

## Closed Asymptotic Couples

Matthias Aschenbrenner and Lou van den Dries<sup>1</sup>

*Department of Mathematics, University of Illinois at Urbana-Champaign,  
Urbana, Illinois 61801*

E-mail: [maschenb@math.uiuc.edu](mailto:maschenb@math.uiuc.edu); [vddries@math.uiuc.edu](mailto:vddries@math.uiuc.edu)

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The derivation of a Hardy field induces on its value group a certain function  $\psi$ . If a Hardy field extends the real field and is closed under powers, then its value group is also a vector space over  $\mathbb{R}$ . Such “ordered vector spaces with  $\psi$ -function” are called *H-couples*. We define *closed H-couples* and show that every *H-couple* can be embedded into a closed one. The key fact is that closed *H-couples* have an elimination theory: solvability of an arbitrary system of equations and inequalities (built up from vector space operations, the function  $\psi$ , parameters, and the unknowns to be solved for) is equivalent to an effective condition on the parameters of the system. The *H-couple* of a maximal Hardy field is closed, and this is also the case for the *H-couple* of the field of logarithmic-exponential series over  $\mathbb{R}$ . We analyze in detail finitely generated extensions of a given *H-couple*. © 2000 Academic Press

### INTRODUCTION

We describe here roughly the main result of the paper, and explain for non-experts the role of model theory in its conception. Precise formulations follow in Section 1, and Sections 2–4 contain the proof of the main result.

We begin with motivating our subject via Hardy fields, and assume some familiarity with its basic theory as developed by Bourbaki [3] and Rosenlicht [14, 15]. This theory is the modern incarnation of ideas on “orders of infinity” originating with Du Bois-Reymond [2] and put on a firm basis by Hardy [6]. Hardy fields are ordered differential fields of germs at  $+\infty$  of real valued differentiable functions defined on half lines  $(a, +\infty)$

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with  $a \in \mathbb{R}$ . A Hardy field  $F$  has valuation ring

$$\mathcal{O}(F) := \{f \in F : |f| \leq r \text{ for some real number } r\}$$

with associated valuation  $v: F^\times \rightarrow V = v(F^\times)$ . This valuation measures the growth of functions at infinity: given  $f, g \in F^\times$  we have

$$v(f) > v(g) \iff \lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 0.$$

A key fact [14] is that  $v(f')$  only depends on  $v(f)$  for  $v(f) \neq 0$ ,  $f \in F^\times$ . Thus the value group  $V$  comes equipped with a natural extra operation  $\psi: V \setminus \{0\} \rightarrow V$  given by

$$\psi(v(f)) := v(f'/f) = v(f') - v(f) \quad \text{for } v(f) \neq 0, f \in F^\times.$$

The pair  $(V, \psi)$  is an “asymptotic couple of Hardy type” in the sense of Rosenlicht [13], who studied especially the situation where the abelian group  $V$  has finite rank; in that case  $\psi$  takes only finitely many different values. We focus on the opposite situation where  $(V, \psi)$  is large in a certain sense. In addition we include a scalar multiplication  $\mathbb{R} \times V \rightarrow V$  among the basic operations. Here is why.

Suppose our Hardy field  $F$  extends  $\mathbb{R}(x)$ , and is closed under powers, that is,  $0 < f \in F \implies f^r \in F$  for all  $r \in \mathbb{R}$ . (All maximal Hardy fields have these properties; see [3] or [14].) Then  $V$  becomes an ordered vector space over  $\mathbb{R}$  by setting  $r \cdot v(f) := v(f^r)$  for  $0 < f \in F$ . Consider the two-sorted structure consisting of the ordered field  $\mathbb{R}$  (first sort), the ordered abelian group  $V$  equipped with the function  $\psi: V \setminus \{0\} \rightarrow V$  as above (second sort), with the scalar multiplication  $\mathbb{R} \times V \rightarrow V$  relating them. This two-sorted structure is completely determined by the structure of  $F$  as ordered differential field:  $\mathbb{R}$  is the field of constants of  $F$ ; the valuation ring, and hence the valuation, is defined in terms of  $\mathbb{R}$  and the ordering as above; the scalar multiplication is then given by  $r \cdot v(f) = v(g)$  whenever  $0 < f, g \in F$  and  $rf'/f = g'/g$ . (The presence of this scalar multiplication is a contrast to the situation with henselian valued fields of equicharacteristic 0, where no “definable interaction” between residue field and value group can exist.)

The “asymptotic couples with scalar multiplication” associated to Hardy fields  $F$  as above belong to a certain elementary class, the class of  $H$ -couples (the “ $H$ ” of Hardy and Hahn). If  $F$  is a maximal Hardy field, its associated  $H$ -couple  $(V, \psi)$  is even *closed*, which implies that the set  $\Psi := \psi(V \setminus \{0\})$  is closed downward in  $V$ . (Precise definitions are given in Section 1.)

Our ultimate aim is to develop a model theory for ordered differential fields such as maximal Hardy fields. At the most basic level this requires

these differential fields to have a common elimination theory for algebraic differential equations and inequalities. We do not yet know if such an elimination theory exists, but our main theorem goes in that direction: it says that the class of closed  $H$ -couples with real closed scalar field has an elimination theory.

Roughly speaking, this means the following. Let  $S$  be any finite system of equations and inequalities built up from symbols for the vector space operations and the function  $\psi$  and from variables, some ranging over scalars and the others over vectors; in addition, some variables are considered as parameters, and the others as the unknowns to be solved for. Let the parameters of the system be given values in a closed  $H$ -couple  $(V, \psi)$  with real closed scalar field. Then the solvability of  $S$  in  $(V, \psi)$  is shown to be equivalent to the *parameters* satisfying a certain finite system  $S'$  of equations and inequalities (in which the unknowns do not occur any longer, they have been *eliminated*). Moreover,  $S'$  only depends on  $S$ , not on  $(V, \psi)$  or the particular values of the parameters. However, this is only true if among the “inequalities” in  $S'$  we allow conditions of the form  $t \in \Psi$  and  $t \notin \Psi$ , with  $\Psi = \psi(V \setminus \{0\})$ . (Such “inequalities” are also allowed in  $S$ .) That is why we deal with  $H$ -triples, not just  $H$ -couples. (A closed  $H$ -couple  $(V, \psi)$  gives rise to the closed  $H$ -triple  $(V, \psi, \Psi)$ .)

One can express this more concisely (and accurately!) using logical terminology where  $S$  and  $S'$  become formulas in a certain language. Relevant here are the notions of *quantifier elimination* (Tarski) and *model completion* (Robinson), which clarify the significance of “having an elimination theory.” For these matters we refer to the first half of [17] (or corresponding parts of other standard texts in model theory, like [9]). Indeed, by model-theoretic generalities the class of closed  $H$ -triples with real closed scalar field has an elimination theory as indicated above if and only if any embedding of a *substructure* of a closed  $H$ -triple  $(V, \psi, \Psi)$  with real closed scalar field into a “sufficiently saturated” closed  $H$ -triple  $(V', \psi', \Psi')$  with real closed scalar field extends to an embedding of  $(V, \psi, \Psi)$  into  $(V', \psi', \Psi')$ .

Thus, rather than directly constructing an elimination theory, we obtain its existence by proving in Section 4 an embedding theorem. The first four sections are mostly algebraic, with model theory as our guide. In Section 5 we address issues of a more intrinsic nature, both algebraically and from the point of view of model theory.

We hope the sketch above is helpful to readers not familiar with the model-theoretic background, which from now on will be assumed. In particular, “ $\subseteq$ ” will be used for the substructure relation as defined in model theory.

## 1. DEFINITIONS AND RESULTS

We now formally introduce the objects studied in this paper.

*Notation.* We put  $S^{>a} := \{s \in S : s > a\}$  for an element  $a$  of a linearly ordered set  $S$ ; similarly for “ $\geq$ ,” “ $<$ ,” or “ $\leq$ ” instead of “ $>$ .”

Recall that an ordered vector space over an ordered field  $k$  is a vector space  $V$  over  $k$  equipped with a linear ordering such that if  $0 < v, w \in V$  and  $0 < \lambda \in k$ , then  $0 < v + w$  and  $0 < \lambda v$ . We then define an equivalence relation on  $V$  by

$$v \sim w : \Longleftrightarrow \exists \lambda \in k^{>1} : \frac{1}{\lambda}|v| \leq |w| \leq \lambda|v|.$$

The equivalence class of  $v \in V$  is written as  $[v]$  (or  $[v]_k$ , if  $k$  is not clear from context), and is called its  $k$ -archimedean class. We let  $[V]$  (or  $[V]_k$ ) be the set of  $k$ -archimedean classes, and linearly order  $[V]$  by

$$\begin{aligned} [v] < [w] : \Longleftrightarrow \forall \lambda \in k^{>0} : \lambda|v| < |w| \\ \Longleftrightarrow [v] \neq [w] \text{ and } |v| < |w|. \end{aligned}$$

Thus  $[0] = \{0\}$  is the smallest  $k$ -archimedean class. For ease of notation we put  $V^* := V \setminus \{0\}$  and  $[V^*] := \{[v] : v \in V^*\}$ .

DEFINITION 1.1. A *Hahn space* is an ordered vector space  $V$  over an ordered field  $k$  such that, for all vectors  $v, w \in V^*$ ,

$$[v] = [w] \implies \exists \lambda \in k : [v - \lambda w] < [w].$$

It is easy to see that any ordered vector space over  $\mathbb{R}$  is a Hahn space. We have chosen the term “Hahn space” since these spaces satisfy an analogue of the Hahn embedding theorem; see Section 2. There we also establish the good behavior of Hahn spaces under scalar extension.

DEFINITION 1.2. An  $H$ -couple  $\mathcal{V} = (V, \psi)$  consists of a Hahn space  $V$  over an ordered field  $k$ , a distinguished positive element  $1 \in V$ , and a function  $\psi : V^* \rightarrow V$  such that, for all  $v, w \in V^*$ ,

- (1)  $\psi(1) = 1$ ,
- (2)  $\psi(v) \leq \psi(w) \iff [v] \geq [w]$  (hence  $\psi(v) = \psi(w) \iff [v] = [w]$ ),
- (3)  $\psi(v) < \psi(w) + |w|$ .

We refer to  $\mathcal{V}$  as an “ $H$ -couple over  $k$ ” if we want to specify the scalar field  $k$ .

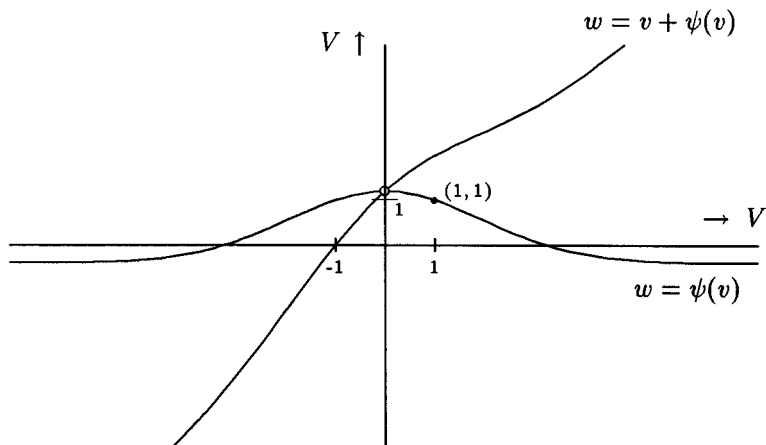
FIG. 1.  $\psi$  and  $\text{id} + \psi$  on  $V^*$ .

Figure 1 shows the qualitative behavior of the functions  $\psi$  and  $\text{id} + \psi$  on  $V^*$ . (In Section 3 we will see that  $\text{id} + \psi$  is strictly increasing.) The picture is quite rough: it cannot show that  $\psi$  is constant on  $k$ -archimedean classes. But it has been a precious guide in our work.

EXAMPLE 1.1. To every Hardy field  $F \supseteq \mathbb{R}(x)$  closed under powers we associate the corresponding  $H$ -couple  $\mathcal{V} = (V, \psi)$  over  $\mathbb{R}$ , as indicated in the Introduction, with  $1 := v(x^{-1})$ . That we actually obtain an  $H$ -couple in this way is clear from the remarks made above and the results given in [12].

EXAMPLE 1.2. Let  $k$  be a logarithmic-exponential ordered field, and let  $F$  be a differential subfield of  $k((t))^{\text{LE}}$  containing  $k(x)$  and closed under powers; that is, if  $0 < f \in F$  and  $r \in k$ , then  $f^r \in F$ . (See [5] for the construction of  $k((t))^{\text{LE}}$ , the field of logarithmic-exponential series over  $k$ .) Let  $v$  be the valuation on  $F$  with valuation ring  $\{f \in F : |f| \leq r \text{ for some } r \in k\}$ , and associate to  $F$  an  $H$ -couple just as we did for the Hardy fields above, with  $1 := v(t) = v(x^{-1})$ . (In Section 3 we show this gives indeed an  $H$ -couple over  $k$ .)

When dealing with  $H$ -couples  $\mathcal{V} = (V, \psi)$  as model-theoretic objects we construe them as  $\mathcal{L}_H$ -structures, where  $\mathcal{L}_H$  is the two-sorted language with

- (1) scalar variables ranging over the extended scalar field  $k_\infty := k \cup \{\infty\}$ ;
- (2) vector variables ranging over the extended vector space  $V_\infty := V \cup \{\infty\}$ ;

and with the following non-logical symbols:

(3)  $<, 0, 1, +, -, \cdot$ , interpreted as usual in the ordered field  $k$  of scalars, with  $\infty$  serving as a default value: the linear ordering on  $k$  is extended to a linear ordering on  $k_\infty$  by setting  $\lambda < \infty$  for all  $\lambda \in k$ , and  $\lambda * \mu := \infty$  for  $*$   $\in \{+, -, \cdot\}$  and all  $\lambda, \mu \in k_\infty$  with  $\lambda = \infty$  or  $\mu = \infty$ ;

(4)  $<, 0, 1, +, -, \psi$ , interpreted in the obvious way in  $V$  and with  $\infty$  serving as a default value: the linear ordering on  $V$  is extended to a linear order on  $V_\infty$  by setting  $a < \infty$  for all  $a \in V$ , and  $a + \infty = \infty + a = a - \infty = \infty - a = \infty$  for all  $a \in V_\infty$ , and  $\psi(0) = \psi(\infty) = \infty$ ;

(5) a symbol  $\cdot$  for the map  $k_\infty \times V_\infty \rightarrow V_\infty$  that is the scalar multiplication on  $k \times V$  and with  $\lambda \cdot v = \infty$  for all  $(\lambda, v) \in (k_\infty \times V_\infty) \setminus (k \times V)$ ;

(6) a symbol  $:$  for the function  $V_\infty^2 \rightarrow k_\infty$  that assigns to each  $(a, b) \in V^2$  with  $[a] \leq [b]$  and  $b \neq 0$  the unique scalar  $a : b = \lambda \in k$  such that  $[a - \lambda b] < [b]$ , and that assigns to all other pairs  $(a, b) \in V_\infty^2$  the default value  $a : b = \infty$ .

*Remarks.* 1. Despite overlap in how we write the symbols of (3), (4), and (5), we actually distinguish them: for example, the symbol  $+$  in (3) is to be regarded as different from the symbol  $+$  in (4). Similarly, the element  $\infty \in k_\infty$  is to be distinguished from the element  $\infty \in V_\infty$ .

2. The default values  $\infty$  are included to make all basic operations totally defined, so that no ambiguities arise in the interpretation of terms.

It is easy to see that the  $H$ -couples in the model-theoretic sense are exactly the models of a *universal* theory in the language  $\mathcal{L}_H$ . Thus each substructure of an  $H$ -couple is also an  $H$ -couple, with possibly smaller scalar field. (That is why we included the division operation of (6).) We will keep writing  $H$ -couples as  $(V, \psi)$ , and so on, even when we regard them as  $\mathcal{L}_H$ -structures.

Let  $(V, \psi)$  be an  $H$ -couple. Then clearly  $\psi(v) < w + \psi(w)$  for all  $v, w \in V^{>0}$ . Thus  $(V, \psi)$  has an “ $H$ -cut” in the following sense.

DEFINITION 1.3. An  $H$ -cut of  $(V, \psi)$  is a set  $P \subseteq V$  such that:

1. For all  $a, b \in V$ , if  $a < b \in P$ , then  $a \in P$ .
2.  $\psi(v) \in P$  and  $w + \psi(w) \notin P$  for all  $v, w \in V^{>0}$ .

We then also call  $(V, \psi, P)$  an  $H$ -triple, and we regard  $(V, \psi, P)$  as an  $\mathcal{L}_{H,P}$ -structure, where  $\mathcal{L}_{H,P}$  extends the language  $\mathcal{L}_H$  by an extra unary predicate  $P$ , to be interpreted by the set  $P \subseteq V$ . Clearly the  $H$ -triples are then exactly the models of a universal theory in the language  $\mathcal{L}_{H,P}$ .

DEFINITION 1.4. The  $H$ -couple  $(V, \psi)$  is *closed* if  $\psi(V^*)$  has no largest element, and

$$\psi(V^*) = \{a \in V : a < w + \psi(w) \text{ for all } w \in V^{>0}\}.$$

In that case  $\Psi := \psi(V^*)$  is clearly the only  $H$ -cut of  $(V, \psi)$ ; we call  $(V, \psi, \Psi)$  a *closed  $H$ -triple*.

In Section 3 we prove that each  $H$ -triple can be embedded (as  $\mathcal{L}_{H,P}$ -structure) into a closed  $H$ -triple with the same scalar field. We also show there that the  $H$ -couples associated to maximal Hardy fields and to the ordered differential field  $\mathbb{R}((t))^{\text{LE}}$  are closed.

We can now state the main result of this paper, to be established in Section 4.

**THEOREM 1.1.** *The theory of closed  $H$ -triples over real closed scalar fields is complete, eliminates quantifiers, and is the model-completion of the theory of  $H$ -triples.*

We actually obtain a relative quantifier elimination where the scalar field is not assumed to be real closed.

In Section 5 we show that no new structure is induced on the scalar field in closed  $H$ -triples and that the underlying ordered vector space in such triples is locally o-minimal. We also determine there the definable closure of a substructure in a closed  $H$ -triple, and study simple extensions of  $H$ -triples.

In Section 6 we indicate a variant of the results above, where there is no scalar field. Here we have a model-completion that is even weakly o-minimal.

### *Remarks on 1 and $P$*

The role of the distinguished positive element 1 with  $\psi(1) = 1$  is to give a convenient normalization. This role is hardly essential, but does affect issues like completeness as stated in the theorem above. To clarify this point further, consider an “ $H$ -couple without 1,” that is, a couple  $(V, \psi)$  consisting of a non-trivial Hahn space  $V$  over an ordered field  $k$  and a function  $\psi: V^* \rightarrow V$  satisfying axioms (2) and (3) for  $H$ -couples. (We do not distinguish a positive element 1 and omit the axiom  $\psi(1) = 1$ .) Then for each vector  $a \in V$  the translate  $(V, a + \psi)$  is clearly also an  $H$ -couple without 1. Choose any vector  $b \in V^{>0}$ , and put  $a := b - \psi(b)$ . Then  $a + \psi(b) = b$ , so by taking  $b$  as our distinguished positive element 1 we make  $(V, a + \psi)$  into an  $H$ -couple over  $k$ .

Similarly, without the predicate  $P$  for an  $H$ -cut we would not have quantifier elimination: using results from Section 3, it is easy to construct closed  $H$ -couples  $(V_1, \psi_1)$ ,  $(V_2, \psi_2)$  over  $\mathbb{R}$ , with common substructure  $(V, \psi)$  containing a vector  $v$ , such that  $v \in \psi_1(V_1^*)$ , but  $v \notin \psi_2(V_2^*)$ .

### Notational Conventions

Let  $(S, <)$  be a linearly ordered set. When  $a \leq b$  in  $S$  and  $(S, <)$  is clear from context we use the following notation:

$$(a, b) := \{x \in S : a < x < b\},$$

$$[a, b] := \{x \in S : a \leq x \leq b\},$$

$$(-\infty, b] := \{x \in S : x \leq b\},$$

$$[a, \infty) := \{x \in S : x \geq a\}.$$

A set  $X \subseteq S$  is called *convex* (in  $S$ ) if  $[a, b] \subseteq X$  for all  $a, b \in X$  with  $a < b$ . For  $X \subseteq S$  and  $a \in X$ , the *convex component of  $a$  in  $X$*  is the (convex) set

$$\{x \in X^{\leq a} : [x, a] \subseteq X\} \cup \{x \in X^{\geq a} : [a, x] \subseteq X\}.$$

(This also depends on  $S$ .) The *convex components of  $X$*  are by definition the convex components of the members of  $X$  in  $X$ . They form a partition of  $X$ . The *convex hull of  $X$*  (in  $S$ ) is the smallest convex subset of  $S$  containing  $X$ . We call a subset  $X$  of  $S$  *closed upward* (in  $S$ ) if  $a \in S$ ,  $a > b \in X$  implies  $a \in X$ , and *closed downward* (in  $S$ ), or a *cut in  $S$* , if  $a \in S$ ,  $a < b \in X$  implies  $a \in X$ . An element  $a$  in an ordered extension of  $(S, <)$  is said to *realize the cut  $X$  in  $S$*  if  $X < a < S \setminus X$ .

Throughout this paper, we let  $m, n$  range over the set  $\mathbb{N} = \{0, 1, 2, \dots\}$  of natural numbers. If  $G$  is an ordered abelian group and  $g \in G$ , we set  $|g| := \max\{-g, g\}$ , and let  $\text{sgn}(g) := -1$  if  $g < 0$ ,  $\text{sgn}(0) := 0$ , and  $\text{sgn}(g) := 1$  if  $g > 0$ .

## 2. HAHN SPACES

The notion of a Hahn space has been already defined in Section 1; see Definition 1.1. In this section we study embedding and scalar extension properties of Hahn spaces. We also include a very useful lemma about functions on ordered abelian groups. Throughout this section, let  $k$  be an ordered field and  $V$  an ordered vector space over  $k$ .

*Properties of  $k$ -Archimedean Classes.* Let  $v, w \in V$  and  $\lambda \in k^\times$ . Then:

- (1)  $[v] = \{0\} \iff v = 0$ ,
- (2)  $[v] = [\lambda v]$ ,
- (3)  $[v + w] \leq \max\{[v], [w]\}$ ,
- (4)  $[v + w] = \max\{[v], [w]\}$ , if  $[v] \neq [w]$ .

We say that  $V$  is  *$k$ -archimedean* if  $[V^*]$  is a singleton. For example,  $k$  as an ordered vector space over itself is  $k$ -archimedean.



For  $\gamma \in [V^*]$ , we define the convex linear subspaces

$$V_{(\gamma)} := \{v \in V : [v] < \gamma\}, \quad V^{(\gamma)} := \{v \in V : [v] \leq \gamma\}.$$

Note that the ordered vector space  $V^{(\gamma)}/V_{(\gamma)}$  is  $k$ -archimedean.

*Remarks.* The following facts are easy consequences of the definitions:

1.  $V$  is a Hahn space if and only if all the vector spaces  $V^{(\gamma)}/V_{(\gamma)}$  have dimension 1.
2. Any linear subspace of a Hahn space over  $k$  is itself a Hahn space over  $k$  with respect to the induced ordering.
3. Any ordered vector space over the field  $\mathbb{R}$  of real numbers is a Hahn space.
4.  $\mathbb{R}$  as an ordered vector space over  $\mathbb{Q}$  is not a Hahn space.

