Finite Automata for the Sub- and Superword Closure of CFLs: Descriptional and Computational Complexity

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joint work with
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Subwords and Closure

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Def.: Subword Closure of a Language

$$\nabla L := \{w \in \Sigma^* : \exists u \in L. w \preccurlyeq u\}$$

Example

$$L = \{a^n c b^n : n \in \mathbb{N}\} \;\rightsquigarrow\; \nabla L = a^*(\varepsilon + c)b^*$$

Subword Closure: Computation

Higman/Haines, 1950s

∇L is regular for any language L.

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Effective regularity for CFLs (van Leeuwen '78, Courcelle '91)

∇L computable for context-free languages (as automaton/regex).

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Question: Size of finite automata representing ∇L (DFA/NFA)?

Size of NFA for the Subword Closure of a CFL

Context-free grammar G, NFA $\mathcal{A}^{\nabla}$ for $\nabla L(G)$,

- Gruber/Holzer/Kutrib '09: $\left|\mathcal{A}^{\nabla}\right| \in 2^{2^{\mathcal{O}(|G|)}}$, based on [vL78].

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Known lower bound $\left|\mathcal{A}^\nabla\right| \in \Omega(2^{|G|})$:

Consider $L_n = \{a^{2^n}\}$, context-free grammar with $|G| \in \mathcal{O}(n)$:

$$\begin{aligned} A_n &\to A_{n-1}A_{n-1} \\ &\vdots \\ A_1 &\to A_0A_0 \\ A_0 &\to a \end{aligned}$$

NFAs for L_n and $\nabla L_n = \{a^i : 0 \leq i \leq 2^n\}$ need 2^n states.

Grammar $\rightarrow$ NFA, pre-processing

1. Assume G in 2-normal form: $X \rightarrow \alpha$, $|\alpha| \leq 2$ (w.l.o.g.).
2. If $X \Rightarrow^* \alpha X \beta X \gamma$ then $\nabla X = \Sigma_X^*$ $\rightsquigarrow$ non-expansive grammar.

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\begin{aligned}
S &\to XA \mid AB \mid AX \mid XB \\
X &\to CY \mid C \\
Y &\to XA \mid A \\
A &\to a \\
B &\to b \\
C &\to c
\end{aligned}
$$

$$
\begin{tikzcd}
 & & S \arrow[dll, bend right] \arrow[dl, bend left] \arrow[dr] & \\
X \arrow[r, bend left] \arrow[d] & Y \arrow[l, bend left] \arrow[d] & & B \\
C & A & &
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3. Contract strongly-connected components.

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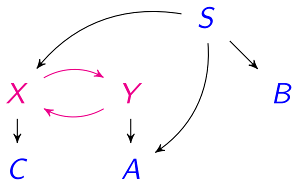

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3. Contract strongly-connected components.
4. Dependency graph is now a DAG (with self-loops)

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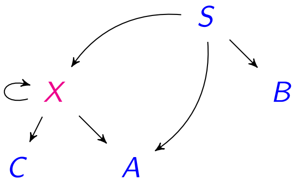

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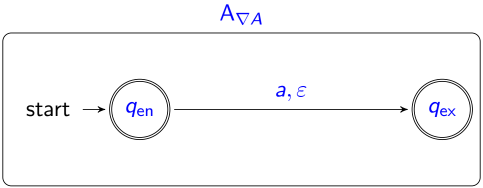

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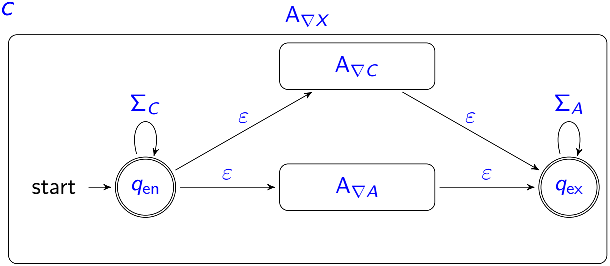

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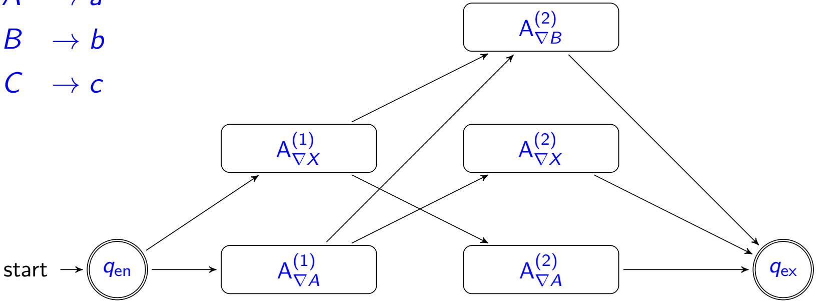

