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Unrecognizability of manifolds

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Abstract

We present a modernized proof, with a modification by M.A. Shtan'ko, of the Markov theorem on the unsolvability of the homeomorphy problem for manifolds. We then discuss a proof of the S.P. Novikov theorem on the unrecognizability of spheres $\mathbb{S}^n$ for $n \geq 5$, from which we obtain a corollary about unrecognizability of all manifolds of dimension at least five. An analogous argument then proves the unrecognizability of stabilizations (i.e. the connected sum with 14 copies of $\mathbb{S}^2 \times \mathbb{S}^2$) of all four-dimensional manifolds. We also give a brief overview of known results concerning algorithmic *recognizability* of three-dimensional manifolds.

Keywords: Unsolvability of the recognition problem for PL manifolds; Adian series of group presentations; Markov manifolds

1. Introduction

A.A. Markov's influential paper "The unsolvability of the homeomorphy problem" [17] was a considerable accomplishment, as it was the first time that the techniques developed in algebra for undecidability problems had been carried over to topology. The paper's main result is the following:

Theorem (*Markov [17]*). *For every natural number $n > 3$ one can create an n-manifold M^n such that the problem of homeomorphy of manifolds to M^n is undecidable.*

This theorem has as corollaries the undecidability of the following three problems: the problem of homeomorphy of n-manifolds for $n > 3$; the problem of homeomorphy for polyhedra of degree no higher than n for $n > 3$; the general problem of homeomorphy.

Markov's three papers [17–19] provide enough information to restore a complete proof, but two factors interfered with its understanding by a wide audience of logicians: advanced (and partly pioneering) topological techniques and the extreme conciseness of Markov's writing. To further complicate matters, the manifold unrecognizability problems were outside the mainstream of manifold topology undergoing a rapid development in the 1950's and 1960's. As a

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result, for a long time this problem has been in relative isolation. As a rare contribution to the field, Stallings [32] in 1962 showed how to apply the general Adian–Rabin theorem to the class of fundamental groups of 3-manifolds proving that this class is algorithmically undecidable.

In 1968, Boone et al. [4] provided a sufficiently full analysis of Markov's original proof. They also extended it to the case of smooth manifolds and claimed "that these problems can be taken to be of any preassigned, recursively enumerable degree of unsolvability".

Only in 1978 was it specified in [9] that the example given by Markov's proof of an unrecognizable manifold was a connected sum of several copies of $\mathbb{S}^2 \times \mathbb{S}^{n-2}$. A.B. Sossinsky gave a coarse upper bound for the number of summands required.

M.A. Shtan'ko recently gave a new detailed proof of the Markov theorem (see [34]). Shtan'ko also used C.M. Gordon's method [23,24] for an Adian sequence [1,2] and proceeded from V.V. Borisov's group [5] to show the unrecognizability of the connected sum of 14 copies of $\mathbb{S}^2 \times \mathbb{S}^{n-2}$, where $n \geq 4$.

After Markov, the only significant advancement in unrecognizability problems in topology was due to S.P. Novikov. By analyzing Markov's proof Novikov was able to establish the unrecognizability of spheres of dimension greater than four. A brief exposition of this result appeared as an appendix of [37].

In this paper, we first present a modernized original proof of the Markov theorem along with Shtan'ko's modification. We then give an exposition of Novikov's proof (in greater detail than we did in [6]) and obtain a corollary about the unrecognizability of *all* manifolds of dimension at least five. Further, we use an analogous argument to prove the unrecognizability of stabilizations (i.e. the connected sum with 14 copies of $\mathbb{S}^2 \times \mathbb{S}^2$) of all four-dimensional manifolds. We conclude with a brief review of the currently known results about algorithmic *recognizability* of 3-manifolds.

2. The Markov theorem

On the one hand, the approach chosen by Markov appears to be rather straightforward: he suggested realizing a sequence of finitely presented groups with undecidable isomorphy problem by fundamental groups $\pi_1(M_i) = G$ of some closed manifolds M_i. It is well known that given a finite presentation of a group G one can effectively build a manifold M^n with $\pi_1(M^n) = G$ for any $n \geq 4$.

However, it was important for Markov's argument that different presentations of the same group G produce the same manifold (at least, that it be the case for the unit group), and the realization procedure use the same algorithm for all presentations in the sequence.

To overcome this problem, Markov ingeniously introduced empty relations. Geometrically this means taking connected unions of the manifold with products of two spheres. This operation does not change the fundamental group, and in the case of the unit group it allows one to define the manifold uniquely. (This so-called "Markov's trick" later played an important role in Smale's work on the study of manifold structure [29].)

Before describing the idea of a modernized proof of the Markov theorem, let us review some topological and algebraical facts essential for this proof.

