Logical Specifications of Infinite Computations*

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Abstract. Starting from an identification of infinite computations with ω-words, we present a framework in which different classification schemes for specifications are naturally compared. Thereby we connect logical formalisms with hierarchies of descriptive set theory (e.g., the Borel hierarchy), of recursion theory, and with the hierarchy of acceptance conditions of ω-automata. In particular, it is shown in which sense these hierarchies can be viewed as classifications of logical formulas by the complexity measure of quantifier alternation. In this context, the automaton theoretic approach to logical specifications over ω-words turns out to be a technique to reduce quantifier complexity of formulas. Finally, we indicate some perspectives of this approach, discuss variants of the logical framework (first-order logic, temporal logic), and outline the effects which arise when branching computations are considered (i.e., when infinite trees instead of ω-words are taken as model of computation).

Keywords. Infinite words, ω-languages, descriptive set theory, Cantor space, Borel hierarchy, recursion theory, Büchi automata, acceptance conditions, regular ω-languages, monadic second-order logic, infinite games, temporal logic, infinite trees

Contents

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1 Introduction

Nonterminating computations have acquired a central role in computer science, supported by the increasing importance of programs whose purpose is not so much a transformation of values, but to maintain an ongoing interaction with their environment. Such *reactive systems* (cf. [Pnu86]) are studied in many areas of computer science, among them operating systems, communication protocols, data base systems, and control systems.

In the *discrete state based approach* to be pursued in this paper, a computation of a nonterminating program is described by the corresponding sequence of program states (or global system states, if the states of the environment are also taken into account). Such a state sequence is assumed to start at some point and to go on indefinitely in discrete time steps. If states are distinguished only up to a finite number of possibilities, we speak of a *finite-state program* (or finite-state system). In the sequel we confine ourselves to infinite computations under this assumption, which are modelled naturally by *ω-words over a finite alphabet.* A program specification, which describes desired properties of the state sequences which a program or system can assume, thus amounts to a definition of a set of ω-words, i.e., of an *ω-language.*

The task of a theory of specifications for infinite computations in this setting is then to study the possibilities of defining and classifying properties of ω-words, and to develop ways of effective manipulation of the considered definitions. Thus, in the present view, a theory of specifications will be a theory of definability over infinite words.

This subject has a long tradition, going back to times when computer science did not yet exist. It originates in *descriptive set theory*, founded by Cantor, Baire, Lebesgue, Suslin, and others around the turn of the century. A central objective in this theory was to confine the general and evasive notion of function (or set, or real number) to more restrictive and mathematically more manageable versions, like that of continuous function or analytic set. A beautiful theory emerged, which offered a rich classification of properties of functions and sets. It is based on definability of functions and sets from certain "finitary conditions" by natural operations. In the special case of the *Cantor space*, which is the appropriate frame for ω-language theory, the aim is to classify properties of sets of natural numbers, the "finitary conditions" are given by the open sets in the so-called Cantor topology, and the admissible operations are countable union and intersection, projection, and complementation. Resulting classification schemes are the Borel hierarchy and the projective hierarchy, which distinguish properties of sets of natural numbers by the complexity of definition in terms of these operations.

There is a simple connection to the theory of ω-words and ω-languages. Clearly, a set of natural numbers may be identified with its characteristic function, mapping a number to 1 if it belongs to the set, and otherwise to 0. A characteristic function (from the set ω of the natural numbers to $\{0,1\}$) can in turn be viewed as an ω-word. Thus, a property of a set of natural numbers corresponds to a set of ω-words, i.e., to an ω-language. We shall adopt this view here and consider hierarchies like the Borel hierarchy or the projective hierarchy as classifications of ω-languages.

A second root of the subject was prepared in the thirties, when the intuitive concept of effectiveness was made precise by the notions of recursive function and recursive set. It turned out in subsequent work of Kleene and Mostowski that an effective version of descriptive set theory could be developed, keeping a rather strict analogy to the classical case. The "finitary conditions" mentioned above were restricted to *effective* conditions (presented, e.g., in terms of Turing machines). The resulting "arithmetical hierarchy" and "analytical hierarchy" of recursion theory corresponded well to the Borel hierarchy and the projective hierarchy of descriptive set theory. Textbooks like [Rog67], [Hin78], [Mos80] give a detailed exposition of the two hierarchy theories and their correspondence.

In the sixties, a surprising new track of definability theory over ω-words was opened within automata theory. Here, instead of effective definitions in terms of Turing machines, the much more special recognizability using finite automata over ω-words was considered. In the fundamental work of Büchi [Bü62], McNaughton [McN66], and Rabin [Rab69], it turned out that central phenomena of descriptive set theory and recursion theory (like diagonalization and the existence of universal sets) fail, but in their place new aspects enter, especially combinatorial ones, which lead to an interesting and beautiful interplay between the two worlds of the infinite and the finite. It was also shown that the analogues of the classical hierarchies are no more strict but collapse at low levels. More important for the purposes of computer science, it was shown that the theory has a strong algorithmic content, given effective presentations of ω-languages in terms of finite automata. (This covers properties of ω-languages like nonemptiness, location within the hierarchies, or transformation from and to other forms of representation.)

The motivation in the investigations of Büchi, McNaughton, and Rabin, however, was not so much a new look at classical hierarchies, but the study of concrete logical theories: the monadic second-order theories S1S and S2S of one, resp. of two successor functions. The aim was to show that these theories are decidable. The key lemmas in the decidability proof provided a reduction of formulas of these theories to finite automata (see e.g. the survey [Tho90]). This step from formulas to automata amounts, in fact, to a version of quantifier elimination. But elimination of quantifiers is the logical meaning of level-reduction within the descriptive hierarchies. In this sense the automata theoretic approach is tightly connected with hierarchy theory.

The purpose of this survey is to present the definability theory of ω-languages in a form which helps to view its three main origins (as sketched above) in a unified way. Thus the paper provides supplementary material to the survey [Tho90], emphasizing a line of thought which has been pursued especially by Büchi. (The reader is invited to trace the ideas starting from the articles of Part 6 of Büchi's *Collected Works* [MLS90], especially from [Bü77] and [Bü83].)

In Section 2 we give a short review of the Borel hierarchy. The treatment is elementary and intended for readers not familiar with descriptive set theory. We do not work in general "Polish spaces" as done in many textbooks, but just treat the concrete case of the Cantor space which underlies ω-language theory. The main purpose here is to recall the essential features of the classical hierarchies, namely diagonalization and the existence of "universal sets", in the most elementary form. On the other hand, we assume basic knowledge of logic, recursion theory, and automata theory.

In Section 3 we present all three definability theories (the "classical", the effective, and the automata theoretic one) in a common logical framework. Technically, we work in three different extensions of Büchi's sequential calculus S1S. This way of presentation might help to clarify the logical meaning of the well-known automata theoretic theorems of Büchi, McNaughton, Landweber, and Rabin. It also sets a path for interesting (and as yet open) generalizations of the theory of regular ω-languages, and we mention some open problems and research directions in this context. We also add a short discussion on related logical frameworks frequently used for the specification of infinite computations: first-order logic and temporal logic.

In the final section, we explain in which way the theory changes fundamentally when the sequential model of computation, captured by infinite words, is replaced by the framework of branching computations, modelled by infinite trees.

The restriction to infinite words and trees as adopted in this paper leaves out a large domain of structures which are important in the study of concurrent systems: infinite *partially ordered structures*, such as pomsets, Mazurkiewicz traces, or event structures. This subject would require an own terminological framework which we cannot include here. Moreover, it seems open whether logical systems over partial orders will lead to essentially new structural properties of hierarchies of definability, or whether such hierarchies will show an analogy to either the case of infinite words or that of infinite trees.

2 The Set Theoretic Classification of ω-Languages

2.1 Notation

We use the letters $i, j, \ldots, m, n, \ldots, r, s, \ldots$ for natural numbers, and ω for the set of natural numbers. The letter $\mathbb{A}$ stands for a finite alphabet with at least two letters $0, 1$. We will use $u, v, w, \ldots$ to indicate finite words, and $\alpha, \beta, \ldots$ for ω-words. Languages (subsets of $\mathbb{A}^*$) are indicated by $U, V, W, \ldots$, and ω-languages (subsets of $\mathbb{A}^\omega$) by $L, L', \ldots$. An ω-word over $\mathbb{A}$ is of the form $\alpha = \alpha(0)\,\alpha(1), \ldots$ with $\alpha(i) \in \mathbb{A}$. We set $\alpha[i] = \alpha(0) \ldots \alpha(i-1)$. If a word u is a proper prefix of v (resp. α) we write $u \sqsubset v$ (resp. $u \sqsubset \alpha$).

For logical connectives and quantifiers we use the standard notation; $\exists^\omega$ and $\forall^\omega$ indicate the quantifiers "there exist infinitely many" and "for almost all" (i.e. "for all but finitely many"), respectively.

Tuples of ω-words are written as $\overline{\alpha} = (\alpha_1, \ldots, \alpha_n)$. Such an n-tuple may be identified with an ω-word in two different ways: Either we proceed to the new alphabet $\mathbb{A}^n$ and identify $\overline{\alpha}$ with the ω-word $\overline{\alpha(0)}\,\overline{\alpha(1)} \ldots$ where $\overline{\alpha(i)} = (\alpha_1(i), \ldots, \alpha_n(i))$. This representation will be adequate in the discussion of logical definability (Section 3). However, in the present section it is convenient to stay with the fixed alphabet $\mathbb{A}$. To achieve this we represent a tuple $\overline{\alpha} = (\alpha_1, \ldots, \alpha_n)$ by the ω-word $\alpha_1(0) \ldots \alpha_n(0)\alpha_1(1) \ldots \alpha_n(1)\alpha_1(2) \ldots$, which we indicate by $\alpha_1{}^\wedge \alpha_2{}^\wedge \ldots{}^\wedge \alpha_n$.

2.2 Cantor Space and Borel Sets

The purpose of this section is to present a brief introduction to the Borel hierarchy in the space of ω-words over a given alphabet. This hierarchy is a classification of properties of ω-words by a measure of complexity which indicates a kind of "distance" from properties concerning just finite words. The classification originates in Cantor's work on his "Continuum Hypothesis" about hundred years ago. Formulated in present language theoretic terminology, his aim was to show that an ω-language over the alphabet $\mathbb{A}$ is either at most countable or of same cardinality as the set of all ω-words over $\mathbb{A}$. Failing to prove the full claim, he approached the problem by proving the hypothesis in special cases. His program was to start with "simple" ω-languages and to work towards the general case by means of an approximation scheme. His "simple" sets (for which he succeeded to prove the Continuum Hypothesis) were the closed sets in what is called the Cantor space today.

In the sequel let $\mathbb{A}$ be a fixed alphabet. The *Cantor metric* on $\mathbb{A}^\omega$ is the distance function $d : \mathbb{A}^\omega \times \mathbb{A}^\omega \to \mathbb{R}$ given by

$$d(\alpha, \beta) = \begin{cases} 0 & \text{if } \alpha = \beta \\ \frac{1}{2^n} & \text{for the minimal } n \text{ with } \alpha(n) \neq \beta(n) \text{ , otherwise} \end{cases}$$

Thus two ω-words are considered "close" to each other if they have a long common prefix.

It is easily shown that d is indeed a metric. Thus we obtain a topological space by declaring as a basis of open sets the $\frac{1}{2^n}$-neighbourhoods of ω-words. The $\frac{1}{2^n}$-neighbourhood of the ω-word α contains all ω-words β with $d(\alpha,\beta) < \frac{1}{2^n}$. Now we have

$$d(\alpha,\beta) < \frac{1}{2^n} \iff \alpha[n] = \beta[n] \iff \beta \in \{\alpha[n]\} \cdot \mathbb{A}^\omega .$$

Thus the $\frac{1}{2^n}$-neighbourhood of α is $\{\alpha[n]\} \cdot \mathbb{A}^\omega$. Unions of such sets $\{w\} \cdot \mathbb{A}^\omega$, i.e. sets of the form

$$\bigcup_{w \in W} \{w\} \cdot \mathbb{A}^\omega = W \cdot \mathbb{A}^\omega$$

with $W \subseteq \mathbb{A}^*$ are the open sets in the topology induced by d. This topology is called the *Cantor topology*, and the set $\mathbb{A}^\omega$ endowed with this topology is the *Cantor space.*

Hence a subset L of the Cantor space $\mathbb{A}^\omega$ is open if and only if for some set $W \subseteq \mathbb{A}^*$, an ω-word α is in L iff some prefix of α is in W. The closed sets are the complements of the open sets. By proceeding from a word-set W to its complement $V = \mathbb{A}^* \setminus W$ we obtain that a set L is closed iff for some set $V \subseteq \mathbb{A}^*$, an ω-word α is in L iff all prefixes of α are in V.

Thus we see that the open and closed sets of ω-words are defined by the simplest way of referring to properties of *finite* words: To verify membership of an ω-word α in an open (resp. closed) ω-language, it suffices to check that some finite prefix (resp. all finite prefixes) of α belong to a given language of finite words. In this sense the open and closed ω-languages are given by "finitary conditions" as indicated in the introduction.

For later use we note special descriptions of the open sets and the "clopen" sets (sets which are closed and open). Call a set W of finite words *prefix-free* if $u \sqsubset v$ does not hold for any distinct words u, $v \in W$.

Lemma 2.1. *(a) For any open set $L = W \cdot \mathbb{A}^\omega$ there is a prefix-free set W_0 with $L = W_0 \cdot \mathbb{A}^\omega$.*
(b) A set $L \subseteq \mathbb{A}^\omega$ is open and closed iff $L = W \cdot \mathbb{A}^\omega$ for a finite set W.

Proof. (a) For $L = W \cdot \mathbb{A}^\omega$ take $W_0 = W \setminus \{w \in W \mid \exists w'\, w' \sqsubset w\}$, i.e. omit all words that have a proper prefix in W.
For the proof of (b), assume $L = W \cdot \mathbb{A}^\omega$ and $\mathbb{A}^\omega \setminus L = V \cdot \mathbb{A}^\omega$ for prefix-free V, W (by (a)). We show that $V \cup W$ is finite. For this it suffices to verify that the set T of words from $\mathbb{A}^*$ without a prefix in $V \cup W$ is finite. Clearly T forms a tree (where u is parent of v if $ua = v$ for some $a \in \mathbb{A}$) and is finitely branching. There are only finite paths in T, because an infinite path would yield an ω-word $\alpha \notin (W \cdot \mathbb{A}^\omega \cup V \cdot \mathbb{A}^\omega)$. By König's Lemma, T is finite.

Conversely, let $L = W \cdot \mathbb{A}^\omega$ for finite W. Let l be the maximal length of words in W. Set $V = \{v \in \mathbb{A}^* \mid |v| = l,\ v \text{ has no prefix in } W\}$. Then $\mathbb{A}^\omega \setminus L = V \cdot \mathbb{A}^\omega$. □

The Borel sets are now constructed from the open and closed ω-languages by means of the operations of countable union, countable intersection, and complement (with respect to $\mathbb{A}^\omega$).

