Termination orderings and complexity characterisations

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1 INTRODUCTION

This paper discusses proof theoretic characterisations of termination orderings for rewrite systems and compares them with the proof theoretic characterisations of fragments of first order arithmetic.

Rewrite systems arise naturally from systems of equations by orienting the equations into rules of replacement. In particular, when a number theoretic function is introduced by a set of defining equations, as is the case in first order systems of arithmetic, this set of equations can be viewed as a rewrite system which *computes* the function.

A termination ordering is a well-founded ordering on terms and is used to prove termination of a term rewriting system by showing that the rewrite relation is a subset of the ordering and hence is also well founded thus guaranteeing the termination of any sequence of rewrites.

The successful use of a specific termination ordering in proving termination of a given rewrite system, R, is necessarily a restriction on the form of the rules in R, and, as we show here in specific cases, translates into a restriction on the proof theoretical complexity of the function computed by R. We shall mainly discuss two termination orderings. The first, the so-called *recursive path ordering* (recently re-christened as the *multiset path ordering*) of [Der79], is widely known and has been implemented in various theorem provers. The second ordering is a derivative of another well known ordering, the *lexicographic path ordering* of [KL80]. This derivative we call the *ramified lexicographic path ordering.* We shall show that the recursive path ordering and the ramified lexicographic path ordering prove termination of different algorithms yet characterise the same class of number-theoretic functions, namely the *primitive recursive functions.* In [DO88], Dershowitz and Okada characterise various termination orderings according to their *order types.* In [Cic90], we showed, in specific cases, how to exploit the embedding of a termination ordering into the ordinals to obtain complexity bounds for lengths of deriva-

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tions of its terminating rewrite systems. In this paper we shall show that both the recursive path ordering and the ramified lexicographic path ordering are of order type up to the same ordinal and that termination proofs for rewrite systems via the recursive path ordering or the ramified lexicographic path ordering imply primitive recursive bounds on derivation lengths. In particular this means that if the rewrite systems compute number theoretic functions, then those functions are primitive recursive.

It is traditional to characterise a formal theory by the class of (recursive) functions that can be proved total in it—the so-called *provably recursive functions* of the theory. It is well known that extending the scheme of primitive recursion to admit recursion with substitution for parameters allows no new functions to be defined, but obviously gives rise to new algorithms. The recursive path ordering cannot prove the termination of algorithms defined in this way. On the other hand, the ramified lexicographic path ordering is able to prove terminating a function defined by primitive recursion with substitution for parameters. The situation is similar for the fragments Σ_1^0-IR and Π_2^0-IR of Peano arithmetic (where induction is restricted to Σ_1^0 and Π_2^0 formulae respectively). These are known to have exactly the same provably recursive functions, namely the primitive recursive ones, but differ in which schemes of definition they allow for these functions. In this respect we shall give examples which suggest that termination of rewrite systems via the recursive path ordering is connected to provability of termination in Σ_1^0-IR, and termination of rewrite systems via the ramified lexicographic path ordering is connected to provability of termination in Π_2^0-IR. A similar connection, based on our results in [Cic90], is shown between the lexicographic path ordering and Σ_2^0-IR.

In the next section we give some basic definitions of term rewriting theory. In sections 3 and 4 we define the recursive path ordering, ramified lexicographic path ordering and the lexicographic path ordering, and we give examples of rewrite systems which contrast their power for termination proofs. Sections 5 and 6 introduce the system $\mathbf{OT^0}$ of ordinal terms and describe their fundamental sequences. In section 7 we discuss some aspects of the slow-growing hierarchy. Our bounding results are obtained using functions from this hierarchy. Section 8 is devoted to the proof of a general bounding theorem. In section 9 we apply the bounding theorem to obtain the complexity characterisations, theorems 9.10 and 9.22.

2 TERM REWRITING SYSTEMS

Here we give a concise description of those aspects of term rewriting theory which are relevant to the discussion in this paper.

2.1 Definition
A set T(F,X) of terms over a finite set F of function symbols and a countable set X of variables is defined by

1. constants (i.e. 0-ary function symbols) and variables are terms.
2. If $t_1, \ldots, t_n$ are terms and f is an n-ary function symbol then $f(t_1, \ldots, t_n)$ is a term.

We write T(F) for the set of terms in T(F,X) which contain no variables. The elements of T(F) are called *ground* or *closed* terms.

2.2 Definition
A Term Rewriting System, $\mathcal{R}$, over a set of terms, T(F,X), is a set of rewrite rules

$$\{l_i \to r_i\},$$

where l_i and r_i are terms belonging to T(F,X) and such that r_i contains only variables already contained in l_i.

A rule $l_i \to r_i$ applies to a term t in T(F,X) if t or some subterm s of t matches l_i after substituting terms of T(F,X) for the variables in l_i. The rule is applied by replacing s by r_i after substitution of the same terms for variables as in l_i.

From an equational calculus i.e. a theory where terms are introduced by defining equations, a rewrite system can be obtained by suitably orienting the equations, and replacing "=" by "→". Then a proof of $t_1 = t_2$ is obtained by showing that t_1 and t_2 rewrite to the same term.

We write

$$t \underset{\mathcal{R}}{\Rightarrow} s$$

to mean that the term t reduces to the term s by application of a rule in $\mathcal{R}$. A sequence $t \underset{\mathcal{R}}{\Rightarrow} t_1 \underset{\mathcal{R}}{\Rightarrow} t_2 \underset{\mathcal{R}}{\Rightarrow} \ldots$ is often referred to as a *derivation*. We also write

$$t \overset{n}{\underset{\mathcal{R}}{\Rightarrow}} s$$

to mean that t reduces to s after n applications of rules in $\mathcal{R}$. So $t \overset{1}{\underset{\mathcal{R}}{\Rightarrow}} s$ and $t \underset{\mathcal{R}}{\Rightarrow} s$ are the same. A term rewriting system $\mathcal{R}$ is said to be *terminating* if the rewrite relation on terms is well founded. All rewrite systems considered here will be finite.

