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Locally cartesian closed categories and type theory

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0. *Introduction.* It is well known that for much of the mathematics of topos theory, it is in fact sufficient to use a category **C** whose slice categories $\mathbf{C}/A$ are cartesian closed. In such a category, the notion of a 'generalized set', for example an 'A-indexed set', is represented by a morphism $B \to A$ of **C**, i.e. by an object of $\mathbf{C}/A$. The point about such a category **C** is that **C** is a **C**-indexed category, and more, is a hyperdoctrine, so that it has a full first order logic associated with it. This logic has some peculiar aspects. For instance, the types are the objects of **C** and the terms are the morphisms of **C**. For a given type A, the predicates with a free variable of type A are morphisms into A, and 'proofs' are morphisms over A. We see here a certain 'ambiguity' between the notions of type, predicate, and term, of object and proof: a term of type A is a morphism into A, which is a predicate over A; a morphism $1 \to A$ can be viewed either as an object of type A or as a proof of the proposition A.

For a long time now, it has been conjectured that the logic of such categories is given by the type theory of Martin-Löf [**5**], since one of the features of Martin-Löf's type theory is that it formalizes 'ambiguities' of this sort. However, to the best of my knowledge, no one has ever worked out the details of the relationship, and when the question again arose in the McGill Categorical Logic Seminar in 1981–82, it was felt that making this precise was long overdue. That is the purpose of this paper. We shall describe the system ML, based on Martin-Löf's system, and show how to construct a locally cartesian closed category from an ML theory, and vice versa. Finally, we show these constructions are inverse.

A somewhat different approach to the question was taken by John Cartmell [**1**], who describes a categorical structure suitable for Martin-Löf's type theory. I have taken greater liberties with the type theory, my purpose being to characterize locally cartesian closed categories; the payoff is that these categories are simpler, and perhaps more natural, than Cartmell's contextual categories. For example, since toposes are locally cartesian closed, there are many familiar locally cartesian closed categories: the category of **Sets**, and more generally Boolean (or Heyting) valued models and Kripke models of set theory, (indeed any other example of a category of sheaves on a site).

These results were first presented at the McGill Categorical Logic Seminar: an early draft, based on the seminar notes, appeared as [**11**] and an abstract was published [**12**].

1. *The type theory ML.* The type theory described here is based on Martin-Löf's, as given in Martin-Löf [**5**]. We adopt some simplifications of Diller [**2**]. In the interests of readability, we present the type theory more or less informally, as in the first section of Martin-Löf [**5**]; a more formal version would follow the second section of

that paper. In particular, a fuller discussion of the 'condition on variables' is given there (§2·2).

An ML system permits the construction of 'terms', 'types', and of expressions of the form $t \in T$ ('t is a term of type T'), $s = t$, when $s \in T$, $t \in T$ have been derived, and $S = T$. We identify expressions differing only by a change of bound variables. If we write $e[x]$, then x denotes all free occurrences of x in the expression e, and $e[a]$ is then the result of replacing these occurrences with a, under the assumption that a is substitutable in e. If $x_1, \ldots, x_n$ is a sequence of variables, we say $x_1, \ldots, x_n$, e satisfy the *condition on variables* (c.o.v.) if for each $i \leqslant n$, x_i does not occur in the type of any free variable of e other than x_k, for $k > i$. If $x_1, \ldots, x_n$ contains all free variables occurring in e, we say the variables are *properly listed* in $e[x_1, \ldots, x_n]$ if $x_1, \ldots, x_n$, e satisfy the c.o.v. It should be noted that 'occur' is used in the following sense: if $x \in X$ occurs in e, then any variables occurring in X also occur in e. We may write

$$x_1 \in X_1, \quad x_2 \in X_2[x_1], \ldots, x_n \in X_n[x_1, \ldots, x_{n-1}],$$

if the variables are properly listed in $e[x_1, \ldots, x_n]$.

1·1. *Definition.* An ML theory is given by a language which includes a set of typed type-valued function constants, a set of typed term-valued function constants, and variables and constants as indicated in the following rules. (By 'typed function constants' we mean that the arguments have types specified, and in the case of term-valued constants, the value has its type specified as well. We assume the arguments are properly listed.)

1·1·1. *Type formation rules.* The following are to be types:

(i) If F is a type-valued function constant, and $a_1, \ldots, a_n$ are terms of the appropriate types, then $F(a_1, \ldots, a_n)$ is a type.

(ii) 1 is a type.

(iii) If $a, b \in A$, then $I(a, b)$ is a type.

(iv) If $A, B[x]$ are types, $x \in A$, where x, B satisfy the c.o.v., then $\Pi_{x \in A} B[x]$ and $\Sigma_{x \in A} B[x]$ are types. If x does not in fact occur in B, these are written $A \supset B$ and $A \times B$ respectively.

1·1·2. *Term formation rules.* The following are to be terms of the indicated types:

(*vbl*) For each type A, there are variables $x \in A$; (such x could also be denoted x_A, if the type of x is not clear from the context.)

(*fcn*) If f is a term-valued function constant, and $a_1, \ldots, a_n$ are terms of the appropriate types, then $f(a_1, \ldots, a_n)$ is a term of the appropriate type.

(1I) $* \in 1$.

(ΠI) If $t[x] \in B[x]$, where x_A, t (and x_A, B) satisfy the c.o.v., then

$$\lambda_{x \in A} t[x] \in \Pi_{x \in A} B[x],$$

(also written $\lambda x_A\, t[x] \in \Pi x_A\, B[x]$).

(ΠE) If $f \in \Pi_{x \in A} B[x]$, $a \in A$, then $f(a) \in B[a]$.

(ΣI) If $a \in A$, $b \in B[a]$, then $\langle a, b \rangle \in \Sigma_{x \in A} B[x]$.

(ΣE) If $c \in \Sigma_{x \in A} B[x]$, then $\pi(c) \in A$, $\pi'(c) \in B[\pi(c)]$.

($=$I) If $a \in A$, then $r(a) \in I(a, a)$.

(= E) If $a, b \in A$, $c \in I(a,b)$, $d \in C[a,a,r(a)]$, where $C[x,y,z]$ is a type depending on $x, y \in A$, $z \in \mathrm{I}(x,y)$, then $\sigma(d)[a,b,c] \in C[a,b,c]$.

1·1·3. *Equality rules.* Using the notation of § 1·1·2, we have the following equations:
(f eq) Any imposed equations on function constants induce the obvious equations.
(1 red) If $t \in 1$, then $t = *$.
(Π red) $(\lambda_{x\in A} t[x])(a) = t[a]$.
(Π exp) $f = \lambda_{x\in A} f(x)$.
(Σ red) $\pi(\langle a,b\rangle) = a$; $\pi'(\langle a,b\rangle) = b$.
(Σ exp) $c = \langle \pi(c), \pi'(c)\rangle$.
(= red) $\sigma(d)[a,a,r(a)] = d$.
(= exp) If $f[a,b,c] \in C[a,b,c]$, then $f = \sigma(f[a,a,r(a)])[a,b,c]$.
(I rule) If $a[x]$, $b[x] \in A$, $t[x] \in I(a[x], b[x])$, then $a[x] = b[x]$, and $t[x] = r(a[x])$.

Furthermore, an ML theory may have axioms of the form $S = T$ for types S, T. (We suppose similar axioms for terms are given by function constants of the appropriate I-type.)

Finally, we have the usual rules for $=$: for types or terms a, b, c (as appropriate):
If $a = b$ then $c[a] = c[b]$. If $a = b$ then $b = a$.
If $a = b$ and $b = c$, then $a = c$. $a = a$.
If $c \in a$ and $a = b$, then $c \in b$. If $a \in c$ and $a = b$, then $b \in c$.

1·2. *Remarks.* There are obvious similarities between ML and first order logic – the types of ML correspond to predicates, and the terms of ML to derivations of the predicates, so that '$t \in T$' can be interpreted as 't proves T'. Under this interpretation, $I(a,b)$ is $a = b$, $\Pi_{x\in A} B[x]$ is $\forall x \in A\, B[x]$, $\Sigma_{x\in A} B[x]$ is $\exists x \in A\, B[x]$, and 1 is T. Furthermore, the term formation rules are then the introduction and elimination rules in a natural deduction system for first order (intuitionistic) logic, as in Prawitz [7] or [8] (also Seely [**10**], where equality is included in the system).

