On Functionality of Visibly Pushdown Transducers

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Abstract. Visibly pushdown transducers form a subclass of pushdown transducers that (strictly) extends finite state transducers with a stack. Like visibly pushdown automata, the input symbols determine the stack operations. In this paper, we prove that functionality is decidable in PSPACE for visibly pushdown transducers. The proof is done via a pumping argument: if a word with two outputs has a sufficiently large nesting depth, there exists a nested word with two outputs whose nesting depth is strictly smaller. The proof uses technics of word combinatorics. As a consequence of decidability of functionality, we also show that equivalence of functional visibly pushdown transducers is EXPTIME-C.

1 Introduction

In [1], it has been shown that visibly pushdown languages (VPL) form a robust subclass of context-free languages. This class strictly extends the class of regular languages and still enjoys strong properties: closure under all Boolean operators and decidability of emptiness, universality, inclusion and equivalence. On the contrary, context-free languages are not closed under complement nor under intersection, moreover universality, inclusion and equivalence are all undecidable.

Visibly pushdown automata (VPA), that characterize VPL, are obtained as a restriction of pushdown automata. In these automata the input symbol determines the stack operation. The input alphabet is partitioned into call, return and internal symbols: if a call is read, the automaton must push a symbol on the stack; if it reads a return, it must pop a symbol; and while reading an internal symbol, it can not touch, not even read, the stack. *Visibly pushdown transducers* have been introduced in [11]. They form a subclass of pushdown transducers, and are obtained by adding output to VPA: each time the VPA reads an input symbol it also outputs a letter. They allow for ϵ-transitions that can produce outputs. In this paper, we consider visibly pushdown transducers where this operation is not allowed. Moreover, each transition can output not only a single letter but a word, and no visibly restriction is imposed on this output word. Therefore in the sequel we call the transducers of [11] ϵ-VPTs, and VPTs will denote the visibly pushdown transducers considered here.

Consider the VPT T of Figure 1. Call (resp. return) symbols are denoted by c (resp. r). The domain of T is $Dom(T) = \{c_1(c_2)^n c_3 r_3 (r_2)^n r_1 \mid n \in \mathbb{N}\}$. For each word of $Dom(T)$, there are two accepting runs, corresponding respectively to the upper and lower part of T. For instance, when reading c_1, it pushes γ_1 and produces either d

(upper part) or dfc (lower part). By following the upper part (resp. lower part), it produces words of the form $dfcab(cabcab)^n gh$ (resp. $dfc(abc)^n ab(cab)^n gh$). Therefore T is functional.

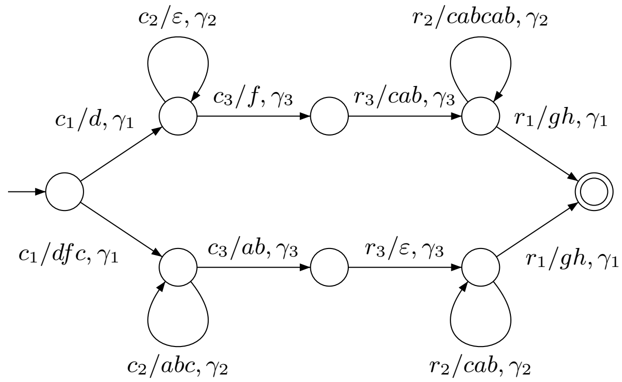

Fig. 1. A functional VPT on $\Sigma_c = \{c_1, c_2, c_3\}$ and $\Sigma_r = \{r_1, r_2, r_3\}$.

In this paper, we prove that the problem of determining if a VPT transduction is functional is decidable. In particular, our algorithm is in PSPACE. Deciding functionality is one of the main problem in transduction theory as it makes deciding equivalence of functional transducers possible. Both problems are undecidable for pushdown transductions. Our proof relies on a pumping argument: if a word is long enough and has two outputs, we show that there is a strictly shorter word with two outputs. We use technics of word combinatorics and in particular, a strong result proved in [8]. As a consequence, we show that the equivalence problem for VPTs is EXPTIME-C.

Related Work ϵ-VPTs have been introduced in [11]. In contrast to VPTs, they allow for ϵ-transitions that produce outputs, so that an arbitrary number of symbols can be inserted. Moreover, each transition of a VPT can output a word while each transition of an ϵ-VPT can output a single letter only. The VPTs we consider here are strictly less expressive than ϵ-VPTs, but functionality and equivalence of functional transducers are decidable, which is not the case for ϵ-VPTs.

The functionality problem for finite state transducers has been extensively studied. The first proof of decidability was given by Schützenberger in [12], and later in [3]. As the proof we give here, the proof of Schützenberger relies on a pumping lemma for functionality. The first PTIME upper bound has been proved in [7], and an efficient procedure has been given in [2].

Deciding equivalence of deterministic (and therefore functional) VPTs is in PTIME [15]. However, functional VPTs are strictly more expressive than deterministic VPTs. In particular, non-determinism is often needed to model functional transformations whose current production depends on some input which may be arbitrary far away from the current input. For instance, the transformation that swaps the first and the last input symbols is functional but non-determinism is needed to guess the last input.

Ordered trees over an arbitrary finite alphabet Σ can be naturally represented by well nested words over the structured alphabet $\Sigma \times \{c\} \cup \Sigma \times \{r\}$. As VPTs can express transductions from well words to well nested words, they are therefore well-suited to model tree tranformations. We distinguish *ranked trees* from *unranked trees*, whose nodes may have an arbitrary number of ordered children. Ranked tree transducers have received a lot of attention. Most notably, *tree transducers* [4] and *macro tree transducers* [6] have been proposed and studied. They are incomparable to VPTs however, as they allow for copy, which is not the case of VPTs, but cannot define any context-free language as codomain, what VPTs can do. Functionality is known to be decidable in PTIME for tree transducers [13]. More generally, finite-valuedness (and equivalence) of tree transducers is decidable [14]. There have been several attempts to generalize ranked tree transducers to unranked tree transducers [9,10]. As mentioned in [5], it is an important problem to decide equivalence for unranked tree transformation formalisms. However, there is no obvious generalization of known results for ranked trees to unranked trees, as unranked tree transformations have to support concatenation of tree sequences, making usual binary encodings of unranked trees badly suited. Considering classical ranked tree transducers, their ability to copy subtrees is the main concern when dealing with functionality. However for VPTs, it is more their ability to concatenate sequences of trees which makes this problem difficult, and which in a way led us to word combinatorics. To the best of our knowledge, VPTs consist in the first (non-deterministic) model of unranked tree transformations for which functionality and equivalence of functional transformations is decidable.

Organization of the paper In Section 2, we define visibly pushdown transducers as a extension of visibly pushdown automata. In Section 3, we recall some notion of word combinatorics. In Section 4, we give a reduction of functionality to a system of word equations. In Section 5, we prove a pumping lemma that preserves non-functionality. Finally, we give a PSPACE algorithm for functionality is Section 6 and prove the EXPTIME completeness of equivalence.

2 Visibly Pushdown Transducers

Let Σ be a finite alphabet partitioned into two disjoint sets Σ_c and Σ_r denoting respectively the *call* and *return* alphabets[1]. We denote by Σ^* the set of words over Σ and by ϵ the empty word. The length of a word u is denoted by $|u|$. The set of *well nested* words Σ^*_{wn} is the smallest subset of Σ^* such that $\epsilon \in \Sigma^*_{\mathsf{wn}}$ and for all $c \in \Sigma^c$, all $r \in \Sigma^r$, all $u, v \in \Sigma^*_{\mathsf{wn}}$, $cur \in \Sigma^*_{\mathsf{wn}}$ and $uv \in \Sigma^*_{\mathsf{wn}}$. The *height* of a well nested word is inductively defined by $h(\epsilon) = 0$, $h(cur) = 1 + h(u)$, and $h(uv) = \max(h(u), h(v))$.

Visibly Pushdown Languages A *visibly pushdown automaton* (VPA) [1] on finite words over Σ is a tuple $A = (Q, I, F, \Gamma, \delta)$ where Q is a finite set of states, $I \subseteq Q$, respectively $F \subseteq Q$, the set of initial states, respectively final states, Γ the (finite) stack

[1] In contrast to [1], we do not consider *internal* symbols i, as they can be simulated by a (unique) call c_i followed by a (unique) return r_i

alphabet, and $\delta = \delta_c \uplus \delta_r$ where $\delta_c \subseteq Q \times \Sigma_c \times \Gamma \times Q$ are the *call transitions*, $\delta_r \subseteq Q \times \Sigma_r \times \Gamma \times Q$ are the *return transitions*. On a call transition $(q, a, q', \gamma) \in \delta_c$, γ is pushed onto the stack and the control goes from q to q'. On a return transition $(q, \gamma, a, q') \in \delta_r$, γ is popped from the stack. Stacks are elements of Γ^*, and we denote by $\perp$ the empty stack. A *run* of a VPA A on a word $w = a_1 \dots a_l$ is a sequence $\{(q_k, \sigma_k)\}_{0 \leq k \leq l}$, where q_k is the state and $\sigma_k \in \Gamma^*$ is the stack at step k, such that $q_0 \in I$, $\sigma_0 = \perp$, and for each $k < l$, we have either: (i) $(q_k, a_{k+1}, \gamma, q_{k+1}) \in \delta_c$ and $\sigma_{k+1} = \sigma_k \gamma$; (ii) $(q_k, a_{k+1}, \gamma, q_{k+1}) \in \delta_r$, and $\sigma_k = \sigma_{k+1}\gamma$. A run is *accepting* if $q_l \in F$ and $\sigma_l = \perp$. A word w is *accepted* by A if there exists an accepting run of A over w. Note that it is necessarily well nested. $L(A)$, the *language* of A, is the set of words accepted by A. A language L over Σ is a *visibly pushdown language* if there is a VPA A over Σ such that $L(A) = L$.

In contrast to [1] and to ease the notations, we do not allow transitions on the empty stack. Therefore the words accepted by a VPA are well-nested (every call symbol has a matching return symbol and conversely).

Visibly Pushdown Transducers As finite-state transducers extend finite-state automata with outputs, visibly pushdown transducers extend VPA with outputs. To simplify notations, we suppose that the output alphabet is Σ, but our results still hold for an arbitrary output alphabet.

Definition 1 (Visibly pushdown transducers). A *visibly pushdown transducer*[2] (VPT) on finite words over Σ is a tuple $T = (Q, I, F, \Gamma, \delta)$ where Q is a finite set of states, $I \subseteq Q$ is the set of initial states, $F \subseteq Q$ the set of final states, Γ is the stack alphabet, $\delta = \delta_c \uplus \delta_r$ the transition relation, with $\delta_c \subseteq Q \times \Sigma_c \times \Sigma^* \times \Gamma \times Q$, $\delta_r \subseteq Q \times \Sigma_r \times \Sigma^* \times \Gamma \times Q$.

A *configuration* of a VPT is a pair $(q, \sigma) \in Q \times \Gamma^*$. A *run* of T on a word $u = a_1 \dots a_l \in \Sigma^*$ from a configuration (q, σ) to a configuration (q', σ') is a finite sequence $\rho = \{(q_i, \sigma_i)\}_{0 \leq k \leq l}$ such that $q_0 = q$, $\sigma = \sigma_0$, $q' = q_n$, $\sigma' = \sigma_n$ and for all $i \in \{1, \dots, l\}$, there exist $v_i \in \Sigma^*$ and $\gamma_i \in \Gamma$ such that $(q_i - 1, a_i, v_i, \gamma_i, q_i) \in \delta_c$ and either $a_i \in \Sigma_c$ and $\sigma_i = \sigma_{i-1}\gamma_i$, or $a_i \in \Sigma_r$ and $\sigma_{i-1} = \sigma_i \gamma_i$. The word $v = v_1 \dots v_l$ is called an *output* of ρ. We write $(q, \sigma) \xrightarrow{u/v} (q', \sigma')$ when there exists a run on u from (q, σ) to (q', σ') producing v as output. The transducer T defines a word binary relation $[\![T]\!] = \{(u, v) \mid \exists q \in I, p \in F,\ (q, \perp) \xrightarrow{u/v} (p, \perp)\}$.

