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Bounds on generalized Frobenius numbers

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ABSTRACT

Let $N \geq 2$ and let $1 < a_1 < \cdots < a_N$ be relatively prime integers. The Frobenius number of this N-tuple is defined to be the largest positive integer that has no representation as $\sum_{i=1}^{N} a_i x_i$ where $x_1, \ldots, x_N$ are nonnegative integers. More generally, the s-Frobenius number is defined to be the largest positive integer that has precisely s distinct representations like this. We use techniques from the geometry of numbers to give upper and lower bounds on the s-Frobenius number for any nonnegative integer s.

1. Introduction

Let $N \geq 2$ be an integer and let $a_1, \ldots, a_N$ be positive relatively prime integers. We say that a positive integer t is *representable* by the N-tuple $\boldsymbol{a} := (a_1, \ldots, a_N)$ if

$$t = a_1 x_1 + \cdots + a_N x_N \tag{1}$$

for some nonnegative integers $x_1, \ldots, x_N$, and we call each such solution $\boldsymbol{x} := (x_1, \ldots, x_N)$ of (1) a *representation for t in terms of* $\boldsymbol{a}$. The *Frobenius number* $g = g(a_1, \ldots, a_N)$ of this N-tuple is defined to be the largest positive integer that has no representations. The condition $\gcd(a_1, \ldots, a_N) = 1$ implies that such g exists. More generally, as defined by Beck and Robins in [6], let s be a nonnegative integer, and define the *s-Frobenius number* $g_s = g_s(a_1, \ldots, a_N)$ of $\boldsymbol{a}$ to be the largest positive integer that has precisely s distinct representations in terms of $\boldsymbol{a}$. Then in particular $g = g_0$.

The Frobenius number has been studied extensively by a variety of authors, starting as early as late 19th century; see [18] for a detailed account and bibliography. More recently, some authors also started studying the more general s-Frobenius numbers; for instance, in [21,5] the authors investigated families of N-tuples $\boldsymbol{a}$ on which the difference $g_s - g_0$ grows unboundedly. This motivates a natural question: how big and how small can g_s be in general?

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The main goal of this note is to extend the geometric method of [13] to obtain general upper and lower bounds on g_s.

Remark 1.1. We should warn the reader that the term s-Frobenius number is also used by some authors to denote not the largest positive integer that has *precisely s* distinct representations in terms of $\boldsymbol{a}$, as we do here, but the largest positive integer that has *at most s* distinct representations in terms of $\boldsymbol{a}$.

Remark 1.2. It should also be mentioned that other generalizations of the Frobenius number of different nature have also been considered by a variety of authors. In particular, see Chapter 6 of [18], as well as more recent works [2,3,22], among others, for further information and references.

2. Results

We start by setting up some notation, following [13]. Let

$$L_{\boldsymbol{a}}(\boldsymbol{X}) = \sum_{i=1}^{N} a_i X_i,$$

be the linear form in N variables with coefficients $a_1, \dots, a_N$, and define the lattice

$$\Lambda_{\boldsymbol{a}} = \left\{ \boldsymbol{x} \in \mathbb{Z}^N : L_{\boldsymbol{a}}(\boldsymbol{x}) = 0 \right\}.$$

Let $V_{\boldsymbol{a}} = \operatorname{span}_{\mathbb{R}} \Lambda_{\boldsymbol{a}}$, then $V_{\boldsymbol{a}}$ is an $(N-1)$-dimensional subspace of $\mathbb{R}^N$ and $\Lambda_{\boldsymbol{a}} = V_{\boldsymbol{a}} \cap \mathbb{Z}^N$ is a lattice of full rank in $V_{\boldsymbol{a}}$. The *covering radius* of $\Lambda_{\boldsymbol{a}}$ is defined to be

$$R_{\boldsymbol{a}} := \inf \left\{ R \in \mathbb{R}_{>0} : \Lambda_{\boldsymbol{a}} + \mathbb{B}_{V_{\boldsymbol{a}}}(R) = V_{\boldsymbol{a}} \right\}, \tag{2}$$

