NORTH-HOLLAND

On the Minors of an Incidence Matrix and Its Smith Normal Form

Jerrold W. Grossman

Department of Mathematical Sciences
Oakland University
Rochester, Michigan 48309-4401

Devadatta M. Kulkarni*

Department of Mathematical Sciences
Oakland University
Rochester, Michigan 48309-4401

and

Department of Mathematics
University of Poona
Pune 411007, India

and

Irwin E. Schochetman*

Department of Mathematical Sciences
Oakland University
Rochester, Michigan 48309-4401

Submitted by Richard A. Brualdi

ABSTRACT

Consider the vertex-edge incidence matrix of an arbitrary undirected, loop-

*Partially supported by an Oakland University Research Fellowship.

655 Avenue of the Americas, New York, NY 10010

0024-3795/95/$9.50
SSDI 0024-3795(93)00173-W

less graph. We completely determine the possible minors of such a matrix. These depend on the maximum number of vertex-disjoint odd cycles (i.e., the *odd tulgeity*) of the graph. The problem of determining this number is shown to be NP-hard. Turning to maximal minors, we determine the rank of the incidence matrix. This depends on the number of components of the graph containing no odd cycle. We then determine the maximum and minimum absolute values of the maximal minors of the incidence matrix, as well as its Smith normal form. These results are used to obtain sufficient conditions for relaxing the integrality constraints in integer linear programming problems related to undirected graphs. Finally, we give a sufficient condition for a system of equations (whose coefficient matrix is an incidence matrix) to admit an integer solution.

---

1. INTRODUCTION AND BACKGROUND

Consider an undirected, loopless graph G with n vertices and m edges. (We will always assume that G has at least one edge.) Let the $n \times m$ vertex edge incidence matrix of G be denoted by $A = A(G) = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1 & \text{if vertex } i \text{ is incident with edge } j, \\ 0 & \text{if not.} \end{cases}$$

Note that an $n \times m$ matrix of 0's and 1's is the incidence matrix of some undirected graph on n vertices and m edges if and only if each of its column sums is 2. (We have imposed the requirement that G contain no loops for simplicity; with appropriate modifications in the definitions and proofs, it could be omitted.)

Suppose that M is a $p \times p$ submatrix of A, where necessarily $1 \leq p \leq \min(m, n)$. In this paper we investigate the possible values that the determinant of M can attain. Specifically, in Section 2 we show that such a minor is either 0 or of the form $\pm 2^k$ for k a nonnegative integer. The exponent k is shown to be bounded by the number of vertex-disjoint odd cycles in G. Recall [6] that the number of disjoint cycles of a graph G has been called the *tulgeity* $\tau = \tau(G)$ (or *point-cycle multiplicity*) of the graph and has been studied by several authors [7, 8, 11, 21]. Thus, the constant k is bounded by the *odd tulgeity* $\tau_0 = \tau_0(G)$ of G, i.e., τ_0 is defined to be the maximum number of vertex-disjoint odd cycles in G. Moreover, for each k, $0 \leq k \leq \tau_0$, there exists a minor equal to $\pm 2^k$. As an immediate corollary, we obtain the well-known [13, p. 7] result that the incidence matrix A of an undirected graph G is totally unimodular (i.e., every minor is 0 or ± 1) if and only if G is bipartite. (Recall that $\tau_0 = 0$, i.e., G has no odd cycles, if and only if G is bipartite. Moreover, if G is assumed to be a *directed* graph, then its (appropriately defined) incidence matrix is well known to be totally unimodular [13, p. 6]. We also show that the problem of determining $\tau_0(G)$ is *NP*-hard.

Next we consider the maximal minors of A. We show that the rank of A is $n - c_0$, where $c_0 = c_0(G)$ is defined to be the number of connected components of G that contain no odd cycles. (See [17, 18] for earlier evaluations of the rank of A.) The maximal minor of A with *maximum* absolute value is shown to be 2^{τ_0}. Analogously, the nonsingular maximal minor of A with *minimum* absolute vlaue is 2^{c_1}, where $c_1 = c_1(G)$ is defined to be the number of connected components of G that contain at least one odd cycle. (Thus, $c_0 + c_1$ is the number of components of G.) Clearly, $\tau_0 \geq c_1$. Moreover, for each k, $c_1 \leq k \leq \tau_0$, there exists a maximal minor whose absolute value is 2^k.

