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THE MODEL THEORY OF DIFFERENTIAL FIELDS WITH FINITELY MANY COMMUTING DERIVATIONS

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Abstract. In this paper we set out the basic model theory of differential fields of characteristic 0, which have finitely many commuting derivations. We give axioms for the theory of differentially closed differential fields with m derivations and show that this theory is ω-stable, model complete, and quantifier-eliminable, and that it admits elimination of imaginaries. We give a characterization of forking and compute the rank of this theory to be $\omega^m + 1$.

§1. Introduction. The results in this paper come out of the second half of my dissertation, *Model-Theoretic Results on Ordinary and Partial Differential Fields* [9]. After giving a brief introduction to differential rings and fields, we will describe axioms for the theory of differentially closed differential fields of characteristic 0 with m commuting derivations, and show that this theory is ω-stable, model complete, and quantifier-eliminable, and that it admits elimination of imaginaries. We will give an algebraic characterization of forking using Kolchin's differential dimension polynomial, and give bounds for the U-rank and Morley rank of this theory. Finally, we will compute the rank of this theory to be $\omega^m + 1$.

It is helpful to compare this treatment of differential fields with David Marker's account of the model theory of differential fields with a single derivation [8].

§2. Differential algebra.

2.1. Differential rings and fields. All rings and fields in this paper will be of characteristic 0.

Let R be a ring. Recall that a derivation on R is a map $\delta : R \to R$ that satisfies the following properties:

$$\delta(a+b) = \delta(a) + \delta(b)$$
$$\delta(ab) = a\delta(b) + \delta(a)b$$

for $a, b \in R$.

Now, suppose R is equipped with a set $\Delta = \{\delta_1, \dots, \delta_m\}$ of m commuting derivations; i.e., for all $a \in R$,

$$\delta_i\delta_j a = \delta_j\delta_i a.$$

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We call R a *differential ring* or a Δ-ring. If k is a field equipped with m derivations, we call k a *differential field.*

If $m = 1$ in the above definition, we call R or k an *ordinary differential ring* or *field.*

NOTATION 2.1.1. For $a \in R$, $\delta_1^{e_1} \dots \delta_m^{e_m} a$ will mean

$$\underbrace{\delta_1 \dots \delta_1}_{e_1 \text{ times}} \dots \underbrace{\delta_m \dots \delta_m}_{e_m \text{ times}} a.$$

We call $\mathscr{C}_R = \{a : \delta_i a = 0 \text{ for } 1 \leq i \leq m\}$ the *ring of constants of* R. If k is a field, it is easy to see that $\mathscr{C}_k$ is a differential subfield of k and hence can be called the *constant field of* k. Moreover, as for ordinary differential fields, if k is algebraically closed, then $\mathscr{C}_k$ is algebraically closed.

Let $\mathscr{I}$ be an ideal (in the usual sense) of R. $\mathscr{I}$ is called a *differential ideal* or Δ-*ideal* of R if whenever $a \in \mathscr{I}$, $\delta_i(a) \in \mathscr{I}$ for all i. If $\mathscr{I}$ is a Δ-ideal of R and is a radical ideal in the usual sense, then $\mathscr{I}$ is called a *radical differential ideal.*

2.2. Differential polynomial rings. Let $\overline{y} = y_1, \dots, y_n$ be a collection of n variables. Let $\Theta = \{\delta_1^{e_1} \dots \delta_m^{e_m} : e_i \geq 0\}$ and let $\Theta\overline{y} = \{\theta y_j : \theta \in \Theta, j = 1, \dots, n\}$ be a set of algebraic indeterminates. Notice that for each j, $y_j = \delta_1^0 \dots \delta_m^0 y_j$, so $y_j \in \Theta\overline{y}$. If $u_1, u_2, \dots, u_\ell$ are in $\Theta\overline{y}$, then $R[u_1, u_2, \dots, u_\ell]$ is the usual polynomial ring generated by $u_1, u_2, \dots, u_\ell$ over R. The *differential polynomial ring* $R\{\overline{y}\}$ is the polynomial ring generated by $\Theta\overline{y}$ over R. We extend the derivations $\delta_1, \dots, \delta_m$ on R to derivations on $R\{\overline{y}\}$ in the obvious way. We call the y_j *differential indeterminates.*

Letting $\overline{y}$ and $\Theta\overline{y}$ be as above, we describe a ranking of $\Theta\overline{y}$. Let $\theta_1 = \delta_1^{e_1} \cdots \delta_m^{e_m}$ and $\theta_2 = \delta_1^{e'_1} \cdots \delta_m^{e'_m}$. Then

$$[\theta_1 y_i < \theta_2 y_{i'}] \text{ if and only if } \left[(\boldsymbol{e}, i, e_m, \dots, e_1) < (\boldsymbol{e}', i', e'_m, \dots, e'_1)\right] \tag{2.2.1}$$

in the lexicographical order of $\mathbb{N}^{m+2}$ where $\boldsymbol{e} = \sum_{j=1}^m e_j$ and $\boldsymbol{e}' = \sum_{j=1}^m e'_j$. We shall call this ranking the *canonical ranking of* $\Theta\overline{y}$. It is easily checked that under this ranking the order type of $\Theta\overline{y}$ is ω.

We will use this ranking in the discussion below.

NOTATION 2.2.1. Let $\{\mathsf{Y}_\ell\}_{\ell=1}^\infty$ be a set of new variables. It is useful to write Y_ℓ interchangeably with $\delta_1^{e_1} \dots \delta_m^{e_m} y_j$, where $\delta_1^{e_1} \dots \delta_m^{e_m} y_j$ is in the ℓ^{th} position in the canonical ranking. Suppose that $\mathsf{Y}_\ell = \delta_1^{e_1} \dots \delta_m^{e_m} y_j$. Then $\delta_i \mathsf{Y}_\ell = \mathsf{Y}_{\hat{\ell}}$, where $\delta_1^{e_1} \dots \delta_i^{e_i+1} \dots \delta_m^{e_m} y_j$ is in the $\hat{\ell}^{\text{th}}$ position in the canonical ranking.

EXAMPLE 2.2.2. The canonical rank gives us the following order on Θy, where y is a single differential indeterminate and $m = 2$.

$$\begin{array}{lll}
\mathsf{Y}_1 = y & \mathsf{Y}_2 = \delta_1 y & \mathsf{Y}_3 = \delta_2 y \\
\mathsf{Y}_4 = \delta_1^2 y & \mathsf{Y}_5 = \delta_1\delta_2 y & \mathsf{Y}_6 = \delta_2^2 y \\
\mathsf{Y}_7 = \delta_1^3 y & \mathsf{Y}_8 = \delta_1^2\delta_2 y & \mathsf{Y}_9 = \delta_1\delta_2^2 y \\
\mathsf{Y}_{10} = \delta_2^3 y & \dots &
\end{array}$$

Using the canonical ranking, we think of $f \in R\{\overline{y}\}$ as a polynomial in the variables $\overline{\mathsf{Y}} = \{\mathsf{Y}_1, \dots, \mathsf{Y}_h\}$ for some h. If $f \in R$, then we say that the *height of* f, denoted h_f, and the *order of* f, denoted $\operatorname{ord}(f)$, are both 0. If $f \notin R$, then we say that the *height of f*, h_f, is the maximal h such that Υ_h appears in f; the *order of f* is

$$\operatorname{ord}(f) = \max\left\{\sum_{i=1}^{m} e_j : \delta_1^{e_1}\cdots\delta_m^{e_m} y_j \text{ appears in } f \text{ for some } j\right\}.$$

Then the *leader of f*, denoted v_f, is Υ_{h_f}, and the degree of Υ_{h_f} in f is denoted by d_f.

There are two Δ-polynomials associated to f. I_f, the coefficient of $\Upsilon_h^{d_f}$, is called the *initial of f*, and S_f, the partial derivative of f with respect to v_f, is called the *separant of f*.

If R is a Δ-ring, we extend the ranking of $\Theta\overline{y}$ to a ranking of the Δ-polynomial ring $R\{\overline{y}\}$. Let $f \in R\{\overline{y}\}$. Define the rank(f) to be the ordered pair (h_f, d_f). For f and $g \in R\{\overline{y}\}$, we say that $\operatorname{rank}(f) < \operatorname{rank}(g)$ if $(h_f, d_f) < (h_g, d_g)$ lexicographically. If $(h_f, d_f) = (h_g, d_g)$, we say that $\operatorname{rank}(f) = \operatorname{rank}(g)$.

DEFINITION 2.2.3. Let $f, g \in R\{\overline{y}\}$. We say that f is *partially reduced with respect to g* if no proper derivative of v_g appears nontrivially in f.

EXAMPLE 2.2.4. Certainly, if $h_f \leq h_g$ for $f, g \in R\{\overline{y}\}$, then f is partially reduced with respect to g.

DEFINITION 2.2.5. Let $f, g \in R\{\overline{y}\}$. We say that f is *reduced with respect to g* if the following conditions hold:

1. f is partially reduced with respect to g;
2. $\deg_{v_g} f < \deg_{v_g} g = d_g$.

We say that $f \in R\{\overline{y}\}$ is *reduced with respect to a subset* $\Lambda \subset R\{\overline{y}\}$ if f is reduced with respect to g for all $g \in \Lambda$.

EXAMPLE 2.2.6. For $f \in R\{\overline{y}\}$, S_f and I_f are reduced with respect to f.

DEFINITION 2.2.7. Let $\Lambda = \{f_1, \ldots, f_s\} \subset R\{\overline{y}\}$. Λ is called *autoreduced* if f_i is reduced with respect to f_j for all $i \neq j$.

Notice that if $\Lambda = \{f_1, \ldots, f_s\}$ is autoreduced then $v_{f_i} \neq v_{f_j}$ for $i \neq j$.

We will always write an autoreduced subset $\Lambda = \{f_1, \ldots, f_s\}$ of Δ-polynomials over R in order of increasing height.

If $A \subset R\{\overline{y}\}$, let $[A]$ denote the differential ideal generated by $A \subset R\{\overline{y}\}$. $[A]$ is equal to the ideal $(\{\theta f : \theta \in \Theta, f \in A\})$.

Theorem 2.2.8 below is the differential analog of the usual division algorithm for polynomials in several variables. But first, we need the following definition.

Suppose that $\Lambda = \{f_1, \ldots, f_s\}$ is an autoreduced subset of δ-polynomials in $R\{\overline{y}\}$. Define $H_\Lambda \in R\{\overline{y}\}$ to be

$$\prod_{i=1}^{s} I_{f_i} S_{f_i}.$$

THEOREM 2.2.8 (Differential Division Algorithm [6, page 79]). *Let R be a differential ring. Suppose $\Lambda = \{f_1, \ldots, f_s\}$ is an autoreduced subset of the differential polynomial algebra $R\{\overline{y}\}$, and $f \in R\{\overline{y}\}$. Then there is $g \in R\{\overline{y}\}$ reduced with* *respect to* Λ, *with the rank of* g *less than or equal to that of* f, *and, more precisely, there is a natural number* ρ *such that*

$$H_\Lambda^\rho f \equiv g \mod [\Lambda];$$

i.e., $H_\Lambda^\rho f - g$ *can be written as a linear combination over* $R[\overline{y}]$ *of derivatives* θf_i *with* $f_i \in \Lambda$, $\theta \in \Theta$, *and* $\theta v_{f_i} \leq v_f$.

We will need the following technical results from Kaplansky.

LEMMA 2.2.9 ([5, page 11-12]). *Let* R *be a* Δ*-ring and let* $\mathcal{I}$ *be a radical differential ideal of* R.

1. *If the product* ab *is in* $\mathcal{I}$, *then* $a\delta_i b \in \mathcal{I}$ *and* $(\delta_i a)b \in \mathcal{I}$ *for all* i.
2. *The radical of* $\mathcal{I}$ *is a differential ideal.*

COROLLARY 2.2.10 ([5, page 13]). *Let* $\mathcal{I} \neq [0]$ *be a differential ideal of a differential ring* R *which is disjoint from a multiplicatively closed set* S. *Then there is a prime differential ideal* $\mathcal{J}$ *containing* $\mathcal{I}$ *which is disjoint from* S.

The Ascending Chain Condition (ACC) does not necessarily hold for δ-ideals in differential fields. The following theorem gives us an important case when the ACC does hold.

THEOREM 2.2.11 (The Ritt-Raudenbush Basis Theorem). [6, *pages* 126-9] *Let* $R \supseteq \mathbb{Q}$ *be a* Δ*-ring such that every radical* Δ*-ideal is finitely generated. Then every radical* Δ*-ideal in* $R\{y\}$ *is finitely generated, equivalently, we have the ACC on radical* Δ*-ideals.*

THEOREM 2.2.12 (Decomposition Theorem). [5, *page* 13] *Let* R *be a* Δ*-ring with ACC on radical* Δ*-ideals. Any radical* Δ*-ideal is the intersection of a finite number of prime* Δ*-ideals.*

Lemma 2.2.9 and Corollary 2.2.10 can be proven in a few lines, but the Basis Theorem is more difficult to prove.

Now we extend the preorder on Δ-polynomials in $R\{\overline{y}\}$ to a ranking of the autoreduced subsets of $R\{\overline{y}\}$. Let $\Lambda = \{f_1, \ldots, f_s\}$ and $\Lambda' = \{f'_1, \ldots, f'_r\}$ be autoreduced. We say that $\Lambda < \Lambda'$ if there is $t \in \mathbb{N}$, $t \leq s$, such that $\operatorname{rank}(f_i) = \operatorname{rank}(f'_i)$ for $i < t$ and $\operatorname{rank}(f_t) < \operatorname{rank}(f'_t)$, or if $s > r$ and $\operatorname{rank}(f_i) = \operatorname{rank}(f'_i)$ for $i \leq r$. If $r = s$ and $\operatorname{rank}(f_i) = \operatorname{rank}(f'_i)$ for all i, then Λ and Λ' have the same rank. In other words, the order is lexicographic in nature, but "humane" comes before "human."

FACT 2.2.13 ([6, pages 81-2]). *Every* Δ*-ideal* $\mathcal{I}$ *of* Δ*-polynomials in* $R\{\overline{y}\}$ *has a lowest ranking autoreduced subset, called a characteristic set of* $\mathcal{I}$.

A Δ-ideal $\mathcal{I}$ of Δ-polynomials in $R\{\overline{y}\}$ does not have a unique characteristic set. However, every characteristic set of $\mathcal{I}$ will have the same set of leaders and corresponding degrees. Hence, given a Δ-ideal $\mathcal{I}$ of Δ-polynomials in $R\{\overline{y}\}$, we will arbitrarily choose one such characteristic set and denote it by $\Lambda_{\mathcal{I}}$.

What is the relationship between a prime Δ-ideals, and its characteristic set? We will need the following definition of a coherent set.

REMARK 2.2.14. Let f_1 and f_2 be Δ-polynomials in $R\{\overline{y}\}$. Notice that if v_{f_1} and v_{f_2} involve the same differential indeterminate, then there are $\theta_1, \theta_2 \in \Theta$ such that $\theta_1 v_{f_1} = \theta_2 v_{f_2} = v$, where $v \in \Theta\overline{y}$ and is a common derivative of v_{f_1} and v_{f_2}.

Indeed, if $v_{f_1} = \delta_1^{e_1} \dots \delta_m^{e_m} y_j$ and $v_{f_2} = \delta_1^{e_1'} \dots \delta_m^{e_m'} y_j$, take $v = \delta_1^{e_1^*} \dots \delta_m^{e_m^*} y_j$ where $e_i^* = \max(e_i, e_i')$.

Of course, if v_{f_1} and v_{f_2} mention different Δ-indeterminates, then there will be no common derivative of v_{f_1} and v_{f_2}.

Let $\Lambda = \{f_1, \dots, f_s\}$ be a finite subset of $R\{\overline{y}\}$. For each h, let $(\Lambda)_h$ denote the ideal generated in $R\{\overline{y}\}$ by the f_i and their derivatives whose heights are less than or equal to h. Notice that $[\Lambda] = \bigcup_{h \in \mathbb{N}} (\Lambda)_h$.

