

Computing Nested Fixpoints in Quasipolynomial Time

DANIEL HAUSMANN, Friedrich-Alexander Universität Erlangen-Nürnberg, Germany

LUTZ SCHRÖDER, Friedrich-Alexander Universität Erlangen-Nürnberg, Germany

It is well known that the winning region of a parity game with n nodes and k priorities can be computed as a k -nested fixpoint of a suitable function; straightforward computation of this nested fixpoint requires $n^{\lceil \frac{k}{2} \rceil + 1}$ iterations of the function. The recent parity game solving algorithm by Calude et al. runs in quasipolynomial time and essentially shows how to compute the same fixpoint using only a quasipolynomial number of iterations. We show that their central idea naturally generalizes to the computation of k -nested fixpoints of *any* set-valued function; hence k -nested fixpoints of set functions that can be computed in quasipolynomial time can be computed in quasipolynomial time as well. While this result is of clear interest in itself, we focus in particular on applications to modal fixpoint logics beyond relational semantics. For instance, the model checking problems for the graded and the (two-valued) probabilistic μ -calculus – with numbers coded in binary – can be solved by computing nested fixpoints of functions that differ from the function for parity game solving, but still can be computed in quasipolynomial time; our result hence implies that model checking for these μ -calculi is in QP. A second implication of our result lies in satisfiability checking for generalized μ -calculi, including the graded, probabilistic and alternating-time variants; in a general setting that covers all the mentioned cases, our result immediately improves the upper time bound for satisfiability checking for fixpoint formulas of size n with alternation-depth k from $2^{O(n^2 k^2 \log n)}$ to $2^{O(nk \log n)}$.

Additional Key Words and Phrases: fixpoints, parity games, μ -calculus, coalgebraic logic, model checking

1 INTRODUCTION

Fixpoints are pervasive in computer science, governing large portions of recursion theory, concurrency theory, logic, and game theory. One famous example are parity games, which are central, e.g., to networks and infinite processes [Bodlaender et al. 2001], tree automata [Zielonka 1998], and μ -calculus model checking [Emerson et al. 2001]. Winning regions in parity games can be expressed as nested fixpoints of particular set functions (e.g. [Bruse et al. 2014; Dawar and Grädel 2008]). In recent breakthrough work on the solution of parity games in quasipolynomial time, Calude et al. [2017] essentially show how to compute this particular fixpoint in quasipolynomial time. Briefly, our contribution in the present work is to show that this result generalizes to nested fixpoints of arbitrary set functions. That is, given a set function $\alpha : (\mathcal{P}(U))^k \rightarrow \mathcal{P}(U)$ on a finite set U , we show that the k -nested fixpoints

$$E_k = \eta_k X_k . \eta_{k-1} X_{k-1} . \dots . \eta_1 X_1 . \alpha(X_1, \dots, X_k)$$

where $\eta_j = \text{GFP}$ (greatest fixpoint) if j is even and $\eta_j = \text{LFP}$ (least fixpoint) if j is odd, are computable in quasipolynomial time, provided that α itself is computable in quasipolynomial time.

In more detail, the method of Calude et al. can be described as annotating nodes of a parity game with histories of quasipolynomial size and then solving this annotated game, but with a safety winning condition instead of the much more involved parity winning condition. It had previously been shown that parity games can be solved by annotating nodes with histories of *exponential* size and then solving the resulting safety games; the fixpoint iteration algorithm (which leaves the histories implicit) [Bruse et al. 2014] and the small progress measures algorithm (using explicit histories) [Jurdziński 2000] both essentially follow this approach, which has also been described

Authors' addresses: Daniel Hausmann, Friedrich-Alexander Universität Erlangen-Nürnberg, Germany, daniel.hausmann@fau.de; Lutz Schröder, Friedrich-Alexander Universität Erlangen-Nürnberg, Germany, lutz.schroeder@fau.de.

as pairing (exponential-sized) *separating automata* with safety games [Czerwinski et al. 2019]. Histories are vectors of natural numbers that encode information about how often certain priorities have been visited. We refer to histories in the style of Calude et al. as *quasipolynomial histories*, and to annotations in progress measure style approaches as *exponential timeouts*.

Hasuo et al. [Hasuo et al. 2016], and more generally, Baldan et al. [Baldan et al. 2019] show that nested fixpoints in highly general settings can be computed by a technique based on progress measures, essentially using exponential timeouts, and obtaining, of course, exponential run time. Our technique is based on showing that one can equivalently use quasipolynomial histories, correspondingly obtaining quasipolynomial run time. In both cases, computation of the nested fixpoint is reduced to a single (least or greatest depending on exact formulation) fixpoint of a function that extends the given set function to keep track of the timeouts or histories, respectively, in analogy to the previous reduction of parity games to safety games. We say that nodes have exponential timeouts or quasipolynomial histories if they are contained in the respective single fixpoint; our central result can then be phrased as saying that nodes have exponential timeouts if and only if they have quasipolynomial histories. In the forward direction of the proof, we use *fixpoint games* [Baldan et al. 2019; Venema 2008]. These games have exponential size but we show how to extract *polynomial-size* witnesses for winning strategies of Eloise, and use these witnesses to show that any node won by Eloise has quasipolynomial timeouts. For the backwards direction, a careful analysis can be used to show that quasipolynomial histories correspond to exponential timeouts in such a way that when both are updated according to the same sequence of priorities, the exponential representation of the quasipolynomial history decreases, in each component, at least as fast as the exponential timeout. Thus nodes with quasipolynomial histories also have exponential timeouts. This proves the quasipolynomial upper bound for computation of nested fixpoints. As a side result, we moreover use our polynomial-sized witnesses for containment in nested fixpoints to show that computing nested fixpoints of functions that can be computed in polynomial time is also in $\text{NP} \cap \text{coNP}$ (a bound that is presently incomparable to the QP bound).

As an immediate application of these results, we improve generic upper complexity bounds on model checking and satisfiability checking in the *coalgebraic μ -calculus* [Cirstea et al. 2011], which serves as a generic framework for fixpoint logics beyond relational semantics. Well-known instances of the coalgebraic μ -calculus include the alternating-time μ -calculus [Alur et al. 2002], the graded μ -calculus [Kupferman et al. 2002], the (two-valued) probabilistic μ -calculus [Cirstea et al. 2011; Liu et al. 2015], and the monotone μ -calculus [Enqvist et al. 2015] (the ambient fixpoint logic of concurrent dynamic logic CPDL [Peleg 1987] and Parikh’s game logic [Parikh 1985]). This level of generality is achieved by abstracting systems types as set functors and systems as coalgebras for the given functor following the paradigm of universal coalgebra [Rutten 2000].

Recent work [Hausmann and Schröder 2019a] has shown that the model checking problem for coalgebraic μ -calculi reduces to the computation of a nested fixpoint. This fixpoint may be seen as a coalgebraic generalization of a parity game winning region but can be literally phrased in terms of small standard parity games (implying quasipolynomial run time) only in restricted cases. Our results show that the relevant nested fixpoint can be computed in quasipolynomial time in all cases of interest. Notably, we thus obtain as new specific upper bounds that even under binary coding of numbers, the model checking problems of both the graded μ -calculus and the probabilistic μ -calculus are in QP, even when the syntax is extended to allow for (monotone) polynomial inequalities.

Similarly, the satisfiability problem of the coalgebraic μ -calculus has been reduced to a computation of a nested fixpoint [Hausmann and Schröder 2019b]; crucially, the nesting depth of this fixpoint always is exponentially smaller than the number of states of the respective carrier set. Our

results imply that this fixpoint is computable in polynomial time, which means that its computation is dominated by other steps of the procedure (especially determinization of Büchi automata). The complexity of satisfiability checking in coalgebraic μ -calculi (satisfying mild conditions on the modalities) thus drops from $2^{O(n^2 k^2 \log n)}$ to $2^{O(nk \log n)}$ for formulas of size n and with alternation depth k .

Related Work. The quasipolynomial bound on parity game solving has in the meantime been realized by a number of alternative algorithms. For instance, Jurdzinski and Lazic [2017] use succinct progress measures to improve to quasilinear (instead of quasipolynomial) space; Fearnley et al. [2019] similarly achieve quasilinear space. Lehtinen [2018] and Boker and Lehtinen [2018] present a quasipolynomial algorithm using register games. Parys [2019] improves Zielonka's algorithm [1998] to run in quasipolynomial time. In particular the last algorithm is of interest as an additional candidate for generalization to nested fixpoints, due to the known good performance of Zielonka's algorithm in practice. Daviaud et al. [2018] generalize quasipolynomial-time parity game solving by providing a pseudo-quasipolynomial algorithm for mean-payoff parity games. On the other hand, Czerwinski et al. [2019] give a quasipolynomial lower bound on universal trees, implying a barrier for prospective polynomial-time parity game solving algorithms.

The computation of unrestricted fixpoints has seen recent progress in the above-mentioned approaches using progress measures [Hasuo et al. 2016] and fixpoint games [Baldan et al. 2019], both with a view to applications in coalgebraic model checking like in the present paper. These approaches are set in a more general framework differing in particular in employing arbitrary complete lattices instead of finite powersets, and moreover in considering systems of fixpoint equations rather than a single nested fixpoint. On the other hand, both approaches have exponential run time. We conjecture that our approach generalizes to systems of fixpoint equations; for the time being, we note that while systems of equations allow a direct encoding of formula evaluation in (coalgebraic) μ -calculi, we still do cover the unrestricted coalgebraic μ -calculus via its translation into nested fixpoints following Hausmann and Schröder [2019a].

2 NOTATION

Let U and V be sets and let $R \subseteq U \times U$ be a binary relation on U ; for $u \in U$, we then put $R(u) := \{v \in U \mid (u, v) \in R\}$. We write $U^* = \{u_0, u_1, \dots, u_n \mid n \in \mathbb{N}, \forall 0 \leq i \leq n. u_i \in U\}$ for the set of finite sequences of elements of U and $U^\omega = \{u_0, u_1, \dots \mid \forall 0 \leq i. u_i \in U\}$ for the set of infinite sequences of elements of U . The *domain* $\text{dom}(f) = \{u \in U \mid \exists v \in V. f(u) = v\}$ of a partial function $f : U \rightarrow V$ is just the set of elements on which the function is defined. We often regard (finite) sequences $\tau = u_0, u_1, \dots \in U^* \cup U^\omega$ of elements of U as (partial) functions of type $\mathbb{N} \rightarrow U$ and then write $\tau(i)$ to denote the element u_i , for $i \in \text{dom}(\tau)$. For a sequence τ of elements of U , we define the set

$$\text{Inf}(\tau) = \{u \in U \mid \forall i \geq 0. \exists j > i. \tau(j) = u\}$$

of elements that occur infinitely often in τ , having $\text{Inf}(\tau) = \emptyset$ if τ is a finite sequence. Given a set U , we use $\mathcal{P}(U)$ to denote the powerset of U and put $U^1 = U$ and $U^{m+1} = U \times U^m$ for $m > 0$. A function $f : \mathcal{P}(U)^k \rightarrow \mathcal{P}(U)$ is *monotone* if we have, $f(V_1, \dots, V_k) \subseteq f(W_1, \dots, W_k)$ for all $V_i \subseteq W_i \subseteq U$ where $1 \leq i \leq k$. For any monotone function $g : \mathcal{P}(U) \rightarrow \mathcal{P}(U)$, we define

$$\text{GFP } g = \bigcup \{V \subseteq U \mid V \subseteq g(V)\} \quad \text{LFP } g = \bigcap \{V \subseteq U \mid g(V) \subseteq V\}$$

which, by the Knaster-Tarski fixpoint theorem, are the greatest and the least fixpoint of g , respectively. Furthermore, we define $g^0(V) = V$ and $g^{m+1}(V) = g(g^m(V))$ for $m \geq 0$, $V \subseteq U$; if U is finite, then we have $\text{GFP } g = g^n(U)$ and $\text{LFP } g = g^n(\emptyset)$ by Kleene's fixpoint theorem. For a function

$g : \mathcal{P}(U) \rightarrow \mathcal{P}(U)$ such that for all $V \subseteq U$, the set $g(V)$ can be computed in time (quasi)polynomial in $|U|$, we say that g can be *iterated* (or *computed*) in (quasi)polynomial time. Given sets A, B and a function $f : A \rightarrow B$, the *image* of $A' \subseteq A$ under f is defined by $f[A'] = \{b \in B \mid \exists a \in A'. f(a) = b\}$.

3 NESTED FIXPOINTS

We now introduce our central notion, *nested fixpoints*. As mentioned above, we work with set-valued functions and consider directly alternating fixpoints of the shape

$$vX_k.\mu X_{k-1}.\dots.\mu X_1.\alpha(X_1, \dots, X_k) \quad \text{or} \quad \mu X_k.vX_{k-1}.\dots.vX_1.\alpha(X_1, \dots, X_k)$$

where α is a monotone k -ary set function, and μ and v take extremal fixpoints. We expect our methods and proofs to generalize naturally (that is, essentially by writing more indices) to the setting where nested fixpoints are defined by systems of fixpoint equations (e.g. [Baldan et al. 2019; Hasuo et al. 2016]), but in this work restrict our attention to directly nested fixpoints of a single function for technical reasons and readability.

Definition 3.1 (Nested fixpoints). Let U be a finite set, let k be some number and let $\alpha : (\mathcal{P}(U))^k \rightarrow \mathcal{P}(U)$ be a monotone function. We define, for $(X_1, \dots, X_k) \in (\mathcal{P}(U))^k$,

$$E_0^\alpha(X_1, \dots, X_k) = A_0^\alpha(X_1, \dots, X_k) = \alpha(X_1, \dots, X_k)$$

and for $0 < i \leq k$,

$$E_i^\alpha(X_{i+1}, \dots, X_k) = \eta_i(\lambda X_i. E_{i-1}^\alpha(X_i, X_{i+1}, \dots, X_k))$$

$$A_i^\alpha(X_{i+1}, \dots, X_k) = \overline{\eta}_i(\lambda X_i. A_{i-1}^\alpha(X_i, X_{i+1}, \dots, X_k)),$$

where $\eta_i = \text{GFP}$ if i is even and $\eta_i = \text{LFP}$ if i is odd and where $\overline{\text{GFP}} = \text{LFP}$ and $\overline{\text{LFP}} = \text{GFP}$. We usually use the more intuitive notation

$$\eta_i X_i. \eta_{i-1} X_{i-1}. \dots. \eta_1 X_1. \alpha(X_1, \dots, X_k) := E_i^\alpha(X_{i+1}, \dots, X_k)$$

$$\overline{\eta}_i X_i. \overline{\eta}_{i-1} X_{i-1}. \dots. \overline{\eta}_1 X_1. \alpha(X_1, \dots, X_k) := A_i^\alpha(X_{i+1}, \dots, X_k).$$

The sets

$$E_k^\alpha = \eta_k X_k. \eta_{k-1} X_{k-1}. \dots. \eta_1 X_1. \alpha(X_1, \dots, X_k)$$

$$A_k^\alpha = \overline{\eta}_k X_k. \overline{\eta}_{k-1} X_{k-1}. \dots. \overline{\eta}_1 X_1. \alpha(X_1, \dots, X_k)$$

are the k -nested fixpoint of α and the dual k -nested fixpoint of α , respectively and we usually omit the index α if no confusion arises. Since fixing arguments of set-valued functions preserves monotonicity, the sets $E_i^\alpha(X_{i+1}, \dots, X_k)$ and $A_i^\alpha(X_{i+1}, \dots, X_k)$ are indeed defined by the Knaster-Tarski fixpoint theorem. The *problem of computing nested fixpoints* consists in deciding, for input number k , input set U and input function $\alpha : (\mathcal{P}(U))^k \rightarrow \mathcal{P}(U)$ and for all $v \in U$ whether we have $v \in E_k^\alpha$ or not.

Example 3.2. For an example of a nested fixpoint, let $A = (U, \Sigma, \delta, \Omega)$ be a parity automaton (e.g. [Grädel et al. 2002]) with set U of states, alphabet Σ , transition relation $\delta \subseteq U \times \Sigma \times U$ and priority function $V \rightarrow \mathbb{N}$ using k priorities, and define $R \subseteq U \times U$ by $R(v) = \{w \mid \exists a \in \Sigma. (v, a, w) \in \delta\}$. We also define $\alpha_{\text{pa}} : (\mathcal{P}(U))^k \rightarrow \mathcal{P}(U)$ by putting

$$\alpha_{\text{pa}}(U_1, \dots, U_k) = \{v \in U \mid \exists 1 \leq i \leq k. \Omega(v) = i, R(v) \cap U_i \neq \emptyset\}$$

for $(U_1, \dots, U_k) \in (\mathcal{P}(U))^k$, that is, $\alpha_{\text{pa}}(U_1, \dots, U_k)$ consists of states that have priority i and a transition to some state from U_i , for some i . Then we have that $E_k^{\alpha_{\text{pa}}}$ is exactly the non-emptiness region of A (that is, the set of states in A that accept some word).

