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Intuitionistic Games: Determinacy, Completeness, and Normalization

Abstract. We investigate a simple game paradigm for intuitionistic logic, inspired by Wajsberg's implicit inhabitation algorithm and Beth tableaux. The principal idea is that one player, ∃ros, is trying to construct a proof in normal form (positions in the game represent his progress in proof construction) while his opponent, ∀phrodite, attempts to build a counter-model (positions or plays can be seen as states in a Kripke model). The determinacy of the game (a proof-construction and a model-construction game in one) implies therefore both completeness and semantic cut-elimination.

Keywords: Intuitionistic logic, Games, Proofs, Inhabitation.

1. Introduction

Using games to explain various aspects of logic has a long and well-established tradition, both from the semantic and proof-theoretic point of view. This tradition originated from works of such authors as Ehrenfeucht, Fraïssé, Lorenzen, Beth, Hintikka and many others. The recent books of van Benthem [19] and Väänanen [18] give broad exposition and bibliography of the subject seen from two different perspectives.

The principal aim of this paper is to use the game paradigm to demonstrate the duality between proof construction and counter-model construction in intuitionistic logic. This duality is a basic phenomenon which seems to be very natural, and is mentioned in various forms in the literature. For instance, the following explicit statement can be found in [19]: *Here we see two faces of tableaus. Read top-down, they are attempts at finding a countermodel, bottom-up (when closed), they are proofs.* From [19] one learns that Lorenzen and Beth must have been aware of the duality already in the 1950's.

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On the other hand, there is certainly a common-sense understanding that logic can be seen as a game: One player is trying to build a proof of a formula, the other attempts to construct a refutation; the latter yields a counter-model. The determinacy of the game thus implies both completeness and semantic cut-elimination (understood as: "every provable formula has a normal proof").
It is very appealing to see both the proof construction and the model construction process as a unified game structure. This appears obvious or folklore to most logicians, yet it turns out difficult to find it directly addressed and studied in detail in the literature. Most game interpretations known to the author are either "proof-oriented" (like e.g. in [15]) or "model-oriented" (as in [18]), somehow missing the simple message that the two players are always playing essentially the same game. On the other hand, if the duality proof-model or proof-refutation is stated, the game flavour is often missing. A few references that seem closest to our way of thinking are [7,10,12,19,21], also the ludics paradigm [8] reveals the duality.

We find it worthwhile to fill this apparent gap, and make the game paradigm fully straightforward and explicit. We define games for intuitionistic logic where two players compete with each other. One is ∃ros, the prover, the other is a reviewer called ∀phrodite. It is quite obvious that any ∃ros' winning strategy yields a proof, and we also demonstrate how a Kripke model (understood as e.g. in [6,15]) can be extracted from ∀phrodite's strategy. This extraction is easy, but not entirely obvious: the natural idea "game position = state" turns out not fully correct. Indeed, we show why some naive attempts must fail. In particular, in the first-order case it turns out that not every strategy of ∀phrodite can actually define a model. We have to consider the largest non-deterministic strategy rather that *any* strategy. In addition, states of our first-order Kripke models must be *plays* of the game rather than just *positions*.

The way our simplest game is defined in Section 2 can be seen as another formulation of a cut-free sequent calculus. But we prefer to use natural deduction, because it comes together with a natural lambda-notation for proofs. Lambda-calculus serves the same purpose as the calculus of strategies postulated in [19, Chap. 18] and seems to make a reasonable choice.

According to the Curry-Howard isomorphism [15], proof search is equivalent to finding type inhabitants. The problem of type inhabitation is a well-studied subject [4,9,22] for the simply-typed lambda-calculus $\lambda_\to$, corresponding to the implicational fragment of propositional intuitionistic logic.

The game-theoretic flavour of the so-called Ben-Yelles algorithm [9] for type inhabitation is widely recognized [1]. Ben-Yelles algorithm differs however from our games in some aspects introducing more efficiency to proof search. Proofs are sought in so-called long normal forms, where all variables are "fully applied". Certain game turns are collapsed to single moves, and the existential player is obliged to directly address the current goal. This resembles focused proofs [11]. Yet another nice aspect of what could be called "Ben-Yelles game" is that it can be easily represented using trees [4].

The Ben-Yelles game does not apply to the full propositional logic; the awkward behaviour of disjunction discourages many authors from addressing the general case, so papers like [10] often treat only the implicational fragment. It is time to give an explicit exposition of how Ben-Yelles should generalize to the full language, and this is the second aim of this paper.

While the issue of inhabitation for the extended lambda-calculus with $\vee$, $\wedge$ and $\perp$ can be seen as under-developed, it may come as a surprise that the principal ideas were published by M. Wajsberg already in 1938 [20] (see Bezhanishvili [2] for correction of a mistake). Inspired by Wajsberg we explain step by step how Ben-Yelles games can be generalized.

We begin with a very simple definition of game rules for intuitionistic propositional logic in Section 2. Under this definition, winning strategies of $\exists$ros correspond to lambda-terms in normal form. In Section 2.2 we propose a definition of a long normal form for the extended lambda-calculus and we adjust games so that only terms in long normal form are permitted as strategies. Then in Section 3 we consider a "fast game" formulation, where certain sequences of game steps can be "collapsed" into single ones, as in Ben-Yelles game. This is close to the proof-search methodology occurring e.g. in Coq [16]. The next step in this direction is to restrict $\exists$ros' strategies to *standard* ones, where the prover may only apply assumptions relevant to the current goal. Under such a restriction we obtain the proof-search technique we call *Wajsberg's algorithm*. In Section 4 we show how the propositional game may be played on a tree representation of a formula. This makes a helpful didactic tool: the construction of a proof can be illustrated by placing tokens on a diagram.

Finally, in Section 5 we attempt to extend the approach to first-order logic. Unfortunately in the first-order case we can no longer support the slogan "every $\forall$phrodite's strategy determines a Kripke model". To define a model we need to gather all winning positions of $\forall$phrodite. But this still suffices to prove completeness for Kripke models as well as normalization for first-order logic.

2. Propositional Logic: The Simple Game and the Long Game

We consider propositional formulas built over the standard set of connectives $\wedge$, $\vee$, $\to$, $\bot$. (Negation is understood as $\neg\alpha = \alpha \to \bot$, and therefore omitted.) We use the convention that implication is right-associative, and of lower priority than $\vee$ and $\wedge$. Thus, $\alpha\vee\beta \to \gamma \to \delta$ means $(\alpha\vee\beta) \to (\gamma \to \delta)$.

A formula is a *target* if it is an atom or a disjunction. If $\varphi = \alpha \vee \beta$ then we say that α and β are *disjuncts* of φ. A formula which is not a disjunction is the only *disjunct* of itself.

If Γ, Δ are sets of formulas then we write Γ, Δ for $\Gamma \cup \Delta$; also Γ, α means $\Gamma \cup \{\alpha\}$, etc. A set of the form $\{\alpha_1, \ldots, \alpha_n\}$ may be occasionally written as $\vec{\alpha}$; also a formula $\alpha_1 \to \cdots \to \alpha_n \to \beta$ may be abbreviated as $\vec{\alpha} \to \beta$. Natural deduction rules, given in Figure 1, derive judgments of the form $\Gamma \vdash \varphi$, where Γ is a set of formulas and φ is a formula. As usual, we write $\vdash \varphi$ for the judgment $\varnothing \vdash \varphi$.

Term assignment: We use term notation for proofs in the spirit if the Curry-Howard correspondence [15] which treats formulas as types. Our terms are generalized λ-expressions, built from *proof variables* by means of the following constructs (where σ, τ are formulas):

- *lambda-abstraction:* $(\lambda \mathrm{x}^\sigma\, M)$;
- *application:* (MN);
- *pair:* $\langle\, M, N\,\rangle$;
- *projections:* $(M\{1\})$ and $(M\{2\})$;

$$\Gamma, \tau \vdash \tau \qquad\qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash \tau}$$

$$\frac{\Gamma, \sigma \vdash \tau}{\Gamma \vdash \sigma \to \tau} \qquad\qquad \frac{\Gamma \vdash \sigma \to \tau \qquad \Gamma \vdash \sigma}{\Gamma \vdash \tau}$$

$$\frac{\Gamma \vdash \tau_i}{\Gamma \vdash \tau_1 \vee \tau_2} \qquad\qquad \frac{\Gamma \vdash \tau \vee \sigma \quad \Gamma, \tau \vdash \rho \quad \Gamma, \sigma \vdash \rho}{\Gamma \vdash \rho}$$

$$\frac{\Gamma \vdash \tau \quad \Gamma \vdash \sigma}{\Gamma \vdash \tau \wedge \sigma} \qquad\qquad \frac{\Gamma \vdash \tau_1 \wedge \tau_2}{\Gamma \vdash \tau_i}$$

Figure 1. Rules for propositional logic

$$\Gamma, \textsc{x}\!:\!\tau \vdash \textsc{x} : \tau \qquad \frac{\Gamma \vdash M : \bot}{\Gamma \vdash \varepsilon_\tau(M) : \tau}$$

$$\frac{\Gamma, \textsc{x}\!:\!\sigma \vdash M : \tau}{\Gamma \vdash (\lambda \textsc{x}^\sigma .\, M) : \sigma \to \tau} \qquad \frac{\Gamma \vdash M : \sigma \to \tau \qquad \Gamma \vdash N : \sigma}{\Gamma \vdash (MN) : \tau}$$

$$\frac{\Gamma \vdash M : \tau_i}{\Gamma \vdash \mathtt{in}_i(M) : \tau_1 \vee \tau_2} \qquad \frac{\Gamma \vdash M : \tau \vee \sigma \quad \Gamma, \textsc{x}\!:\!\tau \vdash N : \rho \quad \Gamma, \textsc{y}\!:\!\sigma \vdash Q : \rho}{\Gamma \vdash \mathtt{case}\ M\ \mathtt{of}\ [\textsc{x}]N\ \mathtt{or}\ [\textsc{y}]Q : \rho}$$

$$\frac{\Gamma \vdash M : \tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash \langle\, M, N\,\rangle : \tau \wedge \sigma} \qquad \frac{\Gamma \vdash M : \tau_1 \wedge \tau_2}{\Gamma \vdash (M\{i\}) : \tau_i}$$

Figure 2. Propositional term assignment

- *injections:* $\mathtt{in}_1(M)$ and $\mathtt{in}_2(M)$;
- *choice:* $\mathtt{case}\ M\ \mathtt{of}\ [\textsc{x}]P\ \mathtt{or}\ [\textsc{y}]Q$;
- *miracle:* $\varepsilon_\tau(M)$.

Figure 2 displays the term assignment rules for propositional intuitionistic logic. Here, Γ is a *type environment*, i.e., a set of variable declarations of the form $(\textsc{x} : \sigma)$, where each variable occurs at most once. (We use small caps for proof variables to distinguish them from object variables used in Section 5.) Observe that erasing term-related information turns these rules into natural deduction rules of Figure 1.

Free variables in terms are defined as usual (e.g., the variable $\textsc{x}$ is bound in $\lambda \textsc{x}^\sigma\, M$ and variables $\textsc{x}$ and $\textsc{y}$ are bound respectively inside P and Q in $\mathtt{case}\ M\ \mathtt{of}\ [\textsc{x}]P\ \mathtt{or}\ [\textsc{y}]Q$), and we take terms up to alpha-conversion (that is, replacement of bound variables). The capture-avoiding substitution of a term N for free occurrences of a variable $\textsc{x}$ in M is denoted by $M[\textsc{x} := N]$. It is required that N be of the same type as $\textsc{x}$.

We apply the common conventions when writing terms. For instance, we skip unnecessary parentheses and assume left-associativity of application: MNP thus stands for $((MN)P)$. We sometimes use the simplified notation $\lambda \textsc{x}\, M$ for lambda-abstraction, and may also informally write e.g. M^ρ to indicate that M is a term of type ρ in a certain environment.

In a type environment, a formula may occur more than once, as there may be many variables of the same type. But for our purposes this is not essential and we take the liberty of confusing type environments with sets of formulas. That is, if Γ is a type environment, we may use the notation Γ for the set of formulas $|\Gamma| = \{\alpha \mid (\textsc{x} : \alpha) \in \Gamma,\ \text{for some variable } \textsc{x}\}$.

Beta-reductions:

- $(\lambda \mathrm{X}\, M)N \;\to\; M[\mathrm{X} := N]$,
- $\langle M, N\rangle\{1\} \;\to\; M, \qquad \langle M, N\rangle\{2\} \;\to\; N$,
- $\mathtt{case}\ \mathtt{in}_1\,(P)\ \mathtt{of}\ [\mathrm{X}]M\ \mathtt{or}\ [\mathrm{Y}]N \;\to\; M[\mathrm{X} := P]$,
- $\mathtt{case}\ \mathtt{in}_2\,(P)\ \mathtt{of}\ [\mathrm{X}]M\ \mathtt{or}\ [\mathrm{Y}]N \;\to\; N[\mathrm{Y} := P]$.

Commuting conversions:

- $\varepsilon_\psi(\varepsilon_\perp(M)) \;\to\; \varepsilon_\psi(M)$,
- $\varepsilon_{\varphi\to\psi}(M)N \;\to\; \varepsilon_\psi(M)$,
- $\varepsilon_{\varphi_1\wedge\varphi_2}(M)\{i\} \;\to\; \varepsilon_{\varphi_i}(M)$,
- $\mathtt{case}\ \varepsilon_{\sigma\vee\tau}(M)\ \mathtt{of}\ [\mathrm{U}]R^\rho\ \mathtt{or}\ [\mathrm{V}]S^\rho \;\to\; \varepsilon_\rho(M)$,
- $\varepsilon_\varphi(\mathtt{case}\ M\ \mathtt{of}\ [\mathrm{X}]P\ \mathtt{or}\ [\mathrm{Y}]Q) \;\to\; \mathtt{case}\ M\ \mathtt{of}\ [\mathrm{X}]\varepsilon_\varphi(P)\ \mathtt{or}\ [\mathrm{Y}]\varepsilon_\varphi(Q)$,
- $(\mathtt{case}\ M\ \mathtt{of}\ [\mathrm{X}]P\ \mathtt{or}\ [\mathrm{Y}]Q)N \;\to\; \mathtt{case}\ M\ \mathtt{of}\ [\mathrm{X}]PN\ \mathtt{or}\ [\mathrm{Y}]QN$,
- $(\mathtt{case}\ M\ \mathtt{of}\ [\mathrm{X}]P\ \mathtt{or}\ [\mathrm{Y}]Q)\{i\} \;\to\; \mathtt{case}\ M\ \mathtt{of}\ [\mathrm{X}]P\{i\}\ \mathtt{or}\ [\mathrm{Y}]Q\{i\}$,
- $\mathtt{case}\ (\mathtt{case}\ M\ \mathtt{of}\ [\mathrm{X}]P\ \mathtt{or}\ [\mathrm{Y}]Q)\ \mathtt{of}\ [\mathrm{U}]R\ \mathtt{or}\ [\mathrm{V}]S \;\to$
  $\mathtt{case}\ M\ \mathtt{of}\ [\mathrm{X}](\mathtt{case}\ P\ \mathtt{of}\ [\mathrm{U}]R\ \mathtt{or}\ [\mathrm{V}]S)\ \mathtt{or}\ [\mathrm{Y}](\mathtt{case}\ Q\ \mathtt{of}\ [\mathrm{U}]R\ \mathtt{or}\ [\mathrm{V}]S)$.

Figure 3. Reduction rules

Also, any set of formulas Γ may be implicitly treated as a type environment $\Gamma = \{(\mathrm{X}_\alpha : \alpha) \mid \alpha \in \Gamma\}$. It is also often convenient to identify a type assignment $\Gamma \vdash M : \tau$ with a logical judgment $\Gamma \vdash \tau$. While it may appear slightly confusing, it is actually quite useful.

Normal forms: Reduction rules (beta reductions and permutation conversions) for our extended lambda-calculus are given in Figure 3. Left-hand sides of rules are *redexes*, and a *normal form* is a term which has no redex as a subterm and therefore cannot be reduced. We have two kinds of redexes: a beta-redex corresponds to a proof where a connective is eliminated right after being introduced. The other redexes represent situations where some elimination step is applied to a conclusion of a $\perp$- or $\vee$-elimination. The commuting conversions permute the "bad" eliminations upwards, and this can be schematically written as $\mathit{Elim}^\sigma(\varepsilon_\tau(M)) \to \varepsilon_\sigma(M)$, and

$$\mathit{Elim}^\sigma(\mathtt{case}\ M\ \mathtt{of}\ [\mathrm{X}]N\ \mathtt{or}\ [\mathrm{Y}]Q) \to \mathtt{case}\ M\ \mathtt{of}\ [\mathrm{X}]\mathit{Elim}^\sigma(N)\ \mathtt{or}\ [\mathrm{Y}]\mathit{Elim}^\sigma(Q).$$

It follows from [5] that proofs in intuitionistic propositional and first-order logic are strongly normalizable. We will not rely on this: our construction yields an alternative argument that every provable formula has a normal proof (Corollary 2.18).
We write $\Gamma \vdash_n M : \tau$ if $\Gamma \vdash M : \tau$ and M is a normal form. Here, M may be omitted, so that $\Gamma \vdash_n \tau$ means "there is a normal form M with $\Gamma \vdash M : \tau$". By inspecting the possible redexes, one can classify the normal forms in our calculus as follows, where the metavariable N stands for normal form and P for proper eliminator:

- *Introductions:* $\lambda \mathrm{X}^\alpha . N, \quad \langle N_1, N_2 \rangle, \quad \mathsf{in}_i(N)$;
- *Proper eliminators:* $\mathrm{X}, \quad PN, \quad P\{i\}$;
- *Improper eliminators:* $\varepsilon_\varphi(P), \quad \mathsf{case}\ P\ \mathsf{of}\ [\mathrm{X}]N_1\ \mathsf{or}\ [\mathrm{Y}]N_2$.

A proper eliminator is *maximal* when its type is a target. The *head variable* of an eliminator P is X when $P = \mathrm{X}$ and it is the head variable of P' when P is $P'N$, $P'\{i\}$, $\mathsf{case}\ P'\ \mathsf{of}\ [\mathrm{X}]M\ \mathsf{or}\ [\mathrm{Y}]N$, or $\varepsilon_\tau(P')$.

