# Bounds for the quantifier depth in two-variable logics

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# Definitions

$G$, $H$, ... will be binary structures (typically, vertex-colored graphs).

A sentence $\Phi$ distinguishes $G$ from $H$ if $G \models \Phi$ while $H \not\models \Phi$.

$$
\begin{aligned}
D^2(G,H) &= \text{the min quantifier depth of such } \Phi \in \mathrm{FO}^2. \\
A^2(G,H) &= \text{the min alternation depth of such } \Phi \in \mathrm{FO}^2.
\end{aligned}
$$

$$
\begin{aligned}
D^2(n) &= \max D^2(G,H), \\
A^2(n) &= \max A^2(G,H),
\end{aligned}
$$

where $\max$ is over $n$-element $G$ and $H$ distinguishable in $\mathrm{FO}^2$.

# Bounds for $A^2(n)$ and $D^2(n)$

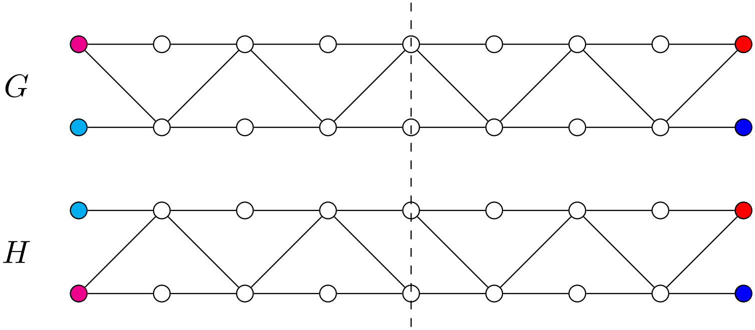
Theorem

$\frac{1}{8}\,n - 2 < A^2(n) \leq D^2(n) \leq n + 1$

Remark

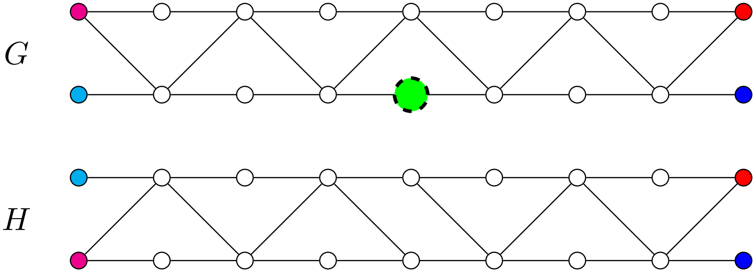
The upper bound due to Immerman and Lander 1990
(stabilization of color refinement)

$A^2(n) > \frac{1}{8}\,n - 2$



## $A^2(n) > \frac{1}{8}\,n - 2$

Assumption: Spoiler pebbles along edges.



moves: $\exists$

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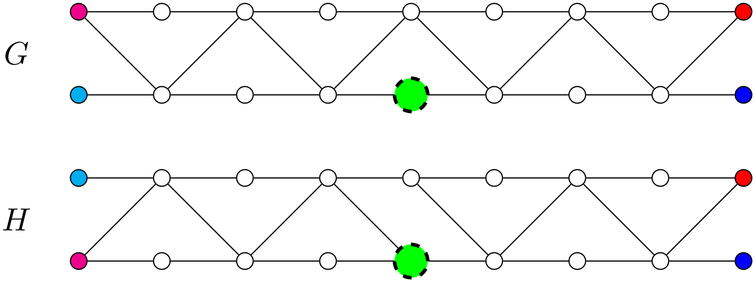


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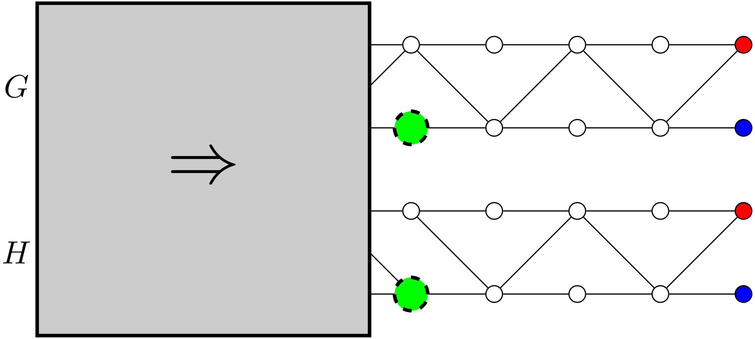


