MARTIN'S AXIOM

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Introduction. During the past decade, many new axioms of set theory have appeared. The principal object of these axioms is to settle important problems which cannot be settled without new axioms. Thus Gödel and Cohen have shown that the Continuum Hypothesis cannot be settled on the basis of the presently accepted axioms; so it is natural to look for reasonable new axioms which will settle it.

A second purpose of new axioms is more mundane: to assist in the proof of independence results. To see how this happens, we need a couple of definitions.

Let ZFC be the usual set of axioms for set theory (including the Axiom of Choice). The reader need not be familiar with these axioms if he will accept the fact that such a set of axioms exists and leads to the usual results of set theory. We shall assume that ZFC is consistent, i.e., that no contradictions can be derived from the axioms of ZFC.

The independence results in which we are interested have the form 'A is not provable from ZFC' (where A is some statement of set theory). We show that such a result can be reformulated as a result on consistency.

LEMMA 1. *The statement A is not provable from ZFC iff the axiom system ZFC + **not**-A is consistent.*

Proof. If A is provable from ZFC, then A and not-A are provable from ZFC + not-A; so ZFC + not-A is inconsistent. Assume ZFC + not-A is inconsistent; we must prove A from the axioms of ZFC. One way to do this is by indirect proof; we assume not-A and derive a contradiction. We can in fact derive such a contradiction from not-A in ZFC because ZFC + not-A is inconsistent. Q.E.D.

It follows that the results of interest to us can be put in the form '$ZFC + A$ is consistent'. Now suppose that we wish to prove several such results; say the statements '$ZFC + A_i$ is consistent' for $i = 1, \cdots, k$. We can do this by devising a new axiom A; proving that $ZFC + A$ is consistent; and proving $A \to A_i$ in ZFC for $i = 1, \cdots, k$. We shall then have proved more than the consistency of each $ZFC + A_i$; we shall have proved the consistency of $ZFC + A_1 + \cdots + A_k$.

This suggests we look for axioms A such that: (a) we can prove that $ZFC + A$ is consistent; (b) we can prove $A \to B$ in ZFC for many interesting statements B. (It does not matter if A itself is not very interesting.) A good example is Gödel's axiom $V = L$. The purpose of this article is to study another such example.

1. Formulation of the Axiom. We shall formulate our axiom in terms of partially ordered sets (henceforth called **posets**). A poset P is a **net** if for every p and q in P, there is an r in P such that $p \leqq r$ and $q \leqq r$. A **subnet** of a poset P is a subset of P which is a net.

A subset D of a poset P is **dense** in P if for every $p \in P$, there is a $q \in D$ such that

$p \leqq q$. (This is actually density in the topological sense for a suitable topology on P, viz., that having as a base all sets of the form $\{p: p \geqq q\}$.) We are interested in subnets of P which intersect many dense subsets of P.

THEOREM 1. *Let P be a non-empty poset, and let $\{D_n\}$ be a sequence of dense subsets of P. Then there is a subnet of P which intersects each D_n.*

Proof. Choose p_n inductively so that $p_{n+1} \geqq p_n$ and $p_{n+1} \in D_n$. This is possible because D_n is dense. Since $n \leqq m$ implies $p_n \leqq p_m$, the p_n constitute a subnet. Q.E.D.

In what follows, κ and λ are infinite cardinals and α and β are ordinals. We identify κ with the first ordinal having κ predecessors. Thus $\aleph_0$ is identified with ω and $\aleph_1$ with the first uncountable ordinal.

Theorem 1 becomes false if $\{D_n\}$ is replaced by a collection $\{D_\alpha: \alpha < \aleph_1\}$ of $\aleph_1$ dense sets. To see this, let A have cardinal $\aleph_0$ and let B have cardinal $\aleph_1$. Let P be the set of mappings from a finite subset of A to B. For $p, q \in P$, let $p \leqq q$ mean that q is an extension of p. If Q is a subnet of P, there is a mapping f from A to B which extends every member of Q. Since f is not onto, there is a $b \in B$ not in the range of f; so Q does not intersect $D_b = \{p: b \in \text{range}(p)\}$. On the other hand, each D_b is dense in P.

