Stochastic Games on Lossy Channels with Finite Memory Players

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Abstract. We consider turn-based $2\frac{1}{2}$-player games, played on infinite graphs induced by probabilistic lossy channel systems (PLCS). We consider Büchi objectives (repeated reachability) and positive probability winning conditions (i.e., almost-sure co-Büchi objectives for the opponent). In general, these games are not memoryless determined, and also undecidable [7].
However, we show that it is decidable whether there exists a finite-state controller that can achieve the positive probability Büchi objective, and pure memoryless controllers (i.e., with only 1 state) do indeed suffice in this case. The existence of such a pure finite-state controller is independent of the memory available to the opponent, and the controller is effectively constructible if it exists. This generalizes previous results on PLCS-induced MDPs/games in [7, 1]. We also identify the general abstract conditions on the infinite game graph under which these results hold, i.e., pointwise finite branching and the existence of a finite attractor.

1 Introduction

Background. 2-player games can be used to model the interaction of a controller (player 0) who makes choices in a reactive system, and a malicious adversary (player 1) who represents an attacker, the environment, or the worst-case of new nondeterminism introduced by abstraction. To model randomness (e.g., unreliability; randomized algorithms) in the system, a third player 'random' is defined who makes choices according to a predefined probability distribution. The resulting stochastic game is called a $2\frac{1}{2}$-player game in the terminology of [9]. The choices of the players induce a run of the system, and the winning conditions of the game are expressed in terms of predicates on runs.

Most classic work on algorithms for stochastic games has focused on finite-state systems (e.g., [19, 11, 12, 9]), but more recently several classes of infinite-state systems have been considered as well. Stochastic games on infinite-state probabilistic recursive systems (i.e., probabilistic pushdown automata with unbounded stacks) were studied in [14, 15, 13]. Another infinite-state probabilistic model, which is incomparable to recursive systems, is probabilistic lossy channel systems (PLCS). In PLCS, finite-state machines communicate by asynchronous message passing via unbounded unreliable (i.e., lossy) FIFO communication channels, and several algorithms for symbolic model checking of these were presented in [8, 5, 2, 17, 6]. A game extension of PLCS is called game probabilistic lossy channel systems (GPLCS), where the players control the transitions in the control graph and message losses are probabilistic [1]. It is also possible to extend this model by allowing probabilistic transitions in the control graph.

Previous work. Markov decision processes (i.e., $1\frac{1}{2}$-player games) on infinite graphs induced by PLCS were studied in [7], where it was shown that $1\frac{1}{2}$-player games with almost-sure Büchi objectives are pure memoryless determined and decidable. This result was later generalized to $2\frac{1}{2}$-player games on GPLCS in [1].

On the other hand, $1\frac{1}{2}$-player games on PLCS with positive probability Büchi objectives (i.e., almost-sure co-Büchi objectives from the (here passive) opponents point of view) can require infinite memory to win and are also undecidable [7]. (The undecidability is really a separate result, since, in general, decidability itself does not imply the existence of finite-memory strategies in infinite-state games). However, the $1\frac{1}{2}$-player game with positive probability Büchi objectives becomes decidable if the player is limited to finite memory [7].

Our contribution. We consider the full $2\frac{1}{2}$-player Büchi-games on GPLCS with positive probability winning condition and a player who is limited to finite memory (without any restriction on the memory of the opponent). I.e., we do not consider the question 'Can the player win the positive probability Büchi-game?' (which is undecidable even for $1\frac{1}{2}$-player games [7]), but rather the question 'Does there exist a finite controller which can achieve the positive probability Büchi objective against every opponent?' This question is significantly more complex for $2\frac{1}{2}$-player games than in $1\frac{1}{2}$-player games. In $1\frac{1}{2}$-player games [7] it suffices to reach, with positive probability, a state which is almost-surely winning. In contrast, this is not necessary in $2\frac{1}{2}$-player games, because the choices of the opponent influence the game and she might be able to lose in a more complex way (see Remark 1).

We show that the latter question is decidable, and that pure memoryless controllers do indeed suffice in this case, i.e., the existence of a winning finite controller implies the existence of a controller with no memory at all. This generalizes the corresponding decidability result on finite-memory $1\frac{1}{2}$-player games in [7]. In particular, the amount of memory available to the opponent (infinite, finite or none) makes no difference at all. We also show that a symbolic representation of the pure positional winning strategies for the player or opponent can be computed, i.e., the pure memoryless controller is effectively constructible.

The problem of deciding positive-Büchi GPLCS is *non-primitive recursive*, since it is harder than the control-state reachability problem (non-primitive recursive by [18]).

Our work is mainly motivated by analyzing lossy channel systems. However, we give our result in general terms and identify the three abstract conditions on the underlying infinite game graph which are required for our construction. Roughly speaking, these conditions are the existence of a finite attractor, pointwise finite branching, and a well-quasi-order on the infinite set of states (see Section 3 for details).

Outline. Section 2 contains basic definitions, and the main problem is defined in Section 3. The solution is presented in three steps. Section 4 solves generalized reachability games and Section 5 solves generalized almost-sure Büchi games (without restricting the memory of the players). Building upon these results, the main problem about positive Büchi games for a finite-memory player is then solved in Section 6. Finally, some related problems are discussed in Section 7.

2 Preliminaries

We use $\mathbb{R}, \mathbb{N}$ for the real and natural numbers. A *probability distribution* on a countable set X is a function $f : X \to [0, 1]$ such that $\sum_{x \in X} f(x) = 1$. We will sometimes need to pick an arbitrary element from a set. To simplify the exposition, we let select (X) denote an arbitrary but fixed element of the nonempty set X.

Turn-based Stochastic Games. A *turn-based stochastic game* (or a *game* for short) is a tuple $\mathcal{G} = (S, S^0, S^1, S^R, \longrightarrow, P)$ where:

- S is a countable set of *states*, partitioned into the pairwise disjoint sets of *random states* S^R, states S^0 of player 0, and states S^1 of player 1.
- $\longrightarrow \subseteq S \times S$ is the *transition relation*. We write $s \longrightarrow s'$ to denote that $(s, s') \in \longrightarrow$. Let $\mathrm{Post}\,(s) := \{s' : s \longrightarrow s'\}$ denote the set of *successors* of s and $\mathrm{Pre}\,(s) := \{s' : s' \longrightarrow s\}$ denote the set of *predecessors* of s. We extend these operations to sets in the standard way. We assume that games are deadlock-free, i.e., each state has at least one successor ($\forall s \in S.\ \mathrm{Post}\,(s) \neq \emptyset$).
- The *probability function* $P : S^R \times S \to [0, 1]$ satisfies both $\forall s \in S^R.\ \forall s' \in S.\ (P(s, s') > 0 \iff s \longrightarrow s')$ and $\forall s \in S^R.\ \sum_{s' \in S} P(s, s') = 1$. Note that for any given state $s \in S^R$, $P(s, \cdot)$ is a probability distribution over $\mathrm{Post}\,(s)$.

For any set $Q \subseteq S$ of states, we let $\overline{Q} := S - Q$ denote its complement. We define $[Q]^R := Q \cap S^R$, $[Q]^0 := Q \cap S^0$, $[Q]^1 := Q \cap S^1$, and $[Q]^{01} := Q \cap (S^0 \cup S^1)$. Furthermore, for $\sigma \in \{0, 1\}$, let $\widetilde{\mathrm{Pre}}^{\sigma}(Q) := [S - \mathrm{Pre}\,(\overline{Q})]^{\sigma}$.

A *run* ρ in a game is an infinite sequence $s_0 s_1 \cdots$ of states s.t. $s_i \longrightarrow s_{i+1}$ for all $i \geq 0$. We use $\rho(i)$ to denote s_i. A *path* π is a finite sequence $s_0 \cdots s_n$ of states s.t. $s_i \longrightarrow s_{i+1}$ for all $i : 0 \leq i < n$. For any $Q \subseteq S$, we use Π_Q to denote the set of paths that end in some state in Q.

Informally, the two players 0 and 1 construct an infinite run $s_0 s_1 \cdots$, starting in some initial state $s_0 \in S$. Player 0 chooses the successor s_{i+1} if $s_i \in S^0$, player 1 chooses s_{i+1} if $s_i \in S^1$, and the successor s_{i+1} is chosen randomly according to the probability distribution $P(s_i, \cdot)$ if $s_i \in S^R$.

Strategies. A *pure strategy* is a partial function $f^\sigma : \Pi_{S^\sigma} \rightharpoonup S$ from sequences of states to states, such that $s_n \to f^\sigma(s_0 \cdots s_n)$ if f^σ is defined. [1] So, in general, the next state depends on the history so far. Consider two total strategies f^0 and f^1 of player 0 and 1. A path $\pi = s_0 \cdots s_n$ in $\mathcal{G}$ is said to be *consistent* with f^0 and f^1 if the following holds. For all $0 \leq i \leq n-1$, $s_i \in S^0$ implies $f^0(s_0 \cdots s_i) = s_{i+1}$ and $s_i \in S^1$ implies $f^1(s_0 \cdots s_i) = s_{i+1}$. We define similarly *consistent* runs. In the sequel, whenever the strategies are known from the context, we assume that all mentioned paths and runs are consistent with them.

We say that a strategy f^σ is *memoryless* (resp., *forgetful*) if it only depends on the current state (resp., on a finite amount of information of the past history). In the latter case, we also say that the strategy uses only a finite amount of memory. A memoryless strategy of player σ can be regarded simply as a function $f^\sigma : S^\sigma \rightharpoonup S$, such that $s \longrightarrow f^\sigma(s)$ whenever f^σ is defined.

[1] One could consider the more liberal definition of *mixed* strategies, i.e., functions from Π_{S^σ} to probability distributions over S. However, this does not affect the results of this paper for two reasons. First, we show that *pure* strategies are indeed sufficient for both players. Second, our proofs can be easily extended to deal with mixed strategies when quantifying over all possible strategies for the opponent player. Furthermore, in the rest of the paper we simply write "strategy" instead of "pure strategy".