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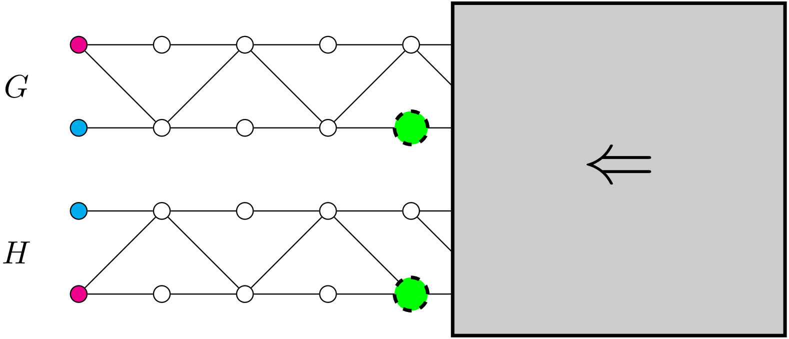


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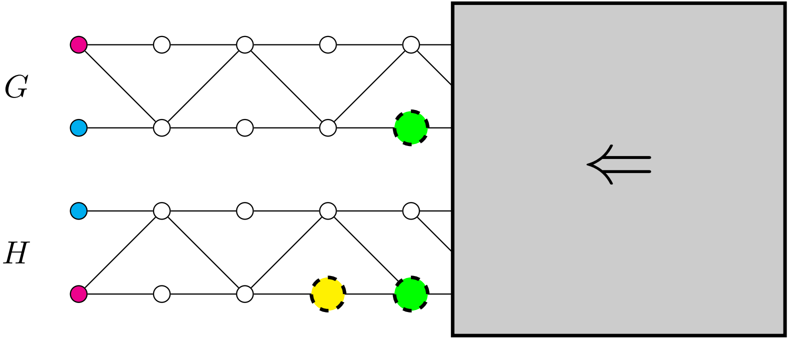


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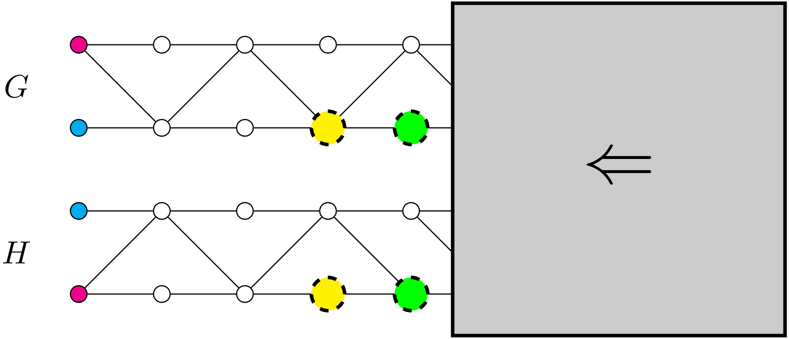


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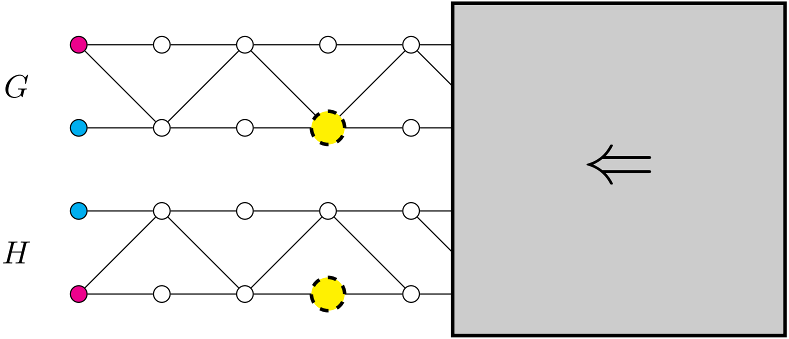


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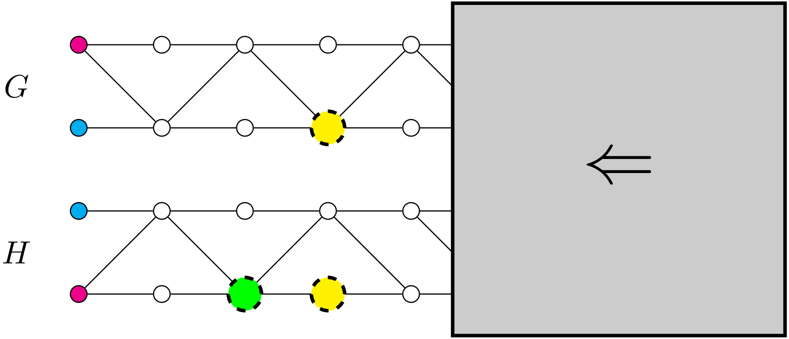


