Geometriae Dedicata **62:** 1–17, 1996.

Penrose Tilings as Coverings of Congruent Decagons

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(Received: 1 July 1994; revised version: 24 March 1995)

Abstract. The open problem of tiling theory whether there is a single aperiodic two-dimensional prototile with corresponding matching rules, is answered for coverings instead of tilings. We introduce admissible overlaps for the regular decagon determining only nonperiodic coverings of the Euclidean plane which are equivalent to tilings by Robinson triangles. Our work is motivated by structural properties of quasicrystals.

Mathematical Subject Classifications (1991): 52C20; 82D25.

Key words: tiling, Penrose tiling, aperiodic tile, quasicrystal

1. Introduction

A *tiling* T is a countable family of regular closed sets (*tiles*) which cover the Euclidean plane without overlaps of nonempty interior. We assume that all tiles are congruent to one of finitely many given sets, so-called *prototiles*. To generate a tiling, the pieces are assembled according to their shape (*puzzle principle*) or, in a more general way, with respect to markings on their sides and corners (*matching rules*). In 1966, a set of 20426 prototiles with corresponding matching rules inducing only nonperiodic tilings, was discovered by R. Berger. Subsequently, the number of *aperiodic* prototiles was reduced to two ([7], [6], [1], [11]). It is still an unsolved problem whether there are matching rules for a single tile, which admit only tilings with no translational symmetry.

In the present paper we show that the well-known *Penrose tilings* can also be generated by a single aperiodic 'prototile'. However, in contrast to *edge-to-edge* matching rules of real tiles the copies of our 'pototile' may intersect in sets with nonempty interior. On the regular decagon we introduce coloured subsets as shown in Figure 1.

We require that the boundary of every decagon is contained in a union of finitely many other pieces, such that all dark subsets are overlapped and colours match. Every constructible covering according to this overlapping rule corresponds to a tiling by Robinson triangles. Moreover, all of these patterns can be generated

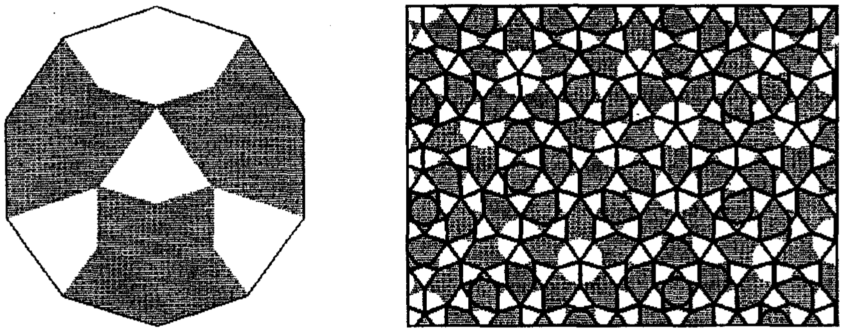


Figure 1. A single aperiodic 'prototile' and a corresponding covering.

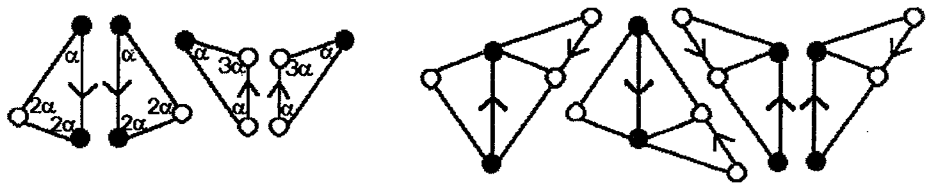
(Section 3). The result is prepared in Section 2 by a detailed investigation of Penrose tilings in their triangle version.

Our work is motivated by the physics of quasicrystals ([5], [9], [13], [16], [14], [15]). Penrose tilings are used as two-dimensional models for the atomic structure of alloys with local five- or ten-fold symmetry. But the shape of the usual pieces does not correspond to real atomic clusters. The description of this classical tiling model by overlapping decagons allows are more realistic explanation of growth processes as well as energetical aspects ([10], Remark 1). In contrast to the structural model of 'interpenetrating decagonal clusters' by S. E. Burkov ([3], [4]) which is related to *random tilings*, our formalism forces aperiodicity.

2. Penrose Tilings as Cartwheel Coverings

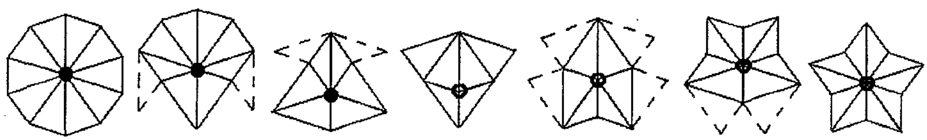
The main idea underlying our result is to define a generalized kind of matching rules, inducing particular local configurations of decagons. To derive them, we consider the version of Penrose tilings using four types of triangles (called *Penrose* or *Robinson triangles*). The length of their sides are one and $\tau = (1+\sqrt{5})/2$, the angles are multiples of $\alpha = \pi/5$. For details we refer to Grünbaum and Shephard [7, ch. 10].

The aperiodicity of these patterns is determined by a fundamental property, the so-called *self-similarity*. The concept of self-similarity is based on the idea of *composition*. A tiling T_1 is called a composition of a tiling T_0 if every tile of T_1 is a union of finitely many tiles of T_0. T_0 is *self-similar* if it contains an infinite family $\{T_i / i \in \{1,2,\ldots\}\}$ of compositions of itself such that the prototiles of each T_i are expanded copies of the prototiles of T_{i-1} (given by similarity mappings $f : \mathbb{R}^2 \to \mathbb{R}^2$ with a fixed expansion factor $r \in \mathbb{R}$, $r > 0$). If this composition process is unique, T_0 is *nonperiodic*. Suppose there is a translation $t : \mathbb{R}^2 \to \mathbb{R}^2$ with $t(T_0) = T_0$. Then t also has to be a translational symmetry of every composed tiling T_i, $i \in \{1,2,\ldots\}$, due to the uniqueness of composition. This leads to a contradiction because for some $k \in \{1,2,\ldots\}$ T_k contains only tiles with a diameter larger than the translation vector of t ([7, p. 524]).



a) b)

Figure 2. Robinson triangles with matching rules and self-similarity.



Sun *Jack* *Deuce* *Ace* *Queen* *King* *Star*

Figure 3. Vertex neighbourhoods of Robinson triangles.

