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The Linear Differential Equation Whose Solutions Are the Products of Solutions of Two Given Differential Equations*

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A linear homogeneous ODE is constructed, among whose solutions are all products of solutions of two given linear homogeneous ODE's $L_m[u]=0$, $M_n[v]=0$, in some classes. Its order is the minimum and its coefficients can be obtained by a finite number of rational operations and differentiations on the coefficients of L_m, M_n. The problem is considered (locally) both in the real and in the complex domain, around an isolated singularity. Examples are also given.

1. INTRODUCTION

The problem of constructing the linear, homogeneous, ordinary differential equation (DE) whose solutions are the products of the solutions of two given linear, homogeneous, ordinary DE's, of orders m, n, respectively, has been considered by many authors, especially in the case $m=n=2$. The aim was to derive the power series expansions for the products of some *Special Functions*, integrating by series the resulting 3rd- or 4th-order DE (see [13; 14a, pp. 144–149; 17; and 21, pp. 147–149] for Bessel functions, and [22, pp. 418–419] for Mathieu functions; [14b, pp. 129–130]). The same procedure also permits proving, as a side-product, some remarkable identities for finite sums of certain quantities [17], integral formulas (see [16] for Laguerre and Hermite polynomials), or other formulas (see [3, 15, 18] for hypergeometric functions, [20, p. 94] for confluent hypergeometric functions and [8, pp. 83–94; 12, pp. 94–103] for spherical harmonics).

Several authors also derive *in general* the 3rd- or 4th-order DE for the products of the solutions of two given 2nd-order DE's [12, 13, 16, 21], and in [16] it is emphasized that the same method could be used for the more

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general case $m = n > 2$. Finally, in [2a] a matrix approach is proposed which turns out to be convenient also for the *general* case of orders m, n.

The knowledge of such DE's "for the products" reduces their solution to that of lower order DE's.

We recall that Appell [1] is generally referred to as the first important contributor to this subject: he constructed, in particular, the DE's having as a fundamental system the set u_1^2, $u_1 u_2$, u_2^2 and the set u_1^3, $u_1^2 u_2$, $u_1 u_2^2$, u_2^3, where u_1, u_2 are two linearly independent solutions of a given 2nd-order DE.

Let

$$L_m[u] := u^{(m)} - \sum_{j=1}^{m} a_j(x)\, u^{(m-j)} = 0, \tag{1.1}$$

$$M_n[v] := v^{(n)} - \sum_{k=1}^{n} b_k(x)\, v^{(n-k)} = 0 \tag{1.2}$$

be the two given DE's, where a_j, b_k are conveniently "smooth" functions in an interval I.

The general procedure indicated in [2a] does *not always* lead to a mnth-order DE, whenever we start from two given DE's of order m, n. This indeed happens *if and only if* the Wronskian determinant of all of the mn functions obtained by multiplying m linearly independent solutions of (1.1) with n linearly independent solutions of (1.2) vanishes nowhere in I (or, at least, in a subinterval of I).

There arise, therefore, the following problems: (a) which is the *minimum* order of the DE "for the products," say, $\mathscr{L}_N[z] = 0$, when the DE's (1.1), (1.2) vary in some class? and (b) which is the *actual* order, when we consider two *fixed* DE's in that class? Moreover, (c) how can the operator $\mathscr{L}_N$ be expressed in terms of the coefficients of L_m, M_n?

In Section 2 we construct $\mathscr{L}_N$, following a method proposed in [16]. In Section 3 we obtain an estimate for the order N, while Theorem 3.2 gives an answer to question (b). In Section 4, finally, we collect some observations and applications.

2. The Construction of the Differential Equation "for the Products"

Throughout this section we assume all the smoothness properties that we need, for the coefficients a_j, b_k in (1.1), (1.2), in some interval I. Moreover, the order N of the DE "for the products" $\mathscr{L}_N[z] = 0$ is taken as *known*.

Question (c), raised at the end of Section 1, is answered in the following

THEOREM 2.1. (i) *The coefficients* $c_h(x)$ $(h = 1, 2, ..., N)$ *of* $\mathscr{L}_N[z]$ *can be obtained by a finite number of rational operations and differentiations on* $a_j(x)$, $b_k(x)$.

(ii) *They can be computed explicitly from* (2.23) *and the recurrent relations* (2.10), (2.12).

Proof. (i) Recall the following lemma (see, e.g., [2a, p. 13]):

LEMMA 2.2. *Suppose that* $\mathbf{U}$, $\mathbf{V}$ *are* $m \times m$ *and* $n \times n$ *matrix-solutions of*

$$\begin{aligned} \mathbf{U}' &= \mathbf{A}(x)\,\mathbf{U}, \qquad \mathbf{U}(x_0) = \mathbf{I}_m, \\ \mathbf{V}' &= \mathbf{B}(x)\,\mathbf{V}, \qquad \mathbf{V}(x_0) = \mathbf{I}_n, \end{aligned} \tag{2.1}$$

respectively, where $\mathbf{A}(x)$ *and* $\mathbf{B}(x)$ *are two given* $m \times m$ *and* $n \times n$ *integrable matrix-valued functions on an interval* I, $x_0 \in I$. $\mathbf{I}_m$ *denotes the* $m \times m$ *identity matrix.*

Then the Cauchy problem

$$\mathbf{Z}' = \mathbf{A}\mathbf{Z} + \mathbf{Z}\mathbf{B}^T, \qquad \mathbf{Z}(x_0) = \mathbf{C}, \tag{2.2}$$

where $\mathbf{Z}$, $\mathbf{C}$ *are* $m \times n$ *matrices,* $\mathbf{C} = \text{const.}$ *has the unique solution*

$$\mathbf{Z} = \mathbf{U}\mathbf{C}\mathbf{V}^T. \tag{2.3}$$

Equation (2.2) is equivalent to *mn scalar* DE's and can be written in *vectorial* form, as follows.

If $\mathscr{S}$ is the linear operator which takes every $m \times n$ matrix $\mathbf{Z}$ into the column-vector $\zeta := \mathscr{S}(\mathbf{Z}) = [Z_1^T Z_2^T \cdots Z_m^T]^T$, where Z_j denotes the jth *row* in $\mathbf{Z}$, then

$$\zeta' = \mathbf{M}(x)\,\zeta, \qquad \zeta(x_0) = \zeta_0, \tag{2.4}$$

where $\mathbf{M} = \mathbf{A} \otimes \mathbf{I}_n + \mathbf{I}_m \otimes \mathbf{B}$, $\zeta_0 = \mathscr{S}(\mathbf{C}^T)$.

