Simple strategies for Banach-Mazur games
and fairly correct systems *

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In 2006, Varacca and Völzer proved that on finite graphs, ω-regular large sets coincide with ω-regular sets of probability 1, by using the existence of positional strategies in the related Banach-Mazur games. Motivated by this result, we try to understand relations between sets of probability 1 and various notions of simple strategies (including those introduced in a recent paper of Grädel and Leßenich). Then, we introduce a generalisation of the classical Banach-Mazur game and in particular, a probabilistic version whose goal is to characterise sets of probability 1 (as classical Banach-Mazur games characterise large sets). We obtain a determinacy result for these games, when the winning set is a countable intersection of open sets.

1 Introduction

Systems (automatically) controlled by computer programs abound in our everyday life. Clearly enough, it is of a capital importance to know whether the programs governing these systems are *correct*. Over the last thirty years, formal methods for verifying computerised systems have been developed for validating the adequation of the systems against their requirements. Model checking is one such approach: it consists first in modelling the system under study (for instance by an automaton), and then in applying algorithms for comparing the behaviours of that model against a specification (modelled for instance by a logical formula). Model checking has now reached maturity, through the development of efficient symbolic techniques, state-of-the-art tool support, and numerous successful applications to various areas.

As argued in [9]: *'Sometimes, a model of a concurrent or reactive system does not satisfy a desired linear-time temporal specification but the runs violating the specification seem to be artificial and rare'*. As a naive example of this phenomenon, consider a coin flipped an infinite number of times. Classical verification will assure that the property stating *"one day, we will observe at least one head"* is false, since there exists a unique execution of the system violating the property. In some situations, for instance when modeling non-critical systems, one could prefer to know whether the system is *fairly correct*. Roughly speaking, a system is fairly correct against a property if the set of executions of the system violating the property is *"very small"*; or equivalently if the set of executions of the system satisfying the property is *"very big"*. A first natural notion of fairly correct system is related to probability: *almost-sure correctness*. A system is almost-surely correct against a property if the set of executions of the system satisfying the property has probability 1. Another interesting notion of fairly correct system is related to topology: *large correctness*. A system is largely correct against a property if the set of executions of the

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system satisfying the property is *large* (in the topological sense). There exists a lovely characterisation of *large sets* by means of the *Banach-Mazur games*. In [8], it has been shown that a set W is large if and only if a player has a winning strategy in the related Banach-Mazur game.

Although, the two notions of *fairly correct systems* do not coincide in general, in [9], the authors proved (amongst other results) the following result: when considering ω-regular properties on finite systems, the *almost-sure correctness* and the *large correctness* coincide, for bounded Borel measures. Motivated by this very nice result, we intend to extend it to a larger class of specifications. The key ingredient to prove the previously mentioned result of [9] is that when considering ω-regular properties, *positional* strategies are sufficient in order to win the related Banach-Mazur game [1]. For this reason, we investigate *simple strategies* in Banach-Mazur games, inspired by the recent work [4] where infinite graphs are studied.

Our contributions. In this paper, we first compare various notions of simple strategies on finite graphs (including *bounded* and *move-counting* strategies), and their relations with the sets of probability 1. Given a set W, the existence of a bounded (resp. move-counting) winning strategy in the related Banach-Mazur game implies that W is a set of probability 1. However there exist sets W of probability 1 for which there is no bounded and no move-counting winning strategy in the related Banach-Mazur game. Therefore, we introduce a generalisation of the classical Banach-Mazur game and in particular, a probabilistic version whose goal is to characterise sets of probability 1 (as classical Banach-Mazur games characterise large sets). We obtain the desired characterisation in the case of countable intersections of open sets. This is the main contribution of the paper. As a byproduct of the latter, we get a determinacy result for our probabilistic version of the Banach-Mazur game for countable intersections of open sets.

2   Banach-Mazur Games on finite graphs

Let $(X, \mathscr{T})$ be a topological space. A notion of topological "bigness" is given by large sets. A subset $W \subset X$ is said to be *nowhere dense* if the closure of W has empty interior. A subset $W \subset X$ is said to be *meagre* if it can be expressed as the union of countably many nowhere dense sets and a subset $W \subset X$ is said to be large if W^c is meagre. In particular, we remark that a countable intersection of large sets is still large and that if $W \subset X$ is large, then any set $Y \supset W$ is large.

If $G = (V,E)$ is a finite directed graph and $v_0 \in V$, then the space of infinite paths in G from v_0, denoted $\mathrm{Paths}(G,v_0)$, can be endowed with the complete metric

$$d((\sigma_n)_{n \geq 0}, (\rho_n)_{n \geq 0}) = 2^{-k} \quad \text{where} \quad k = \min\{n \geq 0 : \sigma_n \neq \rho_n\} \tag{2.1}$$

with the conventions that $\min \emptyset = \infty$ and $2^{-\infty} = 0$. In other words, the open sets in $\mathrm{Paths}(G,v_0)$ endowed with this metric are the countable unions of cylinders, where a cylinder is a set of the form $\{\rho \in \mathrm{Paths}(G,v_0) \mid \pi \text{ is a prefix of } \rho\}$ for some finite path π in G from v_0.

We can therefore study the large subsets of the metric space $(\mathrm{Paths}(G,v_0),d)$. Banach-Mazur games allow us to characterise large subsets of this metric space through the existence of winning strategies.

Definition 2.1. A *Banach-Mazur game* $\mathscr{G}$ on a finite graph is a triplet (G,v_0,W) where $G = (V,E)$ is a finite directed graph where every vertex has a successor, $v_0 \in V$ is the initial state, W is a subset of the infinite paths in G starting in v_0.

A Banach-Mazur game $\mathscr{G} = (G,v_0,W)$ on a finite graph is a two-player game where Pl. 0 and Pl. 1 alternate in choosing a finite path as follows: Pl. 1 begins with choosing a finite[1] path π_1 starting in v_0;

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[1]In this paper, we always assume that a finite path is non-empty.

Pl. 0 then prolongs π_1 by choosing another finite path π_2 and so on. A play of $\mathscr{G}$ is thus an infinite path in G and we say that Pl. 0 wins if this path belongs to W, while Pl. 1 wins if this path does not belong to W. The set W is called the winning condition. It is important to remark that, in general, in the literature, Pl. 0 moves first in Banach-Mazur games but in this paper, we always assume that Pl. 1 moves first in order to bring out the notion of large set (rather than meagre set). The main result about Banach-Mazur games can then be stated as follows:

Theorem 2.2 ([8]). *Let $\mathscr{G} = (G, v_0, W)$ be a Banach-Mazur game on a finite graph. Pl. 0 has a winning strategy for $\mathscr{G}$ if and only if W is large.*

3   Simple strategies in Banach-Mazur games

In a Banach-Mazur game (G, v_0, W) on a finite graph, a strategy for Pl. 0 is given by a function f defined on $\text{FinPaths}(G, v_0)$, the set of finite paths of G starting from v_0, such that for any $\pi \in \text{FinPaths}(G, v_0)$, we have $f(\pi) \in \text{FinPaths}(G, \text{last}(\pi))$. However, we can imagine some restrictions on the strategies of Pl. 0:

1. A strategy f is said to be *positional* if it only depends on the current vertex, i.e f is a function defined on V such that for any $v \in V$, $f(v) \in \text{FinPaths}(G, v)$ and a play ρ is consistent with f if ρ is of the form $(\pi_i f(\text{last}(\pi_i)))_{i \geq 1}$.

