MATHEMATICS

ON PADOA'S METHOD IN THE THEORY OF DEFINITION

BY

E. W. BETH

(Dedicated to Alfred Tarski)

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1. *Introduction.* In order to prove, for a certain postulate set $\mathfrak{a}$, the independence of a primitive notion a with respect to the remaining primitive notions $t_1, t_2, \ldots, t_k, \ldots$, it is often convenient to apply PADOA's method [1]. We exhibit two interpretations $A', T'_1, T'_2, \ldots, T'_k, \ldots$ and $A'', T''_1, T''_2, \ldots, T''_k, \ldots$ of the primitive notions under consideration, such that T'_1 coincides with T''_1, T'_2 with $T''_2, \ldots, T'_k$ with $T''_k, \ldots$, whereas A' is different from A''. We then argue that, if a could be defined in terms of $t_1, t_2, \ldots, t_k, \ldots$, then by interpreting its definition we would obtain similar definitions of A' in terms of $T'_1, T'_2, \ldots, T'_k, \ldots$ and of A'' in terms of $T'_1, T'_2, \ldots, T''_k, \ldots$; hence A' would coincide with A'', contrary to our supposition. PADOA's method is obviously a counter-part to the more familiar method of proving independence for postulates by exhibiting suitable models. With respect to the last-mentioned method it has been asked whether, if a certain postulate is independent of certain other postulates, we are always able to exhibit a model to demonstrate this situation. It is well-known that the answer depends on the choice of the underlying logical system; for classical elementary logic, for instance, we can give a positive answer on account of the theorem of LÖWENHEIM–SKOLEM–GÖDEL.

It seems reasonable to ask the same question with regard to PADOA's method. However, in this case we cannot expect to find an equally satisfactory result, even if we restrict ourselves to elementary logic. Suppose our postulate set $\mathfrak{a}$ to contain one single postulate, namely, $(x)\, a(x)$. Then, if besides a no other primitive notions are admitted, a is trivially independent; but as soon as any other primitive notion t is admitted, a becomes trivially definable in terms of t. This situation may be expected to give rise to certain complications in our analysis. But except for these trivial cases

[1] A. PADOA, Essai d'une théorie algébrique des nombres entiers, précédé d'une introduction logique à une théorie déductive quelconque. *Bibliothèque du Congrès Int. de Philos.* 3 (1900). Cf.:

A. TARSKI et A. LINDENBAUM, Sur l'indépendance des notions primitives dans les systèmes mathématiques. *Annales de la Soc. Polonaise de Math.* 5 (1926).

A. TARSKI, Einige methodologische Untersuchungen über die Definierbarkeit der Begriffe. *Erkenntnis* 5 (1935).

J. C. C. MCKINSEY, On the independence of undefined ideas. *Bull. Amer. Math. Soc.* 41 (1935).

we are able to show that, for elementary logic, our question can be answered in a positive sense; for elementary logic with identity, the answer is positive without any qualification [2]).

2. *Method.* We shall need a few lemmas which derive from an analysis of our proof of the theorem of LÖWENHEIM–SKOLEM–GÖDEL [3]). These lemmas are closely related to certain results by GENTZEN, in particular to his Subformula Theorem and his Extended Hauptsatz [4]); these results are usually proved on the basis of a peculiar formalisation of elementary logic, which also originates with GENTZEN and which has been extensively studied in recent years [5]). It is, perhaps, not without interest that our lemmas can be proved as naturally on the basis of the standard formalisation of elementary logic (it should, however, be noted that our methods of proof are by no means finitary, and that our argument does not extend to intuitionistic elementary logic; but this remains a matter for further investigation) [6]).

It will be useful to sum up the main points in our previous argument [3]).

2.1. Besides the closed formulas of elementary logic—in which $(Ex)X(x)$ is treated as an abbreviation for $\overline{(x)\overline{X(x)}}$—we introduce formulas obtained by replacing, in expressions of elementary logic, all free variables by numerals 1, 2, ..., p, ...; these formulas belong to *reduced logic*, which

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[2]) Two theorems, first announced by TARSKI in a summary ("Autoreferat"), were later proved and amply discussed in a congress report by the same author; both publications have been mentioned in footnote 1.

