+/- BVASS

July 21, 2016

Filip: Hey, what's up?

Ranko: Nothing.

Sławek: Hey, you know Pac-Man?

Lorenzo: I know of him.

1 Hill cut VASS

For 1-dimensional VASS the hill cutting procedure is (in short) as follows. For simplicity assume that we have only +-1 or 0 transitions. Consider the shortest run R witnessing that $(q, 0)$ is reachable from $(p, 0)$. Figure 1 shows the graph of R, where the axis x goes through consecutive configurations in R and the axis y shows their value. Each node has a state and a value. The node c is the node with the biggest value, denote it n. The nodes l_i, r_i are consecutive "nodes with the new smallest value" on the left and on the right of c. Then the values of l_i and r_i are equal to $n - i$ for all i. If n is too big then there are two indexes i and j, where $i < j$, such that l_i and l_j have the same states and r_i and r_j have the same states. Then it is easy to see that we can remove the fragments between l_j and l_i, and between r_i and r_j obtaining a new shorter run (the hill drops down a bit, and l_i and r_i were defined so that nothing bad happens). Therefore, if R is the smallest run, then n is polynomial. This means that R is also polynomial because we can eliminate repeating configurations.

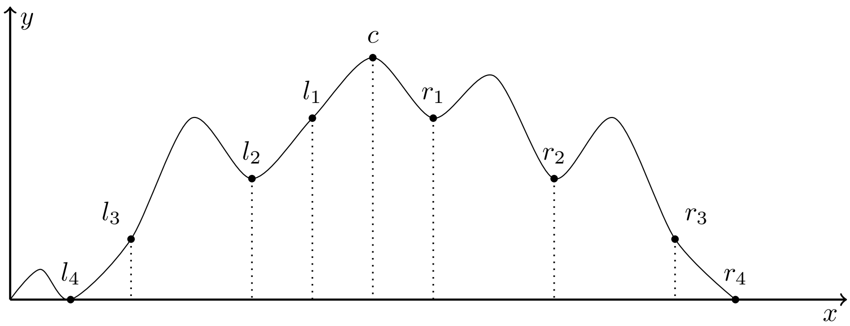

Figure 1: Hill cutting picture

2 Preliminaries

Let $B = (Q, \{q_f\}, \Delta)$ be a 1-BVASS with addition and subtraction, where Q is the set of states, q_f is the final state (in the leaves), $\Delta \subseteq Q^3$ is the transition relation. The intended meaning is that for every $(q, q_l, q_r) \in \Delta$: q is the parent, q_l, q_r are children. For simplicity we assume that q_f is never a parent, i.e., $q \neq q_f$. The set of states is partitioned into three sets: $Q = Q^+ \uplus Q^- \uplus \{q_f\}$.

We consider binary ordered trees such that nodes are labeled with pairs (q, n), where $q \in Q$ and $n \in \mathbb{Z}$. We shall refer to them as the state label and integer label. If a node v has two children we shall refer to them as the left child and right child. For every node v in t we write $t.v$ to denote the subtree of t with v in the root. We say that a tree is a transition tree if:

- if a node v is a leaf then its label is $(q_f, 1)$;
- if a node v labeled with (q, n) is internal then it has two children. Let us denote them v_l, v_r, respectively left and right child, labeled $(q_l, n_l), (q_r, n_r)$ then additionally: $q \neq q_f$; $(q, q_l, q_r) \in \Delta$; and $n = n_l + n_r$ if $q \in Q^+$ or $n = n_l - n_r$ if $q \in Q^-$.

We say that a transition tree t is a *witnessing tree* if for every node its integer label is a natural number $n \geq 0$. The reachability problem for given $(q_0, 0)$ is whether there exists a witnessing tree with the root labeled with $(q_0, 0)$.

