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Half-positional Determinacy of Infinite Games

PhD Thesis [DRAFT]

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March 1, 2008

Abstract

We study infinite games where one of the players always has a positional (memory-less) winning strategy, while the other player may use a history-dependent strategy. We investigate winning conditions which guarantee such a property for all arenas, or all finite arenas. We establish some closure properties of such conditions, which give rise to the XPS class of winning conditions, and discover some common reasons behind several known and new positional determinacy results. We show that this property is decidable in single exponential time for a given prefix independent ω-regular winning condition. We exhibit several new classes of winning conditions having this property: the class of concave conditions (for finite arenas), the classes of monotonic conditions and geometrical conditions (for all arenas).

Keywords

automata, infinite games, omega-regular languages, positional strategies, winning condtions

AMS Classification:

91Axx Game theory
91A05 2-person games
91A43 Games involving graphs
68Qxx Theory of computing
68Q45 Formal languages and automata
68Q60 Specification and verification

Contents

Chapter 1

Introduction

The theory of infinite games is relevant for computer science because of its potential application to verification of interactive systems. In this approach, the system and environment are modeled as players in an infinite game played on a graph (called *arena*) whose vertices represent possible system states. The players (conventionally called Eve and Adam) decide which edge (state transition, or *move*) to choose; each edge has a specific *color*. The desired system's behavior is expressed as a winning condition of the game — the winner depends on the sequence of colors which appear during an infinite play. If a winning strategy exists in this game, the system which implements it will behave as expected. Positional strategies (i.e. depending only on the position, not on the history of play — also called *memoryless*) are of special interest here, because of their good algorithmic properties which can lead to an efficient implementation. Among the most often used winning conditions are the parity conditions, which admit positional determinacy for both players ([Mos91], [EJ91], [McN93]).

Infinite games are also strongly linked to automata theory. Parity condition is a very important notion in both fields — infinite games and automata on infinite structures. Winning conditions in games can often be effectively expressed as ω-regular languages. This allows results from one field to be used in another. For example, positional determinacy of parity games is used in modern proofs of Rabin's complementation theorem for finite automata on infinite trees with parity acceptance condition.

However, not always it is possible to express the desired behavior as a parity condition. An interesting question is, what properties are enough for the winning condition to be positionally determined, i.e. admit positional winning strategies independently on the arena on which the game is played. Recently some interesting characterizations of such positionally determined winning conditions have been found ([CN06], [GZ04], [GZ05]). Another interesting characterization of finitely positional conditions can be found in [GZ04]. For a survey of recent results on positional determinacy see [Gra04].

Our work attempts to obtain similar characterizations and find interesting properties (e.g. closure properties) of half-positionally determined winning conditions, i.e. ones such that all games using such a winning condition are positionally determined for one of the players (us, say), but the other player (environment) can have an arbitrary strategy. We give uniform arguments to prove several known and several new half-positional determinacy results. As we will see, some results on positional determinacy have natural generalizations to half-positional determinacy, but some do not. This makes the theory of half-positional conditions harder than the theory of positional conditions. We also exhibit some large classes of half-positionally determined winning conditions.

1.1 Overview

In the next chapter, Chapter 2, we introduce the basic definitions and notions we will be using throughout the paper.

In Chapter 3 we present tools which then can be used to prove (half-)positional determinacy of many winning conditions in an uniform way. We also use them to give an alternative proof for the positional determinacy of parity conditions. We also recall some recent results regarding parity conditions and other positional winning conditions.

In Chapter 4 we present a simple combinatorial property which guarantees finite half-positional determinacy. In Chapter 5 we generalize the mean payoff game, and investigate whether these generalizations are (finitely) half-positional or not.

In Chapter 6 we explore the links between games and automata theory. We present another half-positional winning condition (monotonic condition), which is defined using a finite automaton. Then, we show some results regarding winning conditions which are ω-regular. For example, we show that finite half-positional determinacy is decidable for them.

In Chapter 7 we present one of the questions which motivated our research: is a finite (countable) union of half-positional winning conditions also half-positional? The conjecture is still open, but we have some partial results. We also present a big class of winning conditions which contains most of the half-positional winning conditions presented here, and is closed under finite union.

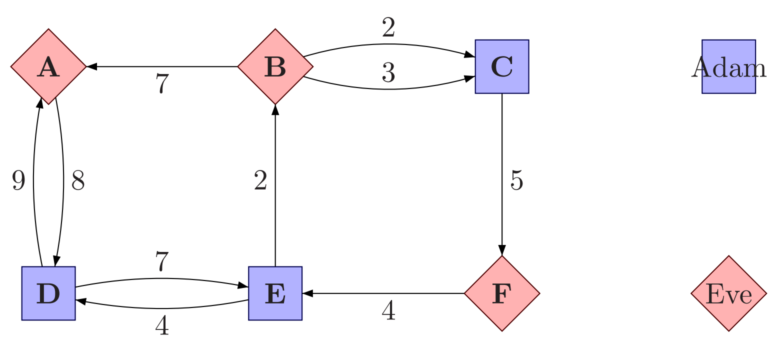
Chapter 2

Preliminaries

In this chapter we define all the basic notions we are working with. We start with an example of a game, then we define games, arenas, and winning conditions in general. Then we proceed to defining plays, strategies, and determinacy. We introduce determinacy types, like positional and half-positional determinacy. Finally, we show three types of arenas which appear in literature, and discuss how these types are different regarding positional strategies.

2.1 Example

Before giving the general definition of an infinite game, we show a typical example of a game.



The picture above shows the *arena* the game is played on. The squares and diamonds are called *positions*; diamonds represents Eve's positions and squares represent Adam's positions.

The game starts by placing a token in one of the available positions. It can be either Eve's position or Adam's position. The owner chooses one of the moves (arrows) available from this position and moves the token to the position which is pointed to by the arrow. For example, if we start in **B**, Eve can choose either to go to **A** (which is also her position), or to Adam's position **C** (either by arrow labelled with 2, or by arrow labelled with 3). Now, this new position can again be either Eve's position or Adam's position — the owner decides the next move to be taken, and so on.

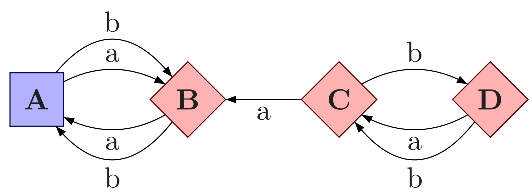
The game never ends: decisions made by both players define an infinite play. Now, there is a *winning condition* which says who will win, depending on the sequence of colors (i.e. labels) of moves which have been used during the infinite play.

In the game above we could use the *parity condition*: Eve wins iff the greatest number appearing infinitely often is even.

By analyzing the game, we can find out that Adam has a winning strategy. In position **C**, always go to **F** (there is no other option anyway); Eve will have to go to **E**. In the position **E**, go to **D**, and in **D**, go to **A**. Now, Eve will have to return to **A**, as it is her only option. In position **D**, Adam always decides to go to **A**; thus, the sequence of color (except the beginning) will be: 8, 9, 8, 9, ... and Adam will win.

Note that this strategy of Adam has the following property: in each position, always the same move is used. This is called a *positional* strategy.

Another example of a game follows. Now Adam now wants both letters a and b to appear infinitely often in the sequence of colors obtained from a play.



By analyzing the game, we get that Adam can win if the game starts in the positions **A** and **B** (an example winning strategy: when moving from **A** to **B**, he alternates between the two moves available, so he wins no matter what Eve is doing), and Eve can win if the game starts in **C** and **D** (in **C** she goes to **D** via b, and in **D** she goes to **C** also via b).

Note that Eve's winning strategy in **C** and **D** is positional, while Adam's winning strategy in **A** and **B** is not. That's what we mean by a *half-positional* game (or winning condition): when Eve can win, we need her strategy to be positional, but when Adam can win, he can use any strategy he wants.

2.2 Games

In this section we formally define games, arenas, and strategies.

We consider perfect information antagonistic infinite games played by two players, called conventionally Adam and Eve. Let C be a set of **colors** (possibly infinite).

An **arena** over C is a tuple $G = (\mathrm{Pos}_A, \mathrm{Pos}_E, \mathrm{Mov})$, where:

- Elements of $\mathrm{Pos} = \mathrm{Pos}_E \cup \mathrm{Pos}_A$ are called **positions**; Pos_A and Pos_E are disjoint sets of Adam's positions and Eve's positions, respectively.
- Elements of $\mathrm{Mov} \subseteq \mathrm{Pos} \times \mathrm{Pos} \times (C \cup \{\epsilon\})$ are called **moves**; (v_1, v_2, c) is a move from v_1 to v_2 colored by c. We denote $\mathrm{source}((v_1, v_2, c)) = v_1$, $\mathrm{target}((v_1, v_2, c)) = v_2$, $\mathrm{rank}((v_1, v_2, c)) = c$. We will write moves as $v_1 \xrightarrow{c} v_2$ instead of (v_1, v_2, c).
- ϵ denotes an empty word, i.e. a colorless move. However, there is a restriction on ϵ-moves: an arena is not allowed to contain infinite paths of them.

A game is a pair (G, W), where G is an arena, and W is a winning condition. A **winning condition** W over C is a subset of C^ω which is *prefix independent*, i.e., $u \in W \iff cu \in W$ for each $c \in C, u \in C^\omega$. We name specific winning conditions WA, WB, ...

Note that, contrary to some other works, when we consider winning conditions in this thesis, we mean prefix independent subsets of C^ω. Occassionally, we might use a game (G, W) where W is not prefix independent; we will then explicitely call W a *prefix dependent winning conditions*.

As in the example above, the game (G, W) carries on in the following way. The play starts in some position v_1. The owner of v_1 (e.g. Eve if $v_1 \in \mathrm{Pos}_E$) chooses one of the moves leaving v_1, say $v_1 \xrightarrow{c_1} v_2$. If the player cannot choose because there are no moves leaving v_1, he or she loses. The next move is chosen by the owner of v_2; denote it by $v_2 \xrightarrow{c_2} v_3$. And so on: in the n-th move the owner of v_n chooses a move $v_n \xrightarrow{c_n} v_{n+1}$. If $c_1 c_2 c_3 \ldots \in W$, Eve wins the infinite play; otherwise Adam wins.

A **play** in the arena G is a sequence of moves, i.e., a path in the arena graph. A play can be finite (the length of play $|\pi|$ is in ω) or infinite ($|\pi| = \omega$). We denote the set of all plays by Play, and $\mathrm{Play}_\infty, \mathrm{Play}_F, \mathrm{Play}_A, \mathrm{Play}_E \subseteq \mathrm{Play}$ are infinite plays, finite plays, and finite plays which end in Adam's and Eve's positions, respectively. We identify finite plays with (some) elements of $\mathrm{Pos} \cup \mathrm{Mov}^+$ (Pos represents plays which have just started and contain no moves yet), and infinite plays with elements of Mov^ω. By $\mathrm{source}(\pi)$ and $\mathrm{target}(\pi)$ we denote the initial and final position of the play, respectively (obviously infinite plays have no target). Thus, for a play of length 0 we have $\mathrm{source}(\pi) = \mathrm{target}(\pi)$, otherwise we have $\mathrm{source}(\pi) = \mathrm{source}(\pi_1)$, $\mathrm{target}(\pi_n) = \mathrm{source}(\pi_{n+1})$, and $\mathrm{target}(\pi_{|\pi|}) = \mathrm{target}(\pi)$.

2.3 Strategies

A **strategy for player** X (i.e. $X \in \{\mathrm{Eve}, \mathrm{Adam}\}$) is a partial function $s : \mathrm{Play}_X \to \mathrm{Mov}$. Intuitively, $s(\pi)$ for π ending in Pos_X says what X should do next. We say that a play π is **consistent** with strategy s for X if for each prefix π' of π such that $\pi' \in \mathrm{Play}_X$ the next move is given by $s(\pi')$.

A strategy s is **winning** (for X) from the position v if $s(\pi)$ is defined for each finite play π starting in v, consistent with s, and ending in Pos_X, and each infinite play starting in v consistent with s is winning for X.

A strategy s is **positional** if it depends only on target(π), i.e., for each finite play π we have $s(\pi) = s(\mathrm{target}(\pi))$.

2.4 Determinacy

A game is **determined** if for each starting position one of the players has a winning strategy. This player may depend on the starting position in the given play. Thus if the game is determined, the set Pos can be split into two sets Win_E and Win_A and there exist strategies s_E and s_A such that each play π with source(π) $\in \mathrm{Win}_X$ and consistent with s_X is winning for X. All games with a Borel winning condition are determined [Mar75], but there exist (exotic) games which are not determined. A winning condition W is **determined** if for each arena G the game (G, W) is determined.

We are interested in games and winning conditions for which one or both of the players have positional winning strategies. Thus, we introduce the notion of a **determinacy type**, given by three parameters: admissible strategies for Eve (positional or arbitrary), admissible strategies for Adam (positional or arbitrary), and admissible arenas (finite or infinite). We say that a winning condition W is (α, β, γ)**-determined** if for every γ-arena G the game (G, W) is (α, β)**-determined**, i.e. for every starting position either Eve has a winning α-strategy, or Adam has a winning β-strategy. Clearly, there are 8 determinacy types in total — for now; we will introduce even more determinacy types, based on other subclasses of arenas (the next section) and memory bounds (Section 8.2) later). We could also use other restrictions for strategies (e.g. persistent or suspendable strategies).

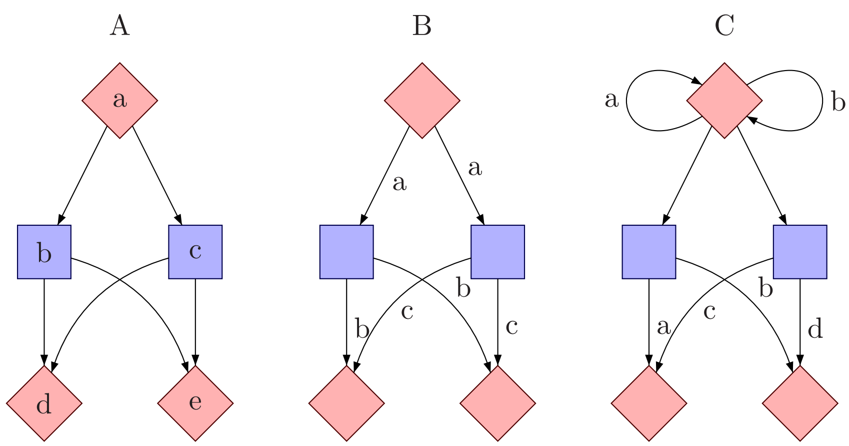
For short, we call (positional, positional, infinite)-determined winning conditions **positionally determined** or just **positional**, (positional, arbitrary, infinite)-determined winning conditions **half-positional**, (arbitrary, positional, infinite)-determined winning conditions **co-half-positional**. If we restrict ourselves to finite arenas, we add **finitely**, e.g. (positional, arbitrary, finite)-determined conditions are called **finitely half-positional**. For a determinacy type $D = (\alpha, \beta, \gamma)$, D-arenas mean γ-arenas, and D-strategies mean α-strategies (if they are strategies for Eve) or β-strategies (for Adam).

