



Positivity of third order linear recurrence sequences[☆]

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ABSTRACT

It is shown that the Positivity Problem for a sequence satisfying a third order linear recurrence with integer coefficients, i.e., the problem whether each element of this sequence is nonnegative, is decidable.

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1. Introduction

Consider a third order linear recurrence of the form

$$u_n = a_1 u_{n-1} + a_2 u_{n-2} + a_3 u_{n-3} \quad (n \geq 3), \quad (1.1)$$

where $a_1, a_2, a_3 (\neq 0)$ are given integers. The recurrence (1.1) defines a unique sequence of integers provided the initial integers u_0, u_1, u_2 are given. We are interested in the *Positivity Problem*: Is it possible to decide whether the sequence $(u_n)_{n \geq 0}$ is nonnegative? Equivalently, is it decidable whether $u_n \geq 0$ for all $n \geq 0$? Let us emphasize that the Positivity Problem considered here is to decide whether *all* elements of a sequence are nonnegative. In contrast, the *Eventual Positivity Problem*, which asks whether the terms u_n are nonnegative for all sufficiently large n is not of interest here. For third order recurrences with integral coefficients, the Eventual Positivity Problem is rather trivial to discern. Deciding the positivity of all sequence elements requires some further considerations to take care of the initial tail of the sequence.

As mentioned in [9], the Positivity Problem is a natural question and well-known in a number of situations, such as for matrices and in the Mortality Problem [12–14,7,11]. A closely related result, known as the Skolem–Mahler–Lech theorem, provides information about when the terms of a linear recurrence sequence are zero ([2,8], Section 2.1.1 of [4]). Another related problem, known as the Orbit Problem, is, in an equivalent form, a question about simultaneous solvability of a system of equations in linear recurrence sequences (see Section 14.2.3 of [4]).

The Positivity Problem for sequences satisfying a second order linear recurrence has already been shown to be decidable by Halava, Harju and Hirvensalo, [9], in 2006; see also [1,3]. This means that there is an algorithmic procedure to decide whether elements in a sequence satisfying a second order linear recurrence with integer coefficients are nonnegative. We show here that the same conclusion holds for each sequence satisfying a third order linear recurrence with integer coefficients. Our underlying methodology is based on the fact that the roots of a third degree algebraic equation with integer coefficients can be explicitly computed, and in one instance on a result in Diophantine approximation; such approximation result has been used in this context before e.g. in [3,5].

Recall that the characteristic polynomial associated with the recurrence (1.1) is

$$p(x) = x^3 - a_1 x^2 - a_2 x - a_3 \in \mathbb{Z}[x].$$

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Following the derivation on pages 169–170 of [6], we start by reviewing some facts. Write

$$p\left(x + \frac{a_1}{3}\right) = x^3 + \alpha x + \beta, \quad \alpha := \frac{-a_1^2 - 3a_2}{3}, \beta := \frac{-2a_1^3 - 9a_2a_1 - 27a_3}{27}.$$

Let x_1, x_2 , and x_3 be all the three roots of $p(x + a_1/3)$. The discriminant of $p(x + a_1/3)$ is

$$D = -4\alpha^3 - 27\beta^2 = (x_1 - x_2)^2(x_1 - x_3)^2(x_2 - x_3)^2.$$

The following facts are known.

Proposition 1.1. Let $(u_n)_{n \geq 0}$ be a sequence of integers satisfying (1.1) with integer coefficients $a_1, a_2, a_3 (\neq 0)$, and initial integral values u_0, u_1, u_2 . Let $p(x)$ be the characteristic polynomial of (1.1) whose roots are λ_i ($i = 1, 2, 3$), and whose discriminant is $D = -4\alpha^3 - 27\beta^2$, where $\alpha = (-a_1^2 - 3a_2)/3$ and $\beta = (-2a_1^3 - 9a_2a_1 - 27a_3)/27$.

I. If $D > 0$, then $\lambda_1, \lambda_2, \lambda_3$ are distinct nonzero real numbers and

$$u_n = A\lambda_1^n + B\lambda_2^n + C\lambda_3^n \quad (n \geq 0),$$

where

$$A = \frac{\lambda_2\lambda_3u_0 - (\lambda_3 + \lambda_2)u_1 + u_2}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} \in \mathbb{R}, \quad B = \frac{-\lambda_1\lambda_3u_0 + (\lambda_3 + \lambda_1)u_1 - u_2}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)} \in \mathbb{R},$$

$$C = \frac{\lambda_1\lambda_2u_0 - (\lambda_2 + \lambda_1)u_1 + u_2}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \in \mathbb{R}.$$

II. (II.1) If $D = 0$ and $\alpha = 0$, then $\lambda_1 = \lambda_2 = \lambda_3 =: \lambda \in \mathbb{R} \setminus \{0\}$ and

$$u_n = (A + Bn + Cn^2)\lambda^n \quad (n \geq 0),$$

where

$$A = u_0 \in \mathbb{R}, \quad B = \frac{-3u_0}{2} + \frac{2u_1}{\lambda} - \frac{u_2}{2\lambda^2} \in \mathbb{R}, \quad C = \frac{u_0}{2} - \frac{u_1}{\lambda} + \frac{u_2}{2\lambda^2} \in \mathbb{R}.$$

(II.2) If $D = 0$ and $\alpha \neq 0$, then $p(x)$ has two distinct real roots $\lambda_1 (\neq 0)$ of multiplicity 1, $\lambda_2 = \lambda_3 (\neq 0)$ of multiplicity 2, and

$$u_n = A\lambda_1^n + (B + Cn)\lambda_2^n \quad (n \geq 0),$$

where

$$A = \frac{\lambda_2^2u_0 - 2\lambda_2u_1 + u_2}{(\lambda_2 - \lambda_1)^2} \in \mathbb{R}, \quad B = \frac{-\lambda_1(2\lambda_2 - \lambda_1)u_0 + 2\lambda_2u_1 - u_2}{(\lambda_2 - \lambda_1)^2} \in \mathbb{R},$$

$$C = \frac{\lambda_1\lambda_2u_0 - (\lambda_2 + \lambda_1)u_1 + u_2}{\lambda_2(\lambda_2 - \lambda_1)} \in \mathbb{R}.$$

