

# On the Power of Unambiguity in Büchi Complementation

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**Abstract.** In this work, we exploit the power of *unambiguity* for the complementation problem of Büchi automata by utilizing reduced run directed acyclic graphs (DAGs) over infinite words, in which each vertex has at most one predecessor. Given a Büchi automaton with  $n$  states and a finite degree of ambiguity, we show that the number of states in the complementary Büchi automaton constructed by the classical rank-based and slice-based complementation constructions can be improved, respectively, to  $2^{\mathcal{O}(n)}$  from  $2^{\mathcal{O}(n \log n)}$  and to  $\mathcal{O}(4^n)$  from  $\mathcal{O}((3n)^n)$ , based on reduced run DAGs. To the best of our knowledge, the improved complexity is exponentially better than best known result of  $\mathcal{O}(5^n)$  in [21] for complementing Büchi automata with a finite degree of ambiguity.

## 1 Introduction

Nondeterministic Büchi automata on words (NBWs), which were originally proposed to prove the decidability of a restricted monadic second-order logic [7], are finite automata accepting infinite words. NBWs now have been widely applied in model checking [3], as they can represent the properties of nondeterministic systems with infinite-length behaviors. For instance, in automata-based model checking [27] framework, when both the system and the specification are given as NBWs, the model-checking problem of verifying whether the behavior of the system satisfies the specification then reduces to a language-containment problem between the corresponding automata [27].

For general NBWs  $A$  and  $B$ , a general approach to checking the containment between  $A$  and  $B$  is to first construct a complementary automaton  $B^c$  such that  $\mathcal{L}(B^c) = \Sigma^\omega \setminus \mathcal{L}(B)$  and then to check language emptiness of  $\mathcal{L}(A) \cap \mathcal{L}(B^c)$ . Various implementations of this approach with optimizations [1, 2, 10, 12] have been proposed to improve its practical performance. All the practical implementations above, however, directly or indirectly, resort to constructing  $B^c$ , which can be exponentially larger than  $B$  [23, 28].

In this work, we focus on the bottleneck of containment checking between NBWs — the complementation of NBWs, whose complexity has been proved to be  $\approx ((0.76n)^n)$  [23, 28]. A classic line of research on complementation aims at developing optimal (or close to optimal) complementation algorithms. Currently there are mainly four types of practical complementation algorithms for NBWs, namely *Ramsey-based* [24],

*determinization-based* [22], *rank-based* [16] and *slice-based* [14] algorithms. These algorithms, however, all unavoidably lead to a super-exponential growth in the size of  $B^c$  in the worst case [28].

With the growing understanding of the worst-case complexity of those algorithms, searching for specialized complementation algorithms for certain subclasses of NBWs with better complexity has become an important line of research. For instance, complementing deterministic and semi-deterministic Büchi automata can be done in  $\mathcal{O}(n)$  [17] and  $\mathcal{O}(4^n)$  [5], respectively. Here we follow this line of research and aim at a subclass of NBWs with restricted nondeterminism. This type of NBWs is important, as in some contexts, especially in probabilistic model checking, unrestricted nondeterminism in the automata representing the properties is problematic for the verification procedure. For instance, general NBWs cannot be used directly to verify properties over Markov chains, as they will cause imprecise probabilities in the product of the system and the property [8]. In turn, it is often necessary to construct their more deterministic counterparts in terms of other types of automata for the properties, for instance semi-deterministic Büchi automata, deterministic Rabin or Parity automata, which, however, adds exponential blowups of states [11].

To avoid state-space exponential blowup, earlier work sought to use of a type of automata called *unambiguous nondeterministic Büchi automata* (UNBW) in probabilistic verification [4, 18], as UNBW can be exponentially smaller than their equivalent deterministic automata [4]. UNBW [9] are a subclass of NBWs that accept with at most one run for each word, while their equivalent NBWs may have more than one accepting run, or even infinitely many accepting runs. For example, by taking advantage of their unambiguity, the language-containment problem of certain proper subclasses of UNBW has been proved to be solvable in polynomial time [6], while this problem is PSPACE-complete for NBWs [15].

The complementation problem of a more general class than UNBW, called *finitely ambiguous nondeterministic Büchi automata* (FANBW), which accept with finitely many runs for each word, was shown to be doable in  $\mathcal{O}(5^n)$  [21], in contrast to  $2^{\Omega(n \log n)}$  for general NBWs [23]. Further, checking whether an NBW is an FANBW can be done in polynomial time [19]. Therefore, once an FANBW has been identified, the specialized complementation construction for FANBW can be applied. Thus, we focus here on an in-depth study of the complementation problem for FANBW.

Our main tool is the construction and study of reduced directed acyclic graphs (DAGs) of runs of FANBW over infinite words called *co-deterministic run DAGs*, in which each vertex has at most one predecessor, as a way to characterize finite unambiguity in automata in this work. We show that such co-deterministic run DAGs can be used to simplify and improve classical complementation constructions. Our contributions are the following.

- First, we introduce the concept of co-deterministic DAGs of FANBW over infinite words as a way to show how unambiguity works in Büchi complementation.
- Second, we show that the construction of co-deterministic DAGs in different complementation algorithms [25] helps to achieve simpler and theoretically better complementation algorithms for FANBW. Given an FANBW with  $n$  states, we show that the number of states of the complementary NBW constructed by the classi-

cal rank-based and slice-based complementation constructions can be improved, respectively, to  $2^{\mathcal{O}(n)}$  from  $2^{\mathcal{O}(n \log n)}$  and to  $\mathcal{O}(4^n)$  from  $\mathcal{O}((3n)^n)$ , which is exponentially better than the result of  $\mathcal{O}(5^n)$  in [21].

- Finally, we reveal that the slice-based algorithm is basically an algorithm based on the construction of co-deterministic DAGs and a specialized complementation algorithm for FANBW. We also provide a simulation relation between states in the complementary NBWs of FANBW, which can be used to improve the containment checking between an NBW and an (FA)NBW.

*Organization of the paper.* In the remainder of this paper, we first recap some definitions about Büchi automata in Section 2 and then introduce the concept of co-deterministic run DAGs in Section 3. We present our improved algorithms for the rank-based and slice-based algorithms in Section 4 and Section 5, respectively. Finally we conclude the paper with some future works in Section 6.

## 2 Preliminaries

We fix an *alphabet*  $\Sigma$ . A *word* is an infinite sequence  $w$  of letters in  $\Sigma$ . We denote by  $\Sigma^\omega$  the set of all (infinite) words. A *language* is a subset of  $\Sigma^\omega$ . Let  $L$  be a language and the complement language of  $L$  is denoted by  $L^c$ , i.e.,  $L^c = \Sigma^\omega \setminus L$ . Let  $\rho$  be a sequence of elements: we denote by  $\rho[i]$  the  $i$ -th element of  $\rho$ . Let  $n$  be a natural number; we denote by  $[n]$  the set of numbers  $\{0, 1, \dots, n\}$ ,  $[n]^{\text{odd}}$  the set of odd numbers in  $[n]$  and  $\langle n \rangle$  the set of numbers  $[n] \setminus \{0\}$ .

A *Büchi automaton on words* (BW) is a tuple  $\mathcal{A} = (Q, I, \delta, F)$ , where  $Q$  is a finite set of states,  $I \subseteq Q$  is a set of initial states,  $\delta : Q \times \Sigma \rightarrow 2^Q$  is a transition function and  $F \subseteq Q$  is a set of accepting states. We extend  $\delta$  to sets of states, by letting  $\delta(S, a) = \bigcup_{q \in S} \delta(q, a)$ . We assume that each BW  $\mathcal{A}$  is *complete* in the sense that for each state  $q \in Q$  and  $a \in \Sigma$ ,  $\delta(q, a) \neq \emptyset$ . A *run* of  $\mathcal{A}$  on a word  $w$  is an infinite sequence of states  $\rho = q_0 q_1 \dots$  such that  $q_0 \in I$  and for every  $i > 0$ ,  $q_i \in \delta(q_{i-1}, a_i)$ . We denote by  $\text{inf}(\rho)$  the set of states that occur infinitely often in the run  $\rho$ . A word  $w \in \Sigma^\omega$  is *accepted* by  $\mathcal{A}$  if there exists a run  $\rho$  of  $\mathcal{A}$  over  $w$  such that  $\text{inf}(\rho) \cap F \neq \emptyset$ . We denote by  $\mathcal{L}(\mathcal{A})$  the *language* recognized by  $\mathcal{A}$ , i.e., the set of words accepted by  $\mathcal{A}$ .