Probability Measures. We use the standard definition of the probability measure for a set of runs [16]. First, we define the measure for total strategies, and then extend it to general (partial) strategies. We let $\Omega^s = sS^\omega$ denote the set of all infinite sequences of states starting from s. Consider a game $\mathcal{G} = (S, S^0, S^1, S^R, \longrightarrow, P)$, an initial state s, and total strategies f^0 and f^1 of player 0 and 1. For a measurable set $\mathfrak{R} \subseteq \Omega^s$, we define $\mathcal{P}^s_{f^0,f^1}(\mathfrak{R})$ to be the probability measure of $\mathfrak{R}$ under the strategies f^0, f^1. It is well-known that this measure is well-defined [16]. When the state s is known from context, we drop the superscript and write $\mathcal{P}_{f^0,f^1}(\mathfrak{R})$. For (partial) strategies f^0 and f^1 of player 0 and 1, $\sim \in \{<, \leq, =, \geq, >\}$, and any measurable set $\mathfrak{R} \subseteq \Omega^s$, we define $\mathcal{P}^s_{f^0,f^1}(\mathfrak{R}) \sim x$ iff $\mathcal{P}^s_{g^0,g^1}(\mathfrak{R}) \sim x$ for all total strategies g^0 and g^1 which are extensions of f^0 resp. f^1. For a single strategy f^σ of player σ, we define $\mathcal{P}^s_{f^\sigma}(\mathfrak{R}) \sim x$ iff $\mathcal{P}^s_{f^0,f^1}(\mathfrak{R}) \sim x$ for all strategies $f^{1-\sigma}$ of player $(1-\sigma)$. If $\mathcal{P}_{f^0,f^1}(\mathfrak{R}) = 1$, then we say that $\mathfrak{R}$ happens *almost surely* (*a.s.*) under the strategies f^0, f^1. Similarly, if $\mathcal{P}_{f^0,f^1}(\mathfrak{R}) > 0$, we say that $\mathfrak{R}$ happens *with positive probability* (*w.p.p.*).

We assume familiarity with the syntax and semantics of the temporal logic CTL* (see, e.g., [10]). We use $(s \models \varphi)$ to denote the set of runs starting in s that satisfy the CTL* path-formula φ. We use $\mathcal{P}_{f^0,f^1}(s \models \varphi)$ to denote the measure of $(s \models \varphi)$ under strategies f^0, f^1, i.e., we measure the probability of those runs which start in s, are consistent with f^0, f^1 and satisfy the path-formula φ. This set is measurable by [20].

Traps. For player $\sigma \in \{0,1\}$ and for $Q \subseteq S$ we say that Q is a σ-trap if the other player $(1-\sigma)$ has a strategy to keep the game forever in Q. Formally, we require that all successors of states in $[Q]^\sigma \cup [Q]^R$ are in Q and that every state in $[Q]^{1-\sigma}$ has some successor in Q.

Winning Conditions. Our main result considers *Büchi* objectives: player 0 wants to visit a given set $F \subseteq S$ infinitely many times. We consider games with *positive probability* winning conditions. More precisely, $s \in S$ is winning for player 0 if she has a strategy f^0 which is winning against all strategies f^1 of player 1: $\mathcal{P}_{f^0,f^1}(s \models \Box\Diamond F) > 0$.

Determinacy and Solvability. We say that a player has a *winning strategy* on $Q \subseteq S$ if there exists a strategy which is winning from each state in Q against any opponent strategy. Memoryless and forgetful winning strategies are defined similarly.

We say that a game is *determined* if for every state $s \in S$ exactly one of the two players has a pure winning strategy on $\{s\}$. Note that if a game is determined, then the winning sets for player 0 and 1, W^0 and W^1, respectively, form a partitioning of S. Similarly, a game is *memoryless determined* if it is determined and there are pure memoryless winning strategies for both players. By *solving* a game, we mean giving an algorithm for constructing a symbolic representation of W^0 and W^1.

3 Games on Infinite Graphs

A *lossy channel system (LCS)* [4] is a finite-state automaton equipped with a finite number of unbounded lossy FIFO channels (queues). The system is *lossy* in the sense that, before and after a transition, an arbitrary number of messages may be lost from the channels. *Probabilistic lossy channel systems (PLCS)* [8, 5, 3] define a probabilistic model for message losses, inducing an infinite-state Markov chain. The standard model assumes that each individual message is lost independently with probability λ in every step, where $\lambda > 0$ is a parameter of the system, but other models (e.g., burst disturbances) can also be handled in this framework. This has been extended to an MDP ($1\frac{1}{2}$-player game) in [7], by letting the player choose transitions in the control graph and keeping message losses probabilistic. A further generalization to $2\frac{1}{2}$-player games (called GPLCS) was defined in [1] by adding a second competing player. The reader is referred to [1] for a formal definition of GPLCS. Here we list the abstract properties of the (GPLCS-induced) game graph which are needed for our constructions.

We consider turn-based infinite-state $2\frac{1}{2}$-player games $(S, S^0, S^1, S^R, \longrightarrow, P)$ satisfying the following conditions.

1. Finite attractor: There exists a finite subset $A \subseteq S$ to which the game returns a.s., regardless of the actions of the players, i.e., for every pair of strategies f^0, f^1 of the two players $\forall s \in S. \mathcal{P}_{f^0,f^1}(s \models \Box\Diamond A) = 1$. ($A$ is not necessarily absorbing.)
2. Finitely branching: Every state has only finitely many successors, i.e., Post (s) is finite.
3. There exists a decidable well-quasi-order $\preceq$ on the set of states S and a formalism γ for representing infinite sets of states s.t.
   - All upward-closed sets are effectively γ-representable. Boolean combinations and Pre () of γ-representable sets are effectively γ-representable.
   - For any set $Q \subseteq S$, the set Pre^R (Q) is upward-closed w.r.t. $\preceq$.

The first two conditions are required for the correctness of the construction, and the third condition is needed to make the construction of the winning sets and strategies effective. GPLCS satisfy these three conditions [1], where the formalism γ is (roughly speaking) regular languages, and $\preceq$ is the substring relation.

To simplify the presentation, we also assume, without restriction, that the game is *bipartite*: every transition goes from a player state to a probabilistic state or the other way around, i.e., $\longrightarrow \subseteq ((S^0 \cup S^1) \times S^R) \cup (S^R \times (S^0 \cup S^1))$.

The following fundamental property of upward-closed sets will be used later.

Lemma 1 *Let $U_0 \subseteq U_1 \subseteq U_2 \ldots$ be a chain of upward-closed sets w.r.t. a well-quasi-order $\preceq$. Then, there exists an $i \in \mathbb{N}$ such that $U_i = U_{i+1}$.*

Problem statement. We consider the following problem. The infinite game graph is one induced by a GPLCS and thus satisfies the three conditions above. The players have conflicting goals: player 0 wants to visit a given target set infinitely often with positive probability (*Büchi objective*), while player 1 wants to visit it at most finitely many times (a.s.). The question is whether player 0 can win the game with only finite memory.

4 Extended Constrained Reachability Games on GPLCS

We address the constrained reachability game with the following winning condition: Given sets of states $T, I, F \subseteq S$, player σ wants to either

1. Reach T with positive probability, or
2. Reach F with positive probability while remaining in I. (We write $I\ \mathcal{U}\ F$ for this property, where we mean that I also holds when F is reached.)

More formally, for an initial state $s \in S$, we want to establish whether player σ has a strategy f^{σ} such that, for any player $(1-\sigma)$ strategy $f^{1-\sigma}$, either

1. $\mathcal{P}_{f^{\sigma},f^{1-\sigma}}(s \models \Diamond T) > 0$, or
2. $\mathcal{P}_{f^{\sigma},f^{1-\sigma}}(s \models I\, \mathcal{U}\, F) > 0$.

Let $\text{Force}_T^{\sigma}(I, F)$ be the set of states which are winning for player σ according to the previous definition.

Remark 1. Note that such objectives are more complex for 2-player games than for the 1-player games of [7]. It is possible that player σ can win even though she cannot enforce either of these two winning conditions, but *only their disjunction*. The opponent player $(1-\sigma)$, while losing, might have the option to choose how she loses, i.e., whether 1) or 2) above is satisfied.

We show that both players can win positionally if they can win at all. We stress the fact that the correctness of the following construction does not rely neither on the system having a finite attractor nor on the system being finitely-branching. We construct monotone increasing sequences of sets of states B_i, D_i, E_i. The states in B_i are those states in which player σ can reach T with positive probability and within i steps.

$$B_0 := T,\ B_{i+1} := B_i \cup \text{Pre}^R(B_i) \cup \text{Pre}^{\sigma}(B_i) \cup \widetilde{\text{Pre}}^{1-\sigma}(B_i)\ .$$

The states in D_i are those states in S^R in which player σ can, with positive probability and within i steps, either 1) force reaching T, or 2) force reaching F while remaining in I. The states in E_i are the states in $S^0 \cup S^1$ satisfying the same property.

$$\begin{aligned}
D_0 &:= [(F \cap I) \cup B_0]^R \\
D_{i+1} &:= D_i \cup (\text{Pre}^R(E_i) \cap I) \cup \text{Pre}^R(B_i) \\
E_0 &:= [(F \cap I) \cup B_0]^{01} \\
E_{i+1} &:= E_i \cup \left(\left(\text{Pre}^{\sigma}(D_i) \cup \widetilde{\text{Pre}}^{1-\sigma}(D_i \cup B_i)\right) \cap I\right) \cup \text{Pre}^{\sigma}(B_i) \cup \widetilde{\text{Pre}}^{1-\sigma}(B_i)\ .
\end{aligned}$$

The set of winning states for player σ is defined as $\text{Force}_T^{\sigma}(I, F) := \bigcup_i (D_i \cup E_i)$. In the following, we will omit the subscript T or the invariant I when they are trivial, i.e., $T = \emptyset$ or $I = S$. (For example, $\text{Force}^{\sigma}(F)$ equals $\text{Force}_{\emptyset}^{\sigma}(S, F)$.) Finally, we note that $\bigcup_i B_i = \text{Force}^{\sigma}(R)$, following directly from the definitions.

