Monadic Logic: Hanf Numbers†

Abstract

This is part of the classification developed in Baldwin Shelah [BSh]. The paper is divided into two parts. In part I we show that $(T_\infty, 2^{nd}) \not\leq (T, \text{mon})$ iff the Hanf number for the theory T in monadic logic is smaller than the Hanf number of second order logic.

For this we deal with partition relations for models of T. The main result is that if T does not have the independence property even after expanding by monadic predicates (or equivalently $(T_\infty, 2^{nd}) \not\leq (T, \text{mon})$) then: $\beth_{\omega+1}(\lambda)^+ \rightarrow_s (\lambda)^{<\omega}_T$. In Part II we analyze such T getting a decomposition theorem like that in [BSh] (but weaker) (This is needed in part I.)

Part I

§1 Preliminaries

We review here some relevant facts and definitions.

1.1. Convention:

T will be a fix complete theory, $\mathfrak{C}$ a $\bar{\kappa}$-saturated model of T, $\bar{\kappa}$ large enough (see [Sh1] I §1); M,N denote elementary submodels of $\mathfrak{C}$ of power $<\bar{\kappa}$, A,B,C subsets of such M, a,b,c,d elements of $\mathfrak{C}$, $\bar{a},\bar{b},\bar{c},\bar{d}$ finite sequences, and I,J denote linear orders. A monadic expansion of M is expansion by monadic predicates; a finite expansion is one by finitely many relations. When dealing with finite monadic expansions of $\mathfrak{C}$, we may mean a $\bar{\kappa}$-saturated one, or any such expansion. We shall not specify, because if $M \subset \mathfrak{C}$, M^+ a finite expansion of M, then we can expand $\mathfrak{C}$ to $\mathfrak{C}^+$, an

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elementary extension of M^+ which is $\bar{\kappa}$-saturated.
This paper has two parts, the major one in part I, but in order to prove an important property of decomposition of models (see claim I2.4(1)) we need a property of types which is lemma 2.3 of Part II. The sole contribution of part II to I is the proof of this lemma.
We quote from [BSh] 1.2, 1.3:

1.2. Definition:

We say $(T_\infty, 2^{nd}) \leq (T, \text{mon})$ *if* in some monadic expansion of $\mathfrak{C}$, there is an infinite set on which a pairing function is defined. (a pairing function on A is a one-to-one function from $A \times A$ onto A).

1.3. Theorem:

1) If T has the independence property (see [Sh1] II §4) then $(T_\infty, 2^{nd}) \leq (T, \text{mon})$. Hence, $(T_\infty, 2^{nd}) \leq (T, \text{mon})$ iff some finite monadic expansion of a model of T has the independence property.

2) If in some finite monadic expansion of $\mathfrak{C}$ for some infinite sets $\{a_t : t \in I\}$, $\{b_t : t \in J\}$ and formula θ, for any $t \in I$, $s \in J$ there is d such that $(\forall u \in I)$ $(\forall v \in J)$ $[\theta(a_u, b_v, d) \longleftrightarrow t = u \wedge s = v]$ *then* $(T_\infty, 2^{nd}) \leq (T, \text{mon})$.

We quote from [Sh1] VII §4:

1.4. Definition:

1) We say p is finitely satisfiable in A if every finite subset of p is realized by elements of A

2) For an ultrafilter D on ${}^I A$, and set B, we define

$Av(D,B) = \{\varphi(\ldots,x_t,\ldots,\bar{b})_{t \in I} : \bar{b} \in B$ and the set

$\{\langle a_t : t \in I\rangle : \models \varphi[\ldots,a_t,\ldots,\bar{b}]_{t \in I}\}$ belong $D\}$

1.5. Lemma:

1) $Av(D,B)$ is a complete type in the variables $\langle x_t : t \in I\rangle$ over B, finitely satisfiable in A; of course $B \subseteq C \Longrightarrow Av(D,B) \subseteq Av(D,C)$

2) If p is finitely satisfiable in A, p a set of formulas in the variables $\{x_t : t \in I\}$, *then* for some ultrafilter D on ${}^I A$, and some set B $p \subseteq Av(D,B)$.

3) If p is finitely satisfiable in A then p does not split over A (i.e., if $\bar{b}, \bar{c}$ realize the same type over A then for no φ, $\varphi(\bar{x}, \bar{b})$, $\neg\varphi(\bar{x}, \bar{c}) \in p$)

4) If p is an m-type over B finitely satisfiable in A, then it can be extended to $p' \in S^m(B)$ finitely satisfiable in A,

5) If $p, q \in \bigcup_{m<\omega} S^m(C)$ are finitely satisfiable in A, $B \subseteq C$, and every m-type over A realized in C is realized in B, then $p \restriction B = q \restriction B \Longrightarrow p = q$

6) If $tp_*(C_0, A \cup B)$ is finitely satisfiable in A, and $tp_*(C_1, A \cup B \cup C_0)$ is finitely satisfiable in $A \cup C_0$ then $tp_*(C_0 \cup C_1, A \cup B)$ is finitely satisfiable in A.

1.6. Observation:

If every $p \in \bigcup_m S^m(A_0)$ is realized in A_1, (hence $A_0 \subseteq A_1$) $tp_*(D \cup C, A_1 \cup B)$ and $tp_*(D, A_1 \cup C)$ are finitely satisfiable in A_0 *then* $tp_*(D, A_1 \cup B \cup C)$ is finitely satisfiable in A_0

Proof: W.l.o.g. $D = \bar{d}$; by 1.5(4) there is $\bar{d}^1$ realizing $tp(\bar{d}, A_1 \cup C)$ such that $tp(\bar{d}^1, A_1 \cup B \cup C)$ is finitely satisfiable in A_0 (remember that $tp(\bar{d}, A_1 \cup C)$ is finitely satisfiable in A_0). By 1.5(6) $tp_*(C \cup \bar{d}^1, A_1 \cup B)$ is finitely satisfiable in A_0.
So $tp_*(C \cup \bar{d}^1, A_1 \cup B)$, $tp_*(C \cup \bar{d}, A_1 \cup B)$ are both finitely satisfiable in A_0, and their restriction to A_1 are equal. By 1.5(5) they are equal. Hence $tp(\bar{d}^1, A_1 \cup B \cup C) = tp(\bar{d}, A_1 \cup B \cup C)$. As $tp(\bar{d}^1, A_1 \cup B \cup C)$ is finitely satisfiable in A_0, necessarily $tp(\bar{d}, A_1 \cup B \cup C)$ is finitely satisfiable in A_0.

1.6A Remark: We can weaken the hypothesis by restricting ourselves to $p \in \bigcup_{m<\omega} S^m(A_0)$ realized in $A_1 \cup B$.

§2 A Weak Decomposition Theorem

Hypothesis: $(T_\infty, 2^{nd}) \not\leq (T, \text{mon})$.

Notation: Let I, J be linear ordering

2.1. Definition:

1) We say that $\bar{A} = \langle A_t : t \in I \rangle$ is a partial decomposition of M over N iff : the A_t's are pairwise disjoint subsets of M and for every $t \in I$, $tp_*(A_t, \bigcup_{s<t} A_s \cup N)$ is finitely satisfiable in N (but not necessarily $N \subseteq M$).

2) $\bar{A}$ is a decomposition of M over N, if it is a partial decomposition of M over N and $M = \bigcup_{t \in I} A_t$.

2.2. Definition:

For partial decomposition $\langle A_t : t \in I \rangle$, $\langle B_t : t \in J \rangle$ of M over N we say $\langle A_t : t \in I \rangle \leq \langle B_t : t \in J \rangle$ if $I \subseteq J$ and for every $t \in I$, $A_t \subseteq B_t$; we say $\langle A_t : t \in I \rangle \leq^* \langle B_t : t \in J \rangle$ if $I = J$ and for every $t \in I$, $A_t \subseteq B_t$.