These theorems are related to the results in Section 4 of our present paper, but they refer to the proof-theory of the (revised) system of "*Principia Mathematica*", for which the situation is much simpler than for elementary logic.

On the basis of HENKIN's well-known results on completeness in the theory of types (*Journal of Symbolic Logic* **15**, (1950)) it is, of course, possible to derive from TARSKI's theorems the corresponding semantic results. On the other hand, it is also possible to derive the last-mentioned results from ours by TARSKI's method of relativising quantifiers.

The present author is indebted to TARSKI for pointing out the connections between his results and our present ones, and for offering valuable suggestions concerning the arrangement of our material; he also wishes to extend his thanks to H. B. CURRY and W. CRAIG, and to S. C. KLEENE, for reading a draft of this paper.

[3]) E. W. BETH, A topological proof of the theorem of LÖWENHEIM–SKOLEM–GÖDEL. These *Proceedings* **54**, (1951).

Some consequences of the theorem of LÖWENHEIM–SKOLEM–GÖDEL–MALCEV. These *Proceedings* **56** (1953).

[4]) G. GENTZEN, Untersuchungen über das logische Schliessen. *Math. Zeitschr.* **39** (1934).

[5]) H. B. CURRY, *A theory of formal deducibility* (Notre Dame, Ind., 1950).

S. C. KLEENE, *Introduction to metamathematics* (Amsterdam–Groningen, 1950).

Two papers on the predicate calculus. *Memoirs Amer. Math. Soc.* No. 10 (1952).

[6]) KURT SCHÜTTE, Schlussweisen-Kalküle der Prädikatenlogik, *Math. Ann.* **122** (1950), also establishes GENTZEN's results on the basis of the standard formalisation of elementary logic, but this method is completely different from the one which is applied in this paper.

is simply sentential logic in a different notation. The formulas of reduced logic are built from certain atomic expressions by means of the operators of sentential logic; hence formulas such as $(x)\,a(x)$, $a(5)$, $(x)[a(x) \rightarrow b(3)]$, $(x)[a(x) \vee (Ey)\,\overline{u(y, 17)}]$ are atomic expressions of reduced logic. It will be clear that every atom of reduced logic is either of form $(x)X(x)$ or of form $f(p_1, p_2, \ldots, p_n)$, where f is any n-ary predicate parameter and $p_1, p_2, \ldots, p_n$ are any numerals.

2.2. We introduce *valuations* w, which (i) assign to each formula X of reduced logic a truth value $w(X) = u$ or v, such that (ii) $w(\overline{X}) \neq w(X)$, (iii) $w(X \vee Y) = v$, if and only if $w(X) = w(Y) = v$, and (iv) $w(X \rightarrow Y) = w(\overline{X} \vee Y)$, $w(X \,\&\, Y) \neq w(\overline{X} \vee \overline{Y})$, $w(X \leftrightarrow Y) = w[(X \rightarrow Y) \,\&\, (Y \rightarrow X)]$. A formula X is called an *identity* (of reduced logic), if $w(X) = u$ for every valuation w.

2.3. A valuation w is called *normal*, if it satisfies the additional condition: (v) $w[(x)X(x)] = u$, if and only if $w[X(1)] = w[X(2)] = \ldots = w[X(p)] = \ldots = u$.

We introduce an enumeration:

$$X_1(x_1),\ X_2(x_2),\ \ldots,\ X_p(x_p),\ \ldots$$

of all expressions $X_k(x_k)$ which are obtained from expressions Y_k of elementary logic when all free variables but one—the variable x_k—are either bound by means of quantifiers or replaced by numerals. Then by taking, for each p, all expressions $(x_1)X_p(x_1)$, $(x_2)X_p(x_2)$, $\ldots$, $(x_m)X_p(x_m)$, $\ldots$, we obtain a complete enumeration of all atoms of reduced logic of the first kind, as mentioned under 2.1 (it does not matter that this enumeration contains repetitions, or that sometimes the expression $(x_m)X_p(x_m)$ does not exist). On the basis of this enumeration, a certain function $s(p)$ is introduced, the peculiar properties of which play an important rôle in our argument [7]).

