Journal of Logic and Computation Advance Access published December 18, 2015

Equivalence closure in the two-variable guarded fragment

EMANUEL KIEROŃSKI, *Institute of Computer Science, University of Wrocław, Poland.*
E-mail: kiero@cs.uni.wroc.pl

IAN PRATT-HARTMANN, *School of Computer Science, University of Manchester, UK; Institute of Mathematics and Informatics, Opole University, Poland.*
E-mail: ipratt@cs.man.ac.uk

LIDIA TENDERA, *Institute of Mathematics and Informatics, Opole University, Oleska 48, 45-052 Opole, Poland.*
E-mail: tendera@math.uni.opole.pl

Abstract

We consider the satisfiability and finite satisfiability problems for the extension of the two-variable guarded fragment in which an equivalence closure operator can be applied to two distinguished binary predicates. We show that the satisfiability and finite satisfiability problems for this logic are 2-EXPTIME-complete. This contrasts with an earlier result that the corresponding problems for the full two-variable logic with equivalence closures of two binary predicates are 2-NEXPTIME-complete.

Keywords: Satisfiability problem, computational complexity, decidability, guarded fragment, equivalence closure.

1 Introduction

The two-variable fragment of first-order logic, FO^2, and the two-variable guarded fragment, GF^2, are widely investigated formalisms whose study is motivated by their close connections to modal, description and temporal logics. It is well-known that FO^2 enjoys the finite model property [15], and that its satisfiability (= finite satisfiability) problem is NEXPTIME-complete [5]. Since GF^2 is contained in FO^2, it too has the finite model property; however, its satisfiability problem is slightly easier, namely EXPTIME-complete [6].

It is impossible, in FO^2, to write a formula expressing the condition that a given binary predicate denotes an equivalence relation (i.e. is reflexive, symmetric and transitive); and the question therefore arises as to whether such a facility could be added at reasonable computational cost. In a series of papers [10, 11, 13], various extensions of FO^2 were investigated in which certain distinguished binary predicates are declared to denote equivalence relations, or in which an operation of equivalence closure can be applied to these predicates. (The equivalence closure of a binary relation is the smallest equivalence relation that includes it.) Denote by EQ^2_k the extension of FO^2 in which k distinguished binary predicates are interpreted as equivalence relations, and by EC^2_k the extension of FO^2 in which we can take the equivalence closure of any of k distinguished binary predicates. To see that EC^2_k

doi:10.1093/logcom/exv075

is more expressive than EQ_k^2 note that EC_1^2 can express graph connectivity, a condition that is not expressible in first-order logic, and hence not expressible in EQ_k^2 for any k.

The following is known: (i) EQ_1^2 and EC_1^2 retain the finite model property, and their satisfiability (= finite satisfiability) problems are NExpTime-complete (cf. [11] for EQ_1^2 and [10] for EC_1^2); (ii) EQ_2^2 and EC_2^2 lack the finite model property, and their satisfiability and finite satisfiability problems are 2-NExpTime-complete (cf. [11] for EQ_2^2-satisfiability, and [10] for the remaining cases); (iii) when $k \geq 3$, the satisfiability and finite satisfiability problems for EQ_k^2 and EC_k^2 are undecidable [11].

Turning now to GF^2, denote by GFEQ_k^2 the extension of GF^2 in which k distinguished binary predicates are interpreted as equivalence relations, and by GFEC_k^2 the extension of GF^2 in which we can take the equivalence closure of any of k distinguished binary predicates. It was shown in [9] that the satisfiability and finite satisfiability problems for GFEQ_1^2 are both NExpTime-hard, whence, by the above-mentioned results on EQ_1^2, both these problems are NExpTime-complete. Taking into account the above-mentioned properties of EC_1^2, the same complexity bounds apply to the (finite) satisfiability problem for GFEC_1^2. In addition, the undecidability results for EQ_k^2 and EC_k^2 ($k \geq 3$) referred to in the previous paragraph employ only guarded formulas, whence satisfiability and finite satisfiability for GFEQ_k^2 and GFEC_k^2 are undecidable when $k \geq 3$.

This leaves only the case $k = 2$. That is: we wish to know the complexity of satisfiability and finite satisfiability for GFEQ_2^2 and GFEC_2^2. It is known that these logics lack the finite model property (see, e.g. the example in Section 2 of [10], and observe that all formulas there are indeed guarded). Furthermore, the satisfiability problem for GFEQ_2^2 was shown to be 2-ExpTime-complete in [9], and the method employed to establish the 2-ExpTime lower-bound can easily be adapted to the finite case. In this article, we solve all the remaining open problems by establishing a 2-ExpTime upper bound on the satisfiability and finite satisfiability problems for GFEC_2^2. It follows that these problems are 2-ExpTime-complete, as is the finite satisfiability problem for GFEQ_2^2.

For GF^2, it also makes sense to study variants in which the distinguished predicates may appear only in guards [4]. In this case, the satisfiability problem for GF^2 with *any* number of equivalence relations appearing only as guards remains NExpTime-complete [9], while GF^2 with *any* number of transitive relations appearing only as guards is 2-ExpTime-complete [8, 16]. The finite satisfiability problem was addressed in [12], giving 2-ExpTime and 2-NExpTime upper bounds, respectively. The 2-ExpTime upper bound is also retained for satisfiability in the more expressive fragment, where one can guard quantifiers using the *transitive closure* of some binary relations [14].

To establish the advertised results on GFEQ_2^2 and GFEC_2^2, we adopt the same strategy as that employed in [10] for the logics EQ_2^2 and EC_2^2; and we briefly review that strategy now. For concreteness, let φ be an EQ_2^2-formula whose satisfiability we are trying to establish, featuring equivalence relations r_1 and r_2. Note that the coarsest common refinement, $r_1 \cap r_2$, of these relations is also an equivalence relation; call the equivalence classes of $r_1 \cap r_2$ *intersections*. We showed in [10] that the intersections arising in any model of φ could, without loss of generality, be assumed to have cardinality exponentially bounded as a function of the size of φ. In any such model, every r_1-class, and also every r_2-class, is the union of some set of such 'small intersections'; and any given r_1-class and r_2-class are either disjoint, or have exactly one common intersection. This decomposition into equivalence classes allowed us to picture such a model as an edge-coloured, bipartite graph: the r_1-classes are the left-hand vertices; the r_2-classes are the right-hand vertices; and two vertices are joined by an edge just in case they share an intersection, with the colour of that edge being the isomorphism type of the intersection concerned. Evidently, the formula φ imposes constraints on the types of intersections that may arise, and on how intersections may be organized into r_1- and r_2-classes; and we showed in [10] how these constraints translated to conditions on the induced bipartite graph of equivalence classes. In this way, the original satisfiability problem for EQ_2^2 was non-deterministically reduced to the problem of determining the existence of an edge-coloured bipartite graph satisfying certain conditions on the local configurations it realises. We called this latter problem BGESC (for 'bipartite graph existence with skew constraints and ceilings'). The reduction in question was non-deterministic, ran in doubly exponential time, and produced instances of BGESC of size doubly exponential in the size of φ. By showing BGESC to be NPTIME-complete, we obtained the sought-after 2-NEXPTIME-upper bound for the satisfiability problem for EQ_2^2. Finite satisfiability was dealt with analogously, via reduction to a finite version of the problem BGESC.

In the present article, we show that, when dealing with the *guarded* sub-fragments, GFEQ_2^2 and GFEC_2^2, a restricted case of BGESC—which we call BGE*—results. We show that the reduction of the satisfiability problem for GFEC_2^2 to BGE* can be improved so that it proceeds in *deterministic* doubly exponential time, and moreover, that BGE* is in PTIME. This yields the sought-after 2-EXPTIME-upper bound on the satisfiability problem for EQ_2^2. Again, finite satisfiability is dealt with analogously. The more stringent requirements on the reduction necessitate various changes at a tactical level throughout the proof, as a by-product of which we obtain some additional results on the model-theoretic differences between EC_2^2 and GFEC_2^2 (see discussion following Theorem 24).

The plan of the article is as follows. In Section 2, we define the logics GFEQ_2^2 and GFEC_2^2, introducing a 'Scott-type' normal form for GFEC_2^2 that allows us to restrict the nesting of quantifiers to depth In Section 3 we define a problem concerning the existence of certain bipartite graphs, called BGE*, and show that both the finite and the general versions of this problem are in PTIME. In Section 4, we prove some technical lemmas that allow us in Section 5 to reduce the (finite) satisfiability problem of a GFEC_2^2-formula to (finite) BGE* in deterministic doubly exponential time.

2 Preliminaries

2.1 Logics, structures and types

We employ standard terminology and notation from model theory throughout this article (see, e.g. [3]). In particular, we refer to structures using Gothic capital letters, and their domains using the corresponding Roman capitals. We denote by GF^2 the guarded two-variable fragment of first-order logic (with equality), without loss of generality restricting attention to signatures of unary and binary predicates (cf. [5]). Formally, GF^2 is the intersection of FO^2 (i.e. the restriction of first-order logic in which only two variables, x and y are available) and the *guarded fragment*, GF [1]. GF is defined as the least set of formulas such that: (i) every atomic formula belongs to GF; (ii) GF is closed under logical connectives $\neg, \vee, \wedge, \rightarrow$; and (iii) quantifiers are appropriately relativized by atoms. More specifically, in GF^2, condition (iii) is understood as follows: if φ is a formula of GF^2, α is an atomic formula containing all the free variables of φ, and u (either x or y) is a free variable in α, then the formulas $\forall u(\alpha \rightarrow \varphi)$ and $\exists u(\alpha \wedge \varphi)$ belong to GF^2. In this context, the atom α is called a *guard*. The predicate $=$ is allowed in guards. We take the liberty of counting as guarded those formulas which can be made guarded by trivial logical manipulations. For instance, we allow unrestricted quantification over formulas containing only one free variable, u, since the atom $(u=u)$ can, in this case, always be inserted as a guard.

We denote by GFEC_k^2 the set of GF^2-formulas over any signature $\tau = \tau_0 \cup \{r_1,\ldots,r_k\} \cup \{r_1^\#,\ldots,r_k^\#\}$, where τ_0 is an arbitrary set containing unary and binary predicates, and $r_1,\ldots,r_k$, $r_1^\#,\ldots,r_k^\#$ are distinguished binary predicates, not present in τ_0. In the sequel, any signature τ is assumed to be of

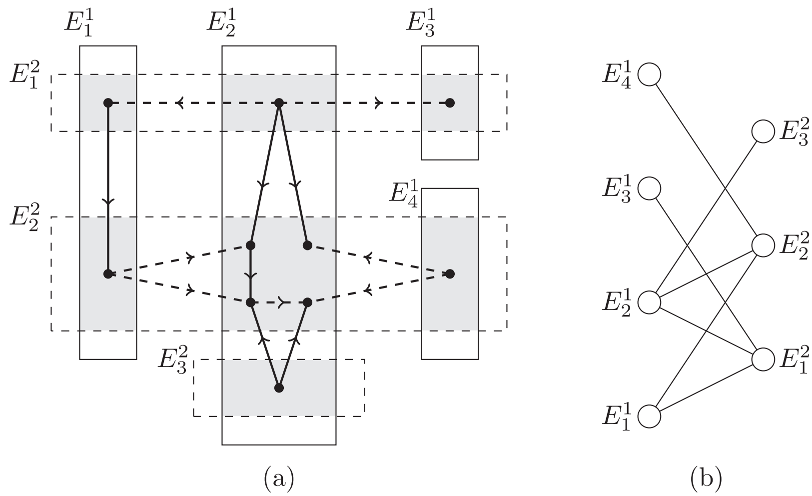

FIGURE 1. The equivalence classes of $r_1^\#$ and $r_2^\#$ in a structure (a), and their representation as a bipartite graph (b).

the above form (for some appropriate value of k). We denote by GFEQ_k^2 the set of GFEC_k^2-formulas in which the predicates $r_1^\#,\ldots,r_k^\#$ do not occur.

The semantics for GFEC_k^2 are standard, subject to the restriction that $r_i^\#$ is always interpreted as the *equivalence closure* of r_i. More precisely: we consider only structures $\mathfrak{A}$ in which, for all i $(1 \le i \le k)$, $(r_i^\#)^\mathfrak{A}$ is the smallest reflexive, symmetric and transitive relation including $r_i^\mathfrak{A}$. Similarly, we require for GFEQ_k^2 that r_i is always interpreted as an equivalence relation. Where a structure is clear from context, we may equivocate between predicates and their extensions, writing, e.g. r_i and $r_i^\#$ in place of the technically correct $r_i^\mathfrak{A}$ and $(r_i^\#)^\mathfrak{A}$.

