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Algorithm for finding structures and obstructions of tree ideals

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Abstract

Let $\mathscr{I}$ be any topological minor closed class of trees (a tree ideal). A classical theorem of Kruskal [Well-quasi-ordering, the Tree Theorem, and Vazsonyi's conjecture, Trans. Am. Math. Soc. 95 (1960) 210–223] states that the set $O(\mathscr{I})$ of minimal non-members of $\mathscr{I}$ is finite. On the other hand, a finite structural description $S(\mathscr{I})$ is developed by Robertson, et al. [Structural descriptions of lower ideals of trees, Contemp. Math. 147 (1993) 525–538]. Given either of the two finite characterizations of $\mathscr{I}$, we present an algorithm that computes the other.

Keywords: Tree ideal; Obstructions; Structure-tree

1. Introduction

A *tree* in this paper is a triple $T = (V, E, r)$, where V is a finite set of vertices, E, the set of edges, is a subset of $V \times V$, and r, the *root* of T, is an element of V, such that for every $t \in V$ there is a unique directed walk from r to t and is denoted by $r = root(T)$. (A sequence $t_0, t_1, \ldots, t_n$ is a *directed walk* from t_0 to t_n if $(t_{i-1}, t_i) \in E$ for all $i, i = 1, 2, \ldots, n$) For $s, t \in V(T)$, let $s \wedge t$ denote the last vertex of the directed walk from $root(T)$ to s which belongs to the directed walk from $root(T)$ to t. Given trees T_1, T_2, we say that T_1 is a *topological minor of* T_2 (and write $T_1 \leqslant_t T_2$) if there exists a 1-to-1 mapping $f: V(T_1) \to V(T_2)$, called a *tree embedding*, such that $f(s \wedge t) = f(s) \wedge f(t)$, for every pair $s, t \in V(T_1)$. The notation $T_1 <_t T_2$ means $T_1 \leqslant_t T_2 \nleqslant_t T_1$. A set Q with relation $\leqslant_q$ is a *quasi-order* if $\leqslant_q$ is reflexive and transitive. If $I \subseteq Q$ and I is closed under $\leqslant_q$, we say I is an ideal. In this note, unless specified otherwise, by ideal we mean ideal of trees ordered by $\leqslant_t$. We denote the set of all trees by $\mathscr{F}$. An ideal $\mathscr{I}$ is *proper* if $\mathscr{I} \subset \mathscr{F}$. A tree T is an*obstruction* of an ideal $\mathscr{I}$, if $T \notin \mathscr{I}$ and for any tree $T' <_t T$, we have $T' \in \mathscr{I}$.

Characterizing an ideal $\mathscr{I}$ by its finite obstruction set O($\mathscr{I}$) is a well-known method [1,2]. In [7] a new method was developed, associating a finite set of "bits" $\mathscr{B}$ with $\mathscr{I}$, a bit B expressed as a finite sequence of proper subideals of $\mathscr{I}$ and a natural number k. The authors in [7] also have shown that the finite set $\mathscr{B}$ leads recursively to a finite structural

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description $S(\mathscr{I})$. The main result of this paper is relating the two types of finite characterizations of ideals by a recursive algorithm.

The proof of existence in [7] uses a set called the set of all "germs of $\mathscr{I}$", that is well-quasi-ordered by a certain relation. The realization of this finite structure involves a theory of decomposing a well-quasi-ordered set in to a finite set of "coherent" ideals. Thanks to this pioneer work done by the authors of [7], in this note we give a pure combinatorial proof of their theorem. A result of uniqueness of $S(\mathscr{I})$ in [5] satisfying certain axioms also implies that both methods (the construction in [7] and what we show below) lead to the same finite structure $S(\mathscr{I})$. For the sake of simplicity, we do not try to obtain uniqueness in this note (see [5]).

