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Quantifier elimination for a class of exponential polynomial formulas ☆

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ABSTRACT

Quantifier elimination is a foundational issue in the field of algebraic and logic computation. In first-order logic, every formula is well composed of atomic formulas by negation, conjunction, disjunction, and introducing quantifiers. It is often made quite complicated by the occurrences of quantifiers and nonlinear functions in atomic formulas. In this paper, we study a class of quantified exponential polynomial formulas extending polynomial ones, which allows the exponential to appear in the first variable. We then design a quantifier elimination procedure for these formulas. It adopts the scheme of cylindrical decomposition that consists of four phases—projection, isolation, lifting, and solution formula construction. For the non-algebraic representation, the triangular systems are introduced to define transcendental coordinates of sample points. Based on that, our cylindrical decomposition produces projections for input variables only. Hence the procedure is direct and effective.

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1. Introduction

Quantifier elimination is a foundational issue in the field of algebraic and logic computation with extremely wide applications, such as computer-aided geometric design, program verification and testing, control synthesis of dynamic and hybrid systems, to name just a few. As is well known, in first-order logic, every formula is well composed of atomic formulas by a finite times of negation, conjunction, disjunction, and introducing quantifiers. It is often made quite complicated by the occurrences of quantifiers and nonlinear functions (particularly transcendental functions) in atomic formulas. Once all quantified variables are eliminated, the resulting formula would be simply understood for both human and machine. Such elimination, however, is not easy in general due to the complex structures of varieties of nonlinear functions. Hence **quantifier elimination** on nonlinear formulas is a significant and challenging job.

In 1930s, Tarski invented a quantifier elimination method for the elementary theory of real closed fields $\mathfrak{T}(\mathbb{R}; <, =; +, \cdot; 0, 1)$ (Tarski, 1951). Seidenberg (1954) and Cohen (1969) offered two alternative methods. However, these methods required too much computation to be practical except for quite trivial instances. In 1973, Collins presented the first practical quantifier elimination method—**cylindrical algebraic decomposition (CAD)** for $\mathfrak{T}(\mathbb{R}; <, =; +, \cdot; 0, 1)$ (Collins, 1975). It splits the whole space into a finite number of connected regions, on each of which the input formula is truth-invariant, and yields the complete solution formula by collecting the defining formulas of feasible regions. (The details will be recalled in Subsection 2.2.) Its complexity is double-exponential w.r.t. the number of input variables. So Collins and his descendants were devoted to improving the efficiency of CADs. An important breakthrough was the **partial CAD** (Collins and Hong, 1991) that utilized three parts of information in the input formula (the quantifiers, the Boolean structure, and the absence of some variables in some atomic formulas) to partially build the decomposition. Besides, an approximate quantifier elimination method was studied in Hong and Safey El Din (2009, 2012). It required the input formula to satisfy a certain extra condition, and allowed the solution formula to be almost equivalent to the input formula. Thus it could successfully tackle some challenging problems, such as stability analysis of the renowned MacCormack's scheme. For the other school, Weispfenning (1988) presented another practical quantifier elimination method for linear polynomial formulas by virtual substitution of linear expressions, and pointed out that the complexity of such linear problems has a double-exponential lower bound w.r.t. the number of input variables too. Thus the theoretical complexity of the quantifier elimination problem for $\mathfrak{T}(\mathbb{R}; <, =; +, \cdot; 0, 1)$ was established. Later Weispfenning (1997) developed this method for the quadratic case by virtual substitution of square-root expressions and the general case by Thom's lemma (Coste and Roy, 1988). The above mature methods were implemented in the computer algebra tools QEPCAD (Brown, 2003) and Redlog (Dolzmann and Sturm, 1997), respectively.

Besides, Tarski also concerned whether the decidability result could be extended to the theory $\mathfrak{T}(\mathbb{R}; <, =; +, \cdot, \exp; 0, 1)$, i.e. introducing the exponential function. Unfortunately, the extended theory was proven to admit no quantifier elimination (van den Dries, 1982), and was shown to be decidable when Schanuel's conjecture holds (Wilkie, 1996). The known positive results thereby focused on some sub-theories of $\mathfrak{T}(\mathbb{R}; <, =; +, \cdot, \exp; 0, 1)$. Richardson (1991) investigated univariate exponential polynomials, and devised the so-called false/pseudo-derivative sequences for estimating the numbers of real roots of them. The result was an overestimate because some non-real roots were not ruled out then. Maignan (1998) applied this method to a class of bivariate exponential polynomial equations. Recently, Achatz et al. (2008) first presented a complete algorithm to isolating distinct real roots of univariate exponential polynomials. Their main idea is to isolate real roots of the original function when real roots of the simpler pseudo-derivative have been isolated. The termination is guaranteed by Lindemann's theorem. McCallum and Weispfenning (2012) varied this isolation algorithm for two similar functions obtained by replacing the exponential function with the logarithm function and the inverse tangent function, respectively, and proposed the decision procedure for multivariate sentences transcendental only in the first variable (i.e. the theories $\mathfrak{T}_{\exp}$, $\mathfrak{T}_{\ln}$, $\mathfrak{T}_{\arctan}$ to be specified later). Meanwhile, Strzeboński (2008, 2009) studied the real root isolation of larger classes of univariate transcendental functions—exp–log functions and tame elementary functions, respectively. But the termination of the isolation algorithm depends on Schanuel's conjecture (Richardson, 1997).

Strzeboński (2011) further established the corresponding cylindrical decomposition for multivariate formulas transcendental only in the first variable.

In this paper, inspired by Strzeboński (2011) and McCallum and Weispfenning (2012), we investigate the quantified exponential polynomial formula of the form

$$\mathrm{Q}_{k+1}\, x_{k+1} \cdots \mathrm{Q}_n\, x_n : \bigwedge_i \bigvee_j \phi_{ij}(x_1, \ldots, x_n) \bowtie_{ij} 0, \tag{1}$$

where

- $k \in \{0, \ldots, n\}$ indicates the number of free variables,
- $\mathrm{Q}_l \in \{\exists, \forall\}$ with $k+1 \le l \le n$ is a quantifier symbol,
- $\phi_{ij} \in \mathbb{Q}[x_1, \ldots, x_n; \exp(x_1)]$ is an exponential polynomial,
- $\bowtie_{ij} \in \{<, \le, =, \ge, >, \neq\}$ is a relational operator symbol,

and study how to construct a quantifier-free formula equivalent to it.

We describe a detailed quantifier elimination procedure for the exponential polynomial formula (1). It consists of four phases—projection, isolation, lifting, and solution formula construction—which are similar to cylindrical algebraic decomposition for polynomial formulas. In the projection phase, we adopt Hong's projection (Hong, 1990) to map higher-dimensional exponential polynomials to lower-dimensional ones (until univariate ones). In the isolation, by continued fraction and interval arithmetic, we modify the algorithm in Achatz et al. (2008) to isolate real roots of univariate exponential polynomials. Then the decomposition is completed in the real line, which sets our theoretical ground. In the lifting phase, we split higher-dimensional real spaces (until the whole space) based on lower-dimensional decompositions. Finally we construct the standard defining formulas of cells in the decomposition using augmented projection (Collins, 1975). The disjunction of defining formulas of feasible cells is the desired quantifier-free formula equivalent to the input formula.

Compared with the existing work, our main contributions lie in:

(1) the triangular systems introduced to exactly define the transcendental samples,
(2) the cylindrical decomposition without additional projection for the exponential (more direct than Strzeboński, 2011 and McCallum and Weispfenning, 2012).

Organization In Section 2, we review some basic notions and notations. Then we describe projection, isolation, lifting, and solution formula construction phases of cylinder decomposition, respectively, in Sections 3–6. Finally the paper is concluded in Section 7.

2. Preliminaries

Here we review some basic notions and notations of exponential polynomials and the background of computational algebra.

2.1. Exponential polynomials

Definition 2.1 *(Exponential polynomials).* Let $\{x_1, \ldots, x_n\}$ be a collection of variables. Then the multivariate exponential polynomial $\phi(x_1, \ldots, x_n)$ (*EP* for short) is an element of $\mathbb{Q}[x_1, \ldots, x_n; \exp(x_1), \ldots, \exp(x_m)]$ with $m \le n$.

Specifically, the aforementioned function is called the (n,m)-variate *EP*, as all n variables $x_1, \ldots, x_n$ appear in polynomial form and the first m variables $x_1, \ldots, x_m$ appear in exponential form.

Given an (n,m)-variate *EP* $\phi(x_1, \ldots, x_n)$, fixed the order on variables $x_1 \prec \cdots \prec x_n$, it can be rearranged as a polynomial mainly in x_n, $\exp(x_n)$:

$$\phi(x_1, \ldots, x_n) = \sum_{i=0}^{I} \sum_{j=0}^{J_i} \phi_{ij}(x_1, \ldots, x_{n-1}) x_n^{l_{ij}} \exp(k_i x_n), \tag{2}$$

where

- $[k_0, \ldots, k_I]$ is an ascending list of nonnegative integers,
- each $[l_{i0}, \ldots, l_{iJ_i}]$ is an ascending list of nonnegative integers,
- each ϕ_{ij} is a nonzero *EP* in $x_1, \ldots, x_{n-1}$.

Then the *leading coefficient* (w.r.t. the leading variable x_n) of ϕ is ϕ_{IJ_I}, denoted by $\mathrm{lc}(\phi)$. The (total) *degree* $\deg(\phi)$ is $I + \sum_{i=0}^{I} l_{iJ_i}$, while the partial degree is k_I in $\exp(x_n)$ and $\max_{i=0}^{I} l_{iJ_i}$ in x_n. For a given *EP* ϕ, we have the factorization $\phi = \mathrm{cont}(\phi)\,\mathrm{prim}(\phi)$, where $\mathrm{cont}(\phi) = \gcd(\bigcup_{i=0}^{I} \bigcup_{j=0}^{J_i} \{\phi_{ij}\})$ is the *content* w.r.t. $\exp(x_n)$ and $\mathrm{prim}(\phi)$ is the *primitive part*. Obviously these notions are compatible with those of ordinary polynomials. At last, we define an operator 'norm' for dropping the redundant divisor $\exp(k_0 x_n)$ of ϕ w.r.t. x_n, i.e.

$$\mathrm{norm}(\phi) = \sum_{i=0}^{I} \sum_{j=0}^{J_i} \phi_{ij}(x_1, \ldots, x_{n-1}) x_n^{l_{ij}} \exp\big((k_i - k_0)x_n\big). \tag{3}$$

(The redundance here means that the divisor is independent of the real variety of ϕ.)

Example 2.2. Consider the $(2,2)$-variate *EP* $\phi(x_1, x_2) = 2x_1^3 \exp(x_1) x_2 \exp(2x_2) - \exp(2x_1) x_2^2 \exp(x_2)$ with the leading variable x_2. Then we have $\mathrm{lc}(\phi) = 2x_1^3 \exp(x_1)$, $\deg(\phi) = 4$, $\mathrm{cont}(\phi) = \exp(x_1)$, and $\mathrm{prim}(\phi) = 2x_1^3 x_2 \exp(2x_2) - \exp(x_1) x_2^2 \exp(x_2)$. The primitive part has a redundant divisor $\exp(x_2)$ w.r.t. x_2, and admits the same real variety as $\mathrm{norm}(\mathrm{prim}(\phi)) = 2x_1^3 x_2 \exp(x_2) - \exp(x_1) x_2^2$.

Throughout this paper, we restrict our attention to the $(n,1)$-variate *EP*, because it has been shown in Richardson (1991) that the $(2,2)$-variate *EP* admits no quantifier elimination. Customarily, we would like to take the Greek letters $\phi, \varphi, \psi, \ldots$ to denote *EP*s and the English letters $f, g, h, \ldots$ to denote ordinary polynomials.

Formally, we are to study the fragment $\mathfrak{L}_{\exp}$ of the *language* $\mathfrak{L}(<, \leq, =, \geq, >, \neq; +, \cdot, \exp; 0, 1)$ under the collection of variables $\{x_1, \ldots, x_n\}$, satisfying:

- the exponential function is operated only to the first variable x_1,
- all variables in the prenex of a given formula are sequentially quantified, i.e. $(\mathsf{Q}_{k+1}\, x_{k+1}) \cdots (\mathsf{Q}_n\, x_n)$ with $\mathsf{Q}_l \in \{\exists, \forall\}$ and $0 \leq k \leq n$. (This implies that the formula in $\mathfrak{L}_{\exp}$ must be a sentence when x_1 is quantified.)

The interpretations of the symbols in $\mathfrak{L}_{\exp}$ are standard over the universe $\mathbb{R}$. The *theory* $\mathfrak{T}_{\exp}$ consists of true sentences in $\mathfrak{L}_{\exp}$. The theories $\mathfrak{T}_{\ln}$, $\mathfrak{T}_{\tan}$, $\mathfrak{T}_{\arctan}$ are defined in the similar manner.

