KÖNIG'S INFINITY LEMMA AND BETH'S TREE THEOREM

Abstract. König [1926] includes a result subsequently called König's Infinity Lemma. König [1927] includes a graph theoretic formulation: an infinite, locally finite and connected (undirected) graph includes an infinite path. Contemporary applications of the infinity lemma in logic frequently refer to a consequence of the infinity lemma: an infinite, locally finite (undirected) tree with a root has a infinite branch. This tree lemma can be traced to Beth [1955]. It is argued that Beth independently discovered the tree lemma in the early 50's and that it was later recognized among logicians that Beth's result was a consequence of the infinity lemma. It appears that the question of whether or not the two lemmas are equivalent was not raised in the logic literature. The equivalence of these lemmas is an easy consequence of a well known result in graph theory: every connected, locally finite graph has among its partial subgraphs a spanning tree.

1. Introduction

The purpose of this paper is to discuss the history of the applications in logic of a graph theoretic result due to Dénes König. The focus of the paper is a theorem whose proof appeared in König [1926]. This theorem has been called König's Lemma, and König's Infinity Lemma. For the purpose of discussion, understand the König Infinity Lemma to be the proposition that any infinite, locally finite connected (undirected) graph has an infinite path; and understand the König Tree Lemma to be the proposition that any infinite, locally finite (undirected) tree has an an infinite path (or branch). Once it is recalled that a tree is a connected acyclic graph, it is obvious that the König Tree Lemma is a consequence of the König Infinity Lemma. It is less obvious that the converse is true. The converse is a consequence of the fact that each connected locally finite graph contains among its partial subgraphs a spanning tree whose nodes are exactly the nodes of the given graph (see Delhommé and Morillon [2006], Theorem 2, page 175). Since the paths in the spanning tree are paths in the graph and the spanning tree is also locally finite, the König Tree Lemma implies that the graph has an infinite path.

König's Tree Lemma was not proved by König. However, there is a related result that was formulated and proved independently by E. V. Beth in the early 50's. This result, Beth's Tree Theorem, appeared in Beth [1955] and Beth [1959]. In both publications the result was used in a proof of the completeness of a tableau formalization of first-order logic. Beth was aware of König [1926] by 1959. However, he did not attribute his tree theorem to König. By the mid 60's it was recognized among logicians that Beth's result was a consequence of König's Infinity Lemma.

Attention is primarily restricted to these results as they apply to undirected simple graphs. However, these results have been formulated for directed simple graphs as well. Intuitively, a graph is composed of a nonempty set of nodes and a set of edges between those nodes. Simple graphs have only one edge type. In contrast,

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those graphs that have more than a single edge type are multigraphs. In directed graphs the edges are directed, while in undirected graphs they are not. The theory of undirected simple graphs has two very simple conceptual foundations. In the first, an undirected simple graph is a binary relational system consisting of a nonempty set of nodes and a binary relation on the set of nodes that is symmetric. In contrast, a directed simple graph is a binary relational system whose binary relation is not symmetric. The second foundation for the theory of undirected simple graphs treats undirected simple graphs as ordered pairs consisting of a nonempty set of nodes and a set of unordered pairs of the nodes. These pairs of nodes are the edges of the graph. Only simple graphs are considered below.

2. Review of some Basic Graph Theory

A *graph* is a binary relational system $\mathfrak{G}$=(G, $\rho_{\mathfrak{G}}$) where G is a non-empty set (*the domain of* $\mathfrak{G}$) and $\rho_{\mathfrak{G}}$ is a binary relation on G that is symmetric on G. Such graphs are *undirected graphs* to distinguish them from those whose relations are not symmetric (*directed graphs*). Members of G are *nodes.* Members of $\rho_{\mathfrak{G}}$ are *arcs.* Nodes x and y are *adjacent in* $\mathfrak{G}$ iff (x, y)$\in \rho_{\mathfrak{G}}$. $\mathfrak{G}$ is *locally finite* iff for all $x \in G$, there are only finitely many nodes in $\mathfrak{G}$ adjacent to x in $\mathfrak{G}$. Every finite graph is locally finite. A *path in* $\mathfrak{G}$ is a sequence of nodes in $\mathfrak{G}$, $z_1 ... z_t$, such that (1) $t > 1$; (2) if $i \neq j$, then $z_i \neq z_j$; and, (3) if $1 \leq i \leq t-1$, then (z_i, z_{i+1})$\in \rho_{\mathfrak{G}}$. $z_1 ... z_t$ *is a path between* x *and* y *in* $\mathfrak{G}$ iff $z_1 ... z_t$ is a path in $\mathfrak{G}$ and either $z_1 = x$ and $z_t = y$ or $z_1 = y$ and $z_t = x$. $\mathfrak{G}$ *is connected* iff for all nodes x and y in G, if $x \neq y$, then there is a path between x and y in $\mathfrak{G}$. Finally, the sequence of nodes in G $z_1 ... z_t z_1$ *is a cycle in* $\mathfrak{G}$ iff $z_1 ... z_t$ is a path in $\mathfrak{G}$, $z_2 ... z_t z_1$ is a path in $\mathfrak{G}$ and $t \geq 3$. The graph $\mathfrak{G}$ *is totally irreflexive* iff $\rho_{\mathfrak{G}}$ is totally irreflexive on G. $\mathfrak{G}$ *is acyclic* iff $\mathfrak{G}$ is totally irreflexive and there are no cycles in $\mathfrak{G}$. Finally, $\mathfrak{G}$ *is a tree* iff $\mathfrak{G}$ is a connected and acyclic graph. $\mathfrak{G}$ *is infinite* iff G is an infinite set. *A branch in* $\mathfrak{G}$ is either a path in $\mathfrak{G}$ or an infinite sequence of members of G, $z_1 ... z_t ...$ where for all i, (z_i, z_{i+1})$\in \rho_{\mathfrak{G}}$ and $z_i \neq z_j$, if $i \neq j$.

