International Journal of Control

ISSN: 0020-7179 (Print) 1366-5820 (Online) Journal homepage: http://www.tandfonline.com/loi/tcon20

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To appear in the *International Journal of Control*
Vol. 00, No. 00, Month 20XX, 1–16

Publisher: Taylor & Francis
Journal: *International Journal of Control*
DOI: https://doi.org/10.1080/00207179.2017.1361042

Degree of Dieudonné determinant defines the order of nonlinear system

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(*v1.0 released June 2016*)

The paper proves that the order of the minimal state-space realisation of a set of irreducible implicit input-output (i/o) difference equations is equal to the degree of the Dieudonné determinant of a certain polynomial matrix. This matrix is defined by the generic linearisation of the i/o equations, and the map that incorporates the knowledge of the i/o equations in the polynomial coefficients. A simple algorithm to compute the degree of the Dieudonné determinant is given. Several examples demonstrate the theoretical result.

Keywords: Nonlinear control system; discrete-time system; Dieudonné determinant

1. Introduction

In the single-input single-output case the order n of the linear time-invariant control system, described by the input-output (i/o) equation $p(z)y = q(z)u$, is given as $n = \deg p(z)$. Moreover, if $p(z)$ and $q(z)$ have no common factors, i.e., if the system description is irreducible, then $\deg p(z)$ defines a state-space realisation with minimal dimension. In the multi-input multi-output (MIMO) case, one can always obtain an observable state-space realisation of order $n = \deg \det P(z)$ (Kailath, 1980). Observe that the global (generic) linearisation of nonlinear i/o equations (Kotta and Tõnso, 2012) defines two polynomial matrices, $P(z)$ and $Q(z)$, whose entries are known to be the elements of a non-commutative polynomial ring that satisfies both left and right Ore conditions (Farb and Dennis, 1993). Note also that the linearised system description $P(z)\mathrm{d}y = Q(z)\mathrm{d}u$ has been a cornerstone in solutions of several problems for nonlinear control systems; see for instance (Belikov et al., 2014) and the references therein. The purpose of this paper is

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(i) to investigate whether the degree of the Dieudonné determinant (Artin, 1957), developed in the non-commutative algebra, could be used to define the order n of the state-space realisation of the nonlinear i/o equations[1], and
(ii) in case the equations are not realizable in the state-space form, define the order of the system. Such definition, that agrees with the known definition from the linear theory is still missing for nonlinear i/o equations.

The results of this paper rely on the recent results from (Kotta et al., 2011), (Bartosiewicz et al., 2016), and (Bartosiewicz et al., 2015), addressing the problem of transformation the set of implicit nonlinear i/o equations into the row-reduced and Popov forms. We will prove that $n = \deg \det_D \bar{P}(z)$, where $\bar{P}(z)$ is closely linked to $P(z)$ via the map that incorporates the knowledge of the i/o equations in the polynomial coefficients. Exactly like in the linear case, once the matrix $\bar{P}(z)$ corresponds to the irreducible system description (Kotta and Tõnso, 2012), $\deg \det_D \bar{P}(z)$ defines the order of the minimal realisation, whenever the system is realizable. We emphasize that the degree of the Dieudonné determinant provides a simple tool to compute the order of the (minimal) realisation without actually finding the state equations nor transforming the i/o equations into some specific forms like the row-reduced or Popov form from which the state dimension may be found almost by inspection. Moreover, if $\deg \det_D \bar{P}(z) \neq -\infty$ (alternatively $\det_D \bar{P}(z) \not\equiv 0$), then the output sequence of the system is uniquely defined by the input sequence under the fixed initial conditions, exactly like in the linear case. This follows from the fact that in such a case the row-reduced form of $\bar{P}(z)$ does not contain zero rows, see (Kotta et al., 2011). The uniqueness guarantees that the set of difference equations has at least one solution (is consistent), which is otherwise, in general, not easy to check.

The paper relies remotely on conference paper (Kotta et al., 2008). In this paper it has been shown that the (left) Ore determinant is of no help to compute the system order since the left Ore determinant is not unique, but more importantly, without additional restrictions, the Ore determinants may be polynomials of different degree. The required restrictions have been found in conference paper for the second-order matrix, but their extension for the general case would be too complicated. Finally, note that Dieudonné determinant has been used in the study of linear time-varying systems, see for instance (Marinescu and Bourlès, 2003), where it was applied to calculate the rank of a matrix over noncommutative polynomial ring.

The degree of the determinant has been used in linear control theory in many different purposes. First, it reveals the system order (Kailath, 1980, p.369), the number of its poles and zeros, and can additionally be used to investigate both the finite and infinite structure of system (Henrion and Šebek, 1999). Some results have been formulated with the help of this concept. For instance, the realization of $P(z)y = Q(z)u$ is minimal only if the degree of the determinant of $P(z)$ is equal to the degree of the rational matrix $[P(z)]^{-1}Q(z)$, see (Chen, 1970). Next, the degree of the determinant of the so-called interactor matrix helps to determine the necessary and sufficient conditions for MIMO linear system to be prime (Baser and Eldem, 1984). In linear control one frequently needs to compute numerically the determinant, inverse or Smith form of a bivariate polynomial matrix using evaluation-interpolation techniques. The number of interpolation points depends on the degree of determinant. The paper (Varsamis and Karampetakis, 2014) presents a recursive formula to find the degree of the determinant of a bivariate polynomial matrix. For earlier results on univariate case, see (Henrion and Šebek, 1999; Hromčík and Šebek, 1999).

The paper is organized as follows. Section 2 serves as a brief overview of the essential notions from the algebraic framework. It follows by description of the generic linearisation of nonlinear system of equations in terms of two polynomial matrices in Section 3. Section 4 is devoted to

[1]Note that the order of the realization of the set of i/o equations may be, in general, of higher or lower order than the order n of the set of i/o equations. However, most realization algorithms find a state-space realisation of order n. In what follows, when we speak about the order of the realization, we mean such order. The minimal realisation is typically constructed from the irreducible system description, and corresponds to the order of the set of irreducible i/o equations.

Dieudonné determinant and its degree. An algorithm complements theoretical material. Section 5 demonstrates how to use the degree of the Dieudonné determinant to compute the order of the system, and its state-space realisation. Conclusions are drawn in Section 6.

2. Preliminaries

Consider a discrete-time MIMO nonlinear control system, described by the set of implicit higher-order i/o difference equations

$$\phi_i(y(t), \dots, y(t+n), u(t), \dots, u(t+n)) = 0, \quad i = 1, \dots, p, \tag{1}$$

where $t \in \mathbb{Z}$, $u(t) \in \mathbb{R}^m$ is the input, $y(t) \in \mathbb{R}^p$ is the output, and ϕ_i's are meromorphic functions, defined on an open and dense subset of $\mathbb{R}^{(n+1)(p+m)}$.

