

Improved Upper Bounds for Finding Tarski Fixed Points

Xi Chen

Columbia University

xichen@cs.columbia.edu

Yuhao Li

Columbia University

yuhaoli@cs.columbia.edu

Abstract

We study the query complexity of finding a Tarski fixed point over the k -dimensional grid $\{1, \dots, n\}^k$. Improving on the previous best upper bound of $O(\log^{2\lceil k/3 \rceil} n)$ [FPS20], we give a new algorithm with query complexity $O(\log^{\lceil (k+1)/2 \rceil} n)$. This is based on a novel decomposition theorem about a weaker variant of the Tarski fixed point problem, where the input consists of a monotone function $f : [n]^k \rightarrow [n]^k$ and a monotone sign function $b : [n]^k \rightarrow \{-1, 0, 1\}$ and the goal is to find an $x \in [n]^k$ that satisfies *either* $f(x) \preceq x$ and $b(x) \leq 0$ *or* $f(x) \succeq x$ and $b(x) \geq 0$.

1 Introduction

In 1955, Tarski [Tar55] proved that every monotone¹ function $f : L \rightarrow L$ over a complete lattice (L, \preceq) has a fixed point, i.e. $x \in L$ with $f(x) = x$. Tarski's fixed point theorem has extensive applications in game theory and economics, where it has been used to establish the existence of important solution concepts such as pure equilibria in supermodular games [Top79, Top98, MR90]. As a byproduct, search problems for these solution concepts naturally reduce to the problem of finding Tarski fixed points, which motivates the investigation of its computational complexity. More compelling motivations for studying Tarski fixed points came from a recent work of Etessami, Papadimitriou, Rubinfeld and Yannakakis [EPRY20], where they discovered new connections of the Tarski fixed point problem with well studied complexity classes such as PPAD and PLS, as well as reductions from Condon's (Shapley's) stochastic games [Con92] to the Tarski fixed point problem. However, as discussed below, our current understanding of the complexity of Tarski fixed points remains rather limited. This is in sharp contrast with Brouwer's fixed point theorem [Pap94, CD08, CD09], the other fixed point theorem that played a major role in economics.

In this paper we study the query complexity of finding a Tarski fixed point in the complete lattice $([n]^k, \preceq)$ over the k -dimensional grid $[n]^k = \{1, \dots, n\}^k$ and equipped with the natural partial order over \mathbb{Z}^k , where $a \preceq b$ if $a_i \leq b_i$ for all $i \in [k]$. An algorithm under this model is given n and k and has query access to an unknown monotone function f over $[n]^k$. Each round the algorithm can send a query $x \in [n]^k$ to reveal $f(x)$ and the goal is to find a fixed point of f using as few queries as possible. We will refer to this problem as TARSKI(n, k).

Back in 2011, Dang, Qi, and Ye [DQY11] gave an $O(\log^k n)$ -query algorithm for TARSKI(n, k) when k is fixed. Their algorithm is based on a natural binary search strategy over coordinates. No progress had been made on the problem until recently. In [EPRY20], Etessami et al. showed that (among other results) the upper bound $O(\log^2 n)$ for TARSKI($n, 2$) [DQY11] is indeed tight, which suggested that the algorithm of Dang et al. might be optimal for all fixed k . However, surprisingly, Fearnley, Pálvölgyi and Savani [FPS20] recently showed that the algorithm of [DQY11] is not optimal by giving an algorithm for TARSKI(n, k) with $O(\log^{2\lceil k/3 \rceil} n)$ queries.

Our contribution. Our main result is an improved upper bound for the complexity of TARSKI:

Theorem 1. *For any fixed k , there is an $O(\log^{\lceil (k+1)/2 \rceil} n)$ -query algorithm for TARSKI(n, k).*

Our algorithm is based on a new variant of the Tarski fixed point problem which we refer to as TARSKI*. It is inspired by the $O(\log^2 n)$ -query algorithm of [FPS20] for TARSKI($n, 3$) (its inner algorithm in particular). Our main contribution is a novel decomposition theorem for TARSKI*, which leads to a more efficient recursive scheme for performing binary search on coordinates of the grid. We discuss the variant TARSKI* and its decomposition theorem next.

1.1 Sketch of the Algorithm

The algorithm of [FPS20] is obtained by combining an $O(\log^2 n)$ -query algorithm for TARSKI($n, 3$) and a decomposition theorem. Their algorithm for TARSKI($n, 3$) consists of an outer algorithm and an $O(\log n)$ -query inner algorithm. Given $f : [n]^3 \rightarrow [n]^3$ as the input function, the outer algorithm starts by running the inner algorithm to solve the following problem:

¹We say f is monotone if $f(a) \preceq f(b)$ whenever $a \preceq b$.

- Find a point $x \in [n]^3$ with $x_3 = \lfloor n/2 \rfloor$ such that x is either prefixed ($f(x) \preceq x$) or postfix ($x \preceq f(x)$). Note that even though we focus on a layer of the grid (with $x_3 = \lfloor n/2 \rfloor$), the condition on x being either prefixed or postfix is about all three dimensions.

Once such a point x is found, the outer algorithm can shrink the search space significantly by only considering $\mathcal{L}_{x,(n,n,n)}$ if x is postfix, or $\mathcal{L}_{(1,1,1),x}$ if x is prefixed, where we write $\mathcal{L}_{a,b}$ to denote the grid with points $c : a \preceq c \preceq b$. In both cases we obtain a grid $\mathcal{L}_{a,b}$ such that $a \preceq b$, $a \preceq f(a)$ and $f(b) \preceq b$. These conditions together guarantee that f maps $\mathcal{L}_{a,b}$ to itself and f has a fixed point in $\mathcal{L}_{a,b}$ (see [Lemma 1](#)). Given that the side length of a dimension goes down by a factor of 2 after each call to the inner algorithm, it takes no more than $O(\log n)$ calls to reduce the search space to a grid $\mathcal{L}_{a,b}$ with $b_i - a_i \leq 1$ and then a fixed point can be found by brute force. The query complexity of the overall algorithm of [\[FPS20\]](#) for $\text{TARSKI}(n, 3)$ is $O(\log^2 n)$.