Prior to a formal definition of a structural description, we would like to familiarize the reader with its basic idea and motive. A structural description $S(\mathscr{I})$ of an ideal $\mathscr{I}$ is a set of rules that shows how to construct every element in $\mathscr{I}$. The finiteness of a structural description allows us to construct a halting Turing machine which accepts precisely the members of $\mathscr{I}$. Either of the two descriptions ($\mathrm{O}(\mathscr{I})$ or $S(\mathscr{I})$) can be used to decide membership of a tree T in $\mathscr{I}$. However, $\mathrm{O}(\mathscr{I})$ is a characterization from external perspective as opposed to $S(\mathscr{I})$ which is internal. From $S(\mathscr{I})$ we have exact structural knowledge of the elements of $\mathscr{I}$, whereas from $\mathrm{O}(\mathscr{I})$ we recognize elements by the fact that they do not allow embedding of any of the obstructions. We also make a note on the computational complexity aspect of having both $\mathrm{O}(\mathscr{I})$ and $S(\mathscr{I})$. We can use an obstruction $T' \in \mathrm{O}(\mathscr{I})$ as a certificate for non-membership of a given tree T in an ideal $\mathscr{I}$, such that $T' \leqslant_t T$. However, for the complementary problem of proving membership of T in $\mathscr{I}$, it would be necessary to verify that all obstructions are not embedable in T unless a different certificate exists. In this case, the structural proof from $S(\mathscr{I})$ can be a more efficient method. An open problem on complexity of a language about deciding membership of a tree in an ideal is given in [4]. We discuss further motivation for our studies in the last section.

The following folklore lemma holds for any quasi-ordered set.

Lemma 1. *Let $(Q, \leqslant_q)$ be a quasi-order and let I and I' be ideals of Q and let $\mathrm{O}(I)$, $\mathrm{O}(I')$ be their respective obstruction set. Then $I \subseteq I'$ if and only if for every $x' \in \mathrm{O}(I')$ there exists $x \in \mathrm{O}(I)$ such that $x \leqslant_q x'$.*

Proof. Suppose there exists $x' \in \mathrm{O}(I')$ such that for all $x \in \mathrm{O}(I)$, $x \not\leqslant_q x'$. Then $x' \in I - I'$, and so $I \not\subseteq I'$. Conversely, suppose every $x' \in \mathrm{O}(I')$ contains some $x \in \mathrm{O}(I)$. Then, by transitivity of $\leqslant_q$, no element of I contains any $x' \in \mathrm{O}(I')$, and so $I \subseteq I'$. □

We now define a basic operation of constructing a new tree by "tree-summing" a finite number of trees. As a notational convenience Γ is defined to be the *null tree*, where $V(\Gamma) = E(\Gamma) = \emptyset$. Clearly, Γ is not a rooted tree. Let $T_1, T_2, \ldots, T_n, n \geqslant 0$, be pairwise vertex disjoint trees or Γ. Then, $T = Tree(T_1, T_2, \ldots, T_n)$ is called the *tree sum* of $T_1, T_2, \ldots, T_n$, where $V(T) = V(T_1) \cup V(T_2) \cup \cdots \cup V(T_n) \cup \{t_0\}$, t_0 is a new vertex, and its set of edges is $E(T_1) \cup E(T_2) \cup \cdots \cup E(T_n) \cup \{(t_0, root(T_i)) : 1 \leqslant i \leqslant n, T_i \neq \Gamma\}$. We say T_i, $1 \leqslant i \leqslant n$ is a *summand* of T. (note that $Tree(T_1, T_2, \Gamma)$ is isomorphic to $Tree(T_1, T_2)$ but allowing the null tree to appear in a tree sum will be convenient later).

Next we define what is called a "proper description" of an ideal $\mathscr{I}$ using "bits". We say that $B = (\mathscr{I}_1, \ldots, \mathscr{I}_n; k; \mathscr{I}_0)$ is a *bit* if $n, k \geqslant 0$ and $\mathscr{I}_i$ is an ideal, $0 \leqslant i \leqslant n$. We say $\mathscr{I}_i$ is a *component* of B and denote this by $\mathscr{I}_i @ B$. If $i \neq 0$, we assume $\mathscr{I}_i \neq \emptyset$ and we call $\mathscr{I}_i$ a *left component of* B. We call $\mathscr{I}_0$ the *right component* of B. Two bits B and B' are assumed to be *equal* if they differ only by a permutation of their left components. The integer k is called the *width* of B and is denoted by $k(B)$.

Let $\mathscr{B} = \{B_1, B_2, \ldots, B_p\}$ be a set of bits. We give two equivalent definitions of a set $I(\mathscr{B})$, called the *span* of $\mathscr{B}$.

