# On Testing the Membership to Differential Ideals

Alexey Zobnin *

Moscow State University,
Department of Mechanics and Mathematics,
Vorobyovy Gory, Moscow, Russia, 119992
al_zobnin@shade.msu.ru

**Abstract.** One of the open problems in differential algebra is testing the membership to differential ideals. This problem is undecidable in general for non-recursive infinitely generated ideals. In the case of ideals of finite type it is solved only in several (but important) cases. These are the cases of radical ideals, isobaric ideals over fields of constants, and ideals for which it is possible to construct a finite standard basis. We give an overview of solutions to the membership problem in these cases and propose new ideas. Some algorithms such as Ollivier's completion process and a reduction w.r.t. a normalizing system are implemented in *Maple*. We obtain interesting results concerning certain non-isobarizable differential ideals using computer algebra system computations.

The paper is organized as follows. In Section 1 we give necessary definitions of constructive differential algebra that can be found in [17, 8]. Then we describe the solutions of the membership problem in three main cases. In Section 3 we generalize these ideas and give algorithms that test the membership to an ideal in the case of isobaric generators. We exhibit a finite standard basis of the ideal $[x^p]$ w.r.t. the so-called $\beta$-ordering. Such orderings are more general than those used by Ollivier [14] and Carrà Ferro [3]. After that we propose the concepts of $M$-basis and normalizing system, which generalize standard bases. Finally, we consider non-isobarizable differential ideals of the form $[x_1^2+\alpha x_0+\beta]$. We show how computer algebra systems helped us to prove some interesting results concerning these ideals.

All mentioned algorithms have been implemented in Maple by the author. They are available online at `http://shade.msu.ru/~difalg/Algorithms`.

## 1 Differential Algebra Preliminaries

A *differential ring* $\mathcal{R}$ is a commutative ring with pairwise commuting *derivation operators* $\delta_1, \ldots, \delta_m$ (they must be linear and satisfy the product rule). Any *derivative operator* on $\mathcal{R}$ can be uniquely written as

$$\theta = \delta^\alpha = \delta_1^{\alpha_1} \ldots \delta_m^{\alpha_m}\,.$$

The *order* of $\theta$ is $\operatorname{ord}\theta := \sum_{i=1}^m \alpha_i$.

Let $\Theta$ denote the monoid of all derivative operators. An ideal $I \subset \mathcal{R}$ is called *differential* iff $I = \Theta I$. If $F$ is any subset of $\mathcal{R}$ then $[F]$ denotes the differential ideal generated by $F$ in $\mathcal{R}$. A differential ideal is called *radical* or *perfect* if it is a radical ideal in the sense of commutative algebra. The radical differential ideal generated by $F$ is denoted by $\{F\}$.

Let $y_1, \ldots, y_n$ be *differential indeterminates*. Then the polynomial ring

$$\mathcal{R}\{y_1, \ldots, y_n\} := \mathcal{R}[\Theta y_1, \ldots, \Theta y_n]$$

in an infinite family of *differential variables* $\theta y_i$, $\theta \in \Theta$, is called a *ring of differential polynomials* over $\mathcal{R}$. It is also a differential ring with the same set of derivation operators.

In this paper we consider only rings of *ordinary* differential polynomials in one variable over a *field of constants* of characteristic zero, i.e., rings of the form $\mathcal{F}\{x\}$, where $\Theta = \{\delta^i,\ i \geqslant 0\}$ and $\delta\mathcal{F} = 0$. Following Ritt [17], we will denote a differential variable $\delta^i x$ by $x_i$.

* The work was partially supported by the Russian Foundation for Basic Research, project No. 02-01-01033.

Any differential polynomial is a finite sum of *differential terms*, i.e., expressions of the form $c \cdot M$, where $c \in \mathcal{F}$, $c \neq 0$, and $M$ is a *differential monomial*, $M = \prod_{i=0}^{k} x_i^{\alpha_i}$, $\alpha_i \geqslant 0$. The coefficient $c$ of the monomial $M$ in $f$ will be denoted by $\mathrm{cf}\,(f, M)$.

Below we will often omit the word "differential" speaking about monomials and polynomials. We will use the letters $M, P, Q$ for monomials and $f, g, h$ for polynomials.

Consider any monomial $M = \prod_{i=0}^{k} x_i^{\alpha_i}$. The *degree* of $M$ is $\deg M := \sum_{i=0}^{k} \alpha_i$ and the *weight* of $M$ is $\mathrm{wt}\, M := \sum_{i=0}^{k} i\alpha_i$. As usual, a polynomial $f$ is called *isobaric* if all monomials in $f$ have the same weight. The *order* of a polynomial (or a set of polynomials) is the maximal order of variables occurring in it.

Now we recall the notions of ranking and admissible ordering.

Let $Y = \{y_1, \ldots, y_n\}$ be a set of differential indeterminates. A *ranking* $\prec$ is a total ordering on the set of differential variables $\Theta Y$ such that for all $y_i, y_j \in Y,\ \theta, \theta_1, \theta_2 \in \Theta$ we have

R1. $\theta_1 y_i \prec \theta_2 y_j \implies \theta\theta_1 y_i \prec \theta\theta_2 y_j$;
R2. $y_i \preccurlyeq \theta y_i$.

Now let $\mathcal{M}$ denote the set of all differential monomials. An *admissible ordering* on $\mathcal{M}$ is a total ordering $\prec_{\mathcal{M}}$ that satisfies the following:

O1. The property of translation: $P \prec_{\mathcal{M}} Q \implies P \cdot M \prec_{\mathcal{M}} Q \cdot M \quad \forall\, P, Q, M \in \mathcal{M}$.
O2. The property of positivity: $1 \preccurlyeq_{\mathcal{M}} P \quad \forall\, P \in \mathcal{M}$.
O3. The property of restriction: the restriction of $\prec_{\mathcal{M}}$ to the set of differential variables $\Theta Y$ must be a ranking.