2. A strategy f is said to be *finite-memory* if it only depends on the current vertex and a finite memory (see [3] for the precise definition of a finite-memory strategy).

3. A strategy f is said to be *b-bounded* if for any $\pi \in \text{FinPaths}(G, v_0)$, $f(\pi)$ has length less than b and a strategy is said to be *bounded* if there is $b \geq 1$ such that f is b-bounded.

4. A strategy f is said to be *move-counting* if it only depends on the current vertex and the number of moves already played, i.e. f is a function defined on $V \times \mathbb{N}$ such that for any $v \in V$, any $n \in \mathbb{N}$, $f(v, n) \in \text{FinPaths}(G, v)$ and a play ρ is consistent with f if ρ is of the form $(\pi_i f(\text{last}(\pi_i), i))_{i \geq 1}$.

5. A strategy f is said to be *length-counting* if it only depends on the current vertex and the length of the prefix already played, i.e. f is a function defined on $V \times \mathbb{N}$ such that for any $v \in V$, any $n \in \mathbb{N}$, $f(v, n) \in \text{FinPaths}(G, v)$ and a play ρ is consistent with f if after a prefix π, the move of Pl. 0 is given by $f(\text{last}(\pi), |\pi|)$.

The notions of positional and finite memory strategies are classical, bounded strategies are present in [9], move-counting and length-counting strategies have been introduced in [4]. We first remark that, by definition, the existence of a positional winning strategy implies the existence of finite-memory/move-counting/length-counting winning strategies. Moreover, since G is a finite graph, a positional strategy is always bounded. In [3], it is proved that the existence of a finite-memory winning strategy implies the existence of a positional winning strategy.

Proposition 3.1 ([3]). *Let $\mathscr{G} = (G, v_0, W)$ be a Banach-Mazur game. Pl. 0 has a finite-memory winning strategy if and only if Pl. 0 has a positional winning strategy.*

Using the ideas of the proof of the above proposition, we can also show that the existence of a winning strategy implies the existence of a length-counting winning strategy.

Proposition 3.2. *Let $\mathscr{G} = (G, v_0, W)$ be a Banach-Mazur game on a finite graph. Pl. 0 has a length-counting winning strategy if and only if Pl. 0 has a winning strategy.*

Proof. Let f be a winning strategy for Pl. 0. Since G is a finite graph, for any $n \geq 0$ and any $v \in V$, we can consider an enumeration $\pi_1, \ldots, \pi_m$ of finite paths in $\text{FinPaths}(G, v_0)$ of length n such that $\text{last}(\pi_i) = v$. We then let

$$h(v, n) = f(\pi_1) f(\pi_2 f(\pi_1)) f(\pi_3 f(\pi_1) f(\pi_2 f(\pi_1))) \ldots f(\pi_m f(\pi_1) f(\pi_2 f(\pi_1)) \cdots).$$

If ρ is a play consistent with h, then ρ is a play where the strategy f is applied infinitely often. Thus such a play ρ can be seen as a play $\sigma_1 \tau_1 \sigma_2 \tau_2 \cdots$ where the τ_i's (resp. the σ_i's) are the moves of Pl. 0 (resp. Pl. 1.) and where $f(\sigma_1 \tau_1 \cdots \sigma_i) = \tau_i$. Each play consistent with h can thus be seen as a play consistent with f, and we deduce that the strategy h is a length-counting winning strategy. □

On the other side, the notions of move-counting winning strategies and bounded winning strategies are incomparable.

Example 3.3 (**Set with a move-counting winning strategy and without a bounded winning strategy**). We consider the complete graph $G_{0,1}$ on $\{0, 1\}$. Let W be the set of any sequences $(\sigma_n)_{n \geq 1}$ in $\{0, 1\}^\omega$ with $\sigma_1 = 0$ such that $(\sigma_n)_{n \geq 1}$ contains a finite sequence of 1 strictly longer than the initial finite sequence of 0. In other words, $(\sigma_n)_{n \geq 1} \in W$ if $\sigma_1 = 0$ and if there exist $j \geq 1$ and $k \geq 1$ such that $\sigma_j = 1$ and $\sigma_{k+1} = \cdots = \sigma_{k+j} = 1$. Let $\mathscr{G} = (G_{0,1}, 0, W)$. The strategy $f(\cdot, n) = 1^n$ is a move-counting winning strategy for Pl. 0 for the game $\mathscr{G}$. On the other hand, there does not exist a bounded winning strategy for Pl. 0 for the game $\mathscr{G}$. Indeed, if f is a b-bounded strategy of Pl. 0, then Pl. 1 can start by playing 0^b and then, always play 0.

Example 3.4 (**Set with a bounded winning strategy and without a move-counting winning strategy**). We consider the complete graph $G_{0,1}$ on $\{0, 1\}$. Let $(\pi_n)_{n \geq 0}$ be an enumeration of $\text{FinPaths}(G)$ with $\pi_0 = 0$. We let W be the set of any sequences in $\{0, 1\}^\omega$ starting by 0 except the sequence $\rho = \pi_0 \pi_1 \pi_2 \ldots$. Let $\mathscr{G} = (G_{0,1}, 0, W)$. It is obvious that Pl. 0 has a 1-bounded winning strategy for $\mathscr{G}$ but we can also prove that Pl. 0 has no move-counting winning strategy. Indeed, if h is a move-counting strategy of Pl. 0, then Pl. 1 can start by playing a prefix π of ρ so that $\pi h(\text{last}(\pi), 1)$ is a prefix of ρ. Afterwards, Pl. 1 can play π' such that $\pi h(\text{last}(\pi), 1) \pi' h(\text{last}(\pi'), 2)$ is a prefix of ρ and so on.

We remark that the sets W considered in these examples are *open* sets, i.e. sets on a low level of the Borel hierarchy. Moreover, by Proposition 3.2, there also exist length-counting winning strategies for these two examples. The relations between the simple strategies are thus completely characterised and are summarised in Figure 1. This Figure also contains other simple strategies which will be discussed later.

4 Link with the sets of probability 1

Let $G = (V, E)$ be a finite directed graph. We can easily define a probability measure P, on the set of infinite paths in G, by giving a weight $w_e > 0$ at each edge $e \in E$ and by considering that for any $v, v' \in V$, $p_w(v, v') = 0$ if $(v, v') \notin E$ and $p_w(v, v') = \frac{w_{(v,v')}}{\sum_{e' \text{ enabled from } v} w_{e'}}$ else, where $p_w(v, v')$ denotes the probability of taking edge (v, v') from state v. Given $v_1 \cdots v_n \in \text{FinPaths}(G, v_1)$, we recall that we denote by $\text{Cyl}(v_1 \cdots v_n)$ the cylinder generated by $v_1 \cdots v_n$ and defined as $\text{Cyl}(v_1 \cdots v_n) = \{\rho \in \text{Paths}(G, v_1) \mid v_1 \cdots v_n \text{ is a prefix of } \rho\}$.

Definition 4.1. Let $G = (V, E)$ be a finite directed graph and $w = (w_e)_{e \in E}$ a family of positive weights. We define the probability measure P_w by the relation

$$P_w(\text{Cyl}(v_1 \cdots v_n)) = p_w(v_1, v_2) \cdot \cdots \cdot p_w(v_{n-1}, v_n) \tag{4.1}$$

and we say that such a probability measure is *reasonable*.

