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On Well-Quasi-Ordering Finite Structures with Labels

Igor Kříž* and Robin Thomas**

Department of Mathematical Analysis, Charles University, Sokolovská 83, 18600 Praha 8, Czechoslovakia, and Bell Communications Research, Inc., 435 South Street, Morristown, NJ 07960, USA

Abstract. A quasi-ordered set A (i.e. one equipped with a reflexive and transitive relation $\leq$) is said to be well-quasi-ordered (wqo) if for every infinite sequence $a_1, a_2, \ldots$ of elements of A there are indices i, j such that $i < j$ and $a_i \leq a_j$.

Various natural wqo sets Q admit "labelling" by another wqo A yielding another quasi-ordered set $Q(A)$, which may or may not be wqo. A suitable concept covering this phenomenon is the notion of a QO-category. We have two conjectures about QO-categories in the effect that labelling QO-categories by a wqo set can always be reduced to labelling by ordinals. We prove these conjectures for a broad class of QO-categories and for general QO-categories we prove weaker forms of these conjectures.

1. Introduction

Let A be a quasi-ordered set (i.e. one equipped with a reflexive and transitive relation $\leq$). A finite or infinite sequence $(a_1, a_2, \ldots)$ of elements of A is called *good* if there are indices i, j such that $i < j$ and $a_i \leq a_j$, and is called *bad* otherwise. The set of bad sequences will be denoted by $Bad(A)$. The set A is called well-quasi-ordered (wqo) if every infinite sequence of elements of A is good, i.e., $Bad(A)$ contains no infinite sequence.

The concept of well-quasi-ordering has been studied for quite a while. The major achievements in the field are Higman's Finite Sequence Theorem [4], Kruskal's Tree Theorem [9] and recently Robertson and Seymour's proof of Wagner's conjecture [13]. We do not wish to go into details of the history, we refer the interested reader to e.g. [10].

In our opinion, there are at least four reasons to be interested in wqo theory, namely

(i) it is fun,
(ii) it implies "Excluded minor theorems", for example it implies Kuratowski type theorems for higher surfaces [14],

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* Current address: Department of Mathematics, The University of Chicago, 5734 S. University Ave., Chicago, IL 60637, USA

** Current address: School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, USA

(iii) it has surprising algorithmic consequences – it implies the existence of certain polynomial-time algorithms without actually constructing a single one, see e.g. [3, 14],
(iv) it has applications in mathematical logic, namely certain wqo theorems can be "miniaturized" to provide statements of finite combinatorics unprovable in relatively strong fragments of second order arithmetic (this is pioneering work of Harvey Friedman [1] – see also [16, 2]).

In this paper we explore the first reason.

Our notation about ordinals is standard, we identify each ordinal with the set of its predecessors. Thus an ordinal is itself a well-quasi-ordered set. Let A be a wqo set. The *type* of A, denoted by $c(A)$, is the least ordinal γ such that there exists a mapping $f\colon Bad(A) \to \gamma$, called a character, such that $f(a_1, \ldots, a_{n-1}) > f(a_1, \ldots, a_n)$ for any $(a_1, \ldots, a_n) \in Bad(A)$. It is worth noting that $c(\alpha) = \alpha$ for any ordinal α. There is an extensive theory of types of well-quasi-ordered sets, see [15, 5, 8]. We shall investigate types of wqo sets in connection with labelling, a concept which we now introduce.

A QO-category is a concrete category Q with finite objects and injective morphisms, the forgetful functors will be denoted by U. In other words every object $q \in Q$ has its finite underlying set $U(q)$ and to every arrow $f\colon q \to q'$ there corresponds an injective mapping $U(f)\colon U(q) \to U(q')$. There is a natural quasi-ordering $\leq$ associated with every QO-category Q, namely $q \leq q'$ if there is an arrow $q \to q'$. Thus every QO-category may be regarded as a quasi-ordered set, and, if it happens to be well-quasi-ordered, it has its type $c(Q)$. Now if Q is a QO-category and A is a quasi-ordered set we define a new QO-category $Q(A)$, Q labelled by A, as follows. Its objects are pairs $z = (u, c)$, where c is an object of Q and $u\colon U(c) \to A$ is a mapping. There is an arrow $(u, c) \to (u', c')$ in $Q(A)$ if there is an arrow $f\colon c \to c'$ in Q such that $u(x) \leq u'(g(x))$ for any $x \in U(c)$, where $g = U(f)$.

