GONZALO E. REYES AND HOUMAN ZOLFAGHARI

BI-HEYTING ALGEBRAS, TOPOSES AND MODALITIES

ABSTRACT. The aim of this paper is to introduce a new approach to the modal operators of necessity and possibility. This approach is based on the existence of two negations in certain lattices that we call bi-Heyting algebras. Modal operators are obtained by iterating certain combinations of these negations and going to the limit. Examples of these operators are given by means of graphs.

The aim of this paper is to present a new approach to modal operators based on the existence of two distinct negations on certain lattices that we call bi-Heyting algebras. Although these structures were introduced more than 70 years ago by Thoralf Skolem (cf. [1]), curiously enough nobody (as far as we know) put one and one together to obtain ... modal operators. This is what we do in this paper.

Our work is divided into two parts, consisting of two sections each. The first part is fairly elementary and the only background presupposed of the reader is some familiarity with the basic notions of lattice theory, including the notion of adjunction (or Galois connections) between pairs of order preserving maps (going in opposite directions) between posets. It is written in a leisurly manner so as to help such a reader along. The second part is much more advanced and presupposes some knowledge of Grothendieck toposes, in particular toposes of presheaves. Consequently, the style is terse and proofs are sometimes given in bare outline. The reason for this forced "co-habitation" of disparate parts is best explained in terms of doctrines or categorical counterparts of logical systems (see [4]). In fact, the two parts deal with the doctrine of propositional and higher order logic, respectively, of the modal operators discussed in this paper. The reader with no specialized knowledge of topos theory, but with some familiarity with the basic notions of category theory including functors and natural transformations, is advised to browse through the second part, replacing "bi-Heyting Grothendieck topos" everywhere by "category of contravariant functors on a poset P". In this case, the forcing relation boils down to the more familiar Kripke's forcing and the reader can specialize the forcing clauses for the modal operators to see what is involved.

Although we do not believe that it is up to an author to point out the philosophical significance of his or her work, perhaps we should call attention to some features of our work that may be relevant in some

Journal of Philosophical Logic **25**: 25–43, 1996.

philosophical discussions. First, we have provided precise mathematical structures in which our model operators are definable. Furthermore, the structures are general enough to avoid such overdeterminations as a priori choice of possible worlds (or situations), of accessibility relations between possible worlds (or situations), etc. to build the usual semantics. In our work, these appear as derived notions, when some further constraints are put on the mathematical structures (namely those of a topos). The choice of Grothendieck toposes as the fundamental structures for modelling modal logic has been motivated by the attempt to conceptualize the fundamental notion of *reference* and the interested reader may consult [10] for a philosophical analysis of that notion and a justification of such a choice.

Besides these general remarks on the broad significance of this work, there are specific features worth mentioning. It seems that both negations which are definable in a bi-Heyting algebra occur naturally in ordinary English, as the following example of J. Macnamara shows: compare the sentences

It is not false that p
It is false that not p.

It is clear that they do not say the same thing, the first suggests the possibility of p whereas the second points rather to its necessity. A plausible explanation may be given by interpreting the colloquial "it is false" as the Heyting negation and "not" as the co-Heyting supplement (in the sense of Section 1) of a bi-Heyting algebra. Under this interpretation, the sentences are indeed interpreted as the (first) possibility and the (first) necessity of p, respectively (see Section 2). A second feature: our system allows a rather neat distinction between contingency and necessity. The logic of contingent propositions turns out to be intuitionistic (or rather bi-intuitionistic), whereas that of necessary propositions turns out to be ordinary classical logic. Indeed, necessary propositions are the only ones that satisfy excluded middle. Thus, our work seems relevant to philosophical discussions by Aristotle, Lukasiewicz and others on the relations between excluded middle and determinism. One of us expects to come back to this question. The final feature that seems worth mentioning is connected with the lack of naturality of our modal operators. From a logical point of view, this means that the two formulas

$$\Box(\phi[t/x]) \quad \text{and} \quad (\Box\phi)[t/x]$$

are not equivalent in general. (A similar remark applies to the corresponding formulas obtained by substituing $\Diamond$ for $\Box$.) This feature seems relevant to discussions about *de re* and *de dicto* distinctions in modal logic, but we shall not elaborate here. On the other hand, if we restrict our interpretation to constant objects or kinds, these formulas turn out to be equivalent. For a philosophical discussion of the notion of kind see again [10].

We now proceed to describe the contents of each of the four sections. In the first we develop the basic theory of bi-Heyting algebras in so far as we need it for the purposes at hand and give examples, mainly from the theory of graphs. This section is largely expository and follows [7]. In the second we combine Heyting negation and co-Heyting supplement in two different ways, iterate and proceed to the limit to obtain operators of necessity ($\Box$) and possibility ($\Diamond$). We study their properties which again we illustrate with examples from graph theory. The third section deals with modal operators in bi-Heyting toposes, i.e., toposes for which the lattice of the subobjects of an object is a bi-Heyting algebra. Our previous results may be applied to define modal operators on the lattice of subobjects of every object of the topos. An important feature of these operators is their lack of functoriality, in contrast to the usual (Heyting) logical operators of a topos. In fact they are only lax functorial. In the fourth section we study these questions of functoriality and show that the modal operators are functorial on subobjects of constant objects. We end our paper by comparing the operators introduced in this paper with those of [5].

1. BI-HEYTING ALGEBRAS

In order to define our modal operators we need to introduce the notion of a bi-Heyting algebra. The notion of a Heyting algebra is well-known and has been extensively used, but its dual, although quite natural, is seldomly seen in the literature. For many years now, Lawvere has advocated its study in order to broaden our insight into the connections between logic and geometry (cf. [6] and [7]).

