THE BULLETIN OF SYMBOLIC LOGIC
Volume 16, Number 2, June 2010

KLEENE'S AMAZING SECOND RECURSION THEOREM

YIANNIS N. MOSCHOVAKIS

CONTENTS

Part 1. Self-reference. ... 192
1. Self reproducing Turing machines ... 192
2. Myhill's characterization of r.e.-complete sets ... 193
3. The Myhill–Shepherdson Theorem ... 194
   3.1. Aside: the two recursion theorems ... 196
4. The Kreisel–Lacombe–Shoenfield–Ceitin Theorem ... 197
5. Incompleteness and undecidability ... 200
6. Solovay's theorem in provability logic ... 204

Part 2. Effective grounded recursion. ... 205
7. Constructive ordinals ... 205
8. Markwald's Theorem ... 208
9. The hyperarithmetical hierarchy ... 209
   9.1. Spector's Uniqueness Theorem ... 210
   9.2. Kleene's Theorem: $\mathsf{HYP} = \Delta^1_1$ ... 214
   9.3. HYP is the smallest effective σ-field ... 218
10. Effective Descriptive Set Theory ... 220
   10.1. The Suslin–Kleene Theorem ... 223
   10.2. The Normed Induction Theorem ... 224
   10.3. The Coding Lemma ... 229
11. Recursion in higher types and Jackson's Theorem ... 230
12. Realizability ... 231

Appendix: preliminaries and notation. ... 231

This little gem is stated unbilled and proved (completely) in the last two lines of §2 of the short note Kleene [1938]. In modern notation, with all the hypotheses stated explicitly and in a strong (uniform) form, it reads as follows:

Received September 5, 2009.

This is an elementary, expository article (with some nods to anecdotal history), written to commemorate the passage of 100 years since the birth of Stephen Cole Kleene. It was the basis of a talk at *Computer Science Logic 2009*, and an extended abstract of it (Moschovakis [2009b]) has been published in the proceedings of that meeting.

1079-8986/10/1602-0002/$6.10

Second Recursion Theorem (SRT). *Fix a set $\mathbb{V} \subseteq \mathbb{N}$, and suppose that for each natural number $n \in \mathbb{N} = \{0, 1, 2, \dots\}$, $\varphi^n : \mathbb{N}^{1+n} \rightharpoonup \mathbb{V}$ is a recursive partial function of $(1+n)$ arguments with values in $\mathbb{V}$ so that* ***the standard assumptions*** (a) *and* (b) *hold with*[1]

$$\{e\}(\vec{x}) = \varphi^n_e(\vec{x}) = \varphi^n(e, \vec{x}) \quad (\vec{x} = (x_1, \dots, x_n) \in \mathbb{N}^n).$$

(a) *Every n-ary recursive partial function with values in $\mathbb{V}$ is φ^n_e for some e.*
(b) *For all m, n, there is a recursive function $S = S^m_n : \mathbb{N}^{m+1} \to \mathbb{N}$ such that*

$$\{S(e, \vec{y})\}(\vec{x}) = \{e\}(\vec{y}, \vec{x}) \quad (e \in \mathbb{N}, \vec{y} \in \mathbb{N}^m, \vec{x} \in \mathbb{N}^n).$$

Then, for every recursive, partial function $f(e, \vec{y}, \vec{x})$ of $(1+m+n)$ arguments with values in $\mathbb{V}$, there is a total recursive function $\tilde{z}(\vec{y})$ of m arguments such that

$$\{\tilde{z}(\vec{y})\}(\vec{x}) = f(\tilde{z}(\vec{y}), \vec{y}, \vec{x}) \quad (\vec{y} \in \mathbb{N}^m, \vec{x} \in \mathbb{N}^n). \tag{1}$$

Proof. Fix $e \in \mathbb{N}$ such that $\{e\}(t, \vec{y}, \vec{x}) = f(S(t, t, \vec{y}), \vec{y}, \vec{x})$ and let $\tilde{z}(\vec{y}) = S(e, e, \vec{y})$.[2] ⊣

We will abuse notation and write $\tilde{z}$ rather than $\tilde{z}(\,)$ when $m = 0$, so that (1) takes the simpler form

$$\{\tilde{z}\}(\vec{x}) = f(\tilde{z}, \vec{x}) \tag{2}$$

in this case (and the proof sets $\tilde{z} = S(e, e)$).

The partial functions φ^n in the standard assumptions are called ***universal*** (for the recursive partial functions into $\mathbb{V}$), with corresponding ***partial evaluation functions*** S^m_n.

Kleene states the theorem with $\mathbb{V} = \mathbb{N}$, relative to specific φ^n, S^m_n, supplied by his *Enumeration Theorem*, $m = 0$ (no *parameters* $\vec{y}$) and $n \geq 1$, i.e., not allowing *nullary partial functions*. And most of the time, this is all we need; but there are some important applications where choosing "the right" φ^n, S^m_n, restricting the values to a proper $\mathbb{V} \subsetneq \mathbb{N}$ or allowing $m > 0$ or $n = 0$ simplifies the proof considerably. With $\mathbb{V} = \{\!\{0\}\!\}$ (singleton 0) and $m = n = 0$, for example, the characteristic equation

$$\{\tilde{z}\}() = f(\tilde{z}) \tag{3}$$

is a rather "pure" form of self-reference, where the number $\tilde{z}$ produced by the proof (as a code of a nullary semirecursive relation) has the *property* $f(\tilde{z})$, at least when $f(\tilde{z})\downarrow$.

[1] I will use both $\varphi^n_e(\vec{x})$ (sometimes without the ^{n}) and Kleene's favorite $\{e\}(\vec{x})$ for the value of the recursive partial function of n arguments with code (Gödel number) e, choosing in each case that notation which makes it easier to see the relevant point.

[2] The proof has always seemed too short and tricky, and some considerable effort has gone into explaining how one discovers it short of "fiddling around" with Hypothesis (b) which is evidently relevant, cf. Rogers [1967]. Some of his students asked Kleene about it once, and his (perhaps facetious) response was that he just "fiddled around with (b)"—but his fiddling may have been informed by similar results in the untyped λ-calculus.

Kleene uses the theorem in the very next page to prove that there is a largest initial segment of the countable ordinals which can be given "constructive notations", in the first application of what we now call *effective grounded* (or *transfinite*) *recursion*, one of the main uses of the result; but there are many others, touching most parts of logic and even classical analysis.

My aim is to list, discuss, explain and in many cases prove some of the most significant applications of the Second Recursion Theorem, in a kind of "retrospective exhibition" of the work that it has done since 1938. Altogether, I will discuss eighteen results in which SRT plays an important role, and I will prove thirteen of them, at least in outline.[3] These were selected (obviously) by what I know and what I like, but also because of their foundational significance and the variety of ways in which they witness how SRT is used. It is quite impressive, actually, the power of such a simple fact with a one-line proof; but part of its wide applicability stems precisely from this simplicity, which makes it easy to formulate and establish useful versions of it in many contexts, even outside ordinary recursion theory on $\mathbb{N}$. Some of the more important applications are in *Effective Descriptive Set Theory*, where the useful versions of SRT are obtained by replacing $\mathbb{N}$ by the Baire space $\mathcal{N} = (\mathbb{N} \to \mathbb{N})$ (or any Polish space, for that matter) and applying various suitably formulated "recursion theories".

Speaking rather loosely, the identity (1) expresses a *self-referential* property of $\tilde{z}(\vec{y})$ and SRT is often applied to justify powerful, self-referential definitions. I will discuss some of these in Part 1, and then in Part 2, I will turn to applications of effective grounded recursion, which was Kleene's favorite (and as far as I know only) way of applying SRT. The Appendix gives a brief account of some basic facts I need which are not as generally known as they should be, especially about recursion and definability in Baire space.[4] The paragraphs in the Appendix are numbered A1, A2, ... , and there are occasional references to them in the text.

Incidentally, Kleene always refers to *the recursion theorem*, even in his [1952] where he also states and proves *the First Recursion Theorem*, and this is the way that the result was generally called for quite some time. To the best of my knowledge, Rogers [1967] was first to suggest that it should be called *the Second Recursion Theorem* in contexts where the first one is also discussed, but I rather like the name and would vote to adopt it in general.[5]

---

[3]Well, some of the proofs are a bit abbreviated and require effort to reconstruct them fully from what is presented; I thought that this is not entirely unsuitable in a work whose aim is primarily to honor Stephen Cole Kleene.

[4]I will assume that the reader knows the basic elements of ordinary (classical) recursion theory on $\mathbb{N}$, but even in this, I will occasionally recall some simple definitions as I need them, partly to set up notation.

[5]Theorem 3.2 below is a simultaneous generalization of both recursion theorems.

Part 1. Self-reference.

Leo Harrington often embarks on the construction of some horrendously complex recursive partial function $\varphi_{\tilde{z}}$ by invoking SRT in the following, all-powerful form:

You need a number $\tilde{z}$? *You got it*!

I do not know a more precise (or more eloquent) quick description which captures all uses of SRT to express self-reference.[6]

§**1. Self reproducing Turing machines.** In one of the standard definitions, a deterministic Turing machine M on the alphabet $\Sigma = \{a_0, \dots, a_{N-1}\}$ and with states $q_0, \dots, q_K$ operates on semi-infinite "tapes" with symbols from Σ in some of their cells. The machine M is determined by a finite sequence of *transitions*, i.e., quintuples of the form

$$q_i \; s \; q_j \; s' \; m$$

where q_i is the *old state*; s is the mark in *the scanned cell*, and it is either in Σ or the fresh (not in Σ) *blank symbol* $\sqcup$; q_j is the *new state*; s' is the new symbol or $\sqcup$, put in the scanned cell; and m is the *move*, one of *left, center, right*. If we assume for simplicity that Σ has at least three symbols, 0, 1 and the comma "," then the transition above is specified uniquely by the non-empty sequence of symbols from Σ (no blanks)

$$i \text{ in binary}, s, j \text{ in binary}, s', mm$$

where s is either a symbol or just omitted, and similarly for s', and the move is coded by one of the two digit combinations 10, 00, 01; for example

$$q_0 \sqcup q_2 \ , \text{ right} \ \text{ is specified by the string } \quad 0,,10,,,01\ .$$

We can then identify every Turing machine M on Σ with a finite sequence of such strings separated by commas, i.e., another string in the set Σ^* of non-empty strings from Σ.

Theorem 1.1. *On every alphabet Σ with $N \geq 3$ symbols, there is a Turing machine M which started on the blank tape outputs itself.*

Proof. For each number u, let $s(u)$ be its unique expansion in base N using the symbols of Σ for digits, and set

$$\begin{aligned}\varphi^n(e, \vec{x}) = w \iff & s(e) \text{ is a Turing machine } M \\ & \text{and if we start } M \text{ on } s(x_1)\sqcup s(x_2)\sqcup \cdots \sqcup s(x_n) \\ & \text{then it stops with the tape starting with } s(w)\sqcup\ .\end{aligned}$$

[6] These applications of SRT are treated extensively in Chapter 11 of Rogers [1967], which includes many of the examples I have chosen to present here.

The standard assumptions hold with these φ^n (with $\mathbb{V} = \mathbb{N}$), because they are all recursive, the codings are effective, and every recursive partial function can be computed by a Turing machine.

A Turing machine M on Σ with code e is *tidy* if whenever $\varphi^n_e(\vec{x}) = w$, then M stops on the input $s(x_1)\sqcup s(x_2)\sqcup \cdots \sqcup s(x_n)$ with just $s(w)$ at the left end of the tape—and just blanks after that.

Lemma. For each n, there is a total recursive function $\mathrm{tidy}^n(e)$, *such that for every code e of a Turing machine,* $\mathrm{tidy}^n(e)$ *is a code of a tidy Turing machine* M' *on the same alphabet such that* $\varphi^n_e = \varphi^n_{\mathrm{tidy}(e)}$.

This is a standard fact about Turing machines.[7]

We now apply SRT with these codings and $m = n = 0$, to get a number $\tilde{z}$ such that

$$\varphi^0_{\tilde{z}}(\,) = \mathrm{tidy}^0(\tilde{z});$$

the machine M with code $\mathrm{tidy}^0(\tilde{z})$ is clearly self-reproducing. $\dashv$

There are more specific, concrete results of this type (cf. Margenstern and Rogozhin [2002]) as well as more general ones in Rogers [1967], but this simple, natural construction illustrates the power of SRT to realize self-reference.

§2. Myhill's characterization of r.e.-complete sets. Recall that a relation $R(\vec{x})$ on $\mathbb{N}$ is *semirecursive* or Σ^0_1 if there is a recursive relation $P(\vec{x}, y)$ which is *monotone in* y and defines it, i.e.,

$$\big[P(\vec{x}, y) \,\&\, y < y'\big] \Longrightarrow P(\vec{x}, y') \text{ and } R(\vec{x}) \iff (\exists y)P(\vec{x}, y). \tag{4}$$

A set $A \subseteq \mathbb{N}$ is *recursively enumerable* (r.e.) if membership in A is semirecursive, or equivalently, if for some e,

$$A = W_e = \{x \in \mathbb{N} \colon \{e\}(x)\!\downarrow\}.$$

An r.e. set A is *complete* if for each r.e. set B there is a recursive (total) function f such that $x \in B \iff f(x) \in A$;[8] and it is *creative* if there is a unary recursive partial function $u(e)$ such that

$$W_e \cap A = \emptyset \Longrightarrow u(e)\!\downarrow \,\&\, u(e) \notin (A \cup W_e). \tag{5}$$

The notions go back to the fundamental Post [1944] who showed (in effect) that every r.e.-complete set is creative and implicitly asked for the converse.

THEOREM 2.1 (Myhill [1955]). *Every creative set is r.e.-complete.*

[7] The Lemma can be proved by quoting basic facts about Turing machines: each φ^n_e is (uniformly) μ-recursive, and each μ-recursive partial function can be (uniformly) computed by a tidy Turing machine.

[8] And then one can find a one-to-one f with the same property, cf. Theorem VII in Rogers [1967], but we will not be concerned with this here.

PROOF. Assume that $u(e)$ satisfies (5), and for a fixed r.e. set B choose a function $\tilde{z}(x)$ by SRT (with $\mathbb{V} = \mathbb{N}, m = n = 1$) so that

$$\{\tilde{z}(x)\}(t) = \begin{cases} 1, & \text{if } x \in B \,\&\, u(\tilde{z}(x))\downarrow \,\&\, t = u(\tilde{z}(x)), \\ \bot & \text{(i.e., undefined), otherwise.} \end{cases}$$

We claim that the function $f(x) = u(\tilde{z}(x))$ is total and that for every x,

$$x \in B \iff f(x) \in A.$$

The first claim is immediate, because if $f(x)\uparrow$, then $W_{\tilde{z}(x)} = \emptyset \subseteq A^c$, and so $u(\tilde{z}(x))\downarrow$ by the hypothesis.

For the second, assume first that $x \notin B$; this implies again that $W_{\tilde{z}(x)} = \emptyset$, so $W_{\tilde{z}(x)} \cap A = \emptyset$ and hence $f(x) = u(\tilde{z}(x)) \notin A$. Finally, if $x \in B$, then

$$W_{\tilde{z}(x)} = \{\!\{f(x)\}\!\},$$

and we must have that $f(x) \in A$—otherwise $u(\tilde{z}(x)) = f(x) \notin \{\!\{f(x)\}\!\}$, which is absurd. $\dashv$

This clever argument of Myhill's has many applications, as we will see in Section §5, but it is also foundationally significant: it identifies *creativeness*, which is an intrinsic property of a set A but depends on the coding of recursive partial functions with *completeness*, which depends on the entire class of r.e. sets but is independent of coding. I believe it is the first important application of SRT in print by someone other than Kleene—except, perhaps, for Spector [1955] which appeared in the same year.

§3. The Myhill–Shepherdson Theorem. One (modern) interpretation of this classical result is that algorithms which call their (computable, partial) function arguments *by name* can be simulated by non-deterministic algorithms which call their function arguments *by value*. It is a rather simple but interesting consequence of SRT.

Let $\mathcal{P}_r^m$ be the set of all m-ary recursive partial functions. A partial operation

$$\Phi\colon \mathbb{N}^n \times \mathcal{P}_r^m \rightharpoonup \mathbb{N} \tag{6}$$

is *effective* if its partial function *associate*

$$f(\vec{x}, e) = \Phi(\vec{x}, \varphi_e^m) \tag{7}$$

is recursive. In programming terms, an effective operation calls its function argument *by name*, i.e., it needs a code of any $p \in \mathcal{P}_r^m$ to compute the value $\Phi(\vec{x}, p)$.

There are cases, however, when we need to compute $\Phi(\vec{x}, p)$ without access to a code of p, only to its values. We can make this precise using a *Turing machine M with an oracle* which can request values of the function argument p on a special *function input tape*: when M needs $p(\vec{y})$, it prints $\vec{y}$ on the function input tape and waits until it is replaced by $p(\vec{y})$ before it can go on—which, in fact, may cause the computation to stall if $p(\vec{y})\uparrow$. In these circumstance we say that M *computes* Φ *by value*, and we do not need to be too fussy in specifying this precisely because of the following characterization in the non-deterministic case with which we are concerned here.

We fix an enumeration $d_0^m, d_1^m, \ldots$ of all unary m-ary functions with finite domain of convergence, such that the enumeration $d^m(s, \vec{y}) = d_s^m(\vec{y})$ and the convergence condition $d^m(s, \vec{y})\downarrow$ are both recursive, and if $d(s, \vec{y})\downarrow$, then $y_1, \ldots, y_m \le s$.

Theorem (Normal form for Turing computable operations). *A partial operation* $\Phi\colon \mathbb{N}^n \times \mathcal{P}_r^m \rightharpoonup \mathbb{N}$ *is computable by a non-deterministic Turing machine with an oracle for the function argument if and only if there exists a semirecursive relation* R *such that*

$$\Phi(\vec{x}, p) = w \iff (\exists s)\big[d_s^m \subseteq p \,\&\, R(\vec{x}, w, s)\big]. \tag{8}$$

This is quite easy to prove, with the correct definition of non-deterministic computation.[9]

Notice that the recursive associate f of an effective operation Φ as in (7) satisfies the following *invariance condition*:

$$\varphi_{e_1} = \varphi_{e_2} \Longrightarrow f(\vec{x}, e_1) = f(\vec{x}, e_2). \tag{9}$$

This is used crucially in the proof below.

Theorem 3.1 (Myhill and Shepherdson [1955]). *A partial operation* Φ *as in* (6) *is effective if and only if it is computable by a non-deterministic Turing machine.*[10]

Proof. Take $m = 1$ and omit it in the notation, for simplicity. One direction is immediate from the normal form theorem above, which gives that

$$f(\vec{x}, e) = w \iff \Phi(\vec{x}, \varphi_e) = w \iff (\exists s)\big[d_s \subseteq \varphi_e \,\&\, R(\vec{x}, w, s)\big]$$

so that the associate of Φ has semirecursive graph and is recursive.

For the converse, we need two facts:

(a) *Every effective operation is monotone, i.e.*,

$$f(\vec{x}, e_1) = w \,\&\, \varphi_{e_1} \subseteq \varphi_{e_2} \Longrightarrow f(\vec{x}, e_2) = w.$$

[9] A non-deterministic machine gives output w on $(\vec{x}, p)$ if at least one computation of M on $(\vec{x}, p)$ terminates and gives w and no terminating computation gives a different value—but divergent computations are allowed.

[10] The use of non-deterministic machines here is essential, because the operation

$$\Phi(p) = \begin{cases} 1, & \text{if } p(0)\downarrow \text{ or } p(1)\downarrow, \\ \bot, & \text{otherwise} \end{cases}$$

is effective but not computable by a deterministic Turing machine.

Proof. Assume $f(\vec{x}, e_1) = w \,\&\, \varphi_{e_1} \subseteq \varphi_{e_2}$ and apply SRT with $\mathbb{V} = \mathbb{N}$, $m = 0$, $n = 1$, to get $\tilde{z}$ such that

$$\varphi_{\tilde{z}}(y) = u \iff \varphi_{e_1}(y) = u \vee \big[f(\vec{x}, \tilde{z}) = w \,\&\, \varphi_{e_2}(y) = u\big].$$

Now $f(\vec{x}, \tilde{z}) = w$—otherwise $\varphi_{\tilde{z}} = \varphi_{e_1}$ so that $f(\vec{x}, \tilde{z}) = f(\vec{x}, e_1) = w$ by the invariance condition (9); and so $\varphi_{\tilde{z}} = \varphi_{e_2}$, which by (9) again gives the required $f(\vec{x}, e_2) = f(\vec{x}, \tilde{z}) = w$.