Here T denotes the transpose and $\otimes$ the Kronecker (or tensor) product of matrices (see, e.g., [2b, p. 238; 11, p. 8]). $\mathbf{M}$ is a $mn \times mn$ matrix.

Now, if $\mathbf{Z} = \{z_{jk}\}_{j=1,2,\ldots,m;\, k=1,2,\ldots,n}$, $\mathbf{U} = \{u_{jk}\}$, $\mathbf{V} = \{v_{jk}\}$, $\mathbf{C} = \{c_{jk}\}$, $\zeta^T = [\zeta_1 \zeta_2 \cdots \zeta_{mn}]$, then:

$$\zeta_1 = z_{11} = \sum_{r=1}^{m} \sum_{s=1}^{n} c_{rs} u_{1r} v_{1s}. \tag{2.5}$$

Suppose that the systems in (2.1) are equivalent to the scalar DE's in (1.1), (1.2), respectively. Let $\mathbf{A}$, $\mathbf{B}$ be, for instance, the *companion* matrices of the characteristic polynomials $\lambda^m - \sum_{j=1}^{m} a_j(x)\,\lambda^{m-j}$,

$\lambda^n - \sum_{k=1}^{n} b_k(x)\,\lambda^{n-k}$, (see, e.g., [11, p. 52]). Then $u_{1r} \equiv u_r$ $(r = 1, 2, ..., m)$, $v_{1s} \equiv v_s$ $(s = 1, 2, ..., n)$, are fundamental systems of solutions of (1.1), (1.2) and (2.5) represents the (generic) linear combination of all their products.

On the other hand, it is well known that *every component* of the vector ζ in (2.4) satisfies a linear homogeneous ODE, whose order is *at most mn*. Such an equation is obtained by an elimination procedure, and this requires only a finite number of rational operations and differentiations on the entries of $\mathbf{M}$ and therefore on a_j, b_k. Thus (i) is proved.

Remark 2.3. This conclusion can be reached, for the case $L_m \equiv M_n$, i.e., for the DE "for the squares," as an application of Appell's Theorem (see [1, p. 212]).

(ii) The method we follow to construct explicitly the DE for ζ_1 is suggested by Palamà in [16, Section 1].

We write $z = uv$ in the most general linear homogeneous ODE of order $p := mn$,

$$\mathscr{L}_p[z] := c_0(x)\,z^{(p)} - \sum_{h=1}^{p} c_h(x)\,z^{(p-h)} = 0 \qquad (p := mn), \tag{2.6}$$

and express the derivatives $u^{(m+r)}$ for $r = 0, 1, 2, ..., p-m$, and $v^{(n+s)}$ for $s = 0, 1, 2, ..., p-n$, in terms of $u^{(m-j)}$ $(j = 1, 2, ..., m)$ and $v^{(n-k)}$ $(k = 1, 2, ..., n)$, respectively, by using (1.1), (1.2).

Setting

$$u^{(m+r)} := \sum_{j=1}^{m} a_j^{[r]}(x)\,u^{(m-j)} \qquad (r = 0, 1, 2, ..., p-m), \tag{2.7}$$

$$v^{(n+s)} := \sum_{k=1}^{n} b_k^{[s]}(x)\,v^{(n-k)} \qquad (s = 0, 1, 2, ..., p-n), \tag{2.8}$$

where

$$\begin{aligned} a_j^{[0]}(x) &:= a_j(x) \qquad (j = 1, 2, ..., m), \\ b_k^{[0]}(x) &:= b_k(x) \qquad (k = 1, 2, ..., n), \end{aligned} \tag{2.9}$$

and, differentiating in (2.7), we obtain

$$\begin{aligned} u^{(m+r+1)} &= \sum_{j=1}^{m} a_j^{[r]} \cdot u^{(m+1-j)} + \sum_{j=1}^{m} a_j^{[r]\prime} \cdot u^{(m-j)} \\ &= \sum_{j=1}^{m-1} \left(a_j a_1^{[r]} + a_j^{[r]\prime} + a_{j+1}^{[r]}\right) \cdot u^{(m-j)} \\ &\quad + \left(a_m a_1^{[r]} + a_m^{[r]\prime}\right) \cdot u \\ &=: \sum_{j=1}^{m} a_j^{[r+1]}(x) \cdot u^{(m-j)}. \end{aligned}$$

Therefore we have the *recurrent relations* (in r, for each fixed j)

$$a_j^{[r+1]} = \begin{cases} a_j a_1^{[r]} + a_j^{[r]\prime} + a_{j+1}^{[r]} & (j = 1, 2,\dots, m-1), \\ a_m a_1^{[r]} + a_m^{[r]\prime} & (j = m), \end{cases} \quad (r = 0, 1, 2,\dots, p-m-1), \tag{2.10}$$

with the *initial values*

$$a_j^{[0]} = a_j \qquad (j = 1, 2,\dots, m). \tag{2.11}$$

Similarly we have for the $b_k^{[s]}$'s:

$$b_k^{[s+1]} = \begin{cases} b_k b_1^{[s]} + b_k^{[s]\prime} + b_{k+1}^{[s]} & (k = 1, 2,\dots, n-1), \\ b_n b_1^{[s]} + b_n^{[s]\prime} & (k = n), \end{cases} \quad (s = 0, 1, 2,\dots, p-n-1), \tag{2.12}$$

$$b_k^{[0]} = b_k \qquad [k = 1, 2,\dots, n). \tag{2.13}$$

Now we proceed to substitute (2.7), (2.8) in (2.6) with $z = uv$. Setting

$$\alpha_j^{[r]}(x) := \begin{cases} \delta_{j,m-r}, & r = 0, 1,\dots, m-1, \\ a_j^{[r-m]}(x), & r = m, m+1,\dots, p, \end{cases} \quad (p := mn), \tag{2.14}$$

where $\delta_{j,k}$ is the Kronecker delta-symbol, and similarly

$$\beta_k^{[s]}(x) := \begin{cases} \delta_{k,n-s}, & s = 0, 1,\dots, n-1, \\ b_k^{[s-n]}(x), & s = n, n+1,\dots, p, \end{cases} \tag{2.15}$$

we obtain

$$\begin{aligned} \mathscr{L}_p[uv] &= c_0(x) \sum_{r=0}^{p} \binom{p}{r} u^{(r)} v^{(p-r)} \\ &\quad - \sum_{h=1}^{p} c_h(x) \sum_{r=0}^{p-h} \binom{p-h}{r} u^{(r)} v^{(p-h-r)} \\ &= c_0(x) \sum_{r=0}^{p} \binom{p}{r} \sum_{j=1}^{m} \sum_{k=1}^{n} \alpha_j^{[r]}(x)\, \beta_k^{[p-r]}(x)\, u^{(m-j)} v^{(n-k)} \\ &\quad - \sum_{h=1}^{p} c_h(x) \sum_{r=0}^{p-h} \binom{p-h}{r} \sum_{j=1}^{m} \sum_{k=1}^{n} \alpha_j^{[r]}(x)\, \beta_k^{[p-h-r]}(x)\, u^{(m-j)} v^{(n-k)} \\ &= 0. \end{aligned} \tag{2.16}$$