Definition 2.2. The finite levels of the *Borel hierarchy* on the Cantor space $\mathbb{A}^\omega$ are the classes $\boldsymbol{\Sigma}_n^0$, $\boldsymbol{\Pi}_n^0$ of ω-languages defined for $n \geq 1$ as follows:

$$\boldsymbol{\Sigma}_1^0 = \{L \subseteq \mathbb{A}^\omega \mid L \text{ open}\}$$
$$\boldsymbol{\Pi}_1^0 = \{L \subseteq \mathbb{A}^\omega \mid L \text{ closed}\}$$
$$\boldsymbol{\Sigma}_{n+1}^0 = \{L \subseteq \mathbb{A}^\omega \mid L \text{ is a union of the form } \bigcup_{i\in\omega} L_i \text{ with } L_i \in \boldsymbol{\Pi}_n^0\}$$
$$\boldsymbol{\Pi}_{n+1}^0 = \{L \subseteq \mathbb{A}^\omega \mid L \text{ is an intersection of the form } \bigcap_{i\in\omega} L_i \text{ with } L_i \in \boldsymbol{\Sigma}_n^0\}$$

In the classical terminology one denotes the levels $\boldsymbol{\Sigma}_1^0$, $\boldsymbol{\Pi}_1^0$, $\boldsymbol{\Sigma}_2^0$, $\boldsymbol{\Pi}_2^0$, $\boldsymbol{\Sigma}_3^0$, $\boldsymbol{\Pi}_3^0, \ldots$ by G, F, F_σ, G_δ, $G_{\delta\sigma}$, $F_{\sigma\delta}, \ldots$. As we see in Theorem 2.3 below, an alternative and equivalent definition for the sets $\boldsymbol{\Pi}_n^0$ would be to set $\boldsymbol{\Pi}_n^0 = \{\mathbb{A}^\omega \setminus L \mid L \in \boldsymbol{\Sigma}_n^0\}$ for $n \geq 1$.

The Borel hierarchy also extends to *transfinite levels*, i.e., to classes $\boldsymbol{\Sigma}_\gamma^0$ and $\boldsymbol{\Pi}_\gamma^0$ for each countable ordinal γ. For successor ordinals $\gamma + 1$ the definition is given as above, referring to $\boldsymbol{\Pi}_\gamma^0$, resp. $\boldsymbol{\Sigma}_\gamma^0$; and for limit ordinals γ one defines

$$\boldsymbol{\Sigma}_\gamma^0 = \boldsymbol{\Pi}_\gamma^0 = \bigcup_{\delta<\gamma} \boldsymbol{\Sigma}_\delta^0.$$

An ω-language is a *Borel set* if it belongs to some class $\boldsymbol{\Sigma}_\gamma^0$ for a countable ordinal γ. In the sequel, however, we confine ourselves to the finite levels of the Borel hierarchy.

Let us verify some easy facts on the classes $\boldsymbol{\Sigma}_n^0$, $\boldsymbol{\Pi}_n^0$.

Theorem 2.3. *Let $n \geq 1$.*

(a) $L \in \boldsymbol{\Sigma}_n^0$ if and only if $\mathbb{A}^\omega \setminus L \in \boldsymbol{\Pi}_n^0$.
(b) The classes $\boldsymbol{\Sigma}_n^0$ and $\boldsymbol{\Pi}_n^0$ are closed under finite intersection and finite union.
(c) The following inclusions (with $A \to B$ indicating $A \subseteq B$) hold:

$$\begin{tikzcd}
\boldsymbol{\Sigma}_n^0 \arrow[r] \arrow[dr] & \boldsymbol{\Sigma}_{n+1}^0 \\
\boldsymbol{\Pi}_n^0 \arrow[r] \arrow[ur] & \boldsymbol{\Pi}_{n+1}^0
\end{tikzcd}$$

Proof. We proceed by induction on n. First we show (a). For $n = 1$ this claim was shown above. In the induction step, for $L \in \boldsymbol{\Sigma}_{n+1}^0$ this holds since from $L = \bigcup_{i\geq 0} L_i$ with $L_i \in \boldsymbol{\Pi}_n^0$ we obtain $\mathbb{A}^\omega \setminus L = \bigcap_{i\geq 0}(\mathbb{A}^\omega \setminus L_i)$, where $\mathbb{A}^\omega \setminus L_i \in \boldsymbol{\Sigma}_n^0$ by induction hypothesis. Thus $\mathbb{A}^\omega \setminus L \in \boldsymbol{\Pi}_{n+1}^0$.

For part (b) the claim is obvious in the case of $n = 1$. For two $\boldsymbol{\Sigma}_{n+1}^0$-sets L, M with $L = \bigcup_{i\geq 0} L_i$ and $M = \bigcup_{i\geq 0} M_i$, where the L_i, M_i are $\boldsymbol{\Pi}_n^0$-sets, we define $R_{ij} = L_i \cap M_j$. By the induction hypothesis we know that all R_{ij} are $\boldsymbol{\Pi}^0_n$-sets. We have

$$\begin{aligned} \alpha \in L \cap M &\iff \exists i\, \alpha \in L_i \wedge \exists j\, \alpha \in M_j \\ &\iff \exists (i,j)\, \alpha \in R_{ij} \end{aligned}$$

Therefore $L \cap M$ is a countable union of $\boldsymbol{\Pi}^0_n$-sets, namely $L \cap M = \bigcup_{i,j \geq 0} R_{ij}$. Iterating this construction we obtain that the intersection of finitely many $\boldsymbol{\Sigma}^0_n$-sets is in $\boldsymbol{\Sigma}^0_n$. Obviously the $\boldsymbol{\Sigma}^0_n$-sets are closed under finite union. We get the analogous results for $\boldsymbol{\Pi}^0_n$-sets by dualization.

Now we show (c). The claim $\boldsymbol{\Sigma}^0_n \subseteq \boldsymbol{\Pi}^0_{n+1}$ is immediate: Let $L \in \boldsymbol{\Sigma}^0_n$; then $L = \left(\bigcap_{i \geq 0} L\right) \in \boldsymbol{\Pi}^0_{n+1}$. Similarly we have $\boldsymbol{\Pi}^0_n \subseteq \boldsymbol{\Sigma}^0_{n+1}$. $\boldsymbol{\Sigma}^0_n \subseteq \boldsymbol{\Sigma}^0_{n+1}$ is obvious for $n = 1$, and $\boldsymbol{\Pi}^0_1 \subseteq \boldsymbol{\Pi}^0_2$ follows by (a). A set $L \in \boldsymbol{\Sigma}^0_{n+1}$ is of the form $L = \bigcup_{i \geq 0} L_i$ with $L_i \in \boldsymbol{\Pi}^0_n$. By induction hypothesis we have $\boldsymbol{\Pi}^0_n \subseteq \boldsymbol{\Pi}^0_{n+1}$ and hence $L_i \in \boldsymbol{\Pi}^0_{n+1}$. Therefore we get $L \in \boldsymbol{\Sigma}^0_{n+2}$. Again by part (a) we have $\boldsymbol{\Pi}^0_{n+1} \subseteq \boldsymbol{\Pi}^0_{n+2}$. □

2.3 The First Two Levels

As mentioned above we can describe ω-languages from $\boldsymbol{\Sigma}^0_1$ and $\boldsymbol{\Pi}^0_1$ by properties of prefixes of ω-words in these ω-languages. So a set L is in $\boldsymbol{\Sigma}^0_1$ iff there exists a set $W \subseteq \mathbb{A}^*$ such that

$$\alpha \in L \iff \exists\, i\ \alpha[i] \in W,$$

and L is in $\boldsymbol{\Pi}^0_1$ iff there is a set W with

$$\alpha \in L \iff \forall\, i\ \alpha[i] \in W.$$

In a classification scheme for specifications of infinite computations introduced by Manna and Pnueli [MP88], [MP92], the $\boldsymbol{\Sigma}^0_1$-properties of ω-words are of the type *guarantee property*, claiming that some initial segment of the computation under consideration exists where a certain condition is met. The $\boldsymbol{\Pi}^0_1$-properties are termed *safety properties*, requiring that all initial segments of a computation stay within a given restriction.

Let us turn to the classes $\boldsymbol{\Sigma}^0_2$ and $\boldsymbol{\Pi}^0_2$. In order to give a characterization similar to that for $\boldsymbol{\Sigma}^0_1$ and $\boldsymbol{\Pi}^0_1$ we need the following lemma:

Lemma 2.4. *Let $L_1, L_2, L_3, \ldots$ be open sets and $L_n = W_n \cdot \mathbb{A}^\omega$. Then there exist sets $W'_n \subseteq \mathbb{A}^*$ such that $L_n = W'_n \cdot \mathbb{A}^\omega$, $W'_m \cap W'_n = \emptyset$ for all $m \neq n$, and for $m < n$ no word of W'_n is prefix of a word in W'_m.*

Proof. By Lemma 2.1 we may assume that all sets W_i are prefix-free. We construct the set W'_{i+1} from the set W_{i+1} and the sets W'_j with $j \leq i$. We take $W'_1 = W_1$. We build up W'_{i+1} in two steps: First we take those words of W_{i+1} that are not prefix of a word in $\bigcup_{j=1}^{i} W'_j$ and, for the remaining words in W_{i+1}, we add their smallest extensions which are not prefix of a word in $\bigcup_{j=1}^{i} W'_j$. Thus we obtain sets as required in the statement of the Lemma. □

Now we are ready for a closer look at the $\boldsymbol{\Sigma}^0_2$- and $\boldsymbol{\Pi}^0_2$-sets.

Theorem 2.5 (Characterization of $\boldsymbol{\Sigma}^0_2$- and $\boldsymbol{\Pi}^0_2$-sets). *Let $L \subseteq \mathbb{A}^\omega$.*

(a) $L \in \boldsymbol{\Pi}^0_2$ iff there is a set $W \subseteq \mathbb{A}^*$ such that

$$\alpha \in L \iff \exists^\omega i\ \alpha[i] \in W\ (i.e.,\ \forall j\, \exists i > j\ \alpha[i] \in W)$$

(b) $L \in \boldsymbol{\Sigma}^0_2$ iff there exists a set $W \subseteq \mathbb{A}^*$ such that

$$\alpha \in L \iff \forall^\omega i\ \alpha[i] \in W\ (i.e.,\ \exists j\, \forall i > j\ \alpha[i] \in W)$$

If L is of the form described in Theorem 2.5(a) one writes $L = \overrightarrow{W}$ (other notations occurring in the literature are *lim* W and W^δ).

Proof. (a): For the direction from left to right we assume a representation

$$L = \bigcap_{n \geq 0} W_n \cdot \mathbb{A}^\omega$$

where $W_n \subseteq \mathbb{A}^*$. From Lemma 2.4 we know that we can find sets $W'_n \subseteq \mathbb{A}^*$ with $W'_n \cdot \mathbb{A}^\omega = W_n \cdot \mathbb{A}^\omega$ and the following property:

$(*)$ $W'_m \cap W'_n = \emptyset$ for all $m \neq n$, and for $m < n$ no word of W'_n is prefix of a word in W'_m.

Set $W = \bigcup_{n \geq 0} W'_n$. Then we have

$$\begin{aligned}
\alpha \in L &\iff \text{for every } n \text{ there exists a prefix of } \alpha \text{ in } W_n \\
&\iff \text{there are prefixes of } \alpha \text{ in } W'_1, W'_2, W'_3, \ldots \text{ where } (*) \text{ holds} \\
&\iff \forall j\, \exists i > j\ \alpha[i] \in \bigcup_{n \geq 0} W'_n \\
&\iff \exists^\omega i\ \alpha[i] \in W \\
&\iff \alpha \in \overrightarrow{W}\,.
\end{aligned}$$

For the reverse direction let $L = \overrightarrow{W}$ and denote by $\mathbb{A}^{\geq n}$ the set of words over $\mathbb{A}$ of length $\geq n$. Then

$$\begin{aligned}
\alpha \in L &\iff \text{for every } n \text{ there exists a prefix of } \alpha \text{ in } W \cap \mathbb{A}^{\geq n} \\
&\iff \alpha \in \bigcap_{n \geq 0} (W \cap \mathbb{A}^{\geq n}) \cdot \mathbb{A}^\omega
\end{aligned}$$

Since $(W \cap \mathbb{A}^{\geq n}) \cdot \mathbb{A}^\omega$ is an open set we obtain $L \in \boldsymbol{\Pi}^0_2$.

For the proof of part (b) we note that the sets of $\boldsymbol{\Sigma}^0_2$ are complements of $\boldsymbol{\Pi}^0_2$-sets. So for $L \in \boldsymbol{\Sigma}^0_2$ we can assume a representation as a union of sets $\mathbb{A}^\omega \setminus (W_n \cdot \mathbb{A}^\omega)$ for $n \geq 0$. Taking V as the complement of W of (a), we have

$$\alpha \in L \iff \neg \exists^\omega j\ \alpha[j] \in W \iff \forall^\omega j\ \alpha[j] \notin W \iff \forall^\omega j\ \alpha[j] \in V$$

The reverse direction of (b) follows similarly from (a). $\square$

Theorem 2.5 yields a characterization of $\boldsymbol{\Sigma}_2^0$ and $\boldsymbol{\Pi}_2^0$ close to that of $\boldsymbol{\Sigma}_1^0$ and $\boldsymbol{\Pi}_1^0$: A property in $\boldsymbol{\Pi}_2^0$ holds for an ω-word α iff infinitely many prefixes of α belong to a certain set of finite words. In the dual case, for a $\boldsymbol{\Sigma}_2^0$ property, the requirement is that from a certain point onwards all initial segments belong to a certain language of finite words. In the scheme of [MP92], these properties are called *response properties* and *persistence properties*, respectively.

The union of a persistence property and response property amounts to the following type of condition on ω-words, referring to two given languages U, V: "sequence α has almost all prefixes in U or infinitely many prefixes in V", which means (by rules of Boolean logic, with U' as the complement of U in the domain of finite words): "if infinitely many prefixes of α are in U', then infinitely many prefixes of it are in V". Requirements of this form (e.g., "a resource requested again and again is granted again and again"), or conjunctions of them are called *fairness properties* or also *reactivity properties* of computations.

2.4 The Hierarchy Theorem

Now we give a short exposition of the hierarchy theorem for the finite levels of the Borel hierarchy. The main idea is to define, for each level, a "complete" set where completeness is defined relative to a suitable type of reduction, given by the *continuous* functions. As complete sets we define "universal" ones which capture the levels of the hierarchy by diagonalization. This idea is the same as for the m-complete recursively enumerable sets in recursion theory. So we start with the definition of continuous function and reducibility.

Definition 2.6. (a) A function $f : \mathbb{A}^\omega \to \mathbb{A}^\omega$ is *continuous* iff $f^{-1}(L)$ is open for every open set L.
(b) We write $L \leq M$ if there exists a continuous function $f : \mathbb{A}^\omega \to \mathbb{A}^\omega$ with $f^{-1}(M) = L$ (in other words $\alpha \in L \iff f(\alpha) \in M$ for $\alpha \in \mathbb{A}^\omega$).

It is useful to note that a function f is continuous iff any finite prefix of an f-image $f(\alpha)$ is determined by a finite prefix of the argument α.

