REGULAR CANONICAL SYSTEMS***

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A special type of canonical systems, called *normal systems* and containing rules of production of form $a\underline{x} \to \underline{x}b$ only, plays a role in Post's version of the theory of effectiveness. One's attention is therefore naturally drawn to another type of canonical system, whose rules are of form $a\underline{x} \to b\underline{x}$. We shall call these *regular systems*.

As Post [2] has shown, normal systems, in spite of their deductive simplicity, produce all canonical (i.e. recursively generable) sets of words. At the same place Post claims that in contrast, regular systems produce only recursive sets of words. While this much is not difficult to prove, it requires a more careful investigation of deductions by regular rules to characterize, among all recursive sets of words, those which can be produced by regular systems. It turns out that they are of surprisingly simple nature. We shall prove that for a set β of words the following are equivalent conditions:

(1) β may be produced from a finite set of axioms by regular rules $a\underline{x} \to b\underline{x}$.

(2) β is periodic in the sense that it may be produced from a finite set of axioms by rules of form $a\underline{x} \to ap\underline{x}$.

(3) β is periodic in the sense that it is the union of some of the equivalence classes of a congruence of concatenation of finite partition.

To fully appreciate (2), the „expansive character" of rules of form $a\underline{x} \to ap\underline{x}$ should be noted (apx is longer than ax). Furthermore, both (2) and (3) are very natural generalizations of the concept of an (ultimately) periodic set of natural numbers, from ordinary (1-ary) arithmetic to the k-ary arithmetic over the words on k letters.

Another equivalent characterization of sets β satisfying (1) is the following:

(4) β is the behaviour of (i.e. the set accepted by) a finite automaton.

This shows that regular systems are related to finite automata in much the same way as normal systems are to Turing machines. The characterization (4) in fact is very similar to (2); finite automata are but another very special type of expansive regular systems.

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1. Terminology and notations.

Words are finite strings formed from the *letters* $1_1, 1_2, 1_3, \ldots$, and *auxiliary letters* usually taken from the list $S_1, S_2, S_3, \ldots$. We will use as syntactic variables with or without subscripts:

$A, B, C, \ldots$	ranging over letters
$a, b, c, \ldots$	ranging over words
$\underline{A}, \underline{B}, \underline{C}, \ldots$	ranging over finite sets and families of words
$\alpha, \beta, \gamma, \ldots$	ranging over sets of words

A notation $\{a_i\}_r$ will be used ambiguously for the set, and also the family of words $a_1, a_2, \ldots, a_r$. *Concatenation* of words is denoted by $x^\frown y$, and often abbreviated as xy. $lg(x)$ denotes the *length* of the word x, o denotes the word of length zero. An *alphabet* is a finite set of letters. $\underline{I}_k$ denotes the alphabet consisting of $1_1, \ldots, 1_k$, and $\underline{S}_n$ denotes the alphabet consisting of $S_1, \ldots, S_n$. N_k denotes the set of all words on the alphabet $\underline{I}_k$, so that N_1 may be interpreted to be the set of natural numbers (zero included).

We add some remarks on the structure of N_k, which generalizes that of the natural number system N_1. The algebra $[N_k, o, {}^\frown]$ is the *free semigroup with identity* [i.e., free relative to the equations $x^\frown 0 = x$, $0^\frown x = x$, $x^\frown(y^\frown z) = (x^\frown y)^\frown z$)] on k generators $1_1, \ldots, 1_k$. More elementary is the following characterization of N_k: If $l_i(x)$ denotes the i-the *left successor* $1_i x$ of the word $x \in N_k$, then $(N_k, l_1, \ldots, l_k)$ is the *absolutely free algebra* (with k unary operations) and one generator o. The same remark, of course, holds for the *right-successor functions* $r_i(x) = xl_i$. Further important relations on N_k are the *left-segment relation* $x \leqq y$, and the *right-segment relation* $y \geq x$, defined by

$$x \leqq y\colon (\exists u)\,[x^\frown u = y]$$

$$y \geqq x\colon (\exists u)\,[y = u^\frown x]$$

The *proper left- and right-segment* relations $x < y$, and $y > x$ are defined by $x \leqq y \wedge x \neq y$, and $y \geqq x \wedge x \neq y$. We will call $[N_k, <]$ the *left-tree* on N_k; valuable intuition is added by interpreting $[N_k, <]$ as the graph which is obtained by starting with a root o and joining to every vertex v, on the level $s = lg(v)$, new vertices $v1_1, \ldots, v1_k$ on the next higher level $s + 1$. For later reference we introduce the following notions. If $\underline{X} \subseteq N_k$ then the *interior* int $(\underline{X})$, and the exterior ext$(\underline{X})$ are defined by:

$$u \in int(\underline{X})\colon (\exists x)\,[x \in \underline{X} \wedge u < x]$$

$$u \in ext(\underline{X})\colon (\exists x)\,[x \in \underline{X} \wedge x < u]$$

Futhermore, $\underline{X}$ will be called a *frontier* of the left-tree on N_k if every $u \,\varepsilon\, N_k$ belongs to exactly one of the sets $\underline{X}$, $int(\underline{X})$, $ext(\underline{X})$.

If a and b are words then the notation $a\underline{x} \to b\underline{x}$ is called *a regular production.* This production *directly produces* from the word u a word v, if there is a word x such that $u = ax$ and $v = bx$. Thus regular productions are a very simple type of canonical productions, studied by Post (see also Rosenbloom [7]).

Definition 1. A *regular system* on the alphabet $\underline{I}_k$ is a finite collection Σ of regular productions, whereby the words a and b occurring in the productions are words on an alphabet $\underline{I}_k \cap \underline{S}$. If $\underline{S} \cup I_k = \Lambda$ then the letters $S \in \underline{S}$ are called the *auxiliary letters* of Σ. If $\underline{S}$ is empty Σ is called a *pure regular system* on $\underline{I}_k$.

If $\underline{A} = \{a_i\}_r$ and $\underline{B} = \{b_i\}_r$ are families of words then $\Sigma(\underline{A}, \underline{B})$ denotes the regular system whose productions are $a_1\underline{x} \to b_1\underline{x}, \ldots, a_r\underline{x} \to b_r\underline{x}$.

Let Σ be a regular system on $\underline{I}_k$ with auxiliary alphabet $\underline{S}$. A sequence $u_1, \ldots, u_r$ of words on the alphabet $\underline{I}_k \cup \underline{S}$ is called a *Σ-deduction*, if, for any $i = 1, \ldots, r$—1 some production in Σ directly produces from u_i the word u_{i+s}. If u and v are words on $\underline{I}_k \cup \underline{S}$ for which there is a Σ-deduction $u, \ldots, v$ then we will write $u \mid \Sigma \vdash v$, and say that Σ *produces from* u the word v.

Without reference we will often use the following property which is characteristic for regular system Σ:

$$(*) \qquad u \mid \Sigma \vdash v \,.\supset.\, u^\frown y \mid \Sigma \vdash v^\frown y.$$

Definition 2. Let Σ be a regular system on $\underline{I}_k$ with auxiliary alphabet $\underline{S}$, and let $\underline{U}$ and $\underline{V}$ be finite sets of words on the alphabet $\underline{I}_k \cup \underline{S}$. The set $\tau(\underline{U}, \Sigma, \underline{V})$ of words *produced by* $[\underline{U}, \Sigma, \underline{V}]$, and the set $\beta(\underline{U}, \Sigma, \underline{V})$ of words *accepted by* $[\underline{U}, \Sigma, \underline{V}]$ are defined by

$$x \,\varepsilon\, \tau(\underline{U}, \Sigma, \underline{V}) \quad .\equiv. \quad x \in N_k \wedge (\exists\, uv)\,[u \in \underline{U} \wedge u \mid \Sigma \vdash vx \wedge v \in \underline{V}]$$

$$x \,\varepsilon\, \beta(\underline{U}, \Sigma, \underline{V}) \quad .\equiv. \quad x \in N_k \wedge (\exists\, uv)\,[u \in \underline{U} \wedge ux \mid \Sigma \vdash v \wedge v \in \underline{V}]$$

The set $\tau(\underline{U}, \Sigma, \{o\})$ is also denoted by $\tau(\underline{U}, \Sigma)$, and will be called the *set of theorems* of $[\underline{U}, \Sigma]$. The elements u of $\underline{U}$ are called the *axioms*.

We remark that even if Σ has auxiliary letters which may come to use in Σ-deductions, τ and β still consist of words x on $\underline{I}_k$ only. Usually in the literature sets of theorems $\tau(\underline{U}, \Sigma)$ of canonical systems are studied. However, also the notion of accepted sets occurs in particular in the theory of Turing machines. If, as in regular systems, all productions of Σ have but one datum, it is a matter of conveniency whether one prefers to work with τ or β. This is a consequence of the following remark. By the *converse production* to $a\underline{x} \to b\underline{x}$ we mean $b\underline{x} \to a\underline{x}$, and by the *converse system* to Σ we mean the system $\overleftarrow{\Sigma}$ consisting of all converses to productions in Σ. Then it is clear that $(\beta(\underline{U}, \Sigma, \underline{V}) = \tau(\underline{V}, \overleftarrow{\Sigma}, \underline{U})$, and if $\Sigma_1 \overset{\leftarrow}{=} \Sigma$ then $\Sigma \overset{\leftarrow}{=} \Sigma_1$.

Definition 3. A *finite automaton* with *input states* $\underline{I}_k$ and *transit states* $\underline{S}_n$, is a regular system Σ on $\underline{I}_k$ with auxiliary alphabet $\underline{S}_n$ which consist of nk productions of form $S_i 1_p \underline{x} \to S_j \underline{x}$, and such that to every pair $S_i \in \underline{S}_n$, $1_p \in \underline{I}_k$, there is exactly one production $S_i 1_p \underline{x} \to S_j \underline{x}$ in Σ.

The functions $tr_p\colon \underline{S}_n \to \underline{S}_n$, defined by $tr_p(S_i) = Sj$ if and only if $S_i l_p \underline{x} \to S_j \underline{x}$

belongs to Σ, are called the ***transition functions*** of the finite automaton Σ; $[\underline{S}_n, tr_1, \ldots, tr_k]$ is called its ***transition system***. The ***rank*** of Σ is the number n of transit states.

For any transit state $X \in \underline{S}_n$ and any sequence of input states $z \in N_k$ of a finite automaton Σ on $\underline{I}_k$ there is exactly one transit state $Y \in \underline{S}_n$ such that $Xz \mid \Sigma \vdash Y$. This Y is called the ***response*** of Σ to z with respect to the initial state X, and will be denoted by X/z (or more precisely $X/_\Sigma z$).