Note that if a game (G, W) is (α, β)-determined, then its dual game obtained by using the complement winning condition and switching the roles of players is (β, α)-determined. Thus, W is (α, β, γ)-determined iff its complement is (β, α, γ)-determined.

2.5 Types of arenas

In the games defined above the moves are colored, and it is allowed to have moves without colors. In the literature, several types of arenas are studied.

- ϵ-arenas (C), like the ones described above.
- *Move-colored arenas* (B). In this setting each move has a color assigned; moves labelled with ϵ are not allowed.
- *Position-colored arenas* (A). In this setting, colors are assigned to positions rather than to moves. Instead of $\mathrm{Mov} \subseteq \mathrm{Pos} \times \mathrm{Pos} \times C$ we have $\mathrm{Mov} \subseteq \mathrm{Pos} \times \mathrm{Pos}$ and a function $\mathrm{rank} : \mathrm{Pos} \to C$. As in (B), each position has a color assigned. The winner of a play in such games is defined similarly.



If we take a position-colored arena and color each move m with the color $\mathrm{rank}(\mathrm{source}(m))$, we obtain an equivalent move-colored arena (this construction is illustrated on the picture). Therefore position-colored arenas are a subclass of move-colored arenas. Obviously, move-colored arenas are also a subclass of ϵ-colored arenas. When speaking about a determinacy type where we restrict arenas to position-colored or move-colored arenas, or we want to emphasize that we allow ϵ-arenas, we add the letter A, B or C (e.g. A-half-positional conditions when we restrict to position-colored arenas).

Hence C-half-positional conditions are a subclass of B-half-positional conditions, and B-half-positional conditions are a subclass of A-half-positional conditions. The inclusion between A-half-positional and B-half-positional conditions is proper: there is no way to transform a move-colored arena into a position-colored one such that nothing changes with respect to positional strategies (we can split a position into several new positions according to colors of moves which come into them, but then we obtain new positional strategies which were not positional previously). Indeed, we know examples of winning conditions which are A-positional but not B-positional. One of them is $C^*(ab)^*$, where $C = \{a, b\}$; for position-colored arenas we know from our current position to which color we should move next (when we are in position of color a, we should move to b, and vice versa), but not for edge-colored arenas. Another example is min-parity [GW06]. B-positional determinacy has been characterized in [CN06]; this result can be easily generalized to ϵ-arenas. ϵ-arenas have been studied in [Zie98].

The question whether the inclusion between C-half-positional conditions and B-half-positional conditions is proper remains open. Note that when we allow ϵ labels there is no difference whether we label positions or moves: we can replace each move $v_1 \to v_2$ colored with c in an ϵ-arena with $v_1 \to v \to v_2$, color v with c, and leave all the original positions colorless.

In this thesis, we concentrate on ϵ-arenas since we think that this class gives the least restriction on arenas. As shown by the example above, $C^*(01)^*$, positional strategies for move-colored games are „more memoryless" than for position-colored games since they do not even remember the last color used, although winning conditions for position-colored games (like min-parity) may also be interesting. As we will see in the sequel, allowing our arenas to contain ϵ-moves — despite potential greater generality of such arenas — usually does not make our proofs harder, and sometimes even makes them easier and more natural.

2.6 Extensions

In some papers a more general situation is investigated, where instead of a winning condition we have a **payoff mapping** $u : C^\omega \to \mathbb{R}$. In such games Eve's and Adam's goals are respectively maximization and minimization of $u(c_1c_2c_3\ldots)$. The payoff mapping can be intuitively interpreted as the quantity of money which Eve wins from Adam. Payoff mapping is a generalization of the winning condition (we can get the equivalent payoff mapping by taking the characteristic function of a winning condition).

Chapter 3

Basic tools

In this chapter we present our basic tools and the most important positional winning conditions. In the first section we prove Lemma 3.1 which will be used in many proofs of half-positional determinacy of various winning conditions. In the next section, we present Büchi and co-Büchi conditions, and a closure property regarding them (Theorem 3.3).

In the last section we show how our results can be used to immediately give an alternative proof for positional determinacy of the parity condition. We also cite and generalize some interesting facts regarding parity conditions.

3.1 An useful lemma

Lemma 3.1 *Let D be a determinacy type. Let $W \subseteq C^\omega$ be a winning condition. (Reminder: by winning condition we mean prefix independent.) Suppose that, for each non-empty D-arena G over C, there exists a non-empty subset $M \subseteq \mathrm{Pos}_G$ such that in game (G, W) one of the players has a D-strategy winning from M. Then W is D-determined.*

Equivalently, instead of taking an non-empty subset M, we could say that there exists a position $v \in \mathrm{Pos}_G$ such that in game (G, W) one of the players has a D-strategy winning from v. Although that wording might be simpler to understand, we will use the wording above, since that's how our lemma will be used. Actually, when we use our lemma to show half-positional determinacy, we will usually show that either Eve has a positional winning strategy everywhere, or Adam has a winning strategy in a non-empty subset.

Proof of Lemma 3.1 Let $G = (\mathrm{Pos}_A, \mathrm{Pos}_E, \mathrm{Mov})$ be a D-arena.

Let P be some subset of Pos, and $G_P = (\mathrm{Pos}_A \cap P, \mathrm{Pos}_E \cap P, \mathrm{Mov} \cap P \times P \times C)$. From our hypothesis we know that in some subset $M = M(P) \subseteq$

$\mathrm{Pos}_A \cap P$ the player X has a winning D-strategy s in G_P. Without loss of generality we can assume that:

(a) No play starting in M consistent with s leaves M. Otherwise, let M' be the set of all positions v such that there is a play starting in M consistent with s which reaches v. Since X has a winning D-strategy in each play which reaches such v, and W is prefix independent, we obtain winning D-strategies for X for plays starting everywhere in M'. Thus, we can replace M with M'.

(b) There is no position outside of M from which X can ensure entering M. Otherwise we can add all such positions to M. The (positional) strategy of X in extended M is first to reach M (which can be done using a positional strategy) and then to use our winning D-strategy starting from the point which was reached (again, we need prefix independence of W here).

As in the proof of the Knaster-Tarski theorem, we construct transfinite sequence (P_α) of subsets of Pos.

Let $P_0 = \mathrm{Pos}$. For each α let $M_\alpha = M(G_{P_\alpha})$, and $P_{\alpha+1} = P_\alpha - M_\alpha$. Let X_α be the player who wins in M_α, and s_α be his or her winning D-strategy in M_α. For a limit ordinal λ, let $P_\lambda = \bigcap_{\alpha<\lambda} P_\alpha$.

For some β the sequences stabilize. We have that $M_\beta = P_{\beta+1} - P_\beta = \emptyset$, hence $P_\beta = \emptyset$ (we know that $M(G)$ is non-empty for each non-empty D-arena G).

Let X be one of the players. Let $F_X = \bigcup_{\alpha:X_\alpha=X} M_\alpha$; we have that $\mathrm{Pos} = F_A \cup F_E$. Let $x \in M_\alpha$ be our starting position. By induction over α, we will show that X_α has a winning D-strategy from each $x \in M_\alpha$.

X_α is able to use the D-strategy s_α as long as the play does not leave M_α. X_α's strategy s_α never forces her to leave M_α, but X_α's opponent is potentially able to do it; however, he cannot go into $P_{\alpha+1}$ (from assumption (a) in the step α) or into M_γ where $X_\gamma \neq X_\alpha, \gamma < \alpha$ (from assumption (b) in the step γ). Hence, he can only go into M_γ for some $\gamma < \alpha$ such that $X_\gamma = X_\alpha$. By induction hypothesis we know that $X_\alpha = X_\gamma$ has a winning strategy in M_γ. X can now continue by using this strategy and win the whole game. ■

3.2 Büchi and co-Büchi conditions

Definition 3.2 *For $S \subseteq C$, WB_S is the set of infinite words where elements of S occur infinitely often, i.e. $(C^*S)^\omega$. Winning conditions of this form are called* Büchi conditions*. Complements of Büchi conditons, $WB'_S = C^*(C-S)^\omega$ are called* co-Büchi conditions*.*

Theorem 3.3 *Let D be a determinacy type. Let $W \subseteq C^\omega$ be a winning condition, and $S \subseteq C$. If W is D-determined, so is $W \cup WB_S$.*

Proof We will show that the assumption of Lemma 3.1 holds. Let our arena be $G = (\mathrm{Pos}_E, \mathrm{Pos}_A, \mathrm{Mov})$. *$S$-moves* are moves m such that $\mathrm{rank}(m) \in S$.

Let G' be G with a new position $\top$ added. The position $\top$ belongs to Adam and has no outgoing moves, hence Adam loses here. For each S-move m we change $\mathrm{target}(m)$ to $\top$.

Since Adam immediately loses after doing an S-move in G', the winning conditions W and $W \cup WB_S$ are equivalent for G', thus we can use D-determinacy of W to find the winning sets $\mathrm{Win}'_E, \mathrm{Win}'_A$ and winning D-strategies s'_E, s'_A in G'.

Suppose $\mathrm{Win}'_A \neq \emptyset$. We can see that since Adam's strategy wins in G' from a starting position in Win'_A, he also wins in G from there by using the same strategy (the game G' is „harder" for Adam than G). Thus the assumption of 3.1 holds (we take $M = \mathrm{Win}'_A$).

Now suppose that $\mathrm{Win}'_A = \emptyset$. We will show that in the game G Eve has a winning D-strategy s in Pos everywhere, hence the assumption of Lemma 3.1 also holds (we take $M = \mathrm{Pos}$).

The strategy is as follows. For a finite play π we take $s(\pi) = s_E(\pi')$, where π' is the longest final segment without any S-moves, unless when $s_E(\pi')$ is a move to $\top$. In this case, there had to be at least one S-move from $\mathrm{target}(\pi')$ in G, and Eve makes one of them.

The strategy s is positional if s_E is positional. Let π be a play consistent with s. There are two possibilities: there is either finite or infinite number of S-moves in π. If the number is infinite, then Eve wins (as she wins WB_S). If the number is finite, then $\pi = \pi_0\pi'$, where π_0 ends with the last S-move (possibly π_0 is empty). Hence, π' does not contain any S-moves and is consistent with s_E, thus Eve also wins. Therefore, s is indeed a winning D-strategy. ∎

Note that, by duality, Thm 3.3 shows that if W is D-determined, then so is $W \cap WB'_S$.

3.3 Parity conditions

The **parity condition** of rank n is the winning condition over $C = \{0, 1, \ldots, n\}$ defined with

$$WP_n = \{w \in C^\omega : \limsup_{i\to\infty} w_i \text{ is even}\}. \tag{3.1}$$

This is one of the most important classical winning conditions. Many proofs of its positional determinacy are already known. Theorem 3.3 immediately gives yet another one: it is enough to start with an empty winning condition (which is positionally determined) and apply Thm 3.3 and its dual n times.

It is worth to remark that in case of infinite arenas the parity conditions are the only ones which admit positional determinacy.

Theorem 3.4 *Let $W \subseteq C^\omega$ be a winning condition. The following properties are equivalent:*

1. $W = h^{-1}(WP_n)$ *for some* $h : C \to \{0, 1, \dots, n\}$ *(we call such a W a* generalized *parity condition);*
2. W *is positionally determined;*
3. (G, W) *is positionally determined for each arena G where either* $\mathrm{Pos}_E = \emptyset$ *or* $\mathrm{Pos}_A = \emptyset$*;*
4. *Let* $W_f = \{u \in C^+ | u^\omega \in W\}$*. We have* $W_f^\omega \subseteq W$ *and, dually,* $C^+ - W_f^\omega \subseteq C^\omega - W$*.*

Note that this theorem works only in case of edge-colored arenas (B) and ϵ-arenas (C), not position-colored arenas (see Section 2.5 for definitions of arena types, and examples of A-positional winning conditions).

Proof

The equivalence of (1) and (2) has been shown in [CN06].

1→2 is a simple generalization of a well known fact — namely, positional determinacy of parity games ([Mos91], [EJ91], [McN93]). As mentioned above, it can be also shown by applying Theorem 3.3 and its dual n times.

2→3 is obvious (a special case).

2→4 is proven in [CN06] (as Lemma 7). Actually, only one-player arenas are used in the proof, so we get $3 \to 4$.

2→1 is proven in [CN06]. However, the assumption (2) is never used except the proof of Lemma 7 (i.e., condition (4)) and Lemma 9. So, to show $4 \to 1$, we only have to prove Lemma 9 using condition (4).

Lemma 3.5 (Lemma 9 from [CN06]) *For any $L, L' \subseteq C^+$ we have*

$$\forall v \in L' \exists u \in L uv \in W_f \text{ iff } \exists u \in L \forall v \in L' uv \in W_f$$

Proof of Lemma 3.5 ($\leftarrow$) is obvious. To prove ($\rightarrow$), assume to the contrary that for each $u \in L$ there exists $v \in L'$ such that $uv \notin W_f$. We define sequences $v_n \in L'$ and $u_n \in L$ by induction. Let u_1 be any element of L. Let $v_n \in L'$ be such that $u_n v_n \notin W_f$. Let u_{n+1} be such that $v_n u_{n+1} \in W_f$. The word $v_1 u_2 v_2 u_3 \ldots \in W_f^\omega \subseteq W$ (by (4)). On the other hand, the word $u_1 v_1 u_2 v_2 \ldots \in C^+ - {W_f}^\omega \subseteq C^\omega - W$ (by dual in (4)). This is a contradiction, since W is prefix independent. ∎

In the case of finite arenas there are more positional winning conditions, and we don't have 2→4 nor 2→1. For example, the winning condition $WF(A)$ from Section 5 below, where A and its complement are both convex sets, is finitely positional. However, we have equivalence of (2) and (3) (a very elegant result from [GZ05]).

Chapter 4

Concave winning conditions

In the following chapters, we give some examples of half-positionally determined winning conditions. We start by giving a simple combinatorial property which guarantees finite half-positional determinacy.

4.1 Definition

Definition 4.1 *A word $w \in \Sigma^* \cup \Sigma^\omega$ is a* **shuffle** *of words w_1 and w_2, iff for some sequence of words (u_n), $u_n \in \Sigma^*$*

- $w = \prod_{k\in\mathbb{N}} u_k = u_0u_1u_2u_3u_4u_5u_6u_7u_8\ldots,$
- $w_1 = \prod_{k\in\mathbb{N}} u_{2k+1} = u_1u_3u_5u_7\ldots,$
- $w_2 = \prod_{k\in\mathbb{N}} u_{2k} = u_0u_2u_4u_6\ldots.$

Definition 4.2 *A winning condition W is* **convex** *if as a subset of C^ω it is closed under shuffles, and* **concave** *if its complement is convex.*

Example 4.3 *Parity conditions (including Büchi and co-Büchi conditions) are both convex and concave.*

Example 4.4 *Let C be an infinite set. The folowing winning conditions are both convex and concave:*

- Exploration condition*: the set of all v in C^ω such that $\{v_n : n \in \omega\}$ is infinite.*
- Unboundedness condition*: the set of all v in C^ω such that no color appears infinitely often.*

Decidability and positional determinacy of these conditions on (infinite) pushdown arenas where each position has a distinct color has been studied in [Gim04] (exploration condition) and [BSW03], [CDT02] (unboundedness condition).