Definition 2. $I(\mathscr{B})$ is the intersection of all ideals $\mathscr{I}$ satisfying the following: (*) if $(\mathscr{I}_1, \ldots, \mathscr{I}_n; k; \mathscr{I}_0) \in \mathscr{B}$, $T_i \in \mathscr{I}_i$, for $i = 1, \ldots, n$; $T_{n+i} \in \mathscr{I}$, for $i = 1, \ldots, k$, and $m \geqslant 0$, $T_{n+k+1}, \ldots, T_{n+k+m} \in \mathscr{I}_0$, then $Tree(T_1, T_2 \ldots, T_{k+n+m})$ belongs to $\mathscr{I}$.

Note that in this definition, T is a tree sum consisting of three types of trees. The three types are depicted in Fig. 1. We call the triangular shaped summands of T the *left part* and *right part*, respectively, and the oval shaped summands the *middle part* of the tree sum. The next definition of $I(\mathscr{B})$ is constructive, that shows how to recursively construct the ideal $I(\mathscr{B})$ starting from the components of the bits in $\mathscr{B}$. We can once more make use of Fig. 1 to see that pictorially this construction is simple.

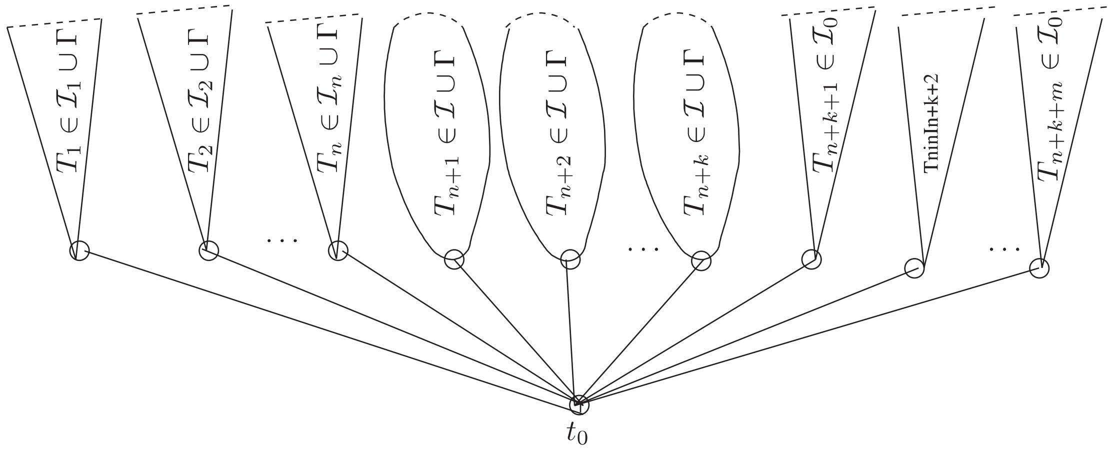

Fig. 1. A typical construction of a tree T using a bit $B=(\mathscr{I}_1,\mathscr{I}_2,\ldots,\mathscr{I}_n;k;\mathscr{I}_0)$ in a set of bits $\mathscr{B}$, where t_0 is a new vertex and the summands are from ideals as shown in the figure.

Definition 3. Let $B_i=(\mathscr{I}^i_1,\mathscr{I}^i_2,\ldots,\mathscr{I}^i_{n_i};k_i;\mathscr{I}^i_0)$, $1\leqslant i\leqslant p$ where $B_i\in\mathscr{B}$, $i=1,2,\ldots,p$. Let $\mathscr{J}_0(\mathscr{B})=\bigcup_{i=1}^p J_0(B_i)$ where $J_0(B_i)=\bigcup_{l=0}^{n_i}\mathscr{I}^i_l$ and recursively, for $j\geqslant 1$, let $\mathscr{J}_j(\mathscr{B})=\mathscr{J}_{j-1}(\mathscr{B})\cup\bigcup_{i=1}^p J_j(B_i)$ where $J_j(B_i)=\{T : T=Tree(T_1,T_2,\ldots,T_{n_i+k_i+m})$, for some integer $m\geqslant 0$ and $T_l\in\mathscr{I}^i_l\cup\{\Gamma\}$, for $l=1,\ldots,n_i$; $T_{n_i+l}\in\mathscr{J}_{j-1}(\mathscr{B})\cup\{\Gamma\}$, for $l=1,\ldots,k_i$ and $T_{n_i+k_i+l}\in\mathscr{I}^i_0$, for $l=1,\ldots,m\}$. We let,

$$I(\mathscr{B})=\bigcup_{j\geqslant 0}\mathscr{J}_j(\mathscr{B})$$

It is trivial to see by inducting on j that $I(\mathscr{B})$ is contained in any ideal $\mathscr{I}$ satisfying (*), and that $I(\mathscr{B})$ itself satisfies (*). Hence, the two definitions are equivalent. We are now ready to give a formal definition of a "proper description" of an ideal $\mathscr{I}$.