2.3 Definition
We define the *rank*, $|t|$, of a term t in T(F,X) as follows:

$$|t| = \begin{cases} 0 & \text{when t is a constant,} \\ 0 & \text{when t is a variable,} \\ \max_{i \in 1..n}\{k, n, |t_i|\} + 1 & \text{when } t = f_k(t_1, \ldots, t_n). \end{cases}$$

3 RECURSIVE PATH ORDERING (rpo)

The definition of the recursive path ordering makes use of an ordering on finite multisets of terms. A *multiset* is a collection of objects in which elements may occur more than once. The number of times an element occurs in a multiset is called its *multiplicity*. For multisets A, B we write $A \cap B$ and $A \setminus B$ to denote the sets formed so that if x occurs m times in A and n times in B then it occurs min(m,n) times in $A \cap B$ and $m \dot{-} n$ times in $A \setminus B$, where $x \dot{-} y =$ x - y if $x > y$, $= 0$ otherwise.

3.1 Definition
Suppose that S is a set ordered by $<$. The induced ordering, $\ll$, on finite multisets of elements of S is defined as follows :
If A and B are finite multisets (of elements of S) then $A \ll B$ if $A \neq B$ and

1. $A = \emptyset$ or
2. $A \cap B \neq \emptyset$ and $A \setminus (A \cap B) \ll B \setminus (A \cap B)$ or
3. $A \cap B = \emptyset$ and for all $a \in A$ there exists $b \in B$ such that $a < b$.

3.2 Definition
We write $f \equiv g$ to mean that f and g belong to the same equivalence class with respect to some quasi-ordered set of function symbols. *Permutative congruence* between terms, $\approx$, is defined:

$$f(s_1, \ldots, s_n) \approx g(t_1, \ldots, t_n) \text{ iff } f \equiv g \text{ and } s_i \approx t_{\pi(i)}$$

for some permutation π of {1,...,n}.

3.3 Definition [Der79]
Suppose that $<$ is a total quasi-ordering on F. The *recursive path ordering*, $<_{rpo}$, is induced on T(F) by

$$s \;=\; f(s_0, \ldots, s_{m-1}) <_{rpo} g(t_0, \ldots, t_{n-1}) \;=\; t$$

if one of the following holds:

1. $s \leq_{rpo} t_i$ for some i = 0,...,n-1,

2. $f < g$ and $s_i <_{rpo} t$ for all $i = 0,\ldots,m-1$,

3. $f \equiv g$ and $\{s_0, \ldots, s_{m-1}\} \ll_{rpo} \{t_0, \ldots, t_{n-1}\}$.

where $\ll_{rpo}$ is the multiset ordering induced by $<_{rpo}$.

The usual definition of the recursive path ordering assumes that $<$ is a quasi-ordering on F. For our purposes we have assumed that this quasi-ordering is total.

3.4 Theorem
If $<$ is well-founded on F, then $<_{rpo}$ is well-founded on T(F).

4 RAMIFIED LEXICOGRAPHIC PATH ORDERING (rlpo)

4.1 Definition
If $<$ is an order on the set F of function symbols and $f \in F$, we define

$$F\lceil f := \{g \in F \ : \ g < f\}$$

4.2 Definition
Suppose, that $<$ is a total quasi-order on the set F of function symbols. The *ramified lexicographic path ordering*, $<_{rlpo}$, is induced on T(F) as follows :

$s = f(s_1, \ldots, s_m) <_{rlpo} g(t_1, \ldots, t_n) = t$ if

1. $s \leq_{rlpo} t_i$ for some $i = 1,\ldots,n$,
   or

2. $f < g$ and $s_i <_{rlpo} t$ for all $i = 1,\ldots,m$,
   or

3. $f \equiv g$ and $(t_1, \ldots, t_n)$ extends $(s_1, \ldots, s_m)$, or, for some $i \leq \min\{m, n\}$, $s_1 \approx t_1, \ldots, s_{i-1} \approx t_{i-1}$, $s_i <_{rlpo} t_i$, and $\{s_{i+1}, \ldots, s_m\} \subseteq T(F\lceil f)$.

The definition of the *lexicographic path ordering* (lpo) of [KL80] is the same as above except for part 3. To obtain the lexicographic path ordering one has only to replace part 3 by

3′. $f \equiv g$ and $(t_1, \ldots, t_n)$ extends $(s_1, \ldots, s_m)$, or, for some $i \leq \min\{m, n\}$, $s_1 \approx t_1, \ldots, s_{i-1} \approx t_{i-1}$, $s_i <_{lpo} t_i$, and $s_{i+1} <_{lpo} t, \ldots, s_n <_{lpo} t$.

Our definition of rlpo induces an ordering on terms which is not total. This is not a cause for concern but it does indicate the limited practical use of the ordering. It is not difficult to see that a rewrite system which has a termination proof using rlpo will also have a termination proof using lpo. The converse is false, however. We shall see later that our modification results in rlpo being considerably weaker for termination proofs than lpo.

4.3 Theorem

If F is well-founded by $<$, then T(F) is well-founded by $<_{\mathrm{rlpo}}$.

4.4 Examples

1. The n^{th} level of the Ackermann Hierarchy is given by the system:

$$A_1(m) \rightarrow s(m)$$

and, for k = 1..n-1,

$$\begin{aligned} A_{k+1}(0) &\rightarrow A_k(s(0)) \\ A_{k+1}(s(m)) &\rightarrow A_k(A_{k+1}(m)) \end{aligned}$$

The termination of this system can be proved using any of rpo, rlpo, lpo where the precedence is $A_n > \ldots > A_1 > s > 0$.

2. Simultaneous Primitive Recursion.

$$\begin{aligned} f_1(0, x_1, \ldots, x_n) &\rightarrow g_1(x_1, \ldots, x_n) \\ &\vdots \\ f_m(0, x_1, \ldots, x_n) &\rightarrow g_m(x_1, \ldots, x_n) \\ f_1(s(y), x_1, \ldots, x_n) &\rightarrow h_1(x_1, \ldots, x_n, y, f_1(y, x_1, \ldots, x_n), \ldots, f_m(y, x_1, \ldots, x_n)) \\ &\vdots \\ f_m(s(y), x_1, \ldots, x_n) &\rightarrow h_m(x_1, \ldots, x_n, y, f_1(y, x_1, \ldots, x_n), \ldots, f_m(y, x_1, \ldots, x_n)) \end{aligned}$$

Any function defined according to these schemes has a termination proof via rpo, rlpo or lpo where the precedence contains $f_1 \equiv \ldots f_m$, $f_1 > g_1, \ldots, g_m$, $f_1 > h_1, \ldots, h_m$.

