and either $h(4) = 0$ or $h(5) = 0$. Now if $h = (h(1), 0, 0, h(4), 0)$ then $h(4) = 0$ by (i′); if $h = (h(1), 0, 0, 0, h(5))$ then $h(5) = 0$ by (ii′). Therefore:

(iii) For $1 \leq i \leq k$, if h is in P_i with $h(1) > 0$ then either $h = (h(1), h(1), 0, 0, 0)$ or $h = (h(1), 0, 0, 0, 0)$.

And by exactly symmetric argument:

(iv) For $1 \leq i \leq k$, if h is in P_i with $h(5) > 0$ then either $h = (0, 0, 0, h(5), h(5))$ or $h = (0, 0, 0, 0, h(5))$.

Let α be the set of those i $(1 \leq i \leq k)$ such that, for every j $(1 \leq j \leq 5)$, P_i contains a period q with $q(j) \neq 0$. Let Z be the finite set $\{c_i(j) \mid 1 \leq i \leq k; 1 \leq j \leq 5\}$; set $m = \max Z + 1$. For p in N^5, let $|p| = \min\{p(j) \mid 1 \leq j \leq 5\}$. Then if p is in $\tau^{-1}(L) = \bigcup_{i=1}^{k} M_i$ with $|p| \geq m$, p must be in an M_i such that i is in α. Let α_1 be the set of those i in α such that P_i contains a period $(\gamma, \gamma, 0, 0, 0)$, $\gamma > 0$; let α_2 be the set of i in α such that P_i contains a period $(0, 0, 0, \delta, \delta)$, $\delta > 0$. If i is in $\alpha_1 \cap \alpha_2$ then there are positive γ and δ such that P_i contains both $(\gamma, \gamma, 0, 0, 0)$ and $(0, 0, 0, \delta, \delta)$. But then Π_i contains $(\gamma, \gamma, 0, \delta, \delta)$; this violates (i). Therefore $\alpha_1 \cap \alpha_2 = \phi$. Now if i is in α then P_i contains periods g and h such that $g(1) > 0$ and $h(5) > 0$. If i is not in α_1 then, by (iii), $g = (g(1), 0, 0, 0, 0)$. If i is not in α_2 then, by (iv), $h = (0, 0, 0, 0, h(5))$. So if i is not in $\alpha_1 \cup \alpha_2$ then $g + h = (g(1), 0, 0, 0, h(5))$ is in Π_i, contradicting (ii). Therefore $\alpha = \alpha_1 \cup \alpha_2$.

[2] $L(c; P)$ is the linear set with constant c and periods the elements of P.

[3] For q in N^n, $q(j)$ denotes the jth coordinate of q, $1 \leq j \leq n$.

Let i be in α_1 with $(\gamma, \gamma, 0, 0, 0)$ in P_i, $\gamma > 0$. Since i is in α, P_i has a period g such that $g(3) > 0$. By (i$'$), $g(2) = g(3)$ or $g(3) = g(4)$. So by stratification, g is either $(0, g(3), g(3), 0, 0)$ or $(0, 0, g(3), g(3), 0)$. But the former is impossible, since it puts $(\gamma, \gamma + g(3), g(3), 0, 0)$ in Π_i and so violates (i). Therefore:

(v) If i is in α_1 then P_i contains $(0, 0, \eta, \eta, 0)$ for some $\eta > 0$.

By exactly symmetric argument:

(vi) If i is in α_2 then P_i contains $(0, \eta, \eta, 0, 0)$ for some $\eta > 0$.

Now let λ be the product of all positive coordinates of elements of $\bigcup_{i=1}^{k} P_i$. $\lambda > 0$. $\tau^{-1}(L)$ contains the quintuple u whose coordinates are all $m + \lambda$. Since $|u| \geq m$, u is in M_r for some r in α. Suppose r is in α_1. $\tau^{-1}(L)$ also contains $v = (m + \lambda, m, m, m, m)$, and since $|v| \geq m$, v is in M_s for some s in α. If s is in α_1 then P_s contains $h = (\gamma, \gamma, 0, 0, 0)$ for some $\gamma > 0$. But then $v + h = (m + \lambda + \gamma, m + \gamma, m, m, m)$ is in M_s, contradicting the fact that $M_s \subseteq \tau^{-1}(L)$. Hence s is in α_2, and P_s contains $g = (0, 0, 0, \delta, \delta)$ for some $\delta > 0$. By (vi), P_s also contains $f = (0, \eta, \eta, 0, 0)$ for some $\eta > 0$. Now δ and η are divisors of λ. Therefore $v + (\lambda/\eta)f + (\lambda/\delta)g = u$ is in M_s. Since u is in M_r and the M_i are pairwise disjoint, it follows that $r = s$. But this contradicts the disjointness of α_1 and α_2. So r cannot be in α_1. The remaining alternative is that r is in α_2. But then a symmetric argument using (v) and the quintuple $(m, m, m, m, m + \lambda)$ leads again to contradiction. Therefore $\tau^{-1}(L)$ is not a stratified FUDLIP, establishing (1).