A direct way to construct such self-similar tilings is to define a sequence of expanded tiles, each of which lies in a fixed position of the next larger one (*blowing up method*). In the case of Penrose tiles there are also matching rules which enforce a self-similar arrangement of the pieces. Furthermore, for every constructible tiling the composition process is uniquely defined.

The matching rules of the Robinson triangles (we follow [7]) and their possible positions in larger prototiles are shown in Figure 2. Since these markings permit only two different neighbours for two edges of the triangles and fix the remaining ones, there are only a few prototypes of *vertex stars* (*vertex configurations*, *vertex neighbourhoods*). Thus, for an arbitrary radius $R > 0$, we have only finitely many different types of finite unions of tiles (*patches*), contained in a ball B_R. This property is used to describe the *local isomorphism class*, the set of all Penrose tilings characterized by the same types of local patterns (with respect to Euclidean motions) for every radius $R > 0$. Moreover, the local isomorphism classes, corresponding to the different versions of Penrose tiles (the first six prototiles *P1*, *kite* and *dart*, *fat* and *skinny rhombus*) can be derived from the Robinson triangles and vise versa (concept of *mutual local derivability* in [2]).

The idea to replace Penrose's tiles by a single decagonal protoset is based on a theorem by Conway ([7, p. 562]): *Every Penrose triangle tiling is covered by congruent decagonal patches (so-called cartwheels).*

The result starts at the property of the triangle matching rules to permit only seven different types of vertex neighbourhood. Conway called them *sun*, *jack*, *deuce*, *ace*, *queen*, *king* and *star* in their kite and dart version (Figure 3).

With the exception of ace all of these vertex neighbourhoods force fixed arrangements of further triangles assembled around it, so called *empires*. The largest one (a patch of 110 tiles) is the king's empire. It contains the *cartwheel decagon* (see

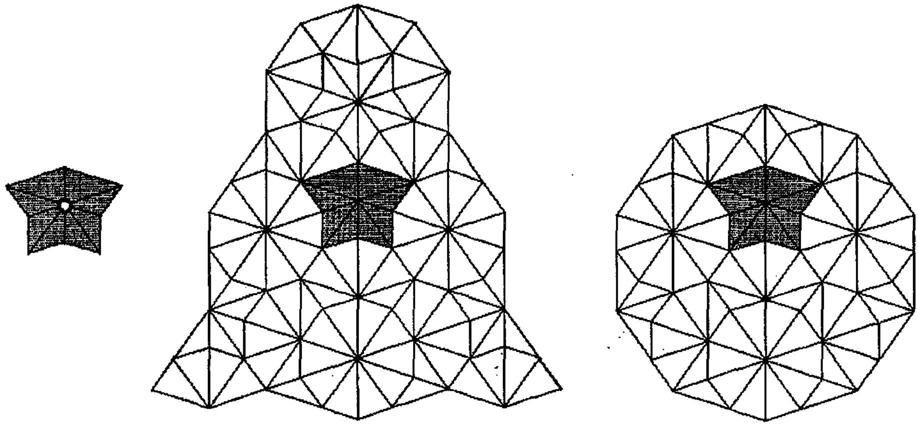


Figure 4. Vertex neighbourhood king, king's empire and cartwheel.

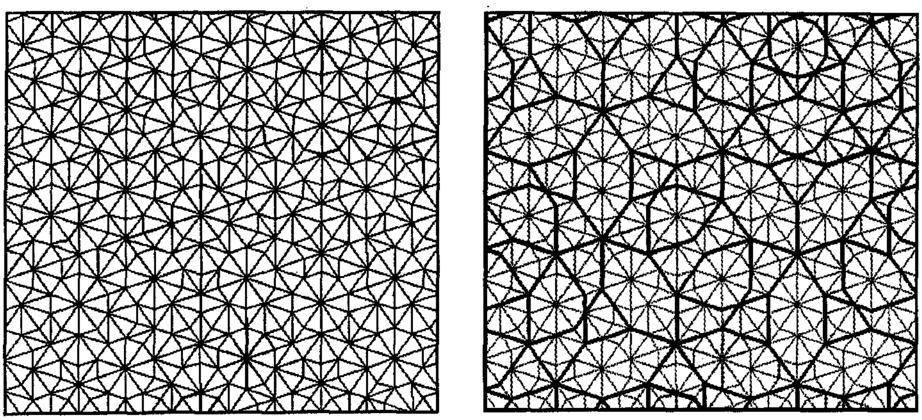


Figure 5. Penrose tiling and corresponding cartwheel covering.

Figure 4). A tiling by Robinson triangles and the corresponding cartwheel covering are drawn in Figure 5.

In the present paper we are interested in the question whether the cartwheel can also be used to *generate* Penrose tilings. Conway has constructed a very special example by self-similarity, the *cartwheel tiling*. It is defined by a sequence of concentric cartwheels C_i, $i \in \mathbb{N}$ with $\text{diam}(C_{i+1}) = \tau \cdot \text{diam}(C_i)$.

We will show that *all* these patterns can be obtained by cartwheel patches using the principle of matching rules. By the local finiteness of Penrose tilings all points $x \in \mathbb{R}^2$ belong to only finitely many distinct cartwheels. Consequently, a gradual covering of cartwheel boundaries by finitely many congruent neighbouring tiles must lead to the full local isomorphism class.

To simplify that method, we derive all local arrangements of cartwheels which may occur in Penrose tilings. Using the triangle matching rules we show that they are determined by only nine different types of neighbourhood (called *A–B*

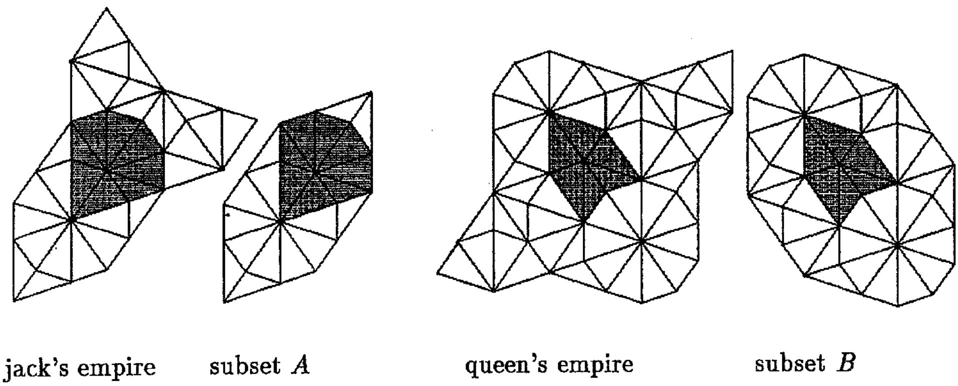


Figure 6.