### Hahn Products

Let  $\Gamma$  be a totally ordered set and  $(V_\gamma)_{\gamma \in \Gamma}$  a system of ordered vector spaces over  $k$ . For each element  $v = (v_\gamma)_{\gamma \in \Gamma}$  of the vector space  $\prod_{\gamma \in \Gamma} V_\gamma$ , we let

$$\text{supp } v := \{\gamma \in \Gamma : v_\gamma \neq 0\}$$

denote the *support* of  $v$ . The subset  $H(\Gamma, (V_\gamma)_{\gamma \in \Gamma})$  of  $\prod_{\gamma \in \Gamma} V_\gamma$  consisting of those elements with anti-well-ordered support is a  $k$ -linear subspace of  $\prod_{\gamma \in \Gamma} V_\gamma$ . It becomes an ordered vector space over  $k$  by setting, for non-zero  $v \in H(\Gamma, (V_\gamma)_{\gamma \in \Gamma})$ ,

$$0 < v: \iff 0 < v_{\mu(v)},$$

where  $\mu(v) := \max(\text{supp } v)$ . We call  $H(\Gamma, (V_\gamma)_{\gamma \in \Gamma})$  the *Hahn product* of  $(V_\gamma)_{\gamma \in \Gamma}$ . Note that if all  $V_\gamma$  are Hahn spaces over  $k$ , then  $H(\Gamma, (V_\gamma)_{\gamma \in \Gamma})$  is a Hahn space over  $k$ . If all  $V_\gamma$  are equal to  $V$ , we also write  $H(\Gamma, V)$ . If  $V$  is  $k$ -archimedean and we put  $H := H(\Gamma, V)$ , then we have a well-defined map

$$[v] \mapsto \max(\text{supp } v): [H^*] \rightarrow \Gamma \quad (v \in H^*),$$

and this map is an isomorphism of linearly ordered sets.

### Hahn Embedding Theorem

Let  $V'$  be an ordered vector space over an ordered field extension  $k'$  of  $k$ . Then by an *embedding*  $V \rightarrow V'$  we mean an injective order preserving  $k$ -linear map  $V \rightarrow V'$ . Such an embedding  $i: V \rightarrow V'$  induces a map  $[v] \mapsto [i(v)]: [V] \rightarrow [V']$  from the set of  $k$ -archimedean classes of  $V$  into the set of  $k'$ -archimedean classes of  $V'$ . This induced map is clearly injective if  $k = k'$ .

PROPOSITION 2.1. *Let  $\Gamma := [V^*]$ . Then there exists an embedding*

$$V \rightarrow H := H(\Gamma, (V^{(\gamma)}/V_{(\gamma)}))$$

*of ordered vector spaces over  $k$  with bijective induced map  $[V] \rightarrow [H]$ .*

*Proof.* The proof is an easy adaptation of Banaschewski's proof [1] of the Hahn embedding theorem as presented in [10, pp. 16–17]. ■

**COROLLARY 2.1 (Hahn Embedding Theorem for Hahn Spaces).** *If  $V$  is a Hahn space, then there exists an embedding  $V \rightarrow H := H(\Gamma, k)$ , where  $\Gamma := [V^*]$ , with bijective induced map  $[V] \rightarrow [H]$ .*

Thus up to isomorphism of ordered vector spaces over  $k$  the Hahn spaces over  $k$  are exactly the ordered linear subspaces of Hahn products  $H(\Gamma, k)$  for linearly ordered sets  $\Gamma$ .

### Scalar Extension

Given a field extension  $k' \supseteq k$ , let  $V_{k'}$  be  $V \otimes_k k'$  viewed as a vector space over  $k'$  in the usual way. We consider  $V$  as a  $k$ -linear subspace of  $V_{k'}$  by identifying  $v \in V$  with  $v \otimes 1 \in V_{k'}$ . The following fact will be used several times:

**LEMMA 2.1.** *Let  $V$  be a Hahn space and  $k' \supseteq k$  a field extension. Then every non-zero vector  $u \in V_{k'}$  can be written as*

$$u = \sum_{i=1}^m \lambda_i u_i \quad \text{with scalars } \lambda_i \in k' \text{ and vectors } u_i \in V^{>0}, \quad (1)$$

such that  $[u_1] > [u_2] > \cdots > [u_m]$  in  $[V^*]$ .

*Proof.* Let  $0 \neq u = \sum_{j=1}^n \mu_j v_j$  ( $\mu_j \in k'$ ,  $v_j \in V$ ). One then shows by induction on  $n$  that  $u$  can be rewritten as in (1), with  $[u_1] \leq \max\{[v_j] : j = 1, \dots, n\}$ . ■

**PROPOSITION 2.2.** *Let  $V$  be a Hahn space, and let  $k \subseteq k'$  be an extension of ordered fields. Then there is a unique linear ordering on  $V_{k'}$  extending the ordering of  $V$ , making  $V_{k'}$  into an ordered vector space over  $k'$ , such that the inclusion  $V \hookrightarrow V_{k'}$  is an embedding with injective induced map  $[V] \rightarrow [V_{k'}]$ .*

*Proof.* Assume that we are given such an ordering on  $V_{k'}$ . We can write each non-zero vector  $u \in V_{k'}$  as in (1), with  $\lambda_1 \neq 0$ . Then  $u > 0$  if and only if  $\lambda_1 > 0$ . This shows uniqueness. For existence, by Corollary 2.1 above, we have an embedding  $V \rightarrow H(\Gamma, k)$  of ordered vector spaces over  $k$ , where  $\Gamma := [V^*]_k$ . Tensoring with  $k'$  gives a  $k'$ -linear injective map  $V_{k'} \rightarrow H(\Gamma, k')$ . This induces an ordering on  $V_{k'}$ , making it into an ordered vector space over  $k'$  with the desired properties, as one easily verifies. ■

*Remarks.* Under the hypothesis of the proposition above we will consider  $V_{k'}$  as being equipped with the unique linear ordering of the proposition. Note then that  $V_{k'}$  is a Hahn space over  $k'$  and that the map  $[V] \rightarrow [V_{k'}]$  induced by the embedding  $V \hookrightarrow V_{k'}$  is a bijection.

**COROLLARY 2.2 (Universal Property).** *Let  $V$  be a Hahn space, and let  $\mathbf{k} \subseteq \mathbf{k}'$  be an extension of ordered fields. Any embedding  $V \rightarrow V'$  into an ordered vector space  $V'$  over  $\mathbf{k}'$  with injective induced map  $[V] \rightarrow [V']$  extends uniquely to an embedding  $V_{\mathbf{k}'} \rightarrow V'$ .*

### *A Lemma about Functions on Ordered Abelian Groups*

We shall say that a function  $f: X \rightarrow Y$  between linearly ordered sets  $X$  and  $Y$  has the *intermediate value property* if for all  $x_1 < x_2$  in  $X$  and all  $y \in Y$  with  $f(x_1) < y < f(x_2)$  or  $f(x_2) < y < f(x_1)$  there is  $x \in X$  such that  $x_1 < x < x_2$  and  $f(x) = y$ .

Let  $G$  be an ordered abelian group, and for  $a, b \in G$  write  $a = o(b)$  to indicate that  $n|a| \leq |b|$  for each positive integer  $n$ .

**LEMMA 2.2.** *Let  $C \subseteq G$  be a convex subset, and assume that the function  $\eta: C \rightarrow G$  has the following properties:*

- (1)  $\eta(x) - \eta(y) = o(x - y)$  for all distinct  $x, y \in C$ .
- (2)  $\eta(y) = \eta(z)$  whenever  $x, y, z \in C$  with  $x < y < z$  and  $z - y = o(z - x)$ .

*Then the function  $x \mapsto x + \eta(x): C \rightarrow G$  is strictly increasing and has the intermediate value property.*

*Proof.* That  $x + \eta(x)$  is strictly increasing is an easy consequence of (1). To prove the intermediate value property, let  $a, b \in C$  with  $a < b$ . Let  $c := b - a$  and define  $\eta_1: [0, c] \rightarrow G$  by  $\eta_1(x) := \eta(a + x) - \eta(a)$ . Then properties (1) and (2) remain valid if  $C$  is replaced by  $[0, c]$  and  $\eta$  by  $\eta_1$ , and it suffices to prove the intermediate value property for the corresponding function  $x \mapsto x + \eta_1(x)$ . So we can assume  $C = [0, c]$  and  $\eta(0) = 0$ . Let  $0 < v < c + \eta(c)$ . It suffices to find  $u \in (0, c)$  such that  $u + \eta(u) = v$ . We distinguish two cases:

If  $c - v = o(c)$ , then we put  $u := v - \eta(c)$ , so  $0 < u < c$ . By (1), we have  $\eta(c) = o(c)$ , and hence  $c - u = (c - v) + (v - u) = (c - v) + \eta(c) = o(c)$ . Therefore, by (2), we have  $\eta(u) = \eta(c)$ , that is,  $u + \eta(u) = v$ .

If  $c - v \neq o(c)$ , then since  $v < c + \eta(c)$  and  $\eta(c) = o(c)$  by (1), we get  $0 < v < c$ . Put  $u := v - \eta(v)$ . Since  $\eta(v) = o(v)$  by (1), we have  $0 < u, v < c$  and  $v - u = o(v)$ , and hence  $\eta(v) = \eta(u)$  by (2), that is,  $u + \eta(u) = v$ , as required. ■

*Remark.* Lemma 2.2 remains of course valid when (2) is replaced by

- (2')  $\eta(y) = \eta(z)$  whenever  $x, y, z \in C$  with  $x > y > z$  and  $z - y = o(z - x)$ .

### 3. $H$ -COUPLES: EXAMPLES AND EMBEDDING PROPERTIES

We refer to Section 1 for various notions concerning  $H$ -couples. In this section  $(V, \psi)$  is an  $H$ -couple over the ordered field  $k$ ,  $\Psi := \psi(V^*)$ , and  $P$  is an  $H$ -cut of  $(V, \psi)$ . So  $(V, \psi, P)$  is an  $H$ -triple over  $k$ .

*Basic properties of  $\psi$ .* (See also [13].)

(1) The map  $v \mapsto \psi(v): V \rightarrow V_\infty$  (with  $\psi(0) = \infty > V$ ) is a valuation on the ordered group  $V$ , that is,  $\psi(v + w) \geq \min\{\psi(v), \psi(w)\}$  for  $v, w \in V$ .

(2)  $\psi(v - w) > \min\{\psi(v), \psi(w)\}$  for all  $v, w \in P$ . In particular,  $\psi(\psi(v) - \psi(w)) > \min\{\psi(v), \psi(w)\}$  for all  $v, w \in V^*$ .

(3)  $[\psi(v) - \psi(w)] < [v - w]$  for  $v, w \in V^*$ ,  $v \neq w$ .

(4) The map  $v \mapsto v + \psi(v): V^* \rightarrow V$  is strictly increasing.

*Proof.* Property (1) follows easily from axiom (2) about  $H$ -couples. For (2), let  $v, w \in P$ ,  $v < w$ . Then  $\psi(w - v) + (w - v) > w$ , and hence  $\psi(v - w) > v$ . Property (3) follows from (1), (2), and axiom (2). Property (4) is now an immediate consequence of (3). ■

Note that Lemma 2.2 and property (3) imply the intermediate value property for the function  $x \mapsto x + \psi(x)$  on  $V^{>0}$ , and also the intermediate value property for  $x \mapsto x + \psi(x)$  on  $V^{<0}$ . A consequence of this is:

LEMMA 3.1. *The set*

$$(\text{id} + \psi)(V^{>0}) = \{x + \psi(x) : x \in V^{>0}\}$$

*is closed upward. The set*

$$(-\text{id} + \psi)(V^{>0}) = \{-x + \psi(x) : x \in V^{>0}\}$$

*is closed downward. Moreover,*

$$(-\text{id} + \psi)(V^{>0}) = (\text{id} + \psi)(V^{<0}) = \{a \in V : a < b \text{ for some } b \in \Psi\}.$$

*Proof.* Let  $a > 1$  in  $V$ . Then  $|\psi(a) - 1| = |\psi(a) - \psi(1)| < a - 1$  by basic property (3), and hence  $|\psi(a)| < a$ . Thus  $2a + \psi(2a) = 2a + \psi(a) > a$ , showing that  $\text{id} + \psi$  takes arbitrarily large values on  $V^{>0}$ . Now use the intermediate value property for  $\text{id} + \psi$  on  $V^{>0}$  to deduce the first statement. For the second, note that  $-2a + \psi(2a) < -a$ , since  $a > \psi(a)$ . Since clearly  $(-\text{id} + \psi)(V^{>0}) = (\text{id} + \psi)(V^{<0})$  it follows as before that  $(-\text{id} + \psi)(V^{>0})$  is closed downward. Let  $a \in V$ ,  $a < \psi(x)$  for some  $x \in V^{>0}$ . Set  $y := \min\{x, \psi(x) - a\} > 0$ ; then  $a \leq \psi(x) - y \leq \psi(y) - y \in (-\text{id} + \psi)(V^{>0})$ . Thus  $a \in (-\text{id} + \psi)(V^{>0})$ . ■

These facts will be tacitly used in the rest of the paper. Next we make the following easy but very useful observation.

PROPOSITION 3.1. *There is at most one element  $v \in V$  such that*

$$\Psi < v < (\text{id} + \psi)(V^{>0}). \quad (2)$$

*Hence  $(V, \psi)$  has at most two  $H$ -cuts, and  $(V, \psi)$  has exactly two  $H$ -cuts if and only if there exists  $v$  such that (2) holds. If  $\Psi$  has a largest element, then  $(V, \psi)$  has only one  $H$ -cut.*

*Proof.* If  $v > v'$  are two elements satisfying (2), choosing  $u := v - v' > 0$  yields  $\psi(u) \leq v' = v - u < (\psi(u) + u) - u = \psi(u)$ , which is a contradiction. If  $\Psi$  has a largest element  $v'$  and  $v$  is supposed to satisfy (2), then the same argument leads to a contradiction. ■

Closed  $H$ -couples have only one  $H$ -cut. In Lemma 3.3 we indicate a class of  $H$ -couples with two  $H$ -cuts. First a general fact that we shall use several times:

LEMMA 3.2. *Let  $k \subseteq k'$  be an ordered field extension, and let  $i: V \rightarrow V'$  be an embedding of  $V$  into a Hahn space  $V'$  over  $k'$ , such that the induced map  $[V] \rightarrow [V']$  is bijective. Then there is a unique function  $\psi': (V')^* \rightarrow V'$  such that  $(V', \psi')$  is an  $H$ -couple over  $k'$ , with  $1 \in V$  as its distinguished positive element, and  $i(\psi(v)) = \psi'(i(v))$  for all  $v \in V^*$ .*

*Proof.* Define  $\psi'(v') := \psi(v)$  for  $v' \in (V')^*$  and  $v \in V^*$  such that  $[v'] = [i(v)]$ . Then  $\psi'$  is well defined, and  $(V', \psi')$  is an  $H$ -couple. The main point to check here is axiom (3) for  $H$ -couples, which follows from the bijectivity of  $[V] \rightarrow [V']$ , and property (3) for  $\psi$  stated at the beginning of this section. ■

Consider now an embedding of  $V$  into the Hahn space  $H := H(\Gamma, k)$  over  $k$  as in Corollary 2.1, with  $\Gamma = [V^*]$ , and identify  $V$  with its image in  $H$  via this embedding. Then the lemma above tells us that  $\psi$  extends uniquely to a function  $\psi_H: H^* \rightarrow H$  such that  $(H, \psi_H)$  is an  $H$ -couple over  $k$  with distinguished element  $1 \in V$ . The next result shows that  $(H, \psi_H)$  has always two  $H$ -cuts if  $\Gamma$  has no least element.

LEMMA 3.3. *Let  $H = H(\Gamma, k)$  for some nonempty linearly ordered set  $\Gamma$  without least element. Then each  $H$ -couple of the form  $(H, \psi)$  has two  $H$ -cuts.*

*Proof.* Let  $\kappa$  be the coinitality of  $\Gamma$  and  $(\gamma_\alpha)_{\alpha < \kappa}$  a strictly decreasing coinital sequence in  $\Gamma$ . Choose  $u_\alpha \in H^{>0}$  with  $[u_\alpha] = \gamma_\alpha$  and set  $w_\alpha := \psi(u_\alpha)$  for all  $\alpha < \kappa$ . Then  $(w_\alpha)_{\alpha < \kappa}$  is cofinal in  $\psi(H^*)$ , and  $(w_\alpha + u_\alpha)_{\alpha < \kappa}$  is coinital in  $(\text{id} + \psi)(H^{>0})$ . Let  $\gamma \in \Gamma$ ; take  $\alpha_0 < \kappa$  such that  $\gamma_{\alpha_0} < \gamma$ . Then  $\alpha_0 < \alpha, \beta < \kappa$  implies  $[w_\alpha - w_\beta] < \gamma$ , that is,  $(w_\alpha)_{\gamma'} = (w_\beta)_{\gamma'}$  for all  $\gamma' \geq \gamma$ . So, for each  $\gamma \in \Gamma$ , the sequence  $((w_\alpha)_{\gamma})_{\alpha < \kappa}$  is eventually constant. Let  $v_\gamma \in k$  be this constant, and set  $v := (v_\gamma)_{\gamma \in \Gamma}$ . One shows that  $v \in H$  and that  $w_\alpha < v < w_\alpha + u_\alpha$  for all  $\alpha < \kappa$ . Therefore  $\psi(H^{>0}) < v < (\text{id} + \psi)(H^{>0})$ . ■

### Examples of $H$ -Couples

EXAMPLE 3.1. Consider  $k$  as an ordered vector space over itself. Then  $(k, \psi)$ , with  $\psi(v) = 1$  for all  $v \in k^\times$  and  $1 \in k$  as distinguished positive element, is an  $H$ -couple over  $k$ . It has a unique embedding (as  $\mathcal{L}_H$ -structure) into any  $H$ -couple over  $k$ . Here,  $k^{\leq 1}$  is the only  $H$ -cut of  $(k, \psi)$ .

EXAMPLE 3.2. Every finite-dimensional Hahn space over  $k$  is isomorphic to the anti-lexicographically ordered vector space  $k^n$  for some  $n$ . (A non-zero vector  $(\alpha_1, \dots, \alpha_n) \in k^n$  is positive in the anti-lexicographic ordering if and only if  $\alpha_i > 0$ , where  $i = \max\{j : 1 \leq j \leq n, \alpha_j \neq 0\}$ .)

Let us fix an  $n > 0$  and some positive vector  $(\lambda_1, \dots, \lambda_n)$  in the anti-lexicographically ordered vector space  $k^n$  over  $k$ . Let

$$e_1 = (1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1)$$

be the standard basis vectors of  $k^n$ , so  $[e_1] < \dots < [e_n]$  are the non-zero  $k$ -archimedean classes of  $k^n$ . Let  $(k^n, \psi)$  be an  $H$ -couple with  $(\lambda_1, \dots, \lambda_n)$  as distinguished positive element. Define the  $n \times n$  matrix  $A = (\alpha_{ij}) \in k^{n \times n}$  by

$$(\alpha_{i1}, \dots, \alpha_{in}) := \psi(e_i) \quad \text{for } i = 1, \dots, n.$$

Then  $A$  has the properties:

- (1)  $(\alpha_{11}, \dots, \alpha_{1n}) > \dots > (\alpha_{n1}, \dots, \alpha_{nn})$ ;
- (2)  $\alpha_{ij} = \alpha_{jj}$  for all  $1 \leq i < j \leq n$ ;
- (3)  $(\lambda_1, \dots, \lambda_n) = (\alpha_{i1}, \dots, \alpha_{in}) > 0$  for  $i := \max\{j : 1 \leq j \leq n, \lambda_j \neq 0\}$ .

Here, (1) and (3) follow from axioms (2) and (1) for  $H$ -couples, respectively, whereas (2) is derived from  $\psi(e_i) - \psi(e_j) < \varepsilon \cdot e_j$  for all  $\varepsilon > 0$  and  $1 \leq i < j \leq n$ , which holds by axiom (3). Conversely, given a matrix  $A = (\alpha_{ij}) \in k^{n \times n}$  with properties (1)–(3), define  $\psi$  by setting

$$\psi(\mu_1, \dots, \mu_n) := (\alpha_{i1}, \dots, \alpha_{in}), \quad \text{where } i := \max\{j : 1 \leq j \leq n, \mu_j \neq 0\},$$

for  $(\mu_1, \dots, \mu_n) \neq 0$  in  $k^n$ , thus obtaining an  $H$ -couple  $(k^n, \psi)$  with distinguished positive element  $(\lambda_1, \dots, \lambda_n)$ . In this way, we get a one-to-one correspondence between  $H$ -couples  $(k^n, \psi)$  with distinguished positive element  $(\lambda_1, \dots, \lambda_n)$  and matrices  $A \in k^{n \times n}$  with the three properties above.

EXAMPLE 3.3. Suppose the Hahn space  $V$  over  $k$  has countable dimension. Then by Brown's argument in [4] there is an embedding  $V \rightarrow H(\Gamma, k)$  with  $\Gamma = [V^*]$  as in Corollary 2.1, whose image is the direct sum

$$k^{(\Gamma)} := \{v \in k^\Gamma : \text{supp } v \text{ finite}\}.$$

For any linearly ordered set  $\Gamma \neq \emptyset$  we can give a description of  $H$ -couples  $(k^{(\Gamma)}, \psi)$  with distinguished positive element  $(\lambda_\gamma)_{\gamma \in \Gamma}$  in terms of certain matrices in  $k^{\Gamma \times \Gamma}$ , similar to the previous example. As an example, consider  $\Gamma := \mathbb{Z}$ ,  $V := k^{(\Gamma)} = \bigoplus_{s \in \mathbb{Z}} k e_s$  with  $e_s > 0$  and  $[e_s] < [e_{s+1}]$  for all  $s \in \mathbb{Z}$ . Define  $\psi: V^* \rightarrow V$  by making  $\psi$  constant on each  $k$ -archimedean class and setting

$$\psi(e_n) := -(e_1 + \cdots + e_{n-1}) \quad \text{if } n > 0$$

(hence  $\psi(e_1) = 0$ ) and

$$\psi(e_{-n}) := e_{-n} + e_{-n+1} + \cdots + e_0 \quad \text{if } n \geq 0.$$

Then  $(V, \psi)$ , with distinguished positive element  $e_0$ , is an  $H$ -couple. In fact, if  $k = \mathbb{R}$ , it is the  $H$ -couple associated with the smallest Hardy field which is closed under powers and contains  $\mathbb{R}(\dots, e^x, x, \log x, \log \log x, \dots)$ . (See [11, p. 263]; [16, Corollary 2].) Note that  $\Gamma$  has no smallest element, but that  $(V, \psi)$  has only one  $H$ -cut. Modifying the definition of  $\psi$  above by letting

$$\psi(e_{-n}) := e_0 + e_{-1} - e_{-n-1} \quad \text{for } n \geq 0,$$

we get an example of an  $H$ -couple with distinguished positive element  $e_0$  and with two  $H$ -cuts, since in this case  $\sup \Psi$  exists and equals  $e_0 + e_{-1} \notin \Psi$ .