We remark that a finite automaton Σ is uniquely determined by its transition system, and that every system $[\underline{S}, tr_1, \ldots, tr_k]$ of function tr_1 from $\underline{S}$ to $\underline{S}$ is the transition system of a finite automaton Σ on $\underline{I}_k$, namely the one whose productions are $S1_i x \to tr_i(S)\, x$. This shows that our concept of a finite automaton is essentially equivalent to others occurring in the literature. Note in this connection that the binary transition function $tr(1_i, S) = tr_i(S)$ could be used in place of $tr_1, \ldots, tr_k$. The response operation X/z can be obtained by the following recursion

$$X/0 = X, \qquad X/1_i z = tr_i(X)/z$$

which perhaps more closely reflects the intuitive idea of how an automaton, originally in transit state X, responds to injection of signals $z = 1_i 1_j \ldots 1_p$ by going successively through the transition states X, $tr_i(X)$, $tr_j tr_i(X)$, $\ldots$ X/z.

Definition 4. A (*binary*) *output* of a finite automaton Σ is a subset $\underline{V}$ of its transition states $\underline{S}$. The behavior of a finite automaton Σ on $\underline{I}_k$ with initial state U and output $\underline{V}$ is the set $\beta(U, \Sigma, \underline{V})$ of input words accepted by $[\{U\}, \Sigma, \underline{V}]$, i.e.,

$$x \in \beta(U, \Sigma, \underline{V}) \equiv U/x \in \underline{V}.$$

The intended interpretation is that if the automaton is in transit state S then the output is either on or off according to whether S does or does not belong to $\underline{V}$. The behavior $\beta(U, \Sigma, \underline{V})$ therefore consists of all input words x which injected into the automaton initially in state U ultimately produce a state S which activates the output $\underline{V}$. In a somewhat modified form this concept of behavior is due to *Kleene* [4].

Let Φ be a finite automaton on $\underline{I}_k$ with transit states $\underline{S}_n$, and let $A \in \underline{S}_n$. We will say that a state $X \in \underline{S}_n$ is *A-accessible* if there is an input word $z \in N_k$ such that $A/z = X$, i.e., $Az \mid \Phi \vdash X$; let $\underline{S}_n(A)$ denote the set of A-accessible states. Suppose that $X_1 I_1 .. I_s, \ldots, X_s I_s, X_{s+1}$ is a Φ-deduction. If $s \geqq n =$ rank of Φ, then a repetition $Xp = Xq$. $1 \leqq p < q \leqq s + 1$ must occur in the deduction. It follows that $X_1 I_1 .. I_p I_{q+1} \ldots I_s, \ldots, X_p I_q I_{q+1} .. I_s, \ldots, X_s I_s, X_{s+1}$ is still a Φ-deduction. Thus, if $Az \mid \Phi \vdash X$ then there must be a y of length $lg(y) \leqq n$ such that $Ay \mid \Phi \vdash Y$. Consequently, X is A-accessible if and only if there is a $y \in N_k$ such that $lg(y) \leqq n =$ rank of Φ and $A/y = X$. Therefore for any finite automaton Φ and transit state A one can effectively select the set $\underline{S}(A)$ of A-accessible states. Furthermore, $\underline{S}(A)$ is closed under the transition functions $tr_1, \ldots, tr_k$ of Φ, and the automaton $\Phi(A)$ obtained by retaining only $\underline{S}(A)$ as transit states has the properties:

(a) Every transit state of $\Phi(A)$ is A-accessible.

(b) Φ and $\Phi(A)$ yield the same response $A/z \in \underline{S}(A)$ to every input word $z \in N_k$.

2. Reduced regular systems.

It turns out that regular systems may be reduced to a simple form. In this section we will therefore present our basic result for this simple type of regular system.

Definition 5. A regular system Σ on $\underline{I}_k$ with auxiliary alphabet $\underline{S}_n$ is called reduced if all its productions are either of the forms:

$$\begin{array}{rl} S_i 1_p \underline{x} \to S_j \underline{x} & \quad \textit{contraction} \\ S_i \underline{x} \to S_j \underline{x} & \quad \textit{neutration} \\ S_i \underline{x} \to S_j 1_p \underline{x} & \quad \textit{expansion} \end{array}$$

The number n of auxiliary letters will be called the *rank* of Σ.

Of special interest are those reduced regular systems Σ which, like finite automata, do not contain expansions. They may be interpreted as *transition-graphs*: Take $\underline{S}_n$ as set of vertices. Join S_i to S_j by an arrow labelled by 1_h (by o), in case $S_i 1_h \underline{x} \to S_j \underline{x}$ ($S_i \underline{x} \to S_j \underline{x}$) belongs to Σ. The set $\beta(\underline{U}, \Sigma, \underline{V})$ consist of those words u, whose letters occur in order as labels on some path through the transition-graph, starting in $\underline{U}$ and ending in $\underline{V}$. Clearly $\beta(\underline{U}, \Sigma, \underline{V})$ is a recursive subset of N_k.
We will now show that to every reduced system Σ there exists a reduced system Σ_0 without expansions, such that $\beta(\underline{U}, \Sigma, \underline{V}) = \beta(\underline{U}, \Sigma_0, \underline{V})$. It follows that $\beta(\underline{U}, \Sigma, \underline{V})$ is recursive for arbitrary reduced systems Σ.

Definition 6. Let Σ be a reduced regular system on $\underline{I}_k$ with auxiliary alphabet $\underline{S}_n$. The system $ctr(\Sigma)$ in the same alphabets $\underline{I}_k$ and $\underline{S}_n$ consist of all contractions and neutrations which are derived rules of Σ, i.e., if $X, Y \in \underline{S}_n$ and $I \in \underline{I}_k$ then

$$\begin{array}{rl} XI\underline{x} \to Y\underline{x} & \quad \text{belongs to } ctr(\Sigma) \textit{ if } XI \mid \Sigma \vdash Y \\ X\underline{x} \to Y\underline{x} & \quad \text{belongs to } ctr(\Sigma) \text{ if } X \mid \Sigma \vdash Y \end{array}$$

Lemma 1. If Σ is a reduced regular system on $\underline{I}_k$ with auxiliary alphabet $\underline{S}_n$, and $\Sigma_0 = ctr(\Sigma)$ then for any $X, Y \in \underline{S}_n$ and $z \in N_k$,

$$Xz \mid \Sigma \vdash Y \quad . \equiv . \quad Xz \mid \Sigma_0 \vdash Y$$

Corollary. For any reduced regular Σ, $\underline{U} \subseteq \underline{S}_n$ and $\underline{V} \subseteq \underline{S}_n$, $\beta(\underline{U}, \Sigma, \underline{V}) = \beta(\underline{U}, ctr(\Sigma), \underline{V})$, and $\beta(\underline{U}, \Sigma, \underline{V})$ is a recursive subset of N_k.

Proof: That $Xz \mid \Sigma_0 \vdash Y$ implies $Xz \mid \Sigma \vdash Y$ clearly follows from the definition of $\Sigma_0 = ctr(\Sigma)$. Suppose now that $Xz \mid \Sigma \vdash Y$. In case $z = 0$ we have

$X \mid \Sigma \vdash Y$ so that, by definition of $\Sigma_0 = \mathrm{ctr}(\Sigma)$, $X \mid \Sigma_0 \vdash Y$, i.e., $Xz \mid \Sigma_0 \vdash Yz$. In case $z = I_1 \ldots I_h \neq o$ there is a Σ-deduction $XI_1 \ldots I_h, \ldots, Y$. It is clear that this deduction must be of form:
(i) $X_1 I_1 ..I_h, \ldots, U_1 I_1 ..I_h, X_2 I_2 ..I_h, ----, U_r I_r ..I_h, X_{r+1} I_{r+1} ..I_h, \ldots, U_{r+1} I_{r+1} ..I_h, X_{r+2} I_{r+2} ..I_h, ----, U_h I_h, X_{h+1}, \ldots, Y$ whereby $X_1 = X$, and for $r = 1, \ldots, h\text{-}1$ ($r = h$) the indicated contraction $U_r I_r ..I_h$, $X_{r+1} I_{r+1} ..I_h$ ($U_h I_h$, X_{h+1}) is the first to the left of any contraction $AI_r ..I_h$, $BI_{r+1} ..I_h$ (AI_h, B) occurring in (i). It follows that for $r = 1, \ldots, h$, $X_r \mid \Sigma \vdash U_r$, and of course $U_r I_r \mid \Sigma \vdash X_{r+1}$, $X_{h+1} \mid \Sigma \vdash Y$. Thus, by definition of $\Sigma_0 = ctr(\Sigma)$, $X_r \underline{x} \to U_r \underline{x}$, $U_r I_r \underline{x} \to X_{r+1} \underline{x}$, and $X_{h+1} \underline{x} \to Y \underline{x}$ are productions belonging to Σ_0. Consequently the modified deduction,
(ii) $X_1 I_1 ..I_h$, $U_1 I_1 ..I_h$, $X_2 I_2 ..I_h$, $----$, $U_r I_r ..I_h$, $X_{r+1} I_{r+1} ..I_h$, $U_{r+1} I_{r+1} ..I_h$, $X_{r+2} I_{r+2} ..I_h$, $----$, $U_h I_h$, X_{h+1}, Y is a Σ_0-deduction. Thus we have shown that $Xz \mid \Sigma \vdash Y$ implies $Xz \mid \Sigma_0 \vdash Y$, which concludes the proof of lemma 1.

The first part of the corollary follows by lemma 1 and definition 2. Furthermore by definition of $\Sigma_0 = ctr(\Sigma)$ it is clear that if $U1_p\underline{x} \to V\underline{x}$, $V\underline{x} \to W\underline{x}$, $W\underline{x} \to Z\underline{x}$ are productions in Σ_0, then also $U1_p\underline{x} \to W\underline{x}$, $V\underline{x} \to Z\underline{x}$ belong to Σ_0. It follows that every Σ_0-deduction $Xz, \ldots, Y$ can be modified to a Σ_0-deduction consisting of $h = lg(z)$ contraction followed by one neutration. Consequently, to check whether or not $x \in \beta(\underline{U}, \Sigma_0, \underline{V})$ one only has to investigate the finite number of Σ_0-deductions of length $\leq lg(x) + 2$. Therefore $\beta(\underline{U}, \Sigma, \underline{V}) = \beta(\underline{U}, \Sigma_0, \underline{V})$ is recursive.

This corollary may be considerably improved in two ways. First, the system $ctr(\Sigma)$ can be effectively constructed (see corollary to lemma 4). Second, one can obtain a finite automaton which accepts the set $\beta(\underline{U}, \Sigma, \underline{V})$. We begin with the second point.

