

PRESERVATION THEOREMS THROUGH THE LENS OF TOPOLOGY

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ABSTRACT. In this paper, we introduce a family of topological spaces that captures the existence of preservation theorems. The structure of those spaces allows us to study the relativisation of preservation theorems under suitable definitions of surjective morphisms, subclasses, sums, products, topological closures, and projective limits. Throughout the paper, we also integrate already known results into this new framework and show how it captures the essence of their proofs.

1. INTRODUCTION

In classical model theory, *preservation theorems* characterise first-order definable sets enjoying some semantic property as those definable in a suitable syntactic fragment [e.g., 6, Section 5.2]. A well-known instance is the Łoś-Tarski Theorem [38, 29]: a first-order sentence φ is preserved under extensions on all structures—i.e., $A \models \varphi$ and A is an induced substructure of B imply $B \models \varphi$ —if and only if it is equivalent to an existential sentence.

A major roadblock for applying these results in computer science is that preservation theorems generally do not relativise to classes of structures, and in particular to the class of all finite structures [see the discussions in 32, Section 2 and 25, Section 3.4]. In fact, the only case where a classical preservation theorem was shown to hold on all finite structures is Rossman’s Theorem [33]: a first-order sentence is preserved under homomorphisms on all finite structures if and only if it is equivalent to an existential positive sentence. This long-sought result has applications in database theory, where existential positive formulæ correspond to unions of conjunctive queries (also known as select-project-join-union queries and arguably the most common database queries in practice [1]). For instance, it is related in [11, Theorem 17] to the existence of homomorphism-universal models (as constructed by chase algorithms) for databases with integrity constraints, in [39, Theorem 3.4] to a characterisation of schema mappings definable via source-to-target tuple-generating dependencies, and in [17, Corollary 4.14] to the naïve evaluation of queries over incomplete databases under open-world semantics. These applications would benefit directly from preservation theorems for more restricted classes of finite structures or for other semantic properties—corresponding to other classes of queries and other semantics of incompleteness—, and this has been an active area of research [4, 5, 8, 21, 16]. Like Rossman’s result, these proofs typically rely on

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careful model-theoretic arguments—typically using Ehrenfeucht-Fraïssé games and locality—and each new attempt at proving a preservation theorem seemingly needs to restart from scratch.

In this paper, we develop a general topological framework for investigating preservation theorems, where preservation theorems, both old and new, can be obtained as byproducts of topological constructions.

As pointed out in the literature, the classical proofs of preservation theorems fail in the finite because the Compactness Theorem does not apply. As we will see in Section 2, one can reinterpret in topological terms the two applications of the Compactness Theorem in the classical proofs of preservation theorems like the Łoś-Tarski Theorem. Here, the topology of interest has the sets of structures closed under extension as its open sets, and one application of the Compactness Theorem shows that the definable open sets are compact (Claim 2.2) while the other shows that the sets definable by existential sentences form a base for the definable open sets (Claim 2.1). In Section 3, we capture these two ingredients in general through the definitions of *logically presented pre-spectral spaces* and *diagram bases* in Section 3, which lead to a generic preservation theorem (Theorem 3.4): under mild hypotheses—which are met in all the preservation results over classes of finite structures in the literature—, preservation holds if and only if the space under consideration is logically presented pre-spectral.

The benefit of this abstract, topological viewpoint, is that preservation results can now be proven by constructing new logically presented pre-spectral spaces from known ones.

Here, the topological core of our definition is the one of *pre-spectral* spaces, which generalise both Noetherian spaces and spectral spaces [18, 12]; see Section 4. To some extent, we can rely on the stability of spectral spaces under various topological constructions to investigate the same constructions for pre-spectral spaces. We focus however in the paper on the *logically presented* pre-spectral spaces, which is where the main difficulty lies when attempting to prove preservation over classes of finite structures, and for which stability must take the logical aspect into account. Accordingly, Section 5 shows the stability of logically presented pre-spectral spaces under typical constructions: under a carefully chosen notion of morphisms, under subclasses provided a sufficient condition is met, and under finite sums and finite products.

Where the topological viewpoint really shines is when it comes to stability for various kinds of ‘limits’ of classes of structures enjoying a preservation property. We show in Section 6 that the limit of a *single* class of structures, when it can be construed as the *closure* in a suitable topology of a logically presented pre-spectral space, is also logically presented pre-spectral. This allows us to show that Rossman’s Theorem—i.e., homomorphism preservation in the finite—extends to the class of structures with the finite model property, and also extends to countable unions of finite structures (the latter was also shown in [31, Chapter 10]). In Section 7, we show that the limit of a *family* of pre-spectral spaces, when built as a *projective limit*, is also pre-spectral. We use this to show that Rossman’s proof of homomorphism preservation in the finite can be re-cast in our framework as building exactly such a projective limit.

2. PRESERVATION THEOREMS

In this section, we revisit classical preservation theorems, whose proofs can be found in many books such as [6, Section 5.2]. We will recall the needed definitions, and illustrate the proof techniques in order to highlight the two ingredients that motivate our definitions of pre-spectral spaces and diagram bases later in Section 3.

2.1. Classical Preservation Theorems.

2.1.1. Notations. A σ -structure A over a finite relational signature σ (without constants) is given by a domain $|A|$ and, for each symbol $R \in \sigma$ of arity n , a relation $\mathbf{R}^A \subseteq |A|^n$; A is finite if $|A|$ is finite. The binary symbol ‘=’ will always be interpreted as equality, and will not be explicitly listed in our signatures. We write $\text{Struct}(\sigma)$ for the set¹ of all the σ -structures and $\text{Fin}(\sigma)$ for the finite ones. We assume the reader is familiar with the syntax and semantics of first-order logic over σ . We write $\text{FO}[\sigma]$ for the set of first-order sentences over σ . For such a sentence φ , we write $\llbracket \varphi \rrbracket_X \triangleq \{A \in X \mid A \models \varphi\}$ for its set of models over a class of structures $X \subseteq \text{Struct}(\sigma)$; by extension, we let $\llbracket \mathbf{F} \rrbracket_X \triangleq \{\llbracket \varphi \rrbracket_X \mid \varphi \in \mathbf{F}\}$ denote the collection of \mathbf{F} -definable subsets of X for a fragment \mathbf{F} of $\text{FO}[\sigma]$.

2.1.2. Abstract Preservation. A preservation theorem over a class of structures $X \subseteq \text{Struct}(\sigma)$ shows that first-order sentences enjoying some semantic property are equivalent to sentences from a suitable syntactic fragment. More precisely, one can model a semantic property as a collection $\mathcal{O} \subseteq \wp(X)$ of ‘semantic observations’ and consider a fragment $\mathbf{F} \subseteq \text{FO}[\sigma]$: we will say that X has the $(\mathcal{O}, \mathbf{F})$ *preservation property* if

- (1) for all $\psi \in \mathbf{F}$, $\llbracket \psi \rrbracket_X \in \mathcal{O}$, and,
- (2) for all $\varphi \in \text{FO}[\sigma]$ such that $\llbracket \varphi \rrbracket_X \in \mathcal{O}$, there exists $\psi \in \mathbf{F}$ such that $\llbracket \varphi \rrbracket_X = \llbracket \psi \rrbracket_X$.

In this definition, item 1 is usually proven by a straightforward induction on the formulæ in \mathbf{F} , and the challenge is to establish item 2. Item 2 is also where *relativisation* to a subset $Y \subseteq X$ might fail, because a set $U \notin \mathcal{O}$ might still be such that $U \cap Y \in \{V \cap Y \mid V \in \mathcal{O}\}$, and thus there might be new first-order sentences enjoying the semantic property and requiring an equivalent sentence in \mathbf{F} .

Put more succinctly, X has the $(\mathcal{O}, \mathbf{F})$ preservation property if

$$\mathcal{O} \cap \llbracket \text{FO}[\sigma] \rrbracket_X = \llbracket \mathbf{F} \rrbracket_X. \quad (1)$$

This formulation explicitly shows how a semantic condition (the left-hand side in (1)) is matched with a syntactic one (the right-hand side). As preservation is of interest beyond first-order logic [e.g., 19, 14, 9, 16], we will say in full generality that a set X equipped with a lattice \mathcal{L} of sets definable in the logic of interest has the $(\mathcal{O}, \mathcal{L}')$ *preservation property* if

$$\mathcal{O} \cap \mathcal{L} = \mathcal{L}' \quad (2)$$

In the rest of this paper we will assume that \mathcal{O} contains \emptyset , contains X , is closed under finite intersections and arbitrary unions. This is equivalent to \mathcal{O} being a collection of open sets and defining a *topology* on X .

¹In order to work over sets instead of proper classes and thereby avoid delicate but out-of-topic foundational issues, every σ -structure in this paper will be assumed to be of cardinality bounded by some suitable infinite cardinal. In particular, the Löwenheim-Skolem Theorem justifies that this is at no loss of generality when working with first-order logic.

TABLE 1. Classical preservation theorems and their relativisations to the finite case.

| preservation theorem | quasi-ordering \leq | fragment F | holds in $\text{Fin}(\sigma)$ |
|----------------------|-----------------------|--------------|-------------------------------|
| homomorphism | \rightarrow | EPFO | yes [33] |
| Tarski-Lyndon | \subseteq | EPFO $^\neq$ | no [3] |
| Łoś-Tarski | \subseteq_i | EFO | no [37, 20, 10] |
| dual Lyndon | \leftarrow | NFO | no [2, 35] |

2.1.3. Monotone Preservation. In a number of cases, which are especially relevant in the applications to database theory mentioned in the introduction [11, 17], the semantic property of interest is a form of monotonicity for some quasi-ordering \leq of $\text{Struct}(\sigma)$. We say that a sentence φ is *monotone* in $X \subseteq \text{Struct}(\sigma)$ if $\llbracket \varphi \rrbracket_X$ is *upwards-closed*, meaning that if $A \in \llbracket \varphi \rrbracket_X$ and B is a σ -structure in X such that $A \leq B$, then $B \in \llbracket \varphi \rrbracket_X$. In terms of abstract preservation, this corresponds to choosing \mathcal{O} as the collection of upwards-closed subsets of X , which is also known as the *Alexandroff topology* and is denoted by τ_\leq .

The quasi-ordering \leq in question is typically defined through some class of homomorphisms. Recall that there is a *homomorphism* between two σ -structures A and B , noted $A \rightarrow B$, if there exists $f: |A| \rightarrow |B|$ such that, for all relation symbols R of σ and all tuples $(a_1, \dots, a_n) \in \mathbf{R}^A$, $(f(a_1), \dots, f(a_n)) \in \mathbf{R}^B$. When f is injective, this entails that A is (isomorphic to) a (not necessarily induced) *substructure* of B and we write $A \subseteq B$; when f is furthermore *strong*—meaning that for all R and $(a_1, \dots, a_n) \in |A|^n$, $(f(a_1), \dots, f(a_n)) \in \mathbf{R}^B$ implies $(a_1, \dots, a_n) \in \mathbf{R}^A$ —, this entails that A is (isomorphic to) an *induced substructure* of B and we write $A \subseteq_i B$; finally, we write $A \twoheadrightarrow B$ when f is surjective.

Table 1 summarises what is known about monotone preservation theorems. In this table, EFO denotes the set of existential first-order sentences, NFO the set of negative ones (namely negative atoms closed under \vee , \wedge , \exists , and \forall), EPFO the set of existential positive ones, and EPFO $^\neq$ the set of existential positive ones extended with atoms of the form $x \neq y$ (interpreted as inequality). Note that Lyndon’s Theorem, which states that a first-order sentence closed under surjective homomorphisms on all structures is equivalent to a positive one, is presented in Table 1 in its *dual* form with inverse surjective homomorphisms and negative sentences. For all these fragments F and associated quasi-orderings \leq , the fact that $\llbracket F \rrbracket_X \subseteq \tau_\leq$ is mostly straightforward.

2.2. The Łoś-Tarski Theorem in Topological Terms. We propose now to inspect the proof of the Łoś-Tarski Theorem on a finite relational signature σ , as found for instance in [6, Theorem 3.2.2] or [23, Section 5.4]. We work here with the collection $\mathcal{O} \triangleq \tau_{\subseteq_i}$ of upwards-closed subsets of $X \triangleq \text{Struct}(\sigma)$ for \subseteq_i (this is the Alexandroff topology of the quasi-order \subseteq_i) and the fragment $F \triangleq \text{EFO}[\sigma]$. The Łoś-Tarski Theorem corresponds to the following instantiation of (1):

$$\tau_{\subseteq_i} \cap \llbracket \text{FO}[\sigma] \rrbracket_{\text{Struct}(\sigma)} = \llbracket \text{EFO}[\sigma] \rrbracket_{\text{Struct}(\sigma)}. \quad (3)$$

The proof of the Łoś-Tarski Theorem can be decomposed into two steps, here corresponding to the upcoming claims 2.1 and 2.2, and each invoking the Compactness Theorem. When translated in topological terms, the first shows that EFO defines a

base for the definable open sets, while the second shows that definable open sets are *compact*.

2.2.1. ‘Syntactic’ Base. Recall that a *base* \mathcal{B} of a topology τ is a collection of open sets such that every open set of τ is a (possibly infinite) union of elements from \mathcal{B} . Equivalently, \mathcal{B} is a base of a topology τ whenever $\forall U \in \tau, \forall A \in U, \exists V \in \mathcal{B}, A \in V \subseteq U$. A *subbase* is a collection of open sets such that every open set of τ is a (possibly infinite) union of finite intersections of elements of the *subbase*. The topology $\langle \mathcal{O} \rangle$ generated by a collection \mathcal{O} of sets is the smallest topology containing those sets; \mathcal{O} is then a subbase of $\langle \mathcal{O} \rangle$.

We first prove a weaker version of Equation (3) by proving the equality on the generated topologies. Because $\llbracket \text{FO}[\sigma] \rrbracket_{\text{Struct}(\sigma)}$ and $\llbracket \text{EFO}[\sigma] \rrbracket_{\text{Struct}(\sigma)}$ are lattices, those generated topologies can be seen as generated by infinite disjunctions of sentences in $\text{FO}[\sigma]$ (resp. $\text{EFO}[\sigma]$).

Claim 2.1. The topologies generated by $\tau_{\subseteq_i} \cap \llbracket \text{FO}[\sigma] \rrbracket_{\text{Struct}(\sigma)}$ and $\llbracket \text{EFO}[\sigma] \rrbracket_{\text{Struct}(\sigma)}$ are the same, i.e., $\langle \tau_{\subseteq_i} \cap \llbracket \text{FO}[\sigma] \rrbracket_{\text{Struct}(\sigma)} \rangle = \langle \llbracket \text{EFO}[\sigma] \rrbracket_{\text{Struct}(\sigma)} \rangle$.

