NORTH-HOLLAND

Construction of a Fundamental Matrix Solution at a Singular Point of the First Kind by Means of the *SN* Decomposition of Matrices

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ABSTRACT

It is known that any matrix can be decomposed into a diagonalizable part and a nilpotent part. We call this the *SN* decomposition. We can derive the *SN* decomposition quite easily with a computer. Generalizing the *SN* decomposition to particular matrices of infinite order, we explain basic steps of construction of a linear transformation which reduces a given system of linear meromorphic ordinary differential equations to a normal form at a singular point of the first kind. Some examples are given utilizing Mathematica. We also show that the same idea produces a block-diagonalization of a given system at a singular point of the second kind.

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0. INTRODUCTION

In order to construct a fundamental matrix solution of a system of linear meromorphic ordinary differential equations at a singular point of the first kind, we utilize traditionally the method of G. F. Frobenius. If we try to use this method with a computer, we face some complicated situations. In particular, we must divide the general case into subcases, and in each subcase we must use a different strategy. We can avoid such unpleasantness by simplifying the given system to a normal form by a linear transformation. There is such a result (Y. Sibuya [5, Chapter 3]). However, on examining the traditional proof of this result, we realize that the Jordan canonical form of matrices plays an important role there. Construction of the Jordan canonical form is quite difficult. If possible, we wish to avoid this difficulty.

It is known that any matrix can be decomposed into a diagonalizable part and a nilpotent part. Let us call this decomposition the *SN* decomposition of matrices (cf. M. W. Hirsch and S. Smale [4, Chapter 6] and N. Bourbaki [1, Chapter 7]). This decomposition looks very sophisticated. However, as we shall show in Sections 1 and 8, we can derive the *SN* decomposition quite easily with a computer. Furthermore, if we examine an idea given in the paper [3] of R. Gérard and A. H. M. Levelt, it is clear that we can construct a linear transformation which reduces the given system to a normal form by utilizing exclusively the *SN* decomposition of matrices.

Gérard and Levelt present their ideas in a sophisticated manner. In this paper, we will explain their ideas simply in terms of matrices. In doing this, we will also show the basic steps in which actual calculations can be carried out by means of a computer (Sections 2–7). We shall give some examples which were obtained by utilizing Mathematica (Version 2.2).

Although our main concern is a system of differential equations at a singular point of the first kind, the same idea can be used also at a singular point of the second kind to produce a block-diagonalization of the given system. We shall discuss this situation together with an example in Section 7. For calculations of this nature, we clearly need a computer.

1. A REVIEW OF THE *SN* DECOMPOSITION OF MATRICES

The basic results concerning the *SN* decomposition of matrices are given in the following theorem.

THEOREM 1.1. *Let A be an $n \times n$ matrix whose entries are complex numbers. Then there exist two $n \times n$ matrices S and N such that*

(a) *S is diagonalizable,*

(b) *N is nilpotent, i.e.* $N^n = 0$,
(c) $A = S + N$,
(d) $SN = NS$.

The two matrices S and N are uniquely determined by these four conditions. If A is real, then S and N are also real. Furthermore, they are polynomials in A with coefficients in the smallest field containing the field $\mathbb{Q}$ of rational numbers, the entries a_{jk} of the matrix A, and the eigenvalues of A.

Since the proof of this theorem is well known (cf. Bourbaki [1, Chapter 7]), we shall show how to construct S and N only. To begin with, let us define the characteristic polynomial $p_A(\lambda)$ of A by

$$p_A(\lambda) = \det[\lambda I - A] = \lambda^n + \sum_{h=1}^{n} p_h(A)\lambda^{n-h},$$

where I is the $n \times n$ identity matrix. Let λ_j ($j = 1, 2, \ldots, k$) be the distinct eigenvalues of A, and let m_j ($j = 1, 2, \ldots, k$) be their respective multiplicities. Then the characteristic polynomial of the matrix A is also given by

$$p_A(\lambda) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_k)^{m_k}.$$

Let us decompose $1/p_A(\lambda)$ into partial fractions:

$$\frac{1}{p_A(\lambda)} = \sum_{j=1}^{k} \frac{Q_j(\lambda)}{(\lambda - \lambda_j)^{m_j}},$$

where, for every j, Q_j is a nonzero polynomial in λ of degree not greater than $m_j - 1$. Hence

$$1 = \sum_{j=1}^{k} Q_j(\lambda)\prod_{h \neq j}(\lambda - \lambda_h)^{m_h}.$$

Let

$$P_j(\lambda) = Q_j(\lambda)\prod_{h \neq j}(\lambda - \lambda_h)^{m_k};$$

then

$$1 = \sum_{j=1}^{k} P_j(\lambda).$$

Note that this is an identity in λ. Let

$$P_j(A) = Q_j(A)\prod_{h\neq j}(A - \lambda_h I)^{m_k} \qquad (j = 1, 2, \dots, k),$$

Then utilizing the Cayley-Hamilton theorem, we can derive the following lemma.

LEMMA 1.2. *The k matrices $P_j(A)$ $(j = 1, 2, \dots, k)$ satisfy the following conditions*:

(i) *A and $P_j(A)$ $(j = 1, 2, \dots, k)$ are commutative,*
(ii) $(A - \lambda I)^{m_j} P_j(A) = 0$ $(j = 1, 2, \dots, k)$,
(iii) $P_j(A) P_h(A) = 0$ *if* $j \neq h$,
(iv) $\sum_{k \geq j \geq 1} P_j(A) = I$,
(v) $P_j(A)^2 = P_j(A)$ $(j = 1, 2, \dots, k)$,
(vi) $P_j(A) \neq 0$ $(j = 1, 2, \dots, k)$.

Now we can define the matrices S and N by

$$S = \lambda_1 P_1(A) + \lambda_2 P_2(A) + \cdots + \lambda_k P_k(A),$$
$$N = A - S.$$

REMARK 1.3.

(1) For our calculations, it is important that the column vectors of the matrix $P_j(A)$ are eigenvectors of S associated with the eigenvalue λ_j.

(2) All of important information for our calculations is given by the polynomials $P_1(\lambda), P_2(\lambda), \dots, P_k(\lambda)$. We shall give practical programs for finding those polynomials $P_j(\lambda)$ and the matrices $P_j(A)$, S, and N with a computer in Section 8.

(3) In the calculation given above, we assumed that the exact factorization of the characteristic polynomial $p_A(\lambda)$ is known for the given matrix A. Therefore, if we want to use the method given above, we need some symbolic method for computing the characteristic polynomial, e.g., the Souriau-Frame algorithm (for example, Cullen [2, pp. 278–282]).

EXAMPLE 1. For the matrix

$$A = \begin{bmatrix} 3 & 4 & 3 \\ 2 & 7 & 4 \\ -4 & 8 & 3 \end{bmatrix}$$

we have $\lambda_1 = 11$, $\lambda_2 = 1$, and

$$p_A(\lambda) = (\lambda - 1)^2(\lambda - 11),$$

$$\frac{1}{p_A(\lambda)} = \frac{1}{100(\lambda - 11)} - \frac{(\lambda + 9)}{100(\lambda - 1)^2}.$$

Hence

$$1 = \frac{(\lambda - 1)^2}{100} - \frac{(\lambda + 9)(\lambda - 11)}{100}.$$

Set

$$P_1(\lambda) = \frac{(\lambda - 1)^2}{100}, \qquad P_2(\lambda) = -\frac{(\lambda + 9)(\lambda - 11)}{100}.$$

Then

$$P_1(A) = \frac{1}{100}\begin{bmatrix} 0 & 56 & 28 \\ 0 & 76 & 38 \\ 0 & 48 & 24 \end{bmatrix}, \qquad P_2(A) = \frac{1}{100}\begin{bmatrix} 100 & -56 & -28 \\ 0 & 24 & -38 \\ 0 & -48 & 76 \end{bmatrix}.$$

Therefore

$$S = 11P_1(A) + P_2(A) = \frac{1}{10}\begin{bmatrix} 10 & 56 & 28 \\ 0 & 86 & 38 \\ 0 & 48 & 34 \end{bmatrix},$$

$$N = A - S = \frac{1}{10}\begin{bmatrix} 20 & -16 & 2 \\ 20 & -16 & 2 \\ -40 & 32 & -4 \end{bmatrix}.$$

In this case $SN = NS = N$ and $N^2 = 0$.

EXAMPLE 2. For the matrix

$$A = \begin{bmatrix} 252 & 498 & 4134 & 698 \\ -234 & -465 & -3885 & -656 \\ 15 & 30 & 252 & 42 \\ -10 & -20 & -166 & -25 \end{bmatrix}$$

we have $\lambda_1 = 4$, $\lambda_2 = 3$, and

$$p_A(\lambda) = (\lambda - 4)^2(\lambda - 3)^2,$$

$$\frac{1}{p_A(\lambda)} = \frac{1}{(\lambda - 4)^2} - \frac{2}{\lambda - 4} + \frac{1}{(\lambda - 3)^2} + \frac{2}{\lambda - 3}.$$

Set

$$P_1(\lambda) = (\lambda - 3)^2 - 2(\lambda - 4)(\lambda - 3)^2,$$

$$P_2(\lambda) = (\lambda - 4)^2 + 2(\lambda - 3)(\lambda - 4)^2.$$

Then

$$P_1(A) = \begin{bmatrix} -1 & -2 & 134 & 198 \\ 1 & 2 & -125 & -186 \\ 0 & 0 & 9 & 12 \\ 0 & 0 & -6 & -8 \end{bmatrix},$$

$$P_2(A) = \begin{bmatrix} 2 & 2 & -134 & -198 \\ -1 & -1 & 125 & 186 \\ 0 & 0 & -8 & -12 \\ 0 & 0 & 6 & 9 \end{bmatrix}.$$

Therefore

$$S = 4P_1(A) + 3P_2(A) = \begin{bmatrix} 2 & -2 & 134 & 198 \\ 1 & 5 & -125 & -186 \\ 0 & 0 & 12 & 12 \\ 0 & 0 & -6 & -5 \end{bmatrix},$$

$$N = A - S = \begin{bmatrix} 250 & 500 & 4000 & 500 \\ -235 & -470 & -3760 & -470 \\ 15 & 30 & 240 & 30 \\ -10 & -20 & -160 & -20 \end{bmatrix}.$$