2.2. Computational algebra

Cylindrical algebraic decomposition (Collins, 1975; Arnon et al., 1984) is one of the most popular methods for quantifier elimination on n-variate polynomial formulas. It achieves the following goals: (i) the whole n-dimensional real space is decomposed into a finite number of connected regions, called *cells*; (ii) each cell is cylindrically indexed and defined by a newly-constructed quantifier-free polynomial formula; (iii) each polynomial, extracted from the original input formula, shares the same sign everywhere in the given cell; (iv) and the quantifiers are replaced with the conjunction/disjunction structures. Some detailed notions should further be explained:

- given a cell $\mathcal{C}$, the *cylinder* over it is exactly $\mathcal{C} \times \mathbb{R}$;
- given a collection of polynomials f_i, whose coefficients are defined on a cell $\mathcal{C}$, the cylinder $\mathcal{C} \times \mathbb{R}$ is decomposed as a *stack* if the real varieties of these polynomials f_i are depicted by continuous real-valued functions $f^{(1)} < \cdots < f^{(k)}$, with $k \geq 0$, on $\mathcal{C}$;

- a cell is a *section* in that cylinder if it is exactly $\{(\hat{\mathbf{s}}, f^{(i)}(\hat{\mathbf{s}})) \mid \hat{\mathbf{s}} \in \mathcal{C}\}$ for some $f^{(i)}$ with $1 \le i \le k$, while a cell is a *sector* if it is exactly $\{(\hat{\mathbf{s}}, s_r) \mid \hat{\mathbf{s}} \in \mathcal{C} \wedge f^{(i)}(\hat{\mathbf{s}}) < s_r < f^{(i+1)}(\hat{\mathbf{s}})\}$ with $0 \le i \le k$, where $f^{(0)} = -\infty$ and $f^{(k+1)} = +\infty$;
- a *sample* is an arbitrary point taken from the given cell.

In the view of cylindrical algebraic decomposition, the collection of polynomials f_i is expected to have the structure of stack over $\mathcal{C}$, i.e. the cylinder $\mathcal{C} \times \mathbb{R}$ is decomposed as a stack consisting of a finite number of sections and sectors, on each of them the polynomials f_i are sign-invariant.

Additionally, some algebraic notations (Collins, 1966) should be recalled. Consider two polynomials $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{i=0}^{m} b_i x^i$ with their Sylvester matrix

$$\mathbf{S} = \begin{bmatrix} a_n & a_{n-1} & \cdots & a_0 & & & \\ & a_n & a_{n-1} & \cdots & a_0 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & a_n & a_{n-1} & \cdots & a_0 \\ b_m & b_{m-1} & \cdots & b_0 & & & \\ & b_m & b_{m-1} & \cdots & b_0 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & b_m & b_{m-1} & \cdots & b_0 \end{bmatrix} \begin{array}{l} \left.\vphantom{\begin{matrix} a \\ a \\ \ddots \\ a \end{matrix}}\right\} m \text{ rows} \\ \left.\vphantom{\begin{matrix} b \\ b \\ \ddots \\ b \end{matrix}}\right\} n \text{ rows} \end{array} \tag{4}$$

Let $\mathbf{S}_{ki}$ be the submatrix obtained from $\mathbf{S}$ by deleting the last k rows of coefficients of f, the last k rows of coefficients of g, and all of the last $2k+1$ columns except the last $(k+i+1)$-st column. Then we have:

- the k-th *subresultant* $\text{res}_k(f, g)$ is $\sum_{i=0}^{k} \det(\mathbf{S}_{ki}) x^i$ for $0 \le k \le \min(m, n)$ with the convention that $\text{res}_m = g$ in the case of $m = n$;
- the k-th *principal subresultant coefficient* $\text{psc}_k(f, g)$ is $\det(\mathbf{S}_{kk})$ for $0 \le k \le \min(m, n)$ with the convention that $\text{psc}_m = b_m$ in the case of $m = n$;
- the k-th *reductum* $\text{red}_k(f)$ is $\sum_{i=0}^{n-k} a_i x^i$ for $0 \le k \le n$.

Hence the identities $\text{res}(f, g) = \text{res}_0(f, g) = \text{psc}_0(f, g)$ hold, where $\text{res}(f, g)$ is the Sylvester resultant of f and g, i.e. $\det(\mathbf{S})$.

3. Projection phase

From this section, we start to describe the entire quantifier elimination procedure for *EP* formulas in the language $\mathfrak{L}_{\text{exp}}$. It adopts the scheme of cylindrical decomposition, which consists of four phases—projection, isolation, lifting, and solution formula construction. First of all, we study how to map a collection of $(r, 1)$-variate *EP*s to a collection of $(r-1, 1)$-variate *EP*s, so that they can be used to establish the nice property of delineability (to be defined below). The variable elimination process is called projection. When the leading variable x_r appears in linear form, the projection would coincide with the Fourier–Motzkin elimination (Dantzig and Curtis Eaves, 1973). By repeated projection, a collection of $(1, 1)$-variate *EP*s can be eventually obtained.

Let $\boldsymbol{s} = (\hat{\boldsymbol{s}}, s_r)$ and $\boldsymbol{x} = (\hat{\boldsymbol{x}}, x_r)$ be short for the r-tuples of constants and variables, respectively, in the rest of the paper. For conciseness, the subscripts x_r will be omitted in most notations (like deg, lc), if they are clear from the context.

Definition 3.1 *(Delineability).* Given a cell $\mathcal{C}$ of $\mathbb{R}^{r-1}$ with $r > 1$, a non-vanishing $(r, 1)$-variate *EP* $\phi(\hat{\boldsymbol{x}}, x_r)$ is delineable on $\mathcal{C}$ if the two conditions are satisfied:

(1) the number and the multiplicities of distinct complex roots of $\phi(\hat{\mathbf{s}}, x_r)$ are constant for every $\hat{\mathbf{s}} \in \mathcal{C}$;
(2) the components of the real variety of $\phi(\hat{\mathbf{x}}, x_r)$ lying in the cylinder $\mathcal{C} \times \mathbb{R}$ consist of k disjoint sections for some $k \geq 0$.

Furthermore, a collection Φ of non-vanishing $(r,1)$-variate *EPs* $\phi(\hat{\mathbf{x}}, x_r)$ is delineable on $\mathcal{C}$ if the product of elements of Φ is delineable on $\mathcal{C}$.

In other words, the delineability gives rise to a stack for Φ by alternately arranging $k+1$ sectors and k sections. So all points of sections are real roots of $\phi(\hat{\mathbf{s}}, x_r)$ for certain $\hat{\mathbf{s}} \in \mathcal{C}$ and $\phi \in \Phi$, and sectors exclude real roots. It is plain that Φ is delineable over a single point. But, if so, uncountable cylinders based single-points are needed to be analyzed. Thus it would fail to be an algorithm. Hence it is expected that the base cells are elaborately designed such that their number is finite while preserving the delineability of Φ. It will be obtained by the following projections.

There exist four milestones in the development history of projections. For a collection $\mathcal{F}$ of r-variate ordinary polynomials, let M be the number of elements of $\mathcal{F}$ and N the maximal degree of any element of $\mathcal{F}$. (The terms 'red' and 'psc' will be explained later.)

(1) Collins (1975)'s projection with $\mathcal{O}(M^2N^3)$ elements is

$$\bigcup_{f \in \mathcal{F}} \bigcup_{g \in \mathrm{red}(f)} \left[\mathrm{lc}(g) \cup \mathrm{psc}\big(g, g'\big)\right] \cup \bigcup_{f_1, f_2 \in \mathcal{F}} \bigcup_{g_1 \in \mathrm{red}(f_1),\, g_2 \in \mathrm{red}(f_2)} \mathrm{psc}(g_1, g_2). \tag{5a}$$

It is the first projection for cylindrical algebraic decomposition, and yields a complete quantifier elimination method on polynomial formulas, which has the double-exponential computational complexity $\mathcal{O}((2N)^{2^{2n+8}} M^{2^{n+6}})$.

(2) McCallum (1988)'s projection with $\mathcal{O}(M^2 + MN)$ elements is

$$\bigcup_{f \in \mathcal{F}} \left[\bigcup_{g \in \mathrm{red}(f)} \mathrm{lc}(g) \cup \mathrm{res}\big(f, f'\big)\right] \cup \bigcup_{f_1, f_2 \in \mathcal{F}} \mathrm{res}(f_1, f_2). \tag{5b}$$

It is a significant improvement on Collins (1975)'s, and reduces the computational complexity of quantifier elimination to $\mathcal{O}((2N)^{n2^{n+7}} M^{2^{n+4}})$. But it requires an additional assumption of **well-orientedness**, i.e. no higher-dimensional projector vanishes identically in any lower-dimensional cell, to ensure the order-invariance.

(3) Hong (1990)'s projection with $\mathcal{O}(M^2N^2)$ elements is

$$\bigcup_{f \in \mathcal{F}} \bigcup_{g \in \mathrm{red}(f)} \left[\mathrm{lc}(g) \cup \mathrm{psc}\big(g, g'\big)\right] \cup \bigcup_{f_1, f_2 \in \mathcal{F}} \bigcup_{g_1 \in \mathrm{red}(f_1)} \mathrm{psc}(g_1, f_2). \tag{5c}$$

It is an unconditional improvement on Collins (1975)'s that cuts down on the number of vast projectors in the last component.

(4) Brown (2001)'s projection with $\mathcal{O}(M^2)$ elements is

$$\bigcup_{f \in \mathcal{F}} \left[\mathrm{lc}(f) \cup \mathrm{res}\big(f, f'\big)\right] \cup \bigcup_{f_1, f_2 \in \mathcal{F}} \mathrm{res}(f_1, f_2). \tag{5d}$$

It is a further improvement on McCallum (1988)'s with the same assumption, and becomes one of the simplest projections up to now. However the computational complexity of quantifier elimination is double-exponential in the number of input variables yet.

To simplify the proof, we adopt the projection for *EP*s as Hong (1990)'s. The formal definition is given below: (regarding Φ as a collection of polynomials mainly in x_r with $(r-1,1)$-variate *EP*s as coefficients)

$$\begin{cases} \mathrm{proj} = \mathrm{cont}(\Phi) \cup \mathrm{proj}_1 \cup \mathrm{proj}_2 \cup \mathrm{proj}_3 \\ \mathrm{proj}_1 = \bigcup\limits_{\phi \in \mathrm{prim}(\Phi)} \bigcup\limits_{\varphi \in \mathrm{red}(\phi)} \mathrm{lc}(\varphi) \\ \mathrm{proj}_2 = \bigcup\limits_{\phi \in \mathrm{prim}(\Phi)} \bigcup\limits_{\varphi \in \mathrm{red}(\phi)} \mathrm{psc}(\varphi, \varphi') \\ \mathrm{proj}_3 = \bigcup\limits_{\phi_1, \phi_2 \in \mathrm{prim}(\Phi)} \bigcup\limits_{\varphi_1 \in \mathrm{red}(\phi_1)} \mathrm{psc}(\varphi_1, \phi_2), \end{cases} \tag{6}$$

where

- $\mathrm{cont}(\Phi) = \{\mathrm{cont}(\phi) \mid \phi \in \Phi\}$,
- $\mathrm{prim}(\Phi) = \{\mathrm{prim}(\phi) \mid \phi \in \Phi\}$,
- $\mathrm{red}(\phi) = \{\mathrm{red}_k(\phi) \mid 0 \leq k < \deg(\phi)\}$,
- $\mathrm{psc}(\phi_1, \phi_2) = \{\mathrm{psc}_k(\phi_1, \phi_2) \mid 0 \leq k < \min(\deg(\phi_1), \deg(\phi_2))\}$.

These $(r-1,1)$-variate *EP*s in $\mathrm{proj}(\Phi)$ suffice to construct disjoint sections and establish the delineability of Φ (see Theorem 3.5). An easy way to reducing the size of the projection is to do the following treatments:

- when $\mathrm{lc}(\mathrm{red}_k(\phi))$ is identically nonzero, the projectors $\mathrm{lc}(\mathrm{red}_l(\phi))$ with $l \geq k$ can be omitted in proj_1, and the projectors $\mathrm{psc}(\mathrm{red}_l(\phi), *)$ (where the star stands for any *EP*) with $l > k$ can be omitted in $\mathrm{proj}_2 \cup \mathrm{proj}_3$;
- when $\mathrm{psc}_k(\varphi, \varphi')$ (resp. $\mathrm{psc}_k(\varphi_1, \phi_2)$) is identically nonzero, the projectors $\mathrm{psc}_l(\varphi, \varphi')$ (resp. $\mathrm{psc}_l(\varphi_1, \phi_2)$) with $l \geq k$ can be omitted in proj_2 (resp. proj_3);
- and all identically nonzero projectors can be omitted in proj.

