# ON TWO-VARIABLE GUARDED FRAGMENT LOGIC WITH EXPRESSIVE LOCAL PRESBURGER CONSTRAINTS

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ABSTRACT. We consider the extension of two-variable guarded fragment logic with local Presburger quantifiers. These are quantifiers that can express properties such as "the number of incoming blue edges plus twice the number of outgoing red edges is at most three times the number of incoming green edges" and captures various description logics up to  $\mathcal{ALCIHb}^{\text{self}}$ . We show that the satisfiability of this logic is EXP-complete. While the lower bound already holds for the standard two-variable guarded fragment logic, the upper bound is established by a novel, yet simple deterministic graph theoretic based algorithm.

## 1. Introduction

In this paper we consider the extension of two-variable guarded fragment logic with the so called local Presburger quantifiers, which we denote by  $\mathsf{GP}^2$ . These are quantifiers that can express local numerical properties such as "the number of outgoing red edges plus twice the number of incoming green edges is at most three times the number of outgoing blue edges." It was first considered in [BOPT21] and captures various description logics up to  $\mathcal{ALCIH}b^{\mathsf{self}}$  [BHLS17, BCM<sup>+</sup>03, Grä98]. Recently Bednarczyk, et. al. [BOPT21] showed that both satisfiability and finite satisfiability are in 3-NEXP by reduction to the two-variable logic with counting quantifiers.

We show that the satisfiability of this logic is EXP-complete. The lower bound is already known for the standard two-variable guarded-fragment logic [Grä99]. The main contribution is the upper bound, which is established by a novel, yet simple deterministic exponential time algorithm. Our approach is graph theoretic based and inspired by the recent work in [LLT21]. It comes with at least three advantages over the standard tableaux method.

Our algorithm has a markedly different flavor from the standard tableaux method usually used to establish the upper bound of guarded fragment logics. It works by exploiting the so called *tree-like model* property, where it tries to construct a tree-like model using polynomial space alternating Turing machine. To apply this, it is essential that there is only a polynomial bound on the number of branching from each node in the tree and it is not clear whether this bound still holds for  $\mathsf{GP}^2$ . In fact, already for  $\mathsf{GC}^2$  (two-variable guarded

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fragment with counting quantifiers), the exponential time upper bound is established by first reducing it to three-variable guarded fragment, before the tableaux method can be applied [Kaz04]. It is not clear a priori how this technique can be extended to  $\mathsf{GP}^2$ .

A recent work by Bednarczyk and Fiuk [BF22] shows that  $\mathsf{GP}^2$  still has the tree-like model property, thus, the tableaux method can be applied. Their proof uses automata theoretic method and yields a rather involved alternating algorithm. In contrast, our method yields a much simpler deterministic algorithm.

Other related work. The guarded fragment is one of the most prominent decidable fragments of first-order logic [ANvB98]. The satisfiability is 2-EXP-complete and drops to EXP-complete when the number of variables or the arity of the signature is fixed [Grä99]. Various DL and ML (description and modal logics) are captured by the fragment when the arity is fixed to two [Grä98, BCM<sup>+</sup>03, BHLS17]. The key reason for the decidability of the guarded fragment is the *tree-like model property* which allows the application of the tableaux method [Var96].

As mentioned earlier, our work is inspired by the approach taken in [LLT21], where the two-variable logic is abstracted as a simple graph theoretic problem called Conditional  $Independent\ Set$ . Among other results, they obtained a deterministic algorithm for the two-variable guarded fragment. However, it is not clear,  $a\ priori$ , how their algorithm can be extended to  $\mathsf{GP}^2$ .

Pratt-Hartmann [Pra07] proposed an elegant reduction of  $\mathsf{GC}^2$  formulas to (exponential size) "homogeneous" instances of Integer Linear Programming (ILP) which are of the form  $A\bar{x} = 0 \land B\bar{x} \geqslant \bar{c}$ , where A and B are matrices with integer entries,  $\bar{x}$  is a (column) vector of variables and  $\bar{c}$  is a vector of integers. Note that to check whether  $A\bar{x} = 0 \land B\bar{x} \geqslant \bar{c}$  admits a solution in  $\mathbb{N}$ , it is sufficient to check whether it admits a solution in  $\mathbb{Q}$ , which is known to be in Ptime. This implies the EXP upper bound for both satisfiability and finite satisfiability of  $\mathsf{GC}^2$ . Adding constant symbols makes the complexity jumps to NEXP-complete [Tob01].

Some logics that allow similar quantifiers as the local Presburger quantifiers were proposed and studied various researchers [Baa17, BBR20, DL10, KP10]. The decidability result is obtained by the tableaux method, but their logics do not allow the inverses of binary relations.