Another example, which justifies the names *convex* and *concave*, is given in Chapter 5 below.

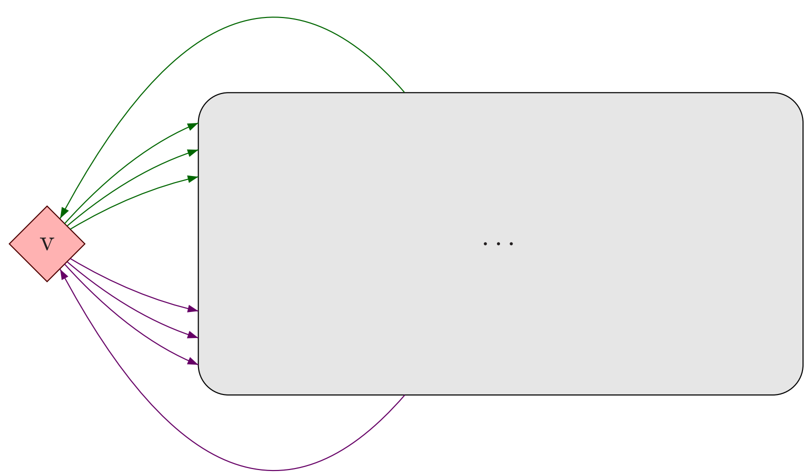
4.2 Half-positional determinacy

Theorem 4.5 *Concave winning conditions are half-positionally finitely determined.*

The proof goes by induction over Mov, and is based on the following idea. Let v be Eve's position, with outgoing moves $m_1, m_2, \ldots$. Suppose that Eve cannot win by using only one of these moves. Then, since the winning condition is concave, she also cannot win by using many of these moves — because it can be written as a shuffle of subplays that appear after each move $m_1, m_2, \ldots$, and Adam wins all of these plays.

Proof of Theorem 4.5 Let $W \subseteq C^\omega$ be a concave winning condition in the game (G, W), where $G = (\mathrm{Pos}_A, \mathrm{Pos}_E, \mathrm{Mov})$. A proof by induction on $|\mathrm{Mov}|$.

Let v be a position belonging to Eve, where she has more than one move. If there are no such positions, the game (G, W) must be half-positionally determined from definition.



Let M be a set of Eve's possible moves from v. Let $M = M_1 \cup M_2$, where M_1 and M_2 are non-empty and disjoint. Let $G_i = (\mathrm{Pos}_A, \mathrm{Pos}_E, \mathrm{Mov} - M_{3-i})$, $G_A = (\mathrm{Pos}_A, \mathrm{Pos}_E, \mathrm{Mov} - M)$.

From the induction hypothesis we know that the games (G_1, W), (G_2, W) and (G_A, W) are half-positionally determined. Let Win_i^E and Win_i^A be winning sets for Eve and Adam, respectively, in the games (G_i, W) for $i \in \{1, 2, A\}$,and let s_i and t_i be the winning strategies of Eve in Win_i^E and Adam in Win_i^A, respectively, in these games. Suppose s_i is a positional strategy for $i \in \{1, 2, A\}$.

First, assume that $v \in \mathrm{Win}_i^E$ for some i. In this case the strategy s_i is also winning for Eve in the set Win_i^E in the arena G (since the only difference between G_i and G is that Eve has more possibilities in G). On the other hand, t_i is a winning strategy for Adam in the set Win_i^A in the arena G, since each play consistent with t_i is winning for Adam and therefore must not go through v (by prefix independence, Eve would win otherwise), hence Eve is unable to use her additional possibilities.

Now, assume that $v \in \mathrm{Win}_1^A$ and $v \in \mathrm{Win}_2^A$. Since $v \in \mathrm{Win}_i^A$, Adam is able to win each play in G_i which goes through v. Therefore the winning sets in G_i are the same as in G_A (again, prefix independence). Therefore, if $v \in \mathrm{Win}_1^A$ and $v \in \mathrm{Win}_2^A$, we have $\mathrm{Win}_1^A = \mathrm{Win}_2^A$ (since both of them are equal to Win_A^A) and $\mathrm{Win}_1^E = \mathrm{Win}_2^E$.

Similarly to Adam's strategy in the first case, Eve's strategy s_1 remains winning for Eve in the set Win_1^E in the game G. We will show a winning strategy for Adam in the set Win_1^A.

Let $\pi = \pi_1 \dots \pi_m$ be a finite play. Let $K = \mathrm{dom}\ \pi = \{1, \dots, m\}$. Let $S_v = \{k \in K : \mathrm{source}(\pi_k) = v\}$. We define the function $f : K \to \{1, 2\}$ in the following way. If $k < \min S_v$, we take $f(k) = 1$. Otherwise, $f(k) = i$ iff $\pi_{k'} \in M_i$, where k' is the greatest element of S_v such that $k' \leq k$.

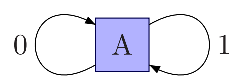
Let $\pi_{(i)} = \Pi_{k \in K} \pi_k^{[f(k)=i]}$, where $v^{[\phi]}$ denotes v if ϕ is true, and the empty word ϵ otherwise. One can easily see that π, as a word over Mov, is then a shuffle of $\pi_{(1)}$ and $\pi_{(2)}$.

It can be easily checked that $\pi_{(i)}$ is a play. For $j = f(m)$ we have $\mathrm{target}(\pi_{(j)}) = \mathrm{target}(\pi)$. Let $t(\pi) = t_j(\pi_{(j)})$. If Adam consistently plays with the strategy t, the plays $\pi_{(i)}$ are consistent with t_i for $i = 1, 2$.

We check that t is indeed a winning strategy for Adam in the set Win_1^A in the game (G, W). Let π be an infinite play consistent with t. Like for finite plays, π is a shuffle of $\pi_{(1)}$ and $\pi_{(2)}$. Hence $\mathrm{rank}(\pi)$, the sequence of colors in the play π, is a shuffle of $\mathrm{rank}(\pi_{(1)})$ and $\mathrm{rank}(\pi_{(2)})$. The plays $\pi_{(i)}$ for $i = 1, 2$ are either finite or winning for Adam (as they are consistent with t_i). If $\pi_{(i)}$ is finite, $\pi_{(3-i)}$ is infinite and winning for Adam; from prefix independence of W we get that π is also winning for Adam. If both plays are infinite, $\mathrm{rank}(\pi_{(1)}) \notin W$ and $\mathrm{rank}(\pi_{(2)}) \notin W$; from concavity of W we get that also $\mathrm{rank}(\pi) \notin W$. ∎

This theorem gives yet another proof of finite positional determinacy of parity games, and also half-positional determinacy of unions of families of parity conditions (where each parity condition may use a different rank for a given color). Half-positional determinacy of Rabin conditions (finite unions of families of parity conditions) over infinite arenas has been proven in [Kla92] and [Gra04].

Note that, in general, concavity does not mean half-positional determinacy over infinite arenas — for examples see Chapter 5 below, and also Example 4.4 and Thm 7.2. Also, half-positional determinacy (even over infinite arenas) does not mean concavity — examples can be found in Chapters 5 (especially see the last note on page 31 — concavity gives false predictions for infinite half-positional determinacy) and Section 6.1 (Proposition 6.6 and the note above it).



Concavity does not force any bound on the memory required by Adam. Indeed, let $x \in [0,1]-\mathbb{Q}$, $C = \{0,1\}$, and consider the game (G, W), where G is the arena with one Adam's position A and two moves $A \to A$ colored 0 and 1 respectively, and let W be the set of sequences (c_n) such that $\sum_{i=1}^{n} c_i/n$ is not convergent to x. This winning condition is concave (Theorem 5.1 in Chapter 5 below), but Adam obviously requires unbounded memory here.

A related property has been shown in [Mar02]: a winning condition W is called *positive* iff its complement is closed under supersequences (i.e., shuffles with C^*). Theorem 3 from [Mar02] says that games with positive winning conditions admit persistent winning strategies for Eve. A winning strategy s is *persistent* iff $s(\pi_1)$ equals $s(\pi_1\pi_2)$ whenever $\mathrm{target}(\pi_1) = \mathrm{target}(\pi_1\pi_2)$ (i.e., Eve always chooses the same move from each position, but she can decide which move she takes not before game, but when the game enters this position). Positiveness is a weaker property than concavity, and persistence is a weaker property than positionality.

4.3 Weakening the Concavity Condition

In [GZ04] a result similar to Thm 4.5 has been obtained in the case of full positional determinacy. To present it, we need the following definition:

Definition 4.6 *A winning condition W is* **weakly convex** *iff for each sequence of words (u_n), $u_n \in C^*$, if*

1. $u_1u_3u_5u_7\ldots \in W$,
2. $u_2u_4u_6u_8\ldots \in W$,
3. $(\star)$ $\forall i\ (u_i)^\omega \in W$,

then $u_1u_2u_3u_4\ldots \in W$.

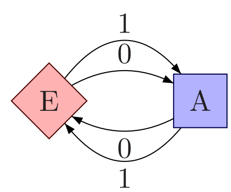
A winning condition W *is* **weakly concave** *iff its complement is weakly convex.*

In the case of normal convexity there is no $(\star)$.

[GZ04] defines *fairly mixing* payoff mappings; in the case of prefix independent winning conditions *fairly mixing* resolves to the conjunction of weak concavity and weak convexity. Theorem 1 from [GZ04] says that games on finite arenas with fairly mixing payoff mappings are positionally determined.

Unfortunately, weak concavity is not enough for half-positional finite determinacy.

Proposition 4.7 *There exists a weakly concave winning condition* WQ *which is not half-positionally finitely determined.*



Proof Let $C = \{0,1\}$. For $w \in C^\omega$ let $P_n(w)$ be the number of 1's among the first n letters of w, divided by n. The winning condition WQ is a set of w such that $P_n(w)$ is convergent and its limit is rational. It can be easily seen that for each $u \in C^+$ we have $u^\omega \in WQ$. Therefore $(\star)$ is never satisfied for the complement of WQ, hence WQ is a weakly concave winning condition. However, WQ is not half-positionally determined. Consider the arena with two positions $E \in \mathrm{Pos}_E$, $A \in \mathrm{Pos}_A$, and moves $E \xrightarrow{0} A$, $E \xrightarrow{1} A$, $A \xrightarrow{0} E$ and $A \xrightarrow{1} E$. If Eve always moves in the same way, Adam can choose the moves 0 and 1 in an irrational proportion, ensuring his victory. However, Eve wins by always moving with the color opposite to Adam's last move — the limit of $P_n(w)$ is then $1/2$. ∎

Note that the given WQ satisfies the even stronger condition obtained by replacing $\forall i$ by $\exists i$ in $(\star)$ in Definition 4.6.

Chapter 5

Geometrical Conditions

In this chapter we show some half-positional determinacy results for *geometrical conditions*, which are based on the ideas similar to that used by the *mean payoff game* (sometimes called *Ehrenfeucht-Mycielski game*). We also show the relations between geometrical conditions and concave winning conditions.

5.1 Definition

Let $C = [0,1]^d$ (where $[0,1]$ is the real interval; we can also use any compact and convex subset of a normed space). For a word $w \in C^+$, let $P(w)$ be the average color of w, i.e., $\frac{1}{|w|}\sum_{k=1}^{|w|} w_k$. For a word $w \in C^\omega$, let $P_n(w) = P(w_{|n})$ ($w_{|n}$ — an n-letter prefix of w).

Let $A \subseteq C$. We want to construct a winning condition W such that $w \in W$ whenever the limit of $P_n(w)$ belongs to A. Since not every sequence has a limit, we have to define the winner for all other sequences.

Let $WF(A)$ be a set of w such that each cluster point of $P_n(w)$ is an element of A. Let $WF'(A)$ be a set of w such that at least one cluster point of $P_n(w)$ is an element of A. Note that $WF'(A) = C^\omega - WF(C - A)$.

As we will see, for half-positional determinacy the important property of A is whether the complement of A is convex — we will call such sets A *co-convex* (as *concave* usually means “non-convex” in geometry).

Geometrical conditions have a connection with the *mean payoff game*, whose finite positional determinacy has been proven in [EM79]. In the mean payoff game, C is a segment in $\mathbb{R}$ and the payoff mapping is $u(w) = \liminf_{n\to\infty} P_n(w)$. If $A = \{x : x \geq x_0\}$ then $u^{-1}(A)$ (“Eve wants x_0 or more”) is exactly the geometrical condition $WF(A)$. Of course, the dual payoff, defined with $u(w) = \limsup_{n\to\infty} P_n(w)$, corresponds to $WF'(A)$. (In case of finite arenas it does not matter whether we take lim sup or lim inf, since if both players use optimal strategies, $(P_n(w))$ will be convergent. However, things change for infinite arenas.)

Geometrical conditions are a generalization of such winning conditions to a larger class of sets A and C.

5.2 Concave and convex

In this section we show how notions of convexity and concavity of winning conditions, introduced in Chapter 4 (Definition 4.2), are related to geometrical convexity of the set A.

Theorem 5.1 *We have:*

1. *$WF'(A)$ is weakly convex iff A is a closed convex subset of C.*
2. *$WF'(A)$ is convex iff A is a trivial subset of C (i.e., $A = \emptyset$ or $A = C$).*
3. *$WF'(A)$ is weakly concave iff A is a co-convex subset of C.*
4. *$WF'(A)$ is concave iff A is a co-convex subset of C.*
5. *$WF(A)$ is weakly convex iff A is a convex subset of C.*
6. *$WF(A)$ is convex iff A is a convex subset of C.*
7. *$WF(A)$ is weakly concave iff A is an open co-convex subset of C.*
8. *$WF(A)$ is concave iff A is a trivial subset of C.*

To prove it, we need the following lemmas:

Lemma 5.2 *If A is a convex subset of C then $WF(A)$ is convex.*

Proof

Now, suppose A is convex; we will show that $WF(A)$ is convex.

Let w_3 be a shuffle of w_1 and w_2, where $w_1, w_2 \in WF(A)$. Let B_k for $k = 1,\ 2$ be a set of cluster points of $P_n(w_k)$, and B_3 be the convex hull of $B_1 \cup B_2$. Since $B_1 \subseteq A$ and $B_2 \subseteq A$, also $B_3 \subseteq A$. All the sets B_1, B_2, B_3 are compact. Let δ_n^k be the distance of $P_n(w_k)$ from the set B_k for $k = 1, 2, 3$. The sequence (δ_n^k) converges to 0 for $k = 1, 2$. We will show that (δ_n^3) also converges to 0.