3. Primitive Recursion with Substitution for Parameters.

$$\begin{aligned} f(0, x_1, \ldots, x_n) &\rightarrow g(x_1, \ldots, x_n) \\ f(s(y), x_1, \ldots, x_n) &\rightarrow h(x_1, \ldots, x_n, y, f(y, p_1(x_1, \ldots, x_{m_1}), \ldots, p_n(x_1, \ldots, x_{m_n}))) \\ & \quad \text{where } m_i \preceq n, \text{ for } i \in 1..n. \end{aligned}$$

With precedence $f > g$, $f > h$, $f > p$, termination is provable using rlpo or lpo. rpo fails to prove termination.

4. Unnested Multiple Recursion

$$\begin{aligned} f(x,0) &\rightarrow g(x,0) \\ f(0,y) &\rightarrow g(0,y) \\ f(s(x),s(y)) &\rightarrow h(x,y,f(x,p(x,y)),f(s(x),y)) \end{aligned}$$

As in example 3, only rpo fails to prove termination.

5. Neither rpo nor rlpo provides a proof of termination of the rewrite system for computing the Ackermann function :

$$
\begin{array}{rcl}
A(0,m) & \to & m \\
A(s(n),0) & \to & A(n,s(0)) \\
A(s(n),s(m)) & \to & A(n,A(s(n),m)).
\end{array}
$$

Rule (iii) is not reducing with respect to rpo and rlpo. This system is, however, provably terminating using lpo.

A traditional way of characterising a formal theory is by its class of *provably recursive functions*. These are the functions that can be proved total in the theory.

The fragments Σ_1^0-IR and Π_2^0-IR of Peano arithmetic (where induction is restricted to Σ_1^0 and Π_2^0 formulae respectively) are known to have exactly the same provably recursive functions, namely the primitive recursive ones. However, Σ_1^0-IR does not prove termination of a function defined by primitive recursion with substitution for parameters, whereas Π_2^0-IR does. Neither Σ_1^0-IR nor Π_2^0-IR prove termination of the Ackermann function as the Ackermann function is not primitive recursive.

The following table gives equivalences between formal theories and termination orderings in terms of their respective classes of provably recursive functions. The connection given here for Σ_2^0-IR and the lexicographic path ordering was worked out in [Cic90].

FORMAL THEORY		TERMINATION ORDERING
Σ_1^0–IR	$\sim$	rpo
Π_2^0–IR	$\sim$	rlpo
Σ_2^0–IR	$\sim$	lpo

5 THE SET OT^0 OF ORDINAL TERMS

In [DO88], Dershowitz and Okada describe the *Ackermann* system of notations for ordinals and demonstrate embeddings of various termination orderings, including the recursive path ordering, into the Ackermann system. This gives a characterisation of these termination orderings according to their *order types*.

In [Cic90], we showed how knowledge of the order type of a termination ordering can be used to obtain complexity bounds for lengths of derivations of

its terminating rewrite systems. This was done for the recursive path ordering and the lexicographic path ordering. We showed that
if $\mathcal{R}$ is a finite rewrite system whose rules are reducing under the recursive path ordering then there is a primitive recursive function f and a constant c (which depend on the rewrite system) such that

$$\textit{length of longest derivation starting from term } t < \mathrm{f}(|\mathrm{t}| + \mathrm{c}).$$

For the lexicographic path ordering the corresponding result is :
if $\mathcal{R}$ is a finite rewrite system whose rules are reducing under the lexicographic path ordering then there is a multiply recursive function f (in the sense of Péter in [Pét67]) and a constant c such that

$$\textit{length of longest derivation starting from term } t < \mathrm{f}(2.|\mathrm{t}| + \mathrm{c}).$$

In this paper we shall show that both the recursive path ordering and the ramified lexicographic path ordering are of order type up to the same ordinal and that termination proofs for rewrite systems via the recursive path ordering or the ramified lexicographic path ordering imply primitive recursive bounds on derivation lengths. In particular this means that if the rewrite systems compute number theoretic functions, then those functions are primitive recursive.

We introduce here the set $\mathbf{OT^0}$ of ordinal terms. In [Cic90] we described a system $\mathbf{OT}$ of ordinal terms. The present system $\mathbf{OT^0}$ is a subsystem of $\mathbf{OT}$, providing notations for a smaller initial segment of the ordinals. Both systems are subsystems of larger systems, details of which can be found in [Buc75] or [Schü80]. Ordinal terms of $\mathbf{OT^0}$ will be used to measure the order types of recursive path orderings and ramified lexicographic path orderings.

5.1 Definition
The set $\mathbf{OT^0}$ of ordinal terms and the ordering $\prec$ on $\mathbf{OT^0}$ are defined simultaneously by the schemes:

1. (a) $0 \in \mathbf{OT^0}$.
   (b) If $\alpha, \beta \in \mathbf{OT^0}$ then $\theta_\alpha \beta \in \mathbf{OT^0}$.
   (c) If $\alpha_1, \dots, \alpha_\mathrm{n} \in \mathbf{OT^0}$ $(\mathrm{n} \in \mathbf{N})$ and $\alpha_1 \succeq \dots \succeq \alpha_\mathrm{n} \succ 0$ where each α_i is of the form $\theta_{\xi_i}\eta_i$, then $\alpha_1 + \cdots + \alpha_\mathrm{n} \in \mathbf{OT^0}$.

