Higher-Order Probabilistic Programming

A Tutorial at POPL 2019

Part I

Ugo Dal Lago

(Based on joint work with *Flavien Breuvart*, *Raphaëlle Crubillé*, *Charles Grellois*, *Davide Sangiorgi*,. . .)

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Probabilistic Models

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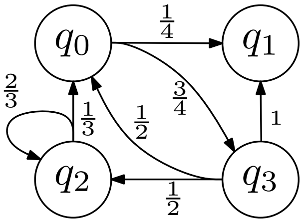

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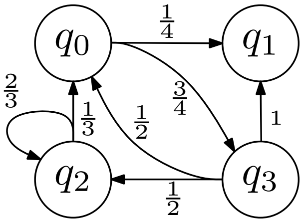

- **Abstractions:**
  - (Labelled) Markov Chains.

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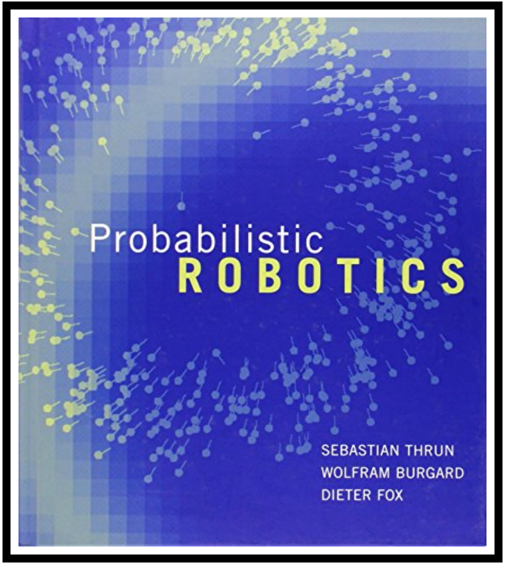
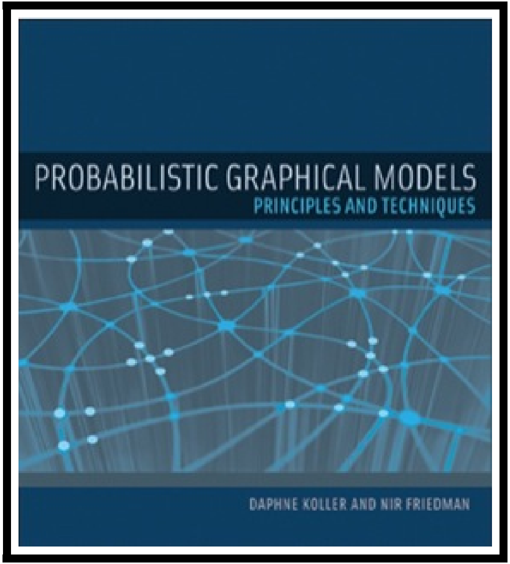


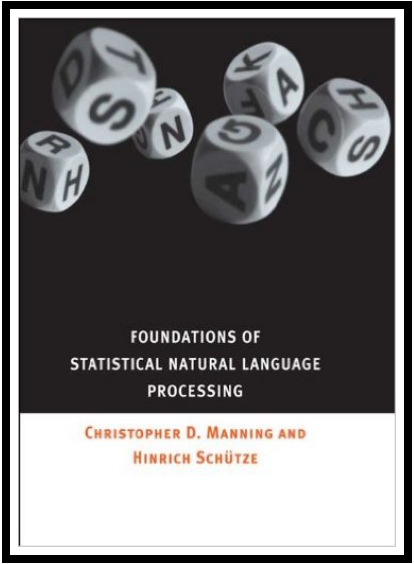
ROBOTICS

Probabilistic Models



ARTIFICIAL INTELLIGENCE

Probabilistic Models



NATURAL LANGUAGE PROCESSING

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```
Input: n > 3, an odd integer to be tested for primality;
Input: k, a parameter that determines the accuracy of the test
Output: composite if n is composite, otherwise probably prime
write n − 1 as 2^s·d with d odd by factoring powers of 2 from n − 1
WitnessLoop: repeat k times:
   pick a random integer a in the range [2, n − 2]
   x ← a^d mod n
   if x = 1 or x = n − 1 then do next WitnessLoop
   repeat s − 1 times:
      x ← x^2 mod n
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- **Abstractions:**
  - Randomized algorithms;
  - Probabilistic Turing machines.
  - Labelled Markov chains.

