On Intersection Problems for Polynomially Generated Sets

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Presburger arithmetic and semi-linear sets: well-known framework for algorithmic applications and verification where aspect of infinity arises from the domain of natural numbers.

System runs induce a semi-linear set of vectors of natural numbers.

Specification is a semi-linear set or, equivalently, a formula of Presburger arithmetic (i.e. FO-formula over the structure $(\mathbb{N}, +)$).

Verification amounts to checking the intersection of both sets for nonemptiness.

Aim

A framework beyond the semi-linear sets or Presburger arithmetic

Outline

Semi-Linear Sets and Presburger Arithmetic

Linear set: $\{\bar{u}_0 + k_1\bar{u}_1 + \cdots + k_m\bar{u}_m \mid k_1, \ldots, k_m \in \mathbb{N}\}$ for some $\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_m \in \mathbb{N}^n$

Semi-linear set: finite union of linear sets

Presburger arithmetic: first-order theory of $(\mathbb{N}, +)$

A Presburger formula $\varphi(x_1, \ldots, x_n)$ defines the set

$$\{(u_1, \ldots, u_n) \in \mathbb{N}^n \mid (\mathbb{N}, +) \models \varphi[u_1, \ldots, u_n]\}$$

of vector of natural numbers.

Ginsburg and Spanier's theorems:

- equivalence between semi-linear and Presburger-definable sets
- effective closure under Boolean operations and projections

Parikh's theorem:

- The image of any context-free language under the Parikh mapping is effectively semi-linear.

$$\Phi(w) = (|w|_{a_1}, \ldots, |w|_{a_n})$$
$$\Phi(L) = \{\Phi(w) \mid w \in L\}$$

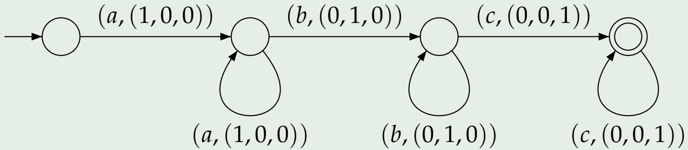
Example: Parikh Automata

Parikh automaton [Klaedtke & Ruess, ICALP 2003] :

- finite automaton $\mathfrak{A}$ over extended alphabet $\Sigma \times D$ ($D \subseteq \mathbb{N}^n$)
- Presburger formula $\varphi(x_1, \ldots, x_n)$ (or semi-linear set $C \subseteq \mathbb{N}^n$)

Example

- $\mathfrak{A}$:



- $\varphi: \quad x_1 \geq 2(x_2 + x_3) \ \wedge \ x_2 = x_3$

Acceptance of a word requires two conditions:

- acceptance by automaton $\mathfrak{A}$ (via a run)
- sum of vectors accumulated along the run must satisfy φ

$L(\mathfrak{A}, \varphi)$: words of the form $a^+b^+c^+$ satisfying "the first half of w consists only of a, and in the second half the number of b's and c's coincide"

Outline

1. Preliminaries
2. Polynomially Generated Sets
3. On the Intersection Problems
4. Quadratic Forms

Aim:

- generalize (semi-)linear sets to sets generated by polynomials
- solve the intersection problem, i.e. the nonemptiness problem for the intersection of two subsets of $\mathbb{N}^n$

$A \subseteq \mathbb{N}^n$ is linear if it is of the form

$$\left\{ \begin{pmatrix} u_{01} + k_1 u_{11} + \cdots + k_m u_{m1} \\ \vdots \\ u_{0n} + k_1 u_{1n} + \cdots + k_m u_{mn} \end{pmatrix} \middle| \; k_1, \ldots, k_m \in \mathbb{N} \right\}$$

linear functions in $k_1, \ldots, k_m$

Definition

$A \subseteq \mathbb{N}^n$ is called polynomial if it is of the form

$$\left\{ \begin{pmatrix} P_1(k_1, \ldots, k_m) \\ \vdots \\ P_n(k_1, \ldots, k_m) \end{pmatrix} \middle| \; k_1, \ldots, k_m \in \mathbb{N} \right\}$$

for some polynomials $P_1, \ldots, P_n \in \mathbb{N}[X_1, \ldots, X_m]$.
Semi-polynomial set: finite union of polynomial sets.

