Positional Determinacy of Parity Games

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October 11, 2017

We present a formalization of parity games (a two-player game on directed graphs) and a proof of their positional determinacy in Isabelle/HOL. This proof works for both finite and infinite games. We follow the proof in [2], which is based on [5].

Contents

1 Introduction

Parity games are games played by two players, called Even and Odd, on labelled directed graphs. Each node is labelled with their player and with a natural number, called its *priority*.

To call this a *parity game*, we only need to assume that the number of different priorities is finite. Of course, this condition is only relevant on infinite graphs.

One reason parity games are important is that determining the winner is polynomial-time equivalent to the model-checking problem of the modal μ-calculus, a logic able to express LTL and CTL* properties ([1]).

1.1 Formal Introduction

Formally, a parity game is $G = (V, E, V_0, \omega)$, where (V, E) is a directed graph, $V_0 \subseteq V$ is the set of Even nodes, and $\omega : V \to \mathbb{N}$ is a function with $|f(V)| < \infty$.

A *play* is a maximal path in G. A finite play is winning for Even iff the last node is not in V_0. An infinite play is winning for Even iff the minimum priority occurring infinitely often on the path is even. On an infinite path at least one priority occurs infinitely often because there is only a finite number of different priorities.

A node v is *winning* for a player p iff all plays starting from v are winning for p. It is well-known that parity games are *determined*, that is, every node is winning for some player.

A more surprising property is that parity games are also *positionally determined*. This means that for every node v winning for Even, there is a function $\sigma : V_0 \to V$ such that all Even needs to do in order to win from v is to consult this function whenever it is his turn (similarly if v is winning for Odd). This is also called a *positional strategy* for the winning player.

We define the *winning region* of player p as the set of nodes from which player p has positional winning strategies. Positional determinacy then says that the winning regions of Even and of Odd partition the graph.

See [3] for a modern survey on positional determinacy of parity games. Their proof is based on a proof by Zielonka [5].

1.2 Overview

Here we formalize the proof from [2] in Isabelle/HOL. This proof is similar to the proof in [3], but we do not explicitly define so-called "σ-traps". Using σ-traps could be worth exploring, because it has the potential to simplify our formalization.

Our proof has no assumptions except those required by every parity game. In particular the parity game

- may have arbitrary cardinality,
- may have loops,
- may have deadends, that is, nodes with no successors.

The main theorem is in section 12.4.

1.3 Technical Aspects

We use a coinductive list of nodes to represent paths in a graph because this gives us a uniform representation for finite and infinite paths. We can then express properties such as that a path is maximal or conforms to a given strategy directly as coinductive properties. We use the coinductive list developed by Lochbihler in [4].

We also explored representing paths as functions *nat* ⇒ *'a option* with the property that the domain is an initial segment of *nat* (and where *'a* is the node type). However, it turned out that coinductive lists give simpler proofs.

It is possible to represent a graph as a function *'a* ⇒ *'a* ⇒ *bool*, see for example in the proof of König's lemma in [4]. However, we instead go for a record which contains a set of nodes and a set of edges explicitly. By not requiring that the set of nodes is *UNIV :: 'a set* but rather a subset of *UNIV :: 'a set*, it becomes easier to reason about subgraphs.

Another point is that we make extensive use of locales, in particular to represent maximal paths conforming to a specific strategy. Thus proofs often start with **interpret** *vmc-path G P* v_0 *p* σ to say that P is a valid maximal path in the graph G starting in v_0 and conforming to the strategy σ for player p.

2 Auxiliary Lemmas for Coinductive Lists

Some lemmas to allow better reasoning with coinductive lists.

theory *MoreCoinductiveList*
imports
  Main
  Coinductive.Coinductive-List
begin

2.1 *lset*

lemma *lset-lnth*: $x \in$ *lset xs* $\Longrightarrow \exists n.$ *lnth xs n* $= x$
  by (*induct rule*: *llist.set-induct*, *meson lnth-0*, *meson lnth-Suc-LCons*)

lemma *lset-lnth-member*: ⟦ *lset xs* ⊆ *A*; *enat n* < *llength xs* ⟧ ⟹ *lnth xs n* ∈ *A*
  using *contra-subsetD*[*of lset xs A*] *in-lset-conv-lnth*[*of - xs*] **by** *blast*

lemma *lset-nth-member-inf*: ⟦ ¬*lfinite xs*; *lset xs* ⊆ *A* ⟧ ⟹ *lnth xs n* ∈ *A*
  by (*metis contra-subsetD inf-llist-lnth lset-inf-llist rangeI*)

lemma *lset-intersect-lnth*: *lset xs* ∩ *A* ≠ {} ⟹ ∃ *n*. *enat n* < *llength xs* ∧ *lnth xs n* ∈ *A*
  by (*metis disjoint-iff-not-equal in-lset-conv-lnth*)

lemma *lset-ltake-Suc*:
  assumes ¬*lnull xs lnth xs 0* = *x lset* (*ltake* (*enat n*) (*ltl xs*)) ⊆ *A*
  shows *lset* (*ltake* (*enat* (*Suc n*)) *xs*) ⊆ *insert x A*
proof−
  have *lset* (*ltake* (*eSuc* (*enat n*)) (*LCons x* (*ltl xs*))) ⊆ *insert x A*
    using *assms*(*3*) **by** *auto*
  moreover from *assms*(*1*,*2*) **have** *LCons x* (*ltl xs*) = *xs*
    by (*metis lnth-0 ltl-simps*(*2*) *not-lnull-conv*)

ultimately show *?thesis* **by** (*simp add*: *eSuc-enat*)
qed

lemma *lfinite-lset*: *lfinite xs* $\Longrightarrow$ $\neg$*lnull xs* $\Longrightarrow$ *llast xs* $\in$ *lset xs*
proof (*induct rule*: *lfinite-induct*)
  case (*LCons xs*)
  show *?case* **proof** (*cases*)
    assume $*$: $\neg$*lnull* (*ltl xs*)
    hence *llast* (*ltl xs*) $\in$ *lset* (*ltl xs*) **using** *LCons.hyps*(*3*) **by** *blast*
    hence *llast* (*ltl xs*) $\in$ *lset xs* **by** (*simp add*: *in-lset-ltlD*)
    thus *?thesis* **by** (*metis* $*$ *LCons.prems lhd-LCons-ltl llast-LCons2*)
  qed (*metis LCons.prems lhd-LCons-ltl llast-LCons llist.set-sel*(*1*))
qed *simp*

lemma *lset-subset*: $\neg$(*lset xs* $\subseteq$ *A*) $\Longrightarrow$ $\exists$ *n*. *enat n* $<$ *llength xs* $\wedge$ *lnth xs n* $\notin$ *A*
  by (*metis in-lset-conv-lnth subsetI*)

2.2 *llength*

lemma *enat-Suc-ltl*:
  assumes *enat* (*Suc n*) $<$ *llength xs*
  shows *enat n* $<$ *llength* (*ltl xs*)
proof−
  from *assms* **have** *eSuc* (*enat n*) $<$ *llength xs* **by** (*simp add*: *eSuc-enat*)
  hence *enat n* $<$ *epred* (*llength xs*) **using** *eSuc-le-iff ileI1* **by** *fastforce*
  thus *?thesis* **by** (*simp add*: *epred-llength*)
qed

lemma *enat-ltl-Suc*: *enat n* $<$ *llength* (*ltl xs*) $\Longrightarrow$ *enat* (*Suc n*) $<$ *llength xs*
  by (*metis eSuc-enat ldrop-ltl leD leI lnull-ldrop*)

lemma *infinite-small-llength* [*intro*]: $\neg$*lfinite xs* $\Longrightarrow$ *enat n* $<$ *llength xs*
  using *enat-iless lfinite-conv-llength-enat neq-iff* **by** *blast*

lemma *lnull-0-llength*: $\neg$*lnull xs* $\Longrightarrow$ *enat 0* $<$ *llength xs*
  using *zero-enat-def* **by** *auto*

lemma *Suc-llength*: *enat* (*Suc n*) $<$ *llength xs* $\Longrightarrow$ *enat n* $<$ *llength xs*
  using *dual-order.strict-trans enat-ord-simps*(*2*) **by** *blast*

2.3 *ltake*

lemma *ltake-lnth*: *ltake n xs* = *ltake n ys* $\Longrightarrow$ *enat m* $<$ *n* $\Longrightarrow$ *lnth xs m* = *lnth ys m*
  by (*metis lnth-ltake*)

lemma *lset-ltake-prefix* [*simp*]: *n* $\leq$ *m* $\Longrightarrow$ *lset* (*ltake n xs*) $\subseteq$ *lset* (*ltake m xs*)
  by (*simp add*: *lprefix-lsetD*)

lemma *lset-ltake*: ($\bigwedge$*m*. *m* $<$ *n* $\Longrightarrow$ *lnth xs m* $\in$ *A*) $\Longrightarrow$ *lset* (*ltake* (*enat n*) *xs*) $\subseteq$ *A*
proof (*induct n arbitrary*: *xs*)
  case *0*
  have *ltake* (*enat 0*) *xs* = *LNil* **by** (*simp add*: *zero-enat-def*)

thus *?case* **by** *simp*
next
case (*Suc n*)
show *?case* **proof** (*cases*)
assume $xs \neq LNil$
then obtain *x xs'* **where** *xs*: $xs = LCons\ x\ xs'$ **by** (*meson neq-LNil-conv*)
{ fix *m* **assume** $m < n$
hence $Suc\ m < Suc\ n$ **by** *simp*
hence $lnth\ xs\ (Suc\ m) \in A$ **using** *Suc.prems* **by** *presburger*
hence $lnth\ xs'\ m \in A$ **using** *xs* **by** *simp*
}
hence $lset\ (ltake\ (enat\ n)\ xs') \subseteq A$ **using** *Suc.hyps* **by** *blast*
moreover have $ltake\ (enat\ (Suc\ n))\ xs = LCons\ x\ (ltake\ (enat\ n)\ xs')$
using *xs ltake-eSuc-LCons*[*of - x xs'*] **by** (*metis* (*no-types*) *eSuc-enat*)
moreover have $x \in A$ **using** *Suc.prems xs* **by** *force*
ultimately show *?thesis* **by** *simp*
qed *simp*
qed

lemma *llength-ltake'*: $enat\ n < llength\ xs \Longrightarrow llength\ (ltake\ (enat\ n)\ xs) = enat\ n$
by (*metis llength-ltake min.strict-order-iff*)

lemma *llast-ltake*:
assumes $enat\ (Suc\ n) < llength\ xs$
shows $llast\ (ltake\ (enat\ (Suc\ n))\ xs) = lnth\ xs\ n$ (**is** *llast ?A* = -)
unfolding *llast-def* **using** *llength-ltake'*[*OF assms*] **by** (*auto simp add*: *lnth-ltake*)

lemma *lset-ltake-ltl*: $lset\ (ltake\ (enat\ n)\ (ltl\ xs)) \subseteq lset\ (ltake\ (enat\ (Suc\ n))\ xs)$
proof (*cases*)
assume $\neg lnull\ xs$
then obtain *v0* **where** $xs = LCons\ v0\ (ltl\ xs)$ **by** (*metis lhd-LCons-ltl*)
hence $ltake\ (eSuc\ (enat\ n))\ xs = LCons\ v0\ (ltake\ (enat\ n)\ (ltl\ xs))$
by (*metis ltake-eSuc-LCons*)
hence $lset\ (ltake\ (enat\ (Suc\ n))\ xs) = lset\ (LCons\ v0\ (ltake\ (enat\ n)\ (ltl\ xs)))$
by (*simp add*: *eSuc-enat*)
thus *?thesis* **using** *lset-LCons*[*of v0 ltake* (*enat n*) (*ltl xs*)] **by** *blast*
qed (*simp add*: *lnull-def*)

2.4 *ldropn*

lemma *ltl-ldrop*: $[\![\ \bigwedge xs.\ P\ xs \Longrightarrow P\ (ltl\ xs);\ P\ xs\ ]\!] \Longrightarrow P\ (ldropn\ n\ xs)$
unfolding *ldropn-def* **by** (*induct n*) *simp-all*

2.5 *lfinite*

lemma *lfinite-drop-set*: $lfinite\ xs \Longrightarrow \exists\, n.\ v \notin lset\ (ldrop\ n\ xs)$
by (*metis ldrop-inf lmember-code*(*1*) *lset-lmember*)

lemma *index-infinite-set*:
$[\![\ \neg lfinite\ x;\ lnth\ x\ m = y;\ \bigwedge i.\ lnth\ x\ i = y \Longrightarrow (\exists\, m > i.\ lnth\ x\ m = y)\ ]\!] \Longrightarrow y \in lset\ (ldropn\ n\ x)$
proof (*induct n arbitrary*: *x m*)

```
  case 0 thus ?case using lset-nth-member-inf by auto
next
  case (Suc n)
  obtain a xs where x: x = LCons a xs by (meson Suc.prems(1) lnull-imp-lfinite not-lnull-conv)
  obtain j where j: j > m lnth x j = y using Suc.prems(2,3) by blast
  have lnth xs (j − 1) = y by (metis lnth-LCons' j(1,2) not-less0 x)
  moreover {
    fix i assume lnth xs i = y
    hence lnth x (Suc i) = y by (simp add: x)
    hence ∃j>i. lnth xs j = y by (metis Suc.prems(3) Suc-lessE lnth-Suc-LCons x)
  }
  ultimately show ?case using Suc.hyps Suc.prems(1) x by auto
qed
```

2.6 *lmap*

```
lemma lnth-lmap-ldropn:
  enat n < llength xs ⟹ lnth (lmap f (ldropn n xs)) 0 = lnth (lmap f xs) n
  by (simp add: lhd-ldropn lnth-0-conv-lhd)

lemma lnth-lmap-ldropn-Suc:
  enat (Suc n) < llength xs ⟹ lnth (lmap f (ldropn n xs)) (Suc 0) = lnth (lmap f xs) (Suc n)
  by (metis (no-types, lifting) Suc-llength ldropn-ltl leD llist.map-disc-iff lnth-lmap-ldropn
                              lnth-ltl lnull-ldropn ltl-ldropn ltl-lmap)
```

2.7 Notation

We introduce the notation $ to denote *lnth*.

```
notation lnth (infix $ 61)

end
```

3 Parity Games

```
theory ParityGame
imports
  Main
  MoreCoinductiveList
begin
```

3.1 Basic definitions

'a is the node type. Edges are pairs of nodes.

```
type-synonym 'a Edge = 'a × 'a
```

A path is a possibly infinite list of nodes.

```
type-synonym 'a Path = 'a llist
```

3.2 Graphs

We define graphs as a locale over a record. The record contains nodes (AKA vertices) and edges. The locale adds the assumption that the edges are pairs of nodes.

record *'a Graph* =
  verts :: *'a set* (V_1)
  arcs :: *'a Edge set* (E_1)
abbreviation *is-arc* :: (*'a*, *'b*) *Graph-scheme* ⇒ *'a* ⇒ *'a* ⇒ *bool* (**infixl** $\rightarrow_1$ *60*) **where**
  $v \rightarrow_G w \equiv (v,w) \in E_G$

locale *Digraph* =
  fixes *G* (**structure**)
  assumes *valid-edge-set*: $E \subseteq V \times V$
begin
lemma *edges-are-in-V* [*intro*]: $v \rightarrow w \Longrightarrow v \in V$ $v \rightarrow w \Longrightarrow w \in V$ **using** *valid-edge-set* **by** *blast+*

A node without successors is a *deadend.*

abbreviation *deadend* :: *'a* ⇒ *bool* **where** *deadend v* ≡ $\neg(\exists w \in V.\ v \rightarrow w)$

3.3 Valid Paths

We say that a path is *valid* if it is empty or if it starts in V and walks along edges.

coinductive *valid-path* :: *'a Path* ⇒ *bool* **where**
  valid-path-base: *valid-path LNil*
| *valid-path-base'*: $v \in V \Longrightarrow$ *valid-path* (*LCons v LNil*)
| *valid-path-cons*: ⟦ $v \in V$; $w \in V$; $v \rightarrow w$; *valid-path Ps*; ¬*lnull Ps*; *lhd Ps* = *w* ⟧
    ⟹ *valid-path* (*LCons v Ps*)

inductive-simps *valid-path-cons-simp*: *valid-path* (*LCons x xs*)

lemma *valid-path-ltl'*: *valid-path* (*LCons v Ps*) ⟹ *valid-path Ps*
  using *valid-path.simps* **by** *blast*
lemma *valid-path-ltl*: *valid-path P* ⟹ *valid-path* (*ltl P*)
  by (*metis llist.exhaust-sel ltl-simps*(*1*) *valid-path-ltl'*)
lemma *valid-path-drop*: *valid-path P* ⟹ *valid-path* (*ldropn n P*)
  by (*simp add*: *valid-path-ltl ltl-ldrop*)

lemma *valid-path-in-V*: **assumes** *valid-path P* **shows** *lset* $P \subseteq V$ **proof**
  fix *x* **assume** $x \in$ *lset P* **thus** $x \in V$
    using *assms* **by** (*induct rule*: *llist.set-induct*) (*auto intro*: *valid-path.cases*)
qed
lemma *valid-path-finite-in-V*: ⟦ *valid-path P*; *enat n* < *llength P* ⟧ ⟹ *P* \$ $n \in V$
  using *valid-path-in-V lset-lnth-member* **by** *blast*

lemma *valid-path-edges'*: *valid-path* (*LCons v* (*LCons w Ps*)) ⟹ $v \rightarrow w$
  using *valid-path.cases* **by** *fastforce*
lemma *valid-path-edges*:
  assumes *valid-path P enat* (*Suc n*) < *llength P*
  shows *P* \$ *n* → *P* \$ *Suc n*
proof–

def $P' \equiv$ *ldropn n P*
have *enat n* $<$ *llength P* **using** *assms(2) enat-ord-simps(2) less-trans* **by** *blast*
hence P' \$ $0 = P$ \$ n **by** *(simp add: P'-def)*
moreover have P' \$ *Suc 0* $= P$ \$ *Suc n*
by *(metis One-nat-def P'-def Suc-eq-plus1 add.commute assms(2) lnth-ldropn)*
ultimately have $\exists$ *Ps.* $P' =$ *LCons (P \$ n) (LCons (P \$ Suc n) Ps)*
by *(metis P'-def ‹enat n < llength P› assms(2) ldropn-Suc-conv-ldropn)*
moreover have *valid-path P'* **by** *(simp add: P'-def assms(1) valid-path-drop)*
ultimately show *?thesis* **using** *valid-path-edges'* **by** *blast*
qed

lemma *valid-path-coinduct [consumes 1, case-names base step, coinduct pred: valid-path]*:
assumes *major: Q P*
and *base:* $\bigwedge v\ P.\ Q\ (LCons\ v\ LNil) \Longrightarrow v \in V$
and *step:* $\bigwedge v\ w\ P.\ Q\ (LCons\ v\ (LCons\ w\ P)) \Longrightarrow v{\rightarrow}w \wedge (Q\ (LCons\ w\ P) \vee valid\text{-}path\ (LCons\ w\ P))$
shows *valid-path P*
using *major* **proof** *(coinduction arbitrary: P)*
case *valid-path*
{ assume $P \neq LNil\ \neg(\exists v.\ P = LCons\ v\ LNil \wedge v \in V)$
then obtain *v w P'* **where** $P = LCons\ v\ (LCons\ w\ P')$
using *neq-LNil-conv base valid-path* **by** *metis*
hence *?case* **using** *step valid-path* **by** *auto*
}
thus *?case* **by** *blast*
qed

lemma *valid-path-no-deadends*:
$[\![$ *valid-path P; enat (Suc i)* $<$ *llength P* $]\!] \Longrightarrow \neg$*deadend (P \$ i)*
using *valid-path-edges* **by** *blast*

lemma *valid-path-ends-on-deadend*:
$[\![$ *valid-path P; enat i* $<$ *llength P; deadend (P \$ i)* $]\!] \Longrightarrow$ *enat (Suc i) = llength P*
using *valid-path-no-deadends* **by** *(metis enat-iless enat-ord-simps(2) neq-iff not-less-eq)*

lemma *valid-path-prefix*: $[\![$ *valid-path P; lprefix P' P* $]\!] \Longrightarrow$ *valid-path P'*
proof *(coinduction arbitrary: P' P)*
case *(step v w P'' P' P)*
then obtain *Ps* **where** *Ps: LCons v (LCons w Ps) = P* **by** *(metis LCons-lprefix-conv)*
hence *valid-path (LCons w Ps)* **using** *valid-path-ltl' step(2)* **by** *blast*
moreover have *lprefix (LCons w P'') (LCons w Ps)* **using** *Ps step(1,3)* **by** *auto*
ultimately show *?case* **using** *Ps step(2) valid-path-edges'* **by** *blast*
qed *(metis LCons-lprefix-conv valid-path-cons-simp)*

lemma *valid-path-lappend*:
assumes *valid-path P valid-path P'* $[\![\ \neg$*lnull P;* $\neg$*lnull P'* $]\!] \Longrightarrow$ *llast P*$\rightarrow$*lhd P'*
shows *valid-path (lappend P P')*
proof *(cases, cases)*
assume $\neg$*lnull P* $\neg$*lnull P'*
thus *?thesis* **using** *assms* **proof** *(coinduction arbitrary: P' P)*
case *(step v w P'' P' P)*
show *?case* **proof** *(cases)*

```
    assume lnull (ltl P)
    thus ?case using step(1,2,3,5,6)
      by (metis lhd-LCons lhd-LCons-ltl lhd-lappend llast-singleton
            llist.collapse(1) ltl-lappend ltl-simps(2))
  next
    assume ¬lnull (ltl P)
    moreover have ltl (lappend P P') = lappend (ltl P) P' using step(2) by simp
    ultimately show ?case using step
      by (metis (no-types, lifting)
          lhd-LCons lhd-LCons-ltl lhd-lappend llast-LCons ltl-simps(2)
          valid-path-edges' valid-path-ltl)
  qed
 qed (metis llist.disc(1) lnull-lappend ltl-lappend ltl-simps(2))
qed (simp-all add: assms(1,2) lappend-lnull1 lappend-lnull2)
```

A valid path is still valid in a supergame.

```
lemma valid-path-supergame:
  assumes valid-path P and G': Digraph G' V ⊆ V_G' E ⊆ E_G'
  shows Digraph.valid-path G' P
using ‹valid-path P› proof (coinduction arbitrary: P
  rule: Digraph.valid-path-coinduct[OF G'(1), case-names base step])
  case base thus ?case using G'(2) valid-path-cons-simp by auto
qed (meson G'(3) subset-eq valid-path-edges' valid-path-ltl')
```

3.4 Maximal Paths

We say that a path is *maximal* if it is empty or if it ends in a deadend.

```
coinductive maximal-path where
  maximal-path-base: maximal-path LNil
| maximal-path-base': deadend v ⟹ maximal-path (LCons v LNil)
| maximal-path-cons: ¬lnull Ps ⟹ maximal-path Ps ⟹ maximal-path (LCons v Ps)

lemma maximal-no-deadend: maximal-path (LCons v Ps) ⟹ ¬deadend v ⟹ ¬lnull Ps
  by (metis lhd-LCons llist.distinct(1) ltl-simps(2) maximal-path.simps)
lemma maximal-ltl: maximal-path P ⟹ maximal-path (ltl P)
  by (metis ltl-simps(1) ltl-simps(2) maximal-path.simps)
lemma maximal-drop: maximal-path P ⟹ maximal-path (ldropn n P)
  by (simp add: maximal-ltl ltl-ldrop)

lemma maximal-path-lappend:
  assumes ¬lnull P' maximal-path P'
  shows maximal-path (lappend P P')
proof (cases)
  assume ¬lnull P
  thus ?thesis using assms proof (coinduction arbitrary: P' P rule: maximal-path.coinduct)
    case (maximal-path P' P)
    let ?P = lappend P P'
    show ?case proof (cases ?P = LNil ∨ (∃v. ?P = LCons v LNil ∧ deadend v))
      case False
      then obtain Ps v where P: ?P = LCons v Ps by (meson neq-LNil-conv)
      hence Ps = lappend (ltl P) P' by (simp add: lappend-ltl maximal-path(1))
```

```
      hence ∃Ps1 P'. Ps = lappend Ps1 P' ∧ ¬lnull P' ∧ maximal-path P'
        using maximal-path(2) maximal-path(3) by auto
      thus ?thesis using P lappend-lnull1 by fastforce
    qed blast
  qed
qed (simp add: assms(2) lappend-lnull1[of P P'])

lemma maximal-ends-on-deadend:
  assumes maximal-path P lfinite P ¬lnull P
  shows deadend (llast P)
proof−
  from ‹lfinite P› ‹¬lnull P› obtain n where n: llength P = enat (Suc n)
    by (metis enat-ord-simps(2) gr0-implies-Suc lfinite-llength-enat lnull-0-llength)
  def P' ≡ ldropn n P
  hence maximal-path P' using assms(1) maximal-drop by blast
  thus ?thesis proof (cases rule: maximal-path.cases)
    case (maximal-path-base' v)
    hence deadend (llast P') unfolding P'-def by simp
    thus ?thesis unfolding P'-def using llast-ldropn[of n P] n
      by (metis P'-def ldropn-eq-LConsD local.maximal-path-base'(1))
  next
    case (maximal-path-cons P'' v)
   hence ldropn (Suc n) P = P'' unfolding P'-def by (metis ldrop-eSuc-ltl ltl-ldropn ltl-simps(2))
    thus ?thesis using n maximal-path-cons(2) by auto
  qed (simp add: P'-def n ldropn-eq-LNil)
qed

lemma maximal-ends-on-deadend': ⟦ lfinite P; deadend (llast P) ⟧ ⟹ maximal-path P
proof (coinduction arbitrary: P rule: maximal-path.coinduct)
  case (maximal-path P)
  show ?case proof (cases)
    assume P ≠ LNil
    then obtain v P' where P': P = LCons v P' by (meson neq-LNil-conv)
    show ?thesis proof (cases)
      assume P' = LNil thus ?thesis using P' maximal-path(2) by auto
    qed (metis P' lfinite-LCons llast-LCons llist.collapse(1) maximal-path(1,2))
  qed simp
qed

lemma infinite-path-is-maximal: ⟦ valid-path P; ¬lfinite P ⟧ ⟹ maximal-path P
  by (coinduction arbitrary: P rule: maximal-path.coinduct)
    (cases rule: valid-path.cases, auto)

end — locale Digraph
```

3.5 Parity Games

Parity games are games played by two players, called Even and Odd.

```
datatype Player = Even | Odd

abbreviation other-player p ≡ (if p = Even then Odd else Even)
```

notation *other-player* ((-**) [*1000*] *1000*)
lemma *other-other-player* [*simp*]: $p{*}{*}{*}{*} = p$ **using** *Player.exhaust* **by** *auto*

A parity game is tuple (V, E, V_0, ω), where (V, E) is a graph, $V_0 \subseteq V$ and ω is a function from $V \to \mathbb{N}$ with finite image.

record *'a ParityGame* = *'a Graph* +
  player0 :: *'a set* (*V0*ı)
  priority :: *'a* ⇒ *nat* (ωı)

locale *ParityGame* = *Digraph G* **for** *G* :: (*'a*, *'b*) *ParityGame-scheme* (**structure**) +
  assumes *valid-player0-set*: *V0* ⊆ *V*
    and *priorities-finite*: *finite* (ω ‘ *V*)
begin

VV p is the set of nodes belonging to player *p*.

