THE COMPLEXITY OF SEMILINEAR SETS

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Abstract: In this paper we shall characterize the computational complexity of two decision problems: the inequality problem and the uniform word problem for semilinear sets. It will be proved that the first problem is log-complete in the second class (Σ_2^p) of the polynomial-time hierarchy and the second problem is log-complete in NP. Moreover we shall show that these problems restricted to the 1-dimensional case have the 'same' computational complexity as the general case.

0. Introduction. Recently, G. Hotz has begun his investigations on invariants of formal languages. He pointed out in [7,8] that necessary criteria for the basic decision problems in language theory such as the equivalence problem, the word problem ... are useful. Considering context-free languages the oldest invariant seems to have appeared in Parikh's theorem, which states that the commutative images of context-free languages are semilinear sets. On the other side semilinear sets play an important role in other research areas of theoretical computer science. Thus it is interesting to study the complexity of the inequality problem and the uniform word problem for semilinear sets.

In section 2 basic definitions and some necessary auxiliary results will be presented. In section 3 we shall derive some results on semilinear sets, especially we shall prove a lemma which gives us some information about the complement of a semilinear set. In the last section we shall prove our main results, which state that the inequality problem resp. the equality problem for semilinear sets is log-complete in Σ_2^p resp. Π_2^p in the polynomial-time hierarchy studied in [11,12].

1. Preliminaries. In this section we review commonly known definitions and give some notations which will be used later.

Let Σ be a finite alphabet. Σ^* denotes the free monoid generated by Σ. Σ^+ is Σ^* without the empty word ε. #w denotes the length of the word w. Let DTIME(C( )) resp. NTIME(C( )) be the class of languages, which are recognizable on TM's resp. NTM's in time C( ). Let DSPACE(C( )) resp. NSPACE(C( )) be the class of languages which are recognizable on TM's resp. NTM's in space C( ).

Notation 1.1. $P := \bigcup_{k=1}^{\infty} \mathrm{DTIME}(n^k)$, $NP := \bigcup_{k=1}^{\infty} \mathrm{NTIME}(n^k)$, $\mathrm{PSPACE} := \bigcup_{k=1}^{\infty} \mathrm{DSPACE}(n^k)$. LOGSPACE denotes the class of functions computable in logarithmic space (logspace).

Definition 1.2. Let Σ and Δ be two finite alphabets, $L_1 \subset \Sigma^*$ and $L_2 \subset \Delta^*$ be two languages. $L_1 \underset{\log}{\leq} L_2$:⇔ [∃f ∈ LOGSPACE : w ∈ L_1 ⇔ f(w) ∈ L_2] . $\underset{\log}{\leq}$ is reflexive and transitive.It is called reduction in logspace.

Let L be a language and Ω be a class of languages. $\Omega \underset{\log}{\leq} L$:⇔ ∀L' ∈ Ω : $L' \underset{\log}{\leq} L$. L is called log-complete in Ω:⇔ L ∈ Ω and $\Omega \underset{\log}{\leq} L$.

Definition 1.3.Let A be a language.NP(A) denotes the class of languages accepted by nondeterministic oracle machines M^A in polynomial time. Let Ω be a class of languages. $NP(\Omega) := \bigcup_{A \in \Omega} NP(A)$.The polynomial-time hierarchy studied in [11,12,14] is the following hierarchy:

Σ_o^p , Π_o^p , Σ_1^p , Π_1^p , Σ_2^p , Π_2^p , ... , where $\Sigma_o^p = \Pi_o^p = P$ and $\Sigma_{k+1}^p = NP(\Sigma_k^p)$, Π_{k+1}^p = co-NP(Σ_k^p) for all k ≥ o.(co-Ω :={$\bar{A}$ | A ∈ Ω }).

Remark 1.4.In [10,11] Meyer and Stockmeyer defined integer expressions and showed that the inequivalence problem for integer expressions is log-complete in Σ_2^p. Our result presents a new combinatorial problem which is log-complete in this class of the polynomial-time hierarchy.

2.Basic definitions and auxiliary results.In this section we give the basic definitions and reproduce some auxiliary results without proofs.

In the following let Z be the set of integers,N_o be the set of nonnegative integers and N the set N_o-{O}.We first define the notion of semilinear sets by the following

Definition 2.1.Let C and Π be two finite subsets of N_o^k and C≠∅.

$L(C;\Pi) := \{c+\sum_{i=1}^{n}\lambda_i p_i \mid c \in C, \lambda_i \in N_o$ and $\Pi=\{p_1,\dots,p_n\}\}$.A subset L of N_o^k is called a linear set,iff L=L({c};Π) for some {c} and Π of N_o^k. c is called the constant,Π the period system,p ∈ Π a period of L.A subset SL ⊂ N_o^k is called a semilinear set (s.l.),iff SL is a finite union of linear sets.If L=L(c;Π) (=L({c};Π)) is a linear set,so we call (c;Π) a representation of L.If SL=$L(c_1;\Pi_1)\cup\dots\cup L(c_m;\Pi_m)$ is a s.l. set,so we call $(c_1;\Pi_1) \cup\dots\cup (c_m;\Pi_m)$ a representation of SL.Let SL_1 and SL_2 be two s.l. set representations.SL_1 and SL_2 are called equivalent,iff SL_1 and SL_2 define the same s.l. set.

Convention 2.2.W.l.o.g. we consider s.l. set representations as words over the finite alphabet Σ:={O,1,{,},(,),,,∪,;}.On our computation models nonnegative integers have binary representations without leading zeros.We now formulate the two decision problems which we shall study.

The equality problem for s.l. sets :It is to decide,whether two s.l. set representations are equivalent,i.e.whether they define the same s.l. set.In a similar way we can formulate the inequality problem for s.l.sets.

The uniform word problem for s.l. sets :For a vector v with nonnegative integer entries and a s.l. set representation SL it is to decide, whether v is a member of the set defined by SL.

Notation 2.3.We define the languages describing these decision problem over the alphabet Σ∪{∫}:

EQ:={$SL_1 \int SL_2$ | SL_1 and SL_2 are equivalent s.l. set representations},
INEQ:={$SL_1 \int SL_2$ | SL_1 and SL_2 are inequivalent s.l. set representations}.
Further let UWP denote the uniform word problem for s.l. sets.