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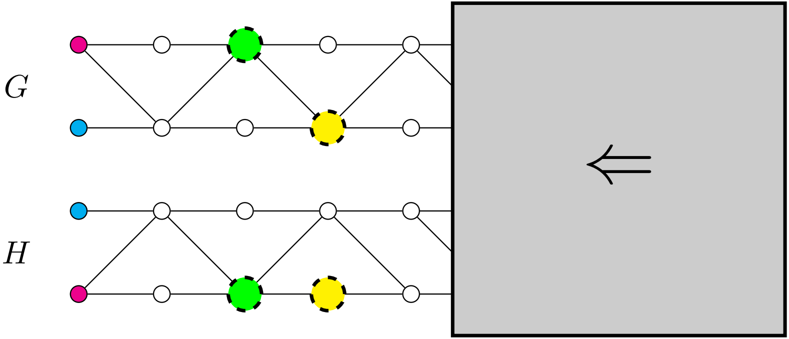


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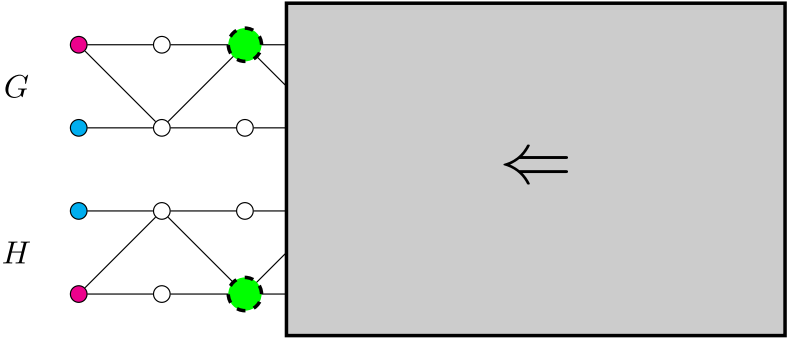


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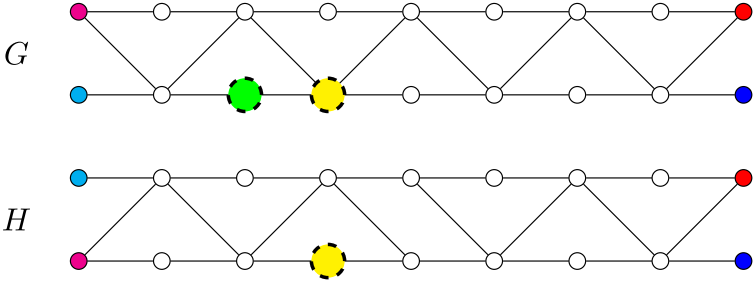


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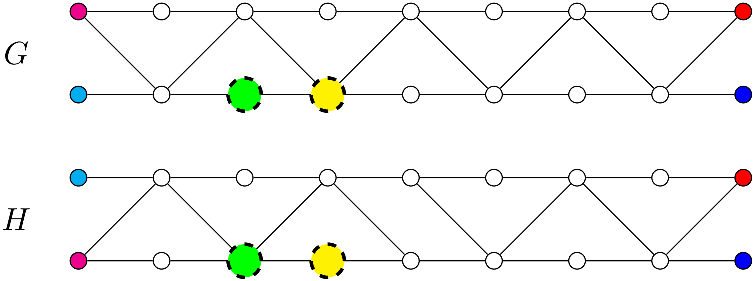


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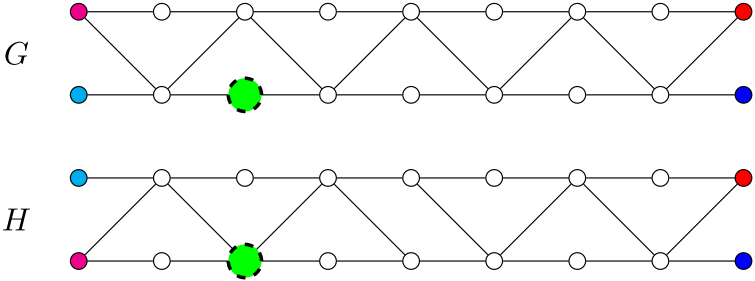