In view of this example, we put a restriction on P. We say that the elements p and q of P are **incompatible** if there is no r in P such that $p \leqq r$ and $q \leqq r$. We say that P satisfies the **countable chain condition** (abbreviated *CCC*) if every pairwise incompatible subset of P is countable.

For each κ, we introduce an axiom.

($A\kappa$) *If P is a non-empty poset satisfying the CCC, and $\{D_i: i \in I\}$ is a family of dense subsets of P having cardinal $\leqq \kappa$, then there is a subnet Q of P which intersects each D_i.*

Theorem 1 shows that $(A\aleph_0)$ is true. We will see in the next section that $(A\kappa)$ is false for $\kappa \geqq 2^{\aleph_0}$. Thus the interesting case is $\aleph_1 \leqq \kappa < 2^{\aleph_0}$.

Martin's Axiom is the statement that $(A\kappa)$ is true for $\aleph_1 \leqq \kappa < 2^{\aleph_0}$. Clearly the Continuum Hypothesis implies Martin's Axiom. The remarkable fact is that many consequences of the Continuum Hypothesis, if properly stated, are also consequences of Martin's Axiom. This is the viewpoint of [4]. From our viewpoint, it is best to consider $(A\kappa)$ for various κ, especially $\kappa = \aleph_1$. According to this viewpoint, we should establish the consistency of $ZFC + (A\aleph_1)$ and then derive interesting consequences of $(A\aleph_1)$.

THEOREM 2 [6]. *The axiom system $ZFC + (A\aleph_1) + 2^{\aleph_0} = \aleph_2$ is consistent.*

Unfortunately, the proof of Theorem 2 is too complicated to even sketch here. It requires a considerable extension of Cohen's forcing technique. We shall devote the rest of the paper to consequences of $(A\kappa)$.

2. Applications to Measure Theory. We know that $(A\aleph_0)$ is provable in ZFC. This suggests the consequences of $(A\kappa)$ will be statements about sets of cardinal $\leqq \kappa$ which are known to be true about countable sets. This is frequently (but not always) the case.

THEOREM 3 [4]. *If* $(A\kappa)$, *then the union of* $\leqq \kappa$ *sets of reals of (Lebesgue) measure zero is of measure zero.*

Proof. Let $\{E_i: i \in I\}$ be the $\leqq \kappa$ sets of measure 0, and let $E = \cup E_i$. Let $\varepsilon > 0$. We must find an open set including E which has measure $\leqq \varepsilon$.

Let P be the collection of all open sets of measure $< \varepsilon$; and for $p, q \in P$, let $p \leqq q$ mean $p \subseteq q$. We first show that the poset P satisfies the CCC. Let Q be a pairwise incompatible subset of P. Let $Q_n = \{p \in Q: m(p) \leqq (1-2^{-n})\varepsilon\}$ (where $m(p)$ is the measure of p). It is enough to show that Q_n is countable.

For each $p \in Q_n$, choose $\bar{p} \subseteq p$ so that $\bar{p}$ is a finite union of open intervals with rational endpoints and $m(p-\bar{p}) < 2^{-n}\varepsilon$. Since there are only countably many such finite unions, it is enough to show that if p and q are distinct members of Q_n, then $\bar{p} \neq \bar{q}$. Suppose $\bar{p} = \bar{q}$. Then $p \cup q \subseteq (p-\bar{p}) \cup q$; so $m(p \cup q) < 2^{-n}\varepsilon + (1-2^{-n})\varepsilon = \varepsilon$. This implies that p and q are compatible, contradicting $p, q \in Q$.

For $i \in I$, let $D_i = \{p \in P: E_i \subseteq p\}$. Using $m(E_i) = 0$, we easily conclude that D_i is dense in P. It follows by $(A\kappa)$ that there is a subnet Q of P which meets every D_i.

Let G be the union of the members of Q. Then G is open. From $D_i \cap Q \neq \emptyset$ we find that $E_i \subseteq G$; so $E \subseteq G$.