DEFINITION 1.1. A *Heyting algebra* is a bounded distributive lattice L with an "implication" operation $\rightarrow: L \times L \rightarrow L$ with the following property

$$x \leqslant y \rightarrow z \quad \text{iff} \quad x \wedge y \leqslant z$$

for all $x, y, z \in L$.

A *co-Heyting algebra* is a bounded distributive lattice L with a "subtraction" operation $\setminus : L \times L \rightarrow L$ with the following property

$$x \setminus y \leqslant z \quad \text{iff} \quad x \leqslant y \vee z$$

for all $x, y, z \in L$. Notice that L is a co-Heyting algebra iff the dual lattice L^0, obtained by reversing the order relation of L, is a Heyting algebra. The operation $\setminus$ in L is simply $\rightarrow$ in L^0.

A *bi-Heyting algebra* is a bounded distributive lattice that is both a Heyting and a co-Heyting algebra.

We shall often write equivalences in the form of Gentzen's rules. Thus the defining properties for $\rightarrow$ and $\setminus$ may be written as

$$\frac{x \leqslant y \rightarrow z}{x \wedge y \leqslant z} \qquad \frac{x \setminus y \leqslant z}{x \leqslant y \vee z}.$$

Each of these operations gives rise to a type of negation: $\neg x = x \rightarrow 0$ and ${\sim}x = 1 \setminus x$, where 0 and 1 are the bottom and top elements of the lattice, respectively. The first is the usual intuitionistic negation. We propose to call ${\sim}x$ the *supplement* of x and $\sim$ itself the *supplementary operation*.

We have the following defining properties

$$\frac{x \leqslant \neg y}{x \wedge y = 0} \qquad \frac{{\sim}x \leqslant y}{1 = x \vee y}.$$

These rules express the fact that $\neg x$ is the largest element disjoint from x, whereas ${\sim}x$ is the smallest element whose join with x gives the top element 1. Notice that $\sim$ of L is just the negation $\neg$ in L^0.

For our purposes we need to develop very little theory. The following simple facts will be used tacitly in the rest of the paper:

PROPOSITION 1.2. *In a Heyting algebra the negation operation $\neg$ is order reversing and satisfies $x \leqslant \neg\neg x$. In a co-Heyting algebra the supplementary operation $\sim$ is also order reversing and ${\sim}{\sim}x \leqslant x$.*

In most examples, the lattices are complete. In this case, we have further distributivity properties which are easily proved.

PROPOSITION 1.3. *In a complete Heyting algebra*

$$a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \wedge b_i).$$

In a complete co-Heyting algebra

$$a \vee \bigwedge_{i \in I} b_i = \bigwedge_{i \in I} (a \vee b_i).$$

EXAMPLES. (1) A Boolean algebra is a bi-Heyting algebra. Define $x \to y = c(x) \vee y$ and $x \setminus y = x \wedge c(y)$, where $c(\,)$ is the operation of Boolean complement. Notice that in this case, $\neg x = {\sim}x = c(x)$. Conversely, a bi-Heyting algebra such that $\neg x = {\sim}x$ for all x is automatically a Boolean algebra.

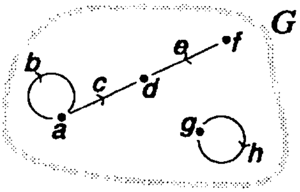
(2) Let X be a topological space. It is well-known that the lattice of open sets of X constitutes a Heyting algebra. We define $U \to V$ (for U and V opens of X) to be the interior of $c(U) \cup V$, where $c(\,)$ is the usual Boolean complement.

Dually, the closed sets of X constitutes a co-Heyting algebra by defining $F \setminus G$ (for F and G closed sets of X) to be the closure of $F \cap c(G)$.

(3) Let G be an oriented irreflexive multigraph. Then the lattice $P(G)$ of subgraphs of G is a bi-Heyting algebra. Although this can be checked directly, this is a consequence of a general result about presheaf toposes that will be established in Section 3. Since this is our main example, let us investigate it closer. G consists of a quadruple $(G_0, G_1, \delta_0, \delta_1)$, where G_0 and G_1 are the set of vertices and (oriented) edges, respectively, and $\delta_0, \delta_1\colon\ G_1 \to G_0$ are the functions that associate with every edge its source and target, respectively.

A *subgraph* X of G consists of subsets X_0, X_1 of G_0 and G_1, respectively, such that every edge in X_1 has its source and target in X_0.

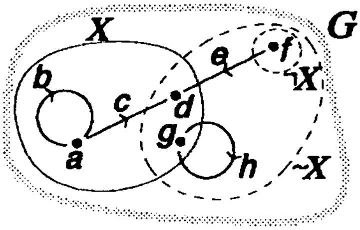
In the rest of the discussion, we will think of G as one set with two kinds of elements structured by the internal incidence relation given by the two maps δ_0 and δ_1. We shall use a single membership relation to simplify the notation, leaving implicit the type of each element. Take, for example, the following graph:



We write $G = \{a, b, c, d, e, f, g, h\}$, keeping in mind the relations $\delta_0(b) = a$, $\delta_1(b) = a$, $\delta_0(c) = a$, $\delta_1(c) = d$, etc. In this notation, a subgraph of a graph G is simply a subset of G closed under the operations of taking source and target of its arrows.

We can clearly take unions and intersections of a subgraphs, but what about complements? Taking the set-theoretical complement $c(X)$ of X will not do, since it is not a graph in general. We may get "problematic" edges, i.e., edges whose sources or targets are missing in $c(X)$. To make a graph we have two options: either disregard problematic edges or, alternatively, keep them and add their sources and targets. The first option leads to the Heyting negation $\neg X$, whereas the second leads to the co-Heyting supplement $\sim X$.