2.1. Topology

We will work in the realm of rather elemental PL topology (PL stands for "piecewise linear"), so our only objects will be polyhedra (i.e. subsets of Euclidean spaces that may be triangulated, decomposed into simplexes) and their PL mappings, which can be made simplicial after triangulating both the domain and image spaces (see [27]). In this theory, manifolds (with boundaries) are polyhedra that have atlases of PL charts (i.e. PL homeomorphisms between their open subsets and open subsets of Euclidean half-space $\mathbb{R}^n_+$ for some fixed n). The boundary of a manifold M will be denoted by ∂M. We will use I^n for the cube $[-1, 1]^n$, and its boundary will be denoted by S^{n-1}.

We will extensively use the following operation of "gluing". If X and Y are two disjoint (compact) spaces, $A \subset X$ and $B \subset Y$ are their (closed) subsets, and $\varphi\colon A \to B$ is a homeomorphism, we can form a new space Z that consists of points from $X \setminus A$, points from $Y \setminus B$, and points associated with pairs (x, y), where $x \in X$, $y = \varphi(x)$. There is a natural mapping $f\colon X \cup Y \to Z$; further, a set U in Z is open iff both $f^{-1}U \cap X$ and $f^{-1}U \cap Y$ are.

The first use of this operation will be in the "*pointed union*" $X \vee Y$; this is a gluing where both subsets A and B are singletons, i.e. consist of a single point. Pointed union of two spaces can be easily generalized to pointed union of a finite family of spaces $\vee_i X_i$. We will use pointed unions of circles S^1_i and discs D^2_i.

Another operation we will be using is attaching a *handle* to a manifold.

Cube I^n in the form of the product $I^i \times I^{n-i}$ is called an i*-handle* of dimension n; the handle's *core* is the subcube $I^i \times 0$; its *sole sphere* is the boundary of the subcube, $\partial I^i \times 0 = \Sigma \times 0$. The *sole* of the handle is $\partial I^i \times I^{n-i}$, and the *lateral surface* of the handle is $I^i \times \partial I^{n-i}$. We have $\partial I^n = \partial I^i \times I^{n-i} \cup I^i \times \partial I^{n-i}$.

Let M^n be an n-dimensional PL manifold with boundary $\partial M^n = N^{n-1}$, and let $\varphi\colon \partial I^i \times I^{n-i} \to N$ be a PL homeomorphism of the sole onto a PL submanifold $\varphi(\partial I^i \times I^{n-i})$ in N with boundary $\varphi(\partial I^i \times \partial I^{n-i})$. If we glue the i-handle $I^i \times I^{n-i}$ to M^n using this homeomorphism we will obtain a new manifold M' with a boundary.

Let us denote the image of the handle in M' by H. We will continue to call it an i-handle of M'. We will also denote the images of $I^i \times 0$, $\Sigma^{i-1} \times 0$, $\Sigma^{i-1} \times I^{n-i}$, and $I^i \times \partial I^{n-i}$ by D^i, S^{i-1}, W, and F respectively; we will continue to call them the core, the sole sphere, the sole, and the lateral surface of handle H. It can be easily seen that the boundary of M' is $(N \setminus W) \cup F$.

A handle glued to a manifold will be called *trivial* if its sole is a submanifold in a subset of the boundary that is PL homeomorphic to the $(n-1)$-cube. (In general, this requirement is not sufficient, but in this case it will satisfy our conditions.)

(Markov's manifold will appear as a boundary of a manifold that is formed from the n-cube by first gluing to it the 1-handles that correspond to generators of the given group presentation, then 2-handles that correspond to its relations, and finally *trivial* 2-handles that correspond to the generators.)

Another important example of gluing is the operation of *connected union* of two manifolds. Suppose we are given two n-manifolds M_1 and M_2. Their connected union is obtained by deleting the interior of an n-ball in each of them and gluing the remaining parts using a homeomorphism of the boundary spheres. We will denote the connected union of M_1 and M_2 by $M_1 \# M_2$.

Remark. If a manifold M' is a result of attaching a trivial i-handle to a manifold M, then the boundary of M' is PL homeomorphic to the connected union of $\partial M'$ and $S^{i-1} \times S^{n-i}$.

The last example of gluing in Markov's proof is the operation of *doubling* the given manifold (with boundary). This operation is performed by gluing the two copies of the manifold together using the identity homeomorphism between their boundaries.

The restriction on the dimension of the manifold, $\dim M > 3$, in the Markov theorem originates from the need to manipulate freely with closed curves, which in a 3-space may be knotted or linked. In a 4-manifold, any homotopy between two (topological) circles can be converted into a PL isotopy, more precisely there always exists a PL isomorphism of the whole manifold that maps a closed PL curve without self-intersections to another such curve homotopic to the first one.

The same is true for finite families of such curves. In addition to that, as we will see, in an n-space with $n \geq 5$, we can manipulate in a similar manner with two-dimensional discs.

2.2. Algebra

We will use finitely presented groups. A presentation of such a group has the form $\langle g_1, \ldots, g_r; R_1, \ldots, R_m\rangle$ where g_i's are the generators of the group and $R_j = 1$ are its relations. It is called an (r, m)*-presentation*. Each R_j is a product $\prod_{s=1}^{t_j} g_{js}^{\varepsilon_{js}}$ where each g_{js} is a generator and each ε_{js} is either 1 or -1. R_j's are called the *relators* of the presentation.