As a consequence of the above, in Section 3 we are able to obtain the Smith normal form of A. It is the (nonsquare) diagonal matrix with diagonal given by a string of $n - c$ copies of 1, followed by c_1 copies of 2, followed by the appropriate number of 0's.

Our results in Section 2 may be applied to integer linear programming problems associated with undirected graphs. In particular, in Section 4 we obtain sufficient conditions on the problem data for the integrality constraints to be relaxed, i.e., for the problem to be an ordinary linear programming problem. Thus, the problem is solvable by linear programming methods, as opposed to *integer* linear programming methods.

It is well known that if $\tau_0 = 0$, i.e., A is totally unimodular, then the system of equations $Ax = b$, for b an integer m-tuple, has an integer solution (if it has a solution). We conclude with a sufficient condition for this system to have an integer solution in the event that $\tau_0 \geq 1$, in which case A is not totally unimodular.

2. RANK AND MINORS

We begin with a special case of our main result in this section. Note that $c_1 > 0$ if and only if G contains an odd cycle.

THEOREM 2.1. *Suppose that G is a loopless, connected graph with n vertices and n edges.*

Then

$$\det A = \begin{cases} \pm 2 & \text{if} \quad c_1 > 0, \\ 0 & \text{if} \quad c_1 = 0. \end{cases}$$

Proof. The hypothesis imply that G is unicyclic, consisting of one cycle with (possibly trivial) rooted trees "growing" out of each vertex of the cycle. Note that every nontrivial rooted tree has a leaf with degree 1. To compute $\det A$, we can expand along any row containing exactly one 1. This corresponds to deleting from G a vertex of degree 1 and its incident edge. Thus, up to sign, the determinant of A is equal to the determinant of the incidence matrix of a smaller unicyclic graph. Ultimately, we must compute the determinant of the incidence matrix $A(C_p)$ of a cycle C_p for some p, $2 \leq p \leq n$. Without loss of generality, we may take

$$A(C_p) = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}.$$

By expanding about the first row, we see that

$$\det A(C_p) = \begin{cases} 2 & \text{if} \quad p \text{ is odd}, \\ 0 & \text{if} \quad p \text{ is even}. \end{cases}$$

Thus, either $\det A = 0$ or $\det A = \pm 2$, according as the unique cycle in A is even or odd. ■

The following is our main result in this section.

THEOREM 2.2. *Let G be a loopless graph. Let M be a $p \times p$ submatrix of A. Then* $\det M$ *is either* 0 *or* $\pm 2^k$, *where k is an integer satisfying* $0 \leq k \leq \tau_0$. *Moreover, for each such k, there exists a minor with absolute value equal to* 2^k.

Proof. The proof of the first claim is by induction on p, for fixed G. For $p = 1$, we have $\det M = 0$ or $\det M = 1 = 2^0$. Suppose that the theorem is true for $1, \ldots, p-1$, where $p > 1$. If M has a row or column containing no 1's, then $\det M = 0$. If M has a row or column containing exactly one 1, then expanding about this element, we obtain $\det M = \pm \det M'$, where M' is the $(p-1) \times (p-1)$ submatrix of M obtained by deleting the appropriate row and column. By inductive assumption, $\det M' = 0$ or $\det M' = \pm 2^k$ for some k, $0 \le k \le \tau_0$.

Thus, we may assume that each row and column of M has at least two 1's. Since the columns of M also have at most two 1's, it follows immediately that each row and column contains exactly two 1's. Thus, M is the incidence matrix of a graph H with p edges and p vertices, where each vertex has degree 2. It follows that H is the disjoint union of k cycles H_i, $i = 1, \ldots, k$, for some k.

Consequently, the incidence matrix M of H can be written in block diagonal form as

$$M = \begin{bmatrix} M_1 & 0 & 0 & \cdots & 0 \\ 0 & M_2 & 0 & \cdots & 0 \\ 0 & 0 & M_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & M_k \end{bmatrix},$$

where each M_i is a square submatrix of $M, i = 1, \ldots, k$, namely, the incidence matrix $A(H_i)$ of H_i. We know that $\det M_i = 0$ (if H_i is an even cycle) or ± 2 (if H_i an odd cycle). Clearly,

$$\det M = \pm \prod_{i=1}^{k} \det M_i.$$

Thus, if one of the H_i is an even cycle, then $\det M = 0$; however, if all the H_i are odd cycles, then $\det M = \pm 2^k$. In the latter case, it is clear that $k \le \tau_0$. Note also that necessarily $k \le p/3$, since each odd cycle has length at least 3.