The extension of one-variable logic with quantifiers of the form  $\exists^S x \ \phi(x)$ , where S is a ultimately periodic set, is NP-complete [Bed20]. Semantically  $\exists^S x \ \phi(x)$  means the number of x where  $\phi(x)$  holds is in the set S. The extension of two-variable logic with such quantifiers is later shown to be 2-NEXP [BKT20] whose proof makes heavy use of the biregular graph method introduced in [KT15] to analyze the spectrum of two-variable logic with counting quantifiers. The proofs and results in [BKT20, KT15] do not apply in our setting since the logics they considered already subsume full two-variable logic.

**Organization.** This paper is organized as follows. In Section 2 we present the formal definition of  $GP^2$ . The main result is presented in Section 3. We conclude with Section 4.

## 2. Two-variable guarded fragment with local Presburger constraints $(GP^2)$

We fix a vocabulary  $\Sigma$  consisting of only unary and binary predicates and there is no constant symbol. As usual, for a vector  $\bar{x}$  of variables, we write  $\varphi(\bar{x})$  to denote that the free variables in the formula  $\varphi$  are exactly those in  $\bar{x}$ .

Let  $\mathcal{A}$  be a structure and a an element in  $\mathcal{A}$ . For a formula  $\varphi(x,y)$ , we denote by  $|\varphi(x,y)|_A^{x/a}$  the number of element b such that  $\mathcal{A}, x/a, y/b \models \varphi(x,y)$ .

Local Presburger (LP) quantifiers. These are quantifiers of the form:

$$\mathcal{P}(x) := \sum_{i=1}^{n} \lambda_i \cdot \#_y^{r_i} [\varphi_i(x, y)] \otimes \delta$$

where  $\lambda_i$ ,  $\delta$  are integers;  $r_i$  is either R(x,y) or R(y,x) for some binary relation R;  $\varphi_i(x,y)$  is a formula with free variables x and y; and  $\circledast$  is one of  $=, \neq, \leq, >, <, >, \equiv_d \text{ or } \not\equiv_d$ , where  $d \in \mathbb{N}$ . Here  $\equiv_d$  denotes the congruence modulo d. Note that  $\mathcal{P}(x)$  has one free variable x.

The semantics is defined as  $\mathcal{A}, x/a \models \mathcal{P}(x)$ , if the (in)equality  $\circledast$  holds when each  $\#_y^{r_i}[\varphi_i(x,y)]$  is substituted with  $|r_i(x,y) \wedge \varphi_i(x,y)|_{\mathcal{A}}^{x/a}$ .

The quantifier  $\mathcal{P}(x)$  is in *basic form*, if it is of the form:

$$\mathcal{P}(x) := \sum_{i=1}^{n} \lambda_i \cdot \#_y^{R_i(x,y)}[x \neq y] \quad \circledast \quad \delta$$

That is, each  $\varphi_i(x,y)$  is simply the inequality  $x \neq y$ . In other words, an LP quantifier in basic form is a linear constraint on the number of its outgoing  $R_i$ -edges.

The guarded fragment class. The class GF of guarded fragment logic is the smallest set of first-order formulas such that the following holds.

- GF contains all atomic formulas  $R(\bar{x})$  and equalities between variables.
- GF is closed under boolean combinations.
- If  $\varphi(\bar{x})$  is in GF,  $R(\bar{z})$  is an atomic predicate and  $\bar{x}, \bar{y} \subseteq \bar{z}$ , then both  $\exists \bar{y} R(\bar{z}) \land \varphi(\bar{x})$  and  $\forall \bar{y} R(\bar{z}) \rightarrow \varphi(\bar{x})$  are also in GF.

We define the class GP to be the extension of GF with LP quantifiers, i.e., by adding the following rule.

•  $\sum_{i=1}^n \lambda_i \cdot \#_y^{r_i}[\varphi_i(x,y)] \circledast \delta$  is in GP if and only if each  $\varphi_i(x,y)$  is in GP.

We denote by  $\mathsf{GP}^2$  the restriction of  $\mathsf{GP}$  to formulas using only two variables: x and y.

The normal form for  $\mathsf{GP}^2$ . Following a routine renaming technique (see, e.g., [Kaz04]) we can convert in linear time a  $\mathsf{GP}^2$  formula into the following equisatisfiable normal form (over an extended signature):

$$\Psi := \forall x \ \gamma(x) \quad \wedge \quad \bigwedge_{i=1}^{k} \forall x \forall y \ \alpha_i(x,y) \wedge \ \bigwedge_{i=1}^{\ell} \forall x \ \left(q_i(x) \to \mathcal{P}_i(x)\right)$$
 (2.1)

where

•  $\gamma(x)$  is quantifier free formula,

• each  $\alpha_i(x,y)$  is a quantifier free formula of the form:

$$(R(x,y) \land x \neq y) \rightarrow \beta(x,y)$$

where R(x,y) is an atomic predicate and  $\beta(x,y)$  is quantifier free formula,

- each  $q_i(x)$  is an atomic predicate,
- each  $\mathcal{P}_i(x)$  is an LP quantifier in basic form.