Considering (2.16) as an *identity* with respect to $u^{(m-j)}v^{(n-k)}$ ($j=1, 2,..., m$; $k=1, 2,..., n$), we determine the $c_h(x)$'s as rational functions of $\alpha_j^{[r]}$ and $\beta_k^{[s]}$ and therefore of a_j, b_k and their derivatives up to some orders (cf. part (i)):

$$\sum_{h=1}^{p}\left[\sum_{r=0}^{p-h}\binom{p-h}{r}\alpha_j^{[r]}(x)\,\beta_k^{[p-h-r]}(x)\right]c_h(x)$$
$$=c_0(x)\sum_{r=0}^{p}\binom{p}{r}\alpha_j^{[r]}(x)\,\beta_k^{[p-r]}(x)$$
$$(j=1, 2,..., m;\ k=1, 2,..., n). \tag{2.17}$$

For each $x\in I$ this is a system of p linear homogeneous equations for the $p+1$ unknowns $c_h(x)$ ($h=0, 1, 2,..., p$).

Writing, for brevity:

$$\begin{aligned}H_{l,h}(x) &:= \sum_{r=0}^{p-h}\binom{p-h}{r}\alpha_j^{[r]}(x)\cdot\beta_k^{[p-h-r]}(x),\\ \Gamma_l(x) &:= \sum_{r=0}^{p}\binom{p}{r}\alpha_j^{[r]}(x)\,\beta_k^{[p-r]}(x) \qquad (l, h=1, 2,..., p),\end{aligned} \tag{2,18}$$

where we mean that the pairs (j, k) are ordered, in an arbitrary way, in a linear array (l), e.g., $l=(k-1)\,m+j$, and

$$\begin{aligned}\mathbf{H}(x) := \{H_{l,h}(x)\}, \qquad \Gamma(x) &:= [\Gamma_1(x)\,\Gamma_2(x)\cdots\Gamma_p(x)]^T,\\ c(x) &:= [c_1(x)\,c_2(x)\cdots c_p(x)]^T,\end{aligned} \tag{2.19}$$

we have

$$\mathbf{H}(x)\,c(x)=c_0(x)\,\Gamma(x). \tag{2.20}$$

If there exists $x_0\in I$ such that $\det\mathbf{H}(x_0)\neq 0$, then $\det\mathbf{H}(x)\neq 0$ in some interval $J(x_0)\subseteq I$ (by continuity) and (2.20) can be solved for $c(x)$, $x\in J(x_0)$, setting $c_0(x)\equiv 1$. In this case we obtain a DE of maximal order $p=mn$. If $\det\mathbf{H}(x)\equiv 0$ in I, we set $c_0(x)\equiv 0$ and consider all minors of order $p-1$ in the $p\times(p-1)$ matrix $\mathbf{K}^{(1)} := [H_2(x)\,H_3(x)\cdots H_p(x)]$, where $H_r(x)$ denotes the rth column in $\mathbf{H}(x)$. If at least one of them does not vanish identically in I, we consider the corresponding $(p-1)\times(p-1)$ matrix in $\mathbf{K}^{(1)}$, say, $\mathbf{H}^{(1)}$. Suppose that this has been obtained by striking out the sth row in $\mathbf{K}^{(1)}$. Consider the system

$$\mathbf{H}^{(1)}(x)\,c^{(1)}(x)=-c_1(x)\,\Gamma^{(1)}(x), \tag{2.21}$$

for $c^{(1)} := [c_2c_3\cdots c_p]^T$, where $\Gamma^{(1)}$ denotes the $(p-1)$-column vector obtained from H_1 by cancellation of the sth element.

Then we set $c_1(x) \equiv 1$ and solve (2.21) for $c^{(1)}$. The DE obtained in this case will have order $mn-1$.

If all the minors above vanish identically in I, we proceed in a similar way, setting $c_1(x) \equiv 0$ in (2.21) and considering all minors of order $p-2$ in the $p \times (p-2)$ matrix $K^{(2)} := [H_3(x)\, H_4(x) \cdots H_p(x)]$, etc.

In general, we will obtain some system

$$\mathbf{H}^{(q)}(x)\, c^{(q)}(x) = -c_q(x)\, \Gamma^{(q)}(x), \tag{2.22}$$

where $\mathbf{H}^{(q)}$ is a $(p-q) \times (p-q)$ submatrix of $\mathbf{K}^{(q)}$, $\mathbf{K}^{(q)}$ being the $p \times (p-q)$ matrix obtained from $\mathbf{H}(x)$ by striking out its first q columns; $c^{(q)} := [c_{q+1} c_{q+2} \cdots c_p]^T$ and $\Gamma^{(q)}$ denotes the qth column in $\mathbf{H}(x)$, after cancellation of the entries cancelled in $\mathbf{K}^{(q)}$ to obtain $\mathbf{H}^{(q)}$.

Suppose one knows that the DE to be constructed has order N ($N \leqslant mn$). Then there exists q such that $\det \mathbf{H}^{(q)}(x) \neq 0$ in some $J \subseteq I$. Setting $c_q(x) \equiv 1$ in (2.22), we can solve it for $c^{(q)}(x)$, $x \in J$, obtaining (as $q = p - N$)

$$c_h(x) = -\frac{\det \mathbf{H}_h^{(p-N)}(x)}{\det \mathbf{H}^{(p-N)}(x)} \qquad (h = p-N+1, p-N+2, ..., p), \tag{2.23}$$

where $\mathbf{H}_h^{(p-N)}(x)$ denotes the matrix obtained from $\mathbf{H}^{(p-N)}$ by replacing its hth column by $\Gamma^{(p-N)}$. ∎

