Theory Comput. Systems **39**, 15–50 (2006)
DOI: 10.1007/s00224-005-1255-x

Theory of Computing Systems

A Burnside Approach to the Finite Substitution Problem*

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Abstract. We introduce a new model of weighted automata: the desert automata. We show that their limitedness problem is PSPACE-complete by solving the underlying Burnside problem. As an application of this result, we give a positive solution to the so-called finite substitution problem which was open for more than 10 years: given recognizable languages K and L, decide whether there exists a finite substitution σ such that $\sigma(K) = L$.

1. Introduction

The present paper is the first one in a series of papers in which we introduce new models of weighted automata to solve important decision problems in the theory of recognizable languages.

Hashiguchi introduced the notion of distance automata motivated by his research on the star height hierarchy in 1982 [11], [12]. *Distance automata* are non-deterministic finite automata with a set of marked edges. The weight of a path is defined as the number of marked transitions in the path. The weight of a word is the minimum of the weights of all successful paths of the word. Distance automata and the more general weighted automata over the tropical semiring became a fruitful concept in theoretical computer science with many applications beyond their impact for the decidability of the star height hierarchy [13], e.g., they have been of crucial importance in the research on the star problem in trace monoids [23], [32], but they are also of interest in industrial

* The main part of the paper was written during a 12 month stay at the Laboratoire d'Informatique Algorithmique, Fondements et Applications (LIAFA, Université Paris 7) which was funded by Grant KI 822/1-1 of the German Research Community (Deutsche Forschungsgemeinschaft).

applications as speech recognition [33], database theory [8], and image compression [6], [18]. Consequently, distance automata and related concepts have been studied by many researchers beside Hashiguchi [14], [15], [24], [28], [31], [39], [41]–[43].

The so-called finite substitution problem was open for more than 10 years [37]: given two recognizable languages K and L, decide whether there exists a finite substitution σ such that $\sigma(K) = L$. By an investigation of this problem and related questions, the author found out that distance automata are insufficient for solving certain types of problems.

This observation led us to introduce a different class of automata, the *desert automata* which are non-deterministic finite automata with a set of marked edges. The weight of a path is defined as the length of a longest subpath which does not contain a marked edge. The weight of a word is the minimum of the weights of all successful paths of the word. The main result of the paper states that it is decidable whether the range of the mapping of a desert automaton is finite which is a counterpart of the corresponding result for distance automata [11], [26], [41]. We also show that this problem is PSPACE-complete. For this, we formulate the limitedness problem for desert automata as a Burnside problem. We solve this Burnside problem by applying results from finite semigroup theory, using techniques from the research on distance automata and developing new ideas.

As an application, we show the decidability of the finite substitution problem.

Remarkably, the notion of a desert automaton was introduced independently by Bala and the author at STACS '04 [1], [21]. Bala showed in a quite different way the decidability of the limitedness problem of desert automata (supremum problem for automata with coloring) in PSPACE [1], [2]. Morover, Bala proved that a slighly more general variant of the finite substitution problem is EXSPACE-complete [1], [2].

The paper is organized as follows. In Section 2 we introduce preliminary notions, and get familiar with three automata concepts: classic automata, Hashiguchi's distance automata, and the new desert automata. We present the main results in Section 2.5. In Section 3 we get familiar with concepts and results from the theory of finite semigroups. In particular, we recall classic notions from ideal theory and we develop the notion of a consistent mapping as an abstraction from concepts due to Simon and Leung. In Section 4 we introduce some algebraic concepts to achieve a deeper understanding for desert automata to regard the limitedness problem of desert automata as a Burnside problem. We solve this Burnside problem in Section 5. In Section 6 we show the decidability of the finite substitution problem. In Section 7 we discuss our approach and point out some open questions concerning desert automata.

2. Overview

2.1. *Preliminaries*

Let $\mathbb{N} = \{0, 1, \ldots\}$ and $\mathbb{N}_1 := \{1, 2, \ldots\}$. For finite sets M, we denote by $|M|$ the number of elements of M. If p belongs to some set M, then we denote by p both the element p and the singleton set consisting of p. For sets M, we denote by $\mathcal{P}(M)$ the power set of M, and we denote by $\mathcal{P}_{\mathrm{f}}(M)$ the set of all finite subsets of M. We denote the union of disjoint sets by $\mathbin{\dot\cup}$.

A *semigroup* $(S, \cdot)$ consists of a set S and a binary associative operation $\cdot$. Usually, we denote $(S, \cdot)$ for short by S, and we denote the operation $\cdot$ by juxtaposition.

Let S be a semigroup. We call S *commutative* if $ab = ba$ for every $a, b \in S$. We call S *idempotent* if $aa = a$ for every $a \in S$. We call an element $1 \in S$ an *identity*, if we have for every $a \in S$, $1a = a1 = a$. If S has an identity, then we call S a *monoid*. We call an element $0 \in S$ a *zero*, if we have for every $a \in S$, $a0 = 0a = 0$. There are at most one identity and at most one zero in a semigroup. We extend the operation of S to subsets of S in the usual way.

Let $\leq$ be a binary relation over some semigroup S. We call $\leq$ *left stable* (resp. *right stable*) if for every $a, b, c \in S$ with $a \leq b$ we have $ca \leq cb$ (resp. $ac \leq bc$). We call $\leq$ *stable* if it is both left stable and right stable.

For subsets $T \subseteq S$, we call the closure of T under the operation of S the *subsemigroup generated by* T and denote it by $\langle T \rangle$. If $\langle T \rangle = T$, then we call T a *subsemigroup* of S.

A *semiring* $(K, +, \cdot)$ consists of a set and two binary operations $+$ and $\cdot$ whereas $(K, +)$ is a commutative monoid with identity 0, $(K, \cdot)$ is a monoid with identity 1 and zero 0, and the distributivity laws hold, i.e., for every $a, b, c \in K$, we have $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$.

For every monoid M, we denote by $\mathcal{P}(M)$ the semiring $(\mathcal{P}(M), \cup, \cdot)$ where $\cdot$ is the extension of the multiplication of M to sets.

The *tropical semiring* is the semiring over the set $\mathbb{N} \cup \infty$ equipped with minimum for the ordering $0 \leq 1 \leq \cdots \leq \infty$ and usual addition of integers extended to $x + \infty = \infty + x = \infty + \infty := \infty$ for every $x \in \mathbb{N}$.

Within the entire paper, we fix some $n \geq 1$ which is used as the dimension of matrices. Whenever we do not explicitly state the range of a variable, then we assume that it ranges over the set $\{1, \dots, n\}$. For example, phrases like "for every i, j" or "there is some l, such that" are understood as "for every $i, j \in \{1, \dots, n\}$," resp. "there is some $l \in \{1, \dots, n\}$, such that".

If $(K, +, \cdot)$ is a semiring, then we denote by $\mathbb{K}_{n\times n}$ the semiring of all $n \times n$ matrices over K equipped with matrix multiplication (defined by $\cdot$ and $+$ as usual) and componentwise operation $+$.

We denote a sequence $A_1, A_2, \dots$ over some set M by $(A_k)_{k\geq 1} \in M$.

Let Σ be a finite set of symbols. We denote by Σ^* the free monoid over Σ, i.e., Σ^* consists of all *words* over Σ with concatenation as operation. We denote the empty word by ε. We denote by Σ^+ the free semigroup over Σ, i.e., $\Sigma^+ := \Sigma^* \backslash \varepsilon$. For every $w \in \Sigma^*$, we denote by $|w|$ the length of w. Subsets of Σ^* are called *languages*.

For $L \subseteq \Sigma^*$, we define $L^* := L^0 \cup L^1 \cup \cdots = \bigcup_{i\in\mathbb{N}} L^i$ and $L^+ := L^1 \cup L^2 \cup \cdots = \bigcup_{i\geq 1} L^i$. Note that regardless of L, we have $L^0 = \{\varepsilon\}$. We call L^* the *iteration of* L.

Note that M^* is defined in two ways, depending on whether M is a set of symbols or M is a language. However, we use the notation M^* in such a way that no confusion arises.

2.2. *Classic Automata*

We recall some standard terminology in automata theory.

A (*non-deterministic*) *automaton* is a tuple $\mathcal{A} = [Q, E, I, F]$ where

1. Q is a finite set of *states*,
2. $E \subseteq Q \times \Sigma \times Q$ is a set of *transitions*, and
3. $I \subseteq Q$ and $F \subseteq Q$ are sets called *initial* and *accepting states*, respectively.

Let $k \geq 1$. A path π in $\mathcal{A}$ of length k is a sequence $(q_0, a_1, q_1)(q_1, a_2, q_2) \cdots (q_{k-1}, a_k, q_k)$ of transitions in E. We say that π starts at q_0 and ends at q_k. We call the word $a_1 \cdots a_k$ the *label of* π. We denote $|\pi| := k$. As usual, we assume for every $q \in Q$ a path which starts and ends at q and is labeled with ε.

We call π *successful* if $q_0 \in I$ and $q_n \in F$. For every $0 \leq i \leq j \leq k$, we denote $\pi(i, j) := (q_i, a_i, q_{i+1}) \cdots (q_{j-1}, a_{i-1}, q_j)$ and call $\pi(i, j)$ a *subpath of* π. For every $p, q \in Q$ and every $w \in \Sigma^*$, we denote by $p \overset{w}{\rightsquigarrow} q$ the set of all paths with the label w which start at p and end at q.

We call $\mathcal{A}$ *unambiguous* if for every word $w \in \Sigma^*$, there is at most one successful path. We call $\mathcal{A}$ *deterministic* if $|I| = 1$ and for every $q \in Q$, $a \in \Sigma$, there is at most one state $q' \in Q$ such that $(q, a, q') \in E$.

We denote the *language of* $\mathcal{A}$ by $L(\mathcal{A})$ and define it as the set of all words in Σ^* which are labels of successful paths. We call some language $L \subseteq \Sigma^*$ *recognizable* if L is the language of some automaton. Every recognizable language is the language of a deterministic automaton. See, e.g., [3], [7], and [44] for surveys on recognizable languages.

2.3. *Distance Automata*

A *distance automaton* is a tuple $\mathcal{A} = [Q, E, I, F, \Delta]$, where

1. $[Q, E, I, F]$ is an automaton and
2. $\Delta\colon E \to \{0, 1\}$ is a mapping called the *distance function*.

Let $\mathcal{A} = [Q, E, I, F, \Delta]$ be a distance automaton. The notions of a path, a successful path, the language of $\mathcal{A}$, unambiguous, resp. deterministic, distance automata... are understood with respect to $[Q, E, I, F]$. The distance function is extended to paths: the distance of a path π is defined as the sum of the distances of all transitions in π and is denoted by $\Delta(\pi)$. The distance of a word $w \in \Sigma^*$ is defined as the minimum over the distances of all successful paths labeled with w. Under the convention that $\min \emptyset = \infty$, we have $\Delta(w) \neq \infty$ iff $w \in L(\mathcal{A})$.

A distance automaton is *limited* if there is a bound $d \in \mathbb{N}$ such that $\Delta(w) \leq d$ for every $w \in L(\mathcal{A})$. If such a bound d exists, then we also say that $\mathcal{A}$ is *limited by* d. Otherwise, we call $\mathcal{A}$ *unlimited*.

The *limitedness problem for distance automata* is the question for an algorithm which decides whether a distance automaton is limited or not. It was already raised by Choffrut in the framework of recognizable power series over the tropical semiring in 1979 [5]. In 1982 Hashiguchi showed its decidability:

Theorem 2.1. *It is decidable whether a distance automaton is limited.*

Today, there are several proofs for this result which can be divided into two independent streams. There are combinatorial proofs mainly by Hashiguchi [11], [14]–[16], [31] and algebraic proofs mainly due to Simon and Leung [26], [27], [30], [41]. The algebraic proofs offer deeper insights into this problem. On the other hand, just the recent combinatorial proofs allow us to show that the limitedness problem for distance automata is in PSPACE [31]. It was already observed in [26] and [28] that it is PSPACE-hard, i.e., the limitedness problem for distance automata is PSPACE-complete [31].

A particular case of the limitedness problem for distance automata is the *finite power problem* (for short *FPP*) which was already raised by J.A. Brzozowski in 1966. It is the question for an algorithm which decides whether for a recognizable language $L \subseteq \Sigma^*$, there is some $k \in \mathbb{N}$ such that $L^* = \bigcup_{i=0}^{k} L^i$. If such a k exists, then L has the *finite power property*. It took 12 years until Simon and Hashiguchi independently showed the existence of such an algorithm [10], [38]. In fact, [10] and [38] are the beginning of the two independent streams of research on the limitedness problem on distance automata. Another approach to the FPP was recently shown by the author [19].

2.4. *Desert Automata*

A *desert automaton* is a tuple $\mathcal{A} = [Q, E, I, F, E^{\curlyvee}]$ where

1. $[Q, E, I, F]$ is an automaton and
2. $E^{\curlyvee} \subseteq E$ are called *water transitions*.

Let $\mathcal{A} = [Q, E, I, F, E^{\curlyvee}]$ be a desert automaton. As for distance automata, the notions of a path, a successful path, the language of $\mathcal{A}$, unambiguous resp. deterministic desert automata ... are understood with respect to $[Q, E, I, F]$.

Imagine that you plan to walk through a desert for a few weeks. You carry a water tank, but this tank does not last the entire way. However, you visit several places during your journey where you can find some water and refill the tank. Clearly, the required capacity of the tank is determined by the maximal distance between two consecutive water places.

If there is no water place within your route, then the required capacity of the tank is determined by the length of the route.

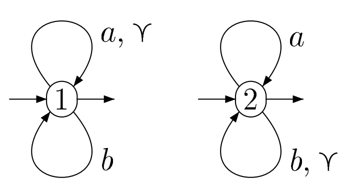
Of course, the water tank has to be sufficient, but, on the other hand, you want to carry a rather small tank. Consequently, if you have the choice between several paths through the desert, then you will probably choose a path for which the required capacity of the tank is minimal. This intuition leads us to the following definition:

Let $k \geq 1$ and π be a path. We denote by $\Delta(\pi)$ the length of a longest subpath of π which does not contain any transition from $E^{\curlyvee}$. For every $w \in \Sigma^*$, let

$$\Delta(w) := \min\{\Delta(\pi) \mid p \in I, q \in F, \pi \in p \overset{w}{\rightsquigarrow} q\}.$$

As for distance automata, a desert automaton is *limited* if there is some $d \in \mathbb{N}$ such that $\Delta(w) \leq d$ for every $w \in L(\mathcal{A})$. Now, it is quite clear that the desert automata are in some sense orthogonal to distance automata.

Example 2.1. Consider the desert automaton $\mathcal{A}$ over $\Sigma = \{a, b\}$, with $Q = I = F = \{1, 2\}$, with E and $E^{\curlyvee}$ as shown in the following picture:



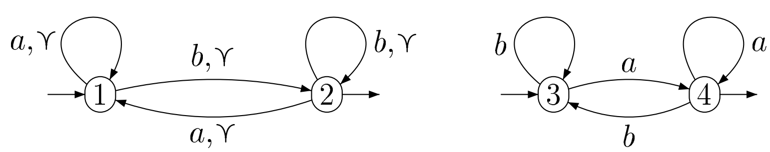
Transitions in $E^{\curlyvee}$ are labeled by, e.g., "$a, \curlyvee$", other transitions are labeled just by their letter.

Obviously, $\mathcal{A}$ accepts every word. Let $w \in \Sigma^*$ be arbitrary. Then $\Delta(w)$ is the largest integer for which w contains subwords of the form $a^{\Delta(w)}$ and $b^{\Delta(w)}$. Consequently, $\Delta(a^k b^k) = k$ for every $k \geq 1$, and, thus, $\mathcal{A}$ is not limited.

Let $u, v, w \in \Sigma^*$ and $k \geq 1$ be arbitrary. If $\Delta(uv^k v) > |uvw|$, then we have $v \in a^+$ and $v \in b^+$, which is a contradiction. Thus, $\Delta(uv^k v) \leq |uvw|$ for every $k \geq 1$. Consequently, it is not possible to prove that $\mathcal{A}$ is not limited by considering some sequence $uvw, uv^2w, uv^3w, \ldots$. More precisely, an easy pumping argument is not sufficient to prove that $\mathcal{A}$ is not limited.

The desert automaton $\mathcal{A}$ is not unambiguous. More precisely, there are exactly two successful paths for every $w \in \Sigma^*$. Assume that the same mapping can be computed by an unambiguous desert automaton $\mathcal{A}'$. Let n be the number of states of $\mathcal{A}'$. Because $\mathcal{A}'$ is unlimited, there is some word $w \in \Sigma^*$ with $\Delta(w) > n$. Let π be the successful path which is labeled by w. We factorize π as $\pi = \pi_1\pi_2\pi_3$ such that $|\pi_2| = \Delta(w) > n$ and π_2 does not contain a water transition. There is a cycle π_2' in π_2, i.e., we can factorize π_2 as $\pi_2 = \pi_1'\pi_2'\pi_3'$ such that $|\pi_2'| > 0$. Let w_1, w_2, w_3 be the labels of $\pi_1\pi_1', \pi_2', \pi_3'\pi_3$, respectively. It is easy to see that for every $k \geq 1$, we have $\Delta(w_1 w_2^k w_3) \geq |\pi_2| + (k-1)|\pi_2'| \geq k$. This contradicts our observation, above. Consequently, $\mathcal{A}$ does not admit an equivalent unambiguous desert automaton.

Example 2.2. There is another interesting desert automaton $\mathcal{A} = [Q, E, I, F, E^{\curlyvee}]$ where $I = \{1, 3\}$ and $F = \{2, 4\}$:



We have $\varepsilon \notin L(\mathcal{A})$. For every word $w \in \Sigma^+$, there is exactly one successful path which is labeled with w. Hence, $\mathcal{A}$ is unambiguous. For every $w \in \Sigma^*$, we have $\Delta(wa) = |wa|$ and $\Delta(wb) = 0$.