We recognize identically nonzero elements simply by positive functions like $\exp(x_1)$ and sums of squares like $x_i^2 + 1$. On the other hand, by the fact that Φ is sign-invariant if and only if the **finest square-free basis** of Φ is sign-invariant, projection on the finest square-free basis of Φ (obtained by factorization over the ring $\mathbb{Q}[x_1, \ldots, x_r; \exp(x_1)]$) is another potential way to reducing the entire computational complexity of quantifier elimination (cf. Collins, 1975).

Example 3.2. Consider the collection of $(4,1)$-variate *EP*s

$$\Phi = \left\{ \begin{array}{l} x_1, x_2, x_3^2 + x_2^2 - 1, x_4 \pm 1, \\ x_1 \exp(x_1) x_4 + \exp(x_1) x_2 + \exp(x_1) - x_1 - 5, \\ x_1^2 \exp(x_1) x_4 + 2\exp(x_1) x_3 + 2x_1 \exp(x_1) x_2 + 2x_1 \exp(x_1) \\ \quad -2x_1^2 - 2x_1 - 10 \end{array} \right\}. \tag{7}$$

Using the projection (6) to eliminate x_4, we have

- $\mathrm{cont}(\Phi) = \{x_1, x_2, x_3^2 + x_2^2 - 1\}$,

- $\mathrm{proj}_1 = \{x_1\exp(x_1), \exp(x_1)x_2 + \exp(x_1) - x_1 - 5, x_1^2\exp(x_1), 2\exp(x_1)x_3 + 2x_1\exp(x_1)x_2 + 2x_1\exp(x_1) - 2x_1^2 - 2x_1 - 10\}$,
- $\mathrm{proj}_2 = \emptyset$,
- $\mathrm{proj}_3 = \{\exp(x_1)x_2 \pm x_1\exp(x_1) + \exp(x_1) - x_1 - 5, 2\exp(x_1)x_3 + 2x_1\exp(x_1)x_2 \pm x_1^2\exp(x_1) + 2x_1\exp(x_1) - 2x_1^2 - 2x_1 - 10, 2\exp(x_1)x_3 + x_1\exp(x_1)x_2 + x_1\exp(x_1) - x_1^2 + 3x_1 - 10\}$.

After the identically nonzero projector $\exp(x_1)$ is omitted, the finest square-free basis of $\mathrm{proj}(\Phi)$ is obtained as

$$\left\{\begin{array}{l} x_1, x_2, x_3^2 + x_2^2 - 1, \exp(x_1)x_2 + \exp(x_1) - x_1 - 5, \\ \exp(x_1)x_2 \pm x_1\exp(x_1) + \exp(x_1) - x_1 - 5, \\ \exp(x_1)x_3 + x_1\exp(x_1)x_2 + x_1\exp(x_1) - x_1^2 - x_1 - 5, \\ 2\exp(x_1)x_3 + 2x_1\exp(x_1)x_2 \pm x_1^2\exp(x_1) + 2x_1\exp(x_1) - 2x_1^2 - 2x_1 - 10, \\ 2\exp(x_1)x_3 + x_1\exp(x_1)x_2 + x_1\exp(x_1) - x_1^2 + 3x_1 - 10. \end{array}\right\} \tag{8}$$

Lemma 3.3. *(See Brown and Traub, 1971.) Given two univariate polynomials $f(x)$, $g(x)$, the k-th subresultant $\mathrm{res}_k(f,g)$ is exactly their greatest common divisor $\gcd(f,g)$ if k is the least index such that $\mathrm{psc}_k(f,g)$ is nonzero.*

Corollary 3.4. *Given two $(r,1)$-variate EPs ϕ_1, ϕ_2 and a cell $\mathcal{C}$ on which ϕ_1, ϕ_2 are degree-invariant and all $\mathrm{psc}_i(\phi_1,\phi_2)$ are sign-invariant, the k-th subresultant $\mathrm{res}_k(\phi_1,\phi_2)$ is exactly their greatest common divisor $\gcd(\phi_1,\phi_2)$ on $\mathcal{C}$ if k is the least index such that $\mathrm{psc}_k(\phi_1,\phi_2)$ does not vanish identically on $\mathcal{C}$.*

The results make a connection between the greatest common divisors and subresultants. It is notable that, in the multivariate case, there must exist a principal subresultant coefficient $\mathrm{res}_k(\phi_1,\phi_2) \neq 0$ unless both $\mathrm{lc}(\phi_1)$ and $\mathrm{lc}(\phi_2)$ vanish identically on $\mathcal{C}$. Then it is necessary and sufficient to introduce some reducta of ϕ_1, one of which has a non-vanishing leading coefficient, to replace ϕ_1. (The technique was first reported in Hong, 1990.)

Theorem 3.5 *(Projection). (See Hong, 1990.) Let Φ be a collection of $(r,1)$-variate EPs with $r > 1$. If $\mathcal{C}$ is a cell of $\mathbb{R}^{r-1}$ on which $\mathrm{proj}(\Phi)$ is sign-invariant, the following statements hold.*

(1) *Every single element of Φ either vanishes identically or is delineable on $\mathcal{C}$.*
(2) *Every pair of sections of elements of Φ (which do not vanish identically on $\mathcal{C}$) are either pairwise disjoint or identical.*
(3) *A stack can be constructed over $\mathcal{C}$ for the collection Φ.*

This theorem has been proven by Collins (1975) and Hong (1990) when Φ is degenerated as a collection of ordinary polynomials. But it is still true when Φ is a collection of *EP*s by noticing that: (i) every element of Φ is a polynomial mainly in x_r; (ii) and its coefficients are continuously defined on $\mathcal{C}$.

After executing such projection process (6) repeatedly, we would obtain a collection $\mathrm{proj}^{r-1}(\Phi)$ of $(1,1)$-variate *EP*s. (The union $\bigcup_{i=0}^{r-1}\mathrm{proj}^i(\Phi)$ is called by the projection closure of Φ.) Thereafter it completes the projection phase.

Example 3.6. Continued to consider Example 3.2. We further eliminate x_3, x_2, and obtain the finest square-free basis of $\mathrm{proj}^2(\Phi)$ as

$$\left\{\begin{array}{l}
x_1, x_2, x_2^2-1, \exp(x_1)x_2+\exp(x_1)-x_1-5,\\
\exp(x_1)x_2 \pm x_1\exp(x_1)+\exp(x_1)-x_1-5,\\
\left[x_1^2+1\right]\exp(2x_1)x_2^2+\left[2x_1^2\exp(2x_1)-\left(2x_1^3+2x_1^2+10x_1\right)\exp(x_1)\right]x_2\\
\quad+\left(x_1^2-1\right)\exp(2x_1)-\left(2x_1^3+2x_1^2+10x_1\right)\exp(x_1)\\
\quad+x_1^4+2x_1^3+11x_1^2+10x_1+25,\\
\left[x_1^2+4\right]\exp(2x_1)x_2^2+\left[2x_1^2\exp(2x_1)+\left(-2x_1^3+6x_1^2-20x_1\right)\exp(x_1)\right]x_2\\
\quad+\left(x_1^2-4\right)\exp(2x_1)+\left(-2x_1^3+6x_1^2-20x_1\right)\exp(x_1)\\
\quad+x_1^4-6x_1^3+29x_1^2-60x_1+100,\\
\left[4x_1^2+4\right]\exp(2x_1)x_2^2+\left[\left(4x_1^3+8x_1^2\right)\exp(2x_1)-\left(8x_1^3+8x_1^2+40x_1\right)\exp(x_1)\right]x_2\\
\quad+\left(x_1^4+4x_1^3+4x_1^2-4\right)\exp(2x_1)-\left(4x_1^4+12x_1^3+28x_1^2+40x_1\right)\exp(x_1)\\
\quad+4x_1^4+8x_1^3+44x_1^2+40x_1+100,\\
\left[4x_1^2+4\right]\exp(2x_1)x_2^2+\left[\left(-4x_1^3+8x_1^2\right)\exp(2x_1)-\left(8x_1^3+8x_1^2+40x_1\right)\exp(x_1)\right]x_2\\
\quad+\left(x_1^4-4x_1^3+4x_1^2-4\right)\exp(2x_1)+\left(4x_1^4-4x_1^3+12x_1^2-40x_1\right)\exp(x_1)\\
\quad+4x_1^4+8x_1^3+44x_1^2+40x_1+100
\end{array}\right\} \tag{9}$$

and that of $\mathrm{proj}^3(\Phi)$ as

$$\left\{\begin{array}{l}
x_1, x_1+5, x_1^2+x_1+5, x_1^2-3x_1+10, 2\exp(x_1)-x_1-5, \exp(x_1)-x_1-5,\\
\pm x_1\exp(x_1)+\exp(x_1)-x_1-5, x_1\exp(x_1)\pm x_1\pm 5, x_1\exp(x_1)\pm 2\exp(x_1)\mp x_1\mp 5,\\
2x_1\exp(x_1)-x_1^2-x_1-5, 2x_1\exp(x_1)-x_1^2+3x_1-10,\\
x_1^2\exp(x_1)\pm 2x_1^2\pm 2x_1\pm 10, \left[x_1^2\pm 4x_1\right]\exp(x_1)\mp 2x_1^2\mp 2x_1\mp 10,\\
\left[2x_1+10\right]\exp(x_1)-17x_1^2+30x_1-50,\\
\left[18x_1^3-6x_1^2+120x_1\right]\exp(x_1)-25x_1^4+46x_1^3-146x_1^2+40x_1-400,\\
\exp(2x_1)+\left[2x_1^3+2x_1^2+10x_1\right]\exp(x_1)-x_1^4-2x_1^3-11x_1^2-10x_1-25,\\
4\exp(2x_1)+\left[2x_1^3-6x_1^2+20x_1\right]\exp(x_1)-x_1^4+6x_1^3-29x_1^2+60x_1-100,\\
\left[x_1^2-1\right]\exp(2x_1)-\left[2x_1^3+2x_1^2+10x_1\right]\exp(x_1)+x_1^4+2x_1^3+11x_1^2+10x_1+25,\\
\left[x_1^2-4\right]\exp(2x_1)+\left[-2x_1^3+6x_1^2-20x_1\right]\exp(x_1)+x_1^4-6x_1^3+29x_1^2-60x_1+100,\\
x_1^4\exp(2x_1)+\left[\pm 16x_1^3\mp 20x_1^2-8x_1-40\right]\exp(x_1)+68x_1^2-120x_1+200,\\
\left[x_1^4+x_1^2+2x_1\right]\exp(2x_1)+\left[-8x_1^3+8x_1^2-12x_1-10\right]\exp(x_1)+17x_1^2-30x_1+50,\\
\left[x_1^4+x_1^2-2x_1\right]\exp(2x_1)+\left[8x_1^3-8x_1^2+8x_1-10\right]\exp(x_1)+17x_1^2-30x_1+50,\\
\left[x_1^4+4x_1^2+8x_1\right]\exp(2x_1)+\left[-16x_1^3+12x_1^2-48x_1-40\right]\exp(x_1)+68x_1^2-120x_1+200,\\
\left[x_1^4+4x_1^2-8x_1\right]\exp(2x_1)+\left[16x_1^3-12x_1^2+32x_1-40\right]\exp(x_1)+68x_1^2-120x_1+200,\\
\left[9x_1^4+4x_1^2+8x_1\right]\exp(2x_1)+\left[-48x_1^3+52x_1^2-48x_1-40\right]\exp(x_1)+68x_1^2-120x_1+200,\\
\left[9x_1^4+4x_1^2-8x_1\right]\exp(2x_1)+\left[48x_1^3-52x_1^2+32x_1-40\right]\exp(x_1)+68x_1^2-120x_1+200,\\
\left[x_1^4+4x_1^3-4\right]\exp(2x_1)-\left[4x_1^4+12x_1^3+28x_1^2+40x_1\right]\exp(x_1)\\
\quad+4x_1^4+8x_1^3+44x_1^2+40x_1+100,\\
\left[x_1^4-4x_1^3-4\right]\exp(2x_1)+\left[4x_1^4-4x_1^3+12x_1^2-40x_1\right]\exp(x_1)\\
\quad+4x_1^4+8x_1^3+44x_1^2+40x_1+100,\\
\left[x_1^4+4x_1^3+4x_1^2-4\right]\exp(2x_1)-\left[4x_1^4+12x_1^3+28x_1^2+40x_1\right]\exp(x_1)\\
\quad+4x_1^4+8x_1^3+44x_1^2+40x_1+100,\\
\left[x_1^4-4x_1^3+4x_1^2-4\right]\exp(2x_1)+\left[4x_1^4-4x_1^3+12x_1^2-40x_1\right]\exp(x_1)\\
\quad+4x_1^4+8x_1^3+44x_1^2+40x_1+100,\\
x_1^6\exp(2x_1)-\left[8x_1^3+8x_1^2+40x_1\right]\exp(x_1)+4x_1^4+8x_1^3+44x_1^2+40x_1+100,\\
\left[x_1^6+x_1^4+8x_1^3\right]\exp(2x_1)-\left[8x_1^4+40x_1^3+72x_1^2+160x_1\right]\exp(x_1)\\
\quad+16x_1^4+32x_1^3+176x_1^2+160x_1+400,\\
\left[x_1^6+x_1^4-8x_1^3\right]\exp(2x_1)+\left[8x_1^4-24x_1^3+8x_1^2-160x_1\right]\exp(x_1)\\
\quad+16x_1^4+32x_1^3+176x_1^2+160x_1+400,\\
\left[x_1^6+4x_1^4+24x_1^3\right]\exp(2x_1)-\left[16x_1^5+4x_1^4+64x_1^3+136x_1^2+480x_1\right]\exp(x_1)\\
\quad+100x_1^4-184x_1^3+584x_1^2-160x_1+1600,\\
\left[x_1^6+4x_1^4-24x_1^3\right]\exp(2x_1)+\left[16x_1^5+4x_1^4-80x_1^3-184x_1^2+480x_1\right]\exp(x_1)\\
\quad+100x_1^4-184x_1^3+584x_1^2-160x_1+1600,
\end{array}\right\} \tag{10}$$

which contains exactly 41 distinct $(1,1)$-variate irreducible *EP*s.