EXAMPLE 3. The characteristic polynomial of the matrix

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & -1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & -1 & 4 \end{bmatrix}$$

is given by

$$p_A(\lambda) = (\lambda - 5)(\lambda - 4)(\lambda - 3)^2(\lambda - 2)^2(\lambda - 1)^2\lambda(\lambda + 1),$$

and

$$\begin{aligned} \frac{1}{p_A(\lambda)} = {} & \frac{1}{17280(\lambda - 5)} - \frac{1}{720(\lambda - 4)} \\ & + \frac{1}{96(\lambda - 3)^2} - \frac{25}{1152(\lambda - 3)} \\ & + \frac{1}{36(\lambda - 2)^2} + \frac{1}{96(\lambda - 1)^2} + \frac{25}{1152(\lambda - 1)} \\ & + \frac{1}{720\lambda} - \frac{1}{17280(\lambda + 1)}. \end{aligned}$$

Set

$$P_1(\lambda) = \frac{(\lambda - 4)(\lambda - 3)^2(\lambda - 2)^2(\lambda - 1)^2\lambda(\lambda + 1)}{17280},$$

$$P_2(\lambda) = -\frac{(\lambda - 5)(\lambda - 3)^2(\lambda - 2)^2(\lambda - 1)^2\lambda(\lambda + 1)}{720},$$

$$P_3(\lambda) = \frac{(\lambda-5)(\lambda-4)(\lambda-2)^2(\lambda-1)^2\lambda(\lambda+1)}{96}$$

$$-\frac{25(\lambda-5)(\lambda-4)(\lambda-3)(\lambda-2)^2(\lambda-1)^2\lambda(\lambda+1)}{1152},$$

$$P_4(\lambda) = \frac{(\lambda-5)(\lambda-4)(\lambda-3)^2(\lambda-1)^2\lambda(\lambda+1)}{36},$$

$$P_5(\lambda) = \frac{(\lambda-5)(\lambda-4)(\lambda-3)^2(\lambda-2)^2\lambda(\lambda+1)}{96}$$

$$+\frac{25(\lambda-5)(\lambda-4)(\lambda-3)^2(\lambda-2)^2(\lambda-1)\lambda(\lambda+1)}{1152},$$

$$P_6(\lambda) = \frac{(\lambda-5)(\lambda-4)(\lambda-3)^2(\lambda-2)^2(\lambda-1)^2(\lambda+1)}{720},$$

$$P_7(\lambda) = -\frac{(\lambda-5)(\lambda-4)(\lambda-3)^2(\lambda-2)^2(\lambda-1)^2\lambda}{17280}.$$

Then

$$S = 5P_1(A) + 4P_2(A) + 3P_3(A) = 2P_4(A) + P_5(A) - P_7(A)$$

$$= \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{64} & \frac{1}{64} & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ -\frac{1}{64} & \frac{1}{64} & \frac{1}{4} & 0 & -1 & 2 & 0 & 0 & 0 & 0 \\ \frac{1}{256} & -\frac{1}{384} & -\frac{1}{64} & \frac{1}{64} & 0 & 0 & 3 & -1 & 0 & 0 \\ \frac{1}{192} & -\frac{1}{256} & -\frac{1}{64} & \frac{1}{64} & \frac{1}{4} & 0 & -1 & 3 & 0 & 0 \\ -\frac{7}{18432} & \frac{7}{36864} & \frac{1}{256} & -\frac{1}{384} & -\frac{1}{64} & \frac{1}{64} & 0 & 0 & 4 & -1 \\ -\frac{25}{36864} & \frac{7}{18432} & \frac{1}{192} & -\frac{1}{256} & -\frac{1}{64} & \frac{1}{64} & \frac{1}{4} & 0 & -1 & 4 \end{bmatrix}$$

and

$$N = A - S$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{64} & -\frac{1}{64} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{64} & -\frac{1}{64} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{256} & \frac{1}{384} & \frac{1}{64} & -\frac{1}{64} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{192} & \frac{1}{256} & \frac{1}{64} & -\frac{1}{64} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{7}{18432} & -\frac{7}{36864} & -\frac{1}{256} & \frac{1}{384} & \frac{1}{64} & -\frac{1}{64} & 0 & 0 & 0 & 0 \\ \frac{25}{36864} & -\frac{7}{18432} & -\frac{1}{192} & \frac{1}{256} & \frac{1}{64} & -\frac{1}{64} & 0 & 0 & 0 & 0 \end{bmatrix}.$$

EXAMPLE 4. The characteristic polynomial of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 8 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 8 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 5 & 8 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 5 & 8 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 3 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 5 & 8 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 & 1 & 0 & -1 \end{bmatrix}$$

is given by

$$p_A(\lambda) = (\lambda - 1)^6(\lambda + 1)^6$$

and

$$\frac{1}{p_A(\lambda)} = \frac{1}{64(\lambda - 1)^6} - \frac{3}{64(\lambda - 1)^5} + \frac{21}{256(\lambda - 1)^4} - \frac{7}{64(\lambda - 1)^3} + \frac{63}{512(\lambda - 1)^2} - \frac{63}{512(\lambda - 1)}$$

$$+\frac{1}{64(\lambda+1)^6}+\frac{3}{64(\lambda+1)^5}+\frac{21}{256(\lambda+1)^4}$$
$$+\frac{7}{64(\lambda+1)^3}+\frac{63}{512(\lambda+1)^2}+\frac{63}{512(\lambda+1)}.$$

Set

$$\begin{aligned}P_1(\lambda) &= p_A(\lambda)\left(\frac{1}{64(\lambda-1)^6}-\frac{3}{64(\lambda-1)^5}+\frac{21}{256(\lambda-1)^4}\right.\\ &\qquad \left.-\frac{7}{64(\lambda-1)^3}+\frac{63}{512(\lambda-1)^2}-\frac{63}{512(\lambda-1)}\right),\\ P_2(\lambda) &= p_A(\lambda)\left(\frac{1}{64(\lambda+1)^6}+\frac{3}{64(\lambda+1)^5}+\frac{21}{256(\lambda+1)^4}\right.\\ &\qquad \left.+\frac{7}{64(\lambda+1)^3}+\frac{63}{512(\lambda+1)^2}+\frac{63}{512(\lambda+1)}\right).\end{aligned}$$

Then

$$\begin{aligned}P_1(\lambda) &= \frac{1}{2}+\frac{693\lambda}{512}-\frac{1155\lambda^2}{512}+\frac{693\lambda^5}{256}-\frac{495\lambda^7}{256}+\frac{385\lambda^9}{512}-\frac{63\lambda^{11}}{512},\\ P_2(\lambda) &= 1-P_1(\lambda),\end{aligned}$$

and hence

$$S = P_1(A) - P_2(A)$$
$$= \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{3}{2} & & 0 & & & & & & & & & \\ -\frac{21}{2} & \frac{21}{8} & 3 & \frac{3}{2} & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{765}{16} & \frac{1149}{16} & -\frac{21}{8} & -\frac{189}{8} & -\frac{3}{2} & \frac{9}{2} & 1 & 1 & 0 & 0 & 0 & 0 \\ \frac{117}{8} & -\frac{765}{16} & -\frac{21}{2} & \frac{21}{8} & 3 & \frac{3}{2} & 0 & -1 & 0 & 0 & 0 & 0 \\ -\frac{31773}{128} & \frac{1827}{128} & \frac{765}{16} & \frac{1149}{16} & -\frac{21}{8} & -\frac{189}{8} & -\frac{3}{2} & \frac{9}{2} & 1 & 1 & 0 & 0 \\ \frac{1659}{16} & \frac{31773}{128} & \frac{117}{8} & -\frac{765}{16} & -\frac{21}{2} & \frac{21}{8} & 3 & \frac{3}{2} & 0 & -1 & 0 & 0 \\ \frac{131403}{256} & -\frac{411849}{256} & -\frac{31773}{128} & \frac{1827}{128} & \frac{765}{16} & \frac{1149}{16} & -\frac{21}{8} & -\frac{189}{8} & -\frac{3}{2} & \frac{9}{2} & 1 & 1 \\ -\frac{119055}{128} & -\frac{131403}{256} & \frac{1659}{16} & \frac{31773}{128} & \frac{117}{8} & -\frac{765}{16} & -\frac{21}{2} & \frac{21}{8} & 3 & \frac{3}{2} & 0 & -1 \end{bmatrix}$$

and

$$N = A - S$$

$$
= \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{13}{2} & \frac{7}{2} & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{21}{8} & \frac{189}{8} & \frac{13}{2} & \frac{7}{2} & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{21}{2} & -\frac{21}{8} & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{765}{16} & -\frac{1149}{16} & \frac{29}{8} & \frac{189}{8} & \frac{13}{2} & \frac{7}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{117}{8} & \frac{765}{16} & \frac{21}{2} & -\frac{13}{8} & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{31773}{128} & -\frac{1827}{128} & -\frac{765}{16} & -\frac{1149}{16} & \frac{37}{8} & \frac{189}{8} & \frac{13}{2} & \frac{7}{2} & 0 & 0 & 0 & 0 \\
-\frac{1659}{16} & -\frac{31773}{128} & -\frac{117}{8} & \frac{765}{16} & \frac{21}{2} & -\frac{5}{8} & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\
-\frac{131403}{256} & \frac{411849}{256} & \frac{31773}{128} & -\frac{1827}{128} & -\frac{765}{16} & -\frac{1149}{16} & \frac{45}{8} & \frac{189}{8} & \frac{13}{2} & \frac{7}{2} & 0 & 0 \\
\frac{119055}{128} & \frac{131403}{256} & -\frac{1659}{16} & \frac{31773}{128} & -\frac{117}{8} & \frac{765}{16} & \frac{21}{2} & \frac{3}{8} & 0 & -\frac{1}{2} & 0 & 0
\end{bmatrix}.
$$

2. THE SN DECOMPOSITION OF A MATRIX OF INFINITE ORDER

In this section, we shall derive the SN decomposition of a matrix of the following form:

$$
A = \begin{bmatrix}
A_{11} & O & O & O & \cdots & \cdots & \cdots \\
A_{21} & A_{22} & O & O & \cdots & \cdots & \cdots \\
A_{31} & A_{32} & A_{33} & O & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \cdots & \cdots & \cdots \\
A_{m1} & A_{m2} & A_{m3} & \cdots & A_{mm} & O & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \ddots
\end{bmatrix},
$$

where, for each (j, k), the quantity A_{jk} is an $n_j \times n_k$ constant matrix (cf. Section 1). Set

$$A_1 = A_{11},$$

$$
A_m = \begin{bmatrix}
A_{11} & O & O & O & \cdots & O \\
A_{21} & A_{22} & O & O & \cdots & O \\
A_{31} & A_{32} & A_{33} & O & \cdots & O \\
\vdots & \vdots & \vdots & \vdots & & \vdots \\
A_{m1} & A_{m2} & A_{m3} & A_{m4} & \cdots & A_{mm}
\end{bmatrix} \qquad (m \geqslant 2).
$$

For $m \geqslant 2$, we can write A_m in the following form:

$$A_m = \begin{bmatrix} A_{m-1} & O \\ B_m & A_{mm} \end{bmatrix} \qquad (m \geqslant 2).$$

Set

$$N_m = \sum_{l=1}^{m} n_l.$$

Then A_m is an $N_m \times N_m$ matrix, while B_m is an $n_m \times N_{m-1}$ matix. We can write A_{mm} and A_m also in the following form:

$$A_{mm} = \mathscr{S}_{mm} + \mathscr{N}_{mm} \quad \text{and} \quad A_m = \mathscr{S}_m + \mathscr{N}_m$$

uniquely, where $\mathscr{S}_{mm}$ is an $n_m \times n_m$ diagonalizable matrix, $\mathscr{S}_m$ is an $N_m \times N_m$ diagonalizable matrix, $\mathscr{N}_{mm}$ is an $n_m \times n_m$ nilpotent matrix, $\mathscr{N}_m$ is an $N_m \times N_m$ nilpotent matrix, and

$$\mathscr{S}_{mm}\mathscr{N}_{mm} = \mathscr{N}_{mm}\mathscr{S}_{mm} \quad \text{and} \quad \mathscr{S}_m\mathscr{N}_m = \mathscr{N}_m\mathscr{S}_m$$

(cf. Theorem 1.1).