Let $\mathfrak{A}$ be a structure over τ and let $a,a' \in A$. We say that there is an r_i-*edge* between a and $a' \in A$ if $\mathfrak{A} \models r_i[a,a']$ or $\mathfrak{A} \models r_i[a',a]$. For any subset $B \subseteq A$, we say that a and a' are r_i-*connected* in B if there exists a sequence $a = a_0, a_1, \ldots, a_{k-1}, a_k = a'$ of elements of B such that for all j $(0 \le j < k)$ there is an r_i-edge between a_j and a_{j+1}. Such a sequence is called an r_i-*path* from a to a' *in* B. In the case $B = A$ we say simply that a and a' are r_i-*connected*. Thus, $\mathfrak{A} \models r_i^\#[a,a']$ if and only if a and a' are r_i-connected. We say that B is r_i-*connected* if every pair of elements of B is r_i-connected in B. Maximal r_i-connected subsets of A are equivalence classes of $r_i^\#$, and are called $r_i^\#$-*classes* of $\mathfrak{A}$. It is obvious that the relation $r_1^\# \cap r_2^\#$ is also an equivalence relation, and we refer to its equivalence classes, simply, as *intersections* of $\mathfrak{A}$. Thus, each $r_i^\#$-class in a structure is the union of the intersections it includes. Furthermore, an $r_1^\#$-class and an $r_2^\#$-class can share at most one intersection. Figure 1a shows an example structure with r_1 depicted by solid arrows and r_2 by dotted arrows. (We have suppressed the signature τ_0 in this diagram.) The $r_1^\#$- and $r_2^\#$-classes of this structure are delineated by solid rectangles and dashed rectangles, respectively, and its intersections are shaded grey.

The decomposition of a structure into equivalence classes allows us to picture it as a bipartite graph: the $r_1^\#$-classes are the left-hand vertices; the $r_2^\#$-classes are the right-hand vertices; and two vertices are joined by an edge just in case they share a (unique) intersection. The bipartite graph induced by the structure of Figure 1a is shown in Figure 1b. Crucially, it can be shown (Lemma 5) that, if a GFEC_2^2-formula φ is (finitely) satisfiable, then it has a (finite) model in which all intersections are bounded in size by a fixed function of the number of symbols in φ. Thus, the number of possible isomorphism-types of intersections occurring in such a model is also bounded as a fixed function of the number of symbols in φ; and we can think of these finitely many isomorphism-types as *colouring* the corresponding edges of the induced bipartite graph. This partial representation of structures as edge-coloured bipartite graphs is one of the principal tools employed in this article: we show that the (finite) satisfiability of φ can be translated into the problem of determining the existence of a (finite) edge-coloured bipartite graph subject to constraints on the colours of the edges on which its vertices can be incident.

An (atomic) 1-*type* (over a given signature) is a maximal satisfiable set of atoms or negated atoms with free variable x. Similarly, an (atomic) 2-*type* is a maximal satisfiable set of atoms and negated atoms with free variables x, y. Note that the numbers of 1-types and 2-types are bounded exponentially in the size of the signature. We often identify a type with the conjunction of its elements.

For a given τ-structure $\mathfrak{A}$, we denote by $\mathrm{tp}^{\mathfrak{A}}(a)$ the 1-type *realised* by a, i.e. the 1-type α such that $\mathfrak{A} \models \alpha[a]$. Similarly, for distinct $a, b \in A$, we denote by $\mathrm{tp}^{\mathfrak{A}}(a,b)$ the 2-type *realised* by the pair a, b, i.e. the 2-type β such that $\mathfrak{A} \models \beta[a,b]$. If φ is a formula, we write $\|\varphi\|$ to denote the number of symbols in φ.

2.2 Normal form and small intersections

In the context of FO^2, it is often convenient to work with formulas in so-called Scott normal form. The following definition adapts this notion to the setting of GFEC^2_2. Recall that τ_0 is the signature of ordinary (non-distinguished) unary and binary predicates occurring in φ.

Definition 1

A *normal guard* is any of the formulas $r_1^\#(x,y) \wedge r_2^\#(x,y)$, $r_1^\#(x,y) \wedge \neg r_2^\#(x,y)$, $\neg r_1^\#(x,y) \wedge r_2^\#(x,y)$, or $\epsilon(x,y) \wedge \neg r_1^\#(x,y) \wedge \neg r_2^\#(x,y)$, where $\epsilon(x,y)$ is a τ_0-atom. A GFEC^2_2-sentence φ is in *normal form* if it is a conjunction of formulas of the forms:

(∃)	$\exists x. \psi(x)$
(∀)	$\forall x. \psi(x)$
(∀∀)	$\forall xy(\eta(x,y) \rightarrow (x \neq y \rightarrow \psi(x,y)))$
(∀∃)	$\forall x(\gamma(x) \rightarrow \exists y(\eta(x,y) \wedge x \neq y \wedge \psi(x,y)))$

where $\gamma(x)$ is a τ_0-atom, $\eta(x,y)$ is a normal guard, and $\psi(x)$, $\psi(x,y)$ are quantifier-free formulas not using $r_1^\#$ and $r_2^\#$, with free variables as indicated. Where a normal-form GFEQ^2_2-formula φ is given, we refer to these four types of formulas as its ∀-, ∃- ∀∀- and ∀∃-*conjuncts*, respectively. We additionally require that the conjuncts of φ include, for $i = 1, 2$,

($\mathrm{K}_{\forall\exists}$)	$\forall x(K_i(x) \rightarrow \exists y(r_i^\#(x,y) \wedge \neg r_{3-i}^\#(x,y) \wedge (r_i(x,y) \vee r_i(y,x))))$
($\mathrm{K}_{\forall\forall}$)	$\forall xy((r_i(x,y) \wedge \neg r_{3-i}^\#(x,y)) \rightarrow (K_i(x) \wedge K_i(y)))$,

where K_1 and K_2 are unary predicates of τ_0 used only in these conjuncts.

The conjuncts ($\mathrm{K}_{\forall\exists}$) and ($\mathrm{K}_{\forall\forall}$) guarantee that, in every model of φ, the elements satisfying K_i are precisely those joined by an r_i-edge to an element in a different intersection.

The following lemma justifies the normal form introduced in Definition 1.

LEMMA 2
Let φ be a GFEC_2^2-sentence over a signature τ. We can compute, in exponential time, a disjunction $\Psi = \bigvee_{i\in I}\psi_i$ of normal form sentences over a signature τ' such that φ is satisfiable if and only if Ψ is satisfiable, $\|\psi_i\| = O(\|\varphi\|\log\|\varphi\|)$ $(i\in I)$ and τ' consists of τ together with some additional unary predicates.

PROOF. The proof uses standard techniques; we describe them briefly for completeness. Without loss of generality we can assume that the two-variable guards in the formula φ are normal. For, if $\theta(x,y)$ is a guard of some quantifier Q we replace $\theta(x,y)$ in φ by an equivalent formula $\theta(x,y)\wedge\bigvee_{s,t\in\{0,1\}}(\neg^s r_1^\#(x,y)\wedge\neg^t r_2^\#(x,y))$, where $\neg^0 r_k^\#(x,y) = r_k^\#(x,y)$ and $\neg^1 r_k^\#(x,y) = \neg r_k^\#(x,y)$, and appropriately rearrange the resulting formula using propositional tautologies and distribution laws for quantifiers.

Now, given a GFEC_2^2-sentence φ, we eliminate proper subsentences of φ one by one, thus reducing the quantifier depth until we obtain a quantifier-free formula φ'. At each step (numbered $i = 1,2\dots$), we consider either (i): a proper subformula of φ of the form $\exists v\theta(v,v')$, where $\theta(v,v')$ is quantifier-free or (ii) a proper subsentence of φ of the form $\exists v\theta(v)$, where $\theta(v)$ is quantifier-free. In either case, we introduce a new unary predicate $p_{i,\theta}$ to τ', and we replace the subformula $\exists v\theta$ in φ by the formula $p_{i,\theta}(v')$. In case (i) we record a new conjunct $\gamma_i = \forall v'(p_{i,\theta}(v')\leftrightarrow\exists v\theta(v,v'))$, and in case (ii) we record $\delta_i = \delta_i^1\wedge\delta_i^2$, where $\delta_i^1 = \forall v'(p_{i,\theta}(v')\leftrightarrow\exists v\theta(v))$ and $\delta_i^2 = \forall x p_{i,\theta}(x)\vee\forall x\neg p_{i,\theta}(x)$ ensuring that the predicate $p_{i,\theta}$ behaves like a boolean variable.

After a linear number of such steps, we obtain from φ a quantifier-free formula φ' without repeating atoms. Let φ'' be obtained by replacing all variables in φ' by x and binding the variable x by an existential quantifier, let $\gamma = \bigwedge_{i\in I}\gamma_i$ and $\delta = \bigwedge_{i\in I}\delta_i$. The formulas have the following properties: (i) $\varphi''\wedge\gamma\wedge\delta\models\varphi$; (ii) every model $\mathfrak{A}\models\varphi$ might be expanded to a model $\mathfrak{A}'\models\varphi''\wedge\gamma\wedge\delta$.

Evidently, every conjunct γ_i can be written as a conjunction of two guarded formulas of the form $\forall v'(p_{i,\theta}(v')\to\exists v\theta(v,v'))$ and $\forall v\forall v'(\theta(v,v')\to p_{i,\theta}(v'))$. Every conjunct δ_i^1 is equivalent to a disjunction of two guarded formulas of the form $(\exists)$ and $(\forall)$. To obtain the required normal form Ψ it suffices to take the disjunctive normal form of $\varphi''\wedge\gamma\wedge\delta$ and add the conjuncts $(\mathrm{K}_{\forall\exists})$, $(\mathrm{K}_{\forall\forall})$. To see that these additional conjuncts do not affect (finite) satisfiability, observe that, since K_1 and K_2 do not occur in $\varphi''\wedge\gamma\wedge\delta$, we can simply expand any model of that formula by interpreting these new predicates as indicated following Definition 1. Thus, Ψ has the properties claimed by the Lemma. Consult e.g. [10] (Lemma 3.1) and [16] (Lemma 2) for more details of the technique. ∎

Since we are going to show that (finite) satisfiability is in 2-EXPTIME, it is enough to consider formulas in normal form (one can e.g. consider each of the disjuncts ψ_i of Lemma 2 in isolation). It is sometimes useful to group together various conjuncts of a normal-form formula.

DEFINITION 3
If φ is as in Definition 1, let us write:

$$\begin{aligned}
\varphi_{\mathrm{univ}} &:= \bigwedge\{\psi \mid \psi \text{ a } \forall\text{-conjunct or } \forall\forall\text{-conjunct of } \varphi\}\\
\varphi_{12} &:= \varphi_{\mathrm{univ}}\wedge\bigwedge\{\psi \mid \psi \text{ a } \forall\exists\text{-conjunct of } \varphi \text{ with } \eta = r_1^\#(x,y)\wedge r_2^\#(x,y)\}\\
\varphi_{1} &:= \varphi_{\mathrm{univ}}\wedge\bigwedge\{\psi \mid \psi \text{ a } \forall\exists\text{-conjunct of } \varphi \text{ with } \eta = r_1^\#(x,y)\wedge\neg r_2^\#(x,y)\}\\
\varphi_{2} &:= \varphi_{\mathrm{univ}}\wedge\bigwedge\{\psi \mid \psi \text{ a } \forall\exists\text{-conjunct of } \varphi \text{ with } \eta = \neg r_1^\#(x,y)\wedge r_2^\#(x,y)\}
\end{aligned}$$

$$\varphi^{-}_{\text{free}} := \bigwedge\{\psi \mid \psi \text{ a } \forall\exists\text{-conjunct of } \varphi \text{ with } \eta = \epsilon(x,y) \wedge \neg r^{\#}_1(x,y) \wedge \neg r^{\#}_2(x,y)\}$$
$$\varphi_{\text{free}} := \varphi_{\text{univ}} \wedge \varphi^{-}_{\text{free}}.$$

FACT 4
If φ is a normal form GFEC^2_2-formula, and $\mathfrak{A}$ a τ-structure, then $\mathfrak{A} \models \varphi$ if and only if the following conditions hold:

(i) $\mathfrak{A} \models \bigwedge\{\psi \mid \psi$ an $\exists$-conjunct of $\varphi\}$;
(ii) for each intersection $\mathfrak{I}$ of $\mathfrak{A}$, $\mathfrak{I} \models \varphi_{12}$;
(iii) for each $r^{\#}_i$-class $\mathfrak{D}$ of $\mathfrak{A}$, $\mathfrak{D} \models \varphi_i$ (for $i = 1, 2$);
(iv) $\mathfrak{A} \models \varphi_{\text{free}}$.

PROOF. Straightforward. ■

It was shown in [10] (Lemma 4.2) that, when considering (finite) satisfiability of EC^2_2-formulas, one can restrict attention to models with exponentially bounded intersections (this technique stems originally from [11]). Moreover, as was also pointed out there, the Löwenheim–Skolem–Tarski theorem applies to EC^2_2. Since these results subsume the case of GFEC^2_2 and there is no essential difference in the normal forms considered, we have:

LEMMA 5
Let φ be a satisfiable GFEC^2_2-formula in normal form over a signature τ. Then there exists a countable model $\mathfrak{A}$ of φ in which the size of each intersection is bounded by $\mathfrak{f}(|\tau|)$, for a fixed exponential function $\mathfrak{f}$. If φ is in fact finitely satisfiable, then we can ensure that $\mathfrak{A}$ too is finite.

