ON MULTISET ORDERINGS

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Abstract: We propose two-well founded orderings on multisets that extend the Dershowitz–Manna ordering. Unlike the Dershowitz–Manna ordering, ours do not have a natural monotonicity property. This lack of monotonicity suggests using monotonicity to provide a new characterization of the Dershowitz–Manna ordering. Section 5 proposes an efficient and correct implementation of that ordering.

Résumé: Dans cette note, on propose deux ordres biend fondés qui étendent l'ordre de Dershowitz et Manna. Ces ordres ne vérifient pas une propriété naturelle de monotonicité que nous définissons. Aussi cela suggère d'utiliser la monotonicité comme une nouvelle caractérisation de l'ordre de Dershowitz et Manna. La cinquième section de cette note propose une implantion efficace et correcte de cet ordre.

Keywords: Multiset ordering, well-founded ordering

1. Introduction

The multiset ordering proposed by Dershowitz and Manna [2] is a basis for many orderings used for proving termination of programs and term rewriting systems [1], and it would be nice to have an efficient implementation of it. Often, deriving an algorithm directly from a mathematical definition is not the best way. Thus a more suitable definition and a proof that this new program implements the desired function must be found. Our approach was as follows. We tried two definitions, both having efficient implementations but both failing to be equivalent. In fact, they are stronger than the Dershowitz–Manna multiset ordering but do not have a monotonicity property. As an explanation of these facts, we give a new definition of the Dershowitz–Manna ordering based on the main characterization of this ordering: no stronger monotonic ordering exists on multisets. In Section 5 of this paper, we propose a correct and efficient implementation of the Dershowitz–Manna ordering.

2. The Dershowitz–Manna ordering

Intuitively, a multiset M (or bag) on E is an unordered collection of elements of E, with possibly multiple occurrences of elements. More formally M is a mapping $E \to \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers, associating with each value in E the number of times it occurs in the multiset. For example, x is in M if $M(x) > 0$. $\mathcal{M}(E)$ denotes the set of all the finite multisets on E, i.e., the multisets M with finite carrier $\{x \in E \mid M(x) \neq 0\}$. The empty multiset $\{\ \}$ has $\{\ \}(x) = 0$, for all x in E. A set is particular case of a multiset such that $S(x) \leq 1$ for each x in E.

Definition 2.1. The *sum of two multisets* M and N is the multiset $M+N$ such that $M+N(x)=M(x)+N(x)$.

$M+N$ is an associative and commutative operation on $\mathcal{M}(E)$ with neutral element { }. If $M_1, M_2, \ldots, M_p$ are multisets $\Sigma_{i=1}^{p} M_i$ is the multiset such that $(\Sigma_{i=1}^{p} M_i)(x)=\Sigma_{i=1}^{p} M_i(x)$. $M+N$ is a set only if M and N are disjoint sets; in this case $+$ is the classical disjoint union or direct sum of sets.

Definition 2.2. Multiset M is *included* in multiset N (written $M \subseteq N$) if and only if $\forall x \in E\ M(x) \leqslant N(x)$.

Definition 2.3. If $M \subseteq N$, the *difference* $N-M$ is defined by $(N-M)(x)=N(x)-M(x)$.

In this paper, an ordering $\prec$ on a set is a partial or total strict ordering, i.e., an irreflexive and transitive relation on E. We use the notation $x \# y$ to mean $\neg(y \prec x$ or $x=y$ or $x \prec y)$. Assume throughout that E is ordered by $\prec$.

Definition 2.4. The *Dershowitz–Manna ordering* is defined as follows:
$M \lll N$ if there exist two multisets X and Y in $\mathcal{M}(E)$ satisfying

(i) $\{\ \} \neq X \subseteq N$,
(ii) $M=(N-X)+Y$,
(iii) X dominates Y: $\forall y \in Y\ \exists x \in X\ y \prec x$.

Using Definition 2.4, it may be difficult to prove that two multisets are not related by $\lll$. In [3], Huet and Oppen give a different and more tractable definition.

Definition 2.5. The *Huet–Oppen definition* is defined as follows:

$M \lll N$ iff $M \neq N$ &
$[M(y)>N(y) \Rightarrow (\exists x \in E\ y \prec x \,\&\, M(x)<N(x)]$.

Lemma 2.6. *The Dershowitz–Manna definition is equivalent to the Huet–Oppen definition.*

Proof. Let $\lll_{DM}$ and $\lll_{HO}$ be the Dershowitz–Manna and Huet–Oppen orderings, respectively.