(2) *$\theta(\Sigma) - L$ is a language.*

It suffices to show that $(\theta(\Sigma) - L) \cap a^*b^*c^*d^*e^*$ is a language, since $\theta(\Sigma) - L$ is the union of this set with the regular set $\theta(\Sigma) - a^*b^*c^*d^*e^*$. By [1, Th. 2.1], this comes to showing that $M = \{(p, q, r, s, t) \mid (p \neq q \vee r \neq s) \wedge (q \neq r \vee s \neq t)\}$ is stratified semilinear. Set $M_1 = \{(p, q, r, s, t) \mid p \neq q \wedge s \neq t\}$, $M_2 = \{(p, q, r, s, t) \mid q \neq r \wedge p + r \neq q + s\}$, $M_3 = \{(p, q, r, s, t) \mid r \neq s \wedge q + s \neq r + t\}$. We first show that $M = M_1 \cup M_2 \cup M_3$.

That $M_1 \subseteq M$ is obvious. If $\pi = (p, q, r, s, t)$ is in M_2 then $q \neq r$ while also either $p \neq q$ or $r \neq s$, so that π is in M. Similarly $M_3 \subseteq M$. Now suppose M contains $\pi = (p, q, r, s, t)$ with $p \neq q$. Either $s \neq t$, putting π in M_1, or $q \neq r$ and $s = t$. If $q \neq r$, $s = t$ and $r = s$ then π is in M_2; if $q \neq r$, $s = t$ and $r \neq s$ then π is in M_3. On the other hand, suppose π is in M with $p = q$ and $r \neq s$. Either $q \neq r$, putting π in M_2; or $q = r$ and $s \neq t$, putting π in M_3. Therefore $M = M_1 \cup M_2 \cup M_3$.

We say that $P \subseteq N^n$ *represents* any condition on n-tuples which is necessary and sufficient for membership in P. Now M_1 is the union of those quintuples in which either (a) $p > q$ and $s > t$; (b) $p > q$ and $t > s$; (c) $q > p$ and $s > t$; or (d) $q > p$ and $t > s$. Condition (a) is represented by $P = L((1, 0, 0, 1, 0); (1, 1, 0, 0, 0), (1, 0, 0, 0, 0), (0, 0, 1, 0, 0), (0, 0, 0, 1, 1), (0, 0, 0, 1, 0))$. Similarly, each of (b) through (d) is represented by a linear set (in fact a permutation of P) with stratified periods. Therefore M_1 is stratified semilinear.

Consider those of M_2's quintuples (p, q, r, s, t) in which $q > r$. These are exactly those in which either (e) $q - r > p - s \geq 0$; (f) $q - r > 0 \geq p - s$; or (g) $p - s > q - r \geq 0$. Condition (e) is represented by $Q = L((0, 1, 0, 0, 0); (0, 1, 1, 0, 0), (1, 0, 0, 1, 0), (0, 0, 0, 0, 1), (0, 1, 0, 0, 0), (1, 1, 0, 0, 0))$; (f) by the like set with $(0, 0, 0, 1, 0)$ as period in place of $(1, 1, 0, 0, 0)$; (g) by the obvious permutation of Q, in fact, by the linear set just like Q except for having $(1, 0, 0, 0, 0)$ as constant and period in place of $(0, 1, 0, 0, 0)$. So the set of those quintuples

(p, q, r, s, t) from M_2 in which $q > r$ is stratified semilinear, and symmetrically for those with $r > q$. Therefore M_2 is stratified semilinear; M_3 is by similar argument, whence M is, establishing (2).