One can construct the inversive difference field $\mathcal{Q}$ of meromorphic functions, related to control system (1). The definition of forward shift operator in this difference field is based on the relations (1), i.e., the expressions on the left hand side of (1) and their shifts are considered to be equal to zero. In the construction of the field, a set $\mathcal{S}$ with the elements required to be non-zero, plays an important role. For instance, some functions ϕ_i in (1) may have denominators that, together with their forward/backward shifts and powers, should be included in $\mathcal{S}$. If there are no denominators, then one may set $\mathcal{S} := \{1\}$. The infinite set $\mathcal{S}$ can be briefly described by its generator set $\mathcal{S}_0$. See more in the Appendix A.

Example 1: Consider the set of i/o equations

$$\begin{aligned} \phi_1 &= y_1(t+2) + \frac{y_2(t+2)y_3(t+1)}{y_1(t)} - \frac{y_2(t+2)u_3(t)}{y_1(t)} - u_1(t) = 0 \\ \phi_2 &= y_1(t+3)u_1(t) + y_2(t+1) - u_1(t+1)u_1(t) - u_2(t) = 0 \\ \phi_3 &= y_3(t+1) - u_3(t) = 0. \end{aligned} \tag{2}$$

Since there is denominator in ϕ_1, define $\mathcal{S}_0 = \{1, y_1(t)\}$.

The field $\mathcal{Q}$ and the operator δ induce the ring of polynomials in a variable z over $\mathcal{Q}$, denoted by $\mathcal{Q}[z;\delta]$. A polynomial $p \in \mathcal{Q}[z;\delta]$ is written as $p = p_l z^l + p_{l-1}z^{l-1} + \dots + p_1 z + p_0$, where $p_i \in \mathcal{Q}$ for $i = 0, \dots, l$. The polynomial p is called monic if $p_l = 1$. The degree of p, denoted by $\deg p$, is defined to be l, if $p_l \not\equiv 0$. For $p = 0$, we set $\deg p = -\infty$. The addition of polynomials from $\mathcal{Q}[z;\delta]$ is defined in a standard manner. The non-commutative multiplication is defined by the linear extension of the following rules

$$z \cdot \alpha := \delta(\alpha)z, \quad \alpha \cdot z := \alpha z, \quad \alpha \in \mathcal{Q}. \tag{3}$$

The ring $\mathcal{Q}[z;\delta]$ satisfies both the left and right Ore conditions[2], i.e., it is an Ore ring (Farb and Dennis, 1993). Moreover, $\deg(p \cdot q) = \deg p + \deg q$.

3. Polynomial matrix description of the nonlinear system

We use differentials to obtain a generic (global) linearisation of the nonlinear system (1). The study below is done mostly via linearised system description and the results on globally linearised system are then lifted to the nonlinear system (1) in Section 5.

[2]Left Ore condition: for all nonzero $a, b \in \mathcal{Q}[z;\delta]$, there exist nonzero $\alpha, \beta \in \mathcal{Q}[z;\delta]$ such that $\alpha b = \beta a$.

We write $\mathcal{Q}[z;\delta]^{p\times q}$ for the set of $p\times q$-dimensional matrices with entries in $\mathcal{Q}[z;\delta]$. A matrix $U(z)\in\mathcal{Q}[z;\delta]^{p\times p}$ is called unimodular if there exists an inverse matrix $U^{-1}(z)\in\mathcal{Q}[z;\delta]^{p\times p}$. We will show how to represent the globally linearised description of the nonlinear system (1) in terms of two polynomial matrices with the elements in $\mathcal{Q}[z;\delta]$. Apply the differential operation d to (1) and use relations $\mathrm{d}y_s(t+j)=z^j\mathrm{d}y_s(t)$, $\mathrm{d}u_k(t+r)=z^r\mathrm{d}u_k(t)$ to obtain

$$P(z)\mathrm{d}y(t)=Q(z)\mathrm{d}u(t), \tag{4}$$

where $P(z)$ and $Q(z)$ are $p\times p$ and $p\times m$-dimensional matrices, respectively, whose elements p_{ij},q_{ij} are defined as $p_{ij}(z)=\sum_{s=0}^{n}\frac{\partial\phi_i}{\partial y_j(t+s)}z^s$, $q_{ij}(z)=-\sum_{s=0}^{n}\frac{\partial\phi_i}{\partial u_j(t+s)}z^s$ and $\mathrm{d}y(t)=[\mathrm{d}y_1(t)\ \cdots\ \mathrm{d}y_p(t)]^{\mathsf{T}}$, $\mathrm{d}u(t)=[\mathrm{d}u_1(t)\ \cdots\ \mathrm{d}u_m(t)]^{\mathsf{T}}$. The coefficients of polynomials in matrices $P(z)$ and $Q(z)$ as given in (4) do not yet incorporate the knowledge that $\phi_i(\cdot)=0$, $i=1,\dots,p$. Let $e_{\mathcal{S}}^{\Phi}$ denote the map that includes such knowledge: $\bar{P}(z):=e_{\mathcal{S}}^{\Phi}(P(z))$, see Appendix B.

Example 2: (Continuation of Example 1). Compute for system (2) the matrix $P(z)$ in the associated polynomial description (4)

$$\begin{bmatrix} z^2-\frac{y_2(t+2)(y_3(t+1)-u_3(t))}{y_1^2(t)} & \frac{y_3(t+1)-u_3(t)}{y_1(t)}z^2 & \frac{y_2(t+2)}{y_1(t)}z\\ u_1(t)z^3 & z & 0\\ 0 & 0 & z\end{bmatrix}.$$

The simplest representative of $\bar{P}(z):=e_{\mathcal{S}}^{\Phi}(P(z))$ is

$$\begin{bmatrix} z^2 & 0 & \frac{y_2(t+2)}{y_1(t)}z\\ u_1(t)z^3 & z & 0\\ 0 & 0 & z\end{bmatrix},$$

since $y_3(t+1)-u_3(t)=0$ by (2).

Example 3: Consider the set of i/o equations

$$\begin{aligned}(y_1(t+2))^3+y_3(t)-u_2(t)&=0,\\ y_1(t+3)-y_2(t+2)+u_1(t)&=0,\\ y_3(t+4)+y_2(t+1)+u_2(t)&=0.\end{aligned}$$

Since there are no denominators in equations, we set $\mathcal{S}_0=\mathcal{S}:=\{1\}$. Compute

$$P(z)=\begin{pmatrix}3(y_1(t+2))^2z^2 & 0 & 1\\ z^3 & -z^2 & 0\\ 0 & z & z^4\end{pmatrix}.$$

Here, $P(z)$ is already a nice representative of $\bar{P}(z)$.