It is very surprising that these properties are sufficient to guarantee that any admissible ordering is a well-ordering (Zobnin [20]).

For a given ordering we can naturally define the *leading monomial* $\mathrm{lm}_{\prec} f$ and the *leading coefficient* of a polynomial $f$. We will omit the subscript $\prec$ if there are no misunderstandings.

In the ordinary case of one indeterminate there is only one ranking: $x_i \prec x_{i+1}$. Therefore, the *lexicographic* (lex) as well as the *graded by degree lexicographic* (deglex) ordering on monomials are uniquely determined.

## 2 The Membership Problem

The membership problem is very important in constructive computer algebra. It is completely solved, for example, for ideals in ordinary polynomial rings as well as for submodules in free finitely generated modules over polynomial rings. The main instrument in testing the membership in these cases is a Gröbner basis computation.

In differential polynomial rings this is not so. Since these rings are not Noetherian, the Gröbner basis technique is in general inapplicable to the differential case. There exist infinitely generated differential ideals. Moreover, there exist non-recursive infinitely generated ideals such that the membership problem is algorithmically undecidable for them (the result of Gallo, Mishra and Ollivier, 1991 [6]). Even for finitely generated ones the problem is still open.

The first attempts to solve it for certain differential ideals generated by monomials were made in 50-es–70-es by Levi [9], O'Keefe [15], Mead [11], Newton [12] and others. As a result, a sufficient criterion for a monomial to membership in the ideal $[x^p]$ has appeared [9, 17, 8]. The situation is better for the ideal $[uv]$ in the ring $\mathcal{F}\{u, v\}$: there is a necessary and sufficient condition for a monomial to be in this ideal [11]. It is remarkable that these criteria can be expressed only in terms of degree and weight, and, thus, they can be applied to arbitrary monomials given by indeterminate powers.

At present, the membership problem is solved in three main cases described below:

### 2.1 Radical Differential Ideals

For this class of ideals the membership problem is *completely* solved by means of characteristic decomposition. The algorithm *Rosenfeld–Gröbner* (based on the ideas of Ritt for prime differential ideals) was presented by Boulier, Lazard, Ollivier and Petitot in 1995 [1, 2]. This algorithm represents a radical differential ideal $\{F\}$ as a finite intersection of *regular* ideals (components) given by their *characteristic sets* $\mathbb{C}_i$:

$$\{F\} = \bigcap_{i=1}^{m} [\mathbb{C}_i] : H_{\mathbb{C}_i}^{\infty} .$$

The membership to a component $[\mathbb{C}_i] : H_{\mathbb{C}_i}^{\infty}$ can be tested by pseudoreducing a given polynomial w.r.t. $\mathbb{C}_i$. Thus, one can also test the membership to an intersection of such components.

These methods develop rapidly nowadays (for example, see [7]). But they are inapplicable for non-radical ideals. Since solutions of differential equations in universal extensions of fields of coefficients are described by radical ideals, the study of non-radical ones may seem useless. But this is not so from the theoretical point of view!

Recall that in differential polynomial rings contaning $\mathbb{Q}$ radical differential ideals are finitely generated [8], but they may be infinitely generated as differential ideals.

Below we will be interested only in non-radical ideals.

### 2.2 Isobaric Ideals

The membership can also be tested for the ideals generated by polynomials that are isobaric w.r.t. some weight function [6].

**Definition 1 ([6]).** *A non-zero mapping $w : \mathcal{F}\{x\} \to \mathbb{N} \cup \{0\}$ is called a weight function iff*

1. $w(fg) = w(f) + w(g)\ \ \forall\, f, g \in \mathcal{F}\{x\} \setminus \mathcal{F}$;
2. *if $f$ consists of the monomials of the same value of $w$ then so does $w(\delta f)\ \ \forall\, f \in \mathcal{F}\{x\} \setminus \mathcal{F}$;*
3. *the set of differential monomials of weight $k$ is finite for any $k$.*

This definition leads to the following constructive description of weight functions. Let $A$ and $B$ be non-negative integers, $B > 0$. Put $w(x) = A$ and $w(x_i) = A + iB$. One can prove that every weight function is equivalent to a weight function of this type[1] If $A = 0$, $B = 1$, we obtain the classical weight wt.

We say that a differential ideal $I$ is *isobaric* w.r.t. a weight function $w$ if, whenever a polynomial $f$ belongs to $I$, the sums of all terms in $f$ of the same weight also belong to $I$.

**Proposition 1 ([6]).** *Let $\mathcal{F}$ be a field of constants. A differential ideal $I$ in $\mathcal{F}\{x\}$ is isobaric iff it has a system of isobaric generators.*

*Example 1.* The ideal $[x_1^2 + x_0^3]$ is isobaric w.r.t. the following weight function:

$$w(x) = 2; \quad w(\delta x) = w(x) + 1 .$$

Let a weight function $w$ be fixed. Consider an isobaric differential ideal $[F]$ and a polynomial $h$. Let

$$G = \{\theta f \mid \theta \in \Theta,\ f \in F,\ w(\theta f) \leqslant w(h)\} .$$

One can prove that $h \in [F] \iff h \in (G)$, where $(G)$ denotes the algebraic ideal in the ring $\mathcal{F}[\{\theta x \mid \operatorname{ord}\theta \leqslant \max_{g \in G} \operatorname{ord} g\}]$. Due to Property (3) of weight functions, the latter ring is an ordinary polynomial ring in finitely many variables. Thus, the membership to isobaric ideals can be checked algorithmically (see Subsection 3.1 below). But there exist differential ideals (such as $[x_1^2 + x]$) that are not *isobarizable*.