In his seminal paper [25], Novikov describes a group G given by a presentation such that no algorithm exists for determining whether a given word in the alphabet $\{g_i, g_i^{-1}\}$ is equal to the unit element of group G. More precisely, a recursive sequence of words in this group exists such that no algorithm can solve the problem of equality to the unit element for the members of this sequence. The simplest known example of such a group is due to Borisov [5]. Its presentation has 4 generators and 12 relators.

Adian [1] used Novikov's group presentation to construct a sequence of group presentations such that there exists no algorithm for determining whether a given presentation from this sequence is a presentation of the unit group. Shortly after that, Rabin [26] obtained an analogous result.

Miller [23,24], using an idea of Gordon, has shown how, given a presentation Π of a group G without an identity algorithm, to construct a so-called "Adian sequence" of presentations with two generators and two additional relations each.

This is how it is done. We take a recursive sequence of words mentioned above. With each word w in the sequence we will associate a presentation: First, we introduce new generators a, b, c, three new relations, and a series of relations having one relation for each generator of the given presentation:

$$a^{-1}ba = c^{-1}b^{-1}cbc \tag{1}$$
$$a^{-2}b^{-1}aba^2 = c^{-2}b^{-1}cbc^2 \tag{2}$$
$$a^{-3}[w,b]a^3 = c^{-3}bc^3 \tag{3}$$
$$a^{-(3+i)}g_iba^{3+i} = c^{-(3+i)}bc^{3+i}. \tag{4}$$

Then we add another generator, $t = ac^{-1}$. It follows from (1) that c is expressible in terms of b and t; also $a = tc$ is expressible in terms of b and t. Using (4), all the old generators g_i can be expressed in terms of a, b, and c and, consequently, in terms of b and t. Thus, the relations of the resulting presentation will be the old relations R_i (with g_i's replaced by their expressions in terms of b and t) and relations (2) and (3).

(In fact, series (4) that allows one to express the old generators in terms of b and t may be omitted. The construction of Markov's manifold depends only on the number of relators; the number of generators is irrelevant.)

Taking Borisov's group as a base yields an Adian sequence of groups with 2 generators and 14 relations each.

2.3. A detailed sketch of a modernized proof of the Markov theorem

Suppose we are given an Adian sequence of (r, m)-presentations. Consider a pointed union Λ of r discs Δ_i with vertex v in Euclidean space $\mathbb{R}^n$, where $n \geq 5$. "Discs" Δ_i will be triangles with one common vertex v. Their boundaries Σ_i form a 1-polyhedron $\vee$, the pointed union of Σ_i. Let M_1 be a *regular neighborhood* of $\vee$, i.e. an n-manifold with boundary in $\mathbb{R}^n$ that contains $\vee$ and that can be deformed in a simple manner into $\vee$. It is constructed in the following way. First, we take a small cube Q with center v such that each of $\Sigma_i \setminus Q$ is a simple broken arc l_i. Then for each l_i we take 1-handle H_i^1 glued to Q with core l_i. Such a manifold is called a (full) *pretzel* of dimension n with r handles. Its fundamental group is free and has rank r (for $n > 4$ this also holds for its boundary).

Remark. The closure of $\Lambda \setminus M_1$ consists of r (disjoint) discs d_i. If we extend each d_i to a 2-handle h_i glued to M_1 with the core d_i, we will obtain PL manifold $\hat{Q} = M_1 \cup \bigcup_i h_i$, which is evidently PL homeomorphic to the n-cube. The boundaries s_i of d_i's are the sole circles of h_i's. They are the intersections of Δ_i's with ∂M_1. We will use them later.

As the next step, we will glue 2-handles to M_1: one handle for every relator $R_j = \prod_{s=1}^{t_j} g_{js}^{\varepsilon_{js}}$.

First, we take handle $I^2 \times I^{n-2}$ and glue the core disc $I^2 \times 0$. We divide the sole circle $S^1 \times 0$ into t_j equal arcs and map the sth arc to the union of lateral surface of 1-handle H_i^1 that corresponds to g_{js} with $\partial Q \cap \partial M_1$. This mapping will keep or invert orientation depending on the sign of ε_{js}, and, of course, the images of dividing points on $\partial Q \cap \partial M_1$ for consecutive mappings must coincide. Then we obtain a mapping $f_j\colon S^1 \times 0 \to \partial M_1$. The dimension of ∂M_1 is sufficiently high, so we can demand this mapping to be a PL embedding (without self-intersections). Since manifold ∂M_1 is orientable, closed curve $f_j(S^1 \times 0)$ has a neighborhood that is homeomorphic to $S^1 \times I^{n-2}$, where the curve itself corresponds to $S^1 \times 0$. Further, we can extend this mapping to a PL homeomorphism $f_j'\colon S^1 \times I^{n-2} \to \partial M_1$ and use it as a sole to glue a 2-handle to M_1. Since we work in $\mathbb{R}^n$ with $n > 4$, we can make the constructed 2-handles H_j^2 be PL embedded into $\mathbb{R}^n$, make them be mutually non-intersecting, and make $H_j^2 \cap M_1 = H_j^2 \cap \partial M_1 = f'(S^1 \times I^{n-2})$. Let M_2 denote the union of M_1 with all m of the constructed 2-handles.