We are interested in characterising the sets W of probability 1 and their links with the different notions of simple winning strategies. We remark that, in general, Banach-Mazur games do not characterise sets of probability 1. In other words, the notions of large sets and sets of probability 1 do not coincide in general on finite graphs. Indeed, there exist some large sets of probability 0. We present here an example of such sets:

Example 4.2 (**Large set of probability** 0). We consider the complete graph $G_{0,1,2}$ on $\{0,1,2\}$ and the set $W = \{(w_i w_i^R)_{i \geq 0} \in \text{Paths}(G_{0,1,2}, 2) : w_i \in \{0,1,2\}^*\}$, where for any finite word $\sigma \in \{0,1,2\}^*$ given by $\sigma = \sigma(1) \cdots \sigma(n)$ with $\sigma(i) \in \{0,1,2\}$, we let $\sigma^R = \sigma(n) \cdots \sigma(1)$. In other words, W is the set of runs ρ starting from 2 that we can divide into a consecutive sequence of finite words and their reverse. It is obvious that Pl. 0 has a winning strategy for the Banach-Mazur game $(G_{0,1,2}, 2, W)$ and thus that W is large. On the other hand, if P is the reasonable probability measure given by the weights $w_e = 1$ for any $e \in E$, then we can verify that $P(W) = 0$. Indeed, we have

$$P(W) \leq \sum_{n=1}^{\infty} P(\{w_0 w_0^R (w_i w_i^R)_{i \geq 1} \in W : |w_0| = n\})$$

$$= \sum_{n=1}^{\infty} P(\{w_0 w_0^R w \in \text{Paths}(G_{0,1,2}, 2) : |w_0| = n\}) \cdot P(W)$$

$$\leq \sum_{n=1}^{\infty} \frac{P(W)}{3^n} = \frac{1}{2} P(W).$$

For certain families of sets, we can however have an equivalence between the notion of large set and the notion of set of probability 1. It is the case for the family of sets W representing ω-regular properties on finite graphs (see [9]). In order to prove this equivalence for ω-regular sets, Varacca and Völzer have in fact used the fact that for these sets, the Banach-Mazur game is positionally determined ([1]) and that the existence of a positional winning strategy for Pl. 0 implies $P(W) = 1$. This latter assertion follows from the fact that every positional strategy is bounded and that, by the Borel-Cantelli lemma, the set of plays consistent with a bounded strategy is a set of probability 1. Nevertheless, if W does not represent an ω-regular properties, it is possible that W is a large set of probability 1 and that there is no positional winning strategy for Pl. 0 and even no bounded or move-counting winning strategy.

Example 4.3 (**Large set of probability** 1 **without a positional/ bounded/ move-counting winning strategy**). We consider the complete graph $G_{0,1}$ on $\{0,1\}$ and the reasonable probability measure P given by $w_e = 1$ for any $e \in E$. Let $a_n = \sum_{k=1}^{n} k$. We let $W = \{(\sigma_k)_{k \geq 1} \in \{0,1\}^{\omega} : \sigma_1 = 0 \text{ and } \sigma_{a_n} = 1 \text{ for some } n > 1\}$ and $\mathcal{G} = (G_{0,1}, 0, W)$. Since Pl. 0 has a winning strategy for $\mathcal{G}$, we deduce that W is a large set. We can also compute that $P(W) = 1$ because if we denote by A_n, $n > 1$, the set

$$A_n := \{(\sigma_k)_{k \geq 1} \in \{0,1\}^{\omega} : \sigma_{a_n} = 1 \text{ and } \sigma_{a_m} = 0 \text{ for any } m < n\},$$

we have:

$$W = \dot{\bigcup}_{n > 1} A_n \quad \text{and} \quad P(A_n) = \frac{1}{2^{n-1}}.$$

On the other hand, there does not exist any positional (resp. bounded) winning strategy f for Pl. 0. Indeed, if f is a positional (resp. bounded) strategy for Pl. 0 such that $f(0)$ (resp. $f(\pi)$ for any π) has length less than n, then Pl. 1 has just to start by playing a_n zeros so that Pl. 1 does not reach the index a_{n+1} and afterwards to complete the sequence by a finite number of zeros to reach the next index a_k, and

so on. Moreover, there does not exist any move-counting winning strategy h for Pl. 0 because Pl. 1 can start by playing a_n zeros so that $|h(0,1)| \leq n$ and because, at each step k, Pl. 1 can complete the sequence by a finite number of zeros to reach a new index a_n such that $|h(0,k)| \leq n$.

On the other hand, we can show that the existence of a move-counting winning strategy for Pl. 0 implies $P(W) = 1$. The key idea is to realise that given a move-counting winning strategy h, the strategy $h(\cdot, n)$ is positional.

Proposition 4.4. *Let $\mathscr{G} = (G, v_0, W)$ be a Banach-Mazur game on a finite graph and P a reasonable probability measure. If Pl. 0 has a move-counting winning strategy for $\mathscr{G}$, then $P(W) = 1$.*

Proof. Let h be a move-counting winning strategy of *Pl*. 0. We denote by f_n the strategy $h(\cdot, n)$. Each set

$$M_n := \{\rho \in \text{Paths}(G, v_0) \ : \ \rho \text{ is a play consistent with } f_n\}$$

has probability 1 since f_n is a positional winning strategy for the Banach-Mazur game (G, v_0, M_n). Moreover, if ρ is a play consistent with f_n for each $n \geq 1$, then ρ is a play consistent with h. In other words, since h is a winning strategy, we get $\bigcap_n M_n \subset W$. Therefore, as $P(M_n) = 1$ for all n, we know that $P(\bigcap_n M_n) = 1$ and we conclude that $P(W) = 1$. $\qquad\square$

Let us notice that the converse of Proposition 4.4 is false in general. Indeed, Example 4.3 exhibit a large set W of probability 1 such that Pl. 0 has no move-counting winning strategy. However, if W is a countable intersection of ω-regular sets, then the existence of a winning strategy for Pl. 0 implies the existence of a move-counting winning strategy for Pl. 0.

Proposition 4.5. *Let $\mathscr{G} = (G, v_0, W)$ be a Banach-Mazur game on a finite graph where W is a countable intersection of ω-regular sets W_n. Pl. 0 has a winning strategy if and only if Pl. 0 has a move-counting winning strategy.*

Proof. Let $W = \bigcap_{n \geq 1} W_n$ where W_n is an ω-regular set and f a winning strategy of Pl. 0 for $\mathscr{G}$. For any $n \geq 1$, the strategy f is a winning strategy for the Banach-Mazur game (G, v_0, W_n). Thanks to [1], we therefore know that for any $n \geq 1$, there exists a positional winning strategy $\tilde{f}_n$ of Pl. 0 for (G, v_0, W_n).

Let $\phi : \mathbb{N} \to \mathbb{N}$ such that for any $k \geq 1$, $\{n \in \mathbb{N} : \phi(n) = k\}$ is an infinite[2] set. We consider the move-counting strategy $h(v, n) = \tilde{f}_{\phi(n)}(v)$. This strategy is winning because each play ρ consistent with h is a play consistent with $\tilde{f}_n$ for any n and thus

$$\{\rho \in \text{Paths}(G, v_0) \ : \ \rho \text{ is a play consistent with } h\}$$
$$\subseteq \bigcap_n \{\rho \in \text{Paths}(G, v_0) \ : \ \rho \text{ is a play consistent with } \tilde{f}_n\}$$
$$\subseteq \bigcap_n W_n = W.$$

$\qquad\square$

Remark 4.6. We cannot extend this result to countable unions of ω-regular sets because the set of countable unions of ω-regular sets contains the open sets and Example 3.4 exhibited a Banach-Mazur game where W is an open set and Pl. 0 has a winning strategy but no move-counting winning strategy.