EXAMPLE 2.4. In order to illustrate the procedure outlined in (ii) above, we consider the case $m = n = 2$, i.e., the DE for the products of solutions of

$$u'' = a_1 u' + a_2 u, \qquad v'' = b_1 v' + b_2 v, \tag{2.24}$$

in I. We know that the order of such a DE is $N \leqslant p := mn = 4$ (cf. part (i) above). From (2.10), (2.11), we have

$$\begin{aligned}
& a_1^{[0]} = a_1, \qquad a_2^{[0]} = a_2; \qquad a_1^{[1]} = a_1^2 + a_1' + a_2, \qquad a_2^{[1]} = a_1 a_2 + a_2'; \\
& a_1^{[2]} = a_1(a_1^2 + a_1' + a_2) + (a_1^2 + a_1' + a_2)' + a_1 a_2 + a_2' \\
& \qquad = a_1^3 + 3a_1 a_1' + 2a_1 a_2 + 2a_2' + a_1'', \\
& a_2^{[2]} = a_2(a_1^2 + a_1' + a_2) + (a_1 a_2 + a_2')' \\
& \qquad = a_1^2 a_2 + a_2^2 + 2a_1' a_2 + a_1 a_2' + a_2'',
\end{aligned} \tag{2.25}$$

while the same relations hold for the $b_k^{[s]}$'s, by replacing b by a in (2.25). $\mathbf{H}(x)$, $c(x)$ and $\Gamma(x)$ in system (2.20) now become (using (2.14), (2.15))

$$\mathbf{H}(x)=\begin{bmatrix} 3(a_1+b_1) & 2 & 0 & 0 \\ 3a_2+b_1^{[1]} & b_1 & 1 & 0 \\ a_1^{[1]}+3b_2 & a_1 & 1 & 0 \\ a_2^{[1]}+b_2^{[1]} & a_2+b_2 & 0 & 1 \end{bmatrix},$$

$$c(x)=[c_1(x)\,c_2(x)\,c_3(x)\,c_4(x)]^T, \tag{2.26}$$

$$\Gamma(x)=\begin{bmatrix} 4a_1^{[1]}+6a_1b_1+4b_1^{[1]} \\ 4a_2^{[1]}+6a_2b_1+b_1^{[2]} \\ a_1^{[2]}+6a_1b_2+4b_2^{[1]} \\ a_2^{[2]}+6a_2b_2+b_2^{[2]} \end{bmatrix}.$$

Computing

$$\begin{aligned} \det \mathbf{H}(x) &= -3(a_1^2-b_1^2)-6(a_2-b_2)+2(a_1^{[1]}-b_1^{[1]}) \\ &= 2(a_1'-b_1')-(a_1^2-b_1^2)-4(a_2-b_2), \end{aligned} \tag{2.27}$$

we observe that, in particular (but not necessarily), $\det \mathbf{H}(x)\equiv 0$ when $a_1(x)\equiv b_1(x)$ *and* $a_2(x)\equiv b_2(x)$ in I, i.e., whenever the two given DE's coincide.

If $\det \mathbf{H}(x_0)\neq 0$ for some $x_0\in I$, the DE for the products will have order $N=4$ and can be constructed in some interval $J(x_0)\subseteq I$. Its coefficients $c_h(x)$ $(h=1,\dots,4)$ can be evaluated from (2.20), (2.26).

If $\det \mathbf{H}(x)\equiv 0$ in I, we set $c_0(x)\equiv 0$ in (2.20) and consider

$$\mathbf{K}^{(1)}:=\begin{bmatrix} 2 & 0 & 0 \\ b_1 & 1 & 0 \\ a_1 & 1 & 0 \\ a_2+b_2 & 0 & 1 \end{bmatrix}. \tag{2.28}$$

Note that there are nonvanishing minors of order 3 for *any* a_j, b_k. This shows that the DE for the products has order $N\geqslant 3$, when $m=n=2$.

Let us choose *any* nonvanishing 3×3 minor in (2.28). If $\mathbf{H}^{(1)}(x)$ denotes the corresponding matrix, e.g.,

$$\mathbf{H}^{(1)}=\begin{bmatrix} 2 & 0 & 0 \\ b_1 & 1 & 0 \\ a_2+b_2 & 0 & 1 \end{bmatrix}, \tag{2.29}$$

we solve the system

$$\mathbf{H}^{(1)}(x)\,c^{(1)}(x)=-c_1(x)\,\Gamma^{(1)}(x) \tag{2.30}$$

for $c^{(1)} := [c_2 c_3 c_4]^T$, being

$$\Gamma^{(1)} := \begin{bmatrix} 3(a_1 + b_1) \\ 3a_2 + b_1^{[1]} \\ a_2^{[1]} + b_2^{[1]} \end{bmatrix}. \tag{2.31}$$

As $\det \mathbf{H}^{(1)}(x) = 2$, setting $c_1(x) \equiv 1$ in (2.30), we obtain

$$\begin{cases} c_2 = -\frac{3}{2}(a_1 + b_1) \\ c_3 = \frac{1}{2}(3a_1 b_1 + b_1^2 - 6a_2 - 2b_2 - 2b_1') \\ c_4 = \frac{1}{2}[a_1 a_2 + b_1 b_2 + 3(a_1 b_2 + a_2 b_1) - 2(a_2' + b_2')]. \end{cases} \tag{2.32}$$

The apparent asymmetry in c_3 vanishes by virtue of condition $\det \mathbf{H}(x) \equiv 0$ (see (2.27)). In fact the c_h's have to be symmetric functions of a_j, b_k, in this case.

Indeed:

$$2c_3 = 3a_1 b_1 - 2(a_2 + b_2) + b_1^2 - 4a_2 - 2b_1',$$

and

$$\gamma := b_1^2 - 4a_2 - 2b_1' \equiv a_1^2 - 4b_2 - 2a_1'.$$

Therefore

$$\gamma = \frac{1}{2}[(a_1^2 + b_1^2) - 4(a_2 + b_2) - 2(a_1' + b_1')],$$

and so

$$c_3 = \frac{1}{4}[a_1^2 + b_1^2 + 6a_1 b_1 - 8(a_2 + b_2) - 2(a_1' + b_1')].$$

In particular, if $a_1(x) \equiv b_1(x) \equiv 0$, we have (cf., e.g., [2a, p. 12])

$$c_2 \equiv 0, \qquad c_3 = -2(a_2 + b_2), \qquad c_4 = -(a_2' + b_2'),$$

i.e.,

$$\mathscr{L}_3[z] = z''' - 2(a_2 + b_2)\, z' - (a_2' + b_2')\, z = 0.$$

3. An Estimate for the Order N

The answer to question (a), raised at the end of Section 1, is contained in the following

THEOREM 3.1. *Let* (1.1), (1.2) *be two linear, homogeneous, ordinary* DE's, *whose orders are* m, n, *respectively, and with real-valued coefficients*

$a_j(x)$ $(j=1,2,\ldots,m)$, $b_k(x)$ $(k=1,2,\ldots,n)$, $a_j \in C^{m(n-1)}(I)$, $b_k \in C^{n(m-1)}(I)$, I *being an open, real interval. Then a linear, homogeneous, ordinary* DE *can be constructed in some interval* $J \subseteq I$, *such that all products of the solutions of* (1.1) *and* (1.2) *are among its solutions and its order* N *is the lowest possible. Moreover, such a* DE *has the following properties*:

(i) $m+n-1 \leqslant N \leqslant mn$;

(ii) *its coefficients are in the class* $C^l(J)$, $l := mn - N$, $0 \leqslant l \leqslant mn - (m+n)+1$.

