Two-Variable First-Order Logic with Equivalence Closure

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***Abstract*—We consider the satisfiability and finite satisfiability problems for extensions of the two-variable fragment of first-order logic in which an equivalence closure operator can be applied to a fixed number of binary predicates. We show that the satisfiability problem for two-variable, first-order logic with equivalence closure applied to two binary predicates is in 2NExpTime, and we obtain a matching lower bound by showing that the satisfiability problem for two-variable first-order logic in the presence of two equivalence relations is 2NExpTime-hard. The logics in question lack the finite model property; however, we show that the same complexity bounds hold for the corresponding finite satisfiability problems. We further show that the satisfiability (=finite satisfiability) problem for the two-variable fragment of first-order logic with equivalence closure applied to a single binary predicate is NExpTime-complete.**

***Index Terms*—computational complexity, decidability.**

I. Introduction

We investigate extensions of the two-variable fragment of first-order logic in which certain distinguished binary predicates are declared to be equivalences, or in which an operation of 'equivalence closure' can be applied to these predicates. (The equivalence closure of a binary relation is the smallest equivalence that includes it.) Denoting the two-variable fragment of first-order logic by FO^2, let EQ^2_k be the extension of FO^2 in which k distinguished binary predicates are interpreted as equivalences; and let EC^2_k be the extension of FO^2 in which we can take the equivalence closure of any of k distinguished binary predicates. (The logic EC^2_k is strictly more expressive than EQ^2_k, because it can define 'non-local' relations such as undirected reachability.) We determine the computational complexity of the satisfiability and finite satisfiability problems for EQ^2_k and EC^2_k.

As is well-known, FO^2 enjoys the finite model property [14], and its satisfiability (= finite satisfiability) problem is NExpTime-complete [3]. It was shown in [10] that EQ^2_1 also has the finite model property, with satisfiability again NExpTime-complete. However, the same paper showed that the finite model property fails for EQ^2_2, and that its satisfiability problem is in 3-NExpTime. An identical upper bound for the finite satisfiability problem was later obtained in [12]. The best currently known corresponding lower bound for these problems is 2-ExpTime hard, obtained from the less expressive two-variable guarded fragment with equivalence relations [8]. It was further shown in [10] that the satisfiability and finite satisfiability problems for EQ^2_3 are undecidable.

In this paper, we show: (i) EC^2_1 retains the finite model property, and its satisfiability problem remains in NExpTime; (ii) the satisfiability and finite satisfiability problems for EC^2_2 are both in 2-NExpTime; (iii) the satisfiability and finite satisfiability problems for EQ^2_2 are both 2-NExpTime-hard. This settles, for all $k \geq 1$, the complexity of satisfiability and finite satisfiability for both EC^2_k and EQ^2_k: all these problems are NExpTime-complete if $k = 1$, 2-NExpTime-complete if $k = 2$, and undecidable if $k \geq 3$. Thus, we close a previously existing gap for EQ^2_2, and extend the complexity bounds for EQ^2_k to the more expressive logic EC^2_k, for $k = 1, 2$.

The most significant of these new results is the upper complexity bound of 2-NExpTime for EC^2_2. Our strategy involves a non-deterministic reduction from the (finite) satisfiability problem for EC^2_2 to the problem of determining the existence of a (finite) edge-coloured bipartite graph subject to constraints on the numbers of edges of each colour incident to its vertices. This reduction runs in doubly-exponential time, and produces a set of constraints doubly-exponential in the size of the given EC^2_2-formula. We then show that this latter problem is in NPTime, by non-deterministic reduction to integer programming. Crucial to our argument is a 'Carathéodory-type' result on integer programming due to [2].

The logic FO^2 embeds, via the standard translation, multi-modal propositional logic, whose good algorithmic and model-theoretic behaviour is characteristically robust both with respect to extensions of its logical syntax (for example, by fixed point operations) and also with respect to restrictions on the class of structures over which it is interpreted (for example, in the form of conditions on the modal accessibility relations). This computational pliability has led to numerous applications in various areas of computer science, including verification of software and hardware, distributed systems, knowledge representation and artificial intelligence.

In respect of robustness under syntactic extensions, FO^2 appears, by contrast, less attractive: with the notable exception of the counting extension [5], [16], [17], most of its syntactic extensions are undecidable [4], [6]. In respect of restrictions on the structures over which it is interpreted, however, the behaviour of FO^2 is more mixed, and to some extent less

1043-6871/12 $26.00 © 2012 IEEE
DOI 10.1109/LICS.2012.53

well-understood. The most salient such restrictions are those featuring (i) linear orders, (ii) transitive relations and (iii) equivalences. In the presence of a single linear order, the satisfiability and finite satisfiability problems for FO^2 remain NEXPTIME-complete [15]. For two linear orders, EXPSPACE-completeness of finite satisfiability is shown, subject to certain restrictions on signatures, in [18]. (The case of unrestricted signatures, and decidability of the general satisfiability problem are currently open.) For three linear orders, both satisfiability and finite satisfiability are undecidable [15], [9]. Turning to transitive relations, the satisfiability problem for FO^2 in the presence of a single transitive relation has recently been shown to be in 2-NEXPTIME [20]. (The corresponding finite satisfiability problem is still open.) In the presence of two transitive relations, however, both problems are known to be undecidable—indeed this is so even for one transitive and one equivalence relation [12]. Restricting attention to interpretations involving equivalences yields the logics EQ^2_k, discussed in this paper.

Closely related to these logics are extensions of FO^2 in which the operations of *transitive closure* or *equivalence closure* can be applied to one or more binary predicates. Decidable fragments of first-order logic augmented with an operation of transitive closure are actually rare. One case is the logic $\exists\forall(\mathrm{DTC}^+[E])$, which has an exponential-size model property [7]. Another is the logic obtained by extending the two-variable guarded fragment [1] with a transitive closure operator applied to binary symbols appearing only in guards; the satisfiability problem for this logic is in 2-NEXPTIME [13]. Adding equivalence closure operators to FO^2 yields the logics EC^2_k, discussed in this paper.

The paper is organized as follows. In Sec. II, we define the logics EC^2_k, in which the distinguished binary predicates $r_1, \ldots, r_k$ are paired with the corresponding predicates $r^\#_1, \ldots, r^\#_k$, representing their respective equivalence closures. We establish a 'Scott-type' normal form for EC^2_2, allowing us to restrict the nesting of quantifiers to depth two, and we recall a *small substructure property* for FO^2 [10], allowing us to replace an arbitrary substructure in a model of some FO^2-formula φ with one whose size is exponentially bounded in the size of φ's signature. Sec. III shows how the normal form of Sec. II can be transformed into so-called *reduced* normal form, producing a syntactically simpler formula at the cost of an exponential increase in size. In Sec. IV, we prove a technical lemma on models of reduced normal-form EC^2_2-formulas, used in the upper complexity bound for EC^2_2 obtained in Sec. VI. As a by-product, we obtain the finite model property for EC^2_1 along with a NEXPTIME-bound on the complexity of satisfiability. In Sec. V, we define two problems concerning bipartite graphs with coloured edges: the *graph existence* problem and *finite graph existence* problem. We show that both problems are in NPTIME, by non-deterministic polynomial-time reduction to integer programming. (This is the most labour-intensive part of the entire proof.) Sec. VI is then able to establish that the (finite) satisfiability problem for EC^2_2 is in 2-NEXPTIME via a non-deterministic doubly exponential-time reduction to the (finite) graph existence problem. Sec. VII shows, using the familiar apparatus of tiling systems, that the satisfiability and finite satisfiability problems for EQ^2_2 are 2-NEXPTIME-hard. These matching bounds establish the 2-NEXPTIME-completeness of satisfiability and finite satisfiability for both EC^2_2 and EQ^2_2.

II. PRELIMINARIES

A. The Logics

We denote by FO^2 the two-variable fragment of first-order logic (with equality), restricting attention to signatures of unary and binary predicates. By EC^2_k, we understand the set of FO^2-formulas over any signature $\tau = \tau_0 \cup \{r_1, r_2, \ldots, r_k\} \cup \{r^\#_1, r^\#_2, \ldots, r^\#_k\}$, where $r_1, r_2, \ldots, r_k$ and $r^\#_1, r^\#_2, \ldots, r^\#_k$ are distinguished binary predicates. In the sequel, any signature τ is assumed to be of the above form (for some appropriate value of k). The semantics for EC^2_k are as for FO^2, subject to the restriction that $r^\#_i$ is always interpreted as the *equivalence closure* of r_i. More precisely: we consider only structures $\mathfrak{A}$ in which, for all i ($1 \leq i \leq k$) $(r^\#_i)^\mathfrak{A}$ is the smallest reflexive, symmetric and transitive relation including $r^\mathfrak{A}_i$. Where a structure is clear from context, we may equivocate between predicates and their extensions, writing, for example, r_i and $r^\#_i$ in place of the technically correct $r^\mathfrak{A}_i$ and $(r^\#_i)^\mathfrak{A}$.

Let $\mathfrak{A}$ be a structure over τ. We say that there is an r_i-*edge* between a and $a' \in A$ if $\mathfrak{A} \models r_i[a, a']$ or $\mathfrak{A} \models r_i[a', a]$. Distinct elements $a, a' \in A$ are r_i-*connected* if there exists a sequence $a = a_0, a_1, \ldots, a_{k-1}, a_k = a'$ in A such that for all j ($0 \leq j < k$) there is an r_i-edge between a_j and a_{j+1}. Such a sequence is called an r_i-*path* from a to a'. Thus, $\mathfrak{A} \models r^\#_i[a, a']$ if and only if a and a' are r_i-connected. A subset B of A is called r_i-*connected* if every pair of distinct elements of B is r_i-connected. Maximal r_i-connected subsets of A are equivalence classes of $r^\#_i$, and are called $r^\#_i$-*classes*. We also say that elements $a, a' \in A$ are *in free position in* $\mathfrak{A}$ if they are not r_i-connected, for any $i \in \{1, \ldots, k\}$. Similarly, subsets B and B' of A are *in free position in* $\mathfrak{A}$ if every two elements $b \in B$ and $b' \in B'$ are in free position in $\mathfrak{A}$.

