SOME PROPERTIES OF CONVERSION*

BY

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Our purpose is to establish the properties of conversion which are expressed in Theorems 1 and 2 below. We shall consider first conversion defined by Church's Rules I, II, III† and shall then extend our results to several other kinds of conversion.‡

1. **Conversion defined by Church's Rules I, II, III.** In our study of conversion we are particularly interested in the effects of Rules II and III and consider that applications of Rule I, though often necessary to prevent confusion of free and bound variables, do not essentially change the structure of a formula. Hence we shall omit mention of applications of Rule I whenever it seems that no essential ambiguity will result. Thus when we speak of replacing $\{\lambda x . \boldsymbol{M}\}(\boldsymbol{N})$§ by $\mathrm{S}_{\boldsymbol{N}}^{x}\boldsymbol{M}|$ it shall be understood that any applications of I are made which are needed to make this substitution an application of II. Also we may write bound variables as unchanged throughout discussions even though tacit applications of I in the discussion may have changed them.

A conversion in which III is not used and II is used exactly once will be called a *reduction*. If II is not used and III is used exactly once, the conversion will be called an *expansion*. "$\boldsymbol{A}$ imr $\boldsymbol{B}$," read "$\boldsymbol{A}$ is immediately reducible to $\boldsymbol{B}$," shall mean that it is possible to go from $\boldsymbol{A}$ to $\boldsymbol{B}$ by a single reduction. "$\boldsymbol{A}$ red $\boldsymbol{B}$," read "$\boldsymbol{A}$ is reducible to $\boldsymbol{B}$," shall mean that it is possible to go from $\boldsymbol{A}$ to $\boldsymbol{B}$ by one or more reductions.‖ "$\boldsymbol{A}$ conv-I $\boldsymbol{B}$," read "$\boldsymbol{A}$ conv $\boldsymbol{B}$ by applications of I only," shall mean just that (including the case of a zero number of applications). "$\boldsymbol{A}$ conv-I–II $\boldsymbol{B}$," read "$\boldsymbol{A}$ conv $\boldsymbol{B}$ by applications of I

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* Presented to the Society, April 20, 1935; received by the editors June 4, 1935.

† By Church's rules we shall mean the rules of procedure given in A. Church, *A set of postulates for the foundation of logic*, Annals of Mathematics, (2), vol. 33 (1932), pp. 346–366 (see pp. 355–356), as modified by S. C. Kleene, *Proof by cases in formal logic*, Annals of Mathematics, (2), vol. 35 (1934), pp. 529–544 (see p. 530). We assume familiarity with the material on pp. 349–355 of Church's paper and in §§1, 2, 3, 5 of Kleene's paper. We shall refer to the latter paper as "Kleene."

‡ The authors are indebted to Dr. S. C. Kleene for assistance in the preparation of this paper, in particular for the detection of an error in the first draft of it and for the suggestion of an improvement in the proof of Theorem 2.

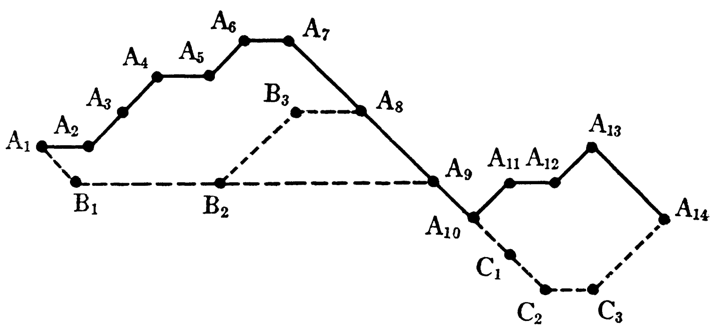
§ Note carefully the convention at the beginning of §3, Kleene, which we shall constantly use.

‖ Our use of "conv" allows us to write "$\boldsymbol{A}$ conv $\boldsymbol{B}$" even in the case that no applications of I, II, or III are made in going from $\boldsymbol{A}$ to $\boldsymbol{B}$ and $\boldsymbol{A}$ is the same as $\boldsymbol{B}$. But we write "$\boldsymbol{A}$ red $\boldsymbol{B}$" only if there is at least one reduction in the process of going from $\boldsymbol{A}$ to $\boldsymbol{B}$ by applications of I and II, and use the notation "$\boldsymbol{A}$ conv-I-II $\boldsymbol{B}$" if we wish to allow the possibility of no reductions.

and II only," shall mean just that (including the case of a zero number of applications).

We shall say that we *contract* or *perform a contraction on* $\{\lambda x . \boldsymbol{M}\}(\boldsymbol{N})$ if we replace it by $\mathrm{S}_{\boldsymbol{N}}^{x}\boldsymbol{M}|$.

It is possible to visualize the process of conversion by drawing a broken line in which the segments correspond to successive steps of the conversion, horizontal segments indicating applications of I, segments of negative slope applications of II, and segments of positive slope applications of III. Thus, in the figure A_1 conv A_2 and B_1 conv B_2 each by a single use of I, A_2 conv A_3 and B_2 conv B_3 each by a single use of III, and A_7 conv A_8 and C_1 conv C_2 each by a single use of II. The dotted lines represent various alternative conversions to the conversion given by the solid line.



A conversion in which no expansions follow any reductions will be called *peak* and one in which no reductions follow any expansions will be called a *valley*. The central theorem of this paper states that if $\boldsymbol{A}$ conv $\boldsymbol{B}$, there is a conversion from $\boldsymbol{A}$ to $\boldsymbol{B}$ which is a valley. We prove it by means of a lemma which states that a peak in which there is a single reduction can always be replaced by a valley. Then the theorem becomes obvious. For example, in the conversion pictured by the solid line in the figure we replace the peak $A_1A_2 \cdots A_8$ by the valley $A_1B_1B_2B_3A_8$, then the peak $B_2B_3A_8A_9$ by the valley B_2A_9, then the peak $A_{10}A_{11} \cdots A_{14}$ by the valley $A_{10}C_1C_2C_3A_{14}$, getting the valley $A_1B_1B_2A_9A_{10}C_1C_2C_3A_{14}$.