2. By $\alpha = \beta$, we mean that α, β are identical as terms.
   (a) $0 \preceq \alpha$.
   (b) $\theta_{\alpha_1}\beta_1 \prec \theta_{\alpha_2}\beta_2$ iff one of the following three conditions holds:
      i. $\alpha_1 \prec \alpha_2$ and $\max\{\alpha_1, \beta_1\} \prec \theta_{\alpha_2}\beta_2$
      ii. $\alpha_1 = \alpha_2$ and $\beta_1 \prec \beta_2$

iii. $\alpha_1 \succ \alpha_2$ and $\theta_{\alpha_1}\beta_1 \preceq \max\{\alpha_2, \beta_2\}$

3. $\alpha_1 + \cdots + \alpha_n \prec \beta_1 + \cdots + \beta_m$ if, for some $i \leq \min\{m, n\}$,
   $\alpha_1 = \beta_1, \ldots, \alpha_{i-1} = \beta_{i-1}, \alpha_i \prec \beta_i$
   or
   $n < m$ and $\alpha_1 = \beta_1, \ldots, \alpha_n = \beta_n$.

5.2 Definition
The *natural sum*, $\alpha \# \beta$, of terms $\alpha, \beta \in \mathbf{OT^0}$.

1. $\alpha \# 0 = 0 \# \alpha := \alpha$
2. If $\alpha = \xi_1 + \cdots + \xi_m$ and $\beta = \xi_{m+1} + \cdots + \xi_{m+n}$ then
$$\alpha \# \beta := \xi_{\pi(1)} + \cdots + \xi_{\pi(m+n)}$$
   where π is a permutation of 1, ..., m+n such that $\xi_{\pi(1)} \succeq \ldots \succeq \xi_{\pi(m+n)}$.

5.3 Remark The function # is commutative and associative. Its main use here will be to enable us to define the embedding of terms ordered by $\prec_{\mathrm{rpo}}$ into $\mathbf{OT^0}$.

We are not able here to give an exhaustive account of the properties of $\mathbf{OT^0}$ but mention only those that suffice for the purposes of this work. Most of the properties of $\mathbf{OT^0}$ follow from work of Buchholz in [Buc75]. In particular, $\mathbf{OT^0}$ is well-ordered by $\prec$. A useful technical result is that

$$\max\{\alpha, \beta\} \prec \theta_\alpha \beta.$$

The term $\theta_0 0$ is the immediate successor of 0 in $\mathbf{OT^0}$ and so, where there is no confusion, we write 1 for $\theta_0 0$. A *successor* term is a term of the form $\alpha \# 1$. $\alpha \# 1$ is the immediate successor of α in $\mathbf{OT^0}$. A *limit* term is a term which is not zero and not a successor term. A limit term has no immediate predecessors and satisfies the property that if $\alpha \prec \lambda$ where λ is a limit term, then $\alpha \# 1 \prec \lambda$. An *additive principle term* is a term λ with the property that if $\alpha, \beta \prec \lambda$ then $\alpha \# \beta \prec \lambda$. In $\mathbf{OT^0}$ all terms of the form $\theta_\xi \eta$ are additive principal terms. Note that if each α_i and β_j ($i \in 1..n$, $j \in 1..m$) is an additive principal term, then $\alpha_1 \# \cdots \# \alpha_n \prec \beta_1 \# \cdots \# \beta_m$ if and only if $\{\alpha_1, \ldots, \alpha_n\} \ll \{\beta_1, \ldots, \beta_m\}$, where $\ll$ is the multiset ordering induced by $\prec$.

Some of the key ordinal correspondences in $\mathbf{OT^0}$ are:

$$\begin{aligned} \theta_0 1 &= \omega \\ &= \text{the first limit ordinal.} \end{aligned}$$

$$\begin{aligned} \theta_1 0 &= \varepsilon_0. \\ &= \text{the first non-zero fixed point of ordinal exponentiation.} \end{aligned}$$

The ordinal term $\theta_\omega 0$ plays a key role in this paper. We shall show that it provides the least upper bound for the possible order types of $(\mathrm{T(F)}, \prec_{\mathrm{rpo}})$ and $(\mathrm{T(F)}, \prec_{\mathrm{rlpo}})$. The precise order types of these orderings depend essentially on the size of the set F.

6 FUNDAMENTAL SEQUENCES IN $\mathbf{OT^0}$

The standard method for defining a number-theoretic hierarchy over the ordinals overcomes the problem of what to do at limit stages by the use of *fundamental sequences* to limit ordinals.

Each limit ordinal α is equipped with its own fundamental sequence, that is, a sequence of ordinals, indexed by integers, which converges to α. The $\mathrm{x^{th}}$ member of of the fundamental sequence is written $\{\alpha\}(\mathrm{x})$. Thus $\{\alpha\}$ is a function : $\mathbf{N} \mapsto \alpha$ such that, for each x, $\{\alpha\}(\mathrm{x}) \prec \{\alpha\}(\mathrm{x}+1)$ and $\sup\{\{\alpha\}(\mathrm{x}) : \mathrm{x} \in \mathbf{N}\} = \alpha$.

Now, in a hierarchy of functions $\{\mathrm{f}_\gamma\}$ indexed by ordinals, when α is a limit ordinal the function f_α can be defined as follows :

$$\mathrm{f}_\alpha(\mathrm{x}) = \textit{term in } \mathrm{f}_{\alpha(\mathrm{x})} \textit{ and } \mathrm{x}.$$

The purpose of this section is to give a definition of fundamental sequences to limit terms on $\mathbf{OT^0}$ and to indicate some of consequences of these definitions.

6.1 Definition

The *rank*, $|\alpha|$, of a term $\alpha \in \mathbf{OT^0}$ is defined

1. $|0| := 0$.
2. $|\theta_\xi \eta| = \max\{|\xi|, |\eta|\} + 1$.
3. $|\alpha_1 + \cdots + \alpha_\mathrm{n}| := \max\{|\alpha_\mathrm{i}|, \mathrm{n}\}$.

6.2 Definition

$$\mathbf{OT^0_n}(\alpha) \;:=\; \{\gamma \in \mathbf{OT^0} \;:\; \gamma \prec \alpha \text{ and } |\gamma| \;\leq\; \mathrm{n}\}$$

6.3 Lemma

The sets $\mathbf{OT^0_n}(\alpha)$ are finite.

This definition of the sets $\mathbf{OT^0_n}(\alpha)$ is a preliminary step to defining fundamental sequences to limit ordinals $\alpha \in \mathbf{OT^0}$.