Simplification. For technical convenience, we now provide an alternative formulation of the previous construction. Let $T' := \text{Force}^{\sigma}(T) = \bigcup_i B_i$. Since $\text{Pre}^R(T') \cup \text{Pre}^{\sigma}(T') \cup \widetilde{\text{Pre}}^{1-\sigma}(T') \subseteq T'$, we can now simplify the previous equations as follows:

$$\begin{aligned}
D'_0 &:= [(F \cap I) \cup T']^R\ ,\ D'_{i+1} := D'_i \cup (\text{Pre}^R(E'_i) \cap I) \\
E'_0 &:= [(F \cap I) \cup T']^{01}\ ,\ E'_{i+1} := E'_i \cup \left(\left(\text{Pre}^{\sigma}(D'_i) \cup \widetilde{\text{Pre}}^{1-\sigma}(D'_i)\right) \cap I\right)
\end{aligned}$$

The previous definitions and the new one are equivalent in the following sense. Clearly, $D_i \subseteq D'_i$ and $E_i \subseteq E'_i$. Moreover, they are equal in the limit, i.e., $\bigcup_i D_i = \bigcup_i D'_i$ (similarly for E_i). Hence, we will use the new formulation in the following and we will drop the prime ($'$) thereafter.

Termination. We now prove that the construction of $\text{Force}_T^\sigma(I,F)$ eventually terminates. First, we show that the construction of $\text{Force}_T^\sigma(I,F)$ terminates if the construction of T' does. Consider the following sequence: $G_0 := D_0 = [(F \cap I) \cup T']^R$, $G_{i+1} := \text{Pre}^R(E_i)$. We have that G_i is upward closed for every $i > 0$. Hence the sequence $G_0 \subseteq G_1 \subseteq \cdots$ eventually terminates by Lemma 1, i.e., there exists an index $n \in \mathbb{N}$ such that $G_{n+1} = G_n$. Now, note that $D_{i+1} = D_0 \cup \left(\bigcup_{j=1}^{i} G_j \cap I\right)$, hence $D_{n+1} = D_n$ and, consequently, $E_{n+2} = E_{n+1}$.

In the previous argument we assumed that T' terminates. Since $T' = \text{Force}_\emptyset^\sigma(S,T)$, we can use the very same argument to show that T' (unconditionally) terminates as well. Therefore, we have the following lemma:

Lemma 2 (Termination) *For any GPLCS and sets $T, I, F \subseteq S$, the sequences $\{D_i\}_{i\in\mathbb{N}}$ and $\{E_i\}_{i\in\mathbb{N}}$ converge.*

Form of Winning Sets. From the termination argument above, we can in fact derive more information about the form of winning sets. Assume that F, I and R are regular (in general, γ-representable). Then, by the assumptions of Section 3, every E_i, D_i is regular. Therefore, since the construction terminates, $\text{Force}_T^\sigma(I,F)$ is regular.

Lemma 3 *Let $Q := \text{Force}_T^\sigma(I,F)$, $Q' := \text{Force}^\sigma(I,F)$ and $T' := \text{Force}^\sigma(R)$. If F, I and R are regular, then Q, Q' and T' are regular.*

Correctness. In the following we argue about the correctness of the above construction by providing pure winning strategies for both players.

Strategy for player σ. We provide a partial pure memoryless winning strategy for player σ, denoted as $\underline{\text{force}}_T^\sigma(I,F)$, which is defined on states in $[\text{Force}_T^\sigma(I,F)]^\sigma$. We define a sequence of strategies $e_0 \subseteq e_1 \subseteq \cdots$ and we let $\underline{\text{force}}_T^\sigma(I,F) := \bigcup_i e_i$. We set $e_0 := \emptyset$ and define e_{i+1} inductively for each s:

1. if $e_i(s)$ is defined, then $e_{i+1}(s) := e_i(s)$; otherwise,
2. if $s \in [E_{i+1} \setminus E_i]^\sigma$, then $e_{i+1}(s) := \text{select}(\text{Post}(s) \cap D_i)$.

Lemma 4 (Uniform bound) *For any GPLCS, set of states $T, I, F \subseteq S$, player $\sigma \in \{0,1\}$ and $s \in \text{Force}_T^\sigma(I,F)$, there exists $\epsilon_s > 0$ such that for any extension $f^\sigma \supseteq \underline{\text{force}}_T^\sigma(I,F)$ of player σ strategy and for any player $(1-\sigma)$ strategy $f^{1-\sigma}$, either*

a) $\mathcal{P}_{f^\sigma,f^{1-\sigma}}(s \models \Diamond T) > 0$, *or*
b) $\mathcal{P}_{f^\sigma,f^{1-\sigma}}(s \models I\,\mathcal{U}\,F) \geq \epsilon_s$.

Remark 2. Notice that in this lemma we state something stronger than simple correctness, i.e., $\mathcal{P}_{f^\sigma,f^{1-\sigma}}(s \models I\,\mathcal{U}\,F) > 0$. However, the uniform bound on the winning probability above (point 4(b)) will be needed in the following sections (for Lemmas 11,12 and 15). Moreover, while in this lemma we use the fact that the system is finitely-branching for proving the uniform bound (in the proof immediately below), we also stress the fact that simple correctness *does not* require this assumption.

Proof (of Lemma 4). We recall that $\text{Force}_T^\sigma(I,F) := \bigcup_i (D_i \cup E_i)$. We prove the lemma for any $i \in \mathbb{N}, s \in (D_i \cup E_i)$. We proceed by induction on i. The base case is easy, as $s \in (D_0 \cup E_0) = (F \cap I) \cup T'$: simply take $\epsilon_s := 1$ and the claim holds by definition. For the inductive step, assume $s \in (D_{i+1} \cup E_{i+1})$. In the following, further assume $s \notin (D_i \cup E_i)$ (the claim already holds by induction hypothesis otherwise). There are three cases to consider, reflecting the membership of s in $(D_{i+1} \cup E_{i+1}) \setminus (D_i \cup E_i)$:

1. $s \in \text{Pre}^R(E_i) \cap I$. Thus, there exists $s' \in E_i$, $s \to s'$ and, either 4(a) holds for s' and hence it also holds for s, or 4(b) holds for s'. In this case, we have $\epsilon_{s'} > 0$ by induction hypothesis. Take $\epsilon_s := P(s,s') \cdot \epsilon_{s'} > 0$.
2. $s \in \text{Pre}^\sigma(D_i) \cap I$. Thus, there exists $s' = \underline{\text{force}}_T^\sigma(I,F) \in D_i$, $s \to s'$ and $\epsilon_{s'} > 0$ by induction hypothesis. Take $\epsilon_s := \epsilon_{s'}$ and apply the same reasoning as in the previous point.
3. $s \in \widetilde{\text{Pre}}^{1-\sigma}(D_i) \cap I$. Then, $\text{Post}(s) \subseteq D_i$ and for each $s' \in \text{Post}(s)$, $\epsilon_{s'} > 0$ by induction hypothesis. Since the system is *finitely branching* (a crucial assumption), $\text{Post}(s)$ is a finite set and there exists an s' that minimizes $\epsilon_{s'} > 0$. Then, take $\epsilon_s := \min_{s' \in \text{Post}(s)} \epsilon_{s'} > 0$ and proceed as before. □

Strategy for player $(1-\sigma)$. Now we show how player $(1-\sigma)$ can win from states in $\text{co-Force}_{\overline{T}}^{1-\sigma}\left(\overline{I},\overline{F}\right) := \overline{\text{Force}_T^\sigma(I,F)}$. We give a partial pure memoryless winning strategy $\underline{\text{co-force}}_{\overline{T}}^{1-\sigma}\left(\overline{I},\overline{F}\right)$ for player $(1-\sigma)$ from $\left[\text{co-Force}_{\overline{T}}^{1-\sigma}\left(\overline{I},\overline{F}\right)\right]^{1-\sigma} \cap I$[2], defined as follows: $\underline{\text{co-force}}_{\overline{T}}^{1-\sigma}\left(\overline{I},\overline{F}\right)(s) := \text{select}\left(\text{Post}(s) \cap \text{co-Force}_{\overline{T}}^{1-\sigma}\left(\overline{I},\overline{F}\right)\right)$. By next Lemma 6, $\text{co-Force}_{\overline{T}}^{1-\sigma}\left(\overline{I},\overline{F}\right)$ is a σ-trap, hence it is always possible to select successor states as specified above. Moreover, the game remains forever outside T and F can't be visited unless $\overline{I}$ is visited before.

Lemma 5 *For any GPLCS, set of states $T, I, F \subseteq S$, player $\sigma \in \{0,1\}$, state $s \in$ co-Force$_{\overline{T}}^{1-\sigma}\left(\overline{I},\overline{F}\right)$ and for any extension $f^{1-\sigma} \supseteq \underline{\text{co-force}}_{\overline{T}}^{1-\sigma}\left(\overline{I},\overline{F}\right)$ of player $(1-\sigma)$ strategy and for any player σ strategy f^σ, both*

a) $\mathcal{P}_{f^\sigma,f^{1-\sigma}}\left(s \models \Box\overline{T}\right) = 1$, and
b) $\mathcal{P}_{f^\sigma,f^{1-\sigma}}\left(s \models I\,\mathcal{U}\,F\right) = 0$.

Lemma 6 *co-Force$_{\overline{T}}^{1-\sigma}\left(\overline{I},\overline{F}\right) := \overline{\text{Force}_T^\sigma(I,F)}$ is a σ-trap.*

Theorem 1 (Determinacy and solvability) *Constrained reachability games on GPLCS are pure memoryless determined and solvable for every regular set I, F and R.*

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[2] If $s \in \text{co-Force}_{\overline{T}}^{1-\sigma}\left(\overline{I},\overline{F}\right)$ but $s \notin I$, then, by definition, s is already winning for player $(1-\sigma)$ and there is nothing to prove.

5 Extended Almost-sure Büchi Games on GPLCS

We now solve the problem in which the objective of player σ is to either 1) reach T with positive probability (and we write $T' := \text{Force}^{\sigma}(T)$ for such states) or 2) visit F infinitely often almost-surely. We show that if player σ can win, then she can win positionally. We prove this under both the assumptions that the system is finitely branching and has a finite attractor (see Section 3). However, we put no restriction on the strategies of the opponent (player $(1-\sigma)$), which can in principle use uncomputable strategies or infinite memory. Additionally, from the point of view of the opponent, we prove the existence of pure memoryless strategies under no assumption on the system, i.e., we do *not* require that the system is either finitely branching or has a finite attractor. This already shows some asymmetry between the two players even for almost-sure Büchi objectives (we will show even more differences in the next section).