$\mathfrak{G}$ *is a subgraph of* $\mathfrak{G}'$ iff

(1) $G \subseteq G'$; and
(2) for all x, $y \in G$, (x, y)$\in \rho_{\mathfrak{G}}$ iff (x, y)$\in \rho_{\mathfrak{G}'}$.

$\mathfrak{G}$ *is a partial subgraph of* $\mathfrak{G}'$ iff

(1) $G \subseteq G'$; and
(2) for all x, $y \in G$, if (x, y)$\in \rho_{\mathfrak{G}}$, then (x, y)$\in \rho_{\mathfrak{G}'}$.

$\mathfrak{G} \subseteq \mathfrak{G}'$ indicates that $\mathfrak{G}$ is a subgraph of $\mathfrak{G}'$. Every subgraph of a graph is a graph and every subgraph of a graph is a partial subgraph. There are partial subgraphs that are not subgraphs. Notice that if $\mathfrak{G}$ is a partial subgraph of $\mathfrak{G}'$, then every branch in $\mathfrak{G}$ is a branch in $\mathfrak{G}'$.

Let $\mathfrak{G}$ be a connected graph and $\mathfrak{G}'$ be a tree, $\mathfrak{G}'$ *is a spanning tree for* $\mathfrak{G}$ iff

(1) $G = G'$; and
(2) $\mathfrak{G}'$ is a partial subgraph of $\mathfrak{G}$.

If $\mathfrak{G}'$ is a spanning tree for $\mathfrak{G}$, and $\mathfrak{G}$ is locally finite, then $\mathfrak{G}'$ is locally finite, and any infinite branch in $\mathfrak{G}'$ is also an infinite branch in $\mathfrak{G}$.

There are several results in graph theory about the existence of spanning tree for various classes of graphs. Delhommé and Morillon [2006] includes an extensive discussion of these results and the equivalence in ZF between these results and various restricted choice principles. Among these results is the Spanning Tree (Principle) for Locally Finite Connected Graphs. Lemma 2.1 below is this result restricted to infinite sets. The proof of the Spanning Tree for Locally Finite Connected Graphs in Delhommé and Morillon [2006] uses a choice principle for countable families of finite sets. In fact, it is shown that in ZF the Spanning Tree for Locally Finite Connected Graphs is equivalent to this choice principle. The reasoning for Lemma 2.1 below is a simple extension of a well known breath first search algorithm for constructing a spanning tree for a finite, connected graph. The idea is to construct the spanning tree in stages starting from a fixed node in the graph. At each stage in the process, a partial subgraph of the initial graph is constructed. This constructed graph is also a tree. At the first stage the nodes of the partial graph consists of the fixed (initial) node and the relation is empty. At stage $n+1$, the nodes of the partial subgraph consists of the nodes of the graph constructed at stage n together with those nodes in the graph, but not in the stage n partial subgraph, that are adjacent to the nodes in the domain of the partial subgraph constructed in stage n. The arcs in the stage $n+1$ partial subgraph are those in that partial subgraph constructed at stage n plus arcs $(x,\, y)$ and $(y,\, x)$ were x is added in stage $n+1$, y is in the domain of the partial subgraph constructed at stage n, and x and y are adjacent in the initial graph. In order for the graph at stage $n+1$ to be a tree, for each x there can be only one such y. Thus, the algorithm requires some way to choose among the nodes in the original graph. A fixed enumeration of the nodes in the initial graph will do. In this case, y is the first element in the enumeration adjacent to x. Since the initial graph is finite the algorithm terminates when all the nodes in the initial graph are in the domain of the partial subgraph. That partial subgraph obtained in the final stage, is the desired spanning tree. When the initial graph is infinite and locally finite, the above algorithm will not terminate. In this case, a family $\{\mathfrak{G}_n\colon\, n \in \omega\}$ of finite partial subgraphs of the initial graph is obtained. Each member is a tree, and $\mathfrak{G}_n \subseteq \mathfrak{G}_{n+1}$, for each $n \in \omega$. The union of this family is the desired spanning tree. The details of this line of reasoning are included below.

Lemma 2.1: Assume that $\mathfrak{G}$ is a connected, infinite and locally finite graph. Then, there is $\mathfrak{G}'$ such that $\mathfrak{G}'$ is a spanning tree for $\mathfrak{G}$.