Lemma 2.1. *If P is a proper eliminator, and M is a normal form, then the substitution $M[\mathrm{X} := P]$ is a normal form. If both P and M are proper eliminators then so is $M[\mathrm{X} := P]$.*

Proof. Simultaneous induction with respect to the definition of M. ■

The following lemma gives an alternative definition of a proper eliminator:

Lemma 2.2. *A term P is a proper eliminator if and only if it has the form*

$$P = \mathrm{X}, \quad \text{or} \quad P = Q[\mathrm{Y} := \mathrm{X}N], \quad \text{or} \quad P = Q[\mathrm{Y} := \mathrm{X}\{i\}], \qquad (*)$$

where N is a normal form and Q is a proper eliminator.

Proof. The right-to-left part follows from Lemma 2.1. From left to right we proceed by induction with respect to the length of P. Let $\mathcal{L}$ be the set of all terms defined by the grammar (*). Case $P = \mathrm{X}$ is obvious. Let $P = P_1 N$. By induction, term P_1 is in $\mathcal{L}$. If $P_1 = \mathrm{X}$ then $P = \mathrm{Y}[\mathrm{Y} := \mathrm{X}N]$, otherwise we have e.g. $P_1 = Q[\mathrm{Y} := \mathrm{X}M]$ for $Q \in \mathcal{L}$. Note that Q must be shorter than P_1. We can assume w.o.l.o.g. that the variable Y is not free in N, whence $P = QN[\mathrm{Y} := \mathrm{X}M]$. The term QN is shorter than $P_1 N$ so by the induction hypothesis $QN \in \mathcal{L}$, and only now we conclude that $P \in \mathcal{L}$. The case $P = P_1\{i\}$ is similar. ■

a1) If α is an assumption $\beta \to \gamma$ then ∀phrodite chooses between positions $\Gamma, \gamma \vdash \tau$ and $\Gamma \vdash \beta$.

a2) If α is an assumption $\beta \vee \gamma$ then ∀phrodite chooses between positions $\Gamma, \beta \vdash \tau$ and $\Gamma, \gamma \vdash \tau$.

a3) If α is an assumption $\beta \wedge \gamma$ then ∀phrodite has no choice and the next position is $\Gamma, \beta, \gamma \vdash \tau$.

b1) If α is an aim $\beta \to \gamma$ then ∀phrodite has no choice and the next position is $\Gamma, \beta \vdash \gamma$.

b2) If α is an aim $\beta \wedge \gamma$ then ∀phrodite chooses between positions $\Gamma \vdash \beta$ and $\Gamma \vdash \gamma$.

b3) If the aim α is a target then ∀phrodite has no choice and the next position is $\Gamma \vdash \alpha$.

Figure 4. Simple game rules

2.1. The Simple Game

A *position* in the game is a judgment $\Gamma \vdash \tau$. Formulas in Γ are called *assumptions*, the formula τ is the *goal* of the position, and disjuncts of τ are called *aims*. (There is always one or two aims.) If position $\Gamma \vdash \tau$ is denoted by $\mathcal{P}$ then we write $\Gamma_{\mathcal{P}}$ for Γ, and $\tau_{\mathcal{P}}$ for τ. An *initial* position has the form $\vdash \varphi$. A *final* position is of the form $\Gamma \vdash \tau$, where Γ contains either the goal τ or the constant $\bot$.

The game is played in *turns*. Each turn in position $\Gamma \vdash \tau$ begins with an ∃ros' move. ∃ros picks a formula α which is either (a) a non-atomic assumption or (b) an aim. Then ∀phrodite determines the next position, depending on the choice of α, see Figure 4.

Observe that some game turns do not change the position, like (b3) in case of an atomic goal. If nothing else is possible this turn can still be played, so that the game cannot get stuck in a non-final position like $p \vdash q$.

A turn of the game from position $\mathcal{P}$ to position $\mathcal{Q}$ is called *static* when $\tau_{\mathcal{P}} = \tau_{\mathcal{Q}}$, otherwise it is *dynamic*. A static turn is *trivial*, when $\Gamma_{\mathcal{P}} = \Gamma_{\mathcal{Q}}$, i.e., the position remains unchanged.

∃ros *wins* a play by reaching a final position. Otherwise the play is infinite (players always can move) and ∀phrodite wins. A position $\mathcal{P}$ is *winning* for one of the players if that player has a winning strategy commencing in position $\mathcal{P}$. The following is a routine observation:

Proposition 2.3. *The game is determined: every position is winning for one of the players.* (*That is, one of the players has a winning strategy.*)

Proof. A position $\mathcal{P}$ is winning for $\exists$ros, when it is final, or:

There is an $\exists$ros move such that every response of $\forall$phrodite leads to a position $\mathcal{P}'$ winning for $\exists$ros.

So if $\mathcal{P}$ is not winning for $\exists$ros then $\mathcal{P}$ is not final, and for each move of $\exists$ros there is a response of $\forall$phrodite resulting in a position $\mathcal{P}'$, which is still not winning for $\exists$ros. This way, $\forall$phrodite can play infinitely long and win: she has a winning strategy, i.e., $\mathcal{P}$ is her winning position. ■

Example 2.4. In the initial position $\vdash p\wedge((p \to q)\vee(p \to r)) \to q\vee r$ the first game turn must be (b1) and it must lead to $p \wedge ((p \to q) \vee (p \to r)) \vdash q \vee r$. Then $\exists$ros may choose the assumption and move (a3) yields the position $p, (p \to q) \vee (p \to r), \psi \vdash q \vee r$, where ψ abbreviates $p \wedge ((p \to q) \vee (p \to r))$. Now $\exists$ros plays (a2) and $\forall$phrodite chooses one of the disjuncts, say the first one. The obtained position is $p, (p \to q) \vee (p \to r), p \to q, \psi \vdash q \vee r$, and $\exists$ros may choose the aim q, which results in $p, (p \to q) \vee (p \to r), p \to q, \psi \vdash q$ (move b3). Then he plays (a1) with the assumption $p \to q$. Now $\forall$phrodite must lose: no matter if she adds q to assumptions or sets p as a new goal, the next position is final.

Example 2.5. Consider the initial position $\vdash ((p \to q) \to p) \to p$. It is easy to see that the first turn must lead to $(p \to q) \to p \vdash p$, and the next reasonable $\exists$ros' move is to choose the assumption. Then $\forall$phrodite moves to the position $(p \to q) \to p \vdash p \to q$, and she can return to this position if $\exists$ros plays the assumption again. If $\exists$ros points to the goal then we get into the position $p, (p \to q) \to p \vdash q$ and then into $p, (p \to q) \to p \vdash p \to q$. From now on only these two positions are possible. The play must be infinite and $\forall$phrodite wins.

The game according to $\exists$ros

Lemma 2.6. *1. If $\Gamma \vdash_n \tau$ then $\exists$ros has a strategy in position $\Gamma \vdash \tau$.*

2. Suppose that $\Gamma \vdash P : \varphi$, where P is a proper eliminator, and that $\exists$ros has a strategy in position $\Gamma, \varphi \vdash \tau$. Then $\exists$ros has a strategy in position $\Gamma \vdash \tau$.

Proof. Simultaneous induction with respect to the length of proof terms.

(1) Let $\Gamma \vdash_n N : \tau$. We consider the possible shapes of N. If $N = \lambda \mathrm{x}^\sigma.\, N'$ then $\tau = \sigma \to \rho$, for some ρ, and $\Gamma, \mathrm{x} : \sigma \vdash N' : \rho$. We claim that $\exists$ros has a strategy in position $\Gamma \vdash \tau$. Indeed, he should point to the goal $\tau = \sigma \to \rho$. This yields the position $\Gamma, \sigma \vdash \rho$, where he has a strategy by the induction hypothesis for N'. The two other cases when N is an introduction are handled similarly: $\exists$ros plays move (b2) or (b3).

If N is a proper eliminator then we observe that ∃ros has a trivial strategy in position $\Gamma, \tau \vdash_n \tau$ and we proceed as in part 2.

Let now $N = \mathsf{case}\ P\ \mathsf{of}\ [\mathrm{X}^\alpha]N_1\ \mathsf{or}\ [\mathrm{Y}^\beta]N_2$. By the induction hypothesis (1) for N_1 and N_2, ∃ros has strategies in positions $\Gamma, \alpha \vdash \tau$ and $\Gamma, \beta \vdash \tau$. In position $\Gamma, \alpha \vee \beta \vdash \tau$ he will point to the assumption $\alpha \vee \beta$ and then play one of these strategies and win. It remains to use the induction hypothesis (2) for P. If $N = \varepsilon_\tau(P)$ then we observe that there is a trivial strategy for $\Gamma, \bot \vdash \tau$ and apply the induction hypothesis (2) for P.

(2) We use Lemma 2.2. If P is a variable then $\Gamma, \varphi = \Gamma$ and the conclusion is obvious. Assume that $P = Q[\mathrm{Z} := \mathrm{Y}M]$, where Q is a proper eliminator. This means that $(\mathrm{Y} : \alpha \to \beta) \in \Gamma$, for some α, β, and that $\Gamma \vdash_n M : \alpha$, and $\Gamma, \mathrm{Z} : \beta \vdash Q : \varphi$.

∃ros' strategy in position $\Gamma \vdash \tau$ is to point to the assumption $\mathrm{Y} : \alpha \to \beta$. If ∀phrodite responds by adding β to assumptions then we are in position $\Gamma, \beta \vdash \tau$, and ∃ros has a strategy in this position by the induction hypothesis (2) for Q. Otherwise, ∀phrodite changes the goal to α, and we apply the induction hypothesis (1) to M.

Let now $P = Q[\mathrm{Z} := \mathrm{Y}\{i\}]$, in which case we have $(\mathrm{Y} : \alpha_1 \wedge \alpha_2) \in \Gamma$ and $\Gamma, \mathrm{Z} : \alpha_i \vdash Q : \varphi$. ∃ros' strategy in position $\Gamma, \varphi \vdash \tau$ can be played as well from position $\Gamma, \varphi, \alpha_i \vdash \tau$, so we can apply the induction hypothesis (2) for Q. This yields a strategy in position $\Gamma, \alpha_i \vdash \tau$, thus ∃ros also wins in $\Gamma \vdash \tau$, by using the assumption Y. ■

PROPOSITION 2.7. ∃ros *has a winning strategy in position* $\Gamma \vdash \tau$ *if and only if* $\Gamma \vdash_n \tau$.

PROOF. The right-to-left direction is Lemma 2.6(1). The proof of the other direction is by induction with respect to the size of the strategy. The trivial cases when $\Gamma \vdash \tau$ is a final position correspond to proofs of the form X or $\varepsilon_\tau(\mathrm{X})$, where X is a variable. Otherwise we consider the initial ∃ros' move according to the strategy. Those are of the form (a1)–(b3) as in Figure 4. In cases (a1)–(a3) we have a variable declaration $\mathrm{X} : \alpha$ in Γ.

(a1) Since ∃ros has a strategy in the position $\Gamma \vdash \tau$, he also has (smaller) strategies in positions $\Gamma, \gamma \vdash \tau$ and $\Gamma \vdash \beta$. Thus $\Gamma, \mathrm{Y} : \gamma \vdash_n M : \tau$ and also $\Gamma \vdash_n N : \beta$. It follows from Lemma 2.1 that $\Gamma \vdash_n M[\mathrm{Y} := \mathrm{X}N] : \tau$.

(a2) By the induction hypothesis there are proofs $\Gamma, \mathrm{Y} : \beta \vdash_n M : \tau$ and $\Gamma, \mathrm{Z} : \gamma \vdash_n N : \tau$. Thus we have $\Gamma \vdash_n \mathsf{case}\ \mathrm{X}\ \mathsf{of}\ [\mathrm{Y}]\, M\ \mathsf{or}\ [\mathrm{Z}]\, N : \tau$.

(a3) Now $\Gamma, \mathrm{Y} : \beta, \mathrm{Z} : \gamma \vdash_n M : \tau$ by induction, whence we obtain that $\Gamma \vdash_n M[\mathrm{Y} := \mathrm{X}\{1\}][\mathrm{Z} := \mathrm{X}\{2\}] : \tau$. Again we use Lemma 2.1.

In cases (b1)–(b3) the formula α is an aim, and it suffices to show that $\Gamma \vdash_n \alpha$. Indeed, if $\tau = \alpha$ then we are done. Otherwise let, say, $\tau = \alpha \vee \beta$ and $\Gamma \vdash_n M : \alpha$. Then $\Gamma \vdash_n \mathsf{in}_1(M) : \tau$.

(b1) From the induction hypothesis $\Gamma, \mathrm{X} : \beta \vdash_n M : \gamma$ it follows that $\Gamma \vdash_n \lambda \mathrm{X}.\, M : \beta \to \gamma$.

(b2) We have $\Gamma \vdash_n M : \beta$ and $\Gamma \vdash_n N : \gamma$ by the induction hypothesis, whence $\Gamma \vdash_n \langle M, N \rangle : \beta \wedge \gamma$.

Case (b3) follows immediately from the induction hypothesis. ■

Example 2.8. The play in Example 2.4 is a part of ∃ros' strategy corresponding to the lambda-term $\lambda \mathrm{X}.\, \mathsf{case}\ \mathrm{X}\{2\}\ \mathsf{of}\ [\mathrm{Y}^{p\to q}]\ \mathrm{Y}(\mathrm{X}\{1\})\ \mathsf{or}\ [\mathrm{Z}^{p\to r}]\ \mathrm{Z}(\mathrm{X}\{1\})$. In Example 2.5, ∃ros attempts to build a proof of shape $\lambda \mathrm{X}^{(p\to q)\to p}.\, \mathrm{X}(\lambda \mathrm{Y}^p \ldots)$.

Remark 2.9. Winning strategies of ∃ros correspond to terms in normal form, but not exactly. One reason is that in a lambda term one can have many variables of the same type while the game does not distinguish between copies of the same assumption. For instance terms $\lambda \mathrm{X}^p\, \lambda \mathrm{Y}^p.\, \mathrm{X}$ and $\lambda \mathrm{X}^p \lambda \mathrm{Y}^p.\, \mathrm{Y}$ represent the same strategy in position $\vdash p \to p \to p$.

On the other hand there is a certain level of redundancy in ∃ros' strategies: two or more strategies are represented by the same lambda-term. For example, in the position

$$\mathrm{X} : p \to q, \mathrm{Y} : q \to r, \mathrm{Z} : p \vdash r,$$

∃ros can play first X then Y, or conversely. These strategies represent two different orders in which the lambda term $\mathrm{Y}(\mathrm{XZ})$ can be assembled (and two different proofs in sequent calculus). One corresponds to the substitution $\mathrm{V}[\mathrm{V} := \mathrm{YU}][\mathrm{U} := \mathrm{XZ}]$, the other to $\mathrm{V}[\mathrm{V} := \mathrm{Y}(\mathrm{U}[\mathrm{U} := \mathrm{XZ}])]$, where $\mathrm{U} : q$ and $\mathrm{V} : r$. The modified game rules in Section 3.2 eliminate this redundancy.

The game according to ∀phrodite

The notion of ∃ros' strategy was used above informally, but we need a more precise definition of an ∀phrodite's winning strategy (in a given position $\mathcal{P}_0$). So we take it to be a tree S labeled by non-final positions, with $\mathcal{P}_0$ as the root, and satisfying the following.

- For every node $\mathcal{P}$ and every ∃ros' move possible in $\mathcal{P}$, there is in S a successor $\mathcal{P}'$ of $\mathcal{P}$ representing a position obtained by that ∃ros' move followed by a response of ∀phrodite.

We often identify nodes with positions. A strategy is *deterministic* if every node has only one successor for every ∃ros' move. In Sections 2–4 we consider only deterministic strategies.

If position $\mathcal{P}'$ is a successor of some $\mathcal{P}$ then we may write $S : \mathcal{P} \to \mathcal{P}'$, or just $\mathcal{P} \to \mathcal{P}'$, if S is known. We write $\mathcal{P} \to_s \mathcal{P}'$ in case of a static turn, and we use the standard notation $\twoheadrightarrow$ and $\twoheadrightarrow_s$ respectively for the reflexive-transitive closures of the relations $\to$ and $\to_s$

We would like to obtain a Kripke counter-model from a winning strategy of ∀phrodite. Seemingly, the most straightforward way to do so is to take all positions as states of the model and define forcing of propositional variables by $\mathcal{P} \Vdash p$ iff $p \in \Gamma_{\mathcal{P}}$. Then we would like to generalize the latter to arbitrary formulas. This does not work (Example 2.13), and it turns out that we need to select a subset of positions. A position $\mathcal{P}$ is *saturated* in a given ∀phrodite's strategy iff every static turn at this position is trivial (i.e. leads back to $\mathcal{P}$ by adding an assumption which is already present in $\Gamma_{\mathcal{P}}$).

Lemma 2.10. *For every position $\mathcal{P}$ there is a saturated position $\mathcal{Q}$ such that $\mathcal{P} \twoheadrightarrow_s \mathcal{Q}$ and $\tau_{\mathcal{Q}} = \tau_{\mathcal{P}}$.*

Proof. The number of subformulas of the assumptions in $\mathcal{P}$ is finite, and only such formulas may be added as new assumptions in static game turns. Thus a play where every turn is static and nontrivial must eventually reach a saturated position. ∎

The model: Given a winning strategy of ∀phrodite, we define a Kripke model:

- States of the model are the saturated positions in S.
- For propositional variables we take $\mathcal{P} \Vdash p$ iff $p \in \Gamma_{\mathcal{P}}$.
- And we define $\mathcal{P} \leq \mathcal{P}'$ iff $\mathcal{P} \twoheadrightarrow \mathcal{P}'$.