Remark 2.1. We stress that if the sentence  $\Psi$  in normal form (2.1) is satisfiable, then it is satisfiable by an infinite model. Indeed, let  $\mathcal{A}$  be a model of  $\Psi$ . We make an infinitely many copies of  $\mathcal{A}$ , denoted by  $\mathcal{A}_1, \mathcal{A}_2, \ldots$ , and disjoint union them all to obtain a new model  $\mathcal{B}$ . For every pair (a, b), where a and b do not come from the same  $\mathcal{A}_i$ , we set (a, b) not to be in any binary relation  $R^{\mathcal{B}}$ . It is routine to verify that  $\mathcal{B}$  still satisfies  $\Psi$ .

## 3. The satisfiability of $GP^2$

We introduce some terminology in Section 3.1. Then, we show how to abstract  $\mathsf{GP}^2$  formulas as graphs in Section 3.2. The algorithm is presented in Section 3.3. Throughout this section we fix a sentence  $\Psi$  in the normal form (2.1) over the signature  $\Sigma$ .

3.1. **Terminology.** A unary type (over  $\Sigma$ ) is a maximally consistent set of unary predicates from  $\Sigma$  or their negations, where each atom uses only one variable x. Similarly, a binary type is a maximally consistent set of binary predicates from  $\Sigma$  or their negations containing the atom  $x \neq y$ , where each atom or its negation uses two variables x and y. We denote by  $\overline{\eta}(x,y)$  the "reverse" of  $\eta(x,y)$ , i.e., the binary type obtained by swapping the variables x and y. The binary type that contains only the negations of atomic predicates is called the null type, denoted by  $\eta_{\text{null}}$ . Otherwise, it is called a non-null type.

Note that both unary and binary types can be viewed as quantifier-free formulae that are the conjunction of their elements. We will use the symbols  $\pi$  and  $\eta$  (possibly indexed) to denote unary and binary type, respectively. When viewed as formula, we write  $\pi(x)$  and  $\eta(x,y)$ , respectively. We write  $\pi(y)$  to denote formula  $\pi(x)$  with x being substituted with y. Let  $\Pi$  denote the set of all unary types over  $\Sigma$  and  $\mathcal K$  the set of all non-null binary types over  $\Sigma$ .

For a structure  $\mathcal{A}$ , the type of an element  $a \in A$  is the unique unary type  $\pi$  that a satisfies in  $\mathcal{A}$ . Similarly, the type of a pair  $(a,b) \in A \times A$ , where  $a \neq b$ , is the unique binary type that (a,b) satisfies in  $\mathcal{A}$ . The configuration of a pair (a,b) is the tuple  $(\pi,\eta,\pi')$  where  $\pi$  and  $\pi'$  are the types of a and b, respectively, and  $\eta$  is the type of (a,b). In this case we will also call (a,b) an  $\eta$ -edge in  $\mathcal{A}$ .

We say that a unary type  $\pi$  is realized in  $\mathcal{A}$ , if there is an element whose type is  $\pi$ . Likewise, a configuration  $(\pi, \eta, \pi')$  is realized in  $\mathcal{A}$ , if there is a pair (a, b) whose configuration is  $(\pi, \eta, \pi')$ .

<sup>&</sup>lt;sup>1</sup>In the standard definition of 2-type, such as in [GKV97, Pra05], a 2-type contains unary predicates or its negation involving variable x or y. For our purpose, we define a binary type as consisting of only binary predicates that strictly use both variables x and y. We view r(x, x) as a unary predicate.

The graph representation. In this paper the term graph means finite directed graph G = (V, E), where  $V \subseteq \Pi$  and  $E \subseteq \Pi \times \mathcal{K} \times \Pi$ . We can think of an edge  $(\pi, \eta, \pi')$  as an edge with label  $\eta$  that goes from vertex  $\pi$  to vertex  $\pi'$ . We write  $\mathcal{N}_G(\pi)$  to denote the set  $\{(\eta, \pi') | (\pi, \eta, \pi') \in E\}$ , i.e., the set of all the edges going out from  $\pi$ . A graph H = (V', E') is a subgraph of G = (V, E) if  $V' \subseteq V$  and  $E' \subseteq E$ . We call H a non-empty subgraph of G, if  $V' \neq \emptyset$ .

**Definition 3.1.** A structure  $\mathcal{A}$  conforms to a graph G, if the following holds.

- If a type  $\pi$  is realized in  $\mathcal{A}$ , then  $\pi$  is a vertex in G.
- If a configuration  $(\pi, \eta, \pi')$  is realized in  $\mathcal{A}$ , where  $\eta$  is not the null type, then  $(\pi, \eta, \pi')$  is an edge in G.

Next, we show how the sentence  $\Psi$  can be represented as a graph. We need the following two definitions.