abbreviation *VV* :: *Player* ⇒ *'a set* **where** *VV p* ≡ (*if p* = *Even then V0 else V* − *V0*)
lemma *VVp-to-V* [*intro*]: *v* ∈ *VV p* ⟹ *v* ∈ *V* **using** *valid-player0-set* **by** (*cases p*) *auto*
lemma *VV-impl1*: *v* ∈ *VV p* ⟹ *v* ∉ *VV p*** **by** *auto*
lemma *VV-impl2*: *v* ∈ *VV p*** ⟹ *v* ∉ *VV p* **by** *auto*
lemma *VV-equivalence* [*iff*]: *v* ∈ *V* ⟹ *v* ∉ *VV p* ⟷ *v* ∈ *VV p*** **by** *auto*
lemma *VV-cases* [*consumes 1*]: ⟦ *v* ∈ *V* ; *v* ∈ *VV p* ⟹ *P* ; *v* ∈ *VV p*** ⟹ *P* ⟧ ⟹ *P* **by** *auto*

3.6 Sets of Deadends

definition *deadends p* ≡ {*v* ∈ *VV p*. *deadend v*}
lemma *deadends-in-V*: *deadends p* ⊆ *V* **unfolding** *deadends-def* **by** *blast*

3.7 Subgames

We define a subgame by restricting the set of nodes to a given subset.

definition *subgame* **where**
  subgame V' ≡ *G*⦇
    verts := *V* ∩ *V'*,
    arcs := *E* ∩ (*V'* × *V'*),
    player0 := *V0* ∩ *V'* ⦈

lemma *subgame-V* [*simp*]: $V_{subgame\ V'} \subseteq V$
  and *subgame-E* [*simp*]: $E_{subgame\ V'} \subseteq E$
  and *subgame-ω*: $\omega_{subgame\ V'} = \omega$
  unfolding *subgame-def* **by** *simp-all*

lemma
  assumes *V'* ⊆ *V*
  shows *subgame-V'* [*simp*]: $V_{subgame\ V'} = V'$
    and *subgame-E'* [*simp*]: $E_{subgame\ V'} = E \cap (V_{subgame\ V'} \times V_{subgame\ V'})$
  unfolding *subgame-def* **using** *assms* **by** *auto*

lemma *subgame-VV* [*simp*]: *ParityGame.VV* (*subgame V'*) *p* = *V'* ∩ *VV p* **proof**−
  have *ParityGame.VV* (*subgame V'*) *Even* = *V'* ∩ *VV Even* **unfolding** *subgame-def* **by** *auto*
  moreover have *ParityGame.VV* (*subgame V'*) *Odd* = *V'* ∩ *VV Odd* **proof**−

```
    have V′ ∩ V − (V0 ∩ V′) = V′ ∩ V ∩ (V − V0) by blast
    thus ?thesis unfolding subgame-def by auto
  qed
  ultimately show ?thesis by simp
qed
corollary subgame-VV-subset [simp]: ParityGame.VV (subgame V′) p ⊆ VV p by simp

lemma subgame-finite [simp]: finite (ω_subgame V′ ‘ V_subgame V′) proof−
  have finite (ω ‘ V_subgame V′) using subgame-V priorities-finite
    by (meson finite-subset image-mono)
  thus ?thesis by (simp add: subgame-def)
qed

lemma subgame-ω-subset [simp]: ω_subgame V′ ‘ V_subgame V′ ⊆ ω ‘ V
  by (simp add: image-mono subgame-ω)

lemma subgame-Digraph: Digraph (subgame V′)
  by (unfold-locales) (auto simp add: subgame-def)

lemma subgame-ParityGame:
  shows ParityGame (subgame V′)
proof (unfold-locales)
  show E_subgame V′ ⊆ V_subgame V′ × V_subgame V′
    using subgame-Digraph[unfolded Digraph-def] .
  show V0_subgame V′ ⊆ V_subgame V′ unfolding subgame-def using valid-player0-set by auto
  show finite (ω_subgame V′ ‘ V_subgame V′) by simp
qed

lemma subgame-valid-path:
  assumes P: valid-path P lset P ⊆ V′
  shows Digraph.valid-path (subgame V′) P
proof−
  have lset P ⊆ V using P(1) valid-path-in-V by blast
  hence lset P ⊆ V_subgame V′ unfolding subgame-def using P(2) by auto
  with P(1) show ?thesis
  proof (coinduction arbitrary: P
    rule: Digraph.valid-path.coinduct[OF subgame-Digraph, case-names IH])
    case IH
    thus ?case proof (cases rule: valid-path.cases)
      case (valid-path-cons v w Ps)
      moreover hence v ∈ V_subgame V′ w ∈ V_subgame V′ using IH(2) by auto
      moreover hence v →_subgame V′ w using local.valid-path-cons(4) subgame-def by auto
      moreover have valid-path Ps using IH(1) valid-path-ltl′ local.valid-path-cons(1) by blast
      ultimately show ?thesis using IH(2) by auto
    qed auto
  qed
qed

lemma subgame-maximal-path:
  assumes V′: V′ ⊆ V and P: maximal-path P lset P ⊆ V′
```

shows *Digraph.maximal-path* (*subgame* V') *P*
proof−
have *lset* $P \subseteq V_{subgame\ V'}$ **unfolding** *subgame-def* **using** *P*(*2*) V' **by** *auto*
with *P*(*1*) V' **show** *?thesis*
by (*coinduction arbitrary*: *P rule*: *Digraph.maximal-path.coinduct*[*OF subgame-Digraph*])
(*cases rule*: *maximal-path.cases*, *auto*)
qed

3.8 Priorities Occurring Infinitely Often

The set of priorities that occur infinitely often on a given path. We need this to define the winning condition of parity games.

definition *path-inf-priorities* :: *'a Path* $\Rightarrow$ *nat set* **where**
path-inf-priorities $P \equiv \{k.\ \forall n.\ k \in$ *lset* (*ldropn n* (*lmap* ω *P*))$\}$

Because ω is image-finite, by the pigeon-hole principle every infinite path has at least one priority that occurs infinitely often.

lemma *path-inf-priorities-is-nonempty*:
assumes *P*: *valid-path P* ¬*lfinite P*
shows $\exists k.\ k \in$ *path-inf-priorities P*
proof−

Define a map from indices to priorities on the path.

def $f \equiv \lambda i.\ \omega\ (P\ \$\ i)$
have *range f* $\subseteq \omega$ ‘ *V* **unfolding** *f-def*
using *valid-path-in-V*[*OF P*(*1*)] *lset-nth-member-inf*[*OF P*(*2*)]
by *blast*
hence *finite* (*range f*)
using *priorities-finite finite-subset* **by** *blast*
then obtain *n0* **where** *n0*: ¬(*finite* $\{n.\ f\ n = f\ n0\}$)
using *pigeonhole-infinite*[*of UNIV f*] **by** *auto*
def $k \equiv f\ n0$

The priority *k* occurs infinitely often.

have *lmap* ω *P* \$ *n0* = *k* **unfolding** *f-def k-def*
using *assms*(*2*) **by** (*simp add*: *infinite-small-llength*)
moreover {
fix *n* **assume** *lmap* ω *P* \$ *n* = *k*
have $\exists n' > n.\ f\ n' = k$ **unfolding** *k-def* **using** *n0 infinite-nat-iff-unbounded* **by** *auto*
hence $\exists n' > n.$ *lmap* ω *P* \$ $n' = k$ **unfolding** *f-def*
using *assms*(*2*) **by** (*simp add*: *infinite-small-llength*)
}
ultimately have $\forall n.\ k \in$ *lset* (*ldropn n* (*lmap* ω *P*))
using *index-infinite-set*[*of lmap* ω *P n0 k*] *P*(*2*) *lfinite-lmap*
by *blast*
thus *?thesis* **unfolding** *path-inf-priorities-def* **by** *blast*
qed

lemma *path-inf-priorities-at-least-min-prio*:
assumes *P*: *valid-path P* **and** *a*: $a \in$ *path-inf-priorities P*

shows *Min* (ω ‘ *V*) $\leq$ *a*
proof−
  have $a \in$ *lset* (*ldropn 0* (*lmap* ω *P*)) **using** *a* **unfolding** *path-inf-priorities-def* **by** *blast*
  hence $a \in \omega$ ‘ *lset P* **by** *simp*
  thus *?thesis* **using** *P valid-path-in-V priorities-finite Min-le* **by** *blast*
qed

lemma *path-inf-priorities-LCons*:
  path-inf-priorities P = *path-inf-priorities* (*LCons v P*) (**is** *?A* = *?B*)
proof
  show *?A* $\subseteq$ *?B* **proof**
    fix *a* **assume** $a \in$ *?A*
    hence $\forall n.$ $a \in$ *lset* (*ldropn n* (*lmap* ω (*LCons v P*)))
      unfolding *path-inf-priorities-def*
      using *in-lset-ltlD*[*of a*] **by** (*simp add*: *ltl-ldropn*)
    thus $a \in$ *?B* **unfolding** *path-inf-priorities-def* **by** *blast*
  qed
next
  show *?B* $\subseteq$ *?A* **proof**
    fix *a* **assume** $a \in$ *?B*
    hence $\forall n.$ $a \in$ *lset* (*ldropn* (*Suc n*) (*lmap* ω (*LCons v P*)))
      unfolding *path-inf-priorities-def* **by** *blast*
    thus $a \in$ *?A* **unfolding** *path-inf-priorities-def* **by** *simp*
  qed
qed
corollary *path-inf-priorities-ltl*: *path-inf-priorities P* = *path-inf-priorities* (*ltl P*)
  by (*metis llist.exhaust ltl-simps path-inf-priorities-LCons*)

3.9 Winning Condition

Let $G = (V, E, V_0, \omega)$ be a parity game. An infinite path $v_0, v_1, \ldots$ in G is winning for player Even (Odd) if the minimum priority occurring infinitely often is even (odd). A finite path is winning for player p iff the last node on the path belongs to the other player.

Empty paths are irrelevant, but it is useful to assign a fixed winner to them in order to get simpler lemmas.

abbreviation *winning-priority p* $\equiv$ (*if p* = *Even then even else odd*)

definition *winning-path* :: *Player* $\Rightarrow$ *'a Path* $\Rightarrow$ *bool* **where**
  winning-path p P $\equiv$
    ($\neg$*lfinite P* $\wedge$ ($\exists\, a \in$ *path-inf-priorities P*.
      ($\forall\, b \in$ *path-inf-priorities P*. $a \leq b$) $\wedge$ *winning-priority p a*))
    $\vee$ ($\neg$*lnull P* $\wedge$ *lfinite P* $\wedge$ *llast P* $\in$ *VV p***)
    $\vee$ (*lnull P* $\wedge$ *p* = *Even*)

Every path has a unique winner.

lemma *paths-are-winning-for-one-player*:
  assumes *valid-path P*
  shows *winning-path p P* $\longleftrightarrow$ $\neg$*winning-path p*** *P*
proof (*cases*)
  assume $\neg$*lnull P*

show *?thesis* **proof** (*cases*)
  assume *lfinite P*
  thus *?thesis*
    using *assms lfinite-lset valid-path-in-V*
    unfolding *winning-path-def*
    by *auto*
next
  assume $\neg$*lfinite P*
  then obtain *a* **where** $a \in$ *path-inf-priorities P* $\bigwedge b.\ b < a \Longrightarrow b \notin$ *path-inf-priorities P*
    using *assms ex-least-nat-le*[*of* $\lambda a.\ a \in$ *path-inf-priorities P*] *path-inf-priorities-is-nonempty*
    by *blast*
  hence $\forall q.$ *winning-priority q a* $\longleftrightarrow$ *winning-path q P*
    unfolding *winning-path-def* **using** ‹$\neg$*lnull P*› ‹$\neg$*lfinite P*› **by** (*metis le-antisym not-le*)
  moreover have $\forall q.$ *winning-priority p q* $\longleftrightarrow \neg$*winning-priority p** q* **by** *simp*
  ultimately show *?thesis* **by** *blast*
qed
qed (*simp add*: *winning-path-def*)

lemma *winning-path-ltl*:
  assumes *P*: *winning-path p P* $\neg$*lnull P* $\neg$*lnull* (*ltl P*)
  shows *winning-path p* (*ltl P*)
proof (*cases*)
  assume *lfinite P*
  moreover have *llast P* = *llast* (*ltl P*)
    using *P*(*2*,*3*) **by** (*metis llast-LCons2 ltl-simps*(*2*) *not-lnull-conv*)
  ultimately show *?thesis* **using** *P* **by** (*simp add*: *winning-path-def*)
next
  assume $\neg$*lfinite P*
  thus *?thesis* **using** *winning-path-def path-inf-priorities-ltl P*(*1*,*2*) **by** *auto*
qed

corollary *winning-path-drop*:
  assumes *winning-path p P enat n* < *llength P*
  shows *winning-path p* (*ldropn n P*)
using *assms* **proof** (*induct n*)
  case (*Suc n*)
  hence *winning-path p* (*ldropn n P*) **using** *dual-order.strict-trans enat-ord-simps*(*2*) **by** *blast*
  moreover have *ltl* (*ldropn n P*) = *ldropn* (*Suc n*) *P* **by** (*simp add*: *ldrop-eSuc-ltl ltl-ldropn*)
  moreover hence $\neg$*lnull* (*ldropn n P*) **using** *Suc.prems*(*2*) **by** (*metis leD lnull-ldropn lnull-ltlI*)
  ultimately show *?case* **using** *winning-path-ltl*[*of p ldropn n P*] *Suc.prems*(*2*) **by** *auto*
qed *simp*

corollary *winning-path-drop-add*:
  assumes *valid-path P winning-path p* (*ldropn n P*) *enat n* < *llength P*
  shows *winning-path p P*
  using *assms paths-are-winning-for-one-player valid-path-drop winning-path-drop* **by** *blast*

lemma *winning-path-LCons*:
  assumes *P*: *winning-path p P* $\neg$*lnull P*
  shows *winning-path p* (*LCons v P*)
proof (*cases*)
  assume *lfinite P*

moreover have *llast P = llast (LCons v P)*
  using *P(2)* **by** *(metis llast-LCons2 not-lnull-conv)*
ultimately show *?thesis* **using** *P* **unfolding** *winning-path-def* **by** *simp*
next
assume $\neg$*lfinite P*
thus *?thesis* **using** *P path-inf-priorities-LCons* **unfolding** *winning-path-def* **by** *simp*
qed

lemma *winning-path-supergame*:
assumes *winning-path p P*
and *G′*: *ParityGame G′ VV p*** $\subseteq$ *ParityGame.VV G′ p*** $\omega = \omega_{G'}$
shows *ParityGame.winning-path G′ p P*
proof−
interpret *G′*: *ParityGame G′* **using** *G′(1)* **.**
have $[\![$ *lfinite P*; $\neg$*lnull P* $]\!] \Longrightarrow$ *llast P* $\in$ *G′.VV p*** **and** *lnull P* $\Longrightarrow$ *p = Even*
  using *assms(1)* **unfolding** *winning-path-def* **using** *G′(2)* **by** *auto*
thus *?thesis* **unfolding** *G′.winning-path-def*
  using *lnull-imp-lfinite assms(1)*
  unfolding *winning-path-def path-inf-priorities-def G′.path-inf-priorities-def G′(3)*
  by *blast*
qed

end — locale ParityGame

3.10 Valid Maximal Paths

Define a locale for valid maximal paths, because we need them often.

locale *vm-path = ParityGame +*
fixes *P v0*
assumes *P-not-null [simp]*: $\neg$*lnull P*
  and *P-valid [simp]*: *valid-path P*
  and *P-maximal [simp]*: *maximal-path P*
  and *P-v0 [simp]*: *lhd P = v0*
begin
lemma *P-LCons*: *P = LCons v0 (ltl P)* **using** *lhd-LCons-ltl[OF P-not-null]* **by** *simp*

lemma *P-len [simp]*: *enat 0 < llength P* **by** *(simp add: lnull-0-llength)*
lemma *P-0 [simp]*: *P $ 0 = v0* **by** *(simp add: lnth-0-conv-lhd)*
lemma *P-lnth-Suc*: *P $ Suc n = ltl P $ n* **by** *(simp add: lnth-ltl)*
lemma *P-no-deadends*: *enat (Suc n) < llength P* $\Longrightarrow$ $\neg$*deadend (P $ n)*
using *valid-path-no-deadends* **by** *simp*
lemma *P-no-deadend-v0*: $\neg$*lnull (ltl P)* $\Longrightarrow$ $\neg$*deadend v0*
by *(metis P-LCons P-valid edges-are-in-V(2) not-lnull-conv valid-path-edges′)*
lemma *P-no-deadend-v0-llength*: *enat (Suc n) < llength P* $\Longrightarrow$ $\neg$*deadend v0*
by *(metis P-0 P-len P-valid enat-ord-simps(2) not-less-eq valid-path-ends-on-deadend zero-less-Suc)*
lemma *P-ends-on-deadend*: $[\![$ *enat n < llength P*; *deadend (P $ n)* $]\!] \Longrightarrow$ *enat (Suc n) = llength P*
using *P-valid valid-path-ends-on-deadend* **by** *blast*

lemma *P-lnull-ltl-deadend-v0*: *lnull (ltl P)* $\Longrightarrow$ *deadend v0*
using *P-LCons maximal-no-deadend* **by** *force*
lemma *P-lnull-ltl-LCons*: *lnull (ltl P)* $\Longrightarrow$ *P = LCons v0 LNil*

using *P-LCons lnull-def* **by** *metis*
lemma *P-deadend-v0-LCons*: *deadend v0* $\Longrightarrow$ *P* = *LCons v0 LNil*
using *P-lnull-ltl-LCons P-no-deadend-v0* **by** *blast*

lemma *Ptl-valid* [*simp*]: *valid-path* (*ltl P*) **using** *valid-path-ltl* **by** *auto*
lemma *Ptl-maximal* [*simp*]: *maximal-path* (*ltl P*) **using** *maximal-ltl* **by** *auto*

lemma *Pdrop-valid* [*simp*]: *valid-path* (*ldropn n P*) **using** *valid-path-drop* **by** *auto*
lemma *Pdrop-maximal* [*simp*]: *maximal-path* (*ldropn n P*) **using** *maximal-drop* **by** *auto*

lemma *prefix-valid* [*simp*]: *valid-path* (*ltake n P*)
using *valid-path-prefix*[*of P*] **by** *auto*

lemma *extension-valid* [*simp*]: *v*$\rightarrow$*v0* $\Longrightarrow$ *valid-path* (*LCons v P*)
using *P-not-null P-v0 P-valid valid-path-cons* **by** *blast*
lemma *extension-maximal* [*simp*]: *maximal-path* (*LCons v P*)
by (*simp add*: *maximal-path-cons*)
lemma *lappend-maximal* [*simp*]: *maximal-path* (*lappend P' P*)
by (*simp add*: *maximal-path-lappend*)

lemma *v0-V* [*simp*]: *v0* $\in$ *V* **by** (*metis P-LCons P-valid valid-path-cons-simp*)
lemma *v0-lset-P* [*simp*]: *v0* $\in$ *lset P* **using** *P-not-null P-v0 llist.set-sel*(*1*) **by** *blast*
lemma *v0-VV*: *v0* $\in$ *VV p* $\vee$ *v0* $\in$ *VV p*** **by** *simp*
lemma *lset-P-V* [*simp*]: *lset P* $\subseteq$ *V* **by** (*simp add*: *valid-path-in-V*)
lemma *lset-ltl-P-V* [*simp*]: *lset* (*ltl P*) $\subseteq$ *V* **by** (*simp add*: *valid-path-in-V*)

lemma *finite-llast-deadend* [*simp*]: *lfinite P* $\Longrightarrow$ *deadend* (*llast P*)
using *P-maximal P-not-null maximal-ends-on-deadend* **by** *blast*
lemma *finite-llast-V* [*simp*]: *lfinite P* $\Longrightarrow$ *llast P* $\in$ *V*
using *P-not-null lfinite-lset lset-P-V* **by** *blast*

If a path visits a deadend, it is winning for the other player.

lemma *visits-deadend*:
assumes *lset P* $\cap$ *deadends p* $\neq$ {}
shows *winning-path p** P*
proof−
obtain *n* **where** *n*: *enat n* < *llength P P* $ *n* $\in$ *deadends p*
using *assms* **by** (*meson lset-intersect-lnth*)
hence *: *enat* (*Suc n*) = *llength P* **using** *P-ends-on-deadend* **unfolding** *deadends-def* **by** *blast*
hence *llast P* = *P* $ *n* **by** (*simp add*: *eSuc-enat llast-conv-lnth*)
hence *llast P* $\in$ *deadends p* **using** *n*(*2*) **by** *simp*
moreover have *lfinite P* **using** * *llength-eq-enat-lfiniteD* **by** *force*
ultimately show *?thesis* **unfolding** *winning-path-def deadends-def* **by** *auto*
qed

end

end

4 Positional Strategies

theory *Strategy*
imports
Main
ParityGame
begin

4.1 Definitions

A *strategy* is simply a function from nodes to nodes We only consider positional strategies.

type-synonym $'a$ *Strategy* $= 'a \Rightarrow 'a$

A *valid* strategy for player p is a function assigning a successor to each node in $VV\ p$.

definition (**in** *ParityGame*) *strategy* :: *Player* $\Rightarrow$ $'a$ *Strategy* $\Rightarrow$ *bool* **where**
strategy $p\ \sigma \equiv \forall v \in VV\ p.\ \neg$*deadend* $v \longrightarrow v{\rightarrow}\sigma\ v$

lemma (**in** *ParityGame*) *strategyI* [*intro*]:
$(\bigwedge v.\ [\![\ v \in VV\ p;\ \neg$*deadend* $v\ ]\!] \Longrightarrow v{\rightarrow}\sigma\ v) \Longrightarrow$ *strategy* $p\ \sigma$
unfolding *strategy-def* **by** *blast*

4.2 Strategy-Conforming Paths

If *path-conforms-with-strategy* $p\ P\ \sigma$ holds, then we call P a σ-*path*. This means that P follows σ on all nodes of player p except maybe the last node on the path.

coinductive (**in** *ParityGame*) *path-conforms-with-strategy*
:: *Player* $\Rightarrow$ $'a$ *Path* $\Rightarrow$ $'a$ *Strategy* $\Rightarrow$ *bool* **where**
path-conforms-LNil: *path-conforms-with-strategy* p *LNil* σ
| *path-conforms-LCons-LNil*: *path-conforms-with-strategy* p (*LCons* v *LNil*) σ
| *path-conforms-VVp*: $[\![\ v \in VV\ p;\ w = \sigma\ v;$ *path-conforms-with-strategy* p (*LCons* w *Ps*) $\sigma\ ]\!]$
$\Longrightarrow$ *path-conforms-with-strategy* p (*LCons* v (*LCons* w *Ps*)) σ
| *path-conforms-VVpstar*: $[\![\ v \notin VV\ p;$ *path-conforms-with-strategy* p *Ps* $\sigma\ ]\!]$
$\Longrightarrow$ *path-conforms-with-strategy* p (*LCons* v *Ps*) σ

Define a locale for valid maximal paths that conform to a given strategy, because we need this concept quite often. However, we are not yet able to add interesting lemmas to this locale. We will do this at the end of this section, where we have more lemmas available.

locale *vmc-path* = *vm-path* +
fixes $p\ \sigma$ **assumes** *P-conforms* [*simp*]: *path-conforms-with-strategy* $p\ P\ \sigma$

Similary, define a locale for valid maximal paths that conform to given strategies for both players.

locale *vmc2-path* = *comp?*: *vmc-path* $G\ P\ v0\ p{**}\ \sigma'$ + *vmc-path* $G\ P\ v0\ p\ \sigma$
for $G\ P\ v0\ p\ \sigma\ \sigma'$

4.3 An Arbitrary Strategy

context *ParityGame* **begin**

Define an arbitrary strategy. This is useful to define other strategies by overriding part of this strategy.

definition σ*-arbitrary* $\equiv \lambda v.\ SOME\ w.\ v{\rightarrow}w$

lemma *valid-arbitrary-strategy* [*simp*]: *strategy p* σ*-arbitrary* **proof**
  fix v **assume** $\neg$*deadend v*
  thus $v \rightarrow \sigma$*-arbitrary v* **unfolding** σ*-arbitrary-def* **using** *someI-ex*[*of* $\lambda w.\ v{\rightarrow}w$] **by** *blast*
qed

4.4 Valid Strategies

lemma *valid-strategy-updates*: $[\![$ *strategy p* σ; $v0{\rightarrow}w0$ $]\!] \Longrightarrow$ *strategy p* $(\sigma(v0 := w0))$
  unfolding *strategy-def* **by** *auto*

lemma *valid-strategy-updates-set*:
  assumes *strategy p* σ $\bigwedge v.\ [\![\ v \in A;\ v \in VV\ p;\ \neg deadend\ v\ ]\!] \Longrightarrow v{\rightarrow}\sigma'\ v$
  shows *strategy p* (*override-on* σ σ' *A*)
  unfolding *strategy-def* **by** (*metis assms override-on-def strategy-def*)

lemma *valid-strategy-updates-set-strong*:
  assumes *strategy p* σ *strategy p* σ'
  shows *strategy p* (*override-on* σ σ' *A*)
  using *assms*(*1*) *assms*(*2*)[*unfolded strategy-def*] *valid-strategy-updates-set* **by** *simp*

lemma *subgame-strategy-stays-in-subgame*:
  assumes σ: *ParityGame.strategy* (*subgame* V') *p* σ
    and $v \in$ *ParityGame.VV* (*subgame* V') *p* $\neg$*Digraph.deadend* (*subgame* V') *v*
  shows $\sigma\ v \in V'$
proof−
  interpret G': *ParityGame subgame* V' **using** *subgame-ParityGame* **.**
  have $\sigma\ v \in V_{subgame\ V'}$ **using** *assms* **unfolding** G'*.strategy-def* G'*.edges-are-in-V*(*2*) **by** *blast*
  thus $\sigma\ v \in V'$ **by** (*metis Diff-iff IntE subgame-VV Player.distinct*(*2*))
qed

lemma *valid-strategy-supergame*:
  assumes σ: *strategy p* σ
    and σ': *ParityGame.strategy* (*subgame* V') *p* σ'
    and G'*-no-deadends*: $\bigwedge v.\ v \in V' \Longrightarrow \neg$*Digraph.deadend* (*subgame* V') *v*
  shows *strategy p* (*override-on* σ σ' V') (**is** *strategy p ?*σ)
proof
  interpret G': *ParityGame subgame* V' **using** *subgame-ParityGame* **.**
  fix v **assume** v: $v \in VV\ p$ $\neg$*deadend v*
  show $v \rightarrow ?\sigma\ v$ **proof** (*cases*)
    assume $v \in V'$
    hence $v \in G'.VV\ p$ **using** *subgame-VV* ‹$v \in VV\ p$› **by** *blast*
    moreover have $\neg G'$*.deadend v* **using** G'*-no-deadends* ‹$v \in V'$› **by** *blast*
    ultimately have $v \rightarrow_{subgame\ V'} \sigma'\ v$ **using** σ' **unfolding** G'*.strategy-def* **by** *blast*
    moreover have $\sigma'\ v = ?\sigma\ v$ **using** ‹$v \in V'$› **by** *simp*
    ultimately show *?thesis* **by** (*metis subgame-E subsetCE*)
  next
    assume $v \notin V'$

thus *?thesis* **using** v σ **unfolding** *strategy-def* **by** *simp*
qed
qed

lemma *valid-strategy-in-V*: $\llbracket$ *strategy* p σ; $v \in VV\ p$; $\neg$*deadend* v $\rrbracket \Longrightarrow \sigma\ v \in V$
unfolding *strategy-def* **using** *valid-edge-set* **by** *auto*

lemma *valid-strategy-only-in-V*: $\llbracket$ *strategy* p σ; $\bigwedge v.\ v \in V \Longrightarrow \sigma\ v = \sigma'\ v$ $\rrbracket \Longrightarrow$ *strategy* p σ'
unfolding *strategy-def* **using** *edges-are-in-V*(*1*) **by** *auto*