Assume $M \lll_{HO} N$, and define X and Y as follows:

$X(x)=\max\{N(x)-M(x),0\}$,

$Y(y)=\max\{M(y)-N(y),0\}$.

Let us prove (i) of Definition 2.4.

(a) $X \subseteq N$ is clear by definition.

(b) Because $M \neq N$ there exist z such that $M(z) \neq N(z)$. If $M(z)<N(z)$, then $z \in X$, if $M(z)>N(z)$, by Definition 2.5 there exist x, such that $z \prec x$ and $M(x)<N(x)$. Hence $x \in X$. In both cases $X \neq \{\ \}$.

(ii) is true by construction. To prove (iii), let $y \in Y$. By hypothesis, there exist x satisfying $y \prec x$ and $M(x)<N(x)$. Hence $\exists\ x \in X\ y \prec x$.

Suppose $M \lll_{DM} N$. $M \neq N$ because $X \neq \{\ \}$ and $Y \neq X$. Without loss of generality we may assume X and Y disjoint, i.e., $X(x)>0 \Rightarrow Y(x)=0$ & $Y(x)>0 \Rightarrow X(x)=0$. Otherwise X would be replaced by $X-Z$ and Y by $Y-Z$ where $Z(x)=\min\{X(x), Y(x)\}$. Assume $M(y)>N(y)$. By (ii) $M(y)=N(y)+Y(y)-X(y)$. This implies $Y(y)>X(y) \geqslant 0$, which means $y \in Y$. By (iii) there exist $x \in X$, i.e., $X(x)>0$, such that $y \prec x$. The value of M in x is $M(x)=(N(x)-X(x))+Y(x)=N(x)-X(x)$ because $Y(x)=0$. Thus $M(x)<N(x)$. □

Another important property of the Dershowitz–Manna ordering is monotonicity.

Definition 2.7. Let $\prec$ be a partial ordering on E and τ a mapping from $E \times E$ into $\mathcal{M}(E) \times \mathcal{M}(E)$. $\tau(\prec)$ is said to be a *monotonic extension* of $\prec$ iff

(1) $\tau(\prec)$ is an ordering,
(2) τ is monotonic, i.e., $\prec \subseteq < \Rightarrow \tau(\prec) \Rightarrow \tau(<)$.

Lemma 2.8 (Monotonicity lemma). *The Dershowitz–Manna ordering $\lll$ is a monotonic extension of $\prec$.*

Proof. Straightforward using the Huet–Oppen definition. □

3. Partition based orderings

We now define two multiset orderings based on partitioning a multiset. We say that $\{M_i \mid i=1,\ldots,p\}$ is a partition of a multiset iff $M=\Sigma_{i=1}^{p} M_i$.

Assume now, we are able to compare the M_i using an ordering $\prec$, and thus to sort them such that $M_p \preccurlyeq M_{p-1} \preccurlyeq \cdots \preccurlyeq M_1$. It is now easy to define a new ordering $\prec_{\mathcal{G}}$ for comparing the multisets $M = \Sigma_{i=1}^{p} M_i$ and $N = \Sigma_{i=1}^{q} N_i$ using a lexicographical extension of $\prec$:

$$M \prec_{\mathcal{G}} N \quad \text{iff} \quad M_1 M_2 \ldots M_p \prec^{\text{lex}} N_1 N_2 \ldots N_q.$$

In practice, we have to define the basic ordering $\prec$ and the method for constructing the partition of a given multiset.

3.1. The multiset ordering $\lll_{\mathcal{M}}$

Assume that the partition $\tilde{M} = (M_i \mid i = 1,\ldots,p)$ of the multiset M satisfies the following properties:

(1) $x \in M_i \Rightarrow M_i(x) = M(x)$.

(2) $x \in M_i$ and $y \in M_i \Rightarrow x$ and y are incomparable.

(3) $\forall\, i \in [2..p]\ x \in M_i \Rightarrow \exists\, y \in M_{i-1}\ x \prec y$.

Intuitively the partition is built by first computing the multiset M_1 of all the maximal elements and then recursively computing the partition of $M - M_1$.

Example 3.1. $M = \{a, a, 2, 2, b, 1, 1\}$ with $a \prec b$, $1 \prec 2$. $M_1 = \{2, 2, b\}$ and $M_2 = \{a, a, 1, 1\}$.