It is easy to construct QO-categories Q which are well-quasi-ordered (even well-ordered) and such that $Q(2)$ is not well-quasi-ordered. We conjectured for some time that $Q(2)$ wqo might imply $Q(A)$ was for any wqo A, but that was recently disproved by Kříž and Sgall [7]. On the other hand, at the time of this writing, we were unable to prove or disprove the following two conjectures

1.1 Conjectures. If Q is a QO-category and A is wqo then
(i) $Q(A)$ is wqo if and only if $Q(c(A))$ is wqo, and
(ii) if $Q(A)$ is wqo then $c(Q(A)) = c(Q(c(A)))$.

However, one implication of 1.1 holds, namely

1.2 Proposition. *If Q is a QO-category and A is wqo then*
(i) *if $Q(A)$ is wqo then $Q(c(A))$ is wqo, and*
(ii) $c(Q(c(A))) \leq c(Q(A))$.

Proof. We can assume without loss of generality that A is partially ordered (otherwise we just factorize through $\equiv$, where $x \equiv y$ if $x \leq y \leq x$). By [6] or [8] there exists an extension $\leq'$ of $\leq$ on A which is linear (hence well-ordered) and of order type $c(A)$. Thus the ordering on $Q(c(A))$ is isomorphic to some extension of the ordering on $Q(A)$ and (i) and (ii) follows. □

Let H be the QO-category of finite linearly ordered sets with strictly increasing mappings as morphisms. As usual, we say that a QO-category Q is a subcategory of a QO-category Q' if every object of Q is an object of Q' and every arrow $q_1 \to q_2$ of Q is an arrow of Q'. For example the QO-category of finite structured trees (i.e. trees with every successor set linearly ordered) with inf-preserving morphisms respecting the successor order may be regarded as a subcategory of H.

Now we can state our results.

1.3 Theorem. *For an arbitrary subcategory Q of H,* 1.1(i) *and* 1.1(ii) *hold, i.e. if A is wqo then*

(i) *$Q(A)$ is wqo if and only if $Q(c(A))$ is wqo, and*

(ii) *if $Q(A)$ is wqo then $c(Q(A)) = c(Q(c(A)))$.*

1.4 Theorem. *Let Q be an arbitrary QO-category and let A be a wqo set. Let $\gamma_1 = min(\omega^{c(A)}, \omega_1)$ and $\gamma_2 = max(\omega, |A|)^{c(A)}$. [$\omega_1$ is the first uncountable ordinal, $|A|$ is the first ordinal with the same cardinality as A]. Then*

(i) *if $Q(\gamma_1)$ is wqo, then $Q(A)$ is wqo, and*

(ii) *if $Q(\gamma_2)$ is wqo, then $Q(A)$ is wqo and $c(Q(A)) \leq c(Q(\gamma_2))$.*

1.5 Corollary. *Let Q be any QO-category. Then the following four conditions are equivalent.*

(i) *$Q(A)$ is wqo for any wqo set A.*

(ii) *$Q(\gamma)$ is wqo for any ordinal γ.*

(iii) *$Q(\omega_1)$ is wqo.*

(iv) *$Q(\gamma)$ is wqo for any ordinal $\gamma \in \omega_1$.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) is obvious. (iv) $\Rightarrow$ (ii) follows from the fact that in each bad sequence of elements of $Q(\gamma)$, only countably many ordinals occur as labels. (ii) $\Rightarrow$ (i) follows from 1.4(i). □

1.6 Corollary. *Let Q be any QO-category and let A be an at most countable wqo set. If $Q(\omega^{c(A)})$ is wqo, then $Q(A)$ is wqo and $c(Q(A)) \leq c(Q(\omega^{c(A)}))$. In particular, $c(Q(A)) \leq (Q(\omega_1))$.*

Proof. Immediate from 1.4. □

1.7. Corollary. *Let Q be any QO-category and let A be an at most countable wqo set such that $c(A)$ is an ε-number (i.e. $c(A) = \omega^{c(A)}$). Then* 1.1(i) *and* 1.1(ii) *hold.*

Proof. Immediate from 1.6 or directly from 1.4. □

Let us make a few comments about the definition of a QO-category. Neither the finiteness nor the injectivity of morphisms may be dropped from the definition, as the two examples below show. We may also consider multi-valued mappings as morphisms, and in that case there are two possibilities how to define the morphisms on $Q(A)$. One of them is to require only the existence of an image with label which dominates the label of the preimage. In fact this is nothing substantially new: we can equivalently consider the QO-category of all mappings obtained from the multi-valued mappings by choosing one of the images at a time.