There is a slight problem with (ΣE): although it specializes properly to ($\wedge$E), it does not seem quite like ($\exists$E), which is usually denoted

$$\frac{\exists x \in A B[x] \quad \begin{array}{c}\{B[x]\}\\ C\end{array}}{C}$$

where {} denote a discharged assumption, and where x must not occur freely in $\exists x \in A\, B[x], C$, or any assumption other than B on which C depends. This is more closely given by Martin-Löf's version of (ΣE) in [**5**]:

(Σ elim) If $C[z]$ is a type depending on $z \in \Sigma_{x\in A} B[x]$ but not on $x \in A$ nor on $y \in B[x]$, and if $t[x,y] \in C[\langle x,y\rangle]$ is a term depending on $x \in A$, $y \in B[x]$, then there is a term $\bar{t}(z) \in C[z]$.
This rule is accompanied by its own reduction and expansion rules:
(Σ red n) $\bar{t}(\langle x,y\rangle) = t[x,y]$.
(Σ exp n) If $f[z] \in C[z]$, and if $t[x,y]$ is $f[\langle x,y\rangle]$, then $f[z] = \bar{t}(z)$.
(Martin-Löf does not give (Σ exp n), nor any other expansions, in his system. The expansions are all based on the ones in Seely [**9**] or [**10**].)

It is easy to see that the (ΣE), (Σ red), (Σ exp) of ML are special cases of (Σ elim), (Σ red n), (Σ exp n), but it is also true that they imply the more general forms. For example, the $\bar{t}$ in (Σ elim) may be given by $\bar{t}(z) = t[\pi(z), \pi'(z)]$; the other equations follow immediately.

The equality rules for ML, under the interpretation of ML as first order logic, correspond to the operations on derivations given in Seely [**10**]. (The (I rule) corresponds to Corollary 1 of §2 of [**10**], and is a consequence of the rule (R Coh) expressing that '$x = x$' is isomorphic to T_X, the terminal 'truth' predicate. It expresses that equality is given by an equalizer; it does not occur in Martin-Löf [**5**].) So, in effect, terms correspond rather to equivalence classes of derivations ('proofs') in first order logic, than to derivations themselves. There is one major difference between ML and first order logic: in regarding types as predicates, we then use both notions at the same time in forming $\Pi_{x\in A} B[x]$, $\Sigma_{x\in A} B[x]$. Equivalently, we are quantifying in some sense over proofs of predicates. It is precisely this ambiguity between types (as 'sets') and predicates that we need to characterize locally cartesian closed categories.

In addition to the analogy with first order logic, the notation suggests the terms and types have a naive interpretation in **Sets**: Π and Σ are the cartesian product and disjoint union of indexed families respectively. 1 is the singleton family. I is the identity: if a, b are indexed families of objects then $I(a,b)$ is the family made up of singletons where a and b are equal and of null sets where they are not. This is the basis for what follows, as will be seen in §4.

1·3. As in Seely [**10**], from the substitution rule for equality (i.e. ($=$ E)) we can derive symmetry and transitivity of equality:

Lemma. *If $a, b, c \in A$, $d \in I(a,b)$, $e \in I(b,c)$, then there are terms $S(a,b,d) \in I(b,a)$ and $T(a,b,c,d,e) \in I(a,c)$. If $f \in A \supset B$, then there is a term* $\mathrm{Ap}\,(a,b,d,f) \in I(f(a), f(b))$.

Proof. (i) Let $C[x,y,u]$ be $I(y,x)$, for $x, y \in A$, $u \in I(x,y)$; then $r(a) \in C[a,a,r(a)]$. Take $S(a,b,d)$ to be $\sigma(r(a))\,[a,b,d]$, of type $C[a,b,d] = I(b,a)$.

(ii) Let $C[y,z,v]$ be $I(a,z)$, for $y, z \in A$, $v \in I(y,z)$. Then $C[b,b,r(b)]$ is $I(a,b)$ and $d \in C[b,b,r(b)]$. Take $T(a,b,c,d,e)$ to be $\sigma(d)\,[b,c,e]$, of type $C[b,c,e] = I(a,c)$.

(iii) Let $C[x,y,u]$ be $I(f(x), f(y))$, for $x, y \in A$, $u \in I(x,y)$;

$$r(f(a)) \in C[a,a,r(a)] = I(f(a), f(a)).$$

Take $\mathrm{Ap}\,(a,b,d,f)$ to be $\sigma(r(f(a)))\,[a,b,d]$.

2. *Categorical preliminaries.* For basics, refer to Mac Lane [**4**].

2·1. *Definition.* A locally cartesian closed category (LCC) **C** is a category **C** with finite limits, such that for any object A of **C**, the slice category $\mathbf{C}/A$ is cartesian closed.

2·1·1. *Remark.* $\mathbf{C}/A$ has as its objects all morphisms $B \to A$ of **C** (for all possible B). Morphisms in $\mathbf{C}/A$ are commutative triangles over A:

$$\begin{tikzcd} B \arrow[rr] \arrow[dr] & & C \arrow[dl] \\ & A & \end{tikzcd}$$

If **C** has finite limits, then each category $\mathbf{C}/A$ also has. However, even if **C** has exponents, the categories $\mathbf{C}/A$ need not have them: that they do is the essential property of an LCC category.

2·2. *Definition.* For **C** a category with finite limits, a **C**-indexed category **P** consists of:

(i) for each object A of **C**, a category $\mathbf{P}(A)$,

(ii) for each morphism $f: A \to B$ of **C**, a functor $f^*: \mathbf{P}(B) \to \mathbf{P}(A)$,

subject to

(i) $(\mathrm{id}_A)^* \cong \mathrm{id}_{\mathbf{P}(A)}$,

(ii) $(gf)^* \cong f^*g^*$,

and standard coherence conditions. (See Paré-Schumacher [**6**].)

2·3. *Definition.* A **C**-indexed category **P** is a hyperdoctrine if

(i) for each object A of **C**, $\mathbf{P}(A)$ is cartesian closed,

(ii) for each $f: A \to B$ of **C**, f^* preserves exponents,

(iii) for each $f: A \to B$ of **C**, f^* has adjoints $\Sigma_f \dashv f^* \dashv \Pi_f$,

(iv) **P** satisfies the Beck condition: if

$$\begin{tikzcd} D \arrow[r, "h"] \arrow[d, "k"'] & C \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

is a pullback in **C**, then for any object ϕ of $\mathbf{P}(C)$, $\Sigma_k h^*\phi \to f^*\Sigma_g\phi$ is an isomorphism in $\mathbf{P}(A)$. (A similar condition for Π follows from this.)

2·4. Any category **C** with finite limits induces a **C**-indexed category (which we shall denote **C** also) given by $\mathbf{C}(A) = \mathbf{C}/A$; f^* is then defined by pullback. One of the basic results of topos theory is the following.

THEOREM. *If* **C** *has finite limits, then* **C** *is LCC iff as a* **C**-*indexed category* **C** *is a hyperdoctrine.*

A *proof* may be found in Freyd [**3**], § 1·3. The point is that for all A, $\mathbf{C}/A$ is cartesian closed iff for all f, f^* has a right adjoint Π_f. For any **C** with finite limits, each f^* of **C** has a left adjoint Σ_f (defined by composition), and the Beck condition for **C** is satisfied (it says the composite of two pullback diagrams is a pullback diagram, which is always true).

2·5. In Seely [**10**] it is shown that the category of hyperdoctrines is equivalent to the category of first order theories (with equality). With the interpretation of ML theories as special first order theories, and of LCC categories as special hyperdoctrines, the connection between ML theories and LCC categories seems natural.