Definition 7. Let Σ be a reduced regular system on I_k with auxiliary alphabet $\underline{S}$, and *Let* $\underline{\underline{S}}$ be the set of all subsets $X \subseteq \underline{S}$. Define the functions sp, tr_1, $\ldots$, tr_k on $\underline{\underline{S}}$, and the subset $\underline{\underline{C}} \subseteq \underline{\underline{S}}$ as follows:

$$Y \in sp(\underline{x}) \quad .\equiv. \quad (\exists X)[X \in \underline{X} \wedge X \mid \Sigma \vdash Y)$$
$$Y \in tr_i(\underline{x}) \quad .\equiv. \quad (\exists X)[X \in X \wedge X1_i \mid \Sigma \vdash Y],$$
$$i = l, \ldots, k$$
$$Y \in \underline{\underline{C}} \quad .\equiv. \quad sp(\underline{X}) = \underline{X}$$

sub (Σ) is the finite automaton on $\underline{I}_k$ whose transit states are the subsets $\underline{X} \in \underline{\underline{C}}$ of $\underline{S}$, and whose productions are:

$$\underline{Y}\, 1_i \underline{x} \to tr_i(\underline{Y})\, \underline{x}, \qquad \underline{Y} \in \underline{\underline{C}},\ i = 1, \ldots, k$$

To see that sub(Σ) is well-defined it is necessary to establish that $tr_i(\underline{Y}) \in \underline{\underline{C}}$ for $\underline{Y} \in \underline{\underline{C}}$. This follows by (6) below. One migth object to using subsets of $\underline{S}$ as letters of an alphabet, in which case one would have to modify the definition

of sub(Σ) by using some other alphabet whose letters are brought into one-to-one correspondence with the elements of $\underline{\underline{C}}$.
The following properties of sp, tr_i, $\underline{\underline{C}}$ are easily established:

(1) $\underline{X} \subseteq sp(\underline{X})$ (2) $\underline{X} \subseteq \underline{Y} \quad . \supset . \quad sp(\underline{X}) \subseteq sp(\underline{Y})$

(3) $sp(sp(\underline{X})) = sp(\underline{X})$ (4) $\underline{X} \subseteq \underline{Y} \quad . \supset . \quad tr_i(\underline{X}) \subseteq tr_i(\underline{Y})$

(5) $tr_i(sp(\underline{X})) = tr_i(\underline{X})$ (7) $\underline{X} \in \underline{\underline{C}} \quad . \equiv . \quad (\exists\, \underline{Y})\,[\underline{X} = sp(\underline{Y})]$

(6) $sp(tr_i(\underline{X})) = tr_i(\underline{X})$

Thus, sp is a closure operation on $\underline{\underline{S}}$, and $\underline{\underline{C}}$ is the collection of closed sets. Furthermore it was pointed out in section 1, following definition 3, that the response operation $\underline{X}/z$ of the automaton sub (Σ) satisfies the recursion:

(8) $$\underline{X}/o = \underline{X}, \qquad \underline{X}/1_i z = tr_i(\underline{X})/z, \quad \text{for } \underline{X} \in \underline{\underline{C}} \text{ and } z \in N_k.$$

We prove next by an induction on z that,

(9) $$\underline{X} \in \underline{\underline{C}} \wedge Y \in \underline{X}/z \quad . \supset . \quad (\exists X)\,[X \in \underline{X} \wedge Xz \mid \Sigma \vdash Y], \quad \text{for all } z \in N_k.$$

That (9) holds for $z = o$ is clear if one notes that $\underline{X}/o = \underline{X}$, $Yo \mid \Sigma \vdash Y$. Now we make the inductive assumption that (9) holds for $z = y$, and assume that $\underline{X} \in \underline{\underline{C}}$ and $Y \in \underline{X}/1_i y$. Then by (8), $Y \in tr_i(\underline{X})/y$, and by (6), $tr_i(\underline{X}) \in \underline{\underline{C}}$. Therefore, by the inductive assumption, there is a U such that $U \in tr_i(\underline{X})$ and $Uy \mid \Sigma \vdash Y$. It follows that there is a $X \in \underline{X}$ such that $X1_i \mid \Sigma \vdash U$, and therefore $X1_i y \mid \Sigma \vdash Uy$. Together with $Uy \mid \Sigma \vdash Y$ this yields $X1_i y \mid \Sigma \vdash Y$. Thus we haves hown, $\underline{X} \in \underline{\underline{C}} \wedge Y \in \underline{X} \mid 1_i y \;.\supset.\; (\exists\, X)\,[X \in \underline{X} \wedge X1_i y \mid \Sigma \vdash Y]$, which completes the proof by induction of (9).
Again by induction on z we show,

(10) $$X \in \underline{X} \in \underline{\underline{C}} \wedge Xz \mid \Sigma \vdash Y \quad . \supset . \quad Y \in \underline{X}/z, \text{ for } z \in N_k.$$

For $z = o$ this is established thus. $X \in \underline{X} \in \underline{\underline{C}}$ and $Xo \mid \Sigma \vdash Y$ implies $Y \in sp(\underline{X}) = \underline{X}$, which by (8) implies $Y \in \underline{X}/o$. Now we make the inductive assumption that (10) holds for $z = y$, and suppose that $X \in \underline{X} \in \underline{\underline{C}}$ and $X1_i y \mid \Sigma \vdash Y$. Then there is a Σ-deduction (i) $X1_i y, \ldots, U1_i y, Vy, \ldots, Y$. Here it may be assumed that $U1_i y$, Vy is the first to the left of any occurence of a contraction of form $A1_i y$, By in (i). This has the effect that the part $X1_i y, \ldots, U1_i y$, Vy of (i) remains a Σ-deduction if one takes of the terminal segment y from each word. Consequently, $X1_i \mid \Sigma \vdash V$, which together with $X \in \underline{X}$ yields $V \in tr_i(\underline{X})$. Furthermore the second part $Vy, \ldots, Y$ of (i) shows that $Vy \mid \Sigma \vdash Y$, so that by inductive assumption $Y \in tr_i(\underline{X})/y$, and by (8), $Y \in \underline{X}/1_i y$. Thus we have shown, $X \in \underline{X} \in \underline{\underline{C}} \wedge X1_i y \mid \Sigma \vdash Y \;.\supset.\; Y \in \underline{X}/1_i y$. This completes the proof by induction of (10).

Lemma 2. Let Σ be a reduced regular system on $\underline{I}_k$ with auxiliary alphabet $\underline{S}$, and let $\underline{U}$ and $\underline{V}$ be any subsets of $\underline{S}$. Define the finite automaton $\Sigma_1 = \text{sub}(\Sigma)$, as in definition 7, take $\underline{U}' = sp(\underline{U})$ as initial state, and define the output $\underline{\underline{V}}$ by $\underline{Y} \in \underline{\underline{V}}$ if and only if $\underline{Y} \in \underline{\underline{C}}$ and $\underline{Y} \cap \underline{V} \neq \Lambda$ (Λ denotes the empty set). Then $\beta(\underline{U}, \Sigma, \underline{V}) = \beta(\underline{U}', \Sigma_1, \underline{\underline{V}})$.

Corollary. If Σ is reduced regular, then every set of words $\beta(\underline{U}, \Sigma, \underline{V})$, accepted by Σ and sets $\underline{U}$ and $\underline{V}$ of auxiliary letters, is equal to the behavior of an output to the finite automaton $\Phi = \text{sub}(\Sigma)$, for proper choice of the initial state. If the rank of Σ is n, then Φ may be taken to have rank $\leq 2^n$.

Proof: Suppose $z \in \beta(\underline{U}, \Sigma, \underline{V})$. Then by definition 2 there are $U \in \underline{U}$ and $V \in \underline{V}$ such that $Uz \mid \Sigma \vdash V$. By (1) it follows that $U \in sp(\underline{U}) = \underline{U}'$ and by (10), $V \in \underline{U}'/z$. Because $V \in \underline{V}$ and $V \in \underline{X}/z$ it follows by the definition of $\underline{\underline{V}}$ that $\underline{U}'/z \in \underline{\underline{V}}$. Therefore, by definition 4, $z \in \beta(\underline{U}, \Sigma_,, \underline{\underline{V}})$. Thus we have shown $\beta(\underline{U}, \Sigma, \underline{V}) \subseteq \beta(\underline{U}', \Sigma_1, \underline{\underline{V}})$.
Suppose next that $z \in \beta(\underline{U}', \Sigma_1, \underline{\underline{V}})$. By definition 4 it follows that $\underline{U}'/z \in \underline{\underline{V}}$, i.e. there is a $V \in \underline{S}$ such that $V \in \underline{V}$ and $V \varepsilon \underline{U}'/z$. Therefore by (9) there is a $X \in \underline{U}'$ such that $Xz \mid \Sigma \vdash V$. Because $X \in \underline{U}' = sp(\underline{U})$ there is a $U \in \underline{U}$ such that $U \mid \Sigma \vdash X$, which together with $Xz \mid \Sigma \vdash V$ implies $Uz \mid \Sigma \vdash V$. Because $U \in \underline{U}$, $V \in V$ this yields, by definition 2, $z \in \beta(\underline{U}, \Sigma, \underline{V})$. Thus we have shown that also $\beta(\underline{U}', \Sigma, \underline{\underline{V}}) \subseteq \beta(\underline{U}, \Sigma, \underline{V})$, which concludes the proof of lemma 2. The corollary is an obvious consequence.

Our definition 7 of the automaton $\text{sub}(\Sigma)$ is a refinement of a subset-construction first used by Myhill [6], and later by Medvedev [8], and Rabin and Scott [1]. In these papers lemma 2 and its corollary are proved in the special case where Σ consists of contractions only. While in their case it is obvious that $\text{sub}(\Sigma)$ can be effectively obtained, the situation is quite different if Σ also contains expansions. We will still be able to show that $\text{sub}(\Sigma)$ is effectively constructable from Σ, however this requires a rather more careful investigation of Σ-deductions. Incidentally we will also obtain the result that $ctr(\Sigma)$, of definition 6 and lemma 1, may be effectively constructed. Because $\text{sub}(\Sigma) = \text{sub}(ctr(\Sigma))$ the constructibility of $\text{sub}(\Sigma)$ actually comes to the same as that of $ctr(\Sigma)$.
We remark that for a reduced system any Σ-deduction contains words of form Sx, $S \in \underline{S}$ and $x \in N_k$ only. We define the *height of a Σ-deducation* to be the maximum of all length $lg(x)$ such that Sx occurs in the deduction. By the *length of a Σ-deduction* we mean the number of words occurring as terms in the deduction. The important result in the process of proving constructibility of $ctr(\Sigma)$ and $sub(\Sigma)$ is:
Lemma 3. Let Σ be a reduced regular system on $\underline{I}_k$ and of rank n. If X and Y are auxiliary letters of Σ and $X \mid \Sigma \vdash Y$, then there is a Σ-deduction $X, \ldots, Sz, \ldots, Y$ of height $h \leq kn^2$, and of length $s \leq n(k^0 + k^1 + \ldots + k^h)$.

