Expressive Completeness of some logics – proof by games

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Introduction

Expressive completeness, used as a name of a field of enquiry, studies the power of some logics with respect to fragments of first-order logic over some class of frames.

A modal logic is said to be expressively complete over some class of frames iff every formula of first-order logic can be expressed by a formula of this modal logic.

Related work

1968 Hans Kamp proved that the temporal logic with connectives Until and Since is expressively complete over the class of all Dedekind complete linear flows of time.

1979 Jonathan Stavi introduced additional connectives and proved that this enriched logic is expressively complete over the class of all linear flows of time.

Related work

1991 Yde Venema proved that CDT (an interval logic with connectives Chop, D, T) is expressively complete over linear flows of time with respect to 3-variable fragment of first-order logic.

2002 Kousha Etessami, Moshe Vardi and Thomas Wilke showed that a temporal logic with connectives Future, Past, Tomorrow and Yesterday is expressively complete over linear flows of time with respect to 2-variable fragment of first-order logic.

Related work

2009 Davide Bresolin, Valentin Goranko, Angelo Montanari and Guido Sciavicco showed that an interval temporal logic Non-strict Propositional Neighbourhood Logic ($PNL^{\pi+}$) is expressively complete over linear flows of time with respect to 2-variable fragment of first-order logic.

Contents

1. Introduction
2. Related work
3. Proof of expressive completeness by games
   - Proof procedure
   - n-pebble game
   - (1) EF theorem for n-pebble games
   - stone game
   - (2 FP) EF theorem for stone games
   - (3 FP) Strategy transfer theorem
   - (4 FP) Equivalence of formulas theorem
4. Contribution

Proof procedure

$$
\begin{tikzcd}
\boxed{\begin{array}{c}\text{Duplicator has a}\\ \text{winning strategy in}\\ \text{an n-pebble game}\end{array}} \arrow[rr, leftrightarrow, "(1)"] \arrow[d, leftrightarrow, "(3)"] & & \boxed{\begin{array}{c}\text{n-variable FO does}\\ \text{not distinguish}\\ \text{between two models}\\ \text{(playing boards)}\end{array}} \arrow[d, leftrightarrow, "(4)"] \\
\boxed{\begin{array}{c}\text{Duplicator has a}\\ \text{winning strategy}\\ \text{in a stone game}\end{array}} \arrow[rr, leftrightarrow, "(2)"] & & \boxed{\begin{array}{c}\text{a temporal logic}\\ \text{does not distinguish}\\ \text{between two models}\\ \text{(playing boards)}\end{array}}
\end{tikzcd}
$$

Contents

1. Introduction
2. Related work
3. Proof of expressive completeness by games
   - Proof procedure
   - n-pebble game
   - (1) EF theorem for n-pebble games
   - stone game
   - (2 FP) EF theorem for stone games
   - (3 FP) Strategy transfer theorem
   - (4 FP) Equivalence of formulas theorem
4. Contribution

n-pebble game

$$P_k^n(\mathfrak{A}, g_0, \mathfrak{B}, h_0)$$

There are n pairs of pebbles ρ_i, π_i $(1 \leq i \leq n)$, k rounds of game, two structures with a finite purely relational signature, $\mathfrak{A}, \mathfrak{B}$, and two initial partial assignments g_0, h_0 of variables $x_1, x_2, ..., x_n$ to elements of each structure.

There are two players Spoiler and Duplicator.

For each round t we define positions of pebbles by (g_t, h_t). $g_t(x_i) = a_i$ indicates that pebble ρ_i is placed on element a_i of structure $\mathfrak{A}$. How?

n-pebble game

At start of round t positions of pebbles: (g_{t-1}, h_{t-1}).

Spoiler selects ρ_i or π_i and places it on a selected element of its structure, Duplicator responds by placing the other pebble on a corresponding element of the other structure. We define the new positions of pebbles by (g_t, h_t).

After k rounds, the game ends. Who wins?

n-pebble game

Let $D_t \subseteq \{x_1, ..., x_n\}$ be the domain of g_t, h_t.