After obtaining the $O(\log^2 n)$ -query algorithm for $\text{TARSKI}(n, 3)$, [\[FPS20\]](#) uses it to solve higher dimensional TARSKI by proving a *decomposition theorem*: if $\text{TARSKI}(n, a)$ can be solved in $q(n, a)$ queries and $\text{TARSKI}(n, b)$ can be solved in $q(n, b)$ queries, then $\text{TARSKI}(n, a + b)$ can be solved in $O(q(n, a) \cdot q(n, b))$ queries. Combined with the $O(\log^2 n)$ -query algorithm for $\text{TARSKI}(n, 3)$, they obtain an $O(\log^{2\lceil k/3 \rceil} n)$ -query algorithm for $\text{TARSKI}(n, k)$.

Our key idea is to develop a new decomposition theorem directly on the problem solved by the inner algorithm of [\[FPS20\]](#), and only apply the outer algorithm at the very end. More formally we refer to the following problem as $\text{TARSKI}^*(n, k)$:²

- Given a monotone function $f : [n]^{k+1} \rightarrow [n]^{k+1}$, find a point x with $x_{k+1} = \lfloor n/2 \rfloor$ such that x is either prefixed or postfix. As mentioned earlier, the condition on x being either prefixed or postfix is about all $k + 1$ dimensions.

Similar to the outer algorithm of [\[FPS20\]](#), any algorithm for $\text{TARSKI}^*(n, k)$ can be used as a subroutine to solve $\text{TARSKI}(n, k + 1)$ with an $O(\log n)$ -factor overhead (see [Lemma 2](#)).

The main technical contribution of this work is the proof of a new decomposition theorem for TARSKI^* : if $\text{TARSKI}^*(n, a)$ can be solved in $q(n, a)$ queries and $\text{TARSKI}^*(n, b)$ can be solved in $q(n, b)$ queries, then $\text{TARSKI}^*(n, a + b)$ can be solved in $O(q(n, a) \cdot q(n, b))$ queries. Now despite sharing the same statement/recursion, the proof of our decomposition theorem requires a number of new technical ingredients compared to that of [\[FPS20\]](#). This is mainly due to the extra coordinate (i.e., coordinate $k + 1$) that appears in TARSKI^* but not in the original TARSKI .

One obstacle is that the solution found by TARSKI^* appears to be too weak to directly prove the new decomposition theorem. In particular, if one gets a postfix point $x \preceq f(x)$ as a solution to $\text{TARSKI}^*(n, k + 1)$, both $x_{k+1} = f(x)_{k+1}$ or $x_{k+1} < f(x)_{k+1}$ could happen, and this uncertainty would cause the proof strategy adopted by [\[FPS20\]](#) to fail. Instead we introduce a stronger variant of TARSKI^* called REFINEDTARSKI^* (see [Definition 3](#)) which poses further conditions on its solution regarding coordinate $k + 1$. Given the same input, REFINEDTARSKI^* asks for two points $p^\ell \preceq p^r$ with $p_{k+1}^\ell = p_{k+1}^r = \lfloor n/2 \rfloor$ such that p^ℓ is postfix in the first k coordinates, p^r is prefixed in the first k coordinates, and one of the following three conditions hold:

1. $p_{k+1}^\ell < f(p^\ell)_{k+1}$;
2. $p_{k+1}^r > f(p^r)_{k+1}$; or

²Note that our formal definition in [Section 3](#) will look different; the problems they capture are the same though.

$$3. f(p^\ell)_{k+1} - p_{k+1}^\ell = f(p^r)_{k+1} - p_{k+1}^r = 0.$$

While REFINEDTARSKI* looks much stronger than TARSKI*, surprisingly we show in [Lemma 4](#) that it can be solved by a small number of calls to TARSKI*. With REFINEDTARSKI* as the bridge, we are able to prove the new decomposition theorem and obtain the improved bound for TARSKI.

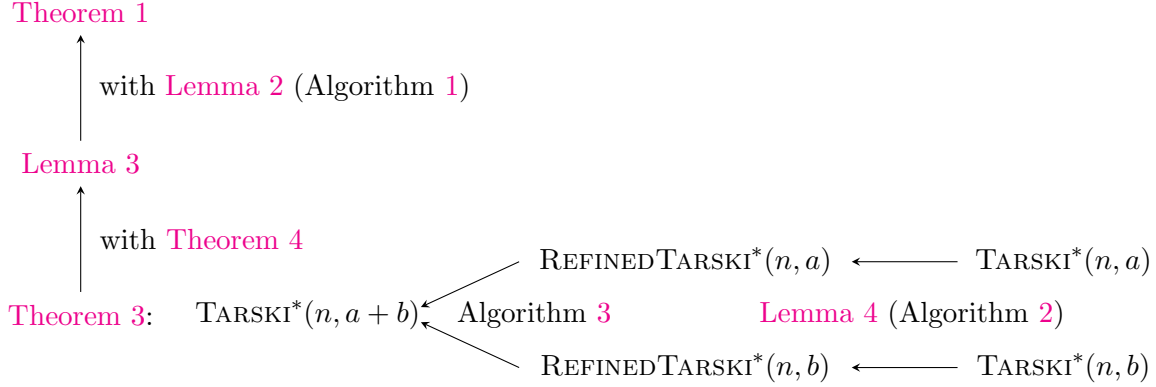


Figure 1: A Proof Sketch

2 Preliminaries

We start with the definition of *monotone* functions and state Tarski's fixed point theorem:

Definition 1 (Monotone functions). *Let (\mathcal{L}, \preceq) be a complete lattice. A function $f : \mathcal{L} \rightarrow \mathcal{L}$ is said to be monotone if $f(a) \preceq f(b)$ for all $a, b \in \mathcal{L}$ with $a \preceq b$.*