Assume that there is an equivalent deterministic desert automaton $\mathcal{A}'$. Let q_0 be its initial state. Because $\Delta(a) = 1$, there is exactly one state q such that (q_0, a, q) is a transition in $\mathcal{A}'$, and, moreover, this transition is not a water transition. Hence, $\Delta(ab) \geq 1$, which is a contradiction. Hence, $\mathcal{A}$ does not admit an equivalent deterministic desert automaton.

2.5. *Main Results*

We have the following theorem.

Theorem 2.2. *The classes of mappings which are computable by deterministic desert automata, unambiguous desert automata, and arbitrary desert automata form a strict hierarchy.*

Proof. See Examples 2.1 and 2.2. □

As a conclusion from Example 2.1, we cannot decide the limitedness problem of desert automata by an easy pumping approach.

As the main result of the present paper we show:

Theorem 2.3. *To decide whether a desert automaton is limited is PSPACE-complete.*

On the one hand, our proof of Theorem 2.3 follows Leung's strategy to show the decidability of the limitedness of distance automata. However, in contrast to distance automata, the distance of some path $\pi = \pi_1\pi_2$ cannot be calculated from $\Delta(\pi_1)$ and $\Delta(\pi_2)$. We just have

$$\max\{\Delta(\pi_1), \Delta(\pi_2)\} \leq \Delta(\pi) \leq (\Delta(\pi_1) + \Delta(\pi_2)).$$

Thus, we cannot use the tropical semiring and we develop the notion of word matrices and related results in Section 4.

To show PSPACE-hardness in Theorem 2.3, we apply Leung's construction to show that the limitedness problem for distance automata is PSPACE-hard [26], [28].

Above, we mentioned the FPP as a particular case of the limitedness problem for distance automata. There is a corresponding weaker version of the limitedness problem for desert automata: the *finite generator problem*, for short *FGP*. It simply means to decide whether for some recognizable language L with $L = L^*$ there is a finite subset $T \subseteq L$ such that $T^* = L$. If such a finite subset $T \subseteq L$ exists, then we say that L is *finitely generated*. The FGP is rather easy to decide. One simply checks by standard techniques of automata theory whether the set $L\backslash((L\backslash\varepsilon)^+(L\backslash\varepsilon))$ is finite.

As an application of the decidability of the limitedness problem for desert automata, we show the decidability of the finite substitution problem. Let Σ_1 and Σ_2 be two alphabets. We call every mapping $\sigma\colon \Sigma_1 \to \mathcal{P}(\Sigma_2^*)$ a *substitution*. Every substitution extends in a natural way to a homomorphism $\sigma\colon \mathcal{P}(\Sigma_1^*) \to \mathcal{P}(\Sigma_2^*)$. Let σ be a substitution. We call σ a *finite substitution* if for every $a \in \Sigma_1$, the language $\sigma(a)$ is finite.

The *finite substitution problem* is the question for an algorithm which decides whether for two recognizable languages $K \subseteq \Sigma_1^*$ and $L \subseteq \Sigma_2^*$, there is a finite substitution σ such that $\sigma(K) = L$.

Consider the particular case $K = a^*$ for some $a \in \Sigma_1$ and $L = L^*$. If σ is a finite substitution with $\sigma(K) = L$, then $\sigma(a)$ is a finite set with $(\sigma(a))^* = L$. Conversely, if there is some finite set $T \subseteq \Sigma_2^*$ with $T^* = L$, then we can define by $\sigma(a) = T$ a finite substitution with $\sigma(K) = L$. Hence, the FGP is a particular case of the finite substitution problem.

Theorem 2.4. *The finite substitution problem is decidable.*

Bala showed that a slightly more general variant of the finite substitution problem is EXPSPACE-complete [1], [2]. In Bala's approach, Σ_1 is a disjoint union of Σ_2 and a finite set of variables $\Xi = \{X_1, X_2, \ldots\}$, and he considers substititions $\sigma\colon (\Sigma_2 \cup \Xi) \to \mathcal{P}(\Sigma_2^*)$ which satisfy the condition $\sigma(a) = a$ for every $a \in \Sigma_2$.

3. Finite Semigroup Theory

We develop some notions on semigoups. In Section 3.1 we get familiar with classic ideas from ideal theory. In Section 3.2, we introduce the notion of a consistent mapping as an abstraction of several particular mappings used by Simon and Leung.

3.1. *Ideal Theory*

We introduce some concepts from ideal theory. This section is far from being a comprehensive overview, for a deeper understanding and for the omitted proofs, the author recommends textbooks, e.g., [9], [25], and [35]. We just state the notions and results which we need in the rest of the paper.

As already mentioned, a semigroup S is a set with a binary associative operation which we denote by juxtaposition. Let S be a semigroup within this section.

If there is no identity in S, then we denote by S^1 the semigroup consisting of the set $S \cup 1$, on which the operation of S is extended in a way that 1 is the identity of S^1. If S has an identity, then we define S^1 to be S.

We call an $e \in S$ an *idempotent* if $e^2 = e$. We denote the set of all idempotents of S by $\mathsf{E}(S)$.

The following relations are called Green's relation. We show several equivalent definitions. Let $a, b \in S$.

1. $a \leq_{\mathscr{J}} b :\Longleftrightarrow a \in S^1 b S^1 \Longleftrightarrow S^1 a S^1 \subseteq S^1 b S^1$.
2. $a \leq_{\mathscr{L}} b :\Longleftrightarrow a \in S^1 b \Longleftrightarrow S^1 a \subseteq S^1 b$.
3. $a \leq_{\mathscr{R}} b :\Longleftrightarrow a \in b S^1 \Longleftrightarrow a S^1 \subseteq b S^1$.

We allow one to denote $a \leq_{\mathscr{J}} b$ by $b \geq_{\mathscr{J}} a$, and similarly for the other relations. The relation $\leq_{\mathscr{L}}$ is right stable and, similarly, $\leq_{\mathscr{R}}$ is left stable. However, we do not have a similar property for $\leq_{\mathscr{J}}$.

Again, let $a, b \in S$. We define:

1. $a =_{\mathscr{J}} b :\Longleftrightarrow a \leq_{\mathscr{J}} b$ and $a \geq_{\mathscr{J}} b \Longleftrightarrow S^1 a S^1 = S^1 b S^1$.
2. $a =_{\mathscr{L}} b :\Longleftrightarrow a \leq_{\mathscr{L}} b$ and $a \geq_{\mathscr{L}} b \Longleftrightarrow S^1 a = S^1 b$.
3. $a =_{\mathscr{R}} b :\Longleftrightarrow a \leq_{\mathscr{R}} b$ and $a \geq_{\mathscr{R}} b \Longleftrightarrow a S^1 = b S^1$.

It is easy to see that $=_{\mathscr{J}}$, $=_{\mathscr{L}}$, and $=_{\mathscr{R}}$ are equivalence relations. We call their equivalence classes $\mathscr{J}$-classes (resp. $\mathscr{L}$-, $\mathscr{R}$-classes). For every $a \in S$, we denote by $\mathscr{J}(a)$, $\mathscr{L}(a)$, and $\mathscr{R}(a)$ the $\mathscr{J}$-, $\mathscr{L}$-, $\mathscr{R}$-class of a, respectively.

Remark 3.1. Let $e \in \mathsf{E}(S)$ and $a \leq_{\mathscr{L}} e$. There is some $p \in S^1$ such that $a = pe$. Hence, $ae = pee = pe = a$. Similarly, if $b \leq_{\mathscr{R}} e$, then $eb = b$.

Let $e, f \in \mathsf{E}(S)$ with $e =_{\mathscr{L}} f$ and $e =_{\mathscr{R}} f$. Then $ef = e$ and $ef = f$, i.e., $e = f$.

On the set of idempotents $\mathsf{E}(S)$, one defines a natural ordering $\leq$ such that for every $e, f \in \mathsf{E}(S)$, we have $e \leq f$ iff $e = ef = fe$. By Remark 3.1, we have $e \leq f$ iff $e \leq_{\mathscr{L}} f$ and $e \leq_{\mathscr{R}} g$. The relation $\leq$ is indeed an ordering: if for some $e, f \in \mathsf{E}(S)$, we have $e \leq f$ and $f \leq e$, then $e =_{\mathscr{L}} f$ and $e =_{\mathscr{R}} f$, and, by Remark 3.1, $e = f$.

As above, $=_{\mathscr{L}}$ and $=_{\mathscr{R}}$ are right stable and left stable, respectively.

For every $a \in S^1$, we denote by $a\cdot$, resp. $\cdot a$, the left, resp. right, multiplication by a.

The following lemma due to Green is of crucial importance to understanding the relations between $\mathscr{J}$-, $\mathscr{L}$-, and $\mathscr{R}$-classes.

Lemma 3.1 (Green's Lemma). Let S be a semigroup, let $a, b \in S$, and let $p, q \in S^1$.

(1) If $b = ap$ and $a = bq$, then $\cdot p$ and $\cdot q$ are mutually inverse, $\mathscr{R}$-class preserving bijections between $\mathscr{L}(a)$ and $\mathscr{L}(b)$.
(2) If $b = pa$ and $a = qb$, then $p\cdot$ and $q\cdot$ are mutually inverse, $\mathscr{L}$-class preserving bijections between $\mathscr{R}(a)$ and $\mathscr{R}(b)$.

The notion *$\mathscr{R}$-class preserving* (and similarly *$\mathscr{L}$-class preserving*) means that we have $c =_{\mathscr{R}} cp$ for every $c \in \mathscr{L}(a)$ and $d =_{\mathscr{R}} dq$ for every $d \in \mathscr{L}(b)$.

We assume from now that S is finite. There are several connections between Green's relations and multiplication.

Lemma 3.2. *Let S be a finite semigroup. Let $a, b \in S$ and $p, q \in S^1$ be arbitrary.*

(1) *If $a =_{\mathscr{J}} paq$, then $pa =_{\mathscr{L}} a =_{\mathscr{R}} aq$.*
(2) *If $a =_{\mathscr{J}} ab$, then $a =_{\mathscr{R}} ab$.*
(3) *If $b =_{\mathscr{J}} ab$, then $b =_{\mathscr{L}} ab$.*
(4) *If $a =_{\mathscr{J}} b$, we have $\mathscr{L}(a) \cap \mathscr{R}(b) \neq \emptyset$.*

The following lemma will be very useful.

Lemma 3.3. *Let S be a finite semigroup, and let $a, b \in S$ satisfying $a =_{\mathscr{J}} b$. We have $a =_{\mathscr{J}} b =_{\mathscr{J}} ab$ iff there is an idempotent $e \in \mathsf{E}(S)$ such that $a =_{\mathscr{L}} e =_{\mathscr{R}} b$.*

Usually, one visualizes a $\mathscr{J}$-class by an "egg-box picture" in which the co**l**umns are $\mathscr{L}$-classes and the **r**ows are $\mathscr{R}$-classes. We can combine Lemmas 3.2(2–4) and 3.3: If $a =_{\mathscr{J}} b =_{\mathscr{J}} ab$, then $ab \in \mathscr{R}(a) \cap \mathscr{L}(b)$ and $\mathscr{L}(a) \cap \mathscr{R}(b) \neq \emptyset$ as shown in the following table:

a		ab	
e		b	

Lemma 3.4. *Let J be a $\mathscr{J}$-class of a finite semigroup S. The following assertions are equivalent*:

(1) $JJ \cap J \neq \emptyset$.
(2) *There is at least one idempotent in J.*
(3) *In every $\mathscr{L}$-class of J and in every $\mathscr{R}$-class of J there is at least one idempotent.*

One distinguishes two kinds of $\mathscr{J}$-classes. If some $\mathscr{J}$-class J satisfies the three equivalent conditions in Lemma 3.4, then we call J a *regular*, $\mathscr{J}$-class, otherwise we call J *non-regular*. We call some element $a \in S$ *regular* if $\mathscr{J}(a)$ is a regular $\mathscr{J}$-class. We denote the set of all regular elements of S by $\mathsf{Reg}(S)$.

Let T be a subsemigroup of S. We have $\mathsf{E}(T) = \mathsf{E}(S) \cap T$ and $\mathsf{Reg}(T) \subseteq \mathsf{Reg}(S)$. However, we do not necessarily have $\mathsf{Reg}(T) = \mathsf{Reg}(S) \cap T$.

For every $k > 0$ and $a_1, \ldots, a_k \in S$, we call $a_1, \ldots, a_k$ a *smooth product* if we have $a_1 =_{\mathscr{J}} a_2 =_{\mathscr{J}} \cdots =_{\mathscr{J}} a_k =_{\mathscr{J}} (a_1 \cdots a_k) \in \mathsf{Reg}(S)$. Note that this is not a classic notion.

The following property will be very useful.

Lemma 3.5. *Let S be a finite semigroup and let $a, b \in S$ such that $a =_{\mathscr{J}} b$. If $ab = a$, then $b \in \mathsf{E}(S)$. If $ab = b$, then $a \in \mathsf{E}(S)$.*

The assumption $a =_{\mathscr{J}} b$ in Lemma 3.5 is crucial. Just assume that S has a zero and consider the case $a = 0$.

Let J_1 and J_2 be two $\mathscr{J}$-classes. There are $a \in J_1$ and $b \in J_2$ satisfying $a \leq_{\mathscr{J}} b$ iff we have $a \leq_{\mathscr{J}} b$ for every $a \in J_1$ and $b \in J_2$. Hence, $\leq_{\mathscr{J}}$ extends to a partial ordering of the $\mathscr{J}$-classes.

In a finite semigroup there is always a maximal $\mathscr{J}$-class, but it is not necessarily unique.

We call some subset of $I \subseteq S$ an *ideal* if $S^1 I S^1 \subseteq I$. Obviously, some subset $I \subseteq S$ is an ideal iff I is closed under $\leq_{\mathscr{J}}$, i.e., iff for every $a \in S$, $b \in I$ with $a \leq_{\mathscr{J}} b$ we have $a \in I$. In particular, every ideal of S is saturated by the $\mathscr{J}$-classes of S.

If S is finite, then there are some $z \geq 1$ and ideals $I_1, \ldots, I_{z+1}$ of S satisfying

$$S = I_1 \supsetneq I_2 \supsetneq \cdots \supsetneq I_z \supsetneq I_{z+1} = \emptyset$$

such that for every $l \in \{1, \ldots, z\}$, the set $I_l \setminus I_{l+1}$ is a $\mathscr{J}$-class. Moreover, z is the number of $\mathscr{J}$-classes of S. It is easy to construct such a chain of ideals: you simply start by $I_1 := S$, then you set $I_2 := I_1 \setminus J_1$ where J_2 is a maximal $\mathscr{J}$-class and so on.

This closes our expedition to the realms of ideal theory. The reader should be aware that ideal theory is just an initial part of the huge field of the structure theory of semigroups. Moreover, the notions and results in this section are just the beginning of ideal theory, and there are many important aspects which are not covered here. For example, there is a deep theorem by Rees and Sushkevich which describes the inner structure of regular $\mathscr{J}$-classes of finite semigroups up to isomorphism.

3.2. *Consistent Mappings*

We develop the notion of a consistent mapping as an abstraction from two concepts of stabilization for matrices over several semirings which play a key role in many articles by Simon and Leung [26]–[30], [39], [40], [41].

Let S be a finite semigroup.

Lemma 3.6. *Let $\overline{}\colon \mathsf{E}(S) \to \mathsf{E}(S)$ be a mapping. The following assertions are equivalent.*

1. *For every $a, b \in S^1$ with $ab, ba \in \mathsf{E}(S)$, we have $(\overline{ab}) = a(\overline{ba})b$.*
2. *For every $a, b \in S^1$ and $e, f \in \mathsf{E}(S)$ with $e =_{\mathscr{J}} f$ and $f = aeb$, we have $\overline{f} = a\bar{e}b$.*

Proof. (2)⇒(1) Let $a, b \in S^1$ with $ab, ba \in \mathsf{E}(S)$. By $ab = abab$, we have $ab \leq_{\mathscr{J}} ba$. By $ba = baba$, we have $ba \leq_{\mathscr{J}} ab$. Thus, $ab =_{\mathscr{J}} ba$. By (2) for $f = ab$ and $e = ba$, we have $(\overline{ab}) = a(\overline{ba})b$.

(1)⇒(2) Let a, b, e, f as in (2). By (1), we have $(\overline{ee}) = e(\overline{ee})e$, i.e., $\bar{e} = e\bar{e}e$.

We have $f = (ae)(eb)$ and $(ae) =_{\mathscr{J}} (eb) =_{\mathscr{J}} e$. We show $e = ebae$. We denote Green's relations between e, f, ae, and eb in the following egg-box picture:

eb		e	
f		ae	

Because the idempotent f belongs to $\mathscr{L}(eb) \cap \mathscr{R}(ae)$, we have $(eb)(ae) =_{\mathscr{J}} e$. Thus, $ebae =_{\mathscr{L}} e$, and $ebae =_{\mathscr{R}} e$. Moreover, we have $(ebae)(ebae) = ebfae = ebae \in \mathsf{E}(S)$. By Remark 3.1, we have $e = ebae$. Because both $(ae)(eb)$ and $(eb)(ae)$ are idempotent, we have by (1) $(\overline{aeeb}) = ae(\overline{ebae})eb$. Thus, $\overline{f} = ae\bar{e}eb = a\bar{e}b$. □

We call a mapping $\overline{}\colon \mathsf{E}(S) \to \mathsf{E}(S)$ *consistent* if it satisfies the two conditions in Lemma 3.6. If $\overline{}$ is a consistent mapping and $e \in \mathsf{E}(S)$, then $e = 1ee = ee1 = eee$ and thus $\bar{e} = \bar{e}e = e\bar{e} = e\bar{e}e$, i.e., $\bar{e} \leq_{\mathscr{L}} e$, and $\bar{e} \leq_{\mathscr{R}} e$. Thus, $\bar{e} \leq e$ in the natural ordering $\leq$ of the idempotents.