The fundamental group of M_2 is G; further, since we have $n \geq 5$, $\pi_1(\partial M_2)$ is also G. (This is an easy topological exercise, which me omit here.)

It would remain for us to prove that different presentations of the same group produce PL homeomorphic manifolds. Indeed, if we could distinguish algorithmically manifolds in general, or manifolds with $\pi_1(\partial M_2) = 1$, or specifically the manifolds with the unit fundamental group in this sequence, then the same would be true of the presentations in the initial Adian sequence because the transition from a group presentation to a manifold is computable. All the topological PL operations used in the construction of M_2 are purely constructive.

But, of course, it could be very difficult (if at all possible) to reach such an ideal situation. It is sufficient, however, to prove PL equivalence for manifolds with $\pi_1(M_2) = 1$ only. If we could algorithmically distinguish this manifold from others, then we would also be able to algorithmically isolate the unit group.

Markov in [19] stated some lemmas about homeomorphy of manifolds; we explain them below reformulating them in terms of handles. These lemmas allow one to mimic elemental word transformations by homeomorphic transformations of the boundaries of the manifolds, which can be easily extended to homeomorphisms of the manifolds themselves. Lemmas 2–4 in [19] are quite evident; they deal with the following word transformations: omitting words $g_s^{\varepsilon} g_s^{-\varepsilon}$, cyclic permutation in one of R_j's, and group inversion of one of R_j's. The first of them follows from our ability to transform any homotopy of closed curves in the boundary of the (five-dimensional) manifold M_2 into an isotopy of the manifold. The other two require some simple extensions of the identity homeomorphism of M_1 to homeomorphisms ψ of the glued handles. (They have the form $\psi = \varphi_2 h \varphi_1^{-1}$, where φ_s is a homeomorphism of the standard handle $I^2 \times I^{n-2}$ onto the corresponding handle, and h is a cyclic homeomorphism of $I^2 \times I^{n-2}$ or a reflection respectively.)

Lemma 5 in [19] considers replacing some of the relators R_j by $P_j P_k$ with $k \neq j$. The sole circles of the corresponding handles are homotopic (one has to deform a part of the sole curve expressing P_j through the lateral surface of the other handle that corresponds to P_k) and, thus, isotopic; therefore, the two gluings produce PL homeomorphic manifolds. This transformation is an instance of a Tietze transformation. Markov did not mention another type of Tietze transformations, where the number of gluings is changed uncontrollably, which might violate the homeomorphy of corresponding manifolds.

Instead, Markov used a trick that now bears his name. Essentially he uses Tietze transformations only to reduce the presentation of the unit group to the trivial presentation. According to this trick, one glues r trivial 2-handles $\bigcup H_s'^2$, where r is the number of generators. Lemma 7 essentially states that for the unit group G the resulting manifold is homeomorphic to manifold M_0 obtained by gluing m 2-handles H_j^2 to the n-cube in $\mathbb{R}^n$. The last manifold is standard, it does not depend on the randomness of gluing (see Lemma 6 in [19]; we could also prove it using the fact that $n \geq 5$). Its boundary is the connected union of m copies of $S^2 \times S^{n-3}$, so *this is the manifold that is unrecognizable in the constructed sequence of manifolds*.

The above mentioned fact that for $G = 1$ the connected union of M_2 with r copies $H_s'^2$ of trivial 2-handles is homeomorphic to M_0 once again follows from our ability to freely convert a homotopy of one-dimensional curves into an isotopy. Indeed, the m curves that are the sole circles of these trivial handles are homotopic in the boundary of M_2 to m sole circles s_i of discs d_i (see the Remark in Section 2.1) since $\pi_1(\partial M_2) = 1$. By converting this homotopy into an isotopy, we will obtain a homeomorphism of ∂M_2 onto itself that maps the former family of curves to the latter. This homeomorphism can be extended to a homeomorphism of the whole M_2. Further, we can require that it maps the soles of our trivial handles $H_s'^2$ to the soles of handles h_i in a way that allows an extension to a homeomorphism from $\bigcup H_s'^2$ to $\bigcup h_i$. Now we see that for $G = 1$ manifold M_0 is a result of gluing m 2-handles to manifold $\hat{Q} = M_1 \cup \bigcup_i h_i$, which is PL homeomorphic to the n-cube (see the Remark in Section 2.1). □

Let us return to the original Markov paper. He begins with a full pretzel in space $\mathbb{R}^{n-1}$ (we adjust the dimension to match the exposition above). Then he takes the doubling of it (it is the boundary of our n-pretzel).

