Linear Equations with Ordered Data

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Abstract

Following a recently considered generalization of linear equations to unordered data vectors, we perform a further generalization to *ordered data* vectors. These generalized equations naturally appear in the analysis of vector addition systems (or Petri nets) extended with ordered data. We show that nonnegative-integer solvability of linear equations is computationally equivalent (up to an exponential blowup) with the reachability problem for (plain) vector addition systems. This high complexity is surprising, and contrasts with NP-completeness for unordered data vectors. Also surprisingly, we achieve polynomial time complexity of the solvability problem when the nonnegative-integer restriction on solutions is dropped.

Keywords Linear equations, Petri nets, Petri nets with data, vector addition systems, sets with atoms, orbit-finite sets

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1 Introduction

Systems of linear equations are useful for approximate analysis of vector addition systems, or Petri nets. For instance, the relaxation of semantics of Petri nets, where the configurations along a run are not required to be nonnegative, yields so called state equation of a Petri net, which is a system of linear equations with nonnegative-integer restriction on solutions. This is equivalent to integer linear programming, a well-known NP-complete problem [12]. If the nonnegative-integer restriction if further relaxed to nonnegative-rational one (or nonnegative-real one), we get a weaker but computationally more tractable approximation, equivalent to linear programming and solvable in polynomial time. We refer to [22] for an exhaustive overview of linear-algebraic and integer-linear-programming techniques in analysis of Petri nets; usefulness of these techniques is confirmed by multiple applications including, for instance, recently proposed efficient tools for the coverability problem of Petri nets [1, 9].

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Motivations. A starting point for this paper is an extension of the model of Petri nets, or vector addition systems, with data [10, 16]. This is a powerful extension of the model, which significantly enhances its expressibility but also increases the complexity of analysis. In case of unordered data (a countable set of data values that can be tested for equality only), the coverability problem is decidable (in non-elementary complexity) [16] but the decidability status of the reachability problem remains still open. In case of ordered data, the coverability problem is still decidable while reachability is undecidable. (Petri nets with ordered data are equivalent to timed Petri nets, as shown in [5].) One can also consider other data domains, and the coverability problem remains decidable as long as the data domain is homogeneous [15] (not to be confused with homogeneous systems of linear equations), but always in non-elementary complexity. In view of these high complexities, a natural need arises for efficient over-approximations.

A configuration of a Petri net with data domain $\mathbb D$ is a nonnegative integer $data\ vector$, i.e., a function $\mathbb D\to\mathbb N^d$ that maps only finitely many data values to a non-zero vector in $\mathbb N^d$. In a search for efficient over-approximations of Petri nets with data, a natural question appears: Can linear algebra techniques be generalised so that the role of vectors is played by data vectors? In case of unordered data, this question was addressed in [11], where first promising results has been shown, namely the nonnegative-integer solvability of linear equations over unordered data domain is NP-complete. Thus, for unordered data, the problem remains within the same complexity class as its plain (data-less) counterpart. The same question for the second most natural data domain, i.e. ordered data, seems to be even more important; ordered data enables modeling features like fresh names creation or time dependencies.

Contributions. In this paper we do a further step and investigate linear equations with ordered data, for which we fully characterise the complexity of the solvability problem. Firstly, we show that nonnegative-integer solvability of linear equations is computationally equivalent (up to an exponential blowup) with the reachability problem for plain Petri nets (or vector addition systems). This high complexity is surprising, and contrasts with NP-completeness for unordered data vectors. Secondly, we prove, also surprisingly, that the complexity of the solvability problem drops back to polynomial time, when the nonnegative-integer restriction on solutions is relaxed to nonnegative-rational, integer, or rational. Thirdly, we offer a conceptual contribution and notice that systems of linear equations with (unordered or) ordered data are a special case of systems of linear equations which are infinite but finite up to an automorphism of data domain. This recalls the setting of sets with atoms [3, 4, 13], with a data domain being a parameter, and the notion of orbit-finiteness relaxing the classical notion of finiteness.

Outline. In Section 2 we introduce the setting we work in, and formulate our results. Then the rest of the paper is devoted to proofs.

First, in Section 3 we provide a lower bound for the nonnegative-integer solvability problem, by a reduction from the VAS reachability problem. Then, in Section 4 we suitably reformulate our problem in terms *multihistograms*, which are matrices satisfying certain combinatorial property. This reformulation is used In the next Section 5 to provide a reduction from the nonnegative-integer solvability problem to the reachability problem of vector addition systems, thus proving decidability of our problem. Finally, in Section 6 we investigate various relaxations of the nonnegative-integer restriction on solutions and work out a polynomial-time decision procedure in each case. In the concluding Section 7 we sketch upon a generalised setting of orbit-finite systems of linear equations.

2 Vector addition systems and linear equations

In this section we introduce the setting of linear equations with data, and formulate our results. For a gentle introduction of the setting, we start by recalling classical linear equations.

Let $\mathbb Q$ denote the set of rationals, and $\mathbb Q_+,\mathbb Z$, and $\mathbb N$ denote the subsets of nonnegative rationals, integers, and nonnegative integers. Classical linear equations are of the form

$$a_1x_1+\ldots a_mx_m=a,$$

where $x_1 \ldots x_m$ are variables (unknowns), and $a_1 \ldots a_m \in \mathbb{Q}$ are rational coefficients. For a system \mathcal{U} of such equations over the same variables x_1, \ldots, x_m , a solution of \mathcal{U} is a vector $(n_1, \ldots, n_m) \in \mathbb{Q}^m$ such that the valuation $x_1 \mapsto n_1, \ldots, x_m \mapsto n_m$ satisfies all equations in \mathcal{U} . In the sequel we are most often interested in nonnegative integer solutions $(n_1, \ldots, n_m) \in \mathbb{N}^m$, but one may consider also other solution domains than \mathbb{N} . It is well known that the nonnegative-integer solvability problem (\mathbb{N} -solvability problem) of linear equations, i.e. the question whether \mathcal{U} has a nonnegative-integer solution, is NP-complete [6]. The complexity remains the same for other natural variants of this problem, for instance for inequalities instead of equations (a.k.a. integer linear programming). On the other hand, for any $\mathbb{X} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{Q}_+\}$, the \mathbb{X} -solvability problem, i.e., the question whether \mathcal{U} has a solution $(n_1, \ldots, n_m) \in \mathbb{X}^m$, is decidable in polynomial time.

Remark 2.1 (integer coefficients). A system of linear equations with rational coefficient can be transformed in polynomial time to a system of linear equations with integer coefficients, while preserving the set of solutions. Thus from now on we allow only for integer coefficients $a_1 \ldots a_m$ in linear equations.

The \mathbb{X} -solvability problem is equivalently formulated as follows: for a given finite set of coefficient vectors $A = \{\mathbf{a}_1, \dots, \mathbf{a}_m\} \subseteq_{\mathrm{fin}} \mathbb{Z}^d$ and a target vector $\mathbf{a} \in \mathbb{Z}^d$ (we use bold fonts to distinguish vectors from other elements), check whether \mathbf{a} is an \mathbb{X} -sum of A, i.e., a sum of the following form

$$\mathbf{a} = \sum_{i=1}^{m} c_i \cdot \mathbf{a}_i,\tag{1}$$

for some $c_1, \ldots, c_m \in \mathbb{X}$. The number d corresponds to the number of equations in \mathcal{U} and is called the *dimension* of \mathcal{U} .

Linear equations may serve as an over-approximation of the reachability set of a Petri net, or equivalently, of a *vector addition* system – we prefer to work with the latter model. A vector addition system (VAS) $\mathcal{V} = (A, \mathbf{i}, \mathbf{f})$ is defined, similarly as above, by a finite set of vectors $A \subseteq_{\text{fin}} \mathbb{Z}^d$ together with two nonnegative vectors $\mathbf{i}, \mathbf{f} \in \mathbb{N}^d$, the initial one and the final one. The set A determines

a transition relation \longrightarrow between configurations, which are non-negative integer vectors $\mathbf{c} \in \mathbb{N}^d$: there is a transition $\mathbf{c} \longrightarrow \mathbf{c}'$ if $\mathbf{c}' = \mathbf{c} + \mathbf{a}$ for some $\mathbf{a} \in A$. The VAS reachability problems asks, whether the final configuration is reachable from the initial one by a sequence of transitions, i.e. $\mathbf{i} \longrightarrow^* \mathbf{f}$. It is important to stress that intermediate configurations are required to be nonnegative. In other words, the reachability problem asks whether there is a sequence $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m \in A$ (called a run) such that

$$\mathbf{i} + \sum_{i=1}^{m} \mathbf{a}_i = \mathbf{f}$$
 $\mathbf{i} + \sum_{i=1}^{j} \mathbf{a}_i \ge \mathbf{0}$, for every $j \in \{1 \dots m\}$

where **0** denotes a zero vector (its length will be always clear from the context). The problem is decidable [14, 19] and ExpSpace-hard [18], and nothing is known about complexity except for the cubic Ackermann upper bound of [17]. For a given VAS, a necessary condition for reachability is that $\mathbf{f} - \mathbf{i} = \sum_{i=1}^m \mathbf{a_i}$, which is equivalent to \mathbb{N} -solvability of a system of linear equations, called (in case of Petri nets) the *state equation*. For further details we refer the reader to an exhaustive overview of linear-algebraic approximations for Petri nets [22], where both \mathbb{N} - and \mathbb{Q}_+ -solvability problems are considered.