Example

- square relation: $\{(x,y) \in \mathbb{N}^2 \mid y = x^2\}$
- product relation: $\{(x,y,z) \in \mathbb{N}^3 \mid z = x \cdot y\}$

Some (trivial) properties of semi-polynomial sets:

- closure under finite union and projection
- decidability of membership problem
- strict hierarchy w.r.t. degree of polynomials
- do not capture sets like $\{(x,y) \in \mathbb{N}^2 \mid y = 2^x\}$

Intersection Problem for Semi-Polynomial Sets ?

Hilbert's Tenth Problem

Given a polynomial $P \in \mathbb{Z}[X_1, \ldots, X_m]$, does the polynomial equation

$$P(k_1, \ldots, k_m) = 0$$

have a solution in natural numbers? $\rightsquigarrow$ undecidable

Hilbert's Tenth Problem

Given polynomials $Q, R \in \mathbb{N}[X_1, \ldots, X_m]$, does the polynomial equation

$$Q(k_1, \ldots, k_m) = R(k_1, \ldots, k_m)$$

have a solution in natural numbers? $\rightsquigarrow$ undecidable

Restriction: No 'Mixed' Terms

Definition

$A \subseteq \mathbb{N}^n$ is a simple polynomial set if it is of the form

$$\left\{ \begin{pmatrix} P_{11}(k_1) + \cdots + P_{1m}(k_m) \\ \vdots \\ P_{n1}(k_1) + \cdots + P_{nm}(k_m) \end{pmatrix} \middle| \, k_1, \ldots, k_m \in \mathbb{N} \right\},$$

where $P_{ij} \in \mathbb{N}[X]$ are (univariate) polynomials.

Simple semi-polynomial set: finite union of simple polynomial sets.

Remark

- Properties of semi-polynomial sets carry over into simple semi-polynomial sets: closure under finite unions, decidability of membership problem, strict hierarchy w.r.t. degree, ...
- Simple semi-polynomial sets form a proper subclass of semi-polynomial sets For example, the product relation is not simple semi-polynomial (see paper for proof).

Outline

Proposition

There are two simple quadratic sets whose intersection is not simple semi-polynomial.

Example (see paper for proof)

Intersection of the quadratic sets

$$\left\{ \begin{pmatrix} (k_1+1)^2 + (k_2+1)^2 \\ k_3 \end{pmatrix} \;\middle|\; k_1, k_2, k_3 \in \mathbb{N} \right\} \quad \text{and} \quad \left\{ \begin{pmatrix} k^2 \\ k \end{pmatrix} \;\middle|\; k \in \mathbb{N} \right\}$$

is not simple semi-polynomial.

Hence, consider intersection of a simple semi-polynomial set and a semi-linear set.

Partial result: decidability for the case of componentwise semi-linear sets

Decidability of the general case remains an open question.

Componentwise Semi-Linear Sets

Definition

$A \subseteq \mathbb{N}^n$ is called componentwise linear if there are linear sets $A_1, \ldots, A_n \subseteq \mathbb{N}$ such that

$$A = A_1 \times \cdots \times A_n \, .$$

union or rectangles

Componentwise semi-linear set: finite union of componentwise linear sets

Theorem

If $A \subseteq \mathbb{N}^n$ is componentwise semi-linear and $B \subseteq \mathbb{N}^n$ is a simple semi-polynomial set of degree d, then $A \cap B$ is a simple semi-polynomial set of degree d.

Moreover, if A and B are given by their generators, generators of $A \cap B$ can be computed and hence nonemptiness of $A \cap B$ be checked effectively.

Remark

The result also holds for the intersection of a componentwise semi-linear sets with a semi-polynomial set.

Proof (Sketch) of the Theorem

Theorem

If $A \subseteq \mathbb{N}^n$ is componentwise semi-linear and $B \subseteq \mathbb{N}^n$ is a (simple) semi-polynomial set of degree d, then $A \cap B$ is a (simple) semi-polynomial set of degree d.

Moreover, if A and B are given by their generators, generators of $A \cap B$ can be computed and hence nonemptiness of $A \cap B$ be checked effectively.

Proof (sketch)

W.l.o.g. A componentwise linear, B simple polynomial.

First, solve the case $n = 1$ (this is the core of the proof).

For $n > 1$, let $A = A_1 \times \cdots \times A_n$, and proceed as follows:

- Analyze $A \cap B$ w.r.t. the first component, i.e. $(A_1 \times \mathbb{N}^{n-1}) \cap B$.
- If this set is nonempty, establish its semi-polynomial representation, say B'.
- Analyze the second component, i.e. $(\mathbb{N} \times A_2 \times \mathbb{N}^{n-2}) \cap B'$.
- ...