Proof. First of all, any sentence in $\text{EFO}[\sigma]$ defines an upwards-closed set for \subseteq_i , and moreover $\text{EFO}[\sigma] \subseteq \text{FO}[\sigma]$, hence $\langle \llbracket \text{EFO}[\sigma] \rrbracket_{\text{Struct}(\sigma)} \rangle \subseteq \langle \tau_{\subseteq_i} \cap \llbracket \text{FO}[\sigma] \rrbracket_{\text{Struct}(\sigma)} \rangle$.

For the converse inclusion, it suffices to show that $\text{EFO}[\sigma]$ defines a base of the topology $\langle \tau_{\subseteq_i} \cap \llbracket \text{FO}[\sigma] \rrbracket_{\text{Struct}(\sigma)} \rangle$. Consider for this a monotone sentence $\varphi \in \text{FO}[\sigma]$ and a structure A such that $A \models \varphi$. Following the classical proofs [e.g., 6, Theorem 3.2.2 or 23, Corollary 5.4.3], define \hat{A} as the expansion of A with one additional constant c_a for each $a \in |A|$, interpreted by $c_a^{\hat{A}} \triangleq a$. The *diagram* $\text{Diag}(A)$ of A is the set of all quantifier-free sentences over this extended signature that hold in \hat{A} . For a structure $\hat{B} \in \text{Struct}(\sigma \cup \{c_a\}_{a \in A})$, we write B for its reduct in $\text{Struct}(\sigma)$ obtained by removing the constants $\{c_a\}_{a \in A}$.

Let $T \triangleq \text{Diag}(A) \cup \{\neg\varphi\}$, and consider $\hat{B} \in \text{Struct}(\sigma \cup \{c_a\}_{a \in A})$ such that $\hat{B} \models T$. Because $\hat{B} \models \text{Diag}(A)$, by construction $A \subseteq_i B$ (in particular, the sentence $\neg(c_a = c_b)$ belongs to $\text{Diag}(A)$ for all $a \neq b$ in $|A|$), and thus $B \models \varphi$ because φ is monotone, and finally $\hat{B} \models \varphi$ because the constants c_a do not occur in φ . Therefore, $\hat{B} \models \varphi \wedge \neg\varphi$, which is absurd: the theory T is inconsistent, and by the Compactness Theorem for first-order logic, there exists a finite conjunction ψ_0 of sentences in $\text{Diag}(A)$, which is already inconsistent with $\neg\varphi$.

Let ψ_A be the existential closure of the formula obtained by replacing each symbol c_a with a variable x_a in ψ_0 ; note that ψ_A is an existential sentence. By construction, $A \models \psi_A$, and if $B \models \psi_A$, there exists an interpretation of the constants $\{c_a\}_{a \in A}$ allowing to build an expansion \hat{B} such that $\hat{B} \models \psi_0$. As we just saw that $\models \psi_0 \implies \varphi$, $\hat{B} \models \varphi$, and since no constant symbol occurs in φ , $B \models \varphi$.

To conclude, for any open set $U \in \langle \tau_{\subseteq_i} \cap \llbracket \text{FO}[\sigma] \rrbracket_{\text{Struct}(\sigma)} \rangle$ and for any $A \in U$, there exists a monotone sentence φ such that $A \in \llbracket \varphi \rrbracket_{\text{Struct}(\sigma)}$, and we have proven that there exists $\llbracket \psi_A \rrbracket_{\text{Struct}(\sigma)} \in \llbracket \text{EFO}[\sigma] \rrbracket_{\text{Struct}(\sigma)}$ such that $A \in \llbracket \psi_A \rrbracket_{\text{Struct}(\sigma)} \subseteq \llbracket \varphi \rrbracket_{\text{Struct}(\sigma)} \subseteq U$. Therefore, $\llbracket \text{EFO}[\sigma] \rrbracket_{\text{Struct}(\sigma)}$ is a base of $\langle \tau_{\subseteq_i} \cap \llbracket \text{FO}[\sigma] \rrbracket_{\text{Struct}(\sigma)} \rangle$. \square

2.2.2. Compactness. The second step relies on the compactness of the sets $\llbracket \varphi \rrbracket_{\text{Struct}(\sigma)}$ for monotone sentences φ . Recall that a subset K is *compact* in a topological space τ if, for any *open cover* $(U_i)_{i \in I}$ of K —i.e., a collection of open sets such that $K \subseteq \bigcup_{i \in I} U_i$ —, there exists a finite subset $I_0 \subseteq I$, such that $K \subseteq \bigcup_{i \in I_0} U_i$ (beware that this definition is also called *quasi-compact* in the literature, because we do not

require any separation property here). If $\tau = \langle \mathcal{O} \rangle$, by *Alexander's Subbase Lemma*, K is compact if and only if, from every open cover of K using only sets from \mathcal{O} , we can extract a finite open cover of K . As open compact sets play a key role in this paper, we introduce here the notation $\mathcal{K}^\circ(X) \triangleq \{U \in \tau \mid U \text{ is compact}\}$. When the topology τ is not clear from the context, we shall write $\mathcal{K}^\circ(X, \tau)$.

Claim 2.2. Every monotone sentence defines a compact open subset in the topology $\langle \tau_{\subseteq_i} \cap \llbracket \text{FO}[\sigma] \rrbracket_{\text{Struct}(\sigma)} \rangle$, i.e., $\tau_{\subseteq_i} \cap \llbracket \text{FO}[\sigma] \rrbracket_{\text{Struct}(\sigma)} \subseteq \mathcal{K}^\circ(\text{Struct}(\sigma))$.

Proof. Consider a monotone sentence $\varphi \in \text{FO}[\sigma]_{\text{Struct}(\sigma)}$. Let $(U_i)_{i \in I}$ be an open cover of $\llbracket \varphi \rrbracket_{\text{Struct}(\sigma)}$. By Alexander's Subbase Lemma, we can assume that for each $i \in I$, $U_i = \llbracket \varphi_i \rrbracket_{\text{Struct}(\sigma)}$ for some monotone sentence φ_i . Consider the theory $T \triangleq \{\neg \varphi_i \mid i \in I\} \cup \{\varphi\}$. Because $(U_i)_{i \in I}$ is an open cover, this theory has no models. By the Compactness Theorem for first order logic, there exists a finite set I_0 such that $T_0 \triangleq \{\neg \varphi_i \mid i \in I_0\} \cup \{\varphi\}$ is not satisfiable, proving that $(U_i)_{i \in I_0}$ is an open cover of $\llbracket \varphi \rrbracket_{\text{Struct}(\sigma)}$. \square

Remark 2.3 (Compact sets in τ_{\subseteq}). As we will often deal with the Alexandroff topology τ_{\subseteq} of a quasi-order (X, \leq) , it is worth noting that $U \in \tau_{\subseteq}$ is compact if and only if it is the upward closure $U = \uparrow F$ of some finite subset $F \subseteq_{\text{fin}} X$; this is equivalent to saying that U has finitely many minimal elements up to \leq -equivalence [see e.g., 18, Exercise 4.4.22]. Thus Claim 2.2 states that any monotone sentence has finitely many \subseteq_i -minimal models in $\text{Struct}(\sigma)$.

2.2.3. Proof of the Łoś-Tarski Theorem. A simple structural induction on the formulæ shows that $\llbracket \text{EFO}[\sigma] \rrbracket_{\text{Struct}(\sigma)} \subseteq \tau_{\subseteq_i} \cap \llbracket \text{FO}[\sigma] \rrbracket_{\text{Struct}(\sigma)}$. Regarding the converse inclusion in Equation (3), consider a sentence $\varphi \in \text{FO}[\sigma]$ defining an open set in τ_{\subseteq_i} . By Claim 2.1, there exists a family $(\psi_i)_{i \in I}$ of existential sentences such that $\llbracket \varphi \rrbracket_{\text{Struct}(\sigma)} = \bigcup_{i \in I} \llbracket \psi_i \rrbracket_{\text{Struct}(\sigma)}$. By Claim 2.2, there is a finite set $I_0 \subseteq_{\text{fin}} I$ for which the equality still holds. Because $\text{EFO}[\sigma]$ is a lattice, this proves the existence of an existential sentence $\psi \triangleq \bigvee_{i \in I_0} \psi_i$ such that $\llbracket \varphi \rrbracket_{\text{Struct}(\sigma)} = \llbracket \psi \rrbracket_{\text{Struct}(\sigma)}$. \square

The two properties singled out in claims 2.1 and 2.2 are of different nature. Claim 2.2 really holds for any topology τ and not only for the Alexandroff topology τ_{\subseteq_i} , as opposed to Claim 2.1. Moreover, Claim 2.1 appears to be the most involved one here, but is often easily proven on classes of finite structures.

3. PRE-SPECTRAL SPACES AND DIAGRAM BASES

Following the two-step decomposition of the proof of the Łoś-Tarski Theorem given in Section 2.2, we define in this section *logically presented pre-spectral spaces* and *diagram bases*, before showing in Theorem 3.4 how they characterise when a preservation theorem holds.

3.1. Pre-spectral Spaces. As a preliminary step toward our main definition, let us first propose a definition of topological spaces (X, τ) where the compact open sets form a *bounded sublattice* of $\wp(X)$ (by which we mean that \emptyset and X belong to the lattice) that generates the topology.

Definition 3.1 (Pre-spectral space). A topological space (X, τ) is a *pre-spectral space* whenever $\mathcal{K}^\circ(X)$ is a bounded sublattice of $\wp(X)$ that generates τ , i.e., $\langle \mathcal{K}^\circ(X) \rangle = \tau$.

The name ‘pre-spectral’ comes from the theory of *spectral* spaces [12], for which the definition is almost identical (see Section 4.2). Pre-spectral spaces will allow us to tap into the rich topological toolset that has been developed for spectral spaces.

3.1.1. Logical presentations. As seen in Claim 2.2, the topology of interest in a preservation theorem is generated by combining a topological space (X, τ) with a bounded sublattice \mathcal{L} of subsets of X , which will be called the *definable* subsets of X . Let us write $\langle X, \tau, \mathcal{L} \rangle$ for the topological space $(X, \langle \tau \cap \mathcal{L} \rangle)$. The following definition is then a direct generalisation of the statement of Claim 2.2.

Definition 3.2 (Logically presented pre-spectral space). Let (X, τ) be a topological space and \mathcal{L} be a bounded sublattice of $\wp(X)$. Then $\langle X, \tau, \mathcal{L} \rangle$ is a *logically presented pre-spectral space* (a *lpps*) if its definable open subsets are compact, i.e., if $\tau \cap \mathcal{L} \subseteq \mathcal{K}^\circ(X)$.

Whenever σ is a finite relational signature, $X \subseteq \text{Struct}(\sigma)$ for a topological space (X, τ) and $\mathcal{L} = \llbracket \text{FO}[\sigma] \rrbracket_X$, we denote it by $\langle X, \tau, \text{FO}[\sigma] \rangle$ for simplicity; e.g., $\langle \text{Struct}(\sigma), \tau_{\subseteq}, \text{FO}[\sigma] \rangle$ is a lpps by Claim 2.2.

As $\tau \cap \mathcal{L}$ is closed under finite intersection, any open set in $\langle \tau \cap \mathcal{L} \rangle$ is a union of sets from $\tau \cap \mathcal{L}$, thus any compact open set in $\mathcal{K}^\circ(X)$ is a finite union of sets from $\tau \cap \mathcal{L}$. As $\tau \cap \mathcal{L}$ is also closed under finite unions, this shows the inclusion $\mathcal{K}^\circ(X) \subseteq \tau \cap \mathcal{L}$. Thus, in a lpps, $\mathcal{K}^\circ(X) = \tau \cap \mathcal{L}$ is a bounded lattice and any lpps is indeed a pre-spectral space. Conversely, $\langle X, \tau, \mathcal{K}^\circ(X) \rangle$ is well-defined whenever (X, τ) is a pre-spectral space; in this case $\langle X, \tau, \mathcal{K}^\circ(X) \rangle$ is a lpps and it equals (X, τ) (they have the same points and opens).

Beware however that $(X, \langle \tau \cap \mathcal{L} \rangle) = \langle X, \tau, \mathcal{L} \rangle$ being pre-spectral does not entail that it is a lpps; see Remark 3.7 at the end of the section. While pre-spectral spaces capture the topological core behind Claim 2.2 with a simple definition, the logically presented ones are the real objects of interest as far as preservation theorems are concerned, and most of the technical difficulties arising in the remainder of the paper will be concerned with those.

3.2. Diagram Bases. Regarding Claim 2.1, we simply turn the statement of the claim into a definition, which is typically instantiated with $\mathcal{L} = \llbracket \text{FO}[\sigma] \rrbracket_X$ and $\mathcal{L}' = \llbracket \text{F} \rrbracket_X$ for a fragment F of $\text{FO}[\sigma]$.

Definition 3.3 (Diagram base). Let (X, τ) be a topological space, and \mathcal{L} be a bounded sublattice of $\wp(X)$. Then $\mathcal{L}' \subseteq \mathcal{L}$ is a *diagram base* of $\langle X, \tau, \mathcal{L} \rangle$ if $\langle \tau \cap \mathcal{L} \rangle = \langle \mathcal{L}' \rangle$.

In particular, if $\text{F} \subseteq \text{FO}[\sigma]$ is stable under finite conjunction, this means that any definable open set in X can be written as an infinite disjunction of F -definable sets. Over $\text{Struct}(\sigma)$, this was the ‘difficult’ step in the classical proof of the Łoś-Tarski Theorem. When $X \subseteq \text{Fin}(\sigma)$, this becomes considerably simpler: for every fragment F in Table 1 and any finite structure A , there exists a *diagram* sentence ψ_A^{F} in F such that $A \leq B$ if and only if $B \models \psi_A^{\text{F}}$ for the corresponding quasi-ordering. Therefore, if φ is monotone and $A \in \llbracket \varphi \rrbracket_X$, then $A \in \llbracket \psi_A^{\text{F}} \rrbracket_X \subseteq \llbracket \varphi \rrbracket_X$, showing that $\llbracket \text{F} \rrbracket_X$ is a base of $\langle \tau_{\subseteq} \cap \llbracket \text{FO}[\sigma] \rrbracket_X \rangle$.

