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# Linearization by generalized input–output injection

F. Plestan*, A. Glumineau

*Laboratoire d'Automatique de Nantes, URA CNRS 823, Ecole Centrale de Nantes-Université de Nantes, 1, rue de la Noë, B.P. 92101, 44321 Nantes cedex 3, France*

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**Abstract**

The problem under interest is the linearization of nonlinear MIMO systems by generalized input–output injection in order to design observers with linear error dynamics. The method is based on the study of the structure of the input–output differential equations; thus, the problem is solved as a realization problem. In this note, one considers the linearization under two kinds of input–output injection. In the first case, the transformation depends on the output and time derivatives of the input, whereas in the second case, derivatives of both the input and the output are considered. Necessary and sufficient conditions are obtained which generalize the ones on standard input–output injection linearization. © 1997 Elsevier Science B.V.

*Keywords:* Nonlinear systems; Input–output injection; Generalized state transformation; Exterior differentiation

## 1. Introduction

The first problem under interest in this paper is the linearization of a nonlinear system by a generalized state coordinates transformation [9] and generalized input–output injection (i.e. injection of the output and a finite number of input time derivatives). Its solution plays a key role in the synthesis of nonlinear observers. The final goal is to build an observer which has exact linear error dynamics and is stable. Some results give observers using time derivatives of the input [1, 2, 12, 22, 24, 26], but no unified and constructive theory has been developed. The exact linearization by input–output injection has been studied with geometric tools in [13, 14, 17, 25] and with more algebraically oriented tools in [10, 18, 20], and yields also practical applications [3, 4, 19]. The exact linearization of MIMO systems by generalized input–output injection has, to our best knowledge, not often been considered [12, 22]. The results given in this note are a direct generalization to MIMO systems of the results for MISO systems in [10, 20].

Time derivatives of the output have been used in [8] for the design of nonlinear observers. Our goal here is to use a minimal number of output derivatives in input–output injection and to take maximal advantage of the inherent linear system structure.

The method is based on the study of the structure of the input–output differential equations; thus, the problem is stated as a realization one. Moreover, the conditions of the existence of a solution to a linearization problem are stated in terms of exterior differential systems. This frame has already been used in [10, 20, 21] and yields

* Corresponding author. Tel.: (+33) 2-40-37-16-36; Fax: (+33) 2-40-37-25-22; e-mail: alain.glumineau@lan.ec-nantes.fr.


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an unified and self-contained theory for the input–output linearization independently from the special class of transformations which is used.

Section 2 briefly recalls results and references on generic observability and state elimination. Section 3 states the problem under interest. Section 4 gives a result on the linearization of MIMO systems by generalized state coordinates transformation and generalized input–output injection: necessary and sufficient condition of linearizability is based on a constructive algorithm (which gives the input–output injection functions). The results in [10, 20, 21] are considered as corollaries of this result. Section 5 gives a necessary and sufficient condition (with a constructive algorithm) on the linearization of MISO systems by generalized state coordinates transformation and generalized input–output injection depending on time derivatives of output and input.

## 2. Preliminaries

Consider the nonlinear system

$$\dot{x} = f(x,u), \qquad y = h(x) \tag{1}$$

with $x \in M$ where $M$ is an open and dense subset of $\mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$. The entries of $f(\cdot,\cdot)$ and $h(\cdot)$ are analytic.

### 2.1. Generic observability

All the definitions and results given in this note can be written locally around a regular point $x_0$ of $M$ because the functions are analytic. If a propriety is generically satisfied, it means that this propriety is satisfied locally around a regular point $x_0$ of $M$. Let $\mathcal{O}$ denote the generic observability space, defined by [27]

$$\mathcal{O} = \mathcal{X} \cap (\mathcal{Y} + \mathcal{U}),$$

where $\mathcal{X} = \mathrm{Span}_{\mathscr{K}}\{\mathrm{d}x\}$, $\mathcal{Y} = \mathrm{Span}_{\mathscr{K}}\{\mathrm{d}y^{(w)},\ w \geqslant 0\}$ and $\mathcal{U} = \mathrm{Span}_{\mathscr{K}}\{\mathrm{d}u^{(w)},\ w \geqslant 0\}$ ($\mathrm{Span}_{\mathscr{K}}$ is a space spanned over the field $\mathscr{K}$ of meromorphic functions of $x$ and a finite number of time derivatives of $u$).

**Definition 1.** The system (1) is generically observable if $\dim \mathcal{O} = n$.

This condition is called *Rank condition of generic observability*. In the sequel, nonlinear systems considered here are supposed to be generically observable and will be called "*observable*".

**Remark 2.** If a system is generically observable, then the state $x$ can be locally written as a function of the output $y$, the input $u$ and a finite number of their time derivatives:

$$x = X(y, \dot{y}, \ldots, y^{(v)}, u, \dot{u}, \ldots, u^{(v')}).$$

### 2.2. State elimination

The results developed in this paper are based on the analysis of the input–output differential equations which have to have a special polynomial structure in $u$ and its derivatives. First, the state variables have to be eliminated to get the input–output representation. For more details on the state elimination, see [6, 5, 23]. The unicity of the input–output differential equations of an observable system is shown in [23] through a necessary and sufficient condition and an algorithm (see also [7]).

Let $(k_1, k_2, \ldots, k_p)$ denote the observability indices [14] of the outputs $(y_1, y_2, \ldots, y_p)$ (with $\sum_{i=1}^{p} k_i = n$ and $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_p$). These indices are uniquely defined for a given output vector. Then, the system of input–output differential equations (2) is associated to (1) in a unique way (up to a renumbering of the output components):

$$y_i^{(k_i)} = P_i(y_1, \dot{y}_1, \ldots, y_1^{(k_1-1)}, \ldots, y_p, \dot{y}_p, \ldots, y_p^{(k_p-1)}, u, \dot{u}, \ldots, u^{(k_1-1)}). \tag{2}$$

## 3. Problems statement

The first problem under interest in this paper can be stated as follows and is a generalization of [25]. System (1) is observable. The problem is to find a generalized state coordinates transformation $\zeta = \phi(x, u, \dot{u}, \ldots, u^{(q-1)})$ ($q \in \mathbb{N}$) such that system (1) is locally equivalent to

$$\begin{aligned} \dot{\zeta} &= A \cdot \zeta + \varphi(y, u, \dot{u}, \ldots, u^{(q)}) = A \cdot \zeta + \begin{pmatrix} \varphi_{11}(y, u, \dot{u}, \ldots, u^{(q)}) \\ \vdots \\ \varphi_{1k_1}(y, u, \dot{u}, \ldots, u^{(q)}) \\ \vdots \\ \varphi_{p1}(y, u, \dot{u}, \ldots, u^{(q)}) \\ \vdots \\ \varphi_{pk_p}(y, u, \dot{u}, \ldots, u^{(q)}) \end{pmatrix}, \\ y &= C \cdot \zeta \end{aligned} \tag{3}$$

with $\zeta \in \mathbb{R}^n$. A solution is provided by Theorem 2 in Section 4.2.

