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Definitions by rewriting in the Calculus of Constructions

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This paper presents general syntactic conditions ensuring the strong normalisation and the logical consistency of the Calculus of Algebraic Constructions, an extension of the Calculus of Constructions with functions and predicates defined by higher-order rewrite rules. On the one hand, the Calculus of Constructions is a powerful type system in which one can formalise the propositions and natural deduction proofs of higher-order logic. On the other hand, rewriting is a simple and powerful computation paradigm. The combination of the two allows, among other things, the development of formal proofs with a reduced size and more automation compared with more traditional proof assistants. The main novelty is to consider a general form of rewriting at the predicate-level that generalises the strong elimination of the Calculus of Inductive Constructions.

1. Introduction

This work aims to define an expressive language allowing us to specify and prove mathematical properties easily. The quest for such a language started with Girard' system F (Girard 1972) on the one hand and De Bruijn's Automath project (De Bruijn 1968) on the other. Later, Coquand and Huet combined both calculi into the Calculus of Constructions (CC) (Coquand 1985). As in system F, data types in CC are defined through impredicative encodings that are difficult to use in practice. So, following Martin-Löf's theory of types (Martin-Löf 1984), Coquand and Paulin-Mohring defined an extension of CC with inductive types and their associated induction principles as first-class objects, the Calculus of Inductive Constructions (CIC) (Coquand and Paulin-Mohring 1988), which is the basis of the proof-assistant Coq (Coq Development Team 2002).

However, defining functions or predicates by induction is not always convenient. Moreover, with such definitions, equational reasoning is not easy and leads to very large proof terms. Yet, for decidable theories, equational proofs do not need to be kept in proof terms. The idea that proving is not only reasoning (which is undecidable) but also computing (which is decidable) has been formalised recently in a general way by Dowek, Hardin and Kirchner with the Natural Deduction Modulo (NDM) for first-order logic (Dowek *et al.* 1998).

A more convenient and powerful way of defining functions and predicates is by using rewrite rules (Dershowitz and Jouannaud 1990). This notion is very old, but its study really began in the 1970's with Knuth and Bendix (Bendix and Knuth 1970), who provided a means of knowing whether, in a given equational theory, an equation is valid or not. Then, rewriting was quickly used as a programming paradigm (see Dershowitz and Jouannaud (1990)) since any computable function can be defined by rewrite rules.

In the following sub-sections, we give more details of our motivations for extending CIC with rewriting, an overview of previous work on the combination of λ-calculus and rewriting, and an outline of our own contributions.

1.1. *Advantages of rewriting*

In CIC, functions and predicates can be defined by induction on inductively defined types. The case of the type *nat* of natural numbers, defined from $0 : nat$ (zero) and $s : nat \Rightarrow nat$ (successor function), yields Gödel's system T: a function $f : nat \Rightarrow \tau$ is defined by giving a pair of terms (u, v), written $(rec\ u\ v)$, where $u : \tau$ is the value of $f(0)$ and $v : nat \Rightarrow \tau \Rightarrow \tau$ is a function that computes the value of $f(n+1)$ from n and $f(n)$. Computation proceeds by applying the following (higher-order) rewrite rules, called ι-*reduction*:

$$\begin{aligned} rec\ u\ v\ 0 &\to_\iota u \\ rec\ u\ v\ (s\ n) &\to_\iota v\ n\ (rec\ u\ v\ n). \end{aligned}$$

For instance, addition can be defined by the term $\lambda xy.(rec\ u\ v\ x)$ with $u = y$ and $v = \lambda nr.s(r)$ (definition by induction on x). Then, one can check that[†]

$$\begin{aligned} 2+2 &\to_\beta^* rec\ 2\ v\ 2 \\ &\to_\iota v\ 1\ (rec\ 2\ v\ 1) \\ &\to_\beta^* s(rec\ 2\ v\ 1) \\ &\to_\iota s(v\ 0\ (rec\ 2\ v\ 0)) \\ &\to_\beta^* s(s(rec\ 2\ v\ 0)) \\ &\to_\iota s(s(2)) \\ &= 4. \end{aligned}$$

Proofs by induction are formalised in the same way: if P is a predicate on natural numbers, u a proof of $P0$ and v a proof of $(n : nat)Pn \Rightarrow P(sn)$[‡], then $rec\ P\ u\ v$ is a proof of $(n : nat)Pn$, and ι-reduction corresponds to the elimination of induction cuts. In fact, $(rec\ u\ v)$ is nothing but a particular case of $(rec\ P\ u\ v)$ with the non-dependent predicate $P = \lambda n.\tau$.

In addition, deduction steps are made modulo $\beta\iota$-equivalence[§], that is, if π is a proof of P and $P =_{\beta\iota} Q$, then π is also a proof of Q. For example, if π is a proof of $P(2+2)$, then

[†] $\to_\beta^*$ is the transitive closure of the β-reduction relation: $(\lambda x.t\ u) \to_\beta u\{x \mapsto t\}$.

[‡] As often in type systems, we use $(x : T)$ to denote universal quantification over a type T.

[§] This is the reflexive, symmetric and transitive closure of the $\beta\iota$-reduction relation (which is the union of the β and ι reduction relations).

it is also a proof of $P(4)$, as one would expect. The verification that a term π is indeed a proof of a proposition P, called type-checking, is decidable since $\beta\iota$ is a confluent (the order of computations does not matter) and strongly normalising (there is no infinite computation) relation (Werner 1994).

Although the introduction of inductive types and their induction principles as first-class objects is a big step towards a greater usability of proof assistants, we are going to see that the restriction of function definitions to definitions by induction, and the restriction of type conversion to $\beta\iota$-equivalence, have several important drawbacks. The use of rewriting, that is, the ability to define functions by giving a set of rewrite rules $\mathscr{R}$, and the possibility of doing deductions modulo $\beta\mathscr{R}$-equivalence, can remedy these problems. It appears that ι-reduction itself is nothing but a particular case of higher-order rewriting (Klop *et al.* 1993; Nipkow 1991) where, in contrast to first-order rewriting, the constructions of the λ-calculus (application, abstraction and product) can be used in the right-hand sides of rules[†]. A common example of a higher-order definition is the function *map*, which applies a function f to each element of a list:

$$\begin{aligned} map\ f\ nil &\to nil \\ map\ f\ (cons\ x\ \ell) &\to cons\ (f\ x)\ (map\ f\ \ell) \end{aligned}$$

where *nil* stands for the empty list and *cons* for the function adding an element at the head of a list.

Easier definitions. First of all, with rewriting, definitions are easier. For instance, addition can be defined by simply giving the rules:

$$\begin{aligned} 0 + y &\to y \\ (s\ x) + y &\to s\ (x + y) \end{aligned}$$

Then, we have $2+2 \to s(2+1) \to s(s(2+0)) \to s(s(2)) = 4$. Of course, one can make some definitions by induction look like this, as is the case in Coq (Coq Development Team 2002), but it is not always possible. For instance, the definition by induction of the comparison function $\leqslant$ on natural numbers requires the use of two recursors,

$$\lambda x.rec\ (\lambda y.true)\ (\lambda nry.rec\ false\ (\lambda n'r'.rn')\ y)\ x,$$

while the definition by rewriting is simply

$$\begin{aligned} 0 \leqslant y &\to true \\ s\ x \leqslant 0 &\to false \\ s\ x \leqslant s\ y &\to x \leqslant y. \end{aligned}$$

More efficient computations. From a computational point of view, definitions by rewriting can be more efficient, although the process of selecting an applicable rule may have a higher cost (Augustsson 1985). For example, since $+$ is defined by induction on its first argument, the computation of $n+0$ requires $n+1$ reduction steps, but with the addition of the rule $x + 0 \to x$, this only takes one step.

[†] We will not consider higher-order pattern-matching here, although it should be possible since we have shown it for the simply-typed λ-calculus in Blanqui (2000).

Quotient types. Rewriting allows us to formalise some quotient types simply, and without requiring any additional extensions (Barthe and Geuvers 1995b; Courtieu 2001), by just considering rewrite rules on constructors, which is forbidden in CIC since constructors must be free in that system. For instance, integers can be formalised by taking 0 for zero, p for predecessor and s for successor, together with the rules

$$s\ (p\ x) \to x$$
$$p\ (s\ x) \to x.$$

This technique applies to any type whose constructors satisfy a set of equations that can be turned into a confluent and strongly normalising rewrite system (Jouannaud and Kounalis 1986).

More automation. We previously saw that, in CIC, if P is a predicate on natural numbers, then $P(2+2)$ is $\beta\iota$-equivalent to $P(4)$ and, hence, that a proof of $P(2+2)$ is also a proof of $P(4)$. This means that proving $P(4)$ from $P(2+2)$ does not require any argument: this is automatically done by the system. But, because functions must be defined by induction, this no longer works for computations on open terms: $P(x+2)$ is not $\beta\iota$-equivalent to $P(s(s(x)))$ since $+$ is defined by induction on its first argument. Proving $P(s(s(x)))$ from $P(x+2)$ requires a user interaction for proving that $x+2$ is equal to $s(s(x))$, which requires induction.

We may even go further and turn some lemmas into simplification rules. Let us, for instance, consider multiplication on natural numbers:

$$0 \times y \to 0$$
$$(s\ x) \times y \to y + (x \times y).$$

Then, the distributivity of the addition over multiplication can be turned into the rewrite rule

$$(x+y) \times z \to (x \times z) + (y \times z),$$

which allows the system to prove more equalities and more lemmas automatically by simply checking the $\beta\mathscr{R}$-equivalence with already proved statements. In the case of an equality $u = v$, it suffices to check whether it is $\beta\mathscr{R}$-equivalent to the instance $u = u$ of the identity axiom, which is the same as checking whether u and v have the same $\beta\mathscr{R}$-normal form.

Smaller proofs. Another important consequence of considering a richer equivalence relation on types is that it reduces the size of proofs, which is currently an important limitation in proof assistants like Coq. For instance, while the proof of $P(s(s(x)))$ requires the application of some substitution lemma in CIC, it is equal to the proof of $P(x+2)$ when rewriting is allowed. The benefit becomes very important with equality proofs, since they require the use of many lemmas in CIC (substitution, associativity, commutativity, and so on), while with rewriting they reduce to reflexivity (if one considers rewriting modulo associativity and commutativity (Peterson and Stickel 1981)).

More typable terms. As one would expect, the fact that some terms are not $\beta\iota$-equivalent has another unfortunate consequence: some apparently well-formed propositions are rejected by the system. Take for instance the type $list : (n : nat)\star$ of lists of length n with

$$P \oplus \bot \to P$$
$$P \oplus P \to \bot$$

$$P \wedge \top \to P$$
$$P \wedge \bot \to \bot$$
$$P \wedge P \to P$$
$$P \wedge (Q \oplus R) \to (P \wedge Q) \oplus (P \wedge R)$$

Fig. 1. Decision procedure for classical propositional tautologies.

the constructors $nil : list0$ and $cons : nat \Rightarrow (n : nat)listn \Rightarrow list(sn)$. Let $app : (n : nat)listn \Rightarrow (n' : nat)listn' \Rightarrow list(n + n')$ be the concatenation function on *list*. If, as usual, *app* is defined by induction on its first argument, then, surprisingly, the following propositions are not typable in CIC:

$$app\ n\ \ell\ 0\ \ell' = \ell$$
$$app\ (n + n')\ (app\ n\ \ell\ n'\ \ell')\ n''\ \ell'' = app\ n\ \ell\ (n' + n'')\ (app\ n'\ \ell'\ n''\ \ell'').$$

In the first equation, the left-hand side is of type $list(n + 0)$ and the right-hand side is of type *listn*. Although one can prove that $n + 0 = n$ holds for any n in *nat*, the equality is not well-typed since $n + 0$ is not $\beta\iota$-convertible to n (only terms of equivalent types can be equal).

In the second equation, the left-hand side is of type $list((n + n') + n'')$ and the right-hand side is of type $list(n + (n' + n''))$. Again, although one can prove that $(n + n') + n'' = n + (n' + n'')$ holds for any n, n' and n'' in *nat*, the two terms are not $\beta\iota$-convertible. Therefore, the proposition is not well-formed.

On the other hand, by adding the rules $x + 0 \to x$ and $(x + y) + z \to x + (y + z)$, the previous propositions become well-typed, as expected.

Integration of decision procedures. One can also define predicates by rewrite rules or by having simplification rules on propositions, hence generalising the definitions by *strong elimination* in CIC. For example, one can consider the set of rules of Figure 1 (Hsiang 1982) where $\oplus$ (exclusive 'or') and $\wedge$ are commutative and associative symbols, $\bot$ represents the proposition always false and $\top$ the proposition always true.

Hsiang (Hsiang 1982) showed that this system is confluent and strongly normalising, and that a proposition P is a tautology (that is, is always true) iff P reduces to $\top$. So, assuming type-checking in CC extended with this rewrite system remains decidable, then, to know whether a proposition P is a tautology, it is sufficient to submit an arbitrary proof of $\top$ to the verification program. We would not only gain in automation but also in the size of proofs (any tautology would have a proof of constant size).

We can also imagine simplification rules on equalities like the ones of Figure 2, where $+$ and $\times$ are associative and commutative, and $=$ is commutative.

1.2. *Problems*

We saw that rewriting has numerous advantages over induction but it is not clear to what extent rewriting can be added to powerful type systems like the Calculus of

$$x = x \to \top$$
$$s\ x = s\ y \to x = y$$
$$s\ x = 0 \to \bot$$
$$x + y = 0 \to x = 0 \wedge y = 0$$
$$x \times y = 0 \to x = 0 \vee y = 0$$

Fig. 2. Simplification rules on equality.

Constructions (CC) without compromising the decidability of type-checking and logical consistency. Furthermore, since rewrite rules are user-defined, it is also not clear whether $\beta\mathscr{R}$-equivalence/normalisation can be made as efficient as a fixed system with $\beta\iota$-reduction only (Grégoire and Leroy 2002), though some work on rewriting seems very promising (Eker 1996; Kirchner and Moreau 2001).

Since we want to consider deductions modulo $\beta\mathscr{R}$-equivalence, we at least need this equivalence to be decidable. The usual way of proving the decidability of such an equivalence relation is by proving confluence and strong normalisation of the corresponding reduction relation. Since these properties are not decidable in general, we will look for the most general decidable sufficient conditions.

As for logical consistency, we can no longer deduce it from normalisation as is the case in CC (Barendregt 1992), since adding function symbols and rewrite rules is like adding hypothesis and equality/equivalence axioms. Therefore, for logical consistency also, we will look for sufficient conditions as general as possible.

In the following sub-section, we present a short history of the different results obtained so far on the combination of β-reduction and rewriting. Then, we will present our own contributions.

1.3. *Previous work*

The first work on the combination of typed λ-calculus and (first-order) rewriting is Breazu-Tannen (1988). This showed that the combination of simply-typed λ-calculus and first-order rewriting is confluent if rewriting is confluent. In 1989, Breazu-Tannen and Gallier (Breazu-Tannen and Gallier 1989), and Okada (Okada 1989) independently, showed that the strong normalisation also is preserved. These results were extended in Dougherty (1991) to any 'stable' set of pure λ-terms. The combination of first-order rewriting and Pure Type Systems (PTS) (Geuvers and Nederhof 1991; Barendregt 1992) has also been studied by several authors (Barbanera 1990; Barthe and Melliès 1996; Barthe and van Raamsdonk 1997; Barthe 1998).

Jouannaud and Okada (1991) extended the result of Breazu-Tannen and Gallier to higher-order rewrite systems satisfying the *General Schema*, an extension of primitive recursion to the simply-typed λ-calculus. With higher-order rewriting, strong normalisation becomes more difficult to prove since there is a strong interaction between rewriting and β-reduction, which is not the case with first-order rewriting.

In 1993, Barbanera, Fernández and Geuvers (Barbanera *et al.* 1994; Fernández 1993) extended the proof of Jouannaud and Okada to the Calculus of Constructions (CC) with

object-level rewriting and simply-typed function symbols. The methods used so far for non-dependent type systems (Breazu-Tannen and Gallier 1989; Dougherty 1991) cannot be applied to dependent type systems like CC since, in this case, rewriting is included in the type conversion rule and, thus, allows more terms to be typable. This was extended to PTS's in Barthe and Geuvers (1995a).

Other methods for proving strong normalisation have appeared. In 1993, Van de Pol (Van de Pol 1993; Van de Pol and Schwichtenberg 1995; Van de Pol 1996) extended the use of monotonic interpretations to the simply-typed λ-calculus. Jouannaud and Rubio (1999) extended Recursive Path Ordering (RPO) to the simply-typed λ-calculus.

In all these papers, even including those on CC, function symbols were always simply typed. It was Coquand (Coquand 1992) who initiated the study of rewriting with dependent and polymorphic symbols. He studied the completeness of definitions with dependent types. He proposed a schema more general than the schema of Jouannaud and Okada since it allows inductive definitions on strictly-positive types, but it does not necessarily imply strong normalisation. In 1996, Giménez (Giménez 1996; Giménez 1998) defined a restriction of this schema for which he proved strong normalisation. In 1999, Jouannaud, Okada and the current author (Blanqui *et al.* 2002; Blanqui *et al.* 1999) extended the General Schema in order to deal with strictly-positive types while still keeping simply-typed symbols. Finally, in 2000, Walukiewicz (Walukiewicz 2000; Walukiewicz-Chrząszcz 2002) extended Jouannaud and Rubio's HORPO to CC with dependent and polymorphic symbols.

All these works share a strong restriction: rewriting is restricted to the object level. In 1998, Dowek, Hardin and Kirchner (Dowek *et al.* 1998) proposed a new approach to deduction for first-order logic: Natural Deduction Modulo (NDM) a congruence $\equiv$ on propositions representing the intermediate computations between two deduction steps. This deduction system consists of replacing the usual rules of Natural Deduction by equivalent rules modulo $\equiv$. For instance, the elimination rule for $\Rightarrow$ (*modus ponens*) becomes

$$\frac{\Gamma \vdash R \quad \Gamma \vdash P}{\Gamma \vdash Q} \quad (R \equiv (P \Rightarrow Q)).$$

They proved that the simple theory of types (Dowek *et al.* 2001) and skolemised set theory can be seen as first-order theories modulo congruences using *explicit substitutions* (Abadi *et al.* 1991). Dowek and Werner (1998) and Dowek and Werner (2000) give several conditions ensuring strong normalisation of cut elimination in NDM.

1.4. *Contributions*

Our main contribution is to establish general conditions ensuring the strong normalisation of the Calculus of Constructions (CC) extended with predicate-level rewriting. We showed in Blanqui (2001) that these conditions are satisfied by most of the Calculus of Inductive Constructions (CIC) and by Natural Deduction Modulo (NDM) a large class of equational theories.

Our work can be seen as an extension of both NDM and CC, where the congruence not only includes first-order rewriting but also higher-order rewriting since, in CC, functions and predicates can be applied to functions and predicates.

It can therefore serve as a basis for a powerful extension of proof assistants like Coq (Coq Development Team 2002) or LEGO (Luo and Pollack 1992), which allow definitions by induction only. For its implementation, it may be convenient to use specialised rewriting-based applications like CiME (Contejean *et al.* 2000), ELAN (Borovanský *et al.* 2000) or Maude (Clavel *et al.* 1999). Furthermore, one can imagine using rewriting-based languages for program extraction (Paulin-Mohring 1989), and hence get more efficient extracted programs.

Interest in predicate-level rewriting is not completely new. A particular case is the 'strong elimination' of CIC, that is, the ability to define predicates by induction on some inductively defined data type. The main novelty here is to consider arbitrary user-defined predicate-level rewrite rules.

Therefore, our proof of the strong normalisation property cannot completely follow the methods of Werner (Werner 1994) and Altenkirch (Altenkirch 1993) since they used, in an essential way, the fact that function definitions are made by induction. And the methods used in the case of non-dependent first-order rewriting (Breazu-Tannen and Gallier 1989; Barbanera 1990; Dougherty 1991) cannot be applied because higher-order rewriting has a strong interaction with β-reduction and because, in dependent type systems, rewriting allows more terms to be typable. Our method is based on the notion of reducibility candidates given by Tait and Girard (Girard *et al.* 1988) and extend Geuvers' method (Geuvers 1994) for dealing with rewriting.

We should mention two other important contributions:

— In order to allow some quotient types (rules on constructors) and matching on function symbols, which is impossible in CIC, we use a notion of constructor that is more general than the usual one (see Section 5.1).
— In order to ensure the subject reduction property, that is, the preservation of typing under reduction, we introduce conditions that are more general than the ones used so far. In particular, these conditions allow us to get rid of non-linearities due to typing, which makes rewriting more efficient and confluence easier to prove (see Section 3).

2. The Calculus of Algebraic Constructions

The Calculus of Algebraic Constructions (CAC) is an extension of the Calculus of Constructions (CC) (Coquand and Huet 1988) with function and predicate symbols defined by rewrite rules.

2.1. *Terms*

CC is a particular Pure Type System (PTS) (Barendregt 1992) defined from a set $\mathscr{S} = \{\star, \Box\}$ of *sorts*. The sort $\star$ is intended to be the type of data types and propositions, while the sort $\Box$ is intended to be the type of predicate types (also called *kinds*). For instance,

the type *nat* of natural numbers is of type $\star$, $\star$ is of type $\Box$, the predicate $\leqslant$ on natural numbers is of type $nat \Rightarrow nat \Rightarrow \star$, and $nat \Rightarrow nat \Rightarrow \star$ is of type $\Box$.

The terms of CC are usually defined by the following grammar rule:

$$t ::= s \mid x \mid [x : t]t \mid (x : t)t \mid tt$$

where s is a sort, x a variable, $[x : t]t$ an abstraction, $(x : t)t$ a (dependent) product, and tt an application. We assume that the set $\mathscr{X}$ of variables is an infinite denumerable set disjoint from $\mathscr{S}$.

We simply extend CC by considering a denumerable set $\mathscr{F}$ of *symbols*, disjoint from $\mathscr{S}$ and $\mathscr{X}$, and by adding the following new construction:

$$t ::= \ldots \mid f \in \mathscr{F}.$$

We use $\mathscr{T}(\mathscr{F}, \mathscr{X})$ to denote the set of terms built from $\mathscr{F}$ and $\mathscr{X}$. Note that, in contrast with Blanqui (2001), function symbols are curried. No notion of arity is required.

2.2. *Notation*

Free and bound variables. A variable x in the scope of an abstraction $[x : T]$ or a product $(x : T)$ is *bound*. As usual, it may be replaced by any other variable. This is *α-equivalence*. A variable that is not bound is *free*. We use $\mathrm{FV}(t)$ to denote the set of free variables of a term t. A term without free variables is *closed*. We often use $U \Rightarrow V$ to denote a product $(x : U)V$ with $x \notin \mathrm{FV}(V)$ (non-dependent product). See Barendregt (1992) for more details on these notions.

Vectors. We often use vectors $(\vec{t}, \vec{u}, \ldots)$ for sequences of terms (or anything else). The size of a vector $\vec{t}$ is denoted by $|\vec{t}|$. For instance, $[\vec{x} : \vec{T}]u$ denotes the term $[x_1 : T_1] \ldots [x_n : T_n]u$ where $n = |\vec{x}|$.

Positions. To designate a subterm of a term, we use a system of *positions à la* Dewey (words over the alphabet of positive integers). Formally, the set $\mathrm{Pos}(t)$ of the positions in a term t is inductively defined as follows:

— $\mathrm{Pos}(f) = \mathrm{Pos}(s) = \mathrm{Pos}(x) = \{\varepsilon\}$,
— $\mathrm{Pos}((x : t)u) = \mathrm{Pos}([x : t]u) = \mathrm{Pos}(tu) = 1.\mathrm{Pos}(t) \cup 2.\mathrm{Pos}(u)$,

where ε denotes the empty word and '.' denotes concatenation. We use $t|_p$ to denote the subterm of t at the position p, and $t[u]_p$ to denote the term obtained by replacing $t|_p$ by u in t. The relation 'is a subterm of' is denoted by $\trianglelefteq$, and its strict part by $\triangleleft$.

We use $\mathrm{Pos}(f, t)$ to denote the set of positions p in t such that $t|_p = f$, and $\mathrm{Pos}(x, t)$ to denote the set of positions p in t such that $t|_p$ is a free occurrence of x in t.

Substitutions. A *substitution* θ is an application from $\mathscr{X}$ to $\mathscr{T}$ whose *domain* $\mathrm{dom}(\theta) = \{x \in \mathscr{X} \mid x\theta \neq x\}$ is finite. Its set of free variables is $\mathrm{FV}(\theta) = \bigcup\{\mathrm{FV}(x\theta) \mid x \in \mathrm{dom}(\theta)\}$. Applying a substitution θ to a term t consists of replacing every variable x free in t by its image $x\theta$ (to avoid variable captures, bound variables must be distinct from free variables). The result is denoted by $t\theta$. We use $\{\vec{x} \mapsto \vec{t}\}$ to denote the substitution that

associates t_i to x_i, and $\theta \cup \{x \mapsto t\}$ to denote the substitution that associates t to x and $y\theta$ to $y \neq x$.

Relations. Let $\to$ be a relation on terms. We use the following denotations:

$\to(t)$ the set of terms t' such that $t \to t'$,
$\leftarrow$ the inverse of $\to$,
$\to^+$ the smallest transitive relation containing $\to$,
$\to^*$ the smallest reflexive and transitive relation containing $\to$,
$\leftrightarrow^*$ the smallest reflexive, transitive and symmetric relation containing $\to$,
$\downarrow$ the relation $\to^* {}^*\!\leftarrow$ ($t \downarrow u$ if there exists v such that $t \to^* v$ and $u \to^* v$).

If $t \to t'$, we say that t *rewrites* to t'. If $t \to^* t'$, we say that t *reduces* to t'. A relation $\to$ is *stable by context* if $u \to u'$ implies $t[u]_p \to t[u']_p$ for all term t and position $p \in \mathrm{Pos}(t)$. The relation $\to$ is *stable by substitution* if $t \to t'$ implies $t\theta \to t'\theta$ for all substitution θ.

The *β-reduction* (respectively, *η-reduction*) relation is the smallest relation stable by context and substitution containing $[x : U]v\ u \to_\beta v\{x \mapsto u\}$ (respectively, $[x : U]tx \to_\eta t$ if $x \notin \mathrm{FV}(t)$). A term of the form $[x : U]v\ u$ (respectively, $[x : U]tx$ with $x \notin \mathrm{FV}(t)$) is a *β-redex* (respectively, *η-redex*).

A relation $\to$ is *weakly normalising* if, for all term t, there exists an irreducible term t' to which t reduces. We say that t' is a *normal form* of t. A relation $\to$ is *strongly normalising* (well-founded, nœtherian) if, for all term t, any reduction sequence issued from t is finite.

The relation $\to$ is *locally confluent* if, whenever a term t rewrites to two distinct terms u and v, we have $u \downarrow v$. The relation $\to$ is *confluent* if, whenever a term t reduces to two distinct terms u and v, we have $u \downarrow v$.

If $\to$ is locally confluent and strongly normalising then $\to$ is confluent (Newman's lemma). If $\to$ is confluent and weakly normalizing, every term t has a unique normal form denoted by $t\downarrow$.

Orderings. A *precedence* is a quasi-ordering on $\mathscr{F}$ whose strict part is well-founded. Let $>_1, \ldots, >_n$ be orderings on the sets $E_1, \ldots, E_n$ respectively. We use $(>_1, \ldots, >_n)_{\mathrm{lex}}$ to denote the *lexicographic* ordering on $E_1 \times \ldots \times E_n$. Now, let $>$ be an ordering on a set E. We use $>_{\mathrm{mul}}$ to denote the ordering on finite multisets on E. An important property of these extensions is that they preserve well-foundedness. See Baader and Nipkow (1998) for more details on these notions.

2.3. *Rewriting*

In first-order frameworks, that is, in a first-order term algebra, a rewrite rule is generally defined as a pair $l \to r$ of terms such that l is not a variable and the variables occurring in r also occur in l (otherwise, rewriting does not terminate). Then, one says that a term t rewrites to a term t' at position p, written $t \to^p t'$, if there exists a substitution σ such that $t|_p = l\sigma$ and $t' = t[r\sigma]_p$. See Dershowitz and Jouannaud (1990) for more details on (first-order) rewriting.

Here we consider a very similar rewriting mechanism by restricting the left-hand sides of rules to be algebraic, while allowing the right-hand sides to be arbitrary. This is a particular case of *Combinatory Reduction System* (CRS) (Klop *et al.* 1993) for which *higher-order pattern-matching* is unnecessary. However, we proved in Blanqui (2000) that, in the case of simply-typed λ-calculus, our termination criteria can be adapted to rewriting with higher-order pattern-matching.

Definition 2.1 (Rewriting). Terms that are only built from variables and applications of the form $f\vec{t}$ with $f \in \mathscr{F}$ are said to be *algebraic*. A *rewrite rule* is a pair of terms $l \to r$ such that l is algebraic and distinct from a variable, and for which $\mathrm{FV}(r) \subseteq \mathrm{FV}(l)$. A rule $l \to r$ is *left-linear* if no variable occurs more than once in l. A rule $l \to r$ is *non-duplicating* if no variable has more occurences in r than in l. A rule $f\vec{l} \to r$ is *compatible with a precedence* $\geqslant$ if, for any symbol g occuring in r, we have $f \geqslant g$.

Let $\mathscr{R}$ be a denumerable set of rewrite rules. The *$\mathscr{R}$-reduction relation* $\to_{\mathscr{R}}$ is the smallest relation containing $\mathscr{R}$ and stable by substitution and context. A term of the form $l\sigma$ with $l \to r \in \mathscr{R}$ is an *$\mathscr{R}$-redex*. We assume that $\to_{\mathscr{R}}$ is finitely branching.

Given a set $\mathscr{G} \subseteq \mathscr{F}$, we use $\mathscr{R}_{\mathscr{G}}$ to denote the set of rules that *define* a symbol in $\mathscr{G}$, that is, whose left-hand side is headed by a symbol in $\mathscr{G}$. A symbol f is *constant* if $\mathscr{R}_{\{f\}} = \varnothing$, otherwise it is (partially) *defined*. We use $\mathscr{C}\mathscr{F}$ to denote the set of constant symbols, and $\mathscr{D}\mathscr{F}$ to denote the set of defined symbols.

2.4. *Typing*

We now define the set of *well-typed* terms. An *environment* Γ is a list of pairs $x : T$ made up of a variable x and a term T. We use $\varnothing$ to denote the empty environment, and $\mathscr{E}(\mathscr{F}, \mathscr{X})$ to denote the set of environments built from $\mathscr{F}$ and $\mathscr{X}$. The *domain* of an environment Γ, $\mathrm{dom}(\Gamma)$, is the set of variables x such that a pair $x : T$ belongs to Γ. If $x \in \mathrm{dom}(\Gamma)$, we use $x\Gamma$ to denote the first term T such that $x : T$ belongs to Γ. The set of *free variables* in an environment Γ is $\mathrm{FV}(\Gamma) = \bigcup\{\mathrm{FV}(x\Gamma) \mid x \in \mathrm{dom}(\Gamma)\}$. Given two environments Γ and Γ', Γ is *included* in Γ', written $\Gamma \subseteq \Gamma'$, if all the elements of Γ occur in Γ' in the same order.

