The Semilinear Home-Space Problem Is Ackermann-Complete for Petri Nets

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Abstract

A set of configurations H is a home-space for a set of configurations X of a Petri net if every configuration reachable from (any configuration in) X can reach (some configuration in) H. The semilinear home-space problem for Petri nets asks, given a Petri net and semilinear sets of configurations X, H, if H is a home-space for X. In 1989, David de Frutos Escrig and Colette Johnen proved that the problem is decidable when X is a singleton and H is a finite union of linear sets with the same periods. In this paper, we show that the general (semilinear) problem is decidable. This result is obtained by proving a duality between the reachability problem and the non-home-space problem. In particular, we prove that for any Petri net and any linear set of configurations L we can effectively compute a semilinear set C of configurations, called a non-reachability core for L, such that for every set X the set L is not a home-space for X if, and only if, C is reachable from X. We show that the established relation to the reachability problem yields the Ackermann-completeness of the (semilinear) home-space problem. For this we also show that, given a Petri net with an initial marking, the set of minimal reachable markings can be constructed in Ackermannian time.

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1 Introduction

Petri nets provide a popular formal method for modelling and analysing parallel processes. The standard model is not Turing-complete, and thus many analyzed properties are decidable; we can refer to [6] as to one of the first survey papers on this issue.

A central algorithmic problem for Petri nets is reachability: given a Petri net A and two configurations $\mathbf{x}$ and $\mathbf{y}$, decide whether there exists an execution of A from $\mathbf{x}$ to $\mathbf{y}$. In fact, many important computational problems in logic and complexity reduce or are even equivalent to this problem (we can refer, e.g., to [19, 9] to exemplify this). It was nontrivial to show that the reachability problem is decidable [16], and recently the complexity of this problem was proved to be extremely high, namely Ackermann-complete (see [15] for the upper-bound and [3, 14, 4] for the lower-bound).

The reachability problem for Petri nets can be generalized to semilinear sets, a class of geometrical sets that coincides with the sets definable in Presburger arithmetic [8]. The semilinear reachability problem for Petri nets asks, given a Petri net A and (presentations of) semilinear sets of configurations $\mathbf{X}$,$\mathbf{Y}$, if there exists an execution from a configuration in $\mathbf{X}$ to a configuration in $\mathbf{Y}$. Denoting by $\textsc{post}^*_A(\mathbf{X})$ the set of configurations reachable from $\mathbf{X}$ and by $\textsc{pre}^*_A(\mathbf{Y})$ the set of configurations that can reach a configuration in $\mathbf{Y}$, the semilinear reachability problem thus asks, in fact, if the intersection $\textsc{post}^*_A(\mathbf{X}) \cap \textsc{pre}^*_A(\mathbf{Y})$ is nonempty. This problem can be easily reduced to the classical reachability problem for Petri nets (where $\mathbf{X}$ and $\mathbf{Y}$ are singletons).

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The semilinear home-space problem is a problem that seems to be similar to the semilinear reachability problem at a first sight. This problem asks, given a Petri net A, and two semilinear sets $\mathbf{X}, \mathbf{H}$, if every configuration reachable from $\mathbf{X}$ can reach $\mathbf{H}$, hence if $\text{POST}_A^*(\mathbf{X}) \subseteq \text{PRE}_A^*(\mathbf{H})$. In 1989, David de Frutos Escrig and Colette Johnen [5] proved that the semilinear home-space problem is decidable for instances where $\mathbf{X}$ is a singleton set and $\mathbf{H}$ is a finite union of linear sets using the same periods; they left the general case open. In fact, the general problem seems close to the decidability/undecidability border, since the reachability set inclusion problem, which can be viewed as asking if $\text{POST}_A^*(\mathbf{x}) \subseteq \text{PRE}_B^*(\mathbf{y})$ where A, B are Petri nets of the same dimension (i.e., with the same sets of places), is undecidable [1, 10], even when the dimension of A, B is fixed to a small value [11].

Our contribution. In this paper, we show that the general semilinear home-space problem is decidable. This result is obtained by proving a duality between the reachability problem and the non-home-space problem. A crucial point consists in proving that for any Petri net A and for any linear set of configurations $\mathbf{L}$, we can effectively compute a semilinear "non-reachability core" $\mathbf{C}$ such that for every set $\mathbf{X}$ the set $\mathbf{L}$ is not a home-space for $\mathbf{X}$ if, and only if, $\mathbf{C}$ is reachable from $\mathbf{X}$. By a technical analysis using the known complexity results for reachability we show that the (semilinear) home-space problem is Ackermann-complete. As an ingredient, we also show that, given a Petri net with an initial marking, the set of minimal reachable markings can be constructed in Ackermannian time.

Organization of the paper. Section 2 states our theorems, after some preliminaries. Section 3 shows the hardness results, and Sections 4 and 5 give a decidability proof. Sections 6 and 7 contain the complexity analysis, and Section 8 adds some concluding remarks.

2 Basic Notions, and Main Results

In this section, we introduce basic notions and notation, and state the main results.

Notation for Vectors of Nonnegative Integers

By $\mathbb{N}$ we denote the set $\{0, 1, 2, \dots\}$ of nonnegative integers. For $i, j \in \mathbb{N}$, by $[i, j]$ we denote the set $\{i, i{+}1, \dots, j\}$.

For (a dimension) $d \in \mathbb{N}$, the elements of $\mathbb{N}^d$ are called (d-dimensional) *vectors*; they are denoted in bold face, and for $\mathbf{x} \in \mathbb{N}^d$ we put $\mathbf{x} = (\mathbf{x}(1), \mathbf{x}(2), \dots, \mathbf{x}(d))$ so that we can refer to the vector components. We use the component-wise sum $\mathbf{x} + \mathbf{y}$ of vectors, and their component-wise order $\mathbf{x} \leq \mathbf{y}$. For $c \in \mathbb{N}$, we put $c \cdot \mathbf{x} = (c \cdot \mathbf{x}(1), c \cdot \mathbf{x}(2), \dots, c \cdot \mathbf{x}(d))$. By the *norm of* $\mathbf{x}$, denoted $\|\mathbf{x}\|$, we mean the sum of components, i.e., $\|\mathbf{x}\| = \sum_{i=1}^{d} \mathbf{x}(i)$.

By $\mathbf{0}$ we denote the zero vector whose dimension is always clear from its context. Occasionally we slightly abuse notation by presenting a vector as a mix of subvectors and integers; in particular, given $\mathbf{x} \in \mathbb{N}^d$ and $y_1, y_2, \dots, y_m \in \mathbb{N}$, we might write $(\mathbf{x}, y_1, y_2, \dots, y_m)$ to denote the $(d{+}m)$-dimensional vector $(\mathbf{x}(1), \mathbf{x}(2), \dots, \mathbf{x}(d), y_1, y_2, \dots, y_m)$.

Given a set $\mathbf{X} \subseteq \mathbb{N}^d$, by $\overline{\mathbf{X}}$ we denote its complement, i.e., $\overline{\mathbf{X}} = \mathbb{N}^d \smallsetminus \mathbf{X}$.

Linear and Semilinear Sets of Vectors

A *set* $\mathbf{L} \subseteq \mathbb{N}^d$ is *linear* if there are d-dimensional vectors $\mathbf{b}$, the *basis*, and $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_k$, the *periods* (for $k \in \mathbb{N}$), such that $\mathbf{L} = \{\mathbf{x} \in \mathbb{N}^d \mid \mathbf{x} = \mathbf{b} + \mathbf{u}(1) \cdot \mathbf{p}_1 + \mathbf{u}(2) \cdot \mathbf{p}_2 \cdots + \mathbf{u}(k) \cdot \mathbf{p}_k$ for some $\mathbf{u} \in \mathbb{N}^k\}$. In this case, by a *presentation of* $\mathbf{L}$ we mean the tuple $(\mathbf{b}, \mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_k)$.

A *set* $\mathbf{S} \subseteq \mathbb{N}^d$ is *semilinear* if it is a finite union of linear sets, i.e. $\mathbf{S} = \mathbf{L}_1 \cup \mathbf{L}_2 \cdots \cup \mathbf{L}_m$ where $\mathbf{L}_i$ are linear sets (for all $i \in [1, m]$). In this case, by a *presentation of* $\mathbf{S}$ we mean the sequence of presentations of $\mathbf{L}_1, \mathbf{L}_2, \dots, \mathbf{L}_m$. When we say that a *semilinear set* $\mathbf{S}$ is given, we mean that we are given a presentation of $\mathbf{S}$; when we say that $\mathbf{S}$ is *effectively constructible* in some context, we mean that there is an algorithm computing its presentation (in the respective context).

We recall that a set $\mathbf{S} \subseteq \mathbb{N}^d$ is semilinear if, and only if, it is expressible in Presburger arithmetic [8]; the respective transformations between presentations and formulas are effective. Hence if $\mathbf{S} \subseteq \mathbb{N}^d$ is semilinear, then also its complement $\overline{\mathbf{S}}$ is semilinear, and $\overline{\mathbf{S}}$ is effectively constructible when (a presentation of) $\mathbf{S}$ is given.

Petri Nets

We use a concise definition of (unmarked place/transition) Petri nets. By a *d-dimensional Petri-net action* we mean a pair $a = (\mathbf{a}_-, \mathbf{a}_+) \in \mathbb{N}^d \times \mathbb{N}^d$. With $a = (\mathbf{a}_-, \mathbf{a}_+)$ we associate the binary relation $\xrightarrow{a}$ on the set $\mathbb{N}^d$ by putting $(\mathbf{x}+\mathbf{a}_-) \xrightarrow{a} (\mathbf{x}+\mathbf{a}_+)$ for all $\mathbf{x} \in \mathbb{N}^d$. The relations $\xrightarrow{a}$ are naturally extended to the relations $\xrightarrow{\sigma}$ for finite sequences σ of (d-dimensional Petri net) actions.