We proceed by building two sequences of sets of states, $\{M_j\}_{j\in\mathbb{N}}$ and $\{X_j\}_{j\in\mathbb{N}}$, as follows: Let $M_0 := S$, $X_0 := \emptyset$ and for $j \in \mathbb{N}$:

$$M_{j+1} := \text{Force}_T^{\sigma}\left(\overline{X_j}, F\right) \tag{1}$$

$$X_{j+1} := \text{Force}^{1-\sigma}\left(\overline{T'}, \overline{M_{j+1}}\right) \tag{2}$$

The set M_j is an over-approximation of the set of winning states for player σ, while X_j is an under-approximation of winning states for the opponent $(1-\sigma)$. We define the set of winning states for player σ (opponent $(1-\sigma)$) as $\text{ForceBüchi}^{\sigma}(T, F) := \bigcap_j M_j$ ($\text{co-ForceBüchi}^{1-\sigma}\left(\overline{T}, \overline{F}\right) := \bigcup_j X_j$, resp.). The following properties are used later:

Lemma 7 *Let X_j and M_j as above and let $T' :=$ Force$^{\sigma}(T)$. Then,*

a) $X_j \subseteq \overline{T'}$ and $\overline{M_j} \subseteq \overline{T'}$, for each $j \in \mathbb{N}$.
b) $X_0 \subseteq \overline{M_1} \subseteq X_1 \subseteq \overline{M_2} \subseteq X_2 \cdots$.

Lemma 8 *For any GPLCS and sets $T, F \subseteq S$, $\{M_j\}_{j\in\mathbb{N}}$ and $\{X_j\}_{j\in\mathbb{N}}$ converge.*

Since in the limit $X_n = \overline{M_n}$, we have that the sets of winning states for the two players form a partition of S, i.e., $\text{co-ForceBüchi}^{1-\sigma}\left(\overline{T}, \overline{F}\right) = \overline{\text{ForceBüchi}^{\sigma}(T, F)}$ and $\text{co-ForceBüchi}^{1-\sigma}\left(\overline{T}, \overline{F}\right) \cup \text{ForceBüchi}^{\sigma}(T, F) = S$. This implies that extended a.s. Büchi games are determined, for any F and R.

Form of Winning Sets. By the assumptions of Section 3 and assuming that F and R are regular, we have that each X_j and M_j is regular (by Lemma 3). Since the construction terminates, also $\text{ForceBüchi}^{\sigma}(T, F)$ and its complement are regular.

The next lemma holds by the definitions and it will be used in the following to show the correctness of the above construction.

Lemma 9 *Let $Q :=$ ForceBüchi$^{\sigma}(T, F)$. Then,*

a) $Q =$ Force$_T^{\sigma}(Q, F)$, and
b) $Q \setminus T'$ is a $(1-\sigma)$-trap.

Correctness. This time it is conceptually simpler to start with the opponent's strategy.

Winning strategy for the opponent $(1-\sigma)$. We recall that the objective of the opponent $(1-\sigma)$ is to always avoid T, while visiting F only a finite number of times (with a positive probability). We build a partial pure memoryless strategy $\underline{\text{co-forceBüchi}}^{1-\sigma}\left(\overline{T}, \overline{F}\right)$, which is winning for the opponent from states in co-ForceBüchi$^{1-\sigma}\left(\overline{T}, \overline{F}\right)$. We proceed by constructing a sequence of strategies $x_0 \subseteq x_1 \subseteq x_2 \subseteq \cdots$, where each x_j is defined on $[X_j]^{1-\sigma}$. Hence, we let $\underline{\text{co-forceBüchi}}^{1-\sigma}\left(\overline{T}, \overline{F}\right) := \bigcup_j x_j$. Correctness follows as $s \in \bigcup_j X_j$ implies $s \in X_j$ for some j and the opponent wins according to x_j.

We define the sequence of strategies by induction on j and we also explain why this is indeed correct. First of all, we notice that $X_j \subseteq \overline{T'}$ (and hence $\subseteq \overline{R}$) by Lemma 7(a), hence the game stays forever outside R. We set $x_0 := \emptyset$ and we inductively define x_{j+1} by case analysis, based on the inclusion $X_j \subseteq \overline{M_{j+1}} \subseteq X_{j+1}$ (see Lemma 7(b)):

1. If $s \in [X_j]^{1-\sigma}$, then $x_{j+1}(s) := x_j(s)$.
   By induction hypothesis, we already know that x_j is a winning strategy.
2. If $s \in \left[\overline{M_{j+1}} \setminus X_j\right]^{1-\sigma}$, then $x_{j+1}(s) :=$ select $\left(\text{Post}\,(s) \cap \overline{M_{j+1}}\right)$.
   This is always possible as $\overline{M_{j+1}}$ is a σ-trap. The idea is that the opponent tries to keep the game forever in $\overline{M_{j+1}} \setminus X_j$, while the player σ can choose whether to remain in $\overline{M_{j+1}} \setminus X_j$ (winning for the opponent as F is never visited again) or to jump into X_j (also winning for the opponent by induction hypothesis). In fact, for any $s \in \overline{M_{j+1}} \setminus X_j$ and player strategy f^σ, $\mathcal{P}_{f^\sigma, x_{j+1}}\left(s \models \neg\left(\overline{X_j}\,\mathcal{U}\,F\right)\right) = 1$, hence, whenever the game reaches F, it has to reach X_j as well, i.e., $\overline{M_{j+1}} \setminus X_j \subseteq \overline{F}$ and $\overline{M_{j+1}} \cap F \subseteq X_j$. Therefore, while losing, the player can choose how to lose and this is not under the control of the (winning) opponent.
3. If $s \in \left[X_{j+1} \setminus \overline{M_{j+1}}\right]^{1-\sigma}$, then $x_{j+1}(s) := \underline{\text{force}}^{1-\sigma}\left(\overline{T'}, \overline{M_{j+1}}\right)(s)$.
   The opponent has a positive probability of reaching $\overline{M_{j+1}}$, while always avoiding R. Then, from $\overline{M_{j+1}}$, she wins according to one of the two previous points.

Lemma 10 *For any GPLCS, sets $T, F \subseteq S$, $\sigma \in \{0,1\}$, $s \in$ co-ForceBüchi$^{1-\sigma}\left(\overline{T}, \overline{F}\right)$, for all player σ strategy g^σ and for any extension $g^{1-\sigma} \supseteq \underline{\text{co-forceBüchi}}^{1-\sigma}\left(\overline{T}, \overline{F}\right)$ of the opponent's $(1-\sigma)$ strategy, both the following two conditions hold:*

a) $\mathcal{P}_{g^\sigma, g^{1-\sigma}}\left(s \models \Box\overline{T}\right) = 1$, *and*
b) $\mathcal{P}_{g^\sigma, g^{1-\sigma}}\left(s \models \Diamond\Box\overline{F}\right) > 0$.

Winning strategy for player σ. The idea is that the player tries to remain forever in the set of winning states $W :=$ ForceBüchi$^\sigma(T, F)$. This is possible as W turns out to be a $(1-\sigma)$-trap. Then, the existence of a finite attractor and the finitely branching property will imply that staying forever in W will lead the game to F infinitely often (a.s.). In this respect, we say that safety (staying in W) implies liveness (reaching F) for stochastic systems which enjoy the finite attractor property and which are finitely branching. This is what is stated in the following lemma: If one stays forever in Force$^\sigma_T(I, F) \setminus T'$, then F is visited infinitely often almost-surely.

Lemma 11 *For $s \in S$, $\mathcal{P}_{\underline{\mathit{force}}^\sigma_T(I,F)}\left(s \models \Box\left(\mathit{Force}^\sigma_T(I,F) \setminus T'\right) \implies \Box\Diamond F\right) = 1$.*

We now give a partial pure memoryless strategy $\underline{\text{forceBüchi}}^{\sigma}(T, F)$ (abbreviated as w in the following) for the player σ, which is winning for states in ForceBüchi$^{\sigma}(T, F)$ (abbreviated as W). We also recall that T' is a force set, i.e., it is defined as $T' :=$ Force$^{\sigma}(R)$, for some R (that will be used in the following section). The idea is to keep the game in W, unless T is reached. If T is reached, then the player wins. Otherwise, the game remains forever in $W \setminus T'$ (this is possible as $W \setminus T'$ is a $(1-\sigma)$-trap by Lemma 9(b)) and, by the previous Lemma 11, which can be applied as W is (also) a force set, F is visited infinitely often almost-surely.

We define $w := \underline{\text{forceBüchi}}^{\sigma}(T, F)$ as follows:

1. Case $s \in [T']^{\sigma}$: $w(s) := \underline{\text{force}}^{\sigma}(R)(s)$ and the player wins by reaching R with positive probability.
2. Case $s \in [(W \setminus T')\setminus F]^{\sigma}$: $w(s) := \underline{\text{force}}_{T}^{\sigma}(W, F)(s)$. This definition is possible since, by Lemma 9(a), $W = \text{Force}_{T}^{\sigma}(W, F)$. In this case, the player either reaches T (and then winning as in the previous point), or visits F once while staying in W.
3. Case $s \in [(W \setminus T')\cap F]^{\sigma}$: $w(s) := \text{select}(\text{Post}(s) \cap W)$. This is possible since, by Lemma 9(b), W is a $(1-\sigma)$-trap. In this case, the player wins according to one of the two previous points.

Lemma 12 *For any GPLCS, sets of states $T, F \subseteq S$, player $\sigma \in \{0,1\}$ and $s \in$ ForceBüchi$^{\sigma}(T, F)$, and for any extension $g^{\sigma} \supseteq \underline{\text{forceBüchi}}^{\sigma}(T, F)$ of player σ strategy and for every player $(1-\sigma)$ strategy $g^{1-\sigma}$, either*

a) $\mathcal{P}_{g^{\sigma}, g^{1-\sigma}}(s \models \Diamond T) > 0$*, or*
b) $\mathcal{P}_{g^{\sigma}, g^{1-\sigma}}(s \models \Box\Diamond F) = 1$.

Theorem 2 (Determinacy and solvability) *Extended almost sure Büchi games on GPLCS are pure memoryless determined and solvable, for every regular $F, R \subseteq Q$.*

6 Positive Büchi Games on GPLCS

We now address the solution of games in which player 0 wins if F is visited infinitely often with positive probability. In the following, when we say the *player* we mean player 0 and the *opponent* is player 1. It has been shown in [7] that, even in the $1\frac{1}{2}$-player case, infinite memory can be needed to win the game, and that determining the existence of such winning strategies is an undecidable problem.