Definition 2.2.15. Let $\Lambda = \{f_1, \dots, f_s\} \subset R\{\overline{y}\}$ be a finite subset of Δ-polynomials of $R\{\overline{y}\}$. Λ is *coherent* if the following conditions are satisfied:

Condition 1: Λ is autoreduced;

Condition 2: For $i \neq j$, suppose there are $\theta_i, \theta_j \in \Theta$ such that $\theta_i v_{f_i} = \theta_j v_{f_j} = v$ where $v \in \Theta\overline{y}$ where $v = \mathrm{Y}_h$ is the least such in the ranking of $\Theta\overline{y}$ (see Remark 2.2.14). It must be the case that $S_{f_j}\theta_i f_i - S_{f_i}\theta_j f_j \in (\Lambda)_{h-1}$. (Observe that by definition, $S_{f_j}\theta_i f_i - S_{f_i}\theta_j f_j$ is already in $(\Lambda)_h$.)

We are now able to characterize the relationship between certain Δ-ideals and their characteristic sets. It turns out that this is a very special relationship if the ideals in question are prime Δ-ideals in a Δ-polynomial ring over a Δ-field k.

Notation 2.2.16. Let $\mathscr{I}$ be an ideal in any ring R and let a be an element of R. Let

$$\mathscr{I} : a^n = \{b \in R : a^n b \in \mathscr{I}\}$$

for $n \in \mathbb{N}$ and let

$$\mathscr{I} : a^\infty = \{b \in R : a^n b \in \mathscr{I} \text{ for some } n \in \mathbb{N}\}.$$

The following lemma will be very important below when we talk about differentially closed fields.

Lemma 2.2.17 (Kolchin [6, page 137]). *Let Λ be a coherent set in $R\{\overline{y}\}$. Then every element of $[\Lambda] : H_\Lambda^\infty$ that is partially reduced with respect to Λ is in $(\Lambda) : H_\Lambda^\infty$.*

A very useful consequence of Lemma 2.2.17 is the following.

Theorem 2.2.18 (Kolchin). [6, *page* 167] *Let k be a differential field and let Λ be a finite subset of $k\{\overline{y}\}$. The following are equivalent*:

1. *Λ is a characteristic set of a prime differential ideal $\mathscr{P}$ of $k\{\overline{y}\}$.*
2. *Λ is coherent, and $(\Lambda) : H_\Lambda^\infty$ is a prime ideal containing no nonzero element reduced with respect to Λ.*

If Λ is a characteristic set of a prime differential ideal $\mathscr{P}$ of $k\{\overline{y}\}$, then $\mathscr{P} = [\Lambda] : H_\Lambda^\infty$.

2.3. Solutions over differential rings. Let R_0 be a Δ-ring and R a Δ-ring extending R_0. Let $\overline{y} = \{y_1, \dots, y_n\}$ and suppose $f \in R_0\{\overline{y}\}$. We consider two types of solutions in R to the equation $f = 0$.

On one hand, an *algebraic solution* is a tuple of elements of R satisfying $f = 0$ as an equation in the variables in $\Theta\overline{y}$. On the other hand, a *differential solution* to the equation $f = 0$ is a tuple of elements of R satisfying $f = 0$ as a differential equation in the variables $\overline{y}$.

Consider $\overline{\eta} = (\eta_1, \dots, \eta_n)$ where $\eta_i \in R$. Define

$$\mathscr{I}(\overline{\eta}/R_0) = \{f \in R_0\{\overline{y}\} : f(\overline{\eta}) = 0\}$$

to be the set of differential polynomials f, in n Δ-indeterminates with coefficients in R_0, for which $\overline{\eta}$ is a differential zero. We call $\mathscr{I}(\overline{\eta}/R_0)$ the *defining differential ideal of* $\overline{\eta}$ *in* $R_0\{\overline{y}\}$. If $\mathscr{I}(\overline{\eta}/R_0) \neq \emptyset$, we say that $\overline{\eta}$ is *differentially algebraic* or Δ-*algebraic over* R_0. If, on the other hand, $\mathscr{I}(\overline{\eta}/R_0) = \emptyset$, then $\overline{\eta}$ is *differentially independent* or Δ-*independent over* R_0.

$R\{\overline{\eta}\}$ is the Δ-*ring generated by* $\overline{\eta}$ *over* R; i.e., $R\{\overline{\eta}\}$ is the ring

$$R[\theta\eta_j : 1 \leq j \leq n, \theta \in \Theta].$$

A differential ring R extending a ring R_0 is said to be *finitely generated over* R_0 if $R = R_0\{\overline{\eta}\}$ where $\overline{\eta}$ is a finite set of elements of R.

If R_0 happens to be a Δ-field, k_0, then $k_0\{\overline{y}\}$ is an integral domain and $\mathscr{I}(\overline{\eta}/k_0)$ is a prime Δ-ideal; $k_0\langle\overline{\eta}\rangle$, the quotient field of $k_0\{\overline{\eta}\}$ is the Δ-*field generated by* $\overline{\eta}$ *over* k_0.

Let k be a Δ-field. Suppose k' is a Δ-field extending k. We say that k' has *differential transcendence degree* d over k if there are $\eta_1, \ldots, \eta_d \in k' \setminus k$ such that $\eta_1, \ldots, \eta_d$ are Δ-algebraically independent over k and there is no $\eta_{d+1} \in k'$ such that $\eta_1, \ldots, \eta_d, \eta_{d+1}$ are Δ-algebraically independent over k. Then we say that $\eta_1, \ldots, \eta_d$ is a differential transcendence basis for k' over k.

§3. Differentially closed differential fields.

3.1. The axioms. In this section we describe the axioms for the theory, m-DF, of differential fields of characteristic 0, which have finitely many commuting derivations. In addition, we introduce axioms for the model completion, m-DCF, of m-DF. m-DCF is then a generalization of DCF and we will see that m-DCF has very similar model-theoretic properties to those of DCF.

Let $\mathscr{L}_F = \{+, -, *, ^{-1}, 0, 1\}$ be the language of fields. Let

$$m\text{-}\mathscr{L}_F = \mathscr{L} \cup \{\delta_1, \ldots, \delta_m\}$$

be an extension of $\mathscr{L}_F$ by a set $\Delta = \{\delta_1, \ldots, \delta_m\}$ of m unary functions. The theory, m-DF, in the language m-$\mathscr{L}_F$ has as axioms the axioms for fields, and axioms describing the additive and multiplicative rules for the derivations $\Delta = \{\delta_1, \ldots, \delta_m\}$. m-DF is the theory of differential fields of characteristic 0 with m commuting derivatives, $\Delta = \{\delta_1, \ldots, \delta_m\}$.

Robinson first introduced the concept of an ordinary differentially closed field. In the same spirit as his original axioms (see [12]), a differential field K is differentially closed if and only if every finite system of the form

$$f_1 = 0, \ldots, f_s = 0, g \neq 0,$$

where g and the f_i are differential polynomials in $K\{y_1, \ldots, y_n\}$, which has a solution in some extension of K, has a solution in K. Seidenberg's elimination theory [13] gives us a formula for axioms that involve only one differential indeterminate, and require that we need only look at coherent systems of Δ-polynomials.

To write down the axioms for differentially closed fields with m derivation operators, we need to express, in a first-order fashion, the condition that a finite set $\Lambda = \{f_1, \ldots, f_s\}$ of Δ-polynomials over a Δ-field k is coherent. The following theorem is due to Hermann [3] and will suffice to show that this is possible.

THEOREM 3.1.1 (Hermann). *Fix $\Delta = \{\delta_1, \dots, \delta_m\}$ a set of m derivation operators. Let $\bar{y} = \{y_1, \dots y_n\}$ be a set of n differential indeterminates. Fix the canonical ranking of $\Theta\bar{y}$. Given natural numbers, h and d, let $\overline{\mathrm{Y}} = (\mathrm{Y}_1, \mathrm{Y}_2, \dots, \mathrm{Y}_h)$. Then there is $d' = d'(h, d) \in \mathbb{N}$, such that for each Δ-field k and all $f_1, \dots, f_t \in k[\overline{\mathrm{Y}}]$ and $\zeta \in k[\overline{\mathrm{Y}}]$ (i.e., the heights of the f_i and ζ are no greater than h), all of total degree less than or equal to d with respect to the variables $\mathrm{Y}_1, \mathrm{Y}_2, \dots, \mathrm{Y}_h$,*

$$\zeta \in (f_1, \dots, f_t) \iff \zeta = \sum_{i=1}^{t} \gamma_i f_i$$

for certain $\gamma_i \in K[\overline{\mathrm{Y}}]$ of total degree no greater than d'.

Theorem 3.1.1 allows us to write every such Δ-polynomial $\zeta = S_{f_j}\theta_i f_i - S_{f_i}\theta_j f_j$ as a linear combination of the f_i and certain of their derivatives over a restricted Δ-polynomial ring. Thus we can express the notion of coherency in a first-order way.

We need to create a template for differential polynomials. Consider the tuple $(s, \bar{h}, \bar{d}, t)$, where $s, t \in \mathbb{N}$, $\bar{h} = (h_1, \dots, h_s, h) \in \mathbb{N}^{s+1}$ and $h_i < h_j$ for $i < j$, and $\bar{d} = (d_1, \dots, d_s, d) \in \mathbb{N}^{s+1}$. Let $h' = \max\{h_s, h\}$. Let $\{y\}$, $\{a_{i,j} : 1 \le i \le s, 1 \le j \le (h')^t\}$ and $\{b_j : 1 \le j \le (h')^t\}$ be sets of variables. For $1 \le i \le s$, let

$$f_i(y) = \sum_{j=1}^{(h')^t} a_{i,j} \prod_{\ell=1}^{h'} (\theta_\ell y)^{e_{i,j,\ell}} \tag{3.1.1}$$

where $\theta_\ell \in \Theta$, $\theta_\ell y = \mathrm{Y}_\ell$ in the canonical ranking of Θy, and $e_{i,j,\ell} \in \mathbb{N}$; for each i, f_i is of height h_i, and of degree d_i in Y_{h_i}. Let

$$g(y) = \sum_{j=1}^{(h')^t} b_j \prod_{\ell=1}^{h'} (\theta_\ell y)^{e_{j,\ell}} \tag{3.1.2}$$

where θ_ℓ and $\theta_\ell y$ are as above, and $e_{j,\ell} \in \mathbb{N}$; g is of height h and degree d in Y_h. Assume further that each of the f_i and g is of total degree no greater than t in the variables $\{\mathrm{Y}_1, \dots, \mathrm{Y}_h\}$.

Consider the following set of conditions.

1. For $i < i'$, the degree of $f_{i'}$ in Y_{h_i} is less than d_i;
2. For $i < i'$, f_i' does not mention any proper derivative of v_{f_i};
3. For each i, the degree of g in Y_{h_i} is less than d_i;
4. For each i, g does not mention any proper derivative of v_{f_i}.

Conditions 1–4 above guarantee that for all parameters

$$\{a_{i,j} : 1 \le i \le s, 1 \le j \le (h')^t\} \text{ and } \{b_j : 1 \le j \le (h')^t\},$$

$\Lambda = \{f_1, \dots, f_s\}$ is autoreduced and g is reduced with respect to Λ.

Let $\mathcal{S}_{(s,\bar{h},\bar{d},t)}$ be the sentence that says: For all $\{a_{i,j}\}$ and $\{b_j\}$, such that f_i and g are of the forms (3.1.1) and (3.1.2), respectively, satisfy conditions 1–4 above, and such that $\Lambda = \{f_1, \dots, f_s\}$ is a coherent set of differential polynomials, the system of differential equations and inequation

$$f_1 = 0, \dots, f_s = 0, H_\Lambda g \ne 0 \tag{3.1.3}$$

has a differential solution for y, if system (3.1.3) has an algebraic solution.

We are now ready to state the axioms for the theory of differentially closed differential fields.

Let m-DCF be the theory having as axioms the axioms for algebraically closed fields, axioms describing the additive and multiplicative rules for the derivations $\Delta = \{\delta_1, \ldots, \delta_m\}$, and

$$\left\{\mathcal{S}_{(s,\bar{h},\bar{d},t)}\right\}_{s,t\in\mathbb{N},\bar{h},\bar{d}\in\mathbb{N}^{s+1}}.$$

We will show that m-DCF is a complete theory and is the model completion of m-DF.

LEMMA 3.1.2. *Every model k of m-DF can be extended to a model K of m-DCF.*

PROOF. Let k be a model of m-DF. Extend k to k', an algebraically closed field. (The derivations extend uniquely to k'.) Suppose $\Lambda \subset k\{y\}$ is a coherent set of Δ-polynomials and $g \in k\{y\}$ is reduced with respect to Λ. Now, suppose the system $\left(\bigwedge_{f\in\Lambda} f = 0\right) \wedge H_\Lambda g \neq 0$ has an algebraic solution in k'. Then $g^n \notin (\Lambda) : H_\Lambda^\infty$ for $n \in \mathbb{N}$. Hence by Lemma 2.2.17, $g^n \notin [\Lambda] : H_\Lambda^\infty$ for $n \in \mathbb{N}$. Now by Corollary 2.2.10, there is a prime differential ideal $\mathcal{I}$ containing Λ but not g. Let F be the fraction field of $k'\{y\}/\mathcal{I}$. (Notice that the quotient rule allows us to extend Δ to derivation operators on F.) Let $a \in F$ be the image of y mod $\mathcal{I}$. Since $\Lambda \subset \mathcal{I}$, $f(a) = 0$ for all $f \in \Lambda$ and $g(a) \neq 0$.

Iterating this two-step process, we can build $K \supseteq k$ so that K is a model of m-DCF. ⊣

REMARK 3.1.3. From now on, when we say that a system of differential equations has a solution we mean that the system has a differential solution. We will continue to use the term algebraic solution to mean just that.

LEMMA 3.1.4. *Let K be a model of m-DCF. Let $\Lambda = \{f_i\}_{i=1}^s \subset K\{y\}$ be a coherent set of Δ-polynomials in one differential indeterminate and suppose $0 \neq g \in K\{y\}$ is reduced with respect to Λ. Then, if K has an extension in which there is a solution to the system*

$$\left(\bigwedge_{f\in\Lambda} f = 0\right) \wedge (H_\Lambda g \neq 0) \tag{3.1.4}$$

then there is $\alpha \in K$ which is a solution to (3.1.4).

PROOF. Suppose that there is a solution to (3.1.4) in some extension L of K, then, there is an algebraic solution to (3.1.4) in L. But, $K \models m$-DCF. Hence, K is algebraically closed and therefore must contain an algebraic solution to (3.1.4). So by the axioms, K must contain a differential solution to (3.1.4). ⊣

LEMMA 3.1.5. *Let K and K' be ω-saturated models of m-DCF. Let $\bar{a} \in K$, $\bar{b} \in K'$, $k = \mathbb{Q}\langle\bar{a}\rangle$, and $k' = \mathbb{Q}\langle\bar{b}\rangle$. Suppose $\sigma: k \to k'$ is an isomorphism such that $\sigma(\bar{a}) = \bar{b}$. For all $\alpha \in K$ there is an extension of σ to an isomorphism σ^* from $k\langle\alpha\rangle$ into K'.*

PROOF. Let $\alpha \in K$ and suppose that α is differentially algebraic over k. Let $\Lambda = \{f_i\}_{i=1}^s \subset k\{y\}$ be a characteristic set of $\mathcal{I}(\alpha/k)$. By Theorem 2.2.18, since Λ is a characteristic set of a prime differential ideal, Λ is coherent. Let Λ' be the image of Λ under σ and $\mathscr{I}'$ the image of $\mathscr{I}(\alpha/k)$ under σ. Clearly, Λ' is in $\mathscr{I}'$ and, since σ is an isomorphism, $\mathscr{I}'$ is prime; furthermore, since coherency is preserved under isomorphism, Λ' is coherent. (One can see this by studying the conditions of coherency.) Since there can be no coherent set of lower rank than Λ' in $\mathscr{I}'$, Λ' must be a characteristic set of $\mathscr{I}'$. By Theorem 2.2.8 any $g \in k'\{y\} \setminus \mathscr{I}'$ is equivalent modulo $\mathscr{I}'$ to some $\hat{g} \in k'\{y\} \setminus \mathscr{I}'$ reduced with respect to Λ'. So it suffices to show that there is some element in K' which vanishes on Λ' but does not vanish on such a $\hat{g}$. So, for any $\hat{g} \in k'\{y\}$, $\hat{g} \notin \mathscr{I}'$, $\hat{g}$ reduced with respect to Λ', there is an extension of k' containing a solution to the system $\left(\bigwedge_{f' \in \Lambda'} f' = 0\right) \wedge (H_\Lambda \hat{g} \neq 0)$. By Lemma 3.1.4, there must be a solution in K'.