3.3. A Generic Preservation Theorem. We have already seen in the proof of the Łoś-Tarski Theorem why logically presented pre-spectral spaces with a diagram base yield preservation. The following theorem also proves the converse direction,

under mild hypotheses on \mathcal{L}' : \mathcal{L}' must be a lattice and must define compact sets in X for the topology generated by \mathcal{L}' . We usually instantiate the theorem with $X \subseteq \text{Struct}(\sigma)$ $\mathcal{L} = \llbracket \text{FO}[\sigma] \rrbracket_X$ and $\mathcal{L}' = \llbracket \mathbf{F} \rrbracket_X$ where \mathbf{F} is a fragment of $\text{FO}[\sigma]$.

Theorem 3.4 (Generic preservation). *Let τ be a topology on X , \mathcal{L} a bounded sublattice of $\wp(X)$, and \mathcal{L}' a sublattice of \mathcal{L} . The following are equivalent:*

- (1) *X has the (τ, \mathcal{L}') preservation property and \mathcal{L}' defines only compact sets for the topology $\langle \mathcal{L}' \rangle$.*
- (2) *$\langle X, \tau, \mathcal{L} \rangle$ is a lpps and \mathcal{L}' is a diagram base of it.*

Proof. We prove the two implications separately.

- (1) Assume that X has the (τ, \mathcal{L}') preservation property. Consider a set $U \in \mathcal{L} \cap \tau$: by the preservation property, $U \in \mathcal{L}'$. This already shows that \mathcal{L}' defines a diagram base of $\langle X, \tau, \mathcal{L} \rangle$. Hence $\langle \mathcal{L}' \rangle = \langle \tau \cap \mathcal{L} \rangle$. Since $U \in \mathcal{L}'$, U is compact in $\langle \mathcal{L}' \rangle$, which means that U is compact in X . Therefore X is a lpps.
- (2) Assume that \mathcal{L}' defines a diagram base of $\langle X, \tau, \mathcal{L} \rangle$. If $U \in \tau \cap \mathcal{L}$, then it is equivalent to a possibly infinite union of elements in \mathcal{L}' . Also assume that $\langle X, \tau, \mathcal{L} \rangle$ is a lpps: then by compactness, U is equivalent to a finite union of elements in \mathcal{L}' , hence equivalent to a single element in \mathcal{L}' since \mathcal{L}' is a lattice. This proves that X has the (τ, \mathcal{L}') preservation property. Finally, sets in \mathcal{L}' define compact sets in $\langle \mathcal{L}' \rangle$ because it is precisely the topology of $\langle X, \tau, \mathcal{L} \rangle$. \square

The additional hypotheses on \mathcal{L}' in items 1 and 2 above are somewhat at odds. Asking for \mathcal{L}' to define a diagram base is asking for $\langle \mathcal{L}' \rangle$ to have enough sets, but asking for \mathcal{L}' to only define compact sets is asking for $\langle \mathcal{L}' \rangle$ not to contain too many sets.

Remark 3.5 (Generic monotone preservation). The condition that \mathbf{F} must define compact sets in X in Theorem 3.4.1 is actually mild. Consider the preservation results from Table 1 for a fragment \mathbf{F} and $\tau = \tau_{\leq}$ the Alexandroff topology of the associated quasi-ordering \leq . Assume that X is a \leq -downwards-closed subset of $\text{Struct}(\sigma)$ —this is the setting of the known preservation results for classes of finite structures [4, 5, 33, 8, 21].

Observe that, in each case, $\llbracket \psi \rrbracket_{\text{Struct}(\sigma)}$ for a sentence $\psi \in \mathbf{F}$ has finitely many \leq -minimal models up to \leq -equivalence. Because X is downwards-closed, $\llbracket \psi \rrbracket_X$ has the same finitely many \leq -minimal models. Thus, by Remark 2.3, $\llbracket \psi \rrbracket_X$ is compact in τ_{\leq} , and since $\llbracket \mathbf{F} \rrbracket_X \subseteq \tau_{\leq}$, it is also compact in the topology generated by $\llbracket \mathbf{F} \rrbracket_X$.

In the case of $X = \text{Fin}(\sigma)$, this downward closure condition is fulfilled and \mathbf{F} defines a base, thus $(\tau_{\leq}, \mathbf{F})$ preservation holds if and only if $\langle \text{Fin}(\sigma), \tau_{\leq}, \text{FO}[\sigma] \rangle$ is a lpps.

Theorem 3.4 is a generic relationship between pre-spectral spaces and preservation theorems. The downward closure hypothesis in Remark 3.5 is necessary for the equivalence between the preservation property and pre-spectral spaces to hold, as will be shown later in Example 4.2. In order to apply Remark 3.5 to a set that is not downwards-closed, it might be tempting to consider its downward closure. The following example shows that downward closures interact poorly with pre-spectrality.

Example 3.6. We provide examples where $X = \downarrow Y$ and X is pre-spectral but not Y (resp. X is not pre-spectral but Y is). Another instance will be given in Example 4.5.

Consider $X \triangleq \mathbb{N} \uplus \{\infty\}$ with the ordering $x \leq y$ if and only if $x = y$ or $y = \infty$. The set $Y \triangleq \{\infty\}$ with the Alexandroff topology is finite and therefore pre-spectral. However, $\downarrow Y = X$, which is not pre-spectral.

Conversely, let $X \triangleq \text{Struct}(\sigma)$, $Y \triangleq \text{Fin}(\sigma)$, and \leq be the reverse ordering \supseteq_i , so that $X = \downarrow Y$. Remark that the space $\langle X, \tau_{\leq}, \text{FO}[\sigma] \rangle$ is pre-spectral thanks to Theorem 3.4 and the ‘dual’ Łoś-Tarski Theorem (a first-order sentence φ is preserved under induced substructures if and only if it is equivalent to a universal sentence). However, Łoś-Tarski Theorem does not relativise to finite structures [37] and UFO generates a basis of the topology of Y , hence using Theorem 3.4 the space $\langle Y, \tau_{\leq}, \text{FO}[\sigma] \rangle$ is not a lpps.

Remark 3.7. For each of the fragments F and associated quasi-orderings \leq of Table 1, $\langle \text{Fin}(\sigma), \tau_{\leq}, \text{FO}[\sigma] \rangle = (\text{Fin}(\sigma), \langle \tau_{\leq} \cap \llbracket \text{FO}[\sigma] \rrbracket_{\text{Fin}(\sigma)} \rangle)$ is a pre-spectral space. Indeed, by Remark 2.3, any compact open K from $\mathcal{K}^\circ(\text{Fin}(\sigma))$ is the upward closure $K = \uparrow F$ of a finite set $F \subseteq_{\text{fin}} \text{Fin}(\sigma)$, thus $K = \llbracket \bigvee_{A \in F} \psi_A^F \rrbracket_{\text{Fin}(\sigma)}$, which shows that $\mathcal{K}^\circ(\text{Fin}(\sigma)) \subseteq \llbracket F \rrbracket_{\text{Fin}(\sigma)}$. As any $\psi \in F$ has finitely many \leq -minimal models in $\text{Fin}(\sigma)$, $\mathcal{K}^\circ(\text{Fin}(\sigma)) \supseteq \llbracket F \rrbracket_{\text{Fin}(\sigma)}$, and since F defines a base, $\langle \text{Fin}(\sigma), \tau_{\leq}, \text{FO}[\sigma] \rangle$ is pre-spectral. However, by Remark 3.5 and the non-preservation results of [37, 20, 3, 2, 35], $\langle \text{Fin}(\sigma), \tau_{\subseteq_i}, \text{FO}[\sigma] \rangle$, $\langle \text{Fin}(\sigma), \tau_{\subseteq}, \text{FO}[\sigma] \rangle$, and $\langle \text{Fin}(\sigma), \tau_{\leftarrow}, \text{FO}[\sigma] \rangle$ are not lpps: the condition $\tau \cap \mathcal{L} \subseteq \mathcal{K}^\circ(X)$ is crucial in order to derive preservation results.

Another way of reaching the topological definitions of this section is to consider a folklore result employed in several proofs of preservation theorems over classes of finite structures for fragments F of EFO [33, 5, 4]: if X is downwards-closed for \leq , a monotone sentence φ is equivalent to a sentence from F if and only if it has finitely many \leq -minimal models in X (up to \leq -equivalence). By Remark 2.3, this says that $\llbracket \varphi \rrbracket_X$ is compact, while the folklore result itself is essentially using the fact that F defines a base.

4. RELATED NOTIONS

Pre-spectral spaces generalise two notions arising from order theory, topology, and logics: Noetherian spaces and spectral spaces.

4.1. Well-Quasi-Orderings and Noetherian Spaces. A topological space in which all subsets are compact, or, equivalently, all open subsets are compact, is called *Noetherian* [see 18, Section 9.7]. A Noetherian space (X, τ) and a bounded sublattice \mathcal{L} of $\wp(X)$ always define a lpps $\langle X, \tau, \mathcal{L} \rangle$. A related notion, considering a quasi-order instead of a topology, leads to the well-known notion of *well-quasi-orders* [26]: a quasi-order is a well-quasi-order if and only if its Alexandroff topology is Noetherian [18, Proposition 9.7.17]. Thus, if (X, \leq) is a well-quasi-order and \mathcal{L} is a bounded sublattice of $\wp(X)$, then $\langle X, \tau_{\leq}, \mathcal{L} \rangle$ is a lpps.

4.1.1. Applications of Noetherian Spaces to Preservation. Let us denote by \mathcal{G} the class of finite simple undirected graphs and by $\sigma_{\mathcal{G}}$ the signature with a single binary edge relation E ; then the induced substructure ordering \subseteq_i coincides with the induced subgraph ordering over \mathcal{G} .

Example 4.1 (Finite graphs of bounded tree-depth). Recall that the *tree-depth* $\text{td}(G)$ of a graph G is the minimum height of the comparability graphs F of partial orders such that G is a subgraph of F [31, Chapter 6]. Let $\mathcal{T}_{\leq n}$ be the set of finite graphs of tree-depth at most n ordered by the induced substructure relation \subseteq_i . This is a well-quasi-order [13], thus $\langle \mathcal{T}_{\leq n}, \tau_{\subseteq_i}, \text{FO}[\sigma_{\mathcal{G}}] \rangle$ is a lpps, and therefore $\mathcal{T}_{\leq n}$ enjoys the $(\tau_{\subseteq_i}, \text{EFO}[\sigma_{\mathcal{G}}])$ -preservation property by Theorem 3.4.

Example 4.2 (Finite cycles). Consider the class $\mathcal{C} \subseteq \mathcal{G}$ of all finite simple cycles. As is well known, $(\mathcal{C}, \subseteq_i)$ is not a well-quasi-order because any two different cycles are incomparable for the induced substructure ordering [13]. In particular, every singleton is an open set: $(\mathcal{C}, \tau_{\subseteq_i})$ is actually a topological space with the *discrete topology*, and its only compact sets are the finite sets: $\langle \mathcal{C}, \tau_{\subseteq_i}, \text{FO}[\sigma_{\mathcal{G}}] \rangle$ is not a lpps.

By standard locality arguments, for any sentence φ , there exists a finite threshold n_0 on the size of cycles, above which φ is either always true or always false.

Claim 4.3. For all φ in $\text{FO}[\sigma_{\mathcal{G}}]$, there exists a threshold n_0 such that, for all $m, n \geq n_0$, $C_n \models \varphi$ if and only if $C_m \models \varphi$.

Proof. Fix d and r two positive integers, and observe that for $n_0 \triangleq \max(2d+2, r+1)$, $n, m \geq n_0$, and t an isomorphism type on d -neighbourhoods of cycles, C_m and C_n have either both zero occurrences of t or both have more than r occurrences of t . Thus, n_0 is such that for all $n, m \geq n_0$, C_m and C_n are (r, d) *threshold equivalent* [28, Definition 4.23]. Fix a sentence $\varphi \in \text{FO}[\sigma_{\mathcal{G}}]$. By [28, Theorem 4.24], there exists (r, d) such that two (r, d) threshold equivalent structures cannot be distinguished by φ . \square

We can directly show that \mathcal{C} has the $(\tau_{\subseteq_i}, \text{EFO}[\sigma_{\mathcal{G}}])$ preservation property using Claim 4.3. For each n , let $\psi_{\supseteq C_n} \in \text{EFO}[\sigma_{\mathcal{G}}]$ be the sentence defining the set of all structures that contain a cycle of size n

$$\psi_{\supseteq C_n} \triangleq \exists x_0, \dots, x_{n-1}. \bigwedge_{i < j} \neg(x_i = x_j) \wedge \bigwedge_{0 \leq i \leq n-1} E(x_i, x_{i+1 \bmod n-1}) \quad (4)$$

(thus $\llbracket \psi_{\supseteq C_n} \rrbracket_{\mathcal{C}} = \{C_n\}$) and let $\psi_{\supseteq P_n} \in \text{EFO}[\sigma_{\mathcal{G}}]$ be the sentence defining the set of all structures that contain a path of size n

$$\psi_{\supseteq P_n} \triangleq \exists x_0, \dots, x_{n-1}. \bigwedge_{i < j} \neg(x_i = x_j) \wedge \bigwedge_{0 \leq i < n-1} E(x_i, x_{i+1}) \quad (5)$$

(thus $\llbracket \psi_{\supseteq P_n} \rrbracket_{\mathcal{C}} = \{C_i \mid i \geq n\}$).

If φ is a monotone sentence, by Claim 4.3 either it has finitely many cycles $\{C_{n_1}, \dots, C_{n_m}\}$ as models and $\bigvee_{1 \leq j \leq m} \psi_{\supseteq C_{n_j}}$ is equivalent to it, or it has finitely many cycles $\{C_{n_1}, \dots, C_{n_m}\}$ as counter-models and $\psi_{\supseteq P_{n_m+1}} \vee \bigvee_{\substack{j \leq n_m \\ j \notin \{n_1, \dots, n_m\}}} \psi_{\supseteq C_j}$ fits.

It is nevertheless enlightening to show preservation using the framework of logically defined pre-spectral spaces, by defining a suitable topology. Let τ_n be the topology over \mathcal{C} generated by the definable co-finite sets and the definable sets containing only cycles of size at most n . This is a variation of the *co-finite topology*, and is also Noetherian.

Claim 4.4. The spaces (\mathcal{C}, τ_n) are Noetherian for all $n \in \mathbb{N}$

Proof. Let $X \subseteq \mathcal{C}$, and using Alexander's Subbase Lemma consider an open cover $(U_i)_{i \in I}$ of X where U_i either contains only cycles of size at most n or is co-finite.