2.3. Claim:

1) For every $<$-increasing sequence of partial decompositions of M over N there is a least upper bound (similarly for $<^*$)

2) If $\langle A_t : t \in I \rangle$ is a partial decomposition of M over N, $I \subseteq J$, and for $t \in J - I$, we let $A_t = \phi$ *then* $\langle A_t : t \in J \rangle$ is a partial decomposition of M over N

Proof: Immediate

2.4. Claim:

1) Suppose $\langle A_t : t \in I \rangle$ is a partial decomposition of M over N and $c \in M$. Then for some $\langle B_t : t \in J \rangle \geq \langle A_t : t \in I \rangle$ (a partial decomposition of M over N), $c \in \bigcup_{t \in I} B_t$

2) If I is a well-ordering with last element then w.l.o.g. $I = J$

Proof:

1) W.l.o.g. $c \notin \bigcup_{t \in I} A_t$. Let I_1 be a maximal initial segment of I [i.e., $(\forall t \in I_1)$ $(\forall s \in I)$ $(s < t \rightarrow s \in I_1)$] such that $tp(c, \bigcup_{t \in I_1} A_t \cup N)$ is finitely satisfiable in N (there is such I_1, as $I_1 = \phi$ satisfies the demand, and by the finitary character of the demand). By 2.3 (2) w.l.o.g. $I_1 = \{s \in I : s < t^*\}$ for some $t^* \in I$. Now we let $J = I$, and let B_t be A_t if $t \neq t^*$, and $A_t \cup \{c\}$ if $t = t^*$. We now check Def 2.1 (1). The main non-obvious point is why for t, $t^* < t \in I$, $tp_*(B_t, \bigcup_{s<t} B_s \cup N)$ is finitely satisfiable in N. If not then for some $\bar{b} \in B_t = A_t$, $\bar{a} \in \bigcup_{s<t} B_s - \{c\} = \bigcup_{s<t} A_s$, $tp(\bar{b}, \bar{a} \cup \{c\} \cup N)$ is not finitely satisfiable in N. However we know that $tp(\bar{b}, \bar{a} \cup N)$ is finitely satisfiable in N (as $\langle A_s : s \in I \rangle$ is a partial decomposition of M over N). Also $tp(c, \bigcup_{s<t} A_s \cup N)$ is not finitely satisfiable in N (by the choice of I_1, as maximal, as $t > t^*$ and as w.l.o.g. we add t^* only if needed), hence w.l.o.g. $tp(c, \bar{a} \cup N)$ is not finitely satisfiable in N. Hence, $tp(\bar{b} \frown \langle c \rangle, \bar{a} \cup N)$ is not finitely satisfiable in N. Together, $N, \bar{a}, \bar{b}, c$ contradict Lemma II 2.3. For $t = t^*$, we should prove for $\bar{b} \in A_{t^*}$, $\bar{a} \in \bigcup_{s<t^*} B_s = \bigcup_{s<t^*} A_s$ that $tp(\bar{b} \frown \langle c \rangle, N \cup \bar{a})$ is finitely satisfieable in

N. Suppose this fails. As $\langle A_s : s \in I \rangle$ is a partial decomposition of M over N, clearly $tp(\bar{b}, N \cup \bar{a})$ is finitely satisfiable in N. By II 2.3 and the last two facts $tp(\bar{b}, (\bar{a} \frown \langle c \rangle) \cup N)$ is finitely satisfiable in N. By the choice of t^*, $tp(c, N \cup \bar{a})$ is finitely satisfiable in N. By 1.5 (6) and the last two facts, $tp(\bar{b} \frown \langle c \rangle, N \cup \bar{a})$ is finitely satisfiable in N, contradiction

2) Either there is t^* as required or $I_1 = I$, and then choose t^* as last.

2.5. Conclusion:

1) Suppose $\langle A_t : t \in I \rangle$ is a partial decomposition of M over N. Then there is a decomposition $\langle B_t : t \in J \rangle \geq \langle A_t : t \in I \rangle$ of M over N

2) If $I = (\alpha + 1, <)$ then w.l.o.g. $J = I$; also $|\{t \in J : B_t \neq \phi\}| \geq |\{t \in I : A_t \neq \phi\}|$

Proof: Immediate by 2.3(1), 2.4

Remember that (see [Sh1]) $Ded_r(\lambda)$ is the first regular cardinal μ, such that every linear order of power λ has strictly less than μ Dedekind cuts.

2.6. Lemma:

1) Suppose $\langle A_t : t \in I \rangle$ is a decomposition of M over N. Then we can find relations $P^n_{\gamma,\alpha}$ $(\alpha < \lambda_N < Ded_r(||N|| + |T|)$, $\gamma < |T|$, hence $\lambda_N \leq 2^{||N|| + |T|})$ such that:

a) $P^n_{\alpha,\gamma}$ is an n-place relation on M.

b) if $\gamma < |T|$, $n < \omega$, and $\alpha \neq \beta$, then $P^n_{\gamma,\alpha} \cap P^n_{\gamma,\beta} = \phi$, and $\bigcup_\alpha P^n_{\gamma,\alpha} = \bigcup_{t \in I} {}^n(A_t)$

c) for a finite sequence $\bar{b}$ from any A_t let $\alpha_\gamma(\bar{b})$, $n(\bar{b})$ be the unique n and α such that $\bar{b} \in P^n_{\gamma,\alpha}$; then if $t_1 < \cdots < t_n$, $\bar{b}_m \in A_{t_m}$ then we can compute the type of $\bar{b}_1 \frown \cdots \frown \bar{b}_n$ from $\langle n(\bar{b}_m) : m = 1, n \rangle$, and $\langle \alpha_\gamma(\bar{b}_m) : m = 1, n \rangle$, for $\gamma < |T|$

2) So as $Ded_r(|T|) \leq (2^T)^+$ we can use just $|T|$ predicates when $|N| \leq |T|$, and we waive the disjointness of the $P^n_{\gamma,\alpha}$'s.

Proof:

1) For any set A, $N \subseteq A$, and formula φ the number of $p \in S^m_\varphi(A)$ finitely satisfiable in N is $< Ded_r(||N||)$ (see [Sh2, p.202], slightly improving a result of Poizat, which suffice for (2) (alternatively use $\alpha < 2^{2^{|T|}}$))
Let N_1 be such that $N \subseteq N_1$, N_1 is $(||N|| + |T|)^+$-saturated and we shall show that w.l.o.g.

(*) for each $t \in I$, $\bar{b} \in A_t$, $tp(\bar{b}, N_1 \cup \bigcup_{s<t} A_s)$ is finitely satisfiable in N

For each $t \in I$ let $A_t = \{b^t_v : v \in I(t)\}$, let $\bar{\mathbf{b}}^t = \langle b^t_v : v \in I(t) \rangle$ and by 1.5(2) we can choose an ultrafilter D_t on ${}^{I(t)}N$ such that:

$$tp(\bar{\mathbf{b}}^t, N \cup \bigcup_{s<t} A_s) = Av(D_t, N \cup \bigcup_{s<t} A_s)$$

It suffice to us that for any $t(0) < \cdots < t(n) \in I$

(*) $tp(\bar{\mathbf{b}}^{t(n)}, N_1 \cup \bigcup_{l<n} \bar{\mathbf{b}}^{t(l)}) = Av(D_{t(n)}, N_1 \cup \bigcup_{l<n} \bar{\mathbf{b}}^{t(l)})$

For any finite set $w \subseteq I$ we define $q_w = q_w(\langle x_c : c \in N_1 \rangle)$, a complete type over $N \cup \bigcup_{t \in w} \bar{\mathbf{b}}^t$ by induction on $|w|$. For $w = \phi$ it is $tp_*(N_1, N)$, if $w \neq \phi$ let $w = \{t(0), \ldots, t(n)\}$, $n \geq 0$, $t(0) < \cdots < t(n)$, and we define it by

(*) if $\langle e_c : c \in N_1 \rangle$ realizes $q_{w - \{t(n)\}}$, we can find $\langle b'_v : v \in I(t(n)) \rangle$ realizing $Av(D_{t(n)}, N \cup \bigcup_{p<n} A_{t(l)} \cup \{e_c : c \in N_1\})$ and let F be an automorphism of $\mathfrak{C}$, F the identiy on $N \cup \bigcup_{l<n} A_{t(l)}$, $F(b'_v) = b^{t(n)}_v$ for $v \in I(t(n))$. Now $q_w = tp(\langle F(e_c) : c \in N_1 \rangle, N \cup \bigcup_{t \in w} A_t)$.

It is easy to prove that for $w_1 \subseteq w_2 (\subseteq I$ finite$)$ $q_{w_1} \subseteq q_{w_2}$, (by induction on $|w_2|$) and obviously q_w is finitely satisfied. Hence $\cup\{q_w : w \subseteq I$ finite$\}$ is finitely satisfiable, hence realized by some $\langle e'_c : c \in N_1 \rangle$. We can use $\{e'_c : c \in N_1\}$ instead N_1 and then (*) holds.