Let $\prec_{\mathcal{M}}$ be the following basic ordering on multisets:

$$M \prec_M N \quad \text{iff} \quad M \neq N \text{ and } \forall\, x \in E\ M(x) \leqslant N(x) \text{ or } \exists\, y \in N\ x \prec y.$$

Let $\mathcal{M}'(E)$ be the subset of $\mathcal{M}(E)$ such that if $M \in \mathcal{M}'(E)$ and $x \in M$ and $y \in M$, then x and y are incomparable. $\prec_{\mathcal{M}}$ is an ordering on $\mathcal{M}'(E)$ equivalent to the restriction of the Dershowitz–Manna ordering to $\mathcal{M}'(E)$.

Definition 3.2. Let M and N be multisets. We say that $M \lll_{\mathcal{M}} N$ iff $\tilde{M} \prec_{\mathcal{M}}^{\text{lex}} \tilde{N}$.

Example 3.3. If $a \prec b$ and $M = \{1, a, b\}$, then $M_1 = \{1, b\}$, $M_2 = \{a\}$. If $N = \{1, 1, b\}$, then $N_1 = \{1, 1, b\}$ and $M \lll_{\mathcal{M}} N$.

It is easy to see that $\lll_{\mathcal{M}}$ is an ordering because lexicographical extension preserves the orderings. We will show that $\lll_{\mathcal{M}}$ is more powerful than $\lll$.

Lemma 3.4. $M \lll N$ *implies* $M \lll_{\mathcal{M}} N$.

Proof. Suppose $M \lll N$, and $\tilde{M} = M_1 \ldots M_p$, and $\tilde{N} = N_1 \ldots N_q$. Then we have $M \lll_{\mathcal{M}} N$. The proof is by induction on $p + q$. The result is obvious if $M = \{\ \}$. Assume that $p \neq 0$ and $q \neq 0$. Three cases have to be distinguished.

Case 1. $M_1 \prec_{\mathcal{M}} N_1$. The result is straightforward.

Case 2. $M_1 = N_1$. Thus $M_2 \ldots M_p \lll_{\mathcal{M}} N_2 \ldots N_q$ and the result follows by the induction hypothesis.

Case 3. $N_1 \prec_{\mathcal{M}} M_1$ or M_1 and N_1 are incomparable. There must exist an $x \in M_1$ such that $M_1(x) > N_1(x)$ and $\forall\, y \in N_1\ x \not\prec y$. This contradicts the hypothesis $M \lll N$. □

We show that the converse is false by using the previous example. We see that $M(a) = 1 > N(a) = 0$. However, the only element in N greater than a, i.e., b has one occurrence in both M and N. Thus M and N are incomparable using the Dershowitz–Manna ordering. On the other hand, $\lll_{\mathcal{M}}$ has a serious drawback, which prevents it from being used in some usual cases requiring incremental orderings; it is not monotonic. Let us go back to Example 3.3 and assume now $1 \prec a \prec b$. This increases the basic ordering by adding the new pair $1 \prec a$. We now get

$$M_1 = \{b\}, \quad M_2 = \{a\}, \quad M_3 = \{1\},$$
$$N_1 = \{b\}, \quad N_2 = \{1,1\}$$

thus $N \lll_{\mathcal{M}} M$.

3.2. The multiset ordering $\lll_{\mathcal{S}}$

A different way to construct a partition of a multiset is to require that each multiset in the partition be a set. Thus the partition $\overline{M} = (S_i \mid i = 1\ldots p)$ must satisfy the following properties:

(1) S_i is a set, that is, $S_i(x) \leqslant 1$.

(2) $x \in S_i$ and $y \in S_i$ implies x and y are incomparable.

(3) $\forall$ i $\in$ [2..p] $x \in S_i$ implies $\exists$ $y \in S_{i-1}$ $x \preccurlyeq y$.

The only difference between the partitions created by $\ll_{\mathfrak{M}}$ and $\ll_S$ is in condition (1). As with $\ll_{\mathfrak{M}}$, the partition here is built by first computing the set S_1 of maximal elements, and then recursively computing the partitions of $M - S_1$.

Example 3.5. $M = \{a, a, 2, 2, b, 1, 1\}$ with $a \prec b$ and $1 \prec 2$ and $S_1 = \{2, b\}$, $S_2 = \{2, a\}$, $S_3 = \{1, a\}$, $S_4 = \{1\}$.

Let $<_S$ be the following ordering on sets:

$S <_S T$ iff $S \neq T$ and $\forall x \in S\ \exists y \in T\ x \preccurlyeq y$.

