On the History of the Theory of Linear Differential Equations

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Communicated by A. P. YOUSHKEVITCH

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1. Introduction

I shall discuss the remarkable analogy between algebraic and linear differential equations, an analogy which largely determined the development of the theory of the latter throughout the 19th century.

This was a heroic period in the history of the theory of algebraic equations. The centuries-old development of the theory, which mainly determined the range of algebra as a whole, was in a sense completed by a series of outstanding achievements (by ABEL, GALOIS, and others). In any case, even in the second half of the 19th century, J. A. SERRET [1, p. 1] in introducing the subject of his well known *Cours d'algèbre supérieure*, stated that *L'algèbre est, proprement parler, l'Analyse des équations*.

By the beginning of our century such opinions had become obsolete. A new understanding of algebra as a science of algebraic structures (N. BOURBAKI) was taking shape and the theory of algebraic equations itself was little by little downgraded to a chapter in university algebra courses. In the 19th century, however, algebraic equations commanded general attention; mathematicians of the highest calibre worked on their theory and each result achieved in this field invariably led to new work both in the theory itself and in other fields that were ready to grasp and use similar ideas.

The theory of linear differential equations was just such a field. Still, to the best of my knowledge, historians of mathematics have not yet studied how the analogy between the two fields brought about an essential advance in the theory of linear differential equations. Note, however, that many contributions on certain isolated aspects of the subject (mainly connected with works of the last three decades of the 19th century, *i.e.*, with the work of L. FUCHS, S. LIE, H. POINCARÉ, and others) were indeed published.

My article is centred on developments that took place in the earlier part of the 19th century. They were devoted to the first steps toward and analogue for linear differential equations of the theory of the greatest common divisor and of elimination theory.

2. Lagrange's Theorem

2.1. Lagrange. In his letter to J. D'ALEMBERT of 26 January 1765, J.-L. LAGRANGE [2, pp. 30–31] communicated his discovery of a method of integrating the equation

$$L_n(y) = y^{(n)} + a_1(x)\, y^{(n-1)} + \dots + a_n(x)\, y = X(x) \tag{1}$$

when $n - 1$ particular solutions of the corresponding homogeneous equation

$$L_n(y) = 0 \tag{2}$$

were known.

LAGRANGE set out his method in the opening paragraphs of a lengthy memoir [3] published in 1766. What he did was to multiply both sides of equation (1) by an undefined function $z(x)$ of the variable x and to integrate by parts the relation thus obtained:

$$\int L_n(y)\, z\, dx = \int X(x)\, z\, dx.$$

This procedure leads to

$$A(x, y, y', \dots, y^{(n-1)}, z, z', \dots, z^{(n-1)}) + \int y L_n^*(z)\, dx = \int X(x)\, z\, dx \tag{3}$$

(I have not followed LAGRANGE's notation), where A is a bilinear function of y, z, and their derivatives up to and including those of order $n - 1$, and

$$L_n^*(z) \equiv (-1)^n z^{(n)} + (-1)^{n-1} (a_1(x)\, z)^{(n-1)} + \dots - (a_{n-1} z)' + a_n z.$$

Now supposing known a particular solution $z = z_1$ of the equation[1]

$$L_n^*(z) = 0 \tag{4}$$

[1] This was the first appearance of an equation subsequently called the adjoint of equation (2). The term itself (*adjungiert*) is due to L. FUCHS (*J. reine angew. Math.*, Bd. **76**, 1877, p. 183).

by substituting that solution into equality (3), LAGRANGE got an equation linear with respect to y and its derivatives:

$$A(x, y, y', \dots, y^{(n-1)}, z_1, z_1', \dots, z_1^{(n-1)}) = \int X(x)\, z_1\, dx,$$

one order lower than the initial equation (1). Thus the integration of this equation was reduced to solving an equation one order lower. Therefore, if p particular solutions

$$z = z_1, \quad z = z_2, \dots, \quad z = z_p \tag{5}$$

of equation (1) are known, substitution of them into equality (3) yields a system

$$A(x, y, y', \dots, y^{(n-1)}, z_i, z_i', \dots, z_i^{(n-1)}) = \int X(x)\, z_i\, dx, \qquad i = 1, 2, \dots, p.$$

Eliminating $y^{(n-1)}, \dots, y^{(n-p+1)}$ from the system one gets a linear equation of order $n-p$. Finally, if n solutions of equation (4) were known, it would be possible, by proceeding in a similar way, to obtain the general solution of equation (1).

By proving property $(L_n^*)^* = L_n$ LAGRANGE was able to obtain the general solution of equation (4) given n particular integrals $y = y_1, y = y_2, \dots, y = y_n$ of equation (2). Consequently, he also determined the n particular solutions (5) of equation (4) and got the desired general solution of equation (1).

Suppose, however, that only $n-1$ particular solutions $y = y_1, y = y_2, \dots, \dots, y = y_{n-1}$ of equation (2) are known. Then application of the procedure described leading to equation (4) will lead to equation

$$Vz + U\frac{dz}{dx} = W,$$

where V, U, and W are functions of x and of arbitrary constants $C_1, C_2, \dots, C_{n-1}$. Integration of this equation furnishes the general solution of equation (4) that in turn enables one to integrate equation (1) by means of LAGRANGE's method.

There is hardly any doubt that LAGRANGE realized that the particular solutions, *e.g.*, solutions (5), must be essentially different, but he did not say so, and even more, did not attempt to go into the essence of this difference. (In point of fact, the particular solutions should be linearly independent.) The possibility of integrating equation (1) by LAGRANGE's method depends, incidentally, on whether the solutions are different (linearly independent).

The method of lowering the order of linear differential equations by means of particular integrals of their adjoint equations, I would note, has subsequently become widely used [4].