The *domain* of T, resp. the *codomain* of T, denoted resp. by *Dom*(T) and *CoDom*(T), is the domain of $[\![T]\!]$, resp. the codomain of $[\![T]\!]$. Note that the domain of T contains only well nested words, which is not the case of the codomain in general.

In this paper, we prove the following theorem:

Theorem 1. *Functionality of* VPTs *is decidable in* PSPACE.

The rest of the paper is devoted to the proof of this theorem.

[2] In contrast to [11], there is no producing ϵ-transitions (inserting transitions) but a transition may produce a word and not a single symbol

3 Preliminaries on Word Combinatorics

The size of a word x is denoted by $|x|$. Given two words $x, y \in \Sigma^*$, we write $x \preceq y$ if x is a prefix of y. If we have $x \preceq y$, then we note $x^{-1}y$ the unique word z such that $y = xz$. A word $x \in \Sigma^*$ is *primitive* if there is no word y such that $|y| < |x|$ and $x \in y^*$. The *primitive root* of a word $x \in \Sigma^*$ is the (unique) primitive word y such that $x \in y^*$. In particular, if x is primitive, then its primitive root is x. Two words x and y are *conjugate* if there exists $z \in \Sigma^*$ such that $xz = zy$. It is well-known that two words are conjugate iff there exist $t_1, t_2 \in \Sigma^*$ such that $x = t_1t_2$ and $y = t_2t_1$. Two words $x, y \in \Sigma^*$ *commute* iff $xy = yx$.

Lemma 1 (folklore). *Let $x, y \in \Sigma^*$ and $n, m \in \mathbb{N}$.*

1. *if x and y commute, then $x, y \in z^*$ for some $z \in \Sigma^*$. Moreover, if xy is primitive, then $x = \epsilon$ or $y = \epsilon$;*
2. *if x^n and y^m have a common subword of length at least $|x| + |y| - d$ (d being the greatest common divisor of $|x|$ and $|y|$), then their primitive roots are conjugate.*

Proof. The first assertion is folklore. For the second, there exists $z \in \Sigma^*$ and $\alpha, \beta \geq 0$ such that $x = z^\alpha$ and $y = z^\beta$. If x and y are non-empty, then $\alpha, \beta > 0$ and $z \neq \epsilon$. Thus $xy = z^{\alpha+\beta}$, which contradicts the primitivity of xy.

Lemma 2 (Hakala, Kortelainen, Theorem 7 of [8]). *Let $v_0, v_1, v_m, v_{\overline{1}}, v_{\overline{0}}, w_0, w_1, w_m, w_{\overline{1}}, w_{\overline{m}} \in \Sigma^*$ and $i \in \mathbb{N}$. If $v_0(v_1)^i v_m (v_{\overline{1}})^i v_{\overline{0}} = w_0(w_1)^i w_m (w_{\overline{1}})^i w_{\overline{0}}$ holds for all $i \in \{0, 1, 2, 3\}$, then it holds for all $i \in \mathbb{N}$.*

Let $x \in \Sigma^*$, we denote by $x^\omega \in \Sigma^\omega$ the infinite (countable) concatenation of x.

Lemma 3. *Let $x, x_1, x_2, y, z, t_1, t_2, p, q \in \Sigma^*$ with t_1t_2, p, q primitive, then:*

1. *if $t_1 \prec p$ and $xpt_1 = ypp$ then $xp^\omega = yp^\omega$*
2. *if $xp^\omega = yp^\omega$ then $\exists \alpha, \beta \geq 0 : xp^\alpha = yp^\beta$*
3. *if $x(t_1t_2)^\omega = y(t_2t_1)^\omega$ and $t_1 \neq \epsilon$, then $\exists \alpha, \beta \geq 0 : x(t_1t_2)^\alpha = y(t_2t_1)^\beta t_2$*
4. *if $x(t_1t_2)^\omega = (t_2t_1)^\omega$ and $t_1 \neq \epsilon$, then $\exists \alpha \geq 0 : x = (t_2t_1)^\alpha t_2$.*
5. *if $\forall i \in \{1, 2\}$, $x_i y(t_1t_2)^\omega = y(t_1t_2)^\omega$ then $\exists \alpha_1, \alpha_2 \geq 0, \exists t_3, t_4 \in \Sigma^* : t_3t_4 = t_1t_2, x_i = (t_4t_3)^{\alpha_i}$*
6. *if $xp^\omega = p^\omega$ then $\exists \alpha \geq 0 : x = p^\alpha$*
7. *if $\exists \alpha > 0$ such that $p^\alpha xp^\omega = xp^\omega$, then $x \in p^*$.*
8. *if $\exists \alpha > 0,\ q^\alpha yp^\omega = yp^\omega$ then $qy = yp$*
9. *if $\exists \alpha, \beta, \gamma \geq 1$ such that $x(t_1t_2)^\alpha y(t_1t_2)^\beta z = (t_2t_1)^\gamma$, then $y \in (t_1t_2)^*$.*

Proof. 1. Let t_2 such that $p = t_1t_2$, then $xt_1t_2t_1 = yt_1t_2t_1t_2$, by Lemma 1 $t_1 = \epsilon$ or $t_2 = \epsilon$ i.e. either $t_1 = \epsilon$ or $t_1 = p$.
2. Direct consequence of the previous property since we have $xp^\alpha t_1 = yp^\beta$ for some $\alpha, \beta > 1$ and $t_1 \prec p$.
3. By applying the previous property to $x(t_1t_2)^\omega = yt_2(t_1t_2)^\omega$.
4. The second assertion is a direct consequence of the first when taking $y = \epsilon$.

5. It is clear if $x_1 = x_2 = \epsilon$. Suppose that $x_1 \neq \epsilon$. Since $x_1 y(t_1t_2)^\omega = y(t_1t_2)^\omega$, we also have $x_1x_1y(t_1t_2)^\omega = y(t_1t_2)^\omega$, and more generally, for all $\beta \geq 1$, $(x_1)^\beta y(t_1t_2)^\omega = y(t_1t_2)^\omega$. By taking β large enough, there exists $\gamma \geq 0$ such that $(x_1)^\beta$ and $(t_1t_2)^\gamma$ have a common factor of length at most $|x_1|+|t_1t_2|-\gcd(|x_1|,|t_1t_2|)$. By the fundamental lemma, there exists $t_3, t_4 \in \Sigma^*$ such that t_3t_4 is primitive, $x_1 \in (t_4t_3)^*$ and $t_1t_2 \in (t_3t_4)^*$. Since t_1t_2 is primitive, we have $t_1t_2 = t_3t_4$. Suppose that $x_2 \neq \epsilon$. Similarly, we can prove that $x_2 = (t_4't_3')^\gamma$ for some $\gamma > 0$ and t_3', t_4' such that $t_1t_2 = t_3't_4'$. We have $x_1y(t_1t_2)^\omega = x_2y(t_1t_2)^\omega$, therefore $t_4t_3 = t_4't_3'$, and $x_2 \in (t_4t_3)^*$.
6. We have $xp^\omega = p^\omega$ so we also have $pxp^\omega = p^\omega$, therefore $xp^\omega = pxp^\omega$ i.e. $xp = px$, and by Lemma 1, $x \in p^*$.
7. We clearly have $xp^\alpha = p^\alpha x$ therefore, by Lemma 1, $x \in p^*$.
8. We have $q^\alpha yp^\omega = yp^\omega$, this implies that for any $x \geq 0$ $q^{x\alpha}yp^\omega = yp^\omega$. Therefore, there exist $\beta \geq 0$ and $t_1 \prec q$ with $y = q^\beta t_1$. Let $t_2 \in \Sigma^*$ such that $q = t_1t_2$, we have $(t_1t_2)^{\alpha+\beta}t_1 = (t_1t_2)^\beta t_1 p^\alpha$. Therefore because $|p| = |q| = |t_1t_2|$ we have $p = t_2t_1$. This concludes the proof.
9. We assume $t_1, t_2 \neq \epsilon$ (otherwise it is obvious). By 1 and 4 we have that $x = (t_2t_1)^a t_2$. By the same argument we have $z = t_1(t_2t_1)^b$ So we have: $t_2(t_1t_2)^{\alpha+a}y(t_1t_2)^{\beta+b}t_1 = (t_2t_1)^\gamma$. Therefore $y \in (t_1t_2)^*$.

$\square$

4 From Functionality to Word Equations

Given some words $u_0, \ldots u_n, u_m, u_{\overline{n}}, \ldots, u_{\overline{0}} \in \Sigma^*$, $k \in \mathbb{N}$, and a function $\pi : \{1, \ldots, k\} \to \{1, \ldots, n\}$, we denote by u_π the word $u_0 u_{\pi(1)} \ldots u_{\pi(j)} u_m u_{\overline{\pi(j)}} \ldots u_{\overline{\pi(1)}} u_{\overline{0}}$. We denote by id_n the identity function on domain $\{1, \ldots, n\}$. The following lemma states that if a word u translated into two words v, w is high enough, u, v and w can be decomposed into subwords that can be removed, repeated, or permutted in parallel in u, v and w, while preserving the transduction relation.

Lemma 4. *Let T be a* VPT *with N states, and $n \geq 1$. Let $u, v, w \in \Sigma^*$ such that $v, w \in T(u)$ (u is thus well nested) and $h(u) > nN^4$. Then there exist $u_m, v_m, w_m \in \Sigma^*$ and $u_i, u_{\overline{i}}, v_i, v_{\overline{i}}, w_i, w_{\overline{i}} \in \Sigma^*$ for all $i \in \{0, \ldots, n\}$ such that $u_{id_n} = u$, $v_{id_n} = v$, $w_{id_n} = w$ and for all $k \in \mathbb{N}$ and all $\pi : \{1, \ldots, k\} \to \{1, \ldots, n\}$: $v_\pi, w_\pi \in T(u_\pi)$ and $u_i, u_{\overline{i}} \neq \epsilon$ for all $i = 1, \ldots, n$.*

Proof. Let T be a VPT, with set of states Q. Let $N = |Q|$, $n \geq 1$, and $u, v, w \in \Sigma^*$ such that $v, w \in T(u)$ and $h(u) > nN^4$. In particular, u is well nested. We denote by ℓ the length of the word u and write $u = (a_j)_{1 \leq j \leq \ell}$, with $a_j \in \Sigma$ for all j. There exists a position $1 \leq j \leq \ell$ in u whose height is equal to $h(u)$. We fix such a position j. Then, for any height $0 \leq k \leq h(u)$, we define two positions, denoted $\alpha(k)$ and $\beta(k)$. $\alpha(k)$ (resp. $\beta(k)$) is the largest (resp. the smallest) index d, such that $d \leq j$ (resp. $d \geq j$) and the height of u in position d is equal to k. The part of the word concerned by mapping α (resp. β) is represented in blue (resp. in red) on Figure 2.

