The determinacy of infinite games specified by automata

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A system in relation with an environment may be specified by an infinite game between two players.

Two players:

- Player 1 : the computer program
- Player 2 : the environment

The possible actions of the players are represented by letters of a finite alphabet A.

INFINITE PLAY

The two players compose an infinite word over the alphabet A:

$$\begin{array}{lccccccccccc}
\text{Player 1 :} & a_1 & & & & a_3 & & & & a_5 & & \\
& & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \ldots \\
\text{Player 2 :} & & & a_2 & & & & a_4 & & & & a_6
\end{array}$$

The infinite word $a_1.a_2.a_3\ldots$ **represents the infinite behaviour of the system.**

A good behaviour **is represented by a set of infinite words** $L \subseteq A^\omega$ **called the winning set for Player 1.**

The above game, with perfect information, is a Gale-Stewart game $G(L)$.

WINNING STRATEGIES

A strategy for Player 1 is a mapping $f : (A^2)^\star \longrightarrow A$. Player 1 follows the strategy f iff $\forall n \geq 1$: $a_{2n+1} = f(a_1 a_2 \ldots a_{2n})$.

The strategy f is winning for Player 1 if it ensures a good behaviour of the system, **i.e. such that :** the infinite word written by the two players belongs to the winning set L:

$$a_1.a_2.a_3 \ldots \in L$$

A winning strategy for Player 2 **is a strategy for Player 2 which ensures** that $a_1.a_2.a_3 \ldots \notin L$.

A Gale-Stewart game $G(L)$ is determined iff one of the two players has a winning strategy.

The important problems to solve in practice are:

(1) Is the game G(L) determined ?

(2) Which player has a winning strategy ?

(3) If Player 1 has a winning strategy, can we effectively construct this winning strategy ? Is it computable ?

(4) What is the complexity of this construction ? What are the necessary amounts of time and space ?

COMPLEXITY OF WINNING SETS

The winning set for Player 1 is often given as the set of infinite behaviours which satisfy a logical formula.

It is also often given as the set of infinite words accepted by a finite automaton, a one-counter automaton, a pushdown automaton, . . . with a Büchi acceptance condition . . .

Regular winning sets

Büchi and Landweber solved the famous Church's Problem posed in 1957, Rabin gave an alternative solution:

Theorem (Büchi-Landweber 1969; Rabin 1972)

If $L \subseteq \Sigma^\omega$ is a regular ω-language then:

- *The game $G(L)$ is determined.*
- *One can decide which Player has a winning strategy.*
- *On can construct effectively a winning strategy given by a finite state transducer.*

This was extended to the case of deterministic context free winning sets ([Walukiewicz 1996]), and to winning sets accepted by deterministic higher-order pushdown automata ([Cachat 2003], [Carayol, Hagues, Meyer, Ong, Serre 2008])

The question of the determinacy

The determinacy of regular or deterministic context-free games follows from the determinacy of Borel games. (Martin 1975).

The question remained open for non-deterministic pushdown automata, one-counter automata, 2-tape automata: these automata accept non-Borel sets.

Complexity of ω-Languages of Non Deterministic Turing Machines

Non deterministic Büchi (or Muller) Turing machines accept **effective analytic sets** (Staiger). The class **Effective-Σ_1^1** is the class of **projections of arithmetical sets**.

There are some non-Borel sets in the class **Effective-Σ_1^1.**

Theorem

- *[Ressayre and F. 2003] There are some non-Borel context-free (and even 1-counter) ω-languages.*

The (effective) analytic determinacy

Theorem (Martin 1970 and Harrington 1978)

The effective analytic determinacy is equivalent to the existence of a particular real called $0^\sharp$.
The existence of the real $0^\sharp$ is known in set theory to be a large cardinal assumption, and is not provable in **ZFC**.

(Viewed as a set of integers, the real $0^\sharp$ is the set of Gödel numbers of formulas which are satisfied by an uncountable set of indiscernible ordinals in **L**, firstly considered by Silver 1966)

The context-free determinacy

Theorem (F. 2011)

The determinacy of games $G(L)$, where L is accepted by a real-time 1-counter Büchi automaton, is equivalent to the effective analytic determinacy, and thus it is not provable in ZFC.

Sketch of the proof

We start from an effective analytic set $L(\mathcal{T})$ accepted by a Büchi Turing machine $\mathcal{T}$.

We construct a real time 1-counter Büchi automaton $\mathcal{A}$ such that Player 1 (resp. Player 2) has a winning strategy in $G(L(\mathcal{T}))$ if and only if that Player 1 (resp. Player 2) has a winning strategy in the game $G(L(\mathcal{A}))$.

The game $G(L(\mathcal{T}))$ is determined iff the game $G(L(\mathcal{A}))$ is determined.

Games with non-recursive strategies when they exist

Theorem (F. 2011)

There exists a 1-counter Büchi automaton $\mathcal{A}$ such that:

(1) There is a model V_1 of **ZFC** *in which Player 1 has a winning strategy σ in the game $G(L(\mathcal{A}))$. But σ cannot be recursive and not even hyperarithmetical.*

(2) There is a model V_2 of **ZFC** *in which the game $G(L(\mathcal{A}))$ is not determined.*

Moreover these are the only two possibilities: there are no models of **ZFC** in which Player 2 has a winning strategy.

Games of maximum strength of determinacy

Theorem (F. 2012)

There exists a 1-counter Büchi automaton $A_\sharp$ such that:
The game $G(A_\sharp)$ is determined iff the effective analytic determinacy holds iff all 1-counter games are determined.

Are there two or more strengths of determinacy ?

A transfinite sequence of 1-counter Büchi automata

The recursive ordinals form an initial segment of the countable ordinals.

The ordinals ω, ω^ω, ω^{ω^ω}, $\ldots$,

$$\varepsilon_0 = \lim_n \underbrace{\omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega}}}}}}_{n}$$

are recursive.

A transfinite sequence of 1-counter Büchi automata

A transfinite sequence of games specified by 1-counter Büchi automata with increasing strength of determinacy.

Theorem (F. 2012)

There is a transfinite sequence of 1-counter Büchi automata $(\mathcal{A}_\alpha)_{\alpha<\omega_1^{\mathrm{CK}}}$, indexed by recursive ordinals, s.t.:

$$\forall \alpha < \beta < \omega_1^{\mathrm{CK}} \ [\ Det(G(L(\mathcal{A}_\beta))) \Longrightarrow Det(G(L(\mathcal{A}_\alpha)))\ ]$$

but the converse is not true:

*For each recursive ordinal α there is a model $\mathbf{V}_\alpha$ of **ZFC** such that in this model the game $G(L(\mathcal{A}_\beta))$ is determined iff $\beta < \alpha$.*

THANK YOU !