(b) *Every effective operation is continuous, i.e.,*

$$\Phi(\vec{x}, \varphi_e) = w \Longrightarrow (\exists \text{ a finite } p \subseteq \varphi_e)\big[\Phi(\vec{x}, p) = w\big].$$

Proof. Let $P(\vec{x}, e, w, t)$ be a (monotone in t) recursive relation such that

$$f(\vec{x}, e) = w \iff (\exists t) P(\vec{x}, e, w, t)$$

as in (4), assume that $f(\vec{x}, e) = w$, and choose $\tilde{z}$ by SRT such that

$$\varphi_{\tilde{z}}(y) = u \iff \varphi_e(y) = u \,\&\, (\forall t \leq y) \neg P(\vec{x}, \tilde{z}, w, t),$$

so that $\varphi_{\tilde{z}} \subseteq \varphi_e$. Check as above that $f(\vec{x}, \tilde{z}) = w$—otherwise $\varphi_{\tilde{z}} = \varphi_e$; and so

$$\varphi_{\tilde{z}}(y)\downarrow \Longrightarrow y < \mu t P(\vec{x}, \tilde{z}, w, t),$$

and $\varphi_{\tilde{z}}$ is the required finite partial function.

Let $d_s = \varphi_{c(s)}$ for a total recursive function c. From (a) and (b) we get immediately that for every recursive $p\colon \mathbb{N}^n \rightharpoonup \mathbb{N}$,

$$\Phi(\vec{x}, p) = w \iff (\exists s)\big[d_s \subseteq p \,\&\, f(\vec{x}, c(s)) = w\big]$$

and then the normal form theorem above supplies a non-deterministic Turing machine which computes Φ. $\dashv$

3.1. Aside: the two recursion theorems.[11] A (partial) *functional* is any partial operation

$$\Phi\colon \mathbb{N}^n \times \mathcal{P}^m \rightharpoonup \mathbb{N},$$

where $\mathcal{P}^m$ is the set of *all* m-ary partial functions, not just the recursive ones; and it is *recursive* (or *Turing computable*) if it is computed by a non-deterministic Turing machine as described above.[12] It is easy to check that this holds exactly when Φ satisfies (8) for all $\vec{x}, w$ and all p.

The next result is a simultaneous extension of both the first and the second recursion theorems:

THEOREM 3.2 (The Recursion Theorem). *For every recursive functional*

$$\Phi\colon \mathbb{N} \times \mathbb{N}^n \times \mathcal{P}^n \rightharpoonup \mathbb{N},$$

there is a number $\tilde{z}$ such that:

[11] Added at the suggestion of the referee.

[12] These should really be called *non-deterministic recursive functionals*, but the terminology has been fixed by long-term usage, including Kleene's.

(a) *For all* $\vec{x}$, $\Phi(\tilde{z}, \vec{x}, \varphi_{\tilde{z}}) = \varphi_{\tilde{z}}(\vec{x})$.
(b) *If* p *is any* m*-ary partial function such that*

$$(\forall \vec{x}, w)\big[\Phi(\tilde{z}, \vec{x}, p) = w \Longrightarrow p(\vec{x}) = w\big],$$

then $\varphi_{\tilde{z}} \subseteq p$.

In particular, $\varphi_{\tilde{z}}$ *is the least* m*-ary partial function which satisfies the identity* $\Phi(\tilde{z}, \vec{x}, p) = p(\vec{x})$.

The Second Recursion Theorem (with $m = 0$) is the version of this result when $\Phi(e, \vec{x}, p) = f(e, \vec{x})$ does not depend on p; and the First Recursion Theorem is the version when $\Phi(e, \vec{x}, p) = \Psi(\vec{x}, p)$ does not depend on e.

Outline of proof. The basic fact is that a recursive functional $\Phi(e, \vec{x}, p)$ is *monotone* and *continuous* in p, in the obvious sense. Using this, fix e, let $\overline{p}_0^e = \emptyset$ be the totally undefined n-ary partial function, and define recursively

$$\overline{p}_{k+1}^e(\vec{x}) = \Phi(e, \vec{x}, \overline{p}_k^e).$$

Now $\overline{p}_0^e \subseteq \overline{p}_1^e \subseteq \overline{p}_2^e \cdots$ (by monotonicity), and (by continuity) the limit partial function

$$\overline{p}^e = \bigcup_k \overline{p}_k^e$$

is the least fixed point of Φ for the given e, i.e.,

$$\Phi(e, \vec{x}, \overline{p}^e) = \overline{p}^e(\vec{x}), \quad (\forall \vec{x}, w)[\Phi(e, \vec{x}, q) = w \Longrightarrow q(\vec{x}) = w] \Longrightarrow \overline{p}^e \subseteq q.$$

The construction of $\overline{p}^e$ from e is effective, and so the partial function

$$f(e, \vec{x}) = \overline{p}^e(\vec{x})$$

is recursive; and then the result follows by applying the Second Recursion Theorem to this f. ⊣

This much is simple. What is not so obvious without some discussion is the foundational significance of the First Recursion Theorem, which Kleene considered as (perhaps) the strongest argument in favor of the *Church–Turing Thesis*—and a discussion of that would take us far from our topic.

§4. The Kreisel–Lacombe–Shoenfield–Ceitin Theorem. Let $\mathcal{F}_r^1$ be the set of all unary total recursive functions. By analogy with operations on $\mathcal{P}_r^1$, a partial operation

$$\Phi\colon \mathbb{N}^n \times \mathcal{F}_r^1 \rightharpoonup \mathbb{N}$$

is *effective* if there is a recursive partial function $f\colon \mathbb{N}^{n+1} \rightharpoonup \mathbb{N}$ such that

$$\varphi_e \in \mathcal{F}_r^1 \Longrightarrow \Phi(\vec{x}, \varphi_e) = f(\vec{x}, e). \tag{10}$$

Notice that such a *recursive associate* f of Φ satisfies the *invariance condition*

$$\varphi_e = \varphi_m \in \mathcal{F}_r^1 \Longrightarrow f(\vec{x}, e) = f(\vec{x}, m), \tag{11}$$

but is not uniquely determined.

The next theorem is a version of the Myhill–Shepherdson Theorem appropriate for these operations. Its proof is not quite so easy, and somewhere in the middle we will appeal to the following, simple result from elementary recursion theory:

LEMMA (Σ^0_1-Selection, XXV in Kleene [1952]). *For each semirecursive relation $R(\vec{x}, t)$, there is a recursive partial function $\nu_R(\vec{x})$ such that*

$$(\exists t) R(\vec{x}, t) \Longrightarrow \nu_R(\vec{x})\downarrow \;\&\; R(\vec{x}, \nu_R(\vec{x})).$$

In effect, $\nu_R(\vec{x})$ searches for some t which satisfies $R(\vec{x}, t)$ and chooses one if one exists, and we convey this by the notation

$$\nu t R(\vec{x}, t) = \nu_R(\vec{x})$$

read *some t such that $R(\vec{x}, t)$*.

THEOREM 4.1 (Kreisel, Lacombe, and Shoenfield [1957], Ceitin [1962]). *Suppose $\Phi\colon \mathbb{N}^n \times \mathcal{F}^1_r \to \mathbb{N}$ is a total effective operation.*

(a) *Φ is effectively continuous: i.e., there is a recursive partial function $g(e, \vec{x})$, such that when φ_e is total, then $g(e, \vec{x})\downarrow$ for all $\vec{x}$, and for all $p \in \mathcal{F}^1_r$,*

$$(\forall t < g(e, \vec{x}))\big[p(t) = \varphi_e(t)\big] \Longrightarrow \Phi(\vec{x}, p) = \Phi(\vec{x}, \varphi_e).$$

(b) *Φ is computed by a deterministic Turing machine.*

OUTLINE OF PROOF. Fix a recursive associate $f(\vec{x}, m)$ of Φ satisfying (10) and a recursive, monotone in t (as in (4)) relation $R(\vec{x}, m, w, t)$ such that

$$f(\vec{x}, m) = w \iff (\exists t) R(\vec{x}, m, w, t).$$

Let $X_0, X_1, \dots$ be an effective enumeration of all (total) functions which are ultimately 0.

Lemma 1. *If φ_e is total and $\Phi(\vec{x}, \varphi_e) = w$, then for every s, there is a k such that*

$$\Phi(\vec{x}, X_k) = w \;\&\; (\forall u < s)\big[X_k(u) = \varphi_e(u)\big],$$

i.e., φ_e can be approximated arbitrarily well by ultimately 0 functions X_k for which $\Phi(\vec{x}, X_k) = \Phi(\vec{x}, \varphi_e)$.

Proof. Assume $f(\vec{x}, e) = w$ and fix s. Choose by SRT some $\tilde{z}_1$ such that

$$\varphi_{\tilde{z}_1}(u) = \begin{cases} \varphi_e(u), & \text{if } u < s \vee (\forall t < u) \neg R(\vec{x}, \tilde{z}_1, w, t), \\ 0, & \text{otherwise,} \end{cases}$$

and verify that $\Phi(\vec{x}, \varphi_{\tilde{z}_1}) = w$ and $\varphi_{\tilde{z}_1}$ is ultimately 0. $\dashv_{\textit{Lemma } 1}$

The next, main lemma is considerably stronger than the first claim (a), and will also help show the second claim (b).

Lemma 2. *There is a total recursive function $\tilde{z}(e, \vec{x})$ such that for fixed $\vec{x}, e$ and $\tilde{z} = \tilde{z}(e, \vec{x})$, if $f(\vec{x}, e) = w$ and $(\forall t)\left[\neg R(\vec{x}, \tilde{z}, w, t) \Longrightarrow \varphi_e(t)\downarrow\right]$, then*

$$g(e, \vec{x}) = \mu t R(\vec{x}, \tilde{z}, w, t)\downarrow, \text{ and} \tag{a}$$

$$\text{for all } p \in \mathcal{F}_r^1, (\forall t < g(e, \vec{x}))\left[p(t) = \varphi_e(t)\right] \Longrightarrow \Phi(\vec{x}, p) = w. \tag{b}$$

This implies, in particular, the first claim (a) in the theorem, since the hypotheses of the lemma are trivially true when φ_e is total.

Proof. Assume the hypothesis and choose by SRT a total recursive $\tilde{z}(e, \vec{x})$ so that with $\tilde{z} = \tilde{z}(e, \vec{x})$,

$$\varphi_{\tilde{z}}(u) = \begin{cases} \varphi_e(u), & \text{if } (\forall t < u)\neg R(\vec{x}, \tilde{z}, w, t), \\ X_{k^*}(u), & \text{otherwise,} \end{cases}$$

where

$$k^* = \nu k\left[\Phi(\vec{x}, X_k) \neq w \,\&\, g(e, \vec{x})\downarrow \,\&\, (\forall t < g(e, \vec{x}))(X_k(t) = \varphi_{\tilde{z}}(t))\right].$$

To prove (a) by contradiction, notice that if $g(e, \vec{x})\uparrow$, then φ_e is total (by the hypothesis) and $\varphi_e = \varphi_{\tilde{z}}$, and so $f(\vec{x}, \tilde{z}) = f(\vec{x}, e) = w$, which means that $g(e, \vec{x})\downarrow$.

To prove (b), also by contradiction, suppose that there is a total, recursive p which agrees with φ_e on all $t < g(e, \vec{x})$, but $\Phi(\vec{x}, p) \neq w$. By Lemma 1 (with $s = g(e, \vec{x})$), there is then an ultimately 0 function X_k which also agrees with φ_e on all $t < g(e, \vec{x})$ and such that $\Phi(\vec{x}, X_k) = \Phi(\vec{x}, p) \neq w$, which implies that $k^*\downarrow$ and $\varphi_{\tilde{z}} = X_{k^*}$; but this is a total recursive function and

$$\Phi(\vec{x}, \varphi_{\tilde{z}}) = w \neq \Phi(\vec{x}, X_{k^*}),$$

which is absurd. $\dashv_{\textit{Lemma } 2}$

To prove (b) of the theorem, we use Lemma 2 to verify that for all total, recursive p,

$$\begin{aligned} &\Phi(\vec{x}, p) = w \\ &\quad \Longleftrightarrow (\exists e)\Big(f(\vec{x}, e) = w \,\&\, g(e, \vec{x})\downarrow \,\&\, (\forall t < g(e, \vec{x}))\left[\varphi_e(t) = p(t)\right]\Big). \end{aligned}$$

This is immediate in the direction $\Rightarrow$, taking e such that $\varphi_e = p$; and for the converse, notice that the hypotheses of Lemma 2 hold of any e which satisfies the right-hand-side, and then the lemma gives the left-hand-side. The equivalence can be used to show (quite easily) that $\Phi(\vec{x}, p)$ is computable by a deterministic Turing machine, on total, recursive p. $\dashv$

The restriction in the theorem to total operations on $\mathcal{F}_r^1$ is necessary, because of a lovely counterexample in Friedberg [1958].

Ceitin [1962] proved independently a general version of (a) of Theorem 4.1: *every recursive operator on one constructive metric space to another is effectively continuous*. His result is the central fact in the school of *constructive analysis* which was flowering in Russia at that time, and it has played an important role in the development of constructive mathematics ever since.

§**5. Incompleteness and undecidability.** We prove in this section two basic theorems which relate SRT to incompleteness and undecidability results: a version of the so-called Fixed Point Lemma, and a beautiful result of Myhill's, which implies most simple incompleteness and undecidability facts about sufficiently strong theories and insures a very wide applicability for the Fixed Point Lemma.

Working in the language of Peano Arithmetic (PA) with symbols $0, 1, +, \cdot$, define first (recursively) for each $x \in \mathbb{N}$ a closed term Δx which denotes x, and for every formula χ, let

$$\#\chi = \text{the code (Gödel number) of } \chi, \quad \ulcorner\chi\urcorner \equiv \Delta\#\chi = \text{the name of } \chi.$$

We assume that the Gödel numbering of formulas is sufficiently effective so that (for example)[13] $\#\varphi(\Delta x_1, \dots, \Delta x_n)$ can be computed from $\#\varphi(v_1, \dots, v_n)$ and $x_1, \dots, x_n$. A *theory* (in the language of PA) is any set of sentences T and

$$\mathrm{Th}(T) = \{\#\theta : \theta \text{ is a sentence and } T \vdash \theta\}$$

is the set of (Gödel numbers of) the theorems of T. A theory T is *sound* if every $\theta \in T$ is true in the standard model $(\mathbb{N}, 0, 1, +, \cdot)$; it is *axiomatizable* if its *proof relation*

$$\begin{aligned}\mathrm{Proof}_T(e, y) \iff\ & e \text{ is the code of a sentence } \sigma \\ & \text{and } y \text{ is the code of a proof of } \sigma \text{ in } T\end{aligned}$$

is recursive, which implies that $\mathrm{Th}(T)$ is recursively enumerable; and it is *sufficiently expressive* if every recursive relation $R(\vec{x})$ is *numeralwise expressible* in T, i.e., for some $\varphi_R(v_1, \dots, v_n)$ whose free variables are in the list $v_1, \dots, v_n$,

$$\begin{aligned} R(x_1, \dots, x_n) &\Longrightarrow T \vdash \varphi_R(\Delta x_1, \dots, \Delta x_n), \\ \neg R(x_1, \dots, x_n) &\Longrightarrow T \vdash \neg\varphi_R(\Delta x_1, \dots, \Delta x_n). \end{aligned}$$

[13]By the usual convention, we view $\varphi(v_1, \dots, v_n)$ as a pair $(\varphi, (v_1, \dots, v_n))$ of a formula and a sequence of variables, and then $\varphi(\Delta x_1, \dots, \Delta x_n)$ is the formula obtained by replacing each v_i in all its free occurrences in φ by Δx_i.

THEOREM 5.1 (Fixed Point Lemma). *If T is axiomatizable in the language of* PA *and* $\mathrm{Th}(T)$ *is r.e.-complete, then for every formula $\theta(v)$ with at most v free, there is a sentence σ such that*

$$T \vdash \sigma \iff T \vdash \theta(\ulcorner\sigma\urcorner). \tag{12}$$

PROOF. Let $\psi^0, \psi^1, \dots$ be recursive partial functions satisfying the standard assumptions, let[14]

$$u \in A \iff (\exists n)\big[\mathrm{Seq}(u)\ \&\ \mathrm{lh}(u) = n+1\ \&\ \psi^n((u)_0, (u)_1, \dots, (u)_n)\downarrow\big],$$

so that A is r.e. and there is a total recursive function $\mathfrak{r}$ such that

$$u \in A \iff \mathfrak{r}(u) \in \mathrm{Th}(T).$$

We will use SRT with $\mathbb{V} = \{\!\{0\}\!\}$ and

$$\varphi^n(e, \vec{x}) = 0 \iff \psi^n(e, \vec{x})\downarrow \iff \mathfrak{r}(\langle e, \vec{x}\rangle) \in \mathrm{Th}(T) \tag{13}$$

which clearly satisfy the standard assumptions; in addition, every semirecursive relation $R(\vec{x})$ satisfies

$$R(\vec{x}) \iff \{e\}(\vec{x}) = 0$$

with some $e \in \mathbb{N}$.

Given $\theta(v)$, SRT (with $m = n = 0$) gives us a number $\tilde{z}$ such that

$$\begin{aligned}&\{\tilde{z}\}(\,) = 0\\ &\qquad\iff \mathfrak{r}(\langle\tilde{z}\rangle) \text{ is not the code of a sentence or } T \vdash \theta(\Delta\mathfrak{r}(\langle\tilde{z}\rangle)).\end{aligned} \tag{14}$$

Now $\mathfrak{r}(\langle\tilde{z}\rangle)$ is the code of a sentence, because if it were not, then the right-hand-side of (14) would be true, which makes the left-hand-side true and insures that $\mathfrak{r}(\langle\tilde{z}\rangle)$ codes a sentence, in fact a theorem of T; and if $\mathfrak{r}(\langle\tilde{z}\rangle) = \#\sigma$, then

$$\begin{aligned}T \vdash \sigma \iff \mathfrak{r}(\langle\tilde{z}\rangle) \in \mathrm{Th}(T) \iff \{\tilde{z}\}(\,) = 0&\\ \iff T \vdash \theta(\Delta\mathfrak{r}(\langle\tilde{z}\rangle)) \iff T \vdash \theta(\ulcorner\sigma\urcorner).&\quad\dashv\end{aligned}$$

The conclusion of the Fixed Point Lemma is usually stated in the stronger form

$$T \vdash \sigma \leftrightarrow \theta(\ulcorner\sigma\urcorner),$$

[14]For any tuple $\vec{x} = (x_0, \dots, x_{n-1}) \in \mathbb{N}^n$, $\langle\vec{x}\rangle$ codes $\vec{x}$ so that for suitable recursive relations and functions,

$$\langle\vec{x}\rangle = f_n(x_0, \dots, x_{n-1}),\ \mathrm{Seq}(w) \iff w \text{ is a sequence code},\ \mathrm{lh}(\langle\vec{x}\rangle) = n,\ (\langle\vec{x}\rangle)_i = x_i;$$

$u \sqsubseteq v \iff \mathrm{lh}(u) \leq \mathrm{lh}(v)\ \&\ (\forall i < \mathrm{lh}(u))\big[(u)_i = (v)_i\big]$, 1 codes the empty sequence, and $\langle u_1, \dots, u_{n-1}\rangle * \langle s\rangle = \langle u_0, \dots, u_{n-1}, s\rangle$.

but (12) is sufficient to yield the applications. For the First Incompleteness Theorem, for example, we assume in addition that T is sufficiently expressive, we choose σ such that

$$T \vdash \sigma \iff T \vdash \neg(\exists u)\mathsf{Proof}_T(\ulcorner\sigma\urcorner, u) \tag{15}$$

where $\mathsf{Proof}_T(v, u)$ numeralwise expresses in T its proof relation, and we check that if T is consistent, then $T \nvdash \sigma$, and if T is also sound, then $T \nvdash \neg\sigma$. The only difference from the usual argument is that (15) does not quite say that σ "expresses its own unprovability"—only that it is is provable exactly when its unprovability is also provable. For the Rosser form of Gödel's Theorem, we need to assume that T is a bit stronger (as we will explain below) and *consistent*, though not necessarily sound, and the classical argument again works with the more complex Rosser sentence and this same, small different understanding of what the Rosser sentence says.

