How to Compute a Puiseux Expansion

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July 29, 2008

Abstract

In this paper, an explanation of the Newton-Peiseux algorithm is given. This explanation is supplemented with well-worked and explained examples of how to use the algorithm to find fractional power series expansions for all branches of a polynomial at the origin.

1 Introduction

Given a polynomial f(x, y), you can always solve for y in terms of x by means of a fractional power series

$$y = c_1 x^{\gamma_1} + c_2 x^{\gamma_1 + \gamma_2} + c_3 x^{\gamma_1 + \gamma_2 + \gamma_3} + \cdots$$

The method for doing this is often explained in books in the context of a proof of the algebraic closure of the field of fractional power series (e.g., [1]). Our purpose here is not to discuss the proof of such a theorem, but to explain clearly, with several examples, the algorithm for obtaining these expansions. Here is a general description of the algorithm.

- 1. Given f(x,y) = 0 draw the Newton polygon of f(x,y). This is done by plotting a point on \mathbb{R}^2 for each term of f with the term kx^ay^b being plotted to the point (b,a) The Newton polygon is the smallest convex shape that contains all the points plotted.
- **2.** Take a segment of the Newton polygon from the set of segments where each point plotted is on, above, or to the right of the segments.
- 3. The first exponent γ_1 will be the negative of the slope of this segment.
- **4.** Find $f(x, x^{\gamma_1}(c_1 + y_1))$.
- **5.** Take the lowest terms in x alone. Since f(x,y) = 0, these must cancel and so you can solve for c_1 ,

- **6.** Taking the values of γ_1 and c_1 and β = the 'x-intercept' on the Newton polygon of the segment we have chosen, we now find $f_1(x, y_1)$.
- 7. $f_1(x,y_1) = x^{-\beta} f(x, x^{\gamma_1}(c_1+y_1))$
- **8.** Now repeat the process for $f_1(x, y_1)$ to find γ_2 and c_2 .
- 9. Continue this process until one of two things happen
 - $f_n(x, y_n)$ has a factor of y_n .
 - The Newton Polygon of $f_n(x, y_n)$ consists of a single segment with only two vertices, one on each axis.

If the former occurs, either the y factor(s) can be factored out and you can continue, or (if you are not left with a segment in the Newton polygon) the series terminates. If the latter occurs, plug in the rest of the series, solving for coefficients by letting lowest powers of x cancel. Use the denominator of the slope of the segment to determine the increase in each γ_i , (e.g., if the slope is $\frac{3}{2}$, each γ_i , should be $\frac{1}{2}$). This will be greatly clarified by the following examples.

Any polynomial of the form f(x,y)=0 can be solved for y in terms of x explicitly around a point. Since by simple translation any point on a curve can be moved to the origin, we will only expand a solution of f(x,y)=0 at the origin. The Newton-Puiseux algorithm gives us that any solution of f(x,y)=0 has the form $y=c_1x^{\gamma_1}+c_2x^{\gamma_1+\gamma_2}+c_3x^{\gamma_1+\gamma_2+\gamma_3}+\cdots$, where $\gamma_i\in\mathbb{Q}^+$ and the $c_i\in\mathbb{Q}^+$. Any time γ_i is referred to, we are speaking of an exponent, and anytime a c_i is referred to, we are referring to a coefficient.

2 Examples

Example 1: In this example we show a more simple way to compute the expansion that can be used if the second case in step 9, mentioned above, occurs where the Newton Polygon has a horizontal intercept of one.

$$f(x,y) = 2x^4 + x^2y + 4xy^2 + 4y^3 = 0 (1)$$

We know that this polynomial has solutions of the form

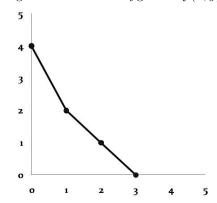
$$y = c_1 x^{\gamma_1} + c_2 x^{\gamma_1 + \gamma_2} + c_3 x^{\gamma_1 + \gamma_2 + \gamma_3} + \cdots$$
 (2)

The possible choices for γ_1 correspond to the negatives of the slopes of the lower segments of the Newton Polygon (Figure 1). The slopes of the lower segments of the Newton Polygon corresponding to (1) are -2, and -1. We will start by considering the case where $\gamma_1 = -(-2) = 2$.

Let $\gamma_1 = 2$. Then by factoring x^{γ_1} (or x^2) out of (2) we obtain,

$$y = x^2(c_1 + y_1), (3)$$

Figure 1: Newton Polygon for f(x, y)



where c_1 is the first coefficient in the series and y_1 is

$$y_1 = c_2 x^{\gamma_2} + c_3 x^{\gamma_2 + \gamma_3} + c_4 x^{\gamma_2 + \gamma_3 + \gamma_4} + \cdots$$
 (4)

i.e. the rest of the series.

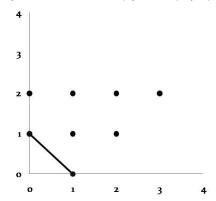
So, substituting (3) into (1) we get,

$$2x^{4} + x^{4}(c_{1} + y_{1}) + 4x^{5}(c_{1} + y_{1})^{2} + 4x^{6}(c_{1} + y_{1})^{3} = 0$$
 (5)

The vertical intercept of the segment we used to find f_1 on the Newton Polygon, which we call β , gives the terms of lowest degree in x alone. These terms must sum to zero because f(x,y) = 0. In this case, $\beta = 4$, so $2x^4 + c_1x^4 = 0$ and solving for c_1 we obtain $c_1 = -2$. Substituting c_1 into (5) and dividing by x^4 gives,

$$x^{-4}f(x, x^{2}(c_{1} + y_{1})) = 2 + (-2 + y_{1}) + 4x(-2 + y_{1})^{2} + 4x^{2}(-2 + y_{1})^{3}$$
$$= y_{1} + 16x - 16xy_{1} + 4xy_{1}^{2} - 32x^{2} + 48x^{2}y_{1} - 24x^{2}y_{1}^{2} + 4x^{2}y_{1}^{3}$$
 (6)

Figure 2: Newton Polygon for $f_1(x, y_1)$



We call this new polynomial $f_1(x,y_1)$, so now that we have our new function $f_1(x,y_1)$ our next goal is to find c_2 . We find the new Newton Polygon for $f_1(x,y_1)$ (Figure 2), from which we get $\beta_2=1$ and $\gamma_2=1$. Again, we let the terms of lowest degree in x alone equal zero. We have now reached a point in our algorithm where the calculations become much simpler. Notice that in the previous Newton polygon the intercept with the horizontal axis was one and we had only one segment. This tells us that we can calculate the rest of the c_i and γ_i directly from the current function, in this case f_1 . This is true because when we obtain such a Newton Polygon, we know that the powers of x are going to increase by at least $1/(\text{denominator of }\gamma_2)$ from now on (with the possibility that c_i could be =0); in other words, $\gamma_3=\gamma_4=\cdots=\gamma_n=\cdots=$. Since γ_2 , we know that

$$y_1 = c_2 x + c_3 x^2 + c_4 x^3 + c_5 x^4 + \cdots$$

So, substituting y_1 into (6) and letting the lowest degree terms of x cancel we see

$$c_2 x + 16x = 0$$

so, $c_2 = -16$

We find c_3 in the same manner. To do this, we let the x^2 terms equal zero. Substituting y_1 into (6) we see that,

$$c_3x^2 - 16c_2x^2 - 32x^2 = 0$$

$$c_3 + 256 - 32 = 0$$
so, $c_3 = -224$

Now, we can let the x^3 terms in f_1 be equal to zero. So,

$$c_4x^3 - 16c_3x^3 + 4c_2^2x^3 + 48c_2x^3 = 0$$

$$c_4 + 3584 + 1024 - 768 = 0$$

so, $c_4 = -3840$

Next, we let the x^4 terms equal zero.

$$c_5x^4 - 16c_4x^4 + 8c_2c_3x^4 + 48c_3x^4 - 24c_1^2x^4 = 0$$

$$c_5 + 61440 + 28672 - 11184 - 10752 = 0$$

so, $c_5 = -39504$

This process can be continued to calculate more terms, but we will stop with c_5 . So, one explicit solution of the polynomial is

$$y = -2x^2 - 16x^3 - 224x^4 - 3840x^5 - 39504x^6 + \cdots$$

We have examined the case where $\gamma_1=2$, and now we must return to the original polynomial f(x,y) and examine the case where $\gamma_1=1$. Recall, $f(x,y)=2x^4+x^2y+4xy^2+4y^3=0$. Let $\gamma_1=1$. Then, $y=x(c_1+y_1)$ and substituting γ_1 and y into (1) we obtain

$$2x^{4} + x^{3}(c_{1} + y_{1}) + 4x^{3}(c_{1} + y_{1})^{2} + 4x^{3}(c_{1} + y_{1})^{3} = 0$$
 (7)

As in the case where $\gamma_1 = 2$,

$$y_1 = c_2 x^{\gamma_2} + c_3 x^{\gamma_2 + \gamma_3} + c_4 x^{\gamma_2 + \gamma_3 + \gamma_4} + \cdots$$

From the Newton Polygon (Figure 1) we see that $\beta_1 = 3$. Therefore, we can let the x^3 terms equal zero.

