

On the Size of Finite Rational Matrix Semigroups

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Abstract

Let n be a positive integer and \mathcal{M} a set of rational $n \times n$ -matrices such that \mathcal{M} generates a finite multiplicative semigroup. We show that any matrix in the semigroup is a product of matrices in \mathcal{M} whose length is at most $2^{n(2n+3)}g(n)^{n+1} \in 2^{O(n^2 \log n)}$, where $g(n)$ is the maximum order of finite groups over rational $n \times n$ -matrices.

2012 ACM Subject Classification Mathematics of computing \rightarrow Discrete mathematics; Theory of computation \rightarrow Algebraic language theory

Keywords and phrases Matrix semigroups, Burnside problem

Funding *Georgina Bumpus*: Work supported by St John's College, Oxford.

Stefan Kiefer: Work supported by a Royal Society University Research Fellowship.

Paul-Ioan Stoienescu: Work supported by St John's College, Oxford.

Jonathan Tanner: Work supported by St John's College, Oxford.

Acknowledgements The authors thank James Worrell for discussing [13] with them.

1 Introduction

The Burnside Problem

An element g of a semigroup G is called *torsion* if $g^i = g^j$ holds for some naturals $i < j$, and G *torsion* if all its elements are torsion. Burnside [5] asked in 1902 a question which became known as the *Burnside problem* for groups: is every finitely generated torsion group finite? Schur [24] showed in 1911 that this holds true for groups of invertible complex matrices, i.e., any finitely generated torsion subgroup of $GL(n, \mathbb{C})$ is finite. This was generalised by Kaplansky [17, p. 105] to matrices over arbitrary fields. The Burnside problem for groups has a negative answer in general: in 1964 Golod and Shafarevich exhibited a finitely generated infinite torsion group [11, 10].

The Maximal Order of Finite Matrix Groups

Schur's result [24] assures that finitely generated torsion matrix groups are finite, but does not bound the group order. Indeed, it is easy to see that any finite cyclic group is isomorphic to a group generated by a matrix in $GL(2, \mathbb{R})$. The same is not true for $GL(n, \mathbb{Q})$: An elementary proof (which we reproduce in the appendix following [19]) shows that any finite subgroup of $GL(n, \mathbb{Q})$ is conjugate to a finite subgroup of $GL(n, \mathbb{Z})$. Another elementary proof shows that the order of any finite subgroup of $GL(n, \mathbb{Z})$ divides $(2n)!$; see, e.g., [23,

Chapter IX]. Thus, denoting the order of the largest finite subgroup of $GL(n, \mathbb{Q})$ by $g(n)$, we have $g(n) \leq (2n)!$. It is shown in a paper by Friedland [9] that $g(n) = 2^n n!$ holds for all sufficiently large n . This bound is attained by the group of signed permutation matrices. Friedland's proof rests on an article by Weisfeiler [29] which in turn is based on the classification of finite simple groups. Feit showed in an unpublished manuscript [7] that $g(n) = 2^n n!$ holds if and only if $n \in \mathbb{N} \setminus \{2, 4, 6, 7, 8, 9, 10\}$.¹ Feit's proof relies on an unpublished manuscript [28], also based on the classification of finite simple groups, which Weisfeiler left behind before his tragic disappearance.

Deciding Finiteness of Matrix Groups

Bounds on group orders give a straightforward, albeit inefficient, way of deciding whether a given set of matrices generates a finite group: starting from the set of generators, enlarge it with products of matrices in the set, until either it is closed under product or the bound on the order has been exceeded. One can do substantially better: it is shown in [2] that, using computations on quadratic forms, one can decide in polynomial time if a given finite set of rational matrices generates a finite group.

Deciding Finiteness of Matrix Semigroups

The Burnside problem has a natural analogue for semigroups. In 1975, McNaughton and Zalcstein [22] positively solved the Burnside problem for matrix semigroups, i.e., they showed, for any field \mathbb{F} , that any finitely generated torsion subsemigroup of $\mathbb{F}^{n \times n}$ is finite, using the result for groups by Schur and Kaplansky as a building block. From a computational point of view, McNaughton and Zalcstein's result suggests an approach for deciding finiteness of the semigroup generated by a given set of rational matrices: finiteness is recursively enumerable, by closing the set of generators under product, as described above for groups. On the other hand, infiniteness is recursively enumerable by enumerating elements in the generated semigroup and checking each element whether it is torsion. By the contrapositive of McNaughton and Zalcstein's result, if the generated matrix semigroup is infinite, it has a non-torsion element, witnessing infiniteness. However, deciding whether a given matrix has finite order is nontrivial. Only in 1980 did Kannan and Lipton [15, 16] show that the so-called *orbit problem* is decidable (in polynomial time), implying an algorithm for checking whether a matrix has finite order.

Avoiding this problem, Mandel and Simon [21] showed in 1977 that there exists a function $f : \mathbb{N}^3 \rightarrow \mathbb{N}$ such that if S is a finite subsemigroup of $\mathbb{F}^{n \times n}$, generated by m of its elements, and the subgroups of S have order at most g , then S has size (cardinality) at most $f(n, m, g)$. For rational matrices, one may use the function $g(n)$ from above for g . By making, in a sense, McNaughton and Zalcstein's proof quantitative, Mandel and Simon explicitly construct such a function f , which implies an algorithm, with bounded runtime, for deciding finiteness of a finitely generated rational matrix semigroup. A similar result about the decidability of this problem was obtained independently and concurrently by Jacob [14].