THEOREM 2. *Let* $L = \{a^p b^q c^r d^s \mid ((10p < q < 12p \vee 10q < p < 12q) \wedge (10r < s < 12r \vee 10s < r < 12s)) \vee (10q < r < 12q \wedge 6p < s < 8p)\}$, $\Sigma = \{a, b, c, d\}$. *L is an unambiguous language and $\theta(\Sigma) - L$ is not a language.*

PROOF. Let τ be the mapping on N^4 defined by $\tau(p, q, r, s) = a^p b^q c^r d^s$. For $x > 0$ let $M(x) = \{(p, q) \mid xp < q < (x + 2)p\}$, $N(x) = \{(p, q) \mid (q, p) \text{ in } M(x)\}$.

(1) *L is an unambiguous language.*

For every $x > 0$, $M(x) = L((1, x + 1); (1, x), (1, x + 1)) \cup L((2, 2x + 3); (1, x + 1), (1, x + 2))$ is a stratified FUDLIP. Symmetrically, so is $N(x)$. Since $M(10)$ and $N(10)$ are clearly disjoint, $M(10) \cup N(10)$ is also a stratified FUDLIP. We use two principles which are easily verified:

(i) If X and Y are stratified FUDLIPS so is $X \times Y$.

(ii) If $X \subseteq N^n$ is a stratified FUDLIP so is $\{(p(n), p(1), \cdots, p(n - 1)) \mid p \text{ in } X\}$.

By (i), both $(M(10) \cup N(10))^2$ and $M(10) \times N(6)$ are stratified FUDLIPS. By (ii), $P = \{(p, q, r, s) \mid (q, r, s, p) \text{ in } M(10) \times N(6)\}$ is also. It is not difficult to verify that $(M(10) \cup N(10))^2$ and P are disjoint. Therefore $(M(10) \cup N(10))^2 \cup P$ is a stratified FUDLIP. But $(M(10) \cup N(10))^2 \cup P = \{(p, q, r, s) \mid ((p, q) \text{ in } M(10) \cup N(10) \wedge (r, s) \text{ in } M(10) \cup N(10)) \vee ((q, r) \text{ in } M(10) \wedge (p, s) \text{ in } M(6))\} = \tau^{-1}(L)$, and (1) now follows by [2, Th. 5.1].

(2) *$\theta(\Sigma) - L$ is not a language.*

If $\theta(\Sigma) - L$ is a language so is its intersection H with the regular set $a^* b^* c^* d^*$. By [1, Th. 2.1], H is a language if and only if $\tau^{-1}(H)$ $(= N^4 - \tau^{-1}(L))$ is stratified semilinear. It will be shown that $\tau^{-1}(H)$ cannot be stratified semilinear.

Suppose that $\tau^{-1}(H) = \bigcup_{i=1}^{l} M_i$, where each $M_i = L(c_i; P_i)$ with stratified P_i. It may be assumed that no P_i contains $(0, 0, 0, 0)$. Consider the set $Q = \{(3n, 2n, m, m) \mid n, m \geq 0\}$. For each n there are infinitely many m, including all $m \geq 24n$, such that $(3n, 2n, m, m)$ is in $\tau^{-1}(H)$. It follows that there exist $t \leq l$ and an infinite set of numbers R such that, for every n in R, $S(n) = \{m \mid (3n, 2n, m, m) \text{ in } M_t\}$ is infinite. Let $p_1, \cdots, p_h$ be the elements of P_t and set $\Pi = L((0, 0, 0, 0); P_t)$. Then $\Pi = \{\sum_{i=1}^{h} e(i)p_i \mid e \text{ in } N^h\}$ is closed under addition, and $M_t = \{c_t + w \mid w \text{ in } \Pi\}$.

Pick n_0 in R and let $E = \{e \text{ in } N^h \mid \text{for some } m \text{ in } S(n_0),\ c_t + \sum_{i=1}^{h} e(i)p_i = (3n_0, 2n_0, m, m)\}$. E is infinite. It follows from a result in [3, p. 168] that E contains distinct elements e_1 and e_2 such that $e_1(i) \leq e_2(i)$ for $1 \leq i \leq h$. Now there are m_1 and m_2 such that $c_t + \sum_{i=1}^{h} e_j(i)p_i = (3n_0, 2n_0, m_j, m_j)$ for $j = 1, 2$, and since no p_i is $(0, 0, 0, 0)$, $m_2 > m_1$. So $\sum_{i=1}^{h} (e_2 - e_1)(i)p_i = c_t + \sum_{i=1}^{h} e_2(i)p_i - (c_t + \sum_{i=1}^{h} e_1(i)p_i) = (3n_0, 2n_0, m_2, m_2) - (3n_0, 2n_0, m_1, m_1) = (0, 0, m_2 - m_1, m_2 - m_1)$. Therefore, with $\gamma = m_2 - m_1 > 0$:

(i) $s = (0, 0, \gamma, \gamma)$ is in Π.