We mostly work with the logic EC^2_2. In any structure $\mathfrak{A}$, the relation $r^\#_1 \cap r^\#_2$ is also an equivalence, and we refer to its equivalence classes, simply, as *intersections*. Thus, an intersection is a maximal set that is both r_1- and r_2-connected. When discussing induced substructures, a subtlety arises regarding the interpretation of the closure operations. If $B \subseteq A$, we take it that, in the structure $\mathfrak{B}$ induced on B, the interpretation of $r^\#_i$ is given by simple restriction: $(r^\#_i)^\mathfrak{B} = (r^\#_i)^\mathfrak{A} \cap B^2$. This means that, while $(r^\#_i)^\mathfrak{B}$ is certainly an equivalence including $r^\mathfrak{B}_i$, it may not be the smallest, since, for some $a, a' \in B$, an r_i-path connecting a and a' in $\mathfrak{A}$ may contain elements which are not members of B. (Such a situation may arise even when B is an intersection.) To reduce notational clutter, we use the (possibly decorated) letter $\mathfrak{A}$ to denote 'full' structures in which we are guaranteed that $(r^\#_i)^\mathfrak{A}$ is the equivalence closure of $r^\mathfrak{A}_i$. For structures denoted by other letters, $\mathfrak{B}$, $\mathfrak{C}$, ... (again, possibly decorated), no such guarantee applies. Typically, but not always, these latter structures will be induced substructures. Since we frequently work with structures induced by intersections in the sequel, the following terminology will be useful. If $\tau = \tau_0 \cup \{r_1, r_2\} \cup \{r_1^\#, r_2^\#\}$, we say that a τ-structure $\mathfrak{I}$ is a *pre-intersection* if for $i = 1, 2$, and for all $a, a' \in I$ we have $\mathfrak{I} \models r_i^\#[a, a']$ (but we do not require $(r_i^\#)^{\mathfrak{I}}$ to be the equivalence closure of $r_i^{\mathfrak{I}}$). Obviously, if I is an intersection of $\mathfrak{A}$, then the induced substructure $\mathfrak{I}$ is a pre-intersection. By the *type* of a pre-intersection, we mean its isomorphism type.

B. Normal Form, Types and Notation

In the sequel, we take the (possibly decorated) letter p to range over unary predicates, and the (possibly decorated) letter θ to range over quantifier-free (but not necessarily equality-free) formulas. If φ is any formula, we write $\neg^0\varphi$ for φ and $\neg^1\varphi$ for $\neg\varphi$. A *normal form* EC_2^2-formula is a sentence

$$\varphi = \chi \wedge \psi_{00} \wedge \psi_{01} \wedge \psi_{10} \wedge \psi_{11}, \tag{1}$$

where χ is of the form $\forall x \forall y.\theta$ and, for $s, t \in \{0, 1\}$, ψ_{st} is a conjunction $\bigwedge_{i\in I} \forall x(p_i(x) \to \exists y(\neg^s r_1^\#(x, y) \wedge \neg^t r_2^\#(x, y) \wedge \theta_i))$ (with index set I depending on s and t).

Lemma 1: Let φ be an EC_2^2-formula over a signature τ. We can compute, in polynomial time, a normal-form EC_2^2-formula φ' over a signature τ' such that φ and φ' are satisfiable over the same domains, and τ' consists of τ together with some additional unary predicates.

Proof sketch: We employ the technique of re-naming subformulas familiar from [19], noting that any formula $\exists y.\theta$ is equivalent to $\bigvee_{s,t\in\{0,1\}} \exists y(\neg^s r_1^\#(x, y) \wedge \neg^t r_2^\#(x, y) \wedge \theta)$. ■

An (atomic) 1-*type* (over a given signature) is a maximal satisfiable set of atoms or negated atoms with free variable x. Similarly, an (atomic) 2-*type* is a maximal satisfiable set of atoms and negated atoms with free variables x, y. Note that the numbers of 1-types and 2-types are bounded exponentially in the size of the signature. We often identify a type with the conjunction of all its elements.

For a given τ-structure $\mathfrak{A}$, we denote by $\mathrm{tp}^{\mathfrak{A}}(a)$ the 1-type *realized* by a, i.e. the 1-type α such that $\mathfrak{A} \models \alpha[a]$. Similarly, for distinct $a, b \in A$, we denote by $\mathrm{tp}^{\mathfrak{A}}(a, b)$ the 2-type *realized* by the pair a, b, i.e. the 2-type β such that $\mathfrak{A} \models \beta[a, b]$. We denote by $\boldsymbol{\alpha}[\mathfrak{A}]$ the set of all 1-types realized in $\mathfrak{A}$, and by $\boldsymbol{\beta}[\mathfrak{A}]$ the set of all 2-types realized in $\mathfrak{A}$. For $S \subseteq A$, we denote by $\boldsymbol{\alpha}[S]$ the set of all 1-types realized in S, and similarly for $\boldsymbol{\beta}[S]$. For $S_1, S_2 \subseteq A$, we denote by $\boldsymbol{\beta}[S_1, S_2]$ the set of all 2-types $\mathrm{tp}^{\mathfrak{A}}(a_1, a_2)$ with $a_i \in S_i$; we write $\boldsymbol{\beta}[a, S_2]$ in preference to $\boldsymbol{\beta}[\{a\}, S_2]$.

C. A Small Substructure Property for FO^2

In [11] it was proved that, for any structure $\mathfrak{A}$ with substructure $\mathfrak{B}$, one may replace $\mathfrak{B}$ by an 'equivalent' structure $\mathfrak{B}'$ of bounded size, in such a way as to preserve the truth of all FO^2-formulas in Scott normal form. (This construction does not in general preserve truth of normal-form EC_2^2-formulas.) Below, we present a precise statement of this lemma, restricted to substructures consisting of realizations of a single 1-type.

Lemma 2: Let $\mathfrak{A}$ be a τ-structure, let B be a subset of A such that $\boldsymbol{\alpha}[B] = \{\alpha\}$ for some 1-type α, and let $C = A \setminus B$. Then there is a τ-structure $\mathfrak{A}'$ with universe $A' = B' \dot\cup C$ for some set B' of size bounded by $3|\boldsymbol{\beta}[\mathfrak{A}]|^3$, such that:

(i) $\mathfrak{A}'\restriction C = \mathfrak{A}\restriction C$;
(ii) $\boldsymbol{\alpha}[B'] = \boldsymbol{\alpha}[B]$, whence $\boldsymbol{\alpha}[\mathfrak{A}'] = \boldsymbol{\alpha}[\mathfrak{A}]$;
(iii) $\boldsymbol{\beta}[B'] = \boldsymbol{\beta}[B]$ and $\boldsymbol{\beta}[B', C] = \boldsymbol{\beta}[B, C]$, whence $\boldsymbol{\beta}[\mathfrak{A}'] = \boldsymbol{\beta}[\mathfrak{A}]$;
(iv) for each $b' \in B'$ there is some $b \in B$ with $\boldsymbol{\beta}[b', A'] \supseteq \boldsymbol{\beta}[b, A]$;
(v) for each $a \in C$: $\boldsymbol{\beta}[a, B'] \supseteq \boldsymbol{\beta}[a, B]$.
(vi) for each $b' \in B'$ we have $\boldsymbol{\beta}[b', B'] = \boldsymbol{\beta}[B]$.

Property (vi) and the bound on the size of B' are not explicitly given in the original statement of the lemma in [11]; they are, however, guaranteed by the construction in its proof.

III. Reduced Normal Form

A *reduced normal form* EC_2^2-formula is a sentence

$$\varphi = \chi \wedge \psi_{00} \wedge \psi_{01} \wedge \psi_{10} \wedge \omega, \tag{2}$$

where χ and the ψ_{st} are as in (1), and ω is a conjunction $\bigwedge_{i\in I} \exists x.p_i(x)$ for some index set I.

Lemma 3: Given any EC_2^2-formula φ over a signature τ, we can compute, in exponential time, an EC_2^2-formula φ' in reduced normal form over a signature τ', such that: (i) $|\tau'|$ is bounded polynomially in $|\varphi|$; and (ii) φ and φ' are satisfiable over the same domains of cardinality greater than $f(|\varphi|)$ for a fixed exponential function f.

This section is devoted to proving Lemma 3. We first fix a normal-form EC_2^2-sentence, φ, as in (1), over a signature τ. Write $\psi_{11} = \bigwedge_{i\in I} \forall x(p_i(x) \to \exists y(\neg r_1^\#(x, y) \wedge \neg r_2^\#(x, y) \wedge \theta_i(x, y)))$ where $I = \{1, \ldots, m\}$. The following terminology will be useful. If $\mathfrak{A} \models \varphi$ and $a \in A$, then any element $b \in A$ such that $\mathfrak{A} \models \neg r_1^\#[a, b] \wedge \neg r_2^\#[a, b] \wedge \theta_i[a, b]$ is called an *ith free witness* for a (*in* $\mathfrak{A}$). Such an ith free witness certainly exists if $\mathfrak{A} \models p_i[a]$.