**Definition 3.2.** A unary type  $\pi$  is compatible with  $\Psi$ , if  $\pi(x) \models \gamma(x)$ .

**Definition 3.3.** A configuration  $(\pi, \eta, \pi')$  is *compatible with*  $\Psi$ , if the following holds.

- Both  $\pi$  and  $\pi'$  are compatible with  $\Psi$ .
- For each  $1 \leq i \leq k$ :

$$\pi(x) \wedge \eta(x, y) \wedge \pi'(y) \models \alpha_i(x, y)$$
  
 $\pi(y) \wedge \overline{\eta}(x, y) \wedge \pi'(x) \models \alpha_i(x, y)$ 

Note that the sentence  $\Psi$  defines a directed graph  $G_{\Psi}$ , where the vertices are the unary types that are compatible with  $\Psi$  and the edges are  $(\pi, \eta, \pi')$ , for every  $(\pi, \eta, \pi')$  compatible with  $\Psi$ . Note that the graph  $G_{\Psi}$  is "symmetric" in the sense that  $(\pi, \eta, \pi')$  is an edge if and only if  $(\pi', \overline{\eta}, \pi)$  is an edge.

Intuitively, the graph  $G_{\Psi}$  contains the information about all the types/configuration that are allowed by the conjunct  $\forall x \ \gamma(x)$  and  $\forall x \forall y \ \alpha_i(x,y)$ . If there is a model  $\mathcal{A} \models \Psi$ , then it is necessary that  $\mathcal{A}$  conforms to the graph  $G_{\Psi}$ . In the next section, we will show how to analyze the graph  $G_{\Psi}$  to infer whether the conjunct  $\bigwedge_{i=1}^{\ell} \forall x \ q_i(x) \to \mathcal{P}_i(x)$  can also be satisfied.

3.2. The characterization for the satisfiability of  $\Psi$ . Recall that  $\Psi$  is a sentence in normal form (2.1). For each  $1 \leq i \leq \ell$ , let the LP quantifier  $\mathcal{P}_i(x)$  be:

$$\mathcal{P}_i(x) := \sum_{i=1}^{t_i} \lambda_{i,j} \cdot \#_y^{R_{i,j}(x,y)} [x \neq y] \quad \circledast_i \quad \delta_i$$

For a graph G, a vertex  $\pi$  in G and  $1 \leq i \leq \ell$ , we define the linear constraint  $\mathcal{Q}_i^{G,\pi}$ :

$$\mathcal{Q}_i^{G,\pi} := \sum_{j=1}^{t_i} \ \lambda_{i,j} \ \cdot \ \left( \sum_{\substack{(\eta',\pi') \in \mathcal{N}_G(\pi) \\ ext{and} \ R_{i,j}(x,y) \in \eta'}} z_{\eta',\pi'} 
ight) \ \circledast_i \ \delta_i$$

The variables in  $\mathcal{Q}_i^{G,\pi}$  are  $z_{\eta',\pi'}$ , for every  $(\eta',\pi')\in\mathcal{K}\times\Pi$ . Intuitively, each  $z_{\eta',\pi'}$  represents the number of  $\eta'$ -edges that goes out from an element with type  $\pi$  to another element with

type  $\pi'$ . This is the reason each  $\#_y^{R_{i,j}(x,y)}[x \neq y]$  is replaced with the sum:

$$\sum_{ \substack{ (\eta',\pi') \in \mathcal{N}_G(\pi) \\ \text{and } R_{i,j}(x,y) \in \ \eta' }} z_{\eta',\pi'}$$

We formalize this intuition as Lemma 3.4.

**Lemma 3.4.** Let G be a graph and  $\mathcal{A}$  be a structure that conforms to G. Suppose there is an element a in  $\mathcal{A}$  whose type is  $\pi$  and that  $\mathcal{A}, x/a \models \mathcal{P}_i(x)$ . Then,  $\mathcal{Q}_i^{G,\pi}$  admits a solution in  $\mathbb{N}$ .

*Proof.* Let G, A, a and  $\pi$  be as in the hypothesis of the lemma. For every  $(\eta', \pi') \in \mathcal{N}_G(\pi)$ , let  $K_{\eta',\pi'}$  be the number of elements c such that the configuration of (a,c) is  $(\pi,\eta',\pi')$ . Formally:

$$K_{\eta',\pi'} = |\{c : \text{ the configuration of } (a,c) \text{ is } (\pi,\eta',\pi')\}|$$

Since  $\mathcal{A}, x/a \models \mathcal{P}_i(x)$ , it is clear that  $z_{\eta',\pi'} = K_{\eta',\pi}$  is a solution to the constraint  $\mathcal{Q}_i^{G,\pi}$ .

Next, we define a system of linear constraints that captures whether a certain configuration can be realized.