III. If $D < 0$, then $p(x)$ has one real root $\lambda_1 (\neq 0)$, two complex conjugate roots $\lambda_2, \lambda_3 = \bar{\lambda}_2 \in \mathbb{C} \setminus \mathbb{R}$ and

$$u_n = A\lambda_1^n + B\lambda_2^n + \bar{B}\bar{\lambda}_2^n \quad (n \geq 0),$$

where

$$A = \frac{\lambda_2\bar{\lambda}_2u_0 - (\bar{\lambda}_2 + \lambda_2)u_1 + u_2}{(\lambda_2 - \lambda_1)(\bar{\lambda}_2 - \lambda_1)} \in \mathbb{C}, \quad B = \frac{-\lambda_1\bar{\lambda}_2u_0 + (\bar{\lambda}_2 + \lambda_1)u_1 - u_2}{(\lambda_2 - \lambda_1)(\bar{\lambda}_2 - \lambda_1)} \in \mathbb{C}.$$

We treat each of the three cases according to $D > 0, D = 0, D < 0$ separately in the remaining sections. In order to facilitate the flow of the proof of the main result we single out two auxiliary lemmas.

Lemma 1.2. Let a, b be positive real numbers belonging to the interval $(0, 1)$ and let \mathcal{B} and \mathcal{C} be two positive real numbers. Consider the following three exponential polynomials with argument $n \in \mathbb{N} \cup \{0\}$,

$$P(n) = a^n (\mathcal{B} - \mathcal{C}b^n), \quad Q(n) = a^n (\mathcal{B} + \mathcal{C}b^n), \quad R(n) = a^n (-\mathcal{B} + \mathcal{C}b^n).$$

The following conclusions hold:

(Type F–Z) there are explicitly computable integers $N_0 \in \mathbb{N} \cup \{0\}$ and $N_1 \in \mathbb{N}$ such that

$$\sup \{P(n); n \geq 0\} = \max \{P(n); N_0 \leq n \leq N_1\};$$

(Type F+Z) $\sup \{Q(n); n \geq 0\} = Q(0)$;

(Type - F+Z) there is an explicitly computable integer $N_2 \in \mathbb{N} \cup \{0\}$ such that

$$\sup \{R(n); n \geq 0\} = \max \{0, R(n); 0 \leq n \leq N_2\}.$$

Proof. (Type F–Z): Since $b^n \downarrow 0$ ($n \rightarrow \infty$), there exists $N_0 \in \mathbb{N} \cup \{0\}$ such that

$$\mathcal{B} - \mathcal{C}b^i \leq 0 < \mathcal{B} - \mathcal{C}b^n \quad (n \geq N_0, 0 \leq i < N_0)$$

so that $P(n) > 0$ for all $n \geq N_0$ and $P(n) \leq 0$ for $0 \leq n < N_0$. Since $P(n) \rightarrow 0$ ($n \rightarrow \infty$), there exists a positive integer $N_1 > N_0$ such that $P(n) < P(N_0)$ for all $n > N_1$, and the desired result follows.

(Type F+Z): Since $Q(n) > 0$ and $Q(n) \downarrow 0$ ($n \rightarrow \infty$), the conclusion is immediate in this case.

(Type -F+Z): Since $b^n \downarrow 0$ ($n \rightarrow \infty$), there exists $N_2 \in \mathbb{N} \cup \{0\}$ such that $-\mathcal{B} + \mathcal{C}b^n < 0$ for all $n > N_2$ and the desired result follows at once noting that the 0 on the right-hand maximal value is to take care of the situation when $R(n) \leq 0$ for all $n \geq 0$. \square

Lemma 1.3. Let $\varphi, \theta \in [-\pi, \pi)$ with $\theta \notin \{-\pi, 0\}$.

(a) If θ is a rational multiple of π , then $\cos(\varphi + n\theta)$ is periodic and takes only finitely many explicitly computable values as n varies over $\mathbb{N} \cup \{0\}$.

(b) If θ is not a rational multiple of π , then as n varies over $\mathbb{N} \cup \{0\}$ the range of values of $\cos(\varphi + n\theta)$ is dense in $[-1, 1]$.

Proof. (a) Assume that θ is a rational multiple of π , say $\theta = s\pi/t$, where $s, t (> 0) \in \mathbb{Z} \setminus \{0\}$ and $\gcd(s, t) = 1$. Since the cosine function is periodic of period 2π , it is easily checked that $\cos(\varphi + n\theta) = \cos(\varphi + ns\pi/t)$ takes at most $2t$ distinct values corresponding to $n = 0, 1, 2, \dots, 2t - 1$.

(b) Assume that θ is not a rational multiple of π , say $\theta = \vartheta\pi$, where $\vartheta \in \mathbb{R} \setminus \mathbb{Q}$. Writing $\vartheta = [\vartheta] + \xi$ where $[\vartheta]$ denotes its integer part, and $\xi := \{\vartheta\} \in (0, 1)$ denotes its fractional part which must be irrational, for even $n = 2k$ ($k \in \mathbb{N} \cup \{0\}$) we have

$$\cos(\varphi + n\theta) = \cos(\varphi + 2k\vartheta\pi) = \cos(\varphi + 2k\xi\pi) = \cos(\varphi + \{k\xi\}2\pi). \quad (1.2)$$

Since ξ is irrational, by the Kronecker's approximation theorem, see e.g. Corollary 6.4 on page 75 of [10], we know the set $\{\{k\xi\}; k \in \mathbb{N} \cup \{0\}\}$ is dense in $[0, 1]$. Consequently, the set $\{\{k\xi\}2\pi; n \in \mathbb{N}\}$ is dense in $[0, 2\pi]$, implying that the range of values of $\cos(\varphi + \{k\xi\}2\pi)$ ($k \in \mathbb{N} \cup \{0\}$) is dense in $[-1, 1]$. \square

2. Case $D > 0$

In this case, the general term of the sequence is

$$u_n = A\lambda_1^n + B\lambda_2^n + C\lambda_3^n \quad (n \geq 0),$$

where $\lambda_1, \lambda_2, \lambda_3$ are distinct nonzero real numbers. There are two possibilities depending on whether there are two λ_i 's having the same absolute value.