2.2. D'Alembert. On 2 March 1765, D'ALEMBERT [2, pp. 33–34] informed LAGRANGE of his own method of integrating equation (1) when $n-1$ solutions of the corresponding equation (2) are known. He published this method in a paper [5] the title of which he agreed upon with LAGRANGE [2, p. 35].

Suppose a particular solution $y = y_1$ of equation (2) is known. Then, as D'ALEMBERT noticed, the substitution

$$y = y_1 \int z\, dx \tag{6}$$

reduces this equation to

$$\frac{1}{y_1} L_n(y_1 \int z\,dx) \equiv L_{n-1}(z) = z^{(n-1)} + b_1(x)\,z^{(n-2)} + \ldots + b_{n-2}(x)\,z' + b_{n-1}(x)\,z = 0, \tag{7}$$

an equation one order lower. (I have introduced my own notation.)

If now a second particular solution $y = y_2$ of equation (2) is known[2], then

$$z = z_1 \equiv \frac{d}{dx}\left(\frac{y_2}{y_1}\right)$$

will be a particular solution of equation (7). In this case substitution of

$$z = z_1 \int u\,dx \tag{8}$$

reduces it, in turn, to

$$\frac{1}{z_1} L_{n-1}(z_1 \int u\,dx) \equiv L_{n-2}(u)$$
$$\equiv \frac{d^{n-2}u}{dx^{n-2}} + c_1(x)\frac{d^{n-3}u}{dx^{n-3}} + \ldots + c_{n-3}(x)\frac{du}{dx} + c_{n-2}(x)\,u = 0, \tag{9}$$

an equation of order lower by one. Thus, if m $(m < n)$ [linearly independent] particular solutions of equation (2) are known, the integration of equation (1) is replaced by the integration of a similar equation of order $n - m$.

D'ALEMBERT next considered the non-homogeneous equation (1), supposing that $n - 1$ [linearly independent] particular solutions $y = y_1, y = y_2, \ldots, y = y_{n-1}$ of the corresponding homogeneous equation (2) are known. He applied the algorithm described above to lower the order of the differential equations by substitutions of type (6) and reduced this problem to the integration of a non-homogeneous equation of the first order. By means of a chain of equalities

$$y = y_1 \int z\,dx = y_1 \int [z_1 \int u\,dx]\,dx = \ldots \tag{10}$$

D'ALEMBERT managed to solve equation (1) when $n - 1$ particular solutions of the corresponding equation (2) were known. D'ALEMBERT'S method was simpler and more convenient than LAGRANGE'S; no wonder it is widely used today [4; 6].

2.3. Lagrange's Theorem. Thus both LAGRANGE and D'ALEMBERT concentrated on the integration of equation (1) when $n - 1$ particular solutions of the corresponding homogeneous equation (2) are known. In essence, each proved (by his own method) the following proposition: if p [linearly independent] particular solutions of equation (2) are known, its integration is reduced to the integration of a linear homogeneous equation of order $n - p$.

Neither LAGRANGE nor D'ALEMBERT precisely singled out this statement (which LIBRI called LAGRANGE'S theorem; see § 3) and, being concerned with a concrete problem, neither pointed out that knowledge of n particular solutions of equation (2) determines its coefficients. These two facts, both of which passed

[2] Like LAGRANGE, D'ALEMBERT did not point out that the solutions must be independent.

unnoticed, taken together suggested an analogy between linear differential equations and algebraic equations. Considering the latter, the following propositions correspond to LAGRANGE's theorem and to the second fact, respectively:

(1) Knowledge of p different roots of an algebraic equation of the n^{th} degree reduces its solution to determination of the roots of an equation of degree $n - p$.

(2) The coefficients of algebraic equations can be determined uniquely (to within a common factor) given their roots.

3. Libri's Memoir (1833)

G. LIBRI was the first to point out the analogy between linear differential equations and algebraic equations and to stress its importance. In a note published in 1836 he [7, p. 10] indicated:

Je crois avoir été le premier à appeler l'attention des géomètres sur les rapports qui existent entre les racines des équations algébriques et les intégrales particulières des équations différentielles linéaires ... . La similitude entre ces deux classes d'équations s'étend très loin, et permet de traiter les équations différentielles linéaires par des méthodes analogues à celles que l'on emploie dans la théorie des équations algébriques.

LIBRI's initial work on the subject is contained in the second part of his memoir [8] the beginning of which, as is apparent from its title, is devoted to the theory of algebraic equations. LIBRI submitted his memoir to the Paris Academy of Sciences in 1830, and he published it in 1833.

First LIBRI proved LAGRANGE's theorem by means of a chain (10) of substitutions of type (6). Notation apart, and without any reference to him, LIBRI's reasoning is the same as D'ALEMBERT's (see § 2.2), and he introduced both the substitution (6) and the proof of LAGRANGE's theorem as his own discoveries. There are, however, very good grounds for supposing (see § 7) that LIBRI, who was erudite in the mathematical literature, knew D'ALEMBERT's paper, which had escaped the notice of other mathematicians.[3] The part of LIBRI's memoir relevant to this matter was not confined to a restatement of D'ALEMBERT's work and to pointing out that the parallel between LAGRANGE's theorem and the corresponding proposition for algebraic equations can serve as the basis for a far-reaching analogy. LIBRI also took the next step in developing this analogy (even two steps; see below). Retracing D'ALEMBERT's line of reasoning, and using a chain of equalities (10), he derived an involved formula that expressed the general solution of equation (1) through n particular solutions.[4]

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[3] Only in 1837 did LIOUVILLE [9] point out that the credit for them belonged to D'ALEMBERT. Note [9] was the beginning of LIOUVILLE's attack on LIBRI; see § 7.