Hence, we will restrict our attention to the case in which the player can use at most a finite amount of memory. So we do not consider the question 'Does there exist a winning strategy?' (which is undecidable [7]), but rather 'Does there exist a finite-memory winning strategy?' (which we show to be decidable). Under this assumption, we show that pure positional strategies are sufficient for both players. In particular, while in general memory helps the player, we also show that memory *does not* help the opponent (against finite memory players).

Let W^0 be the set of winning states for the player, i.e., she can visit F infinitely often with positive probability from every $s \in W^0$. The intuition is that the player tries to either visit F infinitely often almost surely or to reach with positive probability a state

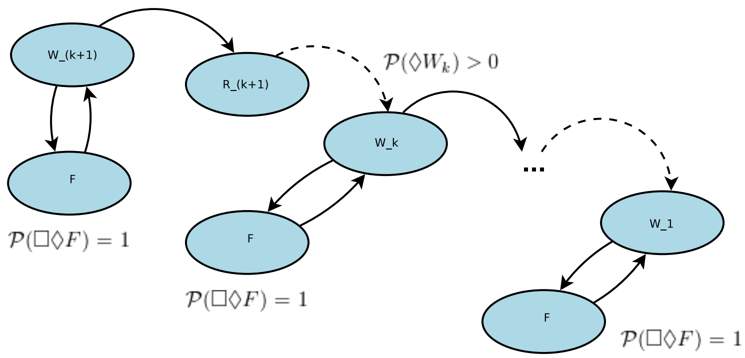

Fig. 1. Example showing the main idea for the player's strategy. If the game is in W_{k+1}, then it is in the opponent's control whether to lose by remaining in W_{k+1} or by going into R_{k+1}. Hence, the opponent can choose how to lose and this is *not* under the player's control. From R_{k+1}, the player can drive the game to W_k with positive probability, and so on.

which is already winning. This is shown in Figure 1. We build two sequences of sets of states $\{R_k\}_{k\in\mathbb{N}}$ and $\{W_k\}_{k\in\mathbb{N}}$ in the following way. Let $W_0 := \emptyset$ and, for $k \in \mathbb{N}$:

$$R_{k+1} := \text{Force}^0(W_k), \quad W_{k+1} := \text{ForceBüchi}^0(R_{k+1}, F)$$

We define $W^0 := \bigcup_k W_k$. The following property follows from the definitions.

Lemma 13 $W_0 \subseteq R_1 \subseteq W_1 \subseteq R_2 \subseteq W_2 \subseteq \cdots$

Lemma 14 *For any GPLCS and for any set $F \subseteq S$, the two sequences $\{R_k\}_{k\in\mathbb{N}}$ and $\{W_k\}_{k\in\mathbb{N}}$ converge and, moreover, they converge to the same set W^0.*

Form of Winning Sets. By the assumptions in Section 3, both W_k and R_k are regular, provided that F is regular. Since the above construction terminates, both W^0 and its complement are regular.

Correctness. We prove the correctness of the above construction by providing pure winning strategies for both players.

The player's strategy (player 0). We define a partial pure memoryless strategy $w^0 := \bigcup_i w_i$ for player 0, which is winning for states in $W^0 := \bigcup_k W_k$, where $w_0 \subseteq w_1 \subseteq w_2 \subseteq \cdots$, and each w_i is winning for states in $W_k = \text{ForceBüchi}^0(R_k, F)$. For each $k \in \mathbb{N}$ and $s \in \left[\text{ForceBüchi}^0(R_k, F)\right]^0$, we define $w_k(s) := \underline{\text{forceBüchi}}^0(R_k, F)(s)$.

The idea is either 1) to visit R_k with positive probability, or 2) to visit F infinitely often almost-surely. As $s \in W^0$ implies $s \in W_k$, for some finite k, then after a finite number of steps the game eventually reaches, with a positive probability $p' > 0$, some state s' in $\{s' \in S \,|\, \mathcal{P}_{g^0,g^1}(s' \models \Box\Diamond F) = 1\}$. Then, from s' the game visits F infinitely often almost-surely. Hence, the probability p of visiting F infinitely often from s is $p := p' \cdot \mathcal{P}_{g^0,g^1}(s' \models \Box\Diamond F) = p' \cdot 1 > 0$. Hence, we have the following lemma, which relies on the existence of a finite-attractor and on the system being finitely branching. (These assumptions are needed when applying Lemma 12 from the previous Section 5 for states in $\text{ForceBüchi}^0(R_i, F)$, $i \leq k$.)

Lemma 15 *For any GPLCS, $F \subseteq S$, $s \in W^0$, and for any extension $g^0 \supseteq w^0$ of player* 0 *strategy and for every player* 1 *strategy g^1, $\mathcal{P}_{g^0,g^1}(s \models \Box\Diamond F) > 0$.*

The opponent's strategy (player 1*).* We now show how the opponent wins positionally from states in $W^1 := \overline{W^0}$. From $W^0 = \text{Force}^0\left(W^0\right)$, it follows that W^1 is a 0-trap. Hence, the opponent can force the game to remain forever in W^1. Let g^1 be such a pure positional strategy for the opponent (or any extension thereof) and let g^0 be any *finite-memory* strategy for the player. From $W^1 = \text{co-ForceBüchi}\left(W^1, F\right)$, we have

$$\forall s \in W^1.\ \mathcal{P}_{g^0,g^1}\left(s \models \Diamond\Box\overline{F}\right) > 0\ . \tag{3}$$

By assumption, the player uses only a finite amount of memory, and by construction the opponent plays positionally. Thus, if we plug both strategies into the game $\mathcal{G}$, we obtain a pure Markov chain which still has a finite attractor A. Let $A' := A \cap W^1 \neq \emptyset$, which is non-empty because the game will remain forever in W^1 and it has to a.s. visit the attractor infinitely often. Let $\mathcal{B} := \{B_1, \ldots, B_k\}$ be the collection of the (finitely many) bottom maximal strongly connected components (BSCCs) of A'. There exists at least one such (non-trivial) BSCC with $\forall B \in \mathcal{B}.\ \forall b \in B.\ \mathcal{P}_{g^0,g^1}\left(b \models \Box\overline{F}\right) = 1$. (By contradiction, if $\mathcal{P}_{g^0,g^1}(b \models \Diamond F) \geq \epsilon > 0$, for some $\epsilon \in \mathbb{R}$, and since the game will visit b infinitely often almost-surely, then it will also visit F infinitely often almost-surely, contradicting $b \in W^1$, i.e., Equation 3.) Our claim follows by the fact that the game will a.s. reach some $B \in \mathcal{B}$ (as $\mathcal{B} \subseteq 2^{A'}$, and A' being an attractor), i.e., $\mathcal{P}_{g^0,g^1}(s \models \Diamond \bigcup \mathcal{B}) = 1$ for any $s \in W^1$, hence $\forall s \in W^1.\ \mathcal{P}_{g^0,g^1}\left(s \models \Diamond\Box\overline{F}\right) = 1$.

Lemma 16 *For any GPLCS, $F \subseteq S$, and $s \in W^1$, there exists a positional strategy g^1 for the opponent s.t. for any* finite memory *player strategy g^0, $\mathcal{P}_{g^0,g^1}\left(s \models \Diamond\Box\overline{F}\right) = 1$.*

Theorem 3 (Solvability) *It is decidable whether the player can win a positive Büchi game on GPLCS with finite memory. If yes, a pure positional player strategy suffices against an arbitrary opponent. If no, then the opponent has a pure positional strategy which is winning against any* finite memory *player. Furthermore, symbolic representations of the winning sets and strategies are computable for any regular target set F.*

7 Extensions

Remark 3. Lemma 16 just shows that the memory *does not* help the opponent against any (finite memory) player. (For an example where memory does help the *player*, see [7].) This situation is rather asymmetrical between the two players. However, this is not true in general, i.e., there are non-GPLCS systems in which the memory *does help* the opponent, as explained below.
Consider the (non-GPLCS) game $\mathcal{G}$ shown in Figure 2. The transition system is infinitely branching and hence not a GPLCS, despite the fact that $\{q\}$ is a finite attractor. Moreover, the player doesn't play in this game, while the opponent controls only q.

We define $F := \{r\}$ and recall that the opponent has the a.s. co-Büchi objective $\Diamond\Box\overline{F}$. For every opponent strategy $f^{1-\sigma}$ and state $s \in S$, the probability of reaching F is positive, i.e., $\mathcal{P}_{f^{1-\sigma}}(s \models \Diamond F) > 0$. Hence, the memoryless opponent strategy

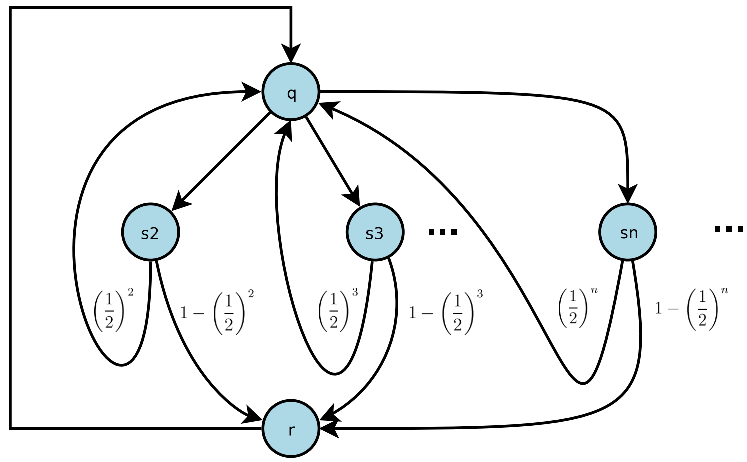

Fig. 2. Example of (non-GPLCS, infinitely branching) game in which the memory actually helps the opponent (almost-sure co-Büchi objective, with $F = \{r\}$). All states are random, except q which is controlled by the opponent and is infinitely branching.

presented previously would be losing for every state in this system. This strategy would try to reach $B := \{s \in S \mid \mathcal{P}_{f^{1-\sigma}}\left(s \models \Box\overline{F}\right) = 1\}$, but $B = \emptyset$ in this example.

