GENERALIZING A THEOREM OF BÈS AND CHOFFRUT

ERIK WALSBERG

ABSTRACT. Bès and Choffrut recently showed that there are no intermediate structures between $(\mathbb{R}, <, +)$ and $(\mathbb{R}, <, +, \mathbb{Z})$. We prove a generalization: if \mathcal{R} is an o-minimal expansion of $(\mathbb{R}, <, +)$ by bounded subsets of Euclidean space then there are no intermediate structures between \mathcal{R} and $(\mathcal{R}, \mathbb{Z})$. It follows there are no intermediate structures between $(\mathbb{R}, <, +, \sin|_{[0,2\pi]})$ and $(\mathbb{R}, <, +, \sin)$.

1. Introduction

Throughout all structures are first order and "definable" means "first-order definable, possibly with parameters". Suppose \mathcal{M} , \mathcal{N} , and \mathcal{O} are structures on a common domain M. Then \mathcal{M} is a reduct of \mathcal{O} if every \mathcal{M} -definable subset of every M^n is \mathcal{O} -definable, \mathcal{M} and \mathcal{O} are interdefinable if each is a reduct of the other, \mathcal{M} is a proper reduct of \mathcal{O} if \mathcal{M} is a reduct of \mathcal{O} and \mathcal{M} is not interdefinable with \mathcal{O} , and \mathcal{N} is intermediate between \mathcal{M} and \mathcal{O} if \mathcal{M} is a proper reduct of \mathcal{N} and \mathcal{N} is a proper reduct of \mathcal{O} . If A is a subset of A then the structure induced on A by \mathcal{M} is the structure on A with an n-ary predicate defining $A^n \cap X$ for every \mathcal{M} -definable $X \subseteq M^{mn}$.

Throughout \mathcal{R} and \mathcal{S} are structures expanding $(\mathbb{R}, <, +)$. Recall that \mathcal{R} is o-minimal if every definable subset of \mathbb{R} is a finite union of open intervals and singletons and \mathcal{R} is locally o-minimal if one of the following equivalent conditions holds.

- (1) If $X \subseteq \mathbb{R}$ is definable then for any $a \in \mathbb{R}$ there is an open interval I containing a such that $I \cap X$ is a finite union of open intervals and singletons.
- (2) If I is a bounded interval then the structure induced on I by \Re is o-minimal.

It is clear that (2) implies (1), the other direction follows by compactness of closed bounded intervals. It follows from independent work of Miller [16] or Weispfenning [27] that $(\mathbb{R},<,+,\mathbb{Z})$ is locally o-minimal, another example of a locally o-minimal non-o-minimal structure is $(\mathbb{R},<,+,\sin)$, see [21].

Bès and Choffrut [3] show that there is no structure intermediate between $(\mathbb{R}, <, +)$ and $(\mathbb{R}, <, +, \mathbb{Z})$. We prove the following.

Theorem 1.1. Suppose \Re is o-minimal and $\alpha > 0$. If $(\Re, \alpha \mathbb{Z})$ is locally o-minimal then there is no structure intermediate between \Re and $(\Re, \alpha \mathbb{Z})$. Furthermore $(\Re, \alpha \mathbb{Z})$ is locally o-minimal if and only if \Re has no poles and rational global scalars.

As $(\mathbb{R}, <, +, \sin|_{[0,2\pi]})$ is o-minimal and $(\mathbb{R}, <, +, \sin|_{[0,2\pi]}, 2\pi\mathbb{Z})$ is interdefinable with $(\mathbb{R}, <, +, \sin)$ there is no intermediate structure between $(\mathbb{R}, <, +, \sin|_{[0,2\pi]})$ and $(\mathbb{R}, <, +, \sin)$. A pole is a definable surjection $\gamma: I \to J$ where I is a bounded interval and J is an unbounded interval and \mathcal{R} has rational global scalars if the

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function $\mathbb{R} \to \mathbb{R}$, $t \mapsto \lambda t$ is only definable when $\lambda \in \mathbb{Q}$.