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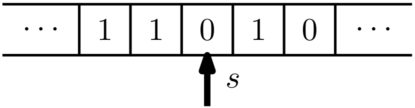

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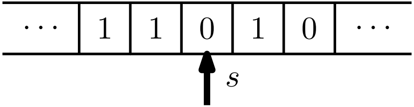

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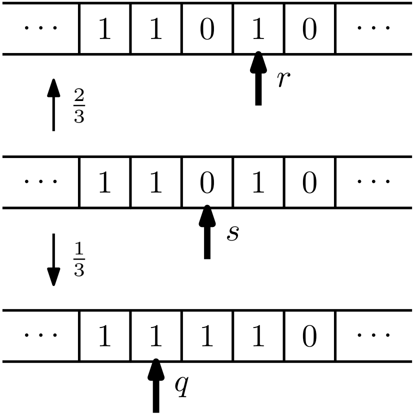

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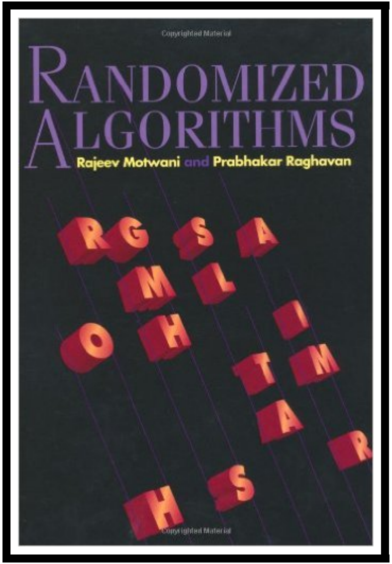

ALGORITHMICS

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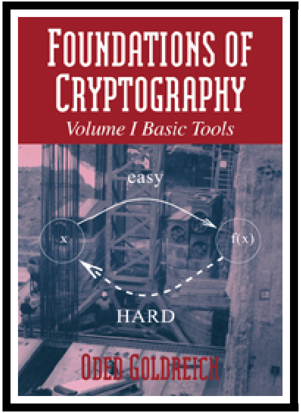

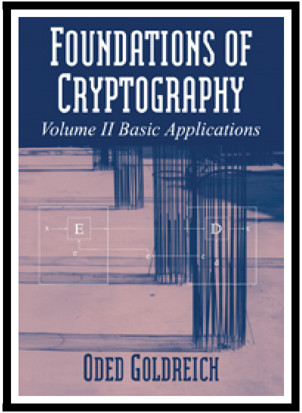

CRYPTOGRAPHY

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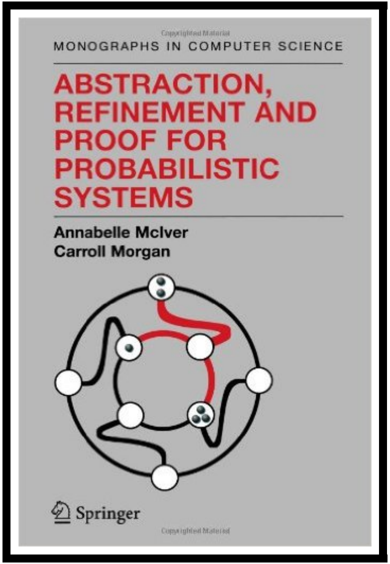

PROGRAM VERIFICATION

What Algorithms Compute

- **Deterministic Computation**
  - For every input x, there is *at most* one output y any algorithm $\mathcal{A}$ produces when fed with x.
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  - For every input x, any algorithm $\mathcal{A}$ outputs y with a probability $0 \leq p \leq 1$.
  - As a consequence:

$$\mathcal{A} \qquad \rightsquigarrow \qquad [\![\mathcal{A}]\!] : \mathbb{N} \rightarrow \mathscr{D}(\mathbb{N}).$$

  - The distribution $[\![\mathcal{A}]\!](n)$ sums to anything between 0 and 1, thus accounting for the probability of divergence.