Let $\epsilon > 0$. Let N be a number such that for all $n \geq N$ we have $\delta_n^1 < \epsilon$ and $\delta_n^2 < \epsilon$. Let $n > ND/\epsilon$, where D is the diameter of C, i.e., the maximum distance between two colors. The word $w_{0|n}$ is a shuffle of $w_{1|m}$ and $w_{2|m'}$ for some $m + m' = n$. One can easily show the following:

$$P_n(w_0) = \frac{m}{n} P_m(w_1) + \frac{m'}{n} P_{m'}(w_2). \tag{5.1}$$

For $k = 1, 2$ let $P_m(w_k) = P(w_{k|m}) = b_k + x_k$, where $b_k \in B_k$ and $|x_k| = \delta^k_m$. Let $b_0 = \frac{m}{n} b_1 + \frac{m'}{n} b_2$, $x_0 = \frac{m}{n} x_1 + \frac{m'}{n} x_2$. From (5.1) we have $P_n(w_0) = b_0 + x_0$. From the definition of B_3, $b_0 \in B_3$. From the definition of x_0 we have that

$$\delta^3_n \le |x_0| \le \frac{m}{n}|x_1| + \frac{m'}{n}|x_2| = \frac{m}{n}\delta^1_m + \frac{m'}{n}\delta^2_{m'}. \tag{5.2}$$

If $m < N$, $\frac{m}{n}\delta^1_m$ is smaller than $\frac{m}{n}D$. Since $m < N$ and $n \ge ND/\epsilon$, we have $\frac{m}{n}\delta^1_m < \epsilon$. If $m \ge N$, we have $\delta^1_m < \epsilon$, so also $\frac{m}{n}\delta^1_m < \epsilon$. By the same reasoning we have that the second component is also smaller than ϵ. Therefore δ^3_n is smaller than 2ϵ for each $n \ge ND/\epsilon$, hence the sequence δ^3_n is indeed convergent do 0. Thus, all cluster points of $(P_n(w_3))$ must be in B_3. ■

Lemma 5.3 *If A is a closed convex subset of C then $WF'(A)$ is weakly convex.*

Proof Let $v_1 = w_1 w_3 \dots$ and $v_2 = w_2 w_4 \dots$ be two words such that v_1, v_2, and w_i^ω are all in $WF'(A)$. We have to show that $v_3 = w_1 w_2 w_3 w_4 \dots$ is also in $WF'(A)$. Let $x_n = P(w_1 w_2 w_3 \dots w_n)$; (x_n) is a subsequence of $(P_n(v_3))$, so to show that $(P_n(v_3))$ has a cluster point in A, it is enough to show that (x_n) has a cluster point in A. However, each element of x_n is in A, since x_n a convex combination of $P(w_1), \dots, P(w_n)$. Since A is closed, (x_n) must have a cluster point in A. ■

Lemma 5.4 *If A is a non-trivial subset of C then $WF'(A)$ is not convex. Moreover, if A is not closed then $WF'(A)$ is not weakly convex.*

Proof Let $x \in A$. If A is not closed and we want to show that $WF(A)$ is not weakly convex, let y_n be a sequence of elements of A convergent to $y \notin A$. Otherwise, just take $y_n = y \notin A$.

Consider the infinite words u, v, w produced by the following algorithm. Start with $u = x$, $v = x$, $w = xx$ (concatenation). For $n = 1, 2, \dots$: Let l be the length of u. Append x^{nl} to u, y_n^{nnl} to v, $(xy_n^n)^{nl}$ to w. Let l be the length of v. Append x^{nl} to v, y_n^{nnl} to u, $(xy_n^n)^{nl}$ to w.

It can be easily seen that w is a shuffle of u and v. However, x is a cluster point of both u and v, but the only cluster point of w is y. ■

Proof of Theorem 5.1 If A is trivial, obviously $WF'(A)$ is convex.

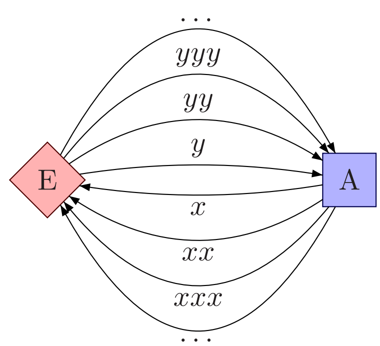
If A is not convex, let $x, y \in A$ such that $z = kx + (1-k)y \notin A$ for $k \in [0, 1]$. Obviously, the infinite words x^ω and y^ω are in $WF(A)$ and $WF'(A)$, but we can shuffle them to obtain a word w such that $P_n(w)$ is convergent to z, thus $w \notin WF(A)$ and $w \notin WF'(A)$. (Note that for weak convexity we had to use colors x and y; $W = WF'([0,1]-\mathbb{Q}) \cap \{0,1\}^\omega$ is weakly convex as a winning condition over $C = \{0,1\}$, since there is no word w such that $w^\omega \in W$.)

These two simple facts, together with the lemmas above, are enough to prove all items above. (Note that items 3, 4, 7, 8 are dual to items 1, 2, 5, 6.) ■

5.3 Positional determinacy

By Theorems 5.1 and 4.5, if A is co-convex then $WF'(A)$ is concave and thus **finitely** half-positionally determined. However, the situation is different for infinite arenas.

Proposition 5.5 *If A is a non-trivial subset of C then $WF'(A)$ is not half-positionally determined.*

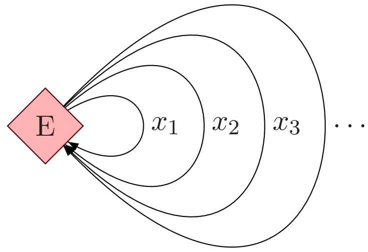


Proof of Proposition 5.5 Let $x \in C - A, y \in A$. Consider the game with two positions where one can choose a move, A and E. A is Adam's position, E is Eve's position. In the position E Eve can choose a path going to A through k edges of color y, for each integer $k > 1$. Similarly, in A Adam can choose a path to E by k edges of color x, for all integers $k > 1$.

If Eve is using a positional strategy, always choosing the move generating the path y^k, Adam can win choosing x^{nk} in the n-th round. In this case the limit of $P_n(w)$ in x, hence Adam wins.

However, Eve can win by using a non-positional strategy. This strategy is to choose the move generating y^{nk} in the round n, where k is the number of x's generated in the last move of Adam. This ensures that y is a cluster point of $P_n(w)$, hence Eve wins. ■

Proposition 5.6 *If A is not open then $WF(A)$ is not half-positionally determined.*



Proof of Proposition 5.6 Let $x = \lim_{n\to\infty} x_n$, where $x \in A$ and $x_n \notin A$. Consider the game with only one Eve's position E and moves E $\to$ E labeled x_n for each positive integer n. Eve has only non-positional winning strategies here. ■

5.4 Simple open set

In this section we show that $WF(A)$ is half-positional for very simple closed sets A. The problem remains unsolved for more complicated sets.

Theorem 5.7 *Let $C = [0, 1]$, $A = [0, 1/2)$. The condition*

$$WF(A) = \{w : \limsup P_n(f(w)) < 1/2\}$$

is half-positional.

Proof of Theorem 5.9

Let $G = (\mathrm{Pos}_A, \mathrm{Pos}_E, \mathrm{Mov})$ be an arena. Consider the following prefix dependent winning condition for $x \in [0, 1]$:

$$WL_x = \{w : \forall_n P_n(w) \le x\} \tag{5.3}$$

Let $L_x = \mathrm{Win}_E(G, WL_x)$, i. e. set of positions v such that there exists a winning strategy for Eve in the game starting from the position v. We will use the following lemma:

Lemma 5.8 *Let $x < 1/2$. In L_x Eve has a positional winning strategy in $(G, WF(A))$.*

To apply Lemma 3.1 it remains to prove that if L_x is empty for each $x < 1/2$ then Adam has a winning strategy everywhere. Let (a_n) be an increasing sequence convergent to $1/2$. The strategy is as follows:

- For each $i = 1,\ 2,\ \dots :$
- Let t be the current time (i.e., length of the play so far), and v be the current position. Since $v \notin L_{a_i}$, we know that Adam has a strategy which guarantees that after some time t' we get $P(w) > a_i$, where w is the color word obtained from time t to t'. Adam uses this strategy until this happens.

If Adam uses this strategy, we get an infinite play whose color word is $w = w_1 w_2 w_3 \dots$, where $P(w_i) > a_i$. One can easily check that, for each i, there will be a t such that $P_t(w) > a_i$. Thus, $\limsup P_n(w)$ is at least $1/2$. ■

Proof of Lemma 5.8
Let $(G^x,\, WP_1)$ (WP_1 is the parity condition) be the game where:

- $\mathrm{Pos}^x_X = \mathrm{Pos}_X \times \mathbb{R}$ for $X \in \{A, E\}$,
- In the position $(v, z) \in \mathrm{Pos}^x$ the move belongs to the same player as in the position $v \in \mathrm{Pos}$;
- For each move $v \xrightarrow{t} w \in \mathrm{Mov}$ and $z \geq 0$ we have a move $(v, z) \xrightarrow{0} (w, z + x - t)$ in Mov^x,
- For each move $v \xrightarrow{t} w \in \mathrm{Mov}$ and $z < 0$ we have a move $(v, z) \xrightarrow{1} (w, z)$ in Mov^x.

The number z in position $(v, z) \in \mathrm{Pos}^x$ defines Eve's reserve. Eve wins all plays where this reserve does not fall beyond 0.

The plays in $(G^x,\, WP_1)$ can be projected to $(G,\, WL_x)$. And vice versa, a play in $(G,\, WL_x)$ starting in v can be raised to a play in $(G^x,\, WP_1)$. One can easily show that projecting and raising plays preserves the winner, provided that in $(G^x,\, WP_1)$ we start in $(v, 0)$ for some v. Hence $L_x = \{v : (v, 0) \in \mathrm{Win}_E(G^x,\, WP_1)\}$.

The parity condition WP_1 is positionally determined, thus the constructed game $(G^x,\, WP_1)$ is positionally determined. Let s' be a positional strategy winning in $\mathrm{Win}_E(G^x,\, WP_1)$. Clearly if $z_1 \leq z_2$ then $(v, z_1) \in \mathrm{Win}_E(G^x,\, WP_1)$ implies $(v, z_2) \in \mathrm{Win}_E(G^x,\, WP_1)$. Let M be the set of v such that $(v, z) \in \mathrm{Win}_E(G^x,\, WP_1)$ for some $z \geq 0$; we have $L_x \subseteq M$. Let $x < y < 1/2$. Consider the following strategy in M:

$$s(v) = \pi(s'(v, z(v) + (y - x))) \tag{5.4}$$

where π is the natural projection, and

$$z(v) = \inf\{z :\ (v, z) \in \mathrm{Win}_E(G^x)\}. \tag{5.5}$$

Let (G^y, WP_1) be a game constructed analogically to (G^x, WP_1). One can easily check that each game starting in M which is consistent with s projects to some play in (G^y, WP_1) winning for Eve and starting in $(v, z(v))$. Hence the play in $(G, WF(A))$ satisfies the winning condition $WF(A)$. ■

This theorem can be generalized to the following:

Proposition 5.9 *Let $A = f^{-1}(\{x \in \mathbb{R} : x < 0\})$ for some affine function $f : C \to \mathbb{R}$. Then, the condition $WF(A)$ is half-positional.*

Proof Let $a_0 = \min f(C), a_1 = \max f(C)$. Let h be such that $0 \leq 1/2 + ha_0 \leq 1/2 + ha_1 \leq 1$. Let G' be the arena like G, except that we replace each color c with $t(c) = 1/2 + hf(c)$. By our assumption, G' is an arena over $[0, 1]$, and one can easily check that Eve wins a play in $(G, WF(A))$ iff she wins the corresponding play in $(G', WF([0, 1/2)))$.

5.5 Summary

The following table summarizes what we know about concavity and half-positional determinacy of geometrical conditions. In every point except No. 0 we assume that A is non-trivial, i.e. $\emptyset \neq A \neq C$. The first two columns specify assumptions about A and whether we consider $WF(A)$ or $WF'(A)$, and the last two answer whether the considered condition is concave and whether it has finite and/or infinite half-positional determinacy. Negative answer means that the answer is negative for all sets A in the given class; the question mark means that the given problem has not been solved yet (but we suppose that the answer is positive).

No.	A	condition	concavity	finite	infinite
0	trivial	$WF'(A)$ or $WF(A)$	yes	yes	yes
1	not co-convex	$WF'(A)$ or $WF(A)$	no	no	no
2	co-convex	$WF'(A)$	yes	yes	no
3	co-convex, not open	$WF(A)$	no	yes?	no
4	co-convex, open	$WF(A)$	weak only	yes?	yes?
5	$[\frac{1}{2}, 1] \subset [0, 1]$	$WF(A)$	weak only	yes	yes

Note that, for any set A which is co-convex and non-trivial, $WF'(A)$ is finitely half-positionally determined, but not infinitely half-positionally determined. This shows a big difference between half-positional determinacy on finite and infinite arenas.

Chapter 6

Games and finite automata

Infinite games are strongly linked to automata theory. An accepting run of an alternating automaton (on a given tree) can be presented as a winning strategy in a certain game between two players. Parity games are related to automata on infinite structures with parity acceptance condition. For example, positional determinacy of parity games is used in modern proofs of Rabin's complementation theorem for finite automata on infinite trees with Büchi or parity acceptance condition. See [GTW02] for more links between infinite games, automata, and logic.

In this chapter we concentrate on the links between our subject and finite automata. We start by presenting the standard definitions from automata theory.

Definition 6.1 *A* **deterministic finite automaton on infinite words with parity acceptance condition** *is a tuple* $A = (Q, q_I, \delta, \mathrm{rank})$*, where* Q *is a finite set of states,* $q_I \in Q$ *the initial state,* $\mathrm{rank} : Q \to \{0, \ldots, d\}$*, and* $\delta : Q \times C \to Q$*. We extend the definition of* δ *to* $\delta : Q \times C^* \to Q$ *by* $\delta(q, \epsilon) = q, \delta(q, wu) = \delta(\delta(q, w), u)$ *for* $w \in C^*, u \in C$*. For* $w \in C^\omega$*, let* $q_0(w) = q_I$ *and* $q_{n+1}(w) = \delta(q_n, w_{n+1}) = \delta(q_I, w_0 \ldots w_{n+1})$*. We say that the word* $w \in C^\omega$ *is* **accepted** *by* A *iff* $\limsup_{n\to\infty} \mathrm{rank}(q_n(w))$ *is even. The set of all words accepted by* A *is called* **language accepted by** A *and denoted* L_A*. We say that a language* $L \subseteq C^\omega$ *is* ω**-regular** *if it is accepted by some automaton.*

It is a well known fact that the class of ω-regular languages, which generalize the class of regular languages of finite words, has very nice properties. It is closed under operations such as union, intersection, negation, and homomorphic preimages and images. It can be defined in many ways, using other kinds of automata, ω-regular expressions, or using notions from logic.

Winning conditions are languages of infinite words over C, and many of those which are used in theory and practice are ω-regular. Examples include parity conditions and Rabin conditions (unions of parity conditions). Thus, finite automata provide nice finite descriptions of these winning conditions, which enables us to give algorithms which check properties of an ω-regular winning condition, given the automaton that accepts it.