2.1. There are two roots λ_i, λ_j ($i, j \in \{1, 2, 3\}, i \neq j$) such that $|\lambda_i| = |\lambda_j|$

Without loss of generality, let the two roots be $\lambda_1 > 0$ and $\lambda_2 = -\lambda_1$. Thus,

$$u_n = \{A + (-1)^n B\} \lambda_1^n + C\lambda_3^n \quad (n \geq 0).$$

Since $\lambda_3 \neq \lambda_1$, we subdivide into two further subcases depending on whether $\lambda_1 > |\lambda_3|$.

2.1.1. $\lambda_1 > |\lambda_3| > 0$

Rewrite the general term of the sequence as

$$u_n = \lambda_1^n \{A + (-1)^n B + C(\lambda_3/\lambda_1)^n\} \quad (n \geq 0).$$

We consider two possibilities corresponding to the signs of λ_3 .

- $\lambda_3 < 0$.

For $k \in \mathbb{N} \cup \{0\}$, we have
$$\begin{cases} u_{2k} = \lambda_1^{2k} (A + B + C(|\lambda_3|/\lambda_1)^{2k}) \\ u_{2k+1} = \lambda_1^{2k+1} (A - B - C(|\lambda_3|/\lambda_1)^{2k+1}). \end{cases}$$

If $C \geq 0$, then the sequence (u_n) is nonnegative if and only if

$$A \geq \max\{-B, B + C|\lambda_3|/\lambda_1\}.$$

If $C < 0$, then the sequence (u_n) is nonnegative if and only if

$$A \geq \max\{B, -B + |C|\}.$$

- $\lambda_3 > 0$.

For $k \in \mathbb{N} \cup \{0\}$, we have
$$\begin{cases} u_{2k} = \lambda_1^{2k} (A + B + C(\lambda_3/\lambda_1)^{2k}) \\ u_{2k+1} = \lambda_1^{2k+1} (A - B + C(\lambda_3/\lambda_1)^{2k+1}). \end{cases}$$

If $C \geq 0$, then the sequence (u_n) is nonnegative if and only if $A \geq |B|$.

If $C < 0$, then the sequence (u_n) is nonnegative if and only if

$$A \geq \max\{-B + |C|, B + |C|\lambda_3/\lambda_1\}.$$

2.1.2. $\lambda_1 < |\lambda_3|$

Rewriting the general term of the sequence as

$$u_n = \lambda_3^n \left\{ (A + (-1)^n B) (\lambda_1/\lambda_3)^n + C \right\} \quad (n \geq 0),$$

we have two choices according to the sign of λ_3 .

- $\lambda_3 < 0$.

$$\text{For } k \in \mathbb{N} \cup \{0\}, \text{ we have } \begin{cases} u_{2k} = |\lambda_3|^{2k} \left\{ (A+B) |\lambda_1/\lambda_3|^{2k} + C \right\} \\ u_{2k+1} = -|\lambda_3|^{2k+1} \left\{ (B-A) |\lambda_1/\lambda_3|^{2k+1} + C \right\}. \end{cases}$$

The sequence (u_n) is nonnegative if and only if, for each $k \geq 0$,

$$(A+B) |\lambda_1/\lambda_3|^{2k} \geq -C \quad \text{and} \quad (B-A) |\lambda_1/\lambda_3|^{2k+1} \leq -C \iff C = 0, \quad B < A, A+B > 0.$$

- $\lambda_3 > 0$.

$$\text{For } k \in \mathbb{N} \cup \{0\}, \text{ we have } \begin{cases} u_{2k} = \lambda_3^{2k} \left\{ (A+B) (\lambda_1/\lambda_3)^{2k} + C \right\} \\ u_{2k+1} = \lambda_3^{2k+1} \left\{ (A-B) (\lambda_1/\lambda_3)^{2k+1} + C \right\}. \end{cases}$$

The sequence (u_n) is nonnegative if and only if, for each $k \geq 0$,

$$C \geq -(A+B) (\lambda_1/\lambda_3)^{2k} \quad \text{and} \quad C \geq -(A-B) (\lambda_1/\lambda_3)^{2k+1}. \quad (2.1)$$

If $A \geq |B|$, then (2.1) holds if and only if $C \geq 0$. If $A < |B|$, then (2.1) holds if and only if $\begin{cases} C \geq -(A-B) (\lambda_1/\lambda_3) & \text{provided } B > 0 \\ C \geq -(A+B) & \text{provided } B \leq 0. \end{cases}$

2.2. All three roots λ_1, λ_2 and λ_3 have different absolute values

Without loss of generality, assume $|\lambda_1| > |\lambda_2| > |\lambda_3| > 0$. Here,

$$u_n = \lambda_1^n \left\{ A + B (\lambda_2/\lambda_1)^n + C (\lambda_3/\lambda_1)^n \right\} \quad (n \geq 0). \quad (2.2)$$

We assume that A, B and C do not vanish simultaneously, for otherwise $u_n \equiv 0$. We treat two separate subcases depending on the sign of λ_1 .

2.2.1. $\lambda_1 < 0$

Note that λ_1^n has alternating signs. Since $(\lambda_2/\lambda_1)^n$ and $(\lambda_3/\lambda_1)^n \rightarrow 0$ ($n \rightarrow \infty$), for n sufficiently large, $u_n \geq 0$ is possible only when $A = 0$. Thus,

$$u_n = B\lambda_2^n + C\lambda_3^n = \lambda_2^n \left\{ B + C (\lambda_3/\lambda_2)^n \right\}.$$

This particular case thus reduces to a sequence satisfying a second order linear recurrence and by the result in [9] it is decidable.