The other possibility is called a "gap-condition" by Friedman [1, 16]: we require all the labels of the images to dominate the label of the preimage. However, it is easy to see that Conjecture 1 fails in this case.

1.8 Example. Let the QO-category S contain objects c_n with $U(c_n) = \{0_n, 1_n\}$ and let identities and constants be the only morphisms. Let $D = \{a, b\}$ be equipped with the discrete quasi-ordering $\leq$ (i.e. $a \leq a$ and $b \leq b$ only). Then $c(D) = 2$, $S(2)$ is wqo and $S(D)$ is not. This shows that the injectivity of morphisms cannot be dropped.

1.9 Example (Based on Rado's [12]). Let the QO-category ψ contain ω (the natural numbers) and all strictly increasing mappings as morphisms. Then $\psi(\gamma)$ is wqo for any ordinal γ: this is a special case of Nash-Williams' transfinite sequence theorem [11]. Let $A = \{(i, j) | \, i, j \in \omega, i < j\}$ be equipped with ordering $\leq$ defined by $(i, j) \leq (k, l)$ if either $i = k$ and $j \leq l$, or $j < k$. Then A is wqo and $c(A) = \omega^2$, but $\psi(A)$ is not wqo: This is the content of the Rado's counterexample. Let $u_n \colon \omega \to A$ be defined by $u_n(i) = (n, n + i + 1)$. Then

$$(u_1, \omega), (u_2, \omega), (u_3, \omega), \ldots$$

is a bad sequence of elements of $\psi(A)$.

2. Proof of 1.3

Throughout this section Q will be a subcategory of H and A will be a wqo set, let us put $\alpha = c(A)$ and let $f \colon Bad(A) \to \alpha$ be a character.

2.1. Lemma. *There exists a mapping $\Phi \colon Bad(Q(A)) \to Bad(Q(\alpha))$ such that if z is an initial segment of $z' \in Bad(Q(A))$, then $\Phi(z)$ is an initial segment of $\Phi(z')$.*

Proof. Let $z = ((u_1, c_1), \ldots, (u_n, c_n)) \in Bad(Q(A))$ be arbitrary, but fixed. Since Q is a subcategory of H, we may assume that for $i = 1, \ldots, n$, $c_i = (x_i(1), \ldots, x_i(m_i))$ and that Q-morphisms preserve order of these sequences. We are going to define mappings $v_i \colon \{x_i(1), \ldots, x_i(m_i)\} \to \alpha$ and then put $\Phi(z) = ((v_1, c_1), \ldots, (v_n, c_n))$. This will be done by induction on n, and for fixed n we will first define $v_n(x_n(m_n))$, then $v_n(x_n(m_n - 1))$, ... etc. In order to state the induction hypothesis we need the following definition.

Let $z = ((u_1, c_1), \ldots, (u_n, c_n)) \in Bad(Q(A))$ be as above, let $m \in \{1, \ldots, m_n\}$. An (n, m)-*tower* is a quadruple (k, I, ϕ, p), where

(T1) $k \in \{1, \ldots, n\}$ is an integer,

(T2) $I = (i_1, \ldots, i_k)$ is a subsequence of $(1, \ldots, n)$ with $i_k = n$,

(T3) $\phi = (\phi_1, \ldots, \phi_{k-1})$ is a sequence of Q-morphisms such that $\phi_j \colon c_{i_j} \to c_{i_{j+1}} (j = 1, \ldots, k - 1)$,

(T4) $p = (p_1, \ldots, p_k)$ is a sequence of natural numbers such that $p_j \in \{1, \ldots, m_{i_j}\}$ for $j = 1, \ldots, k$ and $p_k = m$,

(T5) $\phi_j(x_{i_j}(p_j)) = x_{i_{j+1}}(p_{j+1})$.