Lemma 4: Suppose $\mathfrak{A} \models \varphi$. Then there is a τ-structure $\mathfrak{A}' \models \varphi$ over the same domain, A, with the following property: there exists $B \subseteq A$, of cardinality at most $Z = 2m(m+2)(3m+5)(1+m+m^2)2^{|\tau|}$ such that, if any $a \in A$ has an ith free witness (for any $1 \leq i \leq m$), then a has an ith free witness in B.

Proof: If $\alpha \in \boldsymbol{\alpha}[\mathfrak{A}]$, let A_α be the set of elements of A realizing the 1-type α in $\mathfrak{A}$. Our strategy is to define, for each $\alpha \in \boldsymbol{\alpha}[\mathfrak{A}]$, a subset $B_\alpha \subseteq A_\alpha$ of cardinality at most $2m(m+2)(3m+5)$, and to show that, for every $\ell \leq m$ and every $a \in A$, if a has ℓ distinct free witnesses in A_α, then a is in free position with respect to at least ℓ elements of B_α.

Fixing α, denote by s_i the restriction of $r_i^\#$ to A_α. Thus, s_1, s_2 and $s_1 \cap s_2$ are equivalence relations on A_α: in the remainder of this proof, we refer to the equivalence classes of $s_1 \cap s_2$ as *intersections*, since no confusion will result. We call an s_i-class comprising more than one intersection an s_i-*clique*; we call an intersection which is both an s_1-class and an s_2-class a *loner*; and we use the term *unit* to refer to either an s_1-clique or an s_2-clique or a loner. Thus, the collection of units forms a cover of A_α. Evidently: an s_1- and an s_2-clique have at most one intersection in common; no two s_i-cliques have any intersections in common; and no s_i clique includes any loner. If $a \in A$ is $r_i^\#$-related to any element in an intersection, I, then it is $r_i^\#$-related to every element in I: we simply say that a is *$r_i^\#$-related to I*. The following facts are again obvious: if a is $r_i^\#$-related to any intersection in an s_i-clique, then a is $r_i^\#$-related to every intersection in that s_i-clique; if distinct units C and C' are either s_i-cliques or loners, then a cannot be simultaneously $r_i^\#$-related to an intersection in C and also to an intersection in C'; and a is $r_1^\#$-related to at most one intersection in any s_2-clique, whence there is at least one intersection in that s_2-clique to which a is not $r_1^\#$-related (and similarly with indices exchanged).

To define B_α, select $2(m+2)$ distinct units in $\mathfrak{A}$. (If $\mathfrak{A}$ has fewer units, select them all). Each selected unit C thus contains at most $2(m+2)$ intersections belonging to any other selected unit: select all of these intersections, and, in addition, select $(m+1)$ further intersections in C if possible. (If this is not possible, then C contains fewer than $3m+5$ intersections in total, so select them all). Finally, in any selected intersection I, select up to m elements. (If I contains fewer than m elements, select them all). The set B_α of selected elements in selected intersections in selected units satisfies $|B_\alpha| \leq 2m(m+2)(3m+5)$.

We must show that, for every $a \in A$, if a has $\ell \leq m$ distinct free witnesses in A_α, then a is in free position with respect to at least ℓ elements of B_α. Observe first that, if A_α has $2(m+2)$ or more units, then—switching the indices 1 and 2 in the sequel if necessary—there are $m+2$ selected s_1-cliques or loners. Now fix $a \in A$. At least $m+1$ of these $m+2$ selected units are such that a is not $r_1^\#$-related to them, and at least m of these $m+1$ are not loners to which a is $r_2^\#$-related. Each of these m remaining units therefore contains at least one intersection to which a is in free position. And since distinct s_1-cliques are disjoint, we may choose one element from each, thus obtaining $m \geq \ell$ elements of B_α in free position with respect to a. Henceforth, then, we assume that A_α has fewer than $2(m+2)$ units; and therefore that all units are selected. Again, fix $a \in A$, and suppose first that $a \in A$ has free witnesses in some non-selected intersection. Then that intersection lies in a unit, C, containing at least $m+1$ selected intersections not belonging to any other unit. Without loss of generality, suppose C is an s_1-clique. Then a cannot be $r_1^\#$-related to any intersection in C, and can be $r_2^\#$-related to at most one intersection in C, whence we may find at least m selected intersections in C standing in free position to a. Since distinct intersections are disjoint, we may choose one element from each of these intersections, again obtaining $m \geq \ell$ elements of B_α in free position with respect to a. On the other hand, if all of a's free witnesses lie in selected intersections, then we can obviously replace any non-selected free witness by one of the m selected elements in the same intersection, thus obtaining ℓ elements of B_α in free position with respect to a.

By carrying out this procedure for every 1-type α, we obtain a collection of at most $2m(m+2)(3m+5)|\boldsymbol{\alpha}[\mathfrak{A}]|$ potential free witnesses. Call this set B_1; let B_2 be a set containing the required free witnesses for all elements of B_1; let B_3 be a set containing the required free witnesses for all elements of B_2; and let $B = B_1 \cup B_2 \cup B_3$. Thus, $|B| \leq Z$. We now change the binary predicates of $\mathfrak{A}$ to obtain a structure $\mathfrak{A}'$ as follows. Fix any $a \in A \setminus (B_1 \cup B_2)$. For all i ($1 \leq i \leq m$), if a has an ith free witness, then pick one such witness; and let the (distinct) elements obtained in this way be, in some order, $b_1, \ldots, b_\ell$. Now let $b'_1, \ldots, b'_\ell$ be distinct elements of B_1 in free position with respect to a, with $\text{tp}^{\mathfrak{A}'}[b'_h] = \text{tp}^{\mathfrak{A}}[b_h]$ for all h ($1 \leq h \leq \ell$). By construction of B_1, this is clearly possible. Now set $\text{tp}^{\mathfrak{A}'}[a, b'_h] = \text{tp}^{\mathfrak{A}}[a, b_h]$ for all h ($1 \leq h \leq \ell$). In this way, all elements of $B_1 \cup B_2$ retain their former i-witnesses in B, while all elements of $B \setminus (B_1 \cup B_2)$ acquire (possibly new) i-witnesses in $B_1 \subseteq B$. Furthermore $\boldsymbol{\beta}[\mathfrak{A}'] \subseteq \boldsymbol{\beta}[\mathfrak{A}]$. It follows that we have $\mathfrak{A}' \models \varphi$, so that $\mathfrak{A}'$ and B are as required. ■

Now we can carry out the main task of this section:

Proof of Lemma 3: Let φ be as in (1), and τ the signature of φ. As before, we write $\psi_{11} = \bigwedge_I \forall x(p_i(x) \to \exists y(\neg r_1^\#(x,y) \wedge \neg r_2^\#(x,y) \wedge \theta_i(x,y)))$, where $I = \{1, \ldots, m\}$. We proceed to eliminate the conjuncts of ψ_{11}. Let Z be as in Lemma 4, and write $z = \lceil \log(Z+1) \rceil$ (so that z is bounded by a fixed polynomial function of $|\varphi|$). Now take mz new unary predicates $p_{i,1}, \ldots, p_{i,z}$ ($1 \leq i \leq m$), and a further z unary predicates $q_1, \ldots, q_z$. For all j ($0 \leq j \leq Z$), denote by $\bar{p}_{i,j}(x)$ the formula $\neg^{j[1]} p_{i,1}(x) \wedge \cdots \wedge \neg^{j[z]} p_{i,z}(x)$, where $j[h]$ is the hth digit in the z-bit representation of j; define $\bar{q}_j$ similarly. As an aid to intuition, when $j < Z$, read $\bar{p}_{i,j}(x)$ as "the ith free witness for x (if it exists) is the jth element of a special set" and read $\bar{q}_j(x)$ as "x is the jth element of the special set"; read $\bar{q}_Z(x)$ as "x is not in the special set". The following sentence states that, for all i ($1 \leq i \leq m$), every element satisfies $\bar{p}_{i,j}(x)$ for some j ($0 \leq j < Z$):

$$\chi_a = \forall x \bigwedge_{i=1}^{m} \bigvee_{j=0}^{Z-1} p_{i,j}(x).$$

The following sentence states that, for any pair of elements satisfying, respectively, $\bar{p}_{i,j}$ and $\bar{q}_j$, the second is an ith free witness for the first (if such a free witness exists):

$$\chi_b = \forall x \forall y \bigwedge_{i=1}^{m} \bigwedge_{j=0}^{Z-1} ((p_i(x) \wedge p_{i,j}(x) \wedge q_j(y)) \to$$
$$(\neg r_1^\#(x,y) \wedge \neg r_2^\#(x,y) \wedge \theta_i)).$$

Let $\chi' = \chi_a \wedge \chi_b \wedge \chi$. Observe that all quantification in χ' is universal. Finally, the following sentence states that, for all j ($0 \leq j < Z$), there is an element satisfying $\bar{q}_j(x)$:

$$\omega = \textstyle\bigwedge_{j=0}^{Z-1} \exists x \bar{q}_j(x).$$

Note that $|\chi'|$ and $|\omega|$ are bounded by an exponential function of $|\varphi|$. We claim that φ and $\varphi' = \chi' \wedge \psi_{00} \wedge \psi_{01} \wedge \psi_{10} \wedge \omega$ are satisfiable over the same domains of cardinality at least Z.