For the proofs of our theorems we shall use some known results which are given here without proofs.The interested reader is referred to [3,6].

Auxiliary results.The auxiliary results used later concern:

-Bounds on the minimal positive integer solutions of a linear diophantine equation system.

-Aggregating linear diophantine equations with nonnegative coefficients to a single one without affecting the nonnegative integer solution set.

Let $A=(a_{ij})$, $1\leq i\leq k$, $1\leq j\leq m$, be a kxm matrix with entries in Z, where $k\leq m$. Let $B=(b_i)\in Z^k$, $1\leq i\leq k$ and $X=(x_i)$, $1\leq i\leq m$, be two column vectors. Consider the linear diophantine equation system $A\cdot X=B$ (I). Let S(A,B) denote the set of nonnegative integer solutions of the system (I), i.e. $S(A,B)=\{v\in N_o^m \mid A.v=B\}$. We first show the following lemma.

Lemma 2.4. S(A,B) is a s.l. set in N_o^m of the form $S(A,B)=L(\{c_1,..,c_r\};\{v_1,..,v_s\})$ for some $r,s\in N_o$.

Proof. With the usual partial order relation $\leq$ on N_o^m we can define the notion "minimality" of the elements of some subset of N_o^m. From the well-known theorem on the finiteness of the minimal element number of a set $S\subset N_o^m$ it follows that there are only a finite number of minimal solutions in S(A,B). Let these solutions be $c_1,...,c_r$. Consider the solution set S(A,O), i.e. the solution set of the homogenous system $A\cdot X=O$. It is not hard to show that S(A,O) is a submonoid of the commutative monoid N_o^m. Let $v_1,..,v_s$ be the minimal elements of the set $S(A,O)-\{O\}$. One can easily prove the following fact : $S(A,B)=L(\{c_1,..,c_r\};\{v_1,..,v_s\})$. This completes the proof of the lemma. □

Notation 2.5. For a vector $w\in N_o^m$ let $\|w\|$ be $Max\{w_i \mid w=(w_1,...,w_m)\}$. For a finite set $C=\{c_1,...,c_r\}\subset N_o^m$ we denote by $\|C\|$ the maximum $\underset{1\leq i\leq r}{Max}\{\|c_i\|\}$.

We now give an upper bound for $\|C\|$, where $C=\{c_1,...,c_r\}$ is the minimal solution set of the system (I). Analysing the proof of [3] we get

Theorem 2.6. Let α be the rank of A and M be the maximum of the absolute values of the $\alpha\times\alpha$ subdeterminants of the extended matrix (A|B). The following inequality holds : $\|C\| \leq (m+1)M$. □

Corollary 2.7. With $S(A,B)=L(\{c_1,..,c_r\};\{v_1,..,v_s\})$ the following inequality holds: $\|\{c_1,..,c_r,v_1,..,v_s\}\| \leq (m+1)M$. □

As in the case of s.l. sets we can represent linear diophantine equation systems on TM's in an analogous manner. Thus we can define the size of such an equation system and we denote by #(A,B) the size of the equasystem (I). We can now prove the following

Lemma 2.8. Let S(A,B) be $L(\{c_1,..,c_r\};\{v_1,..,v_s\})$. Then the inequality

$\#\|\{c_1,..,c_r,v_1,..,v_s\}\| \leq d\#(A,B)\cdot\ln(\#(A,B))$ holds, where d is some constant and ln is the logarithm to base 2.

Proof. For an arbitrary real nxn matrix G the following Hadamard's formula holds: $\det(G)^2 \leq \prod_{i=1}^{n}(\sum_{k=1}^{n} g_{ik}^2)$. A simple calculation yields the inequality stated in the lemma. □

The following theorem was proved in [6] .

Theorem 2.9. Let $\sum_{j=1}^{m} a_{1j}x_j = b_1$, $\sum_{j=1}^{m} a_{2j}x_j = b_2$ (II) be a system of two linear diophantine equations, where the a_{ij}'s and b_i's are nonnegative integers and $b_i>0, i=1,2$. Let $t_1,t_2 \in N$ be two natural numbers with the following properties:

(1) $\gcd(t_1,t_2)=1$, (2) $t_1 \nmid b_2$ and $t_2 \nmid b_1$,

(3) $t_1>b_2-a_1$ and $t_2>b_1-a_2$, where $a_i := \min_{1\leq j\leq m}\{a_{ij}>0\}$ for i=1,2 .

Then the nonnegative integer solution set of (II) is the same as the nonnegative integer solution set of the equation

$$\text{(III)} \quad t_1\cdot\sum_{j=1}^{m}a_{1j}x_j + t_2\cdot\sum_{j=1}^{m}a_{2j}x_j = t_1b_1 + t_2b_2 \ . \quad \square$$

3. Some results on semilinear sets. In this section we prove some properties of s.l. sets which are essential in the proof of the upper bounds for the complexity of EQ and INEQ. Especially we are interested in the computing of the complement for a s.l. set. In [5] there is an algorithm due to Ginsburg & Spanier for this problem. Our method is different from theirs and allows us to obtain the derised upper bounds.

For our argument some notions in the theory of convex bodies are needed. A detailed presentation of this topic in connection with the theory of linear inequalities can be found in [12] .

Definition 3.1. Let $L=L(c;\{p_1,..,p_n\}) \subset N_0^k$ be a linear set. The cone K(L) defined by L is the set $K(L):=\{c+\sum_{i=1}^{n}\lambda_i p_i \mid \lambda_i \in Q_+^k\}$, where Q_+ is the set of nonnegative rational numbers.

In the following we are only concerned with linear sets whose constants are the origin O of the space N_0^k. For those cones defined by such linear sets we simply write $K(p_1,..,p_n):=K(L)=\{\sum_{i=1}^{n}\lambda_i p_i \mid \lambda_i \in Q_+\}$.

Remark 3.2. Our definition of cones is not general. On the other side the reader should verify that we can work in the space Q^k instead of R^k as in [12]. All theorems in [12] used later in this paper remain valid in this case.