$$\begin{aligned} c_1x^3 + 4c_1^2x^3 + 4c_1^2x^3 &= 0\\ c_1(1 + 4c_1 + 4c_1^2) &= 0\\ c_1(1 + 2c_1)^2 &= 0\\ c_1 &= 0, -\frac{1}{2}, -\frac{1}{2}\\ c_1 &= 0 \text{ is trivial, so assume, } c_1 &= -\frac{1}{2}. \end{aligned}$$

Substituting c_1 into (7) and dividing by x^3 gives:

$$x^{-3}f(x,x(c_1+y_1)) = 2x + \left(-\frac{1}{2} + y_1\right) + 4\left(-\frac{1}{2} + y_1\right)^2 + 4\left(-\frac{1}{2} + y_1\right)^3$$
$$= 2x + y_1 - 4y_1 + 4y_1^2 + 3y_1 - 6y_1^2 + 4y_1^3$$
$$f_1(x,y) = 2x - 2y_1^2 + 4y_1^3$$

 $\gamma_2 = 2$, so

$$y_1 = x^{\frac{1}{2}}(c_2 + y_2)$$
 so, $2x - 2x(c_2 + y_2)^2 + 4x^{\frac{3}{2}}(c_2 + y_2)^3 = 0$

where c_2 is the second coefficient and y_2 is the rest of the series. Substituting y_1 into $f_1(x, y)$ yields

$$2x - 2x(c_2 + y_2)^2 + 4x^{\frac{3}{2}}(c_2 + y_2)^3 = 0$$

From the new Newton Polygon (Figure 1) we see that $\beta_2 = 1$, so we can let the x terms equal zero.

$$2x - 2xc_2^2 = 0$$

$$c_2 = \pm 1$$
(8)

Let $c_2 = 1$. (We must return later and consider the case where $c_2 = -1$.) Then substituting c_2 into $f_1(x, y)$ and dividing by x we get

$$x^{-1}f(x,y)_1 = 2 - 2(1+y_2)^2 + 4x^{\frac{1}{2}}(1+y_2)^3$$

= $-4y_2 - 2y_2^2 + 4x^{\frac{1}{2}} + 12x^{\frac{1}{2}}y_2 + 12x^{\frac{1}{2}}y_2^2 + 4x^{\frac{1}{2}}y_2^2$

Where $y_2 = c_3 x^{\gamma_3} + c_4 x^{\gamma_3 + \gamma_4} + c_5 x^{\gamma_3 + \gamma_4 + \gamma_5} + \cdots$ The horizontal intercept is one, so $y_2 = c_3 x^{\frac{1}{2}} + c_4 x + c_5 x^{\frac{3}{2}} + \cdots$ Letting the lowest degree terms of x $(x^{\frac{1}{2}})$ equal zero we find

$$-4c_3 + 4 = 0$$
$$c_3 = 1$$

Now, we can let the x terms equal zero.

$$-4c_4x - 2c_3^2x + 12c_3x = 0$$
$$-4c_4 - 2 + 12 = 0$$
$$so, c_4 = \frac{5}{2}$$

We now let the $x^{\frac{3}{2}}$ terms equal zero.

$$-4c_5x^{\frac{3}{2}} - 4c_3c_4x^{\frac{3}{2}} + 12c_4\frac{3}{2} + 12c_3^2x^{\frac{3}{2}} = 0$$
$$-4c_5 - 10 + 30 + 12 = 0$$
so $c_5 = 8$

As in the case where $\gamma_1 = 2$ we could continue this process and calculate more terms. Stopping here, another explicit solution of f(x, y) is

$$y = -\frac{1}{2}x + x^{\frac{3}{2}} + x^2 + \frac{5}{4}x^{\frac{5}{2}} + \frac{27}{4}x^3 + \cdots$$

Now, we must return to (8) and consider the case where $c_1 = -1$. Substituting c_1 into $f_1(x, y)$ and dividing by x^4 gives

$$x^{-4}f_1(x,y) = 2 - 2(-1 + y_2)^2 + 4x^{\frac{1}{2}}(-1 + y_2)^3$$

= $4y_2 - 2y_2^2 - 4x^{\frac{1}{2}} + 12x^{\frac{1}{2}}y_2 - 12x^{\frac{1}{2}}y_2^2 + 4x^{\frac{1}{2}}y_2^3$

Let $y_2 = c_3 x^{\gamma_3} + c_4 x^{\gamma_3 + \gamma_4} + c_5 x^{\gamma_3 + \gamma_4 + \gamma_5} + \cdots$. $\gamma_3 = \frac{1}{2}$, so $y_2 = c_3 x^{\frac{1}{2}} + c_4 x + c_5 x^{\frac{3}{2}} + \cdots$. Letting the $x^{\frac{1}{2}}$ terms equal zero,

$$4c_3x^{\frac{1}{2}} + 4x^{\frac{1}{2}} = 0$$

so, $c_3 = -1$

Letting the x terms equal zero,

$$4c_4x - c_3^2x + 12c_3x = 0$$
$$4c_4 - 1 - 12 = 0$$
$$so, c_4 = \frac{13}{4}$$

Now we let the $x^{\frac{3}{2}}$ terms equal zero and solve for c_5 ,

$$4c_5 + 12c_4 - 12c_3^2 = 0$$

$$4c_5 + 39 - 12 = 0$$

so, $c_5 = -\frac{27}{4}$

So, we find that the last explicit solution for f(x,y) = 0 is

$$y = -\frac{1}{2}x - x^{\frac{3}{2}} - x^2 + \frac{13}{4}x^{\frac{5}{2}} - \frac{27}{4}x^3$$

So, the three solutions for f(x,y) = 0 are

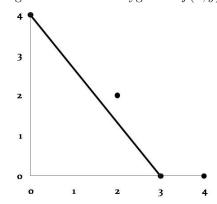
$$y = -2x^{2} - 16x^{3} - 224x^{4} - 3840x^{5} - 39504x^{6} + \cdots$$

$$y = -\frac{1}{2}x + x^{\frac{3}{2}} + x^{2} + \frac{5}{4}x^{\frac{5}{2}} + \frac{27}{4}x^{3} + \cdots$$

$$y = -\frac{1}{2}x - x^{\frac{3}{2}} - x^{2} + \frac{13}{4}x^{\frac{5}{2}} - \frac{27}{4}x^{3}$$

Example 2: In this example we see that the Puiseux algorithm can be used to compute highly accurate fractional powers and expansions with imaginary and complex terms.

Figure 3: Newton Polygon for f(x, y)



$$f(x,y) = x^5 + 8x^4 - 2x^2y^2 - y^3 + 2y^4 = 0$$
 (1)

We know that this polynomial has solutions of the form

$$y = c_1 x^{\gamma_1} + c_2 x^{\gamma_1 + \gamma_2} + c_3 x^{\gamma_1 + \gamma_2 + \gamma_3} + \cdots$$
 (2)

 γ_1 corresponds to the negatives of the slope of the lower segment of the Newton Polygon (Figure 3). The slope of the lower segment of the Newton Polygon corresponding to (1) is $-\frac{4}{3}$. So, $\gamma_1 = -(-\frac{4}{3}) = \frac{4}{3}$.

Let $\gamma_1 = \frac{4}{3}$ Then by factoring x^{γ_1} (or $x^{\frac{4}{3}}$) out of (2) we obtain,

$$y = x^{\frac{4}{3}}(c_1 + y_1),\tag{3}$$

where c_1 is the first coefficient in the series and

$$y_1 = c_2 x^{\gamma_2} + c_3 x^{\gamma_2 + \gamma_3} + c_4 x^{\gamma_2 + \gamma_3 + \gamma_4} + \cdots$$
 (4)

i.e. the rest of the series.

So, substituting (3) into (1) we get,

$$x^{5} + 8x^{4} - 2x^{\frac{14}{3}}(c_{1} + y_{1})^{2} - x^{4}(c_{1} + y_{1})^{3} + 2x^{\frac{16}{3}}(c_{1} + y_{1})^{4} = 0$$
 (5)

The vertical intercept of the segment we use to find γ_1 on the Newton Polygon, which we call β , gives the terms of lowest degree in x alone. These terms must sum to zero because f(x,y) = 0. In this case, $\beta_1 = 4$ so, $8x^4 - c_1^3x^4 = 0$ and solving for c_1 we obtain

$$c_1 = 2, -1 + i\sqrt{3}, -1 - i\sqrt{3} \tag{6}$$

We assume $c_1 = 2$, but we must consider the other two cases later. Substituting c_1 into (5) and dividing by x^4 gives,

$$x^{-4}f(x, x^{2}(c_{1} + y_{1})) = f_{1}(x, y_{1})$$

$$= x - 8x^{\frac{2}{3}} - 8x^{\frac{2}{3}}y_{1} - 2x^{\frac{2}{3}}y_{1}^{2} - 12y_{1} - 6y_{1}^{2} - y_{1}^{3} + 32x^{\frac{4}{3}}$$

$$+ 64x^{\frac{4}{3}}y_{1} + 48x^{\frac{4}{3}}y_{1}^{2} + 16x^{\frac{4}{3}}y_{1}^{3} + 2x^{\frac{4}{3}}y_{1}^{4}$$

$$(7)$$

We call this new polynomial $f_1(x, y_1)$. So, now that we have our new function $f_1(x, y_1)$ our new goal is to find c_2 . We find the new Newton Polygon (Figure 4), from which we get $\beta_2 = \frac{2}{3}$ and $\gamma_2 = \frac{2}{3}$. Similar to (3), we factor x^{γ_2} out of (2) and obtain

$$y_1 = x^{\frac{2}{3}}(c_2 + y_2) \tag{8}$$

where c_2 is the second coefficient in the series and

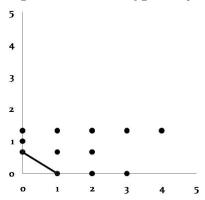
$$y_2 = c_3 x^{\gamma_3} + c_4 x^{\gamma_3 + \gamma_4} + c_5 x^{\gamma_3 + \gamma_4 + \gamma_5} + \cdots$$