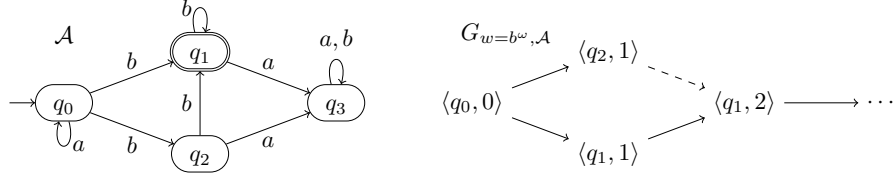
Let  $\mathcal{A}$  be a BW, A complementary BW of  $\mathcal{A}$ , denoted by  $\mathcal{A}^c$ , accepts the complementary language of  $\mathcal{L}(\mathcal{A})$ , i.e.,  $\mathcal{L}(\mathcal{A}^c) = \Sigma^\omega \setminus \mathcal{L}(\mathcal{A})$ ; we denote by  $\mathcal{A}^q$  the automaton  $(Q, \{q\}, \delta, F)$  obtained from  $\mathcal{A}$  by setting its initial state set to the singleton set  $\{q\}$ . We say a state  $q$  of  $\mathcal{A}$  *simulates* a state  $q'$  of  $\mathcal{A}$  if  $\mathcal{L}(\mathcal{A}^{q'}) \subseteq \mathcal{L}(\mathcal{A}^q)$ . We classify  $\mathcal{A}$  into following types of BWs according to their transition structures: (1) *nondeterministic* if  $|I| > 1$  or  $|\delta(q, a)| > 1$  for a state  $q \in Q$  and  $a \in \Sigma$ , (2) *deterministic* if  $|I| = 1$  and for each  $q \in Q$  and  $a \in \Sigma$ ,  $|\delta(q, a)| \leq 1$ , and (3) *reverse deterministic* if for each state  $q' \in Q$ ,  $\mathcal{A}$  has at most one state  $q$  for each  $a \in \Sigma$  such that  $q' = \delta(q, a)$ .

From the perspective of the number of accepting runs of  $\mathcal{A}$ , we have following types of NBWs.

**Definition 1.** Let  $\mathcal{A}$  be an NBW and  $k$  a positive integer. We say  $\mathcal{A}$  is (1) *finitely ambiguous* (an FANBW) if for each  $w \in \mathcal{L}(\mathcal{A})$ , the number of accepting runs of  $\mathcal{A}$  over  $w$

is finite; and (2)  $k$ -ambiguous if for a  $w \in \mathcal{L}(\mathcal{A})$ , the number of accepting runs of  $\mathcal{A}$  over  $w$  is no greater than  $k$ , and unambiguous if  $k = 1$ .

By Definition 1, it holds that both  $k$ -ambiguous NBWs and unambiguous NBWs are special classes of FANBW. For instance, the NBW  $\mathcal{A}$  depicted in Figure 1 is a 2-ambiguous NBW, thus also an FANBW, as  $(q_0)^{i+1}q_1^\omega$  and  $(q_0)^{i+1}q_2q_1^\omega$  are the only two accepting runs for an accepting word  $a^ib^\omega \in \mathcal{L}(\mathcal{A})$  where  $i \geq 0$ .



**Fig. 1.** An FANBW  $\mathcal{A}$  with  $I = \{q_0\}$  and  $F = \{q_1\}$  and the run DAG  $G_{w, \mathcal{A}}$  over  $b^\omega$ .

### 3 Co-Deterministic Run DAGs for FANBW

In this section, we describe the concept of run DAGs of an NBW over a word  $w$ , introduced in [16]. We then describe a construction of co-deterministic run DAGs for FANBW by making use of the finite ambiguity in FANBW, which is the foundation of the results developed in this paper. In the remainder of the paper, we use DAGs as the shorthand for run DAGs.

Let  $\mathcal{A} = (Q, I, \delta, F)$  be an NBW and  $w = a_0a_1 \dots$  be an infinite word. The DAG  $G_{w, \mathcal{A}} = (V, E)$  of  $\mathcal{A}$  over  $w$  is defined as follows:

- Vertices:  $V \subseteq Q \times \mathbb{N}$  is the set of vertices  $\bigcup_{l \geq 0} V_l \times \{l\}$  where  $V_0 = I$  and  $V_{l+1} := \delta(V_l, a_l)$  for every  $l \geq 0$ .
- Edges: There is an edge from  $\langle q, l \rangle$  to  $\langle q', l' \rangle$  iff  $l' = l + 1$  and  $q' \in \delta(q, a_l)$ .

A vertex  $\langle q, l \rangle$  is said to be on level  $l$  and there are at most  $|Q|$  states on each level. A vertex  $\langle q, l \rangle$  is an  $F$ -vertex if  $q \in F$ . A sequence of vertices  $\hat{\rho} = \langle q_0, 0 \rangle \langle q_1, 1 \rangle \dots$  is called a *branch* of  $G_{w, \mathcal{A}}$  if  $q_0 \in I$  and for each  $l \geq 0$ , there is an edge from  $\langle q_l, l \rangle$  to  $\langle q_{l+1}, l+1 \rangle$ . An  $\omega$ -branch of  $G_{w, \mathcal{A}}$  is a branch of infinite length. A finite *fragment*  $\langle q_l, l \rangle \langle q_{l+1}, l+1 \rangle \dots$  of  $\hat{\rho}$  is said to be a branch from the vertex  $\langle q_l, l \rangle$ ; a fragment  $\langle q_l, l \rangle \dots \langle q_{l+k}, l+k \rangle$  of  $\hat{\rho}$  is said to be a *path* from  $\langle q_l, l \rangle$  to  $\langle q_{l+k}, l+k \rangle$ , where  $k \geq 1$ . A vertex  $\langle q_j, j \rangle$  is *reachable* from  $\langle q_l, l \rangle$  if there is a path from  $\langle q_l, l \rangle$  to  $\langle q_j, j \rangle$ . We call a vertex  $\langle q, l \rangle$  is *finite* in  $G_{w, \mathcal{A}}$  if there are no  $\omega$ -branches in  $G_{w, \mathcal{A}}$  starting from  $\langle q, l \rangle$ ; and we call a vertex  $\langle q, l \rangle$   $F$ -free if it is not finite and no  $F$ -vertices are reachable from  $\langle q, l \rangle$  in  $G_{w, \mathcal{A}}$ .

There is a bijection between the set of runs of  $\mathcal{A}$  on  $w$  and the set of  $\omega$ -branches in  $G_{w, \mathcal{A}}$ . To a run  $\rho = q_0q_1 \dots$  of  $\mathcal{A}$  over  $w$  corresponds an  $\omega$ -branch  $\hat{\rho} = \langle q_0, 0 \rangle \langle q_1, 1 \rangle \dots$ . Therefore,  $w$  is accepted by  $\mathcal{A}$  if and only if there exists an  $\omega$ -branch in  $G_{w, \mathcal{A}}$  that visits  $F$ -vertices infinitely often; we say that such an  $\omega$ -branch is *accepting*;  $G_{w, \mathcal{A}}$  is accepting if and only if there exists an accepting  $\omega$ -branch in  $G_{w, \mathcal{A}}$ .

Assume that  $\mathcal{A}$  is an FANBW. Then an accepting  $\omega$ -branch in  $G_{w,\mathcal{A}}$ , if exists, only merges with other (accepting)  $\omega$ -branches for finitely many times. We formalize this property of  $G_{w,\mathcal{A}}$  in Lemma 1.

**Lemma 1 (Separate Levels of Accepting DAGs of FANBW).** *Let  $\mathcal{A}$  be an FANBW and  $G_{w,\mathcal{A}}$  the accepting DAG of  $\mathcal{A}$  over  $w \in \mathcal{L}(\mathcal{A})$ . Then there must exist a separate level  $k \geq 1$  such that all vertices after level  $k$  on an accepting  $\omega$ -branch has exactly one predecessor.*

*Proof.* Since  $\mathcal{A}$  is an FANBW, there are only finitely many accepting  $\omega$ -branches in  $G_{w,\mathcal{A}}$ . Therefore, an accepting  $\omega$ -branch in  $G_{w,\mathcal{A}}$  only merges with other (accepting)  $\omega$ -branches for finitely many times. It follows that given an accepting  $\omega$ -branch  $\hat{\rho}$  in  $G_{w,\mathcal{A}}$ , there must exist a separate level  $h \geq 1$  such that each vertex  $\hat{\rho}[i]$  with  $i \geq h$  has exactly one predecessor. Otherwise, there will be infinitely many accepting branches, contradicting with the assumption that  $\mathcal{A}$  is an FANBW. Assume that there are  $m < \infty$  accepting  $\omega$ -branches in  $G_{w,\mathcal{A}}$ . Then we can set the separate level  $k$  of  $G_{w,\mathcal{A}}$  to  $\max\{h_i \mid 1 \leq i \leq m\}$  where  $h_i$  is the separate level index of  $i$ -th accepting  $\omega$ -branch.  $\square$

For instance, the separate level is 2 in the accepting DAG  $G_{w,\mathcal{A}}$  of  $\mathcal{A}$  over  $b^\omega$  in Figure 1, as each vertex  $\langle q_1, i \rangle$  with  $i \geq 3$  only has the predecessor  $\langle q_1, i - 1 \rangle$ .