Let $m' \in \{1, \ldots, m_{n'}\}$ and assume that $v_1, \ldots, v_{n'-1}$ have already been defined and that $v_{n'}$ has been defined at least on the set $\{x_{n'}(m' + 1), \ldots, x_{n'}(m_{n'})\}$. Consider the following statement $S(n', m')$.

$S(n', m')$: For any (n', m')-tower (k, I, ϕ, p) as above, which for any $j \in \{1, \ldots, k\}$ and any $s \in \{1, \ldots, m_{i_j}\}$ satisfies

(S1) if $s < p_j$ then $u_{i_j}(x_{i_j}(s)) \leq u_{i_{j+1}}(\phi_j(x_{i_j}(s)))$, and
(S2) if $s > p_j$ then $v_{i_j}(x_{i_j}(s)) \leq v_{i_j}(\phi_j(x_{i_j}(s)))$
the sequence $u_{i_1}(x_{i_1}(p_1))$, $u_{i_2}(x_{i_2}(p_2))$, $\ldots$, $u_{i_k}(x_{i_k}(p_k))$ is a bad sequence of elements of A.

Let us emphasize the fact that $S(n, m)$ makes sense even if v_n is not defined on the set $\{x_n(1), \ldots, x_n(m)\}$. This is an easy consequence of (T4), (T5) and the fact that Q-morphisms preserve order of the sequences c_i.

Now we are ready to begin the induction. Let $z \in Bad(Q(A))$ be as above and assume that $v_1, \ldots, v_{n-1}$ have already been defined in such a way that $((v_1, c_1), \ldots, (v_{n-1}, c_{n-1})) \in Bad(Q(\alpha))$ and that $S(n', m')$ holds for every $n' < n$ and every $m' \in \{1, \ldots, m_{n'}\}$. We claim that $S(n, m_n)$ holds true. Indeed, if (k, I, ϕ, p) is an (n, m_n)-tower (with notation as above), then we have $p_j = m_{i_j}$ for $j = 1, \ldots, k$ by (T5) and the fact that Q-morphisms respect the ordering of c_i. So if it was $u_{i_j}(x_{i_j}(p_j)) \leq u_{i_{j'}}(x_{i_{j'}}(p_{j'}))$ for some $1 \leq j < j' \leq k$, then $\phi_j \circ \phi_{j+1} \circ \ldots \circ \phi_{j'-1}: (u_j, c_j) \to (u_{j'}, c_{j'})$ would be a $Q(A)$-morphism, contradicting the badness of z. We are going to define $v_n(x_n(m))$ by induction on $m_n - m$. Assume that $v_n(x_n(m_n)), \ldots, v_n(x_n(m+1))$ have already been defined and that $S(n, m')$ holds true for $m' = m, m+1, \ldots, m_n$. We define

$$v_n(x_n(m)) = \min \{f(u_{i_1}(x_{i_1}(p_1)), \ldots, u_{i_k}(x_{i_k}(p_k))) \mid (k, I, \phi, p) \text{ is an } (n, m)\text{-tower satisfying (S1), (S2)}\}.$$

(Recall that $f: Bad(A) \to \alpha$ is a character). It follows from $S(n, m)$ that $v_n(x_n(m))$ is well-defined. To complete the backwards induction we have to show that $S(n, m-1)$ holds.

So suppose the contrary. Then there exists an $(n, m-1)$-tower (k, I, ϕ, p) (with notation as above) which satisfies the hypothesis, but not the conclusion of $S(n, m-1)$. Since $S(n', m')$ holds for every $n' < n$ and every $m' \in \{1, \ldots, m_{n'}\}$ it follows that $u_{i_j}(x_{i_j}(p_j)) \leq u_{i_k}(x_{i_k}(p_k)) = u_n(x_n(m-1))$ for some $1 \leq j < k$. Let us put $i := i_j$, $\phi := \phi_j \circ \ldots \circ \phi_{k-1}$ and $q := p_j + 1$. Then $\phi: c_i \to c_n$ is a Q-morphism such that

$$u_i(x_i(s)) \leq u_n(\phi(x_i(s))) \text{ for } s < q$$
$$v_i(x_i(s)) \leq v_n(\phi(x_i(s))) \text{ for } s \geq q.$$

