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Notes on equational theories of relations

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Abstract. We describe explicitly the free algebras in the equational class generated by all algebras of binary relations with operations of union, composition, converse and reflexive transitive closure and neutral elements 0 (empty relation) and 1 (identity relation). We show the corresponding equational theory is decidable by reducing the problem to a question about regular sets. Similar results are given for two related equational theories.

1. Introduction

The paper contains two main results. Firstly, Theorem 5.3 gives an explicit description of the free algebras in the equational class $\mathrm{REL}^\vee$ generated by all algebras of binary relations $\mathbf{Rel}_A$ on a set A with operations of union, composition, converse and reflexive transitive closure and neutral elements 0 (the empty relation) and 1 (the identity relation). Second, using the description of the free algebras, we show the corresponding equational theory is decidable by reducing the problem to a question about regular sets, cf. Corollary 5.15. Similar results are given for two related equational theories, namely for the equational theory of *full* relations on a set A, cf. Corollaries 5.21 and 5.22, and for the equational theory of the variety $\mathrm{L}^\vee$ generated by all algebras $\mathbf{L}_X$ of subsets of a free monoid X^* with the operations of union, concatenation, reverse, Kleene star and neutral elements 0 (the empty set) and 1 (the singleton set consisting of the empty word), cf. Theorem 5.1 and Corollary 5.2. In addition, Theorem 5.1 also provides a set of equational axioms for the variety $\mathrm{L}^\vee$.

The equational theory of the variety $\mathrm{REL}^\vee$ is important in theoretical computer science since the algebras $\mathbf{Rel}_A$ form the "Kleenean" part of the standard models of dynamic logic, cf. [12].

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2. Semirings

A *semiring* $S = \langle S, +, \cdot, 0, 1\rangle$ is a set equipped with the binary operation $+$ and and constants $0, 1$ such that $\langle S, +, 0\rangle$ is a commutative monoid, $\langle S, \cdot, 1\rangle$ is a monoid and the distributive laws

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

$$(a + b) \cdot c = a \cdot c + b \cdot c$$

and the zero laws

$$0 \cdot a = 0 = a \cdot 0$$

hold, for all $a, b, c \in S$. Usually we write just ab for the product $a \cdot b$ of the elements a and b. A semiring S is *idempotent* if $a + a = a$, for all $a \in S$. The additive structure of an idempotent semiring determines a semilattice with zero, with partial ordering defined by

$$a \leq b \qquad \text{iff} \qquad a + b = b.$$

An idempotent semiring S is *positive* if $0 \neq 1$ and for all $a, b \in S$, if $ab = 0$ then either $a = 0$ or $b = 0$. Note that in any idempotent semiring, if $a + b = 0$ for some elements a and b, then $a = b = 0$. Thus every positive idempotent semiring is positive in the sense of [5].

Sometimes a semiring S will be equipped with a unary operation denoted $*$ or $^\vee$. In this case we call S a *$*$-semiring* or a *$^\vee$-semiring*. If both $*$ and $^\vee$ are defined on S, then S is a *$*^\vee$-semiring.*

A homomorphism between semirings is a function $\varphi : S \to S'$ which preserves the operations and constants. A homomorphism $\varphi : S \to S'$ is called *positive* if nonzero elements of S are mapped to nonzero elements of S', i.e. $a = 0$ *whenever* $a\varphi = 0$, for all $a \in S$. Note that if S is a positive idempotent semiring and φ is positive and surjective, then S' too is a positive idempotent semiring. Morphisms of $*$ or $^\vee$-semirings also preserve the $*$ or $^\vee$-operation.

3. Complete semirings

Many of the semirings considered here are complete semirings. We say that a semiring $S = \langle S, +, \cdot, 0, 1\rangle$ is *complete*, cf. [5], if for any family $(a_i)_{i \in I}$ of elements of S, there is an element $\sum_{i \in I} a_i$ such that

$$\sum_{i=1}^{n} a_i = a_1 + \cdots + a_n$$

$$\sum_{(i,j) \in I \times J} a_i b_j = \left(\sum_{i \in I} a_i\right) \cdot \left(\sum_{j \in J} b_j\right)$$

$$\sum_{i \in I} a_i = \sum_{j \in J} \left(\sum_{i \in I_j} a_i\right),$$

where I is the union of the pairwise disjoint sets I_j, $j \in J$. It follows that $\sum \varnothing = 0$ and that summation is commutative, associative and completely distributive. A *completely idempotent* semiring is a complete semiring S such that

$$\sum_{i \in I} a_i = a$$

whenever $a_i = a$, for all i in a nonempty set I. In particular, any completely idempotent semiring is idempotent. A homomorphism between complete semirings is *completely additive* if it preserves all sums.

In any complete semiring, we define a unary operation $a \mapsto a^*$ by

$$a^* := \sum_{n=0}^{\infty} a^n.$$

Any completely additive homomorphism between complete semirings automatically preserves $*$.

Two examples of completely additive semirings are the semiring $\mathbf{Rel}_A$ of binary relations on a nonempty set A and the semiring $\mathbf{L}_X$ of subsets of X^*, the set of all words on X. In these semirings, addition is set union and 0 is the empty set. In $\mathbf{Rel}_A$, multiplication is relational composition or relative product with the identity relation as unit. In $\mathbf{L}_X$, multiplication is concatenation with the set containing only the empty word denoted 1 as unit. In $\mathbf{Rel}_A$, a^* is the reflexive transitive closure of the relation a, so that $1 \leq a^*$. In $\mathbf{L}_X$, a^* is the Kleene star of the set a:

$$a^* = \{u_1 \ldots u_n : u_1, \ldots, u_n \in a, n \geq 0\}.$$

A completely additive subsemiring of $\mathbf{Rel}_A$ is the semiring $\mathbf{FRel}_A$ consisting of the empty relation 0 and the full relations. A relation $a \subseteq A \times A$ is *full* if a is defined on the whole set A and its range is A. Note that each of the semirings $\mathbf{L}_X$, $\mathbf{Rel}_A$ and $\mathbf{FRel}_A$ is also completely idempotent and hence idempotent. In addition, $\mathbf{L}_X$ and $\mathbf{FRel}_A$ are positive, but $\mathbf{Rel}_A$ is not, unless A is a singleton set.

We identify each word $v \in X^*$ with the set $\{v\}$. We consider the following proposition well-known.

PROPOSITION 3.1. *$\mathbf{L}_X$ is the free completely idempotent semiring on the set X. More precisely, let $\eta : X \hookrightarrow \mathbf{L}_X$ be the inclusion. If S is any completely idempotent semiring and $\varphi : X \to S$ is any function, there exists a unique completely additive semiring homomorphism*

$$\varphi^{\#} : \mathbf{L}_X \to S$$

with $\eta \circ \varphi^{\#} = \varphi$, i.e. such that $\varphi^{\#}$ agrees with φ on X.

4. Completely idempotent $^\vee$-semirings

In both semirings $\mathbf{Rel}_A$ and $\mathbf{L}_X$, we define an additional operation $a \mapsto a^\vee$. In $\mathbf{Rel}_A$, $a^\vee$ is the *converse* of the relation a:

$$a^\vee = \{(y, x) : (x, y) \in a\}.$$

In $\mathbf{L}_X$, $a^\vee$ is the *reverse* of the set a:

$$a^\vee = \{x_n \dots x_1 : x_1 \dots x_n \in a, x_i \in X, i = 1, \dots, n, n \geq 0\}.$$

Thus, $\mathbf{Rel}_A$ and $\mathbf{L}_X$ are completely idempotent $^\vee$-semirings. The semiring $\mathbf{FRel}_A$ is a completely additive sub $^\vee$-semiring of $\mathbf{Rel}_A$, for if a is a full relation then so is $a^\vee$.

When $X' = \{x' : x \in X\}$ is a disjoint copy of X, there is another way of defining $a^\vee$ for a subset a of $(X \cup X')^*$. We define:

- $x^\vee := x'$ and $(x')^\vee := x$, for all $x \in X$;
- $(x_1 \dots x_n)^\vee := x_n^\vee \dots x_1^\vee$, for all $x_1 \dots x_n \in (X \cup X')^*$ with $x_i \in X \cup X'$, $i = 1, \dots, n$;
- $a^\vee := \{v^\vee : v \in a\}$, for all subsets a of $(X \cup X')^*$.

Thus, $0^\vee = 0$ and $1^\vee = 1$.

Below, if we write $\mathbf{L}_{X \cup X'}$, we will always assume the second definition of $^\vee$ is used. Otherwise, if we write $\mathbf{L}_X$, $a^\vee$ will denote the reverse of a.

PROPOSITION 4.1. *The following equations hold in each of the semirings $\mathbf{Rel}_A$, $\mathbf{L}_X$ and $\mathbf{L}_{X \cup X'}$:*

$$\left(\sum_{i\in I} a_i\right)^{\vee} = \sum_{i\in I} a_i^{\vee} \tag{1}$$

$$(ab)^{\vee} = b^{\vee} a^{\vee} \tag{2}$$

$$a^{\vee\vee} = a. \tag{3}$$

In particular,

$$(a+b)^{\vee} = a^{\vee} + b^{\vee} \tag{4}$$

$$(a^*)^{\vee} = (a^{\vee})^* \tag{5}$$

$$0^{\vee} = 0 \tag{6}$$

$$1^{\vee} = 1. \tag{7}$$

The semiring $\mathbf{L}_{X\cup X'}$ has the following freeness property.

