Chapter 12

THE PROBLEM OF DETERMINACY OF INFINITE GAMES FROM AN INTUITIONISTIC POINT OF VIEW

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Abstract Taking Brouwer's intuitionistic standpoint, we examine finite and infinite games of perfect information for players *I* and *II*. If one understands the disjunction occurring in the classical notion of determinacy constructively, even finite games are not always determinate. We therefore suggest an intuitionistically different notion of determinacy and prove that every subset of Cantor space is determinate in the proposed sense. Our notion is biased and considers games from the viewpoint of player *I*. In Cantor space, both player *I* and player *II* have two alternative possibilities for each move. It turns out that every subset of a space, where player *II* has, for each one of his moves, no more than a finite number of alternative possibilities while player *I* perhaps has infinitely many choices, is determinate in the proposed sense from the viewpoint of player *I*.

'We must have a bit of a fight, but I don't care about going on long,'
said Tweedledum. 'What's the time now?'
Tweedledee looked at his watch, and said, 'Half-past four.'
'Let's fight till six, and then have dinner,' said Tweedledum.

—Lewis Carroll, [1865, page 190]

12.1 Intuitionistic determinacy: the problem, and the case of two-move-games

12.1.1 $\mathbb{N}$ is the set of natural numbers, and *Baire space* $\mathcal{N}$ is the set of all infinite sequences of natural numbers. We use $m, n, p, q, \ldots$ as variables over the set $\mathbb{N}$ and $\alpha, \beta, \ldots$ as variables over the set $\mathcal{N}$.

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Let A be a subset of $\mathcal{N}$. We describe *the game for* A, sometimes called $\mathcal{G}(A)$. There are two players, I and II, who, each time they play the game, together build an infinite sequence α in $\mathcal{N}$, as follows:

$$\begin{array}{ccccccccc}
\text{Player } I \text{ chooses } \alpha(0) & & & & \text{Player } I \text{ chooses } \alpha(2) & & & & \dots \\
& \searrow & & \nearrow & & \searrow & & \nearrow & \\
& & \text{Player } II \text{ chooses } \alpha(1) & & & & \text{Player } II \text{ chooses } \alpha(3) & &
\end{array}$$

The sequence α is called a *play* in the game for A. Player I is the winner if and only if α belongs to A. The set A is sometimes called the *payoff set* of the game.

Following the classical definition, we say that the set A is *determinate* if and only if *either* player I has a method to secure that he wins every play in the game for A, *or* player II has a method to prevent that player I wins any play in the game for A.

We take the intuitionistic point of view advocated by L. E. J. Brouwer. He insisted that every mathematical statement should be considered as a report on what we have been able to prove and that connectives and quantifiers and the corresponding set-theoretic operations should be interpreted constructively. In particular, a disjunctive statement $P \vee Q$ is considered proven if and only if we either have a proof of P or a proof of Q. We not only follow the rules of intuitionistic logic but also make use of some of the new axioms Brouwer proposed as a result of his reflection on the problem how to handle the concept of the continuum. Some of these axioms, the so-called *continuity principles*, are classically unacceptable but our main result, Theorem 3.5, does not depend on any axiom that does not stand a classical reading.

12.1.2 We not only want to study games in Baire space $\mathcal{N}$ but also games that are played in certain subspaces of Baire space $\mathcal{N}$, traditionally called *spreads* in intuitionistic mathematics. To this end we introduce some notations and some terminology.

$\mathbb{N}^*$ is the set of finite sequences of natural numbers. We suppose that a bijective mapping $(a_0, a_1, \dots, a_{n-1}) \mapsto \langle a_0, a_1, \dots, a_{n-1}\rangle$ from $\mathbb{N}^*$ to $\mathbb{N}$ is given, a function *coding* the finite sequences of natural numbers by means of natural numbers. There is a function, *length*, from $\mathbb{N}$ to $\mathbb{N}$ such that, for every natural number a, $m := \mathit{length}(a)$ is the length of the finite sequence coded by a.

Let a be a number of length m. We consider a as a function from the set $\{0, 1, \dots, m-1\}$ to $\mathbb{N}$, and, for each n, if $n < m$, we define $a(n)$ to be the value of this function at n.

$*$ is the binary function on $\mathbb{N}$ which, via the coding, corresponds to the operation of concatenating finite sequences. We assume that for each a, n, $a \leq a * \langle n\rangle$.

For each infinite sequence of natural numbers α, and each natural number n, we define $\overline{\alpha}(n)$ to be (the code number of) the finite sequence $\langle \alpha(0), \dots, \alpha(n-1)\rangle$. If confusion seems unlikely, we write $\overline{\alpha}n$ rather than $\overline{\alpha}(n)$.

For each infinite sequence of natural numbers α, for each natural number s, we define: *α passes through s* if and only if there exists n such that $\overline{\alpha}n = s$.

Let σ belong to $\mathcal{N}$. σ is called a *spread-law* if and only if $\sigma(\langle\ \rangle) = 0$ and, for each a, $\sigma(a) = 0$ if and only if, for some n, $\sigma(a * \langle n\rangle) = 0$. If $\sigma(a) = 0$, we will say that a is *admitted by σ*.

Let σ be a spread-law and let α belong to $\mathcal{N}$. We say: *σ admits α*, if and only if, for each n, $\sigma(\overline{\alpha}n) = 0$. The set of all infinite sequences of natural numbers α admitted by the spread-law σ is called a *spread* and this set is also named σ. The statements "*σ admits α*" and "*α belongs to σ*" are equivalent.

Observe that a subset X of $\mathcal{N}$ coincides with a spread if and only if (i) X is *(sequentially) closed*, that is, every α such that, for each n, some element of X passes through $\overline{\alpha}n$, belongs itself to X, and (ii) X is *located*, that is, there exists σ in $\mathcal{N}$ such that for every s, s contains an element of X if and only if $\sigma(s) = 0$.

Let σ be a spread and let A be a subset of σ. We describe *the game for A in σ*. There are again two players, I and II, who, each time they play the game, join up to build an infinite sequence α in $\mathcal{N}$, but they have to take care that the infinite sequence α will belong to the spread σ:

$$\begin{array}{lll} \text{Player } I \text{ chooses } \alpha(0) \text{ such that } \sigma(\overline{\alpha}(1)) = 0 & & \cdots \\ & \searrow & \nearrow \\ & \text{Player } II \text{ chooses } \alpha(1) \text{ such that } \sigma(\overline{\alpha}(2)) = 0 & \end{array}$$

Player I is the winner if and only if the infinite sequence α belongs to the set A.

Given some spread σ, we want to call a subset A of σ *determinate* if and only if *either* player I has a sure method to win the game for A in σ, *or* player II has a sure method to prevent player I from winning the game for A in σ.