Duplicator wins this play of the game iff for every t, $\{(g_t(x_i), h_t(x_i)) : x_i \in D_t\}$ is a partial isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$; i.e. for every atomic formula α written with variables taken from D_t, we have

$$\mathfrak{A}, g_t \models \alpha \Leftrightarrow \mathfrak{B}, h_t \models \alpha.$$

Contents

1. Introduction
2. Related work
3. Proof of expressive completeness by games
   - Proof procedure
   - n-pebble game
   - (1) EF theorem for n-pebble games
   - stone game
   - (2 FP) EF theorem for stone games
   - (3 FP) Strategy transfer theorem
   - (4 FP) Equivalence of formulas theorem
4. Contribution

Ehrenfeucht-Fräissé theorem for n-pebble games

Let L be a finite purely relational signature, let $\mathfrak{A}, \mathfrak{B}$ be L-structures with domains M, N, respectively, and let $k, n \geq 0$ be integers. Let $D \subseteq \{x_1, ..., x_n\}$.
Then for all assignments $g : D \to M$ and $h : D \to N$, the following are equivalent:

- Duplicator has a winning strategy in $P_k^n(\mathfrak{A}, g, \mathfrak{B}, h)$,
- for every L-formula ψ of quantifier depth at most k and whose free variables are in D and all variables among $x_1, ..., x_n$, we have $\mathfrak{A}, g \models \psi \Leftrightarrow \mathfrak{B}, h \models \psi$.

Contents

1. Introduction
2. Related work
3. Proof of expressive completeness by games
   - Proof procedure
   - n-pebble game
   - (1) EF theorem for n-pebble games
   - stone game
   - (2 FP) EF theorem for stone games
   - (3 FP) Strategy transfer theorem
   - (4 FP) Equivalence of formulas theorem
4. Contribution

stone game

$$S_k(\mathcal{M}, m_0, \mathcal{N}, n_0)$$

There is a pair of stones ρ, π, k rounds of game, two Kripke models $\mathcal{M}, \mathcal{N}$. At the start of the game the position of stones is $(m_0, n_0) \in M \times N$.

There are two players Spoiler and Duplicator.

For each round t we define a position of stones by (m_t, n_t), which corresponds to stone ρ placed on element m_t and stone π placed on element n_t. How? What are the moves?

stone game

Let the current position of the stones be (m, n).

By a forward move in $\mathcal{M}$, we mean selecting $m^* \in M$ such that $m < m^*$ and placing a stone ρ on it.

By a backward move in $\mathcal{M}$, we mean selecting $m^* \in M$ such that $m^* < m$ and placing a stone ρ on it.

Similarly, we define forward and backward moves in $\mathcal{N}$. How to play?

stone game

At start of round t the position of the stones: (m_{t-1}, n_{t-1}).

Spoiler selects either ρ or π, and then makes a move in a chosen direction by placing a selected stone on a selected element. Duplicator responds by making a move in the same direction and choosing an element of the other structure by placing on it the corresponding stone. We define the new position of stones by (m_t, n_t).

After k rounds, the game ends. Who wins?

stone game

Duplicator wins this play of the game iff for every $0 \leq t \leq k$, for every $p \in PROP$, where $PROP$ is a fixed finite set of atoms, we have

$$\mathcal{M}, m_t \models p \Leftrightarrow \mathcal{N}, n_t \models p.$$

FP logic – quick reminder

- Formulas ϕ of FP are

$$\phi ::== p|\top|\neg\phi|\phi \wedge \phi|F\phi|P\phi,$$

  where $p \in PROP$.
- Formulas are evaluated at points in a Kripke model $(T, <, h)$, where the accessibility relation $<$ is the earlier-later relation (linear).
- $F\phi$ means ϕ is true at some future point of time, $P\phi$ similar for the past.

Contents

1. Introduction
2. Related work
3. Proof of expressive completeness by games
   - Proof procedure
   - n-pebble game
   - (1) EF theorem for n-pebble games
   - stone game
   - (2 FP) EF theorem for stone games
   - (3 FP) Strategy transfer theorem
   - (4 FP) Equivalence of formulas theorem
4. Contribution

Ehrenfeucht-Fräïssé theorem for stone games

Let $\mathcal{M}, \mathcal{N}$ be Kripke models of linear flows of time. Let $k \geq 0$ be an integer.
Then for every $m \in \mathcal{M}$ and $n \in \mathcal{N}$, the following are equivalent:

- Duplicator has a winning strategy in $S_k(\mathcal{M}, m, \mathcal{N}, n)$,
- for every *FP*-formula ψ of temporal operator depth at most k written with propositional atoms $p \in PROP$, we have $\mathcal{M}, m \models \psi \Leftrightarrow \mathcal{N}, n \models \psi$.

