A GENERALIZATION OF GINSBURG AND ROSE'S CHARACTERIZATION OF G-S-M MAPPINGS

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Abstract: *We generalize Ginsburg and Rose's characterization of g-s-m mappings to the broader family of so-called subsequential functions, introduced by M.P.Schützenberger*

INTRODUCTION

Let A^* (resp. B^*) be the free monoid generated by the finite non empty set A (resp. B).. For each integer $n \geqslant 0$ let A_n (resp. B_n) be the set of all words of length not greater than n. The empty word shall be denoted by 1.

Consider a partial function $f : A^* \longrightarrow B^*$. We shall set $xf = \emptyset$ whenever xf is undefined, where $\emptyset$ is the zero of the $\mathbb{Z}$ - algebra of B^*. Let $\equiv$ be the right congruence over A^* defined by: $u \equiv v$ iff there exist a partial function $h : A^* \longrightarrow B^*$ and two words $x, y \in B^*$ such that the following holds for each $w \in A^*$: $uwf = x\,(wh)$ and $vwf = y\,(wh)$. With Schützenberger we say that f is *subsequential* iff the right congruence $\equiv$ has finite index (cf. [Sch]).

The main result of this paper is the following characterization of the partial functions which are subsequential:

Main Theorem. *Let* $f : A^* \longrightarrow B^*$ *be a partial function. Then itis subsequential iff it satisfies the two following conditions:*

i) There exists an integer $k > 0$ *such that for all* $u \in A^*$ *and all integer* $n \geqslant 0$ *there exists* $x \in B^*$ *verifying:* $uA_n f \subseteq xB_{k.n}$.

ii) For each rational subset $L \subseteq B^*$, Lf^{-1} *is a rational subset of* A^*.

When f is a function (not just a partial function) and when x equals uf , we obtain - apart from the irrelevant condition $1\,f = 1$ - Ginsburg and Rose's characterization of g-s-m mappings (cf. [Gi Ro]). In order to precise which partial functions are concerned with the theorem, let us recall that a partial function $h : A^* \longrightarrow B^*$ is *rational* iff its *graph* $h_\# = \{(u , v) \in A^* \times B^* \mid v = uh\}$ is a rational subset of the monoid $A^* \times B^*$. Then it is not hard to show that *a partial function* $f : A^* \longrightarrow B^*$ *is subsequential iff there exist a g-s-m mapping* $g : A^* \longrightarrow B^*$ *and a rational partial function of finite image* $h : A^* \longrightarrow B^*$ *such that the following holds for all* $u \in A^*$: $uf = uguh$. (cf. [Ch1] , Proposition VI,1.2.). An example of subsequential partial function is given in section III, §1.

As a consequence of this last remark, every subsequential partial function is rational (cf. [Ei] , Proposition XI,3.1. and Theorem IX,4.1.). The family of subsequential partial functions possess remarkable properties. To quote only two of them: 1) the composition of two subsequential partial functions is subsequential - obvious from the theorem - and 2) given any rational subset of $A^* \times B^*$, it is decidable whether it is the graph of a subsequential partial function (cf.[Ch2]). The notion of subsequential partial function thus provides an interesting class of rational partial functions and it is very likely that its study will benefit the theory of rational partial functions as a whole.

This paper is divided into three sections. Section 1 dispenses some basic notions, slightly altered so that they suit our purpose. For example our sequential transducers are obtained from Ginsburg's generalized sequential machines by introducing final states (a quite natural generalization, now widely accepted, see e.g. [Sa]) and by leaving the possibility for the output function to be into any free monoid but also into any free group. In doing this we can work in more pleasant structures. These machines "realize" in the usual way, sequential partial functions of a free monoid into a free monoid or a free group, which are considered in section2. We verify that these partial functions are rational and we give a characterization of all partial functions of a free monoid into a free group which are sequential. This last result is formal but it makes some verifications easier, as illustrated in section 3.

Subsequential partial functions of a free monoid into another were introduced two years ago in a paper of M.P.Schützenberger (cf. [Sch]). A systematic approach of the problem can be found in [Ch2] Chap.III and VI. In section 3, we recall their definition in terms of "subsequential" transducers and we prove that for each subsequential partial function there exists a subsequential transducer of a standard type (called "normalized") which realizes it. We show that the subsequential partial functions of A^* into B^* are exactly the sequential partial functions of A^* into $B^{(*)}$ whose image is in B^* (we assume B^* embedded into the free group $B^{(*)}$ generated by B). We solve the problem of determining under which conditions a rational partial function of A^* into B^* is the restriction of a sequential function of A^* into B^* . Finally we prove the Main Theorem.

I.PRELIMINARIES

1.Free monoids

Let A^* be the free monoid generated by the set A and 1 its unit - or *empty word* -. An element x of A^* is called a *word*. The length of x is denoted by $|x|$. We set $A^+ = A^* \setminus \{1\}$. As usual we shall denote by Rat A^* the family of all rational subsets of A^* (cf. [Ei], Chap.VII).

