Automatica 90 (2018) 317–320

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Technical communique

Realization of time-delay systems ☆

Arvo Kaldmäe *, Ülle Kotta

Tallinn University of Technology, Akadeemia tee 21b, 12618, Tallinn, Estonia

ARTICLE INFO

Article history:
Received 7 September 2017
Received in revised form 27 November 2017
Accepted 28 November 2017
Available online 15 February 2018

Keywords:
Realization theory
Time-delay systems
Nonlinear control systems
Algebraic approaches
Integrability

ABSTRACT

The paper addresses the problem of transforming a single-input single-output nonlinear retarded time-delay system, described by an input–output equation, in the traditional observable state space form. The solution is generalized from the delay-free case and depends on integrability of certain submodule of differential 1-forms. The integrability conditions are improved to make them constructive. Finally, it is explained why one may obtain two realizations, which are not connected by bi-causal change of state coordinates.

1. Introduction

Numerous papers address the realization problem for nonlinear delay-free control systems, see Belikov, Kotta, and Tõnso (2014, 2015), Zhang, Moog, and Xia (2010) and the references therein. The same cannot be said about nonlinear time-delay systems, where up to the authors knowledge only the paper (Garcia-Ramirez, Moog, Califano, & Márquez-Martínez, 2016) addresses the special case of linear realization up to nonlinear input–output injection term. The problem has been also studied for linear time-delay systems (Glusing-Luersen, 1997) and for the case when the delay depends on the state (Verriest, 2013). However, the reverse problem, i.e., obtaining the i/o equations via state elimination has been already addressed in Anguelova and Wennberg (2009) and Halas and Anguelova (2013). It has been shown in Kotta, Kotta, onso, and Halas (2011) that a nonlinear single-input single-output (SISO) delay-free input–output (i/o) equation is realizable in the state–space form if and only if certain vector space of differential 1-forms is integrable. In the present paper this result is generalized for nonlinear retarded SISO time-delay systems with commensurable delays, i.e., for delays that are multiples of some fixed minimal delay. Extension is, however, not direct since time-delay systems are infinite dimensional. It means that the differential 1-forms have to be viewed as elements of a module, and not a vector space. A consequence is that full rank conditions are not equivalent to invertibility (matrices over ring may have full rank, but nevertheless be non-invertible within the same set of matrices). Therefore, different system properties may be generalized in (often two) different ways. One example is observability property, see Anguelova and Wennberg (2010), Garcia-Ramirez et al. (2016), Xia, Márquez-Martínez, Zagalak, and Moog (2002), Zheng and Richard (2016) and the references therein. In Garcia-Ramirez et al. (2016) one distinguishes weak and strong observability, where weak observability corresponds to full rank of certain matrix, and strong observability to invertibility of the matrix. The same situation happens when one speaks about integrability of the modules of 1-forms. Frobenius theorem is no longer appropriate, since it provides only rather restrictive sufficient conditions. In Kaldmäe, Califano, and Moog (2016) integrability problem is studied for nonlinear time-delay systems. Two notions – weak and strong integrability – are defined and characterized. However, no constructive method is given to check the necessary and sufficient condition of strong integrability. In this paper we improve the results of Kaldmäe et al. (2016) and present a directly verifiable necessary and sufficient condition. This condition is needed to solve the realization problem in the time-delay case.

The aim of this paper is to transform a SISO retarded time-delay system, described by the i/o equation, into a state–space form, which is strongly or weakly observable. As shown already in Garcia-Ramirez et al. (2016), two realizations are not necessarily connected by bi-causal change of coordinates. The reasons are explained in this paper; the source of the problem is in two observability notions. The problem can be avoided when we require the state equations to be strongly observable.

☆ The work of A. Kaldmäe was supported by the Estonian Centre of Excellence in IT (EXCITE), funded by the European Regional Development Fund. The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Zongli Lin under the direction of Editor André L. Tits.

* Corresponding author.
E-mail addresses: arvo@cc.ioc.ee (A. Kaldmäe), kotta@cc.ioc.ee (Ü. Kotta).

https://doi.org/10.1016/j.automatica.2018.01.001

The paper is organized as follows. In Section 2 the sequence of submodules is defined in terms of which the realization problem will be studied. Integrability of 1-forms is studied in Section 3 and the main results on realization problem are presented in Section 4. The paper ends with conclusions.

