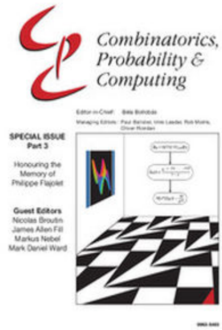

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Ramsey Classes and Homogeneous Structures

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Dedicated to the memory of Walter Deuber

We present a programme of characterizing Ramsey classes of structures by a combination of the model theory and combinatorics. In particular, we relate the classification programme of countable homogeneous structures (of Lachlan and Cherlin) to the classification of Ramsey classes. As particular instances of this approach we characterize all Ramsey classes of graphs, tournaments and partial ordered sets. We fully characterize all monotone Ramsey classes of relational systems (of any type). We also carefully discuss the role of (admissible) orderings which lead to a new classification of Ramsey properties by means of classes of order-invariant objects.

1. Introduction: Ramsey classes

It is often said by students (even good ones) that discrete mathematics and combinatorics are difficult to learn mainly because they seem to be a maze of examples and particular tricks, and that these fields lack 'theories' in the classical (and thus usual) sense. That view is shared by many mathematicians too (see [6] for nice comments related to this problem). Of course, such a view cannot be absolutely true, and several 'counterexamples' suggest themselves. It is our belief that Ramsey Theory (see, *e.g.*, [8, 18]) is one such example which would be listed by many.

Yet this area was certainly not created as a theory, but rather it has developed into one owing to the intensive research of past three decades. This theory is active, as we want to demonstrate on a particular (yet, we believe, central) area of Ramsey Theory devoted to the structural aspects of Ramsey Theory and to Ramsey classes in particular.

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This paper is organized as follows:

1 Ramsey classes
2 subobjects
3 admissible orderings
4 amalgamations
5 classification of Ramsey classes
6 examples
7 concluding remarks.

We are mostly concerned with finite sets. Given a set X and a natural number k, we denote by $\binom{X}{k}$ the set of all k-element subsets of X. Using this notation the Finite Ramsey Theorem takes the following form.

Theorem 1.1. *For every choice p,k,n of natural numbers, there exists N with the following property. If X is a set of size N and $\binom{X}{p} = \mathscr{A}_1 \cup \mathscr{A}_2 \cup \cdots \cup \mathscr{A}_k$ is any partition, then there exists $i, 1 \leqslant i \leqslant k$ and $Y \subseteq X, |Y| \geqslant n$ such that $\binom{X}{p} \subseteq \mathscr{A}_i$.*

Undoubtedly this is formally a complicated statement calling for a special terminology. Some of it is now standard: we speak about colourings (instead of partitions), sets $\mathscr{A}_i$ are usually called colour classes, and the set Y is called homogeneous. This economy culminated in the invention of a special symbol (*partition arrow*) $N \longrightarrow (n)^p_k$ for the core part of Theorem 1.1 and with the definition of A-Ramsey properties and Ramsey classes. In this concise terminology we formulate Ramsey's theorem simply by saying that the class of all finite sets endowed with subsets has the A-Ramsey property for every member A and that the class of all finite sets with isomorphism and subset relations is a Ramsey class. Using the partition arrow, Ramsey's theorem then takes the following concise form: for every choice p,k,n of positive integers there exists an integer N such that

$$N \longrightarrow (n)^p_k.$$

There are other results which can be formulated in a similar way. For example, the Ramsey theorem for finite vector spaces [7] takes a strikingly similar form.

Theorem 1.2. *For every finite field F and for every choice p,k,n of natural numbers, there exists N with the following property: if A,B,C are the vector spaces of dimensions p,n and N and if $\binom{C}{A} = \mathscr{A}_1 \cup \mathscr{A}_2 \cup \cdots \cup \mathscr{A}_k$ is any partition of the set $\binom{C}{A}$ of all p-dimensional vector subspaces of C, then there exists $i, 1 \leqslant i \leqslant k$ and a subspace B' of C, $\dim B' = \dim B = n$ such that $\binom{B'}{A} \subset \mathscr{A}_i$.*

Again, supporting the analogy with Ramsey's theorem, we could simply write: For every finite field F and for every choice of a natural number k and finite vector spaces A and B over F, there exists a vector space C over F such that

$$C \longrightarrow (B)^A_k.$$

Advancing to the abstract definition of a Ramsey class, let us list one more (quite general) example of a Ramsey class. Let $\Delta = (\delta_i; i \in I)$ be a sequence of natural numbers.

Δ is called the type (or signature). For a fixed Δ, we shall consider the class $\vec{\mathrm{Rel}}(\Delta)$ of all finite *ordered relational structures* of type Δ. These are objects of the form $(X, (R_i; i \in I))$ where X is an ordered set and $R_i \subseteq X^{\delta_i}$ (*i.e.*, R_i is a δ_i-ary relation). The class $\vec{\mathrm{Rel}}(\Delta)$ will be considered with embeddings (corresponding to induced substructures): given two relational structures $(X, (R_i; i \in I))$ and $(X', (R'_i; i \in I))$ of type Δ, a mapping $f : X \longrightarrow X'$ is called an *embedding* if it is monotone injection of X into X' satisfying $(f(x_j); j = 1, \dots, \delta_i) \in R'_i$ if and only if $(x_j; j = 1, \dots, \delta_i) \in R_i$. As usual, an inclusion (or bijective) embedding is called a *substructure* (or an *isomorphism*). Given two ordered relational structures A, B, we denote by $\binom{B}{A}$ the class of all substructures A' of B which are isomorphic to A. We have the following [24].

Theorem 1.3. *For every choice of a natural number k, of a type Δ and of structures $A, B \in \vec{\mathrm{Rel}}(\Delta)$ there exists a structure $C \in \vec{\mathrm{Rel}}(\Delta)$ with the following property. For every partition $\binom{C}{A} = \mathscr{A}_1 \cup \mathscr{A}_2 \cup \cdots \cup \mathscr{A}_k$ there exists $i, 1 \leqslant i \leqslant k$ and a substructure $B' \in \binom{C}{B}$ such that $\binom{B'}{A} \subset \mathscr{A}_i$.*

Again we can express the core part of Theorem 1.3 by

$$C \longrightarrow (B)^A_k$$

in the obvious meaning of the symbol.

Thus each of the above results can be expressed by the same formalism which (in hindsight!) is natural and to be expected. Let $\mathscr{K}$ be a class of objects which is isomorphism-closed and endowed with subobjects. Given two objects $A, B \in \mathscr{K}$, we denote by $\binom{B}{A}$ the set of all subobjects A' of B which are isomorphic to A. We say that the class $\mathscr{K}$ has the *A-Ramsey property* if the following statement holds. For every positive integer k and for every $B \in \mathscr{K}$ there exists $C \in \mathscr{K}$ such that $C \longrightarrow (B)^A_k$. Here the last symbol has (in the class $\mathscr{K}$) the following meaning. For every partition $\binom{C}{A} = \mathscr{A}_1 \cup \mathscr{A}_2 \cup \cdots \cup \mathscr{A}_k$ there exists $B' \in \binom{C}{B}$ and an $i, 1 \leqslant i \leqslant k$ such that $\binom{B'}{A} \subset \mathscr{A}_i$.

In the extremal case that a class $\mathscr{K}$ has the A-Ramsey property for every object A, we say that $\mathscr{K}$ is a *Ramsey class*.

These notions crystallized in the early seventies: see, *e.g.*, [16, 22, 23, 4]. This formalism and the natural questions it motivated essentially contributed to the creation of Ramsey Theory (as is nicely put in the introduction to [8]).