It follows immediately from Lemma 1 that for each vertex  $v$  in  $G_{w,\mathcal{A}}$  with more than one incoming edges, keeping only one of incoming edges of  $v$  will not change whether  $G_{w,\mathcal{A}}$  is accepting. Thus we can modify  $G_{w,\mathcal{A}}$  to get an *edge-reduced* DAG  $G_{w,\mathcal{A}}^e = \langle V, E^e \rangle$  called co-deterministic DAG, in which each vertex only has at most one predecessor, by removing redundant edges. Assume that  $Q = \{q_1, q_2, \dots, q_n\}$ . For instance, if there is a vertex with multiple incoming edges in  $G_{w,\mathcal{A}}$ , we can only keep the incoming edge from the predecessor with the minimal index as follows.

- Edges. There is an edge from  $\langle q_k, l \rangle$  to  $\langle q', l' \rangle$  iff  $l' = l + 1$  and  $k = \min\{p \in \langle n \rangle \mid q' \in \delta(q_p, a_{l+1})\}$ .

Lemma 2 ensures that  $G_{w,\mathcal{A}}^e$  is accepting if  $G_{w,\mathcal{A}}$  is accepting.

**Lemma 2 (Acceptance of Co-deterministic DAGs).** *Assume that  $\mathcal{A}$  is an FANBW. Let  $G_{w,\mathcal{A}}^e$  be the co-deterministic DAG of  $\mathcal{A}$  over a word  $w \in \Sigma^\omega$ . Then  $w$  is accepted by  $\mathcal{A}$  if and only if  $G_{w,\mathcal{A}}^e$  is accepting.*

*Proof.* The proof is trivial when  $G_{w,\mathcal{A}}$  is nonaccepting. Assume that  $G_{w,\mathcal{A}}$  is accepting. Let  $\hat{\rho}$  be an accepting  $\omega$ -branch and  $k$  the separate level defined in Lemma 1. According to Lemma 1, the  $\omega$ -branch from  $\hat{\rho}[k + 1]$  must be accepting. Moreover,  $\hat{\rho}[k + 1]$  is reachable from an initial vertex  $\langle q, 0 \rangle$  with  $q \in I$ . Then there must exist an accepting  $\omega$ -branch in  $G_{w,\mathcal{A}}^e$  if  $G_{w,\mathcal{A}}$  is accepting. Thus we conclude that  $w$  is accepted by  $\mathcal{A}$  if and only if  $G_{w,\mathcal{A}}^e$  is accepting.  $\square$

For instance, the co-deterministic DAG of  $G_{w,\mathcal{A}}$  in Figure 1 is still accepting after deleting the edge from  $\langle q_2, 1 \rangle$  to  $\langle q_1, 2 \rangle$ , as denoted by the dashed arrow.

By removing redundant edges, we can now define a reduced transition function  $\delta^e : 2^Q \times \Sigma \rightarrow 2^Q$  over the levels in  $G_{w,\mathcal{A}}^e$ .

**Definition 2 (Transition Function for Co-deterministic DAGs).** Given a set of states  $S \subseteq Q$  at level  $l$  of  $G_{w,\mathcal{A}}^e$ . Let  $S' = \delta(S, a_l)$ . Define  $S_{min} = \{q_m \in S \mid m \in \min\{k \in \langle n \rangle \mid q' \in \delta(q_k, a_l)\}, q' \in S'\}$  as the minimal set of predecessors of  $S'$ . Then, for a set of states  $S_1 \subseteq S$ , we define  $\delta^e(S_1, a_l) = \delta(S_1 \cap S_{min}, a_l)$ . We call  $\delta^e$  the reduced transition function at level  $l$ .

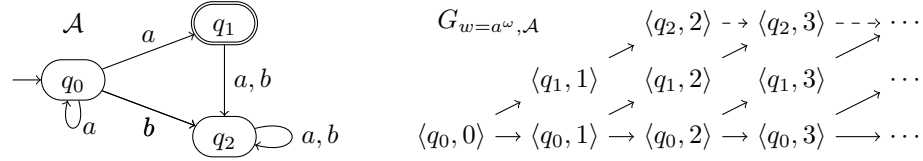
*Example 1.* Consider again  $G_{w,\mathcal{A}}$  in Figure 1 and let  $S = \{q_1, q_2\}$  at level 1: we have  $S' = \delta(S, b) = \{q_1\}$  and  $S_{min} = \{q_1\}$ . Let  $\delta^e$  be the reduced transition function at level 1 defined from  $\delta$  with respect to  $S$ . It follows that  $\delta^e(\{q_1\}, b) = \delta(\{q_1\} \cap S_{min}, b) = \{q_1\}$  and  $\delta^e(\{q_2\}, b) = \delta(\{q_2\} \cap S_{min}, b) = \emptyset$ .

We may write  $\delta^e(q, b)$  instead of  $\delta^e(\{q\}, b)$  for an input singleton set  $\{q\}$ . The transition function  $\delta_e$  will be used in the complementation of FANBW's since the complementation is basically to construct DAGs and then identify accepting DAGs.

One can verify that each vertex in the co-deterministic DAG  $G_{w,\mathcal{A}}^e$  of  $\mathcal{A}$  over  $w$  has at most one predecessor. It follows that the number of  $\omega$ -branches in a non-accepting/accepting  $G_{w,\mathcal{A}}^e$  is at most  $|Q|$ , as stated in Lemma 3.

**Lemma 3 (Finite Number of  $\omega$ -Branches in Co-deterministic DAGs).** Assume that  $\mathcal{A}$  is an FANBW and let  $G_{w,\mathcal{A}}^e$  be the co-deterministic DAG of  $\mathcal{A}$  over  $w$ . Then the number of  $\omega$ -branches in  $G_{w,\mathcal{A}}^e$  is at most  $|Q|$ .

*Proof.* Assume that  $m_i$  with  $i \geq 0$  is the number of vertices which are in the  $\omega$ -branches (not in all branches) on level  $i$ . For instance,  $m_i = 1$  for each  $i \geq 1$  in Fig. 1 while the number of vertices on level 1 is 2. Since each vertex in  $G_{w,\mathcal{A}}^e$  has only one predecessor, we have that  $m_0 \leq m_1 \leq m_2 \leq \dots$ , i.e., the number of vertices in  $\omega$ -branches on each level does not decrease over the levels. In addition, there are at most  $|Q|$  states on each level. Thus there are at most  $|Q|$   $\omega$ -branches since we have  $m_i \leq |Q|$  for each  $i \geq 0$ .  $\square$



**Fig. 2.** Another FANBW  $\mathcal{A}$  with  $I = \{q_0\}$  and  $F = \{q_1\}$  and the run DAG  $G_{w,\mathcal{A}}$  over  $a^\omega$

Consider the DAG  $G_{w,\mathcal{A}}$  in Figure 2: one can verify that there are infinitely many  $\omega$ -branches in the non-reduced DAG  $G_{w,\mathcal{A}}$  over  $a^\omega$ ; while for the co-deterministic DAG of  $G_{w,\mathcal{A}}$  where removed edges are marked with dashed arrows, there is only one  $\omega$ -branch  $\langle q_0, 0 \rangle \langle q_0, 1 \rangle \dots \langle q_0, l \rangle \dots$ .

After redundant edges have been cut off, only finite number of  $\omega$ -branches remain in  $G_{w,\mathcal{A}}^e$ . That is, we obtain a DAG with a finite degree of ambiguity in terms of the number of  $\omega$ -branches. The construction of co-deterministic DAGs with finite ambiguity is

the fundamental idea in this work for exploiting the power of unambiguity for Büchi complementation. By taking advantage of this finite ambiguity, we show in Lemma 4 that there exists a level  $l > 0$  in a nonaccepting co-deterministic DAG  $G_{w,\mathcal{A}}^e$  such that each  $F$ -vertex  $\langle q, l' \rangle$  with  $l' \geq l$  is finite, which can be used for identifying whether  $G_{w,\mathcal{A}}^e$  is accepting in the complementation of FANBW. We call such level  $l$  a *stable level*.