**Definition 3.5.** For an edge  $(\pi_1, \eta, \pi_2)$  in a graph G, let  $\mathcal{Z}_{\pi_1, \eta, \pi_2}^G$  be the following system of linear constraints:

$$z_{\eta,\pi_2} \geqslant 1 \wedge \bigwedge_{i \text{ s.t. } q_i(x) \in \pi_1} \mathcal{Q}_i^{G,\pi_1}$$

Similar to  $\mathcal{Q}_i^{G,\pi}$ , the variables in this system are  $z_{\eta',\pi'}$ , for every  $(\eta',\pi') \in \mathcal{K} \times \Pi$ .

The intuitive meaning of the system  $\mathcal{Z}_{\pi_1,\eta,\pi_2}^G$  is as follows. If it does not admit a solution in  $\mathbb{N}$ , then the configuration  $(\pi,\eta,\pi_2)$  is not realized in any model of  $\Psi$ . This is because either  $z_{\eta,\pi_2}$  must be zero, or one of  $\mathcal{Q}_i^{G,\pi_1}$  is violated for some i where  $q_i(x) \in \pi_1$ . Its formalization is stated as Lemma 3.6.

**Lemma 3.6.** Let G be a graph and  $\mathcal{A}$  be a structure that conforms to G. Suppose there is a pair (a,b) in  $\mathcal{A}$  whose configuration is  $(\pi_1,\eta,\pi_2)$  and that  $\mathcal{A},x/a \models \bigwedge_{i=1}^{\ell} (q_i(x) \to \mathcal{P}_i(x))$ . Then, the system  $\mathcal{Z}_{\pi_1,\eta,\pi_2}^G$  has a solution in  $\mathbb{N}$ .

Proof. The proof is similar to Lemma 3.4. Let G, A, a, b and  $(\pi_1, \eta, \pi_2)$  be as in the hypothesis. For every  $(\eta', \pi') \in \mathcal{K} \times \Pi$ , let  $K_{\eta', \pi'}$  denote the number of elements c such that the configuration of (a, c) is  $(\pi_1, \eta', \pi')$ . Since the configuration of (a, b) is  $(\pi_1, \eta, \pi_2)$ , we have  $K_{\eta, \pi_2} \geq 1$ . Since A,  $x/a \models \bigwedge_{i=1}^{\ell} q_i(x) \to \mathcal{P}_i(x)$ , the assignment  $z_{\eta', \pi'} = K_{\eta', \pi'}$  is a solution to the system  $\mathcal{Z}_{\pi_1, \eta, \pi_2}^G$ .

To infer the satisfiability of  $\Psi$  from the graph  $G_{\Psi}$ , we will need a few more terminology.

**Definition 3.7.** An edge  $(\pi_1, \eta, \pi_2)$  is a *bad* edge in a graph G, if the system  $\mathcal{Z}_{\pi_1, \eta, \pi_2}^G$  does not admit a solution in  $\mathbb{N}$ .

Note that by Lemma 3.6, if  $(\pi_1, \eta, \pi_2)$  is a bad edge in G, then there is no model A that conforms to G such that the configuration  $(\pi_1, \eta, \pi_2)$  is realized in A and that  $A \models \forall x \ (q_i(x) \to \mathcal{P}_i(x))$  for every  $1 \leqslant i \leqslant \ell$ .

Next, we present a similar notion for vertices in G.

**Definition 3.8.** A vertex  $\pi$  is a bad vertex in G, if the following holds.

- It does not have any outgoing edge in G.
- There is  $1 \leq i \leq \ell$  such that  $\pi$  contains  $q_i(x)$ , but the constraint  $\mathcal{Q}_i^{G,\pi}$  does not admit a zero solution, i.e., the solution where all the variables are assigned with zero.

The intuitive meaning of bad vertex is as follows. Suppose  $\mathcal{A}$  is a structure that conforms to a graph G and that  $\pi$  has no outgoing edge in G. This means that for every element a with type  $\pi$  the value  $|R_{i,j}(x,y) \wedge x \neq y|_{\mathcal{A}}^{x/a}$  is zero. So, for every  $1 \leq i \leq \ell$ , if  $\pi$  contains  $q_i(x)$  and if  $\mathcal{A}, x/a \models \mathcal{P}_i(x)$ , then the following must hold.

$$\sum_{i=1}^{t_i} \lambda_{i,j} \cdot |R_{i,j}(x,y) \wedge x \neq y|_{\mathcal{A}}^{x/a} = 0$$

That  $\pi$  is a bad vertex means that  $\delta_i \circledast_i 0$  does not hold for some  $1 \leqslant i \leqslant \ell$ , where  $\pi$  contains  $q_i(x)$ .

We are now ready for the final terminology which will be crucial when deciding the satisfiability of  $\Psi$ .

**Definition 3.9.** Let G be a graph and H be a non-empty subgraph of G. We say that H is a *good* subgraph of G, if the following holds.

- There is no bad vertex and edge in H.
- It is symmetric in the sense that  $(\pi, \eta, \pi')$  is an edge in H if and only if  $(\pi', \overline{\eta}, \pi)$  is an edge.