As $v, w \in T(u)$, there exists two runs ϱ_v, ϱ_w on u in T which produce respectively the outputs v and w. We denote by $(p_i)_{0 \leq i \leq \ell}$ (resp. $(q_i)_{0 \leq i \leq \ell}$) the states we encounter

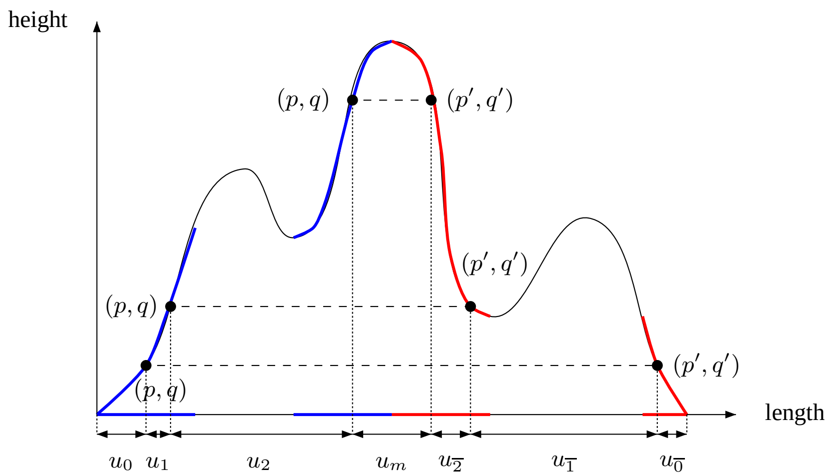

Fig. 2. Form of pumping

along ϱ_v (resp. ϱ_w). As $h(u) > nN^4$, there exists two pairs of states $(p, p'), (q, q') \in Q^2$ such that

$$|\{0 \le k \le h(u) \mid p_{\alpha(k)} = p \text{ and } p_{\beta(k)} = p' \text{ and } q_{\alpha(k)} = q \text{ and } q_{\beta(k)} = q'\}| > n$$

We denote by $0 \le k_1 < \ldots < k_{n+1} \le h(u)$ the $n+1$ different heights associated with the pairs (p, p') and (q, q'). For each $i = 0, \ldots, n-1$, this means that the two runs pass simultaneously in states p and q before a call transition with a height equal to k_i, and that the height of the stack will never be smaller than k_i, until reaching again states p and q with a stack of height k_{i+1}. A symmetric property can be stated for states p' and q'. As a consequence, we obtain n fragments which behave as synchronized "call loops" around p and q with corresponding "return loops" around p' and q'. This situation is described on Figure 2.

Then, we can define the different fragments of u as follows: (see Figure 2)

- $u_0 = a_1 \ldots a_{\alpha(k_1)-1}$,
- $\forall 1 \le i \le n, u_i = a_{\alpha(k_i)} \ldots a_{\alpha(k_{i+1})-1}$,
- $u_m = a_{\alpha(k_{n+1})} \ldots a_{\beta(k_{n+1})-1}$,
- $\forall 1 \le i \le n, u_{\overline{i}} = a_{\beta(k_{i+1})} \ldots a_{\beta(k_i)-1}$,
- $u_{\overline{0}} = a_{\beta(k_1)} \ldots a_\ell$.

We immediately obtain $u = u_{id_n}$ and $u_i, u_{\overline{i}} \neq \epsilon$ for all $i = 1, \ldots, n$. The decompositions of v and w are obtained by considering the outputs produced by the corresponding fragments of u on the two runs ϱ_v and ϱ_w.

Finally, the property of commutativity ($v_\pi, w_\pi \in T(u_\pi)$ for all $\pi : \{1, \ldots, k\} \to \{1, \ldots, n\}$) easily follows from the fact that for each $i \in \{1, \ldots, n\}$, the fragments of the runs associated with u_i and $u_{\overline{i}}$ do not depend on the content of the stack as T is a *visibly* pushdown transducer. $\square$

The following lemma states that if a word u with at least two outputs is high enough, there is a word u' strictly less higher with at least two outputs.

Lemma 5. *Let T be a* VPT *with N states and $u \in Dom(T)$ such that $|T(u)| > 1$ and $h(u) > 8N^4$. There exists $u' \in Dom(T)$ such that $|T(u')| \geq 2$ and $|u'| < |u|$.*

Proof. Let $v, w \in T(u)$ such that $v \neq w$. Thanks to Lemma 4, there exist $u_m, v_m, w_m \in \Sigma^*$, and for all $i \in \{0, \ldots, 8\}$, there exist $u_i, u_{\bar{i}}, v_i, v_{\bar{i}}, w_i, w_{\bar{i}} \in \Sigma^*$, such that $u_{id_8} = u$, $v_{id_8} = v$, $w_{id_8} = w$ and for all $k \in \mathbb{N}$ and all $\pi : \{1, \ldots, k\} \to \{1, \ldots, n\}$: $v_\pi, w_\pi \in T(u_\pi)$ and $u_i, u_{\bar{i}} \neq \epsilon$ for all $i = 1, \ldots, n$. We prove that there exist $k \in \{0, \ldots, 7\}$ and $\pi : \{1, \ldots, j\} \to \{1, \ldots, 8\}$ such that $v_\pi \neq w_\pi$ and $|u_\pi| < |u|$. We proceed by contradiction. Suppose that for all $k \in \{0, \ldots, 7\}$ and for all $\pi : \{1, \ldots, k\} \to \{1, \ldots, 8\}$ such that $|u_\pi| < |u|$ we have $v_\pi = w_\pi$. This defines a system of equations $\mathcal{S} = \{v_\pi = w_\pi \mid \pi : \{1, \ldots, k\} \to \{1, \ldots, 8\},\ |u_\pi| < |u|\}$. We show in the next section that it implies $v = w$ (Theorem 2).

5 Word Equations

In this section, we fix some $n \geq 8$, some words $u_m, v_m, w_m \in \Sigma^*$ and for all $i \in \{0, \ldots, n\}$, we fix $u_i, v_i, w_i, u_{\bar{i}}, v_{\bar{i}}, w_{\bar{i}} \in \Sigma^*$ such that $u_i, u_{\bar{i}} \neq \epsilon$. We consider the system $\mathcal{S} = \{v_\pi = w_\pi \mid \pi : \{1, \ldots, k\} \to \{1, \ldots, n\},\ |u_\pi| < |u_{id_n}|\}$. The main result we prove is the following:

Theorem 2. *If $\mathcal{S}$ holds, then $v_{id_n} = w_{id_n}$.*

We let $\ell \in \{1, \ldots, n\}$ such that $|u_\ell u_{\bar{\ell}}| \leq |u_i u_{\bar{i}}|$ for all $i \in \{1, \ldots, n\}$. We consider several cases to prove Theorem 2:

$$(1)\ |v_\ell| = |w_\ell| \qquad (2)\ |v_\ell| > |w_\ell| \qquad (3)\ |w_\ell| > |v_\ell|$$

Cases 2 and 3 being symmetric, we consider cases 1 and 2 only in the two following subsections.

5.1 Proof of Theorem 2: case $|v_\ell| > |w_\ell|$

We denote by $\mathcal{S}[|v_\ell| > |w_\ell|]$ the system $\mathcal{S}$ with the assumption $|v_\ell| > |w_\ell|$ and from now one we assume that this system holds. We consider the following set of equations, defined for all $a, b \geq 0$ and all $i \in \{1, \ldots, n\}$:

$$\begin{cases} v_0 v_m v_{\bar{0}} = w_0 w_m w_{\bar{0}} & (1) \\ v_0 (v_\ell)^a v_m (v_{\bar{\ell}})^a v_{\bar{0}} = w_0 (w_\ell)^a w_m (w_{\bar{\ell}})^a w_{\bar{0}} & (2) \\ v_0 v_i (v_\ell)^a v_m (v_{\bar{\ell}})^a v_{\bar{i}} v_{\bar{0}} = w_0 w_i (w_\ell)^a w_m (w_{\bar{\ell}})^a w_{\bar{i}} w_{\bar{0}} & (3) \\ v_0 (v_\ell)^a v_i (v_\ell)^b v_m (v_{\bar{\ell}})^b v_{\bar{i}} (v_{\bar{\ell}})^a v_{\bar{0}} = w_0 (w_\ell)^a w_i (w_\ell)^b w_m (w_{\bar{\ell}})^b w_{\bar{i}} (w_{\bar{\ell}})^a w_{\bar{0}} & (4) \end{cases}$$

For $k \in \{1, 2, 3, 4\}$, we denote by $\mathcal{S}_k$ the subsystem that of equations of type k. For instance, $\mathcal{S}_2$ is the system of equations $\{v_0 (v_\ell)^a v_m (v_{\bar{\ell}})^a v_{\bar{0}} = w_0 (w_\ell)^a w_m (w_{\bar{\ell}})^a w_{\bar{0}} \mid a \in \mathbb{N}\}$.

Lemma 6. *For all $k \in \{1, \ldots, 4\}$, $\mathcal{S}_k$ holds.*

Proof. First, $|u_0 u_m u_{\overline{0}}| < |u_{id_n}|$ and $u_0 u_m u_{\overline{0}} = u_\pi$ where π is the function with empty domain. Since $\mathcal{S}$ holds by hypothesis, this equation holds.

We prove that $\mathcal{S}_4$ holds, as $\mathcal{S}_3$ is a particular case of $\mathcal{S}_4$ and $\mathcal{S}_2$ is a similar but easier case. First, $\mathcal{S}_4$ holds for all $a, b \in \{0, 1, 2, 3\}$. Indeed, since $n \geq 8$, there are six pairwise different integers $i_1, \ldots, i_6 \in \{1, \ldots, n\}$ such that $i_k \neq i$ for all $k \in \{1, \ldots, 6\}$ and $6|u_\ell u_{\overline{\ell}}| + |u_i u_{\overline{i}}| \leq |u_i u_{\overline{i}}| + \sum_{k=1}^{6} |u_{i_k} u_{\overline{i_k}}| < |u_{id_n}|$. Second, by Lemma 2, $\mathcal{S}_4$ holds for all $a \in \mathbb{N}$ and $b = 0, 1, 2, 3$. If we fix $a_0 \in \mathbb{N}$, it holds for $a = a_0$ and $b = 0, 1, 2, 3$. Thus by Lemma 2 it holds for $a = a_0$ and all $b \in \mathbb{N}$. □

Proposition 1. *For all $i \in \{1, \ldots, n\}$, $|v_i v_{\overline{i}}| = |w_i w_{\overline{i}}|$.*

Proof. This is implied by $\mathcal{S}_1$ and $\mathcal{S}_4$ (with $a = b = 0$). □

Thanks to $\mathcal{S}_1, \ldots, \mathcal{S}_4$ we can characterize the form of $v_i, w_i, w_{\overline{i}}$ for all i and prove a property on v_m, w_m. This characterization is then used to prove $v_{id_n} = w_{id_n}$. Wlog we assume that $v_0 = \epsilon$ or $w_0 = \epsilon$, and $v_{\overline{0}} = \epsilon$ or $w_{\overline{0}} = \epsilon$. Otherwise we can remove their common prefixes in $\mathcal{S}_1, \ldots, \mathcal{S}_4$.

Lemma 7. *If there exist $k \in \{1, \ldots, n\}$ such that $w_k \neq \epsilon$. Then there exist $t_1, t_2, t_3, t_4 \in \Sigma^*$, $\alpha_0, \beta_0 \geq 0$, $\alpha_i, \beta_i, \beta_{\overline{i}} \geq 0$ for all $i \in \{1, \ldots, n\}$ such that $t_1 t_2$ is primitive and for all $i \in \{1, \ldots, n\}$:*

$$t_1 t_2 = t_3 t_4 \quad t_4 t_3 w_m = w_m t_2 t_1 \quad v_i = (t_1 t_2)^{\alpha_i} \quad w_i = (t_4 t_3)^{\beta_i} \quad w_{\overline{i}} = (t_2 t_1)^{\beta_{\overline{i}}}$$

and if $w_0 = \epsilon$, then $v_0 = (t_4 t_3)^{\alpha_0} t_4$, and if $v_0 = \epsilon$, then $w_0 = (t_3 t_4)^{\beta_0} t_3$.