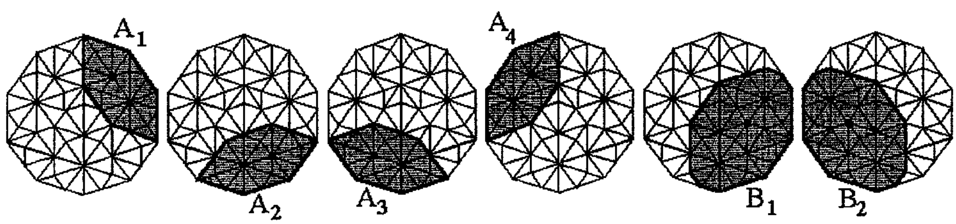


Figure 7. A- and B-subsets of a cartwheel.

neighbourhoods). They are characterized by only two types of intersections with nonempty interior (Propositions 1 to 3). In Lemma 1 we prove that these A–B neighbourhoods do exactly lead to all cartwheel coverings.

In Section 3 we introduce a generalized kind of matching rules (admissible overlaps) for the usual regular decagon forcing exactly the desired prototypes of local configurations. This means that there is an example of a single aperiodic 'prototile', if edge-to-edge correspondence is replaced by intersections of nonempty interior.

2.1. LOCAL CARTWHEEL CONFIGURATIONS

Suppose a Penrose triangle tiling T is given. To derive all prototypes of cartwheel neighbourhoods, we first investigate possible intersections of each two pieces $C_0, C_1 \subseteq T$.

PROPOSITION 1. *The intersection of two different cartwheels is either empty, or a common edge, or a patch of type A or B in Figure 6.*

A cartwheel contains four vertex neighbourhoods of type jack and two of type queen. The corresponding A- and B-subsets are drawn in Figure 7.

Proof of Proposition 1. C_0 and C_1 are assumed to be two distinct cartwheels of a given Penrose triangle tiling T with $C_0 \cap C_1 \neq \emptyset$. Since $C_0 \cup C_1$ is a patch of

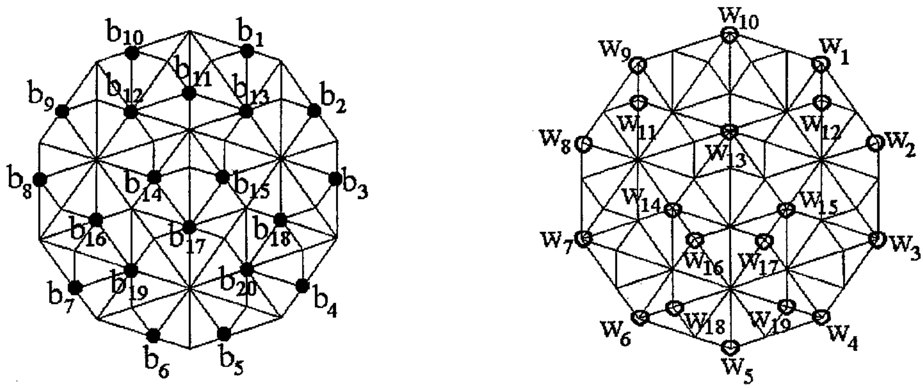


Figure 8.

T fulfilling the matching rules, C_0 and C_1 have at least one corner of a Robinson triangle in common. Using Conway's result, both cartwheels can be described by an isometry $h: \mathbb{R}^2 \to \mathbb{R}^2$ with $C_1 = h(C_0)$. Thus there are at least two triangle vertices $x, y \in C_0$ with $h(y) = x$, $x \in \partial C_0$. We have to mention that only vertices of the same type (marked by black and white dots) may be mapped into each other. They are marked by an indicated symbol *w* for *white* and *b* for *black* in Figure 8.

To prove our statement, we consider for every vertex $x \in \partial C_0$ all possible preimages $y \in C_0$ which lead to an admissible patch of a Penrose tiling. That means, the arrangement of all Robinson triangles around every vertex $x \in C_0 \cap h(C_0)$ has to be a subset of one of the seven vertex neighbourhoods. Except for w_{10} all vertices of ∂C_0 permit two neighbourhood-types (b_1 up to b_{10}: deuce or jack ; w_1, w_4, w_6, w_9: king or queen; w_2, w_5, w_8: ace or queen; w_3, w_7: king or star; w_{10}: star).

Case 1: $\mathrm{int}(C_0 \cap C_1) = \emptyset$
An intersection with empty interior implies $x, y \in \partial C_0$. First we consider the black vertices. It is easy to see that C_0 and C_1 cannot intersect in a single triangle vertex b_1 up to b_{10}. The triangle matching rules permit only rotations of edges $[w_i, w_j]$, $i, j \in \{1, 2, \ldots, 10\}$ into each other, such that a vertex neighbourhood deuce results for the corresponding b_j. Checking admissible vertex neighbourhoods for w_i and w_j we obtain a list of all possible 'edge-to-edge' neighbours, illustrated in the graph of Figure 9. Since the white vertices of ∂C_0 cannot be mapped into each other by other isometries, there are no further prototypes of cartwheel intersections with empty interior.

Case 2: $\mathrm{int}(C_0 \cap C_1) \neq \emptyset$
If two regular decagons intersect in a set of nonempty interior, at least one corner x of the first decagon is an interior point of the second one. Therefore, $C_0 \cap h(C_0)$ contains at least one of the corresponding vertex configurations of x. Consequently, we have to check all possible preimages for the admissible vertex neighbourhoods

Table I.

$x \in \partial C_0$	$y \in \mathrm{int}(C_0)$ with $h(y) = x$	$C_0 \cap h(C_0)$
w_1	w_{13}, w_{14}	A_1
w_2	w_{14}, w_{17}	A_1
	w_{11}	B_1
w_3	w_{13}	B_1
w_4	w_{13}, w_{14}	A_2
	w_{15}	B_1
w_5	w_{14}, w_{17}	A_2
	w_{15}, w_{16}	A_3
	w_{19}	B_1
	w_{18}	B_2
w_6	w_{13}, w_{15}	A_3
	w_{14}	B_2
w_7	w_{13}	B_2
w_8	w_{15}, w_{16}	A_4
	w_{12}	B_2
w_9	w_{13}, w_{15}	A_4
w_{10}	–	

of w_1 up to w_{10}. As shown in Table I, the corresponding intersection is always congruent to a set of type A or B.

