BOREL HIERARCHY AND OMEGA CONTEXT FREE LANGUAGES

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Abstract

We give in this paper additional answers to questions of Lescow and Thomas [Logical Specifications of Infinite Computations, In:"A Decade of Concurrency", Springer LNCS 803 (1994), 583-621], proving topological properties of omega context free languages (ω-CFL) which extend those of [O. Finkel, Topological Properties of Omega Context Free Languages, Theoretical Computer Science, Vol. 262 (1-2), 2001, p. 669-697]: there exist some ω-CFL which are non Borel sets and one cannot decide whether an ω-CFL is a Borel set. We give also an answer to a question of Niwinski [Problem on ω-Powers Posed in the Proceedings of the 1990 Workshop "Logics and Recognizable Sets"] and of Simonnet [Automates et Théorie Descriptive, Ph.D. Thesis, Université Paris 7, March 1992] about ω-powers of finitary languages, giving an example of a finitary context free language L such that L^ω is not a Borel set. Then we prove some recursive analogues to preceding properties: in particular one cannot decide whether an ω-CFL is an arithmetical set. Finally we extend some results to context free sets of infinite trees.

Key words: Context free ω-languages; topological complexity; Borel hierarchy; analytic sets.
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1 Introduction

Since Büchi studied the ω-languages recognized by finite automata to prove the decidability of the monadic second order theory of one successor over

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the integers [Büc60a] the so called ω-regular languages have been intensively studied. See [Tho90] and [PP01] for many results and references.

As pushdown automata are a natural extension of finite automata, Cohen and Gold [CG77] , [CG78] and Linna [Lin76] studied the ω-languages accepted by omega pushdown automata, considering various acceptance conditions for omega words. It turned out that the omega languages accepted by omega pushdown automata were also those generated by context free grammars where infinite derivations are considered , also studied by Nivat [Niv77], [Niv78] and Boasson and Nivat [BN80]. These languages were then called the omega context free languages (ω-CFL). See also Staiger's paper [Sta97a] for a survey of general theory of ω-languages.

Topological properties of ω-regular languages were first studied by Landweber in [Lan69] where he showed that these languages are boolean combinations of G_δ sets. He also characterized the ω-regular languages in each of the Borel classes $\mathbf{F}, \mathbf{G}, \mathbf{F}_\sigma, \mathbf{G}_\delta$, and showed that one can decide, for an effectively given ω-regular language L, whether L is in the Borel class $\mathbf{F}, \mathbf{G}, \mathbf{F}_\sigma$, or $\mathbf{G}_\delta$.
It turned out that an ω-regular language is in the class $\mathbf{G}_\delta$ iff it is accepted by a deterministic Büchi automaton. These results were extended to deterministic ω-CFL by Linna [Lin77]. In the non deterministic case, Cohen and Gold proved in [CG78] that one cannot decide whether an ω-CFL is in the class $\mathbf{F}, \mathbf{G}$ or $\mathbf{G}_\delta$.

We have begun a similar study for ω-CFL in [Fin01a]. we proved that ω-CFL exhaust the finite ranks of the Borel hierarchy and that, for any Borel class $\mathbf{\Sigma^0_n}$ or $\mathbf{\Pi^0_n}$, n being an integer, one cannot decide whether an ω-CFL is in $\mathbf{\Sigma^0_n}$ or $\mathbf{\Pi^0_n}$. Our proof used the Wadge game and the operation of exponentiation of sets defined by Duparc [Dup01].

We pursue this study in this paper. We first show that there exist some ω-CFL which are analytic but non Borel sets. Then we extend the preceding undecidability result to every Borel class (of finite *or* infinite rank) and we prove that one cannot even decide whether an ω-CFL is a Borel set.

The question of the topological complexity of the ω-power of a finitary language is mentioned in [Sta97a] [Sta97b]. Niwinski asked in [Niw90] for an example of a (finitary) language L such that L^ω is not a Borel set. Simonnet asked in [Sim92] for the topological complexity of L^ω where L is a context free language. We proved in [Fin01a] that there exist context free languages L_n such that $(L_n)^\omega$ is a $\mathbf{\Pi^0_n}$-complete set for each integer $n \geq 1$.

We give here an example of a context free language L such that L^ω is an analytic but not Borel set, answering to questions of Niwinski and Simonnet.

Then we derive some new arithmetical properties of omega context free languages from the preceding topological properties. We prove that one cannot decide whether an ω-CFL is an arithmetical set in $\bigcup_{i\geq 1}\Sigma_n$. Then we show that one cannot decide whether the complement of an ω-CFL is accepted by a (non deterministic) Turing machine (or more generally by a non deterministic **X**-automaton as defined in [EH93]) with Büchi (respectively Muller) acceptance condition. The above results give additional answers to questions of Thomas and Lescow [LT94].

Finally we extend some undecidability results to context free sets of infinite trees, as defined by Saoudi [Sao92].

The paper is organized as follows. In sections 2 and 3, we first review some above definitions and results about ω-regular, ω-context free languages, and topology. Then in section 4 we prove our main topological results from which we deduce in section 5 the result about ω-powers and in section 6 arithmetical properties of ω-CFL. Section 7 deals with context free languages of infinite trees.

2 ω-regular and ω-context free languages

We assume the reader to be familiar with the theory of formal languages and of ω-regular languages, see for example [HU69] ,[Tho90]. We first recall some of the definitions and results concerning ω-regular and ω-context free languages and omega pushdown automata as presented in [Tho90] [CG77] , [CG78].
When Σ is a finite alphabet, a finite string (word) over Σ is any sequence $x = x_1 \ldots x_k$, where $x_i \in \Sigma$ for $i = 1, \ldots, k$,and k is an integer ≥ 1. The length of x is k, denoted by $|x|$.
we write $x(i) = x_i$ and $x[i] = x(1) \ldots x(i)$ for $i \leq k$.
If $|x| = 0$, x is the empty word denoted by λ.
$\Sigma^\star$ is the set of finite words over Σ.
The first infinite ordinal is ω.
An ω-word over Σ is an ω -sequence $a_1 \ldots a_n \ldots$, where $a_i \in \Sigma, \forall i \geq 1$.
When σ is an ω-word over Σ, we write $\sigma = \sigma(1)\sigma(2)\ldots\sigma(n)\ldots$
$\sigma[n] = \sigma(1)\sigma(2)\ldots\sigma(n)$ is the finite word of length n, prefix of σ.
The set of ω-words over the alphabet Σ is denoted by Σ^ω.
An ω-language over an alphabet Σ is a subset of Σ^ω.

The usual concatenation product of two finite words u and v is denoted $u.v$ (and sometimes just uv). This product is extended to the product of a finite word u and an ω-word v: the infinite word $u.v$ is then the ω-word such that:
$(u.v)(k) = u(k)$ if $k \leq |u|$, and
$(u.v)(k) = v(k - |u|)$ if $k > |u|$.

For $V \subseteq \Sigma^\star$, $V^\omega = \{\sigma = u_1 \ldots u_n \ldots \in \Sigma^\omega \mid u_i \in V, \forall i \geq 1\}$ is the ω-power of V.
For $V \subseteq \Sigma^\star$, the complement of V (in $\Sigma^\star$) is $\Sigma^\star - V$ denoted V^-.
For a subset $A \subseteq \Sigma^\omega$, the complement of A is $\Sigma^\omega - A$ denoted A^-.

The prefix relation is denoted $\sqsubseteq$: the finite word u is a prefix of the finite word v (denoted $u \sqsubseteq v$) if and only if there exists a (finite) word w such that $v = u.w$.
This definition is extended to finite words which are prefixes of ω-words: the finite word u is a prefix of the ω-word v (denoted $u \sqsubseteq v$) iff there exists an ω-word w such that $v = u.w$.