2.1 Vector addition systems and linear equations, with ordered data

The model of VAS, and linear equations, can be naturally extended with data. In this paper we assume that the data domain $\mathbb D$ is a countable set, ordered by a dense total order \leq with no minimal nor maximal element. Thus, up to isomorphism, $(\mathbb D, \leq)$ is rational numbers with the natural ordering. Elements of $\mathbb D$ we call *data values*. In the sequel we use order preserving permutations (called *data permutations* in short) of $\mathbb D$, i.e. bijections $\rho: \mathbb D \to \mathbb D$ such that $x \leq y$ implies $\rho(x) \leq \rho(y)$.

A *data vector* is a function $\mathbf{v}:\mathbb{D}\to\mathbb{Q}^d$ such that the *support*, i.e. the set $supp(\mathbf{v})\stackrel{\text{def}}{=}\{\alpha\in\mathbb{D}\mid\mathbf{v}(\alpha)\neq\mathbf{0}\}$, is finite (similarly as for vectors, we use bold fonts to distinguish data vectors from other elements). The vector addition + is lifted to data vectors pointwise, so that $(\mathbf{v}+\mathbf{w})(\alpha)\stackrel{\text{def}}{=}\mathbf{v}(\alpha)+\mathbf{w}(\alpha)$. A data vector \mathbf{v} is *nonnegative* if $\mathbf{v}:\mathbb{D}\to(\mathbb{Q}_+)^d$, and \mathbf{v} is *integer* if $\mathbf{v}:\mathbb{D}\to\mathbb{Z}^d$.

Writing \circ for function composition, we see that $\mathbf{v} \circ \rho$ is a data vector for any data vector \mathbf{v} and any order preserving data permutation $\rho: \mathbb{D} \to \mathbb{D}$. For a set V of data vectors we define

Perm
$$(V) = \{ \mathbf{v} \circ \rho \mid \mathbf{v} \in V, \rho \text{ a data permutation} \}.$$

A data vector \mathbf{x} is said to be a *permutation sum* of a finite set of data vectors V if there are $\mathbf{v_1}, \ldots, \mathbf{v_m} \in \text{Perm}(V)$, not necessarily pairwise different, such that $\mathbf{x} = \sum_{i=1}^m \mathbf{v_i}$. In the generalisations of the classical solvability problem, to be defined now, we allow as input only integer data vectors (cf. Remark 2.1):

PERMUTATION SUM PROBLEM.

Input: a finite set V of integer data vectors and an integer data vector \mathbf{x} .

Output: is \mathbf{x} a permutation sum of V?

In the special case when the supports of \mathbf{x} and all vectors in V are all singletons, the Permutation sum problem is just \mathbb{N} -solvability of linear equations and thus the Permutation sum problem is trivially NP-hard. As the first main result, we prove the following inter-reducibility:

Theorem 2.1. The Permutation sum problem and the VAS reachability problem are inter-reducible, with an exponential blowup.

Our setting generalises the setting of *unordered* data, where the data domain $\mathbb D$ is *not* ordered, and hence data permutations are all bijections $\mathbb D \to \mathbb D$. In the case of unordered data the Permutation sum problem is NP-complete, as shown in [11]. The increase of complexity caused by the order in data is thus remarkable.

Similarly as linear equations in the data-less setting, PERMUTA-TION SUM PROBLEM may be used as an overapproximation of the reachability in vector addition systems with ordered data, which are defined exactly as ordinary VAS but in terms of data vectors instead of ordinary vectors. A VAS with ordered data $\mathcal{V} = (V, \mathbf{i}, \mathbf{f})$ consists of $V \subseteq_{\text{fin}} \mathbb{D} \to \mathbb{Z}^d$ a finite set of integer data vectors, and the initial and final nonnegative integer data vectors $\mathbf{i}, \mathbf{f} \in \mathbb{D} \to \mathbb{N}^d$. The configurations are nonnegative integer data vectors, and the set V induces a transition relation between configurations as follows: $\mathbf{c} \longrightarrow \mathbf{c}'$ if $\mathbf{c}' = \mathbf{c} + \mathbf{v}$ for some $\mathbf{v} \in Perm(V)$. The reachability problem asks whether the final configuration is reachable from the initial one by a sequence of transitions, $\mathbf{i} \longrightarrow^* \mathbf{f}$; it is undecidable [16]. (The decidability status of the reachability problem for VAS with unordered data is unknown.) As long as reachability is concerned, VAS with (un)ordered data are equivalent to Petri nets with (un)ordered data [10].

The Permutation sum problem is easily generalised to other domains $\mathbb{X} \subseteq \mathbb{Q}$ of solutions. To this end we introduce scalar multiplication: for $c \in \mathbb{Q}$ and a data vector \mathbf{v} we put $(c \cdot \mathbf{v})(\alpha) \stackrel{\text{def}}{=} c\mathbf{v}(\alpha)$. A data vector \mathbf{x} is said to be a \mathbb{X} -permutation sum of a finite set of data vectors V if there are $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \text{Perm}(V)$, not necessarily pairwise different, and coefficients $c_1, c_2 \ldots c_m \in \mathbb{X}$ such that (cf. (1))

$$\mathbf{x} = \sum_{i=1}^{m} c_i \cdot \mathbf{v_i}.$$

This leads to the following version of Permutation sum problem parametrised by the choice of solution domain \mathbb{X} :

X-Permutation sum problem.

Input: a finite set V of integer data vectors and an integer data vector \mathbf{x} .

Output: is **x** an \mathbb{X} -permutation sum of V?

The Permutation sum problem is a particular case, for $\mathbb{X}=\mathbb{N}.$ Our second main result is the following:

Theorem 2.2. For any $\mathbb{X} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{Q}_+\}$, the \mathbb{X} -Permutation sum problem is in PTIME.

For $\mathbb{X} \in \{\mathbb{Z}, \mathbb{Q}\}$, the above theorem is a direct consequence of a more general fact, where \mathbb{Q} or \mathbb{Z} is replaced by any commutative ring \mathbb{R} , under a proviso that data vectors are defined in a more general way, as finitely supported functions $\mathbb{D} \to \mathbb{R}^d$. With this more general notion, we prove that the \mathbb{R} -Permutation sum problem reduces in polynomial time to the \mathbb{R} -solvability of linear equations with coefficients from \mathbb{R} (cf. Theorem 6.6 in Section 6.2).

The case $\mathbb{X}=\mathbb{Q}_+$ in Theorem 2.2 is more involved but of particular interest, as it recalls continuous Petri nets [8, 21] where fractional firing of a transition is allowed, and leads to a similar elegant theory and efficient algorithms based on \mathbb{Q}_+ -solvability of

linear equations. Moreover, faced with the high complexity of Theorem 2.1, it is expected that Theorem 2.2 may become a cornerstone of linear-algebraic techniques for VAS with ordered data.

3 Lower bound for the PERMUTATION SUM PROBLEM

In this section, all data vectors are silently assumed to be integer data vectors. We are going to show a reduction from the VAS reachability problem to the Permutation sum problem. Fix a VAS $\mathcal{A} = (A, \mathbf{i}, \mathbf{f})$. We are going to define a set of data vectors V and a target data vector \mathbf{x} such that the following conditions are equivalent:

- 1. **f** is reachable from **i** in \mathcal{A} ;
- 2. \mathbf{x} is a permutation sum of V.

W.l.o.g. assume f = 0.

We need some auxiliary notation. First, note that every integer vector $\mathbf{a} \in \mathbb{Z}^d$ is uniquely presented as a difference $\mathbf{a} = \mathbf{a}^+ - \mathbf{a}^-$ of two nonnegative vectors $\mathbf{a}^+ \in \mathbb{N}^d$ and $\mathbf{a}^- \in N^d$ defined as follows:

$$a^{+}(i) = \begin{cases} a(i), & \text{if } a(i) \ge 0 \\ 0, & \text{if } a(i) < 0 \end{cases} \qquad a^{-}(i) = \begin{cases} -a(i), & \text{if } a(i) \le 0 \\ 0, & \text{if } a(i) > 0 \end{cases}$$

For a nonnegative vector $\mathbf{a} \in \mathbb{N}^d$, by a *data spread* of \mathbf{a} we mean any nonnegative integer data vector $\mathbf{v} : \mathbb{D} \to \mathbb{N}^d$ such that

$$\sum_{\alpha \in supp(\mathbf{v})} \mathbf{v}(\alpha) = \mathbf{a}.$$

In words, for every coordinate i, the value $\mathbf{a}(i) \geq 0$ is spread among all values $\mathbf{v}(\alpha)(i)$, for all data values $\alpha \in \mathbb{D}$; clearly, \mathbf{v} is finitely supported.

