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Bottom-up rewriting for words and terms

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ABSTRACT

For the whole class of linear term rewriting systems, we define *bottom-up rewriting* which is a restriction of the usual notion of rewriting. We show that bottom-up rewriting effectively inverse-preserves recognizability.
The *Bottom-Up* class (BU) is, by definition, the set of linear systems for which every derivation can be replaced by a bottom-up derivation. Since membership to BU turns out to be undecidable, we are led to define more restricted classes: the classes $\mathsf{SBU}(k)$, $k \in \mathbb{N}$, of *Strongly Bottom-Up*(k) systems for which we show that membership is decidable. We define the class of *Strongly Bottom-Up* systems by $\mathsf{SBU} = \bigcup_{k\in\mathbb{N}} \mathsf{SBU}(k)$. We give a polynomial-time sufficient condition for a system to be in SBU. The class SBU contains (strictly) several classes of systems which were already known to inverse preserve recognizability: the inverse left-basic semi-Thue systems (viewed as unary term rewriting systems), the linear growing term rewriting systems, the inverse Linear-Finite-Path-Ordering systems.

1. Introduction

General framework. An important concept in rewriting is the notion of *preservation of recognizability* through rewriting. Each identification of a more general class of systems preserving recognizability, yields almost directly a new decidable call-by-need (Durand and Middeldorp, 2005) class, decidability results for confluence, accessibility, joinability. Also, recently, this notion has been used to prove termination of systems for which none of the already known termination techniques work (Geser et al., 2005). Such a preservation property is also a tool for studying the

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recognizable/rational subsets of various monoids which are defined by a presentation $\langle X, \mathcal{R}\rangle$, where X is a finite alphabet and $\mathcal{R}$ a Thue system (see for example Lohrey and Sénizergues, 2008; Kambites et al., 2007). Consequently, the seek of new decidable classes of systems which preserve (or inverse preserve) recognizability is worthwhile.

Many such classes proposed so far have been defined by imposing syntactical restrictions on the rewrite rules. For instance, in *growing* systems (Jacquemard, 1996; Nagaya and Toyama, 2002) variables at depth strictly greater than 1 in the left-hand side of a rule cannot appear in the corresponding right-hand side. Finite-path Overlapping systems (Takai et al., 2010) are also defined by syntactic restrictions on the rules. The class of Finite-path Overlapping systems contains the class of growing systems (Nagaya and Toyama, 2002). Previous works on semi-Thue systems also prove recognizability preservation, under syntactic restrictions: cancellation systems (Benois and Sakarovitch, 1986), monadic systems (Book et al., 1982), basic systems (Benois, 1987), and left-basic systems (Sakarovitch, 1979) (see Sénizergues, 1995 for a survey).

Other works establish that some *strategies* i.e. restrictions on the derivations rather than on the rules, ensure preservation of recognizability. Various such strategies were studied in Fülöp et al. (1998), Réty and Vuotto (2005), Seynhaeve et al. (1999).

We rather follow here this second approach: we define a new rewriting strategy which we call *bottom-up rewriting* for linear term rewriting systems. The bottom-up derivations are, intuitively, those derivations in which the rules are applied, roughly speaking, from the bottom of the term towards the top (this set of derivations contains strictly the bottom-up derivations of Réty and Vuotto (2005) and the one-pass leaf-started derivations of Fülöp et al. (1998); it is incomparable with the *innermost* derivations[1]). An important feature of this strategy, as opposed to the ones quoted above, is that it allows *overlaps* between successive applications of rules. A class of systems is naturally associated with this strategy: it consists of the systems $\mathcal{R}$ for which the binary relation $\to_{\mathcal{R}}^{*}$ coincides with its restriction to the bottom-up strategy. We call "bottom-up" such systems and denote by BU the set of all bottom-up systems.

Overview of the paper. The results proved in this paper were announced in Durand and Sénizergues (2007), which can thus be considered as a medium-scale overview of this paper.

In Section 2, we have gathered all the necessary recalls and notation about words, terms, rewriting and automata.

In Section 3, we define *bottom-up rewriting* for linear term rewriting systems using marking techniques. We first define *bottom-up*(k) derivations for $k \in \mathbb{N}$ ($\mathsf{bu}(k)$ derivations for short) and the classes Bottom-up(k) ($\mathsf{BU}(k)$ for short) of linear systems which consists of those systems which admit $\mathsf{bu}(k)$ rewriting, i.e. such that every derivation between two terms can be replaced by a $\mathsf{bu}(k)$ derivation, and the *Bottom-up* class (BU) of *bottom-up* systems which is the infinite union of the $\mathsf{BU}(k)$ (for k varying in $\mathbb{N}$).

In Section 4, we prove Theorem 4.2 which is the main result of the paper: bottom-up rewriting inverse-preserves recognizability. Our proof consists of a reduction to the preservation of recognizability by finite ground systems, shown in Brainerd (1969), Dauchet and Tison (1990). The proof is constructive i.e. gives an algorithm for computing an automaton recognizing the antecedents of a recognizable set of terms.

It turns out that membership to $\mathsf{BU}(k)$ is undecidable for $k \geq 1$ (Durand and Sénizergues, 2009, Theorem 5.12). In Section 5, we thus define the restricted class of *strongly bottom-up*(k) systems ($\mathsf{SBU}(k)$) for which we show *decidable* membership. We define the class of *strongly bottom-up* systems $\mathsf{SBU} = \bigcup_{k\in\mathbb{N}} \mathsf{SBU}(k)$ and give a polynomial sufficient condition for a system to be in SBU.

2. Preliminaries

This section is mostly devoted to recalling some classical notions and making precise our notation. The reader is referred to Comon et al. (2002) for more details on the subject of tree-automata and to Klop (1992), Terese (2003) for term rewriting.

[1] Which, anyway, do not inverse-preserve nor preserve recognizability.

2.1. Sets, binary relations

Abstract rewriting. For every sets E, F and every binary relation $\to\, \subseteq E \times F$, the inverse binary relation $\to^{-1}$ is defined by $\forall f \in F, \forall e \in E, f \to^{-1} e \Leftrightarrow e \to f$. We note $\to^0 = \mathrm{Id}_E, \to^1 = \to$, for every $n \geq 1$ $\to^{n+1} = \to \circ \to^n$ and finally: $\to^* := \bigcup_{n=0}^{\infty} \to^n$. The relation $\to^*$ is the reflexive and transitive closure of the binary relation $\to$. A finite *derivation* w.r.t. the relation $\to$, is a sequence

$$D = (t_0, t_1, \ldots, t_i, t_{i+1}, \ldots, t_n) \tag{1}$$

such that, for every $i \in [0, n-1]$, $t_i \to t_{i+1}$. Given a subset $T \subseteq E$, we define

$$(\to^*)[T] = \{s \in E \mid s \to^* t \text{ for some } t \in T\}. \tag{2}$$

2.2. Words and terms

A finite *word* over an alphabet A is a map $u : [0, \ell - 1] \to A$ for some $\ell \in \mathbb{N}$. The integer ℓ is the *length* of the word u and is denoted by $|u|$. The set of words over A is denoted by A^* and endowed with the usual *concatenation* operation $u, v \in A^* \mapsto u \cdot v \in A^*$. The *empty* word is denoted by ε. A word u is a prefix of a word v iff there exists some $w \in A^*$ such that $v = uw$. We denote by $u \preceq v$ the fact that u is a prefix of v and by $u \perp v$ the fact that u, v are incomparable for the ordering $\preceq$. Given $w \in A^* \setminus \{\varepsilon\}$, we denote by $\mathsf{last}(w)$ the last (i.e. rightmost) letter of w. We call *signature* a set of symbols $\mathcal{F}$ with fixed arity $\mathsf{ar} : \mathcal{F} \to \mathbb{N}$. The subset of symbols of arity m is denoted by $\mathcal{F}_m$.

As usual, a set $P \subseteq \mathbb{N}^*$ is called a *tree-domain* (or, domain, for short) iff, for every $u \in \mathbb{N}^*, i \in \mathbb{N}$ $(u \cdot i \in P \Rightarrow u \in P)$ & $(u \cdot (i+1) \in P \Rightarrow u \cdot i \in P)$. We call $P' \subseteq P$ a *subdomain* of P iff, P' is a domain and, for every $u \in P, i \in \mathbb{N}$

$$(u \cdot i \in P' \;\&\; u \cdot (i+1) \in P) \quad \Rightarrow \quad u \cdot (i+1) \in P'.$$

A (first-order) *term* on a signature $\mathcal{F}$ is a partial map $t : \mathbb{N}^* \to \mathcal{F}$ whose domain is a tree-domain and which respects the arities. We denote by $\mathcal{T}(\mathcal{F}, \mathcal{V})$ the set of first-order terms built upon the signature $\mathcal{F} \cup \mathcal{V}$, where $\mathcal{F}$ is a denumerable signature and $\mathcal{V}$ is a denumerable set of variables of arity 0.

The domain of t is also called its set of *positions* and denoted by $\mathcal{P}\mathrm{os}(t)$. The set of variable positions (resp. non-variable positions) of a term t is denoted by $\mathcal{P}\mathrm{os}_{\mathcal{V}}(t)$ (resp. $\mathcal{P}\mathrm{os}_{\overline{\mathcal{V}}}(t)$). We write $\mathcal{P}\mathrm{os}^+(t)$ for $\mathcal{P}\mathrm{os}(t) \setminus \{\varepsilon\}$. If $u, v \in \mathcal{P}\mathrm{os}(t)$ and $u \preceq v$, we say that u is an *ancestor* of v in t. Given $v \in \mathcal{P}\mathrm{os}^+(t)$, its *father* is the position u such that $v = uw$ and $|w| = 1$. The *depth* of a term t is defined by: $dpt(t) := \sup\{|u| \mid u \in \mathcal{P}\mathrm{os}_{\overline{\mathcal{V}}}(t)\} + 1$. Given a term t and $u \in \mathcal{P}\mathrm{os}(t)$ the *subterm of t at u* is denoted by t/u and defined by $\mathcal{P}\mathrm{os}(t/u) = \{w \mid uw \in \mathcal{P}\mathrm{os}(t)\}$ and $\forall w \in \mathcal{P}\mathrm{os}(t/u)$, $t/u(w) = t(uw)$. A term s is a *prefix* of the term t iff there exists a substitution σ such that $s\sigma = t$. A term containing no variable is called *ground*. The set of ground terms is abbreviated to $\mathcal{T}(\mathcal{F})$ or $\mathcal{T}$ whenever $\mathcal{F}$ is understood. A term which does not contain twice the same variable is called *linear*. Given a linear term $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, $x \in \mathcal{V}\mathrm{ar}(t)$, we shall denote by $\mathsf{pos}(t, x)$ the position of x in t.

Among all the variables, there is a special one designated by $\square$. A term containing exactly one occurrence of $\square$ is called a *context*. We denote by $\mathcal{C}_1(\mathcal{F})$ the set of all contexts over $\mathcal{F}$.

A context is usually denoted as $C[]$. If u is the position of $\square$ in $C[]$, $C[t]$ denotes the term $C[]$ where t has been substituted at position u. We also denote by $C[]_u$ such a context and by $C[t]_u$ the result of the substitution. Intuitively, the symbol $\square$ denotes a "hole" in C, while $C[t]$ denotes what is obtained by plugging the term t in the hole of $C[]$. We denote by $|t| := \mathsf{Card}(\mathcal{P}\mathrm{os}(t))$ the *size* of a term t.

2.3. Semi-Thue systems

Let A be a set that we take as an alphabet. A *rewrite rule* over the alphabet A is a pair $u \to v$ of words in A^*. We call u (resp. v) the *left-hand side* (resp. *right-hand side*) of the rule (*lhs* and *rhs* for short). A *semi-Thue system* is a pair (S, A) where A is an alphabet and S a set of rewrite rules built upon the alphabet A. When A is clear from the context or contains exactly the symbols occurring in S, we may omit A and write simply S. The one-step derivation generated by S (which is denoted by $\to_S$) is defined by: for every $f, g \in A^*$, $f \to_S g$ iff there exists $u \to v \in S$ and $\alpha, \beta \in A^*$ such that $f = \alpha u \beta$ and $g = \alpha v \beta$. The relation $\to_S^*$ (defined in Section 2.1) is also called the *derivation generated by S*.

2.4. Term rewriting systems

A *rewrite rule* built over the signature $\mathcal{F}$ is a pair $l \to r$ of terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$ which satisfy $\mathcal{V}\mathrm{ar}(r) \subseteq \mathcal{V}\mathrm{ar}(l)$. We call l (resp. r) the *left-hand side* (resp. *right-hand side*) of the rule (*lhs* and *rhs* for short). A rule is *ground* if both its left and right-hand sides are ground. A rule is *linear* if both its left and right-hand sides are linear. A rule is *left-linear* if its left-hand side is linear.

A *term rewriting system* (*system* for short) is a pair $(\mathcal{R}, \mathcal{F})$ where $\mathcal{F}$ is a signature and $\mathcal{R}$ a set of rewrite rules built upon the signature $\mathcal{F}$. When $\mathcal{F}$ is clear from the context or contains exactly the symbols of $\mathcal{R}$, we may omit $\mathcal{F}$ and write simply $\mathcal{R}$. A system is *ground* (resp. *linear*, *left-linear*) if each of its rules is ground (resp. linear, left-linear). A system $\mathcal{R}$ is *shallow* (Godoy et al., 2003) if, in every side of rule, variables can occur only at depth 0 or 1. A system $\mathcal{R}$ is *growing* (Jacquemard, 1996) if every variable of a right-hand side is at depth at most 1 in the corresponding left-hand side. Rewriting is defined as usual: for every $t, t' \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, $t \to_{\mathcal{R}} t'$ means that there exists $C \in \mathcal{C}_1(\mathcal{F} \cup \mathcal{V}), l \to r \in \mathcal{R}, \sigma : \mathcal{V} \to \mathcal{T}(\mathcal{F}, \mathcal{V})$ such that

$$t = C[l\sigma]_u, \qquad t' = C[r\sigma]_u. \tag{3}$$

For this step: $l \to r$ is the rule used, $l\sigma$ is the *redex* and $r\sigma$ is the *contractum*.

2.5. Words viewed as terms

In order to transfer every definition (or statement) about Term Rewriting Systems into a similar one about Semi-Thue systems, we define here precisely an embedding of the set of words (resp. semi-Thue systems) over an alphabet A into the set of terms (resp. Term Rewriting Systems) over some signature $\mathcal{F}$.