Example 2.11. In Example 2.5, ∀phrodite won the play for Peirce's law. Her winning strategy is an infinite tree labeled by only five different positions:

$A : \vdash ((p \to q) \to p) \to p;$

$B : (p \to q) \to p \vdash p;$

$C : (p \to q) \to p \vdash p \to q;$

$D : (p \to q) \to p, p \vdash q;$

$E : (p \to q) \to p, p \vdash p \to q.$

The tree is obtained by unfolding the finite automaton below.

$$\begin{tikzcd} A \arrow[r] & B \arrow[r] & C \arrow[loop above] \arrow[r] & D \arrow[r, bend left] & E \arrow[l, bend left] \arrow[loop right] \end{tikzcd}$$

Since all positions are saturated, the whole strategy makes a model. The atoms forced in the states of the model are as follows:

$$\begin{tikzcd} \varnothing \arrow[r] & \varnothing \arrow[r] & \varnothing \arrow[loop above] \arrow[r] & \{p\} \arrow[r, bend left] & \{p\} \arrow[l, bend left] \arrow[loop right] \end{tikzcd}$$

By identifying equivalent states, the model collapses to the standard counterexample $\varnothing \to \{p\}$.

Lemma 2.12. *Let $\mathcal{P} = (\Gamma \vdash \tau)$ be a saturated position in a winning strategy S. Then $\mathcal{P} \Vdash \Gamma$ and $\mathcal{P} \nVdash \tau$.*

Proof. By simultaneous induction with respect to the length of a formula α we prove two statements:

(1) *If α is an assumption in $\mathcal{P}$ then $\mathcal{P} \Vdash \alpha$;*

(2) *If α is an aim in $\mathcal{P}$ (a disjunct of τ) then $\mathcal{P} \nVdash \alpha$.*

(1) Assume first that α is an assumption. If α is a propositional variable then $\mathcal{P} \Vdash \alpha$ by definition, and if $\alpha = \bot$ then S is not a winning strategy. So let α be a non-atomic assumption; then a possible ∃ros' move is to choose α.

If $\alpha = \beta \vee \gamma$ then one of the positions $\Gamma, \beta \vdash \tau$ and $\Gamma, \gamma \vdash \tau$ belongs to the winning strategy S. Since $\mathcal{P}$ is saturated, we actually have $\beta \in \Gamma$ or $\gamma \in \Gamma$, as otherwise the corresponding game turn would be static and not trivial. By the induction hypothesis, $\mathcal{P} \Vdash \beta$ or $\mathcal{P} \Vdash \gamma$, whence $\mathcal{P} \Vdash \beta \vee \gamma$. The case of $\alpha = \beta \wedge \gamma$ is similar.

Let $\alpha = \beta \to \gamma$ and consider a saturated position $\mathcal{Q} \geq \mathcal{P}$ with $\mathcal{Q} \Vdash \beta$. The formula α is still present in $\mathcal{Q}$ as an assumption, so ∃ros can choose α in position $\mathcal{Q}$. The strategy S provides ∀phrodite with a response to this move, and this response cannot be of the form $\Gamma_{\mathcal{Q}} \vdash \beta$. Indeed, by Lemma 2.10, there would be a saturated position $\mathcal{R} \geq \mathcal{Q}$ with $\tau_{\mathcal{R}} = \beta$. Now $\mathcal{R} \nVdash \beta$ by the induction hypothesis for β, and also $\mathcal{R} \Vdash \beta$, because $\beta \in \Gamma_{\mathcal{Q}} \subseteq \Gamma_{\mathcal{R}}$.

The obtained contradiction implies that ∀phrodite's response to α in position $\mathcal{Q}$ must be to add γ to the assumptions. But that makes a static turn and $\mathcal{Q}$ is saturated, so γ is already there. By the induction hypothesis for γ we get $\mathcal{Q} \Vdash \gamma$.

(2) Now suppose α is an aim. Then α is a possible choice of ∃ros.

If $\alpha = \beta \to \gamma$, this choice yields a position $\mathcal{P}' = (\Gamma, \beta \vdash \gamma)$. Since $\mathcal{P}'$ extends to a saturated position $\mathcal{Q}$ with $\tau_{\mathcal{Q}} = \gamma$ and $\beta \in \Gamma_{\mathcal{Q}}$, we obtain $\mathcal{Q} \Vdash \beta$ and $\mathcal{Q} \nVdash \gamma$ by induction. Thus $\mathcal{P} \nVdash \alpha$.

If α is not an implication then the next position $\mathcal{P}'$ has the same set of assumptions. Since $\mathcal{P}$ is saturated, so is $\mathcal{P}'$.

If α is a target then $\mathcal{P}' = (\Gamma \vdash \alpha)$, and if α is an atom then certainly $\alpha \notin \Gamma$, because a final position cannot occur in $\forall$phrodite's strategy. So let $\alpha = \beta \vee \gamma$. We apply the induction hypothesis to the aims β and γ of $\mathcal{P}' = (\Gamma \vdash \beta \vee \gamma)$. Since $\mathcal{P}' \nVdash \beta$ and $\mathcal{P}' \nVdash \gamma$, the same must hold for $\mathcal{P}$.

Let finally α be an aim of the form $\beta \wedge \gamma$. Then $\mathcal{P}'$ is either $\Gamma \vdash \beta$ or $\Gamma \vdash \gamma$, so we have $\mathcal{P}' \nVdash \beta$ or $\mathcal{P}' \nVdash \gamma$. Then also $\mathcal{P} \nVdash \beta$ or $\mathcal{P} \nVdash \gamma$, whence $\mathcal{P} \nVdash \beta \wedge \gamma$. ∎

EXAMPLE 2.13. Consider the position $\mathcal{P} = (p \to q \vdash p \to r)$. A winning strategy of $\forall$phrodite must include $\mathcal{P}_1 = (p \to q, p \vdash r)$ as a next position. If we define the model so that every position is a state then Lemma 2.12 fails, because $\mathcal{P}_1 \nVdash p \to q$ and thus also $\mathcal{P} \nVdash p \to q$. The problem occurs because $\mathcal{P}_1$ is not saturated.

The following is now immediate:

PROPOSITION 2.14. *If* $\forall$phrodite *has a winning strategy in position* $\Gamma \vdash \tau$ *then there exists a Kripke model* $\mathcal{C}$ *with* $\mathcal{C} \Vdash \Gamma$ *and* $\mathcal{C} \nVdash \tau$.

PROOF. In a model as above, the initial states are saturated positions $\mathcal{P}$ such that $\Gamma \subseteq \Gamma_{\mathcal{P}}$ and $\tau_{\mathcal{P}} = \varphi$ (cf. Lemma 2.10). Therefore $\mathcal{P} \Vdash \Gamma$ and $\mathcal{P} \nVdash \tau$ in every initial state $\mathcal{P}$. ∎

The converse to Proposition 2.14 is also true.

FACT 2.15. *Let* c *be a state in a Kripke model* $\mathcal{C}$. *If* $c \Vdash \Gamma$ *and* $c \nVdash \tau$ *then* $\forall$*phrodite has a winning strategy in position* $\Gamma \vdash \tau$.

PROOF. Let us say that a position $\Gamma \vdash \tau$ is *secure* for $\forall$phrodite if there is a state $c \in \mathcal{C}$ with $c \Vdash \Gamma$ and $c \nVdash \tau$. We show that in a secure position $\forall$phrodite can respond to any move so that she arrives again at a secure position. This way, $\forall$phrodite can play infinitely long.

So let the position $\Gamma \vdash \tau$ be secure. It is not final by definition. We define a response of $\forall$phrodite to every move of $\exists$ros, cf. Figure 4. Moves where $\forall$phrodite has no choice are omitted below.

(a1) Since $c \Vdash \beta \to \gamma$, either $c \Vdash \gamma$ or $c \nVdash \beta$. In the first case $\forall$phrodite moves to $\Gamma, \gamma \vdash \tau$ otherwise she chooses $\Gamma \vdash \beta$.

(a2) We must have either $c \Vdash \beta$ or $c \Vdash \gamma$ and ∀phrodite chooses $\Gamma, \beta \vdash \tau$ or $\Gamma, \gamma \vdash \tau$, respectively.

(b2) ∀phrodite chooses $\Gamma \vdash \beta$ when $c \nVdash \beta$ and $\Gamma \vdash \gamma$ otherwise.

As a result of a single turn we obtain a new position $\Gamma' \vdash \tau'$. We claim that $\Gamma' \vdash \tau'$ is secure. Indeed, in all cases but (b1), we simply have $c \Vdash \Gamma'$ and $c \nVdash \tau'$. In case (b1), since $c \nVdash \beta \to \gamma$, there is a state $c' \geq c$ with $c' \Vdash \beta$ and $c' \nVdash \gamma$. ■

THEOREM 2.16. *Given a formula τ, either there is a normal form of type τ or there is a Kripke counter-model for τ.*

PROOF. Follows from Propositions 2.3, 2.7, and 2.14. ■

COROLLARY 2.17. (Completeness) *A valid formula is provable.*

COROLLARY 2.18. (Semantic normalization) *A provable formula has a normal proof.*

Note that to derive Corollary 2.18 one needs to prove soundness, which is (for non-normal proofs) not implied by our game approach.

2.2. Long Version of the Game

Terms like $\lambda \mathrm{X}^{p\to p}.\,\mathrm{X}$ and $\lambda \mathrm{X}^{p\to p}\,\lambda \mathrm{Y}^{p}.\,\mathrm{XY}$ have essentially the same computational behaviour. The difference lies only in that the latter makes explicit the functionality of X. Such terms are often identified; in lambda-calculus we say that they are "eta-convertible", more precisely that the second one "eta-reduces" to the first one. While eta-reduction leads to shorter terms (faster strategies), it turns out that it is more convenient for a systematic treatment to restrict one's attention to the "fully expanded" terms, called "long normal forms". Gamewise, long normal forms correspond to ∃ros' strategies where he persistently decomposes the problem into atoms, even if a final position has already been reached.

Long normal forms: We generalize the ordinary notion of long normal form to the calculus with $\vee$, $\wedge$, and $\bot$. (Strictly speaking, the notion of a long normal form depends on the environment, but we assume the latter is always implicit.) The idea is that expressions of type τ should be introduced and eliminated in a way that fully exhibits the structure of τ. Therefore, terms of complex types should never be proper eliminators, and should be introductions whenever possible. In addition, every variable should occur as a head variable in a proper eliminator of a target type. For instance, if $\mathrm{X} : r$ and $\mathrm{Y} : r \to p \vee q$ then YX should be replaced by $\mathsf{case}\ \mathrm{YX}\ \mathsf{of}\ [\mathrm{U}]\mathsf{in}_1(\mathrm{U})\ \mathsf{or}\ [\mathrm{V}]\mathsf{in}_2(\mathrm{V})$.

In the following definition, the notation NE is used for both ordinary applications of N to a term E and for projections, where $E = \{1\}$ or $E = \{2\}$. We use the word *coordinate* for the operators $\{1\}$ and $\{2\}$. By a *quasi-long* proper eliminator we mean a term of the form $\mathrm{x}E_1 \dots E_n$, where all the E_i are either long normal forms or coordinates. Our definition of a long normal form is recursive:

- A quasi-long proper eliminator is a long normal form when its type is a propositional variable.
- A constructor $\lambda \mathrm{x}.\, N$, $\langle\, N, M \,\rangle$, $\mathsf{in}_i(N)$, is a long normal form when so are N and M.
- A case-eliminator $\mathsf{case}\ P\ \mathsf{of}\ [\mathrm{x}]M\ \mathsf{or}\ [\mathrm{y}]N$ of a target type is a long normal form when M, N are long normal forms and P is a quasi-long proper eliminator.
- A miracle $\varepsilon_\tau(P)$, of a target type τ, is a long normal form when P is a quasi-long proper eliminator of type $\bot$.

Suppose $\Gamma \vdash M : \tau$. If M is a long normal form then we write $\Gamma \vdash_\ell M : \tau$. If M is a quasi-long proper eliminator then we may write $\Gamma \vdash_p M : \tau$. By convention, M is omitted if not essential.

Lemma 2.19. *If τ is a target, and $\Gamma \vdash_p P : \tau$, then $\Gamma \vdash_\ell \tau$.*

Proof. Let $\Gamma \vdash_p P : \tau$. If τ is a propositional variable then P is a long normal form. If $\tau = \bot$ then $\Gamma \vdash_\ell \varepsilon_\bot(P) : \bot$, and if $\tau = \alpha \vee \beta$ then $\Gamma \vdash_\ell$ $\mathsf{case}\ P\ \mathsf{of}\ [\mathrm{x}]\mathsf{in}_1(\mathrm{x})\ \mathsf{or}\ [\mathrm{y}]\mathsf{in}_2(\mathrm{y}) : \tau$. ∎

The following two easy lemmas correspond to Lemmas 2.1 and 2.2.

Lemma 2.20. *Let P be a quasi-long proper eliminator. If M is a long normal form, then $M[\mathrm{x} := P]$ is a long normal form. If M is quasi-long proper eliminator then so is $M[\mathrm{x} := P]$.*

Lemma 2.21. *A term P is a quasi-long proper eliminator if and only if*

$$P = \mathrm{x}, \qquad \textit{or} \qquad P = Q[\mathrm{y} := \mathrm{x}N], \qquad \textit{or} \qquad P = Q[\mathrm{y} := \mathrm{x}\{i\}],$$

where N is a long normal form and Q is a quasi-long proper eliminator.

The long game: We now consider a long version of our game. The only difference is that we redefine the notion of a *final position*: it is one of the form $\Gamma \vdash \mathbf{a}$, where $\mathbf{a}$ is an atom and either $\mathbf{a}$ or $\bot$ is in Γ. This forces ∃ros to decompose the goal into atoms to win.

EXAMPLE 2.22. Here is a strategy corresponding to the long normal term

$$\mathsf{in}_1(\lambda \mathrm{X}^{p\vee q}.\,\mathsf{case}\ \mathrm{X}\ \mathsf{of}\ [\mathrm{Y}^p]\mathsf{in}_1(\mathrm{Y})\ \mathsf{or}\ [\mathrm{Z}^q]\mathsf{in}_2(\mathrm{Z}))\ :\ (p \vee q \to p \vee q) \vee (q \to r).$$

$\exists$ros begins in the initial position $\vdash (p \vee q \to p \vee q) \vee (q \to r)$ by choosing the aim $p \vee q \to p \vee q$. $\forall$phrodite has no choice and the next position must be $p \vee q \vdash p \vee q$. Now $\exists$ros picks the only assumption $p \vee q$ at the lhs, and $\forall$phrodite must choose a disjunct, say p; the next position is $p, p \vee q \vdash p \vee q$. The next $\exists$ros' move is to pick the aim p at the rhs, and since $\forall$phrodite can but confirm this choice, $\exists$ros wins by reaching the final position $p, p \vee q \vdash p$. Should $\forall$phrodite take the disjunct q, the final position would be $q, p \vee q \vdash q$.

The fundamental properties of the simple game remain valid for the long version with respect to long normal rather than just normal proofs. In particular we have:

PROPOSITION 2.23. *In the long game* $\exists$ros *has a strategy in position* $\Gamma \vdash \tau$ *if and only if* $\Gamma \vdash_\ell \tau$.

PROOF. The same as the proof of Proposition 2.7, using Lemmas 2.20–2.21 instead of Lemmas 2.1 and 2.2. In particular the proof of Lemma 2.6 remains unchanged (in part (1) observe that a long normal proper eliminator must be of an atom type). ■

PROPOSITION 2.24. *If* $\forall$phrodite *has a winning strategy in position* $\Gamma \vdash \tau$ *then there exists a Kripke model* $\mathcal{C}$ *with* $\mathcal{C} \Vdash \Gamma$ *and* $\mathcal{C} \nVdash \tau$.

PROOF. The same as for Proposition 2.14. ■

THEOREM 2.25. *For any formula* τ*, either there is a long normal form of type* τ *or there is a Kripke counter-model for* τ.

COROLLARY 2.26. *A provable formula has a long normal proof.*

In particular, the existence of a normal form of a type implies the existence of a long normal form of the same type.

3. The Fast Game

A long normal form of a type like $\alpha_1 \to \cdots \to \alpha_n \to p$ must be a "maximal" introduction beginning with n lambdas. Thus, to win the long game in position $\Gamma \vdash \alpha_1 \to \cdots \to \alpha_n \to p$, $\exists$ros must, sooner or later, play the goal n times. He might equally well play all these steps at once, "focusing" on the goal, which resembles very much the intros tactic in Coq [16]. In Ben-Yelles algorithm for $\lambda_{\to}$, this step is performed automatically leading to positions $\Gamma \vdash \mathbf{a}$, where $\mathbf{a}$ is an atom.

Similarly, if $\exists$ros wants to use an assumption $\mathrm{X} : \alpha_1 \to \cdots \to \alpha_n \to p$, he will “focus” on the assumption and will challenge $\forall$phrodite with all the α_i in order. Indeed, in a long normal form every variable must be fully applied. This way we obtain a maximal proper eliminator $\mathrm{X}N_1 \ldots N_n$, and we can say that $\exists$ros plays a tactic like Coq's elim. The sequence of steps determined by the choice of a head variable may also be treated as one move.

In this section we consider new game rules where some sequences of $\exists$ros' moves are collapsed, and the responses of $\forall$phrodite are redefined accordingly. Unfortunately, we cannot follow the Ben-Yelles “atom-oriented” style. The “fast” or “focused” moves correspond to maximal proper eliminators, and types of those are not necessarily atoms: they are targets (atoms or disjunctions). For any formula φ, we define the set $\mathsf{Tg}(\varphi)$ of *targets of* φ:

- $\mathsf{Tg}(\mathbf{a}) = \{\mathbf{a}\}$, when $\mathbf{a}$ is an atom (a propositional variable or $\bot$).
- $\mathsf{Tg}(\tau \to \sigma) = \mathsf{Tg}(\sigma)$.
- $\mathsf{Tg}(\tau \vee \sigma) = \{\tau \vee \sigma\}$.
- $\mathsf{Tg}(\tau \wedge \sigma) = \mathsf{Tg}(\tau) \cup \mathsf{Tg}(\sigma)$.