where $\mathbb{B}_{V_{\boldsymbol{a}}}(R)$ is the closed $(N-1)$-dimensional ball of radius R centered at the origin in $V_{\boldsymbol{a}}$. For each $1 \leq m \leq N-1$ define the mth *successive minimum* of $\Lambda_{\boldsymbol{a}}$ to be

$$\lambda_m := \min\{\lambda \in \mathbb{R} : \dim \left(\operatorname{span}_{\mathbb{R}} \left(\mathbb{B}_{V_{\boldsymbol{a}}}(\lambda) \cap \Lambda_{\boldsymbol{a}} \right) \right) \geq m\}, \tag{3}$$

so $0 < \lambda_1 \leq \cdots \leq \lambda_{N-1}$. We also write κ_m for the volume of an m-dimensional unit ball ($\kappa_0 = 1$), and τ_m for the *kissing number* in dimension m, i.e., the maximal number of unit balls in $\mathbb{R}^m$ that can touch another unit ball. Finally, let us write $\boldsymbol{\alpha}_i := (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_N)$. We can now state our main results, starting with the upper bounds on $g_s(\boldsymbol{a})$.

Theorem 2.1. *With the notation above,*

$$g_s(\boldsymbol{a}) \leq \max \left\{ \frac{R_{\boldsymbol{a}}(N-1) \sum_{i=1}^{N} \|\boldsymbol{\alpha}_i\| a_i}{\|\boldsymbol{a}\|} + 1, \left(s(N-1)! \prod_{i=1}^{N} a_i \right)^{\frac{1}{N-2}} \right\}, \tag{4}$$

where $\| \ \|$ stands for the usual Euclidean norm on vectors. If in addition $s \leq \tau_{N-1} + 1$, then

$$g_s(\boldsymbol{a}) \leq \frac{3R_{\boldsymbol{a}} \sum_{i=1}^{N} \|\boldsymbol{\alpha}_i\| a_i}{\|\boldsymbol{a}\|}. \tag{5}$$

Remark 2.1. Note that the quantity $\frac{R_{\boldsymbol{a}}(N-1)\sum_{i=1}^{N} \|\boldsymbol{\alpha}_i\| a_i}{\|\boldsymbol{a}\|} + 1$ in the upper bound (4) above is precisely the upper bound for the Frobenius number g_0 obtained in Theorem 1.1 of [13].

Next we turn to lower bounds. Define the dimensional constant

$$C_N = \frac{2^{N^2 - \frac{7N}{2} + 2} (N-1)^{\frac{N}{2}} ((N-1)!)^{N-1}}{\pi^{\frac{N-2}{2}} \kappa_{N-1}^{N-2}}. \tag{6}$$

Theorem 2.2. *With the notation above,*

$$g_s(\boldsymbol{a}) \geq \left((s+1-N) \prod_{i=1}^{N} a_i \right)^{\frac{1}{N-1}}. \tag{7}$$

Now let $\rho > 1$ be a real number, and suppose that

$$s \geq \frac{\left(\prod_{i=1}^{N} a_i\right)^{N-2}}{(N-1)!} \left(\frac{C_N \lambda_{N-1}^{N-1}}{\rho - 1} \right)^{N-1}, \tag{8}$$

then

$$g_s(\boldsymbol{a}) \geq \left(\frac{s(N-1)!}{\rho} \prod_{i=1}^{N} a_i \right)^{\frac{1}{N-1}}. \tag{9}$$

Remark 2.2. Compare the lower bounds of (7) and (9) above to the lower bound on the Frobenius number obtained by Rødseth [19] (see also Theorem 1.1 of [1]):

$$g_0 \geq \left((N-1)! \prod_{i=1}^{N} a_i \right)^{\frac{1}{N-1}}. \tag{10}$$

In fact, Aliev and Gruber in [1] produced a sharp lower bound for g_0 in terms of the absolute inhomogeneous minimum of the standard simplex, from which a stronger version of (10) (with a strict inequality) follows. It should also be remarked that the quantities $R_{\boldsymbol{a}}$ and λ_{N-1}, present in our inequalities, can be explicitly bounded using standard techniques from the geometry of numbers. Notice that we can assume without loss of generality that no a_i can be expressed as a nonnegative integer linear combination of the rest of the a_j's: otherwise, $g_s(\boldsymbol{a}) = g_s(\boldsymbol{\alpha}_i)$. Then Eqs. (28) and (30) of [13] imply that