Let A be some alphabet. We define the signature $\mathcal{F}(A)$ by

$$\mathcal{F}(A) := A \cup \{\#_0\}, \qquad \forall a \in A, \quad \mathrm{ar}(a) = 1 \quad \text{and} \quad \mathrm{ar}(\#_0) = 0.$$

We define two mappings $\mathcal{F}_i : A^* \to \mathcal{T}(\mathcal{F}(A), \{\Box\})$ ($i \in \{0, 1\}$) by setting:

$$\mathcal{F}_1(\varepsilon) = \Box, \qquad \mathcal{F}_1(a_1 a_2 \cdots a_n) = a_1\bigl(a_2\bigl(\cdots\bigl(a_n(\Box)\bigr)\cdots\bigr)\bigr),$$
$$\mathcal{F}_0(\varepsilon) = \#_0, \qquad \mathcal{F}_0(a_1 a_2 \cdots a_n) = a_1\bigl(a_2\bigl(\cdots\bigl(a_n(\#_0)\bigr)\cdots\bigr)\bigr).$$

Note that, for every word w, $\mathcal{F}_1(w)$ is a context while $\mathcal{F}_0(w)$ is a ground term. We associate with every rewriting rule $u \to v$, the (term) rewriting rule

$$\mathcal{F}(u \to v) := \mathcal{F}_1(u) \to \mathcal{F}_1(v),$$

and with every semi-Thue system (S, A) the term-rewriting system

$$\bigl(\mathcal{F}(S), \mathcal{F}(A)\bigr) \quad \text{where } \mathcal{F}(S) := \bigl\{\mathcal{F}(u \to v) \bigm| u \to v \in S\bigr\}.$$

The following lemma is straightforward.

Lemma 2.1. *Let (S, A) be a semi-Thue system and $w, w' \in A^*$. Then*

$$w \to_S w' \quad \Leftrightarrow \quad \mathcal{F}_0(w) \to_{\mathcal{F}(S)} \mathcal{F}_0\bigl(w'\bigr) \quad \Leftrightarrow \quad \mathcal{F}_1(w) \to_{\mathcal{F}(S)} \mathcal{F}_1\bigl(w'\bigr).$$

In the sequel, the explicit application of $\mathcal{F}_1$ will be sometimes omitted: if $w \in A^*$ and $t \in \mathcal{T}(\mathcal{F}(A), \{\Box\})$, the expression $w(t)$ will denote the unary term $\mathcal{F}_1(w)[t]$.

2.6. Automata

We shall consider bottom-up finite term (tree) automata only (Comon et al., 2002) (which we abbreviate to *f.t.a.*). An *f.t.a.* is a 4-tuple $\mathcal{A} := (\mathcal{F}, Q, Q_f, \Gamma)$ where $\mathcal{F}$ is the signature, Q is a finite set of symbols of arity 0, called the set of *states*, Q_f is the set of *final* states, Γ is the set of *transitions*. Every element of Γ has the form

$$q \to q' \tag{4}$$

for some $q, q' \in Q$, or

$$f(q_1, \ldots, q_m) \to q \tag{5}$$

for some $m \geq 0$, $f \in \mathcal{F}_m$, $q_1, \ldots, q_m \in Q$. The set of rules Γ can be viewed as a rewriting system over the signature $\mathcal{F} \cup Q$. We then denote by $\to_\Gamma$ or by $\to_\mathcal{A}$ (resp. by $\to_\Gamma^*$ or by $\to_\mathcal{A}^*$) the one-step rewriting relation (resp. the rewriting relation) generated by Γ.

Given an automaton $\mathcal{A}$, the set of terms accepted by $\mathcal{A}$ is defined by:

$$L(\mathcal{A}) := \{t \in \mathcal{T}(\mathcal{F}) \mid \exists q \in Q_f,\ t \to_\mathcal{A}^* q\}.$$

A set of terms T is *recognizable* if there exists a finite term automaton $\mathcal{A}$ such that $T = L(\mathcal{A})$.

The automaton $\mathcal{A}$ is called *deterministic* iff

- Γ possesses no rule of the form (4),
- for every $t, u, u' \in \mathcal{T}(\mathcal{F} \cup Q)$, $(t \to u \in \Gamma \ \&\ t \to u' \in \Gamma) \Rightarrow (u = u')$.

The automaton $\mathcal{A}$ is called *complete* iff for every $m \geq 0$, $f \in \mathcal{F}_m$ and m-tuple of states $(q_1, \ldots, q_m) \in Q^m$, either ($m = 0$ and $f \in Q$) or, there exists $q \in Q$ such that

$$f(q_1, \ldots, q_m) \to q \in \Gamma.$$

2.7. Automata and rewriting

A system $\mathcal{R} \subseteq \mathcal{T}(\mathcal{F} \cup \mathcal{V}) \times \mathcal{T}(\mathcal{F} \cup \mathcal{V})$ is *inverse recognizability preserving* if $(\to_\mathcal{R}^*)[T]$ is recognizable for every recognizable language $T \subseteq \mathcal{T}(\mathcal{F})$.

3. Bottom-up rewriting

In order to define *bottom-up rewriting*, we need some marking tools. In the following we assume that $\mathcal{F}$ is a signature. We shall illustrate many of our definitions with the following system $(\mathcal{R}_1, \mathcal{F})$.

Example 3.1. $\mathcal{R}_1 = \{\mathsf{f}(x) \to \mathsf{g}(x), \mathsf{g}(\mathsf{h}(x)) \to \mathsf{i}(x), \mathsf{i}(x) \to \mathsf{a}\}$, $\mathcal{F} = \{\mathsf{a}, \mathsf{f}, \mathsf{g}, \mathsf{h}, \mathsf{i}\}$ with $\mathrm{ar}(\mathsf{a}) = 0$, $\mathrm{ar}(\mathsf{f}) = 1$, $\mathrm{ar}(\mathsf{g}) = 1$, $\mathrm{ar}(\mathsf{h}) = 1$, $\mathrm{ar}(\mathsf{i}) = 1$.

3.1. Marking

As in Geser et al. (2004), we may mark the symbols of a term using natural integers.

3.1.1. Marked symbols

Definition 3.2. We define the (infinite) *signature of marked symbols*:

$$\mathcal{F}^\mathbb{N} = \{f^i \mid f \in \mathcal{F},\ i \in \mathbb{N}\}.$$

For every integer $k \geq 0$ we note: $\mathcal{F}^{\leq k} = \{f^i \mid f \in \mathcal{F},\ 0 \leq i \leq k\}$. The mapping $\mathsf{m} : \mathcal{F}^\mathbb{N} \to \mathbb{N}$ maps every marked symbol into its mark: $\mathsf{m}(f^i) = i$.

3.1.2. Marked terms

Definition 3.3. The terms in $\mathcal{T}(\mathcal{F}^{\mathbb{N}}, \mathcal{V})$ are called *marked* terms.

The mapping m is extended to marked terms by:

if $t \in \mathcal{V}$, $\mathsf{m}(t) = 0$, otherwise, $\mathsf{m}(t) = \mathsf{m}(t(\varepsilon))$.

For every $f \in \mathcal{F}$, we identify f^0 and f; it follows that $\mathcal{F} \subset \mathcal{F}^{\mathbb{N}}$, $\mathcal{T}(\mathcal{F}) \subset \mathcal{T}(\mathcal{F}^{\mathbb{N}})$ and $\mathcal{T}(\mathcal{F}, \mathcal{V}) \subset \mathcal{T}(\mathcal{F}^{\mathbb{N}}, \mathcal{V})$.

Example. $\mathsf{m}(\mathsf{a}^2) = 2$, $\mathsf{m}(\mathsf{i}(\mathsf{a}^2)) = 0$, $\mathsf{m}(\mathsf{h}^1(\mathsf{a})) = 1$, $\mathsf{m}(\mathsf{h}^1(x)) = 1$, $\mathsf{m}(x) = 0$.

Definition 3.4. Given $t \in \mathcal{T}(\mathcal{F}^{\mathbb{N}}, \mathcal{V})$ and $i \in \mathbb{N}$, we define the marked term t^i whose marks are all equal to i:

$$\begin{array}{ll} \text{if } t \text{ is a variable } x & t^i = x, \\ \text{if } t \text{ is a constant } c & t^i = c^i, \\ \text{otherwise } \big(t = f(t_1, \ldots, t_n)\big) \text{ where } n \geq 1 & t^i = f^i\big({t_1}^i, \ldots, {t_n}^i\big). \end{array}$$

This marking extends to sets of terms S ($S^i = \{t^i \mid t \in S\}$) and substitutions σ ($\sigma^i : x \mapsto (x\sigma)^i$).

We use $\mathsf{mmax}(t)$ (resp. $\mathsf{mmin}(t)$) to denote the maximal (resp. minimal) mark of a marked term t.

$$\mathsf{mmax}(t) := \max\big\{\mathsf{m}(t/u) \;\big|\; u \in \mathcal{P}\mathsf{os}(t)\big\}, \qquad \mathsf{mmin}(t) := \min\big\{\mathsf{m}(t/u) \;\big|\; u \in \mathcal{P}\mathsf{os}(t)\big\}.$$

For $u \in \mathcal{P}\mathsf{os}^+(t)$, $\mathsf{mmax}^{\prec u}(t) := \max\{\mathsf{m}(t/v) \mid v \prec u\}$.

Example. $\mathsf{mmax}(\mathsf{i}(\mathsf{a}^2)) = 2$, $\mathsf{mmin}(\mathsf{i}(\mathsf{a}^2)) = 0$, $\mathsf{mmax}^{\prec 1.1}(\mathsf{g}(\mathsf{h}^1(\mathsf{a}^2))) = 1$.

Notation: in the sequel, given a term $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, $\bar{t}$ will always refer to a term of $\mathcal{T}(\mathcal{F}^{\mathbb{N}}, \mathcal{V})$ such that $\bar{t}^0 = t$. The same rule will apply to substitutions and contexts.

Finite automata and marked terms. Given an *f.t.a.* $\mathcal{A} = (\mathcal{F}, Q, Q_f, \Gamma)$ we extend it over the signature $\mathcal{F}^{\leq k}$, by setting

$$\Gamma^{\leq k} := \big\{\big(f^j\big(q_1^{j_1}, \ldots, q_n^{j_n}\big) \to q^j\big) \;\big|\; \big(f(q_1, \ldots, q_n) \to q\big) \in \Gamma,\ j, j_1, \ldots, j_n \in [0, k]\big\},$$

and

$$\mathcal{A}^{\leq k} := \big(\mathcal{F}^{\leq k}, Q^{\leq k}, Q_f^{\leq k}, \Gamma^{\leq k}\big).$$

Since, for every integers k, k', $\mathcal{A}^{\leq k}$ and $\mathcal{A}^{\leq k'}$ have the same action on terms with marks not greater than $\min(k, k')$, we often denote by $\mathcal{A}$ any extension $\mathcal{A}^{\leq k}$ with a sufficiently large k w.r.t. the terms under consideration.

$\mathbb{N}$ *acts on marked terms.* We define a right-action $\odot$ of the monoid $(\mathbb{N}, \max, 0)$ over the set $\mathcal{F}^{\mathbb{N}}$ which just consists in applying the operation max on every mark: for every $\bar{t} \in \mathcal{F}^{\mathbb{N}}, n \in \mathbb{N}$,

$$\mathcal{P}\mathsf{os}(\bar{t} \odot n) := \mathcal{P}\mathsf{os}(\bar{t}), \qquad (\bar{t} \odot n)^0 = \bar{t}^0,$$

$$\forall u \in \mathcal{P}\mathsf{os}(\bar{t}), \quad \mathsf{m}\big((\bar{t} \odot n)/u\big) := \max\big(\mathsf{m}(\bar{t}/u), n\big).$$

Since a marked term can be viewed as a map from its domain to the direct product $\mathcal{F} \times \mathbb{N}$, and since the operation $\odot$ acts on the second component only while every *f.t.a.* acts on the first component only, the following statement is straightforward.

Lemma 3.5. *Let $\mathcal{A}$ be some finite term automaton over $\mathcal{F}$, $\bar{s}, \bar{t} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}})$ and $n \in \mathbb{N}$. If $\bar{s} \to_{\mathcal{A}}^{*} \bar{t}$ then $(\bar{s} \odot n) \to_{\mathcal{A}}^{*} (\bar{t} \odot n)$.*

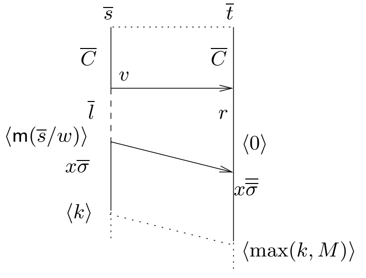

Fig. 1. A marked rewriting step.

3.1.3. *Marked rewriting*

We define here the rewrite relation $\circ\!\!\to$ between marked terms. For every linear marked term $\bar{t} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}}, \mathcal{V})$ and variable $x \in \mathcal{V}\mathrm{ar}(\bar{t})$, we define:

$$\mathsf{M}(\bar{t}, x) = \sup\{\mathsf{m}(\bar{t}/w) \mid w \prec \mathrm{pos}(\bar{t}, x)\} + 1. \tag{6}$$

Let $\mathcal{R}$ be a left-linear system, $\bar{s} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}})$ and $t \in \mathcal{T}$. Let us suppose that $\bar{s} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}})$ decomposes as

$$\bar{s} = \overline{C}[\bar{l}\bar{\sigma}]_v, \quad \text{with } (l, r) \in \mathcal{R}, \tag{7}$$

for some marked context $\overline{C}[]_v$ and substitution $\bar{\sigma}$. We define a new marked substitution $\bar{\bar{\sigma}}$ (such that $\bar{\bar{\sigma}}^0 = \bar{\sigma}^0$) by: for every $x \in \mathcal{V}\mathrm{ar}(r)$,

$$x\bar{\bar{\sigma}} := (x\bar{\sigma}) \odot M(\overline{C}[\bar{l}], x). \tag{8}$$

We then write $\bar{s} \circ\!\!\to \bar{t}$ where

$$\bar{s} = \overline{C}[\bar{l}\bar{\sigma}], \qquad \bar{t} = \overline{C}[r\bar{\bar{\sigma}}]. \tag{9}$$

(This is illustrated in Fig. 1, where M denotes $M(\overline{C}[\bar{l}], x)$ and the marks are written between brackets $\langle \cdots \rangle$.) More precisely, an ordered pair of marked terms $(\bar{s}, \bar{t})$ is linked by the relation $\circ\!\!\to$ iff, there exist $\overline{C}[]_v, (l, r), \bar{l}, \bar{\sigma}$ and $\bar{\bar{\sigma}}$ fulfilling Eqs. (7)–(9). The intuitive idea behind the above definition is that the marks are storing the relevant information concerning the *ordering* of successive positions of redexes during the derivation. A mark k will roughly mean that there were k successive applications of rules, each one with a leaf of the left-hand side at a position strictly greater than a leaf of the previous right-hand side.