¹ A list of the maximal-order finite subgroups of $GL(n, \mathbb{Q})$ for $n \in \{2, 4, 6, 7, 8, 9, 10\}$ can be found in [3, Table 1].

Size Bounds

Unlike the function g for rational matrix groups, Mandel and Simon's function $f(n, m, g)$ depends on m , the number of generators. This is unavoidable: the semigroup generated by the set $\mathcal{M}_m := \left\{ \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} : i \in \{0, \dots, m-1\} \right\}$ is the set \mathcal{M}_m itself, with $|\mathcal{M}_m| = m$ for any $m \in \mathbb{N}$. Further, the growth in n of Mandel and Simon's f is, roughly, a tower of exponentials of height n . They write in [21, Section 3]: "However, it is likely that our upper bound $[f(n, m, g)]$ can be significantly improved."

In [4, Chapter VI], Berstel and Reutenauer also show, for the rational case, the existence of a function in n and m that bounds the semigroup size. They write: "As we shall see, the function [...] grows extremely rapidly." An analysis of their proof shows that the growth of their function is comparable with the growth of Mandel and Simon's function. A related approach is taken in [26]. Further proofs of McNaughton and Zalcstein's result can be found, e.g., in [20, 8, 6, 25], but they do not lead to better size bounds.

Length Bounds

In 1991, Weber and Seidl [27] considered semigroups over *nonnegative* integer matrices. Using combinatorial and automata-theoretic techniques, they showed that if a finite set $\mathcal{M} \subseteq \mathbb{N}^{n \times n}$ generates a finite monoid, then for any matrix M of that monoid there are $M_1, \dots, M_\ell \in \mathcal{M}$ with $\ell \leq \lceil e^2 n! \rceil - 2$ such that $M = M_1 \cdots M_\ell$; i.e., any matrix in the monoid is a product of matrices in \mathcal{M} whose *length* it at most $\lceil e^2 n! \rceil - 2$. Note that this bound does not depend on the number of generators. Weber and Seidl also give an example that shows that such a length bound cannot be smaller than 2^{n-2} .

Almeida and Steinberg [1] proved in 2009 a length bound for rational matrices and expressing the zero matrix: if a finite set $\mathcal{M} \subseteq \mathbb{Q}^{n \times n}$ (with $n > 1$) generates a *finite* semigroup that includes the zero matrix 0, then there are $M_1, \dots, M_\ell \in \mathcal{M}$ with $\ell \leq (2n-1)^{n^2} - 1$ such that $0 = M_1 \cdots M_\ell$. A length bound of n^5 for expressing the zero matrix was recently given in the nonnegative integer case [18]. It is open whether there is a polynomial length bound for expressing the zero matrix in the rational case.

Our Contribution

We prove a $2^{O(n^2 \log n)}$ length bound for the rational case:

► **Theorem 1.** *Let $\mathcal{M} \subseteq \mathbb{Q}^{n \times n}$ be a finite set of rational matrices such that \mathcal{M} generates a finite semigroup $\overline{\mathcal{M}}$. Then for any $M \in \overline{\mathcal{M}}$ there are $M_1, \dots, M_\ell \in \mathcal{M}$ with $\ell \leq 2^{n(2n+3)} g(n)^{n+1} \in 2^{O(n^2 \log n)}$ such that $M = M_1 \cdots M_\ell$. (Here $g(n) \leq (2n)!$ denotes the order of the largest finite subgroup of $GL(n, \mathbb{Q})$.)*

The example by Weber and Seidl mentioned above shows that any such length bound must be at least 2^{n-2} . Theorem 1 implies an exponential-space algorithm for deciding finiteness of a finitely generated rational matrix semigroup.² A length bound trivially implies a size bound: in the rational case we obtain $|\overline{\mathcal{M}}| \leq m^{2^{O(n^2 \log n)}}$, the first significant improvement over the fast-growing function of Mandel and Simon.

The proof of Theorem 1 is largely based on linear-algebra arguments, specifically on the structure of a certain graph of vector spaces obtained from \mathcal{M} . This graph was introduced

² In fact, Theorem 1 implies that deciding finiteness is in $\text{coNEXP}^{\text{NP}}$, the second level of the weak EXP hierarchy (see e.g. [12] for a definition).

and analysed by Hrushovski et al. [13] for the computation of the *Zariski closure* of the generated matrix semigroup.

2 Preliminaries

We write $\mathbb{N} = \{0, 1, 2, \dots\}$. For a finite alphabet Σ , we write $\Sigma^* = \{a_1 \cdots a_k : k \geq 0, a_i \in \Sigma\}$ and $\Sigma^+ = \{a_1 \cdots a_k : k \geq 1, a_i \in \Sigma\}$ for the free monoid and the free semigroup generated by Σ . The elements of Σ^* are called *words*. For a word $w = a_1 \cdots a_k$, its *length* $|w|$ is k . We denote by ε the empty word, i.e., the word of length 0. For $L \subseteq \Sigma^*$, we also write $L^* = \{w_1 \cdots w_k : k \geq 0, w_i \in L\} \subseteq \Sigma^*$ and $L^+ = \{w_1 \cdots w_k : k \geq 1, w_i \in L\} \subseteq \Sigma^*$.