This problem can be extended with insertion of time derivatives of the output, $y, \dot{y}, \ldots, y^{(s)}$, in the input–output injection. For the sake of clarity, only the single output system case is considered in this note for this kind of input–output injection. The problem consists to find a generalized state coordinates transformation $\zeta = \phi(x, u, \dot{u}, \ldots, u^{(v)})$ ($v \in \mathbb{N}$) such that system (1) is locally equivalent to

$$\begin{aligned} \dot{\zeta}_1 &= \zeta_2, \\ \dot{\zeta}_2 &= \zeta_3, \\ &\vdots \\ \dot{\zeta}_s &= \zeta_{s+1}, \\ \dot{\zeta}_{s+1} &= \zeta_{s+2} + \varphi_{s+1}(y, \dot{y}, \ldots, y^{(s)}, u, \dot{u}, \ldots, u^{(q)}), \\ \dot{\zeta}_{s+2} &= \zeta_{s+3} + \varphi_{s+2}(y, \dot{y}, \ldots, y^{(s)}, u, \dot{u}, \ldots, u^{(q)}), \\ &\vdots \\ \dot{\zeta}_{n-1} &= \zeta_n + \varphi_{n-1}(y, \dot{y}, \ldots, y^{(s)}, u, \dot{u}, \ldots, u^{(q)}), \\ \dot{\zeta}_n &= \varphi_n(y, \dot{y}, \ldots, y^{(s)}, u, \dot{u}, \ldots, u^{(q)}), \\ y &= \zeta_1, \end{aligned} \tag{4}$$

where $s$ and $q$ are integers. Note that it will be proved that $v = \mathrm{Max}(s-2, q-1)$. A solution is provided by Theorem 3 in Section 4.3.

## 4. Linearization by generalized input–output injection

The approach proposed here consists in verifying that the input–output differential equations associated to (1) have the same form as the input–output differential equations of a system (3) for the first problem (resp. system (4) for the second problem).

### 4.1. Preliminary example

The following example introduces the approach and shows the role of the input–output differential equations. A problem can appear for the linearization of MIMO systems due to the observability indices: the input–output differential equations cannot be independent. This problem is shown clearly in the following example. This example is displayed in [25], where it has been shown that the conditions given in [14] are not satisfied.

**Example 3.** Consider the nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2, \qquad y_1 = x_1, \\ \dot{x}_2 &= x_2 \cdot x_3, \\ \dot{x}_3 &= x_1, \qquad y_2 = x_3. \end{aligned} \tag{5}$$

The system is under observability canonical form, and the observability indices equal $k_1 = 2$ and $k_2 = 1$. The input–output differential equations of the system (5) are

$$y_1^{(2)} = \dot{y}_1 \cdot y_2, \qquad y_2^{(1)} = y_1. \tag{6}$$

If the system (5) is locally equivalent to a system (3), then the input–output differential equations (6) can be written as

$$y_1^{(2)} = \varphi_{11}^{(1)} + \varphi_{12}, \qquad y_2^{(1)} = \varphi_{21}.$$

One can easily determine the function $\varphi_{21}$ $(= y_1)$. The first differential equation of the previous system becomes

$$y_1^{(2)} = \frac{\partial \varphi_{11}}{\partial y_1}\dot{y}_1 + \frac{\partial \varphi_{11}}{\partial y_2}\dot{y}_2 + \varphi_{12}.$$

The input–output differential equation associated to $y_1$ depends on the input–output differential equation associated to $y_2$. The observability index of $y_2$ equals 1: then, the time derivative $\dot{y}_2$ is not an independent variable of $\{y_1, \dot{y}_1, y_2\}$, and equals $y_1$. Then

$$y_1^{(2)} = \frac{\partial \varphi_{11}}{\partial y_1}\dot{y}_1 + \frac{\partial \varphi_{11}}{\partial y_2}y_1 + \varphi_{12}. \tag{7}$$

If the system (5) is locally equivalent to a system (3), then the first differential equations of (6) and (7) are equal. The difficulty is to determine $\varphi_{11}$ without the explicit knowledge of the time derivative $\dot{y}_2$. One solves

$$\frac{\partial \varphi_{11}}{\partial y_1} = y_2, \qquad \frac{\partial \varphi_{11}}{\partial y_2}y_1 + \varphi_{12} = 0.$$

A solution is $\varphi_{11} = y_1 \cdot y_2$ and $\varphi_{12} = -y_1^2$. The system (5) is then locally equivalent to

$$\begin{aligned} \dot{\zeta}_1 &= \zeta_2 + y_1 \cdot y_2, \qquad y_1 = \zeta_1, \\ \dot{\zeta}_2 &= -y_1^2, \\ \dot{\zeta}_3 &= y_1, \qquad y_2 = \zeta_3. \end{aligned}$$

**Remark 4.** If, for a given system, all observability indices have the same value, then the input–output differential equations describing the system dynamics are independent.

### *4.2. Input–output injection with time derivatives of input*

#### *4.2.1. General case*

The results presented in this paper are based on the analysis of the structure of input–output differential equations associated to (1). To obtain a constructive method, two steps are necessary. First, a necessary condition of the existence of a generalized state coordinates transformation is given. Then, this condition is completed to get a necessary and sufficient condition.

For each output $y_i$ $(1 \leqslant i \leqslant p)$, consider the input–output differential equation

$$y_i^{(k_i)} = P_i(y_1, \dot{y}_1, \ldots, y_1^{(k_1-1)}, \ldots, y_p, \dot{y}_p, \ldots, y_p^{(k_p-1)}, u, \dot{u}, \ldots, u^{(k_1-1)}).$$

**Theorem 5.** *If the nonlinear system* (1) *is locally equivalent to system* (3) *by a generalized state coordinates transformation* $\zeta=\phi(x,u,\dot{u},\dots,u^{(q-1)})$, *then*

$$\frac{\partial P_i}{\partial y_j^{(\alpha)}}=0,\qquad \frac{\partial P_i}{\partial u_k^{(\alpha+q)}}=0 \tag{8}$$

*for* $1\leqslant i\leqslant p$, $1\leqslant j\leqslant p$, $1\leqslant k\leqslant m$ *and* $\alpha\geqslant k_i$.

**Proof.** If system (1) is locally equivalent to system (3), then the input–output differential equation associated to each output $y_i$ $(1\leqslant i\leqslant p)$ reads as

$$y_i^{(k_i)}=\varphi_{i1}^{(k_i-1)}+\varphi_{i2}^{(k_i-2)}+\cdots+\varphi_{ik_i}. \tag{9}$$

The right-hand side of (9) fulfills (8): necessity of Theorem 5 is proved. □

The following method uses the theory of exterior differential systems. A necessary and sufficient condition for the existence of a generalized state coordinates transformation which transforms (1) into (3) is given in Theorem 2. To verify this condition, the following algorithm computes some differential forms which embody the solution, if it exists. This algorithm is the Generalized Input–Output Injection Algorithm (GIOIA).