2.Free groups

Let A^{-1} be a copy of A and denote by a^{-1} the element corresponding to $a \in A$ in the bijection. Denote by $\equiv$ the congruence over the free monoid $(A \cup A^{-1})^*$ generated by all relations : $a\,a^{-1} = a^{-1}a = 1$, where $a \in A$. The *free group* $A^{(*)}$ generated by A is equivalent to the quotient $(A \cup A^{-1})^* \setminus \equiv$ (cf. [MKS]).

We shall make no distinction between A^* and its image in the canonical morphism of $(A \cup A^{-1})^*$ onto $A^{(*)}$. In other words we shall consider A^* as a submonoid of $A^{(*)}$.

All free monoids and free groups considered in the sequel will be supposed finitely generated. As many definitions and properties apply at the samme time to free monoids and free groups, we shall designate by M any free monoid or free Group.

3. Sequential transducers

Definition A *sequential transducer* T consists of:

- a finite non empty set Q, the set of *states*
- an element $q_- \in Q$, the *initial* state
- a subset $Q_+ \subseteq Q$, the subset of *final* states
- a function $\theta : Q \times A \to Q$, the *next state* function
- a function $\lambda : Q \times A \to M$, the *output* function

Observe that the four first data define a finite deterministic automaton which we shall refer to as the *automaton of* T.

For all $q \in Q$ and all $a \in A$ we shall write $q.a$ and $q*a$ instead of $(q,a)\theta$ and $(q,a)\lambda$ respectively. With this convention, and as long as no confusion may arise, we shall write : $T = (Q, q_-, Q_+)$.

The next state and the output functions are extended to $Q \times A^*$ in the usual way, by induction with respect to the length of the words:

For all $q \in Q$ we set : $q.1 = q$ and $q*1 = 1$.

For all $q \in Q$, all $u \in A^*$ and all $a \in A$ we set:

$$q.(ua) = (q.u).a \qquad \text{and} \qquad q*(ua) = (q*u)((q.u)*a)$$

Then for all $q \in Q$ and all $u, v \in A^*$ the following identity holds:

$$(1) \qquad q*uv = (q*u)((q.u)*v)$$

Practically allproofs in this paper involve constructions on sequential transducers. Proposition III.1 asserts that these constructions can be made on standard sequential transducers which are more easily manipulated than the general ones and whose definition is given now:

Definition A sequential transducer $T = (Q, q_-, Q_+)$ is *normalized* iff the following conditions hold:

i) For all $q \in Q$ there exists $u \in A^*$ such that : $q_-.u = q$

ii) For all $q \in Q$ and all $a \in A$ we have : $q.a \neq q_-$

iii) There exists at most one state $q_0 \in Q$ such that : $q_0.A^* \cap Q_+ = \emptyset$

iv) For all $q \in Q$ and all $a \in A$ we have : $q*a = 1$ if $q.a = q_0$

II.SEQUENTIAL PARTIAL FUNCTIONS

Given any partial function f of a set X into M we write $x\,f = \emptyset$ whenever $x\,f$ is undefined, where $\emptyset$ is the zero of the $\mathbb{Z}$-algebra of M.

1.Basic definitions and properties

Definition A partial function $f : A^* \to M$ is *sequential* iff there exists a sequential transducer T verifying for all $u \in A^*$: $u\,f = q_- * u$ if $q_-.u \in Q_+$ and $\emptyset$ otherwise. We say that T *realizes* f.

The usual notion of g-s-m mapping corresponds to the case when $M = B^*$ and when f is a function (cf. [Gi], p.93).

We give two elementary propositions on sequential partial functions (abbreviated *s.p.f.*). We recall that a partial function $f: A^* \longrightarrow M$ is *rational* if its *graph* $f\#= \{(u, v) \in A^* \times M \mid v = u\,f\}$ is a rational subset of the product monoid $A^* \times M$ (cf. [Ei], IX,8.). We first determine which partial sequential functions are sequential partial functions :

Proposition 1. *A partial function* $f : A^* \longrightarrow M$ *is sequential iff it is the restriction of a sequential function* $f': A^* \longrightarrow M$ *to a rational subset of* A^*.

Proof. Let $T = (Q, q_-, Q_+)$ be a sequential transducer realizing f. Denote by f' the sequential function of A^* into M realized by the sequential transducer T' obtained from T by considering every state as final, i.e. by setting : $Q_+ = Q$. Then f is the restriction of f' to the subset $L \subseteq A^*$ recognized by the automaton of T.