2.4. Now let T be any closed formula of elementary logic. Then the following conditions are proved to be equivalent:

(i) There is no model for $\overline{T}$;

(ii) For every normal valuation w, $w(T) = u$;

(iii) For a certain index M, the formula $Z =$

$$\underset{m,p,r \leqq M}{\mathsf{D}} [(x_m)X_p(x_m) \,\&\, \overline{X_p(r)}] \vee \underset{n,q \leqq M}{\mathsf{D}} [X_q(s(q)) \,\&\, \overline{(x_n)X_q(y_n)}] \vee T$$

is an identity of reduced logic [8]);

[7]) Leon Henkin, Some notes on nominalism. *Journal of Symbolic Logic* 18 (1953).

G. Hasenjaeger, Eine Bemerkung zu Henkin's Beweis für die Volständigkeit des Prädikatenkalküls der ersten Stufe. *Ibid.*

From footnote 3 in the last-mentioned article it appears that both Henkin and Hasenjaeger simplified the proofs of the completeness of elementary logic by introducing additional axioms:

$$(Ex_q)X_q(x_q) \rightarrow X_q(s(q)),$$

(in our notation); this method is essentially equivalent to the device which we introduced in our communication mentioned above in footnote 3.

[8]) In metamathematics, the symbol '=' is used as a rule only to denote typo-

(iv) T is a theorem of elementary logic.

To a certain extent, the formula Z corresponds to GENTZEN's midsequent: up to Z, the derivation of T is carried out by means of reduced logic (which is sentential logic in a different notation), whereas from Z onward only quantification theory is applied; in fact, the formula Z provides, so to speak, a lay-out for a derivation of T within elementary logic. There remains, however, a difference: the expression Z has not, automatically, the subformula property (or, better, the property which, in our situation, corresponds to GENTZEN's subformula property).

3. *Statement and proof of lemmas.* It is nevertheless possible to make the analogy complete; instead of the set of all normal valuations, we must only consider a somewhat larger set.

3.1. Let U be any expression of reduced logic; then we call *atoms of* U, those atomic expressions of reduced logic from which U can be constructed by means of operators of sentential logic.

3.2. Let $(x_m)X_p(x_m)$ be an atom of the first kind, as mentioned under 2.1; then we call *constituents of* $(x_m)X_p(x_m)$, the expressions $X_p(1)$, $X_p(2)$, $\ldots$, $X_p(r)$, $\ldots$.

3.3. Let K be any set of expressions of reduced logic. Then K will be called a *CA-set*, if K is closed with respect to taking constituents and atoms. For any set K, $CA(K)$ will be the intersection of all CA-sets K' such that $K \subseteq K'$.

3.4. We have $CA(K) = K$, if and only if K is a CA-set. More generally, CA has the formal properties of a *closure operation* [9]: (i) $K \subseteq CA(K)$, (ii) $CA(CA(K)) \subseteq CA(K)$, (iii) if $K \subseteq K'$, then $CA(K) \subseteq CA(K')$.

3.5. Let U be any expression of reduced logic. Then the expressions in $CA(\{U\})$ will be called *subformulas of* U.

3.6. Let $A = (x_m)X_p(x_m)$ be an atom of reduced logic and let w be any valuation. Then w will be called *A-normal*, if w satisfies the condition: (v^a) $w[(x_m)X_p(x_m)] = u$, if and only if $w[X_p(1)] = w[X_p(2)] = \ldots = w[X_p(r)] = \ldots = u$.

3.7. Let K be any set of expressions of reduced logic and let w be any valuation. Then we will be called *K-normal*, if w is A-normal for every atom A in $CA(K)$.

3.8. Let K be any set of expressions of reduced logic and let w be any K-normal valuation. Then we can find a normal valuation w^* such that, for every expression U in $CA(K)$, we have $w^*(U) = w(U)$.

Proof. Let us take $w^*(A) = w(A)$ for each atom of the second kind, as mentioned under 2.1. Then on account of conditions (ii)–(v), as stated in 2.2 and 2.3, a normal valuation w^* is uniquely defined; and it will be clear that we have $w^*(U) = w(U)$ for every expression U in $CA(K)$.

graphical similarity of formal expressions; but in the present paper it is also used to denote equivalence with respect to reduced logic.

[9]) E. H. MOORE, *Introduction to a form of general analysis* (New Haven, 1910).