Finally, if $1 \le k \le \tau_0$, then there exist k vertex-disjoint odd cycles in G. Let M be the square submatrix of A which is the incidence matrix corresponding to the union of these cycles. Then, as above, the determinant of M is $\pm 2^k$. ■

COROLLARY 1. *The least common multiple of the (absolute values of the) nonsingular minors of A is* 2^{τ_0}.

COROLLARY 2. *If G is Eulerian with an odd number of edges, then A is not totally unimodular.*

Proof. In this case, there must exist an odd cycle, since an Eulerian graph can be decomposed into edge-disjoint cycles. ■

As a consequence of Theorem 2.2, we obtain the following well-known result.

THEOREM 2.3. *The undirected graph G is bipartite if and only if its incidence matrix A is totally unimodular.*

Proof. Suppose G is bipartite, so that $\tau_0 = 0$. Hence, each minor of A is either 0 or $\pm 2^0 = \pm 1$, so A is totally unimodular. Conversely, if G is not bipartite, then necessarily $\tau_0 \geq 1$. Hence, A has a minor of absolute value greater than 1, i.e., A is not totally unimodular. ■

THEOREM 2.4. *The problem of determining the odd tulgeity τ_0 of a graph G is NP-hard.*

Proof. If we could compute τ_0 in polynomial time, then as a special case, we could determine whether $\tau_0 = t$ for a given graph G with $3t$ vertices. This would solve the Partition-into-triangles problem, which is well known to be NP-complete [12, p. 68]. ■

Next we turn to consideration of the *maximal* minors of A, i.e., the determinants of the submatrices of A of maximum rank. This requires knowing the rank of A. Recall [3, p. 30] that, if D is a directed graph with underlying undirected graph G, then the rank of the (oriented) incidence matrix of D is $n-c$, where $c = c(G)$ is the number of connected components of G, i.e., $c = c_0 + c_1$. Note that this rank is independent of the particular orientation D of G. Our objective here is to determine the rank of A and compare it with $n - c$. The following is our undirected version of Theorem 2.3.2 of [3].

THEOREM 2.5. *The rank of A is $n - c_0$. In fact, if we delete c_0 rows from A, each one coresponding to a vertex of each of the c_0 components of G without odd cycles, then the resulting matrix has rank $n - c_0$.*

Proof. Clearly the rank of A is the sum of the ranks of the incidence matrices of the connected components of G, so it suffices to assume that G is connected and show that the rank of A is n if G has an odd cycle and $n - 1$ if it does not. If G has an odd cycle (i.e., $c_0 = 0$), then we can use a standard spanning-tree algorithm to find a connected, unicyclic subgraph of G with n vertices, containing this odd cycle. By Theorem 2.1, the determinant of the incidence matrix of this subgraph is ± 2, so this $n \times n$ submatrix of A is nonsingular; it follows that the rank of A is n.

Next suppose that G has no odd cycle (i.e., $c_0 = 1$). Let T be a rooted spanning tree of G, and let M be the $(n-1) \times (n-1)$ submatrix of A

corresponding to T with its root (but not the edges incident to the root) omitted. We can evaluate $\det M$ by repeatedly expanding along the rows corresponding to leaf vertices of T; so $\det M = \pm 1$. Therefore, the rank of A is at least $n-1$. (The second claim in the statement of the theorem follows from this construction.) To see that this rank cannot be n, it is enough to note that any $n \times n$ submatrix of A necessarily contains all the rows and hence is the incidence matrix for a subgraph of G. By Theorem 2.1, its determinant is 0 if G does not contain an odd cycle. ■

THEOREM 2.6. *The maximal minor of A with maximum absolute value is 2^{τ_0}. The nonsingular maximal minor with minimum absolute value is 2^{c_1}. Furthermore, for each k, $c_1 \leq k \leq \tau_0$, there exists a maximal minor whose absolute value is 2^k.*