Proof. Assume that $\mathfrak{G}$ is an infinite, locally finite, connected graph. Let $v_0 ... v_n ...$ be a well ordering of G of type ω. Reasoning proceeds by constructing an infinite sequence of trees $\{\mathfrak{G}_n ; n \in \omega\}$ such that:

(1) for all n≥ 0, $\mathfrak{G}_n \subseteq \mathfrak{G}_{n+1}$;
(2) for all n≥ 0, $\mathfrak{G}_n$ is a finite partial subgraph of $\mathfrak{G}$: and
(3) $G = \cup\{G_n : n \in \omega\}$.

Suppose that such such family of trees has been constructed. Let $\mathfrak{G}'$=$\cup\{\mathfrak{G}_n : n \in \omega\}$. Reasoning proceeds be showing that $\mathfrak{G}'$ is a spanning tree for $\mathfrak{G}$. First it is shown that $\mathfrak{G}'$ is a graph.

Suppose that $x \in G'$. Suppose further that $(x,\, x)$$\in \rho_{\mathfrak{G}'}$. Thus, there is $n \geq 0$ such that $x \in G_n$. Since $\mathfrak{G}_n \subseteq \mathfrak{G}'$, $(x,\, x)$$\in \rho_{\mathfrak{G}_n}$. Hence, $\mathfrak{G}_n$ is not a totally irreflexive graph. This contradicts the supposition that $\mathfrak{G}_n$ is a tree. Therefore, $(x,$ $x)$$\notin \rho_{\mathfrak{G}'}$; and $\rho_{\mathfrak{G}}$ is totally irreflexive.

Let x and y be in G'. Suppose that $(x, y) \in \rho_{\mathfrak{G}'}$. Thus, $x \neq y$ and there is n such that $(x, y) \in \rho_{\mathfrak{G}_n}$. Since $\mathfrak{G}_n$ is a graph, $(y, x) \in \rho_{\mathfrak{G}_n}$. Since $\mathfrak{G}_n \subseteq \mathfrak{G}'$. $(y, x) \in \rho_{\mathfrak{G}'}$. Hence, $\rho_{\mathfrak{G}'}$ is symmetric. Therefore, $\mathfrak{G}'$ is a graph.

To see that $\mathfrak{G}'$ is a partial subgraph of $\mathfrak{G}$, consider the following. By construction, $\cup\{G_n; n \in \omega\} \subseteq G$. Let x and y be members of G'. Suppose that $(x, y) \in \rho_{\mathfrak{G}'}$. There is n such that $x, y \in G_n$. Since $\mathfrak{G}_n \subseteq \mathfrak{G}'$, $(x, y) \in \rho_{\mathfrak{G}_n}$. As $\mathfrak{G}_n$ is a partial subgraph of $\mathfrak{G}$, $(x, y) \in \rho_{\mathfrak{G}}$. Therefore, $\mathfrak{G}'$ is a partial subgraph of $\mathfrak{G}$.

It remains to show that $\mathfrak{G}'$ is connected and acyclic. Let x and y be different members of G'. There is n such that x and y are in G_n. $\mathfrak{G}_n$ is connected. Thus, there is a path $z_1 ... z_n$ from x to y in $\mathfrak{G}_n$. As $\mathfrak{G}_n \subseteq \mathfrak{G}'$, $z_1 ... z_n$ is a path from x to y in $\mathfrak{G}'$. Hence, $\mathfrak{G}'$ is connected. Suppose that there is a cycle in $\mathfrak{G}'$. Let $x_1 x_2 ... x_m x_1$ be such a cycle. There is n such that $\{x_1, ..., x_m\} \subseteq G_n$. Since $\mathfrak{G}_n \subseteq \mathfrak{G}'$, $x_1 x_2 ... x_m x_1$ is a cycle in $\mathfrak{G}_n$. By supposition, $\mathfrak{G}_n$ is a tree and has no cycles. Recall that $\mathfrak{G}'$ is totally irreflexive. Hence $\mathfrak{G}'$ is acyclic and a spanning tree for $\mathfrak{G}$.

Thus, it suffices to construct the chain $\{\mathfrak{G}_n : n \in \omega\}$. $G_0 = \{v_0\}$ and $\rho_{\mathfrak{G}_0}$ is the null set. $\mathfrak{G}_0$ is a tree and a subgraph of $\mathfrak{G}$. Let

$$T_0 = \{y : y \in G - \{v_0\} \text{ and } (v_0, y) \in \rho_{\mathfrak{G}}\}.$$

$G_1 = G_0 \cup T_0$. Since $\mathfrak{G}$ is locally finite, T_0 is finite and G_1 is also finite. Further, $G_0 \cap T_0$ is empty. $\rho_{\mathfrak{G}_1} = \{(v_0, y) : y \in T_0\} \cup \{(y, v_0) : y \in T_0\}$. $\mathfrak{G}_1$ is a tree and a subgraph of $\mathfrak{G}$. $\mathfrak{G}_0$ is a subgraph of $\mathfrak{G}_1$. Suppose that $\mathfrak{G}_0$, $\mathfrak{G}_1$, ..., $\mathfrak{G}_n$ have been constructed such that:

(1) $\mathfrak{G}_0, \mathfrak{G}_1, \ldots, \mathfrak{G}_n$ are finite trees;
(2) $\mathfrak{G}_0, \mathfrak{G}_1, \ldots, \mathfrak{G}_n$ are partial subgraphs of $\mathfrak{G}$;and
(3) $\mathfrak{G}_0 \subseteq \mathfrak{G}_1 \subseteq .... \subseteq \mathfrak{G}_n$.