4. Dieudonné Determinant

We recall below the definition of Dieudonné determinant and some properties of it (Artin, 1957). Since $\mathcal{Q}[z;\delta]$ is an Ore ring, it can be embedded into a skew field $\mathcal{Q}(z;\delta)$, called the field of left fractions of $\mathcal{Q}[z;\delta]$. We extend the notion of degree to the fractions in a natural way. If $p,q\in\mathcal{Q}[z;\delta]$, then define $\deg p^{-1}q=\deg q-\deg p$. Write $\mathcal{Q}^*(z;\delta)$ for a multiplicative group of $\mathcal{Q}(z;\delta)$, containing all non-zero elements of $\mathcal{Q}(z;\delta)$. If multiplication of matrix elements do not commute, the value of the standard determinant is not unique but depends on the order of factors in multiplication. To overcome the difficulty, a standard approach in algebra is to compute the determinant in some commutative substructure. In case of the Dieudonné determinant this is done with the help of quotient group $\mathcal{Q}^*(z;\delta)/[\mathcal{Q}^*(z;\delta),\mathcal{Q}^*(z;\delta)]$. Note that $[\mathcal{Q}^*(z;\delta),\mathcal{Q}^*(z;\delta)]$ is generated by the elements $aba^{-1}b^{-1} =: \mathcal{K}(a,b)$. From direct computation $ab = \mathcal{K}(a,b)ba$, therefore ab and ba differ from each other only by the multiple $\mathcal{K}(a,b)$. Obviously, $\mathcal{K}(a,a) = 1$. When one constructs the quotient group of $\mathcal{Q}^*(z;\delta)$ modulo $[\mathcal{Q}^*(z;\delta),\mathcal{Q}^*(z;\delta)]$, this means that the elements of the subgroup $[\mathcal{Q}^*(z;\delta),\mathcal{Q}^*(z;\delta)]$ should be taken equal. Since one has to choose the representative in this subgroup, 1 is the simplest choice.

Let $\pi : \mathcal{Q}(z;\delta) \to (\mathcal{Q}^*(z;\delta)/[\mathcal{Q}^*(z;\delta),\mathcal{Q}^*(z;\delta)]) \cup \{0\}$ be a projection from the skew field to the quotient group, defined as follows:

$$\pi(p) = \begin{cases} p[\mathcal{Q}^*(z;\delta),\mathcal{Q}^*(z;\delta)], \text{ if } p \neq 0 \\ \qquad\qquad\qquad\qquad 0, \text{ if } p = 0. \end{cases}$$

Clearly π satisfies

$$\pi(ab) = \pi(a)\pi(b). \tag{5}$$

Note that in the quotient group $\pi(a)\pi(b) = \pi(b)\pi(a)$, i.e., multiplication is commutative. Of course, the value of the Dieudonné determinant is unique only up to multiple $[\mathcal{Q}^*(z;\delta),\mathcal{Q}^*(z;\delta)]$. Although Dieudonné determinant is not single-valued, its degree is, since the degree of the elements in $[\mathcal{Q}^*(z;\delta),\mathcal{Q}^*(z;\delta)]$ is zero. We do not address the computation issues of the Dieudonné determinant, since in what follows we only need to compute its degree, and this can be done without actually computing the determinant itself, but relying on the properties of Dieudonné determinant.

Definition 1: A mapping

$$\det_D : \mathcal{Q}(z;\delta)^{p\times p} \to (\mathcal{Q}^*(z;\delta)/[\mathcal{Q}^*(z;\delta),\mathcal{Q}^*(z;\delta)]) \cup \{0\}$$

is called Dieudonné determinant, if it satisfies the following three axioms:

(d1) if A' is obtained from A by multiplying one row by $\mu \in \mathcal{Q}(z;\delta)$ from the left, then $\det_D A' = \pi(\mu)\det_D A$;
(d2) if A' is obtained from A by replacing the ith row by sum of the ith and the jth rows $i \neq j$, then $\det_D A' = \det_D A$;
(d3) $\det_D I_n = \pi(1)$.

The Dieudonné determinant has a number of desirable properties of the ordinary determinant and the latter is the special case of Dieudonné determinant. The proposition below lists the properties of Dieudonné determinant that will be used later. Note that matrices in proposition are in $\mathcal{Q}(z;\delta)^{w_1\times w_2}$, where $w_1, w_2 \in \{p,q\}$ denote the appropriate dimension.

Proposition 1: (p1) $\det_D A = \pi(0)$ *iff A is singular (non-invertible) matrix;*
(p2) *if we permute two rows in A, then* $\det_D A$ *is multiplied by* $\pi(-1)$*;*
(p3) $\det_D \begin{pmatrix} I_{n-1} & 0 \\ 0 & \mu \end{pmatrix} = \pi(\mu)$*;*
(p4) $\det_D(AB) = \det_D A \cdot \det_D B$*;*
(p5) $\det_D \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \det_D A \cdot \det_D B$*;* $\det_D \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} = \det_D A \cdot \det_D B$*.*

Proposition 2: *If $\pi(p) = \pi(q)$, then $\deg p = \deg q$.*

Proof. The relation $\pi(p) = \pi(q)$ means that p and q are equal up to a multiple in $[\mathcal{Q}^*(z;\delta), \mathcal{Q}^*(z;\delta)]$. Since $[\mathcal{Q}^*(z;\delta), \mathcal{Q}^*(z;\delta)]$ is generated by the *commutators* $a^{-1}b^{-1}ab$, $a, b \in \mathcal{Q}^*(z;\delta)$, we have $p^{-1}q = w_1 \cdots w_k$, where $w_l = a_l^{-1}b_l^{-1}a_l b_l$. Since $\deg a_l^{-1}b_l^{-1}a_l b_l = \deg a_l + \deg b_l - \deg a_l - \deg b_l = 0$, $\deg p^{-1}q = 0$, i.e., $\deg p = \deg q$. □

Proposition 3: $(q1)$ *The degree function* $\deg \det_D : \mathcal{Q}(z;\delta)^{p\times p} \to \mathbb{Z} \cup \{-\infty\}$ *is well-defined and satisfies the property* $\deg \det_D(AB) = \deg \det_D A + \deg \det_D B$, *see (Taelman, 2006).*
$(q2)$ $U \in \mathcal{Q}[z;\delta]^{p\times p}$ *is unimodular iff* $\deg \det_D U = 0$, *see (Giesbrecht and Kim, 2013).*
$(q3)$ $\deg \det_D A = 0$ *if* $A \in \mathcal{Q}$.

Note that the addition and the multiplication of the left fractions $b_1^{-1}a_1$ and $b_2^{-1}a_2$ in $\mathcal{Q}(z;\delta)$ are defined by

$$b_1^{-1}a_1 + b_2^{-1}a_2 = (\beta_2 b_1)^{-1}(\beta_2 a_1 + \beta_1 a_2), \tag{6}$$

where $\beta_2 b_1 = \beta_1 b_2$ and by

$$b_1^{-1}a_1 \cdot b_2^{-1}a_2 = (\beta_2 b_1)^{-1}\alpha_1 a_2, \tag{7}$$

where $\beta_2 a_1 = \alpha_1 b_2$. The degree of Dieudonné determinant of the $p \times p$ matrix can be found by the algorithm described below. Note that in computations the rules (6) and (7) have to be used. Since the degree of the Dieudonné determinant is unique for all elements in the equivalence class (quotient group $\mathcal{Q}^*(z;\delta)/[\mathcal{Q}^*(z;\delta), \mathcal{Q}^*(z;\delta)]$), we make computations not with the equivalence classes but with their representatives since this does not affect the result.