Conversely, let $f' : A^* \longrightarrow M$ be a sequential function and $T' = (Q, q_-, Q_+)$ a sequential transducer realizing f'. Let $L \subseteq A^*$ be the rational subset recognized by a finite automaton (P, p_-, P_+). Denote by $T = (Q \times P, (q_-,p_-), Q_+ \times P_+)$ the sequential transducer whose next state and output functions are defined for all $(q,p) \in Q \times P$ and all $a \in A$ by: $(q,p).a = (q.a,p.a)$ and $(q,p)*a = q*a$. Then the s.p.f. realized by T is the restriction of f' to L $\Box$

The family of s.p.f. is a subfamily of all rational partial functions :

<u>Proposition 2.</u> *Every sequential partial function* $f : A^{*} \longrightarrow M$ *is rational.*

<u>Proof.</u> Let $T = (Q, q_{-}, Q_{+})$ be a sequential transducer realizing f . Let j be a bijection of a set D over the (finite) set of all triples $(q,a,u) \in Q\times A\times M$ such that $q * a = u$. Define a next state function $Q \times D \to Q$ as follows: $q.d = q'$ if $d\,j = (q,a,u)$ and $q.a = q'$. The resulting automaton (Q, q_{-}, Q_{+}) recognizes a rational subset L of D^{*}. Let h be the morphism of D^{*} into $A^{*} \times M$ defined by $d\,h = (a,u)$ if $d\,j = (q,a,u)$. Then in view of Proposition VII,2.4. of [Ei] , the subset $L\,h = f^{\#} \subseteq A^{*} \times M$ is rational □

<u>2.The case when M is a free monoid B^{*}</u>

When in the previous definition, f is supposed to be a function (not just a partial function) and when M is supposed to be a free monoid B^{*}, we obtain the important case of what is known in the literature as g-s-m mappings. Certainly one of the most striking results on these functions is the following characterization (see for ex. [Ei] , Theorem XI,6.3.):

<u>Theorem 3.</u>(Ginsburg and Rose) *Let* $f : A^{*} \longrightarrow B^{*}$ *be a function such that* $1\,f = 1$.*Then it is sequential iff the two following conditions are satisfied :*

- *i) There exists an integer* $k > 0$ *such that for all* $u \in A^{*}$ *and all* $a \in A$ *we have :* $u\,a\,f \in u\,f\,(B^{*} \setminus B^{*}B^{k+1})$
- *ii) For each* $L \in \mathrm{Rat}\, B^{*}$, *we have :* $L\,f^{-1} \in \mathrm{Rat}\, A^{*}$.

As mentionned in the introduction, our main result is a generalization of this theorem to a family of partial functions (the subsequential ones) defined in section 3.

<u>3.The case when M is a free group $B^{(*)}$</u>

Given any partial function $f : A^{*} \longrightarrow B^{(*)}$ consider the right congruence $\underset{f}{\equiv}$ over A^{*} defined by : $u \underset{f}{\equiv} v$ iff there exists $x \in B^{(*)}$ such that for all $w \in A^{*}$ we have : $u\,w\,f = x\,(v\,w\,f)$. The following result generalizes Theorem XII,4.2. of [Ei] :

<u>Proposition 4.</u> *Let* $f : A^{*} \longrightarrow B^{(*)}$ *be a partial function such that* $1\,f = 1$ *or* $\emptyset$. *Then it is sequential iff* $\underset{f}{\equiv}$ *has finite index*

Proof.

Necessity: Let $T = (Q, q_-, Q_+)$ be a sequential transducer realizing f. Denote by $\equiv$ the right congruence over A^* defined by : $u \equiv v$ iff $q_-.u = q_-.v$. Then by identity (1) of I.3., $u \equiv v$ implies for all $w \in A^*$: $u\,w\,f = (q_- * u)((q_-.u) * w)$ and $v\,w\,f = (q_- * v)((q_-.v) * w)$, i.e. $u\,w\,f = x(v\,w\,f)$ with $x = (q_- * u)(q_- * v)^{-1} \in B^{(*)}$. Since $\equiv$ has finite index and is a refinement of $\underset{f}{\equiv}$, the latter right congruence has finite index too.

Sufficiency: For each $u \in A^*$ denote by $[u]$ the class of the right congruence $\underset{f}{\equiv}$ to which it belongs. Denote by Q the set consisting of all $[u]$'s where $u \in A^+$ and of a distinct element q_-. Let Q_+ be the subset of Q consisting of q_- if $1\,f \neq \emptyset$ and of all $[u]$'s such that $u \in A^+$ and $u\,f \neq \emptyset$. Since $\underset{f}{\equiv}$ is a right congruence, we may define as usual a next state function $Q \times A \longrightarrow Q$ by setting :

$$q.a = \begin{cases} [a] & \text{if} \quad q = q_- \\ [ua] & \text{if} \quad [u] = q \quad \text{otherwise} \end{cases}$$

Let q_0 be the unique state of Q (if such a state exists) such that $q_0.A^* \cap Q_+ = \emptyset$. We can suppose $q_- \neq q_0$ since otherwise f is nowhere defined and hence trivially sequential. Assign to every state $q \in Q \setminus \{q_0\}$ two words $v_q, w_q \in A^*$ as follows:

$$v_q = \begin{cases} 1 & \text{if} \quad q = q_- \\ \text{otherwise choose an arbitrary} \\ v \in A^* \text{ such that } [v] = q \end{cases} \qquad w_q = \begin{cases} 1 & \text{if} \quad q \in Q_+ \cup \{q_-\} \\ \text{otherwise choose an arbitrary} \\ w \in A^* \text{ such that } \quad q.w \in Q_+ \end{cases}$$

Define now an output function $Q \times A \longrightarrow B^{(*)}$ by setting :

$$q * a = \begin{cases} 1 & \text{if} \quad q_0 = q.a \\ (aw_{q_-.a})f & \text{if} \quad q = q_- \\ ((v_q w_q)f)^{-1}(v_q a w_{q.a})f & \text{otherwise} \end{cases}$$

In order to prove that the resulting transducer $T = (Q, q_-, Q_+)$ realizes f, we shall verify by induction on the length of $u \in A^+$, that $q_- * u = (uw_{q_-.u})f$ holds whenever $q_-.u \neq q_0$.