Proof. First we infer the form of v_ℓ and $w_{\overline{\ell}}$. Since $|v_\ell| > |w_\ell|$, by $\mathcal{S}_2$, there is $a \geq 0$ such that $(v_\ell)^a$ and $(w_{\overline{\ell}})^a$ have a common factor of length at least $|v_\ell| + |w_{\overline{\ell}}| - \gcd(|v_\ell|, |w_{\overline{\ell}}|)$ (see Fig. 3). Therefore by Lemma 1.2, there exist $t_1, t_2 \in \Sigma^*$ such that $t_1 t_2$ is primitive, $v_\ell = (t_1 t_2)^{\alpha_\ell}$ and $w_{\overline{\ell}} = (t_2 t_1)^{\beta_{\overline{\ell}}}$ for some $\alpha_\ell, \beta_{\overline{\ell}} > 0$.

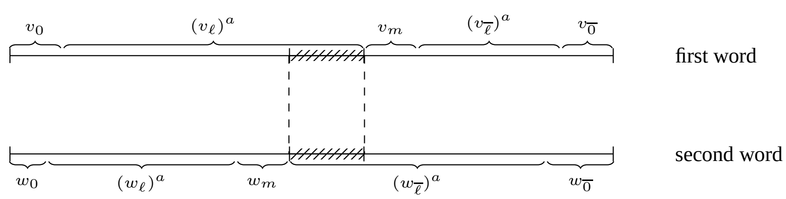

Fig. 3. System $\mathcal{S}_2$ for large values of a, case $|v_\ell| > |w_\ell|$.

Second we derive the form of v_i and $w_{\overline{i}}$ for all $i \in \{1, \ldots, n\}$. As $v_\ell \neq \epsilon$ there is $b_0 \geq 1$ such that $|(v_\ell)^{b_0 - 1} v_m (v_{\overline{\ell}})^{b_0} v_{\overline{i}} v_{\overline{0}}| \geq |w_{\overline{0}}|$. We consider $\mathcal{S}_4$ with $b = b_0$. The size of the suffix $v_\ell v_i (v_\ell)^{b_0} v_m (v_{\overline{\ell}})^{b_0} v_{\overline{i}} (v_{\overline{\ell}})^a v_{\overline{0}}$ is of the form $l_1(a) = k_1 + a|v_{\overline{\ell}}|$ and the size of the suffix $(w_{\overline{\ell}})^a w_{\overline{0}}$ is of the form $l_2(a) = k_2 + a|w_{\overline{\ell}}|$. As $|w_{\overline{\ell}}| > |v_{\overline{\ell}}|$

(by Proposition 1 and $|v_\ell| > |w_\ell|$), there exists $a_0 \geq 1$ such that $l_2(a_0) \geq l_1(a_0)$. Therefore (see Fig. 4) $v_\ell v_i v_\ell$ is a factor of $(w_{\overline{\ell}})^{a_0}$. Thus there is $X, Z \in \Sigma^*$ such that $X(t_1t_2)^{\alpha_\ell} v_i (t_1t_2)^{\alpha_\ell} Z = (t_2t_1)^{a_0\beta_{\overline{\ell}}}$. Since $\alpha_\ell, \beta_{\overline{\ell}} > 0$, we can apply Lemma 3.9 and we get $v_i \in (t_1t_2)^*$. Since $|w_{\overline{\ell}}| > |v_{\overline{\ell}}|$ and $w_{\overline{\ell}} = (t_2t_1)^{\beta_{\overline{\ell}}}$, by symmetry, we also get $w_{\overline{i}} \in (t_2t_1)^*$.

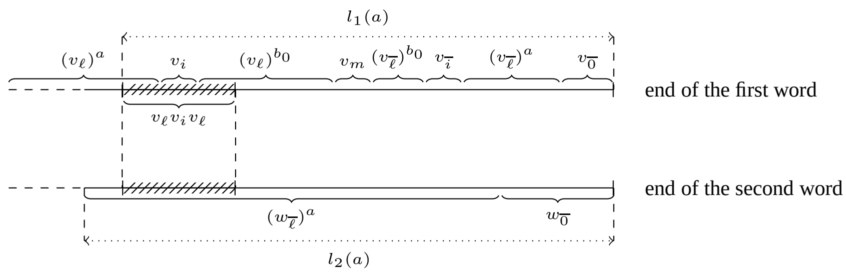

Fig. 4. System $\mathcal{S}_5$ for value b_0 and large values of a, case $|v_\ell| > |w_\ell|$.

Third we determine the form of the words w_i and prove the property on w_m. Since $v_\ell = (t_1t_2)^{\alpha_\ell}$, $w_{\overline{\ell}} = (t_2t_1)^{\beta_{\overline{\ell}}}$ and $v_i = (t_1t_2)^{\alpha_i}$ for some $\alpha_i \geq 0$, $\mathcal{S}_2$ and $\mathcal{S}_3$ can be rewritten as follows:

$$v_0(t_1t_2)^{a.\alpha_\ell} v_m (v_{\overline{\ell}})^a v_{\overline{0}} = w_0 (w_\ell)^a w_m (t_2t_1)^{a.\beta_{\overline{\ell}}} w_{\overline{0}}$$
$$v_0(t_1t_2)^{\alpha_i + a.\alpha_\ell} v_m (v_{\overline{\ell}})^a v_{\overline{i}} v_{\overline{0}} = w_0 w_i (w_\ell)^a w_m (t_2t_1)^{a.\beta_{\overline{\ell}}} w_{\overline{i}} w_{\overline{0}}$$

Since $|v_\ell| > |w_\ell|$, there exist $\alpha, \beta, \gamma, \gamma' \geq 2$ and $t', t'' \prec t_1t_2$ such that

$$v_0(t_1t_2)^\alpha t' = w_0(w_\ell)^\beta w_m t_2 (t_1t_2)^\gamma \qquad v_0(t_1t_2)^\alpha t'' = w_0 w_i (w_\ell)^\beta w_m t_2 (t_1t_2)^{\gamma'}$$

By Lemma 3.1, we get $v_0(t_1t_2)^\omega = w_0 w_i (w_\ell)^\beta w_m (t_2t_1)^\omega$ and

$$v_0(t_1t_2)^\omega = w_0(w_\ell)^\beta w_m (t_2t_1)^\omega \tag{1}$$

Therefore

$$(w_\ell)^\beta w_m (t_2t_1)^\omega = w_i (w_\ell)^\beta w_m (t_2t_1)^\omega \tag{2}$$

Eq. 2 is equivalent to $(w_\ell)^\beta w_m t_2 (t_1t_2)^\omega = w_i (w_\ell)^\beta w_m t_2 (t_1t_2)^\omega$, thus by Lemma 3.5, there exist $t_3, t_4 \in \Sigma^*$ such that $t_1t_2 = t_3t_4$ and for all $i \in \{1, \ldots, n\}$, $w_i = (t_4t_3)^{\beta_i}$ for some $\beta_i \geq 0$. By hypothesis, there is $k \in \{1, \ldots, n\}$ such that $w_k \neq \epsilon$, and therefore $\beta_k > 0$. Eq. 2 gives $(t_4t_3)^{\beta_\ell} w_m (t_2t_1)^\omega = (t_4t_3)^{\beta_\ell + \beta_k} w_m (t_2t_1)^\omega$, i.e. $w_m(t_2t_1)^\omega = (t_4t_3)^{\beta_k} w_m (t_2t_1)^\omega$. By lemma 3.8 we get $w_m t_2 t_1 = t_4 t_3 w_m$.

Finally, we determine the form of v_0 and w_0. If $w_0 = \epsilon$, then Eq. 1 gives $v_0(t_1t_2)^\omega = (t_4t_3)^{\beta.\beta_\ell} w_m (t_2t_1)^\omega$. Since $t_1t_2 = t_3t_4$ and $t_4t_3w_m = w_m t_2 t_1$, $v_0(t_3t_4)^\omega = (t_4t_3)^\omega$. Wlog we can assume that $t_3 \neq \epsilon$. Indeed, $v_\ell \in (t_1t_2)^*$ is non-empty and $t_1t_2 = t_3t_4$, so that $t_3t_4 \neq \epsilon$. By Lemma 3.4, $v_0 \in (t_4t_3)^* t_4$. Alike, if $v_0 = \epsilon$, then wlog we can suppose that $t_4 \neq \epsilon$, and conclude similarly that $w_0 \in (t_3t_4)^* t_3$. $\square$

The mirror of a word $t \in \Sigma^*$ is denoted by $\overline{t}$ and is inductively defined by $\overline{\epsilon} = \epsilon$, $\overline{ct} = \overline{t}c$ for all $c \in \Sigma$. The mirror of an equation $t = t'$ is $\overline{t} = \overline{t'}$. By taking the mirror of the equations $\mathcal{S}_1, \ldots, \mathcal{S}_4$, we obtain a system of equations which has the same form as $\mathcal{S}_1, \ldots, \mathcal{S}_4$. Since $|v_\ell| > |w_\ell|$, by Prop. 1, $|w_{\overline{\ell}}| > |v_{\overline{\ell}}|$. Therefore we can apply Lemma 7 on the mirrors of $\mathcal{S}_1, \ldots, \mathcal{S}_4$ and obtain the following corollary:

Corollary 1. *If there exist $k \in \{1, \ldots, n\}$ such that $v_{\overline{k}} \neq \epsilon$. Then there exist $t_1, t_2, t_5, t_6 \in \Sigma^*$, $\alpha_0, \beta_0 \geq 0$, $\alpha_i, \beta_i, \beta_{\overline{i}} \geq 0$ for all $i \in \{1, \ldots, n\}$ such that t_2t_1 is primitive and for all $i \in \{1, \ldots, n\}$:*

$$t_2t_1 = t_6t_5 \quad t_1t_2v_m = v_mt_5t_6 \quad v_i = (t_1t_2)^{\alpha_i} \quad v_{\overline{i}} = (t_5t_6)^{\alpha_{\overline{i}}} \quad w_{\overline{i}} = (t_2t_1)^{\beta_{\overline{i}}}$$

and if $w_{\overline{0}} = \epsilon$, then $v_{\overline{0}} = t_5(t_6t_5)^{\alpha_{\overline{0}}}$, and if $v_{\overline{0}} = \epsilon$, then $w_{\overline{0}} = t_6(t_5t_6)^{\beta_{\overline{0}}}$

We are now equipped to prove that $v_{id_n} = w_{id_n}$:

Theorem 3. $\mathcal{S}[|v_\ell| > |w_\ell|] \implies v_0 \ldots v_nv_mv_{\overline{n}} \ldots v_{\overline{0}} = w_0 \ldots w_nw_mw_{\overline{n}} \ldots w_{\overline{0}}$

Proof. We consider several cases:

1. *there exist $k, k' \in \{1, \ldots, n\}$ such that $w_{k'} \neq \epsilon$ and $v_{\overline{k}} \neq \epsilon$.*
   By Lemma 7, there exist $t_1, t_2, t_3, t_4 \in \Sigma^*$ and $\alpha_0, \beta_0, \ldots, \alpha_n, \beta_n, \beta_{\overline{n}}, \ldots, \beta_{\overline{1}} \geq 0$ such that:

   $$t_1t_2 = t_3t_4 \quad t_4t_3w_m = w_mt_2t_1 \quad v_i = (t_1t_2)^{\alpha_i} \quad w_i = (t_4t_3)^{\beta_i} \quad w_{\overline{i}} = (t_2t_1)^{\beta_{\overline{i}}}$$

   and if $w_0 = \epsilon$, then $v_0 = (t_4t_3)^{\alpha_0}t_4$, and if $v_0 = \epsilon$, then $w_0 = (t_3t_4)^{\beta_0}t_3$
   By Corollary 1 and the fact that a word is uniquely decomposed as a power of a primitive word, there exist $t_5, t_6 \in \Sigma^*$ and $\alpha_{\overline{n}}, \ldots, \alpha_{\overline{1}} \geq 0$ such that:

   $$t_2t_1 = t_6t_5 \qquad t_1t_2v_m = v_mt_5t_6 \quad v_{\overline{i}} = (t_5t_6)^{\alpha_{\overline{i}}}$$

   and if $w_{\overline{0}} = \epsilon$, then $v_{\overline{0}} = t_5(t_6t_5)^{\alpha_{\overline{0}}}$, and if $v_{\overline{0}} = \epsilon$, then $w_{\overline{0}} = t_6(t_5t_6)^{\beta_{\overline{0}}}$
   We can also suppose that $v_0 = (t_3t_4)^{\alpha_0} = (t_1t_2)^{\alpha_0}$ and $w_0 = (t_3t_4)^{\beta_0}t_3$. Indeed, if $w_0 = \epsilon$, we simply replaced v_0 by t_3v_0 and w_0 by t_3w_0. Similarly, we assume that $w_{\overline{0}} = (t_6t_5)^{\beta_{\overline{0}}}$ and $v_{\overline{0}} = t_5(t_6t_5)^{\alpha_{\overline{0}}}$. By Prop 1, $\alpha_i + \alpha_{\overline{i}} = \beta_i + \beta_{\overline{i}}$ for all $i \in \{1, \ldots, n\}$. Finally:

   $$\begin{aligned}
   & v_0v_1 \ldots v_nv_mv_{\overline{n}} \ldots v_{\overline{0}} \\
   &= (t_1t_2)^{\alpha_0+\cdots+\alpha_n}v_m(t_5t_6)^{\alpha_{\overline{n}}+\cdots+\alpha_{\overline{0}}}t_5 \\
   &= (t_1t_2)^{\alpha_0+\beta_1+\cdots+\beta_n}v_m(t_5t_6)^{\beta_{\overline{n}}+\cdots+\beta_{\overline{1}}+\alpha_{\overline{0}}}t_5 && \text{(since } \alpha_i + \alpha_{\overline{i}} = \beta_i + \beta_{\overline{i}} \text{ and} \\
   & && t_1t_2v_m = v_mt_5t_6) \\
   &= (t_1t_2)^{\beta_1+\cdots+\beta_n}v_0v_mv_{\overline{0}}(t_6t_5)^{\beta_{\overline{n}}+\cdots+\beta_{\overline{1}}} \\
   &= (t_1t_2)^{\beta_1+\cdots+\beta_n}w_0w_mw_{\overline{0}}(t_6t_5)^{\beta_{\overline{n}}+\cdots+\beta_{\overline{1}}} && \text{(by } \mathcal{S}_1) \\
   &= (t_1t_2)^{\beta_1+\cdots+\beta_n}(t_3t_4)^{\beta_0}t_3w_m(t_6t_5)^{\beta_{\overline{0}}}(t_6t_5)^{\beta_{\overline{n}}+\cdots+\beta_{\overline{1}}} \\
   &= (t_3t_4)^{\beta_0+\beta_1+\cdots+\beta_n}t_3w_m(t_2t_1)^{\beta_{\overline{n}}+\cdots+\beta_{\overline{1}}+\beta_{\overline{0}}} && \text{(as } t_1t_2 = t_3t_4 \text{ and } t_2t_1 = t_6t_5) \\
   &= w_0w_1 \ldots w_nw_mw_{\overline{n}} \ldots w_{\overline{1}}w_{\overline{0}}
   \end{aligned}$$ □

2. for all $k \in \{1, \ldots, n\}$, $w_k = v_{\overline{k}} = \epsilon$. As in the proof of Lemma 7, we can characterize the form of v_i and $w_{\overline{i}}$ for all $i \in \{1, \ldots, n\}$. In particular, there exists $t_1, t_2 \in \Sigma^*$ such that $t_1 t_2$ is primitive and $v_i = (t_1 t_2)^{\alpha_i}$ for some $\alpha_i \geq 0$, and $w_{\overline{i}} = (t_2 t_1)^{\beta_{\overline{i}}}$ for some $\beta_i \geq 0$. By Proposition 1, $\alpha_i = \beta_i$ for all i. We let $w'_0 = w_0 w_m$ and $v'_{\overline{0}} = v_m v_{\overline{0}}$. The systems $\mathcal{S}_1, \mathcal{S}_2$ can therefore be rewritten as follows:
$$\begin{cases} v_0 v'_{\overline{0}} = w'_0 w_{\overline{0}} & (1) \\ v_0 (t_1 t_2)^{a\alpha_\ell} v'_{\overline{0}} = w'_0 (t_2 t_1)^{a\alpha_\ell} w_{\overline{0}} & (2) \end{cases}$$
Wlog, we can assume that $v_0 = \epsilon$ or $w'_0 = \epsilon$. Both cases are symmetric, so that we consider only the case $v_0 = \epsilon$. Wlog we can assume that $t_1 \neq \epsilon$. By Lemma 3.4 and $\mathcal{S}_2$, we get $w'_0 = (t_1 t_2)^{\alpha} t_1$ for some $\alpha \geq 0$. Therefore:
$$\begin{aligned} & v_0 v_1 \ldots v_n v_m v_{\overline{n}} \ldots v_{\overline{1}} v_{\overline{0}} \\ = & (t_1 t_2)^{\alpha_1 + \cdots + \alpha_n} v'_0 \\ = & (t_1 t_2)^{\alpha_1 + \cdots + \alpha_n} w'_0 w_{\overline{0}} \text{ by } \mathcal{S}_1 \\ = & (t_1 t_2)^{\alpha_1 + \cdots + \alpha_n + \alpha} t_1 w_{\overline{0}} \\ = & w'_0 (t_2 t_1)^{\alpha_1 + \cdots + \alpha_n} w_{\overline{0}} \\ = & w_0 w_m (t_2 t_1)^{\alpha_1 + \cdots + \alpha_n} w_{\overline{0}} \\ = & w_0 w_1 \ldots w_n w_m w_{\overline{n}} \ldots w_{\overline{1}} w_{\overline{0}} \end{aligned}$$

3. for all $k \in \{1, \ldots, n\}$, $v_{\overline{k}} = \epsilon$ and there exists $p \in \{1, \ldots, n\}$ such that $w_p \neq \epsilon$. By Lemma 7, there exist $t_1, t_2, t_3, t_4 \in \Sigma^*$ and $\alpha_0, beta_0$ and $\alpha_i, \beta_i, \beta_{\overline{i}} \geq 0$ for all $i \in \{1, \ldots, n\}$ such that $t_1 t_2$ is primitive and for all $i \in \{1, \ldots, n\}$, $t_1 t_2 = t_3 t_4$, $t_4 t_3 w_m = w_m t_2 t_1$, $v_i = (t_1 t_2)^{\alpha_i}$, $w_i = (t_4 t_3)^{\beta_i}$ and $w_{\overline{i}} = (t_2 t_1)^{\beta_{\overline{i}}}$. Moreover, if $w_0 = \epsilon$, then $v_0 = (t_4 t_3)^{\alpha_0} t_4$, and if $v_0 = \epsilon$, then $w_0 = (t_3 t_4)^{\beta_0} t_3$. By Proposition 1, since $v_{\overline{k}} = \epsilon$ for all $k \in \{1, \ldots, n\}$, we get $\alpha_k = \beta_k + \beta_{\overline{k}}$. As for the case given in the paper, we can suppose that $v_0 = (t_3 t_4)^{\alpha_0} = (t_1 t_2)^{\alpha_0}$ and $w_0 = (t_3 t_4)^{\beta_0} t_3$. Indeed, if $w_0 = \epsilon$, we simply replaced v_0 by $t_3 v_0$ and w_0 by $t_3 w_0$. Finally:
$$\begin{aligned} & v_0 v_1 \ldots v_n v_m v_{\overline{n}} \ldots v_{\overline{0}} \\ = & (t_1 t_2)^{\alpha_0 + \cdots + \alpha_n} v_m v_{\overline{0}} \\ = & (t_1 t_2)^{\alpha_1 + \cdots + \alpha_n} v_0 v_m v_{\overline{0}} \\ = & (t_1 t_2)^{\alpha_1 + \cdots + \alpha_n} w_0 w_m w_{\overline{0}} \text{ by } \mathcal{S}_1 \\ = & (t_3 t_4)^{\alpha_1 + \cdots + \alpha_n + \beta_0} t_3 w_m w_{\overline{0}} \\ = & w_0 (t_4 t_3)^{\alpha_1 + \cdots + \alpha_n} w_m w_{\overline{0}} \\ = & w_0 (t_4 t_3)^{\beta_1 + \cdots + \beta_n} (t_4 t_3)^{\beta_{\overline{1}} + \cdots + \beta_{\overline{n}}} w_m w_{\overline{0}} \text{ since } \alpha_i = \beta_i + \beta_{\overline{i}} \\ = & w_0 (t_4 t_3)^{\beta_1 + \cdots + \beta_n} w_m (t_2 t_1)^{\beta_{\overline{1}} + \cdots + \beta_{\overline{n}}} w_{\overline{0}} \text{ since } t_4 t_3 w_m = w_m t_2 t_1 \\ = & f w_0 w_1 \ldots w_n w_m w_{\overline{1}} \ldots w_{\overline{n}} w_{\overline{0}} \end{aligned}$$

4. for all $k \in \{1, \ldots, n\}$, $w_k = \epsilon$ and there exists $p \in \{1, \ldots, n\}$ such that $v_{\overline{p}} \neq \epsilon$. This case is symmetric to case 2.

5.2 Proof of Theorem 2: case $|v_\ell| = |w_\ell|$

Remind that we have fixed some $n \geq 8$, some words $u_m, v_m, w_m \in \Sigma^*$ and for all $i \in \{0, \ldots, n\}$, we have fixed $u_i, v_i, w_i, u_{\overline{i}}, v_{\overline{i}}, w_{\overline{i}} \in \Sigma^*$ such that $u_i, u_{\overline{i}} \neq \epsilon$ such that the following system holds: $\mathcal{S} = \{v_\pi = w_\pi \mid \pi : \{1, \ldots, k\} \to \{1, \ldots, n\},\ |u_\pi| < |u_{id_n}|\}$.