2·6. *Definition.* Two **C**-indexed categories $\mathbf{P}_1$ and $\mathbf{P}_2$ are equivalent, $\mathbf{P}_1 \simeq \mathbf{P}_2$, if for each A, there is an equivalence $\mathbf{P}_1(A) \simeq \mathbf{P}_2(A)$, and furthermore, these equivalences commute with the f^*'s.

2·6·1. *Remark.* If $\mathbf{P}_1 \simeq \mathbf{P}_2$ as **C**-indexed categories, and if $\mathbf{P}_1$ is a hyperdoctrine, then so is $\mathbf{P}_2$, and moreover, the equivalences $\mathbf{P}_1(A) \simeq \mathbf{P}_2(A)$ commute with the Σ_f's and Π_f's.

3. *From ML to LCC*. Given an ML theory $\mathbf{M}$, we define a category $\mathbf{C}(\mathbf{M})$, whose objects are all closed types of $\mathbf{M}$ (i.e. types depending on no free variables), and morphisms $A \to B$ are closed terms of type $A \supset B$. (So $f\colon A \to B$ in $\mathbf{C}(\mathbf{M})$ means $f \in A \supset B$ in $\mathbf{M}$.)

3·1. Proposition. $\mathbf{C}(\mathbf{M})$ *is cartesian closed.*

Proof. We can check that $\mathbf{C}(\mathbf{M})$ is a cartesian closed category with finite limits directly; the details are straightforward. (The reader can turn directly to 3·2.)

3·1·1. *Category axioms.* For an object A, $\mathrm{id}_A\colon A \to A$ is $\lambda_{x\in A} x$. Given $f\colon A \to B$, $g\colon B \to C$, $g \circ f\colon A \to C$ is $\lambda_{x \in A} g(f(x))$.

$$\begin{aligned} f \circ \mathrm{id}_A &= f \circ (\lambda_{x\in A} x) = \lambda_{y \in A}\ f((\lambda_{x\in A} x)(y)) && \text{(definitions)}\\ &= \lambda_{y\in A} f(y) && (\Pi\ \text{red})\\ &= f && (\Pi\ \text{exp}). \end{aligned}$$

Similarly $\mathrm{id}_B \circ f = f$, $h \circ (g \circ f) = (h \circ g) \circ f$.

3·1·2. *Products.* 1 is the terminal object of $\mathbf{C}(\mathbf{M})$.
Given any object A, there is a morphism $A \to 1$, namely $\lambda_{x\in A} *$.

Lemma. *For any closed type A, if t is a closed term of type $A \supset 1$, then $t = \lambda_{x\in A} *$.*

Proof. $$\begin{aligned} t &= \lambda_{x\in A} t(x) && (\Pi\ \text{exp})\\ &= \lambda_{x \in A} * && (1\ \text{red}). \end{aligned}$$

For objects A, B, $A \times B$ is given by $A \times B$; pairing $\langle\,,\rangle$, and projections π, π' are likewise given by 'themselves', and that the requisite equations are satisfied is obvious from (Σ red) and (Σ exp).

3·1·3. Lemma (Pullbacks). *Given $t\colon A \to B$, $s\colon C \to B$, the pullback P of s along t*

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "q"'] & A \arrow[d, "t"] \\ C \arrow[r, "s"'] & B \end{tikzcd}$$

is given by $\Sigma_{x\in A} \Sigma_{y \in C} I(t(x), s(y))$, with the evident projections to A and C: p is π and q is $\pi\pi'$.

Proof. Given $f\colon X \to A$, $g\colon X \to C$ such that $tf = sq$, note that there is a term

$$\rho(x) \in I(t(f(x)), s(q(x))), \quad \text{for} \quad x \in X,$$

viz. $r(t(f(x)))$. Define $h\colon X \to P$ by $\lambda_{x\in X} \langle f(x), \langle g(x), \rho(x)\rangle\rangle$. Clearly $ph = f$ and $qh = g$, and (using the (I rule) to see $\rho(x)$ is the only possible term in $I(t(f(x)), s(q(x)))$) h is unique with this property.

3·1·4. *Remark* (*Equalizers*). Given $s, t\colon A \rightrightarrows B$, the equalizer of s, t

$$\mathrm{eq}(s,t) \rightarrowtail A \rightrightarrows B$$

is given by $\Sigma_{x\in A} I(s(x), t(x))$, the inclusion being the projection π. (That it is a monomorphism follows from the (I rule).)

3·1·5. LEMMA (EXPONENTS). *B^A defined as $A \supset B$ makes* $\mathbf{C}(\mathbf{M})$ *cartesian closed.*

Proof. Given $t\colon A \times C \to B$, define $\hat{t} = \lambda_{y\in C}\lambda_{x\in A} t(\langle x, y\rangle)\colon C \to B^A$; given $s\colon C \to B^A$, define $\check{s} = \lambda_{z\in A\times C} s(\pi'(z))(\pi(z))\colon A \times C \to B$. It is a routine exercise to see these operations are inverse. (Note that this correspondence is the usual one; for example, ev: $A \times B^A \to B$ is just $\lambda_{z\in A\times B^A} \pi'(z)(\pi(z))$, so that $\mathrm{ev}(\langle a, f\rangle) = f(a)$.)

3·2. THEOREM. $\mathbf{C}(\mathbf{M})$ *is locally cartesian closed.*

Proof. To see that $\mathbf{C}(\mathbf{M})$ is LCC, we must check that the slice categories $\mathbf{C}(\mathbf{M})/A$ are cartesian closed, or equivalently, that $\mathbf{C}(\mathbf{M})$ is a hyperdoctrine. To do this we define two $\mathbf{C}(\mathbf{M})$-indexed categories, one being $\mathbf{C}(\mathbf{M})$ itself, and show they are equivalent hyperdoctrines.

3·2·1. *Definition.* $\mathbf{P}(\mathbf{M})$ is the $\mathbf{C}(\mathbf{M})$-indexed category defined by:

(i) for an object A of $\mathbf{C}(\mathbf{M})$, $\mathbf{P}(\mathbf{M})(A)$ is the category whose objects are types $B[x]$, depending only on $x \in A$, and whose morphisms are terms $t[x] \in B[x] \supset C[x]$, depending only on $x \in A$.

(ii) for a term $f \in B \supset A$ (i.e. $f\colon B \to A$ in $\mathbf{C}(\mathbf{M})$) f^* is defined by substitution: $e[x] \mapsto e[f(y)]$, $y \in B$, for an expression e.

3·2·2. LEMMA. *For any closed type A,* $\mathbf{P}(\mathbf{M})(A)$ *is cartesian closed.*

Proof. The proof of this fact is exactly like the proof that $\mathbf{C}(\mathbf{M})$ was cartesian closed. (We never really needed to know that the objects were *closed* types, so repeat the arguments with a parameter $x \in A$.)

3·2·3. LEMMA. *For any closed type A,* $\mathbf{C}(\mathbf{M})/A \simeq \mathbf{P}(\mathbf{M})(A)$.

Proof. We define functors $\mathbf{C}(\mathbf{M})/A \underset{\wedge}{\overset{\vee}{\rightleftarrows}} \mathbf{P}(\mathbf{M})(A)$:

(i) For $f\colon B \to A$, $\check{f}$ is the type $f^{-1}(x) =_{df} \Sigma_{y\in B} I(x, f(y))$, $x \in A$, For a morphism h of $\mathbf{C}(\mathbf{M})/A$:

$$\begin{tikzcd} B \arrow[rr, "h"] \arrow[dr, "f"'] & & C \arrow[dl, "g"] \\ & A & \end{tikzcd},$$

so that $f = g \circ h$, $\check{h}$ is the term $\lambda_{z\in f^{-1}(x)} \langle h(\pi(z)), \bar{\rho}\rangle \in f^{-1}(x) \supset g^{-1}(x)$, where $\bar{\rho} \in I(x, g(h(\pi(z))))$ is defined by 'transitivity' from $\pi'(z) \in I(x, f(\pi(z)))$ and $r(f(\pi(z))) \in I(f(\pi(z)), g(h(\pi(z))))$, using Lemma 1·3.

