

192. No Free Inverse Semigroup is Orderable<sup>\*)</sup>

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A semigroup  $S$  is said to be orderable if  $S$  admits a simple order to make  $S$  a simply ordered semigroup. It is known that every free group is orderable ([1] or [4]). On the other hand, it is shown in [3] that a free commutative idempotent semigroup with  $r$  free generators is orderable if and only if  $r \leq 2$ . In this paper we consider the orderability of free inverse semigroups. Since it is easily shown that a free inverse semigroup with  $r$  free generators contains a free commutative idempotent subsemigroup with  $r$  free generators, it follows from the above result that a free inverse semigroup with a set of free generators consisting of more than three elements is not orderable. But we have the following stronger result.

**Theorem.** *No free inverse semigroup is orderable.*

**Proof.** By way of contradiction we assume that  $S$  is a free inverse semigroup which is at the same time an ordered semigroup. We denote by  $\leq$  the natural partial order on the inverse semigroup  $S$ . Let  $F$  be a set of free generators of  $S$  and let  $x \in F$ . Then, by [5] Corollary 2.5, the inverse subsemigroup  $\langle x \rangle$  generated by  $x$  is a free inverse semigroup with a free generator  $x$ . Also in  $S$  the set  $E$  of idempotents of  $S$  forms a semilattice with respect to the partial order  $\leq$  and, by [6] Theorem 3, the semilattice  $E$  is a tree semilattice. Now we have

$$x^2x^{-2} \leq xx^{-1}, \quad xx^{-2}x \leq xx^{-1}$$

and so  $x^2x^{-2}$  and  $xx^{-2}x$  are comparable. If  $x^2x^{-2} \leq xx^{-2}x$ , then  $x^2x^{-2} \leq xx^{-2}x \leq x^{-1}x$ , which contradicts [5] Corollary 2.4. Next we suppose  $xx^{-2}x \leq x^2x^{-2}$ . Then  $xx^{-2}x = (x^2x^{-2})(xx^{-2}x) = x^2x^{-3}x$  and so  $x^2 = x(xx^{-2}x)x = x^3x^{-3}x^2$ . Hence

$$x^{-2}x^2 = x^{-2}x^3x^{-3}x^2 = (x^{-2}x^2)(xx^{-1})(x^{-2}x^2) \leq xx^{-1},$$

which contradicts again [5] Corollary 2.4.

## References

- [1] G. Birkhoff: Review of "Everett, C. J., and Ulam, S., Ordered groups." Math. Rev., 7, 4 (1946).
- [2] A. H. Clifford and G. B. Preston: The algebraic theory of semigroups, Vol. I. Amer. Math. Soc. (1961); Vol. II. Amer. Math. Soc. (1967).

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<sup>\*)</sup> Dedicated to Professor Kiiti Morita on his 60th birthday.

- [3] Е. Я. Габович: Об упорядочиваемости свободных полугрупп некоторых многообразий. Матем. Заметки. (to appear).
- [4] K. Iwasawa: On linearly ordered groups. J. Math. Soc. Japan, **1**, 1-9 (1948).
- [5] N. R. Reilly: Free generators in free inverse semigroups. Bull. Austral. Math. Soc., **7**, 407-424 (1972).
- [6] T. Saitô: Ordered idempotent semigroups. J. Math. Soc. Japan, **14**, 150-169 (1962).
- [7] H. E. Scheiblich: Free inverse semigroups. Proc. Amer. Math. Soc., **38**, 1-7 (1973).