Take, for example, that subgraph $X = \{a, b, c, d, g\}$ of the graph above. The set-theoretical complement is $\{e, f, h\}$ which is not a subgraph, the problematic arrows being e and h. If we disregard them, we obtain $\neg X = \{f\}$, the largest subgraph disjoint from X. On the other hand, if we keep them and add the missing sources and targets, namely d and g we obtain $\sim X = \{d, e, f, g, h\}$, the smallest subgraph whose union with X gives the whole graph G.



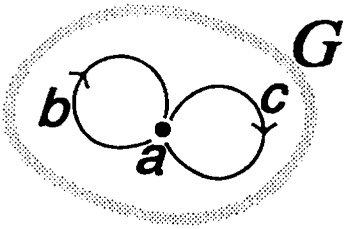
In a co-Heyting algebra, an element and its supplement need not be disjoint. We define the *boundary* $\partial(x)$ of an element x to be $\partial(x) = x \wedge \sim x$. Let us compute boundaries in our examples. In the case of a Boolean algebra, the boundary of any element is 0 and, in fact, this property characterizes Boolean algebras among co-Heyting algebras. In the case of closed sets of a topological space X, $\partial(F)$ is precisely the topological boundary of F. Finally, in the case of a graph G, the boundary $\partial(X)$ of a subgraph X is the subgraph whose set of points are connected to the outside of X. For instance, in the last example, $\partial(X) = \{d, g\}$.

As Lawvere has emphasized, it is precisely the fact that we can recapture the geometric notion of boundary that makes co-Heyting algebras so interesting. In fact this boundary operation is very natural and many of its properties can be guessed by just looking at Venn or Euler diagrams. For instance the diagram of the intersection of two ovals suggests the following result (cf. [7]) that turns out to be correct and whose proof is left to the interested reader.

PROPOSITION 1.4. *Let L be a co-Heyting algebra. Then*

1. $\partial(x \wedge y) = \partial(x) \wedge y \vee x \wedge \partial(y)$
2. $\partial(x \wedge y) \vee \partial(x \vee y) = \partial(x) \vee \partial(y)$.

REMARK 1.5. Recall that a Heyting algebra satisfies the *De Morgan's law for* $\neg$ if $\neg(x \wedge y) = \neg x \vee \neg y$. Dually, a co-Heyting algebra satisfies the *De Morgan's law for* $\sim$ if $\sim(x \vee y) = \sim x \wedge \sim y$. Although it is easy to give examples of bi-Heyting algebras which fail to satisfy both laws, it is rather surprising to notice that they need not be equivalent for these algebras. Take, for example, the following graph:



G is a graph with two loops and one vertex. The lattice L of its subgraphs is a bi-Heyting algebra satisfying the De Morgan's rule for $\neg$. This is easy to check directly or by using the fact that the De Morgan's rule for $\neg$ is equivalent to the validity of $\neg\neg x \vee \neg x = 1$, in other words, to the condition that $\neg x$ is complemented for every x. The negated elements are just G and $\emptyset$, which are obviously complemented. On the other hand, L does not satisfy the De Morgan's law for $\sim$, as shown by taking $x = \{a, b\}$ and $y = \{a, c\}$.

2. MODAL OPERATORS ON BI-HEYTING ALGEBRAS

In this section we show how to build modal operators of necessity ($\Box$) and possibility ($\Diamond$) in bi-Heyting algebras that are *σ-complete* in the sense that countable suprema and infima exist satisfying rules

$$\frac{b \leqslant \bigwedge_n a_n}{b \leqslant a_n \text{ for all } n} \qquad \frac{\bigvee_n a_n \leqslant b}{a_n \leqslant b \text{ for all } n}.$$

The reader will observe that we do not really need bi-Heyting algebras, but only bounded distributive lattices with negation ($\neg$) and supplement ($\sim$). Nevertheless, all the significant examples have both implication ($\to$) and subtraction ($\backslash$) and hence we shall keep the bi-Heyting context.

Let L be a bi-Heyting algebra.

DEFINITION 2.1.

$$\Box_0 = \Diamond_0 = Id,$$
$$\Box_{n+1} = \neg\sim\Box_n, \qquad \Diamond_{n+1} = \sim\neg\Diamond_n.$$

In other words, $\Box_n$ is obtained by iterating n times the composition $\neg\sim$, whereas $\Diamond_n$ is obtained by iterating n times the dual composition $\sim\neg$. Notice that we have $\Box_{n+1} = \Box_n\neg\sim$, and $\Diamond_{n+1} = \Diamond_n\sim\neg$.

Let us recall the notion of adjunction or Galois connections between pairs of order preserving operators. We say that α: $L \to L$ *is left adjoint to* β: $L \to L$ (and write $\alpha \dashv \beta$) when for all $x, y \in L$ $\alpha(x) \leqslant y$ iff $x \leqslant \beta(y)$. We say that β *is right adjoint of* α iff α is left adjoint to β.

PROPOSITION 2.2. 1. $\Box_n$ *and* $\Diamond_n$ *are order preserving, for all* n.

2. $\Box_{n+1} \leqslant \Box_n \leqslant Id \leqslant \Diamond_n \leqslant \Diamond_{n+1}$ *for all* n.

3. $\Diamond_n \dashv \Box_n$ *for all* n.

Proof. Let us first prove the following:

LEMMA 2.3. 1. *For every* $x \in L$, $\neg x \leqslant {\sim} x$.

2. ${\sim}\neg \dashv \neg{\sim}$.