Suppose that a formula $\boldsymbol{A}$ has parts $\{\lambda x_j . \boldsymbol{M}_j\}(\boldsymbol{N}_j)$ which may or may not be parts of each other (cf. Kleene 2VIII (p. 532)). We suppose that, if $p \neq q$, $\{\lambda x_p . \boldsymbol{M}_p\}(\boldsymbol{N}_p)$ is not the same part as $\{\lambda x_q . \boldsymbol{M}_q\}(\boldsymbol{N}_q)$, though it may be the same formula. The $\{\lambda x_j . \boldsymbol{M}_j\}(\boldsymbol{N}_j)$ need not be all of the parts of $\boldsymbol{A}$ which have the form $\{\lambda y . \boldsymbol{P}\}(\boldsymbol{Q})$. We shall define the *residuals* of the $\{\lambda x_j . \boldsymbol{M}_j\}(\boldsymbol{N}_j)$ after a sequence of applications of I and II (these residuals being certain well-formed parts of the formula which results from the sequence of applications of

I and II). If no applications of I or II occur, each part $\{\lambda x_j . M_j\}(N_j)$ is its own residual. If a series of applications of I occur, then each part $\{\lambda x_j . M_j\}(N_j)$ is changed into a part $\{\lambda y_j . M_j'\}(N_j')$ of the resulting formula and this part $\{\lambda y_j . M_j'\}(N_j')$ is the residual of $\{\lambda x_j . M_j\}(N_j)$ in the resulting formula. Clearly the residuals of two parts coincide only if the parts coincide. Next consider the case where a single reduction occurs. Let $\{\lambda x . M\}(N)$ be the part of A which is contracted in performing the reduction and A' be the formula resulting from A by the reduction. Let $\{\lambda x_p . M_p\}(N_p)$ be an arbitrary one of the $\{\lambda x_j . M_j\}(N_j)$.

Case 1. Let $\{\lambda x_p . M_p\}(N_p)$ not be part of $\{\lambda x . M\}(N)$. Then either (a) $\{\lambda x . M\}(N)$ has no part in common with $\{\lambda x_p . M_p\}(N_p)$ or else (b) $\{\lambda x . M\}(N)$ is part of $\{\lambda x_p . M_p\}(N_p)$ (by Kleene 2VIII). If (a) holds, then under the reduction from A to A', $\{\lambda x_p . M_p\}(N_p)$ goes into a definite part of A' which we shall call the residual in A' of $\{\lambda x_p . M_p\}(N_p)$. In this case the residual of $\{\lambda x_p . M_p\}(N_p)$ is the same formula as $\{\lambda x_p . M_p\}(N_p)$. If (b) holds then $\{\lambda x . M\}(N)$ is part either of M_p or of N_p since $\{\lambda x_p . M_p\}(N_p)$ is not part of $\{\lambda x . M\}(N)$ (see Kleene, 2X and 2XII). Hence if we contract $\{\lambda x . M\}(N)$ we perform a reduction on the formula $\{\lambda x_p . M_p\}(N_p)$ which carries it into a formula $\{\lambda x_p' . M_p'\}(N_p')$. Then the reduction from A to A' can be considered as consisting of the replacement of the part $\{\lambda x_p . M_p\}(N_p)$ of A by $\{\lambda x_p' . M_p'\}(N_p')$ and this particular occurrence of $\{\lambda x_p' . M_p'\}(N_p')$ in A' is called the residual in A' of $\{\lambda x_p . M_p\}(N_p)$.

Case 2. Let $\{\lambda x_p . M_p\}(N_p)$ be part of $\{\lambda x . M\}(N)$. By Kleene 2X this case breaks up into three subcases.

(a) Let $\{\lambda x_p . M_p\}(N_p)$ be $\{\lambda x . M\}(N)$. Then we say that $\{\lambda x_p . M_p\}(N_p)$ has no residual in A'.

(b) Let $\{\lambda x_p . M_p\}(N_p)$ be part of $\lambda x . M$ and hence part of M (by Kleene 2XII). Let M' be the result of replacing all free x's of M except those occurring in $\{\lambda x_p . M_p\}(N_p)$ by N. Under these changes the part $\{\lambda x_p . M_p\}(N_p)$ of M goes into a definite part of M' which we shall denote also by $\{\lambda x_p . M_p\}(N_p)$, since it is the same formula. If now we replace $\{\lambda x_p . M_p\}(N_p)$ in M' by $S_N^x \{\lambda x_p . M_p\}(N_p)|$, M' becomes $S_N^x M|$ and we denote by $S_N^x \{\lambda x_p . M_p\}(N_p)|$ the particular occurrence of $S_N^x \{\lambda x_p . M_p\}(N_p)|$ in $S_N^x M|$ that resulted from replacing $\{\lambda x_p . M_p\}(N_p)$ in M' by the formula $S_N^x \{\lambda x_p . M_p\}(N_p)|$. Now the residual in A' of $\{\lambda x_p . M_p\}(N_p)$ in A is defined to be the part $S_N^x \{\lambda x_p . M_p\}(N_p)|$ in the particular occurrence of $S_N^x M|$ in A' that resulted from replacing $\{\lambda x . M\}(N)$ in A by $S_N^x M|$.