2. Preliminaries

In this paper we work with nonlinear retarded single-input single-output (SISO) systems with constant commensurable delays described by the input–output equation of the form

$$\begin{aligned}&\Phi(y^{(n)}(t), y^{(n-1)}(t-i), \ldots, y(t-i), u^{(n-1)}(t-i),\\ &\quad \ldots, u(t-i); 0 \le i \le s) = 0,\end{aligned} \tag{1}$$

where Φ is analytic. Denote by $\mathcal{A}$ the ring of analytic functions depending on finite number of variables from the set $\mathcal{C} = \{y^{(k)}(t-i), u^{(k)}(t-i); i, k \in \mathbb{N}\}$. The delay operator δ is defined on $\mathcal{A}$ as $\delta\varphi(\xi(t)) = \varphi(\xi(t-1))$, where $\varphi \in \mathcal{A}$ and $\xi(t) \in \mathcal{C}$. Let I be the minimal ideal of $\mathcal{A}$ that contains Φ, all the derivatives of Φ and $\delta^i\Phi$ for all $i > 0$. Now, one can construct the quotient ring $\mathcal{A}/I$, where the addition and multiplication are defined in a natural way. We assume that I is prime, which means that $\mathcal{A}/I$ is an integral domain and thus allows to construct the field of fractions of the ring $\mathcal{A}/I$, denoted by $\mathcal{K}$. The operator δ is extended to $\mathcal{K}$ in a natural way.

Define the vector space of 1-forms as $\mathcal{E} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\varphi | \varphi \in \mathcal{K}\}$. The operator δ is extended to $\mathcal{E}$ as $\delta(\sum_j a_j \mathrm{d}\xi_j) = \sum_j \delta(a_j)\mathrm{d}\delta(\xi_j)$, where $a_j \in \mathcal{K}$ and $\xi_j \in \mathcal{C}$. Using the delay operator δ, a non-commutative polynomial ring $\mathcal{K}[\vartheta]$ can be constructed. The addition is defined in $\mathcal{K}[\vartheta]$ as usual, but for multiplication the following rule is used: $\vartheta\varphi = \delta(\varphi)\vartheta$ for $\varphi \in \mathcal{K}$. Now, the 1-forms may be alternatively viewed as elements of the module $\mathcal{M} = \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}\varphi | \varphi \in \mathcal{K}\}$. Unlike a vector space, not every module has a basis. The modules, that do have a basis, are called free modules. Since $\mathcal{K}[\vartheta]$ satisfies the left Ore condition (Xia et al., 2002), any two basis of a free module have the same cardinality, which is called the rank of the free module.

Definition 1 (*Xia et al., 2002*)**.** The closure of a free submodule $\mathcal{F}$ of $\mathcal{M}$, denoted by $cl_{\mathcal{K}[\vartheta]}(\mathcal{F})$, is defined as $cl_{\mathcal{K}[\vartheta]}(\mathcal{F}) = \{\omega \in \mathcal{M} \mid \exists p(\vartheta) \in \mathcal{K}[\vartheta],\ \text{s.t. } p(\vartheta)\omega \in \mathcal{F}\}$.

By definition, the closure of the free submodule $\mathcal{F}$ is the largest free submodule, containing $\mathcal{F}$, and having the same rank as $\mathcal{F}$. If the closure of the submodule $\mathcal{F}$ is equal to itself, then $\mathcal{F}$ is said to be closed.

We also use the set of matrices $\mathcal{K}[\vartheta]^{r\times l}$ defined over the polynomial ring $\mathcal{K}[\vartheta]$. A special subset of $\mathcal{K}[\vartheta]^{r\times r}$ is the set of unimodular matrices, denoted by $\mathcal{U}_r[\vartheta]$. A matrix $U \in \mathcal{K}[\vartheta]^{r\times r}$ is said to be unimodular if it has an inverse in $\mathcal{K}[\vartheta]^{r\times r}$. A useful property for polynomial matrices in $\mathcal{K}[\vartheta]^{r\times l}$ is the Jacobson decomposition, see Cohn (1965).

Theorem 1 (*Cohn, 1965*)**.** *For every $M(\vartheta) \in \mathcal{K}[\vartheta]^{r\times l}$, $r \le l$, there exist matrices $V(\vartheta) \in \mathcal{U}_r[\vartheta]$ and $U(\vartheta) \in \mathcal{U}_l[\vartheta]$ such that*

$$V(\vartheta)M(\vartheta)U(\vartheta) = (\Delta_r, 0_{r,l-r}), \tag{2}$$

where $0_{r,l-r}$ is the matrix with zero entries, Δ_r is square diagonal matrix with elements $(\sigma_1, \ldots, \sigma_k, 0, \ldots, 0)$ such that $\sigma_i \in \mathcal{K}[\vartheta]$, for $i = 1, \ldots, k$, and σ_i is a divisor of σ_{i+1} for all $i = 1, \ldots, k-1$, i.e., $\sigma_{i+1} = \alpha\sigma_i$ for some $\alpha \in \mathcal{K}[\vartheta]$.