Now, we use some results from finite semigroup theory to show that every consistent mapping admits a unique extension to regular elements. Lemma 3.7 was already shown by H. Leung in a more particular framework [26], [27], [30].

Lemma 3.7. *Let $\overline{}$ be a consistent mapping. Let $e, f \in \mathsf{E}(S)$ and $a, b, c, d \in S^1$ satisfying $aeb = cfd =_{\mathscr{J}} e =_{\mathscr{J}} f$. We have $a\bar{e}b = c\bar{f}d$.*

Proof. Let J be the $\mathscr{J}$-class with $aeb = cfd =_{\mathscr{J}} e =_{\mathscr{J}} f \in J$. We have $ae, eb, cf, fd \in J$. As seen in Section 3.1, $ae =_{\mathscr{R}} aeb = cfd =_{\mathscr{R}} cf$ and $eb =_{\mathscr{L}} aeb = cfd =_{\mathscr{L}} fd$.

ae		cf		$aeb = cfd$
pae		f		fd
e		ebr		eb

There is some $p \in S^1$ such that $pcf = f$. By Green's lemma (Lemma 3.1), $p\cdot$ and $c\cdot$ are mutually inverse bijections between $\mathscr{R}(cf)$ and $\mathscr{R}(f)$. By $ae \in \mathscr{R}(cf)$, we have $ae = cpae$.

Similarly, there is some $r \in S^1$ such that $fdr = f$, and, moreover, $eb = ebrd$.

By $cfd = aeb$, we have $pcfdr = paebr$, and thus, $f = paebr$. By (2) in Lemma 3.6, we have $\overline{f} = pa\bar{e}br$. Then we have $f\bar{f}f = pae\bar{e}ebr$, and $cf\bar{f}fd = cpae\bar{e}ebrd$, i.e., $cf\bar{f}fd = ae\bar{e}eb$, and, finally, $c\bar{f}d = a\bar{e}b$. □

Lemma 3.7 allows us to extend consistent mappings to regular elements of S.

Corollary 3.8. *Let S be a finite semigroup and let $\overline{}\colon \mathsf{E}(S) \to \mathsf{E}(S)$ be a consistent mapping. By setting $\overline{aeb} := a\bar{e}b$ for every $a, b \in S^1$, $e \in \mathsf{E}(S)$ satisfying $e =_{\mathscr{J}} aeb$, we define a mapping $\overline{}\colon \mathsf{Reg}(S) \to \mathsf{Reg}(S)$.*

Proof. By Lemma 3.7, it remains to show $a\bar{e}b \in \mathsf{Reg}(S)$. By $aeb =_{\mathscr{J}} e$, we have $ae =_{\mathscr{L}} e =_{\mathscr{R}} eb$. There are $c, d \in S^1$ such that $cae = e = ebd$. We obtain $ca\bar{e}bd = cae\bar{e}ebd = e\bar{e}e = \bar{e}$, and thus $a\bar{e}b =_{\mathscr{J}} \bar{e}$, i.e., $\bar{e}$ is an idempotent in $\mathscr{J}(a\bar{e}b)$. □

Remark 3.2. Let $a \in S$ be arbitrary and let $e, f \in S$ satisfying $e =_{\mathscr{R}} a =_{\mathscr{L}} f$. Then $ea = af = a$ and $\bar{e}a = a\bar{f} = \bar{a}$. Consequently, $\bar{a} \le_{\mathscr{L}} a$ and $\bar{a} \le_{\mathscr{R}} a$.

The next lemma allows us to deal with consistent mappings in a very convenient way.

Lemma 3.9. *Let $a, b, c \in S^1$.*

(1) *If $abc =_{\mathscr{J}} b \in \mathsf{Reg}(S)$, then we have $\overline{abc} = a\bar{b}c$.*
(2) *If $a =_{\mathscr{J}} b =_{\mathscr{J}} ab \in \mathsf{Reg}(S)$, then we have $\overline{ab} = \bar{a}b = a\bar{b} = \bar{a}\bar{b}$.*

Proof. (1) Because $b \in \mathsf{Reg}(S)$, there is some $e \in \mathsf{E}(S)$ with $e =_{\mathscr{L}} b$, i.e., $be = b$. By the extension of $\sharp$, we have $\bar{b} = b\bar{e}$ and $a\bar{b}c = ab\bar{e}c$. For $\overline{abc}$, we obtain $\overline{abc} = \overline{abec} = ab\bar{e}c$.

(2) There is some $e \in \mathsf{E}(S)$ such that $a =_{\mathscr{L}} e =_{\mathscr{R}} b$, i.e., $ae = a$ and $eb = b$. We have $\overline{ab} = a\bar{e}b = \overline{ae}b = \bar{a}b$, $\overline{ab} = a\bar{e}b = a\overline{eb} = a\bar{b}$, and $\overline{ab} = a\bar{e}b = a\bar{e}\bar{e}b = \overline{ae}\,\overline{eb} = \bar{a}\bar{b}$. □

Consider the case $c = 1$ in (1). Then $\overline{ab} = a\bar{b}$. Similarly, $\overline{bc} = \bar{b}c$ if $a = 1$.

If $a, b, c \in S$ are a smooth product, then we can play with a consistent mapping:

$$\overline{abc} = \bar{a}bc = a\bar{b}c = ab\bar{c} = \bar{a}\bar{b}c = \bar{a}b\bar{c} = a\bar{b}\bar{c} = \bar{a}\bar{b}\bar{c} = \overline{ab}\bar{c} = \cdots.$$

For the stabilizations used by Simon and Leung we have $\bar{e} = \bar{\bar{e}}$ for every $e \in \mathsf{E}(S)$. However, this property does not hold for every consistent mapping, as the following example shows.

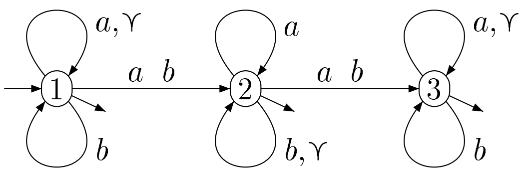
Example 3.1. Consider the monoid over $M = \{1, \ldots, 9\}$ with the maximum operation defined by the usual ordering of the integers. It is easy to verify that the mapping defined by $\bar{x} := x + 1$ for $x \in \{1, \ldots, 8\}$ and $\bar{9} = 9$ is consistent. However, we have, e.g., $\bar{2} = 3 \neq 4 = \bar{\bar{2}}$.

4. An Algebraic Framework for the Limitedness Problem

We develop an algebraic framework to show the decidability of the limitedness problem of desert automata. We achieve a reduction of this problem to a Burnside type problem.

During this section let $\mathcal{A} = [Q, E, I, F, E^{\Upsilon}]$ be a desert automaton, set $n := |Q|$, and assume $Q = \{1, \ldots, n\}$.

Example 4.1. During Sections 4 and 5 we use the following desert automaton $\mathcal{A}_1 = [Q_1, E_1, I_1, F_1, E_1^{\Upsilon}]$ to explain our notions and results:



4.1. *The Semigroup of Word Matrices*

As observed in Section 2.4, $\mathcal{A}$ does not specify a homomorphism from Σ^* to a semigroups of matrices over the tropical semiring. Thus, we introduce the concept of the semigroup of word matrices.

We consider the alphabet $D := \{\Upsilon, \Lambda\!\!\!\Lambda\}$. The words over D represent paths in desert automata. For technical reasons, we avoid the empty word from now on.

We consider the structure $(\mathcal{P}_{\mathrm{f}}(D^+), \cup, \cdot)$. Note that $(\mathcal{P}_{\mathrm{f}}(D^+), \cup, \cdot)$ satisfies all the axioms for a semiring except that there is no identity for multiplication.

We introduce a new element ω by defining $\mathbb{D} := \mathcal{P}_{\mathrm{f}}(D^+) \cup \{\omega\}$, and extend $\cup$ and $\cdot$ to $\mathbb{D}$. For every set $X \in \mathbb{D}\backslash\{\emptyset\}$, we define

1. $\omega \cdot X = X \cdot \omega := \omega$, $\omega \cup X = X \cup \omega := X$, and
2. $\emptyset \cdot \omega = \omega \cdot \emptyset := \emptyset$, $\emptyset \cup \omega = \omega \cup \emptyset := \omega$.

There is a convenient explanation of this strange behavior: You may imagine that some semiring $\mathbb{K}$ has a strong zero 0 and a weak zero 0_{w}. The strong zero 0 is the zero of $\mathbb{K}$ in the usual sense. The weak zero 0_{w} acts as a zero up to the exception that $0 + 0_{\mathrm{w}} := 0_{\mathrm{w}}$ and $0 \cdot 0_{\mathrm{w}} = 0_{\mathrm{w}} \cdot 0 := 0$. Be aware that one cannot simply insert a weak zero 0_{w} into an arbitrary semiring $\mathbb{K}$. If $p, q \in \mathbb{K}$ are divisors of zero, then $0_{\mathrm{w}} \cdot p \cdot q = (0_{\mathrm{w}} \cdot p) \cdot q = 0_{\mathrm{w}} \cdot q = 0_{\mathrm{w}}$ but $0_{\mathrm{w}} \cdot p \cdot q = 0_{\mathrm{w}} \cdot (p \cdot q) = 0_{\mathrm{w}} \cdot 0 = 0$.

In $\mathbb{D}$, $\emptyset$ and ω are "something like a strong and a weak zero". However, $\mathbb{D}$ is not a semiring, because we do not have an identity for concatenation.

For lucidity, we sketch another, more algebraic, approach to define $\mathbb{D}$. Let $D_\omega := D \cup \{\omega\}$. We factorize $(\mathcal{P}_{\mathrm{f}}(D_\omega^+), \cup, \cdot)$ under the congruence $\approx$ induced by $\omega \cdot X \approx X \cdot \omega \approx \omega$ and $\omega \cup X \approx X \cup \omega \approx X$ for every $X \in \mathcal{P}_{\mathrm{f}}(D_\omega^+)\backslash\{\emptyset\}$.

We can easily construct a homomorphism h from $(\mathcal{P}_{\mathrm{f}}(D_\omega^+), \cup, \cdot)$ to $\mathbb{D}$ in a way that $\approx$ is the congruence induced by h.

Let $n \geq 1$. We call an $n \times n$ matrix A with entries in $\mathbb{D}$ a *word matrix* if there is some k such that every word in A has the length k, i.e., for every i, j and every $\pi \in A[i, j]$ we have $|\pi| = k$. We denote by $\mathbb{D}_{n \times n}$ the set of all $n \times n$ word matrices. Clearly, $\mathbb{D}_{n \times n}$ is closed under matrix multiplication, i.e., we can call $\mathbb{D}_{n \times n}$ *the semigroup of word matrices*.

Later, we use the free semigroup $\mathbb{D}_{n\times n}^{+}$ over $\mathbb{D}_{n\times n}$. We denote the unique homomorphism which arises from the identity on letters by $\alpha : \mathbb{D}_{n\times n}^{+} \to \mathbb{D}_{n\times n}$.

4.2. *The Semantics of Desert Automata*

We give another method to define the semantics of desert automata using matrices. We define a mapping $\theta : E \to D$ by

$$\theta(e) := \begin{cases} \wedge\!\!\wedge & \text{if} \quad e \notin E^{\curlyvee}, \\ \curlyvee & \text{if} \quad e \in E^{\curlyvee}. \end{cases}$$

This mapping extends to homomorphisms $\theta\colon\ E^{+} \to D^{+}$ and $\theta\colon\ (\mathcal{P}_{\mathrm{f}}(E^{+}), \cup, \cdot) \to (\mathbb{D}, \cup, \cdot)$. We can assign every word $w \in \Sigma^{+}$ a matrix $\theta(w) \in \mathbb{D}_{n\times n}$ by setting $\theta(w)[i, j] := \theta(i \overset{w}{\rightsquigarrow} j)$. Clearly, $\theta\colon\ \Sigma^{+} \to \mathbb{D}_{n\times n}$ is a homomorphism.

For two paths π, π' with $\theta(\pi) = \theta(\pi')$, we have $\Delta(\pi) = \Delta(\pi')$. Hence, the distance function Δ on paths induces a distance function on D^{+} as

$$\Delta(\pi) := \max\{l \in \mathbb{N} \mid \theta(\pi) \in D^{*}(\wedge\!\!\wedge)^{l} D^{*}\}$$

for every $\pi \in D^{+}$. We extend Δ to $X \in \mathbb{D}$:

1. $\Delta(X) := \min\{\Delta(\pi) \mid \pi \in X\}$ if $X \neq \omega$, and
2. $\Delta(\omega) := \omega$.

We can give another definition of the semantics of a desert automaton. For $w \in \Sigma^{+}$, let

$$\Delta(w) := \min\{\Delta(\theta(w)[i, j]) \mid i \in I,\ j \in F\}.$$

This definition is equivalent to the definition in Section 2.4 up to the empty word.

Example 4.1 (*continued*). For our example $\mathcal{A}_1$, we denote $A := \theta_1(a)$ and $B := \theta_1(b)$. We have

$$A = \begin{pmatrix} \{\curlyvee\} & \{\wedge\!\!\wedge\} & \emptyset \\ \emptyset & \{\wedge\!\!\wedge\} & \{\wedge\!\!\wedge\} \\ \emptyset & \emptyset & \{\curlyvee\} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \{\wedge\!\!\wedge\} & \{\wedge\!\!\wedge\} & \emptyset \\ \emptyset & \{\curlyvee\} & \{\wedge\!\!\wedge\} \\ \emptyset & \emptyset & \{\wedge\!\!\wedge\} \end{pmatrix}.$$

For every $k \geq 2$, we have $A^{k}[1, 3] = \{\curlyvee^{p}(\wedge\!\!\wedge)^{k-p-q}\curlyvee^{q} \mid 0 \leq p, q \leq k-2\}$. We can easily calculate

$$AB = \begin{pmatrix} \{\curlyvee\wedge\!\!\wedge\} & \{\curlyvee\wedge\!\!\wedge, \wedge\!\!\wedge\curlyvee\} & \{\wedge\!\!\wedge\wedge\!\!\wedge\} \\ \emptyset & \{\wedge\!\!\wedge\curlyvee\} & \{\wedge\!\!\wedge\wedge\!\!\wedge\} \\ \emptyset & \emptyset & \{\curlyvee\wedge\!\!\wedge\} \end{pmatrix} \quad \text{and} \quad \Delta_1(ab) = 1.$$

4.3. *The Small Desert Semiring*

Let $\mathcal{D} = \{\curlyvee, \wedge\!\!\wedge, \omega, \infty\}$. Intuitively, $\wedge\!\!\wedge$ represents a path without water, $\curlyvee$ represents a path with water, and ∞ means that there is no path. We define on $\mathcal{D}$ an operation $\cdot$ which describes the concatenation of paths as the maximum over the ordering $\wedge\!\!\wedge \sqsubseteq \curlyvee \sqsubseteq \omega \sqsubseteq \infty$. Clearly, $\cdot$ is commutative and idempotent, and $(\mathcal{D}, \cdot)$ is a monoid with identity $\wedge\!\!\wedge$ and zero ∞.

We define an operation min on $\mathcal{D}$ by the ordering $\curlyvee \le \wedge\!\!\wedge \le \omega \le \infty$. Clearly, $(\mathcal{D}, \min)$ is a commutative monoid with the identity ∞. You can easily verify that $\le$ is stable w.r.t. $\cdot$. Thus, min distributes over $\cdot$, i.e., $(\mathcal{D}, \min, \cdot)$ is a semiring. We denote by $\mathcal{D}_{n\times n}$ the semiring of all $n\times n$-matrices over $\mathcal{D}$. Now, consider the mapping $\Psi\colon \mathbb{D} \to \mathcal{D}$ defined by

$$\Psi(X) = \begin{cases} \curlyvee & \text{if} \quad X \cap \{\curlyvee, \wedge\!\!\wedge\}^* \curlyvee \{\curlyvee, \wedge\!\!\wedge\}^* \neq \emptyset, \\ \wedge\!\!\wedge & \text{if} \quad X \text{ is a non-empty subset of } \wedge\!\!\wedge^+, \\ \omega & \text{if} \quad X = \omega, \\ \infty & \text{if} \quad X = \emptyset. \end{cases}$$

For every $X, Y \in \mathbb{D}$, we have $\Psi(XY) = \Psi(X)\cdot\Psi(Y)$, and $\Psi(X\cup Y) = \min\{\Psi(X),\ \Psi(Y)\}$. Consequently, Ψ extends to a homomorphism $\Psi\colon \mathbb{D}_{n\times n} \to \mathcal{D}_{n\times n}$.

Consider $X = \{\wedge\!\!\wedge\wedge\!\!\wedge,\ \wedge\!\!\wedge\curlyvee\wedge\!\!\wedge\wedge\!\!\wedge\wedge\!\!\wedge\} \in \mathcal{P}_{\mathrm{f}}(D^+)$. We have $\Delta(X) = 2$ and $\Psi(X) = \curlyvee$. However, there is no word $\pi \in X$ with $\Delta(\pi) = 2$ and $\Psi(\pi) = \curlyvee$. This inconvenient effect arises due to the fact that the words in X have different lengths. If $X \neq \omega$, $X \neq \emptyset$, and all words in X have the same length, then there is some $\pi \in X$ with $\Delta(\pi) = \Delta(X)$ and $\Psi(\pi) = \Psi(X)$. In the rest of the paper we will have this convenient property, because we make a careful restriction in the definition of $\mathbb{D}_{n\times n}$.