3 How to do hill cutting on trees?

Let t be a witnessing tree and let $p = p_1 \dots p_k$ be a root-to-leaf path (p_1 is the root, p_k is the leaf), where (q_i, n_i) are corresponding labels. We define the sequence of nodes $p_{i_1} \dots p_{i_k}$ as follows:

- $i_1 = 1$;
- $i_s < i_{s+1}$;
- $n_{i_{s+1}} < n_{i_s}$ and $n_{i_s} \leq n_i$ for every $i_s < i < i_{s+1}$.

In other words p_{i_s} are 'new best minimums' on the path from the root to the leaf p. Comparing to Figure 1 the nodes p_{i_s} correspond to r_i, and the root of t corresponds to c (so t is usually a subtree of a bigger witnessing tree). Notice that the integer label of p_{i_k} is at most 1, since the integer label of the leaves in t is 1.

We define the set of nodes in this sequence $R_p = \{p_{i_1} \dots p_{i_k}\}$ and the the set of all such points is $R_t = \bigcup_p R_p$, where p varies over all paths in t. Notice that by definition if $v \in R_t$ then $v \in R_p$ for every path p that goes through v. For a downward path $p = p_1 \dots p_n$ in t (not necessarily root-to-leaf) we define $R(p) = p_{i_1} \dots p_{i_s}$, which is the subsequence of p of nodes that are also in R_t. Let $n_{i_1}, \dots, n_{i_s}$ be their corresponding integer labels. Notice that this is a decreasing sequence, i.e., $n_{i_k} > n_{i_{k+1}}$. We define $drop_R(p) = \max_k\{n_{i_k} - n_{i_{k+1}}\}$ ($drop_R(p)$ will be useful in the next section).

Let t be a witnessing tree and let v be a node in t. Fix a root-to-leaf path $p = p_1 \dots p_k$ in t that goes through v. Let j be the index such that $p_j = v$. We define the sequence of nodes $p_{i_1} \dots p_{i_k}$ as follows:

- $i_1 = j$;
- $i_s > i_{s+1}$;
- $n_{i_{s+1}} < n_{i_s}$ and $n_{i_s} \leq n_i$ for every $i_s > i > i_{s+1}$.

In other words p_{i_s} are 'new best minimums' from v to the root of t. Comparing to Figure 1 the nodes p_{i_s} correspond to l_i, and v corresponds to c. We define the set of nodes in this sequence $L_p = \{p_{i_1} \dots p_{i_k}\}$. Consider a downward path $p = p_1 \dots p_n$ in t (not necessarily root-to-leaf). It is a root-to-leaf path in $t.p_1$. We define $L(p) = p_{i_1} \dots p_{i_s}$, which is the subsequence of p of nodes that are also in L_p defined on $t.p_1$. Let $n_{i_1}, \dots, n_{i_s}$ be their corresponding integer labels. This is an increasing sequence, i.e., $n_{i_k} < n_{i_{k+1}}$. We define $drop_L(p) = \max_k\{n_{i_{k+1}} - n_{i_k}\}$ (this will be useful in the next section).

For a witnessing tree t we define its positive part t^+ obtained from t by removing some subtrees. For every internal node v we do the following. Suppose (q, n) is the label of v. Let w_l, w_r be its left and right child, and let $(q_l, n_l), (q_r, n_r)$ be their labels. Then:

- if $q \in Q^-$ we remove $t.w_r$ (the subtrahend);
- if $q \in Q^+$ then if $w_l, w_r \in R_t$ then we do not remove anything. Otherwise if $n_l \geq n_r$ then we remove $t.w_r$, and if $n_l < n_r$ then we remove $t.w_l$.