Algorithm:

Step 1. The Step 1 mimics the computation of standard determinant by Gaussian elimination. The matrix A is transformed into the upper triangular form. Let $a_{ij}^{[0]} := a_{ij}$. Then, one can compute recursively

$$a_{ij}^{[k]} = \begin{cases} a_{ij}^{[k-1]} - a_{ik}^{[k-1]} \cdot \left(a_{kk}^{[k-1]}\right)^{-1} \cdot a_{kj}^{[k-1]} & \text{for } i \le j \\ 0 & \text{for } i > j, \end{cases} \tag{8}$$

with $k = 1, \ldots, p-1$, $i = k+1, \ldots, p$, $j = 1, \ldots, p$, and assuming that $a_{kk}^{[k-1]} \not\equiv 0$. In case $a_{kk}^{[k-1]} \equiv 0$, before applying formula (8), one has to exchange the kth row with the lth row where $l \in \{k+1, \ldots, p\}$ such that it has a non-zero element on the currently processed diagonal position. That is, one has to find another element in the kth column, $a_{lk}^{[k-1]} \not\equiv 0$. If it is not possible, then the matrix A is singular and $\deg \det_D A = -\infty$.

Step 2. Although the degree of the Dieudonné determinant can be found by multiplying first the elements on the main diagonal of the triangular form, and then computing the degree of the resultant expression

$$\deg \det_D A = \deg \left(a_{11}a_{22}^{[1]} \cdots a_{pp}^{[p-1]}\right), \tag{9}$$

this is computationally inefficient, since transformation of the product in (9) into the form $b^{-1}a$ requires, according to (7), a large number of symbolic computations. By (7), the multiplication of fractions is based on calculation of the least common left multiple of the numerator of the first fraction and the denominator of the second factor. Using the properties ($p4$) and ($p5$) of Proposition 1

$$\deg \det_D A = \deg a_{11} + \deg a_{22}^{[1]} + \cdots + \deg a_{pp}^{[p-1]}. \tag{10}$$

It follows from Proposition 2 that deg is a constant function on the set $p[\mathcal{Q}^*(z;\delta), \mathcal{Q}^*(z;\delta)]$. Therefore, it is natural to define $\deg \pi(p) = \deg p$.

Proposition 4: *The equation* (10) *holds for any invertible matrix* $A \in \mathcal{Q}(z;\delta)^{p\times p}$. *If* $A \in \mathcal{Q}(z;\delta)^{p\times p}$ *is not invertible, then* $\deg \det_D A = -\infty$.

Proof. In the first part of the proof we transform the matrix A into the upper triangular form as follows, except in the cases when A is already in the upper or lower triangular form when we denote $\mathcal{L} = A$. Denote $A^{[0]} := A$ and its elements by $a_{ij}^{[0]}$, and let the matrix $A^{[k]}$ with the elements $a_{ij}^{[k]}$, $k = 1, \ldots, p-1$ be the one computed from A in Step 1 recursively. For each k, the formula (8) can be rewritten in the matrix form as $A^{[k]} = L^{[k]}T^{[k]}A^{[k-1]}$, where the lower triangular matrix $L^{[k]}$ is defined by

$$L_{ij}^{[k]} = \begin{cases} 1, & \text{if } i = j; \\ -a_{ik}^{[k-1]} \cdot \left(a_{kk}^{[k-1]}\right)^{-1}, & \text{if } i > k \text{ and } j = k; \\ 0, & \text{otherwise,} \end{cases}$$

and $T^{[k]}$ is either identity matrix, if $a_{kk}^{[k-1]} \neq 0$, or the permutation matrix corresponding to permutation of rows k and j ($j > k$) assuming that $a_{jk}^{[k-1]} \neq 0$. In the latter case, the matrix $T^{[k]}$ is obtained from the identity matrix by swapping the kth and jth rows. If $a_{jk}^{[k-1]} = 0$ for all $j \geq k$, then the matrix A is not invertible. For an invertible matrix A we finally have

$$\mathcal{L} = L^{[p-1]}T^{[p-1]} \cdots L^{[1]}T^{[1]}A$$

for some invertible upper triangular matrix $\mathcal{L} \in \mathcal{Q}(z;\delta)^{p\times p}$. Hence,

$$\det_D \mathcal{L} = \det_D \left(L^{[p-1]}T^{[p-1]} \cdots L^{[1]}T^{[1]}A\right) = \prod_{k=1}^{p-1} \left(\det_D L^{[k]} \cdot \det_D T^{[k]}\right) \cdot \det_D A$$

by ($p4$) in Proposition 1. Since $L^{[k]}$ is lower triangular matrix having ones on the main diagonal, we have that $\det_D L^{[k]} = \pi(1)$. This is so because the matrix $L^{[k]}$ is lower triangular, its determinant equals

$$\pi(l_{11}^{[k]} l_{22}^{[k]} \cdots l_{pp}^{[k]}) = \pi(1 \cdot 1 \cdots 1) = \pi(1).$$

Therefore, $\det_D L^{[k]} = \det_D I = \pi(1)$ by ($d2$) and ($p3$) for each k. Further, $\det_D T^{[k]}$ is either $\pi(1)$ (if $T^{[k]}$ is the identity matrix) or $\pi(-1)$. By the fact that $\deg \pi(p) = \deg p$, we have

$$\det_D \mathcal{L} = \pi(\pm 1)\det_D A.$$

In the second part of the proof we compute $\deg \det_D A$ with the help of $\deg \det_D \mathcal{L}$. As $\mathcal{L}$ being an upper triangular matrix, its determinant equals

$$\pi(\ell_{11}\cdots\ell_{pp}) = \pi(a_{11})\pi(a_{22}^{[1]})\cdots\pi(a_{pp}^{[p-1]}).$$

From this we get

$$\deg\det{}_D\mathcal{L} = \deg\pi(a_{11}) + \deg\pi(a_{22}^{[1]}) + \cdots + \deg\pi(a_{pp}^{[p-1]}) = \deg\pi(\pm 1) + \deg\det{}_D A.$$

By the definition of $\deg\pi(\cdot)$ we have $\deg\pi(\pm 1) = 0$ and $\deg\pi(a_{11}) = \deg a_{11}, \ldots, \deg\pi(a_{pp}^{[p-1]}) = \deg a_{pp}^{[p-1]}$. From this we conclude that

$$\deg\det{}_D A = \deg a_{11} + \deg a_{22}^{[1]} + \cdots + \deg a_{pp}^{[p-1]}$$

as desired.