Let $\{\varphi^n_\gamma(\bar{x}, \bar{y}) : \gamma < |T|\}$ be a list of the formulas $\varphi(\bar{x}, \bar{y})$, $l(\bar{x}) = n$, and $\{p^n_{\gamma,\alpha} : \alpha < \lambda_N\}$ be a list of $\{tp(\bar{b}, N_1) \restriction \varphi : \bar{b}, \varphi$ as above$\}$, lastly $\bar{b} \in P^n_{\gamma,\alpha}$ iff $\bar{b} \in \bigcup_{t \in I} {}^n(A_t)$, $n = l(\bar{b})$, and $tp(\bar{b}, N_1) \restriction \varphi_\gamma = p^n_{\gamma,\alpha}$

2) Obvious from (1).

§3 Partition relations for theories

3.1. Definition:

1) $\lambda \to (\mu)^n_T$ mean that for every model M of T of power λ, there are distinct elements a_i $(i < \mu)$ such that $\langle a_i : i < \mu \rangle$ is an n-indiscernible sequence in M.

2) $\lambda \to_s (\mu)^n_T$ means that for every model M of T and $a_i \in M (i < \lambda)$ there is $I \subseteq \lambda$, $|I| = \mu$ such that $\langle a_i : i \in I \rangle$ is an n-indiscernible sequence.

3) $\lambda \rightarrow (\mu)_T^{<\omega}$, $\lambda \rightarrow_s (\mu)_T^{<\omega}$, are defined similarly.

3.2. Discussion

This definition was suggested by R. Grossberg and the author during the winter of 1980/1, but we still know little. We can rephrase [Sh1] I 2.8 as, e.g.: if T is stable, $\lambda = \lambda^{|T|}$ then $\lambda^+ \rightarrow_s (\lambda^+)_T^{<\omega}$. We cannot hope for results on T without the strict order property (see [Sh1] II§4) or even for simple T (see [Sh2].) The reason is as follows: suppose $\lambda \not\rightarrow (\mu)_2^{<\omega}$, and let F be a function from $\{w : w \subseteq \lambda,\ |w| < \aleph_0\}$ to $\{0,1\}$ exemplifying it, let L consist of the predicates R_n (n place) P_n (monadic) for $n < \omega$, and let T be the model completion of $\{(\forall x)(x = x)\}$ in this language. We define an L-model M with universe $\{a_{n,i} : n < \omega, i < \lambda\}$ such that:

(i) for $w \subseteq \lambda$, $|w| = n$, $\langle a_{n,i} : i \in w \rangle \in R_n^M$ iff $F(w) = 0$.

(ii) for every n, i for some k, for every $m > k$ $a_{n,i} \in P_m$ iff m is divisible by the n^{th} prime.

(iii) if $(\forall y_1 \cdots y_m)\ (\exists x)\ [\bigwedge_{l=1}^{m} x \neq y_l \wedge \varphi(x, y_1 \cdots, y_m)]$ belong to T, φ quantifier free, but R_k, P_k do not appear in φ and $a_1 \cdots, a_m \in \{a_{n,i} : n < k,\ i < \lambda\}$, then there is $b \in \{a_{k,i} : i < \lambda\}$ such that $\models \varphi[b, a_1, \ldots, a_m]$.

This is quite easy, M is a model of T (by T's definition and (iii),) and M exemplify $\lambda \not\rightarrow (\mu)_T^{<\omega}$. We can similarly deal with $\lambda \not\rightarrow (\mu)_T^n$.

Now T is simple, and in fact very close to T_{ind}. This leads naturally to:

3.3. Conjecture:

If T does not have the independence property, then for every μ for some λ, $\lambda \rightarrow (\mu)_T^{<\omega}$, or even $\beth_{\omega+\omega}(\mu + |T|) \rightarrow (\mu)_T^{<\omega}$.

3.4. Lemma:

Suppose $(T_\infty, 2^{nd}) \not\leq (T, \mathrm{mon})$, then

$$\beth_{\omega+1}(\lambda + |T|)^+ \rightarrow_s (\lambda)_T^{<\omega}.$$

Proof: W.l.o.g. $\lambda > |T|$, let $\mu = \beth_\omega(\lambda)$, $A = \{a_i : i < (2^\mu)^+\}$, for $i \neq j$

$a_i \neq a_j \in M$, and M is a model of T.

3.5. Fact:

At least one of the following occurs for $A = \{a_i : i < (2^\mu)^+\} \subseteq M$, $|A| = (2^\mu)^+$:

(i) There is an indiscernible sequence of length $(2^\mu)^+$ of distinct members of A (in the same length)

(ii) There is k, and $\bar{a}_i \in {}^k A (i < \mu)$ and θ such that $M \models \theta[\bar{a}_i, \bar{a}_j]$ iff $i < j$;

Proof: Repeat the proof of [Sh1] I 2.12. Let $A_i \stackrel{def}{=} \{a_j : j < i\}$

Let $S = \{\delta < (2^\mu)^+ : cf\ \delta > \mu\}$, clearly S is a stationary subset of $(2^\mu)^+$. For each $\delta \in S$ and formula φ choose if possible a subset $B_{\delta,\varphi} \subseteq A_\delta$, $B_{\delta,\varphi}$ of cardinality $< \mu$ such that: $tp_\varphi(a_\delta A_\delta)$ does not split over $B_{\delta,\varphi}$ [i.e., if $\varphi = \varphi(x, \bar{y})$, $\bar{b}$, $\bar{c}$ sequences from A_δ of length $l(\bar{y})$ realizing the same type over $B_{\delta,\varphi}$ then $\models \varphi[a_\delta, \bar{b}] \equiv \varphi[a_\delta, \bar{c}]$]. Let $S_\varphi = \{\delta \in S : B_{\delta,\varphi}$ is defined$\}$.

Case a: For each φ for some closed unbounded $C \subseteq (2^\mu)^+$, $C \cap S = C \cap S_\varphi$

Then there is a closed unbounded $C \subseteq (2^\mu)^+$ such that for every φ, $C \cap S = C \cap S_\varphi$. For each $\delta \in C \cap S$ choose B_δ, B_δ a subset of A_δ of power μ including $\bigcup_\varphi B_{\delta,\varphi}$ such that for each φ, and $n < \omega$, every n-type over $B_{\delta,\varphi}$ realized in A_δ is realized in B_δ (possible as $|B_{\delta,\varphi}| < \mu$, μ strong limit). Now by Fodor's lemma for some stationary $S^* \subseteq C \cap S$, for all $\delta \in S^*$, B_δ, $\langle B_{\delta,\varphi} : \varphi \in L(T) \rangle$, $tp(a_\delta, B_\delta)$ are the same. Continue as [Sh1] I 2.12.

Case b: For some φ, $S - S_\varphi$ is a stationary subset of $(2^\mu)^+$

So there is $\delta \in S - S_\varphi$ such that for every $B \subseteq A_\delta$, $|B| \leq \mu$, there is $\alpha < \delta$ such that a_α realizes $tp(a_\delta, B)$. So choose by induction on $i < \mu$, $\bar{b}_i$, $\bar{c}_i$, $d_i \in A_\delta$ as follows:

(α) $\bar{b}_\alpha$, $\bar{c}_\alpha$ realizes the same type over $\bigcup_{j<i} \bar{b}_j \frown \bar{c}_j \frown \langle d_j \rangle$

and $\models \varphi[a_\delta, \bar{b}_\alpha] \equiv \neg \varphi[a_\delta, \bar{c}_\alpha]$

($w.l.o.g.$ $\models \varphi[a_\delta, \bar{b}_\alpha] \wedge \neg \varphi[a_\delta, \bar{c}_\alpha]$)

(β) d_i realizes $tp(a_\delta, \bigcup_{j<i} (\bar{b}_j \frown \bar{c}_j \frown \langle d_j \rangle) \cup \bar{b}_i \frown \bar{c}_i$

By the choice of δ this is possible and $\langle \bar{b}_i \frown \bar{c}_i \frown \langle d_i \rangle : i < \mu \rangle$ is as required.