Definition 4. Let $\mathscr{I}$ be a proper ideal and $\mathscr{B}$ be a finite set of bits. We say $\mathscr{B}$ is a proper description of $\mathscr{I}$ if the following two properties hold:

(P1). if $\mathscr{I}'@B$ for any $B\in\mathscr{B}$, then $\mathscr{I}'\subset\mathscr{I}$ (induction axiom), and
(P2). $I(\mathscr{B})=\mathscr{I}$ (spanning axiom).

We offer few simple examples to illustrate the construction.

Example 5. Let $\mathscr{K}_k$ denote the ideal of all trees with out-degree at most k, $k\geqslant 0$, and let $\mathscr{K}_{k,h}\subset\mathscr{K}_k$ be the ideal of trees in $\mathscr{K}_k$ of height at most $h\geqslant 0$. Let $\mathscr{B}=\{(;k;\emptyset)\}$. Then $\mathscr{B}$ is a proper description of $\mathscr{K}_k$. Note that $\mathscr{J}_0(\mathscr{B})=\emptyset$, $\mathscr{J}_1(\mathscr{B})=\mathscr{K}_{k,0}$, $\mathscr{J}_2(\mathscr{B})=\mathscr{K}_{k,1}$, $\mathscr{J}_3(\mathscr{B})=\mathscr{K}_{k,2}$, ...

Example 6. Let $\mathscr{B}=\{(;1;\emptyset),(;0;\{P_0\})\}$. Then $\mathscr{B}$ is a proper description of all finite paths P_k, $k\geqslant 0$ with star glued at their end vertex. It can be seen that $\mathscr{J}_0(\mathscr{B})=\{P_0\}$, $\mathscr{J}_1(\mathscr{B})=\mathscr{S}$ (the star ideal), $\mathscr{J}_2(\mathscr{B})=\mathscr{S}\cup\{T : T=Tree(S), S\in\mathscr{S}\}$, ... $\mathscr{J}_3(\mathscr{B})=\mathscr{J}_2\cup\{T : T=Tree(Tree(S)), S\in\mathscr{S}\}$, ...

Finally, using a proper description $\mathscr{B}$ of an ideal $\mathscr{I}$, we define recursively a *structural description tree* $S(\mathscr{I})$ as follows: $S(\mathscr{I})$ has a root r and $|\mathscr{B}|$ summands $T_1, T_2, \ldots, T_{|\mathscr{B}|}$, where each T_i corresponds to a bit $B_i\in\mathscr{B}$. The root of T_i is labeled by the width $k(B_i)$, and the summands of T_i are the *structural description trees* of the components in B_i.

Note that $S(\mathscr{I})$ is a finitely branching tree. Moreover, the well-quasi-ordering of finite trees [1] means ideals of trees are well founded under set inclusion, and so by (P1), $S(\mathscr{I})$ has no infinite directed walks. Therefore, by König's infinity lemma, $S(\mathscr{I})$ is itself a finite tree, with natural number labels on vertices of odd distance from the root r. We see that the tree $S(\mathscr{I})$ contains all necessary information to construct every tree in $\mathscr{I}$. This is the finite structural description we are referring to.

2. Proper description for tree ideals with a single obstruction

In this section we show that for any tree $T \in \mathscr{F}$, we can give a proper description of $\mathscr{F}/T$, where $\mathscr{F}/T$ denotes the set of all trees not containing T as a topological minor. Note that $\mathscr{F}/T$ is an ideal.