2.2.2. $\lambda_1 > 0$

- If $B = C = 0$, then $u_n = A\lambda_1^n \geq 0$ for all $n \geq 0$ if and only if $A \geq 0$.
- If $B = 0$, then for $C > 0$ the sequence (u_n) is nonnegative if and only if

$$A \geq -C(\lambda_3/\lambda_1)^n \quad (n \geq 0) \iff \begin{cases} A/C \geq 0 & \text{provided } \lambda_3 > 0 \\ A/C \geq |\lambda_3/\lambda_1| & \text{provided } \lambda_3 < 0. \end{cases}$$

For $C < 0$, the sequence (u_n) is nonnegative if and only if

$$A \geq -C(\lambda_3/\lambda_1)^n \quad (n \geq 0) \iff A \geq |C|.$$

- If $C = 0$, then for $B > 0$, the sequence (u_n) is nonnegative if and only if

$$A \geq -B(\lambda_2/\lambda_1)^n \quad (n \geq 0) \iff \begin{cases} A/B \geq 0 & \text{provided } \lambda_2 > 0 \\ A/B \geq |\lambda_2/\lambda_1| & \text{provided } \lambda_2 < 0. \end{cases}$$

For $B < 0$, the sequence (u_n) is nonnegative if and only if

$$A \geq -B(\lambda_2/\lambda_1)^n \quad (n \geq 0) \iff A \geq |B|.$$

- If $B < 0, C < 0$ the sequence (u_n) is nonnegative if and only if $A \geq |B| + |C|$.
- If $B < 0, C > 0$, then the sequence (u_n) is nonnegative if and only if

$$A \geq |B| (\lambda_2/\lambda_1)^n - |C| (\lambda_3/\lambda_1)^n \quad (n \geq 0).$$

- For $\lambda_2 > 0, \lambda_3 > 0$, consider the exponential polynomial

$$P_1(n) := (\lambda_2/\lambda_1)^n \left\{ |B| - |C| (\lambda_3/\lambda_2)^n \right\} \quad (n \geq 0).$$

This exponential polynomial is of the type F–Z, and so [Lemma 1.2](#) shows that (u_n) is nonnegative if and only if

$$A \geq \max \{P_1(n); N_0 \leq n \leq N_1\} = \max \{|B|(\lambda_2/\lambda_1)^n - |C|(\lambda_3/\lambda_1)^n; N_0 \leq n \leq N_1\} > 0, \quad (2.3)$$

for some computable integers $N_1 > N_0 \in \mathbb{N} \cup \{0\}$.

- For $\lambda_2 > 0, \lambda_3 < 0$, we have

$$|B|(\lambda_2/\lambda_1)^n - |C|(\lambda_3/\lambda_1)^n = \begin{cases} |B|(\lambda_2/\lambda_1)^{2k} - |C|(|\lambda_3|/\lambda_1)^{2k} & \text{if } n = 2k \\ |B|(\lambda_2/\lambda_1)^{2k+1} + |C|(|\lambda_3|/\lambda_1)^{2k+1} & \text{if } n = 2k + 1. \end{cases}$$

Since the exponential polynomial

$$P_2(k) := (\lambda_2/\lambda_1)^{2k} \{|B| - |C|(|\lambda_3|/\lambda_2)^{2k}\} \quad (k \geq 0)$$

is of Type F–Z, by [Lemma 1.2](#) there are computable integers $K_1 > K_0 \in \mathbb{N} \cup \{0\}$ such that

$$\sup \{P_2(k); k \geq 0\} = \max \{P_2(k); K_0 \leq k \leq K_1\}.$$

Next consider the exponential polynomial

$$P_3(k) := (\lambda_2/\lambda_1)^{2k+1} \{|B| + |C|(|\lambda_3|/\lambda_2)^{2k+1}\} \quad (k \geq 0),$$

which is of Type F+Z. Thus, by [Lemma 1.2](#) (u_n) is nonnegative if and only if

$$A \geq \max \{P_2(k_1), P_3(0); K_0 \leq k_1 \leq K_1\}.$$

- For $\lambda_2 < 0, \lambda_3 > 0$, we have

$$|B|(\lambda_2/\lambda_1)^n - |C|(\lambda_3/\lambda_1)^n = \begin{cases} |B|(|\lambda_2|/\lambda_1)^{2k} - |C|(\lambda_3/\lambda_1)^{2k} & \text{if } n = 2k \\ -|B|(|\lambda_2|/\lambda_1)^{2k+1} - |C|(\lambda_3/\lambda_1)^{2k+1} & \text{if } n = 2k + 1. \end{cases}$$

The case of odd $n = 2k + 1$ can be ignored as the terms are negative. For even $n = 2k$, whose exponential polynomial is of Type F–Z, by [Lemma 1.2](#) there are computable integers $K_4 > K_3 \in \mathbb{N} \cup \{0\}$ so that (u_n) is nonnegative if and only if

$$A \geq \max \{|B|(|\lambda_2|/\lambda_1)^{2k} - |C|(\lambda_3/\lambda_1)^{2k}; K_3 \leq k \leq K_4\}.$$

- For $\lambda_2 < 0, \lambda_3 < 0$, we have

$$|B|(\lambda_2/\lambda_1)^n - |C|(\lambda_3/\lambda_1)^n = \begin{cases} |B|(|\lambda_2|/\lambda_1)^{2k} - |C|(|\lambda_3|/\lambda_1)^{2k} & \text{if } n = 2k \\ -|B|(|\lambda_2|/\lambda_1)^{2k+1} + |C|(|\lambda_3|/\lambda_1)^{2k+1} & \text{if } n = 2k + 1. \end{cases}$$

In the case of even $n = 2k$, the exponential polynomial

$$P_4(k) := (|\lambda_2|/\lambda_1)^{2k} \{|B| - |C|(|\lambda_3|/|\lambda_2|)^{2k}\} \quad (k \geq 0)$$

is of Type F–Z, and so by [Lemma 1.2](#) there are computable integers $K_6 > K_5 \in \mathbb{N} \cup \{0\}$ such that

$$\sup \{P_4(k); k \geq 0\} = \max \{P_4(k); K_5 \leq k \leq K_6\} > 0.$$

Next for the case of odd $n = 2k + 1$, the exponential polynomial

$$P_5(k) := (|\lambda_2|/\lambda_1)^{2k+1} \{-|B| + |C|(|\lambda_3|/|\lambda_2|)^{2k+1}\} \quad (k \geq 0)$$

is of Type - F+Z. By [Lemma 1.2](#), there exists $K_7 \in \mathbb{N} \cup \{0\}$ such that

$$\sup \{P_5(k); k \geq 0\} = \max \{0, P_5(k); 0 \leq k \leq K_7\}. \quad (2.4)$$