Our soles of 2-handles (that correspond to relators R_j) correspond to his "tunnels" in the doubled pretzel. The soles of our trivial 2-handles correspond to his empty relators, which he had to represent by trivial tunnels (neighborhoods of simple circles); this detail is missing in his original paper. In our construction the soles of 2-handles in the boundary of M_1 were replaced by their lateral surfaces in order to obtain M_2. Markov, instead, replaces the interior of the tunnels by the complements of the tunnels in the $(n-1)$-sphere. But the replacement we used is equivalent to the one used by him because for $n - 1 \geq 4$ the sole circles cannot be knotted in the $(n-1)$-sphere, therefore, the complements of the tunnels are PL homeomorphic to the lateral surfaces of 2-handles of dimension n. Thus, the unrecognizable manifold constructed by Markov is exactly the connected union of m copies of $S^2 \times S^{n-3}$.

3. The S.P. Novikov theorem

Theorem (*S.P. Novikov*). *Given any $n \geq 5$ there is no algorithm that could determine whether a given polyhedron is PL homeomorphic to the n-sphere* (*boundary of* $(n+1)$-*dimensional simplex*).

In fact, it is possible to construct a sequence of PL manifolds such that the n-sphere is indistinguishable among these sequences, which allows one to obtain other similar results (see the last section).

Novikov's idea was effectively replacing an Adian sequence with a new sequence of group presentations for groups with trivial homologies in dimensions one and two, making sure that the presentations of the unit group remain unrecognizable. We will call any sequence of group presentations such that (1) the group homologies of dimensions one and two are zero for each element of the sequence and (2) the presentations of the unit group are algorithmically unrecognizable a *Novikov sequence*.

It turns out that for $n \geq 5$ the two-dimensional homology classes of Markov's manifolds constructed according to a Novikov sequence can be realized with embedded 2-spheres. The neighborhoods of these spheres can be represented as $\mathbb{S}^2 \times D^{n-2}$. This allows us to carry out spherical modifications that "kill" these homologies. For the manifolds with trivial fundamental group this means that manifolds obtained in this way are homotopy spheres and, therefore, they actually are real spheres (according to the generalized Poincaré conjecture proved by Smale for $n \geq 5$ [30]). Thus, an ability to recognize a sphere in this sequence of manifolds would allow one to recognize the unit group in the given sequence of group presentations, which is impossible.

Accordingly, the proof of unrecognizability of spheres of dimension 5 or higher can be divided into two parts. The first part is an effective transition from the given Adian sequence to a Novikov sequence; the second is an effective realization of two-dimensional classes with "correctly" embedded spheres. It may be assumed that the one-dimensional homologies are trivial already in the given Adian sequence (this is simply checked after commuting the group presentations). The proof of the first part is given by the following

Lemma. *Given a finite presentation of a group π with $H_1(\pi) = 0$ one can effectively construct a presentation of a new group $\tilde{\pi}$ with a central extension*

$$1 \to H_2(\pi) \to \tilde{\pi} \to \pi \to 1$$

and $H_1(\tilde{\pi}) = H_2(\tilde{\pi}) = 0$.

To prove that this new group has trivial two-dimensional homologies Novikov resorts to the homology algebra technique for non-commutative groups (see e.g. [8,10]). This makes it harder to verify the effectiveness of the construction. In [6], we provided a direct proof, which will be reproduced here with certain additions.

Proof. Let

$$1 \to R \to F \to \pi \to 1$$

be a presentation of group π, where F is a free group with a finite number k of generators h_j and R is generated as a normal subgroup by a finite number m of elements q_i. Condition $H_1(\pi) = 0$ implies that $F = \{[F, F] \cup R\}$. (The braces are used to denote the normal subgroup generated by the set enclosed by the braces.)

According to Hopf [14], $H_2(\pi) = (R \cap [F, F])/[R, F]$.

We want to construct a group $\tilde{\pi}$ and the central extension

$$1 \to H_2(\pi) \to \tilde{\pi} \to \pi \to 1,$$

so that $H_1(\tilde{\pi}) = 0 = H_2(\tilde{\pi})$.

First, consider group $\approx{\pi} = F/[R, F]$. Here $\overset{\approx}{R} = [R, F]$ lies in R since R is a normal subgroup. Then we have the extension

$$1 \to K \to \overset{\approx}{\pi} \to \pi \to 1,$$

where $K = R/[R, F]$ lies in the center and in addition to that it is commutative. ($[R, R] \subset [R, F] \subset [F, F]$.)

The image of R in $F/[F, F] = \mathbb{Z}^k$ (where k is the number of generators of π) coincides with the whole group $\mathbb{Z}^k$, and the kernel of the epimorphism $R \to \mathbb{Z}^k$ is $[F, F] \cap R$:

$$1 \to [F, F] \cap R \to R \to \mathbb{Z}^k \to 1.$$

Since $[R, F] \subset ([F, F] \cap R)$, we have

$$1 \to ([F, F] \cap R)/[R, F] \to R/[R, F] \to \mathbb{Z}^k \to 1,$$

meaning that $K = H_2(\pi) \oplus \mathbb{Z}^k$ and

$$1 \to H_2(\pi) \oplus \mathbb{Z}^k \to \overset{\approx}{\pi} \to \pi \to 1.$$

Choose elements $g \in R$, whose images generate $\mathbb{Z}^k$ under commuting, and kill them. We have

$$1 \to H_2(\pi) \to \tilde{\pi} \to \pi \to 1.$$

Let $\breve{g}$ denote the union of the chosen elements. Then group $\tilde{\pi}$ is the factor group of F determined by normal subgroup $\tilde{R} = \{\overset{\approx}{R} \cup \breve{g}\}$. It has the following effectively obtained finite presentation: h_j, $1 \le j \le k$, are generators, and the relators are taken to be the generators of $\tilde{R}$ as a normal subgroup. In other words, we consider a finite number of commutators $[h_j, q_i]$, $1 \le j \le k$, $1 \le i \le m$, and the elements of $\breve{g}$.

