On Syntactic Congruences for ω-languages

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Abstract. For ω-languages several notions of syntactic congruence were defined. The present paper investigates relationships between the so-called simple (because it is a simple translation from the usual definition in the case of finitary languages) syntactic congruence and its infinitary refinements investigated by Arnold [Ar85]. We show that in both cases not every ω-language having a finite syntactic monoid is regular and we give a characterization of those ω-languages having finite syntactic monoids. As the main result we derive a condition which guarantees that the simple syntactic congruence and Arnold's syntactic congruence coincide and show that *all* ω-languages in the Borel class $F_\sigma \cap G_\delta$ satisfy this condition.
Finally we define an alternative canonical object for ω-languages, namely a family of right-congruence relations. Using this object we give a necessary and sufficient condition for a regular ω-language to be accepted by its minimal-state automaton.

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1 Introduction

The well-known Kleene-Myhill-Nerode theorem for languages states that a language $U \subseteq \Sigma^*$ is regular (rational), iff its syntactic right-congruence $\sim_U$ defined by

$$x \sim_U y \text{ iff } \forall v \in \Sigma^* : xv \in U \leftrightarrow yv \in U$$

has a finite index. In that case the right-congruence classes correspond to the states of the unique minimal automaton that accepts U. An equivalent condition is that the finer two-sided syntactic congruence $\simeq_U$ defined by

$$x \simeq_U y \text{ iff } \forall u \in \Sigma^* : ux \sim_U uy$$

has a finite index. Here the congruence classes correspond to the elements of the transformation monoid associated with the minimal automaton accepting U.

As already observed by Trakhtenbrot [Tr62] these same observations are no longer true in the case of ω-languages (cf. also [JT83], [LS77] or [St83]). Here the class of ω-languages having a finite syntactic monoid (so-called finite-state ω-languages) is much larger than the class of ω-languages accepted by finite automata (regular or rational ω-languages) [St83].

Recently Arnold [Ar85] investigated a new concept of syntactic congruence for ω-languages. As his results show, this concept yields a characterization of regular ω-languages by finite monoids, but not in the same simple way as for finitary languages.

As we shall see below, despite the fact that Arnold's monoid is indeed more accurate (it is infinite for some ω-languages which are finite-state), yet there are even non-Borel ω-languages for which Arnold's monoid is finite. To this end we shall derive a necessary and sufficient condition for an ω-language for having a finite syntactic monoid in the sense of Arnold. As the main result we give a condition on ω-languages that guarantees that their Arnold's syntactic congruence coincides with the simple one. We show that this condition holds for all (including those which are not finite-state) ω-languages in the Borel-class $F_\sigma \cap G_\delta$ and thus extend the result in [St83].

Finally, we introduce an alternative notion of recognizability by a family of *right*-congruence relations, and give a necessary and sufficient condition for an ω-language to be acceptable by its "minimal-state" automaton, that is, an automaton isomorphic to its syntactic right-congruence.

2 Preliminaries

By Σ^* we denote the set (monoid) of finite words on a finite alphabet Σ, including the empty word e, let Σ^+ denote $\Sigma^* - \{e\}$ and Σ^ω the set of infinite words (ω-words). As usual we call subsets of Σ^* as *languages* and subsets of Σ^ω as *ω-languages*. For $u \in \Sigma^*$ and $\beta \in \Sigma^* \cup \Sigma^\omega$ let $u\beta$ be their concatenation and let u^ω be the ω-word formed by concatenating the word u infinitely often (provided $u \neq e$) . The concatenation product extends in an obvious way to subsets $U \subseteq \Sigma^\omega$ and $B \subseteq \Sigma^* \cup \Sigma^\omega$. For a language $U \subseteq \Sigma^*$ let U^* and U^ω denote respectively the set of finite and infinite sequences formed by concatenating words in U. By $|u|_a$ we denote the number of occurrences of the letter $a \in \Sigma$ in the word $u \in \Sigma^*$. Finally $u \preceq v$ and $u \prec v$ denote the facts that u is a prefix and a proper prefix of v.