Theorem 7. *Let* $T=Tree(T_1, T_2, \dots, T_n) \in \mathscr{F}$, $n\geqslant 0$, *and let* $\mathscr{B}=\{(;k;\mathscr{I}_S) : k=0,1,\dots,n-1, \mathscr{I}_S=\bigcap_{i\in S}\mathscr{F}/T_i, S \subseteq \{1,2,\dots,n\}, |S|=k+1\}$. *Then,* $\mathscr{B}$ *is a proper description of* $\mathscr{F}/T$.

Proof. If $n=0$, then $\mathscr{I}=\emptyset$ and $\mathscr{B}=\emptyset$. Otherwise, for every $S\subseteq\{1,\dots,n\}$, we have $\mathrm{O}(\mathscr{I}_S)\subseteq\{T_1,\dots,T_n\}$ and so $\mathscr{I}_S\subset\mathscr{F}/T$, by Lemma 1. Hence, $\mathscr{B}$ satisfies (P1). It remains to show that $\mathscr{B}$ also satisfies (P2).

Assume $I(\mathscr{B})\nsubseteq\mathscr{F}/T$ and let T' be a minimal height tree in $I(\mathscr{B})-\mathscr{F}/T$. Then, $T\leqslant_t T'$ by a tree embedding f. By definition, $T'\in J_j(B)$ for some bit $(;k;\mathscr{I}_S)=B\in\mathscr{B}$ and some $j\geqslant 0$. If $j=0$ (i.e. $T'\in\mathscr{I}_S$), then $T'\in\mathscr{F}/T$, a contradiction. Suppose $j>0$. By induction, no summand of T' contains T, and so $f(root(T))=f(root(T'))$. Also, the n summands of T are embedded in n distinct summands of T'. Since $\mathscr{I}_S=\bigcap_{i\in S}\mathscr{F}/T_i$, where $|S|=k+1$ and $k(B)=k$, we can embed at most $n-(k+1)+k=n-1$ summands of T, a contradiction. Hence, $I(\mathscr{B})\subseteq\mathscr{I}$.

Conversely, assume $\mathscr{F}/T\nsubseteq I(\mathscr{B})$ and let T'' be a minimal height tree in $\mathscr{F}/T-I(\mathscr{B})$. Then, $T\nleqslant_t T''$. Let $T''=Tree(T_1'',T_2'',\dots,T_p'')$, $p\geqslant 0$. We show $T''\in J_j(B)$ for some $B\in\mathscr{B}$ and $j\geqslant 0$, contrary to our assumption.

Let $[n]=\{1,2,\dots,n\}$ and $[p]=\{1,2,\dots,p\}$. Let $g:[n]\to[p]$, be a mapping such that $g(i)=j$ if and only if $T_i\leqslant_t T_j''$. Since $T\nleqslant_t T''$, g is not injective. Then, by Hall's marriage theorem, there exists a set $S\subseteq[n]$ such that $|S|>|D|$, where $D=\{j\in[p]: T_i\leqslant_t T_j''$ for some $i\in S\}$. Let $|S|=k+1$, $0\leqslant k\leqslant n-1$. Then $|D|\leqslant k$. Let $S=\{i_1,\dots,i_{k+1}\}\subseteq[n]$. Then, for all $j\in[p]-D$ and for all $i=i_1,\dots,i_{k+1}$, we have $T_i\nleqslant_t T_j''$, and so $T_j''\in\bigcap_{i\in S}(\mathscr{F}/T_i)$. Since $(;k;\bigcap_{i\in S}(\mathscr{F}/T_i))=B\in\mathscr{B}$ and since $|D|\leqslant k$ and by the choice of T'' all trees indexed by D are in $I(\mathscr{B})$. It follows that $T''\in J_j(B)$, for some $j\geqslant 1$, acontradiction. Hence (P2) satisfied. $\square$

3. Proper description for tree ideals of several obstructions

In this section we present an algorithm that computes a proper description for arbitrary tree ideal $\mathscr{I}$ and prove its correctness. We assume we have already computed a proper description of an ideal $\mathscr{I}$ which has r obstructions, $r\geqslant 1$ and show how we find the ideal which has $r+1$ obstructions.

As a notational convenience, we sometimes write in the usual integer coefficient notation $B=(\mathscr{I}_1,\dots,m\mathscr{I}_i,\dots,\mathscr{I}_n;k;\mathscr{I}_0)$, to mean that $\mathscr{I}_i$ is repeated m times as a left component of B, $1\leqslant i\leqslant n$. In the previous section, we have seen proper descriptions of ideals of single obstruction do not have bits with left components. However, ideals even with two obstructions can have left components.

