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Polynomial Control Systems: Invariant Sets given by Algebraic Equations/Inequations

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Abstract: Consider a nonlinear input-affine control system $\dot{x}(t) = f(x(t)) + g(x(t))u(t)$, $y(t) = h(x(t))$, where f, g, h are polynomial functions. Let S be a set given by algebraic equations and inequations (in the sense of $\neq$). Such sets appear, for instance, in the theory of the Thomas decomposition, which is used to write a variety as a disjoint union of simpler subsets. The set S is called controlled invariant if there exists a polynomial state feedback law $u(t) = \alpha(x(t))$ such that S is an invariant set of the closed loop system $\dot{x} = (f + g\alpha)(x)$. If it is possible to achieve this goal with a polynomial output feedback law $u(t) = \beta(y(t))$, then S is called controlled and conditioned invariant. These properties are discussed and algebraically characterized, and algorithms are provided for checking them with symbolic computation methods.

Keywords: Nonlinear control systems, Multivariable polynomials, Invariance, State feedback, Output feedback, Algebraic systems theory, Computational methods.

1. INTRODUCTION

The notions of controlled and conditioned invariance go back to Basile and Marro (1992). We study a nonlinear version of these concepts. Let $\mathbb{K}$ denote the field of real or complex numbers and let $\mathcal{P} = \mathbb{K}[x_1, \dots, x_n]$ denote the polynomial ring in n variables over $\mathbb{K}$. Let $f \in \mathcal{P}^n$, $g \in \mathcal{P}^{n\times m}$, and $h \in \mathcal{P}^r$ be given. An input-affine polynomial control system takes the form $\dot{x}(t) = f(x(t)) + g(x(t))u(t)$, $y(t) = h(x(t))$. A set $M \subseteq \mathbb{K}^n$ is called *controlled invariant* if there exists a state feedback law $u(t) = \alpha(x(t))$ with $\alpha \in \mathcal{P}^m$ such that the closed loop system $\dot{x} = (f + g\alpha)(x)$ has M as an invariant set, that is, the solution of the initial value problem with $x(0) = x_0 \in M$ remains within M for all times in the existence interval. If there exists an output feedback law $u(t) = \beta(y(t)) = \beta(h(x(t)))$ such that M is an invariant set of $\dot{x} = (f + g(\beta \circ h))(x)$, then M is called *controlled and conditioned invariant*. The case where M is a variety (that is, the solution set of algebraic equations) has been studied in Christopher et al. (2009); Zerz et al. (2010); Zerz and Walcher (2012); Schilli et al. (2014, 2016). The set $A \subseteq \mathcal{P}^m$ of all feedback laws making a variety invariant has the structure of an affine module, which can be determined by symbolic computation methods (Gröbner bases). For controlled and conditioned invariance, A has to be intersected with $\mathbb{K}[h]^m$, where $\mathbb{K}[h] = \mathbb{K}[h_1, \dots, h_r]$ denotes the subalgebra of $\mathcal{P}$ generated by $h_1, \dots, h_r$. The present paper addresses the case where M is given by algebraic equations and inequations (in the sense of $\neq$). Such sets appear in the theory of the Thomas decomposition, see e.g. Bächler et al. (2012).

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2. SETS GIVEN BY ALGEBRAIC (IN-)EQUATIONS

The results of this section are easy consequences of the contents of (Cox et al., 1992, Ch. 4, §4).

Let $p_1, \dots, p_k, q \in \mathcal{P}$ be given. Consider

$$S = \{x \in \mathbb{K}^n \mid p_1(x) = \dots = p_k(x) = 0, q(x) \neq 0\}. \quad (1)$$

There is no loss of generality in assuming that there is only one inequation (since $q_j(x) \neq 0$ for $1 \leq j \leq l$ is equivalent to $q(x) \neq 0$, where $q = q_1 \cdot \ldots \cdot q_l$). Let $I \subseteq \mathcal{P}$ be the ideal generated by $p_1, \dots, p_k$. If $J \subseteq \mathcal{P}$ is another ideal, the ideal quotient

$$(I : J) := \{p \in \mathcal{P} \mid pJ \subseteq I\}$$

is an ideal in $\mathcal{P}$ that contains I. If J is the principal ideal generated by q, we write

$$(I : q) := (I : \langle q\rangle) = \{p \in \mathcal{P} \mid pq \in I\}.$$

This leads to an ideal chain

$$I \subseteq (I : q) \subseteq (I : q^2) \subseteq \dots,$$

which must become stationary, since $\mathcal{P}$ is Noetherian. The largest ideal in this chain is called the saturation of I by q, and is denoted by

$$(I : q^\infty) = \{p \in \mathcal{P} \mid \exists e \in \mathbb{N} : pq^e \in I\}.$$

We first observe that S is not changed if I is replaced by $(I : q^\infty)$.