The rough idea of the reduction is to simulate every transition $\mathbf{a} \in A$ by a data spread of a such that, intuitively, all positive numbers in a use larger data values than all negative values. By a data realization of a vector $\mathbf{a} \in A$ we mean any data vector of the form $\mathbf{v} = \mathbf{s}^+ - \mathbf{s}^-$, where data vector \mathbf{s}^- is a data spread of \mathbf{a}^- , data vector \mathbf{s}^+ is a data spread of \mathbf{a}^+ , and $supp(\mathbf{s}^-) < supp(\mathbf{s}^+)$ (with the meaning that every element of $supp(\mathbf{s}^-)$ is smaller than every element of $supp(\mathbf{s}^+)$). Intuitively, the effect of \mathbf{v} is like the effect of \mathbf{a} but additionally data values involved are increased. We will shortly write $supp^+(\mathbf{v})$ for $supp(\mathbf{s}^+)$ and $supp^-(\mathbf{v})$ for $supp(\mathbf{s}^-)$. Clearly, a non-zero vector \mathbf{a} has infinitely many different data realizations; on the other hand, there are only finitely many of them up to data permutation. Let $V_{\mathbf{a}}$ be a set of data realizations of a containing representatives up to data permutation. The cardinality of $V_{\mathbf{a}}$ is exponential with respect to the size of \mathbf{a} .

Now we are ready to define V and \mathbf{x} : we put $V = \bigcup_{\mathbf{a} \in A} V_{\mathbf{a}}$, and as the target vector \mathbf{x} we take $\mathbf{x} = -\mathbf{i}$, for some arbitrary data spread \mathbf{i} of \mathbf{i} (recall that $\mathbf{f} = \mathbf{0}$).

It remains to prove the equivalence of conditions 1. and 2. First, 1. easily implies 2. as every run of $\mathcal A$ can be transformed into a permutation sum of V that sums up to $\mathbf x$, using suitable data realisations of the vectors used in the run.

For the converse implication, suppose that $\mathbf{x} = \sum_{i=1}^{n} \mathbf{w}_{i}$, where $\mathbf{w}_{i} = \mathbf{v}_{i} \circ \theta_{i}$ and $\mathbf{v}_{i} \in V$. By construction of V, for every $i \leq n$ the data vector \mathbf{v}_{i} belongs to $V_{\mathbf{a}_{i}}$ for some $\mathbf{a}_{i} \in A$. We claim that the multiset of vectors $\{\mathbf{a}_{i}\}_{i=1}^{n}$ can be arranged into a sequence being a correct run of the VAS \mathcal{A} from \mathbf{i} to \mathbf{f} . For this purpose we define a binary relation of *immediate consequence* on data vectors

 \mathbf{w}_i : we say that \mathbf{w}_j is an immediate consequence of \mathbf{w}_i if the intersection of $supp^+(\mathbf{w}_i)$ and $supp^-(\mathbf{w}_j)$ is non-empty. We observe that the reflexive-transitive closure of the immediate consequence is a partial order. Indeed, antisymmetry follows due to the fact that all data vectors \mathbf{w}_i satisfy $supp^-(\mathbf{w}_i) < supp^+(\mathbf{w}_i)$. Let \leq denote an arbitrary extension of the partial order to a total order, and suppose w.l.o.g. that

$$\mathbf{w}_1 \prec \mathbf{w}_2 \prec \ldots \prec \mathbf{w}_n$$
.

We should prove that the corresponding sequence $\mathbf{a}_1\mathbf{a}_2\ldots\mathbf{a}_n$ of vectors from A is a correct run of the VAS $\mathcal A$ from $\mathbf i$ to $\mathbf f$. This will follow, once we demonstrate that the sequence $\mathbf w_1\mathbf w_2\ldots\mathbf w_n$ is a correct run in the VAS with ordered data with transitions V and the initial configuration $\mathbf i$. We need to prove the data vector $\mathbf u_i=\bar{\mathbf i}+\sum_{i=1}^J\mathbf w_i$ is nonnegative for every $j\in\{0,\ldots,n\}$. To this aim fix $\alpha\in\mathbb D$ and $l\in\{1,\ldots,d\}$, and consider the sequence of numbers

$$\mathbf{u}_0(\alpha, l), \quad \mathbf{u}_1(\alpha, l), \quad \dots \quad \mathbf{u}_n(\alpha, l)$$
 (2)

appearing as the value of the consecutive data vectors $\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_n$ at data value α and coordinate l. We know that the first element of the sequence $\mathbf{u}_0(\alpha,l)=\mathbf{i}(\alpha,l)\geq 0$ and the last element of the sequence $\mathbf{u}_n(\alpha,l)=\mathbf{f}(\alpha,l)\geq 0$. Furthermore, by the definition of the ordering \leq we know that the sequence (2) is first non-decreasing, and then non-increasing. These conditions imply nonnegativeness of all numbers in the sequence.

Remark 3.1. The exponential blowup in the reduction is caused only by binary encoding of numbers in vector addition systems; it can be avoided if numbers are assumed to be encoded in unary or, equivalently, if instead of vector addition systems one uses counter machines without zero tests.

4 Histograms

The purpose of this section is to transform the Permutation sum problem to a more manageable form. As the first step, we eliminate data by rephrasing the problem in terms of matrices. Then, we distinguish matrices with certain combinatorial property, called *histograms*, and use them to further simplify the problem. In Lemma 4.9 at the end of this section we provide a final characterisation of the problem, using *multihistograms*. The characterisation will be crucial for effectively solving the Permutation sum problem in the following Section 5.

In this section, all matrices are integer matrices, and all data vectors are integer data vectors.

Eliminating data. Rational matrices with r rows and c columns we call $r \times c$ -matrices, and r (resp. c) we call row (resp. column) dimension of an $r \times c$ -matrix. We are going to represent any data vector \mathbf{v} as a $d \times |supp(\mathbf{v})|$ -matrix $M_{\mathbf{v}}$ as follows: if $supp(\mathbf{v}) = \{\alpha_1 < \alpha_2 < \ldots < \alpha_n\}$, we put

$$M_{\mathbf{v}}(i,j) \stackrel{\text{def}}{=} \mathbf{v}(i)(\alpha_i).$$

A 0-extension of an $r \times c$ -matrix M is any $r \times c'$ -matrix M', $c' \ge c$, obtained from M by inserting arbitrarily c' - c additional zero columns $\mathbf{0} \in \mathbb{Z}^r$. Thus row dimension is preserved by 0-extension, and column dimension may grow arbitrarily. We denote by 0-ext(M) the (infinite) set of all 0-extensions of a matrix M. In particular, $M \in 0$ -ext(M). For a set M of matrices we denote by 0-ext(M) the set of all 0-extensions of all matrices in M.

Example 4.1. For a data vector \mathbf{v} with support $supp(\mathbf{v}) = \{\alpha_1 < \alpha_2\}$, defined by $\mathbf{v}(\alpha_1) = (1, 3, 0) \in \mathbb{Z}^3$ and $\mathbf{v}(\alpha_2) = (2, 0, 2) \in \mathbb{Z}^3$, here is the corresponding matrix and two its exemplary 0-extension:

$$M_{\mathbf{v}} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 2 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \in 0\text{-ext}(M_{\mathbf{v}})$$

Below, whenever we add matrices we silently assume that they have the same row and column dimensions. For a finite set \mathcal{M} of matrices, we say that a matrix N is a sum of 0-extensions of \mathcal{M} if

$$N = \sum_{i=1}^{m} M_i \tag{3}$$

for some matrices $M_1, \ldots, M_m \in 0$ -ext (\mathcal{M}) , necessarily all of the same row and column dimension. We claim that the Permutation sum problem is equivalent to the question whether some 0-extension of a given matrix X is a sum of 0-extensions of \mathcal{M} .

Up to 0-extension sum problem.

Input: a finite set \mathcal{M} of matrices, and a target matrix X, all of the same row dimension d.

Output: is some 0-extension of X a sum of 0-extensions of \mathcal{M} ?

Lemma 4.2. The Permutation sum problem is polynomially equivalent to the Up to 0-extension sum problem.

Proof. We describe the reduction of Permutation sum problem to the Up to 0-extension sum problem. (The opposite reduction is shown similarly and is omitted here.)

Given an instance \mathbf{x}, V of the former problem, we define the instance

$$X = M_{\mathbf{x}}, \quad \mathcal{M} = \{M_{\mathbf{v}} \mid \mathbf{v} \in V\}$$

of the latter one. We need to show that \mathbf{x} is permutation sum of V if, and only if some 0-extension N of X is a sum of 0-extensions of M. In one direction, suppose \mathbf{x} is a permutation sum of V, i.e.,

$$\mathbf{x} = \sum_{i=1}^{m} \mathbf{v}_i \circ \rho_i \tag{4}$$

and let $\{\alpha_1 < \ldots < \alpha_c\}$ be the union of all supports of data vectors $\mathbf{v}_i \circ \rho_i$ (thus also necessarily including the support of \mathbf{x}). We will define a matrix N and matrices M_1, \ldots, M_m , as required in (3), all of the same column dimension c. Thus their columns will correspond to data values $\alpha_1, \ldots, \alpha_c$. Let N be the unique 0-extension of $M_{\mathbf{x}}$ of column dimension c so that the nonempty columns are exactly those corresponding to element of $supp(\mathbf{x})$. Similarly, let M_i be the unique 0-extension of $\mathbf{v}_i \circ \rho_i$ of column dimension c, whose nonempty columns correspond to elements of $supp(\mathbf{v}_i \circ \rho_i)$. The so defined matrices satisfy the equality (3).