6.4 Definition
For $x \in \mathbf{N}$ and $\alpha \in \mathbf{OT^0}$, $\{\alpha\}$: $\mathbf{N} \mapsto \mathbf{OT^0}(\alpha)$ is defined by

$$\begin{aligned} \{\alpha\}(x) \;&:=\; \max \mathbf{OT}^{\mathbf{0}}_{|\alpha|+x} \\ (\;&=\; \max\{\beta \;:\; \beta \prec \alpha \text{ and } |\beta| \leq |\alpha| + x\} \;). \end{aligned}$$

6.5 Lemma

$$\{\alpha + 1\}(x) \;=\; \alpha, \text{ for all } x \in \mathbf{N}.$$

6.6 Theorem
Suppose x, y $\in \mathbf{N}$ and α is a limit ordinal. Then

1. $x < y \Rightarrow \{\alpha\}(x) \prec \{\alpha\}(y)$
2. $\alpha = \sup\{\{\alpha\}(x) \;:\; x \in \mathbf{N}\}$

6.7 Lemma
If α is a limit ordinal, then, for all $x \in \mathbf{N}$,

$$|\{\alpha\}(x)| \;=\; |\alpha| \;+\; x.$$

6.8 Definition
For $x \in \mathbf{N}$, the *pointwise-at-x ordering*, $\prec_x$, on $\mathbf{OT^0}$ is defined as follows :

$\beta \prec_x \alpha$ if and only if
there is a finite sequence $\gamma_0, \ldots, \gamma_m$ of terms of $\mathbf{OT^0}$ such that

$$\gamma_0 \;=\; \alpha, \; \gamma_m \;=\; \beta,$$

and, for each i = 0, ..., m-1, $\gamma_{i+1} \;=\; \{\gamma_i\}(x)$.

6.9 Lemma
If $\alpha, \beta \in \mathbf{OT^0}$ and $\beta \prec \alpha$, then $\beta \prec_{|\beta| \dot{-} |\alpha|} \alpha$.

6.10 Corollary
If x, y $\in \mathbf{N}$ and α, δ are limit ordinals then

1. If $\{\alpha\}(x) \prec \delta \preceq \{\alpha + 1\}(x)$ then $\{\alpha\}(x) \prec \{\delta\}(0)$.
2. $\{\alpha\}(x) + 1 \preceq_y \{\alpha\}(y)$.

6.11 Remark
Part 1 of corollary 6.10 tells us that, with respect to our definition of fundamental sequences, $\mathbf{OT^0}$ possesses the *Bachmann Property*. Part 2 tells us that the terms in $\mathbf{OT^0}$ provide a set of notations for an initial segment of the *structured tree ordinals* of [D-J,W83].

7 THE SLOW GROWING HIERARCHY

The following defines a version of the *Slow Growing hierarchy* indexed by ordinal terms from $\mathbf{OT^0}$:

7.1 Definition

$$\begin{aligned} G_0(x) &= 0, \\ G_\gamma(x) &= G_{\{\gamma\}(x)}(x) + 1, \text{ when } \gamma \neq 0. \end{aligned}$$

The following lemma is immediate from the definition of the Slow Growing hierarchy:

7.2 Lemma

$$G_\alpha(x) = \textit{cardinality of } \{\gamma : \gamma \prec_x \alpha\}.$$

We shall also need the following technical results:

7.3 Lemma

1. if $\beta \prec_x \alpha$ then $\{\gamma : \gamma \prec_x \beta\} \subset \{\gamma : \gamma \prec_x \alpha\}$, and hence, $G_\beta(x) < G_\alpha(x)$.

2. if $x < y$ then $\{\gamma : \gamma \prec_x \alpha\} \subseteq \{\gamma : \gamma \prec_y \alpha\}$, and hence, $G_\alpha(x) \leq G_\alpha(y)$. Equality occurs only when α is finite.

The Slow Growing hierarchy is important to us because we shall show that functions from the hierarchy can be used to give bounds on the lengths of derivations for rewrite systems. Our results will then depend crucially on the complexity of these functions. We arrive at these complexity characterisations by appealing to the Hierarchy Comparison Theorem which relates the rates of growth of functions in the Slow Growing hierarchy with rates of growth of functions representing the well known Grzegorczyk, or Fast Growing, hierarchy. The Hierarchy Comparison Theorem was first proved by Girard in 1975 and appears in [Gir81]. Other proofs have been given in [Acz80], [Buc80], [CW83], [Jer79], [Sch80] and [Wai89].

The main results of the Hierarchy Comparison Theorem which we use here are with reference to the following version of the Fast Growing hierarchy:

$$\begin{aligned} f_0(x) &= x + 1, \\ f_\gamma(x) &= f^{(x+1)}_{\{\gamma\}(x)}(x), \text{ when } \gamma \neq 0. \end{aligned}$$

where the superscript denotes iterated composition.

These results are:

1. For $n \in \mathbb{N}$, f_{2+n} and $G_{\theta_n 0}$ are elementarily equivalent, and hence $G_{\theta_n 0}$ is primitive recursive.

2. f_ω and $G_{\theta_\omega 0}$ are elementarily equivalent, and so $G_{\theta_\omega 0}$ is a version of the Ackermann function.

8 THE BOUNDING THEOREM

8.1 Theorem

Suppose that $\mathcal{R}$ is a finite set of rewrite rules over a set T(F,X) of terms and that there is an embedding $\pi : T(F) \mapsto OT^0$ such that the rules in $\mathcal{R}$ are reducing under π. Suppose further that there is a constant $C \in \mathbb{N}$ such that $|\pi(t_2)| \dot{-} |\pi(t_1)| \leq C$, whenever $t_1 \overset{1}{\underset{\mathcal{R}}{\Rightarrow}} t_2$. Let $\Lambda = \sup\{\pi(t) : t \in T(F)\}$. We have

$$\text{if } t \overset{n}{\underset{\mathcal{R}}{\Rightarrow}} s \ \text{ then } n \leq G_\Lambda(|\pi(t)| + C)$$

where G_Λ is a function in the slow-growing hierarchy.

Proof We note that s does not appear in the bound for n, thus the bound is for any derivation which starts from t.