Remark. If L, $M \subseteq \mathbb{A}^\omega$ are ω-languages with $L \leq M$ then also $\mathbb{A}^\omega \setminus L \leq \mathbb{A}^\omega \setminus M$ holds. This is clear since from $f^{-1}(M) = L$ we obtain $f^{-1}(\mathbb{A}^\omega \setminus M) = \mathbb{A}^\omega \setminus L$.

Lemma 2.7. *If $L \leq M$ and $M \in \boldsymbol{\Sigma}_n^0$ (resp. $M \in \boldsymbol{\Pi}_n^0$) then also $L \in \boldsymbol{\Sigma}_n^0$ (resp. $L \in \boldsymbol{\Pi}_n^0$).*

Proof. The statement is immediate for $M \in \boldsymbol{\Sigma}_1^0$ (and hence for $\boldsymbol{\Pi}_1^0$) by definition of continuity.

In the induction step with $M \in \boldsymbol{\Sigma}_{n+1}^0$ we know that M is of the form $M = \bigcup_{i\geq 0} M_i$ with $M_i \in \boldsymbol{\Pi}_n^0$. Set $L_i = f^{-1}(M_i)$. Then $L_i \leq M_i$ and hence $L_i \in \boldsymbol{\Pi}_n^0$ by induction hypothesis. So $\alpha \in L_i$ iff $f(\alpha) \in M_i$. Hence we obtain

$$\alpha \in L \iff f(\alpha) \in M \iff \exists i\ f(\alpha) \in M_i \iff \exists i\ \alpha \in L_i \iff \alpha \in \bigcup_{i\geq 0} L_i$$

Therefore $L \in \boldsymbol{\Sigma}_{n+1}^0$; similarly we prove the claim for $M \in \boldsymbol{\Pi}_{n+1}^0$. □

As an application of Lemma 2.7 note that if M is in $\boldsymbol{\Sigma}^0_n$ then $L = \{\alpha \mid \alpha^\wedge\alpha \in M\}$ also belongs to $\boldsymbol{\Sigma}^0_n$. One only has to observe that the function $\alpha \mapsto \alpha^\wedge\alpha$ is continuous.

As a second preparation for the hierarchy theorem we verify a "basis property" for the Borel levels: In each class $\boldsymbol{\Sigma}^0_n$ there is a countable collection of sets such that all other sets in the class are composable as unions of sets from this collection.

Definition 2.8. A class $B \subseteq \boldsymbol{\Sigma}^0_n$ is called *basis* of $\boldsymbol{\Sigma}^0_n$, if every $\boldsymbol{\Sigma}^0_n$-set is a countable union of sets from B. Similarly, a class $B \subseteq \boldsymbol{\Pi}^0_n$ is a *basis* of $\boldsymbol{\Pi}^0_n$ if every $\boldsymbol{\Pi}^0_n$-set is a countable intersection of sets from B.

Note that the complements of the sets of a basis of $\boldsymbol{\Sigma}^0_n$ form a basis of $\boldsymbol{\Pi}^0_n$, and conversely.

Theorem 2.9. *For $n \geq 1$ there exists a countable basis of $\boldsymbol{\Sigma}^0_n$ and a countable basis of $\boldsymbol{\Pi}^0_n$.*

Proof. We show inductively that for $\boldsymbol{\Sigma}^0_n$ ($\boldsymbol{\Pi}^0_n$) there is a countable basis consisting even of $\boldsymbol{\Pi}^0_{n-1}$-sets ($\boldsymbol{\Sigma}^0_{n-1}$-sets), in case $n = 1$ consisting of clopen sets. For $n = 1$ the assertion is immediate because $\{\{w\} \cdot \mathbb{A}^\omega \mid w \in \mathbb{A}^*\}$ is a countable basis of $\boldsymbol{\Sigma}^0_1$ and $\{\mathbb{A}^\omega \setminus \{w\} \cdot \mathbb{A}^\omega \mid w \in \mathbb{A}^*\}$ is a countable basis of $\boldsymbol{\Pi}^0_1$, both consisting of clopen sets.

In the induction step we find a countable basis of $\boldsymbol{\Pi}^0_{n+1}$ (of $\boldsymbol{\Sigma}^0_n$-sets) from a given countable basis $L_0, L_1, \ldots$ of $\boldsymbol{\Sigma}^0_n$ (of $\boldsymbol{\Pi}^0_{n-1}$-sets); the dual case is similar. Let B be the collection of sets of the form

$$\bigcup_{j \in N} L_j$$

where N is a cofinite subset of ω. Clearly B is countable and consists of $\boldsymbol{\Sigma}^0_n$-sets. In order to verify the basis property it suffices to show that any $\boldsymbol{\Sigma}^0_n$-set L is a countable intersection of sets from B. (Then the same will hold for $\boldsymbol{\Pi}^0_{n+1}$-sets, which are countable intersections of $\boldsymbol{\Sigma}^0_n$-sets.) Any set $L \in \boldsymbol{\Sigma}^0_n$ has a representation

$$L = \bigcup_{j \in M} L_j$$

for some $M \subseteq \omega$. Define cofinite sets M_k for $k \geq 0$ by

$$M_k = (M \cap \{0, \ldots, k\}) \cup \{k+1, k+2, \ldots\}.$$

Then

$$L = \bigcap_{k \geq 0} \bigcup_{j \in M_k} L_j$$

whence L is a countable intersection of sets from B. □

The possibility of coding a whole Borel class by a single ω-language is prepared by the following definition.

Definition 2.10. A set L_0 is *universal* for the class $\mathcal{K}$ of ω-languages if for every ω-language L from $\mathcal{K}$ there is an infinite word β_L such that $\alpha \in L \iff \beta_L{}^\wedge\alpha \in L_0$ for all $\alpha \in \mathbb{A}^\omega$.

The following theorem establishes the announced hierarchy result:

Theorem 2.11. *For $n \geq 1$ there exists an ω-language U_n with the following properties:*

(a) $U_n \in \boldsymbol{\Sigma}_n^0$
(b) U_n is universal for the class $\boldsymbol{\Sigma}_n^0$.
(c) $U_n \notin \boldsymbol{\Pi}_n^0$ (and hence, for $n > 1$, $U_n \notin \boldsymbol{\Sigma}_{n-1}^0$)
(d) $L \in \boldsymbol{\Sigma}_n^0$ iff $L \leq U_n$

Part (a) and (d) together justify calling the set U_n "complete for $\boldsymbol{\Sigma}_n^0$ with respect to $\leq$".

Proof. We first prove (a), (b) separately for $n = 1$ and for $n > 1$. To prove the claim for $n = 1$ let $w_0, w_1, w_2, \ldots$ be the words over the alphabet $\mathbb{A}$ in canonical order. Recall that by convention $\mathbb{A}$ contains the two letters $0, 1$. Now set

$$U_1 = \{\beta^\wedge\alpha |\ \beta \in \{0,1\}^\omega,\ \exists i\ (\beta(i) = 1 \wedge w_i \sqsubset \alpha)\}$$

(a): U_1 is open since membership of $\beta^\wedge\alpha$ in U_1 depends only on the existence of a certain prefix of $\beta^\wedge\alpha$ wherein $\beta(i) = 1$ and $w_i \sqsubset \alpha$ are verified for some i.
(b): For an ω-language L define $\beta_L \in \{0,1\}^\omega$ by $\beta_L(i) = 1 \iff w_i \cdot \mathbb{A}^\omega \subseteq L$. Then $\alpha \in L \iff \beta_L{}^\wedge\alpha \in U_1$ holds.

Suppose $n > 1$ and let $L_0, L_1, L_2, \ldots$ be a countable basis of $\boldsymbol{\Pi}_{n-1}^0$. Each set $K \in \boldsymbol{\Pi}_{n-1}^0$ has a representation $K = \bigcap_{j \in N} L_j$ for a certain index set $N \subseteq \omega$. Without loss of generality, N can be taken maximal, i.e. including all j with $K \subseteq L_j$. The set N can be specified by its characteristic 0-1-sequence α_K with $\alpha_K(j) = 1$ iff $j \in N$. Now consider a typical set $L \in \boldsymbol{\Sigma}_n^0$, i.e., a union of sets $K_0, K_1, \ldots$ from $\boldsymbol{\Pi}_{n-1}^0$. L is coded by the corresponding sequences $\alpha_{K_0}, \alpha_{K_1}, \ldots$. We arrange these sequences row by row in a two-dimensional 0-1-array where the entry at position (i,j) is 1 iff $\alpha_{K_i}(j) = 1$, i.e. $K_i \subseteq L_j$.

We associate to such a diagram the ω-word β that results from traversing the entries according to the usual diagonal scheme:

$$\begin{array}{llll} \beta(0) & \beta(1) & \beta(3) & \beta(6) \cdots \\ \beta(2) & \beta(4) & \beta(7) & \\ \beta(5) & \beta(8) & & \\ \beta(9) & & \ddots & \\ \vdots & & & \end{array}$$

Let us write β_{ij} for the entry of this array at position (i,j). We call the ω-word β *induced* by the sets $K_0, K_1, \ldots$. Conversely, given β, we obtain sets

$K_0, K_1, \ldots$ *induced* by β, defining K_i from the i-th row of the 0-1-array given by β. For $\beta \in \{0,1\}^\omega$ define L_β by

$$\begin{aligned} \alpha \in L_\beta &\iff \text{for some } i, \text{ the set } K_i \text{ induced by } \beta \text{ contains } \alpha \\ &\iff \exists i \, \forall j \, (\beta_{ij} = 1 \iff \alpha \in L_j) \end{aligned}$$

Now let $U_n = \{\beta^\wedge \alpha \mid \alpha \in L_\beta\}$. We have to verify (a) and (b). First we show that $U_n \in \boldsymbol{\Sigma}^0_n$: According to our construction we have

$$\beta^\wedge \alpha \in U_n \iff \alpha \in L_\beta \iff \exists i \, [\forall j \; (\beta_{ij} = 1 \iff \alpha \in L_j]$$

where the part $[\ldots]$ is equivalent to $\alpha \in K_i$ as given by β. Thus $U_n = \bigcup_i K_i$ is a countable union of $\boldsymbol{\Pi}^0_{n-1}$-sets, and hence we have $U_n \in \boldsymbol{\Sigma}^0_n$.

U_n is universal for $\boldsymbol{\Sigma}^0_n$ since any $\boldsymbol{\Sigma}^0_n$-set L is a countable union of sets K_i as above, inducing a suitable ω-word β_L. Thus we have $L = \{\alpha \mid \beta_L{}^\wedge \alpha \in U_n\}$.

Now we show part (c), claiming $U_n \notin \boldsymbol{\Pi}^0_n$, by a diagonalization argument: Assuming $U_n \in \boldsymbol{\Pi}^0_n$ we would have $\mathbb{A}^\omega \setminus U_n \in \boldsymbol{\Sigma}^0_n$. Define the "diagonal set" $D_n := \{\alpha \mid \alpha^\wedge \alpha \in \mathbb{A}^\omega \setminus U_n\}$. Since $\alpha \mapsto \alpha^\wedge \alpha$ is continuous we have $D_n \leq \mathbb{A}^\omega \setminus U_n$ and hence, by Lemma 2.7, $D_n \in \boldsymbol{\Sigma}^0_n$. By (b) there is a $\beta (= \beta_{D_n})$ for D_n such that $\alpha \in D_n \iff \beta^\wedge \alpha \in U_n$. Then we get for all $\alpha \in \mathbb{A}^\omega$

$$\beta^\wedge \alpha \in U_n \iff \alpha \in D_n \iff \alpha^\wedge \alpha \notin U_n.$$

For $\alpha = \beta$ we obtain a contradiction.

Finally, we have to prove part (d): $L \in \boldsymbol{\Sigma}^0_n \iff L \leq U_n$. The direction from right to left is clear from part (a) and Lemma 2.7. For the direction from left to right let $L \in \boldsymbol{\Sigma}^0_n$. Then by definition of the universal set U_n there exists an ω-word β_L such that $\alpha \in L \iff \beta_L{}^\wedge \alpha \in U_n$. Consider the function $f : \mathbb{A}^\omega \to \mathbb{A}^\omega$ defined by $f(\alpha) = \beta_L{}^\wedge \alpha$. Since f is continuous and

$$\alpha \in L \iff \beta_L{}^\wedge \alpha \in U_n \iff f(\alpha) \in U_n$$

we have $L \leq U_n$. □

As a supplement to the abstract hierarchy result we present some examples of ω-languages which give a more concrete impression of the first Borel levels. They are more "natural" sets than the sets U_n and are universal for the associated levels. This universality holds for

- $0^* 1 \cdot \mathbb{A}^\omega$ in the class $\boldsymbol{\Sigma}^0_1$,
- $\{0^\omega\}$ in the class $\boldsymbol{\Pi}^0_1$,
- $0^* \cdot \{1^\omega\}$ in the class $\boldsymbol{\Sigma}^0_2$,
- $(0^* 1)^\omega$ in the class $\boldsymbol{\Pi}^0_2$,

and, in general, for $n \geq 1$,

- the ω-language $\{\alpha \in \{0,1\}^\omega \mid \forall^\omega k_1 \ldots \forall^\omega k_{n-1} \; \alpha$ has only finitely many segments $001^{k_1}01^{k_2}\ldots 01^{k_{n-1}}00\}$ is universal for $\boldsymbol{\Sigma}^0_{2n}$,

– the ω-language $\{\alpha \in \{0,1\}^\omega \mid \exists^\omega k_1 \ldots \exists^\omega k_{n-1}\ \alpha$ has infinitely many segments $001^{k_1}01^{k_2}\ldots 01^{k_{n-1}}00\}$ is universal for $\boldsymbol{\Pi}^0_{2n}$.

If we change the quantifiers $\forall^\omega k_1$ and $\exists^\omega k_1$ into $\forall k_1$, resp. $\exists k_1$, we obtain universal sets for $\boldsymbol{\Pi}^0_{2n-1}$, resp. $\boldsymbol{\Sigma}^0_{2n-1}$.

These examples already show a connection with quantifier hierarchies, to be discussed in Section 3.3 below.

2.5 Beyond Borel Sets

The Borel hierarchy is built up from open sets by countable unions and intersections. Typical index sets for such unions and intersections are sets W of finite words. Thus, for example, membership of an ω-word α in a countable union amounts to the existence of a word w in an index set W such that α belongs to the set indexed by w. A more powerful principle of set construction is to apply an existential quantification (or "projection") to an uncountable domain, typically to a set of ω-words.

We say that a set $L \subseteq \mathbb{A}^\omega$ is a *projection* of $L_0 \subseteq \mathbb{A}^\omega$ if for some $k > 0$

$$\alpha \in L \iff \exists \beta_1 \ldots \exists \beta_k \in \mathbb{A}^\omega\ \alpha^\wedge \beta_1{}^\wedge \ldots{}^\wedge \beta_k \in L_0$$

Applying these projections instead of countable unions leads us to the "projective hierarchy". We present the main facts without proofs. For details we refer to [Rog67], [Hin78], [Mos80].