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[2]Such a map ϕ exists because one could build a surjection $\psi : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ and then let $\phi = \psi_1$ where $\psi(n) = (\psi_1(n), \psi_2(n))$.

Remark 4.7. We also notice that if W is a countable intersection of ω-regular sets, then W is large if and only if W is a set of probability 1. Indeed, the notions of large sets and sets of probability 1 are stable by countable intersection and we know that a ω-regular set is large if and only if it is of probability 1 [9].

As a consequence of Remark 4.7, we have that if W is a ωS-regular sets, as defined in [2], the set W is large if and only if W is a set of probability 1. Indeed, it is shown in [6, 7] that ωS-regular sets are countable intersection of ω-regular sets. Nevertheless, the following example shows that, unlike the case of ω-regular sets, positional strategies are not sufficient for ωS-regular sets.

Example 4.8 (ωS-**regular set with a move-counting winning strategy and without a positional/ bounded winning strategy**). We consider the complete graph $G_{0,1}$ on $\{0,1\}$ and the set W corresponding to the ωS-regular expression $((0^*1)^*0^S1)^\omega$, which corresponds to the language of words where the number of consecutive 0 is unbounded. The move-counting strategy which consists in playing n consecutive 0's at the nth step is winning for Pl. 0. However, clearly enough Pl. 0 does not have a positional (nor bounded) winning strategy for W.

Example 4.2 shows that Remark 4.7 does not extend to ω-context-free sets. Another notion of simple strategies, natural inspired by Example 4.2, is the notion of last-move strategy. A strategy f for Pl. 0 is said to be *last-move* if it only depends on the last move of Pl. 1, i.e. for any $v \in V$, for any $\pi \in \text{FinPaths}(G,v)$, $f(\pi) \in \text{FinPaths}(G,\text{last}(\pi))$ and a play ρ is consistent with f if it is of the form $(\pi_i f(\pi_i))_{i \geq 1}$. It is obvious that there exists a last-move winning strategy for Pl. 0 in the game described in Example 4.2. In particular, we deduce that the existence of a last-move winning strategy for W does not imply that W has probability 1. Example 4.2 allows also us to see that the existence of a last-move winning strategy does not imply in general the existence of a move-counting winning strategy or a bounded winning strategy. Indeed, let W be the set $\{(w_i w_i^R)_i \in \text{Paths}(G_{0,1,2},2) : w_i \in \{0,1,2\}^*\}$. Since $P(W) = 0$ (and thus $P(W) \neq 1$), we know that Pl. 0 has no move-counting winning strategy by Proposition 4.4 and no bounded winning strategy.

The notion of last-move winning strategy is in fact incomparable with the notion of move-counting winning strategy and the notion of bounded winning strategy. Indeed, on the complete graph $G_{0,1}$ on $\{0,1\}$, if we denote by W the set of runs in $G_{0,1}$ such that for any $n \geq 1$, the word 1^n appears, then Pl. 0 has a move-counting winning strategy for the game $(G_{0,1},0,W)$ but no last-move winning strategy. In the same way, if we denote by W the set of aperiodic runs on $G_{0,1}$ then Pl. 0 has a 1-bounded winning strategy for the game $(G_{0,1},0,W)$ but no last-move winning strategy (it suffices for Pl. 1 to play at each time the same word).

5   Generalised Banach-Mazur games

Let $\mathscr{G} = (G,v_0,W)$ be a Banach-Mazur game on a finite graph. We know that the existence of a bounded winning strategy or a move-counting winning strategy of Pl. 0 for $\mathscr{G}$ implies that $P(W) = 1$ for every reasonable probability measure P. Nevertheless, it is possible that $P(W) = 1$ and Pl. 0 has no bounded winning strategy and no move-counting winning strategy (Example 4.3). We therefore search a new notion of strategy such that the existence of such a winning strategy implies $P(W) = 1$ and the existence of a bounded winning strategy or a move-counting winning strategy imply the existence of such a winning strategy. To this end, we introduce a new type of Banach-Mazur games:

Definition 5.1. A *generalised Banach-Mazur game* $\mathscr{G}$ on a finite graph is a tuple (G,v_0,ϕ_0,ϕ_1,W) where $G = (V,E)$ is a finite directed graph where every vertex has a successor, $v_0 \in V$ is the initial state, $W \subset \text{Paths}(G,v_0)$, and ϕ_i is a map on $\text{FinPaths}(G,v_0)$ such that for any $\pi \in \text{FinPaths}(G,v_0)$,

$$\phi_i(\pi) \subset \mathscr{P}\big(\text{FinPaths}(G,\text{last}(\pi))\big) \setminus \{\emptyset\} \quad \text{and} \quad \phi_i(\pi) \neq \emptyset.$$

A generalised[3] Banach-Mazur game $\mathscr{G} = (G, v_0, \phi_0, \phi_1, W)$ on a finite graph is a two-player game where Pl. 0 and Pl. 1 alternate in choosing *sets of finite paths* as follows: Pl. 1 begins with choosing a set of finite paths $\Pi_1 \in \phi_1(v_0)$; Pl. 0 selects a finite path $\pi_1 \in \Pi_1$ and chooses a set of finite paths $\Pi_2 \in \phi_0(\pi_1)$; Pl 1. then selects $\pi_2 \in \Pi_2$ and proposes a set $\Pi_3 \in \phi_1(\pi_1\pi_2)$ and so on. A play of $\mathscr{G}$ is thus an infinite path $\pi_1\pi_2\pi_3\ldots$ in G and we say that Pl. 0 wins if this path belongs to W, while Pl. 1 wins if this path does not belong to W.

We remark that if we let $\phi_{\text{ball}}(\pi) := \{\{\pi'\} : \pi' \in \text{FinPaths}(G, \text{last}(\pi))\}$ for any $\pi \in \text{FinPaths}(G, v_0)$, then the generalised Banach-Mazur game given by $(G, v_0, \phi_{\text{ball}}, \phi_{\text{ball}}, W)$ coincides with the classical Banach-Mazur game (G, v_0, W). We also obtain a game similar to the classical Banach-Mazur game if we consider the function $\phi(\pi) = \mathscr{P}(\text{FinPaths}(G, \text{last}(\pi)))$. On the other hand, if we consider $\phi(\pi) := \{\{\pi'\} : \pi' \in \text{FinPaths}(G, \text{last}(\pi)), |\pi'| = 1\}$, we obtain the classical games on graphs such as the ones studied in [5].

We are interested in defining a map ϕ_0 such that Pl. 0 has a winning strategy for $(G, v_0, \phi_0, \phi_{\text{ball}}, W)$ if and only if $P(W) = 1$. To this end, we notice that we can restrict actions of Pl. 0 by forcing each set in $\phi_0(\pi)$ to be "big" in some sense. The idea to characterise $P(W) = 1$ is therefore to force Pl. 0 to play with finite sets of finite paths of conditional probability bigger than α for some $\alpha > 0$.