[4] D'ALEMBERT assumed that $n - 1$ particular solutions of equation (2) were known. LIBRI doubtless understood that particular solutions should, in a sense, "differ" one from another, but, like LAGRANGE and D'ALEMBERT, he did not say so. In 1836 LIBRI [7] once more outlined the proof of LAGRANGE's theorem and indicated that, for example, one must not choose $y_2 = y_1$.

Furthermore, applying the same algorithm, LIBRI obtained expressions for the coefficients of equation (2) given its [linearly independent] particular solutions

$$y = y_1, y = y_2, \ldots, y = y_n.$$

Indeed, substitution (6) transforms equation (2) into equation (7), whence it follows that

$$a_1 = -\frac{n}{y_1}\frac{dy_1}{dx} + b_1.$$

A similar substitution (8) transforms this equation into equation (9) and

$$b_1 = \frac{-(n-1)\dfrac{d}{dx}\left(\dfrac{d}{dx}\left(\dfrac{y_2}{y_1}\right)\right)}{\dfrac{d}{dx}\left(\dfrac{y_2}{y_1}\right)} + c_1.$$

Continuing this procedure step by step, LIBRI finally obtained an awkward expression for a_1:

$$a_1 = -\frac{n}{y_1}\frac{dy_1}{dx} - \frac{(n-1)\dfrac{d^2}{dx^2}\left(\dfrac{y_2}{y_1}\right)}{\dfrac{d}{dx}\left(\dfrac{y_2}{y_1}\right)} - \frac{(n-2)\dfrac{d^2}{dx^2}\left(d\left(\dfrac{y_3}{y_1}\right)\Big/ d\left(\dfrac{y_2}{y_1}\right)\right)}{\dfrac{d}{dx}\left(d\left(\dfrac{y_3}{y_1}\right)\Big/ d\left(\dfrac{y_2}{y_1}\right)\right)} - \ldots.$$

On pourrait, he noted [8, p. 191], *obtenir par une analyse semblable tous les coefficiens* $a_2, a_3, \ldots, a_n$, *en fonction des intégrales particulières* $y_1, y_2, \ldots, y_n$; *et en général étant donnés n* [linearly independent] *fonctions de x de la forme*

$$F_1(x), G_2(x), \ldots, F_n(x),$$

on peut déterminer les coefficiens de l'équation différentielle linéaire de l'ordre n qui aura pour intégrales particulières ces n fonctions; et il n'existera qu'une seule équation différentielle linéaire de l'ordre n qui satisfasse à cette condition, comme il n'y a qu'une équation algébrique d'un degré determiné, qui ait n racines données.

LIBRI also noted that the expressions for the coefficients $a_1. a_2, \ldots, a_n$ of equation (2) in terms of its particular solutions might be derived otherwise, *viz.* by solving the system of algebraic equations

$$\frac{d^n y_i}{dx^n} + a_1\frac{d^{n-1}y_i}{dx^{n-1}} + \ldots + a_n y_i = 0, \quad i = 1, 2, \ldots, n$$

with respect to these coefficients.

LIBRI did not elaborate this system. Analysis of it could have led him to the determinant that came to be called after HOENE-WRONSKI [10, p. 166], and to the notion of linear independence of the solutions of equation (2).

He dwelt in particular on the way the coefficients $a_1, a_2, \ldots, a_n$ of equation (2) depended on its particular solutions $y = y_1, y = y_2, \ldots, y = y_n$. He stressed that these coefficients, regarded as functions of y_i, $i = 1, 2, \ldots, n$, were a special case of symmetric functions: the coefficients persist not only under permutations of the y_i's, but even if the y_i's are replaced by one or another of their linear combinations. [More precisely: by combinations that constitute non-singular transformation.]

LIBRI remarked that the analogy with algebraic equations continues to hold in this case: just as the coefficients of algebraic equations are symmetric functions of their roots, so the coefficients of equation (2) are symmetric functions of their particular solutions. His reasoning was not altogether rigorous; he did not even hint that the particular solutions should, in a sense, be independent [or, rather, should constitute a fundamental system]. Then again, since he did not use determinants, both his intermediate formulas and the final result were extremely involved. In his memoir [8], however, he was the first to introduce formulas expressing the coefficients of the linear equation (2) in terms of its particular solutions and therefore the first to reconstruct the equation from these solutions. He also noted that the coefficients of equation (2) were symmetric functions of a certain [fundamental] system of its particular solutions. Finally, and this is the most important point, he persistently stressed and employed the analogy between linear differential equations and algebraic equations.

This analogy gave LIBRI the idea that *a priori* knowledge (if such exists) of the explicit relation between the particular solutions of equation (2) could be used to lower the order of the equation. His working out of this idea was rather fuzzy, and, if understood literally, could even appear to be wrong, but his reasoning was essentially correct. This is best appreciated by considering his own example. Considering the particular equation

$$\frac{d^2y}{dx^2} = y,$$

he assumed two of its solutions, $y = y_1$ and $y = y_2$, to be connected by the relation $y_1 = 1/y_2$. Then, as he noted, the function $y = y_1$ will also satisfy equation

$$\frac{d^2(1/y)}{dx^2} = \frac{1}{y}$$

or, which is the same thing, equation

$$\frac{d^2y}{dx^2} - \frac{2}{y}\frac{dx}{dy} + y = 0.$$

Eliminating d^2y/dx^2 from the two equations, LIBRI proved that the function y_1 would also satisfy an equation of the first order

$$\left(\frac{dy}{dx}\right)^2 = y^2 \qquad \text{or} \qquad \frac{dy}{dx} = \pm y.$$

LIBRI's remark contained only a hint of a certain unformed idea; still, FROBENIUS (see § 5 below) saw it as the starting point of a line of developments which led him, in 1873, to the theory of the reducibility of differential equations.