The notion of a Ramsey class is highly structured and in a sense it is the top of the line of the Ramsey notions (‘one can partition everything in any number of classes to get anything homogeneous’). Consequently there are not many (essentially different) examples of Ramsey classes known. It is the purpose of this paper to propose a programme which aims at characterizing all Ramsey classes. This is a realistic programme which allows us to characterize all Ramsey classes of some basic classes of combinatorial structures (such as graphs [18], tournaments and posets) and indicate their structure in other examples. However vague this may sound, there is evidence that (up to some singular examples) perhaps we know all Ramsey classes. This is supported by the fact that we shall be able to describe all monotone Ramsey classes (Theorem 5.3).

In such an abstract setting, we have to analyse (and slightly revise) some basic principles which underline Ramsey Theory. Particularly, this leads to a new scheme and classification of Ramsey properties by means of classes of admissible orderings and classes of order-invariant objects.

2. Subobjects

What is a subobject?

We know: a substructure, a subgraph, a subspace... but as we need to discuss subobjects of subobjects this calls for a more functorial approach. That may be easy from a category theory perspective, but in our context it needs a little care. Basically there are two ways to define subobjects in our setting, and we shall see how these approaches complement each other.

2.1. Equivalence classes

An A-subobject of B is defined as an equivalence class $[f]$ of the equivalence $\sim$. This equivalence is defined on the set of all embeddings $f : A \longrightarrow B$: we put $f \sim f'$ if and only if there exists an isomorphism $g : A \longrightarrow A$ with $f' = f \circ g$. In this definition the set $\binom{B}{A}$ is in one-to-one correspondence with a subset of the set $\binom{C}{A}$. For every embedding $g : B \longrightarrow C$,

$$\left\{[gf];[f] \in \binom{B}{A}\right\} \subseteq \binom{C}{A}.$$

This definition is suitable for most of the examples as the embeddings (extremal monomorphisms) are usually clearly defined. In many instances (such as undirected graphs and other symmetric relations) this is also a natural definition of a subobject. Unfortunately in the Ramsey Theory context this definition is less convenient from a very simple reason: most classes have the A-Ramsey property for only a few objects A. We indicate this by means of the following example.

Let X be a finite set, $\Delta = (\delta_i; i \in I)$ a fixed type. A set system of type Δ is the pair $(X, (E_i; i \in I))$ where $E_i \subset \binom{X}{\delta_i}$ for all $i \in I$. An embedding

$$f : (X, (E_i; i \in I)) \longrightarrow (X', (E_i'; i \in I))$$

is an injective mapping $f : X \longrightarrow X'$ which satisfies $f(e) \in E_i'$ if and only if $e \in E_i$ for any $i \in I$ (for short, we write $f(e)$ to denote the set $\{f(x); x \in e\}$). Let $\mathrm{Set}(\Delta)$ be the class of all finite set-systems of type Δ and all their embeddings.

Theorem 2.1. *For any type Δ the class* $\mathrm{Set}(\Delta)$ *has the A-Ramsey property if and only if A is totally symmetric.*

Here a set-system $A = (X, (E_i; i \in I))$ is *totally symmetric* if any permutation π of X which preserves all unary edges (*i.e.*, loops) is an automorphism of A; in other words, the complete or discrete graph for directed graphs (type (2)), and, among others, any complete bipartite graph for set systems of type $(2, 1, 1)$.

In fact we shall show that this result (obtained for graphs in [23]) can be deduced from a stronger statement which uses an alternative view of subobjects explained in the next subsection.

Below we shall define yet another equivalence on the class of all embeddings which is induced by a class of (admissible) orderings.

2.2. Subobjects as mappings

The other way to define subobjects is to consider subobjects as embeddings themselves: a subobject of B isomorphic to A is simply an embedding $A \longrightarrow B$. However, there are natural limitations to this approach. If a class $\mathscr{K}$ has the A-Ramsey property and there exists a non-identical automorphism α of A of order k, then we can partition any set of all embeddings of A into B into k classes such that embedding f and embedding $f \circ \alpha$ will be in a different class of partition. This colouring violates the A-Ramsey property. So we can identify subobjects with embeddings only if A is *rigid* (or asymmetric). This is the case, for example, if part of the structure A is a linear ordering of its vertices.

The role of orderings (and rigidity) in structural Ramsey Theory was recognized early on (see [16, 22]; Leeb speaks about 'Verstarrung'). The reason why the equivalence led to A-Ramsey properties for totally symmetric objects only is the lack of *rigidity*. However, the rigidity can be imposed by enlarging the structure, most conveniently by a linear ordering. Given a class of objects $\mathscr{K}$, we denote by $\vec{\mathscr{K}}$ the class of (*linearly*) *ordered* objects from $\mathscr{K}$.

How this is done usually follows from the concrete definition of the class $\mathscr{K}$ (which we implicitly assume is a concrete category [20]). In the case of graphs and relational systems we can simply consider objects with arbitrary linear ordering of vertices. In the more algebraical systems we have more possibilities. For example, for the class $\mathscr{P}$ of all finite partially ordered sets we can interpret $\vec{\mathscr{P}}$ as the class of all partially ordered sets P with an additional linear extension. Another possibility would be to consider all partially ordered sets P together with an additional linear ordering of its vertices (which does not extend the partial order). In this way we obtain another class, say $\vec{\vec{\mathscr{P}}}$, which is also Ramsey. In a more systematic way we shall discuss the role of various classes of orderings (called admissible orderings) in Section 4. However, usually we try to minimize effort and select as natural an ordering as possible. 'Natural' orderings (also called *canonical*) have been investigated in their own right: see [17, 29, 31, 11].

In any case the crucial property of the addition of the ordering is to guarantee the rigidity, or unicity of embeddings: $\vec{A}$ is a subobject of $\vec{B}$ if and only if $\vec{A} \longrightarrow \vec{B}$ if and only if $|\binom{B}{A}| = 1$.

The introduction of orderings led to many Ramsey classes. Here are some examples.

Theorem 2.2. *The class* $\vec{\mathrm{Gra}}$ *of all ordered graphs is a Ramsey class.*

More generally, we have the following result (which is a reformulation of Theorem 1.3).

Theorem 2.3. *The class* $\vec{\mathrm{Rel}}(\Delta)$ *is a Ramsey class.*

Theorem 2.4. *The class* $\vec{\mathscr{P}}$ *of all finite posets with an additional linear extension is a Ramsey class.*

These results are easy to state and they all fit our general definition. However, the proofs involve *ad hoc* techniques (and results are far from being easy or obsolete). The strongest method available seems to be the *amalgamation technique* [26, 27], which implies many of these results either directly or by a modification of the method; see [19] and references given there. (Thus, for example, Theorem 2.4 is proved in [24] in a more general form for acyclic oriented graphs.) Similarly, an extension of the proofs yields the following result.

Theorem 2.5. *The class* $\vec{\mathscr{M}}$ *of all linearly ordered finite metric spaces with monotone isometric embeddings is a Ramsey class.*

The proof of this will appear in [21]. This result is used in the recent paper [11] in the context of the topological dynamics.

In the next section we investigate the role of orderings in greater detail.

3. Admissible orderings

Let $\mathscr{K}$ be a class of structures. For $A \in \mathscr{K}$ we denote by $|A|$ the set of elements of A. Let $\preceq_A$ be a set of linear orderings of the set $|A|$ (for each $A \in \mathscr{K}$). These orderings are called *admissible orderings* of structures of $\mathscr{K}$; an admissible ordering of A will be denoted by $\leqslant_A$. Let $\preceq$ denote the class of all sets of admissible orderings of elements of $\mathscr{K}$.