**EXAMPLE 3.4.** The  $H$ -couple associated with a maximal Hardy field is closed. More generally, let  $K$  be a Hardy field containing  $\mathbb{R}(x)$  and closed under exponentiation (i.e.,  $f \in K \Rightarrow e^f \in K$ ) and integration (i.e.,  $f \in K \Rightarrow \exists g \in K : g' = f$ ). Then  $K$  is also closed under powers, and the  $H$ -couple associated with  $K$  is closed.

*Proof.* Note that if  $f \in K^{>0}$ , then  $\log f \in K$ , since  $(\log f)' = f'/f \in K$ . The ordered set  $[v(K^\times)^*]$  has no least element since, for any  $f \in K^{>0}$  with  $v(f) > 0$ , we have  $0 < r \cdot v(1/\log f) < v(f)$  for all  $r \in \mathbb{R}^{>0}$ . It remains to show that, for  $f \in K^\times$ , either  $v(f) = v(g'/g)$  for some  $g \in K^\times$ ,  $v(g) > 0$ , or  $v(f) = v(g')$  for some  $g \in K^\times$ ,  $v(g) > 0$ . Take  $g \in K^\times$  with  $g' = f$ . If  $v(g) \geq 0$ , then by subtracting a real constant from  $g$  if necessary, we may assume  $v(g) > 0$ , and we are done. If  $v(g) < 0$ , then, changing  $f$  to  $-f$  and  $g$  to  $-g$  if necessary, we may assume  $g$  is negative infinite, i.e.,  $g < \mathbb{R}$ . Then  $G := e^g$  satisfies  $f = G'/G$ , so  $v(f) = v(G'/G)$  and  $v(G) > 0$ . ■

In the next examples, we assume familiarity with [5].

**EXAMPLE 3.5.** Let  $k$  be an ordered logarithmic-exponential field and  $F$  any differential subfield of  $k((t))^{\text{LE}}$  containing  $k(x)$  and closed under powers. Then with  $V := v(F^\times)$  and  $\psi$  defined as in the Introduction, we get an  $H$ -couple  $(V, \psi)$  with distinguished positive element  $1 = v(t)$ . Moreover, if  $F$  is also closed under exponentiation and integration, then  $(V, \psi)$  is closed.

*Proof.* The valuation  $v$  on  $k((t))^{\text{LE}}$  is defined in terms of the leading monomial map  $\text{Lm}$  via  $v(f) = -\log(\text{Lm}(f))$  for  $f \in k((t))^{\text{LE}}$ ,  $f \neq 0$ ; see [5, (2.9)]. In particular, the ordered  $k$ -linear space  $V$  (the value group) is an ordered  $k$ -linear subspace of  $k((t))^{\text{LE}}$  itself. It is easy to see that the ordered  $k$ -linear space  $k((t))^{\text{LE}}$  is a Hahn space, and thus  $V$  is a Hahn space. It is clear that axiom (1) for  $H$ -couples holds, while axiom (2) is (4.5) in [5]. Axiom (3) is a consequence of Theorem 4(c) in [12], since  $v$  is a differential valuation by [5, (4.1)]. The last part of the statement follows by adapting the proof in the previous example. ■

EXAMPLE 3.6. Let  $k$  be as in the previous example, and let  $F = k((t))^{\text{E}}$  be the differential subfield of exponential series in  $k((t))^{\text{LE}}$ . One shows easily that  $F$  is closed under powers. In the corresponding  $H$ -couple  $(V, \psi)$ , where  $V := v(F^\times)$ , the element  $1 = v(t) = \psi(1)$  is the largest element of  $\psi(V^*) = \Psi$ ; see (2.2) in [5]. It follows from results in [5, Sect. 5] that, for each  $f \in F^\times$ , if  $v(f) > 1$ , then  $f = g'$  for some  $g \in F$  with  $v(g) > 0$ , while, if  $v(f) \leq 1$ , then  $f = g'/g$  for some  $g \in F^\times$  with  $v(g) > 0$ . Thus  $\Psi = V^{\leq 1}$  and  $(\text{id} + \psi)(V^{>0}) = V^{>1}$ .

### Embedding into Closed $H$ -Triples

Besides the  $H$ -triple  $(V, \psi, P)$  over  $k$  we now let  $(V', \psi', P')$  denote a second  $H$ -triple with ordered scalar field  $k'$ . Since we are dealing here with (two-sorted)  $\mathcal{L}_{H,P}$ -structures there is a well-defined notion of embedding  $i: (V, \psi, P) \rightarrow (V', \psi', P')$ . Such an embedding  $i$  is uniquely determined by its *scalar part*  $i_s: k \rightarrow k'$ , an ordered field embedding, and its *vector part*  $i_v: V \rightarrow V'$ , an ordered group embedding between the underlying ordered additive groups of  $V$  and  $V'$ . Conversely, given an ordered field embedding  $i_s: k \rightarrow k'$  and an ordered group embedding  $i_v: V \rightarrow V'$  between the underlying ordered additive groups of  $V$  and  $V'$ , there is an embedding  $i: (V, \psi, P) \rightarrow (V', \psi', P')$  with  $i_s = i_1$  and  $i_v = i_2$  if and only if  $i_1$  and  $i_2$  satisfy the compatibility conditions

$$i_2(\lambda u) = i_1(\lambda)i_2(u), \quad i_2(\psi(v)) = \psi'(i_2(v)), \quad i_2(1_V) = 1_{V'},$$

for all  $u, v \in V$ ,  $v \neq 0$ ,  $\lambda \in k$ , and

$$P' \cap i_2(V) = i_2(P).$$

Here  $1_V, 1_{V'}$  are the distinguished positive elements of  $(V, \psi, P)$  and  $(V', \psi', P')$ , respectively. Given an embedding  $i$  as above, we usually write  $i(\lambda)$  for  $i_s(\lambda)$  when  $\lambda \in k$  and  $i(a)$  for  $i_v(a)$  when  $a \in V$ . Note that  $i$  induces an embedding  $[v] \mapsto [i(v)]: [V] \rightarrow [V']$  of linearly ordered sets.

We will now show that each  $H$ -triple  $(V, \psi, P)$  has an  $H$ -closure  $(V^c, \psi^c, P^c)$  in the following sense:  $(V^c, \psi^c, P^c)$  is a closed  $H$ -triple over



$k$  extending  $(V, \psi, P)$  such that any embedding  $(V, \psi, P) \rightarrow (V', \psi', P')$  into a closed  $H$ -triple (not necessarily over  $k$ ) extends to an embedding  $(V^c, \psi^c, P^c) \rightarrow (V', \psi', P')$ . (We do not require uniqueness.) Later, we will see that any two  $H$ -closures of  $\mathcal{V} = (V, \psi, P)$  are isomorphic over  $\mathcal{V}$ , that is, isomorphic by an isomorphism whose vector part is the identity on  $V$  and whose scalar part is the identity on  $k$ . (See Corollary 5.3.) Toward the existence proof we show three basic extension lemmas:

LEMMA 3.4. *Suppose  $a \in V$ ,  $P < a < (\text{id} + \psi)(V^{>0})$ . Then  $(V, \psi, P)$  extends to an  $H$ -triple  $(V \oplus k\varepsilon, \psi^\varepsilon, P^\varepsilon)$  over  $k$  such that:*

1.  $\varepsilon > 0$ ,  $a = \varepsilon + \psi^\varepsilon(\varepsilon)$ .
2. *Given any embedding  $i: (V, \psi, P) \rightarrow (V', \psi', P')$  of  $H$ -triples and any  $\varepsilon' \in V'$  with  $\varepsilon' > 0$  and  $i(a) = \varepsilon' + \psi'(\varepsilon')$ , there is a unique extension of  $i$  to an embedding  $j: (V \oplus k\varepsilon, \psi^\varepsilon, P^\varepsilon) \rightarrow (V', \psi', P')$  with  $j(\varepsilon) = \varepsilon'$ .*

*Proof.* Take an ordered vector space  $V \oplus k\varepsilon$  over  $k$  extending the ordered vector space  $V$ , such that  $0 < \varepsilon < V^{>0}$ . One verifies immediately that  $V \oplus k\varepsilon$  is a Hahn space. For a non-zero vector  $w = v + \lambda\varepsilon$  ( $v \in V$ ,  $\lambda \in k$ ), we put

$$\psi^\varepsilon(w) := \begin{cases} \psi(v), & \text{if } v \neq 0, \\ a - \varepsilon, & \text{otherwise.} \end{cases}$$

Also let  $P^\varepsilon := \{w \in V \oplus k\varepsilon : w \leq a - \varepsilon\}$ . One verifies easily that  $(V \oplus k\varepsilon, \psi^\varepsilon, P^\varepsilon)$  is an  $H$ -triple extending  $(V, \psi, P)$ . Let  $i: (V, \psi, P) \rightarrow (V', \psi', P')$  be an embedding of  $H$ -triples, and  $\varepsilon' \in V'$ ,  $\varepsilon' > 0$ , with  $i(a) = \varepsilon' + \psi'(\varepsilon')$ . By making the usual identifications we may assume that  $(V, \psi, P) \subseteq (V', \psi', P')$  and that  $i$  is the natural inclusion. Then  $0 < \varepsilon' < V'^{>0}$ . Hence the inclusion  $V \hookrightarrow V'$  extends to an embedding  $V \oplus k\varepsilon \rightarrow V'$  of ordered vector spaces over  $k$  sending  $\varepsilon$  to  $\varepsilon'$ . It is easy to check that this embedding is the vector part of an embedding  $(V \oplus k\varepsilon, \psi^\varepsilon, P^\varepsilon) \rightarrow (V', \psi', P')$  that extends  $i$ . ■

Note that  $P^\varepsilon$  as in Lemma 3.4 has a maximum. In this situation we can apply the next lemma.

LEMMA 3.5. *Suppose  $P$  has a largest element. Then  $(V, \psi, P)$  extends to an  $H$ -triple  $(V \oplus k\varepsilon, \psi^\varepsilon, P^\varepsilon)$  over  $k$  such that:*

1.  $\varepsilon > 0$ ,  $\psi^\varepsilon(\varepsilon) = (\max P) + \varepsilon$ .
2. *Given any embedding  $i: (V, \psi, P) \rightarrow (V', \psi', P')$  of  $H$ -triples and any  $\varepsilon' \in V'$  with  $\varepsilon' > 0$  and  $\psi'(\varepsilon') = i(\max P) + \varepsilon'$ , there is a unique extension of  $i$  to an embedding  $j: (V \oplus k\varepsilon, \psi^\varepsilon, P^\varepsilon) \rightarrow (V', \psi', P')$  with  $j(\varepsilon) = \varepsilon'$ .*

*Proof.* We proceed exactly as in the proof of the previous lemma, except that in the definitions of  $\psi^\varepsilon$  and  $P^\varepsilon$  we put

$$\psi^\varepsilon(w) := \begin{cases} \psi(v), & \text{if } v \neq 0, \\ (\max P) + \varepsilon, & \text{otherwise,} \end{cases}$$

for non-zero  $w = v + \lambda\varepsilon$  ( $v \in V$ ,  $\lambda \in k$ ), and

$$P^\varepsilon := \{w \in V \oplus k\varepsilon : w \leq (\max P) + \varepsilon\}.$$

Then  $\psi^\varepsilon(\varepsilon) = (\max P) + \varepsilon > \max P$ . ■

We note that  $P^\varepsilon$  as in Lemma 3.5 still has a largest element (though larger than  $\max P$ ), so that this lemma can be applied again to the extension  $(V \oplus k\varepsilon, \psi^\varepsilon, P^\varepsilon)$  of  $(V, \psi, P)$ .

LEMMA 3.6. *Suppose  $b \in P \setminus \Psi$ . Then  $(V, \psi, P)$  can be extended to an  $H$ -triple  $(V \oplus ka, \psi^a, P^a)$  over  $k$  such that:*

1.  $a > 0$ ,  $\psi^a(a) = b$ .

2. *Given any embedding  $i: (V, \psi, P) \rightarrow (V', \psi', P')$  of  $H$ -triples and any element  $a' > 0$  in  $V'$  with  $\psi'(a') = i(b)$ , there is a unique extension of  $i$  to an embedding  $j: (V \oplus ka, \psi^a, P^a) \rightarrow (V', \psi', P')$  with  $j(a) = a'$ .*

*Proof.* By Corollary 2.1, we may regard  $V$  as an ordered linear subspace of  $H([V^*], k)$ . We take an object  $\gamma \notin [V^*]$  and extend the linear ordering on  $[V^*]$  to a linear ordering of  $\Gamma := [V^*] \cup \{\gamma\}$  by setting  $\gamma < [v] : \iff b > \psi(v)$  for all  $v \in V^*$ . Next we view  $H([V^*], k)$  as an ordered linear subspace of  $H(\Gamma, k)$  by identifying each function  $f: [V^*] \rightarrow k$  in  $H([V^*], k)$  with its extension to  $\Gamma$  obtained by setting  $f(\gamma) := 0$ . Thus  $V \subseteq H(\Gamma, k)$ . Choose  $a > 0$  in  $H(\Gamma, k)$  with  $\max(\text{supp } a) = \gamma$ . Note that  $V \oplus ka$  is a Hahn space, as an ordered linear subspace of the Hahn space  $H(\Gamma, k)$ . For non-zero  $w = v + \lambda a$  ( $v \in V$ ,  $\lambda \in k$ ), we set

$$\psi^a(w) := \begin{cases} \psi(v), & \text{if } [w] = [v], \\ b, & \text{otherwise, i.e., if } [w] = [a]. \end{cases}$$

Also set  $P^a := \{w \in V \oplus ka : w \leq v \text{ for some } v \in P\}$ . Then we have to check that  $(V \oplus ka, \psi^a, P^a)$  is an  $H$ -triple over  $k$  and that  $(V, \psi, P) \subseteq (V \oplus ka, \psi^a, P^a)$ . It is immediate that axioms (1) and (2) for  $H$ -couple are satisfied. The main point is axiom (3). Let  $w = v + \lambda a$ ,  $w' = v' + \lambda' a$  be positive elements of  $V \oplus ka$  ( $v, v' \in V$ ,  $\lambda, \lambda' \in k$ ); we have to show that  $\psi^a(w') < \psi^a(w) + w$ . We can assume  $[w'] < [w]$ , since otherwise  $\psi^a(w') \leq \psi^a(w) < \psi^a(w) + w$ . We distinguish the following cases:

1.  $[w'] = [v']$ ,  $[w] = [v]$ . Then  $[\psi(v') - \psi(v)] < [v' - v] = [v] = [w]$ ; hence  $\psi^a(w') = \psi(v') < \psi(v) + w = \psi^a(w) + w$ .

2.  $[w'] = [a]$ ,  $[w] = [v]$ . By basic properties (1) and (2) stated at the beginning of this section, we get  $[b - \psi(v)] < [a - v] = [v] = [w]$ ; hence  $\psi^a(w') = b < \psi(v) + w = \psi^a(w) + w$ .

3.  $[w'] = [v']$ ,  $[w] = [a]$ . Similar to the second case,  $[\psi(v') - b] < [v' - a] = [a] = [w]$ ; hence  $\psi^a(w') = \psi(v') < b + w = \psi^a(w) + w$ .

Moreover,  $P^a$  is the unique  $H$ -cut of  $(V \oplus ka, \psi^a)$  such that  $P^a \cap V = P$ . To see this, let  $P_0^a$  be any  $H$ -cut of  $(V \oplus ka, \psi^a)$  with  $P_0^a \cap V = P$ . Assume we are given  $v \in V$  and  $\lambda \in k^\times$ . To determine when  $v + \lambda a \in P_0^a$ , we distinguish several cases:

1.  $\lambda > 0$ ,  $v \geq b$ . Then  $v + \lambda a \geq \psi^a(\lambda a) + \lambda a$ ; hence  $v + \lambda a \notin P_0^a$ .
2.  $\lambda > 0$ ,  $b - v > a$ . Then  $\psi^a(a) > v + \lambda a$ ; hence  $v + \lambda a \in P_0^a$ .
3.  $\lambda > 0$ ,  $0 < b - v < a$ . Choose  $0 < \mu < \lambda$ . Then  $b - v < (\lambda - \mu)a$ ; hence  $v + \lambda a > b + \mu a = \psi^a(\mu a) + \mu a$ , implying  $v + \lambda a \notin P_0^a$ .
4.  $\lambda < 0$ ,  $v - b > a$ . Then  $v - b > (1 - \lambda)a$ ; hence  $v + \lambda a > \psi^a(\lambda a) + \lambda a$ , so we get  $v + \lambda a \notin P_0^a$ .
5.  $\lambda < 0$ ,  $v - b < a$ . Then  $v - b < -\lambda a$ ; hence  $v + \lambda a < \psi^a(a)$ , so  $v + \lambda a \in P_0^a$ .

Therefore  $v + \lambda a \in P_0^a$  if and only if either  $\lambda > 0$  and  $b - v > a$  or  $\lambda < 0$  and  $v - b < a$ . Hence  $P^a = P_0^a$ .

Now let  $i: (V, \psi, P) \rightarrow (V', \psi', P')$  be an embedding of  $H$ -triples and  $a'$  a positive element of  $V'$  with  $\psi'(a') = b$ . We can assume that  $(V, \psi, P) \subseteq (V', \psi', P')$  and that  $i$  is the inclusion. Note that  $a' \notin V$  determines the same cut in  $V$  as  $a$ . Hence the inclusion  $V \hookrightarrow V'$  extends to a unique embedding  $V \oplus ka \rightarrow V'$  of ordered vector spaces over  $k$  mapping  $a$  to  $a'$ . This embedding is the vector part of an embedding  $(V \oplus ka, \psi^a, P^a) \rightarrow (V', \psi', P')$ , by the uniqueness of  $P^a$  proved above. ■

*Remark.* Assume that in the last lemma there is no element  $v \in V$  with  $P < v < (\text{id} + \psi)(V^{>0})$  and that  $P$  has no maximum. Then there is also no  $w \in V \oplus ka$  with  $P^a < w < (\text{id} + \psi^a)((V \oplus ka)^{>0})$ , and  $P^a$  has no largest element.

Starting with  $(V, \psi, P)$  and suitably iterating and alternating the constructions of Lemmas 3.4, 3.5, and 3.6 (possibly transfinitely often), we can build an increasing chain of  $H$ -triples over  $k$  whose union is an  $H$ -closure of  $(V, \psi, P)$ :

COROLLARY 3.1. *Every  $H$ -triple has an  $H$ -closure.*

### Behavior under Scalar Extension

LEMMA 3.7. *Let  $k' \supseteq k$  be an extension of ordered fields. Then there are unique  $\psi_{k'}$  and  $P_{k'}$  such that  $(V_{k'}, \psi_{k'}, P_{k'})$  is an  $H$ -triple over  $k'$  extending  $(V, \psi, P)$ .*

*Proof.* Let  $\psi_{k'} := \psi'$  as in Lemma 3.2 (where  $i: V \rightarrow V_{k'}$  is the natural inclusion). We have to show that there is a unique  $H$ -cut  $P_{k'}$  for  $(V_{k'}, \psi_{k'})$  with  $P_{k'} \cap V = P$ . If  $\Psi$  has a largest element, this is clear since  $\Psi = \psi(V^*) = \psi_{k'}(V_{k'}^*)$ . So assume  $\Psi$  has no largest element. Suppose  $v \in V_{k'}$  satisfies

$$\psi_{k'}(V_{k'}^{>0}) < v < (\text{id} + \psi_{k'})(V_{k'}^{>0}).$$

It suffices to show then that  $v \in V$ . Let  $H := H(\Gamma, k)$ ,  $H' := H(\Gamma, k')$ ,  $\Gamma := [V^*]$ . Consider the following commutative diagram of ordered vector spaces over  $k$  and embeddings between them:

$$\begin{array}{ccccc} V_{k'} & \xrightarrow{\phi \otimes \text{id}} & H_{k'} & \xrightarrow{\mu} & H' \\ \uparrow & & \uparrow & \nearrow \iota & \\ V & \xrightarrow{\phi} & H & & \end{array}$$

Here,  $\phi$  is given by Corollary 2.1, the maps  $V \rightarrow V_{k'}$  and  $H \rightarrow H_{k'}$  are obtained from Proposition 2.2,  $\iota$  is the natural inclusion  $H(\Gamma, k) \hookrightarrow H(\Gamma, k')$ , and  $\mu$  is uniquely determined as an embedding by  $\mu(h \otimes \lambda) = \lambda h$ , for  $\lambda \in k'$ ,  $h \in H$  (using Corollary 2.2). After identifying the  $H$ -couples  $(V, \psi)$ ,  $(V_{k'}, \psi_{k'})$ , and  $(H, \psi_H)$  via these embeddings with  $\mathcal{L}_H$ -substructures of  $(H', \psi_{H'})$  we have  $V_{k'} \cap H = V$ . By Lemma 3.3,  $(H, \psi_H)$  and  $(H', \psi_{H'})$  have two  $H$ -cuts, and hence  $v \in H$ . Thus  $v \in V$ , as desired. ■

In the next section we apply this last result as follows. Let  $\mathcal{V} = (V, \psi, P)$  and  $\mathcal{V}' = (V', \psi', P')$  be  $H$ -triples over ordered fields  $k$  and  $k'$ , respectively. Let  $\mathcal{V}_0 = (V_0, \psi_0, P_0)$  be a substructure of  $\mathcal{V}$ . Thus  $\mathcal{V}_0$  is an  $H$ -triple over an ordered subfield  $k_0$  of  $k$ . Let an embedding  $i_0: \mathcal{V}_0 \rightarrow \mathcal{V}'$  be given, and also an embedding  $e: k \rightarrow k'$  of ordered fields, such that  $e|_{k_0} = (i_0)_s$ .

By the last lemma the Hahn space  $(V_0)_k := V_0 \otimes_{k_0} k$  over  $k$  expands uniquely to an  $H$ -triple  $(\mathcal{V}_0)_k$  over  $k$  such that  $\mathcal{V}_0 \subseteq (\mathcal{V}_0)_k$ . With this notation we have:

LEMMA 3.8. *The embedding  $\mathcal{V}_0 \hookrightarrow \mathcal{V}$  extends uniquely to an embedding  $(\mathcal{V}_0)_k \rightarrow \mathcal{V}$  with scalar part  $\text{id}_k$ . The embedding  $i_0: \mathcal{V}_0 \rightarrow \mathcal{V}'$  extends uniquely to an embedding  $(\mathcal{V}_0)_k \rightarrow \mathcal{V}'$  with scalar part  $e$ .*

*Proof.* By Corollary 2.2 the inclusion  $V_0 \hookrightarrow V$  extends uniquely to an embedding  $(V_0)_k \rightarrow V$  of Hahn spaces over  $k$ . This is actually an embedding  $(\mathcal{V}_0)_k \rightarrow \mathcal{V}$  of  $H$ -triples with scalar part  $\text{id}_k$ , by the uniqueness property in the last proposition. ■

#### 4. ELIMINATION OF QUANTIFIERS

In this section we obtain the main results of this paper. Let  $T_{H,P}$  denote the theory of closed  $H$ -triples in the language  $\mathcal{L}_{H,P}$ . By “formula” we shall mean “ $\mathcal{L}_{H,P}$ -formula.” Let  $x = (x_1, \dots, x_m)$  denote a tuple of distinct scalar variables,  $y = (y_1, \dots, y_n)$  a tuple of distinct vector variables. We call a formula  $\eta(x, y)$  a *scalar formula* if it is of the form  $\zeta(s_1(x, y), \dots, s_N(x, y))$  where  $\zeta(z_1, \dots, z_N)$  is a formula in the language of ordered rings (as specified in part (3) of the description of  $\mathcal{L}_H$  in Section 1), where  $z_1, \dots, z_N$  are scalar variables and  $s_1(x, y), \dots, s_N(x, y)$  are scalar valued terms of  $\mathcal{L}_{H,P}$ .