In the first section we show a class of half-positional winning conditions defined using a finite automaton (on finite words). In the next two sections we show what can be said about finite half-positional determinacy of a winning condition which is ω-regular. Precisely, we show that if an ω-regular winning condition is not half-positional then this is witnessed by a very simple arena, which will lead us to an algorithm which decides whether given winning condition is finitely half-positional. In the last section we show that concavity is also decidable.

6.1 Monotonic automata

In this section we show yet another class of half-positionally determined winning conditions which is based on an idea coming from automata theory, and guarantees half-positional determinacy even for infinite arenas. We need to introduce a special kind of deterministic finite automaton.

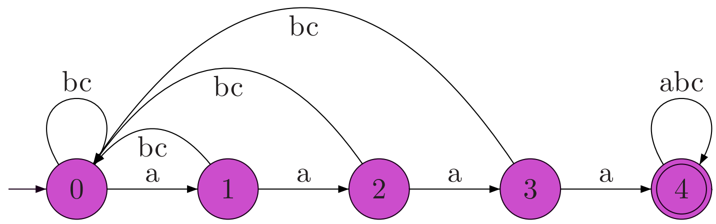
Definition 6.2 *A* **monotonic automaton** $A = (n, \delta)$ *over an alphabet* C *is a deterministic finite automaton (on finite words) where:*

- *the set of states is* $Q = \{0, \ldots, n\}$*;*
- *the initial state is 0, and the accepting state is* n*;*
- *the transition function* $\delta : Q \times C \to Q$ *is monotonic in the first component, i.e., if* $q \leq q'$ *then* $\delta(q, c) \leq \delta(q', c)$*.*

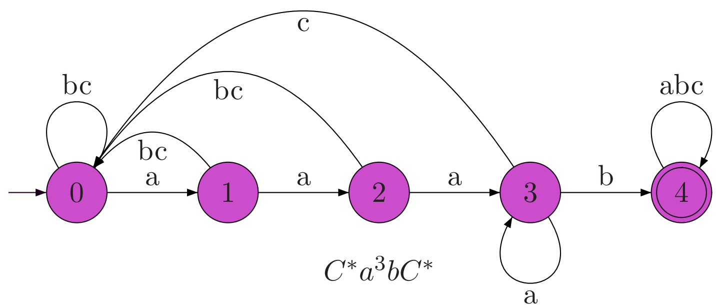
Actually, we need not require that the set of states is finite. All the results presented here except for Thm 7.12 and the remark about finite memory of Adam can be proven with a weaker assumption that Q has a minimum (initial state) and its each non-empty subset has a maximum.

The function δ is extended to C^* as in Definition 6.1; this extension is still monotonic. By L_A we denote *the language accepted (recognized) by* A, i.e., the set of words $w \in C^*$ such that $\delta(0, w) = n$.

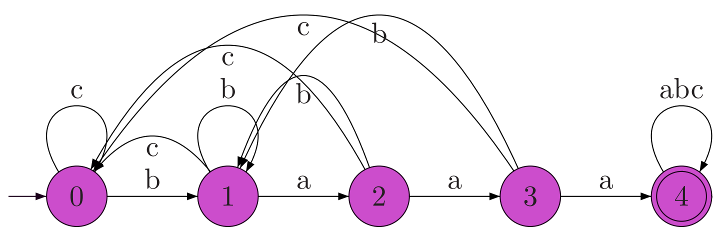
Example 6.3 *Let* $C = \{a, b, c\}$*. Monotonic automata can recognize the following languages:* $C^*a^nC^*$, $C^*a^{n-1}bC^*$, $C^*ba^{n-1}C^*$*. Monotonic automata cannot recognize the following languages:* $C^*a^2b^2C^*$, C^*babC^*, C^*bacC^**.*



$C^*a^4C^*$



$C^*a^3bC^*$



$C^*ba^3C^*$

The pictures illustrate automata recognizing these languages, for $n = 4$. (To show that the other languages are not recognizable by monotonic automata, one can use e.g. Theorem 6.5 or Proposition 6.6 below.)

Definition 6.4 *A* **monotonic condition** *is a winning condition of the form $WM_A = C^\omega - L_A^\omega$ for some monotonic automaton A.*

Note that if $w \in L_A$ then $uw \in L_A$ for each $u \in C^*$. Hence $L_A = C^*L_A$. Therefore L_A^ω is equal to $L_A(C^*L_A)^\omega = (L_AC^*)^\omega$. Hence without affecting WM_A we can assume that $\delta(n, c) = n$ for each c.

Theorem 6.5 *Any monotonic condition is half-positional.*

In Section 8.2 we analyze memory required by Adam to win in his winning set.

Proof of Theorem 6.5

Let $A = (n, \delta)$ be a monotonic automaton, and $W = WM_A = C^\omega - L_A^\omega$. Let $G = (\mathrm{Pos}_A, \mathrm{Pos}_E, \mathrm{Mov})$ be an arena. We will show that the game (G, W) is half-positionally determined.

We will construct a new game on the arena $G' = (\mathrm{Pos}'_A, \mathrm{Pos}'_E, \mathrm{Mov}')$ over the set of colors $C' = \{0, 1\}$ with the parity condition WP_1, where $\mathrm{Pos}'_X = \mathrm{Pos}_X \times Q$. For each move $v_1 \xrightarrow{c} v_2$ in G, in G' we have moves $(v_1, q) \xrightarrow{c'} (v_2, \delta(q, c))$ for each $q \leq n$; $c' = 0$ for $q < n$ and 1 for $q = n$. (We assumed that $\delta(n, c) = n$, which means that after reaching $q = n$ Adam will win, unless the play ends finitely.)

The game (G', WP_1) is positionally determined, therefore Pos' can be split into the winning sets of both players, Win'_A and Win'_E, and in Win'_E we have a positional winning strategy for Eve, $s' : \mathrm{Win}'_E \to \mathrm{Mov}$. Let $M \subseteq \mathrm{Pos}$ be the set of v such that $(v, 0) \in \mathrm{Win}'_E$.

There are two cases:

1. $M = \emptyset$. We will show that Adam has a winning strategy in (G, WM_A) from each position. This strategy is implemented by the following algorithm:

    - Let v_1 be the starting position (after $R_1 = 0$ moves);
    - For $i = 1, 2, 3, \ldots$:
    - After R_i moves we are in position v_i. In G', Adam has a strategy ensuring reaching from $(v_i, 0)$ to some state belonging to the set $N = \{(v, n) : v \in G\}$. Adam uses a projection of this strategy (ignoring R_i moves which have been made before reaching v_i), until he reaches N in G'. Let v_{i+1} be the vertex reached in G.

      The word w created by colors of moves made in meantime satisfies $\delta(0, w) = n$.

    The word created between the R_i-th and R_{i+1}-th move belongs to L_A, therefore the infinite word created during the whole play does not belong to WM_A.

2. $M \neq \emptyset$. We will show that Eve has a positional winning strategy in (G, WM_A) for each starting position $v_0 \in M$.

    Note that if $q_1 < q_2$ and $(v, q_2) \in \mathrm{Win}'_E$ then also $(v, q_1) \in \mathrm{Win}'_E$. (The situation with smaller q is better for Eve.) For $v \in M$ we denote by $H(v)$ the greatest q for each $(v, q) \in \mathrm{Win}'_E$.

    We define Eve's positional strategy in the game (G, WM_A) in the set M in the following way: for $v \in M$, $s(v) = \alpha(s'(v, H(v)))$, where $\alpha(m')$ for a move $m' \in \mathrm{Mov}'$ is the move in Mov such that m' is derived from m (in case if there are many such moves, $\alpha(m')$ can be any one of them).

Let π be a play consistent with the strategy s, and $v_i = \text{target}(\pi_i)$ for $i > 0$. Let $q_i = \delta(0, v_1 v_2 \dots v_i)$ be the state of the automaton when reaching v_i. We will show by induction that for all i we have $q_i \leq H(v_i)$, and therefore $q_i < n$ and $v_i \in M$. Obviously $q_0 = 0 \leq H(v_0)$. Now, assume that $q_i \leq H(v_i)$; we will show that $q_{i+1} \leq H(v_{i+1})$.

Suppose $v_i \in \text{Pos}_E$. This means that $v_{i+1} = \text{target}(s(v_i))$, and thus, $\text{target}(s'(v_i, H(v_i)))$ is (v_{i+1}, q) for some q. Since $q_i \leq H(v_i)$, $q_{i+1} = \delta(v_{i+1}, \text{rank}(s(v_i)))$, $q = \delta(H(v_i), \text{rank}(s(v_i)))$, and δ is monotonic, we have $q_{i+1} \leq q$. On the other hand, we know that $(v_{i+1}, q) \in Win'_E$, therefore $q \leq H(v_{i+1})$. Hence indeed $q_{i+1} \leq H(v_{i+1})$.

Now, suppose $v_i \in \text{Pos}_A$. Then $v_{i+1} = \text{target}(m)$ for some move m from v_i. The move m gives rise to moves $m_1 = ((v_i, q_i) \xrightarrow{0} (v_{i+1}, q_{i+1}))$ and $m_2 = ((v_i, H(v_i)) \xrightarrow{0} (v_{i+1}, q))$ in Mov'. Since $q_i \leq H(v_i)$, by monotonicity of δ we obtain $q_{i+1} \leq q$. We also have $q \leq H(v_{i+1})$, since otherwise Adam could leave Eve's winning set in G' (using the move m_2).

Since for each i we have $q_i \leq H(v_i) < n$, the word $v_1 v_2 \dots$ has to belong to WM_A.

Half-positional determinacy follows from Lemma 3.1.

■

From this theorem and the examples of languages recognized by monotonic automata above one can see that e.g. WA_n, the complement of the set of words containing a^n infinitely many times, is monotonic, and thus half-positionally determined.

For $n = 1$ the set WA_n is just a co-Büchi condition. However, for $n > 1$ it is easily shown that WA_n is not (fully) positionally determined, and also that it is not concave. For example, for $n = 2$ the word $(bababbabab)^\omega$ is a shuffle of $(bbbaa)^\omega$ and $(aabbb)^\omega$.

Proposition 6.6 *All monotonic conditions are weakly concave.*

Proof Let $A = (n, \delta)$ be a monotonic automaton. We will show a stronger propety, namely that, for each sequence of words $w_1, w_2, \dots$, if $\forall_i w_i^\omega \in L_A^\omega$, then $w_1 w_2 w_3 \dots \in L_A^\omega$. (We don't use the assumption that $w_1 w_3 w_5 \dots \in L_A^\omega$ and $w_2 w_4 w_6 \in L_A^\omega$.)

We will assume that $\delta(n, c) = n$ for each c.

Since $w_i^\omega \notin WM_A$, we have that $\delta(q, w_i) > q$ for each $q < n$. Otherwise, if for some q we had $\delta(q, w_i) \leq q$, then, from monotonicity of δ, $\delta(q', w_i) \leq q$ for each $q' \leq q$, thus A will not accept any prefix of w_i^ω, because we will never reach the state n starting from the state $0 \leq q$.

Hence $\delta(0, w_{kn+1} w_{kn+2} w_{kn+3} \dots w_{kn+n}) = n$. This holds for each k, thus the word $w_1 w_2 w_3 \dots$ is indeed in L_A^ω, and is not in WM_A. ■

6.2 Simplifying the Witness Arena

To show that finite half-positional determinacy of winning conditions which are prefix independent ω-regular languages is decidable, we first need to show that if W is not finitely half-positional, then it is witnessed by a simple arena.

Theorem 6.7 *Let W be a winning condition accepted by a deterministic finite automaton with parity acceptance condition $A = (Q, q_I, \delta, \mathrm{rank} : Q \to \{0 \dots d\})$ (see Definition 6.1). If W is not finitely half-positional then there is a witness arena (i.e. such that Eve has a winning strategy, but no positional winning strategy) where there is only one Eve's position, and only two moves from this position. (There is no restriction on Adam's moves and positions.)*

Proof Let G be any finite witness arena. Without loss of generality we can assume that Eve has a winning strategy everywhere (otherwise we restrict our arena to Eve's winning set). First, we will show how to reduce the number of Eve's positions to just one. Then, we will show how to remove unnecessary moves.

Let $G^0 = (\mathrm{Pos}_A \times Q, \mathrm{Pos}_E \times Q, \mathrm{Mov}^0)$ and $G^1 = (\mathrm{Pos}_A \times Q, \mathrm{Pos}_E \times Q, \mathrm{Mov}^1)$ where for each move $v_1 \xrightarrow{c} v_2$ in G and each state q we have corresponding moves $(v_1, q) \xrightarrow{c} (v_2, \delta(q, c))$ in Mov^0 and $(v_1, q) \xrightarrow{\mathrm{rank}(q)} (v_2, \delta(q, c))$ in Mov^1. The three games (G, W), (G^0, W) and (G^1, WP_d) are equivalent: each play in one of them can be interpreted as a play in each another, and the winner does not change for infinite plays.

More specifically, the correspondence between G^0 and G is based on replacing each position v by a set $U(v) = \{(v, q) : q \in Q\}$. For each element w of $U(v)$ if there is a move $v \xrightarrow{c} v'$, then there exists a move $w \xrightarrow{c} w'$ for some $w' \in U(v')$. Also, if there is a move $w \xrightarrow{c} w'$ for some $w \in U(v), w' \in U(v')$, then there is a move $v \xrightarrow{c} v'$.

Between G^0 and G^1, the only difference is the rank function, so plays in one of them can be interpreted as plays in the other in the obvious way. To see that the winner does not change, take a play $\pi = \pi_0 \pi_1 \pi_2 \dots$ in G^0, and let $(v_i, q_i) = \mathrm{source}(\pi_i)$. In the game G^0 the color of π_i is c_i, and the color of the corresponding move in G^1 is $\mathrm{rank}(q_i)$. From definition of G^0, by induction we have $q_i = \delta(q_I, c_1 c_2 \dots c_i)$, so Eve wins iff $\limsup \mathrm{rank}(q_i)$ is even — which agrees with the parity condition in G^1.

Since Eve has a winning strategy in (G, W), she also has a winning strategy in (G^1, WP_d). This game is positionally determined, so she also has a positional strategy here. She can use the corresponding positional strategy in (G^0, W) too.

Let s be Eve's positional winning strategy in G^0. Let

$$N(s) = \{v : \exists q_1 \exists q_2\ \pi_1(\mathrm{target}(s(v, q_1))) \neq \pi_1(\mathrm{target}(s(v, q_2)))\},$$

i.e. the set of positions where s is not positional as a strategy in G. Since the arena is finite, we can assume without loss of generality that there is no positional winning strategy s' in G^0 such that $N(s') \subsetneq N(s)$.

If $N(s)$ was empty, then we could use s as a positional strategy in G, which would contradict our assumption that G is a witness arena. Let $v_0 \in N(s)$. We construct a new arena G^2 from G^0 in two steps.

First, merge $\{v_0\} \times Q$ into a single position v_0. Eve can transform s into a winning (non-positional) strategy s_1 in this new game — the only difference is that in v_0 she needs to remember in what state q she is currently, and move according to $s(v_0, q)$.

Then, for all Eve's positions except v_0, remove all moves which are not used by s (and thus by s_1). Eve still wins by s_1, since she did not lose any options used by s_1. Now, transfer all Eve's positions except v_0 to Adam. Eve still wins by s_1, since there was no choice in these positions.