Definition 5.2. Let $\mathscr{G} = (G, v_0, W)$ be a Banach-Mazur game on a finite graph, P a reasonable probability measure and $\alpha > 0$. An α-*strategy* of Pl. 0 for $\mathscr{G}$ is a strategy of Pl. 0 for the generalised Banach-Mazur game $\mathscr{G}_\alpha = (G, v_0, \phi_\alpha, \phi_{\text{ball}}, W)$ where

$$\phi_\alpha(\pi) = \left\{ \Pi \subset \text{FinPaths}(G, \text{last}(\pi)) : P\left(\bigcup_{\pi' \in \Pi} \text{Cyl}(\pi\pi') \,\Big|\, \text{Cyl}(\pi) \right) \geq \alpha \text{ and } \Pi \text{ is finite} \right\}.$$

We recall that, given two events A, B with $P(B) > 0$, the conditional probability $P(A|B)$ is defined by $P(A|B) := P(A \cap B)/P(B)$.

We notice that every bounded strategy can be seen as an α-strategy for some $\alpha > 0$, since for any $N \geq 1$, there exists $\alpha > 0$ such that for any π of length less than N, we have $P(\{\pi\}) \geq \alpha$. We can also show that the existence of a move-counting winning strategy for Pl. 0 implies the existence of a winning α-strategy for Pl. 0 for every $0 < \alpha < 1$.

Proposition 5.3. *Let $\mathscr{G} = (G, v_0, W)$ be a Banach-Mazur game on a finite graph. If Pl. 0 has a move-counting winning strategy, then Pl. 0 has a winning α-strategy for every $0 < \alpha < 1$.*

Proof. Let P be a reasonable probability measure, h a move-counting winning strategy for Pl. 0 and $0 < \alpha < 1$. We denote by g_n the positional strategy defined by

$$g_n(v) = h(v, 1) \, h\big(\text{last}(h(v, 1)), 2\big) \, \cdots \, h\big(\text{last}(h(v, 1) \, h(\text{last}(h(v, 1)), 2) \cdots), n\big).$$

Let us notice that the definition of the g_n's implies that for any increasing sequence (n_k), a play of the form

$$\pi_1 \, g_{n_1}(\text{last}(\pi_1)) \, \pi_2 \, g_{n_2}(\text{last}(\pi_2)) \, \cdots \, \pi_k \, g_{n_k}(\text{last}(\pi_k)) \, \cdots \tag{5.1}$$

is consistent with h. Since g_n is a positional strategy, we know that each set

$$M_n := \{\rho \in \text{Paths}(G, v_0) \, : \, \rho \text{ is a play consistent with } g_n\}$$

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[3]We only present here a generalisation of Banach-Mazur games on finite graphs but this generalisation could be extended to Banach-Mazur games on topological spaces by asking that for any non-empty open set O, $\phi_i(O)$ is a collection of non-empty open subsets of O.

has probability 1. In particular, for any $\pi_0 \in \text{FinPaths}(G, v_0)$, we deduce that $P(M_n | \text{Cyl}(\pi_0)) = 1$. Since

$$M_n \cap \text{Cyl}(\pi_0) \subseteq \bigcup_{\pi \in \text{FinPaths}(G, \text{last}(\pi_0))} \text{Cyl}\big(\pi_0 \pi g_n(\text{last}(\pi))\big),$$

we have

$$P\Big(\bigcup_{\pi \in \text{FinPaths}(G, \text{last}(\pi_0))} \text{Cyl}\big(\pi_0 \pi g_n(\text{last}(\pi))\big) \Big| \text{Cyl}(\pi_0) \Big) = 1$$

and since $\text{FinPaths}(G, \text{last}(\pi_0))$ is countable, we deduce that for any $n \geq 1$, any $\pi_0 \in \text{FinPaths}(G, v_0)$, there exists a finite subset $\Pi_n(\pi_0) \subset \text{FinPaths}(G, \text{last}(\pi_0))$ such that

$$P\Big(\bigcup_{\pi \in \Pi_n(\pi_0)} \text{Cyl}\big(\pi_0 \pi g_n(\text{last}(\pi))\big) \Big| \text{Cyl}(\pi_0) \Big) \geq \alpha.$$

We denote by $\Pi'_n(\pi_0)$ the set $\{\pi g_n(\text{last}(\pi)) : \pi \in \Pi_n(\pi_0)\}$ and we let

$$f(\pi_0) := \Pi'_{|\pi_0|}(\pi_0).$$

The above-defined strategy f is therefore a winning α-strategy for Pl. 0 since each play consistent with f is of the form (5.1) for some sequence (n_k) and thus consistent with h. $\qquad \square$

Moreover, the existence of a winning α-strategy for some $\alpha > 0$ still implies $P(W) = 1$.

Theorem 5.4. *Let $\mathscr{G} = (G, v_0, W)$ be a Banach-Mazur game on a finite graph and P a reasonable probability measure. If Pl. 0 has a winning α-strategy for some $\alpha > 0$, then $P(W) = 1$.*

Proof. Let f be a winning α-strategy. We consider an increasing sequence $(a_n)_{n \geq 1}$ such that for any $n \geq 1$, any π of length a_n, each $\pi' \in f(\pi)$ has length less than $a_{n+1} - a_n$; this is possible because for any π, $f(\pi)$ is a finite set by definition of α-strategy. Without loss of generality[4], we can even assume that for any $n \geq 1$, any π of length a_n, each $\pi' \in f(\pi)$ has exactly length $a_{n+1} - a_n$. We therefore let

$$A := \{(\sigma_k)_{k \geq 1} \in \text{Paths}(G, v_0) : \#\{n : (\sigma_k)_{a_n+1 \leq k \leq a_{n+1}} \in f((\sigma_k)_{1 \leq k \leq a_n})\} = \infty\}.$$

In other words, $(\sigma_k)_{k \geq 1} \in A$ if (σ_k) can be seen as a play where f has been played on an infinite number of indices a_n. Since f is a winning strategy, A is included in W and it thus suffices to prove that $P(A) = 1$.

We first notice that for any $m \geq 1$, any $n \geq m$, if we let

$$B_{m,n} = \{(\sigma_k)_{k \geq 1} \in \text{Paths}(G, v_0) : (\sigma_k)_{a_j+1 \leq k \leq a_{j+1}} \notin f((\sigma_k)_{1 \leq k \leq a_j}), \forall m \leq j \leq n\},$$

then $P(B_{m,n}) \leq (1 - \alpha)^{n+1-m}$ as f is an α-strategy. We therefore deduce that for any $m \geq 1$,

$$P\Big(\bigcap_{n=m}^{\infty} B_{m,n} \Big) = 0$$

and since $A^c = \bigcup_{m \geq 1} \bigcap_{n=m}^{\infty} B_{m,n}$, we conclude that $P(A) = 1$. $\qquad \square$

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[4]Let π be a finite path and $n_\pi \geq \max\{|\tau| \text{ such that } \tau \in f(\pi)\}$. One can define $\tilde{f}(\pi)$ as the set of finite paths σ of length n_π such that τ is a prefix of σ, for some $\tau \in f(\pi)$. Given a play ρ, one can show that ρ is consistent with f if and only if ρ is consistent with $\tilde{f}$.

If W is a countable intersection of open sets, we can prove the converse of Theorem 5.4 and so obtain a characterisation of sets of probability 1.

Theorem 5.5. *Let $\mathscr{G} = (G, v_0, W)$ be a Banach-Mazur game on a finite graph where W is a countable intersection of open sets and P a reasonable probability measure. Then the following assertions are equivalent:*

1. *$P(W) = 1$,*

2. *Pl. 0 has a winning α-strategy for some $\alpha > 0$,*

3. *Pl. 0 has a winning α-strategy for all $0 < \alpha < 1$.*

Proof. We have already proved 2. $\Rightarrow$ 1., and 3. $\Rightarrow$ 2. is obvious.