(c) Let $\{\lambda x_p . M_p\}(N_p)$ be part of N. And let $\{\lambda y_i . P_i\}(Q_i)$ respectively stand for the particular occurrences of the formula $\{\lambda x_p . M_p\}(N_p)$ in $S_N^x M|$

which are the part $\{\lambda x_p . \boldsymbol{M}_p\}(\boldsymbol{N}_p)$ in each of the particular occurrences of the formula $\boldsymbol{N}$ in $\mathrm{S}_{\boldsymbol{N}}^{x}\boldsymbol{M}|$ that resulted from replacing the free x's of $\boldsymbol{M}$ by $\boldsymbol{N}$. Now the residuals in $\boldsymbol{A}'$ of $\{\lambda x_p . \boldsymbol{M}_p\}(\boldsymbol{N}_p)$ in $\boldsymbol{A}$ are the parts $\{\lambda y_i . \boldsymbol{P}_i\}(\boldsymbol{Q}_i)$ in the particular occurrence of the formula $\mathrm{S}_{\boldsymbol{N}}^{x}\boldsymbol{M}|$ in $\boldsymbol{A}'$ that resulted from replacing $\{\lambda x . \boldsymbol{M}\}(\boldsymbol{N})$ in $\boldsymbol{A}$ by $\mathrm{S}_{\boldsymbol{N}}^{x}\boldsymbol{M}|$.

This completes the definition of the residuals in $\boldsymbol{A}'$ of $\{\lambda x_p . \boldsymbol{M}_p\}(\boldsymbol{N}_p)$ in $\boldsymbol{A}$ in the case that $\boldsymbol{A}'$ is obtained from $\boldsymbol{A}$ by a single reduction. Clearly the residuals in $\boldsymbol{A}'$ of $\{\lambda x_p . \boldsymbol{M}_p\}(\boldsymbol{N}_p)$ in $\boldsymbol{A}$ (if any) have the form $\{\lambda x . \boldsymbol{M}\}(\boldsymbol{N})$. Call them $\{\lambda y_i . \boldsymbol{P}_i\}(\boldsymbol{Q}_i)$. Then if $\boldsymbol{A}'$ imr $\boldsymbol{A}''$, the residuals in $\boldsymbol{A}''$ of $\{\lambda x_p . \boldsymbol{M}_p\}(\boldsymbol{N}_p)$ in $\boldsymbol{A}$ are defined to be the residuals in $\boldsymbol{A}''$ of the $\{\lambda y_i . \boldsymbol{P}_i\}(\boldsymbol{Q}_i)$ in $\boldsymbol{A}'$. We continue in this way, defining the residuals after each successive reduction as the residuals of the formulas that were residuals before the reduction, and noting that the residuals always have the form $\{\lambda x . \boldsymbol{M}\}(\boldsymbol{N})$. We also note that a residual in $\boldsymbol{B}$ of the part $\{\lambda x . \boldsymbol{M}\}(\boldsymbol{N})$ in $\boldsymbol{A}$ cannot coincide with a residual in $\boldsymbol{B}$ of the part $\{\lambda x' . \boldsymbol{M}'\}(\boldsymbol{N}')$ in $\boldsymbol{A}$ unless $\{\lambda x . \boldsymbol{M}\}(\boldsymbol{N})$ coincides with $\{\lambda x' . \boldsymbol{M}'\}(\boldsymbol{N}')$.

We say that a sequence of reductions on $\boldsymbol{A}$, say $\boldsymbol{A}$ imr $\boldsymbol{A}_1$ imr $\boldsymbol{A}_2$ imr $\cdots$ imr $\boldsymbol{A}_{n+1}$, is a *sequence of contractions on the parts* $\{\lambda x_j . \boldsymbol{M}_j\}(\boldsymbol{N}_j)$ *of* $\boldsymbol{A}$ if the reduction from $\boldsymbol{A}_i$ to $\boldsymbol{A}_{i+1}$ ($i=0, \cdots, n$; $\boldsymbol{A}_0$ the same as $\boldsymbol{A}$) is a contraction on one of the residuals in $\boldsymbol{A}_i$ of the $\{\lambda x_j . \boldsymbol{M}_j\}(\boldsymbol{N}_j)$. Moreover, if no residuals of the $\{\lambda x_j . \boldsymbol{M}_j\}(\boldsymbol{N}_j)$ occur in $\boldsymbol{A}_{n+1}$ we say that the sequence of contractions on the $\{\lambda x_j . \boldsymbol{M}_j\}(\boldsymbol{N}_j)$ terminates and that $\boldsymbol{A}_{n+1}$ is the result.

In some cases we wish to speak of a sequence of contractions on the parts $\{\lambda x_j . \boldsymbol{M}_j\}(\boldsymbol{N}_j)$ of $\boldsymbol{A}$ where the set $\{\lambda x_j . \boldsymbol{M}_j\}(\boldsymbol{N}_j)$ may be vacuous. To handle this we shall agree that if the set $\{\lambda x_j . \boldsymbol{M}_j\}(\boldsymbol{N}_j)$ is vacuous, the sequence of contractions shall be a vacuous sequence of reductions.

LEMMA 1. *If* $\{\lambda x_j . \boldsymbol{M}_j\}(\boldsymbol{N}_j)$ *are parts of* $\boldsymbol{A}$, *then a number* m *can be found such that any sequence of contractions on the* $\{\lambda x_j . \boldsymbol{M}_j\}(\boldsymbol{N}_j)$ *will terminate after at most* m *contractions, and if* $\boldsymbol{A}'$ *and* $\boldsymbol{A}''$ *are two results of terminating sequences of contractions on the* $\{\lambda x_j . \boldsymbol{M}_j\}(\boldsymbol{N}_j)$, *then* $\boldsymbol{A}'$ *conv*-I $\boldsymbol{A}''$.

Proof by induction on the number of proper symbols of $\boldsymbol{A}$.