**Lemma 4 (Stable Level in Nonaccepting Co-deterministic DAGs).** *Assume that  $\mathcal{A}$  is an FANBW and  $w \notin \mathcal{L}(\mathcal{A})$ . Let  $G_{w,\mathcal{A}}^e$  be the co-deterministic DAG of  $\mathcal{A}$  over  $w$ . Then there must exist a stable level  $k > 0$  in  $G_{w,\mathcal{A}}^e$  such that each  $F$ -vertex on a level  $l \geq k$  of  $G_{w,\mathcal{A}}^e$  is a finite vertex.*

*Proof.* By Lemma 3, let  $m \leq |Q|$  be the number of  $\omega$ -branches in  $G_{w,\mathcal{A}}^e$ . Since  $w \notin \mathcal{L}(\mathcal{A})$ , all the  $\omega$ -branches in  $G_{w,\mathcal{A}}^e$  is nonaccepting. Therefore, for the  $i$ -th  $\omega$ -branch  $\hat{\rho}_i$ , there is a vertex  $\langle q, k_i \rangle$  such that every vertex of  $\hat{\rho}_i$  reachable from  $\langle q, k_i \rangle$  is not an  $F$ -vertex. It follows that we can set  $k = \max\{k_i \mid i \in \langle m \rangle\}$  and thus all the  $F$ -vertices on a level after  $l \geq k$  are finite and not on  $\omega$ -branches.  $\square$

Consider again the DAG  $G_{w,\mathcal{A}}$  in Figure 2: there does not exist a stable level in the non-reduced DAG  $G_{w,\mathcal{A}}$  since each  $F$ -vertex  $\langle q_1, l \rangle$  with  $l \geq 1$  is not finite; while in the co-deterministic DAG of  $\mathcal{A}$  over  $a^\omega$ , one can verify that the stable level  $k$  is 1.

## 4 Rank-Based Complementation

In this section, we first introduce in Section 4.1 the rank-based complementation (RKC) proposed in [16], which constructs a complementary NBW  $\mathcal{A}^c$  for  $\mathcal{A}$  with at most  $2^{\mathcal{O}(n \log n)}$  states. Then in Section 4.2, we show that if  $\mathcal{A}$  is an FANBW, RKC based on the construction of co-deterministic DAGs can produce a complementary NBW  $\mathcal{A}^c$  with at most  $2^{\mathcal{O}(n)}$  states.

### 4.1 Rank-Based Algorithm for NBWs

RKC was introduced by Kupferman and Vardi in [16] to construct a complementary NBW  $\mathcal{A}^c$  of  $\mathcal{A}$  by identifying the DAGs of  $\mathcal{A}$  over nonaccepting words  $w \notin \mathcal{L}(\mathcal{A})$ . Intuitively, given a word  $w \notin \mathcal{L}(\mathcal{A})$ , all  $\omega$ -branches of the DAG of  $\mathcal{A}$  over  $w$  will eventually stop visiting  $F$ -vertices. Based on this observation, in order to identify the nonaccepting DAG of  $\mathcal{A}$  over  $w$ , they introduced the notion of *level rankings* of  $G_{w,\mathcal{A}}$ . By assigning only even ranks to  $F$ -vertices, they showed that there exists a unique ranking function that assign ranks in  $[2n]$  to the vertices of  $G_{w,\mathcal{A}}$  such that  $w \notin \mathcal{L}(\mathcal{A})$  iff all  $\omega$ -branches of  $G_{w,\mathcal{A}}$  eventually get trapped in odd ranks. Intuitively, if  $w \in \mathcal{L}(\mathcal{A})$ , then there must exist some  $\omega$ -branch of  $G_{w,\mathcal{A}}$  that has infinitely many even ranks; if  $w \notin \mathcal{L}(\mathcal{A})$ , all  $\omega$ -branches in  $G_{w,\mathcal{A}}$  eventually get trapped in odd ranks.

We now define level rankings of a nonaccepting DAG. The level ranking of  $G_{w,\mathcal{A}} = (V, E)$  defines a ranking function  $f : V \rightarrow [2n]$  that satisfies the following conditions:

- (i) for each vertex  $\langle q, i \rangle \in V$  if  $f(\langle q, i \rangle) \in [2n]^{\text{odd}}$ , then  $q \notin F$ ,

(ii) for each edge  $(\langle q, i \rangle, \langle q', i+1 \rangle) \in E$ ,  $f(\langle q', i+1 \rangle) \leq f(\langle q, i \rangle)$

The ranks along a branch decrease monotonically and  $F$ -vertices get only even ranks.

We now define a specific ranking function  $f$  of  $G_{w,\mathcal{A}}$  for a given word  $w \notin \mathcal{L}(\mathcal{A})$ . We define a sequence of DAGs  $G_{w,\mathcal{A}}^0 \supseteq G_{w,\mathcal{A}}^1 \supseteq \dots$ , where  $G_{w,\mathcal{A}}^0 = G_{w,\mathcal{A}}$ , as follows. For each  $i \geq 0$ ,

- $G_{w,\mathcal{A}}^{2i+1}$  is the DAG constructed from  $G_{w,\mathcal{A}}^{2i}$  by removing all finite vertices in  $G_{w,\mathcal{A}}^{2i}$  and the edges associated with them, and
- if  $G_{w,\mathcal{A}}^{2i+1}$  has at least one  $F$ -free vertex, then  $G_{w,\mathcal{A}}^{2i+2}$  is the DAG constructed from  $G_{w,\mathcal{A}}^{2i+1}$  by removing all the  $F$ -free vertices in  $G_{w,\mathcal{A}}^{2i+1}$  and the edges associated with them.

Recall that  $F$ -free vertices cannot reach  $F$ -vertices. It was shown in [16] that  $G_{w,\mathcal{A}}^{2n+1}$  is empty and each vertex  $\langle q, l \rangle$  is either finite in  $G_{w,\mathcal{A}}^{2i}$  or  $F$ -free in  $G_{w,\mathcal{A}}^{2i+1}$ . Thus the sequence of DAGs generated from the definition above defines a ranking function  $f$  over the set of vertices in  $G_{w,\mathcal{A}}$  inductively as follows. For every  $i \geq 0$ ,

- (1)  $f(\langle q, l \rangle) = 2i$  for each vertex  $\langle q, l \rangle$  that is finite in  $G_{w,\mathcal{A}}^{2i}$ , if exists.
- (2)  $f(\langle q, l \rangle) = 2i + 1$  for each  $F$ -free vertex  $\langle q, l \rangle$  in  $G_{w,\mathcal{A}}^{2i+1}$ , if exists.

Consequently, we have Lemma 5 for identifying nonaccepting DAGs.

**Lemma 5 (Nonaccepting DAGs [16]).** *A rejects a word  $w$  iff the ranking function  $f$  defined in (1) and (2) above has  $2n$  as maximum rank, and all  $\omega$ -branches of  $G_{w,\mathcal{A}}$  eventually get trapped in odd ranks.*

We have constructed a unique ranking function above for identifying nonaccepting DAGs. To construct the complementary NBW  $\mathcal{A}^c$  with such a ranking function, we have to guess the ranking level by level. Since the maximum rank is  $2n$ , along an input word  $w$ , we can encode a ranking function for  $G_{w,\mathcal{A}}$  by utilizing a *level-ranking* function  $f : Q \rightarrow [2n] \cup \{\perp\}$  for the states  $S$  at a level in the DAG  $G_{w,\mathcal{A}}$  such that if  $q \in S \cap F$ , then  $f(q)$  is even, and  $f(q) = \perp$  if  $q \in Q \setminus S$ .

**Definition 3 (Coverage Relation for Level Rankings).** *Let  $a$  be a letter in  $\Sigma$  and  $f, f'$  be two level ranking functions. We say  $f$  covers  $f'$  under letter  $a$ , denoted by  $f' \leq_a^\delta f$ , when for all  $q \in Q$  and  $q' \in \delta(q, a)$ , if  $f(q) = \perp$ , then  $f'(q') = \perp$  and  $f'(q') \leq f(q)$ .*

Note here that  $\leq_a^\delta$  is defined based on the transition  $\delta$ . The coverage relation indicates that the level rankings  $f$  and  $f'$  of two consecutive levels of  $G_{w,\mathcal{A}}$  do not increase in ranks. We denote by  $\mathcal{R}$  the set of all possible level ranking functions.

In order to verify that the guess about the ranking of  $G_{w,\mathcal{A}}$  is correct, RKC uses the *breakpoint construction* proposed in [20]. This construction employs a set of states  $O \subseteq Q$  to check that the vertices assigned with even ranks are finite. Similarly to Lemma 4, the nonaccepting DAG  $G_{w,\mathcal{A}}$  with the ranking function defined in (1) and (2) eventually reaches a stable level, after which all  $F$ -vertices are finite. Hence, a breakpoint construction suffices to verify such guesses.

The formal definition of the complementary NBW  $\mathcal{A}^c$  of the input NBW  $\mathcal{A}$  is given in the following definition.