The second claim of Theorem 1.1 is essentially proven in [26]. Indeed, much of the proof of Theorem 1.1 essentially appears in [26]. As that paper is rather long and involves many abstract model-theoretic concepts, and Theorem 1.1 is of independent interest, it seems worthwhile to record a separate proof. For those interested in NIP we note that the following are shown to be equivalent in [26]:

- (1) The primitive relations of S are boolean combinations of closed sets, the primitive functions of S are continuous, and S is strongly dependent,
- (2) S is either o-minimal or interdefinable with $(\mathcal{R}, \alpha \mathbb{Z})$ for some real number $\alpha > 0$ and o-minimal \mathcal{R} with no poles and rational global scalars.

In contrast arbitrary strongly dependent expansions of $(\mathbb{R}, <, +)$ are as complicated as arbitrary strongly dependent structures of cardinality continuum by [12].

The theorem of Bès and Choffrut is analogous to the theorem of Conant [5] that there are no intermediate structures between $(\mathbb{Z},+)$ and $(\mathbb{Z},<,+)$. Conant's proof relies on a detailed analysis of $(\mathbb{Z},<,+)$ -definable subsets of \mathbb{Z}^n . Alouf and d'Elbée [1] give a quicker proof. They use deep model-theoretic machinery to reduce to the unary case and thereby avoid geometric complexity. They apply the earlier theorem of Conant and Pillay [6] that there are no proper stable dp-minimal expansions of $(\mathbb{Z},+)$, which itself depends on work of Palacín and Sklinos [19], who apply the Buechler dichotomy theorem, a high-level result of stability theory. Bès and Choffrut's proof involves a detailed geometric analysis of $(\mathbb{R},<,+,\mathbb{Z})$ -definable subsets of \mathbb{R}^n . Our proof relies on an important o-minimal structure theorem of Edmundo [7] and Kawakami, Takeuchi, Tanaka, and Tsuboi's [15] work on locally o-minimal structures. These theorems reduce the first claim of Theorem 1.1 to the unary case and also show that the unary case follows from the work of Bès and Choffrut.

In Section 3 we describe what we know about $(\mathcal{R}, \mathbb{Z})$ when \mathcal{R} is o-minimal.

Conventions. m, n are natural numbers and $\alpha, \beta, \lambda, s, t, r$ are real numbers.

2. Proof of Theorem 1.1

Fact 2.1 is a special case of a theorem of Edmundo [7].

Fact 2.1. Suppose S is o-minimal. Then the following are equivalent

- (1) S has no poles.
- (2) S does not define $\oplus, \otimes : \mathbb{R}^2 \to \mathbb{R}$ such that $(\mathbb{R}, <, \oplus, \otimes)$ is isomorphic to $(\mathbb{R}, <, +, \cdot)$,
- (3) there is a collection \mathcal{B} of bounded subsets of Euclidean space and a subfield K of $(\mathbb{R}, +, \cdot)$ such that S is interdefinable with $(\mathbb{R}, <, +, \mathcal{B}, (t \mapsto \lambda t)_{\lambda \in K})$.

Also S has no poles and rational global scalars if and only if there is a collection \mathcal{B} of bounded subsets of Euclidean space such that S is interdefinable with $(\mathbb{R}, <, +, \mathcal{B})$.

Let $\tilde{+}$ be the function $[0,1)^2 \to [0,1)$ where $s\tilde{+}t = s+t$ if s+t < 1 and $s\tilde{+}t = s+t-1$ otherwise. We will need two theorems of Kawakami, Takeuchi, Tanaka, and Tsuboi [15] which in particular show that any locally o-minimal expansion of $(\mathbb{R}, <, +)$ which defines \mathbb{Z} is bi-interpretable with the disjoint union of an o-minimal expansion of $([0,1),<,\tilde{+})$ and an arbitrary expansion of $(\mathbb{Z},<,+)$. A special case

is that $(\mathbb{R}, <, +, \mathbb{Z})$ is bi-interpretable with the disjoint union of $([0, 1), <, \tilde{+})$ and $(\mathbb{Z}, <, +)$. Fact 2.2 is a special case of [15, Theorem 18].

Fact 2.2. Suppose \mathfrak{I} is an o-minimal expansion of $([0,1),<,\tilde{+})$ and \mathfrak{Z} is an expansion of $(\mathbb{Z},<,+)$. Then there is an expansion \mathfrak{S} of $(\mathbb{R},<,+)$ such that a subset of \mathbb{R}^n is \mathfrak{S} -definable if and only if it is a finite union of sets of the form $\bigcup_{b\in Y}b+X$ for \mathfrak{I} -definable $X\subseteq [0,1)^n$ and \mathfrak{Z} -definable $Y\subseteq \mathbb{Z}^n$. This \mathfrak{S} is locally o-minimal.