Definition 8. Let $\mathscr{B}$ be a proper description of an ideal $\mathscr{I}$. Let $T\in\mathscr{F}$ and let $\mathscr{B}'$ be a proper description of $\mathscr{F}/T$. For every pair of bits $(B,B')\in\mathscr{B}\times\mathscr{B}'$, where $B=(\mathscr{I}_1,\dots,\mathscr{I}_n;k;\mathscr{I}_0)\in\mathscr{B}$ and $B'=(;k';\mathscr{I}_0')\in\mathscr{B}'$ and for $0\leqslant k''\leqslant\min(k,k')$, $0\leqslant p\leqslant n$, let:

$$\mathscr{I}_i^1=I(\mathscr{B}')\cap\mathscr{I}_i,\quad 1\leqslant i\leqslant p\leqslant k'-k'', \tag{1}$$

$$\mathscr{I}_i^2=\mathscr{I}_0'\cap\mathscr{I}_i,\quad p+1\leqslant i\leqslant n, \tag{2}$$

$$\mathscr{I}^3=I(\mathscr{B})\cap\mathscr{I}_0', \tag{3}$$

$$\mathscr{I}^4=I(\mathscr{B}')\cap\mathscr{I}_0, \tag{4}$$

$$\mathscr{I}^5=\mathscr{I}_0\cap\mathscr{I}_0', \tag{5}$$

$$B\odot B'=\{(\mathscr{I}_1^1,\dots,\mathscr{I}_p^1,\mathscr{I}_{p+1}^2,\dots,\mathscr{I}_n^2,(k-k'')\mathscr{I}^3,(k'-k''-p)\mathscr{I}^4;k'';\mathscr{I}^5)\}. \tag{6}$$

An intertwine of $\mathscr{B}$ and $\mathscr{B}'$ is a set of bits $\mathscr{B}''$ (also denoted by $\mathscr{B} \odot \mathscr{B}'$) such that,

$$\mathscr{B}'' = \bigcup_{(B,B')\in\mathscr{B}\times\mathscr{B}'} B \odot B'.$$

The following is an algorithm A that finds a proper description of any proper ideal:

Algorithm 9. $A=$ "*On input* $\mathrm{O}(\mathscr{I}) = \{T_1, T_2, \ldots, T_q\}$ (*pairwise incomparable trees*):

1. *Obtain a proper description* $\mathscr{B}_i$ *of* $\mathscr{F}/\mathscr{T}_i$ *for each* i, $i = 1, \ldots, q$, *using Theorem* 7. *If* $q = 1$, *output* $\mathscr{B}_1$.
2. *if* $I(\mathscr{B}_1) \cap I(\mathscr{B}_2) \cap \cdots \cap I(\mathscr{B}_r) = I(\mathscr{B}^r)$, $1 \leqslant r \leqslant q-1$, *where* $\mathscr{B}^r$ *is a proper description of* $\bigcap_{l=1}^{r} \mathscr{F}/T_l$, *compute* $\mathscr{B}^{r+1}$ *as* $\mathscr{B}^r \odot \mathscr{B}_{r+1}$ *using Definition* 8.
3. *If* $r = q-1$, *output* $\mathscr{B}^{r+1}$, *otherwise increase* r *and return to step* 2.

Lemma 10. *For any* $r \geqslant 1$, *let* $\mathscr{I} = I(\mathscr{B}^r) \cap I(\mathscr{B}_{r+1})$, *where* $\mathscr{B}^r$ *and* $\mathscr{B}_{r+1}$ *are proper descriptions of* $\bigcap_{l=1}^{r} \mathscr{F}/T_l$ *and* $\mathscr{F}/T_{r+1}$, *respectively. Then*, $\mathscr{B}^r \odot \mathscr{B}_{r+1}$ *is a proper description of* $\mathscr{I}$.

