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ON ψ-INVARIANT SUBVARIETIES OF SEMIABELIAN VARIETIES AND THE MANIN-MUMFORD CONJECTURE

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Abstract

Let A be a semiabelian variety over an algebraically closed field of arbitrary characteristic, endowed with a finite morphism $\psi : A \to A$. In this paper, we give an essentially complete classification of all ψ-invariant subvarieties of A. For example, under some mild assumptions on (A, ψ) we prove that every ψ-invariant subvariety is a finite union of translates of semiabelian subvarieties. This result is then used to prove the Manin-Mumford conjecture in arbitrary characteristic and in full generality. Previously, it had been known only for the group of torsion points of order prime to the characteristic of K. The proofs involve only algebraic geometry, though scheme theory and some arithmetic arguments cannot be avoided.

0. Introduction

Let A be a semiabelian variety over an algebraically closed field K of arbitrary characteristic p, endowed with an isogeny $\varphi : A \to A$. Consider the morphism of schemes $\psi : A \to A$, $x \mapsto \varphi(x) + a$ for some $a \in A$. In this paper, we give an essentially complete classification of all closed algebraic subvarieties $X \subset A$ satisfying $\psi(X) = X$. There are three types of building blocks:

(a) φ-invariant semiabelian subvarieties,
(b) subvarieties on which ψ induces an automorphism of finite order, and
(c) subvarieties on which some power of ψ induces the Frobenius morphism corresponding to a model over a finite subfield of K.

In Theorem 3.1 we show that every ψ-invariant closed subvariety is built up in a precise way from such blocks. Under some mild assumptions on (A, φ) that forbid building blocks of types (b) and (c), we deduce in Theorem 3.4 that every ψ-invariant subvariety is a finite union of translates of semiabelian

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subvarieties. Special cases of this result were proved by Bogomolov [2, Th. 3] and Hrushovski [9, Cor. 4.1.14].

The similarity between this result and the Manin-Mumford conjecture is no accident. Indeed, one of the main purposes of this article is to give another proof of that conjecture, using algebraic geometry alone. (One could also proceed in the other direction; see Proposition 6.1.) In Theorems 3.6 and 3.7, we deduce two versions of the conjecture in arbitrary characteristic. The weaker version 3.7 states that if no nontrivial subquotient of A can be defined over a finite subfield of K, then every closed algebraic subvariety $X \subset A$, whose intersection with the set of torsion points of A is Zariski dense in X, is a finite union of translates of semiabelian subvarieties. The most general previously existing result is due to Hrushovski [8], who proved the conjecture for the group of torsion points of order prime to the characteristic of K.

This setup to prove the Manin-Mumford conjecture actually goes back to Hrushovski [9], [4]. When studying his proof, we came to suspect that one of his main intervening results [9, Cor. 4.1.13], which can be phrased in terms of algebraic geometry, but which he proved with the help of model theory, may be approached with algebraic geometry alone. In an earlier article [13], we achieved this in the case of abelian varieties in characteristic zero. In this article we cover the general case; see Theorem 3.5. It became clear that after several reduction steps inspired by Hrushovski, the central remaining problem is the classification of ψ-invariant subvarieties of semiabelian varieties, i.e., Theorem 3.1 explained above. Therefore, most of this article deals with that problem on its own terms.

The proof of Theorem 3.1 is somewhat involved, but follows natural basic principles. It uses only algebraic geometry, though the modern terminology of schemes and some arguments of arithmetic flavor cannot be avoided. In contrast, no mathematical logic or Arakelov geometry is employed. The proof applies uniformly to all cases, without any distinction between abelian and semiabelian varieties, and with only few case distinctions according to the characteristic p. The case $p > 0$ is substantially more difficult, because of problems arising from inseparable isogenies, but the arguments for $p = 0$ are simply a proper subset of the general ones.

The main results are formulated in Section 3, where they are also reduced to Theorem 3.1. Sections 1, 2, and 4 introduce terminology and some useful ingredients to the proof. After decomposing X and taking quotients, we may assume that X is irreducible and the translation stabilizer $\mathrm{Stab}_A(X)$ is finite. The main part of the proof begins in Section 5, where first consequences of these assumptions are derived. Section 6 deals with torsion points and shows that the finite ψ-orbits in X form a Zariski dense subset of X. This important fact is then used in two ways.

Section 7 deals with the "infinitesimally pure" case where φ differs by a separable isogeny from the identity or Frobenius. In this case, we prove that finite ψ-orbits arise only when some power ψ^n acts trivially on X, or as $\overline{\mathbb{F}}_p$-valued points when ψ^n is the Frobenius morphism corresponding to a model of X over a finite field $\mathbb{F}_{p^r}$. We also show that X is contained in a translate of a φ-invariant semiabelian subvariety on which φ^n is either the identity or the Frobenius morphism corresponding to a model over $\mathbb{F}_{p^r}$. This is why, when A possesses no such subquotient, the original X was a finite union of translates of semiabelian subvarieties. One should note that A may still possess other subquotients that are defined over a finite field, because the determining factor is the relation between φ and Frobenius, not just the presence of Frobenius. Section 7 also finishes the proof in characteristic zero, because there all isogenies are separable.

The next two sections deal with the general case of characteristic $p > 0$. The purpose of Section 8 is to study the action of ψ^n on the formal completion of X at a fixed point. We show that this completion possesses a direct product decomposition according to weights relating ψ and Frob_p. This decomposition can be viewed as an analogue of the decomposition of the tangent space into generalized eigenspaces, which might have been used fruitfully in characteristic zero. It is the central structural result making our proof work in characteristic p. The passage from the infinitesimal decomposition to a global decomposition is achieved in Section 9, thereby finishing the proof of Theorem 3.1 in the general case.

Finally, in Section 10 we discuss the situation for arbitrary connected commutative algebraic groups instead of semiabelian varieties.

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1. Conventions

Throughout this article we fix the following notation. Let K be an algebraically closed field of characteristic $p \geq 0$. If $p > 0$, then $\overline{\mathbb{F}}_p$ denotes the algebraic closure of the prime field in K, and for every integer $r > 0$ we denote by $\mathbb{F}_{p^r}$ the unique subfield of cardinality p^r. The homomorphism $x \mapsto x^p$ on any $\mathbb{F}_p$-algebra is denoted by σ.

Unless indicated otherwise, all schemes are of finite type over K, all morphisms are morphisms of schemes over K, and all fiber products are taken over K. The reduced subscheme of a scheme X is denoted by X^{red}. A reduced scheme of finite type over K is called a *variety over* K and is often identified with the set of its K-valued points.

If $p > 0$, every scheme X over $\mathbb{F}_p$ possesses a natural endomorphism which is the identity on points and σ on local rings. It is called the *absolute Frobenius endomorphism of* X and denoted again by $\sigma : X \to X$. For a scheme X over K the endomorphism $\sigma^r : X \to X$ factors through a natural morphism $\mathrm{Frob}_{p^r} : X \to (\sigma^r)^*X := X \times_{K,\sigma^r} K$ over K, called the *relative Frobenius morphism of* X. If $X = X_0 \times_{\mathbb{F}_{p^r}} K$ for a scheme X_0 over $\mathbb{F}_{p^r}$, the identity on X_0 induces a natural isomorphism $(\sigma^r)^*X \cong X$ over K that identifies Frob_{p^r} on X with the endomorphism $\sigma^r \times \mathrm{id}_K$. The relative Frobenius morphism of any semiabelian variety is an isogeny.

2. Semiabelian varieties with an isogeny

Throughout this article we consider semiabelian varieties A over K together with a fixed isogeny $A \to A$, always denoted φ. A reduced subscheme $X \subset A$ is called *φ-invariant* if $\varphi(X) = X$. A quotient of a φ-invariant semiabelian subvariety $A' \subset A$ by a subgroup scheme $G \subset A'$ satisfying $\varphi(G) \subset G$ is called a *φ-invariant subquotient of* A. The induced isogeny $A'/G \to A'/G$ is then again denoted φ.

A homomorphism $h : A \to A'$ between two semiabelian varieties with fixed isogenies is called *φ-equivariant* if it satisfies $\varphi \circ h = h \circ \varphi$. If there exists a φ-equivariant isogeny, we say that A and A' are *φ-isogenous*. Note that this relation is symmetric, because for any φ-equivariant isogeny $h : A \to A'$ there exists an isogeny $h' : A' \to A$ such that $h' \circ h = n \cdot \mathrm{id}_A$ for some positive integer n, and h' is automatically φ-equivariant.

Definition 2.1. (a) We call *A pure of weight* 0 if φ is an automorphism of finite order on A.

(b) We call *A positive* if A possesses no φ-invariant subquotient of positive dimension that is pure of weight 0.

(c) We call *A pure of weight* $\alpha = r/s$ for positive integers r and s, not necessarily relatively prime, if $p > 0$ and $\varphi^s = \mathrm{Frob}_{p^r}$ for some model of A over $\mathbb{F}_{p^r}$.

(d) We call A *strictly mixed* if A possesses no φ-invariant subquotient of positive dimension that is pure of some weight $\alpha \geq 0$.

Remark 2.2. Condition (c) is invariant under replacing $\mathbb{F}_{p^r}$ by a finite extension, because r and s are then multiplied by the same number. Similarly, if φ is replaced by a power φ^n for $n \geq 1$, the notion of purity is preserved, except that the weight is multiplied by n. Therefore condition (d) does not change.

Remark 2.3. (a) Being pure of positive weight is not invariant under isogenies. For example, a supersingular abelian variety of dimension > 1 is always isogenous to one defined over a finite field, but need not itself be definable over a finite field.

(b) Being pure of some weight is not invariant under extensions, because φ can have a nontrivial unipotent part in $\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$.

(c) If A and A' are pure of distinct weights, then A and A' cannot have a common φ-invariant subquotient.

(d) If A is positive or strictly mixed, then so is any semiabelian variety φ-isogenous to A and any φ-invariant subquotient of A.

Proposition 2.4. *Assume that A is pure of weight r/s, and fix a model A_0 of A over $\mathbb{F}_{p^r}$ such that $\varphi^s = \mathrm{Frob}_{p^r}$. Then every φ-invariant closed subvariety $X \subset A$ arises by base change from a closed subvariety $X_0 \subset A_0$.*

Proof. Let $\mathcal{L}_0$ be an ample invertible sheaf on A_0, and let $\mathcal{L}$ be its pullback to A. Let $\mathcal{I} \subset \mathcal{O}_A$ denote the ideal sheaf of X, and choose $n \gg 0$ so that $\mathcal{I}\mathcal{L}^{\otimes n} = \mathcal{I} \otimes \mathcal{L}^{\otimes n}$ is generated by global sections. The φ-invariance of X implies that $(\sigma^r \times \mathrm{id}_K)(X) = X = (\sigma^r \times \sigma^r)(X)$, and so $X = (\mathrm{id}_{A_0} \times \sigma^r)(X)$. This equality induces a σ^r-linear automorphism on $H^0(A, \mathcal{I}\mathcal{L}^{\otimes n})$; hence this space arises from a subspace $I_0 \subset H^0(A_0, \mathcal{L}_0^{\otimes n})$. If $X_0 \subset A_0$ denotes the closed subscheme defined by the ideal $I_0\mathcal{L}_0^{\otimes -n} \subset \mathcal{O}_{A_0}$, we clearly have $X = X_0 \times_{\mathbb{F}_{p^r}} K$, as desired. q.e.d.

We also consider the morphism $\psi : A \to A$, $x \mapsto \varphi(x) + a$ for some $a \in A$. A subscheme $X \subset A$ is called *ψ-invariant* if $\psi(X) = X$.