Definition 2.2 (Typing). We assume that every variable x is equipped with a sort s_x, that the set $\mathscr{X}^s$ of variables of sort s is infinite, and that α-equivalence preserves sorts. Let $\mathrm{FV}^s(t) = \mathrm{FV}(t) \cap \mathscr{X}^s$ and $\mathrm{dom}^s(\Gamma) = \mathrm{dom}(\Gamma) \cap \mathscr{X}^s$. We also assume that every symbol f is equipped with a sort s_f and a closed type $\tau_f = (\vec{x} : \vec{T})U$ such that, for all rules $f\vec{l} \to r$, we have $|\vec{l}| \leqslant |\vec{x}|$. We often write $f : T$ to say that $\tau_f = T$.

The typing relation of a CAC is the smallest ternary relation $\vdash \subseteq \mathscr{E} \times \mathscr{T} \times \mathscr{T}$ defined by the inference rules of Figure 3 where $s, s' \in \mathscr{S}$. A term t is *typable* if there exists an environment Γ and a term T such that $\Gamma \vdash t : T$ (T is a *type* of t in Γ). In the following, we always assume that $\vdash \tau_f : s_f$ for all $f \in \mathscr{F}$.

An environment is *valid* if a term is typable in it. A substitution θ is *well-typed from* Γ *to* Δ, $\theta : \Gamma \rightsquigarrow \Delta$, if, for all $x \in \mathrm{dom}(\Gamma)$, we have $\Delta \vdash x\theta : x\Gamma\theta$. We use $T \mathscr{C}_\Gamma T'$ to denote the fact that $T \downarrow T'$ and $\Gamma \vdash T' : s'$, and $T \mathbb{C}_\Gamma T'$ to denote the fact that $T \mathscr{C}_\Gamma T'$ and $\Gamma \vdash T : s$.

$$\text{(ax)}\quad \frac{}{\vdash \star : \Box}$$

$$\text{(symb)}\quad \frac{\vdash \tau_f : s_f}{\vdash f : \tau_f}$$

$$\text{(var)}\quad \frac{\Gamma \vdash T : s_x}{\Gamma, x : T \vdash x : T} \qquad (x \notin \mathrm{dom}(\Gamma))$$

$$\text{(weak)}\quad \frac{\Gamma \vdash t : T \quad \Gamma \vdash U : s_x}{\Gamma, x : U \vdash t : T} \qquad (x \notin \mathrm{dom}(\Gamma))$$

$$\text{(prod)}\quad \frac{\Gamma \vdash U : s \quad \Gamma, x : U \vdash V : s'}{\Gamma \vdash (x : U)V : s'}$$

$$\text{(abs)}\quad \frac{\Gamma, x : U \vdash v : V \quad \Gamma \vdash (x : U)V : s}{\Gamma \vdash [x : U]v : (x : U)V}$$

$$\text{(app)}\quad \frac{\Gamma \vdash t : (x : U)V \quad \Gamma \vdash u : U}{\Gamma \vdash tu : V\{x \mapsto u\}}$$

$$\text{(conv)}\quad \frac{\Gamma \vdash t : T \quad \Gamma \vdash T' : s'}{\Gamma \vdash t : T'} \qquad (T \downarrow_{\beta\mathscr{R}} T')$$

Fig. 3. Typing rules.

Compared with CC, we have a new rule, (symb), for typing symbols and, in the type conversion rule (conv), we have $\downarrow_{\beta\mathscr{R}}$ (which we simply denote by $\downarrow$ in the rest of the paper) instead of the β-conversion $\leftrightarrow_\beta^* = \downarrow_\beta$ (since β is confluent).

Well-typed substitutions enjoy the following important substitution property: if $\Gamma \vdash t : T$ and $\theta : \Gamma \rightsquigarrow \Delta$, then $\Delta \vdash t\theta : T\theta$.

The relations $\mathscr{C}_\Gamma$ (not symmetric) and $\mathbf{C}_\Gamma$ (symmetric) are useful when inverting typing judgements. For instance, a derivation of $\Gamma \vdash uv : W'$ necessarily terminates with an application of the (app) rule, possibly followed by applications of the rules (weak) and (conv). Therefore, there exists V and W such that $\Gamma \vdash u : (x : V)W$, $\Gamma \vdash v : V$ and $W\{x \mapsto v\}\ \mathscr{C}_\Gamma^*\ W'$. Since, in the (conv) rule, T is not required to be typable by some sort s (as it is the case for T'), it is not *a priori* the case that $W\{x \mapsto v\}$ is typable and, therefore, that, in fact, $W\{x \mapsto v\}\ \mathbf{C}_\Gamma^*\ W'$.

Many of the well-known basic properties of Pure Type Systems (PTS's) (Barendregt 1992) also hold for CAC's. In Blanqui (2001), we study these properties in an abstract way by considering a PTS equipped with an unspecified type conversion rule (instead of $\downarrow_\beta$ or $\downarrow_{\beta\mathscr{R}}$ for instance), hence factorising several previous proofs for different PTS extensions. The properties we use in this paper are:

Type correctness If $\Gamma \vdash t : T$, then either $T = \Box$ or $\Gamma \vdash T : s$.

Conversion correctness If $\Gamma \vdash T : s$ and $T\ \mathbf{C}_\Gamma^*\ T'$, then $\Gamma \vdash T' : s$.

Convertibility of types If $\Gamma \vdash t : T$ and $\Gamma \vdash t : T'$, then $T\ \mathbf{C}_\Gamma^*\ T'$.

Only convertibility of types requires confluence (conversion correctness is proved in Section 3.2 without using confluence).

Amongst well-typed terms, we distinguish:

— The set $\mathbb{K}$ of *predicate types* or *kinds* made of the terms K such that $\Gamma \vdash K : \Box$. It is easy to check that every predicate type is of the form $(\vec{x} : \vec{T})\star$.
— The set $\mathbb{P}$ of *predicates* made of the terms T such that $\Gamma \vdash T : K$ and $\Gamma \vdash K : \Box$.
— The set $\mathbb{O}$ of *objects* made of the terms t such that $\Gamma \vdash t : T$ and $\Gamma \vdash T : \star$.

3. Subject reduction

Before studying the strong normalisation or the logical consistency of our system, we must make sure that the reduction relation $\rightarrow_{\beta\mathscr{R}}$ is indeed correct with respect to typing, that is, if $\Gamma \vdash t : T$ and $t \rightarrow_{\beta\mathscr{R}} t'$, then $\Gamma \vdash t' : T$. This property is usually called *subject reduction*. Once it holds, it can be easily extended to types, environments and substitutions:

— If $\Gamma \vdash t : T$ and $T \rightarrow T'$, then $\Gamma \vdash t : T'$.
— If $\Gamma \vdash t : T$ and $\Gamma \rightarrow \Gamma'$, then $\Gamma' \vdash t : T$.
— If $\theta : \Gamma \rightsquigarrow \Delta$ and $\theta \rightarrow \theta'$, then $\theta' : \Gamma \rightsquigarrow \Delta$.

In the presence of dependent types and rewriting, the subject reduction for β appears to be a difficult problem. Indeed, in the case of a head-reduction $[x : U']v\ u \rightarrow_\beta v\{x \mapsto u\}$ with $\Gamma \vdash [x : U']v : (x : U)V$ and $\Gamma \vdash u : U$, we must prove that $\Gamma \vdash v\{x \mapsto u\} : V\{x \mapsto u\}$. By inversion, we have $\Gamma, x : U' \vdash v : V'$ with $(x : U')V'\ \mathbb{C}^*_\Gamma\ (x : U)V$. We can conclude that $\Gamma \vdash v\{x \mapsto u\} : V\{x \mapsto u\}$ only if

$$(x : U')V'\ \mathbb{C}^*_\Gamma\ (x : U)V \text{ implies } U'\ \mathbb{C}^*_\Gamma\ U \text{ and } V'\ \mathbb{C}^*_{\Gamma, x:U}\ V,$$

which is a property we call *product compatibility*.

This is immediate as soon as $\rightarrow_{\beta\mathscr{R}}$ is confluent. Unfortunately, there are very few results on the confluence of higher-order rewriting and β-reduction together (see the discussion after Definition 5.12). Fortunately, confluence is not the only way to prove the product compatibility. In Geuvers (1993), Geuvers proves product compatibility for the Calculus of Constructions (CC) with $\leftrightarrow^*_{\beta\eta}$ as type conversion relation, although $\rightarrow_{\beta\eta}$ is not confluent on untyped terms: $[x : T]x_\beta \leftarrow [x : T]([y : U]y\ \ x) \rightarrow_\eta [y : U]y =_\alpha [x : U]x$ (Nederpelt 1973). And, in Barbanera *et al.* (1997), Barbanera, Geuvers and Fernández prove the product compatibility for CC with $\downarrow_\beta \cup \downarrow_{\mathscr{R}}$ as type conversion relation, where $\mathscr{R}$ is a set of simply-typed object-level rewrite rules.

In Section 3.2, we prove the product compatibility, and hence the subject reduction of β, for a large class of rewrite systems, including predicate-level rewriting, without using confluence, but by generalising the proof of Barbanera, Fernández and Geuvers (Barbanera *et al.* 1997). However, before that, we study subject reduction for rewriting.

3.1. *Subject reduction for rewriting*

In first-order sorted algebras, for rewriting to preserve sorts, it suffices that both sides of a rule have the same sort. Carried over to type systems, this condition gives: there exists

an environment Γ and a type T such that $\Gamma \vdash l : T$ and $\Gamma \vdash r : T$. This condition is the one that has been taken in all previous work combining typed λ-calculus and rewriting. However, it has an important drawback. With polymorphic or dependent types, it leads to strongly non left-linear rules, which has two important consequences. First, rewriting is strongly slowed down because of the necessary equality tests. Second, it is more difficult to prove confluence.

Let us take the example of the concatenation of two polymorphic lists (type $list : \star \Rightarrow \star$ with the constructors $nil : (A : \star) listA$ and $cons : (A : \star) A \Rightarrow listA \Rightarrow listA$):

$$app\ A\ (nil\ A)\ \ell' \to \ell'$$
$$app\ A\ (cons\ A\ x\ \ell)\ \ell' \to cons\ A\ x\ (app\ A\ \ell\ \ell').$$

This definition satisfies the usual condition by taking $\Gamma = A : \star, x : A, \ell : listA, \ell' : listA$ and $T = listA$. But one may wonder whether it is really necessary to do an equality test between the first argument of *app* and the first argument of *cons* when one wants to apply the second rule. Indeed, if $app\ A\ (cons\ A'\ x\ \ell)\ \ell'$ is well-typed, then, by inversion, $cons\ A'\ x\ \ell$ is of type $listA$ and, by inversion again, $listA'$ is convertible to $listA$. Thus, A is convertible to A'.

In fact, what is important is not that the left-hand side of a rule be typable, but that, if an instance of the left-hand side of a rule is typable, then the corresponding instance of the right-hand side has the same type. We express this by requiring that there exists an environment Γ in which the right-hand side is typable, and a substitution ρ that replaces the variables of the left-hand side not belonging to Γ by terms typable in Γ. Hence, one can instead consider the rules

$$app\ A\ (nil\ A')\ \ell' \to \ell'$$
$$app\ A\ (cons\ A'\ x\ \ell)\ \ell' \to cons\ A\ x\ (app\ A\ \ell\ \ell')$$

by taking $\Gamma = A : \star, x : A, \ell : listA, \ell' : listA$ and $\rho = \{A' \mapsto A\}$.

Definition 3.1 (Well-typed rule). A rule $l \to r$ with $l = f\vec{l}$, $f : (\vec{x} : \vec{T})U$ and $\gamma = \{\vec{x} \mapsto \vec{l}\}$ is *well-typed* if there exists an environment Γ and a substitution ρ such that[†]:

(S3) $\Gamma \vdash r : U\gamma\rho$,

(S4) $\forall \Delta, \sigma, T$, if $\Delta \vdash l\sigma : T$, then $\sigma : \Gamma \rightsquigarrow \Delta$,

(S5) $\forall \Delta, \sigma, T$, if $\Delta \vdash l\sigma : T$, then $\sigma \downarrow \rho\sigma$.

In the following, we write $(l \to r, \Gamma, \rho) \in \mathscr{R}$ when the previous conditions are satisfied.

An example with dependent types is given by the concatenation of two lists of fixed length (type $list : nat \Rightarrow \star$ with the constructors $nil : list\,0$ and $cons : nat \Rightarrow (n : nat)\ list\,n \Rightarrow list\,(s\,n)$) and the function *map*, which applies a function f to every element of a list:

$$app : (n : nat) list\,n \Rightarrow (n' : nat) list\,n' \Rightarrow list\,(n + n')$$
$$map : (nat \Rightarrow nat) \Rightarrow (n : nat) list\,n \Rightarrow list\,n$$

[†] The conditions **(S1)** $\mathrm{dom}(\rho) \cap \mathrm{dom}(\Gamma) = \varnothing$ and **(S2)** $\Gamma \vdash l\rho : U\gamma\rho$ given in Blanqui (2001) are not necessary for proving the subject reduction property, but they are necessary for proving the strong normalisation property of the higher-order rewrite rules (see Definition 5.9).

where *app* and *map* are defined by

$$\begin{aligned} app\ 0\ \ell\ n'\ \ell' &\to \ell' \\ app\ p\ (cons\ x\ n\ \ell)\ n'\ \ell' &\to cons\ x\ (n+n')\ (app\ n\ \ell\ n'\ \ell') \\ map\ f\ 0\ \ell &\to \ell \\ map\ f\ p\ (cons\ x\ n\ \ell) &\to cons\ (f\ x)\ n\ (map\ f\ n\ \ell) \\ map\ f\ p\ (app\ n\ \ell\ n'\ \ell') &\to app\ n\ (map\ f\ n\ \ell)\ n'\ (map\ f\ n'\ \ell'). \end{aligned}$$

For the second rule of *app*, we take $\Gamma = x : nat, n : nat, \ell : list\ n, n' : nat, \ell' : list\ n'$ and $\rho = \{p \mapsto sn\}$. This avoids checking that p is convertible to sn. For the third rule of *map*, we take $\Gamma = f : nat \Rightarrow nat, n : nat, \ell : list\ n, n' : nat, \ell' : list\ n'$ and $\rho = \{p \mapsto n + n'\}$. This avoids checking that p is convertible to $n + n'$. The reader will find more examples at the end of Section 5.

Lemma 3.2. If $\beta\mathscr{R}$ is product compatible, $f : (\vec{x} : \vec{T})U$, $\theta = \{\vec{x} \mapsto \vec{t}\}$ and $\Gamma \vdash f\vec{t} : T$, then $\theta : \Gamma_f \rightsquigarrow \Gamma$ and $U\theta\ \mathbb{C}^*_\Gamma\ T$.

Proof. By inversion, there is a sequence of products $(x_i : T'_i)U_i$ $(1 \leqslant i \leqslant n = |\vec{x}|)$ such that $\Gamma \vdash f t_1 \ldots t_{n-1} : (x_n : T'_n)U_n$, $\Gamma \vdash t_n : T'_n$, $U_n\theta\ \mathbb{C}^*_\Gamma\ T, \ldots, \Gamma \vdash f : (x_1 : T'_1)U_1$, $\Gamma \vdash t_1 : T'_1$, $U_1\theta\ \mathbb{C}^*_\Gamma\ (x_2 : T'_2)U_2$ and $(\vec{x} : \vec{T})U\ \mathbb{C}^*_\Gamma\ (x_1 : T'_1)U_1$. Let $V_i = (x_{i+1} : T_{i+1}) \ldots (x_n : T_n)U$. By product compatibility, $T_1\theta = T_1\ \mathbb{C}^*_\Gamma\ T'_1$ and $V_1\ \mathbb{C}^*_{\Gamma, x_1 : T_1}\ U_1$. Hence, $V_1\theta = (x_2 : T_2\theta)V_2\theta\ \mathbb{C}^*_\Gamma\ U_1\theta\ \mathbb{C}^*_\Gamma\ (x_2 : T'_2)U_2$. Therefore, by induction, $T_2\theta\ \mathbb{C}^*_\Gamma\ T'_2, \ldots, T_n\theta\ \mathbb{C}^*_\Gamma\ T'_n$ and $U\theta\ \mathbb{C}^*_\Gamma\ U_n\theta\ \mathbb{C}^*_\Gamma\ T$. Hence, by conversion, $\Gamma \vdash t_i : T_i\theta$, that is, $\theta : \Gamma_f \rightsquigarrow \Gamma$. □

Theorem 3.3 (Subject reduction for $\mathscr{R}$). If $\beta\mathscr{R}$ is product compatible and $\mathscr{R}$ is a set of well-typed rules, then $\mathscr{R}$ preserves typing.

Proof. As usual, we prove, by induction on $\Delta \vdash t : T$, that, if $t \to_{\mathscr{R}} t'$, then $\Delta \vdash t' : T$, and if $\Delta \to_{\mathscr{R}} \Delta'$, then $\Delta' \vdash t : T$. We only give details for the (app) case. Assume that $\Delta \vdash l\sigma : T$, $(l \to r, \Gamma, \rho) \in \mathscr{R}$, $l = f\vec{l}$, $f : (\vec{x} : \vec{T})U$ and $\gamma = \{\vec{x} \mapsto \vec{l}\}$. Let $\theta = \gamma\sigma$. After Lemma 3.2, $\theta : \Gamma_f \rightsquigarrow \Delta$ and $U\theta\ \mathbb{C}^*_\Delta\ T$. By **S4**, $\sigma : \Gamma \rightsquigarrow \Delta$. By **S3**, $\Gamma \vdash r : U\gamma\rho$. Therefore, by substitution, $\Delta \vdash r\sigma : U\gamma\rho\sigma$. By **S5**, $\rho\sigma \downarrow \sigma$. Therefore, by conversion, $\Delta \vdash r\sigma : U\theta$ and $\Delta \vdash r\sigma : T$. □

How do we check the conditions S3, S4 and S5? In all their generality, they are certainly undecidable. On the one hand, we do not know whether $\vdash$ and $\downarrow$ are decidable and, on the other hand, in S4 and S5, we arbitrarily quantify over Δ, σ and T. It is therefore necessary to impose additional restrictions. In the following, we consider the three conditions in turn.

Let us look first at S3. In practice, the symbols and their defining rules are often added one after another (or by groups, but the following argument can be generalised). Let $(\mathscr{F}, \mathscr{R})$ be a system in which $\vdash$ is decidable, $f \notin \mathscr{F}$ and $\mathscr{R}_f$ is a set of rules defining f and whose symbols belong to $\mathscr{F}' = \mathscr{F} \cup \{f\}$. Then, in $(\mathscr{F}', \mathscr{R})$, $\vdash$ is still decidable. One can therefore try to check S3 in this system. This does not seem an important restriction: it would be surprising if the typing of a rule requires the use of the rule itself!

We now consider S4.

Definition 3.4 (Canonical and derived types). Let t be a term of the form $l\sigma$ with $l = f\vec{l}$ algebraic, $f : (\vec{x} : \vec{T})U$, $n = |\vec{x}| = |\vec{l}|$ and $\gamma = \{\vec{x} \mapsto \vec{l}\}$. The term $U\gamma\sigma$ will be called the *canonical type* of t. Let $p \in \mathrm{Pos}(l)$ be of the form $(1^*2)^+$. We inductively define the *type of* $t|_p$ *derived from* t, $\tau(t,p)$, as follows:

— if $p = 1^{n-i}2$, then $\tau(t,p) = T_i\gamma\sigma$,
— if $p = 1^{n-i}2q$ and $q \neq \varepsilon$, then $\tau(t,p) = \tau(t_i,q)$.

The type of $t|_p$ derived from t only depends on the term above $t|_p$.

Lemma 3.5 (S4). If, for all $x \in \mathrm{dom}(\Gamma)$, there is $p \in \mathrm{Pos}(x,l)$ such that $x\Gamma = \tau(l,p)$, then S4 is satisfied.

Proof. We prove S4 by induction on the size of l. Assume that $\Delta \vdash l\sigma : T$. We must prove that, for all $x \in \mathrm{dom}(\Gamma)$, $\Delta \vdash x\sigma : x\Gamma\sigma$. By assumption, there is $p \in \mathrm{Pos}(x,l)$ such that $x\Gamma = \tau(l,p)$. Since $l = f\vec{l}$, we have $p = jq$. Assume that $f : (\vec{x} : \vec{T})U$. Let $\gamma = \{\vec{x} \mapsto \vec{l}\}$ and $\theta = \gamma\sigma$. If $q = \varepsilon$, then $x = l_j$ and $x\Gamma = T_j\gamma$. Now, after Lemma 3.2, $\theta : \Gamma_f \rightsquigarrow \Delta$. So, $\Delta \vdash x_j\theta : T_j\theta$, that is, $\Delta \vdash x\sigma : x\Gamma\sigma$. Assume now that $q \neq \varepsilon$. Since $\Delta \vdash l_j\sigma : T_j\theta$, l_j is of the form $g\vec{m}$ and $x\Gamma = \tau(l_j,q)$, by the induction hypothesis, $\Delta \vdash x\sigma : x\Gamma\sigma$. □

For S5, we have no general result. By inversion, S5 can be seen as a unification problem modulo $\downarrow^*$. The confluence of $\to$ (which implies that $\downarrow^* = \downarrow$) can therefore be very useful. Unfortunately, there are very few results on the confluence of the combination of higher-order rewriting and β-reduction (see the discussion after Definition 5.12). On the other hand, one can easily prove that local confluence is preserved.

Theorem 3.6 (Local confluence). If $\mathscr{R}$ is locally confluent on algebraic terms, then $\beta\mathscr{R}$ is locally confluent on any term.

Proof. Assume that $t \to^p t_1$ and $t \to^q t_2$. We prove by induction on t that there exists t' such that $t_1 \to^* t'$ and $t_2 \to^* t'$. There are three cases:

— $p \sharp q$ (p and q have no common prefix). The reductions at p and q can be done in parallel: $t_1 \to^q t_1'$, $t_2 \to^p t_2'$ and $t_1' = t_2'$.
— $p = ip'$ and $q = iq'$. We can conclude by the induction hypothesis on $t|_i$.
— $p = \varepsilon$ or $q = \varepsilon$. By exchanging the roles of p and q, we can assume that $p = \varepsilon$. There are two cases:
  – $t = [x : V]u\ v$ and $t_1 = u\{x \mapsto v\}$. We distinguish three sub-cases:
    • $q = 11q'$ and $V \to^{q'} V'$. Then, $t' = t_1$ works.
    • $q = 12q'$ and $u \to^{q'} u'$. Then, $t' = u'\{x \mapsto v\}$ works.
    • $q = 2q'$ and $v \to^{q'} v'$. Then, $t' = u\{x \mapsto v'\}$ works.
  – $t = l\sigma$, $l \to r \in \mathscr{R}$ and $t_1 = r\sigma$. There exists an algebraic term u of maximal size and a substitution θ such that $t = u\theta$ and $x\theta = y\theta$ implies $x = y$ (u and θ are unique up to the choice of variables and u has the same non-linearities as t). As the left-hand sides of rules are algebraic, $u = l\sigma'$ and $\sigma = \sigma'\theta$. Now we distinguish two sub-cases:

- $q \in \mathrm{Pos}(u)$. As the left-hand sides of rules are algebraic, we have $u \to_{\mathscr{R}} r\sigma'$ and $u \to_{\mathscr{R}} v$. By local confluence of $\to_{\mathscr{R}}$ on algebraic terms, there exists u' such that $r\sigma' \to^* u'$ and $v \to^* u'$. Then $t' = u'\theta$ works.
- $q = q_1 q'$ and $u|_{q_1} = x$. Let $q_2, \ldots, q_n$ be the positions of the other occurrences of x in u. If one reduces t_2 at each position $q_i q'$, one obtains a term of the form $l\sigma'\theta'$ where θ' is the substitution such that $x\theta'$ is the reduct of $x\theta$, and $y\theta' = y\theta$ if $y \neq x$. Then, $t' = r\sigma'\theta'$ works. □

3.2. *Subject reduction for β*

In this section, we prove the product compatibility, and hence the subject reduction of β, for a large class of rewrite systems, including predicate-level rewrite rules, without using confluence, but by generalising the proof of Barbanera, Fernández and Geuvers (Barbanera *et al.* 1997). It is worth noting that no result of this section assumes the subject reduction property for rewriting. They only rely on simple syntactic properties of β-reduction and rewriting with respect to predicates and kinds (Lemma 3.9).

The idea is to β-weak-head normalise all the intermediate terms between $(x : U')V'$ and $(x : U)V$ so that we obtain a sequence of conversions between product terms only. We first show that the subject reduction property can indeed be studied in a system whose conversion relation is like the one used in Barbanera *et al.* (1997).

Lemma 3.7. Let Λ be a CAC with conversion relation $\downarrow_{\beta\mathscr{R}}$ and Λ' be the same CAC but with conversion relation $\downarrow_\beta \cup \downarrow_{\mathscr{R}}$. If $\to_{\beta\mathscr{R}}$ has the subject reduction property in Λ', then $\Lambda = \Lambda'$ (and $\to_{\beta\mathscr{R}}$ has the subject reduction property in Λ).

Proof. Let $\vdash$ (respectively, $\vdash'$) be the typing relation of Λ (respectively, Λ'). Since $\downarrow_\beta \cup \downarrow_{\mathscr{R}} \subseteq \downarrow_{\beta\mathscr{R}}$, we clearly have $\vdash' \subseteq \vdash$. We prove by induction on $\vdash$ that $\vdash \subseteq \vdash'$. The only difficult case is, of course, (conv). By the induction hypothesis, we have $\Gamma \vdash' t : T$ and $\Gamma \vdash' T' : s'$. Furthermore, we have $T \to^*_{r_1} \to^*_{r_2} \cdots {}^*_{s_2}\!\leftarrow {}^*_{s_1}\!\leftarrow T'$ with $r_k, s_k \in \{\beta, \mathscr{R}\}$. By type correctness, either $T = \Box$ or there is a sort s such that $\Gamma \vdash' T : s$. If $T = \Box$, then $T' \to^* \Box$. But, since $\to$ has the subject reduction property in Λ', we get that $\Gamma \vdash' \Box : s'$, which is impossible. Therefore, T and T' are typable in Λ' and, since $\to$ has the subject reduction property in Λ', all the terms between T and T' are also typable in Λ'. Therefore, we can replace the conversion in Λ by a sequence of conversions in Λ'. □

We now prove a series of useful results about kinds and predicates that will allow us to prove the subject reduction property on types for the β-weak-head reduction relation h: $t \to_h t'$ if $t = [\vec{x} : \vec{T}]([x : U]v u\vec{t})$ and $t' = [\vec{x} : \vec{T}](v\{x \mapsto u\}\vec{t})$. The β-internal reduction relation will be denoted by $\not{h}$. To this end, we introduce several sets of terms:

$\mathscr{K}$: terms of the form $(\vec{x} : \vec{T})\star$, usually called *kinds*.
$\mathscr{P}$: the smallest set of terms, called *predicates*, such that $\mathscr{X}^\Box \cup \mathscr{F}^\Box \subseteq \mathscr{P}$ and, if $pt \in \mathscr{P}$ or $[x : t]p \in \mathscr{P}$ or $(x : t)p \in \mathscr{P}$, then $p \in \mathscr{P}$.
$\mathscr{W}$: terms having a subterm of the form $[y : W]K$ or wK, called a *bad kind*.
$\mathscr{B}$: terms containing $\Box$.

Lemma 3.8.

(α) No term in $\mathscr{B}$ is typable.
(β) If $\Gamma \vdash t : \Box$, then $t \in \mathscr{K}$.
(γ) If $t\theta \in \mathscr{B}$, then $t \in \mathscr{B}$ or $x\theta \in \mathscr{B}$ for some x.
(δ) If $t\theta \in \mathscr{K}$, then $t \in \mathscr{K}$ or $x\theta \in \mathscr{K}$ for some x.

Proof.

(α) $\Box$ is not typable and every subterm of a typable term is typable.
(β) The proof is by induction on the size of t (no conversion can take place since $\Box$ is not typable).
(γ) This case is trivial.
(δ) If $t\theta \in \mathscr{K}$ and $t \notin \mathscr{K}$, then $t = (\vec{x} : \vec{T})x$ with $x\theta \in \mathscr{K}$. □

Lemma 3.9. If, for every rule $l \to r \in \mathscr{R}$, $r \notin \mathscr{B} \cup \mathscr{K} \cup \mathscr{W}$, then:

(a) If $t \to t'$ and $t' \in \mathscr{B}$, then $t \in \mathscr{B}$.
(b) If $\Box\ \mathscr{C}^*_\Gamma\ T$, then $T = \Box$.
(c) If $K \in \mathscr{K}$ and $\Gamma \vdash K : L$, then $L = \Box$.
(d) No term in $\mathscr{W}$ is typable.
(e) If $t \to K \in \mathscr{K}$, then $t \in \mathscr{K} \cup \mathscr{W}$.
(f) If $t \to t' \in \mathscr{W}$, then $t \in \mathscr{W}$.
(g) If $\Gamma \vdash T : s$ and $T \to^* K \in \mathscr{K}$, then $T \in \mathscr{K}$ and $s = \Box$.
(h) If $T\ \mathbb{C}^*_\Gamma\ K$ and $\Gamma \vdash K : \Box$, then $\Gamma \vdash T : \Box$ and $T \in \mathscr{K}$.
(i) If $(\vec{x} : \vec{T})\star\ \mathbb{C}^*_\Gamma\ (\vec{y} : \vec{U})\star$, then $|\vec{x}| = |\vec{y}|$ and, for all i, $T_i\ \mathbb{C}^*_{\Gamma_i}\ U_i\{\vec{y} \mapsto \vec{x}\}$ with $\Gamma_i = \Gamma, x_1 : T_1, \ldots, x_i : T_i$.
(j) If $T\ \mathbb{C}^*_\Gamma\ T'$ and $\Gamma \vdash T : \star$, then $\Gamma \vdash T' : \star$.
(k) If $\Gamma \vdash t : T$ and $t \in \mathscr{P}$, then $T \in \mathscr{K}$.
(l) If $\Gamma \vdash t : K$ and $\Gamma \vdash K : \Box$, then $t \in \mathscr{P}$.

Proof.

(a) Assume that $t \to^p t'$ and $t'|_q = \Box$. If $p \not\parallel q$, then $t|_q = \Box$ and $t \in \mathscr{B}$. Otherwise, $p \leqslant q$. If $t|_p = [x : U]v\ u$ and $t'|_p = v\{x \mapsto u\}$, then, by (γ), $v \in \mathscr{B}$ or $u \in \mathscr{B}$. Thus, $t \in \mathscr{B}$. Now, if $t|_p = l\sigma$, $t'|_p = r\sigma$ and $l \to r \in \mathscr{R}$, then, by (γ), $r \in \mathscr{B}$ or $x\sigma \in \mathscr{B}$ for some x. Since $r \notin \mathscr{B}$, $x\sigma \in \mathscr{B}$ and $t \in \mathscr{B}$.
(b) Assume that $\Box \downarrow T'\ \mathbb{C}^*_\Gamma\ T$. Then, $T' \to^* \Box$ and $\Gamma \vdash T' : s$. By (a), $T' \in \mathscr{B}$ and T' cannot be typable. Thus, $T = \Box$.
(c) The proof is by induction on the size of K. If $K = \star$, then, by inversion, $\Box\ \mathscr{C}^*_\Gamma\ L$ and, by (b), $L = \Box$. If $K = (x : T)K'$, then, by inversion, $\Gamma, x : T \vdash K' : s$ and $s\ \mathscr{C}^*_\Gamma\ L$. By the induction hypothesis, $s = \Box$ and, by (b), $L = \Box$.
(d) Assume that $\Gamma \vdash [y : W]K : T$. By inversion, $\Gamma, y : W \vdash K : L$ and $\Gamma \vdash (y : W)L : s$. By (c), $L = \Box$ and $(y : W)L$ cannot be typable. Assume now that $\Gamma \vdash wK : T$. By inversion, $\Gamma \vdash w : (x : L)V$, $\Gamma \vdash K : L$ and $\Gamma \vdash (x : L)V : s$. By (c), $L = \Box$ and $(x : L)V$ cannot be typable.
(e) Assume that $t \to K \in \mathscr{K}$ and $t \notin \mathscr{K}$. We prove that $t \in \mathscr{W}$ by induction on the size of t. The only possible cases are $t = (x : T)u$, $t = [x : U]v\ u$ if $t \to_\beta K$, and $t = l\sigma$

with $l \to r \in \mathscr{R}$ if $t \to_{\mathscr{R}} K$. If $t = (x:T)u$, then $K = (x:T)L$ and $u \to L$. By the induction hypothesis, $u \in \mathscr{W}$. If $t = [x:U]v\ u$, then $K = v\{x \mapsto u\}$. By (δ), either $v \in \mathscr{K}$ or $u \in \mathscr{K}$. In both cases, $t \in \mathscr{W}$. Assume now that $t = l\sigma$ with $l \to r \in \mathscr{R}$. Then, $K = r\sigma$. By (δ), either $r \in \mathscr{K}$ or $x\sigma \in \mathscr{K}$ for some x. Since $r \notin \mathscr{K}$, we have $x\sigma \in \mathscr{K}$ and $t = l\sigma \in \mathscr{W}$ since x is the argument of some symbol (l is algebraic).

