Some Aspects of Equational Categories [1]

By

F. E. J. LINTON *

Introduction

The theory of equationally definable classes of algebras, initiated by BIRKHOFF in the early thirties, is, despite its power, elegance and simplicity, hampered in its usefulness by two defects. The first is its refusal to deal with infinitary operations; the second is the awkwardness inherent in the presentation of an equationally definable class in terms of operations and equations.

Quite recently, LAWVERE [*3*], by introducing the notion — closely akin to the clones (cf. [*1*], Ch. III, § 3, Exer. 3) of P. HALL — of an *algebraic theory*, rectified the second defect without, unfortunately, rectifying the first. Not long before LAWVERE's work, SŁOMINSKI [*5*] rectified the first defect without, however, rectifying the second. It is possible, as it turns out, that both defects can be rectified at once; the present paper will sketch the highlights of the resulting theory. More complete details must, for lack of time, appear elsewhere.

The following pages are divided into seven sections, entitled: 1. Equational theories and their models; 2. The adjointness of structure and semantics; 3. The characterization of varietal categories; 4. Proof of the characterization theorem; 5. Illustrations and applications; 6. Variations on the theme: the question of rank; triples versus theories; 7. When are epimorphisms onto?

1. Equational Theories and their Models

An *equational theory* is a product preserving covariant functor $T : \mathscr{S}^* \to \mathbb{T}$ from the dual $\mathscr{S}^*$ of the category $\mathscr{S}$ of sets and functions to a (large) category $\mathbb{T}$ whose class of objects is put by T in one-one

[1] This paper, written during the author's participation in the 1965 N.S.F. Summer Seminar in Homological Algebra at Bowdoin College, summarises the results of research engaged in with the support of the Air Force Office of Scientific Research (AFOSR 520—64) and the National Science Foundation (NSF GP-2432) while on leave from Wesleyan University at the University of Chicago.

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correspondence with the objects of $\mathscr{S}^*$. It is convenient, then, to identify the classes of objects of $\mathscr{S}$ (or $\mathscr{S}^*$) and of $\mathbb{T}$. The equational theory T is *varietal* if the category $\mathbb{T}$ is locally small, that is, if each class $\mathbb{T}(n, k)$ of $\mathbb{T}$-morphisms from n to k is a set, whatever the sets (i.e., $\mathbb{T}$-objects) n, k.

A *morphism of theories* from T to $T' : \mathscr{S}^* \to \mathbb{T}'$ is any functor $j : \mathbb{T} \to \mathbb{T}'$ satisfying $j \cdot T \cong T'$. The resulting category of varietal theories will be denoted Th.

For each equational theory $T : \mathscr{S}^* \to \mathbb{T}$, we single out, from the category $\mathscr{S}^{\mathbb{T}}$ of all set valued functors on $\mathbb{T}$, the full subcategory $\mathscr{S}^T$ whose objects are those functors $X : \mathbb{T} \to \mathscr{S}$ for which the composite $X \cdot T : \mathscr{S}^* \to \mathscr{S}$ preserves products. Such a functor X is called a *model of T in $\mathscr{S}$*, or a *T-algebra*; any category $\mathscr{X}$ equivalent to the category $\mathscr{S}^T$ of T-algebras and T-homomorphisms is called an *equational category* (*varietal* if T is varietal).

Evaluation at the object $1 \in \mathbb{T}$ provides a functor $U_T : \mathscr{S}^T \to \mathscr{S}$, the *underlying set functor for T-algebras*, which makes $\mathscr{S}^T$ into a concrete category.

Proposition 1. *If the theory T is varietal, the underlying set functor for T-algebras has a left adjoint.*

Proof. For n (resp. k) an object or morphism of $\mathscr{S}$ (resp. $\mathbb{T}$), define

$$F_T(n)(k) = \mathbb{T}(T(n), k).$$

Then each $F_T(n)$ is a T-algebra[2] (called the *free T-algebra freely generated by the set n of free generators*), and F_T is a functor $\mathscr{S} \to \mathscr{S}^T$. That F_T is left adjoint to U_T is established by showing that the maps $\alpha_n : n \to U_T F_T(n)$, defined as the compositions

$$n \cong \mathscr{S}^*(n, 1) \xrightarrow{T} \mathbb{T}(T(n), T(1)) = F_T(n)(1) = U_T F_T(n),$$

serve as the front adjunction ("inclusion of the generators").

