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Algebraic theory of Penrose's non-periodic tilings of the plane. II

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Dedicated to G. Pólya

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9. NEW PARAMETERS FOR PENTAGRIDS

The vector $(\gamma_0, \dots, \gamma_4)$ with zero sum is determined by four real independent variables (e.g. $\gamma_1, \dots, \gamma_4$). It will have some advantages to pass from these to two complex parameters given by

$$(9.1) \qquad \xi = \textstyle\sum_j \gamma_j \zeta^{2j}, \quad \eta = \sum_j \gamma_j \zeta^j$$

with the converse

$$(9.2) \qquad \gamma_j = \tfrac{2}{5}\,\mathrm{Re}(\xi\zeta^{-2j} + \eta\zeta^{-j}).$$

The AR-pattern associated with $\gamma_0, \dots, \gamma_4$ in the regular case depends on ξ only. One way to see this is to put (7.2) and (7.3) in the form $\sum_j x_j \zeta^{2j} = \xi$, another way is to write (8.3) as $(\sum k_j, \sum k_j \zeta^{2j} - \xi) \in V$. A more direct way is to evaluate $K_0(z), \dots, K_4(z)$ where z is solved from (4.4). In the evaluation of $K_h(z)$ we get determinants like

$$(9.3) \qquad \begin{vmatrix} \gamma_h & \gamma_r & \gamma_s \\ \zeta^h & \zeta^r & \zeta^s \\ \zeta^{-h} & \zeta^{-r} & \zeta^{-s} \end{vmatrix}.$$

These can be simplified by remarking that

$$5\gamma_j = \xi\zeta^{-2j} + \bar{\xi}\zeta^{2j} + \eta\zeta^{-j} + \bar{\eta}\zeta^j,$$

and in the evaluation of (9.3) the contributions of η and $\bar{\eta}$ are cancelled by means of row subtractions.

Two pentagrids are called *shift-equivalent* if they can be obtained from each other by a parallel shift. The pentagrids determined by $\gamma_0, \dots, \gamma_4$ and $\gamma_0^*, \dots, \gamma_4^*$, respectively (both with zero sum) are equivalent if and only if there exists $z_0 \in \mathbb{C}$ with

$$\text{(9.4)} \qquad \operatorname{Re}(z_0\zeta^{-j}) + \gamma_j - \gamma_j^* \in \mathbb{Z} \quad (j = 0, \dots, 4).$$

We form ξ, η from $\gamma_0, \dots, \gamma_4$ by (9.1), and similarly ξ^*, η^* from $\gamma_0^*, \dots, \gamma_4^*$. Now shift equivalence can be seen to depend on ξ and ξ^* only:

THEOREM 9.1. The two pentagrids are shift-equivalent if and only if $\xi - \xi^* \in P$ (this P is the ideal defined at the end of Section 1).

PROOF. (i) If (9.4) holds, we put $m_j = \operatorname{Re}(z_0\zeta^{-j}) + \gamma_j - \gamma_j^*$. Then $m_j \in \mathbb{Z}$, $\sum m_j = 0$, whence $\sum m_j \zeta^{2j} \in P$. And $\sum m_j \zeta^{2j} = \xi - \xi^*$.
(ii) If $\xi - \xi^* \in P$ we have $\xi - \xi^* = \sum m_j \zeta^{2j}$ with $\sum m_j = 0$. Hence the vector $(\gamma_0 - \gamma_0^* - m_0, \dots, \gamma_4 - \gamma_4^* - m_4)$ is orthogonal to $(1,1,1,1,1)$, $(1, \zeta^2, \zeta^4, \zeta^6, \zeta^8)$, $(1, \zeta^{-2}, \zeta^{-4}, \zeta^{-6}, \zeta^{-8})$, whence it is a linear combination of $(1, \zeta, \zeta^2, \zeta^3, \zeta^4)$ and $(1, \zeta^{-1}, \zeta^{-2}, \zeta^{-3}, \zeta^{-4})$. This leads to (9.4).

It is sometimes attractive to pass from the complex parameter ξ to two real parameters u and v, related to ξ by

$$\text{(9.5)} \qquad \xi = (1 - \zeta^2)u + (1 - \zeta^3)v.$$

The condition $\xi - \xi^* \in P$ becomes

$$\text{(9.6)} \qquad u - u^* \in J, \quad v - v^* \in J$$

where J is the set of all reals of the form $m + n(\zeta + \zeta^{-1})$ with $m \in \mathbb{Z}$, $n \in \mathbb{Z}$.
Note that

$$\text{(9.7)} \qquad (\gamma_0, \dots, \gamma_4) = (u + v, u, 0, 0, v)$$

gives one of the γ-vectors that lead to ξ by (9.1).

As to AR-patterns associated with pentagrids, we have to restrict ourselves, at least for the time being, to regular cases.

THEOREM 9.2. If the pentagrids determined by (ξ, η) and (ξ^*, η^*) are regular, they produce the same AR-pattern if and only if $\xi = \xi^*$. Their AR-patterns are shift-equivalent if and only if $\xi - \xi^* \in P$.

PROOF. If $\xi - \xi^* \in P$ we have, by the second part of the proof of Theorem 9.1, $\gamma_j - \gamma_j^* - m_j = \operatorname{Re}(z_0\zeta^{-j})$ for some $z_0 \in \mathbb{C}$. A shift by z_0 in the z-plane has no influence at all on the AR-pattern, the m_j's shift the AR-pattern by an element of P (see the end of Section 5). So if $\xi - \xi^* \in P$ the AR-patterns are shift-equivalent, if $\xi = \xi^*$ they are equal.
If (ξ, η) and (ξ^*, η^*) produce the same AR-pattern, we have $\xi = \xi^*$. For if $\xi \neq \xi^*$ we would get a contradiction by Theorem 8.1, taking $k_j \in \mathbb{Z}$ such that

$$(\textstyle\sum k_j, \sum k_j \zeta^{2j} - \xi) \in V, \quad (\sum k_j, \sum k_j \zeta^{2j} - \xi^*) \notin V,$$

since the numbers $\sum k_j \zeta^{2j}$ lie dense in $\mathbb{C}$, even with prescribed $\sum k_j$.