In the sequel, we shall silently assume that structures all are countable. We use the term ‘countable’ in the sense of ‘finite or countably infinite’.

3 A problem concerning bipartite graphs

In this section, we define a pair of problems, called BGE* and finite BGE*, concerning the existence of edge-coloured bipartite graphs satisfying various collections of conditions. It is shown in subsections 3.2 and 3.3 that BGE* and, respectively, finite BGE* are in PTIME.

3.1 Definition of BGE*

Let Δ be a finite, non-empty set. A Δ-*graph* is a triple $H = (U, V, \mathbf{E}_\Delta)$, where U, V are disjoint sets of cardinality at most $\aleph_0$, and $\mathbf{E}_\Delta$ is a collection of pairwise disjoint subsets $E_\delta \subseteq U \times V$, indexed by the elements of Δ. We call the elements of $W = U \cup V$ *vertices*, and the elements of E_δ, δ-*edges*. It helps to think of $\mathbf{E}_\Delta$ as the result of colouring the edges of the bipartite graph (U, V, E), where $E = \bigcup_{\delta \in \Delta} E_\delta$, using the colours in Δ. For any $w \in W$, we define the function $\text{ord}^H_w : \Delta \to \mathbb{N}^*$, called the *order* of w, by

$$\text{ord}^H_u(\delta) = |\{v \in V : (u,v) \in E_\delta\}| \qquad (u \in U)$$
$$\text{ord}^H_v(\delta) = |\{u \in U : (u,v) \in E_\delta\}| \qquad (v \in V).$$

Thus, ord^H_w tells us, for each colour δ, how many δ-edges w is incident to in H. Obviously, if H is finite, the values of ord^H_w all lie in $\mathbb{N}$. When constructing Δ-graphs, it is sometimes more convenient to employ a slightly more general notion. We define a Δ-*multigraph* in the same way as a Δ-graph, except that the E_δ are now *multi*-sets, and are not required to be disjoint. Thus, in a Δ-multigraph $(U, V, \mathbf{E}_\Delta)$, a pair of nodes $u \in U$ and $v \in V$ may be joined by any number of edges of any colours. The order-functions are defined in the obvious way, recording the total number of edges of each colour to which the node in question is incident.

The problem BGE* involves constraints on Δ-graphs. We now set up the apparatus to express those constraints.

Definition 6

Let M be a positive integer. We write $\hat{M}$ to denote the set $\{^{=}0, ^{=}1, \ldots, ^{=}M\} \cup \{^{\geq}0, ^{\geq}1, \ldots, ^{\geq}M\}$. We refer to any function with range $\hat{M}$, for some positive integer M, as a *constraint function*.

Officially, the elements of $\hat{M}$ are simply objects with no internal structure; informally, however, we will use the expressions $^{=}k$ and $^{\geq}k$, where k is a variable $(1 \leq k \leq M)$, to range over this set in the obvious way.

Definition 7

Let M be a positive integer and Δ a finite non-empty set. Given a function $f : \Delta \to \mathbb{N}^*$ and a constraint function $p : \Delta \to \hat{M}$, we say that f *realises* p if, for all $\delta \in \Delta$ and all k $(0 \leq k \leq M)$: $p(\delta) = {}^{=}k$ implies $f(\delta) = k$, and $p(\delta) = {}^{\geq}k$ implies $f(\delta) \geq k$.

Intuitively, a constraint function $p : \Delta \to \hat{M}$ expresses a collection of constraints on functions $f : \Delta \to \mathbb{N}$. To say that f realises p is simply to say that f satisfies the constraint in question. For example, if $p(\delta) = {}^{\geq}4$ and f realises p, then we know that $f(\delta) \geq 4$. We now proceed to define the problem BGE*.

Definition 8

A *BGE*-instance* is a quintuple $\mathcal{P} = (\Delta, \Delta_0, M, P, Q)$, where Δ is a finite, non-empty set, $\Delta_0 \subseteq \Delta$, M is a positive integer, and P and Q are sets of constraint functions $\Delta \to \hat{M}$. A *solution* of $\mathcal{P}$ is a Δ-graph $H = (U, V, \mathbf{E}_\Delta)$ such that:

(G1) for all $\delta \in \Delta_0$, E_δ is non-empty;

(G2) for all $u \in U$, ord_u^H realises some constraint function from P;

(G3) for all $v \in V$, ord_v^H realises some constraint function from Q.

The problem (*finite*) *BGE** is as follows:

Given: a BGE*-instance $\mathcal{P}$.
Output: Yes, if $\mathcal{P}$ has a (finite) solution; No, otherwise.

That is: suppose we are given a set of colours Δ, a distinguished subset $\Delta_0 \subseteq \Delta$ and sets of constraint functions P, Q mapping Δ to the set $\hat{M}$ for some $M \in \mathbb{N}$. We wish to know whether there exists a (finite) Δ-graph $(U, V, \mathbf{E}_\Delta)$ in which each of the colours in Δ_0 is represented by at least one edge, the vertices in U have only order-functions realising constraint functions from P, and the vertices in V have only order-functions realising constraint functions in Q. The following notation and terminology will be useful in the sequel.

Definition 9

Let $p:\Delta \to \hat{M}$ be a constraint function. We denote by $\bar{p}$ the function $\bar{p}:\Delta \to \mathbb{N}$ obtained by erasing decorations from the results of p; that is: for each $\delta \in \Delta$, $\bar{p}(\delta) = k$ if and only if $p(\delta) = {}^{=}k$ or $p(\delta) = {}^{\geq}k$.

Trivially, $\bar{p}$ realises p.

A special case of the problem BGE* is obtained by restricting all constraint functions in P and Q to take values of the form ${}^{=}k$ for $0 \leq k \leq M$. (i.e. the allowed orders of vertices are specified exactly.) This problem was originally defined in [10], under the name *BGE*. The same publication also considered the problem *BGESC*, which—in effect—amounts to extending BGE* by additionally allowing values of the form ${}^{\leq}k$ in constraint functions, interpreted in the obvious way. The problems *finite BGE* and *finite BGESC* are defined analogously. It was shown in [10] that the satisfiability and finite satisfiability problems for EC^2_2 can be non-deterministically reduced, in doubly exponential time, to the respective problems BGESC and finite BGESC; it was further shown there that these problems are both NPTime-complete. (It was also proved, in passing, that the simpler problems BGE and finite BGE are in PTime.) Note that (finite) BGE* lies in between (finite) BGE and (finite) BGESC.

It transpires that, in the reduction just mentioned, constraints involving values ${}^{\leq}k$ arise *only* from non-guarded formulas. Indeed, we show in Sections 4 and 5 that the satisfiability and finite satisfiability problems for GFEC^2_2 can be deterministically reduced, in doubly exponential time, to the respective problems BGE* and finite BGE*. In the remainder of this section, we show that BGE* and finite BGE* remain in PTime.

3.2 Complexity of finite BGE*

In this section, a *linear Diophantine equation* (*inequality*) is a linear equation (inequality) with integer coefficients and with variables ranging over non-negative integers. By a *linear Diophantine clause*, we mean a disjunction of linear Diophantine equations and inequalities. (We allow clauses to have just one disjunct). A *solution* of a system of linear Diophantine clauses is an assignment of non-negative integers to its variables making all its clauses true. If such a solution exists, the system of linear Diophantine clauses in question is said to be *satisfiable*. We proceed to reduce finite BGE* to the satisfiability problem for systems of linear Diophantine clauses of a particular form.

To understand the reduction, let $\mathcal{P} = (\Delta, \Delta_0, M, P, Q)$ be a BGE*-instance, and let us suppose that $\mathcal{P}$ has a finite solution $H = (U, V, \mathbf{E}_\Delta)$. Order the sets of constraint functions P and Q arbitrarily. For every $p \in P$, let U_p be the set of vertices $u \in U$ such that p is the first element of P realised by ord^H_u, and, for every $q \in Q$, let V_q be the set of vertices $v \in V$ such that q is the first element of Q realised by ord^H_u. Now set

$$x_p = |U_p| \text{ for } p \in P \qquad\qquad y_q = |V_q| \text{ for } q \in Q.$$

Thus, the sets U_p form a partition of U (with some cells in the partition allowed to be empty), and similarly the sets V_p form a partition of V.

Using the notation of Definition 9, it is then obvious that, for all $\delta \in \Delta$:

$$\sum_{p \in P} \bar{p}(\delta) x_p \leq |E_\delta| \qquad\qquad \sum_{q \in Q} \bar{q}(\delta) y_q \leq |E_\delta|. \tag{1}$$

Now define, for all $\delta \in \Delta$ and $p \in P$:

$$c_p^\delta = \begin{cases} 1 & \text{if } p(\delta) \neq^= 0 \\ 0 & \text{otherwise.} \end{cases}$$

and similarly for c_q^δ ($q \in Q$). Note that, if $c_p^\delta = 0$, then vertices in U_p cannot be incident to any δ-edges. Likewise, if $c_q^\delta = 0$, then vertices in V_q cannot be incident to any δ-edges. But, since H is a solution of $\mathcal{P}$, E_δ is non-empty for all $\delta \in \Delta_0$, so that there are vertices of both U and V that are incident to δ-edges. Hence the following inequalities hold:

$$\sum_{p \in P} c_p^\delta x_p > 0 \quad \text{for all } \delta \in \Delta_0 \tag{2}$$

$$\sum_{q \in Q} c_q^\delta y_q > 0 \quad \text{for all } \delta \in \Delta_0. \tag{3}$$

Now define, for all $\delta \in \Delta$ and $p \in P$:

$$d_p^\delta = \begin{cases} 1 & \text{if } p(\delta) = {}^{\geq} k \text{ for some } k \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

and similarly for d_q^δ ($q \in Q$). Thus, if $d_p^\delta = 0$, then every vertex in U_p is incident to exactly $\bar{p}(\delta)$ δ-edges. In particular, if $\sum_{p \in P} d_p^\delta x_p = 0$, then $\sum_{p \in P} \bar{p}(\delta) x_p = |E_\delta|$. And likewise, if $\sum_{q \in Q} d_q^\delta y_q = 0$, then $\sum_{q \in Q} \bar{q}(\delta) y_q = |E_\delta|$. It is then a consequence of (1) that the following linear Diophantine clauses hold:

$$\left(\sum_{p \in P} d_p^\delta x_p > 0 \right) \vee \left(\sum_{p \in P} \bar{p}(\delta) x_p \geq \sum_{q \in Q} \bar{q}(\delta) y_q \right) \quad \text{for all } \delta \in \Delta \tag{4}$$

$$\left(\sum_{q \in Q} d_q^\delta y_q > 0 \right) \vee \left(\sum_{q \in Q} \bar{q}(\delta) y_q \geq \sum_{p \in P} \bar{p}(\delta) x_p \right) \quad \text{for all } \delta \in \Delta. \tag{5}$$

Regarding the x_p and y_q as variables ranging over $\mathbb{N}$, let $\mathcal{E}_\mathcal{P}$ be the system of linear Diophantine clauses (2)–(5). Thus, we have shown that, if the BGE*-instance $\mathcal{P}$ has a finite solution, then the system of linear Diophantine clauses $\mathcal{E}_\mathcal{P}$ has a solution over $\mathbb{N}$. Notice that the various constants c_p^δ, c_q^δ, d_p^δ and d_q^δ depend only on $\mathcal{P}$, and not on the supposed finite solution, H.

We now show that, conversely, if the system of linear Diophantine clauses $\mathcal{E}_\mathcal{P}$ has a solution over $\mathbb{N}$, then the BGE*-instance $\mathcal{P}$ has a finite solution. Suppose, then, that the collections of integers $\{x_p\}_{p \in P}$ and $\{y_q\}_{q \in Q}$ satisfy (2)–(5). For each constraint function $p \in P$, we take a set of vertices U_p of cardinality x_p, and let U' be the disjoint union of the U_p; similarly, for each $q \in Q$, we take a set of vertices V_q of cardinality y_q, and let V' be the disjoint union of the V_q. (We assume that U' and V' are disjoint.) Let us imagine each $u \in U_p$ to have $\bar{p}(\delta)$ 'dangling' δ-edges for all $\delta \in \Delta$; and likewise let us imagine each $v \in V_q$ to have $\bar{q}(\delta)$ 'dangling' δ-edges for all $\delta \in \Delta$. Our task is to match up these dangling edges so as to form a bipartite Δ-graph which is a solution of $\mathcal{P}$.