LEMMA 2.1. *The matrices $\mathscr{S}_m$ and $\mathscr{N}_m$ have the following forms*:

$$\mathscr{S}_1 = \mathscr{S}_{11}, \qquad \mathscr{S}_m = \begin{bmatrix} \mathscr{S}_{m-1} & O \\ \mathscr{C}_m & \mathscr{S}_{mm} \end{bmatrix} \qquad (m \geqslant 2),$$

$$\mathscr{N}_1 = \mathscr{N}_{11}, \qquad \mathscr{N}_m = \begin{bmatrix} \mathscr{N}_{m-1} & O \\ \mathscr{F}_m & \mathscr{N}_{mm} \end{bmatrix} \qquad (m \geqslant 2),$$

where $\mathscr{C}_m$ and $\mathscr{F}_m$ are $n_m \times N_{m-1}$ matrices.

Proof. We consider the case $m \geqslant 2$. Since the matrices $\mathscr{S}_m$ and $\mathscr{N}_m$ are polynomials in A_m with constant coefficients, we have

$$\mathscr{S}_m = \begin{bmatrix} \mathscr{B}_{m-1} & O \\ \mathscr{C}_m & \mu_m \end{bmatrix} \quad \text{and} \quad \mathscr{N}_m = \begin{bmatrix} \mathscr{D}_{m-1} & O \\ \mathscr{F}_m & \nu_m \end{bmatrix},$$

where $\mathscr{B}_{m-1}$ and $\mathscr{D}_{m-1}$ are $N_{m-1} \times N_{m-1}$ matrices, $\mathscr{C}_m$ and $\mathscr{F}_m$ are $n_m \times N_{m-1}$ matrices, and μ_m and ν_m are $n_m \times n_m$ matrices. Furthermore, $\mathscr{D}_{m-1}$ and ν_m are nilpotent. Also $\mathscr{B}_{m-1}\mathscr{D}_{m-1} = \mathscr{D}_{m-1}\mathscr{B}_{m-1}$ and $\mu_m \nu_m = \nu_m \mu_m$. Hence it suffices to show that $\mathscr{B}_{m-1}$ and μ_m are diagonalizable.

Note that, since $\mathscr{S}_m$ is diagonalizable, $\mathscr{S}_m$ has N_m linearly independent eigenvectors. An eigenvector of $\mathscr{S}_m$ has one of the following two forms:

$$\begin{bmatrix} \vec{p} \\ \vec{r} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \vec{0} \\ \vec{q} \end{bmatrix},$$

where $\vec{p}$ is an eigenvector of $\mathscr{B}_{m-1}$, whereas $\vec{q}$ is an eigenvector of μ_m. Therefore, if we count those independent eigenvectors, we can show that $\mathscr{B}_{m-1}$ has N_{m-1} linearly independent eigenvectors, while μ_m has n_m linearly independent eigenvectors. Note that $N_m = N_{m-1} + n_m$. This completes the proof of Lemma 2.1. ■

Lemma 2.1 implies that the matrices $\mathscr{S}_m$ and $\mathscr{N}_m$ have the following forms:

$$\mathscr{S}_1 = \mathscr{S}_{11},$$

$$\mathscr{S}_m = \begin{bmatrix} \mathscr{S}_{11} & O & O & O & \cdots & O \\ \mathscr{C}_{21} & \mathscr{S}_{22} & O & O & \cdots & O \\ \mathscr{C}_{31} & \mathscr{C}_{32} & \mathscr{S}_{33} & O & \cdots & O \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \mathscr{C}_{m1} & \mathscr{C}_{m2} & \mathscr{C}_{m3} & \mathscr{C}_{m4} & \cdots & \mathscr{S}_{mm} \end{bmatrix} \qquad (m \geqslant 2)$$

and

$$\mathscr{N}_1 = \mathscr{N}_{11},$$

$$\mathscr{N}_m = \begin{bmatrix} \mathscr{N}_{11} & O & O & O & \cdots & O \\ \mathscr{F}_{21} & \mathscr{N}_{22} & O & O & \cdots & O \\ \mathscr{F}_{31} & \mathscr{F}_{32} & \mathscr{N}_{33} & O & \cdots & O \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \mathscr{F}_{m1} & \mathscr{F}_{m2} & \mathscr{F}_{m3} & \mathscr{F}_{m4} & \cdots & \mathscr{N}_{mm} \end{bmatrix} \qquad (m \geqslant 2),$$

where $\mathscr{C}_{jk}$ and $\mathscr{F}_{jk}$ are $n_j \times n_k$ matrices.

Set

$$\mathcal{S} = \begin{bmatrix} \mathcal{S}_{11} & O & O & O & \cdots & \cdots & \cdots \\ \mathcal{C}_{21} & \mathcal{S}_{22} & O & O & \cdots & \cdots & \cdots \\ \mathcal{C}_{31} & \mathcal{C}_{32} & \mathcal{S}_{33} & O & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & & \\ \mathcal{C}_{m1} & \mathcal{C}_{m2} & \mathcal{C}_{m3} & \cdots & \mathcal{S}_{mm} & O & \cdots \\ \vdots & \vdots & \vdots & \ddots & & \ddots & \ddots \end{bmatrix},$$

$$\mathcal{N} = \begin{bmatrix} \mathcal{N}_{11} & O & O & O & \cdots & \cdots & \cdots \\ \mathcal{F}_{21} & \mathcal{N}_{22} & O & O & \cdots & \cdots & \cdots \\ \mathcal{F}_{31} & \mathcal{F}_{32} & \mathcal{N}_{33} & O & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & & \\ \mathcal{F}_{m1} & \mathcal{F}_{m2} & \mathcal{F}_{m3} & \cdots & \mathcal{N}_{mm} & O & \cdots \\ \vdots & \vdots & \vdots & \ddots & \cdots & \ddots & \ddots \end{bmatrix}.$$

Then we have

$$A = \mathcal{S} + \mathcal{N} \quad \text{and} \quad \mathcal{S}\mathcal{N} = \mathcal{N}\mathcal{S}. \tag{2.1}$$

We call (2.1) the *SN* decomposition of the matrix A.

3. THE *SN* DECOMPOSITION OF A DIFFERENTIAL OPERATOR AT A SINGULAR POINT OF THE FIRST KIND

Let us consider a differential operator

$$\mathcal{L}[\vec{y}] = x\frac{d\vec{y}}{dx} + \Omega(x)\vec{y}, \tag{3.1}$$

where $\Omega(x)$ is an $n \times n$ matrix whose entries are formal power series in x with constant coefficients, i.e.

$$\Omega(x) = \sum_{l=0}^{+\infty} x^l \Omega_l,$$

where the Ω_l are $n \times n$ constant matrices. Denote by $\mathbb{C}[[x]]$ the ring of all formal power series in x with constant coefficients, and set

$$\mathscr{V} = (\mathbb{C}[[x]])^n.$$

Then $\mathscr{V}$ is a vector space over the field $\mathbb{C}$ of complex numbers. Identify a formal power series $\vec{p} = \sum_{l \geqslant 0} x^l \vec{a}_l \in \mathscr{V}$ with a vector

$$\wp = \begin{bmatrix} \vec{a}_0 \\ \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_l \\ \vdots \end{bmatrix}.$$

Then we can represent the operator $\mathscr{L}$ by the matrix

$$A = \begin{bmatrix} \Omega_0 & O & O & O & \cdots & \cdots & \cdots & \cdots \\ \Omega_1 & I + \Omega_0 & O & O & \cdots & \cdots & \cdots & \cdots \\ \Omega_2 & \Omega_1 & 2I + \Omega_0 & O & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & & & & \\ \vdots & \vdots & & \ddots & \ddots & & & \\ \Omega_m & \Omega_{m-1} & \Omega_{m-2} & \cdots & \Omega_1 & mI + \Omega_0 & O & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots \end{bmatrix},$$

where I is the $n \times n$ identity matrix. Let

$$A = \mathscr{S} + \mathscr{N}$$

be the SN decomposition of A. Since

$$A = \begin{bmatrix} \Omega_0 & O \\ \tilde{\Omega} & I_\infty + A \end{bmatrix},$$

where

$$\tilde{\Omega} = \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \vdots \\ \Omega_m \\ \vdots \end{bmatrix}$$

and I_∞ is the $\infty \times \infty$ identity matrix, the matrices $\mathscr{S}$ and $\mathscr{N}$ have the following form:

$$\mathscr{S} = \begin{bmatrix} \mathscr{S}_0 & O & O & O & \cdots & \cdots & \cdots & \cdots \\ \sigma_1 & I + \mathscr{S}_0 & O & O & \cdots & \cdots & \cdots & \cdots \\ \sigma_2 & \sigma_1 & 2I + \mathscr{S}_0 & O & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & & & & \\ \vdots & \vdots & & \ddots & \ddots & & & \\ \sigma_m & \sigma_{m-1} & \sigma_{m-2} & \cdots & \sigma_1 & mI + \mathscr{S}_0 & O & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}$$

and

$$\mathscr{N} = \begin{bmatrix} \mathscr{N}_0 & O & O & O & \cdots & \cdots & \cdots & \cdots \\ \nu_1 & \mathscr{N}_0 & O & O & \cdots & \cdots & \cdots & \cdots \\ \nu_2 & \nu_1 & \mathscr{N}_0 & O & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \cdots & \cdots & \cdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \cdots & \cdots \\ \nu_m & \nu_{m-1} & \nu_{m-2} & \cdots & \nu_1 & \mathscr{N}_0 & O & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}$$

respectively, where the σ_j and ν_j are $n \times n$ matrices and

$$\Omega_0 = \mathscr{S}_0 + \mathscr{N}_0$$

is the SN decomposition of the matrix Ω_0.

