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System BV is NP-complete

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Abstract

System BV is an extension of multiplicative linear logic (MLL) with the rules *mix*, *nullary mix*, and a self-dual, noncommutative logical operator, called *seq*. While the rules *mix* and *nullary mix* extend the deductive system, the operator seq extends the language of MLL. Due to the operator seq, system BV extends the applications of MLL to those where the sequential composition is crucial, e.g., concurrency theory. System FBV is an extension of MLL with the rules *mix* and *nullary mix*. In this paper, by relying on the fact that system BV is a conservative extension of system FBV, I show that system BV is NP-complete by encoding the 3-Partition problem in FBV. I provide a simple completeness proof of this encoding by resorting to a novel proof theoretical method for reducing nondeterminism in proof search, which is also of independent interest.

MSC: 03F99

Keywords: Proof theory; Deep inference; Calculus of structures; System BV; NP-completeness; Nondeterminism

1. Introduction

Since its emergence, the multiplicative fragment of linear logic [5] has remained in the focus of researchers due to its resource conscious features that capture the properties of concurrent computation (see, e.g., [1]). Max Kanovich showed in [12,13] that multiplicative linear logic (MLL) is NP-complete. In [18], Lincoln and Winkler showed that the constant-only fragment of MLL is also NP-complete.

However, from the point of view of applications, multiplicative linear logic lacks a natural notion of sequentiality, which is crucial for expressing many computational phenomena, e.g., sequential composition of processes in concurrency theory. In [7], Guglielmi introduced a system, called BV, which is an extension of MLL with the rules mix, nullary mix (mix0), and a self-dual, noncommutative logical operator, called *seq*. While the rules mix and mix0 extend the deductive system, the operator seq extends the language of MLL. This logic captures sequential and parallel compositions of process algebra naturally by means of logical operators. Bruscoli showed, in [3], that there is a strict correspondence between a fragment of the process algebra CCS [19] and system BV.

System BV cannot be designed in any standard sequent calculus, as it was shown by Tiu in [26]: in the sequent calculus, during bottom-up proof search, inference rules are applied at the main connective; however, in order to obtain all the provable formulae of system BV by means of a deductive system, a notion of deep rewriting is necessary. System

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BV is designed in the proof theoretical formalism, the calculus of structures [7], which allows for such deep rewriting. The calculus of structures is a generalization of the sequent calculus. In the calculus of structures, the notion of main connective disappears and the notions of formula and sequent of the sequent calculus are replaced with the notion of structure. The inference rules can be applied deep inside structures, resulting in one of the distinguishing features of this formalism, that is, *deep inference*. In several other related works (see, e.g., [2,23]), deep inference gives rise to many interesting proof theoretical properties of other logics, e.g., classical logic, linear logic, that are not observable within the sequent calculus presentation of these logics.

The idea of deep inference can be also traced back to categorical logic (see, e.g., [17]). Along these lines, Hughes gave an explicit treatment of the relationship between deep inference proofs and categorical proofs in [9]. Some ideas on categorical semantics for deep inference proof theory and related proof nets can be found in [16,15,14].

Extending multiplicative linear logic with a self-dual, noncommutative operator was also considered in Retoré's pomset logic [20]. In [21], Retoré gave proof nets for the pomset logic, but so far there has been no sequent calculus system for pomset logic with the cut-elimination property. In fact, Guglielmi conjectured, in [7], that pomset logic and system BV are equivalent. Some ideas on this conjecture can be found in [23].

In [8], Guglielmi and Straßburger introduced a system, called NEL, which extends system BV with the exponentials of linear logic. In other words, system NEL is an extension of multiplicative exponential linear logic (MELL) with the rules mix, mix0, and the self-dual, noncommutative logical operator seq. Although it is unknown if multiplicative exponential linear logic is decidable or not, in [24], Straßburger showed that system NEL is undecidable. However, the complexity of the decision problem in system BV remained an open problem.

In this paper, by encoding the 3-Partition problem [4] in multiplicative linear logic extended by the rules mix and mix0, i.e., system FBV, I show the NP-hardness of this logic. This result implies the NP-hardness of system BV, because system BV is a conservative extension of system FBV (MLL + mix + mix0): every provable BV structure, which does not contain any seq structure, is also provable in FBV. Fig. 1 summarizes the relationship between MLL, FBV, BV, MELL, and NEL, and the contribution of this paper.

The encoding in the sequent calculus, which was used in [12] for showing the NP-hardness of MLL, can be used to show the NP-hardness of MLL + mix + mix0. However, in this paper I use an encoding of the 3-Partition problem. This problem was also used by Lincoln and Winkler, in [18], to show the NP-hardness of the constant-only fragment of MLL. By using a similar encoding, I provide a very simple correctness proof within the calculus of structures, by means of an analysis of the proof theory of this logic: in contrast to the sequent calculus, while applying the inference rules in bottom-up proof search, deep applicability of the inference rules in the calculus of structures introduces a greater nondeterminism. I introduce a novel technique for controlling the nondeterminism in proof search, which is also of independent interest from the point of view of applications.

Availability of deep inference does not only provide a richer combinatorial analysis of the logic being studied, but also provides shorter proofs than that in the sequent calculus. For example, there is a class of theorems, called the *Statman's tautologies*, for which the size of proofs in the sequent calculus grows exponentially over the size of the theorems. However, over the same class, there are deep inference proofs that grow polynomially [6]. This is because applicability of the inference rules at any depth inside a structure makes it possible to start the construction of a proof by manipulating and annihilating substructures without any prior branching. In order to see this on an example consider the following two proofs of multiplicative linear logic formula, respectively, in the one-sided sequent system for MLL + mix + mix0 and system FBV of the calculus of structures.

$$\dfrac{\dfrac{\dfrac{}{\vdash a\,,\,\bar a}\;\mathsf{id} \qquad \dfrac{\dfrac{}{\vdash b\,,\,\bar b}\;\mathsf{id}}{\vdash b \parr \bar b}\;\parr}{\vdash a\,,\,\bar a \otimes (b \parr \bar b)}\;\otimes}{\vdash a \parr (\bar a \otimes (b \parr \bar b))}\;\parr \qquad\qquad \mathsf{ai}\!\downarrow\dfrac{\mathsf{ai}\!\downarrow\dfrac{\circ\!\downarrow\dfrac{}{\vdash 1}}{\vdash a \parr \bar a}}{\vdash a \parr (\bar a \otimes (b \parr \bar b))}$$

The inference rules of MLL + mix0 + mix0 can be applied only at the main connective whereas the inference rules of system FBV can be applied at any depth inside a structure, and their application this way results in shorter proofs.[1]

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[1] In the calculus of structures, instead of using the usual infix notation of the logical expressions, the convention is to use mixfix notation. However, in this example I use the infix notation in order to make the contrast between the two proofs more obvious.