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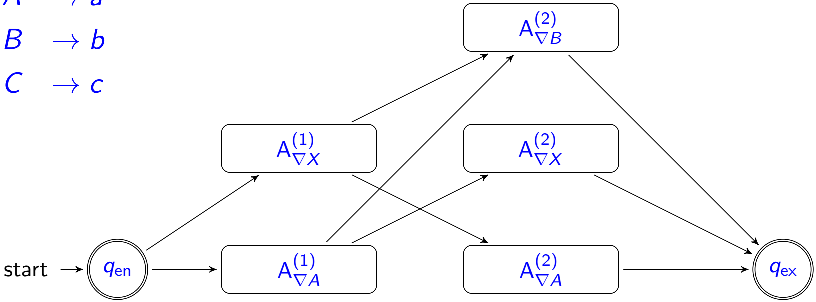

Total size of NFA

At every stage, copy each automaton at most twice
$\rightsquigarrow \left|\mathcal{A}^{\nabla}\right| \in 2^{\mathcal{O}(|G|)}$.

Size of DFA for the Subword Closure of a CFL L

- NFA for ∇L: $\left|\mathcal{A}^{\nabla}\right| \in 2^{\mathcal{O}(|G|)} \rightsquigarrow$ DFA: $\left|\mathcal{D}^{\nabla}\right| \in 2^{2^{\mathcal{O}(|G|)}}$.

Size of DFA for the Subword Closure of a CFL L

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- Bound is tight: $w \in \{0,1\}^{2N+1}$, $w = x0y0z$ with $|y| = N$ (finite language, binary alphabet).

$$L_N := \bigcup_{j=1}^{N} \underbrace{\{0,1\}^{j-1}}_{x}\{0\}\underbrace{\{0,1\}^{N}}_{y}\{0\}\underbrace{\{0,1\}^{N-j}}_{z}.$$

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- $|G| \in \mathcal{O}(\log N)$: let derivation "guess" the location of y.
- $|\mathcal{D}| \geq 2^N$: all $u \in \{0,1\}^N$ inequivalent w.r.t. Myhill-Nerode.
- $\left|\mathcal{D}^{\nabla}\right| \geq 2^N$ (use same witnesses for inequivalence).

Application: Approximate Grammar Equivalence Checking

Given two context-free languages $L_1 = L(G_1), L_2 = L(G_2)$,

$$\nabla L_1 \neq \nabla L_2 \implies L_1 \neq L_2$$

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Questions:

- Complexity of checking $\nabla L_1 \stackrel{?}{=} \nabla L_2$ (given as NFAs)?
- How to build a semi-decision procedure?
  - If $\nabla L_1 \neq \nabla L_2$ output witness $w \in (L_1 \setminus L_2) \cup (L_2 \setminus L_1)$.
  - If $\nabla L_1 = \nabla L_2$ refine grammars and iterate.

Equivalence of NFAs modulo Closure

For NFAs $\mathcal{A}_1, \mathcal{A}_2$,

$$\mathcal{A}_1 \equiv_{\mathsf{cl}} \mathcal{A}_2 \text{ if } \mathsf{cl}(L_1) = \mathsf{cl}(L_2).$$

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- Our result: $\mathcal{A}_1 \stackrel{?}{\equiv}_{\nabla} \mathcal{A}_2$ is (only) coNP-complete.

(I) $\mathcal{A}_1 \overset{?}{\equiv}_{\nabla} \mathcal{A}_2$ is in coNP

Structure of $\mathcal{A}^{\nabla}$ and its powerset-DFA:

- If $s \xrightarrow{a} t$ then $s \xrightarrow{\varepsilon} t$ in $\mathcal{A}^{\nabla}$.
- If $S \xrightarrow{a} T$ in powerset-DFA, then $S \supseteq T$.

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Corollary: In-equivalence modulo ∇ is in NP

If $\mathcal{A}_1 \not\equiv_\nabla \mathcal{A}_2$ then there is a short witness w ($|w| \leq |\mathcal{A}_1^\nabla| + |\mathcal{A}_2^\nabla|$).

(II) $\mathcal{A}_1 \overset{?}{\equiv}_{\nabla} \mathcal{A}_2$ is coNP-hard

Even more: $\mathcal{A}_1 \overset{?}{\equiv}_{\nabla} \mathcal{A}_2$ is coNP-hard for finite languages.

Same idea as hardness of equivalence for regex without Kleene-stars.

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- Last step: $L(\rho) \neq \{0,1\}^n \implies \nabla L(\rho) \neq \{0,1\}^{\leq n}$ since ∇ only adds shorter words.

Summary, Future Work

- Tight bounds on the size of NFAs/DFAs for $\nabla L(G)$.
- $\mathcal{A}_1 \overset{?}{\equiv}_\nabla \mathcal{A}_2$ is (only) coNP-complete.
- All results hold for superword closures as well (see paper).
- Application: Semi-decision procedure for grammar inequivalence (fast in practice – see paper).

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- Simple regex as representation (size $2^{\mathcal{O}(|G|)}$ as well), equivalence in $\mathcal{O}(n^2)$ (Abdulla/Bouajjani/Jonsson '98).
- Develop practical tool for inclusion/equivalence checking.
- Computability of ∇L for larger language classes.

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