Theorem 3.10 states that the satisfiability of  $\Psi$  is equivalent to whether the graph  $G_{\Psi}$  has a good subgraph.

**Theorem 3.10.** Let  $\Psi$  be a  $GP^2$  sentence in normal form (2.1). Then,  $\Psi$  is satisfiable if and only if the graph  $G_{\Psi}$  has a good subgraph.

*Proof.* (only if) Let  $\mathcal{A} \models \Psi$ . Let H be the graph where the vertices are the unary types realized in  $\mathcal{A}$  and the edges are the configurations realized in  $\mathcal{A}$ . Obviously, H is symmetric and a non-empty subgraph of  $G_{\Psi}$  and that  $\mathcal{A}$  conforms to H. It remains to show that there is no bad edge and vertex in H.

Let  $(\pi_1, \eta, \pi_2)$  be an edge in H. By definition, there is a pair (a, b) in  $\mathcal{A}$  whose configuration is  $(\pi_1, \eta, \pi_2)$ . Since  $\mathcal{A} \models \Psi$ , we have  $\mathcal{A}, x/a \models \bigwedge_{i=1}^{\ell} q_i(x) \to \mathcal{P}_i(x)$ . By Lemma 3.6, the system  $\mathcal{Z}_{\pi_1, \eta, \pi_2}^H$  admits a solution in  $\mathbb{N}$ , and hence, by definition,  $(\pi_1, \eta, \pi_2)$  is not a bad edge in H.

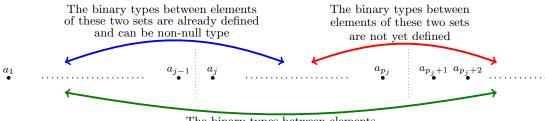
Now, we prove that H does not contain any bad vertex. Let  $\pi$  be a vertex in H without any outgoing edge. By definition, there is an element a in  $\mathcal{A}$  whose type is  $\pi$ . Since  $\mathcal{A}$  conforms to H, for every relation  $R_{i,j}(x,y)$ , we have:

$$|R_{i,j}(x,y) \wedge x \neq y|_A^{x/a} = 0$$

Since  $\mathcal{A}, x/a \models q_i(x) \to \mathcal{P}_i(x)$ , for every  $1 \leqslant i \leqslant \ell$ , if  $\pi$  contains  $q_i(x)$ , the constraint  $\mathcal{Q}_i^{H,\pi}$  admits the zero solution. Hence,  $\pi$  is not a bad vertex in H.

(if) Let H = (V, E) be a good subgraph of  $G_{\Psi}$ . We will show how to construct a model  $\mathcal{A} \models \Psi$  that conforms to H. For each type  $\pi \in V$ , we pick a set  $A_{\pi}$  of infinitely many elements where for every different  $\pi, \pi' \in V$ , the sets  $A_{\pi}$  and  $A_{\pi'}$  are disjoint.

The domain of  $\mathcal{A}$  is the set  $A = \bigcup_{\pi \in V} A_{\pi}$ . We will define the interpretation of each relation symbol by setting the types of each element  $a \in A$  and each pair  $(a,b) \in A \times A$ .



The binary types between elements of these two sets are already defined to be the null type

Figure 1: The relation between the sets  $\{a_1,\ldots,a_{j-1}\},\{a_j,\ldots,a_{p_j}\}$  and  $\{a_{p_j+1},a_{p_j+2},\ldots\}$ before the  $j^{\text{th}}$  iteration starts. The finite sets  $T_{\eta',\pi'}$ 's used to define the type of pairs involving  $a_j$  are all subsets  $\{a_{p_j+1}, a_{p_j+2}, \ldots\}$ .

For unary types, we set the type of each element in  $A_{\pi}$  to be  $\pi$  itself. The type of every pair  $(a,b) \in A \times A$  is defined such that the configuration of (a,b) is an edge in H, i.e., to ensure that  $\mathcal{A}$  conforms to H. Note also that a binary type  $\eta$  uniquely determine its "reverse"  $\overline{\eta}$ . So, when we define a binary type of (a, b), we also define the type of (b, a).

We first enumerate all the elements in A as  $a_1, a_2, \ldots$  The assignment of the binary types is done by iterating the following process starting from j=1 to  $j\to\infty$ . Each index j is associated with an integer  $p_j > j$  such that before the j<sup>th</sup> iteration starts, the following invariant is satisfied:

- (a) The binary types of pairs in  $\{a_1, \ldots, a_{j-1}\} \times \{a_j, \ldots, a_{p_j}\}$  are already defined.
- (b) The binary types of pairs in  $\{a_1,\ldots,a_{j-1}\}\times\{a_{p_j+1},a_{p_j+2},\ldots\}$  are already defined to be the null type.
- (c) The binary types of pairs in  $\{a_j, \ldots, a_{p_j}\} \times \{a_{p_j+1}, a_{p_j+2}, \ldots\}$  are *not* defined yet. (d) For every  $a \in \{a_j, \ldots, a_{p_j}\}$ , there is at most one  $b \in \{a_1, \ldots, a_{j-1}\}$  such that the binary type of (a, b) is not the null type.
- (e) For every  $a \in \{a_1, \ldots, a_{j-1}\}$ , the following already holds.