3.6. Fact:

If $d_i \in M$ are distinct for $i < (2^\lambda)^+$.

$$B_\alpha = \{d_i : i < \alpha\} \ , \ B = \{d_i : i < (2^\lambda)^+\} \ ,$$

then at least one of the following occurs:

(i) for some $\gamma < (2^\lambda)^+$ and $k < \omega$, $\{tp(\bar{d}, B_\gamma)$: $\bar{d}$ a sequence of length k from $B\}$ has power $(2^\lambda)^+$

(ii) (i) does not occur but for some $\varphi = \varphi(x, \bar{y})$ for a stationary set of $\delta < (2^\lambda)^+$, $cf\ \delta > \lambda$ and $tp_\varphi(d_\delta, B_\delta)$ split over B_α for every $\alpha < \delta$

(iii) for some stationary $S \subseteq (2^\lambda)^+$, $\langle d_i : i \in S \rangle$ is an indiscernible sequence

Proof: Again as in [Sh1] I 2.12 (or 3.5 above)

Remark: In the proofs of 3.5, 3.6 we have not used the hypothesis of 3.4.

Continuation of the proof of 3.4:

Clearly if 3.5(i) holds, we finish, so w.l.o.g. 3.5(ii) holds. By Erdos-Rado theorem, for every $m, n < \omega$ there is $I = I_{n,m} \subseteq \mu$, $|I_{n,m}| = \beth_n(\lambda)$, $\{\bar{a}_i : i \in I_{n,m}\}$ is an m-indiscernible sequence. By the proof of [BSh] VIII 1.3, there is a formula θ^1 such that for any n there are $I_n \subseteq (2^\mu)^+$, $|I_n| = \beth_n(\lambda)^+$, and a finite monadic expansion $\mathfrak{C}^+$ of $\mathfrak{C}$ such that (for some distinct $a_i^n (i \in I_n)$):

$$(\forall i, j \in I_n)[\mathfrak{C}^+ \models \theta^1(a_i^n, a_j^n) \quad iff \quad i \leq j]$$

Note that a_i^n belongs to our original A. We now can deal with $\{a_i^1 : i \in I_1\}$ only. W.l.o.g. $I_1 = (2^\lambda)^+$, $\mathfrak{C} = \mathfrak{C}^+$, $a_i^1 = a_i$ and denote $B_\gamma = \{a_i : i < \gamma\}$. Applying 3.6 to $\mathfrak{C}^+$, $A^1 = \{a_i : i < (2^\lambda)^+\}$, if our conclusion fails then one of the following two cases occurs.

Case I: there are $\gamma < (2^\lambda)^+$ and $\bar{b}_\alpha \in A^1 (\alpha < (2^\lambda)^+)$ such that $tp(\bar{b}_\alpha, B_\gamma)$ are distinct (for distinct α's).

W.l.o.g. $l(\bar{b}_\alpha) = k$ for every α. Next we show that w.l.o.g. $k = 1$, otherwise choose an example with minimal k (possibly replacing $\mathfrak{C}$ by a finite monadic expansion). W.l.o.g. the $\bar{b}_\alpha$ form a Δ-system hence by k's minimality are disjoined. If $k > 1$, let $\bar{b}_\alpha = \bar{c}_\alpha ^\frown \langle d_\alpha \rangle$; w.l.o.g. for some $\varphi = \varphi(\bar{x}, y; \bar{z})$, the types $tp_\varphi(\bar{b}_\alpha, B_\gamma)$ are distinct.

Clearly if for some α, $\{tp(d_\beta, B_\gamma \cup \bar{c}_\alpha): \beta < (2^\lambda)^+\}$ has power $> 2^\lambda$, we get contradiction to k's minimality, hence w.l.o.g. $\alpha < \beta < (2^\lambda)^+$, $\sigma < (2^\lambda)^+$ implies $tp_\varphi(\bar{c}_\beta \frown < d_\beta >, B_\gamma) \neq tp_\varphi(\bar{c}_\alpha \frown < d_\sigma >, B_\gamma)$. Similarly w.l.o.g. $\alpha < \beta < (2^\lambda)^+$, $\sigma < (2^\lambda)^+$ implies $tp_\varphi(\bar{c}_\beta \frown < d_\beta >, B_\gamma) \neq tp_\varphi(\bar{c}_\sigma \frown < d_\alpha >, B_\gamma)$. W.l.o.g. for every β there is no $\beta' < \beta$ such that $\bar{c}_\beta \frown < d_{\beta'} >$ satisfies this. W.l.o.g. for some monadic predicate $P, P = \{d_\beta : \beta < (2^\lambda)^+\}$, so d_β is defined from $\bar{c}_\beta$, so we can decrease k.

An alternative way to do it is as follows. Let $\bar{b}_\alpha = \langle a_{i(\alpha,0)}, \ldots, a_{i(\alpha,k-1)} \rangle$, w.l.o.g. $i(\alpha,0) < \cdots < i(\alpha,k-1)$, and as the $\bar{b}_\alpha$'s are pairwise disjoint, w.l.o.g. $\alpha < i(\alpha,k-1) < i(\beta,0)$ for $\alpha < \beta$.
We may expand $\mathfrak{C}$ by $P_m = \{a_{i(\alpha,m)} : \alpha < (2^\lambda)^+\}$, and using the order defined by θ^1 on $\{a_i : i < (2^\lambda)^+\}$ we can define the functions $a_{i(\alpha,0)} \to a_{i(\alpha,m)}$ hence can code $\bar{b}_\alpha$ by $a_{i(\alpha,0)}$.

So there are $\gamma < (2^\lambda)^+$ and $b_\alpha \in A^1$ $(\alpha < (2^\lambda)^+)$, and φ such that $tp_\varphi(b_\alpha, B_\gamma)$ are distinct for distinct α's, and w.l.o.g. γ is minimal. First assume $\varphi = \varphi(x,y)$. Also w.l.o.g. for every $\gamma_1 < \gamma < \alpha < (2^\lambda)^+$, there are $(2^\lambda)^+$ β's such that $tp_\varphi(b_\alpha, B_{\gamma_1}) = tp_\varphi(b_\beta, B_{\gamma_1})$. Hence for any n we can find $\gamma_0 < \gamma_1 < \cdots < \gamma_{2n}$, and $\alpha_\eta < (2^\lambda)^+$ for $\eta \in {}^{2n}2$ such that $\gamma_{2n} < \gamma$, $\gamma < \alpha_\eta$ and for $m \leq 2n$, n, $\nu \in {}^{2n}2$:

$$tp_\varphi(b_{\alpha_\eta}, B_{\gamma_m}) = tp_\varphi(b_{\alpha_\nu}, B_{\gamma_m}) \ iff \ \eta \restriction m = \nu \restriction m \ .$$

Expand $\mathfrak{C}$ by:

$$R = \{b_{\alpha_\eta} : \eta \in {}^{2n}2, \bigwedge_{m<n} (\eta(2m) = 0 \vee \eta(2m+1) = 0)\}$$

$$Q_1 = \{b_{\gamma_{2m}} : m \leq n\}$$

$$Q_2 = \{b_{\gamma_{2m+1}} : m < n\}$$

$$P = B_\gamma \ .$$

Let (remembering θ defines the order on $\{a_i : i < (2^\lambda)^+\}$):

$$\begin{aligned}\psi(x,y) \overset{def}{=} R(x) \wedge Q_2(y) \wedge (\exists\, x_1, y_1)[R(x_1) \wedge Q_1(y_1) \wedge \\ \wedge (\forall y_2)[Q_2(y_2) \wedge \theta^1(y_2,y_1) \to \theta^1(y_2,y)] \wedge \\ \wedge [x, x_1 \text{ realizes the same } \varphi\text{-type over} \\ \{z \in P : \theta^1(z,y)\} \text{ but not over} \\ \{z \in P : \theta^1(z,y_1)\}]\end{aligned}$$

It is easy to see that:

$$\models \psi[b_\eta, a_{\gamma_{2m+1}}] \quad iff \quad \eta(2m+1) = 1$$

Together with compactness this shows that some finite monadic expansion of $\mathfrak{C}$ has the independence property, contradiction.