Since the previous equality holds trivially when $u \in A$, we consider the case $u \in A^+$ and $a \in A$ and we set $q_-.u = q$. By hypothesis of induction we have $q_- * u = (uw_q)f$ and therefore $q_- * ua = (uw_q)f((v_q w_q)f)^{-1}(v_q a w_{q.a})f$. Since $u \equiv v_q$ we have $(uw_q)f((v_q w_q)f)^{-1} = (uaw_{q.a})f\ ((v_q aw_{q.a})f)^{-1}$ which implies $q_- * ua = (uaw_{q_-.ua})f$.

If we observe that $1 f = 1$ holds iff $q_- \in Q_+$, for all $u \in A^*$ we have $q_-.u \in Q_+$ iff $q_- * u = u f \neq \emptyset$, which completes the proof □

III.SUBSEQUENTIAL PARTIAL FUNCTIONS

1.Basic definitions

For all subset $\emptyset \neq X \subsetneq B^*$ we denote by $X\Lambda$ the greatest left common factor of all words in X and we set $\emptyset\Lambda = \emptyset$.

Definition. A *subsequential transducer* is a pair (T, s) where:

- $T = (Q, q_-, Q_+)$ is a sequential transducer
- $s : Q \longrightarrow B^*$ is a partial function whose domain is Q_+

Further (T, s) is *normalized* if T is a normalized sequential transducer and if for all $q \in Q \setminus \{q_-\}$ the following holds:

(2) $\{q * u((q.u)s) \in B^* \mid q.u \in Q_+\}\Lambda = 1$ or $\emptyset$

Definition. A partial function $f : A^* \longrightarrow B^*$ is *subsequential* iff there exists a subsequential transducer (T, s) such that the following holds for all $u \in A^*$: $u f = q_- * u((q_-.u)s)$. We say that (T, s) *realizes* f.

Obviously every p.s.f. is subsequential (take for s the partial function whose image is $\{1\}$). The converse is false:

Example: Let $A = \{a, b\}$ and consider the function $f : A^* \longrightarrow A^*$ defined for all $u \in A^*$ by : $u f = ua$. Then f is subsequential but not sequential.

From now on we shall drop the adjective "partial": by subsequential function we mean subsequential partial function.

Proposition 1. *Any subsequential function can be realized by a normalized subsequential transducer.*

Proof. Let us first verify that the subsequential function $f : A^* \longrightarrow B^*$ can be realized by a subsequential transducer (T, s) where T is normalized. We shall proceed as follows. Starting with any subsequential transducer (T, s) realizing f, we shall assume that the i-th of the four conditions of a normalized transducer may not be satisfied, but that the precedent are. Then we shall prove that there exists a subsequential transducer (T', s') satisfying all i first conditions and realizing f.

i) Denote by $Q' \subseteq Q$ the subset of all elements $q \in Q$ for which there exists $u \in A^*$ with : $q_-.u = q$. Since $Q'.A \subseteq Q'$ we can define a next state function $Q' \times A \longrightarrow Q'$ as the restriction of the next state function of T to $Q' \times A$. Further define an output function $Q' \times A \longrightarrow B^*$ as the restriction of the output function of T to $Q' \times A$. Then $T' = (Q', q_-, Q' \cap Q_+)$ is a sequential transducer satisfying condition i) of the definition of a normalized transducer.

ii) Suppose condition ii) is not satisfied. Consider $Q' = Q \cup \{q'_-\}$ where $q'_- \notin Q_+$ and $Q'_+ = Q_+$ if $q_- \notin Q_+$ and $Q'_+ = Q_+ \cup \{q'_-\}$ if $q_- \in Q_+$. Extend the next state and the output functions of T to $Q' \times A$ by setting: $q'_-.a = q_-.a$ and $q'_- * a = q_- * a$. The resulting sequential transducer $T' = (Q', q'_-, Q'_+)$ satisfies conditions i) and ii). If we denote by $s' : Q' \longrightarrow B^*$ the partial function obtained by extending s by setting $q'_- s' = q_- s$, (T', s') is the desired subsequential transducer satisfying i) and ii).

iii) and iv) Suppose condition iii) or iv) is not satisfied. Consider the equivalence $\sim$ over Q defined by : $q_1 \sim q_2$ iff $q_1 = q_2$ or $q_1.A^* \cap Q_+ = q_2.A^* \cap Q_+ = \emptyset$

Then $\sim$ is a congruence of the automaton of T and we can define a next state function $Q/\!\sim \times A \longrightarrow Q/\!\sim$ and an output function $Q/\!\sim \times A \longrightarrow B^*$ by setting:

$$[q].a = [q.a] \quad \text{where } [q] \text{ designates the class of } q \text{ in } \sim$$

$$\text{and} \qquad [q] * a = \begin{cases} 1 & \text{if } q.A^* \cap Q_+ = \emptyset \\ q * a & \text{otherwise} \end{cases}$$

Then $T' = (Q/\!\sim, q_-, Q_+/\!\sim)$ is normalized. If $s' : Q/\!\sim \longrightarrow B^*$ is the partial function defined by $[q]s' = q\,s$, the subsequential transducer (T', s') realizes f and is therefore the desired one.