An equivalence relation $\simeq$ is a *congruence* on Σ^* if $u \simeq v$ implies $xuy \simeq xvy$ for every $u, v, x, y \in \Sigma^*$. We say that $\simeq$ is a *right-congruence* if $u \simeq v$ implies $uy \simeq vy$ for every $u, v, y \in \Sigma^*$. Clearly every congruence is also a right-congruence. We will denote by $[v] := \{w : w \in \Sigma^* \text{ and } w \simeq v\}$ the equivalence class containing the word v, and we shall use $\langle v \rangle$ instead of $[v]$ if the corresponding relation is a right-congruence. We will say that $\simeq$ is *finite* when it has a finite index (or alternatively, $\Sigma^*/\simeq$ is finite), and that it is *trivial* when $\simeq$ is $\Sigma^* \times \Sigma^*$.

As in [Ar85] we say that a congruence $\simeq$ *covers* an ω-language F provided $F = \bigcup\{[u][v]^\omega : uv^\omega \in F\}$ and we say that an ω-language F is *regular* provided there is a finite congruenc $\simeq$ which covers F.

The natural (*Cantor-*) topology on the space Σ^ω is defined as follows. A set $E \subseteq \Sigma^\omega$ is *open* iff it is of the form $U\Sigma^\omega$, where $U \subseteq \Sigma^*$ (in other words, $\beta \in E$ iff it has a prefix in U). A set is *closed* if its complement is open (or if its elements do not have any prefix in some $U' \subseteq \Sigma^*$). The class G_δ consists of all countable intersection of open sets. A set is in F_σ if its complement is in G_δ, or if it can be written as a countable union of closed sets. The rest of the Borel hierarchy is constructed similarly.

A deterministic Muller automaton is a quintuple $\mathcal{A} = (\Sigma, Q, \delta, q_0, \mathcal{F})$ where Σ is the input alphabet, Q is the state-space, $\delta : Q \times \Sigma \to Q$ is the transition function, q_0 the initial state and $\mathcal{F} \subseteq 2^Q$ is a family of accepting subsets. By $Inf(\mathcal{A}, \alpha)$ we denote the subset of Q which is visited infinitely many times while $\mathcal{A}$ is reading $\alpha \in \Sigma^\omega$. The ω-language accepted/recognized by $\mathcal{A}$ is $\{\alpha \in \Sigma^\omega : Inf(\mathcal{A}, \alpha) \in \mathcal{F}\}$. According to Büchi-McNaughton theorem an ω-language is regular iff it is recognized by some deterministic finite-state Muller automaton. Additional material on ω-languages appears in [Ei74, HR85, St87, Th90, PP91].

Definition 1 (Syntactic Congruences) *Let $E \subseteq \Sigma^\omega$ be an ω-language. We associate with E the following equivalence relations on Σ^*:*

– *Syntactic right-congruence:*

$$x \sim_E y \text{ iff } \forall \beta \in \Sigma^\omega : x\beta \in E \leftrightarrow y\beta \in E \tag{1}$$

– *Simple syntactic congruence:*

$$x \simeq_E y \text{ iff } \forall u \in \Sigma^* : ux \sim_E uy \tag{2}$$

– *Infinitary syntactic-congruence:*

$$x \approx_E y \text{ iff } \forall u, v \in \Sigma^* : u(xv)^\omega \in E \leftrightarrow u(yv)^\omega \in E \tag{3}$$

– *Arnold's syntactic-congruence:*

$$x \cong_E y \text{ iff } x \simeq y \wedge x \approx y \tag{4}$$

By definition $\simeq$ refines $\sim$ and $\cong$ refines both $\simeq$ and $\approx$. In the general case $\simeq$ and $\approx$ are not comparable, since they refer to two different kinds of interchangability of x and y. For example, for $E = \{a, b\}^* a^\omega$, $a \simeq_E b$ but $a \not\approx_E b$. On the other hand for $E = abc^\omega$, $a \not\simeq_E b$ but $a \approx_E b$. We shall see later that some conditions on E imply that $\simeq$ refines $\approx$. An ω-language E such that $\simeq_E$ (or equivalently, $\sim_E$) is finite is called *finite-state*.

3 Some observations on Arnold's congruence

In this section we show that despite the fact that $\cong_E$ provides additional information on E which is missing from $\simeq_E$, still it fails in characterizing regular ω-languages as does $\simeq$ for languages.