The polynomial $\hat{g}$ was chosen arbitrarily. Hence by compactness and ω-saturation, there is some $\beta \in K'$ which has defining ideal exactly $\mathscr{I}'$. We can now extend σ by sending $\alpha \mapsto \beta$. Since the defining ideal, $\mathscr{I}'$, of β is the image of $\mathscr{I}(\alpha/k)$ under σ, $k\langle\alpha\rangle \cong k'\langle\beta\rangle$.

Suppose α is differentially transcendental over K. Since K' is ω-saturated, we can find $\beta \in K'$ such that β is differentially transcendental over k'. We can now extend σ by sending $\alpha \mapsto \beta$. $\dashv$

We will now move towards proving that the theory m-DCF eliminates quantifiers. Once we know that m-DCF admits quantifier elimination, we will know that m-DCF is model complete. We follow Marker [8, pages 46–53].

By the next lemma, to show that a theory has elimination of quantifiers, we need only prove quantifier elimination for formulas of a very simple form. (See [8] for more details.)

LEMMA 3.1.6. *Let L be a language with at least one constant symbol and let T be an L-theory. T has elimination of quantifiers if and only if for any quantifier-free formula $\phi(v, \bar{w})$ and any $M, N \models T$, if $A \subset M$, $A \subset N$ and $\bar{a} \in A$ then $M \models \exists v \phi(v, \bar{a})$ if and only if $N \models \exists v \phi(v, \bar{a})$.*

THEOREM 3.1.7. m-DCF *has elimination of quantifiers.*

PROOF. Let $K, K' \models m$-DCF. Let $k \subset K$, $k \subset K'$ and $\bar{a} \in k$ (without loss of generality, $k = \mathbb{Q}\langle\bar{a}\rangle$). Let $\phi(v, \bar{w})$ be quantifier-free and suppose $K \models \phi(\alpha, \bar{a})$ for some $\alpha \in K$. By Lemma 3.1.6, we must show that $K' \models \exists v \phi(v, \bar{a})$.

Without loss of generality we may assume that K and K' are ω-saturated (if not, we replace them by ω-saturated elementary extensions). By Lemma 3.1.5, we can find $\beta \in K'$ such that $k\langle\alpha\rangle \cong k\langle\beta\rangle$. Therefore $K' \models \phi(\beta, \bar{a})$ and so $K' \models \exists v \phi(v, \bar{a})$. $\dashv$

COROLLARY 3.1.8. m-DCF *is model complete, and hence is the model completion of* m-DF *by definition.*

PROOF. Every quantifier-eliminable theory is model complete. $\dashv$

COROLLARY 3.1.9. m-DCF *is a complete theory.*

PROOF. Let K and K' be models of m-DCF. Then $\mathbb{Q}$, with the trivial derivation, is a substructure of both K and K'. Every sentence ϕ is provably equivalent in

m-DCF to a quantifier-free sentence ψ. Hence

$$\begin{aligned} K \models \phi &\iff K \models \psi \\ &\iff \mathbb{Q} \models \psi \\ &\iff K' \models \psi \\ &\iff K' \models \phi. \end{aligned}$$

So $K \equiv K'$ and m-DCF is a complete theory. ⊣

We get the following result of Seidenberg [13] from quantifier elimination.

THEOREM 3.1.10 (Differential Nullstellensatz). *Let $k \models m$-DF and let Σ be a finite system of Δ-equations and Δ-inequations in finitely many differential indeterminates over k. Suppose that Σ has a solution in some $l \supseteq k$ such that $l \models m$-DF. Then Σ has a solution in every differentially closed $K \supseteq k$.*

PROOF. By quantifier elimination the assertion that there is a solution to Σ is equivalent in m-DCF to a quantifier free formula with parameters from k. Thus if there is any differentially closed $L \supseteq k$ containing a solution to Σ, then every differentially closed $K \supseteq k$ contains a solution to Σ. But if there is any differential field $l \supseteq k$ containing a solution to Σ, then by Lemma 3.1.2 there is a differentially closed $L \supseteq l \supseteq k$. Thus Σ has a solution in every differentially closed $K \supseteq k$. ⊣

3.2. Stability.

THEOREM 3.2.1. *m-DCF is an ω-stable theory.*

PROOF. We must show that if K is a model of m-DCF, then $|S_1(K)| = |K|$. By quantifier elimination, every 1-type over K corresponds to a unique prime differential ideal in $K\{y\}$. But by Theorem 2.2.18, every prime Δ-ideal $\mathscr{P}$ in $K\{y\}$ is uniquely determined by a (finite) characteristic set, $\Lambda_{\mathscr{P}} = \{f_1, \dots, f_s\}$. Hence,

$$|S_1(K)| = |K\{y\}| = |K|. \qquad ⊣$$

It follows immediately from the work of Shelah (see [1, page 503]) that m-DCF has a prime model.

COROLLARY 3.2.2. *m-DCF has a unique prime model.*

We can say explicitly what the isolated types look like.

Let R be a Δ-integral domain. A homomorphism of R into a differential field k is called a *differential specialization of R into k*. If R and k have a common Δ-subring R_0 and the homomorphism leaves invariant each element of R_0, the differential specialization is said to be *over* R_0.

Let $\xi = (\xi_1, \dots, \xi_n)$ and $\xi' = (\xi'_1, \dots, \xi'_n)$ be tuples of elements of R and k, respectively. Suppose there exists a differential specialization $\varphi \colon R \to k$ over R_0 mapping ξ onto ξ'; i.e., $\varphi(\xi_j) = \xi'_j$ for all $j \in J$. We say that ξ' is a *differential specialization of ξ over R_0*. ξ' is a differential specialization of ξ over R_0 if and only if $\mathscr{I}(\xi/R_0) \subseteq \mathscr{I}(\xi'/R_0)$.

If ξ' is a differential specialization of ξ over R_0 and ξ is a differential specialization of ξ' over R_0 we say that ξ' is a *generic differential specialization of ξ over R_0*.

DEFINITION 3.2.3. Let k be a Δ-field. An element $\overline{\eta}$ in a Δ-extension k' of k is *constrained over k*, or *k-constrained*, if there exists a differential polynomial $g \in k\{\overline{y}\}$

such that $g \notin \mathscr{I}(\bar{\eta}/k)$, but $g \in \mathscr{I}(\bar{\eta}'/k)$ for every nongeneric specialization $\bar{\eta}'$ of $\bar{\eta}$ over k.

THEOREM 3.2.4. *Let k be a model of m-DF and let $\bar{\alpha}$ be in some extension k' of k. Then* $\operatorname{tp}(\bar{\alpha}/k)$ *is isolated if and only if $\bar{\alpha}$ is k-constrained.*

PROOF. First, suppose that $\operatorname{tp}(\bar{\alpha}/k)$ is isolated. Then there is some formula $\psi(\bar{y})$ such that for all $\phi(\bar{y})$ in $\operatorname{tp}(\bar{\alpha}/k)$, $\models \psi(\bar{y}) \to \phi(\bar{y})$. Now, by quantifier elimination, $\psi(\bar{y})$ must be equivalent to a formula of the form

$$\left(\bigwedge_{i=1}^{s} f_i = 0\right) \wedge (\neg g = 0)$$

where $f_i \in k\{\bar{y}\}$ for all i and $g \in k\{\bar{y}\}$. Without loss of generality, we may assume that $\Lambda = \{f_1, \dots, f_s\}$ is a characteristic set for $\mathscr{I}(\bar{\alpha}/k)$. Clearly, $g \notin \mathscr{I}(\bar{\alpha}/k)$. Let $\bar{\beta}$ be a nongeneric specialization of $\bar{\alpha}$. Then we must have

$$\models \left(\bigwedge_{i=1}^{s} f_i(\bar{\beta}) = 0\right).$$

But $\mathscr{I}(\bar{\beta}/k) \supsetneq \mathscr{I}(\bar{\alpha}/k)$ since $\bar{\beta}$ is a nongeneric specialization of $\bar{\alpha}$. Therefore, $\models g(\bar{\beta}) = 0$ and $g \in \mathscr{I}(\bar{\beta}/k)$. Hence, $\bar{\alpha}$ is k-constrained.

Conversely, suppose $\bar{\alpha}$ is k-constrained. Then there is some $g \notin \mathscr{I}(\bar{\alpha}/k)$ such that $g \in \mathscr{I}(\bar{\beta}/k)$ for any nongeneric specialization $\bar{\beta}$ of $\bar{\alpha}$. Let $\Lambda = \{f_1, \dots, f_s\}$ be a characteristic set for $\mathscr{I}(\bar{\alpha}/k)$. Then let $\psi(\bar{y})$ be the formula

$$\left(\bigwedge_{i=1}^{s} f_i = 0\right) \wedge (\neg g = 0).$$

By quantifier elimination, ψ isolates $\operatorname{tp}(\bar{\alpha}/k)$. ⊣

REMARK 3.2.5. Phyllis Cassidy has pointed out that separation of variables is not generally possible in this context. Thus, unlike the case with a single derivation, there are consistent systems of equations which have no solution of finite transcendence degree. For an example, see [4].

3.3. Elimination of imaginaries. To show that m-DCF eliminates imaginaries, we use a proposition from Messmer's account of the model theory of separably closed fields [10, pages 146–150].

PROPOSITION 3.3.1 (Messmer). *Let T be the theory of a stable field. Suppose that for every $n \geq 1$ there is a (possibly infinite) set of indeterminates Y_i, $i \in J$, such that for each model K of T there is a one-to-one correspondence between complete n-types over K and certain ideals in the polynomial ring $K[Y_i : i \in J]$, such that every automorphism σ of K fixes the type (setwise) if and only if it fixes the corresponding ideal (setwise). Then T eliminates imaginaries.*

COROLLARY 3.3.2. *The theory m-DCF admits elimination of imaginaries.*

PROOF. Let K be model of m-DCF. For $p \in S_n(K)$, let

$$\mathscr{I}_p = \{f(\bar{y}) \in K\{y_1, \dots y_n\} : \text{“}f(\bar{y}) = 0\text{”} \in p\}.$$

This map is a bijection between n-types and differential prime ideals in $K\{\bar{y}\}$. By Proposition 3.3.1, m-DCF has elimination of imaginaries. ⊣

§4. Forking.

4.1. Special numerical polynomials. In this section, we digress from our discussion of differential algebra.

Let $(a_1, \ldots, a_m), (b_1, \ldots, b_m) \in \mathbb{N}^m$. In the *product order* on $\mathbb{N}^m$,

$$(a_1, \ldots, a_m) \leq (b_1, \ldots, b_m) \iff a_i \leq b_i$$

for $1 \leq i \leq m$. (Notice that this is different from the *lexicographic order* on $\mathbb{N}^m$. The lexicographic order on $\mathbb{N}^m$ is a total order, whereas the product order is not.)

By a *numerical polynomial* we mean a polynomial $f \in \mathbb{R}[X]$ such that $f(t) \in \mathbb{Z}$ for all sufficiently large $t \in \mathbb{N}$. We define the degree of such a polynomial in the usual way except that the polynomial "0" will have degree -1.

If $f \in \mathbb{R}[X]$ is a numerical polynomial and $\deg f \leq m$, then there exist unique $a_0, \ldots, a_m \in \mathbb{Z}$ such that $f = \sum_{0 \leq i \leq m} a_i \binom{X+i}{i}$, where, if $i \neq 0$, $\binom{X}{i}$ denotes the "binomial coefficient" polynomial

$$\frac{X(X-1)\cdots(X-i+1)}{i!} \in \mathbb{Q}[X]$$

of degree i and, if $i = 0$, $\binom{X}{i} = 1$. Notice that the polynomials $\binom{X}{i}$ are numerical, and therefore so is every $\sum_{0 \leq i \leq m} a_i \binom{X+i}{i}$ with $a_0, a_1, \ldots, a_m \in \mathbb{Z}$.

We can put a total order on the numerical polynomials by defining $f \leq g$ to mean that $f(t) \leq g(t)$ for all sufficiently large $t \in \mathbb{N}$. If $f = \sum_{0 \leq i \leq m} a_i \binom{X+i}{i}$ and $g = \sum_{0 \leq i \leq m} b_i \binom{X+i}{i}$, then $f \leq g$ if and only if $(a_m, \ldots, a_0) \leq (b_m, \ldots, b_0)$ relative to the lexicographic order on $(\mathbb{R})^{m+1}$.

LEMMA 4.1.1 ([6, page 51]). *Let E be a subset of $\mathbb{N}^m$, considered as a partially ordered set relative to the product order. Let V denote the set of all points of $\mathbb{N}^m$ that are not greater than or equal to any point of E. Let W denote the set of all points of V that are less than or equal to only finitely many points of V.*

1. *W is a finite set.*
2. *There exists a numerical polynomial ω_E such that, for every sufficiently large $t \in \mathbb{N}$, the number of points $(v_1, \ldots, v_m) \in V$ with $\sum v_i \leq t$ is equal to $\omega_E(t)$.*

The lemma continues to give directions for calculating $\omega_E(t)$. See [6, pages 49–53] for details.

4.2. The differential dimension polynomial. Fix $\Delta = \{\delta_1, \ldots, \delta_m\}$, a set of derivation operators. For $\theta \in \Theta$, if $\theta = \delta_1^{e_1} \ldots \delta_m^{e_m}$, let $\operatorname{ord}\theta = \sum_{i=1}^{m} e_i$. Then let $\Theta(t) = \{\theta \in \Theta : \operatorname{ord}(\theta) \leq t\}$.

THEOREM 4.2.1 ([6, pages 115]). *Let k be a Δ-field. Let $\overline{\eta} = (\eta_1, \ldots, \eta_n)$ be a finite family of elements of an extension k' of k. There exists a numerical polynomial $\omega_{\overline{\eta}/k}$ with the following properties.*

1. *For every sufficiently large $t \in \mathbb{N}$, $\omega_{\overline{\eta}/k}(t)$ equals the transcendence degree of the field $k((\theta\eta_j)_{\theta \in \Theta(t), 1 \leq j \leq n})$ over k.*
2. $\deg \omega_{\overline{\eta}/k} \leq m (= \operatorname{Card}\Delta)$.
3. *If we write $\omega_{\overline{\eta}/k} = \sum_{0 \leq i \leq m} a_i \binom{X+i}{i}$, then a_m equals the differential transcendence degree of $k\langle\overline{\eta}\rangle$ over k.*
4. *If $\mathcal{P}$ is the defining differential ideal of $\overline{\eta}$ in $k\{y_1, \ldots, y_n\}$, if $\Lambda_{\mathcal{P}}$ is a characteristic set of $\mathcal{P}$ relative to the canonical ranking of $\{y_1, \ldots, y_n\}$, and if for each y_j*

we let E_j denote the set of all points $(e_1, \dots, e_m) \in \mathbb{N}^m$ for which $\delta_1^{e_1} \cdots \delta_m^{e_m} y_j$ is a leader of an element of $\Lambda_{\mathscr{P}}$, then $\omega_{\overline{\eta}/k} = \sum_{1 \le j \le n} \omega_{E_j}$.