If there exists $i_0 \in I$ such that U_{i_0} is co-finite, consider $X \setminus U_{i_0}$. This is a finite set, hence there exists $I_0 \subseteq_{\text{fin}} I$ such that $X \setminus U_{i_0} \subseteq \bigcup_{i \in I_0} U_i$, which proves that $(U_i)_{i \in I_0 \cup \{i_0\}}$ is a finite open cover of X . If none of the U_i is co-finite, then X is contained in the set of cycles of size at most n , which proves that X is finite, hence compact. \square

Hence, $\langle \mathcal{T}_{\leq n}, \tau_{\subseteq_i}, \text{FO}[\sigma_{\mathcal{G}}] \rangle$ is a lpps, and as $\text{EFO}[\sigma_{\mathcal{G}}]$ defines a diagram base of it, we can apply Theorem 3.4 to deduce preservation. Now, given a monotone sentence φ , either φ has finitely many models or it has co-finitely many. In both cases, this sentence defines an open set in τ_n for some n that is definable in $\text{EFO}[\sigma_{\mathcal{G}}]$. Thus the set of finite cycles has the $(\tau_{\subseteq_i}, \text{EFO}[\sigma_{\mathcal{G}}])$ preservation property.

The previous example shows that the closure condition of Remark 3.5 was necessary, by proving that a space of structures can enjoy a preservation theorem while not defining a lpps. Example 4.2 also shows that the hypothesis that \mathbf{F} defines compact sets in Theorem 3.4.1 was necessary. Observe that the condition is violated by $\mathbf{F} = \text{EFO}[\sigma_{\mathcal{G}}]$ in \mathcal{C} . Indeed, $\llbracket \psi_{\geq P_3} \rrbracket_{\mathcal{C}} \subseteq \bigcup_{n \geq 3} \llbracket \psi_{\geq C_n} \rrbracket_{\mathcal{C}}$, but there does not exist a finite subcover of it.

Example 4.5 (Downward closure of cycles). If one considers $\langle \downarrow \mathcal{C}, \tau_{\subseteq_i}, \text{FO}[\sigma_{\mathcal{G}}] \rangle$ the space of all the finite graphs that are induced substructures of some finite cycle, then Remark 3.5 can be applied: there is preservation if and only if the space is a lpps. However that space is not a lpps: consider the sentence φ stating that there exists no vertex of degree exactly one; then $\llbracket \varphi \rrbracket_{\downarrow \mathcal{C}} = \mathcal{C}$, which is not compact, but is upwards-closed inside $\downarrow \mathcal{C}$. In particular, although \mathcal{C} enjoys a preservation theorem, $\downarrow \mathcal{C}$ does not.

4.1.2. Relativisation. The following proposition shows that, if we are looking for classes of structures where preservation theorems *always* relativize, then we should endow them with a Noetherian topology.

Proposition 4.6. *Let (X, τ) be a pre-spectral space such that for all $Y \subseteq X$, Y with the induced topology is pre-spectral. Then X is Noetherian.*

Proof. Consider any subset Y of X : by assumption, Y is pre-spectral, hence compact in the induced topology, hence compact in (X, τ) . \square

4.2. Spectral Spaces. Spectral spaces are a class of topological spaces appearing naturally in the study of logics and algebra as a generalisation of the Stone Duality theory. Throughout this section we refer to two books and keep the notations consistent with them [18, 12]. A closed subset F of a topological space X is *irreducible* whenever F is non-empty and is not the disjoint union of two non-empty closed sets. The *closure* of a set Y in a space X is the smallest closed set containing Y and is denoted by \overline{Y}^X or \overline{Y} when X is clear from the context. A topological space X is *sober* whenever any irreducible closed subset F is the closure of exactly one point $x \in X$, which translates formally to $\exists x \in X, \overline{\{x\}} = F$ and $\forall y \in X, \overline{\{y\}} = F \Rightarrow y = x$. A *spectral space* is a pre-spectral space that is sober [12, Definition 1.1.5].

When a space (X, τ) is not sober, it is possible to build a *sobriified* version of this space as follows [18, Definition 8.2.17]: $\mathcal{S}(X)$ is the set of irreducible closed sets of X , and the topology is generated by the sets $\Diamond U \triangleq \{F \in \mathcal{S}(X) \mid F \cap U \neq \emptyset\}$ where U is an open set of X . It can be shown that this construction leads to a sober space, is idempotent up to homeomorphism, and constructs the *free* sober space over X [18, Theorem 8.2.44]. This leads to the following correspondence between pre-spectral spaces and spectral spaces.

Fact 4.7 (Spectral versus pre-spectral). *A space X is pre-spectral if and only if $\mathcal{S}(X)$ is spectral.*

The connection with spectral spaces is of particular interest, because the sobrification functor gives a tool to translate result from the rich theory of spectral spaces to pre-spectral spaces which will be extensively used in Section 5. For instance, spectral spaces are closed under arbitrary products and co-products [12, Theorem 2.2.1 and Corollary 5.2.9], and the sobrification operator commutes with these operations [18, Fact 8.4.3 and Theorem 8.4.8]. This allows to instantly deduce that a finite disjoint union of pre-spectral spaces is pre-spectral and an arbitrary product of pre-spectral spaces is pre-spectral (see lemmas 5.9 and 5.12).

4.2.1. Spectral Spaces of Structures. Spaces of finite structures, such as \mathcal{C} the set of finite cycles, are in general not sober, and the spaces obtained through the constructions $\langle \text{Struct}(\sigma), \tau, \text{FO}[\sigma] \rangle$ are not even T_0 . However, notice that the T_0 quotient of $\text{Struct}(\sigma)$ is sober when $\tau = \tau_{\subseteq_i}$. This proof can be adapted to the case where $\tau = \tau_{\leq}$ and the upwards closure of finite structures are definable in $\text{Struct}(\sigma)$. For a structure $A \in \text{Struct}(\sigma)$, define the *age* of A as $\text{Age}(A) \triangleq \{A_0 \in \text{Fin}(\sigma) \mid A_0 \subseteq_i A\}$.

Claim 4.8. The T_0 quotient of $\langle \text{Struct}(\sigma), \tau_{\subseteq_i}, \text{FO}[\sigma] \rangle$ is a sober space, hence a spectral space.

Proof. Remark that over $\text{Fin}(\sigma)$ the topology we consider is exactly τ_{\subseteq_i} since the upwards closure of a single finite structure is definable in $\text{FO}[\sigma]$. Notice moreover, that the topology over $\text{Struct}(\sigma)$ is exactly the topology generated using sentences in $\text{EFO}[\sigma]$ thanks to the Łoś-Tarski Theorem. In particular the T_0 quotient of $\text{Struct}(\sigma)$ is only equating *infinite* structures, and two infinite structure are equated if and only if they have the same age.

For all $A \in \text{Fin}(\sigma)$, let ψ_A^{EFO} be the diagram sentence such that $\llbracket \psi_A^{\text{EFO}} \rrbracket_{\text{Struct}(\sigma)}$ equals the upwards closure of $\{A\}$ in $\text{Struct}(\sigma)$.

Consider F an irreducible closed subset of $\text{Struct}(\sigma)$. Notice that for all U, V open sets if $U \cap F \neq \emptyset$ and $V \cap F \neq \emptyset$ then $U \cap V \cap F \neq \emptyset$. Assume by contradiction that a pair U, V exists such that $U \cap F \neq \emptyset$ and $V \cap F \neq \emptyset$, but $U \cap V \cap F = \emptyset$. Then $F \subseteq U^c \cup V^c$, but $F \not\subseteq U^c$ and $F \not\subseteq V^c$. Since F is closed, we can write $F = (U^c \cap F) \cup (V^c \cap F)$ and conclude that F is the disjoint union of two non-empty closed sets which is absurd because F is irreducible.

Because F is closed in $\text{Struct}(\sigma)$ it is a downwards closed set for \subseteq_i . Assume that A and B are two finite structures in F , remark that $F \cap \llbracket \psi_A^{\text{EFO}} \rrbracket_{\text{Struct}(\sigma)} \neq \emptyset$ and $F \cap \llbracket \psi_B^{\text{EFO}} \rrbracket_{\text{Struct}(\sigma)} \neq \emptyset$; since F is irreducible, this proves the existence of $C \in F$ such that $C \in F \cap \llbracket \psi_A^{\text{EFO}} \wedge \psi_B^{\text{EFO}} \rrbracket_{\text{Struct}(\sigma)}$. In particular, $A \subseteq_i C$ and $B \subseteq_i C$ and $C \in F$. Note that C might not be finite, but the result can be extended to a finite

set of structures A_1, \dots, A_n : if $A_1, \dots, A_n \in F \cap \text{Fin}(\sigma)$, then there exists C above all of them that is still in F .

Let $T^+ \triangleq \{\psi_A^{\text{EFO}} \mid A \in F \cap \text{Fin}(\sigma)\}$ and $T^- \triangleq \{\neg\psi_A^{\text{EFO}} \mid A \in F^c \cap \text{Fin}(\sigma)\}$. Remark that any finite subset of T^+ has a model in F , this is because a finite subset of T^+ defines the upwards closure of a finite number of finite structures in F , hence has a (possibly infinite) model in F . Remark that an element of F is a model of T^- , because F is downwards closed for \subseteq_i . By the Compactness Theorem, $T^+ \cup T^-$ has a model B . The definition of $T^+ \cup T^-$ immediately ensures that $\text{Age}(B) = F \cap \text{Fin}(\sigma)$.

Let us show that $B \in F$. Assume by contradicton that $B \in F^c$, then there exists an open set U defined by $\varphi \in \text{EFO}[\sigma]$ such that $B \models \varphi$ and $U \subseteq F^c$. As $\varphi \in \text{EFO}[\sigma]$, it has a finite model A_0 such that $A_0 \subseteq_i B$. This is absurd because it shows that $A_0 \notin F$, $A_0 \in \text{Fin}(\sigma)$ but $A_0 \in \text{Age}(B) = F \cap \text{Fin}(\sigma)$.

Let $A \in F \cap \text{Fin}(\sigma)$, let us show that A is in the closure of $\{B\}$. For this, we consider a sentence $\varphi \in \text{EFO}[\sigma]$ such that $A \models \varphi$, and show that $B \models \varphi$. Given such a sentence φ , remark that $\llbracket \psi_A^{\text{EFO}} \rrbracket_{\text{Struct}(\sigma)} \subseteq \llbracket \varphi \rrbracket_{\text{Struct}(\sigma)}$ because φ is monotone. By construction, $B \models \psi_A^{\text{EFO}}$, hence $B \models \varphi$. Hence, $F \cap \text{Fin}(\sigma)$ is included in the closure of B .

Now, consider an infinite structure $B' \in F$, we are going to prove that B' is in the closure of $\{B\}$. For that, consider $\varphi \in \text{EFO}[\sigma]$ such that $B' \models \varphi$, as $\varphi \in \text{EFO}[\sigma]$ there exists a finite structure $A \subseteq_i B'$ such that $A \models \varphi$. Because F is downwards-closed, $A \in F$, hence A is in the closure of $\{B\}$, therefore $B \models \varphi$.

As F is a closed set and $B \in F$, the closure of B is included in F and we have proven that F is the closure of a point in $\text{Struct}(\sigma)$. In the T_0 quotient of $\text{Struct}(\sigma)$ this point is unique by definition.

We have proven that every non-empty irreducible closed set is the closure of exactly one point in the T_0 quotient of $\text{Struct}(\sigma)$, which is the definition of a sober space. \square

5. BASIC CLOSURE PROPERTIES

To study preservation theorems, we not only want to ensure that the space is pre-spectral, but also to see that the lattice of compact open sets is obtained through a restriction of the logic. Therefore, one of our main concerns with closure properties is to characterise the lattice of compact sets, which must use properties of the definable sets and cannot rely solely on topological constructions.

5.1. Morphisms.

5.1.1. Spectral Maps. Let us first introduce the notion of morphism between pre-spectral spaces, inherited from the case of spectral spaces [12, Definition 1.2.2]. A map $f: (X, \tau) \rightarrow (Y, \theta)$ is a *spectral map* whenever it is continuous and the pre-image of a compact-open set of Y is a compact-open set of X . We will write **PreSpec** for the category of pre-spectral spaces and spectral maps.

Fact 5.1. *The image of a pre-spectral space through a spectral map is pre-spectral.*

A crucial role of spectral maps is to guard the definition of *pre-spectral subspaces*, mimicking the one of *spectral subspaces* [12, Section 2.1]. A pre-spectral subspace is not only a subset where the induced topology happens to be pre-spectral, but has the additional property that the *inclusion map* is a spectral map.

5.1.2. Logical Maps. In the case of a lpps, a map $f: \langle X, \tau, \mathcal{L} \rangle \rightarrow \langle Y, \theta, \mathcal{L}' \rangle$ is a *logical map* whenever it is continuous and the pre-image of a definable open set of Y is a definable open set of X . A map between logically defined pre-spectral spaces is logical if and only if it is spectral, since compact open subsets and definable open subsets coincide in that case. However, the use of logical maps is to prove that some spaces are pre-spectral by transferring logical properties rather than topological ones.

Fact 5.2. *The image of a lpps $\langle X, \tau, \mathcal{L} \rangle$ through a logical map is a lpps.*

Of particular interest are the logical maps obtained through syntactic constructions. Let us define an *FO-interpretation* $f: X \rightarrow Y$ where $X \subseteq \text{Struct}(\sigma_1)$ and $Y \subseteq \text{Struct}(\sigma_2)$ through ‘relation’ formulæ ρ_R for all $R \in \sigma_2$, where ρ_R has as many free variables as the arity of R , and an additional ‘domain’ formula $\delta \in \text{FO}[\sigma_1]$ with one free variable. The image of a σ_1 -structure $A \in X$ is the σ_2 -structure $f(A)$ with domain $|f(A)| \triangleq \{a \in |A| \mid A \models \delta(a)\}$ and such that $(a_1, \dots, a_n) \in \mathbf{R}^{f(A)} \iff A \models \rho_R(a_1, \dots, a_n)$. This is a simple model of logical interpretations: many different notions can be found in the literature [see 7].

An FO-interpretation $f: X \rightarrow Y$ allows to transfer logical properties from one class of structures to another: if $\varphi \in \text{FO}[\sigma_2]$ is a formula on the structures of Y , then there exists a formula $f^{-1}(\varphi) \in \text{FO}[\sigma_1]$ such that $A \models f^{-1}(\varphi)(\vec{a})$ if and only if $f(A) \models \varphi(f(\vec{a}))$ [23, Section 4.3]; thus, the pre-image of a definable set is definable.

Fact 5.3. *An FO-interpretation is a logical map if and only if it is continuous.*

This provides us with a proof scheme to show that a space $\langle Y, \tau_2, \text{FO}[\sigma_2] \rangle$ is a lpps: first, build a lpps $\langle X, \tau_1, \text{FO}[\sigma_1] \rangle$, then build a FO-interpretation that is surjective and continuous from X to Y , and conclude that Y is a lpps. This is used for instance by Nešetřil and Ossona de Mendez [31, Corollary 10.7] to show that the class of all p subdivisions of finite graphs enjoys homomorphism preservation (using a slightly more general notion of FO-interpretations).