Thus, (u_n) is nonnegative if and only if

$$A \geq \max \{P_4(k_1), P_5(k_2); K_5 \leq k_1 \leq K_6, 0 \leq k_2 \leq K_7\}.$$

- If $B > 0, C < 0$ the sequence (u_n) is nonnegative if and only if

$$A \geq -|B|(\lambda_2/\lambda_1)^n + |C|(\lambda_3/\lambda_1)^n.$$

- For $\lambda_2 > 0, \lambda_3 > 0$, the exponential polynomial

$$P_6(n) = (\lambda_2/\lambda_1)^n \{-|B| + |C|(\lambda_3/\lambda_2)^n\} \quad (n \geq 0)$$

is of Type - F+Z, and so by [Lemma 1.2](#), there is a computable integer $K_8 \in \mathbb{N} \cup \{0\}$ such that (u_n) is nonnegative if and only if

$$A \geq \max \{0, P_6(n); 0 \leq n \leq K_8\}.$$

- For $\lambda_2 > 0, \lambda_3 < 0$, we have

$$-|B|(\lambda_2/\lambda_1)^n + |C|(\lambda_3/\lambda_1)^n = \begin{cases} -|B|(\lambda_2/\lambda_1)^{2k} + |C|(|\lambda_3|/\lambda_1)^{2k} & \text{if } n = 2k \\ -|B|(\lambda_2/\lambda_1)^{2k+1} - |C|(|\lambda_3|/\lambda_1)^{2k+1} & \text{if } n = 2k + 1. \end{cases}$$

The case of odd $n = 2k + 1$ is ignored because of negative terms. In the case of even $n = 2k$, the corresponding exponential polynomial is of Type - F+Z and so by [Lemma 1.2](#), (u_n) is nonnegative if and only if

$$A \geq \max \{0, -|B|(\lambda_2/\lambda_1)^{2k} + |C|(|\lambda_3|/\lambda_1)^{2k}; 0 \leq k \leq K_9\},$$

where $K_9 \in \mathbb{N} \cup \{0\}$ is a computable integer.

- For $\lambda_2 < 0, \lambda_3 > 0$, we have

$$-|B|(\lambda_2/\lambda_1)^n + |C|(\lambda_3/\lambda_1)^n = \begin{cases} -|B|(|\lambda_2|/\lambda_1)^{2k} + |C|(\lambda_3/\lambda_1)^{2k} & \text{if } n = 2k \\ |B|(|\lambda_2|/\lambda_1)^{2k+1} + |C|(\lambda_3/\lambda_1)^{2k+1} & \text{if } n = 2k + 1. \end{cases}$$

In the case of even $n = 2k$, the exponential polynomial

$$P_7(k) := (|\lambda_2|/\lambda_1)^{2k} \{-|B| + |C|(\lambda_3/|\lambda_2|)^{2k}\} \quad (k \geq 0)$$

is of Type - F+Z and so by Lemma 1.2, there is a computable integer $K_{10} \in \mathbb{N} \cup \{0\}$ such that

$$\sup \{P_7(k); k \geq 0\} = \max \{0, P_7(k); 0 \leq k \leq K_{10}\}.$$

In the case of odd $n = 2k + 1$, the exponential polynomial

$$P_8(k) := (|\lambda_2|/\lambda_1)^{2k+1} \{|B| + |C|(\lambda_3/|\lambda_2|)^{2k+1}\} \quad (k \geq 0)$$

is of Type F+Z, and so by Lemma 1.2

$$\sup \{P_8(k); k \geq 0\} = P_8(0) = (|\lambda_2|/\lambda_1) \{|B| + |C|(\lambda_3/|\lambda_2|)\} > 0.$$

Consequently, the sequence (u_n) is nonnegative if and only if

$$A \geq \max \{P_7(k_1), P_8(0); 0 \leq k_1 \leq K_{10}\}.$$

- For $\lambda_2 < 0, \lambda_3 < 0$, we have

$$-|B|(\lambda_2/\lambda_1)^n + |C|(\lambda_3/\lambda_1)^n = \begin{cases} -|B|(|\lambda_2|/\lambda_1)^{2k} + |C|(|\lambda_3|/\lambda_1)^{2k} & \text{if } n = 2k \\ |B|(|\lambda_2|/\lambda_1)^{2k+1} - |C|(|\lambda_3|/\lambda_1)^{2k+1} & \text{if } n = 2k + 1. \end{cases}$$

For even $n = 2k$, the exponential polynomial

$$P_9(k) := (|\lambda_2|/\lambda_1)^{2k} \{-|B| + |C|(|\lambda_3|/|\lambda_2|)^{2k}\} \quad (k \geq 0)$$

is of Type - F+Z, and so by Lemma 1.2, there is a computable integer $K_{12} \in \mathbb{N} \cup \{0\}$ such that

$$\sup \{P_9(k); k \geq 0\} = \max \{0, P_9(k); 0 \leq k \leq K_{12}\}.$$

For odd n , the exponential polynomial

$$P_{10}(k) := (|\lambda_2|/\lambda_1)^{2k+1} \{|B| - |C|(|\lambda_3|/|\lambda_2|)^{2k+1}\} \quad (k \geq 0)$$

is of Type F-Z and so by Lemma 1.2, there are computable integers $K_{14} > K_{13} \in \mathbb{N} \cup \{0\}$ such that

$$\sup \{P_{10}(k); k \geq 0\} = \max \{P_{10}(k); K_{13} \leq k \leq K_{14}\} > 0.$$

The sequence (u_n) is nonnegative if and only if

$$A \geq \max \{P_9(k_1), P_{10}(k_2); 0 \leq k_1 \leq K_{12}, K_{13} \leq k_2 \leq K_{14}\}.$$

- If $B > 0, C > 0$ the sequence (u_n) is nonnegative if and only if

$$A \geq -B(\lambda_2/\lambda_1)^n - C(\lambda_3/\lambda_1)^n \quad (n \geq 0) \tag{2.5}$$