Note that local o-minimality of S follows directly from the description of S-definable sets. Fact 2.3 is [15, Theorem 24].

Fact 2.3. Suppose S is locally o-minimal and defines \mathbb{Z} . Then every S-definable subset of every \mathbb{R}^n is a finite union of sets of the form $\bigcup_{b\in Y} b+X$ where $X\subseteq [0,1)^n$ and $Y\subseteq \mathbb{Z}^n$ are S-definable.

Proposition 2.4 yields the second claim of Theorem 1.1.

Proposition 2.4. Suppose \Re is o-minimal. If (\Re, \mathbb{Z}) is locally o-minimal then \Re has no poles and rational global scalars. If \Re has no poles and rational global scalars then (\Re, \mathbb{Z}) is locally o-minimal and furthermore every (\Re, \mathbb{Z}) -definable subset of every \mathbb{R}^n is a finite union of sets of the form $\bigcup_{b \in Y} b + X$ for \Re -definable $X \subseteq [0,1)^n$ and $(\mathbb{Z}, <, +)$ -definable $Y \subseteq \mathbb{Z}^n$.

Proof. Suppose \mathcal{R} has a pole $\gamma:I\to J$. Applying the o-minimal monotonicity theorem [24, 3.1.2] and shrinking I and J if necessary we suppose that γ is continuous and strictly increasing or strictly decreasing. After possibly reflecting and translating we suppose that $\mathbb{R}_{>0}\subseteq J$ and γ is strictly increasing. Then $\gamma^{-1}(\mathbb{N})$ is an infinite bounded discrete subset of \mathbb{R} so (\mathcal{R},\mathbb{Z}) is not locally o-minimal. Suppose $\lambda\in\mathbb{R}\setminus\mathbb{Q}$ is such that the function $\mathbb{R}\to\mathbb{R}$ given by $t\mapsto \lambda t$ is \mathbb{R} -definable. Then $\mathbb{Z}+\lambda\mathbb{Z}$ is dense and co-dense, so (\mathcal{R},\mathbb{Z}) is not locally o-minimal.

Now suppose that \mathcal{R} has no poles and rational global scalars. Let \mathcal{I} be the structure induced on [0,1) by \mathcal{R} . Let \mathcal{S} be constructed from \mathcal{I} and $(\mathbb{Z},<,+)$ as in the statement of Fact 2.2. It is enough to show that \mathcal{S} and (\mathcal{R},\mathbb{Z}) are interdefinable. Every \mathcal{I} -definable subset of $[0,1)^n$ is trivially \mathcal{R} -definable, so it follows from Fact 2.2 that \mathcal{S} is a reduct of (\mathcal{R},\mathbb{Z}) . We show that (\mathcal{R},\mathbb{Z}) is a reduct of \mathcal{S} . As \mathcal{S} defines \mathbb{Z} it suffices to show that \mathcal{R} is a reduct of \mathcal{S} . Applying Fact 2.1 let \mathcal{B} be a collection of bounded \mathcal{R} -definable sets such that \mathcal{R} and $(\mathbb{R},<,+,\mathcal{B})$ are interdefinable. Rescaling and translating, we may assume that each $X \in \mathcal{B}$ is a subset of some $[0,1)^n$. So every $X \in \mathcal{B}$ is \mathcal{I} -definable, hence \mathcal{S} -definable. So $(\mathbb{R},<,+,\mathcal{B})$ is a reduct of \mathcal{S} . \square

If \mathcal{M} is a structure with domain M and \mathcal{N} is an expansion of \mathcal{M} , then we say that \mathcal{N} is \mathcal{M} -minimal if every \mathcal{N} -definable subset of M is \mathcal{M} -definable. (\mathcal{R} is o-minimal if and only if \mathcal{R} is (\mathbb{R} , <)-minimal.) Corollary 2.5 follows from Proposition 2.4.

Corollary 2.5. Suppose \mathbb{R} is o-minimal and (\mathbb{R}, \mathbb{Z}) is locally o-minimal. Then (\mathbb{R}, \mathbb{Z}) is $(\mathbb{R}, <, +, \mathbb{Z})$ -minimal.