Definition 3.3 (Complement, dual function). Given a set $V \subseteq U$, we define the *complement* of V in U by $\bar{V} = U \setminus V$, having $\overline{\bar{V}} = V$. Given a monotone function $f : (\mathcal{P}(U))^k \rightarrow \mathcal{P}(U)$, we then define the *dual function* \bar{f} by putting $\bar{f}(U_1, \dots, U_k) = \overline{f(\bar{U}_1, \dots, \bar{U}_k)}$ for $(U_1, \dots, U_k) \in (\mathcal{P}(U))^k$.

LEMMA 3.4 (NESTED FIXPOINT DUALITY). *For all $k \in \mathbb{N}$, all $0 \leq i \leq k$, all monotone functions $\alpha : (\mathcal{P}(U))^k \rightarrow \mathcal{P}(U)$ and all $(X_{i+1}, \dots, X_k) \in (\mathcal{P}(U))^{k-i}$, we have*

$$E_i^\alpha(X_{i+1}, \dots, X_k) = \overline{A_i^{\bar{\alpha}}(\bar{X}_{i+1}, \dots, \bar{X}_k)};$$

in particular, we have $E_k^\alpha = \overline{A_k^{\bar{\alpha}}}$.

PROOF. This proof is standard and can be found in the appendix at the end of this document. \square

4 PARITY GAMES

Next, we recall some basic notions of parity games (see e.g. [Grädel et al. 2002]). Since we essentially prove two striking similarities between parity game solving and computing nested fixpoints, the central contributions of this work rely heavily on certain previous results on parity games which we hence also recall in this section.

Definition 4.1 (Parity games). A *parity game* (V, E, Ω) consists of a set of *nodes* V , a set of *moves* $E \subseteq V \times V$ encoding the rules of the game and a *priority function* $\Omega : V \rightarrow \mathbb{N}$, assigning *priorities* $\Omega(v) \in \mathbb{N}$ to nodes $v \in V$. Furthermore, each node belongs to exactly one of the two players Eloise or Abelard (where we denote Eloise's nodes by V_\exists and Abelard's nodes by V_\forall). A *play* $\rho \in V^* \cup V^\omega$ is a (finite or infinite) sequence of nodes that follows the rules of the game, that is, such that for all $i \geq 0$ such that ρ contains at least $i + 1$ nodes, we have $(\rho(i), \rho(i + 1)) \in E$. We say that an infinite play $\rho = v_0, v_1, \dots$ is *even*, if the largest priority that occurs infinitely often in it is even (formally, if $\max(\text{Inf}(\Omega \circ \rho))$ is an even number), and *odd* otherwise; finite plays are required to end in nodes that have no outgoing move. Player Eloise *wins* all even plays and finite plays that end in an Abelard-node; player Abelard *wins* all other plays. The *size* of a parity game (V, E, Ω) is $|V|$. A (*history-free*) *Eloise-strategy* $s : V_\exists \rightarrow V$ is a partial function that assigns moves $s(x)$ to Eloise-nodes $x \in \text{dom}(s)$. A play ρ *follows* an Eloise-strategy s if for all $i \in \text{dom}(\rho)$ such that $\rho(i) \in V_\exists$, we have $\rho(i + 1) = s(\rho(i))$; then we also say that ρ is an *s-play*. An Eloise-strategy *wins* a node $v \in V$ if Eloise wins all plays that start at v and follow s . We have a dual notion of Abelard-strategies; *solving* a parity game consists in computing the *winning regions* win_\exists and win_\forall of the two players, that is, the sets of states that they respectively win by some strategy. A parity game is *alternating* if $E[V_E] \subseteq V_\forall$ and $E[V_\forall] \subseteq V_\exists$, that is, if all of Eloise's moves lead to Abelard-nodes and vice versa.

A crucial property of parity games is that they are *history-free determined* [Grädel et al. 2002], that is, that every node in a parity game is won by exactly one of the two players and then there is a history-free strategy for the respective player that wins the node. This is reflected by the central fact that the winning regions in parity games can be computed by *fixpoint iteration* (that is, as a nested fixpoint in the sense of Definition 3.1).

LEMMA 4.2 ([BRUSE ET AL. 2014; DAWAR AND GRÄDEL 2008]). *Let (V, E, Ω) be a parity game with k priorities. Then we have*

$$\text{win}_\exists = E_k^{\alpha_\exists} \quad \text{and} \quad \text{win}_\forall = A_k^{\alpha_\forall},$$

where $\alpha_\exists : (\mathcal{P}(V))^k \rightarrow \mathcal{P}(V)$ and $\alpha_\forall : (\mathcal{P}(V))^k \rightarrow \mathcal{P}(V)$ are defined by

$$\begin{aligned} \alpha_\exists(V_1, \dots, V_k) = & \{v \in V_\exists \mid \exists 1 \leq i \leq k. \Omega(v) = i, E(v) \cap V_i \neq \emptyset\} \cup \\ & \{v \in V_\forall \mid \exists 1 \leq i \leq k. \Omega(v) = i, E(v) \subseteq V_i\} \end{aligned}$$

and

$$\begin{aligned}\alpha_V(V_1, \dots, V_k) = & \{v \in V_\exists \mid \exists 1 \leq i \leq k. \Omega(v) = i, E(v) \subseteq V_i\} \cup \\ & \{v \in V_V \mid \exists 1 \leq i \leq k. \Omega(v) = i, E(v) \cap V_i \neq \emptyset\}.\end{aligned}$$

The Lemma also implies determinacy of parity games since we have $\text{win}_\exists = E_k^{\alpha_\exists} = \overline{A_k^{\alpha_V}} = \overline{\text{win}_V}$ where the second equality is by Lemma 3.4 since $\alpha_V = \overline{\alpha_\exists}$ (that is, α_V and α_\exists are dual functions).

It is known that solving parity games is in $\text{NP} \cap \text{coNP}$ (and, more specifically, in $\text{UP} \cap \text{co-UP}$). Recently it has also been shown [Calude et al. 2017] that for parity games with n nodes and k priorities, win_\exists and win_V can be computed in quasipolynomial time $O(n^{\log k+6})$. Towards generalizing both of these results to the computation of nested fixpoints, we now show that applying strategies to alternating parity games can severely reduce the size of the game by removing unused edges and nodes.

Definition 4.3 (Parity game witness). Let (V, E, Ω) be a parity game. A *winning witness* for player Eloise is a graph (W, F) such that $W \subseteq V$ and $F \subseteq E$, for all $v \in V_V \cap W$, we have $F(v) = E(v)$, for all $v \in V_\exists \cap W$, we have $F(v) \neq \emptyset$, and all paths in (W, F) are even.

LEMMA 4.4. *Let (V, E, Ω) be an alternating parity game, let $W = V_\exists \cap \text{win}_\exists$ and let s be a strategy that wins every node in win_\exists . Then there is a winning witness $(W \cup s[W], F)$ of size $|W \cup s[W]| \leq 2|V_\exists| \in O(V_\exists)$.*

PROOF. Let s be an Eloise-strategy that wins every node from win_\exists . Without loss of generality, we assume that s is history-free. Since the game is alternating, we have $W \cap s[W] = \emptyset$. We define F by $F(v) = s(v)$ for $v \in W$ and by $F(v) = E(v)$ for $v \in s[W]$, noting that $E[s[W]] \subseteq W$ since s is a winning strategy, so wherever strategy s moves from an Eloise-node that is won by Eloise, player Abelard can only reply by moving to Eloise-nodes that she wins as well. The properties of winning witnesses are easily verified, noting that s is a winning strategy and that all paths in $(W \cup s[W], F)$ are s -plays of the game and hence even, as required. \square

It is well known that winning strategies in parity games can be verified in polynomial time:

LEMMA 4.5. *Given a parity game (V, E, Ω) , a node $v \in V$ and a strategy s for one of the players, it can be checked in time polynomial in the numbers of nodes and priorities in the game whether the respective player wins v with s .*

PROOF. We define a graph (W, F) with $W \subseteq V$, starting at v and inductively adding nodes as follows: when at a node u that belongs to the player in question, add $s(u)$ to W and put $F(u) = s(u)$; when at a node u that belongs to the opposite player, add $E(u)$ to W and put $F(u) = E(u)$; continue with the newly added nodes until no more new nodes can be added to W . We see (W, F) as a parity automaton with priority $\Omega(u)$ for states $u \in W$. The strategy s wins the node v if and only if all s -plays that start at v are even which is the case if and only if all paths in (W, F) that start at v are even which in turn is the case if and only if the state v in (W, F) is universally accepting, when seen as a state of a parity automaton. We use a standard automata theoretic construction (see e.g. [King et al. 2001]) to translate the automaton (W, F, Ω) to an equivalent Büchi automaton with at most $k \cdot |V|$ states, where k is the number of priorities in the game. Since Büchi automata can be checked for emptiness or universality in time polynomial in their size, we are done. We also note that (W, F) is a winning witness for Eloise if and only if the automaton is universally accepting if and only if s wins v . \square

5 COMPUTING NESTED FIXPOINTS IN $\text{NP} \cap \text{coNP}$

We are now ready to show that the problem of computing nested fixpoints is contained in both NP and coNP. To this end, we first define a slight variation of the fixpoint games from [Baldan et al. 2019; Venema 2008] that characterize containment in nested fixpoints and then can use purely game theoretic methods to obtain the claimed result.

Definition 5.1 (Fixpoint games). The *fixpoint game* for E_k^α is a parity game (V, E, Ω) with set of nodes $V = (U \times \{1, \dots, k\}) \cup (\mathcal{P}(U))^k$, where nodes from $U \times \{1, \dots, k\}$ belong to player Eloise and nodes from $(\mathcal{P}(U))^k$ belong to player Abelard. For nodes $(u, i) \in U \times \{1, \dots, k\}$, we put

$$E(u, i) = \{(U_1, \dots, U_k) \in (\mathcal{P}(U))^k \mid u \in \alpha(U_1, \dots, U_k)\}$$

and for nodes $(U_1, \dots, U_k) \in (\mathcal{P}(U))^k$, we put $E(U_1, \dots, U_k) = \bigcup_{1 \leq j \leq k} (U_j \times \{j\})$. The priority function $\Omega : V \rightarrow \{0, \dots, k\}$ is defined by $\Omega(u, i) = i$ and $\Omega(U_1, \dots, U_k) = 0$. The fixpoint game for A_k^α is defined in the same way, however with all priorities increased by one.

In fact, the fixpoint games in [Baldan et al. 2019] characterize containment in solutions of sets of fixpoint equations of the shape $X_i =_{\eta_i} \alpha_i(X_1, \dots, X_k)$ and hence are more general and slightly more involved than the games that we just defined. In the more general setting, the clause for Eloise-moves is $E(u, i) = \{(U_1, \dots, U_k) \in (\mathcal{P}(U))^k \mid u \in \alpha_i(U_1, \dots, U_k)\}$, where i is used to identify fixpoint equations; the Abelard-moves remain as in Definition 5.1, that is, we have $E(U_1, \dots, U_k) = \bigcup_{1 \leq j \leq k} (U_j \times \{j\})$, also in the more general setting. Since we are working with fixpoints that are nested directly within each other, our definition of fixpoint games in Definition 5.1 is a bit simpler. Nevertheless, our games are equivalent to the games from [Baldan et al. 2019] for fixpoints of the restricted shape: the nested fixpoint E_k^α is just the first component of the solution of the system

$$\begin{aligned} X_1 &=_{\eta_1} \alpha(X_1, \dots, X_k) \\ X_2 &=_{\eta_2} \pi_1(X_1, \dots, X_k) \\ &\dots \\ X_k &=_{\eta_k} \pi_{k-1}(X_1, \dots, X_k), \end{aligned}$$

where $\pi_i(X_1, \dots, X_i, \dots, X_k) = X_i$ for $1 \leq i < k$. In the more general fixpoint game, we then have, for $1 \leq i < k$, moves $E(u, i) = \{(U_1, \dots, U_k) \in (\mathcal{P}(U))^k \mid u \in \pi_{i-1}(U_1, \dots, U_k)\}$ so that Eloise always has the optimal move from (u, i) to $(\emptyset, \dots, \{u\}, \dots, \emptyset)$ where $\{u\}$ is the $i - 1$ -st component in the tuple. Player Abelard in turn has to move from $(\emptyset, \dots, \{u\}, \dots, \emptyset)$ to $(u, i - 1)$. This can be repeated until $i = 1$ so that in the more general games, Eloise can always enforce linear plays from (u, i) to $(u, 1)$ in which i is the highest priority. These plays are collapsed to a single move with priority i in our definition of fixpoint games; for fixpoints of our restricted shape, this collapsing does not affect the existence of winning strategies, showing that the two kinds of fixpoint games indeed agree. A similar construction (however with priorities increased by one) also works for fixpoint games for the dual fixpoint A_k^α . Hence we can import the characterization theorem (Theorem 4.8) from [Baldan et al. 2019].

LEMMA 5.2 ([BALDAN ET AL. 2019]). *We have $u \in E_k^\alpha$ ($u \in A_k^\alpha$) if and only if Eloise wins the node $(u, 1)$ in the fixpoint game for E_k^α (A_k^α);*

We now define a second kind of games, equivalent to fixpoint games but in which the Eloise-nodes are precisely the states from U :

Definition 5.3 (Lean fixpoint games). The *lean fixpoint game* for A_k^α is a parity game (V', E', Ω') with set of nodes $V' = U \cup (\mathcal{P}(U))^k \cup (\mathcal{P}(U) \times \{1, \dots, k\})$, where nodes from U belong to player

Eloise and nodes from $(\mathcal{P}(U))^k$ and nodes from $\mathcal{P}(U) \times \{1, \dots, k\}$ belong to player Abelard. For nodes $u \in U$, we put

$$E'(u) = \{(U_1, \dots, U_k) \in (\mathcal{P}(U))^k \mid u \in \alpha(U_1, \dots, U_k)\},$$

for nodes $(U_1, \dots, U_k) \in (\mathcal{P}(U))^k$, we put $E'(U_1, \dots, U_k) = \{(U_j, j) \mid 1 \leq j \leq k\}$ and for nodes $(U_j, j) \in \mathcal{P}(U) \times \{1, \dots, k\}$, we put $E'(U_j, j) = \{u \mid u \in U_j\}$. The priority function $\Omega' : V' \rightarrow \{0, \dots, k\}$ is defined by $\Omega'(u) = \Omega'(U_1, \dots, U_k) = 0$ and $\Omega'(U_j, j) = j$.

LEMMA 5.4. *Let $u \in U$. Player Eloise wins the node $(u, 1)$ in the fixpoint game for E_k^α (A_k^α) if and only if player Eloise wins the node u in the lean fixpoint game for E_k^α (A_k^α).*

PROOF. We prove the Lemma for E_k^α and note that the proof for A_k^α is completely analogous, but with all priorities in Ω' increased by one. So let s be an Eloise-strategy in the lean fixpoint game for E_k^α such that Eloise wins the node u with strategy s . We define a (history-free) strategy s' in the fixpoint game by putting $s'(v, i) = s(v)$ for all $1 \leq i \leq k$ and all nodes v that s wins in the lean fixpoint game. We show that s' wins the node $(u, 1)$ in the fixpoint game. Let $v_0 = u$, $i_0 = 1$ and let $\tau = (v_0, i_0), s'(v_0, i_0), (v_1, i_1), s'(v_1, i_1), \dots$ be a play of the fixpoint game that adheres to s' and that starts at $(u, 1)$. By definition of the moves in the fixpoint game, we have that for all $j \geq 0$, $v_j \in \alpha(s'(v_j, i_j)) = \alpha(s(v_j))$ and v_{j+1} is contained in the i_{j+1} -th component of $s'(v_j, i_j) = s(v_j)$. Let z denote the highest priority that occurs infinitely often in τ so that we have infinitely many positions j such that $i_j = z$ but only finitely many positions j' such that $i_{j'} > z$. It remains to show that z is even which then implies that Eloise wins τ . Writing $s(v_j) = (U_1^j, \dots, U_k^j)$, the play τ induces the following play in the lean fixpoint game:

$$\rho = v_0, s(v_0), (U_{i_1}^0, i_1), v_1, s(v_1), (U_{i_2}^1, i_2), \dots$$

The infinitely often occurring priorities in ρ and τ are the same so that the highest priority that occurs infinitely often in ρ is z and since s wins u and ρ adheres to s and starts at u , z is even, as required.