3.9. Let T be any closed formula of elementary logic. Then the following conditions are equivalent to each other and to the conditions (i)–(iv) as stated in 2.4:

(v) For every $\{T\}$-normal valuation w, $w(T) = u$;

(vi) For a certain index M, there is a formula Z, as described in 2.4 under (iii), which is an identity of reduced logic, and in which, moreover, all expressions $(x_m)X_p(x_m)$ and $(x_n)X_q(x_n)$ belong to $CA(\{T\})$.

Proof. It is trivial that (ii) follows from (v) and (iii) from (vi).

(I) Suppose that (v) does not hold, and let w be any $\{T\}$-normal valuation such that $w(T) = v$. We apply 3.8 with $K = \{T\}$; then we find a normal valuation w^* such that $w^*(T) = w(T) = v$. So (ii) does not hold either.

(II) Suppose we have (iii). Then we have (ii) by 2.4 and hence (v) by (I). By exactly the same argument as in our previous paper [3]), we obtain (vi).

3.10. For every theorem T of elementary logic, there is a formula Z as described in 2.4 under (iii), which is an identity of reduced logic and in which, moreover, all atoms are subformulas of T. The derivation of T as a theorem of elementary logic can be carried out in two successive steps: first, Z is established by sentential logic, and then only quantification theory is applied [10]).

Our statement is slightly more general than GENTZEN's, as we do not suppose T to be prenex.

GENTZEN's theorem is usually interpreted as stating the possibility, in elementary logic, of carrying out every proof without any detours. It seems that his interpretation is fully corroborated by our analysis, in particular with respect to quantification theory. A theorem T of elementary logic is an identity with regard to normal valuations, but it is not necessarily an identity of reduced logic. The fact that we may have $w(T) = v$ for certain (non-normal) valuations makes it necessary, after having derived the formula Z within sentential logic, to resort to quantification theory. On the other hand, if $w(T) = v$ for a theorem T of elementary logic, then w can never be a normal nor even a $\{T\}$-normal valuation. The additional clauses which appear in Z are connected with such valuations. If $w(T) = v$, then we must either have $w[(x_m)X_p(x_m)\ \&$ $\&\ \overline{X_p(r)}] = u$ or $w[X_q(s(q))\ \&\ \overline{(x_n)X_q(x_n)}] = u$ for some subformula $(x_m)X_p(x_m)$ or $(x_n)X_q(x_n)$ of T; if for a certain clause of this kind we do not find a corresponding valuation w with $w(T) = v$, then this clause is superfluous and its insertion into Z represents a detour in the derivation of T.

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[10]) In the present paper, the application of quantification theory is not, as a rule, explicitly described. But in 4.10 an example is given of the manner in which the applications of quantification theory can be carried out. This is merely a particular case of a general method developed in our previous paper.

4. *A proof-theoretic result concerning definability.* We introduce the following definition.

4.1. Let $\mathfrak{a}$ be any set of closed expressions of elementary logic, containing predicate parameters a (which we suppose to be k-ary), $t_1, t_2, \dots, t_l, \dots$. Then a will be said to be *definable* with respect to $\mathfrak{a}$ and in terms of $t_1, t_2, \dots, t_l, \dots$, if there is an expression $U(x_1, x_2, \dots, x_k)$ containing no predicate parameters except $t_1, t_2, \dots, t_l, \dots$, and in which the free variables $x_1, x_2, \dots, x_k$ and no other ones appear, such that

$$(x_1)(x_2) \dots (x_k)\ [a(x_1, x_2, \dots, x_k) \leftrightarrow U(x_1, x_2, \dots, x_k)]$$

is derivable from $\mathfrak{a}$.

Then our main proof-theoretic result can be stated as follows.

4.2. Let $\mathfrak{a}$ be any set of closed expressions as described under 4.1. For a to be defiable with respect to $\mathfrak{a}$ and in terms of $t_1, t_2, \dots, t_l, \dots$, it is necessary and sufficient that the expression $C =$

$$(x_1)(x_2) \dots (x_k)\ [a(x_1, x_2, \dots, x_k) \leftrightarrow b(x_1, x_2, \dots, x_k)]$$

is derivable from the union $\mathfrak{a} \cup \mathfrak{b}$ of the set $\mathfrak{a}$ with the set $\mathfrak{b}$ of expressions obtained by replacing every occurrence of a in $\mathfrak{a}$ by a k-ary predicate parameter b not occurring in $\mathfrak{a}$.