There are only five prototypes of ‘A–B overlaps’ $(A_1, A_4), (A_1, A_3), (B_1, B_2)$, (A_2, A_4) and (A_2, A_3), which are drawn in Figure 10. With respect to the centres of the decagons they can be described by the following rotations h_1 up to h_5 and their inverse mappings:

$$h_1(z) = z \cdot \omega^2 + l \cdot \omega^1 \quad \text{(Figure 10a)},$$
$$h_2(z) = z \cdot \omega^9 + l \cdot \omega^1 \quad \text{(Figure 10b)},$$
$$h_3(z) = z \cdot \omega^8 + s \cdot \omega^9 \quad \text{(Figure 10c)},$$
$$h_4(z) = z \cdot \omega^1 + l \cdot \omega^4 \quad \text{(Figure 10d)},$$
$$h_5(z) = z \cdot \omega^6 + l \cdot \omega^8 \quad \text{(Figure 10e)},$$

where $z \in \mathbf{C}$, $\omega = \mathrm{e}^{i\pi/5}$, $s = 2 \cdot \tau^2 \cdot \sin(\pi/5)$ and $l = \tau \cdot s$.

We have demonstrated that the matching rules of Robinson triangles permit only three different kinds of cartwheel intersection. Next we show that it is not necessary to pay attention to edge-to-edge neighbours.

PROPOSITION 2. *If two cartwheels C_0 and C_1 of a Penrose triangle tiling T intersect in an edge E of their boundaries, there is always another cartwheel $C_2 \subseteq T$ which contains E and intersects C_0 in a set of type A or B.*

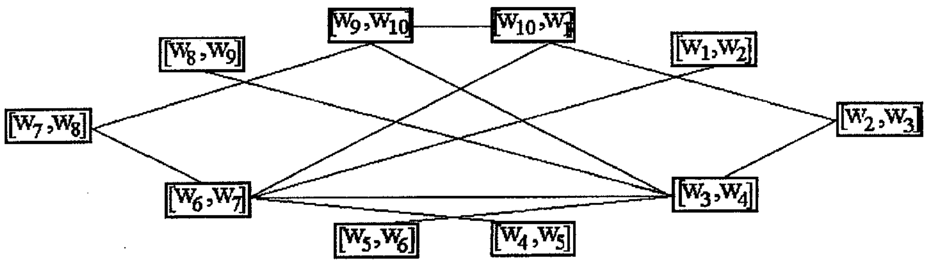


Figure 9. Possible edge-to-edge neighbourhoods of two cartwheels.

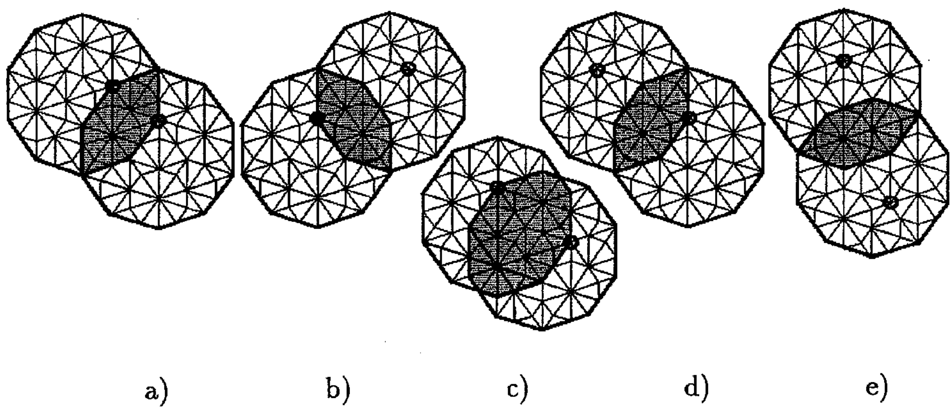


Figure 10. Possible overlaps of two cartwheels in A- and B-sets.

Proof. We suppose two distinct cartwheels C_0 and C_1 (described by an isometry $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ with $C_1 = h(C_0)$) of a Penrose triangle tiling T intersect in an edge E of their boundaries. All admissible edge-to-edge neighbourhoods of C_0 and C_1 are given in the graph of Figure 9. Now we verify that the patch $(C_0 \cup C_1)$ always determines a king-vertex configuration on the boundary of C_0 or C_1. The corresponding cartwheel $C_2 \subseteq T$ intersects C_0 in A_1, A_2, A_3, A_4, B_1 or B_2 and contains the common edge E. In the following we investigate the edges $[w_{10}, w_1]$ up to $[w_4, w_5]$ of ∂C_0 and their possible neighbouring edges in detail. The remaining cases can be obtained from the reflection symmetry of the cartwheel.

(1) $E = [w_{10}, w_1]$. Neighbouring edges of $[w_{10}, w_1] \subset \partial C_0$ may be $[w_6, w_7]$, $[w_9, w_{10}]$ or $[w_2, w_3] \subset \partial C_1$. In both of the first cases the set of Robinson triangles around $w_1 \in \partial C_0$ can only be completed to a vertex neighbourhood of type king. Since we started with a Penrose tiling T, there must be a corresponding cartwheel $C_2 \subseteq T$ with $C_0 \cap C_2 = A_1$. For $h([w_2, w_3]) = [w_{10}, w_1]$ a queen's empire is determined in $w_1 \in \partial C_0$. This leads to a king in $h(w_1)$ and a cartwheel which overlaps C_0 in A_1.

(2) $E = [w_1, w_2]$. There is only one admissible neighbouring edge $[w_6, w_7] \subset \partial C_1$ forcing a king in $w_1 \in \partial C_0$. Thus there exists a cartwheel C_2 with $C_0 \cap C_2 = A_1$.

(3) $E = [w_2, w_3]$. Both possible neighbours $[w_{10}, w_1]$ and $[w_3, w_4] \subset \partial C_1$ determine a queen's empire in $w_2 \in \partial C_0$ which permits only a king in $w_1 \in \partial C_0$. Consequently, C_0 is intersected in A_1 by a third cartwheel C_2.