We still have to deal with the case $\varphi = \varphi(x, \bar{y})$, $l(\bar{y}) > 1$. Let $l(\bar{y}) = m$ let $<^*$ be the lexicographic order on ${}^m(B_\gamma)$, (based on θ^1); so ${}^m B_\gamma = \{\bar{a}_\alpha: \alpha < \gamma_m\}$, $\bar{a}_\alpha <^* \bar{a}_\beta$ iff $\alpha < \beta < \gamma_m$. We then let $\gamma^* \leq \gamma_m$ be minimal such that $\{\{\varphi(x, \bar{a}_\beta): \beta < \gamma^*, \models \varphi(b_\alpha, \bar{a}_\beta)\}: \gamma < \alpha < (2^\lambda)^+\}$ has power $(2^\lambda)^+$. Now again necessarily γ^* is limit and we can find $\gamma_0 < \gamma_1 < \cdots < \gamma^*$ and $\gamma^* < \alpha_\eta < (2^\lambda)^+$ for $\eta \in {}^\omega 2$ which are eventually zero such that

$$\bigwedge_{\beta < \gamma_l} \varphi[b_{\alpha_\eta}, \bar{a}_\beta] \equiv \varphi[b_{\alpha_\nu}, \bar{a}_\beta] \quad iff \quad \eta \restriction l = \nu \restriction l$$

Our only problem is to code $\{\bar{a}_{\gamma_l}: l < \omega\}$ by monadic predicates, which is easy applying Ramsey theorm on the $\bar{a}_{\gamma_l}$'s and using the order on B_j.

Case II: For some finite $\bar{c} \in \mathfrak{C}$ and some $\gamma < (2^\lambda)^+$, $\{tp(\bar{b}, B_\gamma \cup \bar{c}): \bar{b} \subseteq A^1\}$ has power $(2^\lambda)^+$

Like case I.

Case III: Note case II.

We shall prove

$(*)$ if $\bar{c} \in \mathfrak{C}$, $W \subseteq \{\delta: \delta < (2^\lambda)^+, cf\ \delta > \lambda\}$ is stationary, *then* for some closed unbounded $U \subseteq (2^\lambda)^+$, and function f, $Dom\ f = U \cap W$; $f(\alpha) < \alpha$ for $\alpha \in U \cap W$, and for each γ the sequence $\langle tp(a_\alpha, \bar{c} \cup \{a_\beta: \beta < \alpha, f(\beta) = \gamma\}): f(\alpha) = \gamma \rangle$ is increasing.

Now it suffice to prove $(*)$. As then we define by induction on n K_n, and for $t \in K_n$, W_t, U_t, f_t, $\bar{c}_t$ such that:

(a) $K_0 = \{<0>\}$, $W_{<0>} = W \subseteq \{\delta < (2^\lambda)^+: cf\ \delta > \lambda\}$, $\bar{c}_{<0>}$ is the empty sequence.

(b) for $t \in K_n$ $\bar{c}_t \in \mathfrak{C}$ is a sequence of length n, and if $\alpha_1 < \alpha_2 \cdots < \alpha_n$ are in W_t, then

$$tp(\langle a_{\alpha_n}, a_{\alpha_{n-1}}, \ldots, a_{\alpha_2}, a_{\alpha_1} \rangle, \{a_\gamma: \gamma < \alpha_1, \gamma \in W_t\})$$

$$= tp(\bar{c}_t, \{a_\gamma: \gamma < \alpha_1, \gamma \in W_t\})$$

(c) K_n is a family of sequences of length n of ordinals $< (2^\lambda)^+$

(d) for $t \in K_n$, U_t is a closed unbounded subset of $(2^\lambda)^+$, f_t a function with domain $U_t \cap W_t$, $f_t(\alpha) < \alpha$

(e) $K_{n+1} = \{\eta\frown<\gamma>: \eta \in K_n, \gamma \in Rang(f_t)$ for some $t \in K_n\}$ and $W_{\eta\frown<\gamma>} = \{\alpha \in W_\eta: \alpha \in U_\eta$ and $f_\eta(\alpha) = \gamma\}$
For n = 0-no problem, for $n+1$: for each $W_\eta(\eta \in K_n)$ apply (*) (with $\bar{c} = \bar{c}_\eta$).

Now K_0, $W_\eta, \bar{c}_\eta(\eta \in K_0)$ are defined.

If $W_\eta, \bar{c}_\eta$ are defined we can define f_η, U_η by applying (*), then define $W_{\eta\frown<\gamma>}, \bar{c}_{\eta\frown<\gamma>}$ $(\gamma \in Rang(f_\eta))$ by (d). If we do this for every $\eta \in K_n$, we can define K_{n+1} by (e).

For every $\delta \in W_{<>}$, we can define by induction on $l < \omega$, $\eta_l \in K_l$, such that $\eta_l = \eta_{l+1} \restriction l$, $\delta \in W_{\eta_l}$ and $Rang\, \eta_l \subseteq \delta$ and the η_l are unique but maybe for some l, $\delta \notin U_{\eta_l}$ hence η^δ_{l+1} is not defined. Let $\varepsilon(\delta) \le \omega$ be such that η^δ_l is defined iff $l < \varepsilon(\delta)$. If $\{\delta : \varepsilon(\delta) < \omega\}$ is stationary, we get contradiction by Fodor lemma. If $W^* = \{\delta : \varepsilon(\delta) = \omega\}$ is stationary, then $\gamma(\delta) = \sup_{l<\omega} \eta^\delta_{l+1}(l) < \delta$ for $\delta \in W^*$ (as $cf\ \delta > \lambda$) hence for some stationary $W^1 \subseteq W^*$, $\gamma(\delta)$ is constant on W^1. As $(2^\lambda)^{\aleph_0} = 2^\lambda$ w.l.o.g. $\eta^\delta_l = \eta_l$ for every $\delta \in W^1$. Now $\bigcap_{l<\omega} W_{\eta_l}$ is stationary and by (b) $\langle a_i : i \in \bigcap_{l<\omega} W_{\eta_l} \rangle$ is an indiscernible sequence.

Proof of (*): For notational simplicity let $\bar{c} = \phi$. For every $\varphi = \varphi(x,\bar{y})$, and $\gamma < (2^\lambda)^+$, type $p \in S^\varphi_\gamma \overset{def}{=} \{tp_\varphi(a_i, B_\gamma) : \gamma < i < (2^\lambda)^+\}$ and natural number n we define when $Rk_\varphi(p) \ge n$:

For n = 0-always.

For $n = 2m+1$, $Rk(p) \ge n$ *iff* there is β, $\gamma < \beta < (2^\lambda)^+$ and distinct $p_1, p_2 \in S^\varphi_\beta$ extending p with $Rk_\varphi(p_1), Rk_\varphi(p_2) \ge 2m$.

For $n = 2m+2$, $Rk(p) \ge n$ *iff* for every β, $\gamma < \beta < (2^\lambda)^+$ there is $p_1 \in S^\varphi_\beta$ extending p with $Rk_\varphi(p_1) \ge 2m+1$.

If there are p, φ such that $Rk_\varphi(p) \ge n$ for every $n < \omega$, the proof is as in case I. Suppose not, then for every $p \in \bigcup_\gamma S^\varphi_\gamma$ let $Rk_\varphi(p)$ be the maximal n such that $Rk_\varphi(p) \ge n$. Clearly
(*) $p_1 \le p_2$ (both in $\bigcup_\beta S^\varphi_\beta$) implies $Rk_\varphi(p_1) \ge R_\varphi(p_2)$
Now for every $\delta \in W_0 = \{i < (2^\lambda)^+ : cf\ i > \lambda\}$, and α, there is $\gamma(\delta,p) < \delta$ such that:

$$\gamma(\delta,\varphi) \leq \gamma < \delta \Longrightarrow Rk_{\varphi}(tp_{\varphi}(a_{\delta},B_{\gamma(\delta,\varphi)})) = Rk_{\varphi}(tp_{\varphi}(a_{\delta},B_{\gamma}))$$

Let $\gamma(\delta) = \underset{\varphi}{\cup}\, \gamma(\delta,\varphi)$ so $\gamma(\delta) < \delta$. As we can use several f's (by coding) we can restrict ourselves to some stationary $W_1 \subseteq W_0$, such that for some γ^* $(\forall \delta \in W_1)\ [\gamma(\delta) = \gamma^*]$.

As not case II similarly w.l.o.g. for some p $(\forall \delta \in W_1)\ [tp(a_{\delta}, B_{\gamma^*}) = p]$.

Clearly $Rk_{\varphi}(p \restriction \varphi)$ is not even, hence is odd, (for every φ). Suppose $\gamma^* < \delta_1 < \delta_2$ in W_1, $tp(a_{\delta_1},B_{\delta_2}) \not\subseteq tp(a_{\delta_2},B_{\delta_2})$, then for some φ and $\alpha < \delta_1, \alpha > \gamma^*$ and both $tp_{\varphi}(a_{\delta_1},B_{\alpha}) \neq tp_{\varphi}(a_{\delta_2},B_{\alpha})$ have the same rank $(Rk_{\varphi}(-))$ as p, contradiction.