A *Petri net A of dimension d* (with d places in more traditional definitions) is a finite set of d-dimensional Petri-net actions (transitions). Here the vectors $\mathbf{x} \in \mathbb{N}^d$ are also called *configurations* (markings). On the set $\mathbb{N}^d$ of configurations we define the *reachability relation* $\xrightarrow{A^*}$: we put $\mathbf{x} \xrightarrow{A^*} \mathbf{y}$ if there is $\sigma \in A^*$ such that $\mathbf{x} \xrightarrow{\sigma} \mathbf{y}$. For $\mathbf{x} \in \mathbb{N}^d$ and $\mathbf{X} \subseteq \mathbb{N}^d$ we put $\textsc{post}^*_A(\mathbf{x}) = \{\mathbf{y} \in \mathbb{N}^d \mid \mathbf{x} \xrightarrow{A^*} \mathbf{y}\}$, and $\textsc{post}^*_A(\mathbf{X}) = \bigcup_{\mathbf{x}\in\mathbf{X}} \textsc{post}^*_A(\mathbf{x})$. Symmetrically, for $\mathbf{y} \in \mathbb{N}^d$ and $\mathbf{Y} \subseteq \mathbb{N}^d$ we put $\textsc{pre}^*_A(\mathbf{y}) = \{\mathbf{x} \in \mathbb{N}^d \mid \mathbf{x} \xrightarrow{A^*} \mathbf{y}\}$ and $\textsc{pre}^*_A(\mathbf{Y}) = \bigcup_{\mathbf{y}\in\mathbf{Y}} \textsc{pre}^*_A(\mathbf{y})$. By $\mathbf{X} \xrightarrow{A^*} \mathbf{Y}$ we denote that $\mathbf{x} \xrightarrow{A^*} \mathbf{y}$ for some $\mathbf{x} \in \mathbf{X}$ and $\mathbf{y} \in \mathbf{Y}$, i.e. that $\textsc{post}^*_A(\mathbf{X}) \cap \mathbf{Y} \neq \emptyset$, or equivalently $\mathbf{X} \cap \textsc{pre}^*_A(\mathbf{Y}) \neq \emptyset$.

(Semilinear) Reachability Problem

By the (semilinear) *reachability problem* we mean the following decision problem:

> *Instance:* a d-dimensional Petri net A and presentations of two semilinear sets $\mathbf{X}, \mathbf{Y} \subseteq \mathbb{N}^d$, to which we concisely refer as to the triple $A, \mathbf{X}, \mathbf{Y}$.
> *Question:* does $\mathbf{X} \xrightarrow{A^*} \mathbf{Y}$ hold?

In the standard definition of the reachability problem the sets $\mathbf{X}, \mathbf{Y}$ are singletons; the problem is decidable [16], and it has been recently shown to be Ackermann-complete [15, 14, 4]. It is well-known (and easy to show) that the above more general version (the semilinear reachability problem) is tightly related to the standard version, and has thus the same complexity.

▶ Remark 1. We can sketch this tight relation as follows. If $\mathbf{X}$ and $\mathbf{Y}$ are linear, with presentations $(\mathbf{b}, \mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_k)$ and $(\mathbf{b}', \mathbf{p}'_1, \mathbf{p}'_2, \dots, \mathbf{p}'_{k'})$ respectively, then it suffices to ask whether $\mathbf{b} \xrightarrow{(A')^*} \mathbf{b}'$ where A' arises from A by adding the actions $(\mathbf{0}, \mathbf{p}_i)$ for all $i \in [1, k]$ and $(\mathbf{p}'_i, \mathbf{0})$ for all $i \in [1, k']$. Now if $\mathbf{X} = \mathbf{L}_1 \cup \mathbf{L}_2 \cdots \cup \mathbf{L}_m$ and $\mathbf{Y} = \mathbf{L}'_1 \cup \mathbf{L}'_2 \cdots \cup \mathbf{L}'_{m'}$, then it suffices to check if $\mathbf{L}_i \xrightarrow{A^*} \mathbf{L}'_j$ for some $i \in [1, m]$ and $j \in [1, m']$. (In fact, there is a polynomial reduction of the general version to the standard one, which increases the dimension and uses the added vector components to mimic control states.)

Semilinear Home-Space Problem

For a Petri net A of dimension d and two sets $\mathbf{X}, \mathbf{H} \subseteq \mathbb{N}^d$, we call $\mathbf{H}$ a *home-space for* $(A, \mathbf{X})$, or just for $\mathbf{X}$ when A is clear from context, if $\text{POST}^*_A(\mathbf{X}) \subseteq \text{PRE}^*_A(\mathbf{H})$. (A trivial observation is that $\text{POST}^*_A(\mathbf{X})$ is a home-space for $\mathbf{X}$.)

We note that the above (semilinear) reachability problem in fact asks, given $A, \mathbf{X}, \mathbf{Y}$, if $\text{POST}^*_A(\mathbf{X}) \cap \text{PRE}^*_A(\mathbf{Y}) \neq \emptyset$. The *semilinear home-space problem* is defined as follows:

Instance: a triple $A, \mathbf{X}, \mathbf{H}$ where A is a Petri net, of dimension d, and $\mathbf{X}$, $\mathbf{H}$ are two (finitely presented) semilinear subsets of $\mathbb{N}^d$.
Question: is $\text{POST}^*_A(\mathbf{X}) \subseteq \text{PRE}^*_A(\mathbf{H})$ (i.e., is $\mathbf{H}$ a home-space for $\mathbf{X}$) ?

Our main result is stated by Theorem 3. Nevertheless, we first prove the weaker claim, Theorem 2, that answers an open question from [5] and does not need the technicalities related to the complexity analysis.

▶ **Theorem 2.** *The semilinear home-space problem is decidable.*

▶ **Theorem 3.** *The semilinear home-space problem is Ackermann-complete.*

We remark that by [5] we know that the home-space problem is decidable for the instances A, $\mathbf{X}$, $\mathbf{H}$ where $\mathbf{X}$ is a singleton set, and $\mathbf{H}$ is a finite union of linear sets with the same periods; this was established by a Turing reduction to the reachability problem. The decidability in the case where $\mathbf{H}$ is a general semilinear set was left open in [5]; this more general problem indeed looks more subtle but we manage to provide a solution here. Before doing this, we note in Section 3 that the problem has also a high computational complexity, and can be naturally viewed as residing at the decidability/undecidability border.

3 Home-Space Problem is Hard

We first note that even a simple version of the home-space problem is at least as hard as (non)reachability, and thus Ackermann-hard. We use a polynomial reduction that increases the Petri net dimension, by additional vector components that can be viewed as control states. (It would be natural to use the model of *vector addition systems with states* but we do not introduce them formally in this paper.)

▶ **Proposition 4.** *The non-reachability problem is polynomially reducible to the home-space problem restricted to the instances $A, \mathbf{X}, \mathbf{H}$ where $\mathbf{X}$ and $\mathbf{H}$ are singletons.*

Proof. Let us consider a Petri net A of dimension d and two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{N}^d$, as an instance of the (non)reachability problem. We create the $(d{+}3)$-dimensional Petri net A' so that each action $a = (\mathbf{a}_-, \mathbf{a}_+)$ of A is transformed to the action $a' = ((\mathbf{a}_-, 1, 0, 0), (\mathbf{a}_+, 1, 0, 0))$ of A', and A' has also the additional actions $((\mathbf{y}, 1, 0, 0), (\mathbf{0}, 0, 1, 0))$, $((\mathbf{0}, 1, 0, 0), (\mathbf{0}, 0, 0, 1))$, and the actions $((\mathbf{i}_j, 0, 1, 0), (\mathbf{0}, 0, 0, 1))$, $((\mathbf{i}_j, 0, 0, 1), (\mathbf{0}, 0, 0, 1))$ for all $j \in [1, d]$, where $\mathbf{i}_j \in \mathbb{N}^d$ satisfies $\mathbf{i}_j(j) = 1$ and $\mathbf{i}_j(i) = 0$ for all $i \neq j$.

We verify that $\mathbf{x} \xrightarrow{A^*} \mathbf{y}$ if, and only if, $\{(\mathbf{0}, 0, 0, 1)\}$ is not a home-space for $(A', \{(\mathbf{x}, 1, 0, 0)\})$:

- if $\mathbf{x} \xrightarrow{A^*} \mathbf{y}$, then $(\mathbf{x}, 1, 0, 0) \xrightarrow{(A')^*} (\mathbf{y}, 1, 0, 0) \xrightarrow{(A')^*} (\mathbf{0}, 0, 1, 0)$, and $(\mathbf{0}, 0, 0, 1)$ is not reachable from $(\mathbf{0}, 0, 1, 0)$;
- if $\mathbf{x} \not\xrightarrow{A^*} \mathbf{y}$, then any configuration reachable from $(\mathbf{x}, 1, 0, 0)$ in A' is in one of the forms $(\mathbf{y}', 1, 0, 0)$, $(\mathbf{z}, 0, 1, 0)$, $(\mathbf{z}', 0, 0, 1)$ where $\mathbf{y}' \neq \mathbf{y}$ and $\mathbf{z} \neq \mathbf{0}$, and $(\mathbf{0}, 0, 0, 1)$ is clearly reachable from all of them. ◀

Now we note that a slight generalization of the semilinear home-space problem is undecidable; it is the case when instead of semilinear sets $\mathbf{H}$ in the instances $A, \mathbf{X}, \mathbf{H}$ we allow $\mathbf{H}$ to be reachability sets of Petri nets (that are a special case of so called *almost semilinear sets* [13]).

▶ **Proposition 5.** *Given Petri nets A, B of the same dimension d, and two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{N}^d$, it is undecidable if $\textsc{post}^*_B(\mathbf{y})$ is a home-space for $(A, \{\mathbf{x}\})$.*

Proof. We recall that the reachability set inclusion problem is undecidable for Petri nets (and for the equivalent model of vector addition systems); see [1, 10, 11]. Hence it is undecidable, given Petri nets A, B of the same dimension d and $\mathbf{x}, \mathbf{y} \in \mathbb{N}^d$, whether $\textsc{post}^*_A(\mathbf{x}) \subseteq \textsc{post}^*_B(\mathbf{y})$. If A' arises from A by replacing each action $a = (\mathbf{a}_-, \mathbf{a}_+)$ with $a' = ((\mathbf{a}_-, 1), (\mathbf{a}_+, 1))$ and by adding the action $((\mathbf{0}, 1), (\mathbf{0}, 0))$, and B' arises from B by replacing each $b = (\mathbf{b}_-, \mathbf{b}_+)$ with $b' = ((\mathbf{b}_-, 0), (\mathbf{b}_+, 0))$, then we obviously have that $\textsc{post}^*_{B'}((\mathbf{y}, 0))$ is a home-space for $(A', (\mathbf{x}, 1))$ if, and only if, $\textsc{post}^*_A(\mathbf{x}) \subseteq \textsc{post}^*_B(\mathbf{y})$. ◀

▶ Remark 6. Since [11] shows, in fact, that the reachability set inclusion (or equality) problem is undecidable even for some fixed five-dimensional vector addition systems with states (VASSs), we could appropriately strengthen Proposition 5; but we do not pursue this technical issue here.