We denote by $\vec{\mathscr{K}}_{\preceq}$ the class of all objects of $\mathscr{K}$ endowed with admissible orderings; *i.e.*, the objects of $\vec{\mathscr{K}}_{\preceq}$ are pairs $(A, \leqslant_A)$ where $\leqslant_A \in \preceq_A$. The class $\vec{\mathscr{K}}_{\preceq}$ is considered with order-preserving orderings. Clearly every embedding $f : (A, \leqslant_A) \longrightarrow (B, \leqslant_B)$ (in $\vec{\mathscr{K}}_{\preceq}$) is also an embedding $A \longrightarrow B$ (in $\mathscr{K}$). It is nontrivial that sometimes these implications may be reversed. This is based on the following concept.

The class $\mathscr{K}$ is said to have (*admissible*) *ordering property* if, for any structure $A \in \mathscr{K}$, there exists a structure $B \in \mathscr{K}$ such that, for any choice of admissible orderings $\leqslant_A, \leqslant_B \in \preceq$, we have

$$A \longrightarrow B \ \text{ (in } \mathscr{K}\text{) if and only if } \ (A, \leqslant_A) \longrightarrow (B, \leqslant_B) \ \text{ (in } \vec{\mathscr{K}}_{\preceq}\text{).}$$

The *ordering lemma* claims the admissible ordering property for a class $\mathscr{K}$ (with respect to the class of admissible orderings $\preceq$). It is important that the ordering lemma holds in many instances. In fact it holds in more instances than the Ramsey property. We can say that the ordering lemma holds provided that 'sparse (multi)-amalgamation' holds. We illustrate this by means of the ordering property of finite metric spaces (which was applied in [11]). The proof rests on the following (probabilistic) result proved in [25].

Lemma 3.1. *For any choice of positive integers $k, g, k \geqslant 2$ there exists a k-ary relation (X, R) with the following properties:*

(i) *for every linear ordering $\leqslant$ of X there exists an edge $(x_1, \ldots, x_k) \in R$ such that $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_k$,*

(ii) *(X, R) does not contain cycles of length $\leqslant g$.*

Here a cycle in (X, R) is a sequence $z_0, e_1, z_1, e_2, \ldots, e_t, z_t = z_1$ of distinct elements z_i and relations $e_i \in R$ such that any two consecutive z_i, z_{i+1} appear in e_i (in some order).

We use this to prove the following.

Theorem 3.2. *The class $\mathscr{M}$ of all finite metric spaces has the ordering property* (*with respect to all linear orderings*).

Proof. Let (M, ρ), (N, σ) be finite metric spaces. Let $(\tilde{M}, \tilde{\rho})$ be a metric space with ordering $(\tilde{M}, \leqslant)$ which contains (M, ρ) in an arbitrary ordering of M (we can take, for example, the 'disjoint union' of all possible linear orderings of M). Put $|\tilde{M}| = k$ and let $g > D/d$ where D (d, respectively) is the maximal (minimal nonzero, respectively) distance of two points of $\tilde{M}$. Let (X, R) be k-ary relation with properties of Lemma 3.1.

For every $e = (x_1, \ldots, x_k) \in R$, let π_e be the monotone bijection of $(\tilde{M}, \leqslant)$ and of the set $\{x_1, \ldots, x_k\}$ (*i.e.*, we assume that $u < v$ if and only if $i < j$, where $\pi_e(u) = x_i$ and $\pi_e(v) = x_j$). For elements $x, y \in X$, let $\mu(x, y)$ be generated by the transitivity from values $\mu(x_i, x_j) = \tilde{\rho}(\pi_e^{-1}(x_i), \pi_e^{-1}(x_j))$. It follows from our assumptions that the metric μ restricted to any set $\{x_1, \ldots, x_k\}$, $(x_1, \ldots, x_k) \in R$ is monotone isometric with the metric $\tilde{\rho}$ (here we used the fact that (X, R) has no short cycles). It is now easy to check that the space (X, μ) has desired properties: for any ordering $\leqslant_X$ of X there exists $e = (x_1, \ldots, x_k) \in R$ such that the ordering $\leqslant_X$ coincides with the ordering of e on e. However the set $\{x_1, \ldots, x_k\}$ induces a subspace monotone isomorphic to $(\tilde{M}, \tilde{\rho})$, which in turn contains any given linear ordering of (M, ρ). □

It is important that a similar proof can be applied to a wide spectrum of combinatorially interesting classes. We say that a structure A of type Δ is 2-*connected* if it is connected and remains connected by the deletion of any of its elements. (Thus A cannot be written as a free amalgamation by means of the empty or singleton structures.) The following then holds (with a proof analogous to the above proof of Theorem 3.2).

Theorem 3.3. *Let Δ be fixed and let $\mathscr{F}$ be a finite set of* 2*-connected structures. Then the class* $\mathrm{Forb}_\Delta(\mathscr{F})$ *has ordering property* (*with respect to all orderings*).

This result implies the ordering property of many combinatorially interesting classes, including many classes for which the amalgamation property does not hold (such as graphs with girth > 4). On the other hand, it is known [17, 23] that the ordering property is implied by nontrivial Ramsey properties (such as, in the case of graphs, the edge-Ramsey property or the (non-edge)-Ramsey property). (This is a nice trick, as we can use Ramsey's statement for both positive and negative statements.)

Thus the ordering property is closely related to Ramsey properties as a necessary condition. But there is more here than meets the eye. We can use the admissible orderings to classify the Ramsey properties and the Ramsey classes appear naturally as the extremal cases at both ends of the spectrum. This will be clarified by means of the following definitions.

Let $\mathscr{K}$ be a fixed class of structures, $\preceq$ a class of admissible orderings of elements of structures $A \in \mathscr{K}$. For $A \in \mathscr{K}$ we denote by $\preceq_A$ the set of all admissible orderings of A. We assume that $\preceq_A$ is a non-empty set and for isomorphic objects A, A' the sets $\preceq_A$ and $\preceq_{A'}$ are isomorphic (*i.e.*, the admissible orderings are in 1–1 correspondence induced by an isomorphism of A and A'). Denote by $(\mathscr{K}, \preceq)$ the class of all objects of form $(A, \preceq_A)$, where $A \in \mathscr{K}$ and $\preceq_A$ are admissible orderings of $|A|$. The class $(\mathscr{K}, \preceq)$ will be considered with embeddings of $\mathscr{K}$, which are monotone with respect to some admissible orderings. Explicitly, a mapping $f : |A| \longrightarrow |B|$ is an embedding (in $(\mathscr{K}, \preceq)$) if it is an embedding in $\mathscr{K}$, and there exist admissible orderings $\leqslant_A \in \preceq_A$ and $\leqslant_B \in \preceq_B$ such that f is also a monotone mapping (with respect to $\leqslant_A$ and $\leqslant_B$). The existence of embeddings (in $(\mathscr{K}, \preceq)$) will be denoted by $(A, \preceq_A) \longrightarrow (B, \preceq_B)$. This leads to an equivalence on the embeddings $(A, \preceq_A) \to (B, \preceq_B)$: we put $f \sim g$ if and only if $f = gh$ for an isomorphism $h : (A, \preceq_A) \to (A, \preceq_A)$. We also denote by $\binom{C}{A}_{\preceq}$ the set of all subobjects (modulo equivalence $\sim$).