Consider the following equations, defined for all $a \in \mathbb{N}$, for all $i, k \in \{1, \ldots, n\}$:

$$\begin{cases} v_0 v_m v_{\overline{0}} = w_0 w_m w_{\overline{0}} & (1) \\ v_0 (v_\ell)^a v_m (v_{\overline{\ell}})^a v_{\overline{0}} = w_0 (w_\ell)^a w_m (w_{\overline{\ell}})^a w_{\overline{0}} & (2) \\ v_0 v_i (v_\ell)^a v_m (v_{\overline{\ell}})^a v_{\overline{i}} v_{\overline{0}} = w_0 w_i (w_\ell)^a w_m (w_{\overline{\ell}})^a w_{\overline{i}} w_{\overline{0}} & (3) \\ v_0 v_i v_k (v_\ell)^a v_m (v_{\overline{\ell}})^a v_{\overline{k}} v_{\overline{i}} v_{\overline{0}} = w_0 w_i w_k (w_\ell)^a w_m (w_{\overline{\ell}})^a w_{\overline{k}} w_{\overline{i}} w_{\overline{0}} & (4) \\ v_0 \ldots v_{\ell-1} v_{\ell+1} \ldots v_n v_m v_{\overline{n}} \ldots v_{\overline{\ell-1}} v_{\overline{\ell+1}} \ldots v_{\overline{0}} = w_0 \ldots w_{\ell-1} w_{\ell+1} \ldots w_n w_m w_{\overline{n}} \ldots w_{\overline{\ell-1}} w_{\overline{\ell+1}} \ldots w_{\overline{0}} & (5) \end{cases}$$

As done for the case $|v_\ell| > |w_\ell|$, we denoty by $\mathcal{S}_k$ the set of equations of type k, $k = 1, \ldots, 5$. As for the equations given in the paper for the case $|v_\ell| > |w_\ell|$, we can prove similarly the following proposition:

Proposition 2. *For all $k = 1, \ldots, 5$, $\mathcal{S}_k$ holds.*

As for the case $|v_\ell| > |w_\ell|$, we have the following proposition (which is in fact indepent from the cases $|v_\ell| = |w_\ell|$ or not):

Proposition 3. *For all $i \in \{1, \ldots, n\}$, $|v_i v_{\overline{i}}| = |w_i w_{\overline{i}}|$.*

Case study There are four cases:

(i) $|v_\ell| = |w_\ell| = 0$ and $|v_{\overline{\ell}}| = |w_{\overline{\ell}}| = 0$;
(ii) $|v_\ell| = |w_\ell| \neq 0$ and $|v_{\overline{\ell}}| = |w_{\overline{\ell}}| = 0$;
(iii) $|v_{\overline{\ell}}| = |w_{\overline{\ell}}| \neq 0$ and $|v_\ell| = |w_\ell| \neq 0$;
(iv) $|v_{\overline{\ell}}| = |w_{\overline{\ell}}| \neq 0$ and $|v_\ell| = |w_\ell| = 0$;

Cases (iv) is syntactically the same as case (ii) if we consider the mirror of the equations. Therefore we consider only case (i),(ii) and (iii). For each of those three cases, we prove that $v_{id_n} = w_{id_n}$ (Theorem 2).

Similarly as the case $|v_\ell| > |w_\ell|$, we can assume wlog that $v_0 = \epsilon$ or $w_0 = \epsilon$, and $v_{\overline{0}} = \epsilon$ or $w_{\overline{0}} = \epsilon$, otherwise we remove their common prefixes in the systems $\mathcal{S}_1, \ldots, \mathcal{S}_5$.

Subcase $|v_\ell| = |w_\ell| = |v_{\overline{\ell}}| = |w_{\overline{\ell}}| = 0$

Lemma 8. *If $|v_\ell| = |w_\ell| = 0$ and $|v_{\overline{\ell}}| = |w_{\overline{\ell}}| = 0$, then $v_{id_n} = w_{id_n}$.*

Proof. It is an obvious consequence of $\mathcal{S}_5$. □

Subcase $|v_\ell| = |w_\ell| \neq 0$ and $|v_{\overline{\ell}}| = |w_{\overline{\ell}}| \neq 0$

Lemma 9. *There exist $t_1, t_2 \in \Sigma^*$ such that $t_1 t_2$ is primitive and $\alpha_0, \beta_0, \alpha_\ell, \beta_\ell \geq 0$ such that:*

$$v_\ell = (t_1 t_2)^{\alpha_\ell} \qquad w_\ell = (t_2 t_1)^{\beta_\ell} \qquad w_0 = \epsilon \Rightarrow v_0 = (t_2 t_1)^{\alpha_0} t_2 \qquad v_0 = \epsilon \Rightarrow w_0 = (t_1 t_2)^{\beta_0} t_1$$

Proof. Remind that by hypothesis, $v_\ell \neq \epsilon$. Then $w_\ell \neq \epsilon$. By $\mathcal{S}_2$, there exists $a \geq 0$ such that $(v_\ell)^a$ and $(w_\ell)^a$ have a common factor of length at least $|v_\ell|+|w_\ell|-\gcd(|v_\ell|,|w_\ell|)$. By the fundamental lemma, there exist $t_1, t_2 \in \Sigma^*$ such that t_1t_2 is primitive, $v_\ell \in (t_1t_2)^+$ and $w_\ell \in (t_2t_1)^+$. We now infer the form of v_0 when $w_0 = \epsilon$ (the form of w_0 when $v_0 = \epsilon$ can be obtained by symmetry). Wlog, we can assume that $t_1 \neq \epsilon$. Indeed, since $v_\ell \neq \epsilon$, we have $t_1t_2 \neq \epsilon$, so that if $t_1 = \epsilon$, then we take $t_1' = t_2$ and $t_2' = t_1 = \epsilon$, and we have $v_\ell \in (t_1't_2')^+$ and $w_\ell \in (t_2't_1')^+$. By $\mathcal{S}_2$, we get $v_0(t_1t_2)^\omega = (t_2t_1)^\omega$. By Lemma 3.3, $v_0 = (t_2t_1)^{\alpha_0}t_2$ for some $\alpha_0 \geq 0$. □

Since by hypothesis we have $|v_{\overline{\ell}}| = |w_{\overline{\ell}}| \neq 0$, by considering the mirror of the equations, we can prove the following corollary of Lemma 9:

Corollary 2. *There exist $t_3, t_4 \in \Sigma^*$ such that t_3t_4 is primitive and $\alpha_{\overline{0}}, \beta_{\overline{0}}, \alpha_{\overline{\ell}}, \beta_{\overline{\ell}} \geq 0$ such that:*

$$v_{\overline{\ell}} = (t_3t_4)^{\alpha_{\overline{\ell}}} \qquad w_\ell = (t_4t_3)^{\beta_{\overline{\ell}}} \qquad w_{\overline{0}} = \epsilon \Rightarrow v_{\overline{0}} = (t_3t_4)^{\alpha_{\overline{0}}}t_3 \qquad v_{\overline{0}} = \epsilon \Rightarrow w_{\overline{0}} = (t_4t_3)^{\beta_0}t_4$$

Under certain conditions, we can characterize the form of v_i's and w_i's:

Lemma 10. *If there exists $1 \leq k \leq n$ such that $|v_k| \neq |w_k|$ then there exist $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \geq 0$ such that for all $i \neq k$:*

$$v_i = (t_1t_2)^{\alpha_i} \qquad w_i = (t_2t_1)^{\beta_i}$$

Proof. There are two cases: either $v_0 = \epsilon$ or $w_0 = \epsilon$. We consider the second case only, the first being symmetric. By Lemma 9, $v_0 = (t_2t_1)^{\alpha_0}t_2$ for some $\alpha_0 \geq 0$ and $t_1, t_2 \in \Sigma^*$ with t_1t_2 primitive. By $\mathcal{S}_3$ and $\mathcal{S}_4$, we have:

$$(1)\ v_0v_i(t_1t_2)^\omega = w_i(t_2t_1)^\omega \qquad (2)\ v_0v_k(t_1t_2)^\omega = w_k(t_2t_1)^\omega \qquad (3)\ v_0v_kv_i(t_1t_2)^\omega = w_kw_i(t_2t_1)^\omega$$

We again consider two cases:

1. $v_0v_k = w_kw$ for some w. $\mathcal{S}_2$ gives $w(t_1t_2)^\omega = (t_2t_1)^\omega$. By Lemma 3.3, $w = (t_2t_1)^\beta t_2$ for some $\beta \geq 0$. $\mathcal{S}_3$ gives $wv_i(t_1t_2)^\omega = w_i(t_2t_1)^\omega$, and by $\mathcal{S}_1$, we get $wv_i(t_1t_2)^\omega = v_0v_i(t_1t_2)^\omega$, i.e. $(t_2t_1)^\beta t_2v_i(t_1t_2)^\omega = (t_2t_1)^{\alpha_0}t_2v_i(t_1t_2)^\omega$. Since $|v_k| \neq |w_k|$ and $v_0v_k = w_kw$, $|v_0| \neq |w|$, and $\beta \neq \alpha_0$. Thus by taking $\gamma = |\alpha_0 - \beta| > 0$, we get $(t_1t_2)^\gamma v_i(t_1t_2)^\omega = v_i(t_1t_2)^\omega$. By Lemma 3.8, $v_i \in (t_1t_2)^*$.
2. $w_k = v_0v_kv$ for some $v \neq \epsilon$. $\mathcal{S}_2$ gives $(t_1t_2)^\omega = v(t_2t_1)^\omega$, i.e. $(t_1t_2)^\omega = vt_2(t_1t_2)^\omega$. Therefore by Lemma 3.6, $vt_2 \in (t_1t_2)^\omega$. Since $v \neq \epsilon$, we get $v = (t_1t_2)^\eta t_1$ for some $\eta \geq 0$. Now, $\mathcal{S}_3$ gives $v_i(t_1t_2)^\omega = vw_i(t_2t_1)^\omega$, and by $\mathcal{S}_1$, $v_i(t_1t_2)^\omega = vv_0v_i(t_1t_2)^\omega = (t_1t_2)^\eta t_1(t_2t_1)^{\alpha_0}t_2v_i(t_1t_2)^\omega = (t_1t_2)^{\eta+\alpha_0+1}v_i(t_1t_2)^\omega$. By Lemma 3.8, $v_i \in (t_1t_2)^*$.

In both cases $v_i \in (t_1t_2)^*$. By $\mathcal{S}_1$ $v_0v_i(t_1t_2)^\omega = (t_2t_1)^\omega = w_i(t_2t_1)^\omega$ and by Lemma 3.6 $w_i \in (t_2t_1)^*$.

Again by considering the mirror of the equations, we can prove the following corollary of Lemma 10:

Corollary 3. *If there exists* $1 \leq k \leq n$ *such that* $|v_{\overline{k}}| \neq |w_{\overline{k}}|$ *then there exist* $\alpha_{\overline{1}}, \ldots, \alpha_{\overline{n}}, \beta_{\overline{1}}, \ldots, \beta_{\overline{n}} \geq 0$ *such that for all* $i \neq k$*:*

$$v_{\overline{i}} = (t_3 t_4)^{\alpha_{\overline{i}}} \qquad w_{\overline{i}} = (t_4 t_3)^{\beta_{\overline{i}}}$$

Lemma 11. *Let* $\alpha \in \mathbb{N}$*. If for all* $i \in \{1 \ldots n\}$*,* $|v_i| = |w_i|$ *and there exist* $a_i, b_i \in \mathbb{N}$ *such that:*

$$(t_2 t_1)^{\alpha} t_2 v_i (t_1 t_2)^{a_i} = w_i (t_2 t_1)^{b_i} t_2 \tag{3}$$

then

$$(t_2 t_1)^{\alpha} t_2 v_1 \ldots v_n = w_1 \ldots w_n (t_2 t_1)^{\alpha} t_2$$

Proof. From Eq.3, and $|v_i| = |w_i|$ we deduce that $b_i = \alpha + a_i$, so that:

$$(t_2 t_1)^{\alpha} t_2 v_i = w_i (t_2 t_1)^{\alpha} t_2 \tag{4}$$

By induction on n we show that $(t_2 t_1)^{\alpha} t_2 v_1 \ldots v_n = w_1 \ldots w_n (t_2 t_1)^{\alpha} t_2$. Indeed, it is trivial if $n = 0$. So suppose it is true for $n - 1$, we have:

$$\begin{array}{ll} (t_2 t_1)^{\alpha} t_2 v_1 \ldots v_n & \\ = w_1 \ldots w_{n-1} (t_2 t_1)^{\alpha} t_2 v_n & \text{(by induction hypothesis)} \\ = w_1 \ldots w_n (t_2 t_1)^{\alpha} t_2 & \text{(by (4))} \end{array}$$

□

Proposition 4. *One of the following propositions holds:*

1. $\forall i \in \{1, \ldots, n\} : v_i = (t_1 t_2)^{\alpha_i} \wedge w_i = (t_2 t_1)^{\beta_i} \wedge v_{\overline{i}} = (t_3 t_4)^{\alpha_{\overline{i}}} \wedge w_{\overline{i}} = (t_4 t_3)^{\beta_{\overline{i}}}$
2. $\exists k \in \{1, \ldots, n\} \forall i \neq k : |v_i| = |w_i|$ *and* $|v_{\overline{i}}| = |w_{\overline{i}}|$

Proof. Indeed, if there are $k \neq k'$ such that $|v_k| \neq |w_k|$ and $|v_{k'}| \neq |w_{k'}|$, then by Lemma 10 $\forall i : v_i = (t_1 t_2)^{\alpha_i} \wedge w_i = (t_2 t_1)^{\beta_i}$. By Proposition 3, $|v_{\overline{k}}| \neq |w_{\overline{k}}|$ and $|v_{\overline{k'}}| \neq |w_{\overline{k'}}|$ so that by Corollary 3, for all i, $v_{\overline{i}} = (t_3 t_4)^{\alpha_{\overline{i}}}$ and $w_{\overline{i}} = (t_4 t_3)^{\beta_{\overline{i}}}$.