By the definition of $v_i(x_i(q))$ there exists an (i, q)-tower, let us call it (k, I, ϕ, p) again (and let us assume the standard notation for it) such that

$$v_i(x_i(q)) = f(u_{i_1}(x_{i_1}(p_1)), \ldots, u_{i_k}(x_{i_k}(p_k))).$$

We shall define an (n, l)-tower $(k+1, I', \phi', p')$, where l is such that $\phi(x_i(q)) = x_n(l)$ as follows:

$$I' = (i_1, \ldots, i_k, n),$$
$$\phi' = (\phi_1, \ldots, \phi_{k-1}, \phi), \text{ and}$$
$$p' = (p_1, \ldots, p_k, l).$$

It follows that $(k+1, I', \phi', p')$ satisfies the hypothesis of $S(n, l)$, and, since $l \geq m$, $S(n, l)$ holds, and thus $(u_{i_1}(x_{i_1}(p_1)), \ldots, u_{i_k}(x_{i_k}(p_k)), u_n(x_n(l)))$ is a bad sequence of elements of A. We have

$$\begin{aligned} v_i(x_i(q)) &\le v_n(\phi(x_i(q))) = v_n(x_n(l)) \\ &\le f(u_{i_1}(x_{i_1}(p_1)), \ldots, u_{i_k}(x_{i_k}(p_k)), u_n(x_n(l))) \\ &< f(u_{i_1}(x_{i_1}(p_1)), \ldots, u_{i_k}(x_{i_k}(p_k))) = v_i(x_i(q)), \end{aligned}$$

a contradiction, which proves $S(n, m-1)$.

Having thus defined v_n, we have to show that the sequence $(v_1, c_1), \ldots, (v_n, c_n)$ is a bad sequence of elements of $Q(\alpha)$.

So suppose that $\phi: (v_i, c_i) \to (v_j, c_j)$ is a $Q(\alpha)$-morphism. Let m be such that $\phi(x_i(1)) = x_j(m)$. Let (k, I, ϕ, p) (with the usual notation) be an $(i,1)$-tower (observe that $p = (1, \ldots, 1)$) such that

$$v_i(x_i(1)) = f(u_{i_1}(x_{i_1}(1)), \ldots, u_{i_k}(x_{i_k}(1))).$$

Put $I' = (i_1, \ldots, i_k, j)$, $\phi' = (\phi_1, \ldots, \phi_{k-1}, \phi)$, $p' = (1, \ldots, 1, m)$. Then $(k+1, I', \phi', p')$ is a (j,m)-tower which satisfies the presumptions of $S(j,m)$. Hence the sequence $u_{i_1}(x_{i_1}(1)), \ldots, u_{i_k}(x_{i_k}(1)), u_j(x_j(m))$ is bad and we have

$$\begin{aligned} v_i(x_i(1)) \le v_j(x_j(m)) &\le f(u_{i_1}(x_{i_1}(1)), \ldots, u_{i_k}(x_{i_k}(1)), u_j(x_j(m))) \\ &< f(u_{i_1}(x_{i_1}(1)), \ldots, u_{i_k}(x_{i_k}(1))) = v_i(x_i(1)), \end{aligned}$$

a contradiction which proves the badness of $((v_1, c_1), \ldots, (v_n, c_n))$. This completes the induction and hence the proof of the lemma. □

2.2. Proof of 1.3. One part of 1.3 was already established in 1.2. To prove the other part let us assume that $Q(\alpha)$ is wqo, let $F: Bad(Q(\alpha)) \to c(Q(\alpha))$ be a character and let Φ be as in 2.1. Then $G: Bad(Q(A)) \to c(Q(\alpha))$ defined by $G(z) = F(\Phi(z))$ is a character, thus $Q(A)$ is wqo and $c(Q(A)) \le c(Q(\alpha))$, as desired. □

3. Proof of 1.4

3.1. Notation. Let Q be an arbitrary QO-category and let A be a wqo set. We shall assume that the objects of Q are finite sets and that morphisms are injective mappings (i.e. we shall disregard forgetful functors). We may safely assume that no ordinal is an element of A. In this section α denotes an arbitrary ordinal, we define $A_\alpha = \alpha \cup A$ and define a quasi-ordering $\le$ on A_α as follows: $a \le b$ if either $a, b \in \alpha$ and $a \le b$ as ordinals, or $a, b \in A$ and $a \le b$ as elements of A, or $a \in \alpha$ and $b \in A$.