The map $\bar{s} \mapsto \bar{s}^0$ (from marked terms to unmarked terms) extends into a map from marked derivations to unmarked derivations: every

$$\bar{s}_0 = \overline{C}_0[\bar{l}_0\bar{\sigma}_0]_{v_0} \circ\!\!\to \overline{C}_0[r_0\bar{\bar{\sigma}}_0]_{v_0} = \bar{s}_1 \circ\!\!\to \cdots \circ\!\!\to \overline{C}_{n-1}[r_{n-1}\bar{\bar{\sigma}}_{n-1}]_{v_{n-1}} = \bar{s}_n \tag{10}$$

is mapped to the derivation

$$s_0 = C_0[l_0\sigma_0]_{v_0} \to C_0[r_0\sigma_0]_{v_0} = s_1 \to \cdots \to C_{n-1}[r_{n-1}\sigma_{n-1}]_{v_{n-1}} = s_n. \tag{11}$$

The context $\overline{C}_i[]_{v_i}$, the rule (l_i, r_i), the marked version $\bar{l}_i$ of l_i and the substitution $\bar{\sigma}_i$ completely determine $\bar{s}_{i+1}$. Thus, for every fixed pair $(\overline{s_0}, s_0)$, this map is a bijection from the set of derivations (10) starting from $\overline{s_0}$, to the set of derivations (11) starting from s_0.

Example 3.6. With the system $\mathcal{R}_1$ of Example 3.1 we get the following marked derivation:

$$\mathsf{f}(\mathsf{h}(\mathsf{f}(\mathsf{h}(\mathsf{a})))) \circ\!\!\to \mathsf{f}(\mathsf{h}(\mathsf{g}(\mathsf{h}^1(\mathsf{a}^1)))) \circ\!\!\to \mathsf{f}(\mathsf{h}(\mathsf{i}(\mathsf{a}^2))) \circ\!\!\to \mathsf{f}(\mathsf{h}(\mathsf{a})) \circ\!\!\to \mathsf{g}(\mathsf{h}^1(\mathsf{a}^1)) \circ\!\!\to \mathsf{i}(\mathsf{a}^2) \circ\!\!\to \mathsf{a}.$$

From now on, each time we deal with a derivation $s \to^* t$ between two terms $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, we may implicitly decompose it as (11) where n is the length of the derivation, $s = s_0$ and $t = s_n$.

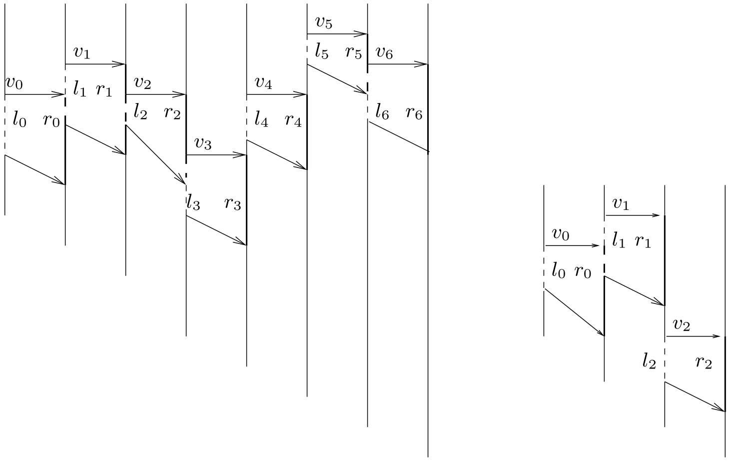

Fig. 2. A wbu (on the left) versus non-wbu (on the right) derivation.

3.2. Bottom-up derivations

Definition 3.7. The marked derivation (10) is *weakly bottom-up* if, for every $0 \leq i < n$,

$$l_i \notin \mathcal{V} \quad \Rightarrow \quad \mathsf{m}(\bar{l_i}) = 0, \tag{12}$$

$$l_i \in \mathcal{V} \quad \Rightarrow \quad \sup\{\mathsf{m}(\bar{s_i}/u) \mid u \prec v_i\} = 0. \tag{13}$$

Definition 3.8. The derivation (11) is *weakly bottom-up* if the corresponding marked derivation (10) starting on the same term $\bar{s} = s$ is *weakly bottom-up* (following the above definition).

We shall abbreviate “weakly bottom-up” to wbu. See Fig. 2. We classify the derivations according to the maximal value of the marks. We abbreviate “bottom-up” to bu.

Definition 3.9. A derivation is $\mathsf{bu}(k)$ (resp. $\mathsf{bu}^-(k)$) if it is wbu and, in the corresponding marked derivation $\forall i \in [0, n]$, $\mathsf{mmax}(\bar{s_i}) \leq k$ (resp. $\forall i \in [0, n-1]$, $\mathsf{mmax}(\bar{l_i}) < k$).

Let us introduce a convenient notation.

Definition 3.10. Let $k \geq 1$. The binary relation ${}_k\!\circ\!\!\to^*_{\mathcal{R}}$ over $\mathcal{T}(\mathcal{F}^{\mathbb{N}})$ is defined by:

$\bar{s} \;{}_k\!\circ\!\!\to^*_{\mathcal{R}}\; \bar{t}$ if and only if there exists a wbu marked derivation from $\bar{s}$ to $\bar{t}$ where all the marks belong to $[0, k]$.

The binary relation ${}_k\!\to^*_{\mathcal{R}}$ over $\mathcal{T}(\mathcal{F})$ is defined by:

$s \;{}_k\!\to^*_{\mathcal{R}}\; t$ if and only if there exists a $\mathsf{bu}(k)$-derivation from s to t.

Example 3.11. For the system $\mathcal{R}_0 = \{\mathsf{f}(\mathsf{f}(x)) \to \mathsf{f}(x)\}$ with the signature $\mathcal{F} = \{\mathsf{a}^{(0)}, \mathsf{f}^{(1)}\}$, although for every k we may get a $\mathsf{bu}(k)$-derivation for a term of the form $\mathsf{f}(\cdots\mathsf{f}(\mathsf{a})\cdots)$ with $k+1$ f symbols:

$$\mathsf{f}\big(\mathsf{f}\big(\mathsf{f}\big(\mathsf{f}(\mathsf{a})\big)\big)\big) \circ\!\!\to \mathsf{f}\big(\mathsf{f}^1\big(\mathsf{f}^1\big(\mathsf{a}^1\big)\big)\big) \circ\!\!\to \mathsf{f}\big(\mathsf{f}^2\big(\mathsf{a}^2\big)\big) \circ\!\!\to \mathsf{f}\big(\mathsf{a}^3\big)$$

we can always achieve a $\mathsf{bu}(1)$-derivation:

$$\mathsf{f}\big(\mathsf{f}\big(\mathsf{f}\big(\mathsf{f}(\mathsf{a})\big)\big)\big) \circ\!\!\to \mathsf{f}\big(\mathsf{f}\big(\mathsf{f}\big(\mathsf{a}^1\big)\big)\big) \circ\!\!\to \mathsf{f}\big(\mathsf{f}\big(\mathsf{a}^1\big)\big) \circ\!\!\to \mathsf{f}\big(\mathsf{a}^1\big).$$

Example *(Comparison with innermost derivation).* Let us consider the signature $\mathcal{F} := \{\mathsf{a}^{(0)}, \mathsf{f}^{(1)}, \mathsf{g}^{(1)}\}$ and the rewriting system $\mathcal{R} := \{\mathsf{fg}(x) \to \mathsf{gf}(x), \mathsf{gf}(x) \to \mathsf{h}(x)\}$. The derivation

$$\mathsf{fggg(a)} \to \mathsf{gfgg(a)} \to \mathsf{ggfg(a)} \to \mathsf{gggf(a)}$$

corresponds to the marked derivation

$$\mathsf{fggg(a)} \circ\!\!\to \mathsf{gfg}^1\mathsf{g}^1\big(\mathsf{a}^1\big) \circ\!\!\to \mathsf{ggfg}^2\big(\mathsf{a}^2\big) \circ\!\!\to \mathsf{gggf}\big(\mathsf{a}^3\big)$$

which is not $\mathsf{BU}(2)$; note however, that this derivation is *innermost* i.e. each derivation step rewrites the innermost redex of the given term.

The derivation

$$\mathsf{fgfgfg(a)} \to \mathsf{fgfhg(a)} \to \mathsf{fhhg(a)}$$

corresponds to the marked derivation

$$\mathsf{fgfgfg(a)} \circ\!\!\to \mathsf{fgfhg}^1\big(\mathsf{a}^1\big) \circ\!\!\to \mathsf{fhh}^1\mathsf{g}^1\big(\mathsf{a}^1\big)$$

which is $\mathsf{BU}(1)$; note however, that this derivation is *not innermost* since the innermost redex of the first term is $\mathsf{fg(a)}$, which is not rewritten in this derivation.

3.3. *Bottom-up systems*

We introduce here a hierarchy of classes of rewriting systems.[2]

Definition 3.12. Let P be some property of derivations w.r.t. Term Rewriting Systems.

1. A Term Rewriting System $(\mathcal{R}, \mathcal{F})$ is called P if for every $s, t \in \mathcal{T}(\mathcal{F})$ such that $s \to_{\mathcal{R}}^{*} t$ there exists a P-derivation from s to t.
2. A semi-Thue system (S, A) is called P if the Term Rewriting System $(\mathcal{F}(S), \mathcal{F}(A))$ is called P.

We shall use the convention that, for a property P denoted by a lower-case acronym for derivations, we use the same acronym, but in upper-case, to denote the property P extended to systems by Definition 3.12. For example, a Term Rewriting System $(\mathcal{R}, \mathcal{F})$ is called $\mathsf{BU}(k)$ if for every $s, t \in \mathcal{T}(\mathcal{F})$ such that $s \to_{\mathcal{R}}^{*} t$ there exists a $\mathsf{bu}(k)$-derivation from s to t.

We denote by $\mathsf{BU}(k)$ the class of $\mathsf{BU}(k)$ systems, by $\mathsf{BU}^-(k)$ the class of $\mathsf{BU}^-(k)$ systems. We define the class of *bottom-up systems*, denoted BU, by:

$$\mathsf{BU} := \bigcup_{k \in \mathbb{N}} \mathsf{BU}(k).$$

Lemma 3.13. *For every* $k > 0$, $\mathsf{BU}(k-1) \subsetneq \mathsf{BU}^-(k) \subsetneq \mathsf{BU}(k)$.

This new class BU encompasses the following classes already known to inverse-preserve recognizability (these classes were studied in Sakarovitch (1979) (see Sénizergues, 1995 for a survey); Godoy et al., 2003; Jacquemard, 1996; Takai et al., 2010).

Lemma 3.14.

Every right-ground system is $\mathsf{BU}(0)$.
Every inverse of a left-basic semi-Thue system is $\mathsf{BU}^-(1)$.

[2] A *class* of TRS is a subset of the set of all TRS (over a fixed denumerable ranked alphabet) which is closed under alphabet isomorphism (i.e. renaming the symbols).

Every shallow system is $\mathsf{BU}^-(1)$.
Every growing linear system is $\mathsf{BU}(1)$.
Every inverse of a linear FPO system is BU.

Precise proofs are given in Durand and Sénizergues (2009, Lemmas 3.21–3.24, Proposition 6.15). In fact the five above subclasses are all fulfilling the sufficient condition presented in Section 5.3, which is polynomially testable.

Example 3.15.

The system $\mathcal{R}_0 = \{\mathsf{f}(\mathsf{f}(x)) \to \mathsf{f}(x)\} \in \mathsf{BU}^-(1)$ and $\mathcal{R}_0$ is not growing.
The system $\mathcal{R}_1$ of Example 3.1 belongs to $\mathsf{BU}^-(2)$ and $\mathcal{R}_1$ is not growing.
The system $\mathcal{R}_2 = \{\mathsf{f}(x) \to \mathsf{g}(x), \mathsf{h}(\mathsf{g}(\mathsf{a})) \to \mathsf{a}\}$ is growing and belongs to $\mathsf{BU}^-(1)$.
The system $\mathcal{R}_3 = \{\mathsf{f}(x) \to \mathsf{g}(x), \mathsf{g}(\mathsf{h}(x)) \to \mathsf{a}\}$ is growing and belongs to $\mathsf{BU}(1)$.

Note that $\mathcal{R}_0$ shows that the fourth inclusion of Lemma 3.14 is strict.

3.4. More about marked derivations

Some more knowledge on marked derivations will be needed in subsequent sections.

The next lemma shows that in the case of a linear system, a derivation can always be replaced by a wbu-derivation.

Lemma 3.16. *Let $\mathcal{R}$ be a linear system. If $s \to^*_{\mathcal{R}} t$ then there exists a* wbu*-derivation between s and t.*

Sketch of proof. We prove by induction on the integer n, that, for every derivation $s \to^n t$, there exists a wbu-derivation from s to t, with the same length n and reducing the same redexes of s

Basis: $n = 0$.

Then $s = t$; the empty derivation is wbu.

Induction step: $n > 0$.

As $\mathcal{R}$ is linear, every redex may have at most one descendant in each term of the derivation. We choose a redex $l\sigma$ ($l \to r \in \mathcal{R}$) of s, whose position is maximal (w.r.t. $\preceq$) among the set of positions of redexes contracted somewhere in the derivation $s \to^n t$; let u be the position of this maximal redex in s. A new derivation can be obtained by transferring the contraction of $l\sigma$ at the beginning of the derivation: we obtain a derivation of equal length

$$s = C[l\sigma]_u \to C[r\sigma]_u \to^{n-1} t. \tag{14}$$

By induction hypothesis, the derivation $C[r\sigma]_u \to^{n-1} t$ can be made wbu while preserving its length $n - 1$ and the set of redexes of $C[r\sigma]_u$ that are contracted. Let us consider the unique marked derivation associated to (14):

$$s = C[l\sigma]_u \circ\!\!\to C[r\bar{\bar{\sigma}}]_u \circ\!\!\to^{n-1} \bar{\bar{t}} \tag{15}$$

and the unique marked derivation associated to the wbu-derivation $C[r\sigma]_u \to^{n-1} t$:

$$C[r\sigma]_u \circ\!\!\to^{n-1} \bar{t}. \tag{16}$$

By assumption and preservation of the redexes, σ does not contain any redex which is contracted inside the derivation $C[r\sigma] \to^{n-1} t$. Hence, for every $j \in [0, n-1[$, the $(j+1)$th step of derivation (15) uses an lhs with a root that possesses the same mark as the root of the lhs of the jth step of derivation (16). Since this mark is always null in (16), it is also null in (15). This shows that (14) is wbu. $\square$

Definition 3.17. A marked term $\bar{s}$ is said to be m-*increasing* iff, for every $u, v \in \mathcal{P}\mathrm{os}(\bar{s}), u \preceq v \Rightarrow \mathrm{m}(\bar{s}/u) \le \mathrm{m}(\bar{s}/v)$.