Example 4.1 (*continued*). By an abuse of notation, we set $a := \Psi(A)$ and $b := \Psi(B)$, i.e.,

$$a = \begin{pmatrix} \curlyvee & \wedge\!\!\wedge & \infty \\ \infty & \wedge\!\!\wedge & \wedge\!\!\wedge \\ \infty & \infty & \curlyvee \end{pmatrix}, \qquad b = \begin{pmatrix} \wedge\!\!\wedge & \wedge\!\!\wedge & \infty \\ \infty & \curlyvee & \wedge\!\!\wedge \\ \infty & \infty & \wedge\!\!\wedge \end{pmatrix}, \quad \text{and} \quad ab = \begin{pmatrix} \curlyvee & \curlyvee & \wedge\!\!\wedge \\ \infty & \curlyvee & \wedge\!\!\wedge \\ \infty & \infty & \curlyvee \end{pmatrix}.$$

Moreover, $a^2[1,3] = \wedge\!\!\wedge$, but for every $k \ge 3$, $a^k[1,3] = \curlyvee$.

4.4. *Strange Limits*

We define the notion of a Ψ-*limit*. It is not a classic limit notion, because it is not based on a metric, and, curiously, the limit of a sequence does not belong to the same algebraic structure as the members of the sequence. A Ψ-limit of some sequence over $\mathbb{D}$ describes in terms of $\mathcal{D}$ how the sequence is bounded. We extend this notion in a natural way to word matrices, and we prove some results which enable us to use Ψ-limits in the same way as a traditional limit concept.

Recall that some sequence $(q_k)_{k\ge1}$ is a *subsequence of* $(p_k)_{k\ge1}$ if there is a strictly increasing mapping $f\colon \mathbb{N} \to \mathbb{N}$ such that $q_k = p_{f(k)}$ for every $k \ge 1$.

A sequence $(x_k)_{k\ge1} \in (\mathbb{N}\cup\{\infty\})$ is said to be *bounded* if there are $l, K \ge 1$ such that $x_k \le K$ for every $k \ge l$. It *tends to infinity* if for every $K \ge 1$ there is some $l \ge 1$ such that for every $k \ge l$ we have $x_k \ge K$.

Let $(X_k)_{k\ge1} \in \mathbb{D}$ be a sequence. We define the Ψ-limit of $(X_k)_{k\ge1}$.

L1. If there is an $l \ge 1$ such that $X_k = \emptyset$ for every $k \ge l$, then $\overline{\Psi}(X_k)_{k\ge1} := \infty$. In this case we call $(X_k)_{k\ge1}$ an ∞-*sequence*.

For every $z \in \{\curlyvee, \curlywedge\}$, $X \in \mathbb{D}$, we define

$$\Delta(X, z) := \min\{\Delta(\pi) \mid \pi \in X,\ \Psi(\pi) = z\},$$

under the convention $\min \emptyset = \infty$. We denote the sequence $(\Delta(X_k, z))_{k\geq 1}$ by $\Delta(X_k, z)_{k\geq 1}$.

Assume that there is some $l \geq 1$ such that $X_k \neq \emptyset$ for every $k \geq l$. In this case we cannot apply (L1) to $(X_k)_{k\geq 1}$, and we define:

L2. If $\Delta(X_k, \curlyvee)_{k\geq 1}$ is bounded, then we define $\overline{\Psi}(X_k)_{k\geq 1} := \curlyvee$.
L3. If $\Delta(X_k, \curlyvee)_{k\geq 1}$ tends to infinity and $\Delta(X_k, \curlywedge)_{k\geq 1}$ is bounded, then we define $\overline{\Psi}(X_k)_{k\geq 1} := \curlywedge$.
L4. If $\Delta(X_k, \curlyvee)_{k\geq 1}$ and $\Delta(X_k, \curlywedge)_{k\geq 1}$ tend to infinity, then $\overline{\Psi}(X_k)_{k\geq 1} := \omega$.

If we can apply one of these four definitions to a sequence $(X_k)_{k\geq 1}$, then we call $(X_k)_{k\geq 1}$ a *convergent sequence*. We denote the set of all convergent sequences by $\mathfrak{C}(\mathbb{D})$. For any other sequence, $\overline{\Psi}(X_k)_{k\geq 1}$ is not defined. It is easy to see that every sequence contains a convergent subsequence. If $X_1 = X_2 = \cdots$ for some sequence $(X_k)_{k\geq 1} \in \mathbb{D}$, then $(X_k)_{k\geq 1} \in \mathfrak{C}(\mathbb{D})$ and $\overline{\Psi}(X_k)_{k\geq 1} = \Psi(X_k)$.

Example 4.1 (*continued*). Let $k \geq 1$. We have $A^k[3,1] = \emptyset$ and by (L1), $\overline{\Psi}(A^k[3,1])_{k\geq 1} = \infty$.

For $k \geq 3$, we have $\Delta(A^k[1,3], \curlyvee) = 2$. Hence, $\Delta(A^k[1,3], \curlyvee)_{k\geq 1}$ is bounded and by (L1) $\overline{\Psi}(A^k[1,3])_{k\geq 1} = \curlyvee$.

For $k \geq 1$, we have $A^k[2,2] = \{\curlywedge^k\}$, i.e., $\Delta(A^k[2,2], \curlyvee) = \infty$ and $\Delta(A^k[2,2], \curlywedge) = k$. Both $\Delta(A^k[2,2], \curlyvee)_{k\geq 1}$ and $\Delta(A^k[2,2], \curlywedge)_{k\geq 1}$ tend to infinity, and, thus, $\overline{\Psi}(A^k[2,2])_{k\geq 1} = \omega$ by (L4).

For $k \geq 1$, let $X_k := \{\curlyvee\curlywedge^k, \curlywedge\curlywedge\}$. We have $\Delta(X_k, \curlyvee) = k$ and $\Delta(X_k, \curlywedge) = 2$, i.e., $\Delta(X_k, \curlyvee)_{k\geq 1}$ tends to infinity but $\Delta(X_k, \curlywedge)_{k\geq 1}$ is bounded. Thus, $\overline{\Psi}(X_k)_{k\geq 1} = \curlywedge$.

Finally, for $k \geq 1$, let

$$Y_k := \begin{cases} \{\curlyvee\curlywedge^k, \curlywedge\} & \text{if } k \text{ is odd,} \\ \{\curlyvee\curlywedge, \curlywedge\} & \text{if } k \text{ is even.} \end{cases}$$

The sequence $\Delta(Y_k, \curlyvee)_{k\geq 1}$ is neither bounded nor does it tend to infinity. Hence, $(Y_k)_{k\geq 1}$ is not convergent.

Lemma 4.1.

(1) *Every subsequence of a convergent sequence over $\mathbb{D}$ is convergent and converges to the same Ψ-limit.*
(2) *The set of convergent sequences over $\mathbb{D}$ is closed under componentwise $\cup$ and $\cdot$, and the Ψ-limit $\overline{\Psi} : (\mathfrak{C}(\mathbb{D}), \cup, \cdot) \to (\mathcal{D}, \min, \cdot)$ is a homomorphism.*

Proof. (1) If $(X_k)_{k\geq 1}$ is an ∞-sequence, then $(X_{f(k)})_{k\geq 1}$ is also an ∞-sequence, i.e., the claim comes up to $\infty = \infty$. We assume that $(X_k)_{k\geq 1}$ is not an ∞-sequence for the rest of the proof of (1). Because $(X_k)_{k\geq 1}$ is convergent, there is some l such that for every $k \geq l$, $X_k \neq \emptyset$. Clearly, we have for every $k \geq l$, $X_{f(k)} \neq l$. Let $z \in \{\curlyvee, \curlywedge\}$. If

$\Delta(X_k, z)_{k\geq 1}$ is bounded (resp. tends to infinity), then $\Delta(X_{f(k)}, z)_{k\geq 1}$ is bounded (resp. tends to infinity). The claim follows from the definition of $\overline{\Psi}$.

For the proof of (2), let $(X_k)_{k\geq 1}$, $(Y_k)_{k\geq 1} \in \mathfrak{C}(\mathbb{D})$ and $z' = \overline{\Psi}(X_k)_{k\geq 1}$ and $z'' = \overline{\Psi}(Y_k)_{k\geq 1}$.

(2a) We show that $(X_kY_k)_{k\geq 1}$ is convergent and $\overline{\Psi}(X_kY_k)_{k\geq 1} = z'z''$. If $z' = \infty$ or $z'' = \infty$, then we easily see that (2a) comes up to $\infty = \infty$. We assume $z' \neq \infty$ and $z'' \neq \infty$ in the rest of the proof of (2a).

We show that for every $z < z'z''$, the sequence $\Delta(X_kY_k, z)_{k\geq 1}$ tends to infinity. By contradiction, let $z \in \{\curlyvee, \curlywedge\!\!\curlywedge\}$ satisfying $z < z'z''$, and assume that $\Delta(X_kY_k, z)_{k\geq 1}$ does not tend to infinity. Hence, there is a $K \in \mathbb{N}$ and there is an infinite set $L \subseteq \mathbb{N}_1$ such that for every $l \in L$ there is some $\pi_l \in X_lY_l$ with $\Delta(\pi) \leq K$ and $\Psi(\pi) = z$. For every $l \in L$, let $\pi_l' \in X_l$ and $\pi_l'' \in Y_l$ such that $\pi_l = \pi_l'\pi_l''$.

We can factorize z into $z = y'y''$ such that we have $\Psi(\pi_l') = y'$ and $\Psi(\pi_l'') = y''$ for infinitely many $l \in L$. By $\Delta(\pi_l') \leq K$, the sequence $\Delta(X_k, y')_{k\geq 1}$ does not tend to infinity. Thus, $y' \geq \overline{\Psi}(X_k)_{k\geq 1} = z'$, and similarly, $y'' \geq z''$, and thus $z = y'y'' \geq z'z''$. This is a contradiction.

If $z' \neq \omega$ and $z'' \neq \omega$, then $\Delta(X_k, z')_{k\geq 1}$ and $\Delta(Y_k, z'')_{k\geq 1}$ are bounded, and, consequently, $\Delta(X_kY_k, z'z'')_{k\geq 1}$ is bounded which completes the proof of (2a).

(2b) We show that $(X_kY_k)_{k\geq 1}$ is convergent and $\overline{\Psi}(X_k \cup Y_k)_{k\geq 1} = \min(z', z'')$.

If $z' = z'' = \infty$, then the claim is obvious. We assume $\min(z', z'') \neq \infty$ in the rest of the proof. Let $z \in \{\curlyvee, \curlywedge\!\!\curlywedge\}$ be arbitrary. The following two assertions are immediate:

1. If one of the sequences $\Delta(X_k, z)_{k\geq 1}$ or $\Delta(Y_k, z)_{k\geq 1}$ is bounded, then $\Delta(X_k \cup Y_k,\ z)_{k\geq 1}$ is bounded.
2. If both the sequences $\Delta(X_k, z)_{k\geq 1}$ or $\Delta(Y_k, z)_{k\geq 1}$ tend to infinity, then $\Delta(X_k \cup Y_k,\ z)_{k\geq 1}$ tends to infinity.

The claim follows from these two assertions. ◻

Note that by (2) of Lemma 4.1 for every sequence $(X_k)_{k\geq 1} \in \mathfrak{C}(\mathbb{D})$ and $Y \in \mathbb{D}$ the sequence $(X_kY)_{k\geq 1}$ is convergent and $\overline{\Psi}(X_kY)_{k\geq 1} = (\overline{\Psi}(X_k)_{k\geq 1})\Psi(Y)$. This holds similarly for $\cup$ and min.

We extend the notion of a Ψ-limit to $\mathbb{D}_{n\times n}$. A sequence $(A_k)_{k\geq 1} \in \mathbb{D}_{n\times n}$ is *convergent* if for every i, j, the sequence $(A_k[i,j])_{k\geq 1} \in \mathbb{D}$ is convergent. In this case, we define $(\overline{\Psi}(A_k)_{k\geq 1})[i,j] := \overline{\Psi}((A_k[i,j])_{k\geq 1})$. We denote the set of all convergent sequences over $\mathbb{D}_{n\times n}$ by $\mathfrak{C}(\mathbb{D}_{n\times n})$.

Example 4.1 (*continued*). We can show

$$\overline{\Psi}(A^k)_{k\geq 1} = \begin{pmatrix} \curlyvee & \curlyvee & \curlyvee \\ \infty & \omega & \curlyvee \\ \infty & \infty & \curlyvee \end{pmatrix} \quad \text{and} \quad \overline{\Psi}(B^k)_{k\geq 1} = \begin{pmatrix} \omega & \curlyvee & \curlyvee \\ \infty & \curlyvee & \curlyvee \\ \infty & \infty & \omega \end{pmatrix}.$$

Lemma 4.2.

(1) *Every subsequence of a convergent sequence over $\mathbb{D}_{n\times n}$ is convergent and converges to the same Ψ-limit.*

(2) *The set of convergent sequences over* $\mathbb{D}_{n\times n}$ *is closed under componentwise* $\cdot$, *and the* Ψ*-limit* $\overline{\Psi} : (\mathfrak{C}(\mathbb{D}), \cdot) \to (\mathcal{D}, \cdot)$ *is a homomorphism.*

Proof. The proof is obvious by Lemma 4.1. □

As above, for every sequence $(A_k)_{k\geq 1} \in \mathfrak{C}(\mathbb{D}_{n\times n})$ and $B \in \mathbb{D}_{n\times n}$ the sequence $(A_k B)_{k\geq 1}$ is convergent and $\overline{\Psi}(A_k B)_{k\geq 1} = (\overline{\Psi}(A_k)_{k\geq 1})\Psi(B)$.

By Lemma 4.2, it is immediate that $\mathfrak{C}(\mathbb{D}_{n\times n})$ is a semigroup. By regarding every word matrix as constant sequence, $\mathbb{D}_{n\times n}$ is a subsemigroup of $\mathfrak{C}(\mathbb{D}_{n\times n})$. By Lemma 4.2(2), $\overline{\Psi} : \mathfrak{C}(\mathbb{D}_{n\times n}) \to \mathcal{D}_{n\times n}$ is a homomorphism.

For every subset $T \in \mathbb{D}_{n\times n}$ we denote by $\overline{\Psi}\langle T\rangle$ the set of all Ψ-limits of all convergent sequences over $\langle T\rangle$. We have $\Psi(\langle T\rangle) \subseteq \overline{\Psi}\langle T\rangle$.

4.5. *The Limitedness Problem of Desert Automata as a Burnside Problem*

We formulate the limitedness problem of desert automata by using the notions of word matrices and Ψ-limits.

Proposition 4.3. *Let* $\mathcal{A} = [Q, E, I, F, E^{\curlyvee}]$ *be a desert automaton and let* $T := \theta(\Sigma)$. *The following assertions are equivalent*:

(1) $\mathcal{A}$ *is not limited.*
(2) *There is a matrix* $a \in \overline{\Psi}\langle T\rangle$ *such that* $\min\{a[i, j] \mid i \in I,\ j \in F\} = \omega$.

Proof. (2) $\Rightarrow$ (1) Let $(A_k)_{k\geq 1} \in \mathfrak{C}\langle T\rangle$ with $\overline{\Psi}(A_k)_{k\geq 1} = a$. Let $(w_k)_{k\geq 1} \in \Sigma^+$ such that for every $k \geq 1$ we have $\theta(w_k) = A_k$. By (2), there are $i \in I$, $j \in F$ such that $a[i, j] = \omega$, i.e., $(A_k[i, j])_{k\geq 1}$ is not an ∞-sequence. Hence, there is some $l \geq 1$ such that $w_k \in L(\mathcal{A})$ for every $k \geq l$. For every $z \in \{\curlyvee, \curlywedge\!\!\curlywedge\}$, $i \in I$, $j \in F$, the sequence $\Delta(A_k[i, j], x)_{k\geq 1}$ tends to infinity. Hence, $\Delta(w_k)_{k\geq 1}$ tends to infinity, i.e., $\mathcal{A}$ is not limited.

(1) $\Rightarrow$ (2) Let $(w_k)_{k\geq 1} \in L(\mathcal{A})$ such that $\Delta(w_1) < \Delta(w_2) < \cdots$. Let $A_k := \theta(w_k) \in \langle T\rangle$. Let $(B_k)_{k\geq 1}$ be some arbitrary convergent subsequence of $(A_k)_{k\geq 1}$ and let $a := \overline{\Psi}(B_k)_{k\geq 1}$.

If for every $i \in I$, $j \in F$, we have $a[i, j] = \infty$, then $(w_k)_{k\geq 1}$ contains words which do not belong to $L(\mathcal{A})$. If for some $i \in I$, $j \in F$, we have $a[i, j] < \omega$, then we can contradict $\Delta(w_1) < \Delta(w_2) < \cdots$. Thus, a proves (2). □

Example 4.1 (*continued*). We examine the sequences $(a^k b^k a^k)_{k\geq 1}$ and $(b^k a^k b^k)_{k\geq 1}$. We leave it to the reader to show for $k \geq 1$, $\Delta_1(a^k b^k a^k) = 1$ and $\Delta_1(b^k a^k b^k) = k$. We consider (1) $\Rightarrow$ (2) in the proof of Proposition 4.3. By Lemma 4.2(2), we obtain

$$\overline{\Psi}(A^k B^k A^k)_{k\geq 1} = \overline{\Psi}(A^k)_{k\geq 1}\overline{\Psi}(B^k)_{k\geq 1}\overline{\Psi}(A^k)_{k\geq 1} = \begin{pmatrix} \omega & \omega & \curlyvee \\ \infty & \omega & \omega \\ \infty & \infty & \omega \end{pmatrix} =: c_1$$

and $\min\{c_1[i,j] \mid i = 1,\ j \in \{1,2,3\}\} = \curlyvee$. It is not surprising that we obtained $\curlyvee$ instead of ω as in Proposition 4.3(2), because the weight of the words $(a^k b^k a^k)_{k \geq 1}$ does not increase. We leave it to the reader to verify $\min\{c_2[i,j] \mid i = 1,\ j \in \{1,2,3\}\} = \omega$ for $c_2 := \overline{\Psi}(B^k A^k B^k)_{k \geq 1}$.