In the following $\overline{N} = (T_i \mid i = 1..q)$ will be the partition of N. If sets are considered as a particular case of multisets, $<_S$ is $\ll$ on the sets of incomparable elements.

Definition 3.6. Let M and N be multisets. We say that $M \ll_S N$ iff $\overline{M} <_S^{lex} \overline{N}$.

Example 3.7. Suppose a and b incomparable. If $N = \{a, b\}$, then $T_1 = \{a, b\}$; if $M = \{b, b\}$, then $S_1 = \{b\}$, $S_2 = \{b\}$ and $M \ll_S N$.

Once more, it is easy to see that $\ll_S$ is an ordering. Let us show that it is more powerful than $\ll$.

Lemma 3.8. $M \ll N$ *implies* $M \ll_S N$.

Proof. By induction on p and q. The result is straightforward if $M = \{\ \}$. Else, we have to distinguish three cases.

Case 1. $S_1 <_S T_1$, the result is straightforward.

Case 2. $S_1 = T_1$, by induction hypothesis.

Case 3. $T_1 <_S S_1$ or S_1 and T_1 are incomparable. There must exist an $x \in S_1$ such that $\forall y \in T_1$ $\neg x \preccurlyeq y$. This contradicts the hypothesis $M \ll N$. $\square$.

Once again, the converse is false, in the same way as shown in the previous example: $M = \{b, b\}$ and $N = \{a, b\}$.

Suppose now $a \prec b$ and M and N are as in Example 3.7. We get $S_1 = \{b\}$, $S_2 = \{b\}$ and $T_1 = \{b\}$, $T_2 = \{a\}$. Thus $N \ll_S M$. Therefore $\ll_S$ is not a monotonic ordering.

Now, let us try to compare $\ll_S$ and $\ll_{\mathfrak{M}}$, using two examples with $a \prec b$:

(i) $M = \{1, a, b\}$, $N = \{1, 1, b\}$, $M \ll_{\mathfrak{M}} N$ and $\neg M \ll_S N$.

(ii) $M = \{b, b\}$, $N = \{1, b\}$, $M \ll_S N$ and $\neg M \ll_{\mathfrak{M}} N$.

Thus $\ll_S$ and $\ll_{\mathfrak{M}}$ are not comparable.

3.3. Well-foundedness

We have the following theorem.

Theorem 3.9. *If* $\prec$ *is well-founded on* E, *then* $\ll$, $\ll_{\mathfrak{M}}$ *and* $\ll_S$ *are well-founded on* $\mathfrak{M}(E)$.

Proof. [2] contains a proof of the well-foundedness of $\ll$. Note that a proof of well-foundedness of $\ll_S$ or $\ll_{\mathfrak{M}}$ is also a proof of well-foundedness of $\ll$ by Lemmas 3.4 and 3.8. A proof of well-foundedness of $\ll_S$ and $\ll_{\mathfrak{M}}$ can be easily obtained by proving that $<_S$ and $<_{\mathfrak{M}}$ are well-founded. This can be done by using Königs lemma as in [2]. On the other hand, note that $<_S$ and $<_{\mathfrak{M}}$ are particular cases of $\ll$. $\square$

4. A property of maximality of the Dershowitz–Manna ordering

In the previous section, we showed two non-monotonic orderings containing the Dershowitz–Manna ordering $\ll$. A natural question arises: Do monotonic orderings exist on $\mathfrak{M}(E)$ that contain $\ll$? The answer is negative and provides a new important characterization of $\ll$. Let us first prove an important lemma.

Lemma 4.1. *Let* $\prec$ *be a partial ordering on* E *and let* M *and* N *be two multisets on* E *such that* $N \not\preccurlyeq\!\!\preccurlyeq M$ *that is* $\neg(N = M$ *or* $N \ll M)$. *Then there exists a partial ordering* $<$ *on* E *such that* $< \supseteq \prec$ (*that is,* $y \prec x \Rightarrow y < x$) *and* $M \lll N$.

Proof. By induction on the set $D = \{(x, y) \in M \times N \mid x \# y\}$.

Basic case: Let $D = \emptyset$. Then $N \lessdot\!\!\not\lessdot M \Rightarrow M \lll N$.

General case: Let D be not empty. Then either $M \lll N$ and the result is proved with $< = \prec$, or $N \lessdot\!\!\not\lessdot M$ and there must exist a pair (x, y) such that

(1) $M(x) > N(x)$ and $\forall z \in E\ x \prec z \Rightarrow N(z) \leqslant M(z)$.