Substituting (8) into f_1 we see that

$$0 = x - 8x^{\frac{2}{3}} - 8x^{\frac{4}{3}}(c_2 + y_2) - 2x^2(c_2 + y_2)^2 - 12x^{\frac{2}{3}}(c_2 + y_2) - 6x^43(c_2 + y_2)^2 - x^2(c_2 + y_2)^3 + 32x^{\frac{4}{3}} + 64x^2(c_2 + y_2) + 48x^{\frac{8}{3}}(c_2 + y_2)^2 + 16x^{\frac{10}{3}}(c_2 + y_2)^3 + 2x^4(c_2 + y_2)^4$$

$$(9)$$

Figure 4: Newton Polygon for f_1



 β_2 gives us the terms of lowest degree in x alone, which we can again set equal to zero.

$$-8x^{\frac{2}{3}} - 12c_2x^{\frac{2}{3}} = 0$$

so, $c_2 = -\frac{2}{3}$

Substituting c_2 into (9) and dividing by $x^{\frac{2}{3}}$ we see

$$x^{-\frac{2}{3}}f_{2}(x,x^{\frac{2}{3}}(c_{2}+y_{2})) = x^{\frac{1}{3}} + \frac{32}{81}x^{\frac{10}{3}} - \frac{1168}{27}x^{\frac{4}{3}} - \frac{128}{27}x^{\frac{8}{3}} + \frac{64}{3}x^{2} + \frac{104}{3}x^{\frac{2}{3}} + \frac{104}{$$

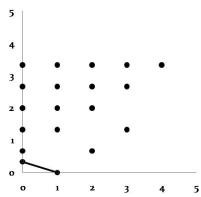
Now, we can see from Figure 5 that $\gamma_3 = \frac{1}{3}$, so

$$y_2 = x^{\frac{1}{3}}(c_3 + y_3) \tag{11}$$

where c_3 is the third coefficient in the series and

$$y_3 = c_4 x^{\gamma_4} + c_5 x^{\gamma_4 + \gamma_5} + c_6 x^{\gamma_4 + \gamma_5 + \gamma_6} + \cdots$$

Figure 5: Newton Polygon for f_2



We substitute y_2 into f_2 and find

$$0 = x^{\frac{1}{3}} + \frac{32}{81}x^{\frac{10}{3}} - \frac{1168}{27}x^{\frac{4}{3}} - \frac{128}{27}x^{\frac{8}{3}} + \frac{64}{3}x^{2} + \frac{104}{3}x^{\frac{2}{3}} + \frac{196}{3}x^{\frac{5}{3}}(c_{3} + y_{3})$$

$$- 12x^{\frac{1}{3}}(c_{3} + y_{3}) - 6x^{\frac{4}{3}}(c_{3} + y_{3})^{2} - x^{\frac{7}{3}}(c_{3} + y_{3})^{3} - 64x^{\frac{7}{3}}(c_{3} + y_{3})$$

$$+ 48x^{\frac{8}{3}}(c_{3} + y_{3})^{2} + \frac{64}{3}x^{3}(c_{3} + y_{3}) - 32x^{\frac{10}{3}}(c_{3} + y_{3})^{2} + 16x^{\frac{11}{3}}(c_{3} + y_{3})^{3}$$

$$- \frac{64}{27}x^{\frac{11}{3}}(c_{3} + y_{3}) + \frac{16}{3}x^{4}(c_{3} + y_{3})^{2} - \frac{16}{3}x^{\frac{13}{3}}(c_{3} + y_{3})^{3} + 2x^{\frac{14}{3}}(c_{3} + y_{3})^{4}$$

$$(12)$$

Since $f_2 = 0$ the $x^{\frac{1}{3}}$ terms must also equal zero, so we can let

$$x^{\frac{1}{3}} - 12c_3x^{\frac{1}{3}} = 0$$

so, $c_3 = \frac{1}{12}$

Substituting c_3 into (12) and dividing by $x^{\frac{1}{3}}$ we see that

$$x^{-\frac{1}{3}}f_{3}(x,x^{\frac{1}{3}}(c_{3}+y_{3})) = -\frac{9353}{216}x - \frac{1}{324}x^{4} - \frac{9217}{1728}x^{2} + \frac{14}{81}x^{3} + \frac{1}{27}x^{\frac{11}{3}} + \frac{16}{9}x^{\frac{8}{3}}$$

$$-\frac{61}{324}x^{\frac{10}{3}} - \frac{119}{27}x^{\frac{7}{3}} + \frac{49}{9}x^{\frac{4}{3}} + \frac{104}{3}x^{\frac{1}{3}} - 12y_{3} + \frac{64}{3}x^{\frac{5}{3}} + \frac{1}{10368}x^{\frac{13}{3}}$$

$$-\frac{16}{3}x^{3}y_{3} - \frac{1}{9}x^{4}y_{3} - \frac{4}{3}x^{4}y_{3}^{2} + \frac{196}{3}x^{\frac{4}{3}}y_{3} + 8x^{\frac{7}{3}}y_{3} + 48x^{\frac{7}{3}}y_{3}^{2} + \frac{64}{3}x^{\frac{8}{3}}y_{3}$$

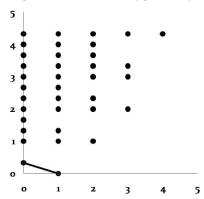
$$-\frac{55}{27}x^{\frac{10}{3}}y_{3} + 4x^{\frac{10}{3}}y_{3}^{2} + \frac{8}{9}x^{\frac{11}{3}}y_{3} + \frac{16}{3}x^{\frac{11}{3}}y_{3}^{2} + \frac{1}{216}x^{\frac{13}{3}}y_{3} + \frac{1}{12}x^{\frac{13}{3}}y_{3}^{2}$$

$$+\frac{2}{3}x^{\frac{13}{3}}y_{3}^{3} - xy_{3} - 6xy_{3}^{2} - \frac{3073}{48}x^{2}y_{3} - \frac{1}{4}x^{2}y_{3}^{2} - x^{2}y_{3}^{3} - 32x^{3}y_{3}^{2}$$

$$+16x^{\frac{10}{3}}y_{3}^{3} - \frac{16}{3}x^{4}y_{3}^{3} + 2x^{\frac{13}{3}}y_{3}^{4}$$

$$(13)$$

Figure 6: Newton Polygon for f_3



Now, we can see from Figure 6 that $\gamma_4 = \frac{1}{3}$ so we have

$$y_3 = x^{\frac{1}{3}}(c_4 + y_4)$$

where c_4 is the next coefficient in the series and y_4 is the rest of the series. Substituting y_3 into f_3 gives

$$0 = \frac{2}{3}x^{\frac{16}{3}}(c_4 + y_4)^3 - x^3(c_4 + y_4)^3 + \frac{64}{3}x^3(c_4 + y_4) - \frac{3073}{48}x^{\frac{7}{3}}(c_4 + y_4)$$

$$- 32x^{\frac{11}{3}}(c_4 + y_4)^2 - 6x^{\frac{5}{3}}(c_4 + y_4)^2 + \frac{16}{3}x^{\frac{13}{3}}(c_4 + y_4)^2 - \frac{4}{3}x^{\frac{14}{3}}(c_4 + y_4)^2$$

$$- \frac{1}{9}x^{\frac{13}{3}}(c_4 + y_4) + 2x^{\frac{17}{3}}(c_4 + y_4)^4 - \frac{1}{4}x^{\frac{8}{3}}(c_4 + y_4)^2 - 12x^{13}(c_4 + y_4)$$

$$+ 48x^{3}(c_4 + y_4)^2 + 4x^{4}(c_4 + y_4)^2 - \frac{9353}{216}x + 16x^{\frac{13}{3}}(c_4 + y_4)^3$$

$$+ 8x^{\frac{8}{3}}(c_4 + y_4) - \frac{16}{3}x^{\frac{10}{3}}(c_4 + y_4) + \frac{196}{3}x^{\frac{5}{3}}(c_4 + y_4) + \frac{1}{216}x^{\frac{14}{3}}(c_4 + y_4)$$

$$- \frac{16}{3}x^{5}(c_4 + y_4)^3 + \frac{1}{12}x^{5}(c_4 + y_4)^2 - \frac{55}{27}x^{\frac{11}{3}}(c_4 + y_4) + \frac{8}{9}x^{4}(c_4 + y_4)$$

$$- x^{\frac{4}{3}}(c_4 + y_4) - \frac{1}{324}x^4 - \frac{9217}{1728}x^2 + \frac{14}{81}x^3 + \frac{1}{27}x^{\frac{11}{3}} + \frac{16}{9}x^{\frac{8}{3}} - \frac{61}{324}x^{\frac{10}{3}}$$

$$- \frac{119}{27}x^{\frac{7}{3}} + \frac{49}{9}x^{\frac{4}{3}} + \frac{104}{3}x^{\frac{1}{3}} + \frac{64}{3}x^{\frac{5}{3}} + \frac{1}{10368}x^{\frac{13}{3}}$$

$$(14)$$

From the Newton Polygon (Figure 6), $\beta_4 = \frac{1}{3}$, so we let the $x^{\frac{1}{3}} = 0$ and solve for c_4 .