**THEOREM 4.1.** *Every formula  $\varphi(x, y)$  is equivalent in  $T_{H,P}$  to a boolean combination of scalar formulas  $\eta(x, y)$  and of atomic formulas  $\alpha(x, y)$ .*

This elimination theorem says in particular that every formula is equivalent in  $T_{H,P}$  to a formula that is free of quantifiers over vector variables. It will be derived from the following embedding result:

**PROPOSITION 4.1.** *Let  $(V, \psi, P)$  and  $(V', \psi', P')$  be closed  $H$ -triples over  $k$  and  $k'$ , respectively. Assume that  $(V', \psi', P')$  is  $\kappa$ -saturated, where  $\kappa := |V|^+$ . Let  $(V_0, \psi_0, P_0)$  be a substructure of  $(V, \psi, P)$ , and thus an  $H$ -triple over a subfield  $k_0$  of  $k$ . Let an embedding  $i_0: (V_0, \psi_0, P_0) \rightarrow (V', \psi', P')$  be given, and also an embedding  $e: k \rightarrow k'$  of ordered fields, such that  $e|_{k_0} = (i_0)_s$ . Then  $i_0$  can be extended to an embedding  $i: (V, \psi, P) \rightarrow (V', \psi', P')$  such that  $e = i_s$ .*

We postpone the proof of this proposition and first deduce Theorem 4.1 from it. To this end we use the following consequence of Proposition 4.1.

**LEMMA 4.1.** *Let  $(V, \psi, P) \subseteq (V', \psi', P')$  be closed  $H$ -triples over  $k$  and  $k'$ , respectively. Then  $k \leq k'$  (as ordered fields) if and only if  $(V, \psi, P) \preceq (V', \psi', P')$ .*

*Proof.* One direction being trivial, we assume  $k \leq k'$ , and shall derive  $(V, \psi, P) \preceq (V', \psi', P')$ . Let  $\varphi(x, y)$  be a formula. By induction on the complexity of  $\varphi$ , one shows, for all  $(V, \psi, P)$  and  $(V', \psi', P')$  as in the hypothesis of the lemma, and all  $\lambda \in k^m$ ,  $v \in V^n$ ,

$$(V, \psi, P) \models \varphi(\lambda, v) \iff (V', \psi', P') \models \varphi(\lambda, v).$$

For the inductive step, let  $\varphi = \exists z\theta$ , where  $\theta(x, y, z)$  is a formula and  $z$  a single variable of the vector or scalar sort. Since  $\theta$  is of lower complexity than  $\varphi$  the direction “ $\Rightarrow$ ” follows from the induction hypothesis. So assume  $(V', \psi', P') \models \varphi(\lambda, v)$ . Choose a  $\kappa$ -saturated elementary extension  $(V'', \psi'', P'')$  of  $(V, \psi, P)$ , where  $\kappa := |V'|^+$ . Let  $k''$  be the scalar field of  $(V'', \psi'', P'')$ . Then there is an elementary embedding of ordered fields  $e: k' \rightarrow k''$  with  $e|_k = \text{id}$ . By Proposition 4.1, there is an embedding  $i: (V', \psi', P') \rightarrow (V'', \psi'', P'')$  with  $i_s = e$  and  $i_v|_V = \text{id}$ . Using the induction hypothesis on  $\theta$ , it follows that  $(V'', \psi'', P'') \models \varphi(\lambda, v)$ . We conclude that  $(V, \psi, P) \models \varphi(\lambda, v)$ . ■

*Proof* (of Theorem 4.1, assuming Proposition 4.1). Suppose  $(V, \psi, P)$  and  $(V', \psi', P')$  are closed  $H$ -triples over  $k$  and  $k'$ , respectively, and let  $(\lambda, v) \in k^m \times V^n$  and  $(\lambda', v') \in (k')^m \times (V')^n$  satisfy the same scalar formulas and the same atomic formulas in  $(V, \psi, P)$  and  $(V', \psi', P')$ , respectively. By a standard model-theoretic argument, it suffices to derive from these assumptions that  $(\lambda, v)$  and  $(\lambda', v')$  satisfy the same formulas in  $(V, \psi, P)$  and  $(V', \psi', P')$ , respectively. We may assume  $(V', \psi', P')$  is  $\kappa$ -saturated, where  $\kappa := |V|^+$ . Let  $(V_0, \psi_0, P_0)$ , with scalar field  $k_0$ , be the substructure of  $(V, \psi, P)$  generated by  $(\lambda, v)$ . Since  $(\lambda, v)$  and  $(\lambda', v')$  satisfy the same atomic formulas, there exists an embedding  $i_0: (V_0, \psi_0, P_0) \rightarrow (V', \psi', P')$  such that  $i_0(\lambda_i) = \lambda'_i$  for  $i = 1, \dots, m$  and  $i_0(v_j) = v'_j$  for  $j = 1, \dots, n$ . Since  $(\lambda, v)$  and  $(\lambda', v')$  satisfy the same scalar formulas, there exists an elementary embedding of ordered fields  $e: k \rightarrow k'$  such that  $e|_{k_0} = (i_0)_s$ . By Proposition 4.1, there is an embedding  $i: (V, \psi, P) \rightarrow (V', \psi', P')$  extending  $i_0$  such that  $e = i_s$ . By the previous lemma  $i$  is an elementary embedding. Thus  $(\lambda, v)$  and  $(\lambda', v')$  satisfy the same  $\mathcal{L}_{H,P}$ -formulas. ■

Let  $T_{H,P,\text{RCF}} \supseteq T_{H,P}$  be the theory of closed  $H$ -triples over real closed scalar fields. The following result was announced in Section 1:

**THEOREM 4.2.** *The theory  $T_{H,P,\text{RCF}}$  is complete, decidable, and admits elimination of quantifiers. It is the model-completion of the theory of  $H$ -triples.*

The proof uses the following consequence of Proposition 4.1:

**LEMMA 4.2.** *Let  $(V, \psi, P)$  and  $(V', \psi', P')$  be closed  $H$ -triples over scalar fields  $k$  and  $k'$ , respectively. Then  $k \equiv k'$  if and only if  $(V, \psi, P) \equiv (V', \psi', P')$ .*

*Proof.* One direction being trivial, we assume  $k \equiv k'$ , and shall derive  $(V, \psi, P) \equiv (V', \psi', P')$ . We can assume that  $(V', \psi', P')$  is  $\kappa$ -saturated, where  $\kappa := |V|^+$ . We may further assume, by Example 3.1, that  $(V, \psi, P)$  and  $(V', \psi', P')$  have as common substructure an  $H$ -triple  $(V_0, \psi_0, P_0)$  over the scalar field  $k_0 := \mathbb{Q}$ . Since  $k \equiv k'$  and  $k'$  is  $|k|^+$ -saturated, there is an elementary embedding of ordered fields  $e: k \rightarrow k'$ . Since necessarily  $e|_{k_0} = \text{id}$ , Proposition 4.1 implies that  $e$  is the scalar part of an embedding

$(V, \psi, P) \rightarrow (V', \psi', P')$ . This is an elementary embedding by Lemma 4.1. Thus  $(V, \psi, P) \equiv (V', \psi', P')$ . ■

*Proof* (of Theorem 4.2). The completeness of the theory RCF of real closed ordered fields, together with Lemma 4.2, implies the completeness of  $T_{H,P,\text{RCF}}$ . By Corollary 3.1, every  $H$ -triple can be embedded into a closed  $H$ -triple over a real closed field. That  $T_{H,P,\text{RCF}}$  admits quantifier elimination follows from Theorem 4.1 and the fact that RCF admits quantifier elimination. ■

*Remark.* By Examples 3.4 and 3.5, the  $H$ -triples of maximal Hardy fields and the  $H$ -triple of the field of LE-series over  $\mathbb{R}$  are models of  $T_{H,P,\text{RCF}}$ . Thus, by Theorem 4.2, a certain fragment of the elementary theories of these ordered differential fields has been fully analyzed at the most basic model-theoretic level.

The rest of this section is devoted to proving Proposition 4.1.

### The Functions $\psi_a$

We need a generalization of the intermediate value property of  $\text{id} + \psi$  on  $V^{>0}$  and  $V^{<0}$ . Below, let  $\mathcal{V} = (V, \psi)$  be an  $H$ -couple (not necessarily closed) over the scalar field  $k$ . For  $a = (a_1, \dots, a_n) \in V^n$ ,  $n > 0$ , we define a function  $\psi_a: V_\infty \rightarrow V_\infty$ . We proceed by induction:

1. For  $n = 1$  (with  $a \in V$ ), we put  $\psi_a(v) := \psi(v - a)$ .
2. For  $n > 1$ , we put  $\psi_a(v) := \psi(\psi_{a'}(v) - a_n)$ , where  $a' := (a_1, \dots, a_{n-1})$ .

We let  $D_a := \{v \in V : \psi_a(v) \neq \infty\}$ . Thus  $D_a = V \setminus \{a\}$  for  $n = 1$ , and  $D_a = \{v \in D_{a'} : \psi_{a'}(v) \neq a_n\}$  for  $n > 1$ . So given  $a_1, a_2, a_3, \dots$  in  $V$ , we get

$$\psi_{(a_1, a_2)}(v) = \psi(\psi(v - a_1) - a_2),$$

$$\psi_{(a_1, a_2, a_3)}(v) = \psi(\psi(\psi(v - a_1) - a_2) - a_3),$$

and so on. One verifies easily by induction on  $n$  that if  $v, v' \in D_a$  with  $v \neq v'$ , then  $[\psi_a(v) - \psi_a(v')] < [v - v']$ .

LEMMA 4.3. Let  $a = (a_1, \dots, a_n) \in V^n$ ,  $\lambda_1, \dots, \lambda_n \in k$ ,  $n > 0$ . The function

$$v \mapsto v + \lambda_1 \psi_{a_1}(v) + \lambda_2 \psi_{(a_1, a_2)}(v) + \dots + \lambda_n \psi_a(v): D_a \rightarrow V$$

is strictly increasing, and has the intermediate value property on each convex component of  $D_a$ .

*Proof.* Let  $\eta: D_a \rightarrow V$  be the function given by

$$\eta(v) := \lambda_1 \psi_{a_1}(v) + \lambda_2 \psi_{(a_1, a_2)}(v) + \cdots + \lambda_n \psi_a(v).$$

Then  $v \mapsto v + \eta(v): D_a \rightarrow V$  is strictly increasing, since for distinct  $v, v' \in D_a$  we have  $[\eta(v) - \eta(v')] < [v - v']$ . Let  $C$  be a convex component of  $D_a$  with  $a_1 < C$ , and let  $x < y < z$  be in  $C$ , with  $z - y \leq y - x$ . Then

$$y - a_1 < z - a_1 = (z - y) + (y - a_1) \leq 2(y - a_1),$$

so  $\psi(y - a_1) = \psi(z - a_1)$ ; thus  $\eta(y) = \eta(z)$ , since  $\eta(v)$  depends only on  $\psi(v - a_1)$ . By Lemma 2.2 the function  $\eta|_C$  has the intermediate value property. For the convex components less than  $a_1$  of  $D_a$  we verify instead condition (2') of the remark following Lemma 2.2. ■

LEMMA 4.4. *Suppose  $\mathcal{V}$  is closed, and let  $a = (a_1, \dots, a_n) \in V^n$ ,  $n > 0$ . Then  $D_a$  has at most  $2^n$  convex components in  $V$ , and on each of these,  $\psi_a$  is monotone and has the intermediate value property.*

*Proof.* We proceed by induction on  $n$ . For  $n = 1$ , the two convex components of  $D_a$  are  $V^{<a}$ , on which  $\psi_a$  is increasing, and  $V^{>a}$ , on which  $\psi_a$  is decreasing; on each of these,  $\psi_a$  has the intermediate value property, since  $\mathcal{V}$  is closed. Suppose the lemma holds for a certain  $a = (a_1, \dots, a_n) \in V^n$ , and let  $\hat{a} = (a_1, \dots, a_n, a_{n+1}) \in V^{n+1}$ . Consider a convex component  $C$  of  $D_a$ . Then  $\psi_a$  is monotone on  $D_a$ , say increasing on  $C$ , and has the intermediate value property on  $C$ . Put

$$C_1 := \{v \in C : \psi_a(v) < a_{n+1}\},$$

$$C_2 := \{v \in C : \psi_a(v) = a_{n+1}\},$$

$$C_3 := \{v \in C : \psi_a(v) > a_{n+1}\}.$$

Thus  $C$  is the disjoint union of its convex subsets  $C_1$ ,  $C_2$ , and  $C_3$ , and  $C_1 < C_2 < C_3$ . Also  $\psi_{\hat{a}}$  is clearly increasing on  $C_1$ , and decreasing on  $C_3$ . If both  $C_1$  and  $C_3$  are nonempty, then also  $C_2$  is nonempty (because of the intermediate value property of  $\psi_a$  on  $C$ ), and thus  $C_1$  and  $C_3$  are the convex components of  $D_{\hat{a}}$  that are contained in  $C$ . Otherwise  $C$  only contributes one convex component to  $D_{\hat{a}}$ , or none at all, depending on whether one or both of  $C_1$  and  $C_3$  are empty. ■

### *Archimedean Classes and Coinitiality*

LEMMA 4.5. *Let  $V \subseteq V'$  be an extension of ordered vector spaces over the ordered field  $\mathbf{k}$ , and let  $b \in V' \setminus V$  be such that*

- (1) *for each  $\varepsilon \in V^{>0}$  there are  $a, c \in V$  with  $a < b < c$  and  $c - a < \varepsilon$ ,*



(2)  $\{a \in V : a < b\}$  has no maximum, and  $\{c \in V : c > b\}$  has no minimum.

Then  $[V] = [V \oplus kb]$  (as subsets of  $[V']$ ).

*Proof.* Assume not. Then there is  $v \in V$  with  $[b - v] \notin [V]$ . Changing  $b$  to  $-b$  and  $v$  to  $-v$ , if necessary, we may assume  $b > v$ . Let  $\varepsilon \in V^{>0}$  be such that  $v + \varepsilon < b$ , by (2), and  $a, c \in V$  with  $v + \varepsilon \leq a < b < c$  and  $c - a < \varepsilon$ , by (1). Then  $b - a < c - a < \varepsilon \leq a - v$  and hence  $[b - a] \leq [a - v]$ . But  $b - v = (b - a) + (a - v)$  and thus  $[a - v] < [b - v] = [b - a]$ , a contradiction. ■

For the proof of Proposition 4.1, and also in Section 5, we shall need the following easy consequence of the lemma above:

**COROLLARY 4.1.** *Let  $(V, \psi) \subseteq (V', \psi')$  be an extension of  $H$ -couples over  $k$  and over  $k' \supseteq k$ , respectively, such that  $[V^*]$  has no minimum. If  $x \in V'$  and  $0 < x < V^{>0}$ , then  $[V]_k = [V \oplus k\psi'(x)]_k$  inside  $[V']_k$ . In particular, if  $[V]_k \neq [V \oplus ky]_k$  for all  $y \in V' \setminus V$ , then  $V^{>0}$  is coinital in  $(V')^{>0}$ .*

*Proof.* Let  $0 < x < V^{>0}$ ,  $x \in V'$ . Then  $b := \psi'(x)$  satisfies the hypothesis of the previous lemma, where  $V'$  is considered as an ordered vector space over  $k$ . Thus we have  $[V]_k = [V \oplus kb]_k$ . ■

*Properties (A) and (B)*

Given an extension  $(V, \psi) \subseteq (V', \psi')$  of  $H$ -couples (not necessarily over the same scalar field), and  $a = (a_1, \dots, a_n) \in V^n$ ,  $n > 0$ , we have functions  $\psi_a: V_\infty \rightarrow V_\infty$ , with  $D_a = \{v \in V : \psi_a(v) \neq \infty\}$ , and  $\psi'_a: V'_\infty \rightarrow V'_\infty$ , with  $D'_a = \{v' \in V' : \psi'_a(v') \neq \infty\}$ . Clearly  $\psi_a$  is the restriction of  $\psi'_a$  to  $V_\infty$ , and thus  $D'_a \cap V = D_a$ . Consider the following two properties of an extension  $(V, \psi) \subseteq (V', \psi')$  of closed  $H$ -couples:

(A) For all  $a \in V^n$ ,  $n > 0$ , and convex components  $C'$  of  $D'_a$ ,  $C' \cap V \neq \emptyset$ .

(B) For all  $x \in V'$ ,  $a = (a_1, \dots, a_n) \in V^n$ ,  $b \in V$ ,  $\lambda_1, \dots, \lambda_n \in k$ ,  $n > 0$ ,

$$x + \lambda_1 \psi'_{a_1}(x) + \lambda_2 \psi'_{(a_1, a_2)}(x) + \dots + \lambda_n \psi'_a(x) = b \implies x \in V.$$

(By Lemmas 4.3 and 4.4 conditions (A) and (B) are clearly satisfied for elementary extensions of closed  $H$ -couples.)

*Remark.* Let  $(V, \psi)$  be a closed  $H$ -couple,  $a \in V^n$ ,  $n > 0$ , and  $E$  a cut in the ordered set  $V$ . Then there exists a convex component  $C$  of  $D_a$  and  $\varepsilon \in \{-1, 1\}$  such that for any extension  $(V', \psi) \supseteq (V, \psi)$  of closed  $H$ -couples satisfying (A) and (B) and any  $v' \in D'_a \setminus D_a$  realizing the cut

$E$ : if  $C'$  is the convex component of  $v'$  in  $D'_a$ , then  $C' \cap V = C$ , and  $\text{sgn}(v' - \psi'_a(v')) = \varepsilon$ . (This follows by an easy induction on  $n$  as in the proof of Lemma 4.7.)

*Proof of Proposition 4.1 Using (A) and (B)*

The hardest part of the proof of Proposition 4.1 consists of showing that all extensions of closed  $H$ -couples satisfy (A) and (B). This was the last difficulty we overcame, and accordingly we postpone this part. Thus in this subsection we assume:

*All Extensions of Closed  $H$ -Couples Satisfy (A) and (B).* Let the hypothesis in the statement of Proposition 4.1 hold. To simplify the notation, we may as well assume that  $(V_0, \psi_0, P_0)$  is a common substructure of  $(V, \psi, P)$  and  $(V', \psi', P')$  and that  $k$  is an ordered subfield of  $k'$ , with  $i_0$  and  $e$  the natural inclusions. We want to extend  $i_0$  to an embedding  $i: (V, \psi, P) \rightarrow (V', \psi', P')$  such that  $i_s = e$ . By scalar extension (Lemma 3.8), we can reduce to the case  $k_0 = k$ . By Corollary 3.1, we can further reduce to the case that  $(V_0, \psi_0, P_0)$  is closed. We may also assume that  $V \neq V_0$ . By a familiar Zorn's lemma argument, it suffices to show that there is some  $H$ -triple  $(V_1, \psi_1, P_1) \subseteq (V, \psi, P)$  strictly containing  $(V_0, \psi_0, P_0)$  as a substructure, such that  $i_0$  extends to an embedding  $(V_1, \psi_1, P_1) \rightarrow (V', \psi', P')$ .

*Case 1.* Assume that we have  $v \in V \setminus V_0$  with  $[V_0 \oplus kv] = [V_0]$ . Then  $\psi(V_0 \oplus kv) = \psi_0(V_0)$ ; in particular,

$$(V_1, \psi_1, P_1) := (V_0 \oplus kv, \psi|(V_0 \oplus kv)^*, P \cap (V_0 \oplus kv))$$

is a substructure of  $(V, \psi, P)$ . We claim that there is an embedding of this substructure into  $(V', \psi', P')$  over  $(V_0, \psi_0, P_0)$ . To see this, we distinguish two subcases:

1. No  $u \in V_0 \oplus kv$  satisfies  $P_0 < u < V_0 \setminus P_0$ . (Thus  $(V_0 \oplus kv, \psi|(V_0 \oplus kv)^*)$  has only one  $H$ -cut, namely  $P \cap (V_0 \oplus kv)$ .) By saturation, we can find  $v' \in V'$  realizing the same cut in  $V_0$  as  $v$ . It follows that we have an isomorphism  $V_0 \oplus kv \rightarrow V_0 \oplus kv'$  of ordered vector spaces over  $k$  that sends  $v$  to  $v'$  and is the identity on  $V_0$ . Hence  $\psi(v_0 + \lambda v) = \psi'(v_0 + \lambda v')$  for all  $v_0 \in V_0$ ,  $\lambda \in k$ , and there is no  $u' \in V_0 \oplus kv'$  with  $P_0 < u' < V_0 \setminus P_0$ . (Thus also  $(V_0 \oplus kv', \psi'|(V_0 \oplus kv')^*)$  has only one  $H$ -cut.) So we have an embedding of  $(V_1, \psi_1, P_1)$  into  $(V', \psi', P')$  as desired.

2. There is  $u \in V_0 \oplus kv$  with  $P_0 < u < V_0 \setminus P_0$ . If  $w \in V_0 \oplus kv$  also satisfies  $P_0 < w < V_0 \setminus P_0$ , then  $\psi(\delta) < u, w < \psi(\delta) + \delta$  for all  $\delta \in V_0^{>0}$ . Therefore  $u = w$  because of  $[V_0 \oplus kv] = [V_0]$ . After renaming, we may also assume  $u = v$ . By saturation, we can find  $v' \in V'$  such that  $P_0 < v' < V_0 \setminus P_0$  and such that  $v \in P \Leftrightarrow v' \in P'$ . It follows as before that we get an embedding of  $(V_1, \psi_1, P_1)$  into  $(V', \psi', P')$  as desired.

*Case 2.* Assume that for every  $v \in V \setminus V_0$  we have  $[V_0 \oplus kv] \neq [V_0]$ . Fix some  $v \in V \setminus V_0$ . Then there is some  $a_1 \in V_0$  such that  $[v - a_1] \notin [V_0]$ , and hence  $\psi(v - a_1) \notin V_0$ . So, for some  $a_2 \in V_0$ ,  $[\psi(v - a_1) - a_2] \notin [V_0]$ , and hence  $\psi(\psi(v - a_1) - a_2) \notin V_0$ . Continuing this way, we obtain elements  $a_1, a_2, a_3, \dots$  in  $V_0$  such that, for all  $n \geq 1$ ,  $\psi_{(a_1, \dots, a_n)}(v) \notin V_0$ . (We use the notation introduced earlier in this section.) Let

$$b_1 := v - a_1, \quad b_n := \psi(b_{n-1}) - a_n \quad \text{for } n > 1.$$

Then  $[b_n] \notin [V_0]$  and  $\psi(b_n) = \psi_{(a_1, \dots, a_n)}(v)$  for all  $n \geq 1$ . We claim that  $\{b_n\}_{n \geq 1}$  is a family of vectors linearly independent over  $V_0$ . Otherwise, we would have a linear relation among the  $b_n$  and elements of  $V_0$ . By changing from  $\{a_n\}_{n \geq 1}$  to  $\{a_{n+k}\}_{n \geq 1}$  and from  $v$  to  $\psi_{(a_1, \dots, a_k)}(v)$ , for some  $k \geq 1$ , if necessary, we can assume it to be of the form

$$v + \lambda_1 \psi_{a_1}(v) + \lambda_2 \psi_{(a_1, a_2)}(v) + \dots + \lambda_n \psi_a(v) = v_0$$

for some  $n > 0$ ,  $a = (a_1, \dots, a_n)$ ,  $\lambda_1, \dots, \lambda_n \in k$ , and  $v_0 \in V_0$ . But then condition (B) would imply  $v \in V_0$ , contrary to our assumption. Thus, in particular,

$$[b_n] \neq [b_m] \quad \text{for all } m > n \geq 1, \quad (3)$$

since otherwise  $b_{n+1} - b_{m+1} = a_{m+1} - a_{n+1} \in V_0$ .