$$\begin{tikzcd}
\mathsf{MLL}\ (\text{NP-complete, [12,18]}) \arrow[r, "+\ \textit{exponentials}\ !,?"] \arrow[d, "\begin{array}{l}+\ \mathsf{mix}\\ +\ \mathsf{mix0}\end{array}"] & \mathsf{MELL} \arrow[dd, "\begin{array}{l}+\ \mathsf{mix}\\ +\ \mathsf{mix0}\\ +\ \textit{seq:}\ \text{self-dual,}\\ \ \ \text{noncommutative}\\ \ \ \text{operator}\end{array}"] \\
\mathsf{FBV}\ (\text{NP-complete, Corollary 6.2}) \arrow[d, "+\ \textit{seq:}\ \text{self-dual, noncommutative operator}"] & \\
\mathsf{BV}\ (\text{NP-complete, Theorem 6.1}) \arrow[r, "+\ \textit{exponentials}\ !,?"] & \mathsf{NEL}\ (\text{undecidable, [24]})
\end{tikzcd}$$

Fig. 1. The relationship between MLL, FBV, BV, MELL and NEL.

However, deep inference causes a greater nondeterminism by providing more premises in a bottom-up application of an inference rule to a structure. This provides many more different proofs of a structure, some of which are shorter than that in the sequent calculus. However, like in the sequent calculus, there are also instances of the inference rules in a bottom-up proof search step, which do not necessarily lead to a proof.

In this paper, I introduce a new technique, in the calculus of structures, that reduces nondeterminism in proof search, and makes these shorter proofs more immediately accessible. Despite the combinatoric explosion in the applicability of the inference rules in the calculus of structures, my method reduces the nondeterminism in proof search without damaging the completeness of the system. This way, it becomes possible to separate the redundant nondeterminism, in my encoding, from the concise nondeterminism, and prove the completeness of the encoding without going into incomprehensible and complicated case analysis.

The rest of the paper is organized as follows: after introducing the calculus of structures and system BV in the next section, I present a method for controlling the nondeterminism in proof search in multiplicative linear logic extended by the rules mix and mix0, i.e., system FBV. I then present an encoding of the 3-Partition problem in FBV, which is an NP-complete problem. Following this, by showing that the length of a proof in BV is bounded by a polynomial in the size of the structure being proved, I show that system BV is NP-complete.

2. The calculus of structures and system BV

This section re-collects some notions and definitions of the calculus of structures and system BV, following [7].

In the language of BV atoms are denoted by $a, b, c, \ldots$. Structures are denoted by $R, S, T, \ldots$ and generated by

$$S ::= \circ \mid a \mid \langle \underbrace{S; \ldots; S}_{>0} \rangle \mid [\underbrace{S, \ldots, S}_{>0}] \mid (\underbrace{S, \ldots, S}_{>0}) \mid \overline{S} \quad ,$$

where $\circ$, the *unit*, is not an atom. $\langle S; \ldots; S\rangle$ is called a *seq structure*, $[S, \ldots, S]$ is called a *par structure*, and $(S, \ldots, S)$ is called a *copar structure*, $\overline{S}$ is the *negation* of the structure S. Structures are considered equivalent modulo the relation $\approx$, which is the smallest congruence relation induced by the equalities shown in Fig. 2. There $\vec{R}$, $\vec{T}$, and $\vec{U}$ stand for finite, nonempty sequence of structures. A *structure context*, denoted as in $S\{\ \}$, is a structure with a hole that does not appear in the scope of negation. The structure R is a *substructure* of $S\{R\}$ and $S\{\ \}$ is its *context*. Context braces are omitted if no ambiguity is possible: for instance $S[R, T]$ stands for $S\{[R, T]\}$. A structure, or a structure context, is in *normal form* when the only negated structures appearing in it are atoms, and no unit $\circ$ appears in it.

We will call the BV structures, which do not involve seq structures, FBV structures. There is a straightforward correspondence between FBV structures and formulae of multiplicative linear logic (MLL), which do not contain the units 1 and $\perp$. For example $[(a, b), \bar{c}, \bar{d}]$ corresponds to $(\,(a \otimes b) \parr c^{\perp} \parr d^{\perp})$, and vice versa.

System BV extends MLL with the rules mix and mix0, and the seq operator.

$$\mathsf{mix}\ \frac{\vdash \Phi \quad \vdash \Psi}{\vdash \Phi, \Psi} \qquad\qquad \mathsf{mix0}\ \frac{}{\vdash}$$

When the rules mix and mix0 are added to MLL, it becomes possible to prove $1 \equiv \perp$. This allows one to map the units 1 and $\perp$ into a single unit $\circ$, which is the unit of system BV. For a more detailed discussion on the proof theory of BV and the precise relation between BV and MLL, the reader is referred to [7].

Associativity

$$\langle \vec{R}; \langle \vec{T} \rangle; \vec{U} \rangle \approx \langle \vec{R}; \vec{T}; \vec{U} \rangle$$
$$[\vec{R}, [\vec{T}]] \approx [\vec{R}, \vec{T}]$$
$$(\vec{R}, (\vec{T})) \approx (\vec{R}, \vec{T})$$

Context Closure

if $R \approx T$ then $S\{R\} \approx S\{T\}$ and $\bar{R} \approx \bar{T}$

Commutativity

$$[\vec{R}, \vec{T}] \approx [\vec{T}, \vec{R}]$$
$$(\vec{R}, \vec{T}) \approx (\vec{T}, \vec{R})$$

Units

$$\langle \circ; \vec{R} \rangle \approx \langle \vec{R}; \circ \rangle \approx \langle \vec{R} \rangle$$
$$[\circ, \vec{R}] \approx [\vec{R}]$$
$$(\circ, \vec{R}) \approx (\vec{R})$$

Negation

$$\bar{\circ} \approx \circ$$
$$\overline{\langle R; T \rangle} \approx \langle \overline{R}; \overline{T} \rangle$$
$$\overline{[R, T]} \approx (\overline{R}, \overline{T})$$
$$\overline{(R, T)} \approx [\overline{R}, \overline{T}]$$
$$\overline{\overline{R}} \approx R$$

Singleton

$$\langle R \rangle \approx [R] \approx (R) \approx R$$

Fig. 2. Equivalence relations underlying BV.

$$\circ\downarrow \frac{}{\circ} \qquad \mathsf{ai}\downarrow \frac{S\{\circ\}}{S[a, \bar{a}]} \qquad \mathsf{s} \frac{S([R, U], T)}{S[(R, T), U]} \qquad \mathsf{q}\downarrow \frac{S\langle [R, U]; [T, V] \rangle}{S[\langle R; T \rangle, \langle U; V \rangle]}$$

Fig. 3. System BV.