The lemma is true if $\boldsymbol{A}$ is a proper symbol, the number m being 0.

Assume the lemma true for formulas with n or less proper symbols. Let $\boldsymbol{A}$ have $n+1$ proper symbols.

Case 1. $\boldsymbol{A}$ is $\lambda x . \boldsymbol{M}$. Then all the parts $\{\lambda x_j . \boldsymbol{M}_j\}(\boldsymbol{N}_j)$ of $\boldsymbol{A}$ must be parts of $\boldsymbol{M}$. However $\boldsymbol{M}$ has only n proper symbols and so we use the hypothesis of the induction.

Case 2. $\boldsymbol{A}$ is $\{\boldsymbol{F}\}(\boldsymbol{X})$.

(a) $\{\boldsymbol{F}\}(\boldsymbol{X})$ is not one of the $\{\lambda x_j . \boldsymbol{M}_j\}(\boldsymbol{N}_j)$. Then any sequence of contractions on the parts $\{\lambda x_j . \boldsymbol{M}_j\}(\boldsymbol{N}_j)$ of $\boldsymbol{A}$ can be replaced* by two sequences of contractions performed successively, one on those of the $\{\lambda x_j . \boldsymbol{M}_j\}(\boldsymbol{N}_j)$ which are parts of $\boldsymbol{F}$ and one on those of the $\{\lambda x_j . \boldsymbol{M}_j\}(\boldsymbol{N}_j)$ which are parts of $\boldsymbol{X}$, in such a way that if the original sequence of contractions carried $\{\boldsymbol{F}\}(\boldsymbol{X})$ into $\{\boldsymbol{F}'\}(\boldsymbol{X}')$ then the two sequences of contractions by which it is replaced carry $\boldsymbol{F}$ into $\boldsymbol{F}'$ and $\boldsymbol{X}$ into $\boldsymbol{X}'$ respectively and the total number of contractions on residuals of parts of $\boldsymbol{F}$ or on residuals of parts of $\boldsymbol{X}$ is the same as before.† Then we use the hypothesis of the induction, since $\boldsymbol{F}$ and $\boldsymbol{X}$ each have n or less proper symbols.

(b) $\{\boldsymbol{F}\}(\boldsymbol{X})$ is one of the $\{\lambda x_j . \boldsymbol{M}_j\}(\boldsymbol{N}_j)$, say $\{\lambda x_p . \boldsymbol{M}_p\}(\boldsymbol{N}_p)$. As long as the residual of $\{\lambda x_p . \boldsymbol{M}_p\}(\boldsymbol{N}_p)$ has not been contracted, the argument of (a) applies. Hence if any sequence of contractions on the $\{\lambda x_j . \boldsymbol{M}_j\}(\boldsymbol{N}_j)$ of $\boldsymbol{A}$ is continued long enough, the residual of $\{\lambda x_p . \boldsymbol{M}_p\}(\boldsymbol{N}_p)$ must be contracted (since we prove readily that one and only one residual of $\{\lambda x_p . \boldsymbol{M}_p\}(\boldsymbol{N}_p)$ occurs, until a contraction on the residual of $\{\lambda x_p . \boldsymbol{M}_p\}(\boldsymbol{N}_p)$). Let a sequence of contractions, μ, on the $\{\lambda x_j . \boldsymbol{M}_j\}(\boldsymbol{N}_j)$ consist of a sequence, ϕ, of contractions on the $\{\lambda x_j . \boldsymbol{M}_j\}(\boldsymbol{N}_j)$ which are different from $\{\lambda x_p . \boldsymbol{M}_p\}(\boldsymbol{N}_p)$, a contraction, β, on the residual of $\{\lambda x_p . \boldsymbol{M}_p\}(\boldsymbol{N}_p)$, and a sequence, θ, consisting of the remaining contractions of μ. Now, as in (a), we can replace ϕ by a sequence, α, of contractions on the $\{\lambda x_j . \boldsymbol{M}_j\}(\boldsymbol{N}_j)$ which are parts of $\lambda x_p . \boldsymbol{M}_p$ (and therefore parts of $\boldsymbol{M}_p$), followed by a sequence, η, of contractions on the $\{\lambda x_j . \boldsymbol{M}_j\}(\boldsymbol{N}_j)$ which are parts of $\boldsymbol{N}_p$, and this without changing the total number of contractions on residuals of parts of $\boldsymbol{M}_p$ or on residuals of parts of $\boldsymbol{N}_p$. Then η followed by β can be replaced by β', a contraction on the residual, $\{\lambda y . \boldsymbol{P}\}(\boldsymbol{N}_p)$, of $\{\lambda x_p . \boldsymbol{M}_p\}(\boldsymbol{N}_p)$, followed by a set of applications of η on each of the occurrences of $\boldsymbol{N}_p$ in $\mathrm{S}_{\boldsymbol{N}_p}^{y}\boldsymbol{P}|$ that arose by substituting $\boldsymbol{N}_p$ for $\boldsymbol{y}$ in $\boldsymbol{P}$. Our sequence of contractions now has a special form, namely a sequence of contractions, α, on parts of $\boldsymbol{M}_p$, followed by a contraction, β', on the residual of $\{\lambda x_p . \boldsymbol{M}_p\}(\boldsymbol{N}_p)$, followed by other contractions. We will now indicate a process whereby this sequence of contractions can be replaced by another having the same special form but having the property that after the contraction on the residual of $\{\lambda x_p . \boldsymbol{M}_p\}(\boldsymbol{N}_p)$ one less contraction on the residuals of parts of $\boldsymbol{M}_p$ occurs. This process can then be successively applied

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* We say that a sequence, μ, of reductions on $\boldsymbol{A}$ can be replaced by a sequence, ν, of reductions on $\boldsymbol{A}$, if both μ and ν give the same end formula $\boldsymbol{B}$ and residuals in $\boldsymbol{B}$ of any part $\{\lambda x_j . \boldsymbol{M}_j\}(\boldsymbol{N}_j)$ are the same under both μ and ν.