Definition 3.3. A subset E of Z^k is called a generating system of a cone K, iff K(E)=K holds. A generating system E of a cone K is called minimal, iff no element of E can be presented as a linear combination of the rest with coefficients in Q_+.

From the above definition one gets easily the following

Lemma 3.4. For every cone K=K(E) there exists a minimal generating system $E' \subset E$ for K and E' is unique up to multiplications with some factors, i.e. if E' and E" are minimal generating systems, the following holds : For every $p' \in E'$ there is exactly one $p'' \in E''$ such that $p' = \lambda p''$ for some $\lambda \in Q_+$

Definition 3.5. If E is a minimal generating system of a cone K, then the cardinality of E is called the rank of K. For a cone $K=K(p_1,..,p_n)$ we define the dimension of K as the dimension of the subspace generated by $p_1,..,p_n$ in the vector space Q^k and we write dim K. Let $A \in Z^k$ be a vector. A hyperplane $H=\{v \in Q^k \mid A \cdot v = 0\}$ is called a boundary plane of the cone K, iff $\sup_{v \in K} A \cdot x = 0$ holds (where v is written as a column vector). A point $v \in K$ is called a boundary point of K, iff $x \in H$ for some boundary plane H of K. The set of all boundary points of K forms the boundary or the frame of K and is denoted by R(K). A point $v \in K-R(K)$ is then an interior point of K. The set of all interior points of K is denoted by $\dot{K}$. A subset S of Q^k is called a face of K, iff $S=K \cap H$ for some boundary plane H. The face of K induced by a boundary plane H is denoted by $S_K(H)$ or shortly $S(H)$.

Remark 3.6. Admitting a cone to be a face of itself the set of faces of K forms a finite complete lattice under set inclusion and we denote it by F(K). One notes that a face of a cone is itself a cone. Therefore dim(s) is well defined for $s \in F(K)$.

A face s' of K covers the face s, iff $s \subsetneq s'$ and there exists no other face $s'' \in F(K)$ such that $s \subsetneq s'' \subsetneq s'$. Let $s, s' \in F(K)$ be two faces of K and $s' \subsetneq s$ Then there exist faces $s_1,..,s_l$ such that $s_1=s, s_l=s'$ and s_i covers s_{i+1} for $i=1,..,l-1$.

We now give another definition of cones, namely the notion of polyhedral cones which will be used later.

Definition 3.7. Let $A \in Z^{m \times k}$ be a mxk matrix with integer entries. The polyhedral cone defined by A is the following set $G(A) := \{v \in Q^k \mid A \cdot v \leq 0\}$. (Let A_i, $1 \leq i \leq m$, be the i-th row of the matrix A. Then there exists a subset $I \subset \{1,..,m\}$ such that the hyperplanes $H_i := \{v \in Q^k \mid A_i \cdot v = 0\}$ are boundary planes of G(A), if we consider G(A) as a cone).

Remark 3.8. Definition 3.7 is also a restriction of the general one. In accordance with our definition of cones we have only to consider such polyhedral cones. It is sufficient for our argument.

Remark 3.9. There is an equivalence between cones and polyhedral cones stated by the theorems of H. Weyl and Minskowski. Weyl's theorem says that every cone is a polyhedral cone. (cf. [12])

Definition 3.10. Let K be a cone with dim K=k', $k' \leq k$. A face s of K is called proper, iff dim s=k'-1.

The proper faces of a cone K form the boundary of K. Every vector in a minimal generating system of K lies in the boundary of K. Now we are interested in computing the number of the proper faces of K. The inductive proof of Weyl' theorem yields a too large upper bound. Using the properties of the lattice $F(K)$ we are able to derive a smaller upper bound for the proper face number of K.

Fact 3.11. Let $K=K(p_1,..,p_n)$ be a cone with minimal generating system $\{p_1,..,p_n\}$ and H be a boundary plane of K. Then we have:

$$S(H) = K(H \cap \{p_1,...,p_n\}) \ .$$

Let K be a cone and H be a boundary plane of K which induces the proper face $S(H)$ of K.H decomposes the space Q^k into two halfspaces denoted by H^l and H^r with the property that $K\subset H^r$.Now let $H_1,...,H_m$ be all boundary planes of K which induce the proper faces of K.Consider the sets G_i, $1\leq i\leq m$,of points in the first octant which lie in the halfspaces H_i^l, i.e. $G_i=Q_+^k\cap H_i^l$.With these notations we get the following

Lemma 3.12. It holds the equality $\bigcup_{i=1}^{m} G_i=Q_+^k-\dot{K}$. Moreover, $H_i^l\cap Q_+^k$ is a cone , $1\leq i\leq m$.

Proof. Trivial. □

In the following we only need to consider cones with dimension k in the space Q^k. The results can be generalized in a straight-forward manner.Before presenting the method for computing the complement of a linear set resp. a s.l. set we show that the cone $Q_+^k\cap H_i^k$ can be generated by a minimal generating system whose vectors have small entries.

Let E_j, $j=1,..,k$,be the hyperplanes $Q^{j-1}x\{0\}xQ^{k-j}$.It is clear that the boundary planes of a cone $Q_+^k\cap H_i^l$ are certain hyperplanes E_j and the hyperplane H_i.This suggests the following lemma.

Lemma 3.13. Let H_i's he the hyperplanes $\{v\in Q^k \mid A_i\cdot v=0\}$, $1\leq i\leq m$,where A_i's are vectors in Z^k.Then the cone $Q_+^k\cap H_i^l$, $1\leq i\leq m$,has a minimal generating system E with the property $\|E\| \leq (k+1)\cdot\|A\|$.

Proof. Consider some fixed cone $Q_+^l\cap H_i^l$.This cone has as boundary planes the hyperplane H_i and some E_j's.Thus $Q_+^k\cap H_i^l$ can be generated by unit vectors in the E_j's and certain vectors in the intersections $H_i\cap E_j=\{v\in Q^k \mid A_i\cdot v=0 \text{ and } e_j\cdot v=0\}$,where e_j is the unit vector whose j-th entry is 1. Now the above formula follows from theorem 2.7. □

Let $K=K(p_1,..,p_n)$ be a cone with minimal generating system $\{p_1,..,p_n\}$ and $\dim K=k$.We are going to give an upper bound for the proper face number of K which depends on n.