4. Libri's Theorem

In 1839 LIBRI submitted another memoir [11] to the Paris Academy of Sciences. Here I consider its subject-matter and, in § 7, I shall return to it in the context of LIBRI's confrontation with LIOUVILLE.

Memoir [11] contained the following proposition, which has come to be known as LIBRI's theorem: Consider two linear homogeneous differential equations

$$L_n(y) = 0 \tag{11}$$

and

$$L_m(y) = 0 \tag{12}$$

of orders n and m ($n > m$) respectively. If each solution of the second is a solution of the first, it is possible (without integration) to construct a linear equation

$$L_{n-m}(y) = 0 \tag{13}$$

of order $n - m$ whose solutions would be the $n - m$ solutions of equation (11) that do not satisfy equation (12). Thus integration of equation (11) is reduced to integrating equations (12) and (13).

LIBRI did not supply a proof of this proposition which is a generalization of LAGRANGE's theorem. He remarked [11, p. 735] only that he arrived at it by means of a certain expression

qui correspond dans la théorie des équations algébriques à la décomposition d'un polynome quelconque en facteurs du premier degré.

LIOUVILLE [12] offered a quite simple proof of LIBRI's theorem at the next sitting of the Academy; see § 7.

LIBRI's contemporaries were well acquainted with his achievements in the theory of linear differential equations. In 1844, in the second volume of his *Leçons* [13], l'Abbé F. N. M. MOIGNO had given a systematic account of LIBRI's findings. In Lesson 36 MOIGNO had paid special attention to the analogy indicated by LIBRI and to his theorem. MOIGNO seems to have been the first to point out that the n particular solutions $y = y_1, y = y_2, \ldots, y = y_n$ of equation (2) provide its general solution only if any values of $x_0, y_0, y_0', \ldots, y_0^{(n-1)}$ can be determined by means of the n arbitrary constants involved.

Apparently following CAUCHY, MOIGNO thus established the possibility of obtaining the solution of the CAUCHY problem with arbitrary initial conditions as a criterion of the generality of the solution sought.

Nowadays, as a result of the works of O. HESSE [14] and E. B. CHRISTOFFEL [15], published in 1857 and 1858, respectively, the conditions for linear independence of the solutions of equation (2) are written in a form that employs the appropriate WRONSKIan [10]. In 1866 L. FUCHS [16, p. 126] introduced the term *Fundamental-system.*

P. E. BRASSINNE (1805–1894), a professor at the Toulouse Artillery School and an author of works on differential and descriptive geometry, analysis and mechanics, expressed the coefficients of equation (2), and reconstructed the equation itself, in terms of its n [linearly] independent solutions by means of determinants using, in particular, the appropriate WRONSKIan. BRASSINNE's contribution to the subject makes up one of the appendices to the second volume of J. C. F. STURM's *Cours d'analyse* [17] (1864, 2nd ed.).

5. Achievements Due to Brassinne

BRASSINNE'S work contained the following generalization of LIBRI'S theorem [17, pp. 331–332]:

Si des équations différentielles linéaires, d'ordre $m, m', m'', \ldots,$ ont p solutions communes, on trouve, par un procédé analogue à la recherche du commun diviseur algébrique, l'équation $X_p = 0$ qui donne ces solutions, et l'intégration des proposées est ramenée à celle de $X_p = 0$ et à celle d'autres équations d'ordre $m - p, m' - p, m'' - p \ldots$.

To prove this proposition he employed an algorithm similar to the EUCLIDean one that is used to determine the greatest common divisor of two polynomials[5].

[5] Thus, for example, a linear homogeneous equation whose solutions are the coinciding integrals of two given equations,

$$L_{m+p}(y) = 0 \qquad (*) \qquad \text{and} \qquad L_m(y) = 0 \qquad (**)$$

can be reconstructed in the following way. Write out a chain of equalities

$$L_{m+p}(y) = \frac{d^p}{dx^p}(L_m(y)) + L_{m+p-1}(y),$$

$$L_{m+p-1}(y) = K\frac{d^{p-1}}{dx^{p-1}}(L_m(y)) + L_{m+p-2}(y),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$L_{m+1}(y) = M\frac{d}{dx}(L_m(y)) + L'_m(y)$$

where $K, \ldots, M$ are functions of x. If

$$L'_m(y) = NL_m(y) \qquad (***)$$

where N is a function of the same argument, then all the solutions of equation (**) are at the same time solutions of equation (*). If, however, equality (***) does not hold, construct another chain

$$L_{m+1}(y) = M\frac{d}{dx}(L_m(y)) + PL_m(y) + L_{m-1}(y),$$

$$L_m(y) = Q\frac{d}{dx}(L_{m-1}(y)) + RL_{m-1}(y) + L_{m-2}(y),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

where $P, Q, R, \ldots$ are functions of x. Either, for some k,

$$L_{k+1}(y) = S\frac{d}{dx}(L_k(y)) + S'L_k(y)$$

where S and S' are functions of the same argument, so that

$$L_k(y) = 0$$

will be the equation sought, or, finally,

$$L_2(y) = S^{(k-1)}(x)\frac{d}{dx}(L_1(y)) + y\varphi(x)$$

with a non-zero remainder $y\varphi(x)$. In this case equations (*) and (**) have no common solutions at all (except solution $y \equiv 0$).

BRASSINNE was guided exclusively by the analogy with the theory of algebraic equations, or, rather, with the theory of the greatest common divisor and the elimination theory for these equations, and he repeatedly stressed this fact. He obtained a whole number of important propositions for linear differential equations. I shall now describe some of these theorems.