We shall say that a set valued functor $U : \mathscr{X} \to \mathscr{S}$ is *tractable* if, whatever the sets n and k, the class $\mathrm{n.t.}(U^n, U^k)$ of natural transformations from the functor U^n (given by $U^n(X) = (U(X))^n$) to the functor U^k is actually a set. Observe that the Yoneda Lemma guarantees the tractability[3] of any set valued functor U having a left adjoint F, since

$$\mathrm{n.t.}(U^n, U^k) \cong U^k(F(n)).$$

Corollary. *If the theory T is varietal, the underlying set functor for T-algebras is tractable.*

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[2] Indeed, $F_T(n)$ preserves all products in T.

[3] See lemma 1, § 3, for a stronger result.

2. The Adjointness of Structure and Semantics

Every functor $U: \mathscr{X} \to \mathscr{S}$ yields an equational theory, varietal if U is tractable, by taking the category $\mathbb{T}_U$ whose morphisms are the k-tuples of n-ary operations on U. That is, $\mathbb{T}_U$ is the category whose class of objects is the class of all sets, whose morphisms are given by

$$\mathbb{T}_U(n, k) = \text{n. t.}\,(U^n, U^k),$$

and whose composition rule is the usual composition of natural transformations. Because each function $f: k \to n$ between sets gives a natural transformation $U^f: U^n \to U^k$, we may define a contravariant functor $\exp_U: \mathscr{S} \to \mathbb{T}_U$ by

$$\exp_U(n) = n, \quad \exp_U(f) = U^f.$$

Product preserving when viewed as a covariant functor from $\mathscr{S}^*$ to $\mathbb{T}_U$, $\exp_U$ is the *equational theory of* the set valued functor U.

Taking as morphisms from one set valued functor $U: \mathscr{X} \to \mathscr{S}$ to another $U': \mathscr{X}' \to \mathscr{S}$ those functors $F: \mathscr{X} \to \mathscr{X}'$ with $U' \cdot F \cong U$, and writing K for the category of tractable set valued functors, we have the following important result.

Proposition 2. *The passages*

$$U \rightsquigarrow \exp_U: K \to Th$$
$$T \rightsquigarrow U_T: Th \to K$$

are contravariantly functorial with respect to morphisms of set valued functors and of equational theories, and are adjoint on the right.

Proof. Given a K-morphism $F: U \to U'$, note that $U'^n \cdot F \cong U^n$, and define $\mathbb{T}_F: \mathbb{T}_{U'} \to \mathbb{T}_U$ by sending $t: U'^n \to U'^k$ to $\mathbb{T}_F(t) = t \cdot F = t_F: U^n \to U^k$. Given a Th-morphism $j: T \to T'$ and a T'-algebra X, observe that $X \cdot j$ is a T-algebra, and define $\mathscr{S}^j: \mathscr{S}^{T'} \to \mathscr{S}^T$ by $\mathscr{S}^j(X) = X \cdot j$. Writing

$$\begin{array}{ll} \mathfrak{S}(U) = \exp_U & (U \in K) \\ \mathfrak{S}(F) = \mathbb{T}_F & (F \text{ a } K\text{-morphism}) \\ \mathfrak{M}(T) = U_T & (T \in Th) \\ \mathfrak{M}(j) = \mathscr{S}^j & (j \text{ a } Th\text{-morphism}) \end{array}$$

we obtain functors $\mathfrak{S}: K \to Th$ and $\mathfrak{M}: Th \to K$, called *equational structure* and *equational semantics*, respectively, extending the terminology of Lawvere.