12.1.3 In order to make the notion of determinacy more precise, we introduce the concept of a strategy.

Let σ be a spread. We let $\mathrm{Strat}_I(\sigma)$, the set of *strategies in σ for player I*, be the set of all functions γ in $\mathcal{N}$ such that for each a, if σ admits a and *length*(a) is even, then σ admits $a * \langle\gamma(a)\rangle$, and, if σ does not admit a or *length*(a) is odd, then $\gamma(a) = 0$. Observe that $\mathrm{Strat}_I(\sigma)$ itself is a spread.

Let α belong to σ and γ to $\mathrm{Strat}_I(\sigma)$. We define: *α I-obeys γ*, if and only if, for each n, $\alpha(2n) = \gamma(\overline{\alpha}(2n))$.

Similarly, we let $\mathrm{Strat}_{II}(\sigma)$, the set of *strategies for player II in σ*, be the set of all functions γ in $\mathcal{N}$ such that for each a, if σ admits a and *length*(a) is odd, then σ admits $a * \langle\gamma(a)\rangle$, and, if σ does not admit a or *length*(a) is even, then $\gamma(a) = 0$.

Let α belong to σ and γ to $\mathrm{Strat}_{II}(\sigma)$. We define: *$\alpha$ II-obeys γ*, if and only if, for each n, $\alpha(2n + 1) = \gamma(\overline{\alpha}(2n + 1))$.

Let A be a subset of σ. We define: A is *strongly determinate in* σ if and only if *either* there is a strategy γ for player I, such that every α in σ I-obeying γ belongs to A, *or* there is a strategy δ for player II in σ such that every α in σ II-obeying δ does not belong to A.

Let A be a subset of σ and let γ be strategy for player I in σ. We say: γ *wins* A *for player* I if and only if every α in σ I-obeying γ belongs to A.

12.1.4 Even games in which players I, II make only finitely many moves, and each move is a choice from finitely many alternatives, need not be strongly determinate.

Consider for instance the game with no moves at all that is won by player I if and only if Riemann's hypothesis holds. To say that this game is strongly determinate is equivalent to deciding Riemann's hypothesis.

Fortunately, the language of intuitionistic mathematics is more refined than the language of classical mathematics, and we may consider formulations of the notion of determinacy that, from a classical point view, would be equivalent to the first formulation, but, from an intuitionistic point of view, are weaker. Here is such a notion.

Let σ be a spread and A a subset of σ. We define: A is *determinate in* σ *from the viewpoint of player* I if and only if: *if* every strategy for player II in σ is II-obeyed by at least one element of A, *then* there is a strategy γ for player I in σ such that every α in σ I-obeying γ belongs to A.

We took the disjunctive formulation of strong determinacy, $P \vee Q$, changed it into $(\neg Q) \to P$, and then replaced the negative antecedent $\neg Q$ by a stronger, positive statement.

The definition is biased, as it considers the problem of the determinacy of A from the viewpoint of player I. It is easy to guess when we want to call a subset A of σ *determinate from the viewpoint of player* II. We will see, in Section 12.1.8, that there exist a spread σ and a subset A of σ such that A is determinate from the viewpoint of player I, while we are unable to prove that A is determinate from the viewpoint of player II.

12.1.5 We interrupt our discussion of the notion of determinacy and ask attention for one of the axioms of intuitionistic analysis.

Let σ be a spread and let ζ belong to $\mathcal{N}$. We define: ζ *codes a continuous function from* σ *to* $\mathcal{N}$ if and only if, for all n, for all α in σ, there exists m such that $\alpha(\langle n \rangle * \overline{\alpha}m) \neq 0$.

Suppose that σ is a spread, and that ζ codes a continuous function from σ to $\mathcal{N}$. For each α in σ we define $\zeta|\alpha$ to be the sequence β such that, for all n, p in $\mathbb{N}$, if p is the least m such that $\zeta(\langle n \rangle * \overline{\alpha}m) \neq 0$, then $\zeta(\langle n \rangle * \overline{\alpha}p) = \beta(n) + 1$.

The following axiom, occurring under its present name in Veldman (2006a), and called *Brouwer's principle for functions* in Kleene and Vesley (1965),

$\mathbf{GAC_{1,1}}$ in Gielen et al. (1981), and **C-C** in Troelstra and van Dalen (1988), is incompatible with a classical reading of the quantifiers. It seems to be the strongest possible formulation of a principle Brouwer is using in his intuitionistic papers.

Second Axiom of Continuous Choice: *Let σ be a spread and let R be a subset of $\sigma \times \mathcal{N}$. If, for all α in σ, there exists β such that $\alpha R\beta$, then there exists ζ coding a continuous function from σ to $\mathcal{N}$ such that, for all α in σ, $\alpha R(\zeta|\alpha)$.*

(We write "$\alpha R\beta$" while intending "(α, β) belongs to R".)

12.1.6 We now continue the discussion of the notion of determinacy.

Let σ be a spread, let A be a subset of σ and suppose that every strategy for player *II* in σ is *II*-obeyed by at least one element of A. Using the Second Axiom of Continuous Choice we find some ζ coding a continuous function from $\mathrm{Strat}_{II}(\sigma)$ to $\mathcal{N}$ such that, for every γ in $\mathrm{Strat}_{II}(\sigma)$, $\zeta|\gamma$ *II*-obeys γ and belongs to A.

An element ζ of $\mathcal{N}$ coding a continuous function from $\mathrm{Strat}_{II}(\sigma)$ to σ such that for every γ in $\mathrm{Strat}_{II}(\sigma)$, $\zeta|\gamma$ belongs to σ and *II*-obeys γ will be called an *anti-strategy* for player *I*. If, in addition, for every γ in $\mathrm{Strat}_{II}(\sigma)$, $\zeta|\gamma$ belongs to A we say that ζ *secures the set A for player I.*

Suppose that there exists an anti-strategy ζ for player *I* that secures the set A for player *I*. What use can player *I* make of it, when actually playing the game? Observe that, when playing the game, player *I* does not know which strategy his opponent is following. In order to win, he should be able, while producing, together with his opponent, a play α, to conjecture a strategy δ for player *II* such that $\zeta|\delta = \alpha$. At first sight, that does not seem to be a very easy task. Observe however that, if $\zeta|\delta = \alpha$, then, for each n there exists m such that for every strategy γ for player *II*, if $\overline{\delta}m = \overline{\gamma}m$, then $(\zeta|\delta)(2n) = (\zeta|\gamma)(2n)$. This means that, for a given n, player *I* may be sure that $\zeta|\delta$ passes through $\overline{\alpha}(2n)$, while he has only a finite piece of information on the strategy player *II* is following. Everyone who has a nephew and once played chess with him, should now imagine this nephew to be player *I*. Player *I*, each time he has to make a move, first asks a number of questions: "What will be your reply if I should make this move? And if I should continue so-and-so and make that move?" Somehow knowing how to make his opponent answer his questions, he collects information and ponders, consulting ζ, and then, at some point, he triumphantly takes his decision. Knowing also how to compel player *II* to act according to the given answers, he is sure that the resulting play α will belong to A.