Theorem 2 (Tarski). *For any complete lattice (\mathcal{L}, \preceq) and any monotone function $f : \mathcal{L} \rightarrow \mathcal{L}$, there must be a point $x_0 \in \mathcal{L}$ such that $f(x_0) = x_0$, i.e., x_0 is a fixed point.*

In this paper we work on the query complexity of $\text{TARSKI}(n, k)$, i.e., the problem of finding a Tarski fixed point over a k -dimensional grid $([n]^k, \preceq)$, where $[n]$ denotes $\{1, 2, \dots, n\}$ and \preceq denotes the natural partial order over \mathbb{Z}^k : $a \preceq b$ if and only if $a_i \leq b_i$ for every $i \in [k]$. For $a, b \in \mathbb{Z}^k$ with $a \preceq b$, we write $\mathcal{L}_{a,b}$ to denote the set of points $x \in \mathbb{Z}^k$ with $a \preceq x \preceq b$. A point $x \in [n]^k$ is called a *prefixed* point of f if $f(x) \preceq x$; a point $x \in [n]^k$ is called a *postfixed* point of f if $x \preceq f(x)$.

Let $S \subseteq \mathbb{Z}^k$ be a finite set of points. A point $p \in \mathbb{Z}^k$ is an *upper bound* of S if $x \preceq p$ for all $x \in S$. We say p is the *least upper bound* of S if p is an upper bound of S and $p \preceq q$ for every upper bound q of S (i.e., $p_i = \max_{x \in S} x_i$ for all $i \in [k]$). Similarly, a point $p \in \mathbb{Z}^k$ is a lower bound of S if $p \preceq x$ for all $x \in S$. We say p is the *greatest lower bound* of S if p is a lower bound of S and $q \preceq p$ for every lower bound q of S (i.e., $p_i = \min_{x \in S} x_i$ for all $i \in [k]$). We write $\text{LUB}(S)$ and $\text{GLB}(S)$ to denote the least upper bound and the greatest lower bound of S , respectively.

We record the following simple fact:

Fact 1. *Let finite $S, T \subseteq \mathbb{Z}^k$ be such that $x \preceq y$ for all $x \in S, y \in T$. Then $\text{LUB}(S) \preceq \text{GLB}(T)$.*

We include a proof of the following simple lemma for completeness:

Lemma 1. *Let $f : [n]^k \rightarrow [n]^k$ be a monotone function. Suppose $\ell, r \in [n]^k$ satisfy $\ell \preceq r$, $\ell \preceq f(\ell)$ and $f(r) \preceq r$. Then f maps $\mathcal{L}_{\ell,r}$ to itself and has a fixed point in $\mathcal{L}_{\ell,r}$.*

Proof. For any $x \in \mathcal{L}_{\ell,r}$, we have from $\ell \preceq x \preceq r$ that

$$\ell \preceq f(\ell) \preceq f(x) \preceq f(r) \preceq r$$

and thus, $f(x) \in \mathcal{L}_{\ell,r}$. The existence of a fixed point in $\mathcal{L}_{\ell,r}$ follows directly from Tarski's fixed point theorem applied on f over $\mathcal{L}_{\ell,r}$. \square

3 Reduction to Tarski*

For convenience we focus on $\text{TARSKI}(n, k+1)$. Our algorithm for $\text{TARSKI}(n, k+1)$ (see Algorithm 1) over a monotone function $f : [n]^{k+1} \rightarrow [n]^{k+1}$ will first set

$$\ell = 1^{k+1} := (1, \dots, 1) \quad \text{and} \quad r = n^{k+1} := (n, \dots, n)$$

and then proceed to find a point $x \in [n]^{k+1}$ with $x_{k+1} = \lfloor n/2 \rfloor$ that is either prefixed ($f(x) \preceq x$) or postfix ($x \preceq f(x)$). Note that such a point x must exist since by Tarski's fixed point theorem, there must be a point x with $x_{k+1} = \lfloor n/2 \rfloor$ such that $f(x)_i = x_i$ for all $i \in [k]$, and such a point must be either prefixed or postfix; on the other hand, it is crucial that the algorithm is not required to find an x with $f(x)_i = x_i$ for all $i \in [k]$ but just an x that is either prefixed or postfix. After finding x , the algorithm replaces r by x if x is prefixed, or ℓ by x if x is postfix. It follows from Lemma 1 that f remains a monotone function from $\mathcal{L}_{\ell,r}$ to itself but one of the $k+1$ dimensions gets cut by one half. The algorithm recurses on $\mathcal{L}_{\ell,r}$.

The key subproblem is to find such a point x with $x_{k+1} = \lfloor n/2 \rfloor$ that is either prefixed or postfix, which we formulate as the following problem called $\text{TARSKI}^*(n, k)$:

Definition 2 ($\text{TARSKI}^*(n, k)$). *Given oracle access to a function $g : [n]^k \rightarrow \{-1, 0, 1\}^{k+1}$ satisfying*

- *For all $x \in [n]^k$ and $i \in [k]$, we have $x_i + g(x)_i \in [n]$; and*
- *For all $x, y \in [n]^k$ with $x \preceq y$, we have $(x, 0) + g(x) \preceq (y, 0) + g(y)$,*

find a point $x \in [n]^k$ such that either $g(x)_i \leq 0$ for all $i \in [k+1]$ or $g(x)_i \geq 0$ for all $i \in [k+1]$.

To see the connection between $\text{TARSKI}^*(n, k)$ and the subproblem described earlier, one can define $g : [n]^k \rightarrow \{-1, 0, 1\}^{k+1}$ using $f : [n]^{k+1} \rightarrow [n]^{k+1}$ by letting, for each $x \in [n]^k$,

$$g(x)_{k+1} = \text{sgn}(f(x, \lfloor n/2 \rfloor)_{k+1} - \lfloor n/2 \rfloor) \quad \text{and} \quad g(x)_i = \text{sgn}(f(x, \lfloor n/2 \rfloor)_i - x_i)$$

for each $i \in [k]$. On the one hand, it is easy to verify that g satisfies both conditions in Definition 2 when f is monotone. On the other hand, every $x \in [n]^k$ with $\{-1, 1\} \not\subseteq \bigcup_{i \in [k+1]} g(x)_i$ must satisfy that $(x, \lfloor n/2 \rfloor)$ is either prefixed or postfix in f .