To construct $\mathfrak{G}_{n+1}$ such that :

(1) $\mathfrak{G}_{n+1}$ is a finite tree;
(2) $\mathfrak{G}_n \subseteq \mathfrak{G}_{n+1}$; and
(3) $\mathfrak{G}_{n+1}$ is a partial subgraph of $\mathfrak{G}$.

Let T_n be the following set:

$$\{y : y \in G - G_n, \exists v \in G_n \text{ and } (v, y) \in \rho_{\mathfrak{G}}\}$$

Since $\mathfrak{G}$ is locally finite and G_n is finite, T_n is finite. $T_n \cap G_n$ is empty. Let $G_{n+1} = G_n \cup T_n$. For $y \in T_n$, $v(y)$ is the least member of G_n (relative to the well ordering of G) such that $(v(y), y) \in \rho_{\mathfrak{G}}$. $\rho_{\mathfrak{G}_{n+1}} = \rho_{\mathfrak{G}_n} \cup \{(y, v(y)) : y \in T_n\} \cup \{(v(y), y) : y \in T_n\}$. Notice that $y \neq v(y)$. By construction, $\rho_{\mathfrak{G}_{n+1}} \subseteq \rho_{\mathfrak{G}}$ and G_{n+1} is a finite subset of G. It suffices to establish the following:

(1) that $\mathfrak{G}_{n+1}$ is a finite graph;
(2) that $\mathfrak{G}_{n+1}$ is a partial subgraph of $\mathfrak{G}$;
(3) that $\mathfrak{G}_n \subseteq \mathfrak{G}_{n+1}$; and
(4) that $\mathfrak{G}_{n+1}$ is a tree.

Notice that, G_{n+1} is the union of two finite sets. It is easily shown that $\rho_{\mathfrak{G}_{n+1}}$ is totally irreflexive. Let $x \in G_{n+1}$. Either $x \in G_n$ or $x \in T_n$. Suppose that $x \in G_n$. By supposition, $(x, x) \notin \rho_{\mathfrak{G}_n}$. By construction, the coordinates of those members of $\rho_{\mathfrak{G}_{n+1}}$ not in $\rho_{\mathfrak{G}_n}$ are different. Thus, $(x, x) \notin \rho_{\mathfrak{G}_{n+1}}$. Suppose that $x \in T_n$. Since G_n and T_n are disjoint, $x \notin G_n$, and $(x, x) \notin \rho_{\mathfrak{G}_n}$. By reasoning as in the first case, $(x, x) \notin \rho_{\mathfrak{G}_{n+1}}$. Therefore, $\rho_{\mathfrak{G}_{n+1}}$ is totally irreflexive. Now, consider showing that $\rho_{\mathfrak{G}_{n+1}}$ is symmetric. Let x and y be members of G_{n+1}. Suppose that $(x, y) \in \rho_{\mathfrak{G}_{n+1}}$.

By construction, either $(x, y) \in \rho_{\mathfrak{G}_n}$ or $(x, y)=(z, v(z))$, for $z \in T_n$ or $(x, y)=(v(z), z)$, for $z \in T_n$. In any case, $(y, x) \in \rho_{\mathfrak{G}_{n+1}}$. Hence, $\rho_{\mathfrak{G}_{n+1}}$ is symmetric and $\mathfrak{G}_{n+1}$ is a totally irreflexive graph.

By construction, $G_{n+1} \subseteq G$. Suppose that $(x, y) \in \rho_{\mathfrak{G}_{n+1}}$. We want to establish that $(x, y) \in \rho_{\mathfrak{G}}$. By supposition, either $(x, y) \in \rho_{\mathfrak{G}_n}$ or there is $z \in T_n$ such that $(x, y)=(z, v(z))$ or $(x, y)=(v(z), z)$. In the first case, by hypothesis, $(x, y) \in \rho_{\mathfrak{G}}$. By construction, both $(z, v(z))$ and $(v(z), z)$ are in $\rho_{\mathfrak{G}}$. Thus, in any case, $(x, y) \in \rho_{\mathfrak{G}}$; and, $\mathfrak{G}_{n+1}$ is a partial subgraph of $\mathfrak{G}$.