Theorem 2.6 finishes the proof of Theorem 1.1.

Theorem 2.6. Suppose \Re is o-minimal, $\alpha > 0$, and $(\Re, \alpha \mathbb{Z})$ is locally o-minimal. Then there are no intermediate structures between \Re and $(\Re, \alpha \mathbb{Z})$.

Proof. Rescaling reduces to the case when $\alpha = 1$. Suppose S is a reduct of $(\mathcal{R}, \mathbb{Z})$ and \mathcal{R} is a reduct of S.

Suppose S is o-minimal. As (S, \mathbb{Z}) is locally o-minimal Proposition 2.4 shows that S has no poles and rational global scalars. Applying Fact 2.1 let \mathcal{B} be a collection of bounded S-definable sets such that S and $(\mathbb{R}, <, +, \mathcal{B})$ are interdefinable. It follows from Proposition 2.4 that every bounded $(\mathcal{R}, \mathbb{Z})$ -definable set is \mathcal{R} -definable. So S is interdefinable with \mathcal{R} .

Suppose S is not o-minimal. Then S defines a subset X of \mathbb{R} which is not \mathcal{R} -definable. By Corollary 2.5 X is definable in $(\mathbb{R}, <, +, \mathbb{Z})$. So $(\mathbb{R}, <, +, X)$ defines \mathbb{Z} by Bès and Choffrut [3]. Thus S and $(\mathcal{R}, \mathbb{Z})$ are interdefinable.

We prove two corollaries.

Corollary 2.7. Suppose \mathbb{R} is o-minimal and has no poles and rational global scalars. Suppose $\alpha, \beta > 0$ and $\alpha/\beta \notin \mathbb{Q}$. Suppose $X \subseteq \mathbb{R}^n$ is definable in both $(\mathbb{R}, \alpha\mathbb{Z})$ and $(\mathbb{R}, \beta\mathbb{Z})$. Then X is definable in \mathbb{R} .

Note that if $\alpha/\beta \in \mathbb{Q}$ then $(\mathcal{R}, \alpha \mathbb{Z})$ and $(\mathcal{R}, \beta \mathbb{Z})$ are interdefinable.

Proof. Suppose X is not \mathbb{R} -definable. By Theorem 2.6 (\mathbb{R}, X) defines $\alpha \mathbb{Z}, \beta \mathbb{Z}$. As $\alpha \mathbb{Z} + \beta \mathbb{Z}$ is dense and co-dense in \mathbb{R} , (\mathbb{R}, X) is not locally o-minimal, contradiction.

A function $f: \mathbb{R} \to \mathbb{R}^n$ is periodic with period $\alpha \neq 0$ if $f(t+\alpha) = f(t)$ for all t.

Corollary 2.8. Suppose $f: \mathbb{R} \to \mathbb{R}^n$ is analytic and periodic with period $\alpha > 0$. Then there is no structure intermediate between $(\mathbb{R}, <, +, f|_{[0,\alpha]})$ and $(\mathbb{R}, <, +, f)$.

So there is no structure intermediate between $(\mathbb{R}, <, +, \sin|_{[0,2\pi]})$ and $(\mathbb{R}, <, +, \sin)$.

Proof. If f is constant then $(\mathbb{R}, <, +, f)$, $(\mathbb{R}, <, +, f|_{[0,\alpha]})$, and $(\mathbb{R}, <, +)$ are all interdefinable. Suppose f is non-constant.

We show that $(\mathbb{R}, <, +, f|_{[0,\alpha]}, \alpha \mathbb{Z})$ and $(\mathbb{R}, <, +, f)$ are interdefinable. We have

$$f(t) = f(t - \max\{a \in \alpha \mathbb{Z} : a < t\})$$
 for all t .