(2) $M(y) < N(y)$ and $\forall z \in E\ y \prec z \Rightarrow M(z) \leqslant N(z)$.

It follows from (1) and (2) that $x \# y$ an thus $(x, y) \in D$. Let now $<$ be the transitive closure of the relation union of $\prec$ and the pair (x, y). $<$ is clearly an ordering strictly containing $\prec$. Therefore $\ll$ is an ordering strictly containing $\lll$. Since $x \prec y$, either $M \ll N$ and the result is true or M and N are incomparable according to $\ll$ and the result follows from the induction hypothesis used with a new D, whose cardinal is strictly less than the previous one. □

Theorem 4.2 (Maximality theorem). *Let* $\prec$ *be a partial ordering on* E *and* $\tau(\prec)$ *a monotonic extension of* $\prec$ *such that* $\lll \subseteq \tau(\prec)$. *Then* $\lll$ *and* $\tau(\prec)$ *are the same ordering.*

Proof. Assume first $\prec$ is total. Then $\lll$ is total on $\mathfrak{M}(E)$ and must coincide with $\tau(\prec)$. Assume now that $\prec$ is partial on E. Then $\lll$ is partial on $\mathfrak{M}(E)$. Suppose that $\tau(\prec) \supset \lll$. Then there must exist two multisets M and N such that $M\tau(\prec)N$ and $M \# N$ for the ordering $\lll$. Using Lemma 4.1 there exist $< \supseteq \prec$ such that $N \ll M$, which implies $N\tau(<)M$ by hypothesis. Using now the monotonicity of the multiset extension τ and the hypothesis $M\tau(\prec)N$, we get $M\tau(<)N$, which is a contradiction. □

This main property of the Dershowitz–Manna ordering can be used to give a simple proof of the equivalence of Definitions 2.4 and 2.5. In the following, we use this technique to present and prove a new definition of $\lll$. If $\prec$ is a total ordering on E, $\prec^{lex}$ is a total ordering on the ordered lists on E which provides a simple definition of the Dershowitz–Manna ordering in that particular case: let $list(M) = \{x_1, x_2, \ldots, x_n\}$ with $j > i \Rightarrow x_j \preccurlyeq x_i$ be the ordered list representation of multiset M. Then $M \lll N$ iff $list(M) \prec^{lex} list(N)$.

Let us now define a new multiset ordering $\lll_{\bullet}$ in the following way.

Definition 4.3. Given a partial ordering $\prec$ on E, let $M \lll_{\bullet} N$ iff for all $<$ that are total orderings containing $\prec$, $list(M) <^{lex} list(N)$.

It is easy to prove that this new ordering is exactly the Dershowitz–Manna ordering, as an application of Theorem 4.2.

Lemma 4.4. $\lll_{\bullet}$ *is a monotonic ordering.*

Proof. Follows obviously from Definition 4.3. □

Lemma 4.5. $\lll \supseteq \lll_{\bullet}$.

Proof. Suppose $M \lll N$ and $\neg(M \lll_{\bullet} N)$. There exists a total ordering $<$ such that $< \supseteq \prec$ and $\neg(list(N) <^{lex} list(M))$. Thus there exist y such that $M(y) > N(y)$ and $y < z \Rightarrow N(z) = M(z)$, which implies $y \prec z \Rightarrow N(z) = M(z)$. Combining this result and Definition 2.5, we infer $\neg(M \lll N)$, which is a contradiction. □

Theorem 4.6. $\lll_{\bullet}$ *and* $\lll$ *are the same multiset orderings.*

Proof. It follows from Theorem 4.2, Lemmas 4.4 and 4.5. □

5. An efficient implementation of the Dershowitz–Manna ordering

It is easy to derive an implementation of the Dershowitz–Manna ordering from the Huet–Oppen definition but it is not efficient because a comparison is performed for each pair of items. Moreover it leads to an algorithm that does not work symmetrically on the data. We propose an implementation based on the following idea. Given a pair of multisets M_k and N_k, build a new pair M_{k+1} and N_{k+1} such that at least one of the two multisets gets smaller and the value of the comparison $comp(M_k, N_k)$ does not change, which means $comp(M_k, N_k) = comp(M_{k+1}, N_{k+1})$. The

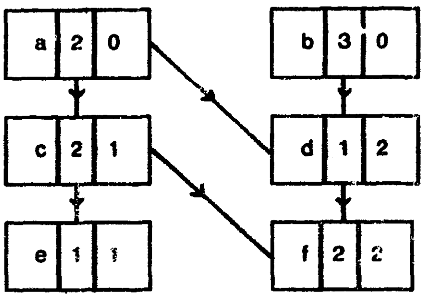

Fig. 1.

process is repeated until it is possible to decide easily whether M is greater than or less than or equal to or incomparable to N. To decrease M_k and N_k choose a pair (a, b) in $M_k \times N_k$ and do the following:

Case 1. If $a=b$ and $M_k(a)=N_k(b)$, a is removed from M_k and N_k.