4.3. It is obvious that the derivability of C is a necessary condition, but we shall have to prove that it is also sufficient. So suppose that C is derivable from $\mathfrak{a} \cup \mathfrak{b}$. Then C must be derivable from some set $\mathfrak{a}' \cup \mathfrak{b}'$ where $\mathfrak{a}'$ and $\mathfrak{b}'$ are finite subsets of $\mathfrak{a}$ and $\mathfrak{b}$ respectively. We can obviously suppose $\mathfrak{a}'$ and $\mathfrak{b}'$ to be symmetric with regard to a and b, and then replace $\mathfrak{a}'$ and $\mathfrak{b}'$ by symmetric axioms A and B. By the deduction theorem for elementary logic, $\overline{A} \vee \overline{B} \vee C$ is a theorem of elementary logic.

4.4. We are now in a position to apply the results of Section 3. For the theorem $T = \overline{A} \vee \overline{B} \vee C$ of elementary logic there must be a corresponding theorem Z of reduced logic; we write $Z = V \vee W \vee X \vee \overline{A} \vee \overline{B} \vee C$, where V is the disjunction of all additional clauses corresponding to subformulas of A, W the disjunction of all those corresponding to subformulas of B, and X the disjunction of all those corresponding to subformulas of C. We suppose, of course, V and W to be symmetric in a and b.

4.5. The expression X must be studied somewhat more closely. It will be convenient to suppose that the enumeration in 2.3 has been carried out in such a manner that $(x_1)X_1(x_1) = C$ (hence $s(1) = 1$), $(x_2)X_2(x_2) = X_1(s(1)) = X_1(1)$ (hence $s(2) = 2$), $\dots$, $(x_k)X_k(x_k) = X_{k-1}(s(k-1)) = X_{k-1}(k-1)$ (hence $s(k) = k$); then we have: $X_k(s(k)) = X_k(k) =$

$$a(1, 2, \dots, k) \leftrightarrow b(1, 2, \dots, k)$$

For the sake of brevity, the last-mentioned expression is written as $a_0 \leftrightarrow b_0$. Now the expression X can be written as follows:

$$[X_1(1) \;\&\; \underset{(=\overline{C})}{\overline{(x_1)X_1(x_1)}}] \vee [X_2(2) \;\&\; \underset{(=\overline{X_1(1)})}{\overline{(x_2)X_2(x_2)}}] \vee \dots \vee [\underset{(=a_0 \leftrightarrow b_0)}{X_k(k)} \;\&\; \underset{(=\overline{X_{k-1}(k-1)})}{\overline{(x_k)X_k(x_k)}}]$$

and the formula Z takes the form:

$$V \vee W \vee [X_1(1) \,\&\, \overline{C}] \vee [X_2(2) \,\&\, \overline{X_1(1)}] \vee \ldots \vee [(a_0 \leftrightarrow b_0) \,\&\, \overline{X_{k-1}(k-1)}] \vee \overline{A} \vee \overline{B} \vee C$$

Indeed, it will be clear that X must contain the clauses which have been written down. On the other hand, we have not written out the clauses $[(y_t)Y_t(y_t) \,\&\, \overline{Y_t(r)}]$ which Z may contain, where $1 \leqq t \leqq k$ and $(y_t)Y_t(y_t)$ is like $(x_t)X_t(x_t)$ except for the bound variables and numerals which it contains. As a matter of fact, these clauses will vanish automatically when we carry out the substitution described under 4.6, provided $(y_t)Y_t(y_t)$ is treated the same way as the corresponding formula $(x_t)X_t(x_t)$.

4.6. As Z is an identity of reduced logic, we can apply the rule of substitution and the modus ponens in order to obtain a somewhat simpler identity. Indeed, if for each of the atoms C, $X_1(1)$, $X_2(2)$, ..., $X_{k-1}(k-1)$ we substitute some contradiction of reduced logic and apply the modus ponens, we obtain $Z_1 =$

$$V \vee W \vee (a_0 \leftrightarrow b_0) \vee \overline{A} \vee \overline{B},$$

which therefore must be a theorem of reduced logic.