(4) $E = [w_3, w_4]$. The five edges of ∂C_1 $[w_5, w_6], [w_8, w_9], [w_6, w_7], [w_9, w_{10}]$ and $[w_2, w_3]$ can match E. Whereas in both first cases a king neighbourhood results in $w_3 \in \partial C_0$, this holds for $w_4 \in \partial C_0$ in the next two examples. The corresponding cartwheels intersect C_0 in B_1 and A_2, respectively. In the last case $(h([w_2, w_3]) = [w_3, w_4])$ we have a queen's empire in $w_4 \in \partial C_0$. This leads to a cartwheel C_2 with a king neighbourhood in $h(w_1)$, such that $C_0 \cap C_2 = A_2$.

(5) $E = [w_4, w_5]$. The only possible neighbouring edge $[w_6, w_7]$ results in a queen's empire of the vertex $w_5 \in \partial C_0$. Two king neighbourhoods are forced in w_4 and w_6. The corresponding cartwheels intersect C_0 in A_2 and A_3, respectively.

Using the results of Propositions 1 and 2, for all Penrose tilings T every boundary point of a cartwheel belongs to a common A- or B-set of at least two distinct decagons $C_0, C_1 \subseteq T$. Consequently, all local configurations are characterized by a set of finitely many cartwheels $\{C_i \subseteq T | i \in \{0, 1, 2, \ldots, n\}; n \in \mathbb{N}\}$ such that $C_0 \cap C_i$ is of type A or B for all $i \in \{1, \ldots, n\}$ and $\partial C_0 \subseteq \bigcup_{i=1}^{n} C_i$. To give a classification we have to check all combinations of 'A–B overlaps' (see Figure 10) with respect to a fixed cartwheel C_0.

PROPOSITION 3. *We suppose C_0 up to C_n are distinct cartwheels of a Penrose triangle tiling T, $n \in \mathbb{N}$, such that $\partial C_0 \subseteq \bigcup_{i=1}^{n} C_i$ and $C_0 \cap C_i$ is of type A or B for all $i \in \{1, \ldots, n\}$. Then n is either* 4, 5 *or* 6. *Furthermore, the union $\bigcup_{i=0}^{n} C_i$ is congruent to one of the nine prototypes of local configurations of Figure* 11 *which we call 'A–B neighbourhoods'.*

Proof. It is obvious that the assumption implies at least four 'A–B neighbours' for every cartwheel $C_0 \subset T$. Whereas there are various intersections for the lower part of C_0, the overlap of both upper subsets A_1 and A_4 is forced, because no points x in $]b_9, w_{10}[$ or $]w_{10}, b_2[$ do belong to another A- or B-set. Since there are two different overlaps for each A_1 and A_4 (Figure 10a,b,d), we have to distinguish between four arrangements $[C_0, h_1(C_0), h_1^{-1}(C_0)]$, $[C_0, h_2(C_0), h_1^{-1}(C_0)]$, $[C_0, h_1(C_0), h_4(C_0)]$, $[C_0, h_2(C_0), h_4(C_0)]$.

To fulfil the conditions of our assertion for the boundary points $x \notin A_1$, $x \notin A_4$ at least two of the subsets A_2, A_3, B_1, B_2 (and at most all of them) have to be covered. Let us first consider minimal sets of cartwheels satisfying our assumption and then all maximal systems.

Minimal neighbourhoods

Case 1 : $[C_0, h_1(C_0), h_1^{-1}(C_0)]$

Since this configuration does not allow neighbours with intersections in $B_1, B_2 \subseteq C_0$, both subsets A_2 and $A_3 \subseteq C_0$ have to be overlapped. Although there are two different kinds of A_2- as well as A_3-overlaps (Figure 10b,d,e), only two pairwise combinations $(h_4^{-1}(C_0), h_2^{-1}(C_0))$ and $(h_5(C_0), h_5^{-1}(C_0))$ are in accordance with Robinson's triangle matching rules. The corresponding A–B neighbourhoods are the first and second prototypes of Figure 11.

Case 2 : $[C_0, h_2(C_0), h_1^{-1}(C_0)]$
The neighbour $h_2(C_0)$ determines a king in $w_3 \in \partial C_0$. Therefore $B_1 \subseteq C_0$ has to be overlapped. The A_4-neighbour of C_0 only compares with an overlap of $A_3 \subseteq C_0$. Since we also have to pay attention to the cartwheel $h_3(C_0)$ intersecting C_0 in B_1, A_3 can only match the subset A_2 of a fourth cartwheel (Figure 10e). The resulting A–B neighbourhood is the third one.

Case 3 : $[C_0, h_1(C_0), h_4(C_0)]$
This example can be deduced from the second case by a reflection with respect to the axis $[w_{10}, w_5]$. The corresponding neighbourhood is prototype five of Figure 11.