Set

$$\sigma(x) = \mathscr{S}_0 + \sum_{l=1}^{+\infty} x^l \sigma_l \quad \text{and} \quad \nu(x) = \mathscr{N}_0 + \sum_{l=1}^{+\infty} x^l \nu_l.$$

Then $\mathscr{S}$ represents a differential operator

$$\mathscr{L}_0[\vec{y}] = x\frac{d\vec{y}}{dx} + \sigma(x)\vec{y},$$

while $\mathscr{N}$ represents multiplication by $\nu(x)$, i.e. the operator: $\vec{y} \to \nu(x)\vec{y}$. Since $A = \mathscr{S} + \mathscr{N}$ and $\mathscr{S}\mathscr{N} = \mathscr{N}\mathscr{S}$, we have

$$\mathscr{L}[\vec{y}] = \mathscr{L}_0[\vec{y}] + \nu(x)\vec{y} \quad \text{and} \quad \mathscr{L}_0[\nu(x)\vec{y}] = \nu(x)\mathscr{L}_0[\vec{y}]. \tag{3.2}$$

We call (3.2) the *SN* decomposition of operator (3.1).

We shall show in the next section that $\mathscr{L}_0[\vec{y}]$ is diagonalizable and that $\nu(x)^n = O$.

4. A NORMAL FORM OF A DIFFERENTIAL OPERATOR AT A SINGULAR POINT OF THE FIRST KIND

Let us again consider a differential operator

$$\mathscr{L}[\vec{y}] = x\frac{d\vec{y}}{dx} + \Omega(x)\vec{y}, \tag{4.1}$$

where

$$\Omega(x) = \sum_{l=0}^{+\infty} x^l \Omega_l.$$

Here the Ω_l are $n \times n$ constant matrices. In Section 3 we derived the *SN* decomposition

$$\mathscr{L}[\vec{y}] = \mathscr{L}_0[\vec{y}] + \nu(x)\vec{y} \quad \text{and} \quad \mathscr{L}_0[\nu(x)\vec{y}] = \nu(x)\mathscr{L}_0[\vec{y}], \tag{4.2}$$

where

$$\mathscr{L}_0[\vec{y}] = x\frac{d\vec{y}}{dx} + \sigma(x)\vec{y},$$

and

$$\sigma(x) = \mathscr{S}_0 + \sum_{l=1}^{+\infty} x^l \sigma_l \quad \text{and} \quad \nu(x) = \mathscr{N}_0 + \sum_{l=1}^{+\infty} x^l \nu_l.$$

Notice that

$$\Omega_0 = \mathscr{S}_0 + \mathscr{N}_0$$

is the *SN* decomposition of Ω_0.

The operator $\mathscr{L}_0[\vec{u}]$ and multiplication by $\nu(x)$ are represented by $\mathscr{S}$ and $\mathscr{N}$ of Section 3 respectively. Set

$$\mathscr{S}_m = \begin{bmatrix} \mathscr{S}_0 & O & O & O & \cdots & \cdots \\ \sigma_1 & I + \mathscr{S}_0 & O & O & \cdots & \cdots \\ \sigma_2 & \sigma_1 & 2I + \mathscr{S}_0 & O & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & & \\ \vdots & \vdots & & \ddots & \ddots & \\ \sigma_m & \sigma_{m-1} & \sigma_{m-2} & \cdots & \sigma_1 & mI + \mathscr{S}_0 \end{bmatrix}.$$

Since $\mathscr{S}_0$ is diagonalizable, there exist an invertible $n \times n$ matrix P_0 and a diagonal matrix Λ_0 such that

$$\mathscr{S}_0 P_0 = P_0 \Lambda_0, \qquad \Lambda_0 = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \lambda_n \end{bmatrix}.$$

Note that the λ_l are eigenvalues of $\mathscr{S}_0$. Hence the λ_l are eigenvalues of Ω_0.

For every positive integer m, $\mathscr{S}_m$ is diagonalizable. Hence there exists an $(m+1)n \times n$ matrix

$$\mathscr{P}_m = \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ \vdots \\ P_m \end{bmatrix}$$

such that

$$\mathscr{S}_m \mathscr{P}_m = \mathscr{P}_m \Lambda_0 .$$

If m is sufficiently large, we can further determine matrices P_l for $l \geqslant m+1$ by the equations

$$(lI + \mathscr{S}_0) P_l + \sum_{h=1}^{l} \sigma_h P_{l-h} = P_l \Lambda_0 . \tag{4.3}$$

Equation (4.3) can be solved with respect to P_l, since the linear operator

$$P_l \to (lI + \mathscr{S}_0) P_l - P_l \Lambda_0$$

is invertible if $l > \max_{j,k=1}^{n} (\lambda_j - \lambda_k)$. In this way, we can find an $\infty \times n$ matrix

$$\mathscr{P} = \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ \vdots \\ P_m \\ \vdots \end{bmatrix}$$

such that

$$\mathscr{S}\mathscr{P} = \mathscr{P}\Lambda_0 .$$

Set

$$P(x) = \sum_{l=0}^{+\infty} x^l P_l.$$

Then the entries of $P(x)^{-1}$ are also formal power series in x, and

$$\mathscr{L}_0[P(x)] = P(x)\Lambda_0. \tag{4.4}$$

Let us define two differential operators and an $n \times n$ matrix by

$$\mathscr{K}[\vec{u}] = P(x)^{-1}\mathscr{L}[P(x)\vec{u}], \qquad \mathscr{K}_0[\vec{u}] = P(x)^{-1}\mathscr{L}_0[P(x)\vec{u}]$$

and

$$\nu_0(x) = P(x)^{-1}\nu(x)P(x)$$

respectively. Then

$$\mathscr{K}[\vec{u}] = \mathscr{K}_0[\vec{u}] + \nu_0(x)\vec{u}.$$

Observe that

$$\begin{aligned}\mathscr{K}_0[\vec{u}] &= P(x)^{-1}\mathscr{L}_0[P(x)\vec{u}] \\ &= P(x)^{-1}\left[P(x)\left(x\frac{d\vec{u}}{dx}\right) + \mathscr{L}_0[P(x)]\vec{u}\right] \\ &= x\frac{d\vec{u}}{dx} + \Lambda_0\vec{u}.\end{aligned} \tag{4.5}$$

This means that the operator $\mathscr{L}_0[\vec{y}]$ is diagonalizable. Furthermore,

$$\begin{aligned}&x\frac{d\nu_0(x)}{dx} + \Lambda_0\nu_0(x) \\ &\quad = P(x)^{-1}\mathscr{L}_0[P(x)\nu_0(x)] = P(x)^{-1}\mathscr{L}_0[\nu(x)P(x)] \\ &\quad = P(x)^{-1}\nu(x)\mathscr{L}_0[P(x)] = \nu_0 P(x)^{-1} fL_0[P(x)] \\ &\quad = \nu_0(x)\Lambda_0.\end{aligned} \tag{4.6}$$

This means that the entry $\nu_{jk}(x)$ on the jth row and kth column of the matrix $\nu_0(x)$ must have the form:

$$\nu_{jk}(x) = \gamma_{jk} x^{\lambda_k - \lambda_j}, \tag{4.7}$$

where γ_{jk} is a constant. Since $\nu_0(x)$ is a formal power series in x, we must have

$$\nu_{jk}(x) = 0 \quad \text{if } \lambda_k - \lambda_j \text{ is not a nonnegative integer.} \tag{4.8}$$

Observe further that the matrix $\nu_0(x)$ can be written in the form

$$\nu_0(x) = x^{-\Lambda_0}\Gamma x^{\Lambda_0}, \qquad \Gamma = \begin{bmatrix} \gamma_{11} & \cdots & \gamma_{1n} \\ \vdots & & \vdots \\ \gamma_{n1} & \cdots & \gamma_{nn} \end{bmatrix}.$$

Hence, for any nonnegative integer p, we have

$$\nu_0(x)^p = x^{-\Lambda_0}\Gamma^p x^{\Lambda_0},$$

where

$$z^{\Lambda_0} = \exp[(\log z)\Lambda_0] = \begin{bmatrix} z^{\lambda_1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & z^{\lambda_2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & z^{\lambda_3} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & z^{\lambda_n} \end{bmatrix}.$$

On the other hand, since $\mathcal{N}_0$ is nilpotent, we can write $\nu_0(x)^p$ in the form

$$\nu_0(x)^p = x^{m_p} Q_p(x),$$

where m_p is a nonnegative integer such that $\lim_{p \to +\infty}, m_p = +\infty$ and the entries of the matrix $Q_p(x)$ are power series in x with constant coefficients. Therefore

$$\nu_0(x)^p = O \quad \text{if } p \text{ is sufficiently large.}$$

This implies that the matrix Γ is nilpotent, i.e.,

$$\Gamma^n = O.$$

Since $\nu(x) = P(x)\nu_0(x)P(x)^{-1}$, we have

$$\nu(x)^n = O.$$

Thus we arrive at the following conclusion.

THEOREM 4.1. *For a given differential operator* (4.1), *let*

$$\Omega_0 = \mathscr{S}_0 + \mathscr{N}_0$$

be the SN decomposition of the matrix Ω_0. *Then there exists an* $n \times n$ *matrix* $P(x)$ *such that*

(1) *the entries of* $P(x)$ *are formal power series in* x *with constant coefficients*

(2) $P(0)$ *is invertible and* $\mathscr{S}_0 P(0) = P(0)\Lambda_0$, *where* Λ_0 *is a diagonal matrix whose diagonal entries are eigenvalues of* Ω_0

(3) *the transformation*

$$\vec{y} = P(x)\vec{u} \tag{4.9}$$

changes the differential operator (4.1) *to another differential operator*:

$$P(x)^{-1}\mathscr{L}[P(x)\vec{u}] = \mathscr{K}_0[\vec{u}] + \nu_0(x)\vec{u} \tag{4.10}$$

where

$$\mathscr{K}_0[\vec{u}] = x\frac{d\vec{u}}{dx} + \Lambda_0\vec{u}, \qquad \nu_0[x] = x^{-\Lambda_0}\Gamma x^{\Lambda_0}$$

and

$$\Gamma = \begin{bmatrix} \gamma_{11} & \cdots & \gamma_{1n} \\ \vdots & & \vdots \\ \gamma_{n1} & \cdots & \gamma_{nn} \end{bmatrix}$$

with constants γ_{jk} *such that*

$$\gamma_{jk} = 0 \quad \text{if } \lambda_k - \lambda_j \text{ is not a nonnegative integer}. \tag{4.11}$$

and that the matrix Γ *is nilpotent.*

REMARK 4.2. Since the entries of $\nu_0(x)$ are polynomials in x, the power series $P(x)$ is convergent if $\Omega(x)$ is convergent. Therefore, in such a case, $\nu(x)$ is convergent and hence $\sigma(x)$ is convergent.