Proof. Intuitively this must be true, since ${\sim}x$ must contain not only $\neg x$ but also what is "between" x and ${\sim}x$ to "add up" to 1 when joined with x. A formal proof proceeds as follows:

$$\dfrac{\dfrac{{\sim}x \leqslant y}{1 \setminus x \leqslant y}}{1 \leqslant x \vee y.}$$

Taking the meet with $\neg x$ and using distributivity we get

$$\neg x \leqslant x \wedge \neg x \vee y \wedge \neg x = y \wedge \neg x.$$

Hence $\neg x \leqslant y$ and this concludes the proof of 1.

To show 2 we verify the following equivalences

$$\begin{array}{c} {\sim}\neg x \leqslant y \\ \hline 1 \leqslant \neg x \vee y \\ \hline {\sim}y \leqslant \neg x \\ \hline {\sim}y \wedge x \leqslant 0 \\ \hline x \leqslant \neg{\sim}y. \end{array}$$

Proof (of Proposition 2.2). (1) is a consequence of Proposition 1.2.

(2) First notice the following consequence of the lemma $\neg{\sim}x \leqslant x \leqslant {\sim}\neg x$. Indeed, $\neg({\sim}x) \leqslant {\sim}({\sim}x)$ by the lemma. On the other hand, ${\sim}{\sim}x \leqslant x$. The second inequality is proved as follows: $x \leqslant \neg\neg x$, but $\neg(\neg x) \leqslant {\sim}(\neg x)$ by the lemma.

(3) Follows from part 2 of the lemma by iterating (recall that adjoints compose).

From now on, we assume that L is a σ*-complete* bi-Heyting algebra.

DEFINITION 2.4.

$$\Box a = \bigwedge_n \Box_n a,$$
$$\Diamond a = \bigvee_n \Diamond_n a.$$

Let us remark that $\Box$ in L is $\Diamond$ in L^0 and vice-versa, a fact that will simplify some of the following proofs.

PROPOSITION 2.5. *Let L be a σ-complete bi-Heyting algebra. The modal operators $\Box$ and $\Diamond$ defined above have the following characterizations:*

$\Box a$ is the largest complemented x such that $x \leqslant a$.
$\Diamond a$ is the smallest complemented x such that $a \leqslant x$.

Proof. This means that these operators have the following properties:

1. $\Box$ and $\Diamond$ are order preserving,
2. $\Box a \leqslant a \leqslant \Diamond a$.
3. $\Box a$ and $\Diamond a$ are complemented.
4. If a is complemented, then $\Box a = \Diamond a = a$.

The first two are immediate from Proposition 2.2.

To show 3, we notice the following equivalences:

$$\begin{array}{l}
b \leqslant \Box_1 \Box a \\ \hline
\Diamond_1 b \leqslant \Box a \\ \hline
\Diamond_1 b \leqslant \Box_n a \text{ for all } n \\ \hline
b \leqslant \Box_1 \Box_n a \text{ for all } n \\ \hline
b \leqslant \Box_{n+1} a \text{ for all } n \\ \hline
b \leqslant \Box a
\end{array}$$

In other words, $\Box a \leqslant \neg{\sim}\Box a$ and this implies that $\Box a \wedge {\sim}\Box a = 0$, i.e., $\Box a$ is complemented. The corresponding property for $\Diamond$ is dual.

The proof of 4 proceeds as follows: a is complemented precisely when ${\sim}a = \neg a$. In this case $a \leqslant \neg\neg a = \neg{\sim}a \leqslant a$, i.e., $a = \Box_1 a$. By induction, $a = \Box a$. Dually for $\Diamond$.

PROPOSITION 2.6. *Let L be a bounded distributive lattice and $\Box$ and $\Diamond$ two operators such that:*

$\Box a$ is the largest complemented x such that $x \leqslant a$
$\Diamond a$ is the smallest complemented x such that $a \leqslant x$.

Then these operators have the following properties:

1. $\Box$ *and* $\Diamond$ *are order preserving.*
2. $\Box a \leqslant a \leqslant \Diamond a$.
3. $\Box a$ *and* $\Diamond a$ *are complemented.*
4. *If* a *is complemented, then* $\Box a = \Diamond a = a$.
5. $\Box^2 = \Box$ *and* $\Diamond^2 = \Diamond$.
6. $\Diamond \dashv \Box$.
7. *If* L *is a Heyting algebra, then* $\Diamond a = \neg\Box\neg a$.
8. *If* L *is a co-Heyting algebra, then* $\Box a = \sim\Diamond\sim a$.

Proof. 1–5 are immediate.

6. From $\Diamond a \leqslant b$ we obtain $\Box\Diamond a \leqslant \Box b$. By 3 and 4, $\Diamond a \leqslant \Box b$ and 2 implies that $a \leqslant \Box b$. Conversely, from $a \leqslant \Box b$ we obtain $\Diamond a \leqslant \Diamond\Box b$. By 3 and 4, $\Diamond a \leqslant \Box b$ and 2 implies that $\Diamond a \leqslant b$.

7. From 2, $\Box\neg a \leqslant \neg a$ or, equivalently, $a \leqslant \neg\Box\neg a$. From 1 it follows that $\Diamond a \leqslant \Diamond\neg\Box\neg a$. Since $\Box\neg a$ is complemented (by 3), so is $\neg\Box\neg a$ and this implies that $\Diamond a \leqslant \neg\Box\neg a$, using 4. To show the inclusion in the other direction, we start from $a \leqslant \Diamond a$. This implies that $\neg\Diamond a \leqslant \neg a$. Using 1, once again, $\Box\neg\Diamond a \leqslant \Box\neg a$. But $\Diamond a$ is complemented (by 3) and hence so is $\neg\Diamond a$. Therefore $\neg\Diamond a \leqslant \Box\neg a$ and this implies that $\neg\Box\neg a \leqslant \Diamond a$, since everything is complemented.

8. This is dual to 7.

Since $\Diamond$ is definable in terms of $\Box$ in the case of a Heyting algebra, we can ask for an axiomatization using $\Box$ only and the following answers this question.