† The reader will easily understand the convention which we use when we say that the same sequence of contractions which carries $\boldsymbol{F}$ into $\boldsymbol{F}'$ will carry $\{\boldsymbol{F}\}(\boldsymbol{X})$ into $\{\boldsymbol{F}'\}(\boldsymbol{X})$ and that the same sequence of contractions which carries $\boldsymbol{X}$ into $\boldsymbol{X}'$ will carry $\{\boldsymbol{F}'\}(\boldsymbol{X})$ into $\{\boldsymbol{F}'\}(\boldsymbol{X}')$. This convention will also be used in (b).

until no contraction on a residual of a part of $\boldsymbol{M}_p$ occurs after the contraction on the residual of $\{\lambda x_p . \boldsymbol{M}_p\}(\boldsymbol{N}_p)$. Moreover this process increases by one the number of contractions which precede the contraction on the residual of $\{\lambda x_p . \boldsymbol{M}_p\}(\boldsymbol{N}_p)$, so that the total number of contractions on residuals of parts of $\boldsymbol{M}_p$ is the same after the process as before, and hence the same as in μ.

Let us consider that sequence of reductions ζ which is composed of β' and the contractions that follow it, up to and including the first contraction on a residual of a part of $\boldsymbol{M}_p$. Denoting the formula on which ζ acts by $\{\lambda y . \boldsymbol{P}\}(\boldsymbol{N}_p)$, we see that ζ can be considered as the act of first replacing the free y's of $\boldsymbol{P}$ by various formulas, $\boldsymbol{N}_{pk}$, got from $\boldsymbol{N}_p$ by sets of reductions, and then contracting on a residual, $\{\lambda z . \boldsymbol{R}\}(\boldsymbol{S})$, of one of the $\{\lambda x_j . \boldsymbol{M}_j\}(\boldsymbol{N}_j)$ which are parts of $\boldsymbol{M}_p$, say $\{\lambda x_q . \boldsymbol{M}_q\}(\boldsymbol{N}_q)$. From this point of view, we see that none of the free z's of $\boldsymbol{R}$ are parts of any $\boldsymbol{N}_{pk}$, and hence ζ can be replaced by a contraction on the residual in $\lambda y . \boldsymbol{P}$ of $\{\lambda x_q . \boldsymbol{M}_q\}(\boldsymbol{N}_q)$ of which $\{\lambda z . \boldsymbol{R}\}(\boldsymbol{S})$ is a residual, followed by a contraction on the residual of $\{\lambda x_p . \boldsymbol{M}_p\}(\boldsymbol{N}_p)$, followed by contractions on residuals of parts of $\boldsymbol{N}_p$. This completes the indication of what the process is.

Hence μ can be replaced by a sequence of contractions, α, on the $\{\lambda x_j . \boldsymbol{M}_j\}(\boldsymbol{N}_j)$ which are parts of $\boldsymbol{M}_p$ (such that α contains as many contractions as there are contractions in μ on residuals of parts of $\boldsymbol{M}_p$), followed by a contraction, β, on the residual of $\{\lambda x_p . \boldsymbol{M}_p\}(\boldsymbol{N}_p)$, followed by a sequence of contractions, γ, on residuals of parts of $\boldsymbol{N}_p$. Moreover, after α and β, $\{\lambda x_p . \boldsymbol{M}_p\}(\boldsymbol{N}_p)$ has become a formula containing several occurrences of $\boldsymbol{N}_p$ and γ is a sequence of contractions on parts of these occurrences of $\boldsymbol{N}_p$. Hence, since $\lambda x_p . \boldsymbol{M}_p$ and $\boldsymbol{N}_p$ each contain n or less proper symbols, we can use the hypothesis of the induction in connection with α and γ to show that if α followed by β followed by γ terminates the result is unique to within applications of I. But if μ terminates, so does α followed by β followed by γ. Hence the results of any two terminating sequences are unique to within applications of I.

It remains to be shown that a number m can be found such that each sequence of contractions terminates after at most m contractions.

By the hypothesis of induction, a number a can be found such that any sequence of contractions on those $\{\lambda x_j . \boldsymbol{M}_j\}(\boldsymbol{N}_j)$ which are parts of $\boldsymbol{M}_p$ terminates after at most a contractions, and a number b can be found such that any sequence of contractions on those $\{\lambda x_j . \boldsymbol{M}_j\}(\boldsymbol{N}_j)$ which are parts of $\boldsymbol{N}_p$ terminates after at most b contractions. Then any sequence of contractions on the $\{\lambda x_j . \boldsymbol{M}_j\}(\boldsymbol{N}_j)$ of $\boldsymbol{A}$ must, if continued, include a contraction on the residual of $\{\lambda x_p . \boldsymbol{M}_p\}(\boldsymbol{N}_p)$, after at most $a+b+1$ contractions. Hence we may confine our attention to sequences of contractions μ on the $\{\lambda x_j . \boldsymbol{M}_j\}(\boldsymbol{N}_j)$ of $\boldsymbol{A}$ which include a contraction on the residual of $\{\lambda x_p . \boldsymbol{M}_p\}(\boldsymbol{N}_p)$ and which can therefore be replaced by a sequence of contractions of the form, α followed by β followed by γ. Moreover it will be clear, on examination of our preceding argument, that, when the sequence μ is replaced by the sequence, α followed by β followed by γ, the total number of contractions in the sequence is either increased or left unchanged (because each step in the process of transforming μ into α followed by β followed by γ has the property that it cannot decrease the total number of contractions). Therefore it is sufficient to find a number m such that any sequence of contractions on the $\{\lambda x_j . \boldsymbol{M}_j\}(\boldsymbol{N}_j)$ of $\boldsymbol{A}$ which has the form, α followed by β followed by γ, must terminate after at most m contractions.