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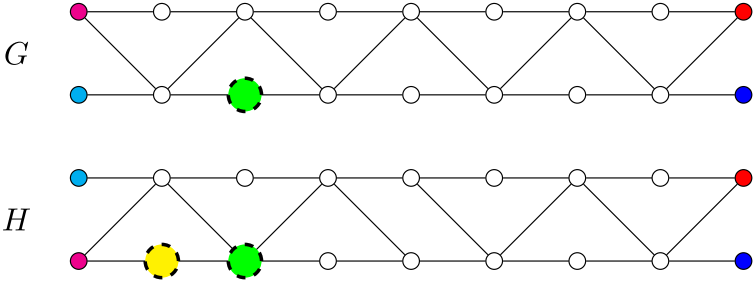


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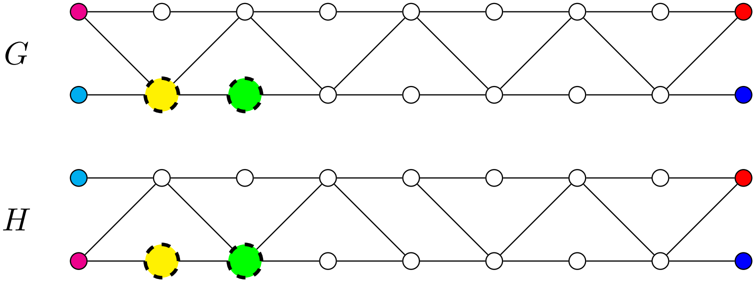


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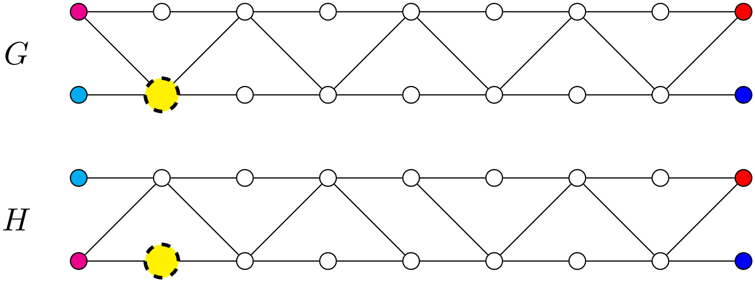


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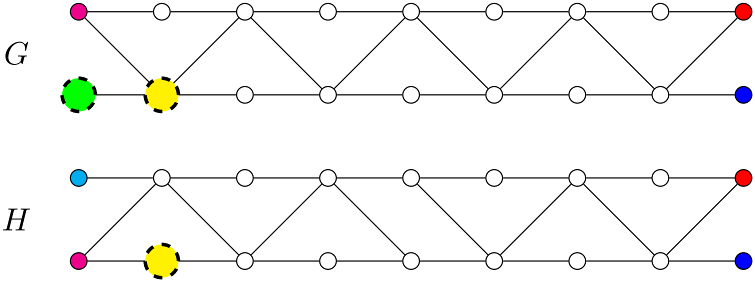

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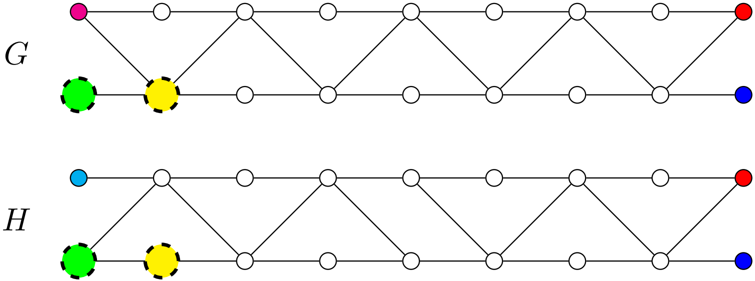
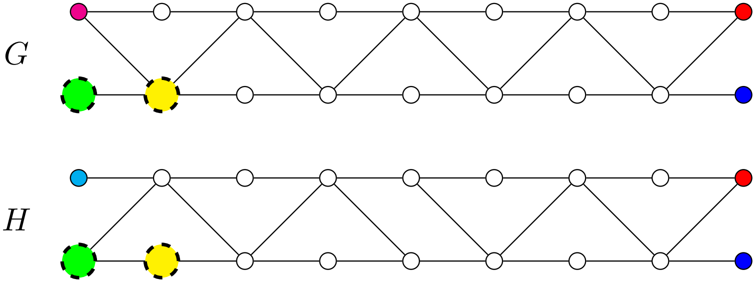



moves: ∃∀∃∀∃

$$A^2(n) > n/4 - 1$$

# $A^2(n) > \frac{1}{8}\,n - 2$

~~Assumption: Spoiler pebbles along edges.~~

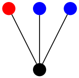



moves: $\exists\forall\exists\forall\exists$

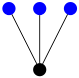
$A^2(n) > n/8 - 2$: Consider $2G$ and $2H$