In the calculus of structures, an *inference rule* is a scheme of the kind $\rho \dfrac{T}{R}$, where ρ is the *name* of the rule, T is its *premise* and R is its *conclusion*. A typical (deep) inference rule has the shape $\rho \dfrac{S\{T\}}{S\{R\}}$ and specifies the implication $T \Rightarrow R$ inside a generic context $S\{\ \}$, which is the implication being modeled in the system.[2] When premise and conclusion in an instance of an inference rule are equivalent, that instance is *trivial*, otherwise it is *nontrivial*. An inference rule is called an *axiom* if its premise is empty. Rules with empty contexts correspond to the case of the sequent calculus.

A (formal) *system* $\mathscr{S}$ is a set of inference rules. A derivation Δ in a certain formal system is a finite chain of instances of inference rules in the system. A derivation can consist of just one structure. The top-most structure in a derivation, if present, is called the *premise* of the derivation, and the bottom-most structure is called its *conclusion*. A derivation Δ whose premise is T, conclusion is R, and inference rules are in $\mathscr{S}$ will be written as $\begin{array}{c} T \\ \Delta \Vert \mathscr{S} \\ R \end{array}$. Similarly, $\begin{array}{c} \Pi \Vert \mathscr{S} \\ R \end{array}$ will denote a *proof* Π, which is a finite derivation, whose top-most inference rule is an axiom. The *length* of a derivation (proof) is the number of instances of inference rules appearing in it.

Two systems $\mathscr{S}$ and $\mathscr{S}'$ are *equivalent* if for every proof of a structure T in system $\mathscr{S}$, there exists a proof of T in system $\mathscr{S}'$, and vice versa.

The system $\{\circ\downarrow, \mathsf{ai}\downarrow, \mathsf{s}, \mathsf{q}\downarrow\}$, shown in Fig. 3, is denoted by BV, and called *basic system* V. The rules of the system are called *unit* ($\circ\downarrow$), *atomic interaction* ($\mathsf{ai}\downarrow$), *switch* (s), and *seq* ($\mathsf{q}\downarrow$). The multiplicative linear logic system extended by mix and mix0, or system $\{\circ\downarrow, \mathsf{ai}\downarrow, \mathsf{s}\}$, is denoted by FBV.

3. Nondeterminism in proof search

In a proof search episode, the inference rules can be applied to a structure, nondeterministically, in many different ways, but only few of these rule instances can provide a proof. While providing a rich combinatorial analysis of the logic being studied, applicability of the inference rules at any depth causes an even greater nondeterminism. However, the mutual dependencies between atoms, which are easily observable due to the notion of structure, provide ways of

[2] Due to duality between $T \Rightarrow R$ and $\bar{R} \Rightarrow \bar{T}$, rules come in pairs of dual rules: a down version and an up version. For instance, the dual of the $\mathsf{ai}\downarrow$ rule in Fig. 3 is the cut rule. In this paper, only the down rules, which provide a sound and complete system are considered.

controlling the nondeterminism without breaking the proof theoretical properties. In this section, I present a system equivalent to system FBV, where the nondeterminism in proof search is reduced by taking these mutual dependencies between dual atoms into consideration.

Definition 3.1. Given a structure R, $\mathsf{at}\, R$ is the set of all the atoms appearing in R.

Definition 3.2. [7] Given a structure R, we talk about *atom occurrences* when considering all the atoms appearing in R as distinct (for example, by indexing them so that two atoms, which are equal, get different indices). $\mathsf{occ}\, R$ is the set of all the atom occurrences appearing in R. The *size* of R is the cardinality of the set $\mathsf{occ}\, R$.

Definition 3.3. [7] Let R be a structure in normal form, the *structural relation* $\downarrow_R$ is the minimal set such that $\downarrow_R \subset (\mathsf{occ}\, R)^2$ and, for every $S\{\ \}$, U and V and for every a in U and b in V, the following holds: if $R = S[U, V]$ then $a \downarrow_R b$. To a structure that is not in normal form we associate the structural relation obtained from any of its normal forms, since they yield the same relation $\downarrow_R$. The notation $|\downarrow_R|$ denotes the cardinality of the set $\downarrow_R$.

Example 3.4. In order to see the above definition at work, consider the following structure: $R = [(\bar{a}, \bar{b}), a, b]$. We have $\mathsf{at}\, R = \mathsf{occ}\, R = \{a, \bar{a}, b, \bar{b}\}$. Then, in $\downarrow_R$, we have $a \downarrow b$, $a \downarrow \bar{b}$, $a \downarrow \bar{a}$, $b \downarrow \bar{b}$, $b \downarrow \bar{a}$, (we omit the symmetric relations, e.g., $b \downarrow a$).

Intuitively, one can consider the relation $\downarrow_R$ as a notion of interaction: the atoms which are related by $\downarrow_R$ are interacting atoms, whereas others are noninteracting. Proofs are constructed by isolating the atoms, by breaking the interaction between some atoms, and this way promoting the interaction between dual atoms, until dual atoms establish a closer interaction in which they can annihilate each other at an application of the atomic interaction rule. During a bottom-up proof search episode, while acting on structures, inference rules perform such an isolation of atoms: in an instance of an inference rule with the conclusion R, a subset of $\downarrow_R$ holds in the premise.

Example 3.5. Consider the following three instances of the switch rule with the structure $[(\bar{a}, \bar{b}), a, b]$ at the conclusion:

$$\text{(i)}\quad \mathsf{s}\,\frac{([\bar{a}, a, b], \bar{b})}{[(\bar{a}, \bar{b}), a, b]} \qquad \text{(ii)}\quad \mathsf{s}\,\frac{[([\bar{a}, b], \bar{b}), a]}{[(\bar{a}, \bar{b}), a, b]} \qquad \text{(iii)}\quad \mathsf{s}\,\frac{[(\bar{a}, \bar{b}, a), b]}{[(\bar{a}, \bar{b}), a, b]}$$

While going up, from conclusion to premise, in (i) $a \downarrow \bar{b}$ and $b \downarrow \bar{b}$; in (ii) $b \downarrow \bar{b}$; in (iii) $a \downarrow \bar{a}$ and $a \downarrow \bar{b}$ cease to hold. However, none of these derivations can lead to a proof.

The following proposition expresses the intuition behind this.

Proposition 3.6. *If a structure R has a proof in BV then, for all the atoms a that appear in R, there is an atom $\bar{a}$ in R such that $a \downarrow_R \bar{a}$.*

Often, inference rules can be applied to a structure in many different ways, however only few of these applications can lead to a proof. For example, to the structure $[(\bar{a}, \bar{b}), a, b]$ switch rule can be applied bottom up in twelve different ways, three of which are given above, but only two of these instances can lead to a proof. With the definition below, we will redesign the switch rule so that only these applications will be possible.

Definition 3.7. A structure R is a *proper par*, if there are two structures R' and R'' with $R = [R', R'']$ and $R' \neq \circ \neq R''$.

Example 3.8. Structures $[a, b]$ and $[a, (b, c)]$ are proper par structures, whereas structures $[a, \circ]$ and (b, c) are not.