1. If two equations, $L_m(y) = 0$ and $L_p(y) = 0$, are given, it is possible to construct a third equation, $L_{m+p}(y) = 0$, the set of whose solutions unites those of the first two equations. Now consider equation

$$L_n(y) \equiv y^{(n)} + a_1(x)\, y^{(n-1)} + \dots + a_n(x)\, y = 0. \tag{2}$$

By substituting $y = uv$ (if $v = y_1$ this expression coincides with D'ALEMBERT'S substitution (6)) BRASSINNE got

$$L_n(v)\, u + Y_{n-1}(v) \frac{du}{dx} + Y_{n-2}(v) \frac{1}{1 \cdot 2} \frac{d^2u}{dx^2} + \dots$$
$$+ \left(n \frac{dv}{dx} + a_1(x)\, v \right) \frac{d^{n-1}u}{dx^{n-1}} + v \frac{d^n u}{dx^n} = 0$$

where $Y_{n-1}(v)$, $Y_{n-2}(v)$, ... are linear differential operators of v of order $n - 1$, $n - 2, \dots$.

2. If

$$L_n(y_1) \equiv Y_{n-1}(y_1) \equiv \dots \equiv Y_{n-p+1}(y_1) \equiv 0$$

then equation (2) will have p solutions

$$y = y_1,\ \ y = xy_1,\ \ y = x^2 y_1, \dots, y = x^{p-1} y_1 .$$

Conversely, if this equation has p such solutions, then the solutions of equation

$$Y_{n-1}(y) = 0,\ \ Y_{n-2}(y) = 0,\ \ etc.$$

are

$$y = y_1,\ \ y = xy_1, \dots, y = x^{p-2} y_1,$$
$$y = y_1,\ \ y = xy_1, \dots, y = x^{p-3} y_1 \quad etc.$$

3. By means of this theorem BRASSINNE proved that if equation (2) has p solutions

$$y = y_1, y = y_2, \dots, y = y_p,$$

$2q$ solutions

$$y = z_1, y = xz_1, y = z_2, y = xz_2, \dots, y = z_q, y = xz_q,$$

and $3r$ solutions

$$y = u_1, y = xu_1, y = x^2 u_1, \dots, y = u_r, y = xu_r, y = x^2 u_r$$

and

$$n = p + 2q + 3r,$$

then its integration is reduced to solution of three linear equations of the order p, q, and r, respectively.

The following is a corollary of the same theorem: if equation (2) has two particular solutions, $y = y_1$ and $y = y_2$, connected by the dependence $y_1 = xy_2$, then there is a linear differential equation of a lower order whose solutions coincide with those of equation (2). This equation, seemingly, will be $X_{n-1}(y) = 0$.

This corollary is also essentially contained in BRASSINNE'S contribution. G. F. FROBENIUS [18, p. 268] put it in a formal way. He called equation (2), whose coefficients are analytic functions uniquely determined in a certain part of the plane, *irreductible*; it has no solutions in common with any differential equation of a lower order whose coefficients satisfy the same condition in the given part of the plane.

Thus, according to FROBENIUS, BRASSINNE proved the reducibility of equation (2) provided it had solutions $y = y_1$ and $y = y_2$ connected by the relation $y_1 = xy_2$. FROBENIUS himself discovered more general conditions for the reducibility of this equation.[6] He referred not only to BRASSINNE, but also to LIBRI, who was the first to notice that the relation between particular solutions of equation (2) could be used to lower its order; see § 3.

6. Conclusion

In mentioning FROBENIUS I have gone outside the period under consideration and into the 1870's, years marked by outstanding discoveries in analysis in general, and in development of the theory of linear differential equations in particular. The analogy of these equations with algebraic ones played an important part in the process. This is how, in 1881, P. APPELL [19, pp. 391–392] described the situation in the opening pages of one of his articles:

La théorie du plus grand commun diviseur de deux polynômes et celle de l'élimination ont conduit MM. Libri, Liouville,[7] *Brassinne à des théories analogues sur les équations différentielles linéaires; et ces questions ont été récemment reprises et complétées par MM. Thomé et Frobenius* (*J. de Crelle* [*J. reine angew. Math.*] *t. 74* ... [and other works]); *M. Frobenius a introduit la notion de l'irréductibilité des équations différentielles linéaires* (*J. de Crelle, t. 76*) [see my § 5] *et a démontré à ce sujet plusieurs théorèmes importants suggérés, sans doute, par les théorèmes analogues de la théorie des équations algébriques. La décomposition des polynômes en facteurs a été l'origine de la théorie de la décomposition du premier membre d'une équation différentielle linéaire en facteurs premiers symboliques* (*voir Floquet, Annales de l'École Normale* ... [(2), *t. 8*, année 1879. *Supplément*]. *Le Mémoire*

[6] For the case of a homogeneous linear equation (2) whose coefficients are single-valued analytic functions, FROBENIUS [18, pp. 268–269] proved that if one of its non-coincident integrals $y = y_1$ and $y = y_2$ is connected with the other by a relation of type

$$y_1 = q_0 \frac{d^k y_2}{dx^k} + q_1 \frac{d^{k-1} y_2}{dx^{k-1}} + \dots + q_k y_2$$

where q_i are single-valued analytic functions of x, this equation is reducible.

[7] LIBRI'S theorem thus mockingly threw the enemies together; see § 7.

fondamental de M. Fuchs (*J. de Crelle, t.* 66 [1866]), *qui depuis a été exposé et complété par M. Tannery* (*Annales de l'École Normale* ... [(2), *t. 3*], *année 1874*), *et qui a pour objet l'étude des fonctions définies par une équation différentielle linéaire, présent plus d'une analogie avec le Mémoire célèbre de M. Puiseux* "*Sur les fonctions algébriques*" (*J. Math. pures et appliquées t. 15* [1850]); *et cette analogie a été poussée à un point inattendu dans un Mémoire récent de M. Fuchs* (*C. r.* [*Acad. sci. Paris*], *t. 90, pp. 678 et 735, et J. de Crelle, t. 89* [1880] "*Sur une classe des fonctions de plusieurs variables tirées de l'inversion des intégrales des solutions des équations différentielles linéaires dont les coefficients sont des fonctions rationelles*"). *Enfin, dans un autre ordre d'idées, la théorie des invariants des formes algébriques a été étendue aux équations différentielles linéaires dans deux Notes présentées par M. Laguerre à l'Académie des Sciences* (*C. r., t. 88, pp. 116 et 224* [1879]).