(f) Assume that $t \to^p t' \in \mathscr{W}$, $t'|_q = wK$ and $K \in \mathscr{K}$ (the case $t'|_q = [x:w]K$ is dealt with in the same way). There are several cases:

— $q \sharp p$. Then, $t|_q = wK$ and $t \in \mathscr{W}$.

— $q < p$.

- $p = q1m$. Then $t|_q = w'K$ with $w' \to w$ and $t \in \mathscr{W}$.
- $p = q2m$. Then $t|_q = wu$ with $u \to K \in \mathscr{K}$. By (e), $u \in \mathscr{K} \cup \mathscr{W}$. Thus $t \in \mathscr{W}$.

— $q \geqslant p$. Then $q = pm$. Assume that $t|_p = l\sigma$, $t'|_p = r\sigma$ and $l \to r \in \mathscr{R}$ (the case $t \to_\beta t'$ is dealt with in the same way). Let $\{p_1, \ldots, p_n\} = \{p \in \mathrm{Pos}(x,r) \mid x \in \mathrm{FV}(r)\}$. There are several cases:

- $m \sharp p_i$ for all i, or $m < p_i$ for some i. Then, $r|_m\sigma = wK$, $r = uv$ and $v\sigma = K$. By (δ), $v \in \mathscr{K}$ or $x\sigma \in \mathscr{K}$ for some x. If $v \in \mathscr{K}$, then $r \in \mathscr{W}$, which is impossible. Thus, $x\sigma \in \mathscr{K}$ and $l\sigma \in \mathscr{W}$.
- $m \geqslant p_i$ for some i. Then there is $x \in \mathrm{FV}(l)$ such that $x\sigma \in \mathscr{W}$. Thus, $t \in \mathscr{W}$.

(g) By (e) and (f), if $T \to^* K \in \mathscr{K}$, then $T \in \mathscr{K} \cup \mathscr{W}$. Since $\Gamma \vdash T : s$, we have $T \notin \mathscr{W}$. Thus, $T \in \mathscr{K}$ and $s = \Box$.

(h) The proof is by induction on the number of conversions between T and K. Assume that $\Gamma \vdash T : s$, $T \downarrow K$ and $\Gamma \vdash K : \Box$. Then there is $K' \in \mathscr{K}$ such that $K \to^* K'$ and $T \to^* K'$. By (g), $T \in \mathscr{K}$ and $s = \Box$.

(i) By (h), all the intermediate well-typed terms between $K = (\vec{x} : \vec{T})\star$ and $L = (\vec{y} : \vec{U})\star$ are kinds and, if $K \downarrow L$, then, clearly, $|\vec{x}| = |\vec{y}|$ and $T_i \downarrow U_i\{\vec{x} \mapsto \vec{y}\}$ for all i.

(j) This case is an immediate consequence of (i).

(k) The proof is by induction on $\Gamma \vdash t : T$.

(l) The proof is by induction on $\Gamma \vdash t : K$. □

Lemma 3.10. Given a rule $l \to r$ with $l = f\vec{l}$, $f : (\vec{x} : \vec{T})U$ and $\gamma = \{\vec{x} \mapsto \vec{l}\}$, we have $r \notin \mathscr{B} \cup \mathscr{K} \cup \mathscr{W}$ if there is an environment Γ and a substitution ρ such that $\Gamma \vdash l\rho : U\gamma\rho$ and $\Gamma \vdash r : U\gamma\rho$.

Proof. Since r is typable, $r \notin \mathscr{B} \cup \mathscr{W}$. We now prove that $r \notin \mathscr{K}$. Since $\Gamma \vdash l\rho : U\gamma\rho$, by inversion, we get that $\gamma\rho : \Gamma_f \rightsquigarrow \Gamma$. Since $\vdash \tau_f : s_f$, by inversion, we get that $\Gamma_f \vdash U : s_f$. So, by substitution, $\Gamma \vdash U\gamma\rho : s_f$. Now, if $r \in \mathscr{K}$, then, by (c), $U\gamma\rho = \Box$ but $\Box$ is not typable. Therefore, $r \notin \mathscr{K}$. □

Theorem 3.11 (Subject reduction for h – Barbanera *et al.* 1997). Assume that no right-hand side is in $\mathscr{B} \cup \mathscr{K} \cup \mathscr{W}$. Then, the restriction $\beta^{P\omega}$ of β to the redexes $[x:T]U\ t \in \mathscr{P}$ preserves typing. Therefore, h preserves typing on terms of type $\star$.

Proof. The proof is, as usual, by induction on $\Gamma \vdash t : T$ and by proving at the same time that, if $\Gamma \to_{\beta^{P\omega}} \Gamma'$, then $\Gamma' \vdash t : T$. The only difficult case is the case of a head-reduction

$[x:U']v\ u \to_{\beta^{P\omega}} v\{x \mapsto u\}$ with $\Gamma \vdash [x:U']v : (x:U)V$ and $\Gamma \vdash u : U$. We must prove that $\Gamma \vdash v\{x \mapsto u\} : V\{x \mapsto u\}$. By inversion, we have $\Gamma, x:U' \vdash v : V'$ with $(x:U')V'\ \mathbb{C}^*_\Gamma\ (x:U)V$. Since $v \in \mathscr{P}$, by (k), $V' \in \mathscr{K}$. Therefore, by Lemma 3.9 (h) and (i), $(x:U)V \in \mathscr{K}$, $U'\ \mathbb{C}^*_\Gamma\ U$ and $V'\ \mathbb{C}^*_{\Gamma,x:U}\ V$. Hence, by environment conversion and type conversion, $\Gamma, x:U \vdash v : V$ and, by substitution, $\Gamma \vdash v\{x \mapsto u\} : V\{x \mapsto u\}$.

Now, if $\Gamma \vdash t : \star$, then, by (l), $t = [x:U]v\vec{u}t \in \mathscr{P}$ and $v \in \mathscr{P}$. So, if $t \to_h t'$, then $t \to_{\beta^{P\omega}} t'$ and $\Gamma \vdash t' : \star$. □

Lemma 3.12 (Commutation). If $t \to_h^* u$ and $t \to_{\mathscr{R}}^* v$, there exists w such that $u \to_{\mathscr{R}}^* w$ and $v \to_h^* w$.

Proof. By induction on the number of h-steps, it suffices to prove that, if $[x:U]v\ u \to_h v\{x \mapsto u\}$ and $[x:U]v\ u \to_{\mathscr{R}}^* t$, there exists w such that $v\{x \mapsto u\} \to_{\mathscr{R}}^* w$ and $t \to_h w$. Since left-hand sides of rules are algebraic, t is of the form $[x:U']v'\ u'$ with $U \to_{\mathscr{R}}^* U'$, $v \to_{\mathscr{R}}^* v'$ and $u \to_{\mathscr{R}}^* u'$. So, it suffices to take $w = v'\{x \mapsto u'\}$. □

Lemma 3.13 (Postponement). Assume that no right-hand side is in $\mathscr{B} \cup \mathscr{K} \cup \mathscr{W}$ and that the right-hand side of every type-level rule is either a product or a predicate symbol application. If $\Gamma \vdash t : \star$ and $t \to_{\mathscr{R}}^* u \to_h^* v$, there exists w such that $t \to_h^* w \to_{\mathscr{R}}^* v$.

Proof. The proof is by induction on the number of h-steps. Assume that $t \to_{\mathscr{R}}^* u \to_h^* u' \to_h v$. By the induction hypothesis, there exists w' such that $t \to_h^* w' \to_{\mathscr{R}}^* u'$. By subject reduction on types, $\Gamma \vdash w' : \star$. So, by (l), w' is either of the form $(x:U)V$, $x\vec{t}$, $f\vec{t}$ with $f \in \mathscr{F}^{\square}$, or $[x:B]ab\vec{t}$. Since $w' \to_{\mathscr{R}}^* u' \to_h v$, we have that w' cannot be of the form $(x:U)V$ or $x\vec{t}$. Since right-hand sides of type-level rules are either a product or a predicate symbol application, w' cannot be of the form $f\vec{t}$. Therefore, $w' = [x:B]ab\vec{t}$, $u' = [x:B']a'b'\vec{t}'$ with $B, a, b, \vec{t} \to_{\mathscr{R}}^* B', a', b', \vec{t}'$, and $v = a'\{x \mapsto b'\}\vec{t}'$. Hence, by taking $w = a\{x \mapsto b\}\vec{t}$, we have $t \to_h w' \to_h w \to_{\mathscr{R}}^* v$. □

Theorem 3.14 (Subject reduction for β). If no right-hand side is in $\mathscr{B} \cup \mathscr{K} \cup \mathscr{W}$ and the right-hand side of every type-level rule is a symbol application, then β preserves typing.

Proof. The proof is, as usual, by induction on $\Gamma \vdash t : T$ and by proving that, if $\Gamma \to_\beta \Gamma'$, then $\Gamma' \vdash t : T$. The only difficult case is the case of a head-reduction $[x:U']v\ u \to_\beta v\{x \mapsto u\}$ with $\Gamma \vdash [x:U']v : (x:U)V$ and $\Gamma \vdash u : U$. We must prove that $\Gamma \vdash v\{x \mapsto u\} : V\{x \mapsto u\}$. We already know that it is true when v is a predicate. We must now prove it when v is an object, that is, when $\Gamma \vdash (x:U)V : \star$. By inversion, we have $\Gamma \vdash [x:U']v : (x:U')V'$ with $(x:U')V'\ \mathbb{C}^*_\Gamma\ (x:U)V$. By Lemma 3.9 (j), we have all the intermediate well-typed terms between $(x:U')V'$ and $(x:U)V$ of type $\star$. Without loss of generality, we can assume that $T_0 = (x:U')V' \downarrow_\beta T_1 \downarrow_{\mathscr{R}} T_2 \downarrow_\beta \ldots T_n = (x:U)V$. Let T_i' be the common reduct between T_i and T_{i+1}. We now prove by induction on the number of conversions that there is a sequence of well-typed product terms $\pi_1, \ldots, \pi_n$ such that $\pi_0 = T_0 \downarrow_\beta \pi_1 \downarrow_{\mathscr{R}} \pi_2 \downarrow_\beta \ldots \pi_n = T_n$.

Since T_0 is a product, $\pi_0' = T_0'$ is also a product. Since $T_1 \to_\beta^* \pi_0'$, by standardisation, there is a product term π_1 such that $T_1 \to_h^* \pi_1 \to_{h'}^* \pi_0'$. Since h has the subject reduction property on types, π_1 is well-typed. Now, since $T_1 \to_{\mathscr{R}}^* T_1'$, by commutation, there is a product term π_1' such that $\pi_1 \to_{\mathscr{R}}^* \pi_1'$ and $T_1' \to_h^* \pi_1'$. Furthermore, since $T_2 \to_{\mathscr{R}}^* T_1'$,

by postponement, there is a term t such that $T_2 \to_h^* t \to_{\mathscr{R}}^* \pi_1'$. Since h has the subject reduction property on types, t is a well-typed term of type $\star$. We now proceed by cases on t.

— If t is an abstraction $[x:T]w$, then, by inversion, there is W such that $(y:T)W \; \mathbb{C}_\Gamma^* \star$. By Lemma 3.9 (h) and (i), this is impossible.
— If t is an application but not a symbol application, then, since left-hand sides of rules are algebraic, π_1' is an application, which is also impossible.
— If t is a symbol application, then, since right-hand sides of type-level rules are symbol applications, π_1' is a symbol application too, which again is impossible.
— Therefore, t is a well-typed product term π_2.

Now, since $T_2 \to_\beta^* T_2'$ and β is confluent, there is a product term π_2' such that $\pi_2 \to_{h\!f}^* \pi_2'$ and $T_2' \to_\beta^* \pi_2'$, and we can now conclude by induction. □

4. Logical consistency

In the case of the pure Calculus of Constructions without symbols and rewrite rules, logical consistency follows easily from normalisation by proving that there can be no normal proof of $\bot = (\alpha:\star)\alpha$ in the empty environment (Barendregt 1992). But, having symbols and rewrite rules is like having hypothesis and axioms. Thus, in this case, logical consistency does not directly follow from normalisation. We can, however, give general conditions ensuring logical consistency.

Theorem 4.1 (Logical consistency). Assume that $\to$ is confluent and that every object symbol f satisfies one of the following conditions:

(1) $f:(\vec{x}:\vec{T})C\vec{v}$ with $C \in \mathscr{C}\mathscr{F}^{\Box}$,
(2) $f:(\vec{x}:\vec{T})T_i$,
(3) $f:(x_1:T_1)\dots(x_n:T_n)U$ with $x_n \notin \mathrm{FV}^{\Box}(U)$ and, for all normal substitutions $\gamma:(\vec{x}:\vec{T}) \rightsquigarrow (\alpha:\star)$, $f\vec{x}\gamma$ is reducible.

Then, there is no normal proof of $\bot = (\alpha:\star)\alpha$ in the empty environment. Therefore, if $\to$ is also normalising, there is no proof of $\bot$ in the empty environment.

Proof. Assume that $\vdash t:\bot$, t is normal and of minimal size, that is, there is no term u smaller than t such that $\vdash u:\bot$. For typing reasons, t cannot be a sort or a product. Assume that t is an application. Since t is typable in the empty environment, it cannot have free variables and, since t is normal, it must be of the form $f\vec{t}$. Assume that $|\vec{t}| = k$ and that f is of type $(\vec{x}:\vec{T})U$ with $|\vec{x}| = n$. Let $\gamma_i = \{x_1 \mapsto t_1, \dots, x_i \mapsto t_i\}$ $(i \leqslant n)$.

(1) In this case, $k \leqslant n$ since f cannot be applied to more than n arguments. Indeed, if f is applied to $n+1$ arguments, we have, by inversion, $\vdash ft_1\dots t_n : (x_{n+1}:T_{n+1})V$. But, since $\vdash ft_1\dots t_n : C\vec{v}\gamma_n$, by convertibility of types and confluence, we must have $(x_{n+1}:T_{n+1})V \downarrow C\vec{v}\gamma_n$, which is impossible. Thus, $k \leqslant n$ and $(x_{k+1}:T_{k+1}\gamma_k)\dots(x_n : T_n\gamma_k)C\vec{v}\gamma_k \downarrow \bot$, which is also impossible.

(2) There are 2 cases:
   — $k < n$. Since $\vdash f\vec{t} : (x_{k+1} : T_{k+1}\gamma_k)\ldots(x_n : T_n\gamma_k)T_i\gamma_k$, we must have $n = k+1$ and, by taking $x_n = \alpha$, $T_n\gamma_k \downarrow \star$ and $T_i\gamma_k \downarrow \alpha$. Hence $T_i\gamma_k \to^* \alpha$, but $T_i\gamma_k$ is closed since $\mathrm{FV}(T_i) \subseteq \{x_1, \ldots, x_{i-1}\}$, γ_k is closed and $i - 1 \leqslant k$. So, $T_i\gamma_k \to^* \alpha$ is impossible.
   — $k \geqslant n$. We have $\vec{t} = \vec{u}\vec{v}$ with $|\vec{u}| = n$. Let $p = k - n$. By inversion, there is a sequence of products $(y_1 : V_1)W_1, \ldots, (y_p : V_p)W_p$ such that $T_i\gamma_n = U\gamma_n \downarrow (y_1 : V_1)W_1$, for all $i < p$, $W_i\{y_i \mapsto v_i\} \downarrow (y_{i+1} : V_{i+1})W_{i+1}$, and $W_p\{y_p \mapsto v_p\} \downarrow \bot$. Then, $\vdash u_i\vec{v} : \bot$ and $u_i\vec{v}$ is smaller than t.

(3) If $k \geqslant n$, then t is reducible, which is impossible. If $k < n$, then $n = k+1$, $x_n = \alpha$ and $U\gamma_k \to^* \alpha$. But $\mathrm{FV}(U) \subseteq \{x_1, \ldots, x_k, \alpha\}$ and γ_k is closed. So, $x_n \in \mathrm{FV}^{\square}(U)$, which is excluded.

Assume now that $t = [\alpha : T]v$. Then, by inversion, we must have $\alpha : T \vdash v : V$ and $(\alpha : T)V \downarrow (\alpha : \star)\alpha$. Therefore, $T = \star$, $V = \alpha$ and $\alpha : \star \vdash v : \alpha$. For typing reasons, v cannot be a sort, a product or an abstraction. Since it is normal, it must be of the form $x\vec{u}$ with x a variable, or of the form $f\vec{t}$. Since α is the only variable that may freely occur in v, $x = \alpha$. Since α can be applied to no argument, $v = \alpha$. Then we get $\alpha : \star \vdash \alpha : \alpha$, which is impossible. Therefore, v is of the form $f\vec{t}$.

(1) In this case, $k \leqslant n$ since f cannot be applied to more than n arguments. Thus, $(x_{k+1} : T_{k+1}\gamma_k)\ldots(x_n : T_n\gamma_k)C\vec{v}\gamma_k \downarrow \alpha$, which is impossible.

(2) If $k < n$, then $(x_{k+1} : T_{k+1}\gamma_k)\ldots(x_n : T_n\gamma_k)T_i\gamma_k \downarrow \alpha$, which is impossible. Thus, $\vec{t} = \vec{u}\vec{v}$ with $|\vec{u}| = n$. Let $p = k - n$. By inversion, there is a sequence of products $(y_1 : V_1)$ $W_1, \ldots, (y_p : V_p)W_p$ such that $T_i\gamma_n = U\gamma_n \downarrow (y_1 : V_1)W_1$, for all $i < p$, $W_i\{y_i \mapsto v_i\} \downarrow$ $(y_{i+1} : V_{i+1})W_{i+1}$, and $W_p\{y_p \mapsto v_p\} \downarrow \alpha$. Then, $\vdash [\alpha : \star]u_i\vec{v} : \bot$ and $[\alpha : \star]u_i\vec{v}$ is smaller than t.

(3) In this case too, $k \geqslant n$. Thus, t is reducible, which is impossible. □

Note that, unlike the third condition, the first two conditions do not care about the rewrite rules defining f.

To see the interest of the third condition, consider the following example. Assume that the only symbols of the calculus are $nat : \star$, $0 : nat$, $s : nat \to nat$ and $rec : (P : nat \to \star)$ $P0 \to ((n : nat)Pn \to P(sn)) \to (n : nat)Pn$ defined by the usual rules for recursors:

$$rec\ P\ u\ v\ 0 \to u$$
$$rec\ P\ u\ v\ (s\ n) \to v\ n\ (rec\ P\ u\ v\ n).$$

This calculus is confluent since the combination of an orthogonal system (the recursor rules) with β-reduction preserves confluence. In this calculus, we can express any function whose existence is provable in intuitionistic higher-order arithmetic.

Now let us look at the normal terms of type nat in the environment $\alpha : \star$. Let $\mathcal{N}$ be the set of these terms. A term in $\mathcal{N}$ cannot be a sort, a product, an abstraction or a variable. It can only be of the form 0, $(s\ t)$ with t itself in $\mathcal{N}$, or of the form $(rec\ P\ u\ v\ t\ \vec{u})$ with $t \in \mathcal{N}$ also. But the last case is impossible since, at some point, the argument t of $(rec\ P\ u\ v\ t)$ must be of the form 0 or $(s\ t')$, and hence $(rec\ P\ u\ v\ t)$ must be reducible. Therefore, all the normal terms of type nat typable in $\alpha : \star$ must be of the form 0 or $(s\ t)$, and if t is such a term, then $(rec\ P\ u\ v\ t)$ is reducible. We also say that functions defined by

induction are *completely defined* (Guttag and Horning 1978; Thiel 1984; Kounalis 1985; Coquand 1992). Therefore, from the previous theorem, this calculus is consistent.

This may certainly be extended to the Calculus of Inductive Constructions and even to the Calculus of Inductive Constructions extended with functions defined by rewrite rules whenever all the symbols are completely defined.

5. Conditions for strong normalisation

We now present the conditions for strong normalisation.

5.1. *Inductive types and constructors*

Up to this point we have made few hypothesis on symbols and rewrite rules. However, Mendler (Mendler 1987) showed that the extension of the simply-typed λ-calculus with recursion on inductive types is strongly normalising if and only if the inductive types satisfy some positivity condition.

A base type T occurs positively in a type U if all the occurrences of T in U are on the left of an even number of $\Rightarrow$. A type T is positive if T occurs positively in the type of the arguments of its constructors. The usual inductive types, such as natural numbers and lists of natural numbers, are positive.

Now we will look at an example of a non-positive type T. Let U be a base type. Assume that T has a constructor c of type $(T \Rightarrow U) \Rightarrow T$. T is not positive because T occurs at a negative position in $T \Rightarrow U$. Now consider the function p of type $T \Rightarrow (T \Rightarrow U)$ defined by the rule $p(cx) \to x$. Let $\omega = \lambda x.(px)x$ of type $T \Rightarrow U$. Then the term $\omega(c\omega)$ of type U is not normalisable:

$$\omega(c\omega) \to_\beta p(c\omega)(c\omega) \to_{\mathscr{R}} \omega(c\omega) \to_\beta \ldots$$

In the case where $U = \star$, we can interpret this as Cantor's theorem: there is no surjection from a set T to the set of its subsets $T \Rightarrow \star$. In this interpretation, p is the natural injection between T and $T \Rightarrow \star$. Saying that p is surjective is equivalent to saying (with the Axiom of Choice) that there exists c such that $p \circ c$ is the identity, that is, such that $p(cx) \to x$. In Dowek (1999), Dowek shows that such a hypothesis is incoherent. Here, we show that this is related to the non-normalisation of non-positive inductive types.

Mendler also gives a condition, strong positivity, in the case of dependent and polymorphic types. A similar but more restrictive notion, called strict positivity, is used by Coquand and Paulin in the Calculus of Inductive Constructions (Coquand and Paulin-Mohring 1988).

In Definition 5.3, we introduce the more general notion of *admissible inductive structure*. In particular, we do not consider that a constructor must be constant: we can have rewrite rules on constructors. This allows us to formalise quotient types like the type *int* of integers by taking $0 : int$ for zero, $s : int \Rightarrow int$ for successor, and $p : int \Rightarrow int$ for predecessor, together with the rules:

$$s\ (p\ x) \to x$$
$$p\ (s\ x) \to x.$$

Definition 5.1 (Inductive structure). An *inductive structure* is given by:

— a precedence $\geqslant_{\mathscr{C}}$ on $\mathscr{C}\mathscr{F}^{\Box}$,
— for every $C : (\vec{x} : \vec{T})\star$ in $\mathscr{C}\mathscr{F}^{\Box}$, a set $\mathrm{Mon}(C) \subseteq \{i \leqslant |\vec{x}| \mid x_i \in \mathscr{X}^{\Box}\}$ for the *monotonic arguments* of C,
— for every $f : (\vec{y} : \vec{U})C\vec{v}$ with $C \in \mathscr{C}\mathscr{F}^{\Box}$, a set $\mathrm{Acc}(f) \subseteq \{1, \ldots, |\vec{y}|\}$ for the *accessible positions* of f.

For convenience, we assume that $\mathrm{Mon}(f) = \varnothing$ if $f \notin \mathscr{C}\mathscr{F}^{\Box}$, and $\mathrm{Acc}(f) = \varnothing$ if f is not of type $(\vec{y} : \vec{U})C\vec{v}$ with $C \in \mathscr{C}\mathscr{F}^{\Box}$.

The accessible positions of f denote the arguments of f that one can use in the right-hand sides of rules. The monotonic arguments of C denote the parameters in which C is monotonic.

Definition 5.2 (Positive and negative positions). The set of *positive positions* in t, $\mathrm{Pos}^{+}(t)$, and the set of *negative positions* in t, $\mathrm{Pos}^{-}(t)$, are simultaneously defined by induction on the structure of t:

— $\mathrm{Pos}^{\delta}(s) = \mathrm{Pos}^{\delta}(x) = \{\varepsilon \mid \delta = +\}$,
— $\mathrm{Pos}^{\delta}((x : U)V) = 1.\mathrm{Pos}^{-\delta}(U) \cup 2.\mathrm{Pos}^{\delta}(V)$,
— $\mathrm{Pos}^{\delta}([x : U]v) = 2.\mathrm{Pos}^{\delta}(v)$,
— $\mathrm{Pos}^{\delta}(tu) = 1.\mathrm{Pos}^{\delta}(t)$ if $t \neq f\vec{t}$,
— $\mathrm{Pos}^{\delta}(f\vec{t}) = \{1^{|\vec{t}|} \mid \delta = +\} \cup \bigcup\{1^{|\vec{t}|-i}2.\mathrm{Pos}^{\delta}(t_i) \mid i \in \mathrm{Mon}(f)\}$,

where $\delta \in \{-, +\}$, $-+ = -$ and $-- = +$ (usual rule of signs).

Definition 5.3 (Admissible inductive structures). An inductive structure is *admissible* if, for all $C \in \mathscr{C}\mathscr{F}^{\Box}$, for all $f : (\vec{y} : \vec{U})C\vec{v}$, and for all $j \in \mathrm{Acc}(f)$:[†]

(I3) $\forall D \in \mathscr{C}\mathscr{F}^{\Box}, D =_{\mathscr{C}} C \Rightarrow \mathrm{Pos}(D, U_j) \subseteq \mathrm{Pos}^{+}(U_j)$
(symbols equivalent to C must be at positive positions),

(I4) $\forall D \in \mathscr{C}\mathscr{F}^{\Box}, D >_{\mathscr{C}} C \Rightarrow \mathrm{Pos}(D, U_j) = \varnothing$
(no symbol greater than C can occur in U_j),

(I5) $\forall F \in \mathscr{D}\mathscr{F}^{\Box}, \mathrm{Pos}(F, U_j) = \varnothing$
(no defined symbol can occur in U_j),

(I6) $\forall Y \in \mathrm{FV}^{\Box}(U_j), \exists \iota_Y, v_{\iota_Y} = Y$
(predicate variables must be parameters of C),

(I2) $\forall Y \in \mathrm{FV}^{\Box}(U_j), \iota_Y \in \mathrm{Mon}(C) \Rightarrow \mathrm{Pos}(Y, U_j) \subseteq \mathrm{Pos}^{+}(U_j)$
(monotonic arguments must be at positive positions).

For instance, with $list : \star \Rightarrow \star$, $nil : (A : \star)listA$ and $cons : (A : \star)A \Rightarrow listA \Rightarrow listA$, we have $\mathrm{Mon}(list) = \{1\}$, $\mathrm{Acc}(nil) = \{1\}$ and $\mathrm{Acc}(cons) = \{1, 2, 3\}$ is an admissible inductive structure. If we add $tree : \star$ and $node : list\ tree \Rightarrow tree$ with $\mathrm{Mon}(list) = \{1\}$, $\mathrm{Mon}(tree) = \varnothing$ and $\mathrm{Acc}(node) = \{1\}$, we still have an admissible structure.

[†] In Blanqui (2001), we give 6 conditions, I1 to I6, for defining an admissible inductive structure. But we found that I1 can be eliminated if we modify I2 slightly, which is why, in the following definition, there is no I1 and I2 is placed after I6.

The condition I6 means that the predicate arguments of a constructor must be parameters of their type. A similar condition appears in the works of Stefanova (Stefanova 1998) ('safeness') and Walukiewicz (Walukiewicz-Chrząszcz 2002) ('$\star$-dependency'). On the other hand, in the Calculus of Inductive Constructions (CIC) (Werner 1994), there is no such restriction.

We can distinguish several kinds of inductive types.

Definition 5.4 (Primitive, basic and strictly-positive predicates). A constant predicate symbol C is:

— *primitive* if for all $D =_{\mathscr{C}} C$, for all $f : (\vec{y} : \vec{U})D\vec{w}$ and for all $j \in \mathrm{Acc}(f)$, we have $U_j = E\vec{t}$ with $E <_{\mathscr{C}} D$ and E primitive, or $U_j = E\vec{t}$ with $E =_{\mathscr{C}} D$.

— *basic* if for all $D =_{\mathscr{C}} C$, for all $f : (\vec{y} : \vec{U})D\vec{w}$ and for all $j \in \mathrm{Acc}(f)$, if $E =_{\mathscr{C}} D$ occurs in U_j, then U_j is of the form $E\vec{t}$.

— *strictly positive* if for all $D =_{\mathscr{C}} C$, for all $f : (\vec{y} : \vec{U})D\vec{w}$ and for all $j \in \mathrm{Acc}(f)$, if $E =_{\mathscr{C}} D$ occurs in U_j, then $U_j = (\vec{z} : \vec{V})E\vec{t}$ and no $D' =_{\mathscr{C}} D$ occurs in $\vec{V}$.

Primitive predicates are basic and basic predicates are strictly positive. Note that primitive predicates not only include the usual first-order data types. They also include some dependent type like the type of lists of fixed length. On the other hand, the type of polymorphic lists is basic but not primitive.

The strictly positive predicates are the predicates allowed in the Calculus of Inductive Constructions (CIC). For example, this includes the type *ord* of Brouwer's ordinals whose constructors are $0 : ord$, $s : ord \Rightarrow ord$ and $lim : (nat \Rightarrow ord) \Rightarrow ord$, the process algebra μCRL which can be formalised as a type *proc* with a choice operator $\Sigma : (data \Rightarrow proc) \Rightarrow proc$ (Sellink 1993), or the type *form* of the formulas of first-order predicate calculus whose constructors are $\neg : form \Rightarrow form$, $\vee : form \Rightarrow form \Rightarrow form$ and $\forall : (term \Rightarrow form) \Rightarrow form$.

For the moment, we do not forbid non-strictly positive predicates but the conditions we describe in the next section do not allow the definition of functions by recursion on such predicates. Yet, these predicates can be very useful, as shown in Matthes (2000) or Abel (2001). In Matthes (2000), a type *cont* with the constructors $D : cont$ and $C : ((cont \Rightarrow list) \Rightarrow list) \Rightarrow cont$, representing continuations, is used to define a breadth-first label listing function on labelled binary trees. In particular, it uses a function $ex : cont \Rightarrow list$ defined by the rules:

$$ex\ D \to nil$$
$$ex\ (C\ f) \to f\ ex.$$

It is not clear how to define a syntactic condition ensuring the strong normalisation of such a definition: in the right-hand side of the second rule, *ex* is explicitly applied to no argument smaller than f. And, although *ex* can only be applied to subterms of reducts of f, not every subterm of a 'computable' term (which is a notion used for proving strong normalisation) is *a priori* computable (see Section 5.2.2).