We denote by I_n the $n \times n$ -identity matrix, and by $\vec{0}$ the zero vector. For vectors v_1, \dots, v_k from a vector space, we denote their span by $\langle v_1, \dots, v_k \rangle$. In this article, we view elements of \mathbb{Q}^n as *row* vectors.

For some $n \in \mathbb{N} \setminus \{0\}$, let $\mathcal{M} \subseteq \mathbb{Q}^{n \times n}$ be a finite set of rational matrices, generating a finite semigroup $\overline{\mathcal{M}}$. For notational convenience, throughout the paper, we associate to \mathcal{M} an alphabet Σ with $|\mathcal{M}| = |\Sigma|$, and a bijection $M : \Sigma \rightarrow \mathcal{M}$ which we extend to the monoid morphism $M : \Sigma^* \rightarrow \overline{\mathcal{M}} \cup \{I_n\}$. Thus we may write $M(\Sigma)$ and $M(\Sigma^*)$ for \mathcal{M} and $\overline{\mathcal{M}} \cup \{I_n\}$, respectively.

We often identify a matrix $A \in \mathbb{Q}^{n \times n}$ with its linear transformation $A : \mathbb{Q}^n \rightarrow \mathbb{Q}^n$ such that $x \mapsto xA$ for row vectors $x \in \mathbb{Q}^n$. To avoid clutter, we extend linear-algebra notions from matrices to words, i.e., we may write $\text{im } w$, $\ker w$, $\text{rk } w$ for the image $\text{im}(M(w)) = \mathbb{Q}^n M(w)$, the kernel $\ker(M(w)) = \{x \in \mathbb{Q}^n : xM(w) = \vec{0}\}$, and the rank of $M(w)$.

If all matrices in $M(\Sigma)$ are invertible and $M(\Sigma^*)$ is finite, then $M(\Sigma^*)$ is a finite subgroup of $GL(n, \mathbb{Q})$. For $n \in \mathbb{N}$, let us write $g(n)$ for the size of the largest finite subgroup of $GL(n, \mathbb{Q})$. As discussed in the introduction, a non-trivial but elementary proof shows $g(n) \leq (2n)!$, and it is known that $g(n) = 2^n n!$ holds for sufficiently large n .

Exterior Algebra

This brief introduction is borrowed and slightly extended from [13, Section 3]. Let V be an n -dimensional vector space over a field \mathbb{F} . (We will only consider $V = \mathbb{Q}^n$.) For any $r \in \mathbb{N}$, let \mathcal{A}_r denote the set of maps $B : V^r \rightarrow \mathbb{F}$ so that B is linear in each argument and further $B(v_1, \dots, v_r) = 0$ holds whenever $v_i = v_{i+1}$ holds for some $i \in \{1, \dots, r-1\}$. These conditions imply that swapping two adjacent arguments changes the sign, i.e.,

$$B(v_1, \dots, v_{i-2}, v_{i-1}, v_{i+1}, v_i, v_{i+2}, v_{i+3}, \dots, v_r) = -B(v_1, \dots, v_r).$$

These properties of \mathcal{A}_r imply that, given an arbitrary basis $\{e_1, \dots, e_n\}$ of V , any $B \in \mathcal{A}_r$ is uniquely determined by all $B(e_{i_1}, \dots, e_{i_r})$ where $1 \leq i_1 < i_2 < \dots < i_r \leq n$. For any $v_1, \dots, v_r \in V$, define the *wedge product*

$$v_1 \wedge \cdots \wedge v_r : \mathcal{A}_r \rightarrow \mathbb{F} \quad \text{by} \quad (v_1 \wedge \cdots \wedge v_r)(B) = B(v_1, \dots, v_r).$$

It follows from the properties of \mathcal{A}_r above that the wedge product is linear in each argument: if $v_i = \lambda u + \lambda' u'$ then

$$\left(\bigwedge_{1 \leq i \leq k} v_i \right)(B) = \lambda \left(\bigwedge_{1 \leq j < i} v_j \wedge u \wedge \bigwedge_{i < j \leq k} v_j \right)(B) + \lambda' \left(\bigwedge_{1 \leq j < i} v_j \wedge u' \wedge \bigwedge_{i < j \leq k} v_j \right)(B)$$

Moreover, $(v_1 \wedge \cdots \wedge v_r)(B) = 0$ if $v_i = v_j$ holds for some i, j with $i \neq j$.

For $r \in \mathbb{N}$ define $\Lambda^r V$ as the vector space generated by the length- r wedge products $v_1 \wedge \cdots \wedge v_r$ with $v_1, \dots, v_r \in V$. For any basis $\{e_1, \dots, e_n\}$ of V , the set $\{e_{i_1} \wedge \cdots \wedge e_{i_r} : 1 \leq i_1 < \dots < i_r \leq n\}$ is a basis of $\Lambda^r V$; hence $\dim \Lambda^r V = \binom{n}{r}$. Note that $\Lambda^1 V = V$ and $\binom{n}{r} = 0$ for $r > n$. One can view the wedge product as an associative operation $\wedge : \Lambda^r V \times \Lambda^\ell V \rightarrow \Lambda^{r+\ell} V$. Define the *exterior algebra of V* as the direct sum $\Lambda V = \Lambda^0 V \oplus \Lambda^1 V \oplus \cdots$. Then also $\wedge : \Lambda V \times \Lambda V \rightarrow \Lambda V$.