To establish that $\mathfrak{G}_n \subseteq \mathfrak{G}_{n+1}$, it suffices to show that for x and y in G_n, $(x, y) \in \rho_{\mathfrak{G}_n}$ iff $(x, y) \in \rho_{\mathfrak{G}_{n+1}}$. By construction, $\rho_{\mathfrak{G}_n} \subseteq \rho_{\mathfrak{G}_{n+1}}$. Suppose that $(x, y) \in \rho_{\mathfrak{G}_{n+1}}$. Thus, either $(x, y) \in \rho_{\mathfrak{G}_n}$ or either x or y are in T_n. Since T_n and G_n are disjoint, and both x and y are in G_n, the latter two cases are impossible. Thus, $\mathfrak{G}_n \subseteq \mathfrak{G}_{n+1}$.

Finally, we show that $\mathfrak{G}_{n+1}$ is a tree. G_{n+1} is the union of two finite disjoint sets. Let $x_1x_2...x_mx_1$ be a cycle in $\mathfrak{G}_{n+1}$, $m \geq 3$. Since, by hypothesis, $\mathfrak{G}_n$ is a tree, the above is not a cycle in $\mathfrak{G}_n$. Thus, at least one of the members of the above cycle are in G_{n+1}, but not in G_n. All such members are in T_n. By construction, no member, y, of T_n is adjacent in $\mathfrak{G}_{n+1}$ to only one member, $v(y)$, of G_{n+1}. However, each of x_1, ..., x_m is adjacent to at least two other members of G_{n+1}. Hence, there are no cycles in $\mathfrak{G}_{n+1}$. It remains to show that $\mathfrak{G}_{n+1}$ is connected. Let x and y be members of G_{n+1}. Suppose that $x \neq y$. recall that G_{n+1} is the union of G_n and T_n and that these sets are disjoint. There are four cases to consider:

(1) both x and y are in G_n;
(2) both x and y are in T_n;
(3) $x \in T_n$ and $y \in G_n$;
(4) $x \in G_n$ and $y \in T_n$.

In the first case, recall that by hypothesis $\mathfrak{G}_n$ is connected. Thus there is a path in $\mathfrak{G}_n$ between x and y. Since $\mathfrak{G}_n \subseteq \mathfrak{G}_{n+1}$, this path in $\mathfrak{G}_n$ is also a path in $\mathfrak{G}_{n+1}$.

In the second case, $xv(x)$ is a path between x and $v(x)$ in $\mathfrak{G}_{n+1}$ and $yv(y)$ is a path between y and $v(y)$ in $\mathfrak{G}_{n+1}$. Either $v(x) = v(y)$ or $v(x) \neq v(y)$. Suppose that $v(x) = v(y)$. $xv(x)y$ is a path between x and y in $\mathfrak{G}_{n+1}$. Suppose that $v(x) \neq v(y)$. By construction, $v(x)$ and $v(y)$ are in G_n. By hypothesis, $\mathfrak{G}_n$ is connected. Let $z_1...z_t$ be a path in $\mathfrak{G}_n$ between $v(x)$ and $v(y)$. Neither x nor y occur in this path. Since $\mathfrak{G}_n \subseteq \mathfrak{G}_{n+1}$, $z_1...z_t$ is a path in $\mathfrak{G}_{n+1}$. Without loss of generality, z_1=$v(x)$. Thus, $xz_1...z_t$ is a path between x and y in $\mathfrak{G}_{n+1}$.

Consider the third case: $x \in T_n$ and $y \in G_n$. $v(x) \in G_n$. Either $v(x) = y$ or $v(x) \neq y$. Suppose that $v(x) = y$. Then, $v(x)x$ is a path between y and x in $\mathfrak{G}_{n+1}$. Suppose that $v(x) \neq y$. Both $v(x)$ and y are in G_n. By hypothesis, $\mathfrak{G}_n$ is connected. Thus, there is $z_1...z_t$, a path in $\mathfrak{G}_n$ between y and $v(x)$. $x \notin \{z_1, ..., z_t\}$. Without loss of generality, $z_t = v(x)$. thus, $z_1...z_tx$ is a path in $\mathfrak{G}_{n+1}$ between y and x. The reasoning in the forth case is analogous. Hence, in any case there is a path in $\mathfrak{G}_{n+1}$ between x and y. Therefore, $\mathfrak{G}_{n+1}$ is connected.

There is one final detail that needs to be verified: $G = \cup\{G_n : n \in \omega\}$. By construction, $\cup\{G_n : n \in \omega\} \subseteq G$. Let x be any member of G. Suppose that $x = v_0$. Then, $x \in G_0$. Suppose that $x \neq v_0$. Since $\mathfrak{G}$ is connected, there is $z_0...z_t$, a path in $\mathfrak{G}$ between v_0 and x. Let $z_0 = v_0$. By construction, for all i, $0 \leq i \leq t$, $z_i \in G_i$.

□

Theorem 8.2.4 of Diestel [2005] (page 205) implies that every countable connected graph (hence, every infinite locally finite connected graph) has a spanning tree. This theorem follows from a characterization of the collection of graphs having a special class of spanning trees in Jung [1969]. The proof of Theorem 8.2.4 outlined by Diestel does not use results from Jung [1969]. Like the reasoning above, Diestel's proof constructs the spanning tree by taking the union of a chain of finite trees.