Notice that we remove at most one subtree for internal nodes, therefore, a leaf in t^+ is a leaf in t. If a node v from t is not removed in t^+ we say that v is a *positive node.*

Claim 1. *Let $v \in R_t$ be a positive node. Then $(t.v)^+ = t^+.v$.*

Proof. □

Lemma 1. *Let p be a root-to-leaf-path in t such that p is also in t^+. Let v be one of the nodes nodes in p. Let p_v be the fragment of p from v to the leaf. Consider the sets R_{p_v} (from v to the leaf) and L_p (from v to the root). Suppose there are nodes $l, l' \in L_p$ and $r, r' \in R_{p_v}$ such that: the labels of l and r are the same, and the labels of l' and r' are the same. Let t' be the tree obtained from t by: removing the fragment between l and l'; removing the fragment between r and r'; and updating the values between l and r. Then t' is a witnessing tree, with the same label in the root as t.*

Proof. In this case one should think that p is like a run in VASS and the sets L_p, R_{p_v} have the nodes that can be safely cut (see Figure 2). The two cuts sum to 0 from the roots perspective (because we cut from t^+). The only scary thing is whether a node with state label $q \in Q^-$ suddenly does not go below 0, but the sets L_p, R_{p_v} are defined to prevent this. □

Lemma 2. *Suppose that v is the root of t such that its state label is $q \in Q^-$. Let w_l, w_r be its left and right child. We denote $t_l = t.w_l$, $t_r = t.w_r$. Let p_l be a root-to-leaf path in t_l^+ and let p_r be a root-to-leaf path in t_r^+. Consider the sets R_{p_1} (from w_l to the leaf), R_{p_2} (from w_r to another leaf). Suppose there are nodes r_1, r_1' and r_2, r_2' such that the labels of r_1 and r_2 are the same, and the labels of r_1' and r_2' are the same. Let t' be the tree obtained from t by: removing the fragment between r_1 and r_1'; removing the fragment between r_2 and r_2'; and*

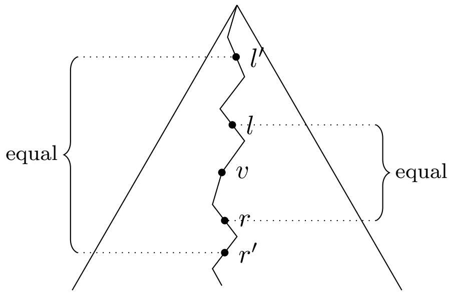

Figure 2: Lemma 1

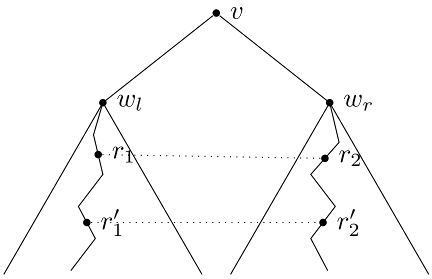

Figure 3: Lemma 2

updating the values on the paths p_1 and p_2. Then t' is a witnessing tree, with the same label in the root as t.

Proof. In this case one should think that starting from the leaf of p_1, then going up through v and then going down to the leaf of p_2 is like a run in VASS and the sets R_{p_i} have the nodes that can be safely cut (see Figure 3). This time R_{p_1} corresponds to l_i and R_{p_2} to r_i. The two cuts also sum to 0, already from v's perspective (because we cut from t_l^+, t_r^+ and the cut values would be subtracted in v anyway). The scary thing follows the ideas of the previous lemma. $\square$

Filip: what we really need in both lemmas is that the cuts are equal on both sides, so the integer labels don't have to be equal, but for the first read lets leave it like this

Notice that Lemma 1 and Lemma 2 have assumptions that are wishful thinking (the existence of two pairs of nodes with the same labels), because on every path the difference between integer labels of consecutive nodes could be arbitrary, not just +-1 or 0 like in VASS case. The hard part will be to find paths for which we could use one of these lemmas.

4 Finding the paths to shorten

The proof follows the ideas of the standard hill cut argument for 1-VASS without branching, but this time the bound on the maximal value in configurations will be double exponential. We need some preparations.