Definition 2.1 *A finite state machine (FSM) is a quadruple $M = (K, \Sigma, \delta, q_0)$, where K is a finite set of states, Σ is a finite input alphabet, $q_0 \in K$ is the initial state and δ is a mapping from $K \times \Sigma$ into 2^K . A FSM is called deterministic (DFSM) iff : $\delta : K \times \Sigma \rightarrow K$.*
A Büchi automaton (BA) is a 5-tuple $M = (K, \Sigma, \delta, q_0, F)$ where $M' = (K, \Sigma, \delta, q_0)$ is a finite state machine and $F \subseteq K$ is the set of final states.
A Muller automaton (MA) is a 5-tuple $M = (K, \Sigma, \delta, q_0, F)$ where $M' = (K, \Sigma, \delta, q_0)$ is a FSM and $F \subseteq 2^K$ is the collection of designated state sets.
A Büchi or Muller automaton is said deterministic if the associated FSM is deterministic.
Let $\sigma = a_1 a_2 \ldots a_n \ldots$ be an ω-word over Σ.
A sequence of states $r = q_1 q_2 \ldots q_n \ldots$ is called an (infinite) run of $M = (K, \Sigma, \delta, q_0)$ on σ, starting in state p, iff: 1) $q_1 = p$ and 2) for each $i \geq 1$, $q_{i+1} \in \delta(q_i, a_i)$.
In case a run r of M on σ starts in state q_0, we call it simply "a run of M on σ ".
For every (infinite) run $r = q_1 q_2 \ldots q_n \ldots$ of M, $In(r)$ is the set of states in K entered by M infinitely many times during run r:
$In(r) = \{q \in K \mid \{i \geq 1 \mid q_i = q\}$ is infinite $\}$.
For $M = (K, \Sigma, \delta, q_0, F)$ a BA , the ω-language accepted by M is $L(M) = \{\sigma \in \Sigma^\omega \mid$ there exists a run r of M on σ such that $In(r) \cap F \neq \emptyset\}$.
For $M = (K, \Sigma, \delta, q_0, F)$ a MA, the ω-language accepted by M is $L(M) = \{\sigma \in \Sigma^\omega \mid$ there exists a run r of M on σ such that $In(r) \in F\}$.

The classical result of R. Mc Naughton [MaN66] established that the expressive power of deterministic MA (DMA) is equal to the expressive power of non deterministic MA (NDMA) which is also equal to the expressive power of non deterministic BA (NDBA) .
There is also a characterization of languages accepted by MA by means of the "ω-Kleene closure" of which we give now the definition:

Definition 2.2 *For any family L of finitary languages over the alphabet Σ,* *the ω-Kleene closure of L, is :*

$$\omega - KC(L) = \{\cup_{i=1}^{n} U_i.V_i^{\omega} \mid U_i, V_i \in L, \forall i \in [1, n]\}$$

Theorem 2.3 *For any ω-language L, the following conditions are equivalent:*

(1) L belongs to $\omega - KC(REG)$, where REG is the class of (finitary) regular languages.
(2) There exists a DMA that accepts L.
(3) There exists a MA that accepts L.
(4) There exists a BA that accepts L.

An ω-language L satisfying one of the conditions of the above Theorem is called an ω-regular language. The class of ω-regular languages will be denoted by REG_ω.

We now define pushdown machines and the class of ω-context free languages.

Definition 2.4 *A pushdown machine (PDM) is a 6-tuple $M = (K, \Sigma, \Gamma, \delta, q_0, Z_0)$, where K is a finite set of states, Σ is a finite input alphabet, Γ is a finite pushdown alphabet, $q_0 \in K$ is the initial state, $Z_0 \in \Gamma$ is the start symbol, and δ is a mapping from $K \times (\Sigma \cup \{\lambda\}) \times \Gamma$ to finite subsets of $K \times \Gamma^\star$.*
If $\gamma \in \Gamma^+$ describes the pushdown store content, the leftmost symbol will be assumed to be on " top" of the store. A configuration of a PDM is a pair (q, γ) where $q \in K$ and $\gamma \in \Gamma^\star$.
For $a \in \Sigma \cup \{\lambda\}$, $\beta, \gamma \in \Gamma^\star$ and $Z \in \Gamma$, if (p, β) is in $\delta(q, a, Z)$, then we write $a : (q, Z\gamma) \mapsto_M (p, \beta\gamma)$.
$\mapsto_M^\star$ is the transitive and reflexive closure of $\mapsto_M$. (The subscript M will be omitted whenever the meaning remains clear).
Let $\sigma = a_1a_2 \ldots a_n \ldots$ be an ω-word over Σ. an infinite sequence of configurations $r = (q_i, \gamma_i)_{i \geq 1}$ is called a complete run of M on σ, starting in configuration (p, γ), iff:

(1) $(q_1, \gamma_1) = (p, \gamma)$
(2) for each $i \geq 1$, there exists $b_i \in \Sigma \cup \{\lambda\}$ satisfying $b_i : (q_i, \gamma_i) \mapsto_M (q_{i+1}, \gamma_{i+1})$ such that $a_1a_2 \ldots a_n \ldots = b_1b_2 \ldots b_n \ldots$

As for FSM, for every such run, $In(r)$ is the set of all states entered infinitely often during run r.
A complete run r of M on σ , starting in configuration (q_0, Z_0), will be simply called " a run of M on σ ".

Definition 2.5 *A Büchi pushdown automaton (BPDA) is a 7-tuple $M = (K, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ where $M' = (K, \Sigma, \Gamma, \delta, q_0, Z_0)$ is a PDM and $F \subseteq K$ is the set of final states.*

The ω-language accepted by M is $L(M) = \{\sigma \in \Sigma^\omega \mid$ *there exists a complete run r of M on σ such that* $In(r) \cap F \neq \emptyset\}$.

Definition 2.6 *A Muller pushdown automaton (MPDA) is a 7-tuple* $M = (K, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ *where* $M' = (K, \Sigma, \Gamma, \delta, q_0, Z_0)$ *is a PDM and* $F \subseteq 2^K$ *is the collection of designated state sets.*
The ω-language accepted by M is $L(M) = \{\sigma \in \Sigma^\omega \mid$ *there exists a complete run r of M on σ such that* $In(r) \in F\}$.

Remark 2.7 *We consider here two acceptance conditions for ω-words, the Büchi and the Muller acceptance conditions, respectively denoted 2-acceptance and 3-acceptance in [Lan69] and in [CG78] and* $(inf, \sqcap)$ *and* $(inf, =)$ *in [Sta97a].*

Cohen and Gold and independently Linna established a characterization Theorem for ω-CFL:

Theorem 2.8 *Let* CFL *be the class of context free (finitary) languages. Then for any ω-language L the following three conditions are equivalent:*

(1) $L \in \omega - KC(CFL)$.
(2) There exists a $BPDA$ *that accepts L.*
(3) There exists a $MPDA$ *that accepts L.*

In [CG77] are also studied ω-languages generated by ω-context free grammars and it is shown that each of the conditions 1), 2), and 3) of the above Theorem is also equivalent to: 4) L is generated by a context free grammar G by leftmost derivations. These grammars are also studied in [Niv77] [Niv78].
Then we can let the following definition:

Definition 2.9 *An ω-language is an ω-context free language (ω-CFL) (or context free ω-language) iff it satisfies one of the conditions of the above Theorem.*

3 Topology

We assume the reader to be familiar with basic notions of topology which may be found in [LT94] [PP01] [Kur66] [Mos80] [Kec95].

Topology is an important tool for the study of ω-languages, and leads to characterization of several classes of ω-languages.
For a finite alphabet X, we consider X^ω as a topological space with the Cantor topology. The open sets of X^ω are the sets in the form $W.X^\omega$, where $W \subseteq X^\star$. A set $L \subseteq X^\omega$ is a closed set iff its complement $X^\omega - L$ is an open set. The

class of open sets of X^ω will be denoted by $\mathbf{G}$ or by $\mathbf{\Sigma^0_1}$. The class of closed sets will be denoted by $\mathbf{F}$ or by $\mathbf{\Pi^0_1}$. Define now the next classes of the Borel Hierarchy:

Definition 3.1 *The classes $\mathbf{\Sigma^0_n}$ and $\mathbf{\Pi^0_n}$ of the Borel Hierarchy on the topological space X^ω are defined as follows:*
$\mathbf{\Sigma^0_1}$ is the class of open sets of X^ω.
$\mathbf{\Pi^0_1}$ is the class of closed sets of X^ω.
$\mathbf{\Pi^0_2}$ or $\mathbf{G}_\delta$ is the class of countable intersections of open sets of X^ω.
$\mathbf{\Sigma^0_2}$ or $\mathbf{F}_\sigma$ is the class of countable unions of closed sets of X^ω.
And for any integer $n \geq 1$:
$\mathbf{\Sigma^0_{n+1}}$ is the class of countable unions of $\mathbf{\Pi^0_n}$-subsets of X^ω.
$\mathbf{\Pi^0_{n+1}}$ is the class of countable intersections of $\mathbf{\Sigma^0_n}$-subsets of X^ω.
The Borel Hierarchy is also defined for transfinite levels. The classes $\mathbf{\Sigma^0_\alpha}$ and $\mathbf{\Pi^0_\alpha}$, for a countable ordinal α, are defined in the following way:
$\mathbf{\Sigma^0_\alpha}$ is the class of countable unions of subsets of X^ω in $\cup_{\gamma<\alpha}\mathbf{\Pi^0_\gamma}$.
$\mathbf{\Pi^0_\alpha}$ is the class of countable intersections of subsets of X^ω in $\cup_{\gamma<\alpha}\mathbf{\Sigma^0_\gamma}$.