On the one hand, φ' evidently entails ψ_{11}, and hence φ. On the other hand, suppose $\mathfrak{A} \models \varphi$, with $|A| \geq Z$. Let $\mathfrak{A}'$ and the set B have the properties guaranteed by Lemma 4, and let $\{b_0, \ldots, b_{Z-1}\}$ include B. We expand $\mathfrak{A}'$ to a structure $\mathfrak{A}''$ interpreting the predicates $p_{i,h}$ and q_h as follows: for all i ($1 \leq i \leq m$) and $a \in A$, if the ith free witness for a exists and is equal to b_j, ensure $\mathfrak{A}'' \models \bar{p}_{i,j}[a]$; for all j ($0 \leq j \leq Z-1$), ensure $\mathfrak{A}'' \models \bar{q}_j[b_j]$; for all $a \notin \{b_0, \ldots, b_{Z-1}\}$, ensure $\mathfrak{A}'' \models \bar{q}_Z[a]$. It is then easy to see that $\mathfrak{A}'' \models \chi' \wedge \omega$. ■

IV. Small Intersection Property for EC_2^2

In this section we prove the following strengthening of Lemma 14 from [11].

Lemma 5: Let φ be a satisfiable EC_2^2-sentence in normal or in reduced normal form, over a signature τ. Then there exists a model of φ in which the size of each intersection is bounded by $K(|\tau|)$, for some exponential function K.

We begin by showing how to bound the size of an intersection consisting of realizations of a single 1-type.

Lemma 6: Let $\mathfrak{A}$ be a τ-structure, $B \subseteq A$ be a maximal r_1- and r_2-connected set such that $\boldsymbol{\alpha}[B] = \{\alpha\}$ is a singleton, D_1, D_2 be the respective $r_1^{\#}$- and $r_2^{\#}$-class of B, and $C = A \setminus B$. Then there is a τ-structure $\mathfrak{A}''$ with universe $A'' = B'' \dot{\cup} C$ for some set B'' of realizations of α with $|B''| \leq 45|\boldsymbol{\beta}[\mathfrak{A}]|^6$, such that:

- (i) $\mathfrak{A}''{\restriction}C = \mathfrak{A}{\restriction}C$;
- (ii) $\boldsymbol{\alpha}[B''] = \{\alpha\} = \boldsymbol{\alpha}[B]$, whence $\boldsymbol{\alpha}[\mathfrak{A}''] = \boldsymbol{\alpha}[\mathfrak{A}]$;
- (iii) $\boldsymbol{\beta}[B''] = \boldsymbol{\beta}[B]$ and $\boldsymbol{\beta}[B'', C] = \boldsymbol{\beta}[B, C]$, whence $\boldsymbol{\beta}[\mathfrak{A}''] = \boldsymbol{\beta}[\mathfrak{A}]$;
- (iv) for each $b'' \in B''$, there is some $b \in B$ with $\boldsymbol{\beta}[b'', A''] \supseteq \boldsymbol{\beta}[b, A]$;
- (v) for each $a \in C$, $\boldsymbol{\beta}[a, B''] \supseteq \boldsymbol{\beta}[a, B]$;
- (vi) $B'' \cup (D_1 \setminus B)$ is an $r_1^{\#}$-class and $B'' \cup (D_2 \setminus B)$ an $r_2^{\#}$-class.

Proof: If $|B| = 1$, then we simply put $B'' = B$ and we are done. Otherwise, our first step is a simple application of Lemma 2. Let p_1, p_2 be fresh unary predicates. Let $\bar{\mathfrak{A}}$ be the expansion of $\mathfrak{A}$ obtained by setting p_1, p_2 true for all elements of D_1, resp. D_2. Let the result of the application of Lemma 2 to $\bar{\mathfrak{A}}$ and the substructure induced by B be a structure $\bar{\mathfrak{A}}'$, in which B' is the replacement of B. By $\mathfrak{A}'$ we denote the restriction of $\bar{\mathfrak{A}}'$ to the original signature, i.e. the structure obtained from $\bar{\mathfrak{A}}'$ by dropping the interpretations of p_1 and p_2. Thus, $\mathfrak{A}'$ is a structure with universe $C \cup B'$ and $|B'|$ is exponentially bounded in the signature.

After applying Lemma 2, $r_i^{\#}$ might no longer be the symmetric transitive closure of r_i in $\mathfrak{A}'$, and we need to repair this defect. To do so, we employ an additional combinatorial construction, yielding a structure $\mathfrak{A}''$ whose universe is $C \dot{\cup} B''$. The restrictions of the structures $\mathfrak{A}$, $\mathfrak{A}'$, and $\mathfrak{A}''$ to C are equal.

Let $D_i' = (D_i \setminus B) \cup B'$ and $D_i'' = (D_i \setminus B) \cup B''$ ($i = 1, 2$). The main goal of the construction of $\mathfrak{A}''$ is to make B'' r_1- and r_2-connected, which will also make D_1'' r_1-connected, and D_i'' r_2-connected. We consider three cases.

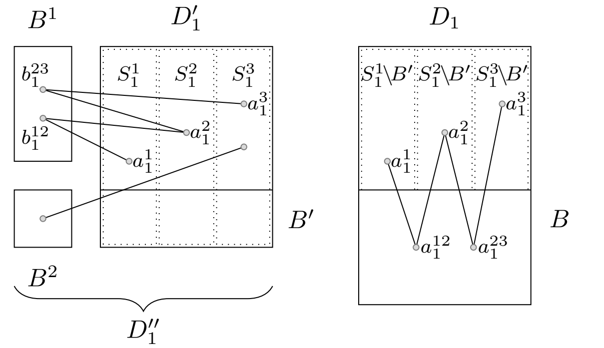

Figure 1. Making D_1'' r_1-connected in Case 2, by means of B^1. Note that $D_1' \setminus B' = D_1 \setminus B$.

Case 1: There is a pair of distinct elements $s, t \in B$ such that $\mathfrak{A} \models r_1[s, t]$, and there is a pair of distinct elements $u, w \in B$ such that $\mathfrak{A} \models r_2[u, w]$.

We build B'' from five pairwise disjoint sets $B_0, \ldots, B_4$. In $\mathfrak{A}''$, we define the substructures $\mathfrak{B}_i$ as copies of $\mathfrak{B}'$ and the substructures induced by $C \cup B_i$ we make isomorphic to $\mathfrak{A}'$. It remains to set connections among $\mathfrak{B}_i$'s. For every pair of elements $b_1 \in B_i$, $b_2 \in B_{i+1 (\mathrm{mod} 5)}$ set $\mathrm{tp}^{\mathfrak{A}''}(b_1, b_2) := \mathrm{tp}^{\mathfrak{A}}(s, t)$. For every pair of elements $b_1 \in B_i$, $b_2 \in B_{i+2 (\mathrm{mod} 5)}$ set $\mathrm{tp}^{\mathfrak{A}''}(b_1, b_2) := \mathrm{tp}^{\mathfrak{A}}(u, w)$. Note that this fully defines $\mathfrak{A}''$.

Case 2: For every pair of distinct elements $s, t \in B$ we have $\mathfrak{A} \models \neg r_1[s, t] \wedge \neg r_2[s, t]$.

Let $\{S_i^k\}_{k \in I_i}$ ($i = 1, 2$) be the partition of D_i' in $\mathfrak{A}'$ into maximal r_i-connected subsets. Observe that each S_i^k contains at least one element from B'. Indeed, $S_i^k \setminus B'$ is a subset of D_i, from which there are no r_i-edges to $D_i \setminus (B \cup (S_i^k \setminus B'))$ in $\mathfrak{A}$, since otherwise, such an edge would be retained in $\mathfrak{A}'$ and S_i^k would not be maximal. Thus, since D_i is r_i-connected in $\mathfrak{A}$, there must be an element $a \in S_i^k \setminus B'$, with an r_i-edge to some $b \in B$ in $\mathfrak{A}$. Now, property (v) of Lemma 2 guarantees that there exists $b' \in B'$ with $\mathrm{tp}^{\mathfrak{A}'}(a, b') = \mathrm{tp}^{\mathfrak{A}}(a, b)$, so b' has an r_i-edge to a, and thus $b' \in S_i^k$. This observation implies that the number of r_i-connected subsets of D_i' in $\mathfrak{A}'$ is bounded by $|B'|$, i.e. exponentially in the signature ($i = 1, 2$). We say that S_i^k and S_i^l are *connected by* B in $\mathfrak{A}$ if and only if there are $a_i^k \in S_i^k \setminus B'$, $a_i^l \in S_i^l \setminus B'$ and $a_i^{kl} \in B$, such that a_i^k, a_i^{kl}, a_i^l is an r_i-path in $\mathfrak{A}$ (Fig. 1). We build B'' from B' and two new sets B^1 and B^2 of additional realizations of α. We define $\mathfrak{A}''{\restriction}C \cup B'$ to be equal to $\mathfrak{A}'$. For S_i^k and S_i^l connected by B, we add a fresh element b_i^{kl} to B^i. For every $c \in C$, and $i = 1, 2$, we set $\mathrm{tp}^{\mathfrak{A}''}(b_i^{kl}, c) := \mathrm{tp}^{\mathfrak{A}}(a_i^{kl}, c)$. The 2-types between b_i^{kl} and B' are set in such a way that $\boldsymbol{\beta}[a_i^{kl}, B] \subseteq \boldsymbol{\beta}[b_i^{kl}, B']$; by part (vi) of Lemma 2 we always have enough elements in B' to secure this property. The 2-types inside $B^1 \cup B^2$ are not relevant and can be set as arbitrary 2-types used in $\mathfrak{B}$.

Case 3: There exists a pair of distinct elements $s, t \in B$ such that $\mathfrak{A} \models r_1[s, t]$, but for all pairs of distinct elements $u, v \in$

B, we have $\mathfrak{A} \models \neg r_2[u, v]$. (Or symmetrically, exchanging r_1 and r_2.)

This construction is a combination of the previous two. We build B'' from three disjoint sets B_0, B_1, B^2 of realizations of α. The role of the sets B_0 and B_1 is similar to the role of the sets $B_0, \ldots, B_4$ from Case 1, while the role of B^2 is similar to the role of B^2 from the Case 2.