Let $\mathsf{Tg}(\Gamma) = \bigcup\{\mathsf{Tg}(\gamma) \mid \gamma \in \Gamma\}$ if Γ is a set of formulas. The set $tr(\alpha, \varphi)$, defined for some occurrences of subformulas α of φ (in particular targets of φ) is called the *trace to α in φ*:

- $tr(\alpha, \alpha) = \varnothing$.
- $tr(\alpha, \tau \to \sigma) = \{\tau\} \cup tr(\alpha, \sigma)$.
- $tr(\alpha, \tau \odot \sigma) = tr(\alpha, \tau)$, when $\odot \in \{\wedge, \vee\}$ and α is a subformula of τ.
- $tr(\alpha, \tau \odot \sigma) = tr(\alpha, \sigma)$ when $\odot \in \{\wedge, \vee\}$ and α is a subformula of σ.

Lemma 3.1. *If $tr(\beta, \alpha)$, $tr(\alpha, \varphi)$ are defined then $tr(\beta, \varphi)$ is defined and $tr(\beta, \varphi) = tr(\beta, \alpha) \cup tr(\alpha, \varphi)$.*

Proof. Induction with respect to the definition of $tr(\alpha, \varphi)$. ∎

A collapsed (b) move (generalized intros) addresses a *component* of the goal (a “deep aim”). The set $\mathsf{Comp}(\varphi)$ of *components* of φ is defined by induction.

- $\mathsf{Comp}(\mathbf{a}) = \{\mathbf{a}\}$, when $\mathbf{a}$ is an atom.
- $\mathsf{Comp}(\tau \to \sigma) = \mathsf{Comp}(\sigma)$.
- $\mathsf{Comp}(\tau \wedge \sigma) = \{\tau \wedge \sigma\}$.
- $\mathsf{Comp}(\tau \vee \sigma) = \{\tau \vee \sigma\} \cup \mathsf{Comp}(\tau) \cup \mathsf{Comp}(\sigma)$.

In position $\Gamma \vdash \varphi$ $\exists$ros plays (A) or (B). Move (B) is mandatory when φ is not a target.

A) He picks a target α of an assumption $\gamma \in \Gamma$.

B) He picks a component α of φ.

In response to $\exists$ros' move (A) or (B), $\forall$phrodite may act as follows:

A1) She either chooses a $\delta \in tr(\alpha, \gamma)$, as a goal, i.e., the new position is $\Gamma \vdash \delta$, or

A2) She picks a disjunct β of α and adds it to assumptions, i.e., the new position is $\Gamma, \beta \vdash \varphi$.

B) She picks a target β of α, and the new position is: $\Gamma, tr(\beta, \varphi) \vdash \beta$.

Figure 5. Fast game rules

In Figure 5 we define modified game rules involving the above mentioned restrictions and using collapsed moves. The notion of a position and a final position is the same as in the long game. As usual, a game turn consists of an $\exists$ros' move followed by $\forall$phrodite's move.

In move (A) $\exists$ros selects a variable of type γ, to be used as a head variable of a maximal proper eliminator of type α. The response given by $\forall$phrodite is the obvious generalization with respect to the simple game: she either agrees to the proposed target (A2), or challenges $\exists$ros with one of the necessary premises (A1).

For example, $\exists$ros in position $\Gamma, x : p \to (q \to r) \vee (s \to r \vee q) \vdash s \vee q$ can choose $\alpha = r \vee q$. One of possible (A1) responses of $\forall$phrodite is to pick $\delta = s$ and a possible (A2) move is to choose $\beta = r$. The results of those are, $\Gamma, p \to (q \to r) \vee (s \to r \vee q) \vdash s$, and $\Gamma, x : p \to (q \to r) \vee (s \to r \vee q), r \vdash s \vee q$, respectively.

Move (B) corresponds to a nested introduction. It must be played when the goal is an implication or a conjunction. ($\exists$ros must decompose the goal until it becomes a target.) For example, in position $\Gamma \vdash \vec{\zeta} \to \tau \wedge \sigma$, $\exists$ros must play move (B). At this stage $\forall$phrodite must have the right to choose one of the conjuncts. We let her go forward and make a final choice of a target. (Otherwise, $\exists$ros would have to play (B) again in the next step, and she will be asked for this choice anyway.)

Note that if the goal is a target then it is a component of itself. ∃ros can always play (B) addressing this component and then ∀phrodite's response is trivial, because the trace is empty. The game cannot get stuck.

Note also that the component chosen in (B) may be arbitrary, ∃ros may choose a "deep aim" but may prefer to "play shallow". A natural question is why we do not restrict ourselves to "final" components, since we did a similar choice in (A). We could have defined $\mathsf{Comp}(\tau \vee \sigma)$ as $\mathsf{Comp}(\tau) \cup \mathsf{Comp}(\sigma)$. However, under this policy our games would be incomplete.

EXAMPLE 3.2. In the position $\mathrm{X} : q \to (q \to p) \vee (q \to r) \vdash s \vee (q \to p \vee r)$ ∃ros should win. But in order to use the assumption X, he must first "unblock" the q at the rhs. So he chooses the "non-final" component $p \vee r$. This corresponds to $\mathsf{in}_2(\lambda \mathrm{Z}^q.\, \mathsf{case}\, \mathrm{XZ}\ \mathsf{of}\ [\mathrm{X}_1]\, \mathsf{in}_1(\mathrm{X}_1\mathrm{Z})\ \mathsf{or}\ [\mathrm{X}_2]\, \mathsf{in}_2(\mathrm{X}_2\mathrm{Z}))$. However, if ∃ros plays p or r at the rhs, trying e.g. "$\mathsf{in}_2(\lambda \mathrm{Z}^q\, \mathsf{in}_1 \ldots$", he will lose.

On the other hand, the response (A2) of ∀phrodite *must* be shallow: she has to pick just a disjunct, and cannot challenge ∃ros into a play referring to a component inside it.

For example, if ∃ros points to the only target of an assumption like $p \to q \vee (r \to s)$, ∀phrodite may add the disjunct $r \to s$ to assumptions. Now she may reasonably expect ∃ros to use that new assumption in the next step. So, instead, she might simply set s as a new goal and r as a new assumption. That is, she could address the component s rather than the disjunct $r \to s$. One may want to consider a game permitting also such deep moves. But it turns out that under this policy not every strategy of ∀phrodite will define a Kripke counter-model. The reason is that *la donna è mobile*: ∀phrodite may change her mind during play.

EXAMPLE 3.3. We modify the game rules of Figure 5 so that ∀phrodite in response to move (A) picks a component β of α, and either

A1') picks a new goal $\delta \in tr(\beta, \gamma)$, i.e., the new position is $\Gamma \vdash \delta$, or

A2') adds β to assumptions, i.e., the new position is $\Gamma, \beta \vdash \varphi$.

Recall that $\neg\alpha$ abbreviates $\alpha \to \bot$ and consider the position $\mathcal{P}$:

$$\mathrm{X} : (p \to q) \vee (r \to s),\ \mathrm{Y} : \neg\neg p \vee \neg\neg r,\ \mathrm{Z} : \neg(c \to d)\ \vdash\ c.$$

Here is a winning strategy S of ∀phrodite:

- If ∃ros plays (A) with the first assumption X, she chooses the component $\beta = s$ and plays (A2') with the new goal r.
- If ∃ros plays (A) with Y, she chooses the component $\bot$ of $\neg\neg p$ and sets $\neg p$ as the new goal (this will later add p to assumptions).

- If $\exists$ros plays (A) with Z, $\forall$phrodite must set $c \to d$ as a new goal.

Only when $\exists$ros has played both Y and Z, and both p and c are already at the lhs, $\forall$phrodite will change her mind and respond to Y using $\neg\neg r$. This way she may reach a position where p, r, c are all assumed, but neither so is q nor s. If $\exists$ros plays now X again, $\forall$phrodite will assert q. (This is her only nontrivial static move in this strategy.) Now, for every reachable position $\mathcal{P}'$:

- The variable s is not in $\mathcal{P}'$;
- If q is in $\Gamma_{\mathcal{P}'}$ then $p, r, c \in \Gamma_{\mathcal{P}'}$;
- There is $\mathcal{P}'' \geq \mathcal{P}'$ with $p, r, c \in \Gamma_{\mathcal{P}''}$.

It follows that if $\mathcal{P}' \Vdash (p \to q) \vee (r \to s)$ then $\mathcal{P}' \Vdash c$ as well, so that Lemma 3.13 fails.

As seen from the above example, we would have to restrict $\forall$phrodite's strategies to "stable" ones: $\forall$phrodite is not supposed to make inconsistent choices in response to the same (A) move. In fact, if $\forall$phrodite has a winning strategy then she has a stable one as well. This follows from the completeness, because a Kripke model yields a stable strategy. We could also consider stable nondeterministic strategies as in Section 5.

One would certainly like to have a simple method to "stabilize" a given deterministic strategy. For instance, in Example 3.3, $\forall$phrodite could safely play the first disjunct each time $\exists$ros refers to Y. It is often safe for her to repeat a previously played move, esp. an (A2') move. But not every strategy can be "stabilized" this way: some (A1') moves cannot be repeated.

Example 3.4. Consider the position

$$(p \vee q) \to [(a \to r) \vee (b \to s) \to p], \neg(a \wedge b) \ \vdash \ p \vee q,$$

and suppose $\forall$phrodite plays as follows: If $\exists$ros chooses the target p of the first assumption, she sets $(a \to r) \vee (b \to s)$ as the new goal (an A1' move). Then $\exists$ros may play $a \to r$ in the next step, which places a at the lhs and r at the rhs. If $\exists$ros keeps referring to the first assumption, $\forall$phrodite cannot repeat her previous (A1') move because $\exists$ros could choose $b \to s$ and eventually win by *ex falso*.

3.1. Completeness in the Fast Game

We show that the fast game of Figure 5 has the desired properties.

Lemma 3.5. *If* $\Gamma, tr(\beta, \alpha) \vdash_\ell \beta$*, for every target* β *of* α*, then also* $\Gamma \vdash_\ell \alpha$*.*

Proof. Induction with respect to α. If α is an atom or a disjunction then it has only one target, namely itself, with an empty trace. The conclusion is immediate.

Let $\alpha = \rho \to \sigma$ and consider any target β of σ. Then $\beta \in \mathsf{Tg}(\alpha)$ as well, and we have $tr(\beta,\alpha) = \{\rho\} \cup tr(\beta,\sigma)$. Thus $\Gamma, \rho, tr(\beta,\sigma) \vdash_\ell \beta$. From the induction hypothesis for σ we obtain $\Gamma, \mathrm{x} : \rho \vdash M : \sigma$, where M is a long normal form. Then $\Gamma \vdash_\ell \lambda \mathrm{x}^\rho.\, M : \alpha$.

Now let $\alpha = \rho \wedge \sigma$. Every target β of α must either be a target of ρ or a target of σ, and we have $tr(\beta,\rho) = tr(\beta,\alpha)$ or $tr(\beta,\sigma) = tr(\beta,\alpha)$, respectively. From the induction hypothesis we get $\Gamma \vdash M : \rho$ and $\Gamma \vdash N : \sigma$, for some long normal M, N. Then $\Gamma \vdash_\ell \langle M, N \rangle : \alpha$. ∎

Lemma 3.6. *If $\Gamma, tr(\alpha,\varphi) \vdash \alpha$, for some $\alpha \in \mathsf{Comp}(\varphi)$, then $\Gamma \vdash_\ell \varphi$.*

Proof. Induction with respect to φ. If $\varphi = \tau \to \sigma$ and $\alpha \in \mathsf{Comp}(\varphi)$ then $\alpha \in \mathsf{Comp}(\sigma)$, and $tr(\alpha,\varphi) = \{\tau\} \cup tr(\alpha,\sigma\}$. Hence $\Gamma, \mathrm{x} : \tau \vdash_\ell M : \sigma$ (i.e. $\Gamma \vdash_\ell \lambda \mathrm{x}\, M : \varphi$) holds for some M by the induction hypothesis.

If $\varphi = \tau \vee \sigma$, and e.g. $\alpha \in \mathsf{Comp}(\tau)$, then $\Gamma \vdash_\ell \tau$ by induction, thus also $\Gamma \vdash_\ell \tau \vee \sigma$. Otherwise we have $\alpha = \varphi$ and $tr(\alpha,\varphi) = \varnothing$, so the conclusion is immediate. ∎

Lemma 3.7. *Let α be a target of γ and let $\Gamma \vdash_\ell \delta$, for all $\delta \in tr(\alpha,\gamma)$. Then $\Gamma, \mathrm{x} : \gamma \vdash_p \alpha$.*

Proof. Induction with respect to γ. If γ is an atom or a disjunction then $\alpha = \gamma$, and we have $\Gamma, \mathrm{x} : \gamma \vdash_p \mathrm{x} : \gamma$.

If $\gamma = \zeta \to \xi$ then $\zeta \in tr(\alpha,\gamma)$, so we have $\Gamma \vdash_\ell M : \zeta$, for some M; then $\mathrm{x}M$ is a quasi-long proper eliminator of type ξ. If $\alpha = \xi$ then we are done, otherwise $\alpha \in \mathsf{Tg}(\xi)$. We apply the induction hypothesis to ξ and we obtain $\Gamma, \mathrm{y} : \xi \vdash_p P : \alpha$. By Lemma 2.21, the term $P[\mathrm{y} := \mathrm{x}M]$ is a quasi-long proper eliminator beginning with x.

Now suppose $\gamma = \zeta \wedge \xi$. Then $\mathrm{x}\{1\}$ has type ζ and $\mathrm{x}\{2\}$ has type ξ. If $\alpha = \zeta$ or $\alpha = \xi$ then that is all, otherwise α is, say, a target of ζ, and we can apply the induction hypothesis to ζ. This yields $\Gamma, \mathrm{y} : \zeta \vdash_p P : \alpha$, and we can take $P[\mathrm{y} := \mathrm{x}\{1\}]$, again by Lemma 2.21. ∎

Lemma 3.8. *If ∃ros has a winning strategy in position $\mathcal{P} = (\Gamma \vdash \varphi)$ then $\Gamma \vdash_\ell \varphi$.*

Proof. Induction with respect to the size of the strategy. Suppose first that ∃ros plays move (A) with $\alpha \in \mathsf{Tg}(\gamma)$, $\gamma \in \Gamma$. Note that φ must then be a target. If that move is winning then ∃ros has winning strategies in all positions reachable for ∀phrodite in the next step, i.e., in all positions of the form $\Gamma, \beta \vdash \varphi$, where β is a disjunct of α, and in all positions of the form $\Gamma \vdash \delta$, where $\delta \in tr(\alpha, \gamma)$. By the induction hypothesis, we have $\Gamma \vdash_\ell tr(\alpha, \gamma)$, and also $\Gamma, \beta \vdash_\ell \varphi$, for the disjuncts β of α. Since $\gamma \in \Gamma$, Lemma 3.7 yields a quasi-long proper eliminator P with $\Gamma \vdash P : \alpha$. Thus $\Gamma \vdash_\ell \alpha$ by Lemma 2.19.

If α is not a disjunction, then $\Gamma, \alpha \vdash_\ell \varphi$, because α is a disjunct of itself. Then $\Gamma \vdash_\ell \varphi$ follows from Lemma 2.20. If $\alpha = \beta_1 \vee \beta_2$ then we have $\Gamma, \mathrm{Y}_1 : \beta_1 \vdash_\ell N_1 : \varphi$ and $\Gamma, \mathrm{Y}_2 : \beta_2 \vdash_\ell N_2 : \varphi$, for some terms N_1, N_2. Therefore $\Gamma \vdash_\ell \mathsf{case}\ P\ \mathsf{of}\ [\mathrm{Y}_1]N_1\ \mathsf{or}\ [\mathrm{Y}_2]N_2 : \varphi$, because φ is a target.

If $\exists$ros plays move (B) choosing an $\alpha \in \mathsf{Comp}(\varphi)$ then he must have strategies in all positions $\Gamma, tr(\beta, \varphi) \vdash \beta$, where $\beta \in \mathsf{Tg}(\alpha)$. From Lemma 3.1 one has $tr(\beta, \varphi) = tr(\beta, \alpha) \cup tr(\alpha, \varphi)$, so it follows from Lemma 3.5 that $\Gamma, tr(\alpha, \varphi) \vdash_\ell \alpha$. Hence $\Gamma \vdash_\ell \varphi$, by Lemma 3.6. ∎

Corollary 3.9. *$\exists$ros has a winning strategy in position $\Gamma \vdash \varphi$ if and only if $\Gamma \vdash_\ell \varphi$.*

Proof. ($\Leftarrow$) By induction with respect to M we define the strategy of $\exists$ros. If M is an introduction, he plays (B); otherwise he refers to the head variable of M. Part ($\Rightarrow$) is Lemma 3.8. ∎

Model construction: A *target* position in $\forall$phrodite's strategy is a position $\mathcal{P}$ where $\tau_{\mathcal{P}}$ is a target. The Kripke model we will now define from a strategy S of $\forall$phrodite differs from the one in Section 2 in one aspect.

- States of the model are the saturated target positions in S.

Otherwise the definition is the same. The reason to change the definition is that Lemma 2.10 is now only applicable to target positions where $\exists$ros is not forced to play (B):

Lemma 3.10. *For every target position $\mathcal{P}$ there is a saturated position $\mathcal{Q}$ such that $\mathcal{P} \twoheadrightarrow_s \mathcal{Q}$ and $\tau_{\mathcal{Q}} = \tau_{\mathcal{P}}$.*

To adjust Lemma 2.12 for the new setting, we begin with some tools.

Lemma 3.11. *Let $\mathcal{C}$ be a Kripke model and let $c \in \mathcal{C}$. Suppose that*

- *For every $\alpha \in \mathsf{Tg}(\gamma)$ and every $c' \geq c$ such that $c' \Vdash tr(\alpha, \gamma)$, there is a disjunct β of α with $c' \Vdash \beta$.*

Then $c \Vdash \gamma$.

Proof. Observe that the assumption immediately implies:

- For every $\alpha \in \mathsf{Tg}(\gamma)$, if $c' \geq c$, and $c' \Vdash tr(\alpha, \gamma)$, then $c' \Vdash \alpha$.