[1] The equivalence of weight funcitons means that the partitions to the sets of monomials of the same weight coincide.

### 2.3 Ideals that Admit Finite Standard Bases

The first generalizations of the concept of Gröbner basis to differential algebra were made by Carrà Ferro [3] in 1989 (differential Gröbner bases) and then by Ollivier [13, 14] in 1990 (standard bases). These two approaches coincide in general. As well as in the algebraic case, finite standard bases allow us to test the membership; but they may be infinite, however. This fact hampered their studies for a long time.

We recall the definitions that can be found in [13, 14]. For simplicity, we give them for the ordinary ring $\mathcal{F}\{x\}$.

Let an admissible ordering $\prec$ be fixed. The polynomial $f$ can be *elementary reduced* w.r.t. the polynomial $g$, if there exists $k \geqslant 0$ such that $M := \operatorname{lm} \delta^k g$ divides some monomial $Q$ in $f$. The result of the reduction is the polynomial

$$\widetilde{f} = f - \frac{\operatorname{cf}(f, Q)}{\operatorname{lc} \delta^k g} \frac{Q}{M} \delta^k g \,.$$

This generalizes the ordinary reduction of polynomials used in the Gröbner bases technique to the differential case. The reduction relation is the reflexive-transitive closure of elementary reductions. We write this relation as $f \underset{g}{\to} \widetilde{f}$. Every chain of elementary reductions always terminates. For given polynomials $f, g$ it is possible to check effectively whether $f$ can be reduced w.r.t. $g$ and to construct the remainder $\widetilde{f}$ [13, 14]. One can also define a reduction w.r.t. a set of polynomials $G$. We extend the algorithm of differential reduction in Subsection 3.2.

A set $G$ is called a *standard basis* of a differential ideal $I$ in the ring $\mathcal{F}\{x\}$ if $G \subset I$ and every element of $I$ can be reduced to zero w.r.t. $G$. One could find equivalent definitions in [14].

In contrast to the differential case, an ordinary Gröbner basis of a non-differential ideal will be called an *algebraic standard basis*.

A standard basis $G$ of an ideal $I$ is called *minimal* if no proper subset of $G$ is a standard basis of $I$. A basis is *reduced* if no element $g \in G$ can be reduced w.r.t. $G \setminus \{g\}$.

**Theorem 1 (Carrà Ferro [4]).** *A differential polynomial $f$ forms a standard basis of the differential ideal $[f]$ w.r.t. the lexicographic ordering iff $f = I_0 + I_1 x_k$, where $I_1 \in \mathcal{F}$ and $x_k = \operatorname{lm} f$.*

If a standard basis is finite and known then it is possible to solve the membership problem effectively. For example, this is the case of linear polynomials, when a standard basis is always finite. In the general case, however, standard bases are infinite. The simplest example is a standard basis of the ideal $[x^2]$ w.r.t. the lexicographical ordering.

Ollivier [13, 14] suggested a completion process that returns a minimal standard basis of an ideal if by chance it stops. We improved this algorithm and implemented it in Maple 7. One can download it from `http://shade.msu.ru~difalg/Algorithms/DGB.mws`.

## 3 Algorithms and Theoretical Generalizations

### 3.1 The Membership to Isobaric Ideals

The following algorithms allow one to test the ideal membership in the case of isobaric generators. They have been implemented in Maple 7. For simplicity, we give them for the case of ordinary polynomials in one differential indeterminate. They can be naturally extended to the partial differential case of several ideterminates, since testing of isobarizability leads to solving a system of linear equations. By Proposition 1 the source polynomials must have constant coefficients.

The first algorithm tries to choose an appropriate weight function to isobarize a given polynomial.

**Algorithm 1: IsIsobaric**
**Input:** $f \in \mathcal{F}\{x\}$, a differential polynomial with constant coefficients.
**Output:** a pair of non-negative integers $A$, $B$ such that either

• $f$ is isobaric w.r.t. the weight function $w$ given by $w(x) = A$ and $w(\delta x) = B > 0$, or
• $f$ is not isobarizable, and then $B = 0$.

$\mathcal{M}$ := the list of all monomials occuring in $f$;
**if** $\mathcal{M} \neq \varnothing$ **then**
  **for** $i$ **from** 2 **to** card $\mathcal{M}$ **do**
    **if** $\deg \mathcal{M}[i] = \deg \mathcal{M}[1]$ **and** $\operatorname{wt} \mathcal{M}[i] = \operatorname{wt} \mathcal{M}[1]$ **then**
      **exclude** $\mathcal{M}[i]$ from $\mathcal{M}$;
    **end if**;
  **next** $i$;
**end if**;
**if** card $\mathcal{M} \leqslant 1$ **then**
  $A := 0;\quad B := 1;$
**else**
  $A := \operatorname{wt} \mathcal{M}[2] - \operatorname{wt} M[1];\ B := \deg \mathcal{M}[2] - \deg \mathcal{M}[1];$
  **if** $B \neq 0$ **then**
    **for** $i$ **from** 3 **to** card $\mathcal{M}$ **do**
      $w := \operatorname{wt} \mathcal{M}[i] - \operatorname{wt} M[1];\ d := \deg \mathcal{M}[i] - \deg \mathcal{M}[1];$
      **if** $Ad \neq Bw$ **then**
        $B := 0;$ **break**;
      **end if**;
    **next** $i$;
  **end if**;
  **if** $B \neq 0$ **then**
    $C := \gcd(A, B);\ A := A/C;\ B := B/C;$
  **end if**;
**end if**;

Obviously, this algorithm terminates. The correctness follows from the following proposition.