$$\frac{104}{3}x^{\frac{1}{3}} - 12c_4x^{\frac{1}{3}} = 0$$
so, $c_4 = \frac{26}{9}$

Substituting c_4 into (14) and dividing by $x^{\frac{1}{3}}$

$$x^{-\frac{1}{3}}f_{4}(x, x^{\frac{1}{3}}(c_{4} + y_{4})) = \frac{23}{9}x - \frac{4}{3}x^{\frac{13}{3}}y_{4}^{2} - \frac{5047}{27}x^{\frac{10}{3}}y_{4} + \frac{35152}{2187}x^{5}$$

$$+ \frac{40119049}{93312}x^{4} - \frac{40901}{216}x^{2} - \frac{5053}{324}x^{3} + 144x^{4}y_{4}^{2} + \frac{11647}{324}x^{\frac{11}{3}}$$

$$+ \frac{319510}{729}x^{\frac{8}{3}} - \frac{66317}{243}x^{\frac{10}{3}} + \frac{1847}{81}x^{\frac{7}{3}} + 160x^{\frac{4}{3}} - \frac{9353}{216}x^{\frac{2}{3}} + \frac{52}{9}x^{5}y_{4}^{2}$$

$$+ \frac{1352}{81}x^{5}y_{4} - \frac{16}{3}x^{3}y_{4} + \frac{11645}{27}x^{4}y_{4} + \frac{2}{3}x^{5}y_{4}^{3} - \frac{16}{3}x^{\frac{14}{3}}y_{4}^{3} + 2x^{\frac{16}{3}}y_{4}^{4}$$

$$- xy_{4} - \frac{3073}{48}x^{2}y_{4} + 16x^{4}y_{4}^{3} - 6x^{\frac{4}{3}}y_{4}^{2} - \frac{1}{4}x^{\frac{7}{3}}y_{4}^{2} - 32x^{\frac{10}{3}}y_{4}^{2}$$

$$- \frac{279695}{2187}x^{\frac{14}{3}} + \frac{2704}{27}x^{\frac{16}{3}}y_{4}^{2} + \frac{140608}{729}x^{\frac{16}{3}}y_{4} + 24x^{\frac{11}{3}}y_{4} + \frac{92}{3}x^{\frac{4}{3}}y_{4}$$

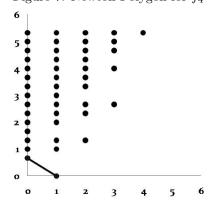
$$- \frac{1663}{216}x^{\frac{13}{3}}y_{4} - 12y_{4} - \frac{9217}{1728}x^{\frac{5}{3}} - \frac{3601}{324}x^{\frac{13}{3}} + \frac{59}{9}x^{\frac{7}{3}}y_{4} + \frac{208}{9}x^{\frac{16}{3}}y_{4}^{3}$$

$$+ \frac{913952}{6561}x^{\frac{16}{3}} + \frac{118}{3}x^{\frac{8}{3}}y_{4}^{2} + \frac{7388}{27}x^{\frac{8}{3}}y_{4} - \frac{10777}{81}x^{\frac{14}{3}}y_{4} - \frac{1661}{36}x^{\frac{14}{3}}y_{4}^{2}$$

$$+ 4x^{\frac{11}{3}}y_{4}^{2} - x^{\frac{8}{3}}y_{4}^{3}$$

$$(15)$$

Figure 7: Newton Polygon for f_4



It can be seen from the new Newton Polygon (Figure 7) that $\gamma_5 = \frac{2}{3}$, so

$$y_4 = x^{\frac{2}{3}}(c_5 + y_5)$$

Substituting y_4 into f_4 we find that

$$0 = \frac{23}{9}x + 4x^{5}(c_{5} + y_{5})^{2} + \frac{92}{3}x^{2}(c_{5} + y_{5}) + \frac{35152}{2187}x^{5} + \frac{40119049}{93312}x^{4}$$

$$- \frac{40901}{216}x^{2} - \frac{5053}{324}x^{3} - 12x^{\frac{2}{3}}(c_{5} + y_{5}) + \frac{11647}{324}x^{\frac{11}{3}} + \frac{319510}{729}x^{\frac{8}{3}}$$

$$- \frac{66317}{243}x^{\frac{10}{3}} + \frac{1847}{81}x^{\frac{7}{3}} + 160x^{\frac{4}{3}} - \frac{9353}{216}x^{\frac{2}{3}} - \frac{279695}{2187}x^{\frac{14}{3}}$$

$$+ 144x^{\frac{16}{3}}(c_{5} + y_{5})^{2} - 6x^{\frac{8}{3}}(c_{5} + y_{5})^{2} - x^{\frac{14}{3}}(c_{5} + y_{5})^{3} - \frac{1663}{216}x^{5}(c_{5} + y_{5})$$

$$- \frac{16}{3}x^{\frac{11}{3}}(c_{5} + y_{5}) - \frac{1}{4}x^{\frac{11}{3}}(c_{5} + y_{5})^{2} + 24x^{\frac{13}{3}}(c_{5} + y_{5})$$

$$- \frac{10777}{81}x^{\frac{16}{3}}(c_{5} + y_{5}) + \frac{2}{3}x^{7}(c_{5} + y_{5})^{3} + \frac{1352}{81}x^{\frac{17}{3}}(c_{5} + y_{5})$$

$$+ \frac{140608}{729}x^{6}(c_{5} + y_{5}) - 32x^{\frac{14}{3}}(c_{5} + y_{5})^{2} - \frac{3073}{48}x^{\frac{8}{3}}(c_{5} + y_{5})$$

$$+ \frac{59}{9}x^{3}(c_{5} + y_{5}) + \frac{52}{9}x^{\frac{19}{3}}(c_{5} + y_{5})^{2} + \frac{208}{9}x^{\frac{22}{3}}(c_{5} + y_{5})^{3} - x^{\frac{5}{3}}(c_{5} + y_{5})$$

$$+ 2x^{8}(c_{5} + y_{5})^{4} + \frac{118}{3}x^{4}(c_{5} + y_{5})^{2} - \frac{1661}{36}x^{6}(c_{5} + y_{5})^{2} + 16x^{6}(c_{5} + y_{5})^{3}$$

$$+ \frac{2704}{27}x^{\frac{20}{3}}(c_{5} + y_{5})^{2} + \frac{11645}{27}x^{\frac{14}{3}}(c_{5} + y_{5}) - \frac{5047}{27}x^{4}(c_{5} + y_{5})$$

$$+ \frac{7388}{27}x^{\frac{10}{3}}(c_{5} + y_{5}) - \frac{16}{3}x^{\frac{20}{3}}(c_{5} + y_{5})^{3} - \frac{4}{3}x^{\frac{17}{3}}(c_{5} + y_{5})^{2} - \frac{9217}{1728}x^{\frac{5}{3}}$$

$$- \frac{3601}{324}x^{\frac{13}{3}} + \frac{913952}{6561}x^{\frac{16}{3}}$$
(16)

We can see from the Newton Polygon (Figure 7) that $\beta_5 = \frac{2}{3}$, so we can set the $x^{\frac{2}{3}}$ terms equal zero and solve for c_5 .

$$\frac{9353}{216}x^{\frac{2}{3}} - 12c_5x^{\frac{2}{3}} = 0$$
so, $c_5 = \frac{9353}{2592}$

This process could be continued to calculate more terms, but we will stop here. So, an explicit solution for f(x,y) is

$$y = \frac{9353}{2592}x^{\frac{10}{3}} + \frac{26}{9}x^{\frac{8}{3}} + \frac{1}{12}x^{\frac{7}{3}} + \frac{2}{3}x^2 + 2x^{\frac{4}{3}}$$

We must now examine the case where the solution to c_1 is imaginary. There are two imaginary solutions, but we will only show the process for one of these cases because they are quite similar. Let $c_1 = -1 + \sqrt{2}i$. We substitute c_1 into

(5) and divide by x^4 .

$$x^{-4}f_1(x, x^2(c_1 + y_1)) = f_1(x, y_1)$$

$$= x + 4i\sqrt{3}x^{\frac{2}{3}} - 3i\sqrt{3}y_1^2 - 16x^{\frac{4}{3}} - 24x^{\frac{4}{3}}y_1^2 - 8x^{\frac{4}{3}}y_1^3$$