Let f be a one-one function mapping the positive integers into R. There is a one-one function g on the positive integers such that $g(j)$ is in $S(f(j))$ for all $j \geq 1$. Let $F = \{e \text{ in } N^h \mid \text{for some } j,\ c_t + \sum_{i=1}^{h} e(i)p_i = (3f(j), 2f(j), g(j), g(j))\}$. F is infinite. Again from [3, p. 168] it follows that F contains distinct elements u_1 and

u_2 such that $u_1(i) \leq u_2(i)$ for $1 \leq i \leq h$. Now there are v_1 and v_2 such that $c_t + \sum_{i=1}^{h} u_j(i)p_i = (3f(v_j), 2f(v_j), g(v_j), g(v_j))$ for $j = 1, 2$. Since no p_i is $(0, 0, 0, 0)$ and f and g are one-one, $f(v_2) > f(v_1)$ and $g(v_2) > g(v_1)$. So $\sum_{i=1}^{h} (u_2 - u_1)(i)p_i = c_t + \sum_{i=1}^{h} u_2(i)p_i - (c_t + \sum_{i=1}^{h} u_1(i)p_i) = (3f(v_2), 2f(v_2), g(v_2), g(v_2)) - (3f(v_1), 2f(v_1), g(v_1), g(v_1)) = (3(f(v_2) - f(v_1)), 2(f(v_2) - f(v_1)), g(v_2) - g(v_1), g(v_2) - g(v_1))$. Therefore, with $\alpha = f(v_2) - f(v_1) > 0$, $\beta = g(v_2) - g(v_1) > 0$ and $u = u_2 - u_1$:

(ii) $\sum_{i=1}^{h} u(i)p_i = (3\alpha, 2\alpha, \beta, \beta)$ is in Π.

Let $M(x)$ and $N(x)$ be as above. Now it is obvious that for all $x > 0$ and y in N^2, if z is in $M(x)$ then $y + nz$ is also in $M(x)$ for sufficiently large n. Similarly for $N(x)$. It follows that if p is in $\tau^{-1}(L)$ then so is $c_t + np$ for n sufficiently large. But if p is in Π, $c_t + np$ is in M_t for every n. So, since $M_t \cap \tau^{-1}(L) = \phi$:

(iii) Π contains no element of $\tau^{-1}(L)$.

Suppose Π contains $(3\lambda, 2\lambda, 0, 0)$ for some $\lambda > 0$. Then in view of (i), Π contains $\xi = 21\lambda s + \gamma(3\lambda, 2\lambda, 0, 0) = (3\lambda\gamma, 2\lambda\gamma, 21\lambda\gamma, 21\lambda\gamma)$. Now $(\xi(2), \xi(3))$ is in $M(10)$ and $(\xi(1), \xi(4))$ is in $M(6)$, so that ξ is in $\tau^{-1}(L)$, violating (iii). Therefore:

(iv) For no $\lambda > 0$ is $(3\lambda, 2\lambda, 0, 0)$ in Π.

Call quadruple p a j-period $(1 \leq j \leq 4)$ if $p(j) > 0$ while $p(i) = 0$ for $i \neq j$. If p is the sum of a j-period and a k-period, $j \neq k$, call p a j-k-period. By stratification of P_t, every p_i $(1 \leq i \leq h)$ is either a j-period or a j-k-period for some $j, k \leq 4$. For $j = 1, 2$, let $\Delta_j = \{i \mid p_i(j) > 0, 1 \leq i \leq h\}$. By (ii), neither Δ_1 nor Δ_2 is empty. With u as in (ii), $\sum_{i \text{ in } \Delta_1 \cup \Delta_2} u(i)p_i = (3\alpha, 2\alpha, \lambda_1, \lambda_2)$ for some $\lambda_1 \leq \beta$ and $\lambda_2 \leq \beta$, and this quadruple is in Π. If every element of $\Delta_1 \cup \Delta_2$ is either a 1-period, a 2-period, or a 1-2-period, then $\lambda_1 = \lambda_2 = 0$ and Π contains $(3\alpha, 2\alpha, 0, 0)$ in violation of (iv). Therefore:

(v) P_t contains either a 1-3-period, a 1-4-period, a 2-3-period or a 2-4-period.