The next lemma shows how to use an algorithm for $\text{TARSKI}^*(n, k)$ to solve $\text{TARSKI}(n, k+1)$.

Lemma 2. *If $\text{TARSKI}^*(n, k)$ can be solved in $q(n, k)$ queries, then $\text{TARSKI}(n, k+1)$ can be solved in $O(2^k + k \log n \cdot q(n, k))$ queries.*

Proof. Suppose that \mathcal{A} is an algorithm for $\text{TARSKI}^*(n, k)$ with $q(n, k)$ queries. We present Algorithm 1 and show that it can solve $\text{TARSKI}(n, k+1)$ in $O(2^k + k \log n \cdot q(n, k))$ queries.

Correctness. The proof of correctness is based on the observation that $\ell \preceq r$, $\ell \preceq f(\ell)$ and $f(r) \preceq r$ at the beginning of each while loop, which we prove below by induction. The basis is

Algorithm 1: Algorithm for $\text{TARSKI}(n, k+1)$ via a reduction to $\text{TARSKI}^*(n, k)$

Input: Oracle access to a monotone function $f : [n]^{k+1} \rightarrow [n]^{k+1}$.

Output: A fixed point $x \in [n]^{k+1}$ of f with $f(x) = x$.

- 1 Let \mathcal{A} be an algorithm for $\text{TARSKI}^*(n, k)$. Let $\ell = 1^{k+1}$ and $r = n^{k+1}$.
 - 2 **while** $|r - \ell|_\infty > 2$ **do**
 - 3 Pick an $i \in [k+1]$ with $r_i - \ell_i > 2$ and let

$$L = (\ell_1, \dots, \ell_{i-1}, \ell_{i+1}, \dots, \ell_{k+1}) \quad \text{and} \quad R = (r_1, \dots, r_{i-1}, r_{i+1}, \dots, r_{k+1}).$$
 - 4 Define a new function $g : \mathcal{L}_{L,R} \rightarrow \{-1, 0, 1\}^{k+1}$ as follows:

$$g(x) := (s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_{k+1}, s_i)$$

where $s_j := \text{sgn}(f(x')_j - x'_j)$ and $x' = (x_1, \dots, x_{i-1}, \lfloor (\ell_i + r_i)/2 \rfloor, x_i, \dots, x_k)$.
 - 5 Run algorithm \mathcal{A} on g to find a point $q \in \mathcal{L}_{L,R}$ with $q_i = \lfloor (\ell_i + r_i)/2 \rfloor$ that is either prefixed or postfix; set $r = q$ if q is prefixed and set $\ell = q$ if q is postfix.
 - 6 Brute-force search $\mathcal{L}_{\ell,r}$ to find a fixed point and return it.
-

trivial. For the induction step, assume that it holds at the beginning of the current while loop. Then f maps $\mathcal{L}_{\ell,r}$ to itself and thus, g satisfies both conditions in [Definition 2](#). As a result, \mathcal{A} can be used to find a point q that is either prefixed or postfix in f . (Formally, one needs to embed g over $\mathcal{L}_{L,R}$ in the subgrid $\mathcal{L}_{1^k, R-L}$ of $[n]^k$ and define $g' : [n]^k \rightarrow \{-1, 0, 1\}^{k+1}$ such that solving $\text{TARSKI}^*(n, k)$ on g' gives us q .) The way ℓ or r is updated at the end of the loop makes sure that the statement holds at the beginning of the next loop.

The last line of the algorithm makes sure that it returns a fixed point at the end.

Query complexity. Each while loop of Algorithm 1 costs $q(n, k)$ queries. After each loop, the side length of a dimension goes down by a factor of 2. So there are no more than $O(k \log n)$ rounds and thus, the query complexity of Algorithm 1 is $O(2^k + k \log n \cdot q(n, k))$. \square

We prove the following upper bound for solving TARSKI^* in the next section.

Lemma 3. *There is an $O(\log^{[k/2]} n)$ -query algorithm for $\text{TARSKI}^*(n, k)$.*

4 A Decomposition Theorem for Tarski*

The proof of [Lemma 3](#) uses the following decomposition theorem for TARSKI^* :

Theorem 3. *If $\text{TARSKI}^*(n, a)$ can be solved in $q(n, a)$ queries and $\text{TARSKI}^*(n, b)$ can be solved in $q(n, b)$ queries, then $\text{TARSKI}^*(n, a+b)$ can be solved in $O((b+1) \cdot q(n, a) \cdot q(n, b))$ queries.*

We prove [Theorem 3](#) in the rest of this section. We also note that the algorithm of [\[FPS20\]](#) can be used to solve the 2-dimensional TARSKI^* (see Theorem 14 in [\[FPS20\]](#)), even though they didn't define TARSKI^* formally in the paper. This leads to the following theorem about TARSKI^* :

Theorem 4 ([\[FPS20\]](#)). *There is an $O(\log n)$ -query algorithm for $\text{TARSKI}^*(n, 2)$.*

[Lemma 3](#) follows directly by combining [Theorem 3](#) and [Theorem 4](#).

4.1 A refined version of Tarski*

We start the proof of our new decomposition theorem ([Theorem 3](#)). To this end we first introduce a refined version of TARSKI*.