Recall some basic results about these classes, [Mos80]:

Proposition 3.2

(a) $\mathbf{\Sigma^0_\alpha} \cup \mathbf{\Pi^0_\alpha} \subsetneq \mathbf{\Sigma^0_{\alpha+1}} \cap \mathbf{\Pi^0_{\alpha+1}}$, for each countable ordinal $\alpha \geq 1$.
(b) $\cup_{\gamma<\alpha}\mathbf{\Sigma^0_\gamma} = \cup_{\gamma<\alpha}\mathbf{\Pi^0_\gamma} \subsetneq \mathbf{\Sigma^0_\alpha} \cap \mathbf{\Pi^0_\alpha}$, for each countable limit ordinal α.
(c) A set $W \subseteq X^\omega$ is in the class $\mathbf{\Sigma^0_\alpha}$ iff its complement is in the class $\mathbf{\Pi^0_\alpha}$.
(d) $\mathbf{\Sigma^0_\alpha} - \mathbf{\Pi^0_\alpha} \neq \emptyset$ and $\mathbf{\Pi^0_\alpha} - \mathbf{\Sigma^0_\alpha} \neq \emptyset$ hold for every countable ordinal $\alpha \geq 1$.

We shall say that a subset of X^ω is a Borel set of rank α, for a countable ordinal α, iff it is in $\mathbf{\Sigma^0_\alpha} \cup \mathbf{\Pi^0_\alpha}$ but not in $\bigcup_{\gamma<\alpha}(\mathbf{\Sigma^0_\gamma} \cup \mathbf{\Pi^0_\gamma})$.

Furthermore, when X is a finite set, there are some subsets of X^ω which are not Borel sets. Indeed there exists another hierarchy beyond the Borel hierarchy, which is called the projective hierarchy and which is obtained from the Borel hierarchy by successive applications of operations of projection and complementation. More precisely, a subset A of X^ω is in the class $\mathbf{\Sigma^1_1}$ of **analytic** sets iff there exists another finite set Y and a Borel subset B of $(X \times Y)^\omega$ such that $x \in A \leftrightarrow \exists y \in Y^\omega$ such that $(x, y) \in B$.
Where (x, y) is the infinite word over the alphabet $X \times Y$ such that $(x, y)(i) = (x(i), y(i))$ for each integer $i \geq 0$.
Now a subset of X^ω is in the class $\mathbf{\Pi^1_1}$ of **coanalytic** sets iff its complement in X^ω is an analytic set.
The next classes are defined in the same manner, $\mathbf{\Sigma^1_{n+1}}$-sets of X^ω are projections of $\mathbf{\Pi^1_n}$-sets and $\mathbf{\Pi^1_{n+1}}$-sets are the complements of $\mathbf{\Sigma^1_{n+1}}$-sets.

Recall also the notion of completeness with regard to reduction by continuous functions.

A set $F \subseteq X^\omega$ is a $\mathbf{\Sigma^0_\alpha}$ (respectively $\mathbf{\Pi^0_\alpha}$)-complete set iff for any set $E \subseteq Y^\omega$ (Y a finite alphabet):
$E \in \mathbf{\Sigma^0_\alpha}$ (respectively $E \in \mathbf{\Pi^0_\alpha}$) iff there exists a continuous function f from Y^ω into X^ω such that $E = f^{-1}(F)$.
A similar notion exists for classes of the projective hierarchy: in particular a set $F \subseteq X^\omega$ is a $\mathbf{\Sigma^1_1}$ (respectively $\mathbf{\Pi^1_1}$)-complete set iff for any set $E \subseteq Y^\omega$ (Y a finite alphabet):
$E \in \mathbf{\Sigma^1_1}$ (respectively $E \in \mathbf{\Pi^1_1}$) iff there exists a continuous function f from Y^ω into X^ω such that $E = f^{-1}(F)$.

A $\mathbf{\Sigma^0_\alpha}$ (respectively $\mathbf{\Pi^0_\alpha}$, $\mathbf{\Sigma^1_1}$)-complete set is a $\mathbf{\Sigma^0_\alpha}$ (respectively $\mathbf{\Pi^0_\alpha}$, $\mathbf{\Sigma^1_1}$)- set which is in some sense a set of the highest topological complexity among the $\mathbf{\Sigma^0_\alpha}$ (respectively $\mathbf{\Pi^0_\alpha}$, $\mathbf{\Sigma^1_1}$)- sets.

4 topological properties of ω-CFL

Recall first previous results. ω-CFL exhaust the finite ranks of the Borel hierarchy.

Theorem 4.1 ([Fin01a]) *For each integer $n \geq 1$, there exist some $\mathbf{\Sigma^0_n}$-complete ω-CFL and some $\mathbf{\Pi^0_n}$-complete ω-CFL.*

Cohen and Gold proved that one cannot decide whether an ω-CFL is in the class $\mathbf{F}, \mathbf{G}$ or $\mathbf{G}_\delta$. We have extended in [Fin01a] this result to all classes $\mathbf{\Sigma^0_n}$ and $\mathbf{\Pi^0_n}$, for n an integer ≥ 1. (We say that an ω-CFL A is effectively given when a MPDA accepting A is given).

Theorem 4.2 ([Fin01a]) *Let n be an integer ≥ 1. Then it is undecidable whether an effectively given ω-CFL is in the class $\mathbf{\Sigma^0_n}$ (repectively $\mathbf{\Pi^0_n}$).*

When considering ω-CFL, natural questions now arise: are all ω-CFL Borel sets of finite rank, Borel sets, analytic sets....? First recall the following:

Theorem 4.3 ([Sta97a]) *Every ω-CFL over a finite alphabet X is an analytic subset of X^ω.*

Proof. we just sketch the proof.
Every ω-CFL $A \subseteq \Sigma^\omega$ is the projection of a deterministic ω-CFL onto Σ^ω but deterministic ω-CFL are Borel sets of rank at most 3, and it is well known that such a projection of a Borel set is an analytic subset of Σ^ω. Remark that in fact each ω-CFL is the projection of an ω-CFL which is accepted by a deterministic Büchi pushdown automaton and therefore which is a $\mathbf{\Pi^0_2}$-set. $\square$

Remark 4.4 *This above theorem is in fact true for ω-languages accepted* *by Turing machines which are much more powerful accepting devices than pushdown automata [Sta97a].*

The following question now arises: are there ω-CFL which are analytic but not Borel sets?

Theorem 4.5 *There exist ω-CFL which are $\mathbf{\Sigma^1_1}$-complete hence non Borel sets.*

Proof. We shall use here results about languages of infinite binary trees whose nodes are labelled in a finite alphabet Σ.
A node of an infinite binary tree is represented by a finite word over the alphabet $\{l, r\}$ where r means "right" and l means "left". Then an infinite binary tree whose nodes are labelled in Σ is identified with a function $t : \{l, r\}^\star \to \Sigma$. The set of infinite binary trees labelled in Σ will be denoted T^ω_Σ.

There is a natural topology on this set T^ω_Σ [Mos80] [LT94][Sim92]. It is defined by the following distance. Let t and s be two distinct infinite trees in T^ω_Σ. Then the distance between t and s is $\frac{1}{2^n}$ where n is the smallest integer such that $t(x) \neq s(x)$ for some word $x \in \{l, r\}^\star$ of length n.
The open sets are then in the form $T_0.T^\omega_\Sigma$ where T_0 is a set of finite labelled trees. $T_0.T^\omega_\Sigma$ is the set of infinite binary trees which extend some finite labelled binary tree $t_0 \in T_0$, t_0 is here a sort of prefix, an "initial subtree" of a tree in $t_0.T^\omega_\Sigma$.