Definition 2.12. The *projective hierarchy* on $\mathbb{A}^\omega$ consists of the classes $\boldsymbol{\Sigma}^1_n, \boldsymbol{\Pi}^1_n$ defined by

$$\begin{aligned}
\boldsymbol{\Sigma}^1_1 &= \{L \subseteq \mathbb{A}^\omega \mid\ L \text{ is a projection of a Borel set}\}\\
\boldsymbol{\Pi}^1_1 &= \{\mathbb{A}^\omega \setminus L \mid L \in \boldsymbol{\Sigma}^1_1\}\\
\boldsymbol{\Sigma}^1_{n+1} &= \{L \subseteq \mathbb{A}^\omega \mid L \text{ is a projection of a } \boldsymbol{\Pi}^1_n\text{-set}\}\\
\boldsymbol{\Pi}^1_{n+1} &= \{\mathbb{A}^\omega \setminus L \mid L \in \boldsymbol{\Sigma}^1_{n+1}\}
\end{aligned}$$

The ω-languages in the union of the classes $\boldsymbol{\Sigma}^1_n$ are called *projective sets*.

The analogues of Theorem 2.3 (inclusions between the classes $\boldsymbol{\Sigma}^1_n, \boldsymbol{\Pi}^1_n$) and of Theorem 2.11 (hierarchy theorem) are valid again.

Let us mention some facts on the relation between Borel sets and projective sets. First, there are projective sets which are not Borel. Let us describe a typical example set in $\boldsymbol{\Pi}^1_1$ which is not a Borel set, the set of (codes of) ω-branching *finite-path trees*. Here we mean by an ω-branching *tree* just a subset T of ω^* (the set of finite sequences of natural numbers) which is closed under forming prefixes. A *path* in T is a subset of T that is linearly ordered by the prefix relation and, moreover, maximal with this property. A *finite-path tree* is an ω-branching tree having only finite paths. These trees represent the behaviour of terminating programs that allow infinite branching.

Let us assume a coding of infinite trees by ω-words. For example, using the alphabet $\{0, 1, \#\}$ we may write an element of ω^* in the form $u_1\#u_2\#\ldots\#u_k\#\#$ where each u_i is a binary representation of a natural number. To code an infinite tree T we concatenate the codes of its nodes by increasing length of these words over $\{0, 1, \#\}$ and for fixed length arrange the codes in lexicographical order, thus obtaining an ω-word α_T representing T. Note that by our coding, the nodes on an infinite path of T have to occur in increasing order within α_T.

Theorem 2.13. *The ω-language*

$$FPT := \{\alpha_T \mid T \textit{ is a finite-path tree in } \omega^*\}$$

is in $\boldsymbol{\Pi}^1_1$ but not a Borel set.

We only indicate the first part of the claim. Clearly, a tree T is finite-path iff there is no infinite sequence of occurrences of the endmarker $\#\#$ in α_T such that the nodes coded in front of these endmarkers form a path. Formally, this can be expressed by a condition $\neg\exists\beta \in \{0,1\}^\omega\, \exists^\omega i\ (\alpha{}^\wedge\beta)[i] \in W$, where a word $(\alpha{}^\wedge\beta)[i]$ belongs to W iff its last letter from β is 1, there is a previous last occurrence of 1 in β, and these two occurrences of 1 mark the ends of codes of nodes u and v from ω^* such that u is parent of v. From this description we can easily infer that FPT belongs to $\boldsymbol{\Pi}^1_1$. We do not show here the more difficult claim that FPT is $\boldsymbol{\Pi}^1_1$-complete with respect to reducibility via continuous functions.

We end with a result that ties the projective sets to the Borel hierarchy. The first part characterizes the Borel sets by the classes $\boldsymbol{\Sigma}^1_1$ and $\boldsymbol{\Pi}^1_1$, the second allows to reduce the complexity of the Borel kernel within a projective representation. We note the second fact in two versions (as parts (b) and (c) of the theorem).

Theorem 2.14. *(a) An ω-language L is a Borel set iff $L \in \boldsymbol{\Sigma}^1_1 \cap \boldsymbol{\Pi}^1_1$.*
(b) For any ω-language $L \in \boldsymbol{\Sigma}^1_1$ there is an ω-language $L_0 \in \boldsymbol{\Pi}^0_2$ such that

$$\alpha \in L \iff \exists\beta_1 \ldots \exists\beta_k\ \alpha{}^\wedge\beta_1{}^\wedge \ldots{}^\wedge\beta_k \in L_0.$$

(c) For any ω-language $L \in \boldsymbol{\Sigma}^1_1$ there is a language W of finite words such that

$$\alpha \in L \iff \exists\beta_1 \ldots \exists\beta_k \exists^\omega i\ (\alpha{}^\wedge\beta_1{}^\wedge \ldots{}^\wedge\beta_k)[i] \in W.$$

The representation in (c), which follows from (b) by Theorem 2.5, can still be sharpened if one works in the so-called Baire space instead of the Cantor space. The Baire space consists of functions from ω to ω instead of ω-words (which are functions from ω to a finite set). In this case, the quantifier $\exists^\omega i$ can be replaced by $\forall i$. For more details, formulated in a language theoretical setting, see e.g. [Tho88a].

3 Logical and Automata Theoretic Descriptions

In Section 2.3 we described the first two levels of the Borel hierarchy by means of quantifications over prefixes in ω-words: In the characterization of $\boldsymbol{\Sigma}_1^0$ and $\boldsymbol{\Pi}_1^0$ one quantifier is applied in connection with a suitable set W of finite words ("some prefix is in W", resp. "all prefixes are in W"), whereas in the characterization of $\boldsymbol{\Sigma}_2^0$ and $\boldsymbol{\Pi}_2^0$ (Theorem 2.5) two such quantifiers are implicit ("for each prefix there is a longer one in W", resp. "there is a prefix such that all longer ones are in W"). In this section we shall see that all finite Borel levels are characterized in terms of quantifier alternation. This view facilitates the comparison with other hierarchies.

3.1 The Formalism S1S[$\mathcal{C}$]

When we specify properties of ω-words by logical formulas, we refer to an identification of an ω-word

$$\alpha = \alpha(0)\alpha(1)\ldots$$

over the alphabet $\mathbb{A} = \{0,1\}^r$ with a relational structure

$$\underline{\alpha} = (\omega, Succ, <, P_1^\alpha, \ldots, P_r^\alpha)$$

where *Succ* and $<$ are the usual successor relation and order relation over the set ω of the natural numbers, respectively, and

$$P_i^\alpha = \{j \in \omega \mid (\alpha(j))_i = 1\}.$$

For example, if

$$\alpha = (0,1)(0,0)(0,1)(0,0)(1,1)(1,0)(1,1)(1,0)(1,1)(1,0)\ldots$$

then $P_1^\alpha = \{4,5,6,7,\ldots\}$ and $P_2^\alpha = \{0,2,4,6,\ldots\}$.

The logical language in which we will formulate properties of ω-words is based on the system S1S ("monadic second-order logic of one successor"), which Büchi [Bü62] also called "Sequential Calculus". It has first-order variables $x, y, \ldots, x_1, x_2, \ldots$ ranging over natural numbers (i.e., positions in ω-words), and monadic second-order variables $X, Y, \ldots, X_1, X_2, \ldots$ ranging over sets of natural numbers (i.e., ω-words over $\{0,1\}$). Atomic formulas are those of the form $x = y$, $Succ(x,y)$, $x < y$, and $X(y)$, meaning that the positions x and y are equal, x has y as successor, x is smaller than y, and y belongs to X, respectively. Formulas of S1S are built up in the usual way from atomic formulas by means of the Boolean connectives $\neg, \vee, \wedge, \rightarrow, \leftrightarrow$, and the quantifiers $\exists, \forall$.

A formula $\varphi(X_1, \ldots, X_r)$ defines a set of ω-words over the alphabet $\{0,1\}^r$, namely the set of those ω-words α such that the corresponding structure $\underline{\alpha} = (\omega, Succ, <, P_1^\alpha, \ldots, P_r^\alpha)$ satisfies φ when X_i is interpreted by the set P_i^α. As an example over $\{0,1\}^2$, consider the formula

$$\exists x\ \forall y\ (x < y \rightarrow X_1(y)) \rightarrow \forall x\ \exists y\ (x < y \wedge \neg X_2(y))).$$

It defines the set of ω-words where the following holds: If from some position onwards the first component is 1, then infinitely often the second component is 0.

In the subsequent discussion it will be convenient to allow a third type of variable besides number variables and set variables, namely *finite-set variables* $F_1, F_2, \ldots$ that range exclusively over finite sets of natural numbers. Of course, their use could be avoided, because finiteness of sets is definable in S1S (a set X of natural numbers is finite iff for some number x, all elements of X are smaller than x). But variables for finite sets of natural numbers correspond directly to variables for finite words and thus will allow for more transparent formalizations.

Thus, in general, an S1S-formula will be of the form

$$\varphi(x_1, \ldots, x_p, F_1, \ldots, F_q, X_1, \ldots, X_r).$$

Considering the components separately, such a formula defines a relation

$$R_\varphi \subseteq \omega^p \times (2^{<\omega})^q \times (2^\omega)^r$$

where $2^{<\omega}$ is the set of finite subsets of ω and 2^ω the set of subsets of ω. We shall identify a number k with the singleton set $\{k\}$, and a set of natural numbers with the associated characteristic function (mapping n to 1 if n belongs to the set, and to 0 otherwise). Then each $(p+q+r)$-tuple in R_φ is given by a $(p+q+r)$-tuple $(\alpha_1, \ldots, \alpha_{(p+r+q)})$ of ω-words. As explained in Section 2.1 (on notations), we view this tuple as a single ω-word over the alphabet $\{0,1\}^{(p+q+r)}$. In order to indicate that in each of the first p components there is exactly one occurrence of 1 and in each of the next q components only finitely many occurrences of 1, we say that the alphabet as well as the ω-words over this alphabet are of *type* (p, q, r). Accordingly, we say that $\varphi(x_1, \ldots, x_p, F_1, \ldots, F_q, X_1, \ldots, X_r)$ defines an ω-language of type (p, q, r). Note that also an ω-word of say type $(0, 0, 1)$ may have just one or finitely many occurrences of 1.

The distinction between components in an alphabet $\{0,1\}^m$ by means of these types leads to some additional expressive power as compared with the "untyped" case, as it occurs in the consideration of formulas $\varphi(X_1, \ldots, X_r)$ where all components are arbitrary. Unless stated otherwise, we keep to the "untyped" case in the sequel.

Let us justify the choice of S1S as a basis of specifications of ω-word properties. First, S1S supplies the basic means for expressing requirements on segments of ω-words: With the free set variables X_i for describing letter occurrences, the successor relation *Succ*, and the order relation $<$ we can write down conditions on the existence and the ordered occurrence of segments of ω-words. This, however, could also be done in the first-order language of the above signature. We allow second-order quantifiers (set quantifiers) in order to obtain a transparent connection with the hierarchies of Section 1; indeed, for this purpose we need to simulate quantifiers over finite words (of possibly unbounded length) and over infinite words. Moreover, if we consider the regular languages and the regular ω-languages as representations of the most basic properties of computation sequences, then S1S is the appropriate logical framework (see Büchi's and McNaughton's Theorem below).

In order to pass beyond regular properties, we shall allow further atomic formulas which state conditions on finite segments of ω-words. For this purpose we refer to a class $\mathcal{C}$ of languages of *finite* words and define a corresponding extension S1S[$\mathcal{C}$] of S1S. More precisely, for any language W from $\mathcal{C}$, we allow conditions of the form "the segment from x to y belongs to W". Let us write such a new atomic formula, involving the language constant W, as

$$[x, y] \in W.$$

If x, y are interpreted in $\underline{\alpha}$ by positions i, j, this atomic formula is satisfied if $\alpha(i) \ldots \alpha(j) \in W$ (for $j < i$ we let $\alpha(i) \ldots \alpha(j)$ be the empty word). Given the class $\mathcal{C}$ of languages of finite words, we denote by S1S[$\mathcal{C}$] the extension of S1S thus defined.

Three natural choices of $\mathcal{C}$ will play a special role below, the class REG of regular languages, the class RCS of recursive languages, and the class ALL of all languages of finite words.

3.2 Prenex Normal Form

In the context of infinite word models, we shall classify formulas by a type of quantifier complexity which puts special emphasis on quantifications with unbounded range within ω-words. Conversely, quantifications within bounded domains will not be counted for an increase of quantifier complexity. This approach aims at separating the infinitary from the finitary aspects in sequence properties. The basis for a proper formalization of this idea is the notion of a *bounded formula*, which is well known from logical systems of arithmetic and was suggested for the study of ω-languages in [Tho81]. Bounded formulas contain only quantifiers which are bounded by the objects interpreting the free variables and which hence (in ω-word models) range only over a finite domain. For example, the formula

$$\exists y \; (y \leq x \wedge X_2(y) \;\rightarrow\; \forall z \; (y \leq z \wedge z \leq x \;\rightarrow\; X_1(z)))$$

is bounded (since the quantifier on y is bounded by the free variable x, and that on z by y and hence by x), whereas

$$\forall y \; (x < y \rightarrow \exists z \; (y < z \wedge X(z)))$$

is not bounded.

In the formal definition of bounded formula, the above mentioned constraint is captured by a syntactic restriction: All quantifiers are "relativized" to numbers and finite sets that are given by the free variables in the formula:

Definition 3.1. A formula $\varphi(x_1, \ldots, x_p, F_1, \ldots, F_q, X_1, \ldots, X_r)$ whose free variables are among $x_1, \ldots, x_p$, $F_1, \ldots, F_q$, $X_1, \ldots, X_r$, is called *bounded* if it only contains quantifiers of the form

$$\begin{aligned}
&\exists z \leq x_i,\, \exists z \leq max(F_j)\\
&\exists F \subseteq [0, x_i],\, \exists F \subseteq [0, max(F_j)],\\
&\forall z \leq x_i,\, \forall z \leq max(F_j)\\
&\forall F \subseteq [0, x_i],\, \forall F \subseteq [0, max(F_j)]
\end{aligned}$$

with $1 \leq i \leq p$ and $1 \leq j \leq q$.

The semantics over ω-word models is defined in the natural way: These quantifiers claim the existence of an element z bounded by the interpretation of x_i, respectively bounded by an element of the set interpreting F_i, the existence of a set F bounded by the interpretation of x_i, respectively bounded by an element of the set interpreting F_i; similarly for the universal quantifiers.

It is clear that we can express all quantifiers within the original syntax of S1S. Henceforth we consider bounded formulas of S1S[$\mathcal{C}$] as special S1S[$\mathcal{C}$]-formulas.

Let us verify that we can put every S1S[$\mathcal{C}$]-formula into a form where a quantifier prefix first contains unrestricted set quantifiers and then quantifiers over finite sets and elements, followed by a "kernel" which is a bounded formula.