§4 From indiscernibles to finitely satisfiable and Hanf numbers

4.1. Lemma:

Suppose $\langle a_t : t \in I \rangle$ is an indiscernible sequence (I infinite). Then we can find a model N of power T such that for every $t \in I$, $tp(a_t, N \cup \{a_s : s < t\})$ is finitely satisfiable in N.

Proof: Let $I \subseteq J$, $t(n) \in J - I$, $(\forall t \in I)\ [t < t(n+1) < t(n)]$.

Let $\{a_t : t \in I\} \subseteq M \subseteq \mathfrak{C}$, and let M^* be an expansion of M by Skolem functions (so M^* is an L^*-model, $L \subseteq L^*$). By Ramsey theorem and the compactness theorem, there is a model M^+ of the theory of M^*, and $b_t \in M^+$ $(t \in J)$ such that:

(*) for every $\varphi(x_1, \ldots, x_n) \in L^*$, and $s_1 < \cdots < s_n \in J$ if

$M^+ \models \varphi[b_{s_1}, \ldots, b_{s_n}]$ *then* for some $t_1 < \cdots < t_n \in I$,

$M^* \models \varphi[a_{t_1}, \ldots, a_{t_n}]$.

Clearly for every $s_1 < \cdots < s_n \in J$, $t_1 < \cdots < t_n \in I$ the L-types of $\langle b_{s_1}, \ldots, b_{s_n} \rangle$ in M^+ and $\langle a_{t_1}, \ldots, a_{t_n} \rangle$ in M are equal, hence w.l.o.g. the L-reduct of M^+ is an elementary submodel of $\mathfrak{C}$ and $a_t = b_t$ for $t \in I$. Lastly let $N \subseteq \mathfrak{C}$ be the model whose universe is the Skolem hull of $\{b_{t(n)} : n < \omega\}$ in M^+, and $a_t \overset{def}{=} b_t$ also for $t \in J - I$.

So let $t \in I$ and we should prove that $tp_L(a_t, N \cup \{a_s : s < t, s \in I\})$ is finitely satisfiable in N. Let $\bar{d} \in N$, $t_0 < t_1 < \cdots < t_n = t \in I$, $\varphi \in L$, $\mathfrak{C} \models \varphi[b_{t_n}, b_{t_{n-1}}, \ldots, b_{t_0}, \bar{d}]$ so for some L^*-term $\bar{\tau}$, and $k < \omega$,

$\bar{d} = \bar{\tau}(b_{t(0)}, \ldots, b_{t(k)})$. As $\langle b_t : t \in J \rangle$ is indiscernible in M^+, and $M^+ \models \varphi[b_{t_n}, b_{t_{n-1}}, \ldots, b_{t_0}, \bar{\tau}(b_{t(0)}, \ldots, b_{t(k)})]$ clearly

$$M^+ \models \varphi[b_{t(k+1)}, b_{t_{n-1}}, \ldots, b_{t_0}, \bar{d}] \ .$$

As $b_{t(k+1)} \in N$, we finish.

4.2. Conclusion:

If $\lambda \rightarrow (\mu)^{<\omega}_{T}$, M a model of power λ, *then* for some N of power $|T|$, M has a decomposition $\langle A_i : i < \alpha \rangle$ over N, $A_i \neq \phi$, $\alpha \in \{\mu, \mu+1\}$

Proof: Immediate by 2.5, 4.1.

Remember $\mathcal{L}^{\delta}_{\infty,\lambda}$ is the set of sentences of $\mathcal{L}_{\infty,\lambda}$ with quantifier depth $< \delta$.

4.3. Theorem:

Suppose $(T_\infty, 2^{nd}) \not\leq (T, \mathrm{mon})$.

1) For limit ordinal δ and every λ the Hanf numbers of the logic $\mathcal{L}^{\delta}_{\infty,\lambda}$,
μ_1 for models of T expanded by $\leq |T|$ monadic predicates, and
μ_2 for linear well ordering expanded by $\leq |T|$ monadic predicates,
satisfies $\beth_{\omega^2}(\mu_1) = \beth_{\omega^2}(\mu_2)$

2) The Lowenheim and Hanf number of $\mathcal{L}^{\delta}_{\infty,\lambda}$, for well ordering expanded by $\leq |T|$ monadic predicates, are equal; so if λ, α are definable in second order logic, then those numbers are smaller than the Hanf number of 2^{nd} order logic.

Proof: 1) By 2.6, 4.2 this is reduced to a problem on monadic theory of sum of models, for complete proof see [Sh4]. However if $(\forall \alpha)(\alpha < \delta \rightarrow \alpha + \alpha < \delta)$, $\beth_\delta > |T|$ there are no problems.

2) See [BSh].

Now by 4.3 and 3.4:

4.4. Conclusion:

For T as above.

1) The Hanf number of $L_{\omega,\omega}(\mathrm{mon})$ for models of T is strictly smaller than the Hanf number of second order logic.

2) Even in $L_{\lambda,\lambda}$ we cannot interpret a pairing function on arbitrarily large sets in models of T.

Part II

Hypothesis: $(T_\infty, 2^{nd}) \not\leq (T,\text{mon})$

§1 On a rude equivalence relation

1.1. Context:

Let M_0 be a fixed model $(\subseteq \mathfrak{C}) M_0 \subseteq M_1 \subseteq \mathfrak{C}$, and in M_1 every type over M_0 (with $< \omega$ variables) is realized. The case $\|M_0\| = |T|$, $\|M_1\| \leq 2^{|T|}$ will suffice. We let $\mathcal{B}$ be an elementary extension of M_0, which is the model we want to analyze: and we assume $tp_*(\mathcal{B}, M_1)$ is finitely satisfiable in M_0 (and $\mathcal{B} \subseteq \mathfrak{C}$).

We usually suppress members of M_0 when used as individual constants.

We further let I be a κ-saturated dense linear order, $\kappa > 2^{|T|}$, and we can find elementary maping $f_t (t \in I)$ such that $Dom\ f_t = \mathcal{B}$, $f_t \restriction M_0 =$ the identity, and for some ultrafilter D on ${}^{\mathcal{B}}M_0$, $tp_*(f_t(\mathcal{B}), M_1 \cup \bigcup_{s<t} f_s(\mathcal{B}))$ is $Av(D, M_1 \cup \bigcup_{s<t} f_s(\mathcal{B}))$ (see for definition I 1.4, 1.5).

We denote by $\mathcal{B}_t$ the image of $\mathcal{B}$ by f_t.

For $a \in \mathcal{B}$ let $a_t = f_t(a)$, $\langle a_1, \ldots, a_n \rangle_t = \langle f_t(a_1), \ldots, f_t(a_n) \rangle$, $0 \in I$, $f_0 =$ the identify.

1.1A Remark:

Except in 2.1, 2.3, we use just the indiscernibility of the $\mathcal{B}_t$'s.

1.2. Definition:

1) On $\mathcal{B} = \mathcal{B}_0$, we define a relation E_0:

aE_0b *iff* in some monadic finite expansion of $\mathfrak{C}$ the set

$\{\langle a_t, b_t \rangle : t \in I\}$ is first order definable.

2) For $a \in \mathcal{B}$, $Od(a)$ hold if in some monadic finite expansion of $\mathfrak{C}$ the set $\{\langle a_t, a_s \rangle : t \in I, s \in I, t < s\}$ is first order definable.

1.3. Claim:

1) E_0 is an equivalence relation

2) aE_0b implies $Od(a) \longleftrightarrow Od(b)$

Proof: Easy

1.3. Claim

If $\bar{a}^k \subseteq b_k/E_0 \subseteq \mathcal{B}(k=1,n)$ and $b_k/E_0 \neq b_m/E_0$ for $k \neq m$ then:

(i) $tp(\bar{a}^1_{t_1} \frown \bar{a}^2_{t_2} \frown \cdots \frown \bar{a}^n_{t_n}, M_1)$ is the same for all $t_1, \ldots, t_n \in I$

(ii) $tp(\bar{a}^n, M_1 \cup \bigcup_{k=1}^{n-1} \bar{a}^k)$ is finitely satisfiable in M_0

(iii) if $a^n = \bar{b} \frown \bar{c}$, $tp(\bar{c}, M_1 \cup \bar{b})$ is finitely satisfiable in M_0, then $tp(\bar{c}, M_1 \cup \bar{b} \bigcup_{k=1}^{n-1} \bar{a}_k)$ is finitely satisfiable in M_0.

