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A COUNTEREXAMPLE TO A CONJECTURE OF SCOTT AND SUPPES

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In [1], it is conjectured that if S is a sentence in the first-order functional calculus with identity, and every subsystem of every finite relational system which satisfies S also satisfies S, then S is finitely equivalent to a universal sentence. (Two sentences are *finitely equivalent* if and only if they are satisfied by the same finite relational systems.) The following sentence S refutes that conjecture, and moreover S is satisfied by all finite subsystems of all (finite or infinite) relational systems which satisfy it.[1] S contains as predicate letters only the two-place predicate letters $\leqq$, R (and the identity symbol $=$).

S is the conjunction of the following three sentences:

$$(1)\quad \forall x \forall y \forall z[[x \leqq y \vee y \leqq x] \,\&\, [[x \leqq y \,\&\, y \leqq x] \supset x = y] \,\&\, [[x \leqq y \,\&\, y \leqq z] \supset x \leqq z]],$$

$$(2)\quad \forall x \forall y \forall z[Rxy \supset [z \leqq x \vee y \leqq z]],$$

$$(3)\quad \forall x \forall y[[Rxy \,\&\, y \leqq x] \supset \exists z[y \leqq z < x \,\&\, \forall u[z \leqq u \supset \sim Rzu]].$$

Sentence (1) asserts that $\leqq$ is a simple ordering. (2) implies that if Rxy and $x \leqq y$, then either $x = y$ or y is the successor of x. Sentence (1) implies $\forall x[x \leqq x]$, and this together with (3) implies

$$(4)\quad \forall x \sim Rxx.$$

Hence (1), (2) and (3) imply that Rxy and $x \leqq y$ hold only if y is the successor of x.

Let $\mathfrak{A} = \langle A, \leqq, R\rangle$ be a relational system which satisfies (1), (2) and (4). Since these sentences are universal, any subsystem of $\mathfrak{A}$ will also satisfy them. A sequence $(a_1, a_2, \ldots, a_n)$ of elements of A is called a *cycle* in $\mathfrak{A}$ if $a_1 \leqq a_2 \leqq \ldots \leqq a_n$, Ra_ja_{j+1} for $j = 1, \ldots, n-1$, and Ra_na_1. $\mathfrak{A}$ satisfies (3) only if it has no cycles. If $\mathfrak{A}$ is finite and satisfies (1), (2) and (4), then it satisfies (3) if and only if it has no cycles. If $\mathfrak{B}$ is a subsystem of $\mathfrak{A}$, then every cycle of $\mathfrak{B}$ is clearly a cycle of $\mathfrak{A}$. Hence, if $\mathfrak{A}$ satisfies S, and $\mathfrak{B}$ is a finite subsystem of $\mathfrak{A}$, then, since $\mathfrak{B}$ satisfies (1), (2) and (4) and has no cycles, it satisfies (3). Thus, every finite subsystem of a relational system satisfying S also satisfies S.

For each $n > 0$, let $\mathfrak{A}_n = \langle\{0, 1, 2, \ldots, n\}, \leqq, R\rangle$, where $\leqq$ is the natural ordering of $\{0, 1, 2, \ldots, n\}$ and Rxy means that $x \neq n$ and $y = x+1$ or

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[1] Both the sentence S and our argument have been simplified due to suggestions by D. Scott.

that $x = n$ and $y = 0$. Since $(0, 1, 2, \ldots, n)$ is a cycle in $\mathfrak{A}_n$, $\mathfrak{A}_n$ does not satisfy S, though it satisfies (1), (2) and (4). On the other hand, no proper subsystem of $\mathfrak{A}_n$ contains cycles. Hence, every proper subsystem of $\mathfrak{A}_n$ satisfies S. But by the remark on page 124 of [1], this proves that S is finitely equivalent to no universal sentence containing precisely n distinct variables; and this holds for every $n > 0$.

Notice that, by Theorem 1.8 of [2], a relational system $\mathfrak{A}$ with an (infinite) subsystem $\mathfrak{B}$ must exist such that $\mathfrak{A}$ satisfies S and $\mathfrak{B}$ does not. Indeed, such a system exists: Let A be the set of ordinal numbers $x \leqq \omega+1$, let $\leqq$ be the natural ordering of these ordinals, and let Rxy mean that $y = x' < \omega$ or that $y = 0$ and $x = \omega+1$. Then $\mathfrak{A} = \langle A, \leqq, R\rangle$ satisfies S, but the subsystem resulting from $\mathfrak{A}$ by dropping the single element ω from A does not satisfy S.

REFERENCES

[1] D. SCOTT and P. SUPPES, *Foundational aspects of theories of measure*, this JOURNAL, vol. 23 (1958), pp. 113–128.

[2] A. TARSKI, *Contributions to the theory of models*, II, ***Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen,*** ser. A, vol. 57 (1954), pp. 582–588.

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