For the adjointness assertion, one of the adjunctions is taken to be the natural *equivalence* id $\to \mathfrak{S} \cdot \mathfrak{M}$ provided by proposition 1 and the Yoneda Lemma:

$$\mathbb{T}(n, k) = F_T(n)(k) = U_T^k(F_T(n)) = \text{n. t.}\,(U_T^n, U_T^k) = \mathbb{T}_{U_T}(n, k);$$

the other, $\Phi\colon \mathrm{id} \to \mathfrak{M} \cdot \mathfrak{S}$, sends the set valued functor $U\colon \mathscr{X} \to \mathscr{S}$ to the functor $\Phi_U\colon \mathscr{X} \to \mathscr{S}^{\exp_U}$ given by

$$\begin{array}{ll} \Phi_U(X)\,(n) = U^n(X) & (X \in \mathscr{X},\quad n \in \mathbb{T}_U) \\ \Phi_U(X)\,(t) = t_X & (X \in \mathscr{X},\quad t \text{ a } \mathbb{T}_U\text{-morphism}) \\ (\Phi_U(f))_n = U^n(f) & (f \text{ an } \mathscr{X}\text{-morphism},\ n \in \mathbb{T}_U). \end{array}$$

Corollary. *The semantics functor* $\mathfrak{M}\colon Th \to K$ *is both full and faithful.*

3. The Characterization of Varietal Categories

We outline here the form of Lawvere's characterisation theorem for algebraic categories appropriate to the present more general context. Some preliminary comments are needed.

Let us remark first that each category $\mathscr{S}^T$ of T-algebras has all set-indexed inverse limits, and that U_T preserves them (and hence is faithful). What is more, the first isomorphism theorem is valid in $\mathscr{S}^T$, in a sense to be explained below, with respect to U_T. When T is varietal, moreover, $\mathscr{S}^T$ has all set-indexed direct limits, as may be proved by straightforward construction as a quotient of a free.

Next, we give two simple but useful lemmas regarding Φ_U.

Lemma 1. *Let* $U\colon \mathscr{X} \to \mathscr{S}$ *be a set valued functor with left adjoint* F. *Then* $\exp_U$ *is varietal and* $\Phi_U \cdot F \cong F_{\exp_U}$.

Proof.

$$\Phi_U(F(n))\,(k) = U^k(F(n)) \cong \mathrm{n.t.}(U^n, U^k) = \mathbb{T}_U(n,k) = F_{\exp_U}(n)\,(k)\,.$$

Lemma 2. *The functor* $\Phi_U\colon \mathscr{X} \to \mathscr{S}^{\exp_U}$ *is faithful if and only if the set valued functor* $U\colon \mathscr{X} \to \mathscr{S}$ *is.*

Proof. This is an immediate consequence of the faithfulness of the underlying set functor for $\exp_U$-algebras and the commutativity[4] of the diagram

$$\begin{tikzcd} & \mathscr{S}^{\exp_U} \arrow[dd] \\ \mathscr{X} \arrow[ur, "\Phi_U"] \arrow[dr, "U"'] & \\ & \mathscr{S} \end{tikzcd}$$

Finally, in any category $\mathscr{X}$, consider three maps in the configuration

$$R \underset{p_1}{\overset{p_2}{\rightrightarrows}} X \xrightarrow{\pi} Q \quad (\pi \cdot p_1 = \pi \cdot p_2)\,. \tag{3.1}$$

[4] Reflecting the fact that Φ_U is a K-morphism.

π is the *difference cokernel* of the pair (p_1, p_2) (loosely speaking, Q is the quotient of X by the relations R) if each map π' from X satisfying

$$\pi' \cdot p_1 = \pi' \cdot p_2$$

is of the form $\pi' = y \cdot \pi$ for a unique map y from Q. (p_1, p_2) is the *kernel pair* of the map π (loosely speaking, R is the congruence relation on X induced by π) if each pair (p_1', p_2') of maps to X satisfying

$$\pi \cdot p_1' = \pi \cdot p_2'$$

determines a unique map x to R with $p_i' = p_i \cdot x$ $(i = 1, 2)$. The configuration (3.1) is *exact* if both the above envisioned relations are valid. The *first isomorphism theorem* is said to hold in $\mathscr{X}$ with respect to a set valued functor $U: \mathscr{X} \to \mathscr{S}$ if