5.2. *General Schema*

5.2.1. *Higher-order rewriting* Which conditions on rewrite rules would ensure the strong normalisation of $\to = \to_{\mathscr{R}} \cup \to_{\beta}$? Since the work of Breazu-Tannen and Gallier

(Breazu-Tannen and Gallier 1989) and Okada (Okada 1989) on the simply-typed λ-calculus or the polymorphic λ-calculus, and later the work of Barbanera (Barbanera 1990) on the Calculus of Constructions and of Dougherty (Dougherty 1991) on the untyped λ-calculus, it is well known that adding first-order rewriting to typed λ-calculi preserves strong normalisation. This comes from the fact that first-order rewriting cannot create β-redexes. We will prove that this result can be extended to predicate-level rewriting if some conditions are fulfilled.

However, there are many useful functions whose definitions do not enter the first-order framework, either because some arguments are not primitive (the concatenation function *app* on polymorphic lists), or because their definition uses higher-order features like the function $map : (A : \star)(B : \star)(A \Rightarrow B) \Rightarrow listA \Rightarrow listB$, which applies a function to every element of a list:

$$\begin{aligned} map\ A\ B\ f\ (nil\ A') &\to nil\ B \\ map\ A\ B\ f\ (cons\ A'\ x\ \ell) &\to cons\ B\ (f\ x)\ (map\ A\ B\ f\ \ell) \\ map\ A\ B\ f\ (app\ A'\ \ell\ \ell') &\to app\ B\ (map\ A\ B\ f\ \ell)\ (map\ A\ B\ f\ \ell'). \end{aligned}$$

This is also the case of recursors like the recursor on natural numbers $natrec : (A : \star)$ $A \Rightarrow (nat \Rightarrow A \Rightarrow A) \Rightarrow nat \Rightarrow A$:

$$\begin{aligned} natrec\ A\ x\ f\ 0 &\to x \\ natrec\ A\ x\ f\ (s\ n) &\to f\ n\ (natrec\ A\ x\ f\ n), \end{aligned}$$

and of induction principles (recursors are just non-dependent versions of the corresponding induction principles), like the induction principle on natural numbers $natind : (P : nat \Rightarrow \star)$ $P0 \Rightarrow ((n : nat)Pn \Rightarrow P(sn)) \Rightarrow (n : nat)Pn$:

$$\begin{aligned} natind\ P\ h_0\ h_s\ 0 &\to h_0 \\ natind\ P\ h_0\ h_s\ (s\ n) &\to h_s\ n\ (natind\ P\ h_0\ h_s\ n). \end{aligned}$$

The methods used by Breazu-Tannen and Gallier (Breazu-Tannen and Gallier 1989) or Dougherty (Dougherty 1991) cannot be applied to our calculus since, on the one hand, higher-order rewriting can create β-redexes and, on the other hand, rewriting is included in the type conversion rule (conv), hence more terms are typable. But there are other methods, which are available just in the simply-typed λ-calculus or in richer type systems, for proving the termination of this kind of definition:

— The *General Schema*, initially introduced by Jouannaud and Okada (Jouannaud and Okada 1991) for the polymorphic λ-calculus and extended to the Calculus of Constructions by Barbanera, Fernández and Geuvers (Barbanera *et al.* 1994), is an extension of the primitive recursion schema: in the right-hand side of a rule $f\vec{l} \to r$, the recursive calls to f must be done on strict subterms of $\vec{l}$. It can treat object-level rewriting with simply-typed symbols defined on primitive types. It has been reformulated and extended to strictly-positive types by Jouannaud, Okada and the author for the simply-typed λ-calculus (Blanqui *et al.* 2002) and the Calculus of Constructions (Blanqui *et al.* 1999).

— The *Higher-Order Recursive Path Ordering* (HORPO) (Jouannaud and Rubio 1999) is an extension of RPO (Plaisted 1978; Dershowitz 1982) to the simply-typed λ-calculus. It has been recently extended by Walukiewicz (Walukiewicz 2000) to the

Calculus of Constructions with strictly positive types (Walukiewicz-Chrząszcz 2002). It can treat object-level rewriting with polymorphic and dependent symbols defined on strictly positive types. The General Schema can be seen as a non-recursive version of HORPO.

— It is also possible to look for an interpretation of the symbols such that the interpretation of a term strictly decreases when a rule is applied. This method, introduced by Van de Pol for the simply-typed λ-calculus (Van de Pol 1996), extends to the higher-order framework the method of interpretations known for the first-order framework (Zantema 1994). This is a very powerful method but difficult to use in practice because the interpretations are themselves higher-order and, also, because it is not modular: adding new rules or new symbols may require us to find new interpretations.

To deal with higher-order rewriting at the predicate-level together with polymorphic and dependent symbols and strictly-positive predicates, we have chosen to extend the method of the General Schema. For first-order symbols, we use other conditions, as in Jouannaud and Okada (1997).

5.2.2. *Definition of the schema* This method is based on Tait and Girard's method of reducibility candidates (Tait 1967; Girard *et al.* 1988) for proving the strong normalisation of simply-typed or polymorphic λ-calculi. This method consists of interpreting each type as a subset of the strongly normalisable terms, the *computable* terms, and proving that each well-typed term is computable. Indeed, a direct proof of strong normalisation by induction on the structure of terms does not go through because of the application case: when u and v are strongly normalisable, it is not clear how we can prove that uv is strongly normalisable too.

The idea of the General Schema is then to define, from the left-hand side $f\vec{l}$ of a rule, a set of terms (called the *computability closure* of $f\vec{l}$), whose elements are computable whenever the l_i's are. Then, to prove the strong normalisation, it suffices to check that, for each rule, the right-hand side belongs to the computability closure of the left-hand side.

To build the computability closure, we first define a subset of the subterms of $\vec{l}$, called the *accessible* subterms of $\vec{l}$, that are computable whenever the l_i's are (not all the subterms of a computable term are *a priori* computable). Then we build the computability closure by closing the set of accessible variables of the left-hand side with computability-preserving operations.

In order to have interesting functions, we must be able to accept recursive calls and, to preserve strong normalisation, recursive calls must decrease in a well-founded ordering. The strict subterm relation $\rhd$ (in fact, this is restricted to accessible subterms in order to preserve computability) is sufficient for dealing with definition on basic predicates. In the definition of *map* for instance, ℓ and ℓ' are accessible subterms of $app\ A'\ \ell\ \ell'$. But, for non-basic predicates, it is not sufficient as exemplified by the following addition on Brouwer's ordinals:

$$x + 0 \to x$$
$$x + (s\ y) \to s\ (x + y)$$
$$x + (lim\ f) \to lim\ ([n : nat]x + fn).$$

Another example is given by the following simplification rule in μCRL (Sellink 1993):

$$(\Sigma\ f) \cdot p \to \Sigma\ ([d : data] f d \cdot p).$$

This is why, in our conditions, we use two distinct orderings. The first, $>_1$, is used for the arguments of basic type and the second, $>_2$, is used for the arguments of strictly-positive type.

Finally, to allow a finer control of the comparison of the arguments, to each symbol, we associate a *status* describing how to compare the arguments by using a simple combination of lexicographic and multiset comparisons (Jouannaud and Okada 1997).

Definition 5.5 (Accessibility). We say that $u : U$ is *accessible modulo* ρ in $t : T$, written $t : T \ \rhd_\rho\ u : U$, if $t = f\vec{u}$, $f : (\vec{y} : \vec{U})C\vec{v}$, $C \in \mathscr{C}\mathscr{F}^{\Box}$, $u = u_j$, $j \in \mathrm{Acc}(f)$, $T\rho = C\vec{v}\gamma\rho$, $U\rho = U_j\gamma\rho$, $\gamma = \{\vec{y} \mapsto \vec{u}\}$ and no $D =_{\mathscr{C}} C$ occurs in $\vec{u}\rho$.

For technical reasons, we take into account not just the terms themselves but also their types. This comes from the fact that we are able to prove that two convertible types have the same interpretation only if the two types are computable. This may imply some restrictions on the types of the symbols.

Indeed, accessibility requires equality (modulo the application of ρ) between canonical types and derived types (see Definition 3.4). More precisely, to get $t : T \rhd_\rho u : U$, T must be equal (modulo ρ) to the canonical type of t, and U must be equal (modulo ρ) to the type of u derived from t. In addition, if $u : U \rhd_\rho v : V$, then U must also be equal (modulo ρ) to the canonical type of u.

Definition 5.6. Let $(x_i)_{i \geqslant 1}$ be an indexed family of variables.

Status A *status* is a term of the form $(lex\ m_1 \dots m_k)$ with $k \geqslant 1$ and each m_i of the form $(mul\ x_{k_1} \dots x_{k_p})$ with $p \geqslant 1$. The *arity* of a status *stat* is the greatest index i such that x_i occurs in *stat*.

Status assignment A *status assignment* is an application *stat* that associates a status $stat_f$ to every $f \in \mathscr{F}$.

Predicate arguments Let $C : (\vec{z} : \vec{V})\star$ and $\vec{u}$ with $|\vec{u}| = |\vec{z}|$. We use $\vec{u}|_C$ to denote the sub-sequence $u_{j_1} \dots u_{j_n}$ such that $j_1 < \dots < j_n$ and $\{j_1, \dots, j_n\} = \{j \leqslant |\vec{z}| \mid z_j \in \mathscr{X}^{\Box}\}$.

Strictly positive positions Let $f : (\vec{x} : \vec{T})U$ with $stat_f = lex\ \vec{m}$. The set of *strictly positive positions* of f, $SP(f)$, is defined as follows. Assume that $m_i = mul\ x_{k_1} \dots x_{k_p}$. Then $i \in SP(f)$ iff there exist $T_f^i = C\vec{a}$ such that C is strictly positive and, for all j, $T_{k_j} = C\vec{u}$ with $C \in \mathscr{C}\mathscr{F}^{\Box}$ and $\vec{u}|_C = \vec{a}|_C$.

Assignment compatibility A status assignment *stat* is *compatible* with a precedence $\geqslant_{\mathscr{F}}$ if $f =_{\mathscr{F}} g$ implies $stat_f = stat_g$, $SP(f) = SP(g)$ and, for all $i \in SP(f)$, $T_f^i = T_g^i$.

Status ordering Let $>$ be an ordering on terms and $stat = lex\ \vec{m}$ be a status of arity n. The *extension* of $>$ to the sequences of terms of length n is the ordering $>_{stat}$ defined as follows:

— $\vec{u} >_{stat} \vec{v}$ if $\vec{m}\{\vec{x} \mapsto \vec{u}\}\ (>^m)_{\mathrm{lex}}\ \vec{m}\{\vec{x} \mapsto \vec{v}\}$,

— $mul\ \vec{u} >^m mul\ \vec{v}$ if $\{\vec{u}\} >_{\mathrm{mul}} \{\vec{v}\}$.

For instance, if $stat = lex(mul\ x_2)(mul\ x_1\ x_3)$, then $(u_1,u_2,u_3) >_{stat} (v_1,v_2,v_3)$ iff $(\{u_2\},\{u_1,u_3\})\ (>_{\mathrm{mul}})_{\mathrm{lex}}\ (\{v_2\},\{v_1,v_3\})$. An important property of $>_{stat}$ is that it is well-founded whenever $>$ is.

We now define the computability closure of a rule $R = (l \to r, \Gamma, \rho)$ with $l = f\vec{l}$, $f : (\vec{x} : \vec{T})U$ and $\gamma = \{\vec{x} \mapsto \vec{l}\}$.

Definition 5.7 (Ordering on symbol arguments). The ordering $>_R$ on arguments of f is an adaptation of $>_{stat_f}$ where the ordering $>$ depends on the type (basic or strictly positive) of the argument. Assume that $stat_f = lex\ m_1 \dots m_k$. Then:

— $\vec{t} : \vec{T} >_R \vec{u} : \vec{U}$ if $\vec{m}\{\vec{x} \mapsto (\vec{t} : \vec{T})\}\ (>^1, \dots, >^k)_{\mathrm{lex}}\ \vec{m}\{\vec{x} \mapsto (\vec{u} : \vec{U})\}$.
— $mul(\vec{t} : \vec{T}) >^i mul(\vec{u} : \vec{U})$ if $i \in SP(f)$ and $\{\vec{t} : \vec{T}\}\ (>^i_R)_{\mathrm{mul}}\ \{\vec{u} : \vec{U}\}$,
— $mul(\vec{t} : \vec{T}) >^i mul(\vec{u} : \vec{U})$ if $i \notin SP(f)$ and $\{\vec{t} : \vec{T}\}\ (\rhd^+_\rho)_{\mathrm{mul}}\ \{\vec{u} : \vec{U}\}$,
— $t : T\ >^i_R\ u : U$ if:
  – $t = f\vec{t}$, $f : (\vec{x} : \vec{T})C\vec{v}$, $\gamma = \{\vec{x} \mapsto \vec{t}\}$ and no $D =_{\mathscr{C}} C$ occurs in $\vec{v}\gamma\rho$,
  – $u = x\vec{u}$ with $x \in \mathrm{dom}(\Gamma)$,
  – $t : T\ \rhd^+_\rho\ x : V$,
  – $V\rho = x\Gamma = (\vec{y} : \vec{U})C\vec{w}$, $\delta = \{\vec{y} \mapsto \vec{u}\}$, $U\rho = C\vec{w}\delta$ and no $D =_{\mathscr{C}} C$ occurs in $\vec{U}\delta$,
  – $\vec{v}\gamma\rho|_C = \vec{w}\delta|_C$.

One can easily check that, for the addition on ordinals, $lim\ f : ord >^1_R f\,n : ord$. Indeed, for this rule, one can take $\Gamma = x : ord, f : nat \Rightarrow ord$ and the identity for ρ. Then, $f \in \mathrm{dom}(\Gamma)$, $f\Gamma = nat \Rightarrow ord$ and $lim\ f : ord \rhd_\rho f : nat \Rightarrow ord$.

Definition 5.8 (Computability closure). Let $\mathscr{F}' = \mathscr{F} \cup \mathrm{dom}(\Gamma)$, $\mathscr{X}' = \mathscr{X} \setminus \mathrm{FV}(l)$, $\mathscr{T} = \mathscr{T}(\mathscr{F}', \mathscr{X}')$ and $\mathscr{E}' = \mathscr{E}(\mathscr{F}', \mathscr{X}')$. The *computability closure* of R with respect to a precedence $\geqslant_{\mathscr{F}}$ and a status assignment *stat* compatible with $\geqslant_{\mathscr{F}}$ is the smallest relation $\vdash_{\mathrm{c}} \subseteq \mathscr{E}' \times \mathscr{T}' \times \mathscr{T}'$ defined by the inference rules of Figure 4 where, for all $x \in \mathrm{dom}(\Gamma)$, we have $\tau_x = x\Gamma$ and $x <_{\mathscr{F}} f$, and where $\delta : \Gamma_g \rightsquigarrow_c \Delta$ means that, for all $y \in \mathrm{dom}(\Gamma_g)$, $\Delta \vdash_{\mathrm{c}} x\delta : x\Gamma_g\delta$.

Note that it is easy to extend the computability closure by adding new inference rules. Then, to preserve strong normalisation, it suffices to complete the proof of Theorem 6.37, where we prove that the rules of the computability closure do indeed preserve computability.

Definition 5.9 (Well-formed rule). R is *well-formed* if:

— $\Gamma \vdash l\rho : U\gamma\rho$,
— $\forall x \in \mathrm{dom}(\Gamma), \exists i, l_i : T_i\gamma\ \rhd^*_\rho\ x : x\Gamma$,
— $\mathrm{dom}(\rho) \subseteq \mathrm{FV}(l) \setminus \mathrm{dom}(\Gamma)$.

For instance, consider the rule:

$$app\ p\ (cons\ x\ n\ \ell)\ n'\ \ell' \to cons\ x\ (n + n')\ (app\ n\ \ell\ n'\ \ell')$$

with $\Gamma = x : nat, n : nat, \ell : listn, n' : nat, \ell' : listn'$ and $\rho = \{p \mapsto sn\}$. We have $\Gamma \vdash l\rho : list(p + n')\rho$. For x, we have $cons\ x\ n\ \ell : listp \rhd_\rho x : nat$. One can easily check that the conditions are also satisfied for the other variables.

$$\text{(ax)}\quad \frac{}{\vdash_{\mathrm{c}} \star : \Box}$$

$$\text{(symb}^{<}\text{)}\quad \frac{\vdash_{\mathrm{c}} \tau_g : s_g}{\vdash_{\mathrm{c}} g : \tau_g} \qquad (g <_{\mathscr{F}} f)$$

$$\text{(symb}^{=}\text{)}\quad \frac{\vdash_{\mathrm{c}} \tau_g : s_g \quad \delta : \Gamma_g \rightsquigarrow_c \Delta}{\Delta \vdash_{\mathrm{c}} g\vec{y}\delta : V\delta} \qquad \begin{array}{l}(g =_{\mathscr{F}} f,\ g : (\vec{y} : \vec{U})V,\\ \vec{y}\delta : \vec{U}\delta <_R \vec{x}\gamma : \vec{T}\gamma)\end{array}$$

$$\text{(var)}\quad \frac{\Delta \vdash_{\mathrm{c}} T : s_x}{\Delta, x : T \vdash_{\mathrm{c}} x : T} \qquad (x \notin \mathrm{dom}(\Delta))$$

$$\text{(weak)}\quad \frac{\Delta \vdash_{\mathrm{c}} t : T \quad \Delta \vdash_{\mathrm{c}} U : s_x}{\Delta, x : U \vdash_{\mathrm{c}} t : T} \qquad (x \notin \mathrm{dom}(\Delta))$$

$$\text{(prod)}\quad \frac{\Delta, x : U \vdash_{\mathrm{c}} V : s}{\Delta \vdash_{\mathrm{c}} (x : U)V : s}$$

$$\text{(abs)}\quad \frac{\Delta, x : U \vdash_{\mathrm{c}} v : V \quad \Delta \vdash_{\mathrm{c}} (x : U)V : s}{\Delta \vdash_{\mathrm{c}} [x : U]v : (x : U)V}$$

$$\text{(app)}\quad \frac{\Delta \vdash_{\mathrm{c}} t : (x : U)V \quad \Delta \vdash_{\mathrm{c}} u : U}{\Delta \vdash_{\mathrm{c}} tu : V\{x \mapsto u\}}$$

$$\text{(conv)}\quad \frac{\Delta \vdash_{\mathrm{c}} t : T \quad \Delta \vdash_{\mathrm{c}} T : s \quad \Delta \vdash_{\mathrm{c}} T' : s}{\Delta \vdash_{\mathrm{c}} t : T'} \qquad (T \downarrow T')$$

Fig. 4. Computability closure of $R = (f\vec{l} \to r, \Gamma, \rho)$ with $f : (\vec{x} : \vec{T})U$ and $\gamma = \{\vec{x} \mapsto \vec{l}\}$.

Definition 5.10 (Computable system). R satisfies the *General Schema* with respect to a precedence $\geqslant_{\mathscr{F}}$ and a status assignment *stat* compatible with $\geqslant_{\mathscr{F}}$ if it is well-formed and if $\vdash_{\mathrm{c}} r : U\gamma\rho$. A set of rules $\mathscr{R}$ is *computable* if there exists a precedence $\geqslant_{\mathscr{F}}$ and a status assignment *stat* compatible with $\geqslant_{\mathscr{F}}$ for which every rule of $\mathscr{R}$ satisfies the General Schema with respect to $\geqslant_{\mathscr{F}}$ and *stat*.

To summarise, the rule $(l \to r, \Gamma, \rho)$ is well-typed and satisfies the General Schema if:

— $\Gamma \vdash l\rho : U\gamma\rho$,
— $\forall \Delta, \sigma, T$, if $\Delta \vdash l\sigma : T$, then $\sigma : \Gamma \rightsquigarrow \Delta$ and $\sigma \downarrow \rho\sigma$,
— $\forall x \in \mathrm{dom}(\Gamma)$, $\exists i$, $l_i : T_i\gamma \ \rhd_\rho^* \ x : x\Gamma$,
— $\mathrm{dom}(\rho) \subseteq \mathrm{FV}(l) \setminus \mathrm{dom}(\Gamma)$,
— $\vdash_{\mathrm{c}} r : U\gamma\rho$.

Because of the (conv) rule, the relation $\vdash_{\mathrm{c}}$ may be undecidable. On the other hand, if we restrict the (conv) rule to a confluent and strongly normalising fragment of $\to$, then $\vdash_{\mathrm{c}}$ becomes decidable (with an algorithm similar to the one for $\vdash$). This is quite reasonable since, in practice, the symbols and the rules are often added one after the other (or by groups, but the argument can be generalised), thus confluence and strong normalisation can be shown incrementally.

For instance, let $(\mathscr{F},\mathscr{R})$ be a confluent and strongly normalising system, $f \notin \mathscr{F}$ and $\mathscr{R}_f$ be a set of rules defining f and whose symbols belong to $\mathscr{F}' = \mathscr{F} \cup \{f\}$. Then $(\mathscr{F}',\mathscr{R})$ is also confluent and strongly normalising. Thus, we can check that the rules of $\mathscr{R}_f$ satisfy the General Schema with the rule (conv) restricted to the case where $T \downarrow_{\beta\mathscr{R}} T'$. This does not seem a big restriction: it would be surprising if the typing of a rule requires the use of the rule itself!

We now give details of the case of

$$app\ p\ (cons\ x\ n\ \ell)\ n'\ \ell' \to cons\ x\ (n+n')\ (app\ n\ \ell\ n'\ \ell').$$

We take $stat_{app} = lex(mul\ x_2)$; $app >_{\mathscr{F}} cons, +$; $cons >_{\mathscr{F}} nat$ and $+ >_{\mathscr{F}} s, 0 >_{\mathscr{F}} nat$. We have already seen that this rule is well-formed. Let us show that $\vdash_{\mathrm{c}} r : list(sn)$.

To apply (symb$^<$), we must show that $\vdash_{\mathrm{c}} \tau_{cons} : \star$, $\vdash_{\mathrm{c}} x : nat$, $\vdash_{\mathrm{c}} n + n' : nat$ and $\vdash_{\mathrm{c}} app\ n\ \ell\ n'\ \ell' : list(n+n')$. The first assertions follow from the fact that the same judgements hold in $\vdash$ without using *app*. Hence, we are left to prove the last assertion.

To apply (symb$^=$), we must show that $\vdash_{\mathrm{c}} \tau_{app} : \star, \vdash_{\mathrm{c}} n : nat, \vdash_{\mathrm{c}} \ell : listn, \vdash_{\mathrm{c}} n' : nat, \vdash_{\mathrm{c}} \ell' : listn'$ and $cons\ x\ n\ \ell : list(sn) \rhd_\rho \ell : listn$. The first assertions follow from the fact that the same judgements hold in $\vdash$ without using *app*. The last assertion has already been shown when proving that the rule is well-formed.

5.3. *Strong normalisation conditions*

Definition 5.11. Let $\mathscr{G} \subseteq \mathscr{F}$. The *rewrite system* $(\mathscr{G},\mathscr{R}_\mathscr{G})$ is:

— *first-order* if every rule of $\mathscr{R}_\mathscr{G}$ has an algebraic right-hand side and, for all $g \in \mathscr{G}$, either $g \in \mathscr{F}^\square$ or $g : (\vec{x} : \vec{T})C\vec{v}$ with $C \in \mathscr{C}\mathscr{F}^\square$ primitive.
— *primitive* if all the rules of $\mathscr{R}_\mathscr{G}$ have a right-hand side of the form $[\vec{x} : \vec{T}]g\vec{u}$ with g a symbol of $\mathscr{G}$ or a primitive constant predicate symbol.
— *simple* if there is no critical pair between $\mathscr{R}_\mathscr{G}$ and $\mathscr{R}$.
— *small* if, for every rule $g\vec{l} \to r \in \mathscr{R}_\mathscr{G}$, $\forall x \in \mathrm{FV}^\square(r)$, $\exists \kappa_x$, $l_{\kappa_x} = x$.
— *positive* if, for every $g \in \mathscr{G}$, for every rule $l \to r \in \mathscr{R}_\mathscr{G}$, $\mathrm{Pos}(g,r) \subseteq \mathrm{Pos}^+(r)$.
— *safe* if for every rule $(g\vec{l} \to r, \Gamma, \rho) \in \mathscr{R}_\mathscr{G}$ with $g : (\vec{x} : \vec{T})U$ and $\gamma = \{\vec{x} \mapsto \vec{l}\}$:
  – $\forall x \in \mathrm{FV}^\square(\vec{T}U)$, $x\gamma\rho \in \mathrm{dom}^\square(\Gamma)$,
  – $\forall x, x' \in \mathrm{FV}^\square(\vec{T}U)$, $x\gamma\rho = x'\gamma\rho \Rightarrow x = x'$.[†]

Definition 5.12 (Strong normalisation conditions).

(A0) All the rules are well-typed.
(A1) The relation $\to = \to_\mathscr{R} \cup \to_\beta$ is confluent on $\mathscr{T}$.
(A2) There exists an admissible inductive structure.
(A3) There exists a precedence $\succeq$ on $\mathscr{D}\mathscr{F}^\square$ that is compatible with $\mathscr{R}_{\mathscr{D}\mathscr{F}^\square}$ and whose equivalence classes form a system that is:
  (p) primitive;
  (q) positive, small and simple; or
  (r) computable, small and simple.

[†] All this means that $\gamma\rho$ is an injection from $\mathrm{FV}^\square(\vec{T}U)$ to $\mathrm{dom}^\square(\Gamma)$.

(A4) There exists a partition $\mathscr{F}_1 \uplus \mathscr{F}_\omega$ of $\mathscr{DF}$ (*first-order* and *higher-order* symbols) such that:

- **(a)** $(\mathscr{F}_\omega, \mathscr{R}_\omega)$ is computable,
- **(b)** $(\mathscr{F}_\omega, \mathscr{R}_\omega)$ is safe,
- **(c)** no symbol of $\mathscr{F}_\omega$ occurs in the rules of $\mathscr{R}_1$,
- **(d)** $(\mathscr{F}_1, \mathscr{R}_1)$ is first-order,
- **(e)** if $\mathscr{R}_\omega \neq \varnothing$, then $(\mathscr{F}_1, \mathscr{R}_1)$ is non-duplicating,
- **(f)** $\to_{\mathscr{R}_1}$ is strongly normalising on first-order algebraic terms.

Condition A1 ensures, among other things, that β preserves typing. This condition may seem difficult to fulfill since confluence is often proved by using strong normalisation and local confluence of critical pairs (Nipkow 1991).

We know that $\to_\beta$ is confluent and that there is no critical pair between $\mathscr{R}$ and β since the left-hand sides of rules are algebraic. Müller (Müller 1992) showed that, in this case, if $\to_{\mathscr{R}}$ is confluent and all the rules of $\mathscr{R}$ are left-linear, then $\to_{\mathscr{R}} \cup \to_\beta$ is confluent. Thus, the possibility we have introduced of linearising some rules (substitution ρ) appears to be very useful (see Definition 3.1).

In the case of left-linear rules, and assuming that $\to_{\mathscr{R}_1}$ is strongly normalizing, as required in (f), how can we prove that $\to$ is confluent? In the case where $\to_{\mathscr{R}_1}$ is non-duplicating, if $\mathscr{R}_\omega \neq \varnothing$, we show in Theorem 6.34 that $\to_{\mathscr{R}_1} \cup \to_{\mathscr{R}_\omega}$ is strongly normalising. Therefore it suffices to check that the critical pairs of $\mathscr{R}$ are confluent (without using any β-reduction).

In A4, in the case where $\mathscr{R}_\omega \neq \varnothing$, we require that the rules of $\mathscr{R}_1$ are non-duplicating. Indeed, strong normalisation is not a modular property (Toyama 1987), even with confluent systems (Drosten 1989). On the other hand, strong normalisation is modular for disjoint and non-duplicating systems (Rusinowitch 1987). Here, $\mathscr{R}_1$ and $\mathscr{R}_\omega$ are not disjoint but hierarchically defined: by (c), no symbol of $\mathscr{F}_\omega$ occurs in the rules of $\mathscr{R}_1$. In Dershowitz (1994), Dershowitz gathers some results on the modularity of strong normalisation for first-order rewrite systems. It would be very interesting to study the modularity of strong normalisation in the case of higher-order rewriting and, in particular, conditions other than non-duplication, which, for example, does not allow us to accept the following definition:

$$\begin{gathered} 0/y \to 0 \\ (s\ x)/y \to s((x-y)/y) \\ \\ 0 - y \to 0 \\ (s\ x) - 0 \to s\ x \\ (s\ x) - (s\ y) \to x - y. \end{gathered}$$

This system is a duplicating first-order system not satisfying the General Schema: it cannot be put in either $\mathscr{R}_1$ or $\mathscr{R}_\omega$. Note that Giménez (Giménez 1998) has developed a termination criterion for the Calculus of Inductive Constructions that accepts this example.

In A3, the smallness condition for computable and positive systems is equivalent in the Calculus of Inductive Constructions to the restriction of strong elimination to 'small'

inductive types, that is, to the types whose constructors have no predicate parameters other than the ones of the type. For example, the type *list* of polymorphic list is small since, in the type $(A:\star)A \Rightarrow listA \Rightarrow listA$ of its constructor *cons*, A is a parameter of *list*. On the other hand, a type T having a constructor c of type $\star \Rightarrow T$ is not small. So, we cannot define a function f of type $T \Rightarrow \star$ with the rule $f(c\ A) \rightarrow A$. Such a rule is not small and does not form a primitive system either. In some sense, primitive systems can always be considered as small systems since they contain no projection and primitive predicate symbols have no predicate argument. This restriction is not only technical: elimination on big inductive types may lead to logical inconsistencies (Coquand 1986).

Finally, in A4, the safeness condition for higher-order symbols means that one cannot do matching or equality tests on predicate arguments that are necessary for typing other arguments. In her extension of HORPO (Jouannaud and Rubio 1999) to the Calculus of Constructions, Walukiewicz (Walukiewicz-Chrząszcz 2002) requires a similar condition. This has to be related to the fact that the polymorphism of CC is essentially parametric, that is, a polymorphic function uses the same algorithm at all types (Reynolds 1983). Girard had already demonstrated in Girard (1971) that normalisation fails if a non-parametric operator $J:(A:\star)(B:\star)A \Rightarrow B$ defined by $J\ A\ A\ x \rightarrow x$ is added to system F. See Harper and Mitchell (1999) for an analysis of Girard's J operator. On the other hand, the rule $map\ A\ A\ [x:A]x\ \ell \rightarrow \ell$, which does not seem problematic, does not satisfy the safeness condition either (note, however, that the left-hand side is not algebraic).

We can now state our main result, whose proof is the subject of Section 6.

Theorem 5.13 (Main result). If a CAC satisfies the conditions of Definition 5.12, its reduction relation $\rightarrow = \rightarrow_{\mathscr{R}} \cup \rightarrow_{\beta}$ preserves typing and is strongly normalising.