For the converse direction, let s be an Eloise-strategy in the fixpoint game for E_k^α such that Eloise wins the node $(u, 1)$ with strategy s . We define a history-dependent strategy $s' : (\mathcal{P}(U) \times \{1, \dots, k\}) \times U \rightarrow (\mathcal{P}(U))^k$ by putting $s'((U_j, j), u) = s(u, j)$ for $(U_j, j) \in \mathcal{P}(U) \times \{1, \dots, k\}$ and $u \in U$ such that $u \in U_j$ and s wins (u, j) in the fixpoint game. Also we put $s'(u) = s(u, 1)$. We show that s' wins the node u in the lean fixpoint game. So let

$$\tau = v_0, s'(v_0), (U_{p_0}^0, p_0), v_1, s'((U_{p_0}^0, p_0), v_1), (U_{p_1}^1, p_1), v_2, \dots$$

be a play of the lean fixpoint game that adheres to s' and starts at the node $v_0 = u$; we write $s'((U_{p_{j-1}}^{j-1}, p_{j-1}), v_j) = (U_1^j, \dots, U_k^j)$ for $j > 0$ and $s'(v_0) = (U_1^0, \dots, U_k^0)$. Then we have that $v_0 \in \alpha(s'(v_0)) = \alpha(s(u, 0))$, for all $j > 0$, $v_j \in \alpha(s'((U_{p_{j-1}}^{j-1}, p_{j-1}), v_j)) = \alpha(s(v_j, p_{j-1}))$ and for all $j \geq 0$, $v_{j+1} \in A_{p_j}^j$. Thus τ induces a play of the fixpoint game:

$$\rho = (v_0, 1), s(v_0, 1), (v_1, p_0), s(v_1, p_0), (v_2, p_1), \dots$$

Let z be the highest priority that occurs infinitely often in ρ . Since s wins $(u, 1)$ in the fixpoint game, z is even. Again, a priority occurs infinitely often in ρ if and only if it occurs infinitely often in τ . Hence player Eloise wins the play τ . \square

Crucially, lean fixpoint games (as well as the normal fixpoint games) are alternating and while their number of nodes is exponential in $|U|$, their number of Eloise-nodes is just $|U|$. By Lemma 4.4, lean fixpoint games hence have winning witnesses of size at most $2|U|$ for Eloise. This allows us to show that the problem of computing nested fixpoints often is contained in $\text{NP} \cap \text{coNP}$:

THEOREM 5.5. *If α can be iterated in polynomial time, then the problem of computing nested fixpoints of α is in $NP \cap coNP$.*

PROOF. Given input k, U and $\alpha : (\mathcal{P}(U))^k \rightarrow \mathcal{P}(U)$ such that α can be iterated in polynomial time, we nondeterministically guess, for each state $u \in U$, whether it is contained in E_k^α or not. If we have $u \in E_k^\alpha$, then Eloise wins (by Lemma 5.2 and Lemma 5.4) the node u in the lean fixpoint game for E_k^α and we nondeterministically guess a winning witness (W, F) for Eloise containing u in its carrier set $W \subseteq U$; such a witness exists by Lemma 4.4 since lean fixpoint games are alternating. Since α can be iterated in polynomial time and since we have $|W| \leq 2|U|$ by Lemma 4.4, the guessed structure can be verified to be a winning witness in polynomial time; in more detail, we have to check, for each $v \in V_\exists \cap W$, whether $v \in \alpha(F(v))$ where $F(v) = (U_1, \dots, U_k)$ for some $(U_1, \dots, U_k) \in (\mathcal{P}(U))^k$. Since α can be iterated in polynomial time, this part of the verification can be done in time polynomial in $|U|$. It remains to verify that all paths in (W, F) are even which can be done, as described in the proof of Lemma 4.5, in time polynomial in $|U|k$. If we have $u \notin E_k^\alpha$, then we have $u \in A_k^{\bar{\alpha}}$ by Lemma 3.4. Hence Eloise wins the node u in the lean fixpoint game for $A_k^{\bar{\alpha}}$ (again by Lemma 5.2 and Lemma 5.4) and we again nondeterministically guess a (polynomially sized) winning witness for Eloise and then verify it in deterministic polynomial time; here we note that since α can be iterated in polynomial time by assumption and since $\bar{\alpha}$ is defined from α by complementation with respect to U , $\bar{\alpha}$ can be iterated in polynomial time as well. \square

6 COMPUTING NESTED FIXPOINTS IN QUASIPOLYNOMIAL TIME

We go on to prove our main result. To this end, we fix a finite set U , an even number k , a monotone function $\alpha : (\mathcal{P}(U))^k \rightarrow \mathcal{P}(U)$ and put $n := |U|$. Next we define our general notions of exponential timeouts and quasipolynomial histories.

Definition 6.1 (Timeouts and histories). The set of *exponential timeouts* is defined by

$$\text{to} = \{(m_{k-1}, m_{k-3}, \dots, m_1) \mid 1 \leq m_i \leq 2^{\lceil \log n \rceil + 1}\}.$$

We write \leq_l for the lexical ordering on to (with m_{k-1} being the most significant letter). A vector $\bar{o} = (o_{\lceil \log n \rceil + 1}, \dots, o_0)$ of $\lceil \log n \rceil + 2$ natural numbers such that $0 \leq o_i \leq k$ and for $i \geq i'$, if $o_i > 0$, then $o_i \geq o_{i'}$ (that is, the sequence of nonzero digits in \bar{o} is not increasing), is a *quasipolynomial history*. We write hi for the set of quasipolynomial histories. We define functions $\beta : \mathcal{P}(U \times \text{to}) \rightarrow \mathcal{P}(U \times \text{to})$ and $\gamma : \mathcal{P}(U \times \text{hi}) \rightarrow \mathcal{P}(U \times \text{hi})$ by

$$\begin{aligned} \beta(X) &= \{(v, \bar{m}) \in (U \times \text{to}) \mid v \in \alpha(X_1^{\bar{m}}, \dots, X_k^{\bar{m}})\} \\ \gamma(Y) &= \{(v, \bar{o}) \in (U \times \text{hi}) \mid v \in \alpha(Y_1^{\bar{o}}, \dots, Y_k^{\bar{o}})\}, \end{aligned}$$

where $X \in \mathcal{P}(U \times \text{to})$ and for $\bar{m} \in \text{to}$, $X_i^{\bar{m}} = \{u \in U \mid (u, \bar{m}@i) \in X\}$ if i is even or $m_i > 1$ and $X_i^{\bar{m}} = \emptyset$ if i is odd and $m_i = 1$; also we define

$$\bar{m}@i = \begin{cases} (m_{k-1}, m_{k-3}, \dots, m_i - 1, 2^{\lceil \log n \rceil + 1}, \dots, 2^{\lceil \log n \rceil + 1}) & \text{if } i \text{ is odd} \\ (m_{k-1}, m_{k-3}, \dots, m_{i+1}, 2^{\lceil \log n \rceil + 1}, \dots, 2^{\lceil \log n \rceil + 1}) & \text{if } i \text{ is even,} \end{cases}$$

that is, when updating exponential timeouts at i , we reset all timeouts below i to $|U|$, and if i is odd, then we reduce the timeout for i by 1. Furthermore, for $\bar{o} \in \text{hi}$ and $Y \subseteq \mathcal{P}(U \times \text{hi})$, we put $Y_i^{\bar{o}} = \emptyset$ if the leftmost digit in $\bar{o}@i$ is not 0 and $Y_i^{\bar{o}} = \{u \in U \mid (u, \bar{o}@i) \in Y\}$ otherwise, where, for $\bar{o} = (o_{\lceil \log n \rceil + 1}, \dots, o_0)$, we define the updating procedure for \bar{o} following [Calude et al. 2017], defining the components of $\bar{o}@i = (o'_{\lceil \log n \rceil + 1}, \dots, o'_0)$ as follows:

- (1) If i is even, then let j be the highest index such that $i > o_j > 0$. If no such index exists, then put $j = *$.

- (2) If i is odd, then let j be the highest index such that
 - a) $i > o_j > 0$ or
 - b) o_j is even and if $o_j > 0$, then $i < o_j$ and for all $j' < j$, $o_{j'}$ is odd.
- (3) If $j = *$, then put $\bar{o}@i = \bar{o}$. Otherwise, put

$$o'_{j'} = \begin{cases} o_{j'} & \text{for } j' > j \\ i & \text{for } j' = j \\ 0 & \text{for } j' < j. \end{cases}$$

For $(x, \bar{m}) \in \text{GFP } \beta$, we say that x has *exponential timeouts* \bar{m} (for α). We say that x has *quasipolynomial histories* \bar{o} if $(x, \bar{o}) \in \text{GFP } \gamma$.

FACT 6.2. We have the following simple properties of the above definitions (where $\bar{m} \in \text{to}$):

- (1) $\bar{m}@j$ depends only on the m_i with $i \geq j$;
- (2) entries m_i with $i > j$ remain unchanged in the transition from \bar{m} to $\bar{m}@j$;
- (3) $X_j^{\bar{m}}$ depends only on the m_i with $i \geq j$

(where the first two claims are immediate and the last follows from the first).

The following is immediate:

FACT 6.3. Let $\bar{o} \in \text{hi}$, and $\bar{o}' = \bar{o}@i$. If $o_j = 0$ and $o'_j > 0$, then $o'_j = i$ and i is odd.

Let $t(\alpha)$ denote the time it takes to compute $\alpha(U_1, \dots, U_k)$ for arbitrary $(U_1, \dots, U_k) \in (\mathcal{P}(U))^k$ and put

$$s_1 = \max(t(\alpha), n \cdot 2^{(\lceil \log n \rceil + 1) \frac{k}{2}}) \quad s_2 = \max(t(\alpha), n \cdot k^{\lceil \log n \rceil + 2})$$

LEMMA 6.4. We have $|\text{to}| \leq (2^{\lceil \log n \rceil + 1})^{\frac{k}{2}} = 2^{(\lceil \log n \rceil + 1) \frac{k}{2}}$ so that $|U \times \text{to}| \leq n \cdot 2^{(\lceil \log n \rceil + 1) \frac{k}{2}}$; hence $\text{GFP } \beta$ can be computed in time $s_1 \cdot n \cdot 2^{(\lceil \log n \rceil + 1) \frac{k}{2}}$. We have $|\text{hi}| \leq k^{\lceil \log n \rceil + 2}$ so that $|U \times \text{hi}| \leq n \cdot k^{\lceil \log n \rceil + 2}$; hence $\text{GFP } \gamma$ can be computed in time $s_2 \cdot n \cdot k^{\lceil \log n \rceil + 2}$.

LEMMA 6.5. The operations $(-)@i$ on to are monotone w.r.t. the lexical ordering.

PROOF. Let $\bar{m} <_l \bar{m}'$, i.e. there is an index j_0 such that $m_j = m'_j$ for $j \geq j_0$ and $m_{j_0} < m'_{j_0}$. If $j_0 > i$, then by Fact 6.2, the relevant parts of \bar{m} and \bar{m}' remain unchanged under $(-)@i$, so that $\bar{m}@i <_l \bar{m}'@i$. If $j_0 = i$ (implying that i is odd), then m_j and m'_j remain unchanged under $(-)@i$ for $j > j_0$, and m_{j_0} and m'_{j_0} both decrease by 1, so $\bar{m}@i <_j \bar{m}'@i$. Finally, if $j_0 < i$, then $\bar{m}@i = \bar{m}'@i$ by Fact 6.2. \square

The updating procedure for quasipolynomial histories preserves the property that the sequence of nonzero digits is not increasing:

LEMMA 6.6. Let $\bar{o} \in \text{hi}$ and let $1 \leq j \leq k$. Then we have $\bar{o}@j \in \text{hi}$.

PROOF. Let $\bar{o} = (o_{\lceil \log n \rceil + 1}, \dots, o_0)$ and $\bar{o}@j = (p_{\lceil \log n \rceil + 1}, \dots, p_0)$, let i, i' be numbers such that $i > i'$ and $p_i > 0$; we have to show $p_i \geq p_{i'}$. First, we consider the case where $p_i = o_i$, that is, the i -th digit does not change when updating \bar{o} with value j . If $p_{i'} = o_{i'}$, then we are done since \bar{o} is a quasipolynomial timeout so that $p_i = o_i \geq o_{i'} = p_{i'}$. Otherwise, if $p_{i'} = 0$, then we trivially have $p_i \geq p_{i'}$. If $p_{i'} \neq 0$, then $p_{i'} = j$ and all digits to the left of $o_{i'}$ have a value of at least j since the updating would otherwise change a digit to the left of $o_{i'}$; thus we are done. Now assume that $p_i \neq o_i$, that is, that the i -th digit is changed when updating \bar{o} with value j . By assumption, $p_i > 0$ so that $p_i = j$ and all digits to the right of p_i are zero, which finishes the proof. \square

LEMMA 6.7. *The greatest fixpoint $\text{GFP } \beta$ is closed under lexical increase of timeouts, i.e. for $\bar{m} \leq_l \bar{m}'$, $(v, \bar{m}) \in \text{GFP } \beta$ implies $(v, \bar{m}') \in \text{GFP } \beta$.*

PROOF. We use fixpoint induction, i.e. we show that the claimed closure property (which holds trivially for the full set $U \times \text{to}$) is stable under β . So let $Y \subseteq U \times \text{to}$ to be closed under lexical increase of timeouts. We have to show that the same holds for $\beta(Y)$; so let $\bar{m}' \geq_l \bar{m}$, and let $(v, \bar{m}) \in \beta(Y)$, i.e. $v \in \alpha(Y_1^{\bar{m}}, \dots, Y_k^{\bar{m}})$. We have to show $(v, \bar{m}') \in \beta(Y)$, i.e. $v \in \alpha(Y_1^{\bar{m}'}, \dots, Y_k^{\bar{m}'})$. Since α is monotone, it suffices to show that $Y_i^{\bar{m}} \subseteq Y_i^{\bar{m}'}$ for all i . This holds trivially if $i = 1$, so assume $i > 1$ and let $u \in Y_i^{\bar{m}}$, i.e. $(u, \bar{m}@i) \in Y$. We have to show that $u \in Y_i^{\bar{m}'}$, i.e. $(u, \bar{m}'@i) \in Y$. But this follows by the assumption on Y and monotonicity of $(-)@i$ (Lemma 6.5). \square

As a first step towards our main result, we now show that nodes are contained in nested fixpoints if and only if they have exponential timeouts. This essentially reproves that the exponential progress measure method can be used to compute nested fixpoints.

LEMMA 6.8. *We have*

$$E_k^\alpha = \pi_1[\text{GFP } \beta].$$

PROOF. To show $\pi_1[\text{GFP } \beta] \subseteq E_k^\alpha$, we define, for $\bar{m} \in \text{to}$ and $1 \leq j \leq k$, the set $Z_j^{\bar{m}} := (\text{GFP } \beta)_j^{\bar{m}}$, where $(\text{GFP } \beta)_j^{\bar{m}}$ is defined as in the definition of the updating procedure of exponential timeouts, that is, if j is odd and $m_j = 1$, then $(\text{GFP } \beta)_j^{\bar{m}} = \emptyset$ and if j is even or $m_j > 1$, then $(\text{GFP } \beta)_j^{\bar{m}} = \{u \in U \mid (u, \bar{m}@j) \in \text{GFP } \beta\}$. Moreover, we put $Z^{\bar{m}} = \{u \in U \mid (u, \bar{m}) \in \text{GFP } \beta\}$, so that $Z_i^{\bar{m}} = Z^{\bar{m}@i}$ if j is even or $m_j > 1$; note that $\pi_1[\text{GFP } \beta]$ is the union of the sets $Z^{\bar{m}}$. We define sets $E_j^{\bar{m}}, F_j^{\bar{m}} \subseteq U$ by mutual recursion (with the lexical ordering on timeouts \bar{m} as the termination measure), putting

$$E_j^{\bar{m}} = \eta_j X_j. \eta_{j-1} X_{j-1}. \dots \eta_1 X_1. \alpha(X_1, \dots, X_j, F_{j+1}^{\bar{m}}, \dots, F_k^{\bar{m}})$$

for $j \geq 0$ (so $E_0^{\bar{m}} = \alpha(F_1^{\bar{m}}, \dots, F_k^{\bar{m}})$), $F_j^{\bar{m}} = Z_j^{\bar{m}}$ if j is even, and $F_j^{\bar{m}} = E_{j-1}^{\bar{m}@j}$ if j is odd and $m_j > 1$. For j odd and $m_j = 1$, we put $F_j^{\bar{m}} = \emptyset$.