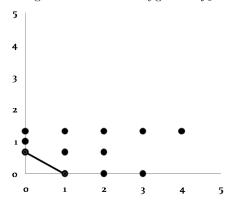
$$+ 2x^{\frac{4}{3}}y_1^4 + 64x^{\frac{4}{3}}y_1 - 24ix^{\frac{4}{3}}3^{\frac{1}{2}}y_1^2 + 6i\sqrt{3}y_1 - 4i\sqrt{3}x^{\frac{2}{3}}y_1$$

$$+ 4x^{\frac{2}{3}}y_1 - 2x^{\frac{2}{3}}y_1^2 + 16i\sqrt{3}x^{\frac{4}{3}} + 8i\sqrt{3}x^{\frac{4}{3}}y_1^3 + 6y_1 + 3y_1^2$$

$$- y_1^3 + 4x^{\frac{2}{3}}$$

$$(17)$$

Figure 8: Newton Polygon for f_1



We can see from the new Newton Polygon (Figure 8) that $\gamma_2 = \frac{2}{3}$, so

$$y_1 = x^{\frac{2}{3}}(c_2 + y_2)$$

where c_2 is the second coefficient in the series and

$$y_2 = c_3 x^{\gamma_3} + c_4 x^{\gamma_3 + \gamma_4} + c_5 x^{\gamma_3 + \gamma_4 + \gamma_5} + \cdots$$

We substitute y_1 into f_1 to get

$$0 = x + 4i\sqrt{3}x^{\frac{2}{3}} - 4i\sqrt{3}x^{\frac{4}{3}}(c_2 + y_2) - 16x^{\frac{4}{3}} - 24x^{\frac{8}{3}}(c_2 + y_2)^2 - 8x^{\frac{10}{3}}(c_2 + y_2)^3$$

$$+ 2x^4(c_2 + y_2)^4 + 64x^2(c_2 + y_2) + 6i\sqrt{3}x^{\frac{2}{3}}(c_2 + y_2) - 3i\sqrt{3}x^{\frac{4}{3}}(c_2 + y_2)^2$$

$$- 24i\sqrt{3}x^{\frac{8}{3}}(c_2 + y_2)^2 + 4x^{\frac{4}{3}}(c_2 + y_2) - 2x^2(c_2 + y_2)^2 + 16i\sqrt{3}x^{\frac{4}{3}}$$

$$+ 8i\sqrt{3}x^{\frac{10}{3}}(c_2 + y_2)^3 + 6x^{\frac{2}{3}}(c_2 + y_2) + 3x^{\frac{4}{3}}(c_2 + y_2)^2 - x^2(c_2 + y_2)^3 + 4x^{\frac{2}{3}}$$

$$(18)$$

From the Newton Polygon (Figure 8), $\beta_2 = \frac{2}{3}$, so we can let the $x^{\frac{2}{3}}$ terms equal zero and solve for c_2 .

$$6c_2x^{\frac{2}{3}} + 6i\sqrt{3}c_2x^{\frac{2}{3}} + 4i\sqrt{3}x^{\frac{2}{3}} + 4x^{\frac{2}{3}} = 0$$

so, $c_2 = -\frac{2}{3}$

Now, substituting c_2 into (18) and dividing by $x^{\frac{2}{3}}$ gives

$$x^{-\frac{2}{3}}f_{2}(x,x^{\frac{2}{3}}(c_{2}+y_{2})) = -\frac{32}{3}x^{2} + \frac{64}{27}x^{\frac{8}{3}} + \frac{32}{81}x^{\frac{10}{3}} - \frac{1168}{27}x^{\frac{4}{3}} + x^{\frac{1}{3}} - \frac{52}{3}x^{\frac{2}{3}}$$

$$+ 6y_{2} + 32x^{2}y_{2} - 24x^{2}y_{2}^{2} - \frac{16}{3}x^{\frac{10}{3}}y_{2}^{3} - \frac{64}{27}x^{\frac{10}{3}}y_{2} + \frac{16}{3}x^{\frac{10}{3}}y_{2}^{2}$$

$$- \frac{32}{3}x^{\frac{8}{3}}y_{2} + 16x^{\frac{8}{3}}y_{2}^{2} + \frac{196}{3}x^{\frac{4}{3}}y_{2} + 3x^{\frac{2}{3}}y_{2}^{2} - x^{\frac{4}{3}}y_{2}^{3} - 8x^{\frac{8}{3}}y_{2}^{3}$$

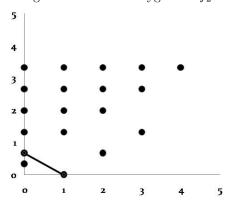
$$+ 2x^{\frac{10}{3}}y_{2}^{4} - 3i\sqrt{3}x^{\frac{2}{3}}y_{2}^{2} - 24i\sqrt{3}x^{2}y_{2}^{2} + 32i\sqrt{3}x^{2}y_{2} + \frac{32}{3}i\sqrt{3}x^{\frac{8}{3}}y_{2}$$

$$+ 8i\sqrt{3}x^{\frac{8}{3}}y_{2}^{3} - 16i\sqrt{3}x^{\frac{8}{3}}y_{2}^{2} + \frac{52}{3}i\sqrt{3}x^{\frac{2}{3}} - \frac{64}{27}i\sqrt{3}x^{\frac{8}{3}} - \frac{32}{3}i\sqrt{3}x^{2}$$

$$+ 6i\sqrt{3}y_{2}$$

$$(19)$$

Figure 9: Newton Polygon for f_2



From Figure 9 we see that $\gamma_3 = \frac{1}{3}$, so

$$y_2 = x^{\frac{1}{3}}(c_3 + y_3)$$

where c_3 is the next coefficient in the series and y_3 is the rest of the series.

Substituting y_2 into f_2 gives

$$0 = \frac{16}{3}x^{4}(c_{3} + y_{3})^{2} - \frac{32}{3}x^{2} + \frac{64}{27}x^{\frac{8}{3}} + \frac{32}{81}x^{\frac{10}{3}} - \frac{1168}{27}x^{\frac{4}{3}} + x^{\frac{1}{3}}$$

$$- \frac{52}{3}x^{\frac{2}{3}} + 6x^{\frac{1}{3}}(c_{3} + y_{3}) - x^{\frac{7}{3}}(c_{3} + y_{3})^{3} - \frac{64}{27}x^{\frac{11}{3}}(c_{3} + y_{3}) - \frac{32}{3}x^{3}(c_{3} + y_{3})$$

$$- 24x^{\frac{8}{3}}(c_{3} + y_{3})^{2} + 2x^{\frac{14}{3}}(c_{3} + y_{3})^{4} - 8x^{\frac{11}{3}}(c_{3} + y_{3})^{3} + \frac{196}{3}x^{\frac{5}{3}}(c_{3} + y_{3})$$

$$+ 16x^{\frac{10}{3}}(c_{3} + y_{3})^{2} + 32x^{\frac{7}{3}}(c_{3} + y_{3}) + 3x^{\frac{4}{3}}(c_{3} + y_{3})^{2} - \frac{16}{3}x^{\frac{13}{3}}(c_{3} + y_{3})^{3}$$

$$+ 6i\sqrt{3}x^{\frac{1}{3}}(c_{3} + y_{3}) + \frac{52}{3}i\sqrt{3}x^{\frac{2}{3}} - \frac{64}{27}i\sqrt{3}x^{\frac{8}{3}} - \frac{32}{3}i\sqrt{3}x^{2}$$

$$+ \frac{32}{3}i\sqrt{3}x^{3}(c_{3} + y_{3}) + 8i\sqrt{3}x^{\frac{11}{3}}(c_{3} + y_{3})^{3} - 16i\sqrt{3}x^{\frac{10}{3}}(c_{3} + y_{3})^{2}$$

$$- 24i\sqrt{3}x^{\frac{8}{3}}(c_{3} + y_{3})^{2} - 3i\sqrt{3}x^{\frac{4}{3}}(c_{3} + y_{3})^{2} + 32i\sqrt{3}x^{\frac{7}{3}}(c_{3} + y_{3})$$
(20)

We can see from the Newton Polygon (Figure 9) that $\beta_3 = \frac{1}{3}$, so we can let the $x^{\frac{1}{3}}$ terms equal zero and solve for c_3 .

