Nash equilibrium in Infinite extensive-form games

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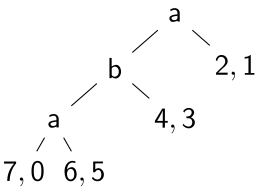
HIGHLIGHTS 2013

September 17, 2013

Nash equilibrium in finite extensive-form games

NE are traditionally obtained by **backward induction**:



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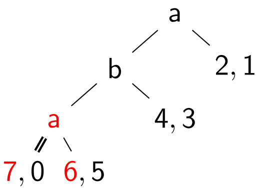

Double lines represent strategical choices.

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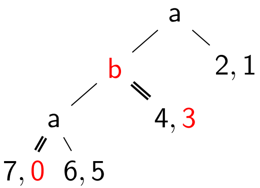

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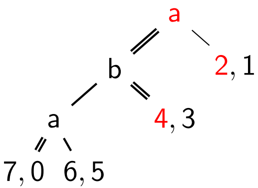

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Gale-Stewart games

Let C be a non-empty set.

Player **a**	$p_0 \in C$		$p_2 \in C$		$\dots$	
Player **b**		$p_1 \in C$		$p_3 \in C$		$\dots$

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Let $W \subseteq C^\omega$, player **a** wins iff $p_0 p_1 p_2 \cdots \in W$.

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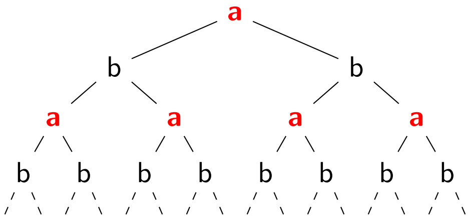
Example with $C = \{\text{left}, \text{right}\}$



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$$\begin{array}{l|lllllll} \text{Player } \mathbf{a} & p_0 \in C & & p_2 \in C & & \ldots & \\ \text{Player } \mathbf{b} & & p_1 \in C & & p_3 \in C & & \ldots \end{array}$$

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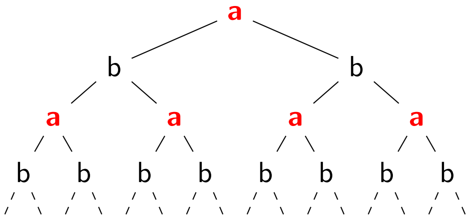

Leaf-free infinite extensive-form games: no backward induction!!!

Quasi-Borel determinacy

Discrete topology on C and product topology on C^ω.

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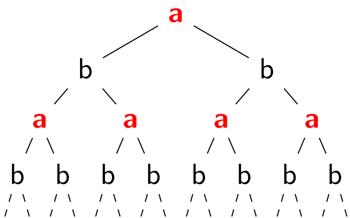
Theorem (Martin 1975, 1990)

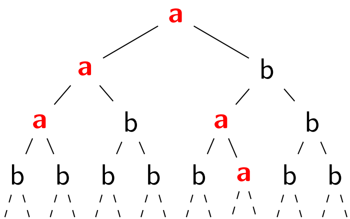
If W is (quasi)-Borel, one player has a winning strategy.

Messy Gale-Stewart games

Gale-Stewart games,
players play alternately:



Messy Gale-Stewart games,
arbitrary order:



First extension of Martin's theorem

Lemma

Let W be the winning set of a **messy** *Gale-Stewart game. If W is quasi-Borel, one player has a winning strategy.*

Proof.

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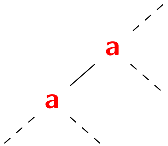

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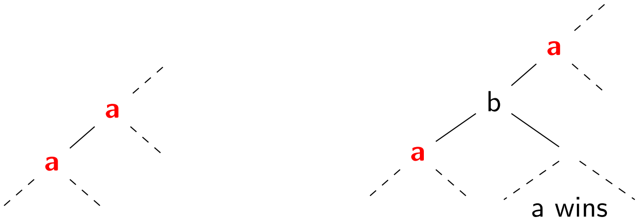
Folklore transformation: insertion of dummy nodes.