To make our task easier, we first construct a Δ-multigraph $G = (U', V', \mathbf{E}'_\Delta)$ satisfying properties (G1)–(G3) of Definition 8. (Recall that a multigraph is like a graph, except that two nodes may be joined by multiple edges.) Fix $\delta \in \Delta$, let $\delta(U')$ be the set of dangling δ-edges attached to the elements of U', and let $\delta(V')$ be the set of dangling δ-edges attached to the elements of V'. Evidently,

$$|\delta(U')| = \sum_{p \in P} \bar{p}(\delta) x_p \qquad |\delta(V')| = \sum_{q \in Q} \bar{q}(\delta) y_q. \tag{6}$$

Suppose first that $|\delta(U')| \leq |\delta(V')|$. Then we identify the elements of $\delta(U')$ with $|\delta(U')|$ elements of $\delta(V')$, thus dealing with all the dangling edges in $\delta(U')$ as well as $|\delta(U')|$ of the dangling edges in $\delta(V')$. If there are any remaining dangling edges in $\delta(V')$, we observe from (6) that $\sum_{p \in P} \bar{p}(\delta) x_p < \sum_{q \in Q} \bar{q}(\delta) y_q$, whence, from (4), $\sum_{p \in P} d_p^\delta x_p > 0$. Now pick some p for which x_p and d_p^δ are both positive. Thus, U_p is non-empty, so we select some $u \in U_p$, and attach all the dangling edges in $\delta(V')$ not yet accounted for to u. Note that, by the definition of d_p^δ, $p(\delta) = {}^{\geq} k$ for some $k \geq 0$. Thus, attaching a collection of δ-edges to u cannot stop the order of u in the resulting multigraph from realising the constraint function p. If, on the other hand, $|\delta(U')| \geq |\delta(V')|$, we apply the mirror-image construction and use (5) in place of (4). Either way, performing the above process for all $\delta \in \Delta$, all dangling edges are linked up, and we obtain a Δ-coloured multigraph, G, with node-sets U' and V'. We see that, for each $p \in P$, and each $u \in U_p$, ord_u^G realises p, securing (G2); likewise, for each $q \in Q$, and each $v \in V_q$, ord_v^G realises q, securing (G3). We make one small change to G before proceeding. Consider any $\delta \in \Delta$, and suppose there are no δ-edges in G. Then, by the construction of G, it is impossible that U_p is empty for all $p \in P$ such that $\bar{p}(\delta) \geq 1$; and it is likewise impossible that V_q is empty for all $q \in Q$ such that $\bar{q}(\delta) \geq 1$. But suppose now that $\delta \in \Delta_0$. From (2), there exists $p \in P$ such that $x_p > 0$, and either $p(\delta) = {}^{\geq} 0$ or $\bar{p}(\delta) \geq 1$, and therefore (since we have just ruled out the second possibility), $p(\delta) = {}^{\geq} 0$. Likewise, from (3), we can find $q \in Q$ such that $y_q > 0$ and $q(\delta) = {}^{\geq} 0$. Since $x_p > 0$ and $y_q > 0$, we may pick $u \in U_p$ and $v \in V_q$, and add a δ-edge between u and v. Furthermore, since $p(\delta) = q(\delta) = {}^{\geq} 0$, this extra edge does not compromise the fact that ord_u^G realises p or that ord_v^G realises q. In this way, we can ensure that, for all $\delta \in \Delta_0$, G contains a δ-edge, thus securing (G1).

It remains to replace the Δ-multigraph G with a Δ-graph H in such a way that the realised order-functions are not disturbed. Let s be the maximum multiplicity of edges in H (i.e. the maximum number of any edges connecting any pair of vertices). Then H can be constructed by taking s replicas of G and appropriately rearranging the multiple edges. More precisely, let $U = \bigcup_{i=0}^{s-1} U_i$ and $V = \bigcup_{i=0}^{s-1} V_i$, where each U_i (V_i) is a fresh copy of U' (respectively, V'). For every $u \in U'$, denote by u_i the copy of u in U_i, and similarly for every $v \in V'$. Now execute the following process for all pairs $u \in U'$ and $v \in V'$ joined by at least one edge: let $e_0, \ldots, e_{r-1}$ $(0 < r \leq s)$ be the collection of (coloured) edges joining u and v in G; for every j $(0 \leq j \leq r-1)$ and i $(0 \leq i \leq s-1)$, let H contain an edge with the same colour as e_j between u_i and $v_{i+j \bmod s}$. Since $r \leq s$, no pair of elements is joined by more than one edge, so that H is a Δ-graph. Moreover, for all $u \in U$, if u is a copy of some element $u' \in U_p$, then $\mathrm{ord}_u^H = \mathrm{ord}_{u'}^G = p$; similarly, if v is a copy of some element $v' \in V_q$, then $\mathrm{ord}_v^H = \mathrm{ord}_{v'}^G = q$. Thus, H is a solution of $\mathcal{P}$. We have proved:

Lemma 10
There is a polynomial-time reduction of finite BGE* to the satisfiability problem for sets of linear Diophantine clauses of the forms (2)–(5).

Theorem 11
Finite BGE* is in PTime.

PROOF. By Lemma 10, it suffices to show that the satisfiability of systems of linear Diophantine clauses of the forms (2)–(5) can be solved in polynomial time. Crucially, these systems of Diophantine clauses involve only constraints of the form $\vec{c}\cdot\vec{z}\geq b$ with $b\geq 0$, a fact which allows us to seek solutions over rationals rather than over integers.

Consider any system of r Diophantine linear inequalities of the form $\vec{a}\cdot\vec{z}\geq b$, in k variables (possibly with negative b); and let C be the maximum absolute value of any of the constants occurring in that system. It is well-known that, if there exists a solution (over $\mathbb{N}$), then there exists such a solution in which all values are bounded by $K=((k+1)C)^r$—i.e. by an exponential function of the size of the system [2]. Evidently, therefore, given a system $\mathcal{E}$ of linear Diophantine *clauses* of the forms (2)–(5), the same bound applies. Now let $R=kCK$, and replace any clause in $\mathcal{E}$ of the form

$$(\vec{a}\cdot\vec{z}>0)\vee(\vec{b}\cdot\vec{z}\geq\vec{c}\cdot\vec{z}) \tag{7}$$

by the corresponding linear equality

$$R\vec{a}\cdot\vec{z}+\vec{b}\cdot\vec{z}-\vec{c}\cdot\vec{z}\geq 0. \tag{8}$$

Let the resulting system of linear inequalities be $\mathcal{E}'$. We first observe that $\mathcal{E}'$ entails $\mathcal{E}$. For if (8) holds, the corresponding instance of (7) clearly does too. Conversely, if $\mathcal{E}$ has a solution, then so has $\mathcal{E}'$. For consider a solution of $\mathcal{E}$ in which all entries are bounded by K, so that the expression $\vec{c}\cdot\vec{z}$ is at most $R=kCK$. If $\vec{a}\cdot\vec{z}=0$, then (7) guarantees $\vec{b}\cdot\vec{z}-\vec{c}\cdot\vec{z}\geq 0$; on the other hand, if $\vec{a}\cdot\vec{z}\geq 1$, then $R\vec{a}\cdot\vec{z}-\vec{c}\cdot\vec{z}\geq 0$. Either way, (8) is satisfied, as required.

Hence, the problem of determining the satisfiability (over $\mathbb{N}$) of a system of linear Diophantine *clauses* of the forms (2)–(5) can be reduced in polynomial time to the problem of determining the corresponding problem for systems of linear Diophantine *inequalities* of the form $\vec{a}\cdot\vec{z}\geq b$ with non-negative b. Evidently such a system has a solution over $\mathbb{N}$ if and only if it has a solution over the non-negative rationals. The result then follows from the fact that linear programming feasibility is in PTIME [7]. ■

3.3 Complexity of BGE*

In this subsection we show that BGE* can be solved in polynomial time. (The technique employed is not essentially different from that used in [10] to show membership in PTIME of BGE.) Rather than introducing infinite values to the systems of equations considered in Section 3.2, we proceed by reduction to the satisfiability problem for propositional Horn clauses. Recall, in this connection, that, if $X_1,\ldots,X_m,X$ are Boolean-valued variables, a *Horn clause* is an implication of either of the forms $X_1\wedge\cdots\wedge X_m\rightarrow X$ or $X_1\wedge\cdots\wedge X_m\rightarrow\bot$, interpreted in the usual way. It is well-known that the problem of determining the satisfiability of a collection of Horn clauses is in PTIME.

THEOREM 12
BGE* is in PTIME.

PROOF. Let $\mathcal{P}=(\Delta,\Delta_0,M,P,Q)$ be an instance of BGE*. For $p\in P$, let X_p be a proposition letter, which we may informally read as 'There are no left-hand vertices whose order-function realises p.' Similarly, for $q\in Q$, let Y_q be a proposition letter, which we may informally read as 'There are no right-hand vertices whose order-function realises q.' Consider the set Γ of the following propositional

Horn clauses

$$\bigwedge_{q\in Q:\, q(\delta)\neq^{=}0} Y_q \quad\rightarrow\quad X_p \quad \text{for all } p\in P, \delta\in\Delta \text{ s.t. } \bar{p}(\delta)>0 \tag{9}$$

$$\bigwedge_{p\in P:\, p(\delta)\neq^{=}0} X_p \quad\rightarrow\quad Y_q \quad \text{for all } q\in Q, \delta\in\Delta \text{ s.t. } \bar{q}(\delta)>0 \tag{10}$$

$$\bigwedge_{p\in P:\, p(\delta)\neq^{=}0} X_p \quad\rightarrow\quad \perp \quad \text{for all } \delta\in\Delta_0 \tag{11}$$

$$\bigwedge_{q\in Q:\, q(\delta)\neq^{=}0} Y_q \quad\rightarrow\quad \perp \quad \text{for all } \delta\in\Delta_0. \tag{12}$$

Intuitively, (9) says 'For all $\delta\in\Delta$, if no vertices in V are allowed to be incident to a δ-edge, then no vertices in U can be required to be incident to a δ-edge'; (10) expresses the mirror-image implication; (11) says 'For all $\delta\in\Delta_0$, some vertices in U are allowed to be incident to some δ-edges'; and (12) expresses the same condition for vertices of V.

Suppose Γ is satisfiable. For each $p\in P$ such that X_p is false, take a countably infinite set U_p, and for each $q\in Q$ such that Y_q is false, take a countably infinite set V_q. Let $U=\bigcup_{p\in P} U_p$ and $V=\bigcup_{q\in Q} V_q$. For each $\delta\in\Delta$, for each $p\in P$, and each $u\in U_p$, attach $\bar{p}(\delta)$ 'dangling' δ-labelled edges to u; and similarly for the elements of V, using the functions $q\in Q$. By (9), if a dangling δ-labelled edge is attached to some vertex of U, then there is $q\in Q$ with $q(\delta)\neq^{=}0$ such that Y_q is false. Now, either we already have a dangling δ-labelled edge attached to some vertex of V or no dangling δ-labelled edge is attached to any vertex of V, but there is $q\in Q$ with $q(\delta)=^{\geq}0$ such that Y_q is false, in which case, we may choose any such q, and for each $v\in V_q$, attach one dangling δ-edge. In this way, we ensure that, if there is a dangling δ-labelled edge attached to some vertex of U (hence infinitely many vertices of U), then there will be a dangling δ-labelled edge attached to infinitely many vertices of V. Using (10), we likewise ensure that if there is a dangling δ-labelled edge attached to some vertex of V, then there will be a dangling δ-labelled edge attached to infinitely many vertices of U. All the resulting dangling edges can then easily be matched up without clashes, thus forming an infinite Δ-graph. Finally, for every $\delta\in\Delta_0$, if there is no δ-edge attached to any vertex of U, then using (11)-(12), we find $p\in P$ and $q\in Q$ such that $p(\delta)\neq^{=}0$, X_p is false, $q(\delta)\neq^{=}0$ and Y_q is false and we find $u\in U_p$ and $v\in V_q$ such that u and v are not connected by any edge and add a δ-edge from u to v. Hence $\mathcal{P}$ is a positive instance of BGE*.

Conversely, if $\mathcal{P}$ is a positive instance of BGE*, let $H=(U,V,\mathbf{E}_\Delta)$ be a solution. Now interpret the variables X_p and Y_q as indicated above. It is obvious that (9)–(12) hold. Thus, Γ is satisfiable. This completes the reduction. ∎

4 Surgery on classes

The material in this section corresponds to the analysis of EC_2^2 in Section 6.1 of [10]. The principal result is Lemma 20, which shows that the $r_1^\#$- and $r_2^\#$-classes in any model of a given GFEC_2^2-formula have 'approximations' whose size is bounded exponentially in the cardinality of the interpreted signature. In the corresponding Lemma 6.2 of [10], approximations of classes were of doubly exponential size, which would not suffice for the 2-ExpTime complexity-bound established here.