If the matrix A is not invertible, then $\det_D A = 0$ and $\deg\det_D A = -\infty$. □

Example 4: (Continuation of Example 3). Consider the matrix $\bar{P}(z)$ from Example 3 with elements in $\mathcal{Q}[z;\delta]$. Using the above algorithm, we first compute iteratively new elements of the upper triangular matrix as

$$\begin{aligned}
a_{22}^{[1]} &= -z^2 - z^3\left(3(y_1(t+2))^2 z^2\right)^{-1}\cdot 0 = -z^2,\\
a_{23}^{[1]} &= 0 - z^3\left(3(y_1(t+2))^2 z^2\right)^{-1}\cdot 1 = -\frac{1}{3(y_1(t+3))^2}z,\\
a_{32}^{[1]} &= z - 0\cdot\left(3(y_1(t+2))^2 z^2\right)^{-1}\cdot 0 = z,\\
a_{33}^{[1]} &= z^4 - 0\cdot\left(3(y_1(t+2))^2 z^2\right)^{-1}\cdot 1 = z^4,\\
a_{33}^{[2]} &= z^4 - z(-z^2)^{-1}\left(-\frac{1}{3(y_1(t+3))^2}z\right) = z^4 - \frac{1}{3(y_1(t+2))^2}.
\end{aligned}$$

In computation of $a_{23}^{[1]}$ we use the multiplication rule (7). Observe that here $b_1 = a_2 = 1$. Now, one has to find α_1 and β_2 that satisfy the condition $\beta_2 a_1 = \alpha_1 b_2$, which are $\beta_2 = 1$ and $\alpha_1 = [3(y_1(t+3))^2]^{-1}z$. Then, the final result follows immediately using the right-hand-side of (7). The computations of the second term in $a_{33}^{[2]}$ can be done as follows. Multiplication of the first two factors $z(-z^2)^{-1}$ results in $-z^{-1}$. Then, the resulting left fraction

$$-z^{-1}\{[-1/(3(y_1(t+3))^2)]z\}$$

is reducible whereas the greatest common left fraction is z. For reduction one may bring z from right to left in the expression $[-1/(3(y_1(t+3))^2)]z$ which results, by (3), in

$$z[-1/(3(y_1(t+2))^2)].$$

Now, from $z^{-1}z = 1$ the result follows.

Note that while transforming the matrix, the set $\mathcal{S}_0$ has to be extended as $\mathcal{S}_0 = \{1, y_1(t)\}$, which guarantees that $y_1(t+2)$ is in the set $\mathcal{S}$. Finally, the representative of the Dieudonné determinant in the equivalence class is equal to

$$\det{}_D\bar{P}(z) = -3(y_1(t+2))^2 z^8 + \frac{(y_1(t+2))^2}{(y_1(t+6))^2}z^4,$$

and its degree is then equal to 8.

5. System Order

For a *non-zero* polynomial row $\bar{P}_i(z) \in \mathcal{Q}^{1\times p}[z;\delta]$, define its degree, denoted by σ_i, as the exponent of the highest power in z presented in $\bar{P}_i(z)$. Obviously, $\sigma_i \geq 0$. If $\bar{P}_i(z) = 0^{1\times p}$, define $\sigma_i = -\infty$. Let $\bar{P}(z)$ be a polynomial matrix in $\mathcal{Q}^{p\times p}[z;\delta]$ with *non-zero* rows $\bar{P}_1(z),\ldots,\bar{P}_p(z) \in \mathcal{Q}^{1\times p}[z;\delta]$. Then, $\bar{P}_i(z) = \bar{P}_{i0}z^{\sigma_i} + \bar{P}_{i1}z^{\sigma_i-1} + \cdots + \bar{P}_{i\sigma_i}$ with $\bar{P}_{ij}$ being a row vector of functions in $\mathcal{Q}$ for $j = 0,\ldots,\sigma_i$. Introduce the vector of row degrees $\sigma := (\sigma_1,\ldots,\sigma_p)$. Let $\deg \bar{P}(z) = \max\{\sigma_1,\ldots,\sigma_p\} := N$. By $\mathrm{diag}\{z^{N-\sigma_1},\ldots,z^{N-\sigma_p}\}$ we denote the diagonal $p\times p$ matrix.

Definition 2: The matrix $L = L(\bar{P}(z)) \in \mathcal{Q}^{p\times p}$ such that

$$\mathrm{diag}\left\{z^{N-\sigma_1},\ldots,z^{N-\sigma_p}\right\}\bar{P}(z) = Lz^N + \text{lower degree terms} \tag{11}$$

is called the *leading row coefficient matrix* of $\bar{P}(z)$.

Definition 3 (Kotta et al. 2011)**:** The matrix $\bar{P}(z) \in \mathcal{Q}^{p\times p}[z;\delta]$ with non-zero rows is called *row-reduced* iff its leading row coefficient matrix $L(\bar{P}(z))$ has full row rank over $\mathcal{Q}$. If $\bar{P}(z)$ contains zero rows, then $\bar{P}(z)$ is called *row-reduced* iff its submatrix consisting of non-zero rows is row-reduced.

Definition 4: The set of i/o difference equations (1) is said to be *row-reduced* if the matrix $\bar{P}(z) = e^{\Phi}_{\mathcal{S}}(P(z))$ is row-reduced over $\mathcal{Q}[z;\delta]$.

Proposition 5: *If* $\bar{P}(z) \in \mathcal{Q}[z;\delta]^{p\times p}$ *is row-reduced and* $\deg\det_D\bar{P}(z) > 0$*, then* $\deg\det_D\bar{P}(z)$ *equals to the sum of row degrees of* $\bar{P}(z)$*.*

Proof. Under the assumption of proposition that $\deg\det_D\bar{P}(z) > 0$, the leading coefficient matrix $L \in \mathcal{Q}^{p\times p}$ of $\bar{P}(z)$ (see relation (11)) is a full row rank matrix. Hence, there exists a matrix $A \in \mathcal{Q}^{p\times p}$, such that AL is in the upper triangular form. Write $M = A\,\mathrm{diag}\{z^{N-\sigma_1},\ldots,z^{N-\sigma_p}\}\bar{P}(z)$, and denote its elements by m_{ij} $(1 \leq i,j \leq p)$. Obviously, $\deg m_{ij} < N$, if $i > j$, since the leading coefficient matrix AL of M is an upper triangular matrix. Denote by $E_{ij}(\bar{p})$ a $p\times p$-matrix having ones at the main diagonal, $\bar{p} \in \mathcal{Q}(z;\delta)$ at the jth column of the ith row, and zeros elsewhere. The matrix M can be transformed to the upper triangular form as well. Let $M' = (m'_{ij})$ and

$$M' = E_{p1}(-m_{p1}m_{11}^{-1})\cdots E_{21}(-m_{21}m_{11}^{-1})M, \tag{12}$$

Recall that the inverse of the left fraction m_{11}, denoted by m_{11}^{-1}, is a left fraction too (by the left Ore condition) and multiplication of left fractions was defined by the formula (7). Clearly, we have that $m'_{i1} = m_{i1} - m_{i1}m_{11}^{-1}m_{11} = 0$ for all $1 < i \leq p$. Compute $\deg m'_{ii}$. If $i = 1$ or $m_{i1} = 0$, then $\deg m'_{ii} = \deg m_{ii}$. Suppose that $m_{i1} \neq 0$ for some $i > 1$. By (12), $m'_{ii} = m_{ii} - m_{i1}m_{11}^{-1}m_{1i}$. Since $\deg m_{ii} = \deg m_{11} = N$, $\deg m_{i1} < N$ and $\deg m_{1i} \leq N$, we have $\deg(m_{i1}m_{11}^{-1}m_{1i}) = \deg m_{i1} - \deg m_{11} + \deg m_{i1} < N - N + N = N$. Hence $\deg(m_{i1}m_{11}^{-1}m_{1i}) < N$ and $\deg m'_{ii} = N$.