DEFINITION 4.2.2. Let k and $\overline{\eta}$ be as in Theorem 4.2.1 above. We define the $\mathscr{K}-\text{type}(\overline{\eta}/k)$ to be the degree of $\omega_{\overline{\eta}/k}$ and the $\mathscr{K}-\deg(\overline{\eta}/k)$ to be a_τ, where $\tau = \mathscr{K}-\text{type}(\overline{\eta}/k)$. Let $\mathscr{P} = \mathscr{I}(\overline{\eta}/k)$; then $\omega_{\mathscr{P}} = \omega_{\overline{\eta}/k}$ and we say that $\mathscr{K}-\text{type}(\mathscr{P}) = \mathscr{K}-\text{type}(\overline{\eta}/k)$ and $\mathscr{K}-\deg(\mathscr{P}) = \mathscr{K}-\deg(\overline{\eta}/k)$. If $\overline{\eta}$ and $\overline{\zeta}$ generate the same Δ-field extension over k, then $\mathscr{K}-\text{type}(\overline{\eta}/k) = \mathscr{K}-\text{type}(\overline{\zeta}/k)$ and $\mathscr{K}-\deg(\overline{\eta}/k) = \mathscr{K}-\deg(\overline{\zeta}/k)$. (Note that $\overline{\eta}$ and $\overline{\zeta}$ need not be tuples of the same length.) Thus we can also talk about the $\mathscr{K}-\text{type}$ and $\mathscr{K}$-degree of a finitely generated extension G of k; i.e., the $\mathscr{K}-\text{type}(G)$ is the $\mathscr{K}-\text{type}$ of a set of k-generators for G over k and the $\mathscr{K}$-degree is defined analogously.

REMARK 4.2.3. Kolchin calls the $\mathscr{K}-\text{type}(\overline{\eta}/k)$ the *differential type* of $k\langle\overline{\eta}\rangle$ over k and the $\mathscr{K}-\deg(\overline{\eta}/k)$ the *typical differential transcendence degree* of $k\langle\overline{\eta}\rangle$ over k.

NOTATION 4.2.4. Let K be a Δ-field. We may consider different differential structures on K as well as the pure field structure of K. In order to distinguish these different points of view, we adopt the following decorations: if no confusion results, the unadorned K will still designate the Δ-structure with constant field $\mathscr{C}_K$; if Δ^* is a linearly independent $\mathscr{C}_K$-linear combination of the elements of Δ or a proper subset of Δ, the corresponding Δ^*-field structure is indicated by K_{Δ^*}, and any other Δ^*-structure will be decorated with a Δ^*; the pure field structure of any differential field K is designated by K_{field}.

THEOREM 4.2.5 ([6, page 119]). *Let k be an infinite Δ-field, and let G be a finitely generated Δ-field extension of k of $\mathscr{K}-\text{type}\ \tau$.*

1. *If $\tau = -1$, then G is an algebraic extension of k of finite degree.*
2. *If $\tau \neq -1$, and d is the $\mathscr{K}$-degree of G over k, then there exists a set Δ^*, consisting of τ linearly independent $\mathscr{C}_k$-linear combinations of the elements of Δ, such that G_{Δ^*} is a finitely generated Δ^*-field extension of k of Δ^*-transcendence degree d.*

The following example will be useful for various model-theoretic rank computations

EXAMPLE 4.2.6. Let k be a Δ-field where $\Delta = \{\delta_1, \dots, \delta_m\}$. Suppose that $G = k\langle\eta\rangle$ where η is a generic solution to the prime Δ-ideal $\mathscr{Q}$ with characteristic set $\Lambda = \{\delta_1^{e_1} \dots \delta_m^{e_m} y - a\}$ where $a \in k$. (Notice that E of Lemma 4.1.1 is $\{(e_1, \dots, e_m)\}$; then W of Lemma 4.1.1 is empty and so $(e_1, \dots, e_m)$ is the least element of $E \cup W$ by default.) Let $e = \sum_{j=1}^m e_j$. Then, by Lemma 4.1.1,

$$
\begin{aligned}
(4.2.1)\quad \omega_{\eta/k} &= \binom{X+m}{m} - \binom{X-e+m}{m} \\
&= \frac{(X+m)\cdots(X+1)}{m!} - \frac{(X-e+m)\cdots(X-e+1)}{m!} \\
&= \frac{X^m + \left(\sum_{\ell=1}^m \ell\right) X^{m-1} + \cdots}{m!} - \frac{X^m + \left(\sum_{\ell=1}^m (-e+\ell)\right) X^{m-1} + \cdots}{m!} \\
&= \frac{me}{m!} X^{m-1} + \cdots .
\end{aligned}
$$

When we write $\omega_{\eta/k}$ as $\sum_{0\leq i\leq\tau} a_i\binom{X+i}{i}$, we see that $\tau = m-1$ and $a_\tau = e$ (recall that $\binom{X+\tau}{\tau} = \frac{1}{\tau!}X^\tau + \cdots$). Then the $\mathscr{K}$ − type of G over k is $m-1$ and the $\mathscr{K}$-degree of G over k is e. In fact, this is true for such a Δ-extension by any generic solution to a linear differential equation whose leader has order e.

DEFINITION 4.2.7. Let R be a Δ-ring and let Δ-Spec(R) be the set of prime Δ-ideals of R. Let $\mathscr{P} \in \Delta$-Spec(R). The Δ-dimension of $\mathscr{P}$, denoted by Δ-Dim$(\mathscr{P})$ is defined inductively by

- $\Delta\text{-Dim}(\mathscr{P}) = 0$ if $\mathscr{P}$ is a maximal element in Δ-Spec(R);
- $\Delta\text{-Dim}(\mathscr{P}) = \sup\{\Delta\text{-Dim}(\mathscr{Q}) + 1 : \mathscr{Q} \in \Delta\text{-Spec}(R), \mathscr{Q} \supset \mathscr{P}\}$.

If R is an integral domain then the Δ-dimension of R, denoted by Δ-Dim(R), is defined to be the Δ-dimension of the zero ideal.

REMARK 4.2.8. For $\mathscr{P} \in \Delta$-Spec(R), the Δ-dimension of $\mathscr{P}$ gives a bound for the ordinals $\alpha + 1$ such that there is a strictly decreasing chain of prime Δ-ideals in Δ-Spec(R),

$$\mathscr{Q}_0 \supset \cdots \supset \mathscr{Q}_\alpha \supset \mathscr{P}.$$

Let k be a Δ-field and let $\mathscr{P}$ be a prime Δ-ideal of $k\{\bar{y}\}$. The following proposition uses $\omega_{\mathscr{P}}$ to give an upper bound for Δ-dimension of $\mathscr{P}$.

PROPOSITION 4.2.9. *Let k be a Δ-field and let $\mathscr{P}$ be a prime Δ-ideal of $k\{\bar{y}\}$. Let $\tau = \mathscr{K}$ − type$(\mathscr{P})$ and $d = \mathscr{K}$ − deg$(\mathscr{P})$. Then*

$$\Delta\text{-Dim}(\mathscr{P}) < \omega^\tau(d+1). \tag{4.2.2}$$

A key ingredient in the proof of Proposition 4.2.9 is the following lemma due to Kolchin. Lemma 4.2.10 tells us that for increasing prime differential ideals, the corresponding differential dimension polynomials are decreasing.

LEMMA 4.2.10 ([6, page 130]). *Let k be a differential field. Let $\mathscr{P}$ and $\mathscr{Q}$ be prime differential ideals of a finitely generated differential polynomial algebra $k\{\bar{y}\}$ over k, with $\mathscr{P} \subset \mathscr{Q}$ and $\mathscr{P} \neq \mathscr{Q}$. Then $\omega_{\mathscr{P}} > \omega_{\mathscr{Q}}$ in the sense described in Section* 4.1 *above.*

The next proposition is a direct consequence of Lemma 4.2.10.

COROLLARY 4.2.11. *Let k be a differential field. Let $\mathscr{P}$ be a prime differential ideal in $k\{\bar{y}\}$. Let $\mathscr{Q}$ be a prime differential ideal containing $\mathscr{P}$. Suppose further that $\Lambda_{\mathscr{Q}}$ has the same leaders as $\Lambda_{\mathscr{P}}$. Then $\mathscr{P} = \mathscr{Q}$.*

PROOF. By Theorem 4.2.1, the differential dimension polynomial for an arbitrary prime differential ideal $\mathscr{I}$ depends only on the leaders of the differential polynomials in the characteristic set. Lemma 4.2.10 implies that $\mathscr{P} = \mathscr{Q}$. ⊣

REMARK 4.2.12. A proof of Proposition 4.2.9 for the case where $m = 1$ and $\tau < 1$ appears in Marker [8]. Pong [11] extends that result to the case where $m = \tau = 1$ and and $d \geq 1$. Suppose that $\mathscr{P}$ is a prime δ-ideal in $k\{\bar{y}\}$, where k is an ordinary differential field with derivation δ. Notice that here, $\deg(\omega_{\mathscr{P}}) \leq 1$ since $m = 1$. Using transcendence degree arguments, Pong shows that if $\omega_{\mathscr{P}} = aX + b$ then both a and b must be nonnegative. (Moreover, for this case, when we write $\omega_{\mathscr{P}} = a_1\binom{X+1}{1} + a_0$, both a_1 and a_0 will be nonnegative.) If $a \neq 0$ then $\mathscr{K}$ − type$(\mathscr{P}) = 1$ and $\mathscr{K}$ − deg$(\mathscr{P}) = a$. Pong uses Lemma 4.2.10 above and the result about the form of the differential dimension polynomial to show that Δ-Dim$(\mathscr{P}) < \omega(a+1)$. See

Theorem 5.2.1 below. (In fact, Pong shows the stronger condition that Δ-Dim($\mathscr{P}$) is no greater that $a\omega + b$. See [11].)

For the general case, where $m \neq 1$, we do not have such a nice situation; i.e., the coefficients of the differential dimension polynomials are not necessarily all nonnegative. To clarify, when we talk about the "coefficients" of a differential dimension polynomial, $\omega_{\mathscr{P}}$, we are referring to the a_i when we write $\omega_{\mathscr{P}} = \sum_{0 \leq i \leq m} a_i \binom{X+i}{i}$. So, to apply the Pong argument to the general case, it would be enough to know, for example, that given coefficients $a_m, \dots, a_{r+1}$ of some $\omega_{\mathscr{P}}$, that we can bound the coefficient a_r from below, say by b; i.e., if $\mathscr{Q} \supset \mathscr{P}$ such that $\omega_{\mathscr{Q}} = \sum_{i=1}^{m} \hat{a}_i \binom{X+i}{i}$, and $\hat{a}_i = a_i$ for $i > r$, then $\hat{a}_r \geq b$.

We first prove a small lemma which states that a_m is nonnegative, as is the next nonzero coefficient (going down from a_m), if such a coefficient exists.

LEMMA 4.2.13. *Suppose $\overline{\eta} = (\eta_1, \dots, \eta_n)$. Let $\omega_{\overline{\eta}/k} = \sum_{0 \leq i \leq m} a_i \binom{X+i}{i}$. Then a_m is nonnegative. Furthermore, suppose there is some $i < m$ such that $a_i \neq 0$; let τ be the greatest such i, i.e., $a_i = 0$ for $\tau < i < m$. Then $a_\tau > 0$.*

PROOF. Let $\overline{y} = (y_1, \dots, y_n)$ and let $\mathscr{P} \subset k\{\overline{y}\}$ be the defining differential ideal of $\overline{\eta}$ over k. By Theorem 4.2.1 (3), a_m is the differential transcendence degree of $k\langle\overline{\eta}\rangle$ over k. Clearly, $a_m \geq 0$.

By Theorem 4.2.1 (4), we may write $\omega_{\mathscr{P}} = \sum_{1 \leq j \leq n} \omega_{E_j}$, where E_j denotes the set of all points $(e_1, \dots, e_m) \in \mathbb{N}^m$ for which $\delta_1^{e_1} \cdots \delta_m^{e_m} y_j$ is a leader of some element of the characteristic set of $\mathscr{P}$. Lemma 4.1.1 implies that $\deg \omega_{E_j} = m$ if and only if E_j is empty, in which case $\omega_{E_j} = \binom{X+m}{m}$. So the ω_{E_j} for which E_j is empty, do not contribute to a_i for $i < m$. Since $a_i = 0$ for all $\tau < i < m$, $\deg(\omega_{E_j}) \leq \tau$ for all nonempty E_j. Furthermore, if $\deg(\omega_{E_j}) = \tau$, then the τ^{th} coefficient of ω_{E_j} must be greater than 0 since by Lemma 4.1.1 (2), for sufficiently large $t \in \mathbb{N}$, $\omega_{E_j}(t)$ is the cardinality of a certain set of points in $\mathbb{N}^m$. Hence, a_τ is the sum of positive integers, and thus must be positive. ⊣

REMARK 4.2.14. Lemma 4.2.13 is enough to show Pong's observation that if $\omega_{\mathscr{P}} = aX + b$, then a and b are nonnegative.

At this point, we must develop some understanding of how the differential dimension polynomials work and what information they provide. Let us first set the stage. Let k be a Δ-field, and let $\overline{\eta} = (\eta_1, \dots, \eta_n)$ be in some extension of k. Consider $k\langle\overline{\eta}\rangle$. Let $\overline{y} = \{y_1, \dots, y_n\}$ and let $\mathscr{P} \subset k\{\overline{y}\}$ be the defining differential ideal of $\overline{\eta}$ over k. Fix the canonical ranking of $\Theta\overline{y}$. Choose a characteristic set $\Lambda_{\mathscr{P}}$ of $\mathscr{P}$.

For each j, let $\mathbb{N}_j^m$ be a copy of $\mathbb{N}^m$ equipped with the product order. We find it convenient to think of each $\mathbb{N}_j^m$ as a lattice. First we need a few definitions about lattices. We use [2] as a reference.

Recall that $E_j \subset \mathbb{N}_j^m$ is the set of points $(e_1, \dots, e_m)$ such that $\delta_1^{e_1} \dots \delta_m^{e_m} y_j$ appears as a leader of some element of $\Lambda_{\mathscr{P}}$. Let the *order of* $(e_1, \dots, e_m)$ be $\operatorname{ord}(e_1, \dots, e_m) = \sum_{i=1}^{m} e_i$.

Suppose first that for some j, E_j is empty. In other words, η_j is Δ-independent over k; i.e., $\{\theta\eta_j : \theta \in \Theta\}$ is algebraically independent over k. In this case, do not make any modifications to $\mathbb{N}_j^m$.

Now suppose $(e_1, \ldots, e_m) \in E_j$. Then $\theta\eta_j = \delta_1^{e_1} \ldots \delta_m^{e_m}\eta_j$ is algebraically dependent (in the usual sense) on $\{\theta'\eta_i : \theta' y_i < \theta y_j\}$ over k. Remove the principal filter generated by $(e_1, \ldots, e_m)$ in $\mathbb{N}_j^m$. This filter corresponds to the set of derivatives of θy_j. Do the same for any other $\theta y_{j'} = \delta_1^{e'_1} \ldots \delta_m^{e'_m} y_j$ that appears as a leader of some element of $\Lambda_{\mathscr{P}}$.

For sake of comparison, consider the simpler case where $m = 1$ and k is an ordinary differential field. Then for each j, we have a copy of $\mathbb{N}$. If E_j is empty, then the number of points in $\mathbb{N}_j$ of order less than or equal to t is $t + 1$. If E_j is nonempty, then only finitely many points of $\mathbb{N}_j$ remain and $\omega_{E_j} = a_0 \in \mathbb{Z}$.

Now, return to the case of arbitrary m.