Proof: Suppose $X, \ldots, Wy, \ldots, Y$ is a Σ-deducation and suppose that $Wy = WI_h \ldots I_1$ is chosen such that $lg(y) = h =$ heights of the deduction. It is easy to verify that the deduction must be of form:

(i) $X, \ldots, U_1, X_1I_1, ----, U_rI_{r-1} \ldots I_1, X_rI_r \ldots I_1, ----, U_hI_{h-1} \ldots I_1, X_h I_h \ldots I_1, \ldots, WI_h \ldots I_1, \ldots, Y_hI_h \ldots I_1, V_hI_{h-1}, \ldots I_1, ----, Y_rI_r \ldots I_1, V_rI_{r-1} \ldots I_1, ----, Y_1I_1, V_1, \ldots, Y.$

Whereby for every $r = 1, \ldots, h$ the indicated expansion $U_rI_r \ldots I_1, X_rI_{r-1} \ldots I_1$ (contraction $Y_rI_r \ldots I_1, V_rI_{r-1} \ldots I_1$) is the first one occurring to the left of (right of) the indicated occurence of $WI_h \ldots I_1$, of any expansion of form $AI_{r-1} \ldots I_1, BI_r \ldots I_1$ (contraction of form $AI_r \ldots I_1, BI_{r-1} \ldots I_1$) occurring in (i). We note that this has the effect that for any $r = 1, \ldots, h$, and any $z \in N_k$ the deduction

(ii) $X_rI_rz, ----, U_hI_{h-1} \ldots I_rz, X_hI_h \ldots I_rz, \ldots, WI_h \ldots I_rz, \ldots, Y_hI_h \ldots I_rz, V_hI_{h-1} \ldots I_rz, ----, X_rI_rz$, obtained from (i) by the indicated modifications is still a Σ-deduction.

Now suppose that $h > kn^2$. Then two of the triples $(X_1, Y_1, I_1), \ldots, (X_h, Y_h, I_h)$ must be identical, say $X_p = X_q$, $Y_p = Y_q$, $I_p = I_q$ for $1 \leqq p < q \leqq h$. Using (ii) with $r = q$, $z = I_{p-1} \ldots I_1$ one obtains that the following modification of (i) still is a Σ-deduction:

(iii) $X, \ldots, U_1, X_1I_1, ----, U_pI_{p-1} \ldots I_1, X_qI_qI_{p-1} \ldots I_1, ----, U_hI_{h-1} \ldots I_qI_{p-1} \ldots I_1, X_hI_h \ldots I_qI_{p-1} \ldots I_1, \ldots, WI_h \ldots I_qI_{p-1} \ldots I_1, \ldots, Y_hI_h \ldots I_qI_{p-1} \ldots I_1, V_hI_h \ldots I_qI_{p-1} \ldots I_1, ----, Y_qI_qI_{p-1} \ldots I_1, V_pI_{p-1} \ldots I_1, ----, Y_1I_1, V_1, \ldots, Y$

Clearly the height of the deduction (iii) is not larger than that of (i), and because $p < q$, (iii) is shorter than (i). Thus we have shown that to every Σ-deduction (i) $X, \ldots, Y$ of height $h > kn^2$ and length s one can construct a Σ-deduction (i_1) $X, \ldots, Y$ of height $h_1 \leqq h$ and length $s_1 < s$. Now if h_1 still is larger than kn^2 one can iterate the procedure to obtain a Σ-deduction (i_2) $X, \ldots, Y$ of height $h_2 \leqq h_1$ and length $s_2 < s_1$, etc. Because $s > s_1 > s_2 > \ldots > 0$ it is clear that the construction must come to an end, say in m steps. Clearly this is possible only if (i_m) is of heights $h_m \leqq kn^2$. Thus we have shown:

(a) If $X \mid \Sigma \vdash Y$ then there is a Σ-deduction $X, \ldots, Y$ of height $h \leqq kn^2$.

Suppose next that (j) $X = X_1u_1, X_2u_2, \ldots, X_su_s = Y$ is a Σ-deduction of height h. Then $lg(u_1), lg(u_2), \ldots, lg(u_s) \leqq h$. But there are just $r = k^0 + k^1 + \ldots + k^h$ words z in N_k such that $lg(z) \leqq h$, and therefore there are just nr words Sz which may occur in (j). Consequently if the length s of (j) is larger than nr there will be a repetition $X_pu_p = X_qu_q$, $p < q$ in (j). Then clearly (j_1) $X, X_2u_2, \ldots, X_pu_p, X_{q+1}\, u_{q+1}, \ldots, X_su_s, = Y$ is still a Σ-deduction of height $h_1 \leqq h$ and length $s_1 < s$. By iteration of this argument, if necessary, one finally arrives at a Σ-deduction $X, \ldots, Y$ of height $\leqq h$ and length $\leqq n \,.\, r = n(k^0 + k^1 \ldots + k^h)$. Together with ($a$) this establishes lemma 3.

Lemma 4. For any reduced regular system Σ on $\underline{I}_k$, and any auxiliary letters X, Y one can effectively decide (*a*) whether or not $X \mid \Sigma \vdash Y$, (*b*) whether or not $X1_i \mid \Sigma \vdash Y$.

Corollary. To every reduced regular system Σ one can effectively construct the system *ctr* (Σ), and the finite automaton sub(Σ), of definitions 6 and 7.

Proof: Let n be the rank of Σ, let $r = kn^2$. Then clearly there are but a finite number of Σ-deductions $X, \ldots, Y$ of height $\leqq r$ and length $\leqq n(k^0 + k^1 + \ldots + k^r)$. Using lemma 3 this means that to decide whether or not $X \mid \Sigma \vdash Y$ one has to investigate but a finite number of Σ-deductions. This establishes (a).

Suppose next $X1_i \mid \Sigma \vdash Y$. Then there is a Σ-deduction $X1_i, \ldots, U1_i, V, \ldots, Y$ whereby the indicated pair $U1_i$, V is the first from the left of any occurrence of a contraction of from $A1_i$, B. Then clearly $X \mid \Sigma \vdash U$, $U1_i\underline{x} \to V\underline{x}$ is a contraction belonging to Σ, and $V \mid \Sigma \vdash Y$. Thus we have shown: (*c*) If $X1_i \mid \Sigma \vdash Y$ then there are auxiliary letters U and V such that $X \mid \Sigma \vdash U$, $U1_i\underline{x} \to V\underline{x}$ belongs to Σ, and $V \mid \Sigma \vdash Y$. The converse to (*c*) is obvious. Therefore, to decide whether or not $X1_i \mid \Sigma \vdash Y$ it is sufficient to check whether or not among the contractions $U1_i\underline{x} \to V\underline{x}$ of Σ there is one such that $X \mid \Sigma \vdash U$ and $V \mid \Sigma \vdash Y$. Because of (*a*) this can be done effectively, which establishes the remaining part (*b*) of the lemma.

By definition 6 and lemma 4 it is clear that $\Sigma_0 = ctr(\Sigma)$ can be effectively obtained. Furthermore, by definition 7 and lemma 4 it is clear that to every set $\underline{X}$ of auxiliaries one can effectively find the sets $sp(\underline{X})$ and $tr_i(\underline{X})$. Consequently the system sub(Σ) also can be effectively constructed from Σ. This establishes the corollary.

Theorem 1. To every reduced regular system Σ on $\underline{I}_k$, and any subsets $\underline{U}$ and $\underline{V}$ of its auxiliary alphabet S_n one can effectively construct a finite automaton Φ on $\underline{I}_k$ with initial state C and output $\underline{D}$ such that the behavior $\beta(C, \Phi, \underline{D})$ is exactly the set $\beta(\underline{U}, \Sigma, \underline{V})$ of accepted words. Moreover Φ can be so constructed that its rank is at most 2^n, whereby n is the rank of Σ.
This statement remains true if $\beta(\underline{U}, \Sigma, \underline{V})$ is replaced by $\tau(\underline{U}, \Sigma, \underline{V})$.

Proof: The first part of this assertion follows by the corollaries to lemmas 2 and 4. To obtain the second part we note (see remark in section 1) that $\tau(\underline{U}, \Sigma, \underline{V}) = \beta(\underline{V}, \overleftarrow{\Sigma}, \underline{U})$, whereby the converse system $\overleftarrow{\Sigma}$ still is reduced regular. Using the first part of theorem 1 we construct the automaton Φ_1 with initial state C and output $\underline{D}_1$ such that $\beta(\underline{V}, \Sigma, \underline{U}) = \beta(C_1, \Phi_1, \underline{D}_1)$. Then clearly, $\tau(\underline{U}, \Sigma, \underline{V}) = \beta(C_1, \Phi_1, \underline{D}_1)$, which proves the second part of theorem 1.

3. *Pure regular Systems*

According to definition 1 a regular system Σ on $\underline{I}_k$ is pure if it does not make use of any auxiliary letters. We will first show that the theorems $\tau(\underline{C}, \Sigma)$ of a pure system Σ with respect to a finite set of axioms $\underline{C}$ must always be the behavior of some finite automaton. The method consists in constructing a reduced system red(Σ) such that $\tau(\underline{C}, \Sigma) = \tau(\underline{U}, \text{red}\ (\Sigma), \underline{V})$, the assertion then follows by rheorem 1.

Lemma 5. To every pure regular system Σ on $\underline{I}_k$ and every finite set of axioms $\underline{C} \subseteq N_k$ one can effectively construct a reduced regular system red(Σ) on $\underline{I}_k$ with auxiliary alphabet $\underline{S}$, and subsets $\underline{U} \subseteq \underline{S}$ and $\underline{V} \subseteq \underline{S}$, such that $\tau(\underline{C}, \Sigma) = \tau(\underline{U}, \text{red}\ (\Sigma), \underline{V})$.