On the basis of this formula, we can rearrange our original derivation of C starting from A & B, so as to carry it out in the following steps:

(i) From A we derive $M = A \,\&\, \overline{V}$, from B we derive $N = B \,\&\, \overline{W}$ by means of quantification theory.

(ii) $Z_1 = (M \,\&\, N) \rightarrow (a_0 \leftrightarrow b_0)$ is a theorem of reduced logic.

(iii) Hence $a_0 \leftrightarrow b_0$ can be derived from M & N by means of sentential logic; this part of the derivation will be further analysed in 4.7–4.9.

(iv) From $a_0 \leftrightarrow b_0$ we derive C by means of quantification theory.

Schematically:

$$\begin{array}{lcl} A\text{----------}M & & \\ & M \,\&\, N\text{----------}a_0 \leftrightarrow b_0\text{----------}C & \\ B\text{----------}N & & \\ \textit{quant. th.} & \textit{sent. log.} & \textit{quant. th.} \end{array}$$

4.7. In the formula $Z_1 = (M \,\&\, N) \rightarrow (a_0 \leftrightarrow b_0)$, M does not contain the predicate b, N does not contain the predicate a; M and N are symmetric in a and b.

Let us write $M = (M_1 \vee a_0) \,\&\, (M_2 \vee \bar{a}_0)$, $N = (N_1 \vee b_0) \,\&\, (N_2 \vee \overline{b}_0)$, where M_1 and M_2 contain neither a_0 nor the predicate b, where N_1 and N_2 contain neither b_0 nor the predicate a, and where M_1 and N_1, M_2 and N_2 are symmetric with regard to a and b.

Then Z_1 can be written as:

$$[(M_1 \vee a_0) \,\&\, (M_2 \vee \bar{a}_0) \,\&\, (N_1 \vee b_0) \,\&\, (N_2 \vee \overline{b}_0)] \rightarrow (a_0 \leftrightarrow b_0)$$

As Z_1 is an identity of reduced logic, it follows by truth table methods that both $\overline{M}_1 \vee \overline{N}_2$ and $\overline{M}_2 \vee \overline{N}_1$ are identities of reduced logic.

These observations give rise to the following conception of part (iii) of our derivation as described under 4.3.

(iii^a) From M, it follows that $a_0 \to M_2$; $M_2 \to \overline{N}_1$ is an identity of reduced logic; from N, it follows that $\overline{N}_1 \to b_0$; hence $a_0 \to b_0$.

(iii^b) From N, it follows that $b_0 \to N_2$; $N_2 \to \overline{M}_1$ is an identity of reduced logic; from M, it follows that $\overline{M}_1 \to a_0$; hence $b_0 \to a_0$.

(iii^c) From the results of (iii^a) and (iii^b), it follows that $a_0 \leftrightarrow b_0$.

It should be noted that, in (iii^a) and (iii^b), the various steps in the argument are fully symmetric in a and b.

4.8. Now the only trouble with this derivation is, that the formulas M_1, M_2, N_1, N_2 still contain the predicates a and b. It will be necessary to replace the derivation by one in which these predicates no longer appear; this is relatively easy.

Let us suppose that M_1 and M_2 contain, besides other atoms in which no predicate a occurs, the atoms $a_1, a_2, \ldots, a_t$ in which this predicate appears. On account of the symmetry, N_1 and N_2 must contain symmetric atoms $b_1, b_2, \ldots, b_t$. We will show how to eliminate a_1 and b_1 from our argument (iii^a)–(iii^c) without introducing any new atoms or destroying symmetry.

4.9. Let us write:

$$M_2 = (M_2' \vee a_1) \,\&\, (M_2'' \vee \bar{a}_1), \qquad N_1 = (N_1' \vee b_1) \,\&\, (N_1'' \vee \overline{b}_1),$$

hence:

$$\overline{N}_1 = (\overline{N}_1' \,\&\, \overline{b}_1) \vee (\overline{N}_1'' \,\&\, b_1)$$

(iii^a) It follows from M that we have $a_0 \to M_2$, hence $a_0 \to (M_2' \vee a_1)$ and $a_0 \to (M_2'' \vee \bar{a}_1)$, and therefore $a_0 \to (M_2' \vee M_2'')$.