The Borel hierarchy and the projective hierarchy on T^ω_Σ are defined from open sets in the same manner as in the case of the topological space Σ^ω.

Let t be a tree. A branch B of t is a subset of the set of nodes of t which is linearly ordered by the tree partial order $\sqsubseteq$ and which is closed under prefix relation, i.e. if x and y are nodes of t such that $y \in B$ and $x \sqsubseteq y$ then $x \in B$.
A branch B of a tree is said to be maximal iff there is not any other branch of t which strictly contains B.

Let t be an infinite binary tree in T^ω_Σ. If B is a maximal branch of t, then this branch is infinite. Let $(u_i)_{i\geq 0}$ be the enumeration of the nodes in B which is strictly increasing for the prefix order.
The infinite sequence of labels of the nodes of such a maximal branch B, i.e. $t(u_0)t(u_1)\ldots t(u_n)\ldots$ is called a path. It is an ω-word over the alphabet Σ.

Let then $L \subseteq \Sigma^\omega$ be an ω-language over Σ. Then we denote $Path(L)$ the set of infinite trees t in T^ω_Σ such that t has (at least) one path in L.

It is well known that if $L \subseteq \Sigma^\omega$ is an ω-language over Σ which is a $\mathbf{\Pi^0_2}$-complete subset of Σ^ω (or a set of higher complexity in the Borel hierarchy) then the set $Path(L)$ is a $\mathbf{\Sigma^1_1}$-complete subset of T^ω_Σ. Hence $Path(L)$ is not a

Borel set, [Niw85] [Sim92] [Sim93].

Whenever B is an ω-CFL we shall find another ω-CFL C and a continuous function

$$h : T_\Sigma^\omega \rightarrow (\Sigma \cup \{A\})^\omega$$

such that $Path(B) = h^{-1}(C)$. For that we will code trees labelled in Σ by words over $\Sigma \cup \{A\} = \Sigma_A$, where A is supposed to be a new letter not in Σ.

Consider now the set $\{l, r\}^\star$ of nodes of binary infinite trees. For each integer $n \geq 0$, call C_n the set of words of length n of $\{l, r\}$. Then $C_0 = \{\lambda\}$, $C_1 = \{l, r\}$, $C_2 = \{ll, lr, rl, rr\}$ and so on. C_n is the set of nodes which appear in the $(n+1)$th level of an infinite binary tree. The number of nodes of C_n is $card(C_n) = 2^n$. We consider now the lexicographic order on C_n (assuming that l is before r for this order). Then, in the enumeration of the nodes with regard to this order, the nodes of C_1 will be: l, r; the nodes of C_3 will be: $lll, llr, lrl, lrr, rll, rlr, rrl, rrr$.
Let $u_1^n, \ldots, u_j^n, \ldots, u_{2^n}^n$ be such an enumeration of C_n in the lexicographic order and let $v_1^n, \ldots, v_j^n, \ldots, v_{2^n}^n$ be the enumeration of the elements of C_n in the reverse order. Then for all integers $n \geq 0$ and i, $1 \leq i \leq 2^n$, it holds that $v_i^n = u_{2^n+1-i}^n$.

We define now the code of a tree t in T_Σ^ω. Let A be a letter not in Σ. We construct an ω-word over the alphabet $(\Sigma \cup \{A\})$ which will code the tree t. We enumerate all the labels of the nodes of a tree in the following manner: firstly the label of the node of C_0 which is $t(u_1^0)$,
followed by an A, followed by the labels of nodes of C_1 in the lexicographic order, i.e. $t(u_1^1)t(u_2^1)$, followed by an A, followed by the labels of the nodes of C_2 in the reverse lexicographic order, followed by an A, followed by the labels of nodes of C_3 in the lexicographic order, and so on ...
For each integer $n \geq 0$, the labels of the nodes of C_n are enumerated before those of C_{n+1} and these two sets of labels are separated by an A. Moreover the labels of the nodes of C_{2n+1}, for $n \geq 0$, are enumerated in the lexicographic order (for the nodes) and the labels of the nodes of C_{2n}, for $n \geq 0$, are enumerated in the reverse lexicographic order (for the nodes).

Then for each tree t in T_Σ^ω, we obtain an ω-word of $\Sigma \cup \{A\}$ which will be denoted $h(t)$. With the preceding notations it holds that:

$$h(t) = t(u_1^0)At(u_1^1)t(u_2^1)At(v_1^2)t(v_2^2)t(v_3^2)t(v_4^2)At(u_1^3)t(u_2^3)t(u_3^3)t(u_4^3)t(u_5^3)t(u_6^3)t(u_7^3)t(u_8^3)A\ldots$$

Let then h be the mapping from T_Σ^ω into $(\Sigma \cup \{A\})^\omega$ such that for every labelled binary infinite tree t of T_Σ^ω, $h(t)$ is the code of the tree as defined above. It is easy to see, from the definition of h and of the order of the enumeration of labels of nodes, that h is a continuous function from T_Σ^ω into $(\Sigma \cup \{A\})^\omega$.

Assume now that B is an ω-CFL accepted by a Büchi pushdown automaton $M = (K, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ where $M' = (K, \Sigma, \Gamma, \delta, q_0, Z_0)$ is a pushdown machine and $F \subseteq K$ is the set of final states.

Now we are looking for another ω-CFL C such that for every tree $t \in T_\Sigma^\omega$, $h(t) \in C$ if and only if t has a path in B. Then we shall have $Path(B) = h^{-1}(C)$.

We shall give a **first description** of such an ω-CFL C by constructing from M another Büchi pushdown automaton $\bar{M}$ which accepts C.
The reader can also skip this description and read a **second description** of the ω-CFL C which will be given below.

Describe first informally the behaviour of the new machine $\bar{M}$. When $\bar{M}$ reads a word in the form $h(t)$, then using the non determinism it guesses a maximal branch of the tree t and simulates on this branch the Büchi pushdown automaton M. Finally the acceptation of $h(t)$ by $\bar{M}$ is related to the acceptation of the ω-word formed by the labels of this branch by M.

More formally $\bar{M} = (\bar{K}, \bar{\Sigma}, \bar{\Gamma}, \bar{\delta}, \bar{q_0}, \bar{Z}_0, \bar{F})$, where

$$\bar{K} = K \cup \{q^1 \mid q \in K\} \cup \{q^2 \mid q \in K\} \cup \{q^3 \mid q \in K\} \cup \{q^4 \mid q \in K\} \cup \{q^5 \mid q \in K\} \cup \{q_r\}$$

$$\bar{\Sigma} = \Sigma \cup \{A\}$$
$$\bar{\Gamma} = \Gamma \cup \{E\}$$

where E is a new letter not in Γ,

$$\bar{q_0} = q_0$$

$$\bar{Z}_0 = Z_0$$
$$\bar{F} = F \cup \{q^5 \mid q \in F\}$$

and the transition relation $\bar{\delta}$ is defined by the following cases which will be explained below:

(a) $(q, \nu) \in \bar{\delta}(q_0, a, Z_0)$ iff $(q, \nu) \in \delta(q_0, a, Z_0)$, for each $a \in \Sigma$ and $\nu \in \Gamma^\star$.
(b) $\bar{\delta}(q_0, A, Z_0) = (q_r, Z_0)$.
(c) $\bar{\delta}(q, a, Z) = (q^1, EZ)$, for each $a \in \Sigma$, $Z \in \Gamma \cup \{E\}$ and $q \in K$.
(d) $\bar{\delta}(q^1, a, E) = (q^1, EE)$, for each $a \in \Sigma$, and $q \in K$.
(e) $\bar{\delta}(q^1, A, Z) = (q^2, Z)$, for each $Z \in \Gamma \cup \{E\}$ and $q \in K$.
(f) $\bar{\delta}(q, A, Z) = (q^2, Z)$, for each $Z \in \Gamma \cup \{E\}$ and $q \in K$.
(g) $\bar{\delta}(q^2, a, E) = (q^3, E)$, for each $a \in \Sigma$, and $q \in K$.