Otherwise we have at most one k with $|v_k| \neq |w_k|$, and for all $i \neq k$, $|v_i| = |w_i|$, and by Prop. 3, $|v_{\overline{i}}| = |w_{\overline{i}}|$. □

We now prove Theorem 2 for each of the cases of Prop. 4. This is done in two lemmas: Lemma 12 and Lemma 13.

Lemma 12. *If for all* $i \in \{1, \ldots, n\}$*,* $v_i = (t_1 t_2)^{\alpha_i}$*,* $w_i = (t_2 t_1)^{\beta_i}$*,* $v_{\overline{i}} = (t_3 t_4)^{\alpha_{\overline{i}}}$ *and* $w_{\overline{i}} = (t_4 t_3)^{\beta_{\overline{i}}}$*, then* $v_0 \ldots v_n v_m v_{\overline{n}} \ldots v_{\overline{0}} = w_0 \ldots w_n w_m w_{\overline{n}} \ldots w_{\overline{0}}$.

Proof. First by Lemma 9 and Corollary 2, we have:

$$\begin{array}{ll} w_0 = \epsilon \Rightarrow v_0 \in (t_2 t_1)^* t_2 & \quad v_0 = \epsilon \Rightarrow w_0 \in (t_1 t_2)^* t_1 \\ w_{\overline{0}} = \epsilon \Rightarrow v_{\overline{0}} \in (t_3 t_4)^* t_3 & \quad v_{\overline{0}} = \epsilon \Rightarrow w_{\overline{0}} \in (t_4 t_3)^* t_4 \end{array}$$

Since $v_0 = \epsilon$ or $w_0 = \epsilon$, and $v_{\overline{0}} = \epsilon$ or $w_{\overline{0}} = \epsilon$, we can assume wlog that $v_0 = (t_1 t_2)^{\alpha_0}$ and $w_0 = (t_1 t_2)^{\beta_0} t_1$ for some $\alpha_0, \beta_0 \geq 0$. Indeed, if $w_0 = \epsilon$, we simply replace in $\mathcal{S}_1, \ldots, \mathcal{S}_5$ v_0 by t_1v_0 and w_0 by t_1w_0 (which is indeed of the form $(t_1t_2)^*t_1$). If $v_0 = \epsilon$, then it is of the form $(t_1t_2)^*$ and w_0 is of the form $(t_1t_2)^*t_1$.

Similarly, we can assume wlog that $v_{\overline{0}} = (t_3t_4)^{\alpha_{\overline{0}}}$ and $w_{\overline{0}} = (t_4t_3)^{\beta_{\overline{0}}}t_4$ for some $\alpha_{\overline{0}}, \beta_{\overline{0}} \geq 0$.

Now, by $\mathcal{S}_1$ and $\mathcal{S}_2$, we have:

$$\begin{aligned} v_0v_\ell v_m v_{\overline{\ell}} v_{\overline{0}} &= w_0w_\ell w_m w_{\overline{\ell}} w_{\overline{0}} \\ v_0v_m v_{\overline{0}} &= w_0 w_m w_{\overline{0}} \end{aligned}$$

So we deduce:

$$\begin{aligned} v_0v_\ell v_m v_{\overline{\ell}} v_{\overline{0}} &= w_0w_\ell w_m w_{\overline{\ell}} w_{\overline{0}} \\ \Leftrightarrow (t_1t_2)^{\alpha_0+\alpha_\ell} v_m (t_3t_4)^{\alpha_{\overline{\ell}}+\alpha_{\overline{0}}} &= w_0w_\ell w_m w_{\overline{\ell}} w_{\overline{0}} \\ \Leftrightarrow (t_1t_2)^{\alpha_\ell}(t_1t_2)^{\alpha_0} v_m (t_3t_4)^{\alpha_{\overline{0}}}(t_3t_4)^{\alpha_{\overline{\ell}}} &= w_0w_\ell w_m w_{\overline{\ell}} w_{\overline{0}} \\ \Leftrightarrow (t_1t_2)^{\alpha_\ell} v_0 v_m v_{\overline{0}} (t_3t_4)^{\alpha_{\overline{\ell}}} &= w_0w_\ell w_m w_{\overline{\ell}} w_{\overline{0}} \\ \Leftrightarrow (t_1t_2)^{\alpha_\ell} w_0 w_m w_{\overline{0}} (t_3t_4)^{\alpha_{\overline{\ell}}} &= w_0w_\ell w_m w_{\overline{\ell}} w_{\overline{0}} \\ \Leftrightarrow (t_1t_2)^{\alpha_\ell+\beta_0} t_1 w_m t_4 (t_3t_4)^{\alpha_{\overline{\ell}}+\beta_{\overline{0}}} &= w_0w_\ell w_m w_{\overline{\ell}} w_{\overline{0}} \\ \Leftrightarrow w_0(t_1t_2)^{\alpha_\ell} w_m (t_3t_4)^{\alpha_{\overline{\ell}}} w_{\overline{0}} &= w_0w_\ell w_m w_{\overline{\ell}} w_{\overline{0}} \\ \Leftrightarrow (t_1t_2)^{\alpha_\ell} w_m (t_3t_4)^{\alpha_{\overline{\ell}}} &= w_\ell w_m w_{\overline{\ell}} \\ \Leftrightarrow (t_1t_2)^{\alpha_\ell} w_m (t_3t_4)^{\alpha_{\overline{\ell}}} &= (t_1t_2)^{\beta_\ell} w_m (t_3t_4)^{\beta_{\overline{\ell}}} \end{aligned}$$

Then we conclude with:

$$\begin{aligned} & v_0v_1\ldots v_m v_{\overline{n}}\ldots v_{\overline{1}} v_{\overline{0}} \\ &= (t_1t_2)^{\alpha_0+\cdots+\alpha_n} v_m (t_3t_4)^{\alpha_{\overline{n}}+\cdots+\alpha_{\overline{0}}} \\ &= (t_1t_2)^{\alpha_\ell} v_0\ldots v_{\ell-1}v_{\ell+1}\ldots v_n v_m v_{\overline{n}}\ldots v_{\overline{\ell+1}} v_{\overline{\ell-1}} (t_3t_4)^{\alpha_{\overline{\ell}}} \\ &= (t_1t_2)^{\alpha_\ell} w_0\ldots w_{\ell-1}w_{\ell+1}\ldots w_n w_m w_{\overline{n}}\ldots w_{\overline{\ell+1}} w_{\overline{\ell-1}} (t_3t_4)^{\alpha_{\overline{\ell}}} \text{ by } \mathcal{S}_5 \\ &= (t_1t_2)^{\alpha_\ell+\beta_0+\cdots+\beta_{\ell-1}+\beta_{\ell+1}\cdots\beta_n} w_m (t_3t_4)^{\beta_{\overline{n}}+\cdots\beta_{\overline{\ell+1}}+\beta_{\overline{\ell-1}}+\cdots+\beta_{\overline{1}}+\alpha_{\overline{\ell}}} \\ &= (t_1t_2)^{\beta_0+\cdots+\beta_{\ell-1}+\beta_{\ell+1}\cdots\beta_n} (t_1t_2)^{\alpha_\ell} w_m (t_3t_4)^{\alpha_{\overline{\ell}}} (t_3t_4)^{\beta_{\overline{n}}+\cdots\beta_{\overline{\ell+1}}+\beta_{\overline{\ell-1}}+\cdots+\beta_{\overline{1}}} \\ &= (t_1t_2)^{\beta_0+\cdots+\beta_{\ell-1}+\beta_{\ell+1}\cdots\beta_n} (t_1t_2)^{\beta_\ell} w_m (t_3t_4)^{\beta_{\overline{\ell}}} (t_3t_4)^{\beta_{\overline{n}}+\cdots\beta_{\overline{\ell+1}}+\beta_{\overline{\ell-1}}+\cdots+\beta_{\overline{1}}} \\ &= w_0w_1\ldots w_n w_m w_{\overline{n}}\ldots w_{\overline{0}} \end{aligned}$$

□

Lemma 13. *If there exists* $\exists k \in \{1, \ldots, n\}$ *such that for all* $i \neq k$, $|v_i| = |w_i|$ *and* $|v_{\overline{i}}| = |w_{\overline{i}}|$, *then* $v_0\ldots v_n v_m v_{\overline{n}}\ldots v_{\overline{0}} = w_0\ldots w_n w_m w_{\overline{n}}\ldots w_{\overline{0}}$.

Proof. By hypothesis, we have assumed that $v_0 = \epsilon$ or $w_0 = \epsilon$, and $v_{\overline{0}} = \epsilon$ or $w_{\overline{0}} = \epsilon$. This leads to four cases:

1. $w_0 = \epsilon$ and $v_{\overline{0}} = \epsilon$;
2. $v_0 = \epsilon$ and $v_{\overline{0}} = \epsilon$;

3. $v_0 = \epsilon$ and $w_{\overline{0}} = \epsilon$;
4. $w_0 = \epsilon$ and $w_{\overline{0}} = \epsilon$.

We have assumed that $|v_\ell| = |w_\ell| \neq 0$ and $|v_{\overline{\ell}}| = |w_{\overline{\ell}}| \neq 0$, and there is k such that for all $i \neq k$, $|v_i| = |w_i|$ and $|v_{\overline{i}}| = |w_{\overline{i}}|$. This assumption is symmetric, so that with respect to the systems $\mathcal{S}_1, \dots, \mathcal{S}_5$, cases 2 and 4 are symmetric, and case 1 and 3 are symmetric. Moreover, the proofs of cases 1 and 2 are very similar, therefore we focus on case 1 only.

From now one, we assume that $w_0 = \epsilon$ and $v_{\overline{0}} = \epsilon$. By $\mathcal{S}_3$ and $v_\ell = (t_1t_2)^{\alpha_\ell}$ (Lemma 9) we have $v_0v_k(t_1t_2)^\omega = w_k(t_2t_1)^\omega$. Wlog we can assume that $t_1 \neq \epsilon$. Therefore by Lemma 3.3, there exist a_k, b_k such that $v_0v_k(t_1t_2)^{a_k} = w_k(t_2t_1)^{b_k}t_2$, equivalently we consider two cases we suppose that either $a_k = 0$ or that $a_k \neq 0, b_k = 0$ i.e. either $v_0v_k = w_k(t_2t_1)^{b_k}t_2$ or $v_0v_k(t_1t_2)^{a_k-1}t_1 = w_k$.