Note that the matrices $U(\vartheta)$ and $V(\vartheta)$ in (2) are not unique whereas Δ_r is. The matrix $(\Delta_r, 0_{r,l-r})$ is called the Jacobson form of the matrix $M(\vartheta)$.

To make the presentation more compact, the following notations are introduced: $\xi_{[s]} = (\xi(t), \ldots, \xi(t-s))$ for all $\xi \in \mathcal{C}$. Thus, the system (1) can be rewritten as

$$\Phi(y^{(n)}, y^{(n-1)}_{[s]}, \ldots, y_{[s]}, u^{(n-1)}_{[s]}, \ldots, u_{[s]}) = 0. \tag{3}$$

Also, for time-derivatives and time-delays the following notations are used: $\mathrm{d}/\mathrm{d}t\xi = \dot{\xi}$, $\mathrm{d}^2/\mathrm{d}t^2\xi = \ddot{\xi}$, $\xi(t-i) = \xi^{[-i]}$ for $i > 0$.

In Xia et al. (2002) a sequence $\{\mathcal{H}_i; i \ge 1\}$ of submodules of $\mathcal{M}$ is used to study the accessibility property of time delay systems. Here we define similar sequence for systems of the form (3) as

$$\begin{aligned}\mathcal{H}_1 &= \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}y^{(n-1)}, \ldots, \mathrm{d}y, \mathrm{d}u^{(n-1)}, \ldots, \mathrm{d}u\}\\ \mathcal{H}_{i+1} &= \{\omega \in \mathcal{H}_i | \omega^{(1)} \in \mathcal{H}_i\}.\end{aligned} \tag{4}$$

It has been shown in Xia et al. (2002) that sequence (4) converges to a submodule, denoted by $\mathcal{H}_\infty$, and all the submodules $\mathcal{H}_i$ are closed. From now on we assume that $\mathcal{H}_\infty = \{0\}$, which guarantees that the system (3) is accessible (Xia et al., 2002). Note that, by definition, $\mathcal{H}_2 = \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}y^{(n-1)}, \ldots, \mathrm{d}y, \mathrm{d}u^{(n-2)}, \ldots, \mathrm{d}u\}$. Now, if we know two consecutive submodules $\mathcal{H}_{i-1}$ and $\mathcal{H}_i$, then Algorithm 1 can be used to compute $\mathcal{H}_{i+1}$.

Algorithm 1. Denote by ρ_i the rank of submodule $\mathcal{H}_i$ and let $\mathcal{H}_{i-1} = \mathrm{span}_{\mathcal{K}[\vartheta]}\{\eta_1, \ldots, \eta_{\rho_i}, \mu_1, \ldots, \mu_{\rho_{i-1}-\rho_i}\}$, $\mathcal{H}_i = \mathrm{span}_{\mathcal{K}[\vartheta]}\{\eta_1, \ldots, \eta_{\rho_i}\}$.

1. Compute $\dot{\eta}_j$ for $j = 1, \ldots, \rho_i$. By the definition of $\mathcal{H}_i$, $\dot{\eta}_j = \sum_{l=1}^{\rho_i} a_{j,l}\eta_l + \sum_{\sigma=1}^{\rho_{i-1}-\rho_i} c_{j,\sigma}\mu_\sigma$ for some $a_{j,l}, c_{j,\sigma} \in \mathcal{K}[\vartheta]$.
2. Construct a matrix $C \in \mathcal{K}[\vartheta]^{\rho_i\times(\rho_{i-1}-\rho_i)}$ whose elements are $c_{j,\sigma}$, $j = 1, \ldots, \rho_i$, $\sigma = 1, \ldots, \rho_{i-1} - \rho_i$.
3. Find the left-kernel $B \in \mathcal{K}[\vartheta]^{(\rho_i-\gamma)\times\rho_i}$ of the matrix C, where γ is the rank of matrix C.
4. Define the basis elements of $\mathcal{H}_{i+1}$ as $B\eta$, where $\eta = (\eta_1, \ldots, \eta_{\rho_i})^T$.

The subspaces $\mathcal{H}_i$ have the following properties.

Lemma 1. *(i) The submodule $\mathcal{H}_i$ of system (3) has rank $2n + 1 - i$.*
(ii) $\omega \in \mathcal{E}$ is an element of $\mathcal{H}_i$ iff $\omega^{(i-1)} \in \mathcal{H}_1$.

Proof. *(i)* The proof is by mathematical induction. Since in Algorithm 1 $\rho_{i-1} = 2n - 1 + 2$ and $\rho_i = 2n - i + 1$, the matrix C has dimension $(2n - i + 1) \times 1$ and the matrix B dimension $(2n - i) \times (2n - i + 1)$.