Lemma 1. Let $I \subseteq \mathcal{P}$ be an ideal and let $q \in \mathcal{P}$ be a polynomial. The following are equivalent for any $x \in \mathbb{K}^n$:

(1) $p(x) = 0$ for all $p \in I$ and $q(x) \neq 0$,
(2) $p(x) = 0$ for all $p \in (I : q^\infty)$ and $q(x) \neq 0$.

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Proof: Since $(I : q^\infty) \supseteq I$, the implication "(2) $\Rightarrow$ (1)" is clear. For "(1) $\Rightarrow$ (2)", let $p \in (I : q^\infty)$ be given. Then $pq^e \in I$ for some nonnegative integer e, and thus $p(x)q(x)^e = 0$. Since $q(x) \neq 0$, we may conclude that $p(x) = 0$ as desired. □

Due to the lemma, we may replace the given generators $p_1, \ldots, p_k$ of I by generators $\tilde{p}_1, \ldots, \tilde{p}_{\tilde{k}}$ of $(I : q^\infty)$. In the following, we will always assume, without loss of generality, that this step has already been performed and that the ideal I generated by the polynomials defining the equations satisfies $(I : q) = I$, where q is the polynomial defining the inequation.

Another important condition which we would like to impose on I is $I = \mathcal{J}(\mathcal{V}(I))$, where

$$\mathcal{V}(I) = \{x \in \mathbb{K}^n \mid p(x) = 0 \text{ for all } p \in I\}$$

and

$$\mathcal{J}(M) = \{p \in \mathcal{P} \mid p(x) = 0 \text{ for all } x \in M\}$$

for arbitrary subsets $I \subseteq \mathcal{P}$ and $M \subseteq \mathbb{K}^n$, respectively. Note that $\mathcal{V}$ and $\mathcal{J}$ are inclusion-reversing and that $I \subseteq \mathcal{J}(\mathcal{V}(I))$ and $M \subseteq \mathcal{V}(\mathcal{J}(M))$ hold for all I, M. This implies that $\mathcal{J} \circ \mathcal{V} \circ \mathcal{J} \equiv \mathcal{J}$ and $\mathcal{V} \circ \mathcal{J} \circ \mathcal{V} \equiv \mathcal{V}$. The following lemma shows that the property $I = \mathcal{J}(\mathcal{V}(I))$ is not destroyed by the transition from I to $(I : q^\infty)$.

Lemma 2. Let $I \subseteq \mathcal{P}$ be an ideal and let $q \in \mathcal{P}$ be a polynomial. If $\mathcal{J}(\mathcal{V}(I)) = I$ holds, then we have

$$\mathcal{J}(\mathcal{V}(I : q)) = (I : q).$$

By induction, this implies that $\mathcal{J}(\mathcal{V}(I : q^\infty)) = (I : q^\infty)$.

Proof: We will use the inclusion

$$\mathcal{V}(I : q) \supseteq \mathcal{V}(I) \setminus \mathcal{V}(q) \tag{2}$$

and the identity

$$(\,\mathcal{J}(\mathcal{V}(I)) : \mathcal{J}(\mathcal{V}(q))\,) = \mathcal{J}(\mathcal{V}(I) \setminus \mathcal{V}(q)) \tag{3}$$

which are standard results, for instance, see (Cox et al., 1992, Ch. 4, §4). The inclusion $(I : q) \subseteq \mathcal{J}(\mathcal{V}(I : q))$ is obvious. For the converse, let $p \in \mathcal{P}$ be such that $p(x) = 0$ for all $x \in \mathcal{V}(I : q) \supseteq \mathcal{V}(I) \setminus \mathcal{V}(q)$. Then

$$\begin{aligned} p \in \mathcal{J}(\mathcal{V}(I) \setminus \mathcal{V}(q)) &= (\,\mathcal{J}(\mathcal{V}(I)) : \mathcal{J}(\mathcal{V}(q))\,) \\ &= (\,I : \mathcal{J}(\mathcal{V}(q))\,) \subseteq (I : q), \end{aligned}$$

which concludes the proof. □

Summing up, we will assume $(I : q) = I = \mathcal{J}(\mathcal{V}(I))$ from now on.