If you are a grown-up, such questioning is no longer allowed, and you have lost the power of making your opponent do as you like. But might not player *I*, by studying his anti-strategy ζ, find a *strategy* γ, such that every α *I*-obeying γ

belongs to A, that is, might he not develop, by some hard thinking, a successful way of playing the game without asking unlawful questions and intimidating player *II*? That question is the main subject of this paper.

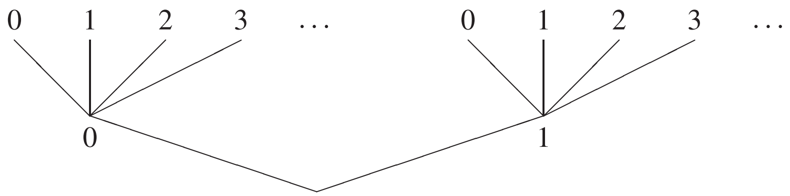
12.1.7 Let σ be a spread, let A be a subset of σ. We define: *A is predeterminate in σ from the viewpoint of player I* if and only if, *if* there is an anti-strategy for player *I* in σ that secures the set A for player *I*, *then* there is a strategy for player *I* in σ that wins the set A for player *I*.

Observe that every subset of σ that is determinate in σ from the viewpoint of player *I*, is also predeterminate in σ from the viewpoint of player *I*.

The Second Axiom of Continuous Choice implies the converse: every subset of σ that is predeterminate in σ from the viewpoint of player *I*, is also determinate in σ from the viewpoint of player *I*.

12.1.8 Disappointingly, if we allow player *II* to choose from countably many alternatives, there exist two-move games that are not determinate from the viewpoint of player *I* in the new weak sense.

We consider games of the following kind: Player *I* chooses either 0 or 1, player *II* chooses a natural number, and the game is over.



A strategy for player *I* in a game of this kind consists of a single number, viz. his first and only move which is either 0 or 1.

A strategy for player *II* is a pair (p, q) of natural numbers, p being the answer player *II* will give to a first move 0, and q being the answer player *II* will give to a first move 1.

A subset A of $\{0, 1\} \times \mathbb{N}$ is determinate from the viewpoint of player *I* in the sense of Section 12.1.4 if and only if: *if*, for all p, for all q, *either* $(0, p)$ belongs to A *or* $(1, q)$ belongs to A, *then: either*, for all p, $(0, p)$ belongs to A, *or*, for all q, $(1, q)$ belongs to A.

A subset A of $\{0, 1\} \times \mathbb{N}$ is predeterminate from the viewpoint of player *I* in the sense of Section 12.1.7 if and only if: *if* there exists α such that for all p, for all q, either $\alpha(\langle p, q\rangle) = 0$ and $(0, p)$ belongs to A, or $\alpha(\langle p, q\rangle) = 1$ and $(1, q)$ belongs to A, *then: either* for all p, $(0, p)$ belongs to A, *or*, for all q, $(1, q)$ belongs to A.

The following axiom, called *2.2 in Kleene and Vesley (1965) and $\mathbf{AC_{0,0}}$ in Gielen et al. (1981), and a weak consequence of the Second Axiom of Continuous Choice, implies that every subset of $\{0, 1\} \times \mathbb{N}$ that is determinate from the viewpoint of player I is also predeterminate from the viewpoint of player I.

First Axiom of Countable Choice: *For each subset R of $\mathbb{N} \times \mathbb{N}$, if for all m there exists n such that mRn, then there exists α such that, for all m, $mR(\alpha(m))$.*

The intuitionistic mathematician will judge this axiom to be true because he allows himself to build an infinite sequence $\alpha = \alpha(0), \alpha(1), \ldots$ step by step, by successive free choices. He does not demand that the future course of the sequence be prescribed by means of an algorithm.

A classical mathematician would say that, if we have a proof that for all m there exists n such that mRn, a suitable α may be *defined*, as follows: let, for each m, $\alpha(m)$ be the least n such that mRn. It may occur, however, that we have a proof of $0R1$ and are uncertain if $0R0$ is true or not. In such a case the given rule is useless for the constructive mathematician.

The unwelcome truth is that not every subset A of $\{0, 1\} \times \mathbb{N}$ is predeterminate from the viewpoint of player I, as we may learn from the following *counterexample in Brouwer's style*:

> Let $p : \mathbb{N} \to \{0, 1, \ldots, 9\}$ be the decimal expansion of π. We let A be the subset of $\{0, 1\} \times \mathbb{N}$ consisting of all pairs (i, n) such that *either* $i = 0$ and *if* there exists $j < n$ such that, for all $k < 99$, $p(j + k) = 9$, *then* the first such j is odd, *or* $i = 1$ and *if* there exists $j < n$ such that, for all $k < 99$, $p(j + k) = 9$, *then* the first such j is even. For all p, for all q, either $(0, p)$ belongs to A or $(1, q)$ belongs to A.
>
> Assuming that A is predeterminate we obtain the conclusion that *either* for all p, $(0, p)$ belongs to A, *or* for all q, $(1, q)$ belongs to A. In the first case we must have a proof that, *if* there exists j such that, for all $k < 99$, $p(j + k) = 9$, *then* the first such j is odd, and in the second case we must have a proof that, *if* there exists j such that, for all $k < 99$, $p(j + k) = 9$, *then* the first such j is even.
>
> The assumption that A is predeterminate from the viewpoint of player I thus leads to a conclusion for which we have no evidence.

A subset C of $\mathbb{N}$ is a *decidable subset of* $\mathbb{N}$ if and only if there exists α such that, for every n, n belongs to C if and only if $\alpha(n) = 1$. The intuitionistic mathematician does not require that α is given by means of an algorithm.

It will be clear how to extend this notion to subsets of $\{0, 1\}\times\mathbb{N}$. Observe that the set A in the counterexample just given is a decidable subset of $\{0, 1\} \times \mathbb{N}$.

Also note that an anti-strategy for player *II* in $\{0, 1\} \times \mathbb{N}$ is the same as a strategy for player *II* in $\{0, 1\} \times \mathbb{N}$. Therefore, every game in $\{0, 1\} \times \mathbb{N}$ is determinate from the viewpoint of player *II*. We may conclude that there are subsets of $\{0, 1\} \times \mathbb{N}$ that are determinate from the viewpoint of player *II* while we are unable to prove that they are determinate from the viewpoint of player *I*.

12.1.9 In Section 12.1.10 we intend to discuss a second class of two-move-games. We will be led to use the *Fan Theorem*. In the literature, the expression "Fan Theorem" is not used unequivocally, and, for this reason, we introduce two precise versions of the theorem in Section 12.1.9.1. In Section 12.1.9.2 we prove a small combinatorial lemma that will be useful in Section 12.1.10.

12.1.9.1 Let σ be a spread-law. σ is called a *finitary spread-law* or a *fan-law* if and only if for each a, if σ admits a, then there are only finitely many numbers n such that σ admits $a * \langle n \rangle$. The set of all infinite sequences obeying a fan-law is called a *fan*.