PROPOSITION 2.7. *Let* $\Box$ *be an operator on a Heyting algebra satisfying the following axioms:*

1. $\Box$ *is order preserving.*
2. $\Box a \leqslant a$.
3. $\Box a$ *is complemented.*
4. *If* a *is complemented, then* $\Box a = a$.

Then there is a unique operator $\Diamond$ *such that* 1–7 *of the previous proposition are satisfied.*

Proof. We define $\Diamond a = \neg\Box\neg a$.

REMARK 2.8. Notice that 1–4 mean that $\Box a$ is the largest complemented x such that $x \leqslant a$ and this means that such an operator, if it exists, is necessarily unique.

Dually, we have the following result about co-Heyting algebras:

PROPOSITION 2.9. *Let $\Diamond$ be an operator on a co-Heyting algebra satisfying the following axioms:*

1. *$\Diamond$ is order preserving.*
2. *$a \leqslant \Diamond a$.*
3. *$\Diamond a$ is complemented.*
4. *If a is complemented, then $\Diamond a = a$.*

Then there is a unique operator $\Box$ such that 1–6 *and* 8 *of the Proposition* 2.6 *are satisfied.*

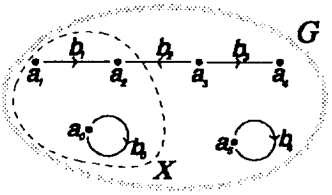
Proof. Define $\Box a = {\sim}\Diamond{\sim}a$ and proceed as in the previous proposition.

REMARK 2.10. In the particular case where we have one of the De Morgan's rules our sequence $\Box_0, \Box_1, \Box_2$, etc. stabilizes after the first step. In fact we have the trivially proved

PROPOSITION 2.11. *If L is a bi-Heyting algebra satisfying the De Morgan's rule for $\neg$, then $\Box = \neg{\sim}$ and $\Diamond = \neg\neg$. Dually, if L satisfies the De Morgan's rule for ${\sim}$, then $\Box = {\sim}{\sim}$ and $\Diamond = {\sim}\neg$.*

EXAMPLES OF GRAPHS. The modal operators $\Box$ and $\Diamond$ as defined in this section have a very nice geometrical interpretation in the case of the bi-Heyting algebras of subgraphs of a given graph.

Let us begin with the following very simple graph G and its subgraph X:



To compute $\Diamond X$ we proceed by steps: $\neg X = \{a_3, b_3, a_4, b_4, a_5\}$, the largest subgraph disjoint from X. Hence $\Diamond_1 X = {\sim}\neg X = X \cup \{b_2, a_3\}$, the smallest subgraph of G whose union with $\neg X$ gives the whole graph. So we have moved one step forward and captured the "layer" of elements connected with X. Going further, we obtain $\Diamond_2 X = {\sim}\neg\Diamond_1 X = X \cup \{b_2, a_3, b_3, a_4\}$ and we have captured the next "layer". We have now reached stability since $\Diamond_2 X$ is complemented. Indeed, $\neg\Diamond_2 X = \{a_5, b_4\}$ and so $\Diamond_3 X = {\sim}\neg\Diamond_2 X = {\sim}\{a_5, b_4\} = \Diamond_2 X$. Therefore $\Diamond X = \Diamond_2 X = X \cup \{b_2, a_3, b_3, a_4\}$. We see that every arrow or vertex that is connected to X through a path will be in $\Diamond X$ after a finite number of iterations.

Once that we have captured these paths entirely, we obtain a subgraph which has no boundary (and no arrow going out). This subgraph is the possibility of X.

To compute $\Box X$, we iterate in the opposite order: going from X to $\Box_1 X = \neg{\sim}X$, we lose the vertex a_2 and hence the arrow b_1 ending in this vertex. If we repeat the process, going from here to $\Box_2 X = \neg{\sim}\Box_1 X$ we loose the vertex a_1. We end up with $\Box_2 X = \{a_0, b_0\}$ which is complemented. This shows that $\Box X = \{a_0, b_0\}$. As Steve Schanuel has suggested, this process is similar to peeling off the layers of an onion.

Going over to a general graph, $\Diamond X$ consists of the elements that can be reached from X through some path, whereas $\Box X$ consists of those elements in X that are not connected to the outside. Both of these subgraphs are complemented sums of connected components.

3. BI-HEYTING TOPOSES

The aim of this section is to apply the results of the previous section to define modal operators on a rather extended class of Grothendieck toposes that we proceed to describe

DEFINITION 3.1. A *bi-Heyting topos* is a topos for which the Heyting algebra of subobjects of any object is a co-Heyting algebra (and hence a bi-Heyting algebra).

Bi-Heyting toposes were introduced by [7] to study intrinsic boundaries in terms of the co-Heyting supplementary operation $\sim$. An important feature of this operation is its lax naturality (rather than strict naturality as in the case of the Heyting operations). This feature will be inherited by our modal operators.

Lawvere (cf. [7]) pointed out that any presheaf topos and, more generally, any essential subtopos of a presheaf topos is bi-Heyting.

We have the following characterization of bi-Heyting toposes that give these results as corollaries

PROPOSITION 3.2. *A topos $\mathcal{E}$ is bi-Heyting iff there is a Boolean topos $\mathcal{B}$ and a surjective geometric morphism Γ: $\mathcal{B} \to \mathcal{E}$ such that the canonical δ: $\Omega_{\mathcal{E}} \to \Gamma(\Omega_{\mathcal{B}})$ has a left lax adjoint.*

Proof. We recall (cf. [11]) that the last condition means that for each $E \in \mathcal{E}$

$$\Delta_E\colon\ Sub_{\mathcal{E}}(E) \longrightarrow Sub_{\mathcal{B}}(\Delta E)$$

has a left adjoint $\Pi_E \dashv \Delta_E$ (where $\Delta \dashv \Gamma$).