(FIT_0) $\mathscr{X}$ has all kernel pairs and difference cokernels;

(FIT_1) a necessary and sufficient condition for an $\mathscr{X}$-morphism π to be a difference cokernel is that $U(\pi)$ be, i.e., that $U(\pi)$ be onto;

(FIT_2) a pair (p_1, p_2) of $\mathscr{X}$-morphisms p_i: $R \to X$ is a kernel pair if and only if (Up_1, Up_2) is, i.e., if and only if the function $UR \to UX \times UX$ induced by the pair of functions Up_i $(i = 1, 2)$ is one-one and has an equivalence relation as image.

Remark. Although it follows from the conjunction of (FIT_1) and (FIT_2) that a necessary and sufficient condition for the exactness of (3.1) is the exactness of

$$U(R) \underset{U(p_1)}{\overset{U(p_2)}{\rightrightarrows}} U(X) \xrightarrow{U(\pi)} U(Q), \tag{3.2}$$

the converse is false, as JON BECK has kindly pointed out in his comments to an earlier draft of this paper.

Proposition 3. *An arbitrary category $\mathscr{X}$ is varietal if and only if the first isomorphism theorem holds in $\mathscr{X}$ with respect to some set valued functor U having a left adjoint F.*

4. Proof of the Characterization Theorem

It being clear from the discussion at the head of § 3 that the conditions stated in proposition 3 for $\mathscr{X}$ to be varietal are necessary, we confine our attention to the proof of their sufficiency. To this end, we agree, throughout this section, to work with a category $\mathscr{X}$, having all kernel pairs and difference cokernels, and a functor U: $\mathscr{X} \to \mathscr{S}$ having a left adjoint $F: \mathscr{S} \to \mathscr{X}$. Then U is tractable; for convenience, we write $\mathbb{T} = \mathbb{T}_U$, $T = \exp_U$, $|\ \ | = U_T$, and $\Phi = \Phi_U : \mathscr{X} \to \mathscr{S}^T$.

The proof of proposition 3 may then be summarised as follows.

Proposition 4. (FIT_1) *guarantees that Φ is full, faithful, and has a left adjoint.* (FIT_2), *in the presence of* (FIT_1), *ensures that the left adjoint to Φ is actually a two sided inverse to Φ, so that, under Φ, $\mathscr{X}$ is equivalent to $\mathscr{S}^T$.*

Proof. Assume (FIT_1). That Φ is faithful is a consequence of lemma 2, the faithfulness of U coming (cf. [2], Ch. II, Prop. 1.5) from the fact that U reflects epimorphisms (every difference cokernel always being epi).

All the rest of the proof depends on the fact that every object X of $\mathscr{X}$ has a presentation as a quotient of a free, in the sense that there is a set n and an $\mathscr{X}$-morphism $\pi\colon F(n) \to X$ which is a difference cokernel. This is proved by taking $n = UX$ and π the back adjunction; π is a difference cokernel, by (FIT_1), since $U(\pi)\colon UF\,UX \to UX$ is onto.

To see that Φ is full, observe first, recalling lemma 1, that it is at least full between free $\mathscr{X}$-objects:

$$\mathscr{X}(F(k), F(n)) \cong U^k(F(n)) = |F_T(n)|^k \cong \mathscr{S}^T(F_T(k), T_T(n)).$$

Then, given X, $Y \in \mathscr{X}$ and $f\colon \Phi X \to \Phi Y$, represent X and Y as quotients of free $\mathscr{X}$-objects, say by maps

$$\pi\colon F(k) \to X, \quad \varrho\colon F(n) \to Y.$$

Using lemma 1 to identify $\Phi F(k)$, lift f over $\Phi(\pi)$ and $\Phi(\varrho)$ to an $\mathscr{S}^T$-morphism $\bar{f}\colon F_T(k) \to F_T(n)$; there is then a unique $\mathscr{X}$-morphism $\overline{\varphi}\colon F(k) \to F(n)$ with $\Phi(\overline{\varphi}) = \bar{f}$. It remains to check that $\varrho\overline{\varphi}p_1 = \varrho\overline{\varphi}p_2$ whenever $\pi \cdot p_1 = \pi \cdot p_2$; granting this, $\varrho\overline{\varphi}$ factors through π to give a unique $\mathscr{X}$-morphism $\varphi\colon X \to Y$ with $\varphi\pi = \varrho\overline{\varphi}$; for this φ, $\Phi(\varphi) = f$, as is readily checked at the level of the underlying sets.