**GIOIA Algorithm**

For $i=1$ to $p$, set $P_i^0:=P_i$ and $\varphi_{i0}(y,u,\dot{u},\dots,u^{(q)}):=0$. For $k=1$ to $k_i$, let

$$P_i^k=P_i^{k-1}-[\varphi_{ik-1}(y,u,\dot{u},\dots,u^{(q)})]^{(k_i-k+1)}. \tag{10}$$

Let $d_i^k$ denote the number of outputs whose observability index is larger than $k_i-k$. The differential form $\omega_i^k$ is defined as

$$\omega_i^k=\sum_{j=1}^{d_i^k}\frac{\partial P_i^k}{\partial y_j^{(k_i-k)}}\,\mathrm{d}y_j+\sum_{j=1}^{m}\frac{\partial P_i^k}{\partial u_j^{(k_i-k+q)}}\,\mathrm{d}u_j^{(q)}. \tag{11}$$

Let $\wedge\,\mathrm{d}u^{[q]}$ denote

$$\wedge\,\mathrm{d}u^{[q]}=\begin{cases}\wedge 1 & \text{if } q=0,\\ \wedge\,\mathrm{d}u\wedge\cdots\wedge\mathrm{d}u^{(q-1)} & \text{if } q>0.\end{cases} \tag{12}$$

- If $d_i^k<p$, then:
  - If $\mathrm{d}\omega_i^k\wedge\mathrm{d}y_{d_i^k+1}\wedge\cdots\wedge\mathrm{d}y_p\wedge\mathrm{d}u^{[q]}=0$, then the function $\varphi_{ik}(y,u,\dot{u},\dots,u^{(q)})$ is a solution of

$$\begin{aligned}&\sum_{j=1}^{d_i^k}\frac{\partial\varphi_{ik}}{\partial y_j}\,\mathrm{d}y_j+\sum_{j=1}^{m}\frac{\partial\varphi_{ik}}{\partial u_j^{(q)}}\,\mathrm{d}u_j^{(q)}=\omega_i^k\quad\text{for } 1\leqslant k\leqslant k_i-1,\\ &\varphi_{ik_i}(y,u,\dot{u},\dots,u^{(q)})=P_i^{k_i}.\end{aligned} \tag{13}$$

  - If $\mathrm{d}\omega_i^k\wedge\mathrm{d}y_{d_i^k+1}\wedge\cdots\wedge\mathrm{d}y_p\wedge\mathrm{d}u^{[q]}\neq 0$, then system (1) is not linearizable.
- If $d_i^k=p$, then:
  - If $\mathrm{d}\omega_i^k\wedge\mathrm{d}u^{[q]}=0$, then the function $\varphi_{ik}(y,u,\dot{u},\dots,u^{(q)})$ is a solution of

$$\begin{aligned}&\sum_{j=1}^{p}\frac{\partial\varphi_{ik}}{\partial y_j}\,\mathrm{d}y_j+\sum_{j=1}^{m}\frac{\partial\varphi_{ik}}{\partial u_j^{(q)}}\,\mathrm{d}u_j^{(q)}=\omega_i^k\quad\text{for } 1\leqslant k\leqslant k_i-1,\\ &\varphi_{ik_i}(y,u,\dot{u},\dots,u^{(q)})=P_i^{k_i}.\end{aligned} \tag{14}$$

  - If $\mathrm{d}\omega_i^k\wedge\mathrm{d}u^{[q]}\neq 0$, then system (1) is not linearizable.

**Remark 6.** The differential form $\omega_i^k$ is used to verify that the coefficient of the higher-order time derivative of $y$ (resp. $u$) in each function $P_i^k$ depends only on $y$ and a finite number of time derivatives of $u$, if system (1) is locally equivalent to a system (3): at each step of the algorithm, this differential form can be computed with (11) and has to satisfy

$$\mathrm{d}\omega_i^k \wedge \mathrm{d}y_{d_i^k+1} \wedge \cdots \wedge \mathrm{d}y_p \wedge \mathrm{d}u^{[q]} = 0$$

or

$$\mathrm{d}\omega_i^k \wedge \mathrm{d}u^{[q]} = 0.$$

These conditions ensures that the solution depends only on $y, u, \ldots, u^{(q)}$, and that the differential form $\omega_i^k$ is integrable. Note also that the differential form takes into account the coupling of input–output differential equations through the integer $d_i^k$.

**Theorem 7.** *The nonlinear system* (1) *is locally equivalent to the system* (3) *by a generalized state coordinates transformation* $\zeta = \phi(x, u, \dot{u}, \ldots, u^{(q-1)})$ $(q \in \mathbb{N})$ *if and only if condition* (8) *of Theorem* 5 *is satisfied and*

$$\mathrm{d}\omega_i^k \wedge \mathrm{d}y_{d_i^k+1} \wedge \cdots \wedge \mathrm{d}y_p \wedge \mathrm{d}u^{[q]} = 0 \quad \textit{if } d_i^k < p, \tag{15}$$

$$\mathrm{d}\omega_i^k \wedge \mathrm{d}u^{[q]} = 0 \quad \textit{if } d_i^k = p \tag{16}$$

*with* $1 \leqslant i \leqslant p$, $1 \leqslant k \leqslant k_i$, *and where the* $\omega_i^k$*'s are defined by* (11), *the* $\wedge\, \mathrm{d}u^{[q]}$ *is defined by* (12).

The generalized state coordinates transformation $\zeta = \phi(x, u, \dot{u}, \ldots, u^{(q-1)})$ is derived from the system (3); for each block associated to an output $y_i$ $(1 \leqslant i \leqslant p)$, one gets

$$\begin{aligned}\zeta = {} & [h_1(x), [h_1(x)]^{(1)} - \varphi_{11}, \ldots, [h_1(x)]^{(k_1-1)} - \varphi_{1k_1-1} - \varphi_{1k_1-2}^{(1)} - \cdots - \varphi_{11}^{(k_1-2)}, \ldots, \\ & h_p(x), [h_p(x)]^{(1)} - \varphi_{p1}, \ldots, [h_p(x)]^{(k_p-1)} - \varphi_{pk_p-1} - \varphi_{pk_p-2}^{(1)} - \cdots - \varphi^{(k_p-2)}]^{\mathrm{T}}.\end{aligned} \tag{17}$$

The proof of Theorem 7 is given in Appendix A. This result is a generalization of [10] which considers only the single output system case.

To show the GIOIA algorithm computations, it is applied to the system (5) without input and with coupling between the input–output differential equations. A second example on a system with two inputs shows the feasibility of the method on a system which is equivalent to a linear system modulo a generalized input–output injection.