Clearly, Proposition 4.3 is rather another formulation of the limitedness problem of desert automata than a solution. To solve the limitedness problem by checking condition (2) in Proposition 4.3, we have to examine the set $\overline{\Psi}\langle T \rangle$. Unfortunately, $\langle T \rangle$ is infinite, and $\mathfrak{C}\langle T \rangle$ is even uncountable for almost every finite set $T \subseteq \mathbb{D}_{n \times n}$ up to a few particular cases. However, $\overline{\Psi}\langle T \rangle$ is finite. To give an algorithm for the limitedness problem of desert automata, we need some method to compute $\overline{\Psi}\langle T \rangle$ by avoiding computing the possibly uncountable set $\mathfrak{C}\langle T \rangle$.

The notion *Burnside problem* originates from Burnside who raised the following question in 1902 [4]: Let G be a finitely generated group and $k \geq 1$ satisfying $x^k = 1$ for every $x \in G$. Is G necessarily finite? It took more than 60 years until Novikov and Adjan showed that G is not necessarily finite [34].

Today, one considers several related problems, e.g., the finite section problem, as Burnside (type) problems (see [36]). The *finite section problem* is the question whether for some $n \geq 1$, some finite set T of $n \times n$ matrices over some semiring $\mathbb{K}$, and some i, j, the set $\{A[i,j] \mid A \in \langle T \rangle\}$ is finite.

Although we are rather interested in whether entries in $\langle T \rangle$ are bounded than the finiteness of $\langle T \rangle$, we can regard the question to decide (2) in Proposition 4.3 as a Burnside problem.

5. The Solution of the Burnside Problem

In this section we solve the Burnside problem for word matrices by showing a method to compute the set $\overline{\Psi}\langle T \rangle$. Our main strategy follows Leung's approach [30] to a similar problem for the tropical semiring. However, there are great differences in the proof details, because we consider a more involved semiring and another notion of stabilization. In Section 5.5 we no longer follow [30] to present the main proof in a more precise fashion.

5.1. *Stabilization*

We define a mapping $\sharp\colon \mathsf{E}(\mathcal{D}_{n \times n}) \to \mathcal{D}_{n \times n}$ which we call *stabilization*. For every $e \in \mathsf{E}(\mathcal{D}_{n \times n})$ and i, j let

$$e^\sharp[i,j] = \begin{cases} \infty & \text{if} \quad e[i,j] = \infty, \\ \curlyvee & \text{if there is some } l \text{ such that} \quad e[i,l] = e[l,l] = e[l,j] = \curlyvee, \\ \omega & \text{otherwise.} \end{cases}$$

We show a small remark and an easy lemma to get familiar with stabilization.

Remark 5.1. Let $e \in \mathsf{E}(\mathcal{D}_{n \times n})$ and let i, l be such that we have $e[i,l] = \wedge\!\!\!\wedge$ and $e[l,l] = \curlyvee$. Then $e^2[i,l] = \curlyvee \neq e[i,l]$. Thus, such i and l cannot exist, and, similarly, it is impossible that for some l, j, we have $e[l,l] = \curlyvee$ and $e[l,j] = \wedge\!\!\!\wedge$.

If for some i, j $e[i, j]^\sharp = \curlyvee$, then $e[i, j] = e^3[i, j] = \curlyvee$.

Lemma 5.1. *Let $e \in \mathsf{E}(\mathcal{D}_{n\times n})$. If $e[i, j] \neq \curlywedge\!\!\curlywedge$ for every i, j, then $e = e^\sharp$.*

Proof. Let i, j be arbitrary. If $e[i, j] \in \{\omega, \infty\}$, then $e[i, j] = e^\sharp[i, j]$. The case $e[i, j] = \curlywedge\!\!\curlywedge$ is not possible. Finally, assume $e[i, j] = \curlyvee$. Because $e^{n+2} = e$, there are $i = i_0, \ldots, i_{n+2} = j$, such that for every $l \in \{1, \ldots, n+2\}$ we have $e[i_{l-1}, i_l] \in \{\curlyvee, \curlywedge\!\!\curlywedge\}$, and by the assumption $e[i_{l-1}, i_l] = \curlyvee$. By counting arguments, there are $p < q \in \{1, \ldots, n+1\}$ satisfying $i_p = i_q$. Then $e[i, i_p] = e[i_p, i_p] = e[i_p, j]$ and $e^\sharp[i, j] = \curlyvee = e[i, j]$. $\square$

We state the main result of Section 5. For subsets $T \subseteq \mathcal{D}_{n\times n}$ we define $\langle T\rangle^\sharp$ as the least subset of $\mathcal{D}_{n\times n}$ which contains T and is closed both under matrix multiplication and stabilization $\sharp$ of idempotent matrices. It is easy to see that $\langle T\rangle^\sharp$ can be effectively computed.

Theorem 5.2. *Let $T \subseteq \mathbb{D}_{n\times n}$ be finite. We have $\overline{\Psi}\langle T\rangle = \langle\Psi(T)\rangle^\sharp$.*

Until we prove Theorem 5.2 in Section 5.5, we need to achieve a deeper understanding of stabilization and relations between stabilization and Ψ-limits.

5.2. *Stabilization Is a Consistent Mapping*

We establish a first connection between stabilization and Ψ-limits of sequences.

Example 4.1 (*continued*). We have

$$\Psi(B^3) = b^3 = \begin{pmatrix} \curlywedge\!\!\curlywedge & \curlyvee & \curlyvee \\ \infty & \curlyvee & \curlyvee \\ \infty & \infty & \curlywedge\!\!\curlywedge \end{pmatrix} \in \mathsf{E}(\mathbb{D}_{3\times 3})$$

and

$$(b^3)^\sharp = \begin{pmatrix} \omega & \curlyvee & \curlyvee \\ \infty & \curlyvee & \curlyvee \\ \infty & \infty & \omega \end{pmatrix} \in \mathsf{E}(\mathbb{D}_{3\times 3}).$$

Surprisingly, we have $\overline{\Psi}(B^{3k})_{k\geq 1} = \overline{\Psi}(B^k)_{k\geq 1} = (b^3)^\sharp$. We study the entry $[1, 3]$ in $(b^3)^\sharp$ and $\overline{\Psi}(B^{3k})_{k\geq 1}$. We have $(b^3)^\sharp[1, 3] = \curlyvee$ because $b^3[1, 2] = b^3[2, 2] = b^3[1, 3] = \curlyvee$, i.e., we use $l = 2$ in the definition of stabilization. For $k \geq 3$, we have

$$\pi_k := (\curlywedge\!\!\curlywedge\ \curlyvee\ \curlyvee)\ \curlyvee^{3(k-2)}\ (\curlyvee\ \curlyvee\ \curlywedge\!\!\curlywedge) \in B^3[1, 2](B^3[2, 2])^{k-2}B^3[2, 3] \subseteq (B^{3k})[1, 3].$$

We have $\Delta(\pi_k) = 1$ and $\Psi(\pi_k) = \curlyvee$, i.e., by the words π_k, the Ψ-limit of $(B^{3k}[1, 3])_{k\geq 1}$ is $\curlyvee$. For our construction of π_k, we used the entry $B^3[1, 3]$, several times we use

$B^3[2,2]$, and we also use $B^3[2,3]$. Above, we used the same entries in b^3 to show that $(b^3)^\sharp[1,3] = \curlyvee$.

We leave it to the reader to verify $\overline{\Psi}(A^{3k})_{k\geq 1} = (a^3)^\sharp$.

We have $T = \{A, B\}$, $c_1 = \overline{\Psi}(A^k B^k A^k)_{k\geq 1} \in \overline{\Psi}\langle T\rangle$, and $c_2 = \overline{\Psi}(B^k A^k B^k)_{k\geq 1} \in \overline{\Psi}\langle T\rangle$. Theorem 5.2 claims that $c_1, c_2 \in \langle \Psi(T)\rangle^\sharp$. Indeed, we have

$$\begin{aligned} c_1 &= \overline{\Psi}(A^k B^k A^k)_{k\geq 1} = \overline{\Psi}(A^k)_{k\geq 1}\overline{\Psi}(B^k)_{k\geq 1}\overline{\Psi}(A^k)_{k\geq 1} \\ &= (a^3)^\sharp (b^3)^\sharp (a^3)^\sharp \in \langle \Psi(T)\rangle^\sharp, \end{aligned}$$

and in the same way $c_2 \in \langle \Psi(T)\rangle^\sharp$.

Proposition 5.3. *Let $E \in \mathbb{D}_{n\times n}$ such that $\Psi(E) \in \mathsf{E}(\mathcal{D}_{n\times n})$. The sequence $(E^k)_{k\geq 1}$ is convergent and $\overline{\Psi}(E^k)_{k\geq 1} = \Psi(E)^\sharp$.*

Proof. Let $e = \Psi(E)$. Let i, j be arbitrary.

If $e^\sharp[i,j] = \infty$, then for every $k \geq 1$, $e^k[i,j] = \infty$ and $E^k[i,j] = \emptyset$, and thus $\overline{\Psi}(E^k)_{k\geq 1} = \infty$. In the rest of the proof we assume $e^\sharp[i,j] \neq \infty$, i.e., $E^k[i,j] \neq \emptyset$ for every $k \geq 1$.

First, we show that if $e^\sharp[i,j] = \curlyvee$ then $\Delta(E^k[i,j], \curlyvee)_{k\geq 1}$ is bounded, i.e., we have $\overline{\Psi}(E^k[i,j])_{k\geq 1} = \curlyvee$. Since $e^\sharp[i,j] = \curlyvee$, there is some l with $e[i,l] = e[l,l] = e[l,j] = \curlyvee$. There are $\pi_1 \in E[i,l]$, $\pi_2 \in E[l,l]$, $\pi_3 \in E[l,j]$ such that $\Psi(\pi_1) = \Psi(\pi_2) = \Psi(\pi_3) = \curlyvee$. For every $k \geq 2$, we have $\pi_1(\pi_2)^{k-2}\pi_3 \in E^k[i,j]$ and $\Delta(\pi_1(\pi_2)^{k-2}\pi_3) < |\pi_1\pi_2\pi_3|$. Thus, $\Delta(E^k[i,j], \curlyvee)_{k\geq 1}$ is bounded.

Finally, we deal with the case $e^\sharp[i,j] = \omega$. We assume by contradiction that the sequence $\Delta(E^k[i,j], \curlyvee)_{k\geq 1}$ does not tend to infinity. There is some $K \in \mathbb{N}$ such that we have $\Delta(E^k[i,j], \curlyvee) = K$ for infinitely many k. Choose some $k \geq (K+1)n + 1$ with $\Delta(E^k[i,j], \curlyvee) = K$. Let $\pi \in E^k[i,j]$ with $\Delta(\pi) = K$ and $\Psi(\pi) = \curlyvee$. There are $i = i_0, \ldots, i_k = j$ and for every $1 \leq l \leq k$, some $\pi_l \in E[i_{l-1}, E_l]$ such that $\pi = \pi_1 \cdots \pi_k$. By a counting argument, there are $p < q \in \{1,\ K{+}2,\ 2K{+}3,\ \cdots,\ n(K{+}1){+}1\}$ such that $i_p = i_q$. We have $|\pi_{p+1}\cdots\pi_q| > K$. If $\Psi(\pi_{p+1}\cdots\pi_q) = \curlywedge\!\!\curlywedge$, then $\Delta(\pi) \geq \Delta(\pi_{p+1}\cdots\pi_q) > K$, which is a contradiction. Hence, $\Psi(\pi_{p+1}\cdots\pi_q)\curlyvee$, i.e., $\curlyvee = e^{q-p}[i_p, i_q] = e[i_p, i_q]$. Similarly, we obtain $e[i_0, i_p], e[i_q, i_k] \in \{\curlyvee, \curlywedge\!\!\curlywedge\}$. By setting $l := i_p = i_q$ and Remark 5.1, we obtain $e[i,l] = e[l,l] = e[l,j] = \curlyvee$, i.e., $e^\sharp[i,j] = \curlyvee$, which is a contradiction.

For every $k \geq 1$, we have $\Delta(E^k[i,j], \curlywedge\!\!\curlywedge) \geq k$, because every word in E^k is at least of length k. Thus, $\Delta(E^k[i,j], \curlywedge\!\!\curlywedge)_{k\geq 1}$ tends to infinity. To sum up, if $e^\sharp[i,j] = \omega$, then both $\Delta(E^k[i,j], \curlyvee)_{k\geq 1}$ and $\Delta(E^k[i,j], \curlywedge\!\!\curlywedge)_{k\geq 1}$ tend to infinity, and thus $\overline{\Psi}(E^k[i,j])_{k\geq 1} = \omega$. $\square$

Lemma 5.4. *Let $T \subseteq \mathbb{D}_{n\times n}$ be finite. For every idempotent $e \in \overline{\Psi}\langle T\rangle$, we have $e^\sharp \in \overline{\Psi}\langle T\rangle$.*

Proof. We show a sequence $(B_k)_{k\geq 1} \in \mathfrak{C}\langle T\rangle$ with $\overline{\Psi}(B_k)_{k\geq 1} = e^\sharp$. There is some sequence $(w_k)_{k\geq 1} \in T^+$ such that $e = \overline{\Psi}(\alpha(w_k))_{k\geq 1}$. By subsequence selection, there is some sequence $(w'_k)_{k\geq 1} \in T^+$ with $\overline{\Psi}(\alpha(w'_k))_{k\geq 1} = e$ such that either $(w'_k)_{k\geq 1}$ is a constant sequence or we have $|w'_k| < |w'_{k+1}|$ for every $k \geq 1$.

If $(w'_k)_{k\geq 1}$ is a constant sequence, then we have $\Psi(\alpha(w'_1)) = e$. For every $k \geq 1$, we set $B_k := \alpha(w'_1)^k$, and by Proposition 5.3, we have $\overline{\Psi}(B_k)_{k\geq 1} = e^\sharp$.

If $(w'_k)_{k\geq 1}$ is length increasing, then we have $e[i, j] \neq \curlywedge$ for every i, j. By Lemma 5.1, we have $e^\sharp = e$, and we can set $(B_k)_{k\geq 1} = (\alpha(w_k))_{k\geq 1}$. □

Proposition 5.3 covers only a very particular case of convergent sequences over $\mathbb{D}_{n\times n}$. In order to prove Theorem 5.2, we need to establish deeper connections between the Ψ-limits of sequences and stabilization. Our next step is to generalize stabilization from $\mathsf{E}(\mathcal{D}_{n\times n})$ to $\mathsf{Reg}(\mathcal{D}_{n\times n})$:

Lemma 5.5. *Stabilization $\sharp$ is a consistent mapping.*

Proof. We show that for every $e \in \mathsf{E}(\mathcal{D}_{n\times n})$, $e^\sharp$ is idempotent, and we show (1) in Lemma 3.6.

Let $e \in \mathsf{E}(\mathcal{D}_{n\times n})$. We show $e^\sharp \in \mathsf{E}(\mathcal{D}_{n\times n})$. Choose some $E \in \mathbb{D}_{n\times n}$ with $\Psi(E) = e$. By Proposition 5.3 and Lemma 4.2, we have

$$e^\sharp e^\sharp = \overline{\Psi}(E^k)_{k\geq 1}\overline{\Psi}(E^k)_{k\geq 1} = \overline{\Psi}(E^k E^k)_{k\geq 1} = \overline{\Psi}(E^k)_{k\geq 1} = e^\sharp.$$

Let $a, b \in \mathcal{D}_{n\times n}$ such that $ab, ba \in \mathsf{E}(\mathcal{D}_{n\times n})$. Let $A, B \in \mathbb{D}_{n\times n}$ such that $a = \Psi(A)$ and $b = \Psi(B)$. Again by Proposition 5.3 and Lemma 4.2, we have

$$(ab)^\sharp = \overline{\Psi}((AB)^k)_{k\geq 1} = \Psi(A)\,\overline{\Psi}((BA)^k)_{k\geq 1}\,\Psi(B) = a(ba)^\sharp b. \qquad \square$$

Due to Lemma 5.5, we have by Corollary 3.8 a natural extension of stabilization to regular elements of $\mathcal{D}_{n\times n}$, and we can use Lemma 3.9 as a very convenient tool whenever we prove some assertion concerning stabilization.

Let $a \in \mathsf{Reg}(\langle T\rangle^\sharp)$. There is some $e \in \mathsf{E}(\langle T\rangle^\sharp)$ with $e =_{\mathscr{L}} a$. Then $ae = a$ and $a^\sharp = ae^\sharp$, and thus $a^\sharp \in (\langle T\rangle^\sharp)$, or, more precisely, $a^\sharp \in \mathsf{Reg}(\langle T\rangle^\sharp)$. Consequently, $\langle T\rangle^\sharp$ is closed under stabilization of matrices in $\mathsf{Reg}(\langle T\rangle^\sharp)$.

For $b \in \mathsf{Reg}(\mathcal{D}_{n\times n})$, it is possible that $b \notin \mathsf{Reg}(\langle T\rangle^\sharp)$ and $b^\sharp \notin \langle T\rangle^\sharp$.

At this point, we have to be very careful. In the definition of $\langle T\rangle^\sharp$ we demand closure under stabilization of idempotents. *After* the definition of $\langle T\rangle^\sharp$, we prove closure under stabilization of matrices which are regular in $\langle T\rangle^\sharp$.

If one defines $\langle T\rangle^\sharp$ in a way that $\langle T\rangle^\sharp$ has to be closed under stabilization of matrices which are regular in $\langle T\rangle^\sharp$, then the definition becomes a mess, because the term "regular matrix in $\langle M\rangle^\sharp$" does not have a meaning unless $\langle T\rangle^\sharp$ is defined.