# $A^2(n) > \log_3 n - 2$ over trees (due to Chandra-Harel 1982)

$G^1$:



$H^1$:



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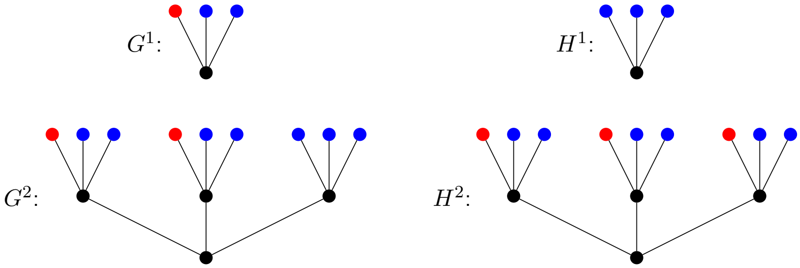

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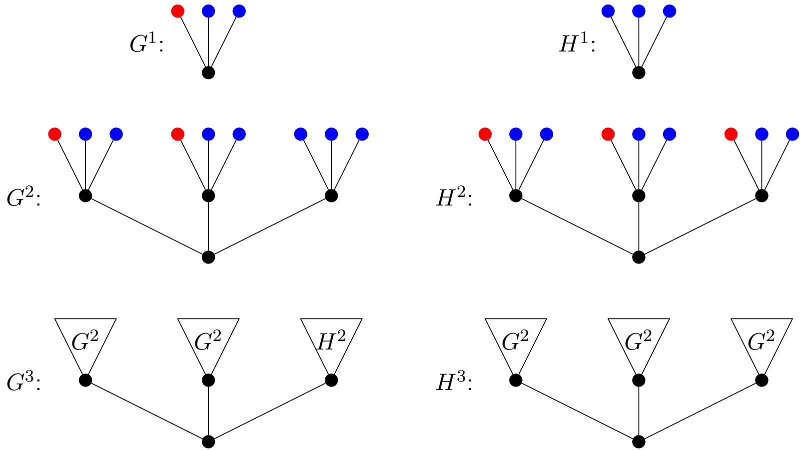

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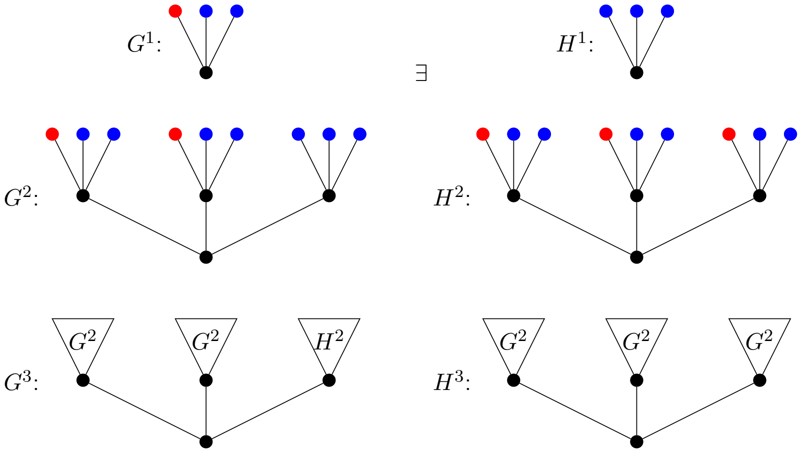

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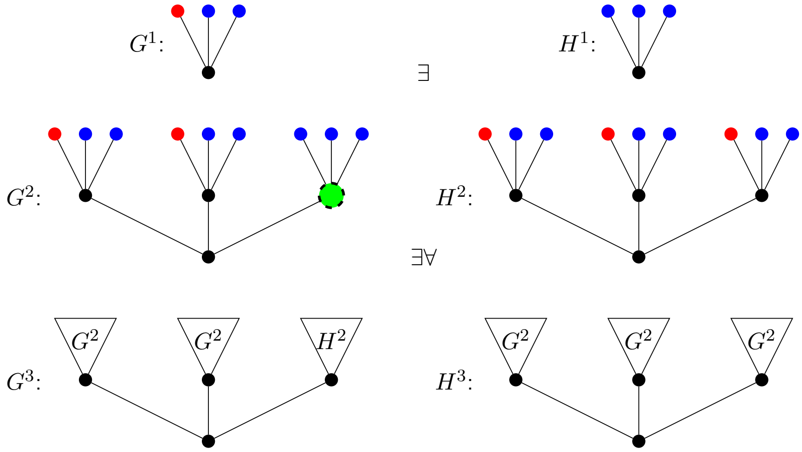