- If $\lambda_2 > 0$ and $\lambda_3 > 0$, then (2.5) holds if and only if $A \geq 0$.
- If $\lambda_2 > 0$ and $\lambda_3 < 0$, then

$$-B(\lambda_2/\lambda_1)^n - C(\lambda_3/\lambda_1)^n = \begin{cases} -B(\lambda_2/\lambda_1)^{2k} - C(|\lambda_3|/\lambda_1)^{2k} & \text{if } n = 2k \\ -B(\lambda_2/\lambda_1)^{2k+1} + C(|\lambda_3|/\lambda_1)^{2k+1} & \text{if } n = 2k + 1. \end{cases}$$

The case of even n is ignored because of negative terms. For odd n , this is of Type - F+Z, and so by Lemma 1.2, there is a computable integer $K_{15} \in \mathbb{N} \cup \{0\}$ such that (u_n) is nonnegative if and only if

$$A \geq \max \{0, -B(\lambda_2/\lambda_1)^{2k+1} + C(|\lambda_3|/\lambda_1)^{2k+1}; 0 \leq k \leq K_{15}\}.$$

- For $\lambda_2 < 0, \lambda_3 > 0$, then

$$-B(\lambda_2/\lambda_1)^n - C(\lambda_3/\lambda_1)^n = \begin{cases} -B(|\lambda_2|/\lambda_1)^{2k} - C(\lambda_3/\lambda_1)^{2k} & \text{if } n = 2k \\ B(|\lambda_2|/\lambda_1)^{2k+1} - C(\lambda_3/\lambda_1)^{2k+1} & \text{if } n = 2k + 1. \end{cases}$$

The case of even n is ignored because of negative terms. For odd n , this is of Type F-Z, and so by Lemma 1.2, there are computable integers $K_{17} > K_{16} \in \mathbb{N} \cup \{0\}$ such that (u_n) is nonnegative if and only if

$$A \geq \max \{B(|\lambda_2|/\lambda_1)^{2k+1} - C(\lambda_3/\lambda_1)^{2k+1}; K_{16} \leq k \leq K_{17}\}.$$

- For $\lambda_2 < 0, \lambda_3 < 0$, then

$$-B(\lambda_2/\lambda_1)^n - C(\lambda_3/\lambda_1)^n = \begin{cases} -B(|\lambda_2|/\lambda_1)^{2k} - C(|\lambda_3|/\lambda_1)^{2k} & \text{if } n = 2k \\ B(|\lambda_2|/\lambda_1)^{2k+1} + C(|\lambda_3|/\lambda_1)^{2k+1} & \text{if } n = 2k + 1. \end{cases}$$

The case of even n is ignored because of negative terms. For odd n , this is of Type F+Z, and so by Lemma 1.2, the sequence (u_n) is nonnegative if and only if

$$A \geq B(|\lambda_2|/\lambda_1) + C(|\lambda_3|/\lambda_1).$$

3. Case $D = 0$

3.1. $\alpha = 0$

The general term of the sequence is of the form

$$u_n = (A + Bn + Cn^2)\lambda^n \quad (n \geq 0),$$

where $\lambda \in \mathbb{R} \setminus \{0\}$. We have two subcases depending on the sign of λ .

3.1.1. $\lambda < 0$

The sequence (u_n) is nonnegative if and only if, for each n , either $A + Bn + Cn^2 = 0$ or $\text{sign}(A + Bn + Cn^2) = \text{sign}(\lambda^n)$. Since $\text{sign}(A + Bn + Cn^2)$ changes at most twice, the sequence (u_n) is nonnegative if and only if $A + Bn + Cn^2 \equiv 0$, i.e., if and only if $A = B = C = 0$.

3.1.2. $\lambda > 0$

The sequence (u_n) is nonnegative if and only if $A + Bn + Cn^2 \geq 0$ for all $n \geq 0$. Since $A + Bn + Cn^2$ is a quadratic polynomial in n , the sequence (u_n) is nonnegative if and only if $C \geq 0$ and either

- the quadratic polynomial $A + Bn + Cn^2$ has no real root, which is equivalent to $B^2 - 4AC < 0$, or
- the quadratic polynomial $A + Bn + Cn^2$ has two (possibly equal) real roots which is equivalent to $B^2 - 4AC \geq 0$, and there are no nonnegative integers in the open interval between the two roots $r_1 = (-B + \sqrt{B^2 - 4AC})/2C$ and $r_2 = (-B - \sqrt{B^2 - 4AC})/2C$.

3.2. $\alpha \neq 0$

The general term of the sequence is of the form

$$u_n = A\lambda_1^n + (B + Cn)\lambda_2^n \quad (n \geq 0),$$

where $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}$, $\lambda_1 \neq \lambda_2$. We distinguish three possibilities according to the absolute values of the two roots.

3.2.1. $|\lambda_1| > |\lambda_2| > 0$

Rewriting

$$u_n = \lambda_1^n \{A + (B + Cn)(\lambda_2/\lambda_1)^n\},$$

we further subdivide into two subcases depending on the sign of λ_1 .

- If $\lambda_1 < 0$, since $(B + Cn)(\lambda_2/\lambda_1)^n \rightarrow 0$ ($n \rightarrow \infty$) and $\text{sign}(\lambda_1^n)$ oscillates, the sequence (u_n) is nonnegative only when $A = 0$, which yields $u_n = (B + Cn)\lambda_2^n$. This is exactly the same as Lemma 4 of [9], which has already been shown to be decidable.
- If $\lambda_1 > 0$, since $\text{sign}\{A + (B + Cn)(\lambda_2/\lambda_1)^n\} = \text{sign}(A)$ when n is large, for the sequence (u_n) to be nonnegative we must have $A \geq 0$ and

$$A \geq -(B + Cn)(\lambda_2/\lambda_1)^n \quad (n \geq 0). \quad (3.1)$$

Clearly, there is a computable integer $T \in \mathbb{N} \cup \{0\}$ such that (3.1) holds if and only if $A \geq \max\{-(B + Cn)(\lambda_2/\lambda_1)^n; 0 \leq n \leq T\}$. Consequently, the sequence (u_n) is nonnegative if and only if

$$A \geq \max\{0, -(B + Cn)(\lambda_2/\lambda_1)^n; 0 \leq n \leq T\}.$$

3.2.2. $|\lambda_1| < |\lambda_2|$

Write

$$u_n = \lambda_2^n \{A(\lambda_1/\lambda_2)^n + (B + Cn)\}.$$

We distinguish two subcases according to the sign of λ_2 .