Let $B = Bad(Q(A))$ and define $b \prec b'$ for $b, b' \in B$ to mean that b is a strict initial segment of b', i.e. if $b' = (b_1, \ldots, b_n)$, then $b = (b_1, \ldots, b_m)$ for some $m < n$. We define Z to be the set of all pairs $z = (x, b)$, where $b = ((u_1, c_1), \ldots, (u_n, c_n)) \in B$ and $x \in c_n$.

We define T to be the set of all nonempty sequences $((x_1, b_1), \ldots, (x_n, b_n))$ of elements of Z such that $b_1 \prec \ldots \prec b_n$ and $(u_1(x_1), \ldots, u_n(x_n)) \in Bad(A)$, where (u_i, c_i) is the last term of b_i. For $t, t' \in T$ we define $t \prec t'$ to mean that t is a strict initial segment of t', i.e. if $t' = ((x_1, b_1), \ldots, (x_n, b_n))$, then $t = ((x_1, b_1), \ldots, (x_m, b_m))$ for some $m < n$. Let us fix, once forever, a bijective mapping $h: Z \to |Z| = max(\omega, |A|)$ and let us define a linear ordering $\ll$ on T as follows. Let $t = ((x_1, b_1), \ldots, (x_n, b_n))$, $t' = ((x'_1, b'_1)), \ldots, (x'_{n'}, b'_{n'}))$ be two elements of T. Let m be maximal such that $(x_i, b_i) = (x'_i, b'_i)$ for $i = 1, \ldots, m$. We define $t \ll t'$ if either $n > m = n'$, or $m < n$,

$m < n'$ and $h((x_{m+1}, b_{m+1})) < h((x'_{m+1}, b'_{m+1}))$. Thus, in particular, $t \prec t'$ implies $t' \ll t$.

3.2. Lemma. *The relation* $\ll$ *is a well-ordering on* T *of order type at most* $\gamma_2 = max(\omega, |A|)^{c(A)}$.

Proof. We proceed by induction on $c(A)$. For $a \in A$ we denote by A/a the set $\{a' \in A | a \not\leq a'\}$. Clearly $c(A/a) < c(A)$ for any $a \in A$. Let $z = (x, ((u_1, c_1), \ldots, (u_n, c_n))) \in Z$, we put $a(z) = u_n(x)$. For an ordinal $\alpha < max(\omega, |A|)$ let z_α be such that $h(z_\alpha) = \alpha$. The set $(T, \ll)$ is isomorphic to the well-ordered sum

$$\sum_{\alpha=0}^{max(\omega, |A|)} T_\alpha,$$

where T_α is the set of those sequences from T, which start with z_α. Now every $T_\alpha \backslash \{z_\alpha\}$ is isomorphic to a subset of a set T'_α defined as T but with $A/a(z_\alpha)$ in place of A. By the induction hypothesis, T'_α has order type at most $max(\omega, |A/a(z_\alpha)|)^{c(A/a(z_\alpha))}$. Hence the order type of T is at most

$$\sum_{\alpha=0}^{max(\omega, |A|)} max(\omega, |A|)^{c(A/a(z_\alpha))} + 1 \leq max(\omega, |A|)^{c(A)}. \qquad \square$$

3.3. Proof of 1.4 (ii). We may assume without loss of generality that A is partially ordered, for otherwise we can identify elements $x, y \in A$ satisfying $x \leq y \leq x$. This assumption is not essential, it will only simplify the inductional invariants.

We are going to define, for $\alpha \geq 0$, mappings Θ_α: $B \to Q(A_\alpha)$ and elements $t_\alpha \in T$. For $\alpha = 0$ and $b = ((u_1, c_1), \ldots, (u_n, c_n))$ we put $u_b^0 = u_n$, $c_b = c_n$ and $\Theta_0(b) = (u_b^0, c_b)$. For $\alpha > 0$ we shall define mappings u_b^α: $c_b \to A_\alpha$ and then put $\Theta_\alpha(b) = (u_b^\alpha, c_b)$. To state the induction hypothesis let λ be an ordinal and assume that Θ_α and t_β are defined for $\beta < \alpha < \lambda$ in such a way that

(U1) $\Theta_\alpha(b) \leq \Theta_\beta(b)$ for all $\beta < \alpha < \lambda$ and all $b \in B$,
(U2) $\Theta_\alpha(b) \not\leq \Theta_\alpha(b')$ for all $\alpha < \lambda$ and all $b \leftarrow b'$ in B, and
(U3) $t_\beta \ll t_\alpha$ for all β, α such that $\beta < \alpha < \gamma$ for some $\gamma < \lambda$.