(h) $\bar{\delta}(q^3, a, E) = (q^2, \lambda)$, for each $a \in \Sigma$, and $q \in K$.
(i) $\bar{\delta}(q^2, A, E) = (q_r, E)$, for each $q \in K$.
(j) $\bar{\delta}(q^3, A, E) = (q_r, E)$, for each $q \in K$.
(k) $\bar{\delta}(q_r, a, Z) = (q_r, Z)$, for each $a \in (\Sigma \cup \{A\})$ and $Z \in \Gamma \cup \{E\}$.
(l) $\bar{\delta}(q^2, a, Z) \ni (q', \nu)$ iff $\delta(q, a, Z) \ni (q', \nu)$, for each $a \in \Sigma$, $q, q' \in K$, $Z \in \Gamma$, and $\nu \in \Gamma^\star$.
(m) $\bar{\delta}(q^2, \lambda, Z) \ni (q'^5, \nu)$ iff $\delta(q, \lambda, Z) \ni (q', \nu)$, for each $q, q' \in K$, $Z \in \Gamma$, and $\nu \in \Gamma^\star$.
(n) $\bar{\delta}(q^5, \lambda, Z) \ni (q'^5, \nu)$ iff $\delta(q, \lambda, Z) \ni (q', \nu)$, for each $q, q' \in K$, $Z \in \Gamma$, and $\nu \in \Gamma^\star$.
(o) $\bar{\delta}(q^5, a, Z) \ni (q', \nu)$ iff $\delta(q, a, Z) \ni (q', \nu)$, for each $a \in \Sigma$, $q, q' \in K$, $Z \in \Gamma$, and $\nu \in \Gamma^\star$.
(p) $\bar{\delta}(q^5, a, Z) \ni (q^4, Z)$, for each $a \in \Sigma$, $Z \in \Gamma$ and $q \in K$.
(q) $\bar{\delta}(q^2, a, Z) \ni (q^4, Z)$, for each $a \in \Sigma$, $Z \in \Gamma$ and $q \in K$.
(r) $\bar{\delta}(q^4, a, Z) \ni (q', \nu)$ iff $\delta(q, a, Z) \ni (q', \nu)$, for each $a \in \Sigma$, $q, q' \in K$, $Z \in \Gamma$, and $\nu \in \Gamma^\star$.
(s) $\bar{\delta}(q^4, A, Z) = (q_r, Z)$, for each $q \in K$, $Z \in \Gamma$.

We describe now more precisely the behaviour of $\bar{M}$.
To the set K of states of M, we add sets of states $K^i = \{q^i \mid q \in K\}$ for each integer $i \in [1, 5]$, and a state q_r which will be a rejecting state.

We firstly consider only the reading by $\bar{M}$ of words in the form $h(t)$ where $t \in T_\Sigma^\omega$. When $\bar{M}$ simulates M on the branch it guesses, it enters in a state of K, as indicated by $(a), (l), (o), (r)$, or of K^5 if it uses a λ-transition, i.e. if it does not read any letter during this transition, as indicated by $(m) - (n)$.

When $\bar{M}$ reads the labels of the nodes of t, it reads successively the labels of nodes of $C_0, C_1, C_2, \ldots, C_i, \ldots$

Let B be the branch which is guessed by $\bar{M}$ during a reading.

After the use of one transition rule of $(a), (l), (o)$ or (r), reading the label of a node u of B in C_n, $n \geq 0$, $\bar{M}$ enters in a state q^1, keeping the memory of q, and then continues the reading of the (labels of) nodes of C_n, pushing an E on the top of the stack for every letter of Σ it reads (transition rules (c), (d)) until it reads an A. Then it enters in state q^2, keeping again the memory of q, (transition rules (e), (f)) and reading the labels of nodes of C_{n+1}, it begins to pop an E from the top of the stack for two letters of Σ it reads, as indicated by transition rules (g), (h) (here are used the two sets of states K^2 and K^3). Thus when the letter at the top of the stack is again a letter of Γ (and not an E) the machine $\bar{M}$ reads the label of one successor of the node u (this is due to the fact that the tree is binary and to the order of the enumeration of the nodes we have chosen in the definition of $h(t)$). It may choose to simulate M on this label, as indicated by the transition rules $(l), (m), (n), (o)$, (perhaps after some λ-transitions). Otherwise it may choose to wait the next label, entering in state q^4, as indicated by the transition rules $(p), (q)$, and then simulates M as indicated by the transition rule (r).

Some other transition rules, $(b), (i), (j), (k), (s)$, lead to the rejecting state q_r in which $\bar{M}$ remains for the rest of the reading. But in fact these transition rules are never used for the reading of ω-words in the form $h(t)$ where $t \in T_\Sigma^\omega$.

Now we can see that when $\bar{M}$ simulates M on the branch B, if M enters in a state $q \in K$, then $\bar{M}$ enters in the state q or in the state q^5 (when a λ-transition is used). Thus the choice of the set of accepting states $\bar{F} = F \cup \{q^5 \mid q \in F\}$ implies the property: for a tree $t \in T_\Sigma^\omega$, $h(t) \in C$ if and only if t has a path in B.

We are going now to give a **second description** of the ω-CFL C.

The ω-language C which we have constructed from the ω-language B can easily be described by means of substitution of context free languages.
Let first D be the following finitary language over the alphabet $(\Sigma \cup \{A\})$:

$$D = \{u.A.v \mid u, v \in \Sigma^\star \textit{ and } (|v| = 2|u|) \textit{ or } (|v| = 2|u| + 1)\}$$

It is easy to see that D is a context free language.

Now an ω-word $\sigma \in C$ may be considered as an ω-word $\sigma' \in B$ to which we add, between two consecutive letters $\sigma'(n)$ and $\sigma'(n+1)$ of σ', a finite word v_n belonging to the context free finitary language D.

Recall now the definition of substitution in languages: A substitution f is defined by a mapping $\Sigma \to P(\Gamma^\star)$, where $\Sigma = \{a_1, \ldots, a_n\}$ and Γ are two finite alphabets, $f : a_i \to L_i$ where $\forall i \in [1; n]$, L_i is a finitary language over the alphabet Γ.
Now this mapping is extended in the usual manner to finite words:

$$f(x(1) \ldots x(n)) = \{u_1 \ldots u_n \quad \mid \quad u_i \in f(x(i)),\ \forall i \in [1; n]\}$$

where $x(1), \ldots, x(n)$ are letters in Σ, and to finitary languages $L \subseteq \Sigma^\star$:

$$f(L) = \cup_{x \in L} f(x)$$

The substitution f is called λ-free if $\forall i \in [1; n]$ L_i does not contain the empty word. In that case the mapping f may be extended to ω-words:

$$f(x(1) \ldots x(n) \ldots) = \{u_1 \ldots u_n \ldots \quad \mid \quad u_i \in f(x(i)),\ \forall i \geq 1\}$$

Let $\mathbb{C}$ be a family of languages, if $\forall i \in [1; n]$ the language L_i belongs to $\mathbb{C}$ the substitution f is called a $\mathbb{C}$-substitution.

Let then g be the substitution $\Sigma \to P((\Sigma \cup \{A\})^\star)$ defined by: $a \to a.D$ where D is the context free language defined above. Then g is a λ-free substitution and $g(B) = C$ holds. But the languages $a.D$ are context free and CFL_ω is closed under λ-free context free substitution [CG77]. Then $B \in CFL_\omega$ implies that $C \in CFL_\omega$.