When discussing induced substructures in EC_2^2, a subtlety arises regarding the interpretation of closure operations. If $B\subseteq A$, we take it that, in the structure $\mathfrak{B}$ induced by B, the interpretation of

$r_i^\#$ is given by simple restriction: $(r_i^\#)^{\mathfrak{B}} = (r_i^\#)^{\mathfrak{A}} \cap B^2$. This means that, while $(r_i^\#)^{\mathfrak{B}}$ is certainly an equivalence relation including $r_i^{\mathfrak{B}}$, it may not be the smallest, since, for some $a, a' \in B$, all r_i-paths connecting a and a' may contain elements which are not members of B. (Indeed, this might be so even if B is an intersection, as exemplified by setting $B = E_2^1 \cap E_2^2$ in the structure of Figure 1a.) To facilitate the treatment of substructures in the sequel, we weaken the restrictions on the interpretation of $r_1^\#$ and $r_2^\#$ given in Section 2. Henceforth, we shall continue to assume that $r_i^\#$ is an equivalence relation that includes the relation r_i; however, we shall not assume that it is the *smallest* such equivalence relation—i.e. we shall not insist that $r_i^\#$ be the equivalence closure of r_i. A structure in which $r_i^\#$ *is* the equivalence closure of r_i will be said to be *perfect*. We shall by convention use the (possibly decorated) letter $\mathfrak{A}$ to denote perfect structures; we use other letters, $\mathfrak{B}$, $\mathfrak{C}$, …(again, possibly decorated), where no such guarantee applies. Typically, but not always, these latter structures will be induced substructures. Thus, in our new terminology, what interests us in this article is the problem of determining whether a given guarded two-variable formula is satisfied in some (finite) perfect structure.

For the rest of this section we fix a normal form GFEC_2^2-formula φ over signature $\tau = \tau_0 \cup \{r_1, r_2\} \cup \{r_1^\#, r_2^\#\}$. Recall from Section 2 that, if $\mathfrak{A}$ is a (perfect) structure, then an *intersection* of $\mathfrak{A}$ is an equivalence class of the relation $r_1^\# \cap r_2^\#$, and an $r_i^\#$-*class of* $\mathfrak{A}$ is an equivalence class of the relation $r_i^\#$. The following definition frees these notions from the containing structure $\mathfrak{A}$.

Definition 13
A τ-structure $\mathfrak{I}$ is a *pre-intersection* if, for all $a, a' \in I$, $\mathfrak{I} \models r_1^\#[a,a'] \wedge r_2^\#[a,a']$. A τ-structure $\mathfrak{D}$ is an $r_i^\#$-*class* if D is r_i-connected.

Clearly, if $\mathfrak{A}$ is a perfect structure, and I is an intersection of $\mathfrak{A}$, then the induced substructure $\mathfrak{I}$ on I is a pre-intersection. Likewise, if D is an $r_i^\#$-class of $\mathfrak{A}$, then the induced sub-structure $\mathfrak{D}$ on D is an $r_i^\#$-class. The slightly different nomenclature here reflects the slightly different character of these notions. If $\mathfrak{D}$ is a $r_i^\#$-class, then, by definition, for all $a, a' \in D$, there is an r_i-path in $\mathfrak{D}$ from a to a', whence $\mathfrak{D} \models r_i^\#[a,a']$. In contrast, if $\mathfrak{I}$ is a pre-intersection, then, by definition, for all $a, a' \in I$, $\mathfrak{I} \models r_1^\#[a,a'] \wedge r_2^\#[a,a']$; however, it is perfectly possible for $\mathfrak{I}$ to be neither r_1- nor r_2-connected. It is, nevertheless, obvious that every $r_i^\#$-class may be unambiguously decomposed into pre-intersections, namely, the equivalence classes of $r_{3-i}^\#$ that it includes.

We will use pre-intersections as building blocks to construct bigger structures in which they will eventually become intersections. By the *type* of a pre-intersection, we mean its isomorphism type. Recalling Lemma 5, let Δ be the set of all types of pre-intersections $\mathfrak{I}$ of size bounded by $\mathfrak{f}(|\tau|)$ such that $\mathfrak{I} \models \varphi_{12}$. Thus, $|\Delta|$ is doubly exponentially bounded as a function of $|\tau|$. Likewise, for $i = 1, 2$, let Ω_i be the set of countable $r_i^\#$-classes $\mathfrak{D}$, all of whose pre-intersections have types from Δ, and which satisfy the condition $\mathfrak{D} \models \varphi_i$. Evidently, if $\mathfrak{A}$ is a countable structure such that $\mathfrak{A} \models \varphi$, and each intersection of $\mathfrak{A}$ is of size at most $\mathfrak{f}(|\tau|)$, then the $r_i^\#$-classes of $\mathfrak{A}$ all lie in Ω_i.

We are now ready to develop the promised apparatus for approximating $r_i^\#$-classes occurring in perfect structures. (The precise sense in which this is an approximation will emerge in the course of this section.) In the sequel, we denote by $\mathbb{N}$ the set of non-negative integers, and by $\mathbb{N}^*$ the set $\mathbb{N} \cup \{\aleph_0\}$. We take it that $n < \aleph_0$ and $n + \aleph_0 = \aleph_0 + n = \aleph_0$, for all $n \in \mathbb{N}$. If $f : X \to \mathbb{N}^*$ is a function with finite domain X, we write $\|f\| = \sum_{x \in X} f(x)$. Note that $\|f\|$ may equal $\aleph_0$.

Definition 14
Let $\mathfrak{D}$ be in Ω_i ($i \in \{1,2\}$). The *characteristic function* of $\mathfrak{D}$ is the function $\mathrm{Ch}^{\mathfrak{D}} : \Delta \to \mathbb{N}^*$ given by:

$$\mathrm{Ch}^{\mathfrak{D}}(\delta) = |\{\mathfrak{I} : \mathfrak{I} \text{ is an intersection of } \mathfrak{D} \text{ of type } \delta\}|.$$

That is, $\mathrm{Ch}^{\mathfrak{D}}$ simply counts the number of pre-intersections of each possible type occurring in $\mathfrak{D}$. We are particularly interested in those characteristic functions $\mathrm{Ch}^{\mathfrak{D}}$ with the property that, for specific $\delta \in \Delta$, there exists some $\mathfrak{D}' \in \Omega_i$ which is just like $\mathfrak{D}$, but which has an additional intersection of type δ. As we might say: $\mathfrak{D}'$ is the result of 'inflating' $\mathfrak{D}$ by a δ-intersection. The following definitions provide an operational characterization of these functions.

Definition 15
For a given isomorphism type of a pre-intersection δ and a 1-type α we say that δ *realises* α if, for $\mathfrak{I}$ of type δ, $\mathfrak{I} \models \exists x \alpha(x)$. For a given function $f : \Delta \to \mathbb{N}^*$ we say that f *realises* α if δ realises α for some $\delta \in \Delta$ such that $f(\delta) \geq 1$.

Definition 16
We say that a pre-intersection type δ is *i-adjoinable* ($i \in \{1,2\}$) to a function $f : \Delta \to \mathbb{N}^*$, and write $\delta \rhd_i f$, if the following three conditions hold:

(i) every 1-type realised by δ is realised by f;
(ii) for any 1-types α realised by δ and α' realised by f (possibly $\alpha = \alpha'$) there exists a 2-type β such that $\beta(x,y) \models \varphi_{\mathrm{univ}}$ and

$$\alpha(x) \cup \alpha'(y) \cup \{r_i^{\#}(x,y) \wedge \neg r_{3-i}^{\#}(x,y)\} \subseteq \beta(x,y);$$

(iii) for any pre-intersection $\mathfrak{I}$ of type δ, any of its r_i-connected components contains an element of 1-type α such that $K_i(x) \in \alpha$.

The above conditions can be motivated by imagining trying to add a pre-intersection of type δ to a structure $\mathfrak{D} \in \Omega_i$, where $f = \mathrm{Ch}^{\mathfrak{D}}$: condition (i) ensures that no new 1-types are introduced; condition (ii) ensures that all relevant 2-types can be filled in in accordance with φ_{univ}; condition (iii) ensures that the resulting structure will be r_i-connected. We mention in passing that i-adjoinability is relatively easy to secure, in the following sense: if a pre-intersection type δ is realised at least twice in $\mathfrak{D}$, then δ is adjoinable to $\mathfrak{D}$. (This fact is not, logically, required for the ensuing argument.)

Fact 17
If $\mathfrak{D} \in \Omega_i$ ($i \in \{1,2\}$) and $\delta \in \Delta$ such that $\mathrm{Ch}^{\mathfrak{D}}(\delta) \geq 2$, then $\delta \rhd_i \mathrm{Ch}^{\mathfrak{D}}$.

Proof. Write $f = \mathrm{Ch}^{\mathfrak{D}}$. Condition (i) follows from the fact that $f(\delta) > 0$. To see (ii) consider any 1-type α realised in δ. Assume that $\mathfrak{I}_1$, $\mathfrak{I}_2$ are distinct pre-intersections of $\mathfrak{D}$ of type δ promised by f. Let $a \in I_1$ be a realisation of α. For any 1-type α' realised by f we can find in D a realisation a' of α' such that $a' \notin I_1$. In particular, if α' is realised only in pre-intersections of type δ then a' can be found in I_2. Now we choose β to be $\mathrm{tp}^{\mathfrak{D}}(a,a')$. We have that $r_i^{\#}(x,y) \in \beta(x,y)$ since $\mathfrak{D}$ is r_i-connected and $\neg r_{3-i}^{\#}(x,y) \in \beta(x,y)$ since a and a' are in different pre-intersections. Obviously $\beta(x,y) \models \varphi_{\mathrm{univ}}$, since φ_{univ} is a fragment of φ_1 and $\mathfrak{D} \models \varphi_1$. Finally, to see (iii), consider a realisation $\mathfrak{I}$ of δ in $\mathfrak{D}$ and one of its r_i-connected components B. Since there are at least two pre-intersections in $\mathfrak{D}$ and $\mathfrak{D}$ is r_i-connected it follows that there must be at least one r_i-edge with one of its endpoints in B and the other in $D \setminus B$, and hence in $D \setminus I$. Both endpoints in such an edge must be marked by K_i due to the ($\mathrm{K}_{\forall\forall}$) subformula of φ. ■

The following definition in effect lifts the notion of i-adjoinability to the level of functions. (It is in this form that this notion gets used in Lemma 19.)

DEFINITION 18
Let $i \in \{1,2\}$. For functions $f, g : \Delta \to \mathbb{N}^*$ we say that g *safely i-extends* f, and write $g \succeq_i f$, if for all $\delta \in \Delta$, (i) $g(\delta) \geq f(\delta)$, and (ii) $g(\delta) > f(\delta)$ implies $\delta \rhd_i f$.

With this apparatus at our disposal, the following two lemmas make possible the basic model-surgery underlying the complexity bound in Theorem 11. The first allows us to 'inflate' $r_i^{\#}$-classes by adjoining pre-intersections. This lemma can be seen as an improved version of Lemma 6.1 from [10], in which a pre-intersection could be added to a class $\mathfrak{D}$ if its type already appeared in $\mathfrak{D}$ at least twice. (In the present paper, we allow some intersections to be added even if their type is not yet realised in $\mathfrak{D}$.) The second lemma is much more involved: it allows us to 'deflate' $r_i^{\#}$-classes by removing intersections. This lemma can be seen as an improved version of Lemma 6.2 of [10], in which a doubly-exponential bound on the characteristic function of the deflated class was obtained. (In the present article, we obtain a singly exponential bound.)

LEMMA 19
Let $\mathfrak{D} \in \Omega_i$ ($i \in \{1,2\}$), and let $g : \Delta \to \mathbb{N}^*$. If $g \succeq_i \mathrm{Ch}^{\mathfrak{D}}$, then there exists $\mathfrak{D}' \models \varphi_i$ such that $\mathrm{Ch}^{\mathfrak{D}'} = g$.

PROOF. We build the structure $\mathfrak{D}'$ by adding to $\mathfrak{D}$ an appropriate number of fresh pre-intersections of each type. Formally, for each $\delta \in \Delta$ such that $g(\delta) > \mathrm{Ch}^{\mathfrak{D}}(\delta)$, construct $g(\delta) - \mathrm{Ch}^{\mathfrak{D}}(\delta)$ fresh pre-intersections of type δ. (Here, we suppose $\aleph_0 - n = \aleph_0$ for all $n \in \mathbb{N}$.) Let $\mathscr{I}$ be the set of all these pre-intersections, and let D' be the result of adding their domains to D, i.e. $D' = D \cup \bigcup\{I \mid \mathfrak{I} \in \mathscr{I}\}$. Let the restriction of $\mathfrak{D}'$ to D be $\mathfrak{D}$; similarly, for each $\mathfrak{I} \in \mathscr{I}$, let the restriction of $\mathfrak{D}'$ to I be $\mathfrak{I}$.