5.2. Relativisation. Preservation theorems do not relativise in general, but the stronger notion of being pre-spectral shows that non trivial sufficient conditions for relativisation exists. However, unlike the theory of spectral spaces, there is not yet a full characterisation of the pre-spectral subsets of a pre-spectral space.

Lemma 5.4 (Sufficient condition for pre-spectral subspaces). *Every positive Boolean combination X of closed subsets of Y and compact-open subsets of Y defines a pre-spectral subspace of Y .*

Proof. It suffices to prove that for all compact-open set U of X the set $U \cap Y$ is a compact-open set of X . Note that in particular this proves $\mathcal{K}^\circ(Y) \supseteq \{U \cap Y \mid U \in \mathcal{K}^\circ(X)\}$. Note that this stronger property is preserved under finite unions and finite intersections. Therefore, we only consider the case of a closed subset, or a compact-open subset. When F is a closed subset of X , we use the fact that the intersection of a compact and a closed subset defines a compact subset. When F is a compact open set of X , we use the fact that $\mathcal{K}^\circ(X)$ is a lattice to conclude. \square

Example 5.5 (Pre-spectral vs. logically presented pre-spectral). An interesting example that will be further discussed in Example 5.7 is $\mathcal{D}_{\leq 2} \subseteq \mathcal{G}$ the set of finite simple graphs of degree bounded by two. It turns out that $\langle \mathcal{D}_{\leq 2}, \tau_{\subseteq}, \text{FO}[\sigma] \rangle$ is a lpps

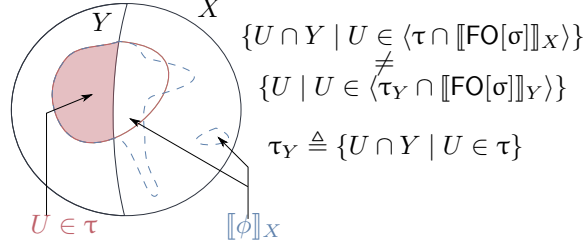


FIGURE 1. Different induced subspace constructions.

by Remark 3.5 and the fact that $\mathcal{D}_{\leq 2}$ is downwards-closed and has the $(\tau_{\subseteq_i}, \text{EFO}[\sigma])$ preservation property [5]. As seen in Example 4.5, the space of downward closures of finite cycles $\langle \downarrow \mathcal{C}, \tau_{\subseteq_i}, \text{FO}[\sigma] \rangle$ is not a lpps.

However, observe that $Y \triangleq \downarrow \mathcal{C}$ is a closed subset of $X \triangleq \mathcal{D}_{\leq 2}$, because $\mathcal{D}_{\leq 2} \setminus \downarrow \mathcal{C}$ is the set of graphs containing the disjoint union of a cycle and an isolated vertex as a subgraph. As we just saw in Lemma 5.4, a closed subset of a pre-spectral space is pre-spectral, thus $(\downarrow \mathcal{C}, \tau_Y)$ where τ_Y is the topology induced by $\downarrow \mathcal{C}$ on $\langle \llbracket \text{FO}[\sigma] \rrbracket_{\mathcal{D}_{\leq 2}} \cap \tau_{\subseteq_i} \rangle$ is a pre-spectral space, albeit not a logically presented one.

The reason behind this apparent discrepancy is that, when restricting the set of structures, more sentences become monotone. In particular, considering Y' as Y with the induced topology from X we have $\mathcal{K}^\circ(Y) = \mathcal{K}^\circ(Y') = \{ \llbracket \varphi \rrbracket_{Y'} \mid \llbracket \varphi \rrbracket_X \in \tau_{\subseteq_i} \} \subsetneq \llbracket \text{FO}[\sigma] \rrbracket_Y \cap \tau_{\subseteq_i}$; see Figure 1 for an illustration.

Proposition 5.6 (Sufficient condition for relativisation). *Let $\langle X, \tau, \text{FO}[\sigma] \rangle$ be a lpps, Y be a Boolean combination of compact-open subsets of X , and θ be the topology induced by τ on Y . Then $\langle Y, \theta, \text{FO}[\sigma] \rangle$ is a lpps.*

Proof. It suffices to prove that any definable open set U of Y is the restriction of some definable open set of X to Y . This stronger hypothesis is stable under finite unions and finite intersections, thus we only need to deal with the cases where Y is a definable open of X or the complement of one.

Let us first consider the case where Y is a definable open set of X . Then $U = U \cap Y$ is the restriction of an open definable set of X to Y . Let us next consider the case where Y is a definable closed set of X . Remark that $V \triangleq U \cup (X \setminus Y)$ is an open set of X , and is still definable. Therefore $U = V \cap Y$ with V a definable open set of X . \square

5.2.1. Pro-constructible Sets. One hope could be more generally to translate the characterisation of spectral subspaces of a spectral space to pre-spectral spaces. In a spectral space, a spectral subspace is a *pro-constructible set*, that is a set written as $\bigcap_{i \in I} (U_i^c \cup V_i)$ where U and V are compact open sets [12, Definition 1.3.10, Proposition 1.3.13, Theorem 2.1.3]. We will use the notation $U \Rightarrow V$ instead of $U^c \cup V$ to understand these sets as satisfying a theory of Horn clauses.

In a lpps $\langle X, \tau, \text{FO}[\sigma] \rangle$, pro-constructible sets correspond precisely to the models of a *first-order theory* of sentences in $\text{FO}[\sigma]$ that are open in τ or closed in τ . Pro-constructible subsets of $\langle X, \tau, \text{FO}[\sigma] \rangle$ are lpps, for any topology τ on the set of structures. This is a consequence of the fact that a pro-constructible set is in particular defined by a first-order theory, hence the Compactness Theorem of

first-order logic still applies to it. However, this relativisation does not generalise, as demonstrated by Example 5.7 below.

Example 5.7 (Pro-constructible sets are not sufficient). Consider again the spaces from Example 5.5. The subset $\mathcal{C} \subseteq \mathcal{D}_{\leq 2}$ consisting of all the finite cycles is definable using a *pro-constructible set*. For instance, one can write that it contains no graph containing a disjoint union of a cycle and a point, and add a constraint that any graph that is not above a cycle of size n contains at least $n + 1$ points. Moreover, the upward closure of a cycle is an open set in $\mathcal{D}_{\leq 2}$, but is a singleton when restricted to \mathcal{C} , proving that the space is not pre-spectral because it has the discrete topology and is infinite.

This counter-example shows that one cannot solely rely on the properties of spectral spaces, but actually have to work to translate their results. Moreover, this example also provides a lpps that is T_0 but not sober. Example 5.7 proves that $\mathcal{D}_{\leq 2}$ is a lpps, and that it has a pro-constructible subset that is not pre-spectral (hence not spectral). However, if $\mathcal{D}_{\leq 2}$ were sober, it would be spectral, and any pro-constructible subset would be spectral, which is a contradiction.

Example 5.7 also shows that we cannot strengthen the hypothesis of Lemma 5.4 to consider arbitrary intersections. One might wonder whether the characterisation of spectral subspaces as sets of the form $\bigcap_{i \in I} U_i \Rightarrow V_i$ where U_i and V_i are compact open sets could lead to a necessary condition on pre-spectral spaces. The following example answers negatively.

Example 5.8 (Pro-constructible sets are not necessary). Consider $X \triangleq \mathbb{N}$ with the Alexandroff topology over its natural order \leq . Consider $Y \triangleq \mathcal{S}(X)$, that is $Y \simeq \mathbb{N} \cup \{\infty\}$ with the Scott topology of the natural order over Y . Both X and Y are Noetherian spaces, hence pre-spectral. Moreover, Y is sober, hence Y is a spectral space. The inclusion of X into Y is spectral because X is Noetherian. Hence X is a pre-spectral subspace of Y . However, notice that if X was obtained as a (possibly infinite) intersection of finite Boolean combinations of compact-open sets of Y , X would be sober [24, Corollary 3.5]. However, $X \neq Y$ and $\mathcal{S}(X) = Y$, which is a contradiction.

5.3. Disjoint Unions and Products. Rather than using an already existing pre-spectral space and considering sub-spaces to build new smaller ones, it can be a rather efficient method to combine existing spaces to build bigger spaces. However, to build preservation theorems out of these constructions, it is necessary to represent those them as spaces of structures over some relational signature, which will be the role of definitions 5.10 and 5.13.

5.3.1. Finite Disjoint Unions. Spectral spaces are closed under arbitrary co-products [12, Corollary 5.2.9], but these co-products are in general not obtained through simple disjoint unions. Indeed, an infinite discrete space is not spectral because it is not compact, but it can be obtained as an arbitrary disjoint union of singletons (which are spectral spaces). The characterisation of co-products of spectral spaces is the goal of an entire section in [12, Section 10.1].

Since we are interested in building new classes of structures, we restrict ourselves to the cases where a concrete representation exists in terms of classes of structures so that we can interpret first-order logic on the co-product; therefore, we only study finite sums. The following lemma is a then direct application of the stability of spectral spaces under finite disjoint union.

Lemma 5.9 (Stability under finite disjoint union). *Let $(X_i, \theta_i)_{i \in I}$ be a family of pre-spectral spaces, where the index set I is finite. The space $\sum_{i \in I} X_i$ with the sum topology is a pre-spectral space.*

Proof. By Fact 4.7, $(\mathcal{S}(X_i), \mathcal{S}(\theta_i))$ is a spectral space. However, spectral spaces are stable under topological sums [12, Corollary 2.4.4], and $\mathcal{S}(\sum_{i \in I} X_i) \simeq \sum_{i \in I} \mathcal{S}(X_i)$ thanks to [18, Fact 8.4.3]. Therefore, by Fact 4.7, $\sum_{i \in I} X_i$ is a pre-spectral space. \square

Definition 5.10 (Logical sum). Let $(\langle X_i, \tau_i, \text{FO}[\sigma_i] \rangle)_{i \in I}$ be a family of spaces. The *logical sum* $\langle X, \tau, \text{FO}[\sigma] \rangle$ is defined as follows:

- (1) The signature σ is the disjoint union of the signatures $(\sigma_i)_{i \in I}$.
- (2) The set X is the union (disjoint by construction) $\bigcup_{i \in I} f_i(X_i)$ where, for all $i \in I$, $f_i : X_i \rightarrow \text{Struct}(\sigma)$ is defined by $|f_i(A)| \triangleq |A|$ and $(a_1, \dots, a_n) \in \mathbf{R}^{f_i(A)}$ if and only if $R \in \sigma_i$ and $(a_1, \dots, a_n) \in \mathbf{R}^A$.
- (3) The topology τ is generated by the sets $f_i(U)$ where $U \in \tau_i$ and $i \in I$.

The logical sum space is a simple translation of the topological sum space, which leads to the following result. In this sense, Lemma 5.9 is mostly a rephrasing.

Proposition 5.11 (Stability under finite logical sum). *Let $(\langle X_i, \tau_i, \text{FO}[\sigma_i] \rangle)_{i \in I}$ be a finite family of lpps. The logical sum of those spaces is a lpps homeomorphic to the sum of those spaces in **PreSpec**.*

Proof. Recall that if $\langle X_i, \tau_i, \text{FO}[\sigma_i] \rangle$ is a lpps, then (X_i, θ_i) is a pre-spectral space, with $\theta_i \triangleq \langle \tau_i \cap \llbracket \text{FO}[\sigma_i] \rrbracket_{X_i} \rangle$ and $\mathcal{K}^\circ(X_i) = \tau_i \cap \llbracket \text{FO}[\sigma_i] \rrbracket_{X_i}$. By Lemma 5.9, the topological sum of these spaces $Y \triangleq \sum_{i \in I} (X_i, \theta_i)$ is a pre-spectral space. Recall that $\langle Y, \tau_Y, \mathcal{K}^\circ(Y) \rangle$ is a lpps. We are going to exhibit a surjective map f from Y to the logical sum X of the X_i .

To a structure $A \in X_i$, we associate $f(A) \triangleq f_i(A)$. By definition, this map is surjective. Notice that this map is actually a *logical map* from $\langle Y, \tau_Y, \mathcal{K}^\circ(Y) \rangle$ to $\langle X, \tau, \text{FO}[\sigma] \rangle$. Indeed, consider a definable open set of X : it is obtained through a sentence $\varphi \in \text{FO}[\sigma]$. However, a simple rewriting allows us to write $\varphi \equiv_X \varphi_1 \vee \dots \vee \varphi_k$ with $\varphi_k \in \text{FO}[\sigma_{i_k}]$ with $i_1, \dots, i_k \in I$. By definition of the sum topology, the sets φ_{i_j} are therefore open in X_{i_j} and thus compact because each $\langle X_{i_j}, \tau_{i_j}, \text{FO}[\sigma_{i_j}] \rangle$ is a lpps. In particular, the pre-image of an open definable set of X is a compact open set of Y , proving that f is continuous, and logical. \square

In the case of products, a sentence over a product is not simply obtained by projecting on each component. This is handled in our proof of Proposition 5.14 by reducing the first-order theory of the product to the first-order theories of its components thanks to Feferman-Vaught decompositions [15, 30].

Again, we first state a direct application of the stability of spectral spaces under products.

Lemma 5.12 (Stability under products). *Let $(X_i, \theta_i)_{i \in I}$ be a family of pre-spectral spaces. The product space $X \triangleq \prod_{i \in I} X_i$ with the product topology is pre-spectral. Moreover, compact open sets of X are obtained as finite union of $p_i^{-1}(K)$ with $i \in I$ and $K \in \mathcal{K}^\circ(X_i)$ where p_i is the projection over the i th component.*

Proof. By Fact 4.7, $(\mathcal{S}(X_i), \mathcal{S}(\theta_i))$ is a spectral space. Since spectral spaces are stable under products [12, Theorem 2.2.1], and $\mathcal{S}(\prod_{i \in I} X_i) \simeq \prod_{i \in I} \mathcal{S}(X_i)$ [18, Theorem 8.4.8]. Therefore, using Fact 4.7, $\prod_{i \in I} X_i$ is a pre-spectral space.