Definition 3 (REFINEDTARSKI*(n, k)). *Given a function $g : [n]^k \rightarrow \{-1, 0, 1\}^{k+1}$ satisfying*

- *For all $x \in [n]^k$ and $i \in [k]$, we have $x_i + g(x)_i \in [n]$; and*
- *For all $x, y \in [n]^k$ with $x \preceq y$, we have $(x, 0) + g(x) \preceq (y, 0) + g(y)$,*

find a pair of points $p^\ell, p^r \in [n]^k$ such that $p^\ell \preceq p^r$,

$$g(p^\ell)_t \geq 0 \quad \text{and} \quad g(p^r)_t \leq 0, \quad \text{for all } t \in [k]$$

and one of the following conditions meets

1. $g(p^\ell)_{k+1} = 1$;
2. $g(p^r)_{k+1} = -1$; or
3. $g(p^\ell)_{k+1} = g(p^r)_{k+1} = 0$.

We note that any solution p^ℓ, p^r of REFINEDTARSKI* would imply a solution of corresponding TARSKI* problem directly by returning either p^ℓ or p^r . The following lemma shows that, in fact, these two problems are computationally equivalent in their query complexity.

Lemma 4. *If TARSKI*(n, k) can be solved in $q(n, k)$ queries, then REFINEDTARSKI*(n, k) can be solved in $O(q(n, k))$ queries.*

Proof. Suppose that \mathcal{A} is an algorithm to solve TARSKI*(n, k) with $q(n, k)$ queries. We show that Algorithm 2 will solve REFINEDTARSKI* in $O(q(n, k))$ queries.

Correctness. It is easy to verify that g^+ over $[n]^k$ satisfies both conditions of [Definition 3](#). So the point p^* returned by algorithm \mathcal{A} on line 3 is either a prefixed or a postfix point of g^+ . If $g(p^*)_{k+1} = 1$, then p^ℓ will be updated to p^* . By the definition of TARSKI* we have $g(p^*)_t \geq 0$ for all $t \in [k]$, which means p^ℓ, p^r will meet the first condition of REFINEDTARSKI*. When $g(p^*)_{k+1} = -1$ p^r will be updated to p^* and p^ℓ, p^r will meet the second condition of REFINEDTARSKI*.

Now we can assume $g(p^*)_{k+1} = 0$. So p^ℓ is updated to p^* and p^r remains n^k . Note that p^* is a solution of TARSKI* under g^+ and $g^+(p^*)_{k+1} = 1$ (because $g(p^*)_{k+1} = 0$). So $g^+(p^*)_t \geq 0$ for all $t \in [k+1]$ and thus, $g(p^\ell)_t \geq 0$ for all $t \in [k]$.

Consider the point q^* returned by algorithm \mathcal{A} on g^- over $\mathcal{L}_{p^\ell, p^r}$ on line 7. If $g(q^*)_{k+1} = 1$, then p^ℓ will be updated to q^* and p^ℓ, p^r will meet the first condition of REFINEDTARSKI*. Otherwise, we have $g(q^*)_{k+1} \leq 0$. Since $p^\ell \preceq q^*$, by the second property of function g , we know $0 = g(p^\ell)_{k+1} \leq g(q^*)_{k+1} \leq 0$, i.e., $g(q^*)_{k+1} = 0$. With the definition of g^- , we know that $g^-(q^*) = -1$. Note that q^* is a solution to TARSKI* on g^- , so we have $g^-(q^*)_t \leq 0$ for all $t \in [k+1]$. In this case, p^r will be updated as q^* , so p^ℓ, p^r will meet the third condition of REFINEDTARSKI*.

Query Complexity. Algorithm 2 just calls the algorithm \mathcal{A} at most two times on line 3 and line 7, so the query complexity of Algorithm 2 is $O(q(n, k))$. \square

Now we are ready to prove [Theorem 3](#).

Algorithm 2: Algorithm for $\text{REFINEDTARSKI}^*(n, k)$ via a reduction to $\text{TARSKI}^*(n, k)$

Input: Oracle access to $g : [n]^k \rightarrow \{-1, 0, 1\}^{k+1}$ that satisfies conditions in [Definition 3](#).

Output: A solution to $\text{REFINEDTARSKI}^*(n, k)$ on g .

- 1 Let \mathcal{A} be an algorithm for $\text{TARSKI}^*(n, k)$. Let $p^\ell = 1^k$ and $p^r = n^k$.
- 2 Construct a new function $g^+ : [n]^k \rightarrow \{-1, 0, 1\}^k \times \{-1, 1\}$ as follows:

$$\begin{cases} g^+(x)_i = g(x)_i, & \text{for all } i \in [k] \\ \text{If } g(x)_{k+1} \geq 0, \text{ then } g^+(x)_{k+1} = 1; \text{ if } g(x)_{k+1} = -1, \text{ then } g^+(x)_{k+1} = -1 \end{cases}$$

- 3 Run algorithm \mathcal{A} to find a solution p^* to $\text{TARSKI}^*(n, k)$ on g^+ over $[n]^k$.
- 4 If $g^+(p^*)_{k+1} = 1$, set $p^\ell \leftarrow p^*$; if $g^+(p^*)_{k+1} = -1$, set $p^r \leftarrow p^*$.
- 5 If $g(p^*)_{k+1} \neq 0$, **return** the pair of points p^ℓ, p^r .
- 6 Construct a new function $g^- : [n]^k \rightarrow \{-1, 0, 1\}^k \times \{-1, 1\}$ as follows:

$$\begin{cases} g^-(x)_i = g(x)_i, & \text{for all } i \in [k] \\ \text{If } g(x)_{k+1} \leq 0, \text{ then } g^-(x)_{k+1} = -1; \text{ if } g(x)_{k+1} = 1, \text{ then } g^-(x)_{k+1} = 1 \end{cases}$$

- 7 Run algorithm \mathcal{A} to find a solution q^* to $\text{TARSKI}^*(n, k)$ on g^- over $\mathcal{L}_{p^\ell, p^r}$. (This can be done by embedding g^- over $\mathcal{L}_{p^\ell, p^r}$ inside $[n]^k$ and running \mathcal{A} .)
 - 8 If $g^-(q^*)_{k+1} = 1$, set $p^\ell \leftarrow q^*$; if $g^-(q^*)_{k+1} = -1$, set $p^r \leftarrow q^*$.
 - 9 **return** the pair of points p^ℓ, p^r .
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Proof of [Theorem 3](#). Suppose \mathcal{A} is a query algorithm to solve $\text{TARSKI}^*(n, a)$ in $q(n, a)$ queries and \mathcal{B} is a query algorithm to solve $\text{TARSKI}^*(n, b)$ in $q(n, b)$ queries. We will show that [Algorithm 3](#) can solve $\text{TARSKI}^*(n, a + b)$ in $O((b + 1) \cdot q(n, a) \cdot q(n, b))$ queries.