It remains to show that $m(G) > \varepsilon$ leads to a contradiction. By Lindelöf's Theorem, G is a countable union of sets in Q. It follows that there is a finite union G_1 of sets in Q such that $m(G_1) \geqq \varepsilon$. But since Q is a net, some member of Q includes G_1 and hence has measure $\geqq \varepsilon$. This is a contradiction. Q.E.D.

Let us see what this means about independence. By Theorem 3 and Theorem 2, the following axiom system is consistent: ZFC + 'the union of $\aleph_1$ sets of measure zero is of measure 0.' Thus by Lemma 1, we cannot prove the following in ZFC: 'there are $\aleph_1$ sets of measure 0 whose union is not of measure 0.' We leave to the reader the reformulation of our remaining results as independence results.

COROLLARY. *If* $(A\kappa)$, *then the union of* $\leqq \kappa$ *measurable sets is measurable.*

Proof. Let $\{E_i: i \in I\}$ be the sets, $E = \cup E_i$. Then E has a measurable kernel, i.e., a measurable subset F such that every measurable subset of $E-F$ has measure zero. In particular, $m(E_i - F) = 0$; so by Theorem 3, $E-F = \cup (E_i-F)$ has measure 0. It follows that $E = F \cup (E-F)$ is measurable. Q.E.D.

Since $[0,1]$ is the union of $2^{\aleph_0}$ sets which contain only one point and hence have measure 0, we see from Theorem 1 that (as previously mentioned) $(A\kappa)$ is false for $\kappa \geqq 2^{\aleph_0}$.

It follows that $(A\kappa)$ is only interesting for $\aleph_0 < \kappa < 2^{\aleph_0}$. Since such κ do not frequently occur in mathematics, one might expect the consequences of $(A\kappa)$ to be only of interest to set theorists. We show this is not so by an example.

If $\boldsymbol{X}$ is a collection of sets of reals, let $C\boldsymbol{X}$ be the collection of complements of sets in $\boldsymbol{X}$ and let $P\boldsymbol{X}$ be the collection of continuous images of sets in $\boldsymbol{X}$. Let A be the collection of continuous images of Borel sets. The sets in A, CA, PCA, $CPCA$, $PCPCA, \cdots$ are called **projective sets,** and have been studied by topologists (see [3]). Lusin proved that every set in A or CA is measurable. Gödel showed that it is consistent with ZFC to assume that there is a non-measurable set in PCA. Now a rather deep result says that every set in PCA is the union of $\aleph_1$ Borel sets. Hence $(A\aleph_1)$ implies that every set in PCA is measurable (by the Corollary to Theorem 3).

This can be carried further. Martin has shown that the existence of a measurable cardinal implies that every set in $PCPCA$ is the union of $\aleph_2$ Borel sets. Thus if there is a measurable cardinal and $(A\aleph_2)$ holds, then every set in $PCPCA$ is measurable.

3. Combinatorial Consequences. Let ω be the set of natural numbers, and let $\boldsymbol{A}$ and $\boldsymbol{B}$ be collections of subsets of ω. An **$(\boldsymbol{A},\boldsymbol{B})$-set** is a subset C of ω such that $C \cap A$ is finite for every $A \in \boldsymbol{A}$ and $C \cap B$ and $C^c \cap B$ are infinite for every $B \in \boldsymbol{B}$. (We use C^c for the complement of C.)

A set D is **$\boldsymbol{A}$-small** if there are sets $A_1, \cdots, A_n$ in $\boldsymbol{A}$ such that $D-(A_i \cup \cdots \cup A_n)$ is finite. If an $(\boldsymbol{A},\boldsymbol{B})$-set exists, then clearly no set in $\boldsymbol{B}$ is $\boldsymbol{A}$-small.

THEOREM 4 [4]. *Let $(A\kappa)$ hold. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be collections of subsets of ω having cardinal $\leqq \kappa$. If no set in $\boldsymbol{B}$ is $\boldsymbol{A}$-small, then there is an $(\boldsymbol{A},\boldsymbol{B})$-set.*

Proof. Let P be the set of all mappings f from an $\boldsymbol{A}$-small subset of ω to $\{0,1\}$ such that $\{i \mid f(i) = 1\}$ is finite. We make P into a poset by letting $f \leqq g$ hold if g is an extension of f.