(ii) By the definition of the sequence $\{\mathcal{H}_i; i \ge 1\}$ $\omega \in \mathcal{H}_i \Leftrightarrow \dot{\omega} \in \mathcal{H}_{i-1} \Leftrightarrow \cdots \Leftrightarrow \omega^{(i-1)} \in \mathcal{H}_1$. ■

3. Integrability

Compared to the results in Kaldmäe et al. (2016), we give a different necessary and sufficient condition for checking the strong integrability of a set of 1-forms. The new condition gives a method to check the condition in Kaldmäe et al. (2016) and is thus more constructive.

Definition 2 (*Kaldmäe et al., 2016*)**.** A set of 1-forms $\{\omega_1, \ldots, \omega_k\}$, independent over $\mathcal{K}[\vartheta]$, is said to be strongly (weakly) integrable if there exist k independent functions $\{\varphi_1, \ldots, \varphi_k\}$, such that $\mathrm{span}_{\mathcal{K}[\vartheta]}\{\omega_1, \ldots, \omega_k\} = \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}\varphi_1, \ldots, \mathrm{d}\varphi_k\}$ ($\mathrm{span}_{\mathcal{K}[\vartheta]}\{\omega_1, \ldots, \omega_k\} \subseteq \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}\varphi_1, \ldots, \mathrm{d}\varphi_k\}$).

If the set of 1-forms $\{\omega_1, \ldots, \omega_k\}$ is strongly (respectively weakly) integrable, then the submodule $\mathrm{span}_{\mathcal{K}[\vartheta]}\{\omega_1, \ldots, \omega_k\}$ is said to be strongly (respectively weakly) integrable.

Next, the conditions for checking strong integrability of a set of 1-forms are developed. Define for $p \ge 0$ the sequence of vector spaces of 1-forms as $I_0^p = \mathrm{span}_{\mathcal{K}}\{\omega_1, \ldots, \omega_k, \ldots, \delta^p\omega_1, \ldots, \delta^p\omega_k\}$, and let $I_\infty^p = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\varphi_{1,p}, \ldots, \mathrm{d}\varphi_{\gamma_p,p}\}$ be the largest integrable (in terms of integration of vector spaces of 1-forms) vector space contained in I_0^p for all $p \geq 0$ and some $\gamma_p \geq 0$. By construction, $\mathrm{d}\varphi_{i,p} \in \mathrm{span}_{\mathcal{K}[\vartheta]}\{\omega_1, \ldots, \omega_k\}$ for $i = 1, \ldots, \gamma_p$ and $p \geq 0$. The exact 1-forms $\mathrm{d}\varphi_{i,p}$, $i = 1, \ldots, \gamma_p$, are independent over $\mathcal{K}$, but may not be independent over $\mathcal{K}[\vartheta]$. We are interested in whether there exist k 1-forms among $\mathrm{d}\varphi_{i,p}$, $i = 1, \ldots, \gamma_p$, that form a basis for $\mathrm{span}_{\mathcal{K}[\vartheta]}\{\omega_1, \ldots, \omega_k\}$. Theorem 2 gives necessary and sufficient conditions whether such k 1-forms exist.

Let $\xi = (y^{(n-1)}, \ldots, y, u^{(n-1)}, \ldots, u)^{\mathrm{T}}$ and $\mathrm{d}\varphi^p$ be the vector of independent (over $\mathcal{K}[\vartheta]$) basis elements of I_∞^p, such that $\mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}\varphi^p\}$ contains I_∞^p. One can always find matrices $M(\vartheta)$ and $N_p(\vartheta)$, for every $p \geq 0$, such that[1]

$$\omega = M(\vartheta)\mathrm{d}\xi \qquad \mathrm{d}\varphi^p = N_p(\vartheta)\mathrm{d}\xi, \tag{5}$$

where $\omega = (\omega_1, \ldots, \omega_k)^{\mathrm{T}}$.

Theorem 2. *A set of 1-forms $\{\omega_1, \ldots, \omega_k\}$, independent over $\mathcal{K}[\vartheta]$, is strongly integrable if and only if the Jacobson forms of matrices $M(\vartheta)$ and $N_p(\vartheta)$, defined by (5), are equal for some $p \geq 0$.*