Overview. At a high level, [Algorithm 3](#) will run \mathcal{B} for $\text{TARSKI}^*(n, b)$ on a function $h : [n]^b \rightarrow \{-1, 0, 1\}^{b+1}$ built on the go using the input $g : [n]^{a+b} \rightarrow \{-1, 0, 1\}^{a+b+1}$ of $\text{TARSKI}^*(n, a + b)$ (that satisfies conditions in [Definition 2](#)). Let $q^1, \dots, q^{i-1} \in [n]^b$ be the $i - 1$ queries that \mathcal{B} has made so far, for some $i \geq 1$, and let $r^1, \dots, r^{i-1} \in \{-1, 0, 1\}^{b+1}$ be the query results on h . Let $q^i \in [n]^b$ be the new query made by \mathcal{B} in the i th round. Our challenge is to use g (its restriction on points with the last b coordinates being q^i) to come up with an $r^i \in \{-1, 0, 1\}^{b+1}$ as the answer $h(q^i)$ to the query such that

1. **Lemma 7:** All results $(q^1, r^1), \dots, (q^i, r^i)$ are consistent with conditions of [Definition 2](#), i.e., $q_i^j + r_i^j \in [n]$ for all $i \in [b]$ and $(q^j, 0) + r^j \preceq (q^{j'}, 0) + r^{j'}$ for all j, j' with $q^j \preceq q^{j'}$; and
2. **Lemma 8:** When q^i is a solution to $\text{TARSKI}^*(n, b + 1)$ on h , i.e., either $r_t^i \geq 0$ for all $t \in [b + 1]$ or $r_t^i \leq 0$ for all $t \in [b + 1]$, we can use q^i to obtain a solution to $\text{TARSKI}^*(n, a + b)$ on the original input function g .

To obtain r^i , we need to run \mathcal{A} for $b + 1$ times to obtain a pair of points $p^{(\ell, i)}, p^{(r, i)} \in [n]^a$ and use $g(p^{(\ell, i)}, q^i)$ and $g(p^{(r, i)}, q^i)$ to determine r^i . A crucial component in the computation of $p^{(\ell, i)}$ and $p^{(r, i)}$ is to initialize the search space using pairs $p^{(\ell, j)}, p^{(r, j)}$, $j \in [i - 1]$, from previous rounds.

Correctness. We prove a sequence of lemmas about [Algorithm 3](#):

Algorithm 3: Algorithm for $\text{TARSKI}^*(n, a + b)$ via $\text{TARSKI}^*(n, a)$ and $\text{TARSKI}^*(n, b)$

Input: Oracle access to $g : [n]^{a+b} \rightarrow \{-1, 0, 1\}^{a+b+1}$ satisfying conditions in [Definition 2](#).

Output: A solution to $\text{TARSKI}^*(n, a + b)$ on g .

1 Let \mathcal{A} be an algorithm for $\text{TARSKI}^*(n, a)$ and \mathcal{B} be an algorithm for $\text{TARSKI}^*(n, b)$.

2 Let $i \leftarrow 1$ be the round number.

3 **do**

4 For each previous round $k \in [i - 1]$, let $q^k \in [n]^b$ be the point queried by \mathcal{B} and $r^k \in \{-1, 0, 1\}^{b+1}$ be the answer.

5 Given the sequence $((q^1, r^1), \dots, (q^{i-1}, r^{i-1}))$, let $q^i \in [n]^b$ be the i th query of \mathcal{B} .

6 Set (when $i = 1$, set $p^{(\ell,1)} = 1^a$ and $p^{(r,1)} = n^a$)

$$p^{(\ell,i)} \leftarrow \text{LUB} \left(\left\{ p^{(\ell,k)} : k \in [i - 1] \text{ and } q^k \preceq q^i \right\} \right) \quad \text{and}$$

$$p^{(r,i)} \leftarrow \text{GLB} \left(\left\{ p^{(r,k)} : k \in [i - 1] \text{ and } q^i \preceq q^k \right\} \right)$$

7 **for** each j from $a + 1$ to $a + b + 1$ **do**

8 Define a new function $g_j : [n]^a \rightarrow \{-1, 0, 1\}^{a+1}$ as follows:

$$g_j(x) = \left(g(x, q^i)_1, \dots, g(x, q^i)_a, g(x, q^i)_j \right), \quad \text{for every } x \in [n]^a.$$

9 Run Algorithm [2](#) with \mathcal{A} to find a solution $p^{(\ell,*)}, p^{(r,*)}$ to $\text{REFINEDTARSKI}^*(n, a)$ on g_j over $\mathcal{L}_{p^{(\ell,i)}, p^{(r,i)}}$; set $p^{(\ell,i)} \leftarrow p^{(\ell,*)}$ and $p^{(r,i)} \leftarrow p^{(r,*)}$.