There is a problem, however, with the key hypothesis in Theorem 5.1 *that* $\mathrm{Th}(T)$ *is r.e.-complete*. This is an immediate consequence of the definitions for sufficiently expressive and sound theories, including, of course, PA, but not so simple to verify for theories which are consistent but not sound. In fact it holds for every consistent, axiomatizable theory T which extends the system Q in Robinson [1950]—which is the standard hypothesis for incompleteness and undecidability results about consistent theories that need not be sound.[15]

The basic facts (and all that we need to know) about Q are that it is sound, axiomatizable, sufficiently expressive, it can prove all true, quantifier-free sentences, and also, for each $k \in \mathbb{N}$,

$$\mathsf{Q} \vdash (\forall x)\big[x \le \Delta k \vee \Delta(k+1) \le x\big],$$
$$\mathsf{Q} \vdash (\forall x)\big[x \le \Delta k \iff x = \Delta 0 \vee x = \Delta 1 \vee \cdots \vee x = \Delta k\big],$$

where $x \le y :\equiv (\exists z)[x + z = y]$.

THEOREM 5.2 (Myhill [1955]). *If T is any consistent, axiomatizable extension of* Q, *then* $\mathrm{Th}(T)$ *is creative, and hence r.e.-complete.*

OUTLINE OF PROOF. We will use SRT with $\mathbb{V} = \{\!\{0, 1\}\!\}$ and the coding

$$\varphi_e(\vec{x}) = \begin{cases} 1, & \text{if } e \text{ codes a formula } \theta(v_1, \ldots, v_n) \text{ whose free variables} \\ & \qquad \text{are in the list } v_1, \ldots, v_n, \text{ and } \mathsf{Q} \vdash \theta(\Delta x_1, \ldots, \Delta x_n), \\ 0, & \text{if } e \text{ codes a formula } \theta(v_1, \ldots, v_n) \text{ whose free variables} \\ & \qquad \text{are in the list } v_1, \ldots, v_n, \text{ and } \mathsf{Q} \vdash \neg\theta(\Delta x_1, \ldots, \Delta x_n), \\ \bot, & \text{otherwise,} \end{cases}$$

[15]For a specification of Q and its properties, see (for example) Boolos, Burgess, and Jeffrey [1974] or even Kleene [1952], §41. Notice also that Theorem 5.2 does not lose much of its foundational interest or its important applications if we replace Q by PA in its statement—and the properties of Q that we use are quite obvious for PA.

where by $\varphi_e(\vec{x}) = \bot$ we mean that $\varphi_e(\vec{x})\uparrow$. This satisfies the standard assumptions of SRT, by a typical (if not entirely trivial) exercise in using the properties of Q—including the fact that it is sound.

Choose a recursive, monotone in y relation $R(m, e, y)$ as in (4) such that

$$e \in W_m \iff (\exists y) R(e, m, y),$$

let $\mathsf{R}(v_1, v_2, v_3)$ numeralwise express $R(e, m, y)$ in Q, set

$$\theta_m(v) :\equiv (\exists y)\big[\mathsf{Proof}_T(v, y) \,\&\, (\forall u \le y) \neg \mathsf{R}(v, \Delta m, u)\big]$$

and choose by SRT a total, recursive $\tilde{z}(m)$ such that

$$\varphi_{\tilde{z}(m)}(\,) = \begin{cases} 1, & \text{if } \tilde{z}(m) \text{ is not the code of a sentence,} \\ & \quad \text{or } \tilde{z}(m) = \#\sigma \text{ and } \mathsf{Q} \vdash \theta_m(\ulcorner \neg\sigma \urcorner), \\ 0, & \text{if } \tilde{z}(m) = \#\sigma \text{ and } \mathsf{Q} \vdash \neg\theta_m(\ulcorner \neg\sigma \urcorner), \\ \bot, & \text{otherwise.} \end{cases}$$

The definition implies that $\tilde{z}(m)$ is always a code of a sentence σ_m, and the coding then gives the following two equivalences:

$$\mathsf{Q} \vdash \sigma_m \iff \mathsf{Q} \vdash \theta_m(\ulcorner \neg\sigma_m \urcorner), \tag{16}$$

$$\mathsf{Q} \vdash \neg\sigma_m \iff \mathsf{Q} \vdash \neg\theta_m(\ulcorner \neg\sigma_m \urcorner). \tag{17}$$

We show that

$$W_m \cap \mathrm{Th}(T) = \emptyset \Longrightarrow \big[T \nvdash \neg\sigma_m \,\&\, \#\neg\sigma_m \notin W_m\big],$$

which establishes that $\mathrm{Th}(T)$ is creative and completes the proof.

Fix m and suppose that $W_m \cap \mathrm{Th}(T) = \emptyset$.

(i) Assume, towards a contradiction, that $T \vdash \neg\sigma_m$, and let k be the code of a proof. Now Q can prove $\mathsf{Proof}_T(\ulcorner \neg\sigma_m \urcorner, \Delta k)$ and it also knows that for every $i \le k$, $\neg\mathsf{R}(\ulcorner \neg\sigma_m \urcorner, \Delta m, \Delta i)$ (since $\#\neg\sigma_m \notin W_m$), and so by its basic properties,

$$\mathsf{Q} \vdash \theta_m(\ulcorner \neg\sigma_m \urcorner);$$

now (16) gives $\mathsf{Q} \vdash \sigma_m$, and hence $T \vdash \sigma_m$ which makes T inconsistent. So $T \nvdash \neg\sigma_m$.

(ii) Assume towards a contradiction that $\#\neg\sigma_m \in W_m$. Again, Q knows that $\mathsf{R}(\ulcorner \neg\sigma_m \urcorner, \Delta m, \Delta_k)$ for some k, and then it can prove the universal sentence

$$\neg\theta_m(\ulcorner \neg\sigma_m \urcorner) \equiv (\forall y)\big[\neg\mathsf{Proof}_T(\ulcorner \neg\sigma_m \urcorner, y) \vee (\exists u \le y)\mathsf{R}(\ulcorner \neg\sigma_m \urcorner, \Delta m, u)\big]$$

by taking cases on whether $y \le \Delta k$ or not and using (i); it follows by (17) that $\mathsf{Q} \vdash \neg\sigma_m$, and so $T \vdash \neg\sigma_m$, which contradicts (i). ⊣

As a consequence, consistent, axiomatizable extensions of Q are undecidable and hence incomplete; moreover, the Fixed Point Lemma Theorem 5.1 applies to them, and so we can construct specific, interesting sentences that they do not decide, a la Rosser.

A minor (notational) adjustment of the proofs establishes Theorems 5.1 and 5.2 for any consistent, axiomatizable theory T, in any recursive language, provided only that Q can be *interpreted in* T,[16] including, for example, ZFC; and then a third fundamental result of Myhill [1955] implies that the sets of theorems of any two such theories are *recursively isomorphic*.[17]

§6. Solovay's theorem in provability logic. The (propositional) *modal formulas* are built up as usual using variables $p_0, p_1, \dots$; a constant $\perp$ denoting falsity; the binary implication operator $\to$ (which we use with $\perp$ to define all the classical propositional connectives); and a unary operator $\Box$, which is usually interpreted by "it is necessary that". Solovay [1976] studies interpretations of modal formulas by sentences of PA in which $\Box$ is interpreted by "it is provable in PA that" and establishes some of the basic results of the logic of provability. His central argument appeals to SRT at a crucial point.

An *interpretation* of modal formulas is any assignment π of sentences of PA to the propositional variables, which is then extended to all formulas by the structural recursion

$$\pi(\perp) \equiv 0 = 1,\ \pi(\varphi \to \psi) \equiv (\pi(\varphi) \to \pi(\psi)),$$
$$\pi(\Box\varphi) \equiv (\exists u)\mathsf{Proof}_{\mathsf{PA}}(\ulcorner \pi(\varphi) \urcorner, u).$$

A *modal formula* φ is PA-*valid* if PA $\vdash \pi(\varphi)$ for every interpretation π.

The axiom schemes of the system GL are:

(A0) All tautologies;
(A1) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ (transitivity of provability);
(A2) $\Box\varphi \to \Box\Box\varphi$ (provable sentences are provably provable); and
(A3) $\big(\Box(\Box\varphi \to \varphi)\big) \to \Box\varphi$ (Löb's Theorem).

The inference rules of GL are:

(R1) $\varphi \to \psi, \varphi \Longrightarrow \psi$ (Modus Ponens); and
(R2) $\varphi \Longrightarrow \Box\varphi$ (Necessitation).

THEOREM 6.1 (Solovay [1976]). *A modal formula φ is* PA-*valid if and only if it is a theorem of* GL.

Solovay shows also that the class of PA-valid modal formulas is decidable, and he obtains a similar decidable characterization of the modal formulas φ such that every interpretation $\pi(\varphi)$ is true (in the standard model), in terms of a related axiom system GL$'$.

The proof of Theorem 6.1 is long, complex, ingenious and depends essentially (and subtly) on the full strength of PA. It is nothing like the one-line

[16] A (weak) interpretation of T_1 in T_2 is any recursive map $\chi \mapsto \chi^*$ of the sentences of T_1 to those of T_2 such that $T_1 \vdash \chi \Longrightarrow T_2 \vdash \chi^*$ and $T_2 \vdash (\neg\chi)^* \leftrightarrow \neg(\chi^*)$.

[17] Pour-El and Kripke [1967] have interesting, stronger results of this type, whose proofs also use the Second Recursion Theorem.

derivations of Theorems 1.1 and 5.1 from standard facts about the relevant objects by a natural application of SRT or even the longer, clever proofs of Theorems 2.1, 3.1, 4.1 and 5.2 in which SRT still yields the punch lines. But I cannot see how one could possibly construct (or even think up) the key, "self-referential" closed term l of Solovay's Lemma 4.1 directly, without appealing to the Second Recursion Theorem,[18] and so, in that sense, SRT is an essential ingredient of his argument.

Part 2. Effective grounded recursion.

Some years back, the French analyst Claude Dellacherie visited UCLA to learn what the logically trained descriptive set theorists there might have to say about potential theory, and after a few days of looking amusingly confused declared that he had finally understood: *Codage*!, he said, "encoding: this is all you guys do!" Logicians, of course, understand well their propensity to *code*—assign to objects names that determine their relevant properties and then *compute*, *decide* or *define* functions and relations on these objects by operating on the codes rather than the objects coded—but I think that it is more prevalent and more important in the results to which we turn now than in most other areas of logic.

It is appropriate to start this part with the first application of SRT in Kleene [1938], which introduced effective grounded (transfinite) recursion.

§7. Constructive ordinals. Ordinal numbers can be viewed as the order types of well ordered sets, but also as *extended number systems*, which go beyond $\mathbb{N}$ and can be used to count (and regulate) transfinite iteration. Church and Kleene developed in the 1930s an extensive theory of such systems, aiming primarily at a *constructive theory of ordinals*; this was never realized fully, but the results of Kleene and his student Spector on ordinal notations had a profound effect on definability theory in first and second order arithmetic and (later) in effective descriptive set theory.

A *notation system for ordinals* or $\mathfrak{r}$-*system* (in Kleene [1938]) is a set $S \subseteq \mathbb{N}$, together with a function $x \mapsto |x|_S$ which assigns to each x in S a countable ordinal so that the following conditions hold:

(ON1) There is a recursive partial function $K(x)$ whose domain of convergence includes S and such that, for $x \in S$,

$$\begin{aligned} |x|_S = 0 &\iff K(x) = 0, \\ |x|_S \text{ is a successor ordinal} &\iff K(x) = 1, \\ |x|_S \text{ is a limit ordinal} &\iff K(x) = 2. \end{aligned}$$

[18] Which Solovay invokes to define a function $h\colon \mathbb{N} \to \{0, \dots, n\}$ by the magical words "Our definition of h will be in terms of a Gödel number e for h. The apparent circularity is handled, using the recursion theorem, in the usual way."

(ON2) There is a recursive partial function $P(x)$, such that if $x \in S$ and $|x|_S$ is a successor ordinal, then $P(x)\downarrow$, $P(x) \in S$ and $|x|_S = |P(x)|_S + 1$.

(ON3) There is a recursive partial function $Q(x, t)$, such that if $x \in S$ and $|x|_S$ is a limit ordinal, then for every t,

$$Q(x,t)\downarrow,\ Q(x,t) \in S,\ |Q(x,t)|_S < |Q(x,t+1)|_S,$$

and $|x|_S = \lim_t |Q(x,t)|_S$.

In short, an $\mathfrak{r}$-system assigns S-*names* (number codes) to some ordinals, so that we can effectively recognize whether a code x names 0, a successor ordinal or a limit ordinal, and we can compute an S-name for the predecessor of each S-named successor ordinal and (S-names for) a strictly increasing sequence converging to each S-named limit ordinal.

A countable ordinal is *constructive* if it gets a name in some $\mathfrak{r}$-system.

The empty set is an $\mathfrak{r}$-system, as is $\mathbb{N}$, which names the finite ordinals, and every $\mathfrak{r}$-system (obviously) assigns names to a countable, initial segment of the ordinal numbers. It is not immediately clear, however, whether the set of constructive ordinals is countable or what properties it may have: the main result in Kleene [1938] clarifies the picture considerably by constructing a single $\mathfrak{r}$-system which names all of them.

Following Kleene, we define the *finite ordinal codes* by the recursion

$$0_O = 1,\ (t+1)_O = 2^{t_O},$$

and for any z, we set $z_t = \varphi_z(t_O)$.

Lemma. *There is an* $\mathfrak{r}$-*system* $(S_1, |\ |)$ *such that*:

(i) $1 \in S_1$ *and* $|1| = 0$.
(ii) *If* $x \in S_1$, *then* $2^x \in S_1$ *and* $|2^x| = |x| + 1$.
(iii) *If for all* t, $e_t\downarrow, e_t \in S_1$ *and* $|e_t| < |e_{t+1}|$, *then* $3 \cdot 5^e \in S_1$ *and* $|3 \cdot 5^e| = \lim_t |e_t|$.

Proof. Call an $\mathfrak{r}$-system$(S, |\ |_S)$ *good* if it satisfies the conditions

(i) If $x \in S$, then $x = 1$, or $x = 2^y$ for some y, or $x = 3 \cdot 5^e$ for some e.
(ii) If $1 \in S$, then $|1|_S = 0$.
(iii) If $2^x \in S$, then $x \in S$ and $|2^x|_S = |x|_S + 1$.
(iv) If $3 \cdot 5^e \in S$, then for every t, $e_t\downarrow, e_t \in S$, $|e_t|_S < |e_{t+1}|_S$, and $|3 \cdot 5^e|_S = \lim_t |e_t|$.

Prove that if S, S' are both good and $x \in S \cap S'$, then $|x|_S = |x|_{S'}$ (by induction on $|x|_S$), and set

$$S_1 = \bigcup \{S : (S, |\ |_S) \text{ is good}\}. \qquad \dashv$$

Theorem 7.1 (Kleene [1938]). *For every* $\mathfrak{r}$-*system* $(S, |\ |_S)$, *there is a unary recursive partial function* ψ *such that*

$$x \in S \Longrightarrow \Big(\psi(x)\downarrow\ \&\ \psi(x) \in S_1\ \&\ |x|_S = |x|\Big).$$

In particular, the system $(S_1, |\ |)$ *names all constructive ordinals.*

PROOF. Let K, P, Q be the recursive partial functions that come with $(S, |\ |_S)$, choose a number e_0 such that

$$\{S(e_0, z, x)\}(t_O) = \{e_0\}(z, x, t_O) = \{z\}(Q(x, t)),$$

fix by SRT (with $\mathbb{V} = \mathbb{N}, m = 0, n = 1$) a number $\tilde{z}$ such that

$$\varphi_{\tilde{z}}(x) = \begin{cases} 1, & \text{if } K(x) = 0, \\ 2^{\varphi_{\tilde{z}}(P(x))}, & \text{if } K(x) = 1, \\ 3 \cdot 5^{S(e_0, \tilde{z}, x)}, & \text{otherwise,} \end{cases}$$

and set $\psi(x) = \varphi_{\tilde{z}}(x)$. The required properties of $\psi(x)$ are proved by a simple (possibly transfinite) induction on $|x|_S$. $\dashv$

In effect, the map from S to S_1 is defined by the obvious transfinite recursion on $|x|_S$, which is made effective by appealing to SRT—hence the name for the method.

The choice of numbers of the form $3 \cdot 5^e$ to name limit ordinals was made for reasons that do not concern us here, but it poses an interesting question: which ordinals get names in the system $(S_1', |\ |_1')$, defined by replacing $3 \cdot 5^e$ by (say) 7^e and e_t by $\varphi_e(t)$ in the definition of S_1? They are the same constructive ordinals, of course, and the proof is by defining by effective grounded recursion (exactly as in the proof of Theorem 7.1) a pair of recursive functions ψ, ψ' such that

$$x \in S_1 \Longrightarrow \Big(\psi'(x) \in S_1' \,\&\, |x|_1 = |\psi'(x)|_1'\Big),$$
$$x \in S_1' \Longrightarrow \Big(\psi(x) \in S_1 \,\&\, |x|_1' = |\psi(x)|_1\Big).$$

(And I do not know how else one could prove this "obvious" fact.)

For a last, elementary application of effective grounded recursion, we include two simple results in *constructive ordinal arithmetic* which will also be useful later:

THEOREM 7.2. (a) *There is recursive function $a +_1 b$ such that*

$$a, b \in S_1 \Longrightarrow a +_1 b \in S_1 \,\&\, |a +_1 b| = |a| + |b| \text{ (ordinal addition).}$$

(b) *There is a recursive function* $\text{ub}(e)$ *such that if φ_e^1 is total and for all t, $\varphi_e(t) \in S_1$, then* $\text{ub}(e) \in S_1$ *and for all t*, $|\varphi_e(t)| < |\text{ub}(e)|$.

PROOF. (a) Fix $\hat{f}$ such that $\{\hat{f}\}(z, a, e, n_O) = \varphi_z(a, \{e\}(n_O))$, choose $\tilde{z}$ by SRT such that

$$\varphi_{\tilde{z}}(a, b) = \begin{cases} a, & \text{if } b = 1, \\ 2^{\varphi_{\tilde{z}}(a, y)}, & \text{if } b = 2^y \text{ for some } y, \\ 3 \cdot 5^{S(\hat{f}, \tilde{z}, a, e)} & \text{if } b = 3 \cdot 5^e \text{ for some } e, \\ 0, & \text{otherwise,} \end{cases}$$

and set $a +_1 b = \varphi_{\bar{z}}(a, b)$. Notice that $a +_1 b\downarrow$, by a simple induction on b; now fix some $a \in S_1$, and check by induction on $|b|$ that if $b \in S_1$, then $a +_1 b \in S_1 \,\&\, |a +_1 b| = |a| + |b|$.

(b) Use (a) to define $\mathrm{ub}(e)$ such that when φ_e satisfies that hypotheses, then $\mathrm{ub}(e) = 3 \cdot 5^z$ for some z such that for all n,

$$z_n = \varphi_e(0) +_1 1_O +_1 \varphi_e(1) +_1 1_O +_1 \cdots +_1 \varphi_e(n) +_1 1_O$$

associating to the left. ⊣

§8. Markwald's Theorem. The constructive ordinals are "constructive analogs" of the classical countable ordinals, and so the constructive analog of the first uncountable ordinal Ω_1 is

$$\omega_1^{\mathrm{CK}} = \sup\{|x| : x \in S_1\} = \text{the least non-constructive ordinal},$$

the superscript standing for *Church–Kleene*. This is one of the most basic "universal constants" which shows up in logic—in many parts of it and under many guises. We establish one of its earliest characterizations.

A countable ordinal ξ is *recursive* if it is the order type of a recursive wellordering on some subset of $\mathbb{N}$.