$$6c_3x^{\frac{1}{3}} + 1x^{\frac{1}{3}} + 6i\sqrt{3}c_3x^{\frac{1}{3}} = 0$$

so, $c_3 = \frac{1}{6(\sqrt{3}+1)}$

Substituting c_3 into (20) and dividing by $x^{\frac{1}{3}}$ gives us

$$x^{-\frac{1}{3}}f_3(x,x^{\frac{1}{3}}(c_3+y_3)) = -\frac{128i\sqrt{3}x^{\frac{13}{3}}y_3^2}{3(i\sqrt{3}+1)^4} + \frac{8i\sqrt{3}x^2y_3^3}{(i\sqrt{3}+1)^4} - \frac{64i\sqrt{3}x^{\frac{7}{3}}y_3}{(i\sqrt{3}+1)^4}$$

$$-\frac{8i\sqrt{3}y_3x}{(i\sqrt{3}+1)^4} - \frac{128x^{\frac{8}{3}}}{9(i\sqrt{3}+1)^4} - \frac{244x^{\frac{19}{3}}}{27(i\sqrt{3}+1)^4} - \frac{1904x^{\frac{7}{3}}}{27(i\sqrt{3}+1)^4}$$

$$-\frac{1568x^{\frac{4}{3}}y_3}{3(i\sqrt{3}+1)^4} - \frac{128x^{\frac{13}{3}}y_3^2}{3(i\sqrt{3}+1)^4} + \frac{128x^{\frac{11}{3}}y_3}{9(i\sqrt{3}+1)^4} - \frac{96xy_3^2}{(i\sqrt{3}+1)^4}$$

$$+\frac{3073x^2y_3}{6(i\sqrt{3}+1)^4} + \frac{96y_3}{(i\sqrt{3}+1)^4} + \frac{8xy_4}{(i\sqrt{3}+1)^4} - \frac{4x^2y_3^2}{(i\sqrt{3}+1)^4}$$

$$+\frac{8x^2y_3^3}{(i\sqrt{3}+1)^4} - \frac{64x^{\frac{7}{3}}y_3}{(i\sqrt{3}+1)^4} - \frac{8x^{\frac{13}{3}}}{27(i\sqrt{3}+1)^4} - \frac{8i\sqrt{3}x^4y_3}{9(i\sqrt{3}+1)^4}$$

$$+\frac{128i\sqrt{3}x^4y_3^3}{3(i\sqrt{3}+1)^4} + \frac{x^{\frac{13}{3}}}{648(i\sqrt{3}+1)^4} + \frac{2x^4}{81(i\sqrt{3}+1)^4} + \frac{784x^{\frac{4}{3}}}{9(i\sqrt{3}+1)^4}$$

$$-\frac{512x^3y_3^2}{(i\sqrt{3}+1)^4} + \frac{440x^{\frac{19}{3}}y_3}{27(i\sqrt{3}+1)^4} - \frac{384x^{\frac{7}{3}}y_3^2}{(i\sqrt{3}+1)^4} + \frac{128x^3y_3}{3(i\sqrt{3}+1)^4}$$

$$-\frac{64x^4y_3^2}{3(i\sqrt{3}+1)^4} + \frac{128x^4y_3^3}{3(i\sqrt{3}+1)^4} - \frac{32x^{\frac{13}{3}}y_3^2}{(i\sqrt{3}+1)^4} + \frac{256x^{\frac{19}{3}}y_3^3}{(i\sqrt{3}+1)^4}$$

$$+\frac{8x^4y_3}{9(i\sqrt{3}+1)^4} - \frac{x^{\frac{13}{3}}}{27(i\sqrt{3}+1)^4} - \frac{16x^{\frac{13}{3}}}{i\sqrt{3}+1)^4} + \frac{32x^{\frac{13}{3}}y_3^3}{3(i\sqrt{3}+1)^4}$$

$$-\frac{2x^{\frac{13}{3}}y_3^2}{3(i\sqrt{3}+1)^4} + \frac{2i\sqrt{3}x^4}{81(i\sqrt{3}+1)^4} + \frac{128i\sqrt{3}x^{\frac{8}{3}}}{9(i\sqrt{3}+1)^4} + \frac{9217i\sqrt{3}x^2}{216(i\sqrt{3}+1)^4}$$

$$+\frac{512i\sqrt{3}x^{\frac{5}{3}}}{3(i\sqrt{3}+1)^4} + \frac{2i\sqrt{3}x^{\frac{13}{3}}y_3^3}{27(i\sqrt{3}+1)^4} - \frac{112i\sqrt{3}x^3}{81(i\sqrt{3}+1)^4} + \frac{8i\sqrt{3}x^{\frac{13}{3}}y_3}{216(i\sqrt{3}+1)^4}$$

$$-\frac{96i\sqrt{3}y_3}{(i\sqrt{3}+1)^4} + \frac{2i\sqrt{3}x^{\frac{13}{3}}y_3^3}{27(i\sqrt{3}+1)^4} - \frac{112i\sqrt{3}x^3}{81(i\sqrt{3}+1)^4} + \frac{8i\sqrt{3}x^{\frac{13}{3}}y_3}{(i\sqrt{3}+1)^4}$$

$$-\frac{96i\sqrt{3}y_3}{6(i\sqrt{3}+1)^4} - \frac{1568i\sqrt{3}x^{\frac{4}{3}}}{3(i\sqrt{3}+1)^4} + \frac{10i\sqrt{3}x^{\frac{19}{3}}y_3}{(i\sqrt{3}+1)^4} + \frac{9217i\sqrt{3}}{216(i\sqrt{3}+1)^4}$$

$$-\frac{128i\sqrt{3}x^3y_3}{3(i\sqrt{3}+1)^4} - \frac{112x^3}{81(i\sqrt{3}+1)^4} + \frac{1024x^{\frac{8}{3}}y_3}{(i\sqrt{3}+1)^4} + \frac{9217x^2}{216(i\sqrt{3}+1)^4}$$

$$-\frac{126i\sqrt{3}x^{\frac{13}{3}}y_3}{6(i\sqrt{3}+1)^4} - \frac{112x^3}{81(i\sqrt{3}+1)^4} + \frac{1024x^{\frac{8}{3}}y_3}{(i\sqrt{3}+1)^4} + \frac{$$

By continuing this process it can be found that $c_4 = \frac{52}{9(-1+i\sqrt{3})}$ and $c_5 = \frac{9353}{1296(1+i\sqrt{3})}$. So, a second explicit solution for f(x,y) is

$$y = -1 + i\sqrt{3}x^{\frac{4}{3}} - \frac{2}{3}x^2 - \frac{x^{\frac{7}{3}}}{6(i\sqrt{3}+1)} + \frac{52x^{\frac{8}{3}}}{9(-1+i\sqrt{3})} + \frac{9353x^{\frac{10}{3}}}{1296(i\sqrt{3}+1)}$$

Using a similar process, the explicit solution of f(x,y) in the case where $c_1 = -1 - i\sqrt{3}$ can be found. So, we have that the explicit solutions for f(x,y) are

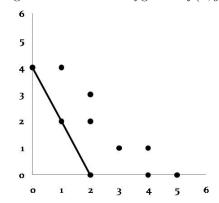
$$y = \frac{9353}{2592}x^{\frac{10}{3}} + \frac{26}{9}x^{\frac{8}{3}} + \frac{1}{12}x^{\frac{7}{3}} + \frac{2}{3}x^{2} + 2x^{\frac{4}{3}} + \cdots$$

$$y = -1 + i\sqrt{3}x^{\frac{4}{3}} - \frac{2}{3}x^{2} - \frac{x^{\frac{7}{3}}}{(6(i\sqrt{3}+1))} + \frac{52x^{\frac{8}{3}}}{9(-1+i\sqrt{3})} + \frac{9353x^{\frac{10}{3}}}{1296(i\sqrt{3}+1)} + \cdots$$

$$y = (-1 - i\sqrt{3})x^{\frac{4}{3}} - \frac{2}{3}x^{2} + \frac{x^{\frac{7}{3}}}{(6(-1+i\sqrt{3}))} - \frac{52x^{\frac{8}{3}}}{9(i\sqrt{3}+1)} - \frac{9353x^{10}3}{1296(-1+i\sqrt{3})} + \cdots$$

Example 3:

Figure 10: Newton Polygon for f(x,y)



In this example, we find that multiple terms of a Puiseux expansion can be calculated before the Puiseux jets differ. Here, the jets will appear the same for the first three terms and then split at $\frac{11}{2}$.

$$f(x,y) = y^2 + 2x^2y + x^4 + x^2y^2 + xy^3 + \frac{1}{4}y^4 + x^4y + x^3y^2 - \frac{1}{2}xy^4 - \frac{1}{2}y^5$$
 (1)

We know that this polynomial has solutions of the form

$$y = c_1 x^{\gamma_1} + c_2 x^{\gamma_1 + \gamma_2} + c_3 x^{\gamma_1 + \gamma_2 + \gamma_3} + \cdots$$
 (2)

The possible choices for γ_1 correspond to the negatives of the slopes of the lower segments of the Newton Polygon (Figure 10). The slopes of the lower segments of the Newton Polygon corresponding to (1) are both -2. We will start by considering this case where $\gamma_1 = -(-2) = 2$.

Let $\gamma_1 = 2$ Then by factoring x^{γ_1} (or x^2) out of (2) we obtain,

$$y = x^2(c_1 + y_1) (3)$$

where c_1 is the first coefficient and y_1 is

$$y_1 = c_2 x^{\gamma_2} + c_3 x^{\gamma_2 + \gamma_3} + c_4 x^{\gamma_2 + \gamma_3 + \gamma_4} + \cdots$$
 (4)

i.e. the rest of the series.