For odd $j \geq 1$, $\bar{m} \in \text{to}$ and $X_j \subseteq U$, we then put

$$f_j^{\bar{m}}(X_j) = \text{GFP } X_{j-1}. \text{LFP } X_{j-2}. \dots \text{LFP } X_1. \alpha(X_1, \dots, X_j, F_{j+1}^{\bar{m}}, \dots, F_k^{\bar{m}})$$

(so $f_1^{\bar{m}}(X_1) = \alpha(X_1, F_2^{\bar{m}}, \dots, F_k^{\bar{m}})$).

We note the following consequences of these definitions:

- FACT 6.9. (1) $F_j^{\bar{m}}$ depends only on the m_i with $i \geq j$;
 (2) $f_j^{\bar{m}}$ depends only on the m_i with $i > j$.

The first of these claims is immediate from Fact 6.2 and (the two cases in) the definition of $F_j^{\bar{m}}$, and the second is immediate from the first by the definition of $f_j^{\bar{m}}$.

For a further important property of the above data, let j be odd, $1 \leq q \leq 2^{\lceil \log n \rceil + 1}$, and let $\bar{m} \in \text{to}$ such that $m_j = q$. Unfolding definitions, we have

$$E_{j-1}^{\bar{m}} = f_j^{\bar{m}}(F_j^{\bar{m}}). \quad (1)$$

We show

$$(f_j^{\bar{m}})^q(\emptyset) = E_{j-1}^{\bar{m}}. \quad (2)$$

by induction over q . If $q = 1$, then we have $(f_j^{\bar{m}})^1(\emptyset) = f_j^{\bar{m}}(\emptyset)$ and, by (1), $E_{j-1}^{\bar{m}} = f_j^{\bar{m}}(F_j^{\bar{m}}) = f_j^{\bar{m}}(\emptyset)$, since $m_j = 1$ so that $F_j^{\bar{m}} = \emptyset$. If $q > 1$, then $F_j^{\bar{m}} = E_{j-1}^{\bar{m}@j}$ since $m_j = q > 1$, and since j is odd, we have

$(\overline{m}@j)_j = q - 1$, so by induction $\overline{E}_{j-1}^{\overline{m}@j} = (f_j^{\overline{m}@j})^{q-1}(\emptyset)$. By Facts 6.9 and 6.2, $f_j^{\overline{m}@j} = f_j^{\overline{m}}$. Thus, we have $(f_j^{\overline{m}})^q(\emptyset) = (f_j^{\overline{m}})((f_j^{\overline{m}})^{q-1}(\emptyset)) = (f_j^{\overline{m}})((f_j^{\overline{m}@j})^{q-1}(\emptyset)) = f_j^{\overline{m}}(\overline{E}_{j-1}^{\overline{m}@j}) = f_j^{\overline{m}}(F_j^{\overline{m}}) = \overline{E}_{j-1}^{\overline{m}}$, using (1) in the last step.

We now tackle the actual proof of the desired inclusion $\pi[\text{GFP } \beta] \subseteq E_k^\alpha$. Let $\epsilon = (2^{\lceil \log n \rceil + 1}, \dots, 2^{\lceil \log n \rceil + 1})$ denote the maximal exponential timeout. By Lemma 6.7, we have $Z^{\overline{m}} \subseteq Z^\epsilon$ for all \overline{m} , and since $E_k^\alpha = E_k^\epsilon$ by definition, it suffices to show that $Z^\epsilon \subseteq E_k^\epsilon$ (recalling that $\pi_1[\text{GFP } \beta]$ is the union of all sets $Z^{\overline{m}}$). We show more generally that for all $\overline{m} \in \text{to}$ and all $0 \leq j \leq k$ such that $m_{j'} = 2^{\lceil \log n \rceil + 1}$ for all odd $j' \leq j$ (that is, all digits from m_j on – or from m_{j-1} on if j is even – have the maximum value), we have

$$Z^{\overline{m}} \subseteq E_j^{\overline{m}}.$$

We proceed by induction over the pair (\overline{m}, j) using the lexical ordering $<_l$ defined by putting $(\overline{m}, j) <_l (\overline{m}', j')$ if and only if $\overline{m} = \overline{m}'$ and $j < j'$ or \overline{m} is less than \overline{m}' with respect to the lexical ordering on to (that is, $\overline{m} <_l \overline{m}'$), distinguishing cases over j :

- If $j > 0$ is even, then we have

$$\overline{E}_j^{\overline{m}} = \text{GFP } X_j. \text{LFP } X_{j-1}. \dots \text{LFP } X_1. \alpha(X_1, \dots, X_j, F_{j+1}^{\overline{m}}, \dots, F_k^{\overline{m}}).$$

By coinduction, it suffices to show

$$\begin{aligned} Z^{\overline{m}} &\subseteq \text{LFP } X_{j-1}. \dots \text{LFP } X_1. \alpha(X_1, \dots, X_{j-1}, Z^{\overline{m}}, F_{j+1}^{\overline{m}}, \dots, F_k^{\overline{m}}) \\ &= \text{LFP } X_{j-1}. \dots \text{LFP } X_1. \alpha(X_1, \dots, X_{j-1}, F_j^{\overline{m}}, F_{j+1}^{\overline{m}}, \dots, F_k^{\overline{m}}) = \overline{E}_{j-1}^{\overline{m}}, \end{aligned} \quad (3)$$

using in the first equality that for even j , the assumptions on \overline{m} imply that $\overline{m}@j = \overline{m}$, so that $F_j^{\overline{m}} = Z_j^{\overline{m}} = Z^{\overline{m}@j} = Z^{\overline{m}}$ by definition. Since all digits from m_{j-1} on have the maximum value, the inclusion (3) holds by the inductive hypothesis.

- If $j > 0$ is odd, then we have to show

$$\begin{aligned} Z^{\overline{m}} &\subseteq \text{LFP } X_j. \text{GFP } X_{j-1}. \dots \text{LFP } X_1. \alpha(X_1, \dots, X_j, F_{j+1}^{\overline{m}}, \dots, F_k^{\overline{m}}) \\ &= (f_j^{\overline{m}})^{2^{\lceil \log n \rceil + 1}}(\emptyset) = \overline{E}_{j-1}^{\overline{m}}, \end{aligned} \quad (4)$$

where the first equation is by Kleene's fixpoint theorem and where the second equality is by (2), since $m_j = 2^{\lceil \log n \rceil + 1}$. Since all digits from m_j on, and hence also all digits from m_{j-2} on, have the maximum value, the inclusion (4) holds by the inductive hypothesis.

- If $j = 0$, then $\overline{E}_0^{\overline{m}} = \alpha(F_1^{\overline{m}}, \dots, F_k^{\overline{m}})$. So let $v \in Z^{\overline{m}}$ so that $(v, \overline{m}) \in \text{GFP } \beta = \beta(\text{GFP } \beta)$ and hence $v \in \alpha(Z_1^{\overline{m}}, \dots, Z_k^{\overline{m}})$. We have to show $v \in \overline{E}_0^{\overline{m}}$; by monotonicity of α , it suffices to show that $Z_i^{\overline{m}} \subseteq F_i^{\overline{m}}$ for $1 \leq i \leq k$. If i is even, then $F_i^{\overline{m}} = Z_i^{\overline{m}}$, and are done. If i is odd and $m_i = 1$, then $Z_i^{\overline{m}} = \emptyset$, and we are done. If i is odd and $m_i > 1$, then $Z_i^{\overline{m}} = Z^{\overline{m}@i}$ and $F_i^{\overline{m}} = \overline{E}_{i-1}^{\overline{m}@i}$, so we are done by the inductive hypothesis since $\overline{m}@i <_l \overline{m}$ and since in $\overline{m}@i$, all digits from $(\overline{m}@i)_{i-2}$ on have the maximal value.

For the converse inclusion $E_k^\alpha \subseteq \pi_1[\text{GFP } \beta]$, we similarly define $\overline{B}_j^{\overline{m}}, \overline{G}_j^{\overline{m}} \subseteq U$ by

$$\overline{B}_j^{\overline{m}} = \eta_j X_j. \eta_{j-1} X_{j-1}. \dots \eta_1 X_1. \alpha(X_1, \dots, X_j, \overline{G}_{j+1}^{\overline{m}}, \dots, \overline{G}_k^{\overline{m}})$$

(so $\overline{B}_0^{\overline{m}} = \alpha(\overline{G}_1^{\overline{m}}, \dots, \overline{G}_k^{\overline{m}})$), $\overline{G}_j^{\overline{m}} = (g_j^{\overline{m}})^{m_j-1}(\emptyset)$ for odd j (so $\overline{G}_j^{\overline{m}} = \emptyset$ in case $m_j = 1$), and $\overline{G}_j^{\overline{m}} = \overline{B}_j^{\overline{m}}$ for even j ; here, we put

$$g_j^{\overline{m}}(X_j) = \text{GFP } X_{j-1}. \text{LFP } X_{j-2}. \dots \text{LFP } X_1. \alpha(X_1, \dots, X_j, \overline{G}_{j+1}^{\overline{m}}, \dots, \overline{G}_k^{\overline{m}})$$

for odd j (so $\bar{g}_1^{\bar{m}}(X_1) = \alpha(X_1, G_2^{\bar{m}}, \dots, G_k^{\bar{m}})$), so that

$$B_{j-1}^{\bar{m}} = g_j^{\bar{m}}(G_j^{\bar{m}}) \quad (5)$$

by unfolding definitions. Moreover, we have

- FACT 6.10. (1) $G_j^{\bar{m}}$ depends only on the m_i with $i \geq j$;
 (2) $B_j^{\bar{m}}$ depends only on the m_i with $i > j$;
 (3) $g_j^{\bar{m}}$ depends only on the m_i with $i > j$.

By Fact 6.10, Item 2, we have that for all \bar{m} ,

$$B_k^{\bar{m}} = E_k^\alpha, \quad (6)$$

as there are no m_i with $i > k$. We now claim that

$$B_j^{\bar{m}} = B_{j-1}^{\bar{m}}$$

for all $1 \leq j \leq k$ and all \bar{m} such that all digits from m_j on (or from m_{j-1} on, if j is even) have the maximum value. We prove this claim by case distinction over j :

- If j is even, then we have

$$\begin{aligned} B_j^{\bar{m}} &= \text{GFP } X_j. \text{LFP } X_{j-1}. \dots \text{LFP } X_1. \alpha(X_1, \dots, X_j, G_{j+1}^{\bar{m}}, \dots, G_k^{\bar{m}}) \\ &= \text{LFP } X_{j-1}. \dots \text{LFP } X_1. \alpha(X_1, \dots, X_{j-1}, B_j^{\bar{m}}, G_{j+1}^{\bar{m}}, \dots, G_k^{\bar{m}}) = B_{j-1}^{\bar{m}}, \end{aligned}$$

using fixpoint unfolding in the second step, and recalling that $B_j^{\bar{m}} = G_j^{\bar{m}}$ by definition since j is even.

- If j is odd, then we have

$$\begin{aligned} B_j^{\bar{m}} &= \text{LFP } X_j. \text{GFP } X_{j-1}. \dots \text{LFP } X_1. \alpha(X_1, \dots, X_j, G_{j+1}^{\bar{m}}, \dots, G_k^{\bar{m}}) \\ &= (g_j^{\bar{m}})^{2^{\lceil \log n \rceil + 1}}(\emptyset) = g_j^{\bar{m}}((g_j^{\bar{m}})^{2^{\lceil \log n \rceil + 1} - 1}(\emptyset)) = g_j^{\bar{m}}(G_j^{\bar{m}}) \stackrel{(5)}{=} B_{j-1}^{\bar{m}}, \end{aligned}$$

where the second equality is by Kleene's fixpoint theorem and the fourth equality is by definition of $G_j^{\bar{m}}$ since j is odd.

Hence we have, for all $1 \leq j \leq k$ and all \bar{m} in which all digits from m_j on (or from m_{j-1} on, if j is even) have the maximum value, that

$$B_j^{\bar{m}} = B_{j-1}^{\bar{m}} = \dots = B_0^{\bar{m}} = \alpha(G_1^{\bar{m}}, \dots, G_k^{\bar{m}}). \quad (7)$$

Next we claim that for all j and all \bar{m} that are maximal from $j-1$ on and that have the property that $m_j > 1$ or j is even, we have

$$G_j^{\bar{m}} = \alpha(G_1^{\bar{m}@j}, \dots, G_k^{\bar{m}@j}). \quad (8)$$

We prove the claim by case distinction over j . If j is even, then we have

$$G_j^{\bar{m}} = B_j^{\bar{m}} \stackrel{(7)}{=} \alpha(G_1^{\bar{m}}, \dots, G_k^{\bar{m}}) = \alpha(G_1^{\bar{m}@j}, \dots, G_k^{\bar{m}@j}),$$

noting that since \bar{m} is maximal from $j-1$ on, $\bar{m} = \bar{m}@j$. If j is odd and $m_j > 1$, then we have

$$\begin{aligned} G_j^{\bar{m}} &= (g_j^{\bar{m}})^{m_j-1}(\emptyset) = g_j^{\bar{m}}((g_j^{\bar{m}})^{m_j-2}(\emptyset)) \\ &= g_j^{\bar{m}}(G_j^{\bar{m}@j}) = g_j^{\bar{m}@j}(G_j^{\bar{m}@j}) \stackrel{(5)}{=} B_{j-1}^{\bar{m}@j} \stackrel{(7)}{=} \alpha(G_1^{\bar{m}@j}, \dots, G_k^{\bar{m}@j}), \end{aligned}$$

noting that $g_j^{\bar{m}} = g_j^{\bar{m}@j}$ since $g_j^{\bar{m}}$ depends only on the m_i with $i > j$ (Fact 6.10), which have the same value in \bar{m} and in $\bar{m}@j$.

Let ϵ be the maximal timeout vector in which each digit has the maximal value. As shown above, we have $E_k^\alpha \stackrel{(6)}{=} B_k^\epsilon$ so that $E_k^\alpha = B_k^\epsilon \stackrel{(7)}{=} \alpha(G_1^\epsilon, \dots, G_k^\epsilon)$. We put $Z = \{(v, \bar{m}) \mid v \in \alpha(G_1^{\bar{m}}, \dots, G_k^{\bar{m}})\}$. It suffices to show that $Z \subseteq \text{GFP } \beta$, which reduces to $Z \subseteq \beta(Z)$ by coinduction. So let $(v, \bar{m}) \in Z$, i.e. $v \in \alpha(G_1^{\bar{m}}, \dots, G_k^{\bar{m}})$. We have to show that $(v, \bar{m}) \in \beta(Z)$, i.e. $v \in \alpha(Z_1^{\bar{m}}, \dots, Z_k^{\bar{m}})$. By monotonicity of α , it suffices to show that $G_j^{\bar{m}} \subseteq Z_j^{\bar{m}}$ for $1 \leq j \leq k$. If j is odd and $m_j = 1$, then we have $G_j^{\bar{m}} = \emptyset$ and there is nothing to show. Otherwise, j is even or $m_j > 1$ so that we have $Z_j^{\bar{m}} = \{v \in U \mid (v, \bar{m}@j) \in Z\}$ by definition. Since $G_j^{\bar{m}}$ depends only on the m_i with $i \geq j$ (Fact 6.10), we can assume that \bar{m} is maximal from $j-1$ on so that we have, as shown above, that $G_j^{\bar{m}} \stackrel{(8)}{=} \alpha(G_1^{\bar{m}@j}, \dots, G_k^{\bar{m}@j})$. So let $v \in \alpha(G_1^{\bar{m}@j}, \dots, G_k^{\bar{m}@j})$. Then we have $(v, \bar{m}@j) \in Z$ by definition of Z , so that $v \in Z_j^{\bar{m}}$, as required. \square

To enable computation of E_k^α in quasipolynomial time, we have to show that $\pi_1[\text{GFP } \beta] = \pi_1[\text{GFP } \gamma]$. To prove $\pi_1[\text{GFP } \gamma] \subseteq \pi_1[\text{GFP } \beta]$ (i.e. that states that have quasipolynomial histories also have exponential timeouts), we map quasipolynomial histories to exponential timeouts by means of a function $\text{exp} : \text{hi} \rightarrow \text{to}$.