$t \overset{n}{\underset{\mathcal{R}}{\Rightarrow}} s$ means that there is a sequence $t_0, \ldots, t_n$ of terms in T(F) such that

$$t = t_0 \overset{1}{\underset{\mathcal{R}}{\Rightarrow}} t_1 \overset{1}{\underset{\mathcal{R}}{\Rightarrow}} \ldots \overset{1}{\underset{\mathcal{R}}{\Rightarrow}} t_n = s.$$

By the supposition, for each i = 0, ..., n-1,

$$|\pi(t_{i+1})| \dot{-} |\pi(t_i)| \leq C.$$

By lemma 6.9,

$$\pi(t_{i+1}) \prec_{|\pi(t_{i+1})| \dot{-} |\pi(t_i)|} \pi(t_i),$$

so that, for each i = 0, ..., n-1, using lemma 7.3.2,

$$\pi(t_{i+1}) \prec_C \pi(t_i).$$

Hence, again using lemma 7.3.2,

$$\{\pi(t_1), \ldots, \pi(t_n)\} \subseteq \{\beta : \beta \prec_C \pi(t)\} \subseteq \{\beta : \beta \prec_{|\pi(t)|+C} \pi(t)\}.$$

By lemma 6.9, since $\pi(t) \prec \Lambda$,

$$\pi(t) \prec_{|\pi(t)| \dot{-} |\Lambda|} \Lambda,$$

hence

$$\pi(t) \prec_{|\pi(t)|} \Lambda,$$

and hence

$$\pi(t) \prec_{|\pi(t)|+C} \Lambda.$$

hence

$$\{\pi(t_1), \ldots, \pi(t_n)\} \subseteq \{\beta : \beta \prec_{|\pi(t)|+C} \Lambda \}.$$

That is, if $t \overset{n}{\underset{\mathcal{R}}{\Rightarrow}} s$ then

$$n \leq \text{cardinality of } \{\beta : \beta \prec_{|\pi(t)|+C} \Lambda\}$$

and hence, by lemma 7.2,

$$n \leq G_\Lambda(|\pi(t)| + C).$$

9 COMPLEXITY CHARACTERISATIONS

In this section we show that theorem 8.1 can be applied to rpo and rlpo and obtain the complexity characterisations. We write $l_i(x_1, \ldots, x_n)$ to indicate the left hand term in the i^{th} rule of a rewrite system $\mathcal{R}$ where $x_1, \ldots, x_n$ are all its variables. $r_i(x_1, \ldots, x_n)$ denotes the corresponding right hand term in the i^{th} rule and its variables are *contained* in $x_1, \ldots, x_n$.

9.1 Definition

The embedding π_1: $T(F) \mapsto \mathbf{OT^0}$, where $F = \{f_0, \ldots, f_p\}$, is given by:

$\pi_1(0) = 0;$

$\pi_1(f_k(t_1, \ldots, t_n)) = \theta_k(\pi_1(t_1) \# \ldots \# \pi_1(t_n))$

9.2 Theorem

$$\text{If } t \prec_{rpo} s \text{ then } \pi_1(t) \prec \pi_1(s).$$

Proof The proof is relatively straightforward and so is omitted.

9.3 Lemma

$$\text{If } t \in T(F) \text{ then } |t| = |\pi_1(t)|$$

Proof The definitions of $|t|$ for $t \in T(F,X)$ and of $|\tau|$ for $\tau \in \mathbf{OT^0}$ were adjusted to produce this result.

9.4 Lemma

For any substitution $d_1, \ldots, d_n$ for $x_1, \ldots, x_n$,

$$|t(d_1, \ldots, d_n)| \leq |t(x_1, \ldots, x_n)| + \max_{i \in 1..n}\{|d_i|\}.$$

Proof The proof is by induction on the term tree for $t(x_1, \ldots, x_n)$. When t is a constant or a variable, the result is trivial.

If $t(x_1, \ldots, x_n) = f_k(t_1(x_1, \ldots, x_n), \ldots, t_m(x_1, \ldots, x_n))$ then

$$\begin{aligned} |t(d_1, \ldots, d_n)| &= \max_{j \in 1..m}\{k, m, |t_j(d_1, \ldots, d_n)|\} \\ &\leq \max_{j \in 1..m}\{k, m, |t_j(x_1, \ldots, x_n)| + \max_{i \in 1..n}\{|d_i|\}\} \\ &\qquad \text{(by induction hypothesis)} \\ &\leq \max_{j \in 1..m}\{k, m, |t_j(x_1, \ldots, x_n)|\} + \max_{i \in 1..n}\{|d_i|\} \\ &= |t(x_1, \ldots, x_n)| + \max_{i \in 1..n}\{|d_i|\}. \end{aligned}$$

9.5 Lemma

There is a $K_R \in \mathbf{N}$ such that $|r_i(x_1, \ldots, x_n)| \leq K_R$

Proof This is obvious since $\mathcal{R}$ contains only finitely many rules.

9.6 Lemma
For any substitution $d_1, \ldots, d_n$ for $x_1, \ldots, x_n$,

$$|r_i(d_1, \ldots, d_n)| \leq |l_i(d_1, \ldots, d_n)| + K_R$$

Proof By lemma 9.4,

$$\begin{aligned} |r_i(d_1, \ldots, d_n)| &\leq |r_i(x_1, \ldots, x_n)| + \max_{i \in 1..n}\{| d_i |\} \\ &\leq K_R + \max_{i \in 1..n}\{| d_i |\}, \text{by lemma 9.5}, \\ &\leq K_R + |l_i(d_1, \ldots, d_n)|. \end{aligned}$$

The next lemma generalises lemma 9.6 to the case where a one step rewrite occurs by applying a rule in $\mathcal{R}$ to a proper subterm of a term t. We shall use the notation t[u] to denote a term t with u as a proper subterm.

9.7 Lemma
For any substitution $d_1, \ldots, d_n$ for $x_1, \ldots, x_n$,

$$|t[r_i(d_1, \ldots, d_n)]| \dot{-} |t[l_i(d_1, \ldots, d_n)]| \leq K_R.$$

Proof This is now a straightforward induction over the term tree for t.