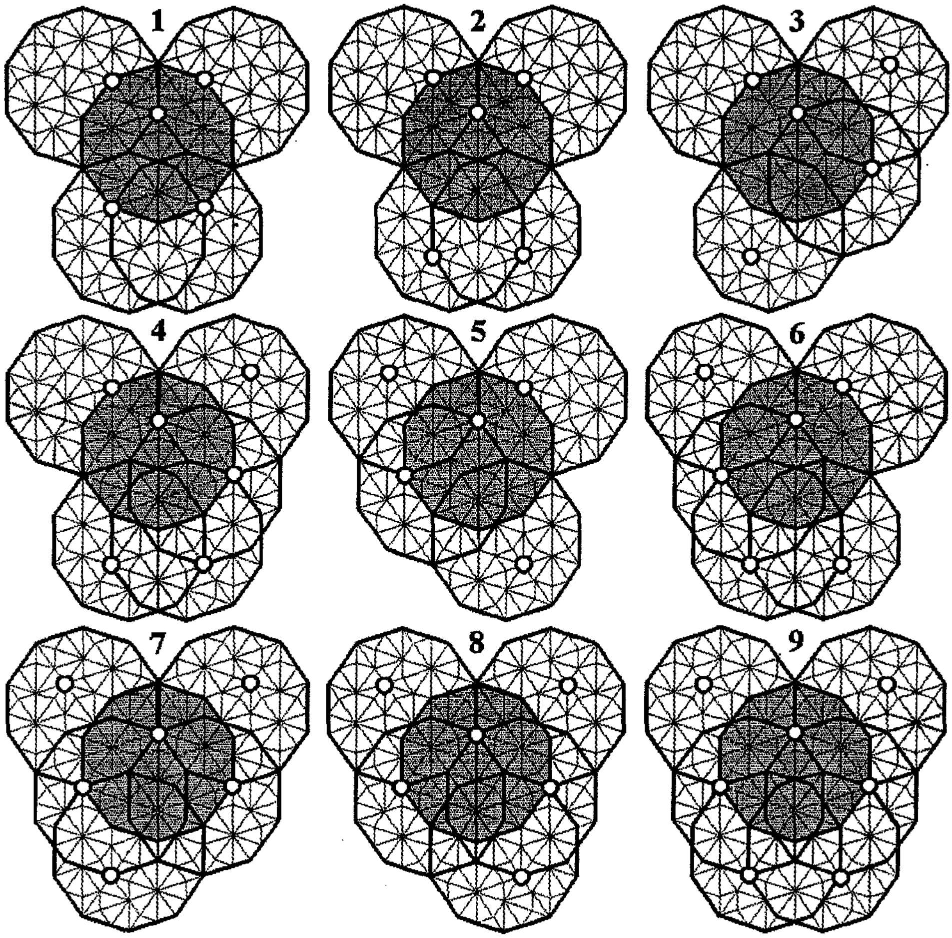
Case 4 : $[C_0, h_2(C_0), h_4(C_0)]$
Two kings are defined in w_3 and $w_7 \in \partial C_0$ by $h_2(C_0)$ and $h_4(C_0)$. Consequently, there are two cartwheels intersecting both subsets $B_1, B_2 \subseteq C_0$. Although $\partial C_0 \subseteq \bigcup_{i=1}^{4} C_i$ and $C_0 \cap C_i$ is of type A or B for all $i \in \{1, \ldots, 4\}$, $\bigcup_{i=0}^{4} C_i$ cannot be a patch of a Penrose tiling. There must be at least one fifth cartwheel overlapping C_0 either in A_2 or A_3, because otherwise an edge-to-edge neighbour would be necessary to cover the boundary of the B_1- and B_2-neighbours. This contradicts Proposition 2. The prototypes seven and eight of Figure 11 are intersected in A_3 and A_2, respectively.

Maximal neighbourhoods

Whereas the minimal A–B neighbourhoods of type 1 and 2 are also maximal, the prototypes 3, 5, 7 and 8 permit an additional neighbouring cartwheel $h_5(C_0)$ or $h_5^{-5}(C_0)$ which overlaps A_2 or A_3 of C_0 according to Figure 10e). The corresponding A–B neighbourhoods are the prototypes 4, 6 and 9.

2.2. CONSTRUCTION OF CARTWHEEL COVERINGS

The nine types of A–B neighbourhood can be used like prototiles to generate a cartwheel covering. We start with a cartwheel C_0 and assign one of the neighbourhood types 1 to 9 to C_0. Next we look for an admissible A–B neighbourhood for every of the neighbouring decagons in dependence on overlaps which are already fixed. Using Conway's result and our three propositions, there exists at least one 'second-level' neighbourhood for every tile of types 1 to 9. Thus a repeated application of this process must lead to all representatives of the family of cartwheel coverings. It is clear that the method works analogously to the puzzle principle of tilings and implies also the construction of finite patches which cannot be complet-



A-B neighbourhoods of a cartwheel. They are characterized by the arrangement of all king configurations

Figure 11.

ed. Now we have to verify that the method leads to nothing else than cartwheel coverings.

LEMMA 1. *Let us assume a covering by congruent cartwheels has been generated according to the nine types of A–B neighbourhood. Then the covering is a Penrose triangle tiling.*

Proof. The generated set of decagons is a union of A–B neighbourhoods, each of which appears in Penrose tilings. Moreover, the whole structure agrees with the matching rules of Robinson's triangles, because each common edge of two such triangles is contained in the interior of at least one cartwheel. But for all A–B neighbourhoods there is at least one point of the boundary of the central decagon

Table II.

neighbourhood-type of the central decagon	neighbourhood-type of the neighbouring decagons
2	(1 3 5 1)
3	(1 5 2 1)
4	(1 8 7 6 1)
5	(1 2 3 1)
6	(1 4 8 7 1)
7	(1 6 4 8 1)
8	(1 7 6 4 1)
9	(1 9 9 9 9 1)

which does not belong to the interior of a neighbouring tile. We have to show that no gaps will occur.

As already mentioned in the proof of Proposition 3, both upper subsets A_1 and A_4 of a cartwheel are always intersected by neighbouring tiles. Using that property, for every of the nine A–B neighbourhoods the assignment of a neighbourhood type to the neighbouring tiles immediately leads to a covering of these 'critical' points. Consequently, every constructible covering is a Penrose triangle tiling.

Let us mention that cartwheels of types 2 to 9 (see Table II) determine a single 'second level' neighbourhood. We start with the right upper neighbour and continue in clockwise order. In contrast, the type-1 neighbourhood permits 13 different neighbourhood vectors.

3. An Aperiodic Decagon

We have described a method to generate the local isomorphism class of Penrose triangle tilings using the nine prototypes of A–B neighbourhood. Now we show that these local arrangements can be forced by markings of a single regular decagon. Whereas for tiles marked edges and vertices are convenient, we colour particular subsets replacing the system of straight lines inside the cartwheel. They correspond to 'rocket-like' patches of every subset of type A (Figure 12).

To construct coverings of the plane we define a generalized kind of matching rules.

OVERLAPPING RULE. *The dark subsets of every marked decagon D have to be overlapped by other tiles, such that the union of these pieces contains the boundary ∂D and colours of overlapping sets are identical.*

We claim that this principle determines the desired prototypes of neighbourhood.

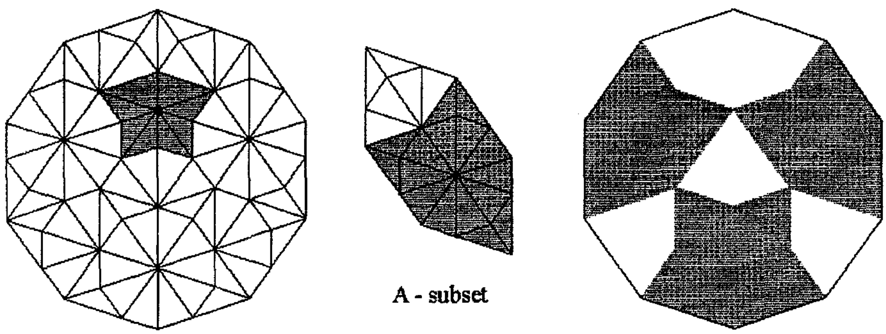


Figure 12. Cartwheel and coloured regular decagon.