If we start with the formula $\boldsymbol{M}_p$ and perform a terminating sequence of contractions on those $\{\lambda x_j . \boldsymbol{M}_j\}(\boldsymbol{N}_j)$ which are parts of $\boldsymbol{M}_p$, the result is a formula $\boldsymbol{M}_p'$, which is unique to within applications of I, and which contains a certain number, c, $\geqq 1$, of occurrences of x_p as a free symbol. And in the case of any sequence of contractions on those $\{\lambda x_j . \boldsymbol{M}_j\}(\boldsymbol{N}_j)$ which are parts of $\boldsymbol{M}_p$, whether terminating or not, the result (that is, the formula into which $\boldsymbol{M}_p$ is transformed) contains at most c occurrences of x_p as a free symbol.

Hence the required number m is $a+1+cb$.

LEMMA 2. *If $\boldsymbol{A}$ imr $\boldsymbol{B}$ by a contraction on the part $\{\lambda x . \boldsymbol{M}\}(\boldsymbol{N})$ of $\boldsymbol{A}$, and $\boldsymbol{A}$ is $\boldsymbol{A}_1$, and $\boldsymbol{A}_1$ imr $\boldsymbol{A}_2$, $\boldsymbol{A}_2$ imr $\boldsymbol{A}_3$, $\cdots$, and, for all k, $\boldsymbol{B}_k$ is the result of a terminating sequence of contractions on the residuals in $\boldsymbol{A}_k$ of $\{\lambda x . \boldsymbol{M}\}(\boldsymbol{N})$, then:*

I. *$\boldsymbol{B}_1$ is $\boldsymbol{B}$.*

II. *For all k, $\boldsymbol{B}_k$ conv-*I–II *$\boldsymbol{B}_{k+1}$.*

III. *Even if the sequence $\boldsymbol{A}_1, \boldsymbol{A}_2, \cdots$ can be continued to infinity, there is a number ϕ_m, depending on the formula $\boldsymbol{A}$, the part $\{\lambda x . \boldsymbol{M}\}(\boldsymbol{N})$ of $\boldsymbol{A}$, and the number m, such that, starting with $\boldsymbol{B}_m$, at most ϕ_m consecutive $\boldsymbol{B}_k$'s occur for which it is not true that $\boldsymbol{B}_k$ red $\boldsymbol{B}_{k+1}$.*

Part I is obvious.

We prove Part II readily as follows. Let $\{\lambda y_i . \boldsymbol{P}_i\}(\boldsymbol{Q}_i)$ be the residuals in $\boldsymbol{A}_k$ of $\{\lambda x . \boldsymbol{M}\}(\boldsymbol{N})$ and let $\boldsymbol{A}_k$ imr $\boldsymbol{A}_{k+1}$ be a contraction on the part $\{\lambda z . \boldsymbol{R}\}(\boldsymbol{S})$ of $\boldsymbol{A}_k$. Then $\boldsymbol{B}_{k+1}$ is the result of a terminating sequence of contractions on $\{\lambda z . \boldsymbol{R}\}(\boldsymbol{S})$ and the parts $\{\lambda y_i . \boldsymbol{P}_i\}(\boldsymbol{Q}_i)$ of $\boldsymbol{A}_k$. Now if $\{\lambda z . \boldsymbol{R}\}(\boldsymbol{S})$ is one of the $\{\lambda y_i . \boldsymbol{P}_i\}(\boldsymbol{Q}_i)$, then no residuals of $\{\lambda z . \boldsymbol{R}\}(\boldsymbol{S})$ occur in $\boldsymbol{B}_k$, and $\boldsymbol{B}_k$ conv-I $\boldsymbol{B}_{k+1}$. If however $\{\lambda z . \boldsymbol{R}\}(\boldsymbol{S})$ is not one of the $\{\lambda y_i . \boldsymbol{P}_i\}(\boldsymbol{Q}_i)$, then a set of residuals of $\{\lambda z . \boldsymbol{R}\}(\boldsymbol{S})$ does occur in $\boldsymbol{B}_k$ and a terminating sequence of contractions on these residuals in $\boldsymbol{B}_k$ gives $\boldsymbol{B}_{k+1}$ by Lemma 1.

In order to prove Part III we note that $\boldsymbol{B}_k$ red $\boldsymbol{B}_{k+1}$ unless the reduction from $\boldsymbol{A}_k$ to $\boldsymbol{A}_{k+1}$ consists of a contraction on a residual in $\boldsymbol{A}_k$ of $\{\lambda x . \boldsymbol{M}\}(\boldsymbol{N})$

(see preceding paragraph). But if we start with any particular A_k this can only be the case a finite number of successive times, by Lemma 1. Hence we define ϕ_m as follows. Perform m successive reductions on A in all possible ways. This gives a finite set of formulas (except for applications of I). In each formula find the largest number of reductions that can occur in a terminating sequence of contractions on the residuals of $\{\lambda x.M\}(N)$. Then let ϕ_m be the largest of these.

THEOREM 1. *If A conv B, there is a conversion from A to B in which no expansion precedes any reduction.*

That is, any conversion can be replaced by a conversion which is a valley. This follows from Lemma 2 by the process already indicated.

COROLLARY 1. *If B is a normal form* of A, then A conv*-I–II B.

For no reductions are possible on a normal form.

COROLLARY 2. *If A has a normal form, its normal form is unique (to within applications of Rule* I).