By saturation we can find  $v' \in V' \setminus V_0$  realizing the same cut in the ordered set  $V_0$  as  $v$ . Put

$$b'_1 := v' - a_1, \quad b'_n := \psi'(b'_{n-1}) - a_n \quad \text{for } n > 1.$$

We now show by induction on  $n \geq 1$  that

$$(1) \quad v' \in D_{(a_1, \dots, a_n)} \text{ and } b'_n \neq \infty,$$

(2) the cut  $C(b_n)$  determined by  $b_n$  in  $V_0$  is the same as the cut  $C(b'_n)$  determined by  $b'_n$  in  $V_0$  (hence  $[b'_n] \notin [V_0]$ ).

This is clear for  $n = 1$ , by choice of  $v'$ . Suppose (1) and (2) hold for a certain  $n \geq 1$ . Then we obtain from  $[b'_n] \notin [V_0]$  that  $\psi'_a(v') = \psi'(b'_n) \notin V_0$ , with  $a = (a_1, \dots, a_n)$ . In particular  $\psi'_a(v') \neq a_{n+1}$ , hence (1) holds for  $n + 1$  in place of  $n$ . Let

$$C_1 := \{\psi_0(v_0) : v_0 \in V_0, [v_0] > [b_n]\},$$

$$C_2 := \{u_0 \in V_0 : u_0 \geq \psi_0(v_0) \text{ for some } v_0 \in V_0 \text{ with } [v_0] < [b_n]\}.$$

Then  $C_1 < \psi(b_n) < C_2$  and  $C_1 < \psi'(b'_n) < C_2$ ,  $C_1 \cup C_2 = V_0$ . Hence  $C_1 - a_{n+1} < b_{n+1} < C_2 - a_{n+1}$ ,  $C_1 - a_{n+1} < b'_{n+1} < C_2 - a_{n+1}$ , thus  $C(b_{n+1}) = C(b'_{n+1})$ . So (2) holds with  $n + 1$  instead of  $n$ , finishing the inductive step.

Now condition (B) implies just as with  $b_1, b_2, \dots$  that  $\{b'_n\}_{n \geq 1}$  is a family of linearly independent vectors over  $V_0$ . From (2), we get

$$\text{sgn}(b_n) = \text{sgn}(b'_n) \quad \text{for all } n \geq 1, \quad (4)$$

and, by the remark preceding this proof,

$$[b_n] < [b_m] \Leftrightarrow [b'_n] < [b'_m] \quad \text{for all } n, m \geq 1. \quad (5)$$

We set

$$V_1 := V_0 \oplus \bigoplus_{n=1}^{\infty} k b_n \subseteq V, \quad \psi_1 := \psi|_{V_1^*}, \quad P_1 := P \cap V_1.$$

Clearly  $(V_1, \psi_1, P_1)$  is the  $\mathcal{L}_{H,P}$ -substructure of  $(V, \psi, P)$  generated by  $v$  over  $(V_0, \psi_0, P_0)$ . Consider the (injective)  $k$ -linear map  $V_1 \rightarrow V'$  that is the identity on  $V_0$  and sends each  $b_n$  to  $b'_n$ . Using the fact that  $[b_n] \notin [V_0]$ ,  $[b'_n] \notin [V_0]$ , and  $C(b_n) = C(b'_n)$  for all  $n \geq 1$ , together with (3)–(5), one sees that this map is also order-preserving. Moreover, it is easily shown to be the vector part of an embedding  $(V_1, \psi_1) \rightarrow (V', \psi')$  of  $\mathcal{L}_H$ -structures with the identity on  $k$  as scalar part. To show that we even have an embedding of  $\mathcal{L}_{H,P}$ -structures  $(V_1, \psi_1, P_1) \rightarrow (V', \psi', P')$ , it suffices to prove that  $(V_1, \psi_1)$  has only one  $H$ -cut. For a contradiction, assume that there exists  $v_1 \in V_1$  with  $\psi_1(V_1^{>0}) < v_1 < (\text{id} + \psi_1)(V_1^{>0})$ . But  $\psi_0(V_0^{>0})$  is cofinal in  $\psi(V^{>0})$ , and  $(\text{id} + \psi)(V_0^{>0})$  is coinital in  $(\text{id} + \psi)(V^{>0})$ , by Corollary 4.1. This implies  $\psi(V^{>0}) < v_1 < (\text{id} + \psi)(V^{>0})$ ; i.e.,  $(V, \psi)$  has two  $H$ -cuts, contradicting the closedness of  $(V, \psi)$ .

This finishes the proof of Proposition 4.1, except that we still have to prove properties (A) and (B) for all extensions  $(V, \psi) \subseteq (V', \psi')$  of closed  $H$ -couples. The remainder of this section is devoted to this task. ■

### *Proof of (A) and (B)*

We first make a more detailed study of the behavior of the functions  $\psi_a$  on the convex components of  $D_a$ , in the case of a closed  $H$ -couple. In the remainder of this section we let  $\mathcal{V} = (V, \psi)$  be a closed  $H$ -couple.

LEMMA 4.6. *Let  $p \in V$ . Then there is  $u \in V$  such that  $\psi(x) = u$  for all sufficiently large  $x \in \Psi + p$ . Moreover, if  $(V', \psi')$  is a closed  $H$ -couple extending  $(V, \psi)$ , the same  $u \in V$  has the property that  $\psi'(x) = u$  for all sufficiently large  $x \in \Psi' + p$ , where  $\Psi' := \psi'((V')^*)$ .*

*Proof.* First assume  $-p \in \Psi$ , so  $\psi(x) + p > 0$  for all sufficiently small  $x > 0$ . Take  $x_0 > 0$  in  $V$  such that  $\psi(x_0) + p > 0$  and  $[x_0] < [\psi(x_0) + p]$ . (Decreasing  $x_0$  makes  $\psi(x_0) + p$  increase, so this is indeed possible.) We claim that

$$[\psi(x_0) + p] = [\psi'(x') + p] \quad \text{for all } 0 < x' < x_0 \text{ in } V'.$$

Otherwise,

$$[\psi(x_0) + p] < [\psi'(x') + p] \leq [\psi(x_0) + x_0 + p] = [\psi(x_0) + p],$$

a contradiction. Thus  $u := \psi(\psi(x_0) + p)$  works. Now assume  $-p \notin \Psi$ . Then  $-p = \psi(x_0) + x_0$  for some  $x_0 > 0$  in  $V$ . We claim that

$$[x_0] = [\psi'(x') + p] \quad \text{for all } 0 < x' < x_0 \text{ in } V'.$$

Otherwise,

$$[x_0] = [\psi(x_0) + p] < [\psi'(x') + p] = [\psi(x_0) + x_0 - \psi'(x')] \leq [x_0],$$

a contradiction. So  $u := \psi(x_0)$  works in this case. ■

*Notation.* We will denote the element  $u$  in Lemma 4.6 by  $\lim_{x \in \Psi + p} \psi(x)$ . (Hence  $\lim_{x \in \Psi + p} \psi(x) = \lim_{x \in \Psi + p} \psi'(x)$ .)

We fix some terminology. Let  $f: V_\infty \rightarrow V_\infty$  be a function, and let  $C$  be a non-empty convex subset of  $V$  on which  $f$  does not take the value  $\infty$ . Let  $p, q \in V$ , and let  $S \subseteq V$  be downward closed. (We only use this for  $f = \psi_a$ ,  $C$  is a convex component of  $D_a$ , and  $S = \Psi$ .)

1.  $f$  increases on  $C$  from  $p$  to  $q$  if  $f|C$  is increasing,  $p \leq q$ , and

$$f(C) = [p, q] = \{v \in V : p \leq v \leq q\}.$$

(We allow  $f|C$  constant and  $p = q$ .)

2.  $f$  increases on  $C$  from  $-\infty$  to  $q$  if  $f|C$  is increasing, and

$$f(C) = (-\infty, q] = \{v \in V : v \leq q\}.$$

3.  $f$  increases on  $C$  from  $p$  to  $S$  if  $f|C$  is increasing, and

$$f(C) = \{v \in S : v \geq p\}.$$

4.  $f$  increases on  $C$  from  $-\infty$  to  $S$  if  $f|C$  is increasing, and  $f(C) = S$ .

Similarly, one defines what it means that  $f$  decreases on  $C$  from  $p$  to  $q$ ,  $f$  decreases on  $C$  from  $p$  to  $-\infty$ ,  $f$  decreases on  $C$  from  $S$  to  $q$ , and  $f$  decreases on  $C$  from  $S$  to  $-\infty$ .

Let now a second closed  $H$ -couple  $\mathcal{V}' = (V', \psi')$  extending  $\mathcal{V} = (V, \psi)$  be given, and let  $a = (a_1, \dots, a_n) \in V^n$ ,  $n > 0$ .

Below, we write “component” instead of “convex component.”

LEMMA 4.7. 1. Each component  $C$  of  $D_a$  is contained in a (necessarily unique) component  $C'$  of  $D'_a$ , and the map  $C \mapsto C'$  is a bijection between the set of components of  $D_a$  and the set of components of  $D'_a$ , with  $C' \cap V = C$  for each component  $C$  of  $D_a$ .

2.  $D_a$  has a (necessarily unique) component  $C_\infty > a_1$  that is unbounded in  $V$ ; the corresponding component  $C'_\infty$  of  $D'_a$  is unbounded in  $V'$ .

3. Let  $C$  be a bounded component of  $D_a$ . Then there are  $p, q \in V$  such that one of the following holds:

- (a)  $\psi_a$  increases on  $C$  from  $p$  to  $q$ , and  $\psi'_a$  increases on  $C'$  from  $p$  to  $q$ .
- (b)  $\psi_a$  decreases on  $C$  from  $p$  to  $q$ , and  $\psi'_a$  decreases on  $C'$  from  $p$  to  $q$ .
- (c)  $\psi_a$  increases on  $C$  from  $p$  to  $\Psi$ , and  $\psi'_a$  increases on  $C'$  from  $p$  to  $\Psi'$ .
- (d)  $\psi_a$  decreases on  $C$  from  $\Psi$  to  $q$ , and  $\psi'_a$  decreases on  $C'$  from  $\Psi'$  to  $q$ .

4. Let  $C_\infty$  be the unbounded component greater than  $a_1$  of  $D_a$ ,  $C'_\infty$  the corresponding component of  $D'_a$ . Then one of the following holds:

- (a)  $\psi_a$  decreases on  $C_\infty$  from  $\Psi$  to  $-\infty$ , and  $\psi'_a$  decreases on  $C'_\infty$  from  $\Psi'$  to  $-\infty$ .
- (b) There is  $p \in V$  such that  $\psi_a$  decreases on  $C_\infty$  from  $p$  to  $-\infty$ , and  $\psi'_a$  decreases on  $C'_\infty$  from  $p$  to  $-\infty$ .

*Proof.* We proceed by induction on  $n$ . The case  $n = 1$  is easy to verify. Suppose the lemma holds for a certain  $a = (a_1, \dots, a_n) \in V^n$ . Let  $\hat{a} = (a_1, \dots, a_n, a_{n+1}) \in V^{n+1}$ . Since the components less than  $a_1$  of  $D_a$  are obtained from the components greater than  $a_1$  of  $D_a$  by reflection at the point  $x = a_1$ , about which the functions  $\psi_a$  and  $\psi'_a$  are symmetric, we only need to consider the case of components greater than  $a_1$ . So let  $C > a_1$  be a component of  $D_a$ , and  $C'$  the corresponding component of  $D'_a$ . Define

$$C_1 := \{v \in C : \psi_a(v) < a_{n+1}\},$$

$$C_2 := \{v \in C : \psi_a(v) = a_{n+1}\},$$

$$C_3 := \{v \in C : \psi_a(v) > a_{n+1}\},$$

and define the sets  $C'_i$  for  $i = 1, 2, 3$  in the same way, by replacing  $C$  by  $C'$  and  $\psi_a$  by  $\psi'_a$ . Hence  $C'_i \cap V = C_i$  for  $i = 1, 2, 3$ . The components of  $D_{\hat{a}}$  that are contained in  $C$  are the nonempty sets among  $C_1$  and  $C_3$ , and, similarly, the components of  $D'_{\hat{a}}$  that are contained in  $C'$  are the nonempty sets among  $C'_1$  and  $C'_3$ .

Assume first  $C$  is bounded in  $V$  (and hence  $C'$  is bounded in  $V'$ ). We shall assume  $\psi_a$  is increasing on  $C$  (hence  $\psi'_a$  increasing on  $C'$ ). The case that  $\psi_a$  is decreasing on  $C$  is similar and left to the reader. We distinguish several cases:

- 1. There exist  $p, q \in V$  such that  $\psi_a$  increases on  $C$  from  $p$  to  $q$ , and  $\psi'_a$  increases on  $C'$  from  $p$  to  $q$ .

(i)  $q \leq a_{n+1}$ . Then  $C_3, C'_3 = \emptyset$ . If  $q < a_{n+1}$ , then  $C_1, C'_1 \neq \emptyset$ ,  $C_2, C'_2 = \emptyset$ , and  $\psi_{\hat{a}}$  increases on  $C_1$  from  $\psi(p - a_{n+1})$  to  $\psi(q - a_{n+1})$ , and

$\psi'_a$  increases on  $C'_1$  from  $\psi(p - a_{n+1})$  to  $\psi(q - a_{n+1})$ . If  $a_{n+1} = q > p$ , then  $C_1, C'_1 \neq \emptyset$ ,  $C_2, C'_2 \neq \emptyset$ , and  $\psi_{\hat{a}}$  increases on  $C_1$  from  $\psi(p - a_{n+1})$  to  $\Psi$ , and  $\psi'_{\hat{a}}$  increases on  $C'_1$  from  $\psi(p - a_{n+1})$  to  $\Psi'$ . If  $a_{n+1} = p = q$ , then  $C_1, C'_1 = \emptyset$ .

(ii)  $a_{n+1} \leq p$ ,  $q \neq a_{n+1}$ . Then  $C_1, C'_1 = \emptyset$  and  $C_3, C'_3 \neq \emptyset$ . If  $a_{n+1} < p$ , then  $C_2, C'_2 = \emptyset$ , and  $\psi_{\hat{a}}$  decreases on  $C_3$  from  $\psi(p - a_{n+1})$  to  $\psi(q - a_{n+1})$ , and  $\psi'_{\hat{a}}$  decreases on  $C'_3$  from  $\psi(p - a_{n+1})$  to  $\psi(q - a_{n+1})$ . If  $a_{n+1} = p$ , then  $C_2, C'_2 \neq \emptyset$ , and  $\psi_{\hat{a}}$  decreases on  $C_3$  from  $\Psi$  to  $\psi(q - a_{n+1})$ , and  $\psi'_{\hat{a}}$  decreases on  $C'_3$  from  $\Psi'$  to  $\psi(q - a_{n+1})$ .

(iii)  $p < a_{n+1} < q$ . Then  $C_1, C'_1 \neq \emptyset$ ,  $C_2, C'_2 \neq \emptyset$ ,  $C_3, C'_3 \neq \emptyset$ . Here,  $\psi_{\hat{a}}$  increases on  $C_1$  from  $\psi(p - a_{n+1})$  to  $\Psi$ , and  $\psi'_{\hat{a}}$  increases on  $C'_1$  from  $\psi(p - a_{n+1})$  to  $\Psi'$ . Similarly,  $\psi_{\hat{a}}$  decreases on  $C_3$  from  $\Psi$  to  $\psi(q - a_{n+1})$ , and  $\psi'_{\hat{a}}$  decreases on  $C'_3$  from  $\Psi'$  to  $\psi(q - a_{n+1})$ .

2. There exists  $p \in V$  such that  $\psi_a$  increases on  $C$  from  $p$  to  $\Psi$ , and  $\psi'_a$  increases on  $C'$  from  $p$  to  $\Psi'$ . (Thus  $p \in \Psi$ .) This case is essentially treated as the first one, using Lemma 4.6. If, for example,  $a_{n+1} < p$ , so that  $C_1, C'_1 = \emptyset$ ,  $C_2, C'_2 = \emptyset$ , and  $C_3, C'_3 \neq \emptyset$ , then  $\psi_{\hat{a}}$  decreases on  $C_3$  from  $\psi(p - a_{n+1})$  to  $\lim_{x \in \Psi - a_{n+1}} \psi(x)$ , and  $\psi'_{\hat{a}}$  decreases on  $C'_3$  from  $\psi(p - a_{n+1})$  to  $\lim_{x \in \Psi' - a_{n+1}} \psi'(x)$ , which equals  $\lim_{x \in \Psi - a_{n+1}} \psi(x)$ . We leave the details to the reader.

Now suppose  $C = C_\infty$  is the unbounded component greater than  $a_1$  of  $D_a$ , and hence  $C' = C'_\infty$  the unbounded component greater than  $a_1$  of  $D'_a$ . We have two cases again:

1.  $\psi_a$  decreases on  $C$  from  $\Psi$  to  $-\infty$ , and  $\psi'_a$  decreases on  $C'$  from  $\Psi'$  to  $-\infty$ . If  $a_{n+1} > \Psi$ , we have  $C_1, C'_1 \neq \emptyset$  and  $C_2, C'_2, C_3, C'_3 = \emptyset$ . Hence  $\psi_{\hat{a}}$  decreases on  $C_1$  from  $\lim_{x \in \Psi - a_{n+1}} \psi(x)$  to  $-\infty$ ,  $\psi'_{\hat{a}}$  decreases on  $C'_1$  from  $\lim_{x \in \Psi' - a_{n+1}} \psi'(x)$  to  $-\infty$ . In this case,  $C_1$  is the unbounded component greater than  $a_1$  of  $D_{\hat{a}}$ ,  $C'_1$  is the unbounded component greater than  $a_1$  of  $D'_{\hat{a}}$ . If, on the other hand,  $a_{n+1} \in \Psi$ , then  $C_1, C'_1 \neq \emptyset$ ,  $C_3, C'_3 \neq \emptyset$ . So  $\psi_{\hat{a}}$  decreases on  $C_1$  from  $\Psi$  to  $-\infty$ ,  $\psi'_{\hat{a}}$  decreases on  $C'_1$  from  $\Psi'$  to  $-\infty$ , and  $\psi_{\hat{a}}$  increases on  $C_3$  from  $\lim_{x \in \Psi - a_{n+1}} \psi(x)$  to  $\Psi$ ,  $\psi'_{\hat{a}}$  increases on  $C'_3$  from  $\lim_{x \in \Psi - a_{n+1}} \psi(x)$  to  $\Psi'$ . The unbounded component greater than  $a_1$  of  $D_{\hat{a}}$  is  $C_1$ , and the unbounded component greater than  $a_1$  of  $D'_{\hat{a}}$  is  $C'_1$ .

2. There is  $p \in V$  such that  $\psi_a$  decreases on  $C$  from  $p$  to  $-\infty$ , and  $\psi'_a$  decreases on  $C$  from  $p$  to  $-\infty$ . This case is treated similarly to the previous one, except that we now have three subcases, according to whether  $a_{n+1} > p$ ,  $a_{n+1} = p$  or  $a_{n+1} < p$ .

This finishes the inductive step and hence the proof of the lemma. ■

COROLLARY 4.2.  $\psi'_a(C') \cap V = \psi_a(C)$  for each component  $C$  of  $D_a$ .

We now also fix scalars  $\lambda_1, \dots, \lambda_n \in k$ , so that we have functions  $\theta: D_a \rightarrow V$  and  $\theta': D'_a \rightarrow V$  given by

$$\begin{aligned}\theta(v) &:= v + \lambda_1 \psi_{a_1}(v) + \lambda_2 \psi_{(a_1, a_2)}(v) + \dots + \lambda_n \psi_a(v), \\ \theta'(v') &:= v' + \lambda_1 \psi'_{a_1}(v') + \lambda_2 \psi'_{(a_1, a_2)}(v') + \dots + \lambda_n \psi'_a(v')\end{aligned}$$

for  $v \in D_a$  and  $v' \in D'_a$ . Note that  $\theta = \theta'|D_a$ .

*Remark.* Let  $C_\infty$  be the unbounded component greater than  $a_1$  of  $D_a$ . Then  $\theta$  is not bounded from above on  $C_\infty$ ; that is, for any  $b \in V$  there exists  $x \in C_\infty$  with  $\theta(x) > b$ . This is because  $[\theta(x) - \theta(y)] = [x - y]$  for all  $x, y \in D_a$  and  $[V]$  has no maximum, by the closedness of  $(V, \psi)$ . Now choose any  $y \in C_\infty$  and  $x > y$  such that  $[x] > [y]$ ,  $[b - \theta(y)]$ . Then  $[\theta(x) - \theta(y)] > [b - \theta(y)]$ ; in particular,  $\theta(x) > b$ . Similarly,  $\theta$  is not bounded from below on the unbounded component less than  $a_1$  of  $D_a$ .

LEMMA 4.8. *Let  $C$  be a component of  $D_a$ , with corresponding component  $C'$  of  $D'_a$ . If  $d \in C' \setminus C$ , then  $\theta'(d) \in V' \setminus V$ .*

*Proof.* We proceed by induction on  $n$ . The case  $n = 1$  is easily checked using Lemma 3.1. Assume the lemma holds for a certain  $a = (a_1, \dots, a_n) \in V^n$  and certain scalars  $\lambda_1, \dots, \lambda_n \in k$ . Let  $\hat{a} = (a_1, \dots, a_n, a_{n+1}) \in V^{n+1}$  and let a further scalar  $\lambda_{n+1} \in k$  be given. Then we have corresponding functions  $\hat{\theta}: D_{\hat{a}} \rightarrow V$  and  $\hat{\theta}': D'_{\hat{a}} \rightarrow V$  given by

$$\begin{aligned}\hat{\theta}(v) &:= \theta(v) + \lambda_{n+1} \psi_{\hat{a}}(v), \\ \hat{\theta}'(v') &:= \theta'(v') + \lambda_{n+1} \psi'_{\hat{a}}(v').\end{aligned}$$

Let  $C$  be a component of  $D_a$  with corresponding component  $C'$  of  $D'_a$ . We define  $C_i$  and  $C'_i$  (for  $i = 1, 2, 3$ ) as in the proof of Lemma 4.7. Then the components of  $D_{\hat{a}}$  that are contained in  $C$  are the nonempty sets among  $C_1$  and  $C_3$ , and the components of  $D'_{\hat{a}}$  that are contained in  $C'$  are the nonempty sets among  $C'_1$  and  $C'_3$ . We assume  $d \in C'_i \setminus C_i$  for  $i = 1$  or  $i = 3$ , and have to show that  $\hat{\theta}'(d) \notin V$ . If  $d$  lies in the convex hull of  $C_i$  in  $C'_i$ , that is, if there are  $p, q \in C_i$  such that  $p < d < q$ , then the injectivity of  $\hat{\theta}'$  and intermediate value property of  $\hat{\theta}|[p, q]$  already guarantee that  $\hat{\theta}'(d) \in V' \setminus V$ , without use of the induction hypothesis. So from now on, we assume that  $d$  does not lie in the convex hull of  $C_i$  in  $C'_i$ .