We now show how memory helps the opponent in this example, and we present an infinite memory strategy that achieves the co-Büchi objective. Starting from state q, the opponent tries to visit the states $s_2, s_3, s_4, \ldots$, in that order. Whenever the game eventually reaches r, then the opponent resets its memory and starts again from q, trying to reach s_2, s_3, etc..., as before. The probability of visiting $F = \{r\}$ at least once starting from q is $1/2^2 + (1 - 1/2^2) \cdot 1/2^3 + (1 - 1/2^3) \cdot 1/2^4 \cdots < 1/2^2 + 1/2^3 + 1/2^4 \cdots = 1/2$. Similarly, the probability of visiting F at least k times is $< 1/2^k$. As $1/2^k \to 0$ when $k \to \infty$, then we conclude that the probability of visiting F infinitely often is 0 and, hence, the opponent satisfies the co-Büchi objective $\Diamond\Box\overline{F}$ almost-surely.

Remark 4. In [7] a reduction was described that transforms a perfect fifo-channel machine into a $1\frac{1}{2}$-player game on a PLCS. Although this was not explicitly mentioned in [7], a closer look at this reduction shows that it has the following properties. (1) If the original fifo-channel machine is space-bounded then the player in the $1\frac{1}{2}$-player game cannot achieve the positive Büchi objective, even with infinite memory, i.e., for every strategy f^0 we have $\mathcal{P}_{f^0}(s \models \Box\Diamond F) = 0$. (2) If the original fifo-channel machine is not space-bounded then for every $\epsilon > 0$ there exists a strategy f^0_ϵ that achieves $\mathcal{P}_{f^0_\epsilon}(s \models \Box\Diamond F) \geq 1 - \epsilon$ (i.e., the player can achieve the limit-sure Büchi objective), but no strategy can achieve the almost-sure Büchi objective. It follows that an optimal strategy that maximizes the probability for the Büchi objective does not exist in general. Furthermore, no non-trivial approximation for the supremum of the achievable values of $\mathcal{P}_{f^0}(s \models \Box\Diamond F)$ can be computed, because every such approximation (except for the trivial upper/lower bounds 1/0) would answer the undecidable question of distinguishing the cases (1) and (2).

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Appendix

Lemma 6 (in Section 4) *co-Force$_{\overline{T}}^{1-\sigma}$ $\left(\overline{I}, \overline{F}\right) := \overline{\text{Force}_T^{\sigma}\left(I, F\right)}$ is a σ-trap.*

Proof. Consider the following construction:

$$\begin{aligned}
coD_0 &:= \left[(\overline{F} \cup \overline{I}) \cap \overline{T'}\right]^R \\
coD_{i+1} &:= coD_i \cap (\widetilde{\text{Pre}}^R\,(coE_i) \cup \overline{I}) \\
coE_0 &:= \left[(\overline{F} \cup \overline{I}) \cap \overline{T'}\right]^{01} \\
coE_{i+1} &:= coE_i \cap (\left(\widetilde{\text{Pre}}^{\sigma}\,(coD_i) \cup \text{Pre}^{1-\sigma}\,(coD_i)\right) \cup \overline{I})
\end{aligned}$$

It is easy to show that $\overline{D}_i \cap \overline{E}_i = coD_i \cup coE_i$ (a boring exercise involving only basic set operations and induction). Hence, co-Force$_{\overline{T}}^{1-\sigma}$ $\left(\overline{I}, \overline{F}\right) = \bigcap_i (coD_i \cup coE_i)$ and by the definitions above it is now easy to see that player $(1-\sigma)$ can always keep the game in co-Force$_{\overline{T}}^{1-\sigma}$ $\left(\overline{I}, \overline{F}\right)$, while player σ and the random player R cannot avoid remaining in co-Force$_{\overline{T}}^{1-\sigma}$ $\left(\overline{I}, \overline{F}\right)$. $\square$

Lemma 7 (in Section 5) *Let X_j and M_j defined as in Section 5, and $T' := \text{Force}^{\sigma}\,(T)$. Then,*

a) $X_j \subseteq \overline{T'}$ *and* $\overline{M_j} \subseteq \overline{T'}$*, for each* $j \in \mathbb{N}$.
b) $X_0 \subseteq \overline{M_1} \subseteq X_1 \subseteq \overline{M_2} \subseteq X_2 \cdots$.

Proof. We first need some auxiliary result about the constructions in Section 4. Let $Q := \text{Force}_T^{\sigma}\,(I, F)$, $Q' := \text{Force}^{\sigma}\,(I, F)$ and $T' := \text{Force}^{\sigma}\,(T)$. Then, the following properties follow directly from the definitions:

1. $T \subseteq T' \subseteq Q, Q' \subseteq I$.
2. $Q \setminus T' \subseteq I$.
3. $I \cap F \subseteq Q'$.

We can now proceed with the proof of Lemma 7.

7(a). A simple application of previous point 1 (twice).

7(b). We prove that $X_j \overset{(1)}{\subseteq} \overline{M_{j+1}} \overset{(2)}{\subseteq} X_{j+1}$, for each $j \in \mathbb{N}$. $\overline{M_{j+1}} \overset{(2)}{\subseteq} X_{j+1}$ follows from the definition of X_{j+1} and point 3, which can be applied since $\overline{M_{j+1}} \subseteq \overline{T'}$. Now, by the definition of M_{j+1} and point 2, we obtain $M_{j+1} \setminus T' \subseteq \overline{X_j}$, which is the same as $X_j \overset{(1)}{\subseteq} \overline{M_{j+1}}$ (since, by the previous part 7(a), $X_j \subseteq \overline{T'}$). □

Lemma 8 (in Section 5) *For any GPLCS and sets $T, F \subseteq S$, the two sequences $\{M_j\}_{j \in \mathbb{N}}$ and $\{X_j\}_{j \in \mathbb{N}}$ converge, i.e., there exist finite indices n and m such that $M_{n+1} = M_n$ and $X_{m+1} = X_m$.*

Proof. Before proving the lemma, we actually need to analyse some properties of force sets, summarised in the following

Preliminary Lemma. Let $T, I \subseteq S$ be any set of states, let $\{F_h\}_{h \in \mathbb{N}}$ be any non-decreasing sequence of set S of states (w.r.t. set inclusion), and let $Q_h := [\text{Force}^{\sigma}_{T}(I, F_h)]^R$. (Note that Q_h is not necessarily upward-closed.) Then, the sequence $\{Q_h\}_{h \in \mathbb{N}}$ is non-decreasing, and

1. For big enough h, the computation of Q_h terminates after a uniformly bounded number of steps, i.e., there exist integers N, l s.t. for any $h \geq N$ the computation of Q_h halts in at most l steps. This means that, even if the sequence $\{Q_h\}_{h \in \mathbb{N}}$ does not necessarily converge, the computation of every individual Q_h does not take arbitrarily long time. Note that the input sequence $\{F_h\}_{h \in \mathbb{N}}$ is possibly infinite and non-converging.
2. Consider the function

$$f(X) := [\text{Force}^{\sigma}_{T}(I, X)]^R$$

Then, there exists an integer N s.t. for all $h \geq N$ the following properties hold:

(a) One can write $f(F_h)$ in the following form

$$f(F_h) := A_{F_N} \cup B \cap [F_h]^R \ , \qquad (*)$$

where $B \subseteq S^R$ is a constant, and $A_{F_N} \subseteq S^R$ depends just on $F_N \subseteq F_h$ and not on the whole F_h. This means that $f(F_h)$ just depends on F_N and on $[F_h]^R$, as stated in the next point.

(b) $f(F_h)$ satisfies the following equality:

$$f(F_h) = f(F_N \cup [F_h]^R)$$

which states that $f(F_h)$ just depends on elements in $F_N \cup [F_h]^R$, i.e., it does not depend on elements in $[F_h]^{01} \setminus F_N$.

(c) Every composition of functions of the form (*) is still of the form (*). I.e., given a non-decreasing sequence $\{X_h\}_{h \in \mathbb{N}}$ and a function $g_1(X_h) = A^1_{X_N} \cup B^1 \cap [X_h]^R$ of the form (*), let $Y_h := g_1(X_h)$ be s.t. $\{Y_h\}_{h \in \mathbb{N}}$ is another non-decreasing sequence. Then, for any function $g_2(Y_h) = A^2_{Y_N} \cup B^2 \cap [Y_h]^R$ of the form (*), the composite $g_2 \circ g_1$ is of the form (*) as well, i.e., it satisfies $g_2(g_1(X_h)) = A^3_{X_N} \cup B^3 \cap [X_h]^R$, for some $A^3_{X_N}, B^3$.

(d) If Q_h is in the form (*) and, moreover, it just depends on Q_{h-1}, then $Q_{h+1} = Q_h$. In particular, $Q_{N+1} = Q_N$.

3. The previous points also hold when considering a non-decreasing sequence $\{I_h\}_{h\in\mathbb{N}}$, any $T, F \subseteq S$ and with $Q_h := f'(I_h) = [\text{co-Force}_T^\sigma(I_h, F)]^R$.

Proof The sequence $\{Q_h\}_{h\in\mathbb{N}}$ is non-decreasing since $\{F_h\}_{h\in\mathbb{N}}$ is non-decreasing and $[\text{Force}_T^\sigma(I, F_h)]^R$ is a monotonic operator.

1. It is sufficient to inspect the definition of the approximants of Q_h in the D sequence (from Section 4). Let $D_{h,i}$ be the i-th such approximant. From Section 4 we know that $D_{h,i}$ can be written in the following form:

$$D_{h,i} = D_{h,0} \cup (\bigcup_{j\leq i} \text{Pre}^R(E_{h,j})) \cap I ,$$

where the term $G_{h,i} := \bigcup_{j\leq i} \text{Pre}^R(E_{h,j})$, which depends on the E sequence, is *upward-closed* for every h, i. This implies, for instance, that the non-decreasing sequence $\{G_{h,i}\}_i$ has to eventually converge (which proves that the computation of Q_h converges for each h), i.e., for any h, there exists N_h s.t. for any $i \geq N_h$, $G_{h,i} = G_{h,i+1}$.
Now consider the diagonal sequence of upward-closed elements $\{G_{h,h}\}_h$. It is non-decreasing: In fact, for any h, i, $G_{h,i} \subseteq G_{h+1,i}$ (by the fact that F_h is non-decreasing and by the monotonicity of the involved operators) and $G_{h,i} \subseteq G_{h,i+1}$, hence $G_{h,i} \subseteq G_{h+1,i+1}$. Thence, $\{G_{h,h}\}_h$ has to eventually converge, i.e., there exists N s.t. $G_{N,N} = G_{N+1,N+1}$. In particular, this implies $G_{i,j} = G_{i+x,j+y}$ for any $i, j \geq N$ and $x, y \in \mathbb{N}$.
As a consequence, for any $h, i \geq N$, $D_{h,i}$ takes the following form:

$$D_{h,i} = D_{h,0} \cup G_{N,N} \cap I . \tag{†}$$

This implies that for any $h \geq N$, the computation of Q_h takes at most N steps.