4.5 Conforming Strategies

lemma *path-conforms-with-strategy-ltl* [*intro*]:
path-conforms-with-strategy p P $\sigma \Longrightarrow$ *path-conforms-with-strategy* p (*ltl* P) σ
by (*drule path-conforms-with-strategy.cases*) (*simp-all add*: *path-conforms-with-strategy.intros*(*1*))

lemma *path-conforms-with-strategy-drop*:
path-conforms-with-strategy p P $\sigma \Longrightarrow$ *path-conforms-with-strategy* p (*ldropn* n P) σ
by (*simp add*: *path-conforms-with-strategy-ltl ltl-ldrop*[*of* $\lambda P.$ *path-conforms-with-strategy* p P σ])

lemma *path-conforms-with-strategy-prefix*:
path-conforms-with-strategy p P $\sigma \Longrightarrow$ *lprefix* P' $P \Longrightarrow$ *path-conforms-with-strategy* p P' σ
proof (*coinduction arbitrary*: P P')
case (*path-conforms-with-strategy* P P')
thus *?case* **proof** (*cases rule*: *path-conforms-with-strategy.cases*)
case *path-conforms-LNil*
thus *?thesis* **using** *path-conforms-with-strategy*(*2*) **by** *auto*
next
case *path-conforms-LCons-LNil*
thus *?thesis* **by** (*metis lprefix-LCons-conv lprefix-antisym lprefix-code*(*1*) *path-conforms-with-strategy*(*2*))
next
case (*path-conforms-VVp* v w)
thus *?thesis* **proof** (*cases*)
assume $P' \neq LNil \wedge P' \neq LCons\ v\ LNil$
hence $\exists\, Q.\ P' = LCons\ v\ (LCons\ w\ Q)$
by (*metis local.path-conforms-VVp*(*1*) *lprefix-LCons-conv path-conforms-with-strategy*(*2*))
thus *?thesis* **using** *local.path-conforms-VVp*(*1*,*3*,*4*) *path-conforms-with-strategy*(*2*) **by** *force*
qed *auto*
next
case (*path-conforms-VVpstar* v)
thus *?thesis* **proof** (*cases*)
assume $P' \neq LNil$
hence $\exists\, Q.\ P' = LCons\ v\ Q$
using *local.path-conforms-VVpstar*(*1*) *lprefix-LCons-conv path-conforms-with-strategy*(*2*) **by** *fastforce*
thus *?thesis* **using** *local.path-conforms-VVpstar path-conforms-with-strategy*(*2*) **by** *auto*
qed *simp*
qed
qed

lemma *path-conforms-with-strategy-irrelevant*:
assumes *path-conforms-with-strategy* p P σ $v \notin$ *lset* P

shows *path-conforms-with-strategy p P* $(\sigma(v := w))$
using *assms* **apply** (*coinduction arbitrary*: *P*) **by** (*drule path-conforms-with-strategy.cases*) *auto*

lemma *path-conforms-with-strategy-irrelevant-deadend*:
assumes *path-conforms-with-strategy p P* σ *deadend v* $\vee$ $v \notin$ *VV p valid-path P*
shows *path-conforms-with-strategy p P* $(\sigma(v := w))$
using *assms* **proof** (*coinduction arbitrary*: *P*)
let $?\sigma = \sigma(v := w)$
case (*path-conforms-with-strategy P*)
thus *?case* **proof** (*cases rule*: *path-conforms-with-strategy.cases*)
case (*path-conforms-VVp v' w Ps*)
have $w = ?\sigma\ v'$ **proof**−
from ‹*valid-path P*› **have** ¬*deadend v'*
using *local.path-conforms-VVp(1) valid-path-cons-simp* **by** *blast*
with *assms(2)* **have** $v' \neq v$ **using** *local.path-conforms-VVp(2)* **by** *blast*
thus $w = ?\sigma\ v'$ **by** (*simp add*: *local.path-conforms-VVp(3)*)
qed
moreover
have $\exists P.$ *LCons w Ps* = *P* $\wedge$ *path-conforms-with-strategy p P* σ $\wedge$ (*deadend v* $\vee$ $v \notin$ *VV p*) $\wedge$ *valid-path P*
proof−
have *valid-path* (*LCons w Ps*)
using *local.path-conforms-VVp(1) path-conforms-with-strategy(3) valid-path-ltl'* **by** *blast*
thus *?thesis* **using** *local.path-conforms-VVp(4) path-conforms-with-strategy(2)* **by** *blast*
qed
ultimately show *?thesis* **using** *local.path-conforms-VVp(1,2)* **by** *blast*
next
case (*path-conforms-VVpstar v' Ps*)
have $\exists P.$ *path-conforms-with-strategy p Ps* σ $\wedge$ (*deadend v* $\vee$ $v \notin$ *VV p*) $\wedge$ *valid-path Ps*
using *local.path-conforms-VVpstar(1,3) path-conforms-with-strategy(2,3) valid-path-ltl'* **by** *blast*
thus *?thesis* **by** (*simp add*: *local.path-conforms-VVpstar(1,2)*)
qed *simp-all*
qed

lemma *path-conforms-with-strategy-irrelevant-updates*:
assumes *path-conforms-with-strategy p P* σ $\bigwedge v.\ v \in$ *lset P* $\Longrightarrow \sigma\ v = \sigma'\ v$
shows *path-conforms-with-strategy p P* σ'
using *assms* **proof** (*coinduction arbitrary*: *P*)
case (*path-conforms-with-strategy P*)
thus *?case* **proof** (*cases rule*: *path-conforms-with-strategy.cases*)
case (*path-conforms-VVp v' w Ps*)
have $w = \sigma'\ v'$ **using** *local.path-conforms-VVp(1,3) path-conforms-with-strategy(2)* **by** *auto*
thus *?thesis* **using** *local.path-conforms-VVp(1,4) path-conforms-with-strategy(2)* **by** *auto*
qed *simp-all*
qed

lemma *path-conforms-with-strategy-irrelevant'*:
assumes *path-conforms-with-strategy p P* $(\sigma(v := w))$ $v \notin$ *lset P*
shows *path-conforms-with-strategy p P* σ
by (*metis assms fun-upd-triv fun-upd-upd path-conforms-with-strategy-irrelevant*)

```
lemma path-conforms-with-strategy-irrelevant-deadend':
  assumes path-conforms-with-strategy p P (σ(v := w)) deadend v ∨ v ∉ VV p valid-path P
  shows path-conforms-with-strategy p P σ
  by (metis assms fun-upd-triv fun-upd-upd path-conforms-with-strategy-irrelevant-deadend)

lemma path-conforms-with-strategy-start:
  path-conforms-with-strategy p (LCons v (LCons w P)) σ ⟹ v ∈ VV p ⟹ σ v = w
  by (drule path-conforms-with-strategy.cases) simp-all

lemma path-conforms-with-strategy-lappend:
  assumes
    P: lfinite P ¬lnull P path-conforms-with-strategy p P σ
    and P': ¬lnull P' path-conforms-with-strategy p P' σ
    and conforms: llast P ∈ VV p ⟹ σ (llast P) = lhd P'
  shows path-conforms-with-strategy p (lappend P P') σ
using assms proof (induct P rule: lfinite-induct)
  case (LCons P)
  show ?case proof (cases)
    assume lnull (ltl P)
    then obtain v0 where v0: P = LCons v0 LNil
      by (metis LCons.prems(1) lhd-LCons-ltl llist.collapse(1))
    have path-conforms-with-strategy p (LCons (lhd P) P') σ proof (cases)
      assume lhd P ∈ VV p
      moreover with v0 have lhd P' = σ (lhd P)
        using LCons.prems(5) by auto
      ultimately show ?thesis
        using path-conforms-VVp[of lhd P p lhd P' σ]
        by (metis (no-types) LCons.prems(4) ‹¬lnull P'› lhd-LCons-ltl)
    next
      assume lhd P ∉ VV p
      thus ?thesis using path-conforms-VVpstar using LCons.prems(4) v0 by blast
    qed
    thus ?thesis by (simp add: v0)
  next
    assume ¬lnull (ltl P)
    hence *: path-conforms-with-strategy p (lappend (ltl P) P') σ
     by (metis LCons.hyps(3) LCons.prems(1) LCons.prems(2) LCons.prems(5) LCons.prems(5)
              assms(4) assms(5) lhd-LCons-ltl llast-LCons2 path-conforms-with-strategy-ltl)
    have path-conforms-with-strategy p (LCons (lhd P) (lappend (ltl P) P')) σ proof (cases)
      assume lhd P ∈ VV p
      moreover hence lhd (ltl P) = σ (lhd P)
        by (metis LCons.prems(1) LCons.prems(2) ‹¬lnull (ltl P)›
              lhd-LCons-ltl path-conforms-with-strategy-start)
      ultimately show ?thesis
        using path-conforms-VVp[of lhd P p lhd (ltl P) σ] * ‹¬lnull (ltl P)›
        by (metis lappend-code(2) lhd-LCons-ltl)
    next
      assume lhd P ∉ VV p
      thus ?thesis by (simp add: * path-conforms-VVpstar)
    qed
    with ‹¬lnull P› show path-conforms-with-strategy p (lappend P P') σ
      by (metis lappend-code(2) lhd-LCons-ltl)
```

qed
qed *simp*

lemma *path-conforms-with-strategy-VVpstar*:
assumes *lset* $P \subseteq VV\ p{*}{*}$
shows *path-conforms-with-strategy* $p\ P\ \sigma$
using *assms* **proof** (*coinduction arbitrary*: P)
case (*path-conforms-with-strategy* P)
moreover have $\bigwedge v\ Ps.\ P = LCons\ v\ Ps \Longrightarrow$ *?case* **using** *path-conforms-with-strategy* **by** *auto*
ultimately show *?case* **by** (*cases* $P = LNil$, *simp*) (*metis lnull-def not-lnull-conv*)
qed

lemma *subgame-path-conforms-with-strategy*:
assumes V': $V' \subseteq V$ **and** P: *path-conforms-with-strategy* $p\ P\ \sigma$ *lset* $P \subseteq V'$
shows *ParityGame.path-conforms-with-strategy* (*subgame* V') $p\ P\ \sigma$
proof−
have *lset* $P \subseteq V_{subgame\ V'}$ **unfolding** *subgame-def* **using** $P(2)$ V' **by** *auto*
with $P(1)$ **show** *?thesis*
by (*coinduction arbitrary*: P *rule*: *ParityGame.path-conforms-with-strategy.coinduct*[*OF subgame-ParityGame*])
(*cases rule*: *path-conforms-with-strategy.cases*, *auto*)
qed

lemma (**in** *vmc-path*) *subgame-path-vmc-path*:
assumes V': $V' \subseteq V$ **and** P: *lset* $P \subseteq V'$
shows *vmc-path* (*subgame* V') $P\ v0\ p\ \sigma$
proof−
interpret G': *ParityGame subgame* V' **using** *subgame-ParityGame* **by** *blast*
show *?thesis* **proof**
show G'*.valid-path* P **using** *subgame-valid-path P-valid* P **by** *blast*
show G'*.maximal-path* P **using** *subgame-maximal-path* V' *P-maximal* P **by** *blast*
show G'*.path-conforms-with-strategy* $p\ P\ \sigma$
using *subgame-path-conforms-with-strategy* V' *P-conforms* P **by** *blast*
qed *simp-all*
qed

4.6 Greedy Conforming Path

Given a starting point and two strategies, there exists a path conforming to both strategies. Here we define this path. Incidentally, this also shows that the assumptions of the locales *vmc-path* and *vmc2-path* are satisfiable.

We are only interested in proving the existence of such a path, so the definition (i.e., the implementation) and most lemmas are private.

context begin

private primcorec *greedy-conforming-path* :: *Player* $\Rightarrow$ *'a Strategy* $\Rightarrow$ *'a Strategy* $\Rightarrow$ *'a* $\Rightarrow$ *'a Path*
where
greedy-conforming-path $p\ \sigma\ \sigma'\ v0$ =
LCons v0 (*if deadend v0*
then LNil
else if $v0 \in VV\ p$

then greedy-conforming-path p σ σ' $(\sigma$ *v0*$)$
else greedy-conforming-path p σ σ' $(\sigma'$ *v0*$))$

private lemma *greedy-path-LNil*: *greedy-conforming-path p* σ σ' *v0* $\neq$ *LNil*
using *greedy-conforming-path.disc-iff llist.discI*(*1*) **by** *blast*

private lemma *greedy-path-lhd*: *greedy-conforming-path p* σ σ' *v0* = *LCons v P* $\Longrightarrow$ *v* = *v0*
using *greedy-conforming-path.code* **by** *auto*

private lemma *greedy-path-deadend-v0*: *greedy-conforming-path p* σ σ' *v0* = *LCons v P* $\Longrightarrow$ *P* = *LNil* $\longleftrightarrow$ *deadend v0*
by (*metis* (*no-types*, *lifting*) *greedy-conforming-path.disc-iff*
greedy-conforming-path.simps(*3*) *llist.disc*(*1*) *ltl-simps*(*2*))

private corollary *greedy-path-deadend-v*:
greedy-conforming-path p σ σ' *v0* = *LCons v P* $\Longrightarrow$ *P* = *LNil* $\longleftrightarrow$ *deadend v*
using *greedy-path-deadend-v0 greedy-path-lhd* **by** *metis*
corollary *greedy-path-deadend-v'*: *greedy-conforming-path p* σ σ' *v0* = *LCons v LNil* $\Longrightarrow$ *deadend v*
using *greedy-path-deadend-v* **by** *blast*

private lemma *greedy-path-ltl*:
assumes *greedy-conforming-path p* σ σ' *v0* = *LCons v P*
shows *P* = *LNil* $\vee$ *P* = *greedy-conforming-path p* σ σ' $(\sigma$ *v0*$)$ $\vee$ *P* = *greedy-conforming-path p* σ σ' $(\sigma'$ *v0*$)$
apply (*insert assms*, *frule greedy-path-lhd*)
apply (*cases deadend v0*, *simp add*: *greedy-conforming-path.code*)
by (*metis* (*no-types*, *lifting*) *greedy-conforming-path.sel*(*2*) *ltl-simps*(*2*))

private lemma *greedy-path-ltl-ex*:
assumes *greedy-conforming-path p* σ σ' *v0* = *LCons v P*
shows *P* = *LNil* $\vee$ $(\exists$ *v*. *P* = *greedy-conforming-path p* σ σ' *v*$)$
using *assms greedy-path-ltl* **by** *blast*

private lemma *greedy-path-ltl-VVp*:
assumes *greedy-conforming-path p* σ σ' *v0* = *LCons v0 P v0* $\in$ *VV p* $\neg$*deadend v0*
shows σ *v0* = *lhd P*
using *assms greedy-conforming-path.code* **by** *auto*

private lemma *greedy-path-ltl-VVpstar*:
assumes *greedy-conforming-path p* σ σ' *v0* = *LCons v0 P v0* $\in$ *VV p*** $\neg$*deadend v0*
shows σ' *v0* = *lhd P*
using *assms greedy-conforming-path.code* **by** *auto*

private lemma *greedy-conforming-path-properties*:
assumes *v0* $\in$ *V strategy p* σ *strategy p*** σ'
shows
greedy-path-not-null: $\neg$*lnull* (*greedy-conforming-path p* σ σ' *v0*)
and *greedy-path-v0*: *greedy-conforming-path p* σ σ' *v0* \$ *0* = *v0*
and *greedy-path-valid*: *valid-path* (*greedy-conforming-path p* σ σ' *v0*)
and *greedy-path-maximal*: *maximal-path* (*greedy-conforming-path p* σ σ' *v0*)
and *greedy-path-conforms*: *path-conforms-with-strategy p* (*greedy-conforming-path p* σ σ' *v0*) σ

and *greedy-path-conforms'*: *path-conforms-with-strategy p∗∗ (greedy-conforming-path p σ σ' v0) σ'*
proof−
def [*simp*]: $P \equiv$ *greedy-conforming-path p σ σ' v0*

show ¬*lnull P P $ 0 = v0* **by** (*simp-all add*: *lnth-0-conv-lhd*)

{
fix *v0* **assume** *v0* ∈ *V*
let *?P = greedy-conforming-path p σ σ' v0*
assume *asm*: ¬(∃ *v*. *?P = LCons v LNil*)
obtain *P'* **where** *P'*: *?P = LCons v0 P'* **by** (*metis greedy-path-LNil greedy-path-lhd neq-LNil-conv*)
hence ¬*deadend v0* **using** *asm greedy-path-deadend-v0* ‹*v0* ∈ *V*› **by** *blast*
from *P'* **have** *1*: ¬*lnull P'* **using** *asm llist.collapse*(*1*) ‹*v0* ∈ *V*› *greedy-path-deadend-v0* **by** *blast*
moreover from *P'* ‹¬*deadend v0*› *assms*(*2*,*3*) ‹*v0* ∈ *V*›
have *v0*→*lhd P'*
unfolding *strategy-def* **using** *greedy-path-ltl-VVp greedy-path-ltl-VVpstar*
by (*cases v0* ∈ *VV p*) *auto*
moreover hence *lhd P'* ∈ *V* **by** *blast*
moreover hence ∃ *v*. *P' = greedy-conforming-path p σ σ' v* ∧ *v* ∈ *V*
by (*metis P' calculation*(*1*) *greedy-conforming-path.simps*(*2*) *greedy-path-ltl-ex lnull-def*)

The conjunction of all the above.

ultimately
have ∃ *P'*. *?P = LCons v0 P'* ∧ ¬*lnull P'* ∧ *v0*→*lhd P'* ∧ *lhd P'* ∈ *V*
∧ (∃ *v*. *P' = greedy-conforming-path p σ σ' v* ∧ *v* ∈ *V*)
using *P'* **by** *blast*
} **note** *coinduction-helper = this*

show *valid-path P* **using** *assms* **unfolding** *P-def*
proof (*coinduction arbitrary*: *v0 rule*: *valid-path.coinduct*)
case (*valid-path v0*)
from ‹*v0* ∈ *V*› *assms*(*2*,*3*) **show** *?case*
using *coinduction-helper*[*of v0*] *greedy-path-lhd* **by** *blast*
qed

show *maximal-path P* **using** *assms* **unfolding** *P-def*
proof (*coinduction arbitrary*: *v0*)
case (*maximal-path v0*)
from ‹*v0* ∈ *V*› *assms*(*2*,*3*) **show** *?case*
using *coinduction-helper*[*of v0*] *greedy-path-deadend-v'* **by** *blast*
qed

{
fix *p'' σ''* **assume** *p''*: (*p'' = p* ∧ *σ'' = σ*) ∨ (*p'' = p∗∗* ∧ *σ'' = σ'*)
moreover with *assms* **have** *strategy p'' σ''* **by** *blast*
hence *path-conforms-with-strategy p'' P σ''* **using** ‹*v0* ∈ *V*› **unfolding** *P-def*
proof (*coinduction arbitrary*: *v0*)
case (*path-conforms-with-strategy v0*)
show *?case* **proof** (*cases v0* ∈ *VV p''*)
case *True*

{ **assume** ¬(∃ *v*. *greedy-conforming-path p σ σ′ v0* = *LCons v LNil*)
with ‹*v0* ∈ *V*› **obtain** *P′* **where**
P′: *greedy-conforming-path p σ σ′ v0* = *LCons v0 P′* ¬*lnull P′ v0*→*lhd P′*
lhd P′ ∈ *V* ∃ *v*. *P′* = *greedy-conforming-path p σ σ′ v* ∧ *v* ∈ *V*
using *coinduction-helper* **by** *blast*
with ‹*v0* ∈ *VV p″*› *p″* **have** *σ″ v0* = *lhd P′*
using *greedy-path-ltl-VVp greedy-path-ltl-VVpstar* **by** *blast*
with ‹*v0* ∈ *VV p″*› *P′*(*1*,*2*,*5*) **have** *?path-conforms-VVp*
using *greedy-conforming-path.code path-conforms-with-strategy*(*1*) **by** *fastforce*
}
thus *?thesis* **by** *auto*
next
case *False*
thus *?thesis* **using** *coinduction-helper*[*of v0*] *path-conforms-with-strategy* **by** *auto*
qed
qed
}
thus *path-conforms-with-strategy p P σ path-conforms-with-strategy p*∗∗ *P σ′* **by** *blast*+
qed

corollary *strategy-conforming-path-exists*:
assumes *v0* ∈ *V strategy p σ strategy p*∗∗ *σ′*
obtains *P* **where** *vmc2-path G P v0 p σ σ′*
proof
show *vmc2-path G* (*greedy-conforming-path p σ σ′ v0*) *v0 p σ σ′*
using *assms* **by** *unfold-locales* (*simp-all add*: *greedy-conforming-path-properties*)
qed

corollary *strategy-conforming-path-exists-single*:
assumes *v0* ∈ *V strategy p σ*
obtains *P* **where** *vmc-path G P v0 p σ*
proof
show *vmc-path G* (*greedy-conforming-path p σ σ-arbitrary v0*) *v0 p σ*
using *assms* **by** *unfold-locales* (*simp-all add*: *greedy-conforming-path-properties*)
qed

end

end

4.7 Valid Maximal Conforming Paths

Now is the time to add some lemmas to the locale *vmc-path*.

context *vmc-path* **begin**
lemma *Ptl-conforms* [*simp*]: *path-conforms-with-strategy p* (*ltl P*) *σ*
using *P-conforms path-conforms-with-strategy-ltl* **by** *blast*
lemma *Pdrop-conforms* [*simp*]: *path-conforms-with-strategy p* (*ldropn n P*) *σ*
using *P-conforms path-conforms-with-strategy-drop* **by** *blast*
lemma *prefix-conforms* [*simp*]: *path-conforms-with-strategy p* (*ltake n P*) *σ*
using *P-conforms path-conforms-with-strategy-prefix* **by** *blast*
lemma *extension-conforms* [*simp*]:

$(v' \in VV\ p \Longrightarrow \sigma\ v' = v0) \Longrightarrow$ *path-conforms-with-strategy p* (*LCons v' P*) σ
by (*metis P-LCons P-conforms path-conforms-VVp path-conforms-VVpstar*)

lemma *extension-valid-maximal-conforming*:
assumes $v' \rightarrow v0$ $v' \in VV\ p \Longrightarrow \sigma\ v' = v0$
shows *vmc-path G* (*LCons v' P*) *v' p* σ
using *assms* **by** *unfold-locales simp-all*

lemma *vmc-path-ldropn*:
assumes *enat n* < *llength P*
shows *vmc-path G* (*ldropn n P*) (*P* $ *n*) *p* σ
using *assms* **by** *unfold-locales* (*simp-all add*: *lhd-ldropn*)

lemma *conforms-to-another-strategy*:
path-conforms-with-strategy p P $\sigma' \Longrightarrow$ *vmc-path G P v0 p* σ'
using *P-not-null P-valid P-maximal P-v0* **by** *unfold-locales blast+*
end

lemma (**in** *ParityGame*) *valid-maximal-conforming-path-0*:
assumes $\neg$*lnull P valid-path P maximal-path P path-conforms-with-strategy p P* σ
shows *vmc-path G P* (*P* $ *0*) *p* σ
using *assms* **by** *unfold-locales* (*simp-all add*: *lnth-0-conv-lhd*)

4.8 Valid Maximal Conforming Paths with One Edge

We define a locale for valid maximal conforming paths that contain at least one edge. This is equivalent to the first node being no deadend. This assumption allows us to prove much stronger lemmas about *ltl P* compared to *vmc-path*.

locale *vmc-path-no-deadend* = *vmc-path* +
assumes *v0-no-deadend* [*simp*]: $\neg$*deadend v0*
begin
definition *w0* $\equiv$ *lhd* (*ltl P*)

lemma *Ptl-not-null* [*simp*]: $\neg$*lnull* (*ltl P*)
using *P-LCons P-maximal maximal-no-deadend v0-no-deadend* **by** *metis*
lemma *Ptl-LCons*: *ltl P* = *LCons w0* (*ltl* (*ltl P*)) **unfolding** *w0-def* **by** *simp*
lemma *P-LCons'*: *P* = *LCons v0* (*LCons w0* (*ltl* (*ltl P*))) **using** *P-LCons Ptl-LCons* **by** *simp*
lemma *v0-edge-w0* [*simp*]: $v0 \rightarrow w0$ **using** *P-valid P-LCons'* **by** (*metis valid-path-edges'*)

lemma *Ptl-0*: *ltl P* $ *0* = *lhd* (*ltl P*) **by** (*simp add*: *lhd-conv-lnth*)
lemma *P-Suc-0*: *P* $ *Suc 0* = *w0* **by** (*simp add*: *P-lnth-Suc Ptl-0 w0-def*)
lemma *Ptl-edge* [*simp*]: $v0 \rightarrow$ *lhd* (*ltl P*) **by** (*metis P-LCons' P-valid valid-path-edges' w0-def*)

lemma *v0-conforms*: $v0 \in VV\ p \Longrightarrow \sigma\ v0 = w0$
using *path-conforms-with-strategy-start* **by** (*metis P-LCons' P-conforms*)

lemma *w0-V* [*simp*]: $w0 \in V$ **by** (*metis Ptl-LCons Ptl-valid valid-path-cons-simp*)
lemma *w0-lset-P* [*simp*]: $w0 \in$ *lset P* **by** (*metis P-LCons' lset-intros*(*1*) *lset-intros*(*2*))

lemma *vmc-path-ltl* [*simp*]: *vmc-path G* (*ltl P*) *w0 p* σ **by** (*unfold-locales*) (*simp-all add*: *w0-def*)
end

context *vmc-path* **begin**

lemma *vmc-path-lnull-ltl-no-deadend*:
  $\neg$*lnull* (*ltl P*) $\Longrightarrow$ *vmc-path-no-deadend G P v0 p* σ
  using *P-0 P-no-deadends* **by** (*unfold-locales*) (*metis enat-ltl-Suc lnull-0-llength*)

lemma *vmc-path-conforms*:
  assumes *enat* (*Suc n*) < *llength P P* \$ *n* $\in$ *VV p*
  shows σ (*P* \$ *n*) = *P* \$ *Suc n*
proof−
  def *P'* $\equiv$ *ldropn n P*
  then interpret *P'*: *vmc-path G P' P* \$ *n p* σ **using** *vmc-path-ldropn assms*(*1*) *Suc-llength* **by** *blast*
  have $\neg$*deadend* (*P* \$ *n*) **using** *assms*(*1*) *P-no-deadends* **by** *blast*
  then interpret *P'*: *vmc-path-no-deadend G P' P* \$ *n p* σ **by** *unfold-locales*
  have σ (*P* \$ *n*) = *P'.w0* **using** *P'.v0-conforms assms*(*2*) **by** *blast*
  thus *?thesis* **using** *P'-def P'.P-Suc-0 assms*(*1*) **by** *simp*
qed

4.9 *lset* Induction Schemas for Paths

Let us define an induction schema useful for proving *lset P* $\subseteq$ *S*.