It follows that for $u_1, \dots, u_r \in V$, we have $u_1 \wedge \cdots \wedge u_r \neq \vec{0}$ if and only if $\{u_1, \dots, u_r\}$ is linearly independent. Furthermore, for $u_1, \dots, u_r, v_1, \dots, v_r \in V$ and $u = u_1 \wedge \cdots \wedge u_r \neq \vec{0}$ and $v = v_1 \wedge \cdots \wedge v_r \neq \vec{0}$, we have that u, v are scalar multiples if and only if $\langle u_1, \dots, u_r \rangle = \langle v_1, \dots, v_r \rangle$.

The Grassmannian $\text{Gr}(n)$ is the set of subspaces of \mathbb{Q}^n . By the above-stated properties of the wedge product there is an injective function

$$\iota : \text{Gr}(n) \rightarrow \Lambda \mathbb{Q}^n$$

such that, for all $W \in \text{Gr}(n)$, we have $\iota(W) = v_1 \wedge \cdots \wedge v_r$ where $\{v_1, \dots, v_r\}$ is an arbitrarily chosen basis of W . Note that the particular choice of a basis for W only changes the value of $\iota(W)$ up to a constant. Given subspaces $W_1, W_2 \in \text{Gr}(n)$, we moreover have $W_1 \cap W_2 = \{\vec{0}\}$ if and only if $\iota(W_1) \wedge \iota(W_2) \neq \vec{0}$.

3 Proof of Theorem 1

It is convenient to state and prove our main result in terms of monoids rather than semigroups:

► **Theorem 2.** *Let $M : \Sigma^* \rightarrow \mathbb{Q}^{n \times n}$ be a monoid morphism whose image $M(\Sigma^*)$ is finite. Then for any $w \in \Sigma^*$ there is $u \in \Sigma^*$ with $M(w) = M(u)$ and*

$$|u| \leq 2^{n(2n+3)} g(n)^{n+1} \in 2^{O(n^2 \log n)}.$$

With this theorem at hand, Theorem 1 follows immediately:

Proof of Theorem 1. Let $M \in \overline{\mathcal{M}}$ be an element of the semigroup generated by \mathcal{M} . If $M \neq I_n$, by Theorem 2, M can be written as a short product. Otherwise, $M = I_n \in G$, where $G = \overline{\mathcal{M}} \cap GL(n, \mathbb{Q})$ is a finite group of order at most $g(n)$. For any product $M_1 \cdots M_\ell$ with $\ell > g(n)$, there are $1 \leq i < j \leq \ell$ such that $M_1 \cdots M_i = M_1 \cdots M_j$, and so $M_1 \cdots M_\ell = M_1 \cdots M_i M_{j+1} \cdots M_\ell$. Hence, there are $\ell \in \{1, \dots, g(n)\}$ and $M_1, \dots, M_\ell \in \mathcal{M}$ such that $M = I_n = M_1 \cdots M_\ell$. ◀

► **Remark 3.** *The same argument as in the proof above shows that in a finite monoid (H, \cdot) , generated by $G \subseteq H$, for any $h \in H$ there are $\ell \in \{0, \dots, |H| - 1\}$ and $g_1, \dots, g_\ell \in G$ with $h = g_1 \cdots g_\ell$.*

In the remainder of this section, we prove Theorem 2. We assume that $M : \Sigma^* \rightarrow \mathbb{Q}^{n \times n}$ is a monoid morphism with finite image $M(\Sigma^*)$.

3.1 The Maximum-Rank Case

In this subsection we prove:

► **Proposition 4.** *Suppose that there is $r \leq n$ with $\text{rk } a = r$ for all $a \in \Sigma$. Let $w \in \Sigma^*$ with $\text{rk } w = r$. Then there is $u \in \Sigma^*$ with $M(w) = M(u)$ and*

$$|u| \leq 2^{2n+3} g(n) - 1 \in 2^{O(n \log n)}.$$

In this subsection we assume that $\text{rk } a = r$ holds for all $a \in \Sigma$. For the proof of Proposition 4, we define a directed labelled graph G whose vertices are the vector spaces $\text{im } w$ for $w \in \Sigma^*$ such that $\text{rk } w = r$, and whose edges are triples (V_1, a, V_2) such that $a \in \Sigma$ and $V_1 M(a) = V_2$. Let (V_1, a, V_2) be an edge; then $V_2 \subseteq \text{im } a$, but $\dim V_2 = r = \text{rk } a = \dim \text{im } a$, hence $V_2 = \text{im } a$, i.e., the edge label determines the edge target. We will implicitly use the fact that any path in G is determined by its start vertex and the sequence of its edge labels. Note that if V_1 is a vertex and $a \in \Sigma$, the edge $(V_1, a, \text{im } a)$ is present in G if and only if $\text{rk } V_1 M(a) = r$ if and only if $V_1 \cap \ker a = \{\vec{0}\}$.

The following two lemmas, which are variants of lemmas in [13, Section 6], are statements about the structure of G in terms of its strongly connected components (SCCs).