$$R_{\boldsymbol{a}} \leq \frac{N-1}{2} \lambda_{N-1} \leq \frac{(N-1)\lambda_{N-1}}{\lambda_1} \left(\frac{\|\boldsymbol{a}\|}{\kappa_{N-1}} \right)^{\frac{1}{N-1}} \leq \frac{(N-1)\|\boldsymbol{a}\|}{\kappa_{N-1}}, \tag{11}$$

while Eqs. (25) and (26) of [13] combined with Minkowski's successive minima theorem (see, for instance, [10], p. 203) imply that

$$2 \left(\frac{\|\boldsymbol{a}\|}{\kappa_{N-1}(N-1)!} \right)^{\frac{1}{N-1}} \leq \lambda_{N-1} \leq \frac{2\|\boldsymbol{a}\|}{\kappa_{N-1}}. \tag{12}$$

In fact, in the situation when the lattice $\Lambda_{\boldsymbol{a}}$ is *well-rounded* (abbreviated WR), meaning that $\lambda_1 = \cdots = \lambda_{N-1}$, inequalities (11) and (12) can clearly be improved:

$$R_{\boldsymbol{a}} \leq (N-1) \left(\frac{\|\boldsymbol{a}\|}{\kappa_{N-1}} \right)^{\frac{1}{N-1}}, \qquad 2 \left(\frac{\|\boldsymbol{a}\|}{\kappa_{N-1}(N-1)!} \right)^{\frac{1}{N-1}} \leq \lambda_{N-1} \leq \left(\frac{2\|\boldsymbol{a}\|}{\kappa_{N-1}} \right)^{\frac{1}{N-1}}, \tag{13}$$

when $\Lambda_{\boldsymbol{a}}$ is WR. The behavior of the Frobenius number $g_0(\boldsymbol{a})$ in this situation was separately studied in [13], where WR lattices were called ESM lattices, which stand for *equal successive minima*. Finally, the kissing number τ_{N-1} can be bounded as follows (see pp. 23–24 of [11]):

$$2^{0.2075\cdots(N-1)(1+o(1))} \leq \tau_{N-1} \leq 2^{0.401(N-1)(1+o(1))}. \tag{14}$$

We prove Theorems 2.1 and 2.2 in Section 4. In Section 3 we develop a lattice point counting mechanism, which is used to derive the lower bound of (9). We are now ready to proceed.

3. Counting lattice points in polytopes

In this section we present an estimate on the number of lattice points in polytopes, which, while also of independent interest, will be used in Section 4 to prove our main result. To start with, let $P \subset \mathbb{R}^N$ be a polytope of dimension $n \leq N$, i.e., $\dim \mathbb{V}(P) = n$ where $\mathbb{V}(P) := \operatorname{span}_{\mathbb{R}} P$, and let $L \subset \mathbb{V}(P)$ be a lattice of rank n. Define the counting function

$$\mathrm{G}(L, P) := |L \cap P| .$$

The Erhart theory studies the properties of $\mathrm{G}(L, tP)$ for $t \in \mathbb{Z}_{>0}$, which is a polynomial in t if P is a lattice polytope and a quasipolynomial in t if P is a rational polytope; very little is known in the irrational case (see for instance [7] for a detailed exposition of the Erhart theory). In fact, even in the case of a lattice or rational polytope the coefficients of the (quasi-)polynomial $\mathrm{G}(L, tP)$ are largely unknown, and hence for many actual applications estimates are needed. Here we record a convenient upper bound on $\mathrm{G}(L, P)$. The basic principle going back to Lipschitz (see p. 128 of [17]) used for such estimates states that when the n-dimensional volume $\mathrm{Vol}_n(P)$ is large comparing to $\det(L)$, then $\mathrm{G}(L, P)$ can be approximated by $\frac{\mathrm{Vol}_n(P)}{\det(L)}$, and so the problem comes down to estimating the error term of such approximations. An upper bound on this error term – not only for polytopes, but for a rather general class of compact domains – has been produced by Davenport [12] and then further refined by Thunder [23]. Here we present a variation of Thunder's bound in the case of polytopes.