We can note that the undecidability of the question if $\textsc{post}^*_B(\mathbf{x}) \subseteq \textsc{post}^*_A(\mathbf{y})$ entails that the question if $\textsc{post}^*_B(\mathbf{x}) \subseteq \textsc{pre}^*_A(\mathbf{y})$ is also undecidable (since $\textsc{post}^*_A(\mathbf{y})$ is equal to $\textsc{pre}^*_{A_{rev}}(\mathbf{y})$ where A_{rev} arises from A by reversing each action $(\mathbf{a}_-, \mathbf{a}_+)$ to $(\mathbf{a}_+, \mathbf{a}_-)$). On the other hand, in the next sections we show that the question if $\textsc{post}^*_A(\mathbf{x}) \subseteq \textsc{pre}^*_A(\mathbf{y})$ is decidable. We will show that, given a d-dimensional Petri net A and $\mathbf{y} \in \mathbb{N}^d$, we can effectively construct a semilinear set (a "non-reachability core") $C \subseteq \mathbb{N}^d$ such that $\textsc{post}^*_A(\mathbf{x}) \not\subseteq \textsc{pre}^*_A(\mathbf{y})$ if, and only if, $\textsc{post}^*_A(\mathbf{x})$ intersects C. The equality of the nets on both sides is crucial, since if $\textsc{post}^*_B(\mathbf{x})$ does not intersect C, then this does not entail $\textsc{post}^*_B(\mathbf{x}) \subseteq \textsc{pre}^*_A(\mathbf{y})$.

4 Decidability of Home-Space via Semilinear Non-Reachability Cores

Now we start to discuss how to decide the semilinear home-space problem. We assume a fixed Petri net A of dimension d if not said otherwise.

We first note that the home-space property can be naturally formulated in terms of "reachability of non-reachability". To this aim we introduce a technical notion and make a related observation.

Given a set $\mathbf{H} \subseteq \mathbb{N}^d$, we say that a set $\mathbf{C} \subseteq \mathbb{N}^d$ is a *non-reachability core for* $\mathbf{H}$ if

1. $\mathbf{C} \stackrel{A^*}{\not\longrightarrow} \mathbf{H}$ (hence $\mathbf{C} \subseteq \overline{\textsc{pre}^*_A(\mathbf{H})}$ where $\overline{\textsc{pre}^*_A(\mathbf{H})} = \mathbb{N}^d \smallsetminus \textsc{pre}^*_A(\mathbf{H})$), and
2. for each $\mathbf{x} \in \mathbb{N}^d$, if $\mathbf{x} \stackrel{A^*}{\not\longrightarrow} \mathbf{H}$ then $\mathbf{x} \xrightarrow{A^*} \mathbf{C}$ (hence $\overline{\textsc{pre}^*_A(\mathbf{H})} \subseteq \textsc{pre}^*_A(\mathbf{C})$).

▶ **Proposition 7.** *If $\mathbf{C}$ is a non-reachability core for $\mathbf{H}$, then for every $\mathbf{X} \subseteq \mathbb{N}^d$ we have that*

$\mathbf{H}$ is not a home-space for $\mathbf{X}$ if, and only if, $\mathbf{X} \xrightarrow{A^*} \mathbf{C}$.

Proof. If $\mathbf{X} \xrightarrow{A^*} \mathbf{C}$, then $\mathbf{x} \xrightarrow{A^*} \mathbf{c}$ for some $\mathbf{x} \in \mathbf{X}$ and $\mathbf{c} \in \mathbf{C}$; since $\mathbf{c} \stackrel{A^*}{\not\longrightarrow} \mathbf{H}$ by condition 1, $\mathbf{H}$ is not a home-space for $\mathbf{X}$.

If $\mathbf{H}$ is not a home-space for $\mathbf{X}$, then we have $\mathbf{x} \xrightarrow{A^*} \mathbf{x}' \stackrel{A^*}{\not\longrightarrow} \mathbf{H}$ for some $\mathbf{x} \in \mathbf{X}$ and some $\mathbf{x}'$. By condition 2, $\mathbf{x}' \xrightarrow{A^*} \mathbf{C}$; hence $\mathbf{x} \xrightarrow{A^*} \mathbf{C}$, which entails $\mathbf{X} \xrightarrow{A^*} \mathbf{C}$. ◀

We note that $\overline{\text{PRE}^*_A(\mathbf{H})}$ is clearly a non-reachability core for $\mathbf{H}$, and the question whether $\mathbf{X} \xrightarrow{A^*} \overline{\text{PRE}^*_A(\mathbf{H})}$ asks, in fact, whether $\mathbf{H}$ is not a home-space for $\mathbf{X}$. To decide the question whether $\mathbf{X} \xrightarrow{A^*} \overline{\text{PRE}^*_A(\mathbf{H})}$ when $\mathbf{H}$ is a semilinear set (given by a standard presentation), a natural idea is to look if there is an effectively constructible semilinear non-reachability core $\mathbf{C}$ for $\mathbf{H}$ (where $\mathbf{C} \subseteq \overline{\text{PRE}^*_A(\mathbf{H})}$); we recall that the question whether $\mathbf{X} \xrightarrow{A^*} \mathbf{C}$ is decidable for semilinear sets $\mathbf{X}, \mathbf{C}$. In fact, we manage to realize this idea directly only in the case when $\mathbf{H}$ is a linear set:

▶ **Lemma 8.** *Given a Petri net A of dimension d, and (a presentation of) a linear set $\mathbf{L} \subseteq \mathbb{N}^d$, there is an effectively constructible semilinear non-reachability core $\mathbf{C}$ for $\mathbf{L}$.*

This crucial lemma will be proved in the next section (Section 5). Here we show the decidability of the semilinear home-space problem when assuming the lemma. To this aim we first note a useful fact captured by the following proposition.

▶ **Proposition 9.** *Assuming a Petri net A of dimension d, if $\mathbf{H} = \mathbf{H}_1 \cup \mathbf{H}_2 \cdots \cup \mathbf{H}_m$ and $\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_m$ are such that $\mathbf{C}_i$ is a non-reachability core for $\mathbf{H}_i$ for each $i \in [1, m]$, then for each $\mathbf{X} \subseteq \mathbb{N}^d$ we have that $\mathbf{H}$ is not a home-space for $\mathbf{X}$ if, and only if, there is an execution*

$$\mathbf{x}_0 \xrightarrow{A^*} \mathbf{x}_1 \xrightarrow{A^*} \mathbf{x}_2 \cdots \xrightarrow{A^*} \mathbf{x}_m \tag{1}$$

where $\mathbf{x}_0 \in \mathbf{X}$, and $\mathbf{x}_i \in \mathbf{C}_i$ for each $i \in [1, m]$. (We do not exclude the cases $\mathbf{x}_i = \mathbf{x}_{i+1}$.)

Proof. Given an execution (1), the facts that $\mathbf{x}_i \in \mathbf{C}_i$ and $\mathbf{C}_i$ is a non-reachability core for $\mathbf{H}_i$ (hence $\mathbf{C}_i \subseteq \overline{\text{PRE}^*_A(\mathbf{H}_i)}$) entail $\mathbf{x}_i \stackrel{A^*}{\not\longrightarrow} \mathbf{H}_i$, for all $i \in [1, m]$. The facts $\mathbf{x}_i \stackrel{A^*}{\not\longrightarrow} \mathbf{H}_i$ and $\mathbf{x}_i \xrightarrow{A^*} \mathbf{x}_m$ entail that $\mathbf{x}_m \stackrel{A^*}{\not\longrightarrow} \mathbf{H}_i$ (for all $i \in [1, m]$). Hence $\mathbf{x}_m \stackrel{A^*}{\not\longrightarrow} \mathbf{H}$ (where $\mathbf{H} = \mathbf{H}_1 \cup \mathbf{H}_2 \cdots \cup \mathbf{H}_m$), and the facts $\mathbf{x}_0 \in \mathbf{X}$ and $\mathbf{x}_0 \xrightarrow{A^*} \mathbf{x}_m \stackrel{A^*}{\not\longrightarrow} \mathbf{H}$ entail that $\mathbf{H}$ is not a home-space for $\mathbf{X}$.

Conversely, we assume a set $\mathbf{X} \subseteq \mathbb{N}^d$ for which $\mathbf{H}$ is not a home-space. Hence there exist configurations $\mathbf{x}_0, \mathbf{x}'_0$ such that $\mathbf{x}_0 \in \mathbf{X}$ and $\mathbf{x}_0 \xrightarrow{A^*} \mathbf{x}'_0 \stackrel{A^*}{\not\longrightarrow} \mathbf{H}$. In particular $\mathbf{x}'_0 \stackrel{A^*}{\not\longrightarrow} \mathbf{H}_1$, and thus $\mathbf{H}_1$ is not a home-space for $\{\mathbf{x}'_0\}$. Since $\mathbf{C}_1$ is a non-reachability core for $\mathbf{H}_1$, we have $\mathbf{x}'_0 \xrightarrow{A^*} \mathbf{x}_1$ for some $\mathbf{x}_1 \in \mathbf{C}_1$. Since $\mathbf{x}'_0 \stackrel{A^*}{\not\longrightarrow} \mathbf{H}$ and $\mathbf{x}'_0 \xrightarrow{A^*} \mathbf{x}_1$, we have $\mathbf{x}_1 \stackrel{A^*}{\not\longrightarrow} \mathbf{H}$, and in particular $\mathbf{x}_1 \stackrel{A^*}{\not\longrightarrow} \mathbf{H}_2$. Since $\mathbf{H}_2$ is not a home-space for $\{\mathbf{x}_1\}$ and $\mathbf{C}_2$ is a non-reachability core for $\mathbf{H}_2$, we get $\mathbf{x}_1 \xrightarrow{A^*} \mathbf{x}_2$ for some $\mathbf{x}_2 \in \mathbf{C}_2$. Continuing in this way, we successively derive the existence of an execution (1). ◀

The next proposition gives us the final ingredient for showing an algorithm deciding the semilinear home-space problem.

▶ **Proposition 10.** *Given a Petri net A of dimension d, and (presentations of) semilinear subsets $\mathbf{X}_0, \mathbf{X}_1, \ldots, \mathbf{X}_m$ of $\mathbb{N}^d$, the existence of an execution*

$$\mathbf{x}_0 \xrightarrow{A^*} \mathbf{x}_1 \xrightarrow{A^*} \mathbf{x}_2 \cdots \xrightarrow{A^*} \mathbf{x}_m \tag{2}$$

where $\mathbf{x}_i \in \mathbf{X}_i$ for each $i \in [0, m]$ is decidable (by a reduction to reachability).