In the other direction, suppose the equality (3) holds for some matrices $N \in 0\text{-ext}(M_{\mathbf{x}})$ and $M_1 \in 0\text{-ext}(M_{\mathbf{v}_1}) \dots M_m \in 0\text{-ext}(M_{\mathbf{v}_m})$, and let c be their common column dimension. Choose arbitrary c data values $\alpha_1 < \alpha_2 < \dots < \alpha_c$ so that $supp(\mathbf{x}) \subseteq \{\alpha_1, \dots, \alpha_c\}$ corresponds to nonempty columns of N, and define data permutations $\rho_1 \dots \rho_m$ so that ρ_i maps the support of \mathbf{v}_i to data values corresponding to nonempty columns in M_i . One easily verifies that (4) holds, as required.

From now on we concentrate on solving the UP to 0-extension sum problem.

Histograms. We write briefly $\sum H(i,1\ldots j)$ as a shorthand for $\sum_{1\leq l\leq j} H(i,l)$. In particular, $\sum H(i,1\ldots 0)=0$ by convention. An integer matrix we call nonnegative if it only contains nonnegative integers. Histograms, to be defined now, are an extension of histograms of [11] to ordered data.

Definition 4.3. A nonnegative integer $r \times c$ -matrix H we call a *histogram* if the following conditions are satisfied:

- there is s > 0 such that $\sum H(i, 1, ..., c) = s$ for every $1 \le i \le r$; s is called the *degree* of H;
- for every $1 \le i < r$ and $0 \le j < c$, the inequality holds:

$$\sum H(i,1\ldots j) \geq \sum H(i+1,1\ldots j+1).$$

Note that the definition enforces $r \leq c$, i.e., the column dimension c of a histogram is at least as large as its row dimension r. Indeed, forcedly

$$H(2,1) = 0$$

 $H(3,1) = H(3,2) = 0$
...
 $H(r,1) = ... = H(r,r-1) = 0$.

Histograms of degree 1 we call simple histograms.

Example 4.4. A histogram of degree 2 decomposed as a sum of two simple histograms:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The following combinatorial property of histograms will be crucial in the sequel:

Lemma 4.5. H is a histogram of degree s if, and only if H is a sum of s simple histograms.

Below, whenever we multiply matrices we silently assume that the column dimension of the first one is the same as the row dimension of the second one. Simple histograms are useful for characterising 0-extensions:

Lemma 4.6. For matrices N and M, $N \in 0$ -ext(M) if, and only if $N = M \cdot S$, for a simple histogram S.

Example 4.7. Recall the matrix $M = M_v$ from Example 4.1. One of the matrices from 0-ext(M) is presented as multiplication of M and a simple histogram as follows:

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \quad = \quad \quad \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

We use Lemmas 4.5 and 4.6 to characterise the UP to 0-extension sum problem:

Lemma 4.8. For a matrix N and a finite set of matrices M, the following conditions are equivalent:

- 1. N is a sum of 0-extensions of \mathcal{M} ;
- 2. $N = \sum_{M \in \mathcal{M}} M \cdot H_M$, for some histograms $\{H_M \mid M \in \mathcal{M}\}$.

Proof. In one direction, assume condition 1. holds, i.e.,

$$N = \sum_{l} N_{l} \tag{5}$$

for $N_l \in 0\text{-ext}(M_l)$, $M_l \in \mathcal{M}$, and then apply Lemma 4.6 to get (simple) histograms H_l with $N_l = M_l \cdot H_l$. Thus $N = \sum_l M_l \cdot H_l$. Now apply the if direction of Lemma 4.5 to get the histograms H_M as required in condition 2. In the other direction, assume condition 2. holds, and use the only if direction of Lemma 4.5 to decompose every H_l into simple histograms. This yields

$$N = \sum_{l} M_l \cdot S_l,$$

where all $M_l \in \mathcal{M}$ all S_l are simple histograms. Finally we apply Lemma 4.6 to get matrices N_l satisfying (5). This completes the proof.

Multihistograms. Using Lemma 4.8 we are now going to work out our final characterisation of the Up to 0-extension sum problem, as formulated in Lemma 4.9 below. We write $H(i, _)$ and $H(_, j)$ for the i-th row and the j-th column of a matrix H, respectively. For an indexed family $\{H_1, \ldots, H_k\}$ of matrices, its j-th column is defined as the indexed family of j-th columns of respective matrices $\{H_1(_, j), \ldots, H_k(_, j)\}$.

Fix an input of the UP to 0-extension sum problem: a matrix X and a finite set $\mathcal{M}=\{M_1,\ldots,M_k\}$ of matrices, all of the same row dimension d. Let c_l stand for the column dimension of M_l . Suppose that some $N\in 0\text{-ext}(X)$ and some family $\mathcal{H}=\{H_1,\ldots,H_k\}$ of histograms satisfy

$$N = M_1 \cdot H_1 + \ldots + M_k \cdot H_k.$$

(The row dimension of every H_l is necessarily c_l .) Boiling down the equation to a single entry of N we get a linear equation:

$$N(i,j) = M_1(i,_) \cdot H_1(_,j) + \ldots + M_k(i,_) \cdot H_k(_,j).$$

By grouping all the equations concerning all entries of a single column $N(_,j) \in \mathbb{Z}^d$ of N we get a system of d (= row dimension of N) linear equations:

$$N(\underline{\hspace{0.1cm}},j) = M_1 \cdot H_1(\underline{\hspace{0.1cm}},j) + \dots + M_k \cdot H_k(\underline{\hspace{0.1cm}},j)$$
$$= \left[M_1 \mid \dots \mid M_k \right] \cdot \begin{bmatrix} H_1(\underline{\hspace{0.1cm}},j) \\ \dots \\ H_k(\underline{\hspace{0.1cm}},j) \end{bmatrix}$$

Therefore, the j-th column of \mathcal{H} , treated as a single column vector of length $s = c_1 + \ldots + c_k$, is a nonnegative-integer solution of a system of d linear equations $\mathcal{U}_{\mathcal{M}, N(_, j)}$, with s unknowns $x_1 \ldots x_s$, of the form:

$$N(\underline{},j) = \begin{bmatrix} M_1 \mid \ldots \mid M_k \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_s \end{bmatrix},$$

Observe that the system $\mathcal{U}_{\mathcal{M},N(_,j)}$ depends on \mathcal{M} and $N(_,j)$ but not on j. For succinctness, for $\mathbf{a}\in\mathbb{Z}^d$ we put

$$C_{\mathbf{a}} := \mathbb{N}\text{-sol}(\mathcal{U}_{M,\mathbf{a}})$$
 (6)

to denote the set of all nonnegative-integer solutions of $\mathcal{U}_{\mathcal{M},\mathbf{a}}$. Therefore, every jth column of the multihistogram \mathcal{H} belongs to $C_{N(_,j)}$.

Now recall that $N \in 0$ -ext(X). Therefore, treating \mathcal{H} as a sequence of its column vectors in \mathbb{N}^c (we call this sequence *the word*

of \mathcal{H}), we arrive at the condition that this sequence belongs to the following language:

$$(C_0)^* C_{X(.,1)} (C_0)^* C_{X(.,2)} \dots (C_0)^* C_{X(.,n)} (C_0)^*$$
 (7)

where n denotes the column dimension of X. If this is the case, we say that \mathcal{H} is an (X, \mathcal{M}) -multihistogram. As the reasoning above is reversible, we have thus shown:

Lemma 4.9. The Up to 0-extension sum problem is equivalent to the following one:

Multihistogram problem.

Input: a finite set \mathcal{M} of matrices, and a matrix X, all of the same row dimension d.

Output: does there exist an (X, \mathcal{M}) -multihistogram?

5 Upper bound for the Permutation sum PROBLEM

We reduce in this section the Multihistogram problem (and hence also the Permutation sum problem, due to Lemmas 4.2 and 4.9) to the VAS reachability problem (with single exponential blowup), thus obtaining decidability. Fix in this section an input to the Multihistogram problem: a matrix X (of column dimension n) and a finite set $\mathcal{M} = \{M_1, \ldots, M_k\}$ of matrices, all of the same row dimension d. We perform in two steps: we start by proving an effective exponential bound on vectors appearing as columns of (X, \mathcal{M}) -multihistograms; then we construct a VAS whose runs correspond to the words of exponentially bounded (X, \mathcal{M}) -multihistograms. For measuring the complexity we assume that all numbers in X and \mathcal{M} are encoded in binary.