$$\mathcal{A}, x/a \models \bigwedge_{i=1}^{\ell} (q_i(x) \rightarrow \mathcal{P}_i(x))$$

Note that (e) makes sense since the binary type of every pair involving the elements in  $\{a_1,\ldots,a_{j-1}\}$  is already defined. Figure 1 illustrates the relation between the sets  ${a_1,\ldots,a_{j-1}}, {a_j,\ldots,a_{p_i}} \text{ and } {a_{p_i+1},\ldots}.$ 

In the  $j^{\text{th}}$  iteration, we set the binary types of each pair  $\{a_j\} \times \{a_{j+1}, a_{j+2}, \ldots\}$ . First, we set the binary type of each pair in  $\{a_j\} \times \{a_{j+1}, \ldots, a_{p_j}\}$  to be the null type. Next, we show how to set the binary type of the pairs in  $\{a_j\} \times \{a_{p_j+1}, a_{p_j+2}, \ldots\}$ . Let  $\pi_1$  be the type of  $a_j$ , i.e.,  $a_j \in A_{\pi_1}$ . There are two cases.

<u>Case 1</u>: There is h < j such that the type  $(a_i, a_h)$  is defined as a non-null type.

By (b), there is at most one such h. Let  $\eta$  be the binary type of  $(a_i, a_h)$  and  $\pi_2$  be the type of  $a_h$ . Since  $(\pi_1, \eta, \pi_2)$  is an edge in H, it is a good edge, i.e., the system  $\mathcal{Z}_{\pi_1, \eta, \pi_2}^H$ admits a solution in N. For every  $\eta', \pi'$ , let  $M_{\eta', \pi'}$  be the solution for the variable  $z_{\eta', \pi'}$ . We fix a set  $T_{\eta',\pi'} \subseteq A_{\pi'} \cap \{a_{p_j+1},a_{p_j+2},\ldots\}$  such that:

$$|T_{\eta',\pi'}| = \begin{cases} M_{\eta',\pi'} & \text{for } (\eta',\pi') \neq (\eta,\pi_2) \\ M_{\eta,\pi_2} - 1 \end{cases}$$

We also require that the sets  $T_{\eta',\pi'}$ 's to be pairwise disjoint, which is possible since each  $A_{\pi'}$  is an infinite set. Note that since  $(\pi_1, \eta, \pi_2)$  is a good edge, by definition,  $M_{\eta,\pi_2} \geqslant 1$ .

Now, for every  $\eta', \pi' \in \mathcal{K} \times \Pi$ , we set the type of every pair in  $\{a_j\} \times T_{\eta',\pi'}$  to be  $\eta'$ . For every element  $c \in \{a_{p_j+1}, a_{p_j+2}, \ldots\}$ , where c is not in any of  $T_{\eta',\pi'}$ , we set the type of  $(a_j, c)$  to be the null type. Finally, we set  $p_{j+1}$  to be the maximal index of the elements in any of the sets  $T_{\eta',\pi'}$ 's.

It is obvious that invariant (a)–(c) still holds after the  $j^{\text{th}}$  iteration. Invariant (d) holds, since the sets  $T_{\eta',\pi'}$ 's are pairwise dijoint. That invariant (e) holds, i.e.,  $\mathcal{A}, x/a_j \models \bigwedge_{i=1}^{\ell} (q_i(x) \to \mathcal{P}_i(x))$ , follows from the fact that  $z_{\eta',\pi'} = M_{\eta',\pi'}$  is a solution for the system  $\mathcal{Z}_{\pi_1,\eta,\pi_2}^H$ .

<u>Case 2</u>: For every h < j, the type  $(a_j, a_h)$  is defined as the null type.

In this case, we pick an arbitrary edge  $(\pi_1, \eta, \pi_2)$  in H. Since it is a good edge, we consider a solution  $z_{\eta',\pi'} = M_{\eta',\pi'}$  for the system  $\mathcal{Z}_{\pi_1,\eta,\pi_2}$ . Then, for every  $\eta',\pi'$ , we define the set  $T_{\eta',\pi'} \subseteq A_{\pi'}$  in similar manner as Case 1, except that  $|T_{\eta',\pi'}| = M_{\eta',\pi'}$ , for every  $\eta',\pi'$ . Setting the binary types of the pairs involving the element  $a_j$  is done exactly as in case 1. That the invariant (a)–(e) still holds also follows in a straightforward manner. This ends the construction for case 2.