5.3. *Stabilization of Regular Matrices*

We show two useful lemmas about stabilization of regular matrices in $\mathcal{D}_{n\times n}$.

Lemma 5.6. *Let $a \in \mathsf{Reg}(\mathcal{D}_{n\times n})$ and let i, j be arbitrary*:

(1) *If $a[i, j] \in \{\omega, \infty\}$, then $a^\sharp[i, j] = a[i, j]$.*
(2) *If $a[i, j] = \curlyvee$, then $a^\sharp[i, j] \in \{\curlyvee, \omega\}$.*
(3) *If $a[i, j] = \curlywedge$, then $a^\sharp[i, j] = \omega$.*

In particular, we have $a^\sharp[i,j] \neq \wedge\!\!\wedge$ *regardless of* $a[i,j]$.

Proof. Let $e \in \mathsf{E}(\mathcal{D}_{n\times n})$ with $e =_{\mathscr{L}} a$, i.e., $a = ae$, $a^\sharp = ae^\sharp$, and in particular

$$a[i,j] = \min\{a[i,l] \cdot e[l,j] \mid l\} \quad \text{and} \quad a^\sharp[i,j] = \min\{a[i,l] \cdot e^\sharp[l,j] \mid l\}.$$

First, note that by the definition of stabilization every entry of $e^\sharp$ is different from $\wedge\!\!\wedge$. Consequently, we have $a^\sharp[i,j] \neq \wedge\!\!\wedge$ regardless of $a[i,j]$.

(1) is immediate. (2) is also immediate, because if $a[i,j] \neq \infty$, then $a^\sharp[i,j] \neq \infty$, and as seen above $a^\sharp[i,j] \neq \wedge\!\!\wedge$.

We show (3). Clearly, $a^\sharp[i,j] \notin \{\wedge\!\!\wedge, \infty\}$. Assume $a^\sharp[i,j] = \Upsilon$. There is some l such that $a[i,l]$, $e^\sharp[l,j] = \Upsilon$, and hence, $a[i,l]$, $e^\sharp[l,j] \in \{\Upsilon, \wedge\!\!\wedge\}$. Because $e^\sharp[l,j] \neq \wedge\!\!\wedge$, we have $e^\sharp[l,j] = \Upsilon$. Thus, $e[l,j] = \Upsilon$. Together with $a[i,l] \in \{\Upsilon, \wedge\!\!\wedge\}$, we obtain $a[i,j] = ae[i,j] = \Upsilon$, which is a contradiction. To sum up, $a^\sharp[i,j] \notin \{\Upsilon, \wedge\!\!\wedge, \infty\}$. □

Lemma 5.7. *Let* $m > 0$, *and let* $a_1, \ldots, a_m \in \mathcal{D}_{n\times n}$ *such that* $a_1, \ldots, a_m$ *is a smooth product. Let* i, j *be such that* $(a_1 \cdots a_m)^\sharp[i,j] = \Upsilon$. *There are* $i = i_0, \ldots, i_m = j$ *such that for every* $l \in \{1, \ldots, m\}$, $a_l[i_{l-1}, i_l] = \Upsilon$.

Proof. By Lemma 3.9(2), we have $(a_1 \cdots a_m)^\sharp = a_1^\sharp \cdots a_m^\sharp$. By $a_1^\sharp \cdots a_m^\sharp[i,j] = \Upsilon$, there are $i = i_0, \ldots, i_m = j$, and for every $l \in \{1, \ldots, m\}$, $a_l^\sharp[i_{l-1}, i_l] \in \{\Upsilon, \wedge\!\!\wedge\}$. By Lemma 5.6, $a_l^\sharp[i_{l-1}, i_l] \neq \wedge\!\!\wedge$, i.e., we have $a_l^\sharp[i_{l-1}, i_l] = \Upsilon$ for every $l \in \{1, \ldots, m\}$. By Lemma 5.6 in contraposition, we have $a_l[i_{l-1}, i_l] = \Upsilon$. □

5.4. *The Growth of Entries*

We call some word $w = A_1 \cdots A_{|w|} \in \mathbb{D}^+_{n\times n}$ a *smooth product* if $\Psi(A_1), \ldots, \Psi(A_{|w|})$ is a smooth product.

We extend the distance function Δ. For a word matrix $A \in \mathbb{D}_{n\times n}$, a word in $\mathbb{D}^+_{n\times n}$, resp. a finite subset $T \subseteq \mathbb{D}^+_{n\times n}$, Δ yields the largest value among the distances of every entry of A, of every letter in w of every word $w \in T$ except ω and ∞:

1. For $A \in \mathbb{D}_{n\times n}$, let $\Delta(A) := \max_{i,j,\, A[i,j] \notin \{\omega, \emptyset\}} \Delta(A[i,j])$.
2. For $w = A_1 \cdots A_{|w|} \in \mathbb{D}^+_{n\times n}$, let $\Delta(w) := \max_{l \in \{1,\ldots,|w|\}} \Delta(A_l)$.
3. For $T = \{w_1, \ldots, w_{|T|}\} \subseteq \mathbb{D}^+_{n\times n}$, let $\Delta(T) := \max_{l \in \{1,\ldots,|T|\}} \Delta(w_l)$.

The operation max is defined by the usual ordering of integers and $\max \emptyset := 0$.

Proposition 5.8. *Let* $w \in \mathbb{D}^+_{n\times n}$ *be a smooth product and let* i, j *be arbitrary*:

(1) *If* $\Psi(\alpha(w))^\sharp[i,j] = \Upsilon$, *then* $\Delta(\alpha(w)[i,j]) \leq 2 \cdot \Delta(w)$.
(2) *If* $\Psi(\alpha(w))^\sharp[i,j] = \omega$, *then* $\Delta(\alpha(w)[i,j]) \geq |w|/4^{n^2} n - 1$.

Proof. Let $w = A_1 \cdots A_{|w|}$ and $a_l := \Psi(A_l)$ for every $l \in \{1, \ldots, |w|\}$.

We show (1). By Lemma 5.7, there are $i = i_0, \ldots, i_{|w|} = j$ such that for every $1 \leq l \leq |w|$ $a_l[i_{l-1}, i_l] = \Upsilon$. Thus, every $A[i_{l-1}, i_l]$ contains a path π_l from i_{l-1} to i_l

with $\Psi(\pi_l) = \curlyvee$ and $\Delta(\pi_l) \leq \Delta(w)$. Then

$$\Delta(\alpha(w)[i, j]) \leq \Delta(\pi_1 \cdots \pi_l) \leq 2 \cdot \Delta(w).$$

Now, we show (2). Let $i = i_0, \dots, i_{|w|} = j$ be arbitrary. For every $1 \leq l \leq |w|$, choose some $\pi_l \in A_l[i_{l-1}, i_l]$ and let $\pi = \pi_1 \cdots \pi_{|w|}$. To show (2), we show $\Delta(\pi) \geq |w|/4^{n^2}n - 1$.

We have $|\mathcal{D}_{n\times n}| = 4^{n^2}$. By a counting argument, there is some $b \in \mathcal{D}_{n\times n}$ such that the set $I := \{l \mid 1 \leq l \leq |w|,\ \Psi(A_1 \cdots A_l) = b\}$ contains at least $|w|/4^{n^2}$ members. By Lemma 3.5, $a_{k+1} \cdots a_l$ is an idempotent for every $k < l \in I$.

Again by a counting argument, there is some p such that the set $I' := \{l \mid l \in I,\ i_l = p\}$ contains at least $|w|/4^{n^2}n$ members. Let k, resp. l, be the least, resp. biggest, number in I'.

We have $(a_{k+1} \cdots a_l)[p, p] \in \{\curlyvee, \curlywedge\!\!\curlywedge\}$. However, if $(a_{p+1} \cdots a_q)[l, l] = \curlyvee$, then we obtain a contradiction as follows:

$$\begin{aligned}\Psi(\alpha(w))^\sharp[i, j] &= (a_1 \cdots a_{|w|})^\sharp[i_0, i_{|w|}]\\ &= (a_1 \cdots a_k)(a_{k+1} \cdots a_l)^\sharp(a_{l+1} \cdots a_{|w|})[i_0, i_{|w|}] = \curlyvee.\end{aligned}$$

Hence, $(a_{k+1} \cdots a_l)[p, p] = \curlywedge\!\!\curlywedge$, and, hence, $\pi_{k+1} \cdots \pi_l \in \curlywedge\!\!\curlywedge^+$. Thus,

$$\Delta(\pi) \geq \Delta(\pi_{k+1} \cdots \pi_l) \geq q - p \geq |I'| - 1 \geq \frac{|w|}{4^{n^2}n} - 1. \qquad \square$$

5.5. *The Proof of Theorem* 5.2

We prove Theorem 5.2.

Proof of Theorem 5.2 (*Part* 1). We show $\overline{\Psi}\langle T\rangle \supseteq \langle\Psi(T)\rangle^\sharp$.

First we note that $\Psi(T) \subseteq \overline{\Psi}\langle T\rangle$, because for every $A \in T$, $\Psi(A)$ is the Ψ-limit of the constant A-sequence. It remains to show that $\overline{\Psi}\langle T\rangle$ is closed under matrix multiplication and stabilization of idempotents. Let $a, b \in \overline{\Psi}\langle T\rangle$ and $e \in \mathsf{E}(\overline{\Psi}\langle T\rangle)$. There are convergent sequences $(A_k)_{k\geq 1}$, $(B_k)_{k\geq 1}$, $(E_k)_{k\geq 1}$ over $\langle T\rangle$ such that $a = \overline{\Psi}(A_k)_{k\geq 1}$, $b = \overline{\Psi}(B_k)_{k\geq 1}$, and $e = \overline{\Psi}(E_k)_{k\geq 1}$. By Lemma 4.2(2), $(A_k B_k)_{k\geq 1}$ is convergent, and

$$ab = \overline{\Psi}(A_k)_{k\geq 1}\overline{\Psi}(B_k)_{k\geq 1} = \overline{\Psi}(A_k B_k)_{k\geq 1} \in \overline{\Psi}\langle T\rangle.$$

By Lemma 5.4, there is a convergent sequence $(E'_k)_{k\geq 1}$ over $\langle T\rangle$ satisfying $\overline{\Psi}(E'_k)_{k\geq 1} = e^\sharp$, and thus $e^\sharp \in \overline{\Psi}\langle T\rangle$. $\square$

In order to complete the proof of Theorem 5.2 by showing $\overline{\Psi}\langle T\rangle \subseteq \langle\Psi(T)\rangle^\sharp$, we define a relation to compare word matrices. Let $K \geq 1$. Let $X, Y \in \mathbb{D}$. We denote $X \preceq_K Y$ if we can transform X into Y by removing paths π with $\Delta(\pi) > K$ from X. More precisely, we denote $X \preceq_K Y$ if we have the following assertions:

1. If $X = \emptyset$, then $Y = \emptyset$.
2. If $X \neq \emptyset$, then $X \supseteq Y \neq \emptyset$.
3. X and Y "agree in their bounded words", i.e., $\{\pi \in X \mid \Delta(\pi) \leq K\} \subseteq Y$.

It is easy to see that $\preceq_K$ is stable with respect to concatenation and union of $\mathbb{D}$. Let $A, B \in \mathbb{D}_{n\times n}$. We denote $A \preceq_K B$ if we have $A[i, j] \preceq_K B[i, j]$ for every i, j. It is easy to see that $\preceq_K$ on $\mathbb{D}_{n\times n}$ is stable with respect to matrix multiplication. For technical reasons, we extend stabilization to word matrices. For matrices $A \in \mathbb{D}_{n\times n}$, we define the stabilization $A^\sharp$ if $\Psi(A) \in \mathsf{Reg}(\mathcal{D}_{n\times n})$ as follows:

$$A^\sharp[i, j] = \begin{cases} A[i, j] & \text{if} \quad \Psi(A)^\sharp[i, j] = \Psi(A)[i, j], \\ \omega & \text{if} \quad \Psi(A)^\sharp[i, j] = \omega. \end{cases}$$

This definition is correct by Lemma 5.6. Note that if $A^\sharp$ is defined, then $A^\sharp \in \mathbb{D}_{n\times n}$ and we have $\Psi(A^\sharp) = \Psi(A)^\sharp$. By Lemma 5.6, we have $\Psi(A^\sharp)[i, j] \neq \curlywedge\!\!\curlywedge$ for every i, j.

Proposition 5.9. *Let T be some finite subset of $\mathbb{D}_{n\times n}$ and let $K \geq 2$. There is some $x_K \geq 1$ such that: For every $w \in T^+$ there is a $B \in \mathbb{D}_{n\times n}$ satisfying $\Psi(B) \in \langle\Psi(T)\rangle^\sharp$, $\alpha(w) \preceq_K B$, and $\Delta(B) \leq x_K$. In particular, this assertion is true for*

$$x_K := (2 \cdot 4^{n^2} n(K + 2))^{4^{n^2}} \cdot \Delta(T).$$

We should pay some attention to the conditions $\Delta(B) \leq x_K$ and $\alpha(w) \preceq_K B$. Let i, j be arbitrary. If $\alpha(w)[i, j] \in \{\omega, \emptyset\}$, then $\alpha(w)[i, j] = B[i, j]$.

If we have $\Delta(\alpha(w)[i, j]) \leq K$, then $\alpha(w)[i, j] \supseteq B[i, j]$ but $\alpha(w)[i, j]$ and $B[i, j]$ agree in their bounded words (see items 2 and 3 in the definition of $\preceq_K$).

If $\Delta(\alpha(w)[i, j]) > x_K$, then $\alpha(w) \preceq_K B$ and $\Delta(B) \leq x_K$ together imply $B[i, j] = \omega$.

However, there is a "gray area": if $K < \Delta(\alpha(w)[i, j]) \leq x_K$, then we know $\alpha(w)[i, j] \supseteq B[i, j]$, but we do not know whether $\Delta(B[i, j]) \leq x_K$ or $\Delta(B[i, j]) = \omega$. This lack of precise information is not really a problem: we apply Proposition 5.9 just on these words w for which no entry in $\alpha(w)$ belongs to this gray area.

We establish the following lemma to prove Proposition 5.9.

Lemma 5.10. *Let T be some finite subset of $\mathbb{D}_{n\times n}$, let $K \geq 2$, and let $x \geq 1$ be arbitrary. Let $I' \subsetneq I \subseteq \langle\Psi(T)\rangle^\sharp$ be two ideals of $\langle\Psi(T)\rangle^\sharp$ such that $I \setminus I'$ is a $\mathscr{J}$-class of $\langle\Psi(T)\rangle^\sharp$.*

There is some $x' \geq 1$ such that for every $w = A_1 \cdots A_{|w|} \in \mathbb{D}^+_{n\times n}$ satisfying

- **A1.** $\Psi(A_1), \ldots, \Psi(A_{|w|}) \in \langle\Psi(T)\rangle^\sharp$,
- **A2.** $\Delta(w) \leq x$,
- **A3.** *for every* $l \in \{1, \ldots, |w| - 1\}$, $\Psi(A_l A_{l+1}) \in I$,

there is some $v = B_1 \cdots B_{|v|} \in \mathbb{D}^+_{n\times n}$ satisfying $\alpha(w) \preceq_K \alpha(v)$ and

- **C1.** $\Psi(B_1), \ldots, \Psi(B_{|v|}) \in \langle\Psi(T)\rangle^\sharp$,
- **C2.** $\Delta(v) \leq x'$,
- **C3.** *for every* $l \in \{1, \ldots, |v| - 1\}$, $\Psi(B_l B_{l+1}) \in I'$.

In particular, this assertion is true for $x' := 2 \cdot 4^{n^2} n(K + 2)x$.

First, note the similarity between assumptions (A1)–(A3) and claims (C1)–(C3). This similarity enables us to apply Lemma 5.10 inductively on a chain of ideals $\emptyset \subsetneq \cdots \subsetneq$

$I_2 \subsetneq I_1 \subseteq \langle \Psi(T) \rangle^\sharp$ to prove Proposition 5.9. In the last step of this induction, I' is the empty ideal, and thus claim (C3) implies that v has length 1, and v is exactly the matrix B which we require to prove Proposition 5.9.

We are not interested in whether $A_1, \ldots, A_{|w|}$ (resp. $B_1, \ldots, B_{|v|}$) belong to $\langle T \rangle$, we just assume (resp. show) that their images under Ψ belong to $\langle \Psi(T) \rangle^\sharp$. In (C3), and similarly in (A3), we are not interested in whether or not the images of the B_l belong to I' or I.

It is important that the bound x' does not depend on the length of w.

Maybe, it is a good idea to skip the proof of Lemma 5.10 for the first reading, and get an insight in the structure of the rest of the section.

Proof of Lemma 5.10. Let K, x, and $w = A_1 \cdots A_{|w|} \in \mathbb{D}^+_{n\times n}$ as in the lemma.

If $|w| = 1$, then claims (C1)–(C3) hold obviously for $v = w$. We assume $|w| > 1$ in the rest of the proof.

A straightforward idea is that whenever $\Psi(A_l A_{l+1}) \in I \setminus I'$ for an $l \in \{1, \ldots, |w|-1\}$, then we merge $A_l A_{l+1}$, i.e., we replace the letters $A_l A_{l+1}$ in w by the result of the matrix multiplication of A_l by A_{l+1}. We do this as many times as necessary, and finally we obtain some word which satisfies (C1) and (C3). However, it will be impossible to show (C2) for some constant x' which is independent on the length of w. We have to be much more careful.

We factorize w into words $v_1, v_2, \ldots, v_m$. If $\Psi(A_1) \in I'$, then let $v_1 := A_1$ and proceed with $A_2 \cdots A_{|w|}$. If $\Psi(A_1) \notin I'$, then let v_1 be the longest prefix of w satisfying $\Psi(\alpha(v_1)) \notin I'$ and proceed with the remaining part of w. Note that this longest prefix may have length 1. The integer m is defined as the number of words which we obtain in this factorization.