lemma *vmc-path-lset-induction* [*consumes 1*, *case-names base step*]:
  assumes *Q P*
    and *base*: *v0* $\in$ *S*
    and *step-assumption*: $\bigwedge$*P v0*. $[\![$ *vmc-path-no-deadend G P v0 p* σ; *v0* $\in$ *S*; *Q P* $]\!]$
      $\Longrightarrow$ *Q* (*ltl P*) $\wedge$ (*vmc-path-no-deadend.w0 P*) $\in$ *S*
  shows *lset P* $\subseteq$ *S*
proof
  fix *v* **assume** *v* $\in$ *lset P*
  thus *v* $\in$ *S* **using** *vmc-path-axioms assms*(*1*,*2*) **proof** (*induct arbitrary*: *v0 rule*: *llist-set-induct*)
    case (*find P*)
    then interpret *vmc-path G P v0 p* σ **by** *blast*
    show *?case* **by** (*simp add*: *find.prems*(*3*))
  next
    case (*step P v*)
    then interpret *vmc-path G P v0 p* σ **by** *blast*
    show *?case* **proof** (*cases*)
      assume *lnull* (*ltl P*)
      hence *P* = *LCons v LNil* **by** (*metis llist.disc*(*2*) *lset-cases step.hyps*(*2*))
      thus *?thesis* **using** *step.prems*(*3*) *P-LCons* **by** *blast*
    next
      assume $\neg$*lnull* (*ltl P*)
      then interpret *vmc-path-no-deadend G P v0 p* σ
        using *vmc-path-lnull-ltl-no-deadend* **by** *blast*
      show *v* $\in$ *S*
        using *step.hyps*(*3*)
              step-assumption[*OF vmc-path-no-deadend-axioms* ‹*v0* $\in$ *S*› ‹*Q P*›]
              vmc-path-ltl
        by *blast*

    qed
  qed
qed

$[\![?Q\ P;\ v0 \in ?S;\ \bigwedge P\ v0.\ [\![vmc\text{-}path\text{-}no\text{-}deadend\ G\ P\ v0\ p\ \sigma;\ v0 \in ?S;\ ?Q\ P]\!] \Longrightarrow ?Q\ (ltl\ P) \wedge vmc\text{-}path\text{-}no\text{-}deadend.w0\ P \in ?S]\!] \Longrightarrow lset\ P \subseteq ?S$ without the Q predicate.

corollary *vmc-path-lset-induction-simple* [*case-names base step*]:
  assumes *base*: $v0 \in S$
    and *step*: $\bigwedge P\ v0.\ [\![\ vmc\text{-}path\text{-}no\text{-}deadend\ G\ P\ v0\ p\ \sigma;\ v0 \in S\ ]\!]$
      $\Longrightarrow vmc\text{-}path\text{-}no\text{-}deadend.w0\ P \in S$
  shows $lset\ P \subseteq S$
  using *assms vmc-path-lset-induction*[*of* $\lambda P.\ True$] **by** *blast*

Another induction schema for proving $lset\ P \subseteq S$ based on closure properties.

lemma *vmc-path-lset-induction-closed-subset* [*case-names VVp VVpstar v0 disjoint*]:
  assumes *VVp*: $\bigwedge v.\ [\![\ v \in S;\ \neg deadend\ v;\ v \in VV\ p\ ]\!] \Longrightarrow \sigma\ v \in S \cup T$
    and *VVpstar*: $\bigwedge v\ w.\ [\![\ v \in S;\ \neg deadend\ v;\ v \in VV\ p{*}{*}\ ;\ v{\rightarrow}w\ ]\!] \Longrightarrow w \in S \cup T$
    and *v0*: $v0 \in S$
    and *disjoint*: $lset\ P \cap T = \{\}$
  shows $lset\ P \subseteq S$
using *disjoint* **proof** (*induct rule*: *vmc-path-lset-induction*)
  case (*step P v0*)
  interpret *vmc-path-no-deadend G P v0 p σ* **using** *step.hyps*(*1*) **.**
  have $lset\ (ltl\ P) \cap T = \{\}$ **using** *step.hyps*(*3*)
    by (*meson disjoint-eq-subset-Compl lset-ltl order.trans*)
  moreover have $w0 \in S \cup T$
    using *assms*(*1*,*2*)[*of v0*] *step.hyps*(*2*) *v0-no-deadend v0-conforms*
    by (*cases* $v0 \in VV\ p$) *simp-all*
  ultimately show *?case* **using** *step.hyps*(*3*) *w0-lset-P* **by** *blast*
qed (*insert v0*)

end

end

5 Attracting Strategies

theory *AttractingStrategy*
imports
  Main
  Strategy
begin

Here we introduce the concept of attracting strategies.

context *ParityGame* **begin**

5.1 Paths Visiting a Set

A path that stays in A until eventually it visits W.

definition *visits-via* $P\ A\ W \equiv \exists n.\ enat\ n < llength\ P \wedge P\ \$\ n \in W \wedge lset\ (ltake\ (enat\ n)\ P) \subseteq A$

lemma *visits-via-monotone*: $[\![\ visits\text{-}via\ P\ A\ W;\ A \subseteq A'\ ]\!] \Longrightarrow visits\text{-}via\ P\ A'\ W$
  unfolding *visits-via-def* **by** *blast*

lemma *visits-via-visits*: $visits\text{-}via\ P\ A\ W \Longrightarrow lset\ P \cap W \neq \{\}$
  unfolding *visits-via-def* **by** (*meson disjoint-iff-not-equal in-lset-conv-lnth*)

lemma (**in** *vmc-path*) *visits-via-trivial*: $v0 \in W \Longrightarrow visits\text{-}via\ P\ A\ W$
  unfolding *visits-via-def* **apply** (*rule exI*[*of - 0*]) **using** *zero-enat-def* **by** *auto*

lemma *visits-via-LCons*:
  assumes *visits-via P A W*
  shows *visits-via* (*LCons v0 P*) (*insert v0 A*) *W*
proof−
  obtain *n* **where** n: $enat\ n < llength\ P\ P\ \$\ n \in W\ lset\ (ltake\ (enat\ n)\ P) \subseteq A$
    using *assms* **unfolding** *visits-via-def* **by** *blast*
  def $P' \equiv LCons\ v0\ P$
  have $enat\ (Suc\ n) < llength\ P'$ **unfolding** *P′-def*
    by (*metis n*(*1*) *ldropn-Suc-LCons ldropn-Suc-conv-ldropn ldropn-eq-LConsD*)
  moreover have $P'\ \$\ Suc\ n \in W$ **unfolding** *P′-def* **by** (*simp add*: *n*(*2*))
  moreover have $lset\ (ltake\ (enat\ (Suc\ n))\ P') \subseteq insert\ v0\ A$
    using *lset-ltake-Suc*[*of P′ v0 n A*] **unfolding** *P′-def* **by** (*simp add*: *n*(*3*))
  ultimately show *?thesis* **unfolding** *visits-via-def P′-def* **by** *blast*
qed

lemma (**in** *vmc-path-no-deadend*) *visits-via-ltl*:
  assumes *visits-via P A W*
    and *v0*: $v0 \notin W$
  shows *visits-via* (*ltl P*) *A W*
proof−
  obtain *n* **where** n: $enat\ n < llength\ P\ P\ \$\ n \in W\ lset\ (ltake\ (enat\ n)\ P) \subseteq A$
    using *assms*(*1*)[*unfolded visits-via-def*] **by** *blast*
  have $n \neq 0$ **using** *v0 n*(*2*) *DiffE* **by** *force*
  then obtain n' **where** n': $Suc\ n' = n$ **using** *nat.exhaust* **by** *metis*
  have $\exists n.\ enat\ n < llength\ (ltl\ P) \wedge (ltl\ P)\ \$\ n \in W \wedge lset\ (ltake\ (enat\ n)\ (ltl\ P)) \subseteq A$
    apply (*rule exI*[*of - n′*])
    using *n n′ enat-Suc-ltl*[*of n′ P*] *P-lnth-Suc lset-ltake-ltl*[*of n′ P*] **by** *auto*
  thus *?thesis* **using** *visits-via-def* **by** *blast*
qed

lemma (**in** *vm-path*) *visits-via-deadend*:
  assumes *visits-via P A* (*deadends p*)
  shows *winning-path p*** *P*
  using *assms visits-via-visits visits-deadend* **by** *blast*

5.2 Attracting Strategy from a Single Node

All σ-paths starting from *v0* visit W and until then they stay in A.

definition *strategy-attracts-via* :: $Player \Rightarrow 'a\ Strategy \Rightarrow 'a \Rightarrow 'a\ set \Rightarrow 'a\ set \Rightarrow bool$ **where**

strategy-attracts-via p σ v0 A W ≡ ∀ *P*. *vmc-path G P v0 p σ* ⟶ *visits-via P A W*

lemma (**in** *vmc-path*) *strategy-attracts-viaE*:
  assumes *strategy-attracts-via p σ v0 A W*
  shows *visits-via P A W*
  using *strategy-attracts-via-def assms vmc-path-axioms* **by** *blast*

lemma (**in** *vmc-path*) *strategy-attracts-via-SucE*:
  assumes *strategy-attracts-via p σ v0 A W v0* ∉ *W*
  shows ∃ *n*. *enat* (*Suc n*) < *llength P* ∧ *P* $ *Suc n* ∈ *W* ∧ *lset* (*ltake* (*enat* (*Suc n*)) *P*) ⊆ *A*
proof−
  obtain *n* **where** *n*: *enat n* < *llength P P* $ *n* ∈ *W lset* (*ltake* (*enat n*) *P*) ⊆ *A*
    using *strategy-attracts-viaE*[*unfolded visits-via-def*] *assms*(*1*) **by** *blast*
  have *n* ≠ *0* **using** *assms*(*2*) *n*(*2*) **by** (*metis P-0*)
  thus *?thesis* **using** *n not0-implies-Suc* **by** *blast*
qed

lemma (**in** *vmc-path*) *strategy-attracts-via-lset*:
  assumes *strategy-attracts-via p σ v0 A W*
  shows *lset P* ∩ *W* ≠ {}
  using *assms*[*THEN strategy-attracts-viaE*, *unfolded visits-via-def*]
  by (*meson disjoint-iff-not-equal lset-lnth-member subset-refl*)

lemma *strategy-attracts-via-v0*:
  assumes *σ*: *strategy p σ strategy-attracts-via p σ v0 A W*
    and *v0*: *v0* ∈ *V*
  shows *v0* ∈ *A* ∪ *W*
proof−
  obtain *P* **where** *vmc-path G P v0 p σ* **using** *strategy-conforming-path-exists-single assms* **by** *blast*
  then interpret *vmc-path G P v0 p σ* **.**
  obtain *n* **where** *n*: *enat n* < *llength P P* $ *n* ∈ *W lset* (*ltake* (*enat n*) *P*) ⊆ *A*
    using *σ*(*2*)[*unfolded strategy-attracts-via-def visits-via-def*] *vmc-path-axioms* **by** *blast*
  show *?thesis* **proof** (*cases n* = *0*)
    case *True* **thus** *?thesis* **using** *n*(*2*) **by** *simp*
  next
    case *False*
    hence *lhd* (*ltake* (*enat n*) *P*) = *lhd P* **by** (*simp add*: *enat-0-iff*(*1*))
    hence *v0* ∈ *lset* (*ltake* (*enat n*) *P*)
      by (*metis* ‹*n* ≠ *0*› *P-not-null P-v0 enat-0-iff*(*1*) *llist.set-sel*(*1*) *ltake.disc*(*2*))
    thus *?thesis* **using** *n*(*3*) **by** *blast*
  qed
qed
corollary *strategy-attracts-not-outside*:
  ⟦ *v0* ∈ *V* − *A* − *W*; *strategy p σ* ⟧ ⟹ ¬*strategy-attracts-via p σ v0 A W*
  using *strategy-attracts-via-v0* **by** *blast*

lemma *strategy-attracts-viaI* [*intro*]:
  assumes ⋀*P*. *vmc-path G P v0 p σ* ⟹ *visits-via P A W*
  shows *strategy-attracts-via p σ v0 A W*
  unfolding *strategy-attracts-via-def* **using** *assms* **by** *blast*

lemma *strategy-attracts-via-no-deadends*:
  assumes $v \in V$ $v \in A - W$ *strategy-attracts-via* $p\ \sigma\ v\ A\ W$
  shows $\neg$*deadend* v
proof
  assume *deadend* v
  def [*simp*]: $P \equiv$ *LCons* v *LNil*
  interpret *vmc-path* $G\ P\ v\ p\ \sigma$ **proof**
    show *valid-path* P **using** ‹$v \in A - W$› ‹$v \in V$› *valid-path-base'* **by** *auto*
    show *maximal-path* P **using** ‹*deadend* v› **by** (*simp add*: *maximal-path.intros*(2))
    show *path-conforms-with-strategy* $p\ P\ \sigma$ **by** (*simp add*: *path-conforms-LCons-LNil*)
  qed *simp-all*
  have *visits-via* $P\ A\ W$ **using** *assms*(3) *strategy-attracts-viaE* **by** *blast*
  moreover have *llength* P = *eSuc* 0 **by** *simp*
  ultimately have P \$ $0 \in W$ **by** (*simp add*: *enat-0-iff*(1) *visits-via-def*)
  with ‹$v \in A - W$› **show** *False* **by** *auto*
qed

lemma *attractor-strategy-on-extends*:
  ⟦ *strategy-attracts-via* $p\ \sigma\ v0\ A\ W$; $A \subseteq A'$ ⟧ $\Longrightarrow$ *strategy-attracts-via* $p\ \sigma\ v0\ A'\ W$
  unfolding *strategy-attracts-via-def* **using** *visits-via-monotone* **by** *blast*

lemma *strategy-attracts-via-trivial*: $v0 \in W \Longrightarrow$ *strategy-attracts-via* $p\ \sigma\ v0\ A\ W$
proof
  fix P **assume** $v0 \in W$ *vmc-path* $G\ P\ v0\ p\ \sigma$
  then interpret *vmc-path* $G\ P\ v0\ p\ \sigma$ **by** *blast*
  show *visits-via* $P\ A\ W$ **using** *visits-via-trivial* **using** ‹$v0 \in W$› **by** *blast*
qed

lemma *strategy-attracts-via-successor*:
  assumes σ: *strategy* $p\ \sigma$ *strategy-attracts-via* $p\ \sigma\ v0\ A\ W$
    and $v0$: $v0 \in A - W$
    and $w0$: $v0 \rightarrow w0$ $v0 \in VV\ p \Longrightarrow \sigma\ v0 = w0$
  shows *strategy-attracts-via* $p\ \sigma\ w0\ A\ W$
proof
  fix P **assume** *vmc-path* $G\ P\ w0\ p\ \sigma$
  then interpret *vmc-path* $G\ P\ w0\ p\ \sigma$ **.**
  def [*simp*]: $P' \equiv$ *LCons* $v0$ P
  then interpret P': *vmc-path* $G\ P'\ v0\ p\ \sigma$
    using *extension-valid-maximal-conforming* $w0$ **by** *blast*
  interpret P': *vmc-path-no-deadend* $G\ P'\ v0\ p\ \sigma$ **using** ‹$v0 \rightarrow w0$› **by** *unfold-locales blast*
  have *visits-via* $P'\ A\ W$ **using** $\sigma(2)$ P'*.strategy-attracts-viaE* **by** *blast*
  thus *visits-via* $P\ A\ W$ **using** P'*.visits-via-ltl* $v0$ **by** *simp*
qed

lemma *strategy-attracts-VVp*:
  assumes σ: *strategy* $p\ \sigma$ *strategy-attracts-via* $p\ \sigma\ v0\ A\ W$
    and v: $v0 \in A - W$ $v0 \in VV\ p$ $\neg$*deadend* $v0$
  shows $\sigma\ v0 \in A \cup W$
proof−
  have $v0 \rightarrow \sigma\ v0$ **using** $\sigma(1)$[*unfolded strategy-def*] $v(2{,}3)$ **by** *blast*
  hence *strategy-attracts-via* $p\ \sigma\ (\sigma\ v0)\ A\ W$

using *strategy-attracts-via-successor* σ *v*(*1*) **by** *blast*
thus *?thesis* **using** *strategy-attracts-via-v0* ‹*v0*→σ *v0*› σ(*1*) **by** *blast*
qed

lemma *strategy-attracts-VVpstar*:
assumes *strategy p* σ *strategy-attracts-via p* σ *v0 A W*
and *v0* ∈ *A* − *W v0* ∉ *VV p w0* ∈ *V* − *A* − *W*
shows ¬*v0* → *w0*
by (*metis assms strategy-attracts-not-outside strategy-attracts-via-successor*)

5.3 Attracting strategy from a set of nodes

All σ-paths starting from A visit W and until then they stay in A.

definition *strategy-attracts* :: *Player* ⇒ *'a Strategy* ⇒ *'a set* ⇒ *'a set* ⇒ *bool* **where**
strategy-attracts p σ *A W* ≡ ∀ *v0* ∈ *A*. *strategy-attracts-via p* σ *v0 A W*

lemma (**in** *vmc-path*) *strategy-attractsE*:
assumes *strategy-attracts p* σ *A W v0* ∈ *A*
shows *visits-via P A W*
using *assms*(*1*)[*unfolded strategy-attracts-def*] *assms*(*2*) *strategy-attracts-viaE* **by** *blast*

lemma *strategy-attractsI* [*intro*]:
assumes ⋀*P v*. ⟦ *v* ∈ *A*; *vmc-path G P v p* σ ⟧ ⟹ *visits-via P A W*
shows *strategy-attracts p* σ *A W*
unfolding *strategy-attracts-def* **using** *assms* **by** *blast*

lemma (**in** *vmc-path*) *strategy-attracts-lset*:
assumes *strategy-attracts p* σ *A W v0* ∈ *A*
shows *lset P* ∩ *W* ≠ {}
using *assms*(*1*)[*unfolded strategy-attracts-def*] *assms*(*2*) *strategy-attracts-via-lset*(*1*)[*of A W*]
by *blast*

lemma *strategy-attracts-empty* [*simp*]: *strategy-attracts p* σ {} *W* **by** *blast*

lemma *strategy-attracts-invalid-path*:
assumes *P*: *P* = *LCons v* (*LCons w P'*) *v* ∈ *A* − *W w* ∉ *A* ∪ *W*
shows ¬*visits-via P A W* (**is** ¬*?A*)
proof
assume *?A*
then obtain *n* **where** *n*: *enat n* < *llength P P* \$ *n* ∈ *W lset* (*ltake* (*enat n*) *P*) ⊆ *A*
unfolding *visits-via-def* **by** *blast*
have *n* ≠ *0* **using** ‹*v* ∈ *A* − *W*› *n*(*2*) *P*(*1*) *DiffD2* **by** *force*
moreover have *n* ≠ *Suc 0* **using** ‹*w* ∉ *A* ∪ *W*› *n*(*2*) *P*(*1*) **by** *auto*
ultimately have *Suc* (*Suc 0*) ≤ *n* **by** *presburger*
hence *lset* (*ltake* (*enat* (*Suc* (*Suc 0*))) *P*) ⊆ *A* **using** *n*(*3*)
by (*meson contra-subsetD enat-ord-simps*(*1*) *lset-ltake-prefix lset-lnth-member lset-subset*)
moreover have *enat* (*Suc 0*) < *llength* (*ltake* (*eSuc* (*eSuc 0*)) *P*) **proof**−
have ∗: *enat* (*Suc* (*Suc 0*)) < *llength P*
using ‹*Suc* (*Suc 0*) ≤ *n*› *n*(*1*) **by** (*meson enat-ord-simps*(*2*) *le-less-linear less-le-trans neq-iff*)
have *llength* (*ltake* (*enat* (*Suc* (*Suc 0*))) *P*) = *min* (*enat* (*Suc* (*Suc 0*))) (*llength P*) **by** *simp*
hence *llength* (*ltake* (*enat* (*Suc* (*Suc 0*))) *P*) = *enat* (*Suc* (*Suc 0*))

using $*$ **by** (*simp add*: *min-absorb1*)
thus *?thesis* **by** (*simp add*: *eSuc-enat zero-enat-def*)
qed
ultimately have *ltake* (*enat* (*Suc* (*Suc 0*))) *P* \$ *Suc 0* $\in$ *A* **by** (*simp add*: *lset-lnth-member*)
hence *P* \$ *Suc 0* $\in$ *A* **by** (*simp add*: *lnth-ltake*)
thus *False* **using** *P*(*1*,*3*) **by** *auto*
qed

If A is an attractor set of W and an edge leaves A without going through W, then v belongs to $VV\ p$ and the attractor strategy σ avoids this edge. All other cases give a contradiction.

lemma *strategy-attracts-does-not-leave*:
assumes σ: *strategy-attracts p* σ *A W strategy p* σ
and *v*: $v \rightarrow w$ $v \in A - W$ $w \notin A \cup W$
shows $v \in VV\ p \wedge \sigma\ v \neq w$
proof (*rule ccontr*)
assume *contra*: $\neg(v \in VV\ p \wedge \sigma\ v \neq w)$

def $\sigma' \equiv \sigma$*-arbitrary*$(v := w)$
hence *strategy p*** σ' **using** ‹$v \rightarrow w$› **by** (*simp add*: *valid-strategy-updates*)
then obtain *P* **where** *P*: *vmc2-path G P v p* σ σ'
using ‹$v \rightarrow w$› *strategy-conforming-path-exists* σ(*2*) **by** *blast*
then interpret *vmc2-path G P v p* σ σ' **.**
interpret *vmc-path-no-deadend G P v p* σ **using** ‹$v \rightarrow w$› **by** *unfold-locales blast*
interpret *comp*: *vmc-path-no-deadend G P v p*** σ' **using** ‹$v \rightarrow w$› **by** *unfold-locales blast*
have $w = w0$ **using** *contra* σ'*-def v0-conforms comp.v0-conforms* **by** (*cases* $v \in VV\ p$) *auto*
hence $\neg$*visits-via P A W*
using *strategy-attracts-invalid-path*[*of P v w ltl* (*ltl P*)] *v*(*2*,*3*) *P-LCons'* **by** *simp*
thus *False* **by** (*meson DiffE* σ(*1*) *strategy-attractsE v*(*2*))
qed

Given an attracting strategy σ, we can turn every strategy σ' into an attracting strategy by overriding σ' on a suitable subset of the nodes. This also means that an attracting strategy is still attracting if we override it outside of $A - W$.

lemma *strategy-attracts-irrelevant-override*:
assumes *strategy-attracts p* σ *A W strategy p* σ *strategy p* σ'
shows *strategy-attracts p* (*override-on* σ' σ $(A - W)$) *A W*
proof (*rule strategy-attractsI*, *rule ccontr*)
fix *P v*
let *?*σ = *override-on* σ' σ $(A - W)$
assume *vmc-path G P v p ?*σ
then interpret *vmc-path G P v p ?*σ **.**
assume $v \in A$
hence *P* \$ *0* $\in$ *A* **using** ‹$v \in A$› **by** *simp*
moreover assume *contra*: $\neg$*visits-via P A W*
ultimately have *P* \$ *0* $\in A - W$ **unfolding** *visits-via-def* **by** (*meson DiffI P-len not-less0 lset-ltake*)
have $\neg$*lset P* $\subseteq A - W$ **proof**
assume *lset P* $\subseteq A - W$
hence $\bigwedge v.\ v \in$ *lset P* $\Longrightarrow$ *override-on* σ' σ $(A - W)$ $v = \sigma\ v$ **by** *auto*
hence *path-conforms-with-strategy p P* σ
using *path-conforms-with-strategy-irrelevant-updates*[*OF P-conforms*] **by** *blast*

hence *vmc-path G P* (*P* $ *0*) *p* σ
using *conforms-to-another-strategy P-0* **by** *blast*
thus *False*
using *contra* ‹*P* $ *0* $\in$ *A*› *assms*(*1*)
by (*meson vmc-path.strategy-attractsE*)
qed
hence $\exists$ *n*. *enat n* < *llength P* $\wedge$ *P* $ *n* $\notin$ *A* − *W* **by** (*meson lset-subset*)
then obtain *n* **where** *n*: *enat n* < *llength P* $\wedge$ *P* $ *n* $\notin$ *A* − *W*
$\bigwedge$*i*. *i* < *n* $\Longrightarrow$ $\neg$(*enat i* < *llength P* $\wedge$ *P* $ *i* $\notin$ *A* − *W*)
using *ex-least-nat-le*[*of* λ*n*. *enat n* < *llength P* $\wedge$ *P* $ *n* $\notin$ *A* − *W*] **by** *blast*
hence *n-min*: $\bigwedge$*i*. *i* < *n* $\Longrightarrow$ *P* $ *i* $\in$ *A* − *W*
using *dual-order.strict-trans enat-ord-simps*(*2*) **by** *blast*
have *n* $\neq$ *0* **using** ‹*P* $ *0* $\in$ *A* − *W*› *n*(*1*) **by** *meson*
then obtain *n'* **where** *n'*: *Suc n'* = *n* **using** *not0-implies-Suc* **by** *blast*
hence *P* $ *n'* $\in$ *A* − *W* **using** *n-min* **by** *blast*
moreover have *P* $ *n'* $\rightarrow$ *P* $ *Suc n'* **using** *P-valid n*(*1*) *n' valid-path-edges* **by** *blast*
moreover have *P* $ *Suc n'* $\notin$ *A* $\cup$ *W* **proof**−
have *P* $ *n* $\notin$ *W* **using** *contra n*(*1*) *n-min* **unfolding** *visits-via-def*
by (*meson Diff-subset lset-ltake subsetCE*)
thus *?thesis* **using** *n*(*1*) *n'* **by** *blast*
qed
ultimately have *P* $ *n'* $\in$ *VV p* $\wedge$ σ (*P* $ *n'*) $\neq$ *P* $ *Suc n'*
using *strategy-attracts-does-not-leave*[*of p* σ *A W P* $ *n' P* $ *Suc n'*]
assms(*1*,*2*) **by** *blast*
thus *False*
using *n*(*1*) *n' vmc-path-conforms* ‹*P* $ *n'* $\in$ *A* − *W*› **by** (*metis override-on-apply-in*)
qed

lemma *strategy-attracts-trivial* [*simp*]: *strategy-attracts p* σ *W W*
by (*simp add*: *strategy-attracts-def strategy-attracts-via-trivial*)

If a σ-conforming path P hits an attractor A, it will visit W.

lemma (**in** *vmc-path*) *attracted-path*:
assumes *W* $\subseteq$ *V*
and σ: *strategy-attracts p* σ *A W*
and *P-hits-A*: *lset P* $\cap$ *A* $\neq$ {}
shows *lset P* $\cap$ *W* $\neq$ {}
proof−
obtain *n* **where** *n*: *enat n* < *llength P P* $ *n* $\in$ *A* **using** *P-hits-A* **by** (*meson lset-intersect-lnth*)
def *P'* $\equiv$ *ldropn n P*
interpret *vmc-path G P' P* $ *n p* σ **unfolding** *P'-def* **using** *vmc-path-ldropn n*(*1*) **by** *blast*
have *visits-via P' A W* **using** σ *n*(*2*) *strategy-attractsE* **by** *blast*
thus *?thesis* **unfolding** *P'-def* **using** *visits-via-visits in-lset-ldropnD*[*of - n P*] **by** *blast*
qed

lemma *attracted-strategy-step*:
assumes σ: *strategy p* σ *strategy-attracts p* σ *A W*
and *v0*: $\neg$*deadend v0 v0* $\in$ *A* − *W v0* $\in$ *VV p*
shows σ *v0* $\in$ *A* $\cup$ *W*
by (*metis DiffD1 strategy-attracts-VVp assms strategy-attracts-def*)

lemma (**in** *vmc-path-no-deadend*) *attracted-path-step*:

assumes σ: *strategy-attracts p* σ *A W*
and *v0*: *v0* $\in$ *A* $-$ *W*
shows *w0* $\in$ *A* $\cup$ *W*
by (*metis* (*no-types*) *DiffD1 P-LCons'* σ *strategy-attractsE strategy-attracts-invalid-path v0*)

end — context ParityGame

end

6 Attractor Sets

theory *Attractor*
imports
Main
AttractingStrategy
begin

Here we define the p-attractor of a set of nodes.

context *ParityGame* **begin**

We define the conditions for a node to be directly attracted from a given set.