The proof is by induction with respect to γ. If γ is an atom or a disjunction then we take $\alpha = \gamma$ and then $tr(\alpha, \gamma) = \varnothing$, so the claim follows immediately for $c' = c$. The case of a conjunction is a routine application of the induction hypothesis. Finally let $\gamma = \zeta \to \xi$. To prove $c \Vdash \gamma$ we assume $c \leq c' \Vdash \zeta$ and observe that

- For every $\alpha \in \mathsf{Tg}(\xi)$, if $c'' \geq c'$ and $c'' \Vdash tr(\alpha, \xi)$ then $c' \Vdash \alpha$,

because $tr(\alpha, \gamma) = \{\zeta\} \cup tr(\alpha, \xi)$. By the induction hypothesis for ξ we obtain $c' \Vdash \xi$. ■

LEMMA 3.12. *Let $\beta \in \mathsf{Tg}(\alpha)$ and let $c \Vdash tr(\beta, \alpha)$ and $c \nVdash \beta$. Then $c \nVdash \alpha$.*

PROOF. Induction with respect to α. Case $\beta = \alpha$ is obvious. If $\alpha = \alpha_1 \to \alpha_2$ then $\beta \in \mathsf{Tg}(\alpha_2)$, and $tr(\beta, \alpha) = \{\alpha_1\} \cup tr(\beta, \alpha_2)$. Then $c \Vdash \alpha_1$ and $c \nVdash \alpha_2$ by the induction hypothesis, whence $c \nVdash \alpha$. If $\alpha = \alpha_1 \wedge \alpha_2$ then assume e.g. $\beta \in \mathsf{Tg}(\alpha_1)$. By the induction hypothesis $c \nVdash \alpha_1$, because the trace is the same. ■

LEMMA 3.13. *Let $\mathcal{P} = (\Gamma \vdash \tau)$ be a position in a winning strategy S.*

1. *If $\mathcal{P}$ is a saturated target position, and $\gamma \in \Gamma$, then $\mathcal{P} \Vdash \gamma$.*
2. *If $\gamma = \tau$ then there exists a saturated target position $\mathcal{P}'$ such that $\mathcal{P} \twoheadrightarrow \mathcal{P}'$ and $\mathcal{P}' \nVdash \gamma$.*

PROOF. Simultaneous induction with respect to γ.

(1) For $\gamma \in \Gamma$, we use Lemma 3.11. Consider any $\alpha \in \mathsf{Tg}(\gamma)$ and let $\mathcal{P} \leq \mathcal{P}' \Vdash tr(\alpha, \gamma)$. In position $\mathcal{P}'$ $\exists$ros can play α. Suppose $\forall$phrodite chooses $\delta \in tr(\alpha, \gamma)$ in response. Then by part (2) of the induction hypothesis we have a state $\mathcal{P}'' \geq \mathcal{P}'$ with $\mathcal{P}'' \nVdash \delta$, a contradiction. So if $\exists$ros plays α in $\mathcal{P}'$ then $\forall$phrodite's response must be (A2) and the next position is $\Gamma, \beta \vdash \tau$, where β is a disjunct of α. But $\mathcal{P}'$ is saturated, so $\beta \in \Gamma_{\mathcal{P}'}$ and thus $\mathcal{P}' \Vdash \beta$ by the induction hypothesis (1) for β.

(2) Now we use Lemma 3.12. Assume first that $\tau = \vec{\zeta} \to \alpha$, where ζ is not empty and α is not an implication (i.e., α is the largest component of τ). Then $\exists$ros can play α in position $\mathcal{P}$. There is a response $\beta \in \mathsf{Tg}(\alpha)$ such that the position $\mathcal{P}' = (\Gamma, \vec{\zeta}, tr(\beta, \alpha) \vdash \beta)$ belongs to the strategy. By Lemma 3.10 there is a saturated target position $\mathcal{P}''$ with $\mathcal{P}' \twoheadrightarrow_s \mathcal{P}''$. Then we have $\mathcal{P}'' \Vdash tr(\beta, \alpha)$ by the induction hypothesis (1) for $tr(\beta, \alpha)$. In addition, $\mathcal{P}'' \nVdash \beta$; indeed, by the induction hypothesis (2) for β there is a state $\mathcal{P}''' \geq \mathcal{P}''$ with $\mathcal{P}''' \nVdash \beta$. Thus, Lemma 3.12 yields $\mathcal{P}'' \nVdash \tau$.

Now suppose that $\tau = \sigma \vee \rho$. Then $\exists$ros may play σ or ρ. In the first case, we have $\beta \in \mathsf{Tg}(\sigma)$ and a position $\mathcal{P}' = (\Gamma, tr(\beta, \sigma) \vdash \beta)$ in $\forall$phrodite's strategy. From the induction hypothesis (2) for β we obtain a state $\mathcal{P}''$ with $\mathcal{P}' \twoheadrightarrow \mathcal{P}''$ and $\mathcal{P}'' \not\Vdash \beta$. Since $\mathcal{P}' \twoheadrightarrow \mathcal{P}''$ we have $tr(\beta, \sigma) \subseteq \Gamma_{\mathcal{P}''}$ whence $\mathcal{P}'' \Vdash tr(\beta, \sigma)$ by the induction hypothesis (1). By Lemma 3.12 we conclude that $\mathcal{P}'' \not\Vdash \sigma$. Similarly we prove that $\mathcal{P} \not\Vdash \rho$. Finally, the case of τ being an atom follows straight from the definition. ■

From Lemmas 3.8 and 3.13 one can obtain another proof of Theorem 2.25.

3.2. Standard Strategies and Non-ambiguity

In the simple game of Section 2 the existential player, ∃ros, constructs a proof term step by step in a quite arbitrary order. As a result, multiple strategies of ∃ros correspond to the same proof term, cf. Remark 2.9. It is even possible that ∃ros plays some moves irrelevant to the obtained proof.

Example 3.14. In position $\mathrm{x} : p \to q, \mathrm{y} : p, \mathrm{z} : p \to r \vdash r$, ∃ros may play X (irrelevant) first and then Z (relevant). This strategy yields the proof term $\mathrm{zy} = \mathrm{zy}[\mathrm{u} := \mathrm{xy}]$, where $\mathrm{u} : q$ is the new assumption variable added by ∀phrodite in response to X.

This ambiguity is only partly eliminated in the fast game. If we look at proof-search from algorithmic point of view, such redundancy is unnecessary and inconvenient. It is therefore natural to require that ∃ros only plays moves properly addressing the purpose of the game. Then a given proof term may only be constructed in one way, namely in a top-down fashion (as in the proof of Corollary 3.9). Thus we define that ∃ros' strategy is *standard* when it obeys the following regulation:

- When the goal φ is an atom, ∃ros never plays (A) with a target which is a variable q other than φ.

In other words, the type of a head variable chosen by ∃ros must be relevant to the current target goal: it must be either $\perp$ or the goal itself or a disjunction.

A standard strategy is (modulo the treatment of conjunction) morally equivalent to *Wajsberg's algorithm*, implicit in [20] and corrected in [2].

For example, in a position of the form $\Gamma \vdash \vec{\zeta} \to \tau \wedge \sigma$, ∃ros must play move (B), asking ∀phrodite to choose a target of τ or of σ as the new goal. And in a position of the form $\Gamma \vdash p$ he cannot refer to assumptions of the form $\vec{\xi} \to q$, where $q \neq p$. (It may seem natural to require ∃ros to use only assumptions of shape $\vec{\xi} \to p$, but that would not work, cf. [2].)

Fact 3.15. *If ∃ros has a winning strategy in the fast game then he also has a standard strategy.*

PROOF. The strategy defined in the proof of Corollary 3.9 is standard. ∎

Unfortunately, if we build such a principle into the rules of the game we will no longer be able to claim that every ∀phrodite's strategy defines a Kripke model.

EXAMPLE 3.16. If ∃ros, as a rule, cannot play $p \to q$ in a position like $p \to q, p, s \to r \vdash r$ then an ∀phrodite's strategy does not have to account for that and q will never be added to assumptions. Therefore, $p \to q$ will not be forced in the model. In other words, a strategy of ∀phrodite, sufficient to win the game against a standard strategy, is not necessarily sufficient to provide a Kripke model.

4. Games on Trees

Under the restriction to standard strategies, we can present our games as played on tree representations of formulas. Generalizing the style of Hanno Nickau [13] (see also [3,14]), we draw formulas as finite trees, with nodes labeled by propositional variables, $\bot$, $\vee$, and $\wedge$ (but not the arrow). In addition, every node is given a parity sign ($+$ or $-$). The root is always positive. If T is a tree then T^- is the same tree with all parities reversed.

- An atom p (resp. $\bot$) is a single node labeled p (resp. $\bot$).
- A formula of the form $\sigma \odot \tau$, with $\odot = \vee, \wedge$, has a root $\odot$ and two immediate subtrees σ and τ (of positive parity).
- A formula of the form $\sigma \to \tau$ is τ expanded by adding σ^- as an immediate subtree of the root (note the reversed parity).

Positive nodes in a tree are called *goals*, and negative nodes are *assumptions*. The order of successors of a node does not matter. A judgment $\gamma_1, \ldots, \gamma_n \vdash \sigma$ is identified with the formula $\gamma_1 \to \cdots \to \gamma_n \to \sigma$. This presentation conveniently identifies certain "isomorphic" formulas, like e.g. $p \to q \to r$ and $q \to p \to r$, which have the same inhabitants up to permutations.

EXAMPLE 4.1. The judgment $p,\ q \to s,\ r \to t,\ p \to q \vee r \ \vdash\ s \vee t$ and the two formulas $(((p \to r) \to ((p \to q) \to r) \to r) \to q) \to q$ and $\neg\neg(((p \to q) \to p) \to p)$ are represented by the trees in Figure 6.

Note that each $\wedge$- or $\vee$-node n has exactly two immediate successor nodes of the same parity as n. These two are called *principal successors* of n.

When talking about disjuncts, components and targets of formulas we refer to their top nodes:

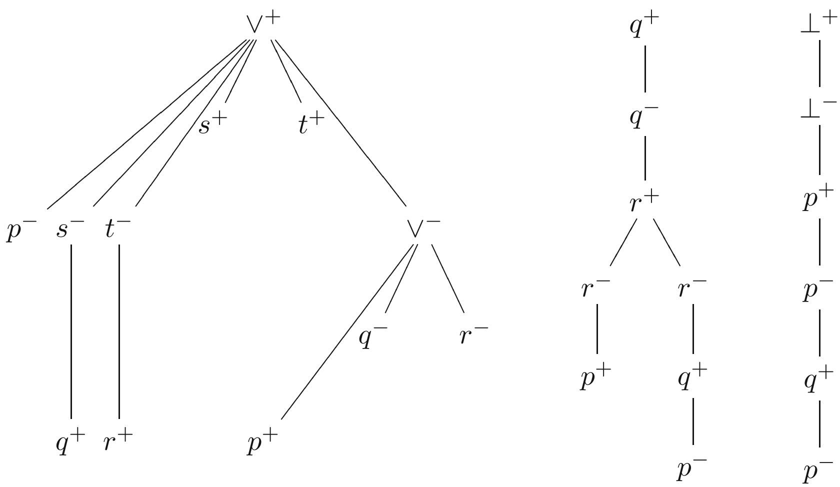

Figure 6. Trees for Examples 4.1 and 4.3

- Disjuncts of a $\vee$-node are its principal successors.
- Any other node is the only disjunct of itself.
- An atom node is the only target and only component of itself.
- A $\vee$-node is the only target of itself.
- A $\wedge$-node is the only component of itself.
- Targets of a $\wedge$-node are targets of its principal successors.
- Components of a $\vee$-node are the node itself and all components of its principal successors.

In other words, components of a $\vee^+$-node n are *all vertices* of the largest subtree rooted at n such that all its internal nodes are labeled $\vee^+$. In contrast, targets of a $\wedge^+$-node n are *leaf* vertices of the largest subtree rooted at n such that all its internal nodes are labeled $\wedge^+$.
The above definition is equivalent to the one in Section 3 if we identify a formula with its root node, cf. Example 4.2. In a similar spirit we adjust the definition of a trace:

- For every node n we take $tr(n,n)$ to be the set of all non-principal children of n.
- If n is a principal child of n', and $tr(n',m)$ is defined, then $tr(n,m) = tr(n,n) \cup tr(n',m)$.

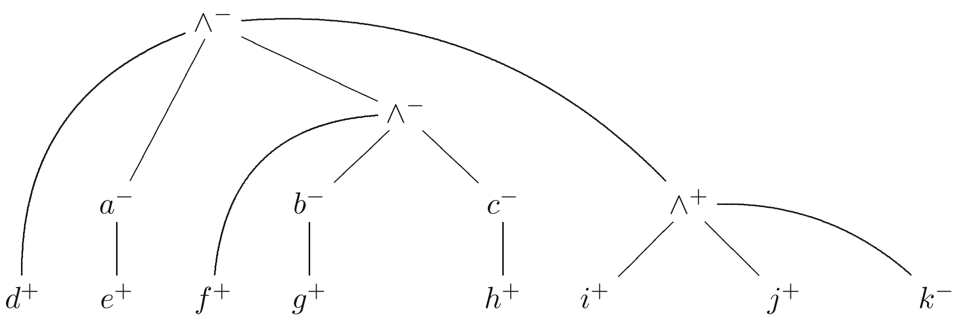

Figure 7. Assumption formula $d \to (k \to i \wedge j) \to (e \to a) \wedge (f \to (g \to b) \wedge (h \to c))$

That is, $tr(n, m)$ is the set of all non-principal children of nodes occurring along the branch to n in the tree of m, provided all these nodes are of the same parity.

Example 4.2. Figure 7 shows a negative $\wedge$-node n, representing an assumption $\varphi = d \to (k \to i \wedge j) \to (e \to a) \wedge (f \to (g \to b) \wedge (h \to c))$. There are three targets of the root, labeled a^-, b^-, and c^-. Then $tr(a, n) = \{d, e, \wedge^+\}$, $tr(b, n) = \{d, f, g, \wedge^+\}$, and $tr(c, n) = \{d, f, h, \wedge^+\}$ (we use labels informally to identify nodes). Observe that the set $tr(a, n)$ corresponds to the traces $tr(e \to a, \varphi) = \{d, k \to i \wedge j\}$ and $tr(a, \varphi) = \{d, k \to i \wedge j, e\}$.

Graph game: A *position* in the game is now a pair (Γ, n) consisting of a single goal n (a positive node) and a set Γ of (negative) assumptions. The goal is never an implication: premises of an implication are automatically added to assumptions, like in the fast game.
The *initial* position has the root as a goal and all negative successors of the root as assumptions. In other words, the initial assumptions are members of the trace of the root in itself. A *final* position has either

– a goal labeled p and a leaf assumption labeled by the same variable p,

– or a $\perp$-leaf node among assumptions, and a goal labeled by an atom.

The fast game rules of Figure 5 are translated to the graph rules in Figure 8. It should be quite clear that this routine is an implementation of the game rules of Section 3, under the additional assumption that $\exists$ros plays a standard strategy.

The restriction to standard behaviour of $\exists$ros is necessary because our interpretation does not distinguish between a target of a formula and the formula itself. A proper implementation of our game in the general case would require an additional complication to handle this difference. However, under

∃ros plays one of the following moves:

A) He picks a target n^- of an assumption a^-.

B) He picks a component n^+ of the current goal g^+.

When the current goal is an atom $\mathbf{a}$, ∃ros must play (A) choosing a target labeled either $\mathbf{a}$, or $\vee$, or $\bot$. Then ∀phrodite responds:

A1) She picks an element $r \in tr(n, a)$, makes r the new goal, and adds all negative children of r to assumptions, or:

A2) She picks a disjunct m^- of n^-, and adds m^- to assumptions.

B) She picks a target m^+ of n^+. Then all nodes in $tr(m, a)$ are added to the set of assumptions, and m becomes a new goal.

Figure 8. Fast game on a tree

the standard strategy rule, this problem does not occur. Indeed, suppose we are in a position $\Gamma, \vec{\zeta} \to p \vdash p$, and ∃ros wants to play $\vec{\zeta} \to p$. Then ∀phrodite must respond (A1), as (A2) gives an immediate win to ∃ros. Thus we can simply ignore the latter option.

Example 4.3. The game can be executed by placing tokens on the graph. There is always one red token representing the current goal, and some green tokens representing assumptions. Green tokens are never removed from the graph. Consider the leftmost tree in Figure 6. In the initial position the red token is placed at the root, and there are four green tokens at p^-, s^-, t^-, and $\vee^-$. Suppose ∃ros begins by choosing node $\vee^-$. If ∀phrodite picks p^+ she loses, so let her choose q^-. This means she places a new green token there. Now ∃ros moves the red token to s^+ and ∀phrodite has no response to this, so he continues by proposing node s^-. As a result, the red token goes to q^+ and ∃ros wins.

On the middle tree the play begins with a red token at q^+ and a green one at q^-. In the first step the red token goes to r^+ and two green ones are placed at the two nodes marked r^-. ∃ros may now point to any of these nodes. He prefers the right one and moves the red token to q^+. ∀phrodite must place a green token at p^-. ∃ros now refers again to q^-, i.e., r^+ is red for the second time. With the green p^- at hand, ∃ros safely chooses the left assumption and wins.