(ii) For $B[x]$ in $\mathbf{P}(\mathbf{M})(A)$, $\hat{B}$ is the morphism in $\mathbf{C}(\mathbf{M})$ given by the projection $\pi\colon \Sigma_{x\in A} B[x] \to A$. For $t[x] \in B[x] \supset C[x]$ in $\mathbf{P}(\mathbf{M})(A)$, $\hat{t}$ is given by

$$\lambda_{z\in\Sigma x_A B[x]} \langle \pi(z), t[\pi(z)](\pi'(z))\rangle \quad \text{of type} \quad \Sigma_{x\in A} B[x] \supset \Sigma_{x\in A} C[x].$$

It is easy to check this gives a morphism in $\mathbf{C}(\mathbf{M})/A$.

We must now show these are functors, and the two composites are isomorphic to the identity functors. We shall not give all the details – the highlights are the following.

3·2·3·1. Sublemma. *If* $f \in B \supset A$ *in* $\mathbf{M}$, *then the objects* B *and* $\Sigma_{x\in A} f^{-1}(x)$ *are isomorphic in* $\mathbf{C}(\mathbf{M})$.

Proof. Morphisms $B \underset{j}{\overset{i}{\rightleftarrows}} \Sigma_{x\in A}\Sigma_{y\in B} I(x,f(y))$ are given by

$$i(y) = \langle f(y), \langle y, r(f(y))\rangle\rangle$$
$$j(\langle x, \langle y, z\rangle\rangle) = y,$$

for $x\in A$, $y\in B$, $z\in I(x,f(y))$. (Actually, there is considerable abuse of language here: i and j should be given using λ terms, and $\langle x,\langle y,z\rangle\rangle$ should be a single variable $w\in\Sigma_{x\in A} f^{-1}(x)$; x, y, z thus stand for projection terms.)

Clearly $j(i(y)) = y$. For the inverse, $i(j(\langle x,\langle y,z\rangle\rangle) = \langle f(y),\langle y, r(f(y))\rangle\rangle$ and thus we are done if we show $x = f(y), z = r(f(y))$. But since $z\in I(x,f(y))$, this is a consequence of the (I rule).

3·2·3·2. Sublemma. *For* $B[x]$ *a type*, $x\in A$, *the objects* $B[x]$ *and* $\Sigma_{y\in\Sigma x_A B[x]} I(x,\pi(y))$ *are isomorphic in* $\mathbf{P}(\mathbf{M})(A)$.

Proof. Morphisms $B[x] \underset{j[x]}{\overset{i[x]}{\rightleftarrows}} \Sigma_{y\in\Sigma x_A B[x]} I(x,\pi(y))$ are given by

$$i(z) = \langle\langle x, z\rangle, r(x)\rangle$$
$$j(\langle\langle x', z'\rangle, v\rangle) = z'$$

for $x, x'\in A, z\in B[x], z'\in B[x'], v\in I(x,x')$, (with the familiar abuse of language.) Clearly $j(i(z)) = z$. Inversely, $i(j(\langle\langle x', z'\rangle, v\rangle)) = \langle\langle x, z'\rangle, r(x)\rangle$, and we are done if $x = x'$, $z'\in B[x]$, and $v = r(x)$. These follow from the (I rule) as before.

3·2·3·3. Sublemma. *If* A *is a closed type*, $B[x_A]$ *a type in* $\mathbf{P}(\mathbf{M})(A)$, *then there is a bijection between the set of terms* $t[x]\in B[x]$ *and the set of (closed) terms* $s\in A\supset\Sigma_{x\in A} B[x]$ *satisfying* $\pi(s(x)) = x$.

Remark. To see the significance of this, recall that $\mathbf{P}(\mathbf{M})(A)$ is cartesian closed and that $A \simeq \Sigma_{x\in A} 1$ in $\mathbf{C}(\mathbf{M})$.

Proof. The functors of 3·2·3 in this case specialize to

$$t[x] \mapsto \hat{t} = \lambda_{x\in A}\langle x, t[x]\rangle,$$
$$s \mapsto \check{s} = \pi'(s(x)).$$

The composites are (i)

$$\begin{aligned}\lambda_{x\in A}\langle x, \pi'(s(x))\rangle &= \lambda_{x\in A}\langle \pi(s(x)), \pi'(s(x))\rangle\\ &= \lambda_{x\in A} s(x)\\ &= s,\end{aligned}$$

and (ii) $\pi'((\lambda_{x\in A}\langle x, t[x]\rangle)(x)) = \pi'(\langle x, t[x]\rangle) = t[x]$.

(These proofs also work if A, B, $B[x]$ have other parameters, and a similar result holds for $B[x]$ with more than one (extra) variable.)

3·2·4. Proposition. *As* $\mathbf{C}(\mathbf{M})$*-indexed categories*, $\mathbf{C}(\mathbf{M}) \simeq \mathbf{P}(\mathbf{M})$.

Proof. The only other point to verify is that the equivalences of Lemma 3·2·3 commute with the f^*'s: given $f: B \to A$ in $\mathbf{C}(\mathbf{M})$, it suffices to show that

$$\begin{tikzcd} \mathbf{C}(\mathbf{M})/A \arrow[r, "f^*"] \arrow[d, "\vee"'] & \mathbf{C}(\mathbf{M})/B \\ \mathbf{P}(\mathbf{M})(A) \arrow[r, "f^*"'] & \mathbf{P}(\mathbf{M})(B) \arrow[u, "\wedge"'] \end{tikzcd}$$

commutes. Let $t: C \to A$ be in $\mathbf{C}(\mathbf{M})/A$; $\check{t} = t^{-1}(x_A)$ in $\mathbf{P}(\mathbf{M})(A)$, $f^*(\check{t}) = t^{-1}(f(y_B))$, and $(f^*(\check{t}))^\wedge$ is $\Sigma_{y \in B} t^{-1}(f(y)) \xrightarrow{\pi} B$, i.e. $\Sigma_{y \in B} \Sigma_{z \in C} I(f(y), t(z)) \xrightarrow{\pi} B$. But this is just the definition of the pullback of t along f, as in Lemma 3·1·3, i.e. $(f^*(\check{t}))^\wedge = f^*(t)$.

3·2·5. Proposition. $\mathbf{C}(\mathbf{M})$ *is a hyperdoctrine.*

Proof. Since $\mathbf{P}(\mathbf{M})(A)$ is cartesian closed, so also is $\mathbf{C}(\mathbf{M})/A$, for each A, and so $\mathbf{C}(\mathbf{M})$ is a hyperdoctrine.

3·3. *Remark.* Although this completes the proof of Theorem 3·2, in fact it is easy to show by direct calculation that $\mathbf{P}(\mathbf{M})$ is a hyperdoctrine (equivalent to $\mathbf{C}(\mathbf{M})$, by the preceding). For example, the construction of Σ_f, Π_f, for $f: B \to A$, is exactly the same as for first order logic, as in Seely [**10**]: for $P[y_B]$ in $\mathbf{P}(\mathbf{M})(B)$,

$$\Sigma_f P[y] = \Sigma_{y \in B}(I(x_A, f(y)) \times P[y]),$$
$$\Pi_f P[y] = \Pi_{y \in B}(I(x_A, f(y)) \supset P[y]).$$

It is easy to check these commute with the equivalences of 3·2·3.

4. *Interpreting ML in LCC.* Given an ML theory $\mathbf{M}$ and an LCC category $\mathbf{C}$, it is possible to define the notion of an interpretation of $\mathbf{M}$ in $\mathbf{C}$, denoted $\mathbf{M} \to \mathbf{C}$.

4·1. *Definition.* An interpretation $^-: \mathbf{M} \to \mathbf{C}$ consists of:

(i) for a type-valued function constant, F, with arguments of types $X_1, \ldots, X_n$, a morphism $\phi: \bar{F} \to \bar{X}_n$ of $\mathbf{C}$, and

(ii) for a term-valued function constant, f, with arguments of type $X_1, \ldots, X_n$, and value of type A, a morphism $\bar{f}: \bar{X}_n \to \bar{A}$ of $\mathbf{C}/\bar{X}$, $\bar{X}$ the codomain of $\bar{X}_n$ and $\bar{A}$. $\bar{X}_n$, $\bar{A}$ must be defined consistently with 4·4. We shall generally write $\bar{F} = \phi: \bar{F} \to \bar{X}_n$ (abusing the notation horribly!).