**Definition 4 ([16]).** Let  $\mathcal{A} = (Q, I, \delta, F)$  be an NBW. We then define an NBW  $\mathcal{A}^c = (Q^c, I^c, \delta^c, F^c)$  of  $\mathcal{A}$  as follows.

- $Q^c \subseteq \mathcal{R} \times 2^Q$ ,
- $I^c = (f, \emptyset)$  where  $f(q) = 2n$  if  $q \in I$  and  $f(q) = \perp$  otherwise.
- $\delta^c$  is defined as follows:
  1. if  $O \neq \emptyset$ , then  $\delta^c((f, O), a) = \{ (f', \delta(O, a) \setminus \text{odd}(f')) \mid f' \leq_a^\delta f \}$  (intuition: breakpoint  $O$  only tracks vertices assigned with even ranks),
  2. if  $O = \emptyset$ , then  $\delta^c((f, O), a) = \{ (f', \text{even}(f')) \mid f' \leq_a^\delta f \}$  (intuition:  $O = \emptyset$  means all previous  $F$ -vertices with even ranks are finite, then verify new vertices with even ranks).
- $F^c = \{ (f, O) \in Q^c \mid O = \emptyset \}$ .

where  $\text{odd}(f) = \{ q \in Q \mid f(q) \text{ is odd} \}$  and  $\text{even}(f) = \{ q \in Q \mid f(q) \text{ is even} \}$ .

Let  $w$  be a word. Intuitively, every state  $(f, O)$  in  $\mathcal{A}^c$  corresponds to a level of the DAG  $G_{w, \mathcal{A}}$  over  $w$ . If  $w$  is accepted by  $\mathcal{A}^c$ , i.e.,  $O$  becomes empty for infinitely many times, then we conclude that all the  $\omega$ -branches of  $G_{w, \mathcal{A}}$  eventually get trapped in odd ranks. It follows that no branches are accepting in  $G_{w, \mathcal{A}}$ , i.e.,  $w \notin \mathcal{L}(\mathcal{A})$ . The other direction is also easy to prove and omitted here. Thus we conclude that  $\mathcal{L}(\mathcal{A}^c) = \Sigma^\omega \setminus \mathcal{L}(\mathcal{A})$ . Since  $f \in \mathcal{R}$  is a function from  $Q$  to  $[2n] \cup \{\perp\}$ , the number of possible  $f$  functions is  $(2n+2)^n \in 2^{\mathcal{O}(n \log n)}$ . Therefore, the number of states in  $\mathcal{A}$  is in  $2^n \times 2^{\mathcal{O}(n \log n)} \in 2^{\mathcal{O}(n \log n)}$ .

**Lemma 6 (The Size and Language of  $\mathcal{A}^c$  [16]).** Let  $\mathcal{A}$  be an NBW with  $n$  states and  $\mathcal{A}^c$  the NBW defined in Definition 4. Then  $\mathcal{L}(\mathcal{A}^c) = \Sigma^\omega \setminus \mathcal{L}(\mathcal{A})$  and  $\mathcal{A}^c$  has at most  $2^{\mathcal{O}(n \log n)}$  states.

*Relation to Construction of Co-deterministic DAGs.* Assume that we have two level-rankings  $f' \leq_a^\delta f$ . A state  $q'$  in the second level can have multiple  $a$ -predecessors defined in the domain of  $f$ . Then  $f'(q') \leq \min\{ f(q) \mid f(q) \neq \perp, q' \in \delta(q, a) \}$ . Thus we can define a co-deterministic DAG out of  $G_{w, \mathcal{A}}$  where each vertex only keeps one predecessor with the minimal rank in the reduced DAG, in contrast to the predecessor with minimal index in Section 3. There may, however, be multiple predecessors with the minimal rank. Consequently, the non-reduced DAG  $G_{w, \mathcal{A}}$  can be mapped to multiple co-deterministic DAGs depending on which ranking function is defined on  $G_{w, \mathcal{A}}$  and how predecessors are chosen. Note here that not every resulting co-deterministic DAG of  $G_{w, \mathcal{A}}$  described above will be accepting if  $G_{w, \mathcal{A}}$  is accepting, since each time the edges in accepting  $\omega$ -branches may be deleted. Thus these co-deterministic DAGs cannot be directly applied in RKC for general NBWs.

## 4.2 Rank-Based Algorithm for FANBW

In the following, we show in Lemma 7 that if  $\mathcal{A}$  is an FANBW, the maximum rank of the vertices in a co-deterministic DAG of  $\mathcal{A}$  is at most 2. It follows that the range of  $f \in \mathcal{R}$  is  $\{0, 1, 2\} \cup \{\perp\}$ . We thus only need the maximum rank to be 2 rather than  $2n$  for the co-deterministic DAG  $G_{w, \mathcal{A}}^e$  of  $\mathcal{A}$ . Therefore, the number of states in  $\mathcal{A}^c$  is in  $2^n \times 4^n \in 2^{\mathcal{O}(n)}$  when the maximum rank is 2.

**Lemma 7 (Maximum Rank of Co-deterministic DAGs for FANBW).** *Assume that  $\mathcal{A}$  is an FANBW and let  $w$  be a word. Let  $G_{w,\mathcal{A}}^e$  be the co-deterministic DAG of  $\mathcal{A}$  over  $w$ . Then  $w \notin \mathcal{L}(\mathcal{A})$  iff  $(G_{w,\mathcal{A}}^e)^3$  is empty.*

*Proof.* Assume that  $w \notin \mathcal{L}(\mathcal{A})$ . Our goal is to prove that starting from  $(G_{w,\mathcal{A}}^e)^0 = G_{w,\mathcal{A}}^e$ ,  $(G_{w,\mathcal{A}}^e)^3$  is empty. By Lemma 4, there exists a stable level, say  $k > 1$ , such that on each level  $l \geq k$ , the  $F$ -vertices are finite. Therefore,  $(G_{w,\mathcal{A}}^e)^1$  contains only non- $F$ -vertices after level  $k$ . It follows that  $(G_{w,\mathcal{A}}^e)^2$  removes all the vertices after level  $k$ . Thus if  $(G_{w,\mathcal{A}}^e)^2$  is not empty,  $(G_{w,\mathcal{A}}^e)^2$  contains only finite vertices. We then conclude that  $(G_{w,\mathcal{A}}^e)^3$  is empty. The other direction is trivial.  $\square$

In order to set the maximum rank to 2 in Definition 4, the underlying DAG  $G_{w,\mathcal{A}}$  constructed for complementing FANBW has to be co-deterministic. Since RKC generates rankings level by level, we have to utilize  $\delta^e$  in Definition 2 for computing successors at next level. For FANBW, the complementation construction in Definition 4 can be improved accordingly:

**Definition 4'** *Let  $\mathcal{A} = (Q, I, \delta, F)$  be an FANBW. We then define an NBW  $\mathcal{A}^c = (Q^c, I^c, \delta^c, F^c)$  where  $Q^c$  and  $F^c$  are as in Definition 4, and  $I^c$  and  $\delta^c$  are defined by:*

- $I^c = (f, \emptyset)$  where  $f(q) = 2$  if  $q \in I$  and  $f(q) = \perp$  otherwise.
- $\delta^c$  is then defined as follows:
  1. if  $O \neq \emptyset$ , then  $\delta^c((f, O), a) = \{ (f', \delta^e(O, a) \setminus \text{odd}(f')) \mid f' \leq_a^{\delta^e} f \}$ ,
  2. if  $O = \emptyset$ , then  $\delta^c((f, O), a) = \{ (f', \text{even}(f')) \mid f' \leq_a^{\delta^e} f \}$ .
 where  $\delta^e$  is the reduced transition function at the level corresponding to current state  $(f, O)$ .

Recall that the coverage relation between two level ranking functions  $f$  and  $f'$ , parameterized with  $\delta^e$ , is defined in Definition 3. Similarly to Definition 2, to compute  $\delta^e(S_1, a)$ , one has to first compute the minimal set  $S_{\min}$  of predecessors of  $S' = \delta(S, a)$  where  $S$  is the domain of  $f$ , i.e., the set of states at current level. Thus we have  $\delta^e(S_1, a) = \delta(S_1 \cap S_{\min}, a)$ . Intuitively, for  $w \in \Sigma^\omega$ ,  $\delta^e$  is used to construct a co-deterministic DAG  $G_{w,\mathcal{A}}^e$  over  $w$  level by level. By Lemma 7, the maximum rank of  $G_{w,\mathcal{A}}^e$  is at most 2, which is sufficient in Definition 4' for constructing a ranking function to identify whether  $G_{w,\mathcal{A}}^e$  is accepting. Therefore, with Definition 4', we can construct a complementary NBW  $\mathcal{A}^c$  with  $2^{\mathcal{O}(n)}$  states, as stated in Theorem 1.