We next define the connections between $\mathfrak{D}$ and each of the newly added pre-intersections. Note that by Definition 18 the type of each of these new pre-intersections, $\mathfrak{I}$, is i-adjoinable to $\mathrm{Ch}^{\mathfrak{D}}$. For each element $a \in I$, using condition (i) of Definition 16, find an element $a' \in D$ realising the same 1-type α as a. For each $b \in D$, $b \neq a'$, set $\mathrm{tp}^{\mathfrak{D}'}(a,b) := \mathrm{tp}^{\mathfrak{D}}(a',b)$. Set $\mathrm{tp}^{\mathfrak{D}'}(a,a') := \beta$, where β is a 2-type promised by condition (ii) of Definition 16 for $\alpha' = \alpha$. This guarantees that any element a has appropriate witnesses inside $\mathfrak{D}$ and that $\mathfrak{D}' \restriction (D \cup I) \models \varphi_{\mathrm{univ}}$. Indeed, by Definition 3, φ_{univ} is a conjunct of φ_{12}, and we recall from earlier in this section that every $\delta \in \Delta$ is the type of some pre-intersection $\mathfrak{I}$ of size bounded by $\mathfrak{f}(|\tau|)$ such that $\mathfrak{I} \models \varphi_{12}$. The r_i-connectedness of $\mathfrak{D}'$ now follows from condition (iii) of Definition 16. To see this, consider any newly-added pre-intersection $\mathfrak{I}$. Any element $a \in I$ satisfying K_i must have an r_i-edge to D due to the $\mathrm{K}_{\forall\exists}$ conjunct of φ; and such elements exist in all connected components of the pre-intersection, as the types of $\mathfrak{I}$ is i-adjoinable to $\mathrm{Ch}^{\mathfrak{D}}$. Yet $\mathfrak{D}$ itself is by assumption r_i-connected.

It remains to define the connections between the newly added pre-intersections. Let $\mathfrak{I}$, $\mathfrak{I}'$ be two distinct newly added pre-intersections, and let $a \in I$, $a' \in I'$ be elements realising 1-types α, α', respectively. Condition (i) of Definition 16 guarantees that α' is realised by $\mathrm{Ch}^{\mathfrak{D}}$. Set $\mathrm{tp}^{\mathfrak{D}'}(a,a') := \beta$, where β is a 2-type promised by part (ii) of Definition 16 . This ensures that $\mathfrak{D}' \models \varphi_{\mathrm{univ}}$. Together with the earlier remarks we have that $\mathfrak{D}' \models \varphi_i$ and $\mathrm{Ch}^{\mathfrak{D}'} = g$. ■

Recall from Section 2 that, for any function $f : \Delta \to \mathbb{N}^*$, $\|f\|$ denotes the sum of the values of f, i.e., if f is a characterictic function of a class $\mathfrak{D}$ then $\|f\|$ is the number of intersections in $\mathfrak{D}$.

LEMMA 20
There exists a fixed exponential function $\mathfrak{g} : \mathbb{N} \to \mathbb{N}$ with the following property. For every $\mathfrak{D} \in \Omega_i$ ($i \in \{1,2\}$), there exists $\mathfrak{D}' \in \Omega_i$ such that $\mathrm{Ch}^{\mathfrak{D}} \succeq_i \mathrm{Ch}^{\mathfrak{D}'}$ and $\|\mathrm{Ch}^{\mathfrak{D}'}\| \leq \mathfrak{g}(|\tau|)$.

PROOF. We prove the lemma for $i = 1$. The proof for $i = 2$ is symmetric. Let m be the number of $\forall\exists$-conjuncts in φ_1. We proceed by first selecting a sub-model $\mathfrak{D}_0 \subseteq \mathfrak{D}$, and then modifying $\mathfrak{D}_0$ to produce the required structure $\mathfrak{D}'$. The selection of $\mathfrak{D}_0$ comprises four steps.

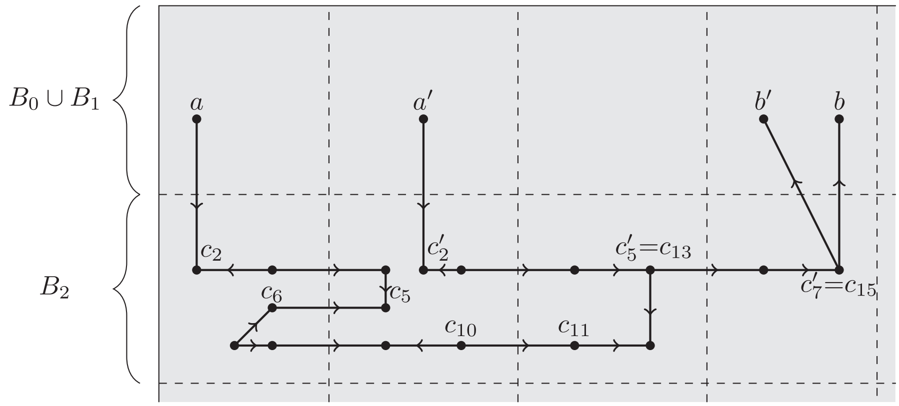

FIGURE 2. Connecting $B_0 \cup B_1$ in *Step 3* of the proof of Lemma 20.

Step 1. For each 1-type α realised in $\mathfrak{D}$, mark m distinct pre-intersections containing a realisation of α (or all such pre-intersections if α is realised in fewer than m of them). Let B_0 be the union of all pre-intersections marked in this step. In the sequel, we will add additional elements to form $\mathfrak{D}'$. Note that, for any such newly added element whose connections with B_0 are not fixed, B_0 will be able to supply any witnesses required by the $\forall\exists$-conjuncts in φ_1.

Step 2. For each $a \in B_0$ and each $(\forall\exists)$-conjunct $\forall x(\gamma(x) \to \exists y(r_1^\#[a,b] \wedge \neg r_2^\#[a,b] \wedge x \neq y \wedge \psi(x,y)))$ in φ_1, if $\mathfrak{D} \models \gamma[a]$, then find a witness $b \in D$ such that $\mathfrak{D} \models r_1^\#[a,b] \wedge \neg r_2^\#[a,b] \wedge \psi[a,b]$, and mark the pre-intersection of b. Let B_1 be the union of all pre-intersections marked in this step. Thus, witnesses have now been found for all elements of B_0.

Step 3. Consider each pair of elements $a, b \in B_0 \cup B_1$ which are not r_1-connected in $\mathfrak{D} \restriction (B_0 \cup B_1)$, but such that there exists an r_1-path of the form $a = c_1, c_2, \ldots, c_l = b$ with $c_2 \ldots, c_{l-1} \notin B_0 \cup B_1$. (See Figure 2.) Choose one such path, call it P_{ab}, and mark the pre-intersections of $c_2, \ldots, c_{l-1}$. (Obviously, an r_1-path in $\mathfrak{D}$ between a and b exists for all $a, b \in B_0 \cup B_1$, as $\mathfrak{D}$ is r_1-connected.) Note that we consider only such paths in which all elements, except a and b, lie outside $B_0 \cup B_1$, since these are sufficient to connect the whole of $B_0 \cup B_1$. Additionally, call the two elements c_2 and c_{l-1} *peripheral connectors* (of P_{ab}). In Figure 2, c_2 and c_{15} are the peripheral connectors of P_{ab} and c_2' and c_7' are the peripheral connectors of $P_{a'b'}$. Note that $c_7' = c_{15}$ is a peripheral connector of both paths. Let B_2 be the set of all pre-intersections marked in this step. Note that after this step any pair of elements from $B_0 \cup B_1$ is r_1-connected in $\mathfrak{D} \restriction (B_0 \cup B_1 \cup B_2)$. This does not mean however that $\mathfrak{D} \restriction (B_0 \cup B_1 \cup B_2)$ is r_1-connected (since some pre-intersections from B_2 may not be internally r_1-connected).

Step 4. For any element a being a peripheral connector or belonging to $B_1 \setminus B_0$, and for each conjunct $\forall x(\gamma(x) \to \exists y(r_1^\#[a,b] \wedge \neg r_2^\#[a,b] \wedge x \neq y \wedge \psi(x,y)))$ from φ_1, if $\mathfrak{D} \models \gamma[a]$, then find a witness $b \in D$ such that $\mathfrak{D} \models r_1^\#[a,b] \wedge \neg r_2^\#[a,b] \wedge \psi[a,b]$, and mark the pre-intersection I of b, if I is not included in $B_0 \cup B_1$. Let B_3 be the union of all pre-intersections marked in this step.

Let $\mathfrak{D}_0 = \mathfrak{D} \restriction (B_0 \cup B_1 \cup B_2 \cup B_3)$. We now modify $\mathfrak{D}_0$ to yield the desired structure $\mathfrak{D}'$. At this moment recall that: (i) elements from B_0, B_1 and peripheral connectors from B_2 have the required witnesses in $\mathfrak{D}_0$; (ii) the sizes of B_0, B_1, B_3 and the number of peripheral connectors are bounded exponentially in $|\tau|$; (iii) all pairs of elements from $B_0 \cup B_1$ are r_1-connected in $\mathfrak{D}_0$. What remains is to decrease the size of B_2 (which at this moment is unbounded), provide any remaining witnesses, and make the whole class r_1-connected.

Consider a path P_{ab} of the form $a = c_1, c_2, \ldots, c_{l-1}, c_l = b$ chosen in *Step 3*. Repeat the following procedure as long as possible: if for some i, j, $3 \le i < j \le l-2$ such that the 1-types of c_i and c_j are identical, c_{i-1} does not belong to the pre-intersection of c_i, c_{j-1} does not belong to the pre-intersection of c_j, and c_{i-1} does not belong the the pre-intersection of c_j, then remove elements $c_i, \ldots, c_{j-1}$ from P_{ab} and make the connection between c_{i-1} and c_j equal to the connection between c_{i-1} and c_i (note that this connection must contain an r_1-edge; thus after this cut, P_{ab} remains an r_1-path). For example, if c_6 and c_{11} from Figure 2 have the same 1-types then P_{ab} can be shortened by removing $c_6, \ldots, c_{10}$ from P_{ab} and linking c_5 directly to c_{11}.

Let us see that the length of the final version of P_{ab} is exponentially bounded. Call an element c_i of P_{ab}, $i \ge 3$, an *entry element*, if c_{i-1} does belong to the intersection of c_i. Note that a pre-intersection may contain more than one entry element. Assume that the total number of entry elements in P_{ab} is greater than $2|\boldsymbol{\alpha}|$, where $\boldsymbol{\alpha}$ is the set of 1-types over τ. We claim that in this case a cut is possible. Indeed, among the entry elements there are at least three, c_p, c_r, c_q, with $p < r < q$, having the same 1-types. Assume to the contrary that a cut is not possible. This implies that c_{p-1} and c_r are in the same pre-intersection, c_{p-1} and c_q are in the same pre-intersection, and c_{r-1} and c_q are in the same pre-intersection. But then, c_{r-1} and c_r are in the same pre-intersection which contradicts the assumpion that c_r is an entry element.

This shows that a path which does not admit a cut has at most $2|\boldsymbol{\alpha}|$ entry elements. Since the number of elements between two consecutive entry elements is bounded by the size of a pre-intersection, and the size of a pre-intersection as well as the size of $\boldsymbol{\alpha}$ are both bounded exponentially, we get that the final version of P_{ab} is also exponentially bounded.

Perform the above process for all paths P_{ab} chosen in *Step 3*. Let B_2' contain only pre-intersections of elements from the paths P_{ab} so shortened. The size of $B_0 \cup B_1 \cup B_2' \cup B_3$ is now exponentially bounded.

For any element a from $B_2' \cup B_3$ which is not a peripheral connector, modify its connections to B_0 in such a way that witnesses for a required by the $\forall\exists$-conjuncts of φ_i can be found in B_0. This is possible due to our choice of B_0 in *Step 1*, since a may require at most m witnesses: we keep the original witnesses unless they are not in B_0; if they are not in B_0, there are at least m elements in B_0 with the same 1-type to choose from, so we can pick alternative witnesses in B_0. We remark that the special treatment of peripheral connectors is important here: they differ from the remaining elements of $B_2 \cup B_3$ in that their connections to some elements from B_0 could earlier have been fixed to contain r_1-edges. Recall in this regard that an element may be a peripheral connector of many chosen paths. Denote the resulting structure (over domain $B_0 \cup B_1 \cup B_2' \cup B_3$) by $\mathfrak{D}'$.

We claim that $\mathfrak{D}'$ has the required properties. Indeed, we have already observed that $|B_0 \cup B_1 \cup B_2' \cup B_3|$ is exponentially bounded in $|\tau|$. Furthermore, all 2-types occurring in $\mathfrak{D}'$ occur in $\mathfrak{D}$, and we took care to provide all required witnesses, whence $\mathfrak{D}' \models \varphi_1$. The r_1-connectivity of $\mathfrak{D}'$ now follows from the fact that r_1-paths connect all pairs of elements from $B_0 \cup B_1$ and each of the remaining elements is r_1-connected to some element in its pre-intersection satisfying K_1 which further must have a witness in B_0 connected to it by a direct r_1-edge due to the ($\mathrm{K}_{\forall\exists}$) conjunct of φ.