We say that $A \in \mathscr{K}$ is *order-invariant* (with respect to $\preceq_A$) if A itself has the order property. Explicitly, A is order-invariant if, for any choice of admissible orderings $\leqslant, \leqslant' \in \preceq_A$ of elements of A, there exists an isomorphism $\psi : A \longrightarrow A$ which is monotone with respect to $\leqslant$ and $\leqslant'$. Denote by $\mathrm{Inv}_{\preceq}(\mathscr{K})$ the class of all $\preceq$ order-invariant structures in $\mathscr{K}$. We have the following.

Theorem 3.4. *Suppose that $\mathscr{K}$ has the $\preceq$-ordering property. Then $A \in \mathrm{Inv}_{\preceq}(\mathscr{K})$ for every A whenever $\mathscr{K}$ has the A-Ramsey property.*

Proof. Let $\mathscr{K}$ be as above and assume that $A \notin \mathrm{Inv}_{\preceq}(\mathscr{K})$. This equivalently means that there are orderings $\leqslant_A, \leqslant'_A \in \preceq_A$ of elements of A such that there is no monotone isomorphism of $(A, \leqslant_A)$ and $(A, \leqslant'_A)$. Let $B \in \mathscr{K}$ have the ordering property for A (with respect to $\preceq$). Now assume that $C \in \mathscr{K}$ is Ramsey for B for the 2-colouring of A-subobjects (thus we assume $C \longrightarrow (B)^A_2$). Fix $\leqslant_C \in \preceq_C$. Define the partition $\binom{C}{A} = \mathscr{A}_1 \cup \mathscr{A}_2$ as follows.

$A' \in \mathscr{A}_1$ if and only if A' with the ordering $\leqslant_C$ restricted to the elements of A' is monotone isomorphic to $(A, \leqslant)$. Otherwise we put $A \in \mathscr{A}_2$. The ordering property of B implies that any subobject $B' \in \binom{C}{B}$ contains A-subobjects of both colours and thus $C \not\longrightarrow (B)^A_2$. $\square$

In the extremal case that every $A \in \mathscr{K}$ has only one admissible ordering (*i.e.*, if $| \preceq_A | = 1$ for every $A \in \mathscr{K}$) the class $\mathscr{K}$ obviously has ordering property and moreover $\mathrm{Inv}_{\preceq}(\mathscr{K}) = \mathscr{K}$. The other extremal case is that $\preceq_A$ is the set of all linear orderings of A. $\mathrm{Inv}_{\preceq}(\mathscr{K})$ in this case corresponds to the set of 'totally symmetric' objects (defined for the classes $\mathrm{Set}(\Delta)$ above). Note that these two extrema are not the only possibilities and we can get the whole scale of partition properties corresponding to various classes $\preceq$ of admissible orderings. We complement this with the following.

Theorem 3.5. *Suppose* $\cup \preceq = \cup \preceq'$ *and* $\preceq_A \subseteq \preceq'_A$ *for every* $A \in \mathscr{K}$. *Then the following holds:*

(i) $\mathrm{Inv}_{\preceq}(\mathscr{K}) \supseteq \mathrm{Inv}_{\preceq'}(\mathscr{K})$,
(ii) *if the class* $(\mathscr{K}, \preceq)$ *has the* $(A, \preceq_A)$*-Ramsey property, then the class* $(\mathscr{K}, \preceq')$ *also has the* $(A, \preceq'_A)$*-Ramsey property.*

Proof. **(i)** This follows easily from the corresponding definitions.

(ii) Assume that $A \in \mathrm{Inv}_{\preceq}\mathscr{K}$. Let $C \in (\mathscr{K}, \preceq)$ satisfy

$$(C, \preceq_C) \longrightarrow (B, \preceq_B)_k^{(A,\preceq_A)} \quad \text{(in } (\mathscr{K}, \preceq)\text{; } (C, \preceq_C) \text{ exists by our assumption).}$$

We claim that also

$$(C, \preceq'_C) \longrightarrow (B, \preceq'_B)_k^{(A,\preceq'_A)} \quad \text{(in } (\mathscr{K}, \preceq')\text{).}$$

To prove this it suffices to observe that any subobject

$$[f] \in \binom{(C, \preceq'_C)}{(A, \preceq'_A)} \quad \text{(in } (\mathscr{K}, \preceq')\text{; this will be denoted as } [f]_{\preceq'}\text{)}$$

is the union of equivalence classes

$$[g] \in \binom{(C, \preceq_C)}{(A, \preceq_A)} \quad \text{(in } (\mathscr{K}, \preceq)\text{; similarly denoted by } [g]_{\preceq}\text{)}$$

for all $g \in [f]$. It follows that any k-colouring (partition)

$$c : \binom{(C, \preceq'_C)}{(A, \preceq'_A)} \longrightarrow \{1, \dots, k\} \quad \text{(in } (\mathscr{K}, \preceq')\text{)}$$

may be viewed as a colouring of the set of all embeddings $\mathrm{Emb}((A, \preceq_A), (C, \preceq_C))$ satisfying the additional property that $c(f) = c(g)$ providing $[f]_{\preceq'} = [g]_{\preceq'}$. But as $\preceq \subseteq \preceq'$ and as $(A, \preceq_A) \in \mathrm{lnv}_{\preceq}(\mathscr{K})$ this colouring induces a colouring $\tilde{c}$ of

$$\binom{(C, \preceq_C)}{(A, \preceq_A)} \quad \text{(in } (\mathscr{K}, \preceq) \text{ by } \tilde{c}([f]_{\preceq}) = c([f]_{\preceq'})\text{).}$$

Let $g\colon (B, \preceq_B) \longrightarrow (C, \preceq_C)$ be an embedding so that

$$\tilde{c} \circ [gf]_{\preceq} = i_0 \quad \text{for every} \quad [f]_{\preceq} \in \binom{(B, \preceq_B)}{(A, \preceq_A)}.$$

However, then also

$$\tilde{c} \circ [f]_{\preceq'} = i_0 \quad \text{for every} \quad [f]_{\preceq'} \in \binom{(B, \preceq'_B)}{(A, \preceq'_A)} \quad \text{(in } (\mathscr{K}, \preceq')\text{).}$$

□

Thus both Theorems 1.3 and 2.1 appear as extremal cases. This also explains why in Ramsey Theory we consider the extremal case of a single ordering for each object. We illustrate these concepts with three examples.

Example 1. Consider the class Gra of all finite graphs. Let $\preceq^1$ be the class of all orderings (of vertices of all graphs) and let $\preceq^2$ be the class of all possible orderings, but such that, for any graph, only one ordering is admissible. (Thus $|\preceq^1_A| = n!$ for A with n vertices, while $|\preceq^2_A| = 1$ for every $A \in \mathrm{Gra}$.) Then Gra has the $\preceq^1$-ordering property (as follows, *e.g.*, from Theorem 3.3) and also Gra has the $\preceq^2$-ordering property. The latter is trivial as any graph has only one admissible colouring. It follows that $\mathrm{Inv}_{\preceq^2}(\mathrm{Gra}) = \mathrm{Gra}$ while $\mathrm{Inv}_{\preceq^2}(\mathrm{Gra})$ is the class of all complete and discrete (= with no edges) graphs.

We have that $\mathrm{Inv}_{\preceq^2}(\mathrm{Gra})$ is a Ramsey class (this is just another form of Ramsey's theorem) while $\mathrm{Inv}_{\preceq^1}(\mathrm{Gra})$ has the A-Ramsey property if and only if $A \in \mathrm{Inv}_{\preceq^1}(\mathrm{Gra})$. This also follows from Theorem 1.3 (specialized to graphs) and from Theorem 3.5.