4. Isolation phase

In this section, we study how to isolate all distinct real roots of the univariate *EP* ϕ into a list of disjoint rational intervals. Let ϕ be the product of elements of $\mathrm{proj}^{r-1}(\Phi)$. Then the sign-invariant decomposition for $\mathrm{proj}^{r-1}(\Phi)$ can be completed in one-dimensional real space via these distinct real roots of ϕ.

Theorem 4.1 *(Lindemann's Theorem). (See Baker, 1975.) The number* $\exp(\alpha)$ *is transcendental for every nonzero algebraic number* α.

Lemma 4.2. *(See Achatz et al., 2008.) Let* $\phi(x) = \sum_{i=0}^{l} p_i(x)\exp(k_i x)$ *be a nonzero univariate EP, the number* $\phi(\alpha)$ *is nonzero for every nonzero algebraic number* α *if the content* $\mathrm{cont}(\phi) = \gcd(p_0(x), \ldots, p_l(x))$ *(w.r.t.* $\exp(x)$*) is a constant.*

Lemma 4.3. *(See Achatz et al., 2008.) If* $\phi(x)$ *is a square-free univariate EP, the only possible multiple root of* $\phi(x)$ *is* 0.

Lemma 4.4. *(See Achatz et al., 2008.) Given a univariate EP* $\phi(x)$*, the sign of* $\phi(x)$ *is decidable at an arbitrary rational point* α.

Here we would like to give another constructive method (Xu et al., 2010a) for the sign determination mentioned in Lemma 4.4. It is sketched as follows. We first determine whether $\phi(\alpha)$ is zero by Lemma 4.2. If not, we can use the **continued fraction** in Olds (1963) that

$$\frac{\exp(x)-1}{\exp(x)+1} = \cfrac{1}{\frac{2}{x} + \cfrac{1}{\frac{6}{x} + \cfrac{1}{\frac{10}{x} + \cdots}}}, \tag{11}$$

to approach the exact value of $\exp(\alpha)$ and that of $\phi(\alpha)$ with increasingly precise rational interval sequences. The intervals would eventually not contain zero by convergence. Thus the sign of $\phi(\alpha)$ is decided.

In what follows, we describe a real root isolation algorithm for univariate *EP*s, which is modified from Achatz et al. (2008). Its rationale is analogous to the differentiation method (Collins and Loos, 1976) that isolates real roots of the original function after those of the simpler pseudo-derivative have been isolated. By the continued fraction (11), the original algorithm is modified in some technical processing (Xu et al., 2010a). The difference between the two will be stressed below the statements of the algorithm.

Subalgorithm 1 *(Bisection).*

$$\mathcal{I}^* := \mathbf{BISECT}(\mathcal{I}, \psi).$$

Input : $\psi(x)$ is a square-free univariate *EP*; and $\mathcal{I} = [a, b]$ is a rational interval containing exactly an irrational root α of ψ.
Output : $\mathcal{I}^* = [a^*, b^*] \subset \mathcal{I}$ is a rational interval containing α, satisfying $b^* - a^* = (b-a)/2$.

We tackle it in two cases.

C1 If $\psi(b)\psi((a+b)/2) < 0$, set $a^* := (a+b)/2$ and $b^* := b$.
C2 If $\psi(a)\psi((a+b)/2) < 0$, set $a^* := a$ and $b^* := (a+b)/2$. $\square$

Some other bisection methods based on continued fractions can be found in Knuth (1997) to possibly accelerate the practical convergence.

Subalgorithm 2 *(Refinement).*

$$\mathcal{I}^* := \textbf{REFINE}(\mathcal{I}, \psi, \tilde{\psi}).$$

Input : $\psi(x)$, $\tilde{\psi}(x)$ are relatively prime and square-free univariate *EP*s; and $\mathcal{I} = [a, b]$ with $ab > 0$ is a rational interval containing exactly an irrational root α of ψ.
Output : $\mathcal{I}^* \subseteq \mathcal{I}$ is a rational interval containing α, satisfying that $\mathcal{I}^*$ contains no real root of $\tilde{\psi}$.

S1 Choose a positive number ϵ as the predefined precision.
S2 Repeatedly invoke $\mathcal{I} := \textbf{BISECT}(\mathcal{I}, \psi)$, until the interval estimate $\mathcal{I} = [a, b]$ of α has length less than ϵ, i.e. $b - a < \epsilon$.
S3 Based on the interval estimate $\mathcal{I} = [a, b]$ of α, the interval estimate of $\exp(\alpha)$ is simply $[\exp(a), \exp(b)]$. Apply the continued fraction (11) to approximate $[\exp(a), \exp(b)]$ by a rational superset within error less than ϵ.
S4 Then an interval estimate $[\mu, \nu]$ of $\tilde{\psi}$ during $\mathcal{I}$ can further be obtained by applying the **interval arithmetic** rules:

$$\begin{aligned}
&[a,b]^n \times \exp\bigl(m \times [a,b]\bigr) = \bigl[a^n \exp(ma), b^n \exp(mb)\bigr] && \text{if } a > 0 \vee 2 \nmid n\\
&[a,b]^n \times \exp\bigl(m \times [a,b]\bigr) \subseteq \bigl[b^n \exp(ma), a^n \exp(mb)\bigr] && \text{if } a < 0 \wedge 2 \mid n\\
&[a,b] \times c = [ac, bc] && \text{if } c > 0\\
&[a,b] \times c = [bc, ac] && \text{if } c < 0\\
&[a_1,b_1] + [a_2,b_2] = [a_1 + a_2, b_1 + b_2] &&\\
&[a,b] + c = [a + c, b + c]. &&
\end{aligned} \tag{12}$$

If $[\mu, \nu]$ does not contain zero, RETURN $\mathcal{I}$ and END; otherwise set $\epsilon := \epsilon/2$ and GOTO S2. $\square$

Algorithm 1 *(Isolation).*

$$\mathcal{L} := \textbf{ISOL}(\phi).$$

Input : $\phi(x)$ is a nonzero univariate *EP*.
Output : $\mathcal{L} = [\mathcal{I}_1, \ldots, \mathcal{I}_K]$ is a list of isolation intervals, where

- K is the number of distinct real roots of ϕ;
- each $\mathcal{I}_k$ is a rational interval $[a_k, b_k]$ containing exactly one distinct real root α_k of ϕ, and those endpoints are ascending, i.e. $a_1 \le b_1 < \cdots < a_K \le b_K$.

S1 (Initialization) Construct the pseudo-derivative sequence $\mathrm{pds}(\phi) = [\psi_0, \ldots, \psi_N]$ as follows:

$$\begin{cases} \psi_0 = \mathrm{gsfd}\bigl(\mathrm{norm}(\phi)\bigr) \\ \psi_i = \mathrm{gsfd}\bigl(\mathrm{norm}\bigl(\psi'_{i-1}\bigr)\bigr) \quad (i > 0), \end{cases} \tag{13}$$

where 'gsfd' is an operator for extracting the greatest square-free divisor (obtained by factorization over the ring $\mathbb{Q}[x, \exp(x)]$). Repeat this process until ψ_i has partial degree 0 in $\exp(x)$ or in x.
S2 (First isolation) Now ψ_i is degenerated as an ordinary polynomial in x or in $\exp(x)$. Isolate all distinct real roots of ψ_i into the isolation list $\mathcal{L}_i = [\mathcal{I}_1, \ldots, \mathcal{I}_K]$, each interval $\mathcal{I}_k$ has rational endpoints a_k, b_k and contains exactly a distinct real root α_k of ψ_i.
S3 (Refining signs) For every $\mathcal{I}_k = [a_k, b_k] \in \mathcal{L}_i$, we discuss it in two cases.
  C1 If α_k is rational, it is either a root of $\mathrm{cont}(\psi_i)$ or 0. It can be easily checked by Eisenstein's criterion. Set $a_k := \alpha_k$ and $b_k := \alpha_k$.
  C2 If α_k is irrational, after invoking $\mathcal{I}_k := \textbf{BISECT}(\mathcal{I}_k, \psi_i)$ a finite number of times, the resulting $\mathcal{I}_k$ would eventually not contain zero.

S4 (Loop control) If $i = 0$, RETURN $\mathcal{L}_0$ and END; otherwise do the following steps to compute the isolation list $\mathcal{L}_{i-1}$ of ψ_{i-1} based on the known $\mathcal{L}_i$ of ψ_i.

S5 (Estimating bounds) Estimate the real root bounds of ψ_{i-1}, saying $[b_0, a_{K+1}]$, satisfying $b_0 \le a_1$ and $b_K \le a_{K+1}$. Utilizing the monotonicity of ψ_{i-1} during $(-\infty, a_1]$ and $[b_K, +\infty)$, the upper and the lower bounds can be easily obtained by one-dimensional search, once

$$\begin{cases} \mathrm{lc}\big(\psi_{i-1}(-x)\big)\psi_{i-1}(b_0) > 0 \\ \mathrm{lc}\big(\psi_{i-1}(x)\big)\psi_{i-1}(a_{K+1}) > 0. \end{cases} \tag{14}$$

S6 (Refining intervals) For every $\mathcal{I}_k = [a_k, b_k] \in \mathcal{L}_i$, we discuss it in three cases.

C1 If $a_k = b_k = 0$, the only possible multiple real root of ψ_{i-1} is found when $\psi_{i-1}(0) = 0$ holds.

C2 If $a_k = b_k \neq 0$, $\mathcal{I}_k$ contains no real roots of ψ_{i-1} by Lemma 4.3.

C3 If $a_k \neq b_k$, by invoking $\mathcal{I}_k := \textbf{REFINE}(\mathcal{I}_k, \psi_i, \psi_{i-1})$, the resulting $\mathcal{I}_k$ would contain no real roots of ψ_{i-1}.

S7 (Lifting isolation) Construct the isolation list

$$\mathcal{L}_{i-1} := \big[[b_k, a_{k+1}] \,\big|\, 0 \le k \le K \wedge \psi_{i-1}(b_k)\psi_{i-1}(a_{k+1}) < 0\big]. \tag{15}$$

Insert $[0, 0]$ into $\mathcal{L}_{i-1}$ when ψ_{i-1} has the only possible even-multiple real root 0. Finally, set $i := i - 1$ and GOTO S3. □

Compared with the real root isolation of univariate *EP*s in Achatz et al. (2008), our main modifications lie in the novel continued fraction method (11) for estimating the value of the exponential $\exp(x)$ at an arbitrary rational point α. This evaluation delivers a rational interval at any precision, whose endpoints are the upper and the lower bounds of $\exp(\alpha)$. Based on that, the interval arithmetic (12) is naturally introduced to estimate the value of the univariate *EP* at any rational point. Then the modified algorithm manipulates only with rational numbers in each step, and hence is absolutely exact. The remaining problem is the termination of the bisection process in S3 and the refinement process in S6 of Algorithm 1. A lemma is given below to justify it.