Definition 6.11. Let $\bar{o} = (o_{\lceil \log n \rceil + 1}, \dots, o_0) \in \text{hi}$ such that $o_{\lceil \log n \rceil + 1} = 0$. We put $\text{exp}(\bar{o}) = (m_{k-1}, m_{k-3}, \dots, m_1)$ where m_i is defined as follows, for $1 \leq i < k$ odd: Let $b_i \leq 2^{\lceil \log n \rceil + 1} - 1$ be the number that is encoded by the bitstring that is obtained from \bar{o} in the following way: let j be the greatest number such that $0 < o_j < i$; if such a number exists, replace every digit to the left of o_j that has an odd value by 1 and every digit to the left of o_j that has an even value, and also o_j itself (be it odd or even) by 0; also, replace all digits to the right of o_j by 1. If no such j exists, then replace every digit with odd value by 1 and every digit with even value by 0. Put $m_i = 2^{\lceil \log n \rceil + 1} - b_i \geq 1$.

Since we assume $o_{\lceil \log n \rceil + 1} = 0$, we have $b_i \leq 2^{\lceil \log n \rceil + 1} - 1$ and hence $1 \leq m_i \leq 2^{\lceil \log n \rceil + 1}$ for all i in the above definition.

LEMMA 6.12. For $1 \leq i \leq k$, if $\bar{m} \geq \text{exp}(\bar{o})$, then we have $\bar{m}@i \geq \text{exp}(\bar{o}@i)$, where \geq is componentwise \geq .

PROOF. • If i is odd, then $\bar{o}@i$ is obtained from \bar{o} by choosing j according to update step 2.a) or 2.b), that is, by taking j to be the largest number such that $i > o_j > 0$ or o_j is even and if $o_j > 0$, then $i < o_j$, but for all $j' < j$, $o_{j'}$ is odd, and by then replacing o_j with i and all $o_{j'}$ with $j' < j$ by 0. We also have $\bar{m}@i = (m_{k-1}, m_{k-3}, \dots, m_i - 1, 2^{\lceil \log n \rceil + 1}, \dots, 2^{\lceil \log n \rceil + 1})$. We show that for all odd $1 \leq i' < k$, we have $(\text{exp}(\bar{o}@i))_{i'} \leq (\bar{m}@i)_{i'}$.

- For $i' < i$, we have $(\bar{m}@i)_{i'} = 2^{\lceil \log n \rceil + 1}$ so that there is nothing to show.
- For $i' = i$, let b and b' be the numbers encoded by the bitstrings obtained when computing $\text{exp}((\bar{o}))_{i'}$ and $\text{exp}((\bar{o}@i))_{i'}$, respectively. By assumption, $(\text{exp}(\bar{o}))_{i'} = 2^{\lceil \log n \rceil + 1} - b \leq m_{i'}$. We have to show that $(\text{exp}(\bar{o}@i))_{i'} = 2^{\lceil \log n \rceil + 1} - b' \leq m_i - 1$. Hence it suffices to show $b < b'$. The digits left of o_j are not changed by the updating procedure. The digit o_j is changed from some even number or some number less than i to i . All digits to the right of o_j are changed to 0. There is no digit left of o_j with a nonzero value less than i , otherwise this digit would be changed to i instead of o_j . So o_j is even or o_j is the leftmost digit that is less than i but greater than 0. In both cases, the j -th bit in the computation of b is 0; in the computation of b' , this bit is 1 since the j -th digit is i after updating, that is, $o'_j \geq i$ and i is odd. Hence the number encoded by the representation of the last j digits of \bar{o} is less than 2^j , while it is exactly 2^j for $\bar{o}@i$ (this is the case since all digits to the right of o'_j are 0).

Since the number that is encoded by the leftmost $j - 1$ digits (which remain unchanged) does not change, we have $b < b'$, as required.

- For $i' > i$, let b and b' again be the numbers encoded by \bar{o} and $\bar{o}@i$, respectively, when looking at i' . We have to show $b \leq b'$. There are two cases where b' can possibly differ from b :

- * If a digit $o_{j'}$ such that $o_{j'} \geq i'$ is changed, then recall that $i' > i$ so that the update from \bar{o} to $\bar{o}@i$ proceeds according to case 2.b), that is, o_j is even, $i < o_j$ and for all $j'' < j$, $o_{j''}$ is odd. Since the digits left of o_j are not changed by the updating procedure, we have that $j' \leq j$ and – since the sequence of nonzero digits in a quasipolynomial timeout is not increasing – all nonzero digits to the left of $o_{j'}$ have a value of at least i' . All digits to the left of o_j remain unchanged, o_j is changed from some even number to i where $i' > i$, and all digits to the right of o_j are changed to 0. We again distinguish two cases: If $o_j \geq i'$, then the j -th bit in the computation of b is 0 since o_j is even. If $o_j < i'$, then we recall that no digit left of o_j has a value less than i' ; hence o_j is the leftmost such digit so that the j -th bit in the computation of b is 0. In both cases, the last j bits encode a number strictly less than 2^j before the updating procedure. Since $i < i'$ and since the digits to the left of o_j are not changed by the updating procedure, j is the leftmost position with a nonzero value less than i' after the updating procedure. Thus in the computation of b' , the j -th bit is 0 while all ensuing bits are 1. Thus the last j bits encode exactly the number $2^j - 1$ after the updating procedure. Hence we have $b \leq b'$, as required.

- * If no digit $o_{j'}$ such that $o_{j'} \geq i'$ is changed, then a change of b can only arise when the leftmost digit, say $o_{j''}$, with a value less than i' but greater than 0 is changed. The digit can either change to i (in which case no digit to the left of $o_{j''}$ is changed) or to 0. If the digit $o_{j''}$ is changed to i and no digit to the left of $o_{j''}$ is changed, then we recall that we have $i' > i$ so that the j'' -th digit still is the leftmost digit with a value less than i' after the updating procedure. Thus we have $b' = b$, as required. If the digit $o_{j''}$ is changed to 0, then we recall that by assumption, no digit to the left of $o_{j''}$ has a nonzero value less than i' . Hence no digit to the left of $o_{j''}$ has a nonzero value less than i . As $o_{j''}$ is changed to 0, we have $j > j''$, so that we have $o_j = 0$ or $o_j \geq i'$. This situation can only occur if the update from \bar{o} to $\bar{o}@i$ proceeds according to case 2.b) where we have that o_j is even and all digits to the right of o_j are odd. The case that o_j is even but nonzero cannot occur since every nonzero digit to the left of $o_{j''}$, including the digit o_j , has a value of at least i' but no digit with a value of at least i' is changed. Hence we have $o_j = 0$. As the update proceeds according to case 2.b), the digit o_{j-1} has an odd value but since no digit with a value of at least i' is changed, we have $0 < o_{j-1} < i'$ so that o_{j-1} is the leftmost digit with a nonzero value less than i' . Hence we have $j - 1 = j''$. Thus the digit $o_{j''+1} = o_j$ is changed to i and all digits to the right of $o_{j''+1}$ are changed to 0. Before updating, the $j'' + 1$ -th and the j'' -th bits both are 0 (since $o_{j''+1} = 0$ and since $o_{j''}$ is the leftmost digit with a value less than i') and all bits to the right of the j'' -th bit are 1; after the updating procedure, the $j'' + 1$ -th digit is the leftmost digit with a value less than i' so that the $j'' + 1$ -th bit is 0 and all bits to the right of the $j'' + 1$ -th bit are 1 after the updating procedure. The last $j'' + 1$ bits hence encode the number $2^{j''} - 1$ before updating and the number $2^{j''+1} - 1$ after updating so that we have $b \leq b'$, as required.

- If i is even, then $\bar{o}@i$ is obtained from \bar{o} by taking j to be the largest number such that $i > o_j > 0$, and, if such a number exists, by replacing o_j with i and all $o_{j'}$ with $j' < j$ by 0. We also have $\bar{m}@i = (m_{k-1}, m_{k-3}, \dots, m_{i+1}, 2^{\lceil \log n \rceil + 1}, \dots, 2^{\lceil \log n \rceil + 1})$. It suffices to show that for all odd $1 \leq i' < k$, $(\bar{m}@i)_{i'} \geq \exp(\bar{o}@i)_{i'}$.

- For $i' < i$, we have $(\overline{m}@i)_{i'} = 2^{\lceil \log n \rceil + 1}$ so that there is nothing to show.
- For $i' > i$, let b and b' denote the numbers represented by the strings in the computation of $(\exp(\overline{o}))_{i'}$ and $(\exp(\overline{o}@i))_{i'}$, respectively. We show $b = b'$ which implies $(\exp(\overline{o}))_{i'} = (\exp(\overline{o}@i))_{i'}$ which in turn implies $(\overline{m}@i)_{i'} \geq \exp(\overline{o}@i)_{i'}$ since $(\overline{m}@i)_{i'} = m_{i'}$ and $m_{i'} \geq (\exp(\overline{o}))_{i'}$ by assumption. For all j' such that $o_{j'} \geq i'$, $o_{j'}$ and the according bit remain unchanged. If there is no j' such that $0 < o_{j'} < i'$, then all bits remain unchanged so that we have $b = b'$. If there is some j' such that $0 < o_{j'} < i'$, then let j'' be the greatest j'' with this property. If the digit $o_{j''}$ remains unchanged, then $o_{j''}$ still is the leftmost digit with a value less than i' so that all bits remain unchanged and we again have $b = b'$. If the digit $o_{j''}$ is changed, then the new value of this digit is i . We have $i' > i$ and all digits to the left of $o_{j''}$ remain unchanged so that j'' is the position of the leftmost digit with a value less than i' , before and after the updating procedure. Thus $b = b'$ and we are done. \square

Thus we have shown that the exponential timeout representation of quasipolynomial histories decreases, in each component, at least as fast as exponential timeouts. Hence we can now prove that nodes have exponential timeouts if they have quasipolynomial histories.

THEOREM 6.13. *We have $\pi_1[\text{GFP } \gamma] \subseteq \pi_1[\text{GFP } \beta]$.*

PROOF. First, we put

$$Z = \{(v, \overline{m}) \in (V \times \text{to}) \mid \exists \overline{o} \in \text{hi}. \overline{m} \geq \exp(\overline{o}), (v, \overline{o}) \in \text{GFP } \gamma\}.$$

Let $(v, \overline{o}) \in \text{GFP } \gamma$. We have to show that there is some \overline{m} such that $(v, \overline{m}) \in \text{GFP } \beta$. Since $(v, \exp(\overline{o})) \in Z$, it suffices to show the more general property that $Z \subseteq \text{GFP } \beta$ which by coinduction is the case if Z is a postfixpoint of β , that is, if $Z \subseteq \beta(Z)$. Let $(v, \overline{m}) \in Z$ so that $(v, \overline{o}) \in \text{GFP } \gamma = \gamma(\text{GFP } \gamma)$ for some \overline{o} such that $\overline{m} \geq \exp(\overline{o})$. Since $(v, \overline{o}) \in \gamma(\text{GFP } \gamma)$, we have $v \in \alpha((\text{GFP } \gamma)_{\overline{o}_1}^{\overline{o}}, \dots, (\text{GFP } \gamma)_{\overline{o}_k}^{\overline{o}})$ by definition of γ . We have to show $(v, \overline{m}) \in \beta(Z)$ which is the case if $v \in \alpha(Z_1^{\overline{m}}, \dots, Z_k^{\overline{m}})$. By monotonicity of α , it suffices to show that for all $1 \leq i \leq k$, we have $(\text{GFP } \gamma)_{\overline{o}_i}^{\overline{o}} \subseteq Z_i^{\overline{m}}$. So let $v \in (\text{GFP } \gamma)_{\overline{o}_i}^{\overline{o}}$, that is, $(v, (\overline{o}@i)) \in \text{GFP } \gamma$ and the leftmost digit in $\overline{o}@i$ is 0. We have to show $v \in Z_i^{\overline{m}}$ which is the case if $(v, (\overline{m}@i)) \in Z$ and if we have $m_i > 1$ when i is odd. The former is the case if there is some \overline{o}' such that $\overline{m}@i \geq \exp(\overline{o}')$ and $(v, \overline{o}') \in \text{GFP } \gamma$. Pick $\overline{o}' = \overline{o}@i$. Since we have $\overline{m} \geq \exp(\overline{o})$ by assumption, we have $\overline{m}@i \geq \exp(\overline{o}@i)$ by Lemma 6.12. If i is odd, then we have $(\exp(\overline{o}@i))_i \geq 1$ and since $\overline{m}@i \geq \exp(\overline{o}@i)$ we also have $(\overline{m}@i)_i \geq 1$; since $(\overline{m}@i)_i = m_i - 1$, we have $m_i > 1$, as required. \square

To prove the converse direction, we use fixpoint games to show that nodes with exponential timeouts have quasipolynomial histories as well:

THEOREM 6.14. *We have $\pi_1[\text{GFP } \beta] \subseteq \pi_1[\text{GFP } \gamma]$.*

PROOF. Let $(u, \overline{m}) \in \text{GFP } \beta$. By Lemmas 6.8, 5.2 and 5.4, player Eloise wins the node u in the lean fixpoint game for E_k^α . Thus there is an Eloise-strategy s that wins u and that is, crucially, history-free. We inductively define a graph (Z, L) with $Z \subseteq (U \times \text{hi})$, $L \subseteq Z \times \{1, \dots, k\} \times Z$, starting at the node $(u, (0, \dots, 0)) \in Z$ where $(0, \dots, 0) \in \text{hi}$. When at (v, \overline{o}) and having that Eloise wins the node v in the lean fixpoint game with the move $s(v) = (U_1, \dots, U_k)$, where Abelard can, for each $1 \leq j \leq k$ move to (U_j, j) and then from there to any $w \in U_j$, where player Eloise wins all nodes from U_j since s is a winning strategy, then we put, for $1 \leq j \leq k$, $L((v, \overline{o}), j) = U_j \times \{\overline{o}@j\}$ and extend Z by these nodes. We note that for any edge $((v, \overline{o}), j, (w, \overline{o}@j)) \in L$ obtained in this way, we have $s(v) = (U_1, \dots, U_k)$, and $w \in U_j$ so that the priority j occurs in the lean fixpoint game

when player Abelard moves from $s(v)$ to (U_j, j) and from there to w . Since $(u, (0, \dots, 0)) \in Z$, it suffices to show that Z is a postfixpoint of γ . So let $(v, \bar{o}) \in Z$; we have to show $(v, \bar{o}) \in \gamma(Z)$, i.e. $v \in \alpha(Z_1^{\bar{o}}, \dots, Z_k^{\bar{o}})$. Now Eloise wins v in the lean fixpoint game with the move $s(v) = (U_1, \dots, U_k)$, and we have $v \in \alpha(U_1, \dots, U_k)$. By monotonicity of α , it thus suffices to show that $U_j \subseteq Z_j^{\bar{o}}$ for all $1 \leq j \leq k$. This certainly holds if the leftmost digit in $\bar{o}@j$ is 0: Let $u \in U_j$. Then $(u, \bar{o}@j) \in Z$ by construction of Z . If the leftmost digit in $\bar{o}@j$ is 0, then $u \in Z_j^{\bar{o}}$ by definition of $Z_j^{\bar{o}}$.

It thus remains to show that Z contains no node (v, \bar{o}) such that the leftmost digit in \bar{o} is not 0, which follows by the construction of L and Fact 6.3 if Z contains no node with a timeout \bar{o} whose leftmost digit is odd. We recall the notion of *i-sequences* from [Calude et al. 2017] and adapt it slightly so that the notion is relative to an L -path: an *i-sequence* (on an L -path π) is a sequence of states a_1, a_2, \dots, a_{2^i} from U that are visited by the path π (that is, occur as first component in π), not necessarily consecutively, but in order, such that for all $q \in \{1, 2, \dots, 2^i - 1\}$, the maximal priority that is passed by the path π between a_q and a_{q+1} is odd.