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- **Example**:

```
foldr                :: (a -> b -> b) -> b -> [a] -> b
foldr f acc []       = acc
foldr f acc (x:xs) = f x (foldr f acc xs)
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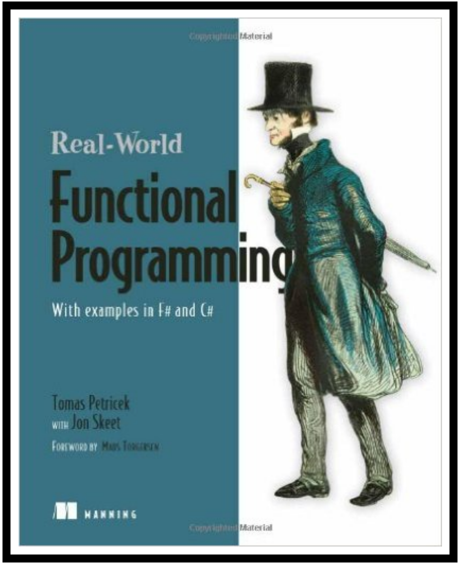
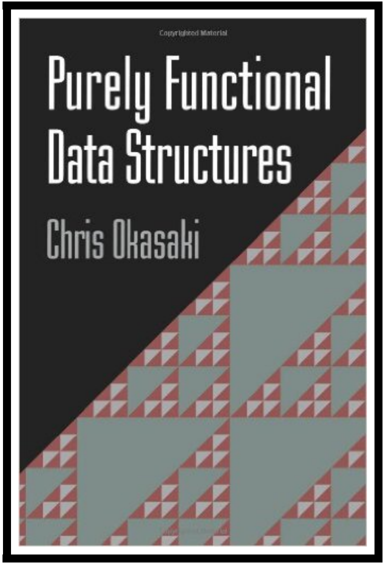
- **Models**:
  - λ-calculus

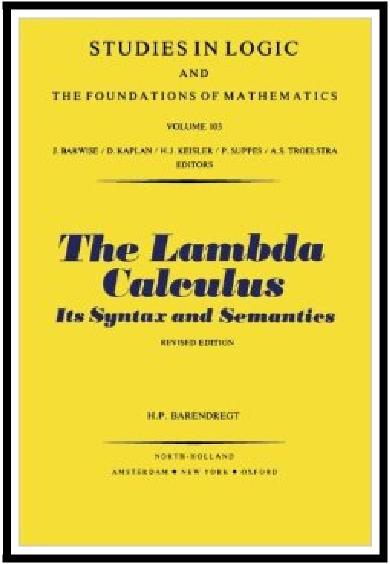
Higher-Order Computation



FUNCTIONAL
PROGRAMMING



FUNCTIONAL
DATA STRUCTURES



λ-CALCULUS

Does it Make Sense?

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Interesting Research Problems?

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1. **Motivating Examples.**
2. **A λ-Calculus Foundation.**
3. **Operational and Denotational Semantics.**
4. **Termination and Resource Analysis.**

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A webpage: `http://www.cs.unibo.it/~dallago/HOPP/`

MergeSort (1)

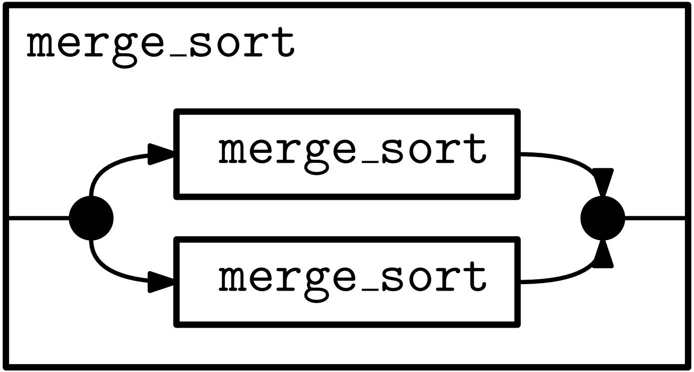
```
let rec merge = function
      | list, []
      | [], list -> list
      | h1::t1, h2::t2 ->
          if h1 <= h2 then
            h1 :: merge (t1, h2::t2)
          else
            h2 :: merge (h1::t1, t2);;

let rec halve = function
      | []
      | [_] as t1 -> t1, []
      | h::t ->
          let t1, t2 = halve t in
            h::t2, t1;;
```

MergeSort (2)

```
let rec merge_sort = function
      | []
      | [_] as list -> list
      | list ->
          let l1, l2 = halve list in
            merge (merge_sort l1, merge_sort l2);;
```

The Structure of MergeSort



MergeSort, HO

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let rec merge = function
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    | ([], list), _ -> list
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let rec halve = function
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          (h::t2, t1),el;;

let rec dac divide conquer = function
    | []
    | [_] as list -> list
    | list ->
        let (l1, l2),el = divide list in
          conquer ((dac divide conquer l1, dac divide conquer l2),el);;

let rec merge_sort = dac halve merge;;
```