Theorem 3.2. *An S1S[$\mathcal{C}$]-formula $\varphi(X_1, \ldots, X_r)$ can be brought into the following prenex normal form:*

$$\underbrace{\exists \overline{Y}_1 \forall \overline{Y}_2 \ldots \overset{\exists}{\forall}\, \overline{Y}_m}_{m \text{ blocks}}\, \underbrace{\exists \overline{F}_1, \overline{x}_1 \forall \overline{F}_2, \overline{x}_2 \ldots \overset{\exists}{\forall}\, \overline{F}_n, \overline{x}_n}_{n \text{ blocks}}$$
$$\underbrace{\varphi_0(X_1, \ldots, X_r, \overline{Y}_1, \ldots, \overline{Y}_m, \overline{F}_1, \overline{x}_1, \ldots, \overline{F}_n, \overline{x}_n)}_{\text{bounded}}$$

where φ_0 is bounded (in the tuples of variables $\overline{F}_1, \overline{x}_1, \ldots, \overline{F}_n, \overline{x}_n$).

Proof. There are three steps: First, one applies the prenex normal form of quantifier logic (where one may ignore the quantifiers that occur in subformulas which are bounded). Second, the quantifiers ranging over finite sets and numbers are shifted towards the inside. The final step consists in forming maximal blocks of quantifiers of the same type (existential, universal). Then the form as given in the statement of the theorem is reached.

Only the second step needs some explanation. Let us consider two adjacent quantifiers where the outer one ranges over numbers or finite sets and the inner one over sets. The quantifiers may be exchanged if both are of the same type (existential or universal). Let us consider one of the remaining cases, say a formula $\exists x \forall Y\ \psi$. The idea is to introduce a new set variable X for a set that contains numbers from x onwards but not up to x (say the singleton $\{x\}$), thus capturing the quantification on x. Namely, we have the equivalence

$$\exists x\, \forall Y\ \psi \iff \exists X\, \forall Y\, \exists x\ (X(x) \wedge \forall z\ (z < x \rightarrow \neg X(z)) \wedge \psi)$$

over all ω-word models. The other cases, with universal and and existential quantifier exchanged, respectively with a finite set variable F in place of x, are handled similarly. □

A formula as in the statement of the theorem with m blocks of set quantifiers and a kernel with quantifiers over finite sets and numbers only is called a $\boldsymbol{\Sigma}^1_m$-formula of S1S[$\mathcal{C}$]. If $m = 0$, we speak of a $\boldsymbol{\Sigma}^0_n$-formula of S1S[$\mathcal{C}$]. Analogously, we speak of $\boldsymbol{\Pi}^1_m$-formulas and $\boldsymbol{\Pi}^0_n$-formulas if the leading quantifiers are universal.

The class of ω-languages definable by a $\boldsymbol{\Sigma}^0_n$-sentence of S1S[$\mathcal{C}$] will be denoted $\boldsymbol{\Sigma}^0_n[\mathcal{C}]$, similarly for $\boldsymbol{\Pi}^0_n$, $\boldsymbol{\Sigma}^1_n$, $\boldsymbol{\Pi}^1_n$.

3.3 S1S[ALL] and S1S[RCS]

The prenex normal form with bounded formulas as kernels provides the link between the set theoretical classification of Section 2 and a logical counterpart:

Theorem 3.3. *Let $\mathbb{A} = \{0,1\}^r$. An ω-language $L \subseteq \mathbb{A}^\omega$ belongs to the class $\boldsymbol{\Sigma}^0_n$ of the Borel hierarchy iff L is defined in* S1S[ALL] *by a $\boldsymbol{\Sigma}^0_n$-formula $\varphi(X_1, \ldots, X_r)$.*

Proof. It will suffice to treat a typical example for each direction. The general claim is proved accordingly by induction.

First consider an ω-language L in $\boldsymbol{\Sigma}^0_3$, which may be presented in the form

$$L = \bigcup_{i \geq 0} \bigcap_{j \geq 0} \bigcup_{k \in N_{ij}} L_k$$

where each N_{ij} is a set of natural numbers and $L_0, L_1, \ldots$ is an enumeration of a basis of open sets; we assume that $L_i = \{w_i\} \cdot \mathbb{A}^\omega$ for an enumeration $w_0, w_1, \ldots$ of $\mathbb{A}^*$.

Call a triple (i,j,k) *good* if $k \in N_{ij}$. We may arrange the good triples in a (generally infinite) sequence, which itself may be coded by an ω-word α_L over $\{0,1\}$ (we do not need to describe the coding in detail). Let W_L be the set of finite prefixes of α_L. Consider the set Q_L of quadruples (v_1, v_2, v_3, v) of finite words such that

- v_1 has exactly one occurrence of 1, say at position i,
- v_2 has exactly one occurrence of 1, say at position j,
- v_3 belongs to W_L and contains a segment coding a good triple (i,j,k) where i, j are as given by v_1, v_2,
- v has w_k (given by the above k) as prefix.

Then we have

$$\alpha \in L \iff \exists v_1 \, \forall v_2 \, \exists v_3 \, \exists l \;\, (v_1, v_2, v_3, \alpha[l]) \in Q_L .$$

Without loss of generality we may assume that the last three components of a quadruple of Q_L are of the length l of the first component $\alpha(0) \ldots \alpha(l-1)$

(note that we can fill the second and third component by 0's and that the fourth component can be adjusted to any sufficiently large length). Thus Q_L specifies a set V_L of words over $\{0,1\}^{r+3}$. Now the $\boldsymbol{\Sigma}_3^0$-formula

$$\exists F_1 \, \forall F_2 \, \exists F_3 \, \exists x \; [0, x] \in V_L$$

of S1S[ALL] defines L.

Conversely, consider an ω-language $L \subseteq (\{0,1\}^r)^\omega$ that is defined by a $\boldsymbol{\Sigma}_3^0$-formula, say by

$$\varphi : \exists F \forall x \exists y \; \varphi_0(F, \, x, \, y, \, X_1, \ldots, X_r)$$

with bounded φ_0. A model of φ_0 is a quadruple $(\alpha_1, \alpha_2, \alpha_3, \alpha)$ with $\alpha \in \{0,1\}^r$ and $\alpha_1, \, \alpha_2, \, \alpha_3 \in \{0,1\}^\omega$, where α_2 has only finitely many occurrences of 1 and each of α_2, α_3 just one occurrence of 1. Let $m(\alpha_1, \alpha_2, \alpha_3)$ be the maximal position in these ω-words where 1 occurs. Since φ_0 is bounded, truth of φ_0 in $(\alpha_1, \alpha_2, \alpha_3, \alpha)$ only depends on the prefix of the model up to position $m(\alpha_1, \alpha_2, \alpha_3)$. Let W_0 be the set of such prefixes in $(\{0,1\}^{r+3})^*$. Thus we obtain that

$$\alpha \in L \Leftrightarrow \exists w_1 \in \{0,1\}^* \forall w_2 \in \{0,1\}^* \exists w_3 \in \{0,1\}^* \exists l \; (w_1, w_2, w_3, \alpha[l]) \in W_0 .$$

From this equivalence and by countability of $\{0,1\}^*$ one obtains a representation of L as a $\boldsymbol{\Sigma}_3^0$-set in the sense of the preceding section. (Strictly speaking, one has to pass from sequence tuples $(\alpha_1, \alpha_2, \alpha_3, \alpha)$ to sequences $\alpha_1{}^\wedge\alpha_2{}^\wedge\alpha_3{}^\wedge\alpha$, and to use a corresponding set $W_0' \subseteq (\{0,1\}^r)^*$ instead of W_0.) □

From Section 2.3 we obtain more refined representations of sets in $\boldsymbol{\Sigma}_n^0$ and $\boldsymbol{\Pi}_n^0$ for $n = 1$ and $n = 2$, namely by formulas $\exists x \, \varphi_0(\overline{X}, \, x)$ and $\forall^\omega x \, \varphi_0(\overline{X}, \, x)$ with bounded φ_0 for sets in $\boldsymbol{\Sigma}_1^0$, $\boldsymbol{\Sigma}_2^0$, respectively, and by corresponding formulas with leading quantifiers $\forall$ and $\exists^\omega$ for sets in $\boldsymbol{\Pi}_1^0$, $\boldsymbol{\Pi}_2^0$. This result may be generalized, as suggested by the example ω-languages mentioned at the end of Section 2.4. There we described universal sets for the finite levels of the Borel hierarchy just using first-order quantifiers $\exists^\omega y$ and $\forall^\omega y$. Without proof we note here that an ω-language L over $\{0,1\}^r$ is in $\boldsymbol{\Pi}_{2n}^0$ iff it is definable by a formula $\exists^\omega x_1 \ldots \exists^\omega x_n \varphi_0(X_1, \ldots X_r, x_1, \ldots, x_n)$ with bounded φ_0. (For the recursion theoretic version of this result see [Rog67], Chapter 14.8.)

The extension of the preceding theorem to the projective hierarchy is straightforward:

Theorem 3.4. *An ω-language $L \subseteq (\{0,1\}^r)^\omega$ belongs to the projective class $\boldsymbol{\Sigma}_m^1$ iff L is defined in* S1S[ALL] *by a $\boldsymbol{\Sigma}_m^1$-formula $\varphi(X_1, \ldots, X_r)$.*

Proof. Let $\mathbb{A} = \{0,1\}^r$. The proof is an easy induction on m, including the corresponding claim on $\boldsymbol{\Pi}_m^1$-formulas.

For the direction from right to left first consider the case $m = 1$. Assume $L \subseteq \mathbb{A}^\omega$ is defined by a formula

$$\exists \overline{Y}_1 \varphi(X_1, \ldots, X_r, \overline{Y}_1)$$

where φ only contains quantifiers over finite sets or numbers and, without loss of generality, we assume that $\overline{Y}_1$ has length $k \cdot r$. Then, by the preceding theorem, φ defines a Borel set $L_0 \subseteq (\{0,1\}^{r+kr})^\omega$. Let L_0' be the ω-language over $\mathbb{A} = \{0,1\}^r$ containing the ω-words $\alpha^\wedge \beta_1^\wedge \dots ^\wedge \beta_k$ where $\alpha, \beta_1, \dots \beta_k$ is in L_0. Also L_0' is a Borel set. Now

$$\alpha \in L \iff \exists \beta_1 \dots \exists \beta_k \, \alpha^\wedge \beta_1{}^\wedge \dots ^\wedge \beta_k \in L_0'.$$

Hence $L \in \boldsymbol{\Sigma}_1^1$. The case of a $\boldsymbol{\Pi}_1^1$-formula is analogous. In the induction step from m to $m+1$ the argument is the same, obtaining by induction hypothesis a $\boldsymbol{\Pi}_m^1$-set, resp. a $\boldsymbol{\Sigma}_m^1$-set L_0'.

For the reverse direction, we start with the representation of Theorem 2.14(b) in order to describe $\boldsymbol{\Sigma}_1^1$-sets by $\boldsymbol{\Sigma}_1^1$-formulas of S1S[ALL]. Again, the induction step is obvious. □

In the remainder of the paper we consider ω-languages defined in S1S[$\mathcal{C}$] for more restricted classes $\mathcal{C}$.

A natural choice is the class RCS of *recursive languages* of finite words, which are recognized by deterministic Turing machines that terminate on each input word. The classes $\boldsymbol{\Sigma}_n^0[RCS]$ (for $n > 1$) form the *arithmetical hierarchy* of ω-languages, and the classes $\boldsymbol{\Sigma}_n^1[RCS]$ the *analytical hierarchy* (see [Rog67], [Hin78], [Mos80]). Both hierarchies are infinite, and the proofs are given along the lines of Section 2, with modifications to meet the effectiveness constraints. For example, the continuous reductions as introduced in the previous section are to be replaced by effectively computable ones (given by Turing machines that for ω-word-inputs produce ω-word-outputs letter by letter from left to right on an output tape).

Let us add some remarks on the notion of *recursive ω-language*, which is treated in different ways in the literature. Usually, the class of *recursively enumerable ω-languages* is identified with $\boldsymbol{\Sigma}_1^0[RCS]$. As a machine oriented characterization one obtains that an ω-language $L \subseteq (\{0,1\}^r)^\omega$ is recursively enumerable iff a Turing machine exists which terminates exactly on the input ω-words α from L. (The input tape can be assumed to be worktape as well.) A typical example is the set of ω-words (α_1, α_2) describing two sets with nonempty intersection, i.e., with some position i where $\alpha_1(i) = \alpha_2(i) = 1$.

In this setting, the *recursive ω-languages* are those in the intersection $\boldsymbol{\Sigma}_1^0[RCS] \cap \boldsymbol{\Pi}_1^0[RCS]$. Thus a recursive ω-language L over a given alphabet $\mathbb{A}$ is both open and closed, and hence, by Lemma 2.1, there is a *finite* set W of finite words with $L = W \cdot \mathbb{A}^\omega$. Equivalently, an ω-language L is recursive iff there is a Turing machine that terminates on each ω-word as input, yielding a decision (say by final states) on membership of the input in L.

The above definitions apply to ω-languages $L \subseteq (\{0,1\}^r)^\omega$ which are *untyped* in the sense of Section 3.1, i.e., where the components of the alphabet are not distinguished into number components, finite-set components, and arbitrary set components. In classical recursion theory, one usually assumes such a distinction, at least between number components and set components (see [Rog67], Chapter 14). While in the untyped approach the property of being a singleton has to be specified explicitly in the corresponding S1S[RCS]-definition or by a defining Turing machine, this is superfluous over typed alphabets. Thus, over an alphabet say of type $(1, 0, 1)$, corresponding to formulas $\varphi(x, X)$, the relation "number x belongs to set X" is definable just by $X(x)$ and hence belongs to the intersection $\boldsymbol{\Sigma}^0_1[RCS] \cap \boldsymbol{\Pi}^0_1[RCS]$, i.e., is recursive, whereas in the setting of untyped alphabets, corresponding to formulas $\varphi(X_1, X_2)$, the associated set of ω-words (α_1, α_2) with $\alpha_1(i) = 1$ for a unique i, for which moreover $\alpha_2(i) = 1$ holds, is only in the Boolean closure of $\boldsymbol{\Sigma}^0_1[RCS]$.

Another variant of the notion "recursive ω-language" refers to a stronger model of Turing machine acceptance, similar to the idea of Büchi automaton to be discussed in the subsequent section. In Staiger's paper [Sta86] and the work surveyed there, an ω-language is called "recursive" if it contains the ω-words accepted by a nondeterministic Turing machine using computations that pass infinitely often through a final state. It is easy to see that this class of recursive ω-languages coincides with $\boldsymbol{\Sigma}^1_1[RCS]$; it strictly contains the recursive ω-languages in either of the above mentioned two versions.

The set theoretic and recursion theoretic hierarchies show strong similarities, which are manifested in the strictness proofs by diagonalization and the characterization of levels by complete sets and suitable notions of reducibility. A completely different situation arises with the ω-languages defined by finite automata, to be discussed in the next section.

3.4 S1S[REG]

As Büchi showed in his pioneering papers [Bü60] and [Bü62], the formalism S1S allows to express just those properties of words and ω-words which can be recognized by finite automata. So S1S is an expressively complete formalism for describing the behaviour of finite-state systems. We first state this connection for sets of finite words, using the notion of bounded formula of Section 3.1.