► **Lemma 5.** *Let $w = w_1 \cdots w_k$ for $w_1, \dots, w_k \in \Sigma^+$ with $\text{rk } w = r$ such that the k vertices $\text{im } w_1, \dots, \text{im } w_k$ are all in different SCCs of G . Then $k \leq 2\binom{n}{r}$.*

Proof. Let $i \in \{2, \dots, k-1\}$. Since $\text{rk } w_i = r = \text{rk}(w_i w_{i+1})$, we have $\text{im } w_i \cap \ker w_{i+1} = \{\vec{0}\}$, thus $\iota(\text{im } w_i) \wedge \iota(\ker w_{i+1}) \neq \vec{0}$. On the other hand, for any $j < i$, since $\text{im } w_i, \text{im } w_j$ are in different SCCs and $\text{im } w_i$ is reachable from $\text{im } w_j$, the vertex $\text{im } w_j$ is not reachable from $\text{im } w_i$; therefore we have $\text{im } w_i \cap \ker w_j \neq \{\vec{0}\}$, thus $\iota(\text{im } w_i) \wedge \iota(\ker w_j) = \vec{0}$. It follows that $\iota(\ker w_{i+1}) \notin \langle \iota(\ker w_j) : j < i \rangle$.

We show by induction on i that $\dim \langle \iota(\ker w_j) : j \in \{1, \dots, i\} \rangle \geq i/2$ for all $i \in \{1, \dots, k\}$. This is clear for $i = 1, 2$. For the induction step, we have $\dim \langle \iota(\ker w_j) : j \in \{1, \dots, i+1\} \rangle \geq \dim \langle \iota(\ker w_{i+1}), \iota(\ker w_j) : j \in \{1, \dots, i-1\} \rangle \geq 1 + (i-1)/2 = (i+1)/2$. Hence $k/2 \leq \dim \langle \iota(\ker w_j) : j \in \{1, \dots, k\} \rangle \leq \dim \Lambda^{n-r} \mathbb{Q}^n = \binom{n}{r}$. ◀

► **Lemma 6.** *Let $a_1 \cdots a_k \in \Sigma^*$ be (the edge labels of) a shortest path in G from a vertex $\text{im } a_0$ to $\text{im } a_k$. Then $k \leq \binom{n}{r}$.*

Proof. Let $i \in \{0, \dots, k-2\}$. We have $\text{im } a_i \cap \ker a_{i+1} = \{\vec{0}\}$, thus $\iota(\text{im } a_i) \wedge \iota(\ker a_{i+1}) \neq \vec{0}$. On the other hand, for any $j > i+1$, since $a_{i+1} \cdots a_j$ is a shortest path from $\text{im } a_i$ to $\text{im } a_j$, there is no edge from $\text{im } a_i$ to $\text{im } a_j$; therefore we have $\text{im } a_i \cap \ker a_j \neq \{\vec{0}\}$, thus $\iota(\text{im } a_i) \wedge \iota(\ker a_j) = \vec{0}$. It follows that $\iota(\ker a_{i+1}) \notin \langle \iota(\ker a_j) : j > i+1 \rangle$.

By induction it follows that $\dim \langle \iota(\ker a_j) : j \in \{i+1, \dots, k\} \rangle \geq k-i$ holds for all $i \in \{0, \dots, k-1\}$. Hence $k \leq \dim \langle \iota(\ker a_j) : j \in \{1, \dots, k\} \rangle \leq \dim \Lambda^{n-r} \mathbb{Q}^n = \binom{n}{r}$. ◀

The next lemmas discuss *cycles* $w \in \Sigma^+$ in G , i.e., (the edge labels of) paths in G such that $\text{im } w \cap \ker w = \{\vec{0}\}$. A cycle w is said to be *around* $\text{im } w_0$ if $\text{im } w = \text{im } w_0$. The following lemma says, loosely speaking, that cycles around a single vertex “generate a group”.

► **Lemma 7.** *Let $w_0 \in \Sigma^+$ with $\text{rk } w_0 = r$, and let $P \in \mathbb{Q}^{r \times n}$ be a matrix with $\text{im } P = \text{im } w_0$. Then for every cycle $w \in \Sigma^+$ around $\text{im } w_0$ there exists a unique invertible matrix $M'(w) \in GL(r, \mathbb{Q})$ such that $PM(w) = M'(w)P$. Moreover, for any nonempty set $C \subseteq \Sigma^+$ of cycles around $\text{im } w_0$, $M'(C^+)$ is a finite subgroup of $GL(r, \mathbb{Q})$.*

Proof. Let $w \in \Sigma^+$ be a cycle around $\text{im } w_0$. Since $\text{im } P \cap \ker(M(w)) = \{\vec{0}\}$, it follows that $\text{im}(PM(w)) = \text{im } w = \text{im } P$. So the rows of $PM(w)$ are linear combinations of rows of P , and vice versa, hence there is a unique $M'(w) \in GL(r, \mathbb{Q})$ with $PM(w) = M'(w)P$.

Let $C \subseteq \Sigma^+$ be a nonempty set of cycles around $\text{im } w_0$. For any $w_1, w_2 \in C$ we have $M'(w_1 w_2)P = PM(w_1 w_2) = PM(w_1)M(w_2) = M'(w_1)PM(w_2) = M'(w_1)M'(w_2)P$, and since the rows of P are linearly independent, it follows that $M'(w_1 w_2) = M'(w_1)M'(w_2)$. Thus, $M'(C^+)$ is a semigroup.

Towards a contradiction, suppose $M'(C^+)$ were infinite. Since the rows of P are linearly independent, it follows that $M'(C^+)P$ is infinite, thus $PM(C^+)$ is infinite. Since $\text{im } w_0 =$

in P , there is a matrix $B \in \mathbb{Q}^{n \times r}$ with $M(w_0) = BP$. Since the columns of B are linearly independent, the set $BPM(C^+)$ is infinite. But this set equals $M(w_0C^+)$, contradicting the finiteness of $M(\Sigma^*)$. Thus the semigroup $M'(C^+)$ is finite. As $M'(C^+) \subseteq GL(r, \mathbb{Q})$, it follows that $M'(C^+)$ is a finite group. \blacktriangleleft

The following lemma allows us, loosely speaking, to limit the number of cycles in a word.