Definition 3.9. The *lazy interaction switch* is the rule

$$\mathsf{lis}\,\frac{S([R, U], T)}{S[(R, T), U]}$$

where structure U is not a proper par structure and $\mathsf{at}\,\overline{R} \cap \mathsf{at}\, U \neq \emptyset$.

Example 3.10. Observe that the rule lis can be applied bottom up to the structure $[(\bar{a}, \bar{b}), a, b]$ only in the following two ways, which lead to proofs.

$$\mathsf{lis}\,\frac{[([\bar{a}, a], \bar{b}), b]}{[(\bar{a}, \bar{b}), a, b]} \qquad \mathsf{lis}\,\frac{[([\bar{b}, b], \bar{a}), a]}{[(\bar{a}, \bar{b}), a, b]}$$

The switch rule can be safely replaced with the lazy interaction switch rule in system FBV without losing completeness. In the following, we will collect some definitions and lemmas which are necessary to prove this result.

Definition 3.11. *System* FBV *with lazy interaction switch*, or system FBVi is the system $\{\circ\!\downarrow, \mathsf{ai}\!\downarrow, \mathsf{lis}\}$.

Proposition 3.12. *In* FBVi (FBV)*, structure* (R, T) *is provable if and only if structures* R *and* T *are provable.*

Definition 3.13. Let R, T be FBV structures such that $R \neq \circ \neq T$. R and T are *independent* if and only if

$$\begin{array}{c} \\ \| \mathsf{FBVi} \\ [R, T] \end{array} \quad \text{implies} \quad \begin{array}{c} \\ \| \mathsf{FBVi} \\ R \end{array} \quad \text{and} \quad \begin{array}{c} \\ \| \mathsf{FBVi} \\ T \end{array} \quad .$$

Otherwise, they are dependent.

Example 3.14. For the structure $S = [a, b, (\bar{a}, \bar{b}), c, \bar{c}]$,

$$R = [a, b, (\bar{a}, \bar{b})] \quad \text{and} \quad T = [c, \bar{c}]$$

are independent, whereas

$$R' = [a, b] \quad \text{and} \quad T' = [(\bar{a}, \bar{b}), c, \bar{c}]$$

are dependent.

Proposition 3.15. *For any* FBV *structures* R *and* T*, if* $\mathsf{at}\,\bar{R} \cap \mathsf{at}\,T = \emptyset$ *then* R *and* T *are independent.*

Proof. Assume that there is a proof Π of $[R, T]$. Construct a proof of R by replacing all the substructures of T in Π with $\circ$: All the instances of the rules s remain intact. Further, from Proposition 3.6 it follows that all the instances of the rule $\mathsf{ai}\!\downarrow$ remain intact, because for every atom $a \in \mathsf{at}\,[R, T]$ there must be an atom $\bar{a} \in \mathsf{at}\,[R, T]$ and we have that $\mathsf{at}\,\bar{R} \cap \mathsf{at}\,T = \emptyset$. This implies that each instance of the rule $\mathsf{ai}\!\downarrow$ in Π annihilates an atom and its dual that are both either in $\mathsf{at}\,R$ or in $\mathsf{at}\,T$. Analogously, construct a proof of T by replacing all the substructures of R in Π with $\circ$. $\square$

Lemma 3.16. *For any* FBV *structures* P*,* U*, and* R*, if* $[P, U]$ *has a proof in* FBVi*, then there is a derivation*

$$\begin{array}{c} R \\ \| \mathsf{FBVi} \\ [(R, P), U] \end{array} \quad .$$

Proof. We label each atom occurring in proof Π of $[P, U]$ such that every pair of atom that is annihilated by an application of the rule $\mathsf{ai}\!\downarrow$ get the same label, and the conclusion of each rule instance in Π consists of pairwise distinct atoms. If U is a proper par, then there must be U_1 and U_2 such that $U = [U_1, U_2]$, and $\mathsf{at}\,[P, U_1] \cap \mathsf{at}\,\overline{U_2} = \emptyset$. If U is not a proper par then it must be that either $U_1 = U$ and $U_2 = \circ$ or $U_1 = \circ$ and $U_2 = U$. Thus, there must be a derivation

$$\begin{array}{c} [(R, [P, U_1]), U_2] \\ \Delta \| \{\mathsf{lis}\} \\ [(R, P), U] \end{array} \quad .$$

Given that $[P, U_1, U_2]$ is provable, from Proposition 3.15, it follows that $[P, U_1]$ and U_2 are independent, which implies that there are proofs $\begin{array}{c} \Pi_1 \| \mathsf{FBVi} \\ [P, U_1] \end{array}$ and $\begin{array}{c} \Pi_2 \| \mathsf{FBVi} \\ U_2 \end{array}$. We can then construct the following derivation:

$$\begin{array}{c} R \\ \Pi_2 \| \mathsf{FBVi} \\ [R, U_2] \\ \Pi_1 \| \mathsf{FBVi} \\ [(R, [P, U_1]), U_2] \\ \Delta \| \mathsf{FBVi} \\ [(R, P), U_1, U_2] \end{array} \quad \square$$

Remark 3.17. Let $R = S[a, \bar{a}]$ and $R' = S\{\circ\}$ be BV structures with pairwise distinct atoms. If $\mathsf{ai}\!\downarrow \frac{R'}{R}$, then $\downarrow_{R'} = \downarrow_R \setminus \{ (a, \bar{a}), (\bar{a}, a) \}$.

Remark 3.18. Let $R = S[(P, T), U]$ and $R' = S([P, U], T)$ be BV structures with pairwise distinct atoms. If $\mathsf{s}\, \frac{R'}{R}$, then

$$\downarrow_{R'} = \downarrow_R \setminus (\{(x, y) \mid x \in \mathsf{occ}\, T \wedge y \in \mathsf{occ}\, U\} \cup \{(x, y) \mid x \in \mathsf{occ}\, U \wedge y \in \mathsf{occ}\, T\}).$$

The following theorem is a specialization of the shallow splitting theorem which was introduced, in [7], for proving cut elimination for system BV. As the name suggests, this theorem splits the context of a structure so that the proof of the structure can be partitioned into smaller pieces in a systematic way. Below we show that splitting theorem can be specialized to system FBVi where the switch rule in system FBV is replaced with the lazy interaction switch rule. In the proof of the theorem, in contrast to Guglielmi's two-dimensional induction measure in [7], I use a one-dimensional induction measure by using Remark 3.17 and Remark 3.18. This results in a simpler proof.

Theorem 3.19 (*Shallow Splitting for* FBVi). *For all structures R, T, and P, if $[(R, T), P]$ is provable in* FBVi *then there exist P_1, P_2, and* $\begin{array}{c} [P_1, P_2] \\ \Delta \| \mathsf{FBVi} \\ P \end{array}$ *such that $[R, P_1]$ and $[T, P_2]$ are provable in* FBVi.