It follows that f is $(\mathbb{R}, <, f|_{[0,\alpha]}, \alpha\mathbb{Z})$ -definable. To show that $(\mathbb{R}, <, +, f|_{[0,\alpha]}, \alpha\mathbb{Z})$ is a reduct of $(\mathbb{R}, <, +, f)$ it suffices to show that $\alpha\mathbb{Z}$ is $(\mathbb{R}, <, +, f)$ -definable. Let P be the set of $r \neq 0$ such that f is r-periodic. Then P is an $(\mathbb{R}, <, +, f)$ -definable subgroup of $(\mathbb{R}, +)$. As f is continuous and non-constant P is not dense in \mathbb{R} , so $P = \lambda\mathbb{Z}$ for some $\lambda > 0$. Then $\alpha \in \lambda\mathbb{Z}$, it follows that $\alpha\mathbb{Z}$ is $(\mathbb{R}, <, +, \lambda\mathbb{Z})$ -definable.

It is well-known that Gabrielov's complement theorem implies $(\mathbb{R}, <, +, g|_I)$ is ominimal for any analytic $g: \mathbb{R} \to \mathbb{R}^n$ and bounded interval I. So $(\mathbb{R}, <, +, f|_{[0,\alpha]})$ is o-minimal. Applying Proposition 2.4 and rescaling we see that $(\mathbb{R}, <, +, f|_{[0,\alpha]}, \alpha\mathbb{Z})$ is locally o-minimal. Now apply Theorem 2.6.

3. Expansions of o-minimal structures by \mathbb{Z}

We survey structures of the form $(\mathcal{R}, \mathbb{Z})$ where \mathcal{R} is o-minimal. This will require fundamental classification results on o-minimal structures and key results from the theory of general expansions of $(\mathbb{R}, <, +)$. This class of structures contains, sometimes in a somewhat disguised form, many interesting structures.

We first describe two opposing examples. It is well-known that $(\mathbb{R},<,+,\cdot,\mathbb{Z})$ defines all Borel subsets of all \mathbb{R}^n . So $(\mathbb{R},<,+,\cdot,\mathbb{Z})$ is totally wild from the model-theoretic viewpoint. Fix $\lambda>0$ and define $\lambda^{\mathbb{Z}}:=\{\lambda^m:m\in\mathbb{Z}\}$. It follows from work of van den Dries [23] that $(\mathbb{R},<,+,\cdot,\lambda^{\mathbb{Z}})$ admits quantifier elimination in a natural expanded language. It follows that this structure is NIP and definable sets are geometrically tame. Let \mathcal{I} be the structure induced on $\mathbb{R}_{>0}$ by $(\mathbb{R},<,+,\cdot)$ and \mathcal{R} be the pushforward of \mathcal{I} by $\log_{\lambda}:\mathbb{R}_{>0}\to\mathbb{R}$. Then \mathcal{R} is an o-minimal expansion of $(\mathbb{R},<,+)$ and (\mathcal{R},\mathbb{Z}) is isomorphic to the structure induced on $\mathbb{R}_{>0}$ by $(\mathbb{R},<,+,\cdot,\lambda^{\mathbb{Z}})$. So in this case (\mathcal{R},\mathbb{Z}) is tame and not locally o-minimal.

We gather some background results. We say that S is field-type if there is a bounded interval I and S-definable $\oplus, \otimes : I^2 \to I$ such that $(I, <, \oplus, \otimes)$ and $(\mathbb{R}, <, +, \cdot)$ are isomorphic. Fact 3.1 is a part of the Peterzil-Starchenko trichotomy theorem [20].

Fact 3.1. Suppose S is o-minimal. Then exactly one of the following holds.

- (1) S is field-type.
- (2) S is a reduct of $(\mathbb{R}, <, +, (t \mapsto \lambda t)_{\lambda \in \mathbb{R}})$.

We will apply the classification of structures intermediate between $(\mathbb{R}, <, +)$ and $(\mathbb{R}, <, +, (t \mapsto \lambda t)_{\lambda \in \mathbb{R}})$. This is probably not original. We associate two subfields of $(\mathbb{R}, +, \cdot)$ to S. The *field of global scalars* is the set of λ such that the function $\mathbb{R} \to \mathbb{R}$, $t \mapsto \lambda t$ is S-definable. The *field of local scalars* is the set of λ for which there is s > 0 such that the function $[0, s) \to \mathbb{R}$, $t \mapsto \lambda t$ is S-definable. It is easy to see that both of these sets are subfields. It is also easy to see that the field of local scalars is the set of λ such that $[0, s) \to \mathbb{R}$, $t \mapsto \lambda t$ is S-definable for any s > 0. Given subfields $K \subseteq L$ of $(\mathbb{R}, +, \cdot)$ we let $\mathcal{V}_{K,L}$ be the expansion of $(\mathbb{R}, <, +)$ by the function $\mathbb{R} \to \mathbb{R}$, $t \mapsto \lambda t$ for all $\lambda \in L$ and by the function $[0, 1) \to \mathbb{R}$, $t \mapsto \lambda t$ for all $\lambda \in K$. We let $\mathcal{V}_{K} := (\mathbb{R}, <, +, (t \mapsto \lambda t)_{\lambda \in K})$ so \mathcal{V}_{K} and $\mathcal{V}_{K,K}$ are interdefinable.