PROPOSITION 4.2. *$\mathbf{L}_{X\cup X'}$ is freely generated by X in the category of completely idempotent ${}^{\vee}$-semirings in which the equations* (1), (2) *and* (3) *hold.*

Proof. Define η as the composite

$$\eta : X \hookrightarrow X\cup X' \overset{\eta'}{\hookrightarrow} \mathbf{L}_{X\cup X'},$$

where both arrows are inclusions. Let S be a completely idempotent ${}^{\vee}$-semiring in which the equations (1), (2) and (3) hold. Let $\varphi : X \to S$ be any function. Extend φ to a function

$$\varphi' : X\cup X' \to S$$

by defining, for $x\in X$,

$$x\varphi' := x\varphi \quad \text{and} \quad x'\varphi' := (x\varphi)^{\vee}.$$

Let $\varphi^{\#} : \mathbf{L}_{X\cup X'} \to S$ be a completely additive semiring homomorphism. Then the diagram

$$\begin{tikzcd} X\cup X' \arrow[r, "\eta'"] \arrow[dr, "\varphi'"'] & \mathbf{L}_{X\cup X'} \arrow[d, "\varphi^{\#}"] \\ & S \end{tikzcd} \tag{8}$$

commutes if and only if $\varphi^{\#}$ preserves the ${}^{\vee}$-operation and the diagram

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[dr, "\varphi"'] & \mathbf{L}_{X \cup X'} \arrow[d, "\varphi^{\#}"] \\ & S \end{tikzcd} \tag{9}$$

commutes. Indeed, if (8) commutes, then so does (9). Further, for all $x_1, \ldots, x_n \in X \cup X'$, $n \geq 0$,

$$\begin{aligned} \varphi^{\#}((x_1 \ldots x_n)^{\vee}) &= \varphi^{\#}(x_n^{\vee} \ldots x_1^{\vee}) \\ &= \varphi^{\#}(x_n^{\vee}) \ldots \varphi^{\#}(x_1^{\vee}) \\ &= \varphi'(x_n^{\vee}) \ldots \varphi'(x_1^{\vee}) \\ &= (\varphi'(x_n))^{\vee} \ldots (\varphi'(x_1))^{\vee} \\ &= (\varphi'(x_1) \ldots \varphi'(x_n))^{\vee} \\ &= (\varphi^{\#}(x_1 \ldots x_n))^{\vee}. \end{aligned}$$

Thus, for all $a \subseteq (X \cup X')^*$,

$$\begin{aligned} \varphi^{\#}(a^{\vee}) &= \sum_{u \in a} \varphi^{\#}(u^{\vee}) \\ &= \sum_{u \in a} (\varphi^{\#}(u))^{\vee} \\ &= (\varphi^{\#}(a))^{\vee}. \end{aligned}$$

Conversely, if (9) commutes, then so does (8).

But, by Proposition 3.1, for any given function $\varphi' : X \cup X' \to S$ there is exactly one completely additive homomorphism $\varphi^{\#} : \mathbf{L}_{X \cup X'} \to S$ such that (8) commutes. □

We note that $\mathbf{L}_{X \cup X'}$ can be embedded in a semiring $\mathbf{L}_Y$, for some set Y.

PROPOSITION 4.3. *Let $\#$ be a symbol not in X and let $Y := X \cup \{\#\}$. There is an injective completely additive ${}^{\vee}$-semiring homomorphism*

$$\mathbf{L}_{X \cup X'} \hookrightarrow \mathbf{L}_Y.$$

Proof. The completely additive ${}^\vee$-semiring homomorphism $\varphi : \mathbf{L}_{X \cup X'} \to \mathbf{L}_Y$ taking each $x \in X$ to $x\#$ will do. Note that φ takes each letter $x' \in X'$ to $\#x$. □

We will be considering an equation which distinguishes between the semirings $\mathbf{Rel}_A$ and the semirings $\mathbf{L}_X$ or $\mathbf{L}_{X \cup X'}$. The equation

$$a + aa^\vee a = aa^\vee a \tag{10}$$

or equivalently, the inequation

$$a \le aa^\vee a$$

holds in each semiring $\mathbf{Rel}_A$, but fails in the semirings $\mathbf{L}_{X \cup X'}$ or $\mathbf{L}_X$ when X is not empty. Also, the following equation will play a prominant part below:

$$1 + aa^\vee = aa^\vee \tag{11}$$

or equivalently, the inequation

$$1 \le aa^\vee,$$

which, when $a \neq 0$, holds in $\mathbf{FRel}_A$ but not in $\mathbf{Rel}_A$ when $|A| \ge 2$.

DEFINITION 4.4. For each pair of words $u, v \in (X \cup X')^*$, define

$$u \quad \Rightarrow_1 \quad v$$

if and only if $u = u_1 ww^\vee wu_2$ and $v = u_1 wu_2$, for some words u_1, w and u_2. Similarly, we define

$$u \quad \Rightarrow_2 \quad v$$

if and only if $u = u_1 ww^\vee u_2$ and $v = u_1 u_2$, for some words u_1, w and u_2. We say that a set $a \subseteq (X \cup X')^*$ is j-closed, for $j = 1, 2$, if $v \in a$ whenever $u \in a$ and $u \Rightarrow_j v$. The smallest j-closed set containing a set $a \subseteq (X \cup X')^*$ is called the j-closure of a denoted $\mathrm{cl}_j(a)$.

Some properties of the closure are summarized in the next two propositions.

LEMMA 4.5. *We have* $\mathrm{cl}_1(a) \subseteq \mathrm{cl}_2(a)$. *Further, for* $j = 1, 2$,

$$
\begin{aligned}
a &\subseteq \mathrm{cl}_j(a)\\
\mathrm{cl}_j(a) &= \mathrm{cl}_j(\mathrm{cl}_j(a))\\
\mathrm{cl}_j\left(\sum_{i \in I} a_i\right) &= \sum_{i \in I} \mathrm{cl}_j(a_i)\\
\mathrm{cl}_j(ab) &= \mathrm{cl}_j(\mathrm{cl}_j(a)\mathrm{cl}_j(b))\\
\mathrm{cl}_j(a^{\vee}) &= (\mathrm{cl}_j(a))^{\vee}\\
\mathrm{cl}_j(1) &= 1.
\end{aligned}
$$

In particular, $\mathrm{cl}_j(0) = 0$.

The *Dyck* set $D_{X \cup X'}$ is the context-free set generated by the grammar whose productions are

$$S \to xSx', \qquad S \to x'Sx, \qquad S \to SS, \quad \text{and} \quad S \to 1,$$

where $x \in X$. (Note that when X is infinite there is an infinite number of productions.)

LEMMA 4.6. *Let* $u, v \in (X \cup X')^*$. *Then* $u \in \mathrm{cl}_2(v)$ *if and only if* v *has a factorization*

$$v = v_0 u_1 v_1 \ldots u_m v_m$$

such that $u = u_1 \ldots u_m$ *and the words* v_i, $i = 0, \ldots, m$, *are in* $D_{X \cup X'}$.

It follows from Lemma 4.5 that the j-closed subsets of $(X \cup X')^*$ form a completely idempotent $^{\vee}$-semiring. A characterization of the set $\mathrm{cl}_1(v)$ will be given later.

PROPOSITION 4.7. *The set of all j-closed subsets of* $(X \cup X')^*$, $j = 1, 2$, *forms a completely idempotent* $^{\vee}$*-semiring, denoted* $\mathbf{CL}^j_{X \cup X'}$. *In* $\mathbf{CL}^j_{X \cup X'}$, $\sum_{i \in I} a_i$ *is the set union and the* $^{\vee}$*-operation as well as the constants are the same as in* $\mathbf{L}_{X \cup X'}$. *However, the product of the j-closed sets a and b is defined to be the j-closure of their product in* $\mathbf{L}_{X \cup X'}$.

The function $\mathrm{cl}_j : a \mapsto \mathrm{cl}_j(a)$ *is a surjective positive completely additive* $^{\vee}$*-semiring homomorphism* $\mathbf{L}_{X \cup X'} \to \mathbf{CL}^j_{X \cup X'}$.

Thus, the equations (1), (2) and (3) hold in $\mathbf{CL}^j_{X\cup X'}$. Further, both $\mathbf{CL}^1_{X\cup X'}$ and $\mathbf{CL}^2_{X\cup X'}$ are positive.

PROPOSITION 4.8. *The equation* (10) *holds in* $\mathbf{CL}^1_{X\cup X'}$. *The equation* (11) *holds in* $\mathbf{CL}^2_{X\cup X'}$, *for all* $a \neq 0$.

LEMMA 4.9. *Let* S *be a completely idempotent* $^\vee$*-semiring and let* $\varphi : \mathbf{L}_{X\cup X'} \to S$ *be a completely additive* $^\vee$*-semiring homomorphism.*

(1) *If* (10) *holds in* S, *then the kernel of* cl_1 *is included in the kernel of* φ. *Thus, since* cl_1 *is surjective, there is a unique completely additive* $^\vee$*-semiring homomorphism* $\psi_1 : \mathbf{CL}^1_{X\cup X'} \to S$ *with* $\varphi = \mathrm{cl}_1 \circ \psi_1$.