We claim that the following properties (that correspond to the invariants I2 and I4 of the updating procedure and to **Property- b_i** in Calude et al.) are satisfied:

FACT 6.15. *For all $(v, \bar{o}) \in Z$ with $\bar{o} = (o_{\lceil \log n \rceil + 1}, \dots, o_0)$ and all L -paths π that lead from $(u, (0, \dots, 0))$ to (v, \bar{o}) , we have the following:*

- (1) *For all $0 \leq j \leq \lceil \log n \rceil + 1$ such that $o_j > 0$, π contains a j -sequence, which we fix and refer to as the tracked j -sequence; and o_j is the largest priority that is visited by π at or after the end of the tracked j -sequence.*
- (2) *For all $\lceil \log n \rceil + 1 \geq j > i \geq 0$ such that $o_j, o_i > 0$ (hence $o_j \geq o_i$), the tracked i -sequence in π starts only after priority o_j has been visited at or after the end of the tracked j -sequence.*

We prove the fact by induction over the length of π . If π has length 0, then we have $(v, \bar{o}) = (u, (0, \dots, 0))$ so that all digits in \bar{o} are 0 and there is nothing to show. If π has length $q + 1$, then there is a path π' of length q that leads from $(u, (0, \dots, 0))$ to some node (v', \bar{o}') such that, for some priority $1 \leq q \leq k$, $((v', \bar{o}'), q, (v, \bar{o})) \in L$ so that $s(v') = (A_1, \dots, A_k)$ and $v \in A_q$; also we have $\bar{o} = \bar{o}'@q$ and $\pi = \pi'; ((v', \bar{o}'), q, (v, \bar{o}))$. By the inductive hypothesis, π' has the claimed properties. We distinguish the cases by which the update from \bar{o}' at q proceeds (recall the cases of the updating procedure for quasipolynomial histories from Definition 6.1).

- If the update proceeds according to case 1), then q is even. By Fact 6.3, every digit in \bar{o} that is nonzero also is nonzero in the previous quasipolynomial timeout \bar{o}' . Moreover, all sequences in π' are also sequences in π . For digits that do not change when updating \bar{o}' at q , we are done since π' has the claimed properties by the inductive hypothesis. If some digit, say the j -th digit o'_j , changes to a nonzero number, then $o_j = q$ by Fact 6.3, and by the definition of case 1), $q > o'_j > 0$. By the inductive hypothesis, o'_j is the largest priority that is visited at or after the end of the tracked j -sequence in π' . Since $q > o'_j$ and since priority q is visited by the last transition in π , $q = o_j$ is the largest priority that is visited at or after the end of the tracked j -sequence in π .
- If the update proceeds according to case 2.a), then q is odd.
 - To show the first claimed property, let j be an index such that $o_j > 0$. If $o'_j = o_j$, then we have $o_j \geq q$. By the inductive hypothesis, π' contains a tracked j -sequence and o_j is the largest priority that is visited by π' at or after the end of the tracked j -sequence. The sequence also is a sequence in π and since $o_j \geq q$, o_j also is the largest priority that is visited by π at or after the end of the sequence. If $o'_j \neq o_j$, then j is the largest index such that $q > o'_j > 0$ and we also have $o_j = q$. By the inductive hypothesis, π' contains a tracked j -sequence and o'_j is the largest priority that is visited by π' at or after the end

- of the sequence. The sequence also is a sequence in π and since $o_j > o'_j$, o_j is the largest priority that is visited by π at or after the end of the sequence.
- To show the second claimed property, let $i > j$ and $o_j, o_i > 0$, hence $o_j \geq o_i$. Since the updating procedure changes at most one digit to a nonzero value and since this digit has a nonzero value before updating, o'_i and o'_j both are nonzero. Moreover, since $o_i > 0$, the j -th digit was not changed in the update, i.e. $o_j = o'_j$. By the inductive hypothesis, π' contains a tracked i -sequence and a tracked j -sequence and the tracked i -sequence in π' starts only after π' has visited priority $o'_j = o_j$ at or after the end the tracked j -sequence. Adding one transition to the end of π' does not change the order of sequences within π' so that π also has the claimed property.
 - If the update proceeds according to case 2.b), then q is again odd.
 - To show the first claimed property, let j be an index such that $o_j > 0$. If $o'_j = o_j$, then we have $o_j \geq q$. By the inductive hypothesis, π' contains a tracked j -sequence and o_j is the largest priority that is visited by π' at or after the end of the sequence. The sequence also is a sequence in π and since $o_j \geq q$, o_j also is the largest priority that is visited by π at or after the end of the sequence. If $o'_j \neq o_j$, then o'_j is even, if $o'_j > 0$, then $q < o'_j$ and for all $j' < j$, $o'_{j'}$ is odd. Also we have $o_j = q$. Since for all $j' < j$, $o'_{j'}$ is odd, we have by the inductive hypothesis that for all $j' < j$, π' contains a tracked j' -sequence and $o'_{j'}$ is the largest priority that is visited by π' at or after the end of that sequence; also, these tracked sequences are ordered, that is, shorter tracked sequences start only after the respective priority in the digits that refer to the longer tracked sequences have been visited at or after the end of longer tracked sequences. We construct a j -sequence (and make it tracked) by concatenating these sequences, starting with the tracked $j-1$ -sequence, adding the tracked $j-2$ -sequence and so on. As last node in the new sequence we chose v so that q is the largest priority that is visited at or after the end of the sequence. If $j = 0$, then we pick the one-element sequence consisting just of v . If $j > 0$, then the highest priority visited between the node of the tracked 0-sequence and v (both inclusive) is, by the inductive hypothesis, either o'_0 or the new last priority q , hence odd (q is odd since we are in case 2.b), and we have observed above that o'_0 is odd). For any $0 \leq j' < j-1$, the highest priority that is visited between the last node of the tracked $j'+1$ -sequence and the first node of the tracked j' -sequence is $o'_{j'+1}$ by the inductive hypothesis. As $o'_{j'+1}$ is odd for all $j' < j$, the newly constructed sequence is indeed is a j -sequence.
 - To show the second claimed property, let i and j be indices such that $j > i$ and $o_j, o_i > 0$ (hence $o_j \geq o_i$). If o'_i and o'_j are both nonzero, then π' contains, by the inductive hypothesis, a tracked i -sequence and a tracked j -sequence, and the tracked i -sequence in π' starts only after π' has visited priority o_j at or after the end of the tracked j -sequence. Adding one transition to the end of π' does not change the order of tracked sequences within π' so that π also has the claimed property. If one of the digits o'_i and o'_j is zero, then o'_i is zero and o'_j is nonzero: the updating procedure changes just one digit to a nonzero number and all digits to the right of this digit are changed to 0. By the inductive hypothesis, there is a tracked j -sequence in π' . The tracked i -sequence in π that is constructed in the previous item starts after the tracked j -sequence in π' since the tracked j -sequence in π' starts before all shorter tracked sequences in π' by the inductive hypothesis and since the tracked i -sequence in π is built up from sequences of length less than i and hence less than j .

Towards a contradiction, we now assume that there is some node $(v, \bar{o}) \in Z$ such that the leftmost digit $o_{\lceil \log n \rceil + 1}$ in \bar{o} is odd. Since all nodes from Z can by construction be reached starting from $(u, (0, \dots, 0))$ using only L -transitions, there is an L -path π through Z that starts at $(u, (0, \dots, 0))$,

that ends at (v, \bar{v}) and that (by Fact 6.15, Item 1) contains a $\lceil \log n \rceil + 1$ -sequence $a_1, a_2, \dots, a_{2^{\lceil \log n \rceil + 1}}$ of at least $n+1$ elements (since $2^{\lceil \log n \rceil + 1} > n$). Thus some node occurs at least twice in the sequence, that is, there are $h, l \in \{1, 2, \dots, 2^{\lceil \log n \rceil + 1}\}$ such that $h < l$ and $a_h = a_l$. The maximum value that occurs in π between a_h and a_l occurs between a_q and a_{q+1} for some $h \leq q < l$. By the definition of i -sequences, the maximum value between a_q and a_{q+1} and hence the maximum value between a_h and a_l is odd, say z . Let $(v_0, \bar{v}_0), p_0, \dots, p_{r-1}, (v_r, \bar{v}_r)$ be the according segment of π that leads from a_h to a_l so that $v_0 = a_h$ and $v_r = a_l$. Writing $s(v_q) = (U_1^q, \dots, U_k^q)$ for $0 \leq q \leq r$, there is an infinite play

$$v_0, s(v_0), (U_{p_0}^0, p_0), v_1, s(v_1), (U_{p_1}^1, p_1), \dots, v_r, s(v_r), (U_{p_r}^r, p_r), v_1, s(v_1), (U_{p_1}^1, p_1) \dots$$

of the lean fixpoint game that follows s but in which the highest priority that occurs infinitely often is z so that the play is won by player Abelard, which is a contradiction to s being a winning strategy. \square

We finally obtain our main result, namely that nested fixpoints can be computed in quasipolynomial time if the function α allows for it:

COROLLARY 6.16. *If α can be iterated in quasipolynomial time, then E_k^α and A_k^α can be computed in quasipolynomial time as well.*

PROOF. For even k , we have $E_k^\alpha = \pi_1[\text{GFP } \beta] = \pi_1[\text{GFP } \gamma]$, where the first equality is by Lemma 6.8 and the second equality is by Theorem 6.13 and Theorem 6.14. By Lemma 6.4, $\pi_1[\text{GFP } \gamma]$ can be computed in time $s_2 \cdot n \cdot k^{\lceil \log n \rceil + 2}$ where $s_2 = \max(t(\alpha), n \cdot k^{\lceil \log n \rceil + 2})$. Since α can be iterated in quasipolynomial time by assumption, we have $t(\alpha) \leq 2^{(\log n)^c}$ for some constant c so that $\pi_1[\text{GFP } \gamma]$ can be computed in quasipolynomial time as well. Define $\alpha' : (\mathcal{P}(U))^{k+1} \rightarrow \mathcal{P}(U)$ by $\alpha'(X_1, \dots, X_k, X_{k+1}) = \alpha(X_1, \dots, X_k)$. For odd k , we have that $E_k^\alpha = E_{k+1}^{\alpha'}$ where the latter can be computed in quasipolynomial time by the first part of this proof. Since we have $A_k^\alpha = E_{k+1}^{\alpha''}$, where $\alpha''(X_1, \dots, X_k, X_{k+1}) = \alpha(X_2, \dots, X_{k+1})$, A_k^α be computed in quasipolynomial time as well. \square

As in Theorem 2.8 in [Calude et al. 2017], we furthermore have that if $k < \log n$, and α can be iterated time polynomial in n , then k -nested fixpoints of α can be computed in time polynomial in n as well:

THEOREM 6.17. *Let $k < \log n$ where n denotes the maximal value of timeouts; then $\text{GFP } \beta$ can be computed in time $O(t(\alpha) \cdot n^4)$, where $t(\alpha)$ is the maximum of n^4 and the time it takes to iterate α .*

PROOF. We note that if $k < \log n$, then all digits in quasipolynomial histories have value less than $\log n$ and – as in the proof of Theorem 2.8. in [Calude et al. 2017] – since the nonzero digits in quasipolynomial histories are non decreasing, hi can be represented by a set of size $O(n^3)$ in such a way that all histories can be recreated from their representation. Hence the fixpoint $\text{GFP } \beta$ can be computed with $O(n^4)$ iterations of β and a single iteration of β can be implemented in time $O(t(\alpha))$. \square

7 APPLICATIONS IN THE COALGEBRAIC μ -CALCULUS

Examples of nested fixpoints that go beyond the computation of winning regions in parity games naturally occur in model checking and satisfiability checking for generalized μ -calculi in the setting of coalgebraic logic, covering, for instance, graded, probabilistic and alternating-time μ -calculi; model checking for coalgebraic μ -calculi can be regarded as computing winning regions in a generalized variant of parity games, where the game arenas are coalgebras instead of Kripke frames. Hence we proceed to recall basic definitions and examples in universal coalgebra [Rutten 2000]

and the coalgebraic μ -calculus [Cirstea et al. 2009] and then continue to show that our main result improves the known upper time bounds for both the model checking problem and the satisfiability problem of the coalgebraic μ -calculus.

The abstraction principle underlying universal coalgebra is to encapsulate system types as functors, for our present purposes on the category of sets. Such a functor $T : \text{Set} \rightarrow \text{Set}$ maps every set X to a set $T(X)$, and every map $f : X \rightarrow Y$ to a map $Tf : T(X) \rightarrow T(Y)$, preserving identities and composition. We think of $T(X)$ as a type of structured collections over X ; a basic example is the covariant powerset functor \mathcal{P} , which assigns to each set its powerset and acts on maps by taking forward image. Systems of the intended type are then cast as T -coalgebras (C, ξ) (or just ξ) consisting of a set C of *states* and a *transition map* $\xi : C \rightarrow T(C)$, thought of as assigning to each state $x \in C$ a structured collection $\xi(x) \in T(C)$ of successors. E.g. a \mathcal{P} -coalgebra $\xi : C \rightarrow \mathcal{P}(C)$ assigns to each state a set of successors; that is, \mathcal{P} -coalgebras are transition systems.

Following the paradigm of *coalgebraic logic* [Cirstea et al. 2011], we fix a set Λ of modal operators, which we interpret over T -coalgebras for a functor T as *predicate liftings*, i.e. natural transformations

$$\llbracket \heartsuit \rrbracket_X : 2^X \rightarrow 2^{T(X)} \quad \text{for } \heartsuit \in \Lambda.$$

Here, the index X , omitted when clear from the context, ranges over all sets; 2^X denotes the set of maps $X \rightarrow 2$ into the two-element set $2 = \{\perp, \top\}$, isomorphic to the powerset of X (i.e. 2^{\cdot} is the *contravariant powerset functor*); and naturality means that $\llbracket \heartsuit \rrbracket_X(f^{-1}[A]) = (Tf)^{-1}[\llbracket \heartsuit \rrbracket_Y(A)]$ for $f : X \rightarrow Y$ and $A \in 2^Y$. Thus, the predicate lifting $\llbracket \heartsuit \rrbracket$ indeed lifts predicates on a base set X to predicates on the set $T(X)$. Two standard examples for $T = \mathcal{P}$ are the predicate liftings for the standard \Box and \Diamond modalities, given by

$$\llbracket \Box \rrbracket_X(A) = \{B \in \mathcal{P}(X) \mid B \subseteq A\} \quad \text{and} \quad \llbracket \Diamond \rrbracket_X(A) = \{B \in \mathcal{P}(X) \mid A \cap B \neq \emptyset\}.$$

Since we mean to form fixpoint logics, we need to require that every $\llbracket \heartsuit \rrbracket$ is *monotone*, that is, $A \subseteq B \subseteq X$ implies $\llbracket \heartsuit \rrbracket_X(A) \subseteq \llbracket \heartsuit \rrbracket_X(B)$. To support negation, we assume moreover that Λ is closed under *duals*, i.e. for each $\heartsuit \in \Lambda$ we have $\overline{\heartsuit} \in \Lambda$ such that $\llbracket \overline{\heartsuit} \rrbracket_X(A) = T(X) \setminus \llbracket \heartsuit \rrbracket_X(X \setminus A)$, chosen so that $\overline{\overline{\heartsuit}} = \heartsuit$ (e.g. $\overline{\Box} = \Diamond$, $\overline{\Diamond} = \Box$).

Given a set Var of *fixpoint variables*, the set of *formulae* of the coalgebraic μ -calculus ϕ, ψ is then defined by the grammar

$$\psi, \phi := \top \mid \perp \mid \psi \vee \phi \mid \psi \wedge \phi \mid \heartsuit \psi \mid X \mid \eta X. \psi \quad (\heartsuit \in \Lambda, X \in \text{Var}).$$

Given a T -coalgebra $\xi : C \rightarrow T(C)$ and a valuation $\sigma : \text{Var} \rightarrow \mathcal{P}(C)$, the extension

$$\llbracket \phi \rrbracket_\sigma \subseteq C$$

of a formula ϕ is defined recursively by the expected clauses for the propositional operators ($\llbracket \top \rrbracket_\sigma = C$; $\llbracket \perp \rrbracket_\sigma = \emptyset$; $\llbracket \phi \wedge \psi \rrbracket_\sigma = \llbracket \phi \rrbracket_\sigma \cap \llbracket \psi \rrbracket_\sigma$; $\llbracket \phi \vee \psi \rrbracket_\sigma = \llbracket \phi \rrbracket_\sigma \cup \llbracket \psi \rrbracket_\sigma$); by $\llbracket X \rrbracket_\sigma = \sigma(X)$ and

$$\llbracket \heartsuit \psi \rrbracket_\sigma = \xi^{-1}[\llbracket \heartsuit \rrbracket(\llbracket \psi \rrbracket_\sigma)];$$

and by $\llbracket \mu X. \psi \rrbracket_\sigma = \text{LFP}[\llbracket \psi \rrbracket_\sigma^X]$, $\llbracket \nu X. \psi \rrbracket_\sigma = \text{GFP}[\llbracket \psi \rrbracket_\sigma^X]$ where the map $\llbracket \psi \rrbracket_\sigma^X : \mathcal{P}(C) \rightarrow \mathcal{P}(C)$ is defined by $\llbracket \psi \rrbracket_\sigma^X(A) = \llbracket \psi \rrbracket_{\sigma[X \mapsto A]}$ for $A \subseteq C$, with $(\sigma[X \mapsto A])(X) = A$ and $(\sigma[X \mapsto A])(Y) = \sigma(Y)$ for $X \neq Y$; monotonicity of $\llbracket \psi \rrbracket_\sigma^X$ is clearly an invariant of the recursive definition.