**Example 8.** Consider the nonlinear system (5). The input–output differential equations are

$$y_1^{(2)} = \dot{y}_1 \cdot y_2, \qquad y_2^{(1)} = y_1.$$

The conditions of Theorem 5 are satisfied. Consider the input–output differential equation associated to $y_2$. Let $P_2^1$ denote the input–output differential equation of $y_2$. The algorithm GIOIA is applied with $q = 0$: there is no input. The integer $d_2^1$ equals 2. The differential form $\omega_2^1$ is derived from (11):

$$\omega_2^1 = \sum_{j=1}^{2} \frac{\partial P_1^2}{\partial y_j}\, \mathrm{d}y_j = \mathrm{d}y_1.$$

The condition $\mathrm{d}\omega_2^1 = 0$, (12)–(16), is satisfied. Then, the function $\varphi_{21}$ is derived from (14): $\varphi_{21} = y_1$. Consider now the input–output differential equation associated to $y_1$. Let $P_1^1$ denote the input–output differential equation of $y_1$. The integer $d_1^1$ equals 1. The differential form $\omega_1^2$ is derived from (11):

$$\omega_1^2 = \sum_{j=1}^{1} \frac{\partial P_1^2}{\partial y_j}\, \mathrm{d}y_j = y_2\, \mathrm{d}y_1.$$

The condition $d\omega_1^2 \wedge dy_2 = 0$, (12)–(16), is satisfied. Then, the function $\varphi_{11}$ is derived from (13): $\varphi_{11} = y_1 \cdot y_2$. The next step of the algorithm gives the function $P_1^2$:

$$P_1^2 = P_1^1 - \varphi_{11} = -y_1^2.$$

The integer $d_1^2$ equals 2. Then, the differential form $\omega_1^2$ reads as

$$\omega_1^2 = -2y_1\, dy_1.$$

The condition $d\omega_1^2 = 0$ is satisfied. Then, the function $\varphi_{12}(y)$ is derived from (14): $\varphi_{12}(y) = -y_1^2$. The system (5) is then locally equivalent to

$$\begin{aligned} \dot\zeta_1 &= \zeta_2 + y_1 \cdot y_2, \quad y_1 = \zeta_1, \\ \dot\zeta_2 &= -y_1^2, \\ \dot\zeta_3 &= y_1, \qquad y_2 = \zeta_3 \end{aligned} \tag{18}$$

by the state coordinates transformation $\zeta = [x_1, x_2 - x_1 \cdot x_3, x_3]^{\mathrm{T}}$ derived from (17).

**Example 9.** Consider the nonlinear system

$$\begin{aligned} \dot x_1 &= x_2 \cdot u_1 + u_2^2 \cdot x_3 \cdot x_1^2, \qquad y_1 = x_1, \\ \dot x_2 &= \sin u_1 \cdot x_1, \\ \dot x_3 &= x_1^2 \cdot u_2, \qquad y_2 = x_3. \end{aligned} \tag{19}$$

The input–output differential equations read as

$$\begin{aligned} \ddot y_1 &= \left(\frac{\dot u_1}{u_1} + 2u_2^2 \cdot y_1 \cdot y_2\right) \cdot \dot y_1 - \left(\frac{u_2^2 \cdot y_2 \cdot y_1^2}{u_1}\right) \cdot \dot u_1 + (2u_2 \cdot y_2 \cdot y_1^2) \cdot \dot u_2 + u_2^3 \cdot y_1^4 + \sin u_1 \cdot y_1, \\ \dot y_2 &= y_1^2 \cdot u_2. \end{aligned} \tag{20}$$

The conditions of Theorem 5 are satisfied for all $q$. The algorithm GIOIA does not converge with $q = 0$ and $q = 1$. One applies the algorithm with $q = 2$.

Set $P_2^1 = y_1^2 \cdot u_2$, and $d_2^1 = 2$ $(=p)$. The differential form $\omega_2^1$ is given by (11):

$$\omega_2^1 = \sum_{j=1}^{2} \frac{\partial P_2}{\partial y_j}\, dy_j + \sum_{j=1}^{2} \frac{\partial P_2}{\partial u_j}\, d\ddot u_j = 2y_1 \cdot u_2 \cdot dy_1.$$

The condition (16) is satisfied: $d\omega_2^1 \wedge du^{[2]} = 0$. One gets $\varphi_{21} = P_2^1 = y_1^2 \cdot u_2$ (14).

For the input–output differential equation associated to $y_1$, set $P_1^1 = \ddot y_1$, and $d_1^1 = 1$. The differential form $\omega_1^1$ is given by (11):

$$\omega_1^1 = \sum_{j=1}^{1} \frac{\partial P_1^1}{\partial \dot y_j}\, dy_j + \sum_{j=1}^{2} \frac{\partial P_1^1}{\partial u_j^{(3)}}\, d\ddot u_j = \left(\frac{\dot u_1}{u_1} + 2u_2^2 \cdot y_1 \cdot y_2\right) dy_1.$$

As $d_1^1 < 2$ $(=p)$, the condition (15),

$$d\omega_1^1 \wedge dy_2 \wedge du^{[2]} = 0,$$

is satisfied. Then, there exists a function $\varphi_{11}(y, u, \dot u)$ defined from (13):

$$\sum_{j=1}^{1} \frac{\partial \varphi_{11}}{\partial y_j}\, dy_j + \sum_{j=1}^{2} \frac{\partial \varphi_{11}}{\partial \ddot u_j}\, d\ddot u_j = \omega_1^1.$$

A solution is $\varphi_1^1 = u_2^2 \cdot y_1^2 \cdot y_2 + \dot{u}_1 \cdot y_1/u_1$. The second step of the algorithm GIOIA gives

$$P_1^2 = P_1^1 - \varphi_1^{(1)} = -\frac{u_2^2 \cdot y_2 \cdot y_1^2}{u_1} \cdot \dot{u}_1 + \sin u_1 \cdot y_1 - y_1 \cdot \frac{\ddot{u}_1}{u_1} + y_1 \cdot \frac{\dot{u}_1^2}{u_1^2}.$$

The integer $d_1^2$ equals 2. The differential form $\omega_1^2$ is derived from (11):

$$\begin{aligned}
\omega_1^2 &= \sum_{j=1}^{2} \frac{\partial P_1^2}{\partial y_j} \mathrm{d}y_j + \sum_{j=1}^{2} \frac{\partial P_1^2}{\partial \ddot{u}_j} \mathrm{d}\ddot{u}_j \\
&= \left( -\frac{2u_2^2 \cdot y_2 \cdot y_1}{u_1} \cdot \dot{u}_1 + \sin u_1 - \frac{\ddot{u}_1}{u_1} + \frac{u_1^2}{u_1^2} \right) \mathrm{d}y_1 + \left( -\frac{u_2^2 \cdot y_1^2 \cdot \dot{u}_1}{u_1} \right) \mathrm{d}y_2 - \frac{y_1}{u_1} \mathrm{d}\ddot{u}_1.
\end{aligned}$$