**Theorem 1 (The Size and Language of  $\mathcal{A}^c$  for FANBW).** *Let  $\mathcal{A}$  be an FANBW with  $n$  states and  $\mathcal{A}^c$  the NBW defined in Definition 4'. Then (1)  $\mathcal{L}(\mathcal{A}^c) = \Sigma^\omega \setminus \mathcal{L}(\mathcal{A})$ ; and (2)  $\mathcal{A}^c$  has at most  $2^{\mathcal{O}(n)}$  states.*

*Proof.* The proof for claim (2) is trivial and thus omitted here. By Lemma 2 and definition of ranking functions, co-deterministic DAGs of  $\mathcal{A}$  over  $w \in \mathcal{L}(\mathcal{A})$  will be rejected in  $\mathcal{A}^c$ , thus  $\mathcal{L}(\mathcal{A}^c) \subseteq \Sigma^\omega \setminus \mathcal{L}(\mathcal{A})$ . According to the proof of Lemma 7, there exists a unique ranking function for each rejecting co-deterministic DAG  $G_{w,\mathcal{A}}^e$  of  $\mathcal{A}$  over  $w \notin \mathcal{L}(\mathcal{A})$ . Since RKC nondeterministically guesses rankings of  $G_{w,\mathcal{A}}^e$ , there must be a guess of such unique ranking function. It follows that  $G_{w,\mathcal{A}}^e$  must be accepting in  $\mathcal{A}^c$ , i.e.,  $\Sigma^\omega \setminus \mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A}^c)$ . Thus it holds that  $\mathcal{L}(\mathcal{A}^c) = \Sigma^\omega \setminus \mathcal{L}(\mathcal{A})$ .  $\square$

In [13], Fogarty and Vardi proved that complementing reverse deterministic BWs with RKC is doable in  $2^{\mathcal{O}(n)}$  as the non-reduced DAGs  $G_{w,\mathcal{A}}$  are already co-deterministic. This is because that if  $\mathcal{A}$  is reverse deterministic, then each vertex  $\langle q, l \rangle$  in  $G_{w,\mathcal{A}}$  has at most one predecessor, as  $q$  has only one  $w[l]$ -predecessor. It follows that  $G_{w,\mathcal{A}}$  is co-deterministic. Similarly to Lemma 3, the number of (accepting)  $\omega$ -branches in  $G_{w,\mathcal{A}}$  is at most  $|Q|$ . According to Definition 1, reverse deterministic BWs are a special class of FANBW, as stated in Corollary 1.

**Corollary 1.** *Let  $\mathcal{A}$  be a reverse deterministic BW. Then  $\mathcal{A}$  is also an FANBW.*

In contrast, note that an FANBW is not necessarily a reverse deterministic BW. For instance, the FANBW  $\mathcal{A}$  of Figure 1 is not reverse deterministic since  $q_1$  has three  $b$ -predecessors, namely  $q_0$ ,  $q_1$  and  $q_2$ . We remark that the construction in [13] just sets the maximum rank to 2 in Definition 4 without modifying the transition function  $\delta^c$ , which turns out to be a special case of our construction according to Corollary 1.

## 5 Slice-Based Algorithm

In Section 5.1, we first recall the *slice-based* complementation construction (SLC) described in [14, 26], adapted using our notations, which produces a complementary NBW  $\mathcal{A}^c$  of  $\mathcal{A}$  with at most  $\mathcal{O}((3n)^n)$  states. Then, in Section 5.2, we show that for FANBW, this construction can be simplified while yielding a complementary NBW with at most  $\mathcal{O}(4^n)$  states.

### 5.1 Slice-Based Algorithm for NBWs

Let  $\mathcal{A}$  be an NBW, and let  $w$  be a word. SLC uses a data structure called *slice* instead of level rankings to encode the set of vertices at the same level in  $G_{w,\mathcal{A}}$ . A slice in [26] is defined as an ordered sequence of disjoint sets of vertices at the same level.

We now describe SLC from the perspective of building co-deterministic DAGs. SLC does the following to construct a co-deterministic DAG  $G_{w,\mathcal{A}}^s$  as it proceeds along the word  $w$ . Here the superscript  $s$  for SLC is used to distinguish the construction of co-deterministic DAGs  $G_{w,\mathcal{A}}^e$  in Section 3. At level 0, we may obtain at most two vertices of  $G_{w,\mathcal{A}}^s$ : a vertex  $\langle S_1, 0 \rangle = \langle I \setminus F, 0 \rangle$  and an  $F$ -vertex  $\langle S_2, 0 \rangle = \langle I \cap F, 0 \rangle$ . Recall that  $I$  and  $F$  are the set of initial states and the set of accepting states of  $\mathcal{A}$ , respectively. Here  $S_1$  and  $S_2$  are disjoint. A vertex  $\langle S_j, i \rangle$  is an  $F$ -vertex if  $S_j \subseteq F$ , where  $j \geq 1$  and  $i \geq 0$ . The vertices  $\langle S_j, i \rangle$  on level  $i$  in  $G_{w,\mathcal{A}}^s$  are ordered from left to right by their indices  $j$  where  $i \geq 0$  and  $1 \leq j \leq n$ . During the construction, empty sets  $S_j$  are removed and the indices of remaining sets are reset according to the increasing order of their original indices.

Assume that on level  $i$ , the sequence of vertices in  $G_{w,\mathcal{A}}^s$  is  $\langle S_1, i \rangle, \dots, \langle S_{k_i}, i \rangle$  where  $i \geq 0$  and  $1 \leq k_i \leq n$ . We now describe how SLC constructs the vertices on level  $i + 1$ . First, for a set  $S_j$  where  $1 \leq j \leq k_i$ , on reading the letter  $w[i]$ , the set of successors of  $S_j$  is partitioned into (1) a non- $F$  set  $S'_{2j-1} = \delta(S_j, w[i]) \setminus F$ , and (2) an  $F$ -set  $S'_{2j} = \delta(S_j, w[i]) \cap F$ , as a possible new  $F$ -vertex.

This gives us a sequence of sets  $S'_1, S'_2, \dots, S'_{2k_i-1}, S'_{2k_i}$ . Note that there can be some states in  $\mathcal{A}$  present in multiple sets  $S'_j$  where  $j \geq 1$ . Here we only keep the rightmost occurrence of a state. Intuitively, different runs of  $\mathcal{A}$  may merge with each other at some level and we only need to keep the right most one and cut off others, as they share the same infinite suffix. This operation does not change whether the co-deterministic DAG  $G_{w,\mathcal{A}}^s$  is accepting, since at least one accepting run of  $\mathcal{A}$  remains and will not be cut off. Formally, for each set  $S'_j$  where  $1 \leq j \leq 2k_i$ , we define a set  $S''_j = S'_j \setminus \bigcup_{j < p \leq 2k_i} S'_p$ . This yields a sequence of disjoint sets  $S''_1, S''_2, \dots, S''_{2k_i-1}, S''_{2k_i}$ . After removing the empty sets in this sequence and reassigning the index of each set according to their positions, we finally obtain the sequence of sets of vertices on level  $i + 1$ , denoted by  $\langle S_1, l + 1 \rangle, \dots, \langle S_{k_{i+1}}, l + 1 \rangle$ . Obviously, the resulting sets at the same level are again pairwise disjoint.

Therefore, we define a co-deterministic DAG  $G_{w,\mathcal{A}}^s = (V, E)$  of  $\mathcal{A}$  over  $w$  for an NBW  $\mathcal{A}$  as follows:

- Vertices.  $V = \bigcup_{l \geq 0, 1 \leq j \leq k_i} \{\langle S_j, l \rangle\}$ .
- Edges. There is an edge from  $\langle S_j, l \rangle$  to  $\langle S_h, l + 1 \rangle$  iff  $S_h$  is either  $S''_{2j-1}$  or  $S''_{2j}$  as defined above where  $1 \leq j \leq k_i$  and  $1 \leq h \leq k_{i+1}$ .