In $\mathfrak{A}''$ we define the substructures $\mathfrak{B}_i$ as copies of $\mathfrak{B}'$ and we make the substructures induced by $C \cup B_i$ ($i = 1, 2$) isomorphic to $\mathfrak{A}'$. For every pair of elements $b_1 \in B_0, b_2 \in B_1$ we set $\mathrm{tp}^{\mathfrak{A}''}(b_1, b_2) := \mathrm{tp}^{\mathfrak{A}}(s, t)$.

Let $\{S_2^k\}_{k \in I}$ be the partition of D_2' in $\mathfrak{A}'$ into maximal r_2-connected subsets. As in Case 2, each S_2^k contains at least one element from B'. This implies that the number of r_2-connected subsets of D_2' is again bounded by $|B'|$. Recall that S_2^k and S_2^l are connected by B if there are $a_2^k \in S_i^k \setminus B'$, $a_2^l \in S_2^l \setminus B'$ and $a_2^{kl} \in B$, such that a_2^k, a_2^{kl}, a_2^l is an r_2-path in $\mathfrak{A}$. Now, if S_2^k and S_2^l are connected by B then we add a fresh element b_2^{kl} to B^2, and set its 1-type to α. For every $c \in C$, we set $\mathrm{tp}^{\mathfrak{A}''}(b_2^{kl}, c) := \mathrm{tp}^{\mathfrak{A}}(a_2^{kl}, c)$. The 2-types between b_2^{kl} and B_i ($i = 0, 1$) are set in such a way that $\boldsymbol{\beta}[a_2^{kl}, B] \subseteq \boldsymbol{\beta}[b_2^{kl}, B_i]$. The 2-types inside B^2 are not relevant and can be set as arbitrary 2-types used in $\mathfrak{B}$.

Finally, for every pair of elements $b_1 \in B^2$, $b_2 \in B_0 \cup B_1$ we set $\mathrm{tp}^{\mathfrak{A}''}(b_1, b_2) := \mathrm{tp}^{\mathfrak{A}}(s, t)$. This makes B^2 r_1-connected to the remaining part of D_1''.

Now we argue that $\mathfrak{A}''$ and B'' are as required. It should be clear that properties (i)-(v) are fulfilled and that the size of B'' is not greater than $5|B'|^2$, which, by the bound on B' from Lemma 2 is not greater than $45|\boldsymbol{\beta}[\mathfrak{A}]|^6$. Now we show that property (vi) also holds.

Case 1: First, note that B'' is both r_1- and r_2-connected. We show that, for any i and $s \in D_i \setminus B$ there is an r_i-path between s and some element $t'' \in B$. As D_i is r_i-connected there must be a path in $\mathfrak{A}$ from s to some $t \in B$. Let $s = s_0, \ldots, s_k = t$ be such a path, with $s_j \notin B$ for all $j < k$. Obviously, s_0 and s_{k-1} are r_i-connected in $\mathfrak{A}''$ as both are in C. We show that s_{k-1} is connected to some element in B''. Indeed, property (v) of Lemma 2 guarantees that there is an r_i-edge between s_{k-1} and some element t' of 1-type $\alpha \cup \{p_1(x), p_2(x)\}$ in $\bar{\mathfrak{A}}$, and property (i) of the same lemma guarantees that there are no such elements outside B'. By our construction, in $\mathfrak{A}''$ there is also an edge between s_{k-1} and t'' - the copy of t' in B_0. Therefore, D_i'' is r_i-connected for $i \in \{1, 2\}$. By property (iii) of Lemma 2, there are no r_i-connections from B' to elements that do not satisfy p_i (i.e. elements from $C \setminus D_i$), and therefore D_i'' is a maximal r_i-connected set.

Case 2: It should be clear that $C \cup B' \cup B^i$ is r_i-connected, for $i = 1, 2$. To see that B^2 is r_1-connected to the remaining part of B'', note that each element of B^2 has at least one r_1-edge to C (as we copied its 2-types from $\mathfrak{A}$ and there where no r_1-edges inside B). Analogously for B^1 and r_2-edges.

Case 3: Here the proof is just a combination of the arguments from the previous cases. ∎

Proof of Lemma 5: We first argue that the structure obtained as an application of Lemma 6 satisfies the same normal form formulas over τ as the original structure. Let $\varphi = \chi \wedge \psi_{00} \wedge \psi_{01} \wedge \psi_{10} \wedge \psi_{11}$ be a formula in normal form over τ, $\mathfrak{A} \models \varphi$, $B \subseteq A$ be a maximal r_1- and r_2-connected set such that $\boldsymbol{\alpha}[B] = \{\alpha\}$ is a singleton set, $C = A \setminus B$, and $\mathfrak{A}''$ with universe $A'' = B'' \dot{\cup} C$ be a result of application of Lemma 6 to $\mathfrak{A}$.

Observe that formula χ is satisfied in $\mathfrak{A}''$ thanks to property (iii) of Lemma 6. For any $c \in C$, properties (i) and (v) guarantee that c has all required witnesses. For any $b \in B''$, the same thing is guaranteed by property (iv).

Now, to find a small replacement of a whole intersection, we apply Lemma 6 iteratively to all 1-types realized in this intersection. Property (vi) guarantees that the obtained substructure is a maximal r_1- and r_2-connected set, so indeed it is an intersection in the new model.

The proof of the Löwenheim-Skolem theorem (every satisfiable formula is satisfiable in a countable model) can easily be extended to EC^2; thus we may restrict our attention to countable structures. Let I_1, $I_2, \ldots$ be a (possibly infinite) sequence of all intersections in a $\mathfrak{A}$, $\mathfrak{A}_0 = \mathfrak{A}$ and $\mathfrak{A}_{j+1}$ be the structure $\mathfrak{A}_j$ modified by replacing intersection I_{j+1} by its small replacement I'_{j+1} as described above. We define the limit structure $\mathfrak{A}_\infty$ with the universe $I'_1 \cup I'_2, \ldots$ such that for all $k < l$ the connections between I'_k and I'_l are defined in the same way as in $\mathfrak{A}_l$. It is easy to see that $\mathfrak{A}_\infty$ satisfies φ and all intersections in $\mathfrak{A}_\infty$ are bounded exponentially in $|\tau|$.

The described construction works also for formulas in reduced normal form because the conjunct ω is satisfied due to property (ii) of Lemma 6. ∎

A Note on EC_1^2

We can now easily get the following *exponential classes property* for EC_1^2.

Lemma 7: Let φ be a satisfiable (reduced) normal form EC_1^2 formula. Then φ is satisfiable in a model in which all $r_1^{\#}$-classes are bounded exponentially.

Proof: Apply Lemma 6 to $\varphi \wedge \forall x \forall y. r_2(x, y)$. ∎

Lemma 7 generalizes the small classes property for FO^2 with one equivalence relation from [11]. We can now repeat the construction from [11] (p. 11, *Few classes*) to show:

Theorem 8: Let φ be a satisfiable EC_1^2 formula. Then φ is satisfiable in a model of at most exponential size. Thus the satisfiability problem (= finite satisfiability problem) is NExpTime-complete.

V. The Graph Existence Problem

Let $\mathfrak{A}$ be any countable EC_2^2-structure over some fixed signature, all of whose intersections are subject to some fixed size bound. (Hence, there is a finite collection of isomorphism types of intersections that $\mathfrak{A}$ can possibly realize.) Let U be the set of $r_1^{\#}$-classes and V the set of $r_2^{\#}$-classes occurring in $\mathfrak{A}$. Since an $r_1^{\#}$-class $u \in U$ may share at most one intersection with any $r_2^{\#}$-class $v \in V$, we may regard U and V as the sets of vertices of a (possibly infinite) bipartite graph by taking (u, v) to be an edge just in case u and v share an intersection. Furthermore, we may consider the edge (u, v) to be coloured by the isomorphism type of the intersection in question. (We count intersections which are both $r_1^\#$-classes and $r_2^\#$-classes—'loners', in the terminology of the proof of Lemma 4—twice: once as an element of U and once as an element of V. Thus, U and V remain disjoint, even when $\mathfrak{A}$ contains loners.) In this section, we define two problems concerning bipartite graphs with coloured edges, and show (Thm. 9) that they are NPTIME-complete. We use this fact in Sec. VI to establish our upper complexity bounds for EC_2^2. In the sequel, we denote by $\mathbb{N}^*$ the set $\mathbb{N} \cup \{\aleph_0\}$. We interpret the arithmetic operations $+$ and $\cdot$ as well as the ordering $<$ over $\mathbb{N}^*$ as expected.

Let Δ be a finite, non-empty set. A Δ-*graph* is a triple $H = (U, V, \mathbf{E}_\Delta)$, where U and V are countable (possibly finite) disjoint sets, and $\mathbf{E}_\Delta$ is a collection of pairwise disjoint subsets $E_\delta \subseteq U \times V$, indexed by $\delta \in \Delta$. We call the elements of $W = U \cup V$ *vertices*, and the elements of E_δ, δ-*edges*. It helps to think of $\mathbf{E}_\Delta$ as the result of 'colouring' an underlying set of *edges* $E = \bigcup_{\delta \in \Delta} E_\delta$ using the 'palette' Δ. We call a pair of edges $e, e' \in E$ *skew* if e and e' share no vertex. For $u \in U$ and $v \in V$, we define the functions $\mathrm{ord}_u^H : \Delta \to \mathbb{N}^*$ and $\mathrm{ord}_v^H : \Delta \to \mathbb{N}^*$ by

$$\mathrm{ord}_u^H(\delta) = |\{v \in V : (u, v) \in E_\delta\}|$$
$$\mathrm{ord}_v^H(\delta) = |\{u \in U : (u, v) \in E_\delta\}|.$$

Thus, for any vertex w, $\mathrm{ord}_w^H(\delta)$ (pronounced: "the δ-order of w") counts the number of δ-edges incident to w. For $M \geq 0$, we define $\lfloor n \rfloor_M = \min(n, M)$, and if f is any function with range $\mathbb{N}^*$, we denote by $\lfloor f \rfloor_M$ the composition $\lfloor \cdot \rfloor_M \circ f$ (i.e., $\lfloor f \rfloor_M$ is the result of applying f and 'capping' at M).