Lemma 3.18. *Suppose that $\bar{s}$ is an* m-increasing *marked term, $\bar{t} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}}, \mathcal{V}) \setminus \mathcal{V}$, $\mathrm{m}(\bar{t}) = 0$, $\bar{C}[]_v$ is a marked context, $\bar{\sigma}$ is a marked substitution and $\bar{s} = \bar{C}[\bar{t}\bar{\sigma}]_v$.*

Then, $\bar{C}[]_v$ has no mark above the position v.

Proof. Let $u \in \mathcal{P}\mathrm{os}(\bar{C})$ such that $u \prec v$ and $\bar{C}(v) = \Box$.

Since $\bar{C}[\bar{t}\bar{\sigma}]$ is m-increasing,

$$\mathrm{m}\big(\bar{C}[\bar{t}\bar{\sigma}]/u\big) \le \mathrm{m}\big(\bar{C}[\bar{t}\bar{\sigma}]/v\big).$$

But $\mathrm{m}(\bar{C}[\bar{t}\bar{\sigma}]/v) = \mathrm{m}(\bar{t}) = 0$. $\Box$

Lemma 3.19. *Let $\bar{s} \circ\!\!\rightarrow \bar{t}$ be a* wbu *marked derivation-step between $\bar{s}, \bar{t} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}})$. If $\bar{s}$ is* m-*increasing, then $\bar{t}$ is* m-*increasing too.*

Proof. Suppose that $\bar{s}$ is m-increasing and that $\bar{s}, \bar{t}$ fulfill (7)–(9). Let us consider $v_1, v_2 \in \mathcal{P}\mathrm{os}(\bar{t})$ such that $v_1 \preceq v_2$. Let us show that

$$\mathrm{m}(\bar{t}/v_1) \le \mathrm{m}(\bar{t}/v_2). \tag{17}$$

We distinguish 3 cases depending on the relative positions of v, v_1, v_2.

Case 1. $v_1 \prec v$.

Since the derivation step is wbu, we have $\mathrm{m}(\bar{s}/v) = 0$. According to the definition of a marked derivation-step, we have $\mathrm{m}(\bar{s}/v_1) = \mathrm{m}(\bar{t}/v_1)$. Moreover, $\bar{s}$ is m-increasing. Hence, $\mathrm{m}(\bar{s}/v) = \mathrm{m}(\bar{s}/v_1) = 0 \le \mathrm{m}(\bar{t}/v_2)$.

Case 2. $v \preceq v_1$.

If $v_1 \in v \cdot \mathcal{P}\mathrm{os}(r)$ then we have $\mathrm{m}(\bar{t}/v_1) = 0 \le \mathrm{m}(\bar{t}/v_2)$. Otherwise, $v_1 = v \cdot w \cdot w_1$, $v_2 = v \cdot w \cdot w_2$, where x is the label of w in r and $w_1, w_2 \in \mathcal{P}\mathrm{os}(x\sigma)$. Let

$$v_1' := v \cdot w' \cdot w_1, \qquad v_2' := v \cdot w' \cdot w_2$$

where $w' = \mathrm{pos}(\ell, x)$.

Since $\bar{s}$ is m-increasing, $\mathrm{m}(\bar{s}/v_1') \le \mathrm{m}(\bar{s}/v_2')$, hence

$$\max\big(\mathrm{m}\big(\bar{s}/v_1'\big), \mathrm{M}\big(C[\bar{l}], x\big)\big) \le \max\big(\mathrm{m}\big(\bar{s}/v_2'\big), \mathrm{M}\big(C[\bar{l}], x\big)\big)$$

i.e. $\mathrm{m}(\bar{t}/v_1) \le \mathrm{m}(\bar{t}/v_2)$.

Case 3. $v_1 \perp v$.

In this case we also have $v_2 \perp v$. It follows that for every $i \in \{1, 2\}$, $\mathrm{m}(\bar{s}/v_i) = \mathrm{m}(\bar{t}/v_i)$, and we can conclude as in Case 1.

In all cases we have established that (17) holds. $\Box$

The previous lemma generalizes to a sequence.

Lemma 3.20. *Let $\bar{s} \circ\!\!\rightarrow^* \bar{t}$ be a* wbu *marked derivation between $\bar{s}, \bar{t} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}})$. If $\bar{s}$ is* m-*increasing, then $\bar{t}$ is* m-*increasing too.*

Proof. Straightforward induction on the length n of the derivation based on Lemma 3.19. $\square$

Remark 3.21. Let us examine the value of $\mathsf{M}(\bar{C}[\bar{l}], x)$ when $s' \circ\!\!\to^* \bar{s} = \bar{C}[\bar{l}\bar{\sigma}] \circ\!\!\to \bar{C}[r\bar{\bar{\sigma}}] = \bar{t}$, and $s' \to^* s$ is wbu:

- if C is the empty context and $l = x$ then $\mathsf{M}(\bar{C}[\bar{l}], x) = 1$,
- otherwise, by Lemma 3.20, $\mathsf{M}(\bar{C}[\bar{l}], x) = \mathsf{m}(\bar{C}[\bar{l}]/f_x) + 1$, where f_x is the father of $\mathsf{pos}(\bar{C}[\bar{l}], x)$.

4. Inverse-preservation of recognizability

Let us recall the following classical result about ground rewriting systems.

Theorem 4.1. *(See Brainerd, 1969.) Every ground system is inverse-recognizability preserving.*

This theorem was further refined and extended in Deruyver and Gilleron (1989), Dauchet and Tison (1990), Dauchet et al. (1990), see Comon et al. (2002) for an exposition. The main theorem of this section (and of the paper) is the following extension of Theorem 4.1 to $\mathsf{bu}(k)$ derivations of linear rewriting systems.

Theorem 4.2. *Let $\mathcal{R}$ be some linear rewriting system over the signature $\mathcal{F}$, let T be some recognizable subset of $\mathcal{T}(\mathcal{F})$ and let $k \geq 0$. Then, the set $({}_k\!\!\to^*_{\mathcal{R}})[T]$ is recognizable too.*

In Section 4.1 we introduce a restricted form of *f.t.a*. We then prove Theorem 4.2 under some mild technical assumptions. We finally show in Section 4.3 that these assumptions can be safely removed.

4.1. Standard automata

Definition 4.3. An *f.t.a.* $\mathcal{A} = (\mathcal{F}, Q, Q_f, \Gamma)$ is called *standard* iff it is deterministic, complete and $\mathcal{F} \cap Q = \mathcal{F}_0$.

Note that, for a standard *f.t.a.* $\mathcal{A}$, the relation $\to_{\mathcal{A}}$ strictly reduces the size of terms. The following lemma extends the property of determinism to tree-domains larger than just a single point.

Lemma 4.4. *Let $\mathcal{A}$ be some standard* f.t.a. *over the signature $\mathcal{F}$. Let $t, t_1, t_2 \in \mathcal{T}(\mathcal{F} \cup Q)$. If $t \to^*_{\mathcal{A}} t_1$, $t \to^*_{\mathcal{A}} t_2$ and $\mathcal{P}\mathsf{os}(t_1) = \mathcal{P}\mathsf{os}(t_2)$, then $t_1 = t_2$.*

We extend to subdomains the usual notion of state reached by some deterministic complete *f.t.a.* from a given term t: we call it the *reduct* of t over the subdomain P.

Definition 4.5 *($\mathcal{A}$-reduct).* Let $\mathcal{A}$ be some standard *f.t.a.* over the signature $\mathcal{F}$. Let $t \in \mathcal{T}(\mathcal{F} \cup Q)$ and let P be some subdomain of $\mathcal{P}\mathsf{os}(t)$. We define $\mathsf{Red}(t, P) = t'$ as the unique element of $\mathcal{T}(\mathcal{F} \cup Q)$ such that

1. $\mathcal{P}\mathsf{os}(t') = P$,
2. $t \to^*_{\mathcal{A}} t'$.

The existence and unicity of such a term $\mathsf{Red}(t, P)$ follows from the technical conditions imposed by Definition 4.3.

Lemma 4.6. *Let $\mathcal{A}$ be some standard* f.t.a. *over the signature $\mathcal{F}$.*
Let $t, t_1, t_2 \in \mathcal{T}(\mathcal{F} \cup Q)$. If $t \to^*_{\mathcal{A}} t_1$, $t \to^*_{\mathcal{A}} t_2$ and $\mathcal{P}\mathsf{os}(t_1) \subseteq \mathcal{P}\mathsf{os}(t_2)$, then $t_2 \to^*_{\mathcal{A}} t_1$.

Proof. Since $t \to_{\mathcal{A}}^{*} t_1$, $t \to_{\mathcal{A}}^{*} t_2 \to_{\mathcal{A}}^{*} \mathsf{Red}(t_2, \mathcal{P}os(t_1))$ and $\mathcal{P}os(t_1) = \mathcal{P}os(\mathsf{Red}(t_2, \mathcal{P}os(t_1)))$, by Lemma 4.4, $t_1 = \mathsf{Red}(t_2, \mathcal{P}os(t_1))$ which implies that $t_2 \to_{\mathcal{A}}^{*} t_1$. □

4.2. Basic construction

In order to prove Theorem 4.2 we have to introduce some technical definitions, and to prove some technical lemmas. Let us fix, from now on and until the end of the subsection, a linear system $(\mathcal{R}, \mathcal{F})$, a language $T \subseteq \mathcal{T}(\mathcal{F})$ recognized by a finite automaton over the extended signature $\mathcal{F} \cup \{\Box\}$, $\mathcal{A} = (\mathcal{F} \cup \{\Box\}, Q, Q_f, \Gamma)$ and an integer $k \geq 0$. In order to make the proofs easier, we assume in this subsection that:

$$\forall l \to r \in \mathcal{R}, \quad l \notin \mathcal{V}, \tag{18}$$

$$\mathcal{A} \text{ is standard.} \tag{19}$$

We postpone to Section 4.3 the proof that these restrictions are not a loss of generality. Let us define the integer

$$\mathsf{d} := \max\{dpt(l) \mid l \to r \in \mathcal{R}\}. \tag{20}$$

Example 4.7. For the system $\mathcal{R}_1$ of Example 3.1, $\mathsf{d} = 2$.

We introduce now a notion of *top* part of a term $\bar{t}$, which is, intuitively, the only part of $\bar{t}$ which can be used in a $\mathsf{bu}(k)$-derivation starting on $\bar{t}$. Everything below this part is merely included in the substitutions used by the derivation-steps, and thus copied (Eq. (20) and Definition 4.8 are tuned for this property). Such copied parts of $\bar{t}$ can be handled just by a state of the *f.t.a.* $\mathcal{A}$. The replacement of terms by their top-part will be used subsequently to show that the full derivations w.r.t. ${}_k\!\!\to_{\mathcal{R}}^{*}$ can be simulated by derivations w.r.t. some "approximating" ground rewriting system (introduced by Definition 4.14): the top-part of a real derivation is a derivation for the ground rewriting system (Lemma 4.21) and, conversely, every derivation for the ground rewriting system is the top-part of some real derivation (Lemma 4.15).

We define, at first, the *top domain* of a term and, later on, the top of a term.

Definition 4.8 *(Top domain of a term).* Let $\bar{t} \in \mathcal{T}((\mathcal{F} \cup Q)^{\leq k}, \{\Box\})$. We define the *top domain* of $\bar{t}$, denoted by $\mathsf{Topd}(\bar{t})$ as: $u \in \mathsf{Topd}(\bar{t})$ iff

1. $u \in \mathcal{P}os(\bar{t})$,
2. $\forall u_1, u_2 \in \mathbb{N}^*$ such that $u = u_1 \cdot u_2$, either $\mathsf{m}(\bar{t}/u_1) = 0$ or $|u_2| \leq (k + 1 - \mathsf{m}(\bar{t}/u_1))\mathsf{d}$.

Lemma 4.9. *For every $\bar{t} \in \mathcal{T}((\mathcal{F} \cup Q)^{\leq k}, \{\Box\})$, $\mathsf{Topd}(\bar{t})$ is a subdomain of $\mathcal{P}os(\bar{t})$.*

Proof. 1. Let $u \in \mathsf{Topd}(\bar{t})$ and let $v \preceq u$. Let $w \in \mathbb{N}^*$ such that $v \cdot w = u$. Suppose that $v = v_1 \cdot v_2$ and that $\mathsf{m}(\bar{t}/v_1) \neq 0$.

Since $u = v_1 \cdot v_2 \cdot w$ and u belongs to $\mathsf{Topd}(\bar{t})$, the inequality $|v_2 w| \leq (k + 1 - \mathsf{m}(\bar{t}/u_1))\mathsf{d}$ holds. But $|v_2| \leq |v_2 w|$, hence

$$|v_2| \leq \big(k + 1 - \mathsf{m}(\bar{t}/u_1)\big)\mathsf{d}.$$

2. Let $u \in \mathbb{N}^*$ and $i, j \in \mathbb{N}$ such that $u \cdot i \in \mathsf{Topd}(\bar{t})$ and $u \cdot j \in \mathcal{P}os(\bar{t})$. Suppose $u_1, u_2 \in \mathbb{N}^*$ such that $u \cdot j = u_1 \cdot u_2$:

– If $u_2 \neq \varepsilon$, since $u \cdot i = u_1 \cdot u_2'$, where $u_2' := u_2(j)^{-1} i$, we know that:

$$\mathsf{m}(\bar{t}/u_1) = 0 \quad \text{or} \quad |u_2'| \leq \big(k + 1 - \mathsf{m}(\bar{t}/u_1)\big)\mathsf{d},$$

which implies, since $|u_2| = |u_2'|$ that:

$$\mathsf{m}(\bar{t}/u_1) = 0 \quad \text{or} \quad |u_2| \leq \bigl(k + 1 - \mathsf{m}(\bar{t}/u_1)\bigr)\mathsf{d}.$$

– If $u_2 = \varepsilon$ the required inequality for $|u_2|$ is obvious. $\square$

Definition 4.10 *(Top of a term).* For every $\bar{t} \in \mathcal{T}((\mathcal{F} \cup Q)^{\leq k}, \{\square\})$, $\mathsf{Top}(\bar{t}) = \mathsf{Red}(\bar{t}, \mathsf{Topd}(\bar{t}))$.

Note that, since $\mathsf{Topd}(\bar{t})$ is a subdomain of $\mathcal{P}\mathsf{os}(\bar{t})$ and is written over the alphabet of the standard automaton $\mathcal{A}$, $\mathsf{Top}(\bar{t})$ is well-defined.

This definition extends naturally, in a pointwise manner, to substitutions.

Lemma 4.11 *(*Top *is morphic). Let* $\bar{C}[]$ *be a context with no mark above the symbol* $\square$ *and let* $\bar{t}$ *be any marked term in* $\mathcal{T}((\mathcal{F} \cup Q)^{\leq k})$*. Then* $\mathsf{Top}(\bar{C}[\bar{t}]) = \mathsf{Top}(\bar{C})[\mathsf{Top}(\bar{t})]$.