The condition (16) is satisfied: $\mathrm{d}\omega_1^2 \wedge \mathrm{d}u^{[2]} = 0$; then, from (14), $\varphi_1^2 = P_1^2$. Theorem 7 is satisfied; then, there exists a generalized state coordinates transformation which changes (19) into

$$\begin{aligned}
\dot{\zeta}_1 &= \zeta_2 + u_2^2 \cdot y_1^2 \cdot y_2 + \frac{\dot{u}_1}{u_1} \cdot y_1, \\
\dot{\zeta}_2 &= -\frac{u_2^2 \cdot y_2 \cdot y_1^2}{u_1} \cdot \dot{u}_1 + \sin u_1 \cdot y_1 - y_1 \cdot \frac{\ddot{u}_1}{u_1} + y_1 \cdot \frac{\dot{u}_1^2}{u_1^2}, \\
\dot{\zeta}_3 &= y_1^2 \cdot u_2, \\
y_1 &= \zeta_1, \\
y_2 &= \zeta_3.
\end{aligned}$$

The generalized state coordinates transformation is derived from (17):

$$\begin{aligned}
\zeta_1 &= x_1, \\
\zeta_2 &= x_2 \cdot u_1 - \frac{\dot{u}_1}{u_1} \cdot x_1, \\
\zeta_3 &= x_3.
\end{aligned}$$

*4.2.2. Special case: linearization by standard input–output injection*

Consider the nonlinear system (1). The problem under interest is to find a state coordinates transformations $\zeta = \phi(x)$ which transforms (1) into

$$\dot{\zeta} = A \cdot \zeta + \varphi(y, u), \qquad y = C \cdot \zeta. \tag{21}$$

This problem has been solved by geometric tools in [25]. The necessary and sufficient condition of the existence of linearizing state transformation is a corollary of Theorem 7 with $q = 0$.

**Corollary 10** (Plestan [21]). *The nonlinear system* (1) *is locally equivalent to the system* (21) *by a generalized state coordinates transformation* $\zeta = \phi(x)$ *if and only if the condition* (8) *of Theorem* 5 *is satisfied with* $q = 0$ *and*

$$\begin{aligned}
&\mathrm{d}\omega_i^k \wedge \mathrm{d}y_{d_{i+1}^k} \wedge \cdots \wedge \mathrm{d}y_p = 0 \quad \text{if } d_i^k < p, \\
&\mathrm{d}\omega_i^k = 0 \quad \text{if } d_i^k = p
\end{aligned}$$

*for* $1 \leqslant k \leqslant n$ *and* $1 \leqslant i \leqslant p$, *and the* $\omega_i^k$*'s are defined by* (11) *with* $q = 0$.

Technically, this condition states that $P$ is a polynomial of a special form in $\dot{y}$, $\dot{u}$ and their time derivatives.

### 4.3. Input–output injection with time derivatives of both output and input

The purpose of this section is to give a necessary and sufficient condition of the existence of a state coordinates transformation changing a nonlinear system into a linear system modulo an input–output injection with time derivatives of input and output. We focus on the case of single output systems for the sake of clarity of notations. Some time derivatives of the output have been used in [8] to design observers: from an interpolation of the output by a polynomial, the successive time derivatives of the output are derived. As the nonlinear system is observable, the state variables can be computed from the $(n-1)$th time derivatives of the output and the input. Then, this type of observer can be very sensitive to measurement noise due to high-order time derivatives of output. Here, the goal is to use a minimal number of output time derivatives.

**Remark 11.** If $s=q=0$, then one gets the problem solved by Corollary 10 for $p=1$. If $s=0$ and $\forall q$, then one gets the problem solved by Theorem 2 for $p=1$.

A necessary and sufficient condition for the existence of a generalized state coordinates transformation which transforms system (1) with $p=1$ into system (4), is given below. Denote $P$ the input–output differential equation of system (1) with $p=1$:

$$y^{(n)}=P(y,\dot{y},\ldots,y^{(n-1)},u,\dot{u},\ldots,u^{(n-1)}).$$

The following algorithm computes differential one forms, functions $\varphi_k$ $(s+1\leqslant k\leqslant n)$ and a linearizing transformation if it exists. It is called Generalized Input–Output Injection Algorithm with output time derivatives (denoted GIOIAd).

**GIOIAd Algorithm**

Set $P_s:=P$ and $\varphi_s(y,\dot{y},\ldots,y^{(s)},u,\dot{u},\ldots,u^{(q)}):=0$. For $k=s+1$ to $n$, let

$$P_k:=P_{k-1}-[\varphi_{k-1}(y,\dot{y},\ldots,y^{(s)},u,\dot{u},\ldots,u^{(q)})]^{(n-k+1)}. \tag{22}$$

The differential form $\omega_k$ is defined as

$$\omega_k=\frac{\partial P_k}{\partial y^{(n-k+s)}}\,\mathrm{d}y^{(s)}+\sum_{j=1}^{m}\frac{\partial P_k}{\partial u_j^{(n-k+q)}}\,\mathrm{d}u_j^{(q)}. \tag{23}$$

Let $\wedge\mathrm{d}u^{[q]}$ denote

$$\wedge\mathrm{d}u^{[q]}=\begin{cases}\wedge 1 & \text{if } q=0,\\ \wedge\mathrm{d}u\wedge\cdots\wedge\mathrm{d}u^{(q-1)} & \text{if } q>0,\end{cases} \tag{24}$$

and $\wedge\mathrm{d}y^{[s]}$ denote

$$\wedge\mathrm{d}y^{[s]}=\begin{cases}\wedge 1 & \text{if } s=0,\\ \wedge\mathrm{d}y\wedge\cdots\wedge\mathrm{d}y^{(s-1)} & \text{if } s>0.\end{cases} \tag{25}$$

If $\mathrm{d}\omega_k\wedge\mathrm{d}y^{[s]}\wedge\mathrm{d}u^{[q]}\neq 0$, then the system (10) is not linearizable.
If $\mathrm{d}\omega_k\wedge\mathrm{d}y^{[s]}\wedge\mathrm{d}u^{[q]}=0$, then the function $\varphi_k(y,\dot{y},\ldots,y^{(s)},u,\dot{u},\ldots,u^{(w)})$ is a solution of

$$\begin{aligned}&\frac{\partial\varphi_k}{\partial y^{(s)}}\,\mathrm{d}y^{(s)}+\sum_{j=1}^{m}\frac{\partial\varphi_k}{\partial u_j^{(q)}}\,\mathrm{d}u_j^{(q)}=\omega_k\quad\text{for } s+1\leqslant k\leqslant n-1,\\ &\varphi_n(y,\dot{y},\ldots,y^{(s)},u,\dot{u},\ldots,u^{(q)})=P_n.\end{aligned} \tag{26}$$

**Theorem 12.** *The nonlinear system* (1) (*with* $p=1$) *is locally equivalent to the system* (4) *by a generalized state coordinates transformation* $\zeta=\phi(x,u,\dot{u},\ldots,u^{(v)})$ ($v\in\mathbb{N}$) *if and only if*

$$\mathrm{d}\omega_k \wedge \mathrm{d}y^{[s]} \wedge \mathrm{d}u^{[q]} = 0 \tag{27}$$

*for* $s+1\leqslant k\leqslant n$, *where the* $\omega_k$*'s are defined by* (23), $\wedge\,\mathrm{d}u^{[q]}$ *and* $\wedge\,\mathrm{d}y^{[s]}$ *are defined by* (24) *and* (25), *and* $v=\mathrm{Max}(s-2,q-1)$.