As we pointed out in [11], $(\Pi_E)_{E\in\mathcal{E}}$ is only lax natural in the sense that for all $f\colon E \to F \in \mathcal{E}$,

$$\Pi_E f^* \leqslant f^*\Pi_F.$$

Assume now that such a $\mathcal{B} \to \mathcal{E}$ exists. Define a co-Heyting structure on $Sub_{\mathcal{E}}(E)$ as follows:

$$F \setminus G = \Pi_E(\Delta_E(F) \setminus \Delta_E(G)).$$

To show the other direction, we use Theorem 6.2.1 of [9] to obtain a surjective geometric morphism $\mathcal{B} \to \mathcal{E}$ such that Δ preserves stable infima of subobjects of any object. But any infima is stable in a bi-Heyting algebra in the sense that

$$a \vee \left(\bigwedge_i b_i\right) = \bigwedge_i (a \vee b_i)$$

as shown in Proposition 1.3.

Therefore, each Δ_E preserves all infima of subobjects and this is equivalent to the existence of $\Pi_E \dashv \Delta_E$.

COROLLARY 3.3. *Any presheaf topos is bi-Heyting.*

Proof. The geometric morphism $Set^{\|C\|} \to Set^{C^0}$ induced by the functor $\|C\| \to C$ from the discrete category of objects of C into C is clearly a surjection from a Boolean topos.

Notice that the graph examples fall in this category, since the topos of (irreflexive) multigraphs may be represented as Set^{C^0}, for $C = (\bullet \rightrightarrows \bullet)$.

For further reference, we give the following explicit description of the forcing relation for supplement in a presheaf topos (cf. [7]).

PROPOSITION 3.4. *Let $E \in Set^{C^0}$, $C \in C$, $e \in E(C)$. Then $C \Vdash {\sim}\phi[e]$ iff $\exists f\colon C \to C' \in C\ \exists e' \in E(C')$ such that $E(f)(e') = e$ and $C' \nVdash \phi[e']$.*

4. MODAL OPERATORS ON BI-HEYTING TOPOSES

Since the lattice of subobjects of any object E of a bi-Heyting topos $\mathcal{E}$ is a complete bi-Heyting algebra (and hence in particular a σ-*algebra*), our results of Section 2 allow us to define modal operators

$$\Box_E, \Diamond_E\colon Sub_{\mathcal{E}}(E) \longrightarrow Sub_{\mathcal{E}}(E)$$

satisfying

1. $\Box_E \leqslant Id \leqslant \Diamond_E$.
2. $\Box_{E^2} = \Box_E$, $\Diamond_{E^2} = \Diamond_E$.
3. $\Diamond_E \dashv \Box_E$.
4. $\Box_E(P) = P$ iff $P \hookrightarrow E$ is complemented.

Recall from Proposition 2.6 that we have the following characterizations of these operators:

$\Box_E(P)$ is the largest complemented subobject of E contained in P
$\Diamond_E(P)$ is the smallest complemented subobject of E containing P.

Notice that, contrary to the previous approaches of [10] and [2], these operators are defined for predicates of *all* objects of the topos $\mathcal{E}$ and not only for predicates of constant objects of the form $\Delta(S)$.

It follows from this that $(\Box_E)_E$ cannot be natural in E. Indeed, we have the following:

PROPOSITION 4.1. *Let $\mathcal{E}$ be any topos and let $\Omega \in \mathcal{E}$ be the sub-object classifier. Assume that $\Box$: $\Omega \to \Omega$ is an operator satisfying $\Box \leqslant Id$ and $\Box T = T$. Then $\Box = Id$.*

Proof. Let p: $X \to \Omega$ be an X-element of Ω. From the assumptions, $\Box \circ p = T$ iff $p = T$. In other words, $T^*(\Box \circ p) = X$ iff $T^*(p) = X$, where $T^*(\ldots)$ is the pull-back of $\ldots$ along T. Applying this to the generic $p = Id_\Omega$, this says that $\Box$ and Id_Ω classify the same subobject of Ω, namely T. Therefore $\Box = Id_\Omega$.

This proposition is a particular case of a more general one for which an elementary proof may be given:

PROPOSITION 4.2. *Let C be a category with pull-backs and let $(\Box_C)_C$ be a natural family of operators $\Box_C$: $Sub(C) \to Sub(C)$ satisfying $\Box_C \leqslant Id_C$ and $\Box_C(T_C) = T_C$ (where T_C is the largest subobject of C). Then $\Box_C = Id_C$ for every $C \in C$.*

Proof. Let i: $A \hookrightarrow X$ be a subobject of X. By naturality of $(\Box_C)_C$, $i^*\Box_X = \Box_A i^*$. Recalling that the pull-back of a subobject of X along i is just the intersection with A, we obtain

$$\begin{aligned} \Box_X(A) &= A \wedge \Box_X(A) \\ &= i^*\Box_X(A) \\ &= \Box_A i^*(A) \\ &= \Box_A(T_A) \\ &= A. \end{aligned}$$

On the other hand, $\Box_E$ and $\Diamond_E$ are lax natural in the following sense:

PROPOSITION 4.3. *Let* $f\colon\ E \to F \in \mathcal{E}$. *Then*

$$f^*\Box_F \leqslant \Box_E f^*,$$

$$\Diamond_E f^* \leqslant f^*\Diamond_F.$$

Proof. In fact, if $Q \in P(F)$, $f^*\Diamond_F(Q)$ is complemented (since so is $\Diamond_F$), and contains f^*Q (since $Q \leqslant \Diamond_F(Q)$). Therefore it contains the smallest complemented subobject of E containing f^*Q, namely $\Diamond_F f^*Q$. The other inclusion (for $\Box$) is similar.