Proof. By a standard construction, we can build a Petri net with a bigger dimension and an initial configuration that first generates m copies of some $\mathbf{x}_0 \in \mathbf{X}_0$, then performs an execution of A from $\mathbf{x}_0$ on all these copies, while at some moment it freezes some configuration $\mathbf{x}_1$ reached in the first copy, later it freezes some $\mathbf{x}_2$ reached in the second copy, etc.; at the end it starts a "testing part" that enables to reach the zero configuration if, and only if, $\mathbf{x}_1 \in \mathbf{X}_1$, $\mathbf{x}_2 \in \mathbf{X}_2$, …, $\mathbf{x}_m \in \mathbf{X}_m$. ◀

We note that a proof of Theorem 2 is now clear: Given a Petri net A of dimension d and two semilinear sets $\mathbf{X}, \mathbf{H} \subseteq \mathbb{N}^d$, we use that $\mathbf{H} = \mathbf{H}_1 \cup \mathbf{H}_2 \ldots \cup \mathbf{H}_m$ where $\mathbf{H}_i$ are linear sets, and by Lemma 8 we can construct a semilinear non-reachability core $\mathbf{C}_i$ for $\mathbf{H}_i$, for each $i \in [1, m]$. Then we ask if there is an execution (1) from Proposition 9; this can be decided effectively by Proposition 10 (since (1) is a particular case of (2) in this case).

5 Effective Semilinear Non-Reachability Core for Linear Set

Before proving Lemma 8 in Section 5.2, in Section 5.1 we recall an important ingredient dealing with computing the minimal elements in some sets $\mathbf{X} \subseteq \mathbb{N}^d$; its use in Petri nets originates in the work by Valk and Jantzen [20].

5.1 Computing $\min(\mathbf{X})$ for $\mathbf{X} \subseteq \mathbb{N}^d$

For $\mathbf{X} \subseteq \mathbb{N}^d$ we call a vector $\mathbf{m} \in \mathbf{X}$ *minimal in* $\mathbf{X}$ if there is no vector $\mathbf{x} \in \mathbf{X}$ such that $\mathbf{x} \leq \mathbf{m}$ and $\mathbf{x} \neq \mathbf{m}$. (We recall that $\mathbf{x} \leq \mathbf{y}$ denotes that $\mathbf{x}(i) \leq \mathbf{y}(i)$ for all $i \in [1, d]$.) By $\min(\mathbf{X})$ we denote *the set of minimal elements in* $\mathbf{X}$. Since $\leq$ is a well-partial-order on $\mathbb{N}^d$ (by Dickson's lemma), the set $\min(\mathbf{X})$ is finite and for every $\mathbf{x} \in \mathbf{X}$ there exists (at least one) $\mathbf{m} \in \min(\mathbf{X})$ such that $\mathbf{m} \leq \mathbf{x}$.

As a basis for computing $\min(\mathbf{X})$ (for special sets $\mathbf{X} \subseteq \mathbb{N}^d$), it is useful to extend the ordered set $(\mathbb{N}, \leq)$ with an extra element $\omega \notin \mathbb{N}$ so that $x \leq \omega$ for every $x \in \mathbb{N}_\omega$, where $\mathbb{N}_\omega$ denotes $\mathbb{N} \cup \{\omega\}$. By $\mathbb{N}_\omega^d$ we denote the set of d-dimensional vectors over $\mathbb{N}_\omega$; the (component-wise) order $\leq$ on $\mathbb{N}^d$ is naturally extended to $\mathbb{N}_\omega^d$. For $\mathbf{v} \in \mathbb{N}_\omega^d$ we put $\downarrow\mathbf{v} = \{\mathbf{y} \in \mathbb{N}^d \mid \mathbf{y} \leq \mathbf{v}\}$. Hence even when $\mathbf{v}$ has some ω-components, $\mathbf{y} \in \downarrow\mathbf{v}$ has none.

For $\mathbf{X} \subseteq \mathbb{N}^d$ we trivially have $\min(\mathbf{X}) = \min(\mathbf{X} \cap \downarrow(\omega, \omega, \ldots, \omega))$. If we want to describe $\min(\mathbf{X} \cap \downarrow\mathbf{v})$, for $\mathbf{v} \in \mathbb{N}_\omega^d$, and we have some $\mathbf{y} \in (\mathbf{X} \cap \downarrow\mathbf{v})$, then we observe that

$$\min(\mathbf{X} \cap \downarrow\mathbf{v}) = \min\Big(\{\mathbf{y}\} \cup \min\left(\mathbf{X} \cap (\downarrow\mathbf{v} \smallsetminus \{\mathbf{x} \mid \mathbf{y} \leq \mathbf{x}\})\right)\Big).$$

To write this more concretely, by $\mathbf{v}[i \leftarrow k]$, where $i \in [1, d]$ and $k \in \mathbb{N}$, we denote the vector $\mathbf{v}' \in \mathbb{N}_\omega^d$ coinciding with $\mathbf{v}$ except that we have $\mathbf{v}'(i) = k$, and we put

$$\delta_{\mathbf{y}}(\mathbf{v}) = \{\mathbf{w} \in \mathbb{N}_\omega^d \mid \mathbf{w} = \mathbf{v}[i \leftarrow (\mathbf{y}(i){-}1)], i \in [1, d], \mathbf{y}(i) > 0\}.$$

▶ **Observation 11.** *For all* $\mathbf{v} \in \mathbb{N}_\omega^d$ *and* $\mathbf{y} \in \downarrow\mathbf{v}$ *we have:*

1. *Each* $\mathbf{w} \in \delta_{\mathbf{y}}(\mathbf{v})$ *is strictly less than* $\mathbf{v}$ *(i.e.,* $\mathbf{w} \leq \mathbf{v}$ *and* $\mathbf{w} \neq \mathbf{v}$*).*
2. $\downarrow\mathbf{v} \smallsetminus \{\mathbf{x} \mid \mathbf{y} \leq \mathbf{x}\} = \bigcup_{\mathbf{w} \in \delta_{\mathbf{y}}(\mathbf{v})} \downarrow\mathbf{w}$.

▶ **Observation 12.** *For all* $\mathbf{X} \subseteq \mathbb{N}^d$*,* $\mathbf{v} \in \mathbb{N}_\omega^d$*, and* $\mathbf{y} \in (\mathbf{X} \cap \downarrow\mathbf{v})$ *we have:*

$$\min(\mathbf{X} \cap \downarrow\mathbf{v}) = \min\left(\{\mathbf{y}\} \cup \bigcup_{\mathbf{w} \in \delta_{\mathbf{y}}(\mathbf{v})} \min(\mathbf{X} \cap \downarrow\mathbf{w})\right).$$

Since each strictly decreasing sequence $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \ldots$ of vectors in $\mathbb{N}_\omega^d$ is finite, we easily observe that there is an algorithm stated in the next lemma. Its inputs are special algorithms that we call *set-related algorithms*. Each set-related algorithm is related to some set $\mathbf{X} \subseteq \mathbb{N}^d$ (for some $d \in \mathbb{N}$); given $\mathbf{v} \in \mathbb{N}_\omega^d$, the algorithm decides if $(\mathbf{X} \cap \downarrow\mathbf{v})$ is nonempty, and in the positive case returns some $\mathbf{y} \in (\mathbf{X} \cap \downarrow\mathbf{v})$.

▶ **Lemma 13.** *There is an algorithm that, given a set-related algorithm related to* $\mathbf{X} \subseteq \mathbb{N}^d$*, computes the set* $\min(\mathbf{X})$*.*

▶ Remark 14. In fact, the algorithm claimed by the lemma does not require to get a code of a set-related algorithm; it suffices to get (black-box) access to such an algorithm.

5.2 Proof of Lemma 8

Now we prove Lemma 8:

> *Given a Petri net A of dimension d, and (a presentation of) a linear set $\mathbf{L} \subseteq \mathbb{N}^d$, there is an effectively constructible semilinear non-reachability core $\mathbf{C}$ for $\mathbf{L}$.*

We assume a fixed Petri net A of dimension d, and we first prove the claim for the case where $\mathbf{L}$ is a singleton; hence $\mathbf{L} = \{\mathbf{b}\}$ (there is a basis $\mathbf{b} \in \mathbb{N}^d$, but no periods). We observe that if $\|\mathbf{x}\| > \|\mathbf{b}\|$ (where $\|\mathbf{x}\| = \sum_{i=1}^{d} \mathbf{x}(i)$), then a necessary condition for reachability of $\mathbf{b}$ from $\mathbf{x}$ is that $\mathbf{x}$ belongs to the set

$$\mathrm{DC} = \{\mathbf{x} \in \mathbb{N}^d \mid \text{ there is } \mathbf{x}' \text{ such that } \mathbf{x} \xrightarrow{A^*} \mathbf{x}' \text{ and } \|\mathbf{x}\| > \|\mathbf{x}'\|\}.$$

For $\mathbf{x} \in \mathrm{DC}$ we say that $\mathbf{x}$ *can Decrease the token-Count*. Since there is no infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ in $\mathbb{N}^d$ where $\|\mathbf{x}_1\| > \|\mathbf{x}_2\| > \|\mathbf{x}_3\| > \cdots$, for $\mathrm{NDC} = \overline{\mathrm{DC}}$ (the complement of DC, i.e. $\mathbb{N}^d \smallsetminus \mathrm{DC}$) we note the following trivial fact:

▶ **Observation 15.** NDC *is a home-space for every* $\mathbf{X} \subseteq \mathbb{N}^d$.

Proposition 16 is a crucial ingredient for Proposition 17 that finishes the proof of Lemma 8 in the special case when $\mathbf{L}$ is a singleton.

▶ **Proposition 16.** *The set* DC *is upward closed and the set* min(DC) *is effectively constructible. Hence both* DC *and* NDC *are effectively constructible semilinear sets.*

Proof. If $\mathbf{x} \xrightarrow{\sigma} \mathbf{x}'$, then $\mathbf{x} + \mathbf{y} \xrightarrow{\sigma} \mathbf{x}' + \mathbf{y}$ (by the monotonicity property of Petri nets). Since $\|\mathbf{x}\| > \|\mathbf{x}'\|$ entails $\|\mathbf{x} + \mathbf{y}\| > \|\mathbf{x}' + \mathbf{y}\|$, it is clear that DC is upward closed (i.e., if $\mathbf{x} \in \mathrm{DC}$ and $\mathbf{x} \leq \mathbf{y}$, then $\mathbf{y} \in \mathrm{DC}$).

Regarding the effective constructability of min(DC), we recall Lemma 13. The question if $(\mathrm{DC} \cap \downarrow\mathbf{v})$ is nonempty, for a given $\mathbf{v} \in \mathbb{N}_\omega^d$, can be reduced to the reachability problem in a standard way (recall the technique sketched for Proposition 10): in the respective net (of a bigger dimension), first some $\mathbf{y} \in \mathbb{N}^d$ belonging to $\downarrow\mathbf{v}$ is generated, and frozen, and then some $\mathbf{y}'$ reachable from $\mathbf{y}$ in the original net is obtained and frozen, and in the final phase a particular place can reach zero if, and only if, $\|\mathbf{y}\| > \|\mathbf{y}'\|$. Hence in the positive case a witness of the respective reachability also yields some $\mathbf{y} \in (\mathrm{DC} \cap \downarrow\mathbf{v})$.