Proof. By the proof of Theorem 2.2, we can find a subgraph G' of G consisting of the vertex-disjoint union of τ_0 odd cycles. The determinant of the incidence matrix of this subgraph will be $\pm 2^{\tau_0}$; furthermore, no minor has larger absolute value. We can then extend G' to a subgraph G'' with n vertices as follows. In each component of G without an odd cycle, adjoin a rooted spanning tree; the number of such components is c_0. In those components of G with one or more odd cycles, successively adjoin an edge of G from an isolated (in G'') vertex to a nonisolated (in G'') vertex until no isolated (in G'') vertices remain. By Theorem, 2.5 $A(G'')$ has rank $n - c_0$. Let M be the maximal rank submatrix of $A(G'')$ obtained by deleting from $A(G'')$ the c_0 rows corresponding to the roots of the spanning trees in the components without odd cycles. It is easy to see that $\det M = \pm 2^{\tau_0}$.

The second part of the theorem is immediate. The last statement can be obtained from the above construction if we modify it as follows. Select k vertex-disjoint odd cycles from the τ_0 such cycles available, making sure that at least one odd cycle is selected from each of the components of G that contain odd cycles (this is possible because $k > c_1$). Let G' be the subgraph of G consisting of the resulting k odd cycles. Then proceed as above. ■

COROLLARY The greatest common divisor of the (absolute values of the) non-singular maximal minors of A is equal to 2^{c_1}.

3. SMITH NORMAL FORM

In Section 2 we determined the rank and the minors of the incidence matrix A of an undirected, loopless graph G. To characterize the structure of A even further, we next determine its Smith normal form.

Interesting applications of the Smith normal form of the incidence matrix of a design may be found in [2, 4, 5]. The Smith normal form of the Laplacian matrix of a graph is studied in [1], and that of its edge version is used in [15]. Several interesting properties of and polynomial-time algorithms to find the Smith normal form are discussed in [19].

In [20], H. J. S. Smith proved that any integer matrix A can be transformed by elementary integral row and column operations into the form

$$\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix},$$

where $D = \operatorname{diag}(\delta_1, \delta_2, \ldots, \delta_k)$, with $\delta_1, \delta_2, \ldots, \delta_k$ positive integers satisfying $\delta_1 \mid \delta_2 \mid \cdots \mid \delta_k$. In this context, elementary row (or column) operations include interchanging rows (or columns), multiplying a row (or column) by -1, and adding an integral multiple of a row (or a column) to another row (or column). Here, one can see that $\delta_1\delta_2\cdots\delta_i$ is the greatest common divisor of all the $i \times i$ minors of the matrix A, and is invariant under the elementary row or column operations.

LEMMA 3.1. *For each i such that $1 \leq i \leq n - c$, there is an $i \times i$ submatrix of A with determinant equal to ± 1.*

Proof. Let G' be a spanning forest of G having a root in each of the c components. Consider the $(n-c) \times (n-c)$ submatrix M of A corresponding to G' with the c roots deleted. We can evaluate $\det M$ by repeatedly expanding along the rows corresponding to the leaf vertices of what remains, since each such row has just one 1. Thus, $\det M = \pm 1$. For each i such that $1 \leq i \leq n - c$, the $i \times i$ submatrix of M obtained in this expansion has determinant ± 1. ∎

COROLLARY For each i such that $1 \leq i \leq n - c$, the greatest common divisor of all nonsingular $i \times i$ minors of A is 1.

LEMMA 3.2. *Each $i \times i$ minor of A is divisible by 2^{i-n+c} for $i \geq n - c + 1$. If, moreover, $i \leq n - c_0$, then there is an $i \times i$ minor of A equal to $\pm 2^{i-n+c}$.*

Proof. For each nonsingular $i \times i$ submatrix M of A, the number of vertices from each component must equal the number of edges from that component. (This follows from the Frobenius-König theorem). For $i \geq n-c+1$, the substructure G' of G corresponding to M contains all vertices of at least $i-n+c$ components of G. Thus G' contains at least $i-n+c$ disjoint unicyclic subgraphs of G. Since M is nonsingular, G' does not contain an even cycle, and must contain at least $i-n+c$ odd cycles. Hence, 2^{i-n+c} divides the determinant of M, thereby proving the first assertion.