By the definition of  $G_{w,\mathcal{A}}^s$ , each vertex  $\langle S_h, l + 1 \rangle$  in which  $S_h$  is either  $S''_{2j-1}$  or  $S''_{2j}$  computed from  $S_j$  has at most one predecessor  $\langle S_j, l \rangle$ . Thus  $G_{w,\mathcal{A}}^s$  is co-deterministic. It follows that number of  $\omega$ -branches in  $G_{w,\mathcal{A}}^s$  is at most  $n$  and  $w \in \mathcal{L}(\mathcal{A})$  if and only if there is an accepting  $\omega$ -branch in  $G_{w,\mathcal{A}}^s$ . Formally:

**Lemma 8 (Finite Ambiguity and Stable Levels [26]).** *Let  $w \in \Sigma^\omega$  and  $G_{w,\mathcal{A}}^s$  be the co-deterministic DAG as defined above. Then (1) the number of (accepting)  $\omega$ -branches in  $G_{w,\mathcal{A}}^s$  is at most the number of states in  $\mathcal{A}$ . (2)  $w$  is accepted by  $\mathcal{A}$  iff  $G_{w,\mathcal{A}}^s$  is accepting. (3) There exists a stable level  $l \geq 1$  in  $G_{w,\mathcal{A}}^s$  such that all  $F$ -vertices after level  $l$  are finite if  $w \notin \mathcal{L}(\mathcal{A})$ .*

SLC for general NBWs can be viewed as consisting of two components: (1) based on the construction of co-deterministic DAGs  $G_{w,\mathcal{A}}^s$  over  $w$  above, NBWs can be translated to FANBW [19] and (2) a specialized complementation algorithm for FANBW. In [26], SLC utilizes these two components at the same time for computing the complementary NBW  $\mathcal{A}^c$ .

A state of  $\mathcal{A}^c$  is an ordered sequence of tuples  $(S_1, l_1), \dots, (S_h, l_h)$  where ordered sequence  $(S_1, \dots, S_h)$  is a slice, and each vertex  $\langle S_j, l \rangle$  is decorated with a label  $l_j \in \{\text{die}, \text{inf}, \text{new}\}$ . The level index  $l$  is omitted during the construction of  $\mathcal{A}^c$ . Intuitively,

- **die**-labelled vertex means that those states in  $S_j$  are currently being inspected. For  $w$  to be accepted (i.e.,  $w \notin \mathcal{L}(\mathcal{A})$ ), **die**-labelled vertices should eventually reach empty set after a finitely many steps, thus become finite. Recall that empty sets will be removed in the construction of  $G_{w,\mathcal{A}}^s$ .
- **inf**-labelled vertex indicates all states never reach accepting states.
- **new**-labelled vertex records new encountered states, that should be inspected later once **die**-labelled vertex becomes empty.

Obviously, here  $h$  is at most the number  $n$  of states in  $\mathcal{A}$ . While for FANBW, thanks to their finite ambiguity, the construction for co-deterministic DAGs can be simplified (see Section 3): we can even use three components  $(N, C, B)$  to compactly encode the slice and their labels. We postpone the details of the construction to the next subsection. Now we recall the complexity of the above slice based construction:

**Lemma 9 (The Size and Language of  $\mathcal{A}^c$  for NBWs [26] ).** *Let  $\mathcal{A}$  be an NBW with  $n$  states and  $\mathcal{A}^c$  the NBW constructed by SLC in Section 5. Then  $\mathcal{L}(\mathcal{A}^c) = \Sigma^\omega \setminus \mathcal{L}(\mathcal{A})$  and  $\mathcal{A}^c$  has at most  $\mathcal{O}((3n)^n)$  states.*

## 5.2 Slice-Based Algorithm for FANBW

In this section, we introduce the specialized complementation construction for FANBW. Recall that, as discussed in Section 5.1, this construction is also the second component of SLC, used for complementing general NBWs.

We first provide some intuitions. According to Lemma 4, given a word  $w \notin \mathcal{L}(\mathcal{A})$ , there exists a stable level  $k$  in the co-deterministic DAG  $G_{w,\mathcal{A}}^e$  such that each  $F$ -vertex on a level after  $k$  is finite. Therefore, in the construction of  $\mathcal{A}^c$ , we can nondeterministically guess level  $k$  and then use breakpoint construction to verify that our guess is correct, in analogy with RKC. More precisely, when constructing the complementary NBW  $\mathcal{A}^c$ , there are the *initial phase* and the *accepting phase*. The initial phase is purely a subset construction to trace the reachable states of each level of the co-deterministic DAG  $G_{w,\mathcal{A}}^e$  over  $w$ . On reading a letter at a state of  $\mathcal{A}^c$  (called *macrostate*) in the initial phase, the run of  $\mathcal{A}^c$  over  $w$  (called *macrorun*) either continues to stay in the initial phase or jumps to the accepting phase. Once entering the accepting phase, we guess that the macrorun of  $\mathcal{A}^c$ , which consists of multiple runs of  $\mathcal{A}$ , has reached the stable level  $k$ . Thus in the accepting phase, we need a breakpoint construction to verify that the guess is correct, i.e., that all  $F$ -vertices after level  $k$  are finite.

In the accepting phase, we use a macrostate, represented as a triple  $(N, C, B)$ , to encode the set of vertices and their labels on a level after  $k$  in the co-deterministic DAG  $G_{w,\mathcal{A}}^e$  (or  $G_{w,\mathcal{A}}^s$  for general NBWs accordingly), where

- the set  $N$  keeps all the reachable vertices on the level, corresponding to the set of all vertices labelled with **die**, **inf** and **new**;
- the set  $C$  keeps all the finite vertices on the level. That means, it contains both **new**-labelled vertices recording new encountered states, and **die**-labelled vertices being inspected now.
- the set  $B \subseteq C$  as a breakpoint construction is used to verify that the guess on the set  $C$  of finite vertices is correct, corresponding to the set of vertices labelled with **die**.

Recall that **die**, **inf** and **new** are three labels of vertices used in SLC for complementing general NBWs, as described in Section 5.1. The specialized complementation algorithm for FANBW is formalized in Definition 5.

**Definition 5.** *Let  $\mathcal{A} = (Q, I, \delta, F)$  be an FANBW. We then define an NBW  $\mathcal{A}^c = (Q^c, I^c, \delta^c, F^c)$  as follows.*

- $Q^c \subseteq 2^Q \cup 2^Q \times 2^Q \times 2^Q$ ;
- $I^c = \{I\}$ ;
- $\delta^c = \delta_1^c \cup \delta_t^c \cup \delta_2^c$  is defined as follows:
  1.  $\delta_1^c(S, a) = \delta^e(S, a)$  for  $S \subseteq Q$  and  $a \in \Sigma$  where  $\delta^e$  is the reduced transition function at current level whose corresponding set of states is  $S$  (intuition: subset construction to organize the macrorun before the guess point).
  2.  $\delta_t^c(S, a) = \delta_2^c(N, C, B)$  where  $N = S, B = S \cap F$  and  $C = B$  (intuition: make the guess point to be the macrostate  $(N, C, B)$ ).
  3.  $\delta_2^c((N, C, B), a) = (N', C', B')$  where  $\delta^e$  is the reduced transition function at current level whose corresponding set of states is  $N$ ,
    - $N' = \delta^e(N, a)$  (intuition: tracing the reachable states correctly),
    - $C' = \delta^e(C, a) \cup (N' \cap F)$  (intuition: tracing the runs which has visited accepting states after the guess point), and
    - if  $B \neq \emptyset$ , then  $B' = \delta^e(B, a)$  and otherwise  $B' = C'$  (intuition:  $B = \emptyset$  means all runs which have visited accepting states are finite and  $B \neq \emptyset$  indicates that previous runs are still under inspection).
- $F^c = \{(N, C, B) \in Q^c \mid B = \emptyset\}$ .

*Remark 1.* As a side remark, we note that the complementary NBW constructed by Definition 5 is *limit deterministic*, as the state set  $Q^c$  of  $\mathcal{A}^c$  can be partitioned into two disjoint sets  $Q_N^c \subseteq 2^Q$  and  $Q_D^c \subseteq 2^Q \times 2^Q \times 2^Q$  such that 1)  $F^c \subseteq Q_D^c$  and 2) for each state  $q \in Q_D^c$  and  $a \in \Sigma$ , we have that  $|\delta^c(q, a)| \leq 1$ .

The complementary NBW  $\mathcal{A}^c$  constructed by Definition 5 for the example FANBW  $\mathcal{A}$  of Figure 1 can be viewed in Appendix A.