Let X be a subset of $\mathcal{N}$ and let B be a subset of $\mathbb{N}$. We say: *B is a bar in X* if and only if every infinite sequence in X has an initial part in B. We say: B is *bounded* if and only if there exists n such that, for each b in B, $length(b) \leq n$. Here are two versions of Brouwer's Fan Theorem:

Unrestricted Fan Theorem: *Let σ be a fan and let B be a subset of $\mathbb{N}$ that is a bar in σ. There exists a bounded subset B' of B that is a bar in σ.*

Strict Fan Theorem: *Let σ be a fan and let B be a decidable subset of $\mathbb{N}$ that is a bar in σ. There exists a bounded subset B' of B that is a bar in σ.*

(The second version occurs as $^*26.6a$ in Kleene and Vesley (1965) and as FAN_D in Troelstra and van Dalen (1988).)

Brouwer's philosophical argument for the bar theorem seems to establish the unrestricted as well as the strict version of the Fan Theorem, see Veldman (2006b). Sometimes, one derives the Unrestricted Fan Theorem from the Strict Fan Theorem by means of the First Axiom of Continuous Choice, a special case of the Second Axiom of Continuous Choice. From a classical point of view, both versions of the Fan Theorem are reformulations of König's lemma. The usual formulation of König's lemma ("*Every infinite finitely-branching tree has an infinite branch*") is not valid intuitionistically.

12.1.9.2 For all natural numbers m, p we let $S(p, m)$ be the set of all numbers a such that $length(a) = m$ and for each $i < m$, $a(i) < p$.

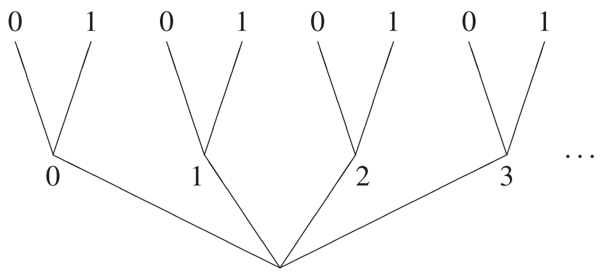
Lemma: *For all m, p, for each subset A of $\mathbb{N} \times \mathbb{N}$, if for all a in $S(p, m)$ there exists $i < m$ such that $(i, a(i))$ belongs to A, then there exists $i < m$ such that, for all $q < p$, (i, q) belongs to A.*

Proof. The proof uses induction on m. The case $m = 1$ is obvious.

Suppose that m is a natural number and that the case m has been established. Let p be a natural number and let A be a subset of $\mathbb{N} \times \mathbb{N}$ such that for all a in $S(p, m + 1)$ there exists $i < m + 1$ such that $(i, a(i))$ belongs to A.

Let a belong to $S(p, m)$. Observe that for each $q < p$, either for some $i < m$, $(i, a(i))$ belongs to A, or (m, q) belongs to A. Therefore, either, for some $i < m$, $(i, a(i))$ belongs to A, or, for all $q < p$, (m, q) belongs to A. Let B be the set of all pairs (i, j) of natural numbers such that either (i, j) belongs to A, or, for all $q < p$, (m, q) belongs to A. Observe that for all a in $S(p, m)$ there exists i such that $(i, a(i))$ belongs to B. Applying the induction hypothesis we find $i < m$ such that for all $q < p$, (i, q) belongs to B, that is, either for all $q < p$, (i, q) belongs to A, or for all $q < p$, (m, q) belongs to A.

12.1.10 We consider two-move-games of the following kind: Player *I* chooses a natural number, player *II* chooses 0 or 1, and the game is over.



A strategy for player *I* in a game like this consists of a single number, player *I*'s first and only move. A strategy for player *II*, on the other hand, is a function from $\mathbb{N}$ to $\{0, 1\}$, assigning to each natural number p the answer player *II* will give if player *I* opens the game with p. The set of strategies for player *II* is the set of all functions from $\mathbb{N}$ to $\{0, 1\}$. This set is a finitary spread, called: the *binary fan*, or: *(intuitionistic) Cantor space* $\mathcal{C}$.

It turns out that every subset of $\mathbb{N} \times \{0, 1\}$ is determinate from the viewpoint of player *I* in the sense of Section 12.1.4:

> Let A be a subset of $\mathbb{N} \times \{0, 1\}$ such that for all α in $\mathcal{C}$ there exists n such that $(n, \alpha(n))$ belongs to A.
>
> Using the unrestricted Fan Theorem, we find m such that for all α in $\mathcal{C}$ there exists $n \leq m$ such that $(n, \alpha(n))$ belongs to A. Therefore, for all a in $S(2, m + 1)$ there exists $n < m + 1$ such that $(n, a(n))$ belongs to A.

> Using the combinatorial lemma from Section 12.1.9.2 we find n such that both $(n, 0)$ and $(n, 1)$ belong to A, and this number obviously is a winning strategy for player I.

In the special case that A is a decidable subset of $\mathbb{N} \times \{0, 1\}$, we obtain the conclusion without using the Fan Theorem, as follows:

> Let A be a decidable subset of $\mathbb{N} \times \{0, 1\}$. We use the First Axiom of Countable Choice and define α such that for every n, $\alpha(n) := 1$ if $(n, 0)$ belongs to A and $\alpha(n) := 0$, if $(n, 0)$ does not belong to A. We determine n such that $(n, \alpha(n))$ belongs to A and conclude that $\alpha(n) = 1$ and that both $(n, 0)$ and $(n, 1)$ belong to A.

The next case to consider is that A is not a decidable, but an *enumerable* subset of $\mathbb{N} \times \{0, 1\}$, that is, there exists a function β in $\mathcal{N}$ such that, for each n, i, (n, i) belongs to A if and only if there exists p such that $\beta(\langle n, i, p\rangle) = 0$. The statement that every enumerable subset of $\mathbb{N} \times \{0, 1\}$ is determinate in the above sense is an *equivalent* of the strict Fan Theorem, see Veldman (2005), that is, in a weak formal system BIM for basic intuitionistic analysis introduced in Veldman (2005) the strict Fan Theorem is equivalent to the statement that every enumerable subset of $\mathbb{N} \times \{0, 1\}$ is determinate from the viewpoint of player I. The stronger statements we are to prove in this paper, Lemma 2.2, Corollaries 2.4, 2.5, Lemma 3.3 and Theorem 3.5 also are equivalents of the strict Fan Theorem, see Veldman (2005).

The result that the Fan Theorem implies that every subset of $\mathbb{N} \times \{0, 1\}$ is determinate from the viewpoint of player I in the sense of 12.1.4 occurs already in Section 4 of Veldman (1982). Following a suggestion by J.R. Moschovakis (see Moschovakis, 1980a), we gave here a slightly different proof.