5. EXAMPLES

In order to illustrate the method of previous sections, let us consider differential equations of the Bessel functions:

$$z\frac{d}{dz}\left(z\frac{dy}{dz}\right) + (z^2 - a^2)y = 0, \tag{5.1}$$

where a is a nonnegative integer. If we change the independent variable z aby $x = z^2$, Equation (5.1) becomes

$$x\frac{d}{dx}\left(x\frac{dy}{dx}\right) + \frac{x - a^2}{4}y = 0.$$

This equation is equivalent to the system

$$x\frac{d\vec{y}}{dx} + \Omega(x)\vec{y} = \vec{0}, \qquad \vec{y} = \begin{bmatrix} y \\ x\,dy/dx \end{bmatrix}, \tag{5.2}$$

where

$$\Omega(x) = \begin{bmatrix} 0 & -1 \\ (x - a^2)/4 & 0 \end{bmatrix} = \Omega_0 + x\Omega_1,$$

$$\Omega_0 = \begin{bmatrix} 0 & -1 \\ -a^2/4 & 0 \end{bmatrix}, \qquad \Omega_1 = \begin{bmatrix} 0 & 0 \\ \frac{1}{4} & 0 \end{bmatrix}.$$

To begin with, let us remark that the SN decomposition of the matrix Ω_0 is given by

$$\Omega_0 = \mathscr{S}_0 + \mathscr{N}_0,$$

where

$$\mathscr{S}_0 = \begin{cases} O & \text{if} \quad a = 0, \\ \Omega_0 & \text{if} \quad a > 0, \end{cases}$$

$$\mathscr{N}_0 = \begin{cases} \Omega_0 & \text{if} \quad a = 0, \\ O & \text{if} \quad a > 0. \end{cases}$$

Set also

$$\Lambda_0 = \begin{bmatrix} -a/2 & 0 \\ 0 & a/2 \end{bmatrix}.$$

Note that two eigenvalues of Ω_0 are $\pm a/2$.

Now we calculate in the following steps:

Step 1. Fix a nonnegative integer m so that $m \geqslant a/2 - (-a/2) = a$.

Step 2. Find three $2(m + 1) \times 2(m + 1)$ matrices A_m, $\mathscr{S}_m$, and $\mathscr{N}_m$ (cf. Sections 3 and 4).

Step 3. Find a $2(m+1) \times 2$ matrix

$$\mathscr{P}_m = \begin{bmatrix} P_0 \\ P_1 \\ \vdots \\ P_m \end{bmatrix},$$

where the P_l are 2×2 matrices such that $\mathscr{S}_m \mathscr{P}_m = \mathscr{P}_m \Lambda_0$.

Step 4. Find two 2×2 matrices

$$M_m(x) = \mathscr{N}_0 + \sum_{l=1}^{m} x^l \nu_l, \qquad Q_m(x) = \sum_{l=0}^{m} x^l P_l$$

(cf. Sections 3 and 4).

Step 5. We must obtain $Q_m(x)^{-1} M_m(x) Q_m(x) = \nu_0(x) + O(x^{m+1})$, where

$$\nu_0(x) = \begin{cases} P_0^{-1} \Omega_0 P_0 & \text{if} \quad a = 0, \\ \begin{bmatrix} 0 & \alpha(a) x^a \\ 0 & 0 \end{bmatrix} & \text{if} \quad a > 0, \end{cases}$$

where $\alpha(a)$ is a real constant depending on a (cf. Theorem 4.1).

After these calculations have been completed, we come to the following conclusion.

CONCLUSION 5.1. *There exists a unique* 2×2 *matrix*

$$P(x) = \sum_{l=0}^{+\infty} x^l P_l$$

such that

(1) *the matrices P_l for $l = 0, 1, \ldots, m$ are given in Step* 3;
(2) *the power series $P(x)$ converges for every x;*
(3) *the transformation $\vec{y} = P(x)\vec{u}$ changes the system* (5.2) *to*

$$x\left(\frac{d\vec{u}}{dx}\right) + [\Lambda_0 + \nu_0(x)]\vec{u} = \vec{0}. \tag{5.3}$$

EXAMPLE 1. Let us apply the scheme to the case when $a = 0$. First of all, we fix $m = 2$. Then

$$A_2 = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ \frac{1}{4} & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & \frac{1}{4} & 0 & 0 & 2 \end{bmatrix},$$

$$\mathcal{S}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & -\frac{1}{2} & 1 & 0 & 0 & 0 \\ \frac{1}{4} & -\frac{1}{4} & 0 & 1 & 0 & 0 \\ -\frac{3}{32} & \frac{3}{32} & \frac{1}{4} & -\frac{1}{2} & 2 & 0 \\ -\frac{1}{16} & \frac{3}{32} & \frac{1}{4} & -\frac{1}{4} & 0 & 2 \end{bmatrix},$$

and

$$\mathcal{N}_2 = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{4} & \frac{1}{2} & 0 & -1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\ \frac{3}{32} & -\frac{3}{32} & -\frac{1}{4} & \frac{1}{2} & 0 & -1 \\ \frac{1}{16} & -\frac{3}{32} & 0 & \frac{1}{4} & 0 & 0 \end{bmatrix}.$$

By finding eigenvectors of $\mathcal{S}_2$ we can find a 6×2 matrix

$$\mathcal{P}_2 = \begin{bmatrix} 192 & -320 \\ 64 & -128 \\ -16 & 16 \\ -32 & 48 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

such that $\mathcal{S}_2\mathcal{P}_2 = O$. Note that, in this case, $\Lambda_0 = O$.

Set

$$\nu_1 = \begin{bmatrix} -\frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{4} \end{bmatrix}, \qquad \nu_2 = \begin{bmatrix} \frac{3}{32} & -\frac{3}{32} \\ \frac{1}{16} & -\frac{3}{32} \end{bmatrix},$$

$$M_2(x) = \Omega_0 + x\nu_1 + x^2\nu_2,$$

and

$$P_0 = \begin{bmatrix} 192 & -320 \\ 64 & -128 \end{bmatrix}, \qquad P_1 = \begin{bmatrix} -16 & 16 \\ -32 & 48 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{5.4}$$

$$Q_2(x) = P_0 + xP_1 + x^2P_2.$$

Then we obtain

$$Q_2(x)^{-1}M_2(x)Q_2(x) = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} + O(x^3).$$

It must be noted that

$$P_0^{-1}\Omega_0 P_0 = \begin{bmatrix} \frac{1}{32} & -\frac{5}{64} \\ \frac{1}{64} & -\frac{3}{64} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 192 & -320 \\ 64 & -128 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}.$$

Thus we arrive at the following conclusion.

CONCLUSION 5.2. *There exists a unique* 2×2 *matrix*

$$P(x) = \sum_{l=0}^{+\infty} x^l P_l$$

such that

(1) *the matrices* P_l *for* $l = 0, 1, 2$ *are given by* (5.4);
(2) *the power series* $P(x)$ *converges for every* x;
(3) *the transformation* $\vec{y} = P(x)\vec{u}$ *changes the system*

$$x\frac{d\vec{y}}{dx} + \begin{bmatrix} 0 & -1 \\ x/4 & 0 \end{bmatrix}\vec{y} = \vec{0} \tag{5.5}$$

to

$$x\frac{d\vec{u}}{dx} + \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}\vec{u} = \vec{0}. \tag{5.6}$$

Furthermore, the transformation $\vec{y} = P(x)P_0^{-1}\vec{v}$ *changes the system* (5.5) *to*

$$x\frac{d\vec{v}}{dx} + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}\vec{v} = \vec{0}. \tag{5.7}$$

EXAMPLE 2. Let us apply the scheme to the case when $a = 2$. This time, we fix $m = 4$. Then A_4, $\mathscr{S}_4$, and $\mathscr{N}_4$ were given in Example 3 of Section 1. By finding eigenvectors of $\mathscr{S}_4$ we can find a 10×2 matrix

$$\mathscr{P}_4 = \begin{bmatrix} 2211840 & -12288 \\ 2211840 & 12288 \\ -184320 & -3072 \\ -368640 & 0 \\ 5760 & 800 \\ 17280 & 1184 \\ -96 & -24 \\ -384 & -80 \\ 1 & 0 \\ 5 & 1 \end{bmatrix}$$

such that $\mathscr{S}_4\mathscr{P}_4 = \mathscr{P}_4\Lambda_0$. Note that, in this case,

$$\Lambda_0 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Set

$$\nu_2 = \begin{bmatrix} \frac{1}{64} & -\frac{1}{64} \\ \frac{1}{64} & -\frac{1}{64} \end{bmatrix}, \qquad \nu_3 = \begin{bmatrix} -\frac{1}{256} & \frac{1}{384} \\ -\frac{1}{192} & \frac{1}{256} \end{bmatrix},$$

$$\nu_4 = \begin{bmatrix} \frac{7}{18432} & -\frac{7}{36864} \\ \frac{25}{36864} & -\frac{7}{18432} \end{bmatrix},$$

$$M_4(x) = x^2\nu_2 + x^3\nu_3 + x^4\nu_4,$$

and

$$P_0 = \begin{bmatrix} 2211840 & -12288 \\ 2211840 & 12288 \end{bmatrix}, \qquad P_1 = \begin{bmatrix} -184320 & -3072 \\ -368640 & 0 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 5760 & 800 \\ 17280 & 1184 \end{bmatrix}, \qquad P_3 = \begin{bmatrix} -96 & -24 \\ -384 & -80 \end{bmatrix}, \tag{5.8}$$

$$P_4 = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix},$$

$$Q_4(x) = P_0 + xP_1 + x^2P_2 + x^3P_3 + x^4P_4.$$

Then we obtain

$$Q_4(x)^{-1}M_4(x)Q_4(x) = \begin{bmatrix} 0 & -x^2/5760 \\ 0 & 0 \end{bmatrix} + O(x^5).$$

Thus we arrive at the following conclusion.