(2) *Suppose that* (11) *holds in* S, *for all* $a \neq 0$, *and that* φ *is positive, so that if* φ *is surjective then* S *too is positive. Then the kernel of* cl_2 *is included in that of* φ. *Thus there is a unique completely additive* $^\vee$*-semiring homomorphism* $\psi_2 : \mathbf{CL}^2_{X\cup X'} \to S$ *such that* $\varphi = \mathrm{cl}_2 \circ \psi_2$. *Further,* ψ_2 *is positive.*

Proof. We only prove the second part. Let $a \subseteq (X \cup X')^*$. We have $\mathrm{cl}_2(a) = \sum a_n$, where

$$a_0 := a$$

$$a_{n+1} := \{u_1 u_2 : (\exists v)[u_1 v v^\vee u_2 \in a_n]\}.$$

We will show that $a\varphi = \mathrm{cl}_2(a)\varphi$, which implies that $a\varphi = b\varphi$ whenever $\mathrm{cl}_2(a) = \mathrm{cl}_2(b)$. Indeed, since $a \subseteq \mathrm{cl}_2(a)$, also $a\varphi \leq \mathrm{cl}_2(a)\varphi$. Conversely, if $u_1 v v^\vee u_2 \in a_n$, then

$$(u_1 u_2)\varphi = u_1\varphi u_2\varphi \leq u_1\varphi v\varphi (v\varphi)^\vee u_2\varphi = (u_1 v v^\vee u_2)\varphi,$$

since $v\varphi \neq 0$ and equation (11) holds in S, for all nonzero elements. It follows that $a_{n+1}\varphi \leq a_n\varphi$, so that $\mathrm{cl}_2(a)\varphi \leq a\varphi$, completing the proof. □

Let $\mathscr{C}_1$ denote the category of completely idempotent $^\vee$-semirings in which the equations (1), (2), (3) and (10) hold. Let $\mathscr{C}_2$ denote the category whose objects are the positive completely idempotent $^\vee$-semirings in which the equations (1), (2), (3) and (11) hold, the latter for all nonzero elements; the morphisms in $\mathscr{C}_2$ are the positive completely additive $^\vee$-semiring homomorphisms.

PROPOSITION 4.10.

(1) *The semiring* $\mathbf{CL}^1_{X\cup X'}$ *is freely generated by* X *in the category* $\mathscr{C}_1$.

(2) *The semiring* $\mathbf{CL}^2_{X\cup X'}$ *if freely generated by* X *in the category* $\mathscr{C}_2$. *In more detail, let* $\eta_2 : X \hookrightarrow \mathbf{CL}^2_{X\cup X'}$ *denote the inclusion. Let* S *be a positive com-*

pletely idempotent $^{\vee}$*-semiring in which the equations* (1), (2), (3) *and* (11) *hold, the latter for all* $a \neq 0$. *If* $\varphi : X \to S$ *is any function such that* $x\varphi \neq 0$, *for all* $x \in X$, *then there is a unique completely additive* $^{\vee}$*-semiring homomorphism* $\varphi^{\#} : \mathbf{CL}^2_{X \cup X'} \to S$ *with* $\eta_2 \circ \varphi^{\#} = \varphi$. *Further,* $\varphi^{\#}$ *is positive.*

Proof. Note that for each letter $x \in X$, $\{x\}$ is both 1-closed and 2-closed and is identified with the letter x. The first statement of the proposition is that if S is a completely idempotent semiring such that the equations (1), (2), (3) and (10) hold in S, and if $\varphi : X \to S$ is any function, then there is a unique completely additive $^{\vee}$-semiring homomorphism $\varphi^{\#} : \mathbf{CL}^1_{X \cup X'} \to S$ with $\eta_1 \circ \varphi^{\#} = \varphi$, where $\eta_1 : X \hookrightarrow \mathbf{CL}^1_{X \cup X'}$ is the inclusion. This follows immediately by Proposition 4.2 and Lemma 4.9.

To prove the second statement, recall that $\mathbf{CL}^2_{X \cup X'}$ is a positive completely idempotent $^{\vee}$-semiring in which the equations (1), (2), (3) hold and in which (11) holds, for all nonzero elements. Suppose that S is another such $^{\vee}$-semiring and that $\varphi : X \to S$ is a function such that $x\varphi \neq 0$, for all $x \in X$. By Proposition 4.2, there is a unique completely additive $^{\vee}$-semiring homomorphism $\bar{\varphi} : \mathbf{L}_{X \cup X'} \to S$ with $\eta \circ \bar{\varphi} = \varphi$, where η is the inclusion of X in $\mathbf{L}_{X \cup X'}$. Since S is positive, it follows that $\bar{\varphi}$ too is positive. By Lemma 4.9, $\bar{\varphi}$ uniquely factors through cl_2. Thus there is a unique completely additive $^{\vee}$-semiring homomorphism $\varphi^{\#} : \mathbf{CL}^2_{X \cup X'} \to S$ with $\eta_2 \circ \varphi^{\#} = \varphi$, where $\eta_2 : X \hookrightarrow \mathbf{CL}^2_{X \cup X'}$ denotes the composite $\eta \circ \mathrm{cl}_2$, so that η_2 is the inclusion of X into $\mathbf{CL}^2_{X \cup X'}$. By Lemma 4.9, $\varphi^{\#}$ is positive. □

Let $v \in (X \cup X')^*$ be a fixed word of length n. Denote the set $\{0, \ldots, n\}$ by A_v. We describe a completely additive $^{\vee}$-semiring homomorphism

$$\varphi_v : \mathbf{L}_{X \cup X'} \to \mathbf{Rel}_{A_v},$$

such that $(0, n) \in v\varphi_v$ and, moreover, for any word u, whenever $(0, n) \in u\varphi_v$, then $v \in \mathrm{cl}_1(u)$. (Recall that each word u is identified with the set $\{u\}$.) It then follows that $a\varphi_v \neq b\varphi_v$ whenever a and b are 1-closed sets and v is in the symmetric difference of a and b.

Let $v = v_1 \ldots v_n$, say, where the v_i's are in $X \cup X'$. In order to define φ_v, by Proposition 4.2 we need to specify the value of φ_v only on each letter $x \in X$. We define

$$x\varphi_v := \{(i-1, i) : v_i = x\} \cup \{(i, i-1) : v_i = x'\}.$$

Thus $(0, n) \in v\varphi_v$. Some properties of φ_v are summarized below. The length of a word u is denoted $|u|$.

(a) For all $x \in X \cup X'$, $i, j \in A_v$, if $(i, j) \in x\varphi_v$ then $|i - j| = 1$.
(b) For all $i, j \in A_v$, there is at most one word $u \in (X \cup X')^*$ with $|u| = |i - j|$ and $(i, j) \in u\varphi_v$.
(c) For all $i, j \in A_v$, $u, w \in (X \cup X')^*$ with $|u| = |w| = |i - j|$, if $(i, j) \in u\varphi_v$ and $(j, i) \in w\varphi_v$ then $w = u^\vee$.

Suppose that u is a word in $(X \cup X')^*$ with $(0, n) \in u\varphi_v$. We show by induction on the length of the word u that $v \in \mathrm{cl}_j(u)$, $j = 1, 2$. By (a) above, the length of u is at least n. So we start the induction with $|u| = n$. In this case, by (b) we have $u = v$, so that $v \in \mathrm{cl}_j(u)$. Suppose that $|u| = m > n$ and that our claim holds for words of length strictly less than the length of u. Write

$$u = y_1 \dots y_m.$$

Then

$$0 = i_0(y_1\varphi_v)i_1 \dots i_{m-1}(y_m\varphi_v)i_m = n.$$

Call a letter y_j *increasing* if $i_j = i_{j-1} + 1$, and *decreasing* if $i_j = i_{j-1} - 1$. Similarly, call a subword w of u increasing (decreasing) if every letter in w is increasing (decreasing). A subword w of u is *monotonic* if it is either increasing or decreasing. We can write

$$u = w_1 \dots w_t$$

as a product of maximal monotonic subwords. Since $m > n$, we know $t \geq 3$, and both w_1 and w_t must be increasing. Let $w = w_j$ be one of these maximal monotonic subwords of *minimum length*. We may assume that $j \neq 1$ and $j \neq t$. Then

$$u = u_1 u_2 w u_3 u_4,$$

where, by definition, u_2 is a suffix of w_{j-1}, u_3 is a prefix of w_{j+1}, and $|u_2| = |w| = |u_3|$. Thus, if w is increasing, u_2 and u_3 are decreasing; if w is decreasing, u_2 and u_3 are increasing. Further,

$$i_r = i_s, \tag{12}$$

where $r := |u_1 u_2|$ and $s := |u_1 u_2 w u_3|$. Also, $u_2 = u_3$ and $w = u_2^\vee$, by (b) and (c). Thus, we have shown:

$$u = u_1 u_2 u_2^\vee u_2 u_4,$$

and $(0, n) \in (u_1 u_2 u_4)\varphi_v$, by (12). Now

$$u_1 u_2 u_4 \in \mathrm{cl}_j(u), \qquad j = 1, 2$$

and

$$v \in \mathrm{cl}_j(u_1 u_2 u_4), \qquad j = 1, 2$$

by the induction hypothesis. The proof is complete.

We are now ready to prove that $\mathbf{CL}^1_{X \cup X'}$ embeds in some direct product of semirings of the form $\mathbf{Rel}_A$. Note that any direct product of complete ${}^\vee$-semirings S_i, $i \in I$, is a complete ${}^\vee$-semiring where the operations, including arbitrary sums, are defined pointwise.