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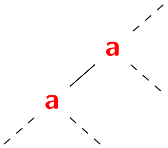
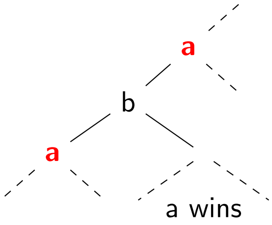


If winning set W is quasi-Borel,



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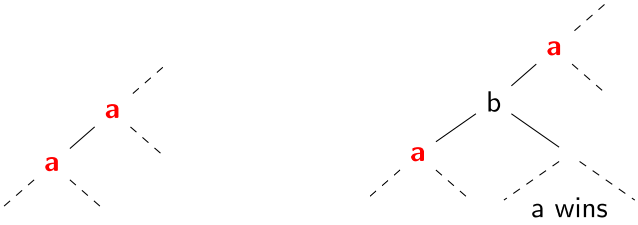
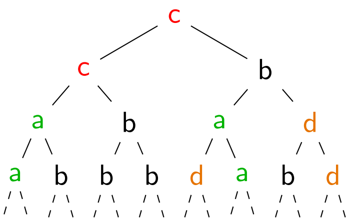


If winning set W is quasi-Borel, modified W is quasi-Borel

Win. strat. after dummy insertion translate back to win. strat.

$\square$

Infinite game in extensive form



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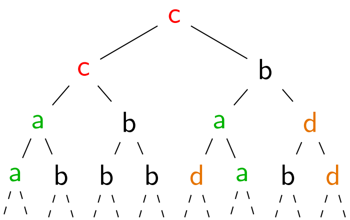

- $v : C^\omega \to O$ from plays to outcomes.

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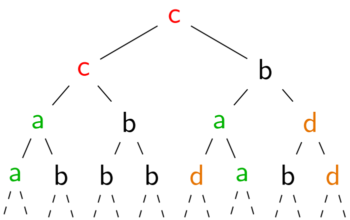

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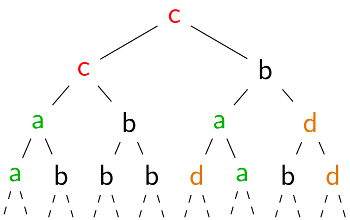

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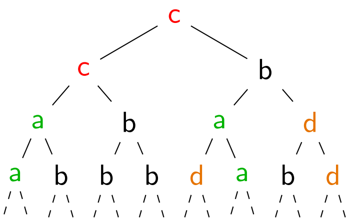
$$
\begin{array}{ccccccccccccc}
\text{start} & \longrightarrow & a & \to & a & \to & a & \text{------} & a & \to & a & \text{------} & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 2 & & 3 & & n & & n+1 & &
\end{array}
$$

Infinite game in extensive form



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Theorem

The game has an NE, if the $\prec_a^{-1}$ are strictly well-founded,

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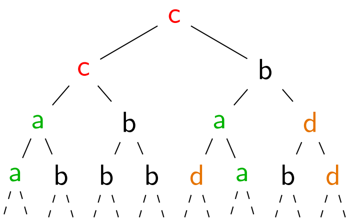

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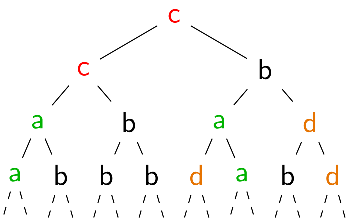

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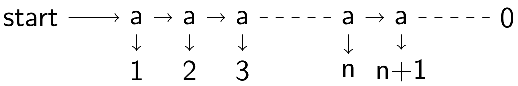

Theorem

The game has an NE, if the $\prec_a^{-1}$ are strictly well-founded, $v^{-1}(o)$ is quasi-Borel for all o in countable O.

Idea of the proof for finite games

Both players get the same payoffs.

```
          a
        /   \
     b         b
    / \       / \
   a   2     4   1
  / \
 3   0
```

- At the root, *a* seeks the best guaranteed payoff.

Idea of the proof for finite games

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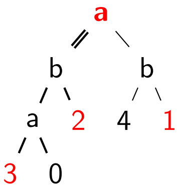

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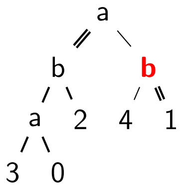

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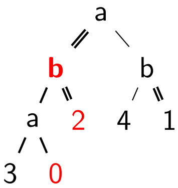

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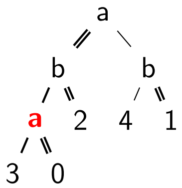

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\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & & 1 & & 1 & & 1 & &
\end{array}
$$

Idea of the proof for infinite games

$$\begin{tikzcd}
a \arrow[r, Rightarrow] \arrow[d] & a \arrow[r] \arrow[d, Rightarrow] & a \arrow[r] \arrow[d] & a \arrow[r, dashed, no head] \arrow[d] & 0 \\
1 & 1 & 1 & 1 &
\end{tikzcd}$$