Lemma 4.5. *The repeated bisection process in S3 and the refinement process in S6 of Algorithm 1 must terminate within a finite number of steps.*

Proof. Suppose that the case C2 in S3 does not terminate. The repeated bisections in $[a_k, b_k]$ define an infinite sequence of intervals $[a_k^{(h)}, b_k^{(h)}]$, satisfying $a_k^{(h)} < 0 < b_k^{(h)}$ and $b_k^{(h)} - a_k^{(h)} = [b_k^{(h-1)} - a_k^{(h-1)}]/2$. Then $\alpha_k = \lim_{h\to\infty} a_k^{(h)} = \lim_{h\to\infty} b_k^{(h)} = 0$, which contradicts that α_k is irrational.

Similarly, if the case C3 in S6 does not terminate, the repeated bisections in ϵ induce an infinite sequence of intervals $[a_k^{(h)}, b_k^{(h)}]$ convergent to α and an infinite sequence of intervals convergent to $\exp(\alpha)$. Based on that, we can further obtain an infinite sequence of intervals $[\mu_k^{(h)}, \nu_k^{(h)}]$, satisfying

$$\mu_k^{(1)} < \cdots < \mu_k^{(h)} < \cdots < 0 < \cdots < \nu_k^{(h)} < \cdots < \nu_k^{(1)}. \tag{16}$$

Then $\psi_{i-1}(\alpha_k) = \lim_{h\to\infty} \mu_k^{(h)} = \lim_{h\to\infty} \nu_k^{(h)} = 0$ by convergence. It implies that α_k is a common real root of ψ_{i-1} and ψ_i, which, however, has been ruled out by Lemma 4.3, since ψ_{i-1} and ψ_i are relatively prime from their construction (13). Hence the two repeated bisection processes terminate. □

Remark 4.6. Strzeboński (2008) studied the real root isolation of more expressive exp–log functions that introduces the logarithm function and composition of functions. He first constructed a semi-Fourier sequence for the given exp–log function. Then he identified real roots by observing the number of sign changes of the semi-Fourier sequence when x varied in its domain. The rationale sources from Budan–Fourier theorem (Basu et al., 2006). This algorithm requires the ability to determine the sign of an exp–log function at roots of another exp–log function. The current known method is based on Schanuel's conjecture (Richardson, 1997). For univariate *EP*s, however, the sign determination at such roots can be computed by an algorithm (see Lemma 5.7), whose termination and correctness is guaranteed by Lindemann's theorem, rather than Schanuel's conjecture. Hence Strzeboński (2008)'s algorithm can be developed as an alternative of Algorithm 1.

In the real line, every distinct real root of $\psi_0(x)$ is exactly a section (with a sample α_k); while every remaining connected segment (α_k, α_{k+1}) (resp. $(-\infty, \alpha_1)$ resp. $(\alpha_K, +\infty)$) of the real line is a sector with a sample $(b_k + a_{k+1})/2$ (resp. b_0 resp. a_{K+1}). Then $\psi_0(x)$ is sign-invariant in every cell, so is $\mathrm{proj}^{r-1}(\Phi)$. Thereafter the stack consisting of K sections and $K+1$ sectors is obtained in one-dimensional real space $\mathbb{R}$.

Example 4.7. Consider the univariate *EP* $\phi = [x^4 - 4x^3 - 4]\exp(2x) + [4x^4 - 4x^3 + 12x^2 - 40x]\exp(x) + 4x^4 + 8x^3 + 44x^2 + 40x + 100$. Its pseudo-derivatives are

$$\begin{aligned}
\psi_0 &= \left[x^4 - 4x^3 - 4\right]\exp(2x) + \left[4x^4 - 4x^3 + 12x^2 - 40x\right]\exp(x) \\
&\quad + 4x^4 + 8x^3 + 44x^2 + 40x + 100 \\
\psi_1 &= \left[x^4 - 2x^3 - 6x^2 - 4\right]\exp(2x) + \left[2x^4 + 6x^3 - 8x - 20\right]\exp(x) \\
&\quad + 8x^3 + 12x^2 + 44x + 20 \\
\psi_2 &= \left[x^4 - 9x^2 - 6x - 4\right]\exp(2x) + \left[x^4 + 7x^3 + 9x^2 - 4x - 14\right]\exp(x) \\
&\quad + 12x^2 + 12x + 22 \\
\psi_3 &= \left[2x^4 + 4x^3 - 18x^2 - 30x - 14\right]\exp(2x) \\
&\quad + \left[x^4 + 11x^3 + 30x^2 + 14x - 18\right]\exp(x) + 24x + 12 \\
\psi_4 &= \left[4x^4 + 16x^3 - 24x^2 - 96x - 58\right]\exp(2x) \\
&\quad + \left[x^4 + 15x^3 + 63x^2 + 74x - 4\right]\exp(x) + 24 \\
\psi_5 &= \left[8x^4 + 48x^3 - 240x - 212\right]\exp(x) + x^4 + 19x^3 + 108x^2 + 200x + 70 \\
\psi_6 &= \left[8x^4 + 80x^3 + 144x^2 - 240x - 452\right]\exp(x) + 4x^3 + 57x^2 + 216x + 200 \\
\psi_7 &= \left[4x^4 + 56x^3 + 192x^2 + 24x - 346\right]\exp(x) + 6x^2 + 57x + 108 \\
\psi_8 &= \left[4x^4 + 72x^3 + 360x^2 + 408x - 322\right]\exp(x) + 12x + 57 \\
\psi_9 &= \left[2x^4 + 44x^3 + 288x^2 + 564x + 43\right]\exp(x) + 6 \\
\psi_{10} &= 2x^4 + 52x^3 + 420x^2 + 1140x + 607,
\end{aligned} \tag{17a}$$

and the corresponding isolation lists are

$$\begin{aligned}
\mathcal{L}_{10} &= \left[[-14, -13], [-8, -7], [-5, -4], \left[-1, -\frac{1}{2}\right]\right] \\
\mathcal{L}_9 &= \left[\left[-\frac{67}{16}, -\frac{3}{4}\right], \left[-\frac{1}{8}, -\frac{1}{16}\right]\right] \\
\mathcal{L}_8 &= \left[\left[-5, -\frac{371}{128}\right], \left[-\frac{2913}{1024}, -\frac{1}{8}\right], \left[\frac{15}{32}, 1\right]\right] \\
\mathcal{L}_7 &= \left[\left[-7, -\frac{9971}{2048}\right], \left[-\frac{4851}{1024}, -\frac{127\,947}{65\,536}\right], \left[-\frac{62\,581}{32\,768}, -\frac{47\,221}{65\,536}\right], \left[\frac{47}{64}, 2\right]\right] \\
\mathcal{L}_6 &= \left[\left[-9, -\frac{454\,387}{65\,536}\right], \left[-\frac{225\,011}{32\,768}, -\frac{6\,101\,991}{2\,097\,152}\right], \left[\frac{269}{256}, 2\right]\right]
\end{aligned}$$

$$\mathcal{L}_5 = \left[\left[-11, -\frac{4\,583\,155}{524\,288}\right], \left[-\frac{36\,529\,803}{4\,194\,304}, -\frac{286\,549\,555}{67\,108\,864}\right],\right.$$

$$\left.\left[-\frac{2\,284\,097\,727}{536\,870\,912}, -\frac{1\,465\,159\,871}{1\,073\,741\,824}\right], \left[\frac{3367}{2048}, 3\right]\right]$$

$$\mathcal{L}_4 = \left[\left[-\frac{6\,033\,355\,325}{2\,147\,483\,648}, -\frac{1\,774\,826\,045}{4\,294\,967\,296}\right], \left[\frac{67\,757}{32\,768}, 3\right]\right]$$

$$\mathcal{L}_3 = \left[\left[\frac{633\,697}{262\,144}, 3\right]\right]$$

$$\mathcal{L}_2 = \left[\left[\frac{11\,513\,767}{134\,217\,728}, \frac{11\,513\,767}{67\,108\,864}\right], \left[\frac{5\,833\,251}{2\,097\,152}, 4\right]\right]$$

$$\mathcal{L}_1 = \left[\left[\frac{209\,662\,245}{67\,108\,864}, 4\right]\right]$$

$$\mathcal{L}_0 = \left[\left[1, \frac{14\,770\,167\,533}{4\,294\,967\,296}\right], \left[\frac{1\,853\,617\,593}{536\,870\,912}, 4\right]\right]. \tag{17b}$$

We can see that ϕ has exactly two distinct real roots (numerically approximated by 1.349913996 and 3.792992799), which are contained in the isolation intervals $\mathcal{I}_1$ and $\mathcal{I}_2$ in $\mathcal{L}_0$ respectively. Let α denote the bigger one.

Furthermore, we consider another univariate *EP* $\tilde{\phi} = [x^2 - 4x]\exp(x) + 2x^2 + 2x + 10$. The *EP*s ϕ and $\tilde{\phi}$ are relatively prime and square-free, since they are elements of the finest square-free basis of $\mathrm{proj}^3(\Phi)$ (see the collection (10)). The isolation intervals $\mathcal{I}_1$ and $\mathcal{I}_2$ would be refined as

$$\left[\frac{44\,834\,938\,605}{34\,359\,738\,368}, \frac{189\,814\,954\,657}{137\,438\,953\,472}\right] \quad \text{and} \quad \left[\frac{130\,092\,302\,097}{34\,359\,738\,368}, \frac{16\,298\,271\,019}{4\,294\,967\,296}\right], \tag{17c}$$

by invoking $\mathcal{I}_k := \mathbf{REFINE}(\mathcal{I}_k, \phi, \tilde{\phi})$. The resulting $\mathcal{I}_1$ and $\mathcal{I}_2$ contain no real root of $\tilde{\phi}$ (numerically approximated by 1.550946458 and 3.701063106).

In fact, none of other 40 univariate *EP*s in the finest square-free basis of $\mathrm{proj}^3(\Phi)$ has a real root greater than α. Hence we have that the interval $(\alpha, +\infty)$ is a sector of $\mathrm{proj}^3(\Phi)$ in $\mathbb{R}$ with a sample 4.

For efficiency, a cell is usually identified by the sign-invariance of projectors plus a sample. It is enough to perform cylindrical decomposition, but not quantifier elimination. For the latter task, a cell should be identified by the standard defining formula, which will be described in Section 6.

Remark 4.8. Richardson (1991) proposed an open problem whether a univariate *EP* $\phi(x)$ with partial degree n in x and partial degree m in $\exp(x)$ would have more than $n+m$ distinct real roots. An example in point is $\phi(x) = 20x\exp(x) + 2x + 9$, which has partial degree 1 in both x and $\exp(x)$. But it has three distinct real roots, respectively, in $[-4, -383/128]$, $[-765/256, -23/16]$ and $[-11/8, -11/16]$ (computed by Algorithm 1). A lemma is given below to derive a bound on the number of distinct real roots.

Lemma 4.9. *Given a univariate EP $\phi(x)$, the number of distinct real roots is bounded from above by its total degree* $\deg(\phi)$.

Proof. Let $\mathrm{pds}(\phi) = [\psi_0, \ldots, \psi_N]$ be the pseudo-derivative sequence of ϕ. It is clear that ψ_i has at most one more distinct real root than ψ_{i+1} by Rolle's theorem, and $\deg(\psi_i) - \deg(\psi_{i+1}) \geq 1$ by the construction of the pseudo-derivative sequence. On the other hand, the last element ψ_N has at most $\deg(\psi_N)$ distinct real roots, since it is an ordinary polynomial in x or in $\exp(x)$ then. Therefore $\deg(\phi) = \deg(\psi_0)$ is an upper bound of the number of distinct real roots of ϕ. $\square$

5. Lifting phase

In the previous section, we have gained a list of isolation intervals containing all critical points, which split the real line $\mathbb{R}$ by the projectors of $\mathrm{proj}^{r-1}(\Phi)$. Now, by tracing these points, we split higher-dimensional real spaces (until $\mathbb{R}^r$), and choose a sample for each cell. Then each cell can be identified by the projection closure of Φ plus its sample, since the projection closure of Φ is sign-invariant on it then.

Definition 5.1 *(Revised sign lists).* Given a univariate polynomial $f(x) = \sum_{i=0}^{n} a_i x^i$ with $a_i \in \mathcal{R}$ and $a_n \neq 0$, the revised sign list $\Delta = [\delta_1, \ldots, \delta_n]$ is constructed as follows:

- initially, set

$$\delta_i = (-1)^{\frac{i(i-1)}{2}} \operatorname{sign}\big(\mathrm{psc}_{n-i}\big(f, f'\big)\big) \quad (1 \le i \le n); \tag{18a}$$

- for every segment $[\delta_i, \ldots, \delta_{i+j}]$ of the above sign list satisfying $\delta_{i+1} = \cdots = \delta_{i+j-1} = 0 \neq \delta_i \delta_{i+j}$, then reset

$$\delta_{i+k} = (-1)^{\lfloor \frac{k+1}{2} \rfloor} \operatorname{sign}(\delta_i) \quad (1 \le k < j). \tag{18b}$$

To be rigorous, we refer the aforementioned $\mathcal{R}$ to computable real numbers that can be effectively computed to within any desired precision. For instance, the value of a univariate *EP* $\phi(x)$ at an arbitrary rational point α is a computable real number.