We first show P satisfies the CCC. If f and g are members of P which have a common extension, then the smallest common extension of f and g is in P. Thus if f and g are incompatible, there is an i such that one of $f(i)$ and $g(i)$ is 0 and the other is 1. It follows that $\{f: f(i) = 1\} \neq \{g: g(i) = 1\}$. Since there are only countably many finite subsets of ω, a pairwise incompatible subset of P must be countable.

For $A \in \boldsymbol{A}$, let D_A be the set of $f \in P$ whose domain includes A. It is clear that D_A is dense in X.

For $B \in \boldsymbol{B}$, let $D_{B,n}$ be the set of $f \in P$ such that $B \cap \{i : f(i) = 0\}$ and $B \cap \{i : f(i) = 1\}$ both have cardinal $\geqq n$. We show that $D_{B,n}$ is dense in P. Let $f \in P$. Since B is not $\boldsymbol{A}$-small, we can find $2n$ numbers $i_1, \cdots, i_n, j_1, \cdots, j_n$ in B but not in the domain of f. Extend f by setting $f(i_\kappa) = 0, f(j_\kappa) = 1$. The extended f is in $D_{B,n}$.

By $(A\kappa)$, there is a subnet Q of P which intersects all the D_A and $D_{B,n}$. Since Q is a net, there is a mapping g from ω to $\{0,1\}$ which extends every member of Q. Let $C = \{x \mid g(x) = 1\}$. If $A \in \boldsymbol{A}$, g extends some $f \in D_A$. Then $C \cap A = \{x \in A \mid f(x) = 1\}$; so $C \cap A$ is finite. If $B \in \boldsymbol{B}$ and $n \in \omega$, then g extends some member of $D_{B,n}$; so $C \cap B$ and $C^c \cap B$ have at least n members. Since this holds for all n, $C \cap B$ and $C^c \cap B$ are infinite. Q.E.D.

We say $\boldsymbol{A}$ is **proper** if ω is not $\boldsymbol{A}$-small and if for each $A \in \boldsymbol{A}$, A is not $(\boldsymbol{A}-\{A\})$-small.

LEMMA 2. *Assume* $(A\kappa)$. *Let* $\boldsymbol{A}$ *be proper and have cardinal* $\leqq \kappa$, *and let* $\boldsymbol{A}_1$ *and* $\boldsymbol{A}_2$ *be disjoint subsets of* $\boldsymbol{A}$. *Then there is an* $(\boldsymbol{A}_1, \boldsymbol{A}_2)$*-set* C *such that* $\boldsymbol{A} \cup \{C\}$ *is proper.*

Proof. Let $\boldsymbol{B}$ consist of all sets $D-(A_1 \cup \cdots \cup A_n)$, where $A_i, \cdots, A_n \in \boldsymbol{A}$ and D is either ω or a member of $\boldsymbol{A}-\boldsymbol{A}_1$ distinct from $A_1, \cdots, A_n$. Since $\boldsymbol{A}$ is proper, no set in $\boldsymbol{A}_2 \cup \boldsymbol{B}$ is $\boldsymbol{A}_1$-small. Since $\boldsymbol{B}$ clearly has cardinal $\leqq \kappa$, it follows from Theorem 4 that there is a $(\boldsymbol{A}_1, \boldsymbol{A}_2 \cup \boldsymbol{B})$-set C. We must show that $\boldsymbol{A} \cup \{C\}$ is proper.