Let $K^p:=\{v\in Q^k \mid \forall w\in K: w^T\cdot v\leq 0\}$ be the polar cone defined by K.If $K=K(p_1,...,p_n)$,one gets the following fact:

$$K^p = K(p_1,...,p_n)^p = \{v\in Q^k \mid \begin{pmatrix} p1 \\ \vdots \\ p_n \end{pmatrix}\cdot v \leq 0\} \ ,$$

where the p_i's are written as row vectors. Thus K^p has at most n proper faces.Further it was proved in [12] that there exists an antiisomorphism between the face lattice of K and the face lattice of K^p. Hence an upper bound for the number of the 1-dimensional faces of K^p is also an upper bound for the proper face number of K. From this fact it is sufficient to derive an upper bound for the number of the 1-dimensional faces of K^p.

In order to achieve this upper bound we show the following claim.

Claim 3.14. Let s and s' be two proper faces of a cone K. Then there exist different proper faces $s_1,..,s_l$, $l\geq 0$, and $(k-2)$-dimensional faces $t_1,\dots,t_{l+1}$ such that $s\cap s_1=t_1$; $s_i\cap s_{i+1}=t_{i+1}$, $1\leq i\leq l-1$; and $s_l\cap s'=t_{l+1}$. That is we have the diagram:

$$\begin{tikzcd}[column sep=small] s \arrow[dr, dash] & & s_1 \arrow[dl, dash] \arrow[dr, dash] & & s_2 & \cdots & s_{l-1} \arrow[dr, dash] & & s_l \arrow[dl, dash] \arrow[dr, dash] & & s' \arrow[dl, dash] \\ & t_1 & & t_2 & & & & t_l & & t_{l+1} & \end{tikzcd}$$

Proof. Let s and s' be two different proper faces of K. Then the intersection $s\cap s'$ is again a face of K. If $\dim(s\cap s')=k-2$, we are finished. So we assume that $\dim(s\cap s')\leq k-2$.

Consider the boundary planes H and H' generated by s and s'. The intersection of H and H' is a linear subspace of dimension $k-2$. Let $q_1,..,q_{k-2}\in Z^k$ be a minimal generating system for $H\cap H'$ which is contained in a halfspace defined by certain boundary plane. Let $\{p_{i,1},..,p_{i,k-1}\}$ resp. $\{p_{j,1},..,p_{j,k-1}\}$ be the generating system for s resp. s'. Look at the cone generated by $\{q_1,..,q_{k-2}\}\cup\{p_{i,1},..,p_{i,k-1}\}\cup\{p_{j,1},..,p_{j,k-1}\}$ we see that H and H' induce two proper faces $\bar{s}$ and $\bar{s}'$. Further $\bar{s}$ and $\bar{s}'$ are generated by $\{q_1,..,q_{k-2}\}$ plus a vector of $\{p_{i,1},..,p_{i,k-1}\}$ resp. of $\{p_{j,1},\dots,p_{j,k-1}\}$. Thus there are $k-2$ vectors of $\{p_{i,1},..,p_{i,k-1}\}$ resp. of $\{p_{j,1},..,p_{j,k-1}\}$ which are interior points of $\bar{s}$ resp. $\bar{s}'$. W.l.o.g. let these sets be $\{p_{i,1},\dots,p_{i,k-2}\}$ and $\{p_{j,1},\dots,p_{j,k-2}\}$. Thus we have the following diagrams in the face lattice $F(K)$:

$$\begin{tikzcd}[column sep=small] & K & & & K \arrow[dr, dash] & \\ s \arrow[ur, dash] \arrow[dr, dash] & & \text{and} & & & s' \\ & K(p_{i,1},..,p_{i,k-2}) & & K(p_{j,1},\dots,p_{j,k-2}) \arrow[urr, dash] & & \end{tikzcd} .$$

Because $F(K)$ is relatively complemented (i.e. if $H,F,G\in F(K)$ and $H\subset F\subset G$, then there exists an $\bar{F}\in F(K)$ such that $F\wedge\bar{F}=H$ and $F\vee\bar{F}=G$ and $\dim F + \dim \bar{F} = \dim H + \dim G$), we can complete the diagrams above:

$$\begin{tikzcd}[column sep=small] & K \arrow[dr, dash] & & & & K \arrow[dr, dash] & \\ s \arrow[ur, dash] \arrow[dr, dash] & & s_1 & \text{and} & s_2 \arrow[ur, dash] \arrow[dr, dash] & & s' \\ & K(p_{i,1},..,p_{i,k-2}) \arrow[ur, dash] & & & & K(p_{j,1},\dots,p_{j,k-2}) \arrow[ur, dash] & \end{tikzcd} ,$$

where s_1 and s_2 are two proper faces.

On the other side the new generating vector for s_1 resp. s_2 is an interior point of $K(\{q_1,..,q_{k-2}\}\cup\{p_{i,1},..,p_{i,k-1}\}\cup\{p_{j,1},..,p_{j,k-1}\})$. Otherwise s resp. s' would be no proper faces of K. Continuing this process we get the situation stated in the claim. □

On the other hand we know that the proper faces of K^p contain all vectors of a minimal generating system of K^p. So we can conclude from claim 3.14 that K^p can be generated by at most $n+k-2$ vectors. Thus one gets the following

Lemma 3.15. A cone $K=K(p_1,..,p_n)$ possesses at most $n+k-2$ proper faces and each of them can be generated by $k-1$ linearly independent vectors of

$\{p_1,\dots,p_n\}$. □

One can easily prove the following lemma.

Lemma 3.16. Let $K=K(p_1,..,p_n)$ be a cone with minimal generating system $\{p_1,..,p_n\}\subset N_o^k$. Let $s_1,\dots,s_m$ be all proper faces of K and v be an interior point of K. Then we have $K = \bigcup_{i=1}^{m} K(v,s_i)$ □

For a cone $K=K(p_1,..,p_n)$ consider the grid points contained in K. The following lemma gives us a property of these points.