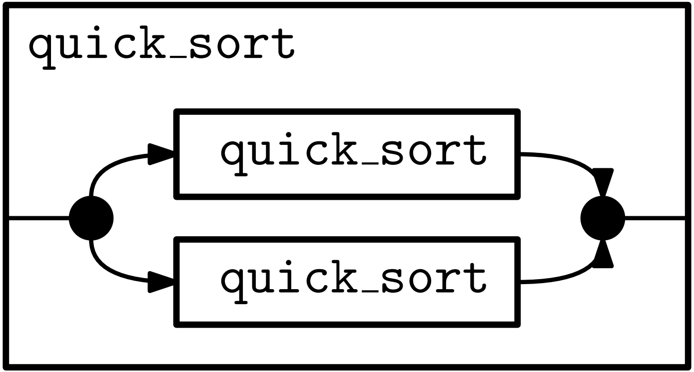
QuickSort

```
let app x y z = x @ (z::y);;

let partition = function
    | pivot :: rest -> (List.partition (( > ) pivot) (rest)),pivot;;

let rec quick_sort = function
    | []
    | [_] as list -> list
    | list ->
      let (l1, l2), el = partition list in
        app (quick_sort l1) (quick_sort l2) el;;
```

The Structure of MergeSort



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let quick_sort = dac partition app;;
```

Randomized QuickSort (1)

```
let app = function
    | (x,y),z -> x @ (z::y);;

let rec extract = function
  | [],_ -> ([],0)
  | hd::tl,n ->
     if n==0 then
       (tl,hd)
     else
       let (l,el) = extract(tl,n-1) in
          (hd::l,el);;

let partition list =
  let (rest,pivot) = extract (list,(Random.int (List.length list))) in
    (List.partition (( > ) pivot) (rest)),pivot;;
```

Randomized QuickSort (2)

```
let rec dac divide conquer = function
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let rand_quick_sort = dac partition app;;
```
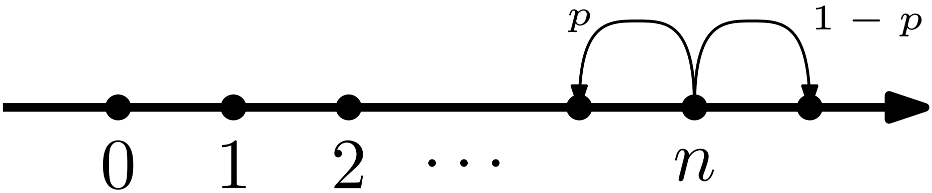

Random Walk



Two Kinds of Random Walks

```
let rec iter f g n = if n==0 then g else let m=pred(n) in f m (iter f g m);;

let mult m n = succ(m)*n;;

let fact = iter mult 1;;

let rec param_iter f g step n =
    if n==0 then g else let m=step(n) in f m (param_iter f g step m);;

let succ_2 m n = n+1;;

let updown_fair x = x+(2*Random.int(2)-1);;

let fair_random_walk = param_iter succ_2 0 updown_fair;;

let updown_biased x = if Random.int(3)==0 then x+1 else x-1;;

let biased_random_walk = param_iter succ_2 0 updown_biased;;
```

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- **Example**: if $n = 5$, you could get the following coupons:

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- Are you guaranteed to win the prize with probability 1? After how many days, on the average?

The Coupon Collector

```
let rec base_param_iter f g step base e =
  if base(e) then g else let d=step(e) in f d (base_param_iter f g step base d);;

let second_zero = function
  | (_,0) -> true
  | _ -> false;;

let succ_2 m n = n+1;;

let step_2 = function
  | (n,m) -> if Random.int(n)<=m then (n,m-1) else (n,m);;

let coupon_collector x = base_param_iter succ_2 0 step_2 second_zero (x,x);;
```

Beyond Randomized Algorithms: The Grass Problem

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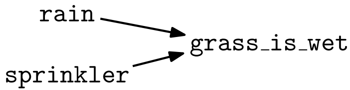
- You further know that:
  - *If it rains*, there is 90% probability that the grass is wet the following morning.
  - *If the sprinkler is activated*, there is 80% probability that the grass is wet the following morning.
  - *In any other case*, there is anyway a 10% probability that the grass is wet.
- In other words, you are in a situation like the following:



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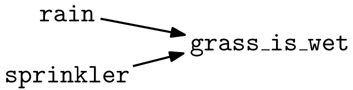

- Now: how likely it is that it rained?

The Grass Model

```
let grass_model () =    (* unit -> bool *)
  let rain = flip 0.3 and sprinkler = flip 0.5 in
  let grass_is_wet =
    (flip 0.9 && rain) || (flip 0.8 && sprinkler) || flip 0.1 in
  if not grass_is_wet then fail ();
  rain
```

Thank You!

Questions?