I would add that the 19th century also witnessed a remarkable analogy between the solubility of algebraic equations in radicals and the problem of integrating linear differential equations by quadratures. Thus LIOUVILLE's theorem (1840) stating that RICCATI's equation[8] could not be solved by quadratures corresponded to ABEL's proof that the algebraic equations of the fifth degree could not be solved by expressions in radicals; and the theory of E. PICARD and E. VESSIOT, which is based on S. LIE's work, primarily on his theory of continuous groups, was analogous to the GALOIS theory.[9]

The fruitful impact of algebra on development of the theory of linear differential equations was not at all one-sided: many ideas that took shape in connection with problems in this theory subsequently proved important elements in the rise of several branches of modern algebra (LIE groups and algebras, differential algebra, *etc.*).

In the opinion of N. BOURBAKI [20, p. 81] the study of linear differential equations

contribuât à mettre en valeur la linéarité et ce qui s'y rattache

and thus contributed to the formation of linear algebra. Inquiry into the interaction of algebraic and linear differential equations is a topical problem of the history of mathematics. Conclusions from it may be important for understanding the interdependence of algebra and analysis. In the course of time this interdependence has manifested itself in various ways; it remains a salient feature of modern mathematics.

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[8] RICCATI's equation can easily be reduced to a linear equation. Mathematicians revealed the correspondence only after the appearance of the works of S. LIE who attempted to construct an analog of the GALOIS theory for differential equations. Up to 1840, the year when LIOUVILLE published his own findings on the subject, GALOIS's work had been all but unknown. GALOIS's main writings appeared in print in 1846, in vol. **11** of LIOUVILLE's journal.

[9] PICARD and VESSIOT's theory (*Ann. École Norm.* (2), t. **10**, 1881, pp. 391–424) is also essentially based on APPELL's memoir [19], quoted above.

7. Appendix: Notes on Libri's Life

Count GUGLIELMO BRUTUS ICILIUS TIMOLÉON LIBRI-CARUCCI DALLA SOMMAIA was born in 1803 in Florence. He descended from an ancient Tuscan family. Even in his early youth he showed a marked aptitude for the sciences, mathematical sciences in particular, and at twenty he held a chair at the University of Pisa. The scientific community came to know LIBRI's memoirs on the theory of numbers, DIOPHANTINE equations, and mathematical analysis, which he published in Italy, mainly as separate booklets.[10]

In 1830 LIBRI became involved in politics and had to emigrate to France. Soon, because of his writings, his brilliant erudition in the history of the mathematical sciences (see below), and his ability to make a good impression and even to be fascinating, he easily gained first-rate standing in Paris. In 1832 he received a chair at the *Collège de France*; and in 1833 the Paris Academy of Sciences elected him as a member in place of the deceased A. M. LEGENDRE. Subsequently LIBRI became a professor of the Sorbonne and *Inspecteur général de l'instruction publique*.

Parisians only gradually discovered the dark side of LIBRI's personality: his inclination for intrigues, unscrupulousness in money matters, morbid vanity, and, finally, his excessive cupidity. Gradually more and more Parisian scholars, members of the Academy included, became hostile to him; among them was F. ARAGO who had initially been extremely well disposed toward him.

For all that, few dared to engage him in open battle. He maneuvered skillfully at the Academy, taking advantage of the support given him by several of its influential members,[11] and also of his vast connections with the government and at court.[12]

Still, the open opposition to LIBRI managed to rise. Its leader was the then young J. LIOUVILLE, an ardent man, who was not afraid of confrontations, and was bent on breaking LIBRI's influence. He decided to strike at LIBRI's reputation as a mathematician. Indeed, he chose his stand happily. He was one of the major analysts of the 19th century, and he easily detected mistakes in LIBRI's work, for LIBRI was a mediocre analyst, and his conclusions and proofs contained many incomplete and insufficiently considered passages.

LIOUVILLE began his attack in 1837 with the publication of a note [9] in his journal. Here he pointed out that both substitution (6) and the ensuing proof of

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[10] In the 1830's CRELLE reprinted many of these memoirs in his journal (*J. reine angew. Math.*).

[11] Including the aged S. F. LACROIX. Discussing the situation at the Sorbonne where STURM and LIBRI were the main claimants for the vacant chair of rational mechanics, LIOUVILLE mentioned LACROIX in a letter to P. G. LEJEUNE DIRICHLET dated 7 July 1840 [21, p. 60]:

> *Je n'ai pas besoin de vous dire que M. Lacroix est tout entier pour Libri. Faut-il qu'après tant de services honorables, un homme digne de tous nos respects termine sa carrière en devenant l'esclave et le jouet d'un vil intrigant?*

[12] He was on friendly terms with FRANÇOIS GUIZOT, a leading statesman and an eminent historian.

LAGRANGE'S theorem that LIBRI had claimed to be his own[13] were D'ALEMBERT'S (see § 2.2). LIOUVILLE spitefully concluded (p. 246):

J'ignore comment ce passage a pu échapper à M. Libri qui s'est occupé si longtemps de l'histoire des sciences mathématiques.