The *alternation depth* $\text{ad}(\eta X. \psi)$ of a fixpoint $\eta X. \psi$ is the depth of alternating nesting of such fixpoints in ψ that depend on X ; we assign *odd* numbers to least fixpoints and *even* numbers to greatest fixpoints. E.g. for $\psi = \nu X. \phi$ and $\phi = \mu Y. (p \wedge \heartsuit X) \vee \heartsuit Y$, we have $\text{ad}(\psi) = 2$, $\text{ad}(\phi) = 1$. For a detailed definition of alternation depth, see e.g. [Niwinski 1986].

Example 7.1. We proceed to see some standard examples of coalgebraic semantics, see [Cirstea et al. 2009; Schröder and Pattinson 2009; Schröder and Venema 2018] for more details.

- (1) *Relational μ -calculus*: As indicated above, we obtain a version of the standard relational μ -calculus [Kozen 1983] without propositional atoms by taking $T = \mathcal{P}$ and $\Lambda = \{\Box, \Diamond\}$ with predicate liftings as described previously. We can add a set At of propositional atoms p, q, \dots by regarding them as nullary modalities, and interpret $p \in \text{At}$ over the extended functor $T = \mathcal{P}(\text{At}) \times \mathcal{P}$ by the nullary predicate lifting $\llbracket p \rrbracket_X = \{(P, B) \in \mathcal{P}(\text{At}) \times \mathcal{P}(X) \mid p \in P\}$. E.g. the formula $\nu X. \mu Y. ((p \wedge \Box X) \vee (q \wedge \Box Y))$ says that on every path from the current state, p holds everywhere except possibly on finite segments where q holds.
- (2) *Graded μ -calculus*: The graded μ -calculus [Kupferman et al. 2002] has modalities $\langle b \rangle, [b]$, indexed over $b \in \mathbb{N}$, read ‘in more than b successors’ and ‘in all but at most b successors’, respectively. These can be interpreted over relational structures but it is technically more convenient to use *multigraphs*, i.e. transition systems with edge weights (*multiplicities*) in $\mathbb{N} \cup \{\infty\}$, which are coalgebras for the multiset functor \mathcal{B} . The latter maps a set X to the set $\mathcal{B}(X) = (\mathbb{N} \cup \{\infty\})^X$ of maps $X \rightarrow (\mathbb{N} \cup \{\infty\})$; we treat elements $\beta \in \mathcal{B}(X)$ as $(\mathbb{N} \cup \{\infty\})$ -valued discrete measures on X , and in particular write $\beta(A) = \sum_{x \in A} \beta(x)$ for $A \subseteq X$. For a map $f : X \rightarrow Y$, the map $\mathcal{B}(f) : \mathcal{B}(X) \rightarrow \mathcal{B}(Y)$ is then given by $\mathcal{B}(f)(\beta)(y) = \beta(f^{-1}[\{y\}])$. Over \mathcal{B} -coalgebras, we interpret $\langle b \rangle$ and $[b]$ by the mutually dual predicate liftings $\llbracket \langle b \rangle \rrbracket_X(A) = \{\beta \in \mathcal{B}(X) \mid \beta(A) > b\}$ and $\llbracket [b] \rrbracket_X(A) = \{\beta \in \mathcal{B}(X) \mid \beta(X \setminus A) \leq b\}$. E.g. the formula $\nu X. (\phi \wedge \Diamond_1 X)$ says that the current state is the root of an infinite tree with branching degree at least 2 (counting multiplicities) on which ϕ holds everywhere.
- (3) *Probabilistic μ -calculus*: Let \mathcal{D} denote the *discrete distribution functor*, defined on sets by $\mathcal{D}(X) = \{\beta : X \rightarrow [0, 1] \mid \sum_{x \in X} \beta(x) = 1\}$. That is, $\mathcal{D}(X)$ is the set of discrete probability distributions on X ; coalgebras for \mathcal{D} are just Markov chains. Similarly as for the graded μ -calculus, we take modalities $[p], \langle p \rangle$ indexed over $p \in [0, 1] \cap \mathbb{Q}$, interpreted over \mathcal{D} by $\llbracket \langle p \rangle \rrbracket_X = \{\beta \in \mathcal{D}(X) \mid \beta(A) > p\}$ and $\llbracket [p] \rrbracket_X = \{\beta \in \mathcal{D}(X) \mid \beta(X \setminus A) \leq p\}$ (using the same measure-theoretic notation as in the previous item). The arising coalgebraic μ -calculus is the *probabilistic μ -calculus* [Cirstea et al. 2009; Liu et al. 2015].
- (4) *Monotone μ -calculus*: The *monotone neighbourhood functor* \mathcal{M} maps a set X to the set $\mathcal{M}(X) = \{\mathfrak{A} \in 2^{2^X} \mid \mathfrak{A} \text{ upwards closed}\}$ of set systems over X that are upwards closed under subset inclusion (i.e. $A \in \mathfrak{A}$ and $A \subseteq B$ imply $B \in \mathfrak{A}$). Coalgebras for \mathcal{M} are *monotone neighbourhood frames* in the sense of Scott-Montague semantics [Chellas 1980]. We take $\Lambda = \{\Box, \Diamond\}$ and interpret \Box over \mathcal{M} by the predicate lifting

$$\llbracket \Box \rrbracket_X(A) = \{\mathfrak{A} \in \mathcal{M}(X) \mid A \in \mathfrak{A}\} = \{\mathfrak{A} \in \mathcal{M}(X) \mid \exists B \in \mathfrak{A}. B \subseteq A\},$$

and \Diamond by the corresponding dual lifting, $\llbracket \Diamond \rrbracket_X(A) = \{\mathfrak{A} \in \mathcal{M}(X) \mid (X \setminus A) \notin \mathfrak{A}\} = \{\mathfrak{A} \in \mathcal{M}(X) \mid \forall B \in \mathfrak{A}. B \cap A \neq \emptyset\}$. The arising coalgebraic μ -calculus is known as the *monotone μ -calculus* [Enqvist et al. 2015]. When we add propositional atoms and actions, and replace \mathcal{M} with its subfunctor \mathcal{M}_s defined by $\mathcal{M}_s(X) = \{\mathfrak{A} \in \mathcal{M}(X) \mid \emptyset \notin \mathfrak{A} \ni X\}$, whose coalgebras are *serial monotone neighbourhood frames*, we arrive at the ambient fixpoint logic of *concurrent dynamic logic* [Peleg 1987] and Parikh’s *game logic* [Parikh 1985]. In game logic, actions are understood as atomic games of Angel vs. Demon, and we read $\Box_a \phi$ as ‘Angel has strategy to enforce ϕ in game a ’. Game logic is then mainly concerned with composite games, formed by the control operators of dynamic logic and additional ones; the semantics can be encoded into fixpoint definitions. For instance, the formula $\nu X. p \wedge \Box_a X$ says that Angel can enforce p in the composite game where a is played repeatedly, with Demon deciding when to stop.

- (5) *Alternating-time μ -calculus*: Fix a set $N = \{1, \dots, n\}$ of *agents*. Using alternative notation from *coalition logic* [Pauly 2002], we present the *alternating-time μ -calculus* (AMC) [Alur et al.

2002] by modalities $[D]$, $\langle D \rangle$ indexed over *coalitions* $D \subseteq N$, read ‘ D can enforce’ and ‘ D cannot prevent’, respectively. We define a functor \mathcal{G} by

$$\mathcal{G}(X) = \{(k_1, \dots, k_n, f) \mid k_1, \dots, k_n \in \mathbb{N} \setminus \{0\}, f : (\prod_{i \in N} [k_i]) \rightarrow X\}$$

where we write $[k] = \{1, \dots, k\}$. We understand $(k_1, \dots, k_n, f) \in \mathcal{G}(X)$ as a one-step concurrent game with k_i available moves for agent $i \in N$, and outcomes in X determined by the *outcome function* f from a joint choice of moves by all the agents. For $D \subseteq N$, we write $S_D = \prod_{i \in D} [k_i]$. Given joint choices $s_D \in S_D$, $s_{\overline{D}} \in S_{\overline{D}}$ of moves for D and $\overline{D} = N \setminus D$ respectively, we write $(s_D, s_{\overline{D}}) \in s_N$ for the joint move of all agents induced in the evident way. In this notation, we interpret the modalities $[D]$ over \mathcal{G} by the predicate lifting

$$\llbracket [D] \rrbracket_X(A) = \{(k_1, \dots, k_n, f) \in \mathcal{G}(X) \mid \exists s_D \in S_D. \forall s_{\overline{D}} \in S_{\overline{D}}. f(s_D, s_{\overline{D}}) \in A\},$$

and the modalities $\langle D \rangle$ by dualization. This captures exactly the semantics of the AMC: \mathcal{G} -coalgebras are precisely *concurrent game structures* [Alur et al. 2002], i.e. assign a one-step concurrent game to each state, and $[D]\phi$ says that the agents in D have a joint move such that however the agents in \overline{D} move, the next state will satisfy ϕ . E.g. $\mu X. p \vee [D]X$ says that coalition D can eventually enforce that p is satisfied (a property expressible already in alternating-time temporal logic ATL [Alur et al. 2002]).

We fix a target formula χ that does not contain free fixpoint variables, assuming w.l.o.g. that χ is *clean*, i.e. that every fixpoint variable is bound by at most one fixpoint operator in χ . For a variable $X \in V$ that is bound in χ , we then write $\theta(X)$ to denote *the* formula $\eta X. \psi$ that is a subformula of χ . Let $\text{Cl}(\chi)$ be the *closure* (that is, the set of subformulae) of χ . We have $|\text{Cl}(\chi)| \leq |\chi|$, where $|\chi|$ denotes the number of operators or variables in χ .

It was shown in [Hausmann and Schröder 2019a] that model checking for the coalgebraic μ -calculus can be reduced to the computation of a nested fixpoint of a particular function; we recall this function:

Definition 7.2 (Coalgebraic model checking function). Let $U = \text{Cl}(\chi) \times C$. We then define the *coalgebraic model checking function* $\alpha_{\text{mc}} : (\mathcal{P}(U))^{k+1} \rightarrow \mathcal{P}(U)$ by putting, for $\mathbf{U} = (U_1, \dots, U_{k+1}) \in (\mathcal{P}(U))^{k+1}$,

$$\begin{aligned} \alpha_{\text{mc}}(\mathbf{U}) = & \{(\top, x) \mid (\top, x) \in U\} \cup \{(\forall \psi, x) \in U \mid \xi(x) \in \llbracket \forall \rrbracket \{y \mid (\psi, y) \in U_1\}\} \cup \\ & \{(\psi \vee \phi, x) \in U \mid \{(\psi, x), (\phi, x)\} \cap U_1 \neq \emptyset\} \cup \\ & \{(\psi \wedge \phi, x) \in U \mid \{(\psi, x), (\phi, x)\} \subseteq U_1\} \cup \\ & \{(\eta X. \psi, x) \in U \mid (\psi, x) \in U_1\} \cup \{(X, x) \mid (\theta(X), x) \in U_{\text{ad}(\theta(X))+1}\}. \end{aligned}$$

LEMMA 7.3 ([HAUSMANN AND SCHRÖDER 2019A]). *Let χ be a coalgebraic μ -calculus formula of alternation depth k , let $\xi : C \rightarrow TC$ be a coalgebra and let $x \in C$ be a state. Then we have*

$$(\chi, x) \in A_k^{\alpha_{\text{mc}}} \text{ if and only if } x \in \llbracket \chi \rrbracket.$$

COROLLARY 7.4. *Model checking for coalgebraic μ -calculus formulas of size n and alternation depth k can be done in time $s_2 \cdot n \cdot k^{\lceil \log n \rceil + 2}$ where $s_2 = \max(t(\alpha_{\text{mc}}), n \cdot k^{\lceil \log n \rceil + 2})$, $n = |\text{Cl}(\chi)| \cdot |C|$.*

PROOF. Immediate since the set $A_k^{\alpha_{\text{mc}}}$ can – by the proof of Corollary 6.16 – be computed in time $s_2 \cdot n \cdot k^{\lceil \log n \rceil + 2}$. \square

The *one-step satisfaction problem* consists in deciding whether $t \in \llbracket \forall \rrbracket(W)$, for given $t \in T(C)$, $\forall \in \Lambda$ and $W \subseteq U$. The time $t(\alpha_{\text{mc}})$ it takes to compute α_{mc} hence depends on the time it takes to solve the one-step satisfaction problem for the modal operators at hand.

Example 7.5. In [Hausmann and Schröder 2019a], Examples 3.2 and 3.3, it was shown, that the upper time bounds for the one-step satisfaction problem in graded and probabilistic cases are $O(\text{size}(\chi) \cdot |C|)$ and $O(\text{size}(C)^2 \cdot |C|^3)$, respectively; here, $\text{size}(\chi)$ denotes the representation size of the formula χ and $\text{size}(C)$ denotes the representation size of the coalgebra C . We hence obtain the following quasipolynomial upper time bounds for the model checking problems of the respective μ -calculi, both with numbers coded in binary:

- (1) for the graded μ -calculus: $O(s_2 \cdot n \cdot k^{\lceil \log n \rceil + 2})$ where $s_2 = \max(\text{size}(\chi) \cdot |C|, n \cdot k^{\lceil \log n \rceil + 2})$;
- (2) for the (two-valued) probabilistic μ -calculus: $O(s_2 \cdot n \cdot k^{\lceil \log n \rceil + 2})$ where $s_2 = \max(\text{size}(C)^2 \cdot |C|^3, n \cdot k^{\lceil \log n \rceil + 2})$.

As a second application of the computation of nested fixpoints, we now consider satisfiability checking for the coalgebraic μ -calculus. It was shown in [Hausmann and Schröder 2019b] that the satisfiability problem of the coalgebraic μ -calculus can be reduced to the computation of nested fixpoints of a particular function. We now recall the essential notions from [Hausmann and Schröder 2019b] that are required to be able to define this function; a detailed description of the construction can be found loc. cit. We fix a formula χ of size n and alternation depth k . We assume a deterministic parity automaton with set D_χ of nodes such that $|D_\chi| \in O(((nk)!)^2)$ where nodes $v \in D_\chi$ are labelled with sets $l(v)$ of formulas; we denote the set of nodes whose labels contain some propositional formula by prestates and the set of nodes whose labels contain only modal formulas by states; for $v \in \text{prestates}$, ψ_v is a fixed propositional formula from the label of v . We assume that the automaton has the transition function δ that tracks sets of formulas according to letters from an alphabet that allows to identify manipulations of formulas; also, selections is a subset of the alphabet and consists of sets of modal formulas. Furthermore, we assume that the automaton has priority function β and that it altogether accepts the *good branches* in prospective models for μ -calculus formulas.

Definition 7.6 (Coalgebraic satisfiability checking function). For sets $U \subseteq D_\chi$ and $\mathbf{U} = U_1, \dots, U_{2nk} \in (\mathcal{P}(U))^{2nk}$, we put

$$\alpha_{\text{sat}}(\mathbf{U}) = \{v \in \text{prestates} \mid \exists b \in \{0, 1\}. \delta(v, (\psi_v, b)) \in U_{\beta(v, (\psi_v, b))}\} \cup \\ \{v \in \text{states} \mid T(\bigcup_{1 \leq i \leq 2nk} U_i(v)) \cap \llbracket l(v) \rrbracket_1 \neq \emptyset\}$$

where $\beta(v, (\psi_v, b))$ abbreviates $\beta(v, (\psi_v, b), \delta(v, (\psi_v, b)))$ and where

$$U_i(v) = \{l(u) \mid u \in X_i, \exists \kappa \in \text{selections}. \delta(v, \kappa) = u, \beta(v, \kappa, u) = i\}.$$

The *one-step satisfiability problem* is to decide whether

$$T(\bigcup_{1 \leq i \leq 2nk} U_i(v)) \cap \llbracket l(v) \rrbracket_1 \neq \emptyset$$

for given U, v . Hence checking whether some $v \in \text{prestates}$ is contained in $\alpha_{\text{sat}}(\mathbf{U})$ for given \mathbf{U} is an instance of the one-step satisfiability problem.