9.8 Corollary

$$\text{If } t \underset{\mathcal{R}}{\overset{1}{\Rightarrow}} s \text{ then } |s| \dot{-} |t| \leq K_R.$$

9.9 Lemma

$$\text{If } t \in T(\{f_0, \ldots, f_{p-1}\}) \text{ where } f_0 < \ldots < f_{p-1} \text{ then } \pi_1(t) \prec \theta_p 0.$$

Proof The proof is by induction over the term tree for t and uses the property that if $\beta \prec \theta_p 0$ and $k < p$ then $\theta_k \beta \prec \theta_p 0$.

9.10 Theorem
If $\mathcal{R}$ is a rewrite system over $T(\{f_1, \ldots, f_p\}, X)$ which admits a termination proof using the recursive path ordering then

$$t \underset{\mathcal{R}}{\overset{n}{\Rightarrow}} s \text{ implies } n \leq G_{\theta_p 0}(|t| + K_R)$$

Proof The theorem follows from corollary 9.8, lemma 9.9, bounding theorem 8.1 and lemma 9.3.

We now analyse the ramified lexicographic path ordering in a similar way.

9.11 Definition

The definition of $\pi_2 : T(F) \mapsto OT^0$, where $F = \{f_0, \ldots, f_{p-1}\}$. Let $M = 1 +$ (maximum of the arities of the function symbols in F).

$$\begin{aligned} \pi_2(0) &= 0 \\ \pi_2(f_k(t_1, \ldots, t_n)) &= \#_{i=1}^{n} \theta_k^{M-i+1} \pi_2(t_i) \text{ for } 0 \le k \le p-1. \end{aligned}$$

9.12 Lemma

$$\text{If } t \in T(\{f_0, \ldots, f_{k-1}\})_{0 \le k \le p-1} \text{ where } f_0 < \ldots < f_{p-1}$$
$$\text{then } \pi_2(t) \prec \theta_k 0.$$

Proof The proof is similar to that of lemma 9.9.

9.13 Theorem

$$\text{If } t \prec_{\mathrm{rlpo}} s \text{ then } \pi_2(t) \prec \pi_2(s).$$

The proof of theorem 9.13 uses the following lemma:

9.14 Lemma

$$\pi_2(t_i) \prec_{\mathrm{rlpo}} \pi_2(f_k(t_1, \ldots, t_n)) \text{ for all } i = 1, \ldots, n.$$

Proof The proof is straightforward as each $\pi_2(t_i)$ is a subterm of $\pi_2(f_k(t_1, \ldots, t_n))$.

9.15 Proof of Theorem 9.13

There are three cases to consider. Using lemma 9.14, the first two cases are straightforward. We give the details in the case $s = g(s_1, \ldots, s_m) \prec_{\mathrm{rlpo}} f(t_1, \ldots, t_n) = t$, where $f = f_k \approx g$ and $s_1 \approx t_1$, ..., $s_{i-1} \approx t_{i-1}$, $s_i \prec_{\mathrm{rlpo}} t_i$, $s_{i+1}, \ldots, s_m \in T(F \lceil f_k)$. Then

$$\pi_2(s) = \sigma = \#_{j=1}^{m} \theta_k^{M-j+1} \pi_2(s_j) \text{ and } \pi_2(t) = \tau = \#_{j=1}^{n} \theta_k^{M-j+1} \pi_2(t_j).$$

From the assumption that $s_1 \approx t_1$, ..., $s_{i-1} \approx t_{i-1}$, we obtain

$$\#_{j=1}^{i-1} \theta_k^{M-j+1} \pi_2(s_j) = \#_{j=1}^{i-1} \theta_k^{M-j+1} \pi_2(t_j).$$

From the assumption that $s_i \prec_{\mathrm{rlpo}} t_i$ we obtain by the induction hypothesis

$$\pi_2(s_i) \prec \pi_2(t_i)$$

and hence
$$\theta_k^{M-i+1}\pi_2(s_i) \prec \theta_k^{M-i+1}\pi_2(t_i).$$
Now, by lemma 9.12, $\pi_2(s_{i+1}) \prec \theta_k 0, \ldots, \pi_2(s_m) \prec \theta_k 0$. Hence
$$\begin{aligned}\#_{j=i+1}^{m}\ \theta_k^{M-j+1}\pi_2(s_j) &\prec \#_{j=i+1}^{n}\ \theta_k^{M-j+1}\theta_k 0\\ &= \#_{j=i}^{n-1}\ \theta_k^{M-j+1}0.\end{aligned}$$

Since $0 \preceq \pi_2(s_i) \prec \pi_2(t_i)$, and since $\theta_k^{M-i+1}\pi_2(t_i)$ is an additive principal term,
$$\theta_k^{M-i+1}\pi_2(s_i)\ \#\ \#_{j=i}^{n-1}\theta_k^{M-j+1}0 \prec \theta_k^{M-i+1}\pi_2(t_i).$$
It follows that $\pi_2(s) \prec \pi_2(t)$.

9.16 Lemma
$$|\pi_2(t)| \leq M.|t|$$
Proof. The proof is by induction on the term tree for t. Note that
$$\begin{aligned}|\theta_k^{M-j+1}\pi_2(t_j)| &= \max\{k, |\pi_2(t_j)|\} + M - j + 1\\ &= \max\{k + M - j + 1, |\pi_2(t_j)| + M - j + 1\} \qquad (*)\end{aligned}$$

and therefore we have
$$\begin{aligned}&|\pi_2(f_k(t_1, \ldots, t_n))|\\ &= |\#_{j=1}^{n}\theta_k^{M-j+1}\pi_2(t_j)|\\ &= \max_{j\in 1..n}\{n, |\theta_k^{M-j+1}\pi_2(t_j)|\}\\ &= \max_{j\in 1..n}\{n, \max\{k + M - j + 1, |\pi_2(t_j)| + M - j + 1\}\}\\ &\qquad \text{by } (*) \text{ above}\\ &= \max_{j\in 1..n}\{n, k + M, |\pi_2(t_j)| + M - j + 1\}\\ &= \max_{j\in 1..n}\{k + M, |\pi_2(t_j)| + M - j + 1\} \qquad (**)\\ &\qquad \text{since } M > n\\ &\leq \max_{j\in 1..n}\{k + M, |\pi_2(t_j)| + M\}\\ &= \max_{j\in 1..n}\{k, |\pi_2(t_j)|\} + M\\ &\leq \max_{j\in 1..n}\{n, k, |\pi_2(t_j)|\} + M\end{aligned}$$