It remains to see that $\mathrm{Ch}^{\mathfrak{D}} \succeq_1 \mathrm{Ch}^{\mathfrak{D}'}$. Since we built $\mathfrak{D}'$ using only pre-intersections from $\mathfrak{D}$ and we did not change the connections inside pre-intersections it is clear that for all $\delta \in \Delta$ we have $\mathrm{Ch}^{\mathfrak{D}}(\delta) \ge \mathrm{Ch}^{\mathfrak{D}'}(\delta)$. Consider δ such that $\mathrm{Ch}^{\mathfrak{D}}(\delta) > \mathrm{Ch}^{\mathfrak{D}'}(\delta)$. We must show that $\delta \rhd_1 \mathrm{Ch}^{\mathfrak{D}'}$. We consider the conditions of Definition 16 in turn. For condition (i), let α be realised in δ. In *Step 1* we choose a pre-intersection containing α and make it a member of B_0 which later becomes a fragment of $\mathfrak{D}'$. Thus α is realised by $\mathrm{Ch}^{\mathfrak{D}'}$. For condition (ii), consider any α realised by δ and any α' realised by

$\mathrm{Ch}^{\mathfrak{D}'}$. Let $\mathfrak{I}_2$ be a pre-intersection of $\mathfrak{D}'$ which contains an element a' of type α'. By our construction $\mathfrak{I}_2$ is also a pre-intersection of $\mathfrak{D}$. Let $\mathfrak{I}_1$ be a pre-intersection of type δ in $\mathfrak{D}$ different from $\mathfrak{I}_2$ (we can choose such $\mathfrak{I}_1$ even if $\mathfrak{I}_2$ is of type δ since we know that $\mathrm{Ch}^{\mathfrak{D}}(\delta) > \mathrm{Ch}^{\mathfrak{D}'}(\delta)$). Let a be an element of type α in $\mathfrak{I}_1$. Set $\beta = \mathrm{tp}^{\mathfrak{D}}(a, a')$. It is readily verified that β is as required. For condition (iii), note that $|\mathrm{Ch}^{\mathfrak{D}'}| \geq 1$ and, since $\mathrm{Ch}^{\mathfrak{D}}(\delta) > \mathrm{Ch}^{\mathfrak{D}'}(\delta)$, it must be that $|\mathrm{Ch}^{\mathfrak{D}}| \geq 2$; the argument then follows from the fact that $\mathfrak{D}$ is r_1-connected, just as in the proof of Fact 17. ■

5 From logic to graphs

We now present a reduction of the (finite) satisfiability problem for normal-form GFEC_2^2-formulas to the problem (finite) BGE*, running in time bounded by a doubly-exponential function of $\|\varphi\|$. The reduction employs some non-deterministic guesses, but all of them are of size exponential in $|\tau|$. This ensures that the eventual decision procedure runs in deterministic doubly-exponential time.

Let a normal-form GFEC_2^2-formula φ, with signature τ, be given. Let the function $\mathfrak{f}$ be as in Lemma 5; let Δ be the set of pre-intersections over τ of size bounded by $\mathfrak{f}(|\tau|)$, as in Section 4; let $\mathfrak{g}$ be the function given in Lemma 20; and let $M = \mathfrak{g}(|\tau|)$. Recall that, if φ has a (finite) model, then it has a (finite) model all of whose intersections are of cardinality bounded by $\mathfrak{f}(|\tau|)$; furthermore, any $r_i^{\#}$-class in such a model may be 'deflated' to one whose characteristic f satisfies $\|f\| \leq M$, in the precise sense of Lemma 20. (Remember that this does not necessarily mean that we can construct a model of φ whose $r_i^{\#}$-classes have characteristic functions bounded in this way.) Note that M is singly exponentially bounded as a function of $\|\varphi\|$, while $|\Delta|$ is doubly exponentially bounded. We proceed to construct a subset $\Delta_0 \subseteq \Delta$ and sets P, Q of constraint functions mapping Δ to $\hat{M}$. The (finite) BGE*-instance $\langle \Delta, \Delta_0, M, P, Q \rangle$ will then give us the desired reduction.

We require two additional pieces of machinery to carry out the construction. The first allows us to transform the characteristic functions of certain $r_i^{\#}$-classes into constraint functions.

Definition 21
Let $f : \Delta \to \{0, \ldots, M\}$ be a function and $i \in \{1, 2\}$. Define $f^{\rhd i} : \Delta \to \hat{M}$ as follows:

$$f^{\rhd i}(\delta) = \begin{cases} {}^{\geq}f(\delta) & \text{if } \delta \rhd_i f \\ {}^{=}f(\delta) & \text{otherwise.} \end{cases}$$

Fact 22
Let $f : \Delta \to \{0, \ldots, M\}$, $g : \Delta \to \mathbb{N}^*$, and $i \in \{1, 2\}$. If $g \succeq_i f$ then g realises $f^{\rhd i}$.

Proof. Follows directly from definitions. ■

To see the point of this definition, suppose $\mathfrak{D}$ is an $r_i^{\#}$-class from Ω_i, and suppose that its characteristic function $f = \mathrm{Ch}^{\mathfrak{D}}$ satisfies $\|f\| \leq M$. (Recall that, since $\mathfrak{D} \in \Omega_i$, $\mathfrak{D} \models \varphi_i$.) According to Lemma 19, if f' realises $f^{\rhd i}$, then there exists an $r_i^{\#}$-class $\mathfrak{D}'$ such that $\mathfrak{D}' \models \varphi_i$ and $\mathrm{Ch}^{\mathfrak{D}'} = f'$. That is, $f^{\rhd i}$ gives us a sufficient condition for a function to be the characteristic function of an $r_i^{\#}$-class $\mathfrak{D}'$ from Ω_i

Our second piece of machinery involves a simple re-arrangement of the material in the normal form of φ. Recall from Definition 3 that φ features the conjuncts φ_{univ} and $\varphi_{\mathrm{free}}^{-}$. Each conjunct of φ_{univ} is either of the form $\forall x.\psi(x)$ or the form $\forall x \forall y.\psi(x,y)$, with $\psi(x)$, $\psi(x,y)$ quantifier-free. Let $\psi_{\forall}(x)$ be the conjunction of all the formulas $\psi(x)$ such that $\forall x.\psi(x)$ is a conjunct of φ_{univ}; and let $\psi_{\forall\forall}(x,y)$ be the conjunction of all those $\psi(x,y)$ such that $\forall x \forall y.\psi(x,y)$ is a conjunct of φ_{univ}. Similarly, $\varphi_{\mathrm{free}}^{-}$ is a conjunction of formulas of the form

$$\forall x(\gamma(x) \to \exists y(\epsilon(x,y) \wedge \neg r_1^{\#}(x,y) \wedge \neg r_2^{\#}(x,y) \wedge x \neq y \wedge \psi(x,y)));$$

let Ψ_{free} be the set of corresponding formulas

$$\gamma(x) \to (\epsilon(x,y) \wedge \neg r_1^{\#}(x,y) \wedge \neg r_2^{\#}(x,y) \wedge x \neq y \wedge \psi(x,y)).$$

Roughly speaking, $\psi_{\forall}(x,y)$, $\psi_{\forall\forall}(x,y)$ and Ψ_{free} represent the result of stripping the quantifiers from φ_{univ} and $\varphi^{-}_{\text{free}}$.

We now present our reduction from the (finite) satisfiability problem for GFEC^2_2 to (finite) BGE*.

Reduction procedure:

(i) Non-deterministically guess a set $\boldsymbol{\alpha}$ of 1-types over τ. Intuitively, $\boldsymbol{\alpha}$ represents the set of 1-types occurring in some putative perfect model of φ.

(ii) Verify that, for each $\alpha \in \boldsymbol{\alpha}$, the quantifier-free formula $\alpha(x) \wedge \psi_{\forall}(x)$ is consistent, failing otherwise. Verify also that for each ($\exists$)-conjunct of φ, $\exists x.\psi(x)$, there exists $\alpha \in \boldsymbol{\alpha}$ such that the quantifier-free formula $\alpha(x) \wedge \psi(x)$ is consistent, failing otherwise. Intuitively, success at this step ensures satisfaction of all the ($\forall$)- and ($\exists$)-conjuncts of φ in any structure realising exactly the collection of 1-types $\boldsymbol{\alpha}$.

(iii) Verify that, for each $\alpha \in \boldsymbol{\alpha}$ and each formula $\psi(x,y) \in \Psi_{\text{free}}$, there exists $\alpha' \in \boldsymbol{\alpha}$ such that the quantifier-free formula

$$\alpha(x) \wedge \alpha'(y) \wedge \psi(x,y) \wedge \psi_{\forall\forall}(x,y) \wedge \psi_{\forall\forall}(y,x)$$

is consistent, failing otherwise. Intuitively, this step ensures that the chosen 1-types $\boldsymbol{\alpha}$ do not preclude finding free witnesses as demanded by $\varphi^{-}_{\text{free}}$, subject to the constraints imposed by φ_{univ}. This clearly can be done in deterministic exponential time.

(iv) For each $\alpha \in \boldsymbol{\alpha}$ guess a pre-intersection type $\delta_\alpha \in \Delta$ such that α is realised in δ_α and δ_α realises only 1-types from $\boldsymbol{\alpha}$, failing if this is not possible; let $\Delta_0 = \{\delta_\alpha : \alpha \in \boldsymbol{\alpha}\}$. Intuitively, Δ_0 is a set of intersection-types realised in our putative model that, between them, account for every 1-type in $\boldsymbol{\alpha}$.

(v) Let $\mathscr{D}_1$ be the set of all $r_1^{\#}$-classes $\mathfrak{D}$ such that: (i) $\mathfrak{D} \models \varphi_1$; (ii) every pre-intersection of $\mathfrak{D}$ has type in Δ; and (iii) the characteristic function $f = \text{Ch}^{\mathfrak{D}}$ satisfies $\|f\| \leq M$. (Note that any such $\mathfrak{D}$ has cardinality at most $M \cdot \mathfrak{f}(|\tau|)$, whence $|\mathscr{D}_1|$ is doubly exponentially bounded.) Now let

$$P = \{f^{\rhd 1} \mid f = \text{Ch}^{\mathfrak{D}} \text{ for some } \mathfrak{D} \in \mathscr{D}_1 \}.$$

Intuitively, P is a set of requirements on the characteristics of $r_1^{\#}$-classes imposed by φ_1, taking into account the possibilities of 'inflation' and 'deflation' allowed by Lemmas 19 and 20.

(vi) Construct Q analogously to P, but using $r_2^{\#}$ in place of $r_1^{\#}$.

The non-determinism (guessing) in this reduction is confined to steps (i) and (iv). As noted above, all guessed data-structures are exponential in $|\tau|$, whence all possibilities may be exhaustively tried in deterministic doubly-exponential time.

Proposition 23

The formula φ is (finitely) satisfiable if and only if, for some guesses of $\boldsymbol{\alpha}$ in (i) and Δ_0 in (iv), the run of the above reduction procedure produces a positive instance $(\Delta, \Delta_0, M, P, Q)$ of (finite) BGE*.

Proof. $\Rightarrow$ Let $\mathfrak{A}$ be a (finite) perfect model of φ with intersections bounded by $\mathfrak{f}(|\tau|)$ as guaranteed by Lemma 5, and recall that $M = \mathfrak{g}(|\tau|)$. In step (i), guess $\boldsymbol{\alpha}$ to be the set of 1-types realised in $\mathfrak{A}$. Step (ii) is successful because every $\exists$-conjunct and every $\forall$-conjunct of

φ is true in $\mathfrak{A}$. Step (iii) is successful because $\mathfrak{A} \models \varphi_{\mathrm{univ}} \wedge \varphi^{-}_{\mathrm{free}}$. Indeed, suppose α is in $\boldsymbol{\alpha}$ and $\psi = \gamma(x) \to (\epsilon(x,y) \wedge \neg r^{\#}_1(x,y) \wedge \neg r^{\#}_2(x,y) \wedge x \neq y \wedge \psi(x,y))$ is in Ψ_{free}. Then the corresponding conjunct $\forall x(\gamma(x) \to \exists y(\epsilon(x,y) \wedge \neg r^{\#}_1(x,y) \wedge \neg r^{\#}_2(x,y) \wedge x \neq y \wedge \psi(x,y)))$ of $\varphi^{-}_{\mathrm{free}}$ has a witness in $\mathfrak{A}$ for a, say b. Let α' be the 1-type of b. In step (iv), for each $\alpha \in \boldsymbol{\alpha}$, select one pre-intersection $\mathfrak{I}$ from $\mathfrak{A}$ containing a realisation of α (the choice is arbitrary), and let δ_α be the type of $\mathfrak{I}$. By assumption, all intersections in $\mathfrak{A}$ are of cardinality bounded by $\mathfrak{f}(|\tau|)$, so that $\delta_\alpha \in \Delta$. Thus, $\Delta_0 = \{\delta_\alpha \mid \alpha \in \boldsymbol{\alpha}\} \subseteq \Delta$. Let P and Q be the sets of constraint-functions generated in steps (v) and (vi). We claim that the BGE*-instance $(\Delta, \Delta_0, M, P, Q)$ has a (finite) solution. Recalling that $\mathfrak{A}$ may be assumed to be countable, we construct a solution, $H = (U, V, \mathbf{E}_\Delta)$, as follows: first define U to be the set of $r^{\#}_1$-classes of $\mathfrak{A}$, and V to be the set of $r^{\#}_2$-classes of $\mathfrak{A}$; then, for each $\mathfrak{D} \in U$ and each $\mathfrak{D}' \in V$, add the edge $(\mathfrak{D}, \mathfrak{D}')$ to E_δ just in case $\mathfrak{D}$ and $\mathfrak{D}'$ have a common intersection of type δ. Condition (G1) of Definition 8 is obviously satisfied. Consider condition (G2). Let $\mathfrak{D} \in U$. We know that $\mathfrak{D} \models \varphi_1$. Let $f = \mathrm{Ch}^{\mathfrak{D}} = \mathrm{ord}^H_{\mathfrak{D}}$. We want to see that f realises a function from P. By Lemma 20 there exists g such that $f \succeq_1 g$, $\|g\| \leq \mathfrak{g}(|\tau|)$ and there is a structure $\mathfrak{D}' \models \varphi_1$ such that $g = \mathrm{Ch}^{\mathfrak{D}'}$. By step (v) of the reduction procedure P contains $g^{\rhd 1}$. By Fact 22, f realises $g^{\rhd 1}$. (G3) follows analogously.