CONCLUSION 5.4. *There exists a unique* 2×2 *matrix*

$$P(x) = \sum_{l=0}^{+\infty} x^l P_l$$

such that

(1) *the matrices* P_l *for* $l = 0, 1, 2, 3, 4$ *are given by* (5.8);
(2) *the power series* $P(x)$ *converges for every* x;
(3) *the transformation* $\vec{y} = P(x)\vec{u}$ *changes the system*

$$x\frac{d\vec{y}}{dx} + \begin{bmatrix} 0 & -1 \\ (x-4)/4 & 0 \end{bmatrix}\vec{y} = \vec{0} \tag{5.9}$$

to

$$x\frac{d\vec{u}}{dx} + \begin{bmatrix} -1 & -x^2/5760 \\ 0 & 1 \end{bmatrix}\vec{u} = \vec{0}. \tag{5.10}$$

6. AN EXAMPLE OF EQUATIONS CONTAINING PARAMETERS

In order to illustrate our method applied to differential equations containing parameters, let us consider the following differential equation:

$$x\frac{d\vec{y}}{dx} + \begin{bmatrix} 0 & a + cx \\ bx & 1 \end{bmatrix}\vec{y} = 0, \tag{6.1}$$

where a, b, and c are parameters. In this case we have

$$\Omega(x) = \begin{bmatrix} 0 & a + cx \\ bx & 1 \end{bmatrix} = \Omega_0 + x\Omega_1,$$

$$\Omega_0 = \begin{bmatrix} 0 & a \\ 0 & 1 \end{bmatrix}, \qquad \Omega_1 = \begin{bmatrix} 0 & c \\ b & 0 \end{bmatrix}.$$

To begin with, let us remark that the SN decomposition of the matrix Ω_0 is given by

$$\mathscr{S}_0 = \Omega_0, \qquad \mathscr{N}_0 = O.$$

Note that two eigenvalues of Ω_0 are 0 and 1. Set

$$\Lambda_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{6.2}$$

Since two eigenvalues of Ω_0 differ by 1, we set $m = 3$. Then

$$A_3 = \begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & 1 & a & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 2 & a & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c & 3 & a \\ 0 & 0 & 0 & 0 & b & 0 & 0 & 4 \end{bmatrix}, \tag{6.3}$$

$$\mathscr{S}_3 = \begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a^2b & 1 & a & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ \frac{1}{2}bd & abd & 0 & a^2b & 2 & a & 0 & 0 \\ 0 & -\frac{1}{2}bd & b & 0 & 0 & 3 & 0 & 0 \\ \frac{5}{12}ab^2d & \frac{1}{12}(5a^2b + 6c)d & \frac{1}{2}bd & abd & 0 & a^2b & 3 & a \\ -\frac{1}{4}b^2d & -\frac{5}{12}ab^2d & 0 & -\frac{1}{2}bd & b & 0 & 0 & 4 \end{bmatrix}, \tag{6.4}$$

and

$$\mathscr{N}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -d & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2}bd & -abd & 0 & -d & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}bd & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{5}{12}ab^2d & -\frac{1}{12}b(5a^2b + 6c)d & -\frac{1}{2}bd & -abd & 0 & -d & 0 & 0 \\ \frac{1}{4}b^2d & \frac{5}{12}ab^2d & 0 & \frac{1}{2}bd & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{6.5}$$

where

$$d = a^2b - c. \tag{6.6}$$

By finding eigenvectors of $\mathscr{S}_3$ we can find an 8×2 matrix

$$\mathscr{P}_3 = \begin{bmatrix} 1 & -a \\ 0 & 1 \\ \frac{1}{2}ab & 0 \\ -\frac{1}{2}b & -ab \\ \frac{1}{12}b(a^2b + 3c) & \frac{1}{4}ab(-3a^2b + 7c) \\ -\frac{1}{6}ab^2 & \frac{1}{4}bd \\ \frac{1}{144}ab^2(a^2b + 11c) & \frac{1}{36}b(-7a^4b^2 + 4a^2bc + 9c^2) \\ -\frac{1}{48}b^2(a^2b + 3c) & \frac{1}{36}ab^2(11a^2b - 23c) \end{bmatrix} \tag{6.7}$$

such that $\mathscr{S}_3\mathscr{P}_3 = \mathscr{P}_3\Lambda_0$.

Set

$$\nu_1 = \begin{bmatrix} 0 & -d \\ 0 & 0 \end{bmatrix}, \qquad \nu_2 = \begin{bmatrix} -\frac{1}{2}bd & -abd \\ 0 & \frac{1}{2}bd \end{bmatrix},$$
$$\nu_3 = \begin{bmatrix} -\frac{5}{12}ab^2d & -\frac{1}{12}b(5a^2b + 6c)d \\ \frac{1}{4}b^2d & \frac{5}{12}ab^2d \end{bmatrix}, \tag{6.8}$$
$$M_3(x) = x\nu_1 + x^2\nu_2 + x^3\nu_3,$$

and

$$P_0 = \begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix}, \qquad P_1 = \begin{bmatrix} \frac{1}{2}ab & 0 \\ -\frac{1}{2}b & -ab \end{bmatrix},$$
$$P_2 = \begin{bmatrix} \frac{1}{12}b(a^2b + 3c) & \frac{1}{4}ab(-3a^2b + 7c) \\ -\frac{1}{6}ab^2 & \frac{1}{4}bd \end{bmatrix},$$
$$P_3 = \begin{bmatrix} \frac{1}{144}ab^2(a^2b + 11c) & \frac{1}{36}(-7a^4b^2 + 4a^2bc + 9c^2) \\ -\frac{1}{48}b^2(a^2b + 3c) & \frac{1}{36}ab^2(11a^2b - 23c) \end{bmatrix}, \tag{6.9}$$
$$Q_3(x) = P_0 + xP_1 + x^2P_2 + x^3P_3.$$

Then we obtain

$$Q_3(x)^{-1}M_3(x)Q_3(x) = x\begin{bmatrix} 0 & -d \\ 0 & 0 \end{bmatrix} + O(x^4).$$

Thus we arrive at the following conclusion.

CONCLUSION 6.1. *There exists a unique* 2×2 *matrix*

$$P(x) = \sum_{l=0}^{+\infty} x^l P_l$$

such that

(1) *the matrices* P_l *for* $l = 0, 1, 2, 3$ *are given by* (6.9);
(2) *the power series* $P(x)$ *converges for every* x;
(3) *the transformation* $\vec{y} = P(x)\vec{u}$ *changes Equation* (6.1) *to*

$$x\frac{d\vec{u}}{dx} + \begin{bmatrix} 0 & -(a^2b - c)x \\ 0 & 1 \end{bmatrix}\vec{u} = \vec{0}. \tag{6.10}$$

7. A DIFFERENTIAL OPERATOR AT A SINGULAR POINT OF THE SECOND KIND

Let us consider a differential operator

$$\mathscr{L}[\vec{y}] = x^{q+1}\frac{d\vec{y}}{dx} + \Omega(x)\vec{y}, \tag{7.1}$$

where q is a positive integer and

$$\Omega(x) = \sum_{l=0}^{+\infty} x^l \Omega_l,$$

and the Ω_l are $n \times n$ constant matrices. We can represent the operator $\mathscr{L}$

by the matrix

$$A = \begin{bmatrix} \Omega_0 & O_1 \\ \tilde{\Omega} & J_\infty + A \end{bmatrix},$$

where

$$\tilde{\Omega} = \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \vdots \\ \Omega_m \\ \vdots \end{bmatrix}$$

and

$$J_\infty = \begin{bmatrix} O_q \\ I_\infty \end{bmatrix}.$$

Here O_r is the $nr \times \infty$ zero matrix, and I_∞ is the $\infty \times \infty$ identity matrix. Let

$$A = \mathscr{S} + \mathscr{N} \quad \text{and} \quad \Omega_0 = \mathscr{S}_0 + \mathscr{N}_0$$

be the SN decompositions of A and Ω_0 respectively. Then we can prove without any difficulty that

$$\mathscr{S} = \begin{bmatrix} \mathscr{S}_0 & O_1 \\ \tilde{\mathscr{S}} & \mathscr{S} \end{bmatrix} \quad \text{and} \quad \mathscr{N} = \begin{bmatrix} \mathscr{N}_0 & O_1 \\ \tilde{\mathscr{N}} & J_\infty + \mathscr{N} \end{bmatrix},$$

where

$$\tilde{\mathscr{S}} = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_m \\ \vdots \end{bmatrix} \quad \text{and} \quad \tilde{\mathscr{N}} = \begin{bmatrix} \nu_1 \\ \nu_2 \\ \vdots \\ \nu_m \\ \vdots \end{bmatrix}.$$

Here the σ_j and ν_j are $n \times n$ matrices.

Set

$$\sigma(x) = \mathscr{S}_0 + \sum_{l=1}^{+\infty} x^l \sigma_l \quad \text{and} \quad \nu(x) = \mathscr{N}_0 + \sum_{l=1}^{+\infty} x^l \nu_l.$$

Then $\mathscr{S}$ represents a multiplication operator $\vec{y} \to \sigma(x)\vec{y}$, and $\mathscr{N}$ represents the differential operator

$$\mathscr{L}_0[\vec{y}] = x^{q+1}\frac{d\vec{y}}{dx} + \nu(x)\vec{y}.$$

Since $A = \mathscr{S} + \mathscr{N}$ and $\mathscr{S}\mathscr{N} = \mathscr{N}\mathscr{S}$, we have

$$\mathscr{L}[\vec{y}] = \mathscr{L}_0[\vec{y}] + \sigma(x)\vec{y} \quad \text{and} \quad \mathscr{L}_0[\sigma(x)\vec{y}] = \sigma(x)\mathscr{L}_0[\vec{y}]. \tag{7.2}$$

We call (7.2) the SN decomposition of the operator (7.1).

Set

$$\mathscr{S}_m = \begin{bmatrix} \mathscr{S}_0 & O & O & O & \cdots & \cdots \\ \sigma_1 & \mathscr{S}_0 & O & O & \cdots & \cdots \\ \sigma_2 & \sigma_1 & \mathscr{S}_0 & O & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ \vdots & \vdots & & \ddots & \ddots & \ddots \\ \sigma_m & \sigma_{m-1} & \sigma_{m-2} & \cdots & \sigma_1 & \mathscr{S}_0 \end{bmatrix}. \tag{7.3}$$

Since $\mathscr{S}_0$ is diagonalizable, there exist an invertible $n \times n$ matrix P_0 and a diagonal matrix Λ_0 such that

$$\mathscr{S}_0 P_0 = P_0 \Lambda_0, \qquad \Lambda_0 = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \lambda_n \end{bmatrix}.$$

Note that the λ_l are eigenvalues of $\mathscr{S}_0$. Hence the λ_l are eigenvalues of Ω_0.