Proof. Let Π be the proof of the structure $[(R, T), P]$ in FBVi. Proof by induction on the cardinality of $\downarrow_{[(R,T),P]}$ by Remark 3.17 and Remark 3.18: Single out the bottom-most rule application ρ in Π. The base case being trivial, the following are the inductive cases for ρ: we assume that $P \neq \circ$ (If this is not the case, take $P_1 = \circ = P_2$). Let us reason on the position of the redex of ρ in $[(R, T), P]$. There are the following possibilities:

(1) $\rho = \mathsf{ai}\!\downarrow$: The following cases exhaust the possibilities:

(a) The redex is inside R:

$$\text{given} \quad \mathsf{ai}\!\downarrow \frac{\begin{array}{c} \| \\ [(R', T), P] \end{array}}{[(R, T), P]} \;, \qquad \text{consider} \quad \mathsf{ai}\!\downarrow \frac{\begin{array}{c} \| \\ [R', P_1] \end{array}}{[R, P_1]} \;.$$

(b) The redex is inside T: Analogous to the previous case.

(c) The redex is inside P:

$$\text{given} \quad \mathsf{ai}\!\downarrow \frac{\begin{array}{c} \| \\ [(R, T), P'] \end{array}}{[(R; T), P]} \;, \qquad \text{consider} \quad \begin{array}{c} [P_1, P_2] \\ \| \\ \mathsf{ai}\!\downarrow \dfrac{P'}{P} \end{array} \;.$$

(2) $\rho = \mathsf{lis}$: If the redex is inside R, T, or P, the proof is analogous to the cases for $\rho = \mathsf{ai}\!\downarrow$; otherwise the following cases exhaust the possibilities:

(a) $R = (R', R'')$, $T = (T', T'')$, $P = (P', P'')$ such that the bottom-most rule instance in Π is of the following form:

$$\mathsf{lis}\,\frac{[([(R',T'),P'],R'',T''),P'']}{[(R',R'',T',T''),P',P'']} \quad .$$

We can apply the induction hypothesis, by Remark 3.18, and we obtain

$$\begin{array}{c}[P'_1,P'_2]\\ \Delta_1\| \\ P''\end{array} \quad , \quad \begin{array}{c}\Pi\,\| \\ [(R',T'),P',P'_1]\end{array} \quad \textit{and} \quad \begin{array}{c}\Pi'\| \\ [(R'',T''),P'_2]\end{array} \quad .$$

Since $|\downarrow_{[(R',T'),P',P'_1]}| < |\downarrow_{[(R',R'',T',T''),P',P'']}| > |\downarrow_{[(R'',T''),P'_2]}|$, we can apply the induction hypothesis both on Π and Π', by Remark 3.18, and obtain the following derivations and proofs:

$$\begin{array}{c}[P''_1,P''_2]\\ \Delta_2\| \\ [P',P'_1]\end{array} \quad , \quad \begin{array}{c}\Pi_1\| \\ [R',P''_1]\end{array} \quad \textit{and} \quad \begin{array}{c}\Pi_2\| \\ [T',P''_2]\end{array} \quad .$$

$$\begin{array}{c}[P'''_1,P'''_2]\\ \Delta_3\| \\ P'_2\end{array} \quad , \quad \begin{array}{c}\Pi_3\| \\ [R'',P'''_1]\end{array} \quad \textit{and} \quad \begin{array}{c}\Pi_4\| \\ [T'',P'''_2]\end{array} \quad .$$

We can now take $P_1 = [P''_1, P'''_1]$, and $P_2 = [P''_2, P'''_2]$ and construct the derivation and the two proofs

$$\begin{array}{c}[P''_1,P''_2,P'''_1,P'''_2]\\ \Delta_3\| \\ [P''_1,P''_2,P'_2]\\ \Delta_2\| \\ [P',P'_1,P'_2]\\ \Delta_1\| \\ [P',P'']\end{array} \quad , \quad \begin{array}{c}\| \\ [R'',P'''_1]\\ \Delta_4\| \\ [(R',R''),P''_1,P'''_1]\end{array} \quad , \text{ and } \quad \begin{array}{c}\| \\ [T'',P'''_2]\\ \Delta_5\| \\ [(T',T''),P''_2,P'''_2]\end{array} \quad ,$$

where Δ_4 is the derivation delivered by Lemma 3.16 with proof Π_3, and Δ_5 is the derivation delivered by Lemma 3.16 with proof Π_4.

(b) $P = [(P', P''), U]$ such that the bottom-most rule instance in Π is of the following form:

$$\mathsf{lis}\,\frac{[([(R,T),P',U],P''),U'']}{[(R,T),(P',P''),U]} \quad .$$

We can apply the induction hypothesis, by Remark 3.18, and we obtain

$$\begin{array}{c}[U_1,U_2]\\ \| \\ U\end{array} \quad , \quad \begin{array}{c}\Pi_1\| \\ [(R,T),P',U_1]\end{array} \quad \textit{and} \quad \begin{array}{c}\Pi_2\| \\ [P'',U_2]\end{array} \quad .$$

Since $|\downarrow_{[(R,T),P',U_1]}| < |\downarrow_{[(R,T),(P',P''),U]}|$ (otherwise the lis instances would be trivial), we can apply the induction hypothesis on Π_1, by Remark 3.18, and obtain

$$\begin{array}{c}[P_1,P_2]\\ \| \\ [P',U_1]\end{array} \quad , \quad \begin{array}{c}\Pi_3\| \\ [R,P_1]\end{array} \quad \textit{and} \quad \begin{array}{c}\Pi_4\| \\ [T,P_2]\end{array} \quad .$$

We can now construct the derivation

$$\begin{array}{c}[P_1, P_2]\\ \Vert \\ [P', U_1] \\ \Delta\Vert \\ [(P', P''), U_1, U_2] \\ \Vert \\ [(P', P''), U]\end{array}$$

where Δ is the derivation delivered by Lemma 3.16 with proof Π_2. □

Since inference rules can be applied at any depth inside a structure, we need the following theorem for accessing the deeper structures. This theorem is a specialization of the context reduction theorem, in [7], for system BV to system FBVi.

Theorem 3.20 (*Context Reduction for* FBVi). *For all structures R and for all contexts $S\{\ \}$ such that $S\{R\}$ is provable in* FBVi*, there exists a structure U such that for all structures X there exist derivations:*

$$\begin{array}{c}[X, U]\\ \Vert\,\mathsf{FBVi} \\ S\{X\}\end{array} \qquad \textit{and} \qquad \begin{array}{c}\overline{\Vert}\,\mathsf{FBVi} \\ [R, U]\end{array} \quad .$$

Proof. By induction on the size of $S\{\circ\}$. The base case is trivial: $U = \circ$. There are three inductive cases:

(1) $S\{\ \} = (S'\{\ \}, P)$, for some $P \neq \circ$: By Proposition 3.12, there are proofs in FBVi of $S'\{R\}$ and of P. By applying the induction hypothesis, we can find a structure U and construct, for all X, the derivation

$$\begin{array}{c}[X, U]\\ \Vert\,\mathsf{FBVi} \\ S'\{X\} \\ \Vert\,\mathsf{FBVi} \\ (S'\{X\}, P)\end{array}$$

such that $[R, U]$ is provable in FBVi.