definition *directly-attracted* :: *Player* $\Rightarrow$ *'a set* $\Rightarrow$ *'a set* **where**
directly-attracted p S $\equiv$ $\{v \in V - S.\ \neg deadend\ v\ \wedge$
$(v \in VV\ p \longrightarrow (\exists w.\ v \to w \wedge w \in S))$
$\wedge\ (v \in VV\ p{**} \longrightarrow (\forall w.\ v \to w \longrightarrow w \in S))\}$

abbreviation *attractor-step p W S* $\equiv$ *W* $\cup$ *S* $\cup$ *directly-attracted p S*

The p-attractor set of W, defined as a least fixed point.

definition *attractor* :: *Player* $\Rightarrow$ *'a set* $\Rightarrow$ *'a set* **where**
attractor p W = *lfp* (*attractor-step p W*)

6.1 *directly-attracted*

Show a few basic properties of *directly-attracted*.

lemma *directly-attracted-disjoint* [*simp*]: *directly-attracted p W* $\cap$ *W* = $\{\}$
and *directly-attracted-empty* [*simp*]: *directly-attracted p* $\{\}$ = $\{\}$
and *directly-attracted-V-empty* [*simp*]: *directly-attracted p V* = $\{\}$
and *directly-attracted-bounded-by-V* [*simp*]: *directly-attracted p W* $\subseteq$ *V*
and *directly-attracted-contains-no-deadends* [*elim*]: *v* $\in$ *directly-attracted p W* $\Longrightarrow$ $\neg$*deadend v*
unfolding *directly-attracted-def* **by** *blast+*

6.2 *attractor-step*

lemma *attractor-step-empty*: *attractor-step p* $\{\}$ $\{\}$ = $\{\}$
and *attractor-step-bounded-by-V*: $[\![$ *W* $\subseteq$ *V*; *S* $\subseteq$ *V* $]\!]$ $\Longrightarrow$ *attractor-step p W S* $\subseteq$ *V*
by *simp-all*

The definition of *attractor* uses *lfp*. For this to be well-defined, we need show that *attractor-step* is monotone.

lemma *attractor-step-mono*: *mono* (*attractor-step p W*)
  unfolding *directly-attracted-def* **by** (*rule monoI*) *auto*

6.3 Basic Properties of an Attractor

lemma *attractor-unfolding*: $attractor\ p\ W = attractor\text{-}step\ p\ W\ (attractor\ p\ W)$
  unfolding *attractor-def* **using** *attractor-step-mono lfp-unfold* **by** *blast*
lemma *attractor-lowerbound*: $attractor\text{-}step\ p\ W\ S \subseteq S \Longrightarrow attractor\ p\ W \subseteq S$
  unfolding *attractor-def* **using** *attractor-step-mono* **by** (*simp add*: *lfp-lowerbound*)
lemma *attractor-set-non-empty*: $W \neq \{\} \Longrightarrow attractor\ p\ W \neq \{\}$
  and *attractor-set-base*: $W \subseteq attractor\ p\ W$
  using *attractor-unfolding* **by** *auto*
lemma *attractor-in-V*: $W \subseteq V \Longrightarrow attractor\ p\ W \subseteq V$
  using *attractor-lowerbound attractor-step-bounded-by-V* **by** *auto*

6.4 Attractor Set Extensions

lemma *attractor-set-VVp*:
  assumes $v \in VV\ p\ v{\rightarrow}w\ w \in attractor\ p\ W$
  shows $v \in attractor\ p\ W$
  apply (*subst attractor-unfolding*) **unfolding** *directly-attracted-def* **using** *assms* **by** *auto*

lemma *attractor-set-VVpstar*:
  assumes $\neg deadend\ v\ \bigwedge w.\ v{\rightarrow}w \Longrightarrow w \in attractor\ p\ W$
  shows $v \in attractor\ p\ W$
  apply (*subst attractor-unfolding*) **unfolding** *directly-attracted-def* **using** *assms* **by** *auto*

6.5 Removing an Attractor

lemma *removing-attractor-induces-no-deadends*:
  assumes $v \in S - attractor\ p\ W\ v{\rightarrow}w\ w \in S\ \bigwedge w.\ [\![\ v \in VV\ p{*}{*};\ v{\rightarrow}w\ ]\!] \Longrightarrow w \in S$
  shows $\exists\, w \in S - attractor\ p\ W.\ v{\rightarrow}w$
proof−
  have $v \in V$ **using** ‹$v{\rightarrow}w$› **by** *blast*
  thus *?thesis* **proof** (*cases rule*: *VV-cases*)
    assume $v \in VV\ p$
    thus *?thesis* **using** *attractor-set-VVp assms* **by** *blast*
  next
    assume $v \in VV\ p{*}{*}$
    thus *?thesis* **using** *attractor-set-VVpstar assms* **by** (*metis Diff-iff edges-are-in-V*(*2*))
  qed
qed

Removing the attractor sets of deadends leaves a subgame without deadends.

lemma *subgame-without-deadends*:
  assumes *V′-def*: $V' = V - attractor\ p\ (deadends\ p{*}{*}) - attractor\ p{*}{*}\ (deadends\ p{*}{*}{*}{*})$
    (**is** $V' = V - ?A - ?B$)
    and *v*: $v \in V_{subgame\ V'}$
  shows $\neg Digraph.deadend\ (subgame\ V')\ v$
proof (*cases*)
  assume *deadend v*
  have *v*: $v \in V - ?A - ?B$ **using** *v* **unfolding** *V′-def subgame-def* **by** *simp*

{ **fix** p' **assume** $v \in VV\ p'{**}$
  hence $v \in$ *attractor* p' (*deadends* $p'{**}$)
    using ‹*deadend v*› *attractor-set-base*[*of deadends* $p'{**}$ p']
    unfolding *deadends-def* **by** *blast*
  hence *False* **using** v **by** (*cases* p'; *cases* p) *auto*
}
thus *?thesis* **using** v **by** *blast*
next
  assume $\neg$*deadend* v
  have v: $v \in V - ?A - ?B$ **using** v **unfolding** V'-*def subgame-def* **by** *simp*
  def $G' \equiv$ *subgame* V'
  interpret G': *ParityGame* G' **unfolding** G'-*def* **using** *subgame-ParityGame* .
  show *?thesis* **proof**
    assume *Digraph.deadend* (*subgame* V') v
    hence G'.*deadend* v **unfolding** G'-*def* .
    have *all-in-attractor*: $\bigwedge w.\ v \rightarrow w \Longrightarrow w \in ?A \lor w \in ?B$ **proof** (*rule ccontr*)
      fix w
      assume $v \rightarrow w$ $\neg(w \in ?A \lor w \in ?B)$
      hence $w \in V'$ **unfolding** V'-*def* **by** *blast*
      hence $w \in V_{G'}$ **unfolding** G'-*def subgame-def* **using** ‹$v \rightarrow w$› **by** *auto*
      hence $v \rightarrow_{G'} w$ **using** ‹$v \rightarrow w$› *assms*(2) **unfolding** G'-*def subgame-def* **by** *auto*
      thus *False* **using** ‹G'.*deadend* v› **using** ‹$w \in V_{G'}$› **by** *blast*
    qed
    { **fix** p' **assume** $v \in VV\ p'$
      { **assume** $\exists w.\ v \rightarrow w \land w \in$ *attractor* p' (*deadends* $p'{**}$)
        hence $v \in$ *attractor* p' (*deadends* $p'{**}$) **using** ‹$v \in VV\ p'$› *attractor-set-VVp* **by** *blast*
        hence *False* **using** v **by** (*cases* p'; *cases* p) *auto*
      }
      hence $\bigwedge w.\ v \rightarrow w \Longrightarrow w \in$ *attractor* $p'{**}$ (*deadends* $p'{****}$)
        using *all-in-attractor* **by** (*cases* p'; *cases* p) *auto*
      hence $v \in$ *attractor* $p'{**}$ (*deadends* $p'{****}$)
        using ‹$\neg$*deadend* v› ‹$v \in VV\ p'$› *attractor-set-VVpstar* **by** *auto*
      hence *False* **using** v **by** (*cases* p'; *cases* p) *auto*
    }
    thus *False* **using** v **by** *blast*
  qed
qed

6.6 Attractor Set Induction

lemma *mono-restriction-is-mono*: *mono* $f \Longrightarrow$ *mono* $(\lambda S.\ f\ (S \cap V))$
  unfolding *mono-def* **by** (*meson inf-mono monoD subset-refl*)

Here we prove a powerful induction schema for *attractor*. Being able to prove this is the only reason why we do not use `inductive_set` to define the attractor set.

See also https://lists.cam.ac.uk/pipermail/cl-isabelle-users/2015-October/msg00123.html

lemma *attractor-set-induction* [*consumes 1*, *case-names step union*]:
  assumes $W \subseteq V$
    and *step*: $\bigwedge S.\ S \subseteq V \Longrightarrow P\ S \Longrightarrow P$ (*attractor-step* $p\ W\ S$)
    and *union*: $\bigwedge M.\ \forall S \in M.\ S \subseteq V \land P\ S \Longrightarrow P\ (\bigcup M)$
  shows P (*attractor* $p\ W$)

```
proof-
  let ?P = λS. P (S ∩ V)
  let ?f = λS. attractor-step p W (S ∩ V)
  let ?A = lfp ?f
  let ?B = lfp (attractor-step p W)
  have f-mono: mono ?f
    using mono-restriction-is-mono[of attractor-step p W] attractor-step-mono by simp
  have P-A: ?P ?A proof (rule lfp-ordinal-induct-set)
    show ⋀S. ?P S ⟹ ?P (W ∪ (S ∩ V) ∪ directly-attracted p (S ∩ V))
      by (metis assms(1) attractor-step-bounded-by-V inf.absorb1 inf-le2 local.step)
    show ⋀M. ∀S ∈ M. ?P S ⟹ ?P (⋃M) proof-
      fix M
      let ?M = {S ∩ V | S. S ∈ M}
      assume ∀S∈M. ?P S
      hence ∀S ∈ ?M. S ⊆ V ∧ P S by auto
      hence *: P (⋃?M) by (simp add: union)
      have ⋃?M = (⋃M) ∩ V by blast
      thus ?P (⋃M) using * by auto
    qed
  qed (insert f-mono)

  have *: W ∪ (V ∩ V) ∪ directly-attracted p (V ∩ V) ⊆ V
    using ‹W ⊆ V› attractor-step-bounded-by-V by auto
  have ?A ⊆ V ?B ⊆ V using * by (simp-all add: lfp-lowerbound)

  have ?A = ?f ?A using f-mono lfp-unfold by blast
  hence ?A = W ∪ (?A ∩ V) ∪ directly-attracted p (?A ∩ V) using ‹?A ⊆ V› by simp
  hence *: attractor-step p W ?A ⊆ ?A using ‹?A ⊆ V› inf.absorb1 by fastforce

  have ?B = attractor-step p W ?B using attractor-step-mono lfp-unfold by blast
  hence ?f ?B ⊆ ?B using ‹?B ⊆ V› by (metis (no-types, lifting) equalityD2 le-iff-inf)

  have ?A = ?B proof
    show ?A ⊆ ?B using ‹?f ?B ⊆ ?B› by (simp add: lfp-lowerbound)
    show ?B ⊆ ?A using * by (simp add: lfp-lowerbound)
  qed
  hence ?P ?B using P-A by (simp add: attractor-def)
  thus ?thesis using ‹?B ⊆ V› by (simp add: attractor-def le-iff-inf)
qed

end — context ParityGame

end
```

7 Winning Strategies

```
theory WinningStrategy
imports
  Main
  Strategy
begin
```

context *ParityGame* **begin**

Here we define winning strategies.

A strategy is winning for player p from $v0$ if every maximal σ-path starting in $v0$ is winning.

definition *winning-strategy* :: *Player* $\Rightarrow$ *'a Strategy* $\Rightarrow$ *'a* $\Rightarrow$ *bool* **where**
  winning-strategy p σ *v0* $\equiv \forall P$. *vmc-path G P v0 p* $\sigma \longrightarrow$ *winning-path p P*

lemma *winning-strategyI* [*intro*]:
  assumes $\bigwedge P$. *vmc-path G P v0 p* $\sigma \Longrightarrow$ *winning-path p P*
  shows *winning-strategy p* σ *v0*
  unfolding *winning-strategy-def* **using** *assms* **by** *blast*

lemma (**in** *vmc-path*) *paths-hits-winning-strategy-is-winning*:
  assumes σ: *winning-strategy p* σ *v*
    and *v*: $v \in$ *lset P*
  shows *winning-path p P*
proof–
  obtain *n* **where** *n*: *enat n* < *llength P P* \$ *n* = *v* **using** *v* **by** (*meson in-lset-conv-lnth*)
  interpret *P'*: *vmc-path G ldropn n P v p* σ **using** *n vmc-path-ldropn* **by** *blast*
  have *winning-path p* (*ldropn n P*) **using** σ **by** (*simp add*: *winning-strategy-def P'.vmc-path-axioms*)
  thus *?thesis* **using** *winning-path-drop-add P-valid n*(*1*) **by** *blast*
qed

There cannot exist winning strategies for both players for the same node.

lemma *winning-strategy-only-for-one-player*:
  assumes σ: *strategy p* σ *winning-strategy p* σ *v*
    and σ': *strategy p*** σ' *winning-strategy p*** σ' *v*
    and *v*: $v \in V$
  shows *False*
proof–
  obtain *P* **where** *vmc2-path G P v p* σ σ' **using** *assms strategy-conforming-path-exists* **by** *blast*
  then interpret *vmc2-path G P v p* σ σ' **.**
  have *winning-path p P*
    using *paths-hits-winning-strategy-is-winning* σ(*2*) *v0-lset-P* **by** *blast*
  moreover have *winning-path p*** *P*
    using *comp.paths-hits-winning-strategy-is-winning* σ'(*2*) *v0-lset-P* **by** *blast*
  ultimately show *False* **using** *P-valid paths-are-winning-for-one-player* **by** *blast*
qed

7.1 Deadends

lemma *no-winning-strategy-on-deadends*:
  assumes $v \in$ *VV p deadend v strategy p* σ
  shows $\neg$*winning-strategy p* σ *v*
proof–
  obtain *P* **where** *vmc-path G P v p* σ **using** *strategy-conforming-path-exists-single assms* **by** *blast*
  then interpret *vmc-path G P v p* σ **.**
  have *P* = *LCons v LNil* **using** *P-deadend-v0-LCons* ‹*deadend v*› **by** *blast*
  hence $\neg$*winning-path p P* **unfolding** *winning-path-def* **using** ‹$v \in$ *VV p*› **by** *auto*
  thus *?thesis* **using** *winning-strategy-def vmc-path-axioms* **by** *blast*

qed

lemma *winning-strategy-on-deadends*:
  assumes $v \in VV\ p$ *deadend v strategy p* σ
  shows *winning-strategy p*∗∗ σ *v*
proof
  fix *P* **assume** *vmc-path G P v p*∗∗ σ
  then interpret *vmc-path G P v p*∗∗ σ **.**
  have $P = LCons\ v\ LNil$ **using** *P-deadend-v0-LCons* ‹*deadend v*› **by** *blast*
  thus *winning-path p*∗∗ *P* **unfolding** *winning-path-def*
    using ‹$v \in VV\ p$› *P-valid paths-are-winning-for-one-player* **by** *auto*
qed

7.2 Extension Theorems

lemma *strategy-extends-VVp*:
  assumes *v0*: $v0 \in VV\ p$ $\neg$*deadend v0*
  and σ: *strategy p* σ *winning-strategy p* σ *v0*
  shows *winning-strategy p* σ (σ *v0*)
proof
  fix *P* **assume** *vmc-path G P* (σ *v0*) *p* σ
  then interpret *vmc-path G P* σ *v0 p* σ **.**
  have $v0 \rightarrow \sigma\ v0$ **using** *v0* $\sigma(1)$ *strategy-def* **by** *blast*
  hence *winning-path p* (*LCons v0 P*)
    using $\sigma(2)$ *extension-valid-maximal-conforming winning-strategy-def* **by** *blast*
  thus *winning-path p P* **using** *winning-path-ltl*[*of p LCons v0 P*] **by** *simp*
qed

lemma *strategy-extends-VVpstar*:
  assumes *v0*: $v0 \in VV\ p$∗∗ $v0 \rightarrow w0$
  and σ: *winning-strategy p* σ *v0*
  shows *winning-strategy p* σ *w0*
proof
  fix *P* **assume** *vmc-path G P w0 p* σ
  then interpret *vmc-path G P w0 p* σ **.**
  have *winning-path p* (*LCons v0 P*)
    using *extension-valid-maximal-conforming VV-impl1* σ *v0 winning-strategy-def*
    by *auto*
  thus *winning-path p P* **using** *winning-path-ltl*[*of p LCons v0 P*] **by** *auto*
qed

lemma *strategy-extends-backwards-VVpstar*:
  assumes *v0*: $v0 \in VV\ p$∗∗
    and σ: *strategy p* σ $\bigwedge w.\ v0 \rightarrow w \Longrightarrow$ *winning-strategy p* σ *w*
  shows *winning-strategy p* σ *v0*
proof
  fix *P* **assume** *vmc-path G P v0 p* σ
  then interpret *vmc-path G P v0 p* σ **.**
  show *winning-path p P* **proof** (*cases*)
    assume *deadend v0*
    thus *?thesis* **using** *P-deadend-v0-LCons winning-path-def v0* **by** *auto*
  next

assume $\neg$*deadend v0*
then interpret *vmc-path-no-deadend G P v0 p* σ **by** *unfold-locales*
interpret *ltlP*: *vmc-path G ltl P w0 p* σ **using** *vmc-path-ltl* **.**
have *winning-path p* (*ltl P*)
using $\sigma(2)$ *v0-edge-w0 vmc-path-ltl winning-strategy-def* **by** *blast*
thus *winning-path p P*
using *winning-path-LCons* **by** (*metis P-LCons' ltlP.P-LCons ltlP.P-not-null*)
qed
qed

lemma *strategy-extends-backwards-VVp*:
assumes *v0*: *v0* $\in$ *VV p* σ *v0* = *w v0*$\to$*w*
and σ: *strategy p* σ *winning-strategy p* σ *w*
shows *winning-strategy p* σ *v0*
proof
fix *P* **assume** *vmc-path G P v0 p* σ
then interpret *vmc-path G P v0 p* σ **.**
have $\neg$*deadend v0* **using** ‹*v0*$\to$*w*› **by** *blast*
then interpret *vmc-path-no-deadend G P v0 p* σ **by** *unfold-locales*
have *winning-path p* (*ltl P*)
using $\sigma(2)$[*unfolded winning-strategy-def*] *v0*(*1*,*2*) *v0-conforms vmc-path-ltl* **by** *presburger*
thus *winning-path p P* **using** *winning-path-LCons* **by** (*metis P-LCons Ptl-not-null*)
qed

end — context ParityGame

end

8 Well-Ordered Strategy

theory *WellOrderedStrategy*
imports
Main
Strategy
begin

Constructing a uniform strategy from a set of strategies on a set of nodes often works by well-ordering the strategies and then choosing the minimal strategy on each node. Then every path eventually follows one strategy because we choose the strategies along the path to be non-increasing in the well-ordering.

The following locale formalizes this idea.

We will use this to construct uniform attractor and winning strategies.

locale *WellOrderedStrategies* = *ParityGame* +
fixes *S* :: *'a set*
and *p* :: *Player*
— The set of good strategies on a node *v*
and *good* :: *'a* $\Rightarrow$ *'a Strategy set*
and *r* :: (*'a Strategy* $\times$ *'a Strategy*) *set*
assumes *S-V*: *S* $\subseteq$ *V*
— *r* is a wellorder on the set of all strategies which are good somewhere.

and *r-wo*: *well-order-on* $\{\sigma.\ \exists\, v \in S.\ \sigma \in good\ v\}$ *r*
— Every node has a good strategy.
and *good-ex*: $\bigwedge v.\ v \in S \Longrightarrow \exists\, \sigma.\ \sigma \in good\ v$
— good strategies are well-formed strategies.
and *good-strategies*: $\bigwedge v\ \sigma.\ \sigma \in good\ v \Longrightarrow strategy\ p\ \sigma$
— A good strategy on v is also good on possible successors of v.
and *strategies-continue*: $\bigwedge v\ w\ \sigma.\ [\![\ v \in S;\ v \rightarrow w;\ v \in VV\ p \Longrightarrow \sigma\ v = w;\ \sigma \in good\ v\ ]\!] \Longrightarrow \sigma \in good\ w$
begin

The set of all strategies which are good somewhere.

abbreviation *Strategies* $\equiv \{\sigma.\ \exists\, v \in S.\ \sigma \in good\ v\}$

definition *minimal-good-strategy* **where**
minimal-good-strategy $v\ \sigma \equiv \sigma \in good\ v \wedge (\forall\, \sigma'.\ (\sigma', \sigma) \in r - Id \longrightarrow \sigma' \notin good\ v)$

no-notation *binomial* (**infixl** *choose 65*)

Among the good strategies on v, choose the minimum.

definition *choose* **where**
choose $v \equiv$ *THE* σ. *minimal-good-strategy* $v\ \sigma$

Define a strategy which uses the minimum strategy on all nodes of S. Of course, we need to prove that this is a well-formed strategy.

definition *well-ordered-strategy* **where**
well-ordered-strategy $\equiv$ *override-on* σ*-arbitrary* $(\lambda v.\ choose\ v\ v)$ S

Show some simple properties of the binary relation r on the set *Strategies*.

lemma *r-refl* [*simp*]: *refl-on Strategies r*
using *r-wo* **unfolding** *well-order-on-def linear-order-on-def partial-order-on-def preorder-on-def* **by** *blast*
lemma *r-total* [*simp*]: *total-on Strategies r*
using *r-wo* **unfolding** *well-order-on-def linear-order-on-def* **by** *blast*
lemma *r-trans* [*simp*]: *trans r*
using *r-wo* **unfolding** *well-order-on-def linear-order-on-def partial-order-on-def preorder-on-def* **by** *blast*
lemma *r-wf* [*simp*]: *wf* $(r - Id)$
using *well-order-on-def r-wo* **by** *blast*

choose always chooses a minimal good strategy on S.

lemma *choose-works*:
assumes $v \in S$
shows *minimal-good-strategy* v (*choose* v)
proof−
have *wf*: *wf* $(r - Id)$ **using** *well-order-on-def r-wo* **by** *blast*
obtain σ **where** $\sigma 1$: *minimal-good-strategy* $v\ \sigma$
unfolding *minimal-good-strategy-def* **by** (*meson good-ex*[*OF* ‹$v \in S$›] *wf wf-eq-minimal*)
hence σ: $\sigma \in good\ v\ \bigwedge\sigma'.\ (\sigma', \sigma) \in r - Id \Longrightarrow \sigma' \notin good\ v$
unfolding *minimal-good-strategy-def* **by** *auto*
{ **fix** σ' **assume** *minimal-good-strategy* $v\ \sigma'$

hence σ': $\sigma' \in$ *good* v $\bigwedge\sigma$. $(\sigma, \sigma') \in r - Id \Longrightarrow \sigma \notin$ *good* v
unfolding *minimal-good-strategy-def* **by** *auto*
have $(\sigma, \sigma') \notin r - Id$ **using** $\sigma(1)$ $\sigma'(2)$ **by** *blast*
moreover have $(\sigma', \sigma) \notin r - Id$ **using** $\sigma(2)$ $\sigma'(1)$ **by** *auto*
moreover have $\sigma \in$ *Strategies* **using** $\sigma(1)$ ‹$v \in S$› **by** *auto*
moreover have $\sigma' \in$ *Strategies* **using** $\sigma'(1)$ ‹$v \in S$› **by** *auto*
ultimately have $\sigma' = \sigma$
using *r-wo Linear-order-in-diff-Id well-order-on-Field well-order-on-def* **by** *fastforce*
}
with $\sigma 1$ **have** $\exists!\sigma$. *minimal-good-strategy* v σ **by** *blast*
thus *?thesis* **using** *theI'*[*of minimal-good-strategy* v, *folded choose-def*] **by** *blast*
qed

corollary
assumes $v \in S$
shows *choose-good*: *choose* $v \in$ *good* v
and *choose-minimal*: $\bigwedge\sigma'$. $(\sigma'$, *choose* $v) \in r - Id \Longrightarrow \sigma' \notin$ *good* v
and *choose-strategy*: *strategy* p (*choose* v)
using *choose-works*[*OF assms*, *unfolded minimal-good-strategy-def*] *good-strategies* **by** *blast+*

corollary *choose-in-Strategies*: $v \in S \Longrightarrow$ *choose* $v \in$ *Strategies* **using** *choose-good* **by** *blast*

lemma *well-ordered-strategy-valid*: *strategy p well-ordered-strategy*
proof−
{
fix v **assume** $v \in S$ $v \in VV$ p $\neg$*deadend* v
moreover have *strategy* p (*choose* v)
using *choose-works*[*OF* ‹$v \in S$›, *unfolded minimal-good-strategy-def*, *THEN conjunct1*] *good-strategies*
by *blast*
ultimately have $v \rightarrow (\lambda v.$ *choose* v $v)$ v **using** *strategy-def* **by** *blast*
}
thus *?thesis* **unfolding** *well-ordered-strategy-def* **using** *valid-strategy-updates-set* **by** *force*
qed

8.1 Strategies on a Path

Maps a path to its strategies.