Fact 1 *There are ω-languages which are finite-state but their Arnold's syntactic monoid is infinite.*

Proof: Let the language $V \subseteq \{a,b\}^*$ be defined by the equation

$$V = a \cup bV^2$$

Alternatively, V may be defined as the language consisting of those words $v \in \Sigma^*$ satisfying $|v|_a = |v|_b + 1$ and $|u|_a \leq |u|_b$ for every $u \prec v$. Let $E = V^\omega$. Then one easily verifies $E = VE = (a \cup bV^2)E = \{a,b\}E$. Thus $u \simeq_E v$ for every $u, v \in \{a,b\}^*$ and $\simeq_E$ is trivial. On the other hand we show that for every i, j such that $0 < i < j$, $b^i \ncong_E b^j$, hence $\cong_E$ is infinite. To this end we show that $(b^i a^{i+1})^\omega \in E$ and $(b^j a^{i+1})^\omega \notin E$. Since $b^i a^{i+1} \in V$, we have $(b^i a^{i+1})^\omega \in E = V^\omega$. Since every word in V contains more occurrences of a than of b, $j > i$ implies that the ω-word $(b^j a^{i+1})^\omega$ has no prefix in V, and consequently $(b^j a^{i+1})^\omega \notin V^\omega = E$. □

The second observation (as already noted in [Ar85]) is that, in general, the finiteness of $\cong_E$ does not guarantee regularity of E:

Fact 2 *The ω-language $Ult = \{uv^\omega : u \in \Sigma^*, v \in \Sigma^+\}$ of all ultimately periodic ω-words has a trivial syntactic monoid, that is $x \cong_{Ult} y$ for every $x, y \in \Sigma^*$, but Ult is not regular.*

Next we investigate the question which ω-languages have a finite syntactic monoid in the sense of Arnold. To this aim we show that with every ω-language E we can associate in a canonical way an ω-language F_E which is covered by $\cong_E$. Define

$$F_E = \bigcup \{[u][v]^\omega : uv^\omega \in E\}$$

where $[\cdot]$ denotes a congruence of $\cong_E$. It holds the following:

Lemma 3 *$E \cap Ult = F_E \cap Ult$.*

Proof: By definition $E \cap Ult \subseteq F_E \cap Ult$. Let $xy^\omega \in F_E$. Then there are u, v such that $uv^\omega \in E$ and $xy^\omega \in [u][v]^\omega$. From this we can obtain words y_1 and y_2 such that $y = y_1 y_2$, and natural numbers i, j, m and n such that $xy^i y_1 \in [u][v]^m$ and $y_2 y^j y_1 \in [v]^n$. Since $\cong_E$ is a congruence, it follows that $xy^i y_1 \cong_E uv^m$ and $y_2 y^j y_1 \cong_E v^n$, and because $uv^m (v^n)^\omega = uv^\omega \in E$, by the definition of $\cong_E$, also $xy^i y_1 (y_2 y^j y_1)^\omega = xy^\omega \in E$. □

Theorem 4 *For every $E \subseteq \Sigma^\omega$, Arnold's syntactic congruence $\cong_E$ is finite iff E is finite-state and there is a regular ω-language F such that $E \cap Ult = F \cap Ult$.*

Proof: Let E be finite-state and let the regular ω-language F satisfy $E \cap Ult = F \cap Ult$. It can be easily verified that $x \simeq_E y$ and $x \cong_{F\cap Ult} y$ imply $x \cong_E y$ and thus $\simeq_E \cap \cong_F \subseteq \cong_E$. But the congruences $\simeq_E$ and $\cong_F$ are both finite and so is $\cong_E$. Conversely, let $\cong_E$ be finite. Then F_E is a regular ω-language satisfying $E \cap Ult = F_E \cap Ult$. □

In [St83] it was shown that the cardinality of the set $\{E : \simeq_E \text{ is finite}\}$ is $2^{2^{\aleph_0}}$, in particular, there are already as many subsets of Σ^ω whose simple syntactic monoid is trivial. The following claim shows that the same is true in the case of $\cong_E$:

Claim 5 *There are $2^{2^{\aleph_0}}$ ω-languages having a trivial syntactic monoid in the sense of Arnold.*

Proof: Since the set $\{E : \simeq_E \text{ is trivial}\}$ is closed under Boolean operations, any such ω-language F splits in a unique way into a disjoint union $(F \cap Ult) \cup (F \setminus Ult)$ where for both parts $\simeq$ is trivial. As Ult is countable, there are at most $2^{\aleph_0}$ distinct parts of the form $F \cap Ult$. Consequently, there are $2^{2^{\aleph_0}}$ ω-languages $E \subseteq \Sigma^\omega \setminus Ult$ such that $\simeq_E$ is trivial. But for every such E $\approx_E$ is trivial and hence Arnold's syntactic congruence of E is trivial, what verifies our assertion. □

Given that a a Borel class in Σ^ω contains only $2^{\aleph_0}$ sets and that there are only countably many Borel classes [Ku66], it follows that there are ω-languages E even beyond the Borel hierarchy for which $\approx_E$ is trivial. This is in sharp contrast with the Myhill-Nerode theorem where the finiteness of the syntactic monoid implies the regularity of the language.