If $\omega_{E_j} = a_0 \in \mathbb{Z}$, then $a_0 \geq 0$ and, as in the case for $m = 1$, we have removed all but finitely many points in $\mathbb{N}_j^m$.

If $\tau_j = \deg(\omega_{E_j}) > 0$, then the portion of $\mathbb{N}_j^m$ that has been removed is the (not necessarily disjoint) union of finitely many principal filters. We are interested in what is left. In particular, given $t \in \mathbb{N}$ sufficiently large, how many points are left with order less than or equal to t? This is exactly what the differential dimension polynomial is counting.

Let $\mathbb{N}_j^m(t)$ denote the points in $\mathbb{N}_j^m$ of order less than or equal to t. Denote the remainder of $\mathbb{N}_j^m$ by R_j. Let a_{τ_j} be the $(\tau_j)^{\text{th}}$ coefficient of ω_{E_j}. In R_j, there are a_{τ_j} maximal convex sublattices of $\mathbb{N}_j^m$, each isomorphic as a lattice to $\mathbb{N}^{\tau_j}$. Denote these a_{t_j} sublattices by Υ_j. Note that the elements of Υ_j may not be pairwise disjoint. The term $\binom{X+\tau_j}{\tau_j}$ in ω_{E_j} has coefficient a_{t_j}; i.e., there is a summand of $\binom{X+\tau_j}{\tau_j}$ for each of the elements of Υ_j. Lemma 4.1.1 (iii) implies that $\binom{t+\tau_j}{\tau_j}$ is the number of points in $\mathbb{N}^{\tau_j}(t)$.

Suppose that $i \leq a_{\tau_j}$ of the elements of Υ_j have nonempty intersection. The intersection is itself a convex sublattice of $\mathbb{N}_j^m$ isomorphic to $\mathbb{N}^{\tau'}$ for some $\tau' < \tau_j$. Then if we simply consider $i\binom{t+\tau_j}{\tau_j}$, the number of points in i copies of $\mathbb{N}^{\tau_j}(t)$, we have counted the points in the intersection i times. So we must subtract $(i-1)\binom{t+\tau'}{\tau'}$, where $\binom{t+\tau'}{\tau'}$ is the number of points in $\mathbb{N}^{\tau'}(t)$. This adjustment affects the $(\tau')^{\text{th}}$ coefficient of ω_{E_j}.

Now, starting over, we assumed that $\omega_{E_j} \neq 0$. Therefore, at least one of the elements of Υ_j must contain the point $(0, \ldots, 0)$, call it L. We say that L is a *"full" sublattice of* $\mathbb{N}_j^m$. Then for $t \in \mathbb{N}$, we know that $\binom{t+\tau_j}{\tau_j}$ is the number of points in $L(t)$.

Now, suppose there is $L' \in \Upsilon_j$ which does not contain the point $(0, \ldots, 0)$. Let e be minimum such that there is a point of order e in L'. Since L' is a maximal sublattice in R_j isomorphic to $\mathbb{N}^{\tau_j}$, $e \leq a_{t_j}$. Notice that there will be exactly one point x in L' of order e. Embed L' in $\mathbb{N}^{\tau_j}$ such that x gets mapped to the point $(0, \ldots, 0, e)$. The image of L' is the set of points of $\mathbb{N}^{\tau_j}$ for which the last coordinate is greater than or equal to e. Let I be the complement of the image of L' in $\mathbb{N}^{\tau}$. Let $t \in \mathbb{N}$. How many points are in $I(t)$? Let $E^{e,\tau_j} = \{(0, \ldots, 0, e)\} \subset \mathbb{N}^{\tau_j}$. Then by Lemma 4.1.1, $\omega_{E^{e,\tau_j}} = \binom{X+\tau_j}{\tau_j} - \binom{X-e+\tau_j}{\tau_j}$. The computations done in Example 4.2.6 with $m = \tau_j$ show that the degree of $\omega_{E^{e,\tau_j}}$ is $\tau_j - 1$ and the $(\tau_j - 1)^{\text{th}}$ coefficient is e. Let $\omega_{E^{e,\tau_j}} = \sum_{i=0}^{\tau_j-1} a_i^{e,\tau_j}\binom{X+i}{i}$ where $a_{\tau_j-1}^{e,\tau_j} = e$. (Note that if $E \subset \mathbb{N}^{\tau_j}$ contains a single point of order e, then $\omega_E = \omega_{E^{e,\tau_j}}$.) $L'(t)$ then contains $\sum_{i=0}^{\tau_j-1} a_i^{e,\tau_j}\binom{t+i}{i}$ fewer points then $L(t)$. This adjustment affects the lower coefficients of ω_{E_j}.

The above analysis is just an example of the complexity of these counting polynomials. Now, it should be clear why, in the general case, that the coefficients are not so well behaved as in the ordinary case.

How can this picture of n copies of $\mathbb{N}^m$, with appropriate filters removed for $\mathscr{P}$, change when we consider a second prime ideal $\mathscr{Q} \supset \mathscr{P}$? In the picture for $\mathscr{Q}$, the same points must be absent since any dependence that appears in $\mathscr{P}$ must also appear in $\mathscr{Q}$. But, if $\mathscr{Q} \neq \mathscr{P}$, then we have shown in Corollary 4.2.11 that the leaders must change. So, in fact, more points will be removed. This could have the effect of dropping some of the coefficients of $\omega_{\mathscr{P}}$ and increasing others. But, as long as we can predict the extreme behavior of these changes, we can bound Δ-Dim$(\mathscr{P})$.

By the discussion in the preceding paragraphs, the question of finding lower bounds for the coefficients of $\omega_{\mathscr{P}}$ can be answered by studying bounds for the coefficients of ω_E for a single copy of $\mathbb{N}^m$ and some E a finite subset of $\mathbb{N}^m$. The proposition below assumes that the degree of $\omega_E < m$, for otherwise there is nothing to prove.

PROPOSITION 4.2.15. *Let E be a finite subset of $\mathbb{N}^m$. Suppose that the degree of ω_E is $\tau < m$. Let $\omega_E = \sum_{i=0}^{\tau} a_i\binom{X+i}{i}$. Let $r < \tau$. If we fix the coefficients $a_m, \dots, a_{r+1}$, then the coefficient a_r is bounded below by a function of $a_\tau, \dots, a_{r+1}$.*

PROOF. Let R be the remainder of $\mathbb{N}^m$ once we have removed the appropriate filters. The calculations below are based on the possibilities for the shape of R. The bounds that we find are not necessarily greatest lower bounds. However, this is enough for our purposes because we need only to know that for a chain of strictly decreasing prime Δ-ideals

$$\mathscr{Q}_0 \supset \mathscr{Q}_1 \supset \cdots \supset \mathscr{Q}_\alpha \supset \cdots \supset \mathscr{P}$$

the corresponding chain of differential dimension polynomials has the property that one coefficient cannot decrease arbitrarily while the coefficients in higher positions remain fixed.

First, we describe the bound for $a_{\tau-1}$. Let $c_\tau = a_\tau$. Let

$$b_{\tau-1} = \left[\sum_{i=2}^{a_\tau}(i-1)\binom{a_\tau}{i}\right] + \sum_{e=2}^{a_\tau-1}|a_{\tau-1}^{e,\tau}|, \tag{4.2.3}$$

where $a_{\tau-1}^{e,\tau}$ is the $(\tau-1)^{\text{th}}$ coefficient of $\omega_{E^{e,\tau}}$ described above.Let Υ be the set of a_τ maximal convex sublattices of $\mathbb{N}^m$, each isomorphic as a lattice to $\mathbb{N}^\tau$. The first summand reflects the number of possible intersections of i elements of Υ. The second summand reflects the possibility that exactly one of the elements of Υ contains the point $(0, \dots, 0)$. Notice that $b_{\tau-1}$ is an upper bound for the number of copies of $\mathbb{N}^{\tau-1}$ which have been overcounted when counting the points in the elements of Υ. Then $a_{\tau-1} \geq -b_{\tau-1}$.

We now describe the bound for $a_{\tau-2}$, keeping in mind that a_m, a_τ and $a_{\tau-1}$ are fixed. Let $c_{\tau-1} = c_\tau + a_{\tau-1} + b_{\tau-1}$. Then $c_{\tau-1}$ is an upper bound for the number of maximal copies of $\mathbb{N}^{\tau-1}$ which are not contained in any element of Υ. Let

$$(4.2.4)\quad b_{\tau-2} = \left[\sum_{i=2}^{c_\tau}(i-1)\binom{c_\tau}{i}\right] + \sum_{e=2}^{a_\tau}|a_{\tau-2}^{e,\tau}| + \left[\sum_{i=2}^{c_{\tau-1}}(i-1)\binom{c_{\tau-1}}{i}\right] + \sum_{e=2}^{c_{\tau-1}-1}|a_{\tau-2}^{e,\tau-1}|.$$

The first summand represents the number of possible intersections of i of the elements of Υ. Each of the points in such an intersection is potentially counted i times. The second summand takes into account the adjustment made for deficits from the full sublattices of elements of Υ. The third and fourth summands are analogs of the summands used in the calculation of the bound for $a_{\tau-1}$. As before, $b_{\tau-2}$ is an upper bound on the copies of $\mathbb{N}^{\tau-2}$ which have been overcounted in some way. Then $a_{\tau-2} \geq -b_{\tau-2}$.

Now, suppose we have fixed $a_\tau, \dots, a_{r+1}$ and have calculated the lower bounds $b_{\tau-1}, \dots, b_{r+1}$ and the corresponding $c_\tau, \dots, c_{r+2}$. Let $c_{r+1} = c_{r+2} + a_{r+1} + b_{r+1}$. Then let

$$(4.2.5)\qquad b_r = \left[\sum_{\ell=r+1}^{\tau}\left(\sum_{i=2}^{c_\ell}(i-1)\binom{c_\ell}{i}\right)\right] + \left[\sum_{\ell=r+1}^{\tau}\left(\sum_{e=2}^{c_\ell}|a_r^{e,\ell}|\right)\right].$$

Here, the first summand takes into account all of the possible adjustments to the r^{th} coefficient from intersections of sublattices isomorphic to $\mathbb{N}^{r'}$ where $r' > r$. The second summand takes into account all of the possible adjustments to the r^{th} coefficient resulting from non-full sublattices isomorphic to $\mathbb{N}^{r'}$ where $r' > r$. Then $a_r \geq -b_r$. ⊣

Now, we are ready to prove Proposition 4.2.9.

Proof of Proposition 4.2.9. Let k be a Δ-field. Suppose that $\mathscr{P}$ is a prime Δ-ideal of $k\{\bar{y}\}$ and suppose that $\tau = \mathscr{K} - \text{type}(\mathscr{P})$ and $d = \mathscr{K} - \deg(\mathscr{P})$. We need to show that the supremum of the set of $\alpha + 1$ such that there is strictly decreasing chain of prime Δ-ideals of $k\{\bar{y}\}$ of the form

$$k\{\bar{y}\} = \mathscr{Q}_0 \supset \mathscr{Q}_1 \supset \cdots \supset \mathscr{Q}_\alpha \supset \mathscr{P}$$

is less than $\omega^\tau(d+1)$.

By our discussion above, this reduces to showing that for any such chain, the corresponding (increasing) chain of differential dimension polynomials has order type less than $\omega^\tau(d+1)$. Recall that for any two numerical polynomials $f = \sum_{0\leq i\leq m} a_i\binom{X+i}{i}$ and $g = \sum_{0\leq i\leq m} b_i\binom{X+i}{i}$, $f \leq g$ if and only if $(a_m, \dots, a_0) \leq (b_m, \dots, b_0)$ relative to the lexicographic order on $(\mathbb{R})^{m+1}$. So in this case we can restrict our attention to the corresponding chains of $(m+1)$-tuples of integers. In fact, since we are dealing with $(m+1)$-tuples of integers which are the coefficients of differential dimension polynomials for prime Δ-ideals, Proposition 4.2.15 implies that there cannot be any infinite strictly decreasing sequences in any such chain.

We will argue by induction on τ and d. For the base case, suppose that $\omega_{\mathscr{P}} = d$. By Lemma 4.2.10 and Corollary 4.2.11 and the fact that $\omega_{\mathscr{P}}(t) \geq 0$ for all sufficiently large $t \in \mathbb{N}$ it should be clear that $\Delta\text{-Dim}(\mathscr{P}) < d+1$.

Now suppose that $\tau > 0$ and for all $\tau' < \tau$ and all $d' \in \mathbb{N}$, if $\mathscr{Q}$ is a prime Δ-ideal of $k\{\bar{y}\}$, and $\mathscr{K}-\text{type}(\mathscr{Q}) = \tau$ and $\mathscr{K}-\deg(\mathscr{Q}) = d'$, then $\Delta\text{-Dim}(\mathscr{Q}) < \omega^{\tau'}(d'+1)$.

First suppose that $\mathscr{K}-\deg(\mathscr{P}) = 1$. We would like to show that $\Delta\text{-Dim}(\mathscr{P}) < \omega^{\tau}2$. Let $a_{\mathscr{P}} = (0,\dots,0,1,a_{\tau-1},\dots,a_0)$ be the coefficients of $\omega_{\mathscr{P}}$. Let $\mathscr{Q}_0$ be a prime Δ-ideal of $k\{\bar{y}\}$ be such that $\mathscr{Q}_0 \supset \mathscr{P}$. There are two possible cases.

For the first case suppose $\mathscr{K}-\text{type}(\mathscr{Q}_0) = \tau' < \tau$ and $\mathscr{K}-\deg(\mathscr{Q}_0) = d'$. Then by induction $\Delta\text{-Dim}(\mathscr{Q}) < \omega^{\tau'}(d'+1) < \omega^{\tau}$. This implies that an increasing chain of $(m+1)$-tuples of integers (with no infinitely decreasing subchains), where the first $(m+1-\tau)$ coordinates are 0 has order type less than $\omega^{\tau-1}n$ for some $n < \omega$. In particular, if $\mathscr{Q}$ is such that $\mathscr{Q}_0 \supset \mathscr{Q} \supset \mathscr{P}$ and $\mathscr{Q}$ is maximal such that $\mathscr{K}-\text{type}(\mathscr{Q}) = \tau$, then $\Delta\text{-Dim}(\mathscr{Q}) \leq \omega^{\tau}$. So it is enough to show that the supremum of the set of $\alpha+1$ such that there is strictly decreasing chain of prime Δ-ideals of $k\{\bar{y}\}$ of the form

$$\mathscr{Q}_0 \supset \cdots \supset \mathscr{Q}_\alpha \supset \mathscr{P},$$

where $\mathscr{Q}_0$ is maximal such that $\mathscr{K}-\text{type}(\mathscr{Q}_0) = \tau$, is less than ω^{τ}. This leads us to the second case.

For the second case suppose that $\mathscr{Q}_0 \supset \mathscr{P}$ is a maximal prime Δ-ideal with $\mathscr{K}-\text{type}\ \tau$. Of course $\mathscr{K}-\deg(\mathscr{Q}) = 1$ since $\mathscr{K}-\deg(\mathscr{P}) = 1$. Represent the coefficients of $\omega_{\mathscr{Q}_0}$ by $b_{\mathscr{Q}_0} = (0,\dots,0,1,b_{\tau-1},\dots,b_0)$. Clearly $b_{\tau-1} \leq a_{\tau-1}$. Now, by induction, we know that if $\mathscr{R}$ is a prime Δ-ideal with $\mathscr{K}-\text{type}\ \tau-1$ and $\mathscr{K}$-degree $a_{\tau-1}$, then $\Delta\text{-Dim}(\mathscr{R}) < \omega^{\tau-1}(a_{\tau-1}+1)$. We have shown that $\Delta\text{-Dim}(\mathscr{Q}_0) \leq \omega^{\tau}$. Consider a chain of $(m+1)$-tuples corresponding to a chain

$$\mathscr{Q}_0 \supset \cdots \supset \mathscr{Q}_\alpha \supset \mathscr{P}.$$

The argument above implies that since the $(m+1-\tau)^{\text{th}}$ coordinate of each of these tuples is 1, such a chain has order type less than $\omega^{\tau-1}s$ for some $s < \omega$. Thus, $\Delta\text{-Dim}(\mathscr{P}) < \omega^{\tau}2$.