The rightmost tree in Figure 6 is similar, but note that this time ∃ros will play *ex falso* by moving the red token from q^+ to $\bot^+$.

$$\frac{\Gamma \vdash M : \varphi}{\Gamma \vdash (\lambda x\, M) : \forall x \varphi}\ (1) \qquad \frac{\Gamma \vdash M : \forall x \varphi}{\Gamma \vdash (My) : \varphi[x := y]}$$

$$\frac{\Gamma \vdash M : \varphi[x := y]}{\Gamma \vdash [y, M] : \exists x \varphi} \qquad \frac{\Gamma \vdash M : \exists x \varphi \quad \Gamma, \textsc{x} : \varphi \vdash N : \psi}{\Gamma \vdash (\texttt{let}\ M\ \texttt{be}\ [x, \textsc{x}]\ \texttt{in}\ N) : \psi}\ (2)$$

Figure 9. Rules for first-order logic. Read (1) as $x \notin \mathrm{FV}(\Gamma)$ and (2) as $x \notin \mathrm{FV}(\Gamma, \psi)$

5. Games for First-Order Logic

We consider first-order intuitionistic logic without function symbols. Individual variables are denoted by $x, y, \ldots$ (recall that we use small caps $\textsc{x}, \textsc{y}, \ldots$ for proof variables). Alpha-equivalent formulas are identified, i.e., bound variables do not matter. The set of all free variables in a formula φ is denoted by $\mathrm{FV}(\varphi)$, and we take $\mathrm{FV}(\Gamma) = \bigcup\{\mathrm{FV}(\varphi) \mid \varphi \in \Gamma\} \cup \{x_0\}$, for some fixed variable x_0. This guarantees that $\mathrm{FV}(\Gamma)$ is never empty.
Term assignment is given in Figures 2 and 9. New reduction rules in Figure 10 for the universal and existential quantifiers should be added to Figure 3. The new beta-redexes correspond to introduction-elimination pairs for the two quantifiers. The commuting conversions permute redexes of the following forms: $\mathit{Elim}^\sigma(\varepsilon_\tau(M))$, $\mathit{Elim}^\sigma(\texttt{case}\ M\ \texttt{of} \ldots)$, and $\mathit{Elim}^\sigma(\texttt{let}\ M\ \texttt{be} \ldots)$. Here is a classification of normal forms:

- *Introductions:* $\lambda \textsc{x}^\alpha .\, N, \quad \lambda x\, N, \quad \langle N_1, N_2 \rangle, \quad \texttt{in}_i(N), \quad [y, N];$
- *Proper eliminators:* $\textsc{x}, \quad PN, \quad P\{i\}, \quad Px;$
- *Improper eliminators:*

$$\varepsilon_\varphi(P),\ \texttt{case}\ P\ \texttt{of}\ [\textsc{x}]N_1\ \texttt{or}\ [\textsc{y}]N_2,\ \texttt{let}\ P\ \texttt{be}\ [x, \textsc{x}]\ \texttt{in}\ N.$$

Lemma 2.1 remains true, and Lemma 2.2 should be adjusted by adding one more case, namely $P = Q[\textsc{y} := \textsc{x}z]$. A useful routine observation is that capture-avoiding substitution of individual variables preserves type inference: if $\Gamma \vdash M : \tau$ then $\Gamma[x := y] \vdash M[x := y] : \tau[x := y]$.
Game playing: Positions, assumptions, etc. are defined as in Section 2. Each turn in position $\Gamma \vdash \tau$ begins with an ∃ros' move. As in Section 2, ∃ros picks a formula α which is either (a) a non-atomic assumption or (b) an aim (a disjunct of the goal). But now in some cases, ∃ros also instantiates the top quantifier in α. In Figure 11 only (a4–a5, b4–b5) are new rules.

Beta-reductions:

- $(\lambda x M)y \ \to\ M[x := y]$, where x and y are object variables;
- $\mathtt{let}\ [y, M]\ \mathtt{be}\ [x, \mathrm{X}]\ \mathtt{in}\ N \ \to\ N[x := y][\mathrm{X} := M]$.

Commuting conversions:

- $\varepsilon_{\forall x\, \sigma}(M)y \ \to\ \varepsilon_{\sigma[x:=y]}(M)$;
- $\mathtt{let}\ \varepsilon_{\exists x\, \sigma}(M)\ \mathtt{be}\ [x, \mathrm{Y}]\ \mathtt{in}\ N^{\rho} \ \to\ \varepsilon_{\rho}(M)$;
- $(\mathtt{case}\ M\ \mathtt{of}\ [\mathrm{X}]R\ \mathtt{or}\ [\mathrm{Y}]Q)y \ \to\ \mathtt{case}\ M\ \mathtt{of}\ [\mathrm{X}]Ry\ \mathtt{or}\ [\mathrm{Y}]Qy$;
- $\mathtt{let}\ (\mathtt{case}\ M\ \mathtt{of}\ [\mathrm{Y}]R\ \mathtt{or}\ [\mathrm{Z}]Q)\ \mathtt{be}\ [x,\, \mathrm{X}]\ \mathtt{in}\ N \ \to$
  $\mathtt{case}\, M\, \mathtt{of}\ [\mathrm{Y}](\mathtt{let}\, R\, \mathtt{be}\ [x,\, \mathrm{X}]\ \mathtt{in}\, N)\ \mathtt{or}\ [\mathrm{Y}](\mathtt{let}\, Q\, \mathtt{be}\ [x,\, \mathrm{X}]\ \mathtt{in}\, N)$;
- $\varepsilon_{\varphi}(\mathtt{let}\ M\ \mathtt{be}\ [x,\, \mathrm{X}]\ \mathtt{in}\ N) \ \to\ \mathtt{let}\ M\ \mathtt{be}\ [x,\, \mathrm{X}]\ \mathtt{in}\ \varepsilon_{\varphi}(N)$;
- $(\mathtt{let}\ M\ \mathtt{be}\ [x,\, \mathrm{X}]\ \mathtt{in}\ N)Q \ \to\ \mathtt{let}\ M\ \mathtt{be}\ [x,\, \mathrm{X}]\ \mathtt{in}\ NQ$;
- $(\mathtt{let}\ M\ \mathtt{be}\ [x,\, \mathrm{X}]\ \mathtt{in}\ N)y \ \to\ \mathtt{let}\ M\ \mathtt{be}\ [x,\, \mathrm{X}]\ \mathtt{in}\ Ny$;
- $(\mathtt{let}\ M\ \mathtt{be}\ [x,\, \mathrm{X}]\ \mathtt{in}\ N)\{i\} \ \to\ \mathtt{let}\ M\ \mathtt{be}\ [x,\, \mathrm{X}]\ \mathtt{in}\ N\{i\}$;
- $\mathtt{case}\ (\mathtt{let}\ M\ \mathtt{be}\ [x,\, \mathrm{X}]\ \mathtt{in}\ N)\ \mathtt{of}\ [\mathrm{Y}]R\ \mathtt{or}\ [\mathrm{Z}]Q \ \to$
  $\mathtt{let}\ M\ \mathtt{be}\ [x,\, \mathrm{X}]\ \mathtt{in}\ \mathtt{case}\ N\ \mathtt{of}\ [\mathrm{Y}]R\ \mathtt{or}\ [\mathrm{Z}]Q$;
- $\mathtt{let}\ (\mathtt{let}\ M\ \mathtt{be}\ [x,\, \mathrm{X}]\ \mathtt{in}\ N)\ \mathtt{be}\ [y,\, \mathrm{Y}]\ \mathtt{in}\ R \ \to$
  $\mathtt{let}\ M\ \mathtt{be}\ [x,\, \mathrm{X}]\ \mathtt{in}\ (\mathtt{let}\ N\ \mathtt{be}\ [y, \mathrm{Y}]\ \mathtt{in}\ R)$.

Figure 10. Reduction rules for first-order proof terms

Note that ∀phrodite only makes actual moves in cases (a1–a2, b2), and she has no choice in cases (a3–a5, b1, b3–b5). We could let her make a choice of a variable in (a5, b4), but a fresh variable suits her best in these cases.

For technical reasons we also permit ∃ros to "pass", i.e., do nothing, in which case ∀phrodite does nothing too and the position remains unchanged. The determinacy of the game is immediate again (cf. Proposition 2.3).

Proposition 5.1. *The play is determined: every position is winning for one of the players.*

Example 5.2. (A double negation). Consider the game beginning in the position $\forall x\,(\mathsf{P}(x) \vee (\mathsf{P}(x) \to \mathsf{Q})) \to \mathsf{Q} \vdash \mathsf{Q}$. The only possible move of ∃ros is to choose the assumption, and the only reasonable response of ∀phrodite is to set the new goal as $\forall x\,(\mathsf{P}(x) \vee (\mathsf{P}(x) \to \mathsf{Q}))$. This goal will have to be pointed to by ∃ros (otherwise ∀phrodite would go back to the same position) a1) If α is an assumption $\beta \to \gamma$ then $\forall$phrodite chooses between positions $\Gamma, \gamma \vdash \tau$ and $\Gamma \vdash \beta$.

a2) If α is an assumption $\beta \vee \gamma$ then $\forall$phrodite chooses between positions $\Gamma, \beta \vdash \tau$ and $\Gamma, \gamma \vdash \tau$.

a3) If α is an assumption $\beta \wedge \gamma$ then $\forall$phrodite has no choice and the next position is $\Gamma, \beta, \gamma \vdash \tau$.

a4) If α is an assumption $\forall x\, \varphi$, then $\exists$ros also chooses an arbitrary variable y. Then the next position is $\Gamma, \varphi[x := y] \vdash \tau$.

a5) If α is an assumption $\exists x\, \varphi$ then the next position is $\Gamma, \varphi[x := z] \vdash \tau$, where z is fresh.

b1) If α is an aim $\beta \to \gamma$ then the next position is $\Gamma, \beta \vdash \gamma$.

b2) If α is an aim $\beta \wedge \gamma$ then $\forall$phrodite chooses between positions $\Gamma \vdash \beta$ and $\Gamma \vdash \gamma$.

b3) If the aim α is an atom or a disjunction then the next position is $\Gamma \vdash \alpha$.

b4) If α is an aim $\forall x\, \varphi$ then the next position is $\Gamma \vdash \varphi[x := z]$, where z is fresh.

b5) If α is an aim $\exists x\, \varphi$ then $\exists$ros chooses some variable y and the next position is $\Gamma \vdash \varphi[x := y]$.

Figure 11. First-order game rules

and the new position is $\forall x\, (\mathsf{P}(x) \vee (\mathsf{P}(x) \to \mathsf{Q})) \to \mathsf{Q} \vdash \mathsf{P}(z_0) \vee (\mathsf{P}(z_0) \to \mathsf{Q})$ with a fresh z_0. Repeating move (a) or selecting the aim $\mathsf{P}(z_0)$ will return the game in at most 2 steps to an earlier position, so $\exists$ros should eventually select the other disjunct $\mathsf{P}(z_0) \to \mathsf{Q}$ which will advance the game to the position $\forall x\, (\mathsf{P}(x) \vee (\mathsf{P}(x) \to \mathsf{Q})) \to \mathsf{Q}, \mathsf{P}(z_0) \vdash \mathsf{Q}$. $\exists$ros may now lead the play to $\forall x\, (\mathsf{P}(x) \vee (\mathsf{P}(x) \to \mathsf{Q})) \to \mathsf{Q}, \mathsf{P}(z_0) \vdash \forall x\, (\mathsf{P}(x) \vee \mathsf{P}(x) \to \mathsf{Q})$, but the quantifier at the right has to be instantiated by a new variable z_1, and not by z_0. With $\mathsf{P}(z_1)$ as the new goal $\exists$ros is losing, so all he can do is introduce another useless assumption $\mathsf{P}(z_1)$ at the left. And so on.

The modest game: For a position $\mathcal{P} = (\Gamma \vdash \tau)$, we define the set of free variables in $\mathcal{P}$ as $\mathrm{FV}(\mathcal{P}) = \mathrm{FV}(\Gamma) \cup \mathrm{FV}(\tau)$. A *play* is a sequence of positions $\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \ldots$ where every $\mathcal{P}_{i+1}$ is a successor of $\mathcal{P}_i$ in the game. If not stated otherwise, a play begins in the initial position.

Let us now make a simple observation that will greatly simplify the next example. In moves (a4, b5) $\exists$ros can in principle choose *any* variable y, but he has got nothing to lose if he restricts himself to variables already known to the players. More precisely, consider the following restriction to game rules:[1] in a play of the form $\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ if $\exists$ros plays an (a4) or a (b5) move in $\mathcal{P}_n$ then the variable y must be a member of $\mathrm{FV}(\mathcal{P}_i)$, for some $i \leq n$. A game so restricted is called *modest*. We now show that without loss of generality one can only consider modest games.

Indeed, we have the following two lemmas. Recall that x_0 is the variable which is assumed to be free in every position.

Lemma 5.3. *If $\exists$ros has a strategy in position $\mathcal{P} = (\Gamma \vdash \tau)$ then he has a strategy (of the same size) in position $\mathcal{P}' = (\Gamma[y := x_0] \vdash \tau[y := x_0])$.*

Proof. The proof is by induction with respect to the size of $\exists$ros' strategy. The base case follows immediately, since a final position remains final after any variable substitution. The induction step is by cases depending on his first move. Most cases are easy, we only discuss two examples.

(a4) Suppose that $\forall x\, \varphi \in \Gamma$, and $\exists$ros in position $\mathcal{P}$ adds the assumption $\varphi[x := z]$ and wins. By the induction hypothesis he also wins in position $\Gamma[y := x_0], \varphi[x := z][y := x_0] \vdash \tau[y := x_0]$. Observe that if $z \neq y$ then $\varphi[x := z][y := x_0] = \varphi[y := x_0][x := z]$, because we can assume that $x \neq y, x_0$. So the above winning position is the same as the position $\Gamma[y := x_0], \varphi[y := x_0][x := z] \vdash \tau[y := x_0]$, reachable by in one step from $\mathcal{P}'$.

If $z = y$ then $\varphi[x := z][y := x_0] = \varphi[x := y][y := x_0] = \varphi[y := x_0][x := x_0]$. Now $\exists$ros can still win in position $\mathcal{P}'$ if he plays (a4) but chooses x_0 rather than y.

(b4) In this case $\forall x\, \varphi$ is a disjunct of τ, and $\exists$ros wins by setting new goal $\varphi[x := z]$, where z is fresh, i.e., $z \notin \mathrm{FV}(\Gamma) \cup \mathrm{FV}(\tau)$. But we can assume $y \in \mathrm{FV}(\Gamma) \cup \mathrm{FV}(\tau)$, as otherwise $\mathcal{P}' = \mathcal{P}$ and the claim follows trivially. In particular we have $y \neq z$, and we commute the substitutions as in case (a4). ■

Lemma 5.4. *If $\exists$ros has a winning strategy then he has a winning strategy in the modest game.*

Proof. Suppose that, as part of his winning strategy, $\exists$ros plays (a4) with a variable $y \notin \mathrm{FV}(\Gamma) \cup \mathrm{FV}(\tau)$. Then the next position is $\Gamma, \varphi[x := y] \vdash \tau$. Substituting x_0 for y yields the position $\Gamma, \varphi[x := x_0] \vdash \tau$, in which he has a strategy by Lemma 5.3. So he could play x_0 as well. Case (b5) is similar. ■

[1] In fact we could impose a more severe restriction on $\exists$ros, namely that $y \in \mathrm{FV}(\mathcal{P}_n)$, but it will turn out inconvenient later.

EXAMPLE 5.5. (Grzegorczyk's constant domain scheme). The game beginning in the position $\forall x\,(\mathsf{P} \vee \mathsf{Q}(x)) \vdash \mathsf{P} \vee \forall x\,\mathsf{Q}(x)$ is winnable by ∀phrodite. Indeed, initially ∃ros has a choice between two moves: (a4) and (b3). By Lemma 5.4 we can assume that ∃ros plays a modest game. Thus the only available variable is x_0, and move (a4) yields the new assumption $\mathsf{P} \vee \mathsf{Q}(x_0)$. If ∃ros then refers to this formula in move (a2) then ∀phrodite must choose the disjunct $\mathsf{Q}(x_0)$. Whether or not this happens, ∃ros may eventually play (b3) and pick one of the aims, either P or $\forall x\,\mathsf{Q}(x)$, and dispose of the other. If he chooses P then all he can do is repeat his previous moves, otherwise he may use (b4) and set a new goal $\mathsf{Q}(y)$. Now he can play (a4) again and introduce the assumption $\mathsf{P} \vee \mathsf{Q}(y)$. But in this case, ∀phrodite chooses P. This is a good choice, because the P at the rhs is already gone.

In any case the game must enter an "idle" position, where ∃ros can but repeat one of his previous moves and receive the answer he has already got before. Assuming that he attempts to use all possible chances, this idle position will be one of the following two:

- $\forall x\,(\mathsf{P} \vee \mathsf{Q}(x)), \mathsf{P} \vee \mathsf{Q}(x_0), \mathsf{Q}(x_0) \vdash \mathsf{P}$;
- $\forall x\,(\mathsf{P} \vee \mathsf{Q}(x)), \mathsf{P} \vee \mathsf{Q}(x_0), \mathsf{Q}(x_0), \mathsf{P} \vee \mathsf{Q}(y), \mathsf{P} \vdash \mathsf{Q}(y)$.

The game according to ∃ros

The identification of normal proofs with ∃ros' strategies is as natural as for the propositional logic. The following extends Lemma 2.6.

LEMMA 5.6. *1. If $\Gamma \vdash_n \tau$ then ∃ros has a strategy in position $\Gamma \vdash \tau$.*

2. Suppose that $\Gamma \vdash P : \varphi$, where P is a proper eliminator, and that ∃ros has a strategy in position $\Gamma, \varphi \vdash \tau$. Then ∃ros has a strategy in position $\Gamma \vdash \tau$.

PROOF. (1) We have some additional cases for the first-order constructs. For example, suppose that $\Gamma \vdash_n [y, M] : \exists x\,\varphi$, where $\Gamma \vdash_n M : \varphi[x := y]$. Then the position $\Gamma \vdash \varphi[x := y]$ is reachable from $\Gamma \vdash \exists x\,\varphi$ by one (b5) move of ∃ros. So all we need is to refer to the induction hypothesis (1) for M.

Now assume that $\Gamma \vdash_n \mathsf{let}\ P\ \mathsf{be}\ [y, \mathrm{Y}]\ \mathsf{in}\ M : \sigma$. Then $\Gamma \vdash P : \exists x\,\varphi$ and $\Gamma, \mathrm{Y} : \varphi[x := y] \vdash_n M : \sigma$, whence ∃ros has a strategy in position $\Gamma, \varphi[x := y] \vdash \sigma$, by the induction hypothesis (1). Therefore he also has a strategy in position $\Gamma, \exists x\,\varphi \vdash \sigma$, and it suffices to apply the induction hypothesis (2) for P.