3. AUTONOMOUS CASE

Consider the autonomous system $\dot{x} = F(x)$ with $F \in \mathcal{P}^n$. Let $M \subseteq \mathbb{K}^n$ be an arbitrary subset. We say that M is F-invariant if the solution of the initial value problem with $x(0) = x_0 \in M$ remains within M, that is, $x(t) \in M$ holds for all t in the existence interval. For $p \in \mathcal{P}$, let

$$L_F(p) := \sum_{i=1}^{n} \frac{\partial p}{\partial x_i} F_i$$

denote the Lie derivative of p along F.

Lemma 3. (1) If $M \subseteq \mathbb{K}^n$ is F-invariant, then we have

$$L_F(p) \in \mathcal{J}(M) \text{ for all } p \in \mathcal{J}(M).$$

(2) Let $I \subseteq \mathcal{P}$ be an ideal and let $V = \mathcal{V}(I)$ be a variety. Then V is F-invariant if and only if

$$L_F(p) \in \mathcal{J}(V) \text{ for all } p \in \mathcal{J}(V).$$

Proof: Let $\varphi(t, x_0)$ denote the solution of the initial value problem $\dot{x} = F(x)$, $x(0) = x_0$ at time $t \in T(x_0)$, where $0 \in T(x_0) \subseteq \mathbb{R}$ is the existence interval of the solution.

(1) Let M be F-invariant and let $p \in \mathcal{J}(M)$ be given. Then we have

$$p(\varphi(t, x_0)) = 0$$

for all $x_0 \in M$ and $t \in T(x_0)$. Differentiation yields

$$\sum_{i=1}^{n} \frac{\partial p}{\partial x_i} F_i(\varphi(t, x_0)) = 0,$$

and plugging in $t = 0$, we get $L_F(p)(x_0) = 0$ for all $x_0 \in M$. Thus $L_F(p) \in \mathcal{J}(M)$.

(2) Only the "if" part needs to be proven. Let $\mathcal{J}(V) = \langle p_1, \ldots, p_k \rangle$ for some $p_i \in \mathcal{P}$. By assumption, we have $L_F(p_i) = \sum_{j=1}^{k} a_{ij} p_j$ for some $a_{ij} \in \mathcal{P}$. Let $x_0 \in V$ be given. Consider $y_i(t) := p_i(\varphi(t, x_0))$. Differentiation yields

$$\dot{y}_i(t) = L_F(p_i)(\varphi(t, x_0)) = \sum_{j=1}^{k} a_{ij}(\varphi(t, x_0)) y_j(t).$$

Set $A_{ij}(t) := a_{ij}(\varphi(t, x_0))$. Then $\dot{y}(t) = A(t)y(t)$ and $y(0) = 0$, which implies that $y \equiv 0$. Hence $\varphi(t, x_0) \in \mathcal{V}(\mathcal{J}(V)) = \mathcal{V}(\mathcal{J}(\mathcal{V}(I))) = \mathcal{V}(I) = V$ for all $t \in T(x_0)$, where we have used $\mathcal{V} \circ \mathcal{J} \circ \mathcal{V} \equiv \mathcal{V}$. Since $x_0 \in V$ was arbitrary, this implies that V is F-invariant. □

Lemma 4. The Zariski closure $\overline{M}^Z := \mathcal{V}(\mathcal{J}(M))$ of a set $M \subseteq \mathbb{K}^n$ is F-invariant if and only if $L_F(p) \in \mathcal{J}(M)$ holds for all $p \in \mathcal{J}(M)$.

Proof: "Only if": Let $\overline{M}^Z$ be F-invariant. According to part (1) of Lemma 3, we have $L_F(p) \in \mathcal{J}(\overline{M}^Z)$ for all $p \in \mathcal{J}(\overline{M}^Z)$. Since $\mathcal{J} \circ \mathcal{V} \circ \mathcal{J} \equiv \mathcal{J}$, we obtain that $L_F(p) \in \mathcal{J}(M)$ for all $p \in \mathcal{J}(M)$.