Proposition 2.5. *If A is positive, then every ψ-invariant subvariety is a translate of a φ-invariant subvariety.*

Proof. Since A is positive, the morphism $\varphi - \mathrm{id}_A : A \to A$ has finite kernel and is therefore surjective. Thus there exists $b \in A$ with $\varphi(b) - b = a$. Therefore $\varphi(X + b) = \varphi(X) + a + b = \psi(X) + b = X + b$, as desired. q.e.d.

Proposition 2.6. (a) *For every irreducible closed subvariety $X \subset A$ there exists a unique smallest semiabelian subvariety $A_X \subset A$ containing a translate of X.*

(b) *If X is ψ-invariant, then A_X is φ-invariant.*

Proof. For $d \gg 0$, consider the morphism

$$X^{2d} \longrightarrow A, \quad (x_1, \ldots, x_{2d}) \mapsto x_1 - x_2 + x_3 - + \cdots - x_{2d}.$$

Since X and hence $X - X := \{x - y \mid x, y \in X\}$ is irreducible, and $0 \in X - X$, the image of this morphism is the closed algebraic subgroup $A_X \subset A$ that is generated by $X - X$ if $d \gg 0$. Now any semiabelian subvariety containing a translate of X contains $X - X$, so it must contain A_X. As A_X contains the translate $X - x$ for any $x \in X$, we find that A_X has the property (a). If X is ψ-invariant, the above morphism is equivariant with respect to the diagonal action $(\psi_X)^{2d}$ on X^{2d} and the action of φ on A; hence A_X is φ-invariant, proving (b). q.e.d.

3. Main results

The following theorem will be proved in the subsequent sections. (In the case where φ is multiplication by an integer > 1, it was shown by Bogomolov [2, Th. 3]. It can also be viewed as a generalization to arbitrary characteristic of the case $\sigma = \mathrm{id}$ of Hrushovski's result [9, Cor. 4.1.14, p. 90].) In this section, we deduce some consequences from it. The definition of $\mathrm{Stab}_A(X)$ is given in Section 4. The ψ-invariance of X implies that $\varphi\big(\mathrm{Stab}_A(X)\big) \subset \mathrm{Stab}_A(X)$.

Theorem 3.1. *Let A be a semiabelian variety over an algebraically closed field, endowed with an isogeny $\varphi : A \to A$. Consider the morphism $\psi : A \to A$, $x \mapsto \varphi(x) + a$ for some $a \in A$, and let $X \subset A$ be an irreducible closed algebraic subvariety satisfying $\psi(X) = X$. Let B denote the identity component of $\mathrm{Stab}_A(X)^{\mathrm{red}}$. Then there exist finitely many φ-equivariant homomorphisms $h_\alpha : A_\alpha \to A/B$, where A_α is pure of weight $\alpha \geq 0$, and irreducible closed subvarieties $X_\alpha \subset A_\alpha$ satisfying $\varphi(X_\alpha) + a_\alpha = X_\alpha$ for some $a_\alpha \in A_\alpha$, such that*

$$h := \sum_\alpha h_\alpha : \ \prod_\alpha A_\alpha \longrightarrow A/B$$

has finite kernel and, for some point $\bar{a} \in A/B$,

$$X/B = \bar{a} + h\Big(\prod_\alpha X_\alpha\Big).$$

Remark 3.2. The subvarieties $X_\alpha \subset A_\alpha$ are determined only up to translation. Thus for $\alpha > 0$ we may, by Proposition 2.5, require that $a_\alpha = 0$, i.e., that X_α is φ-invariant. Then by Definition 2.1 and Proposition 2.4 both $X_\alpha \subset A_\alpha$ are defined over a finite field.

Remark 3.3. If $\varphi : A \to A$ is an arbitrary endomorphism, then $\varphi^n(A)$ for any $n \gg 0$ is the largest φ-invariant semiabelian subvariety on which φ is an isogeny. Any algebraic subvariety $X \subset A$ with $\psi(X) = X$ then satisfies $X = \psi^n(X) = \varphi^n(X) + \psi^n(0) \subset \varphi^n(A) + \psi^n(0)$; hence it can be described by applying Theorem 3.1 to $X - \psi^n(0) \subset \varphi^n(A)$ in place of A.

Theorem 3.4. *Let A be a semiabelian variety over an algebraically closed field, endowed with an isogeny $\varphi : A \to A$. Assume that A is strictly mixed in the sense of 2.1*(d)*. Then any closed algebraic subvariety $X \subset A$ satisfying $\varphi(X) + a = X$ for some $a \in A$ is a finite union of translates of semiabelian subvarieties.*

Proof. Set $\psi(x) := \varphi(x) + a$. Then every irreducible component Y of X is the image under ψ of some irreducible component of X. As the set of these irreducible components is finite, it is therefore permuted by ψ. Thus some positive power ψ^n maps Y to itself. By Remark 2.2 we may replace φ by φ^n, and consequently ψ by ψ^n, and then we may apply Theorem 3.1 to Y in place of X. The resulting homomorphisms h_α must be trivial, because A is strictly mixed, and so $Y = B + a'$ for a semiabelian subvariety B and a point $a' \in A$. q.e.d.

The next consequence generalizes another theorem of Hrushovski to arbitrary characteristic; see [9, Cor. 4.1.13, p. 90] . It follows from Theorem 3.1 and Proposition 2.4 in exactly the same way as [13, Thm. 2.4] was deduced from [13, Thm. 2.1].

Theorem 3.5. *Let A be a semiabelian variety over an algebraically closed field K of characteristic $p \geq 0$, and let X be an irreducible closed algebraic subvariety. Let ρ be an automorphism of K, such that both $X \subset A$ can be defined over the fixed field K^ρ. The automorphism of the abstract group $A(K)$ induced by ρ is again denoted by ρ. Let $P(T) \in \mathbb{Z}[T]$ be a monic polynomial with integral coefficients, no root of which is a root of unity. Let Γ denote the kernel of the homomorphism $P(\rho) : A(K) \to A(K)$, and assume that $X(K) \cap \Gamma$ is Zariski dense in X. Let B denote the identity component of* $\mathrm{Stab}_A(X)^{\mathrm{red}}$.

(a) *If $p = 0$, then X is a translate of B.*
(b) *If $p > 0$, there is a homomorphism with finite kernel $h : A' \to A/B$, a model A'_0 of A' over a finite subfield $\mathbb{F}_{p^r} \subset K$, an irreducible closed subvariety $X'_0 \subset A'_0$, and a point $\bar{a} \in A/B$, such that*

$$X/B = \bar{a} + h\big(X'_0 \times_{\mathbb{F}_{p^r}} K\big).$$

Now let $\mathrm{Tor}(A)$ denote the set of torsion points of $A(K)$. The following two consequences are versions of the Manin-Mumford conjecture. (Compare Bogomolov [3], Raynaud [14], [15], Hrushovski [9], [8], Ullmo, Szpiro, Zhang [17], [19].)

Theorem 3.6. *Let A be a semiabelian variety over an algebraically closed field K of characteristic $p \geq 0$. Let X be an irreducible closed algebraic* *subvariety such that* $X(K) \cap \mathrm{Tor}(A)$ *is Zariski dense in* X. *Let* B *denote the identity component of* $\mathrm{Stab}_A(X)^{\mathrm{red}}$.

(a) *If* $p = 0$, *then* X *is a translate of* B.
(b) *If* $p > 0$, *there is a homomorphism with finite kernel* $h : A' \to A/B$, *a model* A'_0 *of* A' *over a finite subfield* $\mathbb{F}_{p^r} \subset K$, *an irreducible closed subvariety* $X'_0 \subset A'_0$, *and a point* $\bar{a} \in A/B$, *such that*

$$X/B = \bar{a} + h\big(X'_0 \times_{\mathbb{F}_{p^r}} K\big).$$

Proof. (Compare Hrushovski [9, §5], [4], [13].) Choose a subfield $L \subset K$ that is finitely generated over its prime field, such that both $X \subset A$ can be defined over L. Let $\mathcal{S}$ be an integral scheme of finite type over $\operatorname{Spec} \mathbb{Z}$ whose function field is L. After shrinking $\mathcal{S}$, we may assume that A is the generic fiber of a semiabelian scheme $\mathcal{A} \to \mathcal{S}$. For any abelian group G and any prime number ℓ we let $\mathrm{Tor}_\ell(G) \subset \mathrm{Tor}(G)$ denote the group of torsion points of G of ℓ-power order, and $\mathrm{Tor}^\ell(G)$ the subgroup of torsion points of order prime to ℓ. Note that $\mathrm{Tor}(G) = \mathrm{Tor}^\ell(G) \oplus \mathrm{Tor}_\ell(G)$. Now consider a closed point $s \in \mathcal{S}$ of residue characteristic ℓ, and let $\bar{k}_s$ denote an algebraic closure of its residue field k_s. The reduction map then induces an isomorphism of groups $\mathrm{Tor}^\ell(A) \xrightarrow{\sim} \mathrm{Tor}^\ell(\mathcal{A}_s(\bar{k}_s))$.

Suppose first that $p > 0$, then necessarily $\ell = p$. As the p-rank of the fiber $\mathcal{A}_s$ is a constructible function of $s \in \mathcal{S}$, we may choose s so that the p-ranks of $\mathcal{A}_s$ and A coincide. The reduction map then induces an isomorphism $\mathrm{Tor}_p(A) \to \mathrm{Tor}_p(\mathcal{A}_s(\bar{k}_s))$, and hence an isomorphism $\mathrm{Tor}(A) \to \mathrm{Tor}(\mathcal{A}_s(\bar{k}_s))$. Let $q = p^r$ be the cardinality of k_s; then $\sigma^r : x \mapsto x^q$ induces an endomorphism of $\mathcal{A}_s$. It is known [18] that all complex roots of its characteristic polynomial $P(T) \in \mathbb{Z}[T]$ have absolute value $\sqrt{q}$ on the abelian part, respectively q on the toric part of $\mathcal{A}_s$. By construction $P(\sigma^r) = 0$ as an endomorphism of $\mathcal{A}_s$, and so also as an endomorphism of $\mathrm{Tor}(\mathcal{A}_s(\bar{k}_s))$. Consider the unique lift of σ^r on $\bar{k}_s$ to the strict henselization of $\mathcal{O}_{S,s}$ in K, and extend this in any way to an automorphism ρ of K. Then the reduction isomorphism $\mathrm{Tor}(A) \to \mathrm{Tor}(\mathcal{A}_s(\bar{k}_s))$ is equivariant with respect to ρ and σ^r, and we deduce that $P(\rho) = 0$ on $\mathrm{Tor}(A)$. In other words, we have $\mathrm{Tor}(A) \subset \Gamma := \operatorname{Ker} P(\rho)$, and so $X(K) \cap \Gamma$ is Zariski dense in X. Thus Theorem 3.6 is a direct consequence of Theorem 3.5 in this case.