Proof. Let $\mathscr{B}^r = \mathscr{B}$, $\mathscr{B}_{r+1} = \mathscr{B}'$ and $\mathscr{B}^r \odot \mathscr{B}_{r+1} = \mathscr{B}''$. To prove $\mathscr{B}''$ satisfies (P1), take a bit $B'' \in \mathscr{B}''$. Then $B'' \in B \odot B'$ for some bits $B = (\mathscr{I}_1, \ldots, \mathscr{I}_n; k; \mathscr{I}_0) \in \mathscr{B}$ and $B' = (; k'; \mathscr{I}'_0) \in \mathscr{B}'$. We show that the components of B'' listed in (1)–(5) are proper subideals of $\mathscr{I}$. Note that $\mathrm{O}(\mathscr{I}) = \{T_1, \ldots, T_{r+1}\} = \mathrm{O}(I(\mathscr{B})) \cup \mathrm{O}(I(\mathscr{B}'))$. By induction on r, we note that each tree in $\mathrm{O}(\mathscr{I}_i)$, $0 \leqslant i \leqslant n$ is topologically contained in one of the trees in $\{T_1, \ldots, T_r\}$. Moreover, $\mathscr{B}$ satisfies (P1) and so $\mathscr{I}_i \subset I(\mathscr{B})$. Then, by Lemma 1, there is a tree $T' \in \mathrm{O}(\mathscr{I}_i)$ such that $T_j \nleqslant_t T'$, for all j, $1 \leqslant j \leqslant r$. Then $T_{r+1} \nleqslant_t T'$, for otherwise we have $T_{r+1} \leqslant_t T' \leqslant_t T_j$, for some j, $1 \leqslant j \leqslant r$, contrary to $\mathrm{O}(\mathscr{I})$ being an anti-chain. Hence, $\mathscr{I}_i \nsupseteq \mathscr{I}$ by Lemma 1. This implies the components of B'' in (1), (2), (4) and (5) satisfy (P1). Using B', by the same argument we deduce that (3) satisfies (P1). Hence, $\mathscr{B}''$ satisfies (P1). We prove next the spanning axiom (P2) holds.

To prove $I(\mathscr{B}'') \subseteq I(\mathscr{B}) \cap I(\mathscr{B}')$, we may assume every tree of $I(\mathscr{B}'')$ with height less than the height of T is in $I(\mathscr{B}) \cap I(\mathscr{B}')$. Let $T \in J_j(B'')$. By (P1), we have, $J_0(B'') \subseteq I(\mathscr{B}) \cap I(\mathscr{B}')$ and so assume $j > 0$. We write T as a tree sum of at most six types of summands corresponding to the components listed in (1)–(5) and the middle part for B'' (See (6)). The n summands from (1) and (2) are injectively contained in the n left components of B, whereas the summands from (4) and (5) are all in $\mathscr{I}_0$. From (3) and middle part we have $(k - k'') + k'' = k$ summands and by induction these are in $I(\mathscr{B}) \cap I(\mathscr{B}') \subseteq I(\mathscr{B})$. Since $k(B) = k$, we deduce that $T \in J_{j_1}(B) \subseteq I(\mathscr{B})$, for some $j_1 \geqslant 1$. Similarly, from (1), (4) and the middle part we have $p + (k' - k'' - p) + k'' = k(B')$ summands and by induction they are in $I(\mathscr{B}')$, whereas the summands from (2), (3) and (5) are all in $\mathscr{I}'_0$. We deduce $T \in J_{j_2}(B')$ for some $j_2 \geqslant 1$. Hence, $I(\mathscr{B}'') \subseteq I(\mathscr{B}) \cap I(\mathscr{B}')$.

Conversely, let $T' \in I(\mathscr{B}) \cap I(\mathscr{B}')$ be such that every tree in the intersection with shorter height is in $I(\mathscr{B}'')$. We have $T' \in J_{j_1}(B) \subseteq I(\mathscr{B})$ and $T' \in J_{j_2}(B') \subseteq I(\mathscr{B}')$, where $j_1, j_2 \geqslant 0$. Clearly, (1)–(5) shows that if $T' \in J_0(B) \cup J_0(B')$, then $T' \in J_0(B'')$ for some $B'' \in B \odot B' \subseteq \mathscr{B}''$. Hence, we assume j_1 and j_2 are both positive. As $T' \in J_{j_1}(B)$, we write T' as depicted in Fig. 1

$$T' = Tree(T_1, \ldots, T_n, T_{n+1}, \ldots, T_{n+k}, T_{n+k+1}, \ldots, T_{n+k+m}). \tag{7}$$