► Lemma 8. *Let $w_0, w_1, \dots, w_k \in \Sigma^+$ such that w_1, \dots, w_k are cycles around $\text{im } w_0$. Then there exist $\ell \leq g(n) - 1$ and $\{u_1, \dots, u_\ell\} \subseteq \{w_1, \dots, w_k\}$ such that $M(w_0w_1 \cdots w_k) = M(w_0u_1 \cdots u_\ell)$.*

Proof. We can assume $k \geq 1$. Let $C = \{w_1, \dots, w_k\}$. Let P and $M'(w)$ for $w \in C$ as in Lemma 7. By Lemma 7, the set $M'(C^+)$ is a finite subgroup of $GL(r, \mathbb{Q})$, so we have $|M'(C^+)| \leq g(r) \leq g(n)$. By Remark 3, there are $\ell \leq g(n) - 1$ and $u_1, \dots, u_\ell \in C$ such that $M'(w_1) \cdots M'(w_k) = M'(u_1) \cdots M'(u_\ell)$. Since $\text{im } w_0 = \text{im } P$, there is a matrix $B \in \mathbb{Q}^{n \times r}$ with $M(w_0) = BP$. Hence we have $M(w_0w_1 \cdots w_k) = BPM(w_1) \cdots M(w_k) = BM'(w_1) \cdots M'(w_k)P = BM'(u_1) \cdots M'(u_\ell)P = BPM(u_1) \cdots M(u_\ell) = M(w_0u_1 \cdots u_\ell)$. \blacktriangleleft

The following lemma allows us to add cycles to a word.

► Lemma 9. *Let $w \in \Sigma^+$ be a cycle in G . Then there exists $\rho(w) \in \mathbb{N} \setminus \{0\}$ such that $M(w_0) = M(w_0w^{\rho(w)})$ holds for all $w_0 \in \Sigma^+$ with $\text{im } w_0 = \text{im } w$.*

Proof. Let $P \in \mathbb{Q}^{r \times n}$ be a matrix with $\text{im } P = \text{im } w$. By Lemma 7, there exists $M'(w) \in GL(r, \mathbb{Q})$ such that $PM(w) = M'(w)P$ and $\{M'(w)^i : i \in \mathbb{N}\}$ is a finite group. Define $\rho(w)$ to be the order of this group, i.e., $M'(w)^{\rho(w)} = I_r$. Let $w_0 \in \Sigma^+$ with $\text{im } w_0 = \text{im } w$. Since $\text{im } w_0 = \text{im } P$, there is a matrix $B \in \mathbb{Q}^{n \times r}$ with $M(w_0) = BP$. Hence $M(w_0) = BP = BI_rP = BM'(w)^{\rho(w)}P = BPM(w)^{\rho(w)} = M(w_0)M(w)^{\rho(w)} = M(w_0w^{\rho(w)})$. \blacktriangleleft

The following lemma allows us to limit the length of paths within an SCC.

► Lemma 10. *Let $a \in \Sigma$, and let $w \in \Sigma^*$ be a path in G from $\text{im } a$ such that $\text{im } a$ and $\text{im } w$ are in the same SCC. Then there exists $u \in \Sigma^*$ with $M(aw) = M(au)$ and*

$$|u| \leq 2^{n+2}g(n) - 2 \in 2^{O(n \log n)}.$$

Proof. For any $b_1, b_2 \in \Sigma$ such that $\text{im } b_1, \text{im } b_2$ are in the SCC of $\text{im } a$, let $s(b_1, b_2) \in \Sigma^*$ be a shortest path from $\text{im } b_1$ to $\text{im } b_2$. By Lemma 6, we have $|s(b_1, b_2)| \leq \binom{n}{r}$.

Suppose $w = a_1 \cdots a_k$ for $a_i \in \Sigma$. For $i \in \{1, \dots, k\}$ define the cycle $w_i := s(a_i, a)s(a, a_i)$ around $\text{im } a_i$. By Lemma 9, we have $M(aw) = M(aw')$ for

$$w' := a_1w_1^{\rho(w_1)}a_2w_2^{\rho(w_2)} \cdots a_kw_k^{\rho(w_k)}.$$

For $i \in \{1, \dots, k\}$ also define the cycle $v_i := s(a, a_i)s(a_i, a)$ around $\text{im } a$. Then we have:

$$w' = a_1s(a_1, a)v_1^{\rho(w_1)-1}s(a, a_1)a_2s(a_2, a)v_2^{\rho(w_2)-1}s(a, a_2) \cdots a_ks(a_k, a)v_k^{\rho(w_k)-1}s(a, a_k)$$

Define a set of cycles $C \subseteq \Sigma^*$ around $\text{im } a$ by

$$C := \{a_1s(a_1, a), v_1, s(a, a_1)a_2s(a_2, a), v_2, \dots, s(a, a_{k-1})a_ks(a_k, a), v_k\}.$$

Since $w' \in C^*s(a, a_k)$, by Lemma 8, there exist $\ell \leq g(n) - 1$ and $u_1, \dots, u_\ell \in C$ such that $M(aw) = M(aw') = M(au_1u_2 \cdots u_\ell s(a, a_k))$. For all $v \in C$ we have $|v| \leq 2\binom{n}{r} + 1 \leq 2^{n+2}$, and $|s(a, a_k)| \leq \binom{n}{r} \leq 2^n$. Hence the lemma holds for $u := u_1u_2 \cdots u_\ell s(a, a_k)$, as $|u| \leq 2^{n+2}(g(n) - 1) + 2^n \leq 2^{n+2}g(n) - 2$. \blacktriangleleft

We are ready to prove Proposition 4.

