Notre Dame Journal of Formal Logic
Volume 43, Number 4, 2002

Intuitionistic Completeness and Classical Logic

D. C. McCarty

Abstract We show that, if a suitable intuitionistic metatheory proves that consistency implies satisfiability for subfinite sets of propositional formulas relative either to standard structures or to Kripke models, then that metatheory also proves every negative instance of every classical propositional tautology. Since reasonable intuitionistic set theories such as HAS or IZF do not demonstrate all such negative instances, these theories cannot prove completeness for intuitionistic propositional logic in the present sense.

1 Introduction

Since the publication of Kreisel [2], we have known that a strictly intuitionistic metatheory will not suffice for proving even the weak completeness of intuitionistic predicate logic, the completeness of the deductive apparatus for single formulas or finite sets of formulas. It was Gödel who first showed that the weak completeness of intuitionistic predicate logic implies a form of Markov's Principle that intuitionists are reluctant to accept. Because well-known intutionistic set theories are consistent with the statement that Heyting's first-order intuitionistic arithmetic is categorical, proved in McCarty [3], validity for intuitionistic first-order predicate logic need not be arithmetically definable. As for propositional logic, we know from McCarty [4] that a purely intuitionistic metatheory will not prove validity for propositional logic (even in a single sentential variable) with respect to standard structures to be arithmetically definable. It is, however, possible to show intuitionistically that, for finite sets of formulas, intuitionistic propositional logic is complete for interpretations over Kripke models in that, if a finite set of propositional formulas is consistent in intuitionistic logic, then there is a finite Kripke model that satisfies it. To prove this, one can employ either a tableaux system for the logic or a countermodel search procedure on the Jaśkowski sequence, as described in Dummett [1]. It then follows that every propositional formula that is intuitionistically consistent is also satisfiable in

Received July 1, 2002; accepted February 26, 2003; printed December 19, 2003
2001 Mathematics Subject Classification: Primary, 03F55, 03F50
Keywords: intuitionistic logic, completeness, incompleteness

some standard structure. If a formula is forced at any node in a finite Kripke model, it is forced at one of the top nodes, and top nodes are standard structures.

In the present article, we settle an issue of completeness for intuitionistic propositional logic with regard to subfinite sets of formulas, sets that are subsets of finite sets. We show that, if a suitable intuitionistic metatheory proves that consistency implies satisfiability for subfinite sets of propositional formulas relative either to standard structures or to Kripke models, then that metatheory also proves every negative instance of every classical propositional tautology. Since reasonable intuitionistic set theories, for example, **HAS** or **IZF**, do not demonstrate all negative instances of $\varphi \vee \neg\varphi$, these theories will not show that consistency implies satisfiability for subfinite sets of propositional formulas. A fortiori they cannot prove strong completeness in this sense for intuitionistic propositional logic.

2 Definitions

Definition 2.1

1. A set A is *finite* whenever there is a natural number n such that A stands in bijective correspondence to n.
2. A set A is *subfinite* whenever A is a subset of some finite set.

The claim *every subfinite set is finite* is equivalent, even over relatively weak intuitionistic set theories, to the tertium non datur, $\varphi \vee \neg\varphi$.

Definition 2.2

1. A standard structure $\Im$ for propositional logic is a function from the set of propositional atoms into the powerset of $\{0\}$. With respect to such a function, $\Im$ satisfies a propositional formula φ (in symbols, $\Im \models \varphi$) is defined in the usual fashion.
2. Intuitionistic propositional logic is *strongly complete* if and only if, for any set Φ of propositional formulas, if Φ is consistent intuitionistically, then Φ is satisfiable, that is, there is a standard structure $\Im$ in which $\Im \models \Phi$.
3. Intuitionistic propositional logic is *complete for subfinite sets* if and only if, for any subfinite set Φ of propositional formulas, if Φ is consistent intuitionistically, then Φ is satisfiable in a standard structure.
4. Intuitionistic propositional logic is *strongly* **K**-*complete* if and only if, for any set Φ of propositional formulas, if Φ is consistent intuitionistically, then Φ is satisfiable in a Kripke model, that is, there is a Kripke model for intuitionistic logic $\mathbf{K}$ such that $\alpha \Vdash \varphi$, for all nodes $\alpha \in \mathbf{K}$ and all $\varphi \in \Phi$.
5. Intuitionistic propositional logic is **K**-*complete for subfinite sets* if and only if, for any subfinite set Φ of propositional formulas, if Φ is consistent intuitionistically, then Φ is satisfiable in a Kripke model.