4·2. Given an interpretation $^-: \mathbf{M} \to \mathbf{C}$, we shall extend it to all types and terms of $\mathbf{M}$: a type depending on a free variable $x \in A$ will be interpreted as an object in $\mathbf{C}/\bar{A}$ (for suitable $\bar{A}$), and terms will be interpreted as morphisms in the appropriate slice categories. In particular, a closed type A (with no free variables) will correspond to an object $\bar{A}$ of $\mathbf{C}$, and if $t[v] \in A$ depends only on $v \in X$, t will correspond to a morphism $\bar{t}: \bar{X} \to \bar{A}$. The main intuitive idea is that we think of the type $B[x_A]$ as the morphism $\pi: \Sigma_{x \in A} B[x] \to A$, (this will be $\bar{B} \to \bar{A}$), and so $\bar{B} \xrightarrow{f} \bar{A}$ is the type $f^{-1}(x_A)$, as in Lemma 3·2·3; note that via this interpretation, f is the projection π.

4·3. *Remark* (*concerning the condition on variables*). If $B[x_1, \ldots, x_n]$ is a type with the variables properly listed, $x_1 \in X_1$, $x_2 \in X_2[x_1], \ldots, x_n \in X_n[x_1, \ldots, x_{n-1}]$, then under our interpretation X_1, being closed, corresponds to an object $\bar{X}_1$ of $\mathbf{C}$. X_2 should correspond to a morphism $\chi_2: \bar{X}_2 \to \bar{X}_1$, where we think of X_2 as $\Sigma_{x \in X_1} X_2[x]$. Then, in saying $x_2 \in X_2[x_1]$ to form $X_3[x_1, x_2]$, we are implying that the x_1 is $\chi_2(x_2)$ and so X_3 becomes a morphism $\chi_3: \bar{X}_3 \to \bar{X}_2$. This induces the expected morphism $\bar{X}_3 \to \bar{X}_1 \times \bar{X}_2$ as $\langle \chi_2 \chi_3, \chi_3 \rangle$. (Equivalently, we could interpret X_3 as $\chi_3': \bar{X}_3 \to \bar{X}_1 \times \bar{X}_2$; then the c.o.v. would require that

$$\begin{tikzcd} \bar{X}_3 \arrow[r, "\chi'_3"] \arrow[d, "\chi'_3"'] & \bar{X}_1 \times \bar{X}_2 \arrow[r, "\pi"] & \bar{X}_1 \\ \bar{X}_1 \times \bar{X}_2 \arrow[d, "\pi'"'] & & \\ \bar{X}_2 \arrow[uurr, "\chi_2"'] & & \end{tikzcd}$$

commutes, inducing $\chi_3: \bar{X}_3 \to \bar{X}_2$ as $\pi' \chi_3'$). Similarly for the other variables, so that B is interpreted as $\bar{B} \to \bar{X}_n$ with induced 'projections' to $X_1, \ldots, X_{n-1}$. In what follows, we assume variables are properly listed.

4·4. We now extend the notion of interpretation to all terms and types of $\mathbf{M}$. Since $\mathbf{M}$ is (like) a first order theory, and $\mathbf{C}$ is a hyperdoctrine, this follows the ideas of Seely [**10**] closely; we give fairly complete details to allow the reader to verify 4·5.

Definition (*continued*). Given an interpretation $^-: \mathbf{M} \to \mathbf{C}$ (as in 4·1), the extension of $^-$ to all types and terms of $\mathbf{M}$ is defined as follows (i.e. the following equalities must be true of $^-$):

(*Substitution*). The substitution of a term t in a type A is defined via the functor t^* (for the hyperdoctrine $\mathbf{C}$): $\overline{A[t]} = \bar{t}^* \bar{A}$. The substitution of t in a term a is defined by composition: $\overline{a[t]} = \bar{a} \cdot \bar{t}$.

4·4·1. (*Type formation rules*). (i) $\bar{1} = 1$.

(ii) $\overline{I(x_A, y_A)} = \Delta_{\bar{A}}: \bar{A} \to \bar{A} \times \bar{A}$. (In view of the definition of substitution, if $a, b \in A$ are interpreted as $\bar{a}, \bar{b}: \bar{X} \to \bar{A}$, $\bar{X}$ the interpretation of the type(s) of free variable(s) in a, b, then $\overline{I(a,b)}$ is the equalizer $\mathrm{Eq}(\bar{a}, \bar{b}) \to \bar{X}$ of $\bar{a}$ and $\bar{b}$.)

(iii) If $A, B[x_A]$ are types interpreted as $\alpha: \bar{A} \to \bar{X}$, $\beta: \bar{B} \to \bar{A}$, then

$$\overline{\Pi_{x \in A} B[x]} = \Pi_\alpha \beta, \quad \overline{\Sigma_{x \in A} B[x]} = \Sigma_\alpha \beta.$$

(Here Π_α, Σ_α are adjoints to α^*, in the hyperdoctrine $\mathbf{C}$. $\bar{X}$ interprets the type(s) of free variables in A, and also those other than x_A in B.)

Remark. If B is independent of x_A, then $B[x_A]$ is B with a dummy free variable x_A added; i.e. $\bar{B} = \beta': \bar{B} \to \bar{X}$, and $\overline{B[x_A]}$ is the pullback

$$\begin{tikzcd} \bar{A} \times_{\bar{X}} \bar{B} \arrow[r] \arrow[d, "\overline{B[x_A]}"'] & \bar{B} \arrow[d, "\beta'"] \\ \bar{A} \arrow[r, "\alpha"'] & \bar{X} \end{tikzcd}$$

i.e. $\overline{B[x_A]} = \alpha^*\beta'$. Then $\overline{\Pi_{x\in A}B[x]} = \Pi_\alpha \alpha^*\beta' = \alpha \supset \beta'$, and $\overline{\Sigma_{x\in A}B[x]} = \Sigma_\alpha \alpha^*\beta' = \alpha\times\beta'$.

4·4·2. (*Term formation rules*). Variables are interpreted as identity morphisms, as in Seely [**10**].

(1 I). $\bar{*} = \mathrm{id}_1 \colon 1 \to 1$.

(Π I). If $t[x_A] \in B[x_A]$, $\overline{t[x_A]}$ is a morphism $1_{\bar{A}} \to \beta$ in $\mathbf{C}/\bar{A}$, i.e. a triangle

$$\begin{tikzcd} \bar{A} \arrow[rr, "\bar{t}"] \arrow[dr, "\mathrm{id}_A"'] & & \bar{B} \arrow[dl, "\beta"] \\ & \bar{A} & \end{tikzcd}$$

in $\mathbf{C}$. Then $\overline{\lambda_{\alpha\in A}t[x]} = \Pi_\alpha \bar{t}$; since $\Pi_\alpha 1_{\bar{A}} = 1_{\bar{X}}$,t his is a morphism $1_{\bar{X}} \to \Pi_\alpha\beta$ in$\mathbf{C}/\bar{X}$.