Proposition 3.2. Suppose S is a reduct of $(\mathbb{R}, <, +, (t \mapsto \lambda t)_{\lambda \in \mathbb{R}})$. Let K be the field of local scalars of S and L be the field of global scalars of S. Then S is interdefinable with $\mathcal{V}_{K,L}$.

Our proof is rather sketchy.

Proof. Observe that $\mathcal{V}_{K,L}$ is a reduct of \mathcal{S} . We show that \mathcal{S} is a reduct of $\mathcal{V}_{K,L}$. Applying Fact 2.1 we obtain a collection \mathcal{B} of bounded $(\mathbb{R}, <, +, (t \mapsto \lambda t)_{\lambda \in \mathbb{R}})$ -definable sets such that $(\mathbb{R}, <, +, \mathcal{B}, (t \mapsto \lambda t)_{\lambda \in L})$ is interdefinable with \mathcal{S} . Rescaling and translating we may suppose that every $X \in \mathcal{B}$ is a subset of some $[0, 1)^n$. A straightforward application of the semilinear cell decomposition [25, Corollary 7.6] shows that every $X \in \mathcal{B}$ is definable in the expansion of $(\mathbb{R}, <, +)$ by all functions $[0, 1) \to \mathbb{R}$, $t \mapsto \lambda t$ for $\lambda \in K$. So \mathcal{S} is a reduct of $\mathcal{V}_{K,L}$.

We also need the following theorem of Hieronymi [9]. Fact 3.3 is not exactly stated in [9] but follows immediately from the main theorem of that paper.

Fact 3.3. Suppose S is field-type and defines a discrete subset D of \mathbb{R}^n and a function $f: D \to \mathbb{R}$ such that f(D) is somewhere dense. Then S defines all bounded Borel subsets of all \mathbb{R}^n .

Fact 3.4 is due to Hieronymi and Balderrama [2]. It is a corollary to the fundamental Hieronymi-Tychonievich theorem [13].

Fact 3.4. Suppose D, E are S-definable discrete subsets of \mathbb{R}^n , $f: D \to \mathbb{R}$ and $g: E \to \mathbb{R}$ are S-definable, f(D) and g(E) are both somewhere dense, and

$$[f(D) - f(D)] \cap [g(E) - g(E)] = \{0\}.$$

Then S defines all bounded Borel subsets of all \mathbb{R}^n .

3.1. **Pole.** We first consider the case when \mathcal{R} has a pole. Applying Fact 2.1 fix \mathcal{R} -definable $\oplus, \otimes : \mathbb{R}^2 \to \mathbb{R}$ such that $(\mathbb{R}, <, \oplus, \otimes)$ is isomorphic to $(\mathbb{R}, <, +, \cdot)$. Note that there is a unique isomorphism $\iota : (\mathbb{R}, <, \oplus, \otimes) \to (\mathbb{R}, <, +, \cdot)$. Let \mathcal{R}' be the pushforward of \mathcal{R} by ι , $\boxplus : \mathbb{R}^2 \to \mathbb{R}$ be the pushforward of ι by ι , and ι is an o-minimal expansion of $(\mathbb{R}, <, +, \cdot)$ and ι gives an isomorphism $(\mathcal{R}, \mathbb{Z}) \to (\mathcal{R}', Z)$. We now recall the *o-minimal two group question*, see [18].

Question 3.5. Suppose S is an o-minimal expansion of $(\mathbb{R}, <, +, \cdot)$ and $\odot : \mathbb{R}^2 \to \mathbb{R}$ is S-definable such that $(\mathbb{R}, <, \odot)$ is an ordered abelian group. Must there be either an S-definable isomorphism $(\mathbb{R}, <, \odot) \to (\mathbb{R}, <, +)$ or $(\mathbb{R}, <, \odot) \to (\mathbb{R}_{>0}, <, \cdot)$?