Now suppose that $\mathscr{K}-\deg(\mathscr{P}) = d > 1$. Let $\mathscr{Q}_0 \supset \mathscr{P}$ be a prime Δ-ideal of $\mathscr{K}-\text{type}\ \tau$ and $\mathscr{K}$-degree $d-1$. By induction, $\Delta\text{-Dim}(\mathscr{Q}_0) < \omega^{\tau}d$. If $\mathscr{Q}$ is such that $\mathscr{Q}_0 \supset \mathscr{Q} \supset \mathscr{P}$ and $\mathscr{Q}$ is maximal such that $\mathscr{K}-\text{type}(\mathscr{Q}) = \tau$ and $\mathscr{K}-\text{type}(\mathscr{Q}) = d$, then $\Delta\text{-Dim}(\mathscr{Q}) \leq \omega^{\tau}(d)$. So it is enough to show that the supremum of the set of $\alpha+1$ such that there is strictly decreasing chain of prime Δ-ideals of $k\{\bar{y}\}$ of the form

$$\mathscr{Q}_0 \supset \cdots \supset \mathscr{Q}_\alpha \supset \mathscr{P},$$

where $\mathscr{Q}_0$ is maximal such that $\mathscr{K}-\text{type}(\mathscr{Q}_0) = \tau$ and $\mathscr{K}-\text{type}(\mathscr{Q}_0) = d$, is less than ω^{τ}. So, by the argument above, since the corresponding chain of $(m+1)$-tuples have 0's in the first $(m-\tau)$ coordinates and d in the $(m-\tau-1)^{st}$ coordinate, the order type of the chain is less than ω^{τ}. We conclude that $\Delta\text{-Dim}(\mathscr{P}) < \omega^{\tau}(d+1)$. ⊣

4.3. A characterization of forking. If k is a differential field with derivations $\Delta = \{\delta_1,\dots,\delta_s\}$, and $\bar{y} = y_1,\dots,y_n$ is a set of Δ-indeterminates, then any computations will be with respect to the canonical ranking of $\Theta\bar{y}$. If $\mathscr{P}$ is a prime Δ-ideal in $k\{\bar{y}\}$, then $\Lambda_{\mathscr{P}}$ is a characteristic set of $\mathscr{P}$ with respect to the canonical ranking.

For $p \in S_n(k)$, let $\mathscr{I}_p = \{f \in k\{\bar{y}\} : f(\bar{y}) = 0 \in p\}$. Of course, $\mathscr{I}_p$ is a prime differential ideal. Note that $\mathscr{I}_p$ is $\mathscr{I}(\bar{\alpha}/k)$ for any $\bar{\alpha}$ realizing the type p.

The characteristic set of $\mathscr{I}_p$ will be denoted by Λ_p and the differential dimension polynomial of $\mathscr{I}_p$ will be denoted by ω_p.

The following proposition allows us to extend a type p over a Δ-field k to a type p' over the algebraic closure of k such that $\Lambda_{p'}$ and Λ_p have the same set of leaders. We denote the algebraic closure of k by $\widetilde{k}^{\mathrm{alg}}$.

PROPOSITION 4.3.1. *Let k be a Δ-field and let p be an n-type over k. Then there is an extension p' of p over $\widetilde{k}^{\mathrm{alg}}$ such that $\Lambda_{p'}$ and Λ_p have the same leaders.*

PROOF. Let $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$ be a generic solution to p and suppose that $\Lambda_p = \{f_1, \dots, f_s\}$. Let $\Upsilon_{h_1}, \dots \Upsilon_{h_s}$ be the leaders of Λ_p. (By convention, $h_i < h_j$ for $i < j$.) Let $\Sigma = \{\beta_1, \dots \beta_s\}$ where $\beta_i = \theta\alpha_j$ and $\theta y_j = \Upsilon_{h_i}$. Let Σ' be the set of proper derivatives of the members of Σ. Let L be the set of derivatives $\theta\alpha_j$ of $\alpha_1, \dots, \alpha_n$ which are lower in rank than β_s, and are not in $\Sigma \cup \Sigma'$. Let L' be the set of derivatives $\theta\alpha_j$ which are higher in rank than β_s and not in Σ'. Then $k\langle\alpha\rangle = k(L, L', \Sigma, \Sigma')$ where $\Sigma \cup \Sigma'$ is algebraic over $K(L)$.

($*$) Since $L \cup L'$ is a transcendence base for $k\langle\alpha\rangle$ over k, $L \cup L'$ remains a transcendence base for $\widetilde{k}^{\mathrm{alg}}\langle\bar{\alpha}\rangle = \widetilde{k}^{\mathrm{alg}}(L, L', \Sigma, \Sigma')$.

Let f_i^* be a minimal polynomial of β_i over $\widetilde{k}^{\mathrm{alg}}(L \cup \{\beta_j\}_{j<i})$. The set $\{f_1^*, \dots, f_m^*\}$ will give rise to a characteristic set, Λ, of $\mathscr{I}(\bar{\alpha}/\widetilde{k}^{\mathrm{alg}})$. Let $p' \in S_n(\widetilde{k}^{\mathrm{alg}})$ be such that $\mathscr{I}_{p'} = \mathscr{I}(\bar{\alpha}/\widetilde{k}^{\mathrm{alg}})$. ($*$) implies that $\Lambda_{p'}$ must have the same leaders as Λ_p. ⊣

PROPOSITION 4.3.2 ([6, page 19]). *Let R_0 be a subring of a ring R such that R is a free R_0-module; let $\mathscr{P}_0$ be an ideal of R_0. Then $(R\mathscr{P}_0) \bigcap R_0 = \mathscr{P}_0$.*

PROPOSITION 4.3.3 ([6, page 25]). *Let k be a field of characteristic 0 and k' a field extending k. Let $\mathscr{P}$ be a prime ideal of a polynomial algebra $k[(X_i)_{i\in I}]$ and $\overline{\alpha} = (\alpha_i)_{i\in I}$ a generic zero of $\mathscr{P}$. Then $k'\mathscr{P}$ is prime if and only if for every element $\beta \in k'$ that is algebraic over k, the minimal polynomial of β over k is irreducible over $k(\overline{\alpha})$.*

The next theorem allows us to take the extension in Proposition 4.3.1 one step further to a model of m-DCF while still keeping the same leaders. For A a subset of $k\{\overline{y}\}$, we will denote the ideal generated (in the usual sense) by A in $k\{\overline{y}\}$ by $(A)_{k\{\overline{y}\}}$ and the Δ-ideal generated by A in $k\{\overline{y}\}$ by $[A]_{k\{\overline{y}\}}$, i.e., $[A]_\Delta$ is the smallest Δ-ideal of R containing A.

PROPOSITION 4.3.4. *Let k be a Δ-field which is algebraically closed and let $p \in S_n(k)$. Let $K \models m$-DCF be a δ-closed field extending k. Then there is $p' \in S_n(K)$ extending p such that $\Lambda_{p'} = \Lambda_p$.*

PROOF. Let $R_0 = k\{\overline{y}\}$ and $R = K\{\overline{y}\}$. Let $p \in S_n(k)$. By Theorem 2.2.18, Λ_p is coherent in R_0 and is clearly still coherent in R. Again by Theorem 2.2.18, to show that Λ_p is a characteristic set of some prime differential ideal in R, it suffices to show that $(\Lambda_p)_R : H_{\Lambda_p}^\infty$ is a prime ideal, not containing a nonzero element reduced with respect to Λ_p. Since $\mathscr{I}_p$ is a prime differential ideal, $(\Lambda_p)_{R_0} : H_{\Lambda_p}^\infty$ satisfies these two conditions. Proposition 4.3.2 implies that

$$R((\Lambda_p)_{R_0} : H_{\Lambda_p}^\infty) \cap R_0 = (\Lambda_p)_{R_0} : H_{\Lambda_p}^\infty. \tag{4.3.1}$$

Since k is algebraically closed, Proposition 4.3.3 implies that $R((\Lambda_p)_{R_0} : H^{\infty}_{\Lambda_p})$ is prime. Now, clearly,

$$R((\Lambda_p)_{R_0} : H^{\infty}_{\Lambda_p}) \subseteq (\Lambda_p)_R : H^{\infty}_{\Lambda_p}.$$

Let $g \in (\Lambda_p)_R : H^{\infty}_{\Lambda_p}$. By definition, it must be the case that $H^{\sigma}_{\Lambda_p} g \in (\Lambda_p)_R$ for some $\sigma \in \mathbb{N}$. But $(\Lambda_p)_R \subseteq R((\Lambda_p)_{R_0} : H^{\infty}_{\Lambda_p})$ and $R((\Lambda_p)_{R_0} : H^{\infty}_{\Lambda_p})$ is prime; so, either $H_{\Lambda_p} \in R((\Lambda_p)_{R_0} : H^{\infty}_{\Lambda_p})$ or $g \in R((\Lambda_p)_{R_0} : H^{\infty}_{\Lambda_p})$. But H_{Λ_p} is not in $(\Lambda_p)_{R_0} : H^{\infty}_{\Lambda_p}$ by Theorem 2.2.18, and therefore not in $R((\Lambda_p)_{R_0} : H^{\infty}_{\Lambda_p})$ by (4.3.1). Hence, $g \in R((\Lambda_p)_{R_0} : H^{\infty}_{\Lambda_p})$ and we have

$$R((\Lambda_p)_{R_0} : H^{\infty}_{\Lambda_p}) = (\Lambda_p)_R : H^{\infty}_{\Lambda_p}.$$

So $(\Lambda_p)_R : H^{\infty}_{\Lambda_p}$ is a prime ideal.

Now, we need only show that $(\Lambda_p)_R : H^{\infty}_{\Lambda_p}$ does not contain a nonzero element reduced with respect to Λ_p. Let $\Lambda_p = \{f_1, \dots, f_s\}$. Suppose there is $g \in (\Lambda_p)_R : H^{\infty}_{\Lambda_p}$ such that g is reduced with respect to Λ_p. g has leader Υ_{h_g}. Then $g = \sum_{j=1}^{r} b_j m_j$ where each $b_j \in K$, and each m_j is a monomial made up from $\{\Upsilon_1, \dots, \Upsilon_{h_g}\}$ and is reduced with respect to Λ_p. Since $g \in (\Lambda_p)_R : H^{\infty}_{\Lambda_p}$, there is some $\sigma \in \mathbb{N}$ such that $H^{\sigma}_{\Lambda_p} g \in (\Lambda_p)_R$. We write $H_{\Lambda_p} = \sum_{\ell=1}^{t} c_\ell m'_\ell$ where $c_\ell \in k$ and m'_ℓ is a monomial made up from $\{\Upsilon_1, \dots \Upsilon_{h_{f_s}}\}$. Now, $H^{\sigma}_{\Lambda_p} g \in (\Lambda_p)_R$ implies that $H^{\sigma}_{\Lambda_p} g = \sum_{i=1}^{s} \gamma_i(\bar{y}) f_i$ where $\gamma_i(\bar{y}) = \sum_{i'=1}^{r_i} d_{i,i'} m''_{i'}$ where each $d_{i,i'} \in K$ and each $m''_{i'}$ is a monomial made up from $\{\Upsilon_1, \dots \Upsilon_{h_{f_s}}\}$. Let $\varphi(\bar{w}, \bar{u})$ be the sentence

(4.3.2) $\exists w_1, \dots, w_s \text{(not all zero)} \exists u_{1,1}, \dots, u_{1,r_1}, \dots, u_{s,1}, \dots, u_{s,r_s} \text{(not all zero)}$

$$\left(\sum_{\ell=1}^{t} c_\ell m'_\ell\right)^{\sigma} \sum_{j=1}^{r} w_j m_j = \sum_{i=1}^{s} \left(\sum_{i'=1}^{r_i} u_{i,i'} m''_{i'}\right) f_i.$$

$K \models \varphi(\bar{w}, \bar{u})$. Now, since k is an algebraically closed and the condition expressed by φ is an algebraic condition and not a differential condition, $k \models \varphi(\bar{w}, \bar{u})$.

But then, there are $b'_1, \dots, b'_r \in k$ (not all zero) such that

$$g' = \sum_{j=1}^{r} b'_j m_j \in (\Lambda_p)_{R_0} : H^{\infty}_{\Lambda_p}$$

and g' is reduced with respect to Λ_p, a contradiction. Hence, it must be the case that $(\Lambda_p)_R : H^{\infty}_{\Lambda_p}$ does not contain a nonzero element reduced with respect to Λ_p and Λ_p is a characteristic set of the prime differential ideal $\mathscr{Q} = [\Lambda_p]_R : H^{\infty}_{\Lambda_p}$ in R. Let $p' \in S_n(K)$ be such that $\mathscr{I}_{p'} = \mathscr{Q}$. ⊣

We now move on to a characterization of forking for differential fields. We first give two general model-theoretic definitions. See [8] or [7] for more details.

DEFINITION 4.3.5. Let M be a model of a theory T in the language L. Let $p \in S_n(M)$.

1. We say that the L-formula $\phi(\bar{v}, \bar{w})$ is *represented in* p if and only if for some $\bar{a} \in M$, $\phi(\bar{v}, \bar{a}) \in p$.

2. Let N be a model of T such that $M \subseteq N$ and let $q \in S_n(M)$ be such that $q \supseteq p$. We say that q is an *heir of* p if every formula represented in q is represented in p.

DEFINITION 4.3.6. Let $k \subseteq l$ be differential fields, $p \in S_n(k)$, $q \in S_n(l)$ and $p \subseteq q$. We say that q *does not fork over* k if for all $M, N \models m$-DCF such that $k \subseteq M$, $M \cup l \subseteq N$, there are $p_1 \in S_n(M)$ and $q_1 \in S_n(N)$ such that $p \subseteq p_1$, $q \subseteq q_1$, and q_1 is the heir of p_1.

LEMMA 4.3.7 ([8, page 61]). *Let $k \subseteq l$ be differential fields. Let $p \in S_n(k)$ and $q \in S_n(l)$ such that $p \subseteq q$. Suppose for every $K \models m$-DCF with $l \subseteq K$ there is $p_1 \in S_n(K)$ such that $p_1 \supseteq p$ and for all $q_1 \in S_n(K)$, if $q_1 \supseteq q$, then q_1 represents a formula not represented in p_1. Then q forks over k.*

DEFINITION 4.3.8. Given two autoreduced subsets Λ and Λ' of $R\{\overline{y}\}$, we write $\Lambda \ll \Lambda'$ if $\Lambda < \Lambda'$ and the set of leaders of elements of Λ' is not equal to the set of leaders of Λ.

REMARK 4.3.9. Let Λ_1 and Λ_2 be autoreduced subsets of Δ-polynomials in $R\{\overline{y}\}$ and suppose that $\Lambda_1 \ll \Lambda_2$. Let $\Lambda_1 = \{f_1, \dots, f_s\}$ and $\Lambda_2 = \{g_1, \dots, g_r\}$. Then there is some i for which f_i is reduced with respect to Λ_2. Indeed, if $s \leq r$, take the least i for which $v_{f_i} < v_{g_i}$, or for which $v_{f_i} = v_{g_i}$ and $d_{f_i} < d_{g_i}$. For $j < i$, let $v_j = v_{f_j} = v_{g_j}$. f_i is clearly reduced with respect to g_j for $j \geq i$. Furthermore, since Λ_1 is autoreduced and $\Lambda_1 < \Lambda_2$, $\deg_{v_j} f_i < \deg_{v_j} f_j \leq \deg_{v_j} g_j$ for $j < i$. So, f_i is reduced with respect to g_j for $j < i$. Hence f_i is reduced with respect to Λ_2. If $s > r$, take the least i such that $v_{f_i} < v_{g_i}$, or for which $v_{f_i} = v_{g_i}$ and $d_{f_i} < d_{g_i}$, or, if such an i does not exist, take $i = r + 1$. Then a similar argument as above shows that f_i is reduced with respect to Λ_2.