In Blanqui (2001), we prove that most of CIC can be encoded into a CAC satisfying our conditions, and that our conditions can also be applied to prove the cut-elimination property in Natural Deduction Modulo (Dowek and Werner 1998). But let us give a more concrete example:

$$\begin{array}{cc} \neg\top \rightarrow \bot & P \vee \top \rightarrow \top \quad P \wedge \top \rightarrow P \\ \neg\bot \rightarrow \top & P \vee \bot \rightarrow P \quad P \wedge \bot \rightarrow \bot \end{array}$$

$$\begin{array}{rr} x + 0 \rightarrow x & x \times 0 \rightarrow 0 \\ 0 + x \rightarrow x & 0 \times x \rightarrow 0 \\ x + (s\ y) \rightarrow s(x + y) & x \times (s\ y) \rightarrow (x \times y) + x \\ (s\ x) + y \rightarrow s(x + y) & (s\ 0) \times x \rightarrow x \\ (x + y) + z \rightarrow x + (y + z) & x \times (s\ 0) \rightarrow x \end{array}$$

$$\begin{array}{r} eq\ A\ 0\ 0 \rightarrow \top \\ eq\ A\ 0\ (s\ x) \rightarrow \bot \\ eq\ A\ (s\ x)\ 0 \rightarrow \bot \\ eq\ A\ (s\ x)\ (s\ y) \rightarrow eq\ nat\ x\ y \\ app\ A\ (nil\ A')\ \ell \rightarrow \ell \\ app\ A\ (cons\ A'\ x\ \ell)\ \ell' \rightarrow cons\ A\ x\ (app\ A\ \ell\ \ell') \\ app\ A\ (app\ A'\ \ell\ \ell')\ \ell'' \rightarrow app\ A\ \ell\ (app\ A\ \ell'\ \ell'') \end{array}$$

$$
\begin{aligned}
len\ A\ (nil\ A') &\to 0\\
len\ A\ (cons\ A'\ x\ \ell) &\to s\ (len\ A\ \ell)\\
len\ A\ (app\ A'\ \ell\ \ell') &\to (len\ A\ \ell) + (len\ A\ \ell')
\end{aligned}
$$

$$
\begin{aligned}
in\ A\ x\ (nil\ A') &\to \bot\\
in\ A\ x\ (cons\ A'\ y\ l) &\to (eq\ A\ x\ y) \vee (in\ A\ x\ l)
\end{aligned}
$$

$$
\begin{aligned}
incl\ A\ (nil\ A')\ l &\to \top\\
incl\ A\ (cons\ A'\ x\ l)\ l' &\to (in\ A\ x\ l') \wedge (incl\ A\ l\ l')
\end{aligned}
$$

$$
\begin{aligned}
sub\ A\ (nil\ A')\ l &\to \top\\
sub\ A\ (cons\ A'\ x\ l)\ (nil\ A'') &\to \bot\\
sub\ A\ (cons\ A'\ x\ l)\ (cons\ A''\ x'\ l') &\to ((eq\ A\ x\ x') \wedge (sub\ A\ l\ l'))\\
&\quad \vee (sub\ A\ (cons\ A\ x\ l)\ l')
\end{aligned}
$$

$$
\begin{aligned}
eq\ L\ (nil\ A)\ (nil\ A') &\to \top\\
eq\ L\ (nil\ A)\ (cons\ A'\ x\ l) &\to \bot\\
eq\ L\ (cons\ A'\ x\ l)\ (nil\ A) &\to \bot\\
eq\ L\ (cons\ A\ x\ l)\ (cons\ A'\ x'\ l') &\to (eq\ A\ x\ x') \wedge (eq\ (list\ A)\ l\ l')
\end{aligned}
$$

This rewriting system is computable, simple, small, safe and confluent (this can be automatically proved by CiME (Contejean *et al.* 2000)). Since the rules are left-linear, the combination with $\to_\beta$ is also confluent. Therefore, the conditions of strong normalisation are satisfied. For example, for the last rule, take $\Gamma = A:\star, x:A, x':A, \ell:listA, \ell':listA$ and $\rho = \{A' \mapsto A, L \mapsto listA\}$. The rule is well-formed $(cons(A',x',\ell'):L \ \rhd_\rho \ x':A', \ldots)$ and satisfies the General Schema $(\{cons(A,x,\ell):L, cons(A',x',\ell'):L\}\ (\rhd_\rho)_{\rm mul}\ \{x:A, x':A'\}$ and $\{\ell:listA, \ell':listA\})$.

However, the system lacks several important rules to get a complete decision procedure for classical propositional tautologies (Figure 1 in Section 1) or other simplification rules on the equality (Figure 2 in Section 1). To accept these rules, we must consider rewriting modulo associativity and commutativity and get rid of the simplicity conditions. Moreover, the distributivity rule $P \wedge (Q \oplus R) \to (P \wedge Q) \oplus (P \wedge R)$ is not small. Rewriting modulo AC does not seem to be a difficult extension, except perhaps in the case of predicate-level rewriting. On the other hand, confluence, simplicity and smallness seem difficult problems.

From strong normalisation, we can deduce the decidability of the typing relation, which is the essential property on which proof assistants like Coq (Coq Development Team 2002) or LEGO (Luo and Pollack 1992) are based.

Theorem 5.14 (Decidability of type-checking). Let Γ be a valid environment and T be $\Box$ or a term typable in Γ. In a CAC satisfying the conditions of Definition 5.12, checking whether a term t is of type T in Γ is decidable.

Proof. Since Γ is valid, it is possible to say whether t is typable and, if so, it is possible to infer a type T' for t. Since types are convertible, it suffices to check that T and T' have the same normal form. See Coquand (1991) and Barras (1999) for more details. $\Box$

6. Correctness of the conditions

Our strong normalisation proof is based on Tait and Girard's method of computability predicates and reducibility candidates (Girard *et al.* 1988). The idea is to interpret each type T as a set $[\![T]\!]$ of strongly normalisable terms and to prove that every term of type T belongs to $[\![T]\!]$. Those not familiar with these notions should read Chapter 3 of Werner's Ph.D. thesis (Werner 1994) for an introduction, and Gallier (1990) for a more detailed presentation.

It is worth noting several differences with previous strong normalisation proofs:

— The present proof is an important simplification of the proof given in Blanqui (2001), which uses candidates *à la* Coquand and Gallier (Coquand and Gallier 1990) where only well-typed terms are considered. Here, candidates are made of well-typed and not well-typed terms. This leads to simpler notation and fewer properties to take care of.
— In Geuvers (1994), Geuvers uses candidates with possibly not well-typed terms too. However, the way dependent types are interpreted does not allow this proof to be extended to type-level rewriting. Indeed, in this proof, dependencies are simply ignored but, if one has a predicate symbol $F : nat \Rightarrow \star$ defined by $F0 \to nat$ and $F(sn) \to nat \Rightarrow nat$, one expects $F0$ to be interpreted as nat, and $F(sn)$ as $nat \Rightarrow nat$.
— In Werner (1994), Werner uses candidates with (not well-typed) pure λ-terms, that is, terms without type annotation in abstractions, in order to deal with η-conversion, whose combination with β is not confluent on annotated terms. As a consequence, he has to define a translation from annotated terms to pure terms that implies the strong normalisation of annotated terms. Here, we give a direct proof.

6.1. *Reducibility candidates*

We use the following denotations:

$\mathscr{SN}$ for the set of strongly normalisable terms,
$\mathscr{WN}$ for the set of weakly normalisable terms,
$\mathscr{CR}$ for the set of terms from which reductions are confluent.

Definition 6.1 (Neutral terms). A term t is *neutral* if it is not of the following form:

— an abstraction: $[x : T]u$,
— a partial application: $f\vec{t}$ with $f \in \mathscr{DF}$ and $|\vec{t}| < |\vec{l}|$ for some rule $f\vec{l} \to r \in \mathscr{R}$,
— a constructor: $f\vec{t}$ with $f : (\vec{x} : \vec{T})C\vec{v}$ and $C \in \mathscr{CF}^{\square}$.

Let $\mathscr{N}$ be the set of neutral terms.

Note that if t is neutral, tu is neutral and not head-reducible.

Definition 6.2 (Reducibility candidates). We inductively define the set $\mathscr{R}_t$ of the interpretations for the terms of type t, the ordering $\leqslant_t$ on $\mathscr{R}_t$, the element $\top_t \in \mathscr{R}_t$, and the operation $\bigwedge_t$ from the powerset of $\mathscr{R}_t$ to $\mathscr{R}_t$ as follows:

— If $t \neq \square$ and $\Gamma \not\vdash t : \square$ then:
  – $\mathscr{R}_t = \{\varnothing\}$, $\leqslant_t \; = \subseteq$, $\top_t = \varnothing$ and $\bigwedge_t(\mathfrak{R}) = \top_t$.

— Otherwise:

– $\mathscr{R}_s$ is the set of all the subsets R of $\mathscr{T}$ such that:

(R1) $R \subseteq \mathscr{SN}$ (strong normalisation).

(R2) If $t \in R$, then $\to(t) \subseteq R$ (stability by reduction).

(R3) If $t \in \mathscr{N}$ and $\to(t) \subseteq R$, then $t \in R$ (neutral terms).

Furthermore, $\leqslant_s \;=\; \subseteq$, $\top_s = \mathscr{SN}$, $\bigwedge_s(\mathfrak{R}) = \bigcap \mathfrak{R}$ if $\mathfrak{R} \neq \varnothing$, and $\bigwedge_s(\varnothing) = \top_s$.

– $\mathscr{R}_{(x:U)K}$ is the set of functions R from $\mathscr{T} \times \mathscr{R}_U$ to $\mathscr{R}_K$ such that $R(u,S) = R(u',S)$ whenever $u \to u'$, $\top_{(x:U)K}(u,S) = \top_K$, $\bigwedge_{(x:U)K}(\mathfrak{R})(u,S) = \bigwedge_K(\{R(u,S) \mid R \in \mathfrak{R}\})$, and $R \leqslant_{(x:U)K} R'$ iff, for all (u,S), $R(u,S) \leqslant_K R'(u,S)$.

Lemma 6.3. $\mathscr{V} = \{x\vec{t} \in \mathscr{T} \mid x \in \mathscr{X}, \vec{t} \in \mathscr{SN}\} \neq \varnothing$ and, for all $R \in \mathscr{R}_s$, $\mathscr{V} \subseteq R$.

Proof. $\mathscr{V} \neq \varnothing$ since $\mathscr{X} \neq \varnothing$. Let $R \in \mathscr{R}_s$. We prove that $x\vec{t} \in R$ by induction on $\vec{t}$ with $\to_{\mathrm{lex}}$ as well-founded ordering ($\vec{t} \in \mathscr{SN}$). Since $x\vec{t} \in \mathscr{N}$, it suffices to prove that $\to(x\vec{t}) \subseteq R$, which is the induction hypothesis. □

Lemma 6.4.

(a) If $T\ \mathbb{C}^*_\Gamma\ T'$, then $\mathscr{R}_T = \mathscr{R}_{T'}$.
(b) If $\Gamma \vdash T : s$ and $\theta : \Gamma \rightsquigarrow \Delta$, then $\mathscr{R}_T = \mathscr{R}_{T\theta}$.

Proof.

(a) The proof is by induction on the size of T. If $\Gamma \vdash T : \star$, then $\Gamma \vdash T' : \star$ and $\mathscr{R}_T = \{\varnothing\} = \mathscr{R}_{T'}$. Assume now that $\Gamma \vdash T : \Box$. If $T = \star$, then $T' = \star$ and $\mathscr{R}_T = \mathscr{R}_{T'}$. If $T = (x:U)K$, then $T' = (x:U')K'$ with $U\ \mathbb{C}^*_\Gamma\ U'$ and $K\ \mathbb{C}^*_{\Gamma,x:U}\ K'$. By the induction hypothesis, $\mathscr{R}_U = \mathscr{R}_{U'}$ and $\mathscr{R}_K = \mathscr{R}_{K'}$. Therefore, $\mathscr{R}_T = \mathscr{R}_{T'}$.

(b) The proof is by induction on the size of T. If $\Gamma \vdash T : \star$, then $\Delta \vdash T\theta : \star$ and $\mathscr{R}_T = \{\varnothing\} = \mathscr{R}_{T\theta}$. Assume now that $\Gamma \vdash T : \Box$. If $T = \star$, this is immediate. If $T = (x:U)K$, then $T\theta = (x:U\theta)K\theta$. By the induction hypothesis, $\mathscr{R}_U = \mathscr{R}_{U\theta}$ and $\mathscr{R}_K = \mathscr{R}_{K\theta}$. Therefore, $\mathscr{R}_T = \mathscr{R}_{T\theta}$. □

Lemma 6.5 (Completeness of the candidates lattice). $(\mathscr{R}_t, \leqslant_t)$ is a complete lattice with greatest element $\top_t$ and the lower bound of $\mathfrak{R} \subseteq \mathscr{R}_t$ given by $\bigwedge_t(\mathfrak{R})$.

Proof. It suffices to prove that $(\mathscr{R}_t, \leqslant_t)$ is a complete inf-semi-lattice and that $\top_t$ is its greatest element. One can easily check by induction on t that $\leqslant_t$ is an ordering (that is, is reflexive, transitive and anti-symmetric), $\top_t$ is the greatest element of $\mathscr{R}_t$, and $\bigwedge_t(\mathfrak{R})$ is the lower bound of $\mathfrak{R} \subseteq \mathscr{R}_t$. □

Lemma 6.6 (Smallest element). Let $\bot_0 = \varnothing$ and $\bot_{i+1} = \bot_i \cup \{t \in \mathscr{N} \mid \to(t) \subseteq \bot_i\}$. The set $\bot_s = \bigcup\{\bot_i \mid i < \omega\}$ is the smallest element of $\mathscr{R}_s$: $\bot_s = \bigcap \mathscr{R}_s$.

Proof. Let $R \in \mathscr{R}_s$. We prove by induction on i that $\bot_i \subseteq R$. For $i = 0$, this is immediate. Assume that $\bot_i \subseteq R$ and let $t \in \bot_{i+1} \setminus \bot_i$. We have $t \in \mathscr{N}$ and $\to(t) \subseteq R$ by the induction hypothesis. Therefore, by R3, $t \in R$ and $\bot_s \subseteq R$ for all $R \in \mathscr{R}_s$. Thus, $\bot_s \subseteq \bigcap \mathscr{R}_s$.

We now prove that $\bot_s \in \mathscr{R}_s$, and hence that $\bot_s = \bigcap \mathscr{R}_s$.

(R1) We prove that $\bot_i \subseteq \mathscr{SN}$ by induction on i. For $i = 0$, this is immediate. Assume that $\bot_i \subseteq \mathscr{SN}$ and let $t \in \bot_{i+1} \setminus \bot_i$. We have $\to(t) \subseteq \mathscr{SN}$ by the induction hypothesis. Therefore, $t \in \mathscr{SN}$.

(R2) Let $t \in \bot_s$. Since $\bot_0 = \varnothing$, we have $t \in \bot_{i+1} \setminus \bot_i$ for some i. So, $\to(t) \subseteq \bot_i \subseteq \bot_s$.

(R3) Let $t \in \mathscr{N}$ with $\to(t) \subseteq \bot_s$. Since $\to$ is assumed to be finitely branching, $\to(t) = \{t_1, \ldots, t_n\}$. For all i, there exists k_i such that $t_i \in \bot_{k_i}$. Let k be the max of $\{k_1, \ldots, k_n\}$. We have $\to(t) \subseteq \bot_k$ and thus $t \in \bot_{k+1} \subseteq \bot_s$. □

6.2. *Interpretation schema*

The interpretation $[\![t]\!]$ of a term t is defined by using a *candidate assignment* ξ for the free variables and an interpretation I for the predicate symbols. The interpretation of constant predicate symbols will de defined in Section 6.3, and the interpretation of defined predicate symbols in Section 6.5.

Definition 6.7 (Interpretation schema). A *candidate assignment* is a function ξ from $\mathscr{X}$ to $\bigcup\{\mathscr{R}_t \mid t \in \mathscr{T}\}$. A candidate assignment ξ *validates* an environment Γ or is a Γ-*assignment*, written $\xi \models \Gamma$, if, for all $x \in \mathrm{dom}(\Gamma)$, we have $x\xi \in \mathscr{R}_{x\Gamma}$. An *interpretation* of a symbol f is an element of $\mathscr{R}_{\tau_f}$. An *interpretation* of a set $\mathscr{G}$ of symbols is a function that, to each symbol $g \in \mathscr{G}$, associates an interpretation of g.

The *interpretation* of t with respect to a candidate assignment ξ, an interpretation I and a substitution θ, is defined by induction on t as follows:

$$\begin{aligned}
[\![t]\!]^I_{\xi,\theta} &= \top_t \text{ if } t \text{ is an object or a sort},\\
[\![f]\!]^I_{\xi,\theta} &= I_f,\\
[\![x]\!]^I_{\xi,\theta} &= x\xi,\\
[\![(x:U)V]\!]^I_{\xi,\theta} &= \{t \in \mathscr{T} \mid \forall u \in [\![U]\!]^I_{\xi,\theta}, \forall S \in \mathscr{R}_U, tu \in [\![V]\!]^I_{\xi^S_x,\theta^u_x}\},\\
[\![[x:U]v]\!]^I_{\xi,\theta}(u,S) &= [\![v]\!]^I_{\xi^S_x,\theta^u_x},\\
[\![tu]\!]^I_{\xi,\theta} &= [\![t]\!]^I_{\xi,\theta}\big(u\theta, [\![u]\!]^I_{\xi,\theta}\big),
\end{aligned}$$

where $\theta^u_x = \theta \cup \{x \mapsto u\}$ and $\xi^S_x = \xi \cup \{x \mapsto S\}$. In the case where $\Gamma \vdash t : s$, the elements of $[\![t]\!]^I_{\xi,\theta}$ are called *computable*. A substitution θ is *adapted* to a Γ-assignment ξ if $\mathrm{dom}(\theta) \subseteq \mathrm{dom}(\Gamma)$ and, for all $x \in \mathrm{dom}(\theta)$, $x\theta \in [\![x\Gamma]\!]^I_{\xi,\theta}$. A pair (ξ, θ) is Γ-*valid*, written $\xi, \theta \models \Gamma$, if $\xi \models \Gamma$ and θ is adapted to ξ.

After Lemma 6.3, the identity substitution is adapted to any Γ-candidate assignment.

Lemma 6.8 (Correctness of the interpretation schema). If $\Gamma \vdash t : T$ and $\xi \models \Gamma$ then $[\![t]\!]^I_{\xi,\theta} \in \mathscr{R}_T$. Moreover, if $\theta \to \theta'$, then $[\![t]\!]^I_{\xi,\theta} = [\![t]\!]^I_{\xi,\theta'}$.

Proof. The proof is by induction on $\Gamma \vdash t : T$.

(ax) $[\![\star]\!]^I_{\xi,\theta} = \top_\star = \mathscr{SN} \in \mathscr{R}_\Box$ and $[\![\star]\!]^I_{\xi,\theta}$ does not depend on θ.

(symb) $[\![f]\!]^I_{\xi,\theta} = I_f \in \mathscr{R}_{\tau_f}$ by assumption on I and $[\![f]\!]^I_{\xi,\theta}$ does not depend on θ.

(var) $[\![x]\!]^I_{\xi,\theta}$ does not depend on θ. Now, if $x \in \mathscr{X}^\star$, then $[\![x]\!]^I_{\xi,\theta} = \varnothing \in \mathscr{R}_T = \{\varnothing\}$. Otherwise, $[\![x]\!]^I_{\xi,\theta} = x\xi \in \mathscr{R}_T$ since $\xi \models \Gamma, x : T$.

(weak) This case follows from the induction hypothesis.

(prod) $R = [\![(x:U)V]\!]^I_{\xi,\theta} = \{t \in \mathscr{T} \mid \forall u \in [\![U]\!]^I_{\xi,\theta}, \forall S \in \mathscr{R}_U, tu \in [\![V]\!]^I_{\xi^S_x,\theta^u_x}\} \in \mathscr{R}_s$ if it satisfies the properties (R1) to (R3):

(R1) *Strong normalisation.* Let $t \in R$. By the induction hypothesis, $[\![U]\!]^I_{\xi,\theta} \in \mathscr{R}_{s'}$ for some s', and $[\![V]\!]^I_{\xi^S_x,\theta^u_x} \in \mathscr{R}_s$. Therefore, we have $\mathscr{X} \subseteq [\![U]\!]^I_{\xi,\theta}$ and $[\![V]\!]^I_{\xi^S_x,\theta^u_x} \subseteq \mathscr{SN}$. Take $u = x \in \mathscr{X}$. Then $tx \in [\![V]\!]^I_{\xi^S_x,\theta}$ and $t \in \mathscr{SN}$.

(R2) *Stability by reduction.* Let $t \in R$ and $t' \in {\to}(t)$. Let $u \in [\![U]\!]^I_{\xi,\theta}$ and $S \in \mathscr{R}_U$. Then $tu \in [\![V]\!]^I_{\xi^S_x,\theta^u_x}$, which, by the induction hypothesis, is stable by reduction. Therefore, since $t'u \in {\to}(tu)$, we have $t'u \in [\![V]\!]^I_{\xi^S_x,\theta^u_x}$ and $t' \in R$.

(R3) *Neutral terms.* Let t be a neutral term such that ${\to}(t) \subseteq R$. Let $u \in [\![U]\!]^I_{\xi,\theta}$ and $S \in \mathscr{R}_U$. Since t is neutral, tu is neutral and, by the induction hypothesis, $tu \in R' = [\![V]\!]^I_{\xi^S_x,\theta^u_x}$ if ${\to}(tu) \subseteq R'$. We prove this by induction on u with $\to$ as well-founded ordering ($u \in \mathscr{SN}$ by the induction hypothesis). Since t is neutral, tu is not head-reducible and a reduct of tu is either of the form $t'u$ with $t' \in {\to}(t)$, or of the form tu' with $u' \in {\to}(u)$. In the former case, $t'u \in R'$ by assumption. In the latter case, we conclude by the induction hypothesis.

Assume now that $\theta \to \theta'$. Let $R' = [\![(x:U)V]\!]^I_{\xi,\theta'} = \{t \in \mathscr{T} \mid \forall u \in [\![U]\!]^I_{\xi,\theta'}, \forall S \in \mathscr{R}_U, tu \in [\![V]\!]^I_{\xi^S_x,\theta'^u_x}\}$. By the induction hypothesis, we have $[\![U]\!]^I_{\xi,\theta'} = [\![U]\!]^I_{\xi,\theta}$ and $[\![V]\!]^I_{\xi^S_x,\theta'^u_x} = [\![V]\!]^I_{\xi^S_x,\theta^u_x}$. Therefore, $R' = R$.

(abs) Let $R = [\![[x:U]v]\!]^I_{\xi,\theta}$. $R(u,S) = [\![v]\!]^I_{\xi^S_x,\theta^u_x}$. By the induction hypothesis, we have $R(u,S) \in \mathscr{R}_V$. Assume now that $u \to u'$. Then, $R(u',S) = [\![v]\!]^I_{\xi^S_x,\theta^{u'}_x}$. By the induction hypothesis, $[\![v]\!]^I_{\xi^S_x,\theta^{u'}_x} = [\![v]\!]^I_{\xi^S_x,\theta^u_x}$. Therefore, $R \in \mathscr{R}_{(x:U)V}$. Assume now that $\theta \to \theta'$. Let $R' = [\![[x:U]v]\!]^I_{\xi,\theta'}$. $R'(u,S) = [\![v]\!]^I_{\xi^S_x,\theta'^u_x}$. By the induction hypothesis, $R'(u,S) = R(u,S)$. Therefore, $R = R'$.

(app) Let $R = [\![tu]\!]^I_{\xi,\theta} = [\![t]\!]^I_{\xi,\theta}(u\theta, [\![u]\!]^I_{\xi,\theta})$. By the induction hypothesis, $[\![t]\!]^I_{\xi,\theta} \in \mathscr{R}_{(x:U)V}$ and $[\![u]\!]^I_{\xi,\theta} \in \mathscr{R}_U$. Therefore, $[\![tu]\!]^I_{\xi,\theta} \in \mathscr{R}_V = \mathscr{R}_{V\{x\to u\}}$ by Lemma 6.4. Assume now that $\theta \to \theta'$. Then $R' = [\![tu]\!]^I_{\xi,\theta'} = [\![t]\!]^I_{\xi,\theta'}(u\theta', [\![u]\!]^I_{\xi,\theta'})$. By the induction hypothesis, $[\![t]\!]^I_{\xi,\theta'} = [\![t]\!]^I_{\xi,\theta}$ and $[\![u]\!]^I_{\xi,\theta'} = [\![u]\!]^I_{\xi,\theta}$. Finally, since $[\![t]\!]^I_{\xi,\theta}$ is stable by reduction and $u\theta \to^* u\theta'$, we have $R = R'$.

(conv) This case follows from the induction hypothesis since, by Lemma 6.4, $\mathscr{R}_T = \mathscr{R}_{T'}$. □

Lemma 6.9. Let I and I' be two interpretations equal on the predicate symbols occurring in t, let ξ and ξ' be two candidate assignments equal on the predicate variables free in t, and let θ and θ' be two substitutions equal on the variables free in t. If $\Gamma \vdash t : T$ and $\xi \models \Gamma$ then $[\![t]\!]^{I'}_{\xi',\theta'} = [\![t]\!]^I_{\xi,\theta}$.

Proof. The proof is by induction on t. □

Lemma 6.10 (Candidate substitution). If $\Gamma \vdash t : T$, $\sigma : \Gamma \rightsquigarrow \Delta$ and $\xi \models \Delta$, then, for all θ, we have $[\![t\sigma]\!]^I_{\xi,\theta} = [\![t]\!]^I_{\xi',\sigma\theta}$ with $x\xi' = [\![x\sigma]\!]^I_{\xi,\theta}$ and $\xi' \models \Gamma$.

Proof. We first check that $\xi' \models \Gamma$. Let $x \in \mathrm{dom}(\Gamma)$. $x\xi' = [\![x\sigma]\!]_{\xi,\theta}$. By Lemma 6.8, $x\xi' \in \mathscr{R}_{x\Gamma\sigma}$ since $\Delta \vdash x\sigma : x\Gamma\sigma$ and $\xi \models \Delta$. By Lemma 6.4, $\mathscr{R}_{x\Gamma\sigma} = \mathscr{R}_{x\Gamma}$ since $\Gamma \vdash x\Gamma : s_x$

and $\sigma : \Gamma \rightsquigarrow \Delta$. We now prove the lemma by induction on t. If t is an object, then $t\sigma$ is an object too and $[\![t\sigma]\!]^I_{\xi,\theta} = \varnothing = [\![t]\!]^I_{\xi',\sigma\theta}$. If t is not an object, then $t\sigma$ is not an object either. We proceed by cases on t:

— $[\![s\sigma]\!]^I_{\xi,\theta} = \top_s = [\![s]\!]^I_{\xi',\sigma\theta}$.
— $[\![f\sigma]\!]^I_{\xi,\theta} = I_f = [\![f]\!]^I_{\xi',\sigma\theta}$.
— $[\![x\sigma]\!]^I_{\xi,\theta} = x\xi' = [\![x]\!]^I_{\xi',\sigma\theta}$.
— Let $R = [\![(x : U\sigma)V\sigma]\!]^I_{\xi,\theta} = \{t \in \mathscr{T} \mid \forall u \in [\![U\sigma]\!]^I_{\xi,\theta}, \forall S \in \mathscr{R}_{U\sigma} = \mathscr{R}_U, tu \in [\![V\sigma]\!]^I_{\xi^S_x,\theta^u_x}\}$ and $R' = [\![(x : U)V]\!]^I_{\xi',\sigma\theta} = \{t \in \mathscr{T} \mid \forall u \in [\![U]\!]^I_{\xi',\sigma\theta}, \forall S \in \mathscr{R}_U, tu \in [\![V]\!]^I_{\xi'^S_x,(\sigma\theta)^u_x}\}$. By the induction hypothesis, $[\![U\sigma]\!]^I_{\xi,\theta} = [\![U]\!]^I_{\xi',\sigma\theta}$ and $[\![V\sigma]\!]^I_{\xi^S_x,\theta^u_x} = [\![V]\!]^I_{\xi'',\sigma(\theta^u_x)}$ with $y\xi'' = [\![y\sigma]\!]_{\xi^S_x,\theta^u_x}$. Since $\sigma(\theta^u_x) = (\sigma\theta)^u_x$ ($x \notin \mathrm{dom}(\sigma) \cup \mathrm{dom}(\theta) \cup \mathrm{FV}(\sigma)$) and $\xi'' = \xi'^S_x$ ($x \notin \mathrm{dom}(\sigma) \cup \mathrm{FV}(\sigma)$), we have $R = R'$.
— Let $R = [\![[x : U\sigma]v\sigma]\!]^I_{\xi,\theta}$ and $R' = [\![[x : U]v]\!]^I_{\xi',\sigma\theta}$. By Lemma 6.4, R and R' have the same domain $\mathscr{T} \times \mathscr{R}_U$ and the same codomain $\mathscr{R}_V$. Moreover, $R(u, S) = [\![v\sigma]\!]^I_{\xi^S_x,\theta^u_x}$ and $R'(u, S) = [\![v]\!]^I_{\xi'^S_x,(\sigma\theta)^u_x}$. By the induction hypothesis, $R(u, S) = [\![v]\!]^I_{\xi'',\sigma(\theta^u_x)}$ with $y\xi'' = [\![y\sigma]\!]_{\xi^S_x,\theta^u_x}$. Since $\sigma(\theta^u_x) = (\sigma\theta)^u_x$ and $\xi'' = \xi'^S_x$, we have $R = R'$.
— Let $R = [\![t\sigma u\sigma]\!]^I_{\xi,\theta} = [\![t\sigma]\!]^I_{\xi,\theta}(u\sigma\theta, [\![u\sigma]\!]^I_{\xi,\theta})$ and $R' = [\![tu]\!]^I_{\xi',\sigma\theta} = [\![t]\!]^I_{\xi',\sigma\theta}(u\sigma\theta, [\![u]\!]^I_{\xi',\sigma\theta})$. By the induction hypothesis, $[\![t\sigma]\!]^I_{\xi,\theta} = [\![t]\!]^I_{\xi',\sigma\theta}$ and $[\![u\sigma]\!]^I_{\xi,\theta} = [\![u]\!]^I_{\xi',\sigma\theta}$. Therefore, $R = R'$. □

6.3. *Interpretation of constant predicate symbols*

As in Mendler (1987) or Werner (1994), we define the interpretation of constant predicate symbols as the fixpoint of some monotonic function on a complete lattice. The monotonicity is ensured by the positivity conditions of admissible inductive structures (Definition 5.3). The main difference with this previous work is that we have a more general notion of constructor since it includes any function symbol whose output type is a constant predicate symbol. This allows us to define functions and predicates by matching not only on constant constructors but also on defined symbols.

Definition 6.11 (Monotonic interpretation). Let I be an interpretation of $C : (\vec{x} : \vec{T})\star$, $\vec{a} = (\vec{t}, \vec{S})^\dagger$ and $\vec{a}' = (\vec{t}', \vec{S}')$ be arguments of I. Let $\vec{a} \leqslant_i \vec{a}'$ iff $\vec{t} = \vec{t}'$, $S_i \leqslant S'_i$ and, for all $j \neq i$, $S_j = S'_j$. We say that I is *monotonic* if, for all $i \in \mathrm{Mon}(C)$, $\vec{a} \leqslant_i \vec{a}' \Rightarrow I(\vec{a}) \leqslant I(\vec{a}')$.