10 Construct $r^i \in \{-1, 0, 1\}^{b+1}$ as the query result to q^i :

$$r_{t-a}^i = g(p^{(\ell,i)}, q^i)_t, \quad \text{for each } t \in [a + b + 1] \setminus [a].$$

11 If $r_t^i \geq 0$ for all $t \in [b + 1]$, **return** $(p^{(\ell,i)}, q^i)$.

12 If $r_t^i \leq 0$ for all $t \in [b + 1]$, **return** $(p^{(r,i)}, q^i)$.

13 **while**;

Lemma 5. At the end of each round i , we have $p^{(\ell,i)} \preceq p^{(r,i)}$ and

$$g(p^{(\ell,i)}, q^i)_t \geq 0 \quad \text{and} \quad g(p^{(r,i)}, q^i)_t \leq 0, \quad \text{for all } t \in [a]. \quad (1)$$

Proof. We start with the base case for the first round. We have $p^{(\ell,1)} = 1^a \preceq n^a = p^{(r,1)}$ at the beginning so [Equation \(1\)](#) holds. It is easy to prove by induction that both $p^{(\ell,1)} \preceq p^{(r,1)}$ and [Equation \(1\)](#) hold at the beginning of each for loop on line 7, g_j over $\mathcal{L}_{p^{(\ell,1)}, p^{(r,1)}}$ satisfies conditions of TARSKI^* during the for loop and thus, both $p^{(\ell,1)} \preceq p^{(r,1)}$ and [Equation \(1\)](#) hold at the end of the for loop. This shows that both of them hold at the end of the first main loop.

The induction step is similar. Assume that both conditions hold for $p^{(\ell,j)}, p^{(r,j)}$ for $j \in [i - 1]$. We start by showing that $p^{(\ell,i)}, p^{(r,i)}$ on line 6 satisfy both $p^{(\ell,i)} \preceq p^{(r,i)}$ and [Equation \(1\)](#).

To prove $p^{(\ell,i)} \preceq p^{(r,i)}$, we make the following observation: $p^{(\ell,j_1)} \preceq p^{(r,j_2)}$ for all $j_1, j_2 \in [i - 1]$ with $q^{j_1} \preceq q^{j_2}$. We divide the proof into three cases:

Case 0: $j_1 = j_2 = j$. Trivially follow from $p^{(\ell,j)} \preceq p^{(r,j)}$.

Case 1: $j_1 < j_2$. By the inductive hypothesis, we know $p^{(\ell, j_2)} \preceq p^{(r, j_2)}$. Considering the while loop j_2 , by the definition on line 6 and $q^{j_1} \preceq q^{j_2}$, we know $p^{(\ell, j_1)} \preceq p^{(\ell, j_2)}$ before the loop on line 7. Furthermore, by the updating rule on line 9, we know that $p^{(\ell, j_2)}$ is monotonically non-decreasing, which means $p^{(\ell, j_1)} \preceq p^{(\ell, j_2)}$ holds after while loop j_2 . So we can derive that after while loop j_2 , $p^{(\ell, j_1)} \preceq p^{(r, j_2)}$.

Case 2: $j_1 > j_2$. By the inductive hypothesis, we know $p^{(\ell, j_1)} \preceq p^{(r, j_1)}$. Considering the while loop j_1 , by the definition on line 6 and $q^{j_1} \preceq q^{j_2}$, we know $p^{(r, j_1)} \preceq p^{(r, j_2)}$ before the loop on line 7. Furthermore, by the updating rule on line 9, we know that $p^{(r, j_1)}$ is monotonically non-increasing, which means $p^{(r, j_1)} \preceq p^{(r, j_2)}$ holds after while loop j_1 . So we can derive that after while loop j_1 , $p^{(\ell, j_1)} \preceq p^{(r, j_2)}$.

Now we move back to our proof of $p^{(\ell, i)} \preceq p^{(r, i)}$. For every $j_1, j_2 \in [i - 1]$ such that $q^{j_1} \preceq q^i \preceq q^{j_2}$, we know that $p^{(\ell, j_1)} \preceq p^{(r, j_2)}$. So the same partial order relation of the least upper bound of $p^{(\ell, j_1)}$ and the greatest lower bound of $p^{(r, j_2)}$ also holds, i.e., $p^{(\ell, i)} \preceq p^{(r, i)}$ before the loop on line 7.

Next we prove Equation (1) on line 6. For each $t \in [a]$, given that $p^{(\ell, i)}$ is the LUB, there must exist $j^* \in [i - 1]$ such that

$$q^{j^*} \preceq q^i, \quad p^{(\ell, j^*)} \preceq p^{(\ell, i)} \quad \text{and} \quad p_t^{(\ell, j^*)} = p_t^{(\ell, i)}.$$

Since $(p^{(\ell, j^*)}, q^{j^*}) \preceq (p^{(\ell, i)}, q^i)$, we have

$$p_t^{(\ell, j^*)} + g(p^{(\ell, j^*)}, q^{j^*})_t \preceq p_t^{(\ell, i)} + g(p^{(\ell, i)}, q^i)_t,$$

which implies $g(p^{(\ell, j^*)}, q^{j^*})_t \leq g(p^{(\ell, i)}, q^i)_t$. On the other hand, we have $g(p^{(\ell, j^*)}, q^{j^*})_t \geq 0$ by the inductive hypothesis. So $g(p^{(\ell, i)}, q^i)_t \geq 0$. $g(p^{(r, i)}, q^i)_t \leq 0$ can be proved similarly.

Given that both $p^{(\ell, i)} \preceq p^{(r, i)}$ and Equation (1) hold at the beginning of the loop on line 7, the rest of the proof is essentially same as the proof in the base case. It is easy to prove by induction that both $p^{(\ell, i)} \preceq p^{(r, i)}$ and Equation (1) hold at the beginning of each for loop on line 7, g_j over $\mathcal{L}_{p^{(\ell, i)}, p^{(r, i)}}$ satisfies conditions of TARSKI* during the for loop and thus, both $p^{(\ell, i)} \preceq p^{(r, i)}$ and Equation (1) hold at the end of the for loop. This shows that both of them hold at the end of the main while loop.