Let L, M, R be the left middle and right part of T' in (7). As $T' \in J_{j_2}(B')$ and $B' = (; k'; \mathscr{I}'_0)$ we let $L = L_0 \cup L_1$, $M = M_0 \cup M_1$, and $R = R_0 \cup R_1$ such that $\mathscr{I}'_0 \cap (L_0 \cup M_0 \cup R_0) = \emptyset$. Let $0 \leqslant k'' = |M_0| \leqslant \min(k, k')$ and $0 \leqslant p = |L_0| \leqslant n$. Then, $p \leqslant k' - k''$ for otherwise we have $p + k'' > k'$ trees not in $\mathscr{I}'_0$ which implies $T \notin J_{j_2}(B')$, a contradiction. Hence $0 \leqslant |R_0| \leqslant k' - k'' - p$. Now the following one to one correspondence of these partitioned summands with ideals in (6) is easy to see: L_0 with the ideals from (1), L_1 with (2), M_1 with (3) and R_0 with (4). Every tree in R_1 in (5). The k'' trees in M_0 are in $I(\mathscr{B}'')$ by induction. We deduce there is a bit $B'' \in B \odot B' \subseteq \mathscr{B}''$ as in (6) such that $T' \in J_j(B'') \subseteq I(\mathscr{B}'')$, $j \geqslant 0$. The result follows. $\square$

As a result we obtain the following:

Theorem 11. *For every ideal* $\mathscr{I}$, *there exists an algorithm A that outputs the structural description tree* $S(\mathscr{I})$ *from an input* $\mathrm{O}(\mathscr{I})$.

4. Constructing obstructions

We address the reverse question in this section. If a structural description $S(\mathscr{I})$ of an ideal is given, how do we obtain the obstruction set $\mathrm{O}(\mathscr{I})$?

A brute force algorithm is easy to state: Let Δ and h be the maximum degree and maximum height of all trees in $\mathrm{O}(\mathscr{I})$, respectively. Note that these bounds can easily be found from $S(\mathscr{I})$. Then $\mathrm{O}(\mathscr{I})$ is a subset of the finite set of trees of height at most h and degree at most Δ. Using $S(\mathscr{I})$ we can test membership of each of the finite trees and keep only the minimal non-members. Note that it is also easy to check if a given tree is a minimal non-element. That is, we verify $T \notin \mathscr{I}$ and that every proper topological minor of T is in $\mathscr{I}$ by using $S(\mathscr{I})$. On discovering a new obstruction we can run our algorithm in the previous section and see if the description we have is the same as what is given in $S(\mathscr{I})$. Eventually we will find all of the obstructions, since $\mathrm{O}(\mathscr{I})$ is finite. We also know the worst case running time, that the last obstruction we find is among the last trees we check in the finite list. Thus, the problem of finding all obstructions of an ideal $\mathscr{I}$ is a decidable problem.

Theorem 12. *For every ideal $\mathscr{I}$, there exists an algorithm A that outputs the obstruction set* $\mathrm{O}(\mathscr{I})$ *from an input* $S(\mathscr{I})$.

However, we may ask if efficient algorithms exist. Algorithms that are significantly faster than the outlined brute force algorithm indeed exist. For instance one may start from the leaves of $S(\mathscr{I})$ and construct obstructions of the subideals of $\mathscr{I}$ recursively until the obstructions of $\mathscr{I}$ are all found. However, finding efficient algorithm for obstruction is not the theme of this paper.

5. Conclusion and open problems

A similar result for graph minor ideals with bounded tree width is of great interest. Studying topological minor ideals of trees has some advantages. First, we note that minor ideals of trees are topological ideals. In addition, the minor ideal obstruction set is a subset of the topological obstruction set. Hence the result is stronger form these perspectives. On another note, when studying graph minors of bounded tree-width, the tree decomposition of the graphs is a labeled tree [6]. If $T(G)$ and $T(G')$ are the tree decompositions of two graphs G and G' satisfying certain conditions, then it is shown in [7], that topological minor inclusion between the trees implies minor inclusion between the graphs. Hence, analysis of labeled tree ideals, [3], may be used for understanding structures of graph minor ideals as well. Arising from these ideas we present the following problem.

Problem 13. *Can we find a structural description of graph minor ideals of tree-width* 2 (*i.e. K_4-minor-free graphs*)? *Can we generalize the question to any ideal of bounded tree width*?

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