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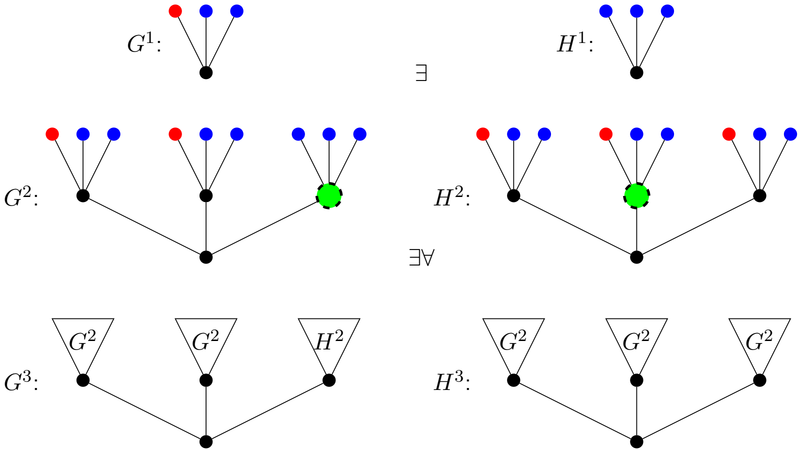

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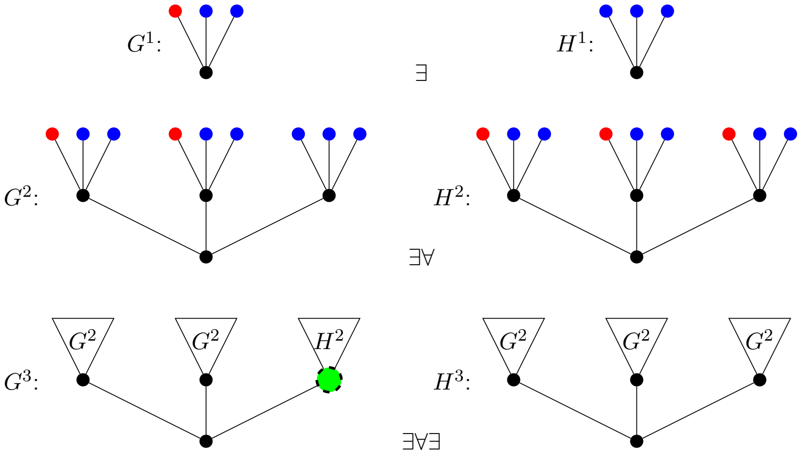

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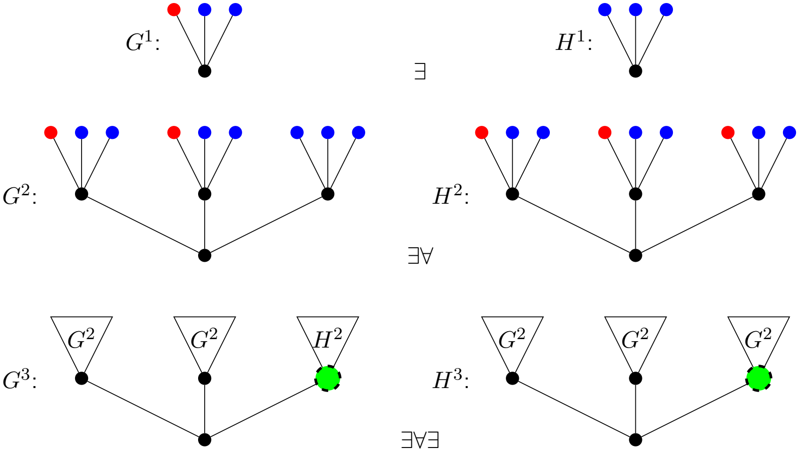


# $A^2(n) > \log_3 n - 2$ over trees

**Question**

How tight is this lower bound?