We now give an algebraic characterization of forking.

THEOREM 4.3.10. *Let $k \subseteq l$ be differential fields, $p \in S_n(k)$, $q \in S_n(l)$ and $p \subseteq q$. Then the following three conditions are equivalent.*

1. *q forks over k;*
2. $\Lambda_q \ll \Lambda_p$;
3. $\omega_q < \omega_p$.

PROOF. Let k, l, p, and q satisfy the hypothesis in the theorem.

$2 \Longrightarrow 1$: Suppose $\Lambda_q \ll \Lambda_p$. Let $K \models DCF$ with $l \subseteq K$. Consider the algebraic closure of k in K. By Proposition 4.3.1, there is $\hat{p} \in S_n(\widetilde{k}^{\text{alg}})$ such that $\hat{p} \supset p$ and $\Lambda_{\hat{p}}$ has the same leaders as Λ_p. Choose a $\hat{p}$ such that $\Lambda_{\hat{p}}$ is of highest rank. Now, by Proposition 4.3.4, extend $\hat{p}$ to $p' \in S_n(K)$ such that $\Lambda = \Lambda_{\hat{p}} = \Lambda_{p'}$. Suppose that $q' \in S_n(K)$ is an extension of q. If $\Lambda_{q'} \ll \Lambda$, then by Remark 4.3.9, $\Lambda_{q'}$ must contain some element g reduced with respect to Λ. Hence, q' contains the formula $g(\overline{y}) = 0$ which cannot be represented in p'. Therefore, by Lemma 4.3.7, q forks over k. Therefore, it is enough to show that $\Lambda_{q'} \ll \Lambda$.

Consider $r = q' \upharpoonright \widetilde{k}^{\text{alg}}$. Since $q' \supset p$, $r \supset p$. Now, we chose $\hat{p}$ such that Λ is of highest rank among the characteristic sets of types over $\widetilde{k}^{\text{alg}}$ extending p. Hence $\Lambda_r \leq \Lambda$. Now, $\Lambda_{q'} \leq \Lambda_r$. Hence, $\Lambda_{q'} \leq \Lambda$ and, since the leaders of Λ_q are not the same as the leaders of Λ_p, $\Lambda_{q'} \ll \Lambda$.

$1 \Longrightarrow 2$: Let $M, N \models m$-DCF such that $k \subseteq M$ and $M \cup l \subseteq N$. Suppose that Λ_p and Λ_q have the same leaders. By Propositions 4.3.1 and 4.3.4, there is $q' \in S_n(N)$ such that $q \subseteq q'$ and $\Lambda_{q'}$ has the same leaders as Λ_q. Now, $p \subseteq q'$. Let $p' \in S_n(M)$ be the restriction of q' to M. By the argument in the proof of Proposition 4.3.4, $\Lambda_{p'}$ is a characteristic set in N. Therefore, $[\Lambda_{p'}]_{N\{\bar{y}\}} : H^{\infty}_{\Lambda_{p'}}$ is a prime Δ-ideal and is contained in the prime Δ-ideal $\mathscr{I}_{q'}$, and by Corollary 4.2.11, it must be the case that $[\Lambda_{p'}]_{N\{\bar{y}\}} : H^{\infty}_{\Lambda_{p'}} = \mathscr{I}_{q'}$. Without loss of generality, we can assume that $\Lambda_{q'} = \Lambda_{p'} = \{f_1, \dots, f_s\}$.

Now, it remains to show that q' is the heir of p'; i.e., every formula that is represented in q' is also represented in p'. Let $\phi(\bar{y}, \bar{a})$ be a formula in q'. By compactness, there is a differential polynomial $g(\bar{y}, \bar{a})$ with coefficients in N, reduced with respect to $\Lambda_{p'}$ such that

$$m\text{-DCF} \vdash \left(\left(\bigwedge_{i=1}^{s} f_i(\bar{y}) = 0 \right) \wedge g(\bar{y}, \bar{w}) \neq 0 \right) \longrightarrow \phi(\bar{y}, \bar{w}).$$

But "$\bigwedge_{i=1}^{s} f_i(\bar{y}) = 0$" $\in p'$ and "$g(\bar{y}, \bar{b}) \neq 0$" $\in p'$ for all $\bar{b}$. Hence, $\phi(\bar{y}, \bar{w})$ is represented in p' and q' is the heir to p'. Therefore, q does not fork over k.

$3 \Longrightarrow 2$: Suppose that $\omega_q < \omega_p$. Since q is an extension of p, $\Lambda_q \leq \Lambda_p$. On the other hand, the differential dimension polynomial of a prime Δ-ideal depends only upon the leaders of the characteristic set; hence, if $\omega_p \neq \omega_q$, then Λ_p and Λ_q cannot have the same leaders. So $\Lambda_q \ll \Lambda_p$.

$2 \Longrightarrow 3$: Suppose that $\Lambda_q \ll \Lambda_p$. Since q is an extension of p, the l-Δ-ideal $\mathscr{I}_q$ contains the k-Δ-ideal $\mathscr{I}_p$. Thus, for sufficiently large $t \in \mathbb{N}$, $\omega_q(t) \leq \omega_p(t)$ (any dependence relationship dictated by $\mathscr{I}_p$ is inherited by $\mathscr{I}_q$). We will show that, for sufficiently large $t \in \mathbb{N}$, this is a strict inequality. Let $\Lambda_q = \{f_1, \dots, f_s\}$ and $\Lambda_p = \{g_1, \dots, g_r\}$. As in Remark 4.3.9, consider the least i for which $v_{f_i} < v_{g_i}$; say, $v_{f_i} = \delta_1^{e_1} \dots \delta_m^{e_m} y_j$. Then the m-tuple $(e_1, \dots, e_m)$ must be included in the count for ω_p, but cannot be included in the count for ω_q. (See Sections 4.1 and 4.2.) Hence, for sufficiently large $t \in \mathbb{N}$, $\omega_q(t) < \omega_p(t)$. ⊣

§5. Rank computations.

5.1. Model theory and differential structures. A ring R can be viewed with several different differential-ring structures; in other words, we may define different (perhaps unrelated) sets of differential operators Δ on R. When we must specify the particular differential structure we have in mind, we will use, for example, R_Δ to denote the Δ-structure on R, and $R_{\Delta'}$ to denote the Δ'-structure on R.

Fix a set $\Delta = \{\delta_1 \dots \delta_m\}$ of derivations inducing a differential-ring structure on R. We mention two ways to define related differential structures on R using Δ.

Let $C = (c_{i,j})$ be an invertible matrix over $\mathscr{C}_R$. Define a new differential operator on R in the following way. Let $a \in R$. For each j, define

$$\delta'_j a = \sum_{i=1}^{m} c_{i,j} \delta_i a. \tag{5.1.1}$$

Let $\Delta' = \{\delta'_1 \dots \delta'_m\}$. Then R acquires a new differential structure from Δ'. Clearly, an ideal $\mathscr{I}$ in R is a Δ-ideal if and only if $\mathscr{I}$ is a Δ'-ideal. Furthermore, $\mathscr{C}_{R_\Delta} = \mathscr{C}_{R_{\Delta'}}$ and for $\bar{\eta}$ in some Δ-extension of R, $R\langle\bar{\eta}\rangle_\Delta = R\langle\bar{\eta}\rangle_{\Delta'}$.

REMARK 5.1.1. Given a Δ-extension $F\langle\overline{\eta}\rangle$ of a Δ-field F and Δ' as above, there is a one-to-one map between the Δ-ideal of $\overline{\eta}$ in $F\{\overline{y}\}_\Delta$ and the Δ'-ideal of $\overline{\eta}$ in $F\{\overline{y}\}_{\Delta'}$.

Starting over, we may consider a subset $\Delta^* \subsetneq \Delta$, and consider the Δ^*-structure on R. Clearly, any Δ-ideal is a Δ^*-ideal, but the converse is not necessarily true.

THEOREM 5.1.2. *Let k be a Δ-field where $\Delta = \{\delta_1, \dots, \delta_m\}$. Choose Δ^* a set of m^* linearly independent linear combinations of the element of Δ. Suppose that we have been given a Δ^*-extension $k\langle\overline{\alpha}\rangle_{\Delta^*}$ of k where $\overline{\alpha} = (\alpha_1 \dots \alpha_n)$. Suppose we are given the information that for $\delta \in \Delta$ and for each $1 \leq i \leq k$, $\delta(\alpha_i)$ is given by a linear combination of $\{\overline{\alpha}\} \cup \{\delta^*\alpha_i\}_{\delta^* \in \Delta^*}$. Then the Δ-extension $k\langle\overline{\alpha}\rangle_\Delta$ is uniquely determined.*

PROOF. By definition,

$$k\langle\overline{\alpha}\rangle_{\Delta^*} = k\left(\overline{\alpha}, \{\theta\overline{\alpha}\}_{\theta\in\Theta^*}\right)$$
$$k\langle\overline{\alpha}\rangle_{\Delta} = k\left(\overline{\alpha}, \{\theta\overline{\alpha}\}_{\theta\in\Theta}\right).$$

So knowing that for $\delta \in \Delta$ and for each $1 \leq i \leq k$, $\delta(\alpha_i)$ is given by a linear combination of $\{\overline{\alpha}\} \cup \{\delta^*\alpha_i\}_{\delta^*\in\Delta^*}$, $(k\langle\overline{\alpha}\rangle_{\Delta^*})_{\text{field}} = (k\langle\overline{\alpha}\rangle_{\Delta})_{\text{field}}$ and we know exactly how to interpret each $\delta \in \Delta$ as a derivation on $(k\langle\overline{\alpha}\rangle_{\Delta^*})_{\text{field}}$. Hence, $k\langle\overline{\alpha}\rangle_\Delta$ is uniquely determined. ⊣

PROPOSITION 5.1.3. *Let R be a Δ-ring. Choose Δ^* a set of m^* linearly independent linear combinations of the elements of Δ. Let $\mathscr{I}$ be a Δ-ideal of R. Suppose that $\mathscr{I}_1 \subsetneq \mathscr{I}_2$ are Δ^*-ideals of R_{Δ^*} and $\mathscr{I}_1 = \mathscr{I} \cap R_{\Delta^*}$. Then $\mathscr{I} \subsetneq (\mathscr{I}_2, \mathscr{I})_\Delta$.*

PROOF. Without loss of generality, $\Delta^* \subseteq \Delta$. Let $\alpha \in \mathscr{I}_2 - \mathscr{I}_1$. Then $\alpha \notin \mathscr{I}$ since $\mathscr{I}_1 = \mathscr{I} \cap R_{\Delta^*}$ but $\alpha \in (\mathscr{I}_2, \mathscr{I})_\Delta$. ⊣

For the rest of this section, let k_Δ be a differential field with set of derivations $\Delta = \{\delta_1, \dots \delta_m\}$. Let $C = (c_{i,j})$ be an invertible matrix over $\mathscr{C}_{k_\Delta}$. Define $\Delta' = \{\delta'_1 \dots \delta'_m\}$ to be a set of m derivations where

$$\delta'_j = \sum_{i=1}^{m} c_{i,j}\delta_i. \tag{5.1.2}$$

PROPOSITION 5.1.4. *Let $K_\Delta \models m$-DF such that $k_\Delta \subseteq K_\Delta$. Let Δ' be as above. Then the Δ'-field $K_{\Delta'}$ is definable in K_Δ. The same is true for any set Δ^* of linearly independent $\mathscr{C}_{k_\Delta}$-linear combination of the derivations in Δ.*

PROOF. The first should be clear from the definition of Δ' in Equation 5.1.2. For the second, extend Δ^* to Δ' as above, and apply the first part of this proof. ⊣

PROPOSITION 5.1.5. *Let $K_\Delta \models m$-DCF such that $k_\Delta \subseteq K_\Delta$. Let Δ^* be a subset of Δ of size τ. Then the τ-differential field K_{Δ^*} definable in K_Δ satisfies the axioms for τ-DCF.*

PROOF. Suppose $\Lambda = \{f_1, \dots, f_s\}$ is a Δ^*- coherent set of Δ^*- polynomials in $K\{y\}_{\Delta^*}$ and g is a Δ^*-polynomial Δ^*-reduced with respect to Λ. Notice that $K\{y\}_{\Delta^*} \subseteq K\{y\}_\Delta$. Then Λ is a Δ-coherent set and g is a Δ-reduced with respect to Λ. Hence, if K contains an algebraic solution to the system

$$f_1 = 0 \wedge \dots \wedge f_s = 0 \wedge H_\Lambda g \neq 0, \tag{5.1.3}$$

then K_Δ contains a differential solution to (5.1.3). But $\Delta^* \subset \Delta$, so K_{Δ^*} contains a differential solution to (5.1.3). Therefore, $K_{\Delta^*} \models \tau$-DCF. $\dashv$

PROPOSITION 5.1.6. *Let $K_\Delta \models m$-DCF such that $k_\Delta \subseteq K_\Delta$. Let Δ' be as defined above. Then the m-differential field $K_{\Delta'}$ definable in K_Δ satisfies the axioms of m-DCF.*

PROOF. We need only check that if $K_{\Delta'}$ contains an algebraic solution to some system of Δ'-polynomial equations and inequations satisfying certain requirements, then $K_{\Delta'}$ has a Δ'-solution to that system. So let $\Lambda' = \{\hat{f}_i\}_{i=1}^{s} \subset K\{y\}_{\Delta'}$ be a Δ'-coherent set of Δ'-polynomials in one differential indeterminate and suppose $0 \neq \hat{g} \in K\{y\}_{\Delta'}$ is Δ'-reduced with respect to Λ. Suppose $K_{\Delta'}$ contains an algebraic solution to the system

$$\left(\bigwedge_{i=1}^{s} \hat{f} = 0\right) \wedge H_{\Lambda'}\hat{g} \neq 0. \tag{5.1.4}$$

By the argument in the proof of Lemma 3.1.2, there is a prime Δ'-ideal $\mathscr{P}'$, containing Λ' and not containing $\hat{g}$. Translate $\mathscr{P}'$ to a Δ-ideal $\mathscr{P}$; $\mathscr{P}$ is a prime Δ-ideal not containing the translate g of $\hat{g}$. (See Remark 5.1.1.) Since K_Δ is Δ-differentially closed, K contains a Δ-solution, $\bar{\alpha}$, to the system

$$\left(\bigwedge_{f \in \Lambda_{\mathscr{P}}} f = 0\right) \wedge H_{\Lambda_{\mathscr{P}}} g \neq 0. \tag{5.1.5}$$

Now, it is easy to see that $\bar{\alpha}$ is also a Δ'-solution to $\mathscr{P}'$ and is not a solution for $\hat{g} = 0$. Therefore, $\bar{\alpha}$ is a solution to (5.1.4) and $K_{\Delta'}$ satisfies the axioms for m-DCF. $\dashv$

COROLLARY 5.1.7. *Let $K_\Delta \models m$-DCF such that $k_\Delta \subseteq K_\Delta$. Let Δ^* be a set of τ linearly independent $\mathscr{C}_{k_\Delta}$-linear combinations of Δ. Then the τ-differential field K_Δ^* definable in K_Δ satisfies the axioms for τ-DCF.*

PROOF. Extend Δ^* to a set Δ' of m linearly independent $\mathscr{C}_{k_\Delta}$-linear combinations of Δ. Then apply Propositions 5.1.6 and 5.1.5. $\dashv$

5.2. Model-theoretic rank computations. Fix K_Δ to be an m-differentially closed field with the set $\Delta = \{\delta_1, \dots \delta_m\}$ of derivation operators.

We will use the following result from Pong [11].