Next assume the AR-patterns to be shift-equivalent. Since all vertices are in $\mathbb{Z}[\zeta]$, the shift vector (i.e. the vector that has to be added to the points of the (ξ, η)-pattern in order to get the (ξ^*, η^*)-pattern) has the form $\sum n_j\zeta^j$ with $n_j \in \mathbb{Z}$. Since for both AR-patterns the index $\sum k_j$ is always in the set $\{1,2,3,4\}$, and since each of these four possibilities occurs at least once (this is trivial from (8.3)), we have $\sum n_j \equiv 0 \pmod 5$. Hence $\sum n_j\zeta^j = \sum m_j\zeta^j$, with $\sum m_j = 0$. So if we take $\gamma_j^{**} = \gamma_j^* - m_j$ (whence $\xi^{**} - \xi^* \in P$), the (ξ^{**}, η^{**})-pattern coincides with the (ξ, η)-pattern. By what we proved before, we have $\xi = \xi^{**}$, and so $\xi - \xi^* \in P$.

10. TRANSFORMATIONS

Here we shall systematically consider some transformations of the parameter vectors $(\gamma_0, \dots, \gamma_4)$ (always with zero sum), and their effect on ξ, u, v (the parameters of Section 9) as well as on the point sets G and U. Here G stands for the pentagrid, considered as a point set in the complex plane, and

(10.1) $\quad U = \{\sum_j K_j(z)\zeta^j \mid z \in \mathbb{C}\}$

where $K_j(z)$ is given by (4.4). In the case that the pentagrid is regular, U is the set of rhombus vertices of the corresponding AR-pattern.

We use the obvious notations for transformed sets in the complex plane: $G - z_0$ stands for $\{z - z_0 \mid z \in G\}$, $\bar{G} = \{\bar{z} \mid z \in G\}$, etc.

(i) Taking any $z_0 \in C$ we pass from the vector γ to the vector γ^* by

$$\gamma_j^* = \gamma_j + \mathrm{Re}(z_0\zeta^{-j}) \quad (j = 0, \dots, 4).$$

Now $\xi^* = \xi$, $u^* = u$, $v^* = v$, $G^* = G - z_0$, $U^* = U$.

(ii) Taking integers $n_0, \dots, n_4$ with $n_0 + \cdots + n_4 = 0$ we define γ^* by

$$\gamma_j^* = \gamma_j + n_j \quad (j = 0, \dots, 4).$$

Now $\quad \xi^* = \xi + \sum n_j\zeta^{2j}, \quad u^* = u - (n_1 + n_2) + n_4(\zeta + \zeta^4), \quad v^* = v - (n_3 + n_4) + n_4(\zeta + \zeta^4)$, $G^* = G$, $U^* = U + \sum n_j\zeta^j$.

(iii) If we pass from γ to γ^* by $\gamma_j^* = \gamma_{5-j}$ $(j = 0, \dots, 4)$ we get $\xi^* = \bar{\xi}$, $u^* = v$, $v^* = u$, $G^* = \bar{G}$, $U^* = \bar{U}$.

(iv) If we take $\gamma_j^* = -\gamma_j$ $(j = 0, \dots, 4)$ then $\xi^* = -\xi$, $u^* = -u$, $v^* = -v$, $G^* = -G$, $U^* = -U$.

(v) The cyclic transform $\gamma_0^* = \gamma_1$, $\gamma_1^* = \gamma_2$, $\gamma_2^* = \gamma_3$, $\gamma_3^* = \gamma_4$, $\gamma_4^* = \gamma_0$ is connected with rotation: $\xi^* = \zeta^{-2}\xi$, $G^* = \zeta^{-1}G$, $U^* = \zeta^{-1}U$. The formulas for u and v are slightly less convenient: $u^* = v$, $v^* = -u + (\zeta^2 + \zeta^3)v$.

11. SINGULAR PENTAGRIDS

The question whether a pentagrid, defined by reals $\gamma_0, \dots, \gamma_4$ with $\gamma_0 + \cdots + \gamma_4 = 0$, is singular, can be answered by means of the complex parameter ξ of (9.1).

THEOREM 11.1. *A pentagrid is singular if and only if its parameter ξ has one of the forms*

(11.1) $\quad iu + \alpha, \quad i\zeta u + \alpha, \quad i\zeta^2 u + \alpha, \quad i\zeta^3 u + \alpha, \quad i\zeta^4 u + \alpha$

with $u \in \mathbb{R}$, $\alpha \in P$ (for P see Section 1).

PROOF. Assume the pentagrid to be singular, so somewhere it has three lines through a single point. If there are more than three lines through that point, we just select three of them. The directions are taken from $\{i, i\zeta, i\zeta^2, i\zeta^3, i\zeta^4\}$ and hence one of the three lines is a line of symmetry for the pair formed by the other two. By shift and rotation (cases (i) and (v) of Section 10) the point becomes 0 and the line of symmetry the imaginary axis. This means that γ_0, γ_1, γ_4 are integers or that γ_0, γ_2, γ_3 are integers (or both). Applying transformation (ii) of Section 10 we get to $\gamma_0=\gamma_2=\gamma_3=0$, $\gamma_4=-\gamma_1$ or to $\gamma_0=\gamma_1=\gamma_4=0$, $\gamma_2=-\gamma_3$. In both cases ξ is purely imaginary. Using what we know about the behavior of ξ under the transformations, we find that, for some $j\in\{0,\dots,4\}$, $\xi\zeta^{-j}$ is congruent mod P to a purely imaginary number.