Hence if B is a Borel set which is a $\mathbf{\Pi^0_2}$-complete subset of Σ^ω (or a set of higher complexity in the Borel hierarchy), the language $h^{-1}(C) = Path(B)$ is a $\mathbf{\Sigma^1_1}$-complete subset of T^ω_Σ. Then the ω-language C is at least $\mathbf{\Sigma^1_1}$-complete because h is a continuous function (note that here h is a continuous function: $T^\omega_\Sigma \to (\Sigma_A)^\omega$ and the preceding definition of $\mathbf{\Sigma^1_1}$-complete set involves continuous reductions: $X^\omega \to Y^\omega$; but the two topological spaces T^ω_Σ and $(\Sigma_A)^\omega$ have good similar properties which enable to extend the previous definition to this new case [Mos80][Kec95]). And C is in fact a $\mathbf{\Sigma^1_1}$-complete subset of $(\Sigma \cup \{A\})^\omega$ because every ω-CFL is an analytic set by Theorem 4.3.
Then in that case C is not a Borel set because a $\mathbf{\Sigma^1_1}$-complete set is not a Borel set [Kur66][Mos80].
Indeed this gives infinitely many non Borel ω-CFL , because there exist infinitely many ω-CFL of borel rank > 2. □

Remark that in the above proof, whenever B is an ω-regular language accepted by a Büchi automaton M, the resulting machine $\bar{M}$ is just a one counter machine, i.e. a pushdown machine having a stack alphabet $\bar{\Gamma} = \{Z_0, E\}$, where Z_0 is the bottom symbol which always remains at the bottom of the pushdown store and appears only there. Then at any moment of any computation the word in the pushdown store is in the form $E^n Z_0$ where n is an integer ≥ 0. Thus it holds that:

Corollary 4.6 *There exist one counter ω-languages which are $\mathbf{\Sigma^1_1}$-complete hence non Borel sets.*

Now we can deduce from the preceding proof the following undecidability result:

Theorem 4.7 *Let Σ be an alphabet containing at least two letters. It is undecidable, for an effectively given ω-CFL B to determine whether B is a Borel subset of Σ^ω.*

Proof. Remark first that $h(T^\omega_\Sigma)$ is the set of ω-words in $(\Sigma_A)^\omega$ which belong to

$$\Sigma.A.\Sigma^2.A.\Sigma^4.A.\Sigma^8.A \ldots A.\Sigma^{2^n}.A\Sigma^{2^{n+1}} \ldots$$

In other words this is the set of words in $(\Sigma_A)^\omega$ which contain infinitely many occurrences of the letter A, and have 2^n letters of Σ between the nth and the (n+1)th occurrences of the letter A. We shall first state the following:

Lemma 4.8 *Let Σ be a finite alphabet. Then $(\Sigma_A)^\omega - h(T_\Sigma^\omega)$ is an omega context free language.*

Proof. Let

$$A_1 = (A \cup \Sigma^2 \cup \Sigma.A.A \cup \Sigma.A.\Sigma.A \cup \Sigma.A.\Sigma^3).(\Sigma_A)^\omega$$

A_1 is the set of words in $(\Sigma_A)^\omega$ which have not any word of $\Sigma.A.\Sigma^2.A$ as prefix. A_1 is clearly an ω-regular language hence it is also an ω-CFL.

Let now B_1 be the set of finite words over the alphabet Σ_A which are in the form $A.u.A.v.A$ where $u, v \in \Sigma$ and $|v| < 2|u|$. And let B_2 be the set of finite words over the alphabet Σ_A which are in the form $A.u.A.v$ where $u, v \in \Sigma$ and $|v| > 2|u|$.

Then it is easy to see that B_1 and B_2 are context free finitary languages, thus the ω-language

$$A_2 = [(\Sigma_A)^\star.B_1.(\Sigma_A)^\omega] \cup [(\Sigma_A)^\star.B_2.(\Sigma_A)^\omega]$$

is an an omega context free language by Theorem 2.8.
But $(\Sigma_A)^\omega - h(T_\Sigma^\omega) = A_1 \cup A_2$ and the class of context free ω-languages is closed under union [CG77] therefore $(\Sigma_A)^\omega - h(T_\Sigma^\omega)$ is an omega context free language. □

We recall now a result established in [Fin01a] in the course of the proof of the above Theorem 4.2. We had seen that:

Lemma 4.9 *There exists a family of (effectively given) context free ω-languages $(A^\sim_{X,Y})^d$ over the alphabet $\{a, b, c, \twoheadleftarrow, d\}$ such that $(A^\sim_{X,Y})^d$ is either $\{a, b, c, \twoheadleftarrow, d\}^\omega$ or an ω-language which is a Borel set but neither a $\mathbf{\Pi^0_2}$-subset nor a $\mathbf{\Sigma^0_2}$-subset of $\{a, b, c, \twoheadleftarrow, d\}^\omega$. But one cannot decide which case holds.*

Consider now these languages. Denote $B(X, Y) = (A^\sim_{X,Y})^d$ and $\Sigma = \{a, b, c, \twoheadleftarrow, d\}$.
Then there are two cases.
In the first case $B(X, Y) = \Sigma^\omega$.
In the second case $B(X, Y)$ is neither a $\mathbf{\Pi^0_2}$-subset nor a $\mathbf{\Sigma^0_2}$-subset of Σ^ω.

Return now to the previous proof.
In the first case $Path(B(X, Y)) = Path(\Sigma^\omega) = T_\Sigma^\omega$.
In the second case $Path(B(X, Y))$ is a $\mathbf{\Sigma^1_1}$-complete subset of T_Σ^ω.

Construct now from $B(X, Y)$ another omega context free language $C(X, Y)$ over the alphabet Σ_A in the same manner as we have constructed C from B

in the above proof.

Let then $D(X,Y) = C(X,Y) \cup [(\Sigma_A)^\omega - h(T_\Sigma^\omega)]$. $D(X,Y)$ is an ω-CFL because it is the union of two ω-CFL and the class of omega context free languages is closed under union.

Then two cases may happen.
In the first case, $Path(B(X,Y)) = T_\Sigma^\omega$ hence $h(T_\Sigma^\omega) \subseteq C(X,Y)$ and $D(X,Y) = (\Sigma_A)^\omega$. Therefore $D(X,Y)$ is a closed and open subset of $(\Sigma_A)^\omega$.

In the second case $h^{-1}(D(X,Y)) = h^{-1}(C(X,Y)) = Path(B(X,Y))$ holds by construction and then $D(X,Y)$ is a $\mathbf{\Sigma_1^1}$-complete subset of $(\Sigma_A)^\omega$, for the same reason as $C(X,Y)$ is $\mathbf{\Sigma_1^1}$-complete .

But one cannot decide which case holds hence one cannot decide whether the context free ω-language is a Borel set.

To see that the result is also true for an alphabet containing two letters, consider the morphism $g : \{a, b, c, \twoheadleftarrow, d, A\}^\star \rightarrow \{a, b\}^\star$ defined by: $a \rightarrow bab$, $b \rightarrow ba^2b$, $c \rightarrow ba^3b$, $(\twoheadleftarrow) \rightarrow ba^4b$, $d \rightarrow ba^5b$, $A \rightarrow ba^6b$.
This morphism is λ-free and may be extended to infinite words in an obvious manner, giving a continuous function $\bar{g} : \{a, b, c, \twoheadleftarrow, d, A\}^\omega \rightarrow \{a, b\}^\omega$.

Let then $F(X,Y) = \bar{g}(D(X,Y))$.
$F(X,Y)$ is an ω-CFL because $D(X,Y)$ is an ω-CFL and the class of context free ω-languages is closed under λ-free morphism [CG77].
There are again two cases.
In the first case, $D(X,Y) = (\Sigma_A)^\omega$, hence $D(X,Y)$ is a compact set and, the image of a compact set by a continuous function being a compact set, $F(X,Y) = \bar{g}(D(X,Y))$ is a compact subset of $\{a,b\}^\omega$, therefore it is a closed subset of $\{a,b\}^\omega$.

In the second case, $D(X,Y) = \bar{g}^{-1}(F(X,Y))$ and $D(X,Y)$ is a $\mathbf{\Sigma_1^1}$-complete subset of T_Σ^ω, thus $F(X,Y)$ is also at least a $\mathbf{\Sigma_1^1}$-complete subset of $\{a,b\}^\omega$, and in fact it is a $\mathbf{\Sigma_1^1}$-complete subset because it is an analytic set as an ω-CFL. $\square$

Remark that we have also extended Theorem 4.2 to all Borel classes:

Theorem 4.10 *Let α be a countable ordinal ≥ 1. Then it is undecidable to determine whether an effectively given ω-CFL is in the class $\mathbf{\Sigma^0_\alpha}$ (respectively $\mathbf{\Pi^0_\alpha}$).*

Proof. The result has been proved for every finite ordinal (integer) ≥ 1 in [Fin01a]. Let then α be a countable infinite ordinal. The above defined ω-CFL $F(X,Y)$ is either a $\mathbf{\Pi_1^0}$-subset or a $\mathbf{\Sigma_1^1}$-complete subset of $\{a,b\}^\omega$. In the first case it is in the class $\mathbf{\Sigma^0_\alpha}$ (respectively $\mathbf{\Pi^0_\alpha}$) and in the second case it is not a Borel set. But one cannot decide which case holds. $\square$

5 ω-powers of finitary languages

We study in this section ω-powers of finitary languages, i.e. ω-languages in the form V^ω where V is a finitary language. ω-powers of finitary languages are always analytic sets because whenever V is finite, V^ω is an ω-regular language and then it is a boolean combination of $\mathbf{\Sigma^0_2}$-sets and whenever V is countably infinite, one can fix an enumeration of V and obtain V^ω as a continuous image of ω^ω (the set of infinite sequences of integers ≥ 0), [Sim92].