**Proposition 2.** *Let $M_1 = \prod_{i=0}^{m} x_i^{\alpha_i}$ and $M_2 = \prod_{j=0}^{n} x_j^{\beta_j}$ be monomials in $\mathcal{F}\{x\}$. Consider a weight function $w$ such that $w(x_0) = A$ and $w(x_1) = A + B,\ A \geqslant 0,\ B \geqslant 1$. Then*

$$w(M_1) = w(M_2) \iff A(\deg M_1 - \deg M_2) = B(\operatorname{wt} M_2 - \operatorname{wt} M_1)\,.$$

*Proof.* We have

$$w(M_1) = \sum_{i=0}^{m} \alpha_i (A + iB) = A \sum_{i=0}^{n} \alpha_i + B \sum_{i=0}^{n} i\alpha_i = A \deg M_1 + B \operatorname{wt} M_1\,,$$

and the same for $w(M_2)$. Now the proof is evident.

The next algorithm tries to find a common weight function to isobarize a *set* of polynomials.

**Algorithm 2: IsIsobaricSet**
**Input:** $G \subset \mathcal{F}\{x\}$, a finite set of differential polynomials with constant coefficients not in $\mathcal{F}$;
**Output:** a pair of non-negative integers $A$, $B$ such that either
• elements of $G$ are isobaric w.r.t. the weight function $w$ given by $w(x) = A$ and $w(\delta x) = B > 0$, or
• $G$ is not isobarizable, and then $B = 0$.

$\mathcal{M} := \varnothing;\ F := G;$
**for all** $f \in F$ **do**
  **if** $f$ is a monomial **then**
    $\mathcal{M} := \mathcal{M} \cup \{f\};\ F := F \setminus \{f\};$
  **end if**;
**next** $f$;

**if** $F \neq \varnothing$ **then**
  **choose** $f \in F$;
  $F := F \setminus \{f\}$;
  $(A, B) :=$ **IsIsobaric** $(f)$;
**else**
  $A := 0;\ B := 1$;
**end if**;
**while** $F \neq \varnothing$ **and** $B \neq 0$ **do**
  **choose** $f \in F$;
  $F := F \setminus \{f\}$;
  **if** $(A, B) \neq$ **IsIsobaric** $(f)$ **then**
    $B := 0$; **break**;
  **end if**;
**end do**;

Algorithm 3 tests the membership to a differential ideal in the case of isobaric generators. We denote by **Weight** $(A, B, f)$ the algorithm that computes the weight function (given as in Proposition 2) of the polynomial $f$.

**Algorithm 3: IsobaricIdealMembership**
**Input:** $f \in \mathcal{F}\{x\}$, a differential polynomial with constant coefficients;
$F = \{f_1, \ldots, f_s\}$, a finite set of differential polynomials with constant coefficients;
$\prec$, an admissible ordering.
**Output:** 'YES', if $f \in [F]$;
'NO', if $f \notin [F]$;
'Non-isobarizable generators', if $F$ is not isobarizable.

**if** $F$ contains a constant polynomial **then**
  **print** 'YES'; **exit**;
**end if**;
$(A, B) :=$ **IsIsobaricSet** $(F)$;
**if** $B \neq 0$ **then**
  $G := \varnothing$;
  $w :=$ **Weight**$(A, B, f)$;
  **for all** $g \in F$ **do**
    $G := G \cup \{\theta g \mid$ **Weight** $(A, B, \theta g) \leqslant w\}$;
  **next** $g$;
  $\widetilde{G} :=$ **GröbnerBasis** $(G, \prec)$;
  $\widetilde{f} :=$ **NormalForm** $(f, \widetilde{G}, \prec)$;
  **if** $\widetilde{f} = 0$ **then print** ('YES');
  **else print** ('NO');
  **end if**;
**else print** 'Non-isobarizable generators';
**end if**;

It is clear that Algorithms 2 and 3 terminate. One can easily check their correctness.

### 3.2 Standard Bases and $\delta$-stability of Admissible Orderings

Let us write $\delta_\prec P := \mathrm{lm}_\prec \delta P$ for any monomial $P$. This notation allows us to define the action of $\delta$ on monomials.

**Definition 2.** *We say that an admissible ordering $\prec$ is* $\delta$-stable *(*strictly $\delta$-stable*) if from $P \preccurlyeq Q$ it follows that $\delta_\prec P \preccurlyeq \delta_\prec Q$ $(P \prec Q \implies \delta_\prec P \prec \delta_\prec Q$, respectively).*

Obviously, lex and deglex are strictly $\delta$-stable orderings.

Introducing standard bases in differential polynomial rings, Ollivier [13] requires the properties of translation and positivity, strict $\delta$-stability and the inequality $P \prec \delta_{\prec} P$ for any monomial $P$ (*domination of differentiations*). One can easily check that the latter inequality is satisfied automatically if it is required just for differential variables. Thus, to satisfy this property it is sufficient to require the ordering to be admissible. Note that in general this property is not satisfied for polynomials.

**Proposition 3 (Zobnin [22]).** *Let $M = \prod_{i=1}^{d} x_{a_i}$, where $a_1 \leqslant a_2 \leqslant \ldots \leqslant a_d$. If the ordering $\prec$ is strictly $\delta$-stable then*

$$\operatorname{lm}_{\prec} \delta M = (\prod_{i=1}^{d-1} x_{a_i}) x_{a_d+1}.$$

*In other words,* $\operatorname{lm}_{\prec} \delta M = \operatorname{lm}_{\text{lex}} \delta M$.

**Corollary 1 (Zobnin [22]).** *Let $\prec$ be a strictly $\delta$-stable ordering, $f$ be a polynomial,* $\deg \operatorname{lm} f > 1$ *and $x_i$ be the smallest variable in* $\operatorname{lm} f$. *Then $x_i$ occurs in every* $\operatorname{lm} \delta^k f$.

Thus, strictly $\delta$-stable orderings resemble lexicographic orderings in some sense (compare with [19]).