The if-part of the theorem can be proved in the same way.

The cases where $\xi\in P$ are *exceptionally singular* in the sense that there is a point that lies on 5 lines. By a shift we get to the grid given by $\gamma_0=\cdots=\gamma_4=0$.

If ξ has simultaneously two of the forms (11.1), i.e., if $\xi=iu_1\zeta^j+\alpha_1=$ $=iu_2\zeta^k+\alpha_2$, with $u_1,u_2\in\mathbb{R}$, $\alpha_1,\alpha_2\in P$, $0\le j<k\le 4$, then we have the exceptionally singular case $\xi\in P$. We can show this by proving that if $i(a\zeta^j-b\zeta^k)\in P$, $a\in\mathbb{R}$, $b\in\mathbb{R}$ then $ia\zeta^j\in P$, $ib\zeta^k\in P$. We shall treat a typical case: $j=0$, $k=1$. We have $i(a-b\zeta)=\sum n_j\zeta^j$. Taking complex conjugates, we get two linear equations for a and b, so a and b can be expressed in terms of the n's. Using $\sum n_j=0$, one finds $ia=n_1(\zeta-\zeta^4)+(n_0+n_2)(\zeta^2-\zeta^3)$, $ib=n_0(\zeta-\zeta^4)+$ $+(n_1+n_4)(\zeta^2-\zeta^3)$, and these values belong to P.

If in a pentagrid three lines pass through a point then one of the lines, namely the one that bisects the angle between the other two, contains infinitely many points of threefold intersection, and no twofold intersection points. Let is call it a *singular line* of the grid. We can study this by transforming to one of the cases $\gamma_0=\gamma_1=\gamma_4=0$, $\gamma_0=\gamma_2=\gamma_3=0$. The singular line has become the imaginary axis, and it is a line of symmetry for the whole pentagrid. In the exceptionally singular case there are five such lines with points of threefold intersection, and these five lines pass through a single point with 10-fold symmetry. Apart from shifts there is only one such pentagrid, namely the one given by $\xi=0$.

12. AR-PATTERNS ASSOCIATED WITH SINGULAR PENTAGRIDS

We consider a singular pentagrid with parameters $\gamma_0^{(0)},\dots,\gamma_4^{(0)}$, with zero sum. We want to know what happens to the singular line if the parameters are varied a little. That is, we consider a perturbed grid with parameters $\gamma_0,\dots,\gamma_4$, also with zero sum.

Let us use the term j-line for all lines of the form $\mathrm{Re}(z\zeta^{-j})=$ constant. Let us assume that the imaginary axis is a 0-line of the unperturbed grid and that some 1-line and some 4-line of that grid intersect on this 0-line. It follows that this 0-line is an axis of symmetry and that the other lines of the grid are arranged in pairs which intersect each other on that axis. These pairs consist either of a 1-line and a 4-line or of a 2-line and a 3-line.

Without loss of generality we assume $\gamma_0^{(0)}=\gamma_1^{(0)}+\gamma_4^{(0)}=\gamma_2^{(0)}+\gamma_3^{(0)}=0$.

Consider a pair of a 1-line and a 4-line intersecting on the singular line. In the perturbed situation, the intersection can be shown to lie on the left of the perturbed 0-line if $\gamma_0 + (\gamma_1 + \gamma_4)(\zeta^2 + \zeta^3)$ is negative, and on the right if that expression is positive. For an intersection of a two-line and a 3-line we get the same answer, but now with $\gamma_0 + (\gamma_2 + \gamma_3)(\zeta + \zeta^4)$. The two expressions have the same sign, however, since

$$(\zeta + \zeta^4)(\gamma_0 + (\gamma_1 + \gamma_4)(\zeta^2 + \zeta^3)) + (\zeta^2 + \zeta^3)(\gamma_0 + (\gamma_2 + \gamma_3)(\zeta + \zeta^4)) = 0.$$

Moreover, that sign is the same as the sign of the real part of ξ (see (9.1)), since (cf. (9.2)).

$$\operatorname{Re} \xi = (1 - \tfrac{1}{2}(\zeta + \zeta^4))(\gamma_0 + (\gamma_1 + \gamma_4)(\zeta^2 + \zeta^3)).$$

We conclude that if the perturbation moves ξ to the left (note that $\operatorname{Re} \xi^{(0)} = 0$), then we get the situation depicted in figure 10: the intersections of 1-lines and 4-lines and the intersections of 2-lines and 3-lines, which were lying on the 0-line, all appear on the left of that line in the perturbed situation. Similarly we get the situation shown in figure 11 if ξ moves to the right. Hence, we can consider the singular pentagrid as the limit of a sequence of regular pentagrid in two ways, and the corresponding AR-patterns have two different limits.

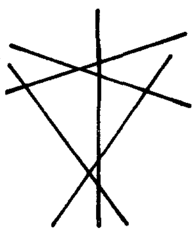

Fig. 10. ξ approaching from the left.

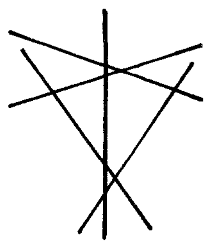

Fig. 11. ξ approaching from the right.

In Section 5 we defined the rhombus patterns of regular pentagrids, but in a singular pentagrid we can still associate a point $\sum K_j(z)\zeta^j$ (with K_j defined by (4.4)) to each mesh, and we can connect these points $\sum K_j(z)\zeta^j$ in a way corresponding to the edges of the meshes. Only, we do not get just rhombuses, but also hexagons (corresponding to threefold intersections), and possibly a regular decagon (corresponding to the fivefold intersection in the exceptionally singular case).

For the time being we shall consider pentagrids which are singular but not exceptionally singular.