1. $\Rightarrow$ 3. Let $0 < \alpha < 1$. Let $W = \bigcap_{n=1}^{\infty} W_n$ where W_n's are open sets. Since $P(W) = 1$, we deduce that for any $n \geq 1$, $P(W_n) = 1$. We can therefore define a winning α-strategy f of Pl. 0 as follows: if $\mathrm{Cyl}(\pi) \subset \bigcap_{k=1}^{n-1} W_k$ and $\mathrm{Cyl}(\pi) \not\subset W_n$, we let $f(\pi)$ be a finite set $\Pi \subset \mathrm{FinPaths}(G, \mathrm{last}(\pi))$ such that $P\left(\bigcup_{\pi' \in \Pi} \mathrm{Cyl}(\pi\pi') | \mathrm{Cyl}(\pi)\right) \geq \alpha$ and for any $\pi' \in \Pi$, $\mathrm{Cyl}(\pi\pi') \subset W_n$. Such a finite set Π exists because W_n has probability 1 and W_n is an open set, i.e. a countable union of cylinders. This concludes the proof. $\qquad\square$

Remark 5.6. We cannot hope to generalise the latter result to any set W. More precisely, there exist sets of probability 1 for which no winning α-strategy exists. Indeed, given a set W, on the one hand, the existence of a winning α-strategy for W implies the existence of a winning strategy for W, and thus in particular such a W is large. On the other hand, we know that there exists some meagre (in particular not large) set of probability 1 (see Example 4.2). However, one can ask whether the existence of a winning α-strategy is equivalent to the fact that W is a large set of probability 1.

When W is a countable intersection of open sets, we remark that the generalised Banach-Mazur game $\mathscr{G}_\alpha = (G, v_0, \phi_\alpha, \phi_{\mathrm{ball}}, W)$ is in fact determined.

Theorem 5.7. *Let $\mathscr{G}_\alpha$ be the generalised Banach-Mazur game given by $\mathscr{G}_\alpha = (G, v_0, \phi_\alpha, \phi_{ball}, W)$ where G is a finite graph, W is a countable intersection of open sets and P a reasonable probability measure. Then the following assertions are equivalent:*

1. *$P(W) < 1$,*

2. *Pl. 1 has a winning strategy for $\mathscr{G}_\alpha$ for some $\alpha > 0$,*

3. *Pl. 1 has a winning strategy for $\mathscr{G}_\alpha$ for all $0 < \alpha < 1$.*

Proof. We deduce from Theorem 5.5 that 2. $\Rightarrow$ 1. because $\mathscr{G}_\alpha$ is a zero-sum game, and 3. $\Rightarrow$ 2. is obvious.

1. $\Rightarrow$ 3. Let $W = \bigcap_{n=1}^{\infty} W_n$ with $P(W) < 1$ and W_n open. We know that there exists $n \geq 1$ such that $P(W_n) < 1$. It then suffices to prove that Pl. 1 has a winning strategy for the generalised Banach-Mazur game $(G, v_0, \phi_\alpha, \phi_{\mathrm{ball}}, W_n)$ for all $0 < \alpha < 1$. Without loss of generality, we can thus assume that W is an open set. We recall that W is open if and only if it is a countable union of cylinders. Since any strategy of Pl. 1 is winning if $W = \emptyset$, we also suppose that $W \neq \emptyset$.

Let $0 < \alpha < 1$. We first show that there exists a finite path $\pi_1 \in \mathrm{FinPaths}(G, v_0)$ such that any set $\Pi_2 \in \phi_\alpha(\pi_1)$ contains a finite path π_2 satisfying

$$P(W | \mathrm{Cyl}(\pi_1 \pi_2)) \leq P(W) < 1. \tag{5.2}$$

Let

$$I_W := \inf\{P(W | \mathrm{Cyl}(\pi)) : \pi \in \mathrm{FinPaths}(G, v_0)\}. \tag{5.3}$$

Since W is a non-empty union of cylinders, there exists $\sigma \in \text{FinPaths}(G, v_0)$ such that $P(W|\text{Cyl}(\sigma)) = 1$. We remark that $P(W) = \sum_{\pi:|\pi|=|\sigma|} P(W|\text{Cyl}(\pi))P(\text{Cyl}(\pi))$ and $\sum_{\pi:|\pi|=|\sigma|} P(\text{Cyl}(\pi)) = 1$. Therefore, since $P(W|\text{Cyl}(\sigma)) > P(W)$, we deduce that there exists $\pi \in \text{FinPaths}(G, v_0)$ with $|\pi| = |\sigma|$ such that $P(W|\text{Cyl}(\pi)) < P(W)$. We conclude that $I_W < P(W)$ and thus, by definition of I_W, there exists $\pi_1 \in \text{FinPaths}(G, v_0)$ such that

$$I_W + \frac{1}{\alpha}(P(W|\text{Cyl}(\pi_1)) - I_W) < P(W). \tag{5.4}$$

Let $\Pi_2 \in \phi_\alpha(\pi_1)$. We consider $\tau_1, \ldots, \tau_n \in \Pi_2$ and $\sigma_1, \ldots, \sigma_m \in \text{FinPaths}(G, \text{last}(\pi_1))$ such that cylinders $\text{Cyl}(\tau_i), \text{Cyl}(\sigma_j)$ are pairwise disjoint, $\bigcup_{\pi \in \Pi_2} \text{Cyl}(\pi) \subset \bigcup_{i=1}^n \text{Cyl}(\tau_i)$ and

$$\text{Paths}(G, \text{last}(\pi_1)) = \bigcup_{i=1}^n \text{Cyl}(\tau_i) \cup \bigcup_{j=1}^m \text{Cyl}(\sigma_j). \tag{5.5}$$

Assume that for all $1 \leq i \leq n$, we have

$$P(W|\text{Cyl}(\pi_1 \tau_i)) > P(W). \tag{5.6}$$

Then, we get

$P(W|\text{Cyl}(\pi_1))$

$$= \sum_{i=1}^n P(W \cap \text{Cyl}(\pi_1 \tau_i)|\text{Cyl}(\pi_1)) + \sum_{j=1}^m P(W \cap \text{Cyl}(\pi_1 \sigma_j)|\text{Cyl}(\pi_1)) \text{ by disjointness and (5.5)}$$

$$= \sum_{i=1}^n P(W|\text{Cyl}(\pi_1 \tau_i))P(\text{Cyl}(\pi_1 \tau_i)|\text{Cyl}(\pi_1)) + \sum_{j=1}^m P(W|\text{Cyl}(\pi_1 \sigma_j))P(\text{Cyl}(\pi_1 \sigma_j)|\text{Cyl}(\pi_1))$$

$$\geq P(W)\sum_{i=1}^n P(\text{Cyl}(\pi_1 \tau_i)|\text{Cyl}(\pi_1)) + I_W \sum_{j=1}^m P(\text{Cyl}(\pi_1 \sigma_j)|\text{Cyl}(\pi_1)) \text{ by (5.6) and (5.3)}$$

$$\geq P(W)\sum_{i=1}^n P(\text{Cyl}(\pi_1 \tau_i)|\text{Cyl}(\pi_1)) + I_W \Big(1 - \sum_{i=1}^n P(\text{Cyl}(\pi_1 \tau_i)|\text{Cyl}(\pi_1))\Big) \text{ by (5.5)}$$

$$\geq P(W)P\Big(\bigcup_{\pi \in \Pi_2} \text{Cyl}(\pi_1 \pi)|\text{Cyl}(\pi_1)\Big) + I_W \Big(1 - P\Big(\bigcup_{\pi \in \Pi_2} \text{Cyl}(\pi_1 \pi)|\text{Cyl}(\pi_1)\Big)\Big) \text{ by properties of } \tau_i\text{'s}$$

$$\geq P(W)\alpha + I_W(1 - \alpha) \quad \text{(because } \Pi_2 \in \phi_\alpha(\pi_1) \text{ and } P(W) > I_W)$$

and thus $P(W) \leq I_W + \frac{1}{\alpha}(P(W|\text{Cyl}(\pi_1)) - I_W)$ which is a contradiction with (5.4). We conclude that if π_1 is given by (5.4), then any set $\Pi_2 \in \phi_\alpha(\pi_1)$ contains a finite path π_2 satisfying (5.2).