The generalized state coordinates transformation $\zeta=\phi(x,u,\dot{u},\ldots,u^{(v)})$ ($v\in\mathbb{N}$) reads as

$$\begin{aligned}\zeta_1 = [&h(x),[h(x)]^{(1)},\ldots,[h(x)]^{(s)},[h(x)]^{(s+1)}-\varphi_{s+1},\ldots,\\ &[h(x)]^{(n-1)}-\varphi_{n-1}-\varphi_{n-2}^{(1)}-\cdots-\varphi_{s+1}^{(n-s-2)}]^{\mathrm{T}}.\end{aligned} \tag{28}$$

The proof of Theorem 12 is given in Appendix B.

## 5. Conclusions

General theoretical results are offered for the design of nonlinear observers depending on time derivatives of input and/or output as they exist in literature. A necessary and sufficient condition for the linearization of MIMO systems by generalized state coordinates transformation and generalized input–output injection has been obtained. This condition states in terms of the input–output differential equation. A constructive algorithm gives differential one forms which embody the solution, if it exists. If $q=0$, one gets the algorithm and the condition proved and derived in [21]; if $p=1$, one gets the algorithm and the condition derived in [10]. The results presented here are the extension of previous work. Next, a necessary and sufficient condition of linearization of MISO systems by generalized state coordinates transformation and input–output injection with time derivatives of output has been obtained. This result generalizes the conditions of [10, 19]. The linearization of MIMO systems by input–output injection with time derivatives of output and input is open for further research.

## Appendix A. Proof of Theorem 7

*Necessity*: Suppose that the generalized state coordinates transformation (17), which transforms (1) into (3), exists. For each output $y_i$ ($1\leqslant i\leqslant p$), the input–output differential equation reads

$$y_i^{(k_i)} = \varphi_{i1}^{(k_i-1)} + \varphi_{i2}^{(k_i-2)} + \cdots + \varphi_{ik_i}.$$

Let $P_i = y_i^{(k_i)}$ denote the input–output differential equation. If system (1) is locally equivalent to (3) then their input–output representations have the same form. Applying the GIOIA Algorithm, one gets

$$\begin{aligned}P_i^1 &= \varphi_{i1}^{(k_i-1)} + \varphi_{i2}^{(k_i-2)} + \cdots + \varphi_{ik_i}\\ &= \sum_{j=1}^{p}\frac{\partial\varphi_{i1}}{\partial y_j}y_j^{(k_i-1)} + \sum_{j=1}^{m}\frac{\partial\varphi_{i1}}{\partial u_j^{(q)}}u_j^{(k_i-1)} + \Theta_{i1}(y,\dot{y},\ldots,y^{(k_i-2)},u,\dot{u},\ldots,u^{(k_i-2+q)}).\end{aligned}$$

Only the $(k_i-1)$th time derivatives of outputs with observability indices larger or equal to $k_i$ are independent of the lower-order time derivatives. Then

$$P_i^1 = \sum_{j=1}^{d_i^1}\frac{\partial\varphi_{i1}}{\partial y_j}y_j^{(k_i-1)} + \sum_{j=1}^{m}\frac{\partial\varphi_{i1}}{\partial u_j^{(q)}}u_j^{(k_i-1+q)} + \bar{\Theta}_{i1}(y_1,\dot{y}_1,\ldots,y_p,\dot{y}_p,\ldots,u,\dot{u},\ldots,u^{(k_i-1)}).$$

From (11), the differential form $\omega_i^1$ equals

$$\omega_i^1 = \sum_{j=1}^{d_i^1} \frac{\partial \varphi_{i1}}{\partial y_j} \mathrm{d}y_j + \sum_{j=1}^{m} \frac{\partial \varphi_{i1}}{\partial u_j^{(q)}} \mathrm{d}u_j^{(q)}.$$

If $d_i^1 < p$, then

$$\begin{aligned}
\mathrm{d}\omega_i^1 = {} & \sum_{j=1}^{d_i^1} \sum_{k=d_i^1}^{p} \frac{\partial^2 \varphi_{i1}}{\partial y_j \partial y_k} \mathrm{d}y_k \wedge \mathrm{d}y_j + \sum_{j=1}^{d_i^1} \sum_{k=1}^{m} \frac{\partial^2 \varphi_{i1}}{\partial y_j \partial u_k} \mathrm{d}u_k \wedge \mathrm{d}y_j \\
& + \sum_{j=1}^{d_i^1} \sum_{k=1}^{m} \frac{\partial^2 \varphi_{i1}}{\partial y_j \partial \dot{u}_k} \mathrm{d}\dot{u}_k \wedge \mathrm{d}\dot{y}_j + \cdots + \sum_{j=1}^{m} \sum_{k=d_i^1+1}^{p} \frac{\partial^2 \varphi_{i1}}{\partial u_j^{(q)} \partial y_k} \mathrm{d}y_k \wedge \mathrm{d}u_j^{(q)} \\
& + \sum_{j=1}^{m} \sum_{k=1}^{m} \frac{\partial^2 \varphi_{i1}}{\partial u_j^{(q)} \partial u_k} \mathrm{d}u_k \wedge \mathrm{d}u_j^{(q)} + \sum_{j=1}^{m} \sum_{k=1}^{m} \frac{\partial^2 \varphi_{i1}}{\partial u_j^{(q)} \partial \dot{u}_k} \mathrm{d}\dot{u}_k \wedge \mathrm{d}u_j^{(q)} \\
& + \cdots + \sum_{j=1}^{m} \sum_{k=1}^{m} \frac{\partial^2 \varphi_{i1}}{\partial u_j^{(q)} \partial u_k^{(q-1)}} \mathrm{d}u_k^{(q-1)} \wedge \mathrm{d}u_j^{(q)}.
\end{aligned}$$

The condition (15) is satisfied.