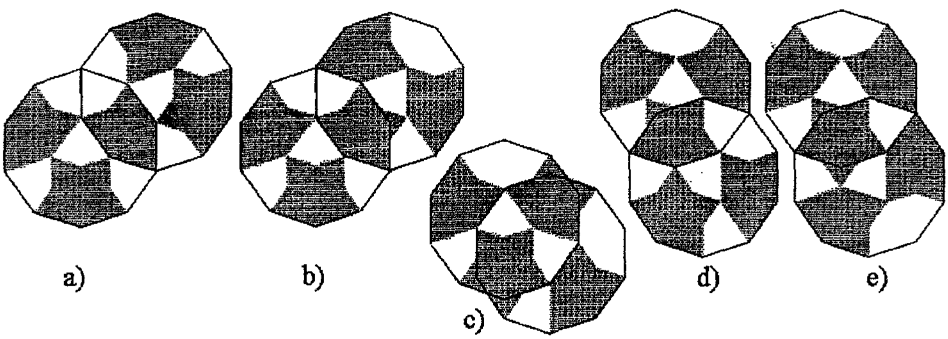


Figure 13.

LEMMA 2. *The overlapping rule permits only nine different types of local configurations which are congruent to the A–B neighbourhoods of type* 1 *up to* 9.

Proof. First we consider admissible intersections of each two marked decagons. It is obvious that an upper left 'rocket' and an upper right one can match (Figure 13a). Furthermore, they may intersect in a proper subset, such that their union gives a 'halfstar' (Figure 13c). Both 'rockets' are also contained in the lower dark 'halfstar' (Figure 13b, 13d). There is only one additional intersection for the dark 'halfstar' by a subset of itself, such that a full 'star' results (Figure 13e). These five prototypes of pairwise intersections are congruent to the 'A–B overlaps' of each two cartwheels. Therefore they can be described by the same rotations h_1 (case a) up to h_5 (case e).

To satisfy the overlapping rule we have to check all combinations of these five intersections with respect to a fixed decagon D. Whereas overlaps of type (a) and (b) both tolerate type (d) and (e), the 'halfstar' of case (c) is not compatible with an intersection of type (a) for the upper right 'rocket'. A type-(b) neighbour is forced. Furthermore, the full dark 'star' of the (c)-overlap does not allow an additional neighbour $h_4^{-1}(D)$ or $h_2^{-1}(D)$. Intersections $(h_4^{-1}(D), h_5^{-1}(D))$ and $(h_2^{-1}(D), h_5(D))$ are also not compatible. Only overlaps of type (c) and (e) may appear together. Since all arguments are passed on the left part of D by reflection, the overlapping rule always forces two upper neighbours $(h_1(D), h_1^{-1}(D))$, $(h_1(D), h_4(D))$, $(h_2(D), h_1^{-1}(D))$ or $(h_2(D), h_4(D))$. The lower part of the boundary ∂D can be overlapped by two, three or four tiles.

Two lower neighbours:
(1) $(h_4^{-1}(D), h_2^{-1}(D))$; (2) $(h_5(D), h_5^{-1}(D))$; (3) $(h_5(D), h_3^{-1}(D))$;
(4) $(h_3(D), h_5^{-1}(D))$; (5) $(h_3(D), h_3^{-1}(D))$.

Three lower neighbours:
(6) $(h_3(D), h_5(D), h_3^{-1}(D))$; (7) $(h_3(D), h_5^{-1}(D), h_3^{-1}(D))$;
(8) $(h_3(D), h_5(D), h_5^{-1}(D))$; (9) $(h_5(D), h_5^{-1}(D), h_3^{-1}(D))$.

Four lower neighbours:
(10) $(h_3(D), h_5(D), h_5^{-1}(D), h_3^{-1}(D))$.

Whereas the first two examples permit all four combinations of the upper neighbours $h_1(D), h_1^{-1}(D), h_2(D)$ and $h_4(D)$ (see Figure 14, examples 1 to 8), the type-(c) overlaps $h_3(D), h_3^{-1}(D)$ of all remaining cases determine one of the upper neighbouring tiles of D. Consequently, our overlapping rule is fulfilled by two possible configurations for each of the cases (3), (4), (8) and (9) (see Figure 14, examples 9 to 12, 16 to 19) and by a single neighbourhood for (5), (6), (7) and (10) (examples 13, 14, 15, 20 of Figure 14). Although we have these 20 different configurations in the first step, a further application of the overlapping rule always leads to our nine types of A–B neighbourhood. The arrows and numbers in Figure 14 stand for the types of A–B neighbourhood which are implied by a given arrangement. The second, third and fourth example have no number, because they lead to a contradiction at $*$.

We have shown that a repeated application of the overlapping rule determines exactly those neighbourhoods which may appear in cartwheel coverings. Using Conway's result and Lemma 1, the local isomorphism class of Penrose triangle tilings can be generated by a single aperiodic 'prototile'.

4. Remarks

(1) Although our work contributes in first respect to the open question for a single aperiodic tile, it seems to be also helpful for the description of two-dimensional quasicrystals. Usual Penrose tilings represent the typical local five- and ten-fold symmetry of these materials and their quasiperiodicity, but the shape of the pieces does not correspond to real atomic clustering. The overlaps of our decagonal protoset allow a more realistic explanation related to the same tiling model.