We say that a set $W \subseteq (\{0,1\}^r)^+$ of finite words is *defined by the bounded formula* $\varphi_0(X_1, \ldots, X_r, x)$ (in which all quantifiers are relativized to the prefix up to position x) if W contains exactly the prefixes $\alpha(0) \ldots \alpha(i)$ of ω-words α such that α satisfies φ when interpreting x by i.

Theorem 3.5. ([Bü60], [Elg61])
A language $W \subseteq (\{0,1\}^r)^+$ is definable by a bounded S1S*-formula iff it is regular.*

An analogous result holds when we consider arbitrary finite segments of ω-words instead of prefixes of ω-words. Adequate S1S formulas involve the parameters x, y as free variables for the minimal and maximal elements of segments, and the notion of bounded formula has to be adjusted, using relativizations to the segment $[x, y]$ instead of $[0, x]$. Thus, the atomic formulas $[x, y] \in W$ for regular W, which are allowed in S1S[REG], can be eliminated, and we have that definability in S1S[REG] is equivalent to definability in S1S.

A transfer of the above theorem to ω-words is built on the notion of *regular ω-language*. We use here a definition based on finite automata over ω-words.

A *Büchi automaton* over $\mathbb{A}$ is a nondeterministic finite automaton of the form $\mathcal{A} = (Q, \mathbb{A}, q_0, \Delta, F)$ where Q is a finite set of states, $q_0 \in Q$ the initial state, $\Delta \subseteq Q \times \mathbb{A} \times Q$ the transition relation, and $F \subseteq Q$ the set of final states. The automaton accepts the ω-word α if there is a sequence ("run") $\beta \in Q^\omega$ such that $\beta(0) = q_0$, $(\beta(i), \alpha(i), \beta(i+1)) \in \Delta$ for $i \geq 0$, and $\beta(i) \in F$ for infinitely many i. An ω-language is *Büchi recognizable* if a Büchi automaton exists which accepts exactly the ω-words of L.

The main result of [Bü62] states that an S1S-definable ω-language is Büchi-recognizable. The crucial point in this transformation of logical formulas into automata is the proof that the class of Büchi recognizable ω-languages is closed under complement. Let us see how this result can be understood as a reduction of S1S-formulas to a special form. (In fact, this is the way in which the result is presented in [Bü62]; automata appear there only implicitly.)

Assuming that $\mathbb{A} = \{0,1\}^r$ and $Q = \{0,1\}^s$, the condition that an ω-word is accepted by a given Büchi automaton is directly expressible in S1S, namely by a formula

$$\exists Y_1 \ldots \exists Y_s \exists^\omega x \varphi_0(X_1, \ldots, X_r, Y_1, \ldots, Y_s, x)$$

where φ_0 is a bounded formula expressing: "the state sequence coded by $Y_1, \ldots, Y_s$ up to x begins with q_0, is compatible with the transition relation Δ, and ends at position x with a state in F". To give a more formal presentation of φ_0, we take an example with $s = 2$: Assume the initial state is $(0,0)$, there is one final state $(0,1)$, and the transition relation contains, for instance, the triples $((0,0),1,(1,0))$ and $((1,0),0,(1,1))$. Then $\varphi_0(X, Y_1, Y_2, x)$ has the following form:

$$\neg Y_1(0) \wedge \neg Y_2(0) \wedge \forall y \leq x\, \forall z \leq x\ (Succ(y,z) \rightarrow \psi(y,z)) \wedge \neg Y_1(x) \wedge Y_2(x)$$

Here a formula $Y(0)$ should be viewed as an abbreviation for the bounded formula $\exists z \leq x(Y(z) \wedge \neg \exists y \leq x(y < z))$, and $\psi(y,z)$ stands for a disjunction over the possibilities that on successor positions y, z an automaton transition is applied. In our example, with the transitions $((0,0),1,(1,0))$ and $((1,0),0,(1,1))$, the disjunction will start as follows:

$$\begin{aligned}&(\neg Y_1(y) \wedge \neg Y_2(y) \wedge X(y) \wedge Y_1(z) \wedge \neg Y_2(z))\\ &\vee((Y_1(y) \wedge \neg Y_2(y) \wedge \neg X(y) \wedge Y_1(z) \wedge Y_2(z)) \vee \ldots.\end{aligned}$$

Summarizing, we can state the main result of [Bü62] as follows:

Theorem 3.6 (Büchi's Theorem).
An S1S-*formula* $\varphi(X_1, \ldots, X_r)$ *can be written as a* $\boldsymbol{\Sigma}_1^1$-*formula*

$$\exists \overline{Y} \exists^\omega x\, \varphi_0(X_1, \ldots, X_r, \overline{Y},\ x)$$

with bounded φ_0.

In other words, the projective hierarchy over REG collapses at the first level.

In a second step, we obtain an additional reduction. The basis is McNaughton's Theorem [McN66], which is usually stated as a determinization result for automata on ω-words. Our formulation uses deterministic Rabin automata. A (deterministic) *Rabin automaton* is of the form $\mathcal{A} = (Q, \mathbb{A}, q_0, \delta, \Omega)$, where $Q, \mathbb{A}, q_0$ are as for Büchi automata, $\delta : Q \times \mathbb{A} \to Q$ is a transition function, and $\Omega = \{(U_1, L_1), \ldots, (U_n, L_n)\}$ is a set of "accepting pairs" with $U_i, L_i \subseteq Q$ for $i = 1, \ldots, n$. The automaton accepts an ω-word $\alpha \in \mathbb{A}^\omega$ if for some $i \in \{1, \ldots, n\}$, the unique state sequence (run) of $\mathcal{A}$ on α, determined by the initial state q_0 and the transition function δ, passes infinitely often through a state of U_i but only finitely often through a state from L_i. This can be formulated by a S1S-formula

$$\bigvee_{i=1}^{n} \left(\exists^\omega x\, \varphi_i(X_1, \ldots, X_r,\, x) \wedge \forall^\omega x\, \varphi'_i(X_1, \ldots, X_r,\, x)\right)$$

where φ_i, φ'_i are bounded formulas which express the existence of a finite run of $\mathcal{A}$ up to x on the input, ending in a state of U_i, resp. in a state outside L_i. Assuming again that $Q = \{0,1\}^s$, the formulas φ_i, φ'_i are of the form

$$\exists F_1 \subseteq [0, x] \ldots \exists F_s \subseteq [0, x]\ \psi(X_1, \ldots, X_r, F_1, \ldots, F_s, x)$$

where ψ is bounded (in x). Note that the determinism of the automaton ensures that the different formulas $\varphi_i(X_1, \ldots, X_r,\, x)$ and $\varphi'_i(X_1, \ldots, X_r,\, x)$ all speak about the *same* run on the input ω-word.

Thus we can formulate McNaughton's Theorem on the equivalence of nondeterministic Büchi automata with deterministic Rabin automata in the following way:

Theorem 3.7 (McNaughton's Theorem).
A $\boldsymbol{\Sigma}^1_1$-formula $\varphi(X_1, \ldots, X_r)$ of S1S *can be written as a Boolean combination of $\boldsymbol{\Sigma}^0_2$- (or $\boldsymbol{\Pi}^0_2$-)formulas, more precisely in the form*

$$\bigvee_{i=1}^{n} \left(\exists^\omega x\, \varphi_i(X_1, \ldots, X_r,\, x) \wedge \forall^\omega x\, \varphi'_i(X_1, \ldots, X_r,\, x)\right)$$

with bounded formulas φ_i, φ'_i.

Thus the class of regular ω-languages, $\boldsymbol{\Sigma}^1_1[REG]$, coincides with the Boolean closure of $\boldsymbol{\Sigma}^0_2[REG]$.

The formula of the theorem can be dualized with respect to the displayed Boolean connectives. The dual form, which is a conjunction of disjunctions with two members each, corresponds to automata with the so-called "Streett acceptance condition", which can be shown to be equivalent to Rabin acceptance in expressive power. Other (strictly weaker) acceptance conditions are obtained if only one formula with leading quantifier $\exists^\omega$ or $\forall^\omega$ is kept, or if these quantifiers are simplified to $\exists$ or $\forall$. The corresponding ω-language classes are $\boldsymbol{\Pi}^0_2[REG]$,

$\boldsymbol{\Sigma}_2^0[REG]$, $\boldsymbol{\Sigma}_1^0[REG]$, and $\boldsymbol{\Pi}_1^0[REG]$, respectively. These classes are characterized by deterministic finite automata with appropriate acceptance types, where a final state is required to occur infinitely often, only finitely often, at least once, or throughout the computation on an infinite word. There are many notations in the literature for these four acceptance types. In a short synopsis, and taking the order $\exists, \forall, \exists^\omega, \forall^\omega$ of quantifiers in front of a bounded formula, the associated acceptance types are called

- 1-, 1′-, 2, 2′-acceptance in [Lan69],
- $(E, \not\subseteq)$, $(E, \subseteq)$, $(U, \not\subseteq)$, $(U, \subseteq)$ in [Sta87],
- $(\mathit{ran}, \sqcap)$, $(\mathit{ran}, \subseteq)$, $(\mathit{inf}, \sqcap)$, $(\mathit{inf}, \subseteq)$ in [EH93].

Landweber [Lan69] established a nice connection between these special classes of regular ω-languages and the unrestricted Borel hierarchy. In our logical formulation, the result is stated as follows:

Theorem 3.8 (Landweber's Theorem).
An ω-language belongs to $\boldsymbol{\Sigma}_1^0[REG]$ iff it is regular and in the Borel class $\boldsymbol{\Sigma}_1^0$; analogous equivalences hold for $\boldsymbol{\Pi}_1^0[REG]$, $\boldsymbol{\Sigma}_2^0[REG]$, and $\boldsymbol{\Pi}_2^0[REG]$. Moreover, membership of a regular ω-language (presented by a Büchi automaton, for example) in any of these four classes is decidable.

In the Boolean closure of $\boldsymbol{\Sigma}_2^0[REG]$ and $\boldsymbol{\Pi}_2^0[REG]$, there is a further classification of regular ω-languages in terms of the so-called "Rabin index". It distinguishes Rabin automata by the number n of accepting pairs (U_i, L_i), or equivalently, distinguishes the S1S-formulas appearing in McNaughton's Theorem by the number n of disjunction members. The *Rabin index* of a regular ω-language L is the minimal such n in a representation of L by a Rabin automaton, resp. S1S-formula. A fundamental paper by Wagner [Wag79] establishes that the Rabin index is effectively computable and induces a strict hierarchy of ω-languages. For a more recent and streamlined treatment see [Car93].

Also the acceptance types of *nondeterministic* automata arise naturally in the logical framework. Here we refer to $\boldsymbol{\Sigma}_1^1$-formulas as they occur in Büchi's Theorem:

$$\exists \overline{Y}\, \exists^\omega x\ \varphi_0(\overline{X}, \overline{Y}, x)$$

with bounded φ_0. In modifying their first-order kernel, we may change the quantifier $\exists^\omega$ following the set quantifiers to $\exists$, $\forall$, or $\forall^\omega$. Let us denote the classes of ω-languages thus defined by by $\boldsymbol{\Sigma}_1^1$-$\boldsymbol{\Sigma}_1^0[REG]$, $\boldsymbol{\Sigma}_1^1$-$\boldsymbol{\Pi}_1^0[REG]$, $\boldsymbol{\Sigma}_1^1$-$\boldsymbol{\Sigma}_2^0[REG]$, respectively, and the original one by $\boldsymbol{\Sigma}_1^1$-$\boldsymbol{\Pi}_2^0[REG]$. Similarly as in Landweber's Theorem, these classes have been characterized as the intersection of the class of regular ω-languages with Borel classes, namely (in the order of the above list) with $\boldsymbol{\Sigma}_1^0$, $\boldsymbol{\Pi}_1^0$, $\boldsymbol{\Sigma}_2^0$, and the Boolean closure of $\boldsymbol{\Pi}_2^0$, respectively. For detailed references see [Sta87], [Wag79].

Büchi's and McNaughton's Theorems are presented here as results on elimination of quantifiers, and the acceptance conditions of automata on ω-words match natural classes of formulas, starting from bounded formulas. In particular, Büchi's Theorem reduces the projective hierarchy over REG to the first level, and McNaughton's Theorem even further to a class within level $\boldsymbol{\Sigma}_3^0[REG]$ (the Boolean closure of $\boldsymbol{\Sigma}_2^0[REG]$). Let us make explicit two important consequences:

Theorem 3.9. *(a) Any regular ω-language is a Borel set.*
(b) Any S1S*-definable ω-language is weakly* S1S*-definable (using only set quantifiers that range over finite sets).*

It may look unusual to view finite automata as a conceptual aid for quantifier elimination, as done in this exposition. However, this approach reflects the historical development of the theory (see e.g. the introduction and historical comments in [Bü83]), and it suggests some interesting generalizations. Later on we mention some of them.

3.5 Aspects of First-Order Definability

Many interesting sequence properties are already formalizable in the first-order fragment of S1S. In this case we use the syntax of S1S, involving the successor relation *Succ* and the order relation $<$, as well as free set variables $X_1, \ldots, X_r$ when ω-languages over $\{0,1\}^r$ are to be described, but leave out set quantifications. Let us denote this formalism by FO($<$). In this section we discuss the role of McNaughton's Theorem in the context of FO($<$), as well as in the related framework of propositional temporal logic. (This is just a single aspect of a wide subject; for a more detailed and complete exposition see the monograph [MP92] or [Em90].)

While statements about the occurrence of fixed words as segments in ω-words are directly formalizable in FO($<$) just as in S1S, it is well-known that S1S-definable properties exist which are not expressible in FO($<$). In particular, properties involving *modular counting* cannot be expressed in FO($<$), for instance the condition that between any two occurrences of 1 there is an even number of occurrences of 0 ([MP71]).

Another difference between S1S and FO($<$) shows up when the order relation $<$ is dropped from the signature and only the successor relation is kept. In S1S, $x < y$ is definable by stating that y belongs to any successor-closed set that contains the successor of x:

$$\exists x' \, (Succ(x,x') \wedge \forall X \; (X(x') \wedge \forall z \, \forall z' \; (X(z) \wedge Succ(z,z') \rightarrow X(z')) \rightarrow X(y))).$$

On the other hand, cancelling $<$ in FO($<$) results in a heavy loss in expressive power: The first-order framework of successor, denoted FO(*Succ*), only allows to express Boolean combinations of statements of the form "there are at least k occurrences of a segment w". Thus only "local properties" of words or ω-words are expressible in FO(*Succ*) (see [Tho82a] for more details). Within S1S, the fragments FO($<$) and FO(*Succ*) are decidable in the sense that it can be checked effectively for an S1S-definable language (or ω-language) whether it is definable in FO($<$), resp. FO(*Succ*). This is obtained from an application of the algebraic (semigroup theoretic) analysis of regular languages and ω-languages. For the latter, these decidability results were proved in [Per84] and [Wil93].