PROPOSITION 4.11. *There is an injective completely additive ${}^\vee$-semiring homomorphism*

$$\varphi_0 : \mathbf{CL}^1_{X \cup X'} \to \prod_{v \in (X \cup X')^*} \mathbf{Rel}_{A_v}.$$

Proof. Let

$$\varphi : \mathbf{L}_{X \cup X'} \to \prod_{v \in (X \cup X')^*} \mathbf{Rel}_{A_v}$$

be the completely additive ${}^\vee$-semiring homomorphism determined by the condition $\varphi \circ \mathrm{pr}_u = \varphi_u$, for all words u. Here, pr_u is the projection $\prod_{v \in (X \cup X')^*} \mathbf{Rel}_{A_v} \to \mathbf{Rel}_{A_u}$. By Lemma 4.9, φ factors through cl_1. Thus there exists a completely additive ${}^\vee$-semiring homomorphism $\varphi_0 : \mathbf{CL}^1_{X \cup X'} \to \prod_{v \in (X \cup X')^*} \mathbf{Rel}_{A_v}$ with $\mathrm{cl}_1 \circ \varphi_0 = \varphi$. φ_0 is unique and is injective by the previous argument. □

REMARK 4.12. We note a simple corollary of the previous proposition. Let $\mathscr{C}$ denote the least class of completely idempotent ${}^\vee$-semirings containing all semirings $\mathbf{Rel}_A$, closed under arbitrary products, completely additive ${}^\vee$-semiring homomorphic images and completely additive substructures. Then $\mathscr{C}_1 = \mathscr{C}$. Indeed, since each of the equations (1), (2), (3) and (10) holds in any semiring $\mathbf{Rel}_A$, we have $\mathscr{C} \subseteq \mathscr{C}_1$. If S is any semiring in $\mathscr{C}_1$, S is a quotient of a free semiring $\mathbf{CL}^1_{X \cup X'}$, which belongs to $\mathscr{C}$, by Proposition 4.11. Thus, $\mathscr{C}_1 \subseteq \mathscr{C}$, proving equality.

A similar argument proves that $\mathbf{CL}^2_{X \cup X'}$, embeds in a direct product of completely idempotent ${}^\vee$-semirings of the form $\mathbf{FRel}_A$. Let $v = v_1 \ldots v_n$ as before.

Define

$$B_v := \{(u, i) : u \in (X \cup X')^*, 0 \le i \le n\}.$$

Let $\varphi_v' : \mathbf{L}_{X \cup X'} \to \mathbf{FRel}_{B_v}$ be the completely additive ${}^\vee$-semiring homomorphism defined by the condition

$$\begin{aligned} x\varphi_v' := &\{((1, i-1), (1, i)) : v_i = x, 0 \le i \le n\} \\ &\cup \{((1, i), (1, i-1)) : v_i = x', 0 \le i \le n\} \\ &\cup \{((u, i), (ux, i)) : u \in (X \cup X')^*, 0 \le i \le n\} \\ &\cup \{((ux', i), (u, i)) : u \in (X \cup X')^*, 0 \le i \le n\}, \end{aligned}$$

for all $x \in X$. Thus

$$\begin{aligned} x'\varphi_v' = &\{((1, i), (1, i-1)) : v_i = x, 0 \le i \le n\} \\ &\cup \{((1, i-1), (1, i)) : v_i = x', 0 \le i \le n\} \\ &\cup \{((ux, i), (u, i)) : u \in (X \cup X')^*, 0 \le i \le n\} \\ &\cup \{((u, i), (ux', i)) : u \in (X \cup X')^*, 0 \le i \le n\}. \end{aligned}$$

It is obvious that each $x\varphi_v'$ is full and that $((1, 0), (1, n)) \in v\varphi_v'$. Indeed, we have

$$((1, i-1), (1, i)) \in v_i\varphi_v'$$

for all $i = 1, \ldots, n$.

Suppose now that $((1, 0), (1, n)) \in u\varphi_v'$, for some word $u \in (X \cup X')^*$. It follows that the word u has a decomposition

$$u := u_0 w_1 u_1 \ldots w_m u_m$$

such that for some integers $i_0 = 0 \le i_1, \ldots, i_{m-1} \le n = i_m$ we have

$$\begin{array}{lcl} ((1, i_0), (1, i_0)) & \in & u_0\varphi_v' \\ ((1, i_0), (1, i_1)) & \in & w_1\varphi_v' \\ ((1, i_1), (1, i_1)) & \in & u_1\varphi_v' \\ & \vdots & \\ ((1, i_{m-1}), (1, i_m)) & \in & w_m\varphi_v' \\ ((1, i_m), (1, i_m)) & \in & u_m\varphi_v'. \end{array}$$

Further, denoting $u_i = u_{i1} \dots u_{ik_i}$ and $w_j = w_{j1} \dots w_{jl_j}$, where the letters u_{it} and w_{js} are in $X \cup X'$, we have

$$\begin{array}{lllll}
(1, i_0) & u_{01}\varphi'_v & (z_{01}, i_0) & u_{02}\varphi'_v \dots u_{0k_0}\varphi'_v & (1, i_0) \\
(1, i_0) & w_{11}\varphi'_v & (1, r_{11}) & w_{12}\varphi'_v \dots w_{1l_1}\varphi'_v & (1, i_1) \\
(1, i_1) & u_{11}\varphi'_v & (z_{11}, i_1) & u_{12}\varphi'_v \dots u_{1k_1}\varphi'_v & (1, i_1) \\
 & & & \vdots & \\
(1, i_{m-1}) & w_{m1}\varphi'_v & (1, r_{m1}) & w_{m2}\varphi'_v \dots w_{ml_m}\varphi'_v & (1, i_m) \\
(1, i_m) & u_{m1}\varphi'_v & (z_{m1}, i_1) & u_{m2}\varphi'_v \dots u_{mk_m}\varphi'_v & (1, i_m),
\end{array}$$

for some words z_{it}, $i = 0, \dots, m$, $t = 1, \dots, k_i - 1$, and some integers r_{js}, $j = 1, \dots, m$, $s = 1, \dots, l_j - 1$. It follows now by construction that each u_i is in the context-free set $D_{X \cup X'}$. Thus, by Lemma 4.6,

$$u' := w_1 \dots w_m \in \mathrm{cl}_2(u).$$

On the other hand, it follows from the previous discussion that

$$(0, n) \in u'\varphi_v,$$

where the function φ_v was defined in the proof of Proposition 4.11. Thus $v \in \mathrm{cl}_1(u')$. Summing up, $u' \in \mathrm{cl}_2(u)$ and $v \in \mathrm{cl}_1(u')$, so that $v \in \mathrm{cl}_2(u)$, since $\mathrm{cl}_1(a) \subseteq \mathrm{cl}_2(a)$, for all sets a.

We thus have the following result.

PROPOSITION 4.13. *There is an injective completely additive* $^\vee$*-semiring homomorphism*

$$\varphi_0 : \mathbf{CL}^2_{X \cup X'} \to \prod_{v \in (X \cup X')^*} \mathbf{FRel}_{B_v}.$$

5. $*^\vee$-semirings

As mentioned above, each complete semiring gives rise to a $*$-semiring by defining a^* as the infinite sum $a^* := \sum_{n=0}^{\infty} a^n$. Thus the semirings $\mathbf{Rel}_A$, $\mathbf{FRel}_A$, $\mathbf{L}_X$, $\mathbf{L}_{X \cup X'}$ and $\mathbf{CL}^j_{X \cup X'}$ have the structure of a $*^\vee$-semiring. In $\mathbf{CL}^j_{X \cup X'}$, the $*$-operation on a j-closed set produces the j-closure of a^* taken in $\mathbf{L}_{X \cup X'}$. In $\mathbf{L}_X$, the smallest $*^\vee$-subsemiring containing the set X consists of the regular subsets of X^*. We will denote this semiring by $\mathbf{R}_X$. Note that $\mathbf{R}_X$ is, at the same time, the smallest $*$-subsemiring of $\mathbf{L}_X$ which contains X as a subset. Let $\mathbf{R}_{X \cup X'}$ (resp. $\mathbf{CR}^j_{X \cup X'}$) denote the smallest $*^\vee$-subsemiring of $\mathbf{L}_{X \cup X'}$ (resp. $\mathbf{CL}^j_{X \cup X'}$) containing X. The elements of $\mathbf{R}_{X \cup X'}$ are the regular subsets of $(X \cup X')^*$. The elements of $\mathbf{CR}^j_{X \cup X'}$ are the j-closures of the regular subsets of $(X \cup X')^*$.