Example 2. Consider the class $\mathscr{P}$ of all finite partially ordered sets. We consider three possible sets of admissible orderings $\preceq^i, i = 1, 2, 3$.

- The set $\preceq^1$ selects for every partially ordered set A unique linear extension $\leqslant_A$ (and thus $\preceq^1_A = \{\leqslant_A\}$).
- The set $\preceq^2$ assigns for every partially ordered set A the set $\preceq^2_A$ of all linear extensions of A.
- The set $\preceq^3$ assigns for every partially ordered set A the set $\preceq^3_A$ of all linear orderings of $|A|$.

We obviously have $\mathrm{Inv}_{\preceq^1}(\mathscr{P}) = (\mathscr{P}, \preceq^1)$ and the set $\mathrm{Inv}_{\preceq^3}(\mathscr{P})$ is just the set of all discrete partial orders (*i.e.*, antichains). It is also not hard to prove that the class $\mathrm{Inv}_{\preceq^2}(\mathscr{P})$ consists exactly of all sums of antichains (which can be described alternatively as those partial orders where all maximal antichains are disjoint).

Each of the classes $\mathrm{Inv}_{\preceq^i}(\mathscr{P})$ is Ramsey. This follows easily for classes $\mathrm{Inv}_{\preceq^2}(\mathscr{P})$ and $\mathrm{Inv}_{\preceq^3}(\mathscr{P})$ (by Ramsey's theorem). The fact that $\mathrm{Inv}_{\preceq^1}(\mathscr{P}) = (\mathscr{P}, \preceq^1)$ is a Ramsey class is stated as Theorem 2.4.

The class $(\mathscr{P}, \preceq^2)$ is also Ramsey and this follows by Theorem 3.5. Also the class $(\mathscr{P}, \preceq^3)$ is Ramsey. However this does not follow from Theorem 3.5 as $\cup \preceq^3 \neq \cup \preceq^1$. $\cup \preceq^3$ is the class of all linear orderings. Let $\preceq^4$ be the class of all linear orderings of vertices of finite partially ordered sets (*i.e.*, not necessarily of linear extensions) and let $\preceq^4_A = \{\leqslant_A\}$ for every A. Then also the class $(\mathscr{P}, \preceq^4)$ is Ramsey and thus by Theorem 3.5 also the class $(\mathscr{P}, \preceq^3)$ is $(A, \preceq^3_A)$-Ramsey for every $A \in \mathrm{lnv}_{\preceq^3}(\mathscr{P})$.

Example 3. Consider the class Eq of all equivalences on finite sets. Equivalently we can consider Eq as the class of all disjoint unions of complete graphs. We consider 4 possible admissible orderings $\preceq^i, i = 1, 2, 3, 4$. $\preceq^1_A$ is an arbitrary linear ordering of A while $\preceq^2_A$ is the set of all linear orderings of vertices of A. $\preceq^3_A$ is any linear ordering induced by a linear ordering of colour classes of A (sometimes called saturated or convex orderings; *cf.* [11]). Let $\preceq^4_A$ be the set of all such convex orderings. Of course $\mathrm{Inv}_{\preceq^1}(\mathrm{Eq}) = (\mathrm{Eq}, \preceq^1)$ and $\mathrm{Inv}_{\preceq^2}(\mathrm{Eq})$ is the class of all equivalences which are either discrete or total (all pairs of distinct points are equivalent). The class $\mathrm{Inv}_{\preceq^3}(\mathrm{Eq}) = (\mathrm{Eq}, \preceq^3)$ while the class $\mathrm{Inv}_{\preceq^4}(\mathrm{Eq})$ consists of all *uniform* equivalences (*i.e.*, those equivalences where all equivalence classes have the same size). All these classes are Ramsey. This can be seen by various applications of Ramsey's theorem (with a little extra work). But the classes $(\mathrm{Eq}, \preceq^i), i = 1, 2, 3, 4$ also have the A-Ramsey property for all $A \in \mathrm{Inv}_{\preceq^i}(\mathrm{Eq}), i = 1, 2, 3, 4$. This common pattern is expressed by Conjecture 5.4.

There seems to be no dichotomy between unordered structures and linearly ordered structures if we define some of the notions introduced in Section 3 at a proper level of generality.

4. Amalgamation

In this section we derive a few general properties of Ramsey classes (with subobjects interpreted as embeddings). Despite their apparent structural complexity there are not many general properties that are common to all Ramsey classes (however, this may be expected in view of the reformulation of Ramsey classes via the chromatic number; see [19]).

We review some basic notions from model theory first (see, *e.g.*, [2, 9, 5]). Let $\mathscr{K}$ be a class of structures endowed with embeddings.

- $\mathscr{K}$ is said to be *isomorphism-closed* if it contains with every $A \in \mathscr{K}$ any object A' isomorphic to A.
- $\mathscr{K}$ is said to be *hereditary* if $B \in \mathscr{K}$ and $A \longrightarrow B$ (in $\mathscr{K}$) implies that $A \in \mathscr{K}$.
- $\mathscr{K}$ is said to have the *joint embedding property* if, for every $A, B \in \mathscr{K}$, there exists $C \in \mathscr{K}$ such that both A and B can be embedded to C.
- $\mathscr{K}$ is said to have the *amalgamation property* if, for every $A, B, B' \in \mathscr{K}$, and every choice of embeddings $f : A \longrightarrow B, f' : A \longrightarrow B'$, there exists $C \in \mathscr{K}$ and embeddings $g : B \longrightarrow C, g' : B' \longrightarrow C$ such that $g \circ f = g' \circ f'$.

(C, g, g') is called an *amalgamation* of (A, B, B', f, f'). Note that it is not assumed that the sets $g \circ f(|B|)$ and $g' \circ f'(|B'|)$ intersect in the set $g \circ f(|A|) = g' \circ f'(|A|)$. An amalgamation with this additional property is called *strong*.

A (finite or infinite) structure A is called *homogeneous* if, for any choice of finite substructures B, B' of A, every isomorphism $f : B \longrightarrow B'$ can be extended to an isomorphism $g : A \longrightarrow A$ (*i.e.*, we demand that g restructured to B is f). For an infinite structure A, the *age* of A (denoted by $age(A)$) is the class of all (isomorphism types of) finite substructures of A. In this case we also say that A is *universal* for $age(A)$ (although this term is usually reserved for countable universality).

Homogeneity is a very strong symmetry property (generalizing vertex-, edge-, path-transitive graphs) and as a result there are just a handful of finite examples (totally symmetric objects are of course among them). But infinite homogeneous structures are more frequent and present important examples. The following model-theoretic result [5] (see also [9]) is the key to the connection of homogeneous structures to classes of finite objects.

Theorem 4.1. *Let $\mathscr{K}$ be a class of structures which is isomorphism-closed and has the joint embedding property. Then the following statements are equivalent:*

(1) *$\mathscr{K}$ has the amalgamation property,*
(2) *there exists (up to an isomorphism) a unique countable structure $\mathscr{U}$ which is homogeneous and universal for $\mathscr{K}$.*

The structure $\mathscr{U}$ is called *generic*, or *Fraïssé limit* for the class $\mathscr{K}$. The generic structure is homogeneous and thus highly structured. This is also reflected in the fact that very often $\mathscr{U}$ has an easy description; compare [1, 10].