Proof: Clearly (ii) follows from (i) (just choose $t_n > t_1, \ldots, t_{n-1}$ in (i)] and also (iii) follows by I 1.6 from (ii).

So let us prove (i), and we prove it by induction on n and then on $k \leq n$, restricting ourselves to $\langle t_1, \ldots, t_n \rangle$ such that $|\{t_1, \ldots, t_n\}| \geq n-k$ (for $k = n$ we get the conclusions)

Suppose we have prove it for $n' < n$ and for $n' = n$, $k' < k$.

1.3A Fact:

By replacing $\mathfrak{C}$ by a monadic finite expansion we can replace $\bar{a}^m$ by a singleton $< a^m >$. Replacing $\mathfrak{C}$ by a finite monadic expansion $\mathfrak{C}^+$ does not preserve the properties of M_0, M_1, $\langle \mathcal{B}_s : s \in I \rangle$. However we can w.l.o.g. assume that $\langle \mathcal{B}_s : s \in I \rangle$ is indiscernible over M_1 in $\mathfrak{C}^+$. We could here also use $L(\mathfrak{C}^+)$-formulas only of the form $\varphi(\cdots, x_n \cdots, F_k(x_n) \cdots)$ where $\varphi \in L(\mathfrak{C})$, F_k are definable in $\mathfrak{C}^+$ and maps each $\mathcal{B}_s$ into itself and commute with the functions f_s.

1.4. Notation:

For non-decreasing sequences $\langle s_1, \ldots, s_n \rangle$, $\langle t_1, \ldots, t_n \rangle$ from I, we say that $\langle s_1, \ldots, s_n \rangle$ is *closed to* $\langle t_1, \ldots, t_n \rangle$ if

either (α) $t_1 < \cdots < t_n$, $s_m = t_{m+1}$ $s_{m+1} = t_m$, $s_i = t_i$ for $i \neq m, m+1$, for some m, $1 \leq m \leq n$

or (β) for some $1 \leq l < m \leq n$

$t_1 \leq \cdots \leq t_{l-1} < t_l = t_{l+1} = \cdots = t_m < t_{m+1} \leq \cdots \leq t_n$, $t_m < s_m < t_{m+1}$

and $(\forall i) \quad [1 \leq i \leq n \wedge i \neq m \rightarrow s_i = t_i]$.

We shall prove:

1.5. Fact:

If $\langle s_1, \ldots, s_n \rangle$ is closed to $\langle t_1, \ldots, t_m \rangle$, both non-decreasing sequences from I, $|\{t_i : i = 1, n\}| = n - k$, *then* $tp(\langle a^1_{t_1}, \ldots, a^n_{t_n} \rangle, M_1) = tp(\langle a^1_{s_1}, \ldots, a^n_{s_n} \rangle, M_1)$

This suffice for proving 1.3 as any equivalence relation E on

$$\{\langle t_1, \ldots, t_n \rangle : t_i \in I, \ |\{t_i : i = 1, n\}| \geq n - k\}$$

satisfying the following has just one class:

(a) if $\bar{s}$ is closed to $\bar{t}$ both non decreasing then $\bar{s}\ E\ \bar{t}$

(b) if $\langle s_1, \ldots, s_n \rangle\ E\ \langle s_{n+1}, \ldots, s_{2n} \rangle$ and $(\forall i,j \in [1,2n])\ [s_i < s_j \equiv t_j < t_j]$ *then* $\langle t_1, \ldots, t_n \rangle\ E\ \langle t_{n+1}, \ldots, t_{2n} \rangle$.

Proof of the Fact 1.5:

Note that 1.4(α) occurs only when $k = 0$, and 1.4(β) occurs only when $k > 0$

Case A: $k = 0$.

So there is a formula φ with parameters from $M_1 \cup \{a^1_{t_1}, \ldots, a^{i-1}_{t_{i-1}}, a^{i+2}_{t_{i+2}}, \ldots, a^n_{t_n}\}$, such that $\models \varphi[a^i_{t_i}, a^{i+1}_{t_{i+1}}]$ but $\models \neg\varphi[a^i_{t_{i+1}}, a^{i+1}_{t_i}]$. So clearly (by the indiscernibility of $\langle \mathcal{B}_t : t \in I \rangle$ over M_1) there is a formula φ with parameters from $\mathfrak{C}$ such that for any $s < t$ in I $\models \varphi[a^i_s, a^{i+1}_t] \wedge \neg\varphi[a^i_t, a^{i+1}_s]$ and w.l.o.g. $\models \varphi[a^i_s, a^{i+1}_s]$.

Adding monadic predicates $P^i = \{a^i_t : t \in I\}$, $P^{i+1} = \{a^{i+1}_t : t \in I\}$, we easily find that:

$$\theta(x,y) = \varphi(x,y) \wedge P^i(x) \wedge P^{i+1}(y) \wedge (\forall z)\,[P^i(z) \wedge x <^i z \rightarrow \neg\varphi(z,y)]$$

define $\{< a^i_s, a^{i+1}_s > : s \in I\}$, where

$$x <^i z \overset{\text{def}}{=} (\forall y)\,[P^{i+1}(y) \wedge \varphi(z,y) \rightarrow \varphi(x,y)] \wedge x \neq z \wedge P^i(x) \wedge P^i(z).$$

Now θ contradict the non E_0-equivalence of a^i, a^{i+1}.

Case B: $k > 0$

So there is a formula φ with parameters from $M_1 \cup \{a^1_{t_1}, \ldots, a^{l-1}_{t_{l-1}}, a^{m+1}_{t_{m+1}}, \ldots, a^n_{t_n}\}$ such that:

$(a)\ \models \varphi[a^l_{t_l}, \ldots, a^{m-1}_{t_{m-1}}, a^m_{s_m}]$

$(b)\ \models \neg\varphi[a^l_{t_l}, \ldots, a^{m-1}_{t_{m-1}}, a^m_{t_m}]$

by the induction hypothesis on k, from (a) it follows

(c) for any $v_l, \ldots, v_m \in \{t \in I : t_{l-1} < t < t_{m+1}\}$,

not all of them equal $\models \varphi[a^l_{v_l}, \ldots, a^m_{v_m}]$
By (b), as $t_l = \cdots = t_m$,

(d) for any $v \in \{t \in I : t \ < t < t_{m+1}\}$

$\models \neg\varphi[a^l_v, \ldots, a^m_v]$
Using the indiscernibility of $\langle \mathcal{B}_t : t \in I \rangle$ over M_1 there is a formula φ' (with parameters from $\mathfrak{C}$) such that (c), (d) holds for any $v_l, \ldots, v_m \in I$ not all equal, and for any $v \in I$ respectively.
Expanding $\mathfrak{C}$ by $P^i = \{a^i_t : t \in I\}$, we find that the formula

$$\theta(x,y) = P^l(x) \wedge P^{l+1}(y) \wedge (\exists z_{l+2}, \ldots, z_m)[\bigwedge_{i=l+2}^{m} P^i(z_i) \wedge \neg\varphi(x,y,z_{l+2}, \ldots, z_n)]$$

define the set $\{< a^l_t, a^{l+1}_t > : t \in I\}$ of pairs, contradicting the non E_0-equivalence of a^l, a^{l+1}.

§2 Extending a pair of finitely satisfiable

We continue to use the context of §1 (of part II)

2.1. Claim:

If $\bar{a}, \bar{b} \in \mathcal{B}$ *then* $tp(\bar{a}\frown\bar{b}, M_1) = tp(\bar{a}_s\frown\bar{b}_t, M_1)$ for some (every) $s < t \in I$ *iff* $tp(\bar{b}, M_1 \cup \bar{a})$ is finitely satisfiable in M_0

Proof: Easy

2.2. Lemma:

There are no $s < t \in I$, $\bar{a}, \bar{b} \in \mathcal{B}$ and $c \in \mathfrak{C}$ and formula φ with parameters from M_1, such that:

$(a)\ \models \varphi[c, \bar{a}_s, \bar{b}_t]$

$(b)\ \models \neg\varphi[c, \bar{a}_s, \bar{b}_{t_1}]$ for every $t_1 > t$ (in I)

$(c)\ \models \neg\varphi[c, a_{s_1}, \bar{b}_t]$ for every $s_1 < s$ (in I)

(d) $\bar{a}\frown\bar{b}$ is included in one E_0-equivalence class .