If λ is a limit ordinal, then for any $b \in B$ there exists by (U1) an ordinal $\beta < \lambda$ such that $\Theta_\beta(b) = \Theta_\alpha(b)$ for any $\beta \leq \alpha < \lambda$ (because c_b is a finite set and there is no infinite decreasing sequence in A_λ). We define $\Theta_\lambda(b) = \Theta_\beta(b)$. Conditions (U1), (U2), (U3) are clearly satisfied.

If λ is a successor ordinal, say $\lambda = \alpha + 1$, we proceed as follows. If $\Theta_\alpha(b) \in Q(\alpha)$ for every $b \in B$, we stop. Otherwise we choose a sequence $t_\alpha = (z_1, \ldots, z_n) = ((x_1, b_1), \ldots, (x_n, b_n))$ of elements of Z such that

(Z1) $b_1 \prec b_2 \prec \cdots \prec b_n$,
(Z2) for $i = 1, \ldots, n-1$ there exists a Q-morphism ϕ_i: $c_{b_i} \to c_{b_{i+1}}$ such that $\phi_i(x_i) = x_{i+1}$ and $u_{b_i}^\alpha(y) \leq u_{b_{i+1}}^\alpha(\phi_i(y))$ for any $y \in c_{b_i} \backslash \{x_i\}$,
(X3) $u_{b_i}(x_i) \in A$ for $i = 1, \ldots, n$,
and [note that (Z1), (Z2), (Z3) and the fact that Θ_α satisfies (U2) imply that $t_\alpha \in T$]
(Z4) t_α is the least element of T with respect to $\ll$ which satisfies (Z1), (Z2), (Z3).

There exists at least one element of T satisfying (Z1), (Z2), (Z3), namely the sequence $((x, b))$, where $\Theta_\alpha(b) \notin Q(\alpha)$ and $u_b^\alpha(x) \in A$. Note also that it follows from (Z4) that

(Z5) there is no z_{n+1} such that $(z_1, \dots, z_n, z_{n+1})$ satisfies (Z1), (Z2), (Z3).

We define, for $b \in B$,

$$u_b^{\alpha+1}(y) = \begin{cases} u_b^\alpha(y) & \text{if } b \neq b_n \text{ or } y \neq x_n \\ \alpha & \text{otherwise} \end{cases}$$

and put $\Theta_{\alpha+1}(b) = (u_b^{\alpha+1}, c_b)$. We claim that (U1), (U2), (U3) are satisfied.

Condition (U1) follows easily from (Z3) and from the definition of $\Theta_{\alpha+1}(b)$. We prove (U2) by way of contradiction. Let $b \ll b'$ and suppose that

$$(u_b^{\alpha+1}, c_b) = \Theta_{\alpha+1}(b) \leq \Theta_{\alpha+1}(b') = (u_{b'}^{\alpha+1}, c_{b'});$$

let $\phi\colon c_b \to c_{b'}$ be the corresponding $Q(A_{\alpha+1})$-morphism. If $b \neq b_n$ then

$$\Theta_\alpha(b) = \Theta_{\alpha+1}(b) \leq \Theta_{\alpha+1}(b') \leq \Theta_\alpha(b')$$

by construction, our assumption and (U1), a contradiction to (U2) at step α. If $b = b_n$ then

$$\alpha = u_{b_n}^{\alpha+1}(x_n) = u_b^{\alpha+1}(x_n) \leq u_{b'}^{\alpha+1}(\phi(x_n)),$$

which implies that $u_{b'}^{\alpha+1}(\phi(x_n)) \in A$. If we let $z' = (\phi(x_n), b')$ then $(z_1, \dots, z_n, z')$ satisfies (Z1), (Z2), (Z3), contrary to (Z5), again a contradiction. This proves (U2).