As $M_2 \to \overline{N}_1$ is an identity of reduced logic, so is

$$[(M_2' \vee a_1) \,\&\, (M_2'' \vee \bar{a}_1)] \to [(\overline{N}_1' \,\&\, \overline{b}_1) \vee (\overline{N}_1'' \,\&\, b_1)]$$

We substitute M_2'' for a_1 and N_1'' for b_1 and obtain the identity:

$$(M_2' \vee M_2'') \to (\overline{N}_1' \,\&\, \overline{N}_1'')$$

It follows from N that we have $\overline{N}_1 \to b_0$, hence $(\overline{N}_1' \,\&\, \overline{N}_1'') \to b_0$.

So (iii^a) can be replaced by a completely similar argument in which the rôles of M_2 and N_1 are taken over by $M_2' \vee M_2''$ and $N_1' \vee N_1''$ respectively, these last-mentioned formulas no longer containing a_1 or b_1. Of course, (iii^b) can be treated symmetrically, while (iii^c) remains unchanged. It will be clear that by and by the atoms $a_2, \ldots, a_t$, $b_2, \ldots, b_t$ can be eliminated so as to finally replace, in (iii^a)–(iii^c), the expressions M_1, M_2, N_1, N_2 by expressions M_1^*, M_2^*, N_1^*, N_2^*, which no longer contain the predicates a and b. However, we then must have, on account of the symmetry: $M_1^* = N_1^*$, $M_2^* = N_2^*$. Accordingly, our argument now takes a much simpler form:

(iii^{a+}) From M, it follows that $a_0 \to M_2^*$ and that $\overline{M}_1^* \to a_0$, hence that $\overline{M}_1^* \to M_2^*$; moreover, $M_2^* \to \overline{M}_1^*$ is an identity of reduced logic; hence it follows from M, that $a_0 \leftrightarrow M_2^*$. Etc.

4.10. In this form, the argument provides at once a suitable definition for the predicate a. For in $a_0 \leftrightarrow M_2^*$ we may replace every index j by a suitable free variable y_j. We then obtain a formula:

$$a(y_1, y_2, \ldots, y_k) \leftrightarrow M_2^{**}$$

which is derivable from A by elementary logic [3]). It may occur that M_2^{**} does not contain the same free variables as $a(y_1, y_2, \ldots, y_k)$, as is required by the theory of definition. But new free variables can be brought in by inserting trivial clauses, and redundant free variables may be eliminated by means of quantification. As a result, we obtain a definition:

$$a(y_1, y_2, \ldots, y_k) \leftrightarrow M_2^{**}(y_1, y_2, \ldots, y_k)$$

of a in terms of the remaining primitive notions in $\mathfrak{a}$.

5. *Trivial cases.* It is now easy to see where the trivial cases, mentioned in Section 1, come in. It may happen—and, if $\mathfrak{a}$ does not contain any other primitive notions besides a, it must happen—that, in eliminating the atoms $a_1, a_2, \ldots, a_k$, $b_1, b_2, \ldots, b_t$, we obtain expressions $M_1^{(f)}$, $M_2^{(f)}$, $N_1^{(f)}$, $N_2^{(f)}$ in which, besides a_f and b_f, no other atoms occur. As $M_2^{(f)} \rightarrow \overline{N_1^{(f)}}$ is a logical identity, we must have either $M_2^{(f)} = a_f \;\&\; \bar{a}_f$ or $\overline{N_1^{(f)}} = b_f \vee \overline{b_f}$. In the first case, it follows from $\mathfrak{a}$ that

$$\text{(I)} \qquad (y_1)(y_2) \ldots (y_k)\, \overline{a(y_1, y_2, \ldots, y_k)}$$

in the second case, it follows from $\mathfrak{a}$ that

$$\text{(II)} \qquad (y_1)(y_2) \ldots (y_k)\, a(y_1, y_2, \ldots, y_k)$$

If $\mathfrak{a}$ contains any other primitive notions besides a, then even in these trivial cases it is possible to find a suitable definition for the primitive notion a.