Next, we transform the second column of M' to the required form. Let $M'' = E_{p2}(-m'_{p2}{m'_{22}}^{-1})\cdots E_{32}(-m'_{32}{m'_{22}}^{-1})M'$ such that $\deg m''_{ii} = N$ and so on. Finally, we obtain $B = \mathcal{E}M$, where $\mathcal{E}$ is a lower triangular (being a product of matrices $E_{ij}(\bar{p})$, $i \neq j$) and B is an upper triangular matrix. Since by ($p3$) and ($p5$) $\det_D\mathcal{E} = \bar{1}$, we have from $B = \mathcal{E}M$, again by ($p3$) and ($p5$) that $\det_D M = \det_D B = \pi(b_{11}\cdots b_{pp})$. From this $\deg\det_D M = pN$. On the other hand, by ($p3$) and ($p4$) $\det_D M = \det_D A\cdot\pi(z^{pN-\sigma_1-\cdots-\sigma_p})\cdot\det_D\bar{P}(z)$. After computing degrees of both sides we get $pN = \deg\det_D A + pN - \sum_{i=1}^p\sigma_i + \deg\det_D\bar{P}(z)$. Since $A \in \mathcal{Q}^{p\times p}$, $\deg\det_D A = 0$. From this we conclude $\deg\det_D\bar{P}(z) = \sum_{i=1}^p\sigma_i$ as required. $\square$

Observe that row-reducedness property of system (1) is actually a property of linearised equations (4): the set of i/o equations (1) is called row-reduced iff the polynomial matrix $\bar{P}(z) = e_{\mathcal{S}}^{\Phi}(P(z))$ is row-reduced. Nevertheless, this property translates back to the i/o equations, though in general, linear transformations defined by the unimodular matrix $U(z)$ as suggested in (Kotta et al., 2011), are not enough and nonlinear transformations are necessary, as shown in (Bartosiewicz et al., 2016). For our purposes it is important that in both cases the row degrees of the set of i/o equations (1) are defined as the row degrees of the matrix $e_{\mathcal{S}}^{\Phi}(P(z))$.

Recall from (Kotta and Tõnso, 2012) two definitions that allow to formulate Theorem 1 below that relates state dimension and the sum of row degrees.

The system (1) is called irreducible, if the greatest common left divisors G_L of matrices $\bar{P}(z)$ and $\bar{Q}(z)$ are unimodular, see more in (Kotta and Tõnso, 2012). Then the irreducible differential form of system (1) is defined as $\omega = \bar{P}(z)\mathrm{d}y - \bar{Q}(z)\mathrm{d}u$. In case the system is reducible (G_L is not unimodular), $\bar{P}(z) = G_L\tilde{P}(z)$ and $\bar{Q}(z) = G_L\tilde{Q}(z)$, and the irreducible differential form is given by $\tilde{\omega} = \tilde{P}(z)\mathrm{d}y - \tilde{Q}(z)\mathrm{d}u$. Application of state elimination to system

$$\begin{aligned} x(t+1) &= f(x(t), u(t)), \quad x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^m,\ y(t) \in \mathbb{R}^p \\ y(t) &= h(x(t)), \end{aligned} \tag{13}$$

results in the set of i/o equations. The irreducible differential form of state equations (11) is defined as the irreducible form of its i/o representation.

Definition 5: The equation (13) is called a state-space realization of the set of i/o equations (1), if their irreducible differential forms, represented as column vectors, are equal, possibly up to multiplication by some unimodular matrices from left. The realization is called minimal if its dimension is minimal among all realizations.

Theorem 1: *The dimension n of the minimal realisation of the irreducible set of i/o equations* (1) *can be computed as* $n = \deg \det_D \bar{P}(z)$, *whenever* $\deg \det_D \bar{P}(z) \neq -\infty$.

Proof. In (Kotta and Tõnso, 2012) it has been proven that the dimension n of the minimal realization of the irreducible set of i/o equations (1) can be computed as the sum of row degrees, $n = \sigma_1 + \cdots + \sigma_p$, once the equations (1) are in the Popov form. Since the equations in the Popov form are also in the row-reduced form by (Bartosiewicz et al., 2015, Def. 6), for systems (1) in the Popov form,

$$n = \deg \det_D \bar{P}(z),$$

according to Proposition 5. The rest of the proof shows that the equations (1) can be transformed by equivalence transformations into the Popov form without changing the degree of the Dieudonné determinant of the matrix $\bar{P}(z)$. First, note that any nonlinear system of the form (1) can be transformed into the equivalent representation being in the row-reduced form; the necessary transformation operator (linear or nonlinear) is defined by the unimodular matrix that transforms the matrix $e_{\mathcal{S}}^{\Phi}P(z) = \bar{P}(z)$ into the row-reduced form (Bartosiewicz et al., 2016; Kotta et al., 2011). Such unimodular matrix $U(z)$ over $\mathcal{Q}[z;\delta]$ always exists and the paper (Kotta et al., 2011) presents an algorithm to compute it. Note that by ($q1$) in Proposition 3,

$$\deg \det_D(U(z)\bar{P}(z)) = \deg \det_D U(z) + \deg \det_D \bar{P}(z)$$

and by ($q2$), $\deg \det_D U(z) = 0$. So, transformation of the equations (1) into the row-reduced form does not change the degree of the Dieudonné determinant of the matrix $\bar{P}(z)$, i.e., $\deg \det_D(\bar{P}(z)) = \deg \det_D(U(z)\bar{P}(z))$. Therefore, one may assume that equations (1) are in the row-reduced form, without the loss of generality.