definition *path-strategies* $\equiv$ *lmap choose*

lemma *path-strategies-in-Strategies*:
assumes *lset* $P \subseteq S$
shows *lset* (*path-strategies* P) $\subseteq$ *Strategies*
using *path-strategies-def assms choose-in-Strategies* **by** *auto*

lemma *path-strategies-good*:
assumes *lset* $P \subseteq S$ *enat* $n <$ *llength* P
shows *path-strategies* P \$ $n \in$ *good* (P \$ n)
by (*simp add*: *path-strategies-def assms choose-good lset-lnth-member*)

lemma *path-strategies-strategy*:
assumes *lset* $P \subseteq S$ *enat* $n <$ *llength* P

shows *strategy p* (*path-strategies P* \$ *n*)
using *path-strategies-good assms good-strategies* **by** *blast*

lemma *path-strategies-monotone-Suc*:
assumes *P*: *lset P* $\subseteq$ *S valid-path P path-conforms-with-strategy p P well-ordered-strategy*
enat (*Suc n*) < *llength P*
shows (*path-strategies P* \$ *Suc n*, *path-strategies P* \$ *n*) $\in$ *r*
proof−
def *P′* $\equiv$ *ldropn n P*
hence *enat* (*Suc 0*) < *llength P′* **using** *P*(*4*)
by (*metis enat-ltl-Suc ldrop-eSuc-ltl ldropn-Suc-conv-ldropn llist.disc*(*2*) *lnull-0-llength ltl-ldropn*)
then obtain *v w Ps* **where** *vw*: *P′* = *LCons v* (*LCons w Ps*)
by (*metis ldropn-0 ldropn-Suc-conv-ldropn ldropn-lnull lnull-0-llength*)
moreover have *lset P′* $\subseteq$ *S* **unfolding** *P′-def* **using** *P*(*1*) *lset-ldropn-subset*[*of n P*] **by** *blast*
ultimately have *v* $\in$ *S w* $\in$ *S* **by** *auto*
moreover have *v*→*w* **using** *valid-path-edges′*[*of v w Ps*, *folded vw*] *valid-path-drop*[*OF P*(*2*)] *P′-def* **by** *blast*
moreover have *choose v* $\in$ *good v* **using** *choose-good* ‹*v* $\in$ *S*› **by** *blast*
moreover have *v* $\in$ *VV p* $\Longrightarrow$ *choose v v* = *w* **proof**−
assume *v* $\in$ *VV p*
moreover have *path-conforms-with-strategy p P′ well-ordered-strategy*
unfolding *P′-def* **using** *path-conforms-with-strategy-drop P*(*3*) **by** *blast*
ultimately have *well-ordered-strategy v* = *w* **using** *vw path-conforms-with-strategy-start* **by** *blast*
thus *choose v v* = *w* **unfolding** *well-ordered-strategy-def* **using** ‹*v* $\in$ *S*› **by** *auto*
qed
ultimately have *choose v* $\in$ *good w* **using** *strategies-continue* **by** *blast*
hence $*$: (*choose v*, *choose w*) $\notin$ *r* − *Id* **using** *choose-minimal* ‹*w* $\in$ *S*› **by** *blast*

have (*choose w*, *choose v*) $\in$ *r* **proof** (*cases*)
assume *choose v* = *choose w*
thus *?thesis* **using** *r-refl refl-onD choose-in-Strategies*[*OF* ‹*v* $\in$ *S*›] **by** *fastforce*
next
assume *choose v* $\neq$ *choose w*
thus *?thesis* **using** $*$ *r-total choose-in-Strategies*[*OF* ‹*v* $\in$ *S*›] *choose-in-Strategies*[*OF* ‹*w* $\in$ *S*›]
by (*metis* (*lifting*) *Linear-order-in-diff-Id r-wo well-order-on-Field well-order-on-def*)
qed
hence (*path-strategies P′* \$ *Suc 0*, *path-strategies P′* \$ *0*) $\in$ *r*
unfolding *path-strategies-def* **using** *vw* **by** *simp*
thus *?thesis* **unfolding** *path-strategies-def P′-def*
using *lnth-lmap-ldropn*[*OF Suc-llength*[*OF P*(*4*)], *of choose*]
lnth-lmap-ldropn-Suc[*OF P*(*4*), *of choose*]
by *simp*
qed

lemma *path-strategies-monotone*:
assumes *P*: *lset P* $\subseteq$ *S valid-path P path-conforms-with-strategy p P well-ordered-strategy*
n < *m enat m* < *llength P*
shows (*path-strategies P* \$ *m*, *path-strategies P* \$ *n*) $\in$ *r*
using *assms* **proof** (*induct m* − *n arbitrary*: *n m*)
case (*Suc d*)

show *?case* **proof** (*cases*)
assume $d = 0$
thus *?thesis* **using** *path-strategies-monotone-Suc*[*OF P*(*1*,*2*,*3*)]
by (*metis* (*no-types*) *Suc.hyps*(*2*) *Suc.prems*(*4*,*5*) *Suc-diff-Suc Suc-inject Suc-leI diff-is-0-eq diffs0-imp-equal*)
next
assume $d \neq 0$
have $m \neq 0$ **using** *Suc.hyps*(*2*) **by** *linarith*
then obtain m' **where** m': *Suc* $m' = m$ **using** *not0-implies-Suc* **by** *blast*
hence $d = m' - n$ **using** *Suc.hyps*(*2*) **by** *presburger*
moreover hence $n < m'$ **using** ‹$d \neq 0$› **by** *presburger*
ultimately have (*path-strategies P* \$ m', *path-strategies P* \$ n) $\in r$
using *Suc.hyps*(*1*)[*of* m' n, *OF - P*(*1*,*2*,*3*)] *Suc.prems*(*5*) *dual-order.strict-trans enat-ord-simps*(*2*) m'
by *blast*
thus *?thesis*
using m' *path-strategies-monotone-Suc*[*OF P*(*1*,*2*,*3*)] **by** (*metis* (*no-types*) *Suc.prems*(*5*) *r-trans trans-def*)
qed
qed *simp*

lemma *path-strategies-eventually-constant*:
assumes $\neg$*lfinite P lset P* $\subseteq$ *S valid-path P path-conforms-with-strategy p P well-ordered-strategy*
shows $\exists n. \forall m \geq n.$ *path-strategies P* \$ n = *path-strategies P* \$ m
proof−
def σ*-set* $\equiv$ *lset* (*path-strategies P*)
have $\exists \sigma. \sigma \in \sigma$*-set* **unfolding** σ*-set-def path-strategies-def*
using *assms*(*1*) *lfinite-lmap lset-nth-member-inf* **by** *blast*
then obtain σ' **where** σ': $\sigma' \in \sigma$*-set* $\bigwedge\tau.\ (\tau, \sigma') \in r - Id \Longrightarrow \tau \notin \sigma$*-set*
using *wfE-min*[*of* $r - Id$ - σ*-set*] **by** *auto*
obtain n **where** n: *path-strategies P* \$ $n = \sigma'$
using σ'(*1*) *lset-lnth*[*of* σ'] **unfolding** σ*-set-def* **by** *blast*
{
fix m **assume** $n \leq m$
have *path-strategies P* \$ n = *path-strategies P* \$ m **proof** (*rule ccontr*)
assume $*$: *path-strategies P* \$ $n \neq$ *path-strategies P* \$ m
with ‹$n \leq m$› **have** $n < m$ **using** *le-imp-less-or-eq* **by** *blast*
with *path-strategies-monotone* **have** (*path-strategies P* \$ m, *path-strategies P* \$ n) $\in r$
using *assms* **by** (*simp add*: *infinite-small-llength*)
with $*$ **have** (*path-strategies P* \$ m, *path-strategies P* \$ n) $\in r - Id$ **by** *simp*
with σ'(*2*) n **have** *path-strategies P* \$ $m \notin \sigma$*-set* **by** *blast*
thus *False* **unfolding** σ*-set-def path-strategies-def*
using *assms*(*1*) *lfinite-lmap lset-nth-member-inf* **by** *blast*
qed
}
thus *?thesis* **by** *blast*
qed

8.2 Eventually One Strategy

The key lemma: Every path that stays in S and follows *well-ordered-strategy* eventually follows one strategy because the strategies are well-ordered and non-increasing along the path.

lemma *path-eventually-conforms-to-σ-map-n*:
  assumes *lset P* ⊆ *S valid-path P path-conforms-with-strategy p P well-ordered-strategy*
  shows ∃ *n*. *path-conforms-with-strategy p* (*ldropn n P*) (*path-strategies P* $ *n*)
proof (*cases*)
  assume *lfinite P*
  then obtain *n* **where** *llength P* = *enat n* **using** *lfinite-llength-enat* **by** *blast*
  hence *ldropn n P* = *LNil* **by** *simp*
  thus *?thesis* **by** (*metis path-conforms-LNil*)
next
  assume ¬*lfinite P*
  then obtain *n* **where** *n*: ⋀*m*. *n* ≤ *m* ⟹ *path-strategies P* $ *n* = *path-strategies P* $ *m*
    using *path-strategies-eventually-constant assms* **by** *blast*
  let *?σ* = *well-ordered-strategy*
  def *P'* ≡ *ldropn n P*
  { **fix** *v* **assume** *v* ∈ *lset P'*
    hence *v* ∈ *S* **using** ‹*lset P* ⊆ *S*› *P'-def in-lset-ldropnD* **by** *fastforce*
      from ‹*v* ∈ *lset P'*› **obtain** *m* **where** *m*: *enat m* < *llength P' P'* $ *m* = *v* **by** (*meson in-lset-conv-lnth*)
    hence *P* $ *m* + *n* = *v* **unfolding** *P'-def* **by** (*simp add*: ‹¬*lfinite P*› *infinite-small-llength*)
    moreover have *?σ v* = *choose v v* **unfolding** *well-ordered-strategy-def* **using** ‹*v* ∈ *S*› **by** *auto*
    ultimately have *?σ v* = (*path-strategies P* $ *m* + *n*) *v*
      unfolding *path-strategies-def* **using** *infinite-small-llength*[*OF* ‹¬*lfinite P*›] **by** *simp*
    hence *?σ v* = (*path-strategies P* $ *n*) *v* **using** *n*[*of m* + *n*] **by** *simp*
  }
  moreover have *path-conforms-with-strategy p P' well-ordered-strategy*
    unfolding *P'-def* **by** (*simp add*: *assms*(*3*) *path-conforms-with-strategy-drop*)
  ultimately show *?thesis*
    using *path-conforms-with-strategy-irrelevant-updates P'-def* **by** *blast*
qed

end — WellOrderedStrategies

end

9 Winning Regions

theory *WinningRegion*
imports
  Main
  WinningStrategy
begin

Here we define winning regions of parity games. The winning region for player p is the set of nodes from which p has a positional winning strategy.

context *ParityGame* **begin**

```
definition winning-region p ≡ { v ∈ V. ∃σ. strategy p σ ∧ winning-strategy p σ v }

lemma winning-regionI [intro]:
  assumes v ∈ V strategy p σ winning-strategy p σ v
  shows v ∈ winning-region p
  using assms unfolding winning-region-def by blast

lemma winning-region-in-V [simp]: winning-region p ⊆ V unfolding winning-region-def by blast

lemma winning-region-deadends:
  assumes v ∈ VV p deadend v
  shows v ∈ winning-region p**
proof
  show v ∈ V using ‹v ∈ VV p› by blast
  show winning-strategy p** σ-arbitrary v using assms winning-strategy-on-deadends by simp
qed simp
```

9.1 Paths in Winning Regions

```
lemma (in vmc-path) paths-stay-in-winning-region:
  assumes σ': strategy p σ' winning-strategy p σ' v0
    and σ: ⋀v. v ∈ winning-region p ⟹ σ' v = σ v
  shows lset P ⊆ winning-region p
proof
  fix x assume x ∈ lset P
  thus x ∈ winning-region p using assms vmc-path-axioms
  proof (induct arbitrary: v0 rule: llist-set-induct)
    case (find P v0)
    interpret vmc-path G P v0 p σ using find.prems(4) .
    show ?case using P-v0 σ'(1) find.prems(2) v0-V unfolding winning-region-def by blast
  next
    case (step P x v0)
    interpret vmc-path G P v0 p σ using step.prems(4) .
    show ?case proof (cases)
      assume lnull (ltl P)
      thus ?thesis using P-lnull-ltl-LCons step.hyps(2) by auto
    next
      assume ¬lnull (ltl P)
      then interpret vmc-path-no-deadend G P v0 p σ using P-no-deadend-v0 by unfold-locales
      have winning-strategy p σ' w0 proof (cases)
        assume v0 ∈ VV p
        hence winning-strategy p σ' (σ' v0)
          using strategy-extends-VVp local.step(4) step.prems(2) v0-no-deadend by blast
        moreover have σ v0 = w0 using v0-conforms ‹v0 ∈ VV p› by blast
        moreover have σ' v0 = σ v0
          using σ assms(1) step.prems(2) v0-V unfolding winning-region-def by blast
        ultimately show ?thesis by simp
      next
        assume v0 ∉ VV p
        thus ?thesis using v0-V strategy-extends-VVpstar step(4) step.prems(2) by simp
      qed
```

thus *?thesis* **using** *step.hyps(3) step(4)* σ *vmc-path-ltl* **by** *blast*
qed
qed
qed

lemma (**in** *vmc-path*) *path-hits-winning-region-is-winning*:
assumes σ': *strategy p* σ' $\bigwedge v.\ v \in$ *winning-region p* $\Longrightarrow$ *winning-strategy p* σ' *v*
and σ: $\bigwedge v.\ v \in$ *winning-region p* $\Longrightarrow \sigma'\ v = \sigma\ v$
and *P*: *lset P* $\cap$ *winning-region p* $\neq \{\}$
shows *winning-path p P*
proof−
obtain *n* **where** *n*: *enat n* $<$ *llength P P* \$ *n* $\in$ *winning-region p*
using *P* **by** (*meson lset-intersect-lnth*)
def $P' \equiv$ *ldropn n P*
then interpret P': *vmc-path G* P' *P* \$ *n p* σ
unfolding P'*-def* **using** *vmc-path-ldropn n(1)* **by** *blast*
have *winning-strategy p* σ' (*P* \$ *n*) **using** $\sigma'(2)$ *n(2)* **by** *blast*
hence *lset* $P' \subseteq$ *winning-region p*
using P'*.paths-stay-in-winning-region*[*OF* $\sigma'(1)$ - σ]
by *blast*
hence $\bigwedge v.\ v \in$ *lset* $P' \Longrightarrow \sigma\ v = \sigma'\ v$ **using** σ **by** *auto*
hence *path-conforms-with-strategy p* P' σ'
using *path-conforms-with-strategy-irrelevant-updates* P'*.P-conforms*
by *blast*
then interpret P': *vmc-path G* P' *P* \$ *n p* σ' **using** P'*.conforms-to-another-strategy* **by** *blast*
have *winning-path p* P' **using** $\sigma'(2)$ *n(2)* P'*.vmc-path-axioms winning-strategy-def* **by** *blast*
thus *winning-path p P* **unfolding** P'*-def* **using** *winning-path-drop-add n(1) P-valid* **by** *blast*
qed

9.2 Irrelevant Updates

Updating a winning strategy outside of the winning region is irrelevant.

lemma *winning-strategy-updates*:
assumes σ: *strategy p* σ *winning-strategy p* σ *v0*
and *v*: $v \notin$ *winning-region p* $v \rightarrow w$
shows *winning-strategy p* $(\sigma(v := w))$ *v0*
proof
fix *P* **assume** *vmc-path G P v0 p* $(\sigma(v := w))$
then interpret *vmc-path G P v0 p* $\sigma(v := w)$ **.**
have $\bigwedge v'.\ v' \in$ *winning-region p* $\Longrightarrow \sigma\ v' = (\sigma(v := w))\ v'$ **using** *v* **by** *auto*
hence $v \notin$ *lset P* **using** *v paths-stay-in-winning-region* σ **unfolding** *winning-region-def* **by** *blast*
hence *path-conforms-with-strategy p P* σ
using *P-conforms path-conforms-with-strategy-irrelevant'* **by** *blast*
thus *winning-path p P* **using** *conforms-to-another-strategy* $\sigma(2)$ *winning-strategy-def* **by** *blast*
qed

9.3 Extending Winning Regions

lemma *winning-region-extends-VVp*:
assumes *v*: $v \in$ *VV p* $v \rightarrow w$ **and** *w*: $w \in$ *winning-region p*
shows $v \in$ *winning-region p*

proof (*rule ccontr*)
  obtain σ **where** σ: *strategy p* σ *winning-strategy p* σ *w*
    using *w* **unfolding** *winning-region-def* **by** *blast*
  let $?\sigma = \sigma(v := w)$
  assume *contra*: $v \notin$ *winning-region p*
  moreover have *strategy p* $?\sigma$ **using** *valid-strategy-updates* $\sigma(1)$ ‹$v \rightarrow w$› **by** *blast*
  moreover hence *winning-strategy p* $?\sigma$ *v*
    using *winning-strategy-updates* σ *contra v strategy-extends-backwards-VVp*
    by *auto*
  ultimately show *False* **using** ‹$v \rightarrow w$› **unfolding** *winning-region-def* **by** *auto*
qed

Unfortunately, we cannot prove the corresponding theorem *winning-region-extends-VVpstar* for *VV p*∗∗-nodes yet. First, we need to show that there exists a uniform winning strategy on *winning-region p*. We will prove *winning-region-extends-VVpstar* as soon as we have this.

end — context ParityGame

end

10 Uniform Strategies

Theorems about how to get a uniform strategy given strategies for each node.

theory *UniformStrategy*
imports
  Main
  AttractingStrategy WinningStrategy WellOrderedStrategy WinningRegion
begin

context *ParityGame* **begin**

10.1 A Uniform Attractor Strategy

lemma *merge-attractor-strategies*:
  assumes $S \subseteq V$
    and *strategies-ex*: $\bigwedge v.\ v \in S \Longrightarrow \exists \sigma.$ *strategy p* $\sigma \wedge$ *strategy-attracts-via p* σ *v S W*
  shows $\exists \sigma.$ *strategy p* $\sigma \wedge$ *strategy-attracts p* σ *S W*
proof−
  def *good* $\equiv \lambda v.$ { $\sigma.$ *strategy p* $\sigma \wedge$ *strategy-attracts-via p* σ *v S W* }
  let $?G = \{\sigma.\ \exists v \in S - W.\ \sigma \in$ *good v*$\}$
  obtain *r* **where** *r*: *well-order-on ?G r* **using** *well-order-on* **by** *blast*

  interpret *WellOrderedStrategies G S* − *W p good r* **proof**
    show $S - W \subseteq V$ **using** ‹$S \subseteq V$› **by** *blast*
  next
    show $\bigwedge v.\ v \in S - W \Longrightarrow \exists \sigma.\ \sigma \in$ *good v* **unfolding** *good-def* **using** *strategies-ex* **by** *blast*
  next
    show $\bigwedge v\ \sigma.\ \sigma \in$ *good v* $\Longrightarrow$ *strategy p* σ **unfolding** *good-def* **by** *blast*
  next
    fix *v w* σ **assume** *v*: $v \in S - W$ $v \rightarrow w$ $v \in VV\ p \Longrightarrow \sigma\ v = w$ $\sigma \in$ *good v*
    hence σ: *strategy p* σ *strategy-attracts-via p* σ *v S W* **unfolding** *good-def* **by** *simp-all*

hence *strategy-attracts-via p σ w S W* **using** *strategy-attracts-via-successor v* **by** *blast*
thus $\sigma \in$ *good w* **unfolding** *good-def* **using** *σ(1)* **by** *blast*
qed (*insert r*)

have *S-W-no-deadends*: $\bigwedge v.\ v \in S - W \Longrightarrow \neg$*deadend v*
using *strategy-attracts-via-no-deadends[of - S W] strategies-ex*
by (*metis* (*no-types*) *Diff-iff S-V rev-subsetD*)

{
fix *v0* **assume** $v0 \in S$
fix *P* **assume** *P*: *vmc-path G P v0 p well-ordered-strategy*
then interpret *vmc-path G P v0 p well-ordered-strategy* **.**
have *visits-via P S W* **proof** (*rule ccontr*)
assume *contra*: ¬*visits-via P S W*

hence *lset P* $\subseteq S - W$ **proof** (*induct rule*: *vmc-path-lset-induction*)
case *base*
show $v0 \in S - W$ **using** ‹$v0 \in S$› *contra visits-via-trivial* **by** *blast*
next
case (*step P v0*)
interpret *vmc-path-no-deadend G P v0 p well-ordered-strategy* **using** *step.hyps(1)* **.**
have *insert v0 S = S* **using** *step.hyps(2)* **by** *blast*
hence ¬*visits-via* (*ltl P*) *S W*
using *visits-via-LCons[of ltl P S W v0, folded P-LCons] step.hyps(3)* **by** *auto*
moreover hence $w0 \notin W$ **using** *vmc-path.visits-via-trivial[OF vmc-path-ltl]* **by** *blast*
moreover have $w0 \in S \cup W$ **proof** (*cases*)
assume $v0 \in VV\ p$
hence *well-ordered-strategy v0 = w0* **using** *v0-conforms* **by** *blast*
hence *choose v0 v0 = w0* **using** *step.hyps(2) well-ordered-strategy-def* **by** *auto*
moreover have *strategy-attracts-via p* (*choose v0*) *v0 S W*
using *choose-good good-def step.hyps(2)* **by** *blast*
ultimately show *?thesis*
by (*metis strategy-attracts-via-successor strategy-attracts-via-v0*
choose-strategy step.hyps(2) ‹$v0 \rightarrow w0$› *w0-V*)
qed (*metis DiffD1 assms(2) step.hyps(2) strategy-attracts-via-successor*
strategy-attracts-via-v0 ‹$v0 \rightarrow w0$› *w0-V*)
ultimately show *?case* **by** *blast*
qed

have ¬*lfinite P* **proof**
assume *lfinite P*
hence *deadend* (*llast P*) **using** *P-maximal* ‹¬*lnull P*› *maximal-ends-on-deadend* **by** *blast*
moreover have *llast P* $\in S - W$ **using** ‹*lset P* $\subseteq S - W$› ‹¬*lnull P*› ‹*lfinite P*› *lfinite-lset*
by *blast*
ultimately show *False* **using** *S-W-no-deadends* **by** *blast*
qed

obtain *n* **where** *n*: *path-conforms-with-strategy p* (*ldropn n P*) (*path-strategies P $ n*)
using *path-eventually-conforms-to-σ-map-n[OF* ‹*lset P* $\subseteq S - W$› *P-valid P-conforms]*
by *blast*
def [*simp*]: $\sigma' \equiv$ *path-strategies P $ n*
def [*simp*]: $P' \equiv$ *ldropn n P*

```
    interpret vmc-path G P′ lhd P′ p σ′ proof
      show ¬lnull P′ unfolding P′-def
        using ‹¬lfinite P› lfinite-ldropn lnull-imp-lfinite by blast
    qed (simp-all add: n)
    have strategy p σ′ unfolding σ′-def
      using path-strategies-strategy ‹lset P ⊆ S − W› ‹¬lfinite P› infinite-small-llength
      by blast
    moreover have strategy-attracts-via p σ′ (lhd P′) S W proof−
      have P $ n ∈ S − W using ‹lset P ⊆ S − W› ‹¬lfinite P› lset-nth-member-inf by blast
      hence σ′ ∈ good (P $ n)
        using path-strategies-good σ′-def ‹¬lfinite P› ‹lset P ⊆ S − W› by blast
      hence strategy-attracts-via p σ′ (P $ n) S W unfolding good-def by blast
      thus ?thesis unfolding P′-def using P-0 by (simp add: ‹¬lfinite P› infinite-small-llength)
    qed
    moreover from ‹lset P ⊆ S − W› have lset P′ ⊆ S − W
      unfolding P′-def using lset-ldropn-subset[of n P] by blast
    ultimately show False using strategy-attracts-via-lset by blast
  qed
}
thus ?thesis using well-ordered-strategy-valid by blast
qed
```

10.2 A Uniform Winning Strategy

Let S be the winning region of player p. Then there exists a uniform winning strategy on S.

```
lemma merge-winning-strategies:
  shows ∃σ. strategy p σ ∧ (∀v ∈ winning-region p. winning-strategy p σ v)
proof−
  def good ≡ λv. { σ. strategy p σ ∧ winning-strategy p σ v }
  let ?G = {σ. ∃v ∈ winning-region p. σ ∈ good v}
  obtain r where r: well-order-on ?G r using well-order-on by blast

  have no-VVp-deadends: ⋀v. ⟦ v ∈ winning-region p; v ∈ VV p ⟧ ⟹ ¬deadend v
    using no-winning-strategy-on-deadends unfolding winning-region-def by blast

  interpret WellOrderedStrategies G winning-region p p good r proof
    show ⋀v. v ∈ winning-region p ⟹ ∃σ. σ ∈ good v
      unfolding good-def winning-region-def by blast
  next
    show ⋀v σ. σ ∈ good v ⟹ strategy p σ unfolding good-def by blast
  next
    fix v w σ assume v: v ∈ winning-region p v→w v ∈ VV p ⟹ σ v = w σ ∈ good v
    hence σ: strategy p σ winning-strategy p σ v unfolding good-def by simp-all
    hence winning-strategy p σ w proof (cases)
      assume v ∈ VV p
      moreover hence σ v = w using v(3) by blast
      moreover have ¬deadend v using no-VVp-deadends ‹v ∈ VV p› v(1) by blast
      ultimately show ?thesis using strategy-extends-VVp σ by blast
    next
      assume v ∉ VV p
      thus ?thesis using strategy-extends-VVpstar σ ‹v→w› by blast
```

```
      qed
      thus σ ∈ good w unfolding good-def using σ(1) by blast
    qed (insert winning-region-in-V r)

    {
      fix v0 assume v0 ∈ winning-region p
      fix P assume P: vmc-path G P v0 p well-ordered-strategy
      then interpret vmc-path G P v0 p well-ordered-strategy .

      have lset P ⊆ winning-region p proof (induct rule: vmc-path-lset-induction-simple)
        case (step P v0)
        interpret vmc-path-no-deadend G P v0 p well-ordered-strategy using step.hyps(1) .
        { assume v0 ∈ VV p
          hence well-ordered-strategy v0 = w0 using v0-conforms by blast
          hence choose v0 v0 = w0 by (simp add: step.hyps(2) well-ordered-strategy-def)
        }
        hence choose v0 ∈ good w0 using strategies-continue choose-good step.hyps(2) by simp
        thus ?case unfolding good-def winning-region-def using ‹w0 ∈ V› by blast
      qed (insert ‹v0 ∈ winning-region p›)

      have winning-path p P proof (rule ccontr)
        assume contra: ¬winning-path p P

        have ¬lfinite P proof
          assume lfinite P
          hence deadend (llast P) using maximal-ends-on-deadend by simp
          moreover have llast P ∈ winning-region p
            using ‹lset P ⊆ winning-region p› ‹¬lnull P› ‹lfinite P› lfinite-lset by blast
          moreover have llast P ∈ VV p
            using contra paths-are-winning-for-one-player ‹lfinite P›
            unfolding winning-path-def by simp
          ultimately show False using no-VVp-deadends by blast
        qed

        obtain n where n: path-conforms-with-strategy p (ldropn n P) (path-strategies P $ n)
        using path-eventually-conforms-to-σ-map-n[OF ‹lset P ⊆ winning-region p› P-valid P-conforms]
by blast
        def [simp]: σ' ≡ path-strategies P $ n
        def [simp]: P' ≡ ldropn n P
        interpret P': vmc-path G P' lhd P' p σ' proof
          show ¬lnull P' using ‹¬lfinite P› unfolding P'-def
            using lfinite-ldropn lnull-imp-lfinite by blast
        qed (simp-all add: n)
        have strategy p σ' unfolding σ'-def
          using path-strategies-strategy ‹lset P ⊆ winning-region p› ‹¬lfinite P› by blast
        moreover have winning-strategy p σ' (lhd P') proof−
          have P $ n ∈ winning-region p
            using ‹lset P ⊆ winning-region p› ‹¬lfinite P› lset-nth-member-inf by blast
          hence σ' ∈ good (P $ n)
            using path-strategies-good choose-good σ'-def ‹¬lfinite P› ‹lset P ⊆ winning-region p›
            by blast
          hence winning-strategy p σ' (P $ n) unfolding good-def by blast
```

```
      thus ?thesis
        unfolding P′-def using P-0 ‹¬lfinite P› by (simp add: infinite-small-llength lhd-ldropn)
    qed
    ultimately have winning-path p P′ unfolding winning-strategy-def
      using P′.vmc-path-axioms by blast
    moreover have ¬lfinite P′ using ‹¬lfinite P› P′-def by simp
    ultimately show False using contra winning-path-drop-add[OF P-valid] by auto
  qed
}
thus ?thesis unfolding winning-strategy-def using well-ordered-strategy-valid by auto
qed
```

10.3 Extending Winning Regions

Now we are finally able to prove the complement of *winning-region-extends-VVp* for *VV p*∗∗ nodes, which was still missing.

```
lemma winning-region-extends-VVpstar:
  assumes v: v ∈ VV p∗∗ and w: ⋀w. v→w ⟹ w ∈ winning-region p
  shows v ∈ winning-region p
proof−
  obtain σ where σ: strategy p σ  ⋀v. v ∈ winning-region p ⟹ winning-strategy p σ v
    using merge-winning-strategies by blast
  have winning-strategy p σ v using strategy-extends-backwards-VVpstar[OF v σ(1)] σ(2) w by
blast
  thus ?thesis unfolding winning-region-def using v σ(1) by blast
qed
```

It immediately follows that removing a winning region cannot create new deadends.

```
lemma removing-winning-region-induces-no-deadends:
  assumes v ∈ V − winning-region p ¬deadend v
  shows ∃w ∈ V − winning-region p. v→w
  using assms winning-region-extends-VVp winning-region-extends-VVpstar by blast

end — context ParityGame

end
```