In part (2) we have the additional case $P = Q[\mathrm{Y} := \mathrm{X}y]$. So assume that ∃ros wins in position $\Gamma, \varphi \vdash \tau$ and that $\Gamma \vdash Q[\mathrm{Y} := \mathrm{X}y] : \varphi$. The latter happens when $(\mathrm{X} : \forall x\,\psi) \in \Gamma$ and $\Gamma, \mathrm{Y} : \psi[x := y] \vdash Q : \varphi$. Since

$\exists$ros has a strategy in position $\Gamma, \varphi \vdash \tau$, he also has a strategy in position $\Gamma, \varphi, \psi[x := y] \vdash \tau$. By the induction hypothesis (2) for Q, he must have a strategy in position $\Gamma, \psi[x := y] \vdash \tau$, reachable from $\Gamma \vdash \tau$. ■

PROPOSITION 5.7. $\exists$ros *has a winning strategy in position* $\Gamma \vdash \tau$ *if and only if* $\Gamma \vdash_n \tau$.

PROOF. The right-to-left direction is Lemma 5.6(1). The added cases for the other direction are the following. In (a4–a5) we assume that $\mathrm{X} : \alpha$ in Γ.

(a4) We have $\Gamma, \mathrm{Y} : \varphi[x := y] \vdash_n M : \tau$, where y is some variable. Then $\Gamma \vdash M[\mathrm{Y} := \mathrm{X}y] : \tau$.

(a5) Now $\Gamma, \mathrm{Y} : \varphi[x := z] \vdash_n M : \tau$, where z is fresh. It follows that $\Gamma \vdash_n \texttt{let}\ \mathrm{X}\ \texttt{be}\ [z, \mathrm{Y}]\ \texttt{in}\ M : \tau$.

(b4) If $\Gamma \vdash_n M : \varphi[x := z]$, with fresh z, then $\Gamma \vdash_n \lambda z\, M : \forall z\, \varphi[x := z]$. By alpha-conversion this is the same as $\Gamma \vdash_n \lambda z\, M : \forall x\, \varphi$.

(b5) If $\Gamma \vdash_n M : \varphi[x := y]$ then $\Gamma \vdash_n [y, M] : \exists x\, \varphi$. ■

The game according to $\forall$phrodite

In what follows we assume that we deal with a modest game where $\forall$phrodite has a winning strategy. In Section 2.1 we defined $\forall$phrodite's strategy as a tree labeled by non-final positions and such that:

- For every node $\mathcal{P}$ and every $\exists$ros' move possible in $\mathcal{P}$, there is a successor $\mathcal{P}'$ of $\mathcal{P}$ representing a position obtained by that $\exists$ros' move followed by a response of $\forall$phrodite.

This definition permits a strategy to contain more than one successor of a node $\mathcal{P}$ corresponding to the same $\exists$ros' move. That is, a strategy may be non-deterministic when $\forall$phrodite has more than one option to win. The common intuition suggests that a strategy should be deterministic or "minimal". But this approach does not work for us, as can be seen from Example 5.22. In a sense, a deterministic strategy may contain less information than is required to define a model. Therefore we choose just the opposite: we consider the *largest strategy* which is the tree of all the winning positions of $\forall$phrodite whose ancestors are winning for $\forall$phrodite. (We disregard winning positions reachable only in case of $\exists$ros' mistake.) This largest strategy will be denoted by $\mathcal{S}$ in the sequel. To be more precise, we mean a tree $\mathcal{S}$ labeled by positions, where some different nodes may have the same label. Whenever we talk below about a position $\mathcal{P} = (\Gamma_{\mathcal{P}} \vdash \tau_P) \in \mathcal{S}$ we actually mean the node, not just the label.

In order to properly define the largest strategy $\mathcal{S}$ we must overcome an embarrassing inconvenience created by the notion of a "fresh variable" introduced in moves (a5) and (b4). We do not want to assume that a "fresh variable" can be any variable at all (whatever that means), as this would result in creating a plethora of positions, equivalent with respect to trivial renamings. We thus prefer a systematic ("deterministic") way of introducing new variables, using indices to identify their origin.[2] We assume that the new variable z is chosen as follows:

- It is $z^i_{\exists x\,\varphi}$ in case of (a5), when $\exists$ros plays the assumption $\exists x\,\varphi$ for the ith time;
- It is $z^i_{\forall x\,\varphi}$ in case of (b4), when $\exists$ros plays the aim $\forall x\,\varphi$ for the ith time.

Note that the number i depends on the node of § and not just the associated judgment.

The model construction must be essentially modified with respect to that of Section 2.1, because first-order logic may require infinite domains (Example 5.21). Therefore we cannot rely on finite positions (in particular saturated positions are not always available) and we use infinite plays (branches in $\mathcal{S}$) instead. (Note that any finite play can be extended to an infinite one by a sequence of trivial turns.)

For a play $\mathfrak{p}$ we take $\Gamma_{\mathfrak{p}} = \bigcup\{\Gamma_{\mathcal{P}} \mid \mathcal{P} \in \mathfrak{p}\}$, $\mathrm{FV}(\mathfrak{p}) = \bigcup\{\mathrm{FV}(\mathcal{P}) \mid \mathcal{P} \in \mathfrak{p}\}$. A play is *static* when almost all turns along $\mathfrak{p}$ are static. In this case almost all positions $\mathcal{P} \in \mathfrak{p}$ have the same goal which is denoted by $\tau_{\mathfrak{p}}$.

A game turn can be identified by a triple $\mathbf{t} = \langle\,\langle\,\alpha, i\,\rangle, j\,\rangle$, where

- α is the formula chosen by $\exists$ros; in cases (a4,b5) we mean the formula $\varphi[x := y]$;
- $i = 1$ when α is an assumption, otherwise $i = 2$;
- $j = 1, 2, \ldots$ identifies the response chosen by $\forall$phrodite.

For example, $\langle\,\langle\,\beta \to \gamma, 1\,\rangle, 2\,\rangle$ is the game turn consisting of $\exists$ros' choice of an assumption formula $\beta \to \gamma$ and $\forall$phrodite's response by choosing β as a new goal. And $\langle\,\langle\,\forall x\,\varphi, 2\,\rangle, 4\,\rangle$ encodes $\exists$ros' choice of an aim $\forall x\,\varphi$, yielding $\varphi[x := z^4_{\forall x\,\varphi}]$ as a new goal.

The pair $\mathbf{m} = \langle\,\alpha, i\,\rangle$ identifies a move of $\exists$ros, and we may also write $\mathbf{t} = \langle\,\mathbf{m}, j\,\rangle$. We write $\mathcal{P} \to_{\mathbf{t}} \mathcal{P}'$ when position $\mathcal{P}'$ is obtained from $\mathcal{P}$ after the game turn $\mathbf{t}$.

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[2] On the $\exists$ros' side the analogous problem is solved by alpha-conversion, a solution not necessarily convenient for infinite structures.

When $\mathcal{P} \to_{\mathbf{t}} \mathcal{P}'$ and $\mathcal{P}'$ is a winning position for ∀phrodite, then the game turn $\mathbf{t}$ is *safe in* $\mathcal{P}$. Turn $\mathbf{t}$ is *safe for a play* $\mathfrak{p}$ when it is safe in almost all positions of $\mathfrak{p}$. An ∃ros' move (possible in position $\mathcal{P}$) is *static safe* in position $\mathcal{P}$ if it can be completed by ∀phrodite into a safe static turn. This is generalized for plays: a move is *static safe* for a play $\mathfrak{p}$ if it is static safe almost everywhere along $\mathfrak{p}$.

Lemma 5.8. *If turn* $\mathbf{t}$ *is safe in* $\mathcal{P} = (\Gamma, \Delta \vdash \tau)$ *and it is possible in position* $\mathcal{P}_0 = (\Gamma \vdash \tau)$*, then it is safe in* $\mathcal{P}_0$*.*

Proof. We have $\mathcal{P} \to_{\mathbf{t}} \mathcal{P}'$ and $\mathcal{P}_0 \to_{\mathbf{t}} \mathcal{P}'_0$ and , where $\mathcal{P}' = (\Gamma', \Delta \vdash \tau')$ and $\mathcal{P}'_0 = (\Gamma' \vdash \tau')$. By Propositions 5.1 and 5.7, if $\mathcal{P}'_0 = (\Gamma' \vdash \tau')$ is winning for ∃ros then the judgment $\Gamma' \vdash \tau'$ has a proof. Hence $\Gamma', \Delta \vdash \tau'$ has a proof as well, and ∀phrodite loses in position $\mathcal{P}'$ as well. ■

A comment before the next two Lemmas: if $\mathcal{P} \twoheadrightarrow \mathcal{P}'$ and $\tau_{\mathcal{P}} = \tau_{\mathcal{P}'}$ then a game turn, which is possible in position $\mathcal{P}$, is also possible in $\mathcal{P}'$, except when it is an (a5) or (b4) turn and the same quantified formula was played by ∃ros between $\mathcal{P}$ and $\mathcal{P}'$. In the latter case, a different fresh variable should be used. It follows that a turn possible in two positions in a play is also possible in between, unless the goal has changed. Therefore we have:

Lemma 5.9. *If a turn* $\mathbf{t}$ *is safe in infinitely many positions along a static play* $\mathcal{P}$ *then it is safe for* $\mathfrak{p}$*.*

Proof. Follows from Lemma 5.8. ■

Lemma 5.10. *Suppose* $\mathcal{P}_1 \to_{\mathbf{t}} \mathcal{P}'_1$ *and* $\mathcal{P}_1 \to_{\mathbf{s}} \mathcal{P}_2$*, where* $\mathbf{s}$ *is a static turn. If turn* $\mathbf{t}$ *is possible in* $\mathcal{P}_2$ *and* $\mathcal{P}_2 \to_{\mathbf{t}} \mathcal{P}'_2$ *then* $\mathcal{P}'_1 \to_{\mathbf{s}} \mathcal{P}'_2$*.*

$$\begin{tikzcd} \mathcal{P}_1 \arrow[r, "\mathbf{t}"] \arrow[d, "\mathbf{s}"'] & \mathcal{P}'_1 \arrow[d, dashed, "\mathbf{s}"] \\ \mathcal{P}_2 \arrow[r, "\mathbf{t}"] & \mathcal{P}'_2 \end{tikzcd}$$

Proof. We have $\Gamma_{\mathcal{P}_1} \subseteq \Gamma_{\mathcal{P}'_1}$, so ∃ros can play the same (a) move as the one in $\mathbf{s}$. ∀phrodite's response can also be the same with one exception: when $\mathbf{t} = \mathbf{s}$ is an (a5) turn (the variable introduced by $\mathbf{t}$ is no longer fresh). But this is excluded by the assumption that $\mathbf{t}$ is possible in $\mathcal{P}_2$. ■

We say that a static play $\mathfrak{p}$ in the largest winning strategy of ∀phrodite is *saturated* when

- Every ∃ros' move (a) which is static safe for $\mathfrak{p}$ is actually played during $\mathfrak{p}$.

LEMMA 5.11. *If $\mathfrak{p} \subseteq \S$ is saturated then the following conditions are fulfilled:*

1. *If $\beta \to \gamma \in \Gamma_{\mathfrak{p}}$, and $\langle \langle \beta \to \gamma, 1 \rangle, 2 \rangle$ is not safe for $\mathfrak{p}$, then $\gamma \in \Gamma_{\mathfrak{p}}$;*
1. *If $\beta \vee \gamma \in \Gamma_{\mathfrak{p}}$ then $\beta \in \Gamma_{\mathfrak{p}}$ or $\gamma \in \Gamma_{\mathfrak{p}}$;*
2. *If $\beta \wedge \gamma \in \Gamma_{\mathfrak{p}}$ then $\beta \in \Gamma_{\mathfrak{p}}$ and $\gamma \in \Gamma_{\mathfrak{p}}$;*
3. *If $\forall x\, \varphi \in \Gamma_{\mathfrak{p}}$ then $\varphi[x := y] \in \Gamma_{\mathfrak{p}}$, for all $y \in \mathrm{FV}(\mathfrak{p})$;*
4. *If $\exists x\, \varphi \in \Gamma_{\mathfrak{p}}$ then $\varphi[x := y] \in \Gamma_{\mathfrak{p}}$, for some y.*

PROOF. We begin with (1). If $\beta \to \gamma \in \Gamma_{\mathfrak{p}}$ and $\mathfrak{p} \subseteq \S$ then $\exists$ros can play move (a1) with $\beta \to \gamma$ anywhere on $\mathfrak{p}$. To this move $\forall$phrodite cannot respond dynamically, because turn $\langle \langle \beta \to \gamma, 1 \rangle, 2 \rangle$ is not safe. So this (a1) move is static safe, and therefore it is actually played during $\mathfrak{p}$, whence $\gamma \in \Gamma_{\mathfrak{p}}$. In part (2), $\exists$ros could refer to the formula $\beta \vee \gamma$ and either β or γ could be safely chosen in response in infinitely many positions of $\mathfrak{p}$. By Lemma 5.9 one of these makes a static turn safe on $\mathfrak{p}$. As $\mathfrak{p}$ is saturated, move (a2) with $\beta \vee \gamma$ is actually played along $\mathfrak{p}$, whence either β or γ is in $\Gamma_{\mathfrak{p}}$. In parts (2–4) there are (a) moves of $\exists$ros, to which $\forall$phrodite must be able to respond (because we are inside $\S$) and this response must be static. So in each case the appropriate (a) move must be played, and the desired formulas are in $\Gamma_{\mathfrak{p}}$. ■

Let $\mathfrak{p} = \mathcal{P}_0\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3\ldots$ and $\mathfrak{p}' = \mathcal{P}'_0\mathcal{P}'_1\mathcal{P}'_2\mathcal{P}'_3\ldots$ be plays beginning with some positions $\mathcal{P}_0$ and $\mathcal{P}'_0$, not necessarily initial. We write $\mathfrak{p} \sqsubseteq \mathfrak{p}'$ when there is an ascending sequence $j_0 < j_1 < j_2 < \ldots$ such that whenever $\mathcal{P}_i \to_{\mathtt{t}} \mathcal{P}_{i+1}$ then also $\mathcal{P}_{j_i} \to_{\mathtt{t}} \mathcal{P}_{j_i+1}$ holds. That is, the position $\mathcal{P}'_{j_i+1}$ is obtained from $\mathcal{P}'_{j_i}$ by the same game turn as $\mathcal{P}_{i+1}$ is obtained from $\mathcal{P}_j$. In summary: the sequence of game turns along $\mathfrak{p}$ is a subsequence of the game turns along $\mathfrak{p}'$. Note that the relation $\sqsubseteq$ is a preorder: it is reflexive and transitive, but not necessarily anti-symmetric. In case $j_i = i$ for all i, we say that $\mathfrak{p}$ and $\mathfrak{p}'$ are *parallel plays*.

LEMMA 5.12. *Let a play $\mathfrak{p}_0\mathcal{P}\mathfrak{p}$ within the strategy $\S$ be such that all turns in $\mathcal{P}\mathfrak{p}$ are static. If $\mathtt{t}$ is safe in all positions of $\mathcal{P}\mathfrak{p}$ and $\mathcal{P} \to_{\mathtt{t}} \mathcal{P}'$ then $\mathfrak{p}' = \mathfrak{p}_0\mathcal{P}\mathcal{P}'\mathfrak{p}''$ is a static play in $\S$, for some $\mathfrak{p}''$ which is parallel to $\mathfrak{p}$.*

PROOF. This follows by repeated application of Lemma 5.10. ■

LEMMA 5.13. *For every static play $\mathfrak{p} \subseteq \S$ there is a saturated play $\mathfrak{r} \subseteq \S$ with $\mathfrak{p} \sqsubseteq \mathfrak{r}$ and $\tau_{\mathfrak{p}} = \tau_{\mathfrak{r}}$.*

PROOF. If a play $\mathfrak{p}$ is not saturated then there is a formula $\varphi \in \Gamma_{\mathfrak{p}}$ such that an $\exists$ros' move addressing φ is static safe, but not played. We will process all such violations one by one, adjusting $\mathfrak{p}$ appropriately. Such adjustments may introduce new variables and formulas, and may create new violations, so we need to determine the set of all formulas under consideration.

Let X be a set of variables and Φ be a set of formulas. We say that the pair $\langle X, \Phi \rangle$ is *rich enough* when $\mathrm{FV}(\mathfrak{p}) \subseteq X$ and:

(1) If $\mathrm{FV}(\varphi) \subseteq X$ then $\varphi \in \Phi$;
(2) If $\varphi \in \Phi$ then $x^i_\varphi \in X$, for all $i \in \mathbb{N}$.

Let Φ be the least set Φ such that some pair $\langle X, \Phi \rangle$ is rich enough, and observe that Φ is a countable set. Take $\Phi = \{\xi_n \mid n \in \mathbb{N}\}$. We say that $n \in \mathbb{N}$ is a *request* for a play $\mathfrak{q}$ if the (a) move $\langle \xi_n, 1 \rangle$ of $\exists$ros is static safe on $\mathfrak{q}$. The request is *satisfied* at $\mathfrak{q}$ if the move is played during $\mathfrak{q}$. Thus a play is saturated if and only if it satisfies every request.

We define a sequence of static plays $\mathfrak{p}_n \in \S$ so that $\mathfrak{p} = \mathfrak{p}_0 \sqsubseteq \mathfrak{p}_1 \sqsubseteq \mathfrak{p}_2 \sqsubseteq \ldots$ and in addition:

– for every $n \in \mathbb{N}$, the first n positions are the same in all $\mathfrak{p}_k$ with $k \geq n$;
– all dynamic turns in $\mathfrak{p}_k$ occur already in $\mathfrak{p}_0$.

Then there exists the "limit" play $\mathfrak{p}_\omega = \mathcal{P}_0\mathcal{P}_1\mathcal{P}_2\ldots$, where $\mathcal{P}_n$ is the nth position of almost all $\mathfrak{p}_k$, and the play $\mathfrak{p}_\omega$ is static.