**Theorem 2 (The Size and Language of  $\mathcal{A}^c$  for FANBW).** *Let  $\mathcal{A}$  be an FANBW with  $n$  states and  $\mathcal{A}^c$  be the NBW defined by Definition 5. Then (1)  $\mathcal{L}(\mathcal{A}^c) = \Sigma^\omega \setminus \mathcal{L}(\mathcal{A})$ ; and (2)  $\mathcal{A}^c$  has at most  $2^n + 4^n$  states.*

*Proof.* We prove claim (1) as follows. Suppose  $w \in \mathcal{L}(\mathcal{A})$ , our goal is to prove  $w$  is not accepted by  $\mathcal{A}^c$ . Assume that the corresponding accepting run of  $\mathcal{A}$  over  $w$  is  $\rho$  and  $\rho'$  is a macrorun of  $\mathcal{A}^c$  over  $w$ . Then for the macrorun  $\rho'$ : (1) if  $\rho'$  only visits states of the form  $s \in 2^Q$ , then  $\rho'$  is not accepted by  $\mathcal{A}^c$  since no accepting  $\mathcal{A}^c$ -states will be visited; (2) if  $\rho'$  is a macrorun of the form  $s_0, \dots, s_{k-1}, (N_k, C_k, B_k)(N_{k+1}, C_{k+1}, B_{k+1}) \dots$ ,  $\rho$  will visit some accepting state, say  $q_f \in F$  infinitely often. Then at some point, say in state  $(N_j, C_j, B_j)$ , we have  $q_f \in B_j$  or  $q_f \in C_j$ . If  $q_f \in B_j$ , then for every  $p \geq j$ , we have  $B_p \neq \emptyset$  according to Lemma 1; otherwise  $q_f \in C_j$ , then either at some point, say  $p > j$ ,  $q_f$  will be moved to  $B_p$  when  $B_{p-1} = \emptyset$ , or  $q_f \in C_p$  for each  $p \geq j$ , which indicates that  $B_p \neq \emptyset$  for  $p \geq j$ . Therefore,  $w$  is not accepted by  $\mathcal{A}^c$ .

Assume that  $w \notin \mathcal{L}(\mathcal{A})$ , our goal is to prove that there exists an accepting macrorun  $\rho'$  of  $\mathcal{A}^c$  over  $w$ . The proof idea is to analyze the co-deterministic DAG  $G_{w, \mathcal{A}}^e$  of  $\mathcal{A}$  over  $w$ . According to Lemma 4, there exists some number  $k \geq 0$  such that every  $F$ -vertex on a level after  $k$  of  $G_w$  is finite. Therefore, the set  $B$  on  $\rho'$  will become empty infinitely often, i.e.,  $w$  is accepted by  $\mathcal{A}^c$ .

We now prove claim (2). According to Definition 5, it is easy to see that the number of possible states of the form  $s \in 2^Q$  is  $2^n$ . For each state  $p = (N, C, B) \in Q^c$  of  $\mathcal{A}^c$ ,

we have that  $C \subseteq N$  and  $B \subseteq C$ . Then for a state  $q \in Q$ : (i) it will either be absent or present in  $N$ ; (ii) for a state  $q \in N$ , one of the following three possibilities holds:  $q$  is only in  $N$ ,  $q$  is both in  $C$  and  $N$  and  $q$  is both in  $B$  and  $C$ . Therefore  $\mathcal{A}^c$  has at most  $2^n + 4^n$  states.  $\square$

As a consequence of Definition 5, we can define a simulation relation between states of  $\mathcal{A}^c$  below.

**Corollary 2 (Simulation Relation between States).** *Let  $\mathcal{A}$  be an FANBW and  $\mathcal{A}^c$  the complement NBW of  $\mathcal{A}$  defined by Definition 5, and  $m_1 = (N_1, C_1, B_1)$  and  $m_2 = (N_2, C_2, B_2)$  are two states of  $\mathcal{A}^c$  such that  $N_1 \subseteq N_2$ . Then  $\mathcal{L}((\mathcal{A}^c)^{m_2}) \subseteq \mathcal{L}((\mathcal{A}^c)^{m_1})$  or  $m_1$  simulates  $m_2$ .*

The proof can be found in Appendix B. Corollary 2 provides the possibility to avoid the exploration of  $m_2$  when  $\mathcal{L}((\mathcal{A}^c)^{m_1})$  has already been found to be empty, when checking the language-containment between an NBW and an FANBW  $\mathcal{A}$ .

## 6 Conclusion and Future Work

This work proposes utilizing co-deterministic DAGs over infinite words to take advantage of the unambiguity in FANBW in Büchi complementation. We have improved the complexity of the classical rank-based and slice-based complementation constructions for FANBW, respectively, to  $2^{\mathcal{O}(n)}$  from  $2^{\mathcal{O}(n \log n)}$  and to  $\mathcal{O}(4^n)$  from  $\mathcal{O}((3n)^n)$ , based on co-deterministic DAGs. To the best of our knowledge, our improved complexity for complementing FANBW is exponentially better than best known result of  $\mathcal{O}(5^n)$  in [21]. As a further contribution, we view the SLC algorithm explicitly as the construction of co-deterministic DAGs and a specialized complementation algorithm for FANBW. We then provide a simulation relation between states in the complementary NBWs of FANBW in hope of improving the containment checking between an NBW and an (FA)NBW.

For future work, we plan to study whether  $\mathcal{O}(4^n)$  is also the lower bound for complementing FANBW. We will also explore a Ramsey-based complementation construction based on co-deterministic DAGs. Another line of future work is studying determinization constructions for FANBW.

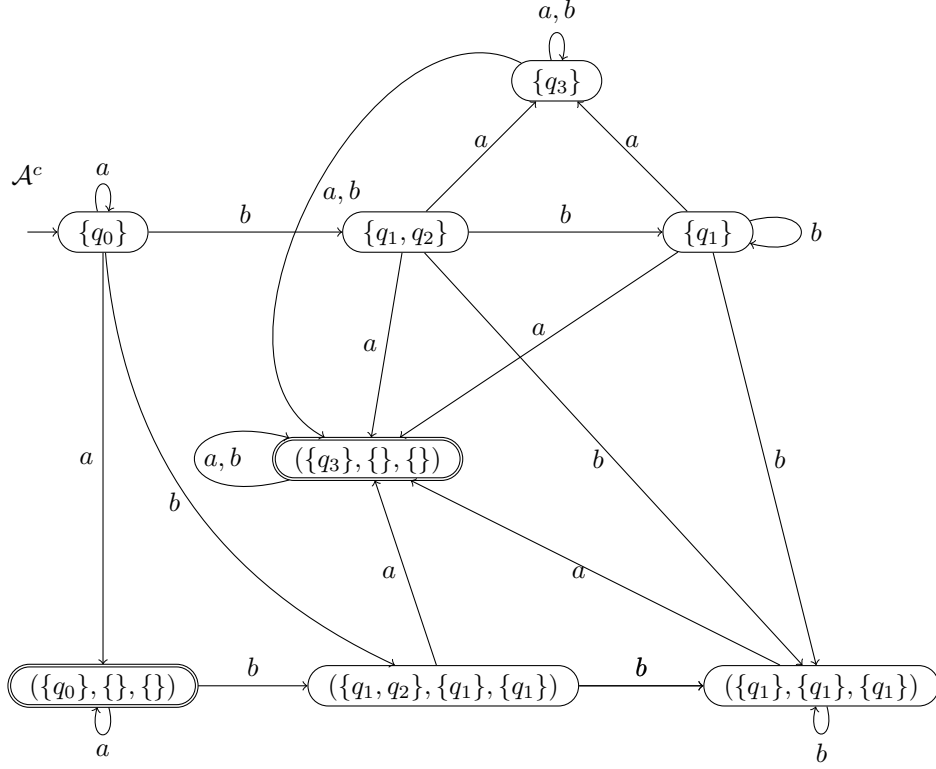
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**A The Complementary BW produced for  $\mathcal{A}$  in Figure 1 by Definition 5**



**B Proof of Corollary 2**

**Corollary 2 (Simulation Relation between States).** *Let  $\mathcal{A}$  be an FANBW and  $\mathcal{A}^c$  the complement NBW of  $\mathcal{A}$  defined by Definition 5, and  $m_1 = (N_1, C_1, B_1)$  and  $m_2 = (N_2, C_2, B_2)$  are two states of  $\mathcal{A}^c$  such that  $N_1 \subseteq N_2$ . Then  $\mathcal{L}((\mathcal{A}^c)^{m_2}) \subseteq \mathcal{L}((\mathcal{A}^c)^{m_1})$  or  $m_1$  simulates  $m_2$ .*

*Proof.* First, we have  $B \subseteq C \subseteq N$ . Assume that  $w \in \Sigma^\omega$ . Since  $N_1 \subseteq N_2$ , in the reduced DAG  $G_{w, \mathcal{A}}$  of  $\mathcal{A}$  over  $w$ , the set of  $\omega$ -branches that visit the vertices in  $N_2$  must contain all  $\omega$ -branches visiting vertices in  $N_1$ . Therefore, if  $w \in \mathcal{L}(\mathcal{A}^c)$ , the macrorun starting from  $m_2$  will be accepted by  $\mathcal{A}^c$ . Consequently, all  $F$ -vertices on the  $\omega$ -branches that visit  $N_2$  are finite. It follows that the macrorun starting from  $m_2$  will be also accepted by  $\mathcal{A}^c$ . Thus  $\mathcal{L}((\mathcal{A}^c)^{m_2}) \subseteq \mathcal{L}((\mathcal{A}^c)^{m_1})$ .  $\square$