Proof of Proposition 4. Decompose the word w into $w = a_1 w_1 a_2 w_2 \cdots a_k w_k$ for $a_i \in \Sigma$ so that for all $i \in \{1, \dots, k\}$ the vertices $\text{im } a_i, \text{im } w_i$ are in the same SCC, and for all $i \in \{1, \dots, k-1\}$ the vertices $\text{im } w_i, \text{im } a_{i+1}$ are in different SCCs. By Lemma 5, we have $k \leq 2^{\binom{n}{r}} \leq 2^{n+1}$. For all $i \in \{1, \dots, k\}$, by Lemma 10, there is $u_i \in \Sigma^*$ with $|u_i| \leq 2^{n+2}g(n) - 2$ such that $M(a_i w_i) = M(a_i u_i)$. Hence the proposition holds for $u := a_1 u_1 a_2 u_2 \cdots a_k u_k$, as $|u| \leq 2^{n+1}(2^{n+2}g(n) - 2 + 1) \leq 2^{2n+3} - 1$. ◀

3.2 The General Case

In this subsection we prove Theorem 2. For $r \in \{0, \dots, n\}$ let $d_r \in \mathbb{N}$ be the smallest number such that for any $w \in \Sigma^*$ with $\text{rk } w \geq r$ there is $u \in \Sigma^*$ with $M(w) = M(u)$ and $|u| \leq d_r$. Also write h for the bound from Proposition 4.

▶ **Proposition 11.** *For any $r \in \{0, \dots, n-1\}$ we have $d_r \leq d_{r+1} + (d_{r+1} + 1)h$.*

Proof. Let $w \in \Sigma^*$ with $\text{rk } w \geq r$. We need to show that there is $u \in \Sigma^*$ with $M(w) = M(u)$ and $|u| \leq d_{r+1} + (d_{r+1} + 1)h$. Decompose w into $w = w_0 a_1 w_1 a_2 w_2 \cdots a_k w_k$ for $a_i \in \Sigma$ such that $\text{rk } w_0 > r$ and for all $i \in \{1, \dots, k\}$ we have $\text{rk}(a_i w_i) = r$ and $\text{rk } w_i > r$. (This decomposition is unique; in particular, $a_k w_k$ is the shortest suffix of w with rank r .) By the definition of d_{r+1} , for all $i \in \{0, \dots, k\}$ there exists $u_i \in \Sigma^*$ with $M(w_i) = M(u_i)$ and $|u_i| \leq d_{r+1}$. Then $M(w) = M(u_0 a_1 u_1 a_2 u_2 \cdots a_k u_k)$.

Define a new alphabet Σ_r and a monoid morphism $M_r : \Sigma_r^* \rightarrow \mathbb{Q}^{n \times n}$ with $M_r(\Sigma_r) = \{M(a_i u_i) : i \in \{1, \dots, k\}\}$, and note that $\text{rk } M_r(b) = r$ for all $b \in \Sigma_r$. Then there is a word $y \in \Sigma_r^*$ such that $M_r(y) = M(a_1 u_1 \cdots a_k u_k)$. By Proposition 4, there is $x \in \Sigma_r^*$ with $M_r(y) = M_r(x)$ and $|x| \leq h$. Obtain the word $v \in \Sigma^*$ from x by replacing each letter $b \in \Sigma_r$ in x by $a_i u_i$ for $i \in \{1, \dots, k\}$ such that $M_r(b) = M(a_i u_i)$. Then $M_r(x) = M(v)$, and thus $M(w) = M(u_0 a_1 u_1 \cdots a_k u_k) = M(u_0) M_r(y) = M(u_0) M_r(x) = M(u_0) M(v) = M(u_0 v)$, where $|u_0 v| = |u_0| + |v| \leq d_{r+1} + (d_{r+1} + 1)|x| \leq d_{r+1} + (d_{r+1} + 1)h$. ◀

We can now prove our main result.

Proof of Theorem 2. We prove by induction that for all $r \in \{0, \dots, n\}$ we have $d_r \leq (h+1)^{n-r} d_n + (h+1)^{n-r} - 1$. For the base case, $r = n$, this is trivial. For the step, let $r < n$. We have:

$$\begin{aligned} d_r &\leq h + (h+1)d_{r+1} && \text{(Proposition 11)} \\ &\leq h + (h+1)((h+1)^{n-r-1} d_n + (h+1)^{n-r-1} - 1) && \text{(induction hypothesis)} \\ &= h + (h+1)^{n-r} d_n + (h+1)^{n-r} - h - 1 \end{aligned}$$

This completes the induction proof. Hence $d_0 \leq (h+1)^n(d_n + 1) = 2^{n(2n+3)}g(n)^n(d_n + 1)$. The rank- n matrices in $M(\Sigma)$ generate a finite subgroup of $GL(n, \mathbb{Q})$. So it follows by Remark 3 that $d_n + 1 \leq g(n)$. Thus $d_0 \leq 2^{n(2n+3)}g(n)^{n+1}$. ◀

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A

 Rational vs. Integer Matrix Groups

We show that any finite subgroup of $GL(n, \mathbb{Q})$ is conjugate to a finite subgroup of $GL(n, \mathbb{Z})$. This implies that rational matrix groups cannot be larger than integer matrix groups.