Coming back to the proof of Theorem 3.10, as  $j \to \infty$ , the binary type of every pair is well defined and due to invariant (e), we have  $\mathcal{A} \models \bigwedge_{i=1}^{\ell} \forall x \ (q_i(x) \to \mathcal{P}_i(x))$ . Since  $\mathcal{A}$  conforms to H and H is a subgraph of  $G_{\Psi}$ , it follows that  $\mathcal{A} \models \Psi$ . This completes the proof of Theorem 3.10.

3.3. The algorithm. Theorem 3.10 tells us that to decide if  $\Psi$  is satisfiable, it suffices to find if  $G_{\Psi}$  contains a good subgraph. It can be done as follows. First, construct the graph  $G_{\Psi}$ . Then, repeatedly delete the bad edges and vertices from the graph  $G_{\Psi}$ . It stops when there is no more bad edge/vertex to delete. If the graph ends up not containing any vertices, then  $\Psi$  is not satisfiable. If the graph still contains some vertices, then it is a good subgraph of  $G_{\Psi}$  and  $\Psi$  is satisfiable. Its formal presentation can be found in Algorithm 3.11. It is worth noting that deleting a bad edge may yield a new bad edge, hence the while-loop.

## Algorithm 3.11.

**Input:** A sentence  $\Psi$  in normal form (2.1).

**Output:** Accept if and only if  $\Psi$  is satisfiable.

- 1:  $G := G_{\Psi}$ .
- 2: while G has a bad edge do
- 3: Delete the bad edge and its inverse from G.
- 4: Delete all bad vertices (if there is any) from G.
- 5: ACCEPT if and only if G is not an empty graph.

The complexity analysis of Algorithm 3.11. Let n and m be the number of unary and binary relation symbols in  $\Psi$ . Thus, there are  $2^{n+m}$  unary types and  $2^{2m}$  binary types. (Recall that we consider atomic R(x,x) as a unary predicate.) So,  $G_{\Psi}$  has at most  $2^{2n+4m}$  edges which bounds the the number of iterations of the while-loop in Algorithm 3.11.

Checking if an edge  $(\pi_1, \eta, \pi_2)$  is a bad edge means checking if the system  $\mathcal{Z}_{\pi_1, \eta, \pi_2}^G$  admits a solution in  $\mathbb{N}$ . Note that  $\mathcal{Z}_{\pi_1, \eta, \pi_2}^G$  contains at most  $\ell + 1$  constraints and  $2^{n+3m}$  variables. Invoking the result in [ES06, Pap81], the complexity of  $\mathcal{Z}_{\pi_1, \eta, \pi_2}^G$  is exponential in  $\ell$ , n, m and b, where b is the maximal number of bits needed to encode each constant. Thus, Algorithm 3.11 runs in time exponential in the length of  $\Psi$ .

Remark 3.12. We note that Algorithm 3.11 starts by constructing the whole graph  $G_{\Psi}$  and it is natural to ask whether the construction is avoidable. It is shown in [LLT21] that a  $\mathsf{GF}^2$  formula  $\Psi$  is a succinct representation (in the sense of [GW83]) of a PTIME-complete problem whose instance is exactly the graph  $G_{\Psi}$ . In general, it is not known how to design algorithms for any of succinctly encoded problems without essentially decompressing the succinct representations, i.e., converting them first into the standard representation [Wil19, Sect. 2.10]. We also note that when succinctly encoded, the complexity of most (if not all) problems jumps exponentially higher: NLog becomes PSPACE, PTIME becomes EXP and NP becomes NEXP [PY86]. Here it is useful to recall that  $\mathsf{GF}^2$  is already EXP-complete [Grä99]. So, in principle, the construction of  $G_{\Psi}$  in Algorithm 3.11 is indeed necessary and unavoidable for the worst case scenario complexity.

There was an attempt to construct algorithms for succinctly encoded problems without the decompression step [ASW09]. However, they are restricted to only AC<sup>0</sup> circuit representation, a rather small class of circuits and on relatively simple problems such as property testing with weak approximation guarantee.

### 4. Concluding remarks

In this paper we consider the extension of  $\mathsf{GF}^2$  with the local Presburger quantifiers which can express rich Presburger constraints while maintaining deterministic exponential time upper bound. It captures various natural DL up to  $\mathcal{ALCIH}b^{\mathsf{self}}$ . The proof is via a novel, yet simple and optimal algorithm. Note also that the proof of Theorem 3.10 relies on the infinity of the model. We leave a similar characterization for the finite models for future work. It is worth noting that the finite satisfiability of  $\mathsf{GP}^2$  has been shown to be decidable in 3-NEXP [BOPT21], though the precise complexity is still not known.

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<sup>&</sup>lt;sup>2</sup>More precisely, Corollary 5 in [ES06] states that if a system  $A\bar{x} = \bar{b}$  has a solution in  $\mathbb{N}$ , then it has a solution such that the number of variables taking non-zero values is at most  $2(d+1)(\log(d+1)+s+2)$ , where d is the number of rows of A and s is the largest size of a coefficient in A and b (in binary representation).

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