12.1.11 We describe the contents of the remaining sections. In Section 12.2 we introduce II-finitary spreads, that is, spreads in which player II has only finitely many possibilities for each one of his moves. Using the Fan Theorem, we show that in such spreads, closed sets and open sets are predeterminate from the viewpoint of player I (in the sense of Section 12.1.7).

In Section 12.3 we prove the much stronger result that *every* subset of a II-finitary spread is predeterminate from the viewpoint of player I. A slightly different version of this main result occurs already in Chapter 16 of Veldman (1981). In Section 12.4, we give two applications of the main result.

The reader who wants to enjoy the classical story of the notion of determinacy may consult Moschovakis (1980b) and Kechris (1995).

12.2 The safe-move-lemma and the determinacy of closed sets and open sets in *II*-finitary spreads

12.2.1 In this subsection we introduce some notations and some terminology.

Let σ be a spread and let a be a natural number admitted by σ, that is, such that $\sigma(a) = 0$. We define the spread-law $\sigma{\downarrow}a$ by: for all b in $\mathbb{N}$, $(\sigma{\downarrow}a)(b) = \sigma(a * b)$.

Let σ be a spread, and let γ be a strategy for player *II* in σ. Let a be a natural number such that $\sigma(a) = 0$ and *length*(a) is even, and let δ be a strategy for player *II* in $\sigma{\downarrow}a$. We define: γ *extends* δ, or: δ *extends to* γ, if and only if, a *II*-obeys γ and, for each b in $\mathbb{N}$, if $(\sigma{\downarrow}a)(b) = 0$ and *length*(b) is odd, then $\delta(b) = \gamma(a * b)$.

Let σ be a spread and let ζ be an anti-strategy for player *I* in σ. Let a be a natural number such that $\sigma(a) = 0$ and *length*(a) is even. We define: a is ζ*-safe* if and only if every strategy δ for player *II* in the spread $\sigma{\downarrow}a$ extends to a strategy γ for player *II* in the spread σ such that $\zeta|\gamma$ passes through a.

Let σ be a spread. We define: σ is *II-finitary* if and only if, for each a, if σ admits a and *length*(a) is odd, then there exists n such that, for every m, if σ admits $a * \langle m\rangle$, then $m < n$.

Observe that, if σ is a *II*-finitary spread, then player *II* has only finitely many possibilities for each one of his moves. Therefore, for each strategy γ for player *II* in σ, for each a in $\mathbb{N}$, if $\sigma(a) \neq 0$ or *length*(a) is even, then $\gamma(a) = 0$, and if $\sigma(a) = 0$ and *length*(a) is odd, then there are finitely many possible values for $\gamma(a)$. This shows that, if σ is a *II*-finitary spread, then $\mathrm{Strat}_{II}(\sigma)$ is a fan.

12.2.2 The safe-move-lemma. *Let σ be a II-finitary spread and let ζ be an anti-strategy for player I in σ. Then:*

(i) *The set of all natural numbers a such that $\sigma(a) = 0$ and* length(a) *is even and a is ζ-safe is a decidable subset of* $\mathbb{N}$.

(ii) *For every natural number a, if $\sigma(a) = 0$,* length(a) *is even and a is ζ-safe, then there exists n such that $\sigma(a * \langle n\rangle) = 0$ and, for all m, if $\sigma(a * \langle n, m\rangle) = 0$, then $a * \langle n, m\rangle$ is ζ-safe.*

Proof: Let σ, ζ fulfill the conditions of the lemma.

(i) Let a be a natural number such that $\sigma(a) = 0$ and *length*(a) is even. Using the strict Fan Theorem, we calculate a natural number N such that for all strategies γ, δ for player *II* in σ, if $\overline{\gamma}N = \overline{\delta}N$, then, for each $i <$ *length*(a), $(\zeta|\gamma)(i) = (\zeta|\delta)(i)$.

Consider the set B consisting of all natural numbers $\overline{\gamma}N$, where γ is a strategy for player *II* in σ, and observe that B is a finite set of natural numbers.

Let δ be a strategy for player *II* in the spread $\sigma{\downarrow}a$. Considering the number $\overline{\delta}N$ and the set B we may decide whether δ extends to a strategy γ for player *II* in σ such that $\zeta|\gamma$ passes through a, or not. If δ does so indeed, we say that δ *fits* a. Observe that, for all strategies δ, ε for player *II* in the spread $\sigma{\downarrow}a$, if δ fits a and $\overline{\delta}N = \overline{\varepsilon}N$, then ε fits a.

Also observe that the set of all natural numbers $\overline{\delta}N$, where δ is a strategy for player *II* in the spread $\sigma{\downarrow}a$, is a finite set of natural numbers. Therefore, we may decide if it is true that every strategy δ for player *II* in the spread $\sigma{\downarrow}a$ fits a, or not. If so, then a is ζ-safe, and, if not, then a is not ζ-safe.

(ii) Let a be a natural number such that $\sigma(a) = 0$, *length*(a) is even and a is ζ-safe. Using the strict Fan Theorem we calculate a natural number N such that for each strategy γ for player *II* in σ, $(\zeta|\gamma)(\mathit{length}(a)) < N$. Observe that for each n, m, if $n \geq N$, then $a * \langle n, m\rangle$ is not ζ-safe.

We have to prove that there exists $n < N$ such that σ admits $a * \langle n\rangle$ and, for all m, if σ admits $a * \langle n, m\rangle$, then $a * \langle n, m\rangle$ is ζ-safe. Because of (i) we may argue by contradiction.

Let us assume that, for every n, if $n < N$ and σ admits $a * \langle n\rangle$, then there exists m such that $a*\langle n, m\rangle$ is not ζ-safe. Let $n_0, n_1, \ldots, n_{k-1}$ be an enumeration of the natural numbers n such that $n < N$ and σ admits $a * \langle n\rangle$. Determine $m_0, m_1, \ldots, m_{k-1}$ in $\mathbb{N}$ such that, for all $i < k$, $\sigma(a*\langle n_i, m_i\rangle) = 0$ and $a*\langle n_i, m_i\rangle$ is not ζ-safe. Determine, for each $i < k$, a strategy δ_i for player *II* in $\sigma{\downarrow}(a*\langle n_i, m_i\rangle)$ such that δ_i does not fit $a * \langle n_i, m_i\rangle$. Let γ be a strategy for player *II* in $\sigma{\downarrow}a$ be such that, for each $i < k$, γ extends δ_i and $\gamma(a * \langle n_i\rangle) = m_i$.

As a is ζ-safe, we may determine a strategy γ' for player *II* in σ, extending the strategy γ, and such that $\zeta|(\gamma')$ passes through a. But then there exists $i < k$ such that $\zeta|(\gamma')$ passes through $a * \langle n_i, m_i\rangle$ and this contradicts the fact that γ' extends δ_i and δ_i does not fit $a * \langle n_i, m_i\rangle$.

We thus see that there exists $n < N$ such that σ admits $a * \langle n\rangle$ and, for all m, if σ admits $a * \langle n, m\rangle$, then $a * \langle n, m\rangle$ is ζ-safe.