Proof. Hereafter $u_j(x)$ $(j=1,2,\ldots,m)$ and $v_k(x)$ $(k=1,2,\ldots,n)$ denote two fundamental systems for (1.1) and (1.2), respectively. Consider the $mn \times mn$ matrix $\mathbf{W}(w_1, w_2,\ldots, w_{mn})(x_0) := \{w_{\mu,\nu}(x_0)\}_{\mu,\nu=1,2,\ldots,mn} = \{w_\nu^{(\mu-1)}(x_0)\}_{\mu,\nu=1,2,\ldots,mn}$, where $x_0 \in I$ and the $w_\nu^{(0)} := w_\nu$ denote the functions $u_j v_k$ $(j=1,\ldots,m;\ k=1,2,\ldots,n)$ arranged in any order, e.g., $u_1v_1, u_1v_2,\ldots, u_1v_n;\ u_2v_1,\ldots,u_2v_n;\ldots;u_mv_1,\ldots,u_mv_n$. Therefore $\mathbf{W}(w_1, w_2,\ldots, w_{mn})(x_0)$ is the *Wronskian matrix* of the mn functions u_jv_k, at x_0. It exists because of the regularity assumed for $a_j(x)$, $b_k(x)$ in I; in its entries $u_j^{(r)}(x_0)$ for $r \geqslant m$ and $v_k^{(s)}(x_0)$ for $s \geqslant n$ are computed by using (1.1), (1.2). The corresponding Wronskian determinant will be denoted by $W(w_1, w_2,\ldots, w_{mn})$.

Suppose that the initial values $u_j^{(r-1)}(x_0)$ $(j, r=1,2,\ldots,m)$, $v_k^{(s-1)}(x_0)$ $(k, s=1,2,\ldots,n)$ are such that $\mathbf{W}(u_1, u_2,\ldots, u_m)(x_0) = \mathbf{I}_m$, $\mathbf{W}(v_1, v_2,\ldots, v_n)(x_0) = \mathbf{I}_n$, where $\mathbf{I}_p$ denotes the $p \times p$ identity matrix.

This choice does not entail loss of generality, as changing fundamental systems in (1.1), (1.2) amounts to multiplying the Wronskian determinants by some nonzero constants. In fact, if the Wronskians of $u_1, u_2,\ldots, u_m$ and $v_1, v_2,\ldots, v_n$ are $W(u_1, u_2,\ldots, u_m)$ and $W(v_1, v_2,\ldots, v_n)$, respectively, then the Wronskians of $\tilde{u}_1, \tilde{u}_2,\ldots, \tilde{u}_m$ and $\tilde{v}_1, \tilde{v}_2,\ldots, \tilde{v}_n$, being $\tilde{u} = \mathbf{C}u$, $\tilde{v} = \mathbf{D}v$, $u := [u_1u_2 \cdots u_m]^T$, $v := [v_1v_2 \cdots v_n]^T$, $\tilde{u} := [\tilde{u}_1\tilde{u}_2 \cdots \tilde{u}_m]^T$, $\tilde{v} := [\tilde{v}_1\tilde{v}_2 \cdots \tilde{v}_n]^T$, where $\mathbf{C}$, $\mathbf{D}$ are $m \times m$ and $n \times n$ constant matrices, with $\det \mathbf{C} \neq 0$, $\det \mathbf{D} \neq 0$, are $(\det \mathbf{C})\ W(u_1, u_2,\ldots, u_m)$ and $(\det \mathbf{D})\ W(v_1, v_2,\ldots, v_n)$.

Now, this turns out to be *a particular* transformation of the fundamental system in the DE for the products. It is easy to verify that $W(w_1, w_2,\ldots, w_{mn})$ is taken into $(\det(\mathbf{C} \otimes \mathbf{D}))\ W(w_1, w_2,\ldots, w_{mn}) = (\det \mathbf{C})^n (\det \mathbf{D})^m\ W(w_1, w_2,\ldots, w_{mn})$, where $\mathbf{C} \otimes \mathbf{D}$ is the Kronecker product of $\mathbf{C}$ and $\mathbf{D}$ (see, e.g., [11, pp. 8–9, 14]).[1]

Investigating the structure of the matrix $\mathbf{W}(w_1, w_2,\ldots, w_{mn})(x_0)$, we see that $(u_1v_k)^{(\mu-1)}(x_0) = \delta_{k,\mu}$ $(k, \mu = 1,2,\ldots,n)$, where $\delta_{k,\mu}$ is the Kronecker delta-

[1] By this observation we can choose x_0 in the interval where the construction outlined in Section 2(ii) can be done and still assume $\mathbf{W}(u_1, u_2,\ldots, u_m)(x_0) = \mathbf{I}_m$, $W(v_1, v_2,\ldots, v_n)(x_0) = \mathbf{I}_n$.

symbol and we have assumed $n \leqslant m$. Moreover, $(u_j v_n)^{(\mu-1)}(x_0) = 0$ ($j = 2, 3, \dots, m$; $\mu = 1, 2, \dots, n+j-2$), while $(u_j v_n)^{(n+j-2)}(x_0) \neq 0$. In fact, in

$$(u_j v_n)^{(k)}(x_0) = \sum_{h=0}^{k} \binom{k}{h} u_j^{(h)}(x_0)\, v_n^{(k-h)}(x_0) \qquad (j = 2, 3, \dots, m),$$

we have $u_j^{(h)}(x_0) = 0$ for $h = 0, 1, \dots, j-2$, and therefore $(u_j v_n)^{(k)}(x_0) = 0$ for $j = 2, 3, \dots, m$ and $k \leqslant j-2$.

For $k \geqslant j-1$, $u_j^{(h)}(x_0)$ *can* be different from zero, but $(u_j v_n)^{(k)}(x_0)$ vanishes again, provided that the indices $k-h$ corresponding to $h \geqslant j-1$, i.e., $k-h = 0, 1, \dots, k-j, k-j+1$ are such that $k-h \leqslant k-j+1 \leqslant n-2$: in fact in this case we have $v_n^{(k-h)}(x_0) = 0$. This happens whenever $k \leqslant n+j-3$, while for $k = n+j-2$ we obtain

$$(u_j v_n)^{(n+j-2)}(x_0) = \binom{n+j-2}{j-1} = \frac{(n+j-2)!}{(j-1)!\,(n-1)!} \neq 0.$$

Therefore we can single out *a nonzero minor of order* $m+n-1$: it is formed by the $n \times n$ principal minor placed at the upper left-hand corner, by the $m-1$ columns $(u_j v_n)^{(\mu-1)}(x_0)$ ($j = 2, 3, \dots, m$; $\mu = 1, 2, \dots, n+j-2$), and completed in an obvious way.