Consider a compact open set U of the product and $i \in I$, the image $p_i(U)$ is an open set of X_i by definition of the product topology, and is compact because p_i is continuous. Conversely, let $i \in I$ and let U_i be a compact open set of X_i , define $V_j \triangleq X_j$ for $j \neq i \in I$, and $V_i \triangleq U_i$. Then $p_i^{-1}(U_i) = \prod_{i \in I} V_i$ is compact as a product of compact spaces thanks to Tychonoff's Theorem. Moreover, it is an open set because p_i is continuous. \square

Definition 5.13 (Logical product). Let $(\langle X_i, \tau_i, \text{FO}[\sigma_i] \rangle)_{i \in I}$ be a family of spaces. The *logical product* $\langle X, \tau, \text{FO}[\sigma] \rangle$ is defined as follows:

- (1) The signature σ is the disjoint union of the signatures $(\sigma_i)_{i \in I}$ with additional unary predicates ε_i for each $i \in I$.
- (2) The set X is the image of $\prod_{i \in I} X_i$ through the map $f: \prod_{i \in I} X_i \rightarrow \text{Struct}(\sigma)$ that associates to each $(A_i)_{i \in I}$ the disjoint union of the structures A_i with ε_i true on the structure A_i for $i \in I$.
- (3) The topology τ generated by the sets U such that $f^{-1}(U)$ is an open set of $\prod_{i \in I} (X_i, \tau_i)$.

Proposition 5.14 (Stability under finite logical product). *Let $(\langle X_i, \tau_i, \text{FO}[\sigma_i] \rangle)_{i \in I}$ be a finite family of lpps. The logical product of those spaces is a lpps homeomorphic to the product of the spaces X_i in **PreSpec**.*

Proof. We consider a binary product and by an immediate induction conclude that the theorem holds for finite products. Consider X the logical product of two lpps $\langle X_1, \tau_1, \text{FO}[\sigma_1] \rangle$ and $\langle X_2, \tau_2, \text{FO}[\sigma_2] \rangle$. We prove that the map $f: X_1 \times X_2 \rightarrow X$ that associates to each (A_1, A_2) the disjoint union $A_1 \uplus A_2$ is a logical map. This will allow us to conclude because $X_1 \times X_2$ is pre-spectral as a product of two pre-spectral spaces by Lemma 5.12.

To this end, we prove that the pre-image of a non-empty definable open set of X is a compact open set of $X_1 \times X_2$. Consider such a definable open set U : by definition of the topology over the logical product, it is the image of an open set of $(X_1, \tau_1) \times (X_2, \tau_2)$. Because U is definable, $U = \llbracket \varphi \rrbracket_X$ for some $\varphi \in \text{FO}[\sigma]$. By the Feferman-Vaught Decomposition Theorem over finite disjoint unions [15], there exists finitely many sentences $(\psi_i^j)_{1 \leq i \leq n}^{j \in \{1, 2\}}$ and a Boolean function $\beta: \{0, 1\}^{2n} \rightarrow \{0, 1\}$ such that

$$\forall x \in X_1, y \in X_2, f(x, y) \models \varphi \iff \beta((x \models \psi_i^1)_{1 \leq i \leq n}; (y \models \psi_i^2)_{1 \leq i \leq n}) = 1. \quad (6)$$

We build two equivalence relations of finite index respectively on X_1 and X_2 . The relation \equiv_1 over X_1 is defined by $x \equiv_1 x'$ if $\forall 1 \leq i \leq n, x \models \psi_i^1 \iff x' \models \psi_i^1$. The relation \equiv_2 over X_2 is defined by $y \equiv_2 y'$ if $\forall 1 \leq i \leq n, y \models \psi_i^2 \iff y' \models \psi_i^2$. We let $(x, y) \equiv (x', y')$ if and only if $x \equiv_1 x'$ and $y \equiv_2 y'$. Thus, by (6), if $f(x, y) \models \varphi$ and $(x, y) \equiv (x', y')$, then $f(x', y') \models \varphi$. This proves that $f^{-1}(U)$ is a subset of $X_1 \times X_2$ that is saturated for \equiv .

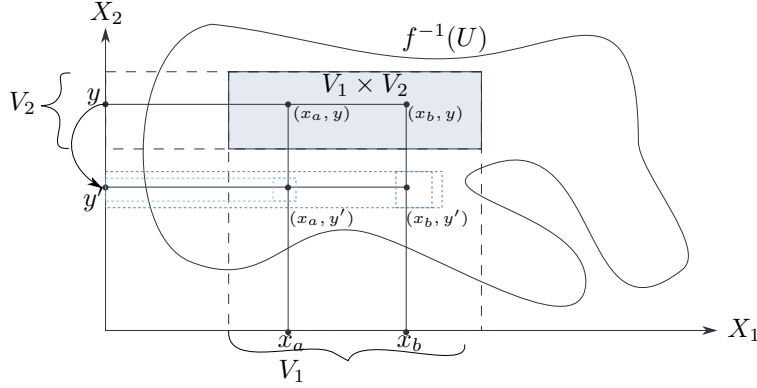


FIGURE 2. Saturation of maximal cylinder open subsets in the case of a binary product $X \times Y$.

For $i \in \{1, 2\}$, a subset $U \subseteq X_i$ that is saturated for \equiv_i is definable in $\text{FO}[\sigma_i]$ since it can be obtained as a finite Boolean combination of the sets $(\llbracket \psi_j^i \rrbracket_{X_i})_{1 \leq j \leq n}$. Thus, if $V_1 \in \tau_1$, $V_2 \in \tau_2$, and $V_1 \times V_2$ is saturated for \equiv , then V_1 (resp. V_2) is definable in X_1 (resp. X_2). Hence using the fact that X_1 and X_2 are lpps, V_1 and V_2 are compact open sets of X_1 and X_2 . By Tychonoff's Theorem the product $V_1 \times V_2$ is compact in $X_1 \times X_2$. We are going to prove that $f^{-1}(U)$ is actually a finite union of \equiv -saturated cylinder sets of this form and conclude using the fact that a finite union of compact open sets is a compact open set.

First of all, because $f^{-1}(U)$ is open in $(X_1, \tau_1) \times (X_2, \tau_2)$, it is obtained as a union of sets of the form $V_1 \times V_2 \subseteq f^{-1}(U)$ with $V_1 \in \tau_1$ and $V_2 \in \tau_2$ and both are non-empty. Consider a set $V_1 \times V_2 \subseteq f^{-1}(U)$ with $V_1 \in \tau_1$ and $V_2 \in \tau_2$ that is *maximal* for inclusion; such a set exists by Zorn's Lemma. We are going to prove that $V_1 \times V_2$ is \equiv -saturated.

Assume by contradiction that $V_1 \times V_2$ is non empty and is not \equiv -saturated; without loss of generality, assume that V_2 is not \equiv_2 -saturated. Following Figure 2, this shows that there exists $y \in V_2$ and $y' \notin V_2$ such that $y \equiv_2 y'$. Remark that for any $x \in V_1$, $(x, y) \equiv (x, y')$, hence $f(x, y) \in U$ implies $f(x, y') \in U$.

Because $f^{-1}(U)$ is open in $(X_1, \tau_1) \times (X_2, \tau_2)$, whenever $(x, y') \in f^{-1}(U)$, there exists a cylinder $V_1^{(x, y')} \times V_2^{(x, y')}$ with $V_1^{(x, y')} \in \tau_1$, $V_2^{(x, y')} \in \tau_2$ such that $(x, y') \in V_1^{(x, y')} \times V_2^{(x, y')} \subseteq f^{-1}(U)$. We illustrate this using two points x_a and x_b in Figure 2. Notice that $V_2^{(x, y')}$ is an open neighborhood of y' for all $x \in V_1$.

Because \equiv_1 is of finite index, there exists a finite set $F \subseteq V_1$ such that $\forall x \in V_1, \exists x' \in F, x \equiv_1 x'$. We define $W \triangleq \bigcap_{x' \in F} V_2^{(x', y')}$, which is an open set since it is a finite intersection of open sets. We claim that $V_1 \times W \subseteq f^{-1}(U)$. To prove this fact notice that if $(a, b) \in V_1 \times W$, then $(a, y) \in V_1 \times V_2$. Consider $a' \in F$ such that $a' \equiv_1 a$, we have $(a', y) \in V_1 \times V_2$ and therefore $(a', y') \in V_1^{(a', y')} \times V_2^{(a', y')} \subseteq f^{-1}(U)$. In particular, as $b \in W \subseteq V_2^{(a', y')}$, we conclude that $(a', b) \in f^{-1}(U)$. Because $a' \equiv_1 a$, this proves that $(a, b) \in f^{-1}(U)$.

Hence, we can build the set $V_1 \times (V_2 \cup W)$ which strictly contains $V_1 \times V_2$, but is still included in $f^{-1}(U)$ which is in contradiction with the maximality of $V_1 \times V_2$.

We have proven that the maximal cylinder sets included in $f^{-1}(U)$ are \equiv -saturated. Remark that there can be only finitely many distinct \equiv -saturated sets in $X_1 \times X_2$. Hence, writing $f^{-1}(U)$ as the finite union of maximal cylinder sets shows that $f^{-1}(U)$ is a finite union of \equiv -saturated cylinder sets, which allows us to conclude that $f^{-1}(U)$ is compact.

We have proven that f is a surjective logical map and therefore that its image is a lpps. We can conclude that the logical product is a lpps.

Let us turn to the statement that X and $X_1 \times X_2$ are homeomorphic. The map f is bijective by construction, and we already know that it is continuous by definition of a logical product, thus it remains to show that f^{-1} is continuous.

Consider a set $K \times X_2$ where K is compact in X_1 . The set K is then definable using a sentence $\varphi \in \text{FO}[\sigma_1]$ since X_1 is a lpps. Build the sentence $\psi \in \text{FO}[\sigma]$ obtained by relativising the quantifiers in φ to elements where ε_1 holds. By construction, $\llbracket \psi \rrbracket_X = f(K \times X_2)$ and by definition of the topology over X , ψ is open in X . A compact open set of $X_1 \times X_2$ is a finite union of sets of the form $K \times X_2$ and $X_1 \times K$ where $K \in \mathcal{K}^\circ(X_1)$ (resp. $K \in \mathcal{K}^\circ(X_2)$) thanks to Lemma 5.12. We conclude that the image through f of a compact open set is a compact open set. In particular, f^{-1} is continuous. Hence X is homeomorphic to the product of X_1 and X_2 in the category **PreSpec**. \square

In particular, Proposition 5.14 proves that logical products provide a concrete representation of products in the category **PreSpec** as a space of structures. One can wonder why the case of infinite products, that is seems similar, cannot be considered. The issue in a generalisation arises from the Feferman-Vaught Theorem, that does not allow us to handle an equality relation over the infinite product but only one distinct equality relation per position in the product. Here is an example where the infinite product is not lpps.

Example 5.15. Consider $I = \mathbb{N}$, $X_i \triangleq \text{Fin}(\sigma_i)$ with $\sigma_i \triangleq \emptyset$, and let τ be the Alexandroff topology for the induced substructure ordering \subseteq_i . Then for any $i \in I$ and $A, B \in X_i$, $A \subseteq B$ if and only if $|A|$ is of cardinality less or equal that of $|B|$. Hence (X_i, \subseteq_i) is order isomorphic to $(\mathbb{N}_{>0}, \leq)$ and is a well-quasi-order. Therefore, each $(X_i, \tau_{\subseteq_i})$ is Noetherian and each $\langle X_i, \tau, \text{FO}[\sigma_i] \rangle$ is thus a lpps.

Consider the logical product space Z of the family $(X_i)_{i \in I}$, thus using the signature $\sigma = \{\epsilon_i \mid i \in I\}$. We define the $\text{FO}[\sigma]$ formulæ $\varphi \triangleq \exists x. \exists y. \neg(x = y)$, $\psi_i \triangleq \exists x. \exists y. \varepsilon_i(x) \wedge \varepsilon_i(y) \wedge \neg(x = y)$ for $i \in I$ and $\theta_{i,j} \triangleq \exists x. \exists y. \varepsilon_i(x) \wedge \varepsilon_j(y)$ for $i < j \in I$.

Notice that $\llbracket \varphi \rrbracket_Z = \bigcup_{i \in I} \llbracket \psi_i \rrbracket_Z \cup \bigcup_{i < j \in I} \llbracket \theta_{i,j} \rrbracket_Z$, that all of the above sentences define open sets in Z , but that there is no finite subcover. Hence there exists a sentence defining a non-compact open set, thus Z is not a lpps.

6. LOGICAL CLOSURE

Consider a set Z equipped with a bounded sublattice \mathcal{L} of $\wp(Z)$. In this section, we provide a way to consider the closure of a space $X \subseteq Z$ in a suitable topology so that if X is a lpps, then its closure also is. Let us write $\tau_{\mathcal{L}} \triangleq \langle \mathcal{L} \cup \{U^c \mid U \in \mathcal{L}\} \rangle$ for the topology generated by the sets of \mathcal{L} and their complements. We call the closure

\overline{X} of X in $(Z, \tau_{\mathcal{L}})$ its *logical closure*. Note that the closure of a set is empty if and only if the set itself is empty, and more generally the following holds.

Fact 6.1. *For all open sets $U \in \tau_{\mathcal{L}}$, $U \cap X$ is empty if and only if $U \cap \overline{X}$ is empty.*

We show that lpps are stable under logical closures. For $X \subseteq Z$ and a sublattice \mathcal{L} of $\wp(Z)$, we write $\mathcal{L}_X \triangleq \{U \cap X \mid U \in \mathcal{L}\}$ for the lattice induced by X .

Proposition 6.2 (Stability under logical closure). *Let $X \subseteq Y \subseteq \overline{X}$ and τ be a topology on Y . If $\langle X, \tau_X, \mathcal{L}_X \rangle$ is a lpps for the topology τ_X induced by τ on X , then so is $\langle Y, \tau, \mathcal{L}_Y \rangle$. If \mathcal{L}' is a sublattice of \mathcal{L} and \mathcal{L}'_X is a diagram base of X then \mathcal{L}'_Y is a diagram base of Y .*

Proof. Let $U \in \mathcal{L}$ such that $U \cap Y \in \tau$. In particular, $U \cap Y \in \mathcal{L}_Y$ is a definable open set of Y . By restriction, $U \cap X \in \mathcal{L}_X$ is a definable open set of X . Using Alexander's Subbase Lemma, let $(U_i \cap Y)_{i \in I}$ be an open cover of U in τ . By restricting to X , we can use the fact that X is a lpps to extract a finite subset $I_0 \subseteq_{\text{fin}} I$ such that $(U_i \cap X)_{i \in I_0}$ is an open cover of $U \cap X$ in τ_X . In particular, this proves that $(U \cap \bigcap_{i \in I_0} U_i^c) \cap X = \emptyset$. Notice that this set is an open set in $\tau_{\mathcal{L}}$, hence we can conclude by Fact 6.1 that $U \cap \overline{X} \subseteq \bigcup_{i \in I_0} U_i \cap \overline{X}$. Since $Y \subseteq \overline{X}$, $(U_i \cap Y)_{i \in I_0}$ is an open cover of $U \cap Y$ in τ . Therefore, the definable open sets of Y are compact and $\langle Y, \tau, \mathcal{L} \rangle$ is a lpps.