Proof: From $[\underline{C}, \Sigma]$ we first construct $[\underline{C}_1, \Sigma_1]$ by introducing the auxiliary letter S_1 and putting S_1c into $\underline{C}_1$, and $S_1a\underline{x} \rightarrow S_1\, b\underline{x}$ into Σ_1 just in case $c \in C$, and $a\underline{x} \rightarrow b\underline{x}$ is in Σ. If $[\underline{C}_m, \Sigma_m]$ with auxiliary letters $S_1, S_2, ----, S_{f(m)}$ has already been obtained, we construct $[\underline{C}_{m+1}, \Sigma_{m+1}]$ according to the following specifications:

(a) If $S_i 1_j c \in \underline{C}_m$, then introduce the next auxiliary letter S which has not yet come to use. Put Sc into $\underline{C}_{m+1}$, and put the production $S\underline{x} \rightarrow S_i 1_j \underline{x}$ into Σ_{m+1}.

(b) If $S_i 1_j a\underline{x} \rightarrow u\underline{x}$ is in Σ^m, then introduce the next auxiliary letter S which has not yet come to use. Put the productions $Sa\underline{x} \rightarrow u\underline{x}$ and $S_i 1_j \underline{x} \rightarrow S\underline{x}$ into Σ_{m+1}.

(c) If $u\underline{x} \rightarrow S_i 1_j a\underline{x}$ is in Σ_m, then introduce the next auxiliary letter S which has not yet come to use. Put the productions $u\underline{x} \rightarrow Sa\underline{x}$ and $S\underline{x} \rightarrow S_i 1_j \underline{x}$ into Σ_{m+1}.

It is clear that these constructions must come to an end with some $[\underline{C}_s, \Sigma_s]$, with auxiliary letters $S_1, S_2, ----, S_{f(s)}$, and having the properties that $\underline{C}_s$ consists of auxiliary letters only (else instruction (a) could still be applied), and that Σ_s consists of reduced regular rules only (else either instruction (b) or (c) could still be applied). Furthermore it is easily seen that $\tau(\underline{C}, \Sigma) - \tau(\underline{C}_1, \Sigma_1, S_1)$, and $\tau(\underline{C}_i, \Sigma_i, S_1) = \tau(\underline{C}_{i+1}, \Sigma_{i+1}, S_1)$ for $i = 1, \ldots, s\text{-}1$. Therefore $\underline{U} = \underline{C}_s$, *red*$(\Sigma) = \Sigma_s$, $\underline{V} = \{S_1\}$ is a triple as required in lemma 5.

Theorem 2. To every pure regular system Σ on $\underline{I}_k$ and any finite set of axioms $\underline{C} \subseteq N_k$ one can effectively construct a finite automaton Φ on I_k, an initial state A, and on output $\underline{U}$, such that the set of theorem $\tau(\underline{C}, \Sigma)$ is exactly the behavior $\beta(A, \Phi, \underline{U})$.

Proof: Construct $[\underline{X}, \text{red}(\Sigma), \underline{Y}]$ according to lemma 5, so that $\tau(\underline{C}, \Sigma) = \tau(\underline{X}, \text{red}(\Sigma), \underline{V})$. Then construct $[A, \Phi, \underline{U}]$ from $\underline{X}$, red(Σ), $\underline{Y}$ according to the second part of theorem 1, so that $\tau(\underline{X}, \text{red}(\Sigma), \underline{Y}) = \beta(A, \Phi, \underline{U})$. Then clearly $[A, \Phi, U]$ is as required in theorem 2.

We shall next establish the converse to theorem 2: every behavior $\beta(A, \Phi, \underline{U})$ of a finite automaton on $\underline{I}_k$ can be generated from a finite set $\underline{C} \subseteq N_k$ of axioms by a pure regular system Σ on $\underline{I}_k$. This comes to showing how the auxiliary letters (transit states) can be eliminated from Φ. We refer to the discussion of the lefttree $[N_k, <]$ at the beginning of section 1, and begin with an investigation of deductions in a particular type of pure systems.

Definition 8. Let $\underline{A} = \{a_i\}_r$ and $\underline{B} = \{b_i\}_r$ be families of elements of N_k, and let Φ be a finite automaton on $\underline{I}_k$ with transitstate A. If $\underline{B}$ is a frontier of $[N_k, <]$ and $\underline{A} \subseteq \text{int}(\underline{B})$ then the pure regular system $\Sigma(\underline{A}, \underline{B})$ is called a *frontier system.* If in addition $A/a_i = A/b_i$ for $i = 1, \ldots, r$, then $\Sigma(\underline{A}, \underline{B})$ is called a $[A, \Phi]$ – *frontier system.*

Lemma 6. If $\Sigma = \Sigma(\underline{A}, \underline{B})$ is a frontier system on $\underline{I}_k$ then to every $z \in N_k$ there is a $z_0 \in \text{int}(\underline{B})$ such that $z_0 \mid \Sigma \vdash z$.

Proof: To every $y \in N_k$ define the excess $ec(y)$ over the frontier $\underline{B}$ as follows:

$$(1) \qquad \begin{aligned} ec(y) &= -1 &&, \text{ if } y \in \text{int}(\underline{B}) \\ ec(y) &= lg(n) &&, \text{ if } y = bu \text{ and } b \in \underline{B} \end{aligned}$$

Note that (1) unambiguously defines $ec(y)$, because by assumption $\underline{B}$ is a frontier. The proof of the lemma now goes by induction on $ec(z)$.
If $ec(z) = -1$ then by (1), $z \in \text{int}(\underline{B})$. Therefore if $z_0 = z$ we have $z_0 \in \text{int}(\underline{B})$ and $z_0 \mid \Sigma \vdash z$. This establishes the base of the induction. Next assume that $ec(z) = s > -1$, and make the inductive assumption:

$$(2) \qquad \text{If } ec(y) < s \text{ then there is a } y_0 \in \text{int}(\underline{B}) \text{ such that } y_0 \mid \Sigma \vdash y.$$

Because $ec(z) = s > -1$ it follows by (1) that there are y and b_i such that $z = b_i y$ and $s = lg(y)$. Because $a_i\underline{x} \to b_i\underline{x}$ is a production of Σ, $a_i y \mid \Sigma \vdash b_i y$, i.e., $a_i y \mid \Sigma \vdash z$. There are two cases to be considered, namely (*a*) $ec(a_i y) = -1$, and (*b*) $ec(a_i y) > -1$. In case (*a*) it follows by (1) that $a_i y \in \text{int}(\underline{B})$. Therefore, if $z_0 = a_i y$, we have $z_0 \in \text{int}(\underline{B})$ and $z_0 \mid \Sigma \vdash z$. In case (*b*) it follows by (1) that there are $v \in N_k$ and $b_p \in \underline{B}$ such that $ec(a_i y) = lg(v)$ and $a_i y = b_p v$. By assumption $a_i \in \text{int}(\underline{B})$, and therefore $b_p \nleqq a_i$. Because $a_i y = b_p v$ this implies that $y > v$, and therefore $s = lg(y) > lg(v)$. Because $lg(v) = ec(a_i y)$ it follows that $ec(a_i y) < s$. Therefore, by inductive assumption (2), there is a $z_0 \in$ *int*$(\underline{B})$ such that $z_0 \mid \Sigma \vdash a_i y$. Because $a_i y \mid \Sigma \vdash z$, it follows that $z_0 \mid \Sigma \vdash z$. Thus also in case (*b*) we have found a $z_0 \in$ *int*(B), $z_0 \mid \Sigma \vdash z$. This concludes the proof of lemma 6.

Lemma 7. Let Φ be a finite automaton on $\underline{I}_k$ with transistate A and output $\underline{U}$. And let $\Sigma = \Sigma(\underline{A}, \underline{B})$ be a $[A, \Phi]$ – frontier system. Then the behavior $\beta(A, \Phi, \underline{U})$ is equal to a set of theorems $\tau(\underline{C}, \Sigma)$. The axioms set is given by $c \in \underline{C} \,.\equiv.\, c \in int(\underline{B}) \wedge A/c \in \underline{U}$.
Proof: Suppose $u \mid \Sigma \vdash v$. Then there is a deduction $u = p_0 z_0$, $q_0 z_0 = p_1 z_1$, $q_1 z_1 = p_2 z_2, \ldots, q_{s-1} z_{s-1} = p_s z_s$, $q_s z_s = v$, whereby $p_i\underline{x} \to q_i\underline{x}$ belongs to Σ. Because Σ is a $[A, \Phi]$–frontier system it follows that $A/p_i = A/q_i$, and by (*) of section (1), $A/p_i z_i = A/q_i z_i$. Therefore, $A/u = A/p_0 z_0 = A/q_0 z_0 = \ldots = A/p_s z_s = A/q_s z_s = A/v$. Thus we have shown

$$(a) \qquad u \mid \Sigma \vdash v \quad . \supset . \quad A/u = A/v$$

Suppose now that $y \in \tau(\underline{C}, \Sigma)$. Then there is a $y_0 \in \underline{C}$ such that $y_0 \mid \Sigma \vdash y$. By (*a*) it follows $A/y_0 = A/y$, and by definition of $\underline{C}$, $A/y_0 \in \underline{U}$. Therefore $A/y \in \underline{U}$, i.e., $y \in \beta(A, \Phi, \underline{U})$. This proves the part $\tau(\underline{C}, \Sigma) \subseteq \beta(A, \Phi, \underline{U})$ of the lemma.

Suppose next that $y \in \beta(A, \Phi, \underline{U})$, i.e., $A/y \in \underline{U}$. By lemma 6 there is a $y_0 \in int(B)$ such that $y_0 \mid \Sigma \vdash y$. It follows by (*a*) that $A/y_0 = A/y$. Thus $A/y_0 \in \underline{U}$ and $Y_0 \in int(\underline{B})$, i.e., by definition of $\underline{C}$, $y_0 \in \underline{C}$. Thus $y_0 \in \underline{C}$ and $y_0 \mid \Sigma \vdash y$ which means that $y \in \tau(\underline{C}, \Sigma)$. This proves the remaining part, $\beta(A, \Phi, \underline{U}) \subseteq \tau(\underline{C}, \Sigma)$, of lemma 7.

Theorem 3. To every finite automaton Φ on $\underline{I}_k$ with transit state A and output $\underline{U}$ one can effectively construct a pure regular system Σ on $\underline{I}_k$ and a finite set of axiom $\underline{C} \subseteq K_k$ such that the behavior $\beta(A, \Phi, \underline{U})$ is exactly the set of theorem $\tau(\underline{C}, \Sigma)$.