2. We recall the definition of f:

$$f(X) := [\text{Force}_T^\sigma(I, X)]^R$$

(a) That $f(F_h)$ takes the special form (*) for h big enough follows directly from Equation (†) in the previous point, with $h \geq N$. In fact, applying the definition of $D_{h,0} := [T \cup F_h \cap I]^R$, we have that

$$f(F_h) = D_{h,N_h} = [T \cup F_h \cap I]^R \cup G_{N,N} \cap I = A_{F_N} \cup B \cap [F_h]^R ,$$

with $A_{F_N} := [T \cup G_{N,N} \cap I]^R$ and $B := I$.

(b) This point follows directly by inspection of the special form (*).

(c) Let X_h be any non-decreasing sequence s.t. there exists N s.t. for any $h \geq N$ the two functions g_1, g_2 satisfy

$$g_1(X_h) = A^1_{X_N} \cup B^1 \cap [X_h]^R \text{ , and}$$
$$g_2(Y_h) = A^2_{Y_N} \cup B^2 \cap [Y_h]^R ,$$

where $Y_h := g_1(X_h)$. Note that Y_h is also a non-decreasing sequence, g_1 being monotonic. Then, the compsite function $g_2 \circ g_1$ is defined as

$$\begin{aligned} g_2(g_1(X_h)) &= A^2_{Y_N} \cup B^2 \cap \left[A^1_{X_N} \cup B^1 \cap [X_h]^R\right]^R = \\ &= A^2_{Y_N} \cup B^2 \cap \left[A^1_{X_N}\right]^R \cup B^2 \cap \left[B^1\right]^R \cap [X_h]^R = \\ &= (A^2_{Y_N} \cup B^2 \cap A^1_{X_N}) \cup (B^2 \cap B^1) \cap [X_h]^R \ , \end{aligned}$$

where the last equality follows from the fact that $A^1_{X_N}, B^1 \subseteq S^R$. Hence, $g_2(g_1(X_h))$ can be put in the form

$$g_2(g_1(X_h)) = A^3_{X_N} \cup B^3 \cap [X_h]^R \ ,$$

with $B^3 := B^2 \cap B^1$, and $A^3_{X_N} := A^2_{Y_N} \cup B^2 \cap A^1_{X_N}$. Notice that $A^3_{X_N}$ truly depends just on X_N, as $Y_N := g_1(X_N) := A^1_{X_N} \cup B^1 \cap [X_N]^R$ does so.

(d) Let $Q_h = A_{Q_N} \cup B \cap [Q_{h-1}]^R$, for $h \geq N$. Then, it is easy to see that

$$\begin{aligned} Q_{h+1} &= A_{Q_N} \cup B \cap [Q_h]^R = \\ &= A_{Q_N} \cup B \cap (A_{Q_N} \cup B \cap [Q_{h-1}]^R) = \\ &= A_{Q_N} \cup B \cap [Q_{h-1}]^R \\ &= Q_h \ . \end{aligned}$$

3. The proof of this point is entirely similar to the previous two points, and it is omitted. □

Proof of Lemma 8. We are now ready for proving Lemma 8. We will prove that $X_{m+1} = X_m$, for some finite m. Notice that when $X_{m+1} = X_m$, then $M_{m+2} = M_{m+1}$ as well by the definition of M_{m+2}. Moreover, in order to prove $X_{m+1} = X_m$, it is enough to prove $[X_{m+1}]^R = [X_m]^R$. In fact, $X_{m+1} = [X_{m+1}]^R \cup [X_{m+1}]^{01}$ and the $[\cdot]^{01}$ part depends just on the $[\cdot]^R$ part. We will show this in more detail in the next paragraph, while in the last paragraph we will finish up the proof.
We first prove $X_{m+1} = X_m$ under the assumption that $[X_{m+1}]^R = [X_m]^R$. Let $X_{m,j}$ be the j-th approximant in the computation of X_m. Let n be a big enough integer such that $[X_{m,n}]^R = [X_{m,n+1}]^R$ and $[X_{m+1,n}]^R = [X_{m+1,n+1}]^R$. Hence, $[X_m]^R = [X_{m,n}]^R$ and $[X_{m+1}]^R = [X_{m+1,n}]^R$. But by the definition of m, $[X_m]^R = [X_{m+1}]^R$, thence the previous terms are all equal. In particular, the following equality holds:

$$[X_{m,n+1}]^R = [X_{m+1,n+1}]^R \tag{‡}$$

We now focus on the behaviour of the $[\cdot]^{01}$ sequence. From the termination argument of Section 4, $[X_m]^{01}$ reaches convergence one step after $[X_m]^R$ does, for each m. In particular, this implies

$$\begin{aligned} [X_{m,n+1}]^{01} &= [X_{m,n+2}]^{01} \\ [X_{m+1,n+1}]^{01} &= [X_{m+1,n+2}]^{01} \end{aligned}$$

Applying the definition given by the D and E sequences from Section 4, we have that

$$[X_{m,n+2}]^{01} := [X_{m,n+1}]^{01} \cup \left(\left(\text{Pre}^{\sigma}\left([X_{m,n+1}]^{R}\right) \cup \widetilde{\text{Pre}}^{1-\sigma}\left([X_{m,n+1}]^{R}\right)\right) \cap \overline{T'}\right)$$
$$[X_{m+1,n+2}]^{01} := [X_{m+1,n+1}]^{01} \cup \left(\left(\text{Pre}^{\sigma}\left([X_{m+1,n+1}]^{R}\right) \cup \widetilde{\text{Pre}}^{1-\sigma}\left([X_{m+1,n+1}]^{R}\right)\right) \cap \overline{T'}\right)$$

Finally, using the previous equality (‡), it is easy to see that $[X_{m,n+2}]^{01} = [X_{m+1,n+2}]^{01}$, implying $[X_m]^{01} = [X_{m+1}]^{01}$ as well. This, together with the assumption $[X_m]^R = [X_{m+1}]^R$, implies $X_m = X_{m+1}$.
We now prove $[X_{m+1}]^R = [X_m]^R$ for a big enough m, concluding our proof. By the definition of X_j, we have that

$$[X_j]^R := \left[\text{Force}^{1-\sigma}\left(\overline{T'}, \text{co-Force}^{1-\sigma}_{\overline{T}}\left(X_{j-1}, \overline{F}\right)\right)\right]^R$$

We define $Y_j := \text{co-Force}^{1-\sigma}_{\overline{T}}\left(X_{j-1}, \overline{F}\right)$. By the Preliminary Lemma, part 3, there exists N s.t., for all $m \geq N$, $[Y_m]^R$ just depends on $X_N \cup [X_{m-1}]^R$ (part 2b), and, moreover, it is of the form (part 2a)

$$[Y_m]^R := \left[\text{co-Force}^{1-\sigma}_{\overline{T}}\left(X_{m-1}, \overline{F}\right)\right]^R = A_{X_N} \cup B \cap [X_{m-1}]^R$$

As $\{Y_m\}_{m\in\mathbb{N}}$ is non-decreasing, we can apply the Preliminary Lemma again to $[X_j]^R$, obtaining that there exists M s.t., for any $h \geq M$, $[X_h]^R$ just depends on $Y_M \cup [Y_{h-1}]^R$ (part 2b), and, moreover, it is of the form (*) (part 2a)

$$[X_h]^R := \left[\text{Force}^{1-\sigma}\left(\overline{T'}, Y_h\right)\right]^R = A'_{Y_M} \cup B' \cap [Y_{h-1}]^R$$

Now we take $L := \max(M, N)$ and, combining the previous two observations for any $n \geq L$, we have that, by the Preliminary Lemma (part 2c), $[X_n]^R$ is in the form (*) *also w.r.t.* X_{n-1}, i.e.,

$$[X_n]^R = A''_{X_L} \cup B'' \cap [X_{n-1}]^R$$

Finally, by the Preliminary Lemma (part 2d), we have that

$$[X_{L+1}]^R = [X_L]^R$$

$\square$

Lemma 11 (in Section 5) $\forall s{\in}S.\mathcal{P}_{\underline{force}^{\sigma}_{T}(I,F)}\left(s{\models}\Box\left(\mathit{Force}^{\sigma}_{T}(I,F)\setminus T'\right) \Longrightarrow \Box\Diamond F\right){=}1.$

Proof. We assume that the player σ uses any extension g^{σ} of $\underline{\text{force}}^{\sigma}_{T}(I,F)$ and the opponent $(1-\sigma)$ uses any strategy $g^{1-\sigma}$. Let $B := \text{Force}^{\sigma}_{T}(I,F) \setminus T'$. We define $x := \mathcal{P}_{g^{\sigma},g^{1-\sigma}}\left(s \models \Box B \wedge \Diamond\Box\overline{F}\right)$ and we want to show that $x = 0$.
Let A be an attractor and let $A' := A \cap B$. If $A' = \emptyset$, then $B \subseteq \overline{A}$, hence it is not possible to stay forever in B: for any $s \in S$, $x \leq \mathcal{P}_{g^{\sigma},g^{1-\sigma}}\left(s \models \Box B\right) \leq \mathcal{P}_{g^{\sigma},g^{1-\sigma}}\left(s \models \Box\overline{A}\right) = 0$, the last equality following from the def. of an attractor.