Second, in (Bartosiewicz et al., 2015) it has been proven that the equations (1) in the row-reduced form can be always transformed into the Popov form, whereas in the process of transformation the row degrees remain unchanged. Like in the transformation of the equations into the row-reduced form, the transformation operator is defined by the unimodular matrix that transforms the matrix $e_{\mathcal{S}}^{\Phi}(P(z)) = \bar{P}(z)$ (in the row-reduced form) into the Popov form. So, transformation of the matrix $\bar{P}(z)$ into the Popov form does not change $\deg \det_D \bar{P}(z)$. Again, we may assume without the loss of generality that the equations (1) are from the beginning already in the Popov form. □

Example 5: (Continuation of Examples 3 and 4). Let us find, for comparison, the order of the system from Example 3 performing calculations according to the approach from (Kotta et al., 2011). From matrix $\bar{P}(z)$ one finds that $N = 4$ and $\sigma = (2, 3, 4)$. Next, according to (11), the leading coefficient matrix L can be calculated as

$$\begin{pmatrix} 3(y_1(t+4))^2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Obviously, L is not of full rank over $\mathcal{Q}$, meaning that the matrix $\bar{P}(z)$ is not in the row-reduced form by Definition 3. One can easily check that the first and the second rows of the matrix L are linearly dependent, resulting in the relation $3(y_1(t+4))^2\lambda_1 + \lambda_2 = 0$ from which $\lambda_1 = -1/(3(y_1(t+4))^2), \lambda_2 = 1$. The set $\mathcal{S}_0$ has to be extended as $\mathcal{S}_0 := \{1, y_1(t)\}$. Performing several steps of the Algorithm from (Kotta et al., 2011), one can find a unimodular matrix

$$U(z) = \begin{pmatrix} 0 & 0 & 1 \\ -u_1(t)z & 1 & \frac{u_1(t)y_2(t+3)}{y_1(t+1)}z \\ 1 & 0 & 0 \end{pmatrix}$$

such that

$$\bar{\bar{P}}(z) = U(z)\bar{P}(z) = \begin{pmatrix} 3(y_1(t+2))^2 z^2 & 0 & 1 \\ 0 & -z^2 & -\frac{1}{3(y_1(t+3))^2}z \\ 0 & z & z^4 \end{pmatrix}.$$

One may easily check that $\operatorname{rank}_{\mathcal{Q}} L(\bar{\bar{P}}(z)) = 3$. Therefore, the order n of the transformed system in the row-reduced form is $n = \sigma_1 + \sigma_2 + \sigma_3 = 8$ as in Example 4, which computes the system order from $\deg \det_D \bar{P}(z)$. In addition, $\mathcal{S}_0$ appears to be the same as in Example 4. Finally, using the method from (Bartosiewicz et al., 2016), the equations in the row-reduced form can be found as

$$\begin{aligned} (y_1(t+2))^3 + y_3(t) - u_2(t) &= 0, \\ y_2(t+2) - u_1(t) &= 0, \\ y_3(t+4) + y_2(t+1) + u_2(t) &= 0 \end{aligned}$$

for which the state-space realisation is given by

$$
\begin{aligned}
x_1(t+1) &= x_2(t)\\
x_2(t+1) &= (u_2(t) - x_5(t))^{\frac{1}{3}}\\
x_3(t+1) &= x_4(t)\\
x_4(t+1) &= -u_1(t)\\
x_5(t+1) &= x_6(t)\\
x_6(t+1) &= x_7(t)\\
x_7(t+1) &= x_8(t)\\
x_8(t+1) &= -u_2(t) - x_4(t)\\
y_1(t) &= x_1(t)\\
y_2(t) &= x_3(t)\\
y_3(t) &= x_5(t).
\end{aligned}
$$

Example 6: Consider the system from (Bartosiewicz et al., 2016)

$$
\begin{aligned}
y_1(t+3) - y_1(t+2) - u_1(t) = 0,\\
\sin(y_1(t+2) + y_2(t+1) - u_2(t)) = 0.
\end{aligned}
$$

Proceeding in the similar manner as in Example 4, one can easily compute a representative of the Dieudonné determinant as

$$\det_D \bar{P}(z) = \cos(u_2(t+3) - y_1(t+5) - y_2(t+4))z^4 - \cos(u_2(t+2) - y_1(t+4) - y_2(t+3))z^3,$$

having the degree equal to 4. This result coincides with that obtained in (Bartosiewicz et al., 2016), relying on nonlinear equivalence transformation.

Finally, note that the discussion above allows to extend the definition of system order for implicit nonlinear MIMO difference equations, not necessarily transformable into the state-space form.

Definition 6: Define the order of system (1) as $n = \deg \det_D \bar{P}(z)$.

Observe that this definition is consistent with linear theory (see Rosenbrock, 1970, p.47) as well as with the definition of the order of state-space realisation of (1) as shown in this paper.

6. Conclusion

The main result of this paper says that the dimension of the minimal realization of the set of implicit irreducible higher order i/o equations (1) equals to the degree of the Dieudonné determinant of the polynomial matrix $\bar{P}(z)$. The matrix $\bar{P}(z)$ is closely linked with the matrix $P(z)$ in the globally linearised system description (4). The map that transforms the elements of $P(z)$ into $\bar{P}(z)$ takes into account the knowledge from equations (1) in the polynomial coefficients. In particular, it relies on the fact that the expressions ϕ_i (as well as their forward and backward shifts) are equal to zero. Using the concept of the degree of the Dieudonné determinant, one can compute the dimension n of the minimal realization of the set of realizable irreducible i/o equations (1) directly. To compute n, there is no need to construct the realization nor to transform the equations into the Popov form that allows to compute n from certain indices of this form. Though such transformation is always possible, it is much easier to compute $\deg \det_D \bar{P}(z)$.

Note that though the concept of non-unique Dieudonné determinant may be not easily understandable for engineers, its degree is unique and can be computed from the simple algorithm, part of which mimics the computation of standard determinant by Gaussian elimination. Moreover, many properties of Dieudonné determinant coincide with those of the standard determinant.

Moreover, recall that not all sets of i/o equations (1) are realizable in the state-space form. The degree of the Dieudonné determinant may be used to define the order of such control systems that agrees with the corresponding definition for realizable systems.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

Ü. Kotta was supported by the ERC grant PUT481. M. Halás was supported by the Slovank Grant Agency VEGA grant No. 1/0276/14.

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Appendix A. Definition of the differential filed $\mathcal{Q}$, associated with control system