Lemma 3.17. Let $K=K(p_1,..,p_n)$ be a cone. Then there exists a constant $h\in N_o$ with the property that $\forall x\in K\cap N_o^k$: $hx\in L(O;\{p_1,\dots,p_n\})$. Moreover, the size of h fulfills the inequality $\# h \le d(\#(p_1,..,p_n))^2$, for some constant d.

Proof. Let $s_1,..,s_m$ be all proper faces of K and $\{p_{i,1},\dots,p_{i,k-1}\}$, $1\le i\le m$, a minimal generating system for s_i, i.e. $K(p_{i,1},\dots,p_{i,k-1})=s_i$. According to lemma 3.16 let $v\in\dot{K}$ be an interior point, $v\in N_o^k$, with the property that $K=\bigcup_{i=1}^{m} K(v,p_{i,1},..,p_{i,k-1})$. $v,p_{i,1},\dots,p_{i,k-1}$ are linearly independent in the vector space Q^k for all i, $1\le i\le m$.

For a fixed i, $1\le i\le m$, consider the cone $K(v,p_{i,1},\dots,p_{i,k-1})$. If x is a grid point in $K(v,p_{i,1},..,p_{i,k-1})$, then from lemma A.2 in [5] there exists a constant $h_i'\in N_o$ such that $h_i'x\in L(O;\{v,p_{i,1},..,p_{i,k-1}\})$. Analysing the proof of this lemma we see that for h_i' one can choose the absolute value of the determinant of the matrix $(v\ p_{i,1}\dots p_{i,k-1})$, where v and the $p_{i,j}$'s are written as column vectors.

Let h_i be the absolute value of $\det(v\ p_{i,1}\dots p_{i,k-1})$ for $1\le i\le m$. Define $h:=\prod_{i=1}^{m} h_i$. One can verify that h has the desired property stated in the lemma above. Now we derive an upper bound for the size of h. From the previous lemmas we have the following facts:

-The proper face number of K is less than 2n.

-$\forall i, 1\le i\le m$: $\#\det(v\ p_{i,1}\dots\ p_{i,k-1}) \le d'\#(v\ p_{i,1}\dots p_{i,k-1})$ for some constant d'.

On the other hand we can choose a vector $v\in L(O;\{p_{i,1},\dots,p_{i,k-1}\})$ with small entries. From these observations lemma 3.17 follows. □

Now we derive an upper bound for the generating vectors of the inverse image of a s.l. set under a linear mapping.

Lemma 3.18. Let $K=K(p_1,..,p_n)$ be a cone, $p_1,\dots,p_n\in N_o^k$, and $h\in N$ be some constant. Let $\mu_h:N_o^k\to N_o^k$ be the mapping defined by $x\mapsto hx$. Then for any linear set $L\subset N_o^k$, $\mu_h^{-1}(L)$ is a s.l. set and an upper bound for the entries of generating vectors of $\mu_h^{-1}(L)$ can be obtained by the inequality $\#(\|\bigcup_{i=1}^{l}\{c_i\}\cup\bigcup_{i=1}^{l}\Pi_i\|) \le d(\ln\# L)(\# L+k\# h)$ for some constant d, where $\mu_h^{-1}(L) = \bigcup_{i=1}^{l} L(c_i;\Pi_i)$.

Proof. Because of the linearity of μ_h, $\mu_h^{-1}(L)$ is a s.l. set. We derive the upper bound stated in the lemma.

Consider the space $N_o^k x N_o^k$. Let M_k be the linear set $M_k:=\{(x,\mu_h(x))\mid x\in N_o^k\} = L(O;\{(e_1,he_1),..,(e_k,he_k)\})$, where the e_i's are unit vectors of N_o^k.

Then we have: $\mu_h^{-1}(L)=\pi_k(M_k \cap N_o^k xL)$, where π_k is the projection on the first k components. W.l.o.g. let be $L=L(0;\{q_1,..,q_m\})$. It follows that $N_o^k xL = L(0;\{(e_1,0),..,(e_k,0),(0,q_1),..,(0,q_m)\})$. From lemma 2.9 we know that the minimal positive integer solutions of the following linear diophantine equation system

$$\begin{bmatrix} 1 & & & & & \\ & 1 & & & 0 & \\ & & \ddots & & & \\ & 0 & & & & 1 \\ h & & & & & \\ & h & & & 0 & \\ & & \ddots & & & \\ & 0 & & & & h \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_k \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ & 1 & & 0 & \\ & & \ddots & & \\ & 0 & & 1 & q_{11}\cdots q_{m1} \\ & & & & \vdots \qquad \vdots \\ & & & & q_{1k}\cdots q_{mk} \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_k \\ y_{k+1} \\ \vdots \\ y_{k+m} \end{bmatrix}$$

have entry size smaller than $\ln(2k+m).(k+h+d'\#(q_1...q_m))$ for some constant d'. A simple calculation yields the desired upper bound. □

It is not hard to show the following lemma.

Lemma 3.19. Let $K=K(p_1,..,p_n)$ be a cone. Then the set L of grid points contained in K, $L=K\cap N_o^k$, is a linear set and the size of the entries of generating vectors for L can be estimated as follows : $\#(\|\{c,q_1,..,q_m\}\|) \leq \#(\|\{p_1,..,p_n\}\|)$, where $L=L(c;\{q_1,..,q_m\})$. □

From the previous lemma the following corollary holds.

Corollary 3.20. Let $K=K(p_1,..,p_n)$ be a cone and $s=K(p_1,..,p_{k-1})$ be a proper face of K. Then the set $SL:=N_o^k\cap(K-s)$ is s.l. and the size of the entries of generating vectors for SL can be estimated as follows : $\#(\|C\cup\Pi\|) \leq \#(\|\{p_1,..,p_n\}\|)$, where $SL=L(C;\Pi)$. □

And we get the following theorem.