Thus, we obtained an arena G^2 with only one Eve's position v_0, where she has a winning strategy from v_0.

Eve has no positional strategy in G^2. Otherwise this strategy could be simulated without changing the winner (in the natural way) by a strategy s_2 in G which is positional in all positions except $N(s) - \{v_0\}$. This means that there is a positional strategy s_3 in G^0 for which $N(s_3) \subseteq N(s) - \{v_0\} \subsetneq N(s)$, which contradicts our assumption that $N(s)$ is minimal. Indeed, let G_* be G without moves which are not used by s_2 — s_2 remains a winning strategy on G_*. Let G^0_* be the arena obtained from G_* in the same way as we obtained G^0 from G. Let s_3 be Eve's positional winning strategy on G^0_* (which exists since Eve had a winning strategy on G_*); as a strategy on G^0, it is also winning, and has $N(s_3) \subsetneq N(s)$.

Hence, we found a witness arena where $|Pos_E| = 1$. (Note that we can assume that Eve has at most $|Q|$ moves here — Eve's positional winning strategy on G_0 cannot use more than $|Q|$ moves from positions derived from v_0, so unused moves can be safely removed.)

Now, suppose that G is a witness arena with only one Eve's position. We will construct a new arena with only two possible moves for Eve. The construction goes as follows:

- We start with $G^3 = G^0$. Let s be Eve's winning strategy in G^3.
- For each of Eve's $|Q|$ positions, we remove all moves except the one which is used by s.
- ($\star$) Let v_1 and v_2 be two Eve's positions in G^3.
- We merge Eve's positions v_1 and v_2 into one, v_0.
- Eve still has a winning strategy everywhere in this new game (by a reasoning similar to one we used for G^2). We check if Eve has a positional winning strategy.

- If yes, we remove the move which is not used in v_0, and go back to $(\star)$. (Two distinct Eve's positions in G^3 must still exist — if we were able to merge all Eve's positions into one, it would mean that G was positionally determined.)

- Otherwise G^3 is now a witness arena. In all Eve's positions except v_0 there is only one move, so we can safely transfer them to Adam, and G^3 will remain a witness arena.

- In G^3 we have now only one Eve's position (v_0) and only two Eve's moves — one inherited from v_1 and one inherited from v_2.

■

6.3 Decidability

Theorem 6.8 *Let W be a (prefix independent) ω-regular winning condition recognized by a DFA with parity acceptance condition $A = (Q, q_I, \delta, \mathrm{rank} : Q \to \{0 \dots d\})$ with n states. Then finite half-positional determinacy of W is decidable in time $n^{O(n^2)}$.*

Proof It is enough to check all possible witness arenas which agree with the hypothesis of Theorem 6.7. Such arena consists of (the only) Eve's position E from which she can move to A_1 by move m_1 or to A_2 by move m_2. Since we are working on ϵ-arenas (see Section 2.5), we can assume that $A_i \neq E$, and also that these two moves are ϵ-moves; otherwise we add a new Adam's position ,,in the middle of the move" and connect it with an ϵ-move. Adam has a choice of word w by which he will return to E from A_i. (In general it is possible that Adam can choose to never return to E. However, if such infinite path was winning for Eve, he would not choose it, and if it would be winning for Adam, Eve would never hope to win by choosing to move to A_i, thus she would always have to choose the other move, and thus our arena wouldn't be a witness.) Let L_i be the set of all possible Adam's return words from A_i to E.

Let $T(w) : Q \to \{0, \dots, d\} \times Q$ be the function defined as follows: $T(w)(q) = (r, q')$ iff $\delta(q, w) = q'$ and the greatest rank visited during these transitions is r. The function $T(w)$ contains all the information about $w \in L_i$ which is important for our game: if $T(w_1) = T(w_2)$ then it does not matter whether Adam chooses to return by w_1 or w_2 (the winner does not change). Thus, instead of Adam choosing a word w from L_i, we can assume that Adam chooses a function t from $T(L_i) \subseteq T(C^*) \subseteq (Q \times \{0, \dots, d\})^Q$.

For non-empty $R \subseteq \{0, \dots, d\}$, let $\mathrm{best}^{\mathrm{A}}(R)$ be the priority which is the best for Adam, i.e. the greatest odd element of R, or the smallest even one if there are no odd priorities in R. We also put $\mathrm{best}^{\mathrm{A}}(\emptyset) = \perp$.

For $T \subseteq (Q \times \{0, \dots, d\})^Q$, let

$$U(T)(q_1, q_2) = \mathrm{best}^{\mathrm{A}}(\{d : \exists t \in T\ t(q_1) = (d, q_2)\}).$$

Again, the function $U_i = U(T(L_i)) : Q \times Q \to \{\perp, 0, \dots, d\}$ contains all the information about L_i which is important for our game — if Adam can go from q_1 to q_2 by one of two words w_1 and w_2 having the highest priorities d_1 or d_2, respectively, he will never want to choose the one which is worse to him.

Our algorithm checks all possible functions U_i. For this, we need to know whether a particular function $U : Q \times Q \to \{\perp, 0, \dots, d\}$ is of form $U(T(L_i))$ for some L_i. This can be done in following way. We start with $V(q, q) = \perp$. Generate all elements of $T(L_i)$. This can be done by doing a search (e.g. breadth first search) on the graph whose vertices are $T(w)$ and edges are $T(w) \to T(wc)$ ($T(wc)$ obviously depends only on $T(w)$). For each of these elements, we check if it does not give Adam a better option than U is supposed to give — i.e. for some q_1 we have $T(wc)(q_1) = (q_2, d)$ and $d = \mathrm{best}^{\mathrm{A}}(d, U(q_1, q_2))$. If it does not, we add $T(w)$ to our set T and update V: for each q_1, $T(wc)(q_1) = (q_2, d)$, we set $V(q_1, q_2) := \mathrm{best}^{\mathrm{A}}(d, V(q_1, q_2))$. If after checking all elements of $T(L_i)$ we get $V = U$, then $U = U(T)$. Otherwise, there is no L such that $U = U(T(L))$.

The general algorithm is as follows:

- Generate all possible functions U of form $U(T(L))$.
- For each possible function U_1 describing Adam's possible moves after Eve's move m_1 such that Eve cannot win by always moving with m_1:
- For each U_2 (likewise):
- Check if Eve can win by using a non-positional strategy. (This is done easily by constructing an equivalent parity game which has $3|Q|$ vertices: $\{E, A_1, A_2\} \times Q$.) If yes, then we found a witness arena.

Time complexity of the first step is $O(d^{O(|Q|^2)}(d|Q|)^{|Q|}|C|)$ (for each of $d^{O(|Q|^2)}$ functions, we have to do a BFS on a graph of size $(d|Q|)^{|Q|}$). The parity game in the fourth step can be solved with one of the known algorithm for solving parity games, e.g. with the classical one in time $O(O(|Q|)^{d/2})$. This is done $O(d^{O(|Q|^2)})$ times. Thus, the whole algorithm runs in time $O(d^{O(|Q|^2)}|Q|^{|Q|}|C|)$. ∎

In the proof above the witness arena we find is an ϵ-arena: we did not assign any colors to moves m_1 and m_2. If we want to check whether the given condition is A-half-positional or B-half-positional (see Section 2.5), similar constructions work. For B-half-positional determinacy, we need to not only choose the sets U_1 and U_2, but also choose specific colors c_1 and

c_2 for both moves m_1 and m_2 in the algorithm above, and take care of the case when $A_1 = E$ or $A_2 = E$. For A-half-positional determinacy, we need to choose specific colors for targets of these two moves, and also a color for Eve's position E.

Once we know that an ω-regular winning condition W is indeed finitely half-positional, we can use the following algorithm to solve a game.

Proposition 6.9 *Suppose that G is an arena with n positions, and W is finitely half-positional and ω-regular, given by a deterministic parity automaton on infinite words using s states and d ranks.*

Then the winning sets in the game (G, W) can be found in time $O((ns)^{d/2})$, and Eve's positional strategy can be found in time $O((ns)^{d/2}t)$, where $t = \sum_{v \in \mathrm{Pos}_E} \log |v\mathrm{Mov}|$, where $|v\mathrm{Mov}|$ is the number of moves outgoing from v.

Proof As in the proof of Theorem 6.7, we transform our game (G, W) (with n positions) into a parity game (G^2, WP_d) (with ns positions). Winning sets and positional strategies in such a game can be determined in time $O((ns)^{d/2})$ (see e.g. [GTW02]).

To obtain Eve's strategy, we use the following reduction of the problem of finding Eve's positional winning strategy to the problem of finding the winning sets for both players (which actually works for all finitely half-positional winning conditions — not only ω-regular ones). If we remove Eve's move which is not used by her winning strategy, Win_E does not change. Thus, we can try to remove half of moves outgoing from one of Eve's positions, and see if Win_E changes — if yes, then Eve should use one of removed moves, otherwise Eve should use one of the remaining moves. We continue doing this until only one move remains in each Eve's position. ■

6.4 ω-regular concave conditions

The following proposition shows that concavity (see Chapter 9) is also decidable for ω-regular language (even in polynomial time). As shown in Theorem 4.5, concave winning conditions are finitely half-positional.

Proposition 6.10 *Suppose that a winning condition W is given by a deterministic parity automaton on infinite words using s states and d ranks. Then there exists a $O(s^6 d^3 |C|)$ algorithm determining whether W is concave (or convex).*

Definition 6.11 *For $q_1, q_2, q_3, r_1, r_2, r_3 \in Q$, $n_1, n_2, n_3 \in \{\perp, 0, \ldots, d\}$, we say that $P(q_1, r_1, n_1,\ q_2, r_2, n_2,\ q_3, r_3, n_3)$ iff there exists a word $w_3 \in C^*$ being a shuffle of w_1 and w_2 such that for each $k \in \{1, 2, 3\}$ we have $\delta(q_k, w_k) = r_k$, and n_k is the greatest rank of states appearing while the automaton works on w^k starting from q^k, i.e. $n^k = \max_{w_k = uv} \mathrm{rank}(\delta(q^k, u))$. In case if $w_k = \epsilon$ we take $n_k = \perp$.*

Lemma 6.12 *L_A is not convex iff for some $q_1, q_2, q_3, m_1, m_2, m_3, n_1, n_2, n_3$ we have $P(q_I, q_1, m_1,\ q_I, q_2, m_2,\ q_I, q_3, m_3)$ and $P(q_1, q_1, n_1,\ q_2, q_2, n_2,\ q_3, q_3, n_3)$ and n_1, n_2 are even and n_3 is odd. ($\perp$ is considered neither even nor odd.)*

Proof

($\leftarrow$) Let u_1, u_2 and u_3 be the words from 6.11 which are witnesses for $P(q_I, q_1, m_1,\ q_I, q_2, m_2,\ q_I, q_3, m_3)$, and v_1, v_2 and v_3 be the words which are witnesses for $P(q_1, q_1, n_1,\ q_2, q_2, n_2,\ q_3, q_3, n_3)$. Let $w_k = u_k v_k^\omega$. It can be easily shown that w_3 is a shuffle of w_1 and w_2 and $w_1, w_2 \in L_A$ but $w_3 \notin L_A$.

($\rightarrow$) Suppose that L_A is not convex, i.e., w^3 is a shuffle of w^1 and w^2, and $w^3 \notin L_A$.

Let $f : \omega \to \{1, 2\}$ be a function such that $w^k = \Pi_n w_n^{3\,[f(n)=k]}$ for $k = 1, 2$. (As on page 21, $w^{[\phi]}$ denotes w if ϕ, ϵ otherwise.)

Let $q_0^3 = q_I$, $q_{n+1}^3 = \delta(q_n^3, w_{n+1}^3)$. For $k = 1, 2$ let $q_0^k = q_I$, $q_{n+1}^k = \delta(q_n^k, w_{n+1}^k)$ if $f(n+1) = k$, and q_n^k otherwise. Let $S^k = \limsup q^k$ for $k = 1, 2, 3$.

Since $w^1, w^2 \in L_A$ and $w^3 \notin L_A$, we have that S^1 and S^2 are both even, but S^3 is not. It can be easily shown that there exist some a, b such that for all $k = 1, 2, 3$ we have $q^k(a) = q^k(b)$, and $\exists m \in \{a \ldots b\} \mathrm{rank}(q_m^k) = S^k$.

Let $q_k = q^k(a)$ and $n_k = S^k$ for $k = \{1, 2, 3\}$. It can be easily seen that our hypothesis holds. ■

Proof of Proposition 6.10 As we can see, to determine if L_A is convex it is enough to compute the predicate P and check the condition given in Lemma 6.12. Now, P satisfies the following rules: ($\vee$ means maximum, where $\perp$ is smaller than everything else)

- (1) For each q_1, q_2, q_3, $P(q_1, q_1, \perp,\ q_2, q_2, \perp,\ q_3, q_3, \perp)$;
- (2) For each $q_1, r_1, n_1,\ q_2, r_2, n_2,\ q_3, r_3, n_3, c$, if the predicate P satisfies $P(q_1, r_1, n_1,\ q_2, r_2, n_2,\ q_3, r_3, n_3)$ and $\delta(r_1, c) = s_1$ and $\delta(r_3, c) = s_3$ then $P(q_1, s_1, n_1 \vee \mathrm{rank}(s_1),\ q_2, r_2, n_2,\ q_3, s_3, n_3 \vee \mathrm{rank}(s_3))$.
- (3) For each $q_1, r_1, n_1,\ q_2, r_2, n_2,\ q_3, r_3, n_3, c$; if the predicate P satisfies $P(q_1, r_1, n_1,\ q_2, r_2, n_2, q_3,\ r_3, n_3)$ and $\delta(r_2, c) = s_2$ and $\delta(r_3, c) = s_3$ then $P(q_1, r_1, n_1,\ q_2, s_2, n_2 \vee \mathrm{rank}(s_2),\ q_3, s_3, n_3 \vee \mathrm{rank}(s_3))$.

Rule (1) corresponds to taking ϵ as the word w_3 from Definition 6.11, and rules (2) and (3) correspond to adding one letter c to w_1 and w_2, respectively.

Now, the algorithm of computing P is as follows: whenever we discover that $P(q_1, r_1, n_1, q_2, r_2, n_2, q_3, r_3, n_3)$ for some parameters, we close it under (2) and (3); our initial knowledge is given by (1). If $P(q_1, r_1, n_1, q_2, r_2, n_2, q_3, r_3, n_3)$, then our algorithm will find it out — by using a sequence of applications of rules (1), (2) and (3) which corresponds to the words w_1, w_2, w_3 (from Definition 6.11). Also, if our algorithm finds out that $P(q_1, r_1, n_1, q_2, r_2, n_2, q_3, r_3, n_3)$, we can reconstruct the words w_1, w_2, w_3 by analizing the sequence of applications of rules which our algorithm used. ■

Chapter 7

Unions of half-positional winning conditions

One of the main open problems which motivated our research was the following problem.