Generalizing the notation of Section 1 to arbitrary lattices, let $\mathbb{B}_{\mathbb{V}(P)}(R)$ be a ball of radius R centered at the origin in $\mathbb{V}(P)$, and for each $1 \leq m \leq n$ define the mth successive minimum of L as in (3) above:

$$\lambda_m = \min\{\lambda \in \mathbb{R} : \dim \left(\operatorname{span}_{\mathbb{R}} \left(\mathbb{B}_{\mathbb{V}(P)}(\lambda) \cap L\right)\right) \geq m\}.$$

Also for each $1 \leq m \leq n$, let

$$V_m(P) := \max\{\mathrm{Vol}_m(F) : F \text{ is an } m\text{-dimensional face of } P\}. \tag{15}$$

With this notation at hand, the following estimate is an immediate implication of Theorem 4 of [23].

Lemma 3.1. *With notation as above,*

$$\mathrm{G}(L, P) \leq \frac{\mathrm{Vol}_n(P)}{\det(L)} + \sum_{m=0}^{n-1} \frac{2^{(n+1)m}\,(mn!)^m}{\kappa_m \kappa_n^m} \binom{n}{m} \frac{V_m(P)}{\lambda_1 \cdots \lambda_m},$$

where the product $\lambda_1 \cdots \lambda_m$ is interpreted as 1 when $m = 0$.

Remark 3.1. Notice that Lemma 3.1, and more generally the counting estimates discussed in Section 5 of [23], provide a mechanism for producing explicit polynomial bounds on the number of points of an arbitrary lattice in a variety of homogeneously expanding compact domains, which is especially easy to use in the case of polytopes (as we do in Section 4 for certain simplices). This observation gives a partial solution to Problem 3.2 of [4], previously formulated by the first author.

In the next section, we apply Lemma 3.1 to derive the lower bound of (9).

4. Bounds on $g_s(\boldsymbol{a})$

In this section we prove Theorems 2.1 and 2.2, deriving inequalities (4), (5), (7) and (9). For a positive integer t, consider the hyperplane $V_{\boldsymbol{a}}(t)$ in $\mathbb{R}^N$ defined by Eq. (1), which is a translate of $V_{\boldsymbol{a}}$, and write $\Lambda_{\boldsymbol{a}}(t) = V_{\boldsymbol{a}}(t) \cap \mathbb{Z}^N$. Fix a point $\boldsymbol{u}_t \in \Lambda_{\boldsymbol{a}}(t)$, and define a translation map $f_t : V_{\boldsymbol{a}} \to V_{\boldsymbol{a}}(t)$ given by $f_t(\boldsymbol{x}) = \boldsymbol{x} + \boldsymbol{u}_t$ for each $\boldsymbol{x} \in V_{\boldsymbol{a}}$. Then f_t is bijective and preserves distance; moreover, it maps $\Lambda_{\boldsymbol{a}}$ bijectively onto $\Lambda_{\boldsymbol{a}}(t)$. The intersection of $V_{\boldsymbol{a}}(t)$ with the positive orthant $\mathbb{R}^N_{\geq 0}$ is an $(N-1)$-dimensional simplex, call it $S(t)$. Then define

$$\mathrm{G}_{\Lambda_{\boldsymbol{a}}}(t) := |\Lambda_{\boldsymbol{a}}(t) \cap S(t)| = |\Lambda_{\boldsymbol{a}} \cap f_t^{-1}(S(t))|, \tag{16}$$

and notice that each point in $\Lambda_{\boldsymbol{a}}(t) \cap S(t)$ corresponds to a solution of (1) in nonnegative integers. Hence for every $t > g_s(\boldsymbol{a})$ we have $\mathrm{G}_{\Lambda_{\boldsymbol{a}}}(t) > s$. Moreover, $g_s(\boldsymbol{a})$ is precisely the smallest among all positive integers m such that for each integer $t > m$, $\mathrm{G}_{\Lambda_{\boldsymbol{a}}}(t) > s$. Therefore, in order to obtain bounds on $g_s(\boldsymbol{a})$, we want to produce estimates on $\mathrm{G}_{\Lambda_{\boldsymbol{a}}}(t)$, which is what we do next.