In the case $p = 0$ we leave aside the ℓ-torsion for a moment. The same arguments yield an automorphism ρ of K over L and a monic polynomial $P(T) \in \mathbb{Z}[T]$ whose roots are no roots of unity, such that $P(\rho) = 0$ on $\mathrm{Tor}^\ell(A)$. Following [4] we repeat the same arguments with another closed point $s' \in \mathcal{S}$ of residue characteristic $\ell' \neq \ell$, obtaining an automorphism ρ' of K over L and a monic polynomial $P'(T) \in \mathbb{Z}[T]$ whose roots are no roots of unity, such that $P'(\rho') = 0$ on $\mathrm{Tor}^{\ell'}(A)$. From these two automorphisms we will construct a third one that possesses the right properties on all of $\mathrm{Tor}(A)$. For this let L^ℓ, $L_\ell \subset K$ be the fields generated over L by the coordinates of all points in $\mathrm{Tor}^\ell(A)$, resp. in $\mathrm{Tor}_\ell(A)$. Both are infinite Galois extensions of L. This intersection is known to be finite over L by Serre. (See [16, pp. 33–34, 56–59] when $L/\mathbb{Q}$ is finite and A is an abelian variety. From [16, p. 54], [4, §3.3] we understand that his arguments extend to the general case.) Thus after replacing L by $L^\ell \cap L_\ell$, we may assume that L^ℓ and L_ℓ are linearly disjoint over L. The subfield of K generated by the coordinates of all points in $\mathrm{Tor}(A)$ is then canonically isomorphic to $L^\ell \otimes_L L_\ell$. The automorphism of $L^\ell \otimes_L L_\ell$ induced by $\rho \otimes \rho'$ then extends to some automorphism ρ'' of K over L. Now $P(\rho)$ vanishes on $\mathrm{Tor}^\ell(A)$; hence so does $P(\rho'')$. Similarly, $P'(\rho')$ vanishes on $\mathrm{Tor}_\ell(A) \subset \mathrm{Tor}^{\ell'}(A)$; hence so does $P'(\rho'')$. Thus with $P''(T) := P(T)P'(T)$ we deduce that $P''(\rho'')$ vanishes on $\mathrm{Tor}(A)$. In other words, we have $\mathrm{Tor}(A) \subset \Gamma := \mathrm{Ker}\, P''(\rho'')$, and so again Theorem 3.6 reduces to Theorem 3.5 in this case. q.e.d.

Theorem 3.7. *Let A be a semiabelian variety over an algebraically closed field K, and let $X \subset A$ be a closed algebraic subvariety. If K has positive characteristic, assume that no nontrivial subquotient of A is defined over a finite subfield of K. Then*

$$X(K) \cap \mathrm{Tor}(A) = \bigcup_{i \in I} X_i(K) \cap \mathrm{Tor}(A),$$

where I is a finite set and each X_i is the translate by an element of A of a semiabelian subvariety of A, immersed in X.

Proof. After replacing X by the Zariski closure of $X(K) \cap \mathrm{Tor}(A)$, we may pass to an irreducible component of X as in the proof of 3.4; the result then follows from Theorem 3.6. q.e.d.

For a discussion of arbitrary connected commutative algebraic groups instead of semiabelian varieties see Section 10.

4. Translation stabilizer

Let A be a semiabelian variety over K, and consider an irreducible closed subvariety $X \subset A$. In this section we recall, resp. prove some basic facts related to the *translation stabilizer of* X. This is the closed subgroup scheme $\mathrm{Stab}_A(X) \subset A$ that is characterized uniquely by the fact that for any scheme S over K and any morphism $a : S \to A$, translation by a on the product $A \times S$ maps the subscheme $X \times S$ to itself if and only if a factors through

$\mathrm{Stab}_A(X)$. It exists by [7, exp. VIII, Ex. 6.5 (e)]. The identity component of the reduced subscheme $\mathrm{Stab}_A(X)^{\mathrm{red}}$ is a semiabelian subvariety.

Proposition 4.1. *For every closed subgroup scheme $G \subset \mathrm{Stab}_A(X)$ we have*

$$\mathrm{Stab}_{A/G}(X/G) = \mathrm{Stab}_A(X)/G.$$

Proof. Consider a morphism $\bar{a} : S \to A/G$. Since the morphism $A \to A/G$ is faithfully flat by [7, exp. VI_B, Prop. 9.2], so is the induced morphism $\mathrm{pr}_1 : T := S \times_{A/G} A \to S$, and the quotient of T by G is S again. Moreover $\bar{a}$ is determined by the morphism $a := \mathrm{pr}_2 : T \to A$. Now $\bar{a}$ is an S-valued point of $\mathrm{Stab}_{A/G}(X/G)$ if and only if translation by $\bar{a}$ on the scheme $(A/G) \times S$ maps the subscheme $(X/G) \times S$ to itself. By faithful flatness this is equivalent to saying that translation by a on $A \times T$ maps the subscheme $X \times T$ to itself. But this means that a is a T-valued point of $\mathrm{Stab}_A(X)$, and taking quotients by G it means that $\bar{a}$ is an S-valued point of $\mathrm{Stab}_A(X)/G$, as desired. q.e.d.

Remark 4.2. In 3.1 it suffices to prove the theorem for $X/B \subset A/B$ in place of $X \subset A$. Thus by Proposition 4.1 it suffices to prove the theorem when $B = 0$, i.e., when $\mathrm{Stab}_A(X)$ is finite.

Remark 4.3. Theorem 3.1 is also invariant under φ-isogenies. Indeed, consider a φ-equivariant isogeny $h : A' \to A$ and a ψ-invariant closed algebraic subvariety $X \subset A$. Then $h^{-1}(X)^{\mathrm{red}}$ is again invariant under a translation of φ, and since X is irreducible, all irreducible components of $h^{-1}(X)^{\mathrm{red}}$ are translates of each other. Thus any single irreducible component X' is invariant under a translation of φ. Furthermore we have $h(X') = X$ and, if B' denotes the identity component of $\mathrm{Stab}_{A'}(X')^{\mathrm{red}}$, we have $h(B') = B$. Thus if Theorem 3.1 holds for $X' \subset A'$, then it follows for $X \subset A$.

Remark 4.4. Theorem 3.1 is invariant under replacing X by an arbitrary translate. Thus for the proof we can replace A by the semiabelian subvariety A_X from Proposition 2.6, where desired.

For later use we recall the jet maps associated to X, following Abramovich [1, §2.1]. For any integer $m \geq 0$ and any point $x \in X(K)$ we have natural K-linear maps

$$L_m := \mathcal{O}_{A,0}/\mathfrak{m}_{A,0}^{m+1} \xrightarrow{\sim} \mathcal{O}_{A,x}/\mathfrak{m}_{A,x}^{m+1} \twoheadrightarrow \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}^{m+1},$$

where the isomorphism on the left-hand side results from translation by x. For $x \in X^{\mathrm{reg}}$, the dimension d_m of the right-hand side is a fixed number depending only on $\dim(X)$ and m. Thus any such point determines a point $f_m(x)$ in the Grassmannian $\mathrm{Grass}_{d_m}(L_m)$ of quotients of L_m of dimension d_m. For the following result see [1, §2.1].

Proposition 4.5. *The map $f_m : X^{\mathrm{reg}} \to \mathrm{Grass}_{d_m}(L_m)$ is a morphism, which for $m \gg 0$ factors through a morphism $X^{\mathrm{reg}}/\mathrm{Stab}_A(X) \to \mathrm{Grass}_{d_m}(L_m)$*

that is a locally closed immersion over an open dense subscheme. In other words, if $Y \subset \mathrm{Grass}_{d_m}(L_m)$ denotes the closure of its image, then the morphism $X^{\mathrm{reg}}/\,\mathrm{Stab}_A(X) \to Y$ is a birational equivalence.

5. Bijectivity

From now on we consider the situation of Theorem 3.1. Let $\psi_X : X \to X$ denote the morphism induced by ψ. By $|S|$ we mean the number of closed points of a scheme S.

Proposition 5.1. *The number of points in every sufficiently general fiber of ψ_X is $|\,\mathrm{Stab}_A(X) \cap \mathrm{Ker}(\varphi)|$.*

Proof. Since $\psi(X) = X$ by assumption, we have $X \subset \psi^{-1}(X)$. For dimension reasons, X is therefore an irreducible component of $\psi^{-1}(X)$. By definition of ψ this is equal to $\varphi^{-1}(X - a)$. Now write φ as the composite of a separable isogeny $\varphi_s : A \to A'$ with a totally inseparable isogeny $\varphi_i : A' \to A$. Then φ_i induces a homeomorphism $X' := \varphi_i^{-1}(X - a) \to X - a$, and φ_s induces an étale Galois covering $\varphi^{-1}(X - a) \to X'$ with Galois group $\mathrm{Ker}(\varphi_s) = \mathrm{Ker}(\varphi)^{\mathrm{red}}$. Since X is an irreducible component of $\varphi^{-1}(X - a)$ and different irreducible components meet at most in a subscheme of codimension ≥ 1, it follows that the restriction of φ_s to $X \to X'$ is an étale Galois covering with Galois group $\mathrm{Stab}_{\mathrm{Ker}(\varphi)^{\mathrm{red}}}(X) = \mathrm{Stab}_A(X) \cap \mathrm{Ker}(\varphi)^{\mathrm{red}}$ over an open dense subscheme of X'. This implies the desired assertion. q.e.d.

Proposition 5.2. *If $\mathrm{Stab}_A(X)$ is finite, then ψ_X is generically bijective.*

Proof. (Compare [2], [12, §1.2].) By induction on n the preceding proposition shows that the number of points in every sufficiently general fiber of ψ_X^n is $|\,\mathrm{Stab}_A(X) \cap \mathrm{Ker}(\varphi)|^n$. On the other hand, by applying the proposition directly to ψ_X^n instead of ψ_X this number is also equal to $|\,\mathrm{Stab}_A(X) \cap \mathrm{Ker}(\varphi^n)|$. Letting $n \to \infty$ shows that $|\,\mathrm{Stab}_A(X) \cap \mathrm{Ker}(\varphi)| = 1$ if $\mathrm{Stab}_A(X)$ is finite. By the preceding proposition ψ_X is then generically bijective. q.e.d.

Proposition 5.3. *If $\mathrm{Stab}_A(X)$ is finite, then ψ_X is bijective.*

Proof. By the preceding proposition ψ_X is generically bijective. Let X' be the normalization of X. Then ψ_X induces a finite morphism $\psi_{X'} : X' \to X'$ that is again generically bijective. The corresponding function field extension is purely inseparable; hence it is dominated by Frobenius Frob_{p^r} for some r. By normality, it follows that Frob_{p^r} on X' actually factors through $\psi_{X'}$. Like Frob_{p^r}, it follows that $\psi_{X'}$ is bijective everywhere.

Now let X'' be the fiber product of X' with itself over X, and let $\psi_{X''} : X'' \to X''$ be the restriction of $\psi_{X'} \times \psi_{X'}$. Then we have a commutative diagram:

$$\begin{tikzcd}
X'' \arrow[r, shift left, "\mathrm{pr}_1"] \arrow[r, shift right, "\mathrm{pr}_2"'] \arrow[d, "\psi_{X''}"'] & X' \arrow[r] \arrow[d, "\psi_{X'}"] & X \arrow[d, "\psi_X"] \\
X'' \arrow[r, shift left, "\mathrm{pr}_1"] \arrow[r, shift right, "\mathrm{pr}_2"'] & X' \arrow[r] & X.
\end{tikzcd}$$

As $\psi_{X'}$ is proper and bijective, and X'' is a closed subscheme of $X' \times X'$, the morphism $\psi_{X''}$ is proper and injective. On the one hand it is therefore injective on the finite set of irreducible components of X'', and hence surjective on this set. But on the other hand its image is closed, so it is equal to X'', and $\psi_{X''}$ is bijective. Since set-theoretically X is the quotient of X' by the equivalence relation defined by X'', we deduce that the ψ_X is bijective, as desired. q.e.d.

Proposition 5.4. *Let $A_X \subset A$ be the φ-invariant semiabelian subvariety associated to X by Proposition 2.6. If $\mathrm{Stab}_A(X)$ is finite, then the endomorphism $A_X \to A_X$ induced by φ is bijective.*

Proof. For $d \gg 0$, consider the morphism $X^{2d} \twoheadrightarrow A_X$ from the proof of 2.6. Setting $Y := X^{2d}$ and $\tilde{Y} := Y \times_{A_X} Y \subset X^{4d}$, we obtain a commutative diagram:

$$\begin{tikzcd}
\tilde{Y} \arrow[r, shift left, "\mathrm{pr}_1"] \arrow[r, shift right, "\mathrm{pr}_2"'] \arrow[d, "(\psi_X)^{4d}"'] & Y \arrow[r] \arrow[d, "(\psi_X)^{2d}"] & A_X \arrow[d, "\varphi"] \\
\tilde{Y} \arrow[r, shift left, "\mathrm{pr}_1"] \arrow[r, shift right, "\mathrm{pr}_2"'] & Y \arrow[r] & A_X.
\end{tikzcd}$$

As $\psi_X : X \to X$ is proper and bijective by Proposition 5.3, so are the vertical morphism in the middle and the morphism $(\psi_X)^{4d} : X^{4d} \to X^{4d}$. Thus the vertical morphism on the left is proper and injective. On the one hand it is therefore injective on the finite set of irreducible components of $\tilde{Y}$, and hence surjective on this set. But on the other hand its image is closed, so it is equal to $\tilde{Y}$. The vertical morphism on the left is therefore bijective. Since set-theoretically A_X is the quotient of Y by the equivalence relation defined by $\tilde{Y}$, we deduce that the vertical map on the right is bijective, as desired. q.e.d.