There are two kinds of hexagons, which we shall call D-hexagons and Q-hexagons, respectively. In the case that the singular line is a 0-line, the D-hexagons are obtained from intersections with 1-lines and 4-lines, the Q-hexagons from intersections with 2-lines and 3-lines. If we modify the pentagrids by moving ξ to the left (figure 10) the D-hexagons and Q-hexagons are filled as in figure 12, if we move ξ to the right (figure 11) we get the pictures of figure 13. (The names D and Q are chosen according to the type of the point inside, which can be a deuce or a queen.) This way we see how the rhombus-hexagon pattern (belonging to the singular pentagrid) can be filled in two ways to form an AR-pattern. One of them is obtained by taking the limit of the AR-pattern of the perturbed pentagrid with ξ tending to its limit from the left. The other one is obtained if ξ approaches from the right.

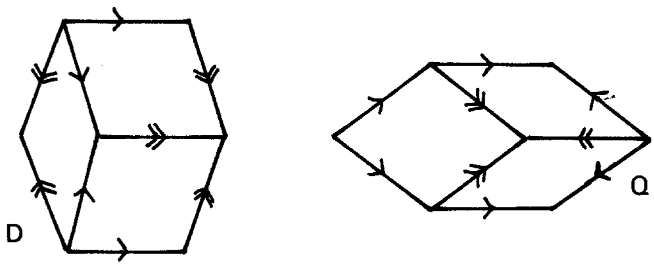

Fig. 12. Hexagons corresponding to figure 10.

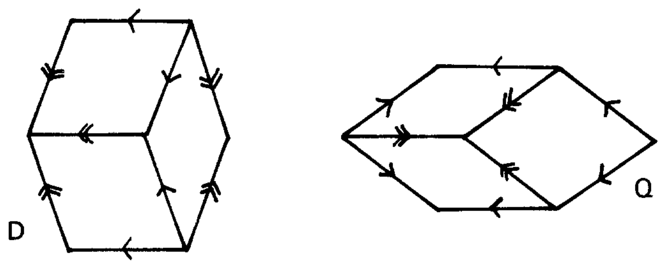

Fig. 13. Hexagons corresponding to figure 11.

The two AR-patterns produced by our singular pentagrid (with vertical singular line) are mirror twins. In the middle they have an infinite vertical chain of D's and Q's, either all as in figure 12, or all as in figure 13. In Section 17 we shall discuss the question of what the sequence of D's and Q's can be.

Apart from the chain of D's and Q's, the AR-patterns are symmetric with respect to the vertical line.

For the exceptionally singular pentagrid ($\xi = 0$, say) the above discussion is not entirely valid. The figures formed by small variations of the five lines through the fivefold point are not of the type suggested in figures 10 and 11. What happens is determined by which one of the 10 angles formed by the lines $\mathrm{Re}(z\zeta^{-j}) = 0$ contains ξ. This means that there are 10 different ways to approach $\xi = 0$, and these 10 are obtained from each other by rotation. A typical case of the situation around the point $z = 0$ in a perturbed pentagrid is given in figure 14, and figure 15 shows the decagon filling corresponding to it. From figure 14 it can be derived what happens with the threefold points on the singular lines. E.g., since the 1-line and the 4-line intersect on the left of the 0-line, we have the situation of figure 10 for all threefold points along that 0-line. This is also clear from the rhombus pattern. On each side of the decagon of figure 14 there grows an infinite chain of D- and Q-hexagons, and whether these are filled according to figure 12 or to figure 13 depends on the direction of the arrow on the side of the decagon.

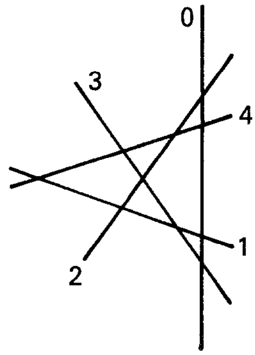

Fig. 14. Five lines which almost pass through a point.

Altogether, to the exceptionally singular pentagrid there correspond 10 different AR-patterns. All these are congruent. Each one has just one axis of symmetry (orthogonal to one of the grid lines).

If we pass from AR-patterns to kites and darts, sequences of D's and Q's turn into sequences of what were called in [2] long and short bow ties. It is not hard to prove there is at most one Q between consecutive D's, whence the sequence can be broken up into pieces $(\frac{1}{2}D, Q, \frac{1}{2}D)$ and $(\frac{1}{2}D, \frac{1}{2}D)$. These pieces correspond to long and short bow ties, respectively.

The essentially unique kite-and-dart pattern belonging to the exceptionally singular pentagrids was described in [2] and called the *cartwheel.*

13. SYMMETRIES OF PENTAGRIDS

Here we shall investigate symmetries of pentagrids irrespective of whether they are regular or not. Symmetries of regular pentagrids carry over at once to the corresponding AR-patterns (and therefore to the kite-and-dart patterns).

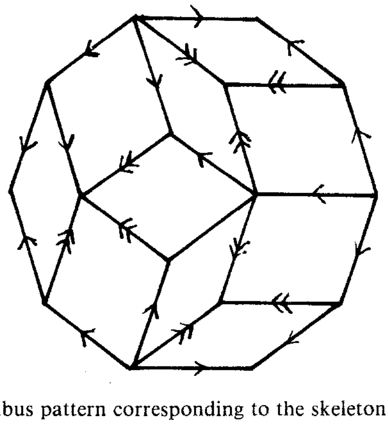

Fig. 15. The rhombus pattern corresponding to the skeleton shown in figure 14.

For singular pentagrids, however, the constructions of Section 12 may distort the symmetry.