We shall verify now that condition (2) at the beginning of this paragraph, may as well be satisfied. Assume f is realized by (T, s) where T is normalized. Let $h : Q \longrightarrow B^*$ be defined by :

$$q\,h = \begin{cases} 1 & \text{if } q = q_o \text{ or } q_- \\ \bigwedge \{q * u((q.u)s) \in B^* \mid q.u \in Q_+\} & \text{otherwise} \end{cases}$$

Let T' be the normalized sequential transducer obtained from T by defining a new output function $\circ$ as follows:

$$q \circ a = \begin{cases} 1 & \text{if} \quad q.a = q_o \\ (qh)^{-1} q*a\ (q.a)h & \text{otherwise} \end{cases}$$

We have : $q*a\ \{(q.a)*u((q.au)s) \in B^*|\ q.au \in Q_+\} = \{q*au((q.au)s) \in B^*|\ q.au \in Q_+\}$

$\subseteq \{q*u((q.u)s) \in B^*| q.u \in Q_+\}$, which shows that qh is a left factor of $q*a(q.a)h$. This implies : $(qh)^{-1}q*a(q.a)h \in B^*$. Let $s' : Q \longrightarrow B^{(*)}$ be defined by : $qs' = (qh)^{-1}(qs)$. As qh is a left factor of qs , s' applies Q into B^*. Since (T', s') is a normalized subsequential transducer, it suffices to prove that it realizes f . But this follows from the equalities:

$$q_- o u(q_-.u)s' = q_- * u\ (q_-.u)h \quad ((q_-.u)h)^{-1}(q_-.u)s = uf \ \square$$

The following proposition is a verification that subsequential functions constitute a subfamily of all rational partial functions.

Proposition 2. *Each subsequential function is rational*

Proof. Let (T, s) be a subsequential transducer realizing the subsequential function $f : A^* \longrightarrow B^*$, and set $T = (Q, q_-, Q_+)$. For every $q_+ \in Q_+$ denote by f_{q_+} the sequential partial function realized by the transducer obtained from T by considering q_+ as unique final state. Then by Proposition II.2., the graph $f\#$ is rational since we have: $f\# = \bigcup_{q_+ \in Q_+} f_{q_+}\# . \{(1, q_+ s)\}$ $\square$

2.Subsequential functions and sequential partial functions into a free group

The relationship between the family of subsequential functions of A^* into B^* and the family of sequential partial functions of A^* into $B^{(*)}$ is given by the following result which justifies our extension of the classical notion of sequential transducer to the free group.

Proposition 3. *Let* $f : A^* \longrightarrow B^{(*)}$ *be a partial function such that* $A^* f \subseteq B^*$. *Then it is sequential iff there exists a subsequential function* $g : A^* \longrightarrow B^*$ *such that* $ug = uf$ *holds for each* $u \in A^+$.

Proof.

Sufficiency: Let (T, s) be a nomalized subsequential transducer realizing g . According to Proposition 1., we may suppose that $q_- s = 1$ or $\emptyset$. Denote by h the function of Q into B^* defined by :

$$qh = \begin{cases} qs & \text{if } qs \neq \emptyset \\ 1 & \text{otherwise} \end{cases}$$

Let $T' = (Q, q_-, Q_+)$ - where $Q_+ = \mathrm{Dom}\, s \subseteq Q$ - be the sequential transducer obtained from T by replacing the output function $*$ by the new output function $\circ$ defined for all $q \in Q$ and $a \in A$ by: $q \circ a = (qh)^{-1} q * a(q.a)h \in B^{(*)}$. Then one easily verifies by induction on the length of $u \in A^*$ that the following holds: $q_- \circ u = q_- * u(q_-.u)h$. In other words, the partial function $f : A^* \longrightarrow B^{(*)}$ such that for all u A, $uf = ug$ holds, is sequential $\square$

<u>Necessity:</u> Let $T = (Q, q_-, Q_+)$ be a normalized sequential transducer realizing f. Without loss of generality, we may suppose that for all $u \in A^*$ the following holds : (3) $q_- * u \in B^*$. Indeed assign to every $q \in Q' = Q \setminus (Q_+ \cup \{q_-\} \cup \{q_o\})$, any word $w_q \in A^*$ such that $q.w_q \in Q_+$ and consider the function h of Q into B^* defined by :

$$qh = \begin{cases} q * w_q & \text{if } q \in Q' \\ 1 & \text{otherwise} \end{cases}$$