(2) $S\{\ \} = [S'\{\ \}, P]$, for some $P \neq \circ$ such that $S'\{\ \}$ is not a proper par: If $S'\{\circ\} = \circ$ then the theorem is proved; otherwise it must be that $S'\{\ \} = (S''\{\ \}, P')$, for some $P \neq \circ$. By Theorem 3.19 there exist the derivation and the two proofs

$$\begin{array}{c}[P_1, P_2]\\ \Delta\Vert\,\mathsf{FBVi} \\ P\end{array} \quad , \qquad \begin{array}{c}\Pi_1\overline{\Vert}\,\mathsf{FBVi} \\ [S''\{R\}, P_1]\end{array} \qquad \textit{and} \qquad \begin{array}{c}\Pi_2\overline{\Vert}\,\mathsf{FBVi} \\ [P', P_2]\end{array} \quad .$$

By applying the induction hypothesis to Π_1, we can construct the derivation

$$\begin{array}{c}[X, U]\\ \Vert\,\mathsf{FBVi} \\ [S''\{X\}, P_1] \\ \Delta'\Vert\,\mathsf{FBVi} \\ [(S'\{X\}, P'), P_1, P_2] \\ \Delta\Vert\,\mathsf{FBVi} \\ [(S''\{X\}, P'), P]\end{array} \qquad \textit{and} \qquad \begin{array}{c}\Pi\overline{\Vert}\,\mathsf{FBVi} \\ [R, U]\end{array}$$

where Δ' is the derivation delivered by Lemma 3.16 with proof Π_2. □

Theorem 3.21. *System* FBV *and* FBVi *are equivalent.*

Proof. Observe that every proof in FBVi is also a proof in FBV. For the other direction, single out the upper-most instance of the switch rule in the FBV proof, which is not an instance of the interaction switch rule:

$$\begin{array}{c} \Vert\,\mathsf{FBVi} \\ \mathsf{s}\,\dfrac{S([R,U],T)}{S[(R,T),U]} \end{array}$$

From Theorem 3.20, we have

$$\begin{array}{c} [\{\ \},V] \\ \Vert\,\mathsf{FBVi} \\ S\{\ \} \end{array} \qquad \text{such that} \qquad \begin{array}{c} \Vert\,\mathsf{FBVi} \\ [([R,U],T),V] \end{array} .$$

Then, from Theorem 3.19, we obtain

$$\begin{array}{c} [K_1,K_2] \\ \Vert\,\mathsf{FBVi} \\ V \end{array} , \qquad \begin{array}{c} \Vert\,\mathsf{FBVi} \\ [R,U,K_1] \end{array} \qquad \text{and} \qquad \begin{array}{c} \Pi\,\Vert\,\mathsf{FBVi} \\ [K_2,T] \end{array} .$$

We can then construct the following proof

$$\begin{array}{c} \Vert\,\mathsf{FBVi} \\ [R,U,K_1] \\ \Delta\,\Vert\,\mathsf{FBVi} \\ [(R,T),U,K_1,K_2] \\ \Vert\,\mathsf{FBVi} \\ [(R,T),U,V] \\ \Vert\,\mathsf{FBVi} \\ S[(R,T),U] \end{array}$$

where Δ is the derivation delivered by Lemma 3.16 with the proof Π. Repeat the above procedure inductively until all the instances of the switch rule, which are not instances of interaction switch rule, are removed. □

It is important to observe that proof search in FBVi involves much less nondeterminism than FBV.

Example 3.22. Consider the following six proofs of the structure $[(\bar{a},\bar{b}),a,b]$ in FBVi, which are the only possible derivations in system FBVi with this structure at the conclusion:

$$\mathsf{lis}\,\dfrac{\mathsf{ai}{\downarrow}\,\dfrac{\mathsf{ai}{\downarrow}\,\dfrac{\circ{\downarrow}\,\dfrac{-}{\circ}}{[b,\bar{b}]}}{[([a,\bar{a}],\bar{b}),b]}}{[(\bar{a},\bar{b}),a,b]} \qquad \mathsf{lis}\,\dfrac{\mathsf{lis}\,\dfrac{\mathsf{ai}{\downarrow}\,\dfrac{\mathsf{ai}{\downarrow}\,\dfrac{\circ{\downarrow}\,\dfrac{-}{\circ}}{[a,\bar{a}]}}{([a,\bar{a}],[b,\bar{b}])}}{[([a,\bar{a}],\bar{b}),b]}}{[(\bar{a},\bar{b}),a,b]} \qquad \mathsf{lis}\,\dfrac{\mathsf{lis}\,\dfrac{\mathsf{ai}{\downarrow}\,\dfrac{\mathsf{ai}{\downarrow}\,\dfrac{\circ{\downarrow}\,\dfrac{-}{\circ}}{[b,\bar{b}]}}{([a,\bar{a}],[b,\bar{b}])}}{[([a,\bar{a}],\bar{b}),b]}}{[(\bar{a},\bar{b}),a,b]}$$

$$\mathsf{lis}\,\dfrac{\mathsf{ai}{\downarrow}\,\dfrac{\mathsf{ai}{\downarrow}\,\dfrac{\circ{\downarrow}\,\dfrac{-}{\circ}}{[a,\bar{a}]}}{[a,(\bar{a},[b,\bar{b}])]}}{[(\bar{a},\bar{b}),a,b]} \qquad \mathsf{lis}\,\dfrac{\mathsf{lis}\,\dfrac{\mathsf{ai}{\downarrow}\,\dfrac{\mathsf{ai}{\downarrow}\,\dfrac{\circ{\downarrow}\,\dfrac{-}{\circ}}{[b,\bar{b}]}}{([a,\bar{a}],[b,\bar{b}])}}{[a,(\bar{a},[b,\bar{b}])]}}{[(\bar{a},\bar{b}),a,b]} \qquad \mathsf{lis}\,\dfrac{\mathsf{lis}\,\dfrac{\mathsf{ai}{\downarrow}\,\dfrac{\mathsf{ai}{\downarrow}\,\dfrac{\circ{\downarrow}\,\dfrac{-}{\circ}}{[a,\bar{a}]}}{([a,\bar{a}],[b,\bar{b}])}}{[a,(\bar{a},[b,\bar{b}])]}}{[(\bar{a},\bar{b}),a,b]}$$

However, in system FBV, in the proof search space of structure $[(\bar{a}, \bar{b}), a, b]$, there are 358 derivations including the 6 proofs above, and no other proofs.