Proof. *Necessity.* Let Λ_1 be the Jacobson form of the matrix $M(\vartheta)$. Since the set of 1-forms $\{\omega_1, \ldots, \omega_k\}$ is strongly integrable, $\omega = A(\vartheta)\mathrm{d}\varphi$ for some unimodular matrix $A(\vartheta)$ and a vector $\varphi = (\varphi_1, \ldots, \varphi_k)^{\mathrm{T}}$. Let $\bar{N}(\vartheta)$ be such that $\mathrm{d}\varphi = \bar{N}(\vartheta)\mathrm{d}\xi$. Now, $M(\vartheta) = A(\vartheta)\bar{N}(\vartheta)$ and $\Lambda_1 = V(\vartheta)M(\vartheta)U(\vartheta) = V(\vartheta)A(\vartheta)\bar{N}(\vartheta)U(\vartheta)$ for some unimodular matrices $V(\vartheta)$, $U(\vartheta)$. Since $V(\vartheta)A(\vartheta)$ is also unimodular, then Λ_1 is also the Jacobson form of $\bar{N}(\vartheta)$. Also, because $\mathrm{d}\varphi = A^{-1}(\vartheta)\omega$, then $\mathrm{d}\varphi = \mathrm{d}\varphi^p$ and $\bar{N}(\vartheta) = N_p(\vartheta)$ for some $p \geq 0$.
Sufficiency. Since the Jacobson forms of $M(\vartheta)$ and $N_p(\vartheta)$ are equal, there exist unimodular matrices $V_1(\vartheta)$, $V_2(\vartheta)$, $U_1(\vartheta)$, $U_2(\vartheta)$ such that

$$V_1(\vartheta)M(\vartheta)U_1(\vartheta) = V_2(\vartheta)N_p(\vartheta)U_2(\vartheta). \tag{6}$$

Because $N_p(\vartheta) = A(\vartheta)M(\vartheta)$ for some full rank matrix $A(\vartheta)$, then one can take $U_1(\vartheta) = U_2(\vartheta)$ in (6). Now, from (6) one gets $M(\vartheta) = V_1^{-1}(\vartheta)V_2(\vartheta)N_p(\vartheta)$. The matrix $V_1^{-1}(\vartheta)V_2(\vartheta) =: A^{-1}(\vartheta)$ is unimodular and $\omega = A^{-1}(\vartheta)N_p(\vartheta)\mathrm{d}\xi = A^{-1}(\vartheta)\mathrm{d}\varphi^p$. Therefore $\omega_i \in \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}\varphi^p\}$ for $i = 1, \ldots, k$ and since the elements of the vector of 1-forms $\mathrm{d}\varphi^p$ belong to $\mathrm{span}_{\mathcal{K}[\vartheta]}\{\omega_1, \ldots, \omega_k\}$ the set of 1-forms $\{\omega_1, \ldots, \omega_k\}$ is strongly integrable. ■

The condition of Theorem 2 can be checked step-by-step, increasing the index p at every step. By Theorem 1 in Kaldmäe et al. (2016) the index p satisfies $p \leq s(k-1)$, where s is the largest delay (in coefficients or differentials) that appears in the given set of k 1-forms.

Example 1. Check the strong integrability of the 1-forms $\omega_1 = \mathrm{d}x_2$, $\omega_2 = x_3\mathrm{d}x_1 + \mathrm{d}x_2^{[-1]}$, $\omega_3 = x_3^{[-1]}\mathrm{d}x_1^{[-1]} + x_5\mathrm{d}x_4$. Compute,[2] for $p = 0, 1, 2$, the subspaces I_∞^p: $I_\infty^0 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_2\}$, $I_\infty^1 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_2, \mathrm{d}x_2^{[-1]}, \mathrm{d}x_1\}$ and $I_\infty^2 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_2, \mathrm{d}x_2^{[-1]}, \mathrm{d}x_2^{[-2]}, \mathrm{d}x_1, \mathrm{d}x_1^{[-1]}, \mathrm{d}x_4\}$. Clearly, $\mathrm{d}\varphi^2 = (\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_4)^{\mathrm{T}}$. Now, the matrices $M(\vartheta)$ and $N_2(\vartheta)$ are

$$M(\vartheta) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ x_3 & \vartheta & 0 & 0 & 0 \\ x_3^{[-1]}\vartheta & 0 & 0 & x_5 & 0 \end{pmatrix} \; N_2(\vartheta) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

One can check that the Jacobson form of both matrices $M(\vartheta)$ and $N_2(\vartheta)$ is $(I_3, 0)$. Thus, the set of 1-forms $\{\omega_1, \omega_2, \omega_3\}$ is strongly integrable.

Lemma 2 (Kaldmäe et al., 2016). *A set of 1-forms $\{\omega_1, \ldots, \omega_k\}$ is weakly integrable if and only if the closure of the submodule, generated by $\{\omega_1, \ldots, \omega_k\}$, is strongly integrable.*

4. Realization

Problem statement. Given the system (3), transform it into the strongly observable[3] state–space form

$$\dot{x} = f(x_{[q]}, u_{[q]}) \qquad y = h(x_{[q]}) \tag{7}$$

for some $q \in \mathbb{N}$. An overview of the observability problem for systems of the form (7) is presented next.