Exponentially bounded multihistograms. We need to recall first a characterisation of nonnegative-integer solution sets of systems of linear equations as (effectively) exponentially bounded hybrid-linear sets, i.e., of the form $B+P^{\oplus}$, for $B,P\subseteq\mathbb{N}^k$, where k is the number of variables and P^{\oplus} stands for the set of all finite sums of vectors from P (see e.g. [6] (Prop. 2), [7], [20]). By $\mathcal{U}_{M,\mathbf{a}}$ denote a system of linear equations determined by a matrix M and a column vector \mathbf{a} , and by $\mathcal{U}_{M,\mathbf{0}}$ the corresponding *homogeneous* systems of linear equations. Again, for measuring the size $|\mathcal{U}_{M,\mathbf{a}}|$ of $\mathcal{U}_{M,\mathbf{a}}$ we assume that all numbers in M and \mathbf{a} are encoded in binary.

Lemma 5.1 ([6] Prop. 2). \mathbb{N} -sol($\mathcal{U}_{M,\mathbf{a}}$) = $B + P^{\oplus}$, where $B, P \subseteq \mathbb{N}^k$ such that all vectors in $B \cup P$ are bounded exponentially w.r.t. $|\mathcal{U}_{M,\mathbf{a}}|$ and $P \subseteq \mathbb{N}$ -sol($\mathcal{U}_{M,\mathbf{0}}$).

We will use Lemma 5.1 together with the following operation on multihistograms. A *j-smear* of a histogram H is any nonnegative matrix H' obtained by replacing j-th column $H(_,j)$ of H by two columns that sum up to $H(_,j)$. Here is an example (j=5):

$$\begin{bmatrix} 3 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Formally, a j-smear of H is any nonnegative matrix H' satisfying:

$$H'(_,l) = H(_,l)$$
 for $l < j$
 $H'(_,j) + H'(_,j+1) = H(_,j)$
 $H'(_,l+1) = H(_,l)$ for $l > j$.

One easily verifies that smear preserves the defining condition of histogram:

Claim 5.1. A smear of a histogram is a histogram.

Finally, a j-smear of a family of matrices $\{H_1, \ldots, H_k\}$ is any indexed family of matrices $\{H'_1, \ldots, H'_k\}$ obtained by applying a j-smear simultaneously to all matrices H_l . We omit the index j when it is irrelevant.

So prepared, we claim that every (X, \mathcal{M}) -multihistogram $\mathcal{H} = \{H_1, \ldots, H_k\}$ can be transformed by a number of smears into an (X, \mathcal{M}) -multihistogram containing only numbers exponentially bounded with respect to X, \mathcal{M} . Indeed, recall (7) and let

$$N = \sum_{l=1,\ldots,k} M_l \cdot H_l \in 0\text{-ext}(X).$$

Take an arbitrary (say j-th) column $\mathbf{w} \in C_{\mathbf{a}}$ of \mathcal{H} (recall (6)), where $\mathbf{a} = N(_, j)$, treated as a single column vector $\mathbf{w} \in \mathbb{N}^s$ (for s the sum of row dimensions of H_1, \ldots, H_k), and present it (using Lemma 5.1) as a sum

$$\mathbf{w} = \mathbf{b} + \mathbf{p}_1 + \ldots + \mathbf{p}_m,$$

for some exponentially bounded $\mathbf{b} \in C_{\mathbf{a}}$ and $\mathbf{p}_1, \ldots, \mathbf{p}_m \in C_0$. Apply smear m times, replacing the j-th column by m+1 columns $\mathbf{b}, \mathbf{p}_1, \ldots, \mathbf{p}_m$. As \mathbf{b} is a solution of the system $\mathcal{U}_{M,\mathbf{a}}$ and every \mathbf{p}_l is a solution of the homogeneous system $\mathcal{U}_{M,\mathbf{0}}$,

$$\begin{bmatrix} M_1 \mid \dots \mid M_k \end{bmatrix} \cdot \mathbf{b} = \begin{bmatrix} M_1 \mid \dots \mid M_k \end{bmatrix} \cdot \mathbf{w}$$
$$\begin{bmatrix} M_1 \mid \dots \mid M_k \end{bmatrix} \cdot \mathbf{p}_l = \mathbf{0},$$

the so obtained family $\mathcal{H}' = \{H'_1, \dots, H'_k\}$ still satisfies the condition $\sum_{l=1,\dots,k} M_l \cdot H'_l \in 0\text{-ext}(X)$. Using Claim 5.1 we deduce that \mathcal{H}' is an (X,\mathcal{M}) -multihistogram. Repeating the same operation for every column of \mathcal{H} yields the required exponential bound.

Construction of a VAS. Given X and M we now construct a VAS whose runs correspond to the words of exponentially bounded (X, \mathcal{M}) -multihistograms. Think of the VAS as reading (or nondeterministically guessing) consecutive column vectors (i.e., the word) of a potential (X, \mathcal{M}) -multihistogram $\mathcal{H} = \{H_1, \dots, H_k\}$. The VAS has to check two conditions:

- 1. the word of \mathcal{H} belongs to the language (7);
- 2. the matrices H_1, \ldots, H_k satisfy the histogram condition.

The first condition, under the exponential bound proved above, amounts to the membership in a regular language and can be imposed by a VAS in a standard way. The second condition is a conjunction of k histogram conditions, and again the conjunction can be realised in a standard way. We thus focus, from now on, only on showing that a VAS can check that its input is a histogram.

To this aim it will be profitable to have the following characterisation of histograms. For an arbitrary $r \times c$ -matrix H, define the $(r-1) \times c$ -matrix Δ_H as:

$$\Delta_H(i,j) \stackrel{\text{def}}{=} \sum H_0(i,1\ldots j) - \sum H_0(i+1,1\ldots j+1),$$

where H_0 is an $r \times (c + 1)$ -matrix which extends H by the (c + 1)-th zero column.

Lemma 5.2. A nonnegative $r \times c$ -matrix H is a histogram if, and only if Δ_H is nonnegative and $\Delta_H(_, c) = \mathbf{0}$.

Proof. Indeed, nonnegativeness of Δ_H is equivalent to saying that

$$\sum H(i,1\ldots j) \ \geq \ \sum H(i+1,1\ldots j+1)$$

for every $1 \le i < r$ and $0 \le j < c$; moreover, $\Delta_H(_,c) = \mathbf{0}$ is equivalent to saying that $\sum H(i,1\ldots c)$ is the same for every $i=1,\ldots,r$.

For technical convenience, we always extend Δ_H with an additional very first zero column ${\bf 0}$; in other words, we put $\Delta_H(_,0)={\bf 0}$. Here is a formula relating two consecutive columns $\Delta_H(_,j-1)$ and $\Delta_H(_,j)$ of Δ_H and two consecutive columns $H(_,j)$ and $H(_,j+1)$ of $H(_,j)$

$$\Delta_H(i,j) = \Delta_H(i,j-1) + H(i,j) - H(i+1,j+1), \tag{8}$$

that will lead our construction.

We now define a VAS of dimension 2(r-1) that reads consecutive columns $\mathbf{w} \in \mathbb{N}^r$ of an exponentially bounded matrix and accepts if, and only if the matrix is a histogram. According to the convention that $\Delta_H(_,0)=\mathbf{0}$, all the 2(r-1) counters are initially set to 0. Counters $1,\ldots,r-1$ of the VAS are used as a buffer to temporarily store the input; counters $r,\ldots,2(r-1)$ ultimately store the current column $\Delta_H(_,j)$. According to (8), the VAS obeys the following invariant: after j steps,

$$\Delta_H(i,j) = \text{counter}_i + \text{counter}_{r-1+i}.$$
 (9)

Let $C \subseteq \mathbb{N}^r$ denote the exponential set of all column vectors that can appear in a histogram, as derived above. For every $\mathbf{w} = (w_1, \dots, w_r) \in C$, the VAS has a 'reading' transition that adds $(w_1, \dots, w_{r-1}) \in \mathbb{N}^{r-1}$ to its counters $1, \dots, r-1$, and subtracts $(w_2, \dots, w_r) \in \mathbb{N}^{r-1}$ from its counters $r, \dots, 2(r-1)$ (think of $\mathbf{w}(i+1) = H(i+1,j+1)$ in the equation (8)). Furthermore, for every $i=1,\dots,r-1$ the VAS has a 'moving' transition that subtracts 1 from counter i and adds 1 to counter r-1+i, i.e., moves 1 from counter i to counter r-1+i. (recall the '+H(i,j)' summand in the equation (8)). Observe that these transitions preserve the invariant (9).

Relying on Lemma 5.2 we claim that the so defined VAS reaches nontrivially (i.e., along a nonempty run) the zero configuration (all counters equal 0) if, and only if its input, treated as an $r \times c$ -matrix H, is a histogram with all entries belonging to C. In one direction, the invariant (9) assures that Δ_H is nonnegative and the final zero configuration assures that $\Delta_H(_,c)=0$. In the opposite direction, if a histogram is input, the VAS has a run ending in the zero configuration. The VAS is computable in exponential time (as the set C above is so).

We have shown that, given X and \mathcal{M} , one can effectively built a VAS which admits reachability if, and only if there exists an (X,\mathcal{M}) -multihistogram. The (exponential-blowup) reduction of the Permutation sum problem to the VAS reachability problem is thus completed.