LEMMA 7.7 ([HAUSMANN AND SCHRÖDER 2019B]). *Let χ be a coalgebraic μ -calculus formula of size n and alternation depth k . Then we have*

$$v_0 \in A_{2nk}^{\alpha_{\text{sat}}} \text{ if and only if } \chi \text{ is satisfiable.}$$

COROLLARY 7.8. *If the one-step satisfiability problem of a coalgebraic logic can be solved in time $2^{O(nk \log n)}$, then the satisfiability problem of the μ -calculus over this logic can be solved in time $2^{O(nk \log n)}$ as well.*

PROOF. By the previous Lemma, it suffices to show that $A_{2nk}^{\alpha_{\text{sat}}}$ can be computed in time $2^{O(nk \log n)}$. Since we have $2nk < \log |D_\chi|$, $A_{2nk}^{\alpha_{\text{sat}}}$ can – by Theorem 6.17 – be computed in time $O(t(\alpha_{\text{sat}}) \cdot |D_\chi|^4)$,

where $t(\alpha_{\text{sat}})$ denotes the maximum of $|D_\chi|^4$ and the time it takes to iterate α_{sat} ; by assumption, α_{sat} can be iterated in time $2^{O(nk \log n)}$ so that we have $t(\alpha_{\text{sat}}) \in 2^{O(nk \log n)}$. \square

Example 7.9. It has been shown (e.g. in [Hausmann and Schröder 2019b]) that the one-step satisfiability problems of all logics from Example 7.1 can be solved in time $2^{O(nk \log n)}$. Hence we obtain an upper bound $2^{O(nk \log n)}$ for the satisfiability problems of all these logics, in particular including the monotone μ -calculus, the alternating-time μ -calculus, the graded μ -calculus and the (two-valued) probabilistic μ -calculus, even when the latter two are extended with (monotone) polynomial inequalities as described in [Hausmann and Schröder 2019b].

8 CONCLUSION

We have shown how to compute k -nested fixpoints of functions $\alpha : \mathcal{P}(U)^k \rightarrow \mathcal{P}(U)$ (over a finite set U) in time $O(s(\alpha) \cdot n \cdot k^{\lceil \log n \rceil + 2})$, where $n = |U|$, $s(\alpha) = \max(t(\alpha), n \cdot k^{\lceil \log n \rceil + 2})$ and where $t(\alpha)$ denotes the time it takes to compute one application of α . In particular, if α can be computed in quasipolynomial time, then its k -nested fixpoint can be computed in quasipolynomial time as well. One consequence of this is that upper time bounds for model checking and satisfiability checking for the coalgebraic μ -calculus improve; in particular, model checking in the coalgebraic μ -calculus can be performed in quasipolynomial time under very mild assumptions on the modalities. In terms of concrete instances, we obtain, e.g., quasipolynomial-time model checking for the graded μ -calculus [Kupferman et al. 2002] and the probabilistic μ -calculus [Cirstea et al. 2011; Liu et al. 2015] as new results (corresponding results for, e.g., the alternating-time μ -calculus [Alur et al. 2002] and the monotone μ -calculus [Enqvist et al. 2015] follow as well but have already been obtained in recent work by Hausmann and Schröder [Hausmann and Schröder 2019a]), as well as improved upper bounds for satisfiability checking in the graded μ -calculus, the probabilistic μ -calculus, the monotone μ -calculus, and the alternating-time μ -calculus.

As in the case of parity games, a natural open question that remains is whether nested fixpoints can be computed in polynomial time (which would of course imply that parity games can be solved in polynomial time). A more immediate perspective for further investigation is to generalize recent quasipolynomial variant [Parys 2019] of Zielonka’s algorithm [Zielonka 1998] for solving parity games to computing nested fixpoints, with a view to improving efficiency in practice.

ACKNOWLEDGMENTS

We would like to thank Barbara König for bringing fixpoint games to our attention.

REFERENCES

- Rajeev Alur, Thomas Henzinger, and Orna Kupferman. 2002. Alternating-time temporal logic. *J. ACM* 49 (2002), 672–713.
- Paolo Baldan, Barbara König, Christina Mika-Michalski, and Tommaso Padoan. 2019. Fixpoint games on continuous lattices. *PACMPL* 3, POPL (2019), 26:1–26:29.
- Hans Bodlaender, Michael Dinneen, and Bakhadyr Khoussainov. 2001. On Game-Theoretic Models of Networks. In *Algorithms and Computation, ISAAC 2001 (LNCS)*, Vol. 2223. Springer, 550–561.
- Udi Boker and Karoliina Lehtinen. 2018. On the Way to Alternating Weak Automata. In *Foundations of Software Technology and Theoretical Computer Science, FSTTCS 2018 (LIPIcs)*, Vol. 122. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 21:1–21:22. <http://www.dagstuhl.de/dagpub/978-3-95977-093-4>
- Florian Bruse, Michael Falk, and Martin Lange. 2014. The Fixpoint-Iteration Algorithm for Parity Games. In *Games, Automata, Logics and Formal Verification, GandALF 2014 (EPTCS)*, Vol. 161. 116–130.
- Cristian Calude, Sanjay Jain, Bakhadyr Khoussainov, Wei Li, and Frank Stephan. 2017. Deciding parity games in quasipolynomial time. In *Theory of Computing, STOC 2017*. ACM, 252–263.
- Brian F. Chellas. 1980. *Modal Logic*. Cambridge University Press.
- Corina Cirstea, Clemens Kupke, and Dirk Pattinson. 2009. EXPTIME Tableaux for the Coalgebraic μ -Calculus. In *Computer Science Logic, CSL 2009 (LNCS)*, Vol. 5771. Springer, 179–193.

- Corina Cirstea, Clemens Kupke, and Dirk Pattinson. 2011. EXPTIME Tableaux for the Coalgebraic μ -Calculus. *Log. Meth. Comput. Sci.* 7 (2011).
- Corina Cirstea, Alexander Kurz, Dirk Pattinson, Lutz Schröder, and Yde Venema. 2011. Modal logics are coalgebraic. *Comput. J.* 54 (2011), 31–41.
- Wojciech Czerwinski, Laure Daviaud, Nathanaël Fijalkow, Marcin Jurdzinski, Ranko Lazic, and Pawel Parys. 2019. Universal trees grow inside separating automata: Quasi-polynomial lower bounds for parity games. *SIAM*, 2333–2349.
- Laure Daviaud, Marcin Jurdzinski, and Ranko Lazic. 2018. A pseudo-quasi-polynomial algorithm for mean-payoff parity games. In *Logic in Computer Science, LICS 2018*. ACM, 325–334. <https://doi.org/10.1145/3209108>
- Anuj Dawar and Erich Grädel. 2008. The Descriptive Complexity of Parity Games. In *Computer Science Logic, CSL 2008 (LNCS)*, Vol. 5213. Springer, 354–368.
- E. Allen Emerson, Charanjit Jutla, and A. Prasad Sistla. 2001. On model checking for the μ -calculus and its fragments. *Theor. Comput. Sci.* 258 (2001), 491–522.
- Sebastian Enqvist, Fatemeh Seifan, and Yde Venema. 2015. Monadic Second-Order Logic and Bisimulation Invariance for Coalgebras. In *Logic in Computer Science, LICS 2015*. IEEE.
- John Fearnley, Sanjay Jain, Bart de Keijzer, Sven Schewe, Frank Stephan, and Dominik Wojtczak. 2019. An ordered approach to solving parity games in quasi-polynomial time and quasi-linear space. *STTT* 21, 3 (2019), 325–349. <https://doi.org/10.1007/s10009-019-00509-3>
- Erich Grädel, Wolfgang Thomas, and Thomas Wilke (Eds.). 2002. *Automata, Logics, and Infinite Games: A Guide to Current Research*. LNCS, Vol. 2500. Springer.
- Ichiro Hasuo, Shunsuke Shimizu, and Corina Cirstea. 2016. Lattice-theoretic Progress Measures and Coalgebraic Model Checking. In *Principles of Programming Languages, POPL 2016*. ACM, 718–732.
- Daniel Hausmann and Lutz Schröder. 2019a. Game-Based Local Model Checking for the Coalgebraic μ -Calculus. In *Concurrency Theory, CONCUR 2019 (LIPIcs)*. Schloss Dagstuhl – Leibniz-Zentrum für Informatik.
- Daniel Hausmann and Lutz Schröder. 2019b. Optimal Satisfiability Checking for Arithmetic μ -Calculi. In *Foundations of Software Science and Computation Structures, FOSSACS 2019 (LNCS)*, Vol. 11425. Springer, 277–294. <https://doi.org/10.1007/978-3-030-17127-8>
- Marcin Jurdzinski. 2000. Small Progress Measures for Solving Parity Games. In *Symposium on Theoretical Aspects of Computer Science, STACS 2000*, Horst Reichel and Sophie Tison (Eds.). Springer Berlin Heidelberg, Berlin, Heidelberg, 290–301.
- Marcin Jurdzinski and Ranko Lazic. 2017. Succinct progress measures for solving parity games. In *Logic in Computer Science, LICS 2017*. IEEE Computer Society, 1–9. <http://ieeexplore.ieee.org/xpl/mostRecentIssue.jsp?punumber=7999337>
- V. King, O. Kupferman, and M.Y. Vardi. 2001. On the complexity of parity word automata. In *Foundations of Software Science and Computation Structures, FoSSaCS 2001 (LNCS)*, Vol. 2030. Springer, 276–286.
- Dexter Kozen. 1983. Results on the propositional μ -calculus. *Theor. Comput. Sci.* 27 (1983), 333–354.
- Orna Kupferman, Ulrike Sattler, and Moshe Vardi. 2002. The Complexity of the Graded μ -Calculus. In *Automated Deduction, CADE 02 (LNCS)*, Vol. 2392. Springer, 423–437.
- Karoliina Lehtinen. 2018. A modal μ perspective on solving parity games in quasi-polynomial time. In *Logic in Computer Science, LICS 2018*. ACM, 639–648. <https://doi.org/10.1145/3209108>
- Wanwei Liu, Lei Song, Ji Wang, and Lijun Zhang. 2015. A Simple Probabilistic Extension of Modal μ -calculus. In *International Joint Conference on Artificial Intelligence, IJCAI 2015*. AAAI Press, 882–888.
- Damian Niwinski. 1986. On Fixed-Point Clones (Extended Abstract). In *Automata, Languages and Programming, ICALP 1986 (LNCS)*, Vol. 226. Springer, 464–473.
- Rohit Parikh. 1985. The logic of games and its applications. *Ann. Discr. Math.* 24 (1985), 111–140.
- Pawel Parys. 2019. Parity Games: Zielonka’s Algorithm in Quasi-Polynomial Time. In *Mathematical Foundations of Computer Science, MFCS 2019 (LIPIcs)*. Schloss Dagstuhl – Leibniz-Zentrum für Informatik.
- Marc Pauly. 2002. A Modal Logic for Coalitional Power in Games. *J. Logic Comput.* 12 (2002), 149–166.
- David Peleg. 1987. Concurrent dynamic logic. *J. ACM* 34 (1987), 450–479.
- Jan Rutten. 2000. Universal Coalgebra: A Theory of Systems. *Theor. Comput. Sci.* 249 (2000), 3–80.
- Lutz Schröder and Dirk Pattinson. 2009. Strong completeness of coalgebraic modal logics. In *Theoretical Aspects of Computer Science, STACS 09*. Schloss Dagstuhl – Leibniz-Zentrum für Informatik; Dagstuhl, Germany, 673–684.
- Lutz Schröder and Yde Venema. 2018. Completeness of Flat Coalgebraic Fixpoint Logics. *ACM Trans. Comput. Log.* 19, 1 (2018), 4:1–4:34.
- Yde Venema. 2008. Lectures on the modal μ -calculus. Lecture notes, Institute for Logic, Language and Computation, Universiteit van Amsterdam.
- Wieslaw Zielonka. 1998. Infinite Games on Finitely Coloured Graphs with Applications to Automata on Infinite Trees. *Theor. Comput. Sci.* 200, 1-2 (1998), 135–183.

A APPENDIX

A detailed proof of Lemma 3.4:

PROOF. We first show that for all monotone functions $f : \mathcal{P}(U) \rightarrow \mathcal{P}(U)$, we have $\eta f = \overline{\overline{\eta f}}$. We consider just the case where $\eta = \text{LFP}$, where the proof of the case for $\eta = \text{GFP}$ is dual. We first show that $\overline{\overline{\text{GFP } f}}$ is a fixpoint and hence also a prefixpoint of f :

$$\overline{\overline{\text{GFP } f}} = \overline{\overline{f(\text{GFP } f)}} = \overline{\overline{f(\overline{\overline{\text{GFP } f}})}} = f(\overline{\overline{\text{GFP } f}}).$$

Hence we have $\text{LFP}(f) \subseteq \overline{\overline{\text{GFP } f}}$. For the converse direction, let Z be a prefixpoint of f , that is, let $f(Z) \subseteq Z$. We have $\overline{\overline{f(Z)}} = \overline{\overline{f(Z)}} \supseteq \overline{\overline{Z}}$, that is, $\overline{\overline{Z}}$ is a postfixpoint of $\overline{\overline{f}}$ so that $\overline{\overline{Z}} \subseteq \text{GFP } \overline{\overline{f}}$ which implies $\overline{\overline{\text{GFP } f}} \subseteq Z$. Thus we have shown $\overline{\overline{\text{GFP } f}} \subseteq \text{LFP } f$.

The proof of the Lemma then is by induction on k . For $k = 0$, we have $E_0^\alpha(X_1, \dots, X_k) = \alpha(X_1, \dots, X_k)$ and $\overline{\overline{A_0^\alpha(\overline{X_1}, \dots, \overline{X_k})}} = \overline{\overline{\alpha(\overline{X_1}, \dots, \overline{X_k})}} = \alpha(X_1, \dots, X_k)$, where the last equality is by definition of $\overline{\overline{\alpha}}$. For $k > 0$, we have

$$E_k^\alpha(X_{i+1}, \dots, X_k) = \eta_i X_i. \eta_{i-1} X_{i-1}. \dots. \eta_1 X_1. \alpha(X_1, \dots, X_i, X_{i+1}, \dots, X_k) = \eta_i f$$

where $f(X_i) = \eta_{i-1} X_{i-1}. \dots. \eta_1 X_1. \alpha(X_1, \dots, X_i, X_{i+1}, \dots, X_k)$ for $X_i \subseteq U$. As we have shown above, we have $\eta_i f = \overline{\overline{\eta_i f}}$ so that we are done if

$$\overline{\overline{\eta_i f}} = \overline{\overline{\eta_i X_i. \eta_{i-1} X_{i-1}. \dots. \eta_1 X_1. \overline{\overline{\alpha(X_1, \dots, X_i, \overline{X_{i+1}}, \dots, \overline{X_k})}}}} = \overline{\overline{A_k^\alpha(\overline{X_{i+1}}, \dots, \overline{X_k})}}$$

which is the case if $\overline{\overline{f(X_i)}} = \overline{\overline{\eta_{i-1} X_{i-1}. \dots. \eta_1 X_1. \overline{\overline{\alpha(X_1, \dots, X_i, \overline{X_{i+1}}, \dots, \overline{X_k})}}}} for $X_i \subseteq U$. We have $\overline{\overline{f(X_i)}} = \overline{\overline{f(\overline{X_i})}} = \overline{\overline{\eta_{i-1} X_{i-1}. \dots. \eta_1 X_1. \alpha(X_1, \dots, X_{i-1}, \overline{X_i}, X_{i+1}, \dots, X_k)}} = E_{i-1}^\beta(X_i, \dots, X_k)$, where we define $\beta(X_1, \dots, X_k) = \alpha(X_1, \dots, X_{i-1}, \overline{X_i}, X_{i+1}, \dots, X_k)$. Since we have $\overline{\overline{A_{i-1}^\beta(\overline{X_i}, \dots, \overline{X_k})}} = \overline{\overline{\eta_{i-1} X_{i-1}. \dots. \eta_1 X_1. \overline{\overline{\alpha(X_1, \dots, X_i, \overline{X_{i+1}}, \dots, \overline{X_k})}}}}$, the inductive hypothesis finishes the case. $\square$$