Otherwise, assume $A' \neq \emptyset$. By Lemma 4(b) (Note that Lemma 4(a) does not hold, as $\mathcal{P}_{g^\sigma,g^{1-\sigma}}\left(s \models \Diamond T\right) = 0$ for $s \in \overline{T'}$, by the definition of T'.) we have that for each state $s' \in A'$ there exists a lower bound $\epsilon_{s'} > 0$ on the probability of reaching F, i.e., $\mathcal{P}_{g^\sigma,g^{1-\sigma}}\left(s' \models \Diamond F\right) \geq \epsilon_{s'}$. By the finiteness of A (and hence of A'), we have $\epsilon := \min_{s' \in A'} \epsilon_{s'} > 0$. Since runs in $(\Box B \wedge \Diamond \Box \overline{F})$ must have an infinite suffix in $\overline{F}$ and since each time we visit the attractor we have a chance $\leq (1-\epsilon)$ of not visiting F, then we have $x \leq (1-\epsilon)^\infty = 0$.
In both cases we have $x = 0$, hence the main claim follows: For each $s \in S$,

$$\mathcal{P}_{g^\sigma,g^{1-\sigma}}\left(s \models \Box \left(\text{Force}_T^\sigma\left(I, F\right) \setminus T'\right) \implies \Box \Diamond F\right) = 1 - x = 1 \; .$$

$\square$

Lemma 14 (in Section 6) *For any GPLCS and for any set $F \subseteq S$, the two sequences $\{R_k\}_{k \in \mathbb{N}}$ and $\{W_k\}_{k \in \mathbb{N}}$ converge and, moreover, they converge to the same set W^0.*

Proof. We first prove that there exists a finite index n s.t. $[R_n]^R = [R_{n+1}]^R$, and then we will prove that even $R_{n+1} = R_n$.
By the definition of the two sequences $\{R_k\}_{k \in \mathbb{N}}$ and $\{W_k\}_{k \in \mathbb{N}}$, we have that R_{i+1} can be written in the following form.

$$\begin{aligned} R_{i+1} &= \text{Force}^0\left(W_i\right), \text{ with} \\ W_i &= \text{ForceBüchi}^\sigma\left(R_i, F\right) \end{aligned}$$

It follows from the properties of $\text{ForceBüchi}^\sigma\left(R_i, F\right)$, established in Section 5, that for every given R_i, the set W_i is the *greatest* fixpoint of the following equation.

$$W_i = \text{Force}_{R_i}^0\left(\text{co-Force}^0\left(R_i, W_i\right), F\right) \tag{4}$$

To simplify the presentation, we define an auxiliary sequence Y_i, where

$$Y_i := \text{co-Force}^0\left(R_i, W_i\right)$$

By combining these equations and restricting to random states, $[R_{i+1}]^R$ can be written in the following form.

$$\begin{aligned} [R_{i+1}]^R &= \left[\text{Force}^0\left(\text{Force}_{R_i}^0\left(\text{co-Force}^0\left(R_i, W_i\right), F\right)\right)\right]^R = \left[\text{Force}^0\left(W_i\right)\right]^R, \text{ with} \\ [W_i]^R &= \left[\text{Force}_{R_i}^0\left(\text{co-Force}^0\left(R_i, W_i\right), F\right)\right]^R = \left[\text{Force}_{R_i}^0\left(Y_i, F\right)\right]^R \\ [Y_i]^R &= \left[\text{co-Force}^0\left(R_i, W_i\right)\right]^R \end{aligned}$$

By Lemma 7(b), for any i, $R_i \subseteq W_i \subseteq R_{i+1}$, thence $\{R_i\}_{i \in \mathbb{N}}$ and $\{W_i\}_{i \in \mathbb{N}}$ are non-decreasing, and

$$[R_i]^R \subseteq [W_i]^R \subseteq [R_{i+1}]^R \; .$$

Note that this entails that $\{[Y_i]^R\}_i$ is non-decreasing too (by the monotonicity of co-Force). By using arguments as in the Preliminary Lemma (see the proof of Lemma 8), one can prove that there exists N_1 s.t. for any $h \geq N_1$, $[Y_h]^R$ can be written in the form

$$[Y_h]^R = [R_h]^R \cup [W_h]^R \cap A^1_{R_{N_1},W_{N_1}} \ ,$$

for some $A^1_{R_{N_1},W_{N_1}}$ which just depends on R_{N_1} and W_{N_1}. Notice that $R_{N_1} \subseteq W_{N_1}$, hence $A^1_{R_{N_1},W_{N_1}}$ actually just depends on W_{N_1}, and we will simply write $A^1_{W_{N_1}}$ thenceforth. By a similar argument applied to $[W_i]^R$, we obtain that there exists N_2 s.t. for any $h \geq N_2$,

$$[W_h]^R = [R_h]^R \cup [Y_h]^R \cap A^2_{R_{N_2},Y_{N_2}} \ ,$$

where, again, $A^2_{R_{N_2},Y_{N_2}}$ is some set which just depends on R_{N_2} and Y_{N_2}. Notice that $R_{N_2} \subseteq Y_{N_2}$, hence we just write $A^2_{Y_{N_2}}$ instead of $A^2_{R_{N_2},Y_{N_2}}$. Finally, the same argument applied to $[R_{i+1}]^R$ yields that there exists N_3 s.t. for any $h \geq N_3$,

$$[R_{h+1}]^R = [W_h]^R \cup A^3_{W_{N_3}} \ ,$$

with $A^3_{W_{N_3}}$ just depending on W_{N_3}.
We define $N := \max\{N_1, N_2, N_3\}$. By combining the previous three formulae (and observations thereof), we have that, for $h \geq N$

$$\begin{aligned} [R_{h+1}]^R &= [W_h]^R \cup A^3_{W_N} \\ [W_h]^R &= [R_h]^R \cup [Y_h]^R \cap A^2_{Y_N} \\ [Y_h]^R &= [R_h]^R \cup [W_h]^R \cap A^1_{W_N} \end{aligned}$$

We now eliminate $[Y_h]^R$ by substituting the right hand side of the third equation into the second one.

$$\begin{aligned} [R_{h+1}]^R &= [W_h]^R \cup A^3_{W_N} \\ [W_h]^R &= [R_h]^R \cup [W_h]^R \cap A^4_{W_N} \end{aligned}$$

where $A^4_{W_N} := A^2_{Y_N} \cap A^1_{W_N}$ just depends on W_N (because $Y_N \subseteq W_N$).
We now recall the earlier observation that (for every given R_h) W_h is the greatest fixpoint of Equation 4. Hence, $[W_h]^R$ is the maximal solution of the second equation above. In other words, $[W_h]^R$ is the biggest subset of S^R s.t. $[W_h]^R = [R_h]^R \cup [W_h]^R \cap A^4_{W_N}$ holds. Hence, $[W_h]^R = [R_h]^R \cup A^4_{W_N}$ indeed. By substituting this into the above equation for $[R_{h+1}]^R$, we obtain

$$[R_{h+1}]^R = [R_h]^R \cup A^5_{R_{N+1}} \ ,$$

with $A^5_{R_{N+1}} := A^4_{W_N} \cup A^3_{W_N}$ depending just on $R_{N+1} \supseteq W_N$. Then, if we let $n := N+1$, this implies $[R_{n+1}]^R = [R_n]^R$, thus proving that $\{[R_i]^R\}_i$ eventually converges.
Now we show that even $R_{n+1} = R_n$. The inclusion $R_n \subseteq R_{n+1}$ follows directly from Lemma 13. For the other inclusion, we let $x \in R_{n+1}$ and show that $x \in R_n$ by case distinction.

1. If $x \in [R_{n+1}]^R$ then $x \in R_n$, because $[R_{n+1}]^R = [R_n]^R \subseteq R_n$.
2. If $x \in [R_{n+1}]^0 = \left[\text{Force}^0(W_n)\right]^0$ then there are two cases.
   (a) If $x \in W_n = \text{ForceBüchi}^0(R_n, F)$ then one of the following three cases holds.
      i. The first case is that $x \in R_n$, and our claim holds directly.
      ii. The second case is that there exists some successor x' of x such that $x' \in R_n = \text{Force}^0(W_{n-1})$. Thus we have that $x \in \text{Force}^0(W_{n-1}) = R_n$.
      iii. The last case is that there exists some successor x' of x such that $x' \in \text{ForceBüchi}^0(R_n, F)$. Since the graph is bipartite, it follows that $x' \in \left[\text{ForceBüchi}^0(R_n, F)\right]^R = [W_n]^R \subseteq [R_{n+1}]^R = [R_n]^R \subseteq R_n = \text{Force}^0(W_{n-1})$. Thus we have that $x \in \text{Force}^0(W_{n-1}) = R_n$.
   (b) If $x \notin W_n$ then there must exist some successor x' of x such that $x' \in \text{Force}^0(W_n)$. Since the graph is bipartite we have $x' \in \left[\text{Force}^0(W_n)\right]^R = [R_{n+1}]^R = [R_n]^R \subseteq R_n = \text{Force}^0(W_{n-1})$. Therefore $x \in \text{Force}^0(W_{n-1}) = R_n$.
3. If $x \in [R_{n+1}]^1 = \left[\text{Force}^0(W_n)\right]^1$ then there are two cases.
   (a) If $x \in W_n = \text{ForceBüchi}^0(R_n, F)$ then one of the following three cases holds.
      i. The first case is that $x \in R_n$, and our claim holds directly.
      ii. The second case is that $\text{Post}(x) \subseteq R_n = \text{Force}^0(W_{n-1})$. Thus we have that $x \in \text{Force}^0(W_{n-1}) = R_n$.
      iii. The last case is that $\text{Post}(x) \subseteq \text{ForceBüchi}^0(R_n, F)$. Since the graph is bipartite, it follows that $\text{Post}(x) \subseteq \left[\text{ForceBüchi}^0(R_n, F)\right]^R = [W_n]^R \subseteq [R_{n+1}]^R = [R_n]^R \subseteq R_n = \text{Force}^0(W_{n-1})$. Thus we have that $x \in \text{Force}^0(W_{n-1}) = R_n$.
   (b) If $x \notin W_n$ then $\text{Post}(x) \subseteq \text{Force}^0(W_n)$. Since the graph is bipartite we have $\text{Post}(x) \subseteq \left[\text{Force}^0(W_n)\right]^R = [R_{n+1}]^R = [R_n]^R \subseteq R_n = \text{Force}^0(W_{n-1})$. Therefore $x \in \text{Force}^0(W_{n-1}) = R_n$.

So we have that $R_{n+1} = R_n$. It follows from Lemma 13 that $W_n = R_n$. Furthermore, it follows from the definition of the sequences $\{R_k\}_{k\in\mathbb{N}}$ and $\{W_k\}_{k\in\mathbb{N}}$ that $W_i = R_{i'} = R_n = W_n$ for all $i, i' \geq n$, i.e., the sequences converge to the same set W_n. Thus the common limit of both sequences is $W^0 := \bigcup_k W_k = W_n$. $\square$