Lemma 5.2. *(See Yang et al., 1996.) Given a univariate polynomial $f(x) = \sum_{i=0}^{n} a_i x^i$ with $a_i \in \mathcal{R}$ and $a_n \neq 0$, if the number of nonzero elements of its revised sign list Δ is l, it has exactly l distinct complex roots. Furthermore, if the number of sign changes in Δ is m, it has exactly $l - 2m$ distinct real roots.*

By Lemma 3.3, we have known that the greatest common divisor of f and f' is exactly $\mathrm{res}_{n-l}(f, f')$. Then the number and the multiplicities of distinct complex/real roots of f can be computed, once those for $\mathrm{res}_{n-l}(f, f')$ are known. The degree of $\mathrm{res}_{n-l}(f, f')$ is less than that of f. Hence, within a finite depth of recursions, we can obtain:

Theorem 5.3 *(Complete discrimination system). (See Yang et al., 1996.) Given a univariate polynomial $f(x) = \sum_{i=0}^{n} a_i x^i$ with $a_i \in \mathcal{R}$ and $a_n \neq 0$, the number and the multiplicities of its distinct complex/real roots are computable.*

Note that the complete discrimination system is an off-line algorithm. One can utilize the existing real root classification of certain order, some of which have been demonstrated in Yang (1999), to reduce the on-line computational complexity.

Corollary 5.4. *Given an $(r, 1)$-variate EP $\phi(\hat{\boldsymbol{x}}, x_r)$ and an $(r-1)$-dimensional sample $\hat{\boldsymbol{s}}$, the number and the multiplicities of distinct complex/real roots of $\phi(\hat{\boldsymbol{s}}, x_r)$ are computable if the signs of $(r-1, 1)$-variate EPs are decidable at $\hat{\boldsymbol{s}}$.*

The complete discrimination system is directly applicable for polynomials with nonzero leading coefficients. So, in the multivariate case, we again have to: (i) guess the degree, saying k, of $\phi(\hat{\boldsymbol{s}}, x_r)$; (ii) replace ϕ with $\mathrm{red}_{\deg(\phi)-k}(\phi)$; (iii) and then apply the complete discrimination system on $\mathrm{red}_{\deg(\phi)-k}(\phi)$.

Lemma 5.5. *Given an $(r, 1)$-variate EP $\phi(\hat{\boldsymbol{x}}, x_r)$ and an $(r-1)$-dimensional sample $\hat{\boldsymbol{s}}$, all distinct real roots of $\phi(\hat{\boldsymbol{s}}, x_r)$ are separable if the signs of $(r-1, 1)$-variate EPs are decidable at $\hat{\boldsymbol{s}}$.*

Proof. The number and the multiplicities of distinct real roots of $\phi(\hat{\boldsymbol{s}}, x_r)$ are computable by Corollary 5.4. By interpolations with rational numbers, all distinct real roots of $\phi(\hat{\boldsymbol{s}}, x_r)$ of odd multiplicity can be isolated. Meanwhile all distinct real roots of $\phi(\hat{\boldsymbol{s}}, x_r)$ of even multiplicity amount to all distinct real roots of $\gcd(\phi(\hat{\boldsymbol{s}}, x_r), \phi'(\hat{\boldsymbol{s}}, x_r))$ of odd multiplicity, thus they can be similarly isolated. After a finite number of bisections, these isolation intervals can be pairwise disjoint. Hence all distinct real roots are separable. □

For efficiency, the aforementioned interpolations and bisections could incorporate with Sturm's sequences (Knuth, 1997), which can count the number of distinct real roots of $\phi(\hat{\boldsymbol{s}}, x_r)$ during any open interval. If an interval has no real root, the interpolations and bisections are redundant, thus they could be avoided.

Definition 5.6 *(Triangular definability).* Given an r-tuple of real numbers $\boldsymbol{s}$, it is triangularly definable if it is the unique real solution of a triangular system of $(r, 1)$-variate *EP* equations of the form

$$\begin{cases} \psi_1(x_1) = 0 \\ \psi_2(x_1, x_2) = 0 \\ \qquad \vdots \\ \psi_r(x_1, \ldots, x_r) = 0 \end{cases} \tag{19}$$

confined in the rectangular region $[u_1, v_1] \times \cdots \times [u_r, v_r] \subseteq \mathbb{R}^r$.

It is obvious that real roots of univariate *EP*s are triangularly definable. In fact, we can further require the irreducibility of $\psi_i(x_1, \ldots, x_i)$ over the ring $\mathbb{Q}[x_1, \ldots, x_i; \exp(x_1)]$ and $\psi_i(s_1, \ldots, s_{i-1}, u_i) \times \psi_i(s_1, \ldots, s_{i-1}, v_i) \leq 0$ in technical processing. It implies that every coordinate s_i is either the unique real root of $\psi_r(s_1, \ldots, s_{i-1}, x_i)$ in (u_i, v_i) or the unique rational root at $u_i = v_i$. Then one can efficiently approach the exact values of coordinates of such triangularly definable samples.

Lemma 5.7. *Given an $(r, 1)$-variate EP $\phi(\boldsymbol{x})$, the sign of $\phi(\boldsymbol{x})$ is decidable at an arbitrary triangularly definable sample $\boldsymbol{s}$.*

Proof. (It is a generalized version of Lemma 4.4.) Assume that $\boldsymbol{s}$ is defined by the triangular system (19), we complete this proof by induction on r.

(1) If $r = 1$, by Lemma 4.3, we have that $\phi(\boldsymbol{s}) = 0$ only when $\psi_r(\boldsymbol{x})$ is a divisor of $\phi(\boldsymbol{x})$ or $\boldsymbol{s} = 0$. Both are easy to check. For the remaining cases, the sign determination can be finished by invoking $[u_r, v_r] :=$ **REFINE**$([u_r, v_r], \psi_r, \phi)$ (see Subalgorithm 2), which delivers an interval estimate of ϕ at the unique real root of ψ_r in $[u_r, v_r]$.
(2) The sign of any element of $\mathbb{Q}[x_1, \ldots, x_{r-1}; \exp(x_1)]$ is decidable at $\hat{\boldsymbol{s}}$ by inductive hypothesis.
(3) Since all distinct real roots of the product $\phi(\hat{\boldsymbol{s}}, x_r) \times \psi_r(\hat{\boldsymbol{s}}, x_r)$ are computable by Lemma 5.5, it is decidable whether $\phi(\boldsymbol{s})$ is zero. If not, we can approach the exact value of $\phi(\boldsymbol{s})$ by repeatedly bisecting $[u_i, v_i]$ and estimating the range of $\phi(\boldsymbol{s})$ then. Hence its sign is decidable, which is ensured by convergence. □

Algorithm 2 *(Lifting).*

$$\mathcal{S} := \textbf{LIFT}(\Phi, \mathcal{C} \ni \hat{\boldsymbol{s}}).$$

Input : Φ is a collection of $(r, 1)$-variate *EP*s with $r > 1$; $\mathcal{C}$ is a cell of $\mathbb{R}^{r-1}$ on which the projection closure of $\mathrm{proj}(\Phi)$ is sign-invariant; and $\hat{\boldsymbol{s}}$ is a triangularly definable sample of $\mathcal{C}$.
Output : $\mathcal{S} = [\mathcal{C}_1 \ni \boldsymbol{s}_1, \ldots, \mathcal{C}_{2K+1} \ni \boldsymbol{s}_{2K+1}]$ is a stack over $\mathcal{C}$ such that
- the projection closure of Φ is sign-invariant on every $\mathcal{C}_k$;

- every $\mathcal{C}_{2k-1}$ is a sector while every $\mathcal{C}_{2k}$ is a section;
- every $\boldsymbol{s}_k$ is a triangularly definable sample of $\mathcal{C}_k$.

S1 (Initialization) Assume that $\hat{\boldsymbol{s}}$ is uniquely determined by

$$\begin{cases} \psi_1(x_1) = 0 \\ \psi_2(x_1, x_2) = 0 \\ \qquad \vdots \\ \psi_{r-1}(x_1, \ldots, x_{r-1}) = 0 \end{cases} \tag{20}$$

plus $x_1 \in [u_1, v_1], \ldots, x_{r-1} \in [u_{r-1}, v_{r-1}]$. Let $\varphi(\hat{\boldsymbol{x}}, x_r) = \sum_{i=0}^{I} \varphi_i(\hat{\boldsymbol{x}}) x_r^i$ be the product of elements of prim(Φ), i.e. $\prod\{\phi \mid \phi \in \text{prim}(\Phi)\}$.

S2 (Estimating bound) Regard $\varphi(\hat{\boldsymbol{s}}, x_r) = \sum_{i=0}^{I} \varphi_i(\hat{\boldsymbol{s}}) x_r^i$ as a real polynomial in x_r. The sign of every element of $\mathbb{Q}[x_1, \ldots, x_{r-1}; \exp(x_1)]$ is decidable at $\hat{\boldsymbol{s}}$ by Lemma 5.7. Let J be the greatest index such that $\varphi_J(\hat{\boldsymbol{s}})$ is nonzero. Then the absolute values of real roots of $\varphi(\hat{\boldsymbol{s}}, x_r)$ are bound by

$$\mathrm{B} = 1 + \max_{j=0}^{J-1} \left| \frac{\varphi_j(\hat{\boldsymbol{s}})}{\varphi_J(\hat{\boldsymbol{s}})} \right| \tag{21}$$

by a corollary of Descartes' sign rule (Xu et al., 2010b).

S3 (Choosing samples) Let $[[a_1, b_1], \ldots, [a_K, b_K]]$, with $a_1 \geq -\mathrm{B}$ and $b_K \leq \mathrm{B}$, be a list of disjoint rational intervals that isolate all distinct real roots α_k of $\varphi(\hat{\boldsymbol{s}}, x_r)$ by Lemma 5.5. Take $K+1$ rational numbers β_k, each $\beta_k \in (b_{k-1}, a_k)$ where $b_0 < -\mathrm{B}$ and $a_{K+1} > \mathrm{B}$. Every $\boldsymbol{s}_{2k-1} = (\hat{\boldsymbol{s}}, \beta_k)$ can be a sample of the sector $\mathcal{C}_{2k-1}$, and it is defined by the triangular system (20) plus $x_r - \beta_k = 0 \wedge x_r \in [\beta_k, \beta_k]$. Meanwhile every $(\hat{\boldsymbol{s}}, \alpha_k)$ can be a sample of the section $\mathcal{C}_{2k}$, and it is defined by the triangular system (20) plus $\varphi(\boldsymbol{x}) = 0 \wedge x_r \in [a_k, b_k]$ or $\gcd(\varphi(\boldsymbol{x}), \varphi'(\boldsymbol{x})) = 0 \wedge x_r \in [a_k, b_k]$ depending on the multiplicity of α_k. (It is preferable that such α_k is defined by an irreducible divisor of $\varphi(\boldsymbol{x})$.)

S4 (Constructing stack) The sign of every element of Φ is decidable at $\boldsymbol{s}_k$ by Lemma 5.7. It amounts to deciding the sign of every element of Φ on every cell $\mathcal{C}_k$ by sign-invariance. Finally a stack $\mathcal{S}$ over $\mathcal{C}$ is constructed by alternately arranging those sectors and sections. □

The sign-invariance of proj(Φ) entails that Φ is delineable on $\mathcal{C}$, and hence has a stack over $\mathcal{C}$ by Theorem 3.5. Then we have known that the number of sections over $\mathcal{C}$ is constant, which amounts to the number of distinct real roots of $\varphi(\hat{\boldsymbol{s}}, x_r)$ for an arbitrary sample $\hat{\boldsymbol{s}} \in \mathcal{C}$. In the cylinder $\mathcal{C} \times \mathbb{R}$, every section $\mathcal{C}_{2k}$ is assigned with a sample $(\hat{\boldsymbol{s}}, \alpha_k)$, where α_k is a real root of $\varphi(\hat{\boldsymbol{s}}, x_r)$; and every sector $\mathcal{C}_{2k-1}$ is assigned with a sample $(\hat{\boldsymbol{s}}, \beta_k)$, where β_k is an arbitrary rational number between two successive real roots of $\varphi(\hat{\boldsymbol{s}}, x_r)$. Hence Algorithm 2 meets its specification. Besides, we require a stricter condition on the input—the sign-invariance of the projection closure of proj(Φ), which implies that the base cell $\mathcal{C}$ can be recursively constructed by Algorithm 2.