Theorem 8.1 (Markwald [1954], Spector [1955]). *A countable ordinal ξ is constructive if and only if it is recursive.*

Proof. Suppose first that $\xi = \text{order type}(\leq^*)$ where $\leq^*$ is a recursive binary relation which well orders its field $F = \{x : x \leq^* x\}$, and let $\rho : F \to \xi$ be its rank function, a bijection of F with ξ such that

$$x \leq^* y \iff \rho(x) \leq \rho(y) \quad (x, y \in F).$$

The idea is to construct an $\mathfrak{r}$-system $(S, |\ |_S)$ which names all the ordinals below $\omega \cdot \xi$, including ξ, and to do this, we set first

$$S = \text{the closure of } \{3^y : y \in F\} \text{ under the operation } t \mapsto 2^t,$$

and on S we define

$$|x|_S = \begin{cases} \omega \cdot \rho(y) & \text{if } x = 3^y \text{ for some } y \in F, \\ |t|_S + 1 & \text{if } x = 2^t \text{ for some } t \in S. \end{cases}$$

For example, if a, b are the first two elements of F (relative to $\leq^*$), then

$$|3^a|_S = 0,\ |2^{(3^a)}|_S = 1,\ |2^{2^{(3^a)}}|_S = 2,\ \ldots\ |3^b|_S = \omega,\ \ldots$$

It is clear that $(S, |\ |_S)$ satisfies properties (ON1) and (ON2) in the definition of an $\mathfrak{r}$-system. To verify (ON3), we first check (easily) that the ordering

$$x \leq_S y \iff x \in S \,\&\, y \in S \,\&\, |x|_S \leq |y|_S$$

is recursive, and we define $Q(3^y, t)$ by the recursion

$$Q(3^y, 0) = 3^a \text{ (where } a \text{ is the } \leq^*\text{-least member of } F),$$

$$Q(3^y, t+1) = \begin{cases} t, & \text{if } Q(3^y, t) <_S t <_S 3^y, \\ 2^{Q(3^y,t)} & \text{otherwise.} \end{cases}$$

It is quite easy to prove that $Q(x, t)$ satisfies (ON3).

For the converse,[19] set first

$$x \leq_e y \iff \varphi_e^2(x, y) = 0, \tag{18}$$

$$W = \{e : \varphi_e^2 \text{ is total and } \leq_e \text{ well orders } \{x : \varphi_e(x, x) = 1\}\}, \tag{19}$$

so that W codes all recursive wellorderings and also the recursive ordinals, by

$$||e|| = \text{order type}(\leq_e) \quad (e \in W).$$

The idea now is to define by effective grounded recursion a recursive function $\varphi_{\tilde{z}}$ such that

$$x \in S_1 \Longrightarrow \Big(\varphi_{\tilde{z}}(x) \in W \;\&\; |x| \leq ||\varphi_{\tilde{z}}(x)||\Big). \tag{20}$$

Choose some $e_0 \in W$ such that $||e_0|| = 0$; define (easily, by "pasting" linear orderings together) recursive (total) functions g_s and g_l such that

$$e \in W \Longrightarrow \Big(g_s(e) \in W \;\&\; ||g_s(e)|| = ||e|| + 1\Big),$$

$$(\forall t)\big[\varphi_m(e_t) \in W\big] \Longrightarrow \Big(g_l(m, e) \in W \;\&\; (\forall t)\big[||\varphi_m(e_t)|| < ||g_l(m, e)||\big]\Big);$$

and choose $\tilde{z}$ by SRT such that

$$\varphi_{\tilde{z}}(x) = \begin{cases} e_0, & \text{if } x = 1, \\ g_s(\varphi_{\tilde{z}}(t)) & \text{if } x = 2^t \text{ for some } t, \\ g_l(\tilde{z}, e) & \text{if } x = 3 \cdot 5^e \text{ for some } e, \\ 0, & \text{otherwise.} \end{cases}$$

The verification of (20) is by an easy induction on $|x|$. ⊣

§9. The hyperarithmetical hierarchy. Kleene [1955a] associates with each $a \in S_1$ a set $H_a \subseteq \mathbb{N}$ so that:

(H1) $H_1 = \mathbb{N}$.

(H2) $H_{2^b} = H_b'$ ($=$ the jump of H_b, see A4 in the Appendix).

(H3) If $a = 3 \cdot 5^e$, then $x \in H_a \iff (x)_0 \in H_{e_{(x)_1}}$.

[19]This was already proved (by a different method) in Kleene [1938].

A set $A \subseteq \mathbb{N}$ is *hyperarithmetical* (HYP) if it is recursive in some H_a, and if $A \leq_e^T H_a$, then the pair $\langle e, a \rangle$ is a HYP-code of A.[20] A relation $P \subseteq \mathbb{N}^n$ is hyperarithmetical if

$$P(\vec{x}) \iff f(\vec{x}) \in A$$

for some $A \in$ HYP and a recursive function $f(\vec{x})$.

It is easy to check, directly from the definition that

(i) a set A is recursive in some H_a with finite $|a|$ if and only if A is arithmetical; and
(ii) if $|a| = \omega$, then $H_a \equiv^T \{\#\theta : \theta$ is a true PA-sentence $\}$,

so that HYP is a proper extension of the arithmetical hierarchy. It was also defined independently by Davis [1950a, 1950b] and (in a different way) by Mostowski [1951], both of whom knew most of its basic properties, but not the two central facts that we will prove in this section.

9.1. Spector's Uniqueness Theorem. This early, spectacular result extends the *uniqueness property* of ω in (ii) above to all constructive ordinals and shows that HYP truly ramifies into an ordinal-indexed hierarchy of degrees of unsolvability. The proof is a bit easier to follow if we notice first a simple, two-variable version of SRT:

SECOND RECURSION THEOREM, 2 (SRT2). *For any two recursive partial functions $f(u, v, \vec{x}), g(u, v, \vec{y})$ on $\mathbb{N}$, there exist $\tilde{u}, \tilde{v} \in \mathbb{N}$ such that*

$$\{\tilde{u}\}(\vec{x}) = f(\tilde{u}, \tilde{v}, \vec{x}), \ \{\tilde{v}\}(\vec{y}) = g(\tilde{u}, \tilde{v}, \vec{y}).$$

PROOF. Choose e_1, e_2 such that

$$\{e_1\}(z, \vec{x}) = \{z\}(0, \vec{x}, \vec{0}), \ \{e_2\}(z, \vec{y}) = \{z\}(1, \vec{0}, \vec{y});$$

fix $\tilde{z}$ by SRT so that

$$\{\tilde{z}\}(t, \vec{x}, \vec{y}) = \begin{cases} f(S(e_1, \tilde{z}), S(e_2, \tilde{z}), \vec{x}), & \text{if } t = 0, \\ g(S(e_1, \tilde{z}), S(e_2, \tilde{z}), \vec{y}), & \text{otherwise;} \end{cases}$$

and take $\tilde{u} = S(e_1, \tilde{z}), \tilde{v} = S(e_2, \tilde{z})$. ⊣

To simplify notation, set

$$a \prec b \iff a, b \in S_1 \,\&\, |a| < |b|, \ a \precsim b \iff a, b \in S_1 \,\&\, |a| \leq |b|,$$

and for each $b \in S_1$, let

$$\mathsf{IS}(b) = \{a : a \prec b\}$$

[20] Actually, Kleene defines H_a only when $a \in O$, a subsystem of S_1 which has more structure and is "more constructive", while still providing notations for all constructive ordinals. I will disregard this fine point here, as many basic facts about O can only be proved classically and the attempt to prove them constructively whenever it is possible clouds and complicates the arguments. In any case, the results we will formulate and prove about S_1 imply immediately their versions about O.

be the set of S_1-codes of ordinals below $|b|$. To avoid towers of exponents, we also set (with Kleene)

$$x^* = 2^x.$$

THEOREM 9.1 (The Uniqueness Theorem, Spector [1955]). *There are recursive functions $v(b)$ and $u(a, b)$, such that*

(A) *If $b \in S_1$, then $\mathsf{IS}(b)$ is recursive in H_{b^*} with code $v(b)$.*

(B) *If $a \precsim b$, then H_a is recursive in H_b with code $u(a, b)$.*

In particular, if $|a| = |b|$, then H_a and H_b have the same degree of unsolvability.[21]

OUTLINE OF PROOF. We will define $v(b)$ and $u(a, b)$ by appealing to SRT2. Officially, the definitions of these two functions are by cases (in terms of their codes $\tilde{v}, \tilde{u}$) and then (A) and (B) are proved together by induction on $|b|$. We will be rather informal, however, and simply explain how to decide membership in $\mathsf{IS}(b)$ by asking questions about membership in H_{b^*}, and similarly, whether $x \in H_a$ or not by asking questions about H_b, when $a \precsim b$. We will avoid index constructions and make few explicit references to the functions $v(b)$ and $u(a, b)$ or their codes, and we will mix the construction with the proof, as we would normally do in a set theoretic *definition by transfinite recursion* of functions which satisfy specified conditions. Finally, we will repeatedly appeal to the fact that H_b is uniformly recursive in $H_{b^*} = H'_b$, and that if $b = 3 \cdot 5^z$, then each H_{z_n} is uniformly recursive in H_b, because

$$t \in H_{z_n} \iff \langle t, n \rangle \in H_{3 \cdot 5^z}.$$

The more interesting of the many cases that follow are 2A.4 and 3B.3.

Case 1, *$|b|$ is finite*. In this case $\mathsf{IS}(b) = \{1, 1^*, 1^{**}, \ldots, 1^{*\cdots*}\}$ is a finite set of "towers of 2", and the definitions of $v(b)$ and $u(a, b)$ are quite easy, if a bit messy; we will skip them.

Case 2, *$|b|$ is infinite and $b = y^*$, for some y.*

(A) We consider subcases on a and y, to decide whether $a \in \mathsf{IS}(y^*)$ recursively in H_{b^*}.

Subcase 2A.1, $a = 1$. Now $a \prec y^*$ is true.

Subcase 2A.2, $a = x^*$, for some x. Now

$$x^* \prec y^* \iff x \prec y \iff x \in \mathsf{IS}(y),$$

and the induction hypothesis for (A) supplies us with a code of $\mathsf{IS}(y)$ from $H_{y^*} = H'_y$; from this we can construct a code of $\mathsf{IS}(y)$ from $H_{b^*} = H''_y$, and then use this code to decide from H_{b^*} whether $x \prec y$.

[21] The "uniqueness problem" was posed by Davis [1950a, 1950b], who answered it positively for $|a| < \omega^2$. Davis showed in fact that if $|a| = |b| < \omega^2$, then H_a and H_b are *many-one equivalent*, i.e., $t \in H_a \iff f(t) \in H_b$ with a recursive function f, and vice versa. This stronger form of uniqueness holds only for constructive ordinals of the form $\xi + \eta$ with $\eta < \omega^2$, cf. Moschovakis [1966], Nelson [1974].

Subcase 2A.3, $a = 3 \cdot 5^w$ *for some* w *and* y *is not of the form* $3 \cdot 5^z$. In this case $|y|$ is a successor ordinal while $|a|$ is a limit ordinal (if $a \in S_1$), and so we cannot have $|a| = |y|$; thus

$$a \in \mathsf{IS}(y^*) \iff a \in \mathsf{IS}(y),$$

and we can use the induction hypothesis for (A) on y to decide from H_{b^*} whether $a \in \mathsf{IS}(y^*)$ as above.

Subcase 2A.4, $a = 3 \cdot 5^w$ *for some* w *and* $y = 3 \cdot 5^z$ *for some* z. In this case

$$\begin{aligned} a \in \mathsf{IS}(b) &\iff 3 \cdot 5^w \precsim 3 \cdot 5^z \\ &\iff (\forall m)\big[w_m\!\downarrow \;\&\, w_{m+1}\!\downarrow \;\&\, (\exists n)\big[w_{m+1} \in \mathsf{IS}(z_n) \,\&\, w_m \in \mathsf{IS}(w_{m+1})\big]\big] \\ &\iff (\forall m)(\exists s)(\exists t)\big[w_m = s \,\&\, w_{m+1} = t \,\&\, (\exists n)\big[t \in \mathsf{IS}(z_n) \,\&\, s \in \mathsf{IS}(t)\big]\big] \\ &\iff (\forall m)(\exists s)(\exists t)(\exists n)\big[w_m = s \,\&\, w_{m+1} = t \,\&\, t \in \mathsf{IS}(z_n) \,\&\, s \in \mathsf{IS}(t)\big]. \end{aligned}$$

Now $\mathsf{IS}(z_n)$ is (uniformly) recursive in $H_{z_n^*}$ by the inductive hypothesis for (A); and so (uniformly) recursive in $H_{z_{n+1}}$ by the inductive hypothesis for (B), since $z_n^* \precsim z_{n+1}$; and so (uniformly) recursive in H_y. Similarly, $\mathsf{IS}(t)$ is (uniformly) recursive in H_{t^*} if $t \in \mathsf{IS}(z_n)$, so that $t \in S_1$; and so (uniformly) recursive in H_y, by the same argument if $t \in \mathsf{IS}(z_n)$. Also, the relation $w_m = s$ is semirecursive and so uniformly recursive in H_y, since $|y|$ is infinite. If we now combine these uniformities, we get a fixed number e such that if $b \in S_1$, then

$$a \in \mathsf{IS}(b) \iff (\forall m)(\exists s)(\exists t)(\exists n)\big[\{e\}^0(b, a, m, s, t, n, H_y) = 0\big];$$

and if we now contract like quantifiers and apply (32) and (33) (in the Appendix), we get an e' such that

$$a \in \mathsf{IS}(b) \iff \big\{e'\big\}^0(b, a, H_{y^{**}}) = 0 \iff \big\{S(e', b)\big\}^0(a, H_{b^*}) = 0$$

which is what we need.

Case 2A.5, *otherwise*. In this case $a \in \mathsf{IS}(b)$ is false.

(B) We define $u(a, b)$ taking cases on the form of a, and show that it has the required property when $a \prec b$.

Case 2B.1, $a = 1$. Now $H_a = \mathbb{N}$, and it is trivially recursive in H_b.

Case 2B.2, $a = x^*$ *for some* x. Now $a \precsim b$ gives $x \precsim y$, if $a \in S_1$; the induction hypothesis gives us a code of H_x from H_y; and from this we can construct a code of $H_a = H_x'$ from $H_b = H_y'$.

Case 2B.3, $a = 3 \cdot 5^w$. In this case, if $a \precsim b$, then $a \precsim y$, since $|a|$ is a limit ordinal while $|b|$ is a successor, and so they cannot be equal. The induction hypothesis supplies us with a code of H_a from H_y, from which we can compute a code of H_a from $H_b = H_y'$.

Case 2B.4, *otherwise*. In this case $a \notin S_1$ and we can define $u(a,b)$ arbitrarily, say set $u(a,b) = 0$.

This completes the definition of $v(b)$ and $u(a,b)$ and the proof of (A) and (B) when $b = y^*$ for some y.

Case 3, $b = 3 \cdot 5^z$, *for some* z.

(A) In this case,

$$a \in \mathsf{IS}(b) \iff (\exists n)\big[a \in \mathsf{IS}(z_n)\big],$$

and the induction hypothesis for (A) supplies us with a code of $\mathsf{IS}(z_n)$ from $H_{z_n^*}$; and then the induction hypothesis for (B) supplies us with a code of $H_{z_n^*}$ from $H_{z_{n+1}}$, and the definition of H_b gives us directly a code of $H_{z_{n+1}}$ from H_b. So finally we have an equivalence of the form

$$a \in \mathsf{IS}(b) \iff (\exists n)\big[\{e\}(a,b,n,H_b) = 0\big]$$

with some e; from which we can get a code of $\mathsf{IS}(b)$ from H_{b^*} using (32).

(B) For this we need again to take cases on the form of a.

Subcases 3B.1, 3B.2, $a = 1$ or $a = x^*$ for some x. The first is trivial, and in the second case we know that if $a \in S_1$, then

$$a \precsim b \Longrightarrow a \prec z_n$$

for some n, since $|a|$ is a successor ordinal. The induction hypothesis for (A) supplies us (uniformly) a code of $\mathsf{IS}(z_n)$ from $H_{z_n^*}$, from which we can compute a code of $\mathsf{IS}(z_n)$ from $H_{z_{n+1}}$ using the induction hypothesis for (A) (since $z_n^* \precsim z_{n+1}$), and then a code of the relation

$$R(a,b,n) \iff a \in \mathsf{IS}(z_n)$$

from H_b; and so we can compute the least n such that $a \in \mathsf{IS}(z_n)$ recursively in H_b. We can now decide whether $t \in H_a$ recursively from H_{z_n}, for this n, and so from H_b.

Subcase 3B.3, $a = e \cdot 5^w$ for some w. The assumption that $a \precsim b$ means that

$$(\forall m)(\exists n)[w_m{\downarrow}\ \&\, w_m \in \mathsf{IS}(z_n)],$$

and the induction hypothesis supplies us with a code of the relation

$$R(a,b,m,n) \iff w_m{\downarrow}\ \&\, w_m \in \mathsf{IS}(z_n)$$

from H_b, as in Case 3B.2. It follows that the function

$$f(m) = \mu n\big[w_m \in \mathsf{IS}(z_n)\big]$$

is recursive in H_b, and we can compute a code of it from H_b. Now, to reduce H_a to H_b, it is enough to reduce (uniformly) every H_{w_m} to H_b: and we do this by reducing H_{w_m} to H_{z_n} with $n = f(m)$, and then using the reduction of this to H_b. ⊣

Spector's proof was a big thing and, in particular, impressed Kleene immensely.[22] I don't think this was because of its technical difficulty—the combinatorial messiness: the earlier Kleene [1955a, 1955b, 1955c] include very intricate proofs by effective grounded recursion, and Kleene, most likely, already had some of the truly hairy applications of the method (like the λ-substitution theorem XXII) in Kleene [1959b] which was submitted less than three years later. Moreover, Spector's proof was considerably less technical than the argument above, because he was working with the subsystem $O \subsetneq S_1$ for which part (A) of Theorem 9.1 can be pulled out, shown separately, and then used as a lemma in the proof of (B). I think that the radically new ingredient in Spector's argument was the free-willing use of *relative recursion*: as he needs to formulate it in order for the proof to go through, Theorem 9.1 is about the relation $A \leq_e^T B$ where all three of the variables (A, e, B) vary, while (as far as I know) all arguments involving relative recursion before then involved using $A \leq_e^T B_0$ with a fixed B_0. It is a standard move ("*vary the parameter*"), but not always easy to make in connection with new notions, as relative recursion still was in 1954.

9.2. Kleene's Theorem: $\mathsf{HYP} = \Delta^1_1$**.** Above all the arithmetical relations in (35) and at the bottom of the analytical hierarchy sits Δ^1_1, whose characterization was an obvious challenge:

THEOREM 9.2 (Kleene [1955a, 1955c]). *A set $A \subseteq \mathbb{N}$ is Δ^1_1 if and only if it is hyperarithmetical. In fact*:

(a) *There is a recursive function $u(e)$, such that if e is a* HYP-*code of A, then $u(e)$ is a Δ^1_1-code of A, and*

(b) *there is a recursive function $v(m)$, such that if m is a Δ^1_1-code of A, then $v(m)$ is a* HYP-*code of A.*[23]

Theorem 9.2 is the most significant, foundational result in the sequence of articles Kleene [1935, 1943, 1944, 1955a, 1955b, 1955c, 1959a] in which Kleene developed the theory of arithmetical, hyperarithmetical and analytical relations on $\mathbb{N}$, surely one of the most impressive bodies of work in the theory of definability.[24] Starting with the [1944] article, Kleene uses effective,

[22]One of Kleene's favorite stories was how he "acted improperly" when Spector explained to him his proof: "I accepted it as Part 2 of his Ph.D. Thesis, as his supervisor, I urged him to write it up and submit it to me, as a JSL editor, and then I appointed myself to referee it and accepted it immediately; very improper—but ultimately for the good of logic, I think!" The story terrified his later students: how could our own, meager efforts measure up against this historic past?

[23]Kleene does not claim this uniform version of (b), only that $\Delta^1_1 \subseteq \mathsf{HYP}$, and Joe Shoenfield once gave me a spirited argument that *Kleene's Theorem* should properly refer to this weaker, non-uniform version. But the uniformity is just under the surface in Kleene's argument, as will become clear in the outline of the proof we give below.

[24]Kleene was the first logician to receive in 1983 the *Steele Prize for a seminal contribution to research* of the American Mathematical Society, specifically for the articles Kleene [1955a, 1955b, 1955c].

grounded recursion in practically every argument: it is the key, indispensable technical tool for this theory.

We will use (A) of Spector's Theorem 9.1 to show Theorem 9.2, reversing the historical order: Spector solved the uniqueness problem after Kleene proved his basic result—and got a (much) simpler proof of his teacher's theorem at the same time. I will outline here a proof which lies somewhere between Kleene's and Spector's, partly to be faithful to the spirit of Kleene's work but also because it illustrates more clearly the dependence of this fundamental result on the Second Recursion Theorem.

Outline of proof of Theorem 9.2. The argument depends heavily on the (uniform) closure properties of the classes of Π^1_1, Δ^1_1 and HYP relations which we have not presented in any detail here, and it involves some considerable computation.