The effective semilinearity of DC and NDC follows trivially. ◀

▶ **Proposition 17.** *Given a Petri net A of dimension d and a vector $\mathbf{b} \in \mathbb{N}^d$, the set*

$$\mathbf{C} = \mathrm{NDC} \cap \big(\{\mathbf{x} \in \mathbb{N}^d \mid \|\mathbf{x}\| > \|\mathbf{b}\|\} \cup \{\mathbf{x} \in \mathbb{N}^d \mid \|\mathbf{x}\| \leq \|\mathbf{b}\| \textit{ and } \mathbf{x} \not\xrightarrow{A^*} \mathbf{b}\}\big)$$

is an effectively constructible semilinear non-reachability core for $\{\mathbf{b}\}$.

Proof. We first show that $\mathbf{C}$ is a non-reachability core for $\{\mathbf{b}\}$:

1. We have $\mathbf{C} \not\xrightarrow{A^*} \{\mathbf{b}\}$, since $\mathbf{b}$ is clearly not reachable from any element of $\mathbf{C}$.
2. For each $\mathbf{x} \in \mathbb{N}^d$, if $\mathbf{x} \not\xrightarrow{A^*} \mathbf{b}$, then $\mathbf{x} \xrightarrow{A^*} \mathbf{x}' \not\xrightarrow{A^*} \mathbf{b}$ for some $\mathbf{x}' \in \mathrm{NDC}$ (recall Observation 15); the facts $\mathbf{x}' \in \mathrm{NDC}$ and $\mathbf{x}' \not\xrightarrow{A^*} \mathbf{b}$ obviously entail $\mathbf{x}' \in \mathbf{C}$, and thus $\mathbf{x} \xrightarrow{A^*} \mathbf{C}$.

The effective semilinearity of $\mathbf{C}$ follows from Proposition 16 and from the fact that the finite set $\{\mathbf{x} \in \mathbb{N}^d \mid \|\mathbf{x}\| \leq \|\mathbf{b}\| \text{ and } \mathbf{x} \not\xrightarrow{A^*} \mathbf{b}\}$ can be constructed by repeatedly using an algorithm deciding reachability. ◀

Now we proceed to prove Lemma 8 in general. We have a Petri net A of dimension d, and a linear set $\mathbf{L}$ presented by a basis $\mathbf{b} \in \mathbb{N}^d$ and periods $\mathbf{p}_1, \mathbf{p}_2 \ldots, \mathbf{p}_k \in \mathbb{N}^d$; we aim to construct a semilinear non-reachability core for $\mathbf{L}$. We would like to generalize the above special-case proof with the upward closed set DC, which is, in fact, closely related to the approach in [5]. But here is a subtle problem, as we already mentioned. Our solution is not working with configurations $\mathbf{x} \in \mathbb{N}^d$ directly, but rather via their $\mathbf{L}$-like presentations.

We note that each configuration $\mathbf{x} \in \mathbb{N}^d$ can be presented as

$$\mathbf{x} = \mathbf{y} + \mathbf{u}(1) \cdot \mathbf{p}_1 + \mathbf{u}(2) \cdot \mathbf{p}_2 \cdots + \mathbf{u}(k) \cdot \mathbf{p}_k$$

for at least one (but often more) pairs $(\mathbf{y}, \mathbf{u}) \in \mathbb{N}^d \times \mathbb{N}^k$. For $\mathbf{y} \in \mathbb{N}^d$ and $\mathbf{u} \in \mathbb{N}^k$ we put

$$\textsc{conf}(\mathbf{y}, \mathbf{u}) = \mathbf{y} + \mathbf{u}(1) \cdot \mathbf{p}_1 + \mathbf{u}(2) \cdot \mathbf{p}_2 \cdots + \mathbf{u}(k) \cdot \mathbf{p}_k.$$

Hence $\mathbf{L} = \{\textsc{conf}(\mathbf{b}, \mathbf{u}) \mid \mathbf{u} \in \mathbb{N}^k\}$.

Let $\textsc{dcb-pr}$ (determined by the Petri net A and the sequence of periods of $\mathbf{L}$) be the set of presentation pairs that present configurations that *can Decrease the token-Count in the presentation Basis*:

$$\textsc{dcb-pr} = \{(\mathbf{y}, \mathbf{u}) \in \mathbb{N}^d \times \mathbb{N}^k \mid \exists (\mathbf{y}', \mathbf{u}') : \|\mathbf{y}\| > \|\mathbf{y}'\|, \textsc{conf}(\mathbf{y}, \mathbf{u}) \xrightarrow{A^*} \textsc{conf}(\mathbf{y}', \mathbf{u}')\}.$$

We note that if $\mathbf{y} \geq \mathbf{p}_i$, for some $i \in [1, k]$, then we trivially have $(\mathbf{y}, \mathbf{u}) \in \textsc{dcb-pr}$ since $\textsc{conf}(\mathbf{y}, \mathbf{u}) = \textsc{conf}(\mathbf{y} - \mathbf{p}_i, \mathbf{u}')$ where $\mathbf{u}'$ arises from $\mathbf{u}$ by adding 1 to $\mathbf{u}(i)$. (As expected, we assume that all $\mathbf{p}_i$ are nonzero vectors.)

▶ **Proposition 18.** $\textsc{dcb-pr}$ *is upward closed and the set* $\min(\textsc{dcb-pr})$ *is effectively constructible.*

Proof. As expected, we compare the elements of $\textsc{dcb-pr}$ component-wise. To show that $\textsc{dcb-pr}$ is upward closed, we assume that $(\mathbf{y}_1, \mathbf{u}_1) \in \textsc{dcb-pr}$ and $(\mathbf{y}_1, \mathbf{u}_1) \leq (\mathbf{y}_2, \mathbf{u}_2)$. To demonstrate that $(\mathbf{y}_2, \mathbf{u}_2) \in \textsc{dcb-pr}$ as well, we again use monotonicity of Petri nets: Since $\textsc{conf}(\mathbf{y}_1, \mathbf{u}_1) \xrightarrow{\sigma} \textsc{conf}(\mathbf{y}'_1, \mathbf{u}'_1)$ (for some sequence σ) where $\|\mathbf{y}_1\| > \|\mathbf{y}'_1\|$, and $\textsc{conf}(\mathbf{y}_1, \mathbf{u}_1) \leq \textsc{conf}(\mathbf{y}_2, \mathbf{u}_2)$, we have $\textsc{conf}(\mathbf{y}_2, \mathbf{u}_2) \xrightarrow{\sigma} \textsc{conf}(\mathbf{y}'_1 + (\mathbf{y}_2 - \mathbf{y}_1), \mathbf{u}'_1 + (\mathbf{u}_2 - \mathbf{u}_1))$; $\|\mathbf{y}_1\| > \|\mathbf{y}'_1\|$ entails $\|\mathbf{y}_2\| > \|\mathbf{y}'_1 + (\mathbf{y}_2 - \mathbf{y}_1)\|$.

The effective constructability of $\min(\textsc{dcb-pr})$ is again based on Lemma 13, when we identify $\mathbb{N}^d \times \mathbb{N}^k$ with $\mathbb{N}^{d+k}$. It is again a technical routine to show that the question whether $(\textsc{dcb-pr} \cap \downarrow \mathbf{v})$ is nonempty, for a given $\mathbf{v} \in \mathbb{N}_\omega^{d+k}$, can be reduced to the reachability problem, so that in the positive case a witness of this reachability also yields some $(\mathbf{y}, \mathbf{u}) \in (\textsc{dcb-pr} \cap \downarrow \mathbf{v})$. ◀

We now define the set of configurations that can be presented so that the presentation basis cannot be decreased:

$$\text{NDCB} = \{\mathbf{x} \in \mathbb{N}^d \mid \mathbf{x} = \textsc{conf}(\mathbf{y}, \mathbf{u}) \text{ for some } (\mathbf{y}, \mathbf{u}) \notin \textsc{dcb-pr}\}.$$

▶ **Observation 19.** NDCB *is a home-space for every* $\mathbf{X} \subseteq \mathbb{N}^d$.

(Suppose there is some $\mathbf{x} \in \mathbb{N}^d$ such that $\mathbf{x} \not\xrightarrow{A^*} \text{NDCB}$; we fix one such $\mathbf{x}$ that can be written as $\mathbf{x} = \textsc{conf}(\mathbf{y}, \mathbf{u})$ for $\mathbf{y}$ with the least norm $\|\mathbf{y}\|$. Since $\mathbf{x} \notin \text{NDCB}$, we have $(\mathbf{y}, \mathbf{u}) \in \textsc{dcb-pr}$, which entails a contradiction by the definition of $\textsc{dcb-pr}$.)

▶ **Proposition 20.** NDCB *is an effectively constructible semilinear set.*

Proof. By Proposition 18, DCB-PR is an effectively constructible semilinear set. Since semilinear sets (effectively) coincide with the sets definable in Presburger arithmetic, the claim is clear. ◀

The next proposition finishes a proof of Lemma 8, and thus also of Theorem 2.

▶ **Proposition 21.** *Given a Petri net A of dimension d and a linear set $\mathbf{L} \subseteq \mathbb{N}^d$ presented by* $(\mathbf{b}, \mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_k)$*, the set*

$$\mathbf{C} = \{\mathbf{x} \in \mathbb{N}^d \mid \mathbf{x} = \textsc{conf}(\mathbf{y}, \mathbf{u}) \textit{ where } (\mathbf{y}, \mathbf{u}) \notin \textsc{dcb-pr} \textit{ and}$$
$$\textit{either } \|\mathbf{y}\| > \|\mathbf{b}\|, \textit{ or } \|\mathbf{y}\| \leq \|\mathbf{b}\| \textit{ and } \textsc{conf}(\mathbf{y}, \mathbf{u}) \overset{A^*}{\not\longrightarrow} \mathbf{L}\}.$$

is an effectively constructible semilinear non-reachability core for $\mathbf{L}$.

Proof. We note that $\mathbf{C}$ is a subset of NDCB, and we recall that $\mathbf{x} \in \mathbf{L}$ iff $\mathbf{x} = \textsc{conf}(\mathbf{b}, \mathbf{u})$ for some $\mathbf{u} \in \mathbb{N}^k$. We verify that $\mathbf{C}$ is a non-reachability core for $\mathbf{L}$:

1. By definition of $\mathbf{C}$ we clearly have $\mathbf{C} \overset{A^*}{\not\longrightarrow} \mathbf{L}$.
2. For each $\mathbf{x} \in \mathbb{N}^d$, if $\mathbf{x} \overset{A^*}{\not\longrightarrow} \mathbf{L}$, then $\mathbf{x} \xrightarrow{A^*} \mathbf{x}' \overset{A^*}{\not\longrightarrow} \mathbf{L}$ for some $\mathbf{x}' \in$ NDCB (recall Observation 19); the facts $\mathbf{x}' \in$ NDCB and $\mathbf{x}' \overset{A^*}{\not\longrightarrow} \mathbf{L}$ obviously entail $\mathbf{x}' \in \mathbf{C}$, and thus $\mathbf{x} \xrightarrow{A^*} \mathbf{C}$.