For every positive integer m, $\mathscr{S}_m$ is diagonalizable. Hence there exists an $(m+1)n \times n$ matrix

$$\mathscr{P}_m = \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ \vdots \\ P_m \end{bmatrix}$$

such that

$$\mathscr{S}_m \mathscr{P}_m = \mathscr{P}_m \Lambda_0 .$$

It can be shown by examining the structure of linearly independent eigenvectors of $\mathscr{S}_m$ that we can choose the matrix $\mathscr{P}_m$ in such a way that the $n \times n$ matrices $P_0, \dots, P_m$ are independent of m. In this way, we can find an $\infty \times n$ matrix

$$\mathscr{P} = \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ \vdots \\ P_m \\ \vdots \end{bmatrix}$$

such that

$$\mathscr{S}\mathscr{P} = \mathscr{P}\Lambda_0 .$$

Set

$$P(x) = \sum_{l=0}^{+\infty} x^l P_l .$$

Then the entries of $P(x)^{-1}$ are also formal power series in x, and

$$\sigma(x)P(x) = P(x)\Lambda_0, \qquad \text{or} \quad P(x)^{-1}\sigma(x)P(x) = \Lambda_0 . \quad (7.4)$$

Let us define two differential operators by

$$\mathscr{K}[\vec{u}] = P(x)^{-1}\mathscr{L}[P(x)\vec{u}] \quad \text{and} \quad \mathscr{K}_0[\vec{u}] = P(x)^{-1}\mathscr{L}_0[P(x)\vec{u}]$$

respectively. Then

$$\mathscr{K}[\vec{u}] = \mathscr{K}_0[\vec{u}] + \Lambda_0\vec{u}.$$

Observe that

$$\begin{aligned}\mathscr{K}_0[\vec{u}] &= P(x)^{-1}\mathscr{L}_0[P(x)\vec{u}]\\ &= P(x)^{-1}\left[P(x)\left(x^{q+1}\frac{d\vec{u}}{dx}\right) + \mathscr{L}_0[P(x)]\vec{u}\right]\\ &= x^{q+1}\frac{d\vec{u}}{dx} + \nu_0(x)\vec{u},\end{aligned} \tag{7.5}$$

where

$$\nu_0(x) = P(x)^{-1}\mathscr{L}_0[P(x)] = x^{q+1}P(x)^{-1}\frac{dP(x)}{dx} + P(x)^{-1}\nu(x)P(x). \tag{7.6}$$

Furthermore,

$$\begin{aligned}\Lambda_0\nu_0(x) &= P(x)^{-1}\sigma(x)P(x)\nu_0(x) = P(x)^{-1}\sigma(x)\mathscr{L}_0[P(x)]\\ &= P(x)^{-1}\mathscr{L}_0[\sigma(x)P(x)] = P(x)^{-1}\mathscr{L}_0[P(x)\Lambda_0]\\ &= \nu_0(x)\Lambda_0.\end{aligned} \tag{7.7}$$

This means that the entry $\nu_{jk}(x)$ on the jth row and kth column of the matrix $\nu_0(x)$ must satisfy the condition

$$\nu_{jk}(x) = 0 \qquad \text{if} \quad \lambda_j \neq \lambda_k. \tag{7.8}$$

Hence the operator $\mathscr{K}$ has a block-diagonal form.

EXAMPLE. Let us apply the scheme to the following operator:

$$\mathscr{L}[\vec{y}] = x^3 \frac{d\vec{y}}{dx} + (\Omega_0 + x\Omega_1)\vec{y},$$

where

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \qquad \Omega_0 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \qquad \Omega_1 = \begin{bmatrix} 5 & 8 \\ 3 & 1 \end{bmatrix}.$$

This operator is represented by the matrix

$$A = \begin{bmatrix} \Omega_0 & O_1 \\ \tilde{\Omega} & J_\infty + A \end{bmatrix}.$$

Then A_5 and the SN decomposition $A_5 = \mathscr{S}_5 + \mathscr{N}_5$ were already obtained in Example 4 in Section 1. By finding eigenvectors of $\mathscr{S}_5$ we can find a 12×2 matrix

$$\mathscr{P}_5 = \begin{bmatrix} 512 & 512 \\ -1024 & 0 \\ 6272 & 3200 \\ -7168 & 768 \\ 17952 & 6688 \\ -17088 & 2688 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 117789 & 7755 \end{bmatrix}$$

such that $\mathscr{S}_5\mathscr{P}_5 = \mathscr{P}_5\Lambda_0$. Note that, in this case,

$$\Lambda_0 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Set

$$\nu_1 = \begin{bmatrix} \frac{13}{2} & \frac{7}{2} \\ 0 & -\frac{1}{2} \end{bmatrix}, \quad \nu_2 = \begin{bmatrix} \frac{21}{8} & \frac{189}{8} \\ \frac{21}{2} & -\frac{21}{8} \end{bmatrix}, \quad \nu_3 = \begin{bmatrix} -\frac{765}{16} & -\frac{1149}{16} \\ -\frac{117}{8} & \frac{765}{16} \end{bmatrix},$$

$$\nu_4 = \begin{bmatrix} \frac{31773}{128} & -\frac{1827}{128} \\ -\frac{1659}{16} & -\frac{31773}{128} \end{bmatrix}, \quad \nu_5 = \begin{bmatrix} -\frac{131403}{256} & \frac{411849}{256} \\ \frac{119055}{128} & \frac{131403}{256} \end{bmatrix},$$

$$M_5(x) = x\nu_1 + x^2\nu_2 + x^3\nu_3 + x^4\nu_4 + x^5\nu_5,$$

and

$$P_0 = \begin{bmatrix} 512 & 512 \\ -1024 & 0 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 6272 & 3200 \\ -7168 & 768 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 17952 & 6688 \\ -17088 & 2688 \end{bmatrix}, \quad P_3 = O, \quad P_4 = O,$$

$$P_5 = \begin{bmatrix} 0 & 0 \\ 117789 & 7755 \end{bmatrix},$$

$$Q_5(x) = P_0 + xP_1 + x^2P_2 + x^5P_5.$$

Then we obtain

$$Q_5(x)^{-1}x^3\frac{dQ_5(x)}{dx} + Q_5(x)^{-1}M_5(x)Q_5(x)$$

$$= \begin{bmatrix} \lambda_1(x) & 0 \\ 0 & \lambda_2(x) \end{bmatrix} + O(x^6),$$

where

$$\lambda_1(x) = -\frac{x}{2} - \frac{63x^2}{8} + \frac{553x^3}{16} - \frac{9875x^4}{128} - \frac{9401x^5}{256},$$

$$\lambda_2(x) = \frac{13x}{2} + \frac{63x^2}{8} - \frac{329x^3}{16} + \frac{6627x^4}{128} - \frac{10199x^5}{256}.$$

REMARK. As we have shown above, our method is useful even for the case of a singular point of the second kind. However, our method yields only a block-diagonalization of the given system in this case. This means that, if the matrix Ω_0 on the right-hand side of (7.1) admits at least two distinct eigenvalues, then our method reduces (7.1) to at least two systems of smaller sizes. In particular, in the case when Ω_0 admits n distinct eigenvalues as in the example given above, our method decomposes the given system into a diagonalized system. However, for example, in case when Ω_0 is nilpotent, our method does not yield anything.

8. PROGRAMS

First we try to calculate the polynomials $P_1(\lambda), \ldots, P_k(\lambda)$ of Section 1 for a given polynomial

$$p_A(\lambda) = (\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_k)^{m_k}, \qquad n = m_1 + \cdots + m_k.$$

As in Section 1, we look at the decomposition into partial fractions:

$$\frac{1}{p_A(\lambda)} = \sum_{l=1}^{k} \frac{Q_l(\lambda)}{(\lambda - \lambda_l)^{m_l}}$$

and define

$$P_j(\lambda) = Q_j(\lambda) \frac{p_A(\lambda)}{(\lambda - \lambda_j)^{m_j}}.$$

Since $1 = \sum_{l=1}^{k} P_l(\lambda)$ and $P_l(\lambda) = O((\lambda - \lambda_j)^{m_j})$ for $l \neq j$, we have

$$1 = P_j(\lambda) + O\left((\lambda - \lambda_j)^{m_j}\right).$$

Keeping this in mind, we set

$$\frac{p_A(\lambda)}{(\lambda - \lambda_j)^{m_j}} = \sum_{h=0}^{n-m_j} w_{j,h}(\lambda - \lambda_j)^h \quad \text{and} \quad Q_j(\lambda) = \sum_{h=0}^{m_j-1} a_{j,k}(\lambda - \lambda_j)^h.$$

Then

$$1 = w_{j,0}a_{j,0}, \qquad 0 = w_{j,0}a_{j,p} + \sum_{r=0}^{p-1} w_{j,p-r}a_{j,r} \quad (p = 1, \ldots, m_j - 1).$$

Hence

$$a_{j,0} = \frac{1}{w_{j,0}}, \qquad a_{j,p} = -\frac{\sum_{r=0}^{p-1} w_{j,p-r}a_{j,r}}{w_{j,0}} \quad (p = 1, \ldots, m_j - 1).$$

Thus we arrive at the following program for Mathematica:

```
L[j_] := L[j]; M[j_] := M[j];
F[x_, j_] := (x - L[j])^M[j];
G[x_, 1] := F[x, 1]; G[x_, k_] := G[x, k - 1]F[x, k];
G[x_, k_, j_] := G[x, k]/F[x, j];
H[x_, k_, j_, 0] := G[x, k, j];
H[x_, k_, j_, h_] := D[H[x, k, j, h - 1], x];                    (8.1)
W[k_, j_, h_] := Limit[H[x, k, j, h]/(h!), x -> L[j]];
B[k_, j_, 0] := 1/W[k, j, 0];
B[k_, j_, p_] := -Sum[W[k, j, p - r]B[k, j, r], {r, 0, p - 1}]
/W[k, j, 0];
Q[x_, k_, j_] := Sum[(x - L[j])^rB[k, j, r], {r, 0, M[j] - 1}];
P[x_, k_, j_] := Q[x, k, j]G[x, k, j];
```

Here we specify the $L[j]$ and $M[j]$ then our computer will produce the polynomials $P_j(\lambda)$.

For calculating the matrices $P_j(A)$, S, and N, the main problem is how to replace λ by A. To do this, we first expand these matrices in a suitable way and then we replace powers of some quantities by powers of corresponding matrices. For example, after the program (8.1), we expand $P_j(\lambda)$ in powers of $\lambda - \lambda_j$. Actually, after (8.1), we start with

$$\begin{aligned} &T[j\_, p\_] := \mathrm{Min}[p, M[j] - 1]; \\ &Y[k\_, j\_, p\_] := \mathrm{Sum}[W[k, j, p - r]B[k, j, r], \{r, 0, T[j, p]\}]; \end{aligned} \tag{8.2}$$

Then we specify the $L[j]$, $M[j]$, and A. For example, in case of Example 1 of Section, we set

$$\begin{aligned} &L[1] = 1, \quad L[2] = 11, \quad M[1] = 2, \quad M[2] = 1, \\ &A = \{\{3, 4, 3\}, \{2, 7, 4\}, \{-4, 8, 3\}\}, \quad r = R[2]. \end{aligned} \tag{8.3}$$

Then

$$\begin{aligned} &X[h\_] := A - L[h]\mathrm{IdentityMatrix}[r]; \\ &Z[h\_, 0] := \mathrm{IdentityMatrix}[r]; Z[h\_, 1] \\ &:= X[h]; Z[h\_, q\_] := X[h].Z[h, q - 1]; \\ &P[j\_] := \mathrm{Sum}[Y[2, j, p]Z[j, p], \{p, 0, r - 1\}]; \\ &S = \mathrm{Sum}[L[j]P[j], \{j, 1, 2\}]; V = A - S; \\ &\{P[1], P[2], S, V\} \end{aligned} \tag{8.4}$$

In this way, P_1, P_2, S, and N of Example 1 of Section 1 were obtained.