As noted in [2], when considered to be $*$-semirings, the semirings $\mathbf{Rel}_A$ and the semirings $\mathbf{L}_X$ generate the same equational class that we denote REL. The free $*$-semiring in REL, freely generated by the set X, can be identified as the semiring $\mathbf{R}_X$, considered without the $^\vee$-operation. An equational axiomatization of REL is given in [2, 3]. The axioms consist of the semiring axioms, the equations

$$(a+b)^* = (a^*b)^*a^* \tag{13}$$

$$(ab)^* = 1 + a(ba)^*b \tag{14}$$

$$1^* = 1 \tag{15}$$

and two axiom schemes called the *commutative* identity and the *dual commutative* identity. (These conditions already imply the identity $a + a = a$.) Independently, another equational axiomatization of the variety REL was given by D. Krob in [10]. The Krob axioms consist of the same equations except that the commutative and dual commutative identities are replaced by the simpler *group equations.*

In [9], D. Kozen exhibited a simple universal Horn theory whose equational part is the equational theory of the variety REL. His Horn theory is equivalent to the system obtained by adding the implications (16) and (17) to the semiring axioms and the equations (13)–(15):

$$ab \leq b \quad \Rightarrow \quad a^*b \leq b \tag{16}$$

$$ba \leq b \quad \Rightarrow \quad ba^* \leq b \tag{17}$$

Another instance of a universal Horn theory of this sort was found by D. Krob in [10] as a consequence of his equational axiomatization result. He showed that, in addition to the equations of the Kozen axioms, the following implication suffices:

$$aa = a \quad \Rightarrow \quad a^* \leq 1 + a, \tag{18}$$

which, in virtue of the rest of the axioms, can be viewed as a particular subcase of either Kozen implication. (We note that a result whose strength lies between

Kozen's and Krob's appears without proof in [4].) The following Ng–Tarski axioms (19)–(21) for star are theorems of this Horn theory:

$$1 + a \leq a^* \tag{19}$$

$$a^*a^* \leq a^* \tag{20}$$

$$1 + a \leq b \quad \& \quad bb \leq b \quad \Rightarrow \quad a^* \leq b \tag{21}$$

(More exactly, in [11] Ng and Tarski treat transitive closure rather than reflexive transitive closure. The relation of these theories to one another as well as to some other universal Horn theories is studied in [6].)

Let $\mathrm{REL}^\vee$ denote the equational class generated by the $*^\vee$-semirings $\mathbf{Rel}_A$, and let $\mathrm{L}^\vee$ be the variety generated by all $*^\vee$-semirings $\mathbf{L}_X$.

5.1. *The variety* $\mathrm{L}^\vee$

In this subsection we describe the free $*^\vee$-semirings in the variety $\mathrm{L}^\vee$. We also provide an equational axiomatization of $\mathrm{L}^\vee$.

THEOREM 5.1. *$\mathbf{R}_{X \cup X'}$ is the free $*^\vee$-semiring in $\mathrm{L}^\vee$ on the set X. An equational axiomatization of $\mathrm{L}^\vee$ consists of the axioms for* REL *and the equations*

$$(a + b)^\vee = a^\vee + b^\vee \tag{22}$$

$$(ab)^\vee = b^\vee a^\vee \tag{23}$$

$$(a^*)^\vee = (a^\vee)^* \tag{24}$$

$$a^{\vee\vee} = a \tag{25}$$

Proof. First we note that the equations

$$0^\vee = 0$$

$$1^\vee = 1.$$

are consequences of those given in Theorem 5.1.

We have shown, cf. Proposition 4.3, that there is an injective completely additive $^\vee$-semiring homomorphism

$$\varphi : \mathbf{L}_{X \cup X'} \to \mathbf{L}_Y,$$

for some set Y. Thus the restriction of φ to $\mathbf{R}_{X \cup X'}$ is an injective $*^{\vee}$-semiring homomorphism $\mathbf{R}_{X \cup X'} \to \mathbf{L}_Y$. Thus $\mathbf{R}_{X \cup X'}$ is in $\mathrm{L}^{\vee}$.

The axioms of REL and the equations given in the theorem hold in each semiring $\mathbf{L}_Y$. Thus, to complete the proof, we must show that if $\eta : X \hookrightarrow \mathbf{R}_{X \cup X'}$ is the inclusion and if $\varphi : X \to S$ is any function to a $*^{\vee}$-semiring S in which all valid equations of the variety REL and the equations given in the theorem hold, then there is a unique $*^{\vee}$-semiring homomorphism $\varphi^{\#} : \mathbf{R}_{X \cup X'} \to S$ with $\eta \circ \varphi^{\#} = \varphi$. Our argument is similar to the proof of Proposition 4.2. Let $\eta' : X \cup X' \hookrightarrow \mathbf{R}_{X \cup X'}$ be the inclusion. Define $\varphi' : X \cup X' \to S$ by

$$x\varphi' := x \quad \text{and} \quad x'\varphi' := (x\varphi)^{\vee},$$

for all $x \in X$. Let $\varphi^{\#} : \mathbf{R}_{X \cup X'} \to S$ be a $*$-semiring homomorphism. Then the diagram

$$\begin{tikzcd} X \cup X' \arrow[r, "\eta'"] \arrow[dr, "\varphi'"'] & \mathbf{R}_{X \cup X'} \arrow[d, "\varphi^{\#}"] \\ & S \end{tikzcd}$$

commutes if and only if $\varphi^{\#}$ preserves $^{\vee}$ and the diagram

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[dr, "\varphi"'] & \mathbf{R}_{X \cup X'} \arrow[d, "\varphi^{\#}"] \\ & S \end{tikzcd}$$

is commutative. But since $\mathbf{R}_{X \cup X'}$ as a $*$-semiring is freely generated by $X \cup X'$ in REL, there is a unique such homomorphism. □

We note that, due to the axioms on $^{\vee}$, the dual commutative identity can be deleted from the resulting system. □

COROLLARY 5.2. *The equational theory of the variety* $\mathrm{L}^{\vee}$ *is decidable.*

Proof. This follows from Theorem 5.1 and the fact that the equality problem for regular sets is decidable.

5.2. *The variety* $\mathrm{REL}^{\vee}$

In this subsection we give an explicit description of the free $*^{\vee}$-semirings in the variety $\mathrm{REL}^{\vee}$ and show that the corresponding equational theory is decidable.

THEOREM 5.3. *The semiring* $\mathbf{CR}^1_{X\cup X'}$ *is the free* $*^\vee$*-semiring on* X *in the equational class* $\mathrm{REL}^\vee$.

Proof. That $\mathbf{CR}^1_{X\cup X'}$ is isomorphic to a $*^\vee$-subsemiring of a direct product $\prod_{i\in I}\mathbf{Rel}_{A_i}$ follows by Proposition 4.11. Thus $\mathbf{CR}^1_{X\cup X'}$ is in $\mathrm{REL}^\vee$. Let η and η' denote the inclusions

$$X \hookrightarrow \mathbf{CR}^1_{X\cup X'} \quad \text{and} \quad X \hookrightarrow \mathbf{CL}^1_{X\cup X'},$$

respectively. If $\varphi : X \to \mathbf{Rel}_A$ is any function, there is a unique completely additive $^\vee$-semiring homomorphism $\bar{\varphi} : \mathbf{CL}^1_{X\cup X'} \to \mathbf{Rel}_A$ with $\eta' \circ \bar{\varphi} = \varphi$. Let $\varphi^\#$ denote the restriction of $\bar{\varphi}$ to $\mathbf{CR}^1_{X\cup X'}$. Then $\varphi^\#$ is a $*^\vee$-semiring homomorphism $\mathbf{CR}^1_{X\cup X'} \to \mathbf{Rel}_A$ with $\eta \circ \varphi^\# = \varphi$. Since the $*^\vee$-semiring $\mathbf{CR}^1_{X\cup X'}$ is generated by X, it follows now that if S is any $*^\vee$-semiring in $\mathrm{REL}^\vee$ and if $\varphi : X \to S$ is any function, there is a unique $*^\vee$-semiring homomorphism $\varphi^\# : \mathbf{CR}^1_{X\cup X'} \to S$ with $\eta \circ \varphi^\# = \varphi$. □

We have described the free semirings in two varieties of $*^\vee$-semirings, namely in the variety $\mathrm{L}^\vee$ generated by all $*^\vee$-semirings $\mathbf{L}_X$ of subsets of X^* and in the variety $\mathrm{REL}^\vee$ generated by the $*^\vee$-semirings of binary relations. It follows from our characterization that the equational theory of $\mathrm{L}^\vee$ is decidable. The decidability of the equational theory of the variety $\mathrm{REL}^\vee$ reduces to the problem of deciding if two regular languages have the same 1-closure. Below we will show that this question is decidable. Tarski [13] showed that the equational theory of relations with the Boolean operations, product, converse and constructs 0 and 1 is undecidable. We have obtained equational axioms for $\mathrm{L}^\vee$. We conjecture that a set of equational axioms for $\mathrm{REL}^\vee$ can be obtained by adding the equation (10) to the axioms of $\mathrm{L}^\vee$. To prove this, one has to show that the least $*^\vee$-congruence on the semiring $\mathbf{R}_{X\cup X'}$ such that (10) holds in the quotient collapses any two regular sets with the same 1-closure.

It follows from Kozen's result [9] that the system consisting of the semiring axioms, (13)–(15), (22)–(25) and one of Kozen's (16) and (17) is a universal Horn theory whose equational part is the equational theory of $\mathrm{L}^\vee$. Further, if $\mathrm{REL}^\vee$ is completely axiomatized by the axioms of $\mathrm{L}^\vee$ and the equation (10), as we believe, then the equational part of the Horn theory consisting of the semiring axioms, the equations (13)–(15), (22)–(25), (10) and one of (16) and (17) or Krob's (18) above coincides with the equational theory of the variety $\mathrm{REL}^\vee$. Nevertheless there are semirings in $\mathrm{REL}^\vee$ which are not models of Kozen's (16) or (17) or Krob's (18). For one example, take the semiring $S := \mathbf{R}_{\{x\}\cup\{x'\}}/\rho$ considered in [12], where ρ is the $*^\vee$-semiring congruence defined by $a\rho b$ if and only if $a = b = 0$, or $a = b = 1$, or a and b are both finite or both infinite.