We define the monotonic interpretation I of $\mathscr{CF}^\square$ by induction on $>_{\mathscr{C}}$ **A2**. Let $C \in \mathscr{CF}^\square$ and assume that we have already defined a monotonic interpretation K for every symbol smaller than C. Let $\mathscr{I}$ (respectively, $\mathscr{I}^m$) be the set of (respectively, monotonic) interpretations of $\{D \in \mathscr{CF}^\square \mid D =_{\mathscr{C}} C\}$, and $\leqslant$ be the relation on $\mathscr{I}$ defined by $I \leqslant I'$ iff, for all $D =_{\mathscr{C}} C$, $I_D \leqslant_{\tau_D} I'_D$. For simplicity, we denote $[\![t]\!]^{K \cup I}$ by $[\![t]\!]^I$.

Lemma 6.12. $(\mathscr{I}^m, \leqslant)$ is a complete lattice.

Proof. First, $\leqslant$ is an ordering since, for all $D =_{\mathscr{C}} C$, we have $\leqslant_{\tau_D}$ is an ordering.

† For simplicity, we write $(\vec{t}, \vec{S})$ instead of $(t_1, S_1), \ldots, (t_n, S_n)$.

The function $I^\top$ defined by $I^\top_D = \top_{\tau_D}$ is the greatest element of $\mathscr{I}$. We show that it belongs to $\mathscr{I}^m$. Let $D =_{\mathscr{C}} C$ with $D : (\vec{x} : \vec{T})U$, $i \in \mathrm{Mon}(D)$ and $\vec{a} \leqslant_i \vec{a}'$. Then, $I^\top_D(\vec{a}) = \top_U = I^\top_D(\vec{a}')$.

We now show that every part of $\mathscr{I}^m$ has an inf. Let $\mathfrak{I} \subseteq \mathscr{I}^m$ and $I^\wedge$ be the function defined by $I^\wedge_D = \bigwedge_{\tau_D}(\mathfrak{R}_D)$ where $\mathfrak{R}_D = \{I_D \mid I \in \mathfrak{I}\}$. We show that $I^\wedge \in \mathscr{I}^m$. Let $D =_{\mathscr{C}} C$ with $D : (\vec{x} : \vec{T})U$, $i \in \mathrm{Mon}(D)$ and $\vec{a} \leqslant_i \vec{a}'$. Then, $I^\wedge_D(\vec{a}) = \bigwedge_U\{I_D(\vec{a}) \mid I \in \mathfrak{I}\}$ and $I^\wedge_D(\vec{a}') = \bigwedge_U\{I_D(\vec{a}') \mid I \in \mathfrak{I}\}$. Since each I_D is monotonic, $I_D(\vec{a}) \leqslant_U I_D(\vec{a}')$. Therefore, $I^\wedge_D \leqslant_{\tau_D} I^\wedge_D$.

We are left to show that $I^\wedge$ is the inf of $\mathfrak{I}$. For all $I \in \mathfrak{I}$, $I^\wedge \leqslant I$ since, for all $D =_{\mathscr{C}} C$, $I^\wedge_D$ is the inf of $\mathfrak{R}_D$. Assume now that there exists $I' \in \mathscr{I}^m$ such that, for all $I \in \mathfrak{I}$, $I' \leqslant I$. Then $I' \leqslant I^\wedge$ since $I^\wedge_D$ is the inf of $\mathfrak{R}_D$. □

Definition 6.13 (Interpretation of constant predicate symbols). Let φ be the function that associates to $I \in \mathscr{I}^m$ the interpretation $\varphi^I \in \mathscr{I}^m$ such that $\varphi^I_D(\vec{t}, \vec{S})$ is the set of terms $u \in \mathscr{SN}$ such that if u reduces to $f\vec{u}$ with $f : (\vec{y} : \vec{U})D\vec{v}$ and $|\vec{u}| = |\vec{y}|$, we have, for all $j \in \mathrm{Acc}(f)$, $u_j \in [\![U_j]\!]^I_{\xi,\theta}$ with $\theta = \{\vec{y} \mapsto \vec{u}\}$ and $y\xi = S_{\iota_y}$. We show hereafter that φ is monotonic. Therefore, we can take $I = \mathrm{lfp}(\varphi)$, the least fixpoint of φ.

Since $\varphi^I_D(\vec{t}, \vec{S})$ does not depend on $\vec{t}$, we may sometimes write $I_D(\vec{S})$ instead of $I_D(\vec{t}, \vec{S})$. The aim of this definition is to ensure the correctness of the accessibility relations (Lemma 6.23): if $f\vec{u}$ is computable, then each accessible u_j is computable. This will allow us to ensure the computability of the variables of the left-hand side of a rule if the arguments of the left-hand side are computable, and thus the computability of the right-hand sides that belong to the computability closure.

Lemma 6.14. φ^I is a well-defined interpretation.

Proof. We first prove that φ^I is well-defined. The existence of ι_y is the hypothesis **I6**. The interpretations necessary for computing $[\![U_j]\!]_{\xi,\theta}$ are all well-defined. The interpretation of constant predicate symbols smaller than D is K. The interpretation of constant predicate symbols equivalent to D is I. By **I4** and **I5**, constant predicate symbols greater than D and defined predicate symbols do not occur in U_j. Finally, we must make sure that $\xi \models \Gamma$ where Γ is the environment made up of the declarations $y_i : U_i$ such that $y_i \in \mathrm{FV}^\square(U_j)$ for some j. Let $y \in \mathrm{dom}(\Gamma)$. We must prove that $y\xi \in \mathscr{R}_{y\Gamma}$. Assume that $D : (\vec{x} : \vec{T})U$. Then, $y\xi = S_{\iota_y} \in \mathscr{R}_{T_{\iota_y}}$. Let $\gamma = \{\vec{x} \mapsto \vec{v}\}$. Since $\gamma : \Gamma_D \rightsquigarrow \Gamma_f$, by Lemma 6.4, $\mathscr{R}_{T_{\iota_y}} = \mathscr{R}_{T_{\iota_y}\gamma}$. By **I6**, $v_{\iota_y} = y$. So, $\Gamma_f \vdash y : T_{\iota_y}\gamma$ and $T_{\iota_y}\gamma \ \mathbf{C}^*_{\Gamma_f}\ y\Gamma$. Therefore, by Lemma 6.4, $\mathscr{R}_{T_{\iota_y}\gamma} = \mathscr{R}_{y\Gamma}$ and $y\xi \in \mathscr{R}_{y\Gamma}$.

We now prove that $\varphi^I_D \in \mathscr{R}_{\tau_D}$. It is clearly stable by reduction since it does not depend on $\vec{t}$. Furthermore, $R = \varphi^I_D(\vec{t}, \vec{S})$ satisfies the properties R1 to R3:

- **(R1)** *Strong normalisation.* This follows by definition.
- **(R2)** *Stability by reduction.* Let $u \in R$ and $u' \in {\to}(u)$. Since $u \in \mathscr{SN}$, $u' \in \mathscr{SN}$. Assume, furthermore, that $u' \to^* f\vec{u}$ with $f : (\vec{y} : \vec{U})D\vec{v}$. Then $u \to^* f\vec{u}$. Therefore, for all $j \in \mathrm{Acc}(f)$, we have $u_j \in [\![U_j]\!]_{\xi,\theta}$ and $u' \in R$.
- **(R3)** *Neutral terms.* Let u be a neutral term such that ${\to}(u) \subseteq R$. Then $u \in \mathscr{SN}$. Assume now that $u \to^* f\vec{u}$ with $f : (\vec{y} : \vec{U})D\vec{v}$. Since u is neutral, $u \neq f\vec{u}$ and there exists

$u' \in \rightarrow(u)$ such that $u' \rightarrow^* f\vec{u}$. Therefore, for all $j \in \mathrm{Acc}(f)$, we have $u_j \in [\![U_j]\!]_{\xi,\theta}$ and $u \in R$. □

Lemma 6.15. Let $\leqslant^+ = \leqslant$, $\leqslant^- = \geqslant$ and $\xi \leqslant_x \xi'$ iff $x\xi \leqslant x\xi'$ and, for all $y \neq x$, $y\xi = y\xi'$. If I is monotonic, $\xi \leqslant_x \xi'$, $\mathrm{Pos}(x,t) \subseteq \mathrm{Pos}^\delta(t)$, $\Gamma \vdash t : T$ and $\xi, \xi' \models \Gamma$, then $[\![t]\!]^I_{\xi,\theta} \leqslant^\delta [\![t]\!]^I_{\xi',\theta}$.

Proof. The proof is by induction on t:

— $[\![s]\!]^I_{\xi,\theta} = \top_s = [\![s]\!]^I_{\xi',\theta}$.
— $[\![x]\!]^I_{\xi,\theta} = x\xi \leqslant x\xi' = [\![x]\!]^I_{\xi',\theta}$ and $\delta = +$ necessarily.
— $[\![y]\!]^I_{\xi,\theta} = y\xi = y\xi' = [\![y]\!]^I_{\xi',\theta}$ $(y \neq x)$.
— Let $R = [\![F\vec{t}]\!]^I_{\xi,\theta}$ and $R' = [\![F\vec{t}]\!]^I_{\xi',\theta}$. $R = I_F(\vec{a})$ with $a_i = (t_i\theta, [\![t_i]\!]^I_{\xi,\theta})$ and $R' = I_F(\vec{a}')$ with $a'_i = (t_i\theta, [\![t_i]\!]^I_{\xi',\theta})$. $\mathrm{Pos}^\delta(F\vec{t}) = \{1^{|\vec{t}|} \mid \delta = +\} \cup \bigcup\{1^{|\vec{t}|-i}2.\mathrm{Pos}^\delta(t_i) \mid i \in \mathrm{Mon}(F)\}$. If $i \in \mathrm{Mon}(F)$, then $\mathrm{Pos}(x,t_i) \subseteq \mathrm{Pos}^\delta(t_i)$ and, by the induction hypothesis, $[\![t_i]\!]^I_{\xi,\theta} \leqslant^\delta [\![t_i]\!]^I_{\xi',\theta}$. Otherwise, $\mathrm{Pos}(x,t_i) = \varnothing$ and $[\![t_i]\!]^I_{\xi,\theta} = [\![t_i]\!]^I_{\xi',\theta}$. Therefore, in both cases, $R \leqslant^\delta R'$ since I_F is monotonic.
— Let $R = [\![(x:U)V]\!]^I_{\xi,\theta}$ and $R' = [\![(x:U)V]\!]^I_{\xi',\theta}$. $R = \{t \in \mathscr{T} \mid \forall u \in [\![U]\!]^I_{\xi,\theta}, \forall S \in \mathscr{R}_U, tu \in [\![V]\!]^I_{\xi_x^S,\theta_x^u}\}$. $R' = \{t \in \mathscr{T} \mid \forall u \in [\![U]\!]^I_{\xi',\theta}, \forall S \in \mathscr{R}_U, tu \in [\![V]\!]^I_{\xi'^S_x,\theta_x^u}\}$. Since $\mathrm{Pos}^\delta((x:U)V) = 1.\mathrm{Pos}^{-\delta}(U) \cup 2.\mathrm{Pos}^\delta(V)$, $\mathrm{Pos}(x,U) \subseteq \mathrm{Pos}^{-\delta}(U)$ and $\mathrm{Pos}(x,V) \subseteq \mathrm{Pos}^\delta(V)$. Therefore, by the induction hypothesis, $[\![U]\!]^I_{\xi,\theta} \leqslant^{-\delta} [\![U]\!]^I_{\xi',\theta}$ and $[\![V]\!]^I_{\xi_x^S,\theta_x^u} \leqslant^\delta [\![V]\!]^I_{\xi'^S_x,\theta_x^u}$. So $R \leqslant^\delta R'$. Indeed, if $\delta = +$, $t \in R$ and $u \in [\![U]\!]^I_{\xi',\theta} \subseteq [\![U]\!]^I_{\xi,\theta}$, then $tu \in [\![V]\!]^I_{\xi_x^S,\theta_x^u} \subseteq [\![V]\!]^I_{\xi'^S_x,\theta_x^u}$ and $t \in R'$. If $\delta = -$, $t \in R'$ and $u \in [\![U]\!]^I_{\xi,\theta} \subseteq [\![U]\!]^I_{\xi',\theta}$, then $tu \in [\![V]\!]^I_{\xi'^S_x,\theta_x^u} \subseteq [\![V]\!]^I_{\xi_x^S,\theta_x^u}$ and $t \in R$.
— Let $R = [\![[x:U]v]\!]^I_{\xi,\theta}$ and $R' = [\![[x:U]v]\!]^I_{\xi',\theta}$. R and R' have the same domain $\mathscr{T} \times \mathscr{R}_U$ and the same codomain $\mathscr{R}_V$. $R(u,S) = [\![v]\!]^I_{\xi_x^S,\theta_x^u}$ and $R'(u,S) = [\![v]\!]^I_{\xi'^S_x,\theta_x^u}$. Since $\mathrm{Pos}^\delta([x:U]v) = 2.\mathrm{Pos}^\delta(v)$, $\mathrm{Pos}(x,v) \subseteq \mathrm{Pos}^\delta(v)$. Therefore, by the induction hypothesis, $R(u,S) \leqslant^\delta R'(u,S)$ and $R \leqslant^\delta R'$.
— Let $R = [\![tu]\!]^I_{\xi,\theta}$ and $R' = [\![tu]\!]^I_{\xi',\theta}$ $(t \neq f\vec{t})$. $R = [\![t]\!]^I_{\xi,\theta}(u\theta, S)$ with $S = [\![u]\!]^I_{\xi,\theta}$. $R' = [\![t]\!]^I_{\xi',\theta}(u\theta, S')$ with $S' = [\![u]\!]^I_{\xi',\theta}$. Since $\mathrm{Pos}^\delta(tu) = 1.\mathrm{Pos}^\delta(t)$, $\mathrm{Pos}(x,t) \subseteq \mathrm{Pos}^\delta(t)$ and $\mathrm{Pos}(x,u) = \varnothing$. Therefore, $S = S'$ and, by the induction hypothesis, $[\![t]\!]^I_{\xi,\theta} \leqslant^\delta [\![t]\!]^I_{\xi',\theta}$. So $R \leqslant^\delta R'$. □

Lemma 6.16. φ^I is monotonic.

Proof. Let $D =_{\mathscr{C}} C$ with $D : (\vec{x} : \vec{T})U$, $i \in \mathrm{Mon}(D)$ and $\vec{a} \leqslant_i \vec{a}'$ with $\vec{a} = (\vec{t}, \vec{S})$ and $\vec{a}' = (\vec{t}, \vec{S}')$. We have to show that $\varphi^I_D(\vec{a}) \subseteq \varphi^I_D(\vec{a}')$. Let $u \in \varphi^I_D(\vec{a})$. We prove that $u \in \varphi^I_D(\vec{a}')$. First we have $u \in \mathscr{SN}$. Assume now that u reduces to $f\vec{u}$ with $f : (\vec{y} : \vec{U})D\vec{v}$. Let $j \in \mathrm{Acc}(f)$. We have to prove that $u_j \in [\![U_j]\!]_{\xi',\theta}$ with $\theta = \{\vec{y} \mapsto \vec{u}\}$ and, for all $y \in \mathrm{FV}^\square(U_j)$, $y\xi' = S'_{\iota_y}$. Since $u \in \varphi^I_D(\vec{a})$, we have $u_j \in [\![U_j]\!]_{\xi,\theta}$ with, for all $y \in \mathrm{FV}^\square(U_j)$, $y\xi = S_{\iota_y}$. If, for all $y \in \mathrm{FV}^\square(U_j)$, $\iota_y \neq i$, then ξ and ξ' are equal on $\mathrm{FV}^\square(U_j)$. Therefore, $[\![U_j]\!]_{\xi,\theta} = [\![U_j]\!]_{\xi',\theta}$ and $u_j \in [\![U_j]\!]_{\xi',\theta}$. If there exists $y \in \mathrm{FV}^\square(U_j)$ such that $\iota_y = i$, then $\xi \leqslant_y \xi'$. By **I2**, $\mathrm{Pos}(y, U_j) \subseteq \mathrm{Pos}^+(U_j)$. Therefore, by Lemma 6.15, $\varphi^I_D(\vec{a}) \subseteq \varphi^I_D(\vec{a}')$ and $u_j \in [\![U_j]\!]_{\xi',\theta}$. □

Lemma 6.17. Let $I \leqslant_F I'$ iff $I_F \leqslant I'_F$ and, for all $G \neq F$, $I_G = I'_G$. If I is monotonic, $I \leqslant_F I'$, $\mathrm{Pos}(F,t) \subseteq \mathrm{Pos}^\delta(t)$, $\Gamma \vdash t : T$ and $\xi \models \Gamma$, then $[\![t]\!]^I_{\xi,\theta} \leqslant^\delta [\![t]\!]^{I'}_{\xi,\theta}$.

Proof. The proof is by induction on t:

— $[\![s]\!]^I_{\xi,\theta} = \top_s = [\![s]\!]^{I'}_{\xi,\theta}$.
— $[\![x]\!]^I_{\xi,\theta} = x\xi = [\![x]\!]^{I'}_{\xi,\theta}$.
— Let $R = [\![G\vec{t}]\!]^I_{\xi,\theta}$ and $R' = [\![G\vec{t}]\!]^{I'}_{\xi,\theta}$. $R = I_G(\vec{a})$ with $a_i = (t_i\theta, [\![t_i]\!]^I_{\xi,\theta})$. $R' = I'_G(\vec{a}')$ with $a'_i = (t_i\theta, [\![t_i]\!]^{I'}_{\xi,\theta})$. $\mathrm{Pos}^\delta(G\vec{t}) = \{1^{|\vec{t}|} \mid \delta = +\} \cup \bigcup\{1^{|\vec{t}|-i}2.\mathrm{Pos}^\delta(t_i) \mid i \in \mathrm{Mon}(G)\}$. If $i \in \mathrm{Mon}(G)$, then $\mathrm{Pos}(F, t_i) \subseteq \mathrm{Pos}^\delta(t_i)$ and, by the induction hypothesis, $[\![t_i]\!]^I_{\xi,\theta} \leqslant^\delta [\![t_i]\!]^{I'}_{\xi,\theta}$. Otherwise, $\mathrm{Pos}(F, t_i) = \varnothing$ and $[\![t_i]\!]^I_{\xi,\theta} = [\![t_i]\!]^{I'}_{\xi,\theta}$. Therefore, $I_G(\vec{a}) \leqslant^\delta I_G(\vec{a}')$ since I_G is monotonic. Now, if $G = F$, then $\delta = +$ and $I_G(\vec{a}) \leqslant I_G(\vec{a}') = I_F(\vec{a}') \leqslant I'_F(\vec{a}') = I'_G(\vec{a}')$. Otherwise, $I_G(\vec{a}) \leqslant^\delta I_G(\vec{a}') = I'_G(\vec{a}')$.
— Let $R = [\![(x : U)V]\!]^I_{\xi,\theta}$ and $R' = [\![(x : U)V]\!]^{I'}_{\xi,\theta}$. $R = \{t \in \mathscr{T} \mid \forall u \in [\![U]\!]^I_{\xi,\theta}, \forall S \in \mathscr{R}_U, tu \in [\![V]\!]^I_{\xi_x^S,\theta_x^u}\}$ and $R' = \{t \in \mathscr{T} \mid \forall u \in [\![U]\!]^{I'}_{\xi,\theta}, \forall S \in \mathscr{R}_U, tu \in [\![V]\!]^{I'}_{\xi_x^S,\theta_x^u}\}$. Since $\mathrm{Pos}^\delta((x : U)V) = 1.\mathrm{Pos}^{-\delta}(U) \cup 2.\mathrm{Pos}^\delta(V)$, we have $\mathrm{Pos}(F, U) \subseteq \mathrm{Pos}^{-\delta}(U)$ and $\mathrm{Pos}(F, V) \subseteq \mathrm{Pos}^\delta(V)$. Therefore, by the induction hypothesis, $[\![U]\!]^I_{\xi,\theta} \leqslant^{-\delta} [\![U]\!]^{I'}_{\xi,\theta}$ and $[\![V]\!]^I_{\xi_x^S,\theta_x^u} \leqslant^\delta [\![V]\!]^{I'}_{\xi_x^S,\theta_x^u}$. So $[\![t]\!]^I_{\xi,\theta} \leqslant^\delta [\![t]\!]^{I'}_{\xi,\theta}$. Indeed, if $\delta = +$, $t \in R$ and $u \in [\![U]\!]^{I'}_{\xi,\theta} \subseteq [\![U]\!]^I_{\xi,\theta}$, we have $tu \in [\![V]\!]^I_{\xi_x^S,\theta_x^u} \subseteq [\![V]\!]^{I'}_{\xi_x^S,\theta_x^u}$ and $t \in R'$. If $\delta = -$, $t \in R'$ and $u \in [\![U]\!]^I_{\xi,\theta} \subseteq [\![U]\!]^{I'}_{\xi,\theta}$, we have $tu \in [\![V]\!]^{I'}_{\xi_x^S,\theta_x^u} \subseteq [\![V]\!]^I_{\xi_x^S,\theta_x^u}$ and $t \in R$.
— Let $R = [\![[x : U]v]\!]^I_{\xi,\theta}$ and $R' = [\![[x : U]v]\!]^{I'}_{\xi,\theta}$. R and R' have the same domain $\mathscr{T} \times \mathscr{R}_U$ and same codomain $\mathscr{R}_V$. $R(u, S) = [\![v]\!]^I_{\xi_x^S,\theta_x^u}$ and $R'(u, S) = [\![v]\!]^{I'}_{\xi_x^S,\theta_x^u}$. Since $\mathrm{Pos}^\delta([x : U]v) = 2.\mathrm{Pos}^\delta(v)$, we have $\mathrm{Pos}(F, v) \subseteq \mathrm{Pos}^\delta(v)$. Therefore, by the induction hypothesis, $R(u, S) \leqslant^\delta R'(u, S)$ and $R \leqslant^\delta R'$.
— Let $R = [\![tu]\!]^I_{\xi,\theta}$ and $R' = [\![tu]\!]^{I'}_{\xi,\theta}$ $(t \neq f\vec{t})$. $R = [\![t]\!]^I_{\xi,\theta}(u\theta, S)$ with $S = [\![u]\!]^I_{\xi,\theta}$. $R' = [\![t]\!]^{I'}_{\xi,\theta}(u\theta, S')$ with $S' = [\![u]\!]^{I'}_{\xi,\theta}$. Since $\mathrm{Pos}^\delta(tu) = 1.\mathrm{Pos}^\delta(t)$, we have $\mathrm{Pos}(F, t) \subseteq \mathrm{Pos}^\delta(t)$ and $\mathrm{Pos}(F, u) = \varnothing$. Therefore, $S = S'$ and, by the induction hypothesis, $[\![t]\!]^I_{\xi,\theta} \leqslant^\delta [\![t]\!]^{I'}_{\xi,\theta}$. So $R \leqslant^\delta R'$. □

Lemma 6.18. φ is monotonic.

Proof. Let $I, I' \in \mathscr{I}^m$ such that $I \leqslant I'$. We have to prove that, for all $D =_{\mathscr{C}} C$, $\varphi^I_D \leqslant \varphi^{I'}_D$, that is, $\varphi^I_D(\vec{a}) \subseteq \varphi^{I'}_D(\vec{a})$ for all $\vec{a}$. Let $u \in \varphi^I_D(\vec{a})$. We prove that $u \in \varphi^{I'}_D(\vec{a})$. First we have $u \in \mathscr{SN}$. Assume now that u reduces to $f\vec{u}$ with $f : (\vec{y} : \vec{U})D\vec{v}$. Let $j \in \mathrm{Acc}(f)$. We have to prove that $u_j \in [\![U_j]\!]^{I'}_{\xi,\theta}$ with $\theta = \{\vec{y} \mapsto \vec{u}\}$ and, for all $y \in \mathrm{FV}^\square(U_j)$, $y\xi = S_{\iota_y}$. Since $u \in \varphi^I_D(\vec{a})$, we have $u_j \in [\![U_j]\!]^I_{\xi,\theta}$. Since $j \in \mathrm{Acc}(f)$, by **I3** , for all $E =_{\mathscr{C}} D$, we have $\mathrm{Pos}(E, U_j) \subseteq \mathrm{Pos}^+(U_j)$. Now, only a finite number of symbols $E =_{\mathscr{C}} D$ can occur in U_j, say $E_0, \ldots, E_{n-1}$. Let $I^0 = I$ and, for all $i < n$, $I^{i+1}_D = I^i_D$ if $D \neq E_i$, and $I^{i+1}_D = I'_{E_i}$ otherwise. We have $I = I^0 \leqslant_{E_0} I^1 \leqslant_{E_1} \ldots I^{n-1} \leqslant_{E_{n-1}} I^n = I'$. Hence, by Lemma 6.17, $[\![U_j]\!]^I_{\xi,\theta} \leqslant [\![U_j]\!]^{I'}_{\xi,\theta}$ and $u \in \varphi^{I'}_D(\vec{a})$. □

Since $(\mathscr{I}^m, \leqslant)$ is a complete lattice, φ has a least fixpoint I, which is an interpretation for all the constant predicate symbols equivalent to C. Hence, by induction on $>_{\mathscr{C}}$, we obtain an interpretation I for all the constant predicate symbols.

In the case of a primitive constant predicate symbol, the interpretation is simply the set of strongly normalisable terms of this type:

Lemma 6.19 (Interpretation of primitive constant predicate symbols). If C is a primitive constant predicate symbol, then $I_C = \top_{\tau_C}$.

Proof. Since $I_C \leqslant \top_{\tau_C}$, it suffices to prove that $\top_{\tau_C} \leqslant I_C$. Since, by assumption, $\vdash \tau_C : \Box$, we have τ_C is of the form $(\vec{x} : \vec{T})\star$. If $\vec{a}$ are arguments of $\top_{\tau_C}$, then $\top_{\tau_C}(\vec{a}) = \top_\star = \mathscr{SN}$, and it suffices to prove, for all $u \in \mathscr{SN}$, C primitive and $\vec{a}$ arguments of I_C, that $u \in I_C(\vec{a})$, by induction on u with $\rightarrow \cup \rhd$ as well-founded ordering. Assume that $u \rightarrow^* f\vec{u}$ with $f : (\vec{y} : \vec{U})C\vec{v}$. If $u \rightarrow^+ f\vec{u}$, we can conclude by the induction hypothesis. So assume that $u = f\vec{u}$. In this case, we have to prove, for all $j \in \mathrm{Acc}(f)$, that $u_j \in [\![U_j]\!]_{\xi,\theta}$ with $\theta = \{\vec{y} \mapsto \vec{u}\}$ and, for all $y \in \mathrm{FV}^{\Box}(U_j)$, $y\xi = S_{l_y}$. By definition of primitive constant predicate symbols, for all $j \in \mathrm{Acc}(f)$, we have U_j is of the form $D\vec{w}$ with D primitive too. Hence, $[\![U_j]\!]_{\xi,\theta} = I_D(\vec{a}')$ with $a'_i = (w_i\theta, [\![w_i]\!]_{\xi,\theta})$. Since $u_j \in \mathscr{SN}$, by the induction hypothesis, $u_j \in I_D(\vec{a}')$. Therefore, $u \in I_C(\vec{a})$. □

6.4. *Computability ordering*

In this section we assume that we are given an interpretation J for defined predicate symbols and denote $[\![T]\!]^{I \cup J}$ by $[\![T]\!]$. The fixpoint of the function φ defined in the previous section can be reached by transfinite iteration from the smallest element of $\mathscr{I}^m$, $\bot_C(\vec{t}, \vec{S}) = \bot_\star$. Let $I^{\mathfrak{a}}$ be the interpretation reached after $\mathfrak{a}$ iterations of φ.

Definition 6.20 (Order of a computable term). The *order* of a term $t \in I_C(\vec{S})$, written $o_{C(\vec{S})}(t)$, is the smallest ordinal $\mathfrak{a}$ such that $t \in I_C^{\mathfrak{a}}(\vec{S})$.

This notion of order will enable us to define a well-founded ordering in which recursive definitions on strictly positive predicates strictly decrease. Indeed, in this case, the subterm ordering is not sufficient. In the example of the addition on ordinals, we have the rule

$$x + (lim\ f) \rightarrow lim\ ([n : nat]x + fn).$$

We have a recursive call with (fn) as argument, which is not a subterm of $(lim\ f)$. However, thanks to the definition of the interpretation for constant predicate symbols and products, we can say that, if $(lim\ f)$ is computable, then f is computable and thus we can say that, for all computable n, (fn) is computable. So the order of $(lim\ f)$ is greater than that of (fn): $o(lim\ f) > o(fn)$.

Definition 6.21 (Computability ordering). Let $f \in \mathscr{F}$ with $stat_f = lex\ m_1 \ldots m_k$. Let Θ_f be the set of tuples (g, ξ, θ) such that $g =_{\mathscr{F}} f$ and $\xi, \theta \models \Gamma_g$. We equip Θ_f with the ordering $\sqsupset_f$ defined by:

— $(g, \xi, \theta)\ \sqsupset_f\ (g', \xi', \theta')$ if $\vec{m}\theta\ (\sqsupset_f^{1,m}, \ldots, \sqsupset_f^{k,m})_{\mathrm{lex}}\ \vec{m}\theta'$,
— $mul\ \vec{t}\ \sqsupset_f^{i,m}\ mul\ \vec{t}'$ if $\{\vec{t}\}\ (\sqsupset_f^i)_{\mathrm{mul}}\ \{\vec{t}'\}$,
— $t \sqsupset_f^i t'$ if $i \in SP(f)$, $T_f^i = C\vec{a}$, $[\![\vec{a}]\!]_{\xi,\theta} = [\![\vec{a}]\!]_{\xi',\theta'} = \vec{S}$ and $o_{C(\vec{S})}(t) > o_{C(\vec{S})}(t')$,
— $t \sqsupset_f^i t'$ if $i \notin SP(f)$ and $t\,(\rightarrow \cup \rhd)\,t'$.

We equip $\Theta = \bigcup\{\Theta_f \mid f \in \mathscr{F}\}$ with the *computability ordering* $\sqsupset$ defined by $(f, \xi, \theta) \sqsupset (f', \xi', \theta')$ if $f >_{\mathscr{F}} f'$ or, $f =_{\mathscr{F}} f'$ and $(f, \xi, \theta)\ \sqsupset_f\ (f', \xi', \theta')$.

Lemma 6.22. The computability ordering is well-founded and compatible with $\to$, that is, if $\theta \to \theta'$, then $(g,\xi,\theta) \sqsupseteq (g,\xi,\theta')$.

Proof. The computability ordering is well-founded since ordinals are well-founded and lexicographic and multiset orderings preserve well-foundedness. It is compatible with $\to$ by definition of the interpretation of constant predicate symbols. □

We check hereafter that the accessibility relation is correct, that is, an accessible subterm of a computable term is computable. Then we check that the ordering on arguments is correct too, that is, if $t >_R^i u$ and t is computable, then u is computable and $o(t) > o(u)$.

Lemma 6.23 (Correctness of accessibility). If $t : T \rhd_\rho u : U$ and $t\sigma \in [\![T\rho]\!]^{I^{\mathfrak{a}}}_{\xi,\sigma}$ with $\mathfrak{a}$ as small as possible, then $\mathfrak{a} = \mathfrak{b}+1$ and $u\sigma \in [\![U\rho]\!]^{I^{\mathfrak{b}}}_{\xi,\sigma}$.