Assume that \mathcal{L}'_X defines a diagram base of X . Consider a definable open set $U \cap Y \in \mathcal{L}_Y$ of Y where $U \in \mathcal{L}$. Remark that $U \cap X \in \mathcal{L}_X$ is a definable open set in X , hence there exists a family $(V_i)_{i \in I}$ of elements of \mathcal{L}'_X such that $U \cap X = \bigcup_{i \in I} V_i \cap X$. Since X is a lpps, $U \cap X$ is compact, and therefore $U \cap X = \bigcup_{i \in I_0} V_i \cap X$ for some finite $I_0 \subseteq_{\text{fin}} I$. Because \mathcal{L}'_X is a lattice, we conclude that $U \cap X = V \cap X$ for $V \triangleq (\bigcup_{i \in I_0} V_i)$ such that $V \cap X \in \mathcal{L}'_X$. Using Fact 6.1, this proves that $U \cap \overline{X} = V \cap \overline{X}$, and because $Y \subseteq \overline{X}$ we have $U \cap Y = V \cap Y$ where $V \cap Y \in \mathcal{L}'_Y$. Hence, \mathcal{L}'_Y is a diagram base of Y . \square

We now show that Proposition 6.2 allows to restate known preservation theorems and derive new ones. We consider the case where $Z = \text{Struct}(\sigma)$ and $\mathcal{L} = \llbracket \text{FO}[\sigma] \rrbracket_{\text{Struct}(\sigma)}$, and we write τ_{FO} for the topology $\tau_{\mathcal{L}}$.

Let us define $\text{FMP}(\sigma) \subseteq \text{Struct}(\sigma)$ as the set of structures whose first-order theory satisfies the *finite model property*: any definable subset of $\text{FMP}(\sigma)$ has a finite model. We prove that homomorphism preservation can be lifted from $\text{Fin}(\sigma)$ (where it holds by Rossman's Theorem) to $\text{FMP}(\sigma)$ in Corollary 6.3. To our knowledge this is a new result.

This follows from Proposition 6.2 and the fact that $\text{FMP}(\sigma)$ is the closure of $\text{Fin}(\sigma)$ in the topology τ_{FO} . Indeed, consider a structure $A \in \overline{\text{Fin}(\sigma)}$ and a sentence $\varphi \in \text{FO}[\sigma]$ such that $A \models \varphi$. Because $\llbracket \varphi \rrbracket_{\text{Struct}(\sigma)}$ is an open set of $\text{Struct}(\sigma)$ for τ_{FO} , this means that $\llbracket \varphi \rrbracket_{\text{Struct}(\sigma)} \cap \text{Fin}(\sigma) \neq \emptyset$ by definition of the topological closure, hence φ has a finite model. Conversely, consider a structure A enjoying the finite model property and let U be a definable open set of τ_{FO} that contains A . By the finite model property, there exists $B \in \text{Fin}(\sigma)$ such that $B \in U$. Hence, A is in the closure of $\text{Fin}(\sigma)$.

Corollary 6.3 (Homomorphism preservation for structures with the finite model property). *FMP(σ) has the $(\tau_{\rightarrow}, \text{EPFO}[\sigma])$ preservation property.*

Proof. By Rossman's Theorem [33], $\text{Fin}(\sigma)$ has the $(\tau_{\rightarrow}, \text{EPFO}[\sigma])$ preservation property. Since $\text{Fin}(\sigma)$ is downwards-closed for \rightarrow inside $\text{Struct}(\sigma)$, Remark 3.5 shows that the space $\langle \text{Fin}(\sigma), \tau_{\rightarrow}, \text{FO}[\sigma] \rangle$ is a lpps. Moreover, notice that $\text{FMP}(\sigma) = \overline{\text{Fin}(\sigma)}$ and $\text{EPFO}[\sigma]$ defines a diagram base of $\text{Fin}(\sigma)$. Leveraging Proposition 6.2, we conclude that $\langle \text{FMP}(\sigma), \tau_{\rightarrow}, \text{FO}[\sigma] \rangle$ is a lpps and that $\text{EPFO}[\sigma]$ defines a diagram base of $\text{FMP}(\sigma)$. Now, by Theorem 3.4, $\text{FMP}(\sigma)$ has the $(\tau_{\rightarrow}, \text{EPFO}[\sigma])$ preservation property. \square

Let $\text{Fin}^{\sqcup}(\sigma)$ be the set of countable disjoint unions of finite structures over a finite relational signature σ . We state in Corollary 6.4 another consequence of Rossman's Theorem and Proposition 6.2, using the fact $\text{Fin}(\sigma) \subsetneq \text{Fin}^{\sqcup}(\sigma) \subsetneq \text{FMP}(\sigma) = \overline{\text{Fin}(\sigma)}$; the same result was first shown by Nešetřil and Ossona de Mendez in [31, Theorem 10.6].

Corollary 6.4 (Homomorphism preservation for countable unions of finite structures). *$\text{Fin}^{\sqcup}(\sigma)$ has the $(\tau_{\rightarrow}, \text{EPFO}[\sigma])$ preservation property.*

Proof. Clearly, $\text{Fin}(\sigma) \subsetneq \text{Fin}^{\sqcup}(\sigma)$. Regarding $\text{Fin}^{\sqcup}(\sigma) \subsetneq \text{FMP}(\sigma)$, we want to prove that for all $A \in \text{Fin}^{\sqcup}(\sigma)$ and for all $\varphi \in \text{FO}[\sigma]$, if $A \models \varphi$, then there exists a finite structure A_{fin} such that $A_{\text{fin}} \models \varphi$.

This is an application of the locality of first-order logic. By Gaifman's Locality Theorem [27, Theorem 4.22], it suffices to prove this assuming that φ is a Boolean combination of *basic local sentences*, i.e., of sentences of the form

$$\exists x_1, \dots, x_s . \left(\bigwedge_{1 \leq i \leq s} \alpha^{(r)}(x_i) \wedge \bigwedge_{1 \leq i < j \leq s} d^{>2r}(x_i, x_j) \right), \quad (7)$$

where the $\alpha^{(r)}(x_i)$ formula is r -local around x_i [see e.g., 28, Section 4.5]. Notice that if a basic local sentence of the form (7) is satisfied on a structure B , then for all structures C , the local sentence is also satisfied in the disjoint union $B \sqcup C$. Conversely, if a basic local sentence of quantifier rank less than s is satisfied on a structure B , then it is satisfied in a union of at most s connected components of B . In particular, for a structure $A \in \text{Fin}^{\sqcup}(\sigma)$, consider the sequence $(A_n)_{n \in \mathbb{N}_{\geq 1}}$ where each A_n is the disjoint union of all components of A of size at most n . For a basic local sentence β , if $A \models \beta$ then $\exists n_0, \forall n \geq n_0, A_n \models \beta$. Conversely, if $A_n \models \beta$ for some n , then $A \models \beta$, hence if $A \models \neg\beta$, then $\forall n \geq 1, A_n \models \neg\beta$. We have proven that when β is a basic local sentence or the negation of a basic local sentence, $A \models \beta$ implies $\exists n_0, \forall n \geq n_0, A_n \models \beta$. Notice that this property is stable under finite disjunction and finite conjunctions: this shows the announced result for Boolean combinations of basic local sentences.

As a consequence, $\text{Fin}(\sigma) \subsetneq \text{Fin}^{\sqcup}(\sigma) \subsetneq \overline{\text{Fin}(\sigma)}$ and we apply the same reasoning as in Corollary 6.3. \square

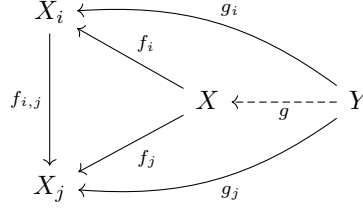


FIGURE 3. The commutative diagram of a projective system.

7. PROJECTIVE SYSTEMS

A natural construction in the category of topological spaces is the *projective limit*, and the category **Spec** of spectral spaces and spectral maps is closed under this construction [12, Corollary 2.3.8]. As an illustration, we show in Section 7.2 that $\langle \text{Fin}(\sigma), \tau_{\rightarrow}, \text{FO}[\sigma] \rangle$ is the projective limit of a system of Noetherian spaces, which provides an alternative understanding of Rossman's Theorem [33]. In fact, as we show in Section 7.3, any pre-spectral space is the limit of a projective system of Noetherian spaces.

7.1. Projective Systems. A *projective system* \mathcal{F} in a category **C** assigns to each element i of a directed partially ordered set I an object X_i and to each ordered pair $i \leq j$ a so-called *bonding map* $f_{i,j}: X_i \rightarrow X_j$ so that, for all $i, j, k \in I$ with $k \leq j \leq i$, we have $f_{i,i} = \text{id}_{X_i}$ and $f_{j,k} \circ f_{i,j} = f_{i,k}$. The *projective limit* of a projective system \mathcal{F} is an object X with maps $f_i: X \rightarrow X_i$ compatible with the system \mathcal{F} , which means that, for all $i \geq j$, $f_{i,j} \circ f_i = f_j$. Moreover, X satisfies a universal property: whenever $\{g_i: Y \rightarrow X_i\}_{i \in I}$ is a family of maps compatible with \mathcal{F} , there exists a unique map $g: Y \rightarrow X$ such that $g_i = f_i \circ g$ for all $i \in I$.

Recall that we write **PreSpec** for the category of pre-spectral spaces and spectral maps. The category **Spec** of spectral spaces and spectral maps is closed under projective systems [12, Corollary 2.3.8], meaning that every projective system in **Spec** has a limit in **Spec**. This is not the case for **PreSpec**, as can be seen by the following adaptation of [36, Example 3].

Example 7.1. Consider as in Example 4.2 the spaces (\mathcal{C}, τ_n) of finite cycles with the topology generated using sentences having models of size at most n or finitely many counter-models (when $n = 0$, this is the co-finite topology). Recall that (\mathcal{C}, τ_n) is a Noetherian space for all $0 \leq n < \infty$, and is therefore pre-spectral. Observe that the identity maps between (\mathcal{C}, τ_i) and (\mathcal{C}, τ_j) are defining a projective system in **PreSpec**. Assume that $\{f_i: X \rightarrow (\mathcal{C}, \tau_i)\}_{i \in I}$ is the limit of this projective system in **PreSpec**. The commutation property $\text{id}_{i,j} \circ f_i = f_j$ shows that all $f_i = f_j$ for $i, j \in I^2$ and we let $f \triangleq f_i$ for some $i \in I$. The function f is continuous when endowing \mathcal{C} with the discrete topology. Indeed, consider $C \in \mathcal{C}$ a given cycle, then $\{C\}$ is open in $(\mathcal{C}, \tau_{|C|})$ and therefore $f^{-1}(\{C\})$ is open in X . Since X is a pre-spectral space, it is compact, hence its image through a continuous function is compact, but \mathcal{C} with the discrete topology is not compact which is a contradiction.

Let us introduce here the category of topological spaces and continuous maps, denoted by **Top**. A projective system in **PreSpec** is a projective system of topological spaces in **Top**. A projective system in **PreSpec** always has a limit when considered as a projective system in **Top**; we give a sufficient condition for this space to be the limit in **PreSpec**.

Lemma 7.2 (Transfer of projective limits). *Let \mathcal{F} be a projective system of pre-spectral spaces in **PreSpec**. If $\{f_i: X \rightarrow X_i\}_{i \in I}$ is the limit of \mathcal{F} in **Top** where the maps $f_i: X \rightarrow X_i$ are spectral, then it is the limit of \mathcal{F} in **PreSpec**. Moreover, $\mathcal{K}^\circ(X) = \bigcup_{i \in I} \{f_i^{-1}(V) \mid V \in \mathcal{K}^\circ(X_i)\}$.*

Proof. Since X is the projective limit in **Top**, we can assume that it is the standard projective limit, that is $X \triangleq \{\vec{x} \in \prod_{i \in I} X_i \mid \forall i \leq j \in I, x_j = f_{i,j}(x_i)\}$ with the topology inherited from $\prod_{i \in I} X_i$, and that f_i is the projection over component i , that is $f_i(\vec{x}) = x_i$ [18, Exemple 4.12.8]. We will now prove that the set of compact-open sets of X is a bounded sublattice generating the topology and thus that X is pre-spectral.

Consider an open set U of X : by definition of the topology of X , it is a union $\bigcup_{i \in I} f_i^{-1}(U_i)$ where each U_i for $i \in I$ is open in X_i . For every $i \in I$, X_i is pre-spectral, meaning that $\mathcal{K}^\circ(X_i)$ generates the topology of X_i , thus each U_i can be written as a union of compact opens of X_i . Note that for all $i \in I$ and all compact open subsets K of X_i , $f_i^{-1}(K)$ is a compact open set in X since f_i is spectral. Therefore, U itself can be written as a union of pre-images $f_i^{-1}(K)$, which are compact open in X .

Consider now a compact open set U of X . We have just seen that it is a union of compact sets of X , and since U is compact, we can extract a finite cover, so U is a finite union of compact open sets of the form $f_{i_j}^{-1}(K_j)$ for $1 \leq j \leq n$ where $\forall 1 \leq j \leq n, i_j \in I$ and $\forall 1 \leq j \leq n, K_j \in \mathcal{K}^\circ(X_{i_j})$. Since I is directed, there is an index i above all i_j for $1 \leq j \leq n$. Let $K'_j \triangleq f_{i,i_j}^{-1}(K_j)$ for $1 \leq j \leq n$; it is a compact open of X_i because f_{i,i_j} is spectral. Notice that $f_{i,i_j} \circ f_i = f_{i_j}$ hence $f_i^{-1}(K'_j) = f_{i_j}^{-1}(K_j)$. Hence $U = f_i^{-1}(\bigcup_{j=1}^n K'_j)$ where $\bigcup_{j=1}^n K'_j \in \mathcal{K}^\circ(X_i)$ since X_i is pre-spectral and therefore $\mathcal{K}^\circ(X_i)$ is a lattice. This shows that $\mathcal{K}^\circ(X) = \{f_i^{-1}(K) \mid \forall i \in I, \forall K \in \mathcal{K}^\circ(X_i)\}$, and that $\mathcal{K}^\circ(X)$ generates the topology of X .