Suppose there exists an element  $c \in V$  lying strictly between  $d$  and  $a_1$ , and set  $\varepsilon := \frac{1}{2}|c - a_1| > 0$ . Then  $\psi'_{a_1}$  is constant on the segment

$$I = I_c := \{x \in V' : d - \varepsilon \leq x \leq d + \varepsilon\},$$

since  $[x - a_1] = [d - a_1]$  for all  $x \in I$ . By an easy induction on  $k$ , one shows that  $I \subseteq D'_{(a_1, \dots, a_k)}$  and that  $\psi'_{(a_1, \dots, a_k)}$  is constant on  $I$  for  $k = 1, \dots, n + 1$ .



In particular,  $I \subseteq C'_i$ , and  $\psi'_a$  is constant on  $I$ , and  $\hat{\theta}'(x) = \hat{\theta}'(d) + x - d$  for all  $x \in I$ . If  $I \cap C_i \neq \emptyset$ , say  $e \in I \cap C_i$ , then

$$\hat{\theta}'(d) = \theta'(d) + \lambda_{n+1}\psi'_{\hat{a}}(d) = \theta'(d) + \lambda_{n+1}\psi_{\hat{a}}(e) \notin V,$$

since  $\theta'(d) \notin V$ , by the induction hypothesis. Thus for the rest of the proof we shall assume that whenever  $c \in V$  lies strictly between  $d$  and  $a_1$ , and  $I = I_c$  is defined as above, then  $I \cap C_i = \emptyset$ .

Next we observe that the situation is symmetric about  $a_1$ , that is, the reflection  $a_1 + x \mapsto a_1 - x: V' \rightarrow V'$  maps  $D'_a$  onto itself, and  $\hat{\theta}'$  is invariant under this reflection. Therefore we shall assume in addition that  $d$ ,  $C$ , and  $C'$  are all greater than  $a_1$ .

We now first consider the case that  $C$  is bounded in  $V$ ,  $\psi_a$  is increasing on  $C$  (hence  $\psi'_a$  increasing on  $C'$ ), and  $i = 1$ . Then  $C_1 < C_2 < C_3$  and  $C'_1 < C'_2 < C'_3$ . The following possibilities arise (see the proof of Lemma 4.7):

(1)  $\psi_{\hat{a}}$  increases on  $C_1$  from  $p$  to  $\Psi$ , and  $\psi'_{\hat{a}}$  increases on  $C'_1$  from  $p$  to  $\Psi'$  for some  $p \in V$ . By the proof of Lemma 4.7, this implies  $C_2 \neq \emptyset$ . Since  $d$  is not in the convex hull of  $C_1$  in  $C'_1$ , either  $d > C_1$  or  $d < C_1$ .

(i)  $d > C_1$ . Then there exists an element  $c \in V$  with  $a_1 < c < d$  (take any  $c \in C_1$ ), and thus  $C_1 < I < C_2$ , where  $I = I_c$  as defined above. We can choose  $b \in C_1$  so large that  $|a_{n+1} - \psi_a(b)| \leq \varepsilon$ , with  $\varepsilon \in V^{>0}$  as above. Hence, in  $[V']$ ,

$$[\psi'_{\hat{a}}(d) - \psi_{\hat{a}}(b)] < [\psi'_{\hat{a}}(d) - \psi_a(b)] \leq [a_{n+1} - \psi_a(b)] \leq [\varepsilon].$$

Let  $f(x) := \theta'(x) + \lambda_{n+1}\psi_{\hat{a}}(b)$  for  $x \in I$ . Then

$$\begin{aligned} \hat{\theta}'(d) - f(d - \varepsilon) &= \varepsilon + \hat{\theta}'(d - \varepsilon) - f(d - \varepsilon) \\ &= \varepsilon + \lambda_{n+1}(\psi'_{\hat{a}}(d) - \psi_{\hat{a}}(b)). \end{aligned}$$

Thus  $\hat{\theta}'(d) > f(d - \varepsilon)$  and, similarly,  $\hat{\theta}'(d) < f(d + \varepsilon)$ . Hence, by the intermediate value property for  $f$  on  $I$  (Lemma 4.3), there exists  $x \in I$  with  $f(x) = \hat{\theta}'(d)$ . Since  $I \cap C_1 = \emptyset$ , we have  $x \notin V$ , and hence  $f(x) \notin V$ , by the induction hypothesis. Therefore  $\hat{\theta}'(d) \notin V$ , as required.

(ii)  $d < C_1$ . Then  $\psi'_{\hat{a}}(x) = p$  for all  $d \leq x < C_1$ . In particular,  $\hat{\theta}'(d) = \theta'(d) + \lambda_{n+1}p \notin V$ , by the induction hypothesis.

(2)  $\psi_{\hat{a}}$  increases on  $C_1$  from  $p$  to  $q$ , and  $\psi'_{\hat{a}}$  increases on  $C'_1$  from  $p$  to  $q$  for certain  $p, q \in V$ . Again either  $d < C_1$  or  $d > C_1$ . Both subcases are treated as in (1)(ii).

Next we consider the case that  $C$  is bounded in  $V$ ,  $\psi_a$  is increasing on  $C$  (hence  $\psi'_a$  increasing on  $C'$ ), and  $i = 3$ . Then either  $\psi_{\hat{a}}$  decreases on  $C_3$  from  $\Psi$  to  $q$ , and  $\psi'_{\hat{a}}$  decreases on  $C'_3$  from  $\Psi'$  to  $q$  for some  $q \in V$ , or

$\psi_{\hat{a}}$  decreases on  $C_3$  from  $p$  to  $q$ , and  $\psi'_{\hat{a}}$  decreases on  $C'_3$  from  $p$  to  $q$  for some  $p, q \in V$ . The latter subcase is treated as in (2) above. In the first subcase, suppose that  $d < C_3$ . Then  $C_2 \neq \emptyset$ , hence there exists  $c \in V$  with  $a_1 < c < d$ , and thus  $C_2 < I < C_3$ , where  $I = I_c$  as previously defined. Now for any  $\varepsilon \in V^{>0}$ , in particular for  $\varepsilon = \frac{1}{2}(c - a_1)$ , we can choose  $b \in C_3$  so small that  $|a_{n+1} - \psi_a(b)| \leq \varepsilon$ . Now continue as in (1)(i) above. If  $d > C_3$ , argue as in (1)(ii).

The case that  $C$  is bounded in  $V$  and  $\psi_a$  is decreasing on  $C$  can be handled in a similar way, and is left to the reader.

Now assume  $C$  is unbounded in  $V$  and  $i = 1$ . Then we have the following possibilities:

1.  $\psi_{\hat{a}}$  decreases on  $C_1$  from  $p$  to  $-\infty$ , and  $\psi'_{\hat{a}}$  decreases on  $C'_1$  from  $p$  to  $-\infty$  for some  $p \in V$ . Again, either  $d < C_1$  or  $d > C_1$ . The first option is treated as in (1)(ii) above, whereas, in the second case,  $\hat{\theta}'(d) \notin V$  follows from the remark preceding this lemma and Lemma 4.3.

2.  $\psi_{\hat{a}}$  decreases on  $C_1$  from  $\Psi$  to  $-\infty$ , and  $\psi'_{\hat{a}}$  decreases on  $C'_1$  from  $\Psi'$  to  $-\infty$ . If  $d < C_1$ , we see, by inspection of the proof of Lemma 4.7, that necessarily  $C_2 \neq \emptyset$ . Hence there exists  $c \in V$  with  $a_1 < c < d$ . Now adopt the argument in 1(i) above. If  $d > C_1$ , we again apply the remark preceding the lemma.

Finally, consider the case that  $C$  is unbounded and  $i = 3$ . Then  $\psi_{\hat{a}}$  increases on  $C_3$  from  $p$  to  $\Psi$ , and  $\psi'_{\hat{a}}$  increases on  $C_3$  from  $p$  to  $\Psi'$  for some  $p \in V$ . If  $d > C_3$ , note that any  $c \in C_3$  will satisfy  $a_1 < c < d$ , and continue as in 1(i). If  $d < C_3$ , argue as in 1(ii). This finishes the induction. ■

*Remark.* Property (A) follows from Lemma 4.7, and property (B) from the previous lemma.

## 5. MODEL-THEORETIC PROPERTIES

The results of the previous section constitute a model-theoretic analysis of closed  $H$ -couples on the most basic level, namely that of “elimination theory.” In this section we deal with the more intrinsic properties of  $H$ -couples to which this analysis gives access. This concerns in the first place the shape of the definable sets in a closed  $H$ -couple; see Corollary 5.1 and Proposition 5.1 below. Here and in the rest of the paper “definable” will mean “definable with parameters.” We also determine the definable closure of an  $H$ -triple in a closed extension, and prove uniqueness of  $H$ -closures. Finally, we analyze simple extensions of  $H$ -couples, and use this analysis to show that in a finitely generated  $H$ -couple the set  $\Psi$  is well ordered.

### Induced Structure on the Scalar Field

We first show that in a closed  $H$ -couple no new structure is induced on the scalar field. More precisely:

**COROLLARY 5.1.** *Let  $(V, \psi)$  be a closed  $H$ -couple over  $k$ , and let  $S \subseteq k^n$  be definable in  $(V, \psi)$ . Then  $S$  is already definable in the ordered field  $k$ .*

*Proof.* Let the  $H$ -couples  $(V_1, \psi_1)$  over  $k_1$  and  $(V_2, \psi_2)$  over  $k_2$  be elementary extensions of  $(V, \psi)$ . (In particular, the ordered fields  $k_1$  and  $k_2$  are elementary extensions of the ordered field  $k$ .) Suppose that  $\lambda = (\lambda_1, \dots, \lambda_n) \in k_1^n$  and  $\mu = (\mu_1, \dots, \mu_n) \in k_2^n$  realize the same type over  $k$  in  $k_1$  and  $k_2$ , respectively. It suffices to show that then they realize the same type over  $(V, \psi)$  in  $(V_1, \psi_1)$  and  $(V_2, \psi_2)$ , respectively. We may assume that  $(V_2, \psi_2)$  is  $\kappa$ -saturated, where  $\kappa := |V_1|^+$ . It follows that there is an elementary embedding  $e: k_1 \rightarrow k_2$  that is the identity on  $k$  and that sends each  $\lambda_i$  to  $\mu_i$ . By Proposition 4.1 and Lemma 4.1,  $e$  is the scalar part of an elementary embedding  $(V_1, \psi_1) \rightarrow (V_2, \psi_2)$  over  $(V, \psi)$ . Hence  $\lambda$  and  $\mu$  realize the same type over  $(V, \psi)$  in  $(V_1, \psi_1)$  and  $(V_2, \psi_2)$ . ■

### Induced Structure on the Vector Space

Let  $(V, \psi)$  be a closed  $H$ -couple over the scalar field  $k$ . To discuss the induced structure on the underlying vector space  $V$  we introduce the one-sorted language  $\mathcal{L}_{k,v}$  that extends the language  $\{0, +, -, <\}$  of ordered abelian groups by an  $n$ -ary relation symbol  $R_{\lambda, \varphi}$  for every  $\lambda \in k^m$  and  $\mathcal{L}_H$ -formula  $\varphi = \varphi(x, y)$ , where  $x = (x_1, \dots, x_m)$  is a tuple of scalar variables and  $y = (y_1, \dots, y_n)$  is a tuple of vector variables. We make  $V$  into an  $\mathcal{L}_{k,v}$ -structure by interpreting  $0, +, -, <$  as usual, and  $R_{\lambda, \varphi}$  as

$$\{v \in V^n : (V, \psi) \models \varphi(\lambda, v)\}.$$

Thus a set  $S \subseteq V^n$  is definable in the one-sorted  $\mathcal{L}_{k,v}$ -structure  $V$  if and only if it is definable in the two-sorted  $\mathcal{L}_H$ -structure  $(V, \psi)$ .

Let  $\mathbf{A} = (A, <, \dots)$  be a structure (in some one-sorted language  $\mathcal{L}$  containing a binary relation symbol  $<$ ) that expands a linearly ordered nonempty set  $(A, <)$ , dense without endpoints. Following Marker and Steinhorn we say that  $\mathbf{A}$  is *locally o-minimal* if for each definable set  $S \subseteq A$  and each  $a \in A$  there exist  $a_1, a_2 \in A$  such that  $a_1 < a < a_2$ , and  $(a_1, a)$  is either disjoint from  $S$  or contained in  $S$ , and  $(a, a_2)$  is either disjoint from  $S$  or contained in  $S$ . The structure  $\mathbf{A}$  is called *weakly o-minimal* if every definable subset of  $A$  is a finite union of convex subsets; see [8]. Clearly, if  $\mathbf{A}$  is weakly o-minimal, then it is locally o-minimal.

For a closed  $H$ -couple  $(V, \psi)$ , the  $\mathcal{L}_{k,v}$ -structure  $V$  is never weakly o-minimal: consider the definable subset  $k \cdot 1$  of  $V$ ; it is not a finite union of convex subsets. However, we have:

PROPOSITION 5.1. *Let  $(V, \psi)$  be a closed  $H$ -couple over  $k$ . Then  $V$  is locally  $\mathcal{L}_{k,v}$ -minimal as an  $\mathcal{L}_{k,v}$ -structure.*

*Proof.* Below we consider  $V$  as an  $\mathcal{L}_{k,v}$ -structure. Take a  $\kappa$ -saturated elementary extension  $(V', \psi')$  of  $(V, \psi)$  where  $\kappa = |V|^+$ . Thus  $V'$  is then naturally a  $\kappa$ -saturated  $\mathcal{L}_{k,v}$ -structure elementary extending  $V$ . Below we consider  $V'$  as an  $\mathcal{L}_{k,v}$ -structure in this way.

By familiar model-theoretic reasoning, it now suffices to show that, given  $v \in V$ , any two vectors  $v_1, v_2 \in V'$  such that  $v < v_i < v + \varepsilon$  for all  $\varepsilon > 0$  in  $V$ ,  $i = 1, 2$ , realize the same type over  $V$  in  $V'$ . By translation over  $-v$  we reduce to the case  $v = 0$ . Then  $\Psi < \psi'(v_i) < (\text{id} + \psi)(V^{>0})$ ; hence  $[V \oplus k\psi'(v_i)]_k = [V]_k$  inside  $[V']_k$ , for  $i = 1, 2$ , by Corollary 4.1. After embedding  $(V, \psi)$  into  $(H, \psi_H)$  with  $H = H([V^*], k)$ , cf. Lemma 3.3, we see that in some  $H$ -couple over  $k$  extending  $(V, \psi)$ , there is an element  $u$  such that  $\Psi < u < (\text{id} + \psi)(V^{>0})$ . But  $V \oplus ku$  and  $V \oplus k\psi'(v_i)$  are isomorphic over  $V$  as ordered vector spaces over  $k$ . Since  $V \oplus ku$  is a Hahn space over  $k$ , so is  $V_i := V \oplus k\psi'(v_i)$  for  $i = 1, 2$ . Let  $P_i := V_i^{\leq \psi'(v_i)}$  for  $i = 1, 2$ . Since  $[V_i] = [V]$ , we have  $\psi'(V_i^*) \subseteq V$ ; let  $\psi_i := \psi'|_{V_i^*}$ . As in the first part of the proof of Proposition 4.1 above, we obtain an isomorphism  $(V_1, \psi_1, P_1) \rightarrow (V_2, \psi_2, P_2)$  over  $(V, \psi, \Psi)$ , mapping  $\psi'(v_1)$  to  $\psi'(v_2)$ . Since  $[v_i] < [V^*] = [V_i^*]$ , the cut in  $V_1$  realized by  $v_1$  corresponds, under this isomorphism, to the cut in  $V_2$  realized by  $v_2$ . Hence we can extend the vector part of this isomorphism to an isomorphism  $V'_1 := V_1 \oplus kv_1 \rightarrow V_2 \oplus kv_2 =: V'_2$  of ordered vector spaces over  $k$ , mapping  $v_1$  to  $v_2$ . Note that the image  $\Psi \cup \{\psi'(v_i)\}$  of  $\psi'_i := \psi'|_{(V'_i)^*}$  has a largest element  $\psi'(v_i)$ ; hence  $(V'_i, \psi'_i)$  has only one  $H$ -cut. Therefore the map under consideration is the vector part of an isomorphism  $(V'_1, \psi'_1, \Psi' \cap V'_1) \rightarrow (V'_2, \psi'_2, \Psi' \cap V'_2)$  of  $\mathcal{L}_{H,P}$ -structures, whose scalar part is the identity on  $k$ . Thus, by relative quantifier elimination,  $v_1$  and  $v_2$  have the same type over  $V$  in  $V'$ . ■

*Remark.* The  $\mathcal{L}_{k,v}$ -structure  $V$  is even “ $\mathcal{o}$ -minimal at infinity”: for any definable set  $S \subseteq V$  there exists  $a \in V$  such that either  $V^{>a} \subseteq S$  or  $V^{>a} \cap S = \emptyset$ . This can be verified easily along the lines of the preceding proof.

### Definable Closure

Let  $\mathcal{V} = (V, \psi, P)$  be an  $H$ -triple with scalar field  $k$ , and let  $\mathcal{V}' = (V', \psi', P')$  be a closed  $H$ -triple extending  $\mathcal{V}$ , with the same scalar field  $k$ . An element  $v' \in V'$  is said to be *definable over  $\mathcal{V}$*  if there is an  $\mathcal{L}_{H,P}$ -formula  $\varphi(x, y, z)$ , where  $x = (x_1, \dots, x_m)$  is a tuple of scalar variables,  $y = (y_1, \dots, y_n)$  a tuple of vector variables, and  $z$  a vector variable, and there are  $\lambda \in k^m$   $v \in V^n$ , such that  $v'$  is the unique element in  $V'$  with  $\mathcal{V}' \models \varphi(\lambda, v, v')$ . The *definable closure of  $\mathcal{V}$  in  $\mathcal{V}'$*  is the substructure of  $\mathcal{V}'$  that extends  $\mathcal{V}$  and whose underlying vector space consists of all  $v' \in V'$

that are definable over  $\mathcal{V}$ . If  $\mathcal{V}'' \supseteq \mathcal{V}$  is another closed  $H$ -triple over  $k$ , the definable closure of  $\mathcal{V}$  in  $\mathcal{V}'$  is isomorphic to the definable closure of  $\mathcal{V}$  in  $\mathcal{V}''$ , by a unique isomorphism that is the identity on  $\mathcal{V}$ . We say that  $\mathcal{V}$  is *definably closed* in  $\mathcal{V}'$  if every  $v' \in V'$  definable over  $\mathcal{V}$  belongs to  $V$ . In that case  $\mathcal{V}$  is definably closed in *every* closed  $H$ -triple over  $k$  extending  $\mathcal{V}$ , and we also just say then that  $\mathcal{V}$  is *definably closed*. More generally, if  $\mathcal{W} = (W, \dots) \supseteq \mathcal{V}$  is any  $H$ -triple over  $k$ , we say that  $\mathcal{V}$  is *definably closed* in  $\mathcal{W}$  if  $W \cap \bar{V} = V$ , where  $\bar{\mathcal{V}} = (\bar{V}, \dots)$  is the definable closure of  $\mathcal{V}$  in an  $H$ -closure of  $\mathcal{W}$ .

**LEMMA 5.1.** *Suppose there is no  $a \in V$  with  $P < a < (\text{id} + \psi)(V^{>0})$ , and  $P$  has no largest element. Then  $\mathcal{V}$  is definably closed in  $\mathcal{V}'$ .*

*Proof.* By iterating the construction of Lemma 3.6 we obtain an increasing continuous chain  $\{(V_\alpha, \psi_\alpha, P_\alpha)\}_{\alpha < \mu}$  ( $\mu$  an ordinal) of  $H$ -triples contained in  $\mathcal{V}'$  as substructures, with  $(V_0, \psi_0, P_0) = (V, \psi, P)$ , such that the union

$$\mathcal{V}^c = (V^c, \psi^c, P^c) = \bigcup_{\alpha < \mu} (V_\alpha, \psi_\alpha, P_\alpha)$$

is  $H$ -closed. The reference to Lemma 3.6 means that for  $\alpha < \alpha + 1 < \mu$  we have  $V_{\alpha+1} = V_\alpha \oplus k a_\alpha$  with  $a_\alpha > 0$  and  $\psi_\alpha(a_\alpha) \in P_\alpha \setminus \psi_\alpha(V_\alpha^*)$ . That the chain is continuous means that  $(V_\delta, \psi_\delta, P_\delta) = \bigcup_{\alpha < \delta} (V_\alpha, \psi_\alpha, P_\alpha)$  for limit ordinals  $\delta < \mu$ . Since  $\mathcal{V}^c \preceq \mathcal{V}'$ , it suffices to show: for any  $v \in V^c \setminus V$  there exists an element  $w \neq v$  in  $V^c$  and an automorphism of  $\mathcal{V}^c$  that is the identity on  $\mathcal{V}$  and sends  $v$  to  $w$ . Now, given such  $v$ , take  $\alpha$  with  $0 \leq \alpha < \alpha + 1 < \mu$  and  $v \in V_{\alpha+1} \setminus V_\alpha$ . Write

$$v = v_\alpha + \lambda a_\alpha \quad \text{with } v_\alpha \in V_\alpha, \lambda \in k^\times.$$

Let  $a \in V_{\alpha+1}^{>0}$  be any element not equal to  $a_\alpha$  with the same  $k$ -archimedean class as  $a_\alpha$ , and let  $w := v_\alpha + \lambda a$ . By Lemma 3.6, there is a unique automorphism  $\sigma$  of  $(V_{\alpha+1}, \psi_{\alpha+1}, P_{\alpha+1})$  that is the identity on  $(V_\alpha, \psi_\alpha, P_\alpha)$  and satisfies  $\sigma(a_\alpha) = a$ ; hence  $\sigma(v) = w$ . Applying once more Lemma 3.6 iteratively, we can extend  $\sigma$  to an automorphism of  $(V^c, \psi^c, P^c)$  that sends  $v$  to  $w$ , as desired. ■

In general we define an  $H$ -triple  $\bar{\mathcal{V}} = (\bar{V}, \bar{\psi}, \bar{P})$  with  $\mathcal{V} \subseteq \bar{\mathcal{V}} \subseteq \mathcal{V}'$  as follows, distinguishing three mutually exclusive cases:

1. There is  $a \in V$  with  $P < a < (\text{id} + \psi)(V^{>0})$ . This element  $a$  determines a sequence  $\{\varepsilon_n\}$  of positive elements of  $V'$  with

$$[V^*] > [\varepsilon_0] > [\varepsilon_1] > [\varepsilon_2] > \dots$$

and a corresponding sequence  $\{V_n\}$  of linear subspaces of  $V'$  with

$$V_n = V \oplus k\varepsilon_0 \oplus \dots \oplus k\varepsilon_n$$

by requiring  $a = \varepsilon_0 + \psi'(\varepsilon_0)$  and  $\psi'(\varepsilon_{n+1}) = \max(P' \cap V_n) + \varepsilon_{n+1}$  (cf. Lemmas 3.4 and 3.5). Then  $\tilde{V} := \bigcup_n V_n$ , so that  $[\tilde{V}] = [V] \cup \{[\varepsilon_n] : n \geq 0\}$ .