For if B and B' are both normal forms of A, then B' is a normal form of B. Hence B conv-I–II B'. Hence B conv-I B', since no reductions are possible on the normal form B.

Note that only parts I and II of Lemma 2 are needed for Theorem 1 and its corollaries.

THEOREM 2. *If B is a normal form of A, then there is a number m such that any sequence of reductions starting from A will lead to B (to within applications of Rule* I) *after at most m reductions.*

We prove by induction on n that, if a formula B is a normal form of some formula A, and there is a sequence of n reductions leading from A to B, then there is a number $\psi_{A,n}$ depending on the formula A and the number n such that any sequence of reductions starting from A will lead to a normal form of A (which will be B to within applications of I by Theorem 1, Corollary 2) in at most $\psi_{A,n}$ reductions.

If $n=0$, we take $\psi_{A,0}$ to be 0.

Assume our statement for $n=k$. Let A imr C, C imr C_1, C_1 imr $C_2, \cdots,$ C_{k-1} imr B. Let A be the same as A_1, A_1 imr A_2, A_2 imr $A_3, \cdots$. By Lemma 2 there is a sequence (D_1 the same as C) D_1 conv-I–II D_2, D_2 conv-I–II $D_3, \cdots,$ such that A_j conv-I–II D_j for all j's for which A_j exists, and also, if the reduction from A to D_1 (or C) is a contraction on $\{\lambda x.M\}(N)$, such that, starting with D_m, at most ϕ_m consecutive D_j's occur for which it is not true that

* Kleene §5, p. 535.

$\boldsymbol{D}_j$ red $\boldsymbol{D}_{j+1}$. The sequence $\boldsymbol{C}$ imr $\boldsymbol{C}_1$, $\boldsymbol{C}_1$ imr $\boldsymbol{C}_2$, $\cdots$ leads to $\boldsymbol{B}$ in k reductions and so by the hypothesis of the induction there is a number $\psi_{C,k}$ such that any sequence of reductions on $\boldsymbol{C}$ leads to a normal form (i.e., terminates) after at most $\psi_{C,k}$ reductions. Hence there are at most $\psi_{C,k}$ reductions in the sequence $\boldsymbol{D}_1$ conv-I–II $\boldsymbol{D}_2$, $\boldsymbol{D}_2$ conv-I–II $\boldsymbol{D}_3$, $\cdots$, and hence this terminates after at most $f(\psi_{C,k})$ steps where $f(p)$ is defined as follows:

$$f(0) = \phi_1,$$

$$f(p+1) = f(p) + M + 1,$$

where M is the greatest of the numbers $\phi_1, \phi_2, \cdots, \phi_{f(p)+1}$.*

Then, since the sequence of $\boldsymbol{D}_j$'s continues as long as there are $\boldsymbol{A}_j$'s on which reductions can be performed, it follows that after at most $f(\psi_{C,k})$ reductions we come to an $\boldsymbol{A}_j$ on which no reductions are possible. But this is equivalent to saying that this $\boldsymbol{A}_j$ is a normal form. Hence any sequence of $k+1$ reductions from $\boldsymbol{A}$ to $\boldsymbol{B}$ determines an upper bound which holds for all sequences of reductions starting from $\boldsymbol{A}$. We then take all sequences of $k+1$ reductions starting from $\boldsymbol{A}$ (this is a finite set of sequences, since we reckon two sequences as the same if they differ only by applications of I) and find the upper bounds determined by each one of them that leads to a normal form. Then we define $\psi_{A,k+1}$ to be the least of these upper bounds.

This completes our induction. But, by Theorem 1, Corollary 1, if $\boldsymbol{B}$ is a normal form of $\boldsymbol{A}$, there is a sequence of some finite number of reductions leading from $\boldsymbol{A}$ to $\boldsymbol{B}$. Hence Theorem 2 follows.

COROLLARY. *If a formula has a normal form, every well-formed part of it has a normal form.*

2. **Other kinds of conversion.** There are also other systems of operations on formulas, similar to the system which we have been discussing and in which there can be distinguished reductions and expansions and possibly neutral operations (such as applications of Rule I). For convenience, we speak of the operations of any such system as conversions, and we define a normal form to be a formula on which no reductions are possible.

A kind of conversion which appears to be useful in certain connections is obtained by taking a new undefined term δ (restricting ourselves by never using δ as a bound symbol) and adding to Church's Rules I, II, III the following rules:

IV. *Suppose that* $\boldsymbol{M}$ *and* $\boldsymbol{N}$ *contain no free symbols other than* δ, *that there*

* $f(p)$ depends, of course, on the formula $\boldsymbol{A}$ and the part $\{\lambda \mathbf{x}.\boldsymbol{M}\}(\boldsymbol{N})$ of $\boldsymbol{A}$, as well as on p, because ϕ_m depends on $\boldsymbol{A}$ and $\{\lambda \mathbf{x}.\boldsymbol{M}\}(\boldsymbol{N})$.

is no part $\{\lambda z.\,\boldsymbol{P}\}(\boldsymbol{Q})$ *of either* $\boldsymbol{M}$ *or* $\boldsymbol{N}$, *and that there is no part* $\delta(\boldsymbol{R},\boldsymbol{S})$ *of either* $\boldsymbol{M}$ *or* $\boldsymbol{N}$ *in which* $\boldsymbol{R}$ *and* $\boldsymbol{S}$ *contain no free symbols other than* δ. *Then we may pass from a formula* $\boldsymbol{J}$ *to a formula* $\boldsymbol{K}$ *obtained from* $\boldsymbol{J}$ *by substituting for a particular occurrence of* $\delta(\boldsymbol{M},\boldsymbol{N})$ *in* $\boldsymbol{J}$ *either* $\lambda fx.f(f(x))$ *or* $\lambda fx.f(x)$ *according to whether it is or is not true that* $\boldsymbol{M}$ *conv*-I $\boldsymbol{N}$.