3. König's Infinity Lemma

There are several results in the literature referred to as 'the König Infinity Lemma' or 'the König Tree Lemma'. Some of these are formulated set theoretically (e.g. Berge [1958], König [1926], [1927]), some are formulated graph theoretically (e.g. König [1927], Beth [1955], [1959], Devlin [1993], Kuratowski and Mostowski [1976] and Wilson [1996]), and others are formulated number theoretically (e.g. Simpson [1999]) or combinatorially (e.g. Kaye [2007]). Among the results formulated in graph theory, there are those formulated for directed graphs (e.g. Devlin [1993], and Kuratowski and Mostowski [1976]) and those formulated for undirected graphs (e.g. Beth [1955], [1959], and König [1927]).

Franchella [1997] provides a wealth of information on the origins of König's Infinity Lemma. The translation of the versions of the lemma contained in König [1926] and [1927] presented below are from Franchella [1997]. König [1927] contains three statements of the lemma. The set theoretic version was presented as Lemma E in König [1926].

Lemma E: Let E_1, E_2, E_3,... be a denumerable sequence of finite, non empty sets, and R a relationship such that to each element x_{n+1} of E_{n+1}at least one element x_n of E_n corresponds (written $x_n R x_{n+1}$, where n=1, 2, ...). Then it is possible to choose from each E_n an element a_n such that $a_n R a_{n+1}$ (n=1, 2,...) holds for the infinite sequence S a_1, a_2, a_3,... .

In addition, there are two graph theoretic versions. Both involve undirected graphs.

The First Graph Theoretic Lemma: If a denumerable point set of an infinite graph splits into denumerably many sets E_1, E_2,.. which are finite and non-empty and such that every point of E_{n+1} (n=1,2,...) is joined with a point of E_n by an edge, then there is, in the graph, an infinite path a_1, a_2,... that contains from each set E_n a point a_n (here it is not necessary that the sets E_n 's be disjoint).

The Second Graph Theoretic Lemma: If every point of a connected infinite graph has only finitely many edges going into it, then the graph contains an infinite path.

4. Beth's Tree Theorem

Beth's Tree Theorem appears in section 69 of Beth [1959]. It had appeared earlier in Beth [1955]. It is clear from comments in both publications that Beth was aware of this theorem in 1954. There is evidence that Beth discovered this result independently. Beth was aware, by 1959, of König's Infinity Lemma. Both König [1927] and König [1936] appear among the references of Beth [1959]. Further, König's Infinity Lemma is mentioned twice in the text: page 262 and page 377. However, neither the Infinity Lemma nor König [1927] are mentioned in section 69.

Beth's definition of a tree was not the standard one from graph theory (compare Berge [1958], page 152). For Beth, a tree consists of points and edges between the points. Each point is assigned a unique positive integer called its *rank*. There is one point of rank 1, *the origin*. There are a finite number of points of each rank and each point of rank n+1 is connected with exactly one point of rank n. Points whose ranks are not consecutive are not connected by an edge. A *terrminal (end) point* is a point of rank n not connected to a point of rank n+1. *A branch* is a sequence of points of successive ranks, starting with the origin, such that each is connected to the point that immediately precedes it and which cannot be further extended without violating one of the first two conditions. Hence, branches can be finite or infinite. The notion of a tree presented by Beth is well suited as a representation of the semantic tableaux.

In graph theoretic terms, Beth's trees are locally finite, connected, acyclic, undirected graphs with an origin or root. The statement of the Tree Theorem in Beth [1955] is different, but equivalent to the statement in Beth [1959]. In the following the Tree Theorem is taken to be that of Beth [1959]. The statement of Beth [1955] is the following; given any tree, either it is finite and every branch is finite or the tree is infinite and there is an infinite branch.

Beth's Tree Theorem: A tree with infinitely many points contains at least one branch that is infinite.

A tree, in Beth's sense, is a tree in the graph theoretic sense that is locally finite and has an origin. However, since any node in a connected graph (in the graph theoretic sense) can be taken as the origin of the graph, trees in Beth's sense are exactly the locally finite, trees in the graph theoretic sense. It follows that Beth's Tree Theorem and König's Tree Lemma are equivalent, given the appropriate definitions of the two notions of what constitutes a tree.

The last paragraph of section 69 provides examples of disciplines in which trees play a part. There is no mention of graph theory in this paragraph. In the next to last paragraph of this section Beth repeats remarks form Beth [1955] that indicate the origins of the Tree Theorem and his proof of that theorem. He credits discussions with L. Henkin, A. Heyting and J.J de Iongh during a seminar on intuitionistic mathematics held in November of 1954 and later discussions with Henkin and P. Erdös.

The two references to König [1927] in Beth [1959] provide no indication that König's Infinity Lemma implies the Tree Theorem. There is no mention of a specific result. Rather, the discussion focuses on a method of proof or inference. On page 262 Beth formulates the satisfiability formulation of the compactness theorem (i.e. if every finite subset of a set has a model, then the set has a model) for classical sentential logic and observes on the next page that the method of proof for this theorem was pointed out in König [1927]. On page 377, Beth observes that "the conclusive character of the method of inference from the finite to the infinite" follows from the fact that every chain in a partially ordered set is included in a maximal chain in that partially ordered set.