We have to show that $\breve{g}$ can be chosen effectively. Indeed, using the identity $ab = ba[a^{-1}, b^{-1}]$, it is possible to express each relation $q_i = 1$ in the free group in the form $q_i = h_1^{n_1} \dots h_k^{n_k} \Pi_i$, where h_j are the generators in a certain order and Π_i are products of the commutators. Then, as in the case of a usual matrix diagonalization, we have

$$g_j = h_j \Pi'_j,\ 1 \le j \le k; \qquad g_j = \Pi'_j,\ \ j > k,$$

where g_j are words in the alphabet q_i, Π'_j are new products of commutators. Then $\breve{g}$ consists of g_j, $1 \le j \le k$. It is clear that products Π'_j, $j \le k$, are effectively expressed in terms of generators h_l, so we have effectively constructed a presentation of $\tilde{\pi}$ from the given presentation of π.

Clearly, $H_1(\tilde{\pi}) = \mathbb{Z}^k/\mathrm{im}\, R = 0$. Let us prove $H_2(\tilde{\pi}) = (\tilde{R} \cap [F, F])/[\tilde{R}, F] = 0$, where kernel $\tilde{R} = \{\overset{\approx}{R} \cup \breve{g}\} = \{[R, F] \cup \breve{g}\}$. We have

$$\tilde{R} \cap [F, F] = \{[R, F] \cup \breve{g}\} \cap [F, F] = [R, F]$$

since $[R, F] \subset [F, F]$, and g_j, $1 \le j \le k$, are not products of commutators.

Also $[\tilde{R}, F] = [\{[R, F] \cup \breve{g}\}, F] \subset [R, F]$ because $[[R, F], F] \subset [R, F]$ since R is a normal subgroup and $[\{\breve{g}\}, F] \subset [R, F]$.

It remains to show that $[R, F] \subset [\tilde{R}, F]$.

Each generator $h_j \in F$ is written as $g_j \Pi_j$, where g_j is one of the chosen elements of R, and Π_j is a product of commutators. Thus, it is sufficient to prove that $[r, h_j] = [r, g_j \Pi_j]$ lies in $[\tilde{R}, F]$ for any $r \in R$. Write $[r, g_j \Pi_j]$ as $[r, g_j] g_j [r, \Pi_j] g_j^{-1}$. The first cofactor lies in $[g_j, F] \subset [\tilde{R}, F]$. Since $[\tilde{R}, F]$ is a normal subgroup, we need to show that $[r, \Pi] \in [\tilde{R}, F]$ for any r and for any product of commutators Π. We first show that $[r, [f_1, f_2]] \in [[R, F], F]$.

$$\begin{aligned}
[r, [f_1, f_2]] = {} & \\
& r f_1 f_2 f_1^{-1} f_2^{-1} r^{-1} f_2 f_1 f_2^{-1} f_1^{-1} = \\
& f_1 r [r^{-1}, f_1^{-1}] f_2 f_1^{-1} f_2^{-1} r^{-1} f_2 f_1 f_2^{-1} f_1^{-1} = \\
& f_1 r f_2 [r^{-1}, f_1^{-1}] \big[[r^{-1}, f_1^{-1}]^{-1}, f_2^{-1}\big] f_1^{-1} f_2^{-1} r^{-1} f_2 f_1 f_2^{-1} f_1^{-1} = \\
& f_1 f_2 r [r^{-1}, f_2^{-1}][r^{-1}, f_1^{-1}] \big[[r^{-1}, f_1^{-1}]^{-1}, f_2^{-1}\big] f_1^{-1} f_2^{-1} r^{-1} f_2 f_1 f_2^{-1} f_1^{-1} =
\end{aligned}$$

The second half is treated analogously and we have (after canceling the central members):

$$\begin{aligned}
& f_1 f_2 r (r^{-1} f_2^{-1} r f_2)(r^{-1} f_1^{-1} r f_1) \big[[r^{-1}, f_1^{-1}]^{-1}, f_2^{-1}\big] \\
& \quad r^{-1} (r f_1^{-1} r^{-1} f_1)(r f_2^{-1} r^{-1} f_2) \big[[r, f_2^{-1}]^{-1}, f_1^{-1}\big] f_2^{-1} f_1^{-1} = \\
& f_1 f_2 \big([f_2^{-1}, r][f_1^{-1}, r] r \big[[r^{-1}, f_1^{-1}]^{-1}, f_2^{-1}\big] r^{-1} [r, f_1^{-1}][r, f_2^{-1}] \\
& \quad \big[[r, f_2^{-1}], f_1^{-1}\big]\big) f_2^{-1} f_1^{-1},
\end{aligned}$$

i.e. we have a product of two elements from $[[R, F], F]$ modulo a conjugation.