Proof of (a). We need the following uniform closure properties of Δ^1_1, which are not hard to verify:

(i) *There is a recursive function $u_1(e)$, such that if e is a Δ^1_1-code of a set A, then $u_1(e)$ is a Δ^1_1-code of the jump A'.*

(ii) *There is a recursive function $u_2(e)$ such that if for each n, $\{e\}(n)\downarrow$ and $\{e\}(n)$ is a Δ^1_1-code of a set A_n, then $u_1(e)$ is a Δ^1_1-code of the set*

$$B_e = \{t : (t)_0 \in A_{(t)_1}\}.$$

(iii) *There is a recursive function $u_3(e, m)$, such that if m is a Δ^1_1-code of A and B is recursive in A with code e, then $u_3(e, m)$ is a Δ^1_1-code of B.*

Let e_0 be a fixed Δ^1_1-code of $\mathbb{N}$, fix e_1 so that

$$\{e_1\}(z, e, n) = \varphi_z(\{e\}(n_O)),$$

and choose $\tilde{z}$ by SRT such that

$$\varphi_{\tilde{z}}(a) = \begin{cases} e_0, & \text{if } a = 1, \\ u_1(\varphi_{\tilde{z}}(y)), & \text{if } x = 2^y \text{ for some } y, \\ u_2(S(e_1, \tilde{z}, e)), & \text{if } a = 3 \cdot 5^e \text{ for some } e, \\ 0, & \text{otherwise.} \end{cases}$$

To complete the proof, check first by induction on $|a|$ that if $a \in S_1$, then $\varphi_{\tilde{z}}(a)$ is a Δ^1_1-code of H_a; and then set $u(\langle e, a\rangle) = u_3(e, \varphi_{\tilde{z}}(a))$.

Proof of (b). We need two basic lemmas about ordinal notations.

Lemma 1. *The notation system S_1 is Π^1_1.*

Proof. As in (18), let

$$x \leq_e y \iff \varphi^2_e(x, y) = 0,$$

and set

$$e \in \mathrm{LO} \iff \varphi^2_e \text{ is total and } \leq_e \text{ is a linear ordering.}$$

The set LO is obviously arithmetical, and its subset W of codes of wellorderings is Π^1_1. Let also

$$Z_\alpha = \{t : \alpha(t) = 0\}$$

be the 0-set of each $\alpha \in \mathcal{N}$. Put

$$\begin{aligned} P(\alpha, \beta) \iff & (\forall x \in Z_\alpha)\,[\beta(x) \in \mathrm{LO}] \\ & \&\, 1 \in Z_\alpha \,\&\, \leq_{\beta(1)} \text{ is the empty relation} \\ & \&\, (\forall x)\,[x \in Z_\alpha \Longrightarrow x^* \in Z_\alpha \,\&\, \leq_{\beta(x^*)} \simeq \mathrm{Succ}(\leq_{\beta(x)})] \\ & \&\, (\forall z)\Big((\forall n)\,[z_n{\downarrow} \,\&\, z_n \in Z_\alpha \,\&\, P_1(z, n)] \\ & \qquad \Longrightarrow 3 \cdot 5^z \in Z_\alpha \,\&\, P_2\,(\beta(3 \cdot 5^z), z)\Big) \end{aligned}$$

where $\simeq$ indicates similarity (order isomorphism) of linear orderings and

(i) $\mathrm{Succ}(\leq_e)$ is the linear ordering constructed by putting one new point at the top of $\leq_e$;

(ii) $P_1(z, n) \iff \leq_{z_{n+1}}$ is not similar with any initial segment of $\leq_{z_n}$; and

$$\begin{aligned} P_2(e, z) \iff & e \in \mathrm{LO} \,\&\, (\forall n)[z_n \in \mathrm{LO}] \\ & \&\, (\forall n)[\, \leq_{z_n} \text{ is similar with a proper initial segment of } \leq_e \\ & \quad \&\, \text{ every proper initial segment of } \leq_e \\ & \qquad \text{is similar with an initial segment of some } \leq_{z_n} \,]. \end{aligned} \tag{iii}$$

(a) The relation $P(\alpha, \beta)$ is Σ^1_1. This is because a similarity between two linear orderings on subsets of $\mathbb{N}$ can be witnessed by some $\gamma \in \mathcal{N}$ which satisfies some arithmetical properties.

(b) If $Z_\alpha = S_1$ and β chooses some wellordering $\leq_{\beta(a)}$ of order type $|a|$ for each $a \in S_1$, then $P(\alpha, \beta)$.

(c) If $P(\alpha, \beta)$, then for every $a \in S_1$, $a \in Z_\alpha$ and $\leq_{\beta(a)}$ is a wellordering of order type $|a|$.

These are all quite simple (the last by induction on $|a|$), and together they imply that

$$a \in S_1 \iff (\forall \alpha)(\forall \beta)\big[P(\alpha, \beta) \Longrightarrow \alpha(a) = 0\big],$$

which gives the required Π^1_1 definition of S_1. $\dashv_{\mathit{Lemma}\ 1}$

Lemma 2. S_1 *is* Π^1_1*-complete: i.e., for every* Π^1_1 *set* A*, there is a total recursive function* $g(x)$ *such that*

$$x \in A \iff g(x) \in S_1. \tag{21}$$

Proof. Directly from the definition, every Π^1_1 set B satisfies an equivalence of the form

$$x \in B \iff (\forall\alpha)(\exists t)R(x, \overline{\alpha}(t)),$$

where $R \subseteq \mathbb{N}^{n+1}$ is recursive and $\big[u \sqsubseteq v \,\&\, R(\vec{x}, u)\big] \Longrightarrow R(\vec{x}, v)$. Let

$$Q(x,u) \iff (\forall\alpha)\big[u \sqsubseteq \alpha \Longrightarrow (\exists t)R(x, \overline{\alpha}(t))\big]$$

so that

$$x \in B \iff Q(x, 1),$$

and for any x, u, let

$$T_{x,u} = \Big\{v \colon \big[v \sqsubseteq u \vee u \sqsubseteq v\big] \,\&\, \neg R(x, v)\Big\}.$$

Thinking of $T_{x,u}$ as a set of sequences (rather than sequence codes), it is clearly closed under initial segments (it is a tree) because of the monotonicity condition on R, and easily

$$Q(x,u) \iff T_{x,u} \text{ is well founded.}$$

Fix $\hat{h}$ such that

$$\{\hat{h}\}(e, x, u, s) = \{e\}(x, u * \langle s\rangle)$$

and choose by SRT a function $\{\tilde{z}\}(x, u)$ such that

$$\{\tilde{z}\}(x,u) = \begin{cases} 1, & \text{if } R(x,u), \\ \mathrm{ub}(S(\hat{h}, \tilde{z}, x, u)), & \text{otherwise,} \end{cases}$$

where $\mathrm{ub}(e)$ is the upper bound function of Theorem 7.2. It follows that if $\{\tilde{z}\}(x, u * \langle s\rangle) \in S_1$ for every s, then $\{\tilde{z}\}(x,u) \in S_1$ and for every s, $|\{\tilde{z}\}(x, u * \langle s\rangle)| < |\{\tilde{z}\}(x,u)|$. The claim (21) now holds with $g(x) = \{\tilde{z}\}(x, 1)$, because

$$Q(x,u) \iff \{\tilde{z}\}(x,u) \in S_1;$$

this is easy to check, by induction on the rank of the wellfounded tree $T_{x,u}$ in the direction ($\Longrightarrow$) and by induction on $|\tilde{z}(x,u)|$ in the converse direction ($\Longleftarrow$). $\dashv_{\textit{Lemma 2}}$

To finish the proof, we will need a consequence of (A) in the Uniqueness Theorem 9.1 and Part (a) above: *there is a* Σ^1_1 *relation* $R^\Sigma(a, b)$ *such that*

$$b \in S_1 \Longrightarrow (\forall a)\Big[a \prec b \iff R^\Sigma(a,b)\Big];$$

because $\mathsf{IS}(b)$ is uniformly recursive in H_{b^*} and hence uniformly Δ^1_1—in particular, Σ^1_1.

Let $G(e,t)$ be a universal Π^1_1 relation for unary relations on $\mathbb{N}$ (for example $G = \mathsf{P}^{1,1,1,0}$ in the awful notation set in A5), and choose a recursive $g(t)$ by Lemma 2 such that

$$G(t,t) \iff g(t) \in S_1.$$

Suppose A is Δ^1_1, and by Lemma 2 again, choose a recursive $f(x)$ such that

$$x \in A \iff f(x) \in S_1.$$

The relation

$$Q(t) \iff (\exists x)\Big[x \in A \,\&\, R^\Sigma(g(t), f(x))\Big]$$

is evidently Σ^1_1, and so (by the universality of G) there is a number $\bar{e}$ such that

$$Q(t) \iff \neg G(\bar{e}, t).$$

Notice that $G(\bar{e},\bar{e})$; because if $\neg G(\bar{e},\bar{e})$, then $Q(\bar{e})$ holds, and so there is some x such that $x \in A$ and $R^\Sigma(g(\bar{e}), f(x))$, which means that $f(x) \in S_1$ and $g(\bar{e}) \prec f(x)$; which in turn implies that $g(\bar{e}) \in S_1$, and so $G(\bar{e},\bar{e})$. Now this implies that $\neg Q(\bar{e})$, which gives

$$\sup\big\{|f(x)|\colon x \in A\big\} < |g(\bar{e})|,$$

and so

$$x \in A \iff f(\vec{x}) \prec g(\bar{e})$$

and A is recursive in $H_{g(\bar{e})^*}$ by (A) of Theorem 9.1. The required uniformity $v(m)$ which computes a HYP-code of A from a given Δ^1_1-code m of it can be extracted from this argument. ⊣

9.3. HYP **is the smallest effective σ-field.** We formulate here a somewhat different, structural characterization of HYP, which clarifies its place in effective descriptive set theory, to which we will turn next.

An *effective σ-field* on $\mathbb{N}$ is a pair $(\mathcal{F}, c)$ where $\mathcal{F}$ is a class of subsets of $\mathbb{N}$; $c\colon C \twoheadrightarrow \mathcal{F}$ is a *number coding* of $\mathcal{F}$, a map from some $C \subseteq \mathbb{N}$ onto $\mathcal{F}$; and the following conditions hold with suitable recursive partial functions $u_s, u_c, u_\cup$, where, for simplicity, we write

$$F_a = c(a) \quad (a \in C).$$

(i) $\mathcal{F}$ contains uniformly all singletons: i.e., $u_s(n) \in C$ for every n and $F_{u_s(n)} = \{\!\{n\}\!\}$.

(ii) $\mathcal{F}$ is uniformly closed under complementation: i.e.,

$$a \in C \Longrightarrow \big[u_c(a)\!\downarrow \;\&\, u_c(a) \in C \,\&\, F_{u_c(a)} = F^c_a = \mathbb{N} \setminus F_a\big].$$

(iii) $\mathcal{F}$ is uniformly closed under recursive unions: i.e.,

$$(\forall t)\big[\varphi_m(t) \in C\big] \Longrightarrow \big[u_\cup(m)\!\downarrow \;\&\, u_\cup(m) \in C \,\&\, F_{u_\cup(m)} = \textstyle\bigcup_t F_{\varphi_m(t)}\big].$$

THEOREM 9.3. *The set of* HYP *subsets of* $\mathbb{N}$ *is the smallest effective σ-field; i.e., it is an effective σ-field* (with its natural coding), *and if* $(\mathcal{F}, c)$ *is any effective σ-field, then there is a recursive function* $\imath(x)$ *such that for every* HYP *set* A *with code* $\langle e, a\rangle$, $\imath(\langle e, a\rangle) \in C$ *and* $A = F_{\imath(\langle e,a\rangle)}$.

OUTLINE OF PROOF. From the functions $u_s, u_c, u_\cup$ that are needed to show that HYP is an effective σ-field, u_s and u_c are trivial. To define $u_\cup(m)$, fix recursive functions $e(m)$ and $a(m)$ such that

$$\{e(m)\}(t) = \big(\{m\}(t)\big)_0, \ \{a(m)\}(t) = \big(\{m\}(t)\big)_1,$$

so that if $\{m\}(t) = \langle\{e(m)\}(t), \{a(m)\}(t)\rangle$ is a HYP code of some set A_t, then $\{a(m)\}(t) \in S_1$ and $\{e(m)\}(t)$ is a code of A_t from $H_{\{a(m)\}(t)}$. By (b) of Theorem 7.2 then, $\overline{m} = \mathrm{ub}(a(m)) \in S_1$ and $\{a(m)\}(t) \prec \overline{m}$ for every t, if $\varphi^1_{a(m)}$ is total with values in S_1, so that by (B) of the Uniqueness Theorem (and standard properties of relative recursiveness), A_t is recursive in $H_{\overline{m}}$ with code $v(t, m)$, with a recursive function v. From this and (32), we easily get a recursive $w(m)$ such that

$$\begin{aligned} x \in \textstyle\bigcup_t A_t \iff & (\exists t)\big[\big\{v(t,m)\big\}\big(x, H_{\overline{m}}\big) = 0\big] \\ & \iff \big\{w(m)\big\}\big(x, H'_{\overline{m}}\big) = 0 \iff \big\{w(m)\big\}\big(x, H_{\overline{m}^*}\big) = 0, \end{aligned}$$

and then $u_\cup(m) = \langle w(m), \overline{m}^*\rangle$ is a HYP-code of $\bigcup_t A_t$, as required.

To prove the converse, suppose first that $(\mathcal{F}, c)$ is an effective σ-field which is *uniformly closed under recursive preimages*: i.e., there is a recursive function $u(a, m)$, such that if $a \in C$ and φ^1_m is total, then $u(a, m) \in C$ and

$$F_{u(a,m)} = \varphi_m^{-1}[F_a].$$

A relation $R \subseteq \mathbb{N}^n$ *is in* $\mathcal{F}$ *with code* $\langle n, a\rangle$ if $a \in C$ and

$$R(x_1, \dots, x_n) \iff \langle x_1, \dots, x_n\rangle \in F_a = c(a).$$

Using the additional hypothesis on $(\mathcal{F}, c)$, it is now routine to show:

(i) *The* (coded) *collection* $\mathcal{F}^r$ *of relations in* $\mathcal{F}$ *contains uniformly every* Σ^0_1 *relation and is uniformly closed under recursive substitutions,* $\&, \vee, \neg, (\exists t \in \mathbb{N})$ *and* $(\forall t \in \mathbb{N})$.

For example, this means that for some recursive function $v_\exists(n, a)$,

$$(\exists t)\big[\langle x_1, \dots, x_n, t\rangle \in F_a\big] \iff \langle x_1, \dots, x_n\rangle \in F_{v_\exists(n,a)},$$

and this $v_\exists(n, a)$ can be easily constructed using the uniform closure of $\mathcal{F}$ under recursive substitutions and recursive unions.

One of the consequences of (i) is that $\mathcal{F}$ is uniformly closed under the jump operation. Moreover:

(ii) $\mathcal{F}^r$ *is uniformly closed under diagonalization*: i.e., for some recursive $u_d(n, m)$, if φ^1_m is total and for every t, $\varphi_m(t)$ codes in $\mathcal{F}^r$ an n-ary relation

R_t, then the relation

$$R(t,\vec{x}) \iff R_t(\vec{x})$$

is also in $\mathcal{F}^r$ with code $u_d(n,m)$. This is because

$$R(t,\vec{x}) \iff \langle t,\vec{x}\rangle \in \big\{\langle t,\vec{x}\rangle : \vec{x} \in R_t\big\} \iff \langle t,\vec{x}\rangle \in \bigcup_s \big\{\langle s,\vec{x}\rangle : \vec{x} \in R_s\big\},$$

the relation $\{\langle t,\vec{x}\rangle : \vec{x} \in R_t\}$ is in $\mathcal{F}^r$ uniformly in t, because $\mathcal{F}^r$ is uniformly closed under recursive substitutions, and then $\mathcal{F}^r$ is uniformly closed under recursive unions, because $\mathcal{F}$ is.

Using (i) and (ii) now, we can define the required function ι which imbeds HYP in $\mathcal{F}$ by a routine effective grounded recursion.

To prove[25] the result for an arbitrary effective σ-field $(\mathcal{F}, c)$, let $\mathcal{F}^*$ be the set of all $A \in \mathcal{F}$ such that for every total φ_e, the inverse image $\varphi^{-1}[A] \in \mathcal{F}$, *uniformly*: i.e., there is a $\hat{v}$ such that

$$\text{if } \varphi_e \text{ is total, then } \{\hat{v}\}(e)\downarrow \ \& \ \{\hat{v}\}(e) \in C \ \& \ \varphi^{-1}[A] = c(\{\hat{v}\}(e));$$

any $\hat{v}$ with this property is an $\mathcal{F}^*$-code of A—and it determines an $\mathcal{F}$-code of A, namely $\{\hat{v}\}(i_0)$ where i_0 is any code of the identity function. It is quite easy to check that $\mathcal{F}^*$ is an effective σ-field with this coding, and it is uniformly closed under recursive preimages: so the proof of the special case gives us an imbedding ι of HYP into $\mathcal{F}^*$ whose composition with $\hat{v} \mapsto \{\hat{v}\}(i_0)$ finally embeds HYP into $(\mathcal{F}, c)$. $\dashv$

§10. Effective Descriptive Set Theory. Kleene was primarily interested in relations on $\mathbb{N}$, and he was more-or-less dragged into introducing quantification over $\mathcal{N}$ and the analytical hierarchy in order to find *explicit forms* for the hyperarithmetical relations. Once they were defined, however, the analytical relations on Baire space naturally posed new problems: is there, for example, a *construction principle* for the Δ^1_1 subsets of $\mathcal{N}$—a useful and interesting generalization of Theorem 9.2?

In fact, these were very old problems, initially posed (and sometimes solved, in different form) by Borel, Lebesgue, Lusin, Suslin and many others, primarily analysts and topologists who were working in *Descriptive Set Theory* in the first third of the 20th century. The similarity between what they had been doing and Kleene's work was first noticed by Mostowski [1946] and (especially) Addison [1954, 1959], and later work by many people created a common generalization of the classical and the new results now known as *Effective Descriptive Set Theory*, cf. Moschovakis [2009a].

After introducing the necessary definitions in this preamble, I will discuss just three results from effective descriptive set theory, which are proved by effective, grounded recursion and witness the power of the method and the breadth of its applicability.

[25] This wrinkle is needed because I do not know if every effective σ-field is uniformly closed under recursive preimages; I suspect it is not.

A topological space $\mathcal{X}$ is *Polish* if it is homeomorphic with a separable, complete metric space. Standard examples are the real numbers $\mathbb{R}$ and the Baire space $\mathcal{N}$, but $\mathbb{N}$ (with the discrete topology) is also Polish, albeit trivial, and the category of Polish spaces is closed under finite and countable products as well as many other natural operations.

Basic to the study of Polish spaces is the family $\mathbf{B} = \mathbf{B}\restriction\mathcal{X}$ of *Borel subsets* of each $\mathcal{X}$, the smallest σ-field of subsets of $\mathcal{X}$ which contains all the open sets. A function

$$f : \mathcal{X} \to \mathcal{Y}$$

is *Borel* (*measurable*) if the inverse image $f^{-1}[G]$ of any open set in $\mathcal{Y}$ is a Borel subset of $\mathcal{X}$, and it can be shown that *any two uncountable Polish spaces are Borel isomorphic*; so for most of the intersting results, it is enough to prove them for $\mathbb{R}$ or $\mathcal{N}$.

The effective theory of Polish spaces starts with the definition of a *presentation of* $\mathcal{X}$: this is any triple $(\{r_0, r_1, \dots, \}, P, Q)$ such that for some distance function $d : \mathcal{X} \to \mathbb{R}$ which generates the topology of $\mathcal{X}$:

(i) $\{r_0, r_1, \dots, \}$ is dense in $\mathcal{X}$,
(ii) $P(i, j, m, k) \iff d(r_i, r_j) \leq \frac{m}{k+1}$, and
(ii) $Q(i, j, m, k) \iff d(r_i, r_j) < \frac{m}{k+1}$.

Every presentation of $\mathcal{X}$ determines a compatible metric structure up to isometry and, in particular, it determines $\mathcal{X}$ up to homeomorphism. A presentation is *recursive* (in $\varepsilon \in \mathcal{N}$) if P and Q are recursive (in ε), and one typically gives proofs for *recursively presented* (r.p.) Polish spaces whose (trivial) *relativizations* apply then to all Polish spaces.