If $d_i^1 = p$, then

$$\begin{aligned}
\mathrm{d}\omega_i^1 = {} & \sum_{j=1}^{p} \sum_{k=1}^{m} \frac{\partial^2 \varphi_{i1}}{\partial y_j \partial u_k} \mathrm{d}u_k \wedge \mathrm{d}y_j + \sum_{j=1}^{p} \sum_{k=1}^{m} \frac{\partial^2 \varphi_{i1}}{\partial y_j \partial \dot{u}_k} \mathrm{d}\dot{u}_k \wedge \mathrm{d}y_j + \cdots + \sum_{j=1}^{p} \sum_{k=1}^{m} \frac{\partial^2 \varphi_{i1}}{\partial y_j \partial u_k^{(q-1)}} \mathrm{d}u_k^{(q-1)} \wedge \mathrm{d}y_j \\
& + \sum_{j=1}^{m} \sum_{k=1}^{m} \frac{\partial^2 \varphi_{i1}}{\partial u_j^{(q)} \partial u_k} \mathrm{d}u_k \wedge \mathrm{d}u_j^{(q)} + \sum_{j=1}^{m} \sum_{k=1}^{m} \frac{\partial^2 \varphi_{i1}}{\partial u_j^{(q)} \partial \dot{u}_k} \mathrm{d}\dot{u}_k \wedge \mathrm{d}u_j^{(q)} \\
& + \cdots + \sum_{j=1}^{m} \sum_{k=1}^{m} \frac{\partial^2 \varphi_{i1}}{\partial u_j^{(q)} \partial u_k^{(q-1)}} \mathrm{d}u_k^{(q-1)} \wedge \mathrm{d}u_j^{(q)}.
\end{aligned}$$

The condition (16) is satisfied.

The necessity of the condition of Theorem 7 is then proved for the first step, and is proved by the same way for the following steps.

*Sufficiency*: Suppose that condition of Theorem 7 is satisfied. Functions $\varphi_{ik}(y, u, \dot{u}, \ldots, u^{(q)})$ are derived from GIOIA Algorithm. The generalized state coordinates transformation is derived from (17). For each associated block to an output variable of (1) and from (17), dynamics of state variables of system (3) are known. System (3) is then fully known: system (1) is then locally equivalent to the system (3) by a generalized state coordinates transformation (17). Sufficiency of Theorem 7 is then proved.

Suppose que the state coordinates transformation is defined by $\zeta = \phi(x, \tilde{u})$, with $\tilde{u} = (u, \dot{u}, \ldots, u^{(k)})$ $(k \in \mathbb{N})$. One gets

$$\dot{\zeta} = \frac{\partial \phi}{\partial x} f(x, u) + \frac{\partial \phi}{\partial \tilde{u}} \dot{\tilde{u}}.$$

Using a state coordinates transformation, one gets

$$\frac{\partial \phi_{11}}{\partial x} f(x, u) + \frac{\partial \phi_{11}}{\partial \tilde{u}} \dot{\tilde{u}} - \phi_{12}(x, \tilde{u}) = \varphi_{11}(y, u, \dot{u}, \ldots, u^{(q)}),$$

$$\vdots$$

$$\frac{\partial \phi_{1k_1}}{\partial x} f(x,u) + \frac{\partial \phi_{1k_1}}{\partial \tilde{u}} \dot{\tilde{u}} = \varphi_{1k_1}(y,u,\dot{u},\ldots,u^{(q)}),$$

$$\vdots$$

$$\frac{\partial \phi_{p1}}{\partial x} f(x,u) + \frac{\partial \phi_{p1}}{\partial \tilde{u}} \dot{\tilde{u}} - \phi_{p2}(x,\tilde{u}) = \varphi_{p1}(y,u,\dot{u},\ldots,u^{(q)}),$$

$$\vdots$$

$$\frac{\partial \phi_{pk_p}}{\partial x} f(x,u) + \frac{\partial \phi_{pk_p}}{\partial \tilde{u}} \dot{\tilde{u}} = \varphi_{pk_p}(y,u,\dot{u},\ldots,u^{(q)}).$$

The right-hand side of the previous equations is a function of $(y,u,\dot{u},\ldots,u^{(q)})$. There is equality between the two sides if the left-hand side is also a function of $(y,u,\dot{u},\ldots,u^{(q)})$. Given that $y=h(x)$, consider the last equation of each block: there is equality between the two sides of the equations if $\phi_{ik_i}$ $(1\leqslant i\leqslant p)$ is a function of $(x,u,\dot{u},\ldots,u^{(q-1)})$. From this remark, one gets easily that $\phi_{ij}$ $(1\leqslant i\leqslant p, 1\leqslant j\leqslant k_i)$ is a function of $(x,u,\dot{u},\ldots,u^{(q-1)})$. By this way, it is proved that the linearizing state coordinates transformation is a function of $(x,u,\dot{u},\ldots,u^{(q-1)})$.

## Appendix B. Proof of Theorem 12

*Necessity*: Suppose that the generalized state coordinates transformation (17), which transforms (1) with $p=1$ into (4), exists. Then, for each output $y_i$ $(1\leqslant i\leqslant p)$, the input–output differential equation reads

$$y^{(n)} = \varphi_{s+1}^{(n-s-1)} + \varphi_{s+2}^{(n-s-2)} + \cdots + \varphi_n.$$

Let $y^{(n)}=P$ denote the input–output differential equation of system (1) with $p=1$. If system (1) is locally equivalent to (4), then their input–output differential equations have the same form. Applying the GIOIAd Algorithm for $k=s+1$, one gets

$$P_{s+1} = \varphi_{s+1}^{(n-s-1)} + \varphi_{s+2}^{(n-s-2)} + \cdots + \varphi_n$$

$$= \frac{\partial \varphi_{s+1}}{\partial y^{(s)}} y^{(n-1)} + \sum_{j=1}^{m} \frac{\partial \varphi_{s+1}}{\partial u_j^{(q)}} u_j^{(n-1-s+q)} + \Theta_1(y,\dot{y},\ldots,y^{(n-2)},u,\dot{u},\ldots,u^{(n-2-s+q)}).$$

The function $\Theta_1$ depends only on terms of $P$ which do not depend on $y^{(n-1)}$ and $u^{(n-1-s+q)}$. The differential form $\omega_{s+1}$, derived from (23), reads as

$$\omega_{s+1} = \frac{\partial P_{s+1}}{\partial y^{(n-1)}} \mathrm{d}y^{(s)} + \sum_{j=1}^{m} \frac{\partial P_{s+1}}{\partial u_j^{(n-1-s+q)}} \mathrm{d}u_j^{(q)} = \frac{\partial \varphi_{s+1}}{\partial y^{(s)}} \mathrm{d}y^{(s)} + \sum_{j=1}^{m} \frac{\partial \varphi_{s+1}}{\partial u_j^{(q)}} \mathrm{d}u_j^{(q)}.$$

The necessity of the condition of Theorem 12 is proved for the step $k=s+1$ and is proved by the same way for $k>s+1$.

*Sufficiency*: Suppose that condition of Theorem 12 is satisfied. Then, functions $\varphi_k$ are derived from GIOIAd Algorithm. The generalized state coordinates transformation is derived from (28). For each associated block to an output variable of (1) with $p=1$ and from (28), dynamics of state variables of system (4) are known. System (4) is then determined: system (1) with $p=1$ is then locally equivalent to system (4) by a generalized state coordinates transformation (28). Sufficiency of Theorem 12 is then proved.