McNaughton's Theorem says that all ω-languages definable by FO($<$)-formulas are definable by Boolean combinations of formulas $\exists^\omega x\varphi_0$ with bounded φ_0 from S1S. Within FO($<$), we may ask for a representation by Boolean combinations of formulas $\exists^\omega x\varphi_0$ with bounded *first-order* formula φ_0. The following result establishes this "McNaughton normal form" for FO($<$):

Theorem 3.10. ([Tho81])
A formula $\varphi(X_1, \ldots, X_r)$ *of* FO($<$) *can be written in the form*

$$\bigvee_{i=1}^{n} \left(\exists^\omega x\ \varphi_i(X_1, \ldots, X_r, x) \wedge \forall^\omega x\ \varphi_i'(X_1, \ldots, X_r, x)\right)$$

with bounded φ_i *and* φ_i'.

In the specification of computation properties a variant of FO($<$) is frequently used, which is expressively equivalent but rather different in practical use: *propositional temporal logic* of linear time. Here, instead of quantifiers and variables for "positions" or "time instances" in ω-words, one finds temporal operators applied to propositional formulas. Often this leads leads to readable and short formulas; on the other hand there are cases where FO($<$) is more flexible. In particular, a variable x for a time instant may be referenced more than once in a first-order formula, while the syntax of temporal operators allows only one such reference. In order to illustrate the connection with FO($<$), we introduce here the system PTL of propositional temporal logic as a fragment of FO($<$), using formulas with one free variable x (which is to be interpreted by position 0 in an ω-word model). Let us denote this first-order version of PTL by f-PTL. In the defining clauses we also add the standard notation for the used temporal operators "next" (X), "eventually" (F), "always" (G), and "until" (U).

Definition 3.11. The f-PTL-formulas over $\{0,1\}^r$ are defined inductively by the following clauses:

- $X_i(x)$ is a f-PTL-formula for $i = 1, \ldots, r$;
- if $\varphi(x), \psi(x)$ are f-PTL-formulas, so are $\neg\varphi(x)$, $\varphi(x) \vee \psi(x)$, $\varphi(x) \wedge \psi(x)$, and $\varphi(x) \rightarrow \psi(x)$;
- if $\varphi(y), \psi(z)$ are f-PTL-formulas, so are

$$\begin{array}{rl}
(\mathrm{X}\varphi) & \exists y\ \big(Succ(x,y) \wedge \varphi(y)\big) \\
(\mathrm{F}\varphi) & \exists y\ \big(x \leq y \wedge \varphi(y)\big) \\
(\mathrm{G}\varphi) & \forall y\ \big(x \leq y \rightarrow \varphi(y)\big) \\
(\varphi \mathrm{U} \psi) & \exists y\ \big(x \leq y \wedge \varphi(y) \wedge \forall z\big(x \leq z < y \rightarrow \psi(z)\big)\big)
\end{array}$$

Consider, for instance, the following property of ω-words over $\{0,1\}^2$: "after any letter with first component 1 there appears another letter with first component 1 such that between them only letters with second component 0 occur".

This property is formalized in classical PTL with two atomic propositions p_1, p_2 as follows:

$$\mathrm{G}(p_1 \to \mathrm{X}((\neg p_2)\mathrm{U}p_1))$$

or, written in f-PTL,

$$\forall x(X_1(x) \to \exists y(Succ(x,y) \land \exists z(y \leq z \land X_1(z) \land \forall z'(y \leq z' < z \to \neg X_2(z')))))$$

or, slightly shorter,

$$\forall x(X_1(x) \to \exists z(x < z \land X_1(z) \land \forall z'(y \leq z' < z \to \neg X_2(z')))).$$

A difficult theorem due to Kamp establishes the expressive equivalence of PTL and FO($<$) over finite as well as infinite words (for references see [Em90]). The result is of deeper interest because the satisfiability problem of PTL is in PSPACE while that of FO($<$) is nonelementary. This indicates that expressing certain properties needs excessively long formulas in PTL compared with formalizations in FO($<$), contrary to the superficial impression one might have from the above example. A detailed analysis of these phenomena still seems to be open.

Each operator of propositional temporal logic involves an unbounded quantifier. The only point where bounded quantifiers occur appears in the "until"-operator. Thus it is impossible to represent bounded formulas in PTL directly. A natural approach to incorporate bounded formulas into PTL is to allow *past operators*, as suggested by Lichtenstein, Pnueli, Zuck in [LPZ85]. These operators are X^- ("previous"), F^- ("once"), G^- ("has always been"), and U^- ("since"). We extend the syntax of f-PTL by the clause

– If $\varphi(y), \psi(z)$ are PTL-formulas, so are

$$\begin{array}{rl}
(\mathrm{X}^-\varphi) & \exists y \ \big(Succ(y,x) \land \varphi(y)\big) \\
(\mathrm{F}^-\varphi) & \exists y \ \big(y \leq x \land \varphi(y)\big) \\
(\mathrm{G}^-\varphi) & \forall y \ \big(y \leq x \to \varphi(y)\big) \\
(\varphi\mathrm{U}^-\psi) & \exists y \ \big(y \leq x \land \psi(y) \land \forall z \ \big(y < z \leq x \to \varphi(z)\big)\big)
\end{array}$$

If a formula of the extended framework only contains past operators (viz. their descriptions in first-order logic) we speak of a *past formula*. By an adaptation of Kamp's theorem mentioned above, a bounded formula of FO($<$) with free variable x can be expressed by a past formula of f-PTL, also with the free variable x. Now, using the McNaughton normal form of FO($<$) (Theorem 3.10), one obtains

Theorem 3.12. ([LPZ85])
Any formula $\varphi(X_1, \ldots, X_r)$ of FO($<$) *is equivalent to a* PTL-*formula of the form*

$$\bigvee_{i=1}^{n} (\mathrm{GF}\varphi_i \land \mathrm{FG}\varphi'_i)$$

where φ_i, φ'_i are past formulas.

The special cases which are obtained from this normal form by keeping only one formula $\mathrm{GF}\psi$ or $\mathrm{FG}\psi$ or just $\mathrm{G}\psi$ or $\mathrm{F}\psi$ correspond to the special sublogics of S1S introduced by Landweber (see Theorem 3.8). An analysis of this correspondence is carried out in [Tho88b]; in particular, it is shown that one can decide effectively whether a given formula of PTL (or FO($<$)) is expressible in one of the forms $\mathrm{GF}\psi$, $\mathrm{FG}\psi$, $\mathrm{G}\psi$, $\mathrm{F}\psi$ with past formula ψ.

3.6 Some Perspectives

We end our discussion of descriptive hierarchies by a number of open questions; they are mainly concerned with S1S($\mathcal{C}$)-definable ω-languages for choices of $\mathcal{C}$ "between" REG and ALL. The first three problems deal with the issue of expressiveness. (Partly they can also be studied in the restricted framework of first-order logic FO($<$).) The remaining questions aim at applications concerning *infinite games*. Such games are a central subject in classical descriptive set theory; in computer science (and with the proper restrictions) they define a promising approach for the construction of reactive programs.

In studying the language classes $\mathcal{C} = ALL$, RCS, and REG, we have seen two kinds of results on the hierarchies of ω-language classes $\boldsymbol{\Sigma}^1_n[\mathcal{C}]$ and $\boldsymbol{\Sigma}^0_n[\mathcal{C}]$: either these hierarchies strictly increase at each level (as with ALL and RCS), or they collapse below $\boldsymbol{\Sigma}^0_3[\mathcal{C}]$ (as in case REG). A natural question asks which happens for other choices of $\mathcal{C}$, for example for the class of context-free languages. As shown by Seibert [Sei92], the closure of the class CFL of context-free languages (of finite words) under projections and complementations exhausts the arithmetical hierarchy of sets of finite words. From this it is easy to infer that the classes $\boldsymbol{\Sigma}^1_n[CFL]$ and $\boldsymbol{\Sigma}^0_n[CFL]$ form infinite hierarchies exhausting the arithmetical hierarchy and the analytical hierarchy.

Problem 1. Study the hierarchies $\boldsymbol{\Sigma}^1_n[\mathcal{C}]$ and $\boldsymbol{\Sigma}^0_n[\mathcal{C}]$ for subclasses of CFL or classes incompatible with CFL, such as the union of REG with a Dyck language, or classes defined by automata supplied with storage types of various kinds (as considered, e.g., in [EH93]).

Problem 2 (see also [Bü83])**.** Is there any natural class $\mathcal{C}$ of languages of finite words where the hierarchies $\boldsymbol{\Sigma}^1_n[\mathcal{C}]$ and $\boldsymbol{\Sigma}^0_n[\mathcal{C}]$ are neither both infinite nor collapse below level $\boldsymbol{\Sigma}^0_3[\mathcal{C}]$? In particular, is there such a class $\mathcal{C}$ where the classes $\boldsymbol{\Sigma}^1_n[\mathcal{C}]$ only contain Borel sets, however not bounded by $\boldsymbol{\Sigma}^0_n[\mathcal{C}]$ for any fixed n?

A related question is motivated by the S1S-formulas which describe acceptance of ω-words by nondeterministic automata, starting from the S1S-formulas as they appear in Büchi's Theorem. Let us introduce the ω-language classes $\boldsymbol{\Sigma}^1_1$-$\boldsymbol{\Pi}^0_n[\mathcal{C}]$, $\boldsymbol{\Sigma}^1_1$-$\boldsymbol{\Sigma}^0_n[\mathcal{C}]$ for $n = 1, 2$ and a language class $\mathcal{C}$, analogously to the special case $\mathcal{C} = REG$ treated after Landweber's Theorem 3.8 above. The step from $\mathcal{C}$ to these ω-language classes associated with $\mathcal{C}$ is similar to the step from an automaton model recognizing finite words to the associated automata on ω-words, equipped with various acceptance conditions. Engelfriet and Hoogeboom [EH93] give a detailed study of this subject in a general context. They consider

"X-automata", where X is a storage type, supplied with different acceptance conditions on ω-words. Related to the results and methods of [EH93], we ask for a corresponding logical investigation:

Problem 3. Analyze the relation between the classes $\boldsymbol{\Sigma}^1_1$-$\boldsymbol{\Pi}^0_n[\mathcal{C}]$, $\boldsymbol{\Sigma}^1_1$-$\boldsymbol{\Sigma}^0_n[\mathcal{C}]$, in particular for $n = 1, 2$, in an abstract framework, using suitable properties of the classes $\mathcal{C}$. (In comparison with [EH93], however, note that bounded formulas of S1S[$\mathcal{C}$] usually define a Boolean closed class of languages, which is not necessarily true for automaton defined classes.)

The possibility of reducing sets from a class $\boldsymbol{\Sigma}^1_n[\mathcal{C}]$ to levels within the Borel hierarchy, as it appears in McNaughton's Theorem for $\mathcal{C} = REG$, is essential for applications concerning reactive finite-state systems. Here the objective is to transform a given specification to a reactive program realizing the specification. A natural way to model reactive systems uses a notion of infinite game, the so-called *Gale-Stewart game* (see e.g. [Tho90], Section 10). A Gale-Stewart game with "moves" or "actions" from an alphabet $\mathbb{A}$ is specified by an ω-language $L \subseteq \mathbb{A}^\omega$ and played between two players, which we identify here with "program" and "environment". A play of the game consists of choices of actions from $\mathbb{A}$, alternatively by the program and the environment, starting say with a choice by the program, and resulting in an ω-word $\alpha \in \mathbb{A}^\omega$. The player "program" has won if this action sequence satisfies the specification, i.e., if $\alpha \in L$, otherwise the environment has won. A *winning strategy* for a player of the game is a function which determines a letter from $\mathbb{A}$ for any initial segment of a play, i.e. for any word $w \in \mathbb{A}^*$ (of even length if the program is concerned, of odd length otherwise). One of the most prominent results of descriptive set theory, due to Martin [Mar75], states that a Borel set L specifies a game which is *determined*, i.e., where one of the two players has a winning strategy. A stronger result was obtained by Büchi and Landweber [BL69] for games given by regular ω-languages L: From the definition of L (say in terms of a S1S-formula or a Rabin automaton) it can be decided effectively which player has a winning strategy, and, moreover, such a winning strategy can be executed by a deterministic finite automaton. Such an automaton can be constructed effectively from the definition of L, and its state upon input w indicates the choice of an appropriate letter from the alphabet after the initial play w.

Problem 4. Generalize the result of Büchi and Landweber to games L in classes $\boldsymbol{\Sigma}^0_n[\mathcal{C}]$ or $\boldsymbol{\Pi}^0_n[\mathcal{C}]$ where $\mathcal{C}$ properly includes REG. As an example, consider the class $DCFL$ of deterministic context-free languages; moreover, if there are no winning strategies executable by pushdown automata in such games, try to find e.g. recursive winning strategies.

A closely related question is concerned with succinct representations of winning strategies:

Problem 5. Find minimization algorithms for the finite automata that execute winning strategies in infinite games of the Büchi-Landweber type. (This problem is connected with minimal representations of infinite regular trees; see e.g. [Tho90], Theorem 10.1.) Can strategies be represented more succinctly when the automata are extended by more powerful storage types?

In the analysis of infinite games it is useful to transform logical specifications into automata and to set up winning strategies by studying the automaton state sequences which result from infinite plays. This is the approach of [BL69]; see also the lucid paper [McN92]. Considering specifications in formalisms S1S[$\mathcal{C}$], it is of interest whether they can be transformed into transition systems (as S1S-formulas can be transformed into Rabin automata). For $\mathcal{C} = ALL$ and $\mathcal{C} = RCS$, levels of the $\boldsymbol{\Sigma}^0_n[\mathcal{C}]$-hierarchy and the $\boldsymbol{\Sigma}^1_n[\mathcal{C}]$-hierarchy have been characterized by infinite transition systems with appropriate acceptance conditions (cf. Arnold [Arn83], Staiger [Sta93]). As another example for sets of finite words, we mention the work of Muller and Schupp [MS85], who showed that context-free languages of finite words can be characterized by transition systems in the form of "context-free graphs".

Problem 6. Find more classes $\mathcal{C}$ for which the S1S[$\mathcal{C}$]-definable ω-languages (or ω-languages in certain subclasses like $\boldsymbol{\Sigma}^1_1[\mathcal{C}]$, $\boldsymbol{\Pi}^0_2[\mathcal{C}]$) are characterized by special infinite transition systems with appropriate acceptance conditions. Apply this to the construction of winning strategies in S1S[$\mathcal{C}$]-definable infinite games.

4 Hierarchies of Tree Languages

The results of Büchi and McNaughton on the formalism S1S were strengthened considerably by Rabin [Rab69], who succeeded in giving an automata theoretic characterization of S2S, a formalism to specify properties of infinite labelled trees. It is instructive to compare the results on ω-languages with their counterparts in the domain of infinite trees. In a first step, we will provide the necessary definitions. Then we review the known results on the fine-structure of S2S[REG] (the tree version of S1S[REG]), showing that many quantifier reduction results of the previous section fail over infinite trees.

4.1 Infinite Trees and S2S[$\mathcal{C}$]

Instead of ω-words, which can be considered as functions $\alpha : \omega \to \mathbb{A}$ for a given alphabet $\mathbb{A}$, we consider infinite binary trees whose nodes are labelled in $\mathbb{A}$. We represent the nodes by the words over the alphabet $\{l, r\}$ ("left", "right"), and thus identify trees labelled in $\mathbb{A}$ with functions $t : \{l, r\}^* \to \mathbb{A}$. The set of binary trees labelled in $\mathbb{A}$ is denoted $T^\omega_{\mathbb{A}}$.