So, substituting (3) into f we get,

$$0 = x^{4}(c_{1} + y_{1})^{2} + 2x^{4}(c_{1} + y_{1}) + x^{4} + x^{6}(c_{1} + y_{1})^{2} + x^{7}(c_{1} + y_{1})^{3}$$

$$+ \frac{1}{4}x^{8}(c_{1} + y_{1})^{4} + x^{6}(c_{1} + y_{1}) + x^{7}(c_{1} + y_{1})^{2} - \frac{1}{2}x^{9}(c_{1} + y_{1})^{4}$$

$$- \frac{1}{2}x^{10}(c_{1} + y_{1})^{5}$$

$$(5)$$

The vertical intercept on the Newton Polygon (Figure 10), which we call β , gives the terms of lowest degree in x alone. These terms must sum to zero because f(x,y)=0. In this case, $\beta=4$ so, $x^4c_1^2+x^4c_1+x^4=0$. The x^n part of the terms can be divided out and will not be included in subsequent steps. Solving for c_1 we obtain $c_1=-1$ Substituting c_1 into (5) and dividing by x^4 gives,

$$x^{-4}f(x, x^{2}(c_{1} + y_{1})) = f_{1}(x, y_{1}) =$$

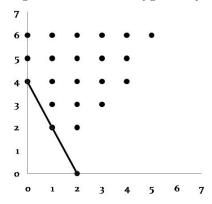
$$= -\frac{1}{2}x^{5} + \frac{1}{2}x^{6} + \frac{1}{4}x^{4} + y_{1}^{2} - x^{2}y_{1} + x^{2}y_{1}^{2} + x^{3}y_{1}$$

$$-2x^{3}y_{1}^{2} + x^{3}y_{1}^{3} - x^{4}y_{1} + \frac{3}{2}x^{4}y_{1}^{2} - x^{4}y_{1}^{3} + \frac{1}{4}x^{4}y_{1}^{4} + 2x^{5}y_{1} - 3x^{5}y_{1}^{2}$$

$$+2x^{5}y_{1}^{3} - \frac{1}{2}x^{5}y_{1}^{4} - \frac{5}{2}x^{6}y_{1} + 5x^{6}y_{1}^{2} - 5x^{6}y_{1}^{3} + \frac{5}{2}x^{6}y_{1}^{4} - \frac{1}{2}x^{6}y_{1}^{5}$$

$$(6)$$

Figure 11: Newton Polygon for f_1



We call this new polynomial $f_1(x, y_1)$, so now that we have our new function $f_1(x, y_1)$, and our new goal is to find c_2 . We find the new Newton Polygon (Figure 11), from which we get $\beta_2 = 4$ and $\gamma_2 = 2$. Similar to (3), we then have

$$y_1 = x^2(c_2 + y_2) (7)$$

where c_2 is the coefficient of the second term in the series and y_2 is

$$y_2 = c_3 x^{\gamma_3} + c_4 x^{\gamma_3 + \gamma_4} + \dots$$
(8)

We substitute (7) into f_1 to get

$$0 = -\frac{1}{2}x^{5} + \frac{1}{2}x^{6} + \frac{1}{4}x^{4} + x^{4}(c_{2} + y_{2})^{2} - x^{4}(c_{2} + y_{2}) + x^{6}(c_{2} + y_{2})^{2}$$

$$+ x^{5}(c_{2} + y_{2}) - 2x^{7}(c_{2} + y_{2})^{2} + x^{9}(c_{2} + y_{2})^{3} - x^{6}(c_{2} + y_{2})$$

$$+ \frac{3}{2}x^{8}(c_{2} + y_{2})^{2} - x^{10}(c_{2} + y_{2})^{3} + \frac{1}{4}x^{12}(c_{2} + y_{2})^{4} + 2x^{7}(c_{2} + y_{2})$$

$$- 3x^{9}(c_{2} + y_{2})^{2} + 2x^{11}(c_{2} + y_{2})^{3} - \frac{1}{2}x^{13}(c_{2} + y_{2})^{4} - \frac{5}{2}x^{8}(c_{2} + y_{2})$$

$$+ 5x^{10}(c_{2} + y_{2})^{2} - 5x^{12}(c_{2} + y_{2})^{3} + \frac{5}{2}x^{14}(c_{2} + y_{2})^{4} - \frac{1}{2}x^{16}(c_{2} + y_{2})^{5}$$
 (9)

Again, lowest terms must cancel so, since $\beta_2 = 4$, we see that $\frac{1}{4} + c_2^2 - c_2 = 0$.

Solving for c_2 gives us $c_2 = \frac{1}{2}$, and we substitute this c_2 into (9) and divide by $x^{-\beta}$ giving

$$x^{-4}f(x,x^{2}(c_{2}+y_{2})) = f_{2}(x,y_{2}) =$$

$$= -\frac{5}{8}x^{5} + \frac{9}{8}x^{6} + \frac{1}{4}x^{7} - \frac{1}{32}x^{9} - \frac{39}{64}x^{8} - \frac{1}{64}x^{12}$$

$$+ \frac{5}{32}x^{10} + \frac{1}{4}x^{2} - \frac{7}{8}x^{4} + \frac{1}{2}x^{3} + y_{2}^{2} + x^{2}y_{2}^{2} + xy_{2} - 2x^{3}y_{2}^{2} - \frac{9}{4}x^{5}y_{2}$$

$$- \frac{3}{2}x^{5}y_{2}^{2} + x^{5}y_{2}^{3} - x^{4}y_{2} + \frac{3}{2}x^{4}y_{2}^{2} + \frac{17}{4}x^{6}y_{2} + \frac{7}{2}x^{6}y_{2}^{2} - x^{6}y_{2}^{3} - \frac{29}{8}x^{8}y_{2}$$

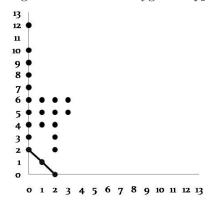
$$- \frac{57}{8}x^{8}y_{2}^{2} - \frac{9}{2}x^{8}y_{2}^{3} + \frac{1}{4}x^{8}y_{2}^{4} + \frac{3}{2}x^{7}y_{2} + 3x^{7}y_{2}^{2} + 2x^{7}y_{2}^{3} - \frac{1}{4}x^{9}y_{2}$$

$$- \frac{3}{4}x^{9}y_{2}^{2} - x^{9}y_{2}^{3} - \frac{1}{2}x^{9}y_{2}^{4} + \frac{5}{4}x^{10}y_{2} + \frac{15}{4}x^{10}y_{2}^{2} + 5x^{10}y_{2}^{3} + \frac{5}{2}x^{10}y_{2}^{4}$$

$$- \frac{5}{32}x^{12}y_{2} - \frac{5}{8}x^{12}y_{2}^{2} - \frac{5}{4}x^{12}y_{2}^{3} - \frac{5}{4}x^{12}y_{2}^{4} - \frac{1}{2}x^{12}y_{2}^{5}$$

$$(10)$$

Figure 12: Newton Polygon for f_2



We must now find c_3 . From the Newton Polygon for f_2 (Figure 12), we can see that $\beta_3 = 2$ and $\gamma_3 = 1$. Therefore,

$$y_2 = x(c_3 + y_3) (11)$$

where c_3 is the coefficient of the third term in the series and y_3 is

$$y_3 = c_4 x^{\gamma_4} + c_5 x^{\gamma_4 + \gamma_5} + \dots \tag{12}$$

Substituting (11) into f_2 we get

$$0 = \frac{15}{4}x^{12}(c_3 + y_3)^2 - \frac{57}{8}x^{10}(c_3 + y_3)^2 - 298x^9(c_3 + y_3) - x^{12}(c_3 + y_3)^3$$

$$- 2x^5(c_3 + y_3)^2 - \frac{3}{4}x^{11}(c_3 + y_3)^2 - x^9(c_3 + y_3)^3 + x^2(c_3 + y_3)$$

$$+ \frac{3}{2}x^8(c_3 + y_3) - \frac{5}{4}x^{16}(c_3 + y_3)^4 - \frac{3}{2}x^7(c_3 + y_3)^2 + \frac{7}{2}x^8(c_3 + y_3)^2$$

$$- \frac{1}{4}x^{10}(c_3 + y_3) + \frac{5}{2}x^{14}(c_3 + y_3)^4 - \frac{5}{32}x^{13}(c_3 + y_3) - \frac{1}{2}x^{13}(c_3 + y_3)^4$$

$$+ \frac{5}{4}x^{11}(c_3 + y_3) + 3x^9(c_3 + y_3)^2 + \frac{1}{4}x^{12}(c_3 + y_3)^4 + x^8(c_3 + y_3)^3$$

$$- x^5(c_3 + y_3) + \frac{3}{2}x^6(c_3 + y_3)^2 - \frac{1}{2}x^{17}(c_3 + y_3)^5 - \frac{5}{8}x^{14}(c_3 + y_3)^2$$

$$- \frac{9}{2}x^{11}(c_3 + y_3)^3 + \frac{1}{4}x^2 - \frac{7}{8}x^4 + \frac{1}{2}x^3 - \frac{9}{4}x^6(c_3 + y_3) - \frac{5}{4}x^{15}(c_3 + y_3)^3$$

$$+ 2x^{10}(c_3 + y_3)^3 - \frac{5}{8}x^5 + \frac{9}{8}x^6 + \frac{1}{4}x^7 - \frac{1}{32}x^9 - \frac{39}{64}x^8 + \frac{17}{4}x^7(c_3 + y_3)$$

$$- \frac{1}{64}x^{12} + \frac{5}{32}x^{10} + x^4(c_3 + y_3)^2 + x^2(c_3 + y_3)^2 + 5x^{13}(c_3 + y_3)^3$$
 (13)