Theorem 3.21. Let $K=K(p_1,..,p_n)$ be a cone. Then the set $SL:=N_o^k\cap(Q_+^k-K)$ is s.l. and the entry size of generating vectors for SL can be estimated as follows: $\#(\|\{c_1,..,c_l\}\cup\bigcup_{i=1}^{l}\Pi_i\|) \leq \#\|\{p_1,...,p_n\}\|$. □

Theorem 3.19 gives us an upper bound for the maximal size of the vectors in certain representation for the complement of a linear set where this complement is outside the cone induced by this linear set. We have not yet derived an upper bound for the size of the vectors of some generating system for the complement inside the induced cone. This will be done in the following theorem.

Theorem 3.22. Let $L=L(0;\{p_1,..,p_n\})$ be a linear set. Then the set $\bar{L} := N_o^k\cap K(L) - L$ is a s.l. set and we have an upper bound for the size of the generating vectors for $\bar{L}$ as follows : $\#\|\{c_1,..,c_l\}\cup\bigcup_{i=1}^{l}\Pi_i\| \leq d\#(p_1,..,p_n)^4$ for some constant d, where $\bar{L}=\bigcup_{i=1}^{l}L(c_i;\Pi_i)$.

Proof. W.l.o.g. we assume that $\dim K(L)=k$ and $\{p_1,..,p_n\}$ is a minimal generating system of K(L). Let h be the constant given in lemma 3.17. Consider the set $N_o^k x N_o^n$. For all i, $1\leq i\leq n$, define the vectors $\bar{p}_i$'s by $\bar{p}_i := (p_i,0,..,0,\underset{(k+i)\text{-th position}}{1},0,..,0)\in N_o^k x N_o^n$. The following facts are evident:

(a) As vectors in Q^{k+n} , $\bar{p}_1,..,\bar{p}_n$ are linearly independent .

(b) $x=\sum_{i=1}^{n}\lambda_i p_i \in L(0;\{p_1,..,p_n\}) \subset N_0^k \Leftrightarrow (x,\lambda_1,..,\lambda_n) \in L(0;\{\bar{p}_1,..,\bar{p}_n\})$.

Let $\bar{L}_h$ be the set $\bar{L}_h:=L(0;\{\bar{p}_1,..,\bar{p}_n\})-L(0;\{h\bar{p}_1,..,h\bar{p}_n\})$.

<u>Claim</u>. $\mu_h^{-1}(\pi_k(\bar{L}_h)) = \bar{L}$, where μ_h is the linear mapping defined in lemma 3.18.

<u>Proof of the claim</u>. We have : $x \in \bar{L} \Leftrightarrow x \in (N_0^k \cap K(L))-L$

$\Leftrightarrow \exists\lambda_1,..,\lambda_n \in N_0$: $hx=\sum_{i=1}^{n}\lambda_i p_i$ and $\forall\beta_1,..,\beta_n \in N_0$: $x \neq \sum_{i=1}^{n}\beta_i \bar{p}_i$

$\Leftrightarrow \exists\lambda_1,..,\lambda_n \in N_0$: $(hx,\lambda_1,..,\lambda_n) = \sum_{i=1}^{n}\lambda_i\bar{p}_i$ and $\forall\beta_1,..,\beta_n \in N_0$:

$$(hx,h\beta_1,..,h\beta_n) \neq \sum_{i=1}^{n}\beta_i(h\bar{p}_i)$$

$\Leftrightarrow$ $hx \in \pi_k((L(0;\{\bar{p}_1,..,\bar{p}_n\}) - L(0;\{h\bar{p}_1,..,h\bar{p}_n\}))$ $\Leftrightarrow$ $x \in \mu_h^{-1}(\pi_k(\bar{L}_h))$.

This completesthe proof of the claim . □

Now define the following linear mapping $\varphi:N_0^n \longrightarrow N_0^{n+k}$ by : $\forall i, 1\leq i\leq n$: $e_i \longmapsto \bar{p}_i$, where the e_i's are unit vectors in N_0^n. It is evident that φ is a bijection from N_0^n onto $L(0;\{\bar{p}_1,..,\bar{p}_n\})$ and $\bar{L}_h=\varphi(N_0^n-hN_0^n)$.

Let $C=\{c_1,..,c_s\}$ be the set $\{x\in N_0^n \mid 0<x<(h,h,..,h)\}$. One can easily verify that $\bigcup_{i=1}^{s} c_i+hN_0^n = N_0^n-hN_0^n$. Hence we have :

$$\bar{L} = \mu_h^{-1}(\pi_k(\varphi(N_0^n-hN_0^n))) \quad .$$

From this fact we conclude that L is a s.l. set,because semilinearity is closed under linear mappings,inverse linear mappings and projections.

From lemma 3.18 one gets the desired upper bound.This completes the proof of the theorem . □

From theorem 3.21 and theorem 3.22 we obtain the following

<u>Theorem 3.23</u>. Let $L=L(0;\{p_1,..,p_n\})$ be a linear set.Then the complement $\bar{L}=N_0^k-L$ of L is a s.l. set and $\bar{L}$ can be generated by vectors whose maximal entry size is bounded by $\#(\|\{c_1,..,c_m\}\cup\bigcup_{i=1}^{m}\Pi_i\|) \leq d\#(p_1,..,p_n)^4$, for some constant d,where $\bar{L}=\bigcup_{i=1}^{m}L(c_i;\Pi_i)$. □

From the previous theorem the following corollary holds.

<u>Corollary 3.24</u>. Let $L=L(c;\{p_1,..,p_n\})$ be a linear set.Then the complement $\bar{L}=N_0^k-L$ is a s.l. set and $\bar{L}$ can be generated by vectors whose maximal entry size is bounded by $\#(\|\{c_1,..,c_m\}\cup\bigcup_{i=1}^{m}\Pi_i\|) \leq d\#(p_1,..,p_n)^4$ for some constant d,where $\bar{L} =\bigcup_{i=1}^{m}L(c_i;\Pi_i)$.