The symmetries we have to consider are (with the notation of Section 10) of the kind where some rotation turns G into something that is either shift-equivalent to G or to $\bar{G}$. That means either (cf. Theorem 9.1)

(13.1) $\quad \xi-(-1)^h\zeta^{2j}\xi\in P$

with $h=0$ or $1, j=0, 1, 2, 3, 4$ ($h=j=0$ excluded) or

(13.2) $\quad \xi-(-1)^h\zeta^{2j}\bar{\xi}\in P$

with $h=0$ or 1, $j=0$, 1, 2, 3, 4. From every class of mutually congruent pentagrids it will suffice to indicate just one element.

We first consider (13.1) with $j\neq 0$. In $\mathbb{Z}[\zeta]$ the factor $1-(-1)^h\zeta^{2j}$ divides $1-\zeta^{4j}$ and therefore $1-\zeta$, so (13.1) implies $(1-\zeta)\xi\in P$. Since P consists of all $(1-\zeta)\theta$ with $\theta\in\mathbb{Z}[\zeta]$, we infer $(1-\zeta)(\xi-\theta)=0$ for some θ, whence $\xi\in\mathbb{Z}[\zeta]$. Every element of $\mathbb{Z}[\zeta]$ is congruent to 0, ± 1, ± 2 mod P. The cases with 1 and -1 produce congruent pentagrids (passage from G to $-G$), and so do 2 and -2. So we only have to consider $\xi=0$, 1 and 2. We know that $\xi=0$ is the exceptionally singular case (Section 12), but $\xi=1$ and $\xi=2$ are regular. Passing from AR-patterns to kite and darts, $\xi=1$ and $\xi=2$ correspond to the "infinite star" pattern and the "infinite sun" pattern (see [2]), respectively.

The case of (13.1) with $h=1$, $j=0$, i.e., the relation $2\xi\in P$, gives $\xi=\frac{1}{2}\sum n_j\zeta^j$ with $n_j\in\mathbb{Z}$, $\sum n_j=0$. Hence zero or two or four of the n_j are odd. If all n_j are even then $\xi\in P$, i.e., the pentagrid is congruent to the one with $\xi=0$. If four of the n_i are odd, $n_1,\dots,n_4$, say, we write $(n_0,\dots,n_4)=2(m_0,\dots,m_4)+$ $+(4,-1,-1,-1,-1)$, and we get $\xi=5/2$. By rotation we get the five cases $\xi=5\zeta^j/2$. The vectors with n_2, n_3 odd, n_0, n_1, n_4 even, give ξ's with $\xi\equiv\frac{1}{2}\zeta^2-\frac{1}{2}\zeta^3 (\operatorname{mod} P)$, since $\frac{1}{2}n_0$, $\frac{1}{2}n_1$, $\frac{1}{2}(n_2-1)$, $\frac{1}{2}(n_3+1)$, $\frac{1}{2}n_4$ have zero sum. The cases with n_1, n_4 odd, n_0, n_2, n_3 even, produce $\xi\equiv\frac{1}{2}(\zeta-\zeta^4) \pmod P$. The further cases are reduced to these by rotation.

The three values 5/2, $\frac{1}{2}\zeta^2-\frac{1}{2}\zeta^3$, $\frac{1}{2}\zeta-\frac{1}{2}\zeta^4$ give essentially different cases. All three are singular, since $\zeta^2-\zeta^3$ and $\zeta-\zeta^4$ are purely imaginary, and 5/2 is congruent to the sum of the two others.

We now turn to (13.2), and we first remark that $\xi\equiv\bar{\xi} \pmod P$ if and only if $\xi\in P+\mathbb{R}$ (if $\xi-\bar{\xi}=\sum n_j\zeta^j$ with $n_j\in\mathbb{Z}$, $\sum n_j=0$, we easily derive $n_0=0$, $n_1=-n_4$, $n_2=-n_3$, whence $\xi+n_1(1-\zeta)+n_2(1-\zeta^2)\in\mathbb{R}$). Similarly, we have $\xi\equiv-\bar{\xi}$ $\pmod P$ if and only if $\xi\in P+i\mathbb{R}$ (if $\xi+\bar{\xi}=\sum n_j\zeta^j$ with $n_j\in\mathbb{Z}$, $\sum n_j=0$, we derive $n_1=n_4$, $n_2=n_3$, $n_0=-2n_1-2n_2$, whence $\xi+n_1(1-\zeta)+n_2(1-\zeta^2)\in i\mathbb{R}$).

If (13.2) holds, we put $\xi_1=\xi\zeta^{-j}$ and we get $\xi_1=(-1)^h\bar{\xi}_1$. It follows that $\xi_1\in i^h\mathbb{R}+P$, whence $\xi\in i^h\zeta^j\mathbb{R}+P$.

The cases with $\xi\in i\zeta^j\mathbb{R}+P$ are all singular (Section 11). The cases with $\xi\in\zeta^j\mathbb{R}+P$ are not necessarily singular. We investigate the cases with $\xi\in\mathbb{R}+P$ (the others are obtained by rotation). Assume that such a ξ is singular, so also $\xi\in i\zeta^j\mathbb{R}+P$. If $j=1,2,3,4$ this leads to $\xi\in P$ only. This can be shown by calculation, but also geometrically: if there are two non-parallel axes of symmetry then there is a point of ten-fold symmetry. We finally consider $\xi \in i\mathbb{R}+P$. Since also $\xi \in \mathbb{R}+P$, we have $\xi = a_1 + p_1 = ia_2 + p_2$ with $a_1, a_2 \in \mathbb{R}$, $p_1, p_2 \in P$. We deduce $a_2 = \mathrm{Im}(p_1 - p_2)$. Therefore ξ has modulo P the form $\frac{1}{2}m(\zeta-\zeta^4)+\frac{1}{2}n(\zeta^2-\zeta^3)$, whence $2\xi \equiv 0 \pmod{P}$. The only essentially different cases are 0, $\frac{1}{2}(\zeta-\zeta^4)$, $\frac{1}{2}(\zeta^2-\zeta^3)$ and these were all mentioned before.