We can now exhibit a winning strategy for Pl. 1. We assume that Pl. 1 begins with playing a finite path π_1 satisfying (5.4). Let f be an α-strategy. We know that Pl. 1 can select a finite path $\pi_2 \in f(\pi_1)$ satisfying (5.2), i.e. $P(W|\text{Cyl}(\pi_1 \pi_2)) \leq P(W)$. By repeating the above method from $\pi_1 \pi_2$, we also deduce the existence of a finite path π_3 such that any set $\Pi_4 \in \phi_\alpha(\pi_1 \pi_2 \pi_3)$ contains a finite path π_4 satisfying $P(W|\text{Cyl}(\pi_1 \pi_2 \pi_3 \pi_4)) \leq P(W)$. We can thus assume that Pl. 1 plays such a finite path π_3 and then selects $\pi_4 \in f(\pi_1 \pi_2 \pi_3)$ such that $P(W|\text{Cyl}(\pi_1 \pi_2 \pi_3 \pi_4)) \leq P(W)$. This strategy is a winning strategy for Pl. 1. Indeed, as W is an open set and thus a countable union of cylinders, if $P(W|\text{Cyl}(\pi_1 \cdots \pi_{2n})) \leq P(W) < 1$ for any n, then $\pi_1 \pi_2 \pi_3 \cdots \notin W$. $\qquad \square$

Corollary 5.8. *Let* $0 < \alpha < 1$. *The generalised Banach-Mazur game* $\mathscr{G}_\alpha = (G, v_0, \phi_\alpha, \phi_{ball}, W)$ *is determined when W is a countable intersection of open sets. More precisely, Pl. 0 has a winning strategy for* $\mathscr{G}_\alpha$ *if and only if $P(W) = 1$, and Pl. 1 has a winning strategy for* $\mathscr{G}_\alpha$ *if and only if $P(W) < 1$.*

Since the existence of a bounded winning strategy for Pl. 0 implies the existence of a winning α-strategy for Pl. 0 and the existence of a move-counting winning strategy for Pl. 0 implies the existence of a winning α-strategy for Pl. 0, we deduce from Example 3.3 and Example 3.4 that in general, the existence of a winning α-strategy for Pl. 0 does not imply the existence of a move-counting winning strategy Pl. 0 and the existence of a bounded winning strategy for Pl. 0. On the other hand, we know that there exists a Banach-Mazur game for which Pl. 0 has a bounded winning strategy and no last-move winning strategy. The existence of a winning α-strategy thus does not imply in general the existence of a last-move winning strategy. Conversely, if we consider the game $(G_{0,1}, 0, W)$ described in Example 4.2, Pl. 0 has a last-move winning strategy but no winning α-strategy (as $P(W) = 0$). The notion of α-strategy is thus incomparable with the notion of last-move strategy.

6   More on simple strategies

We finish this paper by considering the crossings between the different notions of simple strategies and the notion of bounded strategy i.e. the bounded length-counting strategies, the bounded move-counting strategies and the bounded last-move strategies. Obviously, the existence of a bounded length-counting winning strategy for Pl. 0 implies the existence of a length-counting winning strategy for Pl. 0, and we have this implication for each notion of bounded strategies and their no bounded counterpart. We start by noticing that the existence of a bounded move-counting winning strategy is equivalent to the existence of a positional winning strategy.

Proposition 6.1. *Let $\mathcal{G} = (G, v_0, W)$ be a Banach-Mazur game on a finite graph. Pl. 0 has a bounded move-counting winning strategy if and only if Pl. 0 has a positional winning strategy.*

Proof. Let h be a bounded move-counting winning strategy for Pl. 0. We denote by $C_1, \ldots, C_N$ the bottom strongly connected components (BSCC) of G. Let $1 \leq i \leq N$. Since h is a bounded strategy and G is finite, there exist some finite paths $w_1^{(i)}, \ldots, w_{k_i}^{(i)} \subset C_i$ such that for any $v \in C_i$, for any $n \geq 1$,

$$h(v, n) \in \{w_1^{(i)}, \ldots, w_{k_i}^{(i)}\}.$$

Let $v \in V$. If $v \in C_i$, we let $f(v) = \sigma_0 w_1^{(i)} \sigma_1 w_2^{(i)} \sigma_2 \ldots w_{k_i}^{(i)}$ where σ_l are finite paths in C_i such that $f(v)$ is a finite path in C_i starting from v. If $v \notin \bigcup_i C_i$, we let $f(v) = \sigma_v$ where σ_v starts from v and leads into a BSCC of G. The positional strategy f is therefore winning as each play ρ consistent with f can be seen as a play consistent with h.   □

The other notions of bounded strategies are not equivalent to any other notion of simple strategy.

Example 6.2 (**Set with a bounded length-counting winning strategy and without a positional winning strategy**)**.** Let $G_{0,1}$ be the complete graph on $\{0,1\}$, (ρ_n) an enumeration of finite words in $\{0,1\}$ and $\rho_{\text{target}} = 0\rho_1\rho_2\cdots$. We consider the set $W = \{\sigma \in \{0,1\}^\omega : \#\{i \geq 1 : \sigma(i) = \rho_{\text{target}}(i)\} = \infty\}$. It is evident that Pl. 0 has a bounded length-counting winning strategy for the game $(G_{0,1}, 0, W)$. However, Pl. 0 has no positional winning strategy. Indeed, if f is a positional strategy such that $f(0) = a(1)\cdots a(k)$, then Pl. 1 can play according to the strategy h defined by $h(\sigma(1)\cdots\sigma(n)) = \sigma(n+1)\cdots\sigma(N)0$ such that for any $n+1 \leq i \leq N$, $\sigma(i) \neq \rho_{\text{target}}(i)$, $\rho_{\text{target}}(N+1) \neq 0$ and for any $1 \leq i \leq k$, $a(i) \neq \rho_{\text{target}}(N+i+1)$.