Combining (16) with bounds by Blichfeldt [9] (see also equation 3.2 of [15]) and by Gritzmann [14] (see also equation 3.3 of [15]), we have

$$\frac{\mathrm{Vol}_{N-1}(S(t)) - R_{\boldsymbol{a}} A_{N-1}(S(t))}{\det \Lambda_{\boldsymbol{a}}} \leq \mathrm{G}_{\Lambda_{\boldsymbol{a}}}(t) \leq \frac{\mathrm{Vol}_{N-1}(S(t))}{\det \Lambda_{\boldsymbol{a}}}(N-1)! + (N-1), \tag{17}$$

where $\mathrm{Vol}_{N-1}(S(t))$ is the volume and $A_{N-1}(S(t))$ is the surface area of $S(t)$, and $R_{\boldsymbol{a}}$ is the covering radius of $\Lambda_{\boldsymbol{a}}$ as defined in (2) above. Eqs. (17) and (18) of [13] state that

$$\mathrm{Vol}_{N-1}(S(t)) = \frac{t^{N-1}\|\boldsymbol{a}\|}{(N-1)!\prod_{i=1}^{N} a_i}, \qquad A_{N-1}(S(t)) = \frac{t^{N-2}\sum_{i=1}^{N}\|\boldsymbol{\alpha}_i\| a_i}{(N-2)!\prod_{i=1}^{N} a_i}. \tag{18}$$

In addition, by Eq. 25 of [13], $\det \Lambda_{\boldsymbol{a}} = \|\boldsymbol{a}\|$. Combining these observations with (17), we obtain

$$\mathrm{G}_{\Lambda_{\boldsymbol{a}}}(t) \geq \frac{t^{N-2}}{(N-2)!\prod_{i=1}^{N} a_i}\left(\frac{t}{N-1} - \frac{R_{\boldsymbol{a}}\sum_{i=1}^{N}\|\boldsymbol{\alpha}_i\| a_i}{\|\boldsymbol{a}\|}\right), \tag{19}$$

and

$$\mathrm{G}_{\Lambda_{\boldsymbol{a}}}(t) \leq \frac{t^{N-1}}{\prod_{i=1}^{N} a_i} + (N-1). \tag{20}$$

Notice however that Blichfeldt's upper bound of (17) is weaker than the bound of Lemma 3.1 for large t, hence our next goal is to produce an explicit upper bound on $\mathrm{G}_{\Lambda_{\boldsymbol{a}}}(t)$ from Lemma 3.1. Since each m-dimensional face of $S(t)$ is an m-dimensional simplex for each $0 \leq m \leq N-1$, Eq. (17) of [13] implies that

$$V_m(S(t)) \leq \frac{t^m\|\boldsymbol{a}\|}{m!}. \tag{21}$$

On the other hand, Minkowski's successive minima theorem implies that for each $1 \leq m \leq N-2$,

$$\lambda_1 \cdots \lambda_m \geq \frac{2^{N-1}\det \Lambda_{\boldsymbol{a}}}{(N-1)!\lambda_{m+1}\cdots\lambda_{N-1}} \geq \frac{2^{N-1}\|\boldsymbol{a}\|}{(N-1)!\lambda_{N-1}^{N-1-m}}. \tag{22}$$

Also notice that for all $1 \leq m \leq N-1$,

$$\frac{m^m}{\kappa_m m!} = \frac{m^m\Gamma\left(1+\frac{m}{2}\right)}{\pi^{m/2}m!} = \begin{cases} \dfrac{(2k)^{2k}k!}{\pi^k(2k)!} & \text{if } m = 2k \\[2ex] \dfrac{(2k+1)^{2k+1}}{\pi^k 2^{2k+1}k!} & \text{if } m = 2k+1 \end{cases} \leq \left(\frac{2m}{\pi}\right)^{m/2}, \tag{23}$$

where Γ stands for the Γ-function. Finally, $\binom{N-1}{m} \leq (N-1)\binom{N-2}{m}$. Define