THEOREM 5.2.1 (Pong). *Let $K_\delta \models DCF$. Let p be a type in $S_n(K_\delta)$. Suppose that the differential dimension polynomial of p is $\omega_p = dX + a$. Then*

$$\omega d \leq \mathrm{RU}(p) \leq \mathrm{RM}(p) \leq \delta\text{-}\mathrm{Dim}(\mathscr{I}_p) \leq \omega d + a < \omega(d+1). \tag{5.2.1}$$

THEOREM 5.2.2. *Let $K_\Delta \models m$-DCF and let p be in $S_n(K_\Delta)$. Let $\tau = \tau_p$ be the $\mathscr{K}$ – type of $\mathscr{I}_p$ and let $d = d_p$ be the $\mathscr{K}$-degree of $\mathscr{I}_p$. Then*

$$\omega^\tau d \leq \mathrm{RU}(p) \leq \mathrm{RM}(p) \leq \Delta\text{-}\mathrm{Dim}(\mathscr{I}_p) < \omega^\tau(d+1). \tag{5.2.2}$$

REMARK 5.2.3. Of course $\mathrm{RU}(p) \leq \mathrm{RM}(p)$ always. And Proposition 4.2.9 proves that $\Delta\text{-}\mathrm{Dim}(\mathscr{I}_p) < \omega^\tau(d+1)$.

Without loss of generality, we assume that K is sufficiently saturated. We work in the universe $\mathscr{M}$ where $\mathscr{M} \models m$-DCF. We prove the remainder of Theorem 5.2.2 in two steps.

LEMMA 5.2.4. *Let K_Δ and p be as stated in the theorem. Then $\omega^\tau d \leq \mathrm{RU}(p)$.*

Proof. We must first set the stage for the proof. Let $\bar{\eta} = (\eta_1, \ldots, \eta_n) \in \mathscr{M}^n$ be a generic realization of p. Let G_Δ be the Δ-extension $K\langle\bar{\eta}\rangle_\Delta$ of K_Δ. By Theorem 4.2.5, there exists a set Δ^* of τ linearly independent $\mathscr{C}_{K_\Delta}$-linear combinations of the elements of the elements of Δ, and $\bar{\gamma} = (\gamma_1, \ldots, \gamma_d, \ldots, \gamma_{n^*})$, an n^*-tuple of elements of G_{field} such that $G_{\text{field}} = (K\langle\bar{\gamma}\rangle_{\Delta^*})_{\text{field}}$ and such that $\gamma_1, \ldots, \gamma_d$ are Δ^*-independent over K_{Δ^*} and $\gamma_{d+1}, \ldots, \gamma_{n^*}$ are Δ^*-dependent over $K\langle\gamma_1, \ldots, \gamma_d\rangle_{\Delta^*}$. Since $K_\Delta \models m$-DCF, by Corollary 5.1.7, $K_{\Delta^*} \models \tau$-DCF. Notice here that the underlying language has changed. Since Δ^* is a $\mathscr{C}_{K_\Delta}$-linear combination of the elements of Δ, we can extend Δ^* to a set Δ' of m derivations such that for each $\delta' \in \Delta'$, δ' is a $\mathscr{C}_{K_\Delta}$-linear combination of the elements of Δ. By Proposition 5.1.6, $K_{\Delta'} \models m$-DCF, and $K\langle\bar{\gamma}\rangle_\Delta = K\langle\bar{\gamma}\rangle_{\Delta'}$. Hence, without loss of generality, we may assume that $\Delta^* \subset \Delta$. Let $p^* = \text{tp}(\bar{\gamma}/K_{\Delta^*})$ and $p' = \text{tp}(\bar{\gamma}/K_\Delta)$. It is clear that $\mathscr{K} - \text{type}(p^*) = \tau$ and $\mathscr{K} - \deg(p^*) = d$. Moreover, since $\gamma_1, \ldots, \gamma_{n^*}$ are elements of the Δ-field $K\langle\bar{\eta}\rangle_\Delta$ and $G_{\text{field}} = (K\langle\bar{\gamma}\rangle_{\Delta^*})_{\text{field}}$, $K\langle\bar{\gamma}\rangle_\Delta = K\langle\bar{\eta}\rangle_\Delta$. Hence $\mathscr{K} - \text{type}(p') = \tau$ and $\mathscr{K} - \deg(p') = d$. By the same token, $\text{RU}(p') = \text{RU}(p)$. So, without loss of generality, we may assume that $p = p'$ and $n = n^*$.

We will now proceed to prove the theorem by induction on m, the number of derivations present in our language.

For the base case, suppose that $m = 1$. The result follows from previous results found in [8] and Theorem 5.2.1.

Now consider the case where $m > 1$. We will assume that the theorem holds for differentially closed fields $F_{\Delta'}$ where Δ' is a set of $m' < m$ derivations.

Suppose first that $\tau < m$. Consider K_{Δ^*} as above. Recall that $K_{\Delta^*} \models \tau$-DCF.

Claim 5.2.5. *For this claim, let $p \in S_n(K_\Delta)$ and $p^* \in S_n(K_{\Delta^*})$ be any types as above for which p^* is the reduct of p to the language containing Δ^*. Let $L_{\Delta^*} \models \tau$-DCF be a Δ^*-extension of K_{Δ^*}. Suppose that there is $q^* \in S_n(L_{\Delta^*})$ such that q^* is a forking extension of p^* and $\text{RU}(q^*) \geq \beta$, for some ordinal β. Then there is some $F_\Delta \models m$-DCF, a Δ-extension of K_Δ, such that the reduct F_{Δ^*} extends L_{Δ^*}; and there is $r \in S_n(F_\Delta)$ such that r is a forking extension of p and $r|_{\Delta^*} \supset q^*$, and such that $\text{RU}(r) \geq \beta$.*

Proof of Claim. We prove the claim by induction on β.

For the base case, let $\beta = 0$. Suppose that $L_{\Delta^*} \models \tau$-DCF is a Δ^*-extension of K_{Δ^*} and $q^* \in S_n(L_{\Delta^*})$ such that q^* is a forking extension of p^*. Expand L_{Δ^*} to a model F_Δ of m-DCF in such a way that F_Δ is a Δ extension of K_Δ, and the Δ-partial type q^* is consistent in F_Δ. Then F_{Δ^*} is a Δ^*-extension of K_{Δ^*}, and by Corollary 5.1.7, $F_{\Delta^*} \models \tau$-DCF. Let $r^* \in S_n(F_{\Delta^*})$ be the nonforking extension of q^*. Now, down in K_Δ and K_{Δ^*}, p and p^* are intimately connected. p contains the information needed to extend p^* uniquely from K_{Δ^*} to K; more to the point, p contains linear instructions on how to define the action of Δ on the generic realizations of p^*. Hence, we know exactly how to extend r^* uniquely from F_{Δ^*} to a type $r \in S_n(F_\Delta)$, so that $r \supseteq p$. We claim that r is a forking extension of p. Let $p' \in S_n(F_\Delta)$ be the nonforking extension of p. By Lemma 4.2.10 and Theorem 4.3.10, it is enough to show that $\mathscr{I}_r \supsetneq \mathscr{I}_{p'}$. The containment is obvious. Let $\hat{p} \in S_n(F_{\Delta^*})$ be the nonforking extension of p^*. Then $\mathscr{I}_{r^*} \supsetneq \mathscr{I}_{\hat{p}}$. But Proposition 5.1.3 now implies that $\mathscr{I}_r \supsetneq \mathscr{I}_{p'}$. So r is a forking extension of p.

Now suppose that the theorem holds for all $\alpha < \beta$. Let p^*, L_{Δ^*}, q^*, F_{Δ^*}, F_Δ, r, and r^* be as in the case above and suppose that $\mathrm{RU}(q^*) \geq \beta$. As above, r is a forking extension of p. We claim that $\mathrm{RU}(r) \geq \beta$. Now consider r and r^*. By the way in which we chose r, r^* is the reduct of r to the language containing Δ^*. By the induction hypothesis, for each $\alpha < \beta$, we can find an $M_\Delta \models m$-DCF such that M_Δ is a Δ-extension of F_Δ and $r' \in S_n(M_\Delta)$, a forking extension of r, such that $\mathrm{RU}(r') \geq \alpha$. Hence, $\mathrm{RU}(r) \geq \beta$. The claim is proved. ⊣

We return to the proof of the theorem. We supposed that $\tau < m$. $K_{\Delta^*} \models m^*$-DCF and $K_\Delta \models m$-DCF. The claim shows that $\mathrm{RU}(p^*) \leq \mathrm{RU}(p)$. But, by the induction hypothesis, $\mathrm{RU}(p^*) \geq \omega^\tau d$. Hence, $\mathrm{RU}(p) \geq \omega^\tau d$.

Now suppose that $\tau = m$. We will argue by induction on d. For the base case, suppose that $d = 1$. Then γ_1 is Δ-independent over K_Δ and $\gamma_2, \ldots, \gamma_n$ are Δ-dependent over $K\langle\gamma_1\rangle_\Delta$. The additivity of U-rank implies that

$$\mathrm{RU}(\gamma_1/K_\Delta) + \mathrm{RU}(\gamma_2, \ldots, \gamma_n/K_\Delta \cup \{\gamma_1\}) \leq \mathrm{RU}(\gamma_1, \ldots, \gamma_n/K_\Delta).$$

So it is enough to show that $\mathrm{RU}(\gamma_1/K_\Delta) \geq \omega^\tau d$.

Since γ_1 is Δ-independent over K_Δ, Theorem 4.2.1 and Lemma 4.1.1 imply that $\omega_{\gamma_1/K} = \binom{X+m}{m}$. Let $p_1 = \mathrm{tp}(\gamma_1/K)$. Let $\delta \in \Delta$. Consider $L_\Delta \models m$-DCF, a Δ-extension of K_Δ, and some element $a \in L_\Delta \setminus K_\Delta$ such that $p_1 \cup \{\delta^e y = a\}$ is consistent where $e \in \mathbb{N}$ and $e > 1$. Let α_e be a generic realization of $p_1 \cup \{\delta^e y = a\}$ and let $p_e = \mathrm{tp}(\alpha_e/L)$. Certainly, p_e is a forking extension of p_1. We have shown in Example 4.2.6 that $\mathscr{K} - \mathrm{type}(\omega_{\alpha_e/L}) = m - 1$ and that $\mathscr{K} - \deg(\omega_{\alpha_e/L}) = e$. By the induction hypothesis, $\mathrm{RU}(p_e) \geq \omega^{m-1}e$. Hence $\mathrm{RU}(p_1) \geq \omega^{m-1}e$. Since e was chosen arbitrarily, $\mathrm{RU}(p_1) \geq \omega^m$.

Now consider the case where $d > 1$. Then the elements $\gamma_1, \ldots, \gamma_d$ are Δ-independent over K_Δ and $\gamma_{d+1}, \ldots \gamma_n$ are Δ-independent over $K\langle\gamma_1, \ldots, \gamma_d\rangle_\Delta$. The additivity of U-rank implies that

$$\mathrm{RU}(\gamma_1, \ldots, \gamma_{d-1}/K_\Delta) + \mathrm{RU}(\gamma_d, \ldots, \gamma_n/K_\Delta) \leq \mathrm{RU}(\bar{\gamma}/K_\Delta).$$

The induction hypothesis implies that $\mathrm{RU}(\gamma_1, \ldots, \gamma_{d-1}/K_\Delta) \geq \omega^m(d-1)$ and we have just shown above that $\mathrm{RU}(\gamma_d, \ldots, \gamma_n/K_\Delta) \geq \omega^m$. Hence $\mathrm{RU}(p) \geq \omega^m d$. ⊣

LEMMA 5.2.6. *Let K_Δ and p be as above. Then* $\mathrm{RM}(p) \leq \Delta\text{-}\mathrm{Dim}(\mathscr{I}_p)$.

PROOF. Let α be an ordinal. Recall that $\mathrm{RM}(p) \geq \alpha + 1$ if and only if p is a limit point of types of Morley rank at least α. We make use of the fact that in ω-stable theories, we may compute Morley rank inside a sufficiently saturated model.

Let $M^\alpha(K_\Delta)$ be the types in $S_n(K_\Delta)$ of rank at least α. Clearly, if $p \in M^0(K_\Delta)$ then $\Delta\text{-}\mathrm{Dim}(\mathscr{I}_p) \geq 0$. By induction suppose that for some $\alpha > 0$, if $p \in M^\alpha(K_\Delta)$ then $\Delta\text{-}\mathrm{Dim}\,\mathscr{I}_p \geq \alpha$. Suppose that $p \in M^\alpha(K_\Delta)$ and $\Delta\text{-}\mathrm{Dim}(\mathscr{I}_p) = \alpha$. Let Λ_p be a characteristic set of $\mathscr{I}_p$ and H_{Λ_p} be defined as usual. Notice that $H_{\Lambda_p} \notin \mathscr{I}_p$. Otherwise, since $\mathscr{I}_p$ is prime, I_f or S_f would be in $\mathscr{I}_p$ for some $f \in \Lambda_p$. But I_f and S_f are reduced with respect to Λ_p and so cannot be in $\mathscr{I}_p$. Suppose $q \in M^\alpha(K_\Delta)$ and $\left(\left(\bigwedge_{f \in \Lambda_p} f(\bar{y}) = 0\right) \wedge H_{\Lambda_p}(\bar{y}) \neq 0\right) \in q$. Then $\mathscr{I}_q \supseteq \mathscr{I}_p$. Since $\Delta\text{-}\mathrm{Dim}(\mathscr{I}_p) = \alpha$ and by induction $\Delta\text{-}\mathrm{Dim}(\mathscr{I}_q) \geq \alpha$, $\Delta\text{-}\mathrm{Dim}(\mathscr{I}_p) = \Delta\text{-}\mathrm{Dim}(\mathscr{I}_q)$ and it must be the case that $p = q$. Thus "$\left(\bigwedge_{f \in \Lambda} f(\bar{y}) = 0\right) \wedge H_\Lambda(\bar{y}) \neq 0$" isolates p in $M^\alpha(K_\Delta)$ and $\mathrm{RM}(p) = \alpha$. ⊣

COROLLARY 5.2.7. *Let $K_\Delta \models m$-DCF. Let $p \in S_1(K_\Delta)$ be the type of a generic realization of the equation $\delta_1^{e_1} \ldots \delta_m^{r_1} y = 0$. Let $e = \sum_{i=1}^{m} e_i$. Then*

$$\omega^{m-1}e \leq \mathrm{RM}(p) < \omega^{m-1}(e+1).$$

PROOF. We showed in Example 4.2.6 that $\mathscr{K} - \mathrm{type}(\omega_p) = m - 1$ and that

$$\mathscr{K} - \deg(\omega_p) = e.$$

The result follows immediately from Theorem 5.2.2. ⊣

COROLLARY 5.2.8. *Let $K_\Delta \models m$-DCF. Let $p \in S_1(K_\Delta)$ be the type of a Δ-transcendental over K_Δ. Then*

$$\mathrm{RM}(p) = \omega^m.$$

PROOF. By Corollary 5.2.7, for every $\in \mathbb{N}$, p has a forking extension q with

$$\omega^{m-1}e \leq \mathrm{RM}(p) < \omega^{m-1}(e+1).$$

So, we see that $\mathrm{RM}(p) \geq \omega^m$. But every forking extension q of p has U-rank less than $\omega^{m-1}(e+1)$ for some e. Hence $\mathrm{RM}(p) \leq \omega^m$. ⊣

REMARK 5.2.9. The Morley rank of a structure M is defined to be the Morley rank of the formula $x = x$. The Morley rank of a complete theory T is the Morley rank of any model of T. For a model K of m-DCF, it should be clear that the Morley rank of K is the Morley rank of the type of a differential transcendental plus one.

THEOREM 5.2.10. *The Morley rank of m-DCF is $\omega^m + 1$.*

PROOF. The result follows from Corollary 5.2.8 and Remark 5.2.9. ⊣

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