The most important fact is that if $|v_l| > 1$ for some $l \in \{1, \ldots, m\}$, then $\Psi(\alpha(v_l)) \in I \setminus I'$, because $\Psi(\alpha(v_l)) \notin I'$ by construction and $\Psi(\alpha(v_l)) \in I$ by (A3).

In this way, we achieve some $m \geq 1$ and $v_1, \ldots, v_m \in \mathbb{D}^+_{n\times n}$ such that:

1. $A_1 \cdots A_{|w|} = v_1 \cdots v_m$ (concatenation of words).
2. $\Psi(\alpha(v_1)), \ldots, \Psi(\alpha(v_m)) \in \langle \Psi(T) \rangle^\sharp$.
3. For every $l \in \{1, \ldots, m-1\}$, $\Psi(\alpha(v_l v_{l+1})) \in I'$ (by construction of v_l).
4. For every $l \in \{1, \ldots, m\}$ with $|v_l| > 1$, we have $\Psi(\alpha(v_l)) \in I \setminus I'$.

Let $l \in \{1, \ldots, m\}$ be arbitrary. We define B_l from v_l and derive some information on B_l. We distinguish two cases. We show (C1), (C2), and $\alpha(v_l) \preceq_K B_l$ within these cases, but we show (C3) after both cases are completed.

Case 1: $|v_l| < 2 \cdot 4^{n^2} n(K+2)$. We set $B_l := \alpha(v_l)$. This implies in particular that $\alpha(v_l) \preceq_K B_l$ and

$$\Delta(B_l) \leq |v_l| \cdot \Delta(v_l) \leq 2 \cdot 4^{n^2} n(K+2)x = x',$$

i.e., B_l satisfies (C2). Clearly, B_l satisfies (C1).

Case 2: $|v_l| \geq 2 \cdot 4^{n^2} n(K+2)$. We are caught on the horns of a dilemma. If we set $B_l := \alpha(v_l)$, then B_l satisfies (C1). However, we cannot use the length argument on v_l to show (C2). Thus, we need to eliminate the entries in $\alpha(v_l)$ which are not bounded by x'. We could simply solve this problem by setting these entries to ω. Then it quite easy to show (C2), but we probably violate (C1). We could try to eliminate the large entries of $\alpha(v_l)$ by setting $B_l = \alpha(v_l)^\sharp$. Unfortunately, this raises a bunch of serious problems: is $\alpha(v_l)^\sharp$ defined and is $\alpha(v_l)^\sharp \in \langle\Psi(T)\rangle^\sharp$, i.e., is $\Psi(\alpha(v_l))$ regular in $\langle\Psi(T)\rangle^\sharp$? Can we really eliminate any entry which is not bounded by x'? Moreover, if we accidentally change an entry which is bounded by less then K to ω, then we will not be able to show $\alpha(w) \preceq_K \alpha(v)$.

We choose the only way out: we define $B_l := \alpha(v_l)^\sharp$ and face these serious problems.

We denote v_l as $v_l = V_1 \cdots V_{|v_l|}$. We transform v_l into a word u. If $|v_l|$ is even, then we set $u := \alpha(V_1V_2)\alpha(V_3V_4)\cdots\alpha(V_{|v_l|-1}V_{|v_l|})$. If $|v_l|$ is odd, then we set $u := \alpha(V_1V_2)\alpha(V_3V_4)\cdots\alpha(V_{|v_l|-2}V_{|v_l|-1}V_{|v_l|})$. Clearly, $\alpha(v_l) = \alpha(u)$.

We have $|u| \geq (4^{n^2}n + 1)(K + 2)$. We denote the letters of u by $u := U_1 \cdots U_{|u|}$.

By (A3), we have for every $k \in \{1, \ldots, |u|\}$, $\Psi(U_k) \in I$. If $\Psi(U_k) \in I'$, then $\Psi(\alpha(u)) \in I'$ and $\Psi(\alpha(u)) = \Psi(\alpha(v_l)) \in I'$ which contradicts condition 4 above. Hence, $\Psi(U_k) \in I\backslash I'$ for every $k \in \{1, \ldots, |u|\}$. Consequently, $I\backslash I'$ is a regular $\mathscr{J}$-class of $\langle\Psi(T)\rangle^\sharp$, and u is a smooth product. Moreover, $\Psi(\alpha(u)) = \Psi(\alpha(v_l)) \in \mathsf{Reg}(\langle\Psi(T)\rangle^\sharp)$, and thus $\alpha(v_l)^\sharp$ is defined, and

$$\Psi(B_l) = \Psi(\alpha(v_l)^\sharp) = \Psi(\alpha(v_l))^\sharp \in \langle\Psi(T)\rangle^\sharp.$$

Now we show $\alpha(u) \preceq_K \alpha(u)^\sharp$. We know that $\alpha(u)$ and $\alpha(u)^\sharp$ are only different in entries i, j for which $\Psi(\alpha(u))^\sharp[i, j] = \omega$. So let i, j satisfy $\Psi(\alpha(u))^\sharp[i, j] = \omega$. By Proposition 5.8(2),

$$\Delta(\alpha(u)[i, j]) \geq \frac{|u|}{4^{n^2}n} - 1 \geq \frac{4^{n^2}n(K + 2)}{4^{n^2}n} - 1 > K,$$

and thus $\alpha(u) \preceq_K \alpha(u)^\sharp$, and moreover $\alpha(v_l) = \alpha(u) \preceq_K \alpha(u)^\sharp = \alpha(v_l)^\sharp = B_l$.

Now we take care of (C2) for B_l. Let i, j be arbitrary. Assume that $B_l[i, j]$ contains some path. By the definition of B_l as $\alpha(u)^\sharp$, $B_l[i, j]$ contains some water transition, i.e., $\Psi(B_l)[i, j] = \curlyvee$. By Proposition 5.8(1), we obtain

$$\Delta(B_l[i, j]) = \Delta(\alpha(u)[i, j]) \leq 2 \cdot \Delta(u) \leq 6 \cdot \Delta(w) \leq 6 \cdot x < x'.$$

(Note that $\Delta(u) \leq 3 \cdot \Delta(w)$ because the letters of u arise by merging two or three adjacent letters of w.) Hence, B_l satisfies (C2).

We show (C3). By condition 3 above, we have $\Psi(\alpha(v_l))\Psi(\alpha(v_{l+1})) \in I'$ for every $l \in \{1, \ldots, m-1\}$. By the definition of B_l and Remark 3.2, we have $\Psi(B_l) \leq_{\mathscr{L}} \Psi(\alpha(v_l))$ and $\Psi(B_{l+1}) \leq_{\mathscr{R}} \Psi(\alpha(v_{l+1}))$. Thus,

$$\Psi(B_l)\Psi(B_{l+1}) \leq_{\mathscr{R}} \Psi(B_l)\Psi(\alpha(v_{l+1})) \leq_{\mathscr{L}} \Psi(\alpha(v_l))\Psi(\alpha(v_{l+1})),$$

i.e., $\Psi(B_l)\Psi(B_{l+1}) \leq_{\mathscr{J}} \Psi(\alpha(v_l))\Psi(\alpha(v_{l+1})) \in I'$ and $\Psi(B_l)\Psi(B_{l+1}) = \Psi(B_lB_{l+1}) \in I'$.

It remains to show $\alpha(w) \preceq_K \alpha(v)$. In both cases 1 and 2, we have seen $\alpha(v_l) \preceq_K \alpha(B_l)$ for every $l \in \{1, \ldots, m\}$. By the stability of $\preceq_K$ with respect to matrix multiplication it follows that $\alpha(w) \preceq_K \alpha(v)$. □

Proof of Proposition 5.9. Let z be the number of $\mathscr{J}$-classes of $\langle\Psi(T)\rangle^\sharp$. We have $z \leq 4^{n^2}$. Let $\langle\Psi(T)\rangle^\sharp = I_1 \supsetneq I_2 \supsetneq \cdots \supsetneq I_z \supsetneq I_{z+1} = \emptyset$ be ideals of $\langle\Psi(T)\rangle^\sharp$ such that for every $l \in \{1, \dots, z\}$ the set $I_l \setminus I_{l+1}$ is a $\mathscr{J}$-class of $\langle\Psi(T)\rangle^\sharp$.

Let $w \in T^+$. We apply Lemma 5.10 z times over the ideals $I_1, I_2, \dots, I_{z+1}$. Initially, we set $x := \Delta(T)$, and it is quite clear that (A1)–(A3) are satisfied. By Lemma 5.10, we obtain a bound x' and a word v. Then we apply Lemma 5.10 for I_2 and I_3 on v and x' instead of w and x, and so on. After the last application of Lemma 5.10 we achieved a bound x' and a word v which is by (C3) just a single letter. Thus, we set $B = v$, and x' is the constant x_K in Proposition 5.10. $\square$

Proof of Theorem 5.2 (*Part* 2). We show $\overline{\Psi}\langle T\rangle \subseteq \langle\Psi(T)\rangle^\sharp$. Let $(w_k)_{k\geq 1} \in T^+$ be some arbitrary sequence. To complete the proof of Theorem 5.2, it suffices to show $\overline{\Psi}(\alpha(w_k))_{k\geq 1} \in \langle\Psi(T)\rangle^\sharp$, provided that $(\alpha(w_k))_{k\geq 1} \in \mathfrak{C}\langle T\rangle$. So we assume $(\alpha(w_k))_{k\geq 1} \in \mathfrak{C}\langle T\rangle$ and denote $a := \overline{\Psi}(\alpha(w_k))_{k\geq 1}$.

By subsequence selection arguments, we can assume some bound $K \geq 1$ such that every matrix in $(\alpha(w_k))_{k\geq 1}$ satisfies the following properties. Let i, j, and $l \geq 1$ be arbitrary:

1. If $a[i,j] = \infty$, then $\alpha(w_l)[i,j] = \emptyset$. If $a[i,j] \neq \infty$, then $\alpha(w_l)[i,j] \neq \emptyset$.
2. If $a[i,j] = \curlyvee$, then $\Delta(\alpha(w_l)[i,j]) \leq K$ and $\Psi(\alpha(w_l))[i,j] = \curlyvee$.
3. If $a[i,j] = \curlywedge\!\!\curlywedge$, then $\Delta(\alpha(w_l)[i,j]) \leq K$ and $\Psi(\alpha(w_l))[i,j] = \curlywedge\!\!\curlywedge$.

Let x_K be the constant provided by Proposition 5.9. There is some w among $(w_k)_{k\geq 1}$ which satisfies

4. If $a[i,j] = \omega$, then $\Delta(\alpha(w)[i,j]) > x_K$.

We apply Proposition 5.9 on w and obtain a matrix $B \in \mathbb{D}_{n\times n}$. We derive some information on B. Let i, j be arbitrary.

Case 1: $a[i,j] = \infty$. By $\alpha(w)[i,j] = \emptyset$ due to property 1 and $\alpha(w) \preceq_K B$ due to Proposition 5.9, we have $B_l[i,j] = \emptyset$, i.e., $\Psi(B[i,j]) = \infty = a[i,j]$.

Case 2: $a[i,j] = \curlyvee$. By property 2 above, there is some $\pi \in \alpha(w)[i,j]$ with $\Psi(\pi) = \curlyvee$ and $\Delta(\pi) \leq K$. Again by $\alpha(w) \preceq_K B$, we have $\pi \in B[i,j]$, i.e., $\Psi(B[i,j]) = \curlyvee = a[i,j]$.

Case 3: $a[i,j] = \curlywedge\!\!\curlywedge$. Similarly to Case 2, there is some $\pi \in B[i,j]$ such that $\Psi(\pi) = \curlywedge\!\!\curlywedge$ and $\Delta(\pi) \leq K$, and thus $B[i,j] \in \{\curlyvee, \curlywedge\!\!\curlywedge\}$.

By contradiction, assume $B[i,j] = \curlyvee$. Then there is some $\pi' \in B[i,j]$ with $\Psi(\pi') = \curlyvee$. By $\alpha(w) \preceq_K B$, we have $\pi' \in \alpha(w)[i,j]$, i.e., $\Psi(\alpha(w))[i,j] = \curlyvee$ which contradicts property 3 above.

Consequently, $\Psi(B[i,j]) = \curlywedge\!\!\curlywedge = a[i,j]$.

Case 4: $a[i,j] = \omega$. By property 1, we have $\alpha(w)[i,j] \neq \emptyset$. By property 4, we have $\Delta(\pi) > x_K$ for every $\pi \in \alpha(w)[i,j]$. Consequently, $\alpha(w) \preceq_K B$ and $\Delta(B) \leq x_K$ imply $B[i,j] = \omega$, i.e., $\Psi(B[i,j]) = \omega = a[i,j]$.

If we combine these four cases, we obtain $\Psi(B) = a = \overline{\Psi}(\alpha(w_k))_{k\geq 1}$. Due to Proposition 5.9, we have $\Psi(B) \in \langle\Psi(T)\rangle^\sharp$. $\square$

5.6. *The Proof of Theorem* 2.3

First we show that nested stabilizations are not required to compute the closure $\langle \Psi(T) \rangle^\sharp$.

Lemma 5.11. *Let* $p, q \in \mathcal{D}_{n\times n}$, $e \in \mathsf{E}(\mathcal{D}_{n\times n})$, *and assume* $peq \in \mathsf{E}(\mathcal{D}_{n\times n})$. *We have* $(pe^\sharp q)^\sharp = pe^\sharp q$.

Proof. By the definition of stabilization, we have for every i, j, $e^\sharp[i, j] \neq \curlywedge\!\!\!\curlywedge$. By the definition of multiplication in $\mathcal{D}$, we have for every i, j, $(pe^\sharp q)[i, j] \neq \curlywedge\!\!\!\curlywedge$. By Lemma 5.1, we have $(pe^\sharp q)^\sharp = (pe^\sharp q)$. $\square$

We denote the identity matrix in $\mathcal{D}_{n\times n}$ by $1_{\mathcal{D}_{n\times n}}$.

Lemma 5.12. *Let* $M \subseteq \mathcal{D}_{n\times n}$ *and assume* $1_{\mathcal{D}_{n\times n}} \in M$. *For every* $a \in \langle M \rangle^\sharp$, *there are* $a_1, \ldots, a_{|\mathcal{D}_{n\times n}|} \in \mathcal{D}_{n\times n}$ *such that for every* $k \in \{1, \ldots, |\mathcal{D}_{n\times n}|\}$,

(1) $a_k \in M$, *or*
(2) *there are* $b_1, \ldots, b_{|\mathcal{D}_{n\times n}|} \in M$ *such that* $b := (b_1 \cdots b_{|\mathcal{D}_{n\times n}|}) \in \mathsf{E}(\mathcal{D}_{n\times n})$ *and* $a_k = b^\sharp$.

Proof. The lemma is a straightforward conclusion from Lemma 5.11 along with some counting and cancellation arguments. $\square$

Proof of Theorem 2.3. First we show that the problem is decidable in PSPACE. Let $\mathcal{A} = [Q, E, I, F, E^\curlyvee]$ be a desert automaton, set $n := |Q|$. Let $T := \theta(\Sigma)$. We want to decide whether $\mathcal{A}$ is limited. By Proposition 4.3 and Theorem 5.2, it suffices to decide whether there is a matrix $a \in \langle \Psi(T) \rangle^\sharp$ with $\min\{a[i, j] \mid i \in I, j \in F\} = \omega$. We regard I and F as a row (and a column) matrix, respectively, and denote $\min\{a[i, j] \mid i \in I, j \in F\}$ by $I \cdot a \cdot F$. Below, we give a non-deterministic algorithm which decides the existence of a matrix a for which $I \cdot a \cdot F = \omega$.

The input of the algorithm is the integer n, the set $\Psi(T)$, and the sets I and F. The function call `guess(0,1)` causes the algorithm to branch into two instances and returns the result 0, resp. 1, in these instances. The function call `guess(`$\Psi(T) \cup 1_{\mathcal{D}_{n\times n}}$`)` causes the algorithm to branch into $|\Psi(T) \cup 1_{\mathcal{D}_{n\times n}}|$ instances and returns a matrix in $\Psi(T) \cup 1_{\mathcal{D}_{n\times n}}$.

```
a:= 1_{𝔻_{n×n}}
for k:=1 to |𝒟_{n×n}| do begin
  if guess(0,1) = 0 then a:= a·guess(Ψ(T) ∪ 1_{𝒟_{n×n}})
  else begin
    b := 1_{𝔻_{n×n}}
    for l :=1 to |𝒟_{n×n}| do b := b·guess(Ψ(T) ∪ 1_{𝒟_{n×n}})
    if b = b·b then a := a·b^♯
  end
end
if I·a·F = ω then output('𝒜 is unlimited.')
  else fail
```

If there is a run of the algorithm which outputs "$\mathcal{A}$ `is unlimited.`", then there is a matrix $a \in \langle \Psi(T) \rangle^\sharp$ with $I \cdot a \cdot F = \omega$, and, hence, $\mathcal{A}$ is indeed unlimited. Conversely, if $\mathcal{A}$ is unlimited then there is a matrix $a \in \langle \Psi(T) \rangle^\sharp$ with $I \cdot a \cdot F = \omega$, and by Lemma 5.12 there is a run of the algorithm with the output "$\mathcal{A}$ `is unlimited.`"

The algorithm requires $\mathcal{O}(n^2)$ space to store the matrices `a` and `b`. Moreover, it requires an additional matrix and three integer variables which range over $\{1, \ldots, n\}$ to compute matrix multiplication, stabilization, and the comparison of matrices. Finally, it requires $\mathcal{O}(n^2)$ space to store `k` and `l`. Hence, the algorithm requires $\mathcal{O}(n^2)$ space. By Savitch's theorem, the limitedness problem of desert automata is decidable in PSPACE.