The proof is easy and therefore omitted.

Lemma 4.12 *(*Top *preserves unmarked terms). If* $t \in \mathcal{T}(\mathcal{F} \cup Q, \mathcal{V})$ *and* $\bar{\sigma} : \mathcal{V} \to \mathcal{T}((\mathcal{F} \cup Q))^{\mathbb{N}}$ *then* $\mathsf{Top}(t\bar{\sigma}) = t\mathsf{Top}(\bar{\sigma})$.

Proof. The proof is easy and therefore omitted. $\square$

Lemma 4.13 *(*Top *is decreasing). Let* $\bar{s}, \bar{t} \in \mathcal{T}((\mathcal{F} \cup Q)^{\mathbb{N}})$ *be such that* $\mathcal{P}\mathsf{os}(\bar{s}) = \mathcal{P}\mathsf{os}(\bar{t})$ *and, such that, for every* $u \in \mathcal{P}\mathsf{os}(\bar{s})$, $\mathsf{m}(\bar{s}/u) \leq \mathsf{m}(\bar{t}/u)$*.*

Then $\mathcal{P}\mathsf{os}(\mathsf{Top}(\bar{s})) \supseteq \mathcal{P}\mathsf{os}(\mathsf{Top}(\bar{t}))$.

Definition 4.14. We consider the following ground rewriting system $\mathcal{S}$ over $\mathcal{T}((\mathcal{F} \cup Q)^{\leq k})$ consisting of all the rules of the form:

$$\bar{l}\bar{\tau} \to r\bar{\bar{\tau}} \tag{21}$$

where $l \to r$ is a rule of $\mathcal{R}$

$$\mathsf{m}(\bar{l}) = 0 \tag{22}$$

and $\bar{\tau} : \mathcal{V} \to \mathcal{T}((\mathcal{F} \cup Q)^{\leq k})$ is a marked substitution such that, $\forall x \in \mathcal{V}\mathsf{ar}(l)$

$$x\bar{\bar{\tau}} = x\bar{\tau} \odot M(\bar{l}, x), \qquad dpt(x\bar{\tau}) \leq k \cdot \mathsf{d}. \tag{23}$$

(Recall that the number d was defined by (20).)

Lemma 4.15 *(Lifting* $\mathcal{S} \cup \mathcal{A}$ *to* $\mathcal{R}$*). Let* $\bar{s}, \bar{s'}, \bar{t} \in \mathcal{T}((\mathcal{F} \cup Q)^{\leq k})$ *such that* $\bar{s'}$ *is* m*-increasing. If* $\bar{s}' \to^*_{\mathcal{A}} \bar{s}$ *and* $\bar{s} \to^*_{\mathcal{S} \cup \mathcal{A}} \bar{t}$ *then there exists a term* $\bar{t'} \in \mathcal{T}((\mathcal{F} \cup Q)^{\leq k})$ *such that* $\bar{s}' \;{}_{k}{\circ\!\!\to}^*_{\mathcal{R}} \bar{t'}$ *and* $\bar{t'} \to^*_{\mathcal{A}} \bar{t}$*. See Fig. 3.*

Proof. 1. Let us prove that the lemma holds for $\bar{s} \to_{\mathcal{S} \cup \mathcal{A}} \bar{t}$. Let us suppose that $\bar{s}' \to^*_{\mathcal{A}} \bar{s} \to_{\mathcal{A}} \bar{t}$. Let us then choose $\bar{t}' := \bar{s}'$. It satisfies: $\bar{s}' \;{}_{k}{\circ\!\!\to}^0_{\mathcal{R}} \bar{t'}$ and $\bar{t'} = \bar{s}' \to^*_{\mathcal{A}} \bar{s} \to_{\mathcal{A}} \bar{t}$. Hence the conclusion of the lemma holds.

Suppose now that $\bar{s} \to_{\mathcal{S}} \bar{t}$. This means that

$$\bar{s} = \bar{C}[\bar{l}\bar{\tau}], \qquad \bar{t} = \bar{C}[r\bar{\bar{\tau}}]$$

for some rule $l \to r \in \mathcal{R}$, marked context $\bar{C}$, and marked substitution $\bar{\tau}$, satisfying (22)–(23).

Since $\bar{s}' \to^*_{\mathcal{A}} \bar{s}$, $\bar{s}'$ must be of the following form

$$\bar{s}' = \bar{C}\bigl[\bar{l}\bar{\tau}'\bigr]$$

$$\begin{tikzcd} \bar{s}' \arrow[r, dotted, "*", "{}_k\circ\ \mathcal{R}"'] \arrow[d, "\mathcal{A}"', "*"] & \bar{t}' \arrow[d, dotted, "\mathcal{A}"', "*"] \\ \bar{s} \arrow[r, "*", "\mathcal{S}\cup\mathcal{A}"'] & \bar{t} \end{tikzcd}$$

Lifting $\mathcal{S} \cup \mathcal{A}$

$$\begin{tikzcd} \bar{s} \arrow[r, "*", "{}_k\circ\ \mathcal{R}"'] \arrow[d, dotted, "\mathcal{A}"', "*"] & \bar{t} \arrow[d, dotted, "\mathcal{A}"', "*"] \\ \bar{s}' \arrow[r, dotted, "*", "\mathcal{S}\cup\mathcal{A}"'] & \bar{t}' \end{tikzcd}$$

Projecting $\mathcal{R}$

Fig. 3. Lemmas 4.15 and 4.21.

where, for every $x \in \mathcal{V}\mathrm{ar}(l)$, $x\bar{\tau}' \to^*_{\mathcal{A}} x\bar{\tau}$. Let us set

$$x\bar{\bar{\tau}}' := x\bar{\tau}' \odot M(\bar{l}, x), \qquad \bar{t}' := \bar{C}\big[r\bar{\bar{\tau}}'\big].$$

Since $\bar{s'}$ is m-increasing, $M(\bar{l}, x) = M(C[\bar{l}], x)$. Hence, by the definition of $\circ\!\!\to$, $\bar{s}' \circ\!\!\to_{\mathcal{R}} \bar{t}'$ and by condition (22) this step is wbu, i.e.

$$\bar{s}' \ {}_k\!\circ\!\!\to_{\mathcal{R}} \bar{t}'.$$

By Lemma 3.5, for every x,

$$x\bar{\bar{\tau}}' = x\bar{\tau}' \odot M(\bar{l}, x) \to^*_{\mathcal{A}} x\bar{\tau} \odot M(\bar{l}, x) = x\bar{\bar{\tau}}.$$

Hence $\bar{t}' = \bar{C}[r\bar{\bar{\tau}}'] \to^*_{\mathcal{A}} \bar{C}[r\bar{\bar{\tau}}] = \bar{t}$.

2. Let us prove, by induction over the integer $n \geq 0$, the statement

$$\begin{aligned} &\forall n \in \mathbb{N},\ \forall \bar{s}, \bar{s'}, \bar{t} \in \mathcal{T}\big((\mathcal{F} \cup Q)^{\leq k}\big) \\ &\big(\bar{s'} \text{ m-increasing} \ \&\ \bar{s}' \to^*_{\mathcal{A}} \bar{s} \ \&\ \bar{s} \to^n_{\mathcal{S}\cup\mathcal{A}} \bar{t}\big) \Rightarrow \exists \bar{t'},\ \big(\bar{s}' \ {}_k\!\circ\!\!\to^*_{\mathcal{R}} \bar{t'} \ \&\ \bar{t'} \to^*_{\mathcal{A}} \bar{t}\big). \end{aligned} \tag{24}$$

(Here $\bar{t'}$ is implicitly quantified over $\mathcal{T}((\mathcal{F} \cup Q)^{\leq k})$.)

Basis: $n = 0$.

In this case $\bar{s}' = \bar{s}$. Choosing $\bar{t'} := \bar{t}$, the conclusion of implication (24) holds.

Induction step: $n \geq 1$.

Let us suppose that the hypothesis of implication (24) holds. There exists a term $\hat{t} \in \mathcal{T}((\mathcal{F} \cup Q)^{\leq k})$ such that

$$\bar{s} \to^{n-1}_{\mathcal{S}\cup\mathcal{A}} \hat{t} \to^1_{\mathcal{S}\cup\mathcal{A}} \bar{t}.$$

By induction hypothesis, there exists some $\hat{t'} \in \mathcal{T}((\mathcal{F} \cup Q)^{\leq k})$ such that

$$\bar{s}' \ {}_k\!\circ\!\!\to^*_{\mathcal{R}} \hat{t'} \ \&\ \hat{t'} \to^*_{\mathcal{A}} \hat{t}. \tag{25}$$

By Lemma 3.20 $\hat{t'}$ is m-increasing and by point 1 of this proof, there exists some $\bar{t'} \in \mathcal{T}((\mathcal{F} \cup Q)^{\leq k})$ such that

$$\hat{t'} \ {}_k\!\circ\!\!\to^*_{\mathcal{R}} \bar{t'} \ \&\ \bar{t'} \to^*_{\mathcal{A}} \bar{t}. \tag{26}$$

Putting together statements (25) and (26), we obtain the conclusion of implication (24). □

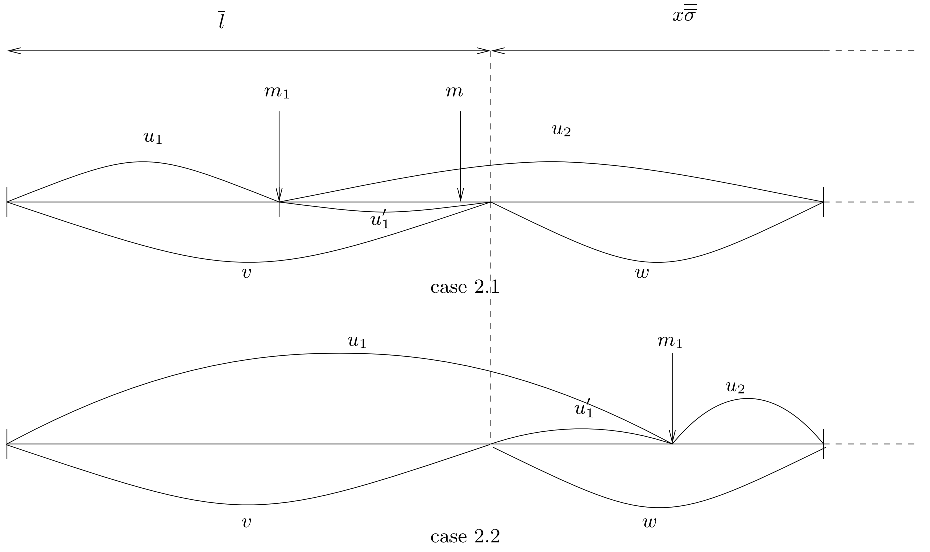

Fig. 4. Fact 4.17.

Lemma 4.16 *(Projecting one step of $\mathcal{R}$ on $\mathcal{S} \cup \mathcal{A}$). Let $\bar{s}, \bar{t} \in \mathcal{T}((\mathcal{F} \cup Q)^{\leq k})$ such that:*

1. $\bar{s} \circ\!\!\to_{\mathcal{R}} \bar{t}$.
2. *The marked rule $(\bar{l}, r)$ used in the above rewriting-step is such that* $\mathsf{m}(\bar{l}) = 0$.
3. $\bar{s}$ *is* m*-increasing.*

Then, $\mathsf{Top}(\bar{s}) \to_{\mathcal{A}}^{*} \to_{\mathcal{S}} \mathsf{Top}(\bar{t})$.

Proof. Let us assume hypotheses 1, 2, 3 of Lemma 4.16. In particular:

$$\bar{s} = \overline{C}[\bar{l}\overline{\sigma}], \qquad \bar{t} = \overline{C}[r\overline{\overline{\sigma}}]$$

for some $\overline{C}, \overline{\sigma}, \bar{l}, r, \overline{\overline{\sigma}}$ fulfilling (7)–(9) and $\mathsf{m}(\bar{l}) = 0$. Let us then define a context $\overline{D}$ and marked substitutions $\overline{\tau}, \overline{\overline{\tau}}$ by:

$$\overline{D}[] = \mathsf{Top}\big(\overline{C}[]\big), \tag{27}$$

$$\forall x \in \mathcal{V}, \quad x\overline{\overline{\tau}} = \mathsf{Top}(x\overline{\overline{\sigma}}), \qquad x\overline{\tau} = \mathsf{Red}\big(x\overline{\sigma}, \mathcal{P}\mathsf{os}(x\overline{\overline{\tau}})\big). \tag{28}$$

We claim that

$$\mathsf{Top}(\bar{s}) \to_{\mathcal{A}}^{*} \overline{D}[\bar{l}\overline{\tau}] \to_{\mathcal{S}} \overline{D}[r\overline{\overline{\tau}}] = \mathsf{Top}(\bar{t}). \tag{29}$$

We cut into four facts the detailed verification of this claim.

Fact 4.17. $\mathcal{P}\mathsf{os}(\bar{l}\mathsf{Top}(\overline{\overline{\sigma}})) \subseteq \mathcal{P}\mathsf{os}(\mathsf{Top}(\bar{l}\overline{\overline{\sigma}}))$. See Fig. 4.

Let $u \in \mathcal{P}\mathsf{os}(\bar{l}\mathsf{Top}(\overline{\overline{\sigma}}))$.

Case 1. $u \in \mathcal{P}\mathsf{os}_{\overline{\mathcal{V}}}(\bar{l})$.

In this case $|u| \leq \mathsf{d}$. Hence, for every factorization $u = u_1 \cdot u_2$, since $\mathsf{m}(\bar{t}/u_1) \leq k$,

$$|u_2| \leq |u| \leq \mathsf{d} \leq \big(k + 1 - \mathsf{m}(\bar{t}/u_1)\big)\mathsf{d}.$$

Case 2.

$$u = v \cdot w$$

for some $x \in \mathcal{V}\mathrm{ar}(l)$, $v = \mathrm{pos}(\bar{l}, x)$, $w \in \mathrm{Topd}(x\overline{\overline{\sigma}})$.

Let us consider any decomposition $u = u_1 \cdot u_2$ and show that it fulfills condition 2 of Definition 4.8. We use the notation

$$m_1 = \mathrm{m}(\bar{l}\overline{\overline{\sigma}}/u_1), \qquad m = \mathrm{m}(\bar{l}\overline{\overline{\sigma}}/f)$$

where f is the father of v. If $m_1 = 0$ this condition 2 is clearly true. Let us assume that $m_1 \geq 1$.

Case 2.1. $u_1 \preceq v$.