6 PTIME decision procedures

In this section we prove Theorem 2.2, namely we provide polynomial-time decision procedures for the \mathbb{X} -Permutation sum problem, where $\mathbb{X} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{Q}_+\}$. The most interesting case $\mathbb{X} = \mathbb{Q}_+$ is treated in Section 6.1. The remaining ones are in fact special cases of a more general result, shown in Section 6.2, that applies to an arbitrary commutative ring.

6.1 $X = \mathbb{O}_+$

We start by noticing that the whole development of (multi)-histograms in Section 4 is not at all specific for $X = \mathbb{N}$ and works equally well for $\mathbb{X} = \mathbb{Q}_+$. It is enough to relax the definition of histogram: instead of nonnegative integer matrix, let histogram be now a nonnegative rational matrix satisfying exactly the same conditions as in Definition 4.3 in Section 4. In particular, the degree of a histogram is now a nonnegative rational. Accordingly, one adapts the Up to 0-extension sum problem and considers a sum of 0-extensions of M multiplied by nonnegative rationals. The same relaxation as for histograms we apply to multihistograms, and in the definition of the latter (cf. the language (7) at the end of Section 4) we consider nonnegative-rational solutions of linear equations instead of nonnegative-integer ones. With these adaptations, the \mathbb{Q}_+ -Per-MUTATION SUM PROBLEM is equivalent to the following decision problem (whenever a risk of confusion arises, we specify explicitly which matrices are integer ones, and which rational ones):

Q+-Multihistogram problem.

Input: a finite set \mathcal{M} of integer matrices, and an integer matrix X, all of the same row dimension d.

Output: does there exist a rational (X, \mathcal{M}) -multihistogram?

From now on we concentrate on the polynomial-time decision procedure for this problem. We proceed in two steps. First, we define homogeneous linear Petri nets, a variant of Petri nets generalising continuous PNs [21], and show how to solve its reachability problem by \mathbb{Q}_+ -solvability of a slight generalisation of linear equations (linear equations with implications), following the approach of [8]. Next, using a similar construction as in Section 5, combined with the above characterisation of reachability, we encode \mathbb{Q}_+ -MULTIHISTOGRAM PROBLEM as a system of linear equations with implications.

Homogeneous linear Petri nets. A homogeneous linear Petri net (homogeneous linear PN) of dimension d is a finite set of homogeneous systems of linear equations $\mathcal{V} = \{\mathcal{U}_1, \dots, \mathcal{U}_m\}$, called transition rules, all over the same 2d variables x_1, \dots, x_{2d} . The transition rules determine a transition relation \longrightarrow between configurations, which are nonnegative rational vectors $\mathbf{c} \in (\mathbb{Q}_+)^d$, as follows: there is a transition $\mathbf{c} \longrightarrow \mathbf{c}'$ if, for some $i \in \{1, \dots, m\}$ and $\mathbf{v} \in \mathbb{Q}_+$ -sol(\mathcal{U}_i), the vector $\mathbf{c} - \pi_{1...d}(\mathbf{v})$ is still a configuration, and

$$\mathbf{c'} = \mathbf{c} - \pi_{1...d}(\mathbf{v}) + \pi_{d+1...2d}(\mathbf{v}).$$

(The vectors $\pi_{1...d}(\mathbf{v})$ and $\pi_{d+1...2d}(\mathbf{v})$ are projections of \mathbf{v} on respective coordinates.) The reachability relation $\mathbf{c} \longrightarrow^* \mathbf{c}'$ holds, if there is a sequence of transitions (called a run) from \mathbf{c} to \mathbf{c}' .

A class of *continuous PN* [21] is a subclass of homogeneous linear PN, where every system of linear equations \mathcal{U}_i has a 1-dimensional solution set of the form $\{cv \mid c \in \mathbb{Q}_+\}$, for some fixed $v \in \mathbb{N}^{2d}$.

Linear equations with implications. A \Rightarrow -system is a finite set of linear equations, all over the same variables, plus a finite set of implications of the form

$$x > 0 \implies y > 0$$
,

where x,y are variables appearing in the linear equations. The solutions of a \Rightarrow -system are defined as usually, but additionally they must satisfy all implications. The \mathbb{Q}_+ -solvability problem asks

¹ If non-homogeneous systems were allowed, the model would subsume (ordinary)

if there is a nonnegative-rational solution. In [8] (Algorithm 2) it has been shown (within a different notation) how to solve the problem in PTIME; another proof is derivable from [2], where a polynomial-time fragment of existential $FO(\mathbb{Q}, +,<)$ has been identified that captures \Rightarrow -system:

Lemma 6.1 ([2, 8]). The \mathbb{Q}_+ -solvability problem for \Rightarrow -systems is decidable in PTIME.

Due to [8], the reachability problem for continuous PNs reduces to the \mathbb{Q}_+ -solvability of \Rightarrow -systems. We generalise this result and prove the reachability relation of a homogeneous linear PN to be effectively described by a \Rightarrow -system:

Lemma 6.2. Given a homogeneous linear PN V of dimension d (with numbers encoded in binary) one can compute in polynomial time $a \Rightarrow$ -system whose \mathbb{Q}_+ -solution set, projected onto a subset of 2d variables, describes the reachability relation of V.

We return to the proof of this lemma, once we first use it in the decision procedure for our problem.

Polynomial-time decision procedure. Now, we are ready to describe a decision procedure for the \mathbb{Q}_+ -Multihistogram problem, by a polynomial-time reduction to the \mathbb{Q}_+ -solvability problem of \Rightarrow -systems.

Fix an input to the \mathbb{Q}_+ -Multihistogram problem, i.e., X and $\mathcal{M} = \{M_1, \dots, M_k\}$. Analogously as in (6) in Section 4 we put for succinctness, for $\mathbf{a} \in \mathbb{Z}^d$,

$$C_{\mathbf{a}} := \mathbb{Q}_{+}\text{-sol}(\mathcal{U}_{\mathcal{M},\mathbf{a}}) \subseteq (\mathbb{Q}_{+})^{r}$$

to denote the set of all nonnegative-rational solutions of the system $\mathcal{U}_{\mathcal{M},a}$ of linear equations determined by the matrix

$$M_1 \mid \ldots \mid M_k$$

and the column vector **a**. Recall the language (7):

$$(C_0)^* C_{X(\cdot,1)} (C_0)^* C_{X(\cdot,2)} \dots (C_0)^* C_{X(\cdot,n)} (C_0)^*,$$
 (10)

where n is the column dimension of X. Our aim is to check existence of an (X, \mathcal{M}) -multihistogram, i.e., of a family $\mathcal{H} = \{H_1, \dots, H_k\}$ of nonnegative-rational matrices, such that the following conditions are satisfied:

- 1. the word of \mathcal{H} belongs to the language (10);
- 2. the matrices H_1, \ldots, H_k satisfy the histogram condition.

 \mathcal{H} has $r = r_1 + \ldots + r_k$ rows, where r_l is the row dimension of H_l , equal to the column dimension of M_l for $l = 1, \ldots, k$. Given X, \mathcal{M} , we construct in polynomial time $a \Rightarrow$ -system \mathcal{S} which is solvable if, and only if some (X, \mathcal{M}) -multihistogram exists. The solvability of \mathcal{S} is decidable in PTIME according to Lemma 6.1.

The idea is to characterise an (X, \mathcal{M}) -multihistogram by a sequence of runs in a homogeneous linear PN interleaved by single steps described by non-homogeneous systems of linear equations; then, using Lemma 6.2, the sequence is translated to a \Rightarrow -system. Conceptually, the construction is analogous to the construction of a VAS in Section 5. We define a homogeneous linear PN \mathcal{V}_0 , recognizing the language $(C_0)^*$. (Think of \mathcal{V}_0 as if it reads consecutive nonnegative-rational column vectors belonging to C_0 .) The dimension of \mathcal{V}_0 is

$$2(r-k) = 2(r_1-1) + \ldots + 2(r_k-1),$$

i.e., V_0 has $2(r_i - 1)$ counters corresponding to each H_i .

Concerning the 'reading' transitions, the construction is essentially the same, except that we do not restrict the input to the finite set C as in Section 5, but we allow for all infinitely many solutions C_0 of the homogeneous system of linear equations $\mathcal{U}_{M,0}$ as inputs. Moreover, we deal with all histogram conditions for H_1,\ldots,H_k simultaneously. Thus, the 'reading' transition rule of \mathcal{V}_0 is described by a homogeneous system \mathcal{U}_0 over 4(r-k) variables, half of them describing subtraction and half describing addition in a transition, derived from $\mathcal{U}_{M,0}$ as follows. Let x_1,\ldots,x_{r_j} be the variables of $\mathcal{U}_{M,0}$ corresponding to rows of some H_j . As \mathcal{V}_0 has $2(r_j-1)$ dimensions corresponding to H_j , the system \mathcal{U}_0 has $4(r_j-1)$ corresponding variables $z_1,\ldots,z_{4(r_j-1)}$, half of them, say $z_1,\ldots,z_{2(r_j-1)}$, describing subtraction and the other half $z_{2(r_j-1)+1},\ldots,z_{4(r_j-1)}$ addition. Imitating the construction of a VAS in Section 5, the system \mathcal{U}_0 is obtained by adding to $\mathcal{U}_{M,0}$, for every j, the following equations:

$$(z_{2(r_j-1)+1},\ldots,z_{3(r_j-1)}) = (x_1,\ldots,x_{r_j-1})$$

describing addition in dimensions $1, \ldots, r_j - 1$; and the following equations:

$$(z_{(r_i-1)+1},\ldots,z_{2(r_i-1)}) = (x_2,\ldots,x_{r_i})$$

describing subtraction in dimensions $(r_j - 1) + 1, \dots, 2(r_j - 1)$; and then by eliminating the (redundant) variables x_1, \dots, x_{r_i} .