4.1 What do we want to find?

Consider a BVASS $B = (Q, \{q_f\}, \Delta)$. We will need a function $f : \mathbb{N} \to \mathbb{N}$ satisfying the following recursive property:

$$f(x) > (f(y) \cdot |Q|)^2 + f(y) \tag{1}$$

for all $y < x$. Such a function exists, for example $f(x) = 2^{2^{|Q| \cdot (x+1)}}$ should be fine. We assume that f is given from now on. The function f is probably at least double exponential in x, because a simpler function defined as

$$f(x) = f(x-1)^2$$
$$f(0) = 2$$

is equivalently $f(x) = 2^{2^x}$.

The following two lemmas show why we need to have values like $f(x)$ in the registers. The idea is that if we make sure that the drops are small ($f(y)$ for $y < x$), then everything should be ok.

Lemma 3. *Let $x > y \geq 0$ be natural numbers. Suppose there is a downward path (not necessarily from the root) $p = p_1 \dots p_k$, which is in $(t.p_1)^+$, where p_k is a leaf and $n_1 \dots n_k$ are their integer labels. Additionally suppose there is a node $v = p_j$ on the path p such that:*

- $n_1 < f(y)$;
- $n_j \geq f(x)$;
- $drop_L(p_1 \dots p_j) < f(y)$;
- $drop_R(p_j \dots p_n) < f(y)$;

then we can apply Lemma 1 to p in the subtree $t.p_1$

Proof. Let $p_v = p_j \dots p_n$ be the fragment of p from v. Consider the sets L_p in $t.p_1$ and R_{p_v} in $t.v$. The difference between the biggest integer label and the smallest integer label in L_p is more than $f(x) - f(y)$, because $n_j \geq f(x)$ and $n_1 < f(y)$. In R_{p_v} it is at least $f(x) - 1$, because $n_k = 1$, thus we may assume that for both it is at least $f(x) - f(y)$. Notice that by property (1) $f(x) > (f(y) \cdot |Q|)^2 + f(y)$. Thus, $f(x) - f(y) > (f(y) \cdot |Q|)^2$.

We also have that $drop_L(p_1 \dots p_j) < f(y)$ and $drop_R(p_j \dots p_n) < f(y)$. Think of these drops as 1-drops but with $f(y)$ additional dummy states. Then by a pigeon hole argument there has to be a repetition of pairs on the same level in R_{p_v} and L_p (actually the original nodes in R_{p_v} and L_p that we are interested in might be on different levels but the pairs will have the same integer difference, which is good enough). □

Lemma 4. *Let $x > y \geq 0$ be natural numbers. Suppose there is a node v labeled (q, n) such that $q \in Q^-$. Suppose w_l, w_r are its children, and consider root-to-leaf paths $p_l = p^l_1 \dots p^l_k$ in $t.w_l$, $p_r = p^r_1 \dots p^r_{k'}$ in $t.w_r$, where $p^l_1 = w_l$, $p^r_1 = w_r$, and $n^l_1 \dots n^l_k, n^r_1 \dots n^r_{k'}$ are their integer labels. Suppose they satisfy:*

- $n \geq f(x)$;
- $drop_R(p_l) < f(y)$;
- $drop_R(p_r) < f(y)$;

then we can apply Lemma 2 to p_l, p_r in the subtree $t.v$

Proof. Similar as in Lemma 3. □

The remaining part is to show that we can find such paths to use either Lemma 3 or Lemma 4.

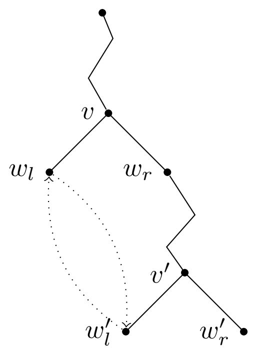

Figure 4: Lemma 5

4.2 Preparing the model to be balanced

Let $\delta = (q, q_l, q_r) \in \Delta$. We say that a triplet of nodes (v, w_l, w_r) from t is a δ-occurrence if: $v, w_l, w_r \in t^+$; their state labels are q, q_l, q_r, respectively; w_l is the left child of v, and w_r is the right child of v. Notice that a δ-occurrence is possible only if $q \in Q^+$ and $w_l, w_r \in R_t$ (by definition of t^+). A tree t is *δ-free* if there are no δ-occurrences in t^+. A δ-occurrence (v, w_l, w_r) is *balanced* if either $t.w_l$ is δ-free or $t.w_r$ is δ-free.