V. *The inverse operation of that described in Rule* IV *is allowable. That is, we may pass from* $\boldsymbol{K}$ *to* $\boldsymbol{J}$ *under the same circumstances.*

We call an application of Rule II together with applications of Rule I, or an application of Rule IV together with applications of Rule I, a reduction, and the reverse operations (involving Rule III or Rule V) expansions. We call any sequence of applications of various ones of the five rules a conversion. Also we say that we contract $\{\lambda x.\boldsymbol{M}\}(\boldsymbol{N})$ if we replace it by $\mathrm{S}_{\boldsymbol{N}}^{x}\boldsymbol{M}|$, and that we contract $\delta(\boldsymbol{M},\boldsymbol{N})$ if we replace it by $\lambda fx.f(f(x))$ or $\lambda fx.f(x)$ in accordance with Rule IV.

We define the residuals of $\{\lambda x.\boldsymbol{M}\}(\boldsymbol{N})$ after an application of I or II in the same way as before, and after an application of IV as what $\{\lambda x.\boldsymbol{M}\}(\boldsymbol{N})$ becomes (the restrictions in IV ensure that it becomes something of the form $\{\lambda y.\,\boldsymbol{P}\}(\boldsymbol{Q})$). The residuals of $\delta(\boldsymbol{M},\boldsymbol{N})$ after an application of I, II, or IV are defined only in the case that $\boldsymbol{M}$ and $\boldsymbol{N}$ are in normal form and contain no free symbols other than δ. In that case the residuals of $\delta(\boldsymbol{M},\boldsymbol{N})$ are whatever part or parts of the entire resulting formula $\delta(\boldsymbol{M},\boldsymbol{N})$ becomes, except that after an application of IV which is a contraction of $\delta(\boldsymbol{M},\boldsymbol{N})$ itself, $\delta(\boldsymbol{M},\boldsymbol{N})$ has no residual. Thus residuals of $\delta(\boldsymbol{M},\boldsymbol{N})$ are always of the form $\delta(\boldsymbol{P},\boldsymbol{Q})$, where $\boldsymbol{P}$ and $\boldsymbol{Q}$ are in normal form and contain no free symbols other than δ. We define a sequence of contractions on the parts $\{\lambda x_i.\boldsymbol{M}_i\}(\boldsymbol{N}_i)$ and $\delta(\boldsymbol{P}_i,\boldsymbol{Q}_i)$ of $\boldsymbol{A}$, where $\boldsymbol{P}_i$ and $\boldsymbol{Q}_i$ are in normal form and contain no free symbols other than δ, by analogy with our former definition. Similarly for a terminating sequence of such contractions. Then we prove Lemma 1 by an obvious extension of our former argument. Lemma 2 and Theorems 1 and 2 then follow as before. Of course we replace "conv-I–II" by "conv-I–II–IV" in Lemma 2 and in general throughout the proofs of Theorems 1 and 2. In Lemma 1 we allow that the set of parts of $\boldsymbol{A}$ on which a sequence of contractions is taken should include not only parts of the form $\{\lambda x_i.\boldsymbol{M}_i\}(\boldsymbol{N}_i)$ but also parts of the form $\delta(\boldsymbol{P}_k,\boldsymbol{Q}_k)$ in which $\boldsymbol{P}_k$ and $\boldsymbol{Q}_k$ are in normal form and contain no free symbols other than δ. And in Lemma 2 we consider also the case that $\boldsymbol{A}$ imr $\boldsymbol{B}$ by a contraction on the part $\delta(\boldsymbol{P},\boldsymbol{Q})$ of $\boldsymbol{A}$.

We may also consider a third kind of conversion, namely the conversion that results if we modify Kleene's definitions of *well-formed*, *free*, and *bound* by omitting the requirement that x be a free symbol of $\boldsymbol{R}$ from (3) of the definition (see Kleene, top of p. 530) and then modify Church's rules I, II, III by using the new meanings of *well-formed*, *free* and *bound* and by omitting "if the proper symbol x occurs in M" from both II and III.* Then we call an application of the modified II together with applications of the modified I a reduction, and the reverse operation an expansion, and applications of all three rules conversion. Also we say that we contract $\{\lambda x . M\}(N)$ if we replace it by $S_N^x M|$. If A has the form $\{\lambda x . M\}(N_1, N_2, \cdots, N_r)$, then $\{\lambda x . M\}(N_1)$ is said to be of order one in A and a contraction of $\{\lambda x . M\}(N_1)$ is said to be a reduction of order one of A. Then Lemma 1 and parts I and II of Lemma 2 hold (although some modification is required in the proofs). Hence Theorem 1 and its corollaries hold. But Theorem 2 is false. Instead a weaker form of Theorem 2 can be proved, namely:

THEOREM 3. *If A has a normal form, then there is a number m such that at most m reductions of order one can occur in a sequence of reductions on A.*

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* The first kind of conversion which we have considered is essentially equivalent to a certain portion of the combinatory axioms and rules of H. B. Curry (American Journal of Mathematics, vol. 52 (1930), pp. 509–536, 789–834), as has been proved by J. B. Rosser (see Annals of Mathematics, (2), vol. 36 (1935), p. 127). In terms of Curry's notation, our third kind of conversion can be thought of as differing from the first kind by the addition of the constancy function K.

In dealing with properties of conversion, use of the Schönfinkel-Curry combinatory analysis appears in certain connections to be an important, even indispensable, device. But to recast the present discussion and results entirely into a combinatory notation would, it is thought, be awkward or impossible, because of the difficulty in finding a satisfactory equivalent, for combinations, of the notions of *reduction* and *normal form* as employed in this paper.

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