Consider two compact open sets of X , U and U' . They are obtained as pre-images $U = f_i^{-1}(V)$ and $U' = f_j^{-1}(V')$ with $V \subseteq X_i$ and $V' \subseteq X_j$ compact open. Because I is directed, there exists $k \in I$ such that $k \geq i, j$. Because $f_j \circ f_{k,j} = f_k$, we can see that $U = f_k^{-1}(f_{k,i}^{-1}(V))$ and $U' = f_k^{-1}(f_{k,j}^{-1}(V'))$. Because the maps are morphisms and compact open sets of the pre-spectral space X_k form a lattice, we can see that their intersection and union form compact open sets of X_k , and therefore compact open sets are stable under binary intersection and binary unions in X . This shows that $\mathcal{K}^\circ(X)$ is indeed a lattice. It remains to prove that it is a bounded one. Since I is non empty, let us pick $i \in I$ and notice that $f_i^{-1}(X_i) = X$ since X_i is compact and f_i is spectral: this proves that X is compact. Hence $\mathcal{K}^\circ(X)$ is a bounded sublattice of $\wp(X)$.

We finally prove that X is the limit in **PreSpec**. Consider $\{g_i: Y \rightarrow X_i\}_{i \in I}$ any family of spectral maps compatible with \mathcal{F} , namely such that $g_j \circ f_{i,j} = g_i$ for all $i \leq j \in I$. In particular, the maps g_i are continuous. By the universal property of X as a limit in **Top**, there exists a unique map $g: Y \rightarrow X$ such that $f_i \circ g = g_i$ for every $i \in I$. In particular, consider a compact open set of X : it can

written as $f_i^{-1}(K)$ for some $i \in I$ and $K \in \mathcal{K}^\circ(X)$, hence $g^{-1}(f_i^{-1}(K)) = g_i^{-1}(K)$ is compact because g_i is spectral. Hence, g is spectral as well. The uniqueness of g follows from the uniqueness in **Top**. This shows that $\{f_i: X \rightarrow X_i\}_{i \in I}$ is the limit of \mathcal{F} in **PreSpec**. \square

7.2. Application to the Homomorphism Preservation Theorem. Throughout this section, we fix a finite relational signature σ and a downwards-closed subset X of $\text{Fin}(\sigma)$ for the homomorphism ordering \rightarrow , i.e., X is co-homomorphism closed. We will see how Rossman's Theorem can be explained as the existence of a projective limit.

7.2.1. n -Homomorphisms. Let us define the *tree-depth* $\text{td}(A)$ of a finite structure A as the tree-depth $\text{td}(\mathcal{G}(A))$ of its associated *Gaifman graph* $\mathcal{G}(A)$ [28, Definition 4.1]. Following the idea of the original proof by Rossman [33, Section 3.2], we are going to use quasi-orders that are *coarser* than the homomorphism quasi-order, and refine those progressively. For every $n \in \mathbb{N}$, we define $A \rightarrow_n B$ if for every structure C of tree-depth at most n , $C \rightarrow A$ implies $C \rightarrow B$. Note that on finite structures, $A \rightarrow B \iff A \rightarrow_{\text{td}(A)} B$. Then the intersection of all the \rightarrow_n relations is \rightarrow . Let us consider the corresponding Alexandroff topologies: $X \triangleq \langle X, \tau_{\rightarrow}, \text{FO}[\sigma] \rangle$ and for $n \in \mathbb{N}$, let $X_n \triangleq \langle X, \tau_{\rightarrow_n}, \text{FO}[\sigma] \rangle$.

For $A \in \text{Fin}(\sigma)$ and $n \geq 1$, there exists a structure $\text{Core}^n(A)$ of tree-depth at most n such that $A \rightarrow_n \text{Core}^n(A)$, $\text{Core}^n(A) \rightarrow_n A$, and furthermore $A \rightarrow_n B$ if and only if $\text{Core}^n(A) \rightarrow B$ [33, definitions 3.6 and 3.10 and Lemma 3.11]. Notice that for all $n \geq 1$, if $A \in X$ then $\text{Core}^n(A) \in X$ since X is downwards closed.

7.2.2. Rossman's Lemma. In his paper [33], Rossman provides a function $\rho: \mathbb{N} \rightarrow \mathbb{N}$ and relates indistinguishability in the fragment $\text{FO}_n[\sigma]$ of first-order logic with at most n quantifier alternations with $\rho(n)$ -homomorphism equivalence [33, Corollary 5.14]. We state this result in a self-contained manner below [see also 31, Theorem 10.5].

Lemma 7.3 (Rossman's Lemma [33]). *There exists $\rho: \mathbb{N} \rightarrow \mathbb{N}$ such that, for all $n \in \mathbb{N}$, if $\varphi \in \text{FO}_n[\sigma]$ is closed under homomorphisms, then it is closed under $\rho(n)$ -homomorphisms.*

Proof. We prove that the actual statement of Rossman's Lemma in [33, Corollary 5.14] implies Lemma 7.3. Corollary 5.14 in [33] states that for every pair $A, B \in \text{Fin}(\sigma)$ such that $A \rightarrow_{\rho(n)} B$ and $B \rightarrow_{\rho(n)} A$ there exists two finite structures \tilde{A} homomorphically equivalent to A and \tilde{B} homomorphically equivalent to B (more precisely, A and B are *retracts* of \tilde{A} and \tilde{B}), such that \tilde{A} and \tilde{B} satisfy the same $\text{FO}_n[\sigma]$ sentences.

Assume now that $\varphi \in \text{FO}_n[\sigma]$ is closed under homomorphisms, $A \models \varphi$, and $A \rightarrow_{\rho(n)} C$. Let us define $B \triangleq \text{Core}^{\rho(n)}(A)$: then $A \rightarrow_{\rho(n)} B$, $B \rightarrow_{\rho(n)} A$ and furthermore $B \rightarrow C$. Apply [33, Corollary 5.14] to A and B . Then $\tilde{A} \models \varphi$ because $A \rightarrow \tilde{A}$ and φ is closed under homomorphisms. Therefore $\tilde{B} \models \varphi$ because \tilde{A} and \tilde{B} are n -elementary equivalent. Finally, $\tilde{B} \rightarrow B \rightarrow C$, thus $C \models \varphi$ because φ is closed under homomorphisms. This shows that φ is closed under $\rho(n)$ -homomorphisms. \square

Rossmann's Lemma is the combinatorial heart of Rossmann's Theorem, so the developments in this section are really meant to show how the pre-spectral framework can capture Rossmann's arguments translating the technical statement from Lemma 7.3 into a proof of homomorphism preservation in the finite.

7.2.3. Projective System. We are now ready to obtain X as a limit of a projective system in **PreSpec**. We are going to exploit Lemma 7.3 through the definition of the topological spaces $Y_n \triangleq \langle X, \tau_{\rightarrow}, \mathbf{FO}_n[\sigma] \rangle$ for all n . We will use the following consequence of Rossmann's Lemma.

Claim 7.4. $\forall n \geq 1, \mathcal{K}^\circ(Y_n) \subseteq \mathcal{K}^\circ(X_{\rho(n)}) \subseteq \mathcal{K}^\circ(X)$.

Proof. The proof is split in three parts: we first show that X_m is Noetherian, then that a compact open set of X_m is compact open in X , and finally that a compact open in Y_n is a compact open in $X_{\rho(n)}$.

For the first step, let U be an open subset of X_m . By [33, Lemma 3.11], for $A, B \in X$, $A \rightarrow_m B \iff \text{Core}^m(A) \rightarrow B$. Consider the associated set of cores $F \triangleq \{\text{Core}^m(A) \mid A \in U\}$: then U is the upward closure of F under homomorphisms. Furthermore, F is finite up to homomorphic equivalence by [33, Lemma 3.9], and thus also finite up to m -homomorphic equivalence. Finally, still by [33, Lemma 3.11], U is also the upward closure of F under m -homomorphisms. By Remark 2.3, U is thus compact in X_m . This shows that X_m is Noetherian.

As we have seen, any open subset U of X_m is the upward closure for \rightarrow of a finite set F of cores. Thus U is a compact open set in X by Remark 2.3. We have proven that a compact open set of X_m is also compact open in X .

Finally, consider a compact open set U of Y_n . By Lemma 7.3, U is open in $X_{\rho(n)}$ as well. Furthermore, U is compact in $X_{\rho(n)}$ since $X_{\rho(n)}$ is Noetherian. \square

The following theorem was famously first shown by Rossmann in [33, Corollary 7.1]. A more recent proof by Rossmann [34] uses lower bounds from circuit complexity. Similar results were shown by Nešetřil and Ossona de Mendez [31, Section 10.7] when assuming essentially the same statement as Lemma 7.3; in fact, carefully unwrapping the hypotheses of the *topological preservation theorem* of Nešetřil and Ossona de Mendez [31, Theorem 10.3] leads to the very definition of a projective system.

Theorem 7.5. *Let σ be a finite relational signature and X be a non-empty downwards-closed subset of $\text{Fin}(\sigma)$ for \rightarrow . Then X has the $(\tau_{\rightarrow}, \mathbf{EPFO}[\sigma])$ preservation property.*

Proof. Consider the projective system $\mathcal{F} \triangleq \{\text{id}_{i,j} : Y_i \rightarrow Y_j\}_{i \leq j \in I}$ indexed by $I \triangleq \mathbb{N} \setminus \{0\}$. Each space Y_i is Noetherian for all $i \in I$ because $\mathbf{FO}_i[\sigma]$ contains finitely many non-equivalent sentences, hence Y_i contains finitely many open sets. Hence $\mathcal{K}^\circ(Y_i) = \tau_{\rightarrow} \cap \llbracket \mathbf{FO}_i[\sigma] \rrbracket_X$. Also, the maps $\text{id}_{i,j}$ are spectral and \mathcal{F} is a projective system in **PreSpec**. Claim 7.4 shows that the identity map $\text{id}_i : X \rightarrow Y_i$ is a spectral map for all $i \in I$.

Assume that $\{g_i : Z \rightarrow Y_i\}_{i \in I}$ is a collection of morphisms in **Top** such that $\forall i \geq j \in I, g_j = \text{id}_{i,j} \circ g_i$. Since $\text{id}_{i,j}$ is the identity map, all the maps $(g_i)_{i \in I}$ are equal. In particular, one can build $g : Z \rightarrow X$ defined by any one of them. Let us show that g is a continuous map. If U is a definable open set of X , then U is a definable open set in Y_n for some n , hence $g^{-1}(U) = g_n^{-1}(U)$ is open. Since X has a base of definable open sets, this proves that g is continuous.

Assume that g' is an other continuous map making the diagram commute. As I is non empty, consider some $i \in I$, we have $g_i = \text{id}_i \circ g = \text{id}_i \circ g'$. Since f_i is the identity map we conclude $g = g'$.

We have shown that X is the limit of \mathcal{F} in **Top**. Since the maps $\text{id}_i: X \rightarrow X_i$ are spectral, Lemma 7.2 shows that X is a pre-spectral space such that $\mathcal{K}^\circ(X) = \bigcup_{i \in I} \mathcal{K}^\circ(Y_i) = \tau_{\rightarrow} \cap \llbracket \text{FO}[\sigma] \rrbracket_X$. In particular, X is a lpps. As X is downwards-closed, by Remark 3.5 it has the $(\tau_{\rightarrow}, \text{EPFO}[\sigma])$ -preservation property. \square

7.3. Completeness. We are now going to prove that any pre-spectral space can be obtained as a solution to a projective system of pre-spectral spaces, showing that the proof method of the previous sub-section is in some sense complete. In fact, this system is going to contain only Noetherian spaces. It is analogous to the fact that any spectral space is a projective limit of finite T_0 spaces [22, Proposition 10].

Proposition 7.6 (Pre-spectral spaces are limits of Noetherian spaces). *Let (X, τ) be a pre-spectral space, there exists a projective system of Noetherian spaces in **PreSpec** such that X is the limit of this projective system.*

Proof. We index our projective system by the finite subsets of $\mathcal{K}^\circ(X)$ ordered by inclusion. Whenever $K \subseteq_{\text{fin}} \mathcal{K}^\circ(X)$, let us define $X_K \triangleq (X, \langle K \rangle)$, that is, X with the topology generated by the finite collection K of compact open sets of X . Note that X_K is Noetherian and that the maps $\text{id}_{K, K'}: X_K \rightarrow X_{K'}$ are spectral maps whenever $K' \subseteq K$, so this is a projective system in **PreSpec**. Moreover, $\text{id}_K: X \rightarrow X_K$ is also spectral.

It remains to check that X is the limit. Assume that Y is a solution to the projective system in **PreSpec**: there exists spectral maps $g_K: Y \rightarrow X_K$ for all $K \subseteq_{\text{fin}} \mathcal{K}^\circ(X)$. Remark that $\text{id}_{K, K'} \circ g_K = g_{K'}$ by definition, hence $g_K = g_{K'}$. Define $g: Y \rightarrow X$ as $g(y) \triangleq g_\emptyset(y)$; notice that g is spectral as well. Finally, g is unique, as an other map $h: Y \rightarrow X$ making the diagram commute satisfies $g_\emptyset = f_\emptyset \circ g = f_\emptyset \circ h$, hence $g = h$. \square

Proposition 7.6 should be contrasted with Example 7.1, which showed that the projective limit of Noetherian spaces can be a non pre-spectral space. The existence of preservation theorems can therefore be reduced to the existence of a certain projective limit.

8. CONCLUDING REMARKS

In this paper, we have introduced a general framework for preservation results, mixing topological and model-theoretic notions. The key notion here is the one of *logically presented pre-spectral spaces*, which requires the (topological) compactness of the definable sets of interest. This definition captures simultaneously the classical proofs of preservation theorems over the class of all structures (we detailed the case of the Łoś-Tarski Theorem in Section 2.2) and all the known preservation results over classes of finite structures in the literature (see Remark 3.5). Our approach is comparable to the one adopted in the *topological preservation theorem* of Nešetřil and Ossona de Mendez [31, Theorem 10.3], in that we employ topological concepts to present a generic preservation theorem; however we believe our formulation to be considerably simpler and more flexible.

We have developed a mathematical toolbox for working with logically presented pre-spectral spaces, allowing to build new spaces from known ones. Besides relatively

mundane stability properties under suitable notions of morphisms, subspaces, finite sums, and finite products—which still required quite some care in order to account for first-order definability—, we have shown that more exotic constructions through topological closures or projective limits of topological spaces could also be employed. Those last two constructions give an alternative viewpoint on Rossman’s proof of homomorphism preservation over the class of finite structures (Theorem 7.5), and a new homomorphism preservation result over the class of structures with the finite model property (Corollary 6.3).

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