Suppose that a static play $\mathfrak{p}_k \in S$ has already been defined. If it is saturated then $\mathfrak{p}_{k+1} = \mathfrak{p}_k$. Otherwise, let n be the least request for $\mathfrak{p}_k$ which is not satisfied. It means that $\forall$phrodite could safely respond to move $\langle \xi_n, 1 \rangle$ in a static way almost everywhere along $\mathfrak{p}_k$, yet she never does so. In most cases this response is unique by definition. In particular, when ξ_n is existential, and move (a5) is not played along $\mathfrak{p}_k$, then the same variable $z^1_{\xi_n}$ is available at every position. The only exception is when ξ_n is a disjunction, say $\beta \vee \gamma$. Then either β or γ can be chosen in response infinitely many times, and thus almost everywhere in $\mathfrak{p}_k$ (Lemma 5.9). Thus, also in this case, we can select a single static turn $\mathfrak{t}$, safe along $\mathfrak{p}_k$, which could handle the request by adding a certain formula to assumptions. By Lemma 5.12, if we play $\mathfrak{t}$ at a sufficiently late position $\mathcal{P}$ of $\mathfrak{p}_k$, we obtain a static play $\mathfrak{p}_{k+1}$ satisfying the request n. Clearly, we have $\mathfrak{p}_k \sqsubseteq \mathfrak{p}_{k+1}$, in particular all requests which were already satisfied at $\mathfrak{p}_k$ are still satisfied at $\mathfrak{p}_{k+1}$. If we ensure that $\mathcal{P}$ is at least the kth position of $\mathfrak{p}_k$ then the limit play $\mathfrak{p}_\omega$ is well-defined. It consists of winning positions only, and it is saturated: every request for $\mathfrak{p}_\omega$ must occur as a request for some $\mathfrak{p}_k$. ■

Let $\mathfrak{p}$ and $\mathfrak{p}'$ be static plays, and suppose that $\mathfrak{p} = \mathfrak{p}_1\mathcal{P}\mathfrak{p}_2$ with $\mathcal{P}\mathfrak{p}_2$ containing no dynamic turns. If $\mathfrak{p}' = \mathfrak{p}_1\mathcal{P}\mathcal{P}'\mathfrak{p}'_2$, where $\mathfrak{p}_2 \sqsubseteq \mathfrak{p}'_2$ and $\mathcal{P} \to_{\mathfrak{t}} \mathcal{P}'$, for some dynamic turn $\mathfrak{t}$, then we write $\mathfrak{p} <_1 \mathfrak{p}'$. The relation $\leq$ on states is the reflexive and transitive closure of $<_1$. It is clearly a partial order. Observe that $\mathfrak{p} \leq \mathfrak{p}'$ implies $\Gamma_\mathfrak{p} \subseteq \Gamma_{\mathfrak{p}'}$ and $\mathrm{FV}(\mathfrak{p}) \subseteq \mathrm{FV}(\mathfrak{p}')$.

The following is an immediate consequence of Lemmas 5.12 and 5.13:

LEMMA 5.14. *Let $\mathfrak{p} = \mathfrak{p}_1\mathcal{P}\mathfrak{p}_2$ be a saturated play such that $\mathcal{P}\mathfrak{p}_2$ does not contain any dynamic turn. Assume that a dynamic turn $\mathbf{t}$ is safe in all positions of $\mathcal{P}\mathfrak{p}_2$ and that $\mathcal{P} \to_\mathbf{t} \mathcal{P}'$. Then there is a saturated play $\mathfrak{p}'$ such that $\mathfrak{p} <_1 \mathfrak{p}'$ and $\tau_{\mathfrak{p}'} = \tau_{\mathcal{P}'}$*

Let v be a substitution mapping variables to variables. We write $\alpha[v]$ for the formula obtained from α by a simultaneous replacement of all variables $x \in \mathrm{FV}(\alpha)$ by $v(x)$.

The model: From the largest winning strategy of ∀phrodite (in a position $\mathcal{P}_0$) we define a Kripke model.

- States of the model are saturated plays starting in $\mathcal{P}_0$ and ordered by $\leq$.
- The object domain in state $\mathfrak{p}$ is $D_\mathfrak{p} = \mathrm{FV}(\mathfrak{p})$.
- For atoms $\mathbf{a}$, and valuations v, we take $\mathfrak{p}, v \Vdash \mathbf{a}$ if and only if $\mathbf{a}[v] \in \Gamma_\mathfrak{p}$.

In every state $\mathfrak{p}$ of our model we consider the trivial valuation w defined on $D_\mathfrak{p}$ as $w(x) = x$. From our definition it follows that for atomic formulas we have $\mathfrak{p}, w \Vdash \mathbf{a}$ if and only if $\mathbf{a} \in \Gamma_\mathfrak{p}$.

Write $v' = v[x \mapsto a]$, when $v'(x) = a$ and $v'(y) = v(y)$ when $y \neq x$. The following is a well-known fact:

LEMMA 5.15. *In a Kripke model, $c, v \Vdash \varphi[x := u]$ iff $c, v[x \mapsto v(u)] \Vdash \varphi$.*

LEMMA 5.16. *Let $\mathfrak{p} \subseteq \S$ be a saturated play. If α is in $\Gamma_\mathfrak{p}$ then $\mathfrak{p}, w \Vdash \alpha$, and if α is an aim in $\tau_\mathfrak{p}$ then $\mathfrak{p}, w \nVdash \alpha$.*

PROOF. We prove the two statements by simultaneous induction with respect to the length of α. Assume first that α is an assumption in some $\Gamma_\mathcal{P}$ with $\mathcal{P} \in \mathfrak{p}$. If α is an atom then $\mathfrak{p}, w \Vdash \alpha$ by definition (note that α cannot be $\perp$ in a winning position). So let α be a non-atomic assumption; then a possible ∃ros' move in position $\mathcal{P}$ is to choose α.

Let $\alpha = \beta \to \gamma$ and consider a saturated play $\mathfrak{r} \geq \mathfrak{p}$ with $\mathfrak{r}, w \Vdash \beta$. Since $\alpha \in \Gamma_\mathfrak{p} \subseteq \Gamma_\mathfrak{r}$, ∃ros can play α at almost every position $\mathcal{R} \in \mathfrak{r}$. First suppose that ∀phrodite can safely play β in response, i.e., the turn $\mathbf{t} = \langle\, \langle\, \beta \to \gamma, 1\,\rangle, 2\,\rangle$ is safe for $\mathfrak{r}$.

Choose $\mathcal{R}$ so that there is no dynamic turn in $\mathfrak{r}$ after $\mathcal{R}$. By Lemma 5.14 there is a saturated play $\mathfrak{q} \geq \mathfrak{r}$ with $\tau_\mathfrak{q} = \beta$. Now $\mathfrak{q}, w \nVdash \beta$ by the induction hypothesis for β. This is a contradiction, because $\mathfrak{q}, w \Vdash \beta$ follows from $\mathfrak{r}, w \Vdash \beta$ and $\mathfrak{r} \leq \mathfrak{q}$.

Therefore $\mathbf{t}$ is not safe, so by Lemma 5.11(1) we have $\gamma \in \Gamma_{\mathfrak{r}}$. Thus $\mathfrak{r}, w \Vdash \gamma$, by the induction hypothesis for γ. This proves $\mathfrak{p}, w \Vdash \alpha$.

The cases of disjunction, conjunction, and the existential quantifier follow immediately from parts 1,2, and 4 of Lemma 5.11 applied to $\mathfrak{p}$.

If $\alpha = \forall x\, \varphi$ then we must consider an arbitrary saturated extension $\mathfrak{r} \geq \mathfrak{p}$ and any $y \in D_{\mathfrak{r}}$. We apply Lemma 5.11(3) to $\mathfrak{r}$.

Now suppose α is an aim, i.e., a disjunct of $\tau_{\mathfrak{p}}$. Then α is a possible choice of $\exists$ros in almost every $\mathcal{P} \in \mathfrak{p}$. If it is an atom or a disjunction then the resulting position is $\Gamma_{\mathcal{P}} \vdash \alpha$. Assume first that α is an atom. As we play a winning strategy, we know that α cannot be an assumption in $\Gamma_{\mathcal{P}}$. Since $\mathcal{P} \in \mathfrak{p}$ is (almost) arbitrary, we have that $\alpha \notin \Gamma_{\mathfrak{p}}$, i.e., $\mathfrak{p}, w \nVdash \alpha$.

If $\alpha = \beta \vee \gamma$ then we choose $\mathcal{P} \in \mathfrak{p}$ such that no dynamic turn occurs in $\mathfrak{p}$ after $\mathcal{P}$. By Lemma 5.14 there is $\mathfrak{r} \geq \mathfrak{p}$ with $\tau_{\mathfrak{r}} = \beta \vee \gamma$. We apply the induction hypothesis to the aims β and γ. Thus $\mathfrak{r}, w \nVdash \beta$ and $\mathfrak{r}, w \nVdash \gamma$, and the same must hold for $\mathfrak{p}$.

If α is an aim $\beta \to \gamma$ then, by Lemma 5.14, there is $\mathfrak{r} \geq \mathfrak{p}$ with $\beta \in \Gamma_{\mathfrak{r}}$ and $\gamma = \tau_{\mathfrak{r}}$.

Let now α be an aim of the form $\beta \wedge \gamma$. For almost every $\mathcal{P} \in \mathfrak{p}$, one of the positions $\Gamma_{\mathcal{P}} \vdash \beta$ and $\Gamma_{\mathcal{P}} \vdash \gamma$ is winning for $\forall$phrodite. One of these two cases, say the first one, holds for infinitely many positions $\mathcal{P}$, and thus it is a safe choice at almost all positions. Choose one such position $\mathcal{P}$ after all dynamic turns and use Lemma 5.14.

Consider an aim $\alpha = \forall x\, \varphi$, and let $\mathfrak{p} = \mathfrak{p}_1 \mathcal{P} \mathfrak{p}_2$, where no dynamic turn occurs in $\mathcal{P}\mathfrak{p}_2$. Then, for some i, the turn $\mathbf{t} = \langle \langle \alpha, 2 \rangle, i \rangle$ is safe in all positions of $\mathcal{P}\mathfrak{p}_2$. By Lemma 5.14 there is $\mathfrak{r} \geq \mathfrak{p}$ with $\tau_{\mathfrak{r}} = \varphi[x := z^i_\alpha]$. Then $\mathfrak{r}, w \nVdash \varphi[x := z^i_\alpha]$, whence $\mathfrak{r}, w[x \mapsto z^i_\alpha] \nVdash \varphi$. It follows that $\mathfrak{p}, w \nVdash \forall x\, \varphi$.

For an aim $\alpha = \exists x\, \varphi$, $\exists$ros may play $\varphi[x := y]$, for any $y \in D_{\mathfrak{p}}$, and this must be safe at almost every point of $\mathfrak{p}$. Again we use Lemma 5.14, and we get $\mathfrak{r} \geq \mathfrak{p}$ with $\mathfrak{r}, w \nVdash \varphi[x := y]$, i.e., $\mathfrak{r}, w[x \mapsto y] \nVdash \varphi$. It follows that also $\mathfrak{p}, w[x \mapsto y] \nVdash \varphi$. Since y was arbitrary, we have $\mathfrak{p}, w \nVdash \exists x\, \varphi$. ■

PROPOSITION 5.17. *If* $\forall$phrodite *has a winning strategy in position* $\Gamma \vdash \tau$ *then there exists a Kripke model* $\mathcal{C}$*, a state* c*, and a valuation* w *such that* $\mathcal{C}, c, w \Vdash \Gamma$ *and* $\mathcal{C}, c, w \nVdash \tau$*.*

PROOF. We have constructed the Kripke model, and we have defined w. All we need is a state c satisfying $c, w \Vdash \Gamma$ and $c, w \nVdash \tau$. Take any static play $\mathfrak{p}$ containing no dynamic turns and expand it to a saturated play $\mathfrak{r}$ using Lemma 5.13. Then $\mathfrak{r}, w \Vdash \Gamma$ and $\mathfrak{r}, w \nVdash \tau$, because $\mathfrak{p} \sqsubseteq \mathfrak{r}$ and $\tau_{\mathfrak{p}} = \tau_{\mathfrak{r}} = \tau$. ■

We can now derive an analogue of Theorem 2.16:

THEOREM 5.18. *Given τ, either there is a normal form of type τ or there is a Kripke counter-model for τ.*

COROLLARY 5.19. *A valid formula has a normal proof.*

EXAMPLE 5.20. In Example 5.5, the largest strategy of ∀phrodite has four infinite saturated plays $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$, $\mathfrak{s}$, ordered so that $\mathfrak{p} \leq \mathfrak{q}$, and $\mathfrak{p} \leq \mathfrak{r} \leq \mathfrak{s}$.

- $\Gamma_{\mathfrak{p}} = \{\forall x\,(\mathsf{P} \vee \mathsf{Q}(x)), \mathsf{P} \vee \mathsf{Q}(x_0), \mathsf{Q}(x_0)\}$ and $\tau_{\mathfrak{p}} = \mathsf{P} \vee \forall x\,\mathsf{Q}(x)$;
- $\Gamma_{\mathfrak{q}} = \{\forall x\,(\mathsf{P} \vee \mathsf{Q}(x)), \mathsf{P} \vee \mathsf{Q}(x_0), \mathsf{Q}(x_0)\}$ and $\tau_{\mathfrak{q}} = \mathsf{P}$;
- $\Gamma_{\mathfrak{r}} = \{\forall x\,(\mathsf{P} \vee \mathsf{Q}(x)), \mathsf{P} \vee \mathsf{Q}(x_0), \mathsf{Q}(x_0)\}$ and $\tau_{\mathfrak{r}} = \forall x\,\mathsf{Q}(x)$;
- $\Gamma_{\mathfrak{s}} = \{\forall x\,(\mathsf{P} \vee \mathsf{Q}(x)), \mathsf{P} \vee \mathsf{Q}(x_0), \mathsf{Q}(x_0), \mathsf{P} \vee \mathsf{Q}(y), \mathsf{P}\}$ and $\tau_{\mathfrak{s}} = \mathsf{Q}(y)$.

To refute Grzegorczyk's scheme states $\mathfrak{p}$ and $\mathfrak{s}$ are sufficient.

EXAMPLE 5.21. Consider the unprovable judgment

$$\forall x \exists y\, \mathsf{R}(x,y),\ \forall xyz\,(\mathsf{R}(x,y) \to \mathsf{R}(y,z) \to \mathsf{R}(x,z)) \vdash \exists x\, \mathsf{R}(x,x).$$

A Kripke countermodel for this judgment must use a state with an infinite domain. This corresponds to a saturated play in which there is no "finite saturation" i.e., no finite initial segment includes all safe static moves. If we add $\forall x\,(\mathsf{P}(x) \vee \neg\mathsf{P}(x))$ to assumptions, we also force the Kripke model to have infinitely many states (cf. Example 5.2).

EXAMPLE 5.22. This example shows why building a model from a deterministic strategy of ∀phrodite is problematic. Consider ∀phrodite's winning position $\forall x \exists y(\mathsf{S}(x,y) \wedge (\mathsf{L}(x) \vee \mathsf{R}(x))) \vdash \mathsf{P} \to \mathsf{Q}$, and the following deterministic winning strategy. When ∃ros plays the assumption with $x := x_0$, ∀phrodite introduces a new variable x_1 for y. Then, if ∃ros explores the formula $\mathsf{S}(x_0,x_1) \wedge (\mathsf{L}(x_0) \vee \mathsf{R}(x_0))$, ∀phrodite chooses $\mathsf{L}(x_0)$. When ∃ros plays the assumption again with $x := x_1$, she introduces x_2 etc., choosing $\mathsf{L}(x_i)$ each time. But when ∃ros plays the goal, ∀phrodite changes her mind, and when she is introducing new variables x_i, she keeps choosing $\mathsf{R}(x_i)$. Now consider a saturated play $\mathfrak{p}$ where ∃ros never chooses the goal. Observe that there is no $\mathfrak{r} \geq \mathfrak{p}$ with $\tau_{\mathfrak{r}} = \mathsf{Q}$, because any extension of the dynamic turn (b) leads to a choice inconsistent with $\mathfrak{p}$. (To make the inconsistency more explicit, one can add $\forall x\,\neg(\mathsf{L}(x) \wedge \mathsf{R}(x))$ to assumptions.)

For a complete exposition we also show that a Kripke counter-model yields a strategy of ∀phrodite.

LEMMA 5.23. *Let c be a state in a Kripke model $\mathcal{C}$ and v be a valuation. If $c, v \Vdash \Gamma$ and $c, v \nVdash \tau$ then ∀phrodite has a winning strategy in position $\Gamma \vdash \tau$.*

Proof. We say that a position $\Gamma \vdash \tau$ is *secure* when $c, v \Vdash \Gamma$, and $c, v \nVdash \tau$, for some c, v. The proof is similar to the proof of Fact 2.15. The new cases that are not obvious are (a5), (b4). In case (a5) we have $c, v \Vdash \exists x\, \varphi$, that is $c, v[x \mapsto a] \Vdash \varphi$, for some a. Hence $c, v[z \mapsto a] \Vdash \varphi[x := z]$, and the position $\Gamma, \varphi[x := z] \vdash \tau$ is secure.

In case (b4) there is $c' \geq c$ and some a in the domain of c' such that $c', v[x \mapsto a] \nVdash \varphi$. Thus, for a fresh z we have $c', v[z \mapsto a] \nVdash \varphi[x := z]$, whence the position $\Gamma \vdash \varphi[x := z]$ is secure. ∎

Conclusion

We have demonstrated a unified approach to the essentials of intuitionistic logic: every formula defines a game where one of the players has a strategy that either yields a normal proof or a counter-model. Completeness and normalization can therefore be seen as two faces of the same coin: the determinacy. Adjusting game rules appropriately yields a generalized Ben-Yelles algorithm for the extended lambda-calculus. We have also shown how the process of proof or model construction can be illustrated by playing a token game on a graph. Extending this didactically convenient approach to various other logical formalisms (in particular classical logic, modal logic, and various resource-conscious logics, e.g. linear logic) may turn out into an interesting field of research.

When trying to connect the notion of a strategy to Kripke semantics one discovers a particular inconvenience. In a sense, the finitary information in a deterministic winning strategy of ∀phrodite may be not sufficient to directly define a Kripke counter-model, cf. Examples 3.16 and 5.22. On the other hand, the existence of *any* strategy is perfectly sufficient to refute a formula. It means that a Kripke model may be more rich than needed: one can define complete semantics using less information; in addition a strategy-based semantics is more finitary than Kripke semantics (positions are finite).

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