The key observation in our programme is the fact that Ramsey classes are closed on amalgamation (this has been noted already in [18, 19] see also [11]). We formulate this in a stronger form by means of admissible orderings. We say that a class $\preceq$ of admissible orderings of objects in $\mathscr{K}$ is *rich* if the following holds. For any objects A, B_1, B_2 of $\mathscr{K}$ and with embeddings $f_i : A \longrightarrow B_i$ there exist objects $B_i' \in \mathscr{K}$, embeddings $f_i' : A \longrightarrow B_i'$ and admissible orderings $\leqslant, \leqslant_i, i = 1, 2$ of $A, B_i', i = 1, 2$ such that f_i' extends f_i (thus B_i is a substructure of B_i') and such that every amalgamation (in $\mathscr{K}, \preceq$)) of $(A, \leqslant), (B_1', \leqslant_1'), (B_2', \leqslant_2'), f_1', f_2')$ is strong.

Most classes of admissible orderings introduced in Section 3 are rich. For example, the class of all linear orderings is rich, as well as the class of all linear extensions of partial orders.

Theorem 4.2. *Let $\mathscr{K}$ be any hereditary, isomorphism-closed class with the joint embedding property. Then we have the following.*

(i) *If $\mathscr{K}$ is a Ramsey class* (*with embeddings as subobjects*) *then $\mathscr{K}$ has the amalgamation property.*
(ii) *If $\preceq$ is a rich class of admissible orderings of $\mathscr{K}$ and $\vec{\mathscr{K}}_{\preceq}$ is a Ramsey class then $\mathscr{K}$ has a strong amalgamation property.*

Proof. **(i)** Let A, B_1, B_2 be objects of $\mathscr{K}$ and let embeddings $f_i : A \longrightarrow B_i, i = 1, 2$, be given. Using the joint embedding property let C be an object for which there are embeddings $g_i : B_i \longrightarrow C, i = 1, 2$. If $g_1 \circ f_1 = g_2 \circ f_2$ then we are done. So assume $g_1 \circ f_1 \neq g_2 \circ f_2$. Let $D \longrightarrow (C)^A_2$. In this situation we define a partition $\binom{D}{A} = \mathscr{A}_1 \cup \mathscr{A}_2$ as follows. $A' \in \mathscr{A}_1$ if and only if there exist embeddings $h : C \longrightarrow D$ such that the embedding $f : A \longrightarrow D$ which represent A' as member of the set $\binom{D}{A}$ can be written as $f = h \circ g_1 \circ f_1$. Otherwise we put $A' \in \mathscr{A}_2$. By the Ramsey property of D there exists $C' \in \binom{D}{C}$ represented by an embedding $h' : C \longrightarrow D$ such that $\{h' \circ f; f \in \binom{C'}{A}\}$ is a subset of $\mathscr{A}_{i_0}$. Now any subobject of D isomorphic to C contains a subobject isomorphic to A which is coloured by 1. Thus $i_0 = 1$. Consider the embedding $h' \circ g_2 \circ f_2 \in \binom{D}{A}$. This subobject of D is also a subobject of C' and thus it is also coloured by $i_0 = 1$. Thus there must exist an embedding $h'' : C \longrightarrow D$ corresponding to a subobject $C'' \in \binom{D}{C}$ such that $h'' \circ g_1 \circ f_1 = h' \circ g_2 \circ f_2$. But this means that object D together with embeddings $h'' \circ g_1 : B_1 \longrightarrow D$ and $h' \circ g_2 : B_2 \longrightarrow D$ is an amalgamation of (A, B_1, B_2, f_1, f_2).

(ii) To prove (ii) we can proceed similarly. We stress only differences: given $A, B_1, B_2 \in \mathscr{K}$ and embeddings f_1, f_2, we find objects B_1', B_2', admissible orderings $\leqslant, \leqslant_1, \leqslant_2$ and embeddings f_1', f_2' by the richness of admissible class $\preceq$ (we preserve the notation of the above definition). By the proof of (i), for the class $(\vec{\mathscr{K}}_{\preceq})$ there exists an amalgamation (C, g_1, g_2) of $(A, \leqslant), (B_1', \leqslant_1), (B_2', \leqslant_2), f_1, f_2)$ (in $(\vec{\mathscr{K}}_{\preceq})$). Consider the restrictions $\tilde{g}_i = g_i \mid |B_i|, i = 1, 2$, and denote by $\tilde{C}$ the subobject of C induced by the set $|B_1| \cup |B_2|$. We see easily that $(\tilde{C}, \tilde{g}_1, \tilde{g}_2)$ is a strong amalgamation of (A, B_1, B_2, f_1, f_2) in $\mathscr{K}$. $\square$

It is perhaps surprising that we can prove this result on this level of generality. Despite its simplicity, this is a basic and powerful tool as it allows the study of (seemingly too diverse) Ramsey classes by means of single objects of high symmetry. In this way we reduce one part of the problem of characterization of Ramsey classes to the characterization problem of homogeneous structures. This will be done in Section 5.

5. Characterization of Ramsey classes

The following is a consequence of Theorems 4.1 and 4.2.

Corollary 5.1. *Let* $\mathscr{K}$ *be a Ramsey class* (*with embeddings as subobjects*) *which is hereditary, isomorphism-closed and has the joint embedding property. Then* $\mathscr{K}$ *is the age of a generic* (*homogeneous and universal*) *structure* $\mathscr{U}(\mathscr{K})$.

Proof. By Theorem 4.2 the class $\mathscr{K}$ has amalgamations, and thus by Theorem 4.1 $\mathscr{K}$ is the age of the unique homogeneous structure (which is denoted by $\mathscr{U}(\mathscr{K})$). $\square$

This relates two seemingly unrelated things: Ramsey classes and homogeneous structures. This allows us to use known results about homogeneous structures (in the cases when their classification programme is completed) and to check whether the corresponding classes (*i.e.*, their ages) are Ramsey. Schematically we want to proceed as follows:

I Ramsey classes $\Rightarrow$ amalgamation classes
II amalgamation classes $\Rightarrow$ generic structures
III generic structure $\Rightarrow$ special generic structures
IV special generic structures $\Rightarrow$ Ramsey classes

$$\begin{tikzcd}
\text{Ramsey classes} \arrow[r, Rightarrow] & \text{amalgamation classes} \arrow[d, Rightarrow] \\
\text{special structures} \arrow[u, Rightarrow] & \text{generic structures} \arrow[l, Rightarrow]
\end{tikzcd}$$

Statement I is provided by Theorem 4.2, and II is provided by the Fraïssé theorem (Theorem 4.1). The bottlenecks of this programme are obviously statements III and IV. Statement III symbolizes the *classification programme* (of Lachlan and Cherlin; see, *e.g.*, [14, 2]). The Ramsey Theory context here presents some interesting problems (see Section 6). Also it is hopeful that we might be able to deduce and strengthen Corollary 5.1 in such a way that the classification programme will be easier. In fact this is possible, as we shall characterize all *monotone* Ramsey classes of relational structures; *cf.* Theorem 5.3. Statement IV symbolizes the *Ramsey classes problem* where we have to decide for each (special) generic structure whether the corresponding age is a Ramsey class. This is nontrivial, and achieved by special *ad hoc* techniques. Yet it seems that a discussion at this end is possible (particularly for stable theories [2]).

Yet we believe that this *Ramsey classes classification programme* is realistic, at least in some cases. This is documented in [19] and in the next section by a few examples. But before doing so let us introduce the following general approach to monotone classes.