To prove (U3) let us first observe that $t_\beta \neq t_\alpha$ for $\beta < \alpha$. For if (x, b) is the last term of t_β, then $u_b^{\beta+1}(x) = \beta$, hence $u_b^\alpha(x) \in \alpha$ by (U1) and thus t_β does not satisfy (Z3) at step α. There exists a $\beta_0 < \alpha$ such that

(1) $\Theta_\beta(b) = \Theta_\alpha(b)$ for all $b \ll b_n$ and all $\beta \geq \beta_0$, $\beta < \alpha$.

Indeed, this follows from (U1) if α is limit and from the construction if α is a successor ordinal. It is enough to prove that $t_\beta \ll t_\alpha$ for $\beta \geq \beta_0$. So let $\beta \geq \beta_0$ and let $t_\beta = (z'_1, \dots, z'_{n'})$, recall that $t_\alpha = (z_1, \dots, z_n) = ((x_1, b_1), \dots, (x_n, b_n))$. Let m be maximal such that $z_1 = z'_1, \dots, z_m = z'_m$. We want to prove that either $m = n < n'$, or $m < n$ and $m < n'$ and $h(z_{m+1}) > h(z'_{m+1})$. So suppose the contrary. Then (since we already know that $t_\alpha \neq t_\beta$) $m < n$ and either $m = n'$, or $m < n'$ and $h(z'_{m+1}) > h(z_{m+1})$. We claim that

(2) $(z_1, \dots, z_m, z_{m+1})$ satisfies (Z1), (Z2), (Z3) at step β,

which will be a contradiction to (Z4) at step β in either case.

So it remains to prove (2). We use the fact that $(z_1, \dots, z_{m+1})$ is an initial segment of t_α, which immediately implies that $(z_1, \dots, z_{m+1})$ satisfies (Z1). From the fact that $(z_1, \dots, z_{m+1})$ satisfies (Z2) at step α there exist morphisms $\phi_i\colon c_{b_i} \to c_{b_{i+1}}$ $(i = 1, \dots, m)$ such that $\phi_i(x_i) = x_{i+1}$ and $u_{b_i}^\alpha(y) \leq u_{b_{i+1}}^\alpha(\phi_i(y))$ for any $y \in c_{b_i} \backslash \{x_i\}$. We have, for $y \in c_{b_i} \backslash \{x_i\}$ and any $i = 1, \dots, m$,

$$u_{b_i}^\beta(y) = u_{b_i}^\alpha(y) \leq u_{b_{i+1}}^\alpha(\phi(y)) \leq u_{b_{i+1}}^\beta(\phi(y))$$

by (1) and (U1), which proves that ϕ_i satisfy the requirements of (Z2) at step β as well. To verify (Z3) we first note that $u_{b_i}^\alpha(x_i) \in A$ by the fact that t_α satisfies (Z3), hence $u_{b_i}^\beta(x_i) \in A$ by (U1).

This proves (3) and hence completes the proof of (U3), thus completing the induction.

It follows from 3.2 and (U3) that this transfinite process will stop after $\alpha \leq \gamma_2$ steps. At the last step we have a mapping $\Theta_\alpha\colon Z \to Q(\alpha) \subseteq Q(\gamma_2)$ satisfying (U1). Now if $f\colon Bad\, Q(\gamma_2) \to c(Q(\gamma_2))$ is a character, we can define a character $g\colon B \to c(Q(\gamma_2))$ by

$$g((b_1,\ldots,b_n)) = f(\Theta_\alpha(b_1), \Theta_\alpha(b_1,b_2),\ldots,\Theta_\alpha(b_1,\ldots,b_n)).$$

This proves 1.4 (ii). □

3.4. Proof of 1.4 (i). Let $Q(\gamma_1)$ be wqo. It is enough to prove that an arbitrary countable subset Q' of $Q(A)$ is wqo. But $Q' \subseteq Q(A')$ for some countable set $A' \subseteq A$. We have $max(\omega, |A'|)^{c(A')} = \omega^{c(A')} \leq min(\omega^{c(A')}, \omega_1)$ and since $Q(\gamma_1)$ wqo and $\gamma_2 \leq \gamma_1$ implies $Q(\gamma_2)$ wqo, we may use 1.4 (ii) to infer that $Q(A')$ is wqo, and, consequently that $Q(A)$ is wqo. □

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