It remains to show that the limitedness problem for desert automata is PSPACE-hard. We follow the same idea as Leung's proof for PSPACE-hardness of the limitedness problem of distance automata. Let $\mathcal{A} = [Q, E, I, F]$ be a non-deterministic automaton. The problem whether $L(\mathcal{A}) = \Sigma^*$ is known to be PSPACE-hard [17]. We construct a desert automaton which is limited iff $L(\mathcal{A}) = \Sigma^*$.

Let $c \notin \Sigma$ be a new letter. We can construct an automaton which accepts $L(\mathcal{A})c^+$ by adding just one state to $\mathcal{A}$. By defining every transition to be a water transition, we obtain a desert automaton $\mathcal{A}'$ which accepts $L(\mathcal{A})c^+$ such that $\Delta'(wc^k) = 0$ for every $w \in L(\mathcal{A})$, $k \geq 1$.

By adding two more states to $\mathcal{A}'$, we can construct a desert automaton $\mathcal{A}''$ which accepts $\Sigma^* c^+$ such that for every $w \in \Sigma^*$, $k \geq 1$, we have $\Delta''(wc^k) = k$ if $w \notin L(\mathcal{A})$ but $\Delta''(wc^k) = 0$ if $w \in L(\mathcal{A})$. Obviously, $\mathcal{A}''$ is limited iff $L(\mathcal{A}) = \Sigma^*$ and the size of $\mathcal{A}''$ is polynomial in the size of $\mathcal{A}$. □

6. The Finite Substitution Problem

As an application of the decidability of the limitedness problem for desert automata, we show the decidability of the finite substitution problem.

We call every mapping $\sigma : \Sigma_1 \to \mathcal{P}(\Sigma_2^*)$ a *substitution*. Every substitution extends in a natural way to a homomorphism $\sigma : \mathcal{P}(\Sigma_1^*) \to \mathcal{P}(\Sigma_2^*)$. Let σ be a substitution. We call σ a *finite substitution* if for every $a \in \Sigma_1$, the language $\sigma(a)$ is finite. We call σ a *non-erasing substitution* if for every $a \in \Sigma_1$, we have $\varepsilon \notin \sigma(a)$.

We fix two alphabets Σ_1 and Σ_2 and two recognizable languages $K \subseteq \Sigma_1^*$ and $L \subseteq \Sigma_2^*$ for this section. We denote by $M(L)$ and $\eta : \Sigma_2^* \to M(L)$ the syntactic monoid and the syntactic homomorphism of L. Clearly, η extends in a unique way to a homomorphism $\eta : \mathcal{P}(\Sigma_2^*) \to \mathcal{P}(M(L))$.

We call a substitution $\sigma\colon \Sigma_1 \to \mathcal{P}(\Sigma_2^*)$ *non-erasing* if for every $a \in \Sigma_1$, we have $\varepsilon \notin \sigma(a)$.

We call every homomorphism $\tau : \mathcal{P}(\Sigma_1^*) \to \mathcal{P}(M(L))$ with $\tau(K) = \eta(L)$ a *type* (*for* K). There are only finitely many types, and there is an algorithm which computes the list of all types.

Let σ be a substitution and let τ be a type. We say that σ *is of type* τ if for every $a \in \Sigma_1$, we have $\eta(\sigma(a)) \subseteq \tau(a)$. If σ is of type τ, then $\eta(\sigma(K)) \subseteq \tau(K) \subseteq \eta(L)$, i.e., $\sigma(K) \subseteq L$.

Let σ be a substitution with $\sigma(K) = L$. Then $\eta \circ \sigma : \mathcal{P}(\Sigma_1^*) \to \mathcal{P}(M(L))$ is a type and σ is of type $\eta \circ \sigma$.

6.1. *The Existence of Non-Erasing Finite Substitutions*

We show that the existence of a non-erasing finite substitution is decidable.

Lemma 6.1. *Let τ be a type. It is decidable whether there is a non-erasing finite substitution σ of type τ such that $\sigma(K) = L$.*

Proof. For every $d \geq 1$ and $a \in \Sigma_1$, let $\tilde{\sigma}_d(a) := \{w \in \Sigma_1^* \mid 1 \leq |w| \leq d,\ \eta(w) \in \tau(a)\}$ and

$$\tilde{\sigma}(a) := \bigcup_{d \geq 1} \tilde{\sigma}_d(a) = \eta^{-1}(\tau(a)) \backslash \varepsilon.$$

We have $\tilde{\sigma}(K) \subseteq L$.

We construct a desert automaton $\mathcal{A}'$, such that there is a non-erasing finite substitution σ of type τ with $\sigma(K) = L$ iff $L(\mathcal{A}') = L$ and $\mathcal{A}'$ is limited.

Let $\mathcal{A} = [Q, E, I, F]$ be an automaton which recognizes K. For every transition $t = (p, a, q) \in E$, we construct a desert automaton $\mathcal{A}_t = [Q_t, E_t, p, q', E_t^{\curlyvee}]$ with the following properties:

- $L(\mathcal{A}_t) = \tilde{\sigma}(a) = \eta^{-1}(\tau(a)) \backslash \varepsilon$,
- $E_t \subseteq (Q_t \backslash q') \times \Sigma_2 \times (Q_t \backslash p)$, and
- $E_t^{\curlyvee} = E_t \cap (Q_t \times \Sigma_2 \times q')$.

We call the accepting state q' instead of q to avoid some problems in the particular case $p = q$. The construction of $\mathcal{A}_t$ is straightforward by setting $Q_t := M(L) \cup \{p, q'\}$ and defining E_t in the obvious way.

For every $t \in E$, let $\mathcal{A}_t$ be such a desert automaton and assume that the sets of states of all these automata are mutually disjoint.

Now we define $\mathcal{A}' = [Q', E', I, F, E'^{\curlyvee}]$. We replace in $\mathcal{A}$ every transition $t = (p, a, q) \in E$ by $\mathcal{A}_t$ and identify the initial and accepting state of $\mathcal{A}_t$ with p (resp. q). In this way the states Q', transitions E', and water transitions $E'^{\curlyvee}$ of $\mathcal{A}$ arise as unions of the states, transitions, and water transitions of the automata $\mathcal{A}_t$. The initial (resp. accepting) states of $\mathcal{A}'$ are the initial (resp. accepting) states of $\mathcal{A}$. We denote the mapping of $\mathcal{A}'$ by Δ'.

We show the following assertions:

(1) For every $w \in L(\mathcal{A}')$, we have $w \in \tilde{\sigma}_{\Delta'(w)+1}(K) \subseteq L$. Thus, $L(\mathcal{A}') \subseteq L$.
(2) If $L(\mathcal{A}') = L$ and $\mathcal{A}'$ is limited by some d, then we have $\tilde{\sigma}_{d+1}(K) = L$.
(3) For every $d \in \mathbb{N}$ and every $w \in \tilde{\sigma}_{d+1}(K)$, we have $w \in L(\mathcal{A}')$ and $\Delta'(w) \leq d$.
(4) If there is a non-erasing finite substitution σ of type τ with $\sigma(K) = L$, then $L(\mathcal{A}') = L$ and $\mathcal{A}'$ is limited.

(1) Let π be a successful path in $\mathcal{A}$ with the label w and $\Delta'(\pi) = \Delta'(w)$. If $|\pi| = 0$, then $w = \varepsilon$, $d = 0$, $I \cap F \neq \emptyset$, and $\varepsilon \in K$. Then $w \in \tilde{\sigma}_1(K)$. We assume $|\pi| \geq 1$ in the rest of the proof of (1).

There is some $k \geq 1$ such that we can factorize π into $\pi = \pi_1 \cdots \pi_k$ such that every path $\pi_1, \ldots, \pi_k$ starts and ends at some state from Q, but there is no state from Q inside $\pi_1, \ldots, \pi_k$. From now on, let $1 \leq i \leq k$ be arbitrary. Let $q_0, \ldots, q_k \in Q$ such that every π_i is a path from q_{i-1} to q_i. Let w_i be the label of π_i, i.e., $w = w_1 \cdots w_k$. By the construction of $\mathcal{A}$, we have $|w_i| = |\pi_i| = \Delta'(\pi_i) + 1 \leq \Delta'(\pi) + 1 = \Delta'(w) + 1$. Moreover, there is a transition $(q_{i-1}, a_i, q_i) \in E$ such that $w_i \in \eta^{-1}(\tau(a_i))$, i.e., $w_i \in \tilde{\sigma}_{\Delta'(w)+1}(a_i)$, and thus $w \in \tilde{\sigma}_{\Delta'(w)+1}(a_1 \cdots a_k)$. The transitions (q_{i-1}, a_i, q_i) form a successful path in $\mathcal{A}$. Hence, $a_1 \cdots a_k \in K$, and, finally, $w \in \tilde{\sigma}_{\Delta'(w)+1}(K)$.

(2) We have $\tilde{\sigma}_{d+1}(K) \subseteq L$. Let $w \in L$ be arbitrary. By $L(\mathcal{A}') = L$, we have $w \in L(\mathcal{A}')$. By (1), we have $w \in \tilde{\sigma}_{\Delta'(w)+1}(L) \subseteq \tilde{\sigma}_{d+1}(K)$.

(3) Let $v \in K$ such that $w \in \tilde{\sigma}_{d+1}(v)$. We denote the letters of v by $v = a_1 \cdots a_{|v|}$. Moreover, let $(q_0, a_1, q_1)\,(q_1, a_2, q_2) \cdots (q_{|v|-1}, a_{|v|}, q_{|v|})$ be a successful path in the automaton for K. From now on, let $1 \leq i \leq |v|$. There is some word $w_i \in \tilde{\sigma}_{d+1}(a_i)$ such that $w = w_1 \cdots w_{|v|}$. Because σ_{d+1} is of type τ, we have $w_i \in \eta^{-1}(\tau(a_i))$. We have $w_i \neq \varepsilon$. By the construction of $\mathcal{A}'$, there is a path π_i in $\mathcal{A}'$ which starts in q_{i-1}, ends in q_i, and is labeled with w_i. The concatenation $\pi_1 \cdots \pi_{|v|}$ is a successful path in $\mathcal{A}'$ with the label w. The last transition of every π_i is a water transition, and we have $|\pi_i| = |w_i| \leq (d+1)$. Thus, $\Delta'(w) \leq \Delta'(\pi_1 \cdots \pi_{|v|}) \leq d$.

(4) Let $d \in \mathbb{N}$ such that any word in $\sigma(\Sigma_1)$ has a length of at most $d+1$. Then $L = \sigma(K) \subseteq \tilde{\sigma}_{d+1}(K) \subseteq L$, i.e., $L = \tilde{\sigma}_{d+1}(K)$. By (3), $\tilde{\sigma}_{d+1}(K) \subseteq L(\mathcal{A}') \subseteq L$, and thus $\tilde{\sigma}_{d+1}(K) = L(\mathcal{A}') = L$. Let $w \in L(\mathcal{A}')$ be arbitrary. We have $w \in \tilde{\sigma}_{d+1}(K)$, and by (3), $\Delta'(w) \leq d$. Hence, $\mathcal{A}'$ is limited.

By (2) and (4), there is a non-erasing finite substitution of type τ with $\sigma(K) = L$ iff $L(\mathcal{A}') = L$ and $\mathcal{A}'$ is limited. We can effectively construct $\mathcal{A}'$, it is decidable whether $L(\mathcal{A}') = L$, and, by Theorem 2.3, it is decidable whether $\mathcal{A}'$ is limited. □

6.2. *The Existence of Finite Substitutions*

The homomorphisms which we need in this section are shown in the following diagram:

$$\begin{tikzcd}
\mathcal{P}(\Sigma_1^*) \arrow[rr, "\sigma"] \arrow[dr, "\sigma_\tau"] \arrow[ddrr, bend right, "\tau"'] & & \mathcal{P}(\Sigma_2^*) \arrow[dd, "\eta"] \\
& \mathcal{P}(\Sigma_1^*) \arrow[ur, "\sigma'"] \arrow[dr, "\tau"] & \\
& & \mathcal{P}(M(L))
\end{tikzcd}$$

For every type τ, we define a substitution $\sigma_\tau\colon \mathcal{P}(\Sigma_1^*) \to \mathcal{P}(\Sigma_1^*)$ by setting for every $a \in \Sigma_1$

$$\sigma_\tau(a) := \begin{cases} a & \text{if} \quad 1 \notin \tau(a), \\ a \cup \varepsilon & \text{if} \quad 1 \in \tau(a). \end{cases}$$

For every $a \in \Sigma_1$, we have $\tau(a) = \tau(\sigma_\tau(a))$. Thus, we have $\tau(K) = \tau(\sigma_\tau(K))$. Hence, if τ is a type for K, then τ is also a type for $\sigma_\tau(K)$.

Lemma 6.2. *Let τ be a type. The following assertions are equivalent*:

(1) *There is a finite substitution σ of type τ such that $\sigma(K) = L$.*
(2) *There is a non-erasing finite substitution σ' of type τ such that $\sigma'(\sigma_\tau(K)) = L$.*

Proof. (2) $\Rightarrow$ (1) Let $\sigma := \sigma' \circ \sigma_\tau$. We show that $\sigma' \circ \sigma_\tau$ is of type τ. Let $a \in \Sigma_1$. Because σ' is of type τ, we have for every $R \subseteq \Sigma_1^*$, $\eta(\sigma'(R)) \subseteq \tau(R)$. In the particular case $R = \sigma_\tau(a)$, we have $\eta(\sigma'(\sigma_\tau(a))) \subseteq \tau(\sigma_\tau(a)) = \tau(a)$.

(1) $\Rightarrow$ (2) For every $a \in \Sigma_1$, let $\sigma'(a) := \sigma(a)\backslash\varepsilon$. For every $a \in \Sigma_1$, we have $\sigma'(a) \subseteq \sigma(a)$. Thus, σ' is of type τ. As seen above, τ is a type for $\sigma_\tau(K)$. Hence, $\sigma'(\sigma_\tau(K)) \subseteq L$.

Let $w \in L$ be arbitrary. There is some $v \in K$ such that $w \in \sigma(v)$. Denote $v = a_1 \cdots a_{|v|}$. There are $w_1 \in \sigma(a_1), \ldots, w_{|v|} \in \sigma(a_{|v|})$ such that $w = w_1 \cdots w_{|v|}$. Let $I := \{i \mid 1 \leq i \leq |v|, w_i = \varepsilon\}$. For every $i \in I$, we have $1 \in \tau(a_i)$, and thus $\sigma_\tau(a_i) = a_i \cup \varepsilon$.

We transform v into v' by erasing for every $i \in I$ the letter at the ith position in v. We have $v' \in \sigma_\tau(v) \subseteq \sigma_\tau(K)$. We have $w \in \sigma'(v')$, i.e., $w \in \sigma'(\sigma_\tau(K))$. Thus, $L \subseteq \sigma'(\sigma_\tau(K))$. □

Proof of Theorem 2.4. To determine whether a finite substitution σ with $\sigma(K) = L$ exists, an algorithm determines for every type τ whether there is a substitution σ of type τ with $\sigma(K) = L$. To check this, an algorithm constructs an automaton for $\sigma_\tau(K)$, and determines whether there is a non-erasing finite substitution σ' of type τ with $\sigma'(\sigma_\tau(K)) = L$. This is decidable by Lemma 6.1. Such a non-erasing finite substitution σ' exists iff there is a finite substitution σ of type τ with $\sigma(K) = L$. □

7. Conclusions and Next Research Steps

We have shown that the limitedness problem of desert automata is PSPACE-complete. As an application we have seen that the finite substitution problem is decidable. Moreover, we observed that the classes of mappings which are computable by deterministic desert automata, unambiguous desert automata, resp. arbitrary desert automata, form a strict hierarchy. Remarkably, the strategies to show the decidability of the limitedness problems of desert and distance automata follow similar strategies, although the concepts of desert and distance automata are orthogonal.

One aim of this paper is to introduce notions and to develop proof techniques which will be used in [22].

The next step is to develop a general automata concept which includes desert and distance automata as two extremal cases and to solve the limitedness problem of the new automata concept by a fusion of the two underlying Burnside problems and their solutions. This new automata concept allows a new proof and gives an upper complexity bound for the decidability of the star height problem [22].

The author is not sure whether one can prove the main results in a simpler way. The reader should be aware that the proof of Theorem 5.2 is presented in a fashion that promotes the fusion of the solutions of the two Burnside problems in [20]. Concerning a simplification of the proof of Theorem 2.3, there are two interesting observations: in Section 2.4 we compared the finite power problem and the finite generator problem as two particular cases of the limitedness problems for distance, resp. desert, automata. The finite power problem was open for 12 years, but the finite generator problem is rather easy. From this observation, one could conjecture that the limitedness problem for desert automata should be easier than the limitedness problem for distance automata.

In Part (2) of the proof of Theorem 5.2, it would be sufficient to consider the two extremal cases that the sequence $(w_k)_{k\geq 1}$ is either constant or strictly length increasing. The case of a constant sequence is obvious. If $(w_k)_{k\geq 1}$ is strictly length increasing and convergent, then we have $\overline{\Psi}(\alpha(w_k)[i, j])_{k\geq 1} \neq \wedge\!\!\!\wedge$ for every i, j. However, this easy observation does not play a crucial role in the proof of Theorem 5.2.

Finally, we would like to address some open problems on desert automata:

1. One of the most examined problems on distance automata is to find a sharp upper bound on the range of the distance function of limited distance automata [14]–[16], [31], [42], [43]. We do not know such a bound for desert automata.
2. It is undecidable whether two given distance automata compute the same mapping [24], but this problem is open for desert automata.

Acknowledgments

The author acknowledges the discussions with Jean-Éric Pin on the paper. The author thanks the anonymous referees of the present paper and the conference version [21] for their careful work. The author thanks Volker Diekert for the invitation to publish in this special issue.

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Received March 18, 2004, *and in final form May* 6, 2005. *Online publication October* 31, 2005.