Let $A_1, \cdots, A_n \in \boldsymbol{A}$. Since $\omega-(A_1 \cup \cdots \cup A_n)$ is in $\boldsymbol{B}$, $C-(A_1 \cup \cdots \cup A_n)$ and $\omega-(A_1 \cup \cdots \cup A_n \cup C)$ are infinite. This shows that C is not $\boldsymbol{A}$-small and that ω is not $(\boldsymbol{A} \cup \{C\})$-small. We must now show that if $A \in \boldsymbol{A}$, then A is not $(\boldsymbol{A} \cup \{C\}-\{A\})$-small. Let $A_1, \cdots, A_n \in \boldsymbol{A}-\{A\}$. If $A \notin \boldsymbol{A}_1$, then $A-(A_1 \cup \cdots \cup A_n)$ is in $\boldsymbol{B}$ and hence $A-(A_1 \cup \cdots \cup A_n \cup C)$ is infinite. If $A \in \boldsymbol{A}_1$, $A \cap C$ is finite and $A-(A_1 \cup \cdots \cup A_n)$ is infinite; so again, $A-(A_1 \cup \cdots \cup A_n \cup C)$ is infinite. Q.E.D.

THEOREM 5 [4]. *If* $(A\kappa)$, *then* $2^\kappa = 2^{\aleph_0}$.

Proof. Using Lemma 2 and transfinite induction, we can choose A_α for $\alpha < \kappa$ so that $\{A_\alpha : \alpha < \kappa\}$ is proper. For each $I \subseteq \{\alpha : \alpha < \kappa\}$, we may choose a $(\{A_\alpha : \alpha \in I\}, \{A_\alpha : \alpha \notin I\})$-set C_I by Theorem 4. Since $\alpha \in I$ iff $C_I \cap A_\alpha$ is finite, the mapping from I to C_I is one-one. Since there are 2^κ possible I's and $2^{\aleph_0}$ subsets of ω, $2^\kappa \leqq 2^{\aleph_0}$; so $2^\kappa = 2^{\aleph_0}$. Q.E.D.

We now consider a problem which appeared in this MONTHLY (Advanced Problem 5845). Let $O_\kappa = \{\alpha : \alpha < \kappa\}$, and let $\boldsymbol{S}_\kappa$ be the σ-ring in $O_\kappa \times O_\kappa$ generated by the set of all rectangles (i.e., sets of the form $A \times B$). Let $(R\kappa)$ be the statement that every subset of $O_\kappa \times O_\kappa$ is in $\boldsymbol{S}_\kappa$. For what uncountable κ is $(R\kappa)$ true? The published answer gives references to show that $(R\aleph_1)$ is true and that $(R\kappa)$ is false if $\kappa > 2^{\aleph_0}$. This does not completely solve the problem unless we assume the Continuum Hypothesis.

THEOREM 6 [2]. *If* $(A\lambda)$ *holds for all* $\lambda < \kappa$, *then* $(R\kappa)$ *holds.*

Proof. Let $W \subseteq O_\kappa \times O_\kappa$. Using transfinite induction, we select subsets A_α and B_α of ω for $\alpha < \kappa$ so that $A_\alpha \cap B_\beta$ is infinite iff $(\alpha, \beta) \in W$. Suppose A_α and B_α have been selected for $\alpha < \gamma$, and suppose (as an induction hypothesis) that $\{A_\alpha : \alpha < \gamma\} \cup \{B_\alpha : \alpha < \gamma\}$ is proper. Using Lemma 2, we first select A_γ so that $A_\gamma \cap B_\alpha$ is infinite iff $(\gamma, \alpha) \in W$ and $\{A_\alpha : \alpha \leqq \gamma\} \cup \{B_\alpha : \alpha < \gamma\}$ is proper; we then select B_γ so that $A_\alpha \cap B_\gamma$ (for $\alpha \leqq \gamma$) is infinite iff $(\alpha, \gamma) \in W$ and $\{A_\alpha : \alpha \leqq \gamma\} \cup \{B_\alpha : \alpha \leqq \gamma\}$ is proper.

Set $X_n = \{\alpha \mid n \in A_\alpha\}$, $Y_n = \{\alpha \mid n \in B_\alpha\}$. Then $(\alpha, \beta) \in X_n \times Y_n$ iff $n \in A_\alpha \cap B_\beta$; so

$(\alpha, \beta) \in W$ iff $(\alpha, \beta) \in X_n \times Y_n$ for infinitely many n. Hence

$$W = \bigcap_m \bigcup_{n \geqq m} (X_n \times Y_n);$$

so $W \in \boldsymbol{S}_\kappa$. Q.E.D.