We say that t is *balanced* if for every subtree for every $\delta \in \Delta$ all δ-occurrences in t^+ are balanced.

Lemma 5. *Let t be a witnessing tree. There exists a balanced witnessing tree t' with the same number nodes as t.*

but possibly deeper

To prove this lemma we introduce swapping to modify the tree. Consider two positive nodes v, w in a transition tree t labeled respectively $(q, n), (q, m)$ (they have the same state) such that v and w are not on the same path. Let a be their least common ancestor. Then by $t[v \leftrightarrow w]$ we denote the tree obtained by:

- Swapping $t.v$ with $t.w$;
- adding $m - n$ to the integer label on nodes on the path from v to a and $n - m$ to the integer label on the path from w to a.

In particular the integer label of a remains the same. It is easy to see that $t[v \leftrightarrow w]$ is a transition tree (but not necessarily a witnessing tree).

Proof of Lemma 5. Let $\delta \in \Delta$. Suppose there is a δ-occurrence (v, w_l, w_r) in t^+ and let $(q, n), (q_l, n_l), (q_r, n_r)$ be the labels of respectively v, w_l, w_r. W.l.o.g. we can assume that $n_l \geq n_r$. Now suppose that there are δ-occurrences in both $t.w_l$ and $t.w_r$. Since t is finite then there exists a δ-occurrence (v', w_l', w_r') in $t^+.w_r$ such that $t^+.w_l'$ is δ-free. Let (q_l, n_l') be the label of w_l' (it has the same state as w_l by definition of δ-occurrence).

Let us denote $t' = t[w_l \leftrightarrow w_l']$. We will replace t with t' (see Figure 4). It is easy to see that the number of nodes in t and t' is the same. We prove that t' is still a witnessing tree, and that it does not introduce new δ-occurrences. Let $g : t \to t[w_l \leftrightarrow w_l']$ be the bijection such that $g(v)$ is the new copy of the node v in t.

Filip: This proof turned out to be more complicated then I thought, I'll try to finish and polish it later. I think we need to do this swapping top-down in t^+ and show that we do not add new δ'-occurrences. It seems quite clear on the paper, but there are many cases.

First, let us prove that t' is a witnessing tree. Since v is the parent of w_l then the only nodes whose integer label was modified are on the path from w_r to w_l'. They were modified by adding $n_l - n_l'$. We assumed $n_l \geq n_r$. We also know that $n_r > n_l'$ because $w_r, w_l' \in R_t$ and both are on the same path. Therefore, $n_l - n_l' > 0$ and $t[w_l \leftrightarrow w_l']$ is a witnessing tree.

Second, we prove that this swapping does not add new δ' occurrences. It is easy to see that it suffices to show that for every node u if $g(u)$ is in t'^+ then u is in t^+. The construction does not affect the tree t outside $t.v$, therefore we can focus on nodes in $t'.g(v)$. Let $p = p_1 \dots p_k$ be the downward path from v to v', and let (q_i, n_i) be their corresponding labels. Since $p_1 \dots p_k \in t^+$ then we do not need to show anything for $g(p_1), \dots, g(p_n)$. We also know that $g(p) = g(p_1) \dots g(p_n)$ is the path from $g(v)$ to $g(v')$).

Claim 2. *Let u be a child of p_i for some i such that u is not on the path p. If $g(u) \in t'^+$ then $u \in t^+$ and $g(u) \in R_{t'}$.*

First, let us argue that Claim 2 would conclude the proof. Since the construction does not affect the subtree $t.u$ then $(t'.g(u))^+ = (t.u)^+$. By Claim 1 $t'^+.g(u) = (t'.g(u))^+ = (t.u)^+ = t^+.u$, which means we do not add new elements to the subtree $t'^+.g(u)$. Since all nodes in $t'.g(v)$ are either on the path p or below it this concludes the proof.