Contents

1. Introduction
2. Related work
3. Proof of expressive completeness by games
   - Proof procedure
   - n-pebble game
   - (1) EF theorem for n-pebble games
   - stone game
   - (2 FP) EF theorem for stone games
   - (3 FP) Strategy transfer theorem
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4. Contribution

Strategy transfer theorem

Let $\mathcal{M}, \mathcal{N}$ be Kripke models of linear flows of time. Let $\mathfrak{A}(\mathcal{M}) = (M, \{<^{\mathfrak{A}}\} \cup \{P^{\mathfrak{A}} : p \in PROP\})$ and $\mathfrak{B}(\mathcal{N}) = (N, \{<^{\mathfrak{B}}\} \cup \{P^{\mathfrak{B}} : p \in PROP\})$ be the first-order $L(PROP)$-structures constructed from $\mathcal{M}, \mathcal{N}$, respectively. Let $k \geq 0$ be an integer.

Then for all $m \in \mathcal{M}$ and $n \in \mathcal{N}$, the following two are equivalent:

(stone) Duplicator has a winning strategy in $S_k(\mathcal{M}, m, \mathcal{N}, n)$,

(pebble) Duplicator has a winning strategy in $P_k^2(\mathfrak{A}(\mathcal{M}), \{(x_1, m)\}, \mathfrak{B}(\mathcal{N}), \{(x_1, n)\})$.

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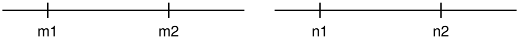
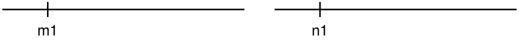
Fragment of a proof (stone) $\Rightarrow$ (pebble)

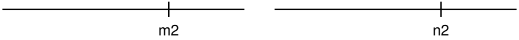
Public game



Private game s_1

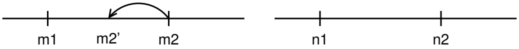


Private game s_2



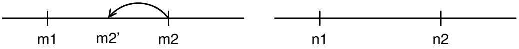
Fragment of a proof (stone) ⇒ (pebble)

Spoiler's move in the public game

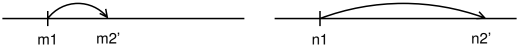


Fragment of a proof (stone) $\Rightarrow$ (pebble)

Spoiler's move in the public game

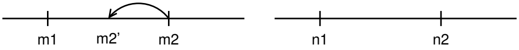


Duplicator establishes her response in the private game due to her winning strategy (s_1 copied and replaces s_2)



Fragment of a proof (stone) $\Rightarrow$ (pebble)

Spoiler's move in the public game



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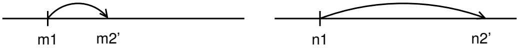
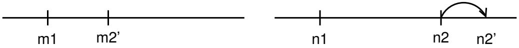


Duplicator's response in the public game



Contents

1 Introduction

2 Related work

3 Proof of expressive completeness by games
- Proof procedure
- n-pebble game
- (1) EF theorem for n-pebble games
- stone game
- (2 FP) EF theorem for stone games
- (3 FP) Strategy transfer theorem
- (4 FP) Equivalence of formulas theorem

4 Contribution

Equivalence of formulas theorem

Let $\mathcal{C}$ be a class of linear flows of time $\mathcal{F} = (T, <)$.
Then for every finite set of propositional atoms $PROP$, for every FO^2 formula $\psi(x, L(PROP))$, there is a formula $A(PROP)$ of FP logic, whose standard translation $A^x(x, L(PROP))$ is equivalent to $\psi(x, L(PROP))$ over the class of $L(PROP)$-structures constructed from models of linear flows of time in $\mathcal{C}$.

Contribution

Proofs by games – uniform approach

first order logics	all linear flows of time	all Kripke frames
	point logics	
2-variable	FP	$PLOZ$
	interval logics	
2-variable	$PNL^{\pi+}$	SD
3-variable	CDT	E

Thank you. Questions?