6. Torsion points and ψ-orbits

Consider the following groups of torsion points:

$$\mathrm{Tor}^{\varphi}(A) := \{a \in \mathrm{Tor}(A) \mid \exists n \geq 1 : \varphi^n(a) = a\},$$

$$\mathrm{Tor}_{\varphi}(A) := \{a \in \mathrm{Tor}(A) \mid \exists n \geq 1 : \varphi^n(a) = 0\}.$$

Using the fact that $\mathrm{Ker}(m \cdot \mathrm{id}_A)$ is finite for every $m \geq 1$ one easily shows that

$$\mathrm{Tor}(A) = \mathrm{Tor}^{\varphi}(A) \oplus \mathrm{Tor}_{\varphi}(A).$$

Proposition 6.1. *If A is positive and X is φ-invariant, then $X \cap \mathrm{Tor}^{\varphi}(A)$ is Zariski dense in X.*

This result shows that the classification of φ-invariant subvarieties and the Manin-Mumford conjecture are intimately related. In fact, Theorem 3.1 can probably be deduced from Theorem 3.6. In our setup, however, we proceed in the reverse direction.

Proof. Suppose first that $K = \overline{\mathbb{F}}_p$ for $p > 0$ and that $\mathrm{Stab}_A(X)$ is finite. Then every closed point of X is torsion. Choose a finite subfield $\mathbb{F}_{p^r} \subset \overline{\mathbb{F}}_p$ over which $X \subset A$ and $\varphi = \psi$ can be defined. Since ψ_X is bijective by Proposition 5.3, it is bijective on the finite set $X(\mathbb{F}_{p^{rn}})$ for every $n \geq 1$. It follows that $X(\mathbb{F}_{p^{rn}}) \subset \mathrm{Tor}^{\varphi}(A)$ and hence $X(\overline{\mathbb{F}}_p) \subset \mathrm{Tor}^{\varphi}(A)$, which is trivially Zariski dense.

If $\mathrm{Stab}_A(X)$ is not finite, let B be the identity component of $\mathrm{Stab}_A(X)^{\mathrm{red}}$. One easily shows that the sequence

$$0 \to \mathrm{Tor}^{\varphi}(B) \to \mathrm{Tor}^{\varphi}(A) \to \mathrm{Tor}^{\varphi}(A/B) \to 0$$

is exact. Let ℓ be any prime number not dividing $\deg(\varphi)$ and not equal to p. Then the group $\mathrm{Tor}_{\ell}(B)$ of ℓ-power torsion points of B is contained in $\mathrm{Tor}^{\varphi}(B)$ and Zariski dense in B. Let $Y \subset X$ be the Zariski closure of $X \cap \mathrm{Tor}^{\varphi}(A)$. From $\mathrm{Tor}_{\ell}(B) \subset \mathrm{Stab}_A(Y)$ we can then deduce $B \subset \mathrm{Stab}_A(Y)$. But Y/B is the Zariski closure of $(X/B) \cap \mathrm{Tor}^{\varphi}(A/B)$, which by the earlier case is X/B. Thus $Y = X$, as desired.

In the general case there exist a finitely generated integral $\mathbb{Z}$-algebra $R \subset K$, a semiabelian scheme $\mathcal{A} \to \mathcal{S} := \mathrm{Spec}\, R$, an isogeny $\varphi : \mathcal{A} \to \mathcal{A}$, and a closed subscheme $\mathcal{X} \subset \mathcal{S}$ satisfying $\varphi(\mathcal{X}) = \mathcal{X}$, such that $(A, \varphi, X) \cong (\mathcal{A}_\eta, \varphi_\eta, \mathcal{X}_\eta)$ for the K-valued point η corresponding to the embedding $R \hookrightarrow K$. After shrinking $\mathcal{S}$, if necessary, we may assume that $\mathcal{S}$ is regular, that $\mathcal{X} \to \mathcal{S}$ is flat, and that every geometric fiber $\mathcal{X}_s$ is irreducible. Let s be a geometric point above a closed point of $\mathcal{S}$.

Lemma 6.2. *Every point in $\mathcal{X}_s^{\mathrm{reg}} \cap \mathrm{Tor}^{\varphi_s}(\mathcal{A}_s)$ lifts to a point in $X^{\mathrm{reg}} \cap \mathrm{Tor}^{\varphi}(A)$.*

Proof. For $n \geq 1$, let $\mathcal{X}_n^{\mathrm{reg}}$ denote the intersection of $\mathrm{diag}(\mathcal{X}^{\mathrm{reg}})$ and $\mathrm{graph}(\varphi^n)$ within $\mathcal{X}^{\mathrm{reg}} \times_{\mathcal{S}} \mathcal{X}^{\mathrm{reg}}$. By construction $\mathcal{X}^{\mathrm{reg}} \times_{\mathcal{S}} \mathcal{X}^{\mathrm{reg}}$ is regular of dimension $\dim(\mathcal{S}) + 2\dim(X)$, and we have taken the intersection of two closed subschemes of codimension $\dim(X)$. Thus every irreducible component of $\mathcal{X}_n^{\mathrm{reg}}$ has dimension $\geq \dim(\mathcal{S})$. On the other hand $\varphi^n - \mathrm{id}_A$ is an isogeny because A is positive; hence $\varphi^n - \mathrm{id}_{\mathcal{A}}$ is an isogeny in every fiber. Since $\mathcal{X}_n^{\mathrm{reg}}$

can be embedded in $\mathrm{Ker}(\varphi^n - \mathrm{id}_{\mathcal{A}})$, it is quasifinite over $\mathcal{S}$. For dimension reasons we thus find that every irreducible component of $\mathcal{X}_n^{\mathrm{reg}}$ has dimension $\dim(\mathcal{S})$ and dominates $\mathcal{S}$. Therefore every point in every fiber of $\mathcal{X}_n^{\mathrm{reg}}$ can be lifted to a point in the generic fiber. Letting $n \to \infty$ proves the lemma. q.e.d.

Now let $Y \subset X$ be the Zariski closure of $X \cap \mathrm{Tor}^{\varphi}(A)$, and let $\mathcal{Y}$ be its closure in the whole family $\mathcal{X}$. If $Y \neq X$, then $\mathcal{Y}_s \neq \mathcal{X}_s$ for all sufficiently general points s. But by the lemma, $\mathcal{Y}_s$ contains $\mathcal{X}_s^{\mathrm{reg}} \cap \mathrm{Tor}^{\varphi_s}(\mathcal{A}_s)$, and for $\overline{\mathbb{F}}_p$-valued points we have already seen that the latter is Zariski dense in $\mathcal{X}_s$. Thus $\mathcal{Y}_s = \mathcal{X}_s$ for all points s defined over some $\overline{\mathbb{F}}_p$. This shows that $Y = X$, as desired. q.e.d.

The next result concerns the opposite case where A has no nontrivial positive subquotient.

Proposition 6.3. *Assume that A is a successive extension of φ-invariant subquotients that are pure of weight* 0. *If* $\mathrm{Stab}_A(X)$ *is finite, then some power ψ_X^n for $n \geq 1$ is the identity.*

Proof. After raising φ and ψ to a positive power, we may assume that φ acts trivially on the given subquotients of A. Then the endomorphism $\varphi - \mathrm{id}_A$ is nilpotent. Next suppose that the proposition has been proved for $X/\mathrm{Stab}_A(X) \subset A/\mathrm{Stab}_A(X)$ in place of $X \subset A$. Then some power ψ_X^n induces the identity on $X/\mathrm{Stab}_A(X)$. It therefore permutes the $d := |\mathrm{Stab}_A(X)|$ points in every $\mathrm{Stab}_A(X)$-orbit; hence $\psi_X^{nd!}$ fixes every point of X, as desired. Using Proposition 4.1, we are thus reduced to the case that $\mathrm{Stab}_A(X)$ is trivial.

Then by Proposition 4.5 we may fix $m \gg 0$ so that the jet map $f_m : X^{\mathrm{reg}} \to \mathrm{Grass}_{d_m}(L_m)$ is a birational equivalence to a closed subvariety $Y \subset \mathrm{Grass}_{d_m}(L_m)$. As $\varphi - \mathrm{id}_A$ is nilpotent, φ induces a unipotent automorphism of $L_m := \mathcal{O}_{A,0}/\mathfrak{m}_{A,0}^{m+1}$. For simplicity we denote this automorphism and the associated automorphism of $\mathrm{Grass}_{d_m}(L_m)$ again by φ. By construction we then have $f_m \circ \psi_X = \varphi \circ f_m$, and so Y is invariant under φ. We now distinguish two cases.

If $p > 0$, every unipotent automorphism of L_m has finite order. Thus φ has finite order on Y; hence ψ_X has finite order, as desired. If $p = 0$ we let $G \subset \mathrm{Aut}_K(L_m)$ be the Zariski closure of the subgroup generated by φ. Then the action of φ on the Grassmannian factors through the natural algebraic action of G, and the φ-invariant closed subvariety Y is also invariant under G. Suppose that φ acts nontrivially on Y. Then so does G, and since G is the Zariski closure of the subgroup generated by a nontrivial unipotent automorphism in characteristic zero, we have $G \cong \mathbb{G}_{a,K}$. Every nontrivial G-orbit in Y is therefore isomorphic to G. Choose such an orbit which meets the open subset $U \subset Y$ over which $f_m : X^{\mathrm{reg}} \to Y$ is an isomorphism.

The equivariance of f_m implies that U is invariant under φ; hence the entire orbit is contained in U. Lifting this orbit via f_m yields an embedding $\mathbb{A}^1_K \cong G \hookrightarrow X^{\mathrm{reg}} \subset A$. But a semiabelian variety A does not possess a nonconstant morphism $\mathbb{A}^1_K \to A$. Thus φ must act trivially on Y, and so ψ_X acts trivially on X, as desired. q.e.d.

Proposition 6.4. *If* $\mathrm{Stab}_A(X)$ *is finite, then the union of all finite* ψ*-orbits in* X *is Zariski dense in* X*.*

Remark. The assumption that $\mathrm{Stab}_A(X)$ is finite in the two preceding propositions cannot be dropped, because ψ might involve a translation by a point of infinite order.

Proof. Let $A' \subset A$ be the largest φ-invariant semiabelian subvariety contained in $\mathrm{Ker}(\varphi^{n!} - \mathrm{id}_A)^n$ for some $n \gg 0$, and consider the φ-invariant short exact sequence

$$0 \longrightarrow A' \longrightarrow A \xrightarrow{\pi} A'' \longrightarrow 0.$$

Then A'' is positive and $X'' := \pi(X)$ is invariant under the induced morphism $\psi : A'' \to A''$. Let us write it in the form $\psi(x'') = \varphi(x'') + a''$. The proof of Proposition 2.5 shows that $\varphi(X'' + b'') = X'' + b''$ for some $b'' \in A''$ with $\varphi(b'') - b'' = a''$. From Proposition 6.1 we deduce that the finite φ-orbits form a Zariski dense subset of $X'' + b''$. Therefore the finite ψ-orbits form a Zariski dense subset of X''. To prove the proposition, it suffices to show that for all $n \geq 1$ and all fixed points x'' of ψ^n in some dense open subset $U'' \subset X''$, the finite ψ^n-orbits form a Zariski dense subset of $X \cap \pi^{-1}(x'')$.