Now we aim to show that $\mathbf{C}$ is an effectively constructible semilinear set. We recall Propositions 20 and 18, and the fact that for any concrete $\mathbf{y}$ and $\mathbf{u}$ we can decide if $\textsc{conf}(\mathbf{y}, \mathbf{u}) \xrightarrow{A^*} \mathbf{L}$. Though there are only finitely many $\mathbf{y}$ to consider, namely those satisfying $\|\mathbf{y}\| \leq \|\mathbf{b}\|$, we are not done: it is not immediately obvious how to express $\textsc{conf}(\mathbf{y}, \mathbf{u}) \overset{A^*}{\not\longrightarrow} \mathbf{L}$ in Presburger arithmetic, even when $\mathbf{y}$ is fixed. To this aim, for any fixed $\mathbf{y} \in \mathbb{N}^d$ we define the set

$$\mathbf{U}_\mathbf{y} = \{\mathbf{u} \in \mathbb{N}^k \mid \textsc{conf}(\mathbf{y}, \mathbf{u}) \xrightarrow{A^*} \mathbf{L}\} = \{\mathbf{u} \in \mathbb{N}^k \mid \exists \mathbf{u}' \in \mathbb{N}^k : \textsc{conf}(\mathbf{y}, \mathbf{u}) \xrightarrow{A^*} \textsc{conf}(\mathbf{b}, \mathbf{u}')\}.$$

For each fixed $\mathbf{y} \in \mathbb{N}^d$, the set $\mathbf{U}_\mathbf{y}$ is clearly upward closed (by monotonicity of Petri nets). Moreover, the set $\min(\mathbf{U}_\mathbf{y})$ is effectively constructible, again by using Lemma 13: Given a fixed $\mathbf{y}$, for each $\mathbf{v} \in \mathbb{N}_\omega^k$ we can decide whether $(\mathbf{U}_\mathbf{y} \cap {\downarrow}\mathbf{v})$ is nonempty by a reduction to the reachability problem, so that in the positive case a witness of this reachability also yields some $\mathbf{u} \in (\mathbf{U}_\mathbf{y} \cap {\downarrow}\mathbf{v})$.

Now it is clear that we can effectively construct a Presburger formula defining $\mathbf{C}$; hence $\mathbf{C}$ is a semilinear set for which we can effectively construct a presentation. ◀

6 Minimal Reachable Configurations

In this section we provide several Ackermannian-time algorithms. The first one is given a Petri net A of dimension d and a configuration $\mathbf{x} \in \mathbb{N}^d$, and it computes the set $\min(\textsc{post}_A^*(\mathbf{x}))$, i.e. the set of minimal configurations in the respective reachability set. The second algorithm computes $\min(\textsc{post}_A^*(\mathbf{x}) \cap \mathbf{S})$ when given (a presentation of) a semilinear set $\mathbf{S} \subseteq \mathbb{N}^d$ besides A and $\mathbf{x}$. The third algorithm is given $A, \mathbf{x}$, and (a presentation of) a semilinear predicate $P \subseteq \mathbb{N}^h \times \mathbb{N}^d \times \mathbb{N}^d$ (for some $h \in \mathbb{N}$), and it computes the set

$$\min(\{\mathbf{x} \in \mathbb{N}^h \mid \exists \alpha, \beta \in \mathbb{N}^d : \alpha \xrightarrow{A^*} \beta \wedge (\mathbf{x}, \alpha, \beta) \in P\}).$$

The complexity of computing the above mentioned minimal configurations can be derived by using the approach by Hsu-Chun Yen and Chien-Liang Chen in [21]; they observed that complexity bounds on a set-related algorithm related to some set $\mathbf{X} \subseteq \mathbb{N}^d$ (recall the definition before Lemma 13) allow us to derive complexity bounds on the computation of $\min(\mathbf{X})$. As a crucial ingredient here, we recall the known complexity upper bound for reachability in Section 6.1. In Section 6.2 we derive an Ackermannian bound on the size of minimal configurations in Petri net reachability sets, and we extend this bound in Section 6.3 and in Section 6.4 to obtain the mentioned second algorithm and the third algorithm, respectively.

▶ Remark 22. Mayr and Meyer described in [17] a family of Petri nets that exhibits finite reachability sets whose size grows as the Ackermann function; hence also the size of the maximal configurations in these sets grows similarly. Concerning the size of minimal configurations, we cannot deduce any interesting size properties using the same family. However, by using the family of Petri nets recently introduced in [14, 4, 12] for proving that the reachability problem is Ackermann-hard, we can observe that the maximal size of minimal configurations in Petri net reachability sets grows at least as the Ackermann function.

6.1 Petri Net Reachability Problem in Fixed Dimension

Here we recall some definitions in order to state that the Petri net reachability problem is primitive-recursive when restricted to a fixed dimension, and Ackermannian in general.

The *fast-growing functions* $F_d : \mathbb{N} \to \mathbb{N}$, $d \in \mathbb{N}$, are defined inductively: $F_0(n) = n + 1$, and $F_{d+1}(n) = F_d^{(n+1)}(n)$; by $f^{(n)}$, for a function $f : \mathbb{N} \to \mathbb{N}$, we denote the iteration of f by itself n times (i.e., $f^{(n+1)} = f^{(n)} \circ f$). Following [18], we introduce the class $\mathbf{F}_d$ of functions computable in time $O(F_d(F_{d-1}^{(c)}(n)))$ where n is the size of the input and $c \in \mathbb{N}$ is any constant. We recall that $\bigcup_{d\in\mathbb{N}} \mathbf{F}_d$ is the class of *primitive-recursive functions*. We also introduce the function $F_\omega : \mathbb{N} \to \mathbb{N}$ defined by $F_\omega(n) = F_n(n)$, which is a variant of the Ackermann function; by $\mathbf{F}_\omega$ we denote the class of functions computable in time $O(F_\omega(F_d(n)))$ where $d \in \mathbb{N}$ is any constant and n is the size of the input. A function in $\mathbf{F}_\omega$ is said to be computable in *Ackermannian time*. (We note that Ackermannian time coincides with Ackermannian space.)

For $\mathbf{x} \in \mathbb{N}^d$ we have defined the norm of $\mathbf{x}$ as $\|\mathbf{x}\| = \sum_{i=1}^{d} \mathbf{x}(i)$. Now we extend the notion of norm to other objects. For a *Petri net action* $a = (\mathbf{a}_-, \mathbf{a}_+)$, by its *norm* we mean $\|a\| = \max\{\|\mathbf{a}_-\|, \|\mathbf{a}_+\|\}$. For a *Petri net* A, by its *norm* we mean $\|A\| = \max_{a\in A} \|a\|$. The *norm* of a linear set $\mathbf{L} \subseteq \mathbb{N}^d$ implicitly given by a presentation $(\mathbf{b}, \mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_k)$ is defined by $\|\mathbf{L}\| = \max\{\|\mathbf{b}\|, \|\mathbf{p}_1\|, \|\mathbf{p}_2\|, \ldots, \|\mathbf{p}_k\|\}$. The *norm* of a semilinear set $\mathbf{S} \subseteq \mathbb{N}^d$ implicitly given by a sequence of presentations of $\mathbf{L}_1, \mathbf{L}_2, \ldots, \mathbf{L}_m$ is defined by $\|\mathbf{S}\| = \max_{1\le n\le m} \|\mathbf{L}_n\|$.

Now we recall a result showing that the reachability problem restricted to Petri nets of dimension d is in $\mathbf{F}_{d+4}$, and that the general Petri net reachability problem is in $\mathbf{F}_\omega$. (We view a decision problem as a function with the co-domain $\{0, 1\}$.) This result is crucial for us to derive the upper bound in Theorem 3.

▶ **Theorem 23** ([15]). *There is a constant $c > 0$ such that for all $d, n, A, \mathbf{x}, \mathbf{y}$ where $d, n \in \mathbb{N}$, A is a Petri net of dimension d, $\mathbf{x}, \mathbf{y} \in \mathbb{N}^d$, and the norms of $A, \mathbf{x}, \mathbf{y}$ are bounded by n, we have that if $\mathbf{x} \xrightarrow{A^*} \mathbf{y}$, then $\mathbf{x} \xrightarrow{\sigma} \mathbf{y}$ for a word $\sigma \in A^*$ such that $|\sigma| \le F_{d+4} \circ F_{d+3}^{(c)}(n)$.*

We remark that in what follows we formulate some results in the form

“There is a constant $c' > 0$ such that...”

Naturally we could replace c' with c without changing the meaning of the respective statements, but we prefer keeping the difference in order to highlight the special role of the constant c introduced in Theorem 23.

6.2 Minimal Reachable Configurations

We provide an algorithm computing the set of minimal reachable configurations, by following the approach of [21]. To ease notation, we introduce the functions $f_d = F_{d+4} \circ F_{d+3}^{(c)}$ $(d \in \mathbb{N})$ where c is the constant introduced in Theorem 23, and we first prove the following proposition; for $\mathbf{v} \in \mathbb{N}_\omega^d$, by its *norm* we mean $\|\mathbf{v}\| = \sum_{i:\mathbf{v}(i)\neq\omega} \mathbf{v}(i)$.

▶ **Proposition 24.** *For all d, n, A, $\mathbf{x}$, $\mathbf{v}$, where $d, n \in \mathbb{N}$, A is a Petri net of dimension d, $\mathbf{x} \in \mathbb{N}^d$, $\mathbf{v} \in \mathbb{N}_\omega^d$, and the norms of $A, \mathbf{x}, \mathbf{v}$ are bounded by n, we have that if $(\textsc{post}_A^*(\mathbf{x}) \cap \downarrow\mathbf{v})$ is nonempty, then there is $\mathbf{y} \in (\textsc{post}_A^*(\mathbf{x}) \cap \downarrow\mathbf{v})$ such that $\mathbf{x} \xrightarrow{\sigma} \mathbf{y}$ for some $\sigma \in A^*$ where $|\sigma| \leq f_d(n)$.*

Proof. For $n = 0$ the claim is trivial, so we assume $n \geq 1$.

For each $j \in [1, d]$ we define the Petri net action $b_j = (\mathbf{i}_j, \mathbf{0})$ where $\mathbf{i}_j(j) = 1$ and $\mathbf{i}_j(i) = 0$ for all $i \in [1, d] \setminus \{j\}$; this action decrements the jth component of configurations. We put $I_\omega = \{j \mid j \in [1, d], \mathbf{v}(j) = \omega\}$, and by B we denote the Petri net $\{b_j \mid j \in I_\omega\}$. Since $n \geq 1$, we derive $\|A \cup B\| \leq n$.