We now set out to prove that the equational theory of the variety $\mathrm{REL}^{\vee}$ is decidable. This will be accomplished by showing that the 1-closure of a regular set is effectively regular.

The set of all prefixes of a word $u \in (X \cup X')^*$ will be denoted pre(u). If $v \in \mathrm{pre}(u)$ and $v \neq u$, then v is a proper prefix of u. If v is a suffix of u then we define $u/v := w$ iff $wv = u$. Below we use the notation u/v only if v is a suffix of u.

DEFINITION 5.4. Let u, v be words in $(X \cup X')^*$. A function $\gamma : \mathrm{pre}(u) \to \mathrm{pre}(v)$ is called admissible (with respect to u and v) if $\gamma(1) = 1$ and for all $u_1 \in (X \cup X')^*$ and $x \in X \cup X'$ with $u_1 x \in \mathrm{pre}(u)$ either

$$\gamma(u_1 x) = \gamma(u_1)x$$

or $\gamma(u_1)$ ends in $x^{\vee}$ and

$$\gamma(u_1 x) = \gamma(u_1)/x^{\vee}.$$

DEFINITION 5.5. Let $v \in (X \cup X')^*$. We denote by $F(v)$ the set of all words $u \in (X \cup X')^*$ such that there is an admissible function $\gamma : \mathrm{pre}(u) \to \mathrm{pre}(v)$ with $\gamma(u) = v$. Similarly, we define $u \in G(v)$ if and only if there is an admissible function $\gamma : \mathrm{pre}(u) \to \mathrm{pre}(v)$ with $\gamma(u) = 1$.

It is clear that $v \in F(v)$ and $1 \in G(v)$. We will show that $u \in F(v)$ if and only if $v \in \mathrm{cl}_1(u)$.

LEMMA 5.6. *The following hold for the operators F and G*:
(1) $F(1) = G(1) = 1$;
(2) $F(v)G(v^{\vee}) = F(v)$;
(3) $G(v)G(v) = G(v)$.

Proof. We only prove the second statement. Suppose that $u_1 \in F(v)$ and $u_2 \in G(v^{\vee})$, so that there exist admissible functions

$$\gamma_1 : \mathrm{pre}(u_1) \to \mathrm{pre}(v) \quad \text{and} \quad \gamma_2 : \mathrm{pre}(u_2) \to \mathrm{pre}(v^{\vee})$$

with $\gamma_1(u_1) = v$ and $\gamma_2(u_2) = 1$. Define

$$\gamma : \mathrm{pre}(u_1 u_2) \to \mathrm{pre}(v)$$
$$z \mapsto \gamma_1(z),$$

for all $z \in \mathrm{pre}(u_1)$,

$$u_1 z \mapsto v/(\gamma_2(z))^{\vee},$$

for all $z \in \mathrm{pre}(u_2)$, $z \neq 1$. Note that $\gamma(u_1) = \gamma_1(u_1) = v = v/1 = v/(\gamma_2(1))^{\vee}$. To see that γ is admissible, suppose $zx \in \mathrm{pre}(u_2)$, where $z \in (X \cup X')^*$ and $x \in X \cup X'$. If $\gamma_2(zx) = \gamma_2(z)x$ then

$$\begin{aligned}\gamma(u_1 zx) &= v/(\gamma_2(z)x)^{\vee}\\ &= v/(x^{\vee}\gamma_2(z)^{\vee})\\ &= (v/\gamma_2(z)^{\vee})/x^{\vee}\\ &= \gamma(u_1 z)/x^{\vee}.\end{aligned}$$

If $\gamma_2(zx) = \gamma_2(z)/x^{\vee}$ then

$$\begin{aligned}\gamma(u_1 zx) &= v/\gamma_2(zx)^{\vee}\\ &= v/(\gamma_2(z)/x^{\vee})^{\vee}\\ &= (v/\gamma_2(z)^{\vee})x\\ &= \gamma(u_1 z)x.\end{aligned}$$

Thus γ is admissible with respect to $u_1 u_2$ and v. Since also

$$\gamma(u_1 u_2) = v/(\gamma_2(u_2))^{\vee} = v/1 = v,$$

we have $u_1 u_2 \in F(v)$. □

COROLLARY 5.7. *The following two equations hold*:
(1) $F(v) = F(v)(G(v^{\vee}))^*$;
(2) $G(v) = (G(v))^*$.

LEMMA 5.8. *Suppose that* $\gamma : \mathrm{pre}(u) \to \mathrm{pre}(v)$ *is admissible with* $\gamma(u) = v$. *If* $u = u_1 u_2$ *and* $\gamma(u_1) = v$ *then* $u_1 \in F(v)$ *and* $u_2 \in G(v^{\vee})$.

Proof. Let $\gamma_1 : \mathrm{pre}(u_1) \to \mathrm{pre}(v)$ be the restriction of γ to $\mathrm{pre}(u_1)$. It is obvious that γ_1 is admissible. Since $\gamma_1(u_1) = \gamma(u_1) = v$, we have $u_1 \in F(v)$. Now define $\gamma_2 : \mathrm{pre}(u_2) \to \mathrm{pre}(v^{\vee})$ by

$$\gamma_2(z) := v^{\vee}/\gamma(u_1 z)^{\vee}.$$

It is easy to show that γ_2 is admissible. Also

$$\gamma_2(u_2) = v^\vee/\gamma(u_1u_2)^\vee = v^\vee/v^\vee = 1,$$

so that $u_2 \in G(v^\vee)$. $\square$

LEMMA 5.9. *Let $\gamma : \mathrm{pre}(u) \to \mathrm{pre}(v)$ be admissible with $\gamma(u) = 1$. If $u = u_1u_2$ and $\gamma(u_1) = 1$ then $u_1, u_2 \in G(v)$.*

LEMMA 5.10. *Suppose $\gamma : \mathrm{pre}(u) \to \mathrm{pre}(v)$ is admissible and $\gamma(u) = 1$. Let w be the longest prefix of v in the range of γ. If $u = u_1u_2$ with $\gamma(u_1) = w$, then $u_1 \in F(w)$ and $u_2 \in F(w^\vee)$.*

Proof. Let $\gamma_1 : \mathrm{pre}(u_1) \to \mathrm{pre}(w)$ denote the restriction of γ to $\mathrm{pre}(u_1)$. Since γ_1 is clearly admissible and $\gamma_1(u_1) = w$, we have $u_1 \in F(w)$. Define

$$\gamma_2(z) := w^\vee/\gamma(u_1z)^\vee,$$

for all $z \in \mathrm{pre}(u_2)$. Then γ_2 is an admissible function $\mathrm{pre}(u_2) \to \mathrm{pre}(w^\vee)$ and $\gamma_2(u_2) = w^\vee$. $\square$

PROPOSITION 5.11. *Let $v \in (X \cup X')^*$ and $x \in X \cup X'$. Then:*
(1) $F(vx) = F(v)x(x^\vee G(v^\vee)x)^*$;
(2) $G(xv) = (xG(v)x^\vee)^*$.

Proof. It is straightforward to prove that $F(v)x \subseteq F(vx)$ and $xG(v)x^\vee \subseteq G(xv)$. Thus, by Corollary 5.7,

$$\begin{aligned} F(v)x(x^\vee G(v^\vee)x)^* &\subseteq F(vx)G(x^\vee v^\vee)^* \\ &= F(vx)G((vx)^\vee)^* \\ &= F(vx). \end{aligned}$$

Also

$$(xG(v)x^\vee)^* \subseteq (G(xv))^* = G(xv),$$

again by Corollary 5.7. We now prove that $G(xv) \subseteq (xG(v)x^\vee)^*$. Suppose $u \in G(xv)$. If $u = 1$ then clearly $u \in (xG(v)x^\vee)^*$. We proceed by induction on the length of u. Suppose $|u| > 0$ and let $\gamma : \mathrm{pre}(u) \to \mathrm{pre}(xv)$ be an admissible function with $\gamma(1) = 1$. If there is a nontrivial decomposition $u = u_1 u_2$ such that $\gamma(u_1) = 1$, then by Lemma 5.9 we have $u_1, u_2 \in G(xv)$. Thus

$$u = u_1 u_2 \in (xG(x)^\vee)^*(xG(v)x^\vee)^* = (xG(v)x^\vee)^*$$

by the induction hypothesis. Otherwise, i.e. when u has no nontrivial decomposition $u = u_1 u_2$ with $\gamma(u_1) = 1$, u is of the form $u = xwx^\vee$. Further, the function

$$\begin{aligned}\gamma' : \mathrm{pre}(w) &\to \mathrm{pre}(v)\\ z \longmapsto s \quad &\Leftrightarrow \quad \gamma(xz) = xs\end{aligned}$$

is admissible and $\gamma'(w) = 1$. Thus $w \in G(v)$ and $u = xwx^\vee \in xG(v)x^\vee$.