4 The case when $\simeq$ and $\cong$ coincide

In Theorem 21 of [St83] it was proved that every finite-state ω-language E which is simultaneously in the Borel classes F_σ and in G_δ Σ^ω is regular. Our aim is to show that this very condition also guarantees that Arnold's syntactic congruence of E coincides with the simple syntactic congruence of E. It is remarkable that this condition holds for all ω-languages in $F_\sigma \cap G_\delta$ not only for those which are finite-state.

First let us mention the following simple properties of the congruences $\simeq_E$ and $\cong_E$:

Fact 6 *For every $u \in \Sigma^*$, $x, y \in \Sigma^+$: 1) If $x \simeq_E y$ then $u\{x,y\}^* x^\omega \cap E \neq \emptyset$ implies $u\{x,y\}^* x^\omega \subseteq E$ 2) If $x \cong_E y$ then $u\{x,y\}^* x^\omega \cap E \neq \emptyset$ implies $u\{x,y\}^* y^\omega \subseteq E$.*

Now we obtain the following necessary and sufficient condition under which the congruences $\simeq_E$ and $\cong_E$ coincide:

Lemma 7 *Let $E \subseteq \Sigma^\omega$. Then $\simeq_E = \cong_E$ if and only if the following condition holds*

$$\forall u \in \Sigma^* x, y \in \Sigma^+ : x \simeq_E y \rightarrow (u\{x,y\}^* x^\omega \subseteq E \rightarrow u\{x,y\}^* y^\omega \cap E \neq \emptyset)$$

Proof: Clearly, the condition is necessary. In order to show its sufficiency we assume $x \simeq y$, and we show that then

$$\forall u, v \in \Sigma^* : u(xv)^\omega \in E \rightarrow u(yv)^\omega \in E)$$

that is, the additional condition for $\cong_E$ is satisfied.

If $x \simeq y$ and $u(xv)^\omega \in E$ then $xv \simeq yv$, and by the above claim it holds also $u\{xv, yv\}^*(xv)^\omega \subseteq E$. Now our condition implies $u\{xv, yv\}^*(yv)^\omega \cap E \neq \emptyset$. Again the above claim shows that $u(xv)^\omega \in E$. □

As an immediate consequence we obtain the following:

Corollary 8 *If for every $u \in \Sigma^* x, y \in \Sigma^+$ the inclusion $u\{x,y\}^*x^\omega \subseteq E$ implies that $u\{x,y\}^*y^\omega \cap E \neq \emptyset$ then $\simeq_E = \cong_E$.*

In order to prove the announced statement for ω-languages in the Borel-class $F_\sigma \cap G_\delta$ we recall that for every ω-language $E \in G_\delta$ there exists a language $U \in \Sigma^*$ such that for every $\beta \in \Sigma^\omega$, $\beta \in E$ iff β has infinitely many prefixes in U.

Theorem 9 *For every ω-language $E \in F_\sigma \cap G_\delta$, and every $x, y \in \Sigma^*$ $x \simeq_E y$ iff $x \cong_E y$.*

Proof: It suffices to show that every ω-language E in the Borel-class $F_\sigma \cap G_\delta$ satisfies the premise of corollary 8.

Since both E and its complement are in G_δ, there exist two languages U and U' such that every ω-word in E has infinitely many prefixes in U and every ω-word not in E has infinitely many prefixes in U'. Suppose that for some $u, x, y \in \Sigma^*$, $u\{x,y\}^*x^\omega \subseteq E$ and $u\{x,y\}^*y^\omega \subseteq \Sigma^\omega \setminus E$.