(Π E). If $f \in \Pi_{x\in A}B[x]$, $a \in A$ are interpreted by morphisms $\bar{f}\colon \zeta \to \Pi_\alpha\beta$, $\bar{a}\colon \zeta\to\alpha$ in $\mathbf{C}/\bar{X}$, for some $\zeta\colon \bar{Z}\to\bar{X}$, then $\overline{f(a)}\colon 1_{\bar{Z}} \to \bar{a}^*\beta$ in $\mathbf{C}/\bar{Z}$ is defined as follows: (without loss in generality, we suppose a,f depend on the same free variables 'Z'; add dummy variables if not). Since $\bar{a}\colon \zeta\to\alpha$ is a morphism in $\mathbf{C}/\bar{X}$, $\alpha\bar{a} = \zeta$, and so there is a morphism ('id') $\Sigma_\alpha \bar{a} \to \zeta$ in $\mathbf{C}/\bar{X}$. By adjointness, there is a morphism $\bar{a}\to\alpha^*\zeta$ in $\mathbf{C}/\bar{A}$. Similarly, $\bar{f}\colon \zeta\to\Pi_\alpha\beta$ in $\mathbf{C}/\bar{X}$ induces a morphism $\alpha^*\zeta\to\beta$ in $\mathbf{C}/\bar{A}$. Composing, we have a morphism $\bar{a}\to\beta$; i.e. a morphism $\Sigma_{\bar{a}}1_{\bar{Z}}\to\beta$ in $\mathbf{C}/\bar{A}$, and again adjointness gives the required $\overline{f(a)}\colon 1_{\bar{Z}} \to \bar{a}^*\beta$.

(Σ I). If $a\in A$, $b \in B[a]$, $\bar{a}\colon \zeta\to\alpha$ in $\mathbf{C}/\bar{X}$, $\bar{b}\colon 1_{\bar{Z}}\to\bar{a}^*\beta$ in $\mathbf{C}/\bar{Z}$, for some $\zeta\colon\bar{Z}\to\bar{X}$ (again supposing a, b depend on the same free variables), then $\overline{\langle a,b\rangle}\colon \zeta\to\Sigma_\alpha\beta$ in $\mathbf{C}/\bar{X}$ is obtained from the morphism induced by $\bar{b}\colon 1_{\bar{Z}}\to\bar{a}^*\beta$: *viz.* $\Sigma_{\bar{a}}1_{\bar{Z}}\to\beta$, in $\mathbf{C}/\bar{A}$. Since $\Sigma_{\bar{a}}1_{\bar{Z}}$ is $\bar{a}$, we have $\bar{a}\to\beta$. Apply Σ_α to get $\Sigma_\alpha\bar{a}\to\Sigma_\alpha\beta$. But again the morphism ('id') $\zeta\to\Sigma_\alpha\bar{a}$ gives us, by composition, $\overline{\langle a,b\rangle}$.

(Σ E). As indicated in 4·2, $\bar{\pi} = \beta\colon \Sigma_\alpha\beta\to\alpha$. As for π', essentially $\bar{\pi}'$ is the identity map. (This is because of a peculiarity of category theory, whereby for a map whose 'codomain' depends on the argument of the map, these 'partial codomains' are replaced by their disjoint sum. Here we want a map $\Sigma x_A B[x] \to$ '$B[x]$'; we replace this with $\Sigma x_A B[x] \to \Sigma x_A B[x]$.)

To see how this works, suppose $c\in\Sigma_{x\in A}B[x]$, $\bar{c}\colon\zeta\to\Sigma_\alpha\beta$ in $\mathbf{C}/\bar{X}$ (for some $\zeta\colon\bar{Z}\to\bar{X}$); we derive $\overline{\pi'(c)}\colon 1_{\bar{Z}}\to\overline{(\pi(c))}^*\beta$ via the following correspondence. Note that $\overline{\pi(c)} = \beta\bar{c}\colon \zeta\to\alpha$, and so $\overline{(\pi(c))}^*\beta = (\beta\bar{c})^*\beta = \bar{c}^*\beta^*\beta$.

$$\frac{\dfrac{\bar{c}\colon \beta\bar{c}\to\beta \quad \text{in} \quad \mathbf{C}/\bar{A}}{\bar{c}\colon \Sigma_{\beta\bar{c}}1_{\bar{Z}}\to\beta \quad \text{in} \quad \mathbf{C}/\bar{A}}}{\overline{\pi'(c)}\colon 1_{\bar{Z}}\to(\beta\bar{c})^*\beta \quad \text{in} \quad \mathbf{C}/\bar{Z}}$$

($=$ I). $\overline{r(x_A)} = \mathrm{id}_{\bar{A}}\colon \bar{A}\to\bar{A}$. Hence if $a\in A$, $\bar{a}\colon\zeta\to\alpha$, then $\overline{I(a,a)}$ is

$$(\mathrm{Eq}\,(a,a) \rightarrowtail \bar{Z}) = (\mathrm{id}_Z\colon \bar{Z}\to\bar{Z}), \quad \text{and} \quad \overline{r(a)} \quad \text{is} \quad \mathrm{id}_{\bar{Z}}.$$

($=$ E). In Seely [**10**] (§5) it is shown that any hyperdoctrine $\mathbf{P}$ has 'substitution' morphisms $B[a]\times I(a,b)\to B[b]$, where $B[x_A]$ is a predicate in $\mathbf{P}(A)$, I is the 'equality predicate', and a, b are terms of type A. This is essentially (= E), since by the c.o.v. $C[x_A, y_A, z_{I(x,y)}]$ must be interpreted as a morphism $\bar{C} \to \bar{A}$; recall that $\overline{I(x,y)}$ is $\Delta_{\bar{A}}$: $\bar{A} \to \bar{A} \times \bar{A}$.) Briefly the details are as follows: If, for some ζ: $\bar{Z} \to \bar{X}$, $a, b \in A$ are interpreted as $\bar{a}, \bar{b}$: $\zeta \to \alpha$ in $\mathbf{C}/\bar{X}$, the substitution morphism is

$$\bar{a} * \beta \times \overline{\langle a, b \rangle} * \bar{I} \to \bar{b} * \beta \quad \text{in } \mathbf{C}/\bar{Z}$$

given by the following. $\bar{a} * \beta \times \overline{\langle a, b \rangle} * \bar{I}$ is the morphism $\bar{P} \to \bar{Z}$ defined by the top left pullback in

$$\begin{tikzcd}
\bar{P} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & \bar{I}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & \bar{A} \arrow[d, "\Delta_A"] \\
\bar{a}^*\bar{B} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & \bar{Z} \arrow[r, "<\bar{a},\, \bar{b}>"'] \arrow[d, "\bar{a}"] & \bar{A} \times \bar{A} \\
\bar{B} \arrow[r, "\beta"'] & \bar{A} &
\end{tikzcd}$$

$\bar{b} * \beta$ is the pullback of β along $\bar{b}$:

$$\begin{tikzcd}
\bar{b}^*\bar{B} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & \bar{B} \arrow[d, "\beta"] \\
\bar{Z} \arrow[r, "\bar{b}"'] & \bar{A}
\end{tikzcd}$$

The morphism between these pullbacks is induced by the commutative square

$$\begin{tikzcd}
\bar{P} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & \bar{a}^*\bar{B} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & \bar{B} \arrow[d, "\beta"] \\
\bar{I}' \arrow[r] \arrow[d] & \bar{Z} \arrow[r, "\bar{a}"] & \bar{A} \arrow[d, "\mathrm{id}_A"] \\
\bar{Z} \arrow[rr, "\bar{b}"'] & & \bar{A}
\end{tikzcd}$$

by the pullback's universal property.

4·5. Proposition (Soundness). *For any interpretation* $^-$: $\mathbf{M} \to \mathbf{C}$, *as given in* 4·1 *and* 4·4, *the equality rules are all valid under the interpretation.*

The proof of the soundness theorem is a straightforward matter of checking definitions. Since a similar sort of result is discussed in Seely [**10**], I omit most of the details here.

(1 red) is valid since 1 is a terminal object.

(Π red). If in the notation of 4·4·2 for the Π rules, $\bar{Z} = \bar{X}$, $\zeta = \mathrm{id}_{\bar{X}}$, $\bar{f} = \overline{\lambda x_A t[x]}$. $\bar{a}$: $1_{\bar{X}} \to \alpha$, then one can check that of the two morphisms whose composite $\bar{a} \to \beta$ induces $\overline{f(a)}$: $1_{\bar{X}} \to \bar{a} * \beta$, the first $\bar{a} \to \alpha * \zeta$ is $\bar{a}$ itself ($\alpha * \zeta = 1_A$), and the second $\alpha * \zeta \to \beta$ is $\bar{t}$, so the composite is $\bar{t} \cdot \bar{a} = \overline{t[a]}$.