In fact, for the formula (1), if the truth of $\bigwedge_i \bigvee_j \phi_{ij}(x_1, \ldots, x_n) \bowtie_{ij} 0$ on the sample $\hat{\boldsymbol{s}} \in \mathcal{C}$, a partial assignment, is already known, the cell $\mathcal{C}$ is unnecessary to be lifted, since the truth is invariant on the cylinder $\mathcal{C} \times \mathbb{R}$. So we can construct the partial cylinder decomposition (Collins and Hong, 1991) for efficiency.

Note that those samples form the structure of trees, where

- every one-dimensional sample is a root;
- every $\hat{\boldsymbol{s}}$ is the parent of section samples $(\hat{\boldsymbol{s}}, \alpha_k)$ and sector samples $(\hat{\boldsymbol{s}}, \beta_k)$;
- every n-dimensional sample is a leaf.

Let cld$(\hat{\boldsymbol{s}})$ be the collection $\bigcup_{k=1}^{K} \{(\hat{\boldsymbol{s}}, \alpha_k)\} \cup \bigcup_{k=1}^{K+1} \{(\hat{\boldsymbol{s}}, \beta_k)\}$.

Example 5.8. Continued to consider Example 4.7. Now we show how to lift the sector $(\alpha, +\infty) \ni 4$ of $\mathbb{R}$. After substituting $x_1 = 4$ into proj$^2(\Phi)$ (see the collection (9)), we have 10 real roots w.r.t. x_2

totally. Since $(\alpha, +\infty)$ is a sector, no multiple root occurs. Thus all real roots are distinct. Some of them are α_1 (the real root of $\exp(4)x_2 - 3\exp(4) - 9$), 1 and 0, and others are negative transcendental. Hence we obtain a stack of $\mathrm{proj}^2(\Phi)$ over $(\alpha, +\infty)$, which has 10 sections and 11 sectors. For instance, $(\alpha, +\infty) \times (0, 1)$ is a sector of $\mathbb{R}^2$ with a sample $(4, 1/4)$, and $\mathcal{C}_1 = \{(x_1, x_2) \mid \alpha < x_1 \wedge \exp(x_1)x_2 - x_1\exp(x_1) + \exp(x_1) - x_1 - 5 = 0\}$ is a section of $\mathbb{R}^2$ with a sample $(4, \alpha_1)$. The sample $(4, \alpha_1)$ is triangularly defined by

$$\begin{cases} x_1 - 4 = 0 \\ \exp(x_1)x_2 - x_1\exp(x_1) + \exp(x_1) - x_1 - 5 = 0, \end{cases} \tag{22a}$$

plus $x_1 \in [4, 4], x_2 \in [3, 4]$.

Furthermore, we proceed to lift the cell $(\alpha, +\infty) \times (0, 1)$. After substituting $x_1 = 4, x_2 = 1/4$ into $\mathrm{proj}(\Phi)$ (see the collection (8)), we have 6 distinct real roots w.r.t. x_3. They are $3 + 25\exp(-4)$, $\sqrt{15}/4$, $-\sqrt{15}/4$, $-5/2 + 7\exp(-4)$, $-5 + 25\exp(-4)$, $-13 + 25\exp(-4)$ descendingly. Hence we obtain a stack of $\mathrm{proj}(\Phi)$ over $(\alpha, +\infty) \times (0, 1)$, which has 6 sections and 7 sectors. For instance, $\mathcal{C}_2 = \{(x_1, x_2, x_3) \mid \alpha < x_1 \wedge 0 < x_2 < 1 \wedge -\sqrt{1 - x_2^2} < x_3 < \sqrt{1 - x_2^2}\}$ is a sector of $\mathbb{R}^3$ with a sample $(4, 1/4, 0)$.

Finally, we proceed to lift the cell $\mathcal{C}_2$. After substituting $x_1 = 4, x_2 = 1/4, x_3 = 0$ into Φ (see the collection (7)), we have 4 distinct real roots w.r.t. x_4. They are 1, $9\exp(-4)/4 - 5/16$, $25\exp(-4)/8 - 5/8$, -1 descendingly. Hence we obtain a stack of Φ over $\mathcal{C}_2$, which has 4 sections and 5 sectors. For instance, $\mathcal{C}_2 \times [1, 1]$ is a section of $\mathbb{R}^4$ with a sample $(4, 1/4, 0, 1)$.

Besides, we can also lift the section $\mathcal{C}_1$. After substituting $x_1 = 4, x_2 = \alpha_1$ into $\mathrm{proj}(\Phi)$, we have 4 real roots w.r.t. x_3. But only three of them are distinct, because $2\exp(4)x_3 + 8\exp(4)\alpha_1 - 8\exp(4) - 50$ and $2\exp(4)x_3 + 4\exp(4)\alpha_1 + 4\exp(4) - 14$ have a common real root α_2. Hence we obtain a stack of $\mathrm{proj}(\Phi)$ over $\mathcal{C}_1$, which has 3 sections and 4 sectors. The sample $(4, \alpha_1, \alpha_2)$ of a section of $\mathbb{R}^3$ is triangularly defined by

$$\begin{cases} x_1 - 4 = 0 \\ \exp(x_1)x_2 - x_1\exp(x_1) + \exp(x_1) - x_1 - 5 = 0 \\ 2\exp(x_1)x_3 + x_1\exp(x_1)x_2 + x_1\exp(x_1) - x_1^2 + 3x_1 - 10 = 0, \end{cases} \tag{22b}$$

plus $x_1 \in [4, 4], x_2 \in [3, 4], x_3 \in [-9, -8]$.

By Algorithm 2, the sign-invariant decompositions of higher-dimensional real spaces can be induced from those of lower-dimensional real spaces step by step. Thereafter it completes the lifting phase.

6. Solution formula construction phase

In this section, we describe how to construct the solution formula for the formula (1). The technical hardness is the definition of each cell simply by a quantifier-free formula, rather than by a quantifier-free formula plus a sample (shown in the lifting phase). We will augment the projection to resolve it.

Definition 6.1 *(Standard defining formulas).* For a cell $\mathcal{C}$ of $\mathbb{R}^r$, its standard defining formula is a quantifier-free formula in disjunctive normal form ρ such that $\mathcal{C} = \{\boldsymbol{s} \in \mathbb{R}^r \mid \rho(\boldsymbol{s})\}$.

Recall that, for a cell of $\mathbb{R}$ as the output of Algorithm 1, we have that every section α_k is defined by

$$\psi_0(x) = 0 \wedge a_k \le x \le b_k, \tag{23a}$$

and every sector (α_k, α_{k+1}) (with virtual sections $\alpha_0 = -\infty$ and $\alpha_{K+1} = +\infty$) is defined by

$$\psi_0\left(\frac{b_k + a_{k+1}}{2}\right)\psi_0(x) > 0 \wedge a_k < x < b_{k+1}. \tag{23b}$$

So their standard defining formulas are obtained by replacing $\psi_0(x)$ with elements of the finest square-free basis of $\mathrm{proj}^{r-1}(\Phi)$, the divisors of $\psi_0(x)$.

However, for a higher-dimensional cell $\mathcal{C}$, the projection (6) does not always suffice to yield the standard defining formula of $\mathcal{C}$. For instance, we assume that the irreducible *EP* $\phi(\hat{\boldsymbol{x}}, x_r)$ has a stack over $\mathcal{C}$ (with standard defining formula ρ). If the stack consists of more than three cells, we would fail to assign each cell with a distinct standard defining formula merely by ρ plus $\phi > 0$, $\phi = 0$ or $\phi < 0$. Fortunately, Collins (1975) found that constructing the standard defining formula could resort to derivatives of such ϕ. That is, the projection component proj_2 could be augmented as

$$\mathrm{proj}_2^+ = \bigcup_{\phi \in \mathrm{prim}(\Phi)} \bigcup_{\varphi \in \mathrm{red}(\phi)} \bigcup_{\psi \in \mathrm{der}(\varphi)} \mathrm{psc}\big(\psi, \psi'\big), \tag{24}$$

where $\mathrm{der}(\varphi) = \{\varphi^{(k)} \mid 0 \le k < \deg(\varphi) - 1\}$. So the total augmented projection is

$$\mathrm{proj}^+ = \mathrm{cont}(\Phi) \cup \mathrm{proj}_1 \cup \mathrm{proj}_2^+ \cup \mathrm{proj}_3\,. \tag{25}$$

Example 6.2. Continued to consider Example 5.8. In the cylinder over $(\alpha, +\infty) \times (0, 1) \ni (4, 1/4)$, the (exponential) polynomial $x_3^2 + x_2^2 - 1$ has the same sign in the top and the bottom sectors with samples $(4, 1/4, 1)$ and $(4, 1/4, -1)$, respectively. Hence the derivative $\partial(x_3^2 + x_2^2 - 1)/\partial x_3 = 2x_3$ is necessary to be introduced to differentiate them.

Lemma 6.3. *Let $\mathcal{S}$ be the output stack of Algorithm 2 on the input Φ and $\mathcal{C}$, and $\mathcal{C}_k$ be a cell in $\mathcal{S}$. Given the standard defining formula ρ of $\mathcal{C}$, the collection* $\mathrm{der}(\Phi)$ *is sufficient to construct a standard defining formula ρ_k of $\mathcal{C}_k$ if* $\mathrm{proj}_2^+(\Phi)$ *is sign-invariant on $\mathcal{C}$.*

Proof. Every sector $\mathcal{C}_{2k-1}$ in $\mathcal{S}$ is the intersection of sectors of elements of Φ; while every section $\mathcal{C}_{2k}$ is a section of some elements of Φ. It suffices to show that every cell of a single projector $\phi \in \Phi$ can be defined by $\mathrm{der}(\phi)$.

Since $\mathrm{proj}_1(\phi)$ is sign-invariant on $\mathcal{C}$, the degree of ϕ is constant on $\mathcal{C}$, saying K. Let $\varphi = \mathrm{red}_{\deg(\phi)-K}(\mathrm{prim}(\phi))$, and $\varphi, \varphi', \ldots, \varphi^{(K-1)}$ be all derivatives with positive degree in x_r of φ. Each single derivative is delineable on $\mathcal{C}$ by the sign-invariance of $\mathrm{proj}_2^+(\phi)$ and Theorem 3.5. Then we construct the standard defining formulas ρ_k^i of the cells $\mathcal{C}_k^i$ in the stack $\mathcal{S}^i$ for $\varphi^{(i)}$ with auxiliary derivatives by induction on i.

(1) Since $\varphi^{(K-1)}$ is a linear polynomial in x_r, the stack $\mathcal{S}^{K-1}$ for $\varphi^{(K-1)}$ consists of three cells (two sectors $\mathcal{C}_1^{K-1}$, $\mathcal{C}_3^{K-1}$ and one section $\mathcal{C}_2^{K-1}$) in the cylinder $\mathcal{C} \times \mathbb{R}$. Their respective standard defining formulas are

$$\begin{cases} \rho_1^{K-1}(\hat{\boldsymbol{x}}, x_r) \equiv \rho(\hat{\boldsymbol{x}}) \wedge \varphi^{(K-1)}(\hat{\boldsymbol{x}}, x_r) < 0 \\ \rho_2^{K-1}(\hat{\boldsymbol{x}}, x_r) \equiv \rho(\hat{\boldsymbol{x}}) \wedge \varphi^{(K-1)}(\hat{\boldsymbol{x}}, x_r) = 0 \\ \rho_3^{K-1}(\hat{\boldsymbol{x}}, x_r) \equiv \rho(\hat{\boldsymbol{x}}) \wedge \varphi^{(K-1)}(\hat{\boldsymbol{x}}, x_r) > 0. \end{cases} \tag{26}$$

(2) By inductive hypothesis, let $\mathcal{C}_k^1$ be a cell in $\mathcal{C} \times \mathbb{R}$, on which φ' is sign-invariant, with its standard defining formula $\rho_k^1(\hat{\boldsymbol{x}}, x_r)$, and $\mathcal{S}^1$ be the stack for φ' with auxiliary derivatives.