A union of co-Büchi and co-half-positional conditions does not need to be co-half-positional ($WB'_{\{a\}} \cup WB'_{\{b\}}$ is not). What about a union of co-Büchi and a half-positional condition, does it have to be half-positional? We have no proof nor counterexample for this yet. This conjecture can be generalized to the following:

Conjecture 7.1 *Let $\mathcal{W}$ be a (finite, countable, ...) family of half-positionally (finitely) determined winning conditions. Then $\bigcup \mathcal{W}$ is a half-positionally (finitely) determined winning condition.*

Note again that we assume prefix independence here. It is very easy to find two prefix dependent winning conditions which are positionally determined, but their union is not half-positionally determined.

In the first section, we show that this conjecture fails for non-countable unions and infinite arenas, even for such simple conditions as Büchi and co-Büchi conditions. In the second section, we present a broad class of winning conditions which is closed under countable union, and includes some of the previously mentioned winning conditions. In the third section we present a yet broader class of winning conditions, which has another closure properties (although is known to be closed only under finite union). In the last section we say more about links between this conjecture and monotonic and concave conditions.

7.1 Uncountable unions

Theorem 7.2 *There exists a family of 2^ω Büchi (co-Büchi) conditions such that its union is not a half-positionally determined winning condition.*

Proof Let $I = \omega^\omega$. Our arena A consists of one Eve's position E and infinitely many Adam's positions $(A_n)_{n\in\omega}$. In E Eve can choose $n \in \omega$ and go to A_n by move $E \xrightarrow{\star} A_n$. In each A_n Adam can choose $r \in \omega$ and return to E by move $A_n \xrightarrow{(n,r)} E$.

For each $y \in I$, let $S_y = \{(n, y_n) : n \in \omega\} \subseteq C$, and $S'_y = C - S_y - \{\star\}$. Let $WA_1 = \bigcup_{y\in I} WB_{S_y}$, $WA_2 = \bigcup_{y\in I} WB'_{S'_y}$.

The games (A, WA_1) and (A, WA_2) are not half-positionally determined. Let (n_k) and (r_k) be n and r chosen by Eve and Adam in the k-th round, respectively. If Eve always plays $n_k = k$, she will win both the conditions WB_{S_y} and $WB'_{S'_y}$, where $y_k = r_k$. However, if Eve plays with a positional strategy $n_k = n$, Adam can win by playing $r_k = k$. ■

7.2 Positional/suspendable conditions

Definition 7.3 *A* **suspendable winning strategy** *for player X is a pair (s, Σ), where $s : \mathrm{Play}_X \to \mathrm{Mov}$ is a strategy, and $\Sigma \subseteq \mathrm{Play}_F$, such that:*

- *s is defined for every finite play π such that $\mathrm{target}(\pi) \in \mathrm{Pos}_X$.*
- *every infinite play π that is consistent with s from some point t [1] has a prefix longer than t which is in Σ;*
- *Every infinite play π that has infinitely many prefixes in Σ is winning for X.*

We say that X has a **suspendable winning strategy in** Win_X *when he has a suspendable winning strategy in the arena $(\mathrm{Pos}_A \cap \mathrm{Win}_X, \mathrm{Pos}_E \cap \mathrm{Win}_X, \mathrm{Mov} \cap \mathrm{Win}_X \times \mathrm{Win}_X \times C)$.*

A winning condition W is **positional/suspendable** *if for each arena G in the game (G, W) Eve has a positional winning strategy in Win_E and Adam has a suspendable winning strategy in Win_A.*

Intuitively, if at some moment X decides to play consistently with s, the play will eventually reach Σ; Σ is the set of moments when X can temporarily suspend using the strategy s and return to it later without a risk of ruining his or her victory.

A suspendable winning strategy is a winning strategy, because from the conditions above we know that each play which is always consistent with s has infinitely many prefixes in Σ, and thus is winning for X. The co-Büchi condition is positional/suspendable; more examples will be given in Theorems 5.9 and 6.5. However, the parity condition WP_2 is positional,

[1] That is, for each prefix u of π which is longer than t and such that $\mathrm{target}(u) \in \mathrm{Pos}_X$, the next move is given by $s(u)$.

but not positional/suspendable, because a suspendable strategy cannot be winning for Adam — it is possible that the play enters state 2 infinitely many times while it is suspended.

We will now show that the positional/suspendable winning conditions are common. Some of conditions which we have previously shown to be half-positional are actually positional-suspendable.

Theorem 7.4 *The condition $WF(A) = \{w : \limsup P_n(f(w)) < 0\}$ given in Theorem 5.9 is positional/suspendable.*

Proof Consider Adam's strategy given in the proof of 5.9. That strategy led us to a word $w = w_1 w_2 w_3 \ldots$, where $P(w_i) > a_i$. If we allow an initial segment to be played not according to this strategy, we will get a word $w = u w_i w_{i+1} \ldots$ instead. Still, there will be a t such that $P_t(w) > a_i$; and we can suspend at time t. Thus, $\limsup P_n(w)$ is still at least $1/2$. ∎

Note that $WF(A_1) \cup WF(A_2)$ usually is not equal to $WF(A_1 \cup A_2)$, so a union of positional/suspendable conditions given above usually is not of form $WF(A)$ itself.

Theorem 7.5 *Any monotonic condition (Theorem 6.5) is positional/suspendable.*

Proof Again, Adam's strategy given in the proof of 6.5 is suspendable, because he can suspend his strategy after each step of the iteration above.
∎

Theorem 7.6 *A union of countably many positional/suspendable conditions is also positional/suspendable.*

Proof of Theorem 7.6 A lemma similar to Lemma 3.1 can be used to prove that a winning condition is positional/suspendable. Its proof is also similar.

Lemma 7.7 *Let W be a winning condition. Suppose that, for each non-empty arena G, either there exists a non-empty subset $M \subseteq G$ where Eve has a positional winning strategy in M, or Adam has an suspendable winning strategy everywhere. Then W is positional/suspendable.*

Let $\{W_1, W_2, \ldots\}$ be a countable set of positional/suspendable conditions.

If for some i we have $v \in \mathrm{Win}_E(G, W_i)$, then Eve wins from v in G also, by using the same positional strategy.

Now assume that for each i we have $\mathrm{Win}_E(G, W_i) = \emptyset$, hence for every i Adam has a suspendable strategy (s_i, Σ_i) in (G, W_i). We will show that Adam can win everywhere in $(G, \bigcup_i W_i)$ then.

Let $(i_k)_{k\in\omega}$ be a sequence where every index i appears infinitely often. The strategy of Adam is to first play consistent with s_{i_1} until Σ_{i_1} happens, then play consistent with s_{i_2} until Σ_{i_2} happens, and so on. Since every Σ_i happens infinitely many times, Adam wins each W_i.

This proves that for each arena G either Eve wins somewhere positionally, or Adam wins everywhere, which from Lemma 3.1 means that $\bigcup_i W_i$ is half-positional. To get that it it is positional/suspendable, we have to use Lemma 7.7 and note that Adam's strategy given above is suspendable (Σ is the set of moments when Adam reaches Σ_{i_k} while playing consistently with s_{i_k}). ■

7.3 Extended positional/suspendable conditions

In this section we present a class of half-positional winning conditions which generalizes both positional/suspendable conditions and Rabin conditions. The proof is a modification and generalization of proof of half-positional determinacy of Rabin conditions from [Gra04]. However, it is by induction, hence we need to restrict ourselves to finite union.

Definition 7.8 *The class of* **extended positional/suspendable (XPS)** *conditions over C is the smallest set of winning conditions that contains all Büchi and positional/ suspendable conditions, is closed under intersection with co-Büchi conditions, and is closed under finite union.*

Theorem 7.9 *All XPS conditions are half-positional.*

Proof Let W be an XPS condition. The proof is by induction over construction of W.

We know that Büchi conditions and positional/suspendable conditions are half-positional.

If W is a finite union of simpler XPS conditions, and one of them is a Büchi condition WB_S, then $W = W' \cup WB_S$. Then W' is half-positional since it is a simpler XPS condition, and from Theorem 3.3 we get that W is also half-positional.

Otherwise, $W = W' \cup \bigcup_{k=1}^{n} (W_k \cap WB'_{S_k})$, where W' is a positional/ suspendable condition, W_k is a simpler XPS condition, and WB'_{S_k} is a co-Büchi condition. (It is also possible that there is no W', but it is enough to consider this case since it is more general. A union of several positional/suspendable conditions is also positional/suspendable.) To apply Lemma 3.1 we need to show that either Eve has a positional winning strategy from some position in the arena, or Adam has a winning strategy everywhere.

For $m = 1, \ldots, n$ let $W^{(m)} = W' \cup W_m \cup \bigcup_{k \neq m} (W_k \cap WB'_{S_k})$. We know that $W^{(m)}$ is half-positional since it is a simpler XPS condition.

Let G be an arena. Let H_m be the greatest subgraph of G which has no moves colored with any of the colors from S_m, Adam's positions where he can make a move not in H_m, and moves which lead to positions which are not in H_m (i.e. it is the subgraph where Adam is unable to force doing a move colored with a color from S_m). If Eve has a positional winning strategy from some starting position v in $(H_m, W^{(m)})$, then she can use the same strategy in (G, W) and win (Eve has more options in G hence she can use the same strategy, and this strategy forces moves colored with S_m to never appear).

Assume that Eve has no positional strategy for any starting position and m. Then Adam has a following winning strategy in (G, W):

- Adam uses his suspendable strategy (s, Σ) for the game (G, W'), until the play reaches Σ.
- For $m = 1, \ldots, n$:
  - Let v be the current position.
  - If $v \in H_m$ then Adam uses his winning strategy s'_m in $(H_m, W^{(m)})$. (Adam forgets what has happened so far in order to use s'_m.) If Eve never makes a move which does not belong to H_m then Adam wins. Otherwise, he stops using s'_m in some position v.
  - If $v \notin H_m$ then Adam performs a sequence of moves which finally lead to a move colored with S_m. (He does not need to do that if Eve made a move which is not in H_m, since it is already colored with S_m.)
- Repeat.

If finally the game remains in some H_m, then Adam wins since he is using a winning strategy in $(H_m, W^{(m)})$. Otherwise, Adam wins W' and all the co-Büchi conditions WB'_{S_k} for $k = 1, \ldots, n$, hence he also wins $W \subseteq W' \cup \bigcup_{k=1}^{n} WB'_{S_k}$. ■

7.4 Between concave and monotonic conditions

In this section we investigate how our conjecture works for concave and monotonic conditions. First, we will state the following:

Proposition 7.10 *Concave winning conditions are closed under union. Convex winning conditions are closed under intersection.*

This is obvious from the definition.

Proposition 7.11 *Monotonic conditions are closed under finite union.*

Proof It can be easily shown that $C^\omega - W_{A_1} \cup W_{A_2} = L^\omega_{A_1} \cap L^\omega_{A_2} = (C^* L_{A_1})^\omega \cap (C^* L_{A_2})^\omega = (C^* L_{A_1} C^* L_{A_2})^\omega = (L_{A_1} L_{A_2})^\omega$. The language $(L_{A_1} L_{A_2})^\omega$ is recognized by the monotonic automaton $A_s = (n_1 + n_2, \delta)$, where $\delta(q, c) = \delta_1(q, c)$ for $0 \le q < n_1$ and $\delta(n_1 + q, c) = n_1 + \delta_2(q, c)$ for $0 \le q \le n_2$. ■

A countable union of monotonic conditions is not necessarily defined by a single monotonic automaton, but it is still positional/suspendable; however, a union of cardinality 2^ω of monotonic conditions does not have to be half-positionally determined, since co-Büchi conditions are monotonic. Monotonic conditions are not closed under other Boolean operations.

Theorem 7.12 *Let $W_1 \subseteq C^\omega$ be a concave winning condition, and A be a monotonic automaton. Then the union $W = W_1 \cup WM_A$ is a half-positionally finitely determined winning condition.*

Proof of Theorem 7.12 Let $G = (\mathrm{Pos}_A, \mathrm{Pos}_E, \mathrm{Mov})$. A proof by induction on $|\mathrm{Mov}|$. We define v, $M = M_1 \cup M_2$, G_1, G_2, Win^A_i, and t_i exactly like in the proof of Theorem 4.5. We will show a winning strategy for Adam in the set Win^A_1 in the case when $v \in \mathrm{Win}^A_1 = \mathrm{Win}^A_2$; all other cases are done just like in the proof of Theorem 4.5.

The sequence $(q_k(\pi))$ and $(r_k(\pi))$ are defined by induction: $q_0(\pi) = 0$; $r_k(\pi) = 0$ if $q_k(\pi) = n$ and $\mathrm{source}(\pi_{k+1}) = v$, and $q_k(\pi)$ otherwise; and $q_{k+1}(\pi) = \delta(r_k(\pi), \mathrm{rank}\ \pi_{k+1})$. If the play π visits v infinitely many times, then $\mathrm{rank}(\pi) \notin WM_A$ iff $q_k(\pi) = n$ for infinitely many values of k.

Let $\pi = \pi_1 \ldots \pi_m$ be a finite play. Let $K = \mathrm{dom}\ \pi = \{1, \ldots, m\}$. Let $S_v = \{k \in K : \mathrm{source}(\pi_k) = v\}$. We define the function $f : K \to \{1, 2\} \times Q$ in the following way. If $k < \min S_v$, we take $f(k) = (1, 0)$. Otherwise, let k' be the greatest element of S_v such that $k' \le k$, and $f(k) = (i, q_{k'})$ iff $\pi_{k'} \in M_i$.

Let $\pi_{(i,q)} = \Pi_{k \in K} \pi_k^{[f(k)=(i,q)]}$. One can easily see that π, as a word over Mov, is then a shuffle of all $\pi_{(i,q)}$. (A shuffle of more than two words is defined in an obvious way.)

It can be easily checked that $\pi_{(i,q)}$ is a play. For $(j, q) = f(n)$ we have $\mathrm{target}(\pi_{(j,q)}) = \mathrm{target}(\pi)$. Let $t(\pi) = t_j(\pi_{(j,q)})$. If Adam consistently plays with the strategy t, all plays $\pi_{(i,q)}$ are consistent with t_i for $i = 1, 2$ and each $q \in Q$.

We check that t is indeed a winning strategy for Adam in the set Win^A_1. Let π be an infinite play consistent with t; we have to show that $\mathrm{rank}(\pi) \notin W_1$ and $\mathrm{rank}(\pi) \notin WM_A$.

Like for finite plays, π is a shuffle of $\pi_{(i,q)}$ for all $i = 1, 2$ and $q \in Q$. For each (i, q) we have $\mathrm{rank}(\pi_{(i,q)}) \notin W_1$, hence from concavity of W_1 we have that $\mathrm{rank}(\pi) \notin W_1$.

Let $S \in \{1,2\} \times Q$ be the set of all (j,q) such that $f(m) = (j,q)$ for infinitely many values of m. Let (j_s, q_s) be the element of S with the greatest value of q_s. Assume $q_s < n$, otherwise $\mathrm{rank}(\pi) \notin WM_A$ is obvious.