"If": Suppose that $L_F(p) \in \mathcal{J}(M)$ for all $p \in \mathcal{J}(M)$. Using $\mathcal{J} \circ \mathcal{V} \circ \mathcal{J} \equiv \mathcal{J}$ and part (2) of Lemma 3, we conclude that $V := \mathcal{V}(\mathcal{J}(M)) = \overline{M}^Z$ is F-invariant. □

Combining the previous two lemmas, we have the following immediate consequence.

Theorem 5. If M is F-invariant, then so is its Zariski closure $\overline{M}^Z = \mathcal{V}(\mathcal{J}(M))$.

Now we return to semi-algebraic sets $S \subseteq \mathbb{K}^n$ as in Equation (1). Under the (nonrestrictive) assumptions discussed in Section 2, the F-invariance of S can be characterized in terms of the F-invariance of two varieties associated to S as follows.

Theorem 6. Let $F \in \mathcal{P}^n$ and $p_1, \dots, p_k, q \in \mathcal{P}$ be given. Without loss of generality, assume that the ideal $I := \langle p_1, \dots, p_k \rangle$ satisfies $(I : q) = I = \mathcal{J}(\mathcal{V}(I))$. Then the following are equivalent:

(1) $S = \mathcal{V}(I) \setminus \mathcal{V}(q)$ is F-invariant,
(2) $\mathcal{V}(I)$ and $\mathcal{V}(I) \cap \mathcal{V}(q)$ are F-invariant.

Proof: We first note that the complement of an F-invariant set is F-invariant, and that the intersection of two F-invariant sets is F-invariant. For $M \subseteq \mathbb{K}^n$, let $\complement M := \mathbb{K}^n \setminus M$ denote its complement.

"(1) $\Rightarrow$ (2)": Using Equation (3) and $(I : q) = I = \mathcal{J}(\mathcal{V}(I))$, we have

$$\mathcal{J}(S) = \mathcal{J}(\mathcal{V}(I) \setminus \mathcal{V}(q)) = (\,\mathcal{J}(\mathcal{V}(I)) : \mathcal{J}(\mathcal{V}(q))\,) = (\,I : \mathcal{J}(\mathcal{V}(q))\,) \subseteq (I : q) = I.$$

Since $I = \mathcal{J}(\mathcal{V}(I)) \subseteq \mathcal{J}(S)$, we may conclude that $\mathcal{J}(S) = I$. Thus

$$\overline{S}^Z = \mathcal{V}(\mathcal{J}(S)) = \mathcal{V}(I).$$

Using Theorem 5, we obtain the F-invariance of $\mathcal{V}(I)$. On the other hand, the complement of S, which is given by

$$\complement S = \mathbb{K}^n \setminus (\mathcal{V}(I) \cap \complement\mathcal{V}(q)) = (\,\mathbb{K}^n \setminus \mathcal{V}(I)\,) \cup \mathcal{V}(q),$$

is F-invariant as well. Therefore, also the intersection

$$\mathcal{V}(I) \cap \complement S = \mathcal{V}(I) \cap \mathcal{V}(q)$$

is F-invariant.

"(2) $\Rightarrow$ (1)": The complement of $\mathcal{V}(I) \cap \mathcal{V}(q)$ is given by

$$M := \complement(\mathcal{V}(I) \cap \mathcal{V}(q)) = \complement\mathcal{V}(I) \cup \complement\mathcal{V}(q)$$

and it is F-invariant. Thus the intersection

$$\mathcal{V}(I) \cap M = \mathcal{V}(I) \cap \complement\mathcal{V}(q) = S$$

is also F-invariant. □

Corollary 7. In the situation of Theorem 6, let $\mathcal{M}$ denote the set of all vector fields $F \in \mathcal{P}^n$ such that S is F-invariant. Define $J := I + \langle q \rangle$. Then S is F-invariant if and only if both $\mathcal{V}(I)$ and $\mathcal{V}(J) = \mathcal{V}(I) \cap \mathcal{V}(q)$ are F-invariant. Thus we have $F \in \mathcal{M}$ if and only if

$$L_F(p) \in I \text{ for all } p \in I, \text{ and}$$
$$L_F(\bar{q}) \in \mathcal{J}(\mathcal{V}(J)) \text{ for all } \bar{q} \in \mathcal{J}(\mathcal{V}(J)).$$

In particular, $\mathcal{M}$ is a submodule of $\mathcal{P}^n$.