In the same way as above, it is proved that the linearizing state coordinates transformation is a function of $(x,u,\dot{u},\dots,u^{(v)})$ with $v=\mathrm{Max}(s-2,q-1)$. Suppose that the linearizing state coordinates transformation is defined by $\zeta=\phi(x,\tilde{u})$, with $\tilde{u}=(u,\dot{u},\dots,u^{(k)})$ $(k\in\mathbb{N})$. One gets

$$\dot{\zeta}=\frac{\partial\phi}{\partial x}f(x,u)+\frac{\partial\phi}{\partial\tilde{u}}\dot{\tilde{u}}.$$

Using the state coordinates transformation, one gets

$$\begin{aligned}
&\frac{\partial\phi_1}{\partial x}f(x,u)+\frac{\partial\phi_1}{\partial\tilde{u}}\dot{\tilde{u}}-\phi_2(x,\tilde{u})=\varphi_1(y,\dot{y},\dots,y^{(s)},u,\dot{u},\dots,u^{(q)}),\\
&\vdots\\
&\frac{\partial\phi_{n-1}}{\partial x}f(x,u)+\frac{\partial\phi_{n-1}}{\partial\tilde{u}}\dot{\tilde{u}}-\phi_n(x,\tilde{u})=\varphi_{n-1}(y,\dot{y},\dots,y^{(s)},u,\dot{u},\dots,u^{(q)}),\\
&\frac{\partial\phi_n}{\partial x}f(x,u)+\frac{\partial\phi_n}{\partial\tilde{u}}\dot{\tilde{u}}=\varphi_n(y,\dot{y},\dots,y^{(s)},u,\dot{u},\dots,u^{(q)}).
\end{aligned}$$

The right-hand side of these equations depends on $(y,\dot{y},\dots,y^{(s)},u,\dot{u},\dots,u^{(q)})$. There is an equality between the two sides if the left-hand side is also a function of $(h,h^{(1)},\dots,h^{(s)},u,\dot{u},\dots,u^{(q)})$, i.e. depends on $(x,u,\dot{u},\dots,u^{(l)})$, with $l=\mathrm{Max}(s-1,q)$. Consider the last equation: the two sides are equal if the function $\phi_n$ depends on $(x,u,\dot{u},\dots,u^{(v)})$ with $v=\mathrm{Max}(s-2,q-1)$. From this remark, one gets easily that $\phi_i$ $(1\leqslant i\leqslant n)$ is a function of $(x,u,\dot{u},\dots,u^{(v)})$ with $v=\mathrm{Max}(s-2,q-1)$. In this way, it is proved that the linearizing state coordinates transformation is a function of $(x,u,\dot{u},\dots,u^{(v)})$ with $v=\mathrm{Max}(s-2,q-1)$.

## References


[1] D. Bestle, M. Zeitz, Canonical form observer for nonlinear time-variable systems, Internat. J. Control 38 (1983) 419–431.

[2] J. Birk, M. Zeitz, Extended Luenberger observer for nonlinear multivariable systems, Internat. J. Control 47 (1988) 1823–1836.

[3] J.N. Chiasson, R.T. Novotnak, Nonlinear speed observer for the PM stepper motor, IEEE Trans. Automat. Control 38 (1993) 1584–1588.

[4] J.N. Chiasson, Nonlinear differential-geometric techniques for control of a series DC motor, IEEE Trans. Control System Technol. 2 (1994) 35–42.

[5] G. Conte, C.H. Moog, A.M. Perdon, Un théorème sur la représentation entrée-sortie d'un système non linéaire, C.R. Acad. Sci., Paris 307 (1988) 363–366.

[6] S. Diop, Elimination in control theory, Math. Control Signals Systems 4 (1991) 17–32.

[7] S. Diop, M. Fliess, On nonlinear observability, Proc. ECC'91, Grenoble, France, 1991, pp. 152–157.

[8] S. Diop, J.W. Grizzle, P.E. Moraal, A. Stefanopoulou, Interpolation and numerical differentiation for observer design, Proc. ACC'94, Evanston, IL, 1994, pp. 1329–1333.

[9] M. Fliess, Generalized controller canonical forms for linear and nonlinear dynamics, IEEE Trans. Automat. Control 35 (1990) 994–1001.

[10] A. Glumineau, C.H. Moog, F. Plestan, New algebro-geometric conditions for the linearization by input–output injection, IEEE Trans. Automat. Control 41 (1996) 598–603.

[11] A. Glumineau, F. Plestan, C.H. Moog, Symbolic nonlinear analysis and control package: Analysis, control and observers design, Proc. IMACS Multiconference on Computational Engineering in Systems Applications CESA'96, Lille, France, 1996, pp. 30–32.

[12] H. Keller, Nonlinear observer design by transformation into a generalized observer canonical form, Internat. J. Control 46 (1987) 1915–1930.

[13] A.J. Krener, A. Isidori, Linearization by output injection and nonlinear observers, Systems Control Lett. 3 (1983) 47–52.

[14] A.J. Krener, W. Respondek, Nonlinear observers with linearizable error dynamics, SIAM J. Control Optim. 23 (1985) 197–216.

[15] D.G. Luenberger, Observing the state of a linear system, IEEE Trans. Mil. Electron. 6 (1964) 74–80.

[16] D.G. Luenberger, An introduction to observers, IEEE Trans. Automat. Control 16 (1971) 596–602.

[17] R. Marino, P. Tomei, Dynamic output feedback linearization and global stabilisation, Systems Control Lett. 17 (1991) 115–121.

[18] A.R. Phelps, On constructing nonlinear observers, SIAM J. Control Optim. 29 (1991) 516–534.

[19] F. Plestan, B. Cherki, An observer for a one flexible joint robot by an algebraic method, Proc. IFAC Workshop on New Trends in Design of Control Systems NTDCS'94, Smolenice, Slovakia, 1994, pp. 41–46.

[20] F. Plestan, A. Glumineau, C.H. Moog, Algebraic conditions for linearization by input-output injection, Proc. IFAC Conf. on System Structure and Control, Nantes, France, 1995, pp. 161–166.

[21] F. Plestan, A new approach for the linearization of MIMO systems by input–output injection, Proc. 34th IEEE Conf. on Decision and Control, New Orleans, LA, 1995, pp. 999–1000.
[22] T.Ph. Proychev, R.L. Mishkov, Transformation of nonlinear systems in observer canonical form with reduced dependency on derivatives of the input, Automatica 29 (1993) 495–498.
[23] A.J. Van der Schaft, Representing a nonlinear state space system as a set of higher-order differential equations in the inputs and outputs, Systems Control Lett. 12 (1989) 151–160.
[24] D. Williamson, Observation of bilinear systems with application to biological systems, Automatica 13 (1977) 243–254.
[25] X.H. Xia, W.B. Gao, Nonlinear observer design by observer error linearization, SIAM J. Control Optim. 1 (1989) 199–216.
[26] M. Zeitz, The extended Luenberger observer for nonlinear systems, Systems Control Lett. 9 (1987) 149–156.
[27] Y.F. Zheng, L. Cao, Reduced inverse for controlled systems, Math. Control Signals Systems 6 (1993) 363–379.