Proof: Let n be the rank of Φ, let $m = k^n$, and let $z_1, \ldots, z_m$ be an enumeration of all words $z \in N_k$, such that $lg(z) = n$. Now choose (a_i, p_i) such that $a_i p_i$ is the first left-segment of z_i with a repetition $A/a_i = A/a_i p_i$. That this is possible follows by the accessibility argument at the end of section 1. Furthermore, it is clear that for $i \neq j$ it is not possible that $a_i p_i < a_j p_j$, because else $a_j p_j$ would not be the first segment of z_j of the kind considered. It is also clear that to every $u \in N_k$ there is $a_i p_i$ such that either $u < a_i p_i$ or $u = a_i p_i$ or $a_i p_i < u$. Consequently, $\underline{B} = \{a_i p_i\}_m$ is a frontier of $[N_k, <]$. Furthermore, $a_i \in int(\underline{B})$ and $A/a_i = A/a_i p_i$. Therefore $\Sigma = \Sigma(\underline{A}, \underline{B})$ is a $[A, \Phi]$ – frontier system. Thus we have indicated how one can effectively construct a $[A, \Phi]$ – frontier system to any finite automaton Φ and transitstate A. Theorem 3 now follows by lemma 7.

Remark. The system $\Sigma(\underline{A}, \underline{B})$ constructed in the proof of theorem 3 consist of as many as $m = k^n$ productions. However many among the pairs $(a_1, p_1), \ldots, (a_m, p_m)$ will usually turn out to be equal to each other, so that Σ can be greatly reduced to Σ_0 by dropping repetitions (see end of next section for improved construction). In fact, only for more trivial automata Φ this does not happen. That the construction leads to a rather economical Σ_0 is indicated by the fact that Σ_0 is independent in the sense that none of its productions may be omitted without changing $\tau(\underline{C}, \Sigma_0)$. If one is not interested in this additional feature of economy one can obtain theorem 3 more directly, without using lemma 6 and frontier systems.

Note also that Σ constructed in the proof of theorem 3 is a frontier system, which has the effect that it is *strictly expanding* in the sense that $u \mid \Sigma \vdash v$, $u \neq v$ implies that the excess (see proof of lemma 6)of v is larger than that of u. In section 5 we will analyze sets of theorems of such systems Σ. Via theorem 3 this leads to a better understanding of the structure of behaviors of finite automata.

It is possible to modify the construction as to obtain a $[A, \Phi]$ – frontier system $\Sigma(\underline{A}, \underline{B})$ with the property that $A/u \neq A/v$ for $u, v \in int(\underline{B})$. This has the effect that $\Sigma(\underline{A}, \underline{B})$ becomes minimal in the sense that the lengths of words a, *b* of productions $a\underline{x} \to b\underline{x}$ in the system are small. Also the number of axioms needed will be minimal.

One may consider a triple $[\underline{C}, \underline{A}, \underline{B}]$, consisting of finite families $\underline{C}, \underline{A}, \underline{B}$ of words in N_k, to be a syntactic objet denoting the set $\tau(\underline{C}, \Sigma(\underline{A}, \underline{B})) \subseteq N_k$. Theorem 2 then is a *synthesis result*: to every notation $[\underline{C}, \underline{A}, \underline{B}]$ one can construct a finite automaton Φ with initial state A and output $\underline{U}$ such that the behavior $\beta(A, \Phi, \underline{U})$ is just the set denoted by $[\underline{C}, \underline{A}\ \underline{B}]$. Theorem 3 is the corresponding *analysis result*.

4. *Periodic sets of words.*

In its own right the following seems to be a reasonable extension of the concept of a (ultimately) periodic set of natural numbers to the arithmetic of words in N_k:

Definition 9. A set $\beta \subseteq N_k$ is *closed* with respect to the production $a\underline{x} \to b\underline{x}$ if for any $u \epsilon N_k$, $au \epsilon \beta\ .\supset.\ bu \epsilon \beta$. If β is closed with respect to $a\underline{x} \to ap\underline{x}$, we will say that (a, p) is a *phase-period* of β.

A set $\beta \subseteq N_k$ is called *k-periodic* if there are a finite set $\underline{C} \subseteq N_k$ and a finite set of pairs $\underline{P} \subseteq N_k \times N_k$, such that β is the smallest subset of N_k containing $\underline{C}$ and having the phase-periods $(a, p) \epsilon P$. The pair $(\underline{C}, \underline{P})$ then is called a *periodic description* of β.

A *periodic regular system* on $\underline{I}_k$ is a pure regular system Σ on $\underline{I}_k$ all of whose productions are of the form $a\underline{x} \to ap\underline{x}$.

Later we will indicate more evidence in the direction of showing that k-periodicity is a natural generalization of periodicity on N_1.

Lemma 8. A set $\beta \subseteq N_k$ is k-periodic if and only if there is a periodic regular system Σ and a finite set of axioms $\underline{C} \subseteq N_k$ such that β is the set of theorems $\tau(\underline{C}, \Sigma)$. Furthermore, if $(\underline{C}, P)$ is a periodic description of β, one can take Σ to consist of all productions $a\underline{x} \to ap\underline{x}$, $(a, p) \epsilon \underline{P}$.

Proof: If Σ is a system of productions on $\underline{I}_k$ and $\underline{C} \subseteq N_k$ is a system of axioms, then $\tau(\underline{C}, \Sigma)$ is the intersection of (smallest of) all set $\gamma \subseteq N_k$ such that $\underline{C} \subseteq \gamma$ and γ is closed under the productions in Σ. (This is the well known fact used already by Dedekind in obtaining explicit set-theoretic definitions for recursively defined sets and relations on natural numbers). With this remark in mind one easily obtains the lemma from definition 9.

Theorem 4. A set $\beta \subseteq N_k$ is k-periodic if and only if it is the behavior $\beta(A, \Phi, \underline{U})$ of some finite automaton Φ on $\underline{I}_k$, initial state A, and output $\underline{U}$. Furthermore from a periodic description $(\underline{C}, \underline{P},)$ of β one can effectively construct $[A, \Phi, \underline{U}]$ and conversely.

Proof: Suppose that $(\underline{C}, \underline{P})$ is a periodic description of β. Then by lemma 8 and theorem 2 one can obtain an automaton Φ, initial state A, and output U such that $\beta = \beta(A, \Phi, \underline{U})$. This establishes one direction of theorem 4.

Suppose now that $[A, \Phi, \underline{U}]$ is given. In the proof to theorem 3 we constructed $\Sigma = \Sigma(\underline{A}, \underline{B})$ and $\underline{C}$ such that $\beta(A, \varnothing, \underline{U}) = \tau(\underline{C}, \Sigma)$, and furthermore the family $\underline{B}$ is of form $\{a_i p_i\}_r$, whereby $\underline{A} = \{a_i\}_r$. Therefore, if $\underline{P}$ is the set consisting of the pairs $(a_1, p_1), \ldots, (a_r, p_r)$, then by lemma 8 the pair $(\underline{C}, \underline{P})$ is a periodic description of $\beta(A, \Phi, \underline{U})$. This establishes theorem 4 in the other direction.

Remark. A periodic description $(\underline{C}, \underline{P})$ of a set β of words seems to give a ratther clear picture of how the set β is built up. Therefore the value of theorem 4 is that it provides a good idea of how behaviors of finite automata look like. This is not the case for other characterizations of behaviors of finite automata, which have appeared in the literature. Let us therefore recapitulate the construction of (C, P), in the proof of theorem 3, in more intuitive terminology:

Construction of a periodic description for the behavior $\beta(A, \Phi, \underline{U})$. Let n be the rank of Φ. Construct the finite part of the left tree $[N_k, <)$ starting with o at level o, and ending with level n, consisting of the words $z_1 = 1_1 1_1 .. 1_1, \ldots, z_m = 1_k 1_k .. 1_k$ of length n, in lexicographic order. Attach the label A/u to each vertex u in the tree. Find $y_1, y_2, \ldots$, and $(a_1, p_1), (a_2, p_2), \ldots$ according to the following instructions

(1) Take y_1 to be z_1.

(2) If y_i has been found then search along the path o to y_i for the first repetition $A/a = A/ap$ of a label, and let $(a_i, p_i) = (a, p)$.

(3) If (a_i, p_i) has been found then search for the first element y in the list $y_i = z_j, z_{j+1}, \ldots, z_m$ such that $a_i p_i < y$, and let $y_{i+1} = y$.

This procedure clearly comes to an end in $s \leqq m$ steps, and $\underline{B} = \{a_1 p_1, \ldots, a_s p_s\}$ is a frontier. Among the vertices c occurring between o and the frontier $\underline{B}$ (i. e., $c \in int(\underline{B})$) select those which carry a label $A/c \in \underline{U}$, and put them into the set $\underline{C}$. Let $\underline{P} = \{(a_1, p_1), \ldots, (a_s, p_s)\}$. Then $(\underline{C}, \underline{P})$ is a periodic description of $\beta(A, \Phi, \underline{U})$.

For a discussion of economy of this $(\underline{C}, \underline{P})$ see the remark following theorem 3. Going through, in an example, the construction of $(\underline{C}, \underline{P})$ in case Φ is given by a transition diagram $[\underline{S}, tr_1, \ldots, tr_k]$ will show the workability of this procedure, and will provide an idea of how much better one knows the behavior $\beta(A, \Phi, \underline{U})$ once the periodic description $(\underline{C}, \underline{P})$ is found.

5. *The structure of the set of theorems of a frontier system.*

In this section we will analyze how the sets $\tau(\underline{C}, \Sigma)$ for frontier systems are built up. Our analysis applies equally well to expansive systems (definition 11) which are slightly more general than frontier systems. The fundamental concept in our analysis is that of an elementary set:

Definition 10. Let R be a binary relation on the numbers $1, 2, \ldots, r$. By an R-*family* $\underline{E} = \{e_{i,j}\}_R$ in N_k we mean a doubly indexed family of words $e_{i,j} \in N_k$, whereby $e_{i,j}$ is defined just in case iRj holds.

Let $\underline{E} = \{e_{i,j}\}_R$ be an *R-family* in N_k. Then the *$\underline{E}$-elementary sets* $\varrho_{p,q}(\underline{E}) \subseteq N_k$ are defined, for $p, q = 1, \ldots, r$, as follows:
$\varrho_{p,q}(\underline{E})$ consist of all words of form $e_{i_0, i_1} e_{i_1, i_2} \ldots e_{i_{s-2}, i_{s-1}} e_{i_{s-1}, i_s}$ whereby $i_0 = p$, $i_s = q$, and $s = 1, 2, 3, \ldots$ The word o is included in $\varrho_{p,q}(\underline{E})$ just in case $p = q$
Note that a sequence $i_0, i_1, \ldots, i_s$ gives rise to a word $e_{i_0, i_1} e_{i_1, i_2} \ldots e_{i_{s-1}, i_s}$ just in case it is an R-sequence, i. e., in case $i_0 R i_1, \ldots, i_{s-1} R i_s$ holds.