Since $ux^\omega \in E$ there is a number k_1 such that ux^{k_1} has a prefix in U, and since $ux^{k_1}y^\omega \notin E$, the word $ux^{k_1}y^{l_1}$ has a prefix in U' for some l_1. Next we consider $ux^{k_1}y^{l_1}x^\omega \in E$: there must be some k_2 such that $ux^{k_1}y^{l_1}x^{k_2}$ has at least two prefixes in U, etc. Repeating this alternating argument, we construct an infinite sequence $ux^{k_1}y^{l_1}\ldots x^{k_i}y^{l_i}\ldots$ having infinitely many prefixes in U and infinitely many prefixes in U' and thus belonging simultaneously to E and to its complement. □

It is worth mentioning that this result does not hold for higher Borel classes, e.g., $\{a,b\}^*a^\omega$ which is in F_σ but not in G_δ. On the other hand, from claim 5 it follows that $\simeq_E$ and $\cong_E$ coincide for some non-Borel sets.

5 Acceptance by the minimal-state automaton

In this section we will give a necessary and sufficient condition for a regular ω-language E to be acceptable by its minimal-state automaton. We will start with a necessary condition which is based on a relation between $\approx_E$ and a refinement of $\simeq_E$. Then we introduce more subtle definitions in order to arrive to a condition which is also sufficient.

First we define a congruence relation based on $\sim_E$ which refines $\simeq_E$ by considering two words equivalent only if they have the same set of right-factors (modulo $\sim_E$).

Definition 2 (Factorized congruence) *The factorization of $\sim_E$ is a congruence $\sim_E^*$ defined as*

$$\begin{array}{llll} x \sim_E^* y \text{ iff } & \forall u \in \Sigma^* & ux \sim_E uy & \\ & \text{and} & (\forall v \prec x)(\exists v' \prec y)\; uv \sim_E uv' & \qquad (5) \\ & \text{and} & (\forall v' \prec y)(\exists v \prec x)\; uv \sim_E uv' & \end{array}$$

It is more intuitive to see the meaning of this relation in terms the minimal-state automaton $\mathcal{A}$ isomorphic to $\sim_E$. Here $x \sim^* y$ iff from every state q both x and y lead to the same state while visiting the same set of states. One can see that $u \sim_E v$ and $x \sim_E^* y$ imply that for every z, $Inf(\mathcal{A}, u(xz)^\omega) = Inf(\mathcal{A}, v(yz)^\omega)$.

Claim 10 *An ω-regular set E can be accepted by its minimal-state automaton $\mathcal{A}$ using Muller condition only if for every $x, y \in \Sigma^*$, $x \sim_E^* y \to x \approx_E y$.*

Proof: Suppose $x \sim_E^* y \not\to x \approx_E y$, that is, for some $x \sim_E^* y$, there exist u, v such that $u(xv)^\omega \in E$ and $u(yv)^\omega \notin E$. But $xv \sim_E^* yv$, hence $Inf(\mathcal{A}, u(xv)^\omega) = Inf(\mathcal{A}, u(yv)^\omega)$ and $\mathcal{A}$ cannot accept E. □

The condition of the previous claim fails to be sufficient because two ω-words can have the same Inf in $\mathcal{A}$ without having the same $(\sim, \sim^*)$-factorization (for example, consider $E = \{ab^\omega\} \cup \{b, c\}^* c^\omega$ and the ω-words a^ω and ba^ω that reach their common Inf by different paths). For the sufficiency conditions we need different definitions. We consider $\sim_E$ as before and with *each* of its right-congruence classes $\langle u \rangle$ we associate two corresponding right-congruence relations: $\approx_u$ (syntactic) and $\sim_u^*$ (automatic).

Definition 3 (Syntactic induced right-congruence) *The syntactic right congruence induced by $\langle u \rangle$ is defined as:*

$$\begin{aligned} x \approx_u y \text{ iff } \quad & ux \sim_E uy \\ \text{and } \quad & (\forall v \in \Sigma^*)(uxv \sim_E u) \to (u(xv)^\omega \in E \leftrightarrow u(yv)^\omega \in E) \end{aligned} \tag{6}$$

One can see that $\approx_u$ is coarser than $\approx_E$ in two respects: 1) It does not quantify over all u (just those in $\langle u \rangle$), and 2) It does not quantify over all v, only over those for which v makes a cycle from $\langle u \rangle$. In fact $\sim_E$ and the induced family $\{\approx_u\}_{u \in \Sigma^*/\sim_E}$ can be considered as an alternative canonical object for E which satisfies the following saturation property:

Lemma 11 *For any regular ω-language E, let $\langle u \rangle$ be a class of $\sim_E$ and $[v]_u$ a class of $\approx_u$ satisfying $uv \sim u$. Then $\langle u \rangle([v]_u)^\omega \cap E \neq \emptyset$ implies $\langle u \rangle([v]_u)^\omega \subseteq E$*