Note that Theorem 6 gives a new proof of $(R\aleph_1)$. Kunen [**2**] has also shown that $(R\aleph_2)$ is not provable in ZFC from $2^{\aleph_0} \geqq \aleph_2$.

4. Topological consequences. Let X be a topological space. Recall that a set A in X is **nowhere dense** if every non-empty open set includes a non-empty open set disjoint from A. A set is **meager** if it is the union of countably many nowhere dense sets. A set A has the **property of Baire** if there is an open set B such that $A-B$ and $B-A$ are meager.

There is an analogue between meager sets and sets of measure zero; likewise between sets having the property of Baire and measurable sets. We shall prove the analogue of Theorem 3.

THEOREM 7 [**4**]. *Assume* $(A\kappa)$. *If X is a topological space having a countable base, then the union of $\leqq \kappa$ meager sets in X is meager.*

Proof. It is clearly enough to show that if $\{E_i : i \in I\}$ is a collection of $\leqq \kappa$ nowhere dense sets, then their union E is meager. Let $\{G_n : n \in \omega\}$ be a sequence in which every non-empty set in some countable base of X appears infinitely often (and no other set appears). For $i \in I$, let $A_i = \{n : E_i \cap G_n \neq \varnothing\}$. For $n \in \omega$, let $B_n = \{m : G_m \subseteq G_n\}$. Let $\boldsymbol{A} = \{A_i : i \in I\}$, $\boldsymbol{B} = \{B_n : n \in \omega\}$.

We show no B_n is $\boldsymbol{A}$-small. If I_0 is a finite subset of I, $E_{I_0} = \bigcup_{i \in I_0} E_i$ is nowhere dense. Hence there are infinitely many m such that $G_m \subseteq G_n$ and $G_m \cap E_{I_0} = \varnothing$. This means there are infinitely many m in B_n which are not in A_i for any $i \in I_0$.

By Theorem 4, there is an $(\boldsymbol{A}, \boldsymbol{B})$-set C. Let H_k be the union of the G_m for $m \geqq k$ and $m \in C$. Then H_k is open. We show that H_k^c is nowhere dense. It suffices to show that for each n, there is a G_m included in $H_k \cap G_n$, i.e., an $m \geqq k$ such that $m \in C$ and $m \in B_n$. This holds because $C \cap B_n$ is infinite.

We must show now that $E \subseteq \cup_k H_k^c$. It will suffice to show that for each i, there is a k such that $E_i \cap H_k = \varnothing$. Since $A_i \cap C$ is finite, we can choose k greater than every member of $A_i \cap C$. If $m \geqq k$ and $m \in C$, then $m \notin A_i$; so $E_i \cap G_m = \varnothing$. It follows that $E_i \cap H_k = \varnothing$. Q.E.D.

COROLLARY. *Assume* $(A\kappa)$. *If X is a topological space having a countable base, then the union of $\leqq \kappa$ sets in X having the property of Baire has the property of Baire.*

The Corollary to Theorem 7 has consequences similar to the Corollary to Theorem 3 (for X the space of reals).

Suppose that in the topological space X, x is a limit point of the sequence $\{x_n\}$. Then x need not be the limit of a subsequence of $\{x_n\}$. This will be the case, however, if there is a countable base at x.

THEOREM 8. *Assume* $(A\kappa)$. *In the topological space* X, *let* x *be a limit point of the sequence* $\{x_n\}$. *Suppose that there is a base at* x *having cardinal* $\leqq \kappa$. *Then there is a subsequence of* $\{x_n\}$ *converging to* x.

Proof. Let $\{G_i : i \in I\}$ be the base at x. Let $A_i = \{n : x_n \notin G_i\}$, $A = \{A_i : i \in I\}$. We show that ω is not A-small. Let I_0 be a finite subset of I, $G = \bigcap_{i \in I_0} G_i$. Then G is a neighborhood of x; so $x_n \in G$ for infinitely many n. Each such n is in A_i^c for $i \in I_0$.

By Theorem 4, there is an infinite set C such that $C \cap A_i$ is finite for all i. The subsequence $\{x_n : n \in C\}$ then converges to x. Q.E.D.