König's proof of Lemma E can easily be extended to a proof of the Tree Theorem. However, Beth followed a different line of reasoning. König's proof of Lemma E in

1926 (see Franchella [1977], page 20) is a simple combinatorial argument, avoiding graph theoretic terminology. Let Γ be the collection of all finite sequences $a_1...a_n$ where $n\geq 1$ and for all i, between 1 and n, $a_i \in E_i$ and if i is strictly less than n, then $a_i R a_{i+1}$. It follows from the hypothesis of Lemma E that there is no bound on the length of the members of Γ. Hence, Γ is infinite. Let $a \in E_1$. Γ_a is the collection of all sequences in Γ of length at least 2 whose first member is a. Let $a_1...a_n \in \Gamma$. $\Gamma_{a_1...a_n}$ is the collection of all members of Γ of length at least n+1 whose first n members are, in order, $a_1...a_n$. Since E_1 is finite and Γ is infinite, there is $a \in \Gamma$ such that Γ_a is infinite. Further, since E_{n+1} is finite, if $\Gamma_{a_1...a_n}$ is infinite, then there is $b \in E_{n+1}$ such that $a_n R b$ and $\Gamma_{a_1...a_n b}$ is infinite. Let a_{n+1} be b. In this way the sequence $a_1...a_n...$ can be chosen that satisfies the conclusion of Lemma E.

There are minor differences between the proof in Beth [1955] and the proof in Beth [1959]. Both use terminology used in graph theory. In contrast to König's proof of Lemma E, Beth's proof of the Tree Theorem has an algebraic flavor; it constructs a subtree of the given infinite tree in which there are no terminal points. In such a tree, any point is the initial point of an infinite branch. Since the origin of the given tree is in this subtree, there is an infinite branch whose first point is the origin. Let $\mathfrak{T}$ be an infinite tree with origin 1. For each x, a point in $\mathfrak{T}$, let P(x) be the collection of all points in $\mathfrak{T}$ occurring in any branch in $\mathfrak{T}$ that includes the point x together with the point x. P(1) is the set of points in $\mathfrak{T}$. The point x *is of the second kind in* $\mathfrak{T}$ provided P(x) is an infinite set. Let S be the set of all points in $\mathfrak{T}$ of the second kind in $\mathfrak{T}$. Let $\mathfrak{S}$ be that subgraph of $\mathfrak{T}$ whose points are exactly the members of S. $1 \in S$. Further, $\mathfrak{S}$ is a tree whose origin is 1. Each branch in $\mathfrak{S}$ (finite or infinite) is a branch in $\mathfrak{T}$. As $\mathfrak{T}$ is infinite, there are no terminal points in $\mathfrak{S}$. Therefore, any point in $\mathfrak{S}$ is the a member of an infinite branch in $\mathfrak{S}$ that begins with 1, hence a member of an infinite branch in $\mathfrak{T}$ that begins with 1.

5. The Recognition of König's Contribution Among Logicians

Even in 1959, Beth appears not to have been aware that his Tree Theorem was a consequence of König's Infinity Lemma in any of its forms. By the mid 60's it was recognized among logicians that Beth's Tree Theorem was a consequence of König's Infinity Lemma. During this period Beth's Tree Theorem was referred to as 'König's Lemma' or' König's Tree Lemma'. Some care should be taken to understand the claim the Beth's Tree Theorem was referred to by one term or another. The claim is intended to indicate that the notion of tree used in the reference resembles the notion of a tree found in Beth as opposed to that notion of tree from graph theory. It is useful to contrast a *graphic/pictorial* notion of a tree in Beth with the *graph theoretic* notion of a tree in graph theory.

Mendelson [1964] (page 97) mentions a version of the Infinity Lemma from König [1927]. While the lemma was not stated explicitly, it was clear from the context that the focus was on graphs and not trees. In addition, Mendelson uses 'König's unendlichkeitslemma'. This terminology is found in König [1936] (page 81). It appears that the term was used by Mendelson to refer to The First Graph Theoretic Lemma from the 1927 paper. This 'unendlichkeitslemma' terminology also appears in Kreisel [1958]. Kreisel does not state the lemma, does not attribute it to König in either the text or the footnotes and there is no citation to König in the references. The lemma is mentioned in the course of a sketch of a proof of the strong completeness theorem for first order logic for sentences in prenex normal form. It is clear from the context that the application involves undirected trees with an origin. Kreisel indicates in a footnote that his exposition was suggested by Beth's tableaux. He cites Beth [1957], but does not cite Beth [1955]. This paper is very close to the explicit recognition that the Tree Theorem is a separate result that is a consequence of the Infinity Lemma.