In the above proof, we have used the splitting technique to prove the completeness of system FBVi. The splitting technique was originally introduced, in [7], to prove cut elimination for system BV. Due to the central importance of cut elimination in the design of deductive systems, system FBVi remains clean from a proof theoretic point of view: Cut elimination for system FBVi follows from the splitting theorem, analogous to the cut-elimination proof in [7].

Proposition 3.23. *System* BV *is a conservative extension of system* FBV*, that is, if a structure R, not containing any seq structures, is provable in* BV*, then it is also provable in* FBV.

Proof. Let R be a BV structure that does not contain any seq structures. By induction on the length of the proof Π of R in BV, construct the proof Π' of R in FBV. Since the only rule that involves seq structures is the rule $\mathsf{q}\!\downarrow$, it must be $\Pi = \Pi'$. □

4. BV is NP-hard

In this section, I present an encoding of the 3-Partition Problem in system FBV to show the NP-hardness of this logic, and system BV. This problem was also used by Lincoln and Winkler, in [18], to show the NP-hardness of the constant-only fragment of MLL. By providing a similar encoding, and resorting to the proof theoretical ideas developed in the previous section, I provide a very simple correctness proof without going into a complicated case analysis.

Problem 4.1. [4] (3-Partition) Given a set of $A = \{a_1, a_2, \ldots, a_{3m}\}$ of elements, a bound $B \in \mathbb{Z}^+$, and a size $S(a) \in \mathbb{Z}^+$ for each $a \in A$ such that $\frac{1}{4}B < S(a) < \frac{1}{2}B$ and $\sum_{a \in A} S(a) = Bm$, does there exist a partition of A into m disjoint subsets A_i so that $\sum_{a \in A_i} S(a) = B$ for each A_i in the partition?

The constraints on the $S(a)$ imply that such a partition must have exactly three elements in each of its sets. This problem is NP-complete in the strong sense, which implies that even when the input is represented in unary, the problem is NP-hard. This property of 3-Partition is essential for my encoding, because I represent the input problem by using atoms.

4.1. Encoding the 3-Partition problem in FBV

Given an instance of 3-Partition equipped with a set $A = \{a_1, a_2, \ldots, a_{3m}\}$, a unary function S, and a natural number B, presented as a tuple $\langle A, m, B, S \rangle$, the encoding function θ is defined as $\theta(\langle A, m, B, S \rangle) =$

$$[(k, [\underbrace{c, \ldots, c}_{\times S(a_1)}]), \ldots, (k, [\underbrace{c, \ldots, c}_{\times S(a_{3m})}]), (\underbrace{[\bar{k}, \bar{k}, \bar{k}, (\underbrace{\bar{c}, \ldots, \bar{c}}_{\times B})], \ldots, [\bar{k}, \bar{k}, \bar{k}, (\underbrace{\bar{c}, \ldots, \bar{c}}_{\times B})]}_{\times m})]$$

Lemma 4.2. *Let $S(a_1)$, $S(a_2)$ and $S(a_3)$ be natural numbers such that, for some natural number B, it holds that $\frac{1}{4}B < S(a_1), S(a_2), S(a_3) < \frac{1}{2}B$. If $S(a_1) + S(a_2) + S(a_3) = B$, then*

$$\dfrac{[R, Q]}{[R, (k, [\underbrace{c, \ldots, c}_{\times S(a_1)}]), (k, [\underbrace{c, \ldots, c}_{\times S(a_2)}]), (k, [\underbrace{c, \ldots, c}_{\times S(a_3)}]), (Q, [\bar{k}, \bar{k}, \bar{k}, (\underbrace{\bar{c}, \ldots, \bar{c}}_{\times B})])]}\ \Delta \Big\Vert \mathsf{FBV} \; .$$

Proof. Take the following derivation where the redex in the conclusion of the applied rule is highlighted.

$$
\mathsf{s}\,\dfrac{\mathsf{s}\,\dfrac{\mathsf{ai}\!\downarrow\dfrac{\mathsf{s}\,\dfrac{\mathsf{ai}\!\downarrow\dfrac{\mathsf{s}\,\dfrac{\mathsf{ai}\!\downarrow\dfrac{[R,Q]}{\vdots}}{[R,(Q,[\underbrace{c,\dots,c}_{\times S(a_1)},\underbrace{c,\dots,c}_{\times S(a_2)},\underbrace{c,\dots,c}_{\times S(a_3)},(\underbrace{\bar{c},\dots,\bar{c}}_{\times B})])]}}{\vdots}}{[R,(k,[c,\dots,c]),(k,[c,\dots,c]),(Q,[\bar{k},\bar{k},c,\dots,c,(\bar{c},\dots,\bar{c})])]}}{[R,(k,[c,\dots,c]),(k,[c,\dots,c]),(Q,[([k,\bar{k}],[c,\dots,c]),\bar{k},\bar{k},(\bar{c},\dots,\bar{c})])]}}{[R,(k,[c,\dots,c]),(k,[c,\dots,c]),(Q,[(k,[c,\dots,c]),\bar{k},\bar{k},\bar{k},(\bar{c},\dots,\bar{c})])]}}{[R,(k,[\underbrace{c,\dots,c}_{\times S(a_1)}]),(k,[\underbrace{c,\dots,c}_{\times S(a_2)}]),(k,[\underbrace{c,\dots,c}_{\times S(a_3)}]),(Q,[\bar{k},\bar{k},\bar{k},(\underbrace{\bar{c},\dots,\bar{c}}_{\times B})])]} \quad \square
$$

Theorem 4.3. *If a 3-Partition problem $\langle A, m, B, S\rangle$ is solvable, then there is a proof of $\theta(\langle A, m, B, S\rangle)$ in* FBV.

Proof. By induction on m, using Lemma 4.2. $\square$

4.2. Completeness of the encoding

Theorem 4.4. *For A, m, B, and S satisfying the constraints of 3-Partition, if there is a proof of $\theta(\langle A, m, B, S\rangle)$ in* FBV*, then the 3-Partition problem $\langle A, m, B, S\rangle$ is solvable.*

Proof. By induction on m: the case for $m = 0$ corresponds to empty problem. Let $\langle A, m+1, B, S\rangle$ be such that $A = \{a_1, a_2, \dots, a_{3m}, a_{3m+1}, a_{3m+2}, a_{3m+3}\}$. Assuming that we have a proof of $\theta(\langle A, m+1, B, S\rangle)$, we show that $\langle A, m+1, B, S\rangle$ is solvable. Let

$$
R = [(k,[\underbrace{c,\dots,c}_{\times S(a_1)}]),(k,[\underbrace{c,\dots,c}_{\times S(a_2)}]),\dots,(k,[\underbrace{c,\dots,c}_{\times S(a_{3m+2})}]),(k,[\underbrace{c,\dots,c}_{\times S(a_{3m+3})}])]
$$

and

$$
Q = (\underbrace{[\bar{k},\bar{k},\bar{k},(\underbrace{\bar{c},\dots,\bar{c}}_{\times B})],\dots,[\bar{k},\bar{k},\bar{k},(\underbrace{\bar{c},\dots,\bar{c}}_{\times B})]}_{\times m})
$$

such that

$$
\theta(\langle A, m+1, B, S\rangle) = [R,(Q,[\bar{k},\bar{k},\bar{k},(\underbrace{\bar{c},\dots,\bar{c}}_{\times B})])] \quad .
$$