Definition 11. Let $\underline{A} = \{a_i\}_r$ and $\underline{B} = \{b_i\}r$ be families in N_k. The pure regular system $\Sigma = \Sigma(\underline{A}, \underline{B})$ on $\underline{I}_k$ is called *expansive* in case, $b_i \leqq a_j$ for $i, j = 1, \ldots, r$. The R-family $\underline{E} = \{e_{i,j}\}$ of such a system Σ is defined as follows:

$$iRj \quad . \equiv . \quad a_i < b_j \qquad , \text{ for } i, j = l, \ldots, r$$
$$iRj \quad . > . \quad a_i e_{i,j} = b_j$$

The elementary sets of Σ are the $\underline{E}$-elementary sets, and will also be denoted by $\varrho_{i,j}(\Sigma)$.

Lemma 9. Let $\Sigma = \Sigma(\underline{A}, \underline{B})$ be an expansive system on $\underline{I}_k$, and let $\underline{E} = \{e_{i,j}\}_R$ be its R-family. Suppose that $c, \ldots, y$ is a Σ-deducation of length greater than 2. Then there is a $d \epsilon N_k$ and *a* (i, j) such that $c = a_i d$ and $y \in b_j{}^\frown \varrho_{j,i}{}^\frown d$.

Proof: The Σ-deducation clearly must be of form $c = a_{i_0} d_0$, $b_{i_0} d_0 = a_{i_1} d_1$, $b_{i_1} d, = a_{i_2} d_2, \ldots, b_{i_{s-1}} d_{s-1} = a_{i_s} d_s$, $b_{i_s} d_s = y$. Because Σ is expansive it follows from $b_{i_0} d_0 = a_{s_1} d_1$ that $a_{i_1} < b_{i_0}$, and therefore, $b_{i_0} = a_{i_1} e_{i_1, i_0}$ and $e_{i_1, i_0} d_0 = d_1$. Similarly one obtains $e_{i_2, i_1} d_1 = d_2, \ldots, e_{i_s, i_{s-1}} d_{s-1} = d_s$. Therefore $d_s = e_{i_s, i_{s-1}} \ldots e_{i_2, i_1} e_{i_1, i_0} d_0$. By definition, of $\varrho_{j,i}$ this implies $d_s \in \varrho_{j,i}{}^\frown d_0$, if $j = i_s$ and $i = i_0$. Because $y = b_{i_s} d_s$ we therefore have $y \in b_j{}^\frown \varrho_{j,i}{}^\frown d_0$. Because $c = a_i d_0 (i = i_0)$, this completes the proof of the lemma.

Theorem 5. If Σ is an expansive regular system on $\underline{I}_k$ and $\underline{C}$ is any finite set of axioms then the set of theorems $\tau(\underline{C}, \Sigma)$ is a finite union of sets of form $b^\frown \varrho^\frown d$, whereby ϱ is elementary. More exactly, if $\Sigma = \Sigma(\underline{A}, \underline{B})$, $\underline{A} = \{a_i\}_r$, $\underline{B} = \{b_i\}_r$, and $\varrho_{i,j} = p_{i,j}(\Sigma)$ are the elementary sets of Σ, then

$$\tau(\underline{C}, \Sigma) = \underline{C} \cup \bigvee_{\substack{i = 1, \ldots, r \\ j = 1, \ldots, r}} b_i{}^\frown \varrho_{i,j}{}^\frown \underline{D}_j$$

whereby $\underline{D}_j$ consists of all d such that $a_j d \epsilon \underline{C}$.

Proof: Let $\{e_{i,j}\}_R$ be the R-family of Σ (see definition 11). Then clearly Σ contains (among others) the following productions:

$$(1) \qquad a_i \underline{x} \to a_j e_{j,i} \underline{x}, \qquad \text{whenever } jRi$$

Suppose first that y belongs to the indicated union. If $y \epsilon \underline{C}$ then clearly $y \epsilon \tau(\underline{C}, \Sigma)$. If $y \notin \underline{C}$ then $y \in b_{i_0}{}^\frown \varrho_{i_0, i_s}{}^\frown d$ for some i_0, i_s and some d such that $a_{i_s} d \epsilon \underline{C}$. Therefore, by definition of $\varrho_{i,j}$, either $i_0 = i_s$ and $y = b_{i_s} d$. or $y = b_{i_0} e_{i_0, i_1} e_{i_1, i_2} \ldots e_{i_{s-1}, i_s} d$. In thefi rst case it is clear that $a_i\, d \mid \Sigma \vdash b_{i_s} d$, and because $a_{i_s} d \epsilon \underline{C}$, $y = b_{i_s} d \epsilon \tau(\underline{C}, \Sigma)$. In the second case it follows by (1) that

$$a_{i_s}d,\ a_{i_{s-1}}e_{i_{s-1},\ i_s}d,\ \ldots\ldots,\ a_{i_0}e_{i_0,\ i_1}\ldots e_{i_{s-1},\ i_s}d$$

is a Σ-deduction. Because $a_{i_s}\ d\epsilon \underline{C}$, and $a_{i_0}\underline{x} \to b_{i_0}\underline{x}$ is a production of Σ, this implies that $y = b_{i_0}e_{i_0,\ i_1}\ldots e_{i_{s-1},\ i_s}d\epsilon\tau(\underline{C}, \Sigma)$. Thus we have shown that the union indicated in theorem 5 is contained in $\tau(\underline{C}, \Sigma)$.

Assume now that $y\epsilon\tau(\underline{C}, \Sigma)$, i.e., $\underline{C} \mid \Sigma \vdash y$. Then either $y\epsilon\underline{C}$ or there is an $c\epsilon\underline{C}$ and a Σ-deduction $c, \ldots, y$ of length greater than one. In case the length of the deduction is 2, it must be of form $c = a_i d$, $b_i d = y$. By definition of $\underline{D}_i$ it follows that $d\epsilon\underline{D}_i$, and because $o\epsilon\varrho_{i,\ i}$ we obtain $y\epsilon b_i{}^\frown\varrho_{i,\ i}{}^\frown\underline{D}i$. In case the length of the deduction, is greater than 2 it follows by lemma 9 that $c = a_i d$ and $y\epsilon b_j{}^\frown\varrho_{j,\ i}{}^\frown d$, for some i, j and $d\epsilon N_k$. By definition of $\underline{D}_i$, $c = a_i d$ implies $d\epsilon\underline{D}_i$, so that also in this case y belong to some $b_j{}^\frown\varrho_{j,\ i}{}^\frown d$. Consequently $\tau(\underline{C}, \Sigma)$ is contained in the union indicated in theorem 5, which completes the proof.

Remark. By their very definition one has a clear idea of how elementary sets are constructed. Theorem 5 therefore explains much about the structure of sets of theorems of expansive systems. Because the construction of theorem 3 (see also end of section 4) yields an expansive system, this explanation carries over to behaviors of finite automata, and, via theorems 1 and 2, to sets of theorems of other regular systems.

For an elementary set $\varrho_{i,\ j}(\underline{E})$ it is easy to set up a regular system Σ and axioms such that $\varrho_{i,\ j}(\underline{E})$ becomes the set of theorems. More generally, every finite union of terms of form $a^\frown\varrho^\frown b$ with elementary ϱ, is easily seen to be the set of theorems of some regular system and finitely many axioms.

If $\underline{E} = \{e_{i,\ j}\}_R$ is an R-family on N_k, one may introduce a different letter $1_{i,\ j}$ corresponding to every pair (i, j) such that iRj. The R-family $\underline{E}_0 = \{1_{i,\ j}\}_R$ then clearly is such that $\varrho_{i,\ j}(\underline{E})$ may be obtained from $\varrho_{i,\ j}(\underline{E}^0)$ by replacing in each word $u\epsilon\varrho_{i,\ j}(\underline{E}^0)$ every occurrence of $1_{p,\ q}$ by the word $e_{i,\ j}$.

6. *Concluding remarks.*

a) It remains to generalize theorem 2 to arbitrary regular systems. This however does not pose new difficulties and can be handled thus:

Let Σ be a regular system on $\underline{I}_k$ with auxiliary alphabet $\underline{S}_n$, and let $\underline{C}$ be a finite set of words on $\underline{I}_k \cup \underline{S}_n$. We take Σ_1 to be the pure regular system on the alphabet $\underline{I}_k \cup \underline{S}_n$ and consisting of the same productions as Σ. Then clearly

$$y\epsilon\tau(\underline{C}, \Sigma) \quad . \equiv . \quad a\epsilon N_k \wedge y\epsilon\tau(\underline{C}, \Sigma_1)$$

By theorem 2 one can construct a finite automaton Φ_1 with input alphabet $\underline{I}_k \cup \underline{S}_n$, transition alphabet $\underline{A} = \{A_1, \ldots, A_m\}$, such that

$$y\epsilon\tau(\underline{C}, \Sigma_1) \quad . \equiv . \quad y\epsilon\beta(A_1, \Phi_1, \underline{U})$$

whereby $\underline{U} \subseteq \underline{A}$. Now let Φ be the automaton on $\underline{I}_k$ with transit alphabet $\underline{A}$ obtained from Φ_1 by simply dropping the productions $A_i S_p \underline{x} \to A_j \underline{x}$. Then clearly:

$$y \in \beta(A_1, \Phi, \underline{U}) \quad .\equiv. \quad y \in N_k \wedge y \in \beta(A_1, \Phi_1, \underline{U})$$

It is now clear that $\beta(A_1, \Phi, \underline{U}) = \tau(\underline{C}, \Sigma)$.

b) Up to this point we have called regular, what should more appropriately be termed *right- regular*. Of course our results also hold for *left-regular systems*, whose rules are of form $\underline{x}a \to \underline{x}b$. However, additional investigation is necessary to prove the following stronger form of theorem 2:
If the system Σ consists of right- and left-regular rules and $\underline{C}$ is a finite set of axioms, then the set of theorems $\tau(\underline{C}, \Sigma)$ is the behavior $\beta(A, \Phi, \underline{U})$ of a finite automaton Φ with initial state A and output U. Again, $[A, \Phi, \underline{U}]$ can be effectively constructed.

c) Theorem 2 can also be generalized to systems Σ which contain productions of form $a_1\underline{x}, \ldots, a_n\underline{x} \to b\underline{x}$. Let us call these *general-right-regular systems*. However, it does not seem possible to derive this result from theorem 2, one rather has to modify the proof, which is mainly contained in the lemmas of section 1. The difficulty is that deductions now are not any more linear, but of tree-form. This does not really require new ideas, however, the presentation of the proofs gets clumsy.
The fact that, also for general-right-regular systems, $\tau(C, \Sigma)$ is the behavior of a finite automaton, generalizes a result of Rabin and Scott [1] concerning "two way non erasing automata". Inspection will show that such automata, as well as other special types of Turing machines, may be interpreted as reduced general-right-regular systems.