$\Leftarrow$ In the opposite direction, suppose that the BGE*-instance $(\Delta, \Delta_0, M, P, Q)$ constructed by some run of our procedure has a (finite) solution $H = (U, V, \mathbf{E}_\Delta)$. We assemble a (finite) perfect structure $\mathfrak{A}'$ of φ, whose bipartite graph of $r^{\#}_1$- and $r^{\#}_2$-classes is exactly H, and we show that $\mathfrak{A}' \models \varphi$. Thus, in terms of Figure 1, we are moving from right to left: starting with a bipartite graph whose edges are coloured with the isomorphism types of pre-intersections, we build a perfect structure whose intersections are precisely of the types labelling the corresponding edges.

To make our job easier, we first construct an auxiliary structure $\mathfrak{A}$ satisfying conditions (i)–(iii) of Fact 4, i.e., a structure satisfying all conjuncts of φ except possibly those of φ_{free}. Let $\boldsymbol{\alpha}$ be the set of 1-types chosen in step (i) of this run. We build $\mathfrak{A}$ in several steps. In the first step, for each $\delta \in \Delta$ and each $e \in E_\delta$, we construct a fresh pre-intersection, $\mathfrak{I}_e$, of type δ. Let A be the union of all the (disjoint) domains I_e of these sets, and partially define $\mathfrak{A}$ on A by setting the substructure induced on each I_e by $\mathfrak{A}$ to be $\mathfrak{I}_e$. Property (G1) of Definition 8 then ensures that, however $\mathfrak{A}$ is completed, the structure $\mathfrak{A}$ will satisfy all $\exists$-conjuncts; i.e., condition (i) of Fact 4 holds. Moreover, the fact that all intersections have types from Δ ensures condition (ii).

Consider now any vertex $u \in U$. Let $\mathcal{U}$ be the set of all pre-intersections corresponding to the edges incident to u. Our task is to compose from them an $r^{\#}_1$-class $\mathfrak{D}'$ of $\mathfrak{A}$ satisfying φ_1. Writing f for ord^H_u, let $p \in P$ be a function which is realised by f, guaranteed by (G2). By the construction of P in step (v), there exists an $r^{\#}_1$-class, $\mathfrak{D}$, such that $\mathfrak{D} \models \varphi_1$ and $\mathrm{Ch}^{\mathfrak{D}} = \bar{p}$. Note that $f \succeq_1 \bar{p}$. Indeed, if for any $\delta \in \Delta$, $f(\delta) > \bar{p}(\delta)$ then (since f realises p) $p(\delta) = {}^{\geq}k$ for some $k \in \mathbb{N}$, whence, by Definition 21, $\delta \rhd_1 \bar{p}$. Having established $f \succeq_1 \bar{p}$, we apply Lemma 19: the structure $\mathfrak{D}'$ guaranteed by that lemma amounts to adjoining all the remaining intersections of $\mathcal{U}$ to $\mathfrak{D}$, forming the desired $r^{\#}_1$-class. This ensures that every pre-intersection is r_1-connected and that condition (iii) of Fact 4 holds for $i = 1$. Notice that, when executing this step for some vertex $u \in U$, any newly-established connections involve elements of some pair of pre-intersections corresponding to distinct edges incident to u. Thus, these new connections cannot over-write each other, and cannot interfere with the internal structures of the pre-intersections.

Similarly, from any vertex $v \in V$, we form a $r^{\#}_2$-class consisting of all pre-intersections corresponding to edges incident to v, using (G3) and the construction of Q. This ensures that every pre-intersection is r_2-connected and that condition (iii) of Fact 4 holds for $i = 2$. Thus, all pre-intersections become both r_1- and r_2-connected in $\mathfrak{A}$. Again, when executing this step for some vertex $v \in V$, any newly-established connections involve elements of some pair of pre-intersections corresponding to distinct edges incident to v. Thus, these new connections cannot over-write each other, and cannot interfere with the internal structures of the pre-intersections. Moreover, no two edges incident to a common $v\in V$ can also be incident to a common $u\in U$, so that these new connections cannot overwrite those established when assembling the $r_1^\#$-classes. For the same reason, no two pre-intersections can become connected by both r_1 and r_2 through these new connections; hence, every pre-intersection becomes an intersection of $\mathfrak{A}$, as required.

At this point, we have specified the 2-type in $\mathfrak{A}$ of any pair of elements which belong to the same $r_1^\#$- or $r_2^\#$-class. To complete the definition of $\mathfrak{A}$, for any pair of elements which do not share an $r_1^\#$- nor $r_2^\#$ class join them by the unique compatible 2-type which does not contain any positive binary literals. Note that each 2-type in $\mathfrak{A}$ either does not contain any positive binary literals or has been copied from a structure which satisfied φ_{univ}. This ensures that $\mathfrak{A}\models\varphi_{\mathrm{univ}}$.

Now we take care of the $\forall\exists$-conjuncts from φ_{free} using a circular witnessing pattern known from [5]. Let m be the number of formulas of type $\forall\exists$ in φ_{free}. We build $\mathfrak{A}'$ out of $3m$ disjoint copies of $\mathfrak{A}$ which are divided into three groups X_0, X_1, X_2 consisting of m copies each. This guarantees that in each group every 1-type from $\boldsymbol{\alpha}$ is realised at least m times. Now, for every element $a\in X_i$ $(i=0,1,2)$ and every conjunct from φ_{free} of the form $\forall x(\gamma(x)\to\exists y(\eta(x,y)\wedge x\neq y\wedge\psi(x,y)))$, if the 1-type of a is α, and $\alpha(x)\models\gamma(x)$, then let α' be as guaranteed by step (iii) of the reduction procedure, and let β be some 2-type consistent with $\alpha(x)\wedge\alpha'(y)\wedge\psi(x,y)\wedge\psi_{\forall\forall}(x,y)\wedge\psi_{\forall\forall}(y,x)$. By step (iv), $\delta_\alpha\in\Delta_0$, whence α' is certainly realised in $\mathfrak{A}$. Take a fresh copy b of such a realisation from $X_{i+1 \mod 3}$ and set $\mathrm{tp}^{\mathfrak{A}'}(a,b)=\beta$. As in $X_{i+1 \mod 3}$ each necessary 1-type is realised at least m types, every element from X_i can choose its witnesses without conflicts. All 2-types in $\mathfrak{A}'$ which have not been fixed yet are set in such a way that they contain no positive binary literals. One can see that $\mathfrak{A}'$ satisfies all conditions of Fact 4, hence $\mathfrak{A}'\models\varphi$. ■

Recalling Lemma 2 and Theorems 11 and 12 we obtain the main result of our article.

THEOREM 24
The (finite) satisfiability problem for GFEC_2^2 is in 2-EXPTIME.

Finally, we observe a model-theoretic difference between the logics EC_2^2 and the GFEC_2^2 (which also holds between EQ_2^2 and GFEQ_2^2). Recall that all these logics allow one to construct infinity axioms. Moreover, in EC_2^2 or EQ_2^2 it is even possible to write a satisfiable formula all of whose models must contain infinite equivalence classes (see Example 5.1.1 in [11]). A closer inspection of our constructions shows that this is not the case in GFEC_2^2 (a similar observation for GFEQ_2^2 follows also from the proof of the upper bound in Theorem 5 from [9]). Indeed, in the model $\mathfrak{A}$ obtained in the proof of $\Leftarrow$ from Proposition 23, elements correspond to vertices, and intersections correspond to edges of the graph H, which we can assume is obtained as in the proof of Theorem 12. The maximal number of intersections in a class of $\mathfrak{A}$ is thus equal to the maximal degree of a node of H. Each such node is incident to precisely $\|f\|$ edges for some function f, for which $f^{\triangleright 1}$ was added to P in step (v) or $f^{\triangleright 2}$ to Q in step (vi) of our reduction procedure. Note that $\|f\|\leq M$, and M is exponentially bounded in $\|\varphi\|$. Hence we can formulate the following *exponential classes property*.

COROLLARY 25
Every satisfiable GFEC_2^2 formula φ has a model in which all $r_1^\#$- and $r_2^\#$-classes are bounded exponentially in $\|\varphi\|$.

It is worth recalling that if we consider satisfiability over finite models then it is possible to enforce in GFEC_2^2 classes of doubly exponential size. It is even possible in a restricted fragment of GFEQ_2^2 in which equivalence relations are used only as guards (see Example 1 in [12]).

Funding

This work is supported by the Polish National Science Centre grant [DEC-2013/09/B/ST6/01535 to E.K. and L.T.] and the British Engineering and Physical Sciences Research Council grant [EP/K017438/1 to I.P-H.]. The authors would like to express their appreciation to the anonymous referees for their valuable comments.

References

[1] H. Andréka, J. van Benthem and I. Németi. Modal languages and bounded fragments of predicate logic. *Journal of Philosophical Logic*, **27**, 217–274, 1998.

[2] I. Borosh, M. Flahive and B. Treybig. Small solutions of linear Diophantine equations. *Discrete Mathematics*, **58**, 215–220, 1986.

[3] C. C. Chang and H. J. Keisler. *Model Theory*, 3rd edn. North-Holland, 1990.

[4] H. Ganzinger, Ch. Meyer and M. Veanes. The two-variable guarded fragment with transitive relations. In *LICS*, pp. 24–34. IEEE Computer Society, 1999.

[5] E. Grädel, P. Kolaitis and M. Vardi. On the decision problem for two-variable first-order logic. *Bulletin of Symbolic Logic*, **3**, 53–69, 1997.

[6] E. Grädel. On the restraining power of guards. *Journal of Symbolic Logic*, **64**, 1719–1742, 1999.

[7] L. G. Khachiyan. A polynomial algorithm in linear programming. *Soviet Mathematics Doklady*, **20**, 191–194, 1979.

[8] E. Kieroński. The two-variable guarded fragment with transitive guards is 2EXPTIME-hard. In *Foundations of Software Science and Computational Structures, 6th International Conference, FOSSACS 2003 Warsaw, Poland, April 7–11, 2003, Proceedings*, pp. 299–312, 2003.

[9] E. Kieroński. Results on the guarded fragment with equivalence or transitive relations. In *Computer Science Logic*. Vol. 3634 of *LNCS*, pp. 309–324. Springer, 2005.

[10] E. Kieroński, J. Michaliszyn, I. Pratt-Hartmann and L. Tendera. Two-variable first-order logic with equivalence closure. *SIAM Journal of Computing*, **43**, 1012–1063, 2014.

[11] E. Kieroński and M. Otto. Small substructures and decidability issues for first-order logic with two variables. *Journal of Symbolic Logic*, **77**, 729–765, 2012.

[12] E. Kieroński and L. Tendera. On finite satisfiability of the guarded fragment with equivalence or transitive guards. In *Logic for Programming, Artificial Intelligence, and Reasoning*. Vol. 4790 of *LNAI*, pp. 318–332. Springer, 2007.

[13] E. Kieroński and L. Tendera. On finite satisfiability of two-variable first-order logic with equivalence relations. In *Logic in Computer Science*, pp. 123–132. IEEE, 2009.

[14] J. Michaliszyn. Decidability of the guarded fragment with the transitive closure. In *Automata, Languages and Programming, 36th Internatilonal Collogquium, ICALP 2009, Rhodes, Greece, July 5–12, 2009, Proceedings, Part II*, pp. 261–272, 2009.

[15] M. Mortimer. On languages with two variables. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, **21**, 135–140, 1975.

[16] W. Szwast and L. Tendera. The guarded fragment with transitive guards. *Annals of Pure and Applied Logic*, **128**, 227–276, 2004.

Received 10 March 2015