$$C_N' = \frac{(N-1)(N-1)!}{2^{N-1}}. \tag{24}$$

Combining (21)–(23) with Lemma 3.1, we obtain

$$\begin{aligned}
\mathrm{G}_{\Lambda_{\boldsymbol{a}}}(t) &\le \frac{t^{N-1}}{(N-1)!\prod_{i=1}^{N} a_i} + C'_N\lambda_{N-1}\sum_{m=0}^{N-2}\binom{N-2}{m}\left(\frac{2^N(N-2)^{1/2}(N-1)!\,t}{\kappa_{N-1}\sqrt{2\pi}}\right)^m \lambda_{N-1}^{N-2-m} \\
&\le \frac{t^{N-1}}{(N-1)!\prod_{i=1}^{N} a_i} + C'_N\lambda_{N-1}\left(\frac{2^N(N-2)^{1/2}(N-1)!\,t}{\kappa_{N-1}\sqrt{2\pi}} + \lambda_{N-1}\right)^{N-2} \\
&\le \frac{t^{N-1}}{(N-1)!\prod_{i=1}^{N} a_i} + \frac{2^{N^2-\frac{7N}{2}+2}(N-1)^{\frac{N}{2}}((N-1)!\lambda_{N-1})^{N-1}t^{N-2}}{\pi^{\frac{N-2}{2}}\kappa_{N-1}^{N-2}}. \qquad (25)
\end{aligned}$$

Then for any $\rho > 1$,

$$\mathrm{G}_{\Lambda_{\boldsymbol{a}}}(t) \le \frac{\rho t^{N-1}}{(N-1)!\prod_{i=1}^{N} a_i}, \quad \text{when } t \ge \frac{C_N\lambda_{N-1}^{N-1}\prod_{i=1}^{N} a_i}{\rho-1}, \qquad (26)$$

where C_N is as in (6).

Remark 4.1. Similarly to the observations in Remark 2.2, inequality (22) can be improved in case $\Lambda_{\boldsymbol{a}}$ is WR. As a result in this case, inequalities (25) and (26) can also be made stronger.

A different technique can be used to produce a lower bound on $\mathrm{G}_{\Lambda_{\boldsymbol{a}}}(t)$ for small t. Notice that an open ball of radius $R_{\boldsymbol{a}}$ in $V_{\boldsymbol{a}}$ contains at least one point of $\Lambda_{\boldsymbol{a}}$, hence one can estimate the number of such balls in $S(t)$ to obtain a lower bound on $\mathrm{G}_{\Lambda_{\boldsymbol{a}}}(t)$. The kissing number τ_{N-1} is the maximal number of balls of radius $R_{\boldsymbol{a}}$ that can touch another ball of radius $R_{\boldsymbol{a}}$ without overlap, hence each ball of radius $3R_{\boldsymbol{a}}$ in $V_{\boldsymbol{a}}$ contains an arrangement of $\tau_{N-1}+1$ non-overlapping balls of radius $R_{\boldsymbol{a}}$. Now a standard isoperimetric identity (see, for instance, Eq. (1.3) of [8]) implies that the inradius $r(t)$ of the simplex $S(t)$ satisfies

$$r(t) = \frac{(N-1)\,\mathrm{Vol}_{N-1}(S(t))}{A_{N-1}(S(t))} = \frac{t\|\boldsymbol{a}\|}{\sum_{i=1}^{N}\|\boldsymbol{\alpha}_i\|a_i}, \qquad (27)$$

and so if $t \ge \frac{3R_{\boldsymbol{a}}\sum_{i=1}^{N}\|\boldsymbol{\alpha}_i\|a_i}{\|\boldsymbol{a}\|}$, then $S(t)$ contains a ball of radius $3R_{\boldsymbol{a}}$, and hence at least $\tau_{N-1}+1$ points of $\Lambda_{\boldsymbol{a}}$. In other words,

$$\mathrm{G}_{\Lambda_{\boldsymbol{a}}}(t) \ge \tau_{N-1}+1, \quad \text{when } t \ge \frac{3R_{\boldsymbol{a}}\sum_{i=1}^{N}\|\boldsymbol{\alpha}_i\|a_i}{\|\boldsymbol{a}\|}. \qquad (28)$$

Now, equipped with these inequalities on $\mathrm{G}_{\Lambda_{\boldsymbol{a}}}(t)$, we can easily derive the bounds of Theorems 2.1 and 2.2.