d) In view of remarks *b*) and *c*) one might except that also systems Σ containing both general-right-regular and general-left-regular rules produce recursive sets of words. However, as noted by Post [2], this is not the case. In fact every recursively generable set of words can be produced from a single axiom by rules of form $a_1\underline{x}, a_2\underline{x} \to b\underline{x}$ and $\underline{x}a_1, \underline{x}a_2 \to \underline{x}b$.
We intend to present elsewhere our proofs of the assertions in *b*, *c*, and *d*. It seems likely that *b* and *c* may be improved to,
Conjecture: If the system Σ consist of right-regular and general-left regular rules and $\underline{C}$ is finite, then the set of theorems $\tau(\underline{C}, \Sigma)$ is the behavior $\beta(A, \Phi, \underline{U})$ of a finite automaton Φ. Furthermore $[A, \Phi, \underline{U}]$ can be effectively constructed from $[\underline{C}, \Sigma]$.

e) A right-regular system Σ on $\underline{I}_k$ may be called *symmetric* if with every production $a\underline{x} \to b\underline{x}$ it also contains the converse $b\underline{x} \to a\underline{x}$. If Σ is pure, note that the relation $x \mid \overline{\Sigma} \vdash y$ is a *right-congruence* on $[\overline{N}_k, {}^\frown]$, i.e., it is an equivalence relation and

$$x \mid \Sigma \vdash y \quad .>. \quad x^\frown u \mid \Sigma \vdash y^\frown u$$

In contrast to the analogous word problem for semi-groups, formulated by Thue [10], and solved in the negative sense by Post [3], our results show that $x \mid \Sigma \vdash y$ is recursive, for any symmetric regular system. We have shown even more: there is a general method applying to all x, Σ, y and yielding a decision as to whether or not $x \mid \Sigma \vdash y$.

f) Let us say that a right-congruence $x \sim y$ on N_k is of finite rank n, if its partition consists of n classes. The (right-) congruences of finite rank play the same role in the artihmetic on N_k as do the relations $x \equiv y$ $(mod\, n)$ in ordinary arithmetic. In particular:
A set $\beta \subseteq N_k$ is k-periodic if and only if it is the union of congruence classes of some (right-) congruence of finite rank.
An equivalent condition is that the *induced* right-congruence

$$x \sim y \ (\mathrm{mod}\, \beta) \quad . \equiv . \quad (\forall u)\ (x^\frown u \in \beta \equiv y^\frown u \in \beta)$$

has finite rank. These assertions follow by theorem 4 and the remark that to every right congruence $\sim$ of finite rank n there is a finite automaton Φ (of rank n) with transit state A, such that $A/x = A/y$ just in case $x \sim y$. (This has been noted also by Rabin and Scott [7]).
Using our results one can show that to every (right-) congruence $\sim$ of finite rank on N_k there is a symmetric pure regular system Σ on $\underline{I}_k$ such that $x \sim y$ if and only if $x \mid \Sigma \vdash y$.
Perhaps of special interest are the congruences $x \sim y$ (mod ϱ) for elementary sets ϱ. Then there are the meet-irreducible congruences of finite rank, which essentially correspond to congruences modulo powers of primes. To investigate in which manner the fundamental facts about primes generalize to the k-array arithmetic on words, would seem to be of importance for a better understanding of k-periodicity, i.e., the behavior of finite automata. Some remarks on k-array arithmetic are contained in Buchi [9].

g) Simplifying the work of Kleene [4], Myhill [6], and Copi, Elgot, and Wright [5] have shown that:

Synthesis: To every regular expression E one can construct a finite automata Φ, A, and $\underline{U}$ such that E denotes the behavior $\beta(A, \Phi, \underline{U})$.

Analysis: To every finite automaton Φ, A, and $\underline{U}$ one can construct a regular expression E which denotes $\beta(A, \Phi, \underline{U})$.

Let Σ_1 and Σ_2 be reduced regular system without expansions on $\underline{I}_k$ and auxiliary alphabets $\underline{S}_1$ respectively $\underline{S}_2$. If $\underline{S}_1 \cap \underline{S}_2 = \Lambda$, $\underline{U}_1, \underline{V}_1 \subseteq \underline{S}_1$ and $\underline{U}_2, \underline{V}_2 \subseteq \underline{S}_2$, one can construct $\Sigma, \underline{U}, \underline{V}$ such that

(a) $\beta(\underline{U}, \Sigma, V) = \beta(\underline{U}_1, \Sigma_1, \underline{V}_1) \cup \beta(\underline{U}_2, \Sigma_2, \underline{V}_2)$

(b) $\beta(\underline{U}, \Sigma, V) = \beta(\underline{U}_1, \Sigma_1, \underline{V}_1)^\frown \beta(\underline{U}_2, \Sigma_2, \underline{V}_2)$

(c) $\beta(\underline{U}, \Sigma, V) = \beta(\underline{U}_1, \Sigma_1, \underline{V}_1)^*$

Namely as follows:

(a) Let $\Sigma = \Sigma_1 \cup \Sigma_2$, $\underline{U} = \underline{U}_1 \cup \underline{U}_2$, $\underline{V} = \underline{V}_1 \cup \underline{V}_2$

(b) Let Σ consist of Σ_1, Σ_2, and $V\underline{x} \to U\underline{x}$ for $V \in \underline{V}_1$, $U \in \underline{U}_2$.
Let $\underline{U} = \underline{U}_1$, and $\underline{V} = \underline{V}_1$.

(c) Let Σ consist of Σ_1 and $V\underline{x} \to U\underline{x}$, for $V \in \underline{V}$ and $U \in \underline{U}$
Let $\underline{U} = \underline{U}_1$, and $\underline{V} = \underline{V}_1$.

These remarks together with theorem 1 can be used to prove the synthesis-result.

By theorem 5 it is sufficient to prove the analysis result for elementary sets $\varrho_{i,j}(\underline{E})$. That to every elementary set $\varrho_{i,j}(\underline{E})$ one can obtain a regular expression is best shown by an induction on the cardinality of $\underline{E}$. This result also follows by the remark concerning $\underline{E}^0$ at the end of section 5, and the fact that the sets $\varrho_{i,j}(\underline{E}^0)$ are regular, which is simply arest atement of lemma 1 of Copi, Elgot, and Wright. (This lemma was also used by Myhill [6] and is implicit already in Kleene [4]).

h) Among the right- and left-regular systems there are right- and left-automata. Let u^0 denote the word obtained from u by a left-right-inversion, and let β^0 consist of all u^0, $u \in \beta$. Then clearly β is the behavior of a left-automaton just in case β^0 is the behavior of the corresponding right-automaton. However, a priori it seems likely that a behavior β of some right-automaton might not be the behavior of a left-automaton, i.e., right-periodicity might not coincide with left-periodicity. That this is not actually the case follows from the theory of regularity; the primitive operations $\cup, {}^\frown, {}^*$ on regular sets do not distinguis right from left.

While for automata behavior this remark is non-trivial, it is on the other hand quite easy to construct a reduced left-regular system Σ_l to any right-automaton Φ such that $\beta(A, \Phi, \underline{U}) = \beta(\underline{U}, \Sigma_l, A)$, namely

$$\underline{x}\, 1_\varrho S_\mu \quad \to \quad \underline{x} S_\nu \text{ in } \Sigma_l$$

if and only if

$$S_\nu 1_\varrho \underline{x} \quad \to \quad S_\mu \underline{x} \text{ in } \Phi.$$

The desired result then follows by theorem 1.

Theorem. The following condition on a set β of words in N_k are equivalent:

(1) β is the behavior $\beta(A, \Phi, \underline{U})$ of some finite right-automaton Φ $\underline{I}_k$ on with initial state A and output $\underline{U}$.

(2) β is the set $\beta(\underline{U}, \Sigma, \underline{V})$ accepted by a reduced right-regular system Σ on $\underline{I}_k$, and sets $\underline{U}$ and $\underline{V}$ of auxiliary letters of Σ.

(3) β is the set of theorems $\tau(\underline{C}, \Sigma)$ of a pure right-regular system Σ on $\underline{I}_k$ and a finite number of axioms $\underline{C} \subseteq N_k$.

(4) β is right-k-periodic, i.e., β has a right-periodic description $(\underline{C}, \underline{P})$.

(5) The right-congruence generated by β is of finite rank.

(6) β is the union of equivalence classes modulo a right-congruence of finite rank.

(7) β is the set of theorems $\tau(\underline{C}, \Sigma)$ of a general right-regular system Σ on $\underline{I}_k$ and finite set $\underline{C}$ of axioms.

(8) β is a regular set, i.e., β is the set $\beta(E)$ denoted by a regular expression E.

(i′) standing for (i) with "right" replaced by "left"; $i = l, \ldots, 7$.

BIBLIOGRAPHY

[1] Rabin, M., and Scott, D., "Finite Automata and Their Decision Problems", *IBM Journal*, April, 1959, pp. 114–125.

[2] Post, Emil L., "Formal Reductions of the General Combinatorial Decision Problem". *American Journal of Mathematics* Vol. 65, pp. 197–215. (1943).

[3] Post, Emil L., "Recursive Unsolvability of a Problem of Thue", *Journal of Symbolic Logic*, Vol 12, pp. 1–11 (1947).

[4] Kleene, S. C., "Representation of Events in Nerve Nets and Finite Automata", *Automata Studies*, Princeton University Press, 1956, pp. 3–41.

[5] Copi, I. M., Elgot, C. C., and Wright, J. B. "Realization of Events by Logical Nets", *Journal of the Association for Computing Machinery*, 5, pp. 181–196 (1958).

[6] Myhill, John, "Finite Automata and Representation of Events", WADC Report TR 57–624, *Fundamental Concepts in the Theory of Systems*, October, 1957, pp. 112–137.

[7] Rosenbloom, P. C., "*The Elements of Mathematical Logic*". Dover (1950).

[8] Medvedev, I. T., "On a Class of Events Representable in a Finite Automaton", MIT Lincoln Laboratory Group Report, 34–73, translated from the Russian by J. Schorr-Kon, June 30, 1958.

[9] Büchi, J. R., "Weak Second Order Arithmetic and Finite Automata", *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, Vol. 6 (1960), pp. 66–92.

[10] Thue, Axel, "Probleme über Veränderungen von Zeichenreihen nach gegebenen Regeln", *Skrifter utgit av Videnskaps selskapet i Kristiania*, I. Matematisknaturvidenskabelig klasse 1914, no. 10, 34 pp.