- $\lambda_2 < 0$.
Since $\text{sign}(A(\lambda_1/\lambda_2)^n + (B + Cn)) = \text{sign}(C)$ when n is large enough, and $\text{sign}(\lambda_2^n)$ oscillates, the sequence (u_n) is nonnegative only when $C = 0$, and so $u_n = \lambda_2^n (A(\lambda_1/\lambda_2)^n + B)$. Since $\text{sign}(A(\lambda_1/\lambda_2)^n + B) = \text{sign}(B)$ when n is sufficiently large, we conclude that the sequence (u_n) is nonnegative only when $B = 0$ and so $u_n = A\lambda_1^n$. Hence, the sequence (u_n) is nonnegative if and only if $A \geq 0$ and $\lambda_1 > 0$.
- $\lambda_2 > 0$.
The sequence (u_n) is nonnegative if and only if

$$B \geq R(n) := -A(\lambda_1/\lambda_2)^n - Cn \quad (n \geq 0). \quad (3.2)$$

It is now a matter of finding $\sup_{n \geq 0} R(n)$. Clearly, we can rule out the situation where $C < 0$ because no real number B satisfies (3.2) for all $n \geq 0$.

- If $C = 0$, then $R(n) = -A(\lambda_1/\lambda_2)^n \rightarrow 0$ ($n \rightarrow \infty$) and so there is a computable integer $T_2 \in \mathbb{N} \cup \{0\}$ such that $\sup \{R(n); n \geq 0\} = \max \{0, R(n); 0 \leq n \leq T_2\}$; the 0 on the right-hand maximum is to take care of the case when $R(n) \leq 0$ for all $n \geq 0$. Thus, (3.2) holds if and only if

$$B \geq \max \{0, R(n); 0 \leq n \leq T_2\}.$$

- If $C > 0$, since $R(n) \rightarrow -\infty$ ($n \rightarrow \infty$), there is a computable integer $T_3 \in \mathbb{N} \cup \{0\}$ such that $\max \{R(n); n \geq 0\} = \max \{R(n); 0 \leq n \leq T_3\}$ and so (3.2) holds if and only if

$$B \geq \max \{R(n); 0 \leq n \leq T_3\}.$$

3.2.3. $|\lambda_1| = |\lambda_2|$

- $\lambda_1 = -\lambda_2 > 0$. The general term of the sequence is

$$u_n = \{A + (-1)^n(B + Cn)\} \lambda_1^n = \begin{cases} \{A + (B + 2Ck)\} \lambda_1^{2k} & \text{if } n = 2k \\ \{A - (B + C + 2Ck)\} \lambda_1^{2k+1} & \text{if } n = 2k + 1. \end{cases}$$

Then the sequence (u_n) is nonnegative if and only if for all $k \geq 0$ we have

$$A - B - C \geq 2Ck \geq -A - B.$$

Observe that C must vanish for if $C > 0$, the left-hand inequality cannot hold for sufficiently large k , while for $C < 0$, the right-hand inequality is untenable for large k . Thus, the sequence (u_n) is nonnegative if and only if

$$A - B \geq 0 \geq -A - B \iff A \geq |B|.$$

- $\lambda_1 = -\lambda_2 < 0$. Here,

$$u_n = \{(-1)^n A + B + Cn\} \lambda_2^n = \begin{cases} (A + B + 2Ck) \lambda_2^{2k} & \text{if } n = 2k \\ (-A + B + C + 2Ck) \lambda_2^{2k+1} & \text{if } n = 2k + 1. \end{cases}$$

The sequence (u_n) is nonnegative if and only if for all $k \geq 0$, we must have

$$A + B + 2kC \geq 0 \quad \text{and} \quad -A + B + (2k + 1)C \geq 0.$$

Now, $C \geq 0$ since both inequalities do not hold if $C < 0$ when k is large enough. Thus, the two inequalities hold for all $k \geq 0$ if and only if

$$C \geq 0, \quad A + B \geq 0 \quad \text{and} \quad C \geq A - B.$$

4. Case $D < 0$

There remains the case of one real and two complex conjugate roots. Note first that the coefficient A is in fact a real number because by direct checking in Proposition 1.1, we find $A = \bar{A}$. The general term of the sequence is of the form

$$u_n = A\lambda_1^n + B\lambda_2^n + \bar{B}\bar{\lambda}_2^n,$$

where $A, \lambda_1 \in \mathbb{R}$, $B \in \mathbb{C}$ and $\lambda_2 \in \mathbb{C} \setminus \mathbb{R}$. Let $\lambda_2 = |\lambda_2|e^{i\theta}$, $B = |B|e^{i\varphi}$, where $\theta, \varphi \in [-\pi, \pi)$, $\theta \notin \{-\pi, 0\}$ so that

$$u_n = A\lambda_1^n + 2|B||\lambda_2|^n \cos(\varphi + n\theta).$$

If $A = 0$, then $u_n = 2|B||\lambda_2|^n \cos(\varphi + n\theta)$. Since $\theta \neq 0$, when n varies over $\mathbb{N} \cup \{0\}$, by the same arguments as in the proof of Lemma 5 in [9], $\cos(\varphi + n\theta)$ takes both positive and negative values implying that the sequence (u_n) is nonnegative only when $B = 0$, i.e. $u_n \equiv 0$ for every n . Assume henceforth that $A \neq 0$. Our two main cases correspond to the signs of λ_1 .

4.1. $\lambda_1 < 0$

If $B = 0$, then $u_n = A\lambda_1^n$ oscillates between positive and negative values and so the sequence (u_n) is never nonnegative. For the rest of this subsection we assume $B \neq 0$.