Assume that $\mathcal{J}(\mathcal{V}(J)) = \langle p_1, \dots, p_k, q_1, \dots, q_l \rangle$ for suitable polynomials $q_1, \dots, q_l \in \mathcal{P}$. Then the module $\mathcal{M}$ can be determined as follows: First, compute the kernel $\mathcal{N}$ of the matrix

$$\begin{bmatrix} \dfrac{\partial p}{\partial x}\,, & p_1 I_k\,, \ \dots\,, \ p_k I_k & & \\ \dfrac{\partial q}{\partial x}\,, & & p_1 I_l\,, \ \dots\,, \ p_k I_l\,, & q_1 I_l\,, \ \dots\,, \ q_l I_l \end{bmatrix}$$

where $\frac{\partial p}{\partial x} \in \mathcal{P}^{k \times n}$, $\frac{\partial q}{\partial x} \in \mathcal{P}^{l \times n}$ denote the Jacobians of $p = (p_1, \dots, p_k)^T$, $q = (q_1, \dots, q_l)^T$, and I_s is the identity matrix of size s. Then $\mathcal{M}$ is the projection of $\mathcal{N}$ onto the first n components.

4. CONTROLLED INVARIANCE

We return to control systems

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t), \tag{4}$$

where $f \in \mathcal{P}^n$, $g \in \mathcal{P}^{n \times m}$ are given. Let $p_1, \dots, p_k, q \in \mathcal{P}$ be given and consider S as in Equation (1). Let $\mathcal{M}$ be the module of all vector fields $F \in \mathcal{P}^n$ such that S is F-invariant. Then S is controlled invariant if and only if there exists $\alpha \in \mathcal{P}^m$ such that $f + g\alpha \in \mathcal{M}$. Such an α will be called an admissible feedback law (for f, g, and S).

Corollary 8. The set S is controlled invariant for the system from Equation (4) if and only if

$$f \in \mathcal{M} + \operatorname{im}(g).$$

If S is controlled invariant, then the set A of all $\alpha \in \mathcal{P}^m$ such that $f + g\alpha \in \mathcal{M}$ has the structure of an affine module, that is, $A = \alpha^* + A_0$, where α^* is a particular element of A, and

$$A_0 = \{\alpha \in \mathcal{P}^m \mid g\alpha \in \mathcal{M}\}$$

is a submodule of $\mathcal{P}^m$.

Example: Let $p = x_1^2 + x_2^2 - 1, q = x_3^2 - 1 \in \mathbb{R}[x_1, x_2, x_3]$. Then S is a cylinder with two circles taken out. Using the computer algebra system SINGULAR, see Greuel and Pfister (2008) and Decker et al. (2015), we compute the module $\mathcal{M}$ of all vector fields F such that S is F-invariant, which is generated by the columns of the matrix

$$\begin{bmatrix} p & 0 & 0 & x_2 & 0 \\ 0 & p & 0 & -x_1 & 0 \\ 0 & 0 & p & 0 & q \end{bmatrix}.$$

Now let

$$f = \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \\ x_3 & 0 \end{bmatrix}.$$

Then S is controlled invariant and the set of all admissible feedback laws α is given by

$$A = \begin{bmatrix} -x_1x_3 \\ -x_1x_3 - 1 \end{bmatrix} + \operatorname{im} \begin{bmatrix} p & 0 & x_2^2 q \\ 0 & p & -x_1^2 q \end{bmatrix}.$$

For instance, consider the following specific solutions of the closed loop system

$$\dot{x} = f + g \begin{bmatrix} -x_1x_3 \\ -x_1x_3 - 1 \end{bmatrix} = \begin{bmatrix} (1 - x_1^2)x_3 \\ -x_1x_2x_3 \\ x_1(1 - x_3^2) \end{bmatrix},$$