The notions of completeness here defined should be distinguished from the various forms of the completeness concept "valid if and only if derivable" examined in [4].

Definition 2.3 A formula in the language of a formal theory T is *negative* just in case it is equivalent in T to a formula φ without $\vee$ or $\exists$ and such that every atomic subformula ψ of φ is stable: in T, $\neg\neg\psi$ implies ψ.

Definition 2.4 Unless otherwise indicated, $\Phi \vdash \varphi$ designates the formal derivability of φ from the set of formulas Φ in intuitionistic propositional logic. $\Phi \vdash_{\mathrm{cl}} \varphi$ designates formal derivability in classical propositional logic.

3 Completeness for Subfinite Sets

Let T be an intuitionistic theory extending PRA, primitive recursive arithmetic. We ask that T be powerful enough to define standard structures for propositional formulas and to prove basic results about them. For example, T should prove that, whenever $\Im$ is a standard structure in which propositional formula $\alpha \wedge \beta$ holds, both α and β hold in $\Im$ as well, and similarly for the other connectives. T should also be able to define Kripke and Beth models for propositional logic and to prove fundamental results about them. We assume further that T can show that, whenever A and B are finite sets, and φ and ψ statements in the language of T, such sets as

$$\{x \in A \mid \varphi\}$$

and

$$\{x \in A \mid \varphi\} \cup \{x \in B \mid \psi\}$$

exist and have their usual simple properties. The formal theories **HAS**, which is second-order Heyting arithmetic, and **IZF**, intuitionistic Zermelo-Fraenkel set theory, could serve as metatheories of this sort.

Fix some language $\mathcal{L}$ for propositional logic with p, q and perhaps other propositional variables. We assume that $\mathcal{L}$ and the language of T employ the same propositional connective symbols.

Definition 3.1 Let $\varphi(p, q)$ be a formula in $\mathcal{L}$ whose sole propositional variables are p and q, which are distinct. For any statements ψ and χ in the language of T, $\varphi(\psi, \chi)$ is the formula of the latter language obtained by simultaneously substituting ψ for p and χ for q in $\varphi(p, q)$.

That φ should contain two, rather than ninety-nine, propositional variables is inessential throughout.

Definition 3.2 For any object a and formula φ in the language of T, $\{a \mid \varphi\}$ is the set $\{x \mid x = a \wedge \varphi\}$.

Theorem 3.3 *Let T be an intuitionistic metatheory as specified above. If T proves that intuitionistic propositional logic is complete for subfinite sets, T proves every negative instance of every classical propositional tautology.*

Proof Let T be a suitable metatheory, $\varphi(p, q)$ a classical propositional tautology, and ψ and χ negative formulas in the language of T. Consider this subfinite collection Φ of propositional formulas

$$\{\varphi(p, q)\} \cup \{p \mid \psi\} \cup \{q \mid \chi\} \cup \{\neg p \mid \neg\psi\} \cup \{\neg q \mid \neg\chi\}.$$

First, T proves Φ to be consistent intuitionistically, that is, T proves that $\Phi \nvdash \bot$. Working in T, assume that $\Phi \vdash \bot$. Since T extends PRA, T knows that $\Phi \vdash_{\mathrm{cl}} \bot$. It follows, in T, that the set Ψ

$$\{p \mid \psi\} \cup \{q \mid \chi\} \cup \{\neg p \mid \neg\psi\} \cup \{\neg q \mid \neg\chi\}$$

is classically inconsistent, that is, $\Psi \vdash_{\mathrm{cl}} \bot$. Now, assume in T that both ψ and χ hold. In that case, Ψ is just the set $\{p, q\}$. This set cannot be classically inconsistent and T suffices to prove that. If, on the other hand, we assume that both ψ and $\neg\chi$ hold, then Ψ reduces to the set $\{p, \neg q\}$. And it would be equally absurd to think that this set is classically inconsistent, as T also recognizes. The same result, that

Ψ is not classically inconsistent, follows in T from either of the remaining available assumptions: that both $\neg\psi$ and χ hold, and that both $\neg\psi$ and $\neg\chi$ hold.