2.  $P$  has a largest element. Proceeding as in case 1, except that we restrict to  $n \geq 1$ , we define a sequence  $\{\varepsilon_n\}_{n \geq 1}$  of positive elements of  $V'$  with  $[V^*] > [\varepsilon_1] > [\varepsilon_2] > \dots$ , and a corresponding sequence  $\{V_n\}_{n \geq 1}$  of linear subspaces of  $V'$  with  $V_n = V \oplus k\varepsilon_1 \oplus \dots \oplus k\varepsilon_n$  by  $\psi'(\varepsilon_1) = (\max P) + \varepsilon_1$ , and  $\psi'(\varepsilon_{n+1}) = \max(P' \cap V_n) + \varepsilon_{n+1}$ . Then  $\tilde{V} := \bigcup_{n \geq 1} V_n$ , so that  $[\tilde{V}] = [V] \cup \{[\varepsilon_n] : n \geq 1\}$ .

3. There is no  $a \in V$  with  $P < a < (\text{id} + \psi)(V^{>0})$  and  $P$  has no largest element. Then we put  $\tilde{V} := V$ .

The previous lemma now easily implies:

COROLLARY 5.2. *The definable closure of  $\mathcal{V}$  in  $\mathcal{V}'$  is  $\tilde{\mathcal{V}}$ .*

In the next lemma we continue to use the notation introduced in the definition of  $\tilde{\mathcal{V}}$  above.

LEMMA 5.2. *The only  $H$ -triples  $\mathcal{W}$  with  $\mathcal{V} \subseteq \mathcal{W} \subseteq \tilde{\mathcal{V}}$  are  $\mathcal{V}$ ,  $\tilde{\mathcal{V}}$ , and*

1.  $(V_n, \bar{\psi}|V_n^*, \bar{P} \cap V_n)$  for  $n \geq 0$ , in case 1 above,
2.  $(V_n, \bar{\psi}|V_n^*, \bar{P} \cap V_n)$  for  $n \geq 1$ , in case 2 above.

*Proof.* First assume we are in case 1. Let  $\mathcal{W}$  be an  $H$ -triple such that  $\mathcal{V} \subseteq \mathcal{W} \subseteq \tilde{\mathcal{V}}$ , and let  $W$  denote the underlying ordered vector space of  $\mathcal{W}$ . Let  $w \in W \setminus V$ . After subtracting from  $w$  a vector in  $V$  we have

$$w = \lambda_m \varepsilon_m + \dots + \lambda_n \varepsilon_n \text{ with } n \geq m, \quad \lambda_m, \dots, \lambda_n \in k, \quad \lambda_m \neq 0, \lambda_n \neq 0.$$

By induction on  $i$  we shall obtain  $\varepsilon_i \in W$  for  $i = 0, \dots, n$ , which immediately implies the lemma in case 1. Note that  $\bar{\psi}(w) = \bar{\psi}(\varepsilon_m) = a - \varepsilon_0 + \varepsilon_1 + \dots + \varepsilon_m \in W$ , hence  $\varepsilon_0 = a - \bar{\psi}(\bar{\psi}(w) - a) \in W$ , which proves our claim for  $i = 0$ . So assume  $0 \leq i < n$  and  $\varepsilon_0, \dots, \varepsilon_i \in W$ .

1. Suppose  $i < m$ . Then

$$\varepsilon_{i+1} + \dots + \varepsilon_m = \bar{\psi}(w) - (a - \varepsilon_0 + \varepsilon_1 + \dots + \varepsilon_i) \in W.$$

Hence

$$\varepsilon_{i+1} = \bar{\psi}(\varepsilon_{i+1} + \dots + \varepsilon_m) - (a - \varepsilon_0 + \varepsilon_1 + \dots + \varepsilon_i) \in W.$$

2. Suppose  $i \geq m$ . Take  $j$  minimal such that  $i < j \leq n$  and  $\lambda_j \neq 0$ . Then

$$\lambda_j \varepsilon_j + \dots + \lambda_n \varepsilon_n = w - (\lambda_m \varepsilon_m + \dots + \lambda_i \varepsilon_i) \in W.$$

Hence

$$\varepsilon_{i+1} + \dots + \varepsilon_j = \bar{\psi}(\lambda_j \varepsilon_j + \dots + \lambda_n \varepsilon_n) - (a - \varepsilon_0 + \varepsilon_1 + \dots + \varepsilon_i) \in W,$$

and therefore

$$\varepsilon_{i+1} = \bar{\psi}(\varepsilon_{i+1} + \cdots + \varepsilon_j) - (a - \varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_i) \in W.$$

This finishes the induction step and thus the proof of the lemma in case 1. For case 2 one argues similarly. ■

### Uniqueness of $H$ -Closure

Let  $\mathcal{L}$  be a one-sorted language and  $\mathbf{A} = (A, \dots)$  an  $\mathcal{L}$ -structure. A *construction* of  $\mathbf{A}$  is an enumeration  $\{a_\alpha\}_{\alpha < \gamma}$  of  $A$  ( $\gamma$  an ordinal), such that, with  $A_\alpha := \{a_\beta : \beta < \alpha\}$ , the type of  $a_\alpha$  over  $A_\alpha$  in  $\mathbf{A}$  is isolated for each  $\alpha < \gamma$ . Let such a construction of  $\mathbf{A}$  be given. Choose for each  $\alpha < \gamma$  an  $\mathcal{L}$ -formula  $\varphi_\alpha(y_\alpha, z)$ , with  $y_\alpha = (y_{\alpha 1}, \dots, y_{\alpha n(\alpha)})$  a tuple of variables and  $z$  a single variable, and a tuple  $b_\alpha \in A_\alpha^{n(\alpha)}$ , such that  $\varphi_\alpha(b_\alpha, z)$  isolates the type of  $a_\alpha$  over  $A_\alpha$ . We also choose by recursion on  $\alpha$  a finite set  $D_\alpha \subseteq A_\alpha$  as follows:  $D_0 := \{a_0\}$ , and for  $0 < \alpha < \gamma$ , put  $D_\alpha := \{a_\alpha\} \cup D_{\beta_1} \cup \cdots \cup D_{\beta_{n(\alpha)}}$ , where  $b_\alpha = (a_{\beta_1}, \dots, a_{\beta_{n(\alpha)}})$  for certain  $\beta_1, \dots, \beta_{n(\alpha)} < \alpha$ . An elementary substructure  $\mathbf{C} = (C, \dots)$  of  $\mathbf{A}$  is said to be *closed in  $\mathbf{A}$*  (relative to the given construction and the further choices made) if, for all  $\alpha < \gamma$ ,  $a_\alpha \in C$  implies  $D_\alpha \subseteq C$ . In that case a theorem of Ressayre [9, Lemme 10.15, Théorème 10.18] implies that  $\mathbf{A} \cong \mathbf{C}$ .

**COROLLARY 5.3.** *Let  $\mathcal{V}$  be an  $H$ -triple over  $k$ . Then any two  $H$ -closures of  $\mathcal{V}$  are isomorphic over  $\mathcal{V}$ .*

*Proof.* We may assume that  $\mathcal{V}$  is definably closed. Then we build an  $H$ -closure  $\mathcal{V}^c$  of  $\mathcal{V}$  as in the proof of Lemma 5.1. Let  $\mathcal{W} \supseteq \mathcal{V}$  be another  $H$ -closure of  $\mathcal{V}$ . We have to show that  $\mathcal{V}^c \cong \mathcal{W}$  over  $\mathcal{V}$ . By the defining property of  $H$ -closure we can assume  $\mathcal{V} \subseteq \mathcal{W} \subseteq \mathcal{V}^c$ . Write  $\mathcal{V}^c = \bigcup_{\alpha < \mu} (V_\alpha, \psi_\alpha, P_\alpha)$  as in the proof of Lemma 5.1. We now consider the underlying vector spaces  $W$  and  $V^c$  of  $\mathcal{W}$  and  $\mathcal{V}^c$  as structures for the language  $\mathcal{L}_{k, \mathbf{v}}(V)$  obtained from  $\mathcal{L}_{k, \mathbf{v}}$  by adding names for the vectors in  $V$ ; see Proposition 5.1. By Lemma 3.6, the type of  $a_\alpha$  over  $V_\alpha$  in  $V^c$  (for  $\alpha < \alpha + 1 < \mu$ ) is isolated by the formula  $\varphi(\psi_\alpha(a_\alpha), z)$ , where  $\varphi(y, z)$  is “ $y = \psi(z)$  and  $z > 0$ .” It follows easily that  $V^c$  has a construction. By Lemma 4.1 we have  $W \leq V^c$ . If  $\alpha < \alpha + 1 < \mu$  and  $a_\alpha \in W$ , then  $\psi_\alpha(a_\alpha) \in W$ , so  $W$  is closed in  $V^c$  (relative to a suitable construction of  $V^c$  and associated choices of isolating formulas and so on). Thus by Ressayre’s theorem  $V^c \cong W$ , which implies  $\mathcal{V}^c \cong \mathcal{W}$  over  $\mathcal{V}$ . ■

*Remark.* We do not know whether the  $H$ -closure  $\mathcal{V}^c$  of an  $H$ -triple  $\mathcal{V}$  is always minimal over  $\mathcal{V}$ , i.e., whether or not for some  $\mathcal{V}$  there exists a closed  $H$ -triple  $\mathcal{W} \supseteq \mathcal{V}$  strictly contained in  $\mathcal{V}^c$  as a substructure.

### Analysis of Simple Extensions

Let an  $H$ -triple  $\mathcal{V} = (V, \psi, P)$  with scalar field  $k$  be given, and a *simple extension* of  $\mathcal{V}$ , that is, an  $H$ -triple  $\mathcal{V}' = (V', \psi', P')$  over the same scalar field  $k$  and extending  $\mathcal{V}$  for which there exists a vector  $c \in V'$  such that  $\mathcal{V}'$  is generated as  $\mathcal{L}_{H,P}$ -structure over  $\mathcal{V}$  by  $c$ . (This state of affairs is also indicated by writing  $\mathcal{V}' = \mathcal{V}\langle c \rangle$ , and we put  $V' = V\langle c \rangle$  for the underlying ordered vector spaces in that case.)

Consider the following five properties that this simple extension with its distinguished generator  $c$  may or may not have:

$$(I) \quad 0 < c < V^{>0} \text{ and } c + \psi'(c) \in V.$$

$$(II) \quad 0 < c < V^{>0} \text{ and } -c + \psi'(c) \in V.$$

$$(III) \quad \psi'(c) \in P \setminus \Psi.$$

$$(IV) \quad c \notin V \text{ and } [V \oplus kc] = [V].$$

$$(V) \quad V' = V \oplus \bigoplus_{n=1}^{\infty} kb_n \text{ for vectors } b_n \in V' \text{ that are } k\text{-linearly independent over } V, \text{ with } [b_n] \notin [V] \text{ for all } n, \text{ and such that there are vectors } a_n \in V \text{ with } b_1 = c - a_1 \text{ and } b_{n+1} = \psi'(b_n) - a_{n+1} \text{ for all } n \geq 1.$$

*Remarks.* If (I), respectively (II), holds, then  $\mathcal{V}' \cong (V \oplus k\varepsilon, \psi^\varepsilon, P^\varepsilon)$ , as in Lemma 3.4, respectively Lemma 3.5, by an isomorphism that is the identity on  $\mathcal{V}$  and sends  $c$  to  $\varepsilon$ . If (III) holds, then  $\mathcal{V}' \cong (V \oplus ka, \psi^a, P^a)$ , as in Lemma 3.6, by an isomorphism that is the identity on  $\mathcal{V}$  and sends  $c$  to  $a$ . If (IV) holds, then  $V' = V \oplus kc$ . Note that if (V) holds, then  $[b_n] \neq [b_m]$  for all  $n \neq m$ . (Otherwise  $b_{n+1} - b_{m+1} = a_{m+1} - a_{n+1} \in V$ , contradicting the linear independence of  $\{b_i\}_{i \geq 1}$  over  $V$ .)

One sees easily that these properties are mutually exclusive. We call  $\mathcal{V} \subseteq \mathcal{V}\langle c \rangle$  a *simple extension of type (I)*, respectively (II), (III), (IV), (V), if (I), respectively (II), (III), (IV), (V), holds. Here the generator  $c$  figuring in the definition of these properties has been specified. If we do not want to specify the generator we simply say that  $\mathcal{V}'$  is a *simple extension of type (I)*, respectively (II), (III), (IV), (V), to mean that for *some*  $c \in V'$  we have  $\mathcal{V}' = \mathcal{V}\langle c \rangle$  and  $\mathcal{V} \subseteq \mathcal{V}\langle c \rangle$  is a simple extension of type (I), respectively (II), (III), (IV), (V).

We now show that if  $\mathcal{V}$  is definably closed in  $\mathcal{V}'$ , then we can obtain  $\mathcal{V}'$  by a finite number of simple extensions of types (I)–(V). More precisely:

**PROPOSITION 5.2.** *Suppose  $\mathcal{V}$  is definably closed in its simple extension  $\mathcal{V}' = \mathcal{V}\langle c \rangle$ , with  $c \notin V$ . Then either*

1.  $\mathcal{V} \subseteq \mathcal{V}\langle c \rangle$  is a simple extension of type (V) or
2. there is a finite chain of  $H$ -triples

$$\mathcal{V} = \mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \cdots \subseteq \mathcal{V}_n = \mathcal{V}' \quad (n \geq 1)$$



such that  $\mathcal{V}_1$  is a simple extension of  $\mathcal{V}_0$  of type (III) or type (IV), and each  $\mathcal{V}_{i+1}$  is a simple extension of  $\mathcal{V}_i$  of type (III), for  $i = 1, \dots, n-1$ .

*Proof.* If  $\mathcal{V} \subseteq \mathcal{V}\langle c \rangle$  is of type (IV), we are done. Suppose  $\mathcal{V} \subseteq \mathcal{V}\langle c \rangle$  is not of type (IV). Then there is  $a_1 \in V$  such that, with  $b_1 := c - a_1$ , we have  $[b_1] \notin [V]$ . This is the starting point for an inductive construction of elements  $a_i \in V$  and  $b_i \in V'$ . Suppose we have already constructed  $a_1, \dots, a_n \in V$  and non-zero vectors  $b_1, \dots, b_n \in V'$  with  $n \geq 1$ , where  $a_1$  and  $b_1$  are as above,  $b_{i+1} = \psi'(b_i) - a_{i+1}$  for  $i = 1, \dots, n-1$ , such that  $[b_i] \notin [V]$  for  $i = 1, \dots, n$ .

We claim that then  $[b_i] \neq [b_j]$  for  $1 \leq i < j \leq n$  (hence  $b_1, \dots, b_n$  are linearly independent over  $V$ ). Otherwise  $[b_i] = [b_j]$  for certain  $1 \leq i < j \leq n$ , so  $\psi'(b_j) = \psi'(b_i)$ . But also  $\psi'(b_j) = \psi'_{(a_{i+1}, \dots, a_j)}(\psi'(b_i))$  (see Sect. 4), and hence  $\psi'(b_i) = \psi'_{(a_{i+1}, \dots, a_j)}(\psi'(b_i))$ . Thus by Lemma 4.3 the vector  $\psi'(b_i)$  is definable over  $\mathcal{V}$ . Therefore  $b_{i+1} \in V$ , contradicting  $b_{i+1} \notin [V]$ .

If  $[\psi'(b_n) - a_{n+1}] \notin [V]$  for some  $a_{n+1} \in V$ , we take such a vector  $a_{n+1}$  and put  $b_{n+1} := \psi'(b_n) - a_{n+1}$ . If there is no such  $a_{n+1}$ , the construction breaks off, with  $a_n$  and  $b_n$  as the last vectors.

First assume that the construction goes on indefinitely, that is, we obtain infinite sequences  $\{a_i\}_{i \geq 1}$  in  $V$  and  $\{b_i\}_{i \geq 1}$  in  $V'$  such that  $b_1 = c - a_1$ ,  $b_{i+1} = \psi'(b_i) - a_{i+1}$ , and  $[b_i] \notin [V]$  for all  $i \geq 1$ . Then one easily sees that  $V' = V \oplus \bigoplus_{i=1}^{\infty} kb_i$  and that  $\mathcal{V} \subseteq \mathcal{V}\langle c \rangle$  is of type (V).

Now suppose our construction stops after the vectors  $a_n$  and  $b_n$  have been obtained. There are two ways in which this could happen:

1.  $\psi'(b_n) \in V$ ,
2.  $\psi'(b_n) \notin V$ , but  $[V \oplus k\psi'(b_n)] = [V]$ .

In the first case we put  $\mathcal{V}_0 := \mathcal{V}$ , and for  $i = 1, \dots, n$  we let  $\mathcal{V}_i$  be the substructure of  $\mathcal{V}'$  with underlying vector space

$$V_i := V \oplus \bigoplus_{j=n-i+1}^n kb_j.$$

Then  $\mathcal{V}_{i+1}$  is a simple extension of type (III) of  $\mathcal{V}_i$  for  $i = 0, \dots, n-1$ , and  $\mathcal{V}_n = \mathcal{V}'$ . In the second case, take  $\mathcal{V}_0 := \mathcal{V}$ , and for  $i = 1, \dots, n+1$  let  $\mathcal{V}_i$  be the substructure of  $\mathcal{V}'$  with underlying vector space

$$V_i := V \oplus k\psi'(b_n) \oplus \bigoplus_{j=n-i+2}^n kb_j.$$

Then  $\mathcal{V}_0 \subseteq \mathcal{V}_1$  is a simple extension of type (IV), whereas  $\mathcal{V}_i \subseteq \mathcal{V}_{i+1}$  for  $i = 1, \dots, n$  is a simple extension of type (III), and  $V_{n+1} = V \oplus k\psi'(b_n) \oplus \bigoplus_{j=1}^n kb_j = V'$ . ■

*Remark.* In the previous proposition, if  $\mathcal{V}$  is a closed  $H$ -triple and  $\mathcal{V} \subseteq \mathcal{V}'$  is not of type (V), the extension  $\mathcal{V}_0 \subseteq \mathcal{V}_1$  will be of type (IV), since  $\mathcal{V}$  admits no simple extensions of type (III).

Suppose that  $\mathcal{V} \subseteq \mathcal{V}'$  is a simple extension such that  $\mathcal{V}$  is not definably closed in  $\mathcal{V}'$ . To reduce to a situation where we can apply the last proposition, we let  $\bar{\mathcal{V}} = (\bar{V}, \dots)$  be the definable closure of  $\mathcal{V}$  in an  $H$ -closure of  $\mathcal{V}'$ , and let  $\mathcal{W} = (W, \dots)$  be the  $H$ -triple with  $\mathcal{V} \subset \mathcal{W} \subseteq \mathcal{V}'$  and  $W = V' \cap \bar{V}$ .

Then  $\mathcal{W}$  is definably closed in  $\mathcal{V}'$ , so that Proposition 5.2 is applicable to the simple extension  $\mathcal{W} \subseteq \mathcal{V}'$ . The possibilities for the proper extension  $\mathcal{V} \subset \mathcal{W}$  are described by Lemma 4.5, but can it actually happen that  $\mathcal{W} = \bar{\mathcal{V}}$ ? The following example shows that this case indeed occurs, and also shows that there are simple extensions that cannot be obtained by a finite number of simple extensions of types (I)–(V).

*Example.* Let  $\mathcal{V} = (V, \psi, P)$  be the  $H$ -triple over  $k$  with  $V := ke_0$  ( $e_0 > 0$ ), distinguished positive element  $1 = e_0$ , and  $\max P = e_0$ . (See Example 3.1.) Let  $\mathcal{V}' = (V', \psi', P')$  be the  $H$ -triple over  $k$  with  $V' = \bigoplus_{n \in \mathbb{N}} ke_{-n}$  and  $\psi'(e_{-n}) := e_0 + e_{-1} - e_{-n-1}$  for all  $n \in \mathbb{N}$ , and  $P' := \{v' \in V' : v' < e_0 + e_{-1}\}$ , as in Section 3, end of Example 3.3. Note that  $\mathcal{V}' = \mathcal{V}\langle e_{-1} \rangle$ . Let  $\bar{\mathcal{V}} = (\bar{V}, \dots)$  be the definable closure of  $\mathcal{V}$  in an  $H$ -closure of  $\mathcal{V}'$ . Put  $\varepsilon_n := e_{-n} - e_{-n-1}$  for  $n \geq 1$ . One sees easily (using Corollary 5.2) that then  $\bar{\mathcal{V}} \subseteq \mathcal{V}\langle e_{-1} \rangle$  with

$$\bar{V} := ke_0 \oplus \bigoplus_{n=1}^{\infty} k\varepsilon_n.$$

It can be shown that  $\mathcal{V}' = \mathcal{V}\langle e_{-1} \rangle$  cannot be obtained from  $\mathcal{V}$  by finitely many simple extensions of types (I)–(V). One proves that whenever  $\mathcal{V}_1 = (V_1, \psi_1, P_1)$  is an  $H$ -triple with  $\mathcal{V} \subseteq \mathcal{V}_1 \subset \bar{\mathcal{V}}$  and  $c \in V\langle e_{-1} \rangle \setminus V_1$ , then  $\mathcal{V}_1 \subseteq \mathcal{V}_1\langle c \rangle$  is not of type (III), (IV), or (V). We leave the details to the reader.

### Well-Orderedness of $\Psi$

We now use our analysis of simple extensions to show that in a finitely generated  $H$ -couple  $(V, \psi)$  the set  $\Psi = \psi(V^*)$  is always well ordered. We first need to take a closer look at type (V) extensions.

**LEMMA 5.3.** *Let  $V_0 \subseteq V$  be ordered vector spaces over the ordered field  $k$ , and  $v \in V \setminus V_0$  such that  $[v] \notin [V_0]$ . Then  $[V_0 \oplus kv] = [V_0] \cup \{[v]\}$  and  $[v] \leq [w]$  for all  $w \in (V_0 \oplus kv) \setminus V_0$ .*

This follows easily from the properties of  $k$ -archimedean classes listed in the beginning of Section 2, especially property (4).

PROPOSITION 5.3. *Let  $\mathcal{V} \subseteq \mathcal{V}\langle c \rangle = \mathcal{V}'$  be a simple extension of  $H$ -triples of type (V), with  $V' = V \oplus \bigoplus_{n=1}^{\infty} kb_n$  as in the definition of type (V) extensions. Then*

- (1)  $[V'] = [V] \cup \{[b_n] : n = 1, 2, 3, \dots\}$ .
- (2)  $V'^{>0}$  is coinital in  $(V')^{>0}$ .
- (3)  $\mathcal{V}$  is definably closed in  $\mathcal{V}'$ .
- (4) The sequence  $\{[b_n]\}$  is strictly decreasing.