Lastly we prove the inclusion $F(vx) \subseteq F(v)x(x^\vee G(v^\vee)x)^*$. Suppose $u \in F(vx)$. Since the length of each word in $F(vx)$ is at least $|vx|$, we start the induction with $|u| = |vx|$. However, in this case $u = vx$, so that $u \in F(v)x$. Suppose for the induction step that $|u| > |vx|$ and that $\gamma : \mathrm{pre}(u) \to \mathrm{pre}(v)$ is admissible with $\gamma(u) = vx$. There are two cases. If there is no proper prefix z of u with $\gamma(z) = vx$, then $u = u_1 x$ and the restriction γ' of γ to $\mathrm{pre}(u_1)$ is admissible with respect to u_1 and v. Also $\gamma'(u_1) = v$, so that $u_1 \in F(v)$ and $u \in F(v)x$. If u_1 is a proper prefix of u with $\gamma(u_1) = vx$, then let u_2 denote the corresponding suffix. By Lemma 5.8 we have $u_1 \in F(vx)$ and $u_2 \in G((vx)^\vee) = G(x^\vee v^\vee) = (x^\vee G(v^\vee)x)^*$. Thus, by the induction assumption,

$$\begin{aligned}u = u_1 u_2 &\in F(v)x(x^\vee G(v^\vee)x)^*(x^\vee G(v^\vee)x)^*\\ &= F(v)x(x^\vee G(v^\vee)x)^*. \qquad \square\end{aligned}$$

LEMMA 5.12. *If* $u_1 w u_2 \in F(v)$ *then* $u_1 ww^\vee wu_2 \in F(v)$.

Proof. Let $\gamma : \mathrm{pre}(u_1 wu_2) \to \mathrm{pre}(v)$ be an admissible function with $\gamma(u_1 wu_2) = v$. Define

$$\gamma' : \mathrm{pre}(u_1 ww^\vee wu_2) \to \mathrm{pre}(v)$$

as follows. For words $z \in \mathrm{pre}(u_1 w)$, let $\gamma'(z) := \gamma(z)$. If $w_1 \in \mathrm{pre}(w)$, let w_2 be determined by the condition $w_1 w_2 = w$. Define

$$\begin{aligned}\gamma'(u_1 ww_2^\vee) &:= \gamma(u_1 w_1)\\ \gamma'(u_1 ww^\vee w_1) &:= \gamma(u_1 w_1).\end{aligned}$$

Finally, for words $z \in \mathrm{pre}(u_2)$, let $\gamma'(u_1 ww^{\vee} wz) := \gamma(u_1 wz)$. A straightforward calculation shows that γ' is admissible with respect to $u_1 ww^{\vee} wu_2$ and v. Since also $\gamma'(u_1 ww^{\vee} wu_2) = \gamma(u_1 wu_2) = v$, we have $u_1 ww^{\vee} wu_2 \in F(v)$ as claimed. □

PROPOSITION 5.13. *For words* $u, v \in (X \cup X')^*$, $u \in F(v)$ *if and only if* $v \in \mathrm{cl}_1(u)$.

Proof. Suppose $v \in \mathrm{cl}_1(u)$. Then there is a sequence

$$u = u_0 \quad \Rightarrow_1 \quad u_1 \quad \Rightarrow_1 \quad \cdots \quad \Rightarrow_1 \quad u_n = v.$$

Since $v \in F(v)$, it follows by a straightforward induction argument using Lemma 5.12 that $u_i \in F(v)$, for all $i = n, n-1, \ldots, 0$.

Suppose for the converse that $u \in F(v)$. We will prove $v \in \mathrm{cl}_1(u)$ by induction on the length of u. When $|u| = |v|$ we have $u = v$ and thus $v \in \mathrm{cl}_1(u)$. Suppose for the induction step that $|u| > |v|$ and that our claim holds for all words of length strictly less than the length of u. Since $u \in F(v)$, there is an admissible function $\gamma : \mathrm{pre}(u) \to \mathrm{pre}(v)$. If there is no proper prefix u_1 of u with $\gamma(u_1) = v$, then, as in the proof of Proposition 5.11, it follows that $v = v_1 x$, $u = u_1 x$ and $u_1 \in F(v_1)$ for some words u_1, v_1 and a letter $x \in X$. Thus $v_1 \in \mathrm{cl}_1(u_1)$ and

$$v = v_1 x \in \mathrm{cl}_1(u_1)x \subseteq \mathrm{cl}_1(u)$$

by the induction assumption. To complete the proof suppose that there is a proper prefix $z \in \mathrm{pre}(u)$ with $\gamma(z) = v$. Let u_1 denote the longest such prefix and write u in the form $u = u_1 u_2$. We have $u_1 \in F(v)$ and $u_2 \in G(v^{\vee})$ by Lemma 5.8. Further, by Lemma 5.10 u_2 has a decomposition $u_2 = u_3 u_4$ such that for some prefix $w^{\vee}$ of $v^{\vee}$ we have $u_3 \in F(w^{\vee})$ and $u_4 \in F(w)$. Define $s := v/w$. Since by the induction hypothesis $v \in \mathrm{cl}_1(u_1)$, $w^{\vee} \in \mathrm{cl}_1(u_3)$ and $w \in \mathrm{cl}_1(u_4)$, we have

$$v = sw \in \mathrm{cl}_1(sww^{\vee} w) \subseteq \mathrm{cl}_1(u_1 u_3 u_4) = \mathrm{cl}_1(u). \qquad \square$$

THEOREM 5.14. *If* $a \subseteq (X \cup X')^*$ *is regular, then* $\mathrm{cl}_1(a)$ *is also regular. Further, there is an algorithm that, given a deterministic finite automaton (dfa) accepting the set* a, *produces a dfa accepting* $\mathrm{cl}_1(a)$.

Proof. Let $\mathscr{A} = (Q, Z \cup Z', \delta, q_0, Q_f)$ be a dfa accepting the set a, where Q is the finite set of states, $Z \subseteq X$ is a finite set such that $Z \cup Z'$ contains all letters appearing in the words belonging to a, where $Z' = \{x' : x \in Z\}$, and where δ is the transition, q_0 is the initial state and Q_f is the set of accepting states. Recall that the monoid of $\mathscr{A}$ consists of the transformations $u_{\mathscr{A}} : Q \to Q$ induced by the words

$u \in (Z \cup Z')^*$. The set of all subsets of the monoid of $\mathscr{A}$ is a completely idempotent semiring in a natural way: multiplication is complex product and addition is set union. Let us denote this semiring by $S_{\mathscr{A}}$. Now let

$$\mathscr{A}' = (Q', Z \cup Z', \delta', q_0', Q_f')$$

be the following dfa:

$$Q' := \{(F, G) : F, G \in S_{\mathscr{A}}\}$$

$$q_0' := (\{1_{\mathscr{A}}\}, \{1_{\mathscr{A}}\})$$

$$Q_f' := \{(F, G) \in Q' : (\exists u_{\mathscr{A}} \in F)[q_0 u_{\mathscr{A}} \in Q_f]\}$$

Finally, for all $(F, G) \in Q'$ and $x \in Z \cup Z'$, let

$$\delta'((F, G), x) := (F x_{\mathscr{A}} (x^{\vee}_{\mathscr{A}} G x_{\mathscr{A}})^*, (x^{\vee}_{\mathscr{A}} G x_{\mathscr{A}})^*).$$

Note that the formulas used to define δ' are those appearing in Proposition 5.11. It follows by an easy induction argument that

$$\delta'(q_0', v) = (F(v)_{\mathscr{A}}, G(v^{\vee})_{\mathscr{A}}),$$

for all $v \in (Z \cup Z')^*$, where if Y is a set of words, $Y_{\mathscr{A}}$ is the set of all functions $u_{\mathscr{A}}$, $u \in Y$. Thus, by Proposition 5.13, $\mathscr{A}'$ accepts v if and only if $v \in \mathrm{cl}_1(a)$. □

COROLLARY 5.15.
(1) *It is decidable if two regular sets have the same 1-closure.*
(2) *The equational theory of the variety* $\mathrm{REL}^{\vee}$ *is decidable.*

5.3. *Full relations*

In the rest of the paper, we will consider a particular subcase of the problems outlined above for the variety $\mathrm{REL}^{\vee}$. We will show that the equational theory of full relations with the operations of sum, composition, converse, star and constants 0 and 1 is decidable (provided that no variable is evaluated 0). Further, we will prove that a set of equational axioms consists of the axioms of $\mathrm{L}^{\vee}$ and the equation (11), which holds for nonzero elements. These facts will follow from the following two propositions.

PROPOSITION 5.16. *Let $a \subseteq (X \cup X')^*$ be a regular set. Then* $\mathrm{cl}_2(a)$ *is regular.*

Proof. Suppose that a is accepted by the dfa

$$\mathscr{A} = (Q, Z \cup Z', \delta, q_1, Q_f)$$

where $Z \subseteq X$ is a finite set such that $Z \cup Z'$ contains all letters appearing in the words belonging to a, where $Z' = \{x' : x \in Z\}$. We construct a nondeterministic finite state automaton (nfa)

$$\mathscr{A}' = (Q, Z \cup Z', \delta', q_1, Q_f)$$

with 1-moves which accepts the set $\mathrm{cl}_2(a)$. We define, for all $q \in Q$ and $x \in Z \cup Z'$,

$$\delta'(q, x) := \{\delta(q, x)\}$$

$$\delta'(q, 1) := \{\delta(q, w) : w \in D_{Z \cup Z'}\}.$$

Using the fact that a word $w \in (Z \cup Z')^*$ belongs to the Dyck set $D_{Z \cup Z'}$ if and only if $1 \in \mathrm{cl}_2(w)$, it is easy to prove by induction on the length of the word v that

$$q' \in \delta'(q, v) \quad \Leftrightarrow \quad (\exists u)[v \in \mathrm{cl}_2(u) \;\&\; \delta(q, u) = q'].$$

It follows that $\mathscr{A}$ accepts the set $\mathrm{cl}_2(a)$. □

REMARK 5.17. Let $L \subseteq X^*$ be a language. Call a set a of words *L-closed* if whenever $u_1 v u_2 \in a$ and $v \in L$, then $u_1 u_2 \in a$. The proof of the previous proposition can be modified to yield the following result. Let a be any regular language, and let L be any language. Then $\mathrm{cl}_L(a)$ is also regular, where $\mathrm{cl}_L(a)$ is the least L-closed set containing a. This fact is contained in the statement of Theorem 2.2.1 in [7], attributed to J. Berstel.