Remark 1. For time-delay systems, unlike the delay-free case, the state is not $x(t)$ but a function, corresponding to the past time interval $[t-s, t]$ (Richard, 2003). Nevertheless, some authors (Anguelova & Wennberg, 2010; Halas & Anguelova, 2013; Xia et al., 2002) call the vector x the state of the system (7) as well as the problem statement above the realization problem (Garcia-Ramirez et al., 2016).

Consider the system (7) and let $\xi = (y, y^{(1)}, \ldots, y^{(n-1)})^{\mathrm{T}}$, $\zeta = (u, u^{(1)}, \ldots, u^{(n-2)})^{\mathrm{T}}$. Then, one can write

$$\mathrm{d}\xi = P(\vartheta)\mathrm{d}x + Q(\vartheta)\mathrm{d}\zeta. \tag{8}$$

Recall from Garcia-Ramirez et al. (2016) the definitions of weak and strong observability of system (7).

Definition 3. The system (7) is said to be weakly observable if the matrix $P(\vartheta)$ in (8) has rank n over the polynomial ring $\mathcal{K}[\vartheta]$, and strongly observable if the matrix $P(\vartheta)$ is unimodular.

The strong observability yields that one can represent the state x as a function of variables $\{y_{[r]}, \ldots, y_{[r]}^{(n-1)}, u_{[r]}, \ldots, u_{[r]}^{(n-2)}\}$ for some $r \in \mathbb{N}$. Really, since $P(\vartheta)$ is unimodular, one gets from (8) $\mathrm{d}x = P^{-1}(\vartheta)\mathrm{d}\xi - Q(\vartheta)\mathrm{d}\zeta$. On the other hand, the weak observability yields that $a_i(\vartheta)x_i$, $i = 1, \ldots, n$, $a_i(\vartheta) \in \mathcal{K}[\vartheta]$, can be written as a function of variables $\{y_{[r]}, \ldots, y_{[r]}^{(n-1)}, u_{[r]}, \ldots, u_{[r]}^{(n-2)}\}$.

The solution to the realization problem is given in terms of the sequence $\{\mathcal{H}_i; i > 0\}$ of submodules of system (3). Since these submodules are closed, then strong and weak integrability definitions coincide for them and from now on we talk about integrability of $\mathcal{H}_i$.

Theorem 3. *The realization problem is solvable for system (3) if and only if $\mathcal{H}_{n+1}$ is integrable.*

Proof. *Sufficiency.* By (*i*) of Lemma 1 the submodule $\mathcal{H}_{n+1}$ has n basis elements, which, since $\mathcal{H}_{n+1}$ is integrable, can be chosen as locally exact 1-forms. Let $\mathcal{H}_{n+1} = \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}\varphi_1, \ldots, \mathrm{d}\varphi_n\}$. By (*ii*) of Lemma 1, $\mathrm{d}u \in \mathcal{H}_n$, but $\mathrm{d}u \notin \mathcal{H}_{n+1}$. Then, since $\mathcal{H}_{n+1}$ is closed, the submodule $\mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}\varphi_1, \ldots, \mathrm{d}\varphi_n, \mathrm{d}u\}$ is closed and contained in $\mathcal{H}_n$, thus equals $\mathcal{H}_n$. Choose $x_i = \varphi_i$. Then $\mathrm{d}\dot{x}_i \in \mathcal{H}_n$, i.e., $\dot{x}_i = f_i(x_{[q]}, u_{[q]})$ for some function $f_i \in \mathcal{K}$ and $q \in \mathbb{N}$. Similarly, by (*ii*) of Lemma 1 and (3), $\mathrm{d}y \in \mathcal{H}_{n+1}$, there exists a function $h \in \mathcal{K}$ such that $y = h(x_{[q]})$ for some $q \in \mathbb{N}$.
Necessity. Because (7) is assumed to be strongly observable, one can write the state x as $x = \psi(y_{[p]}, \ldots, y_{[p]}^{(n-1)}, u_{[p]}, \ldots, u_{[p]}^{(n-1)})$ and since from (7), $x^{(n)} = F(x_{[p]}, u_{[p]}, \ldots, u_{[p]}^{(n-1)})$, then $\mathrm{d}x^{(n)} \in \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}y^{(n-1)}, \ldots, \mathrm{d}y, \mathrm{d}u^{(n-1)}, \ldots, \mathrm{d}u\} = \mathcal{H}_1$. Thus, by (*ii*) of Lemma 1, $\mathrm{d}x = \mathrm{d}\psi(\cdot) \in \mathcal{H}_{n+1}$. Since $\mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}x\}$ is closed and the rank of $\mathcal{H}_{n+1}$ is n, then $\mathcal{H}_{n+1} = \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}x\}$, which means that $\mathcal{H}_{n+1}$ is integrable. ■

[1] Note that matrices $N_p(\vartheta)$ are not unique since $\mathrm{d}\varphi^p$ are not unique.