Let us now assume a configuration $\mathbf{z} \in (\textsc{post}_A^*(\mathbf{x}) \cap \downarrow\mathbf{v})$. Let $\mathbf{c}$ be the configuration arising from $\mathbf{z}$ by replacing the components in I_ω with zero; we thus have $\|\mathbf{c}\| \leq \|\mathbf{v}\| \leq n$ (using the fact that $\mathbf{c} \leq \mathbf{z}$, and thus $\mathbf{c} \in \downarrow\mathbf{v}$).

From $\mathbf{x} \xrightarrow{A^*} \mathbf{z}$ and $\mathbf{z} \xrightarrow{B^*} \mathbf{c}$ we derive $\mathbf{x} \xrightarrow{(A\cup B)^*} \mathbf{c}$. By Theorem 23 we deduce that $\mathbf{x} \xrightarrow{u} \mathbf{c}$ for some word $u \in (A \cup B)^*$ for which $|u| \leq f_d(n)$. Since Petri net actions in B only decrease some components, we can assume that all these actions in u are at the end; hence $u = \sigma v$ where $\sigma \in A^*$ and $v \in B^*$, and we have $\mathbf{x} \xrightarrow{\sigma} \mathbf{y} \xrightarrow{v} \mathbf{c}$ for a configuration $\mathbf{y} \in \textsc{post}_A^*(\mathbf{x})$. Since $\mathbf{c} \leq \mathbf{z}$, $\mathbf{z} \in\downarrow\mathbf{v}$, and $\mathbf{y} \xrightarrow{v} \mathbf{c}$ only decreases the components that are ω in $\mathbf{v}$, we deduce that $\mathbf{y} \in\downarrow\mathbf{v}$. ◀

To ease the formulation of the next proposition, we define the functions $g_d : \mathbb{N} \to \mathbb{N}$ by $g_d(n) = n \cdot (\,2 + f_d(n)\,)$, for all $d \in \mathbb{N}$.

▶ **Proposition 25.** *For all $d, n, A, \mathbf{x}, \mathbf{v}, \mathbf{m}$, where $d, n \in \mathbb{N}$, A is a Petri net of dimension d, $\mathbf{x} \in \mathbb{N}^d$, $\mathbf{v} \in \mathbb{N}_\omega^d$, $\mathbf{m}$ belongs to $\min(\textsc{post}_A^*(\mathbf{x}) \cap \downarrow\mathbf{v})$, and the norms of $A, \mathbf{x}, \mathbf{v}$ are bounded by n, there exists a word $\sigma \in A^*$ such that $\mathbf{x} \xrightarrow{\sigma} \mathbf{m}$ and $|\sigma| \leq f_d \circ g_d^{(k)}(n)$ where $k = |\{i \mid \mathbf{v}(i) = \omega\}|$.*

Proof. The strict version $<$ of the relation $\leq$ on $\mathbb{N}_\omega^d$ (defined by $\mathbf{w} < \mathbf{v}$ if $\mathbf{w} \leq \mathbf{v}$ and $\mathbf{w} \neq \mathbf{v}$) is clearly well-founded. We use this property for an inductive proof.

We aim to show the claim for a considered tuple $d, n, A, \mathbf{x}, \mathbf{v}, \mathbf{m}$, while we can assume that the claim is valid for $d, n', A, \mathbf{x}, \mathbf{w}, \mathbf{m'}$ for all $\mathbf{w} < \mathbf{v}$ and all $\mathbf{m'} \in \min(\textsc{post}_A^*(\mathbf{x}) \cap \downarrow\mathbf{w})$.

Since $\mathbf{m}$ is in $(\textsc{post}_A^*(\mathbf{x}) \cap \downarrow\mathbf{v})$, we deduce from Proposition 24 that we can fix $\mathbf{y} \in (\textsc{post}_A^*(\mathbf{x}) \cap \downarrow\mathbf{v})$ and a word $\sigma \in A^*$ such that $\mathbf{x} \xrightarrow{\sigma} \mathbf{y}$ and $|\sigma| \leq f_d(n)$; we thus have $\|\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{A}\| \cdot |\sigma| \leq g_d(n) - n$. If $\mathbf{m} = \mathbf{y}$, then the claim is proved; so we assume that $\mathbf{m} \neq \mathbf{y}$.

By Observation 12 we can fix $\mathbf{w} \in \delta_{\mathbf{y}}(\mathbf{v})$ such that $\mathbf{m} \in \min(\textsc{post}_A^*(\mathbf{x}) \cap \downarrow\mathbf{w})$; since $\mathbf{w} \in \delta_{\mathbf{y}}(\mathbf{v})$, we have $\mathbf{w} < \mathbf{v}$. By the induction hypothesis, there is a word $\sigma' \in A^*$ such that $\mathbf{x} \xrightarrow{\sigma'} \mathbf{m}$ and $|\sigma'| \leq f_d \circ g_d^{(k')}(n')$ where $n' = \max\{\|A\|, \|\mathbf{x}\|, \|\mathbf{w}\|\})$ and $k' = |\{i \mid \mathbf{w}(i) = \omega\}|$.

Putting $k = |\{i \mid \mathbf{v}(i) = \omega\}|$, we observe that $k' = k$ or $k' = k - 1$. If $k' = k$, then $\|\mathbf{w}\| < \|\mathbf{v}\|$ and we are done by monotonicity of f_d and g_d. Otherwise $k' = k - 1$ and in that case $\|\mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{y}\| \leq g_d(n)$ since in that case $\mathbf{w}$ is obtained from $\mathbf{v}$ by replacing component i of $\mathbf{v}$ for some i such that $\mathbf{v}(i) = \omega$ and $\mathbf{y}(i) > 0$ by $\mathbf{y}(i) - 1$. It follows that $n' \leq g_d(n)$ and we are done also in that case by monotonicity of f_d and g_d. ◀

Finally, by instantiating the previous proposition with $\mathbf{v} = (\omega, \omega, \ldots, \omega)$, and by bounding $f_d \circ g_d^{(d)}(n)$ by $F_{d+5}(c'n)$ for some constant $c' > 0$ independent of d, n, we deduce the following two corollaries.

▶ **Corollary 26.** *There is a constant $c' > 0$ such that for all $d, n, A, \mathbf{x}, \mathbf{m}$, where $d, n \in \mathbb{N}$, A is a Petri net of dimension d, $\mathbf{x} \in \mathbb{N}^d$, $\mathbf{m}$ belongs to $\min(\textit{POST}^*_A(\mathbf{x}))$, and the norms of $A, \mathbf{x}$ are bounded by n, there exists a word $\sigma \in A^*$ such that $\mathbf{x} \xrightarrow{\sigma} \mathbf{m}$ and $|\sigma| \leq F_{d+5}(c'n)$.*

▶ **Corollary 27.** *There is a constant $c' > 0$ such that for all $d, n, A, \mathbf{x}$, where $d, n \in \mathbb{N}$, A is a Petri net of dimension d, $\mathbf{x} \in \mathbb{N}^d$, and the norms of $A, \mathbf{x}$ are bounded by n, the set $\min(\textit{POST}^*_A(\mathbf{x}))$ is computable in time exponential in $F_{d+5}(c'n)$ and the norms of vectors in that set are bounded by $n \cdot (1 + F_{d+5}(c'n))$.*

Proof. In fact, the set of minimal reachable configurations can be obtained by exploring configurations reachable from $\mathbf{x}$ by sequences of at most $F_{d+5}(c'n)$ actions in A. We note that the norms of configurations reachable in this way are bounded by $\|\mathbf{x}\| + F_{d+5}(c'n) \cdot \|A\| \leq n \cdot (1 + F_{d+5}(c'n))$. ◀

6.3 Extension to Semilinear Sets

The algorithm computing minimal reachable configurations can be also simply used for computing the set $\min(\text{POST}^*_A(\mathbf{x}) \cap \mathbf{S})$ where $\mathbf{S}$ is a semilinear set; we thus formulate this fact as a corollary (though with providing a proof). We recall that the norm of a semilinear set is the maximum norm of vectors occurring in its (implicitly assumed) presentation.

▶ **Corollary 28.** *There is a constant $c' > 0$ such that for all $d, n, A, \mathbf{x}, \mathbf{S}$, where $d, n \in \mathbb{N}$, A is a Petri net of dimension d, $\mathbf{x} \in \mathbb{N}^d$, $\mathbf{S}$ is (a presentation of) a semilinear set $\mathbf{S} \subseteq \mathbb{N}^d$, and the norms of $A, \mathbf{x}, \mathbf{S}$ are bounded by n, the set $\min(\textit{POST}^*_A(\mathbf{x}) \cap \mathbf{S})$ is computable in time exponential in $F_{2d+6}(c'n)$ and the norms of vectors in that set are bounded by $n \cdot (1 + F_{2d+6}(c'n))$.*

Proof. Let us consider a d-dimensional Petri net A, an initial configuration $\mathbf{x}$, and a semilinear set $\mathbf{S} \subseteq \mathbb{N}^d$ given as the union of linear sets $\mathbf{L}_1, \mathbf{L}_2, \ldots, \mathbf{L}_m$. Since $\min(\text{POST}^*_A(\mathbf{x}) \cap \mathbf{S}) = \min(\bigcup_{j=1}^m \min(\text{POST}^*_A(\mathbf{x}) \cap \mathbf{L}_j))$ we can reduce the problem of computing $\min(\text{POST}^*_A(\mathbf{x}) \cap \mathbf{S})$ to the special case of a linear set $\mathbf{S}$, denoted as $\mathbf{L}$ in the sequel. So, let $\mathbf{L}$ be a linear set presented by a basis $\mathbf{b} \in \mathbb{N}^d$ and a sequence of periods $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_k \in \mathbb{N}^d$, and let us provide an algorithm for computing $\min(\text{POST}^*_A(\mathbf{x}) \cap \mathbf{L})$.