Next, to construct the left adjoint to Φ, take a T-algebra $X \in \mathscr{S}^T$. Represent it as a quotient of a free T-algebra,

$$E \underset{p_1}{\overset{p_2}{\rightrightarrows}} F_T(n) \xrightarrow{\pi} X \quad (\pi = \operatorname{cok}(p_1, p_2), \quad (p_1, p_2) = \ker(\pi)),$$

and represent E in turn as a quotient of a free

$$F_T(k) \xrightarrow{p} E;$$

then $\pi = \operatorname{cok}(p_1 p,\ p_2 p)$. Now take $\mathscr{X}$-morphisms $\varrho_1, \varrho_2\colon F(k) \to F(n)$ with $\Phi(\varrho_i) = p_i \cdot p$ $(i = 1, 2)$, and form the difference cokernel (in $\mathscr{X}$)

$$t\colon F(n) \to \bar{X}$$

of the pair (ϱ_1, ϱ_2). It must then be checked that $\Phi(t) \cdot p_1 = \Phi(t) \cdot p_2$; granting that, there is a unique T-homomorphism $\alpha_X\colon X \to \Phi(\bar{X})$ making commutative the diagram

$$\begin{tikzcd} F_T(n) \arrow[rr, "\pi"] \arrow[dr, "\Phi(t)"'] & & X \arrow[dl, "\alpha_X"] \\ & \Phi(\bar{X}) & \end{tikzcd}$$

These maps α_X constitute the front adjunction making the passage $X \rightsquigarrow \bar{X} : \mathscr{S}^T \to \mathscr{X}$ a functor left adjoint to Φ.

Finally, assuming the validity of (FIT_2) in addition, Φ is an equivalence because each T-homomorphism α_X is an isomorphism, as can be seen by comparing the congruence relation of $U(t)$ with that of $|\pi|$ (they're the same).

This completes the proof.

5. Illustrations und Applications

It should be clear that both SLOMINSKI's equationally definable classes of algebras and LAWVERE's algebraic categories, taken with the standard underlying set functors, are varietal. There are, however, other varietal categories of interest, some of which we list here.

1. Compact Hausdorff spaces, with the usual underlying set functor, form a varietal category. The first isomorphism theorem is easily proved, using compactness; the adjoint to the underlying set functor is provided by the Stone-Čech compactification. (The first isomorphism theorem fails so badly for just plain topological spaces that, for them, the functor Φ *is* the underlying set functor.)

2. An equational category that is not varietal is provided by the category of complete boolean algebras (and complete homomorphisms). In fact, the obvious underlying set functor fails to be tractable, as is shown by GAIFMANN's proof[5] of the nonexistence of a free complete boolean algebra freely generated by a countable set.

3. Modifying example 2 slightly by taking the full subcategory of complete *atomic* boolean algebras does yield a varietal category (the adjoint to the underlying set functor assigns to the set n the boolean ring of subsets of the set of 2-valued functions on n).

4. Compact abelian groups form a varietal category. The first isomorphism theorem is not hard; the adjoint to the underlying set functor is obtained by passing from the set n to the character group $((G(n)\hat{\ })_d)\hat{\ }$ of the character group made discrete $(G(n)\hat{\ })_d$ of the free abelian group $G(n)$ generated by n.

5. More generally, the category of all compact groups is varietal — indeed, whatever the varietal theory T, the category of all compact T-algebras is varietal. Again, the first isomorphism theorem is clear; the adjoint to the underlying set functor may be obtained, for example, by use of the adjoint functor theorem (the only conceivable stumbling block is the solution set condition, but a counting argument disposes of that).