Proof: By (d) and 1.3A, replacing $\mathfrak{C}$ by a monadic finite expansion w.l.o.g. $\bar{a} = <a>$, $\bar{b} = <b>$. By Ramsey theorem and compactness we can assume that if $\langle v_1, \ldots, v_m \rangle$, $\langle u_1, \ldots, u_m \rangle$, are increasing sequences from I, $(\exists k)(v_k = u_k = s)$, $(\exists k)(v_k = u_k = t)$ then

$$tp_*(\langle \mathcal{B}_{v_1}, \ldots, \mathcal{B}_{v_m} \rangle , M_1 \cup \{c\}) =$$

$$tp_*(\langle \mathcal{B}_{u_1}, \ldots, \mathcal{B}_{u_m} \rangle , M_1 \cup \{c\}) .$$

By II. 1.3A, w.l.o.g. $\mathfrak{C}$ has predicates for $\{a_t : t \in I\}$, $\{b_t : t \in I\}$, and $\{<a_t, b_t> : t \in I\}$. We shall try to use c for coding $\{s, t\}$ (i.e., $\{a_s, b_t\}$), which contradict $(T_\infty, 2^{nd}) \not\leq (T, \mathrm{mon})$ (see I. 1.3(2)).

Case A: not $0d(a)$

Subcase A1: For any $v \in I$, $s < v < t$, $\models \varphi[c, a_v, b_t]$.

Then we can fix t, and define $\{<a_v, b_u> : v < u < t\}$ as in the proof of 1.5 Case A and then define $\{<a_v, a_u> : v < u \in I\}$, contradicting not $0d$ (a).

Subcase A2: Not A1 but for any $v \in I$, if $v > t$, then $\models \varphi[c, a_v, b_t]$

Similar contradiction: fix s, and using the function $\{<a_v, b_v> : v \in I\}$ define $\{<b_v, b_u> : s < v < u>\}$.

Subcase A3: For $v \in I$, if $s < v < t$ then $\models \varphi[c, a_s, b_v]$
like subcase A1 (interchanging a and b)

Subcase A4: Note A3 but if $v \in I$, $v < s$ then $\models \varphi[c, a_s, b_v]$
like A2 (interchanging a and b)

Subcase A5: Not A1-A4

Here c code the pair $<a_s, b_t>$: a_s is unique for t such that $s \neq t$ and $\varphi(c, a_s, b_t)$ (by not A1, A2). By symmetry (i.e., as not A3, A4) t is unique for s, by the indiscernibility we have over c and as I is dense this shows that c determine $<s, t>$, so we get the contradiction to the hypothesis of Part II.

Case B: $0d(a)$

Let $\theta(x,y,z)$ says all the relevant things on $<a,b,c>$: $x \in \{a_v : v \in I\}$, $y \in \{b_v : v \in I\}$, $\varphi(x,y,z)$, $\neg\varphi(z,x',y)$ where $x' < x$ [i.e., $(\exists v < u)$ $(x' = a_v \wedge x = a_u)$] and $\neg\varphi(z,x,y')$ where $y' < y$ [i.e., $(\exists v < u)$ $(y = b_v \wedge y' = b_u)$] and the amount of $\varphi(z,-,-)$-indiscernibility of $\langle <a_v, b_v> : v \in I \rangle$ over $\{c\}$ which holds.

Clearly $\models \theta[a_s, b_t, c]$

It suffices to prove that

(*) If $\theta[a_{s(k)}, b_{t(k)}, c]$ for $k=1,2$ then $s(1) = s(2)$, $t(1) = t(2)$.

By symmetry we can assume $t(1) < t(2)$ (if $t(2) < t(1)$ interchange the order, if $t(2) = t(1)$ neccessarily $s(1) \neq s(2)$ and invert the order). Below u, v denote elements of I.

Suppose $s(2) < u < v$, we can find u_1, v_1, $t(1) < u_1 < v_1$ such that $s(2) < u_1$, $u < t(2) \iff u_1 < t(2)$, $u = t(2) \iff u_1 = t(2)$, $v < t(2) \iff v_1 < t(2)$, and $v = t(2) \iff v_1 = t(2)$.

As $\models \theta[a_{s(2)}, b_{t(2)}, c]$, it follows that

(i) $\varphi(c, a_{u_1}, b_{v_1}) \equiv \varphi(c, a_u, b_v)$

Now choose $u_2 > v_2 > t(2)$, as $\models \theta[a_{s(1)}, b_{t(1)}, c]$, clearly

(ii) $\varphi(c, a_{u_1}, b_{v_1}) \equiv \varphi(c, a_{u_2}, b_{v_2})$

By transitivitiy of $\equiv$

(iii) the truth value of $\varphi(c, a_u, b_v)$ is the same for all $v > u > s(2)$.

Now (iii) is a property of c and $s(2)$, and it fails for any $s' < s(2)$ as $\models \varphi[a_{s(2)}, b_{t(2)}, c]$ but $\models \neg\varphi[a_{s(2)}, b_v, c]$ when $v > t(2)$; so $a_{s(2)}$ is definable from c, and then we can easily define $b_{t(2)}$, and so get the desired contraction.

2.3. Lemma:

If $\bar{a}, \bar{b}, c \in \mathfrak{C}$, $tp(\bar{b}, M_0 \cup \bar{a})$ is finitely satisfiable in M_0 *then:*

$tp(\bar{b} \frown \langle c \rangle, M_0 \cup \bar{a})$ is finitely satisfiable in M_0 or

$tp(\bar{b}, M_0 \cup \bar{a} \frown \langle c \rangle)$ is finitely satisfiable in M_0

Proof: Suppose $\bar{a}$, $\bar{b}$, c form a counterexample. W.l.o.g. $\mathcal{B}$ is $\|M_0\|^+$-saturated. Choose $\bar{a}' \in \mathcal{B}$ realizing $tp(\bar{a}, M_0)$, then choose $\bar{b}'$ such that $tp(\bar{a}' \frown \bar{b}', M_0) = tp(\bar{a} \frown \bar{b}, M_0)$. Then choose $\bar{b}''$ realizing $tp(\bar{b}', M_0 \cup \bar{a}')$ such that $tp(\bar{b}'', M_1 \cup \bar{a}')$ is finitely satisfiable in M_0; now $tp(\bar{a}' \frown \bar{b}'', M_1)$ is finitely satisfiable in M_0, so we could have chosen $\mathcal{B}$, D such that $\bar{a}' \frown \bar{b}'' \subseteq \mathcal{B}$.

Now choose $c' \in \mathcal{B}$ such that $tp(\bar{a}' \frown \bar{b}'' \frown \langle c' \rangle, M_0) = tp(\bar{a} \frown \bar{b} \frown \langle c' \rangle, M_0)$; hypothesis 2.2 (d) may fail for $\bar{a}'$, $\bar{b}''$, c', but by 1.3 (iii) we get it by replacing $\bar{a}'$, $\bar{b}''$ by $\bar{a}' \cap (c'/E_0)$, $\bar{b}'' \cap (c/E_0)$.

We can choose c'', such that $c''\frown\bar{a}'_s\frown\bar{b}''_t$, $c\frown\bar{a}\frown\bar{b}$ realizes the same type over M_0, and $tp_*(\{c''\} \cup\cup \{\mathcal{B}_v : s \le v \le t\}, M_1 \cup\cup \{\mathcal{B}_v : v < s\})$ is finitely satisfiable in M_0. We can furthermore assume as in the proof of I. 2.6 that for $v > t$ $tp_*(\mathcal{B}_v, \bigcup_{u<v} \mathcal{B}_u \cup \{c''\} \cup M_1)$ is finitely satisfiable in M_0, so $tp_*(\bigcup_{v>t} \mathcal{B}_v, M_1 \cup \bigcup_{u\le t} \mathcal{B}_u \cup \{c'\})$ is finitely satisfiable in M_1. Now $\bar{a}'_s$, $\bar{b}''_t$, c'' satisfies (a) (b) (c) (d) of 2.2 if $\bar{a}$, $\bar{b}$, c where a counterexample to 2.2, where $s < t \in I$. So by 2.2 we have proved 2.3.

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