[2] Note that $s(k-1) = 2$ in this example.

[3] See the definition of strong observability below.

Based on the proof of Theorem 3, the following algorithm can be used to transform Eq. (3) into the form (7), whenever possible.

Algorithm 2. Given the system (3),

1. Compute the submodule $\mathcal{H}_{n+1}$ using the Algorithm 1.
2. Check by Theorem 2 whether $\mathcal{H}_{n+1}$ is integrable, i.e., has locally an exact basis $\mathrm{d}\psi = (\mathrm{d}\psi_1, \dots, \mathrm{d}\psi_n)$. If yes, then continue, otherwise, the problem is not solvable.
3. Define $x = \psi(y_{[p]}, \dots, y_{[p]}^{(n-1)}, u_{[p]}, \dots, u_{[p]}^{(n-1)})$.

Example 2. Consider the i/o equation

$$\ddot{y} = \dot{u} + y + y^{[-1]}. \tag{9}$$

By (4), compute $\mathcal{H}_3 = \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}(\dot{y} - u), \mathrm{d}y\}$. Thus, by the proof of Theorem 3, define $x_1 = y, x_2 = \dot{y} - u$. Therefore,

$$\dot{x}_1 = x_2 + u \quad \dot{x}_2 = x_1 + x_1^- \quad y = x_1. \tag{10}$$

Example 3. Consider the i/o equation

$$\ddot{y} = \dot{y}y^{[-1]} + y\dot{y}^{[-1]} + y^{[-1]}u^{[-3]}. \tag{11}$$

Compute $\mathcal{H}_3 = \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}\dot{y}, \mathrm{d}y\}$, which is integrable. Again, by the proof of Theorem 3, $x_1 = y, x_2 = \dot{y}$. This gives the state equations $\dot{x}_1 = x_2, \dot{x}_2 = x_2x_1^{[-1]} + x_1x_2^{[-1]} + x_1^{[-1]}u^{[-3]}, y = x_1$. Of course, one can choose the exact basis elements of $\mathcal{H}_3$ differently. For example, $\mathrm{d}y$ and $\mathrm{d}(\dot{y} - yy^{[-1]})$ form also an exact basis of $\mathcal{H}_3$ and taking $x_1 = y, x_2 = \dot{y} - yy^{[-1]}$ yields linear realization up to input–output injection as found in Garcia-Ramirez et al. (2016).

4.1. Generalized problem statement

It has been already shown that in time-delay case two systems of the form (7), not related by any invertible change of state coordinates, may have the same input–output representation (Garcia-Ramirez et al., 2016). Here we explain why this can happen by showing how two such realizations can be obtained. In the problem statement above, we required the system (7) to be *strongly* observable, which means that $x = \psi(y_{[p]}, \dots, y_{[p]}^{(n-1)}, u_{[p]}, \dots, u_{[p]}^{(n-1)}) \in \mathcal{K}^n$. Under the assumption of weak observability, x is not necessarily a function of variables from the set $\mathcal{C}$, though $a_i(\vartheta)x_i = \psi_i(y_{[p]}, \dots, y_{[p]}^{(n-1)}, u_{[p]}, \dots, u_{[p]}^{(n-1)})$ for some polynomials $a_1(\vartheta), \dots, a_n(\vartheta)$. Now, if it happens that $\dot{\psi}_i(\cdot) = b_i(\vartheta)f_i(\psi_{[p]}, u_{[p]})$, where $b_i(\vartheta)$ satisfies $\mathrm{d}/\mathrm{d}t(a_i(\vartheta)x_i) = b_i(\vartheta)\dot{x}_i$, then $b_i(\vartheta)\dot{x}_i = b_i(\vartheta)f_i(\psi_{[p]}, u_{[p]})$ and one can factor out the polynomial $b_i(\vartheta)$ to define $\dot{x}_i = f_i(x_{[p]}, u_{[p]})$.