which has $(0, \pm 1, 0)^T \in S$ as fixed points. Choosing $x_0 = (1, 0, 0)^T \in S$, we get $x_1(t) = 1$, $x_2(t) = 0$, $x_3(t) = \tanh(t)$ and we note that $x_3(t) \in (-1, 1)$ for all $t \in T(x_0) = \mathbb{R}$. Choosing $x_0 = (1, 0, 2)^T \in S$, we obtain $x_1(t) = 1$, $x_2(t) = 0$, $x_3(t) = \coth(t + \frac{1}{2}\ln(3))$ and we note that $x_3(t) > 1$ for all $t \in T(x_0) = (-\frac{1}{2}\ln(3), \infty)$. These examples illustrate the invariance of S. In $\mathcal{V}(p, q)$, we have the fixed points $(\pm 1, 0, \pm 1)^T$. Starting in $x_0 = (0, 1, \pm 1)^T \in \mathcal{V}(p, q)$, we get the trajectory $x_1(t) = \pm\tanh(t)$, $x_2(t) = \sqrt{1 - \tanh(t)^2}$, $x_3(t) = \pm 1$, which illustrates the invariance of $\mathcal{V}(p, q)$.

5. CONTROLLED AND CONDITIONED INVARIANCE

Finally, consider control systems

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t), \quad y(t) = h(x(t)) \tag{5}$$

where $f \in \mathcal{P}^n$, $g \in \mathcal{P}^{n\times m}$, $h \in \mathcal{P}^r$ are given. Let $p_1, \ldots, p_k, q \in \mathcal{P}$ be given and consider S as in Equation (1). Let $\mathcal{M}$ be the module of all vector fields $F \in \mathcal{P}^n$ such that S is F-invariant. Let $A \subseteq \mathcal{P}^m$ denote the set of all admissible feedback laws and let $\mathcal{R} = \mathbb{K}[y_1, \ldots, y_r]$. Then S is controlled and conditioned invariant if and only if there exists $\beta \in \mathcal{R}^m$ such that $f + g(\beta \circ h) \in \mathcal{M}$, or equivalently, $\beta \circ h \in A$. Recall that $\mathbb{K}[h] = \mathbb{K}[h_1, \ldots, h_r] \subseteq \mathcal{P}$ denotes the subalgebra of $\mathcal{P}$ generated by $h_1, \ldots, h_r$.

Corollary 9. The set S is controlled and conditioned invariant for the system from Equation (5) if and only if there exists $\alpha \in A$ (that is, $f + g\alpha \in \mathcal{M}$) and $\beta \in \mathcal{R}^m$ such that $\alpha = \beta \circ h \in \mathbb{K}[h]^m$. In other words, S is controlled and conditioned invariant if and only if

$$A \cap \mathbb{K}[h]^m \neq \emptyset.$$

An algorithm for checking this condition, and for finding a concrete element of $A \cap \mathbb{K}[h]^m$ can be found in Schilli et al. (2014). An improved version of the algorithm was presented in Schilli et al. (2015).

Example: Consider f, g, and S from the previous example and let $h = x_1x_3$. Since

$$\alpha := \begin{bmatrix} -x_1x_3 \\ -x_1x_3 - 1 \end{bmatrix} \in A \cap \mathbb{K}[h]^2,$$

we conclude that S is controlled and conditioned invariant. Choosing $h = (h_1, h_2)^T$ with $h_1 = x_1^2x_3^2 + x_1x_3 - x_1^2$, $h_2 = x_3$, the situation is less obvious, because $\alpha \notin \mathbb{K}[h_1, h_2]^2$. However,

$$\tilde{\alpha} := \alpha - q \begin{bmatrix} p \\ 0 \end{bmatrix} + \begin{bmatrix} x_2^2 q \\ -x_1^2 q \end{bmatrix} \in A$$

and

$$\tilde{\alpha} = \begin{bmatrix} h_2^2 - h_1 - 1 \\ -h_1 - 1 \end{bmatrix} \in \mathbb{K}[h_1, h_2]^2.$$

Thus S is still controlled and conditioned invariant.

6. CONCLUSION

In this paper, the question of (controlled/conditioned) invariance of sets given by algebraic equations and inequations (in the sense of $\neq$) has been completely reduced to the known case of invariant varieties, for which algorithmic solutions have been given in Christopher et al. (2009); Zerz et al. (2010); Zerz and Walcher (2012); Schilli et al. (2014, 2016). Controlled and conditioned invariant varieties have also been studied by Yuno and Ohtsuka (2014, 2015). An interesting generalization to sets given by algebraic equations and inequalities (in the sense of $<$) is treated in Yuno and Ohtsuka (2016). There, dynamic compensators are used instead of static feedback laws. Stability issues have not yet been addressed in this algebraic framework. This remains an open topic for future research.

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