12.2.3 Let σ be a spread and let A be a subset of σ. We define: A is an *open* subset of σ if and only if there exists a decidable subset C of $\mathbb{N}$ such that for every α in σ, α belongs to A if and only if, for some n, $\overline{\alpha}n$ belongs to C.

We define: A is a *closed* subset of σ if and only if there exists a decidable subset C of $\mathbb{N}$ such that for all α, α belongs to A if and only if, for each n, $\overline{\alpha}n$ belongs to C.

If A itself is a spread, then A is a closed subset of σ, but not every closed subset of σ is a spread, see Veldman (1981). The reason is that, given a decidable subset C of $\mathbb{N}$, it is not always possible to decide if there exists α such that for every n, $\overline{\alpha}n$ belongs to C.

For each a, n such that $n \leq\ \mathit{length}(a)$, we let $\overline{a}(n)$ be the code number of the finite sequence $(a(0), a(1), \ldots, a(n-1))$.

For each a, for each γ, we define: *a I-obeys γ*, if and only if, for each n, if $2n < length(a)$, then $a(2n) = \gamma(\overline{a}(2n))$.

Similarly, for each a, for each γ, we define: *a II-obeys γ*, if and only if, for each n, if $2n + 1 < length(a)$, then $a(2n + 1) = \gamma(\overline{a}(2n + 1))$.

12.2.4 Corollary. *In II-finitary spreads, closed sets are predeterminate from the viewpoint of player I.*

Proof: Let σ be a II-finitary spread and let A be a closed subset of σ. Let C be a decidable subset of $\mathbb{N}$ such that for all α, α belongs to A if and only if, for all n, $\overline{\alpha}n$ belongs to C. Let ζ be an anti-strategy for player I in σ such that for every strategy γ for player II in σ, $\zeta|\gamma$ belongs to A.

We apply the safe-move-lemma 3.6 and determine a strategy γ for player I in σ such that for every a, if σ admits a and $length(a)$ is even and a is ζ-safe, then σ admits $a * \langle\gamma(a)\rangle$, and, for each m, if σ admits $a * \langle\gamma(a), m\rangle$, then $a * \langle\gamma(a), m\rangle$ is ζ-safe.

As the empty sequence $\langle\ \rangle$ is ζ-safe, every α that I-obeys γ will have the property that, for each n, $\overline{\alpha}(2n)$ is ζ-safe. Observe that, for each a, if $length(a)$ is even and a is ζ-safe, then every initial part of a belongs to C. It follows that every α that I-obeys γ belongs to A.

12.2.5 Corollary. *In II-finitary spreads, open sets are determinate from the viewpoint of player I.*

Proof: Let σ be a II-finitary spread and let A be an open subset of σ. Let C be a decidable subset of $\mathbb{N}$ such that, for every α in σ, α belongs to A if and only if, for some n, $\overline{\alpha}n$ belongs to C. Suppose that every strategy for player II in σ II-governs at least one element of A. Note that, for every strategy γ for player II in σ,v there exists a in C such that a II-obeys γ. Applying the strict Fan Theorem, we find N in $\mathbb{N}$ such that for every strategy γ for player II in σ, there exist a such that $a \leq N$ and a belongs to C and a II-obeys γ.

Let B be the set of all α in σ such that, for some n, $\overline{\alpha}n \leq N$ and $\overline{\alpha}n$ belongs to C. observe that B is a closed subset of σ and a subset of A. We now define an anti-strategy ζ for player I in σ, as follows. Let γ be a strategy for player II in σ. Let b be the least a such that a II-obeys γ and a belongs to C. Let $\zeta|\gamma$ be the sequence β passing through b such that β II-obeys γ and for each n, if $2n \geq length(b)$, then $\beta(2n)$ is the least p such that σ admits $\overline{\beta}(2n) * \langle p\rangle$.

It will be clear that, for each strategy γ for player II in σ, the sequence $\zeta|\gamma$ II-obeys γ and belongs to B. Applying Corollary 2.4 we conclude that there is a strategy for player I in σ that wins the set B for player I, and therefore also the set A.

12.3 The safe-conjecture-lemma and the intuitionistic determinacy theorem

12.3.1 In Section 12.2 we have seen that, in a II-finitary spread σ, if the payoff set A is closed or open, every anti-strategy securing the set A for player I may be effectively transformed in a strategy winning the set A for player I.

In this section, we will strengthen this result considerably: we show that, in any II-finitary spread σ, any anti-strategy ζ for player I may be effectively transformed in a strategy γ for player I with the property that to any play α in σ I-obeying γ one may effectively construct a strategy δ for player II such that $\alpha = \zeta|\delta$.

It is not difficult to see that this result solves the determinacy problem for II-finitary spreads: every subset of a II-finitary spread is predeterminate from the viewpoint of player I.

12.3.2 Let σ be a spread and let ζ be an anti-strategy for player I in σ. We want to refine the notion of a "ζ-safe position", introduced in Section 12.2.1.

Let a be a natural number admitted by σ such that *length*(a) is even, and let c be a natural number. We define: *a is ζ-safe with conjecture c* if and only if each strategy for player II in the spread $\sigma{\downarrow}a$ extends to a strategy γ for player II in the spread σ *passing through c* such that $\zeta|\gamma$ passes through a.

12.3.3 The safe-conjecture-lemma. *Let σ be a II-finitary spread and let ζ be an anti-strategy for player I in σ.*

(i) *For each c, the set of all natural numbers a such that* length(a) *is even and $\sigma(a) = 0$ and a is ζ-safe with conjecture c is a decidable subset of* $\mathbb{N}$.

(ii) *For all natural numbers a, c, if $\sigma(a) = 0$,* length(a) *is even and a is ζ-safe with conjecture c, then there exists n such that $\sigma(a * \langle n \rangle) = 0$ and, for all m, if $\sigma(a * \langle n, m \rangle) = 0$, then $a * \langle n, m \rangle$ is ζ-safe with conjecture c.*

(iii) *For all natural numbers a, c, if $\sigma(a) = 0$,* length(a) *is even and a is ζ-safe with conjecture c, then, for every strategy δ for player II in the spread $\sigma{\downarrow}a$ there exists d, n such that* length(d) *is even and d II-obeys δ and $a * d$ is ζ-safe with conjecture $c * \langle n \rangle$.*

Proof: Let σ, ζ fulfill the conditions of the lemma.

(i), (ii): We omit the proofs, as they are similar to the proofs of the corresponding statements in Lemma 2.2.

(iii) Let a, c be natural numbers such that $\sigma(a) = 0$ and *length*(a) is even and a is ζ-safe with conjecture c. Let δ be a strategy for player II in the spread $\sigma{\downarrow}a$. We determine a strategy γ for player II in the spread σ such that γ extends δ

and γ passes through c and $\zeta|\gamma$ passes through a. We determine p such that for every strategy ε for player *II* in σ, if ε passes through $\overline{\gamma}p$, then $\zeta|\varepsilon$ passes through a.