The o-minimal two group question is open. However, it is shown in [18] that if the Pfaffian closure of S is exponentially bounded then the two group question has a positive answer for S. Recall that an expansion of $(\mathbb{R}, <, +)$ is exponentially bounded if every definable function $\mathbb{R} \to \mathbb{R}$ is eventually bounded above by a compositional iterate of the exponential, the Pfaffian closure of an o-minimal expansion is o-minimal, and every known o-minimal expansion of $(\mathbb{R}, <, +, \cdot)$ is exponentially bounded. So we assume that the two group question has a positive answer over \mathcal{R}' .

Suppose $\tau: (\mathbb{R}, <, \boxplus) \to (\mathbb{R}, <, +)$ is an \mathcal{R}' -definable isomorphism. Then $\tau \circ \iota$ is an isomorphism $(\mathbb{R}, <, +) \to (\mathbb{R}, <, +)$ so there is $\lambda > 0$ such that $(\tau \circ \iota)(t) = \lambda t$ for all t. So $\tau(Z) = \lambda \mathbb{Z}$. It follows that (\mathcal{R}', Z) defines \mathbb{Z} and is therefore totally wild, hence $(\mathcal{R}, \mathbb{Z})$ is totally wild.

Suppose $\tau: (\mathbb{R}, <, \boxplus) \to (\mathbb{R}_{>0}, <, \cdot)$ is an \mathcal{R}' -definable isomorphism. Then $\tau \circ \iota$ is an isomorphism $(\mathbb{R}, <, +) \to (\mathbb{R}_{>0}, <, \cdot)$ so there is $\lambda > 0$ such that $(\tau \circ \iota)(t) = \lambda^t$ for all t. Then $\tau(Z) = \lambda^{\mathbb{Z}}$ and $(\mathcal{R}'Z)$ is interdefinable with $(\mathcal{R}', \lambda^{\mathbb{Z}})$. We say that \mathcal{R}' has rational exponents if the function $\mathbb{R}_{>0} \to \mathbb{R}_{>0}$ given by $t \mapsto t^r$ is only \mathcal{R}' -definable when $r \in \mathbb{Q}$. If $r \in \mathbb{R} \setminus \mathbb{Q}$ then $\{ab^r: a, b \in \lambda^{\mathbb{Z}}\}$ is dense in $\mathbb{R}_{>0}$. So it follows from Fact 3.3 that if \mathcal{R}' has irrational exponents then $(\mathcal{R}', \lambda^{\mathbb{Z}})$ defines all Borel subsets of all \mathbb{R}^n , so in this case $(\mathcal{R}, \mathbb{Z})$ is totally wild. Suppose \mathcal{R}' has rational exponents. Generalizing [23] Miller and Speissegger have shown that in this case $(\mathcal{R}', \lambda^{\mathbb{Z}})$ admits quantifier elimination in a natural expanded language [17, Section 8.6]. It follows that $(\mathcal{R}', \lambda^{\mathbb{Z}})$ is NIP (see [8]) and that $(\mathcal{R}', \lambda^{\mathbb{Z}})$ -definable sets are geometrically tame (see [17, 22]). So in this case $(\mathcal{R}, \mathbb{Z})$ is tame.

3.2. No pole. We now assume that \mathcal{R} has no poles. If \mathcal{R} has rational global scalars then $(\mathcal{R}, \mathbb{Z})$ is locally o-minimal, we have a good description of $(\mathcal{R}, \mathbb{Z})$ -definable sets by Proposition 2.4, and $(\mathcal{R}, \mathbb{Z})$ is strongly dependent (see [26]).

Suppose \mathcal{R} has irrational global scalars and is field type. Fix an irrational element λ of the field of global scalars of \mathcal{R} . Then $\mathbb{Z} + \lambda \mathbb{Z}$ is dense in \mathbb{R} . Fact 3.3 shows that $(\mathcal{R}, \mathbb{Z})$ defines all bounded Borel sets and is therefore totally wild.