11 Attractor Strategies

```
theory AttractorStrategy
imports
  Main
  Attractor UniformStrategy
begin
```

This section proves that every attractor set has an attractor strategy.

```
context ParityGame begin

lemma strategy-attracts-extends-VVp:
  assumes σ: strategy p σ strategy-attracts p σ S W
```

and *v0*: *v0* ∈ *VV p v0* ∈ *directly-attracted p S v0* ∉ *S*
shows ∃σ. *strategy p* σ ∧ *strategy-attracts-via p* σ *v0* (*insert v0 S*) *W*
proof −
from *v0*(*1*,*2*) **obtain** *w* **where** *v0*→*w w* ∈ *S* **using** *directly-attracted-def* **by** *blast*
from ‹*w* ∈ *S*› σ(*2*) **have** *strategy-attracts-via p* σ *w S W* **unfolding** *strategy-attracts-def* **by** *blast*
let *?*σ = σ(*v0* := *w*) — Extend σ to the new node.
have *strategy p ?*σ **using** σ(*1*) ‹*v0*→*w*› *valid-strategy-updates* **by** *blast*
moreover have *strategy-attracts-via p ?*σ *v0* (*insert v0 S*) *W* **proof**
fix *P*
assume *vmc-path G P v0 p ?*σ
then interpret *vmc-path G P v0 p ?*σ **.**
have ¬*deadend v0* **using** ‹*v0*→*w*› **by** *blast*
then interpret *vmc-path-no-deadend G P v0 p ?*σ **by** *unfold-locales*

def [*simp*]: *P''* ≡ *ltl P*
have *lhd P''* = *w* **using** *v0*(*1*) *v0-conforms w0-def* **by** *auto*
hence *vmc-path G P'' w p ?*σ **using** *vmc-path-ltl* **by** (*simp add*: *w0-def*)

have ∗: *v0* ∉ *S* − *W* **using** ‹*v0* ∉ *S*› **by** *blast*
have *override-on* (σ(*v0* := *w*)) σ (*S* − *W*) = *?*σ
by (*rule ext*) (*metis* ∗ *fun-upd-def override-on-def*)
hence *strategy-attracts p ?*σ *S W*
using *strategy-attracts-irrelevant-override*[*OF* σ(*2*,*1*) ‹*strategy p ?*σ›] **by** *simp*
hence *strategy-attracts-via p ?*σ *w S W* **unfolding** *strategy-attracts-def*
using ‹*w* ∈ *S*› **by** *blast*
hence *visits-via P'' S W* **unfolding** *strategy-attracts-via-def*
using ‹*vmc-path G P'' w p ?*σ› **by** *blast*
thus *visits-via P* (*insert v0 S*) *W*
using *visits-via-LCons*[*of ltl P S W v0*] *P-LCons* **by** *simp*
qed
ultimately show *?thesis* **by** *blast*
qed

lemma *strategy-attracts-extends-VVpstar*:
assumes σ: *strategy-attracts p* σ *S W*
and *v0*: *v0* ∉ *VV p v0* ∈ *directly-attracted p S*
shows *strategy-attracts-via p* σ *v0* (*insert v0 S*) *W*
proof
fix *P*
assume *vmc-path G P v0 p* σ
then interpret *vmc-path G P v0 p* σ **.**
have ¬*deadend v0* **using** *v0*(*2*) *directly-attracted-contains-no-deadends* **by** *blast*
then interpret *vmc-path-no-deadend G P v0 p* σ **by** *unfold-locales*
have *visits-via* (*ltl P*) *S W*
using *vmc-path.strategy-attractsE*[*OF vmc-path-ltl* σ] *v0 directly-attracted-def* **by** *simp*
thus *visits-via P* (*insert v0 S*) *W* **using** *visits-via-LCons*[*of ltl P S W v0*] *P-LCons* **by** *simp*
qed

lemma *attractor-has-strategy-single*:
assumes *W* ⊆ *V*
and *v0-def*: *v0* ∈ *attractor p W* (**is** - ∈ *?A*)
shows ∃σ. *strategy p* σ ∧ *strategy-attracts-via p* σ *v0 ?A W*

using *assms* **proof** (*induct arbitrary*: *v0 rule*: *attractor-set-induction*)
  case (*step S*)
  have $v0 \in W \Longrightarrow \exists\sigma.$ *strategy p* $\sigma \wedge$ *strategy-attracts-via p* σ *v0* {} *W*
    using *strategy-attracts-via-trivial valid-arbitrary-strategy* **by** *blast*
  moreover {
    assume $*$: $v0 \in$ *directly-attracted p S* $v0 \notin S$
    from *assms*(*1*) *step.hyps*(*1*) *step.hyps*(*2*)
      have $\exists\sigma.$ *strategy p* $\sigma \wedge$ *strategy-attracts p* σ *S W*
      using *merge-attractor-strategies* **by** *auto*
    with $*$
      have $\exists\sigma.$ *strategy p* $\sigma \wedge$ *strategy-attracts-via p* σ *v0* (*insert v0 S*) *W*
      using *strategy-attracts-extends-VVp strategy-attracts-extends-VVpstar* **by** *blast*
  }
  ultimately show *?case*
    using *step.prems step.hyps*(*2*)
    attractor-strategy-on-extends[*of p - v0 insert v0 S W W* $\cup$ *S* $\cup$ *directly-attracted p S*]
    attractor-strategy-on-extends[*of p - v0 S*   *W W* $\cup$ *S* $\cup$ *directly-attracted p S*]
    attractor-strategy-on-extends[*of p - v0* {}   *W W* $\cup$ *S* $\cup$ *directly-attracted p S*]
    by *blast*
next
  case (*union M*)
  hence $\exists S.\ S \in M \wedge v0 \in S$ **by** *blast*
  thus *?case* **by** (*meson Union-upper attractor-strategy-on-extends union.hyps*)
qed

11.1 Existence

Prove that every attractor set has an attractor strategy.

theorem *attractor-has-strategy*:
  assumes $W \subseteq V$
  shows $\exists\sigma.$ *strategy p* $\sigma \wedge$ *strategy-attracts p* σ (*attractor p W*) *W*
proof–
  let *?A* = *attractor p W*
  have *?A* $\subseteq$ *V* **by** (*simp add*: ‹$W \subseteq V$› *attractor-in-V*)
  moreover
    have $\bigwedge v.\ v \in$ *?A* $\Longrightarrow \exists\sigma.$ *strategy p* $\sigma \wedge$ *strategy-attracts-via p* σ *v ?A W*
    using ‹$W \subseteq V$› *attractor-has-strategy-single* **by** *blast*
  ultimately show *?thesis* **using** *merge-attractor-strategies* ‹$W \subseteq V$› **by** *blast*
qed

end — context ParityGame

end

12 Positional Determinacy of Parity Games

theory *PositionalDeterminacy*
imports
  Main
  AttractorStrategy

begin

context *ParityGame* **begin**

12.1 Induction Step

The proof of positional determinacy is by induction over the size of the finite set ω ‘ V, the set of priorities. The following lemma is the induction step.

For now, we assume there are no deadends in the graph. Later we will get rid of this assumption.

lemma *positional-strategy-induction-step*:
  assumes $v \in V$
    and *no-deadends*: $\bigwedge v.\ v \in V \Longrightarrow \neg deadend\ v$
    and *IH*: $\bigwedge(G :: ('a, 'b)\ ParityGame\text{-}scheme)\ v.$
      $[\![\ card\ (\omega_G\ ‘\ V_G) < card\ (\omega\ ‘\ V);\ v \in V_G;$
        $ParityGame\ G;$
        $\bigwedge v.\ v \in V_G \Longrightarrow \neg Digraph.deadend\ G\ v\ ]\!]$
      $\Longrightarrow \exists p.\ v \in ParityGame.winning\text{-}region\ G\ p$
  shows $\exists p.\ v \in winning\text{-}region\ p$
proof−

First, we determine the minimum priority and the player who likes it.

  def *min-prio* $\equiv$ *Min* (ω ‘ V)
  have $\exists p.$ *winning-priority p min-prio* **by** *auto*
  then obtain *p* **where** *p*: *winning-priority p min-prio* **by** *blast*

Then we define the tentative winning region of player $p{*}{*}$. The rest of the proof is to show that this is the complete winning region.

  def *W1* $\equiv$ *winning-region* $p{*}{*}$

For this, we define several more sets of nodes. First, U is the tentative winning region of player p.

  def $U \equiv V - W1$
  def $K \equiv U \cap (\omega -‘\ \{min\text{-}prio\})$
  def $V' \equiv U -$ *attractor p K*

  def [*simp*]: $G' \equiv$ *subgame* V'
  interpret G': *ParityGame* G' **using** *subgame-ParityGame* **by** *simp*

  have *U-equiv*: $\bigwedge v.\ v \in V \Longrightarrow v \in U \longleftrightarrow v \notin winning\text{-}region\ p{*}{*}$
    unfolding *U-def W1-def* **by** *blast*

  have $V' \subseteq V$ **unfolding** *U-def V'-def* **by** *blast*
  hence [*simp*]: $V_{G'} = V'$ **unfolding** *G'-def* **by** *simp*

  have $V_{G'} \subseteq V\ E_{G'} \subseteq E\ \omega_{G'} = \omega$ **unfolding** *G'-def* **by** (*simp-all add*: *subgame-ω*)
  have $G'.VV\ p = V' \cap VV\ p$ **unfolding** *G'-def* **using** *subgame-VV* **by** *simp*

  have *V-decomp*: $V =$ *attractor p K* $\cup\ V' \cup W1$ **proof**−

have $V \subseteq$ *attractor p K* $\cup$ $V' \cup$ *W1*
  unfolding *V'-def U-def* **by** *blast*
moreover have *attractor p K* $\subseteq V$
  using *attractor-in-V[of K]* **unfolding** *K-def U-def* **by** *blast*
ultimately show *?thesis*
  unfolding *W1-def winning-region-def* **using** ‹$V' \subseteq V$› **by** *blast*
qed

have *G'-no-deadends*: $\bigwedge v.\ v \in V_{G'} \Longrightarrow \neg G'.deadend\ v$ **proof**−
  fix v **assume** $v \in V_{G'}$
  hence $*$: $v \in U -$ *attractor p K* **using** ‹$V_{G'} = V'$› *V'-def* **by** *blast*
  moreover hence $\exists w \in U.\ v \to w$
    using *removing-winning-region-induces-no-deadends[of v p**] no-deadends U-equiv U-def*
    by *blast*
  moreover have $\bigwedge w.\ [\![\ v \in VV\ p{*}{*};\ v \to w\ ]\!] \Longrightarrow w \in U$
    using $*$ *U-equiv winning-region-extends-VVp* **by** *blast*
  ultimately have $\exists w \in V'.\ v \to w$
    using *U-equiv winning-region-extends-VVp removing-attractor-induces-no-deadends V'-def*
    by *blast*
  thus $\neg G'.deadend\ v$ **using** ‹$v \in V_{G'}$› ‹$V' \subseteq V$› **by** *simp*
qed

By definition of *W1*, we obtain a winning strategy on *W1* for player $p{*}{*}$.

obtain σ*W1* **where** σ*W1*:
  *strategy p*** σ*W1* $\bigwedge v.\ v \in W1 \Longrightarrow$ *winning-strategy p*** σ*W1 v*
  unfolding *W1-def* **using** *merge-winning-strategies* **by** *blast*

{
  fix v **assume** $v \in V_{G'}$

Apply the induction hypothesis to get the winning strategy for v in G'.

have *G'-winning-strategy*: $\exists p.\ v \in G'.winning\text{-}region\ p$ **proof**−
  have *card* $(\omega_{G'}\ \text{‘}\ V_{G'}) <$ *card* $(\omega\ \text{‘}\ V)$ **proof**−
    { assume *min-prio* $\in \omega_{G'}\ \text{‘}\ V_{G'}$
      then obtain v **where** v: $v \in V_{G'}$ $\omega_{G'}\ v =$ *min-prio* **by** *blast*
      hence $v \in \omega -\text{‘}\ \{$*min-prio*$\}$ **using** ‹$\omega_{G'} = \omega$› **by** *simp*
      hence *False* **using** *V'-def K-def attractor-set-base* ‹$V_{G'} = V'$› *v(1)*
        by (*metis DiffD1 DiffD2 IntI contra-subsetD*)
    }
    hence *min-prio* $\notin \omega_{G'}\ \text{‘}\ V_{G'}$ **by** *blast*
    moreover have *min-prio* $\in \omega\ \text{‘}\ V$
      unfolding *min-prio-def* **using** *priorities-finite Min-in assms(1)* **by** *blast*
    moreover have $\omega_{G'}\ \text{‘}\ V_{G'} \subseteq \omega\ \text{‘}\ V$ **unfolding** *G'-def* **by** *simp*
    ultimately show *?thesis* **by** (*metis priorities-finite psubsetI psubset-card-mono*)
  qed
  thus *?thesis* **using** *IH[of G']* ‹$v \in V_{G'}$› *G'-no-deadends G'.ParityGame-axioms* **by** *blast*
qed

It turns out the winning region of player $p{*}{*}$ is empty, so we have a strategy for player p.

have $v \in G'.winning\text{-}region\ p$ **proof** (*rule ccontr*)
  assume $\neg$*?thesis*

moreover obtain p' σ **where** p': *G'.strategy* p' σ *G'.winning-strategy* p' σ v
  using *G'-winning-strategy* **unfolding** *G'.winning-region-def* **by** *blast*
ultimately have $p' \neq p$ **using** ‹$v \in V_{G'}$› **unfolding** *G'.winning-region-def* **by** *blast*
hence $p' = p{*}{*}$ **by** (*cases p*; *cases p'*) *auto*
with p' **have** σ: *G'.strategy* $p{*}{*}$ σ *G'.winning-strategy* $p{*}{*}$ σ v **by** *simp-all*

have $v \in$ *winning-region* $p{*}{*}$ **proof**
  show $v \in V$ **using** ‹$v \in V_{G'}$› ‹$V_{G'} \subseteq V$› **by** *blast*
  def $\sigma' \equiv$ *override-on* (*override-on* σ*-arbitrary* $\sigma W1$ $W1$) σ V'
  thus *strategy* $p{*}{*}$ σ'
    using *valid-strategy-updates-set-strong valid-arbitrary-strategy* $\sigma W1(1)$
        valid-strategy-supergame $\sigma(1)$ *G'-no-deadends* ‹$V_{G'} = V'$›
    unfolding *G'-def* **by** *blast*
  show *winning-strategy* $p{*}{*}$ σ' v
  proof (*rule winning-strategyI*, *rule ccontr*)
    fix P **assume** *vmc-path* G P v $p{*}{*}$ σ'
    then interpret *vmc-path* G P v $p{*}{*}$ σ' **.**
    assume $\neg$*winning-path* $p{*}{*}$ P

First we show that P stays in V', because if it stays in V', then it conforms to σ, so it must be winning for $p{*}{*}$.

    have *lset* $P \subseteq V'$ **proof** (*induct rule*: *vmc-path-lset-induction-closed-subset*)
      fix v **assume** $v \in V'$ $\neg$*deadend* v $v \in VV$ $p{*}{*}$
      hence $v \in$ *ParityGame.VV* (*subgame* V') $p{*}{*}$ **by** *auto*
      moreover have $\neg$*G'.deadend* v **using** *G'-no-deadends* ‹$V_{G'} = V'$› ‹$v \in V'$› **by** *blast*
      ultimately have σ $v \in V'$
        using *subgame-strategy-stays-in-subgame* $p'(1)$ ‹$p' = p{*}{*}$›
        unfolding *G'-def* **by** *blast*
      thus σ' $v \in V' \cup W1$ **unfolding** σ'*-def* **using** ‹$v \in V'$› **by** *simp*
    next
      fix v w **assume** $v \in V'$ $\neg$*deadend* v $v \in VV$ $p{*}{*}{*}{*}$ $v{\rightarrow}w$
      show $w \in V' \cup W1$ **proof** (*rule ccontr*)
        assume $w \notin V' \cup W1$
        hence $w \in$ *attractor* p K **using** *V-decomp* ‹$v{\rightarrow}w$› **by** *blast*
        hence $v \in$ *attractor* p K **using** ‹$v \in VV$ $p{*}{*}{*}{*}$› *attractor-set-VVp* ‹$v{\rightarrow}w$› **by** *auto*
        thus *False* **using** ‹$v \in V'$› *V'-def* **by** *blast*
      qed
    next
      have $\bigwedge v.\ v \in W1 \Longrightarrow \sigma W1\ v = \sigma'\ v$ **unfolding** σ'*-def* *V'-def* *U-def* **by** *simp*
      thus *lset* $P \cap W1 = \{\}$
        using *path-hits-winning-region-is-winning* $\sigma W1$ ‹$\neg$*winning-path* $p{*}{*}$ P›
        unfolding *W1-def*
        by *blast*
    next
      show $v \in V'$ **using** ‹$V_{G'} = V'$› ‹$v \in V_{G'}$› **by** *blast*
    qed

This concludes the proof of *lset* $P \subseteq V'$.

    hence *G'.valid-path* P **using** *subgame-valid-path* **by** *simp*
    moreover have *G'.maximal-path* P
      using ‹*lset* $P \subseteq V'$› *subgame-maximal-path* ‹$V' \subseteq V$› **by** *simp*

moreover have G'*.path-conforms-with-strategy* $p{*}{*}$ P σ **proof**−
have G'*.path-conforms-with-strategy* $p{*}{*}$ P σ'
using *subgame-path-conforms-with-strategy* ‹$V' \subseteq V$› ‹*lset* $P \subseteq V'$›
by *simp*
moreover have $\bigwedge v.\ v \in$ *lset* $P \Longrightarrow \sigma'\ v = \sigma\ v$ **using** ‹*lset* $P \subseteq V'$› σ'*-def* **by** *auto*
ultimately show *?thesis*
using G'*.path-conforms-with-strategy-irrelevant-updates* **by** *blast*
qed
ultimately have G'*.winning-path* $p{*}{*}$ P
using $\sigma(2)$ G'*.winning-strategy-def* G'*.valid-maximal-conforming-path-0* *P-0* *P-not-null*
by *blast*
moreover have $G'.VV\ p{*}{*}{*}{*} \subseteq VV\ p{*}{*}{*}{*}$ **using** *subgame-VV-subset* G'*-def* **by** *blast*
ultimately show *False*
using G'*.winning-path-supergame*[*of* $p{*}{*}$] ‹$\omega_{G'} = \omega$›
‹$\neg$*winning-path* $p{*}{*}$ P› *ParityGame-axioms*
by *blast*
qed
qed
moreover have $v \in V$ **using** ‹$V_{G'} \subseteq V$› ‹$v \in V_{G'}$› **by** *blast*
ultimately have $v \in W1$ **unfolding** *W1-def winning-region-def* **by** *blast*
thus *False* **using** ‹$v \in V_{G'}$› **using** *U-def* V'*-def* ‹$V_{G'} = V'$› ‹$v \in V_{G'}$› **by** *blast*
qed
} **note** *recursion* = *this*

We compose a winning strategy for player p on $V - W1$ out of three pieces.

First, if we happen to land in the attractor region of K, we follow the attractor strategy. This is good because the priority of the nodes in K is good for player p, so he likes to go there.

obtain $\sigma 1$
where $\sigma 1$: *strategy* p $\sigma 1$
strategy-attracts p $\sigma 1$ (*attractor* p K) K
using *attractor-has-strategy*[*of* K p] *K-def* *U-def* **by** *auto*

Next, on G' we follow the winning strategy whose existence we proved earlier.

have G'*.winning-region* $p = V_{G'}$ **using** *recursion* **unfolding** G'*.winning-region-def* **by** *blast*
then obtain $\sigma 2$
where $\sigma 2$: $\bigwedge v.\ v \in V_{G'} \Longrightarrow G'$*.strategy* p $\sigma 2$
$\bigwedge v.\ v \in V_{G'} \Longrightarrow G'$*.winning-strategy* p $\sigma 2$ v
using G'*.merge-winning-strategies* **by** *blast*

As a last option we choose an arbitrary successor but avoid entering *W1*. In particular, this defines the strategy on the set K.

def *succ* $\equiv \lambda v.$ *SOME* $w.\ v \rightarrow w \wedge (v \in W1 \vee w \notin W1)$

Compose the three pieces.

def $\sigma \equiv$ *override-on* (*override-on succ* $\sigma 2$ V') $\sigma 1$ (*attractor* p $K - K$)

have *attractor* p $K \cap W1 = \{\}$ **proof** (*rule ccontr*)
assume *attractor* p $K \cap W1 \neq \{\}$

then obtain v **where** v: $v \in$ *attractor p K* $v \in$ *W1* **by** *blast*
hence $v \in V$ **using** *W1-def winning-region-def* **by** *blast*
obtain P **where** *vmc2-path G P v p* $\sigma 1$ $\sigma W1$
using *strategy-conforming-path-exists* $\sigma W1(1)$ $\sigma 1(1)$ ‹$v \in V$› **by** *blast*
then interpret *vmc2-path G P v p* $\sigma 1$ $\sigma W1$ **.**
have *strategy-attracts-via p* $\sigma 1$ v (*attractor p K*) K **using** $v(1)$ $\sigma 1(2)$ *strategy-attracts-def* **by** *blast*
hence *lset* $P \cap K \neq \{\}$ **using** *strategy-attracts-viaE visits-via-visits* **by** *blast*
hence $\neg$*lset* $P \subseteq W1$ **unfolding** *K-def U-def* **by** *blast*
thus *False* **unfolding** *W1-def* **using** *comp.paths-stay-in-winning-region* $\sigma W1$ $v(2)$ **by** *auto*
qed

On specific sets, σ behaves like one of the three pieces.

have σ-$\sigma 1$: $\bigwedge v.\ v \in$ *attractor p K* $- K \Longrightarrow \sigma\ v = \sigma 1\ v$ **unfolding** σ*-def* **by** *simp*
have σ-$\sigma 2$: $\bigwedge v.\ v \in V' \Longrightarrow \sigma\ v = \sigma 2\ v$ **unfolding** σ*-def* V'*-def* **by** *auto*
have σ-K: $\bigwedge v.\ v \in K \cup W1 \Longrightarrow \sigma\ v =$ *succ* v **proof**−
fix v **assume** $v \in K \cup W1$
moreover hence $v \notin V'$ **unfolding** V'*-def U-def* **using** *attractor-set-base* **by** *auto*
ultimately show $\sigma\ v =$ *succ* v **unfolding** σ*-def U-def* **using** ‹*attractor p K* $\cap W1 = \{\}$›
by (*metis* (*mono-tags*, *lifting*) *Diff-iff IntI UnE override-on-def override-on-emptyset*)
qed

Show that *succ* succeeds in avoiding entering *W1*.

{ fix v **assume** v: $v \in VV\ p$
hence $\neg$*deadend* v **using** *no-deadends* **by** *blast*
have $\exists w.\ v \rightarrow w \wedge (v \in W1 \vee w \notin W1)$ **proof** (*cases*)
assume $v \in W1$
thus *?thesis* **using** *no-deadends* ‹$\neg$*deadend* v› **by** *blast*
next
assume $v \notin W1$
show *?thesis* **proof** (*rule ccontr*)
assume $\neg(\exists w.\ v \rightarrow w \wedge (v \in W1 \vee w \notin W1))$
hence $\bigwedge w.\ v \rightarrow w \Longrightarrow$ *winning-strategy p*∗∗ $\sigma W1$ w **using** $\sigma W1(2)$ **by** *blast*
hence *winning-strategy p*∗∗ $\sigma W1$ v
using *strategy-extends-backwards-VVpstar* $\sigma W1(1)$ ‹$v \in VV\ p$› **by** *simp*
hence $v \in W1$ **unfolding** *W1-def winning-region-def* **using** $\sigma W1(1)$ ‹$\neg$*deadend* v› **by** *blast*
thus *False* **using** ‹$v \notin W1$› **by** *blast*
qed
qed
hence $v \rightarrow$*succ* v $v \in W1 \vee$ *succ* $v \notin W1$ **unfolding** *succ-def*
using *someI-ex*[*of* $\lambda w.\ v \rightarrow w \wedge (v \in W1 \vee w \notin W1)$] **by** *blast+*
} note *succ-works* = *this*

have *strategy p* σ **proof**
fix v **assume** v: $v \in VV\ p$ $\neg$*deadend* v
hence $v \in$ *attractor p K* $- K \Longrightarrow v \rightarrow \sigma\ v$ **using** σ-$\sigma 1$ $\sigma 1(1)$ v **unfolding** *strategy-def* **by** *auto*
moreover have $v \in V' \Longrightarrow v \rightarrow \sigma\ v$ **proof**−
assume $v \in V'$
moreover have $v \in V_{G'}$ **using** ‹$v \in V'$› ‹$V_{G'} = V'$› **by** *blast*
moreover have $v \in G'.VV\ p$ **using** ‹$G'.VV\ p = V' \cap VV\ p$› ‹$v \in V'$› ‹$v \in VV\ p$› **by** *blast*
moreover have $\neg$*Digraph.deadend* G' v **using** G'*-no-deadends* ‹$v \in V_{G'}$› **by** *blast*

ultimately have $v \rightarrow_{G'} \sigma 2\ v$ **using** $\sigma 2(1)$ *G'.strategy-def*[*of* $p\ \sigma 2$] **by** *blast*
with ‹$v \in V'$› **show** $v{\rightarrow}\sigma\ v$ **using** ‹$E_{G'} \subseteq E$› σ-$\sigma 2$ **by** (*metis subsetCE*)
qed
moreover have $v \in K \cup W1 \Longrightarrow v{\rightarrow}\sigma\ v$ **using** *succ-works*(*1*) v σ-K **by** *auto*
moreover have $v \in V$ **using** ‹$v \in VV\ p$› **by** *blast*
ultimately show $v{\rightarrow}\sigma\ v$ **using** *V-decomp* **by** *blast*
qed

have σ-*attracts*: *strategy-attracts* $p\ \sigma$ (*attractor* $p\ K$) K **proof**−
have *strategy-attracts* p (*override-on* $\sigma\ \sigma 1$ (*attractor* $p\ K - K$)) (*attractor* $p\ K$) K
using *strategy-attracts-irrelevant-override* $\sigma 1$ ‹*strategy* $p\ \sigma$› **by** *blast*
moreover have σ = *override-on* $\sigma\ \sigma 1$ (*attractor* $p\ K - K$)
by (*rule ext*) (*simp add*: *override-on-def* σ-$\sigma 1$)
ultimately show *?thesis* **by** *simp*
qed

Show that σ is a winning strategy on $V - W1$.