For an IBM PC the following program in REDUCE will work.

We first make a data file, in which we write the degree N of a characteristic polynomial $p_A(\lambda)$, its eigenvalues $L[j]$, and their multiplicities $M[j]$. For example (cf. Example 1 of Section 1)

```
%%%%% SIBUYA1.DAT %%%%%
ARRAY L(15), M(15), C(15), A(15), D(15);
N := 3$
K := 2$
L(1) := 1$
L(2) := 11$
M(1) := 2$
M(2) := 1$
END;
```

We make a file whose name is SN. RED. This is the main program which calculates the polynomials $P_j(\lambda)$ and also makes another datafile. Actually we start with

```
1: OUT "YS1.DAT";
2: IN "SIBUYA1.DAT"$
3: IN "SN.RED"$
4: QUIT;
```

YS1.DAT is the next input file. Here we have to correct a little the file YS1.DAT so that it becomes a complete REDUCE input file. For example, we have to replace a scalar in the polynomials $P_j(\lambda)$ by a scalar multiplied by the identity matrix E. In this file, we put the data of a matrix $A = X$. We proceed as follows:

```
1: OUT "YS1.OUT";
2: IN "YS1.DAT"$
3: QUIT;
```

We then obtain the required results in the file YS1.OUT.

The following are the file SN.RED and the derived YS1.DAT:

```
%%%%% SN.RED SN DECOMPOSITION %%%%%
%%%%%%%%%%%%%%%%%%%%
PROCEDURE POCH(R,N);
BEGIN
SCALAR PRO;
PRO := 1;
IF N=0 THEN RETURN PRO$
PRO := (FOR J := 1:N PRODUCT (R-J+1));
RETURN PRO$
END;
%%%%%%%%%%%%%%%%%%%%
PROCEDURE PHI(L,N);
BEGIN
SCALAR PRO;
PRO=1;
IF N=0 THEN RETURN PRO$
PRO := (FOR J := 1:N PRODUCT (X-L));
RETURN PRO$
END;
%%%%%%%%%%%%%%%%%%%%
DEPEND CH,X;
```

```
DEPEND PHI, X; DEPEND F, X; DEPEND G, X;
ARRAY P(15);

ON FACTOR;
CH := (FOR J := 1:K PRODUCT PHI(L(J),M(J)))$

OFF NAT;
J := 1:K DO
BEGIN
SCALAR WA, MM, LL, NN, PROJECTION;
MM := M(J);
LL := L(J);
G := PHI(LL,MM);
F := CH / G;
NN := N-1;
C(0) := SUB(X=LL,F);
FOR I := 1:NN DO
<<H := DF(F,X,I);
C(I) := SUB(X=LL,H) / POCH(I,I);
>>;

A(0) := 1 / C(0);
MM := M(J)-1;
FOR R := 1:MM DO
<<WA := (FOR I := 1:R SUM C(I)*A(R-I));
A(R) := -WA / C(0);
>>;

PROJECTION := 1;
MM := M(J);
FOR I := MM:NN DO
<<D(I) := (FOR K := 1:MM SUM A(K-1)*C(I-K+1));
PROJECTION := PROJECTION+D(I)*(X-L(J))**I;
>>;
P(J) := PROJECTION$

END;

S := (FOR J := 1:K SUM L(J)*P(J))$

WRITE %%%%%%%%%%%%%%%%%%%%"
WRITE "PROCEDURE TANGEN(N);";
WRITE "BEGIN";
WRITE "CLEAR E;";
```

```
WRITE "MATRIX E(N,N);";
WRITE "FOR J := 1:N DO E(J,J) := 1;";
WRITE "RETURN E$";
WRITE "END;";
WRITE "%%%%%%%%%%%%%%%%%%%%"

WRITE "MATRIX X, SPART, NPART";
FOR J := 1:K DO WRITE "MATRIX P",J;
WRITE "N := ",N";
WRITE "TANGEN(N)";
WRITE "%X := MAT( )";
FOR J := 1:K DO WRITE "P", J," := ",P(J);
WRITE "SPART := ",S;
WRITE "NPART := X- SPART";
WRITE "END;";

END;

%%%%% ysl.dat %%%%%

%%%%%%%%%%%%%%%%%%%%$
PROCEDURE TANGEN(N);
BEGIN
CLEAR E;
MATRIX E(N,N);
FOR J := 1:N DO E(J,J) := 1;
RETURN E$
END;
%%%%%%%%%%%%%%%%%%%%$

MATRIX P1$
MATRIX P2$
N := 3$
TANGEN(N)$
%X := MAT( )$
%Here we put the data of the matrix A as follows;
X := MAT((3,4,3),(2,7,4),(-4,8,3));
P1 := (- (x + 9*E)*(x- 11*E)) / 100;
P2 := ((x- 1*E)**2) / 100;
SPART := ((x**2- 2*x + 11*E)) / 10;
NPART := X- SPART;
END$
```

For Example 3 in Section 1, if we use the following YS3.DAT, it takes about 3 hours and 40 minutes to complete the calculation of the $P_j(A)$ ($j = 1, 2, 3, 4, 5, 6, 7$), S, and N:

```
%%%%% ys3.dat %%%%%

%%%%%%%%%%%%%%%$
PROCEDURE TANGEN(N);
BEGIN
CLEAR E;
MATRIX E(N,N);
FOR J := 1:N DO E(J,J) := 1;
RETURN E$
END;
%%%%%%%%%%%%%%%$
MATRIX X,SPART,NPART$
MATRIX P1$
MATRIX P2$
MATRIX P3$
MATRIX P4$
MATRIX P5$
MATRIX P6$
MATRIX P7$

N := 10$
TANGEN(N)$
X := MAT((0,-1,0,0,0,0,0,0,0,0),(-1,0,0,0,0,0,0,0,0,0),
(0,0,1,-1,0,0,0,0,0,0),(1/4,0,-1,1,0,0,0,0,0,0),
(0,0,0,0,2,-1,0,0,0,0),(0,0,1/4,0,-1,2,0,0,0,0),
(0,0,0,0,0,0,3,-1,0,0),(0,0,0,0,1/4,0,-1,3,0,0),
(0,0,0,0,0,0,0,0,4,-1),(0,0,0,0,0,0,1/4,0,-1,4));
P1 := (-(x-1*E)**2*(x-2*E)**2*(x-3*E)**2*(x-4*E)*
(x-5*E)*x)/17280;
P2 := ((x+1*E)*(x-1*E)**2*(x-2*E)**2*(x-3*E)**2*
(x-4*E)*(x-5*E))/720;
P3 := ((25*x-13*E)*(x+1*E)*(x-2*E)**2*(x-3*E)**2*
(x-4*E)*(x-5*E)*x)/1152;
P4 := ((x+1*E)*(x-1*E)**2*(x-3*E)**2*(x-4*E)*
(x-5*E)*x)/36;
P5 := (-(25*x-87*E)*(x+1*E)*(x-1*E)**2*(x-2*E)**2*
(x-4*E)*(x-5*E)*x)/1152;
P6 := (-(x+1*E)*(x-1*E)**2*(x-2*E)**2*(x-3*E)**2*
(x-5*E)*x)/720;
```

```
P7 := ((x+1*E)*(x-1*E)**2*(x-2*E)**2*(x-3*E)**2*
(x-4*E)*x)/17280;
```

```
SPART := (-(7*x**8-126*x**7+906*x**6-3276*x**5
+5943*x**4-3654*x**3-3976*x**2+7056*x-3024*E)*x)/144;
```

```
NPART := X-SPART;
```

```
END$
```

However, if one wants only the matrices SPART and NPART, one can use

```
SPART := (-(7*x**8-126*x**7+906*x**6-3276*x**5+
5943*x**4-3654*x**3-3976*x**2+7056*x-3024*E)*x)/144;
```

```
N := X-SPART;
```

and REDUCE calculates them instantly. The computer is always obedient. In order to calculate P1, P2, P3, P4, P5, P6, and P7, it follows commands from the beginning no matter what.

Hence we change YS3.DAT as follows:

```
%%%%% ys31.dat %%%%%
MATRIX X,SPART,NPART$
MATRIX P1,P2,P3,P4,P5,P6,P7$
MATRIX X1,X2,X22,X3,X32,X4,X42,X5,X6$

X := MAT((0,-1,0,0,0,0,0,0,0,0),(-1,0,0,0,0,0,0,0,0,0),
(0,0,1,-1,0,0,0,0,0,0),(1/4,0,-1,1,0,0,0,0,0,0),
(0,0,0,0,2,-1,0,0,0,0),(0,0,1/4,0,-1,2,0,0,0,0),
(0,0,0,0,0,0,3,-1,0,0),(0,0,0,0,1/4,0,-1,3,0,0),
(0,0,0,0,0,0,0,0,4,-1),(0,0,0,0,0,0,1/4,0,-1,4));

X1 := X+E;
X2 := X-E;
X22 := (X-E)**2;
X3 := X-2*E;
X32 := (X-2*E)**2;
X4 := X-3*E;
X42 := (X-3*E)**2;
X5 := X-4*E;
X6 := X-5*E;

P1 := (-X22*X32*X42*X5*X6*x)/17280;
P2 := (X1*X22*X32*X42*X5*X6)/720;
P3 := ((25*x-13*e)*X1*X32*X42*X5*X6*x)/1152;
```

```
P4 := (X1*X22*X42*X5*X6*x)/36;
P5 := (-(25*x-87*e)*X1*X22*X32*X5*X6*x)/1152;
P6 := (-X1*X22*X32*X42*X6*x)/720;
P1 := (X1*X22*X32*X42*X5*x)/17280;

SPART := -1*P1+P3+2*P4+3*P5+4*P6+5*P7;

NPART := X-SPART;

END$
%%%%%%%%%%%%%%
```

Then it takes about 2 minutes to obtain `YS3.OUT`. In this way, if the matrix $X(=A)$ considered is big, it would be better to calculate parts of factors in the polynomials P_j in advance.

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