For the second part, let G' be a substructure of G consisting of unicyclic spanning subgraphs having an odd cycle in $i-n+c$ components of G, and rooted spanning trees in the remaining $n-i$ components of G with the roots deleted. There corresponds an $i \times i$ submatrix of A to G' whose determinant is $\pm 2^{i-n+c}$. ■

COROLLARY For each i such that $n-c+1 \leq i \leq n-c_0$, the greatest common divisor of all $i \times i$ minors of A is 2^{i-n+c}.

THEOREM 3.3. *The Smith normal form of A is*

$$\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix},$$

where

$$D = \mathrm{diag}(1, 1, \ldots, 1, 2, 2, \ldots, 2),$$

$n-c$ *is the number of* 1's, *and* c_1 *is the number of* 2's.

Proof. Since the ith diagonal entry of D is the quotient of the greatest common divisor of the $i \times i$ minors of A divided by the greatest common divisor of the $(i-1) \times (i-1)$ minors of A, the theorem follows from the previous corollaries. ■

4. APPLICATION TO INTEGER PROGRAMMING

Suppose A is as above and b is an integer n-tuple. We consider integer linear programs of the form

$$\max\ c^t x$$

subject to

$$\begin{aligned} Ax &\leq b \qquad (\text{or} \quad Ax = b), \\ x &\geq 0, \qquad \text{integer}. \end{aligned}$$

Such a program of inequality type is called a *b-matching problem* [16].

We would like to know when the feasible regions

$$\mathcal{E}(A,b) = \{x \in \Re^m : Ax = b,\ x \geq 0\}$$

and

$$\mathcal{I}(A,b) = \{x \in \Re^m : Ax \leq b,\ x \geq 0\}$$

have the property that all extreme points are integer. If, in addition, the vector variable x is bounded above by some integer m-tuple u, then the program (of inequality type) is called a *bounded b-matching problem.* In this event, we investigate the integer extreme point question for the feasible regions.

$$\mathcal{E}(A,b,u) = \{x \in \Re^m\ :\ Ax = b,\ 0 \leq x \leq u\}$$

and

$$\mathcal{I}(A,b,u) = \{x \in \Re^m\ :\ Ax \leq b,\ 0 \leq x \leq u\},$$

with coefficient matrix $\begin{bmatrix} A \\ I_m \end{bmatrix}$.

These are fundamental problems in combinatorial optimization which are genuine integer programming problems, i.e., not all basic feasible solutions are automatically integer. Each requires additional linear constraints (*matching* or *blossom constraints*) to yield a system for which all the basic feasible solutions are integer. In each case, J. Edmonds [9, 10] has given a polynomially bounded algorithm which shows that the resulting set of constraints defines a polyhedron with all-integer extreme points.

Our objective in this section is to exhibit matching problems where Edmonds's blossom constraints are not required to guarantee integer solutions. In [14], sufficient conditions on the right-hand-side data vectors of general feasible regions were given for them to have the integer extreme-point property. As we shall see, this will be the case here when the right-hand-side data vectors are multiples of 2^{τ_0}.

Recall that the incidence matrix A is totally unimodular (resp. unimodular) if and only if $\tau_0 = 0$, i.e., G contains no odd cycles (equivalently, G is bipartite). However, in general $\tau_0 \geq 0$. Hence, in our b-matching problems, the coefficient matrix A need not be unimodular.

THEOREM 4.1. (i) *If b is a vector multiple of 2^{τ_0} then $\mathcal{I}(A,b)$ and $\mathcal{E}(A,b)$ have integer extreme points only.*

(ii) *If b and u are vector multiples of 2^{τ_0} then $\mathcal{I}(A,b,u)$ and $\mathcal{E}(A,b,u)$ have integer extreme points only.*

Proof. This follows from the results of [14] once it has been observed from Section 2 that the least common multiple of the absolute values of all the nonzero minors of A is 2^{τ_0}. ∎

Finally, we consider the question of when the equation $Ax = b$ has an integer solution. For this purpose, we require the following additional notation. Let $\mathcal{N}$ denote the set of all square, nonsingular submatrices of A. For convenience, let

$$\gamma(A) = \gcd\{\, |\det M| : M \in \mathcal{N}\}.$$

Recall [16, p. 84] that, in general, $Ax = b$ has a solution (in $\Re^m$) if and only if the rank of A is equal to the rank of $[A\ b]$. In addition, by Kronecker's theorem [19, p. 51], $Ax = b$ has an *integer* solution if and only if $\gamma(A) = \gamma([A\ b])$. If $c_1 = 0$ then $\tau_0 = 0$, i.e., A is totally unimodular. Hence, $Ax = b$ has an integer solution (if it has a solution at all), since $\gamma(A) = \gamma([A\ b]) = 1$.