Proof: We prove it similarly to lemma 2.2 in [Ar85]. Suppose the contrary, then by regularity there exists $uv^\omega \in E$ and $xy^\omega \in \langle u \rangle([v]_u)^\omega \setminus E$. By finiteness there exist some m, n such that $xy^\omega = zx_1 \ldots x_m(y_1 \ldots y_n)^\omega$ with $z \sim u$ and $x_i \approx_u y_j \approx_u v$ for every $i \leq m, j \leq n$. This implies that $zx_1 \ldots x_m \sim u$ and $y_1 \ldots y_n \approx_u v^n$ and thus by the definition of $\approx_u$, $zx_1 \ldots x_m(y_1 \ldots y_n)^\omega \in E$ if $u(v^n)^\omega \in E$ which means $xy^\omega \in E$ – a contradiction. □

Definition 4 (Automatic induced right-congruence) *The automatic right congruence induced by $\langle u \rangle$ is defined as:*

$$\begin{aligned} x \sim_u^* y \text{ iff } \quad & ux \sim_E uy \\ \text{and } \quad & (\forall v \prec x)(((\exists z) uxz \sim_E uv) \to ((\exists v' \prec y) uv \sim_E uv')) \\ \text{and } \quad & (\forall v' \prec y)(((\exists z) uyz \sim_E uv') \to ((\exists v \prec x) uv \sim_E uv')) \end{aligned} \tag{7}$$

Intuitively this means that in an automaton $\mathcal{A}$ isomorphic to $\sim_E$, $x \sim_u^* y$ iff $\delta(q_0, ux) = \delta(q_0, uy)$ and both ux and uy visit the same set of states in the strongly-connected component (SCC) of $\delta(q_0, ux)$. We do not care here if ux and uy visit different states outside that SCC.

Theorem 12 *Let E be a regular ω-language, let $\sim_E$ be its syntactic right-congruence* (1) *and let $\approx_u$ and $\sim_u^*$ be respectively the induced syntactic* (6) *and automatic* (7) *right congruences. E can be accepted by an automaton $\mathcal{A}$ isomorphic to its syntactic right-congruence $\sim_E$ if and only if for every $u, x, y \in \Sigma^*$, $x \sim_u^* y \rightarrow x \approx_u y$.*

Proof: It can be easily seen that all the ω-words having the same Inf in $\mathcal{A}$ admit the same factorization $\langle u\rangle([v]_u)^\omega$ where $\langle u\rangle$ and $[v]_u$ are classes of $\sim$ and $\sim_u^*$ respectively and $uv \sim u$. Our condition implies that they have also the same factorization $\langle u\rangle([v]_u)^\omega$ where $[v]_u$ is a class of $\approx_u$ and $uv \sim u$. According to lemma 11 all these ω-words are either in E or in $\Sigma^\omega \setminus E$ and thus $\mathcal{A}$ can accept E using Muller condition. Conversely, suppose $x \sim_u^* y \not\rightarrow x \approx_u y$, that is, for some $u, x, y \in \Sigma^*$ satisfying $x \sim_u^* y$, there exists v such that $uxv \sim u$, $u(xv)^\omega \in E$ and $u(yv)^\omega \notin E$. But in this case since $\delta(q_0, u)$ and $\delta(q_0, ux)$ share the same SCC, $x \sim_u^* y$ implies $Inf(\mathcal{A}, u(xv)^\omega) = Inf(\mathcal{A}, u(yv)^\omega)$ and thus $\mathcal{A}$ cannot accept E. $\square$

As an illustration consider again $E = \{a, b\}^* a^\omega$. The relation $\sim_E$ is trivial, hence $\sim_u^*$ is trivial as well. On the other hand, $\approx_u$ has two classes a^+ and $(a^*b)^+$ and E cannot be accepted by its minimal-state automaton.

The introduction of the syntactic family of right-congruences $\{\approx_u\}_{u \in \Sigma^*/\sim_E}$ may have significance beyond the proof of the above theorem. Up to now the only syntactic characterization of ω-languages was by means of a two-sided congruence and the lack of the other half of a Myhill-Nerode theorem was believed to be an inherent feature of the theory of ω-languages. From a practical point of view, although Arnold's congruence $\cong_E$ (which is the intersection of $\{\approx_u\}$) has a simpler definition, its size might be exponentially larger, and there are situations[3] where *the right-congruences are the right congruences.*

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