Denote by T' the (normalized) sequential transducer obtained from T by replacing the output function $*$ by the new output function $\circ$ defined for all $a \in A$ and all $q \in Q$ by:

$$q \circ a = \begin{cases} (qh)^{-1} q * a(q.a)h & \text{if } q.a \neq q_o \\ 1 & \text{otherwise} \end{cases}$$

For all $u \in A^*$ we have:

$$q_- \circ u = \begin{cases} 1 & \text{if } q_-.u = q_- \\ q_- * u(q_-.u)h = q_- * u((q_-.u) * w_{q_-.u}) = q_- * uw_{q_-.u} \in B^* & \text{if } q_-.u \in Q' \\ q_- * u = uf \in B^* & \text{if } q_-.u \in Q_+ \end{cases}$$

Since $q_-.u = q_o$ implies $u = u'a$ with $u' \in A^*$ and $a \in A$, and $q_-.u' = q \neq q_o$, we have $q_- \circ u = q_- \circ u' \in B^*$. Thus, in all cases, equality (3) holds.

From now on we shall assume that (3) holds. For every $q \in Q \setminus \{q_-\}$ denote by $x_q \in B^*$ the longest right common factor of all words of the set $\{q_- * w \in B^* \mid w \in A^*, q_-.w = q\}$, and set $x_{q_-} = 1$. Let T' be the (normalized) sequential transducer obtained from T by replacing the output function $*$ by the new output function $\circ$ defined for all $a \in A$ and $q \in Q$ by : $q \circ a = 1$ if $q.a = q_o$ and $x_q\,(q*a)(x_{q.a})^{-1}$ otherwise. One easily verifies by induction on the length of

$u \in A^*$ that $q\_ o u = q \underline{*} u\, (x_{q\_ .u})^{-1}$ holds, i.e. that (3) still holds. Let us verify now that for every $q \in Q$ and every $a \in A$, $q \circ a$ belongs to B^*.

Let $q \in Q$ and $u, v \in A^*$ be two words satisfying $q\_ .u = q\_ .v = q$ and such that $q\_ \circ u$ and $q\_ \circ v$ have no right common factor different from the empty word. If $q.a \neq q_o$ we have: $q\_ \circ ua = (q \underline{*} u (x_{q\_ .u})^{-1})\ x_{q\_ .u}\ ((q\_ .u) * a)\ (x_{q\_ .ua})^{-1} = y \in B^*$

and $q\_ \circ va = (q \underline{*} v (x_{q\_ .v})^{-1})\ x_{q\_ .v}\ ((q\_ .v) * a)\ (x_{q\_ .va})^{-1} = y' \in B^*$

Setting: $q \underline{*} u\ (x_{q\_ .u})^{-1} = z \in B^*$ and $q \underline{*} v (x_{q\_ .v})^{-1} = z' \in B^*$ we obtain:

$z\ (q \circ a) = y$ and $z'\ (q \circ a) = y'$ i.e. $z\ z'^{-1} = y\ y'^{-1}$. Since z and z' have no right common factor different from 1, the word $z\ z'^{-1}$ is reduced and there exists a word $t \in B^*$ such that $y = zt$ and $y' = z't$ hold. Thus: $q \circ a = t \in B^*$.

Let $s : Q \longrightarrow B^*$ be the partial function defined by $q\ s = x_q$ if $q \in Q_+$ and $\emptyset$ otherwise. If we denote by $g : A^* \longrightarrow B^*$ the subsequential function realized by the subsequential transducer (T', s), for all $u \in A^+$ we have $uf = ug$ □

3.The problem of the extension of a rational partial function to a sequential function

As another application of Proposition 1., we characterize all rational partial functions $f: A^* \longrightarrow B^*$ which are the restriction of a sequential function.

Proposition 4. *Let* $f : A^* \longrightarrow B^*$ *be a rational partial function such that* $1\ f = 1$ *or* $\emptyset$. *It can be extended to a sequential function iff the following conditions hold:*

i) f *is subsequential*

ii) For every $u,v \in A^*$ *such that* $uf \neq \emptyset$ *and* $uvf \neq \emptyset$ *we have:* $uvf \in uf\ B^*$.

Proof.

Necessity: Condition ii) is necessary since it is implied by the identity (1) of I.3. Further, if f is rational, then $\mathrm{Dom}\, f = (f^{\sharp})p$ (where p is the natural projection of $A^* \times B^*$ over A^*) is rational in view of Proposition VII,2.4. of [Ei]. It suffices now to use Proposition II.2.

Sufficiency: Let (T, s) be a normalized subsequential transducer realizing f. We have for each $q\_ .u = q \in Q_+$: $q \underline{*} u(qs) = uf = \{ uvf \in B^* | q.v \in Q_+ \} \wedge$

$= q \underline{*} u \{ q * v((q.v)s) \in B^* | q.v \in Q_+ \} \wedge = q\_ * u$ which implies

$qs = 1$. Consider now all states of T as final and denote by $g : A^* \longrightarrow B^*$ the sequential function realized by the new sequential transducer. Then g is the desired extension of f $\square$

4.A characterisation of subsequential functions: proof of the Main Theorem

We can now turn to the proof of the Main Theorem. We recall that given any integer $n \geqslant 0$, A_n (resp. B_n) denotes the set of all words of A^* (resp. B^*) of length less or equal to n.