Fix a r.p. Polish space $\mathcal{X}$ and for each $s \in \mathbb{N}$, let

$$B_s = B_s^{\mathcal{X}} = \left\{ \mathbf{x} \in \mathcal{X} : d(r_{(s)_0}, \mathbf{x}) < \frac{(s)_1}{(s)_2 + 1} \right\},$$

so that the sequence of *open balls* $B_0, B_1, \dots$ generates the topology of $\mathcal{X}$.

A set $G \subseteq \mathcal{X}$ is *recursively open* or Σ^0_1 if for some r.e. set $W_e \subseteq \mathbb{N}$,

$$\mathbf{x} \in G \iff (\exists s)\big[\mathbf{x} \in B_s \,\&\, s \in W_e\big], \tag{22}$$

which means that G is a *recursive union* of open balls (or empty).

A function $f : \mathcal{X} \to \mathcal{Y}$ on one r.p. Polish space to another is *recursive*, if its *neighborhood diagram*

$$G_f(\mathbf{x}, s) \iff f(\mathbf{x}) \in B_s^{\mathcal{Y}} \quad (\mathbf{x} \in \mathcal{X}, s \in \mathbb{N})$$

is semirecursive. The class of these functions has all the natural closure properties one would expect, and they coincide with those defined in A2 and A6 when $\mathcal{X}$ is a "simple product space" and $\mathcal{Y}$ is $\mathbb{N}$ or $\mathcal{N}$. One may think of recursiveness in this context as a "computable refinement" of continuity.[26]

Starting with Σ^0_1, we define the *arithmetical pointclasses*[27] in r.p. Polish spaces by the obvious extension of the definitions in A5: Π^0_l is the pointclass of complements of sets in Σ^0_l; Σ^0_{l+1} comprises all sets $P \subseteq \mathcal{X}$ such that for some Π^0_l set $Q \subseteq \mathcal{X} \times \mathbb{N}$

$$P(\mathbf{x}) \iff (\exists t)Q(\mathbf{x}, t);$$

and $\Delta^0_l = \Sigma^0_l \cap \Pi^0_l$. For the *analytical pointclasses*, again: a set $P \subseteq \mathcal{X}$ is Σ^1_1 if

$$P(\mathbf{x}) \iff (\exists \alpha)Q(\mathbf{x}, \alpha) \tag{23}$$

with some Π^0_1 set $Q \subseteq \mathcal{X} \times \mathcal{N}$; it is Π^1_s if it is the complement of a Σ^1_s set; it is Σ^1_{s+1} if it satisfies (23) with a Π^1_s set $Q \subseteq \mathcal{X} \times \mathcal{N}$; and $\Delta^1_s = \Sigma^1_s \cap \Pi^1_s$. The definitions agree with those in A5 when $\mathcal{X}$ is a simple product space, and the extended pointclasses have all the standard closure properties of the arithmetical and analytical pointclasses on $\mathbb{N}$ an $\mathcal{N}$. In particular, they are all closed under *recursive substitutions*, i.e., if Γ is any one of them, $Q \subseteq \mathcal{Y}$ is in Γ, $f\colon \mathcal{X} \to \mathcal{Y}$ is recursive and

$$P(\mathbf{x}) \iff Q(f(\mathbf{x})) \quad (\mathbf{x} \in \mathcal{X}),$$

then $P(\mathbf{x})$ is also in Γ.

The *boldface associate* $\utilde{\boldsymbol{\Gamma}}$ of a pointclass Γ comprises all subsets $P \subseteq \mathcal{X}$ of a r.p. Polish space such that for some $R \subseteq \mathcal{N} \times \mathcal{X}$, $R \in \Gamma$ and some $\delta_0 \in \mathcal{N}$,

$$P(\mathbf{x}) \iff R(\delta_0, \mathbf{x}).$$

The pointclasses $\utilde{\boldsymbol{\Sigma}}^1_s, \utilde{\boldsymbol{\Pi}}^1_s$ are constructed in this way from Σ^1_s, Π^1_s, and then we set $\utilde{\boldsymbol{\Delta}}^1_s = \utilde{\boldsymbol{\Sigma}}^1_s \cap \utilde{\boldsymbol{\Pi}}^1_s$. It turns out that these pointclasses are exactly the *projective pointclasses* which had been introduced by Luzin and Sierpinski in topological terms in the 1920s, one of the "similarities" between classical descriptive set theory and the "hierarchy theory" of Kleene (as it was then called) which caught the attention of Mostowski [1946] and Addison [1954, 1959]; the other was the following, central result of the classical theory:

Theorem (Suslin [1917]). *On every Polish space $\mathcal{X}$, $\utilde{\boldsymbol{\Delta}}^1_1 = \boldsymbol{B}$.*

Once Suslin's Theorem is expressed in this notation,[28] it is not possible to miss the similarity between it and Kleene's Theorem 9.2.

[26]One can also define *recursive partial functions* $f\colon \mathcal{X} \rightharpoonup \mathcal{Y}$ on one r.p. Polish space to another, cf. Section 7A of Moschovakis [2009a]. It is a bit tricky, however, and we only need here the special case in A6, when $\mathcal{X}$ is a simple product space and $\mathcal{Y}$ is $\mathbb{N}$ or $\mathcal{N}$.

[27]A *pointclass* is a collection of subsets of sets in some family which is typically closed under Cartesian products—the family of all r.p. Polish spaces in this case.

[28]The $\Sigma, \Pi, \utilde{\boldsymbol{\Sigma}}, \utilde{\boldsymbol{\Pi}}$ notations were introduced by Addison, who employed them precisely to bring out the analogies between the work of Kleene and classical descriptive set theory. (Shoenfield added later the $\Delta, \utilde{\boldsymbol{\Delta}}$ symbols).

10.1. The Suslin–Kleene Theorem. Suslin's Theorem says nothing about $\mathbb{N}$, since every $A \subseteq \mathbb{N}$ is Borel, and Kleene's Theorem says nothing about subsets of $\mathbb{R}$. The first substantial achievement of effective descriptive set theory was the derivation of a simple, general result which implies both of them. To state it precisely, we need to code the $\underset{\sim}{\mathbf{\Delta}}^1_1$ and the Borel subsets of r.p. Polish spaces, and we might as well do this in a natural way that will also be useful later on.

Suppose Γ is a pointclass of subsets of r.p. Polish spaces. A *good parametrization* of Γ (in $\mathcal{N}$) is an assignment of a Γ-relation $G^{\mathcal{X}} \subseteq \mathcal{N} \times \mathcal{X}$ to every r.p. Polish space, so that the following hold with

$$G^{\mathcal{X}}_\varepsilon(\mathbf{x}) \iff G^{\mathcal{X}}(\varepsilon, \mathbf{x}) \quad (\varepsilon \in \mathcal{N}, \mathbf{x} \in \mathcal{X}).$$

(i) A set $P \subseteq \mathcal{X}$ is in $\underset{\sim}{\mathbf{\Gamma}}$ if and only if $P = G^{\mathcal{X}}_\varepsilon$ for some $\varepsilon \in \mathcal{N}$; and P is in Γ if and only if $P = G^{\mathcal{X}}_\varepsilon$ with a recursive ε.

(ii) For every k and every r.p. space $\mathcal{X}$, there is a recursive function $S^k_{\mathcal{X}} = S : \mathcal{N}^{1+k} \to \mathcal{N}$ such that

$$G^{\mathcal{N}^k \times \mathcal{X}}(\varepsilon, \vec{\alpha}, \mathbf{x}) \iff G^{\mathcal{X}}(S(\varepsilon, \vec{\alpha}), \mathbf{x}).$$

Lemma. *The pointclasses Σ^0_l, Π^0_l, Σ^1_s, Π^1_s are all well parametrized.*

Proof is very easy, by induction on l and then s, starting with a good parametrization of Σ^0_1 which can be read-off the definition (22). ⊣

We now fix a good parametrization of Π^1_1 and we use it to code the $\underset{\sim}{\mathbf{\Delta}}^1_1$ sets: ε is a $\underset{\sim}{\mathbf{\Pi}}^1_1$-code of $G^{\mathcal{X}}_\varepsilon$, and if $A = G^{\mathcal{X}}_{\varepsilon_1} = \mathcal{X} \setminus G^{\mathcal{X}}_{\varepsilon_2}$, then $\langle \varepsilon_1, \varepsilon_2 \rangle$ is a $\underset{\sim}{\mathbf{\Delta}}^1_1$ code of A (with the tupling function $\langle \varepsilon_1, \varepsilon_2 \rangle$ defined in A6).

To code the Borel sets, we first let BC be the least subset of $\mathcal{N}$ which contains $\{\alpha : \alpha(0) = 0\}$ and satisfies the following implication, for every $\alpha \in \mathcal{N}$, with the notations in A6, A7:

$$\Big(\alpha(0) \neq 0 \,\&\, (\forall t)\big[\{\alpha^*\}^1(t) \in \mathrm{BC}\big]\Big) \Longrightarrow \alpha \in \mathrm{BC}.$$

This is the set of *Borel codes*. For each r.p. Polish space $\mathcal{X}$ then, we assign to each $\alpha \in \mathrm{BC}$ a set $B_\alpha = B^{\mathcal{X}}_\alpha$ such that

$$\alpha(0) = 0 \Longrightarrow B_\alpha = B_{\alpha(1)} \quad \text{(the open ball)},$$
$$\alpha \in \mathrm{BC} \,\&\, \alpha(0) \neq 0 \Longrightarrow B_\alpha = \textstyle\bigcup_t \big(\mathcal{X} \setminus B_{\{\alpha^*\}^1(t)}\big).$$

One needs to verify that the definitions make sense and that the family $\{B^{\mathcal{X}}_\alpha : \alpha \in \mathrm{BC}\}$ comprises precisely the Borel subsets of $\mathcal{X}$, but nothing difficult is involved in this.

Theorem 10.1 (The Suslin–Kleene Theorem, see Moschovakis [2009a]). *For each r.p. Polish space $\mathcal{X}$, there are recursive functions $u, v : \mathcal{N} \to \mathcal{N}$ such that if α is a Borel code of a set $A \subseteq \mathcal{X}$, then $u(\alpha)$ is a $\underset{\sim}{\mathbf{\Delta}}^1_1$-code of A, and if β is a $\underset{\sim}{\mathbf{\Delta}}^1_1$-code of A, then $v(\beta)$ is a Borel code of A.*

In particular, the Δ^1_1 subsets of $\mathcal{X}$ are exactly the Borel subsets of $\mathcal{X}$ which have recursive codes.

The Suslin–Kleene Theorem implies immediately Suslin's Theorem and also Kleene's Theorem 9.2 as follows: if we set

$$C = \Big\{e : \varphi^1_e : \mathbb{N} \rightharpoonup \mathbb{N} \text{ is total and } \varphi^1_e \in \mathrm{BC}\Big\},$$

then Theorem 10.1 for $\mathcal{X} = \mathbb{N}$ implies that every Δ^1_1 subset of $\mathbb{N}$ is

$$B_e = B^{\mathbb{N}}_{\varphi^1_e} \text{ for some } e \in C;$$

and a routine effective grounded recursion imbeds then $\{B_e : e \in C\}$ into every effective σ-field, including HYP, as required.[29]

Beyond its foundational significance, the Suslin–Kleene Theorem has many applications, as did the classical theorem of Suslin. For example: *if*

$$f : \mathcal{X} \rightarrowtail \mathcal{Y}$$

is one-to-one and Borel measurable, then the image $f[\mathcal{X}]$ is a Borel subset of $\mathcal{Y}$ and f has a Borel measurable inverse. This is the famous theorem that Lebesgue thought he had proved in his [1905], until Suslin discovered the error ten years later and proved his result, which is a key ingredient of a correct proof of Lebesgue's claim. There is a natural effective version of this claim which, understandably, needs the Suslin–Kleene Theorem for its proof.

The Suslin–Kleene Theorem is proved in Moschovakis [2009a] by adapting one of the classical proofs of Suslin's Theorem and a version of effective grounded recursion justified by the Second Recursion Theorem for simple product spaces in A7.[30] A somewhat novel slant on this proof (which also uses SRT) is discussed in Moschovakis [2010].[31]

10.2. The Normed Induction Theorem. Inductive definitions are prevalent in many parts of definability theory, including (emphatically) effective descriptive set theory. We will prove here a very easy result—it is called a "remark" in Moschovakis [2009a]—which provides explicit definitions of inductively defined sets in a great variety of circumstances. It is an immediate consequence of the following

[29] By adjusting the proof of Theorem 9.3 and with the proper definitions, it is easy to show that for every r.p. Polish space $\mathcal{X}$, *the family of Δ^1_1 subsets of $\mathcal{X}$ is the smallest effective σ-field which contains uniformly all the open balls B_s.*

[30] The Suslin–Kleene Theorem is only claimed in Moschovakis [2009a] for products of discrete and perfect r.p. Polish spaces, as are all the effective results in that book; but the proof works for all r.c. Polish spaces, as do most but not all proofs of effective versions of basic, classical theorems. For example, *every perfect r.p. space is Δ^1_1-isomorphic with $\mathcal{N}$*, but not every uncountable r.p. space has this property; cf. Gregoriades [2009], which identifies the main results of the effective theory that do not hold for all r.p. Polish spaces.

[31] It is not clear who (if anyone) deserves credit for noticing that the proof in Kuratowski [1966] can be "effectivized" by using SRT; the idea was certainly in the air in the early seventies.

Second Recursion Theorem for relations. *Suppose Γ is a well parametrized pointclass of r.p. Polish spaces which is closed under recursive substitutions, $G \subseteq \mathcal{N} \times \mathcal{X}$ is the universal set for $\mathcal{X}$, and $P \subseteq \mathcal{N} \times \mathcal{X}$ is in Γ; then there exists a recursive $\tilde{\varepsilon} \in \mathcal{N}$, such that*

$$G(\tilde{\varepsilon}, \mathbf{x}) \iff P(\tilde{\varepsilon}, \mathbf{x}).$$

Proof is the same as always. $\dashv$

Although easy to prove, the Normed Induction Theorem is unfortunately not elementary. We will illustrate the background we need to state it using the following effective version of the Cantor–Bendixson theorem, which is a trivial consequence of it.[32]

Theorem (Kreisel [1959]). *Suppose F is a Σ^1_1 subset of the r.p. space $\mathcal{X}$ which is closed, and*

$$F = k(F) \cup s(F)$$

is the canonical (unique) *decomposition of F into a perfect kernel $k(F)$ and a countable scattered part $s(F)$; then $k(F)$ is Σ^1_1.*

Recall that an operator $\Phi\colon \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$ on the subsets of a space $\mathcal{X}$ is *monotone* if

$$A \subseteq B \Longrightarrow \Phi(A) \subseteq \Phi(B) \quad (A, B \subseteq \mathcal{X}).$$

Every monotone operator on $\mathcal{P}(\mathcal{X})$ has a *least fixed point* $\Phi^{(\infty)}$ and a *largest fixed point* $\Phi_{(\infty)}$, defined by

$$\Phi^{(\infty)} = \bigcup\nolimits_\xi \Phi^{(\xi)}, \quad \Phi_{(\infty)} = \bigcap\nolimits_\xi \Phi_{(\xi)},$$

where for each ordinal ξ, we set recursively

$$\Phi^{(\xi)} = \Phi\Big(\bigcup\nolimits_{\eta<\xi} \Phi^{(\eta)}\Big) \text{ and } \Phi_{(\xi)} = \Phi\Big(\bigcap\nolimits_{\eta<\xi} \Phi^{(\eta)}\Big)$$

(with the usual conventions that $\bigcup \emptyset = \emptyset$ and $\bigcap \emptyset = \mathcal{X}$). These are related via the *dual operator*

$$\check{\Phi}(A) = (\Phi(A^c))^c$$

which is also monotone and such that

$$\check{\Phi}^{(\infty)} = \Big(\Phi_{(\infty)}\Big)^c, \quad \check{\Phi}_{(\infty)} = \Big(\Phi^{(\infty)}\Big)^c;$$

this identity holds in fact *locally*, i.e., for every ξ, not just ∞.

For the Cantor–Bendixson decomposition, the relevant operator is the *Cantor derivative*

$$D_F(A) = \big\{\mathbf{x}\colon (\forall s)\big[\mathbf{x} \in B_s \Longrightarrow (\exists \mathbf{z})\big[\mathbf{z} \in B_s \,\&\, \mathbf{z} \neq \mathbf{x} \,\&\, \mathbf{z} \in F \cap A\big]\big]\big\} \quad (24)$$

[32] Kreisel actually proved much more, including the fact that this estimate of the complexity of $k(F)$ is best possible: there are, in fact, Π^0_1 sets of reals whose kernel is not Δ^1_1.

which assigns to each $A \subseteq \mathcal{X}$ the set of limit points of $F \cap A$. By one of the standard proofs of the Cantor–Bendixson Theorem, the kernel of F is the largest fixed point of D_F,

$$D_{F(\infty)} = k(F);$$

and so what we need to prove is that $D_{F(\infty)}$ is Σ^1_1, presumably because of some properties of D_F and Σ^1_1.

We take up first the pertinent properties of D_F.

Suppose Γ is a pointclass on r.p. Polish spaces, for example one of the arithmetical or analytical pointclasses we have been studying, and Φ is a monotone operator on $\mathcal{P}(\mathcal{X})$ for some space $\mathcal{X}$. For any $\mathcal{Y}$ and any set $P \subseteq \mathcal{Y} \times \mathcal{X}$, we put

$$P_\Phi(\mathbf{y}, \mathbf{x}) \iff \mathbf{x} \in \Phi\big(\{\mathbf{x}' : P(\mathbf{y}, \mathbf{x}')\}\big),$$

and we say that Φ *is* Γ *on* Γ if for any such P,

$$\text{if } P \in \Gamma, \text{ then } P_\Phi \in \Gamma.$$

By (24),

$$P_{D_F}(\mathbf{y}, \mathbf{x}) \iff (\forall s)\big[\mathbf{x} \in B_s \Longrightarrow (\exists \mathbf{z})\big[\mathbf{z} \in B_s \,\&\, \mathbf{z} \neq \mathbf{x} \,\&\, \mathbf{z} \in F \,\&\, P(\mathbf{y}, \mathbf{z})\big]\big],$$

and so the Cantor Derivative $D_F(A)$ is evidently Σ^1_1 on Σ^1_1; which implies that the dual operator

$$\Phi(A) = \check{D}_F(A) = D_F(A^c)^c$$

is Π^1_1 on Π^1_1. Thus, what we need to show for the Kreisel result is that

Π^1_1 *on* Π^1_1 *monotone operators have* Π^1_1 *least fixed points.*

The *prewellordering property* of Π^1_1 which insures this fact is very basic and widely applicable.

A *norm* on a set $P \subseteq \mathcal{X}$ is any function

$$\rho : P \to \text{Ordinals},$$

which we view as an abstract (ordinal valued) "complexity measure"; and it is a Γ-*norm* if the following two binary relations on $\mathcal{X}$ associated with ρ are in Γ:

$$\mathbf{x} \leq^*_\rho \mathbf{y} \iff \mathbf{x} \in P \,\&\, \big[\mathbf{y} \notin P \vee \rho(\mathbf{x}) \leq \rho(\mathbf{y})\big],$$
$$\mathbf{x} <^*_\rho \mathbf{y} \iff \mathbf{x} \in P \,\&\, \big[\mathbf{y} \notin P \vee \rho(\mathbf{x}) < \rho(y)\big].$$

A pointclass Γ is *normed on* $\mathcal{X}$ if every $P \subseteq \mathcal{X}$ which is in Γ carries a Γ-norm; Γ is *normed* (or has the *prewellordering property*) if it is normed on every $\mathcal{X}$.

The simplest, classical example is Σ^0_1: when $\mathcal{X}$ is a simple product space (as these are defined in the Appendix), then the norms $\rho : P \to \omega$ come directly from the appropriate *Kleene Normal Form Theorem*—i.e., $\rho(\mathbf{x})$ is "the code of the computation which verifies that $\mathbf{x} \in P$".[33]

More significantly, for the applications we are pursuing here, Π^1_1 *and* Σ^1_2 *are normed on every r.p. space* $\mathcal{X}$: these are classical results which are proved in one form or another at the very beginning of an exposition of effective (or classical, for that matter) descriptive set theory.