Niwinski asked in [Niw90] for an example of finitary language W such that W^ω is an analytic but non Borel set.
From the results of preceding section, we can easily find an example of a context free language W such that W^ω is not a Borel set.

Consider the construction of the ω-language C from the ω-language $B \subseteq \Sigma^\omega$ in the proof of Theorem 4.5. As stated above, if g is the substitution $\Sigma \rightarrow P((\Sigma \cup \{A\})^\star)$ defined by $a \rightarrow a.D$ where

$$D = \{u.A.v \ \mid \ u, v \in \Sigma^\star \ and \ (|v| = 2|u|) \ \ or \ \ (|v| = 2|u| + 1) \ \}$$

then D is a context free language over the alphabet $(\Sigma \cup \{A\})$ and $g(B) = C$ holds.

Assume now that B is an ω-power in the form V^ω. Then $g(B) = (g(V))^\omega$ is also an ω-power.

Let then $\Sigma = \{0, 1\}$ be an alphabet containing two letters 0 and 1 and $W = 0^\star.1$. Then $W^\omega = (0^\star.1)^\omega$ is the set of ω-words over the alphabet Σ which contain infinitely many occcurrences of the letter 1. It is a well known example of an ω-regular language which is a $\mathbf{\Pi^0_2}$-complete subset of Σ^ω.

Thus the language $g(W)$ is a finitary context free language such that $(g(W))^\omega$ is an analytic but non Borel set.
This language $g(W)$ is in fact a one counter language.

This gives an answer to Niwinski's question and additional answer to questions of Simonnet who asked in [Sim92] for the topological complexity of the ω-powers of context free languages.

6 Arithmetical properties

We are going to deduce from the previous proofs some new results about ω-context free languages and the Arithmetical hierarchy. We recall first the definition of the Arithmetical hierarchy of ω-languages, [Sta97a].

Let X be a finite alphabet. An ω-language $L \subseteq X^\omega$ belongs to the class Σ_n if and only if there exists a recursive relation $R_L \subseteq (\mathbb{N})^{n-1} \times X^\star$ such that

$$L = \{\sigma \in X^\omega \mid \exists a_1 \ldots Q_n a_n \quad (a_1, \ldots, a_{n-1}, \sigma[a_n + 1]) \in R_L\}$$

where Q_i is one of the quantifiers $\forall$ or $\exists$ (not necessarily in an alternating order). An ω-language $L \subseteq X^\omega$ belongs to the class Π_n if and only if its complement $X^\omega - L$ belongs to the class Σ_n.
The inclusion relations that hold between the classes Σ_n and Π_n are the same as for the corresponding classes of the Borel hierarchy.

Proposition 6.1 ([Sta97a]) *a)* $\Sigma_n \cup \Pi_n \subsetneq \Sigma_{n+1} \cap \Pi_{n+1}$*, for each integer* $n \geq 1$*.*

b) A set $W \subseteq X^\omega$ *is in the class* Σ_n *if and only if its complement* W^- *is in the class* Π_n*.*

c) $\Sigma_n - \Pi_n \neq \emptyset$ *and* $\Pi_n - \Sigma_n \neq \emptyset$ *hold for each integer* $n \geq 1$*.*

The classes Σ_n and Π_n are strictly included in the respective classes $\mathbf{\Sigma^0_n}$ and $\mathbf{\Sigma^0_n}$ of the Borel hierarchy:

Theorem 6.2 ([Sta97a]) *For each integer* $n \geq 1$, $\Sigma_n \subsetneq \mathbf{\Sigma^0_n}$ *and* $\Pi_n \subsetneq \mathbf{\Pi^0_n}$.

Recall now preceding results of [Fin01a]:

Theorem 6.3 *Let n be an integer ≥ 1. Then it is undecidable whether an effectively given ω-CFL is in the class Σ_n (respectively Π_n).*

As in the case of the Borel hierarchy, projections of arithmetical sets (of the second Π-class) lead beyond the Arithmetical hierarchy, to the Analytical hierarchy of ω-languages. The first class of this hierarchy is the class Σ^1_1. An ω-language $L \subseteq X^\omega$ belongs to the class Σ^1_1 if and only if there exists a recursive relation $R_L \subseteq (\mathbb{N}) \times \{0,1\}^\star \times X^\star$ such that:

$$L = \{\sigma \in X^\omega \mid \exists \tau (\tau \in \{0,1\}^\omega \wedge \forall n \exists m ((n, \tau[m], \sigma[m]) \in R_L))\}$$

Then an ω-language $L \subseteq X^\omega$ is in the class Σ^1_1 iff it is the projection of an ω-language over the alphabet $X \times \{0,1\}$ which is in the class Π_2 of the arithmetical hierarchy.

It turned out that an ω-language $L \subseteq X^\omega$ is in the class Σ_1^1 iff it is accepted by a non deterministic Turing machine (reading ω-words) with a Muller acceptance condition [Sta97a]. This class is denoted $NT(inf, =)$ (where $(inf, =)$ indicates the Muller condition) in [Sta97a] and also called the class of recursive ω-languages REK_ω. [1]

With the above definitions, one can state the following:

Theorem 6.4 ([Sta97a]) *The class CFL_ω is strictly included into the class REK_ω of recursive ω-languages.*

A natural question arises: are there ω-CFL which are in the class Σ_1^1 but in not any class of the arithmetical hierarchy? The answer can be easily derived from the preceding corresponding results about the Borel Hierarchy.

Theorem 6.5 *There exist some context free ω-languages in $\Sigma_1^1 - \bigcup_{n\geq 1} \Sigma_n$.*

Proof. It follows from Theorems 4.5 and 6.2. □

We now obtain a recursive analogue to Theorem 4.7:

Theorem 6.6 *Let Σ be an alphabet containing at least two letters. It is undecidable, for an effectively given context free ω-language B to determine whether B is in $\Sigma_1^1 - \bigcup_{n\geq 1} \Sigma_n$.*

Proof. Recall that we had found (see proof of Theorem 4.7) a family of context free ω-languages $D(X, Y)$ over the alphabet $\Gamma = \{a, b, c, \twoheadleftarrow, d, A\}$ such that $D(X, Y)$ is either a $\boldsymbol{\Sigma_1^1}$-complete subset of Γ^ω, or equal to Γ^ω.

Whenever $D(X, Y)$ is $\boldsymbol{\Sigma_1^1}$-complete, it is not in $\bigcup_{n\geq 1} \Sigma_n$ because each arithmetical class Σ_n (respectively Π_n) is included in the Borel class $\boldsymbol{\Sigma_n^0}$ (respectively $\boldsymbol{\Pi_n^0}$).

Whenever $D(X, Y)$ is equal to Γ^ω, $D(X, Y)$ is in the class Σ_1 because of the characterization of ω-languages in Σ_1 [Sta97a]: an ω-language $L \subseteq X^\omega$ belongs to the class Σ_1 if and only if there exists a recursive finitary language $W \subseteq X^\star$ such that $L = W.X^\omega$.

But we had proved that one cannot decide which of these two cases holds, hence the result is proved for the alphabet Γ. (And we can use similar methods as in the proof of Theorem 4.7 to obtain the result for an alphabet of cardinal ≥ 2). □

Considering Turing machines, we get the following:

[1] In another presentation, as in [Rog67], the recursive ω-languages are those which are in the intersection $\Sigma_1 \cap \Pi_1$, see also [LT94].

Theorem 6.7 *It is undecidable to determine whether the complement of an effectively given ω-CFL is accepted by a non deterministic Turing machine with Büchi (respectively Muller) acceptance condition.*

Proof. As in the preceding proof consider the family of context free ω-languages $D(X,Y)$ over the alphabet $\Gamma = \{a, b, c, \twoheadleftarrow, d, A\}$ such that $D(X,Y)$ is either a $\mathbf{\Sigma^1_1}$-complete subset of Γ^ω, or equal to Γ^ω.