Example 6.3 (**Set with a bounded last-move winning strategy and without a positional winning strategy**)**.** Let $G_{0,1,2}$ be the complete graph on $\{0,1,2\}$. For any $\phi : \{0,1,2\}^* \to \{0,1\}$, if we consider the set $W := \{(\pi_i\phi(\pi_i))_{i\geq 1} : \pi_i \in \{0,1,2\}^*\}$, then Pl. 0 has a 1-bounded last-move winning strategy given

by ϕ for the game $(G_{0,1,2}, 2, W)$. On the other hand, we can choose ϕ such that Pl. 0 has no positional winning strategy. Indeed, it suffices to choose $\phi : \{0,1,2\}^* \to \{0,1\}$ such that for any $\pi \in \{0,1,2\}^*$, any $n \geq 1$, any $\sigma(1), \ldots, \sigma(n) \in \{0,1,2\}$, there exists $k \geq 1$ such that $\phi(\pi 2^k) \neq \sigma(1)$ and for any $1 \leq i \leq n-1$, $\phi(\pi 2^k \sigma(1) \cdots \sigma(i)) \neq \sigma(i+1)$. Such a function exists because the set $\{0,1,2\}^*$ is countable. Therefore, Pl. 0 has no positional winning strategy for the game $(G_{0,1,2}, 2, W)$ because, if f is a positional strategy and $f(2) = \sigma(1) \ldots \sigma(n)$, then Pl. 1 can play consistent with the strategy h defined by $h(\pi) = 2^k$ such that $\phi(\pi 2^k) \neq \sigma(1)$ and for any $1 \leq i \leq n-1$, $\phi(\pi 2^k \sigma(1) \cdots \sigma(i)) \neq \sigma(i+1)$. Pl. 0 has thus a 1-bounded last-move winning strategy and no positional winning strategy for the game $(G_{0,1,2}, 2, W)$.

Example 6.4 (**Set with a bounded winning strategy and without a bounded length-counting winning strategy**). Let $G_{0,1,2,3}$ be the complete graph on $\{0,1,2,3\}$. For any $\phi : \{0,1,2,3\}^* \to \{0,1\}$, if we denote by W the set of runs ρ such that $\#\{n \geq 1 : \phi(\rho(1) \ldots \rho(n)) = \rho(n+1)\} = \infty$, then Pl. 0 has a 1-bounded winning strategy given by ϕ for the game $(G_{0,1,2,3}, 2, W)$. We now show how we can define ϕ so that Pl. 0 has no bounded length-counting winning strategy. Let $n_k = \sum_{i=1}^{k} 3i$. We choose $\phi : \{0,1,2,3\}^* \to \{0,1\}$ such that for any $k \geq 1$, any $\pi \in \{0,1,2,3\}^*$ of length n_k and any $\sigma(1), \ldots, \sigma(k) \in \{0,1,2,3\}$, there exists $\tau \in \{2,3\}^*$ of length $2k$ such that $\phi(\pi \tau 2) \neq \sigma(1)$ and for any $1 \leq i \leq k-1$, $\phi(\pi \tau 2 \sigma(1) \cdots \sigma(i)) \neq \sigma(i+1)$. Such a function exists because the cardinality of $\{2,3\}^{2k}$ is equal to the cardinality of $\{0,1,2,3\}^k$ and the length of $\pi \tau 2 \sigma(1) \cdots \sigma(k) < n_{k+1}$. Therefore, Pl. 0 has no bounded length-counting winning strategy because if f is a k-bounded length-counting strategy (for some $k \in \mathbb{N}$) and $f(2, n_k + k + 1) = \sigma$, then Pl. 1 can start by playing $2^{n_k} \tau 2$, where $\tau \in \{2,3\}^*$ of length $2k$ such that $\phi(\pi \tau 2) \neq \sigma(1)$ and for any $1 \leq i \leq k-1$, $\phi(\pi \tau 2 \sigma(1) \cdots \sigma(i)) \neq \sigma(i+1)$, and if Pl. 1 keep playing with same philosophy, then Pl. 1 wins the play. Pl. 0 has thus a 1-bounded winning strategy and no bounded length-counting winning strategy for the game $(G_{0,1,2}, 2, W)$.

The relations between the different notions of simple strategies on a finite graph can be summarised as depicted in Figure 1. We draw attention to the fact that the situation is very different in the case of infinite graphs. For example, a positional strategy can be unbounded, the notion of length-counting winning strategy is not equivalent to the notion of winning strategy (except if the graph is finitely branching), and the notion of bounded move-counting winning strategy for Pl. 0 is not equivalent to the notion of positional winning strategy.

Example 6.5 (**Set on an infinite graph with a bounded move-counting winning strategy and without a positional winning strategy**). We consider the complete graph $G_{\mathbb{N}}$ on $\mathbb{N}$ and the game $\mathscr{G} = (G_{\mathbb{N}}, 0, W)$ where $W = \{(\sigma_k) \in \mathbb{N}^\omega : \forall n \geq 1, \exists k \geq 1, (\sigma_k, \sigma_{k+1}) = (n, n+1)\}$. Pl. 0 has a bounded move-counting winning strategy given by $h(v, n) = n\, n + 1$ but no positional winning strategy.

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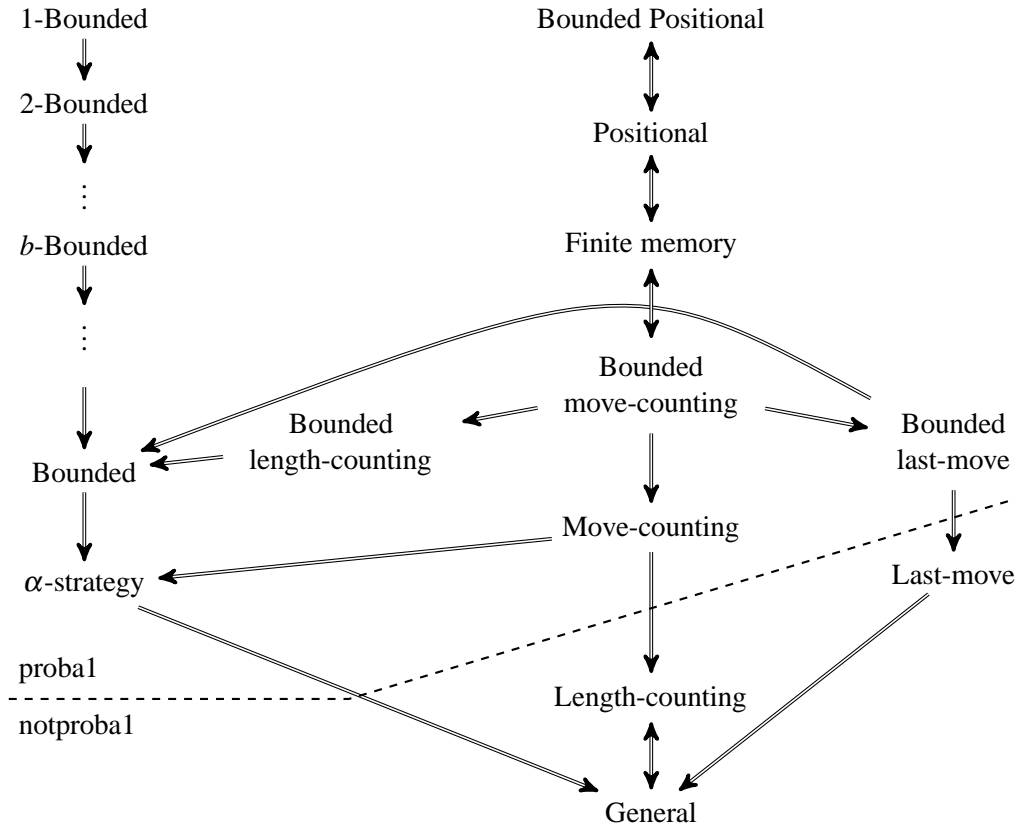
Figure 1: Winning strategies for Player 0 on finite graphs.

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