

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

***Introduction to the
Modern Theory of Dynamical Systems***

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