Now, the intuitionistic metatheory T proves

$$\neg\neg[(\psi \wedge \chi) \vee (\psi \wedge \neg\chi) \vee (\neg\psi \wedge \chi) \vee (\neg\psi \wedge \neg\chi)].$$

Hence, T recognizes that $\Psi \vdash_{\mathrm{cl}} \bot$ is absurd. Therefore, T proves Φ's consistency: $\Phi \nvdash \bot$.

Second, suppose in T that intuitionistic propositional logic is complete for subfinite sets. It follows that, since Φ is consistent, there is a standard propositional structure $\Im$ such that $\Im \models \Phi$. Given the definition of Φ, if ψ holds, then p belongs to Φ and $\Im \models p$. Similarly, if ψ fails to hold, then $\Im \not\models p$. Since ψ is negative, we can combine these facts and conclude that ψ holds if and only if $\Im \models p$. Parallel reasoning shows that χ holds just in case $\Im \models q$.

Finally, because $\varphi(p, q) \in \Phi$, $\Im \models \varphi(p, q)$. T treats standard structures in the usual way. This means that, in T, the predicate '$\Im \models$' commutes with all the connectives. So, $\varphi(\psi, \chi)$ holds. $\square$

Corollary 3.4 *With T as above, if T proves that intuitionistic logic is strongly complete, then T proves every negative instance of every classical tautology.*

Corollary 3.5 *For T as above, if some negative instance of the tertium non datur is independent of T, then T neither proves that intuitionistic propositional logic is strongly complete nor proves that it is complete for subfinite sets.*

Corollary 3.6 *Neither* **HAS** *nor* **IZF** *proves that intuitionistic propositional logic is strongly complete. Neither theory proves propositional completeness for subfinite sets.*

Proof Soundness for Kleene's number realizability (as explained in Troelstra [5]) shows that neither theory derives all negative instances of the tertium non datur. Both theories fail to derive $\neg\exists n K(m, n) \vee \neg\neg\exists n K(m, n)$, where $K(m, n)$ is the negative arithmetic predicate 'Turing machine number m halts in precisely n computation steps if m is input'. $\square$

I think there are some who would question the significance of the above results, insisting that a proper intuitionistic account of interpretation or structure $\Im$ for propositional logic does not require that the associated '$\Im \models$' predicate commute with the connectives, as is standardly the case. For example, they would maintain that, because intuitionistic negation is to be "stronger" than conventional negation, an intuitionist need not assume that $\Im \models \neg\varphi$ just in case $\Im \not\models \varphi$. I must admit that I have always believed this point of view unsatisfactory. First, the extent to which the intuitionist has available a notion of *truth*, rather than some distinct but truth-like notion, may be gauged by the intuitionist's willingness to treat the connectives standardly. Second, when it concerns classical logic and its interpretation, we logic pedagogues like to inform our students that the familiar interpretative clauses requiring that $\Im \models (\varphi \vee \psi)$ just in case either $\Im \models \varphi$ or $\Im \models \psi$, and that $\Im \models \neg\varphi$ just in case $\Im \not\models \varphi$ record, at least in part, our intention that $\vee$ mean 'or' and $\neg$ mean 'not'. Doesn't the committed intuitionist wish to contend much the same: that, in intuitionistic mathematics, the sign $\vee$ means (intuitionistic) 'or' and $\neg$ means (intuitionistic) 'not'? And I would think that a suitable metatheory T, employed by the intuitionist and treated as adequate, would be a theory in which he or she would wish to interpret

$\vee$ as 'or' and $\neg$ as 'not' and to prove that this interpretation makes sense. Surely, defining standard interpretative structures for propositional logic so that the familiar connectives commute with the '$\Im \models$' predicate would be one way of ensuring that $\vee$ mean 'or', $\neg$ mean 'not' and so on, and by such means confirming the internal semantic coherence of the intuitionistic vision.

4 Completeness, Kripke Models, and Subfinite Sets

One may be able to avoid such worries by putting the proof ideas of the current section to work on Kripke and Beth models.

Theorem 4.1 *Let T be an intuitionistic metatheory as specified above. If T proves* **K***-completeness for subfinite sets, T proves every negative instance of every classical propositional tautology.*

Proof Again, let T be a suitable metatheory, $\varphi(p, q)$ a classical propositional tautology, and ψ and χ negative formulas in the language of T. The definition of Φ and the proof that T shows Φ consistent are the same as in Theorem 3.3.