To show this we set $X'_x := (X - x) \cap A'$ for every $x \in X$; we then have $X'_x + x = X \cap \pi^{-1}(\pi(x))$. Let B'_x be the identity component of $\mathrm{Stab}_{A'}(X'_x)^{\mathrm{red}}$. This is a semiabelian subvariety of A' which varies in a flat family for x in some dense open subset $U \subset X$. Now recall that every flat family of semiabelian subvarieties of a semiabelian variety is locally constant for the Zariski topology. Indeed, for every n relatively prime to p, the kernel of $n \cdot \mathrm{id}$ is finite étale over the base and hence locally constant, and thus so is the set of all torsion points of order prime to p; but this set is dense in the fiber, and so the whole fiber is locally constant. As X is irreducible, it follows that all B'_x are equal for $x \in U$. Therefore translation by these B'_x maps a dense open subset of X to itself; hence it stabilizes X. But $\mathrm{Stab}_A(X)$ is finite by assumption, so we may deduce that $\mathrm{Stab}_{A'}(X'_x)$ is finite for all $x \in U$. Any nonempty open subset $U'' \subset \pi(U)$ will suffice for our purposes.

Indeed, suppose that $x \in U$ and that $\psi^n(\pi(x)) = \pi(x)$ for $n \geq 1$. Then ψ^n maps $X'_x + x = X \cap \pi^{-1}(\pi(x))$ to itself. Thus a translate of ψ^n, and hence of φ^n, maps X'_x to itself. Since $\mathrm{Stab}_{A'}(X'_x)$ is finite, we can now apply Proposition 6.3 to $X'_x \subset A'$, with φ^n in place of φ. It follows that some power

ψ^{nm} for $m \geq 1$ acts trivially on $X'_x + x$, and so its fixed points are Zariski dense, as desired. q.e.d.

7. The infinitesimally pure case

In this section we prove Theorem 3.1 when A and φ behave infinitesimally as if they were pure according to 2.1, i.e., when some power φ^s differs from the identity or from Frobenius only by a separable isogeny. The two remaining sections will reduce the general case to this infinitesimally pure case.

The arguments are based on the jet map $f_m : X^{\mathrm{reg}} \to \mathrm{Grass}_{d_m}(L_m)$. We assume that $\mathrm{Stab}_A(X)$ is finite. Then by Proposition 4.5 we may fix $m \gg 0$ so that f_m induces a quasifinite dominant morphism from X^{reg} to a closed subvariety $Y \subset \mathrm{Grass}_{d_m}(L_m)$. Let $A_X \subset A$ be the φ-invariant semiabelian subvariety from Proposition 2.6, which is the unique smallest one containing a translate of X. The first result finishes the proof in characteristic zero:

Proposition 7.1. *If* $\mathrm{Stab}_A(X)$ *is finite and* φ *is separable, then* A_X *is pure of weight* 0 *in the sense of 2.1. In particular, Theorem 3.1 is true when* φ *is separable.*

Proof. If φ is separable, then so is the induced isogeny on any φ-invariant quotient of A. Thus by Remark 4.2 the theorem in the separable case is reduced to the case that $\mathrm{Stab}_A(X)$ is finite. In that case, we simply set $A_0 := A_X$ and let X_0 be any translate of X that is contained in A_0; Theorem 3.1 is then clearly true in this case. Thus it suffices to prove the first assertion.

So assume now that φ is separable and $\mathrm{Stab}_A(X)$ is finite. Then φ induces an automorphism of $L_m = \mathcal{O}_{A,0}/\mathfrak{m}_{A,0}^{m+1}$ and thus of $\mathrm{Grass}_{d_m}(L_m)$, which for simplicity we again denote by φ. By construction we then have $f_m \circ \psi_X = \varphi \circ f_m$, and so Y is φ-invariant. As $\mathrm{Stab}_A(X)$ is trivial, Proposition 6.4 implies that the union of all finite φ-orbits in Y is Zariski dense in Y.

Lemma 7.2. *Some power* φ^n *for* $n \geq 1$ *acts trivially on* Y*.*

Proof. Let $G \subset \mathrm{Aut}_K(L_m)$ be the Zariski closure of the subgroup generated by φ. Then the action of φ on the Grassmannian factors through the natural algebraic action of G, and by construction every finite φ-orbit is a finite G-orbit and vice versa. Now G is an algebraic group of finite type over K, so its number of connected components is finite, say n. The stabilizer of any point from a finite orbit contains the identity component; hence the length of every finite G-orbit divides n. Therefore the length of every finite φ-orbit in $\mathrm{Grass}_{d_m}(L_m)$ divides n.

As the union of all finite φ-orbits in Y is Zariski dense in Y, we find that the same assertion holds for all φ-orbits of length dividing n. Thus the set of fixed points of φ^n in Y is Zariski dense in Y. It is also closed, so it is equal to Y; hence φ^n acts trivially on Y. q.e.d.

Now recall that the morphism $X^{\mathrm{reg}} \to Y$ is equivariant and quasifinite, say of degree r. Since ψ_X^n permutes the, at most, r points in every fiber, we deduce that $\psi_X^{nr!}$ acts trivially on X^{reg}, and hence also on X. The equivariance of the morphism $X^{2d} \twoheadrightarrow A_X$ from the proof of Proposition 2.6 implies that $\varphi^{nr!}$ acts trivially on A_X. Thus A_X is pure of weight 0, as desired. q.e.d.

Proposition 7.3. *If* $\mathrm{Stab}_A(X)$ *is finite and there exist positive integers* r *and* s *such that* φ^s *on* A *is the composite of* Frob_{p^r} *with a separable isogeny* $\lambda : (\sigma^r)^*A \to A$*, then* A_X *is pure of weight* r/s *in the sense of 2.1. In particular, Theorem 3.1 is true in this case.*

Proof. If A_X is pure of weight $\alpha := r/s$, we simply set $A_\alpha := A_X$ and let X_α be any translate of X that is contained in A_α; then clearly Theorem 3.1 is true in this case. Thus it suffices to prove the first assertion.

Since ψ^s is a translate of φ^s, the assumption implies that ψ^s is the composite of $\mathrm{Frob}_{p^r} : A \to (\sigma^r)^*A$ with a separable morphism $\mu : (\sigma^r)^*A \to A$ that is a translate of λ. Here μ is a finite étale covering. On the other hand $\psi_X : X \to X$ is bijective by Proposition 5.3. Thus μ restricts to a bijective morphism $\mu_X : (\sigma^r)^*X \to X$ which is also an étale covering; hence μ_X is an isomorphism. On the other hand, since λ is separable, it induces an isomorphism $(\sigma^r)^*L_m \xrightarrow{\sim} L_m$ and hence an isomorphism

$$(\sigma^r)^* \mathrm{Grass}_{d_m}(L_m) \cong \mathrm{Grass}_{d_m}\big((\sigma^r)^*L_m\big) \xrightarrow{\sim} \mathrm{Grass}_{d_m}(L_m),$$

denoted again by λ. Since

$$(\sigma^r)^*f_m\colon\ (\sigma^r)^*X^{\mathrm{reg}} \longrightarrow (\sigma^r)^* \mathrm{Grass}_{d_m}(L_m) \cong \mathrm{Grass}_{d_m}\big((\sigma^r)^*L_m\big)$$

is the jet map for $(\sigma^r)^*X \subset (\sigma^r)^*A$, we have a commutative diagram:

$$\begin{tikzcd}
(\sigma^r)^*X^{\mathrm{reg}} \arrow[r, "(\sigma^r)^*f_m"] \arrow[d, "\mu_X", "\wr"'] & (\sigma^r)^* \mathrm{Grass}_{d_m}(L_m) \arrow[d, "\lambda", "\wr"'] \\
X^{\mathrm{reg}} \arrow[r, "f_m"] & \mathrm{Grass}_{d_m}(L_m).
\end{tikzcd}$$

Thus by construction of Y the morphism λ induces an isomorphism $(\sigma^r)^*Y \xrightarrow{\sim} Y$. q.e.d.

Lemma 7.4. *There exists a model* X_0^{reg} *of* X^{reg} *over* $\mathbb{F}_{p^r}$ *such that* $\mu_X|X^{\mathrm{reg}} = \mathrm{id}_{X_0^{\mathrm{reg}}} \times \sigma^r$ *on* $X^{\mathrm{reg}} \cong X_0^{\mathrm{reg}} \times_{\mathbb{F}_{p^r}} K$.

Proof. Let X' be the normalization of Y in the function field of X. Since X^{reg} is normal and $X^{\mathrm{reg}} \to Y$ is quasifinite, X' contains X^{reg} as an open dense subscheme. We first prove the assertion for X' in place of X^{reg}.

As the Picard group of $\mathrm{Grass}_{d_m}(L_m)$ is $\mathbb{Z}$, its smallest ample invertible sheaf $\mathcal{L}$ is unique up to automorphism. In particular, we have $\lambda^*\mathcal{L} \cong \mathcal{L} \cong (\sigma^r)^*\mathcal{L}$. Since the morphism $f'_m : X' \to Y \hookrightarrow \mathrm{Grass}_{d_m}(L_m)$ is finite, the invertible sheaf $f'^*_m\mathcal{L}$ is again ample. Thus with $S_n := H^0(X', f'^*_m\mathcal{L}^{\otimes n})$, which is a finite dimensional K-vector space for every $n \geq 0$, we have $X' = \mathrm{Proj}\bigoplus_n S_n$.

Next observe that μ_X induces an isomorphism $\mu_{X'} : (\sigma^r)^*X' \xrightarrow{\sim} X'$. By construction, for every n it induces an isomorphism

$$\begin{aligned} S_n \;=\; H^0(X', f'^*_m\mathcal{L}^{\otimes n}) &\;\longrightarrow\; H^0\big((\sigma^r)^*X', \mu^*_{X'}f'^*_m\mathcal{L}^{\otimes n}\big) \\ &\;\cong\; H^0\big((\sigma^r)^*X', (\sigma^r)^*f'^*_m\mathcal{L}^{\otimes n}\big) \\ &\;\cong\; (\sigma^r)^*S_n. \end{aligned}$$

Viewing it as a σ^{-r}-linear automorphism of S_n, the set of fixed points S_{n0} is an $\mathbb{F}_{p^r}$-subspace such that $S_{n0} \otimes_{\mathbb{F}_{p^r}} K \cong S_n$. Clearly the direct sum of these spaces is a graded algebra turning $X'_0 := \mathrm{Proj}\bigoplus_n S_{n0}$ into a model of X' over $\mathbb{F}_{p^r}$, and by construction $\mu_{X'} = \mathrm{id}_{X'_0} \times \sigma^r$ on $X' \cong X'_0 \times_{\mathbb{F}_{p^r}} K$.

Finally, X^{reg} is a $\mu_{X'}$-invariant open dense subscheme of X'. Applying the same construction as above to the reduced closed complement, or by repeating the arguments in the proof of 2.4, we deduce that this complement comes from a closed subvariety of X'_0. Thus X^{reg} comes from an open subvariety $X^{\mathrm{reg}}_0 \subset X'_0$ having the desired properties. q.e.d.