Case 2. If $a \prec b$ or $a=b$ and $M_k(a)<N_k(b)$, then we would like to remove a from M_k without changing the value of comp(M_k, N_k). This is possible if a is maximal in M_k, in which case $a \prec b$ or $a=b$ and $M(a)<N(b)$ implies $M_k \ll N_k$ or M_k and N_k are incomparable (written $M_k \# N_k$). Thus comp (M_k, N_k) is not changed by removing a from M_k. Note that in this case one may remove with a all the elements in M_k which are less than a.

Thus, instead of choosing any elements a and b in M_k and N_k, choose maximal elements in M_k and N_k. Thus, after removing a from M_k, we have to compute the set of maximal elements of M_{k+1}. This is not difficult. Let us first define the function *succ* as

```
while "possible" do
  choose a new pair (a, b) in Maximal(M)×Maximal(N)
  if b≺a or [a=b and M(a)>N(b)] then
    for each x in succ*(b, N) do remove(x, N)
    end for
  end if

  if a≺b or [a=b and M(a)<N(b)] then
    for each x in succ*(a, M) do remove(x, M)
    end for
  end if

  if a=b and M(a)=N(b) then
    for each x in succ(a, M) do
      nb_ant(x, N):=nb_ant(x, N)−1
      if nb_ant(x, N)=0 then ad(x, Maximal(M))
    end for

    remove(a, Maximal(M))

    for each x in succ(b, N) do
      nb_ant(x, N):=nb_ant(x, N)−1
      if nb_ant(x, N)=0 then ad(x, Maximal(N))
    end for

    remove (b, Maximal(N))
  end if
end while
if Maximal(M)={ } then if Maximal(N)={ } then return ("M=N")
                                          else return ("M<<N") end if
                  else if Maximal(N)={ } then return ("N<<M")
                                          else return ("M#N") end iff
end if
```

Fig. 2. An algorithm to implement the Dershowitz–Manna ordering.

$(y \in succ(x) \Rightarrow y \prec x)$

$\quad \& (y \prec x \Rightarrow [\exists\, z \in succ(x)]\, y \preccurlyeq z).$

In Case 1, $M_{k+1} = M_k - \{a\}$, then

$$\mathrm{Maximal}(M_{k+1}) = \mathrm{Maximal}(M_k) - \{a\} + \{x \mid x \in succ(a) \,\&\, x \notin succ(a') \text{for } a \neq a'\}.$$

In Case 2, $M_{k+1} = M_k - \{x \mid x \in succ^*(a)\}$ where $succ^*(a) = \Sigma_{i=1}^{n} succ^i(a)$, then

$$\mathrm{Maximal}(M_{k+1}) = \mathrm{Maximal}(M_k) - \{a\}.$$

This suggests representing a multiset M as a directed acyclic graph representing the relation *succ*. Each node contains a triple: the value x of the element, M(x), the number of antecedents nb_ant(x, M) of x in M, i.e., card$\{y \mid succ(y) = x\}$. If this last number is 0, x belongs to Maximal(M).

The arrows deduced by transitivity are not necessary and the algorithm is more efficient if they are not present. For examples if the definition of *succ* is the following minimal one:

$$y \in succ(x) \Leftrightarrow y \prec x \,\&\, \neg(\exists z \in M) x \prec z \prec y.$$

For example if $c \prec a$, $d \prec a$, $d \prec b$, $e \prec c$, $f \prec c$, $f \prec d$ and $M = \{a, a, b, b, b, c, c, d, e, f, f\}$, a representation of M is given in Fig. 1.

Fig. 2 gives an algorithm describing our implementation. Note that if it is not possible to choose a new pair in the body of the loop, then all the elements present in Maximal(M) and Maximal(N) are incomparable. Then easily $M \#_{\mathfrak{M}} N$ and by Lemma 3.4, $M \# N$. The only problem is to prove termination of the algorithm, but it is easy to see that, although Maximal(M) and Maximal(N) can increase, they remain included in M and N which do decrease. Thus the algorithm terminates for any choice which computes a new pair (a, b) at each iteration.

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