THEOREM 4.2. *Suppose that the rank of A equals the rank of* $[A\ b]$. *If G contains an odd cycle, i.e.,* $c_1 \geq 1$, *and b is a vector multiple of* 2^{c_1}, *then* $Ax = b$ *has an integer solution.*

Proof. By hypothesis, $\gamma(A) = 2^{c_1}$ (Theorem 2.6). It is clear that $\gamma([A\ b])$ divides $\gamma(A)$ in general. Conversely, by hypothesis $\gamma(A)$ also divides $\gamma([A\ b])$ (Corollary to Theorem 2.6). Thus, $\gamma([A\ b]) = \gamma(A)$ and the proof is complete. ∎

COROLLARY Suppose that G is connected and the rank of A equals the rank of $[A\ b]$. If G contains an odd cycle and b is even, then $Ax = b$ has an integer solution.

REFERENCES

1 K. A. Berman, Bicycles and spanning trees, *SIAM J. Algebraic Discrete Methods* 7:1–12 (1986).
2 A. Blockhuis and A. R. Calderbank Quasi-symmetric designs and the Smith normal form, *Des. Codes Cryptogr.* 2:189–206 (1992).
3 R. A. Brualdi and H. J. Ryser, *Combinatorial Matrix Theory*, Cambridge U. P., Cambridge, 1991.
4 A. R. Calderbank, The application of invariant theory to the existence of quasi-symmetric designs, *J. Combin. Theory Ser. A* 44:94–109 (1987).

5 A. R. Calderbank, Geometric invariants for quasi-symmetric designs, *J. Combin. Theory Ser. A* 47:101–110 (1988).
6 G. Chartrand, H. V. Kronk, and C. E. Wall, The point-arboricity of a graph, *Israel J. Math.* 6:169–175 (1968).
7 K. Corrádi and A. J. Hajnal, On the maximal number of independent circuits in a graph, *Acta Math.* 14:423–439 (1963).
8 G. Dirac and P. Erdös, On the maximal number of independent circuits in a graph, *Acta Math.*14:79–90 (1963).
9 J. Edmonds, Maximum matching and a polyhedron with 0, 1 vertices, *J. Nat. Bur. Stand. Sect. B* 69:125–130 (1965).
10 J. Edmonds, Paths, trees and flowers, *Canad. J. Math.* 17:449–467 (1965).
11 P. Erdös and L. Pósa, On the maximal number of disjoint circuits of a graph, *Publ. Math. Debrecen* 9:3–12 (1962).
12 M. R. Garey and D. S. Johnson, *Computers and Intractability*, Freeman, San Francisco, 1979.
13 M. Gondran and M. Minoux, *Graphs and Algorithms*, Wiley-Interscience, New York, 1984.
14 D. M. Kulkarni and I. E. Schochetman, Multimodularity and integer extreme points with application to b-matching, submitted for publication.
15 R. Merris, Unimodular equivalence of graphs, *Linear Algebra Appl.* 173:181–189 (1992).
16 G. L. Nemhauser and L. A. Wolsey, *Integer and Combinatorial Optimization,* Wiley Interscience, New York, 1988.
17 C. Van Nuffelen, On the rank of the incidence matrix of a graph, *Cahiers Centre Études Rech. Oper.* 15:363–365 (1973).
18 H. Sachs, Über Teiler, Faktoren and charakteristische Polynome von Graphen. Teil I, *Wiss. Z. Techn. Hochsch. Ilmenau,* 12:7–12 (1966).
19 A. Schrijver, *Theory of Linear and Integer Programming,* Wiley-Interscience, New York, 1986.
20 H. J. S. Smith, On systems of linear indeterminate equations and congruences, *Philos. Trans. Roy. Soc. London* 151:293–326 (1861).
21 J. L. Szwarcfiter, On maximum sets of cycles and maximum disjoint cycles, in *Graphs and Algorithms* (R. B. Richter, Ed.), Contemp. Math. 89, Amer. Math. Soc., 1989, 153–166.

Received 8 September 1992; final manuscript accepted 7 July 1993