In this case there exists u'_1 such that

$$v = u_1 u'_1, \qquad u_2 = u'_1 w, \qquad |u'_1| \geq 0.$$

As $w \in \mathrm{Topd}(x\overline{\overline{\sigma}})$,

$$|w| \leq \big(k + 1 - \mathrm{m}(x\overline{\overline{\sigma}})\big)\mathrm{d} \tag{30}$$

but $\mathrm{m}(x\overline{\overline{\sigma}}) \geq M(\bar{l}, x) = m + 1$, hence

$$|w| \leq (k + 1 - m - 1)\mathrm{d}. \tag{31}$$

Using the fact that $|u'_1| \leq dpt(\bar{l}) \leq \mathrm{d}$ we obtain that

$$|u'_1 w| \leq (k + 1 - m - 1)\mathrm{d} + \mathrm{d} = (k + 1 - m)\mathrm{d} \tag{32}$$

and, since the marks increase from top to leaves, $m \geq m_1$, so that

$$|u'_1 w| \leq (k + 1 - m_1)\mathrm{d} \tag{33}$$

which can be reformulated as

$$|u_2| \leq \big(k + 1 - \mathrm{m}(\bar{l}\overline{\overline{\sigma}}/u_1)\big)\mathrm{d}. \tag{34}$$

Case 2.2. $v \prec u_1$.

In this case there exists u'_1 such that

$$u_1 = v u'_1, \qquad u'_1 u_2 = w, \qquad |u'_1| \geq 1.$$

As $w \in \mathrm{Topd}(x\overline{\overline{\sigma}})$

$$|u_2| \leq \big(k + 1 - \mathrm{m}\big(x\overline{\overline{\sigma}}/u'_1\big)\big)\mathrm{d} \tag{35}$$

which can be rewritten as

$$|u_2| \leq \big(k + 1 - \mathrm{m}(\bar{l}\overline{\overline{\sigma}}/u_1)\big)\mathrm{d}. \tag{36}$$

Since in all cases condition 2 of Definition 4.8 is fulfilled, Fact 4.17 is established.

Fact 4.18. $\mathsf{Top}(\bar{l}\bar{\sigma}) \to^*_{\mathcal{A}} \bar{l}\bar{\tau}$.

We know that

$$\bar{l}\bar{\sigma} \to^*_{\mathcal{A}} \mathsf{Top}(\bar{l}\bar{\sigma}) \tag{37}$$

(by the definition of Top) and that

$$\bar{l}\bar{\sigma} \to^*_{\mathcal{A}} \bar{l}\bar{\tau} \tag{38}$$

because, by (28), every $x\bar{\tau}$ is a reduct of the corresponding $x\bar{\sigma}$. Moreover, by Fact 4.17,

$$\mathcal{P}\mathsf{os}(\bar{l}\bar{\tau}) = \mathcal{P}\mathsf{os}\big(\bar{l}\mathsf{Top}(\bar{\bar{\sigma}})\big) \subseteq \mathcal{P}\mathsf{os}\big(\mathsf{Top}(\bar{l}\bar{\bar{\sigma}})\big),$$

and by Lemma 4.13 $\mathcal{P}\mathsf{os}(\mathsf{Top}(\bar{l}\bar{\bar{\sigma}})) \subseteq \mathcal{P}\mathsf{os}(\mathsf{Top}(\bar{l}\bar{\sigma}))$, so that

$$\mathcal{P}\mathsf{os}(\bar{l}\bar{\tau}) \subseteq \mathcal{P}\mathsf{os}\big(\mathsf{Top}(\bar{l}\bar{\sigma})\big). \tag{39}$$

Lemma 4.6 applied to (37)–(39) shows that $\mathsf{Top}(\bar{l}\bar{\sigma}) \to^*_{\mathcal{A}} \bar{l}\bar{\tau}$.

Fact 4.19. $\bar{D}[\bar{l}\bar{\tau}] \to_{\mathcal{S}} \bar{D}[r\bar{\bar{\tau}}]$.

By hypothesis 2 of the lemma, $\mathsf{m}(\bar{l}) = 0$.
By the general assumption (18) and hypothesis 3 of the lemma,

$$\forall x \in \mathcal{V}\mathsf{ar}(l), \quad \mathsf{M}(\bar{l}, x) = \mathsf{M}\big(\bar{C}[\bar{l}], x\big),$$

hence

$$x\bar{\bar{\tau}} := x\bar{\tau} \odot \mathsf{M}(\bar{l}, x).$$

Moreover, $dpt(x\bar{\tau}) \le (k + 1 - \mathsf{M}(\bar{l}, x)) \cdot \mathsf{d} \le k \cdot \mathsf{d}$, since $\mathsf{M}(\bar{l}, x) \ge 1$. Hence $\bar{l}\bar{\tau} \to r\bar{\bar{\tau}}$ is a rule of $\mathcal{S}$.

Fact 4.20. $\bar{D}[r\bar{\bar{\tau}}] = \mathsf{Top}(\bar{t})$.

This fact follows from Lemma 4.11 and Lemma 4.12.
Using these facts we obtain that

$$\mathsf{Top}(\bar{s}) = \bar{D}\big[\mathsf{Top}(\bar{l}\bar{\sigma})\big] \quad \text{(by Lemma 3.18 and Lemma 4.11)},$$
$$\bar{D}\big[\mathsf{Top}(\bar{l}\bar{\sigma})\big] \to^*_{\mathcal{A}} \bar{D}[\bar{l}\bar{\tau}] \quad \text{(by Fact 4.18)},$$
$$\bar{D}[\bar{l}\bar{\tau}] \to_{\mathcal{S}} \bar{D}[r\bar{\bar{\tau}}] \quad \text{(by Fact 4.19)},$$
$$\bar{D}[r\bar{\bar{\tau}}] = \mathsf{Top}(\bar{t}) \quad \text{(by Fact 4.20)}.$$

Thus claim (29) is verified, which proves the lemma. □

Lemma 4.21 *(Projecting $\mathcal{R}$ on $\mathcal{S} \cup \mathcal{A}$). Let $\bar{s}, \bar{t} \in \mathcal{T}(\mathcal{F}^{\le k})$ and assume that $\bar{s}$ is* m*-increasing. If $\bar{s} \;_k{\circ\!\!\to}^*_{\mathcal{R}}\; \bar{t}$ then there exist terms $\bar{s}', \bar{t}' \in \mathcal{T}((\mathcal{F} \cup Q)^{\le k})$ such that*

$$\bar{s} \to^*_{\mathcal{A}} \bar{s}' \to^*_{\mathcal{S}\cup\mathcal{A}} \bar{t'} \quad \textit{and} \quad \bar{t} \to^*_{\mathcal{A}} \bar{t}'.$$

Proof. The marked derivation $\bar{s} \circ\!\!\to^*_{\mathcal{R}} \bar{t}$ is wbu, hence it can be decomposed into n successive steps where the hypothesis 2 of Lemma 4.16 is valid. Hypothesis 3 of Lemma 4.16 will also hold, owing to our assumption and to Lemma 3.20. We can thus deduce, inductively, from the conclusion of Lemma 4.16, that $\mathsf{Top}(\bar{s}) \to^*_{\mathcal{S}\cup\mathcal{A}} \mathsf{Top}(\bar{t})$. The choice $\bar{s}' := \mathsf{Top}(\bar{s})$, $\bar{t}' := \mathsf{Top}(\bar{t})$ fulfills the conclusion of the lemma. □

Lemma 4.22. *Let $s \in \mathcal{T}(\mathcal{F})$. Then $s\ {}_k{\to}^*_{\mathcal{R}}\ T$ iff $s \to^*_{\mathcal{S}\cup\mathcal{A}} Q_f^{\leq k}$.*

Proof. ($\Rightarrow$): Suppose $s\ {}_k{\to}^*_{\mathcal{R}}\ t$ and $t \in T$. Let us consider the corresponding marked derivation

$$\bar{s}\ {}_k{\circ\!\!\to}^*_{\mathcal{R}}\ \bar{t} \tag{40}$$

where $\bar{s} := s$. Derivation (40) is wbu and lies in $\mathcal{T}(\mathcal{F}^{\leq k})$. Let us consider the terms $\bar{s}', \bar{t}'$ given by Lemma 4.21:

$$\bar{s} \to^*_{\mathcal{A}} \bar{s}' \to^*_{\mathcal{S}\cup\mathcal{A}} \bar{t}' \tag{41}$$

and $\bar{t} \to^*_{\mathcal{A}} \bar{t}'$. Since $\bar{t} \to^*_{\mathcal{A}} Q_f^{\leq k}$, by Lemma 4.6,

$$\bar{t}' \to^*_{\mathcal{A}} Q_f^{\leq k}. \tag{42}$$

Combining (41) and (42) we obtain

$$s \to^*_{\mathcal{S}\cup\mathcal{A}} Q_f^{\leq k}.$$

($\Leftarrow$): Suppose $s \to^*_{\mathcal{S}\cup\mathcal{A}} q^j \in Q_f^{\leq k}$.

The hypotheses of Lemma 4.15 are met by $\bar{s} := s$, $\bar{s}' := s$ and $\bar{t} := q^j$. By Lemma 4.15 there exists some $\bar{t}' \in \mathcal{T}((\mathcal{F} \cup Q)^{\leq k})$ such that

$$\bar{s}\ {}_k{\circ\!\!\to}^*_{\mathcal{R}}\ \overline{t'} \to^*_{\mathcal{A}} q^j \in Q_f^{\leq k}.$$

These derivations are mapped (by removal of the marks) into:

$$s\ {}_k{\to}^*_{\mathcal{R}}\ t' \to^*_{\mathcal{A}} q \in Q_f,$$

which shows that $t' \in T$ hence that $s\ {}_k{\to}^*_{\mathcal{R}}\ T$. □

We can now prove Theorem 4.2.

Proof of Theorem 4.2. By Lemma 4.22, $({}_k{\to}^*_{\mathcal{R}})[T] = (\to^*_{\mathcal{S}\cup\mathcal{A}})[Q_f^{\leq k}] \cap \mathcal{T}(\mathcal{F})$. The rewriting systems $\mathcal{S}$ and $\mathcal{A}$ being ground are inverse-recognizability preserving (Theorem 4.1). So $(\to^*_{\mathcal{S}\cup\mathcal{A}})[Q_f^{\leq k}]$ is recognizable and thus $({}_k{\to}^*_{\mathcal{R}})[T]$ is recognizable. □

Corollary 4.23. *Every linear rewriting system of the class* BU *is inverse-recognizability preserving.*

Proof. If $\mathcal{R}$ belongs to $\mathsf{BU}(k)$, then $(\to^*_{\mathcal{R}})[T] = ({}_k{\to}^*_{\mathcal{R}})[T]$. □

4.3. General construction

We show here that Theorem 4.2 still holds when the restrictions (18)–(19) are removed.

4.3.1. Allowing variable lhs

Let $\mathcal{R}$ be some left-linear finite rewriting system over the signature $\mathcal{F}$. We show here how to reduce the properties of this TRS $\mathcal{R}$ to properties of a TRS which has no variable on the left-hand side nor any variable on the right-hand side.

Let us introduce a new unary symbol $\#_1 \notin \mathcal{F}$ and consider the signature $\mathcal{F}_1 := \mathcal{F} \cup \{\#_1\}$. We consider the map $H : \mathcal{T}(\mathcal{F}, \mathcal{V}) \to \mathcal{T}(\mathcal{F}_1, \mathcal{V})$ defined inductively by: $\forall v \in \mathcal{V}$, $\forall a \in \mathcal{F}_0$, $\forall n \geq 1$, $\forall f \in \mathcal{F}_n$, $\forall t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{V})$

$$H(v) = v, \qquad H(a) = \#_1(a), \qquad H\big(f(t_1, \ldots, t_n)\big) = \#_1\big(f\big(H(t_1), \ldots, H(t_n)\big)\big)$$

and the map $E_1 : \mathcal{T}(\mathcal{F}, \mathcal{V}) \to \mathcal{T}(\mathcal{F}_1, \mathcal{V})$ defined by

$$E_1(v) = \#_1(v) \quad (\text{if } v \in \mathcal{V}), \qquad E_1(t) = H(t) \quad (\text{if } t \notin \mathcal{V}).$$

It is clear that E_1 is an injective map. Since H is an injective term-homomorphism, for every subset $T \subseteq \mathcal{T}(\mathcal{F}, \mathcal{V})$, T is recognizable if and only if $H(T)$ is recognizable, iff $E_1(T)$ is recognizable (because $E_1(T) = \#_1(T \cap \mathcal{V}) \cup H(T \setminus \mathcal{V})$). We define a new TRS

$$\mathcal{R}_1 := \left\{ E_1(l) \to E_1(r) \;\middle|\; l \to r \in \mathcal{R} \right\}.$$

The system $\mathcal{R}_1$ is a left-linear finite rewriting system over the signature $\mathcal{F}_1$ and every rule $(l_1, r_1) \in \mathcal{R}_1$ is such that $l_1 \notin \mathcal{V}, r_1 \notin \mathcal{V}$.

Lemma 4.24 *($\mathcal{R}$ embeddable in $\mathcal{R}_1$). For every $s, t \in \mathcal{T}(\mathcal{F})$ and integer $k \geq 0$,*

1. $s \to^*_{\mathcal{R}} t \Leftrightarrow E_1(s) \to^*_{\mathcal{R}_1} E_1(t)$.
2. $s \;{}_k\!\!\to^*_{\mathcal{R}} t \Leftrightarrow E_1(s) \;{}_k\!\!\to^*_{\mathcal{R}_1} E_1(t)$.

Sketch of proof. 1. One easily checks that $s \to_{\mathcal{R}} t \Leftrightarrow E_1(s) \to_{\mathcal{R}_1} E_1(t)$. Point 1 follows.

2. Let

$$s_0 \to s_1 \to \cdots \to s_n \tag{43}$$

be some derivation and

$$\bar{s}_0 \circ\!\!\to \bar{s}_1 \circ\!\!\to \cdots \circ\!\!\to \bar{s}_n \tag{44}$$

be the associated marked derivation. By point 1

$$E_1(s_0) \to E_1(s_1) \to \cdots \to E_1(s_n). \tag{45}$$

The associated marked derivation

$$\overline{E_1(s_0)} \circ\!\!\to \overline{E_1(s_1)} \circ\!\!\to \cdots \circ\!\!\to \overline{E_1(s_n)} \tag{46}$$

has the following property:

if $(m_0, m1, \ldots, m_\ell)$ is the sequence of marks of the branch of $\bar{s_i}$, ending at node $i_1 \cdot i_2 \cdots i_\ell$, then $(m_0, m_0, m_1, m_1, \ldots, m_\ell, m_\ell)$ is the sequence of marks of the branch of $\overline{E_1(s_i)}$ ending at node $0 \cdot i_1 \cdot 0 \cdot i_2 \cdots 0 \cdot i_\ell \cdot 0$. We deduce that:
- derivation (44) is wbu iff derivation (46) is wbu,
- the maximum mark used in derivations (44) and (46) is the same.