This completes the induction and the proof of the lemma. \square

Lemma 6. *At the end of every round i , we have $g(p_1, q^i)_j = g(p_2, q^i)_t$ for all $p_1, p_2 \in \mathcal{L}_{p^{(\ell, i)}, p^{(r, i)}}$ and all $t \in [a + b + 1] \setminus [a]$,*

Proof. Consider the end of round t on line 7. If $g(p^{(\ell, *)}, q^i)_t = 1$, then for every $p^{(\ell, *)} \preceq p \preceq p^{(r, *)}$, we have $1 = g(p^{(\ell, *)}, q^i)_t \leq g(p, q^i)_t$, i.e., $g(p, q^i)_t = 1$. Similarly, if $g(p^{(r, *)}, q^i)_t = -1$, then we have $g(p, q^i)_t = -1$ for every $p^{(\ell, *)} \preceq p \preceq p^{(r, *)}$. For the last case of $g(p^{(\ell, *)}, q^i)_t = g(p^{(r, *)}, q^i)_t = 0$, for every $p^{(\ell, *)} \preceq p \preceq p^{(r, *)}$, we have $0 = g(p^{(\ell, *)}, q^i)_t \leq g(p, q^i)_t \leq g(p^{(r, *)}, q^i)_t = 0$, i.e., $g(p, q^i)_t = 0$.

For subsequent round $j + 1, j + 2, \dots$, we know that $\mathcal{L}_{p^{(\ell, i)}, p^{(r, i)}}$ can only shrink. So the property remains. This finishes the proof of the lemma. \square

We are now ready to prove the two lemmas needed for the correctness of Algorithm 3:

Lemma 7. *For any two rounds j_1 and j_2 , if $q^{j_1} \preceq q^{j_2}$, then $(q^{j_1}, 0) + r^{j_1} \preceq (q^{j_2}, 0) + r^{j_2}$.*

Proof. We consider two cases when $j_1 < j_2$ and when $j_1 > j_2$.

Case 1: $j_1 < j_2$. Considering the round j_2 , by the definition on line 6 and $q^{j_1} \preceq q^{j_2}$, we know that $p^{(\ell, j_1)} \preceq p^{(\ell, j_2)}$ before the loop on line 7. In the loop on line 7, when $p^{(\ell, j_2)}$ is updated by $p^{(\ell, *)}$, $p^{(\ell, j_2)}$ is monotonically non-decreasing. So at the end of the loop we still have $p^{(\ell, j_1)} \preceq p^{(\ell, j_2)}$. Given that $(p^{(\ell, j_1)}, q^{j_1}) \preceq (p^{(\ell, j_2)}, q^{j_2})$, we have for every $t \in [a + b + 1] \setminus [a]$:

$$q_{t-a}^{j_1} + r_{t-a}^{j_1} = q_{t-a}^{j_1} + g(p^{(\ell, j_1)}, q^{j_1})_t \leq q_{t-a}^{j_2} + g(p^{(\ell, j_2)}, q^{j_2})_t = q_{t-a}^{j_2} + r_{t-a}^{j_2}.$$

Case 2: $j_1 > j_2$. Case 2 is analogous to Case 1. Considering the round j_1 , by the definition on line 6 and $q^{j_1} \preceq q^{j_2}$, we know that $p^{(r, j_1)} \preceq p^{(r, j_2)}$ before the loop on line 7. In the loop on line 7, when $p^{(r, j_1)}$ is updated by $p^{(r, *)}$, $p^{(r, j_1)}$ is monotonically non-increasing, so that $\forall t \in [a + b + 1] \setminus [a]$, $g(p^{(r, j_1)}, q^{j_1})_t$ is non-decreasing. So we have $p^{(r, j_1)} \preceq p^{(r, j_2)}$ at the end of the loop on line 7. Using $(p^{(r, j_1)}, q^{j_1}) \preceq (p^{(r, j_2)}, q^{j_2})$ and Lemma 6, we have for every $t \in [a + b + 1] \setminus [a]$:

$$\begin{aligned} q_{t-a}^{j_2} + r_{t-a}^{j_2} &= q_{t-a}^{j_2} + g(p^{(\ell, j_2)}, q^{j_2})_t = q_{t-a}^{j_2} + g(p^{(r, j_2)}, q^{j_2})_t \\ &\geq q_{t-a}^{j_1} + g(p^{(r, j_1)}, q^{j_1})_t = q_{t-a}^{j_1} + g(p^{(\ell, j_1)}, q^{j_1})_t = q_{t-a}^{j_1} + r_{t-a}^{j_1}. \end{aligned}$$

This completes the proof of the lemma. \square

Lemma 8. *At the end of each round i , if $r_t^i \geq 0$ for $t \in [b + 1]$, then $(p^{(\ell, i)}, q^i)$ is a solution to $\text{TARSKI}^*(n, a + b)$ on g ; if $r_t^i \leq 0$ for $t \in [b + 1]$, then $(p^{(r, i)}, q^i)$ is a solution to $\text{TARSKI}^*(n, a + b)$ on g .*

Proof. By Lemma 5 we have $g(p^{(\ell, i)}, q^i)_t \geq 0$ and $g(p^{(r, i)}, q^i)_t \leq 0$ for all $t \in [a]$. So if $r_t^i \geq 0$ for all $t \in [b + 1]$, then $g(p^{(\ell, i)}, q^i)_t = r_{t-a}^i \geq 0$ for all $t \in [a + b + 1] \setminus [a]$ and thus, $(p^{(\ell, i)}, q^i)$ is a solution to $\text{TARSKI}^*(n, a + b)$ on g . Similarly if $r_t^i \leq 0$ for all $t \in [b + 1]$, then $(p^{(r, i)}, q^i)$ is a solution to $\text{TARSKI}^*(n, a + b)$ on g . \square

Query complexity. For each round of Algorithm 3, Algorithm 2 is called $b + 1$ times on line 9 and each call of Algorithm 2 will use $O(q(n, a))$ queries. The outer algorithm \mathcal{B} has no more than $q(n, b)$ rounds, which means the query complexity of Algorithm 3 is $O((b + 1) \cdot q(n, a) \cdot q(n, b))$. \square

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