To do so, we build from A a new Petri net B of dimension $2d+1$ defined as follows and an initial configuration $(\mathbf{x}, 1, \mathbf{0})$. We associate to each Petri net action $a \in A$ of the form $(\mathbf{a}_-, \mathbf{a}_+)$ the action $((\mathbf{a}_-, 1, \mathbf{0}), (\mathbf{a}_+, 1, \mathbf{0}))$ in B that intuitively executes a on the first d counters and check that the middle counter (the counter $d+1$) is at least 1. We also add in B for each $j \in [1, k]$ an action $((\mathbf{p}_j, 0, \mathbf{0}), (\mathbf{0}, 0, \mathbf{p}_j))$ that removes the period $\mathbf{p}_j$ on the first d counters and adds it on the last d counters. Finally, we add to B the action $((\mathbf{b}, 1, \mathbf{0}), (\mathbf{0}, 0, \mathbf{b}))$ that decrements the middle counter and simultaneously removes $\mathbf{b}$ from the first d counters, and adds $\mathbf{b}$ on the last d counters. Since for any set $\mathbf{X} \subseteq \mathbb{N}^d$ and any set $I \subseteq [1, d]$, the set $\min(\{\mathbf{x} \in \mathbf{X} \mid \bigwedge_{i \in I} \mathbf{x}(i) = 0\})$ is equal to $\{\mathbf{m} \in \min(\mathbf{X}) \mid \bigwedge_{i \in I} \mathbf{m}(i) = 0\}$), one can observe that $\{\mathbf{0}\} \times \{0\} \times \min(\text{POST}^*_A(\mathbf{x}) \cap \mathbf{L})$ is equal to $\min(\text{POST}^*_B(\mathbf{x}, 1, \mathbf{0})) \cap (\{\mathbf{0}\} \times \{0\} \times \mathbb{N}^d)$. ◀

6.4 Extension to Semilinear Predicates

By another corollary (with a proof) we also note that the algorithm computing minimal reachable configurations can be used for computing minimal vectors in sets of the following form

$$\mathbf{X} = \{\mathbf{x} \in \mathbb{N}^h \mid \exists \alpha, \beta \in \mathbb{N}^d : \alpha \xrightarrow{A^*} \beta \wedge (\mathbf{x}, \alpha, \beta) \in P\} \tag{3}$$

where $P \subseteq \mathbb{N}^h \times \mathbb{N}^d \times \mathbb{N}^d$ is a semilinear predicate given by a presentation. Notice that we use Greek letters α and β in the definition of $\mathbf{X}$ in order to emphasise vectors that act as configurations of the Petri net A.

▶ **Corollary 29.** *There is a constant $c' > 0$ such that for all d, h, n, A, P, where $d, h, n \in \mathbb{N}$, A is a Petri net of dimension d, $\mathbf{x} \in \mathbb{N}^d$, P is (a presentation of) a semilinear predicate $P \subseteq \mathbb{N}^h \times \mathbb{N}^d \times \mathbb{N}^d$, and the norms of $A, \mathbf{x}, P$ are bounded by n, the set of minimal elements of the set $\mathbf{X}$ denoted by equation (3) is computable in time exponential in $F_{2h+4d+6}(c'n)$ and the norms of these minimal elements are bounded by $n \cdot (1 + F_{2h+4d+6}(c'n))$.*

Proof. We first introduce the set Y defined as $Z \cap P$ where

$$Z = \{(\mathbf{x}, \alpha, \beta) \in \mathbb{N}^h \times \mathbb{N}^d \times \mathbb{N}^d \mid \alpha \xrightarrow{A^*} \beta\}.$$

Since $\min(\mathbf{X}) = \min\{\mathbf{x} \in \mathbb{N}^k \mid \exists \alpha, \beta \in \mathbb{N}^d : (\mathbf{x}, \alpha, \beta) \in \min(Y)\}$ it is sufficient to provide an algorithm computing $\min(Y)$.

Our algorithm is based on the fact that Z is the reachability set of a $(h + 2d)$-dimensional Petri net B starting from the zero configuration and defined as follows from A. By $\mathbf{i}_i$ we denote the vector in $\mathbb{N}^h$ defined by $\mathbf{i}_i(i) = 1$ and $\mathbf{i}_i(j) = 0$ if $j \in [1, h] \backslash \{i\}$. The Petri net B is defined as the actions $((\mathbf{0}, \mathbf{0}, \mathbf{0}), (\mathbf{i}_j, \mathbf{0}, \mathbf{0}))$ where $j \in [1, h]$ that increment the counters corresponding to $\mathbf{x}$, actions $((\mathbf{0}, \mathbf{0}, \mathbf{0}), (\mathbf{0}, \mathbf{i}_j, \mathbf{i}_j))$ that increment simultaneously by the same amount the counters corresponding to α and β, and actions obtained from A that simulate the computation of A on the counters β and defined for each action a of A of the form $(\mathbf{a}_-, \mathbf{a}_+)$ by the action $((\mathbf{0}, \mathbf{0}, \mathbf{a}_-), (\mathbf{0}, \mathbf{0}, \mathbf{a}_+))$ in B. Notice that $Z = \textsc{post}_B^*(\mathbf{0}, \mathbf{0}, \mathbf{0})$ and we are done by Corollary 28. ◀

7 Complexity of the Semilinear Home-Space Problem

In this section we provide an Ackermannian complexity upper-bound for deciding the semilinear home-space problem; Theorem 3 will thus be proven.

So let $A, \mathbf{X}, \mathbf{H}$ be an instance of the semilinear home-space problem where A is a Petri net, of dimension d, and $\mathbf{X}, \mathbf{H}$ are two (presentations of) semilinear subsets of $\mathbb{N}^d$. Since $\mathbf{H}$ can be decomposed, in elementary time, into a finite union of linear sets using presentations with at most d periods [7, Lemma 6.6], we can assume that each linear set $\mathbf{L}$ of the presentation of $\mathbf{H}$ satisfies this constraint. We put $n = \max\{\|A\|, \|\mathbf{X}\|, \|\mathbf{H}\|\}$.

We first consider the problem of computing a semilinear non-reachability core for each linear set $\mathbf{L}$ of the presentation of $\mathbf{H}$. Such a linear set $\mathbf{L}$ is presented with a basis $\mathbf{b}$ and a sequence of k periods $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_k$ with $k \leq d$. As previously shown, this computation reduces to the computation of the minimal elements of the upward closed set DCB-PR and the upward-closed sets $\mathbf{U}_\mathbf{y}$ where $\mathbf{y}$ belongs to the finite set of vectors in $\mathbb{N}^d$ satisfying $\|\mathbf{y}\| \leq \|\mathbf{b}\|$. The computation of those minimal elements can be obtained by rewriting the definitions of DCB-PR and $\mathbf{U}_\mathbf{y}$ to match the statement of Corollary 29. To do so, we note that DCB-PR and $\mathbf{U}_\mathbf{y}$ can be described in the following way:

$$\textsc{dcb-pr} = \{(\mathbf{y},\mathbf{u}) \in \mathbb{N}^d \times \mathbb{N}^k \mid \exists \alpha,\beta \in \mathbb{N}^d : \alpha \xrightarrow{A^*} \beta \wedge (\mathbf{y},\mathbf{u},\alpha,\beta) \in P\}$$
$$\mathbf{U_y} = \{\mathbf{u} \in \mathbb{N}^k \mid \exists \alpha,\beta \in \mathbb{N}^d : \alpha \xrightarrow{A^*} \beta \wedge (\mathbf{u},\alpha,\beta) \in P_\mathbf{y}\}$$

where:

$$P = \left\{ (\mathbf{y},\mathbf{u},\alpha,\beta) \in \mathbb{N}^d \times \mathbb{N}^k \times \mathbb{N}^d \times \mathbb{N}^d \mid \exists (\mathbf{y}',\mathbf{u}') \in \mathbb{N}^d \times \mathbb{N}^k : \begin{array}{l} \|\mathbf{y}\| > \|\mathbf{y}'\| \wedge \\ \alpha = \textsc{conf}(\mathbf{y},\mathbf{u}) \wedge \\ \beta = \textsc{conf}(\mathbf{y}',\mathbf{u}') \end{array} \right\}$$
$$P_\mathbf{y} = \{(\mathbf{u},\alpha,\beta) \in \mathbb{N}^k \times \mathbb{N}^d \times \mathbb{N}^d \mid \alpha = \textsc{conf}(\mathbf{y},\mathbf{u}) \wedge \beta \in \mathbf{L}\}.$$

Since the sets P and $P_\mathbf{y}$ are clearly expressible by formulas in Presburger arithmetic, we can effectively construct, in elementary time, semilinear presentations of those sets [8]. We introduce an elementary function E (independent of any instance) corresponding to that computation. We deduce that for some constant $c' > 0$, independent of any input, we can compute, in time exponential in $F_{8d+6}(c'E(n))$, the sets $\min(\textsc{dcb-pr})$ and $\min(\mathbf{U_y})$ for $\|\mathbf{y}\| \leq \|\mathbf{b}\|$. Moreover, the norms of vectors in those sets are bounded by $F_{8d+6}(c'E(n))$. It follows from the proof of Proposition 21 that there exists an elementary function E' (independent of any instance) such that we can compute, in time $E'(F_{8d+6}(c'E(n)))$, a (presentation of a) semilinear non-reachability core $\mathbf{C}$ for each linear set $\mathbf{L}$ of the presentation of $\mathbf{H}$.

Let $\mathbf{L}_1, \mathbf{L}_2, \ldots, \mathbf{L}_m$ be the presentation sequence of $\mathbf{H}$, and let $\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_m$ be the respective semilinear non-reachability cores computed for $\mathbf{L}_1, \mathbf{L}_2, \ldots, \mathbf{L}_m$, respectively, as shown in the previous paragraph. Proposition 9 shows that $\mathbf{H}$ is not a home-space for $\mathbf{X}$ if, and only if, there is an execution

$$\mathbf{x}_0 \xrightarrow{A^*} \mathbf{x}_1 \xrightarrow{A^*} \mathbf{x}_2 \cdots \xrightarrow{A^*} \mathbf{x}_m \tag{4}$$

where $\mathbf{x}_0 \in \mathbf{X}$, and $\mathbf{x}_i \in \mathbf{C}_i$ for each $i \in [1,m]$.

The existence of such an execution can be decided by Proposition 10, by a reduction to the reachability problem for a Petri net of a dimension that is elementary in $\max\{d,m,n\}$. Theorem 23 thus entails that the semilinear home-space problem is decidable in Ackermannian time, which finishes the proof of Theorem 3.

8 Concluding Remarks

There are various issues that can be elaborated and added to the presented material. One such issue was mentioned in Remark 6, dealing with strengthening the lower bound.

We can also look for positive witnesses of the home-space property; e.g., we anticipate that given a Petri net A of dimension d, and two semilinear sets $S_0, S_1 \subseteq \mathbb{N}^d$, we have $\textsc{post}^*_A(S_0) \subseteq \textsc{pre}^*_A(S_1)$ (i.e., S_1 is a home-space for S_0) iff there is a semilinear set S' such that $\textsc{post}^*_A(S_0) \subseteq S' \subseteq \textsc{pre}^*_A(S_1)$.

Best and Esparza [2] consider the "existential" home-space problem that asks, given a Petri net A of dimension d and an initial configuration $\mathbf{x}$, if there exists a singleton home-space for $\{\mathbf{x}\}$; the main result of [2] shows that this existential problem is decidable. We can consider a related problem that asks, given A and $\mathbf{x}$, if there is a semilinear home-space included in $\textsc{post}^*_A(\mathbf{x})$; currently we have no answer to the respective decidability question.

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