Hence (44) is bu(k) iff (46) is bu(k). □

In particular: $s \;{}_k\!\!\to^*_{\mathcal{R}} T \Leftrightarrow E_1(s) \;{}_k\!\!\to^*_{\mathcal{R}_1} E_1(T)$ and $\mathcal{R}$ is BU(k) iff $\mathcal{R}_1$ is BU(k).
Hence Theorem 4.2 and Corollary 4.23 still hold, without assuming (18).

4.3.2. Allowing non-standard automata

Let $\mathcal{R}$ be some left-linear finite rewriting system over the signature $\mathcal{F}$ fulfilling restriction (18) and let $\mathcal{A} = (\mathcal{F}, Q, Q_f, \Gamma)$ be some *f.t.a.* It is well known that, w.l.o.g., we can assume that this *f.t.a.* is deterministic, complete and such that $\mathcal{F} \cap Q = \emptyset$. Let us define

$$\hat{Q} := Q \cup \mathcal{F}_0,$$

$$\begin{aligned}\hat{\Gamma} &:= \big(\Gamma \setminus \big\{(f,q) \in \Gamma \bigm| f \in \mathcal{F}_0\big\}\big) \\ &\quad \cup \big\{ f(\hat{p}_1,\dots,\hat{p}_m) \to q \bigm| m \geq 1,\ f \in \mathcal{F}_m,\ \forall i \in [1,m]\ \hat{p}_i \in \hat{Q},\ \exists p_i \in Q,\ \hat{p}_i \to^*_{\Gamma} p_i \\ &\quad \text{and } f(p_1,\dots,p_m) \to_{\Gamma} q \big\}, \\ \hat{Q}_f &:= Q_f \cup \{ f \in \mathcal{F}_0 \mid \exists q \in Q_f,\ f \to_{\Gamma} q \}\end{aligned}$$

and finally $\hat{\mathcal{A}} := (\mathcal{F}, \hat{Q}, \hat{Q}_f, \hat{\Gamma})$. It should be clear that $\hat{\mathcal{A}}$ is a standard *f.t.a.* and $L(\mathcal{A}) = L(\hat{\mathcal{A}})$. Hence Theorem 4.2 and Corollary 4.23 still hold, without assuming (18), (19).

5. Strongly bottom-up systems

It turns out that $\mathsf{BU}(k)$ conditions are undecidable (Durand and Sénizergues, 2009, Theorem 5.12). We are thus led to define some stronger but *decidable* conditions. We study in Section 5.1 the *strongly bottom-up* (SBU for short) restriction. We introduce in Section 5.2 a technical tool that will be used in Section 5.3 and Section 5.4 for giving a polynomially decidable condition implying the SBU condition.

5.1. Strongly bottom-up systems

We abbreviate strongly bottom-up to sbu.

Definition 5.1. A system $(\mathcal{R}, \mathcal{F})$ is said to be $\mathsf{SBU}(k)$ iff for every derivation $D : s \to^*_{\mathcal{R}} t$, from a term $s \in \mathcal{T}(\mathcal{F})$ to a term $t \in \mathcal{T}(\mathcal{F})$,

$$D \text{ is wbu} \quad \Leftrightarrow \quad D \text{ is bu}(k).$$

We denote by $\mathsf{SBU}(k)$ the class of $\mathsf{SBU}(k)$ systems and by $\mathsf{SBU} = \bigcup_{k \in \mathbb{N}} \mathsf{SBU}(k)$ the class of *strongly bottom-up* systems.

In other words: instead of requiring that the binary relations $\to^*_{\mathcal{R}}$ and ${}_k{\to}^*_{\mathcal{R}}$ over $\mathcal{T}(\mathcal{F})$ are equal, we require that *all* wbu marked derivations starting on an unmarked term use only marks smaller or equal to k. The following lemma is obvious.

Lemma 5.2. *Every* $\mathsf{SBU}(k)$ *linear system is* $\mathsf{BU}(k)$.

This follows easily from Lemma 3.16. This stronger condition over term rewriting systems is interesting because of the following property.

Proposition 5.3. *For every* $k \geq 0$*, it is decidable whether a finite term rewriting system* $(\mathcal{R}, \mathcal{F})$ *is* $\mathsf{SBU}(k)$*.*

Proof. Note that every marked derivation starting from some $s \in \mathcal{T}(\mathcal{F})$ and leading to some $\bar{t} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}}) \setminus \mathcal{T}(\mathcal{F}^{\leq k})$ must decompose as

$$s \ {}_{k+1}{\circ\!\!\to}^*_{\mathcal{R}} \ \bar{s}' \ {\circ\!\!\to}^*_{\mathcal{R}} \ \bar{t},$$

with $\bar{s}' \in \mathcal{T}(\mathcal{F}^{\leq k+1}) \setminus \mathcal{T}(\mathcal{F}^{\leq k})$. A necessary and sufficient condition for $\mathcal{R}$ to be $\mathsf{SBU}(k)$ is thus that:

$$\big({}_{k+1}{\circ\!\!\to}^*_{\mathcal{R}}\big)\big[\mathcal{T}\big(\mathcal{F}^{\leq k+1}\big) \setminus \mathcal{T}\big(\mathcal{F}^{\leq k}\big)\big] \cap \mathcal{T}(\mathcal{F}) = \emptyset. \tag{47}$$

By Theorem 4.2 the left-hand side of equality (47) is a recognizable set for which we can construct an *f.t.a.*; we then just have to test whether this *f.t.a.* recognizes the empty set or not. □

According to the results of Knapik and Calbrix (1999) it seems likely that the property [$\exists k \geq 0$ such that $(\mathcal{R}, \mathcal{F})$ is $\mathsf{SBU}(k)$] is undecidable for term rewriting systems. It is then interesting to look for a decidable sufficient condition. Our condition is based on a finite graph that we define in the next subsection.

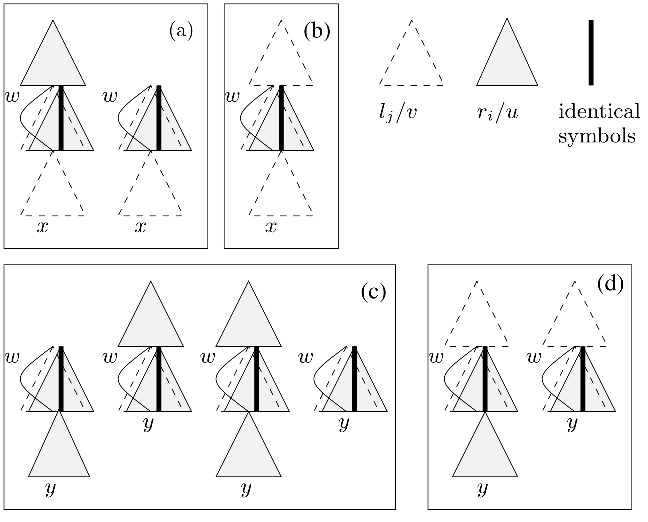

Fig. 5. Sticking-out cases.

5.2. *The sticking-out graph* $\mathsf{SG}(\mathcal{R})$

Let us associate with every Term Rewriting System a *graph* whose vertices are the rules of the system and whose arcs (R, R') express some kind of overlap between the right-hand side of R and the left-hand side of R'. Every arc has a *label* indicating the category of overlap that occurs and a *weight* which is an integer (0 or 1). The intuitive meaning of the weight is that any derivation step using the corresponding overlap would increase some mark by this weight. The precise graph is defined below and is directly inspired by the one of Takai et al. (2010), though slightly different.

Definition 5.4. Let $s \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, $t \in \mathcal{T}(\mathcal{F}, \mathcal{V}) \setminus \mathcal{V}$ and $w \in \mathcal{P}\mathrm{os}_{\mathcal{V}}(t)$. We say that s *sticks out of* t *at* w if

1. $\forall v \in \mathcal{P}\mathrm{os}(t)$ s.t. $\varepsilon \preceq v \prec w$, $v \in \mathcal{P}\mathrm{os}(s)$ and $s(v) = t(v)$,
2. $w \in \mathcal{P}\mathrm{os}(s)$ and $s/w \notin \mathcal{T}(\mathcal{F})$.

If in addition $s/w \notin \mathcal{V}$ then s *strictly* sticks out of t at w.

Definition 5.5. Let $\mathcal{R} = \{l_1 \to r_1, \ldots, l_n \to r_n\}$ be a system. The *sticking-out graph* is the directed graph $\mathsf{SG}(\mathcal{R}) = (V, E)$ where $V = \{1, \ldots, n\}$ and E is defined as follows:

(a) if l_j strictly sticks out of a subterm of r_i at w, $i \stackrel{(a)}{\to} j \in E$;
(b) if a strict subterm of l_j strictly sticks out of r_i at w, $i \stackrel{(b)}{\to} j \in E$;
(c) if a subterm of r_i sticks out of l_j at w, $i \stackrel{(c)}{\to} j \in E$;
(d) if r_i sticks out of a strict subterm of l_j at w, $i \stackrel{(d)}{\to} j \in E$.

Fig. 5 shows all the possibilities in the four categories (a), (b), (c), (d).

Example 5.6. The graph of the system $\mathcal{R}_0 = \{\mathsf{f}(\mathsf{f}(x)) \to \mathsf{f}(x)\}$ contains one vertex and two loops labeled (d) and (a).

It can be shown with an *ad hoc* proof that $\mathcal{R}_0 \in \mathsf{BU}$ (actually in $\mathsf{BU}^-(1)$). We have already seen in Example 3.11 that $\mathcal{R}_0 \notin \mathsf{SBU}$.

$1 : \mathsf{f}(x) \to \mathsf{g}(x)$
$2 : \mathsf{g}(\mathsf{h}(x)) \to \mathsf{i}(x)$
$3 : \mathsf{i}(x) \to \mathsf{a}$

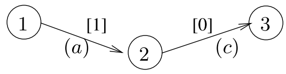

Fig. 6. The sticking-out graph of $\mathcal{R}_1$.

Example 5.7. The graph of system $\mathcal{R}_4 = \{\mathsf{g}(\mathsf{f}(\mathsf{g}(x))) \to \mathsf{f}(x)\}$ contains one vertex and a simple loop labeled (b). $\mathcal{R}_4$ is not inverse recognizability preserving since the set

$$\left(\to^*_{\mathcal{R}_4}\right)\left[\left\{f(a)\right\}\right] = \left\{g^n\left(f\left(g^n(a)\right)\right) \mid n \geq 0\right\}$$

is not recognizable.

The *weight* of each arc of $\mathsf{SG}(\mathcal{R})$ is defined by:

- arcs (a) or (b) have weight 1,
- arcs (c) or (d) have weight 0.

The *weight of a path* in the graph is the sum of the weights of its arcs. The *weight of a graph* is the maximal weight of a path in the graph; it is infinite if the graph contains a cycle with an arc of weight 1.

The sticking-out of $\mathcal{R}_1$ of Example 3.1 is given in Fig. 6.

5.3. A sufficient condition for semi-Thue systems

Let us fix a semi-Thue system $\mathcal{R}$ over an alphabet Y. The main result of this subsection is that, if every path of $\mathsf{SG}(\mathcal{R})$ has a weight $\leq k$, then $\mathcal{R}$ has the property $\mathsf{SBU}(k+1)$. We prove some lemmas establishing some links between wbu-derivations, on one hand, and paths of $\mathsf{SG}(\mathcal{R})$, on the other hand.

More notation for derivations. The general notion of derivation which was given in Section 2.1 for binary relations $\to$, turns out not to be precise enough for an analysis in the case where the binary relation is defined through combinatorial means, as it is the case for derivations induced by semi-Thue systems or term rewriting systems. We thus borrow from Cremanns and Otto (1994), Lafont (1995) a more precise notion of derivation together with some useful notation.

We assume that some semi-Thue system $\mathcal{R}$ over an alphabet Y is given. For every rule $R = l \to r$ and words $v, w \in Y^*$, we note $\partial^+((v, R, w)) := vrw$, $\partial^-((v, R, w)) := vlw$. We call derivation any non-empty sequence of triples of the form

$$D = \big((v_1, R_1, w_1), \ldots, (v_i, R_i, w_i), \ldots, (v_n, R_n, w_n)\big) \tag{48}$$

such that, for every $1 \leq i \leq n-1$, $\partial^+(v_i, R_i, w_i) = \partial^-(v_{i+1}, R_{i+1}, w_{i+1})$ and also the triples

$$D_v := (v, \mathsf{ID}, \varepsilon), \tag{49}$$

where ID is a special symbol that we view as the *Identity* rule. We extend the notations ∂^α ($\alpha \in \{+, -\}$) by defining for D given in (48):

$$\partial^+(D) := \partial^+(v_n, R_n, w_n), \qquad \partial^-(D) := \partial^-(v_1, R_1, w_1)$$

and for D_v given in (49):

$$\partial^+(D_v) := v, \qquad \partial^-(D_v) := v.$$

The length $\ell(D)$ is defined as n for the derivation (48) and 0 for the derivation (49). Given derivations D, D' such that $\partial^+(D) = \partial^-(D')$, their composition $D \otimes D'$ is just their concatenation (when they both have non-null length), D when $\ell(D') = 0$, and D' when $\ell(D) = 0$.

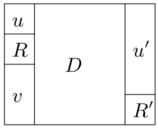

Fig. 7. Downwards derivation.

The words of Y^* act on the right and on the left over derivations: for D defined by (48) we set

$$D \cdot v := \big((v_1, R_1, w_1 v), \dots, (v_i, R_i, w_i v), \dots, (v_n, R_n, w_n v)\big)$$

and $v \cdot D$ is defined similarly; $D_u \cdot v := D_{uv}$ and $v \cdot D_u := D_{vu}$.

One can easily check that, for every derivations D, D', F, F', words u, v and signs $\alpha \in \{+, -\}$:

$$u\big(D \otimes D'\big)v = uDv \otimes uD'v, \qquad \partial^{\alpha}(uDv) = u\partial^{\alpha}(D)v,$$
$$\partial^{+}\big(D \otimes D'\big) = \partial^{+}\big(D'\big), \qquad \partial^{-}\big(D \otimes D'\big) = \partial^{-}(D).$$

We are now ready for giving a sufficient condition for semi-Thue systems.

Lemma 5.8 *(Downwards derivations). Let $R, R' \in \mathcal{R}, u, v, u' \in Y^*$ and D be a derivation such that $uRv \otimes D \otimes u'R'$ is a* wbu*-derivation. Then, there exists a path from R to R' in* SG$(\mathcal{R})$.

(See Fig. 7.)

Proof. Let us consider the following property $P(n)$:

> for every wbu-derivation $(u_i R_i v_i)_{0 \le i \le n+1}$ for the system $\mathcal{R}$, if $|v_{n+1}| = 0$ then there exists a path from R_0 to R_{n+1} in SG$(\mathcal{R})$.

We show by induction over n that, for every $n \in \mathbb{N}$, $P(n)$ holds.

Basis: $n = 0$.