Proof. By the definition of $\rhd_\rho$, we have $t = f\vec{u}$, $f : (\vec{y} : \vec{U})C\vec{v}$, $C \in \mathscr{C}\mathscr{F}^{\square}$, $u = u_j$, $j \in \mathrm{Acc}(f)$, $T\rho = C\vec{v}\gamma\rho$, $U\rho = U_j\gamma\rho$, $\gamma = \{\vec{y} \mapsto \vec{u}\}$ and no $D =_{\mathscr{C}} C$ occurs in $\vec{u}\rho$. Hence, $t\sigma \in [\![C\vec{v}\gamma\rho]\!]^{I^{\mathfrak{a}}}_{\xi,\sigma} = I^{\mathfrak{a}}_C(\vec{S})$ with $\vec{S} = [\![\vec{v}\gamma\rho]\!]^{I^{\mathfrak{a}}}_{\xi,\sigma}$. Assume that $\mathfrak{a} = 0$. Then $I^{\mathfrak{a}}_C(\vec{S}) = \bot_\star$. But $f\vec{u} \notin \bot_\star$ since $f\vec{u}$ is not neutral (see Lemma 6.6). So $\mathfrak{a} \neq 0$. Assume now that $\mathfrak{a}$ is a limit ordinal. Then $I^{\mathfrak{a}}_C(\vec{S}) = \bigcup\{I^{\mathfrak{b}}_C(\vec{S}) \mid \mathfrak{b} < \mathfrak{a}\}$ and $t\sigma \in I^{\mathfrak{b}}_C(\vec{S})$ for some $\mathfrak{b} < \mathfrak{a}$, which is impossible since $\mathfrak{a}$ is as small as possible. Therefore, $\mathfrak{a} = \mathfrak{b}+1$ and, by definition of I_C, $u_j\sigma \in [\![U_j]\!]^{I^{\mathfrak{b}}}_{\xi',\gamma\rho\sigma}$ with $y\xi' = S_{\iota_y}$. By **I6**, $v_{\iota_y} = y$. Thus $y\xi' = [\![y\gamma\rho]\!]^{I^{\mathfrak{a}}}_{\xi,\sigma}$. Now, since no $D =_{\mathscr{C}} C$ occurs in $\vec{u}\rho$, we have $y\xi' = [\![y\gamma\rho]\!]^{I^{\mathfrak{b}}}_{\xi,\sigma}$. Hence, by candidate substitution, $[\![U_j]\!]^{I^{\mathfrak{b}}}_{\xi',\gamma\rho\sigma} = [\![U_j\gamma\rho]\!]^{I^{\mathfrak{b}}}_{\xi,\sigma}$ and $u\sigma \in [\![U\rho]\!]^{I^{\mathfrak{b}}}_{\xi,\sigma}$ since $U\rho = U_j\gamma\rho$. □

Lemma 6.24 (Correctness of the ordering on arguments). Assume that $t : T >_R^i u : U$ as in Definition 5.7, $t\sigma \in [\![T\rho]\!]_{\xi,\sigma}$ and $\vec{u}\sigma \in [\![\vec{U}\delta]\!]_{\xi,\sigma}$. Then, $u\sigma \in [\![U\rho]\!]_{\xi,\sigma}$ and $o_{C(\vec{S})}(t\sigma) > o_{C(\vec{S})}(u\sigma)$ with $\vec{S} = [\![\vec{v}\gamma\rho]\!]_{\xi,\sigma}$.

Proof. Since $t : T \rhd^+_\rho x : V$, we have $T\rho = C\vec{v}\gamma\rho$. Hence $t\sigma \in I^{\mathfrak{a}}_C(\vec{S})$ with $\mathfrak{a} = o_{C(\vec{S})}(t\sigma)$. By Lemma 6.23, $\mathfrak{a} = \mathfrak{b}+1$ and $x\sigma \in [\![V\rho]\!]^{I^{\mathfrak{b}}}_{\xi,\sigma}$. Since no $D =_{\mathscr{C}} C$ occurs in $\vec{U}\delta$, we have $[\![\vec{U}\delta]\!]_{\xi,\sigma} = [\![\vec{U}\delta]\!]^{I^{\mathfrak{b}}}_{\xi,\sigma}$. Since $V\rho = (\vec{y} : \vec{U})C\vec{w}$ and $\vec{u}\sigma \in [\![\vec{U}\delta]\!]^{I^{\mathfrak{b}}}_{\xi,\sigma}$, we have $u\sigma \in [\![C\vec{w}]\!]^{I^{\mathfrak{b}}}_{\xi^{\vec{R}}_{\vec{y}},\sigma^{\vec{u}\sigma}_{\vec{y}}}$ with $\vec{R} = [\![\vec{u}]\!]^{I^{\mathfrak{b}}}_{\xi,\sigma}$. By candidate substitution, $[\![C\vec{w}]\!]^{I^{\mathfrak{b}}}_{\xi^{\vec{R}}_{\vec{y}},\sigma^{\vec{u}\sigma}_{\vec{y}}} = [\![C\vec{w}\delta]\!]^{I^{\mathfrak{b}}}_{\xi,\sigma} = I^{\mathfrak{b}}_C(\vec{S}')$ with $\vec{S}' = [\![\vec{w}\delta]\!]^{I^{\mathfrak{b}}}_{\xi,\sigma}$. Since $\vec{w}\delta|_C = \vec{v}\gamma\rho|_C$, we have $\vec{S}' = [\![\vec{v}\gamma\rho]\!]^{I^{\mathfrak{b}}}_{\xi,\sigma}$. Since no $D =_{\mathscr{C}} C$ occurs in $\vec{v}\gamma\rho$, we have $\vec{S}' = \vec{S}$. Therefore, $u\sigma \in I^{\mathfrak{b}}_C(\vec{S})$ and $o_{C(\vec{S})}(t\sigma) > o_{C(\vec{S})}(u\sigma)$. □

6.5. *Interpretation of defined predicate symbols*

We define the interpretation J for defined predicate symbols by induction on $\succ$ (**A3**). Let F be a defined predicate symbol and assume that we have already defined an interpretation K for every symbol smaller than F. There are three cases depending on whether the equivalence class of F is primitive, positive or computable. For simplicity, we denote $[\![T]\!]^{I\cup K\cup J}$ by $[\![T]\!]^J$.

6.5.1. *Primitive systems*

Definition 6.25. For every $G \simeq F$, we take $J_G = \top_{\tau_G}$.

6.5.2. *Positive, small and simple systems* Let $\mathscr{J}$ be the set of interpretations of the symbols equivalent to F and $\leqslant$ be the relation on $\mathscr{J}$ defined by $J \leqslant J'$ if, for all $G \simeq F$, $J_G \leqslant_{\tau_G} J'_G$. Since $(\mathscr{R}_{\tau_G}, \leqslant_{\tau_G})$ is a complete lattice, it is easy to see that $(\mathscr{J}, \leqslant)$ is a complete lattice too.

Definition 6.26. Let ψ be the function that associates to $J \in \mathscr{J}$ and $G \simeq F$ with $G : (\vec{x} : \vec{T})U$, the interpretation ψ^J_G defined by

$$\psi^J_G(\vec{t}, \vec{S}) = \begin{cases} [\![r]\!]^J_{\xi,\sigma} & \text{if } \vec{t} \in \mathscr{WN} \cap \mathscr{CR},\ \vec{t}\downarrow = \vec{l}\sigma \text{ and } (G\vec{l} \to r, \Gamma, \rho) \in \mathscr{R} \\ \top_U & \text{otherwise} \end{cases}$$

where $x\xi = S_{\kappa_x}$. We show below, in Lemma 6.28, that ψ is monotonic. So we can take $J = \mathrm{lfp}(\psi)$.

Lemma 6.27. ψ^J is a well-defined interpretation.

Proof. By simplicity, at most one rule can be applied at the top of $G(\vec{t}\downarrow)$. The existence of κ_x is the smallness condition **q**. We now prove that $\psi^J_G \in \mathscr{R}_{\tau_G}$. By **S3**, $\Gamma \vdash r : U\gamma\rho$ with $\gamma = \{\vec{x} \mapsto \vec{l}\}$. Now we prove that $\xi \models \Gamma$. Let $x \in \mathrm{FV}^{\square}(r)$. $x\xi = S_{\kappa_x} \in \mathscr{R}_{x\sigma}$ since $S_{\kappa_x} \in \mathscr{R}_{t_{\kappa_x}}$ and, by smallness, $t_{\kappa_x} = l_{\kappa_x}\sigma = x\sigma$. Therefore, by Lemma 6.8, $[\![r]\!]_{\xi,\sigma} \in \mathscr{R}_{U\gamma\rho} = \mathscr{R}_U$. We are left to check that ψ^J_G is stable by reduction. Assume that $\vec{t} \to \vec{t}'$. By **A1**, $\to$ is confluent. Therefore, $\{\vec{t}\} \subseteq \mathscr{WN}$ iff $\{\vec{t}'\} \subseteq \mathscr{WN}$. Furthermore, if $\{\vec{t}\} \subseteq \mathscr{WN}$, we have $\vec{t}\downarrow = \vec{t}'\downarrow$ and $\psi^J_G(\vec{t}, \vec{S}) = \psi^J_G(\vec{t}', \vec{S})$. □

Lemma 6.28. ψ is monotonic.

Proof. The proof is as in Lemma 6.18. □

6.5.3. *Computable, small and simple systems* Let $\mathscr{D}$ be the set of tuples $(G, \vec{t}, \vec{S})$ such that $G \simeq F$, and $\{\vec{x} \mapsto \vec{S}\}, \{\vec{x} \mapsto \vec{t}\} \models \Gamma_G$. We equip $\mathscr{D}$ with the well-founded ordering $(G, \vec{t}, \vec{S}) \sqsupset_{\mathscr{D}} (G', \vec{t}', \vec{S}')$ iff $(G, \{\vec{x} \mapsto \vec{S}\}, \{\vec{x} \mapsto \vec{t}\}) \sqsupset (G', \{\vec{x} \mapsto \vec{S}'\}, \{\vec{x} \mapsto \vec{t}'\})$ (see Definition 6.21).

Definition 6.29. We first define J' on $\mathscr{D}$ by induction on $\sqsupset_{\mathscr{D}}$. Let $G \simeq F$ with $G : (\vec{x} : \vec{T})U$.

$$J'_G(\vec{t}, \vec{S}) = \begin{cases} [\![r]\!]^{J'}_{\xi,\sigma} & \text{if } \vec{t} \in \mathscr{WN} \cap \mathscr{CR},\ \vec{t}\downarrow = \vec{l}\sigma \text{ and } (G\vec{l} \to r, \Gamma, \rho) \in \mathscr{R} \\ \top_U & \text{otherwise} \end{cases}$$

where $x\xi = S_{\kappa_x}$. Then $J_G(\vec{t}, \vec{S}) = J'_G(\vec{t}\downarrow, \vec{S})$ if $\vec{t} \in \mathscr{WN} \cap \mathscr{CR}$, and $J_G(\vec{t}, \vec{S}) = \top_U$ otherwise.

Lemma 6.30. J is a well-defined interpretation.

Proof. The proof is as in Lemma 6.27. The well-foundedness of the definition comes from Lemma 6.38 and Theorem 6.37. In Lemma 6.38, we will show that, starting from a sequence in $\mathscr{D}$, we can apply Theorem 6.37 where we will show that, in a recursive call $G'\vec{t}'$, $(G, \vec{t}, \vec{S}) \sqsupset (G', \vec{t}', \vec{S}')$ for some $\vec{S}'$. □

6.6. *Correctness of the conditions*

Definition 6.31 (Cap and aliens). Let ζ be an injection from classes of terms modulo $\leftrightarrow^*$ to $\mathscr{X}$. The *cap* of a term t with respect to a set $\mathscr{G}$ of symbols is the term $cap_{\mathscr{G}}(t) = t[x_1]_{p_1} \dots [x_n]_{p_n}$ such that, for all i, $x_i = \zeta(t|_{p_i})$ and $t|_{p_i}$ is not of the form $g\vec{t}$ with $g \in \mathscr{G}$. The $t|_{p_i}$'s are the *aliens* of t. We use $aliens_{\mathscr{G}}(t)$ to denote their multiset.

Lemma 6.32 (Pre-computability of first-order symbols). If $f \in \mathscr{F}_1$ and $\vec{t} \in \mathscr{SN}$, then $f\vec{t} \in \mathscr{SN}$.

Proof. We prove that every reduct t' of $t = f\vec{t}$ is in $\mathscr{SN}$. From now on, $cap = cap_{\mathscr{F}_1}$. There are two cases:

— $\mathscr{R}_\omega \neq \varnothing$: We use induction on $(aliens(t), cap(t))_{\text{lex}}$ with $((\to \cup \rhd)_{\text{mul}}, \to_{\mathscr{R}_1})_{\text{lex}}$ as well-founded ordering (the aliens are strongly normalisable and, by **f**, $\to_{\mathscr{R}_1}$ is strongly normalising on first-order algebraic terms).
If the reduction takes place in $cap(t)$, this is a $\mathscr{R}_1$-reduction. By **c**, no symbol of $\mathscr{F}_\omega$ occurs in the rules of $\mathscr{R}_1$. And, by **d**, the right-hand sides of the rules of $\mathscr{R}_1$ are algebraic. Therefore, $cap(t) \to_{\mathscr{R}_1} cap(t')$. By **e**, the rules of $\mathscr{R}_1$ are non-duplicating. Therefore, $aliens(t) \rhd_{\text{mul}} aliens(t')$, and we can conclude by the induction hypothesis.
If the reduction takes place in an alien, then $aliens(t)$ $(\to \cup \rhd)_{\text{mul}}$ $aliens(t')$ and we can conclude by the induction hypothesis.

— $\mathscr{R}_\omega = \varnothing$: Since the t_i's are strongly normalisable and no β-reduction can take place at the top of t, t has a β-normal form. Let $cap\beta(t)$ be the cap of its β-normal form. We prove that every immediate reduct t' of t is strongly normalisable by induction on $(\beta cap(t), aliens(t))_{\text{lex}}$ with $(\to_{\mathscr{R}_1}, (\to \cup \rhd)_{\text{mul}})_{\text{lex}}$ as well-founded ordering (the aliens are strongly normalisable and, by **f**, we have $\to_{\mathscr{R}_1}$ is strongly normalising on first-order algebraic terms).
If the reduction takes place in $cap(t)$, this is an $\mathscr{R}_1$-reduction. By **d**, the right-hand sides of the rules of $\mathscr{R}_1$ are algebraic. Therefore, t' has a β-normal form and $cap\beta(t) \to_{\mathscr{R}_1} cap\beta(t')$. Hence we can conclude by the induction hypothesis. If the reduction is a β-reduction in an alien, then $cap\beta(t) = cap\beta(t')$ and $aliens(t)$ $(\to \cup \rhd)_{\text{mul}}$ $aliens(t')$. Hence we can conclude by the induction hypothesis.
We are left with the case where the reduction is an $\mathscr{R}_1$-reduction taking place in an alien u. Then, $aliens(t) \to_{\text{mul}} aliens(t')$, $cap\beta(t) \to^*_{\mathscr{R}_1} cap\beta(t')$, and we can conclude by the induction hypothesis. To see that $cap\beta(t) \to^*_{\mathscr{R}_1} cap\beta(t')$, it suffices to remark that if we β-normalise u, then all the residuals of the $\mathscr{R}_1$-redex are still reducible (left- and right-hand sides of first-order rules are algebraic). □

Lemma 6.33 (Computability of first-order symbols). For all $f \in \mathscr{F}_1$, we have $f \in [\![\tau_f]\!]$.

Proof. Assume that $f : (\vec{x} : \vec{T})U$. $f \in [\![\tau_f]\!]$ iff, for all Γ_f-valid pair (ξ, θ), we have $f\vec{x}\theta \in R = [\![U]\!]_{\xi,\theta}$. For first-order symbols, $U = \star$ or $U = C\vec{v}$ with C primitive. If $U = \star$, then $R = \top_\star = \mathscr{SN}$. If $U = C\vec{v}$ with $C : (\vec{y} : \vec{U})V$, then $R = I_C(\vec{a})$ with $a_i = (v_i\theta, [\![v_i]\!]_{\xi,\theta})$. Since C is primitive, by Lemma 6.19, $I_C = \top_{\tau_C}$ and $R = \top_V$. By assumption, $\vdash \tau_C : \square$ and $\vdash \tau_f : s_f$. After Lemma 3.9, $s_f = \star$ and $V = \star$. Therefore $R = \top_\star = \mathscr{SN}$. Now,

since $\xi, \theta \models \Gamma_f$, we have $x_i\theta \in [\![T_i]\!]_{\xi,\theta} \subseteq \mathscr{SN}$ by **R1**. Hence, by the pre-computability of first-order symbols, we have $f\vec{x}\theta \in [\![U]\!]_{\xi,\theta}$. □

Theorem 6.34 (Strong normalisation of $\to_{\mathscr{R}}$). The relation $\to_{\mathscr{R}} = \to_{\mathscr{R}_1} \cup \to_{\mathscr{R}_\omega}$ is strongly normalising.

Proof. We use induction on the structure of terms. The only difficult case is $f\vec{t}$. If f is first-order, we use the Lemma of pre-computability of first-order symbols. If f is higher-order, we have to show that, if $\vec{t} \in \mathscr{SN}_{\mathscr{R}}$, then $t = f\vec{t} \in \mathscr{SN}_{\mathscr{R}}$, where $\mathscr{SN}_{\mathscr{R}}$ is the set of terms that are strong normalisable with respect to $\to_{\mathscr{R}}$.

Let $\varpi(t) = 0$ if t is not of the form $g\vec{u}$ and $\varpi(t) = 1$ otherwise. We now prove that every reduct t' of t is strongly normalisable by induction on $(f, \varpi(\vec{t}), \vec{t}, \vec{t})$ with $(>_{\mathscr{F}}, (>_{\mathbb{N}})_{stat_f}, (\rhd \cup \to_{\mathscr{R}})_{stat_f}, (\to_{\mathscr{R}})_{\text{lex}})_{\text{lex}}$ as well-founded ordering. Assume that $t' = f\vec{t}'$ with $t_i \to_{\mathscr{R}} t'_i$ and, for all $j \neq i$, $t_j = t'_j$. Then, $\vec{t}\ (\to_{\mathscr{R}})_{\text{lex}}\ \vec{t}'$ and $\varpi(t_i) \geqslant \varpi(t'_i)$ since if t_i is not of the form $g\vec{u}$, then t'_i is not of the form $g\vec{u}$ either.

Assume now that there exists $f\vec{l} \to r \in \mathscr{R}_\omega$ such that $\vec{t} = \vec{l}\sigma$ and $t' = r\sigma$. By **a**, r belongs to the computability closure of l. It is then easy to prove that $r\sigma$ is strongly normalisable by induction on the structure of r. Again, the only difficult case is $g\vec{u}$. But then, either g is smaller than f, or g is equivalent to f and its arguments are smaller than $\vec{l}$. If $l_i >_1 u_j$, then $l_i \rhd u_j$ and $\text{FV}(u_j) \subseteq \text{FV}(l_i)$. Therefore $l_i\sigma \rhd u_j\sigma$ and $\varpi(l_i\sigma) = 1 \geqslant \varpi(u_j\sigma)$. If now $l_i >_2 u_j$, then u_j is of the form $x\vec{v}$ and $\varpi(l_i\sigma) = 1 > \varpi(u_j\sigma) = 0$. □

Lemma 6.35 (Invariance by reduction). If $\Gamma \vdash t : T$, $t \to t'$, $\xi \models \Gamma$ and $t\theta \in \mathscr{WN}$, then $[\![t]\!]_{\xi,\theta} = [\![t']\!]_{\xi,\theta}$.

Proof. We use induction on t. If t is an object, then t' is an object too and $[\![t]\!]_{\xi,\theta} = \varnothing = [\![t']\!]_{\xi,\theta}$. Otherwise, we proceed by cases on t and t':

— Let $R = [\![F\vec{l}\sigma]\!]_{\xi,\theta}$ and $R' = [\![r\sigma]\!]_{\xi,\theta}$ with $(F\vec{l} \to r, \Gamma_0, \rho) \in \mathscr{R}$. $R = I_F(\vec{a})$ with $a_i = (l_i\sigma\theta, [\![l_i\sigma]\!]_{\xi,\theta})$. By **A3**, there are two sub-cases:

  – F **belongs to a primitive system**. Then, $I_F = \top_{\tau_F}$ and r is of the form $[\vec{x} : \vec{T}]\ G\vec{u}$ with $G \simeq F$ or G a primitive constant predicate symbol. In both cases, $I_G = \top_{\tau_G}$. Therefore, $R = R'$.

  – F **belongs to a positive or computable, small and simple system**. Since $l_i\sigma\theta \in \mathscr{WN}$, by **A1**, $l_i\sigma\theta$ has a unique normal form t_i. By simplicity, the symbols in $\vec{l}$ are constant. Therefore, t_i is of the form $l_i\theta'$ with $\sigma\theta \to^* \theta'$, and $R = [\![r]\!]_{\xi',\theta'}$ with $x\xi' = [\![l_{\kappa_x}\sigma]\!]_{\xi,\theta}$. By smallness, $l_{\kappa_x} = x$ and $x\xi' = [\![x\sigma]\!]_{\xi,\theta}$. By Lemma 6.8, $[\![r]\!]_{\xi',\theta'} = [\![r]\!]_{\xi',\sigma\theta}$. By **S4**, $\sigma : \Gamma_0 \rightsquigarrow \Gamma$. Therefore, by candidate substitution, $R = R'$.

— Let $R = [\![[x : U]v\ u]\!]_{\xi,\theta}$ and $R' = [\![v\{x \mapsto u\}]\!]_{\xi,\theta}$. Let $S = [\![u]\!]_{\xi,\theta}$. $R = [\![[x : U]v]\!](u\theta, S) = [\![v]\!]_{\xi^S_x,\theta'}$ with $\theta' = \theta^{u\theta}_x = \{x \mapsto u\}\theta$. Since $\{x \mapsto u\} : (\Gamma, x : U) \to \Gamma$, by candidate substitution, $R' = [\![v]\!]_{\xi^S_x,\theta'} = R$.

— Let $R = [\![tu]\!]_{\xi,\theta}$ and $R' = [\![t'u']\!]_{\xi,\theta}$ with $t \to t'$ and $u \to u'$. $R = [\![t]\!]_{\xi,\theta}(u\theta, [\![u]\!]_{\xi,\theta})$ and $R' = [\![t']\!]_{\xi,\theta}(u'\theta, [\![u']\!]_{\xi,\theta})$. By the induction hypothesis, $[\![t]\!]_{\xi,\theta} = [\![t']\!]_{\xi,\theta}$ and $[\![u]\!]_{\xi,\theta} = [\![u']\!]_{\xi,\theta}$. Finally, since candidates are stable by reduction, $R = R'$.

— Let $R = [\![[x : U]v]\!]_{\xi,\theta}$ and $R' = [\![[x : U']v']\!]_{\xi,\theta}$ with $U \to U'$ and $v \to v'$. Since $\mathscr{R}_U = \mathscr{R}_{U'}$, R and R' have the same domain $\mathscr{T} \times \mathscr{R}_U$ and codomain $\mathscr{R}_V$, where V is

the type of v. $R(u,S) = [\![v]\!]_{\xi^S_x,\theta^u_x}$ and $R'(u,S) = [\![v']\!]_{\xi^S_x,\theta^u_x}$. By the induction hypothesis, $R(u,S) = R'(u,S)$. Therefore, $R = R'$.

— Let $R = [\![(x:U)V]\!]_{\xi,\theta}$ and $R' = [\![(x:U')V']\!]_{\xi,\theta}$. $R = \{t \in \mathscr{T} \mid \forall u \in [\![U]\!]_{\xi,\theta}, \forall S \in \mathscr{R}_U, tu \in [\![V]\!]_{\xi^S_x,\theta^u_x}\}$ and $R' = \{t \in \mathscr{T} \mid \forall u \in [\![U']\!]_{\xi,\theta}, \forall S \in \mathscr{R}_U, tu \in [\![V']\!]_{\xi^S_x,\theta^u_x}\}$. By the induction hypothesis, $[\![U]\!]_{\xi,\theta} = [\![U']\!]_{\xi,\theta}$ and $[\![V]\!]_{\xi^S_x,\theta^u_x} = [\![V']\!]_{\xi^S_x,\theta^u_x}$. Therefore, $R = R'$. □

Lemma 6.36 (Pre-computability of well-typed terms). Assume, for all f, that $f \in [\![\tau_f]\!]$. If $\Gamma \vdash t : T$ and $\xi, \theta \models \Gamma$, then $t\theta \in [\![T]\!]_{\xi,\theta}$.

Proof. We use induction on $\Gamma \vdash t : T$:

(ax) $\star\theta = \star \in [\![\Box]\!]_{\xi,\theta} = \top_\Box = \mathscr{SN}$.

(symb) This follows by assumption.

(var) $x\theta \in [\![T]\!]_{\xi,\theta}$ since θ is adapted to ξ.

(weak) This follows from the induction hypothesis.

(prod) We have to prove that $(x:U\theta)V\theta \in [\![s']\!]_{\xi,\theta} = \top_{s'} = \mathscr{SN}$. By the induction hypothesis, $U\theta \in [\![s]\!]_{\xi,\theta} = \mathscr{SN}$. Now let $\xi' = \xi^{\top_U}_x$. Since $\xi', \theta \models \Gamma, x : U$, by the induction hypothesis, $V\theta \in [\![s']\!]_{\xi',\theta} = \mathscr{SN}$.

(abs) Let $t = [x : U]v$. We have to prove that $t\theta \in [\![(x : U)V]\!]_{\xi,\theta}$. First note that $U\theta, v\theta \in \mathscr{SN}$. Indeed, let $\xi' = \xi^{\top_U}_x$. Since $\xi', \theta \models \Gamma, x : U$, by the induction hypothesis, $v\theta \in [\![V]\!]_{\xi',\theta}$. Furthermore, by inversion, $\Gamma \vdash U : s$ for some s. So, by the induction hypothesis, $U\theta \in [\![s]\!]_{\xi,\theta} = \mathscr{SN}$. Now let $u \in [\![U]\!]_{\xi,\theta} \subseteq \mathscr{SN}$ and $S \in \mathscr{R}_U$. We must prove that $t\theta u \in S' = [\![V]\!]_{\xi^S_x,\theta^u_x}$. Since $t\theta u$ is neutral, it suffices to prove that $\to(t\theta u) \subseteq S'$. We prove it by induction on $(U\theta, v\theta, u)$ with $\to_{\mathrm{lex}}$ as well-founded ordering. We have $t\theta u \to v\theta\{x \mapsto u\} = v\theta'$. Since $\xi^S_x, \theta^u_x \models \Gamma, x : U$, by the induction hypothesis, $v\theta' \in S'$. For the other cases, we can conclude by the induction hypothesis on $(U\theta, v\theta, u)$.

(app) We have to prove that $t\theta u\theta \in [\![V\{x \mapsto u\}]\!]_{\xi,\theta}$. By the induction hypothesis, $t\theta \in [\![(x : U)V]\!]_{\xi,\theta}$ and $u\theta \in [\![U]\!]_{\xi,\theta}$. Since $S = [\![u\theta]\!]_{\xi,\theta} \in \mathscr{R}_{U\theta} = \mathscr{R}_U$, by definition of $[\![(x : U)V]\!]_{\xi,\theta}$, $t\theta u\theta \in [\![V]\!]_{\xi^S_x,\theta'}$ with $\theta' = \theta^{u\theta}_x$. By candidate substitution, $[\![V\{x \mapsto u\}]\!]_{\xi,\theta} = [\![V]\!]_{\xi',\{x\mapsto u\}\theta}$ with $y\xi' = [\![y\{x \mapsto u\}]\!]_{\xi,\theta}$. Since $\xi' = \xi^S_x$ and $\{x \mapsto u\}\theta = \theta'$, we have $t\theta u\theta \in [\![V\{x \mapsto u\}]\!]_{\xi,\theta}$.

(conv) In Blanqui (2001), we show that adding the hypothesis $\Gamma \vdash T : s$ does not change the typing relation. Therefore, by the induction hypothesis, $t\theta \in [\![T]\!]_{\xi,\theta}$, $T\theta \in [\![s]\!]_{\xi,\theta} = \top_s = \mathscr{SN}$ and $T'\theta \in [\![s]\!]_{\xi,\theta} = \mathscr{SN}$. Hence, by invariance and reduction, $[\![T]\!]_{\xi,\theta} = [\![T']\!]_{\xi,\theta}$ and $t\theta \in [\![T']\!]_{\xi,\theta}$. □

Theorem 6.37 (Computability closure correctness). Let $(f\vec{l} \to r, \Gamma, \rho)$ be a well-formed rule with $f \in \mathscr{F}_\omega$, $f : (\vec{x} : \vec{T})U$ and $\gamma = \{\vec{x} \mapsto \vec{l}\}$. Assume that $\eta, \gamma\sigma \models \Gamma_f$, $\xi, \sigma \models \Gamma$, $x\eta = [\![x\gamma\rho]\!]_{\xi,\sigma}$, and $\vec{l}\sigma \in [\![\vec{T}\gamma\rho]\!]_{\xi,\sigma}$. Assume also that:

— $\forall g <_{\mathscr{F}} f$, $g \in [\![\tau_g]\!]$,

— $\forall g =_{\mathscr{F}} f$, if $g : (\vec{y} : \vec{U})V$ and $(f, \eta, \gamma\sigma) \sqsupset (g, \xi'', \theta)$, then $g\vec{y}\theta \in [\![V]\!]_{\xi'',\theta}$.

If $\Delta \vdash_{\mathrm{c}} t : T$ and $\xi\xi', \sigma\sigma' \models \Gamma, \Delta$, then $t\sigma\sigma' \in [\![T]\!]_{\xi\xi',\sigma\sigma'}$.

Proof. By induction on $\Delta \vdash_{\mathrm{c}} t : T$, we prove that $t\sigma\sigma' \in [\![T]\!]_{\xi\xi',\sigma\sigma'}$ as in the previous lemma. We only give details for the case (symb$^=$). Let $\vec{u} = \vec{y}\delta$. By the induction hypothesis,

$\vec{u}\sigma\sigma' \in [\![\vec{U}\delta]\!]_{\xi\xi',\sigma\sigma'}$. By candidate substitution, there exists ξ'' such that $[\![\vec{U}\delta]\!]_{\xi\xi',\sigma\sigma'} = [\![\vec{U}]\!]_{\xi'',\delta\sigma\sigma'}$, $[\![V\delta]\!]_{\xi\xi',\sigma\sigma'} = [\![V]\!]_{\xi'',\delta\sigma\sigma'}$ and $\xi'' \models \Gamma_g$. Therefore, $\xi'', \delta\sigma\sigma' \models \Gamma_g$.

We now prove that $(f,\eta,\gamma\sigma) \sqsupset (g,\xi'',\delta\sigma\sigma')$. If $l_i : T_i\gamma \rhd_\rho^+ u_j : U_j\delta$, then $l_i \rhd u_j$ and $\mathrm{FV}(u_j) \subseteq \mathrm{FV}(l_i)$. Therefore, $l_i\sigma = l_i\sigma\sigma' \rhd u_j\sigma\sigma'$. Assume now that $l_i : T_i\gamma >_R^k u_j : U_j\delta$, $k \in SP(f)$ and $T_f^k = C\vec{a}$. By definition of $>_R^k$, we have $l_i = h\vec{t}'$, $h : (\vec{x}' : \vec{T}')C\vec{v}$, $u_j = x\vec{u}'$, $x \in \mathrm{dom}(\Gamma)$, $l_i : T_i\gamma \rhd_\rho^+ x : V$ and $V\rho = x\Gamma = (\vec{y}' : \vec{U}')C\vec{w}$, where $\gamma' = \{\vec{x}' \mapsto \vec{t}'\}$ and $\delta' = \{\vec{y}' \mapsto \vec{u}'\}$. We must prove that $[\![\vec{a}]\!]_{\eta,\gamma\sigma} = [\![\vec{a}]\!]_{\xi'',\delta\sigma\sigma'} = \vec{S}$ and $o_{C(\vec{S})}(l_i\sigma) > o_{C(\vec{S})}(u_j\sigma\sigma')$.