4.1.1. $|\lambda_1| > |\lambda_2|$

Here,

$$u_n = \lambda_1^n \{A + 2|B|(|\lambda_2|/|\lambda_1|)^n \cos(\varphi + n\theta)\}.$$

Since $\text{sign}\{A + 2|B|(|\lambda_2|/|\lambda_1|)^n \cos(\varphi + n\theta)\} = \text{sign}(A)$ when n is large enough, and λ_1^n oscillates between $\pm |\lambda_1|$, the sequence (u_n) is nonnegative only when $A = 0$, which is not the case under consideration.

4.1.2. $|\lambda_1| = |\lambda_2|$

Here,

$$u_n = |\lambda_2|^n \{(-1)^n A + 2|B| \cos(\varphi + n\theta)\}.$$

The sequence (u_n) is nonnegative if and only if

$$-2|B| \cos(\varphi + 2k\theta) \leq A \leq 2|B| \cos(\varphi + (2k + 1)\theta) \quad (k \geq 0).$$

By the same arguments as in the proof of Lemma 5 in [9], as k varies over the nonnegative integers, since $\theta \neq 0$, the functions $\cos(\varphi + 2k\theta)$ and $\cos(\varphi + (2k + 1)\theta)$ take both positive and negative values. Thus, the two inequalities hold only when $A = B = 0$ which is not the case here.

4.1.3. $|\lambda_1| < |\lambda_2|$

Here,

$$u_n = A\lambda_1^n + 2|B\lambda_2^n| \cos(\varphi + n\theta) = |\lambda_2^n| \{A(\lambda_1/|\lambda_2|)^n + 2|B| \cos(\varphi + n\theta)\}.$$

Observe that $A(\lambda_1/|\lambda_2|)^n \rightarrow 0$ ($n \rightarrow \infty$). Next, by Lemma 1.3, $\cos(\varphi + n\theta)$ is either periodic or interval-filling. The periodic case occurs when $\theta = s\pi/t$ ($s, t(> 0) \in \mathbb{Z} \setminus \{0\}$, $\gcd(s, t) = 1$) is a multiple of π . Since $\theta \in [-\pi, \pi) \setminus \{-\pi, 0\}$, we have $t \geq 2$, and so in the periodic case $\cos(\varphi + n\theta)$ takes both positive and negative values. Thus, the sequence (u_n) is nonnegative only when $B = 0$ yielding $u_n = A\lambda_1^n$ which is oscillating (as $\lambda_1 < 0$) and so is nonnegative only when $A = 0$, which is not tenable here.

4.2. $\lambda_1 > 0$

If $B = 0$, then $u_n = A\lambda_1^n$ and so the sequence (u_n) is nonnegative if and only if $A \geq 0$. From now on, we assume that $B \neq 0$.

4.2.1. $|\lambda_1| > |\lambda_2|$

Here,

$$u_n = \lambda_1^n \{A + 2|B|(|\lambda_2|/\lambda_1)^n \cos(\varphi + n\theta)\}.$$

Since $\text{sign}\{A + 2|B|(|\lambda_2|/\lambda_1)^n \cos(\varphi + n\theta)\} = \text{sign}(A)$ when n is sufficiently large, for the sequence (u_n) to be nonnegative we must have $A \geq 0$ and

$$A \geq -2|B|(|\lambda_2|/\lambda_1)^n \cos(\varphi + n\theta) \quad (n \geq 0).$$

By Lemma 5 of [9], there is a least $N_L \in \mathbb{N} \cup \{0\}$ such that $\cos(\varphi + N_L\theta) < 0$. Since $(|\lambda_2|/\lambda_1)^n \rightarrow 0$ ($n \rightarrow \infty$) there is $N_M > N_L$ such that for all $n \geq N_M$ we have

$$-2|B|(|\lambda_2|/\lambda_1)^n \cos(\varphi + n\theta) < -2|B|(|\lambda_2|/\lambda_1)^{N_L} \cos(\varphi + N_L\theta).$$

Consequently, the sequence (u_n) is nonnegative if and only if

$$A \geq \max\{-2|B|(|\lambda_2|/\lambda_1)^n \cos(\varphi + n\theta); N_L \leq n \leq N_M\}.$$

4.2.2. $|\lambda_1| = |\lambda_2|$

Here,

$$u_n = \lambda_1^n \{A + 2|B| \cos(\varphi + n\theta)\}.$$

The sequence (u_n) is nonnegative if and only if

$$A \geq \max\{-2|B| \cos(\varphi + n\theta); n \geq 0\}. \quad (4.1)$$

If θ is a rational multiple of π , by Lemma 1.3(a), $\cos(\varphi + n\theta)$ takes on only finitely many explicitly computable values, say C_1, \dots, C_m , and so (4.1) holds if and only if

$$A \geq \max\{-2|B|C_1, \dots, -2|B|C_m\}.$$

If θ is not a rational multiple of π , by Lemma 1.3(b), (4.1) holds if and only if $A \geq 2|B|$.

4.2.3. $|\lambda_1| < |\lambda_2|$

Here,

$$u_n = |\lambda_2^n| \{A(\lambda_1/|\lambda_2|)^n + 2|B| \cos(\varphi + n\theta)\}.$$

The situation in this case is similar but simpler than that in Section 4.1.3, and by analogous arguments, we deduce that the sequence (u_n) is never nonnegative.

Summing up, we have:

Theorem 4.1. *The Positivity Problem is decidable for each sequence of integers satisfying a linear third order recurrence with integer coefficients.*

5. Final remark

Using the approach presented here, in order to settle the Positivity Problem for sequences satisfying linear recurrences of higher orders, it is necessary to obtain explicit shapes of all elements in the sequence. Yet from Galois theory, we know that general quintics or equations of higher degrees are not solvable by radicals. It is thus unlikely that the approach given here can be used to solve the Positivity Problem for general recurrence sequences whose characteristic polynomials are of degree

five or higher. However, there are some special equations of degree five or higher that are solvable by radicals and it may be of interest to determine whether the Positivity Problem for the special recurrence sequences with such corresponding characteristic polynomials is decidable by our approach. We hope to return to this question and the remaining case of general fourth order linear recurrence soon.

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