This shows that there exists a Wronskian matrix, submatrix of $\mathbf{W}(w_1, w_2, \dots, w_{mn})$, of order $N \geqslant m+n-1$, whose determinant does not vanish at x_0. Let $\mathbf{W}(z_1, \dots, z_N)(x_0)$ be this matrix, being $(z_1, z_2, \dots, z_N)$ some subset of $(w_1, w_2, \dots, w_{mn})$. *By continuity* there exists an interval $J(x_0) \subseteq I$ where $W(z_1(x), z_2(x), \dots, z_N(x)) := \det(\mathbf{W}(z_1, z_2, \dots, z_N)(x)) \neq 0$. Therefore (see, e.g., [4, pp. 83–84]) there exists a unique normalized DE, with *continuous* coefficients, having the $z_j(x)$'s as a fundamental system:

$$\begin{aligned} \mathscr{L}_N(z) &:= \frac{W(z_1(x), z_2(x), \dots, z_N(x), z)}{W(z_1(x), z_2(x), \dots, z_N(x))} \\ &= z^{(N)} - \sum_{h=1}^{N} c_h(x)\, z^{(N-h)} = 0 \qquad (\text{in } J). \end{aligned} \tag{3.1}$$

Here $c_h = W_h / W$, W_h being obtained from W by replacing $z_j^{(N-h)}$ by $z_j^{(N)}$. Therefore the lower estimate in (i) holds.

Equations $u^{(m)} = 0$, $v^{(n)} = 0$ provide an example in which the minimum order occurs.

This estimate is "the best possible" and when $N = m+n-1$ the set $u_1 v_1, u_1 v_2, \dots, u_1 v_n$; $u_2 v_n$; $u_3 v_n$;...; $u_m v_n$ above can be taken as fundamental.

Observe finally that (3.1) contains the z_j's ($j = 1, 2, \dots, N$) and their derivatives up to the Nth order, and therefore it involves (because of (1.1), (1.2)) polynomial combinations of the a_j's and their derivatives up to the

$(N-m)$th order, as well as the b_k's and their derivatives up to the $(N-n)$th. It follows from the assumptions in Theorem 3.1 that $a_j^{(N-m)}, b_k^{(N-n)} \in C^l(I)$, where $l := mn - N$, and therefore the coefficients in (3.1) belong to $C^l(J)$ and $0 \leqslant l \leqslant mn - (m+n) + 1$. Thus (ii) is also proved. ∎

From the proof of Theorem 3.1 it is clear that N is the maximum order of the nonvanishing minors of $\mathbf{W}(w_1, w_2,..., w_{mn})(x_0)$ which are Wronskians of *some* of the functions $w_1, w_2,..., w_{mn}$. In other words, we have considered all the *principal* minors in $\mathbf{W}(w_1, w_2,..., w_{mn})$ and in every other Wronskian matrix obtained from it by permuting the functions $w_1, w_2,..., w_{mn}$ in all possible ways.

It results that

$$N \leqslant \operatorname{rank} \mathbf{W}(w_1, w_2,..., w_{mn})(x_0). \tag{3.2}$$

However, we can go further:

THEOREM 3.2. *Under the hypotheses in Theorem* 3.1, for two given *DE's the* actual *order is determined as*

$$N = \operatorname{rank} \mathbf{W}(w_1, w_2,..., w_{mn})(x), \tag{3.3}$$

where x is an arbitrary point in the interval J in Theorem 3.1.

Proof. For any set of n functions $u_1, u_2,..., u_n$, *differentiable q times* in the interval J, $q \geqslant n-1$, define

$$\mathbf{M}_q(u_1, u_2,..., u_n) := \{u_{j,k}\}_{\substack{j=1,2,...,q+1\\ k=1,2,...,n}} = \{u_k^{(j-1)}\}_{\substack{j=1,2,...,q+1\\ k=1,2,...,n}}.$$

Thus $\mathbf{M}_{n-1}(u_1, u_2,..., u_n) = \mathbf{W}(u_1, u_2,..., u_n)$.

A theorem by D. R. Curtiss [7, Theorem VII, p. 293] states that, if $W(u_1, u_2,..., u_n)(x) = \det \mathbf{M}_{n-1}(u_1, u_2,..., u_n)(x) \equiv 0$ in J, then *every n*-rowed determinant of $\mathbf{M}_q(u_1, u_2,..., u_n)(x)$ also vanishes identically in J.

From this follows (3.3). In fact, suppose that there exists a submatrix $\mathbf{M}$ of $\mathbf{W}(w_1, w_2,..., w_{mn})(x)$, of order $r > N$, whose determinant does *not* vanish identically in J. Considering the matrix $\mathbf{M}_{mn-1}(w_{j_1}, w_{j_2},..., w_{j_r})(x)$, where $j_1, j_2,..., j_r$ are the column numbers of $\mathbf{M}$, $W(w_{j_1}, w_{j_2},..., w_{j_r})(x)$ *cannot* be identically zero in J (by Curtiss' theorem) and therefore $W(w_{j_1}, w_{j_2},..., w_{j_r})(x_1) \neq 0$ for some $x_1 \in J$ and hence in some interval $J_1 \subseteq J$, by continuity. This contradicts the assumption that N is the maximum order of the Wronskians nonvanishing in J. ∎

In particular, the matrix $\mathbf{W}(w_1, w_2,..., w_{mn})(x)$ has *constant rank* in J.

Some observations are now in order.

(A) The lower estimate in (i), $N \geqslant m+n-1$, can also be obtained by a theorem due to I. Connell [6] and concerned with *generic* vector spaces. Assume $k=\mathbf{C}$ and $K=\mathbf{C}(u_1,..., u_m, v_1,..., v_n)$, in his notations. Here we mean that $u_1,..., u_m$ and $v_1,..., v_n$ are $\mathbf{C}$-bases for the vector spaces spanned by the solutions of (1.1), (1.2), respectively.