(Π exp) is checked similarly; the point here is that the adjunction correspondences are just (Π I). $(\alpha^* \gamma \to \beta) \mapsto (\gamma \to \Pi_\alpha \beta)$ and (Π E): $(\gamma \to \Pi_\alpha \beta) \mapsto (\alpha^* \gamma \to \beta)$.

(Σ red) and (Σ exp) follow similarly, from the remark that essentially (Σ I) is the adjunction correspondence $(\beta \to \alpha^* \gamma) \mapsto (\Sigma_\alpha \beta \to \gamma)$, and ($\Sigma$ E) is $(\Sigma_\alpha \beta \to \gamma) \mapsto (\beta \to \alpha^* \gamma)$.

(= red) and (= exp) are similar to the proof in Seely **[10]** (§5).

(I rule) is valid, essentially since equalizers are monomorphisms: If $a, b \in A$ are interpreted as $\bar{a}, \bar{b}\colon \bar{Z} \to \bar{A}$ ($\bar{Z}$ interpreting the type of the variable x in $a[x]$, $b[x]$), then $i\colon \overline{I(a,b)} \rightarrowtail \bar{Z}$ is the equalizer. If $t[x] \in I(a[x], b[x])$, and x satisfies the c.o.v., then $\bar{t}$ is a morphism in $\mathbf{C}/\bar{Z}$:

$$\begin{tikzcd} \bar{Z} \arrow[r, "\bar{t}"] \arrow[dr, "\mathrm{id}_{\bar{Z}}"'] & \overline{I(a,b)} \arrow[d, "i"] \\ & \bar{Z} \end{tikzcd}$$

It is easy to see that there can be at most one such $\bar{t}$, since $i \cdot \bar{t} = \mathrm{id}_{\bar{Z}}$, and i is mono; and that the existence of such a $\bar{t}$ implies i is an isomorphism, so that $\bar{a} = \bar{b}$. (Hint: $\bar{t}$ must be mono since i is, and so also $\bar{t} \cdot i = \mathrm{id}_{\bar{Z}}$.)

4·6. Given an ML theory $\mathbf{M}$, there is a canonical interpretation $\tilde{}\colon \mathbf{M} \to \mathbf{C}(\mathbf{M})$, essentially using the ideas of Lemma 3·2·3; types are interpreted as their 'extensions', and terms are interpreted according to 4·4·2. So if $A[x_1, \ldots, x_n]$ is a type depending on $x_1 \in X_1$, $x_2 \in X_2[x_1], \ldots, x_n \in X_n[x_1, \ldots, x_{n-1}]$, then X_1 is interpreted as itself, X_2 is interpreted as the projection term $\pi\colon \Sigma_{x_1 \in X_1} X_2[x_1] \to X_1$ (denoted $\tilde{X}_2 \to \tilde{X}_1$), and so on, A being interpreted as the projection

$$\Sigma_{x_1 \in X_1} \Sigma_{x_2 \in X_2[x_1]} \ldots \Sigma_{x_n \in X_n[x_1 \ldots x_{n-1}]} A[x_1 \ldots x_n] \\ \to \Sigma_{x_1 \in X_1} \Sigma_{x_2 \in X_2[x_1]} \ldots \Sigma_{x_{n-1} \in X_{n-1}[x_1 \ldots x_{n-2}]} X_n[x_1 \ldots x_{n-1}],$$

(denoted $\tilde{A} \to \tilde{X}_n$). The main fact we must check is that this is compatible with 4·4·1.

(i) Since 1 is closed, $\tilde{1} = 1$.

(ii) $I(x, y)$ is interpreted as $\Sigma_{x \in A} \Sigma_{y \in A} I(x, y) \to A \times A : \langle x, y, r \rangle \mapsto \langle x, y \rangle$ (supposing for simplicity that A is closed). We must show an isomorphism

$$\begin{tikzcd} \Sigma_{x \in A} \Sigma_{y \in A} I(x,y) \arrow[dr, "\pi"'] & & A \arrow[dl, "\Delta_A"] \\ & A \times A & \end{tikzcd}$$

making the triangle commute. The evident maps will do, because of the (I rule): $\langle x, y, r \rangle \mapsto x$ and $x \mapsto \langle x, x, r(x) \rangle$ are the required inverses.

(iii) If for $x \in A$, $B[x]$ is a type, then $B[x]$ is interpreted as $\Sigma_{x \in A} B[v] \to A$ (again supposing A closed). $\Pi_{x \in A} B[x]$ and $\Sigma_{x \in A} B[x]$ are interpreted as themselves (since they are closed): we must show they are (isomorphic to) $\Pi_A \pi$ and $\Sigma_A \pi$ (where we identify A with $A \to 1$). The latter is obvious, since $\Sigma_A \pi = \Sigma_{x \in A} B[x] \to 1$. For the former, note that $\Pi_A \pi = \Sigma_{f \in A \supset \Sigma x_A B[x]} I(\pi f, \mathrm{id}_A)$, the pullback of

$$\pi^A\colon A \supset \Sigma x_A B[x] \to A \supset A \quad \text{along} \quad \ulcorner \mathrm{id} \urcorner\colon 1 \to A \supset A.$$

To see this is isomorphic to $\Pi_{x \in A} B[x]$, we can copy the proof of 3·2·3·3: to t in $\Pi_{x \in A} B[x]$

associate $\hat{t} = \langle \lambda_{x\in A} \langle x, t(x)\rangle,\ r(\mathrm{id}_A)\rangle$, and to s in $\Pi_A \pi$, associate $\check{s} = \lambda_{x\in A}\, \pi'(\pi s)\,(x)$. It is easy to check these are inverse.

4·6·1. Given an ML theory $\mathbf{M}$ and an LCC $\mathbf{C}$, an interpretation $^-\colon \mathbf{M} \to \mathbf{C}$ induces a functor $F\colon \mathbf{C}(\mathbf{M}) \to \mathbf{C}$ preserving all LCC structure. Under $^-$, closed types of $\mathbf{M}$ are interpreted as objects of $\mathbf{C}$, and closed terms as morphisms; it can then be checked that this is in fact functorial. That LCC structure is preserved follows from soundness (4·5). Hence:

4·6·1. PROPOSITION. *There is a canonical interpretation* $^\sim\colon \mathbf{M} \to \mathbf{C}(\mathbf{M})$; *given any other interpretation* $^-\colon \mathbf{M} \to \mathbf{C}$, *there is a unique functor* $F\colon \mathbf{C}(\mathbf{M}) \to \mathbf{C}$ *making*

$$\begin{tikzcd} \mathbf{M} \arrow[r, "\sim"] \arrow[dr, "-"'] & \mathbf{C}(\mathbf{M}) \arrow[d, "F"] \\ & \mathbf{C} \end{tikzcd}$$

commute (*in the evident sense*).

4·7. *Definitions.* (i) **LCC** is the category of all LCC categories and structure preserving functors.

(ii) For any ML theory $\mathbf{M}$ and any LCC category $\mathbf{C}$, $\mathrm{Int}\,(\mathbf{M}, \mathbf{C})$ is the set of interpretations $\mathbf{M} \to \mathbf{C}$.

(iii) For ML theories $\mathbf{M}, \mathbf{M}'$, an interpretation $\mathbf{M} \to \mathbf{M}'$ is an interpretation $\mathbf{M} \to \mathbf{C}(\mathbf{M}')$.

(iv) **ML** is the category of all ML theories and interpretations between them.

Remark. It is straightforward to check that the definition (iii) is equivalent to the standard syntactical description one would expect of the notion 'interpretation of ML theories'.

4·8. COROLLARY. (i) *For any ML theory* $\mathbf{M}$, *and any LCC category* $\mathbf{C}$,

$$\mathrm{Int}\,(\mathbf{M}, \mathbf{C}) \cong \mathbf{LCC}\,(\mathbf{C}(\mathbf{M}), \mathbf{C}).$$

(ii) *For ML theories* $\mathbf{M}, \mathbf{M}'$, $\mathbf{ML}(\mathbf{M}, \mathbf{M}') \cong \mathbf{LCC}(\mathbf{C}(\mathbf{M}), \mathbf{C}(\mathbf{M}'))$; *each is isomorphic to* $\mathrm{Int}\,(\mathbf{M}, (\mathbf{CM}'))$.