REMARK 5.18. Since the intersection of a regular set with a context-free set is context-free and since the emptiness of context-free sets is decidable, the proof of Proposition 5.16 is effective, i.e. there is an algorithm that, given the dfa $\mathscr{A}$, constructs the nfa $\mathscr{A}'$. Thus, if a is regular, $\mathrm{cl}_2(a)$ is effectively regular.

Let S be a semiring, so that the collection of all $n \times n$, $n \geq 1$, matrices over S is also a semiring denoted $S^{n \times n}$. When S is a *-semiring, it is possible to define an *-operation on $n \times n$ matrices turning $S^{n \times n}$ to a *-semiring.

DEFINITION 5.19. Let $M \in S^{n \times n}$. We define M^* by induction on n:
(1) When $n = 1$, so that $M = [a]$ for an element a, $M^* := [a^*]$.
(2) When $n = k + 1$, $k \geq 1$, let us partition M as

$$M := \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where a is $k \times k$, b is $k \times 1$, etc. Let

$$\alpha := (a + bd^*c)^*$$

$$\beta := \alpha b d^*$$

$$\gamma := \delta c a^*$$

$$\delta := (d + ca^*b)^*.$$

We define

$$M^* := \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}. \tag{26}$$

It is known that if the equations (13) and (14) hold in S, then (26) holds in $S^{n \times n}$, regardless how the matrix M is partitioned to four submatrices, cf. [4] and [1].

Below, for regular sets $a, b \subseteq (X \cup X')^*$, we will write $a \sim b$ if $\mathrm{cl}_2(a) = \mathrm{cl}_2(b)$. Further, we will denote by $\approx$ the smallest ${}^{*\vee}$-congruence θ on $\mathbf{R}_{X \cup X'}$ such that the equation (11) holds in the quotient $\mathbf{R}_{X \cup X'}/\theta$, for all $a \neq 0$. Thus, $a \approx b \Rightarrow a \sim b$.

LEMMA 5.20. *Let $a, b \subseteq (X \cup X')^*$ be regular sets. Then $a \approx \mathrm{cl}_2(a)$. Thus $a \approx b$ if and only if $a \sim b$.*

Proof. Let a be the regular set accepted by the dfa $\mathscr{A}$ given in the proof of Proposition 5.16. Suppose that $Q = \{q_1, \ldots, q_n\}$, where q_1 is the initial state. Let M and M' be the $n \times n$ transition matrices of the automata $\mathscr{A}$ and $\mathscr{A}'$, i.e.

$$M_{ij} := \{x \in Z \cup Z' : \delta(q_i, x) = q_j\}$$

$$M'_{ij} := \{x \in Z \cup Z' : q_j \in \delta'(q_i, x)\}$$
$$\cup \{1 : i \neq j, q_j \in \delta'(q_i, 1)\}.$$

Further, define

$$\beta_i := \begin{cases} 1 & \text{if } q_i \in Q_f \\ 0 & \text{otherwise} \end{cases}$$

$$\alpha := [1 \quad 0 \ldots 0]$$

$$\beta := \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}.$$

Thus

$$a = \alpha M^* \beta \quad \text{and} \quad \mathrm{cl}_2(a) = \alpha M'^* \beta.$$

We define yet two more $n \times n$ (transition) matrices A and B. First, for all $i \neq j$ such that $q_j \in \delta'(q_i, 1)$ choose a nonempty word $w_{ij} \in D_{Z \cup Z'}$ with $\delta(q_i, w_{ij}) = q_j$. Then let

$$A_{ij} := \begin{cases} M_{ij} \cup \{w_{ij}\} & \text{if } i \neq j \quad \text{and} \quad q_j \in \delta'(q_i, 1) \\ M_{ij} & \text{otherwise} \end{cases}$$

$$B_{ij} := \begin{cases} M_{ij} \cup \{w_{ij}, 1\} & \text{if } i \neq j \quad \text{and} \quad q_j \in \delta'(q_i, 1) \\ M_{ij} & \text{otherwise.} \end{cases}$$

Since clearly

$$a = \alpha A^* \beta$$

$$\mathrm{cl}_2(a) = \alpha B^* \beta$$

and $A \approx B$, i.e. $A_{ij} \approx B_{ij}$, for all $1 \leq i, j \leq n$, we have $a \approx \mathrm{cl}_2(a)$ by the formula (26). □

COROLLARY 5.21. *Let* $\eta : X \hookrightarrow \mathbf{CR}^2_{X \cup X'}$ *be the inclusion. Let* S *be a positive idempotent* $*^\vee$*-semiring in which the valid equations of the equational class* $\mathrm{L}^\vee$ *hold as well as the equation* (11), *for all* $a \neq 0$. *If* $\varphi : X \to S$ *is any function with the property that the elements* $x\varphi$, $x \in X$, *are nonzero, then there is a unique* (*positive*) $*^\vee$*-semiring homomorphism* $\varphi^\# : \mathbf{CR}^2_{X \cup X'} \to S$ *with* $\eta \circ \varphi^\# = \varphi$.

COROLLARY 5.22. *It is decidable whether an equation* (*involving the operations of addition, product,* $*$ *and* $^\vee$ *as well as the constants* 0 *and* 1) *holds in all semirings*

FRel$_A$, *both when the variables are interpreted arbitrarily or when the variables are interpreted to be nonzero relations.*

Proof. By Corollary 5.21, Remark 5.18 and Proposition 4.13. □

In fact, when the variables are interpreted as nonzero relations, an equation holds in all semirings **FRel**$_A$ if and only if the 2-closures of the regular sets determined by the two sides of the equation are equal. In this case the equation can be derived from the axioms of $L^\vee$ and the axiom (11) using equational calculus. However, (11) can be used only to instantiate equations $1 + tt^\vee = tt^\vee$ for 'nonzero terms' t. A term t is nonzero if $t = 0$ is not a valid equation in $L^\vee$. Note that nonzero terms may be characterized syntactically. When the variables are interpreted arbitrarily, we may generate a set of equations from a given equation by substituting 0 for some variables in all possible ways. The equation holds in all semirings **FRel**$_A$ if and only if all generated equations hold when the variables have nonzero interpretations.

Note added in proof. One of the principal results of the paper, Theorem 5.3 gives a description of the free algebras in the variety $\mathrm{REL}^\vee$. Since the 1-closure of a regular set is regular, these free algebras can be described alternatively as certain $*^\vee$-semirings of 1-closed regular sets. On the basis of this characterization, we conjectured that a set of equational axioms for the variety $\mathrm{REL}^\vee$ consists of equational axioms for the variety $L^\vee$ and the equation (10). Recently, this conjecture has been proved in [6].

REFERENCES

[1] BLOOM, S. L. and ÉSIK, Z., *Matrix and matricial interaction theories, Part I*, JCSS, *46* (1993), 381–408.
[2] BLOOM, S. L. and ÉSIK, Z., *Matrix and matricial interaction theories of regular sets*, Stevens Technical Report, December 1990.
[3] BLOOM, S. L. and ÉSIK, Z., *Equational axioms for regular sets*, Mathematical Structures in Computer Science, *3* (1993), 1–24.
[4] CONWAY, J., *Regular Algebra and Finite Machines*, Chapman and Hall, London, 1971.
[5] EILENBERG, S., *Automata, Languages and Machines, vol. A*, Academic Press, New York 1974.
[6] ÉSIK, Z. and BERNÁTSKY, L., *Equational properties of Kleene algebras of relations with conversion*, Theoretical Computer Science, to appear.
[7] JANTZEN, M., *Confluent String Rewriting*, EATCS Monographs on Theoretical Computer Science, Vol. 14, Springer-Verlag, 1988.
[8] JÓNSSON, B., *The theory of binary relations*, in: *Algebraic Logic*, Colloquia Mathematica Societatis János Bolyai, 54, North Holland, 1988, pp. 245–292.
[9] KOZEN, D., *A completeness theorem for Kleene algebras and the algebra of regular events*, Proc. 6th IEEE Symp. Logic in Comput. Sci., July 1991, pp. 214–225 and Information and Computation, to appear.

[10] KROB, D., *Complete systems of B-rational identities*, Theoret. Comput. Sci., *89* (1991), 207–343.

[11] NG, K. C. and TARSKI, A., *Relation algebras with transitive closure*, Abstract 742-02-09, Notices Amer. Math. Soc., *24* (1977), A29–A30.

[12] PRATT, V., *Dynamic algebras as a well-behaved fragment of relation algebras*, in: *Algebraic Logic and Universal Algebra in Computer Science*, LNCS 425, Springer-Verlag, 1990, pp. 74–110.

[13] TARSKI, A., *On the calculus of relations*, Journal of Symbolic Logic, *6* (1941), 73–89.

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