Theorem 10.2 (Moschovakis [1974] and (7C.8) in [2009a]). *Suppose* Γ *is a pointclass on r.p. Polish spaces which is well parametrized, closed under recursive substitutions and normed on* $\mathcal{N} \times \mathcal{X}$, *and suppose that* Φ *is a monotone operator on* $\mathcal{P}(\mathcal{X})$ *which is* Γ *on* Γ; *then* $\Phi^{(\infty)}$ *is in* Γ, *and so* $\check{\Phi}_{(\infty)}$ *is in the dual pointclass* $\neg\Gamma$ *of complements of* Γ*-sets.*

Proof. Let ρ be a Γ-norm on the set $G \subseteq \mathcal{N} \times \mathcal{X}$ which is universal for the Γ-subsets of $\mathcal{X}$. The relation

$$P(\varepsilon, \mathbf{x}) \iff \mathbf{x} \in \Phi\big(\{\mathbf{y} \colon (\varepsilon, \mathbf{y}) <^*_\rho (\varepsilon, \mathbf{x})\}\big)$$

is (easily) in Γ by the hypotheses, and so by SRT for relations above, there is a recursive $\tilde{\varepsilon} \in \mathcal{N}$ such that

$$G(\tilde{\varepsilon}, \mathbf{x}) \iff \mathbf{x} \in \Phi\big(\{\mathbf{y} \colon (\tilde{\varepsilon}, \mathbf{y}) <^*_\rho (\tilde{\varepsilon}, \mathbf{x})\}\big). \tag{25}$$

We complete the proof by showing that $G_{\tilde{\varepsilon}} = \Phi^{(\infty)}$.

(a) *For all* $\mathbf{x} \in \mathcal{X}$, $G(\tilde{\varepsilon}, \mathbf{x}) \Longrightarrow \mathbf{x} \in \Phi^{(\infty)}$. By induction on $\rho(\tilde{\varepsilon}, \mathbf{x})$:

$$\begin{aligned} G(\tilde{\varepsilon}, \mathbf{x}) &\Longrightarrow \mathbf{x} \in \Phi\big(\{\mathbf{y} \colon (\tilde{\varepsilon}, \mathbf{y}) <^*_\rho (\tilde{\varepsilon}, \mathbf{x})\}\big) && \text{(by (25))} \\ &\Longrightarrow \mathbf{x} \in \Phi\big(\{\mathbf{y} \colon \mathbf{y} \in \Phi^{(\infty)}\}\big) && \text{(ind. hyp. and monotonicity of } \Phi) \\ &\Longrightarrow \mathbf{x} \in \Phi^{(\infty)}. && \text{(because } \Phi^{(\infty)} \text{ is a fixed point of } \Phi) \end{aligned}$$

(b) *For all* ξ *and* $\mathbf{x}$, $\mathbf{x} \in \Phi^{(\xi)} \Longrightarrow G(\tilde{\varepsilon}, \mathbf{x})$. By induction on ξ: suppose the claim holds for all $\eta < \xi$ but for some $\mathbf{x} \in \Phi^{(\xi)}$, $\neg G(\tilde{\varepsilon}, \mathbf{x})$. By the definition of $<^*_\rho$, this means that

$$\big[\mathbf{y} \in \Phi^{(\eta)} \;\&\; \eta < \xi\big] \Longrightarrow (\tilde{\varepsilon}, \mathbf{y}) <^*_\rho (\tilde{\varepsilon}, \mathbf{x}),$$

and by the monotonicity of Φ,

$$\begin{aligned} \mathbf{x} \in \Phi^{(\xi)} &\Longrightarrow \mathbf{x} \in \Phi\Big(\textstyle\bigcup_\eta \Phi^{(\eta)}\Big) \\ &\Longrightarrow \mathbf{x} \in \Phi\big(\{\mathbf{y} \colon (\tilde{\varepsilon}, \mathbf{y}) <^*_\rho (\tilde{\varepsilon}, \mathbf{x})\}\big) \Longrightarrow G(\tilde{\varepsilon}, \mathbf{x}), \end{aligned}$$

which is a contradiction. ⊣

The Normed Induction Theorem yields *structural characterizations* for various pointclasses which are foundationally interesting, e.g., Π^1_1 *is the least pointclass on r.p. Polish spaces which includes the basic relations* $\mathbf{x} \in B_s$, *is closed under recursive substitutions,* $\&, \vee, (\exists t \in \mathbb{N}), (\forall t \in \mathbb{N})$ *and is well*

[33] A Σ^0_1 set of real numbers which carries a Σ^0_1 norm is (easily) either empty or the whole space, and so Σ^0_1 is not (fully) normed—the only interesting example of this kind.

parametrized and normed; this was the immediate motivation for proving it. More recently it has also been used in applications of the effective theory to analysis, cf. Debs [1987, 2009].

There is an interesting connection between Theorem 10.2 and the infamous error in Kleene [1944], where it was claimed that O is Π^0_2, while, in fact, O (like S_1) is Π^1_1-complete and hence not even Σ^1_1, let alone arithmetical: this correct placement of O in the Kleene hierarchy is the main result of Kleene [1955b].

In the first ordinal paper, Kleene introduced a method for finding an explicit form for an inductively defined relation which is not unlike the way that differential equations are often solved: one guesses a form of the solution (for example as a power series) and then computes, by plugging in, a specific function of this form which satisfies the equation. Without doing too much violence to the facts,[34] we can paraphrase Kleene's application of this method as follows.

The ordinal notation system O is the least fixed point of a specific monotone operator Φ_O on $\mathcal{P}(\mathbb{N})$. Using our terminology, Kleene proved (in effect) that Φ_O is Π^0_2 on Π^0_2, so that if $H \subseteq \mathcal{N} \times \mathbb{N}$ is the universal Π^0_2 set relative to a good parametrization of Π^0_2, then SRT yields a recursive $\tilde{\varepsilon}$ such that

$$H(\tilde{\varepsilon}, a) \iff a \in \Phi_O(H_{\tilde{\varepsilon}}), \tag{26}$$

i.e., a Π^0_2 fixed point of Φ_O; and so—he said—$H_{\tilde{\varepsilon}} = O$ and O is Π^0_2. But there is no reason for $H_{\tilde{\varepsilon}}$ to be *the least fixed point* of Φ_O—and in fact it cannot be, since O is not Π^0_2. This is the false step in the argument. (It is a little like assuming that a formal, power series solution of a differential equation is the solution we want without checking that it satisfies the relevant initial and boundary conditions.)

In fact, it is *the largest fixed point* of Φ_O which is Π^0_2, by Theorem 10.2: because (by Kleene's computation), the dual operator $\check{\Phi}_O$ is Σ^0_2 on Σ^0_2 and Σ^0_2 is normed—which Π^0_2 is not. Moreover, Theorem 10.2 also implies that O is Π^1_1, because Φ_O is (easily) Π^1_1 on Π^1_1. The argument[35] suggests that

[34] Kleene did not explicitly use SRT in the first ordinal paper, for reasons that I do not quite understand: it simplifies the argument without interfering with any of its sensitive parts, including the error.

[35] The proof in Kleene [1955b] that O is Π^1_1 is so complicated as to be practically unreadable, but the appeal to Theorem 10.2 is also not needed: the result can be easily read off the equivalence

$$a \in O \iff (\forall A \subseteq \mathbb{N})\big[(\Phi_O(a) = A) \Longrightarrow a \in A\big]$$

more-or-less as we did for S_1 in the proof of Theorem 9.2, cf. Wang [1958]. The Normed Induction Theorem is useful in deriving tight explicit forms for relations which are defined inductively over $\mathcal{N}$, where this simple computation is not useful because it requires quantification over $\mathcal{P}(\mathcal{N})$.

the equivalence we should use to "solve" a recursive definition is not (26) but (25) which, to make sense, requires Γ to be normed on $\mathcal{N} \times \mathcal{X}$.

From this point of view, one may view Theorem 10.2 as one "correct version" of what Kleene had in mind in the first ordinal paper.

10.3. The Coding Lemma. The last example is from the exotic world of *determinacy*, about as far from recursion theory as one could go—or so it seems at first.

THEOREM 10.3 (The Coding Lemma, Moschovakis [1970, 2009a]). *In the theory* ZFDC+AD: *if there exists a surjection* $f : \mathbb{R} \twoheadrightarrow \kappa$ *of the continuum onto a cardinal* κ, *then there exists a surjection* $g : \mathbb{R} \twoheadrightarrow \mathcal{P}(\kappa)$ *of the continuum onto the powerset of* κ.

Here ZFDC stands for ZFC with the Axiom of (full) Choice AC replaced by the weaker *Axiom of Dependent Choices* DC, and AD is the *Axiom of Determinacy*, which is inconsistent with AC. It has been shown by Martin and Steel [1988] and Woodin [1988] that (granting the appropriate large cardinal axioms), AD holds in $L(\mathbb{R})$, the smallest model of ZFDC which contains all ordinals and all real numbers. Long before that great (and reassuring!) result, however, AD was used systematically to uncover the structure of the analytical and projective hierarchies, see Moschovakis [2009a].

It is not possible to give here a brief, meaningful explanation of all that goes into the statement of Theorem 10.3 which, in any case, is only a corollary of a substantially stronger result and a precursor of more general, later theorems. Notice, however, that in a world where it holds, $\mathbb{R}$ is immense in size, if we measure size by surjections: it can be mapped onto $\aleph_1$ (classically), and so onto $\aleph_2$, and inductively onto every $\aleph_n$ and so onto $\aleph_\omega$, etc., all the way onto every $\aleph_\xi$ for $\xi < \aleph_1$; and it can also be mapped onto the powerset of each of these cardinals. This *surjective size* of $\mathbb{R}$ is actually immense in the world of AD and the Coding Lemma is one of the important tools in proving this, as well as many other consequences of AD.

The Coding Lemma is also one of the basic tools used in Jackson's Theorem 11.2 in the next section, one of the most spectacular results connecting two seemingly totally unrelated parts of logic.

The proof of the Coding Lemma is by effective transfinite recursion supported by a version of SRT which is a bit like that in A7, except that $\{\varepsilon\}^1(x)$ is computed relative to some arbitrary $A \subseteq \mathcal{N}$; and it does not appear possible to prove the Coding Lemma without some version of SRT, which in this way creeps into the study of cardinals, perhaps the most purely set-theoretic part of set theory.

The hypothesis AD of full determinacy is explored in Section 7D, *The completely playful universe* of Moschovakis [2009a], part of Chapter 7 whose title is *The Recursion Theorem*.

§11. Recursion in higher types and Jackson's Theorem. Starting with his [1959b], Kleene developed a theory of recursive partial functions with arguments objects of arbitrary (simple, finite) type over $\mathbb{N}$, i.e., members of the sets T_n defined by

$$T_0 = \mathbb{N}, \quad T_{n+1} = (T_n \to \mathbb{N}) = \{\alpha^{n+1} \colon \alpha^{n+1} \colon T_n \to \mathbb{N}\}.$$

Simple examples which illustrate how these objects can be "called" as arguments include

$$f(\alpha^2, \beta^1) = \alpha^2(\beta^1) + 1, \quad g(\alpha^2, \beta^1, s) = \alpha^2(\lambda n(\mu t[\beta^1(sn+t) = 0])),$$

where we have used superscripts to indicate the types of function arguments— $\beta^1 \in T_1$, $\alpha^2 \in T_2$, etc. There was no obvious "machine model" which could capture the notion of computability which seemed natural for these partial functions, and so Kleene developed an entirely novel approach. Letting

$$\mathfrak{a} = \mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_n \quad (\text{type}(\mathfrak{a}_1) \le \text{type}(\mathfrak{a}_2) \le \dots \le \text{type}(\mathfrak{a}_n))$$

vary over finite sequences of objects of any fixed, non-decreasing sequence of types, he defines directly the basic relation

$$\begin{aligned}&\{z\}(\mathfrak{a}) = w \\ &\quad \iff z \text{ codes a recursive partial function which on } \mathfrak{a} \text{ gives output } w\end{aligned}$$

by an inductive definition much like that of S_1 above. A partial function

$$f \colon T_{a_1} \times \cdots T_{a_n} \rightharpoonup \mathbb{N}$$

is recursive, if there is some z such that for all $\mathfrak{a}$ with type sequence $a_1, \dots, a_n$ and all $w \in \mathbb{N}$,

$$f(\mathfrak{a}) = w \iff \{z\}(\mathfrak{a}) = w.$$

The set-theoretic interpretation of this inductive definition closes at the (relatively) big cardinal $\beth_\omega$, but the definition is effective enough so that results about these partial functions can be established reasonably constructively—by effective grounded recursion in almost every case, justified by the appropriate version of SRT which is (essentially) the first result in Kleene [1959b].

This is a beautiful theory with many applications to several areas of logic, and there is no way that we can do it justice here, so I will confine myself to the statements of just one of Kleene's basic results and a much more recent (and very different) theorem of Jackson.

For each $k \ge 1$, let

$$^{k+1}\mathbf{E}(\alpha^k) = \begin{cases} 0, & \text{if } (\exists \beta \in T_{k-1})[\alpha(\beta) = 0], \\ 1, & \text{otherwise,} \end{cases} \tag{27}$$

be the type-$(k+1)$ object which "embodies" (in Kleene's description) quantification over type $k-1$, so that $^2\mathbf{E}$ models quantification over $\mathbb{N}$.

THEOREM 11.1 (Kleene [1959b]). *A relation* $R \subseteq \mathbb{N}^n$ *is* Δ^1_1 *if and only if it is recursive in* ${}^2\mathsf{E}$.

The theorem expresses in a different way the basic foundational claim of Theorem 9.2, that the Δ^1_1 relations on $\mathbb{N}$ are "essentially" first-order: they are the sets for which membership can be *decided effectively, if we are allowed to quantify over* $\mathbb{N}$.

Recursion in the higher-type quantifiers ${}^k\mathsf{E}$ has also been the object of intensive study, cf. Kechris and Moschovakis [1977] and Sacks [1990]. But one more recent result about ${}^3\mathsf{E}$ deserves stating here. It is about *the Kleene ordinal* associated with ${}^3\mathsf{E}$,

$$o({}^3\mathsf{E}) = \sup\left\{\operatorname{rank}(\prec) : \ \prec \text{ is a well founded relation on } \mathcal{N} \text{ in } \mathbf{sec}({}^3\mathsf{E})\right\},$$

where $\mathbf{sec}({}^3\mathsf{E})$ is the *boldface section of* ${}^3\mathsf{E}$, the class of all relations recursive in ${}^3\mathsf{E}$ and some $\alpha \in \mathcal{N}$.

THEOREM 11.2 (Jackson [1989]). *In* ZFDC+AD:

$$o({}^3\mathsf{E}) = \textit{the least weakly inaccessible cardinal.}$$

The proof of this amazing identity has not yet been published in full. It is known to be very complex and to involve a great deal of set theory—and the Coding Lemma.

§12. Realizability. In the early 1940s, Kleene initiated a program of constructing classical interpretations of intuitionistic mathematics by modelling the "constructions" in the Brouwer–Kolmogorov explication of intuitionistic truth with recursive partial functions. The initial *number realizability* had little to do with the Second Recursion Theorem, which is not mentioned at all in the first publications on the topic Kleene [1945], Nelson [1947] and Kleene [1952]; but various forms of SRT are used essentially in the later *function realizability* (Kleene and Vesley [1965]), especially to validate definition by *bar recursion*, proof by *bar induction* and various *continuity principles*. This connection between realizability theory and SRT is the main topic of Moschovakis [2010], which is a companion article to this one, and so I will not go into it here.

Appendix: preliminaries and notation.

We collect some elementary facts that we need about recursion in Baire space and the arithmetical and analytical hierarchies, partly to explain those notations that we use in the main article which are not entirely standard.

To formulate the relevant facts in easily applicable form, it is useful to consider partial functions

$$f : X_1 \times \cdots \times X_n \rightharpoonup X_{n+1}, \tag{28}$$

where each X_i is $\mathbb{N}$ or $\mathcal{N} = (\mathbb{N} \to \mathbb{N})$. As usual,

$$\overline{\alpha}(t) = \langle \alpha(0), \dots, \alpha(t-1) \rangle \quad (\alpha \in \mathcal{N}),$$

and perhaps less usual,

$$\overline{s}(t) = \langle \overbrace{s, \dots, s}^{t} \rangle \quad (s \in \mathbb{N}),$$

so that $\overline{x}(t)$ makes sense if $x \in \mathcal{N}$ or $x \in \mathbb{N}$ and it codes both t and the first t values of x—which is all of x if $x \in \mathbb{N}$. If

$$\mathbf{x} = (x_1, \dots, x_n) \in \mathcal{X} = X_1 \times \cdots \times X_n$$

as above, then for every t,

$$\overline{\mathbf{x}}(t) = (\overline{x}_1(t), \dots, \overline{x}_n(t)) \in \mathbb{N}^n$$

is a tuple which codes n (in its length) and the first t values of each x_i.

Here we will just call these products $\mathcal{X} = X_1 \times \cdots \times X_n$ *simple product spaces*—they are the Polish spaces *of type* ≤ 1 in Moschovakis [2009a].

A1. Σ^0_1. A relation $P(\mathbf{x})$ on a simple product space $\mathcal{X}$ is *semirecursive* or Σ^0_1 if

$$P(\mathbf{x}) \iff (\exists t) R(\overline{\mathbf{x}}(t)) \quad (\mathbf{x} \in \mathcal{X})$$

for some recursive $R \subseteq \mathbb{N}^n$ which is *monotone*, i.e.,

$$\big[R(\overline{\mathbf{x}}(t)) \,\&\, t < s\big] \Longrightarrow R(\overline{\mathbf{x}}(s)). \tag{29}$$

This pointclass[36] extends the collection of semirecursive relations on $\mathbb{N}$, and it is (easily) closed under $\&$, $\vee$, existential quantification $(\exists t)$ over $\mathbb{N}$ and $(\exists \alpha)$ over $\mathcal{N}$, bounded quantification of both kinds $(\exists t \leq s)$, $(\forall t \leq s)$, and recursive substitutions of number-theoretic functions.

A2. **Recursion with values in** $\mathbb{N}$. A partial function $f \colon \mathcal{X} \rightharpoonup \mathbb{N}$ is *recursive* if its *graph*

$$G_f(\mathbf{x}, w) \iff f(\mathbf{x}) = w$$

is Σ^0_1. It is not difficult to extend to these partial functions the classical *Enumeration Theorem* of Kleene:

THEOREM. *For every space $\mathcal{X}$, there is a recursive partial function*

$$\varphi^{\mathcal{X}}(e, \mathbf{x}) = \varphi^{\mathcal{X}}_e(\mathbf{x}) = \{e\}^0(\mathbf{x}), \tag{30}$$

with values in $\mathbb{N}$, such that every recursive $f \colon \mathcal{X} \rightharpoonup \mathbb{N}$ is $\varphi^{\mathcal{X}}_e$ for some e, and for all $\mathcal{X}, m$, there is a recursive function $S = S^m_{\mathcal{X}} \colon \mathbb{N}^{(1+m)} \to \mathbb{N}$ such that

$$\{S(e, \mathbf{y})\}^0(\mathbf{x}) = \{e\}^0(\mathbf{y}, \mathbf{x}) \quad (\mathbf{y} \in \mathbb{N}^m, \mathbf{x} \in \mathcal{X}).$$

[36] A *pointclass* is a collection of subsets of sets in some family which is typically closed under Cartesian products—the family of all simple product spaces in this Appendix.

A3. Universal sets and uniform closure. As with relations on $\mathbb{N}$, a relation on a simple product space $\mathcal{X}$ is (quite trivially) Σ^0_1 exactly when it is the domain of convergence of a recursive partial function $f: \mathcal{X} \rightharpoonup \mathbb{N}$. The *universal* Σ^0_1 relations for each $\mathcal{X}$ are defined by

$$\mathsf{S}^{\mathcal{X}}(e, \mathbf{x}) \iff \mathsf{S}^{\mathcal{X}}_e(\mathbf{x}) \iff \{e\}^0(\mathbf{x})\downarrow, \tag{31}$$

and the partial evaluation functions for the recursive partial functions clearly work with them too,

$$\mathsf{S}^m_{\mathcal{X}}(e, \mathbf{y}, \mathbf{x}) \iff \mathsf{S}^X(S(e, \mathbf{y}), \mathbf{x}) \quad (\mathbf{y} \in \mathbb{N}^m, \mathbf{x} \in \mathcal{X});$$

as usual, *e is a Σ^0_1-code* of $\mathsf{S}^{\mathcal{X}}_e$.

The notation is pretty awful, but we will not ever need to refer to it explicitly: it is used to prove *uniform closure* of Σ^0_1 under various operations as in this simple but typical example:

Lemma. *The pointclass of Σ^0_1 relations on simple product spaces is uniformly closed under bounded number quantification; i.e., for each $\mathcal{X}$, there is a recursive function $u(e)$, such that if $R(i, \mathbf{x})$ is Σ^0_1 with code e, then the relation*

$$P(s, \mathbf{x}) \iff (\forall i \leq s) R(i, \mathbf{x})$$

is also Σ^0_1 with code $u(e)$.