After n steps, we obtain a semi-polynomial representation of $A \cap B$.

Proof (Sketch): the One-Dimensional Case

Suppose

$$A = \{x_0 + k_1 x_1 + \cdots + k_m x_m \mid k_1, \ldots, k_m \in \mathbb{N}\}$$
$$B = \{P(\ell_1, \ldots, \ell_r) \mid \ell_1, \ldots, \ell_r \in \mathbb{N}\}$$

Let $g := \gcd(x_1, \ldots, x_m)$. Then, there is some $z_0 \in \mathbb{N}$ such that

- $C := \{z_0 + kg \mid k \in \mathbb{N}\} \subseteq A$ and
- $A \setminus C$ is finite.

Hence, it suffices to consider only $C \cap B$.

$x \in C \cap B$ iff $x = P(\ell_1, \ldots, \ell_r)$ for some solution $\ell_1, \ldots, \ell_r$ of the congruence equation

$$z_0 \equiv P(\ell_1, \ldots, \ell_r) \pmod g$$

If a solution exists, then also such with $\ell_1, \ldots, \ell_r < g$.

Such a solution $s_1, \ldots, s_r < g$ then generates elements of $C \cap B$ of the form

$$x = P(s_1 + \ell'_1 g, \ldots, s_r + \ell'_r g)$$

where $\ell'_i \in \mathbb{N}$ (up to finitely many exceptions). □

Outline

Solvability of Quadratic Equations

Hilbert's Tenth Problem revisited:

- Undecidability holds already for solving polynomial equations of degree four and systems of quadratic equations in integers.
- Solvability of quadratic equations in integers is decidable (Siegel, 1972).

Solvability of quadratic equations in natural numbers:

Theorem (Grunewald & Segal, 2004)

Consider quadratic form $Q \in \mathbb{Z}[X_1, \ldots, X_n]$, linear forms $L_1, \ldots, L_k \in \mathbb{Z}[X_1, \ldots, X_n]$, and the system

$$
\begin{aligned}
&Q(x_1, \ldots, x_n) = 0\,, \\
&L_j(x_1, \ldots, x_n) \# c_j, \text{where } c_j \in \mathbb{Z} \text{ and } \# \in \{<, \leq\}, \text{ for } j = 1, \ldots, k\,, \\
&(x_1, \ldots, x_n) \equiv (h_1, \ldots, h_n) \pmod{m}, \text{ where } h_1, \ldots, h_n \in \mathbb{Z}, m \in \mathbb{N}\,,
\end{aligned}
$$

It is decidable whether a solution in $\mathbb{Z}^n$ exists.

Remark: Linear constraints $-x_i \leq 0$ restrict to solutions in natural numbers.

Applications of Grunewald & Segal's Theorem

Scenario 1

- system: semi-linear set
- specification: quadratic equation $Q(x_1, \ldots, x_n) = 0$

$\rightsquigarrow$ decidable, as a semi-linear set is the solution set of a linear (in)equation systems

Scenario 2

- system: semi-linear set (semi-one-quadratic set)
- specification: (semi-)one-quadratic set

quadratic

linear

$$\left\{ \begin{pmatrix} Q(k_1, \ldots, k_m) \\ L_2(k_1, \ldots, k_m) \\ \vdots \\ L_n(k_1, \ldots, k_m) \end{pmatrix} \;\middle|\; k_1, \ldots, k_m \in \mathbb{N} \right\}$$

$\rightsquigarrow$ decidable; transform nonemptiness question to (in)equation system described previously.

Conclusions

- Some possibilities of extending the framework of semi-linear sets with polynomials
- Some restrictions needed in order to retain decidability results w.r.t. intersection problem, e.g. simple semi-polynomial sets, componentwise semi-linear sets, one-quadratic sets
- Number-theoretical results and methods are required.

Overview of present results:

system	specification	intersection problem
semi-linear	semi-polynomial	undecidable
semi-linear	simple semi-polynomial	?
componentwise semi-linear	(simple) semi-polynomial	decidable
semi-linear	semi-one-quadratic	decidable (Grunewald & Segal, 2004)

Further Prospects

- algorithmic and complexity analysis of Grunewald & Segal's results
- closure properties of (simple) semi-polynomial sets, e.g. under additive operations

Open question

- intersection problem for simple semi-polynomial sets and semi-linear sets