*Proof.* Using the lemma above and the fact that  $[b_i] \neq [b_j]$  for  $i \neq j$ , one shows by induction on  $n$  that  $[V \oplus \bigoplus_{i=1}^n kb_i] = [V] \cup \{[b_i] : i = 1, \dots, n\}$ . This proves (1).

For (2) we first note that if  $y \in V' \setminus V$ , then  $[V] \neq [V \oplus ky]$ : write  $y = v + \lambda_1 b_1 + \dots + \lambda_n b_n$  with  $v \in V$ ,  $\lambda_1, \dots, \lambda_n \in k$ , and some  $\lambda_i \neq 0$ ; hence  $[y - v] = [b_i] \in [V \oplus ky] \setminus [V]$  for some  $i$ , by (1). Thus, by the last part of Corollary 4.1, if  $[V^*]$  has no minimum, then (2) holds, and hence (3) holds as well, by Lemma 5.2. We now prove (2) and (3) in the remaining case that  $[V^*]$  has a minimum. Equivalently, we assume  $\Psi$  has a maximum. We first show that then (3) holds. If it does not hold then, by Lemma 5.2, there are  $v \in V$ ,  $\lambda_1, \dots, \lambda_n \in k$  ( $n > 0$ ) such that  $v' := v + \sum_{i=1}^n \lambda_i b_i > 0$  and  $\max \Psi = \psi'(v') - v'$ . If  $[v] > [b_i]$  for all  $i \in \{1, \dots, n\}$  with  $\lambda_i \neq 0$ , then  $\psi'(v') = \psi(v)$ , so  $\max \Psi = \psi(v) - v' < \max \Psi$ , a contradiction. Now assume  $[v] < [b_i]$  for some  $i \in \{1, \dots, n\}$  with  $\lambda_i \neq 0$ , and let  $j \in \{1, \dots, n\}$  be such that  $[b_j] = \max\{[b_i] : 1 \leq i \leq n, \lambda_i \neq 0\}$ . Then

$$\max \Psi = \psi'(v') - v' = b_{j+1} + a_{j+1} - v - \sum_{i=1}^n \lambda_i b_i;$$

hence  $\lambda_j = 0$ , a contradiction. We have now established (3). To obtain (2), suppose that  $\psi'(v') > \max \Psi$  for some  $v' \in (V')^{>0}$ . By Lemma 3.1, there is  $w \in (V')^{>0}$  with  $\psi'(w) - w = \max \Psi$ . So  $w$  is definable over  $\mathcal{V}$ , and hence  $w \in V$  and  $\psi(w) = \max \Psi + w > \max \Psi$ , which is impossible. This finishes the proof of (2).

As to (4), given any  $n > 0$ , we can choose by (2) an  $a \in V^*$  with  $|b_n| > |a|$ . By Lemma 5.3 above and basic property (3) of  $\psi$  listed at the beginning of Section 3, we have

$$[b_{n+1}] = [\psi'(b_n) - a_{n+1}] \leq [\psi'(b_n) - \psi(a)] < [b_n - a] = [b_n],$$

as required. ■

*Remark.* In the situation of this proposition the sequence  $\{[b_n]\}$  enumerates the set  $[V'] \setminus [V]$  in strictly decreasing order. Thus the sequence  $\{[b_n]\}$  is independent of the choice of the sequence  $\{b_n\}$ . It also follows that  $\Psi' \setminus \Psi$  is enumerated in strictly increasing order by the sequence  $\{\psi'(b_n)\}$ .

**THEOREM 5.1.** *Let  $\mathcal{V} \subseteq \mathcal{V}'$  be a finitely generated extension of  $H$ -couples over the same scalar field, such that  $\Psi$  is well ordered. Then  $\Psi'$  is also well ordered.*

*Proof.* We first equip  $\mathcal{V}'$  and  $\mathcal{V}$  with suitable  $H$ -cuts so that we are dealing with an extension of  $H$ -triples. By induction on the number of generators of  $\mathcal{V}'$  over  $\mathcal{V}$  we then reduce to the case that  $\mathcal{V} \subseteq \mathcal{V}'$  is a simple extension.

Let  $\tilde{\mathcal{V}} = (\tilde{V}, \dots)$  be the definable closure of  $\mathcal{V}$  in an  $H$ -closure of  $\mathcal{V}'$ , and let  $\mathcal{W} = (W, \dots)$  be the  $H$ -triple with  $\mathcal{V} \subseteq \mathcal{W} \subseteq \mathcal{V}'$  and  $W = V' \cap \tilde{V}$ . By Lemma 5.2,  $\psi'(W^*)$  is well ordered. Moreover,  $\mathcal{W}$  is definably closed in  $\mathcal{V}'$ , and  $\mathcal{W} \subseteq \mathcal{V}'$  is a simple extension. By Proposition 5.2 we then further reduce to the case that  $\mathcal{W} \subseteq \mathcal{V}'$  is a simple extension of one of the types (III), (IV), and (V). If  $\mathcal{W} \subseteq \mathcal{V}'$  is of type (III) or type (IV),  $\Psi' \setminus \psi'(W^*)$  has at most one element, so  $\Psi'$  is well ordered. If  $\mathcal{W} \subseteq \mathcal{V}'$  is of type (V), it follows from the remark preceding the theorem that  $\Psi'$  is well ordered. ■

**COROLLARY 5.4.** *For any  $H$ -couple  $(V, \psi)$  over  $k$  that is finitely generated over its substructure with vector space  $k \cdot 1 \subseteq V$ , the set  $\Psi = \psi(V^*)$  is well ordered.*

Another issue is whether in a finitely generated  $H$ -couple  $(V, \psi)$  the set  $\Psi = \psi(V^*)$  always has a supremum in  $V$ . This turns out to be false:

*Example.* We take  $V = \bigoplus_{n \in \mathbb{N}} k e_{-n}$  as in the example preceding Lemma 5.3, but define  $\psi: V^* \rightarrow V$  by making it constant on  $k$ -archimedean classes of  $V$  and setting

$$\psi(e_0) := e_0, \quad \psi(e_{-n}) := e_0 + e_{-1} + \cdots + e_{-n} - e_{-n-1} \quad \text{if } n > 0.$$

It is easy to check that  $(V, \psi)$  is an  $H$ -couple with distinguished positive element  $1 = e_0$ . It is generated over its substructure with vector space  $k \cdot 1$  by its vector  $e_{-1}$ . The set  $\Psi$  has no supremum in  $V$ , as is easily verified.

However, we note that if  $\mathcal{V} \subseteq \mathcal{V}' \langle c \rangle = \mathcal{V}'$  is a simple extension of  $H$ -triples of one of the types (I)–(V), and  $\sup \Psi$  exists, so does  $\sup \Psi'$ . This is clear for simple extensions of types (I)–(IV), while for type (V) extensions, it follows from part (2) of Proposition 5.3.

## 6. REMOVING SCALARS

The goal of this section is Proposition 6.2. It strengthens the local o-minimality result, Proposition 5.1, to a global weak o-minimality for sets whose definition does not involve scalars. Another motive for this section

is that, in attempting to construct a model theory of Hardy fields, it appears useful to have analogues of the previous theorems in a setting where no scalar field is present.

**DEFINITION 6.1.** An  $H_0$ -couple is a pair  $\mathcal{V} = (V, \psi)$ , consisting of a divisible ordered abelian group  $V$ , a distinguished positive element  $1 \in V$ , and a function  $\psi: V^* \rightarrow V$ , such that, for all  $v, w \in V^*$ ,

1.  $\psi(1) = 1$ ,
2.  $\psi(nv) = \psi(v)$  for all  $n > 0$ ,
3.  $\psi(v) < \psi(w) + |w|$ ,
4.  $|v| \leq |w| \implies \psi(v) \geq \psi(w)$  (hence  $\psi(v) = \psi(-v)$ ).

We consider a divisible ordered abelian group as an ordered vector space over  $\mathbb{Q}$  in the usual way.

*Examples.* 1. Each  $H$ -couple becomes an  $H_0$ -couple by “forgetting” the scalar field.

2. If  $F \supseteq \mathbb{R}(x)$  is a real closed Hardy field,  $V := v(F^\times)$  its value group,  $1 := v(x^{-1})$ , and  $\psi: V^* \rightarrow V$  is defined as in the Introduction, then  $(V, \psi)$  is an  $H_0$ -couple, with distinguished positive element 1.

**DEFINITION 6.2.** An  $H_0$ -cut of an  $H_0$ -couple  $(V, \psi)$  is a set  $P \subseteq V$  which is closed downward, contains  $\Psi := \psi(V^*)$ , and is disjoint from  $(\text{id} + \psi)(V^{>0})$ . We then call  $(V, \psi, P)$  an  $H_0$ -triple. An  $H_0$ -couple  $(V, \psi)$  is *closed* if  $\Psi$  has no maximum and

$$\psi(V^*) = \{a \in V : a < w + \psi(w) \text{ for all } w \in V^{>0}\}.$$

In this case  $\Psi = \psi(V^*)$  is the only  $H_0$ -cut of  $(V, \psi)$ , and we call  $(V, \psi, \Psi)$  a *closed  $H_0$ -triple*. Note that a closed  $H$ -couple (closed  $H$ -triple) becomes a closed  $H_0$ -couple (closed  $H_0$ -triple) by forgetting the scalar field.

When dealing with  $H_0$ -couples  $\mathcal{V} = (V, \psi)$  as model-theoretic objects we construe them as  $\mathcal{L}_{H_0}$ -structures, where  $\mathcal{L}_{H_0}$  is the (one-sorted) language with (vector) variables ranging over the extended vector space  $V_\infty := V \cup \{\infty\}$ . The non-logical symbols of  $\mathcal{L}_{H_0}$  are:

1. those listed under part (4) of the description of  $\mathcal{L}_H$  in Section 1, to be interpreted as relations and functions on  $V_\infty$ , as indicated there;
2. unary function symbols  $\delta_n$  for each  $n > 0$ , to be interpreted on  $V$  as the scalar multiplication by  $1/n$  (and  $\delta_n(\infty) := \infty$ ).

Adding to  $\mathcal{L}_{H_0}$  a unary predicate symbol  $P$  we obtain the language  $\mathcal{L}_{H_0, P}$ , and  $H_0$ -triples  $(V, \psi, P)$  are then construed as  $\mathcal{L}_{H_0, P}$ -structures. The  $H_0$ -couples are easily seen to be the models of a universal theory in  $\mathcal{L}_{H_0}$ , and the same is true for the  $H_0$ -triples with respect to the language  $\mathcal{L}_{H_0, P}$ .

*Remark.* The division symbols  $\delta_n$  are included to guarantee quantifier elimination for the theory of  $H_0$ -triples; see Corollary 6.2 below. (There is an example which shows that if we omit them, then in the resulting smaller language the theory of  $H_0$ -triples would not eliminate quantifiers.)

*Notation.* If  $\mathcal{V} = (V, \psi)$  is an  $H_0$ -couple, we set

$$[v] := \{w \in V : \psi(w) = \psi(v)\} \quad \text{for } v \in V.$$

We let  $[V] := \{[v] : v \in V\}$  and make it into a linearly ordered set by defining

$$\begin{aligned} [v] < [w] &: \Longleftrightarrow [v] \neq [w] \text{ and } |v| < |w| \\ &\Longleftrightarrow \psi(v) > \psi(w). \end{aligned}$$

In the case that  $\mathcal{V}$  is obtained from an  $H$ -couple over  $k$  by “forgetting the scalar field,”  $[v]$  (or  $[v]_k$ ) also denotes the  $k$ -archimedean class of a vector  $v \in V$ . Fortunately, this agrees with  $[v]$  as just defined. Note also that the four properties of  $k$ -archimedean classes stated in the beginning of Section 2 go through for the classes  $[v] \subseteq V$  of an  $H_0$ -couple  $\mathcal{V}$  as above, with  $\lambda \in \mathbb{Q}^\times$  in property (2). For any  $H_0$ -couple  $\mathcal{V}$  and vectors  $v, w \in V^*$ ,  $[v]_{\mathbb{Q}} \leq [w]_{\mathbb{Q}}$  implies  $[v] \leq [w]$ .

### Basic Properties

The beginning of Section 3 up to and including Proposition 3.1 goes through for  $H_0$ -triples  $(V, \psi, P)$ , with  $[v]$  interpreted according to the definition just given, and with  $H_0$ -cuts instead of  $H$ -cuts in Proposition 3.1. The proofs are the same.

### Embedding into Closed $H_0$ -Triples

An  $H_0$ -closure of the  $H_0$ -triple  $\mathcal{V} = (V, \psi, P)$  is any closed  $H_0$ -triple  $\mathcal{V}^c = (V^c, \psi^c, P^c)$  extending  $\mathcal{V}$ , such that any embedding  $\mathcal{V} \rightarrow \mathcal{V}'$  into a closed  $H_0$ -triple  $\mathcal{V}'$  extends to an embedding  $\mathcal{V}^c \rightarrow \mathcal{V}'$ . We want to show that each  $H_0$ -triple  $\mathcal{V} = (V, \psi, P)$  has an  $H_0$ -closure. This will follow, just as for  $H$ -triples, by iterated application of three basic extension lemmas. These are exactly Lemmas 3.4, 3.5, and 3.6, modified as follows:  $H$ -triples become  $H_0$ -triples,  $V \oplus k\varepsilon$  becomes  $V \oplus \mathbb{Q}\varepsilon$ , and the phrase “over  $k$ ” should be omitted. The proofs go through, with similar trivial changes. We get:

COROLLARY 6.1. *Every  $H_0$ -triple has an  $H_0$ -closure.*

### *Elimination of Quantifiers*

We have the following counterpart of Proposition 4.1:

**PROPOSITION 6.1.** *Let  $\mathcal{V} = (V, \psi, P)$  and  $\mathcal{V}' = (V', \psi', P')$  be closed  $H_0$ -triples, where  $\mathcal{V}'$  is  $\kappa$ -saturated,  $\kappa := |V|^+$ . Let  $\mathcal{V}_0 = (V_0, \psi_0, P_0)$  be a substructure of  $\mathcal{V}$ , so again an  $H_0$ -triple. Any embedding  $i_0$  of  $\mathcal{V}_0$  into  $\mathcal{V}'$  can be extended to an embedding of  $\mathcal{V}$  into  $\mathcal{V}'$ .*

*Proof.* One can basically copy the proof in Section 4, changing  $k$  to  $\mathbb{Q}$  and making other obvious modifications. ■

Let  $T_{H_0, P}$  be the theory of closed  $H_0$ -triples in the language  $\mathcal{L}_{H_0, P}$ .

**COROLLARY 6.2.** *The theory  $T_{H_0, P}$  is complete, decidable, and has elimination of quantifiers. It is the model completion of the theory of  $H_0$ -triples.*

*Proof.* Elimination of quantifiers follows from Proposition 6.1 and a variant of the well-known Robinson–Shoenfield–Blum criterion for quantifier elimination, see, e.g., [17, Theorem 17.2]. The  $H_0$ -triple  $(V_0, \psi_0, P_0)$ , with  $V_0 := \mathbb{Q}$ ,  $P_0 := \mathbb{Q}^{\leq 1}$ ,  $\psi_0(x) := 1$  for all  $x \in \mathbb{Q}^*$ , and  $1 \in \mathbb{Q}^{>0}$  as distinguished element, can be embedded into any  $H_0$ -triple. This implies completeness of  $T_{H_0, P}$ . The rest now follows from Corollary 6.1. ■

*Definable Closure. Uniqueness of  $H_0$ -Closure. Analysis of Simple Extensions. Well-Orderedness of  $\Psi$*

The correspondingly named subsections of Section 5 go through for  $H_0$ -triples and  $H_0$ -closures with the following changes:  $H$ -triples (over  $k$ ) become  $H_0$ -triples,  $\mathcal{L}_{H, P}$ -formulas become  $\mathcal{L}_{H_0, P}$ -formulas (without scalar variables  $x_1, \dots, x_m$ ),  $k$ -linear spaces  $kw$  become  $\mathbb{Q}$ -linear spaces  $\mathbb{Q}w$ , more generally, vector spaces over  $k$  become vector spaces over  $\mathbb{Q}$ , scalars from  $k$  (as in the proofs of Lemmas 5.1 and 5.2) become scalars from  $\mathbb{Q}$ , and, finally,  $k$ -linear independence (as in property (V) of the analysis of simple extensions) becomes  $\mathbb{Q}$ -linear independence. Also, the equivalence classes  $[v]$  of vectors  $v$  should of course be interpreted in the sense of the present section.

### *Weak o-Minimality*

We now use the  $H_0$ -version of Proposition 5.2 to show:

**PROPOSITION 6.2.** *The theory of closed  $H_0$ -triples is weakly o-minimal; i.e., each closed  $H_0$ -triple is weakly o-minimal.*

(Compare with Proposition 5.1.) In the proof we also need the following.

### A Criterion for Weak o-Minimality

Let  $\mathcal{L}$  be a language containing a binary relation symbol  $<$ , and let  $\mathbf{A} = (A, <, \dots)$  be an  $\mathcal{L}$ -structure expanding a nonempty linearly ordered set  $(A, <)$ , dense without endpoints. A *cut* in  $\mathbf{A}$  is just a downward closed set  $C \subseteq A$ . To such a cut  $C$  we associate the set

$$\Phi_C(y) := \{c < y : c \in C\} \cup \{y < d : d \in A \setminus C\}$$

of  $\mathcal{L}_A$ -formulas in the variable  $y$ .

**LEMMA 6.1 (Kulpeshov [7]).** *An  $\mathcal{L}$ -structure  $\mathbf{A} = (A, <, \dots)$  as above is weakly o-minimal if and only if for all cuts  $C$  in  $A$  there exist at most two complete  $y$ -types over  $A$  extending  $\Phi_C(y)$ , and for each of these types, its set of realizations in any elementary extension  $\mathbf{B} = (B, <, \dots)$  of  $\mathbf{A}$  is convex in  $B$ .*

#### Proof of Proposition 6.2

Let  $\mathcal{V} = (V, \psi, P)$  be a closed  $H_0$ -triple,  $\mathcal{V}' = (V', \psi', P')$  an elementary extension of  $\mathcal{V}$ , and  $C$  a cut in  $V$ . By quantifier elimination, the complete  $y$ -types over  $V$  extending  $\Phi_C(y)$  correspond bijectively to isomorphism classes over  $\mathcal{V}$  of simple extensions  $\mathcal{V}\langle c \rangle$  of  $\mathcal{V}$  with distinguished generator  $c$  such that  $C < c < V \setminus C$ . We claim:

1. Up to isomorphism over  $\mathcal{V}$ , there exist at most two simple extensions  $\mathcal{V}\langle c \rangle$  of  $\mathcal{V}$  with distinguished generator  $c$  such that  $C < c < V \setminus C$ .
2. If  $c$  is an element of  $V'$  with  $C < c < V \setminus C$ , then the set of all  $d \in V'$ , such that  $\mathcal{V}\langle c \rangle \cong \mathcal{V}\langle d \rangle$  by an isomorphism over  $\mathcal{V}$  that maps  $c$  to  $d$ , is a convex subset of  $V'$ .

By Kulpeshov's criterion it will then follow that  $\mathcal{V}$  is weakly o-minimal. So assume  $\mathcal{V} \subseteq \mathcal{V}\langle c \rangle$  is a simple extension with  $C < c < V \setminus C$ . We may (for our purpose) assume that  $c \in V'$ . By our analysis of simple extensions, either

1.  $\mathcal{V} \subseteq \mathcal{V}\langle c \rangle$  is of type (IV) or type (V), or
2. there are  $n \geq 1$ ,  $a_1, \dots, a_n \in V$ , and non-zero  $b_1, \dots, b_n \in V'$  such that  $b_1 = c - a_1$ ,  $b_{j+1} = \psi'(b_j) - a_{j+1}$  for  $1 \leq j < n$ , the vectors  $\psi'(b_n)$ ,  $b_1, \dots, b_n$  are  $\mathbb{Q}$ -linearly independent over  $V$ ,  $[b_j] \notin [V]$  for  $1 \leq j < n$ ,  $[b_i] \neq [b_j]$  for  $1 \leq i < j \leq n$ , and  $V\langle c \rangle = V \oplus \mathbb{Q}\psi'(b_n) \oplus \bigoplus_{j=1}^n \mathbb{Q}b_j$ .

In all three cases, an argument as in the proof of Proposition 4.1 shows that then for any simple extension  $\mathcal{V}\langle d \rangle$  of  $\mathcal{V}$  with  $C < d < V \setminus C$ , the  $H_0$ -couples  $(V\langle c \rangle, \psi'|V\langle c \rangle^*)$  and  $(V\langle d \rangle, \psi'|V\langle d \rangle^*)$  are isomorphic over  $(V, \psi)$  by an isomorphism mapping  $c$  to  $d$ . Also,  $(V\langle c \rangle, \psi'|V\langle c \rangle^*)$  has at most two



$H_0$ -cuts, and hence can be expanded in at most two ways to an  $H_0$ -triple. This proves the first part of the claim.

For the second part we need some notation: let  $n \in \mathbb{N}$ ,  $b \in V$ ,  $a = (a_1, \dots, a_n) \in V^n$ , and  $\lambda = (\lambda_0, \dots, \lambda_n) \in \mathbb{Q}^{n+1}$ . Consider the map

$$y \mapsto \theta_{b,a,\lambda}(y) := b + \lambda_0 y + \sum_{j=1}^n \lambda_j \psi'_{(a_1, \dots, a_j)}(y): D'_a \rightarrow V'.$$

It is monotone on each convex component of  $D'_a$  (by the analogues of Lemmas 4.3 and 4.4 for  $H_0$ -couples). In particular, for each convex component  $D$  of  $D'_a$ , the set  $D \cap \theta_{b,a,\lambda}^{-1}(P')$  is downward or upward closed in  $D$ .

Now assume first that  $(V\langle c \rangle, \psi'|V\langle c \rangle^*)$  has only one  $H_0$ -cut. Then each  $H_0$ -triple  $\mathcal{V}\langle d \rangle$ , where  $C < d < V \setminus C$ , is isomorphic to  $\mathcal{V}\langle c \rangle$  by an isomorphism over  $\mathcal{V}$  mapping  $c$  to  $d$ . So assume that  $(V\langle c \rangle, \psi'|V\langle c \rangle^*)$  has two  $H_0$ -cuts. This means that there exists  $w \in V\langle c \rangle$  such that

$$\psi'(V\langle c \rangle^*) < w < (\text{id} + \psi')(V\langle c \rangle^{>0}).$$

In all three cases for  $\mathcal{V} \subseteq \mathcal{V}\langle c \rangle$ , we find  $n \in \mathbb{N}$ ,  $b \in V$ ,  $a \in V^n$ , and  $\lambda \in \mathbb{Q}^{n+1}$  such that  $w = \theta_{b,a,\lambda}(c)$ . Observe that, for any  $d \in V'$  with  $C < d < V \setminus C$ ,  $d$  lies in  $D'_a$ . In fact, it lies in the same convex component  $D$  of  $D'_a$  as  $c$ , and  $\mathcal{V}\langle c \rangle \cong \mathcal{V}\langle d \rangle$  by an isomorphism over  $\mathcal{V}$  with  $c \mapsto d$  if and only if either both  $c$  and  $d$  are in  $D \cap \theta_{b,a,\lambda}^{-1}(P')$  or both are not in  $D \cap \theta_{b,a,\lambda}^{-1}(P')$ . Thus also in this case, the second part of the claim follows.

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