Now consider the commutation of $r \in R$ with a product of commutators. Note that

$$[r, ab] = [r, a] a [r, b] a^{-1}.$$

It follows by induction that commuting with a product can be expressed through commuting with cofactors by means of multiplication and conjugation. So, in our case $[r, \Pi] \in [[R, F], F]$. $\square$

The second part of the proof is carried out through the following sequence of steps. For each of them the effectiveness can be verified directly.

- Choosing a sequence of triangulations with meshes approaching zero (e.g. the sequence of barycentric subdivisions for a given triangulation).
- Choosing free generators for the (free abelian) group of two-dimensional homologies.
- Realizing these generators by simplicial cycles in a triangulation of the constructed sequence of triangulations.
- Transforming this cycle into a two-dimensional surface (this is a standard operation).
- Converting this surface into an embedded sphere by embedded discs, whose boundary circles form cuts of the surface's handles. Such discs can be constructed by a brute-force search in a finite number of steps because all such cycles are known to be spherical.
- The regular neighborhoods of the resulting spheres are products of a sphere by a disc of the complementary dimension. It is well known that, since the $(n+1)$-manifold, whose boundary is Markov's manifold, lies in $\mathbb{R}^{n+1}$, which makes it parallelizable, Markov's manifold itself is almost-parallelizable (i.e. parallelizable in the complement to any one of its points). Then the normal fibering of every embedded 2-sphere in it must be trivial (given that the spheres are almost-parallelizable). A trivialization of the normal fibering can be specified effectively through the structure of a regular neighborhood (a simplicial star in the second barycentric subdivision of the chosen triangulation) because its triviality is known.

4. Unrecognizability of n-dimensional manifolds for $n > 4$

The following argument (a modified form of the argument in [6]) shows that the Novikov theorem implies unrecognizability of all manifolds in a rather straightforward way. The main idea of this argument is to use the Grushko theorem ([11], see [20]). (This method was originally used by Adian [1] to establish a similar result in group theory.)

Theorem. *Given any manifold of dimension $n \geq 5$, no algorithm exists that would recognize this manifold among the class of all n-dimensional manifolds.*

Proof. Suppose M_0 is a (connected[1]) n-dimensional manifold (possibly with a boundary or non-compact), which can be effectively recognized among the class of all n-dimensional manifolds, $n \geq 5$. We will show that in this case it would be possible to recognize the n-dimensional sphere $\mathbb{S}^n$, which would contradict the Novikov theorem.

Let M be a compact manifold of dimension n in a Novikov sequence of manifolds. Let M_1 be its connected union with M_0, $M_1 = M\#M_0$. Apply the algorithm for M_0-recognition to M_1.

If the answer is No, it is clear that M is not a sphere.

If the answer is Yes, note that the fundamental group of M is the unit group. Indeed, the fundamental group of M_1 is a free product of the fundamental groups of M and M_0, at the same time it must coincide with the fundamental group of M_0. This is possible only if M is simply connected because, according to the Grushko theorem [11], the minimal number of generators of the fundamental group is additive relative to the free product. But only the sphere is simply connected in the Novikov sequence.

Thus, the recognizability of M_0 implies the recognizability of sphere, which is impossible by the Novikov theorem. $\square$

5. Unrecognizability in dimension 4

Let us apply this argument to the four-dimensional manifolds. We will call a connected union of a 4-manifold M^4 with 14 copies of $\mathbb{S}^2 \times \mathbb{S}^2$ (in other words $M^4\#14(\mathbb{S}^2 \times \mathbb{S}^2)$) the *stabilization* of M^4.

Theorem. *Given any (connected) 4-manifold, no algorithm exists that would recognize the stabilization of this manifold among the class of all 4-manifolds.*

[1] This assumption is added for simplicity only.

Proof. Suppose M_0 is a (connected) 4-manifold, whose stabilization can be effectively recognized among the class of all 4-manifolds. We will show that this would imply the existence of an algorithm that recognizes $14\#\mathbb{S}^2 \times \mathbb{S}^2$, which contradicts the results of Section 2.

Let M be any 4-manifold from the Markov sequence. Let M_1 be its connected union with M_0, $M_1 = M\#M_0$. Apply the algorithm recognizing stabilization of M_0 to M_1.

If the answer is No, obviously M is not $14\#\mathbb{S}^2 \times \mathbb{S}^2$.

If the answer is Yes, note that M has the unit fundamental group; this may be proved as in the preceding case using the Grushko theorem. So M is $14\#\mathbb{S}^2 \times \mathbb{S}^2$ because this is the only simply connected manifold in the Markov sequence.