First notice that if we pick t greater than the maximal expression in the upper bound of (4), then (19) implies $\mathrm{G}_{\Lambda_{\boldsymbol{a}}}(t) > s$. In addition, (28) implies that for $s \le \tau_{N-1}+1$, $g_s(\boldsymbol{a})$ satisfies (5). As for lower bounds on $g_s(\boldsymbol{a})$, if we pick

$$t \le \left((s+1-N)\prod_{i=1}^{N} a_i\right)^{\frac{1}{N-1}},$$

then (20) implies $\mathrm{G}_{\Lambda_{\boldsymbol{a}}}(t) \le s$, and so produces the lower bound of (7). Finally, (26) implies that when s satisfies (8), $g_s(\boldsymbol{a})$ satisfies (9). This completes the proof of Theorems 2.1 and 2.2. □

Remark 4.2. For comparison purposes with (25), we mention another upper bound on $\mathrm{G}_{\Lambda_{\boldsymbol{a}}}(t)$, which is given by Eq. 3.3 of [15]:

$$\mathrm{G}_{\Lambda_{\boldsymbol{a}}}(t) \leq \frac{\mathrm{Vol}_{N-1}(S(t)+C(\Lambda_{\boldsymbol{a}}))}{\det \Lambda_{\boldsymbol{a}}} \leq \frac{\mathrm{Vol}_{N-1}(S(t)+\mathbb{B}_{N-1}(R_{\boldsymbol{a}}))}{\det \Lambda_{\boldsymbol{a}}}, \tag{29}$$

where

$$C(\Lambda_{\boldsymbol{a}}) := \{\boldsymbol{y} \in V_{\boldsymbol{a}} : \|\boldsymbol{y}\| \leq \|\boldsymbol{y}-\boldsymbol{x}\| \ \forall\, \boldsymbol{x} \in \Lambda_{\boldsymbol{a}}\} \tag{30}$$

is the Voronoi cell of the lattice $\Lambda_{\boldsymbol{a}}$. Now the right-hand side of (29) can be expanded using mixed volumes (see for instance [20]), i.e.:

$$\mathrm{Vol}_{N-1}(S(t)+\mathbb{B}_{N-1}(R_{\boldsymbol{a}})) = \sum_{m=0}^{N-1} \kappa_m R_{\boldsymbol{a}}^m \mathcal{V}_{N-m-1}(S(t)), \tag{31}$$

where $\mathcal{V}_k(S(t))$ denotes the kth mixed volume of $S(t)$. In particular,

$$\mathcal{V}_{N-1}(S(t)) = \mathrm{Vol}_{N-1}(S(t)), \qquad \mathcal{V}_{N-2}(S(t)) = \frac{1}{2}A_{N-1}(S(t)),$$

as given by (18), and $\mathcal{V}_0(K) = 1$. Then combining (29), (31) and (18), we obtain an upper bound on $\mathrm{G}_{\Lambda_{\boldsymbol{a}}}(t)$ in terms of the covering radius $R_{\boldsymbol{a}}$, analogous to the lower bound of (19):

$$\mathrm{G}_{\Lambda_{\boldsymbol{a}}}(t) \leq \frac{t^{N-1}}{(N-1)!\prod_{i=1}^N a_i} + \frac{t^{N-2}R_{\boldsymbol{a}}\sum_{i=1}^N \|\boldsymbol{\alpha}_i\| a_i}{(N-2)!\|\boldsymbol{a}\|\prod_{i=1}^N a_i} + \sum_{m=2}^{N-1} \frac{\kappa_m R_{\boldsymbol{a}}^m \mathcal{V}_{N-m-1}(S(t))}{\|\boldsymbol{a}\|}. \tag{32}$$

The bound of (32) is similar in spirit to that of (25), although the mixed volumes may generally be hard to compute. An expansion similar to (31) has recently been used by Henk and Wills to obtain a strengthening of Blichfeldt's upper bound as in (17), at least in the case of the integer lattice $\mathbb{Z}^N$ (see Theorem 1.1 and Conjecture 1.1 of [16]).

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