Let us prove Claim 2. If $u = w_l'$ then this follows directly from the construction. By definition $w_l' \in t^+$. Notice that the integer value of v and nodes above did not change when we swapped w_l' with w_l. Since $n_l' < n_l$ then $g(w_l') \in R_{t'}$, which concludes this case.

Now, suppose u is a child of p_i for some $i < n$ and the other child of p_i is p_{i+1}. Let (q_u, n_u) be the label of u. Consider the following cases:

- if $u \in t^+$. Since $p_{i+1} \in t^+$ then $u, p_{i+1} \in R_t$ by definition of t^+. It is easy to see that $g(u) \in R_{t'}$, because the integer label of $g(u)$ remains n_u, and the integer labels of ancestors of u were increased.

- if $u \notin t^+$. We show that $g(u) \notin t'^+$. Since $p_{i+1} \in t^+$ then $p_{i+1} \notin R_t$ and $n_{i+1} \geq n_u$ by definition of t^+. We show that $g(p_{i+1}) \notin R_{t'}$, which finishes the proof because then the integer label of $g(p_{i+1})$ is $n_{i+1} + n_l - n_l' > n_w$, and therefore, $t'.g(u)$ is removed from t'^+. Let us show that $g(p_{i+1}) \notin R_{t'}$. Since $p_{i+1} \notin R_t$, then there exists u' with integer label m, such that u' is an ancestor of p_{i+1}, such that $m \leq n_{i+1}$. The integer label of $g(u')$ is either m (if $u' = v$ or an ancestor of v) or $m + n_l - n_l'$ (otherwise). Then comparing the integer label of $g(p_{i+1})$ with the integer label of $g(u')$ we get: $n_{i+1} + n_l - n_l' \geq m + n_l - n_l' > m$, which proves that $g(p_{i+1}) \notin R_{t'}$.

It remains to consider $u = w_l$ and $u = w_r'$. By definition $w_l, w_r' \in t^+$ and $w_l, w_r' \in R_t$. We show that $g(w_l), g(w_r') \in R_{t'}$ (regardless of whether $g(w_l), g(w_r') \in t'^+$). For $g(w_r')$ this is obvious because the integer label of $g(w_r')$ remains the same as the integer label of w_r', while its ancestors integer label was either increased or remained the same. Now, recall that w_l's integer label is n_l. Suppose there is a node $g(u')$, an ancestor of $g(w_l)$ in t' with integer label m, such that $m \leq n_l$. If u' is v or its ancestor then its integer label was not changed and it is m. But then u' is also the ancestor of w_l, which is a contradiction because $w_l \in R_t$, and therefore $n_l < m$. This means that u' is on path p below v and its integer label is $m - n_l + n'_l$. Since u' is an ancestor of w'_l then $n'_l < m - n_l + n'_l$, which implies $n_l < m$, which is a contradiction. ☐

4.3 Finding the paths

Lemma 6 (Going down)**.** *Let t be a balanced witnessing tree for the BVASS B and let $c \in R_t$ such that c is not a leaf. Suppose that the integer label of c is $n \geq f(x)$ for some given $x > 0$. Let $0 \leq y < x$. There exists a downward path $p = p_1 \dots p_k$ in $(t.v)^+$, where $p_1 = c$, such that one of the following holds:*

(i) (p_{k-1}, w_l, w_r) is a δ-occurrence, $p_k \in \{w_l, w_r\}$, the tree $t.p_k$ is δ-free, and the integer label of p_k is at least $f(y)$;

(ii) p_k is a leaf, and $drop_R(p) < f(y)$.