The field K above, i.e., the *quotient field* of the rational functions in the indeterminates $u_1,..., u_m, v_1,..., v_n$ with coefficients in $\mathbf{C}$, satisfies all hypotheses required by Connell's theorem: in particular k is algebraically closed in K, with these choices (see, e.g., [5, pp. 214, 216; 10, pp. 163–164]). However, Theorem 3.1 gives more information in the case considered here, since analytical results, in addition to those purely algebraic, are derived. Moreover, the *actual* dimension N is also computed (Theorem 3.2): this answers question (b) at the end of Section 1.

(B) The regularity hypotheses on a_j, b_k in Theorem 3.1 can be somewhat relaxed, as shown in the following

COROLLARY 3.3. *Conditions* $a_j \in C^{m(n-1)}(I)$, $b_k \in C^{n(m-1)}(I)$ *in Theorem* 3.1 *can be replaced by the weaker* $a_j \in C^{N-m}(I)$, $b_k \in C^{N-n}(I)$. *In this case we obtain* $l=0$ *in Theorem* 3.1(ii).

In fact, above we wrote *the whole* $mn \times mn$ Wronskian matrix, while only a $N \times N$ particular submatrix is needed; however, we did not know this N from the beginning.

Consider, for $q=2, 3,..., mn$, all the subsets of q functions among the mn $u_j v_k$'s ($j=1, 2,..., m$; $k=1, 2,..., n$). We will find a number $q=N$ and N functions (if $N<mn$) such that their Wronskian determinant is different from zero at x_0, while those relative to *any* $N+1$ functions obtained adding to them *any* of the remaining functions vanish. Thus we have determined N and we can form the DE by simple insertion of these N functions for the z_j's in (3.1). This involves only the derivatives of a_j and b_k up to the $(N-m)$th and $(N-n)$th orders, respectively, and therefore we can replace the regularity hypotheses in Theorem 3.1 by $a_j \in C^{N-m}(I)$, $b_k \in C^{N-n}(I)$. In (ii) we have $l=0$, in this case. ∎

We can assert further

COROLLARY 3.4. *Suppose that the regularity conditions on* a_j, b_k *are reduced to* $a_j^{(N-m-1)}$, $b_k^{(N-n-1)} \in AC(I^*)$, *where* $I^* \subseteq I$ *is an arbitrary compact interval. Then Theorem* 3.1 *holds, by interpreting the DE for the products in the sense of Carathéodory.*

In fact, in this case the proof above holds, but the coefficients of (3.1) are just in the class $L^1(I^*)$, as $a_j^{(N-m)}$, $b_k^{(N-n)}$ exist a.e. in I^* and belong to

$L^1(I^*)$, while the other $a_j^{(r)}$, $b_k^{(s)}$ are at least in $C^0(I^*)$. We have to interpret this DE "in the sense of Carathéodory" (see, e.g., [4, pp. 42–43]). ∎

COROLLARY 3.5. *Denote by* c_h $(h = 1, 2, ..., N)$ *the coefficients in the DE* (3.1). *If* $a_j, b_k \in C^\infty(I)$, *then* $c_h \in C^\infty(J)$; *if* $a_j, b_k \in H(\Omega)$, *the set of the holomorphic functions in the open simply connected set* $\Omega \subseteq \mathbf{C}$, Ω *replacing* I *in Theorem* 3.1, *then* $c_h \in H(\Omega^*)$, *for some* $\Omega^* \subseteq \Omega$.

COROLLARY 3.6. *Theorem* 3.1 *holds also for analytic DE's, with* $a_j, b_k \in H(\Omega \setminus \{a\})$, *where* Ω *is a disk with center* a, *and* a *is a regular* (*or fuchsian*) *singularity.*[2] *The DE for the products also has a fuchsian singularity there.*

In fact, the result first proved in a neighborhood of x_0, $x_0 \neq a$, contained in $\Omega \setminus \{a\}$, can be extended in a whole annulus around a, by analytic continuation.

Moreover, recall that a *necessary and sufficient* condition in order that a given DE has at $x = a$ (at most) a fuchsian singularity is that no solution becomes unbounded faster than some negative power of x, as $x \to a$ [4, pp. 124–125]. As products of such functions also become unbounded no faster than some negative power, we conclude that the DE for the products also has a fuchsian singularity at $x = a$. ∎

The construction of Section 2 can also be carried out.

(C) Consider, near (1.1), (1.2), two other DE's of the same orders m, n, respectively, with $a_j(x)$, $b_k(x)$ replaced by

$$a_j^*(x) := \sum_{r=0}^{N'-1} \frac{a_j^{(r)}(x_0)}{r!} (x - x_0)^r \qquad (j = 1, 2, ..., m; N' = N - m),$$

$$b_k^*(x) := \sum_{s=0}^{N''-1} \frac{b_k^{(s)}(x_0)}{s!} (x - x_0)^s \qquad (k = 1, 2, ..., n; N'' = N - n).$$

They have *polynomial coefficients* and assume at x_0 the same values as $a_j(x)$, $b_k(x)$. The same happens for the derivatives at x_0, up to the orders $N' - 1$, $N'' - 1$, respectively.

Therefore, if we consider also for these DE's two sets of linearly independent solutions $u_j^*(x)$ $(j = 1, 2, ..., m)$, $v_k^*(x)$ $(k = 1, 2, ..., n)$, with $\mathbf{W}(u_1^*, u_2^*, ..., u_m^*)(x_0) = \mathbf{I}_m$, $\mathbf{W}(v_1^*, v_2^*, ..., v_n^*)(x_0) = \mathbf{I}_n$, we obtain *the same matrix* $\mathbf{W}(w_1^*, w_2^*, ..., w_{mn}^*)(x_0) = \mathbf{W}(w_1, w_2, ..., w_{mn})(x_0)$, where the w_ν^*'s $(\nu = 1, 2, ..., mn)$ denote the $u_j^* v_k^*$'s and, in particular, *the same order* $N^* = N$ for the DE's for the products.

[2] We mean here that $x = a$ is a *singularity of the first kind*, in the terminology of [4].

In other words, the **C**-vector spaces spanned by the $u_j^* v_k^*$'s and by the $u_j v_k$'s are *isomorphic*, having the same dimension. The interval J in the proof of Theorem 3.1 and its analogue J^*, however, can be different, and the coefficients in the DE for the products are polynomials, in the second case.

4. Some Further Observations

(A) The *necessary and sufficient* condition for which the order of the DE for the products, in the case $m = n = 2$, is the minimum (i.e., the 3rd one) in I can be easily stated.