There is a natural transfer of the Cantor topology from the set of ω-words over $\mathbb{A}$ to the set of infinite trees over $\mathbb{A}$. Here the distance between two distinct trees s, t is $\frac{1}{2^n}$ for the smallest n such that $s(u) \neq t(u)$ for some word $u \in \{l, r\}^*$ of length n. The corresponding open sets we write in the form $T_0 \cdot T^\omega_{\mathbb{A}}$ where T_0 is a set of finite trees; they consist of trees that extend some finite tree $t_0 \in T_0$. Thus, prefixes of ω-words correspond to finite "initial subtrees" in infinite trees.

Now the definition of the Borel hierarchy and the projective hierarchy are defined in complete analogy to Section 2 above.

The logical description uses the formalism S2S ("second-order theory of two successors"), which is defined analogously to S1S, however with two successor relations $Succ_l$ and $Succ_r$ instead of $Succ$. The order relation $<$ between positions in ω-words, which was present in our version of S1S, is replaced by the prefix relation $\sqsubset$ over $\{l,r\}^*$. As usual, we allow also $\sqsubseteq$ as an abbreviation. We also extend the prefix relation $\sqsubseteq$ to pairs consisting of a node u and a set V of nodes by setting $u \sqsubseteq V$ iff there is $v \in V$ such that $u \sqsubseteq v$. In S2S we use variables $x, \ldots$ for tree nodes, $F, \ldots$ for finite sets of tree nodes, and $X, \ldots$ for arbitrary sets of tree nodes. Also the notion of bounded formula is introduced as before; here quantifiers are bounded with respect to $\sqsubseteq$ by variables $x, \ldots, F, \ldots$ occurring free in the formula.

For a given class $\mathcal{C}$ of sets of finite trees we introduce the formalism S2S[$\mathcal{C}$] as the extension of S2S by formulas $[x, F] \in T$ where T is considered as a constant for a tree language from $\mathcal{C}$. When $v \in \{l,r\}^*$ and $V \subseteq \{l,r\}^*$ are the interpretations of x, X, respectively, this formula is true in a tree t iff the finite tree within t whose domain is $\{u \in \{l,r\}^* \mid v \sqsubseteq u \sqsubseteq V\}$ belongs to T.

There are also natural versions of REG, RCS, and ALL, considered as classes of sets of finite trees. The class REG consists of the *regular tree languages* (cf. e.g. [GS84]), the class RCS contains the *recursive tree languages*, for example defined in terms of terminating Turing machines that accept finite trees, using $\{l,r\}^*$ as "worktape", and ALL is the class of all sets of finite trees.

It is clear that the results of Section 2 on the strictness of the set theoretic and recursion theoretic hierarchies are valid again, because we can embed ω-words into infinite trees and copy the arguments of Section 2. For example, we may restrict all considerations to trees where a trivial label occurs everywhere, excepting only the leftmost path (where an ω-word is coded). So we concentrate on the case S2S[REG] in the subsequent considerations, where we know that over ω-words the hierarchies $\boldsymbol{\Sigma}^1_n[REG]$ and $\boldsymbol{\Sigma}^0_n[REG]$ collapse.

4.2 A Tour Through S2S[REG]

As a preliminary remark, let us note that the regular tree languages (of finite trees) are characterized by bounded formulas of S2S. This tree analogue to Theorem 3.5 above is a straightforward consequence of the results of [Don70] and [TW68], characterizing regular tree languages in weak monadic second-order logic.

Rabin [Rab69] succeeded in describing S2S-definable sets of infinite trees by finite tree automata, so-called "Rabin tree automata". As with the case of Büchi automata on ω-words, the main (and very difficult) step is to show closure under complement for the Rabin recognizable sets of trees. A *Rabin tree automaton* over $\mathbb{A}$ is of the form $\mathcal{A} = (Q, \mathbb{A}, q_0, \Delta, \Omega)$, where $Q, \mathbb{A}, q_0, \Omega$ are given as for Rabin automata on ω-words, in particular Ω is a finite set of "accepting pairs" (U_i, L_i) of subsets of Q; in addition, $\Delta \subseteq Q \times \mathbb{A} \times Q \times Q$ is the transition relation. The automaton accepts a tree t if there is a "run" $\rho : \{l,r\}^* \to Q$ with $\rho(\epsilon) = q_0$

(where ϵ is the root node), and $(\rho(u), t(u), \rho(ul), \rho(ur)) \in \Delta$ for $u \in \{l, r\}^*$, such that ρ is "successful" in the following sense: for each path π through $\{l, r\}^*$ there is an accepting pair (U_i, L_i) such that for infinitely many $u \in \pi$ we have $\rho(u) \in U_i$ and for almost all $u \in \pi$ we have $\rho(u) \notin L_i$. We denote by RABIN the class of tree sets which are recognizable by Rabin tree automata.

Let us assume, as in Section 3, $\mathbb{A} = \{0,1\}^r$ and $Q = \{0,1\}^s$. We express that a tree t is accepted by a Rabin automaton with state set Q similarly as for Büchi automata in Theorem 3.6 . The auxiliary formula "Z is a path" is useful (abbreviating "any two nodes in Z are comparable with respect to $\sqsubseteq$, for any $z \in Z$ all prefixes are in Z, and any $z \in Z$ has a successor in Z"). The desired formula expresses the existence of a state assignment to the tree nodes (formulated by existential quantifiers over $Y_1, \dots, Y_s$) which satisfies the acceptance condition along any path Z:

$$(*) \qquad \begin{aligned} &\exists Y_1 \dots \exists Y_s \,\forall Z\, (\forall F\ \psi_0(\overline{X}, \overline{Y}, F) \\ &\quad \wedge (Z \text{ is a path} \rightarrow \bigvee_{i=1}^{n} (\exists^\omega x\ \varphi_i(\overline{X}, \overline{Y}, Z, x) \wedge \forall^\omega x\ \varphi_i'(\overline{x}, \overline{Y}, Z, x)))). \end{aligned}$$

Here $\psi_0(\overline{X}, \overline{Y}, F)$ says that for the nodes bounded by an element of F the initial state condition and the transition relation of the automaton are respected by the state assignment as given by $\overline{Y}$. The formulas $\varphi_i(\overline{X}, \overline{Y}, Z, x)$ and $\varphi_i'(\overline{X}, \overline{Y}, Z, x)$ express that the state coded by $\overline{Y}$ at x belongs to U_i, respectively does not belong to L_i. Note that the formulas φ_i and φ_i' are bounded. We obtain that the subformula of $(*)$ following $\forall Z$ is a Boolean combination of $\boldsymbol{\Pi}^0_2$-formulas. Thus Rabin's deep theorem on the reduction of S2S-formulas to Rabin tree automata, which we also indicate by the equality S2S$[REG]$ = RABIN, can be stated as follows:

Theorem 4.1 (Rabin's Theorem).
A S2S*-formula* $\varphi(X_1, \dots, X_r)$ *can be written as a* $\boldsymbol{\Sigma}^1_2$*-formula*

$$\exists \overline{Y} \forall Z \varphi_0(X_1, \dots, X_r, \overline{Y},\, Z)$$

where φ_0 *is a Boolean combination of* $\boldsymbol{\Pi}^0_2$*-formulas.*

Since φ_0 defines a Borel set, the projective hierarchy over the class of regular tree languages becomes stationary at the second level. From Rabin's complementation theorem for his tree automata we even obtain a reduction of S2S$[REG]$ to the class $\boldsymbol{\Sigma}^1_2[REG] \cap \boldsymbol{\Pi}^1_2[REG]$.

Let us consider modifications in the Boolean term occurring in the above formula $(*)$. If we keep only one member $\exists^\omega \varphi_1$ from the whole disjunction, the acceptance by *Büchi tree automata* is described, where the item Ω consists of a single accepting pair $(U, \emptyset)$. Rabin [Rab70] showed that these restricted formulas (and thus the Büchi tree automata) are strictly weaker in expressive power than S2S. In fact, the class BÜCHI of Büchi recognizable tree sets is not closed under complement.

More sets of trees become definable if we allow disjunctions as in $(*)$ with n members (for $n = 1, 2 \ldots$). The minimal n that suffices for the definition of a given set T of trees is called the (nondeterministic) *Rabin index* of T. Niwinski [Niw88] showed that the Rabin index induces indeed a strict hierarchy of tree language classes (called RABIN_1, RABIN_2, etc. below). This is in contrast to the case of ω-words where this hierarchy collapses (because S1S is equivalent already to nondeterministic sequential Büchi automata).

Another approach to modify the class BÜCHI is to consider Boolean combinations of tree sets from BÜCHI. For example, consider the class Bool(BÜCHI) of all Boolean combinations of sets in BÜCHI. This amounts to applying Boolean connectives outside the formulas describing Büchi acceptance (and not inside, as done in $(*)$). Hafer [Haf87] proved that this proper extension of BÜCHI does not exhaust the class RABIN. In order to obtain a smaller class than BÜCHI, a natural choice is the intersection of BÜCHI with co-BÜCHI (the class of complements of Büchi recognizable tree sets). This intersection coincides with the class of tree sets which are definable in *weak* S2S (short WS2S), where only quantifiers over tree nodes and finite sets of nodes are allowed ([Rab70]). Call this class WS2S[REG].

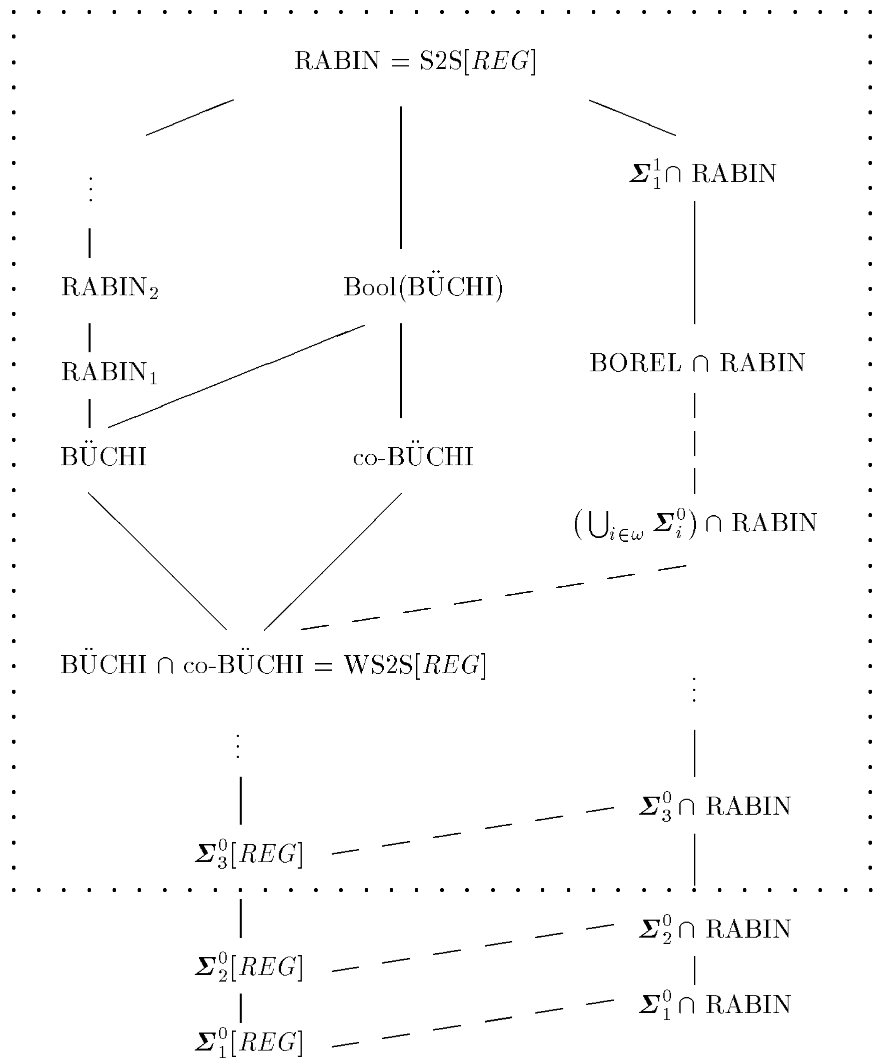
In a further step, let us look inside WS2S[REG]. From McNaughton's Theorem 3.7 we know that S1S is captured by Boolean combinations of formulas with just two unbounded quantifiers over finite sets or numbers. Over trees, the corresponding hierarchy of classes $\boldsymbol{\Sigma}^0_n[REG]$ is strict ([Tho82b]). This hierarchy is closely related to the classical ones: First, the strictness proof of [Tho82b] uses a reduction to the strictness of the arithmetical hierarchy of recursion theory. Secondly, there is a simple connection to the Borel hierarchy (of sets of infinite trees). Namely, we have (by an argument analogous to Theorem 3.3) that

$$\boldsymbol{\Sigma}^0_n[REG] \subseteq \boldsymbol{\Sigma}^0_n \cap \mathrm{RABIN}.$$

Supplementing the hierarchy proof for the left-hand side, Skurzyński [Sku93] showed that the classes on the right-hand side form also an infinite hierarchy. One might suspect that in analogy to Landweber's Theorem 3.8 we have $\boldsymbol{\Sigma}^0_n[REG] = \boldsymbol{\Sigma}^0_n \cap \mathrm{RABIN}$, but this is still open. Also it is not known whether the finite levels of the Borel hierarchy exhaust all Borel sets in RABIN or whether there are Rabin recognizable sets of transfinite Borel level. However, as shown by Niwinski, there are non-Borel sets of trees in the class RABIN. (An example is obtained by coding the set FPT of Theorem 2.14 as set of labelled binary trees.)

The diagram below summarizes these results. We use the symbols $\boldsymbol{\Sigma}^0_n$ and $\boldsymbol{\Sigma}^1_1$ for the Borel levels (consisting of sets of infinite trees here), BOREL for the class of Borel sets of trees, and WS2S[REG], S2S[REG] for the classes of tree languages definable by WS2S-formulas, respectively S2S-formulas. Straight lines indicate strict inclusions, whereas interrupted lines show inclusions whose strictness is still open.

It is interesting to compare this diagram to the situation in the theory of ω-languages. In that context the whole picture collapses to three classes: all the classes in the box encircled with the dotted line (including the three infinite hierarchies displayed there) collapse to the single class of regular ω-languages, and the two columns below the box form only one class per level (by Landweber's Theorem).



This may serve as an illustration how complicated the definability theory for general properties of branching computations is, compared with that of linear computations.

As mentioned in the introduction, the structure of the hierarchy diagram is so far not systematically analyzed over other types of infinite models (different from infinite words and infinite trees). In particular, it seems open over domains of infinite partial orders with labelled nodes and edges whether (with an appropriate notion of bounded formula) a hierarchy diagram of monadic second-order definable properties may result which is essentially different from the diagram for ω-languages or that for sets of infinite trees.

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