Let us consider the class $\vec{\mathrm{Rel}}(\Delta)$ of all relational systems of type Δ. Let the class $\preceq$ of admissible orderings be the class of all linear orders. This is a rich class according to the above definition. Consequently, if $\vec{\mathscr{K}}_{\preceq} = (\mathscr{K}, \preceq)$ is a Ramsey subclass of $\vec{\mathrm{Rel}}(\Delta)$ then $\mathscr{K}$ has a strong amalgamation property. We say that $\mathscr{K}$ is *monotone* if it is closed on *monomorphisms*. Explicitly, $\mathscr{K}$ is monotone if $A' = (X', (R_i'; i \in I)) \in \mathscr{K}$, provided that there exists $A = (X, (R_i; i \in I)) \in \mathscr{K}$ such that $X' \subseteq X, R_i' \subseteq R_i, i \in I$. In this case we write simply $A' \subseteq A$. For graphs a monotone class corresponds to a class closed on (not necessarily induced) subgraphs.

Let $\mathscr{F}$ be a class of structures (of type Δ). Denote by $\mathrm{Forb}_{m,\Delta}(\mathscr{F})$ the class of all structures $A \in \mathrm{Rel}(\Delta)$ which do not contain any F' isomorphic to an $F \in \mathscr{F}$.

We also say that a structure $F \in \mathrm{Rel}(\Delta)$ is (*amalgamation*)-*irreducible* if F is not a free amalgamation of two proper substructures of F. Alternatively F is irreducible if any two its vertices appear in one of the edges of F.

We have the following.

Proposition 5.2. *Let $\mathscr{K}$ be a class of structures of type Δ which is monotone and which has strong amalgamation. Then there exists a class $\mathscr{F}$ of irreducible structures such that $\mathscr{K} = \mathrm{Forb}_{m,\Delta}(\mathscr{F})$.*

Proof. As $\mathscr{K}$ is monotone there exists a class $\mathscr{F}'$ such that $\mathscr{K} = \mathrm{Forb}_{m,\Delta}(\mathscr{F}')$ (for example we can put $\mathscr{F}' = \mathrm{Rel}(\Delta) \setminus \mathscr{K}$). Let $\mathscr{F}$ be a subclass of all non-isomorphic inclusion-minimal systems. It suffices to prove that every $A \in \mathscr{F}$ is irreducible. But if elements x, y of an $F \in \mathscr{F}$ do not belong to any edge of A then both systems $F_x = F - x, F_y = F - y$ belong to $\mathscr{K}$ and F is then contained (via a monomorphism) in any strong amalgamation of F_x and F_y. $\square$

Combining all the above definitions we obtain the following *characterization of monotone Ramsey classes*.

Theorem 5.3. *For every type Δ and every monotone, isomorphism-closed class $\mathscr{K}$ of Δ-systems with the join embedding property, the following two statements are equivalent:*

(i) *the class $(\mathscr{K}, \preceq)$ is a Ramsey class,*

(ii) *$\mathscr{K} = \mathrm{Forb}_{m,\Delta}(\mathscr{F})$ for a class of irreducible Δ-systems.*

Proof. We follow the above scheme. Suppose that $(\mathscr{K}, \preceq)$ is a Ramsey class (all orderings are admissible). By Theorem 4.2(ii) $\mathscr{K}$ has strong amalgamation (I $\Rightarrow$ II). By Proposition 5.2 there exists a class $\mathscr{F}$ of irreducible Δ-systems such that $\mathscr{K} = \mathrm{Forb}_{m,\Delta}(\mathscr{F})$ (II $\Rightarrow$ IV) (thanks to the stronger assumption of monotonicity, we can bypass III). Any class of the form $(\mathrm{Forb}_{m,\Delta}(\mathscr{F}), \preceq)$ is a Ramsey class (IV $\Rightarrow$ I). This is the most difficult part; it generalizes Theorem 1.3 and will appear elsewhere. $\square$

The general setting of Section 3 allows us to relate the characterization of A-Ramsey properties to Ramsey classes. Based on the above examples and all examples of known Ramsey classes, we formulate this (perhaps too boldly) as a 'metaconjecture'.

Conjecture 5.4. *For any class of structures $\mathscr{K}$ and for any set $\preceq$ of admissible orderings, the following two statements are equivalent.*

(i) $(\mathscr{K}, \preceq)$ *has the A-Ramsey property if and only if* $A \in \mathrm{Inv}_{\preceq}(\mathscr{K})$,
(ii) $\mathrm{Inv}_{\preceq}(\mathscr{K})$ *is a truncated Ramsey class.*

(We say that $\mathscr{R}$ is a *truncated Ramsey class* if there exists a Ramsey class $\mathscr{R}'$ such that $\mathscr{R} = \mathscr{R}' \cap \mathscr{K}$.) Note that, interestingly, for 'small' sets of admissible orderings (for example when $|\preceq_A| = 1$ for every $A \in \mathscr{K}$) the conjecture holds. It also holds for all examples mentioned here.

6. Examples

First consider the class Gra of all finite undirected graphs. Also, denote by Gra_k the class of all K_k-free graphs ($k \geqslant 2$), by Compl the class of all complete graphs, and by Eq the class of all finite equivalences (*i.e.*, disjoint unions of complete graphs). For a class $\mathscr{K}$ of graphs, denote by $\mathscr{K}_c$ the class of all complements of graphs belonging to $\mathscr{K}$. The following result characterizes all Ramsey classes of undirected graphs. This was proved in [18] and provided a motivation for this paper. We sketch the proof for completeness.

Theorem 6.1. *The following are all Ramsey classes of undirected graphs:*

(i) *the class* $\{K_1\}$,
(ii) *the class of all complete graphs,* Compl,
(iii) *the class of all equivalences* $\vec{\mathrm{Eq}}$,
(iv) *each of the classes* $\vec{\mathrm{Gra}}_k$,
(v) *the class* $\mathscr{K}_c$ *for each of the above classes,*
(vi) *the class of all finite graphs* $\vec{\mathrm{Gra}}$.

(Note that in each of the cases (iii), (iv) and (vi) we have to consider classes of ordered graphs; see Section 2.)

Proof. We follow the scheme of Section 4: Let $\mathscr{K}$ be a Ramsey class. Let $\preceq$ be the class of all admissible orderings. Then both classes $\mathscr{K}$ and $\vec{\mathscr{K}}$ have the amalgamation property, and thus let $\mathscr{U}(\mathscr{K})$ be the corresponding generic graph. The countable homogeneous undirected graphs were characterized in [15] as the generic graphs for classes stated in (i)–(vi), and in addition generic graphs for a few other classes of the graphs (such as finite examples and classes of equivalences where either the number of equivalence classes or their size is bounded). But none of these additional classes is Ramsey. This is obvious for finite examples (with the single exception of (i)) and also for equivalences with bounded number of classes (with the single exception of the class (ii)).

So there are no other Ramsey classes. Now we have to prove that each of the classes (i)–(vi) is Ramsey. This is sometimes easy (as in (i) or (v): this follows as all our embeddings correspond to induced subgraphs). Class (ii) is the Ramsey theorem and (iii) is the Product Ramsey Theorem [8, 19]. Case (iv) is not easy, providing one of the motivations for structural Ramsey Theory, and this was finally proved in [24]. Case (vi) then follows from (iv). □

Next we characterize the Ramsey classes of tournaments (*i.e.*, orientations of complete graphs). Denote by $\mathscr{T}$ the class of all finite tournaments, by $\mathscr{O}$ the class of all linear orders (which we interpret as transitive tournaments) and by $\mathscr{L}\mathscr{O}$ the class of all local orders. The last notion needs a definition: (X, R) is a *local order* if, for every vertex $x \in X$, the set $\{y; (x, y) \in R\}$ (and also the set $\{y; (y, x) \in R\}$) induces a transitive subtournament of (X, R).