In fact, the transformation $u = U \exp\{\frac{1}{2} \int_{x_0}^x a_1(\xi)\, d\xi\}$, $v = V \exp\{\frac{1}{2} \int_{x_0}^x b_1(\xi)\, d\xi\}$, $x \in I$, takes (1.1), (1.2) with $m = n = 2$ into $U'' + P(x)\, U = 0$, $V'' + Q(x)\, V = 0$, where

$$P = \tfrac{1}{2} a_1' - \tfrac{1}{4} a_1^2 - a_2, \qquad Q = \tfrac{1}{2} b_1' - \tfrac{1}{4} b_1^2 - b_2$$

and preserves the linear dependence *and* independence of the solutions, as well as of their products, in I (and in *every* subinterval of I). It follows immediately that the required condition is $P(x) \equiv Q(x)$ in I (see, e.g., [21, p. 145]).

Therefore, the DE for the products of the solutions of (1.1), (1.2) with $m = n = 2$ reduces to the 3rd order *iff*:

$$a_2 - b_2 = \tfrac{1}{2}(a_1' - b_1') - \tfrac{1}{4}(a_1^2 - b_1^2) \qquad (\text{in } I), \tag{4.1}$$

(cf. Example 2.4).

Remark 4.1. It is *not* necessary that the two DE's coincide to have the minimal order.

If this is the case, $b_j(x) \equiv a_j(x)$ in I $(j = 1, 2)$, then (4.1) is trivially satisfied, but there are other possibilities, e.g.:

(i) If $b_j(x) \equiv 0$ in I $(j = 1, 2)$, then (by (4.1)) $a_2 = \frac{1}{2} a_1' - \frac{1}{4} a_1^2$.

(ii) While $b_1(x) \equiv a_1(x)$ in I implies $b_2(x) \equiv a_2(x)$ (by (4.1)), the converse is *not* true. In fact $b_2(x) \equiv a_2(x)$ in I requires

$$\delta' - b_1(x)\, \delta + \delta^2 = 0 \qquad (\text{in } I), \tag{4.2}$$

where $\delta(x) := \frac{1}{2}[b_1(x) - a_1(x)]$, and therefore $a_1(x) \equiv b_1(x)$ in I *or* $a_1(x) = b_1(x) - 2 \exp\{\int_{x_0}^x b_1(\xi)\, d\xi\} \{\int^x \exp\{\int_{x_0}^{\xi} b_1(\eta)\, d\eta\}\, d\xi\}^{-1}$, in some interval $I' \subseteq I$.

(B) Theorem 3.1 can be applied repeatedly to several DE's (having conveniently regular coefficients). This *iteration* yields the DE satisfied by the products of the solutions of *all* such DE's.

If we denote by U_m and $U_m V_n$ the vector spaces spanned by the u_j's and by the $u_j v_k$'s, respectively, we have

$$\dim U_m V_n =: N, \qquad m+n-1 \leqslant N \leqslant mn$$

(cf. [6]). Similarly,

$$\dim U_m V_n W_p =: M, \qquad m+n+p-2 \leqslant M \leqslant mnp,$$

so that, in general,

$$\dim U_{m_1}^{(1)} U_{m_2}^{(2)} \cdots U_{m_k}^{(k)} =: N, \qquad \sum_{r=1}^{k} m_r - k + 1 \leqslant N \leqslant \prod_{r=1}^{k} m_r. \tag{4.3}$$

In particular, taking k DE's *of the same order* m, we have

$$\dim U_m^{(1)} U_m^{(2)} \cdots U_m^{(k)} =: N, \qquad k(m-1)+1 \leqslant N \leqslant m^k. \tag{4.4}$$

Considering k times *the same* mth-order DE, we obtain the DE for the kth *powers* of the solutions of such an equation. In this case the *upper bound* for the order can be *lowered*:

$$N \leqslant N_{k,m} := \binom{k+m-1}{m-1} = \binom{k+m-1}{k} = \frac{(k+m-1)!}{k!(m-1)!}. \tag{4.5}$$

Here the symmetries in $(\sum_{j=1}^{m} c_j u_j)^k$ have been exploited.

In [19] a DE is constructed, whose solutions are the kth powers of the solutions of a given 2nd-order DE. From (4.4), we have $k+1 \leqslant N \leqslant 2^k$. In [19] it has been proved *directly* that the minimal case $N = k+1$ occurs. Here we have an alternative proof, because (4.5) yields $N \leqslant N_{k,2} = k+1$.

Remark 4.2. The estimate $N \leqslant N_{2,m} = m(m+1)/2$ is *not* redundant. In fact, *not* even when the two DE's coincide do we *necessarily* attain the minimum, $2m-1$ (see Examples 4.3 below, with $v_j = u_j$ ($j = 1, 2,..., n = m$)). This is true for $m = 2$ (cf. Remark 4.1).

(C) Examples 4.3. We have the following *classes of* examples for the "maximal" and the "minimal" cases:

(a) $u_j = u_1 U^{j-1}$ ($j = 1, 2,..., m$), $v_k = v_1 U^{m(k-1)}$ ($k = 1, 2,..., n$), and therefore $u_j v_k = u_1 v_1 U^{j+m(k-1)-1}$;

(b) $u_j = u_1 U^{j-1}$ ($j = 1, 2,..., m$), $v_k = v_1 U^{k-1}$ ($k = 1, 2,..., n$), and therefore $u_j v_k = u_1 v_1 U^{j+k-2}$. Here u_1, v_1 are *arbitrary* and U is *any function* *nonconstant* in every interval $I' \subset I$, of class $C^N(I)$, $u_1(x) \neq 0$, $v_1(x) \neq 0$ in some interval $J_1 \subseteq I$, with $N = mn$ in (a) and $N = m + n - 1$ in (b).

In fact, there are N functions $u_j v_k$, say $z_1, z_2, \ldots, z_N$, such that

$$W(z_1, z_2, \ldots, z_N) = (u_1 v_1)^N \, W(1, U, U^2, \ldots, U^{N-1}).$$

On the other hand, $\sum_{i=0}^{N-1} c_i U^i(x) \not\equiv 0$ in J_1, for any constants c_i, $\sum |c_i| > 0$. Then $\sum_{i=0}^{N-1} c_i U^i(x) \neq 0$ for some $x = x_0 \in J_1$ and hence in some interval $J_2(x_0) \subseteq J_1$. As this is also true *in every interval* $J^* \subseteq J_2$, we conclude (see, e.g., [9, p. 48]) that $W(1, U, U^2, \ldots, U^{N-1}) \neq 0$ in J_2 and therefore $W(z_1, z_2, \ldots, z_N) \neq 0$ in J_2.

Remark 4.4. Taking, in particular, $u_1 \equiv v_1 \equiv 1$, $U = e^x$ in the Examples above, we see that *all* cases are represented also confining ourselves to DE's with *constant coefficients.*

The Examples show that the maximal and the minimal cases are *not unique.*

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