One of the possible generalizations of standard bases can be obtained by considering only essential properties (O1)–(O3) of admissible orderings. Chains of reductions will terminate as well, since these three axioms guarantee that the set of all monomials is well ordered (see [20]).

**Definition 3.** *The ordering* degrevlex *on differential monomials is such that the degrees of monomials are compared first, and if $\sum_{i=0}^{k} \alpha_i = \sum_{i=0}^{k} \beta_i$ then*

$$\prod_{i=0}^{k} x_i^{\alpha_i} \prec_{\text{degrevlex}} \prod_{i=0}^{k} x_i^{\beta_i} \iff$$

$$\iff \textit{the left-most non-zero entry of the vector } \alpha - \beta \textit{ is positive.}$$

*In contrast to the algebraic case [5], here we compare the vectors $\alpha$ and $\beta$ starting from the left to provide the admissibility.*

Let us define a more general class of orderings:

**Definition 4 (Levi [9]).** *A monomial $M = \prod_{i=0}^{k} x_i^{\alpha_i}$ is called a $\beta$-term (w.r.t. an integer $p$) if there exists $i$, $1 \leqslant i \leqslant k$ such that $\alpha_{i-1} + \alpha_i \geqslant p$. All other monomials are called $\alpha$-terms.*

**Definition 5 (Zobnin [22]).** *An admissible ordering such that the leading monomial* $\operatorname{lm} \delta^m x^p$ *is a $\beta$-term w.r.t. $p$ for all $m > 0$ is called a $\beta$-*ordering.

One can check that for all $k \geqslant 0$ the polynomial $\delta^k x^p$ contains only one $\beta$-term w.r.t. $p$. If $k = ap + b$, where $0 \leqslant b < p$, then this $\beta$-term is equal to $x_a^{p-b} x_{a+1}^b$. Of course, if $b = 0$ we get the monomials $x_a^p$.

The examples of $\beta$-orderigns are degrevlex and wt-degrevlex (monomials are compared first by weight).

It turns out that some differential ideals acquire a finite standard basis w.r.t. generalized orderings. Using Levi's work [9], the author proved in [22] that $\{x^p\}$ is a standard basis of the ideal $[x^p]$ for any $p \geqslant 1$ and any $\beta$-ordering. It should be pointed out that it is not only finite, but it consists of one element.

At the same time we stress that $\beta$-orderings *are not strictly $\delta$-stable*:

$$\operatorname{lm} x_0 x_2 \prec_{\beta} \operatorname{lm} x_1^2, \quad \text{but} \quad \operatorname{lm} \delta(x_0 x_2) = x_1 x_2 = \operatorname{lm} \delta x_1^2.$$

In this connection the following proposition can be proved.

**Proposition 4 (Zobnin [22]).** *For any strictly $\delta$-stable ordering the ideal $[x^p]$ has no finite standard basis.*

Certainly, this fact is not very challenging from the computational viewpoint, as the membership to the ideal generated by isobaric polynomials[2] can always be effectively checked [13]. But in one way or another it can shed light on the solution of the membership problem for finitely generated differential ideals. It occurs that Olliver's standard basis is not such a hopeless theoretical tool in the non-linear case as it may seem at first sight. Nevertheless, there are a lot of problems.

Firstly, Ollivier's process for construction of standard bases (suggested in [14]), applied to the ideal $[x^p]$ with any $\beta$-ordering, never stops, though it returns at each step of the loop the set $\{x^p\}$. (In fact, it does not always stop even for strictly $\delta$-stable orderings.) It is necessary to obtain an effective criterion for this process to stop.

Secondly, it is quite possible that there may exist finitely generated differential ideals that do not admit finite standard bases for any ordering.

### 3.3 Cancellation of Monomials in Derivatives

It is clear that if all coefficients of $f$ are of the same sign then no cancellations of monomials in derivatives of $f$ can occur. This means that sooner or later *all* monomials of degree $d$ and weight $n+w$ in variables starting from $x_{n_0}$ will be presented in $\delta^n f$ for some $n_0$. Nevertheless, if the coefficients of $f$ have different signs, disappearing sequences can exist. For example, let

$$f = 2\,x_0x_2^2 - x_0x_1x_3 - x_1^2x_2\,.$$

As it is proved in [22], in the derivatives of this polynomial no $\beta$-term of the form $\{x_r^3\}$ occurs. The monomials $M_r := x_{r-1}x_r^2$ also cancel in the derivatives of $f$ because we can obtain the monomials $x_r^3$ only differentiating $M_r$.

As a consequence, we obtain the following result: *if* strict *$\delta$-stability is not required then, in general, the operators $\delta^i$ and* lm *do not commute.* This distinguishes Ollivier's standard bases from the algebraic ones. If $G$ is a standard basis of a differential ideal $I$ then the result of the autoreduction of $G$ may be not a standard basis of $I$ at all! In fact, the reduction of lower terms of some element in $G$ w.r.t. the other elements may result in a polynomial with another set of leading monomials of derivatives. Thus, although a reduced basis still can be defined as before, not every basis can be turned into a reduced one.

Let us present a modified differential reduction algorithm that is adapted to the most general orderings satisfying just main properties O1–O3. The main improvement is that possible cancellations of leading monomials in derivatives are taken into account. Denote by $\min\deg g$ and by $\min\operatorname{wt} g$ the minimal degree and the minimal weight of the monomials occuring in $g$, respectively.