Based on the concept that the structure of a solid can be determined by the lowest-energy clusters, Jeong and Steinhardt explain in [10] the formation of perfect quasicrystalline ordering by an assignment of low energy to a few typical

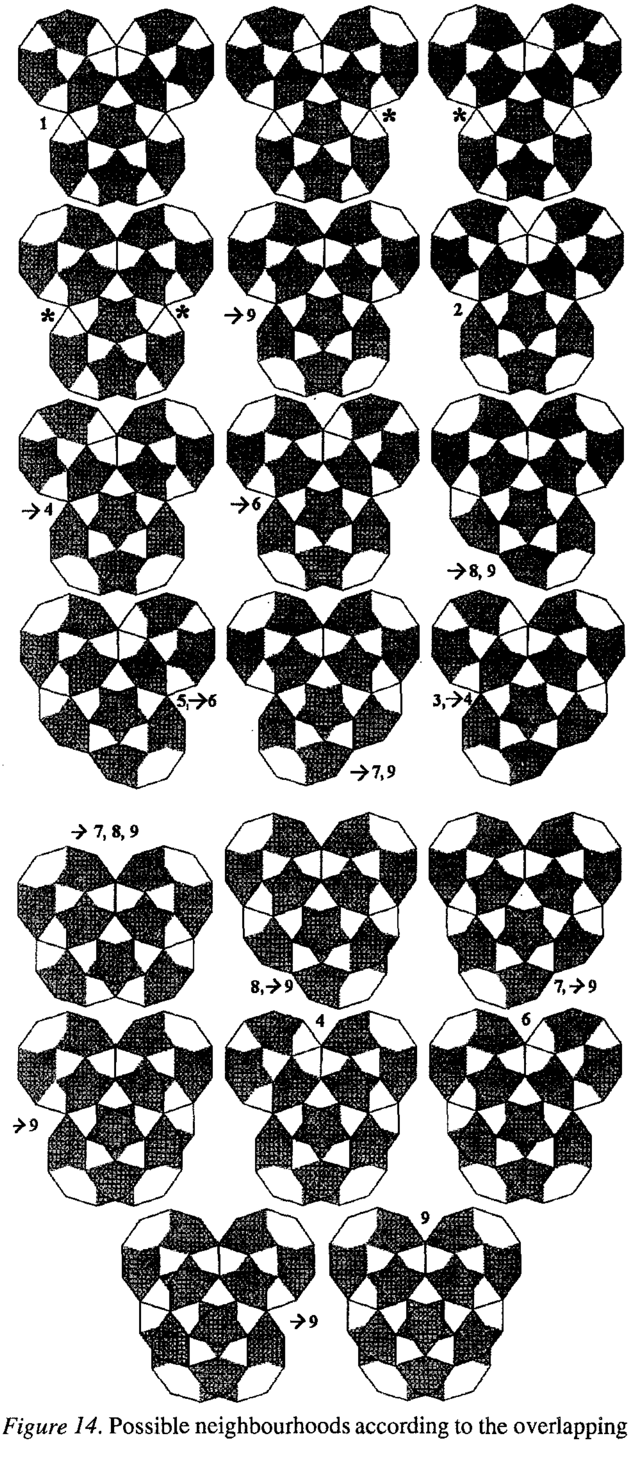


Figure 14. Possible neighbourhoods according to the overlapping rule.

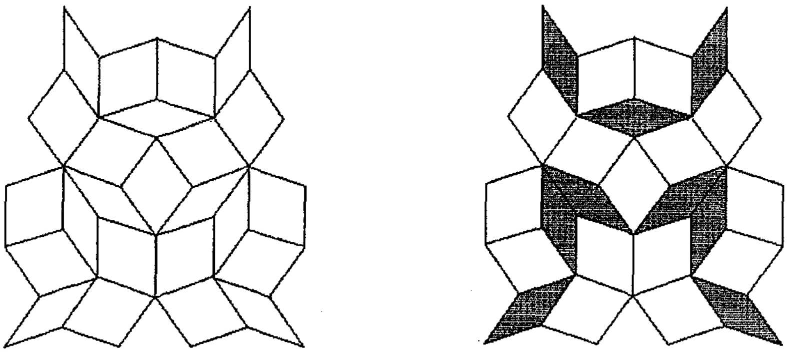


Figure 15. Rhombus version of the cartwheel and corresponding markings.

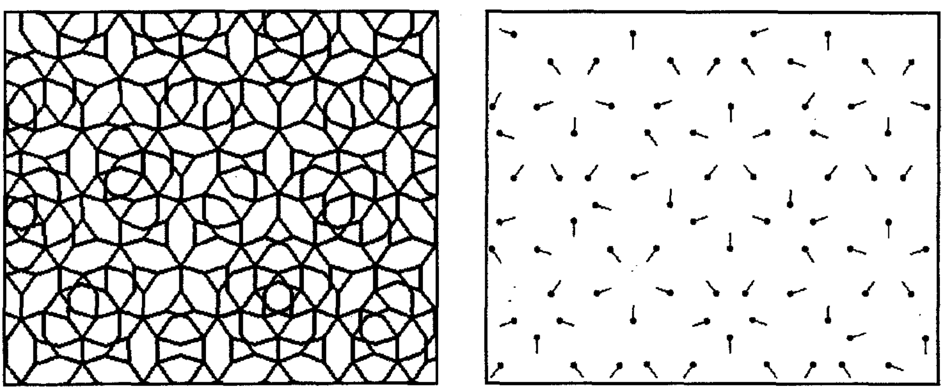


Figure 16. Decagonal covering and corresponding discrete pattern.

clusters. The minimization of free energy favours their overlap to further increase the density. It seems, that our A–B neighbourhoods are induced by that principle.

(2) By the mutual local derivability of Penrose tilings our construction principle is not only restricted to Robinson triangles. Whereas it is obvious that all other versions can be derived starting with a decagonal covering, direct matching rules for the corresponding cartwheels are not implied. The kite and dart version also leads to this patch, but the rhombuses define a corresponding empire of slightly different shape (Figure 15). In that case we suggest a colouring of all thin rhombuses to get the same result by the same overlapping rule.

(3) In 1986 V. Sasisekharan introduced a similar result using both kinds of rhombuses and admissible numbers of intersecting edges [12]. Whereas we need only one protoset, he generates aperiodic coverings by two decagonal patches. One of them contains a pentagonal star of thick rhombuses and five thin rhombuses. The other one is a subset of Figure 15.

(4) The pattern of all centres of the decagons gives a tiling of the first six Penrose tiles if an edge is assigned to all intersections of type A and B. In Figure 16 we

show a covering and the corresponding discrete pattern of their centres (with 'flags' describing the orientation).

(5) The self-similarity property, applied to construct the very special *cartwheel tiling*, can also be used to generate all the cartwheel coverings. A generalized blowing-up method can be described, using the definition of a large cartwheel of expansion factor τ by the type-1 neighbourhood.

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