We now define the problem GE ("graph existence"). A *GE-instance* is a sextuple $\mathcal{P} = (\Delta, \Delta_0, M, F, G, X)$, where Δ is a finite, non-empty set, $\Delta_0 \subseteq \Delta$, M is a positive integer, F and G are sets of functions $\Delta \to [0, M]$, and $X \subseteq \Delta^2$ is a symmetric binary relation on Δ. A *solution* of $\mathcal{P}$ is a Δ-graph $H = (U, V, \mathbf{E}_\Delta)$, such that:

(G1) for all $\delta \in \Delta_0$, E_δ is non-empty;
(G2) for all $u \in U$, $\lfloor \mathrm{ord}_u^H \rfloor_M \in F$;
(G3) for all $v \in V$, $\lfloor \mathrm{ord}_v^H \rfloor_M \in G$;
(G4) for all $e \in E_\delta$ and $e' \in E_{\delta'}$, if e and e' are skew, then $(\delta, \delta') \in X$.

The problem *GE* is as follows: given a GE-instance $\mathcal{P}$, determine whether $\mathcal{P}$ has a solution. Call a Δ-graph $H = (U, V, \mathbf{E}_\Delta)$ *finite* if $U \cup V$ is finite. The problem *finite GE* is as follows: given a GE-instance $\mathcal{P}$, determine whether $\mathcal{P}$ has a finite solution. The main result of this section is:

Theorem 9: GE and finite GE are NPTIME-complete.

Proof outline: The difficulty is to show membership in NPTIME. We present a non-deterministic, polynomial time procedure which, given any instance $\mathcal{P}$ of GE, produces an integer programming problem $\mathcal{E}$. The variables of $\mathcal{E}$ represent—simplifying somewhat—the numbers of vertices with given order-functions. That is, for each $f \in F$ and $g \in G$, $\mathcal{E}$ features variables x_f and y_g. We ensure that, if $H = (U, V, \mathbf{E}_\Delta)$ is a solution of $\mathcal{P}$, then setting x_f to be the number of vertices $u \in U$ with $\lfloor \mathrm{ord}_u^H \rfloor_M = f$, and y_g the number of vertices $v \in V$ with $\lfloor \mathrm{ord}_v^H \rfloor_M = g$ yields a solution of $\mathcal{E}$ over $\mathbb{N}^*$. Conversely, if $\mathcal{E}$ has a solution over $\mathbb{N}^*$, we can construct a solution $H = (U, V, \mathbf{E}_\Delta)$ of $\mathcal{P}$ in which the number of vertices $u \in U$ with $\lfloor \mathrm{ord}_u^H \rfloor_M = f$ is given by the value of x_f, and the number of vertices $v \in V$ with $\lfloor \mathrm{ord}_v^H \rfloor_M = g$ by the value of y_g. We prove an analogous result for *finite* solutions of $\mathcal{P}$ and solutions of $\mathcal{E}$ over $\mathbb{N}$. The theorem follows from the fact that integer programming, and also its variant in which solutions are sought over $\mathbb{N}^*$, are in NPTIME. ■

It is shown in [2, Theorem 1] that a Carathéodory-type result holds for integer programming: if $\mathcal{E}$ features m linear equations and inequalities whose coefficients (each) have at most k bits, and $\mathcal{E}$ has a solution over $\mathbb{N}$, then $\mathcal{E}$ has a solution in which the number of non-zero values is bounded by a polynomial function of m and k regardless of the number of variables or the total size of $\mathcal{E}$. (The proof extends easily to solutions over $\mathbb{N}^*$.) Because of this, the proof of Thm. 9 yields the following corollary, which we put to use in Sec. VI.

Corollary 10: If $(\Delta, \Delta_0, M, F', G', X)$ is a positive instance of (finite) GE, then there exist subsets $F \subseteq F'$, $G \subseteq G'$, both of cardinality bounded by a polynomial function h_0 of $|\Delta|$ and M, such that $(\Delta, \Delta_0, M, F, G, X)$ is also a positive instance of (finite) GE.

VI. Upper Bound for EC_2^2

The purpose of this section is to establish that the satisfiability and finite satisfiability problems for EC_2^2 are both in 2-NEXPTIME. We proceed by transforming a reduced normal-form EC_2^2-formula φ, non-deterministically, into a GE-instance, $\mathcal{P}$, and showing that φ is (finitely) satisfiable if and only if this transformation can be carried out in such a way that $\mathcal{P}$ is a positive instance of (finite) GE. Any solution of $\mathcal{P}$ is a bipartite graph in which the left-hand vertices represent $r_1^\#$-classes, the right-hand vertices represent $r_2^\#$-classes and the edges represent intersections; incidence of an edge on a vertex represents inclusion of the corresponding intersection in the corresponding $r_1^\#$- or $r_2^\#$-class. The main work in this reduction is performed in Sec. VI-B; Sec. VI-A is devoted to establishing technical results allowing us to manipulate structures built from collections of intersections.

We introduce some additional notation. Let Δ be a set of types of pre-intersections, and $f : \Delta \to \mathbb{N}^*$ a function not uniformly 0 on Δ. For each $\delta \in \Delta$, take $f(\delta)$ fresh sets $D_{\delta,0}, D_{\delta,1}, \ldots$ all having the same cardinality as any pre-intersection of type δ; and let $D = \bigcup\{D_{\delta,i} \mid \delta \in \Delta,\ 0 \leq i < f(\delta)\}$. We write $\mathfrak{D} \approx [\![f]\!]_1$ to indicate that $\mathfrak{D}$ is a structure on D satisfying the following properties: (i) $\mathfrak{D}$ is a single $r_1^\#$-class, with $r_1^\#$ the equivalence closure of r_1; (ii) no elements from different sets $D_{\delta,i}$ are related by r_2; (iii) for all $\delta \in \Delta$ and all $i < f(\delta)$, $\mathfrak{D} \restriction D_{\delta,i}$ is a pre-intersection of type δ. The notation $\mathfrak{D} \approx [\![f]\!]_2$ is defined symmetrically, with r_1 and r_2

exchanged. Obviously, f does not determine $\mathfrak{D}$; on the other hand, it does determine how many pre-intersections of type δ there are in $\mathfrak{D}$—namely, $f(\delta)$.

A. Approximating Classes

Fix a reduced normal-form EC_2^2-formula $\varphi = \chi \wedge \psi_{00} \wedge \psi_{01} \wedge \psi_{10} \wedge \omega$ over signature τ. We take φ_1 to denote $\chi \wedge \psi_{00} \wedge \psi_{01}$, and φ_2 to denote $\chi \wedge \psi_{00} \wedge \psi_{10}$. Thus, φ_1 incorporates the universal requirements of φ, as well as its existential requirements in respect of the relation $r_1^{\#}$; similarly, *mutatis mutandis*, for φ_2. We employ the exponential function $K : \mathbb{N} \to \mathbb{N}$ of Lemma 5. In addition, we take $N : \mathbb{N} \to \mathbb{N}$ to be a doubly exponential function such that $N(|\tau|)$ bounds number of isomorphism types of τ-structures consisting of two pre-intersections of size at most $K(|\tau|)$. We define the function $L(n) = 45(N(n))^6$, corresponding to the size bound obtained in Lemma 6. We prove two simple facts regarding the $r_i^{\#}$-classes in a model of φ. The first allows us to add pre-intersections to an existing $r_1^{\#}$- or $r_2^{\#}$-class.

Lemma 11: Let Δ be a finite set of isomorphism types of pre-intersections. Let f and f' be functions $\Delta \to \mathbb{N}^*$, such that, for all $\delta \in \Delta$, $f(\delta) \leq 1$ implies $f'(\delta) = f(\delta)$, and $f(\delta) \geq 2$ implies $f'(\delta) \geq f(\delta)$. For $i \in \{1, 2\}$, if $\mathfrak{D} \approx [\![f]\!]_i$ is such that $\mathfrak{D} \models \varphi_i$, then there exists $\mathfrak{D}' \approx [\![f']\!]_i$ such that $\mathfrak{D}' \models \varphi_i$.

Proof: We prove the result for $i = 1$; the case $i = 2$ follows by symmetry. Consider first the case where, for some $\delta \in \Delta$, $f'(\delta) = f(\delta)+1$, with $f'(\delta') = f(\delta')$ for all $\delta' \neq \delta$. By assumption, $f(\delta) \geq 2$. We show how to add to $\mathfrak{D}$ a single pre-intersection of type δ to obtain a model $\mathfrak{D}' \models \varphi_1$. Let I_1, I_2 be pre-intersections in $\mathfrak{D}$ of type δ; and let $\mathfrak{D}'$ extend $\mathfrak{D}$ by a new pre-intersection I of type δ. For every pre-intersection I' of $\mathfrak{D}$, $I' \neq I_1$, set the connection between I and I', i.e. the 2-types realized by pairs of elements from, respectively, I and I', isomorphically to the connection between I_1 and I'. This ensures all the required witnesses for I inside $\mathfrak{D}'$, and, as I_1 has to be r_1-connected to the remaining part of $\mathfrak{D}$, this also makes $\mathfrak{D}'$ r_1-connected. Complete $\mathfrak{D}'$ by setting the connection between I and I_1 isomorphically to the connection between I_1 and I_2. Note that all 2-types in $\mathfrak{D}'$ are also realized in $\mathfrak{D}$, so $\mathfrak{D}' \models \chi$. Observe that, in this construction, $\mathfrak{D} \subseteq \mathfrak{D}'$.