Thus, the recognizability of M_0 implies the recognizability of $14\#\mathbb{S}^2 \times \mathbb{S}^2$. $\square$

6. Decision problems for 3-manifolds

In dimension 3 unrecognizability results are absent (except for the work of Stallings mentioned above). The question of the existence of some unrecognizable 3-manifold becomes pertinent in connection with the (one hopes) resolved Poincaré conjecture (cf. [31]), and construction of an algorithm recognizing the 3-sphere (see below). On the other hand, some remarkable results were obtained recently concerning algorithmic *recognizability*. We list here the most interesting and important of them.

- In 1961, Haken [12] presented a recognition algorithm for the trivial knot in the 3-sphere. This algorithm has extremely high complexity and is certainly not suitable for computer realization. But in this work a remarkable technique was developed for the study of 3-manifolds (the so-called "normal surfaces"). This technique proved to be one of the most productive for further research. Later Haken's work was analyzed by Schubert who obtained a recognition algorithm for decomposition of a link. Further, an algorithm was devised for the irreducibility (i.e. undecomposability into the connected union) of manifolds problem, but *modulo the recognition of the sphere*, which was an open question at that time.
- For the class of manifolds of genus 2 (results of different gluings of two full pretzels of genus 2) a useful algorithm for the sphere recognition was proposed by Volodin et al. in 1974 [37]. It was proved in 1980 by Japanese topologists [13].
- W. Waldhausen in [38] introduced a notion of a "sufficiently large" manifold (such a manifold contains a two-sided surface different from a sphere or a projective plane, and its fundamental group embeds monomorphically). In 1984 Jaco and Oertel [15] invented an algorithm which determines whether a given irreducible manifold is sufficiently large.
- Since the 1980's hyperbolical structure has played an important role in the study of manifold topology. A recognition algorithm for homeomorphism of hyperbolic manifolds was developed in [35].
- A major breakthrough occurred between the 1980's and 1990's: in 1991 Thompson developed a recognition algorithm for a sphere based on ideas of Jaco and Rubinstein [36].
- A recognition algorithm for homeomorphism between sufficiently large irreducible manifolds was proposed in 1976 by W. Waldhausen, K. Johansson, and G. Hemion (see [38]), but the proof turned out to have an essential gap. S.V. Matveev [21] was able in 1997 to complete this proof by using the results of W.P. Thurston (1982) [35] and M. Bestvina and M. Handel (1995) [3].
- In 2000 S.V. Matveev presented an algorithm that determines whether a given manifold is a Seifert manifold [22].
- All results mentioned and the recognition algorithm for a sphere, in particular, are described in the recently published book of Matveev [22].
- It is also worth noting the paper by Dynnikov [7] that has appeared very recently. It contains a new algorithm for the recognition of triviality of a knot; it also features a review of other known algorithms in this area.

7. Markov on effective settings in manifold topology

We refer once more to the work of Markov, namely to the third paper in his series ("On unsolvability of certain problems of topology" [18]). He returns there to the term "*homeomorphy*", which he refines in the preceding work as simplicial and, accordingly, discusses combinatorial manifolds in the sense of Newman and Alexander. Such manifolds have finite presentations: they are simplicial complexes, where stars of vertices are combinatorially equivalent to the cube (see [27]). (Here the relation of combinatorial equivalence is obtained by chains of elementary transformations: star subdivisions and their inverses. It is worth noting that, according to S.P. Novikov's result, the property of a complex of being a combinatorial manifold is algorithmically unrecognizable for dimension 6 and higher.)

Properly formulated, the Markov theorem proves unrecognizability of a specific manifold ($14\#\mathbb{S}^2 \times \mathbb{S}^2$) in the class of combinatorial manifolds. The relation of "affinity" (i.e. of equality of fundamental groups) considered by Markov in fact almost doubles the relation of group isomorphism because for every group presentation of G it is possible to effectively build a manifold M_G for any dimension higher than 3 such that $\pi_1(M_G) = G$, whereas the unit group is the only one which produces simply connected manifold $M_1 = 14\#\mathbb{S}^2 \times \mathbb{S}^2$. For the set of all combinatorial presentations of manifolds, Markov also considered other relations in between affinity and combinatorial equivalence. The main candidate for such a role is the purely continuous homeomorphy of manifolds.

What can be said about "continuous homeomorphisms", which arise naturally in the theory of 4-manifolds? The main problem is the lack of a constructive definition of this notion.

Markov suggested a novel idea that unfortunately so far has not found its audience, despite the fact that it is very natural. The idea is using the projective spectra in P.S. Alexandrov's sense for a constructive definition of a topological (not necessarily simplicial) manifold (essentially of a larger class of spaces, e.g. compact). For example, the Cantor set or the cube have obvious constructive descriptions of their projective constructions. But the general projective definition of a compact topological manifold is less evident and requires certain theoretical elaboration.

Note, however, that the result about homeomorphy can be formulated also in the framework of combinatorial manifolds without appealing to the notion of a topological manifold: the relation of topological equivalence lies between affinity and combinatorial equivalence, and thus, it is algorithmically unrecognizable. The same is true also for homotopy equivalence.

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