First we need a basic fact about finitely generated Abelian groups. Let A be an (additively written) Abelian (= commutative) group. The group A is isomorphic to \mathbb{Z}^n (“free of rank n ”) if and only if there is a set $\{a_1, \dots, a_n\} \subseteq A$, called *basis* of A , such that for each $g \in A$ there are unique coefficients $k_1, \dots, k_n \in \mathbb{Z}$ such that $g = k_1 a_1 + \dots + k_n a_n$. For any $g_1, \dots, g_m \in A$ we write

$$\langle g_1, \dots, g_m \rangle := \{k_1 g_1 + \dots + k_m g_m : k_1, \dots, k_m \in \mathbb{Z}\}.$$

Using elementary arguments, we prove the following proposition, which is related to the Fundamental Theorem of Finitely Generated Abelian Groups.

► **Lemma 12.** *Let A be isomorphic to \mathbb{Z}^n , and let H be a subgroup of A . Then H is isomorphic to \mathbb{Z}^m for some $m \leq n$.*

Proof. We proceed by induction on n , the rank of A . For the base case, if $n = 0$, then $H = A = \{0\}$. For the induction step, suppose the theorem holds for ranks less than n . Fix a basis $\{a_1, \dots, a_n\}$ of A and define $\phi : H \rightarrow \mathbb{Z}$ by $\phi(k_1 a_1 + \dots + k_n a_n) := k_1$.

If $\phi(H) = \{0\}$ then $H \subseteq \langle a_2, \dots, a_n \rangle$, so H is isomorphic to \mathbb{Z}^m for some $m \leq n - 1$ by the induction hypothesis. So assume $\phi(H) \neq \{0\}$. Let $m \in \mathbb{N} \setminus \{0\}$ be the smallest positive value in $\phi(H)$ and let $h_1 \in H$ be such that $\phi(h_1) = m$. Note that $h_1 \notin \ker(\phi)$.

Consider any $h \in H$. Let $\ell, j, k_2, \dots, k_n \in \mathbb{Z}$ with $j \in \{0, \dots, m - 1\}$ such that $h = (\ell m + j)a_1 + k_2 a_2 + \dots + k_n a_n$. Then $\phi(h - \ell h_1) = j < m$. By the definition of m it follows that $j = 0$, and thus $h = \ell h_1 + h_2$ with $h_2 \in \ker(\phi)$.

Let $H_1 := \langle h_1 \rangle$ and $H_2 := \ker(\phi)$. They are subgroups of H . By the argument in the previous paragraph, we have $H = H_1 + H_2$. Further $H_1 \cap H_2 = \{0\}$, as $h_1 \notin \ker(\phi)$. Since H_2 is a subgroup of $\langle a_2, \dots, a_n \rangle$, by the induction hypothesis, H_2 is isomorphic to $\mathbb{Z}^{m'}$ for some $m' \leq n - 1$. It follows that H is isomorphic to $H_1 \times \mathbb{Z}^{m'}$, and so to $\mathbb{Z}^{1+m'}$. ◀

For any group $G \subseteq GL(n, \mathbb{Q})$ and any $C \in GL(n, \mathbb{Q})$, the set CGC^{-1} is a group that is conjugate, hence isomorphic, to G . Following [19] we show that any finite rational matrix group is conjugate to an integer matrix group.

► **Proposition 13.** *Let $G \subseteq GL(n, \mathbb{Q})$ be finite. Then there is $C \in GL(n, \mathbb{Q})$ such that $CGC^{-1} \subseteq GL(n, \mathbb{Z})$.*

Proof. Observe that for any $M \in G$, we have $GM = G$. Define $A := \sum_{M \in G} \mathbb{Z}^n M \subseteq \mathbb{Q}^n$. By the observation, $AM = A$ holds for all $M \in G$.

The set A forms a group with respect to vector addition. It is finitely generated by the rows of the matrices in G , i.e., $A = \langle e_i M : M \in G, i \in \{1, \dots, n\} \rangle$, where $\{e_1, \dots, e_n\} \subseteq \{0, 1\}^n$ is the standard basis. Let $d \in \mathbb{N}$ be a common denominator of all entries of all those generators. Then $dA \subseteq \mathbb{Z}^n$, so by Lemma 12 the group A is isomorphic to $\mathbb{Z}^{n'}$ for some $n' \leq n$. On the other hand, since the identity matrix is in G , we have $\mathbb{Z}^n \subseteq A$. Hence, $n \leq n'$, and so A is isomorphic to \mathbb{Z}^n . Let $\gamma : \mathbb{Z}^n \rightarrow A$ be such an isomorphism. Then there is a matrix $C \in GL(n, \mathbb{Q})$ such that $vC = \gamma(v)$ holds for all $v \in \mathbb{Z}^n$.

Let $M \in G$. Since $\mathbb{Z}^n CMC^{-1} = AMC^{-1} = AC^{-1} \subseteq \mathbb{Z}^n$, all entries of CMC^{-1} are integers. ◀