Example 4. Consider again the system (9). Now, we do not define $x = \psi(\cdot)$ as in Example 2, but $a_1(\vartheta)x_1 = y$ and $a_2(\vartheta)x_2 = \dot{y} - u$ for some polynomials $a_i(\vartheta) \in \mathcal{K}[\vartheta]$, $i = 1, 2$. To find $a_i(\vartheta)$, compute the time-derivatives of ψ (i.e., y and $\dot{y} - u$): $\dot{\psi}_1 = \dot{y} = a_2(\vartheta)x_2 + u$, $\dot{\psi}_2 = \ddot{y} - \dot{u} = (1 + \vartheta)y = (1 + \vartheta)a_1(\vartheta)x_1$. Clearly, one can take $a_1(\vartheta) = 1$ and $a_2(\vartheta) = (1 + \vartheta)a_1(\vartheta) = 1 + \vartheta$. Therefore, the transformation $z_1 = y, (1 + \vartheta)z_2 = \dot{y} - u$ yields

$$\dot{z}_1 = z_2 + z_2^- + u \quad \dot{z}_2 = z_1 \quad y = z_1. \tag{12}$$

Note that the two realizations (10) and (12) of the i/o equation (9) are not related through a bi-causal transformation, i.e., one cannot express z variables in terms of x variables and their delays. Also note that the system (12) is weakly, while the system (10) is strongly observable.

5. Conclusion

A necessary and sufficient condition for possibility to transform a SISO nonlinear time-delay i/o equation into the strongly or weakly observable state space form was given. The condition depends on integrability of certain module of differential 1-forms. The integrability conditions for time-delay case were made more constructive. The paper also explains why and how in the time-delay case one can obtain realizations, which are not connected by bi-causal change of state coordinates. The explicit formulas for computation of the differentials of state coordinates from Belikov et al. (2015) can be extended for time-delay case. These formulas are found from the left quotients computed by the left Euclidean division algorithm. Though Euclidean division is known not to work for polynomials defined over rings, in the special case, necessary for the realization problem, when we just divide by $s = \mathrm{d}/\mathrm{d}t$, the algorithm works.

References

Anguelova, M., & Wennberg, B. (2009). Input–output representation and identifiability of delay parameters for nonlinear systems with multiple time-delays. In *Lecture notes in control and information science*: *vol. 388*. *Topics in time delay systems. analysis, algorithms and control* (pp. 243–253). Berlin, Heidelberg: Springer.

Anguelova, M., & Wennberg, B. (2010). On analytic and algebraic observability of nonlinear delay systems. *Automatica*, *46*, 682–686.

Belikov, J., Kotta, Ü., & Tõnso, M. (2014). Adjoint polynomial formulas for nonlinear state-space realization. *IEEE Transactions on Automatic Control*, *59*(1), 256–261.

Belikov, J., Kotta, Ü., & Tõnso, M. (2015). Realization of nonlinear MIMO system on homogeneous time scales. *European Journal of Control*, *23*, 48–54.

Cohn, R. M. (1965). *Difference algebra*. New York: Wiley-Interscience.

Garcia-Ramirez, E., Moog, C. H., Califano, C., & Márquez-Martínez, L. A. (2016). Linearisation via input–output injection of time delay systems. *International Journal of Control*, *89*(6), 1125–1136.

Glusing-Luersen, H. (1997). Realization of behaviors given by delay-differential equations. In *Proceedings of the 1997 European control conference*, Brussels, Belgium (pp. 1171–1176).

Halas, M., & Anguelova, M. (2013). When retarded nonlinear time-delay systems admit an input–output representation of neutral type. *Automatica*, *49*, 561–567.

Kaldmäe, A., Califano, C., & Moog, C. H. (2016). Integrability for nonlinear time-delay systems. *IEEE Transactions on Automatic Control*, *61*(7), 1912–1917.

Kotta, Ü., Kotta, P., Tõnso, M. T., & Halas, M. (2011). State-space realization of nonlinear input–output equations: unification and extension via pseudo-linear algebra. In *9th IEEE Conference on control and automation*, Santiago, Chile (pp. 354–359).

Richard, J. P. (2003). Time-delay systems: an overview of some recent advances and open problems. *Automatica*, *39*(10), 1667–1694.

Verriest, E. I. (2013). State space realization for systems with state dependent delay. In *11th IFAC workshop on time-delay systems*, Grenoble, France (pp. 451–456).

Xia, X., Márquez-Martínez, L. A., Zagalak, P., & Moog, C. H. (2002). Analysis of nonlinear time-delay systems using modules over non-commutative rings. *Automatica*, *38*(9), 1549–1555.

Zhang, J., Moog, C. H., & Xia, X. (2010). Realization of multivariable nonlinear systems via the approaches of differential forms and differential algebra. *Kybernetika*, *46*(5), 799–830.

Zheng, G., & Richard, J. P. (2016). Identifiability and observability of nonlinear time-delay systems with unknown inputs. In *Recent results on nonlinear delay control systems* (pp. 385–403). Springer.