(3) For every section $\mathcal{C}_{2k}^0$ of φ, if the multiplicity of $\mathcal{C}_{2k}^0$ is more than 1, then $\mathcal{C}_{2k}^0$ is also a section $\mathcal{C}_{2l}^1$ of φ' and its standard defining formula is

$$\rho_{2k}^0(\hat{\boldsymbol{x}}, x_r) \equiv \rho_{2l}^1(\hat{\boldsymbol{x}}, x_r); \tag{27a}$$

and define two sectors $\mathcal{C}_{2k-1}^0$ and $\mathcal{C}_{2k+1}^0$ adjacent to $\mathcal{C}_{2k}^0$ respectively by

$$\begin{cases} \rho_{2k-1}^0(\hat{\boldsymbol{x}}, x_r) \equiv \rho_{2l-1}^1(\hat{\boldsymbol{x}}, x_r) \\ \rho_{2k+1}^0(\hat{\boldsymbol{x}}, x_r) \equiv \rho_{2l+1}^1(\hat{\boldsymbol{x}}, x_r), \end{cases} \tag{27b}$$

since there must be a section of φ' between two successive sections of φ by Rolle's theorem and hence the two sectors $\mathcal{C}^1_{2l-1}$ and $\mathcal{C}^1_{2l+1}$ of φ' contain no section of φ. Otherwise $\mathcal{C}^0_{2k}$ lies in a sector $\mathcal{C}^1_{2l-1}$ of φ' and its standard defining formula is

$$\rho^0_{2k}(\hat{\boldsymbol{x}}, x_r) \equiv \rho^1_{2l-1}(\hat{\boldsymbol{x}}, x_r) \wedge \varphi(\hat{\boldsymbol{x}}, x_r) = 0, \tag{28a}$$

since every sector of φ' contains at most one section of φ by Rolle's theorem; and define the rest region in $\mathcal{C}^1_{2l-1}$ as two sectors $\mathcal{C}^0_{2k-1,1}$ and $\mathcal{C}^0_{2k+1,1}$ respectively by

$$\begin{cases} \rho^0_{2k-1}(\hat{\boldsymbol{x}}, x_r) \equiv \rho^1_{2l-1}(\hat{\boldsymbol{x}}, x_r) \wedge \varphi(\hat{\boldsymbol{x}}, x_r) < 0 \\ \rho^0_{2k+1}(\hat{\boldsymbol{x}}, x_r) \equiv \rho^1_{2l-1}(\hat{\boldsymbol{x}}, x_r) \wedge \varphi(\hat{\boldsymbol{x}}, x_r) > 0. \end{cases} \tag{28b}$$

For other cell $\mathcal{C}^1_l$ of φ', the standard defining formula is $\rho^1_l(\hat{\boldsymbol{x}}, x_r)$ plus $\varphi(\hat{\boldsymbol{x}}, x_r) < 0$ or $\varphi(\hat{\boldsymbol{x}}, x_r) > 0$ depending on the sample. Thus we have the stack $\mathcal{S}^0$ for φ with auxiliary derivatives.

(4) Finally we proceed to refine $\mathcal{S}^0$ to be the stack for φ. For three adjacent cells (two sectors $\mathcal{C}^0_{2k-1}$, $\mathcal{C}^0_{2k+1}$ and one section $\mathcal{C}^0_{2k}$) in $\mathcal{S}^0$, if $\mathcal{C}^0_{2k}$ is a section of certain-order derivative of φ, rather than a section of φ, combine the three cells into one sector

$$\mathcal{C}^0 = \left\{(\hat{\boldsymbol{s}}, s_r) \;\middle|\; \rho^0_{2k-1}(\hat{\boldsymbol{s}}, s_r) \vee \rho^0_{2k}(\hat{\boldsymbol{s}}, s_r) \vee \rho^0_{2k+1}(\hat{\boldsymbol{s}}, s_r)\right\}. \tag{29}$$

After applying the **resolution** rule $[(\varrho \wedge \psi < 0) \vee (\varrho \wedge \psi = 0) \vee (\varrho \wedge \psi > 0)] \equiv \varrho$, the defining formula $\rho^0_{2k-1} \vee \rho^0_{2k} \vee \rho^0_{2k+1}$ results in a simpler one, which is the standard defining formula of $\mathcal{C}^0$. Reset $\mathcal{S}^0 := \mathcal{S}^0 \cup \{\mathcal{C}^0\} \setminus \{\mathcal{C}^0_{2k-1}, \mathcal{C}^0_{2k}, \mathcal{C}^0_{2k+1}\}$. By executing such combination a finite number of times, the stack $\mathcal{S}^0$ for φ would be refined step by step.

Hence every cell $\mathcal{C}_k$ in the stack $\mathcal{S} = \mathcal{S}^0$ is determined by the newly-constructed standard defining formula ρ_k. □

The augmented projection (25) has more $\mathcal{O}(MN^3)$ elements than the projection (6), so it brings a large amount of computation. For efficiency, we use

- the augmented projection (25) to eliminate free variables, since the desired quantifier-free formula is dependent to free variables;
- and the projection (6) to eliminate quantified variables, since the desired quantifier-free formula is independent of quantified variables.

Two interesting observations are: (i) higher-dimensional cells are **globally** projected into lower-dimensional cells to ensure the delineability and construct the standard defining formulas, thus the size of the projection is huge; (ii) and lower-dimensional cells are **locally** lifted to higher-dimensional cells to construct stacks, thus there are many useless projectors in the local lifting. How to utilize the local information to simplify the projection is an ongoing work.

Remark 6.4. Brown (1999) invented the extended language that allows reference to indexed roots of polynomials to identify cells. It can avoid the expensive augmented projection in cylindrical algebraic decomposition. The formula in the extended language can be immediately transformed in the original language by introducing some quantified variables to represent indexed roots. But, if so, the solution formula either is expressed in a middle (extended) language or has some quantified variables. Hence we would like to adopt the augmented projection to do the quantifier elimination here.

Theorem 6.5 *(Quantifier elimination). Let $\mathcal{D}^k$ be the cylindrical decomposition of $\mathbb{R}^k$, in which the standard defining formulas ρ^k_l of the cells $\mathcal{C}^k_l$ are known, and $\mathcal{D}^n$ be the cylindrical decomposition of $\mathbb{R}^n$ in which the samples $\mathbf{s}^n_l$ of the cells $\mathcal{C}^n_l$ are known. Then the formula (1) is equivalent to*

$$\bigvee_{\mathcal{C}_l^k \in \mathcal{D}^k} \left\{ \rho_l^k \wedge \left[\underset{\boldsymbol{s}_l^{k+1} \in \mathrm{cld}(\boldsymbol{s}_l^k)}{\mathrm{Q}_{k+1}^*} \cdots \underset{\boldsymbol{s}_l^{n} \in \mathrm{cld}(\boldsymbol{s}_l^{n-1})}{\mathrm{Q}_{n}^*} : \bigwedge_i \bigvee_j \phi_{ij}\left(\boldsymbol{s}_l^n\right) \bowtie_{ij} 0 \right] \right\}, \tag{30}$$

where $\mathrm{Q}^* \equiv \bigwedge$ *in the case of* $\mathrm{Q} \equiv \forall$ *and* $\mathrm{Q}^* \equiv \bigvee$ *in the case of* $\mathrm{Q} \equiv \exists$.

Proof. It follows immediately from the sign-invariant decomposition $\mathcal{D}^n$ and the standard defining formulas ρ_l^k of cells $\mathcal{C}_l^k$ in the decomposition $\mathcal{D}^k$. □

Example 6.6. Consider the quantifier elimination problem

$$\begin{aligned} &\exists x_1 \geq 0 \forall x_2 \forall x_3 \exists x_4 : \left[x_3^2 + x_2^2 \geq 1\right] \vee [x_2 \leq 0] \\ &\quad \vee \left[-1 \leq x_4 \leq 1 \wedge (x_1 x_4 + x_2 + 1)\exp(x_1) - x_1 > 5 \right. \\ &\quad \left. \wedge \left(x_1^2 x_4 + 2x_3 + 2x_1 x_2 + 2x_1\right)\exp(x_1) - 2x_1^2 - 2x_1 > 10\right]. \end{aligned} \tag{31}$$

All *EP*s extracted from the formula (31) are $\Phi = \{x_1, x_2, x_3^2 + x_2^2 - 1, x_4 \pm 1, x_1 \exp(x_1) x_4 + \exp(x_1) x_2 + \exp(x_1) - x_1 - 5, x_1^2 \exp(x_1) x_4 + 2\exp(x_1) x_3 + 2x_1 \exp(x_1) x_2 + 2x_1 \exp(x_1) - 2x_1^2 - 2x_1 - 10\}$. In Examples 3.2 and 3.6, the projection closure of Φ is demonstrated. In Example 4.7, we isolate real roots of a univariate *EP* of $\mathrm{proj}^3(\Phi)$, which yields the rightmost cell of $\mathrm{proj}^3(\Phi)$ with a sample 4 in the real line. Then we partially construct the cylindrical decomposition over this cell in Example 5.8. Based on the partial decomposition, the formula (31) can be decided to be true as follows.

(1) The existential quantifier $\exists x_1$ can be replaced with the disjunction on a finite number of nonnegative samples of x_1. The formula (31) is true if the disjunct on $x_1 = 4$ holds, i.e.

$$\begin{aligned} &\forall x_2 \forall x_3 \exists x_4 : \left[x_3^2 + x_2^2 \geq 1\right] \vee [x_2 \leq 0] \\ &\quad \vee \left[-1 \leq x_4 \leq 1 \wedge (4x_4 + x_2 + 1)\exp(4) > 9 \right. \\ &\quad \left. \wedge (8x_4 + x_3 + 4x_2 + 4)\exp(4) > 25\right]. \end{aligned} \tag{32a}$$

(2) The universal quantifier $\forall x_2$ can be replaced with the conjunction on a finite number of samples of x_2. Since $1/4$ is the sample of the only feasible cell satisfying $x_2^2 + x_3^2 < 1 \wedge x_2 > 0$, the formula (32a) is true if and only if the conjunct on $x_2 = 1/4$ holds, i.e.

$$\begin{aligned} &\forall x_3 \exists x_4 : \left[16x_3^2 \geq 15\right] \vee \left[-1 \leq x_4 \leq 1 \right. \\ &\quad \left. \wedge (16x_4 + 5)\exp(4) > 36 \wedge (8x_4 + x_3 + 5)\exp(4) > 25\right]. \end{aligned} \tag{32b}$$

(3) The universal quantifier $\forall x_3$ can be replaced with the conjunction on a finite number of samples of x_3. Since 0 is the sample of the only feasible cell satisfying $16x_3^2 < 15$, the formula (32b) is true if and only if the conjunct on $x_3 = 0$ holds, i.e.

$$\exists x_4 : \left[-1 \leq x_4 \leq 1 \wedge (16x_4 + 5)\exp(4) > 36 \wedge (8x_4 + 5)\exp(4) > 25\right]. \tag{32c}$$

(4) The existential quantifier $\exists x_4$ can be replaced with the disjunction on a finite number of samples of x_4. The formula (32c) is true since the disjunct on $x_4 = 1$ holds.

Remark 6.7. It has been proven by Lindemann–Weierstrass theorem (Baker, 1975) that the number $\tan(\alpha)$ is transcendental for every nonzero algebraic number α. We can use the continued fraction in Olds (1963) that

$$\tan(x) = \cfrac{x}{1 - \cfrac{x^2}{3 - \cfrac{x^2}{5 - \cdots}}} = \cfrac{1}{\frac{1}{x} - \cfrac{1}{\frac{3}{x} - \cfrac{1}{\frac{5}{x} - \cdots}}}, \tag{33}$$

to approach the exact value of $\tan(\alpha)$. On the other hand, the rational approximations of the inverse functions $\ln(x)$ and $\arctan(x)$ can again be obtained. Thereby, the real root isolations of $\mathbb{Q}[x, \tan(x)]$, $\mathbb{Q}[x, \ln(x)]$, $\mathbb{Q}[x, \arctan(x)]$ and the quantifier elimination procedures for $\mathfrak{T}_{\ln}$, $\mathfrak{T}_{\tan}$, $\mathfrak{T}_{\arctan}$ can be similarly established as Strzeboński (2011) and McCallum and Weispfenning (2012).

Unlike the cylindrical decomposition on $\mathbb{Q}[x_1, \ldots, x_n; \mathrm{tran}(x_1)]$ in Strzeboński (2011) and McCallum and Weispfenning (2012) that

(1) introduces an additional variable y to represent the transcendental function $\mathrm{tran}(x_1)$, which yields a polynomial formula in $x_1, \ldots, x_n, y$;
(2) performs the cylindrical algebraic decomposition on $\mathbb{Q}[x_1, \ldots, x_n; y]$, which needs the additional expensive projection to eliminate the variable x_1 or y;
(3) and then checks the reality of the output feasible cells $\mathcal{C}$ of $\mathbb{R}^2$ in x_1, y, i.e. whether there is a point in $\mathcal{C}$ satisfying $y = \mathrm{tran}(x_1)$;

our procedure avoids the additional projection, and hence is more direct, effective, and potentially efficient.

7. Conclusion

In this paper, we have proposed the quantifier elimination procedure for the *EP* formulas. It adopts the scheme of cylindrical decomposition consisting of four phases—projection, isolation, lifting, and solution formula construction. For future work, we are interested in exploring three aspects.

(1) How to perform a non-cylindrical decomposition on certain (n, m)-variate *EP*s? The Schanuel's conjecture is a useful tool.
(2) How to improve the efficiency of the proposed procedure for the *EP* formulas? A promising way is to combine the DPLL algorithm.
(3) How to apply these decision procedures to other fields, such as program verification and hybrid systems?

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