Mostowski [1966] contains a passage (page 56) that explicitly indicates an awareness that Beth's Tree Theorem (in the form from Beth [1955]) is a consequence of "a well-known theorem due to König " . Mostowski sketched a proof of the strong completeness of first-order logic that "differs but inessentially from the one used by Beth". The trees considered here are undirected and have a origin. Interestingly, Mostowski did not state the König Infinity Lemma or mention either of the papers in which it appeared and these papers were not cited in the bibliography. Mostowski does not introduce terminology to distinguish between the Infinity Lemma and the Tree Theorem. And, he does not consider the question of the equivalence of the Infinity Lemma and the Tree Theorem. There is evidence that Mostowski was aware that Beth's Tree Theorem is a consequence of König's result as early as the summer of 1964. Mostowski [1966] was based on a series of lectures he gave in the summer of 1964 at the Vassa Summer School in Finland. Mostowski was encouraged by Professors Ketonen, Hintikka and von Wright to work on these lectures and publish them in *Acta Philosophia Fennica*.

It appears that after 1966 there is no clear terminological distinction between König's Infinity Lemma and König's Tree Lemma in the logic literature. Each of the following appear in the literature: 'König's Lemma', 'König's Infinity Lemma' and 'König's Tree Lemma'. However, these terms are not used uniformly. Kleene [1967] uses 'König's Lemma' to refer to a version of Beth's Tree Theorem for directed trees with origin. As in Mostowski [1966], this result is introduced as part of the proof of the strong completeness of first-order logic. Jeffery [1967] also uses 'König's Lemma' to refer to Beth's Tree Theorem (for undirected trees with origin). Here the lemma is introduced as part of a proof of the satisfiability formulation of the compactness theorem for classical sentential logic. Neither Beth nor König are cited among the references. The contrast with the graph theory literature is instructive. For example, in Diestel [2005] (page 245) 'König's Lemma' is used as a short hand for 'König's Infinity Lemma'.

'König's Infinity Lemma' appears in both Kuratowski and Mostowski [1968] and [1976]. In Kuratowski and Mostowski [1976] the term is used for a theorem about directed trees without an origin. In Kuratowski and Mostowski [1968] the term is used for a set theoretic formulation in the spirit of König's Lemma E. Thomson [1967] contains a related usage: 'König's Infinity Lemma on Trees' appears to refer to a special case of König's Infinity Lemma. However, there is no explicit indication of the notion of tree used here. It seems likely that the notion is graphic/pictorial rather than graph theoretical. There is no reference to either König or Beth in the paper.

While the author has not found the use of 'König's Lemma' to refer to Beth's Tree Theorem in the literature prior to 1967, there is evidence this terminology was used somewhat earlier. In 1965 Raymond Smullyan taught a course on recursive function theory at the University of Pennsylvania. In these lectures, which the author attended, Smullyan used 'König's Lemma' to refer to Beth's Tree Theorem (for undirected trees with origin). This result was used in a proof of completeness for Smullyan's tableau formulation of first-order logic. Smullyan's usage suggests the explicit recognition, prior to 1966, that Beth's Tree Theorem was a consequence of König's Infinity Lemma.

Hodges [1983] (page 63) uses 'König's Tree Lemma' to refer to Beth's Tree Theorem. Interestingly, the lemma is not stated and the exposition relies on a graphic/pictorial notion of a tree. Devlin [1993] (page 110) uses the same terminology for a version of Beth's theorem for directed trees. As in the case of 'König's Lemma', there is evidence that this terminology was used earlier. In 1966 John Corcoran taught a course on sentential logic at the University of Pennsylvania. In these lectures, which the author also attended, Corcoran used 'König's Tree Lemma' to refer to Beth's Tree Theorem.

6. Concluding Remarks

It has been argued above that Beth independently formulated and proved the Tree Theorem and that logicians were not aware that Beth's theorem was a consequence of König's Infinity Lemma until the mid 60's. While explicit mention of Beth's contribution appears to have disappeared from discussions in the logic literature, the graphic/pictoral notion of a tree dominates discussions of the tree result (now attributed to König) and its applications.

By 1966 König's Infinity Lemma was distinguished in the logic literature from Beth's Tree Theorem in the sense that Beth's Theorem was recognized as a consequence of König's Lemma. While there is evidence that this recognition occurred prior to 1966, the author has not found this recognition in the literature prior to 1966.

Was the question of whether Beth's Theorem implies König's Lemma raised in the logic literature? The author has found no such discussion. Without a clear terminological distinction in the logic literature between these results, the question would have been difficult to even formulate. By 1969 there was a proof of the Spanning Tree Theorem for the class of infinite connected, locally finite undirected graphs in the graph theory literature. The ease with which spanning trees can be constructed for the class of countable connected graphs by merely extending the construction for finite connected graphs (compare page 16 and page 205 of Diestel [2005]) suggests that proving the Infinity Lemma from König's Tree Lemma is not a hard problem, once one is familiar with spanning trees. Of course, it is unlikely that this problem would even arise in graph theory as it appears that König's Infinity Lemma was not distinguished from König's Tree Lemma in graph theory.

There appears to have been an obstacle to establishing the Infinity Lemma from Beth's Theorem along the lines of the proof of the Infinity Lemma from König's Tree Lemma. The introduction of spanning trees appears to be conceptually difficult without adopting some version of the graph theoretic notion of a tree. And it is precisely this move that seems to be missing from discussions of Beth's Tree Theorem and it applications in logic.

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