Theorem 5. *Let $f : A^* \longrightarrow B^*$ be a partial function. Then it is subsequential iff it satisfies the two following conditions:*

i) There exists an integer $k > 0$ such that for all $u \in A^*$ and all integer $n \geqslant 0$ there exists $x \in B^*$ with : $uA_n f \subseteq x B_{k.n}$

ii) For each rational subset $L \subseteq B^*$, Lf^{-1} is a rational subset of A^*.

Proof.

Necessity. Let (T , s) be a subsequential transducer realizing f and $k > 0$ an integer greater than all $|q*a|$ and $|qs|$ where $q \in Q$ and $a \in A$. For every $u \in A^*$ and every $v \in A_n$ we have: $uvf = q_-*u((q_-.u)*v)((q_-.uv)s) \subseteq q_-*u\ B_{2k.n}$. This proves condition i). In view of Proposition 2., f is rational. Then by Theorem IX,3.1. of [Ei], for each $L \in \text{Rat } B^*$ we have $Lf^{-1} \in \text{Rat } A^*$, which proves condition ii).

Sufficiency: Condition ii) implies that: $B^* f^{-1} = \text{Dom } f \in \text{Rat } A^*$. Let n be the number of states of an automaton recognizing $\text{Dom } f$. We set $m = k.n$ (where k is as in condition i)), $F = A^* \setminus A^* A^{n+1}$ and $G = B^* \setminus B^* B^{m+1}$. We denote by R the set of all partial functions $r : F \longrightarrow G$ such that $(Fr)\wedge = 1$ or $\emptyset$. We shall prove the following facts:

A) there exists a function ρ assigning with each $u \in A^*$ an element $u\rho \in R$.

B) for each $r \in R$, $r\rho^{-1}$ is a rational subset of A^*. As a consequence, since R is finite, there exists a finite automaton $\mathcal{A}$ recognizing each $r\rho^{-1}$ (with $r \in R$)

C) the right congruence associated with the automaton $\mathcal{A}$ is a refinement of the congruence $\underset{f}{\equiv}$ defined in II.4.

A) Let $g : A^* \longrightarrow B^*$ be the partial function defined by: $ug = (uFf)\wedge$. To each $u \in A^*$ assign the partial function $u\rho = r : F \longrightarrow B^*$ defined by:

$$xr = \begin{cases} \emptyset & \text{if} \quad Fr = \emptyset \\ (ug)^{-1}uxf & \text{otherwise} \end{cases}$$

In view of condition i) we have $Fr \subsetneq G$. Since $ug = (uFf)\Lambda$, we have $(Fr)\,\Lambda = \emptyset$ or 1 and therefore $r \in R$.

B) For each $z \in G$ and $|y| \leqslant 3m$ we set:

$$L_z^y = \begin{cases} yzf^{-1} & \text{if} \quad |y| < m \\ (B^{2m+1})^* yzf^{-1} & \text{if} \quad m \leqslant |y| \leqslant 3m \end{cases}$$

It is convenient to write $L_\emptyset^y = A^* \setminus B^* f^{-1}$ for all $|y| \leqslant 3m$.

For each $x \in A^*$, the partial function $t_x : A^* \longrightarrow A^*$ which to every $ux \in A^*$ associates $u \in A^*$ is rational since its graph is $(\bigcup_{a \in A} (a,a))^* (x,1)$. By theorem IX,3.1. of [Ei] , for all $r \in R$ we have: $X_r = \bigcup_{|y| \leqslant 3m} (\bigcap_{x \in F} L_{xr}^y\, t_x) \in \operatorname{Rat} A^*$. In order to prove B) it suffices to verify: $r\rho^{-1} = X_r$.

1) $r\rho^{-1} \subsetneq X_r$

If $u\rho = r \in R$, then for all $x \in F$ we have $uxf = ug(xr)$. In particular, if $r = r_o$ where $r_o \in R$ is the partial function of F into G which is nowhere defined, for all $x \in F$ we have: $ux \in A^* \setminus B^* f^{-1}$. By defintion of $L_\emptyset^y$, this implies: $u \in \bigcap_{x \in F} (A^* \setminus B^* f^{-1})\, t_x = \bigcap_{x \in F} L_\emptyset^y\, t_x \subsetneq X_{r_o}$ with $|y| \leqslant 3m$ arbitrary.

If $r \neq r_o$, two different cases must be distinguished:

<u>Case 1.</u> $|ug| < m$. This implies for all $x \in F$:
$ux \in L_{xr}^{ug}$ i.e. $u \in \bigcap_{x \in F} L_{xr}^{ug}\, t_x \subsetneq X_r$.

<u>Case 2.</u> $m \leqslant |ug|$. Let $s \geqslant 0$ be the unique integer such that $(2m+1)s \leqslant |ug| - m < (2m+1)(s+1)$. There exist two words $b \in (B^{2m+1})^*$ and $y \in B^*$ such that we have: $ug = by$ and $m \leqslant |y| \leqslant 3m$. Thus, for all $x \in F$ we have $ux \in L_{xr}^y$ which implies: $u \in \bigcap_{x \in F} L^y{}_{xr}\, t_x \subsetneq X_r$.