Idea of the proof for infinite games

$$\begin{array}{ccccccccc}
a & \boldsymbol{\rightarrow} & \color{red}{\mathbf{a}} & \boldsymbol{\rightarrow} & a & \rightarrow & a & \text{- - - - - -} & 0 \\
\downarrow & & \downarrow & & \boldsymbol{\downarrow} & & \downarrow & & \\
1 & & 1 & & 1 & & 1 & &
\end{array}$$

Idea of the proof for infinite games

$$
\begin{array}{ccccccccc}
a & \Rightarrow & a & \Rightarrow & \color{red}{\mathbf{a}} & \Rightarrow & a & \text{- - - - - -} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \Downarrow & & \\
1 & & 1 & & 1 & & 1 & &
\end{array}
$$

Idea of the proof for infinite games

$$
\begin{tikzcd}
a \arrow[r] \arrow[d] & a \arrow[r] \arrow[d] & \color{red}{a} \arrow[r] \arrow[d] & a \arrow[d] \arrow[rr, dashed, no head] & & 0 \\
1 & 1 & 1 & 1 & &
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$$\begin{tikzcd}
a \arrow[r] \arrow[d] & a \arrow[r] \arrow[d] & {\color{red}\mathbf{a}} \arrow[r] \arrow[d] & a \arrow[r, dashed, no head] \arrow[d] & 0 \\
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- To fix it, change subprofiles only if improving the guarantee.

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$$\begin{array}{ccccccccc} a & \rightarrow & a & \rightarrow & a & \rightarrow & a & ====== & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & & 1 & & 1 & & 1 & & \end{array}$$

- To fix it, change subprofiles only if improving the guarantee.
- No infinite ascending preference chain, so each player improves guarantee only finitely many times.

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$$\begin{array}{ccccccccc} a & \dashrightarrow & a & \dashrightarrow & \color{red}{\mathbf{a}} & \dashrightarrow & a & \cdots\cdots & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & & 1 & & 1 & & 1 & & \end{array}$$

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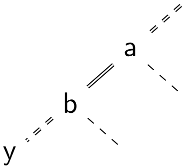

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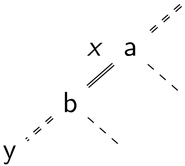

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$$\begin{array}{ccccccc} a & \rightarrow & a & \rightarrow & \color{red}{\mathbf{a}} & \rightarrow & a \; {=}\,{=}\,{=}\,{=}\,{=}\,{=} \; 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 1 & & 1 & & 1 & & 1 \end{array}$$

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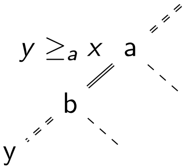

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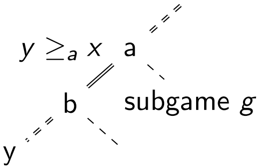

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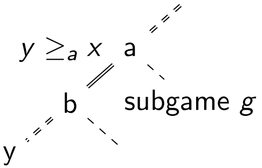

Transform g into a messy Gale-Stewart game:
a's opponents team up and win if a gets $x' \leq_a x$.

Idea of the proof for infinite games

$$\begin{tikzcd} a \arrow[r] \arrow[d] & a \arrow[r] \arrow[d] & \color{red}{\mathbf{a}} \arrow[r] \arrow[d] & a \arrow[r, dashed, equal] \arrow[d] & 0 \\ 1 & 1 & 1 & 1 & \end{tikzcd}$$

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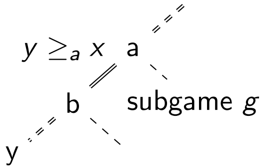

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Player a has no winning strategy, by definition of the guarantee.
The winning set is quasi-Borel, so a's opponents have a win. strat.

Idea of the proof for infinite games

$$\begin{tikzcd} a \arrow[r, dashed] \arrow[d] & a \arrow[r, dashed] \arrow[d] & \color{red}{\mathbf{a}} \arrow[r, dashed] \arrow[d] & a \arrow[r, equal, dashed] \arrow[d] & 0 \\ 1 & 1 & 1 & 1 & \end{tikzcd}$$

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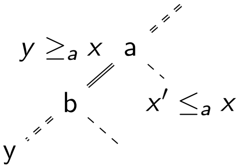

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Conclusion

The actual result works for more than just quasi-Borel sets:

- If a pointclass with reasonnable closure properties yields determined games, it yields also games with NE.

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Questions:

- Several proofs in logic invoke Borel determinacy, can the results be extended by invoking existence of NE?

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Questions:

- Several proofs in logic invoke Borel determinacy, can the results be extended by invoking existence of NE?
- Is anyone familiar with the proof of Borel determinacy?