Whenever $D(X,Y)$ is $\mathbf{\Sigma^1_1}$-complete, its complement is $\mathbf{\Pi^1_1}$-complete thus it is not a $\mathbf{\Sigma^1_1}$ set (because a set which is both $\mathbf{\Sigma^1_1}$ **and** $\mathbf{\Pi^1_1}$ is a Borel set) and therefore it is not a Σ^1_1-set (because the class Σ^1_1 is included in the class $\mathbf{\Sigma^1_1}$) then it is not accepted by any Turing machine with Büchi (respectively Muller) acceptance condition.

In the other case $D(X,Y)$ is equal to Γ^ω, then its complement is the empty-set and it is accepted by a Turing machine with Büchi (respectively Muller) acceptance condition.

But we had proved that one cannot decide which of these two cases holds, hence the result is proved for the alphabet Γ. (And we can use similar methods as in the proof of Theorem 4.7 to obtain the result for an alphabet of cardinal ≥ 2). □

In fact this result can be extended to other deterministic machines. Consider $\mathbf{X}$-automata as defined in [EH93] which are automata equipped with a storage type $\mathbf{X}$.

Theorem 6.8 *Let $\mathbf{X}$ be a storage type as defined in [EH93]. Then it is undecidable to determine whether the complement of an effectively given ω-CFL is accepted by a non deterministic $\mathbf{X}$-automaton with Büchi (respectively Muller) acceptance condition.*

Proof. It is similar to the previous one because every $\mathbf{X}$-automaton is less expressive than a Turing machine hence it cannot accept any $\mathbf{\Pi^1_1}$-complete set. And conversely Γ^ω is accepted by every $\mathbf{X}$-automaton. □

7 Context free languages of infinite trees

The theory of automata reading infinite words have been extended to automata reading infinite binary trees labelled in a finite alphabet, i.e. trees in a space T^ω_Σ where Σ is a finite alphabet (and one may also consider infinite k-ary trees labelled in Σ but we shall restrict ourselves here to binary trees), see [Tho90][Tho96][LT94][Sim92] for many results and references.

It is known that regular languages of infinite binary trees exhaust the hierarchy of Borel sets of finite rank as shown by Skurczynski [Sku93]. Niwinski proved that there exist some regular set of trees which are non Borel sets, [Niw85].

Some regular sets of trees are $\mathbf{\Sigma_1^1}$-complete, as $Path(B)$ where B is any $\mathbf{\Pi_2^0}$-complete regular subset of Σ^ω. $Path(B)$ (defined in the proof of Theorem 4.5) is accepted by a non deterministic tree automaton which guesses a branch of a tree (using the non determinism) and then simulates a finite automaton on the path associated with this branch.
One can also define, for each ω-language $B \subseteq \Sigma^\omega$, the following sets of trees. Let

$$\forall - Path(B)$$

be the set of trees t in T_Σ^ω such that every path of t is in B, and let

$$Left - Path(B)$$

be the set of trees t in T_Σ^ω such that the leftmost path of t is in B (the nodes of the leftmost branch are the words of $\{l, r\}^\star$ which are in the form l^n for an integer $n \geq 0$).

It is then well-known that whenever $B \subseteq \Sigma^\omega$ is an ω-regular language , the sets $\forall - Path(B)$ and $Left - Path(B)$ are regular sets of trees. Then if B is a $\mathbf{\Pi_2^0}$-complete subset of Σ^ω it holds that:

$$\forall - Path(B^-) = T_\Sigma^\omega - (Path(B))$$

hence $\forall - Path(B^-)$ is a $\mathbf{\Pi_1^1}$-complete subset of T_Σ^ω.

The Theorem of complementation of Rabin implies that every regular set of trees is in $\mathbf{\Sigma_2^1} \cap \mathbf{\Pi_2^1}$, and it has been shown that there exist regular sets of trees which are not in $\mathbf{\Sigma_1^1} \cup \mathbf{\Pi_1^1}$, see [LT94] for a view of a hierarchy of regular sets of trees.

As finite automata have been extended to (top-down) automata on infinite trees, pushdown automata have been extended to (top-down) pushdown automata on infinite trees by Saoudi [Sao92]. Denote, as in [Sao92], CF_3 the family of languages of infinite (binary) trees accepted by (top-down) pushdown automata with Muller acceptance condition.

It is easy to see from the definition of these automata that, as in the case of tree automata, if B is an ω-CFL, then the sets of trees $Path(B)$ and $Left - Path(B)$ are accepted by tree pushdown automata. Then we can extend our preceding undecidability results of Theorems 4.7 and 4.10.

Theorem 7.1 *(a) Let α be a countable ordinal ≥ 1. Then it is undecidable to determine whether an effectively given language in CF_3 is in the Borel class $\mathbf{\Sigma_\alpha^0}$ (respectively $\mathbf{\Pi_\alpha^0}$).*

(b) It is undecidable to determine whether an effectively given language in CF_3 is a Borel set.
(c) It is undecidable to determine whether an effectively given language in CF_3 is in the class $\mathbf{\Pi^1_1}$.
(d) It is undecidable to determine whether an effectively given language in CF_3 is a $\mathbf{\Sigma^1_1}$ but non Borel set.

Proof. The proofs are easily derived from the proof of Theorem 4.7. Recall we had got a family of omega context free languages $D(X,Y)$ over the alphabet Σ_A such that: either $D(X,Y) = (\Sigma_A)^\omega$, or $D(X,Y)$ is a $\mathbf{\Sigma^1_1}$-complete subset of $(\Sigma_A)^\omega$. But one cannot decide which case holds.
It is easy to see that $Left - Path(D(X,Y))$ has the same topological complexity as the ω-language $D(X,Y)$.
Indeed let f be the function: $(\Sigma_A)^\omega \rightarrow T^\omega_{\Sigma_A}$ defined by $f(\sigma) = t_\sigma$ where t_σ is the tree in $T^\omega_{\Sigma_A}$ with σ as leftmost path and the letter A labelling the other nodes. Then f is continuous and $f^{-1}(Left - Path(D(X,Y))) = D(X,Y)$. Assume first that $D(X,Y)$ is a $\mathbf{\Sigma^1_1}$-complete subset of $(\Sigma_A)^\omega$, then $Left - Path(D(X,Y))$ is also at least $\mathbf{\Sigma^1_1}$-complete and not a Borel set.
Now let j be the function $T^\omega_{\Sigma_A} \rightarrow (\Sigma_A)^\omega$ defined by: $j(t)$ is the leftmost path of the tree t. Then j is a continuous function and $j^{-1}(D(X,Y)) = Left-Path(D(X,Y))$. Hence when $D(X,Y)$ is a $\mathbf{\Sigma^1_1}$-complete subset of $(\Sigma_A)^\omega$, $Left-Path(D(X,Y))$ is a $\mathbf{\Sigma^1_1}$-set because the class $\mathbf{\Sigma^1_1}$ is closed under inverse of continuous functions. Thus $Left-Path(D(X,Y))$ is a $\mathbf{\Sigma^1_1}$-complete subset of $T^\omega_{\Sigma_A}$ and not a $\mathbf{\Pi^1_1}$-set.
In the other case $D(X,Y) = (\Sigma_A)^\omega$ and $Left - Path(D(X,Y)) = T^\omega_{\Sigma_A}$ then $Left - Path(D(X,Y))$ is in every Borel class and also in the class $\mathbf{\Pi^1_1}$. But one cannot decide which case holds. This proves (a), (b), (c) and (d). □

8 Concluding remarks and further work

We have proved in [Fin01a] that the class of ω-CFL exhausts the finite ranks of the Borel hierarchy and in this paper (Theorem 4.5) that there exist some analytic but non Borel ω-CFL.
The question to know whether there exist some ω-CFL which are Borel sets of infinite rank is still open.

There exists a refinement of the Borel hierarchy which is called the Wadge hierarchy of Borel sets. We proved in [Fin01b] that the length of the Wadge hierarchy of ω-CFL is an ordinal greater than or equal to the Cantor ordinal ε_0. And it remains to find the exact length of the Wadge hierarchy of Borel ω-CFL.

Mention that on the other side, the Wadge hierarchy of deterministic ω-CFL has been determined. its length is the ordinal $\omega^{(\omega^2)}$. It has been recently studied in [DFR01] [Dup99] [Fin99].

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