Adam wins the play $\pi' = \pi_{(j_s,q_s)}$ since it is consistent with t_{j_s}. The play π' is infinite. Let $S'_v = \{k \in \omega : \text{source } \pi'_k = v\}$. If S'_v is finite, this means that π and π' have a common suffix (as we don't return to v we are stuck in $\pi_{(j_s,q_s)}$), and from the prefix independence of WM_A Adam wins π. Otherwise π' visits v infinitely many times, and hence $q_k(\pi') = n$ for infinitely many values of k.

For $m \in S'_v$ let $m^+ = \min\{m^+ \in S'_v : m^+ > m\}$, and P_m be the segment of play from $m+1$-th to m^+-th move.

For each $m_0 \in S'_v$ there exists $m > m_0$ such that $r_m(\pi') \leq q_s$ and $q_{m^+}(\pi') = \delta(r_m(\pi'), P_m) > q_s$. Otherwise by induction we would have that $r_m(\pi') \leq q_s$ for each m_0, which is impossible since $q_k(\pi') = n$ for infinitely many values of k.

The segment P_m appears also in play π after some m'-th move, where $q_{m'}(\pi) = q_s$. Afterwards we have $q_{m'+|P_m|}(\pi) = \delta(q_{m'}(\pi), P_m) = \delta(q_s, P_m) \geq \delta(r_m(\pi'), P_m) > q_s$, and we are back in v. Hence we have found that in π after $m' + |P_m|$ moves we are back in v with the automaton state greater than q_s. since there is an infinite number of such m's, we are back in v with the automaton state greater than q_s infinitely many times, which contradicts the definition of S.

■

Chapter 8

Beyond positional strategies

When it is impossible to win the game using no memory, we can still hope to use the smallest possible amount of memory states. In this chapter we estimate memory required by the other player for the winning conditions which were introduced before.

8.1 Definition

Below is the definition of memory and a strategy with memory which is commonly used in literature, e.g. in [DJW97] where memory required to win a game using a Müller condition is calculated.

Definition 8.1 *A* **memory** *for a game* (G, W) *is a triple* $\mathcal{M} = (M, m_I, \mu)$*, where* M *represents possible memory states,* $m_I \in M$ *is the* **initial memory state***, and* $\mu : M \times \mathrm{Mov} \to M$ *is the* **memory update function***. We define* $m_{\mathcal{M}} : \mathrm{Play}_F \to M$ *inductively by* $m_{\mathcal{M}}(v) = m_0$ *for* $v \in \mathrm{Pos}$*, and* $m_{\mathcal{M}}(\pi e) = \mu(m_{\mathcal{M}}(\pi), e)$*.*

A **strategy with memory** $\mathcal{M}$ *is a strategy* s *such that* $s(\pi)$ *depends only on* $\mathrm{target}(\pi)$ *and* $m_{\mathcal{M}}(\pi)$*.*

8.2 Chromatic memory

As we can see, this standard definition of memory is strongly dependant on the arena. Since in this paper we are interested in properties of winning conditions rather than games, we hope for memory which could be defined regardless of the arena. Below is the natural definition.

Definition 8.2 *We say that a memory* $\mathcal{M}$ *is* **chromatic** *if it depends only on colors of the moves, i.e. there exists a function* $\widehat{\mu} : M \times C \to M$ *such that* $\mu(m, e) = m$ *when* $\mathrm{rank}(e) = \epsilon$*, and* $\mu(m, e) = \widehat{\mu}(m, \mathrm{rank}(e))$ *otherwise. We extend* $\widehat{\mu}$ *to* $\widehat{\mu}^* : M \times C^* \to M$ *in the natural way.*

A **strategy with chromatic memory** $(M, m_I, \widehat{\mu})$ *is a strategy with memory* (M, m_I, μ) *where* μ *is a chromatic memory given by* $\widehat{\mu}$*.*

Example 8.3 *Let WQ be the winning condition from the proof of Proposition 4.7, and G be an arena with one Adam's position and two moves, which are colored 0 and 1. Then Adam has a winning strategy in* (G, WQ)*, but has no winning strategy with memory of any finite size.*

Example 8.4 *Let* W *be an* ω*-regular winning condition, recognized by a DFA* $A = (Q, q_I, \delta, \mathrm{rank})$*. Then both players have strategies with chromatic memory* $\mathcal{M} = (Q, q_I, \delta)$ *in their winning sets.*

Proof Positional winning strategies in the game (G^1, WP_d) defined in Theorem 6.7 can be interpreted as strategies with such a chromatic memory. ∎

Example 8.5 *Let* $W = WM_A$ *be a monotonic winning condition, where* $A = (n, \delta)$*. Then Adam has a winning strategy with chromatic memory* $\mathcal{M} = (\{0, \ldots, n-1\}, 0, \widehat{\mu})$ *in his winning set, where* $\widehat{\mu}(k, c) = \delta(k, c) \bmod n$*.*

Proof Adam's strategy given in the proof of Theorem 6.5 can be interpreted as a strategy with such a chromatic memory. ∎

Our definitions allow us to construct new determinacy types, which require one or both players to have a strategy with (chromatic) memory of finite size, or a strategy with (chromatic) memory of size n. Lemma 3.1 still works for these new determinacy types.

For a winning condition W, let $\mathrm{mm}_X(W)$ be the smallest n such that the player X can win with a strategy with memory of size n, and mm^χ_X be respectively the smallest size of chromatic memory. Obviously $\mathrm{mm}^\chi_X(W) \geq \mathrm{mm}_X(W)$; a natural question arises whether $\mathrm{mm}^\chi_X(W) = \mathrm{mm}_X(W)$.

The following example does not answer this question.

Example 8.6 *For the winning condition* $W = C^\omega - (b^* a^N)^\omega$ *we have* $\mathrm{mm}_A(W) = \mathrm{mm}^\chi_A(W) = N$*.*

Note that the winning condition given above is half-positional since it is a monotonic condition (Example 6.3).

Proof We will construct our arena from *gadgets*, i.e. subgraphs which perform required simple operations. By memory state we mean state of memory $\mathcal{M}$ from Example 8.5 (i.e. the number of as at the end of our word so far).

Let s_n be a *synchronizer*, i.e. a gadget which sets the memory state to n. (This is just a sequence of moves of colors ba^n.)

Let w_n be a gadget such that when w_i is entered by Eve in state $\geq i$ infinitely many times, then Eve wins. (Again, this is just a sequence of colors a^{N-n}.)

In our arena, we will have only one Eve's position, E. Eve has to decide between N moves, numbered 0 to $N-1$.

When Eve decides to do move n, then Adam can decide between $w_n s_0$ and (for $n < N-1$) s_{n+1}.

Adam's optimal strategy is as follows. If Eve is in E in state i, and decides to use move $j \leq i$, then Adam will choose s_{j+1} (if possible, i.e. $j < N-1$). If she decides to use move $j > i$, then Adam will choose $w_j s_0$.

Eve wins iff the play enters w_i in state $\geq i$ infinitely many times. We can easily see that Eve has to make use of all her possible moves to win against Adam's optimal strategy given above. Thus, she needs memory of size N, since there are so many available moves.

This construction also works for some winning conditions other than the one given in hypothesis (by using appropriate gadgets, which can be more complicated than for $W = C^\omega - (b^* a^N)^\omega$). ■

8.3 Chromatic memory requirements

In this section, we extend Theorems 6.7 and 6.8 to calculate chromatic memory requirements.

Definition 8.7 *Let $\mathcal{M} = (M, m_I, \widehat{\mu})$ be a chromatic memory. An arena G* **adheres to** $\mathcal{M}$ *iff there is a function $\phi : G \to M$ such that for each move $(v, w, c) \in G$ we have $\widehat{\mu}(\phi(v), c) = \phi(w)$.*

Proposition 8.8 *If G adheres to $\mathcal{M}$, and $\phi(v_I) = m_I$, then in (G, W) each player has a positional strategy from v_I iff he or she has a strategy with chromatic memory $\mathcal{M}$.*

Theorem 8.9 *Let W be a winning condition accepted by a deterministic finite automaton with parity acceptance condition $A = (Q, q_I, \delta, \mathrm{rank} : Q \to \{0 \dots d\})$ (see Definition 6.1), and $\mathcal{M}$ be a chromatic memory. Let a* witness arena *be an arena G such that in game (G_0, W) Eve has a winning strategy, but not a winning strategy which uses $\mathcal{M}$ as memory. Then, if there exists a witness arena, then there exists a witness arena G such that G adheres to $\mathcal{M}$, and Eve has only one position in $G(v_0)$, and only two moves from here.*

Proof Without loss of generality we can assume that $\mathcal{M}$ is connected, i.e. such that for each $m \in \mu$ there is a word $w \in C^*$ such that $\widehat{\mu}^*(m_I, w) = m$. Otherwise we can remove from M memory states which are not accessible.

Let $G_0 = (\mathrm{Pos}_A, \mathrm{Pos}_E, \mathrm{Mov})$ be a witness arena.

First, we construct an arena G_1 such that G_1 adheres to $\mathcal{M}$. Let $G_1 = (\mathrm{Pos}^1_A, \mathrm{Pos}^1_E, \mathrm{Mov}^1)$, where $\mathrm{Pos}^1_X = Pos_X \times M$ and $\mathrm{Mov}^1 = \{((v_1, m_1), (v_2, m_2), c) : (v_1, v_2, c) \in Mov \wedge \widehat{\mu}(m_1, c) = m_2\}$. One can easily check that G_1 is also a witness arena. (To show that G_1 adheres to $\mathcal{M}$, take $\phi(v, m) = m$.)

According to Proposition 8.8, G_1 is also a witness arena against positionality of W. Thus, we can now apply to it the same simplification which we used in Theorem 6.7, obtaining a new arena G_2 with only one Eve's position and two moves, and adhering to $\mathcal{M}$, since simplification preserves adherence.

Now, if the play is in position v_0 and Eve's memory is in state $m = \phi(v_0)$, then she has no winning $\mathcal{M}$-strategy for continuation (Proposition 8.8 again). To obtain our hypothesis, we only have to force that memory actually will be in state m when the play is in v_0. To obtain this, we add an (Adam's) initial position v_I and connect it to v_0 via a color word w such that $\widehat{\mu}^*(m_I, w) = m$. ■

Theorem 8.10 *Let W be a winning condition accepted by a deterministic finite automaton with parity acceptance condition $A = (Q, q_I, \delta, \mathrm{rank} : Q \to \{0 \dots d\})$. Then $\mathrm{mm}^{\chi}_E(A)$ can be calculated in single exponential time.*

Proof We check all possible chromatic memories of size up to $|Q|$ until we find one which works. There are $O(|Q|^{|Q||C|})$ such memories, and checking whether memory works can be done in a way analogous to Theorem 6.8. ■

Chapter 9

Conclusion

As a conclusion, we recollect open problems which are related to the results presented in this paper.

9.1 Closure under union

We would like to know more closure properties of the class of half-positionally determined winning conditions. Specifically, conjecture 7.1 asks whether an union (finite or countable) of (finitely) half-positional conditions remains half-positional. It is known that an uncountable union does not need to be half-positional (Theorem 7.2). In many special cases it is known that unions of specific half-positional conditions are half-positional (Theorem 3.3, Chapter 7).

Specifically, we know that all XPS conditions have this property (Theorem 7.9), and that this class is closed under finite union. All half-positional winning conditions constructed in this paper (except concave conditions – this is a property rather than a construction) fall in this class; it is possible that XPS captures all the reasons for a winning condition to be half-positional, and therefore all half-positional conditions are there.

9.2 ω-regular conditions

In Theorem 6.8 we have shown that *finite* half-positional determinacy of winning conditions is decidable. We used the fact that if a winning condition is not half-positional, then there is a very simple arena witnessing it (Theorem 6.7); this fact was obtained via induction over the number of Eve's positions where her move is not decided. However, what about *infinite* half-positional determinacy? In this case we can no longer use our inductive argument. One can easily create an infinite arena where applying the method used in proof of Theorem 6.7 leads to an arena where Eve no longer can win (after an infinite number of steps).

Also, the algorithm given in Theorem 6.8 is exponential, which is not satisfying. It is possible that there is a simpler property which also answers whether a given ω-regular winning condition is (finitely) half-positional.

9.3 Types of arenas

In section 2.5 we have introduced three types of arenas. We have shown examples of winning conditions which are half-positional when restricted to position-colored arenas, but not on all edge-colored arenas. We gave some arguments why we regard the broader classes of arenas as more natural when talking about positional determinacy.

The problem remains whether a winning condition which is half-positional with respect to edge-colored arenas has to be half-positional for all ϵ-arenas.

9.4 Chromatic memory

Section 8.2 raises a problem about strategies which are allowed to use memory, but want to use as few memory states as possible. Is it always possible to create a memory of the smallest possible size which also has a nice property of being independent from the arena, i.e. a chromatic memory of size $\mathrm{mm}_X(W)$? We already have an algorithm for calculating $\mathrm{mm}_X^\chi(W)$ (Theorem 8.10), but not for $\mathrm{mm}_X(W)$ — a positive answer would mean that we don't need another one. This result could potentially simplify proofs of further results about strategies with memory.

9.5 Geometrical conditions

The results in Chapter 5 do not cover all possible cases. We still do not know whether $WF(A)$ is finitely half-positionally determined for all co-convex sets A, and whether it is half-positionally determined for all co-convex open sets A. For As which are unions of a finite number of half-spaces, e.g. $A = A_1 \cup A_2$, we cannot obtain half-positional determinacy via Theorem 5.9 and Theorem 7.6 (union of positional/ suspendable conditions), because this does not leads to $WF(A)$ but to another set which is a bit different: $WF(A_1 \cup A_2)$ says that each cluster point is element of $A_1 \cup A_2$, and $WF(A_1) \cup WF(A_2)$ says that either each cluster point is an element of A_1 or each cluster point is an element of A_2.

9.6 Extensions

Another area of research is to extend our results to more general settings. There are several possible extensions.

One of them is examining *payoff mappings* or *preference relations* instead of winning conditions, which allow a game to have a wide spectrum of results instead of win or lose. See Section 2.6, and also [GZ04], [GZ05], [EM79].

We can also try to relax our requirement of prefix independence. In we hope for positional strategies, then the most important thing about prefix independence is that the past should not alter what is good for us in the future. For payoff mappings and preference relations it can be written as $u(w_1) \leq u(w_2)$ if and only if $u(cw_1) \leq u(cw_2)$. We could also use an even weaker version where we have only one implication instead of equivalence. *Monotone preference relations* are defined in [GZ05] in a similar way.

So far, we have either considered all possible arenas or restricted to finite arenas only. However, there are more examples of interesting classes of arenas. One example is push-down graphs, which are infinite, but have a finite representation. Another one is infinite arenas with finite branching. And of course, more research of position-colored arenas would be useful.

Another generalization is *stochastic games*. In addition to Eve's positions (where a ,,good" player decides) and Adam's positions (where a ,,bad" player decides), these games also allow random positions, where a move is decided randomly. In this setting we are interested in *optimal* strategies, which lead to the greatest possible probability of winning, or the greatest expected value of payoff in case of payoff mappings. Several new papers by Gimbert and Zielonka, e.g. [GZ07], explore this setting.

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