have $\forall v \in V - W1$. *winning-strategy* $p\ \sigma\ v$ **proof** (*intro ballI winning-strategyI*)
fix $v\ P$ **assume** P: $v \in V - W1$ *vmc-path* $G\ P\ v\ p\ \sigma$
interpret *vmc-path* $G\ P\ v\ p\ \sigma$ **using** $P(2)$ **.**

have *lset* $P \subseteq V - W1$ **proof** (*induct rule*: *vmc-path-lset-induction-closed-subset*)
fix v **assume** $v \in V - W1$ $\neg$*deadend* v $v \in VV\ p$
show $\sigma\ v \in V - W1 \cup \{\}$ **proof** (*rule ccontr*)
assume $\neg$*?thesis*
hence $\sigma\ v \in W1$
using ‹*strategy* $p\ \sigma$› ‹$\neg$*deadend* v› ‹$v \in VV\ p$›
unfolding *strategy-def* **by** *blast*
hence $v \notin K$ **using** *succ-works*(*2*)[*OF* ‹$v \in VV\ p$›] ‹$v \in V - W1$› σ-K **by** *auto*
moreover have $v \notin$ *attractor* $p\ K - K$ **proof**
assume $v \in$ *attractor* $p\ K - K$
hence $\sigma\ v \in$ *attractor* $p\ K$
using *attracted-strategy-step* ‹*strategy* $p\ \sigma$› σ-*attracts* ‹$\neg$*deadend* v› ‹$v \in VV\ p$›
attractor-set-base
by *blast*
thus *False* **using** ‹$\sigma\ v \in W1$› ‹*attractor* $p\ K \cap W1 = \{\}$› **by** *blast*
qed
moreover have $v \notin V'$ **proof**
assume $v \in V'$
have $\sigma 2\ v \in V_{G'}$ **proof** (*rule* *G'.valid-strategy-in-V*[*of* $p\ \sigma 2\ v$])
have $v \in V_{G'}$ **using** ‹$V_{G'} = V'$› ‹$v \in V'$› **by** *simp*
thus $\neg$*G'.deadend* v **using** *G'-no-deadends* **by** *blast*
show *G'.strategy* $p\ \sigma 2$ **using** $\sigma 2(1)$ ‹$v \in V_{G'}$› **by** *blast*
show $v \in$ *G'.VV* p **using** ‹$v \in VV\ p$› ‹*G'.VV* $p = V' \cap VV\ p$› ‹$v \in V'$› **by** *simp*
qed
hence $\sigma\ v \in V_{G'}$ **using** ‹$v \in V'$› σ-$\sigma 2$ **by** *simp*
thus *False* **using** ‹$V_{G'} = V'$› ‹$\sigma\ v \in W1$› *V'-def U-def* **by** *blast*
qed
ultimately show *False* **using** ‹$v \in V - W1$› *V-decomp* **by** *blast*
qed
next
fix $v\ w$ **assume** $v \in V - W1$ $\neg$*deadend* v $v \in VV\ p{*}{*}$ $v{\rightarrow}w$

show $w \in V - W1 \cup \{\}$ **proof** (*rule ccontr*)
  assume $\neg$*?thesis*
  hence $w \in W1$ **using** ‹$v \rightarrow w$› **by** *blast*
  let $?\sigma = \sigma W1(v := w)$
  have *winning-strategy p*** $\sigma W1$ w **using** ‹$w \in W1$› $\sigma W1(2)$ **by** *blast*
  moreover have $\neg(\exists \sigma.$ *strategy p*** $\sigma \wedge$ *winning-strategy p*** σ $v)$
    using ‹$v \in V - W1$› **unfolding** *W1-def winning-region-def* **by** *blast*
  ultimately have *winning-strategy p*** $?\sigma$ w
    using *winning-strategy-updates*[*of p*** $\sigma W1$ w v w] $\sigma W1(1)$ ‹$v \rightarrow w$›
    unfolding *winning-region-def* **by** *blast*
  moreover have *strategy p*** $?\sigma$ **using** ‹$v \rightarrow w$› $\sigma W1(1)$ *valid-strategy-updates* **by** *blast*
  ultimately have *winning-strategy p*** $?\sigma$ v
    using *strategy-extends-backwards-VVp*[*of v p*** $?\sigma$ w]
      ‹$v \in VV$ *p***› ‹$v \rightarrow w$›
    by *auto*
  hence $v \in W1$ **unfolding** *W1-def winning-region-def*
    using ‹*strategy p*** $?\sigma$› ‹$v \in V - W1$› **by** *blast*
  thus *False* **using** ‹$v \in V - W1$› **by** *blast*
qed
qed (*insert* $P(1)$, *simp-all*)

This concludes the proof of *lset* $P \subseteq V - W1$.

hence *lset* $P \subseteq$ *attractor p K* $\cup V'$ **using** *V-decomp* **by** *blast*

have $\neg$*lfinite P*
  using *no-deadends lfinite-lset maximal-ends-on-deadend*[*of P*] *P-maximal P-not-null lset-P-V*
  by *blast*

Every σ-conforming path starting in $V - W1$ is winning. We distinguish two cases:

1. P eventually stays in V'. Then P is winning because $\sigma 2$ is winning.
2. P visits K infinitely often. Then P is winning because of the priority of the nodes in K.

show *winning-path p P* **proof** (*cases*)
  assume $\exists n.$ *lset* (*ldropn n P*) $\subseteq V'$

The first case: P eventually stays in V'.

  then obtain n **where** n: *lset* (*ldropn n P*) $\subseteq V'$ **by** *blast*
  def $P' \equiv$ *ldropn n P*
  hence *lset* $P' \subseteq V'$ **using** n **by** *blast*
  interpret *vmc-path* G' P' *lhd* P' p $\sigma 2$ **proof**
    show $\neg$*lnull* P' **unfolding** P'*-def*
      using ‹$\neg$*lfinite P*› *lfinite-ldropn lnull-imp-lfinite* **by** *blast*
    show G'*.valid-path* P' **proof**−
      have *valid-path* P' **unfolding** P'*-def* **by** *simp*
      thus *?thesis* **using** *subgame-valid-path* ‹*lset* $P' \subseteq V'$› G'*-def* **by** *blast*
    qed
    show G'*.maximal-path* P' **proof**−
      have *maximal-path* P' **unfolding** P'*-def* **by** *simp*
      thus *?thesis* **using** *subgame-maximal-path* ‹*lset* $P' \subseteq V'$› ‹$V' \subseteq V$› G'*-def* **by** *blast*
    qed

```
    show G'.path-conforms-with-strategy p P' σ2 proof−
      have path-conforms-with-strategy p P' σ unfolding P'-def by simp
      hence path-conforms-with-strategy p P' σ2
        using path-conforms-with-strategy-irrelevant-updates ‹lset P' ⊆ V'› σ-σ2
        by blast
      thus ?thesis
        using subgame-path-conforms-with-strategy ‹lset P' ⊆ V'› ‹V' ⊆ V› G'-def
        by blast
    qed
  qed simp
  have G'.winning-strategy p σ2 (lhd P')
    using ‹lset P' ⊆ V'› ‹¬lnull P'› σ2(2)[of lhd P'] ‹V_G' = V'› llist.set-sel(1)
    by blast
  hence G'.winning-path p P' using G'.winning-strategy-def vmc-path-axioms by blast
  moreover have G'.VV p** ⊆ VV p** unfolding G'-def using subgame-VV by simp
  ultimately have winning-path p P'
    using G'.winning-path-supergame[of p P' G] ‹ω_G' = ω› ParityGame-axioms by blast
  thus ?thesis
    unfolding P'-def
    using infinite-small-llength[OF ‹¬lfinite P›]
          winning-path-drop-add[of P p n] ‹valid-path P›
    by blast
next
  assume asm: ¬(∃n. lset (ldropn n P) ⊆ V')
```

The second case: P visits K infinitely often. Then *min-prio* occurs infinitely often on P.

```
  have min-prio ∈ path-inf-priorities P
  unfolding path-inf-priorities-def proof (intro CollectI allI)
    fix n
    obtain k1 where k1: ldropn n P $ k1 ∉ V' using asm by (metis lset-lnth subsetI)
    def k2 ≡ k1 + n
    interpret vmc-path G ldropn k2 P P $ k2 p σ
      using vmc-path-ldropn infinite-small-llength ‹¬lfinite P› by blast
    have P $ k2 ∉ V' unfolding k2-def
      using k1 lnth-ldropn infinite-small-llength[OF ‹¬lfinite P›] by simp
    hence P $ k2 ∈ attractor p K using ‹¬lfinite P› ‹lset P ⊆ V − W1›
      by (metis DiffI U-def V'-def lset-nth-member-inf)
    then obtain k3 where k3: ldropn k2 P $ k3 ∈ K
      using σ-attracts strategy-attractsE unfolding G'.visits-via-def by blast
    def k4 ≡ k3 + k2
    hence P $ k4 ∈ K
      using k3 lnth-ldropn infinite-small-llength[OF ‹¬lfinite P›] by simp
    moreover have k4 ≥ n unfolding k4-def k2-def
      using le-add2 le-trans by blast
    moreover have ldropn n P $ k4 − n = P $ (k4 − n) + n
      using lnth-ldropn infinite-small-llength ‹¬lfinite P› by blast
    ultimately have ldropn n P $ k4 − n ∈ K by simp
    hence lset (ldropn n P) ∩ K ≠ {}
      using ‹¬lfinite P› lfinite-ldropn in-lset-conv-lnth[of ldropn n P $ k4 − n]
      by blast
    thus min-prio ∈ lset (ldropn n (lmap ω P)) unfolding K-def by auto
  qed
```

thus *?thesis* **unfolding** *winning-path-def*
using *path-inf-priorities-at-least-min-prio*[*OF P-valid*, *folded min-prio-def*]
⟨*winning-priority p min-prio*⟩ ⟨¬*lfinite P*⟩
by *blast*
qed
qed
hence $\forall v \in V.\ \exists p\ \sigma.$ *strategy p σ* ∧ *winning-strategy p σ v*
unfolding *W1-def winning-region-def* **using** ⟨*strategy p σ*⟩ **by** *blast*
hence $\exists p\ \sigma.$ *strategy p σ* ∧ *winning-strategy p σ v* **using** ⟨$v \in V$⟩ **by** *simp*
thus *?thesis* **unfolding** *winning-region-def* **using** ⟨$v \in V$⟩ **by** *blast*
qed

12.2 Positional Determinacy without Deadends

theorem *positional-strategy-exists-without-deadends*:
assumes $v \in V$ $\bigwedge v.\ v \in V \Longrightarrow \neg$*deadend v*
shows $\exists p.\ v \in$ *winning-region p*
using *assms ParityGame-axioms*
by (*induct card* (ω ‘ *V*) *arbitrary*: *G v rule*: *nat-less-induct*)
(*rule ParityGame.positional-strategy-induction-step*, *simp-all*)

12.3 Positional Determinacy with Deadends

Prove a stronger version of the previous theorem: Allow deadends.

theorem *positional-strategy-exists*:
assumes $v0 \in V$
shows $\exists p.\ v0 \in$ *winning-region p*
proof−
{ **fix** *p*
def $A \equiv$ *attractor p* (*deadends p*∗∗)
assume *v0-in-attractor*: $v0 \in$ *attractor p* (*deadends p*∗∗)
then obtain σ **where** σ: *strategy p σ strategy-attracts p σ A* (*deadends p*∗∗)
using *attractor-has-strategy*[*of deadends p*∗∗ *p*] *A-def deadends-in-V* **by** *blast*

have $A \subseteq V$ **using** *A-def* **using** *attractor-in-V deadends-in-V* **by** *blast*
hence $A -$ *deadends p*∗∗ $\subseteq V$ **by** *auto*

have *winning-strategy p σ v0* **proof** (*unfold winning-strategy-def*, *intro allI impI*)
fix *P* **assume** *vmc-path G P v0 p σ*
then interpret *vmc-path G P v0 p σ* **.**
show *winning-path p P*
using *visits-deadend*[*of p*∗∗] σ(*2*) *strategy-attracts-lset v0-in-attractor*
unfolding *A-def* **by** *simp*
qed
hence $\exists p\ \sigma.$ *strategy p σ* ∧ *winning-strategy p σ v0* **using** σ **by** *blast*
} **note** *lemma-path-to-deadend* = *this*
def $A \equiv \lambda p.$ *attractor p* (*deadends p*∗∗)

Remove the attractor sets of the sets of deadends.

def $V' \equiv V - A\ Even - A\ Odd$
hence $V' \subseteq V$ **by** *blast*

```
show ?thesis proof (cases)
  assume v0 ∈ V'
  def G' ≡ subgame V'
  interpret G': ParityGame G' unfolding G'-def using subgame-ParityGame .
  have V_G' = V' unfolding G'-def using ‹V' ⊆ V› by simp
  hence v0 ∈ V_G' using ‹v0 ∈ V'› by simp
  moreover have V'-no-deadends: ⋀v. v ∈ V_G' ⟹ ¬G'.deadend v proof−
    fix v assume v ∈ V_G'
    moreover have V' = V − A Even − A Even** using V'-def by simp
    ultimately show ¬G'.deadend v
      using subgame-without-deadends ‹v ∈ V_G'› unfolding A-def G'-def by blast
  qed
  ultimately obtain p σ where σ: G'.strategy p σ G'.winning-strategy p σ v0
    using G'.positional-strategy-exists-without-deadends
    unfolding G'.winning-region-def
    by blast

  have V'-no-deadends': ⋀v. v ∈ V' ⟹ ¬deadend v proof−
    fix v assume v ∈ V'
    hence ¬G'.deadend v using V'-no-deadends ‹V' ⊆ V› unfolding G'-def by auto
    thus ¬deadend v unfolding G'-def using ‹V' ⊆ V› by auto
  qed

  obtain σ-attr
    where σ-attr: strategy p σ-attr strategy-attracts p σ-attr (A p) (deadends p**)
    using attractor-has-strategy[OF deadends-in-V] A-def by blast
  def σ' ≡ override-on σ σ-attr (A Even ∪ A Odd)
  have σ'-is-σ-on-V': ⋀v. v ∈ V' ⟹ σ' v = σ v
    unfolding V'-def σ'-def A-def by (cases p) simp-all

  have strategy p σ' proof−
    have σ' = override-on σ-attr σ (UNIV − A Even − A Odd)
      unfolding σ'-def override-on-def by (rule ext) simp
    moreover have strategy p (override-on σ-attr σ V')
      using valid-strategy-supergame σ-attr(1) σ(1) V'-no-deadends ‹V_G' = V'›
      unfolding G'-def by blast
    ultimately show ?thesis by (simp add: valid-strategy-only-in-V V'-def override-on-def)
  qed
  moreover have winning-strategy p σ' v0 proof (rule winning-strategyI, rule ccontr)
    fix P assume vmc-path G P v0 p σ'
    then interpret vmc-path G P v0 p σ' .
    interpret vmc-path-no-deadend G P v0 p σ'
      using V'-no-deadends' ‹v0 ∈ V'› by unfold-locales
    assume contra: ¬winning-path p P

    have lset P ⊆ V' proof (induct rule: vmc-path-lset-induction-closed-subset)
      fix v assume v ∈ V' ¬deadend v v ∈ VV p
      hence v ∈ G'.VV p unfolding G'-def by (simp add: ‹v ∈ V'›)
      moreover have ¬G'.deadend v using V'-no-deadends ‹v ∈ V'› ‹V_G' = V'› by blast
      moreover have G'.strategy p σ'
        using G'.valid-strategy-only-in-V σ'-def σ'-is-σ-on-V' σ(1) ‹V_G' = V'› by auto
      ultimately show σ' v ∈ V' ∪ A p using subgame-strategy-stays-in-subgame
```

```
      unfolding G′-def by blast
    next
      fix v w assume v ∈ V′ ¬deadend v v ∈ VV p** v→w
      have w ∉ A p** proof
        assume w ∈ A p**
        hence v ∈ A p** unfolding A-def
          using ⟨v ∈ VV p**⟩ ⟨v→w⟩ attractor-set-VVp by blast
        thus False using ⟨v ∈ V′⟩ unfolding V′-def by (cases p) auto
      qed
      thus w ∈ V′ ∪ A p unfolding V′-def using ⟨v→w⟩ by (cases p) auto
    next
      show lset P ∩ A p = {} proof (rule ccontr)
        assume lset P ∩ A p ≠ {}
        have strategy-attracts p (override-on σ′ σ-attr (A p − deadends p**))
                                 (A p)
                                 (deadends p**)
          using strategy-attracts-irrelevant-override[OF σ-attr(2) σ-attr(1) ⟨strategy p σ′⟩]
          by blast
        moreover have override-on σ′ σ-attr (A p − deadends p**) = σ′
          by (rule ext, unfold σ′-def, cases p) (simp-all add: override-on-def)
        ultimately have strategy-attracts p σ′ (A p) (deadends p**) by simp
        hence lset P ∩ deadends p** ≠ {}
          using ⟨lset P ∩ A p ≠ {}⟩ attracted-path[OF deadends-in-V] by simp
        thus False using contra visits-deadend[of p**] by simp
      qed
    qed (insert ⟨v0 ∈ V′⟩)

    then interpret vmc-path G′ P v0 p σ′
      unfolding G′-def using subgame-path-vmc-path[OF ⟨V′ ⊆ V⟩] by blast
    have G′.path-conforms-with-strategy p P σ proof−
      have ⋀v. v ∈ lset P ⟹ σ′ v = σ v
        using σ′-is-σ-on-V′ ⟨V_G′ = V′⟩ lset-P-V by blast
      thus G′.path-conforms-with-strategy p P σ
        using P-conforms G′.path-conforms-with-strategy-irrelevant-updates by blast
    qed
    then interpret vmc-path G′ P v0 p σ using conforms-to-another-strategy by blast
    have G′.winning-path p P
      using σ(2)[unfolded G′.winning-strategy-def] vmc-path-axioms by blast
    from ⟨¬winning-path p P⟩
         G′.winning-path-supergame[OF this ParityGame-axioms, unfolded G′-def]
         subgame-VV-subset[of p** V′]
         subgame-ω[of V′]
      show False by blast
  qed
  ultimately show ?thesis unfolding winning-region-def using ⟨v0 ∈ V⟩ by blast
next
  assume v0 ∉ V′
  then obtain p where v0 ∈ attractor p (deadends p**)
    unfolding V′-def A-def using ⟨v0 ∈ V⟩ by blast
  thus ?thesis unfolding winning-region-def
    using lemma-path-to-deadend ⟨v0 ∈ V⟩ by blast
qed
```

qed

12.4 The Main Theorem: Positional Determinacy

Prove the main theorem: The winning regions of player EVEN and ODD are a partition of the set of nodes V.

theorem *partition-into-winning-regions*:
  shows V = *winning-region Even* ∪ *winning-region Odd*
    and *winning-region Even* ∩ *winning-region Odd* = {}
proof
  show V ⊆ *winning-region Even* ∪ *winning-region Odd*
    by (*rule subsetI*) (*metis* (*full-types*) *Un-iff other-other-player positional-strategy-exists*)
next
  show *winning-region Even* ∪ *winning-region Odd* ⊆ V
    by (*rule subsetI*) (*meson Un-iff subsetCE winning-region-in-V*)
next
  show *winning-region Even* ∩ *winning-region Odd* = {}
    using *winning-strategy-only-for-one-player*[*of Even*]
    unfolding *winning-region-def* **by** *auto*
qed

end — context ParityGame

end

13 Defining the Attractor with `inductive_set`

theory *AttractorInductive*
imports
  Main
  Attractor
begin

context *ParityGame* **begin**

In section 6 we defined *attractor* manually via *lfp*. We can also define it with `inductive_set`. In this section, we do exactly this and prove that the new definition yields the same set as the old definition.

13.1 *attractor-inductive*

The attractor set of a given set of nodes, defined inductively.

inductive-set *attractor-inductive* :: *Player* ⇒ *'a set* ⇒ *'a set*
  for *p* :: *Player* **and** *W* :: *'a set* **where**
  Base [*intro!*]: $v \in W \Longrightarrow v \in$ *attractor-inductive p W*
| *VVp*: ⟦ $v \in VV\ p$; $\exists w.\ v \rightarrow w \wedge w \in$ *attractor-inductive p W* ⟧
    $\Longrightarrow v \in$ *attractor-inductive p W*
| *VVpstar*: ⟦ $v \in VV\ p{*}{*}$; $\neg$*deadend* v; $\forall w.\ v \rightarrow w \longrightarrow w \in$ *attractor-inductive p W* ⟧
    $\Longrightarrow v \in$ *attractor-inductive p W*

We show that the inductive definition and the definition via least fixed point are the same.

lemma *attractor-inductive-is-attractor*:
  assumes $W \subseteq V$
  shows *attractor-inductive* p W = *attractor* p W
proof
  show *attractor-inductive* p $W \subseteq$ *attractor* p W **proof**
    fix v **assume** $v \in$ *attractor-inductive* p W
    thus $v \in$ *attractor* p W **proof** (*induct rule*: *attractor-inductive.induct*)
      case (*Base* v) **thus** *?case* **using** *attractor-set-base* **by** *auto*
    next
      case (*VVp* v) **thus** *?case* **using** *attractor-set-VVp* **by** *auto*
    next
      case (*VVpstar* v) **thus** *?case* **using** *attractor-set-VVpstar* **by** *auto*
    qed
  qed
  show *attractor* p $W \subseteq$ *attractor-inductive* p W **proof**−
    def $P \equiv \lambda S.\ S \subseteq$ *attractor-inductive* p W
    from ‹$W \subseteq V$› **have** P (*attractor* p W) **proof** (*induct rule*: *attractor-set-induction*)
      case (*step* S)
      hence $S \subseteq$ *attractor-inductive* p W **using** *P-def* **by** *simp*
      have $W \cup S \cup$ *directly-attracted* p $S \subseteq$ *attractor-inductive* p W **proof**
        fix v **assume** $v \in W \cup S \cup$ *directly-attracted* p S
        moreover
        { assume $v \in W$ **hence** $v \in$ *attractor-inductive* p W **by** *blast* **}**
        moreover
        { assume $v \in S$ **hence** $v \in$ *attractor-inductive* p W
            by (*meson* ‹$S \subseteq$ *attractor-inductive* p W› *set-rev-mp*) **}**
        moreover
        { assume *v-attracted*: $v \in$ *directly-attracted* p S
          hence $v \in V$ **using** ‹$S \subseteq V$› *attractor-step-bounded-by-V* **by** *blast*
          hence $v \in$ *attractor-inductive* p W **proof** (*cases rule*: *VV-cases*)
            assume $v \in VV\ p$
            hence $\exists w.\ v \rightarrow w \wedge w \in S$ **using** *v-attracted directly-attracted-def* **by** *blast*
            hence $\exists w.\ v \rightarrow w \wedge w \in$ *attractor-inductive* p W
              using ‹$S \subseteq$ *attractor-inductive* p W› **by** *blast*
            thus *?thesis* **by** (*simp add*: ‹$v \in VV\ p$› *attractor-inductive.VVp*)
          next
            assume $v \in VV\ p{*}{*}$
            hence $*$: $\forall w.\ v \rightarrow w \longrightarrow w \in S$ **using** *v-attracted directly-attracted-def* **by** *blast*
            have $\neg$*deadend* v **using** *v-attracted directly-attracted-def* **by** *blast*
            show *?thesis* **proof** (*rule ccontr*)
              assume $v \notin$ *attractor-inductive* p W
              hence $\exists w.\ v \rightarrow w \wedge w \notin$ *attractor-inductive* p W
                by (*metis attractor-inductive.VVpstar* ‹$v \in VV\ p{*}{*}$› ‹$\neg$*deadend* v›)
              hence $\exists w.\ v \rightarrow w \wedge w \notin S$ **using** ‹$S \subseteq$ *attractor-inductive* p W› **by** (*meson subsetCE*)
              thus *False* **using** $*$ **by** *blast*
            qed
          qed
        }
        ultimately show $v \in$ *attractor-inductive* p W **by** (*meson UnE*)
      qed

thus P ($W \cup S \cup$ *directly-attracted* p S) **using** *P-def* **by** *simp*
qed (*simp add*: *P-def Sup-least*)
thus *?thesis* **using** *P-def* **by** *simp*
qed
qed

end

end

14 Compatibility with the Graph Theory Package

theory *Graph-TheoryCompatibility*
imports
ParityGame
Graph-Theory.Digraph
Graph-Theory.Digraph-Isomorphism
begin

In this section, we show that our *Digraph* locale is compatible to the *nomulti-digraph* locale from the graph theory package from the Archive of Formal Proofs.

For this, we will define two functions converting between the different types and show that with these conversion functions the locales interpret each other. Together, this indicates that our definition of digraph is reasonable.

14.1 To Graph Theory

We can easily convert our graphs into *pre-digraph* objects.

definition *to-pre-digraph* :: ($'a$, $'b$) *Graph-scheme* $\Rightarrow$ ($'a$, $'a \times 'a$) *pre-digraph*
where *to-pre-digraph* $G \equiv$ (|
pre-digraph.verts = *Graph.verts* G,
pre-digraph.arcs = *Graph.arcs* G,
tail = *fst*,
head = *snd*
|)

With this conversion function, our *Digraph* locale contains the locale *nomulti-digraph* from the graph theory package.

context *Digraph* **begin**
interpretation *is-nomulti-digraph*: *nomulti-digraph to-pre-digraph* G **proof**
fix e **assume** $*$: $e \in$ *pre-digraph.arcs* (*to-pre-digraph* G)
show *tail* (*to-pre-digraph* G) $e \in$ *pre-digraph.verts* (*to-pre-digraph* G)
by (*metis* $*$ *edges-are-in-V(1) pre-digraph.ext-inject pre-digraph.surjective prod.collapse to-pre-digraph-def*)
show *head* (*to-pre-digraph* G) $e \in$ *pre-digraph.verts* (*to-pre-digraph* G)
by (*metis* $*$ *edges-are-in-V(2) pre-digraph.ext-inject pre-digraph.surjective prod.collapse to-pre-digraph-def*)
qed (*simp add*: *arc-to-ends-def to-pre-digraph-def*)
end

14.2 From Graph Theory

We can also convert in the other direction.

definition *from-pre-digraph* :: ($'a$, $'b$) *pre-digraph* $\Rightarrow$ $'a$ *Graph*
  where *from-pre-digraph G* $\equiv$ (|
    Graph.verts = *pre-digraph.verts G*,
    Graph.arcs = *arcs-ends G*
  |)

context *nomulti-digraph* **begin**
interpretation *is-Digraph*: *Digraph from-pre-digraph G* **proof**−
  {
    fix *v w* **assume** $(v,w) \in E_{\textit{from-pre-digraph } G}$
    then obtain *e* **where** *e*: $e \in$ *pre-digraph.arcs G tail G e* = *v head G e* = *w*
      unfolding *from-pre-digraph-def* **by** *auto*
    hence $(v,w) \in V_{\textit{from-pre-digraph } G} \times V_{\textit{from-pre-digraph } G}$
      unfolding *from-pre-digraph-def* **by** *auto*
  }
  thus *Digraph* (*from-pre-digraph G*) **by** (*simp add*: *Digraph.intro subrelI*)
qed
end

14.3 Isomorphisms

We also show that our conversion functions make sense. That is, we show that they are nearly inverses of each other. Unfortunately, *from-pre-digraph* irretrievably loses information about the arcs, and only keeps tail/head intact, so the best we can get for this case is that the back-and-forth converted graphs are isomorphic.

lemma *graph-conversion-bij*: *G* = *from-pre-digraph* (*to-pre-digraph G*)
  unfolding *to-pre-digraph-def from-pre-digraph-def arcs-ends-def arc-to-ends-def* **by** *auto*

lemma (**in** *nomulti-digraph*) *graph-conversion-bij2*: *digraph-iso G* (*to-pre-digraph* (*from-pre-digraph G*))
proof−
  def *iso* $\equiv$ (|
    iso-verts = *id* :: $'a \Rightarrow 'a$,
    iso-arcs = *arc-to-ends G*,
    iso-head = *snd*,
    iso-tail = *fst*
  |)

  have *inj-on* (*iso-verts iso*) (*pre-digraph.verts G*) **unfolding** *iso-def* **by** *auto*
  moreover have *inj-on* (*iso-arcs iso*) (*pre-digraph.arcs G*)
    unfolding *iso-def arc-to-ends-def* **by** (*simp add*: *arc-to-ends-def inj-onI no-multi-arcs*)
  moreover have $\forall a \in$ *pre-digraph.arcs G*.
    iso-verts iso (*tail G a*) = *iso-tail iso* (*iso-arcs iso a*)
    $\wedge$ *iso-verts iso* (*head G a*) = *iso-head iso* (*iso-arcs iso a*)
    unfolding *iso-def* **by** (*simp add*: *arc-to-ends-def*)

  ultimately have *digraph-isomorphism iso*

unfolding *digraph-isomorphism-def* **using** *arc-to-ends-def wf-digraph-axioms* **by** *blast*

moreover have *to-pre-digraph* (*from-pre-digraph G*) = *app-iso iso G*
unfolding *to-pre-digraph-def from-pre-digraph-def iso-def app-iso-def* **by** (*simp-all add*: *arcs-ends-def*)

ultimately show *?thesis* **unfolding** *digraph-iso-def* **by** *blast*
qed

end

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