Consider now the case where, for some $\delta \in \Delta$, $f'(\delta) > f(\delta) \geq 2$, with $f'(\delta') = f(\delta')$ for all $\delta' \neq \delta$. If $f'(\delta)$ is finite, iterating the above procedure $f'(\delta) - f(\delta)$ times yields the required $\mathfrak{D}'$. If $f'(\delta) = \aleph_0$, we define a sequence $\mathfrak{D}_1 \subseteq \mathfrak{D}_2 \subseteq \cdots$ of models of φ_1 with increasing numbers of copies of pre-intersections of type δ, and set $\mathfrak{D}' = \bigcup_i \mathfrak{D}_i$. The statement of the lemma is then obtained by applying the above construction successively for all $\delta \in \Delta$. ■

In the next lemma we show that, from a local point of view, every class can be 'approximated' by a class in which the number of realizations of each pre-intersection type is bounded doubly exponentially in τ. (In fact, exponentially many realizations of each type suffice; however, a doubly exponential bound makes for a simpler proof.) This lemma is a counterpart of Lemma 16 from [11].

Lemma 12: Let Δ be the set of all types of pre-intersections of size bounded by $K(|\tau|)$. Let f be a function $\Delta \to \mathbb{N}^*$, and let $f' = \lfloor f \rfloor_{L(|\tau|)}$. For $i \in \{1, 2\}$, if $\mathfrak{D} \approx [\![f]\!]_i$ is such that $\mathfrak{D} \models \varphi_i$, then there exists $\mathfrak{D}' \approx [\![f']\!]_i$ such that $\mathfrak{D}' \models \varphi_i$.

Proof: Again, we prove the result for $i = 1$; the case $i = 2$ follows by symmetry. We translate $\mathfrak{D}$ into a structure $\mathfrak{F}$ whose universe is the set of all pre-intersections of $\mathfrak{D}$, atomic 1-types in $\mathfrak{D}$ represent isomorphism types of pre-intersections, and atomic 2-types represent connections among them. The signature σ of $\mathfrak{F}$ contains a binary symbol r_1', corresponding to r_1 from τ, a dummy binary symbol r_2' and some sets of unary and binary predicates bounded logarithmically in $N(|\tau|)$. We build $\mathfrak{F}$ in such a way that:

- I_1, I_2 have the same 1-type in $\mathfrak{F}$ if and only if I_1 and I_2 are isomorphic in $\mathfrak{D}$;
- pairs of pre-intersections I_1, I_2 and I_1', I_2' have the same 2-types in $\mathfrak{F}$ if and only if $\mathfrak{D}\restriction(I_1 \cup I_2)$ is isomorphic to $\mathfrak{D}\restriction(I_1' \cup I_2')$;
- $\mathfrak{F} \models r_1'(I_1, I_2)$ if and only if there exist $a_1 \in I_1$, $a_2 \in I_2$ such that $\mathfrak{D} \models r_1(a_1, a_2)$;
- r_2' is the universal relation: $\mathfrak{F} \models r_2'[I_1, I_2]$ for all $I_1, I_2 \in F$.

Note that $\mathfrak{F}$ is r_1'-connected, and thus forms a single ${r_1'}^{\#}$-class, and, as ${r_2'}^{\#}$ is universal, $\mathfrak{F}$ is actually an intersection. Note also that $|\boldsymbol{\beta}[\mathfrak{F}]|$, i.e. the number of 2-types in $\mathfrak{F}$, is bounded by $N(|\tau|)$.

Let α be a 1-type realized in $\mathfrak{F}$. Let F_α be the set of realizations of α. If $|F_\alpha| > 45|\boldsymbol{\beta}[\mathfrak{F}]|^6$ then apply Lemma 6, taking $\mathfrak{A} := \mathfrak{F}$, $B := F_\alpha$, $D_1 := D_2 := F$. Repeat this step for all 1-types of $\mathfrak{F}$. Let $\mathfrak{F}'$ be the structure thus obtained.

Since, by Lemma 6 (ii) and (iii), no new 1-types or 2-types can appear in $\mathfrak{F}'$, it has a natural translation back into a structure $\mathfrak{D}''$, with elements of $\mathfrak{F}'$ corresponding to pre-intersections in $\mathfrak{D}''$. Thus, each isomorphism type δ is realized in $\mathfrak{D}''$ at most $45|\boldsymbol{\beta}[\mathfrak{F}]|^6 \leq L(|\tau|)$ times. If δ is realized fewer than $\min(f(\delta), L(|\tau|))$ times in $\mathfrak{D}''$, then we can use Lemma 11 to add an appropriate number of realizations of δ to $\mathfrak{D}''$ to obtain a model $\mathfrak{D}' \models \varphi_1$ with $\mathfrak{D}' \approx [\![f']\!]_1$. ■

B. The (Finite) Satisfiability Problem for EC_2^2 and (Finite) GE

Let φ, φ_1, φ_2, τ and the function L be as in Sec. VI-A. (Recall: $\varphi = \chi \wedge \psi_{00} \wedge \psi_{01} \wedge \psi_{01} \wedge \omega$, $\varphi_1 = \chi \wedge \psi_{00} \wedge \psi_{01}$ and $\varphi_2 = \chi \wedge \psi_{00} \wedge \psi_{10}$.) We now explain how to transform φ non-deterministically into a GE-instance $\mathcal{P} = (\Delta, \Delta_0, M, F, G, X)$. We show that φ is (finitely) satisfiable if and only if this transformation has a run in which the resulting tuple $\mathcal{P}$ is a positive instance of the problem (finite) GE.

We first define the components Δ, M, and X of $\mathcal{P}$. Let Δ be the set of isomorphism types of pre-intersections over the signature τ satisfying $\chi \wedge \psi_{00}$, and of size at most $K(|\tau|)$. Let $M = \max(L(|\tau|), 2)$, and let X be the set of pairs $(\delta, \delta') \in \Delta^2$ for which there exists a model $\mathfrak{D} \models \chi$ consisting of exactly one pre-intersection of type δ and another of type δ', each forming its own $r_1^{\#}$-class and its own $r_2^{\#}$-class. Thus, $|\Delta|$, M and $|X|$ are all bounded by a doubly exponential function of $|\tau|$.

The remaining components of $\mathcal{P}$, namely, Δ_0, F and G, will be guessed. The following terminology and notation will prove useful. Say that a set of pre-intersection types $\Delta' \subseteq \Delta$ *certifies* ω if, for every conjunct $\omega_i = \exists x.p_i(x)$ of ω we can find δ in Δ' such that p_i is instantiated in any structure consisting of a single pre-intersection of type δ. Now let F^* be the set of functions $f : \Delta \to [0, M]$ for which there exists a structure $\mathfrak{D} \approx [\![f]\!]_1$ such that $\mathfrak{D} \models \varphi_1$. Similarly, let G^* be the set of functions $g : \Delta \to [0, M]$ for which there exists a structure $\mathfrak{D} \approx [\![g]\!]_2$ such that $\mathfrak{D} \models \varphi_2$. (Note that $|F^*|$ and $|G^*|$ are bounded by a *triply* exponential function of $|\varphi|$.)

Lemma 13: Let φ, Δ, F^*, G^*, X be as defined above, and let h_0 be the polynomial guaranteed by Corollary 10. Then φ is (finitely) satisfiable if and only if there exist $\Delta_0 \subseteq \Delta$ certifying ω, and collections of functions $F \subseteq F^*$, $G \subseteq G^*$, both of cardinality bounded by $h_0(|\Delta|, M)$, such that $\mathcal{P} = (\Delta, \Delta_0, M, F, G, X)$ is a positive instance of the problem (finite) GE.

Proof: $\Rightarrow$ By Lemma 5, let $\mathfrak{A} \models \varphi$ be a model with intersections bounded by $K(|\tau|)$. Let E be the set of intersections in $\mathfrak{A}$. For each conjunct ω_i of ω choose one element of E satisfying ω_i. Let Δ_0 be the set of isomorphism types of the chosen intersections. Clearly Δ_0 certifies ω. We show that the GE-instance $\mathcal{P}^* = (\Delta, \Delta_0, M, F^*, G^*, X)$ is positive. (Of course: F^* and G^* do not satisfy the cardinality bounds of the lemma.) Let U be the set of $r_1^{\#}$-classes in $\mathfrak{A}$, and V the set of $r_2^{\#}$-classes. (As before, any 'loner' contributes one element of U and a distinct element of V.) Since each intersection is contained in exactly one $r_1^{\#}$-class and exactly one $r_2^{\#}$-class, and indeed is determined by those classes, we may regard the intersections in E as *edges* in a bipartite graph (U, V, E). Denoting by E_δ the set of intersections in E having any type $\delta \in \Delta$, we obtain a Δ-graph $H = (U, V, \{E_\delta\}_\Delta)$. We show that H is a solution of $\mathcal{P}^*$ by checking properties (**G1**)–(**G4**). Property (**G1**) is obvious. For (**G2**), we show that, for each $\mathfrak{D} \in U$, $\lfloor \mathrm{ord}^H_{\mathfrak{D}} \rfloor_M \in F^*$. Since $\mathfrak{A} \models \varphi$, and $\mathfrak{D}$ is an $r_1^{\#}$-class in $\mathfrak{A}$, $\mathfrak{D} \models \varphi_1$; moreover, by definition, $\mathfrak{D} \approx [\![\mathrm{ord}^H_{\mathfrak{D}}]\!]_1$. Setting $f = \mathrm{ord}^H_{\mathfrak{D}}$ and $f' = \lfloor f \rfloor_M$, Lemma 12 then states that there exists a model $\mathfrak{D}' \models \varphi_1$ such that $\mathfrak{D}' \approx [\![f']\!]_1$. Thus by the definition of F^*, $\lfloor \mathrm{ord}^H_{\mathfrak{D}} \rfloor_M \in F^*$ as required. Property (**G3**) follows symmetrically. For property (**G4**), consider any pair (I, I') of skew edges in H, $I \in E_\delta$, $I' \in E_{\delta'}$. Observe that the structure $\mathfrak{A}\restriction(I \cup I')$ consists of two pre-intersections of types δ, δ', each forming its own $r_1^{\#}$- and $r_2^{\#}$-class. Thus (δ, δ') is a member of X. Applying Corollary 10, we may find $F \subseteq F^*$ and $G \subseteq G^*$, of size bounded by $h_0(|\Delta|, M)$, such that $\mathcal{P} = (\Delta, \Delta_0, M, F, G, X)$ is positive.

