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TWO THEORIES WITH AXIOMS BUILT BY MEANS OF PLEONASMS

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This paper contains examples T_1 and T_2 of theories which answer the following questions:

(1) Does there exist an essentially undecidable theory with a finite number of non-logical constants which contains a decidable, finitely axiomatizable subtheory?[1]

(2) Does there exist an undecidable theory categorical in an infinite power which has a recursive set of axioms? (Cf. [2] and [3].)

The theory T_1 represents a modification of a theory described by Myhill [7]. The common feature of theories T_1 and T_2 is that in both of them pleonasms[2] are essential in the construction of the axioms.

Let T_1 be a theory with identity $=$ which contains one binary predicate $R(x, y)$ and is based on the axioms A_1, A_2, A_3, B_1, B_2, B_3, B_4, C_{nm} which follow.

A_1: $x=x$. A_2: $x=y \supset y=x$. A_3: $x=y \wedge y=z \supset x=z$.

(Axioms of identity.)

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[1] Cf. [8] p. 19. A number of similar problems were recently suggested in the literature. In order to systematize them let us consider the following hypotheses.

H_1: Every axiomatizable, essentially undecidable theory T contains a finitely axiomatizable, essentially undecidable subtheory.

H_2: If T_1 and T_2 are compatible axiomatizable theories with the same constants and if T_2 is essentially undecidable, then T_1 is undecidable ([8] p. 19).

H_3: Every recursive extension T_2 of a decidable theory T_1 is decidable ([6] p. 384).

H_4: Every finitely axiomatizable subtheory of an axiomatizable essentially undecidable theory T is undecidable.

Further hypotheses H_1^*, H_2^*, H_3^* (cf. [6]), H_4^* are obtained from H_1—H_4 by assuming that all theories concerned are based on a finite number of constants.

One can easily check the following connections:

$$H_1 \supset H_2 \supset H_4, \quad H_3 \supset H_4, \quad H_1^* \supset H_2^* \supset H_4^*, \quad H_3^* \supset H_4^*,$$
$$H_i \supset H_i^* \quad (i = 1, 2, 3, 4).$$

Kreisel [6] gave an example disproving H_4 and hence H_1, H_2, H_3. He also noticed that the theory **R** described in [8] p. 52 is a counterexample for H_1^*.

In [7] Myhill gave another beautiful counterexample for H_1^*. However he wrote incorrectly that Kreisel [6] left this question open. Myhill stated also that his example disproves H_2^*. This however is not obvious and the proof is lacking. One could obtain this proof if one could show that there exists a decidable theory compatible with the theory of Myhill and having the same constants. (Added October 28, 1956: According to the referee, a paper by Putnam forthcoming in this JOURNAL contains an example of a theory which satisfies these conditions.)

The theory T_1 to be defined below disproves H_4^*.

[2] Here: repetition of one and the same formula in a single axiom.

B_1: $R(x, x)$. B_2: $R(x, y) \supset R(y, x)$. B_3: $R(x, y) \wedge R(y, z) \supset R(x, z)$.

(Axioms of equivalence.)

B_4: $x=y \supset [R(z, x) \equiv R(z, y)]$.

(Axiom of extensionality.)

Let ϕ_n be the formula

$$(\exists x_1, \ldots, x_n)\{x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge \ldots \wedge x_{n-1} \neq x_n \wedge$$
$$R(x_1, x_2) \wedge R(x_1, x_3) \wedge \ldots \wedge R(x_{n-1}, x_n) \wedge$$
$$(y)[R(x_1, y) \supset (y=x_1 \vee y=x_2 \vee \ldots \vee y=x_n)]\},$$

which express that there is an abstraction class of the relation R which has exactly n elements.

Let $f(n)$ and $g(n)$ be two recursive functions which enumerate two recursively inseparable sets [5], and call these sets X_1 and X_2.

We now specify the axioms C_{nm}.

$$C_{nm}: \quad \underbrace{\phi_m \wedge \ldots \wedge \phi_m}_{n \text{ times}} \qquad \text{if } f(n) = m,$$

$$\underbrace{\sim\phi_m \wedge \ldots \wedge \sim\phi_m}_{n \text{ times}} \qquad \text{if } g(n) = m,$$

$$x=x \qquad \text{if } g(n) \neq m \neq f(n).$$

It is obvious that the set composed of the formulas A_1—A_3, B_1—B_4, C_{nm} $(n, m = 1, 2, \ldots)$ is recursive.

The theory T_1 is essentially undecidable; for if there were a complete and decidable extension T_1' of it, then the recursive sets $Z = \{n : \phi_n$ is provable in $T_1'\}$ and $Z' = \{n : \sim\phi_n$ is provable in $T_1'\}$ would separate the sets X_1 and X_2.

By a result of Janiczak [4], every finitely axiomatizable theory T which has the same constants as T_1 and satisfies the condition that A_1—A_3, B_1—B_4 are provable in T is decidable. Thus T_1 has all the properties required in (1).

Let T_2 be the theory which has only one constant = (the predicate of identity) and which is based on the axioms A_1—A_3 as well as on the axioms β_{nm} given below.

Let ψ_n be the formula

$$(\exists x_1, \ldots, x_n)[x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge \ldots \wedge x_{n-1} \neq x_n \wedge$$
$$(y)(y=x_1 \vee y=x_2 \vee \ldots \vee y=x_n)],$$

which means that there exist exactly n elements; and let $h(n)$ be a recursive function which enumerates a non-recursive set X. We specify β_{nm} as follows.

$$\beta_{nm}\colon\ \begin{cases} \underbrace{\sim\psi_m \wedge \ldots \wedge \sim\psi_m}_{n \text{ times}} & \text{if } h(n) = m, \\ x{=}x & \text{if } h(n) \neq m. \end{cases}$$

It is obvious that the set of axioms of T_2 is recursive, and that T_2 is categorical in the power $\aleph_0$.

T_2 is undecidable; for $\sim\psi_m$ is provable in T_2 if and only if m is in X. Thus T_2 gives a positive answer to the problem (2).[3]

LITERATURE

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[3] (Added October 28, 1956, at the suggestion of the referee.)

All complete extensions of the theory T_2 are decidable (cf. Behmann [1]). Thus T_2 solves a problem of Mostowski, who asked whether an undecidable theory always possesses at least one undecidable complete extension.

Pleonasms can also be used to obtain an example disproving H_4. This example is simpler than the example given by Kreisel in [6]. It is sufficient to consider a theory whose non-logical symbols are a monadic predicate P and an infinite number of constants $c_1, c_2, \ldots$, and whose axioms are

$$\underbrace{P(c_m) \wedge \ldots \wedge P(c_m)}_{n \text{ times}} \quad \text{if} \quad f(n) = m,$$

$$\underbrace{\sim P(c_m) \wedge \ldots \wedge \sim P(c_m)}_{n \text{ times}} \quad \text{if } g(n) = m.$$