We now suppose that, in T, intuitionistic propositional logic is **K**-complete for subfinite sets. It follows that, since Φ is consistent intuitionistically and this is provable in T, there is a Kripke model $\mathbf{K}$ such that $\mathbf{K} \models \Phi$, that is, the set Φ is forced at every node α of $\mathbf{K}$. Now, given the definition of Φ, if ψ holds, then p belongs to the set Φ and $\mathbf{K} \models p$. Hence, if α is any node of $\mathbf{K}$ and if ψ holds, $\alpha \Vdash p$. Conversely, if $\alpha \Vdash p$, then ψ holds, thanks to the negativity of ψ. For, if $\neg\psi$ were to hold, then $\alpha \Vdash \neg p$, contrary to the assumption. By parallel reasoning, χ holds just in case $\alpha \Vdash q$.

Since $\varphi(p, q) \in \Phi$, $\mathbf{K} \models \varphi(p, q)$. If we use the definition of Kripke forcing to work out in full the **K**-forcing condition D of $\varphi(p, q)$, we can, given the results of the preceding paragraph, replace each expression of the forms $\alpha \Vdash p$ or $\alpha \Vdash q$ in D, where α is any node, by ψ or χ, respectively. We can certainly assume that ψ and χ contain no appearance of any variable, such as 'α', ranging over nodes of $\mathbf{K}$. Therefore, since each node α is such that $\alpha \preceq \alpha$ in the order $\preceq$ on $\mathbf{K}$, any quantification in D over nodes is now redundant and can be dropped without loss. What remains of D is simply $\Phi(\psi, \chi)$, which holds since D does. □

Corollary 4.2 *With T as above, if T proves that intuitionistic logic is strongly* **K***-complete, then T proves every negative instance of every classical tautology.*

Corollary 4.3 *For T as above, if some negative instance of the* tertium non datur *is independent of T, then T does not prove that intuitionistic propositional logic is strongly* **K***-complete nor does it prove* **K***-completeness for subfinite sets.*

Corollary 4.4 *Neither* **HAS** *nor* **IZF** *proves that intuitionistic propositional logic is strongly* **K***-complete. Neither theory proves propositional* **K***-completeness for subfinite sets.*

Since Beth satisfiability and Kripke satisfiability are provably equivalent in T (see Troelstra and van Dalen [6] and [7]), strictly analogous results hold for completeness with respect to Beth models.

References

[1] Dummett, M., *Elements of Intuitionism*, Clarendon Press, 1977. Zbl 0358.02032. MR 58:16194. 243

[2] Kreisel, G., "On weak completeness of intuitionistic predicate logic," *The Journal of Symbolic Logic*, vol. 27 (1962), pp. 139–58. Zbl 0117.01005. MR 28:5000. 243

[3] McCarty, C., "Constructive validity is nonarithmetic," *The Journal of Symbolic Logic*, vol. 53 (1988), pp. 1036–41. Zbl 0724.03034. MR 90f:03095. 243

[4] McCarty, D. C., "Incompleteness in intuitionistic metamathematics," *Notre Dame Journal of Formal Logic*, vol. 32 (1991), pp. 323–58. Zbl 0751.03030. MR 92j:03053. 243, 244

[5] Troelstra, A. S., editor, *Metamathematical Investigation of Intuitionistic Arithmetic and Analysis*, vol. 344, Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1973. Zbl 0275.02025. MR 48:3699. 246

[6] Troelstra, A. S., and D. van Dalen, *Constructivism in Mathematics. An Introduction, Vol. I*, vol. 121 of *Studies in Logic and the Foundations of Mathematics*, North-Holland Publishing Co., Amsterdam, 1988. Zbl 0653.03040. MR 90e:03002a. 247

[7] Troelstra, A. S., and D. van Dalen, *Constructivism in Mathematics. An Introduction, Vol. II*, vol. 123 of *Studies in Logic and the Foundations of Mathematics*, North-Holland Publishing Co., Amsterdam, 1988. Zbl 0661.03047. MR 90e:03002b. 247

Acknowledgments

The author thanks Nathan Carter and the anonymous referee for their helpful comments and suggestions.

Department of Philosophy
Sycamore 026
Indiana University
1033 E Third St
Bloomington IN 47405-7005
dmccarty@indiana.edu