Assume that $T_i = C\vec{t}$ and $U_j = C\vec{u}$. Since $k \in SP(f)$, $\vec{t}|_C = \vec{u}|_C = \vec{a}|_C$. By definition of $\rhd_\rho$, we have $T_i\gamma\rho = C\vec{v}\gamma'\rho$. Hence $\vec{a}\gamma\rho|_C = \vec{v}\gamma'\rho|_C$. By the definition of $>_R^k$, we have $\vec{v}\gamma'\rho|_C = \vec{w}\delta'|_C$ and $U_j\delta\rho = C\vec{w}\delta'$. Therefore $\vec{a}\gamma\rho|_C = \vec{w}\delta'|_C = \vec{u}\delta\rho|_C = \vec{a}\delta\rho|_C = \vec{a}\delta|_C$ since $\mathrm{dom}(\rho) \subseteq \mathrm{FV}(l)$, $\mathrm{FV}(\delta) \subseteq \mathrm{dom}(\Delta)$ and $\mathrm{dom}(\Delta) \cap \mathrm{FV}(l) = \varnothing$. By **S5**, $[\![\vec{a}]\!]_{\eta,\gamma\sigma} = [\![\vec{a}]\!]_{\eta,\gamma\rho\sigma}$. Since $x\eta = [\![x\gamma\rho]\!]_{\xi,\sigma}$, by candidate substitution, $[\![\vec{a}]\!]_{\eta,\gamma\rho\sigma} = [\![\vec{a}\gamma\rho]\!]_{\xi,\sigma}$. So $[\![\vec{a}]\!]_{\eta,\gamma\sigma} = [\![\vec{a}\delta]\!]_{\xi,\sigma} = [\![\vec{a}\delta]\!]_{\xi\xi',\sigma\sigma'} = [\![\vec{a}]\!]_{\xi'',\delta\sigma\sigma'}$. Now, by the induction hypothesis, $\vec{u}'\sigma\sigma' \in [\![\vec{U}'\delta']\!]_{\xi\xi',\sigma\sigma'}$. Therefore, since $l_i\sigma = l_i\sigma\sigma' \in [\![T_i\gamma\rho]\!]_{\xi,\sigma} = [\![T_i\gamma\rho]\!]_{\xi\xi',\sigma\sigma'}$, by Lemma 6.24, $u_j\sigma\sigma' \in [\![U_j\delta\rho]\!]_{\xi\xi',\sigma\sigma'}$ and $o_{C(\vec{R})}(l_i\sigma) > o_{C(\vec{R})}(u_j\sigma\sigma')$ where $\vec{R} = [\![\vec{v}\gamma'\rho]\!]_{\xi\xi',\sigma\sigma'} = \vec{S}$. □

Lemma 6.38 (Computability of higher-order symbols). For all $f \in \mathscr{F}_\omega$, $f \in [\![\tau_f]\!]$.

Proof. Assume that $f : (\vec{x} : \vec{T})U$. $f \in [\![\tau_f]\!]$ iff, for all Γ_f-valid pair (η,θ), $f\vec{x}\theta \in [\![U]\!]_{\eta,\theta}$. We prove this by induction on $((f,\eta,\theta),\theta)$ with $(\sqsupset,\to)_{\mathrm{lex}}$ as well-founded ordering. Let $t_i = x_i\theta$ and $t = f\vec{t}$. By assumption (see Definition 2.2), for all rules $f\vec{l} \to r \in \mathscr{R}$, $|\vec{l}| \leqslant |\vec{t}|$. So, if $U \neq C\vec{v}$ with $C \in \mathscr{C}\mathscr{F}^\square$, we have t is neutral and it suffices to prove that $\to(t) \subseteq [\![U]\!]_{\eta,\theta}$. Otherwise, $[\![U]\!]_{\eta,\theta} = I_C(\vec{a})$ with $a_i = (v_i\theta, [\![v_i]\!]_{\eta,\theta})$. Since $\eta,\theta \models \Gamma_f$, we have $t_j \in [\![T_j]\!]_{\eta,\theta}$. Therefore, in this case too, it suffices to prove that $\to(t) \subseteq [\![U]\!]_{\eta,\theta}$.

If the reduction takes place in one t_i, we can conclude by the induction hypothesis since reducibility candidates are stable by reduction and $\sqsupset$ is compatible with reduction. Assume now that there exist $(l \to r,\Gamma,\rho) \in \mathscr{R}$ and σ such that $l = f\vec{l}$ and $t = l\sigma$. Then $\theta = \gamma\sigma$ with $\gamma = \{\vec{x} \mapsto \vec{l}\}$. Furthermore, by **S5**, $\sigma \downarrow \rho\sigma$. Hence, by Lemma 6.8, $[\![U]\!]_{\eta,\theta} = [\![U]\!]_{\eta,\gamma\rho\sigma}$ and $[\![\vec{T}]\!]_{\eta,\theta} = [\![\vec{T}]\!]_{\eta,\gamma\rho\sigma}$. Now, since rules are well-formed, $\Gamma \vdash l\rho : U\gamma\rho$. Therefore, by inversion, $\Gamma \vdash l_i\rho : T_i\gamma\rho$ and $\gamma\rho : \Gamma_f \rightsquigarrow \Gamma$.

We now define ξ such that $[\![U]\!]_{\eta,\gamma\rho\sigma} = [\![U\gamma\rho]\!]_{\xi,\sigma}$ and $[\![\vec{T}]\!]_{\eta,\gamma\rho\sigma} = [\![\vec{T}\gamma\rho]\!]_{\xi,\sigma}$. By safeness **b**, for all $x \in \mathrm{FV}^\square(\vec{T}U)$, $x\gamma\rho \in \mathrm{dom}(\Gamma)$ and, for all $x,x' \in \mathrm{FV}^\square(\vec{T}U)$, $x\gamma\rho = x'\gamma\rho \Rightarrow x = x'$. Let $y \in \mathrm{dom}^\square(\Gamma)$. If there exists $x \in \mathrm{dom}(\Gamma_f)$ (necessarily unique) such that $y = x\gamma\rho$, we take $y\xi = x\eta$. Otherwise, we take $y\xi = \top_{y\Gamma}$. We check that $\xi \models \Gamma$. If $y \neq x\gamma\rho$, $y\xi = \top_{y\Gamma} \in \mathscr{R}_{y\Gamma}$. If $y = x\gamma\rho$, then $y\xi = x\eta$. Since $\eta \models \Gamma_f$, we have $x\eta \in \mathscr{R}_{x\Gamma_f}$. Since $\gamma\rho : \Gamma_f \rightsquigarrow \Gamma$, we have $\Gamma \vdash y : x\Gamma_f\gamma\rho$. Therefore $y\Gamma\ \mathbb{C}_\Gamma^*\ x\Gamma_f\gamma\rho$ and, by Lemma 6.4, $y\xi = x\eta \in \mathscr{R}_{x\Gamma_f} = \mathscr{R}_{x\Gamma_f\gamma\rho} = \mathscr{R}_{y\Gamma}$. So $\xi \models \Gamma$. Now, by candidate substitution, $[\![U\gamma\rho]\!]_{\xi,\sigma} = [\![U]\!]_{\xi',\gamma\rho\sigma}$ with $x\xi' = [\![x\gamma\rho]\!]_{\xi,\sigma}$. Let $x \in \mathrm{FV}(\vec{T}U)$. By **b**, $x\gamma\rho = y \in \mathrm{dom}^\square(\Gamma)$ and $x\xi' = y\xi = x\eta$. Since ξ' and η are equal on $\mathrm{FV}^\square(\vec{T}U)$, we have $[\![U]\!]_{\xi',\gamma\rho\sigma} = [\![U]\!]_{\eta,\gamma\rho\sigma} = [\![U\gamma\rho]\!]_{\xi,\sigma}$ and $[\![\vec{T}]\!]_{\xi',\gamma\rho\sigma} = [\![\vec{T}]\!]_{\eta,\gamma\rho\sigma} = [\![\vec{T}\gamma\rho]\!]_{\xi,\sigma}$.

We now prove that σ is adapted to ξ. Let $x \in \mathrm{dom}(\Gamma)$. Since rules are well-formed, there exists i such that $l_i : T_i\gamma \rhd_\rho^* x : x\Gamma$ and $\mathrm{dom}(\rho) \subseteq \mathrm{FV}(l) \setminus \mathrm{dom}(\Gamma)$. Since $l_i\sigma \in [\![T_i\gamma\rho]\!]_{\xi,\sigma}$, by correctness of accessibility, $x\sigma \in [\![x\Gamma\rho]\!]_{\xi,\sigma}$. Since $\mathrm{dom}(\rho) \cap \mathrm{dom}(\Gamma) = \varnothing$, we have

$x\Gamma\rho = x\Gamma$ and $x\sigma \in [\![x\Gamma]\!]_{\xi,\sigma}$. Therefore σ is adapted to ξ and, by correctness of the computability closure, $r\sigma \in [\![U\gamma\rho]\!]_{\xi,\sigma} = [\![U]\!]_{\eta,\theta}$. □

Lemma 6.39 (Computability of well-typed terms). If $\Gamma \vdash t : T$ and $\xi, \theta \models \Gamma$, then $t\theta \in [\![T]\!]_{\xi,\theta}$.

Proof. The proof follows from Lemmas 6.33, 6.36 and 6.38. □

Theorem 6.40 (Strong normalisation). Every typable term is strongly normalisable.

Proof. Assume that $\Gamma \vdash t : T$. Let $x\xi = \top_{x\Gamma}$ for all $x \in \mathrm{dom}(\Gamma)$. Since $\xi \models \Gamma$ and the identity substitution ι is adapted to ξ, we have $t \in S = [\![T]\!]_{\xi,\iota}$. Now, we have either $T = \Box$ or $\Gamma \vdash T : s$ for some s. If $T = \Box$, then $S = \top_\Box = \mathscr{SN}$. If $\Gamma \vdash T : s$, then $S \in \mathscr{R}_s$ and $S \subseteq \mathscr{SN}$ by **R1**. So, in both cases, $t \in \mathscr{SN}$. □

7. Future directions of research

We conclude by giving some directions of research to improve our conditions for strong normalisation.

Rewriting modulo. We did not consider rewriting modulo some equational theories such as associativity and commutativity. While this does not create too many difficulties at the object level (Blanqui 2003b), it is less clear for rewriting at the type level.

Quotient types. We have seen that rewrite rules on constructors allow us to formalise some quotient types. However, to prove properties by induction on such types requires us to know what the normal forms are (Jouannaud and Kounalis 1986) and may also require a particular reduction strategy (Courtieu 2001) or conditional rewriting.

Confluence. Among our strong normalisation conditions, we not only require rewriting to be confluent but also its combination with β-reduction. This is a strong condition since we cannot rely on strong normalisation for proving confluence (Nipkow 1991; Blanqui 2000). Apart from first-order rewriting systems without dependent types (Breazu-Tannen and Gallier 1994) and left-linear higher-order rewrite systems (Müller 1992; Van Oostrom 1994), few results are known on modularity of confluence for the combination of higher-order rewriting and β-reduction. Therefore, it would be interesting to study this problem more deeply.

Local confluence. We believe that local confluence is sufficient for establishing strong normalisation since local confluence and strong normalisation together imply confluence. But then it seems necessary to prove many properties simultaneously (subject reduction, strong normalisation and confluence), which seems difficult.

Simplicity. For non-primitive predicate symbols, we require that their defining rules have no critical pairs between them or with the other rules. These strong conditions allow us to define a valid interpretation in a simple way. It is important to be able to weaken these conditions in order to capture more decision procedures.

Local definitions. In our work, we only considered globally defined symbols, that is, symbols whose type is typable in the empty environment. However, in practice, during a formal proof in a system like Coq (Coq Development Team 2002), it may be very useful to introduce symbols and rules using some hypothesis. We should study the problems arising from local definitions and how our results can be used to solve them. Local abbreviations are studied in Poll and Severi (1994) and local definitions by rewriting are considered in Chrząszcz (2000).

HORPO. For higher-order definitions, we have chosen to extend the General Schema of Jouannaud and Okada (1997). But the Higher-Order Recursive Path Ordering (HORPO) of Jouannaud and Rubio (1999), which is an extension of RPO to the simply typed λ-calculus, is naturally more powerful. Walukiewicz recently extended this ordering to the Calculus of Constructions with symbols at the object level only (Walukiewicz 2000; Walukiewicz-Chrząszcz 2002). Combining these approaches should allow us to extend RPO to the Calculus of Constructions with type-level rewriting too.

η-Reduction. Among our conditions, we require the confluence of $\rightarrow_{\mathscr{R}} \cup \rightarrow_{\beta}$. Hence our results cannot be directly extended to η-reduction, which is well known to create important difficulties (Geuvers 1993), since $\rightarrow_{\beta} \cup \rightarrow_{\eta}$ is not confluent on terms that are not well-typed.

Non-strictly positive predicates. The ordering used in the General Schema for comparing the arguments of function symbols can capture recursive definitions on basic and strictly-positive types, but cannot capture recursive definitions on non-strictly positive types (Matthes 2000). However, Mendler (1987) showed that such definitions are strongly normalising. In Blanqui (2003a), we recently showed how to deal with such definitions in the Calculus of Algebraic Constructions.

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References

Abadi, M., Cardelli, L., Curien, P.-L. and Lévy, J.-J. (1991) Explicit substitutions. *Journal of Functional Programming* **1** (4) 375–416.

Abel, A. (2001) A third-order representation of the $\lambda\mu$-calculus. In: Proc. of the Workshop on Mechanized Reasoning about Languages with Variable Binding. *Electronic Notes in Theoretical Computer Science* **58** (1).

Altenkirch, T. (1993) *Constructions, Inductive Types and Strong Normalization*, Ph.D. thesis, University of Edinburgh, United Kingdom.

Augustsson, L. (1985) Compiling pattern matching. In: Proc. of the Conf. on Functional Programming Languages and Computer Architecture. *Springer-Verlag Lecture Notes in Computer Science* **201**.

Baader, F. and Nipkow, T. (1998) *Term Rewriting and All That*, Cambridge University Press.

Barbanera, F. (1990) Adding algebraic rewriting to the Calculus of Constructions: strong normalization preserved. In: Proc. of the 2nd Int. Work. on Conditional and Typed Rewriting Systems. *Springer-Verlag Lecture Notes in Computer Science* **516**.

Barbanera, F., Fernández, M. and Geuvers, H. (1994) Modularity of strong normalization and confluence in the algebraic-λ-cube. In: *Proc. of the 9th IEEE Symp. on Logic in Computer Science*.

Barbanera, F., Fernández, M. and Geuvers, H. (1997) Modularity of strong normalization in the algebraic-λ-cube. *Journal of Functional Programming* **7** (6) 613–660.

Barendregt, H. (1984) *The Lambda Calculus: Its Syntax and Semantics*, 2nd edition, North-Holland.

Barendregt, H. (1992) Lambda calculi with types. In: Abramski, S., Gabbay, D. and Maibaum, T. (eds.) *Handbook of logic in computer science*, volume 2, Oxford University Press.

Barras, B. (1999) *Auto-validation d'un système de preuves avec familles inductives*, Ph.D. thesis, Université Paris VII, France.

Barthe, G. (1998) The relevance of proof-irrelevance. In: Proc. of the 25th Int. Colloq. on Automata, Languages and Programming. *Springer-Verlag Lecture Notes in Computer Science* **1443**.

Barthe, G. and Geuvers, H. (1995a) Modular properties of algebraic type systems. In: Proc. of the 2nd Int. Work. on Higher-Order Algebra, Logic and Term Rewriting. *Springer-Verlag Lecture Notes in Computer Science* **1074**.

Barthe, G. and Geuvers, H. (1995b) Congruence types. In: Proc. of the 9th Int. Work. on Computer Science Logic. *Springer-Verlag Lecture Notes in Computer Science* **1092**.

Barthe, G. and Melliès, P.-A. (1996) On the subject reduction property for algebraic type systems. In: Proc. of the 10th Int. Work. on Computer Science Logic. *Springer-Verlag Lecture Notes in Computer Science* **1258**.

Barthe, G. and van Raamsdonk, F. (1997) Termination of algebraic type systems: the syntactic approach. In: Proc. of the 6th Int. Conf. on Algebraic and Logic Programming. *Springer-Verlag Lecture Notes in Computer Science* **1298**.

Bendix, P. and Knuth, D. (1970) *Computational problems in abstract algebra*, chapter entitled 'Simple word problems in universal algebra', Pergamon Press.

Blanqui, F. (2000) Termination and confluence of higher-order rewrite systems. In: Proc. of the 11th Int. Conf. on Rewriting Techniques and Applications. *Springer-Verlag Lecture Notes in Computer Science* **1833**.

Blanqui, F. (2001) *Théorie des Types et Récriture*, Ph.D. thesis, Université Paris XI, Orsay, France. (Available in English as "Type Theory and Rewriting".)

Blanqui, F. (2001) Definitions by rewriting in the Calculus of Constructions (extended abstract). In: *Proc. of the 16th IEEE Symp. on Logic in Computer Science*.

Blanqui, F. (2003a) Inductive types in the Calculus of Algebraic Constructions. In: Proceedings of the 6th International Conference on Typed Lambda Calculi and Applications. *Springer-Verlag Lecture Notes in Computer Science* **2701**.

Blanqui, F. (2003b) Rewriting modulo in Deduction modulo. In: Proceedings of the 14th International Conference on Rewriting Techniques and Applications. *Springer-Verlag Lecture Notes in Computer Science* **2706**.

Blanqui, F., Jouannaud, J.-P. and Okada, M. (1999) The Calculus of Algebraic Constructions. In: Proc. of the 10th Int. Conf. on Rewriting Techniques and Applications. *Springer-Verlag Lecture Notes in Computer Science* **1631**.

Blanqui, F., Jouannaud, J.-P. and Okada, M. (2002) Inductive-data-type Systems. *Theoretical Computer Science* **272** 41–68.

Borovanský, P., Cirstea, H., Dubois, H., Kirchner, C., Kirchner, H., Moreau, P.-E., Ringeissen, C. and Vittek, M. (2000) *ELAN User Manual*, INRIA Nancy, France. (Available at `http://elan.loria.fr/`.)

Breazu-Tannen, V. (1988) Combining algebra and higher-order types. In: *Proc. of the 3rd IEEE Symp. on Logic in Computer Science.*

Breazu-Tannen, V. and Gallier, J. (1989) Polymorphic rewriting conserves algebraic strong normalization. In: Proc. of the 16th Int. Colloq. on Automata, Languages and Programming. *Springer-Verlag Lecture Notes in Computer Science* **372**.

Breazu-Tannen, V. and Gallier, J. (1994) Polymorphic rewriting conserves algebraic confluence. *Information and Computation* **114** (1) 1–29.

Chrząszcz, J. (2000) Modular rewriting in the Calculus of Constructions. Presented at the International Workshop on Types for Proofs and Programs.

Clavel, M., Durán, F., Eker, S., Lincoln, P., Martí-Oliet, N., Meseguer, J. and Quesada, J. (1999) *Maude: Specification and Programming in Rewriting Logic*, Computer Science Laboratory, SRI International, United States. (Available at `http://maude.csl.sri.com/`.)

Contejean, E., Marché, C., Monate, B. and Urbain, X. (2000) CiME. (Available at `http://cime.lri.fr/`.)

Coq Development Team (2002) *The Coq Proof Assistant Reference Manual – Version 7.3*, INRIA Rocquencourt, France. (Available at `http://coq.inria.fr/`.)

Coquand, T. (1985) *Une théorie des constructions*, Ph.D. thesis, Université Paris VII, France.

Coquand, T. (1986) An analysis of Girard's paradox. In: *Proc. of the 1st IEEE Symp. on Logic in Computer Science.*

Coquand, T. (1991) An algorithm for testing conversion in type theory. In: Huet, G. and Plotkin, G. (eds.) *Logical Frameworks*, Cambridge University Press 255–279.

Coquand, T. (1992) Pattern matching with dependent types. In: *Proc. of the Int. Work. on Types for Proofs and Programs.* (Available at `http://www.lfcs.informatics.ed.ac.uk/research/types-bra/proc/`.)

Coquand, T. and Gallier, J. (1990) A proof of strong normalization for the Theory of Constructions using a Kripke-like interpretation. (Paper presented at the First International Workshop on Logical Frameworks, but not published in the proceedings – available at `ftp://ftp.cis.upenn.edu/pub/papers/gallier/sntoc.dvi.Z`.)

Coquand, T. and Huet, G. (1988) The Calculus of Constructions. *Information and Computation* **76** (2–3) 95–120.

Coquand, T. and Paulin-Mohring, C. (1988) Inductively defined types. In: Proc. of the Int. Conf. on Computer Logic. *Springer-Verlag Lecture Notes in Computer Science* **417**.

Courtieu, P. (2001) Normalized types. In: Proc. of the 15th Int. Work. on Computer Science Logic. *Springer-Verlag Lecture Notes in Computer Science* **2142**.

de Bruijn, N. (1968) The mathematical language AUTOMATH, its usage, and some of its extensions. In: Proc. of the Symp. on Automatic Demonstration. *Springer-Verlag Lecture Notes in Mathematics.* (Reprinted in: Geuvers, H., Nederpelt, R. and de Vrijer, R. (eds.) (1994) Selected Papers on Automath. *Studies in Logic and the Foundations of Mathematics* **133**, North-Holland.)

Dershowitz, N. (1982) Orderings for term rewriting systems. *Theoretical Computer Science* **17** 279–301.

Dershowitz, N. (1994) Hierarchical termination. In: Proc. of the 4th Int. Work. on Conditional and Typed Rewriting Systems. *Springer-Verlag Lecture Notes in Computer Science* **968**.

Dershowitz, N. and Jouannaud, J.-P. (1990) Rewrite systems. In: van Leeuwen, J. (ed.) *Handbook of Theoretical Computer Science* **B** Chapter 6, North-Holland.

Dougherty, D. (1991) Adding algebraic rewriting to the untyped lambda calculus. In: Proc. of the 4th Int. Conf. on Rewriting Techniques and Applications. *Springer-Verlag Lecture Notes in Computer Science* **488**.

Dowek, G. (1999) La part du calcul, Mémoire d'habilitation.

Dowek, G. and Werner, B. (1998) Proof normalization modulo. In: Proc. of the Int. Work. on Types for Proofs and Programs. *Springer-Verlag Lecture Notes in Computer Science* **1657**.

Dowek, G. and Werner, B. (2000) An inconsistent theory modulo defined by a confluent and terminating rewrite system.

Dowek, G., Hardin, T. and Kirchner, C. (1998) Theorem proving modulo. Technical Report 3400, INRIA Rocquencourt, France.

Dowek, G., Hardin, T. and Kirchner, C. (2001) HOL-$\lambda\sigma$: an intentional first-order expression of higher-order logic. *Mathematical Structures in Computer Science* **11** 21–45.

Drosten, K. (1989) *Termersetzungssysteme*, Ph.D. thesis, Universität Passau, Germany.

Eker, S. (1996) Fast matching in combinations of regular equational theories. In: Proceedings of the 1st International Workshop on Rewriting Logic and Applications. *Electronic Notes in Theoretical Computer Science* **4**.

Fernández, M. (1993) *Modèles de calculs multiparadigmes fondés sur la réécriture*, Ph.D. thesis, Université Paris XI, Orsay, France.

Gallier, J. (1990) On Girard's "Candidats de Réductibilité". In: Odifreddi, P.-G. (ed.) *Logic and Computer Science*, North-Holland.

Geuvers, H. (1994) A short and flexible proof of strong normalization for the Calculus of Constructions. In: Proc. of the Int. Work. on Types for Proofs and Programs. *Springer-Verlag Lecture Notes in Computer Science* **996**.

Geuvers, H. (1993) *Logics and Type Systems*, Ph.D. thesis, Katholiecke Universiteit Nijmegen, The Netherlands.

Geuvers, H. and Nederhof, M.-J. (1991) A modular proof of strong normalization for the Calculus of Constructions. *Journal of Functional Programming* **1** (2) 155–189.

Geuvers, H., Nederpelt, R. and de Vrijer, R. (eds.) (1994) Selected Papers on Automath. *Studies in Logic and the Foundations of Mathematics* **133**, North-Holland.

Giménez, E. (1996) *Un Calcul de Constructions infinies et son application à la vérification de systèmes communiquants*, Ph.D. thesis, ENS Lyon, France.

Giménez, E. (1998) Structural recursive definitions in type theory. In: Proc. of the 25th Int. Colloq. on Automata, Languages and Programming. *Springer-Verlag Lecture Notes in Computer Science* **1443**.

Girard, J.-Y. (1971) Une extension de l'interprétation de Gödel à l'analyse et son application à l'élimination des coupures dans l'analyse et la théorie des types. In: Fenstad, J. (ed.) Proc. of the 2nd Scandinavian Logic Symposium. *Studies in Logic and the Foundations of Mathematics* **63**, North-Holland.

Girard, J.-Y. (1972) *Interprétation fonctionelle et élimination des coupures dans l'arithmetique d'ordre supérieur*, Ph.D. thesis, Université Paris VII, France.

Girard, J.-Y., Lafont, Y. and Taylor, P. (1988) *Proofs and Types*, Cambridge University Press.

Grégoire, B. and Leroy, X. (2002) A compiled implementation of strong reduction. In: *Proceedings of the 7th ACM SIGPLAN International Conference on Functional Programming*.

Guttag, J. and Horning, J. (1978) The algebraic specification of abstract data types. *Acta Informatica* **10** (1) 27–52.

Harper, R. and Mitchell, J. (1999) Parametricity and variants of Girard's J operator. *Information Processing Letters* **70** 1–5.

Hsiang, J. (1982) *Topics in Automated Theorem Proving and Program Generation*, Ph.D. thesis, University of Illinois, Urbana-Champaign, United States.

Jouannaud, J.-P. and Kounalis, E. (1986) Proof by induction in equational theories without constructors. In: *Proc. of the 1st IEEE Symp. on Logic in Computer Science*.

Jouannaud, J.-P. and Okada, M. (1991) Executable higher-order algebraic specification languages. In: *Proc. of the 6th IEEE Symp. on Logic in Computer Science*.

Jouannaud, J.-P. and Okada, M. (1997) Abstract Data Type Systems. *Theoretical Computer Science* **173** (2) 349–391.

Jouannaud, J.-P. and Rubio, A. (1999) The Higher-Order Recursive Path Ordering. In: *Proc. of the 14th IEEE Symp. on Logic in Computer Science*.

Kirchner, H. and Moreau, P.-E. (2001) Promoting rewriting to a programming language: A compiler for non-deterministic rewrite programs in associative-commutative theories. *Journal of Functional Programming* **11** (2) 207–251.

Klop, J.-W., van Oostrom, V. and van Raamsdonk, F. (1993) Combinatory reduction systems: introduction and survey. *Theoretical Computer Science* **121** 279–308.

Kounalis, E. (1985) Completeness in data type specifications. In: Proc. of the European Conf. on Computer Algebra. *Springer-Verlag Lecture Notes in Computer Science* **204**.

Luo, Z. and Pollack, R. (1992) *LEGO Proof Development System: User's manual*, University of Edinburgh, United Kingdom. (Available at `http://www.dcs.ed.ac.uk/home/lego/`.)

Martin-Löf, P. (1984) *Intuitionistic type theory*, Bibliopolis, Napoli, Italy.

Matthes, R. (2000) Lambda calculus: A case for inductive definitions.

Mendler, N. P. (1987) *Inductive Definition in Type Theory*, Ph.D. thesis, Cornell University, United States.

Müller, F. (1992) Confluence of the lambda calculus with left-linear algebraic rewriting. *Information Processing Letters* **41** (6) 293–299.

Nederpelt, R. (1973) *Strong normalization in a typed lambda calculus with lambda structured types*, Ph.D. thesis, Technische Universiteit Eindhoven, The Netherlands.

Nipkow, T. (1991) Higher-order critical pairs. In: *Proc. of the 6th IEEE Symp. on Logic in Computer Science*.

Okada, M. (1989) Strong normalizability for the combined system of the typed lambda calculus and an arbitrary convergent term rewrite system. In: *Proc. of the 1989 Int. Symp. on Symbolic and Algebraic Computation*, ACM Press.

Paulin-Mohring, C. (1989) Extracting Fω's programs from proofs in the Calculus of Constructions. In: *Proc. of the 16th ACM Symp. on Principles of Programming Languages*.

Peterson, G. and Stickel, M. (1981) Complete sets of reductions for some equational theories. *Journal of the ACM* **28** (2) 233–264.

Plaisted, D. A. (1978) A recursively defined ordering for proving termination of term rewriting systems. Technical report, University of Illinois, Urbana-Champaign, United States.

Poll, E. and Severi, P. (1994) Pure Types Systems with definitions. In: Proc. of the 3rd Int. Symp. on Logical Foundations of Computer Science. *Springer-Verlag Lecture Notes in Computer Science* **813**.

Reynolds, J. (1983) Types, abstraction and parametric poymorphism. In: *Proc. of the 9th IFIP World Computer Congress*, North-Holland.

Rusinowitch, M. (1987) On termination of the direct sum of term-rewriting systems. *Information Processing Letters*.

Sellink, M. P. A. (1993) Verifying process algebra proofs in type theory. In: *Proc. of the Int. Work. on Semantics of Specification Languages*, Workshops in Computing.

Stefanova, M. (1998) *Properties of Typing Systems*, Ph.D. thesis, Katholiecke Universiteit Nijmegen, The Netherlands.

Tait, W. W. (1967) Intensional interpretations of functionals of finite type I. *Journal of Symbolic Logic* **32** (2) 198–212.

Thiel, J.-J. (1984) Stop losing sleep over incomplete specifications. In: *Proc. of the 11th ACM Symp. on Principles of Programming Languages*.

Toyama, Y. (1987) Counterexamples to termination for the direct sum of term rewriting systems. *Information Processing Letters* **25** (3) 141–143.

Van de Pol, J. (1993) Termination proofs for higher-order rewrite systems. In: Proc. of the 1st Int. Work. on Higher-Order Algebra, Logic and Term Rewriting. *Springer-Verlag Lecture Notes in Computer Science* **816**.

Van de Pol, J. (1996) *Termination of higher-order rewrite systems*, Ph.D. thesis, Utrecht Universiteit, Nederlands.

van de Pol, J. and Schwichtenberg, H. (1995) Strict functionals for termination proofs. In: Proc. of the 2nd Int. Conf. on Typed Lambda Calculi and Applications. *Springer-Verlag Lecture Notes in Computer Science* **902**.

van Oostrom, V. (1994) *Confluence for Abstract and Higher-Order Rewriting*, Ph.D. thesis, Vrije Universiteit Amsterdam, The Netherlands.

Walukiewicz, D. (2000) Termination of rewriting in the Calculus of Constructions (extended abstract). In: *Proc. of the Workshop on Logical Frameworks and Meta-languages*.

Walukiewicz-Chrząszcz, D. (2002) Termination of rewriting in the Calculus of Constructions. (To appear in *Journal of Functional Programming*.)

Werner, B. (1994) *Une Théorie des Constructions Inductives*, Ph.D. thesis, Université Paris VII, France.

Zantema, H. (1994) Termination of term rewriting: interpretation and type elimination. *Journal of Symbolic Computation* **17** (1) 23–50.