So the only case in which we do not find a suitable definition for a occurs, if $\mathfrak{a}$ enables us to prove (I) or (II), while $\mathfrak{a}$ does not contain any other primitive notions besides a.

If we replace elementary logic without identity by elementary logic with identity, then even in the last-mentioned trivial cases a suitable definition for a can be found.

6. *Semantic consequences.* We now return to the problem discussed in Section 1: suppose that, for a set $\mathfrak{a}$, the predicate parameter a is independent with respect to the remaining predicate parameters $t_1, t_2, \ldots, t_l, \ldots$ (or, using the terminology of Section 4: suppose that a is not definable with respect to $\mathfrak{a}$ and in terms of $t_1, t_2, \ldots, t_l, \ldots$); is it, apart from trivial cases, always possible to exhibit interpretations to demonstrate this situation?

It has been observed that this statement of our problem remains vague unless we define the notion of an *interpretation*. Therefore, let us agree that we have an interpretation whenever a model $[S, A, T_1, T_2, \ldots, T_l, \ldots]$

of $\mathfrak{a}$ is given, where S is the range of the individual variables and where $A, T_1, T_2, \ldots, T_l, \ldots$ are the values assigned to the predicate parameters a, $t_1, t_2, \ldots, t_l, \ldots$, respectively.

It may be asked whether perhaps it would be possible to admit interpretations of another kind, not connected with models of $\mathfrak{a}$; but this question turns out to be immaterial, as it proves sufficient to take into account interpretations in the above-mentioned sense. Indeed, we have the following theorem.

6.1. For a not to be definable with respect to $\mathfrak{a}$ and in terms of $t_1, t_2, \ldots$. $t_l, \ldots$, it is necessary and sufficient that the set $\mathfrak{a} \cup \mathfrak{b} \cup \{\bar{C}\}$ has a model,

(i) Suppose a to be definable. Then, on account of Theorem 4.2, the set $\mathfrak{a} \cup \mathfrak{b} \cup \{\bar{C}\}$ is inconsistent. It follows that the last-mentioned set cannot have a model.

(ii) Suppose a not to be definable. Then it follows from Theorem 4.2 that the set $\mathfrak{a} \cup \mathfrak{b} \cup \{\bar{C}\}$ is consistent, and hence, on account of the theorem of LÖWENHEIM–SKOLEM–GÖDEL, this set must have a model.

Let $[S, A, B, T_1, T_2, \ldots, T_l, \ldots]$ be this model, where $A, B, T_1, T_2, \ldots, T_l, \ldots$ are the values assigned to the predicate parameters $a, b, t_1, t_2, \ldots, t_l, \ldots$, respectively. It will be clear that, by taking $A' = A$, $A'' = B$, $T_1' = T_1'' = T_1$, $T_2' = T_2'' = T_2, \ldots, T_l' = T_l'' = T_l, \ldots$, we obtain interpretations to demonstrate the independence of a, as discussed in Section 1.—The trivial cases have been neglected, of course.

7. *Finitary validity of results.* The results of Sections 3–5 can be established by the methods of finitary proof theory. It will be clear, for instance, that, if we are given a derivation of the formula C, we are able, on account of Theorem 4.2, effectively to rearrange this derivation in such a manner as to obtain, either a suitable definition of the notion a, or a proof of (I) or (II); in the case of elementary logic with identity, we always obtain a suitable definition of the notion a. However, we hope to return later to this the general problem of the finitistic validity of the methods applied in paper.

8. *Application.* As an application we consider Peano's arithmetic [11]), in which addition and multiplication are taken as primitive motions. As is well known, the theory of natural numbers in which addition is the only primitive notion is decidable, while Peano's arithmetic is undecidable. Hence multiplication is not definable in terms of addition. It follows from our present result that two models of Peano's arithmetic can be found such that the corresponding sets of "numbers" and the corresponding operations of addition coincide, whereas the operations of multiplication in the two models are different.

Instituut voor Grondslagenonderzoek en Philosophie der Exacte Wetenschappen, Universiteit van Amsterdam.

[11]) For a description of this theory, cf. A. TARSKI, A. MOSTOWSKI and R. M. ROBINSON, *Undecidable Theories*, p. 31 (Amsterdam, 1953). For the application described in Section 8, we are indebted to TARSKI.