4·8·1. *Remark.* $\mathrm{Int}\,(-,-)$ is in fact functorial: given an interpretation $f\colon \mathbf{M} \to \mathbf{M}'$ in **ML**, and a functor $g\colon \mathbf{C} \to \mathbf{C}'$ in **LCC**, we have maps

$$\mathrm{Int}\,(f, \mathbf{C})\colon \mathrm{Int}\,(\mathbf{M}', \mathbf{C}) \to \mathrm{Int}\,(\mathbf{M}, \mathbf{C}) \quad \text{and} \quad \mathrm{Int}\,(\mathbf{M}, g)\colon \mathrm{Int}\,(\mathbf{M}, \mathbf{C}) \to \mathrm{Int}\,(\mathbf{M}, \mathbf{C}'),$$

with the expected properties. The isomorphisms of 4·8 are then natural in each variable.

5. *From LCC to ML.* Given an LCC category $\mathbf{C}$, we define an ML theory $\mathbf{M}(\mathbf{C})$, basically by mimicking the definition (4·1, 4·4) of an interpretation. The objects of $\mathbf{C}$ are to be the type constants of $\mathbf{M}(\mathbf{C})$ (i.e. type-valued function symbols with no argument), and the morphisms of $\mathbf{C}$ are to be term-valued function symbols, with argument and value of types given by the domain and codomain respectively. Following the steps of 4·4, the other types and terms are defined as objects and

morphisms of appropriate slice categories, so that, e.g., a type with a free variable x_A is an object of $\mathbf{C}/A$. We have to make certain that the syntactical constructions give the correct results; for instance, if A, B are objects of $\mathbf{C}$ (and so types of $\mathbf{M}(\mathbf{C})$) then there is an object $A \times B$ (which is a type of $\mathbf{M}(\mathbf{C})$) as well as a (different!) type $A \times B$: these should be equal. So we add all such equations to $\mathbf{M}(\mathbf{C})$. (In Seely [**10**] a similar construction is carried out, adding appropriate terms (or derivations) and equations (or operations) forcing the terms to be the required isomorphisms.) We omit the details.

5·1. The point about $\mathbf{M}(\mathbf{C})$ is that its types are the predicates of the first order theory $\mathbf{T}(\mathbf{C})$ corresponding to the hyperdoctrine $\mathbf{C}$, and its terms are the 'proofs' (equivalence classes of derivations) in $\mathbf{T}(\mathbf{C})$.

5·2. PROPOSITION. $\mathbf{M}(\mathbf{C})$ *is an ML theory.*

The *proof* is essentially the same as the proof of the Soundness Theorem 4·5.

5·3. *Remark.* In defining $\mathbf{M}(\mathbf{C})$ above, we could regard an ML theory as made up of a set of types, a set of terms, a relation '$\in$' between them, and equality relations '$=$' on each, the type and term formation rules being regarded as closure conditions on these sets, and the equality rules as conditions on '$=$'. This viewpoint produces a somewhat more economical structure for $\mathbf{M}(\mathbf{C})$. In adopting the more usual viewpoint, regarding the formation rules as a procedure for generating new terms and types from old, we produce a 'fatter' structure for $\mathbf{M}(\mathbf{C})$; this does not present too much of a problem. First we note that an ML theory $\mathbf{M}$ has a natural categorical structure (essentially $\mathbf{C}(\mathbf{M})$). Then note that the 'extra' types and terms of the fat version of $\mathbf{M}(\mathbf{C})$ are isomorphic to ones in the more economical version of $\mathbf{M}(\mathbf{C})$, and so the two versions are in fact equivalent theories; this is essentially Proposition 3·2·4.

5·4. *Remark.* An LCC functor $F: \mathbf{C} \to \mathbf{C}'$ canonically induces an interpretation $\mathbf{M}(\mathbf{C}) \to \mathbf{C}'$, and furthermore $\mathrm{Int}(\mathbf{M}(\mathbf{C}), \mathbf{C}') \simeq \mathbf{LCC}(\mathbf{C}, \mathbf{C}')$ (naturally in $\mathbf{C}$ and $\mathbf{C}'$). A direct proof may be done as an exercise. The result is also a corollary of 6·1.

5·5. *Definition.* Two ML theories $\mathbf{M}$, $\mathbf{M}'$ are equivalent if the functors $\mathrm{Int}(\mathbf{M}, -)$, $\mathrm{Int}(\mathbf{M}', -)$ are naturally isomorphic.

Remark. This means that for any $\mathbf{C}$ in $\mathbf{LCC}$, there is an isomorphism

$$\mathrm{Int}(\mathbf{M}, \mathbf{C}) \simeq \mathrm{Int}(\mathbf{M}', \mathbf{C}),$$

and moreover, for any functor f: $\mathbf{C} \to \mathbf{C}'$,

$$\begin{tikzcd}
\mathrm{Int}(\mathbf{M}, \mathbf{C}) \arrow[r, phantom, "\simeq"] \arrow[d] & \mathrm{Int}(\mathbf{M}', \mathbf{C}) \arrow[d] \\
\mathrm{Int}(\mathbf{M}, \mathbf{C}') \arrow[r, phantom, "\simeq"] & \mathrm{Int}(\mathbf{M}', \mathbf{C}')
\end{tikzcd}$$

commutes. By 4·8 this is equivalent to $\mathbf{C}(\mathbf{M}) \simeq \mathbf{C}(\mathbf{M}')$ in $\mathbf{LCC}$, or that $\mathbf{M} \simeq \mathbf{M}'$ in $\mathbf{ML}$.

There is also an equivalent 'standard syntactical description' of equivalence of ML theories: it may be left as an exercise to write this out, showing it equivalent to 5·5.

6. *Equivalences*

6·1. THEOREM. *If* $\mathbf{C}$ *is LCC, then* $\mathbf{C} \simeq \mathbf{C}(\mathbf{M}(\mathbf{C}))$.

Proof. By construction of $\mathbf{M}(\mathbf{C})$, the closed types and terms of $\mathbf{M}(\mathbf{C})$ are the predicates and proofs of $\mathbf{C}/1$ (i.e. without free variables). But $\mathbf{C}/1 \simeq \mathbf{C}$. Hence $\mathbf{C}(\mathbf{M}(\mathbf{C})) \simeq \mathbf{C}$.

6·1·1. *Remark.* By 4·8 and 5·4, $\mathbf{LCC}(\mathbf{C}, \mathbf{C}') \simeq \mathbf{LCC}(\mathbf{C}(\mathbf{M}(\mathbf{C})), \mathbf{C}')$, giving an alternative proof. (Or: 6·1 and 4·8 imply 5·4.)

6·2. THEOREM. If $\mathbf{M}$ is an ML theory, $\mathbf{M}(\mathbf{C}(\mathbf{M}))$ is an equivalent ML theory.

Proof. Immediate from 6·1 and the definition 5·5.

6·3. THEOREM. The categories $\mathbf{ML}$ and $\mathbf{LCC}$ are equivalent.

Proof. Immediate from 6·1, 6·2. Note that, from 5·4 and 6·1,

$$\mathbf{LCC}(\mathbf{C}, \mathbf{C}') \simeq \mathrm{Int}\,(\mathbf{M}(\mathbf{C}), \mathbf{C}') \simeq \mathrm{Int}\,(\mathbf{M}(\mathbf{C}), \mathbf{C}(\mathbf{M}(\mathbf{C}'))) = \mathbf{ML}(\mathbf{M}(\mathbf{C}), \mathbf{M}(\mathbf{C}')),$$

and we have already seen in 4·8 that $\mathbf{ML}(\mathbf{M}, \mathbf{M}') \simeq \mathbf{LCC}(\mathbf{C}(\mathbf{M})), \mathbf{C}(\mathbf{M}'))$.

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