Concerning the 'moving' transitions, there are r - k of them in \mathcal{V}_0 , each one described by a separate system of homogeneous linear equations \mathcal{W}_i , consisting of just one equation of the form (using the same indexing as above)

$$z_l = z_{3(r_i-1)+l}$$
, where $1 \le l \le r_j - 1$, $1 \le j \le k$.

In addition, both in \mathcal{U}_0 and in \mathcal{W}_i , all variables z_l not mentioned above are equalised to 0.

Summing up, $\mathcal{V}_0 \stackrel{\text{def}}{=} \{\mathcal{U}_0, \mathcal{W}_1, \dots, \mathcal{W}_{r-k}\}$. By Lemma 6.2 one can compute in polynomial time $a \Rightarrow$ -system \mathcal{S}_0 such that the projection A_0 of \mathbb{Q}_+ -sol (\mathcal{S}_0) to some $2 \cdot 2(r-k)$ of its variables describes the reachability relation of \mathcal{V}_0 .

According to (10), we aim at constructing a \Rightarrow -system S whose solvability is equivalent to existence of the following sequence of n+1 runs of \mathcal{V}_0 :

$$\mathbf{0} \xrightarrow{C_0^*} \mathbf{c}_1 \quad \mathbf{c}_2 \xrightarrow{C_0^*} \mathbf{c}_3 \dots \mathbf{c}_{2n-2} \xrightarrow{C_0^*} \mathbf{c}_{2n-1} \quad \mathbf{c}_{2n} \xrightarrow{C_0^*} \mathbf{0}$$
 (11)

where the relation between every ending configuration \mathbf{c}_{2i-1} and every next starting configuration \mathbf{c}_{2i} is determined by $C_{X(_,i)}$. The required \Rightarrow -system $\mathcal S$ is constructed as follows: we introduce 2(r-k) variables per each intermediate configuration \mathbf{c}_i ($\mathbf{c}_0 = \mathbf{c}_{2n+1} = \mathbf{0}$), and impose the constraints:

- 1. there is a run from \mathbf{c}_{2i} to \mathbf{c}_{2i+1} in \mathcal{V}_0 , i.e., $(\mathbf{c}_{2i}, \mathbf{c}_{2i+1})$ belongs to the projection A_0 of \mathbb{Q}_+ -sol(\mathcal{S}_0);
- 2. $\mathbf{c}_{2i} \mathbf{c}_{2i-1} \in \mathbb{Q}_+$ -sol(\mathcal{U}_i), where the (nonhomogeneous) system \mathcal{U}_i is obtained from $\mathcal{U}_{\mathcal{M},X(\cdot,i)}$ similarly as \mathcal{U}_0 above.

S is solvable iff some (X, \mathcal{M}) -multihistogram exists.

Proof of Lemma 6.2. We start by observing that the reachability of a homogeneous PN can be simulated by a continuous PN:

Lemma 6.3. For a given homogeneous PNV one can construct a continuous PNN such that $\mathbf{c} \to \mathbf{c}'$ in V iff $\mathbf{c} \to^* \mathbf{c}'$ in N.

Thus, knowing that homogeneous linear PN subsume continuous PN, the former are potentially (exponentially) more succinct representations of the latter.

We need to recall a crucial observation on continuous PN, made in Lemma 12 in [8], namely that whenever the initial and final configuration have positive values on all coordinates used by the transitions in a run, the reachability reduces to Q₊-solvability of state equation. Using Lemma 6.3, we translate this observation to homogeneous linear PN. Configurations below are understood to be elements of $(\mathbb{Q}_+)^d$, where *d* is the dimension. Recall that a homogeneous linear PN is determined by a finite set of homogeneous systems of linear equations \mathcal{U} ; and that its transition is determined by a solution $v \in \mathbb{Q}_+$ -sol(\mathcal{U}) of some of the systems, namely the transition first subtracts $\mathbf{v}^- = \pi_{1...d}(\mathbf{v})$ from the current configuration, and then adds $\mathbf{v}^+ = \pi_{d+1...2d}(\mathbf{v})$ to it.

Lemma 6.4. Let $V = \{U_1 \dots U_k\}$ be a d-dimensional homogeneous linear PN. A configuration f is reachable from a configuration i in V whenever there are some solutions $\mathbf{u}_i \in \mathbb{Q}_+$ -sol(\mathcal{U}_i), for i = 1..., k, satisfying the following conditions:

- 1. $\mathbf{f} \mathbf{i} = \sum_{i=1}^{k} -\mathbf{u}_{i}^{-} + \mathbf{u}_{i}^{+};$ 2. $if \mathbf{u}_{i}^{-}(j) > 0 \ then \ \mathbf{i}(j) > 0;$ 3. $if \mathbf{u}_{i}^{+}(j) > 0 \ then \ \mathbf{f}(j) > 0;$

with i ranging over $1 \dots k$ and j over $1 \dots d$.

Thus Lemma 6.4 provides a sufficient (but not necessary) condition for $i \longrightarrow^* f$. We now use the lemma to fully characterise the reachability relation in homogeneous linear PN:

Lemma 6.5. Let $V = \{U_1 \dots U_k\}$ be d-dimensional homogeneous linear PN. A configuration f is reachable from a configuration i if, and only if for some two configurations i' and f':

- 1. i' is reachable from i in at most d steps;
- 2. \mathbf{f} is reachable from \mathbf{f}' in at most d steps;
- 3. i' and f' satisfy the sufficient condition of Lemma 6.4.

Proof. The if direction is immediate. For the only if direction, consider a fixed run from i to f, i.e., a sequece of \mathbb{Q}_+ -solutions of systems $\mathcal{U}_1, \ldots, \mathcal{U}_k$. For every $i = 1, \ldots, m$, let \mathbf{u}_i be the sum of the multiset of \mathbb{Q}_+ -solutions of \mathcal{U}_i that take part in the run; clearly $\mathbf{u}_i \in \mathbb{Q}_+$ -sol(\mathcal{U}_i) as the solution set is additive.

As the only requirement we demand for i' (and f') is its positivity on certain coordinates, we modify the run by requiring that in its few first steps it executes transitions with shrinking quantities guarantying that the number of coordinates equal 0 in the intermediate configurations is monotonically non-increasing. Furthermore, we may also require that every of the few first steps increases the number of nonzero coordinates. Thus, after at most d steps a configuration i' is reached that achieves the maximum number of nonzero coordinates. Likewise, reasoning backward, we obtain a configuration f' with the same property. Thus i' and f'satisfy the conditions 2. and 3. of Lemma 6.4. Moreover, we may assume that the sum of solutions $\mathbf{v}_i \in \mathbb{Q}_+$ -sol (\mathcal{U}_i) that take part in the runs $\mathbf{i} \longrightarrow^* \mathbf{i}'$ and $\mathbf{f}' \longrightarrow^* \mathbf{f}$ satisfy $\mathbf{v}_i \leq \mathbf{u}_i$, for $i = 1, \dots, k$, as arbitrary small quantities are sufficient for positiveness of i' and f'. The inequalities allow us to derive condition 1. of Lemma 6.4, as $\mathbf{u}_i - \mathbf{v}_i \in \mathbb{Q}_+$ for $i = 1, \dots, k$. Indeed, the run $\mathbf{i} \longrightarrow^* \mathbf{f}$ implies the first equality below, and the two runs $i \longrightarrow^* i'$ and $f' \longrightarrow^* f$ imply

the second one:

$$f - i = \sum_{i=1}^{k} -u_i^- + u_i^+$$
 $(i' - i) + (f - f') = \sum_{i=1}^{k} -v_i^- + v_i^+.$

Subtraction of the two equalities yields:

$$\mathbf{f} - \mathbf{i} = \sum_{i=1}^{k} -(\mathbf{u}_i - \mathbf{v}_i)^- + (\mathbf{u}_i - \mathbf{v}_i)^+,$$

as required in condition 1. of Lemma 6.4.

Now we are prepared to complete the proof of Lemma 6.2. The \Rightarrow -system S is constructed relying directly on the characterisation of Lemma 6.5. Linear equations are used to express the two runs of length bounded by d, as well as the condition 1. of Lemma 6.4 (which appears in condition 3. in Lemma 6.5); and implications are used to express conditions 2. and 3. of Lemma 6.4. The size of S is clearly polynomial in \mathcal{V} .