Since $\beta_3 = 2$ (which can be seen from Figure 12), when lowest terms cancel, we find that $c_3 + \frac{1}{4} + c_3^2 = 0$ so $c_3 = -\frac{1}{2}$. We substitute this value for c_3 into (12) and divide by $x^{-\beta}$ to get

$$x^{-2}f_3(x, x(c_3 + y_3)) = f_3(x, y_3) =$$

$$= \frac{1}{2}x - 58x^2 + \frac{21}{8}x^4 - \frac{5}{8}x^3 - \frac{9}{4}x^5 - \frac{39}{64}x^6 + \frac{85}{32}x^7 - \frac{1}{4}x^9$$

$$- \frac{7}{4}x^8 + \frac{17}{16}x^{10} - \frac{37}{64}x^{11} + \frac{1}{64}x^{15} - \frac{5}{64}x^{14} + \frac{5}{32}x^{13} + y_3^2 - \frac{5}{8}x^{12}y_3$$

$$+ \frac{25}{8}x^{12}y_3^2 - \frac{37}{8}x^{10}y_3 + \frac{45}{8}x^{10}y_3^2 - \frac{11}{8}x^9y_3 + \frac{23}{4}x^5y_3 - \frac{3}{2}x^5y_3^2 + 6x^9y_3^2$$

$$- \frac{9}{2}x^9y_3^3 + \frac{67}{8}x^8y_3 - \frac{59}{8}x^7y_3 + \frac{9}{2}x^7y_3^2 - \frac{81}{8}x^8y_3^2 + \frac{5}{8}x^{14}y_3 - \frac{15}{8}x^{14}y_3^2$$

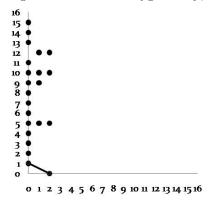
$$+ \frac{5}{2}x^{14}y_3^3 - \frac{5}{4}x^{14}y_3^4 + \frac{123}{32}x^{11}y_3 - 5x^{12}y_3^3 + \frac{5}{2}x^{12}y_3^4 + 2x^8y_3^3 - \frac{5}{4}x^6y_3$$

$$+ 2x^6y_3^2 - \frac{33}{4}x^{11}y_3^2 + 6x^{11}y_3^3 - \frac{3}{2}x^{10}y_3^3 - \frac{15}{4}x^4y_3 + \frac{3}{2}x^4y_3^2 - x^2y_3 + x^2y_3^2$$

$$- \frac{15}{16}x^{13}y_3 + \frac{15}{8}x^{13}y_3^2 - \frac{5}{4}x^{13}y_3^3 + x^3y_3 - 2x^3y_3^2 - x^7y_3^3 - \frac{1}{2}x^{11}y_3^4$$

$$+ \frac{1}{4}x^{10}y_3^4 + x^6y_3^3 - \frac{5}{32}x^{15}y_3 + \frac{5}{8}x^{15}y_3^2 - \frac{5}{4}x^{15}y_3^3 + \frac{5}{4}x^{15}y_3^4 - \frac{1}{2}x^{15}y_3^5$$
 (14)

Figure 13: Newton Polygon for f_3



To find c_4 , we repeat the process for f_3 . Looking at the Newton Polygon (Figure 13), we see that $\beta_4=1$ and $\gamma_4=\frac{1}{2}$. From this we have

$$y_3 = x^{\frac{1}{2}}(c_4 + y_4) \tag{15}$$

where c_4 is the coefficient of the fourth term in the Puiseux series and y_4 is

$$y_4 = c_4 x^{\gamma_4} + c_5 x^{\gamma_4 + \gamma_5} + \dots {16}$$

Substituting (15) into f_3 , we get

$$0 = \frac{3}{2}x^{5}(c_{4} + y_{4})^{2} + \frac{123}{32}x^{\frac{23}{2}}(c_{4} + y_{4}) + \frac{1}{2}x + 6x^{10}(c_{4} + y_{4})^{2} - \frac{11}{8}x^{\frac{19}{2}}(c_{4} + y_{4})$$

$$- \frac{15}{16}x^{\frac{27}{2}}(c_{4} + y_{4}) - 2x^{4}(c_{4} + y_{4})^{2} + 2x^{\frac{19}{2}}(c_{4} + y_{4})^{3} - \frac{5}{32}x^{\frac{31}{2}}(c_{4} + y_{4})$$

$$- \frac{5}{4}x^{\frac{33}{2}}(c_{4} + y_{4})^{3} + x^{\frac{7}{2}}(c_{4} + y_{4}) - x^{\frac{17}{2}}(c_{4} + y_{4})^{3} + \frac{5}{8}x^{16}(c_{4} + y_{4})^{2}$$

$$+ \frac{5}{8}x^{\frac{29}{2}}(c_{4} + y_{4}) + \frac{9}{2}x^{8}(c_{4} + y_{4})^{2} - \frac{1}{2}x^{13}(c_{4} + y_{4})^{4} - \frac{1}{2}x^{\frac{35}{2}}(c_{4} + y_{4})^{5}$$

$$+ \frac{45}{8}x^{11}(c_{4} + y_{4})^{2} - \frac{5}{4}x^{\frac{13}{2}}(c_{4} + y_{4}) + \frac{1}{4}x^{12}(c_{4} + y_{4})^{4} - 5x^{\frac{27}{2}}(c_{4} + y_{4})^{3}$$

$$- \frac{59}{8}x^{\frac{15}{2}}(c_{4} + y_{4}) - \frac{15}{4}x^{\frac{9}{2}}(c_{4} + y_{4}) - \frac{81}{8}x^{9}(c_{4} + y_{4})^{2} - x^{\frac{5}{2}}(c_{4} + y_{4})^{3}$$

$$- \frac{3}{2}x^{\frac{23}{2}}(c_{4} + y_{4})^{3} + \frac{5}{2}x^{14}(c_{4} + y_{4})^{4} - \frac{33}{4}x^{12}(c_{4} + y_{4})^{2} - \frac{5}{4}x^{\frac{29}{2}}(c_{4} + y_{4})^{3}$$

$$+ \frac{67}{8}x^{\frac{17}{2}}(c_{4} + y_{4}) + \frac{25}{8}x^{13}(c_{4} + y_{4})^{2} + x^{\frac{15}{2}}(c_{4} + y_{4})^{3} - \frac{3}{2}x^{6}(c_{4} + y_{4})^{2}$$

$$+ \frac{15}{8}x^{14}(c_{4} + y_{4})^{2} - \frac{5}{8}x^{2} + \frac{21}{8}x^{4} - \frac{5}{8}x^{3} + \frac{23}{4}x^{\frac{11}{2}}(c_{4} + y_{4})$$

$$- \frac{9}{2}x^{\frac{21}{2}}(c_{4} + y_{4})^{3} - \frac{5}{4}x^{16}(c_{4} + y_{4})^{4} - \frac{9}{4}x^{5} - \frac{39}{64}x^{6} + \frac{85}{32}x^{7} + \frac{5}{4}x^{17}(c_{4} + y_{4})^{4}$$

$$- \frac{37}{8}x^{\frac{21}{2}}(c_{4} + y_{4}) - \frac{1}{4}x^{9} - \frac{7}{4}x^{8} + \frac{5}{2}x^{\frac{31}{2}}(c_{4} + y_{4})^{3} + \frac{17}{16}x^{10}$$

$$- \frac{37}{64}x^{11} + \frac{1}{64}x^{15} - \frac{5}{64}x^{14} + \frac{5}{32}x^{13} + 2x^{7}(c_{4} + y_{4})^{2} - \frac{5}{8}x^{\frac{25}{2}}(c_{4} + y_{4})$$

$$+ x^{3}(c_{4} + y_{4})^{2} + 6x^{\frac{25}{2}}(c_{4} + y_{4})^{3} - \frac{15}{8}x^{15}(c_{4} + y_{4})^{2} + x(c_{4} + y_{4})^{2}$$

$$+ x^{3}(c_{4} + y_{4})^{2} + 6x^{\frac{25}{2}}(c_{4} + y_{4})^{3} - \frac{15}{8}x^{15}(c_{4} + y_{4})^{2} + x(c_{4} + y_{4})^{2}$$

$$+ \frac{1}{3}x^{\frac{15}{2}}(c_{4} + y_{4}) + \frac{1}{4}x^{\frac{15}{2}}(c_{4} + y_{4}$$

Since $\beta_4 = 1$, when lowest terms cancel, we find that $\frac{1}{2} + c_4^2 = 0$ so $c_4 = \pm \frac{\sqrt{2}i}{2}$. So the two solutions given by the Newton Puiseux Algorithm are

$$y = -x^2 + \frac{x^4}{2} - \frac{x^5}{2} + \frac{(-2x)^{\frac{11}{2}}}{64} + \cdots$$

$$y = -x^2 + \frac{x^4}{2} - \frac{x^5}{2} - \frac{(-2x)^{\frac{11}{2}}}{64} + \cdots$$

as one of the jets. It can be shown that the the possible values of c_4 are equal to the roots of $\frac{(-2)^{\frac{11}{2}}}{64}$, and our solutions for c_4 seem to be imaginary. However, $x^{\frac{11}{2}}$ forms a cusp, and the fact that there is a -x in the jet simply means that the cusp opens left, so when x is negative, the coefficient of the $\frac{11}{2}$ term in the jet is real.

References

[1] Robert John Walker, *Algebraic curves*, Princeton Mathematical Series, no. 13, Princeton University Press, Princetion, N.J.