**Algorithm 4: DifferentialReduction**
**Input:** $f \in \mathcal{F}\{x\}$, a differential polynomial;
$G = \{g_1, \dots, g_s\}$, a finite set of differential polynomials;
$\prec$, an admissible ordering.
**Output:** $h = \tilde{f}$, a differential remainder of $f$ w.r.t. $G$ and $\prec$.
$\widetilde{G} := \varnothing$;
**for all** $g \in G$ **do**
  **if** $\min\deg g \leqslant \deg f$ **and** $\min\operatorname{wt} g \leqslant \operatorname{wt} f$ **then**
    **for all** $\theta \in \Theta$ such that $\operatorname{ord}\theta \leqslant \operatorname{wt} f - \min\operatorname{wt} g$ **do**
      $\widetilde{G} := \widetilde{G} \cup \{\theta g\}$;
    **next** $\theta$
  **end if**;
**next** $g$;
**while** exist $g \in \widetilde{G}$ and a term $c \cdot Q$ of $h$ such that $\operatorname{lm}_\prec g$ divides $Q$ **do**
  **choose** the first such $g$;
  $h := h - \frac{\operatorname{cf}(f,Q)}{\operatorname{lc}_\prec g} \frac{Q}{\operatorname{lm}_\prec g} g$;
**end do**;

[2] Recall that ideals generated by a monomial are isobaric w.r.t. any weight function.

Lines 1–8 fill the set $\tilde{G}$ with the derivatives of the source polynomials $g_i$ that may be used in reductions of $f$. Lines 9–12 are similar to those of ordinary reduction algorithms used in the Gröbner bases technique.

### 3.4 *M*-Bases and Normalizing Systems

Standard bases in $\mathcal{F}\{X\}$ can be viewed as a suitable parametric representation of infinite Gröbner bases in the algebraic ring $\mathcal{F}[\Theta X]$. This representation is provided by derivation operators and is compatible with the structure of the differential ring $\mathcal{F}\{X\}$. Therefore, we deal with one element $f$ instead of the family $\Theta f$. In other words, we can define a differential standard basis in the following way:

**Definition 6.** *A pair $(G, \prec)$, where $G \subset I$ is a set of polynomials and $\prec$ is an admissible ordering, is a differential standard basis of an ideal $I$, if $\Theta G$ forms an algebraic standard basis of $I$.*

How can we generalize such suitable representations? Here we give some ideas.

**Definition 7.** *Let us call a system $(G, H, \prec)$, where $G$ and $H$ are finite sets of polynomials and $\prec$ is an admissible ordering, an $M$-basis*[3] *of the set $F$, if $\Theta G \cdot \Theta H$ forms an algebraic standard basis of $F$.*

**Definition 8.** *Let $S$ be any subset of $\mathcal{F}\{x\}$. An ordered set of $M$-bases*

$$\mathcal{G} = \{(G_1, H_1, \prec_1), (G_2, H_2, \prec_2), \ldots\},$$

*where $G_i$ and $H_i$ are finite subsets and $\prec_i$ are admissible orderings, is said to be a* normalizing system *of $S$ if*

$$f \in S \iff \exists\, k \geqslant 1 :$$
$$f \xrightarrow[\{G_1,H_1,\prec_1\}]{} h_1,\ \ h_1 \xrightarrow[\{G_2,H_2,\prec_2\}]{} h_2,\ \ \ldots,\ \ h_{k-1} \xrightarrow[G_k,H_k,\prec_k]{} 0\,.$$

We see that if an ideal $I$ has a finite standard basis $G$ w.r.t. $\prec$ then $I$ also has the $M$-basis $(G, \{1\}, \prec)$ as well as a finite normalizing system. Evidently, the concepts of $M$-basis and normalizing system are wider than that of standard basis.

Since the orderings $\prec_i$ might be different, one cannot guarantee that $h_{i+1} \preccurlyeq h_i$ for $f \in \mathcal{F}\{x\}$. But if all $\prec_i$ are equal, $H_i = \{1\}$ and the union of all $G_i$ is finite then the reduction w.r.t. $\mathcal{G}$ can be viewed as a special normal form algorithm for the standard basis $\cup_i G_i$ of $S$.

Any differential ideal $[F]$ admits the following infinite normalizing system: the orderings $\prec_i$ are arbitrary but fixed and $G_i$ are the Gröbner bases (w.r.t. $\prec_i$) of the algebraic ideals $J_i$ generated by the elements and the derivatives of $F$ of order lower than $i$.

Consider the following basic examples.

*Example 2.* Let $I = [\{x_i^2 \mid i \geqslant 0\}]$. This is the simplest example of an infinitely generated differential ideal [6]. For any admissible ordering $\prec$ the system $(\{x_0\}, \{x_0\}, \prec)$ is an $M$-basis of $I$.

*Example 3.* Consider the ideal $I = [x_0x_1]$ obtained from $[x^2]$ simply by differentiating the generator. It seems that this ideal does not have a finite standard basis for any ordering [22]. But one can prove that a system

$$(\{x_0x_1\}, \{1\}, \prec_{\text{degrevlex}})$$

reduces every element of this ideal to the form $x_0^2 f$, where every term of $f$ contains some $x_i$, $i \geqslant 1$. On the other hand the monomials $x_0^2 x_i$ belong to $I$ for all $i \geqslant 1$ [22]. Thus,

$$\{(\{x_0x_1\}, \{1\}, \prec_{\text{degrevlex}}),\ (\{x_0^2\}, \{x_1\}, \prec)\}$$

is a normalizing system of $I$ for any $\prec$.

It is not clear yet whether there exist finitely generated differential ideals that have no finite normalizing systems. Even more so that we do not know how to construct such systems effectively. But they could become a good theoretical generalization of standard bases in the future research.

[3] $M$ stands for “multiplicative”.