<u>Proof</u>. This corollary follows directly from theorem 3.21 and the proof of lemma A.5 in [5] . □

Until now we have derived an upper bound for generating vectors of the complement of a linear set and we are going to do that for the general case,i.e. the case of s.l. sets. This is done in the following

<u>Theorem 3.25</u>. Let $SL=L_1\cup L_2\cup\ldots\cup L_n$ be a s.l. set,where $L_i=L(c^i;\{p_1^i,\ldots,p_{m_i}^i\})$ for $1\leq i\leq n$. Then the complement $\overline{SL}:=N_0^k-SL$ of SL is a s.l. set and $\overline{SL}$ can be generated by vectors whose maximal entry size is bounded by $\#(\|\{c_1,..,c_l\}\cup\bigcup_{i=1}^{l}\Pi_i\|) \leq P(\#SL)$ for some fixed polynomial P,where $\overline{SL} = \bigcup_{i=1}^{l}L(c_i;\Pi_i)$.

<u>Proof</u>. We have $SL=L_1\cup\ldots\cup L_n$. Because of the semilinearity of the

L_i's it follows that $\overline{SL}$ is s.l.,too.(Semilinearity is closed under intersections). From corollary 3.24. the following fact holds :

$$(*)\quad \bar{L}_i=\bar{L}_1^i\cup\ldots\cup\bar{L}_{l_i}^i \text{ and } =(\|\{\bar{c}_1^i,\ldots,\bar{c}_{l_i}^i\}\cup\bigcup_{j=1}^{l_i}\bar{\Pi}_j^i\|)\leq d(\#(p_1^i,\ldots,p_{m_i}^i))^4$$

for some constant d,where $\bar{L}_j^i=L(\bar{c}_j^i;\bar{\Pi}_j^i)$, $1\leq j\leq l_i$, $1\leq i\leq n$.

Now we want to derive an upper bound for the maximal entry size of the generating vectors for $\overline{SL}$. From (*) it follows that the cardinality $t_{i_j}=|\bar{\Pi}_j^i|$ of the period system of $\bar{L}_j^i$, $1\leq j\leq l_i$, $1\leq i\leq n$, is bounded by $\left[2^{d[\#(p_1^i,\ldots,p_{m_i}^i)]^4}\right]^k$ .In order to determine generating vectors for $\overline{SL} = \bar{L}_1\cap\ldots\cap\bar{L}_n$ we consider the minimal positive integer solutions of the following linear diophantine equation system

$$\begin{bmatrix} \bar{\Pi}_{j_1}^1 & -\bar{\Pi}_{j_2}^2 & & & & \\ & \bar{\Pi}_{j_2}^2 & -\bar{\Pi}_{j_3}^3 & & 0 & \\ & & \ddots & \ddots & & \\ & 0 & & \ddots & \ddots & \\ & & & & \bar{\Pi}_{j_{n-2}}^{n-2} & -\bar{\Pi}_{j_{n-1}}^{n-1} & \\ & & & & & \bar{\Pi}_{j_{n-1}}^{n-1} & -\bar{\Pi}_{j_n}^{n} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{t_{1j_1}} \\ \vdots \\ \vdots \\ x_{\sum_{i=1}^n t_{i_{j_i}}} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_k \\ \vdots \\ \vdots \\ b_{(n-1)\cdot k} \end{bmatrix} \quad (**)$$

where $1\leq j_i\leq l_i$, $1\leq i\leq n$, the vectors of $\bar{\Pi}_{j_i}^i$ are written as column vectors, $-\{v_1,\ldots,v_n\}$ means $\{-v_1,\ldots,-v_n\}$ for a subset $\{v_1,\ldots,v_n\}$ of N_o^k , and for $0\leq r\leq n-1$:

(i) to determine the constants for $\overline{SL}$ we set

$$\begin{bmatrix} b_{rk+1} \\ \vdots \\ b_{rk+k} \end{bmatrix} = \begin{bmatrix} \bar{c}_{j_{r+1},1}^{r+1} - \bar{c}_{j_r,1}^{r} \\ \\ \bar{c}_{j_{r+1},k}^{r+1} - \bar{c}_{j_r,k}^{r} \end{bmatrix}$$

(ii) to determine the periods for $\overline{SL}$ we set

$$\begin{bmatrix} b_{rk+1} \\ \vdots \\ b_{rk+k} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

The determinants of the $[(n-1)k \times (n-1)k]$ submatrices of the extended matrix defined by (**) can be estimated by the Hadamard's formula.The maximal length of their binary representations is bounded by

$$k\cdot\underbrace{\ln[(n-1)k]}_{\leq \#SL}\cdot\sum_{i=1}^n \underbrace{d(\#(p_1^i,\ldots,p_{m_i}^i,c_{m_i}^i))^4}_{\leq d'(\#SL)^4 \text{ for some } d'} \leq d''\cdot(\#SL)^6 \text{ for some constant } d''.$$

On the other hand we have $\sum_{i=1}^n t_{i j_i} \leq \sum_{i=1}^n \left[2^{d(\#(p_1^i,\ldots,p_{m_i}^i))^4}\right]^k$

Hence : $\#(\sum_{i=1}^n t_{i j_i}) \leq \ln(n) + dk[\max_{1\leq i\leq n}\{\#(p_1^i,\ldots,p_{m_i}^i)\}]^4 \leq d'''(\#SL)^5$

for some constant d''' .

Thus the maximal entry size of generating vectors for $\overline{SL}$ can be bounded as follows : $\#(\|\{\tilde{c}_1,..,\tilde{c}_l\}\cup\bigcup_{i=1}^{l}\tilde{\Pi}_i\|) \leq \tilde{d}(\#SL)^8$, for some constant $\tilde{d}$. This completes the proof of theorem 3.24 . □

Remark 3.26. In order to classify the complexity of EQ resp. INEQ , we have deriveda number of upper bounds and did not pay any attention to the optimality of these bounds.The reader should note that we have repeatedly used lemma 2.8 which is not sharp.For our purpose it suffices to obtain polynomial bounds.

4.The complexity of the equality problem for s.l. sets. As the main result of this paper we shall show in this section that the inequality problem resp. the equality problem for s.l. sets is log-complete in Σ_2^p resp. in Π_2^p .

We first prove that INEQ is in Σ_2^p by the following lemma .

Lemma 4.1. INEQ is in Σ_2^p .