$\Leftarrow$ Assume now that there exist Δ_0 certifying ω, $F \subseteq F^*$ and $G \subseteq G^*$, such that $\mathcal{P} = (\Delta, \Delta_0, M, F, G, X)$ is positive. Let $H = (U, V, \{E_\delta\}_\Delta)$ be an edge-coloured bipartite graph which is a solution of $\mathcal{P}$. Thus, H satisfies (**G1**)–(**G4**). We show how to construct a model $\mathfrak{A} \models \varphi$ from the graph H. Intersections of $\mathfrak{A}$ correspond to the edges of H: for each $\delta \in \Delta$ and each $e \in E_\delta$, we put into $\mathfrak{A}$ a pre-intersection I_e of type δ. Property (**G1**) ensures that $\mathfrak{A} \models \omega$; and the fact that all intersections have types from Δ ensures that $\mathfrak{A} \models \psi_{00}$.

Consider now any vertex $u \in U$. Let $\mathcal{J}$ be the set of all pre-intersections corresponding to the edges incident to u. Our task is to compose from them an $r_1^{\#}$-class $\mathfrak{D}_u$ satisfying φ_1. First, writing f for ord^H_u and f' for $\lfloor f \rfloor_M$, we form from some subset of $\mathcal{J}$ a class $\mathfrak{D} \approx [\![f']\!]_1$ such that $\mathfrak{D} \models \varphi_1$. This is possible by (**G2**) and the construction of F^*. For each of the remaining intersections from $\mathcal{J}$ of type δ, note that the number of intersections of type δ realized in $\mathfrak{D}$ is bigger than $M \geq 2$ and thus the preconditions of Lemma 11 are fulfilled. Thus all the remaining intersections of $\mathcal{J}$ can be joined to $\mathfrak{D}$ using Lemma 11, forming a desired $\mathfrak{D}_u$. We repeat this construction for all vertices in U. This ensures that $\mathfrak{A} \models \psi_{01}$. It also makes every pre-intersection r_1-connected.

Similarly, from any vertex $v \in V$, we form a $r_2^{\#}$-class consisting of all pre-intersections corresponding to edges incident on v, using (**G3**) and the construction of G. This step ensures that $\mathfrak{A} \models \psi_{10}$ and makes every pre-intersection r_2-connected. Thus, all pre-intersections become both r_1- and r_2-connected; moreover, no two pre-intersections can be connected to each other by both r_1 and r_2 (because no two edges of H can have common vertices in both U and V); hence, every pre-intersection becomes an intersection of $\mathfrak{A}$, as required.

At this point, we have specified the 2-type in $\mathfrak{A}$ of any pair of elements not in free position. To complete the definition of $\mathfrak{A}$, consider a pair of intersections $I_e, I_{e'}$ which are in free position, i.e. are not members of the same $r_1^{\#}$-class or $r_2^{\#}$-class. But then the edges e and e' are skew in H. Assume that $e \in E_\delta$ and $e' \in E_{\delta'}$, so that I_e and $I_{e'}$ have respective isomorphism types δ and δ'. By (**G4**), $(\delta, \delta') \in X$. By the definition of X, there is a structure $\mathfrak{D} \models \chi$ consisting of exactly one intersection of type δ and another of type δ', each forming its own $r_1^{\#}$-class and its own $r_2^{\#}$-class. We make $\mathfrak{A}\restriction I_e \cup I_{e'}$ isomorphic to $\mathfrak{D}$. Finally, we point out that each pair of intersections in $\mathfrak{A}$ has been connected by copying the connections between a pair of intersections from a structure which satisfied χ. This ensures that $\mathfrak{A} \models \chi$. ■

C. Main Theorem

Theorem 14: The satisfiability and finite satisfiability problems for EC^2_2 are in 2-NEXPTIME.

Proof: Let $\varphi \in \mathrm{EC}^2_2$ be given. By Lemma 3, we may assume that $\varphi = \chi \wedge \psi_{00} \wedge \psi_{01} \wedge \psi_{10} \wedge \omega$ is in reduced normal form, since satisfiability of φ over models of at most exponential size can be tested in doubly exponential time. We continue to write φ_1 for $\chi \wedge \psi_{00} \wedge \psi_{01}$, and φ_2 for $\chi \wedge \psi_{00} \wedge \psi_{10}$. Let M, Δ, F^*, G^* and X be as in Sec. VI-B. To determine the (finite) satisfiability of φ', execute the following procedure. Non-deterministically guess a subset $\Delta_0 \subseteq \Delta$, and sets of functions F and G of type $\Delta \to [0, M]$, such that $|F|$ and $|G|$ are bounded by $h_0(|\Delta|, M)$, where h_0 is the polynomial guaranteed by Corollary 10. Check, in deterministic doubly exponential time, that Δ_0 certifies ω, and fail if not. For each $f \in F$, guess a structure $\mathfrak{D} \approx [\![f]\!]_1$, and check that $\mathfrak{D} \models \varphi_1$, failing if not; and similarly, for each $g \in G$, guess a structure $\mathfrak{D} \approx [\![g]\!]_2$, and check that $\mathfrak{D} \models \varphi_2$, failing if

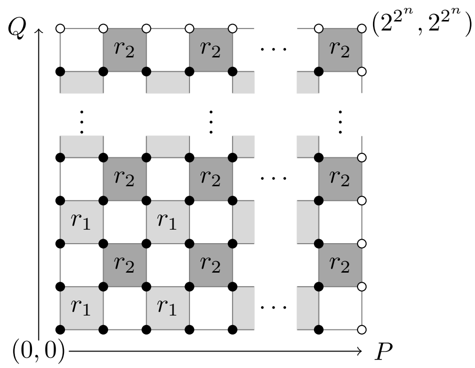

Figure 2. A doubly-exponential toroidal grid of intersections: light grey squares indicate r_1-classes, and dark grey squares, r_2-classes.

not. This non-deterministic process runs in doubly exponential time, and has a successful run just in case $F \subseteq F^*$ and $G \subseteq G^*$. Let $\mathcal{P}$ be the GE-instance $(\Delta, \Delta_0, M, F, G, X)$; thus the size of $\mathcal{P}$ is bounded doubly exponentially in $|\tau|$. Check the existence of a (finite) solution of $\mathcal{P}$ using the NPTIME-algorithm guaranteed by Thm. 9, and report the result. This non-deterministic procedure runs in time bounded by a doubly exponential function of $|\varphi|$. By Lemma 13, it has a successful run if and only if φ is (finitely) satisfiable. ■

VII. Lower Bound for FO^2 with Two Equivalences

In this section we show that the satisfiability and finite satisfiability problems for EQ^2_2, the two variable first-order logic in which two distinguished predicates, r_1 and r_2, are required to denote equivalences, are both 2-NEXPTIME-hard. It follows that the satisfiability and finite satisfiability problems for both EQ^2_2 and EC^2_2 are 2-NEXPTIME-complete.

Theorem 15: The satisfiability and finite satisfiability problems for EQ^2_2 are 2-NEXPTIME-hard.

Proof Sketch: We proceed to reduce the doubly-exponential tiling problem to the satisfiability and finite satisfiability problems for EQ^2_2. The crux of the proof is a succinct axiomatization of a toroidal grid structure of doubly exponential size by means of an EQ^2_2-formula φ. The nodes of this grid are intersections of some r_1-class and some r_2-class; we can easily write φ so as to ensure that each such intersection contains 2^n elements. By regarding these elements as indices of binary digits, we can endow each intersection with a pair of (x, y)-coordinates in the range $[0, 2^{2^n} - 1]$. By adding further conjuncts to φ, we can ensure that each intersection has a vertical and a horizontal successor, with appropriate coordinates, joined by r_1 and r_2 in the pattern shown in Fig. 2. This ensures that for each pair of coordinates there exists a corresponding intersection. To guarantee that there is at most one such intersection it is sufficient to say explicitly that there is at most one intersection having coordinates $(2^{2^n}-1, 2^{2^n}-1)$ and to prevent two intersections from having a common horizontal or a common vertical successor. To enforce the latter condition we write a conjunct which says that if two elements are joined by one of the equivalence relations and if the parities of their (x, y)-coordinates agree, then they are also joined by the other equivalence relation, and hence are members of the same intersection. Having established a grid, encoding an instance of the tiling problem can be done in a standard fashion. ■

Acknowledgments

The first, the second, and the fourth authors are supported by the Polish Ministry of Science and Higher Education grant N N206 371339. The third author gratefully acknowledges the generous support of the University of Wrocław.

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