Suppose \mathcal{R} has irrational scalars and is not field type. Applying Fact 3.1 and Proposition 3.2 we may suppose that $\mathcal{R} = \mathcal{V}_{K,L}$ for subfields $K, L \neq \mathbb{Q}$. If L is not quadratic then it follows from a theorem of Hieronymi and Tychonievich [13, Theorem B] that $(\mathcal{V}_L, \mathbb{Z})$ defines all Borel sets.

Suppose $L=\mathbb{Q}(\alpha)$ for a quadratic irrational α . Suppose K is not $\mathbb{Q}(\alpha)$. (This case is a slight extension of [13].) Fix positive $\beta \in K \setminus \mathbb{Q}(\alpha)$. Let $f: \mathbb{Z}^2 \to \mathbb{R}$ be given by $f(k,k')=k+\alpha k'$. Then f is $(\mathcal{V}_{K,L},\mathbb{Z})$ -definable and $f(\mathbb{Z}^2)$ is dense in \mathbb{R} . Let E be the set of $(k,k')\in\mathbb{Z}^2$ such that $0\leq f(k,k')<1$. Let $g:E\to\mathbb{R}$ be given by $g(k,k')=\beta k+\beta \alpha k'$. Then E and g are $(\mathcal{V}_{K,L},\mathbb{Z})$ -definable and g(E) is dense in $[0,\beta)$. Observe that $f(\mathbb{Z}^2)-f(\mathbb{Z}^2)=f(\mathbb{Z}^2)=\mathbb{Z}+\alpha\mathbb{Z}$ and $g(E)-g(E)\subseteq\beta\mathbb{Z}+\beta\alpha\mathbb{Z}$. As β is not in $\mathbb{Q}(\alpha)$ elementary algebra yields $(\mathbb{Z}+\alpha\mathbb{Z})\cap(\beta\mathbb{Z}+\beta\alpha\mathbb{Z})=\{0\}$. An application of Fact 3.4 shows that $(\mathcal{V}_{K,L},\mathbb{Z})$ defines all bounded Borel sets.

One case remains, when K = L is quadratic. In this case something remarkable happens. Consider \mathcal{V}_K for a quadratic subfield K of $(\mathbb{R}, <, +, \cdot)$. Let \mathcal{B} be the standard model $(\mathcal{P}(\mathbb{N}), \mathbb{N}, \in, +1)$ of the monadic second order theory of one successor, i.e. we have a sort $\mathcal{P}(\mathbb{N})$ for the power set of \mathbb{N} , a sort for \mathbb{N} , the membership relationship \in between these two sorts, and the successor function +1 on \mathbb{N} . It is a theorem of Büchi [4] that \mathcal{B} is decidable. By [10, Theorem D] or [14, Theorem C] $(\mathcal{V}_K, \mathbb{Z})$ defines an isomorphic copy of \mathcal{B} . Hieronymi has shown that \mathcal{B} defines an isomorphic copy of $(\mathcal{V}_K, \mathbb{Z})$, see [11, Theorem D] and [10, Theorem C].

If an expansion of $(\mathbb{R}, <, +)$ defines all bounded Borel sets then it defines an isomorphic copy of $(\mathbb{R}, <, +, \cdot, \mathbb{Z})$. It is also easy to see that $(\mathcal{R}, \mathbb{Z})$ is a reduct of $(\mathbb{R}, <, +, \cdot, \mathbb{Z})$ for any o-minimal \mathcal{R} . So we see that if \mathcal{R} is a *known* o-minimal expansion of $(\mathbb{R}, <, +)$ then $(\mathcal{R}, \mathbb{Z})$ is, up to mutual interpretation, one of the following:

- (1) $(\mathcal{R}, \mathbb{Z})$ where \mathcal{R} has no poles and rational scalars,
- (2) $(S, \lambda^{\mathbb{Z}})$, S an o-minimal expansion of $(\mathbb{R}, <, +, \cdot)$ with rational exponents,
- $(3) (\mathcal{P}(\mathbb{N}), \mathbb{N}, \in, +1),$
- $(4) (\mathbb{R}, <, +, \cdot, \mathbb{Z}).$

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Department of Mathematics, Statistics, and Computer Science, Department of Mathematics, University of California, Irvine, 340 Rowland Hall (Bldg.# 400), Irvine, CA 92697-3875

E-mail address: ewalsber@uci.edu

URL: http://www.math.illinois.edu/~erikw