It is very improbable that the passage in question had escaped the sharp eye of Count LIBRI, the more so because the third volume of the *Miscellanea Taurinensia* was very likely well known to him: LAGRANGE'S memoir [3] (see § 2.1 above) was included in that volume, directly before an extract from D'ALEMBERT'S letter [5].

In the next (third) volume of his journal, which appeared in 1838, LIOUVILLE criticized LIBRI'S old memoir on the theory of heat[14], which he had submitted to the *Institut de France* in 1825, and on which FOURIER had commented approvingly.[15]

LIBRI delayed his reply, waiting for a favorable opportunity. A chance came in the spring of 1839, when LIOUVILLE stood for election to a vacancy in the astronomy section of the Academy of Sciences. On May 13 LIBRI submitted a memoir [11] to the Academy smearing LIOUVILLE (see § 4). Attack is the best defence and his concrete objective was to block LIOUVILLE'S election. LIBRI first had to strengthen his own position, his reputation as a mathematician having been shaken by LIOUVILLE, and, second, to throw doubt on his adversary's competence, so sparing himself the dire need to answer LIOUVILLE'S criticisms.

LIBRI had no alternative. His works did contain errors that he was unable to correct, in particular since by then he had given up contemporary science and was mostly engaged in the history of mathematics. Nevertheless, when he found an answer, he did not neglect to use it [11, p. 733]:

J'avouerai qu'il est parfaitement vrai que j'ignorais ce passage: mais je ferai remarquer que ni M. Lacroix, dont l'érudition est si vaste et si profonde, ni M. Liouville lui-même, lorsqu'il publiait ma Note, n'avait aucune connaissance de ce fragment de d'Alembert .... L'histoire de la science est si vaste qu'il est fort difficile de tout savoir: M. Liouville l'a prouvé lui-même à propos de certains théorèmes d'Abel ....[16]

LIBRI began his memoir [11] by repeating the main findings of his work [8] of 1833 specially stressing his priority in discovering and developing the analogy

[13] LACROIX supported LIBRI'S claim. In the fifth edition of his *Traité* [22, *Corrections et additions*] he wrote:

M. Libri ... a repris d'une manière très élégante et très féconde, la théorie des équations différentielles linéaires.

[14] ... *J'ai reconnu,* LIOUVILLE contended [23, p. 351], *que les formules données par M. Libri sont inexactes et que le principe général sur lequel il s'appuie est inadmissible ....*

[15] Quoting FOURIER, LIBRI [11, p. 740] wrote:

... la méthode qu'il a suivie, et les résultats auxquels il est parvenu, méritent toute l'attention des géomètres.

[16] Right on target!

between linear differential and algebraic equations. LIBRI no longer appreciated either the substitution (6) or the ensuing proof of LAGRANGE's theorem, on which he had put such a high value some three years earlier, since they had proved to be D'ALEMBERT's discoveries. But he did not scruple to claim that the whole progress in the general theory of linear differential equations beginning from the appearance of LAGRANGE's memoir [3] to the present day (the 1830's) boiled down to the findings contained in his own contribution [8]:

> *... je n'ai pas oublié,* LIBRI noted [11, p. 736], *que M. Liouville, professeur à l'École Polytechnique, s'est occupé à plusieurs reprises de questions analogues.*

However, LIBRI continued, LIOUVILLE's achievements were based on an *inadmissible* classification of elementary transcendental functions.[17] Thus, soiling the latter's reputation as a mathematician, LIBRI asserted (p. 738):

> *je n'avais pas dû mentionner les recherches de M. Liouville parmi celles qui ont pu contribuer aux progrès de la théorie générale des équations différentielles linéaires.*

Then came the decisive move (on p. 734):

> *Cependant, je n'ai jamais cessé de m'occuper de ce genre de recherches dans l'espoir de parvenir enfin à donner une théorie complète des équations différentielles linéaires.*

And, further,

> *Il serait impossible d'énumérer ici tous les théorèmes qui se trouvent démontrés dans le Mémoire*[18] *... je me bornerai à citer celui qui sert de base à mes recherches* ... .

LIBRI then formulated the theorem that is called after him (see § 4). As to *all these theorems,* they either did not exist at all, or, at best, were LIBRI's intuitive assumptions. At any rate, nobody had ever seen them.

Having played his trump, LIBRI cited FOURIER's review of his contribution on the theory of heat (see note 15) and ended his memoir in the tone of outraged innocence: I am overburdened with my research on the history of mathematics, but I have not abandoned mathematics proper, a fact that my confreres will now have been able to convince themselves of; and I simply have no time to defend myself in more detail against the unfounded swipes of a young professor whose own results do not inspire special confidence.

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[17] In 1833, a special commission of the Paris Academy of Sciences (S. D. POISSON, LACROIX, and LIBRI himself) had rejected LIOUVILLE's classification. Still, in 1840, using it as a basis, LIOUVILLE demonstrated it impossible for RICCATI's equation to be solved in general by quadrature.

[18] He refers to a memoir he claimed to have written and was about to publish.

Rien ne presse donc, he wrote [10, pp. 740–741], *et je crois pouvoir continuer à me livrer à mes travaux habituels: la réponse arrivera toujours à temps.*

Yes, LIBRI made his move *in time*. Although LIOUVILLE was already widely known as an outstanding analyst, the venom could well influence the views of the members of the Academy (primarily, of course, non-mathematicians) and prevent his election. LIOUVILLE had to find the correct tactical response.

At the very next meeting of the Academy LIOUVILLE first offered an extremely simple proof [12] of 'LIBRI's theorem'.[19] He then went on to answer LIBRI's criticisms, noting that [25, p. 792]

... le théorème que M. Libri donne comme la base de son travail, et qu'il déduit, dit-il, de ses nouveaux principes, peut être établi directement de la manière la plus simple, sans aucune théorie préliminaire.