We now consider $n := \gamma(length(c))$. Observe that γ passes through $c * \langle n \rangle$.

Let m be the greatest one of the two numbers p, $length(c) + 1$. Observe that for every strategy β for player *II* in the spread $\sigma{\downarrow}a$, if β passes through $\overline{\delta}m$, then there exists a strategy η for player *II* in the spread σ such that η extends β and η passes through $\overline{\gamma}m$.

We define $k := 2m + length(a)$. We let B be the set of all numbers $\overline{(\zeta|\eta)}k$, where η is a strategy for player *II* in the spread σ passing through $\overline{\gamma}m$ and extending a strategy δ' for player *II* in the spread $\sigma{\downarrow}a$ with the property: for each e, if $\sigma(a * e) = 0$ and $length(e) < 2m$, then $\delta'(e) = \delta(e)$. The set of all such strategies η is a fan and it follows from the strict Fan Theorem that B is a finite subset of $\mathbb{N}$. Remark that for each b in B there exists d such that $b = a * d$ and $length(d)$ is even and d *II*-obeys δ.

We claim that some member of B must be ζ-safe with conjecture $c * \langle n \rangle$. Because of (i) we may argue by contradiction.

Assume that no member of B is ζ-safe with conjecture $c * \langle n \rangle$. We then choose for each b in B a strategy δ_b for player *II* in the spread $\sigma{\downarrow}b$ such that δ_b does not extend to a strategy η for player *II* in σ with the property that η passes through $c * \langle n \rangle$ and $\zeta|\eta$ passes through b. We then form a strategy β for player *II* in $\sigma{\downarrow}a$ passing through $\overline{\delta}m$ such that, for each b in B, β extends δ_b, and, for each e, if $\sigma(a * e) = 0$ and $length(e) < 2m$, then $\beta(e) = \delta(e)$. We let ε be a strategy for player *II* in σ extending β and passing through $c * \langle n \rangle$ such that $\zeta|\varepsilon$ passes through a.

Consider $b := \overline{(\zeta|\varepsilon)}(k)$ and remark: b belongs to B and ε extends δ_b and ε passes through $c * \langle n \rangle$ and $\zeta|\varepsilon$ passes through b. Contradiction.

We conclude that some element of B must be ζ-safe with conjecture $c * \langle n \rangle$. Let b be such an element of B. Determine d such that $b = a * d$. Observe that $length(d)$ is even and d *II*-obeys δ and that we have obtained the desired conclusion.

12.3.4 For each α, for each n, we let α^n be the element β of $\mathcal{N}$ such that, for all m, $\beta(m) = \alpha(\langle n, m \rangle)$. In the proof of our main theorem, we use the following axiom:

Second Axiom of Countable Choice: *For each subset R of $\mathbb{N} \times \mathcal{N}$, if for each n there exists α such that $nR\alpha$, then there exists α such that, for each n, $nR\alpha^n$.*

This axiom, occurring as $^*2.1$ in Kleene and Vesley (1965), as $\mathbf{AC_{01}}$ in Gielen et al. (1981) and as *AC-NF* in Troelstra and van Dalen (1988), is a consequence of the Second Axiom of Continuous Choice, that we mentioned in Section 12.1.5.

Unlike the Second Axiom of Continuous Choice, the Second Axiom of Countable Choice is, from a classical point of view, a sensible assumption.

12.3.5 Intuitionistic Determinacy Theorem. *Let σ be a II-finitary spread. Every subset of σ is predeterminate from the viewpoint of player I.*

Proof: Let σ be a II-finitary spread. Let A be a subset of σ and let ζ be an anti-strategy for player I in σ securing the set A for player I. We prove that there exist a strategy for player I in σ with the property that, for every α in σ, if α I-obeys γ, then there exists a strategy δ for player II in σ such that α coincides with $\zeta|\delta$. Obviously, the strategy γ then wins the set A for player I.

According to Lemma 3.3 and Corollary 2.5 we may determine, for each a, c such that $\sigma(a) = 0$, $length(a)$ is even and a is ζ-safe with conjecture c, a strategy γ for player I in $\sigma\downarrow a$ with the property that for every α in σ I-obeying γ there exist p, n such that $a * \overline{\alpha}(2p)$ is ζ-safe with conjecture $c * \langle n\rangle$.

Let B be the set of all numbers $\langle a, c\rangle$ in $\mathbb{N}$ such that $\sigma(a) = 0, length(a)$ is even and a is ζ-safe with conjecture c. According to Lemma 3.3, B is a decidable subset of $\mathbb{N}$.

Using the Second Axiom of Countable Choice we determine ε in $\mathcal{N}$ with the property that, for each $\langle a, c\rangle$ in B, $\varepsilon^{\langle a,c\rangle}$ is a strategy for player I in $\sigma\downarrow a$ such that for every α in $\sigma\downarrow a$ I-obeying $\varepsilon^{\langle a,c\rangle}$ there exist p, n such that $a * \overline{\alpha}(2p)$ is ζ-safe with conjecture $c * \langle\lambda(\langle a, c\rangle)\rangle$.

We now describe informally the strategy γ that player I should obey in σ.

> Observe that $\langle\ \rangle$ is ζ-safe with conjecture $\langle\ \rangle$. Define $\delta(0) = \lambda(\langle\langle\ \rangle, \langle\ \rangle\rangle)$. Follow the strategy $\varepsilon^{\langle\langle\ \rangle,\langle\ \rangle\rangle}$, until, in cooperation with player II a position $\overline{\alpha}(2n_0)$ is reached such that $n_0 > 0$ and, for some n, $\overline{\alpha}(2n_0)$ is ζ-safe with conjecture $\langle n\rangle$. Let $\delta(0)$ be the least such n.
>
> Follow the strategy $\varepsilon^{\langle\overline{\alpha}(2n_0),\ \langle\delta(0)\rangle\rangle}$ until, in cooperation with player II, a position $\overline{\alpha}(2n_1)$ is reached such that $n_1 > n_0$ and, for some n, $\overline{\alpha}(2n_1)$ is ζ-safe with conjecture $\langle\delta(0), n\rangle$. Let $\delta(1)$ be the least such n.
>
> And so on.

Lemma 3.2(ii) ensures that it is indeed possible for player I to ensure that $n_1 > n_0$ and $n_2 > n_1$, and so on.

Suppose that α belongs to σ and is played by player I according to this strategy and that δ is the sequence of conjectures formed by player I during the play. Observe that, for all n, there exists a strategy β for player II in σ passing through $\overline{\delta}n$ such that $\zeta|\beta$ passes through $\overline{\alpha}(2n)$. It follows that δ is a strategy for player II in σ with the property: $\zeta|\delta = \alpha$.

12.4 Two applications

12.4.1 For each a, b in $\mathbb{N}$ we define: *the finite sequence (coded by) a is an initial part of the finite sequence (coded by) b* , notation: $a \sqsubseteq b$, if and only if there exists $n \leq length(b)$ such that $a = \overline{b}n$.