From Theorem 3.21 we have that $\theta(\langle A, m+1, B, S\rangle)$ has a proof in FBV if and only if it has a proof in FBVi. It follows from Theorem 3.19 that

$$
\begin{array}{c}[K_1,K_2]\\ \Delta\,\Big\Vert\,\mathsf{FBVi}\\ R\end{array}
\quad\text{such that}\quad
\begin{array}{c}\Pi\,\Big\Vert\,\mathsf{FBVi}\\ {[K_1,Q]}\end{array}
\quad\text{and}\quad
\begin{array}{c}\Big\Vert\,\mathsf{FBVi}\\ {[K_2,\bar{k},\bar{k},\bar{k},(\bar{c},\dots,\bar{c})]}\end{array}
\quad .
$$

Since there are only positive atoms in R, it follows that none of the rules $\mathsf{ai}\!\downarrow$ and is can be applied in Δ, hence the derivation Δ must be the structure R. This implies that $[K_1, K_2]$ are two partitions of R. Observe that in K_2 there must be exactly 3 occurrences of k, which implies that

$$
K_2 = [(k,[\underbrace{c,\dots,c}_{\times S(a_i)}]),(k,[\underbrace{c,\dots,c}_{\times S(a_j)}]),(k,[\underbrace{c,\dots,c}_{\times S(a_k)}])]
$$

and $S(a_i) + S(a_j) + S(a_k) = B$, and Π is the proof delivered by the induction hypothesis. $\square$

Corollary 4.5. *System* FBV *is NP-hard.*

Proof. Follows immediately from Theorems 4.3 and 4.4. □

Since system BV is a conservative extension of system FBV, this result implies the NP-hardness of system BV.

Corollary 4.6. *System* BV *is NP-hard.*

Proof. Follows immediately from Proposition 3.23 and Corollary 4.5. □

5. System BV is in NP

In this section, I show that the size of any proof of a BV structure is bounded by a polynomial in the size of this structure.

Remark 5.1. Let $R = S[\langle P; T\rangle, \langle U; V\rangle]$ and $R' = S\langle [P, U]; [T, V]\rangle$ be BV structures with pairwise distinct atoms. If $\mathsf{q}{\downarrow}\,\dfrac{R'}{R}$, then

$$\begin{aligned}\downarrow_{R'} = \downarrow_R \setminus \quad & (\{(x, y) | x \in \mathsf{occ}\, P \ \wedge\ y \in \mathsf{occ}\, V\} \cup \{(x, y) | x \in \mathsf{occ}\, V \ \wedge\ y \in \mathsf{occ}\, P\} \cup \\ & \{(x, y) | x \in \mathsf{occ}\, U \ \wedge\ y \in \mathsf{occ}\, T\} \cup \{(x, y) | x \in \mathsf{occ}\, T \ \wedge\ y \in \mathsf{occ}\, U\}).\end{aligned}$$

Proposition 5.2. *The length of any proof of a* BV *structure R is bounded by $\mathcal{O}(|\mathsf{occ}\, R|^2)$.*

Proof. With Remarks 3.17, 3.18 and 5.1; observe that $\downarrow_R \subset (\mathsf{occ} R)^2$, hence $| \downarrow_R | < |\mathsf{occ} R|^2$. For each (nontrivial) application of an inference rule such that $\rho\,\dfrac{R'}{R}$, we have that $| \downarrow_{R'} | < | \downarrow_R |$. □

6. Main result

The main result of the paper follows from the results in Sections 4 and 5:

Theorem 6.1. *System* BV *is NP-complete.*

Proof. Follows immediately from Corollary 4.6 and Proposition 5.2. □

Corollary 6.2. *Multiplicative linear logic extended by the rules* mix *and* mix0*, or System* FBV*, is NP-complete.*

Proof. Follows immediately from Corollary 4.5 and Proposition 5.2. □

7. Discussions

The 3-Partition problem was previously used by Lincoln and Winkler, in [18] for showing the NP-hardness of the constant-only fragment of MLL. By providing a similar encoding of this problem, in this paper I showed the NP-hardness of system BV. In order to show the correctness of the encoding, I developed a new proof theoretical technique for reducing nondeterminism in proof search. The use of this technique provided a very simple correctness proof.

The technique that I introduced in this paper for reducing nondeterminism in proof search is also of interest for applications. Since proofs are constructed by annihilating dual atoms, the restrictions imposed by this technique do not only reduce the breadth of the search space drastically, but also make the shorter proofs more immediately accessible. In this paper, I applied this technique on system FBV (MLL+mix+mix0) to obtain the system FBVi. In this equivalent system, the inference rules can be applied only in certain ways that promote the interaction, in the sense of a specific mutual relation, between dual atoms. The splitting argument, that I used to show the completeness of the resulting system, is strongly related to cut elimination. For this reason, the system obtained by this technique remains clean from a proof theoretic point of view.

The seq rule of system BV (i.e., rule q ↓ in Fig. 3) is also common to system NEL, the Turing-complete [24] extension of system BV presented in [8]. The rules s and q ↓ manage the context of the commutative and noncommutative contexts, respectively, in proof construction in a similar way. In fact, Guglielmi obtained these two rules in [7] as the instances of the same rule in different contexts. The interaction scheme that we exploit in rule lis applies also to the seq rule. However the interleaving between commutative and noncommutative contexts in BV proofs makes it difficult to extend the techniques of the switch rule to the seq rule. Some preliminary ideas along these lines can be found in [10].

The calculus of structures also provides systems which bring new insights to the proof theories of other logics: in [2], Brünnler presented systems in the calculus of structures for classical logic; in [23], Straßburger presented systems for different fragments of linear logic. In [22], Stewart and Stouppa gave systems for a collection of modal logics. Tiu presented, in [25], a local system for intuitionistic logic. All these systems follow a scheme in which atomic interaction rule and switch rule (i.e., rules ai↓ and s in Fig. 3), are common to all these systems. These two rules give the multiplicative linear logic, whereas a system for classical logic is obtained by adding the contraction and weakening rules to these two rules. By using this common scheme, and the splitting technique which is general to systems of the calculus of structures, this new technique can be analogously applied to other systems of the calculus of structures. However, the generalization of this technique to these other systems is not in the scope of this paper. [11] discussed these ideas for classical logic, and implementations of this technique.

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