Proof. Let $SL_1=L(c_1^1;\Pi_1^1)\cup\ldots\cup L(c_m^1;\Pi_m^1)$ and $SL_2=L(c_1^2;\Pi_1^2)\cup\ldots\cup L(c_n^2;\Pi_n^2)$ be two s.l. set representations.(It is not hard to see that a TM can verify in polynomial time,whether a word $w\in(\Sigma\cup\{\int\})^*$is syntactically correct,i.e.whether it is of the form $SL_1\int SL_2$,where SL_1 and SL_2 are two representations of s.l. sets in N_o^k for some $k\in N$).

Now we have : $(c_1^1;\Pi_1^1)\cup\ldots\cup(c_m^1;\Pi_m^1)\int(c_1^2;\Pi_1^2)\cup\ldots\cup(c_n^2;\Pi_n^2)\in INEQ \Leftrightarrow$ $SL_1-SL_2\neq\emptyset$ or $SL_2-SL_1\neq\emptyset \Leftrightarrow$ (*) $SL_1\cap(N_o^k-SL_2)\neq\emptyset$ or $SL_2\cap(N_o^k-SL_1)\neq\emptyset$. Consider the set $SL_1\cap(N_o^k-SL_2)$.(The case $SL_2\cap(N_o^k-SL_1)$ is symmetric).From theorem 3.23 it follows that $N_o^k\cap SL_2$ is s.l. and it can by generated by vectors of which the maximal entry size is bounded by $P(\#SL_2)$ for some fixed polynomial P.Let $N_o^k-SL_2=\overline{SL}_2=L(\bar{c}_1^2;\bar{\Pi}_1^2)\cup\ldots\cup L(\bar{c}_{\bar{n}}^2;\bar{\Pi}_{\bar{n}}^2)$ be some representation for $\overline{SL}_2$ and $\#(\|\{\bar{c}_1^2,..,\bar{c}_{\bar{n}}^2\}\cup\bigcup_{i=1}^{\bar{n}}\bar{\Pi}_i^2\|)\leq P(\#SL_2)$. We have :

$$SL_1\cap(N_o^k-SL_2) = [L(c_1^1;\Pi_1^1)\cup\ldots\cup L(c_m^1;\Pi_m^1)]\cap[L(\bar{c}_1^2;\bar{\Pi}_1^2)\cup\ldots\cup L(\bar{c}_{\bar{n}}^2;\bar{\Pi}_{\bar{n}}^2)]$$
$$= \bigcup_{\substack{1\leq i\leq m\\ 1\leq j\leq\bar{n}}} L(c_i^1;\Pi_i^1)\cap L(\bar{c}_j^2;\bar{\Pi}_j^2) \quad .$$

Now the intersection $L(c_i^1;\Pi_i^1)\cap L(\bar{c}_j^2;\bar{\Pi}_j^2)$, $1\leq i\leq m, 1\leq j\leq n$, are s.l. sets and they can be represented by vectors of which the maximal entry size is bounded by : $\#(\|\{$generating vectors for $L(c_i^1;\Pi_i^1)\cap L(\bar{c}_j^2;\bar{\Pi}_j^2)\}\|)$ $\leq d\cdot Max(\#(c_i^1,\Pi_i^1),\#(\bar{c}_j^2,\bar{\Pi}_j^2))^5 \leq P'(Max(\#SL_1,\#SL_2))$ for some fixed polynomial P'.(The intersection of two linear sets can be computed in the usual way.One has to solve certain linear diophantine equation system again). Thus we have the following facts :

$(SL_1\int SL_2)\in INEQ \Leftrightarrow$ There exists a vector $v\in N_o^k$ with $\#v\leq P'(Max(\#SL_1,\#SL_2))$ such that ($v\in SL_1$ and $v\notin SL_2$) or ($v\in SL_2$ and $v\notin SL_1$).

On the other side we have : $v\in SL_1$ and $v\notin SL_2 \Leftrightarrow$

[There exist some i,$1\leq i\leq m$,and some coefficient vector Λ with $\|\Lambda\|\leq\|v\|$ such that $v=c_i^1+\Pi_i^1\cdot\Lambda$ and for all j,$1\leq j\leq n$,for all coefficient vector Δ with $\|\Delta\|\leq\|v\|$ it holds : $v\neq\bar{c}_j^2+\bar{\Pi}_j^2\cdot\Delta$].

Thus INEQ is in Σ_2^p and the proof is complete . □

Corollary 4.2. EQ is in Π_2^p. □

Modifying the proof of Stockmeyer in [11] for the Log-completeness of the inequivalence problem for integer expressions in Σ_2^p we can prove the following lemma.

Lemma 4.3. INEQ is $\leq_{\log}$-hard for Σ_2^p .

Proof. We omit the proof. □

From lemma 4.1 and lemma 4.3 one obtains the main

Theorem 4.4. INEQ is log-complete in Σ_2^p . □

and the following corollary.

Corollary 4.5. EQ is log-complete in Π_2^p . □

With 1-EQ resp. 1-INEQ we denote the sublanguages of EQ resp. INEQ, which describe the equality problem and the inequality problem for s.l. sets in N_o.Using theorem 2.9 the following result can be proved.We omit the proof .

Theorem 4.6. 1-INEQ resp. 1-EQ is log-complete in Σ_2^p resp. Π_2^p . □

For the uniform word problem for s.l. sets we have the following results.

Theorem 4.7. UWP is log-complete in NP . □

Theorem 4.8. UWP in N_o is log-complete in NP . □

Remark. Recently,E.M. Gurari and O.H. Ibara have achieved an $2^{2^{c\cdot N^2}}$ upper bound for the equality problem,where c is some constant and N is the input size.(E.M. Gurari and O.H. Ibara:"The Complexity of the Equivalence Problem for Counter Machines,Semilinear Sets,and Simple Programs", Proc. of the 11-th Annual ACM Symposium on the Theory of Computing,1979, pp.142-152) From our main theorem in this paper one can verify that EQ and INEQ are in PSPACE.Thus we can recognize EQ resp. INEQ in DTIME($2^{P(N)}$) where P is a fixed polynomial and N is the input size.

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