We thus have $|v_0| + |\partial^{+}(R_0)| + |u_0| = |\partial^{-}(R_1)| + |u_1|$. Since this derivation is wbu we also have $|v_0| < |\partial^{-}(R_1)|$ or $|v_0| = |\partial^{-}(R_1)| = 0$ (i.e. the *lhs* of R_1 is the empty word). From these inequalities it follows that (R_0, R_1) is an edge of SG$(\mathcal{R})$.

Induction step: $n \ge 1$.

We define

$$i := \min\big\{ j \in [1, n+1] \,\big|\, |v_j| < |v_0| + \big|\partial^{+}(R_0)\big| \big\}.$$

Since the given derivation is wbu, $|v_0| < |v_i| + |\partial^{-}(R_i)|$ or $(|v_0| = |v_i|$ and $|\partial^{-}(R_i)| = 0)$. Hence (R_0, R_i) is an edge of SG$(\mathcal{R})$ and $(u_j R_j v_j)_{i \le j \le n+1}$ is a wbu-derivation fulfilling $|v_{n+1}| = 0$. By induction hypothesis, there exists a path p from R_i to R_{n+1} in SG$(\mathcal{R})$. The edge (R_0, R_i) followed by the path p is a path from R_0 to R_{n+1} in SG$(\mathcal{R})$.

Let R, R', u, v, u', D fulfill the hypothesis of the lemma. Let us note:

$$u_0 := u, \qquad R_0 := R, \qquad v_0 := v, \qquad D := (u_i R_i v_i)_{1 \le i \le n}, \qquad u_{n+1} := u',$$
$$R_{n+1} := R', \qquad v_{n+1} := \varepsilon.$$

Applying $P(n)$ to the derivation $(u_i R_i v_i)_{0 \le i \le n+1}$, we obtain the conclusion of the lemma. $\square$

Lemma 5.9 *(Strict downwards derivations). Let $R, R' \in \mathcal{R}, u, v, u' \in Y^*$ and D be a derivation such that $uRv \otimes D \otimes u'R'$ is a* wbu*-derivation and $|v| \geq 1$. Then, there exists a path with non-null weight from R to R' in* $\mathsf{SG}(\mathcal{R})$*.*

Proof. Let us consider the following property $Q(n)$:

> for every wbu-derivation $(u_i R_i v_i)_{0 \leq i \leq n+1}$ for the system $\mathcal{R}$, if $|v_0| \geq 1$ and $|v_{n+1}| = 0$, then there exists a path with non-null weight from R_0 to R_{n+1} in $\mathsf{SG}(\mathcal{R})$.

We show by induction over n that, for every $n \in \mathbb{N}$, $Q(n)$ holds.

Basis: $n = 0$.

$Q(0)$: we thus have $|v_0| + |\partial^+(R_0)| + |u_0| = |\partial^-(R_1)| + |u_1|$. Since this derivation is wbu we also have $|v_0| < |\partial^-(R_1)|$. From these inequalities it follows that (R_0, R_1) is an edge of type (a) or (b) of $\mathsf{SG}(\mathcal{R})$.

Induction step: $n \geq 1$.

We define

$$i := \min\big\{ j \in [1, n+1] \;\big|\; |v_j| < |v_0| + \big|\partial^+(R_0)\big| \big\}.$$

Case 1. $|v_i| \geq |v_0|$.

In this case (R_0, R_i) is an edge of $\mathsf{SG}(\mathcal{R})$ and $(u_j R_j v_j)_{i \leq j \leq n+1}$ is a wbu-derivation fulfilling $|v_i| \geq 1$ and $|v_{n+1}| = 0$. Hence, by induction hypothesis, there exists a path p from R_i to R_{n+1}, with non-null weight, in $\mathsf{SG}(\mathcal{R})$. The edge (R_0, R_i) followed by the path p is a path with non-null weight from R_0 to R_{n+1}.

Case 2. $|v_i| < |v_0|$.

In this case, since the given derivation is wbu, $|v_0| < |v_i| + |\partial^-(R_i)|$. Hence (R_0, R_i) is an edge of weight 1 of $\mathsf{SG}(\mathcal{R})$. By Lemma 5.8 there exists a path p from R_i to R_{n+1} in $\mathsf{SG}(\mathcal{R})$. We can conclude as in Case 1.

From $Q(n)$ we can deduce the lemma. □

Lemma 5.10 *(History of a mark). Let D be some marked* wbu*-derivation and let $y \in Y$, $w_1, w_2 \in Y^*$ such that $\partial^-(D)$ is unmarked, $\partial^+(D) = w_1 y w_2$ and the mark of y in the corresponding marked word is $k > 0$. Then, there exist $u, v \in Y^*$, $R \in \mathcal{R}$ and some derivations D', D'' such that*

1. *$D = D' \otimes uRvyw_2 \otimes D''yw_2$,*
2. *the mark of y in every step of $D''yw_2$ is k,*
3. *the mark of y in $\partial^+(D')$ is $< k$. (See Fig. 8.)*

Proof. Let us remark that every derivation D fulfilling the hypothesis of the lemma must have a length $\ell(D) = n + 1$ for some integer $n \geq 0$ (since its result $\partial^+(D)$ has some non-null mark). We prove the lemma by induction on this integer n.

Basis: $n = 0$.

Thus $D = v_1 R v_2$ for some $v_1, v_2 \in Y^*$, $R \in \mathcal{R}$. Since the given occurrence of y has a non-null mark, it must be a position of v_2. It follows that $D = uRvyw_2$ for some words $u, v \in Y^*$. Let us define:

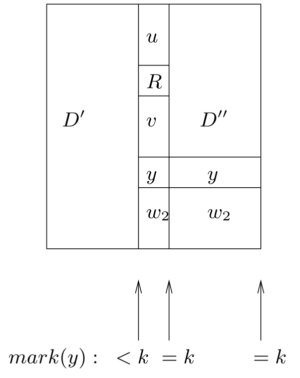

Fig. 8. History of a mark.

$$D' := D_{u\partial^-(R)vyw_2}, \qquad D'' := D_{w_1}.$$

These derivations fulfill conclusions 1–3 of the lemma.

Induction step: $n \geq 1$.

By the same arguments, $D = E \otimes uRvyw_2$ for some $u, v \in Y^*$, $R \in \mathcal{R}$ and some derivation E of length n.

Case 1. The mark of y in $\partial^+(E) = u\partial^-(R)vyw_2$ is k.

By induction hypothesis E has some decomposition as $E = E' \otimes u'R'v'yw_2 \otimes E''yw_2$ such that the mark of y in every step of $E''yw_2$ is k and the mark of y in $\partial^+(E')$ is $< k$. Taking $D' := E'$ and $D'' := E'' \otimes uRv$, the conclusion of the lemma is fulfilled.

Case 2. The mark of y in $\partial^+(E) = u\partial^-(R)vyw_2$ is $< k$.

Taking $D' := E$ and $D'' := D_{u\partial^+(R)v}$, the conclusion of the lemma is fulfilled. □

Let $\mathcal{R}$ be a semi-Thue system and $k \in \mathbb{N}$. We consider the following property $\mathsf{PATH}(k)$: for every $v_1, w_2 \in Y^*$, $R' \in \mathcal{R}$ and wbu-derivations D, E such that

$$E = D \otimes v_1R'w_2 \;\&\; \mathsf{m}\big(\mathsf{last}\big(v_1\partial^-\big(R'\big)\big)\big) = k \tag{50}$$

there exists a path in $\mathsf{SG}(\mathcal{R})$ with weight $\geq k$ and with extremity R'.

Lemma 5.11. *Let $\mathcal{R}$ be a semi-Thue system. For every $k \in \mathbb{N}$, the property $\mathsf{PATH}(k)$ holds.*

Proof. We prove by induction over $k \in \mathbb{N}$ the statement: $\forall k \in \mathbb{N}$, $\mathsf{PATH}(k)$.

Basis: $k = 0$.

There exists a path of length 0, thus of weight ≥ 0, in $\mathsf{SG}(\mathcal{R})$, with extremity R'.

Induction step: $k \geq 1$.

Let us assume (50). Applying Lemma 5.10 to the derivation D, to the letter $y = \mathsf{last}(v_1\partial^-(R'))$ and to the words $w_1 := v_1\partial^-(R')y^{-1}$, w_2, we obtain u, v, R, D', D'' such that:

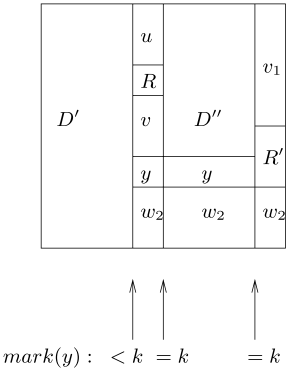

Fig. 9. From marks (in derivations) to weights (in paths).

$$E = D' \otimes uRvyw_2 \otimes D''yw_2 \otimes v_1R'w_2,$$

the mark of y in every step of $D''yw_2$ is k and the mark of y in $\partial^+(D')$ is $< k$ (see Fig. 9). By the definition of a marked rewriting-step, we must have:

$$k = M(\bar{u}\bar{l}, x)$$

where $R = l \to r$, x is the variable of l and $\bar{u}$ is the marked word corresponding to the context where R is applied. Let us consider $E' := D' \otimes uRvyw_2$. It fulfills

$$E' = D' \otimes uRvyw_2 \,\&\, \mathsf{m}\big(\mathsf{last}\big(v\partial^-(R)\big)\big) = k - 1.$$

By induction hypothesis, there exists a path p in $\mathsf{SG}(\mathcal{R})$ with weight $\geq k-1$ and with extremity R; by Lemma 5.9, there exists a path q with non-null weight from R to R' in $\mathsf{SG}(\mathcal{R})$. The concatenation $p \cdot q$ is a path with weight $\geq k$ in $\mathsf{SG}(\mathcal{R})$. $\square$

Proposition 5.12. *Let $\mathcal{R}$ be a semi-Thue system and $k \geq 1$. If $W(\mathsf{SG}(\mathcal{R})) = k - 1$ then $\mathcal{R} \in \mathsf{SBU}(k)$.*

Proof. Suppose that $\mathcal{R} \notin \mathsf{SBU}(k)$. This means that some wbu-derivation (w.r.t. $\hat{\mathcal{R}}$) starting from a non-marked (unary) term over $Y \cup \{\#\}$ reaches a marked term with the mark $k+1$. Let $\hat{E}$ be a wbu derivation (w.r.t. $\hat{\mathcal{R}}$) with minimal length reaching the mark $k+1$. Let us consider the derivation E (w.r.t. $\mathcal{R}$) corresponding to $\hat{E}$ (it is obtained from $\hat{E}$ just by erasing all occurrences of the nullary symbol $\#$). This derivation E must have a decomposition of the form (50). By Lemma 5.11 $\mathsf{PATH}(k)$ holds, hence there exists a path in $\mathsf{SG}(\mathcal{R})$ with weight $\geq k$. By contraposition, if $W(\mathsf{SG}(\mathcal{R})) \leq k-1$ then $\mathcal{R} \in \mathsf{SBU}(k)$, which proves the proposition. $\square$

5.4. A sufficient condition for term rewriting systems

Proposition 5.13. *Let $\mathcal{R}$ be a linear system and $k \geq 1$. If $W(\mathsf{SG}(\mathcal{R})) = k - 1$ then $\mathcal{R} \in \mathsf{SBU}(k)$.*

Sketch of proof. Let us associate to $\mathcal{R}$ the semi-Thue system T corresponding to the "branch-rewriting" induced by $\mathcal{R}$: it consists of all the rules

$$u \to v \in \mathcal{F}^* \times \mathcal{F}^*$$

such that there exist a rule $l \to r \in \mathcal{R}$, and a variable $x \in \mathcal{V}$, such that ux labels a branch of l and vx labels a branch of r. Suppose that the mark $k+1$ appears in an $\mathcal{R}$-derivation. Since the marking-mechanism is defined branch by branch, the mark $k+1$ also appears in a T-derivation. By Proposition 5.12, there exists a path in $\mathsf{SG}(T)$ with weight $\geq k$. Let us fix some total ordering on $\mathcal{R}$ and define the map $h : T \to \mathcal{R}$ by:

$$h(u \to v) = l \to r$$

iff $l \to r$ is the smallest rule of $\mathcal{R}$ such that ux (resp. vx) labels a branch of l (resp. r) and x is a variable. This map h is a homomorphism of labeled graphs from $\mathsf{SG}(T)$ to $\mathsf{SG}(\mathcal{R})$, i.e. it is compatible with the labels. It follows that it is also compatible with the weights. Hence there exists a path of weight $\geq k$ in $\mathsf{SG}(\mathcal{R})$. $\square$

Corollary 5.14. *Let $\mathcal{R}$ be a linear system. If $W(\mathsf{SG}(\mathcal{R}))$ is finite then $\mathcal{R} \in \mathsf{SBU}$.*

Proposition 5.15. $\mathsf{LFPO}^{-1} \subsetneq \mathsf{SBU}$.

Proof. Let $\mathcal{R} \in \mathsf{LFPO}^{-1}$. By definition the sticking-out graph of Takai et al. (2010) does not contain a cycle of weight 1, hence from Corollary 5.14, $\mathcal{R} \in \mathsf{SBU}$. So $\mathsf{LFPO}^{-1} \subseteq \mathsf{SBU}$. $\mathcal{R}_0 \in \mathsf{SBU}$ but $\mathcal{R}_0 \notin \mathsf{LFPO}^{-1}$. We conclude that $\mathsf{LFPO}^{-1} \subsetneq \mathsf{SBU}$. $\square$

6. Related works and perspectives

In parallel with the results reported here the authors (Durand and Sénizergues, 2009) started to investigate what is the *complexity* of the constructions described here and the decidability of properties $\mathsf{BU}(k)$, BU. Some other natural perspectives of development for this work are the following:

1. the method developed here for the sake of showing a property of recognizability preservation might be used, also, for testing some termination properties; this idea is implemented in Durand et al. (2010);
2. it is tempting to extend the notion of *bottom-up* rewriting (resp. system) to left-linear but non-right-linear systems. This class would extend the class of growing systems studied in Nagaya and Toyama (2002); this idea is implemented in Durand and Sylvestre (2011);
3. a dual notion of *top-down* rewriting and a corresponding class of top-down systems should be defined; this class would presumably extend the class of Layered Transducing systems defined in Seki et al. (2002);
4. we know that the condition $\mathsf{BU}(k)$ is undecidable (for every $k \geq 1$) and that the condition $\mathsf{SBU}(k)$ is decidable (for every $k \geq 1$); whether the condition SBU is decidable is thus a natural question;
5. the systems considered in Geser et al. (2004) and the systems considered here might be treated in a unified manner; such a unified approach should lead to an even larger class of rewriting systems with still good algorithmic properties.

Some work in directions 1, 2, 3 has been undertaken by the authors.

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