A topological space X is **sequentially compact** if every sequence in X has a convergent subsequence. It is well known that a countable product of sequentially compact spaces is sequentially compact. On the other hand, it is easy to show that the product of $2^{\aleph_0}$ two-point discrete spaces is not sequentially compact.

COROLLARY. *If* $(A\kappa)$, *then the product of* $\leqq \kappa$ *compact metric spaces is sequentially compact.*

Proof. Let $\{X_i : i \in I\}$ be the collection of spaces, X their product. Since X is compact, every sequence in X has a limit point. In view of Theorem 8, it is enough to show that X has a base of cardinal $\leqq \kappa$. This is a simple exercise in cardinal arithmetic (noting that each X_i has a countable base). Q.E.D.

This corollary generalizes a theorem of Booth. Booth [**1**] has also applied Martin's axiom to questions about dimension.

We state without proof an unpublished result of Silver bearing on a famous topological problem.

THEOREM 9 (Silver). *If* $(A\aleph_1)$, *then there is a separable normal non-metrizable Moore space.*

5. Souslin's Problem. We now discuss briefly the problem which inspired Theorem 2. Souslin's problem was originally stated in terms of axioms for the ordered set of reals. We use instead a problem about posets, which was proved equivalent to the original problem by Miller [**5**].

A poset P is a **chain** $\{$**antichain**$\}$ if every two distinct elements in P are comparable $\{$incomparable$\}$. A **tree** is a poset P such that for each $x \in P$, $\{y : y \leqq x\}$ is a chain. A **Souslin tree** is an uncountable tree P such that every chain or antichain in P is countable. We let (SH) be the statement that there is no Souslin tree.

Soon after the invention of forcing, Tennenbaum and Jech (independently) showed that SH is not provable in ZFC. The next theorem, in conjunction with Theorem 2, shows that not-(SH) is not provable in ZFC.

LEMMA 3. *If there is a Souslin tree, there is a Souslin tree Q having cardinal $\aleph_1$ such that for every $x \in Q$, $\{y \in Q: y \geqq x\}$ is uncountable.*

Proof. Since every uncountable subset of a Souslin tree is a Souslin tree, we may assume that we have a Souslin tree P of cardinal $\aleph_1$. Let Q be the set of $x \in P$ such that $\{y \in P: y \geqq x\}$ is uncountable.

Using Zorn's lemma, choose a maximal antichain Z in $P-Q$. Since P is a Souslin tree, Z is countable. If $z \in Z$, $\{y \in P: y \geqq z\}$ is countable and $\{y \in P: y \leqq z\}$ is a chain and hence is countable. Thus only countably many y are comparable with some $z \in Z$. By choice of Z, these include all elements of $P-Q$; so $P-Q$ is countable. Thus Q has cardinal $\aleph_1$ and hence is a Souslin tree. If $x \in Q$, $\{y \in Q: y \geqq x\}$ includes all members of $\{y \in P: y \geqq x\}$ not in $P-Q$; so $\{y \in Q: y \geqq x\}$ is uncountable. Q.E.D.

THEOREM 10 [6]. *If* $(A\aleph_1)$, *then* (SH).

Proof. Suppose otherwise. Let Q be as in Lemma 3. We can write $Q=\{x_\alpha: \alpha < \aleph_1\}$. Let $Q_\beta = \{x_\alpha: \beta \leqq \alpha < \aleph_1\}$. Our choice of Q implies that each Q_β is dense in Q. Every pairwise incompatible set is an antichain; so Q satisfies the CCC. Hence by $(A\aleph_1)$, there is a subnet N of Q such that N intersects each Q_β. Since Q is a tree, the subnet N is a chain. But clearly N is uncountable; so we have a contradiction. Q.E.D.

Since $(A\aleph_1)$ implies that the Continuum Hypothesis is false, Theorem 10 leaves open the question of whether not-(SH) is provable in ZFC from the Continuum Hypothesis. Recently Jensen has shown that it is not; his proof requires a considerable extension of the methods used to prove Theorem 2.

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