To see a concrete example of this inequality take the inclusion of graphs

The subgraph $\{a\}$ is not complemented in F and $\Diamond\{a\} = F$. Taking the pull-back we obtain $f^*\Diamond\{a\} = E$. On the other hand, $f^*(\{a\}) = \{a\}$ is complemented in E and hence $\Diamond f^*(\{a\}) = \{a\}$.

As we pointed out already, an outstanding example of a bi-Heyting topos is a presheaf topos, i.e., a topos of the form Set^{C^0}, where C is a small category. In this case we can give explicit forcing conditions for our modal operators:

PROPOSITION 4.4. *Let* $e \in E(C)$. *Then*

1. $C \Vdash (\Box_n)_E\phi[e]$ *iff for any chain in* C

$$\begin{tikzcd}[column sep=small]
C = C_0 \arrow[dr] & & C_2 \arrow[dl] \arrow[dr] & & C_{2n-2} \arrow[dl] \arrow[dr] & & C_{2n} \arrow[dl] \\
& C_1 & & \cdots & & C_{2n-1} &
\end{tikzcd}$$

and any elements $e_i \in E(C_i)$ $(0 \leqslant i \leqslant 2n)$ *such that* $e_0 = e$, $E(C_{2k} \to C_{2k-1})(e_{2k-1}) = e_{2k}$, $E(C_{2k} \to C_{2k+1})(e_{2k+1}) = e_{2k}$, *we have* $C_{2n} \Vdash \phi[e_{2n}]$.

2. $C \Vdash (\Diamond_n)_E\phi[e]$ *iff there is a chain in* C

$$\begin{tikzcd}[column sep=small]
C = C_0 \arrow[dr] & & C_2 \arrow[dl] \arrow[dr] & & C_{2n-2} \arrow[dl] \arrow[dr] & & C_{2n} \arrow[dl] \\
& C_1 & & \cdots & & C_{2n-1} &
\end{tikzcd}$$

and there are elements $e_i \in E(C_i)$ $(0 \leqslant i \leqslant 2n)$ *such that* $e_0 = e$, $E(C_{2k} \to C_{2k-1})(e_{2k-1}) = e_{2k}$, $E(C_{2k} \to C_{2k+1})(e_{2k+1}) = e_{2k}$, *and* $C_{2n} \Vdash \phi[e_{2n}]$.

Proof. We just prove 2, the other being similar. It is enough to check it for $n = 1$ and then iterate. But $C \Vdash \sim\neg\phi[e]$ iff there is some $C \to C_1 \in \mathcal{C}$ such that $C_2 \Vdash \phi[E(C_2 \to C_1)(e_1)]$. Let $e_2 = E(C_2 \to C_1)(e_1)$.

COROLLARY 4.5. *Let* $e \in E(C)$. *Then*

1. $C \Vdash \Box\phi[e]$ *iff for every* $C' \in \Pi_0(C)$ *and for every* $e' \in E(C')$ *such that* $\eta_C(e) = \eta_{C'}(e')$, $C' \Vdash \phi[e']$.
2. $C \Vdash \Diamond\phi[e]$ *iff there is some* $C' \in \Pi_0(C)$ *and there is some* $e' \in E(C')$ *such that* $\eta_C(e) = \eta_{C'}(e')$ *and* $C' \Vdash \phi[e']$,

where $\eta_C\colon E(C) \to \operatorname{colim} E$ *is the canonical map and* $\Pi_0(C)$ *is the connected component of* $C \in \mathcal{C}$.

Proof. Immediate from the previous proposition and the explicit description of $\operatorname{colim} E = \coprod_{C\in\mathcal{C}} E(C)/\sim$, where $\sim$ is the smallest equivalence relation R such that $(e, E(C' \to C)(e)) \in R$.

Although the modal operators are only lax natural, they turn out to be natural on constant objects as one can easily check starting from the explicit description of the forcing relation.

COROLLARY 4.6. *The modal operators on* $Set^{\mathcal{C}^0}$ *are functorial on constant objects of the form* ΔS, *i.e., if* $S \to T \in Set$, *then*

$$\Box_{\Delta T}(\Delta f)^*Q = \Delta f^*\Box_{\Delta S}Q,$$

$$\Diamond_{\Delta T}(\Delta f)^*Q = \Delta f^*\Diamond_{\Delta S}Q.$$

In fact, a more general result holds

PROPOSITION 4.7. *Let* $\Gamma\colon \mathcal{E} \to Set$ *be a bi-Heyting Grothendieck topos. Then the substraction operator is natural on constant objects* ΔS *(where* $S \in Set$ *and* $\Delta \dashv \Gamma$*).*

Proof. Given $F \hookrightarrow \Delta S$, define F_s $(s \in S)$ as the result of pulling-back F along $\Delta s : 1 = \Delta 1 \to \Delta S$ and let Fam/S be the bi-Heyting algebra of the families $(F_s)_{s\in S}$ with point-wise operations. The map

$$P(\Delta S) \longrightarrow Fam/S$$

which sends F into $(F_s)_{s\in S}$ is easily seen to be an isomorphism of bi-Heyting algebras (essentially because Δs^* preserves $\vee$, $\wedge$, 0, 1). Furthermore it is clearly functorial: given $f\colon T \to S \in Set$, the following diagram commutes

$$\begin{tikzcd}
P(\Delta S) \arrow[d, "(\Delta f)^*"'] & \simeq & Fam\,/S \arrow[d, "f^*"] \\
P(\Delta T) & \simeq & Fam\,/T
\end{tikzcd}$$

where f^* is the operation of re-indexing: $f^*(F_s)_{s\in S} = (F_{f(t)})_{t\in T}$. Since f^* is obviously a co-Heyting morphism, so is $(\Delta f)^*$. In particular, for $F, G \hookrightarrow \Delta S$, $(\Delta f)^*(F \setminus G) = (\Delta f)^*(F) \setminus (\Delta f)^*(G)$ and this concludes the proof.