Summarizing, we have the following essentially different cases of symmetry:

$$\xi=0,\quad \xi=1,\quad \xi=2,\quad \xi=5/2,\quad \xi=\tfrac{1}{2}(\zeta^2-\zeta^3),$$
$$\xi=\tfrac{1}{2}(\zeta-\zeta^4),\quad \xi\in\mathbb{R},\quad \xi\in i\mathbb{R},$$

apart from the fact that the latter two can be equivalent to one of the others in exceptional cases.

14. DEFLATION AND INFLATION

A decisive point in the construction of kite-and-dart patterns is the operation of deflation and its inverse, inflation. By an ingenious subdivision rule for the separate kites and darts, a kite-and-dart pattern is turned into a new one, where the pieces have a smaller size, $-\frac{1}{2}+\frac{1}{2}\sqrt{5}$ times the original one. It is called the *deflation* of the old one. The construction can already be applied to a finite set of kites and darts that covers just a part of the plane. Conversely, if we have a tiling of the entire plane with kites and darts, it can be shown to be the deflation of a uniquely defined kite-and-dart pattern with bigger pieces, $\frac{1}{2}+\frac{1}{2}\sqrt{5}$ times the size of the old one. That pattern is called the *inflation* of the old one. We do not present details (which can be found in [2]) since deflation and inflation have their equivalents for AR-patterns, and it is for those that we shall give a full description. In figure 16 we depict the thick rhombus and its deflation. In figure 17 we show the same thing for the thin rhombus. It is quite easy to check that the AR-pattern turns into a new one with smaller pieces.

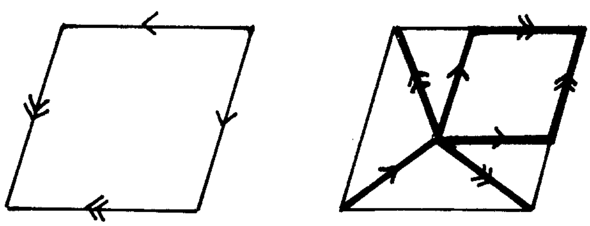

Fig. 16. Deflation of the thick rhombus.

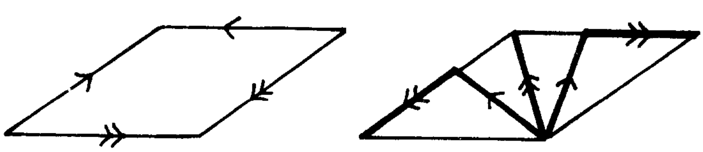

Fig. 17. Deflation of the thin rhombus.

It is interesting to note what happens to the types (cf. figure 7) of the vertices. Let us call the original pattern ϕ, and the deflated pattern (with smaller pieces)

ψ. Every J of ϕ becomes a K in ψ. We denote this fact by $J_\phi \to K_\psi$. Similarly we have $D_\phi \to Q_\psi$, $K_\phi \to S4_\psi$, $Q_\phi \to S3_\psi$, $S3_\phi \to S_\psi$, $S4_\phi \to S_\psi$, $S5_\phi \to S_\psi$, $S_\phi \to S5_\psi$. No vertex of ϕ turns into a J_ψ, but a J_ψ stems from a point in the interior of a thick rhombus. And D_ψ's arise from points on red arrows of ϕ. Actually every thick rhombus produces one J_ψ, and every red arrow one D_ψ.

It is now easy to describe the inflation χ of an AR-pattern ϕ. Just omit all J_ϕ's and D_ϕ's; the remaining vertices are the vertices of χ. We connect two vertices of χ if their distance is $\frac{1}{2}+\frac{1}{2}\sqrt{5}$ times the edge-length in ϕ. If such a connection passes through a D_ϕ it is coloured red, otherwise green. We can now orient the arrows such as to get the proper orientation in the thick and thin rhombuses.

Thus far we studied general AR-patterns in this section, but we now turn our attention to patterns generated by pentagrids. If ϕ is the AR-pattern generated by a regular pentagrid with parameters $\gamma_0, \dots, \gamma_4$, then its inflation χ admits a very simple description. We form $\delta_0, \dots, \delta_4$ by

(14.1) $\quad \delta_j = \gamma_{j+1} + \gamma_{j-1} \quad (j = 0, \dots, 4)$

(where $\gamma_5 = \gamma_0$, $\gamma_{-1} = \gamma_4$). Note the converse $\gamma_j = \delta_{j-1} + \delta_j + \delta_{j+1}$. And note what happens to the parameter ξ:

$$\textstyle\sum_j \delta_j \zeta^{2j} = -p \sum_j \gamma_j \zeta^{2j}$$

where $p = -(\zeta^2 + \zeta^{-2}) = \frac{1}{2} + \frac{1}{2}\sqrt{5}$ (see (1.1)).

We claim that the inflation χ of ϕ satisfies $\chi = p\phi_\delta$, if ϕ_δ is the AR-pattern generated by the pentagrid with parameters $\delta_0, \dots, \delta_4$, and $p\phi_\delta$ is the pattern obtained from the points and lines of ϕ_δ by multiplication with p (in the sense of multiplication in the complex plane).

A nice way to establish this result is provided by the pentagons of Section 8. Let V be the set (8.2). We define an injection $H: V \to V$ by

$$H(h, z) = (3h \bmod 5,\ -z/p) \quad (h \in \{1,2,3,4\}, z \in \mathbb{C}),$$

where $3h \bmod 5$ stands for the $h' \in \{1,2,3,4\}$ with $h' \equiv 3h \pmod 5$. From the fact that the types of the vertices can be derived from figures 8 and 9 it follows that $V \setminus H(V)$ is just the set of J's and D's on levels 1 and 4.