We have the following.

Theorem 6.2. *The following are Ramsey classes of ordered tournaments:*

(i) $\{K_1\}$,
(ii) *the class of all transitive tournaments* $\mathscr{O}$,
(iii) *the class of all ordered tournaments* $\vec{\mathscr{T}}$.

Proof. Let $\mathscr{K}$ be a Ramsey class of tournaments. Let $\mathscr{U}(\mathscr{K})$ be the generic tournament. The homogeneous tournaments were characterized by [12]: see [2] for a simple proof. These generic tournaments correspond to the cases (i), (ii), (iii) and to two more cases: the oriented cycle C_3 and the special generic graph $S(2)$, which is generic for the class of all local orders (the ordering of rational numbers is generic for the class of all finite orders). It can be shown that local orders are not a Ramsey class. Case (i) is the only finite Ramsey class; case (ii) is equivalent to the Ramsey theorem. All finite ordered tournaments are Ramsey. This is a particular case of Ramsey's theorem for finite oriented graphs proved in [24]. □

Next we shall consider partially ordered sets. An ordered partially ordered set is a partial ordered set $A = (X, R)$ together with a linear extension $\leqslant_A$ the (admissible) ordering. We denote by $\mathscr{P}$ the class of all finite partially ordered sets together with their embeddings. The class of all finite ordered partially ordered sets together with their embeddings which are monotone with respect to their admissible orderings will accordingly be denoted by $\vec{\mathscr{P}} = \vec{\mathscr{P}}_{\leq}$. We have the following.

Theorem 6.3. *The following are Ramsey classes of ordered partially ordered sets:*

(i) $\{K_1\}$ ($\{K_1\}$ *here means the singleton partial ordered set*),
(ii) *the class* $\vec{\mathscr{O}}$ *of all finite linear orders,*
(iii) *the class of all chain-sums of finite antichains* $\mathscr{O}\vec{\mathscr{A}}$,
(iv) *the class* $\vec{\mathscr{A}}$ *of all ordered antichains,*
(v) *the class* $\vec{\mathscr{P}}$ *of all ordered finite partially ordered sets.*

Proof. Let $\vec{\mathscr{K}}$ be an (isomorphism-closed, hereditary) Ramsey subclass of $\vec{\mathscr{P}}$. It is easy to see that the corresponding class $\mathscr{K}$ (of all partially ordered sets that have a linear extension in $\vec{\mathscr{K}}$) is hereditary, isomorphism-closed and with amalgamation. It follows that the corresponding generic partial order $\mathscr{U}(\mathscr{P})$ is one of the few types that were characterized by Schmerl [30]. These are the generic partial order for the classes listed in the statement of Theorem 6.3 together with the classes of cardinality-restricted classes in (iii) and in (iv) (*i.e.*, in (iv) the classes of disjoint unions of at most k chains, and in (iii) the classes of all chain-sums of antichains of size $\leqslant k$). For $k > 1$ none of these classes is Ramsey. Thus no other Ramsey classes exist. The fact that all the above classes are Ramsey is far from evident. While the case (ii) is just finite Ramsey theorem, the cases (iii) and (iv) may be proved using the Product Ramsey Theorem, and lastly (v) needs more care and general techniques. This is stated as Theorem 2.4. □

In the Introduction we stated that there is some evidence that perhaps we know all Ramsey classes. This can be illustrated by considering the case of type $\Delta = (2)$ and by the case of general Δ.

The classification programme for the type $\Delta = (2)$ amounts to characterizing homogeneous directed graphs. This has been completed by extremely complicated and lengthy arguments [2]. The list contains all homogeneous (undirected) graphs, tournaments, as well as posets. But there are other (countably many) special (stable in the classification of [14]) oriented graphs. Ages of some of these special homogeneous graphs are Ramsey classes and some are not. These special classes can perhaps all be discussed and it seems that there are only finitely many Ramsey classes here. But there are total continuum many homogeneous oriented graphs, and apart from these special graphs these are graphs of type $\mathrm{Forb}(\mathscr{F})$ for a class $\mathscr{F}$ (of possibly infinitely many) tournaments. However, all these classes are Ramsey by (the more general) Theorem 6.4. The same is true for ages of graphs Γ_n, which are characterized as all oriented graphs that do not contain a set of n independent vertices. We hope to give the full discussion of Ramsey classes of oriented graphs elsewhere.

For the general type Δ (and even for the type $\Delta = (2, 2)$ corresponding to the directed graphs with 2-coloured edges) the classification programme is very much unsolved and (in view of the complexity of [2]) not expected to be solved by the current methods. Yet it has been indicated that, apart from special countably many structures, all (continuum many) ages of homogeneous structures of type Δ are of the form $\mathrm{Forb}(\mathscr{F})$ for a set of (*amalgamation*)-*irreducible* structures. However, all such classes are Ramsey. Let us formulate this more precisely.

Let Δ be a fixed type. In the previous section we introduced the notion of an irreducible structure. (F is irreducible if any two of its vertices appear in one of the edges of F.) Let $\mathrm{Forb}_\Delta(\mathscr{F})$ denote the class of all structures of type Δ which do not contain any $F \in \mathscr{F}$ as a substructure. Finally, let $\overrightarrow{\mathrm{Forb}}_\Delta(\mathscr{F})$ denote the class of linearly ordered structures belonging to $\mathrm{Forb}_\Delta(\mathscr{F})$ together with their monotone embeddings. We have the following [27].

Theorem 6.4. *The class* $\overrightarrow{\mathrm{Forb}}_\Delta(\mathscr{F})$ *is Ramsey for every type* Δ.

This is the strengthening of Theorem 1.3, and perhaps it covers most Ramsey classes of relational structures of type Δ.

7. Concluding remarks

(1) This paper is based on a lecture delivered by the author at the Walter Deuber memorial meeting held in Berlin in October 2002. The author thanks H.-J. Prömel for organizing this meeting, which prompted this author to rethink now classical approaches to Ramsey Theory. The author is pleased that structural Ramsey Theory (exactly in its most abstract setting explained in this paper) recently got into a new context of characterizing *extremely amenable groups* and *minimal G-flows* of topological dynamics, [11]. This in turn inspired new problems, some of which are reported here as Theorems 2.5 and 3.2. Perhaps some additional Ramsey classes included in this paper may be of some interest in the setting of [11].

(2) We will try to complete the characterization programme of Ramsey classes (explained in Section 4) for other structures with the solved characterization of homogeneous structures (for example, oriented graphs, n-tournaments). This seems routine, but on the positive side it involves a study of interesting classes of structures which have to be tested for Ramsey properties.

(3) It would be nice to complete the characterization programme of Ramsey classes for other structures (such as vector-space systems and combinatorial number theory examples, for instance (m, p, c)-set systems; see [3]). Note that for such richer structures the characterization problem for homogeneous structures is wide open.

(4) The same is true for dual structures where a merely model theory part (*i.e.*, the Fraïssé Theory) has to be developed. Sometimes, rarely, we can view dual structures by means of embeddings (see, for example, the case of Boolean algebras [11, 19]), but this seems to be an exception.

(5) It remains to be seen how much (or, rather, how little) of this theory can be generalized for the infinite objects: see, *e.g.*, [28].

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