Recall briefly the algebraic formalism from (Kotta et al., 2011). Let $\mathcal{A}$ be the ring of real analytic functions in a finite number of variables from the sets $\mathcal{Y} = \{y(t+k), k \in \mathbb{Z}\}$ and $\mathcal{U} = \{u(t+l), l \in \mathbb{Z}\}$. Let $\delta : \mathcal{A} \to \mathcal{A}$ and $\delta^{-1} : \mathcal{A} \to \mathcal{A}$ be the forward- and backward-shift operators, defined respectively as $\delta(y(t+k)) = y(t+k+1)$, $\delta(u(t+l)) = u(t+l+1)$ and $\delta^{-1}(y(t+k)) = y(t+k-1)$, $\delta^{-1}(u(t+l)) = u(t+l-1)$ for $k, l \in \mathbb{Z}$. When one applies the forward- or backward-shift operator to a function from the ring $\mathcal{A}$, then all its arguments have to be shifted. Note that $\mathcal{A}$ is a difference ring with the shift operator, being an automorphism. Let $\mathcal{S}$ be a multiplicative subset of the ring $\mathcal{A}$, meaning that $1 \in \mathcal{S}, 0 \notin \mathcal{S}$ and if $\alpha \in \mathcal{S}, \beta \in \mathcal{S}$, then $\alpha\beta \in \mathcal{S}$. Since $\mathcal{A}$ is a difference ring, we additionally require that $\mathcal{S}$ is invariant with respect both to δ and δ^{-1}. Then, $\mathcal{A}_{\mathcal{S}} := \mathcal{S}^{-1}\mathcal{A} = \{\beta^{-1}\alpha \mid \alpha \in \mathcal{A} \text{ and } \beta \in \mathcal{S}\}$ defines the localization of the ring $\mathcal{A}$ with respect to $\mathcal{S}$. Observe that $\mathcal{A}_{\mathcal{S}}$ is an inversive difference ring with the shift operator δ given by $\delta(\beta^{-1}\alpha) := [\delta(\beta)]^{-1}\delta(\alpha)$ and via the natural injection $\alpha \mapsto \alpha/1$, $\mathcal{S}$ may be interpreted as a subset of $\mathcal{A}_{\mathcal{S}}$. Let $\Phi = \{\phi_1, \dots, \phi_p\}$ be a finite subset of $\mathcal{A}_{\mathcal{S}}$; Φ may be interpreted as a system of implicit i/o equations. Let $\mathcal{I}_{\mathcal{S}}$ be the difference ideal of $\mathcal{A}_{\mathcal{S}}$, containing all forward and backward shifts of ϕ_i, $i = 1, \dots, p$. Observe that Φ may be considered as a subset of $\widetilde{\mathcal{S}}^{-1}\mathcal{A}$ for some other multiplicative set $\widetilde{\mathcal{S}}$. For that reason we put $\mathcal{S}$ in the notation of the ideal $\mathcal{I}_{\mathcal{S}}$.

Assumption 1: *$\mathcal{I}_{\mathcal{S}}$ is prime, i.e., if $\alpha, \beta \in \mathcal{A}_{\mathcal{S}}$ and $\alpha\beta \in \mathcal{I}_{\mathcal{S}}$, then $\alpha \in \mathcal{I}_{\mathcal{S}}$ or $\beta \in \mathcal{I}_{\mathcal{S}}$, and proper, i.e., different from the entire ring.*

Properness of the ideal $\mathcal{I}_{\mathcal{S}}$ is equivalent to the condition $\mathcal{S} \cap \mathcal{I}_{\mathcal{S}} = \emptyset$. In particular, numerators of ϕ_i's do not belong to $\mathcal{S}$. Note that the assumption is an analogue of the submersivity assumption for explicitly defined systems. In (Halás et al., 2009) it has been proved that explicitly defined discrete-time nonlinear control system is submersive, iff the associated ideal is prime, proper and reflexive. Note that $\mathcal{I}_{\mathcal{S}}$ is reflexive.

Observe that $\mathcal{S}$ is constructed for system (1). However, when applying equivalence transformations with equations (1), $\mathcal{S}$ may have to be extended to $\widetilde{\mathcal{S}}$ by including possible expressions that do not equal to zero, restricting this way domain of definition. When we start, some functions ϕ_i in (1) may have denominators that, together with their forward/backward shifts and powers, should be included in the set $\mathcal{S}$. If the functions are analytic, then one may set $\mathcal{S} := \{1\}$, meaning that $\mathcal{S}^{-1}\mathcal{A} = \mathcal{A}$. Of course, additional denominators in the polynomial coefficients that show up in computation of degree of the Dieudonné determinant by Algorithm 1 should be also included in $\mathcal{S}$ together with their shifts and powers. That is, we extend our initial $\mathcal{S}$ by adding an infinite number of elements. The infinite $\mathcal{S}$ can be briefly described by its generator $\mathcal{S}_0$. The set $\mathcal{S}_0$ generates $\mathcal{S}$ if each element of $\mathcal{S}$ can be obtained from a finite number of elements of $\mathcal{S}_0$ by applying a finite number of multiplications and backward/forward shifts to these elements.

Let $\mathcal{A}_{\mathcal{S}}/\mathcal{I}_{\mathcal{S}}$ be the quotient ring. It consists of cosets $\bar{\varphi} = \varphi + \mathcal{I}_{\mathcal{S}}$ for $\varphi \in \mathcal{A}_{\mathcal{S}}$. We define addition and multiplication in this new ring by $\bar{\varphi} + \bar{\psi} := \overline{\varphi + \psi}$ and $\bar{\varphi} \cdot \bar{\psi} := \overline{\varphi \cdot \psi}$. These definitions do not depend on the choice of a representative in a coset. In particular, $\bar{\phi}_i = 0$, for $i = 1, \dots, p$. Since $\mathcal{I}_{\mathcal{S}}$ is a prime ideal, $\mathcal{A}_{\mathcal{S}}/\mathcal{I}_{\mathcal{S}}$ is an integral ring. Let $\mathcal{Q}$ denote the field of fractions of the ring $\mathcal{A}_{\mathcal{S}}/\mathcal{I}_{\mathcal{S}}$. Since δ can be naturally extended to the field of fractions, $\mathcal{Q}$ is an inversive difference field.

Appendix B. Definition of the map $e_{\mathcal{S}}^{\Phi}$

Assume that for equation (1) exists a multiplicative set $\mathcal{S}$ such that for all $i = 1, \dots, p$, $\phi_i \in \mathcal{A}_{\mathcal{S}}$ and Assumption 1 is satisfied. Then, the matrices $P(z)$ and $Q(z)$ in the associated polynomial description (4) have elements that are polynomials in z with coefficients in $\mathcal{A}_{\mathcal{S}}$. Let $e_{\mathcal{S}}^{\Phi}$ denote the map: $\mathcal{A}_{\mathcal{S}} \to \mathcal{Q} : \phi \mapsto \frac{\phi + \mathcal{I}_{\mathcal{S}}}{1}$. If $p(z) = \sum_i p_i z^i$ with $p_i \in \mathcal{A}_{\mathcal{S}}$, then we define $e_{\mathcal{S}}^{\Phi}(p(z)) = \sum_i e_{\mathcal{S}}^{\Phi}(p_i) z^i$.

This is a polynomial in $\mathcal{Q}[z;\delta]$. For a matrix $P(z)$ with elements in $\mathcal{A}_{\mathcal{S}}[z;\delta]$, by $\bar{P}(z) := e_{\mathcal{S}}^{\Phi}(P(z))$ we mean the matrix defined by $e_{\mathcal{S}}^{\Phi}(P(z))_{ij} = e_{\mathcal{S}}^{\Phi}(P(z)_{ij})$. We will also often use notation $e_{\mathcal{S}}^{\Phi}(p(z)) = p(z) + \mathcal{I}_{\mathcal{S}}$ for polynomials and $e_{\mathcal{S}}^{\Phi}(P(z)) = P(z) + \mathcal{I}_{\mathcal{S}}$ for polynomial matrices.