By the induction hypothesis, for each $j = 1,\ldots,n$, $|\pi_2(t_j)| \le M.|t_j|$ and so

$$\begin{aligned}|\pi_2(f_k(t_1,\ldots,t_n))| &\le \max_{j\in 1..n}\{n,k,M.|t_j|\}+M\\ &\le M.\max_{j\in 1..n}\{n,k,|t_j|\}+M\\ &= M.|f_k(t_1,\ldots,t_n)|\end{aligned}$$

9.17 Lemma
There is a constant $C_R \in \mathbf{N}$ such that, for each rule $l_i \to r_i \in \mathcal{R}$,

$$|\pi_2(r_i(0,\ldots,0))| \le C_R.$$

where $r_i(0,\ldots,0)$ denotes the replacement of all variables in r_i by 0.
Proof. This is clear since there are finitely many terms in $\mathcal{R}$.

9.18 Lemma
For any substitution $d_1,\ldots,d_n$ for $x_1,\ldots,x_n$,

$$|\pi_2(t(d_1,\ldots,d_n))| \le |\pi_2(t(0,\ldots,0))| + \max_{i\in 1..n}\{|\pi_2(d_i)|\}$$

Proof. The proof is by induction on the term tree for t. The result is true when $t = 0$. When $t = x$, $|\pi_2(t(d))| = |\pi_2(d)|$, and the result follows easily. When $t(d_1,\ldots,d_n) = f_k(t_1(d_1,\ldots,d_n),\ \ldots,\ t_m(d_1,\ldots,d_n))$,

$$\begin{aligned}&|\pi_2(t(d_1,\ldots,d_n))|\\ &= \max_{j\in 1..n}\{k+M, |\pi_2(t_j(d_1,\ldots,d_n))|+M-j+1\}\\ &\qquad \text{from } (**) \text{ in the proof of lemma 9.16}\\ &\le \max_{j\in 1..m}\{k+M, |\pi_2(t_j(0,\ldots,0))|+\max_{i\in 1..n}\{|\pi_2(d_i)|\}+M-j+1\}\\ &\qquad \text{(by induction hypothesis)}\\ &\le \max_{j\in 1..m}\{k+M, |\pi_2(t_j(0,\ldots,0))|+M-j+1\}+\max_{i\in 1..n}\{|\pi_2(d_i)|\}\\ &= |\pi_2(t(0,\ldots,0))|+\max_{i\in 1..n}\{|\pi_2(d_i)|\} \text{ as required.}\end{aligned}$$

9.19 Lemma
For any substitution $d_1,\ldots,d_n$ for $x_1,\ldots,x_n$,

$$|\pi_2(r_i(d_1,\ldots,d_n))| \dot{-} |\pi_2(l_i(d_1,\ldots,d_n))| \le C_R$$

Proof. By lemma 9.18,

$$|\pi_2(r_i(d_1,\ldots,d_n))| \leq |\pi_2(r_i(0,\ldots,0))| + \max_{i\in 1..n}\{|\pi_2(d_i)|\}.$$

So, by lemma 9.17,

$$|\pi_2(r_i(d_1,\ldots,d_n))| \leq C_R + \max_{i\in 1..n}\{|\pi_2(d_i)|\}.$$

Therefore, since $|\pi_2(l_i(d_1,\ldots,d_n))| \geq \max_{i\in 1..n}\{|\pi_2(d_i)|\}$,

$$|\pi_2(r_i(d_1,\ldots,d_n))| \leq C_R + |\pi_2(l_i(d_1,\ldots,d_n))|$$

and the result follows.

9.20 Lemma
For any substitution $d_1,\ldots,d_n$ for $x_1,\ldots,x_n$,

$$|\pi_2(t[r_i(d_1,\ldots,d_n)])| \dot{-} |\pi_2(t[l_i(d_1,\ldots,d_n)])| \leq C_R.$$

Proof. If $t[x]$ is the variable x, the result is lemma 9.19.
If $t[x] = f_k(t_1,\ldots,t_{n-1}, t_n[x])$, then

$$\begin{aligned}
&|\pi_2(t[r_i(d_1,\ldots,d_n)])| \\
&= \max_{j\in 1..m-1}\{k+M, |\pi_2(t_j)|+M-j+1, |\pi_2(t_n[r_i(d_1,\ldots,d_n)])|+M-n+1\} \\
&\qquad \text{from } (**) \text{ in the proof of lemma 9.16} \\
&\leq \max_{j\in 1..m-1}\{k+M, |\pi_2(t_i)|+M-j+1, |\pi_2(t_n[l_i(d_1,\ldots,d_n)])|+C_R+M-n+1\} \\
&\qquad \text{by induction hypothesis,} \\
&\leq \max_{i\in 1..m-1}\{k+M, |\pi_2(t_i)|, |t_n[l_i(s_1,\ldots,s_n)]|\}+C_R \\
&= |\pi_2(t[l_i(d_1,\ldots,d_n)])|+C_R, \\
&\qquad \text{and hence the result.}
\end{aligned}$$

9.21 Corollary

$$\text{If } t \overset{1}{\underset{\mathcal{R}}{\Rightarrow}} s \text{ then } |\pi_2(s)| \dot{-} |\pi_2(t)| \leq C_R.$$

9.22 Theorem
If $\mathcal{R}$ is a rewrite system over $T(\{f_1,\ldots,f_p\},X)$ which admits a termination proof using the ramified lexicographic path ordering then

$$t \overset{n}{\underset{\mathcal{R}}{\Rightarrow}} s \text{ implies } n \leq G_{\theta_p 0}(M.|t|+C_R)$$

Proof The theorem follows from bounding theorem 8.1, lemma 9.13, lemma 9.16 and corollary 9.21.

When a function f is computed by a rewrite system $\mathcal{R}$, and $\mathcal{R}$ can be proved terminating using the recursive path ordering (or the ramified lexicographic path ordering), then theorem 9.10 (or theorem 9.22) can be used to show that f is primitive recursive.

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