2) $X_r \subsetneq r\rho^{-1}$

Let $u \in \bigcap_{x \in F} L_{xr}^y\, t_x$ where $|y| \leqslant 3m$. If $r = r_o$ then for each $x \in F$ we have $uvf = \emptyset$ i.e. $ug = \emptyset$ and therefore $u\rho = r_o$. We shall assume now $r \neq r_o$

Two different cases must be considered:

Case 1. $|y| < m$. For every $x \in F$ we have $uxf = y(xr)$. Since $(Fr)\Lambda = 1$ holds, we obtain $y = ug$ and thus: $u\rho = r$.

Case 2. $m \leqslant |y| \leqslant 3m$. Let $x_1, x_2 \in \mathrm{Dom}\, r$. According to condition i) there exist $b_1, b_2 \in (B^{2m+1})^*$, $z, t_1, t_2 \in B^*$ with $|t_1| \leqslant m$ and $|t_2| \leqslant m$ such that the following equalities hold: $\{t_1, t_2\}\Lambda = 1$, $ux_1f = b_1y(x_1r) = zt_1$ and $ux_2f = b_2y(x_2r) = zt_2$. For $i = 1, 2$ we have: $|z| - m \leqslant |b_iy| \leqslant |z| + m$, which implies $||b_2y| - |b_1y|| \leqslant 2m$ and consequently $|b_1| = |b_2|$ since $||b_2y| - |b_1y|| = 1.(2m+1)$ where $1 \geqslant 0$ is an integer. Further we have $b_1 = b_2$, since $b_1 \neq b_2$ implies $|z| < |b_1|$ and thus $|z| + m < |b_1y|$, which contradicts the previous inequalities. As a consequence, there exists a word $b \in B^*$ such that $uxf = by(xr)$ holds for all $x \in \mathrm{Dom}\, r$. This signifies precisely that: $u\rho = r$.

C) Let $\mathcal{Q} = (Q, q\_)$ be a finite deterministic automaton recognizing each $r\rho^{-1} \in \mathrm{Rat}\, A^*$, where r ranges over R , and let $\equiv$ be the right congruence over A^* defined by: $u \equiv v$ iff $q\_.u = q\_.v$. Thus $u \equiv v$ implies $u\rho = v\rho$.

Given any integer $1 > 0$, consider the following predicate:

P_1: For each $u, v \in A^*$ such that $u \equiv v$ there exists an element $x \in B^{(*)}$ satisfying for all $w \in A_1$: $uwf = x\,(vwf)$.

Assume there exists a greatest integer $L > 0$ for which P_1 is true. Then certainly $L \geqslant n$. For every $a, b \in A$ and $u, v \in A^*$ satisfying $u \equiv v$, there exist $x, y \in B^{(*)}$ such that for all $w \in A_L$ we have: $uawf = x\,(vawf)$ and $ubwf = y\,(vbwf)$. We may suppose that there exist $w_1, w_2 \in A_L$ such that $uaw_1f \neq \emptyset$ and $ubw_2f \neq \emptyset$ since otherwise x or y or both x and y may be chosen arbitrarily and therefore x and y may be chosen equal. Then there exist two words $t_1, t_2 \in A_{n-1}$ (because $\mathrm{Dom}\, f = B^*f^{-1}$ is recognized by an automaton having n states) such that $uat_1f \neq \emptyset$ and $ubt_2f \neq \emptyset$. Since we have $at_1, bt_2 \subsetneq A_n \subsetneq A_L$ there exists $z \in B^*$ with: $uat_1f = z\,(vat_1f)$ and $ubt_2f = z\,(vbt_2f)$ i.e. $x = y$; This shows that P_{L+1} is true, which contradicts the fact that L is maximal.

Since $\equiv$ is a refinement of $\underset{f}{\equiv}$, by Propositions 3. and II.4., the proof is completed □

<u>Corollary 6.</u> *If* $f : A^* \longrightarrow B^*$ *and* $g : B^* \longrightarrow C^*$ *are subsequential functions, then the composition* $f\,g : A^* \longrightarrow C^*$ *is subsequential*

<u>Proof.</u> For all $L \in \text{Rat } C^*$ we have $Lg^{-1} \in \text{Rat } B^*$ and therefore $(Lg^{-1})f^{-1} = L(fg)^{-1} \in \text{Rat } A^*$ which proves that condition ii) of the theorem is satisfied.

Let $k, l > 0$ be the integers satisfying condition i) for f and g respectively. For each $u \in A^*$ and each integer $n > 0$ there exists $x \in B^*$ with $uA_n f \subseteq xB_{k.n}$. Applying the same condition to $x \in B^*$ and $k.n$, we have: $xB_{k.n}\, g \subseteq tC_{l.k.n}$ where $t \in C^*$. Thus: $uA_n fg \subseteq tC_{l.k.n}$ □

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