As a further preparation we define the set W by

(14.2) $\quad W = \{(k_0, \dots, k_4) \in \mathbb{Z}^5 \mid 1 \le \sum k_j \le 4\}$

and the bijection $\Phi : W \to W$ by $\Phi(k) = m$, with

$$m_j = k_{j-1} + k_j + k_{j+1} - c \quad (j = 0, \dots, 4)$$

where $k_{-1} = k_4$, $k_5 = k_0$, and $c = 0, 1, 1, 2$ according to $\sum k_j = 1, 2, 3, 4$. If $k \in W$ we have indeed $m \in W$, and $\sum m_j \equiv 3 \sum k_j \pmod 5$. The inverse mapping is given by $k_j = m_{j-1} + m_j - d$, where $d = 0, 0, 1, 1$ according to $\sum m_j = 1, 2, 3, 4$. We note that if $m = \Phi(k)$ then

(14.3) $\quad \sum_j m_j \zeta^j = p \sum_j k_j \zeta^j.$

We now get back to the AR-patterns. If $k \in W$ we have by Theorem 8.1 that $\sum k_j \zeta^j$ is a vertex of ϕ_γ if and only if $f(\gamma, k) \in V$, where

$$f(\gamma, k) = (\sum k_j, \sum (k_j - \gamma_j)\zeta^{2j}).$$

And $\sum k_j \zeta^j$ is a vertex of ϕ_δ if and only if $f(\delta, k) \in V$.

It is easy to check that for all $k \in W$

(14.4) $\quad f(\gamma, \Phi(k)) = H(f(\delta, k)).$

If we delete from ϕ_γ the J's and D's, the remaining vertices are the $\sum k_j \zeta^j$ with $k \in W$, $f(\gamma, k) \in H(V)$. Replacing k by m and then m by $\Phi(k)$, we get (by (14.3)) the $p \sum k_j \zeta^j$ with $k \in W$, $f(\gamma, \Phi(k)) \in H(V)$. The latter condition is equivalent to $f(\delta, k) \in V$. So if we delete the J's and D's from ϕ we get ϕ_δ, and that is what was claimed.

REMARK. We mention how the δ-pentagrid can be obtained from the γ-pentagrid. Take any intersection of a $(j+1)$-line and a $(j-1)$-line in the γ-pentagrid, and draw a j-line through the point we get if that intersection point is multiplied by p^{-1}. It easily follows from (4.1) that this is a j-line of the δ-pentagrid, and that all j-lines of that grid are obtained in this way.

15. ALL AR-PATTERNS ARE PRODUCED BY PENTAGRIDS

We shall use the results about inflation and deflation for showing that every AR*-pattern is produced by a regular or a singular pentagrid, in the sense of Sections 5 and 12. We use the term *AR*-pattern* for AR-patterns (with side length 1, as always) whose vertices all have the form $\sum k_j \zeta^j$ with $k \in W$ (see (14.2)). According to what we proved about the index in Section 6, every AR-pattern can be turned into an AR*-pattern by rotation and shift, and then the index turns out to be equal to $\sum k_j$. And we know (Sections 5 and 12) that the patterns generated by regular or singular pentagrids are AR*-patterns.

If ϕ is an AR-pattern then its deflation has the form $p^{-1}\phi^{(1)}$, where $p = \frac{1}{2} + \frac{1}{2}\sqrt{5}$, and $\phi^{(1)}$ is again an AR*-pattern. This is easily derived by means of the following remarks: (i) we can establish by means of what we know about the index and its relation to the position of the red and green arrows, that the new points introduced in figures 16 and 17 (inside the thick rhombus and on the red arrows) are again in W, and (ii) we have $pW = W$.

Similarly, the inflation of ϕ has the form $p\phi^{(-1)}$, where $\phi^{(-1)}$ is again an AR*-pattern.

We define $\phi^{(2)}, \phi^{(3)}, \dots$ by $\phi^{(n+1)} = (\phi^{(n)})^{(1)}$, and similarly $\phi^{(-2)}, \phi^{(-3)}, \dots$ by $\phi^{(-n+1)} = (\phi^{(-n)})^{(1)}$.

Consider two AR*-patterns ϕ and ψ. Assume that they have a vertex z_0 in common, and that the set of neighbors of z_0 in ϕ is the same as in ψ. Denote by K the union of the closed interiors of the rhombuses of ϕ and ψ that meet in z_0. It is easy to check that the deflations of ϕ and ψ coincide at least inside K, and the same thing holds for the deflations of the deflations, etc. Therefore $\phi^{(n)}$ and $\psi^{(n)}$ coincide inside $p^n K$, for $n = 1, 2, \dots$.

It is now easy to show that for any $R>0$ and for any AR*-pattern ϕ there exists an AR*-pattern ψ which is generated by a regular pentagrid, such that ϕ and ψ coincide in the region given by $|z|<R$. Take $n\in\mathbb{N}$ such that $p^n \sin 36° > 2R$ and consider $\phi^{(-n)}$. In $\phi^{(-n)}$ the point 0 belongs to a closed rhombus. Let z_0 be the vertex of that rhombus that is closest to 0 and again let K denote the union of the closed rhombuses meeting at z_0. The distance of 0 to the boundary of K_1 is at least $\frac{1}{2}\sin 36°$.

We can find a regular pentagrid that generates an AR*-pattern χ that coincides with $\phi^{(-n)}$ as far as z_0 and its neighbors are concerned. (This can be established by taking an arbitrary regular pentagrid, and verifying that all types of figure 7 occur at least once in its AR-pattern.) Therefore the n-th deflation of $\phi^{(n)}$ and χ coincide in a circle with center 0 and radius $\frac{1}{2}\sin 36°$. According to Section 14, the n-th deflation has the form $p^{-n}\psi$, where ψ is also generated by a regular pentagrid. So ϕ and ψ coincide inside the region given by $|z|<\frac{1}{2}p^n \sin 36°$ and therefore in $|z|<R$.