- Case $v_0v_k = w_k(t_2t_1)^{b_k}t_2$:
  First, we know that $|v_i| = |w_i|$ for all $i < k$ and that there are $a_i, b_i \in \mathbb{N}$ with $v_0v_i(t_1t_2)^{a_i} = w_i(t_2t_1)^{b_i}t_2$ (by $\mathcal{S}_3$ and Lemma 3.3) where $v_0 = (t_1t_2)^{\alpha_0}t_2$, so by Lemma 11 we have:
  $$v_0v_1 \dots v_{k-1} = w_1 \dots w_{k-1}v_0 \tag{5}$$
  Second we have $v_0v_k = w_k(t_2t_1)^{b_k}t_2$ by hypothesis (the case we are considering). Third, again by $\mathcal{S}_3$ and Lemma 3.3 we know that $|v_i| = |w_i|$ for all $i > k$ and that there are $a'_i, b'_i \in \mathbb{N}$ with $v_0v_kv_i(t_1t_2)^{a'_i} = w_kw_i(t_2t_1)^{b'_i}t_2$ i.e. by replacing v_0v_k with $w_k(t_2t_1)^{b_k}t_2$ we have $(t_2t_1)^{b_k}t_2v_i(t_1t_2)^{a'_i} = w_i(t_2t_1)^{b'_i}t_2$, so by Lemma 11 we have:
  $$(t_2t_1)^{b_k}t_2v_{k+1} \dots v_n = w_{k+1} \dots w_n(t_2t_1)^{a_k}t_2 \tag{6}$$
  As a consequence we have:
  $$\begin{aligned} & v_0 \dots v_n \\ = {} & v_0 \dots v_{k-1}v_kv_{k+1} \dots v_n \\ = {} & w_1 \dots w_{k-1}v_0v_kv_{k+1} \dots v_n \\ = {} & w_1 \dots w_{k-1}w_k(t_2t_1)^{b_k}t_2v_{k+1} \dots v_n \\ = {} & w_1 \dots w_{k-1}w_kw_{k+1} \dots w_n(t_2t_1)^{b_k}t_2 \\ = {} & w_1 \dots w_n(t_2t_1)^{b_k}t_2 \end{aligned} \tag{7}$$
- Case $v_0v_k(t_1t_2)^at_1 = w_k$: We can show that $v_0 \dots v_n(t_2t_1)^{a_k-1}t_2 = w_1 \dots w_n$ with a very similar proof.

By symmetry (since $v_{\overline{1}} \neq \epsilon$ and $w_{\overline{1}} \neq \epsilon$), we have either $t_3(t_4t_3)^{d_k}v_{\overline{k}} = w_{\overline{k}}w_{\overline{0}}$ or $v_{\overline{k}} = t_4(t_3t_4)^{c_k}w_{\overline{k}}w_{\overline{0}}$.

We conclude the proof by putting this together and showing that $v_0v_1 \dots v_mv_{\overline{n}} \dots v_{\overline{1}} = w_1 \dots w_nw_mw_{\overline{n}} \dots w_{\overline{0}}$:

– Subcase $t_3(t_4t_3)^{d_k}v_{\overline{k}} = w_{\overline{k}}w_{\overline{0}}$: this implies that $t_3(t_4t_3)^{d_k}v_{\overline{n}}\dots v_{\overline{1}} = w_{\overline{n}}\dots w_{\overline{0}}$. Moreover we know that $v_0v_kv_mv_{\overline{k}} = w_kw_mw_{\overline{k}}w_{\overline{0}}$ i.e. $(t_2t_1)^{b_k}t_2v_m = w_mt_3(t_4t_3)^d$. We can deduce:

$$\begin{aligned}
&v_0v_1\dots v_mv_{\overline{n}}\dots v_{\overline{1}}\\
&= w_1\dots w_n(t_1t_2)^{b_k}t_1v_mv_{\overline{n}}\dots v_{\overline{1}}\\
&= w_1\dots w_nw_mt_3(t_4t_3)^{d_k}v_{\overline{n}}\dots v_{\overline{1}}\\
&= w_1\dots w_nw_mw_{\overline{n}}\dots w_{\overline{0}}
\end{aligned}$$

– Subcase $v_{\overline{k}} = t_4(t_3t_4)^cw_{\overline{k}}w_{\overline{0}}$: this implies that $v_{\overline{n}}\dots v_{\overline{1}} = t_4(t_3t_4)^cw_{\overline{n}}\dots w_{\overline{0}}$. Moreover we know that $v_0v_kv_mv_{\overline{k}} = w_kw_mw_{\overline{k}}w_{\overline{0}}$ i.e. $(t_2t_1)^bt_2v_mt_4(t_3t_4)^c = w_m$. We can deduce:

$$\begin{aligned}
&v_0v_1\dots v_mv_{\overline{n}}\dots v_{\overline{1}}\\
&= w_1\dots w_n(t_1t_2)^bt_1v_mv_{\overline{n}}\dots v_{\overline{1}}\\
&= w_1\dots w_n(t_1t_2)^bt_1v_mt_4(t_3t_4)^cw_{\overline{n}}\dots w_{\overline{0}}\\
&= w_1\dots w_nw_mw_{\overline{n}}\dots w_{\overline{0}}
\end{aligned}$$

□

Subcase $|v_\ell| = |w_\ell| \neq 0$ and $|v_{\overline{\ell}}| = |w_{\overline{\ell}}| = 0$ Similarly as Proposition 4, one can prove the following proposition:

Proposition 5. *One of the following propositions holds:*

1. $\forall i \in \{1,\dots,n\} : v_i = (t_1t_2)^{\alpha_i} \wedge w_i = (t_2t_1)^{\beta_i}$
2. $\exists k \in \{1,\dots,n\}\forall i \neq k : |v_i| = |w_i|$.

Lemma 14. *If $|v_\ell| = |w_\ell| \neq 0$ and $|v_{\overline{\ell}}| = |w_{\overline{\ell}}| = 0$, then $v_{id_n} = w_{id_n}$.*

Proof. Let pose $V_1 = v_0\dots v_{\ell-1}v_{\ell+1}\dots v_n$, resp. $W_1 = w_0\dots w_{\ell-1}w_{\ell+1}\dots w_n$, and $V = v_mv_{\overline{n}}\dots v_{\overline{\ell+1}}v_{\overline{\ell-1}}\dots v_{\overline{0}} = v_mv_{\overline{n}}\dots v_{\overline{0}}$, resp. $W = w_mw_{\overline{n}}\dots w_{\overline{\ell+1}}w_{\overline{\ell-1}}\dots w_{\overline{0}} = w_mw_{\overline{n}}\dots w_{\overline{0}}$. By $\mathcal{S}_5$ we have $V_1V = W_1W$. We can suppose wlog that $W_1 = V_1W'$, i.e. we have:

$$V = W'W \tag{8}$$

Now let $V_2 = v_0\dots v_n$ and $W_2 = w_0\dots w_n$. We have $v_{id_n} = V_2V$ and $w_{id_n} = W_2W$. We will show that $W_2 = V_2W'$. This will conclude the proof as with Eq. 8 we have $v_{id_n} = V_2V = V_2W'W = W_2W = w_{id_n}$.

First note that Lemma 9 is valid in this context and therefore we have $w_0 = \epsilon \Rightarrow v_0 \in (t_2t_1)^*t_2$ and $v_0 = \epsilon \Rightarrow w_0 \in (t_1t_2)^*t_1$, as above we can consider that $v_0 = (t_2t_1)^{\alpha_0}t_2$ and $w_0 = (t_2t_1)^{\beta_0}$.

We consider two cases following Proposition 5:

1. $\forall i \in \{1,\dots,n\} : v_i = (t_1t_2)^{\alpha_i} \wedge w_i = (t_2t_1)^{\beta_i}$: Let write $\alpha = \alpha_0 + \dots + \alpha\ell - 1 + \alpha\ell + 1 + \dots + \alpha_n$ and $\beta_0 + \dots + \beta\ell - 1 + \beta\ell + 1 + \dots + \beta_n$ we have $V_1 = v_0\dots v_{\ell-1}v_{\ell+1}\dots v_n = (t_2t_1)^\alpha t_2$ and $W_1 = w_0\dots w_{\ell-1}w_{\ell+1}\dots w_n = (t_2t_1)^\beta$, therefore $W' = (t_2t_1)^{\alpha-\beta}t_2$. Moreover $V_2 = V_1(t_1t_2)^{\alpha_\ell}$ and $W_2 = W_2(t_1t_2)^{\alpha_\ell}$, as a result $W_2 = V_2W'$.

2. $\exists k \in \{1, \ldots, n\} \forall i \neq k : |v_i| = |w_i|$: By using the same construction as for Eq. 7 of Lemma 13, we can show that there exists α_k such that $W' = (t_2 t_1)^{a_k} t_2$ with $W_1 = V_1 W'$ and $W_2 = V_2 W'$.

□

6 A PSPACE algorithm for functionality

We now show how the pumping lemma for functionality can be used to decide functionality in PSPACE. It relies an NLOGSPACE algorithm for functionality of FSTs, which is a consequence of the following pumping argument by Schützenberger:

Theorem 4 (Schützenberger, 1975 [12]). *Let T be an* FST *with m states. If T is non-functionnal then there exists a word w of length at most $3 * m^2$ that admits two different outputs.*

As a consequence, we obtain:

Theorem 5. *Functionality of* FSTs *is decidable in* NLOGSPACE.

Proof. We give a CO-NLOGSPACE algorithm. The result follows as CO-NLOGSPACE = NLOGSPACE.

Note that each transition outputs a sequence of letters of bounded length, therefore one can bound polynomially the length of the two different outputs for a single input that witnesses non-functionality. Let us point out that two outputs differ either because one is a strict prefix of the other, or on a common position their letters differ. By a small trick and a new dummy symbol in the input alphabet, it is easy to reduce the first case to the second one with an augmentation of the FST of constant size.

We consider a non-deterministic algorithm for deciding non-functionality, operating as follows: one guesses a position i in the output where two outputs differ. Then using only logarithmic space, one can check that this guess is correct. At each step, this algorithm guesses itself one letter of the input and the two transitions of the two runs computing the two different outputs. Therefore at each step, this algorithm keeps two counters and the two states reached by the two runs so far. The first (resp. second) counter counts the length of the first (resp. second) output. When one of the outputs has reached position i, the algorithm stores the i-th letter of this output, and continue until the other output reaches the i-th position. At this point, the two runs are in two states p, q, and one just has to check whether the two letters at the i-th position are different. Finally, the algorithm checks whether the two runs can be continued into successful runs (from p and q) on the same input. This can be again done in non-deterministic logarithmic space.

By Schützenberger's Theorem, one can take $i \leq 3m^2$, and therefore the two counters are represented in logarithmic space in the size of the FST. □

We can now give a PSPACE algorithm for functionality. We devise a construction which given a VPT A, builds an FST B that simulates A for nested input words of small height. The height of the input word being polynomially bounded (Lemma 5), one can bound similarly the height of the stack of the VPT. Then, as runs cross only finitely many stacks, one can incorporate these stacks into a finite-control part, turning the VPT into an FST. This construction is correct in the following sense:

Proposition 6. *For all* VPT A *with* n *states, one can construct an* FST B *of exponential size wrt* n*, such that* $Dom(B) = \{u \in Dom(A) \mid h(u) \leq 8n^4\}$ *and for all* $w \in Dom(B)$*,* $B(w) = A(w)$*. Moreover,* A *is functional iff* B *is functional.*

The idea is to apply the NLOGSPACE algorithm of Theorem 5 on B. However, building this FST B of exponential size wrt to the size of the VPT A as the first step of an algorithm will not yield a PSPACE algorithm. Therefore, the construction of the transition rules of B has to be performed on-demand when such a transition is needed. Altogether, this gives a PSPACE algorithm for deciding functionality of VPTs.

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