Next recall that φ induces a bijection $A_X \to A_X$ by Proposition 5.4. Thus the homomorphism $\lambda' : (\sigma^r)^*A_X \to A_X$ induced by λ is both separable and bijective, so it is an isomorphism. In order to perform constructions as in the preceding lemma, we need to know:

Lemma 7.5. *There exists a λ'-invariant ample divisor on A_X.*

Proof. Let $u : (X^{\mathrm{reg}})^{2d} \to A_X$ be the restriction of the morphism $X^{2d} \twoheadrightarrow A_X$ from the proof of Proposition 2.6. Choose any ample effective divisor $D \subset A_X$. After replacing it by a translate, if necessary, we may assume that the closure of $u(u^{-1}(D))$ is D. Now $E := u^{-1}(D)$ is a closed subscheme of $(X^{\mathrm{reg}})^{2d} = (X^{\mathrm{reg}}_0)^{2d} \times_{\mathbb{F}_{p^r}} K$, defined over K. Let E'_0 be a specialization of E that is defined over $\overline{\mathbb{F}}_p$, and let $D' \subset A_X$ be the closure of $u(E'_0 \times_{\overline{\mathbb{F}}_p} K)$. If the specialization is sufficiently general, D' will still have codimension 1. Moreover, since the translation stabilizer lies in a constructible family, for E'_0 near E the translation stabilizer of D' is no bigger than that of D, which is finite. Thus D' is again an ample divisor. If E'_0 is defined over $\mathbb{F}_{p^{rs}}$, then D' is invariant under $(\lambda')^s$. The sum of all s distinct translates $(\lambda')^i(D')$ is then a λ'-invariant ample divisor on A_X. q.e.d.

The next lemma then finishes the proof of Proposition 7.3:

Lemma 7.6. *There exists a model A_{X0} of A_X over $\mathbb{F}_{p^r}$ such that $\lambda' = \mathrm{id}_{A_{X0}} \times \sigma^r$ on $A_X \cong A_{X0} \times_{\mathbb{F}_{p^r}} K$. In particular, A_X is pure of weight r/s.*

Proof. The assertion about λ' is complementary to that in Definition 2.1; hence the second assertion follows from the first. To prove the first, let D be the λ'-invariant ample divisor on A_X given by Lemma 7.5. If A_X is an abelian variety, i.e., if it is proper, we can argue as in Lemma 7.4: The space $S_n := H^0(A_X, \mathcal{O}_{A_X}(nD))$ is then finite dimensional with a semilinear automorphism induced by λ'. It thus acquires a model S_{n0} over $\mathbb{F}_{p^r}$. Thus $A_X = \operatorname{Proj} \bigoplus_n S_n$ acquires the model $A_{X0} := \operatorname{Proj} \bigoplus_n S_{n0}$ over $\mathbb{F}_{p^r}$ with all the desired properties. However, in the general case, the space S_n can be infinite dimensional, so we need a different argument.

For this recall that the morphism $u : (X^{\mathrm{reg}})^{2d} \to A_X$ above is dominant. Let $U \subset (X^{\mathrm{reg}})^{2d}$ be the largest open subscheme on which u is flat. Then the morphism $v := u + u : V := U \times U \to A_X$ is flat and surjective, and therefore faithfully flat of finite presentation. We are interested in the descent diagram:

$$\mathrm{W} := V \times_{A_X} V \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} V \xrightarrow{\quad v \quad} A_X.$$

By equivariance, the subscheme U is invariant under μ_X^{2d}; so as in the proof of Lemma 7.4 we deduce that it comes from an open subscheme $U_0 \subset (X_0^{\mathrm{reg}})^{2d}$. Thus V acquires the model $V_0 := U_0 \times U_0$ over $\mathbb{F}_{p^r}$. Moreover, the morphism v is equivariant in the sense that $v \circ \mu_X^{4d} = \lambda' \circ v$. Therefore the closed subscheme $W \subset V \times V$ is invariant under μ_X^{8d}, and so it, too, acquires a model W_0 over $\mathbb{F}_{p^r}$. By construction the above morphisms pr_i yield morphisms over $\mathbb{F}_{p^r}$,

$$W_0 \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} V_0.$$

We can use this to construct a descent datum for A_X, as follows. Let $\tilde{A}_X$, $\tilde{V}$, and $\tilde{W}$ be derived from A_X, V, and W, respectively, by taking the fiber product $\times_{\operatorname{Spec} \mathbb{F}_{p^r}} \operatorname{Spec} K$. Then we can construct the following commutative diagram:

$$\begin{tikzcd}
\tilde{W} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \tilde{V} \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \tilde{A}_X \arrow[d, shift left] \arrow[d, shift right] \\
W \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & V \arrow[r, "v"] \arrow[d] & A_X \\
W_0 \arrow[r, shift left] \arrow[r, shift right] & V_0 &
\end{tikzcd}$$

Here the middle and left columns describe how V_0 and W_0 are obtained from V and W by descent. The two vertical morphisms on the right-hand side are obtained by descent of morphisms from the vertical morphisms in the middle. Using the fact that v is faithfully flat, one easily shows that these morphisms constitute a descent datum for A_X relative to $\operatorname{Spec} K \to \operatorname{Spec} \mathbb{F}_{p^r}$. This descent datum is effective by [6, éxp. VIII Cor. 7.8], because it leaves invariant the ample divisor given by Lemma 7.5. The resulting model A_{X0} of A_X over $\mathbb{F}_{p^r}$ has the desired properties. q.e.d.

8. Infinitesimal decomposition

In this section we assume $p > 0$. We are interested in the behavior of the formal completions of A and X under φ. To this end we derive some results on p-divisible formal groups with an isogeny φ and on φ-invariant formal subschemes. The main point will be a direct product decomposition according to the different weights relating φ and Frob_p.

We begin by recalling the classification of Dieudonné modules (see [11], [5]). Let W denote the ring of Witt vectors of K. This is a complete discrete valuation ring containing $\mathbb{Z}$, whose maximal ideal is generated by p and whose residue field is K. Let $W_{\mathbb{Q}}$ denote its quotient field. The Frobenius automorphism $\sigma : x \mapsto x^p$ of K lifts in a canonical way to automorphisms of W and $W_{\mathbb{Q}}$ that are again denoted by σ.

For our purposes, a *Dieudonné module* is a free W-module of finite type M together with an injective σ-linear endomorphism F, i.e., an injective endomorphism of additive groups satisfying $F(xm) = {}^{\sigma}xF(m)$ for all $x \in W$ and $m \in M$. Similarly, a *rational Dieudonné module* is a finite-dimensional $W_{\mathbb{Q}}$-vector space $M_{\mathbb{Q}}$ together with a σ-linear automorphism F. A *homomorphism* of Dieudonné modules (rational or not) is simply a W-linear map that commutes with F. Typical examples of rational Dieudonné modules are:

$$M_{\mathbb{Q}}^{r,s} := W_{\mathbb{Q}}[F] \bigm/ W_{\mathbb{Q}}[F](F^s - p^r)$$

for relatively prime integers $s > 0$ and r. The rational number $\beta = r/s$ is called the *slope of* $M_{\mathbb{Q}}^{r,s}$. In fact, all rational Dieudonné modules are classified by slopes (see [11, Thm. 2.1]):

Theorem 8.1. (a) *Every rational Dieudonné module is isomorphic to a direct sum* $\bigoplus_i M_{\mathbb{Q}}^{r_i,s_i}$ *for suitable pairs* (r_i, s_i).

(b) $\mathrm{Hom}(M_{\mathbb{Q}}^{r,s}, M_{\mathbb{Q}}^{r',s'}) = 0$ *for* $(r,s) \neq (r',s')$.

Now let $M_{\mathbb{Q}}$ be a rational Dieudonné module endowed with a bijective homomorphism $\varphi : M_{\mathbb{Q}} \to M_{\mathbb{Q}}$. Theorem 8.1 implies that the isotypic decomposition $M_{\mathbb{Q}} = \bigoplus_\beta M_{\mathbb{Q}}^\beta$ according to slopes β is invariant under φ. On the other hand, we can look at the decomposition of $M_{\mathbb{Q}}^\beta$ under φ. For this, note that by Hensel's lemma, all algebraic conjugates over $W_{\mathbb{Q}}$ of an eigenvalue of φ have the same p-adic valuation. Thus there exists a unique φ-invariant decomposition $M_{\mathbb{Q}}^{\beta} = \bigoplus_{\gamma} M_{\mathbb{Q}}^{\beta,\gamma}$ such that all eigenvalues of φ on $M_{\mathbb{Q}}^{\beta,\gamma}$ have valuation γ.

Lemma 8.2. *Assume that $\beta > 0$ and $\gamma \geq 0$, and consider integers $r \geq 0$ and $s > 0$ with $r/s = \gamma/\beta$. Then there exists a free W-submodule $M^{\beta,\gamma}$ generating $M_{\mathbb{Q}}^{\beta,\gamma}$, which is mapped to itself by F and φ, such that $\varphi^s(M^{\beta,\gamma}) = F^r(M^{\beta,\gamma})$. If $\beta \leq 1$, we can require in addition that $pM^{\beta,\gamma} \subset F(M^{\beta,\gamma})$.*

Proof. We take any W-submodule of finite type N that generates $M_{\mathbb{Q}}^{\beta,\gamma}$ and set

$$M^{\beta,\gamma} := \sum_{\substack{i,j\in\mathbb{Z} \\ ir+js\geq 0}} \varphi^i F^j(N).$$

If $\beta \leq 1$, we choose N so that $pN \subset F(N)$; this then implies that $pM'_\alpha \subset F(M'_\alpha)$. We must prove that $M^{\beta,\gamma}$ is finitely generated over W; all the other desired properties follow easily from the construction. Choose an integer $n \geq 1$ such that $n\beta$ and $n\gamma$ are integers, and set $W' := W(\sqrt[n]{p})$ and $W'_{\mathbb{Q}} := W_{\mathbb{Q}}(\sqrt[n]{p})$. Extend σ to these rings by setting ${}^{\sigma}(\sqrt[n]{p}) = \sqrt[n]{p}$. In terms of any chosen basis of N we can write $\varphi = p^{\gamma} \cdot C \cdot \varphi' \cdot C^{-1}$, where C is an invertible matrix over $W'_{\mathbb{Q}}$ and φ' an invertible matrix over W'. Similarly, we can write $F = p^{\beta} \cdot B \cdot F'\sigma \cdot B^{-1}$, where B is an invertible matrix over $W'_{\mathbb{Q}}$ and F' is an invertible matrix over W'. Thus

$$\begin{aligned} \varphi^i F^j &= p^{i\gamma} \cdot C \cdot \varphi'^i \cdot C^{-1} \cdot p^{j\beta} \cdot B \cdot (F'\sigma)^j \cdot B^{-1} \\ &= p^{(ir+js)\beta/s} \cdot C \cdot \varphi'^i \cdot C^{-1} \cdot B \cdot (F'\sigma)^j \cdot B^{-1}. \end{aligned}$$

Since $ir + js \geq 0$ in the above sum and the remaining terms have bounded denominators, we deduce that $M^{\beta,\gamma}$ is contained in $p^{-m}N$ for some $m \gg 0$ and is therefore finitely generated over W, as desired. q.e.d.

Proposition 8.3. *Let M be a Dieudonné module satisfying $F^n(M) \subset pM$ for some $n \geq 1$, endowed with an injective homomorphism $\varphi : M \to M$. Then there exists a Dieudonné submodule $M' \subset M$ containing $p^n M$ for some n, which possesses a direct sum decomposition $M' = \bigoplus_{\alpha} M'_\alpha$ indexed by certain rational numbers $\alpha = r/s \geq 0$ such that $\varphi(M'_\alpha) \subset M'_\alpha$ and $\varphi^s(M'_\alpha) = F^r(M'_\alpha)$. If, moreover, $pM \subset F(M)$, we can require in addition that $pM'_\alpha \subset F(M'_\alpha)$.*

Proof. The assumption on M implies that all slopes β of $M_{\mathbb{Q}} := M \otimes_W W_{\mathbb{Q}}$ are > 0; besides, the valuations γ of all eigenvalues of φ are ≥ 0. Thus Lemma 8.2 applies to all constituents $M_{\mathbb{Q}}^{\beta,\gamma}$. After rescaling every $M^{\beta,\gamma}$ by a power of p the submodule $M' := \bigoplus_{\beta} \bigoplus_{\gamma} M^{\beta,\gamma}$ is contained in M, and its direct summands $M'_\alpha := \bigoplus_{\beta} M^{\beta,\alpha\beta}$ have all the desired properties. q.e.d.