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[5] GAIFMANN shows that n.t. $(U^{\aleph_0}, U)$ is a proper class, where U is the underlying set functor for complete boolean algebras.

These examples should suffice. We are tempted to point out that examples 3 and 4 arise as the duals of the algebraic categories of sets and discrete abelian groups, respectively, and to speculate when the dual of a varietal category need itself be varietal. Of greater interest, however, is the following very useful blanket assertion regarding the existence of adjoint functors between varietal categories.

Proposition 5. *Let $U_i \colon \mathscr{X}_i \to \mathscr{S}\ (i = 1, 2)$ be functors by virtue of which $\mathscr{X}_i$ is varietal, and suppose $T \colon \mathscr{X}_1 \to \mathscr{X}_2$ is a functor satisfying $U_2 \cdot T \cong U_1$. Then T has a left adjoint $S \colon \mathscr{X}_2 \to \mathscr{X}_1$.*

Proof. Writing F_i for the left adjoint to U_i, pass from $X \in \mathscr{X}_2$ to the quotient of $F_1(U_2(X))$ by the intersection of the congruence relations induced on $F_1(U_2(X))$ by those $\mathscr{X}_1$-morphisms $F_1(U_2(X)) \to X_1$ that arise as values of the transformation

$$\mathscr{X}_2(X, TX_1) \xrightarrow{U_2} \mathscr{S}(U_2X, U_2TX_1) \cong \mathscr{S}(U_2X, U_1X_1) \cong \mathscr{X}_1(F_1U_2X, X_1).$$

The details of the proof that this passage provides the desired adjoint are very similar to standard arguments.

Corollary. *Let T be a varietal theory. The forgetful functor from the category of compact T-algebras to $\mathscr{S}^T$, as well as that from compact T-algebras to compact spaces, has a left adjoint.*

In particular, one has compact abelian groups freely generated (in a manner of speaking) by compact spaces, compact groups freely generated by (or the universal compactifications of) discrete groups, and so on. The full import of proposition 5 awaits further exploitation.

6. Variations on the Theme: The Question of Rank; Triples Versus Theories

Call a cardinal number $\mathfrak{r}$ *regular* if each sum of fewer than $\mathfrak{r}$ cardinals each less than $\mathfrak{r}$ is itself less than $\mathfrak{r}$. The *rank* of a theory $T \colon \mathscr{S}^* \to \mathbb{T}$ is the least regular cardinal $\mathfrak{r} \geqq 2$ (if any; ∞ otherwise) for which it is the case that each $\mathbb{T}$-morphism $t \colon n \to 1$ can be represented as a composition

$$n \xrightarrow{T(f)} k \xrightarrow{t'} 1$$

with $f \in \mathscr{S}^*(n, k)$, $t' \in \mathbb{T}(k, 1)$, and $\operatorname{card}(k) < \mathfrak{r}$.

If $\mathbb{T}$ is any category whose objects are all sets, and if $\mathfrak{r}$ is a cardinal, ${}_{\mathfrak{r}}\mathbb{T}$ will denote the full subcategory of $\mathbb{T}$ whose objects have cardinality $< \mathfrak{r}$. Each equational theory $T \colon \mathscr{S}^* \to \mathbb{T}$ induces, by restriction, a product-preserving functor

$$ {}_{\mathfrak{r}}T \colon {}_{\mathfrak{r}}\mathscr{S}^* \to {}_{\mathfrak{r}}\mathbb{T}, \tag{6.1}$$

the $\mathfrak{r}$-*truncation* of T.

Proposition 6. *For any varietal theory $T \colon \mathscr{S}^* \to \mathbb{T}$ and any regular cardinal $\mathfrak{r} \geqq 2$, the following three assertions are equivalent:*

1. $\operatorname{rank}(T) \leq \mathfrak{r}$;

2. *each functor* $X\colon {}_{\mathfrak{r}}\mathbb{T} \to \mathscr{S}$ *whose composition with* ${}_{\mathfrak{r}}\mathbb{T}$ *preserves products has a unique extension to a T-algebra*;

3. $U_T F_T(n) = \bigcup_{\substack{\operatorname{card}(k) < \mathfrak{r} \\ f \in \mathscr{S}(k,n)}} U_T F_T(f)\,(U_T F_T(k))$.