## 4 Ideals of the Form $[x_1^2 + \alpha\, x_0 + \beta]$

The simplest example of a non-isobarizable ideal is $I := [x_1^2 + x_0]$. It is interesting to study ideals of this kind. First of all, one can prove that $I$ is non-radical: $x_3^2 \in I$, but $x_3 \notin I$. The natural problem is to represent $x_3^2$ via the generator $f_1 := x_1^2 + x_0$ and its derivatives. For purposes of this kind, the extended Buchberger algorithm has been implemented in Maple. It allows one to express new elements of the Gröbner basis via the source elements. Therefore, one can compute the Gröbner basis of $(f_1, \dots, f_k)$ where $f_i := \delta^{i-1} f_1$, for a sufficiently large $k$. It turns out that for $x_3^2$ one can choose $k = 5$. Unfortunately, the representation given by the extended Buchberger algorithm is often too complicated. In our case it consists of more than 100 summands.

At the same time there exist more simple representations for $x_3^2$. For instance, Prof. M.V. Kondratieva has found the following one:

$$
\begin{aligned}
x_3^2 = (-10\, x_2 x_3 x_4 - \frac{5}{3}\, x_3 x_4 - \frac{1}{3}\, x_2 x_5 + 30\, x_3^3)\, f_2 + \\
+ (10\, x_1 x_3 x_4 - 9\, x_3^2 + \frac{1}{3}\, x_1 x_5)\, f_3 + \\
+ (x_3 + 3\, x_2 x_3 - 10\, x_1 x_3^2)\, f_4 - \\
- \frac{1}{3}\, x_1 x_3\, f_5 .
\end{aligned}
$$

It appears that all monomials of the form $x_i x_j$, $i, j \geqslant 3$ are in $I$. In fact, the ideal $[\{x_i x_j\}]$ coincides with $[\{x_i^2\}]$ considered above. These statements were proved and generalized after numerous computations of Gröbner bases of $(f_1, \dots, f_k)$ for different $k$.

**Lemma 1.** *The families of differential polynomials*

1. $g_1 := x_0 x_3$,
2. $g_2 := x_0^2 x_4$,
3. $h := 2\, x_2^2 + x_2 - 2\, x_0\, x_4$,
4. $h_k^0 := 4\, x_0 x_k + (k-2)(k-3) x_{k-2}, \;\; k \geqslant 5$,
5. $h_k^1 := 2\, x_1 x_k - (k-2) x_{k-1}, \;\; k \geqslant 4$,
6. $h_k^2 := 2\, x_2 x_k + x_k, \;\; k \geqslant 3$,
7. $h_k^3 := x_3 x_{k-1}, \;\; k \geqslant 4$

*belong to the ideal* $[x_1^2 + x_0]$.

*Proof.* Let us use the induction on $k$. One can directly check that the reduced Gröbner basis of $(f_1, f_2, f_3, f_4, f_5)$ w.r.t. the deglex ordering contains the polynomials $g_1, g_2, h$ and (4) – (7) for admissible $k \leqslant 5$. We have

$$\delta h_k^0 = 4\, x_1 x_k + 4\, x_0 x_{k+1} + (k-2)(k-3) x_{k-1} = h_{k+1}^0 + 4\, x_1 x_k - 2(k-2) x_{k-1} = h_{k+1}^0 + 2 h_k^1 .$$

Hence, $h_{k+1}^0 \in I$. Similarly, $\delta h_k^1 = h_{k+1}^1 + h_k^2$ and $h_{k+1}^1 \in I$.

Consider the $S$-polynomial $x_2 \delta h_k^2 - x_{k+1} h = 2\, x_2 x_3 x_k + 2\, x_0 x_4 x_{k+1} \in I$. Reducing it w.r.t. $h_{k+1}^0$ and $h_k^2$, we obtain

$$x_3 x_k + \frac{(k-1)(k-2)}{2} x_4 x_{k-1} \in I .$$

At the same time $\delta h_k^3 = x_3 x_k + x_4 x_{k-1} \in I$. From this for $k \geqslant 4$ we get $x_4 x_{k-1} \in I$ and, therefore, $h_{k+1}^3 = x_3 x_{k+1} \in I$.

Thus, $h_{k+1}^2 = \delta h_k^2 - 2\, x_3 x_k \;\in\; I$. The inductive step is completed and the lemma is proved.

**Proposition 5.** $[x_1^2 + x_0] \cap \mathcal{F}\{x_3\} = [x_i\, x_j,\; i, j \geqslant 3]$.

*Proof.* By the previous lemma, $x_3x_k \in I$ for all $k \geqslant 3$. Assume that $x_ix_j \in I$ for given $i$ and all $j \geqslant 3$. Then we have $x_{i+1}x_j = \delta(x_ix_j) - x_ix_{j+1} \in I$ for all $j \geqslant 3$. By induction we conclude that $x_ix_j \in I \,\forall\, i,j \geqslant 3$, i.e., $[x_i\,x_j,\ i,j \geqslant 3] \subset [x_1^2 + x_0] \cap \mathcal{F}\{x_3\}$.

The converse inclusion holds true due to the following proposition.

**Proposition 6.** *Polynomials (1) – (7) given above, the polynomials $x_ix_j$, $i,j \geqslant 3$, and the polynomials $f_1, f_2$ and $\delta g_1 = x_0x_4 + x_1x_3$ form the reduced algebraic standard basis w.r.t. the* degrevlex *ordering of the ideal $I = [x_1^2 + x_0]$ in the non-differential ring $\mathcal{F}[\{x_i \mid i \geqslant 0\}]$.*

*Proof.* One can write down explicitly all essential $S$-polynomials of these elements and check that they reduce to zero.

**Theorem 2.** *For all $\alpha, \beta \in \mathcal{F}$, $\alpha \neq 0$, we have $[x_1^2 + \alpha\, x_0 + \beta] \cap \mathcal{F}\{x_3\} = [x_i\,x_j,\ i,j \geqslant 3]$.*

*Proof.* Consider a differential automorphism $\phi$ of the ring $\mathcal{F}\{x\}$:

$$\phi(x_0) = \alpha\, x_0 + b\,;$$

$$\phi(\delta) = \frac{1}{a}\,\delta\,.$$

This automorphism maps the ideal $I = [x_1^2 + x_0]$ to the ideal $\phi(I) = [x_1^2 + \alpha\, x_0 + \beta]$. Moreover, $\phi(x_i) = a^{1-i}\,x_i$ for $i \geqslant 1$. Thus, $a^{i+j-2}\,\phi(x_ix_j) = x_ix_j \in \phi(I)$ for $i,j \geqslant 3$ by Proposition 5.