We translate this result into one of formal groups. Let $R := K[[x_1, \ldots, x_d]]$ denote the ring of power series in d variables over K. For our purposes, a *commutative formal group of dimension d over K* is a formal scheme $\mathcal{G} \cong \operatorname{Spf} R$ together with a morphism $\mu : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ representing the group operation and a morphism $\iota : \mathcal{G} \to \mathcal{G}$ representing the inverse, which satisfy the usual axioms for commutative groups. Turning $\operatorname{Spf} R$ into a formal group is equivalent to giving the dual ring homomorphism $\mu^* : R \to R \hat{\otimes} R$, which is described by power series and is called a *formal group law.* A *homomorphism of commutative formal groups* $f : \mathcal{G} \to \mathcal{G}'$ is a morphism of formal schemes that commutes with the group operations. A homomorphism of commutative formal groups of the same dimension that is also an epimorphism is called an *isogeny.* The kernel of an isogeny is a finite formal subscheme; so it is a finite commutative group scheme in the usual sense.

One example of a homomorphism $\mathcal{G} \to \mathcal{G}$ is the multiplication by p. In the following, we assume throughout that this is an isogeny, i.e., we restrict attention to *p-divisible* formal groups.

There is a contravariant equivalence of categories $\mathcal{G} \mapsto M(\mathcal{G})$ from the category of (p-divisible) commutative formal groups over K to the full subcategory of Dieudonné modules M satisfying $F^n(M) \subset pM \subset F(M)$ for some $n \geq 1$ (see [11, Thm. 1.4] or [5, ch. III, §1.4]). The extra condition on M implies that all slopes β of $M_{\mathbb{Q}}$ satisfy $0 < \beta \leq 1$. A homomorphism of commutative formal groups is an isogeny if and only if it induces an isomorphism on the associated rational Dieudonné modules.

Now let $\mathcal{G}$ be a (p-divisible) commutative formal group over K, endowed with an isogeny $\varphi : \mathcal{G} \to \mathcal{G}$.

Definition 8.4. We call $\mathcal{G}$ *pure of weight* $\alpha = r/s$ for integers $r \geq 0$ and $s > 0$, not necessarily relatively prime, if φ^s is the composite of $\operatorname{Frob}_{p^r} : \mathcal{G} \to (\sigma^r)^*\mathcal{G}$ with an isomorphism $(\sigma^r)^*\mathcal{G} \xrightarrow{\sim} \mathcal{G}$.

Theorem 8.5. *There exists a φ-invariant isogeny $\mathcal{G} \to \prod_\alpha \mathcal{G}_\alpha$, where the $\mathcal{G}_\alpha$ are finitely many formal groups with an isogeny φ that are pure of weight $\alpha \geq 0$.*

Proof. The Dieudonné module M of $\mathcal{G}$ carries an injective endomorphism induced by φ, to which we can apply Proposition 8.3. The resulting Dieudonné submodule $\bigoplus_\alpha M'_\alpha \subset M$ then corresponds to an isogeny of formal groups $\mathcal{G} \to \prod_\alpha \mathcal{G}_\alpha$, which is equivariant with respect to isogenies φ induced on all factors $\mathcal{G}_\alpha$. To prove that $\mathcal{G}_\alpha$ is pure of weight $\alpha = r/s$, set ${}^r M'_\alpha := M'_\alpha$ with the new action of $x \in W$ by $(x, m) \mapsto {}^{\sigma^{-r}}x \cdot m$ and the same action of F. This is the Dieudonné module of the formal group $(\sigma^r)^*\mathcal{G}_\alpha$, and the homomorphism of Dieudonné modules ${}^r M'_\alpha \to M'_\alpha$ induced by F^r corresponds by the equivalence of categories to the Frobenius isogeny $\operatorname{Frob}_{p^r} : \mathcal{G}_\alpha \to (\sigma^r)^*\mathcal{G}_\alpha$. The relation

$\varphi^s(M'_\alpha) = F^r(M'_\alpha) = F^r({}^rM'_\alpha)$ now means that φ^s factors through F^r and an isomorphism

$$M'_\alpha \xrightarrow{\sim} {}^rM'_\alpha \xrightarrow{F^r} M'_\alpha$$

and so the corresponding isogeny φ factors through Frob_{p^r} and an isomorphism

$$\mathcal{G}_\alpha \xrightarrow{\mathrm{Frob}_{p^r}} (\sigma^r)^*\mathcal{G}_\alpha \xrightarrow{\sim} \mathcal{G}_\alpha,$$

as desired. q.e.d.

Next we will exploit purity.

Proposition 8.6. *If $\mathcal{G}$ is pure of weight $r/s > 0$, there exists an isomorphism $\mathcal{G} \cong \operatorname{Spf} K[[x_1, \dots, x_d]]$ such that $(\varphi^s)^*(x_i) = x_i^{p^r}$ for all $1 \leq i \leq d$.*

Proof. To begin with, choose any identification $\mathcal{G} = \operatorname{Spf} R$ with $R = K[[x_1, ..., x_d]]$. By assumption there exists a σ^{-r}-linear automorphism ψ of R such that $(\varphi^s)^*(f) = \psi(f)^{p^r}$ for all $f \in R$. Let $\mathfrak{m}$ be the maximal ideal of R. We claim that there exist power series $f_1, \dots, f_r \in \mathfrak{m}$ generating $\mathfrak{m}$ and satisfying $\psi(f_i) = f_i$ for all $i = 1, \dots, r$. Then, after replacing the x_i by these f_i, the desired assertion holds.

To find the f_i, note first that ψ induces a σ^{-r}-linear automorphism of the finite-dimensional K-vector space $\mathfrak{m}/\mathfrak{m}^2$. Since $r > 0$, Lang's theorem [10] for $\mathrm{GL}_d(K)$ implies that there is a basis of vectors fixed by ψ. Thus the corresponding elements $f_i \in \mathfrak{m}$ form a system of formal parameters of R and satisfy $\psi(f_i) \equiv f_i \mod \mathfrak{m}^2$. Next we use successive approximation:

Lemma 8.7. *Let $n \geq 2$ and let $f_i \in \mathfrak{m}$ be a system of formal parameters of R such that $\psi(f_i) \equiv f_i \mod \mathfrak{m}^n$ for all i. Then there exist $g_i \in \mathfrak{m}^n$ such that $\psi(f_i + g_i) \equiv f_i + g_i \mod \mathfrak{m}^{n+1}$ for all i.*

Proof. The desired congruence is equivalent to $\psi(g_i) - g_i \equiv f_i - \psi(f_i) \mod \mathfrak{m}^{n+1}$, where both sides now lie in $\mathfrak{m}^n$. In other words, we must solve the equation $\psi(\bar{g}_i) - \bar{g}_i = v_i := \big(f_i - \psi(f_i) \mod \mathfrak{m}^{n+1}\big)$ within the finite-dimensional vector space $V := \mathfrak{m}^n/\mathfrak{m}^{n+1}$. Here ψ is a σ^{-r}-linear automorphism of V, and so Lang's theorem [10] for the vector group V implies the existence of a solution. q.e.d.

Clearly, the elements $f_i + g_i$ again form a system of formal parameters of R. Thus after replacing f_i by $f_i + g_i$ and repeating the process for all $n \geq 2$, we obtain a convergent sequence of formal parameters, whose limit has the desired properties. q.e.d.

Theorem 8.8. *Assume that $\mathcal{G} \cong \prod_\alpha \mathcal{G}_\alpha$, where every $\mathcal{G}_\alpha$ is pure of weight α. Let $\mathcal{X} \subset \mathcal{G}$ be a formally smooth closed formal subscheme satisfying $\varphi(\mathcal{X}) \subset \mathcal{X}$. Then $\mathcal{X} = \prod_\alpha \mathcal{X}_\alpha$ for closed formal subschemes $\mathcal{X}_\alpha \subset \mathcal{G}_\alpha$.*

Proof. After replacing φ by a suitable power we may assume that for all $\alpha = r/s$ occurring in the product we have $s = 1$. Identify every $\mathcal{G}_r \cong \operatorname{Spf} R_r$, where $R_r = K[[x_{r,1}, \ldots, x_{r,d_r}]]$. By Proposition 8.6 we may assume that $\varphi^*(x_{r,i}) = x_{r,i}^{p^r}$ whenever $r > 0$. For $r = 0$, the map $\varphi^* : R_0 \to R_0$ is an automorphism. By assumption we have $\mathcal{G} = \operatorname{Spf} R$, where R is the completed tensor product of the rings R_r. Let $I \subset R$ be the ideal of $\mathcal{X}$; the φ-invariance of $\mathcal{X}$ means that $\varphi^*(I) \subset I$.

Let $\mathfrak{m}_r$ denote the maximal ideal of R_r, and $\mathfrak{m}$ the maximal ideal of R. We first consider the decomposition $R = R_0 \oplus \sum_{r>0} R\mathfrak{m}_r$. On the first summand φ^* is an automorphism; on the second it is nilpotent modulo $\mathfrak{m}^n$ for every n. Since $\varphi^*(I) \subset I$, it follows that $I \mod \mathfrak{m}^n$ is a direct sum of an ideal in $R_0 \mod \mathfrak{m}_0^n$ and an ideal in $\sum_{r>0} R\mathfrak{m}_r \mod \mathfrak{m}^n$. Letting $n \to \infty$, by completeness we find that $I = I_0 \oplus I^0$ for $I_0 = I \cap R_0$ and an ideal $I^0 \subset \sum_{r>0} R\mathfrak{m}_r$.

To find the desired decomposition of $\mathcal{X}$, recall that R/I is formally smooth. Thus a subset of our chosen parameters $x_{r,i}$ maps to a complete system of formal parameters of R/I. After reindexing the $x_{r,i}$ for every r, if necessary, we may assume that this subset consists of the $x_{r,i}$ for all $1 \leq i \leq e_r$. Set $R'_r := K[[x_{r,1}, \ldots, x_{r,e_r}]]$ for $r > 0$, and $R'_0 := R_0/I_0$, and let R' be the completed tensor product of all R'_r. By construction, every R'_r inherits an endomorphism φ^*; hence so does R', making the natural map $R' \to R/I$ an equivariant isomorphism. Setting $\mathcal{X}_r := \operatorname{Spf} R'_r$, this isomorphism determines a product decomposition $\mathcal{X} = \prod_r \mathcal{X}_r$.

It remains to prove that each $\mathcal{X}_r$ is contained in $\mathcal{G}_r$. For this consider the morphism $\mathcal{X}_r \to \mathcal{G}_{r'}$ for any $r \neq r'$. By construction it is φ-equivariant, so by comparing the behavior of φ on both sides one easily finds that this morphism is zero. This shows that $\mathcal{X}_r \to \mathcal{G}$ factors through a closed embedding $\mathcal{X}_r \hookrightarrow \mathcal{G}_r$, as desired. q.e.d.

9. The general case

In this section we finish the proof of Theorem 3.1 in the general case. If $p = 0$, we are already done by Proposition 7.1, so we assume $p > 0$. We proceed by induction on $\dim(A)$. By Remark 4.2, we may thus assume that $\mathrm{Stab}_A(X)$ is finite. Let $\mathcal{G}$ denote the formal completion of A at 0, and denote its isogeny induced by φ again by φ. Let $\mathcal{G} \to \prod_\alpha \mathcal{G}_\alpha$ be the isogeny provided by Theorem 8.5. Its kernel is a finite subgroup scheme of A, by which we can also divide A. Thus the isogeny of formal groups comes from an isogeny of semiabelian varieties $A \to A'$ (compare [11, Prop. 1.6]). After replacing A by A' using Remark 4.3, we may therefore assume that $\mathcal{G} = \prod_\alpha \mathcal{G}_\alpha$ with $\mathcal{G}_\alpha$ pure of weight $\alpha \geq 0$ in the sense of 8.4. In the rest of the proof we will keep this A fixed.