**Remark**

If $k \geq 3$, then over trees

$$\log_{k+1} n - 2 < A^k(n) \leq D^k(n) < (k+3)\log_2 n.$$

# Existential-positive two-variable logic

Let $D^2_{\exists,+}(n)$ be the variant of $D^2(n)$ for existential-positive $\mathrm{FO}^2$.

### Theorem

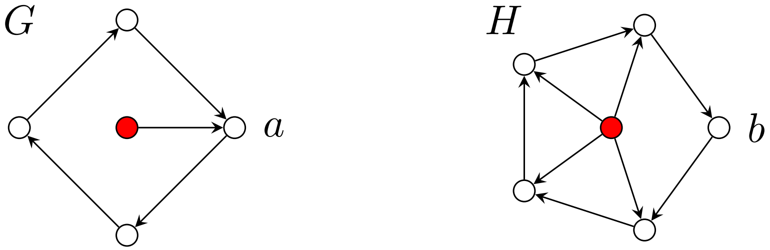
$\frac{1}{6}\,(n-10)^2 < D^2_{\exists,+}(n) \leq n^2+1.$

### Remarks

- The result can be extended to any fragment of $\mathrm{FO}^2$ with bounded number of alternations.
- Upper bound: If Spoiler is going to move one of the pebbles, the rest of the game is determined by the position $(u,v) \in V(G) \times V(H)$ of the other pebble. If the play is optimal and finite, the same position $(u,v)$ never occurs twice.

# $D^2_{\exists,+}(n) = \Omega(n^2)$

$G$ and $H$ are "co-wheels" with coprime lenghts $n-1$ and $n$



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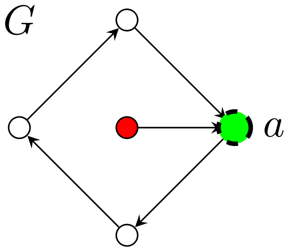


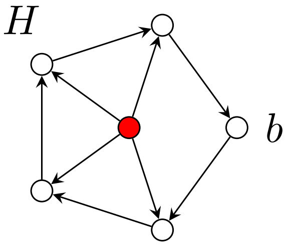

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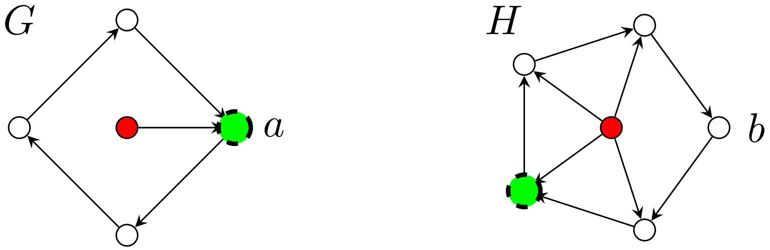

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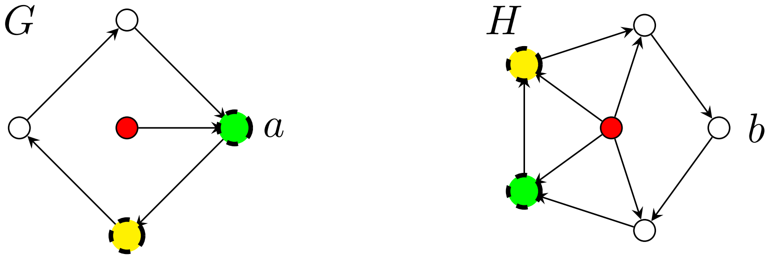

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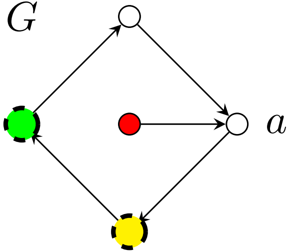


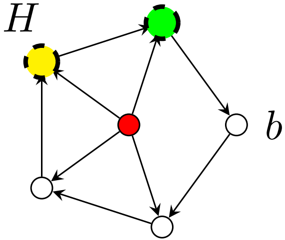

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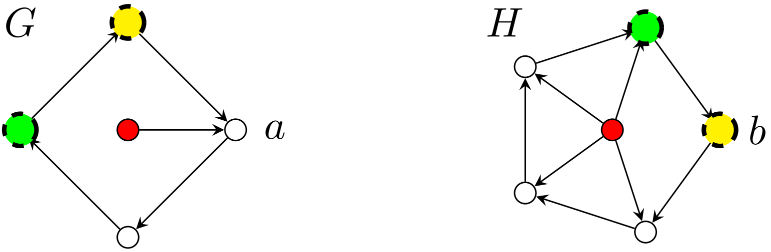