Motivated by the salient features of (6.1), one makes the obvious modification in the definition of a varietal theory and comes up with a definition of *$\mathfrak{r}$-ary theory*. Calling a set valued functor U *$\mathfrak{r}$-tractable* if n.t.(U^n, U^k) is a set whenever the sets n and k have cardinality less than $\mathfrak{r}$, one can easily extend the definition of structure and semantics to provide a pair of contravariant functors, adjoint on the right, between the category $K_{\mathfrak{r}}$ of $\mathfrak{r}$-tractable set valued functors and the category $Th_{\mathfrak{r}}$ of $\mathfrak{r}$-ary theories. It then becomes possible to identify $Th_{\mathfrak{r}}$ with the full subcategory of Th whose objects are the varietal theories of rank $\leq \mathfrak{r}$. Bearing this in mind, we combine propositions 3 and 6 to obtain a sharper generalization of LAWVERE's characterization theorem.

Corollary. *The category $\mathscr{X}$ is equivalent to the category of all T-algebras, for some $\mathfrak{r}$-ary theory T, if and only if the first isomorphism theorem holds in $\mathscr{X}$ with respect to a set valued functor U having a left adjoint F of such a sort that each $\mathscr{X}$-morphism $F(1) \to F(n)$ has a factorization*

$$F(1) \to F(k) \xrightarrow{F(f)} F(n),$$

where $f \in \mathscr{S}(k, n)$ *and* $\operatorname{card}(k) < \mathfrak{r}$.

(LAWVERE's theorem is recovered from this corollary by taking $\mathfrak{r} = \aleph_0$.)

So much for the question of rank. Turning to another direction, we note that each varietal theory $T\colon \mathscr{S}^* \to \mathbb{T}$ has a left adjoint, namely the functor $\mathbb{T}(-, 1)$. If we now replace $\mathscr{S}$ by any category $\mathscr{A}$, we may speak of a *theory over $\mathscr{A}$* as a functor $T\colon \mathscr{A}^* \to \mathbb{T}$ having a left adjoint and setting up a one-one correspondence between the objects of $\mathscr{A}$ and the objects of $\mathbb{T}$. In the same way can be justified the definition of a T-algebra as a functor $X\colon \mathbb{T} \to \mathscr{S}$ whose composition XT with T is a representable contravariant functor on $\mathscr{A}$.

It turns out that this notion of theory is entirely equivalent with BECK's notion (exposed elsewhere in these Proceedings) of triple, and that the algebras in the sense of theories are, modulo this equivalence, the same as BECK's algebras over triples. In particular, proposition 3 serves to characterise those categories that, in BECK's terminology, are triplable over $\mathscr{S}$. The obvious question, whether (or under what hypotheses on $\mathscr{A}$) this proposition is still valid when $\mathscr{S}$ is replaced by $\mathscr{A}$, remains unanswered. (Added in proof: BECK has just answered this question. Details will appear elsewhere.)

7. When are Epimorphisms onto?

This last section is devoted to the elucidation of a condition on a varietal category, necessary and sufficient for every epimorphism to be onto. In § 2, we have already vaguely referred to a natural transformation $U^n \to U^k$ as a k-tuple of n-ary operations on the set valued functor U. Here we shall deal with k-tuples of partial n-ary operations, that is to say, with natural transformations $V \to U^k$ (with V a subfunctor of U^n) of a special sort to be called *k-fold clusters of implicit n-ary operations*[6] (briefly, *implicit clusters*) on U. The main result is this:

Proposition 7. *Let T be a varietal theory and U_T: $\mathscr{S}^T \to \mathscr{S}$ the underlying set functor for T-algebras. $U_T(f)$ is onto for every epimorphism f in $\mathscr{S}^T$ if and only if every implicit cluster on U_T is explicit.*

It can be proved (cf. [*4*]) that the optimistic but precarious assumption of the existence of a non-trivial injective in the category of boolean σ-algebras forces every epimorphism of σ-algebras to be onto. It is to be hoped the technique of implicit clusters will permit a more satisfactory solution to the question, whether a σ-epimorphism must be onto.