Let $\psi(x) = \varphi(x) + a$ for some $a \in A$, and let $X \subset A$ be an irreducible closed algebraic subvariety satisfying $\psi(X) = X$, such that $\mathrm{Stab}_A(X)$ is finite. By Proposition 6.4, we may choose a point $x_0 \in X^{\mathrm{reg}}$ satisfying $\psi^n(x_0) = x_0$ for some $n \geq 1$. After replacing X by the translate $X - x_0$ and a by $a + \varphi(x_0) - x_0$, which does not affect Theorem 3.1, we may assume that $x_0 = 0$. Thus $0 \in X^{\mathrm{reg}}$ and $\psi^n(0) = 0$, and hence $\varphi^n(X) = \psi^n(X) = X$. Now observe that, if we were to replace φ by φ^n, then the decomposition $\mathcal{G} = \prod_\alpha \mathcal{G}_\alpha$ would remain the same except that the indices α would be multiplied by n. Thus we may apply Theorem 8.8 to the formal completion $\mathcal{X}$ of X at 0, with φ^n in place of φ. We deduce that $\mathcal{X} = \prod_\alpha \mathcal{X}_\alpha$ for closed formal subschemes $\mathcal{X}_\alpha \subset \mathcal{G}_\alpha$. It remains to algebraize each $\mathcal{X}_\alpha$ and the formal subgroup of $\mathcal{G}_\alpha$ generated by it.

Lemma 9.1. *For every α, there exists an irreducible closed subvariety $X_\alpha \subset X$ such that $0 \in X_\alpha^{\mathrm{reg}}$ and whose formal completion at 0 equals $\mathcal{X}_\alpha$.*

Proof. Fix α and set $\mathcal{G}^\alpha := \prod_{\alpha' \neq \alpha} \mathcal{G}_{\alpha'}$ and $\mathcal{X}^\alpha := \prod_{\alpha' \neq \alpha} \mathcal{X}_{\alpha'}$. For every $n \geq 0$, let $\mathcal{X}_n^\alpha$ denote the nth infinitesimal neighborhood of $0 \in \mathcal{X}^\alpha$. Consider the transporter,

$$Y_\alpha^n := \mathrm{Transp}_A\big(\mathcal{X}_n^\alpha, X\big).$$

This is a closed subscheme of A that is characterized uniquely by the fact that for any scheme S over K and any morphism $a : S \to A$, translation by a on the product $A \times S$ maps the subscheme $\mathcal{X}_n^\alpha \times S$ to $X \times S$ if and only if a factors through Y_α^n. Its existence is guaranteed by [7, exp. VIII, Ex. 6.5 (e)], because $\mathcal{X}_n^\alpha$ is finite and S is the spectrum of a field. As $n \to \infty$, the Y_α^n form a decreasing sequence of closed subschemes. Since A is noetherian, their intersection Y_α is therefore equal to Y_α^n for all $n \gg 0$.

As $\mathcal{X}_n^\alpha \subset X$, we clearly have $0 \in Y_\alpha^n$. Its formal completion $\mathcal{Y}_\alpha^n$ at 0 can be characterized as the corresponding transporter within the formal group $\mathcal{G}$, and it decomposes:

$$\begin{aligned}
\mathcal{Y}_\alpha^n &= \mathrm{Transp}_{\mathcal{G}}\big(\mathcal{X}_n^\alpha, \mathcal{X}_\alpha \times \mathcal{X}^\alpha\big) \\
&= \mathrm{Transp}_{\mathcal{G}_\alpha}\big(\{0\}^{\mathrm{red}}, \mathcal{X}_\alpha\big) \times \mathrm{Transp}_{\mathcal{G}^\alpha}\big(\mathcal{X}_n^\alpha, \mathcal{X}^\alpha\big) \\
&= \mathcal{X}_\alpha \times \mathrm{Transp}_{\mathcal{G}^\alpha}\big(\mathcal{X}_n^\alpha, \mathcal{X}^\alpha\big).
\end{aligned}$$

For $n \gg 0$, this is independent of n and hence

$$0 \in \mathrm{Transp}_{\mathcal{G}^\alpha}\big(\mathcal{X}_n^\alpha, \mathcal{X}^\alpha\big) = \mathrm{Stab}_{\mathcal{G}^\alpha}(\mathcal{X}^\alpha) \subset \mathrm{Stab}_{\mathcal{G}}(\mathcal{X}).$$

Since X is irreducible, the latter is simply the formal completion of $\mathrm{Stab}_A(X)$, which by assumption is finite. Now we have

$$\mathcal{X}_\alpha \subset \mathcal{Y}_\alpha^n \subset \mathcal{X}_\alpha \cdot (\text{finite}),$$

which implies that the formal completion of some irreducible component X_α of Y_α^{red} is equal to $\mathcal{X}_\alpha$, as desired. q.e.d.

Following Proposition 2.6, let $A_\alpha \subset A$ be the smallest φ-invariant semiabelian subvariety containing a translate of X_α. Since $0 \in X_\alpha$, we already have $X_\alpha \subset A_\alpha$. By the proof of 2.6, A_α is the image of the morphism

$$X_\alpha^{2d} \longrightarrow A, \quad (y_1, \ldots, y_{2d}) \mapsto y_1 - y_2 + y_3 - + \cdots - y_{2d}$$

for any $d \gg 0$. As the corresponding morphism of formal completions $\mathcal{X}_\alpha^{2d} \to \mathcal{G}$ factors through $\mathcal{G}_\alpha$, it follows that the formal completion of A_α is contained in $\mathcal{G}_\alpha$.

Lemma 9.2. *For every α, the semiabelian subvariety A_α is φ-invariant.*

Proof. The semiabelian subvariety of A generated by all the A_α is the smallest one whose formal completion at 0 contains $\mathcal{X}$; hence it is the smallest semiabelian subvariety containing X. It thus coincides with the smallest semiabelian subvariety A_X containing a translate of X. Since A_X is φ-invariant by Proposition 2.6, and so is the decomposition $\mathcal{G} = \prod_\alpha \mathcal{G}_\alpha$, we deduce that the formal completion of every A_α is φ-invariant. Thus A_α is φ-invariant, as desired. q.e.d.

Let now $h : \prod_\alpha A_\alpha \to A$ be the homomorphism induced by the inclusions $A_\alpha \hookrightarrow A$. Since h is injective on the formal completions at 0, its kernel is finite. Moreover, since $h\big(\prod_\alpha \mathcal{X}_\alpha\big) = \mathcal{X}$, and both $\prod_\alpha X_\alpha$ and X are irreducible, we must have $h\big(\prod_\alpha X_\alpha\big) = X$. As in Remark 4.3, we can deduce that $\prod_\alpha X_\alpha \subset \prod_\alpha A_\alpha$ is invariant under a translate of φ; and this implies that every single $X_\alpha \subset A_\alpha$ is invariant under a translate of φ.

Furthermore, by 8.4, for every α there are integers $r \geq 0$ and $s > 0$ with $\alpha = r/s$, such that $\varphi^s|\mathcal{G}_\alpha$ is the composite of Frob_{p^r} with an isomorphism. Thus $\mathrm{Ker}(\varphi^s|\mathcal{G}_\alpha) = \mathrm{Ker}(\mathrm{Frob}_{p^r}|\mathcal{G}_\alpha)$, and the same is true for the restriction to the formal completion of A_α. The identity component of $\mathrm{Ker}(\varphi^s|A_\alpha)$ is therefore equal to $\mathrm{Ker}(\mathrm{Frob}_{p^r}|A_\alpha)$. This in turn implies that $\varphi^s|A_\alpha$ is the composite of Frob_{p^r} with a separable isogeny.

Depending on the case we can now apply Proposition 7.1 or 7.3 to $X_\alpha \subset A_\alpha$, showing that A_α is pure of weight α. This finishes the proof of Theorem 3.1 in the general case. q.e.d.

10. Arbitrary commutative algebraic groups

It is natural to ask whether the results of Section 3 extend from semiabelian varieties to arbitrary connected commutative algebraic groups. It turns out that Theorems 3.1 and 3.4 do not generalize and that Theorems 3.6 and 3.7 generalize only in characteristic zero.

Counterexample 10.1. Let A be a product of n copies of the additive group. As an algebraic variety, this is simply the affine space $\mathbb{A}_K^n$. Let $X \subset A$ be the affine cone over an arbitrary irreducible projective variety in $\mathbb{P}_K^{n-1}$. Let $t \in K^*$ be an arbitrary scalar and define $\varphi : A \to A$, $(x_1, \ldots, x_n) \mapsto (tx_1, \ldots, tx_n)$. This is an automorphism of A that obviously satisfies $\varphi(X) = X$. If t is not a root of unity, then all assumptions of Theorems 3.1 and 3.4 are satisfied, except that A is now a unipotent group. Thus these theorems do not generalize.

The same example shows that Theorem 3.6 does not generalize in characteristic $p > 0$, because all points of A are (p-)torsion but not all subvarieties can be defined over a finite subfield of K. Theorem 3.7 is, of course, vacuous in characteristic $p > 0$ unless A is an abelian variety.

However, the Manin-Mumford conjecture itself does generalize in characteristic zero, as was already known by Hrushovski [9]:

Theorem 10.2. *Let A be a connected commutative algebraic group over an algebraically closed field K of characteristic* 0*.*

(a) *Let $X \subset A$ be an irreducible closed algebraic subvariety such that $X(K) \cap \mathrm{Tor}(A)$ is Zariski dense in X. Then X is a translate of an algebraic subgroup of A.*
(b) *Let $X \subset A$ be a closed algebraic subvariety. Then*

$$X(K) \cap \mathrm{Tor}(A) = \bigcup_{i \in I} X_i(K) \cap \mathrm{Tor}(A),$$

where I is a finite set and each X_i is the translate by an element of A of an algebraic subgroup of A, immersed in X.

Proof. Construct ρ and $P(T)$ as in the proof of 3.6. Note that for this it suffices to look at the semiabelian part of A, because that determines $\mathrm{Tor}(A)$ completely. Next, if $\deg P(T) = n$, then as in [13] one passes to A^n and the Zariski closure $Y \subset A^n$ of

$$X^n \cap \left\{ (a, \rho(a), \ldots, \rho^{n-1}(a)) \;\middle|\; a \in \mathrm{Tor}(A) \right\},$$

and one must prove the conclusion for Y in place of X. As in [13], one shows that Y is invariant under an isogeny $\varphi : A^n \to A^n$ which does not have roots of unity as eigenvalues. Thus $Y \subset A^n$ and φ satisfy the assumptions of 3.4, except that A^n is not necessarily semiabelian, and in addition we know that $Y(K) \cap \mathrm{Tor}(A^n)$ is Zariski dense in Y. Renaming $Y \subset A^n$ as $X \subset A$, it suffices to prove the generalization of Theorem 3.4 to arbitrary connected commutative algebraic groups under the additional assumption that $X(K) \cap \mathrm{Tor}(A)$ is Zariski dense in X.

The proof of this proceeds exactly as in the preceding sections. In fact, the reduction to irreducible X, and all the results of Sections 4 and 5, extend directly to arbitrary A. For the results of Section 6 this is not so, but the conclusion of Proposition 6.4 holds anyway by our additional assumption, because all torsion points lie in finite ψ-orbits. Then, to finish, we only need Proposition 7.1 which again directly generalizes to arbitrary A. q.e.d.

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