Proof. It suffices to go down from c in such a way to stay in $(t.v)^+$. Let v be the currently processed node and let (q, n) be its label. If v has one child in $(t.v)^+$ then we go there, by definition the integer label is not decremented. If v is a leaf then (ii) works. Additionally, let w_l, w_r be v's left and right child and let (q_l, n_l), (q_r, n_r) be their labels. By definition of $(t.v)^+$ this possible only if $w_l, w_r \in R_t$. Recall that t is balanced. W.l.o.g. suppose w_l is the δ-free node. Then if $n_l < f(y)$ then go to w_r (we decreased the integer label by at most $f(y)$). Otherwise we finish the path in w_l and (i) works. ☐

Lemma 7 (Going up)**.** *Let t be a witnessing tree for the BVASS B and let c be a node in t that is not the root of t, such that its integer label is $n \geq f(x)$ for some given $x > 0$. Let $0 \leq y < x$. There exists a node w and a downward path $p = p_1 \dots p_k$, where $p_k = c$ and the children of p_1 are p_2 and w, such that $p_2 \dots p_k$ is in $(t.p_2)^+$ and one of the following holds:*

(i) The state label of p_1 is $q \in Q^-$ and the integer label of w is at least $f(y)$;

(ii) The integer label of p_1 is less than $f(y)$ and $drop_L(p_2 \dots p_k) < f(y)$.

Proof. It suffices to go up from c with the following rules. Let w be the currently processed node and let (q, n) be its label. Additionally, let v be the parent of w and let (q_v, n_v) be its label. And let w' be the other child of v labeled (q', n').

- If $q_v \in Q^+$ then go to v;
- if $q_v \in Q^-$ and $n' \geq f(y)$ then STOP (condition (i) works);
- if $q_v \in Q^-$ and $n' < f(y)$ then if w is the left child of v then go to v. Otherwise STOP (condition (ii) works because $n \leq n'$).

It is clear that the integer label here is decreased only by at most $f(y)$. We also go up only if c is in $(t.v)^+$. ☐

Lemma 8. *Suppose there is a witnessing tree of reachability to $(q_0, 0)$ and let t be a balanced tree with a minimal number of nodes. Then the integer labels of all nodes in t are at most $f(2 \cdot b + 1)$, where $b = |\Delta|$.*

Proof. Let r be the root of t. Recall that the integer value of r is 0, and it is 1 for all leafs. We show that if there exists a node v in t, whose integer label is bigger than $f(2 \cdot b + 1)$ then t can be shortened.

First do the following: set $x = 2b + 1$ and $y = 2b$. By Lemma 5 we may assume that $t.v$ is balanced. Apply Lemma 6 for $c = v$ in $t.v$ for these x and y (notice that obviously $v \in R_{t_v}$). If (i) is true then keep applying with the following modifications: $x := y$, $y := y - 1$, $c = p_n$. After at most b steps the condition (ii) has to be true (because each time (i) is true, it cannot be true later for the same transition δ).

Now: set $x_2 = 2b + 1$ and $y_2 = y$ (the final y from previous step). Apply Lemma 7 for $c = v$ and x_2, y_2. If (ii) is true then we can apply Lemma 3. Otherwise we continue with w from (i). By Lemma 5 we can assume that $t.w$ is balanced. Keep applying Lemma 6 to $c = w$ and $t.w$ like in the first step, modifying $x_2 := y_2$, $y_2 := y_2 - 1$, $c = p_n$ until condition (ii) is true.

Now lets go back to $t.v$ let c be the last node to which we applied Lemma 6 in $t.v$, with the current value y. Then lets apply Lemma 6 again but this time with $y = y_2$ (y_2 was shortened so $y_2 < y$). If condition (ii) is true then we can use Lemma 4. If not then we keep applying Lemma 6 until condition (ii) is true.

We go back to $t.w$ and change the last y_2 to the current y and so on... After at most $2b$ steps we will have condition (ii) true in both subtrees with $y = y_2$, because in every subtree we can do at most b drops. Then we apply Lemma 4. ☐

Theorem 1. *The reachability problem for +/-BVASS is in 2EXPTIME.*

Proof. AEXPSPACE = 2EXPTIME. ☐