Proof. The relation[37]

$$Q(e, s, \mathbf{x}) \iff (\forall i \leq s) \mathsf{S}^{\mathbb{N} \times \mathcal{X}}(e, i, \mathbf{x})$$

is evidently Σ^0_1, and so for some fixed $\bar{z}$,

$$(\forall i \leq s) \mathsf{S}^{\mathbb{N} \times \mathcal{X}}(e, i, \mathbf{x}) \iff \mathsf{S}^{\mathbb{N}^2 \times \mathcal{X}}(\bar{z}, e, s, \mathbf{x}) \iff \mathsf{S}^{\mathbb{N} \times \mathcal{X}}(S(\bar{z}, e), s, \mathbf{x});$$

and so $u(e) = S(\bar{z}, e)$ is the required *uniformity*. ⊣

In the article we often appeal to similar uniform closure results for various pointclasses (of relations or partial functions) without much ado and sometimes silently, but it should be noted that they are often at the heart of proofs in this area: in many cases, SRT just provides the glue that puts the proofs together from such simple, uniform closure properties.

A4. Relative recursion. A function $\beta \in \mathcal{N}$ is *recursive* (Turing computable) *in $\alpha_1, \ldots, \alpha_m$ with code e* if

$$\beta(t) = \{e\}^0(t, \alpha_1, \ldots, \alpha_m) \quad (t \in \mathbb{N});$$

[37] By convention, if $\mathcal{X} = X_1 \times \cdots \times X_n$ and $\mathcal{Y} = Y_1 \times \cdots \times Y_m$, then

$$\mathcal{Y} \times \mathcal{X} = Y_1 \times \cdots \times Y_m \times X_1 \times \cdots \times X_n.$$

sometimes we say instead that *e is a code of β from $\alpha_1, \dots, \alpha_m$*. Most often we use this notion for sets via their *characteristic functions*

$$\chi_A(t) = \begin{cases} 1, & \text{if } t \in A, \\ 0, & \text{otherwise,} \end{cases}$$

and we may even identify A with χ_A and write

$$\{e\}^0(\mathbf{x}, A) := \{e\}^0(\mathbf{x}, \chi_A).$$

We set

$$\begin{aligned} A \leq_e^T B &\iff (\forall t)\big[\chi_A(t) = \{e\}^0(t, B)\big], \\ A \leq^T B &\iff (\exists e)\big[A \leq_e^T B\big] \\ A \equiv^T B &\iff A \leq^T B \,\&\, B \leq^T A. \end{aligned}$$

The equivalence classes of this last *Turing equivalence* relation are the much studied (Turing) degrees of sets of natural numbers.

The *jump* of A is the set

$$A' = \big\{t \colon \{t\}^0(t, A)\!\downarrow\big\}.$$

We use many standard, elementary properties of relative recursion and the jump operation: for example,

$$\begin{gathered} A \leq_e^T B \,\&\, B \leq_m^T C \Longrightarrow A \leq_{u_1(e,m)}^T C, \\ A \leq_{e_0}^T A', \quad A \leq_e^T B \Longrightarrow A' \leq_{u_2(e)}^T B' \end{gathered}$$

with a fixed number e_0 and fixed recursive functions $u_1(e, m), u_2(e)$; and (more importantly),

$$(\exists t)\big[\{e\}^0(\mathbf{x}, t, A) = 0\big] \iff \big\{v_1(e)\big\}^0(\mathbf{x}, A') = 0, \tag{32}$$

$$(\forall t)\big[\{e\}^0(\mathbf{x}, t, A) = 0\big] \iff \big\{v_2(e)\big\}^0(\mathbf{x}, A') = 0, \tag{33}$$

for fixed, recursive functions v_1, v_2 (which depend only on the length n of the tuple $\mathbf{x} = (x_1, \dots, x_n)$).

A5. The Kleene pointclasses. Starting with Σ_1^0, recursively: a relation $P(\mathbf{x})$ on a simple product space $\mathcal{X}$ is Π_ℓ^0 if its negation is Σ_ℓ^0, and it is $\Sigma_{\ell+1}^0$ if

$$P(\mathbf{x}) \iff (\exists s)R(\mathbf{x}, s)$$

where R is Π_ℓ^0. A relation $P(\mathbf{x})$ is Σ_1^1 if for some Π_1^0 relation $R(\mathbf{x}, \alpha)$,

$$P(\mathbf{x}) \iff (\exists \alpha)R(\mathbf{x}, \alpha); \tag{34}$$

it is Π_s^1 if its negation is Σ_s^1, and it is Σ_{s+1}^1 if it satisfies (34) with a Π_s^1 relation R. Starting with the *universal Σ_1^0 relations* in (31), we can define inductively *universal* $\Sigma_l^0, \Pi_l^0, \Sigma_s^1, \Pi_s^1$ *relations*

$$\mathsf{S}^{0,l,\mathcal{X}}(e, \mathbf{x}), \quad \mathsf{P}^{0,l,\mathcal{X}}(e, \mathbf{x}), \quad \mathsf{S}^{1,s,\mathcal{X}}(e, \mathbf{x}), \quad \mathsf{P}^{1,s,\mathcal{X}}(e, \mathbf{x})$$

so that (for example), a relation $P(\mathbf{x})$ is Π^1_3 exactly when it is $\mathsf{P}^{1,3,\mathcal{X}}_e$ for some e; and then we say that e is a Π^1_3-*code* of P. Moreover, there are partial evaluation functions associated with these universal relations (which can be easily constructed starting with those for Σ^0_1), and so when one of them is closed under some operation Φ, then it is uniformly closed under Φ, as above.

A relation $P(\mathbf{x})$ is Δ^1_s if it is both Σ^1_s and Π^1_s, and a Δ^1_s-code of P is any pair $\langle e, m\rangle$ of a Σ^1_s and a Π^1_s code of it.

The *arithmetical* and *analytical relations* which occur in these pointclasses were introduced in Kleene [1943,1955a] which established their basic properties, including the fact that they fall into two *hierarchies*, i.e., for all $\ell, s \geq 1$,

$$\Sigma^0_\ell \cup \Pi^0_\ell \subsetneq \Delta^0_{\ell+1} \subsetneq \cdots \subsetneq \Delta^1_1 \subsetneq \cdots \subsetneq \Sigma^1_s \cup \Pi^1_s \subsetneq \Delta^1_{s+1} \subsetneq \cdots \tag{35}$$

This is shown by using the universal relations $\mathsf{S}^{0,l,\mathcal{X}}, \ldots$ above—and it is, basically, the only result whose proof appeals to this rather overwhelming notation.

A6. Recursion with values in $\mathcal{N}$. A partial function $f\colon \mathcal{X} \rightharpoonup \mathcal{N}$ is *recursive* if there is a recursive $f^*\colon \mathcal{X} \times \mathbb{N} \rightharpoonup \mathbb{N}$ such that

$$f(\mathbf{x}) = \lambda t f^*(\mathbf{x}, t);$$

in particular, $f(\mathbf{x})\!\downarrow \iff (\forall t)\big[f^*(\mathbf{x}, t)\!\downarrow\big]$, and the domain of convergence of f is not (in general) Σ^0_1 (as one might expect) but Π^0_2,

$$f(\mathbf{x})\!\downarrow \iff (\forall t)(\exists w)\big[f^*(\mathbf{x}, t) = w\big].$$

The pointclass PR^1 of recursive partial functions with values in $\mathcal{N}$ is closed under composition, primitive recursion, (suitably defined) minimalization, etc., and it also "respects relative recursion", for example,

$$\text{if } f(s, \alpha, \beta) = \gamma, \text{ then } \gamma \text{ is recursive in } \alpha, \beta. \tag{36}$$

Mostly we use total recursive functions into $\mathcal{N}$ and, to set notation, it is worth listing here the *shift*

$$\alpha^* = \lambda t \alpha(t+1),$$

and the *tupling* and *projection functions*:

$$\langle \alpha_0, \ldots, \alpha_{k-1}\rangle = \lambda t \begin{cases} \alpha_i(s), & \text{if } t = \langle i, s\rangle \text{ for some } i < k \text{ and some } s,\\ 0, & \text{otherwise,}\end{cases}$$

$$(\beta)_i = \lambda t \beta(\langle i, t\rangle),$$

so that for $i < k$, $(\langle \alpha_0, \ldots, \alpha_{k-1}\rangle)_i = \alpha_i$. These are all recursive.

A7. Continuous partial functions. A partial function $f\colon \mathcal{X} \rightharpoonup \mathcal{N}$ on a simple product space is *continuous*, if

$$f(\mathbf{x}) = g(\delta_0, \mathbf{x})$$

with a recursive $g : \mathcal{N} \times \mathcal{X} \rightharpoonup \mathcal{N}$ and some $\delta_0 \in \mathcal{N}$.[38]

This pointclass of partial functions has a good uniformization in the sense of Section 10.1, and so it satisfies an appropriate version of SRT as follows:

SECOND RECURSION THEOREM FOR SIMPLE PRODUCT SPACES. *For every simple product space $\mathcal{X}$, there is a recursive partial function*

$$\psi^{\mathcal{X}} : \mathcal{N} \times \mathcal{X} \rightharpoonup \mathcal{N},$$

such that (a) *and* (b) *hold with*

$$\{\varepsilon\}^1(\mathbf{x}) = \psi^{\mathcal{X}}_{\varepsilon}(\mathbf{x}) = \psi^{\mathcal{X}}(\varepsilon, \mathbf{x}).$$

(a) *Every continuous $f : \mathcal{X} \rightharpoonup \mathcal{N}$ is $\psi^{\mathcal{X}}_{\varepsilon}$ for some ε, and every recursive $f : \mathcal{X} \rightharpoonup \mathcal{N}$ is $\psi^{\mathcal{X}}_{\varepsilon}$ for some recursive ε.*

(b) *For any two simple product spaces $\mathcal{Y}, \mathcal{X}$, there is a recursive function $S = S^{\mathcal{Y}}_{\mathcal{X}} : \mathcal{N} \times \mathcal{Y} \to \mathcal{N}$ such that*

$$\{S(\varepsilon, \mathbf{y})\}^1(\mathbf{x}) = \{\varepsilon\}^1(\mathbf{y}, \mathbf{x}) \quad (\varepsilon \in \mathcal{N}, \mathbf{y} \in \mathcal{Y}, \mathbf{x} \in \mathcal{X}).$$

(c) *For every recursive $f : \mathcal{N} \times \mathcal{X} \rightharpoonup \mathcal{N}$, there is a recursive $\varepsilon^* \in \mathcal{N}$ such that*

$$\{\varepsilon^*\}^1(\mathbf{x}) = f(\varepsilon^*, \mathbf{x}).$$

There is just computation to the proof of (a) and (b), and (c) follows as before. (And it also holds with parameters, of course, but this version is the most useful.)

REFERENCES

[1954] J. W. ADDISON, ***On some points of the theory of recursive functions***, Ph.D. thesis, University of Wisconsin, 1954.

[1959] ———, *Separation principles in the hierarchies of classical and effective descriptive set theory*, ***Fundamenta Mathematicae***, vol. 46 (1959), pp. 123–135.

[1974] GEORGE S. BOOLOS, JOHN P. BURGESS, and RICHARD C. JEFFREY, ***Computability and logic***, Cambridge University Press, 1974.

[1962] G. CEITIN, *Algorithmic operators in constructive metric spaces*, ***Trudy Matematicheskogo Instituta imeni V. A. Steklova***, vol. 67 (1962), pp. 295–361.

[1950a] MARTIN DAVIS, ***On the theory of recursive unsolvability***, Ph.D. thesis, Princeton University, 1950.

[1950b] ———, *Relatively recursive functions and the extended Kleene hierarchy*, ***Proceedings of the International Congress of Mathematicians, Cambridge, Massachusetts, 1950***, 1950, p. 723.

[1987] GABRIEL DEBS, *Effective properties in compact sets of Borel functions*, ***Mathematika***, vol. 34 (1987), pp. 64–68.

[38]It is not difficult to check that a partial $f : \mathcal{X} \rightharpoonup \mathcal{N}$ is continuous if its domain of convergence D_f is a G_δ ($\mathbf{\Pi}^0_2$) subset of $\mathcal{X}$ and f is topologically continuous on D_f.

[2009] ———, *Borel extractions of converging sequences in compact sets of Borel functions*, ***Journal of Mathematical Analysis and Applications***, vol. 350 (2009), pp. 731–744.

[1958] Richard M. Friedberg, *Un contre-example relatif aux fonctionnelles récursives*, ***Comptes Rendus de l'Académie des Sciences, Paris***, vol. 247 (1958), pp. 852–854.

[2009] Vassilis Gregoriades, ***Effective descriptive set theory and applications in analysis***, Ph.D. thesis, University of Athens, 2009.

[1989] S. Jackson, AD *and the very fine structure of* $L(\mathbb{R})$, ***Bulletin of the American Mathematical Society***, vol. 21 (1989), pp. 77–81.

[1977] A. S. Kechris and Y. N. Moschovakis, *Recursion in higher types*, ***Handbook of mathematical logic*** (J. Barwise, editor), Studies in Logic, No. 90, North Holland, Amsterdam, 1977, pp. 681–737.

[1935] Stephen C. Kleene, *General recursive functions of natural numbers*, ***Mathematische Annalen***, vol. 112 (1935), pp. 727–742.

[1938] ———, *On notation for ordinal numbers*, ***The Journal of Symbolic Logic***, vol. 3 (1938), pp. 150–155.

[1943] ———, *Recursive predicates and quantifiers*, ***Transactions of the American Mathematical Society***, vol. 53 (1943), pp. 41–73.

[1944] ———, *On the form of predicates in the theory of constructive ordinals*, ***American Journal of Mathematics***, vol. 66 (1944), pp. 41–58.

[1945] ———, *On the interpretation of intuitionistic number theory*, ***The Journal of Symbolic Logic***, vol. 10 (1945), pp. 109–124.

[1952] ———, ***Introduction to metamathematics***, D. Van Nostrand Company, North Holland Company, 1952.

[1955a] ———, *Arithmetical predicates and function quantifiers*, ***Transactions of the American Mathematical Society***, vol. 79 (1955), pp. 312–340.

[1955b] ———, *On the form of predicates in the theory of constructive ordinals* (second paper), ***American Journal of Mathematics***, vol. 77 (1955), pp. 405–428.

[1955c] ———, *Hierarchies of number theoretic predicates*, ***Bulletin of the American Mathematical Society***, vol. 61 (1955), pp. 193–213.

[1959a] ———, *Quantification of number-theoretic functions*, ***Compositio Mathematica***, vol. 14 (1959), pp. 23–40.

[1959b] ———, *Recursive functionals and quantifiers of finite types I*, ***Transactions of the American Mathematical Society***, vol. 91 (1959), pp. 1–52.

[1965] Stephen C. Kleene and Richard Eugene Vesley, ***The foundations of intuitionistic mathematics***, North Holland, Amsterdam, 1965.

[1959] Georg Kreisel, *Analysis of Cantor-Bendixson Theorem by means of the analytic hierarchy*, ***Bulletin de l'Academie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques***, vol. 7 (1959), pp. 621–626.

[1957] Georg Kreisel, Daniel Lacombe, and Joseph R. Shoenfield, *Partial recursive functionals and effective operations*, ***Constructivity in mathematics: proceedings of the colloquium held in Amsterdam, 1957*** (A. Heyting, editor), North Holland Publishing Company, 1957, pp. 195–207.

[1966] K. Kuratowski, ***Topology***, vol. 1, Academic Press, New York and London, 1966, translated from the French *Topologie*, vol. 1, PWN, Warsaw, 1958.

[1905] Henri Lebesgue, *Sur les fonctions represéntables analytiquement*, ***Journal de Mathématiques 6^e serie***, vol. 1 (1905), pp. 139–216.

[2002] Maurice Margenstern and Yurii Rogozhin, *Self-describing Turing machines*, ***Fundamenta Informatica***, vol. 50 (2002), no. 3, pp. 285–303.

[1954] Werner Markwald, *Zur Theorie der konstruktiven Wohlordnungen*, ***Mathematische Annalen***, vol. 127 (1954), pp. 135–149.

[1988] D. A. Martin and J. R. Steel, *Projective determinacy*, ***Proceedings of the National Academy of Sciences of the United States of America***, vol. 85 (1988), no. 18, pp. 6582–6586.

[1966] Yiannis N. Moschovakis, *Many-one degrees of the predicates $H_a(x)$*, ***Pacific Journal of Mathematics***, vol. 18 (1966), pp. 329–342.

[1970] ———, *Determinacy and prewellorderings of the continuum*, ***Mathematical logic and foundations of set theory*** (Y. Bar-Hillel, editor), North-Holland, Amsterdam, 1970, pp. 24–62.

[1974] ———, *Structural characterizations of classes of relations*, ***Generalized recursion theory*** (J. E. Fenstad and P. G. Hinman, editors), North-Holland, Amsterdam, 1974, pp. 53–81.

[2009a] ———, ***Descriptive set theory, second edition***, Mathematical Surveys and Monographs., vol. 155, American Mathematical Society, 2009.

[2009b] ———, *Kleene's amazing second recursion theorem (extended abstract)*, ***CSL 2009*** (E. Grädel and R. Kahle, editors), Lecture Notes in Computer Science, no. 5771, Springer-Verlag, Berlin - Heidelberg, 2009, pp. 24–39.

[2010] ———, *Classical descriptive set theory as a refinement of effective descriptive set theory*, (2010), submitted.

[1946] A. Mostowski, *On definable sets of positive integers*, ***Fundamenta Mathematicae***, vol. 34 (1946), pp. 81–112.

[1951] ———, *A classification of logical systems*, ***Studia Philosophica***, vol. 4 (1951), pp. 237–274.

[1955] John Myhill, *Creative sets*, ***Zeitschrift für Mathematische Logik und Grundlagen der Mathematik***, vol. 1 (1955), pp. 97–108.

[1955] John Myhill and John C. Shepherdson, *Effective operations on partial recursive functions*, ***Zeitschrift für Mathematische Logik und Grundlagen der Mathematik***, vol. 1 (1955), pp. 310–317.

[1947] David Nelson, *Recursive functions and intuitionistic number theory*, ***Transactions of the American Mathematical Society***, vol. 61 (1947), pp. 308–368.

[1974] George C. Nelson, *Many-one reducibility within the Turing degrees of the hyperarithmetic sets $Ha(x)$*, ***Transactions of the American Mathematical Society***, vol. 191 (1974), pp. 1–44.

[1944] Emil L. Post, *Recursively enumerable sets of positive integers and their decision problems*, ***Bulletin of the American Mathematical Society***, vol. 50 (1944), pp. 284–316.

[1967] Marian Boykan Pour-El and Saul Kripke, *Deduction-preserving "recursive isomorphisms" between theories*, ***Fundamenta Mathematicae***, vol. 61 (1967), pp. 141–163.

[1950] Raphael M. Robinson, *An essentially undecidable axiom system*, ***Proceedings of the International Congress of Mathematics***, 1950, pp. 729–730.

[1967] Hartley Rogers, Jr., ***Theory of recursive functions and effective computability***, McGraw-Hill, 1967.

[1990] Gerald E. Sacks, ***Higher recursion theory***, Perspectives in Mathematical Logic, Springer, 1990.

[1976] Robert M. Solovay, *Provability interpretations of modal logic*, ***Israel Journal of Mathematics***, vol. 25 (1976), pp. 287–304.

[1955] Clifford Spector, *Recursive wellorderings*, ***The Journal of Symbolic Logic***, vol. 20 (1955), pp. 151–163.

[1917] M. Suslin, *Sur une definition des ensembles measurables B sans nombres transfinis*, ***Comptes Rendus de l'Académie des Sciences, Paris***, vol. 164 (1917), pp. 88–91.

[1958] Hao Wang, *Alternative proof of a theorem of Kleene*, ***The Journal of Symbolic Logic***, vol. 23 (1958), p. 250.

[1988] W. H. Woodin, *Supercompact cardinals, sets of reals, and weakly homogeneous trees*, ***Proceedings of the National Academy of Sciences of the United States of America***, vol. 85 (1988), pp. 6587–6591.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
LOS ANGELES, CA 90095-1555, USA
and
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ATHENS, GREECE
E-mail: ynm@math.ucla.edu