Clearly, the proof of proposition 7 requires some definitions. A *k-fold cluster of implicit n-ary operations* on a set valued functor U: $\mathscr{X} \to \mathscr{S}$ is a natural transformation

$$\eta : V \to U^k, \tag{7.1}$$

with V a subfunctor of U^n, subject to two conditions whose description requires introduction of the class E of all pairs (e_1, e_2) of natural transformations e_i: $U^{n+k} \to U$ for which both compositions

$$V \xrightarrow{\text{inclu} \times \eta} U^n \times U^k \cong U^{n+k} \xrightarrow{e_i} U \qquad (i = 1, 2)$$

are equal. The conditions are that for each $X \in \mathscr{X}$ and $x \in U^n X$,

(IC$_1$) there is at most one $y \in U^k X$ with

$$(e_1)_X (x, y) = (e_2)_X (x, y) \quad \text{for all} \quad (e_1, e_2) \in E;$$

(IC$_2$) $x \in VX$ iff there is a y as envisioned in (IC$_1$).

(Necessarily, $\eta_X(x)$ is that element y. Moreover, the assumptions that V is a sub*functor* and that η is a *natural* transformation are superfluous; any collection of subsets VX and functions η_X satisfying (IC$_1$)and (IC$_2$) will in fact be natural. Finally, given a class of pairs of natural trans-

[6] To illustrate what is envisioned by this notion, let us say that inversion, in the category of semigroups (or rings) with unit, the identity element, in the category of possibly empty groups, and countable union, in the category of countably intersection-complete boolean rings (not necessarily having units), are examples of single implicit unary, nullary, and $\aleph_0$-ary operations, respectively.

formations, E, satisfying (IC_1), one obtains an implicit cluster whose subfunctor V is defined by (IC_2), and whose η is forced.)

An implicit cluster (7.1) is *explicit* if the following necessary condition (7.2) for the relation $x \in VX$ is also sufficient. The inclusion function $n \to n+k$ induces a projection $U^{n+k} \to U^n$, composition with which sends $\mathrm{n.t.}(U^n, U)$ to $\mathrm{n.t.}(U^{n+k}, U)$. Let E' be those pairs of natural transformations $U^n \to U$ sent in this way to pairs in E. The necessary condition that we have in mind is (that $x \in U^n X$ and)

$$(e_1')_X(x) = (e_2')_X(x) \quad \text{for all} \quad (e_1', e_2') \in E'. \tag{7.2}$$

Turning to the proof of proposition 7, let $\mathscr{X} = \mathscr{S}^T$ and $U = U_T$, T a varietal theory. The sets $\mathrm{n.t.}(U^{n+k}, U)$ and $\mathrm{n.t.}(U^n, U)$ can be reinterpreted as $UF(n+k)$ and $UF(n)$, respectively, and the sets of pairs E and E' are in fact congruence relations. The inclusion $F(n) \to F(n+k)$ happily drops through to a T-homomorphism

$$F(n)/_{E'} \to F(n+k)/_E \tag{7.3}$$

which is both monic and epic.

In the same setting, each epimorphism $f: A \to B$ gives rise to an implicit cluster by setting $n = f(A)$, $k = UB - n$, and taking E in $F(n+k) \times F(n+k)$ to be the kernel pair (i.e., congruence relation) of the projection $F(n+k) \to B$ afforded by the back adjunction. (IC_1) holds because f is epi, and the map (7.3) obtained from the induced implicit cluster is just the inclusion in B of the image of f (precisely f, if f is mono).

The proof of proposition 7 is clinched by the lemma below, which shows explicitly how non onto epimorphisms must always arise. The proof of the lemma can safely be omitted.

Lemma. *The map* (7.3) *is onto if and only if the implicit cluster* (7.1) *is explicit.*

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Department of Mathematics
Wesleyan University
Middletown, Connecticut