For each a, b in $\mathbb{N}$ we define: *a, b form a branching*, notation: $a \perp b$, if and only if a is not an initial part of b and b is not an initial part of a.

Let A be a subset of $\mathcal{N}$. We consider the following game, sometimes called $\mathcal{G}^*(A)$, that has been devised by Morton Davis in Davis (1964).

Player I chooses $\langle \ell_0, r_0 \rangle$ in $\mathbb{N} \times \mathbb{N}$ such that $\ell_0 \perp r_0$.

$\searrow$

Player II chooses i_0 in $\{0, 1\}$.
We define $a_0 := \ell_0$ if $i_0 = 0$, and
$a_0 := r_0$ if $i_0 = 1$.

$\swarrow$

Player I chooses $\langle \ell_1, r_1 \rangle$ in $\mathbb{N} \times \mathbb{N}$ such that $a_0 \sqsubseteq \ell_1, a_0 \sqsubseteq r_1$ and $\ell_1 \perp r_1$.

$\searrow$

Player II chooses i_1 in $\{0, 1\}$.
We define : $a_1 := \ell_1$ if $i_1 = 0$, and
$a_1 := r_1$ if $i_1 = 1$.

$\swarrow$

Player I chooses $\langle \ell_2, r_2 \rangle$ in $\mathbb{N} \times \mathbb{N}$ such that $a_1 \sqsubseteq \ell_2, a_1 \sqsubseteq r_2$ and $\ell_2 \perp r_2$.

$\searrow$

Player II chooses i_2 in $\{0, 1\}$
We define $a_2 := \ell_2$ if $i_2 = 0$, and
$a_2 := r_2$ if $i_2 = 1$.

And so on.

In the end, we determine α in $\mathcal{N}$ such that, for all n, α passes through a_n. Player I wins if and only if α belongs to A.

It will be clear that $\mathcal{G}^*(A)$ may be described as a game in a II-finitary spread σ. It follows that for every subset A of $\mathcal{N}$, the game $\mathcal{G}^*(A)$ is predeterminate from the viewpoint of player I.

One may prove constructively that player I has a winning strategy in the game $\mathcal{G}^*(A)$ if and only if there exists an *embedding* of Cantor space $\mathcal{C}$ into A, that is: an element ζ of $\mathcal{N}$ coding a continuous function from $\mathcal{C}$ into A such that for all α, β in $\mathcal{C}$, if there exists n such that $\alpha(n) \neq \beta(n)$, then there exists p such that $(\zeta|\alpha)(p) \neq (\zeta|\beta)(p)$.

One may prove constructively that, if the set A is *enumerable*, that is, if there exists an element α of $\mathcal{N}$ such that every element of A occurs in the sequence $\alpha^0, \alpha^1, \alpha^2, \ldots$, then player II has a strategy ensuring that the result of a play in $\mathcal{G}^*(A)$ will not belong to A: he makes his n-th move such that the result will differ from α^n.

Classically, it is also true that if player *II* has a successful strategy in $\mathcal{G}^*(A)$, then the set A is at most enumerable. The argument is unconstructive, but, as we hope to show in a future paper, one may prove an intuitionistic counterpart to this result, using Brouwer's Thesis on bars.

The statement that all games $\mathcal{G}^*(A)$ are predeterminate from the viewpoint of player *I* is an intuitionistic theorem that is in some sense related to the continuum hypothesis, like the different theorem in Section 2 of Gielen et al. (1981), to which it forms a kind of counterpart.

12.4.2 Let A be a subset of the set $\mathbb{Q}$ of rational numbers. We consider the following game that we call $\mathcal{H}(A)$, the letter $\mathcal{H}$ honouring F. Hausdorff.

Player *I* chooses q_0 in $\mathbb{Q}$.

$\searrow$

Player *II* chooses i_0 in $\{0, 1\}$.
We define $H_0 := \{q \in \mathbb{Q} \mid q < q_0\}$ if $i_0 = 0$, and
$H_0 := \{q \in \mathbb{Q} \mid q > q_0\}$ if $i_0 = 1$.

$\swarrow$

Player *I* chooses q_1 in H_0.

$\searrow$

Player *II* chooses i_1 in $\{0, 1\}$.
We define $H_1 := H_0 \cap \{q \in \mathbb{Q} \mid q < q_1\}$ if $i_1 = 0$, and
$H_1 := H_0 \cap \{q \in \mathbb{Q} \mid q > q_1\}$ if $i_1 = 1$.

$\swarrow$

Player *I* chooses q_2 in H_1.

$\searrow$

Player *II* chooses i_2 in $\{0, 1\}$.
We define $H_2 := H_1 \cap \{q \in \mathbb{Q} \mid q < q_2\}$ if $i_2 = 0$, and
$H_2 := H_1 \cap \{q \in \mathbb{Q} \mid q > q_2\}$ if $i_2 = 1$.

and so on.

In the end, player *I* wins if and only if, for each n, q_n belongs to A.

The game $\mathcal{H}(A)$ may be described as a game in a *II*-finitary spread σ. Thus, Theorem 3.3 applies, and, for every subset A of $\mathbb{Q}$, the game $\mathcal{H}(A)$ is predeterminate from the viewpoint of player *I*.

Observe that player *I* has a winning strategy in $\mathcal{H}(A)$ if and only if there exists an order-preserving embedding of $(\mathbb{Q}, <)$ into $(A, <)$.

From a classical point of view, the game $\mathcal{H}(A)$ is determinate as it is a closed game, and the class of all subsets A of $\mathbb{Q}$ such that player *II* has a winning strategy in the game $\mathcal{H}(A)$ coincides with the class of all *scattered* subsets of $\mathbb{Q}$, that is, the class of all subsets A of $\mathbb{Q}$ such that it is impossible to embed $(\mathbb{Q}, <)$ into $(A, <)$.

Intuitionistically, it seems wise to restrict oneself to *decidable* subsets A of $\mathbb{Q}$. The statement "*player II has a strategy in the game $\mathcal{H}(A)$, such that, for any resulting sequence $q_0, q_1, q_2, \ldots$, some n may be found with the property $q_n \notin A$*" turns out to be equivalent to "A is *very discrete*", as we hope to show in a future paper. The argument uses Brouwer's Thesis on bars, see Veldman (2006a). The notion of a very discrete subset of $\mathbb{Q}$ is defined inductively, see Rosenstein (1982). A subset A of $\mathbb{Q}$ is *very discrete* if either $A = \emptyset$ or A contains exactly one number, or there exists a sequence $\ldots, A_{-2}, A_{-1}, A_0, A_1, A_2, \ldots$ of very discrete sets such that, for each i in $\mathbb{Z}$, for each q in A_i, for each r in A_{i+1}, $q < r$, and $A = \bigcup\limits_{i \in \mathbb{Z}} A_i$. Scattered sets were first studied by F. Hausdorff (see Hausdorff, 1908) and Rosenstein (1982).

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