Let $\{\mu, \nu\} = \{1, 2\}$ and $\{\eta, \delta\} = \{3, 4\}$. Suppose that q is a μ-η-period contained in P_t and that P_t contains no ν-δ-period. Then for every i in Δ_ν, p_i is either a ν-period, a ν-μ-period or a ν-η-period. Now $r = \sum_{i \text{ in } \Delta_\nu} u(i)p_i$ is in Π, with $r(1) \leq 3\alpha$, $r(2) \leq 2\alpha$, $r(\eta) \leq \beta$ and $r(\delta) = 0$. If $\nu = 1$ then $r(\nu) = 3\alpha$; if $\nu = 2$ then $r(\nu) = 2\alpha$. So $11r(\nu) \geq 22\alpha > 3\alpha \geq r(\mu)$. By use of (i), Π contains $\rho = 10\gamma(11r(\nu) - r(\mu))q + 10\gamma q(\mu)r + [q(\eta)(11r(\nu) - r(\mu)) + q(\mu)r(\eta)]s$. Now $\rho(\mu) = 10\gamma(11r(\nu) - r(\mu))q(\mu) + 10\gamma q(\mu)r(\mu) = 110\gamma q(\mu)r(\nu)$; $\rho(\nu) = 10\gamma q(\mu)r(\nu)$; $\rho(\eta) = 10\gamma(11r(\nu) - r(\mu))q(\eta) + 10\gamma q(\mu)r(\eta) + [q(\eta)(11r(\nu) - r(\mu)) + q(\mu)r(\eta)]\gamma = 11[q(\eta)(11r(\nu) - r(\mu)) + q(\mu)r(\eta)]\gamma$; $\rho(\delta) = [q(\eta)(11r(\nu) - r(\mu)) + q(\mu)r(\eta)]\gamma$. So $\rho(\mu) = 11\rho(\nu)$, $\rho(\eta) = 11\rho(\delta)$ and $\rho(j) > 0$ for $1 \leq j \leq 4$, whereby $(\rho(1), \rho(2))$ and $(\rho(3), \rho(4))$ are in $M(10) \cup N(10)$. Then ρ is in $\tau^{-1}(L)$, contradicting (iii). This establishes that if P_t contains a μ-η-period then it must also contain a ν-δ-period. But since P_t is stratified, it cannot contain both a 1-3-period and a 2-4-period. Therefore, applying (v):

(vi) P_t contains both a 1-4-period and a 2-3-period.

Now let x and y be a 1-4-period and 2-3-period, respectively, in P_t. If $x(4)y(2) \geq x(1)y(3)$ let $\varphi = x$, $\psi = y$, $(\mu, \eta) = (1, 4)$ and $(\nu, \delta) = (2, 3)$; if $x(1)y(3) > x(4)y(2)$ let $\varphi = y$, $\psi = x$, $(\mu, \eta) = (2, 3)$ and $(\nu, \delta) = (1, 4)$. Then in either case φ is a μ-η-period, ψ is a ν-δ-period, and $\varphi(\eta)\psi(\nu) \geq \varphi(\mu)\psi(\delta)$. By use of (i), Π contains $\zeta = 110\gamma\psi(\nu)\varphi + 10\gamma\varphi(\mu)\psi + 11[\varphi(\eta)\psi(\nu) - \varphi(\mu)\psi(\delta)]s$. Now $\zeta(\mu) = 110\gamma\psi(\nu)\varphi(\mu)$; $\zeta(\nu) = 10\gamma\varphi(\mu)\psi(\nu)$; $\zeta(\eta) = 110\gamma\psi(\nu)\varphi(\eta) + 11[\varphi(\eta)\psi(\nu) - \varphi(\mu)\psi(\delta)]\gamma = [121\varphi(\eta)\psi(\nu) - 11\varphi(\mu)\psi(\delta)]\gamma$; $\zeta(\delta) = 10\gamma\varphi(\mu)\psi(\delta) +$

$11[\varphi(\eta)\psi(\nu) - \varphi(\mu)\psi(\delta)]\gamma = [11\varphi(\eta)\psi(\nu) - \varphi(\mu)\psi(\delta)]\gamma$. So $\zeta(\mu) = 11\zeta(\nu)$, $\zeta(\eta) = 11\zeta(\delta)$ and $\zeta(j) > 0$ for $1 \leq j \leq 4$. Hence ζ is in $\tau^{-1}(L)$, violating (iii). Therefore (vi) is impossible, $\tau^{-1}(H)$ is not stratified semilinear, and (2) is established.

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