COROLLARY 4.8. *Under the same assumptions (of the previous propositions), $\Box$ and $\Diamond$ are natural on constant objects.*

We saw that $\Box$ cannot be functorial on all objects, unless it reduces to the identity operator. More precisely, we have

PROPOSITION 4.9. *For a bi-Heyting topos $\mathcal{E}$ the following conditions are equivalent:*

1. *$\sqsupset$ is functorial.*
2. *$\sqsupset = Id$.*
3. *$\mathcal{E}$ is Boolean.*
4. *$\mathcal{C}$ is a groupoid in case that $\mathcal{E} = Set^{\mathcal{C}^{0}}$.*

Proof. $1 \to 2$ is just Proposition 4.1. On the other hand, $2 \to 3$ is obvious: any subobject $A \hookrightarrow X$ is complemented, since $A = \Box(A)$. It is clear that $3 \to 1$ since $\sim\, = \neg$ in a Boolean topos. Finally the equivalence of 3 and 4 is well-known and can be easily checked.

Nevertheless this does not apply to $\Diamond$, although $\Diamond \dashv \sqsupset$. In fact, we have the following result about the naturality of the possibility operator

PROPOSITION 4.10. *For a bi-Heyting topos $\mathcal{E}$ the following conditions are equivalent:*

1. *$\Diamond$ is functorial.*
2. *The De Morgan's rule for Heyting negation holds in the topos.*
3. *$\Diamond = \neg\neg$.*
4. *The Ore condition holds in the category $\mathcal{C}$ in case that $\mathcal{E} = Set^{\mathcal{C}^{0}}$.*

Proof. To show that $1 \to 2$, assume that $\Diamond$ is functorial. Then so is $\neg\Diamond\neg$. But this operator satisfies the conditions of Proposition 4.1 and hence $\neg\Diamond\neg = Id$. But $\neg\Diamond\neg p = p$ implies that $\Diamond\neg p = \neg p$, since $\Diamond q$ is complemented for all q. Therefore $\neg p$ is complemented for every p and this is equivalent to the De Morgan's law. We know by Proposition 2.11 that 2 implies 3. Now 3 implies 1 is clear, since $\neg$ is functorial. Finally, the equivalence between 2 and 4 is well-known (cf. [3]).

We finish our paper by comparing the modal operators $\Box$ and $\Diamond$ of [5] with those introduced in this paper, $\Box_{RZ}$ and $\Diamond_{RZ}$, when restricted to constant objects

PROPOSITION 4.11. *Let* Γ: $\mathcal{E} \to Set$ *be a bi-Heyting Grothendieck topos. Then the following are equivalent:*

1. Γ *is connected.*
2. $\Box = \Box_{RZ}$ *on predicates of constant objects.*
3. $\Diamond = \Diamond_{RZ}$ *on predicates of constant objects.*

Proof. First notice that since Set is a Boolean topos, the operators $\Box$ and $\Diamond$ of [5] are definable. Furthermore, notice that if Γ is connected and $F \hookrightarrow \Delta S$ is complemented, then $F \simeq \Delta S' \hookrightarrow \Delta S$, for some $S' \hookrightarrow S$. In fact, considering the isomorphism of Proposition 4.6, $P(\Delta S) \to Fam\,/S$, we conclude that F_s is complemented. By connectedness, $F_s \simeq 1$ or $F_s \simeq \emptyset$. Letting $S' = \{s \in S\colon\ F_s \simeq 1\}$, one easily checks that $F \simeq \Delta S'$.

Let us return to the proof. By adjunction, 2 and 3 are equivalent and we turn to

1 implies 2: Let $F \hookrightarrow \Delta S$ be given. Then $\Box F = \Delta S' \hookrightarrow \Delta S$ (cf. [11]) and so $\Box F$ is a complemented subobject of ΔS. Hence $\Box_{RZ}\Box F = \Box F$.

On the other hand, $\Box_{RZ} F \hookrightarrow \Delta S$ is complemented and hence $\Box_{RZ} F = \Delta S' \hookrightarrow \Delta S$ for some $S' \hookrightarrow S$, as we just proved. But $\Box\Delta S' = \Delta S' \hookrightarrow \Delta S$ by functoriality of $\Box$ and this shows that $\Box\Box_{RZ} F = \Box_{RZ} F$. Putting together these relations between the two modal operators we conclude 1.

2 implies 1: Let $1 = U \vee V$ with $U \wedge V = \emptyset$. Then $\Box_{RZ} U = U \hookrightarrow 1$. By hypothesis, $\Box U = \Box_{RZ} U$ and since $\Box U = \Delta S' \hookrightarrow \Delta 1 = 1$ for some $S' \hookrightarrow 1$, this shows that $U \simeq \Delta S' \hookrightarrow 1$. But $S' = 1$ or $S' = \emptyset$ and this implies that $U \simeq 1$ or $U \simeq \emptyset$.

ACKNOWLEDGEMENTS

This paper was presented in talks given at Montréal and Bloomington (Indiana) and we are grateful to the people who attended these talks for their comments. We are also grateful to an anonymous referee for helpful comments and suggestions which resulted in a simplification of the exposition. The first author was partially supported by an individual grant from the Canada's National Science and Engineering Research Council (NSERC).

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Department of Mathematics,
Université de Montréal,
C.P. 6128 Succursale A,
Montréal H3C 3J7,
Canada