Now start from some AR*-pattern ϕ. Let $\psi_1, \psi_2, \dots$ be AR*-patterns arising from regular pentagrids and such that ψ_n coincide with ϕ for all points in the circle with center 0 and radius n.

Let $\gamma_{0n}, \dots, \gamma_{4n}$ be the parameters of the pentagrid producing ψ_n. Take a fixed vertex $\sum k_j\zeta^j$ of ϕ, with $k\in W$. For n sufficiently large it is a vertex of ψ_n, so $z_n\in\mathbb{C}$ exists such that $\lceil \mathrm{Re}(z_n\zeta^{-j})+\gamma_{jn}\rceil = k_j$. Replacing γ_{jn} by $\mathrm{Re}(z_n\zeta^{-j})+\gamma_{jn}$ we get the same AR*-pattern (cf. Section 10(i)), and so we may assume that we had $|\gamma_{jn}|\le|k_j|+1$ from the start. It follows that there is a subsequence $(\gamma_{01},\dots,\gamma_{41})$, $(\gamma_{02},\dots,\gamma_{42}),\dots$ converging to some $(\gamma_0,\dots,\gamma_4)$. Obviously γ has zero sum too.

If γ produces a regular AR-pattern then it is easy to check that it coincides with ϕ. If it produces a singular AR-pattern its pentagrid is the limit of a sequence of regular pentagrids, and we get one of the singular patterns corresponding to the singular pentagrid (cf. Section 12).

16. QUASI-PERIODICITY OF AR-PATTERNS

Two of the most amazing things about kite-and-dart patterns are (see [2]): (i) none of these patterns is periodic, and (ii) if we have two patterns, then any portion of the first one can be found in the second one, just applying a parallel shift.

These things are quite easy to understand since we know that all AR-patterns are produced by pentagrids. Let us use as pentagrid parameters the ξ and η of Section 9. The η has no influence at all on the AR-patterns. According to the index properties ($\sum k_j$ being 1, 2, 3 or 4, see Section 6) the only possible periods can be $\sum n_j\zeta^j$ with $n_j\in\mathbb{Z}$, $\sum n_j=0$. By Section 10(ii) this means that we want to have $\sum n_j\zeta^{2j}\neq 0$, $\sum n_j\zeta^j=0$, and that is impossible.

If ϕ_1 and ϕ_2 are AR-patterns with parameters ξ_1, ξ_2, then ξ_2 can be approximated with arbitrary precision by numbers congruent to ξ_1 mod P (since P is dense in $\mathbb{C}$). From this we can derive the statement on finite portions of ϕ_1 which are repeated in ϕ_2. Needless to say, the singular cases require some extra attention.

17. RELATION OF PENROSE PATTERNS TO SEQUENCES OF ZEROS AND ONES GENERATED BY SPECIAL REWRITING RULES

In a previous paper [1] we dealt with a kind of sequences which form a paradigm for the Penrose tilings. We took a doubly infinite sequence of zeros and ones (i.e., a mapping of $\mathbb{Z}$ into $\{0,1\}$). Its "deflation" is obtained by replacing each 0 by 10 and each 1 by 100. Not every sequence is the deflation of another one, but there exist sequences s which have inflations (in [1] the term "predecessor" was used) of all orders, by which we mean the following. The sequence s is the deflation of a sequence $s^{(1)}$, this $s^{(1)}$ is the deflation of $s^{(2)}$, etc. In [1] Sections 8 and 9 it was shown that such sequences can be characterized by means of a procedure that in a two-dimensional square lattice mimics the things we have described in Section 7 of the present paper for a five-dimensional cubic lattice, with a similar rôle of deflation. The Penrose–Conway construction of arbitrary patterns by application of a sequence of shifts and deflations, starting with a single piece, has its direct analog in those sequences. Even the distinction between regular and singular cases occurs in the paradigm.

One thing is missing in the paradigm. The Penrose patterns are forced upon us by means of the rules for fitting the pieces together, but this does not seem to have an analog in the case of the sequences.

In [1] the rewriting rule $1\to 100$, $0\to 10$, was a special case of a class of rewriting rules. The simplest case is $1\to 10$, $0\to 1$, and this is related to the golden section. The sequences with inflations of all orders with respect to the rule $1\to 10$, $0\to 1$ are actually present in our singular AR-patterns. Consider the singular pentagrids with an infinite chain of D's and Q's as in Section 12. For convenience we write the D's and Q's from left to right instead of from bottom to top. We cut the D's into a left part D_L and a right part D_R. ("Left" and "right" refer to the figures we get if figures 12 and 13 are rotated over 90° to the right.) Similarly each Q splits into Q_L and Q_R. Now we study deflation. Deflation of a D gives D_RQD_L, and deflation of a Q gives D_RD_L. A doubly infinite sequence of Q's and D's deflates into a doubly infinite sequence of D_RQD_L's and D_RD_L's. In this sequence each D_RQD_L and each D_RD_L is preceded by a D_L. So if for each D_RQD_L and each D_RD_L we take away the D_L on the right and add it on the left we still have the same doubly infinite sequence. So instead of D_RQD_L we have DQ, and instead of D_RD_L we have D. Hence the deflation of the AR-pattern causes in the central chain of D's and R's just the same thing as the rewriting rule $D\to DQ$, $Q\to D$ (this corresponds to $D=1$, $Q=0$). The D-Q-sequences occurring in singular pentagrids have inflations of all orders, and indeed, every D-Q-sequence with this property occurs in some singular AR-pattern. This is easily verified by selecting the pentagrid parameters so as to check with the algebraic formula of [1] for those sequences.

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