**Proposition 7.** *If $\beta \neq 0$ then $[x_1^2 + \beta] \cap \mathcal{F}\{x_2\} = [x_2]$.*

*Proof.* For $f = x_1^2 + \beta$ we have $2\,x_2 f - x_1\delta f = 2\,\beta\, x_2 \in [f]$.

## 5 Conclusions

We gave an overview of the membership problem for differential ideals in ordinary differential polynomial rings in one variable. This puzzling problem is far from being solved except for several important cases. Since the case of radical ideals has been well studied, we were interested in ideals of the form $[F]$ given by their differential generators. We presented the algorithms to test whether the set $F$ is isobarizable. In the case of the positive answer it is possible to test the membership to $[F]$.

We extended the notion of differential Gröbner basis to the more general class of admissible orderings and presented the advantages and disadvantages of this extension. In order to work with these general orderings we improved the algorithm of differential reduction.

We suggested the concepts of $M$-basis and normalizing system that generalize Ollivier's standard bases. Using computations in Maple we studied certain non-isobarizable differential ideals such that $[x_1^2 + x]$.

The author hopes that the ideas described in this paper can be generalized to the case of any finitely generated ideals and lead to the complete solution of the membership problem.

## 6 Acknowledgements

I am grateful to my scientific advisor Prof. E.V. Pankratiev and to all the participants of the seminar on computer and differential algebra at the Faculty of Mechanics and Mathematics of MSU, especially to Prof. M.V. Kondratieva and to A.I. Ovchinnikov. I also would like to thank Prof. François Ollivier for his valuable comments.

## References


1. Boulier, F.: *Étude et implantation de quelques algorightmes en algèbre différentielle.* Thése de l'Université des Sciences et Technologies de Lille (1994)
2. Boulier, F., Lazard, D., Ollivier, F., Petitot, M.: Representation for the Radical of a Finitely Generated Differential Ideal. In: *Proc. ISSAC'95*, A.H.M. Levelt (Ed.), ACM Press, New York (1995) 158–166
3. Carrà Ferro, G.: Gröbner bases and differential algebra. *Lecture Notes in Computer Science* **356** (1989) 129–140
4. Carrà Ferro, G.: Differential Gröbner bases in one variable and in the partial case. *Math. Comput. Modelling*, Pergamon Press **(25)** (1997) 1–10
5. Cox, D., Little, J., O'Shea, D.: *Ideals, Varieties and Algorithms*, 2nd edition, Springer-Verlag, New York (1998)
6. Gallo, G., Mishra, B., Ollivier, F.: Some constructions in rings of differential polynomials. *Lecture Notes in Computer Science* **539** (1991) 171–182
7. Hubert E.: Factorization-free decomposition algorithm in differential algebra. *J. Symbolic Computation* **29** (2000) 641–662
8. Kolchin, E.R.: *Differential Algebra and Algebraic Groups*, Academic Press, New York (1973)
9. Levi, H.: On the structure of differential polynomials and on their theory of ideals. *Trans. AMS* **51** (1942) 532–568
10. Mansfield, E.: *Differential Gröbner Bases*, Ph.D. Thesis, University of Sydney (1991)
11. Mead, D.G.: A necessary and sufficient condition for membership in $[uv]$. *Proc. AMS* **17** (1966) 470–473
12. Mead, D.G., Newton, M.E.: Syzygies in $[y^p z]$. *Proc. AMS* **43 (2)** (1974) 301–305
13. Ollivier, F.: *Le problème de l'identifiabilité structurelle globale.* Doctoral Dissertation, Paris (1990)
14. Ollivier, F.: Standard bases of differential ideals. *Lecture Notes in Computer Science* **508** (1990) 304–321
15. O'Keefe, K.B.: A property of the differential ideal $[y^p]$. *Trans. AMS* **94** (1960) 483–497
16. Pankratiev, E.V.: Some approaches to construction of standard bases in commutative and differential algebra. In: *Proc. CASC 2002*, V.G. Ganzha, E.W. Mayr, E.V. Vorozhtsov (Eds.), Technische Universität München, Garching, Germany (2002) 265–268
17. Ritt, J.F.: *Differential Algebra*, volume XXXIII of Colloquium Publications, AMS, New York (1950)
18. Sit, W.: The Ritt–Kolchin theory for differential polynomials. In: *Differential Algebra and Related Topics*, World Scientific, Singapore (2001)
19. Weispfenning, V.: Differential term-orders. In: *Proc. ISSAC'93*, M. Bronstein (Ed.), ACM Press, New York (1993) 245–253
20. Zobnin, A.: Essential properties of admissible orderings and rankings. In: *Contributions to General Algebra 14* (2004) 205–221
    Available online at `http://shade.msu.ru/~difalg/Articles/Our/Zobnin/Zobnin_CGA14.ps`
21. Zobnin, A.: Generalized reduction in rings of differential polynomials. *Programming and Computer Software* **30 (2)** (2004) 88–94
    Available online at `http://shade.msu.ru/~difalg/Articles/Our/Zobnin/Zobnin_Prog2004.ps`
22. Zobnin, A.: On standard bases in differential polynomial rings. To appear in *Journal of Mathematical Sciences*, New York (2004)
    Available online at `http://shade.msu.ru/~difalg/Articles/Our/Zobnin/dgb-fam-eng.ps`