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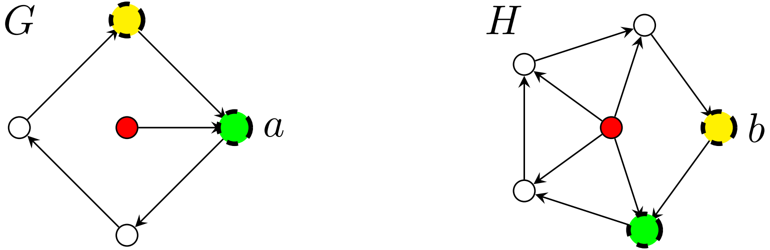

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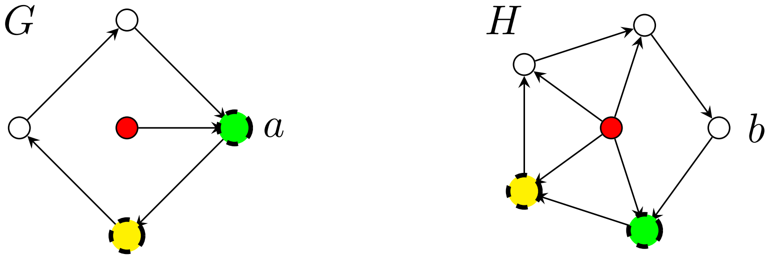

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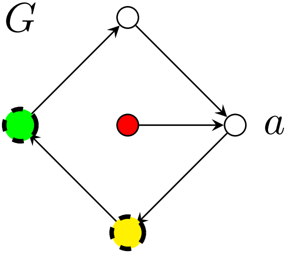


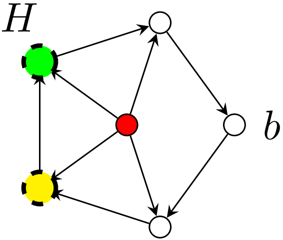

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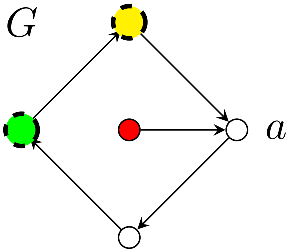


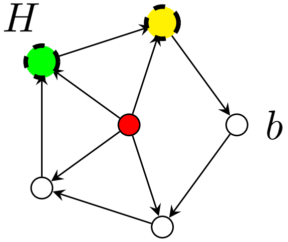

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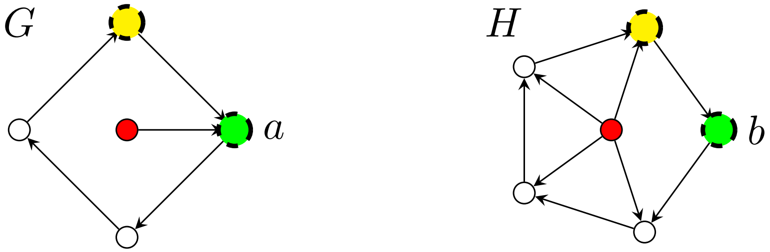

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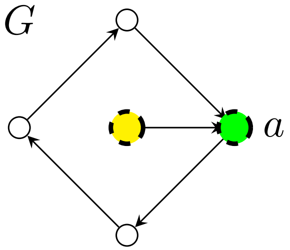


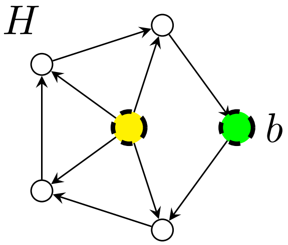

# Application of $D^2_{\exists,+}(n) = \Omega(n^2)$

**Theorem**

*All algorithms for the Arc Consistency problems that are based on constraint propagation*

*(like AC-1, AC-3, AC-3.1 / AC-2001, AC-3.2, AC-3.3, AC-3$_d$, AC-4, AC-5, AC-6, AC-7, AC-8, AC-$*$)*

*take time $\Omega(n^3)$ (and this bound is tight).*

# Further work

What about $\mathrm{FO}^3$? Well,

$$\begin{aligned} A^3(n) \leq \quad D^3(n) \quad &\leq n^2 + 1, \\ D^3_{\exists,+}(n) \quad &\leq n^4 + 1 \end{aligned}$$

and we are working hard on lower bounds...

# Further work

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Thank you for your attention!