At the same time, as he justly maintained, the memoir [11] contained nothing that was not included in LIBRI's former writing [8]. In this way LIOUVILLE rejected LIBRI's criticisms of his classification of transcendences, putting the latter in an unfavorable light as a mathematician, and, of course, not bothering to be overly objective in doing so.

The fierce polemic, in which J. C. F. STURM [26] was also involved, sometimes became an inadmissible wrangle. One can sense the tenseness of the atmosphere even from the obviously toned down minutes published in Volume **8** of *Comptes Rendus*.

I shall not go into the atmosphere of this scandal in which each of the adversaries felt himself in his element. Suffice it to say that LIOUVILLE carried the day: he was, after all, elected to the Academy. The whole episode appreciably damaged LIBRI's prestige.[20] Still, his position remained firm and his scientific reputation continued to be quite high. In 1837–1841 he published his *Histoire des sciences mathématiques*

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[19] In connection with LIOUVILLE's proof LIBRI [24, p. 799] stated that

M. Liouville a envoyé à l'Académie la démonstration d'un des théorèmes que j'avais énoncés dans la dernière séance. Au moment même ou M. le Secrétaire perpétuel annonçait cela à l'Académie, j'ai rédigé à la hâte ma démonstration sur une petite feuille de papier: cette démonstration a été paraphée immédiatement par M. Arago; je dois dire que celle de M. Liouville, qui m'a été communiquée le lendemain, ne diffère de la mienne que dans quelques détails qui n'ont aucune importance.

In 1844, formulating LIBRI's theorem, MOIGNO [13, p. 579] wrote:

M. Liouville a publié le premier la démonstration de ce théorème fondamental.

He then gave precisely this proof; see note 7.

[20] In 1840, sometime between May 6 and July 7, LIOUVILLE [21, pp. 55–56] informed DIRICHLET about LIBRI's position at the Academy:

Il y a là, sans doute, de quoi compenser cent fois pour une l'ennui que j'éprouve à m'occuper de M. Libri, c'est-a-dire d'un homme qui, dans l'Académie du moins, commence à être méprisé presque autant qu'il le mérite.

en Italie (in four volumes) and thus became one of the most eminent historians of mathematics of the 19^{th} century. His behavior was as aggressive as ever, and, at the sittings of the Academy, he entered into verbose discussions, often sharp in tone, with M. CHASLES on the history of mathematics and astronomy.

Nothing, it seemed, foreshadowed that catastrophe that overtook him in 1848.

It suddenly came to light that, over the years, while inspecting archives and libraries for the government, LIBRI had purloined a great many valuable books and documents, to a total value of half a million francs. Influential friends warned him in time and he managed to get across the Channel with his loot, and he watched the court proceedings from there.

He was sentenced *in absentia* to ten years' imprisonment. From his refuge in England, where he made a fortune from selling his loot, LIBRI continued to maintain his innocence in letters to the Ministers of Justice and Education, and to members of the *Institut de France*, unsuccessfully petitioning for review of a verdict that (he pleaded) had made him the victim of a miscarriage of justice in the turmoil of the revolution. He died in Italy in 1869.

The fuss around the 'LIBRI affair' was so great that it impugned his fame as a historian of mathematics and his more modest mathematical achievements. Nevertheless, mathematicians and historians of mathematics in the first place did not forget him. CANTOR gave a significant place to LIBRI's research on the history of mathematics in his opening address to the Second International Congress of Mathematicians in Paris in 1900.[21] As for LIBRI's mathematical work, his achievements in the field of the theory of linear differential equations have enjoyed most attention.[22]

I have mentioned above (§§ 3 and 4) that l'Abbé MOIGNO had described them in his *Leçons*, and that BRASSINNE had developed them, while FROBENIUS had referred to his memoir [8]. VESSIOT, too, did not overlook LIBRI'S and BRASSINNE'S theorems in his fundamental *Étude* [28; 29]. He also mentioned LIBRI in connection with reconstructing linear homogeneous equations given their particular integrals.

Nevertheless, from an analytical standpoint, LIBRI's achievements taken by themselves are rather simple. As I said before (§§ 3 and 4), by far more important for the development of the theory of linear differential equations was the analogy

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[21] Here is what CANTOR [27, pp. 36–37] wrote:

il est indiscutable que Libri a rendu des services énormes à l'Historiographie des Mathématiques. Il a étudié nombre de manuscrits dont il donne des extraits pour la plus grande partie très exacts ... il manie la langue avec un art tout à fait hors ligne. Son "Histoire des sciences mathématiques en Italie" se lit comme un roman, même dans les parties où elle n'en est pas un.

It must also be said, in all fairness, that LIBRI quoted long passages from medieval Latin manuscripts in his writings thus putting into scientific circulation many contributions, all but unknown at the time, of medieval European and Arabic scholars, including, for example, MUḤAMMAD IBN MŪSĀ and LEONARDO PISANO, alias FIBONACCI.

[22] Recently E. P. OZHIGOVA [30, pp. 24–25] considered LIBRI's claim to have proved GAUSS's assertion about partitioning the lemniscate.

between these and algebraic equations, the analogy that LIBRI was the first to discover and to develop in a systematic way.

From the viewpoint of psychology of creative work, LIBRI was an interesting type of mathematician, *viz.*, a weak analyst capable of a kind of sixth sense that enabled him to guess future developments in a certain field and even to foresee some essential bits of future theories (like FROBENIUS'S theory of irreducibility).

Ackknowledgements. It is my pleasure and duty to express cordial thanks to A. P. YOUSHKEVITCH and A. N. PARSHIN for their valuable comments on this paper. O. B. SHEYNIN translated the manuscript into English and introduced a number of corrections, and H. C. CREIGHTON read and edited the translation.

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(Received November 15, 1982)