6.2
$$\mathbb{X} \in {\mathbb{Z}, \mathbb{Q}}$$

In this, and only in this section we generalise slightly our setting and consider a fixed commutative ring \mathbb{R} , instead of just the ring \mathbb{Q} or rationals. Accordingly, by a data vector we mean in this section a function $\mathbb{D} \to \mathbb{R}^d$ from data values to *d*-tuples of elements of \mathbb{R} that map almost all data values to the vector of zero elements $\mathbf{0} \in \mathbb{R}^d$. With this more general notion of data vectors, we define R-permutation sums and the R-Permutation sum problem analogously as in Section 2.1. Furthermore, we define analogously \mathbb{R} -sums and consider linear equations with coefficients from $\mathbb R$ and their $\mathbb R$ solvability problem.

Theorem 6.6. For any commutative ring \mathbb{R} , the \mathbb{R} -Permutation SUM PROBLEM reduces polynomially to the \mathbb{R} -solvability problem of linear equations.

Clearly, Theorem 6.6 implies the remaining cases of Theorem 2.2, namely $\mathbb{X} \in \{\mathbb{Z}, \mathbb{Q}\}$, as in these cases the \mathbb{X} -solvability of linear equations is in PTIME. Theorem 6.6 follows immediately by Lemma 6.7, stated below, whose proof is strongly inspired by Theorem 15 in [11]. For a data vector v, we define the vector $sum(v) \in \mathbb{R}^d$

and a finite set of vectors
$$\operatorname{vectors}(\mathbf{v}) \subseteq_{\operatorname{fin}} \mathbb{R}^d$$
:
$$\operatorname{SUM}(\mathbf{v}) \stackrel{\operatorname{def}}{=} \sum_{\alpha \in \operatorname{supp}(\mathbf{v})} \mathbf{v}(\alpha)$$

$$\operatorname{vectors}(\mathbf{v}) \stackrel{\operatorname{def}}{=} \{\mathbf{v}(\alpha) \mid \alpha \in \operatorname{supp}(\mathbf{v})\}.$$

Clearly both operations commute with data permutations: $SUM(\mathbf{v}) =$ $SUM(\mathbf{v} \circ \theta)$ and $VECTORS(\mathbf{v}) = VECTORS(\mathbf{v} \circ \theta)$, and can be lifted naturally to finite sets of data vectors:

$$\begin{split} \text{SUM}(\mathcal{V}) & \stackrel{\text{def}}{=} & \{\text{SUM}(\mathbf{v}) \mid \mathbf{v} \in \mathcal{V}\} \\ \text{VECTORS}(\mathcal{V}) & \stackrel{\text{def}}{=} & \bigcup_{\mathbf{v} \in \mathcal{V}} \text{VECTORS}(\mathbf{v}). \end{split}$$

Lemma 6.7. Let x be a data vector and V be a finite set of data vectors V. Then \mathbf{x} is an \mathbb{R} -permutation sum of V if, and only if

- 1. $SUM(\mathbf{x})$ is an \mathbb{R} -sum of SUM(V), and
- 2. every $\mathbf{a} \in \text{VECTORS}(\mathbf{x})$ is an \mathbb{R} -sum of VECTORS(V).

Proof. The only if direction is immediate: if $\mathbf{x} = z_1 \cdot \mathbf{w}_1 + \ldots + z_n \cdot \mathbf{w}_n$ $z_n \cdot \mathbf{w}_n$ for $z_1, \dots, z_n \in \mathbb{R}$ and $\mathbf{w}_1, \dots, \mathbf{w}_n \in \text{Perm}(V)$, then clearly $SUM(\mathbf{x}) = z_1 \cdot SUM(\mathbf{w}_1) + \ldots + z_n \cdot SUM(\mathbf{w}_n)$ and hence $SUM(\mathbf{x})$ is a \mathbb{R} -sum of sum(V) (using the fact that $sum(_)$ commutes with data permutations). Also $\mathbf{x}(\alpha)$ is necessarily an \mathbb{R} -sum of vectors(V) for every $\alpha \in supp(\mathbf{x})$.

For a vector $\mathbf{a} \in \mathbb{R}^d$, we define an \mathbf{a} -move as an arbitrary data vector that maps some data value to \mathbf{a} , some other data value to $-\mathbf{a}$, and all other data values to $\mathbf{0}$.

Claim 6.1. Every a-move, for $a \in VECTORS(v)$, is an \mathbb{R} -permutation sum of $\{v\}$.

Indeed, for $\mathbf{a} = \mathbf{v}(\alpha)$, consider a data permutation θ which preserves all elements of $supp(\mathbf{v})$ except that it maps α to a data value α' related in the same way as α by the order \leq to other data values in $supp(\mathbf{v})$. Then a-moves are exactly data vectors $(\mathbf{v} - \mathbf{v} \circ \theta) \circ \rho = \mathbf{v} \circ \rho - \mathbf{v} \circ (\theta \circ \rho)$.

For the if direction, suppose point 1. holds: $\operatorname{sum}(\mathbf{x})$ is an \mathbb{R} -sum of $\operatorname{sum}(V)$. Treat the vector $\operatorname{sum}(\mathbf{x})$ and the vectors in $\operatorname{sum}(V)$ as data vectors with the same singleton support. Observe that $\operatorname{sum}(\mathbf{v})$ for any $\mathbf{v} \in V$ is an \mathbb{R} -permutation sum of $\{\mathbf{v}\}$; indeed we can use a-moves to transfer all nonzero vectors for data in $\operatorname{supp}(\mathbf{v})$ into one datum. With this view in mind we have:

• $SUM(\mathbf{x})$ is an \mathbb{R} -permutation sum of V.

Furthermore, suppose point 2. holds: every $\mathbf{a} \in \text{VECTORS}(\mathbf{x})$ is an \mathbb{R} -sum of VECTORS(V). Thus every \mathbf{a} -move, for $\mathbf{a} \in \text{VECTORS}(\mathbf{x})$, is an \mathbb{R} -sum of $\{\mathbf{b}$ -move $| \mathbf{b} \in \text{VECTORS}(V)\}$. By Claim 6.1 we know that every element of the latter set is an \mathbb{R} -permutation sum of V. Thus we entail:

 every a-move, for a ∈ VECTORS(x), is an ℝ-permutation sum of V.

We have shown that $SUM(\mathbf{x})$, as well as all a-moves (for all $\mathbf{a} \in VECTORS(\mathbf{x})$), are \mathbb{R} -permutation sums of V. We use the a-moves to transform $SUM(\mathbf{x})$ into \mathbf{x} , which proves that \mathbf{x} is an \mathbb{R} -permutation sum of V as required.

7 Concluding remarks

The main result of this paper is determining the computational complexity of solving linear equations with integer (or rational) coefficients, over ordered data. We observed the huge gap: while the \mathbb{N} -solvability problem is equivalent (up to an exponential blowup) to the VAS reachability problem, the \mathbb{Z} -, \mathbb{Q} -, and \mathbb{Q}_+ -solvability problems are all in PTIME.

Except for the last Section 6.2, we assumed in this paper that the coefficients and solutions come from the ring $\mathbb Q$ of rationals, but clearly one can consider other commutative rings as well. There is another possible axis of generalisation, which we want to mention now, namely orbit-finite systems of linear equations over an orbit-finite set of variables.

Fix a commutative ring \mathbb{R} . Let X, \mathcal{Y} be arbitrary, possibly infinite sets. By an X-vector we mean any function $X \to \mathbb{R}$ which maps almost all elements of X to $0 \in \mathbb{R}$. An $X \times \mathcal{Y}$ -matrix is an X-indexed family of (column) \mathcal{Y} -vectors,

$$M = (M_x \in \mathbb{R}^{\mathcal{Y}})_{x \in \mathcal{X}}.$$

Such a matrix M, together with a (column) \mathcal{Y} -vector \mathbf{a} , determines a system of linear equations $\mathcal{U}_{M,\mathbf{a}}$, whose solutions are those \mathcal{X} -vectors which, treated as coefficients of a linear combination of vectors $M_{\mathcal{X}}$, yield $\mathbf{a} \in \mathcal{Y}$:

$$\operatorname{sol}(\mathcal{U}_{M,\mathbf{a}}) = \{\mathbf{v} \in \mathbb{R}^{\mathcal{X}} \mid \sum_{x \in \mathcal{X}} \mathbf{v}(x) \cdot M_x = \mathbf{a}\}.$$

Note that the sum is well defined as $\mathbf{v}(x) \neq 0$ for only finitely many elements $x \in \mathcal{X}$. The setting of this paper is nothing but a special case, where $\mathbb{R} = \mathbb{Q}$; and where

$$X = Perm(V)$$
 and $Y = \mathbb{D} \times \{1, \dots, d\}$

are *orbit-finite* sets, i.e., sets which are finite up to the natural action of automorphisms of the data domain (\mathbb{D}, \leq) : data vectors are clearly elements of $\mathbb{Z}^{\mathbb{D} \times \{1, \dots, d\}}$, and solutions we seek for are essentially elements of $\mathbb{R}^{\mathrm{PERM}(V)}$. The natural action of a monotonic bijection $\theta: \mathbb{D} \to \mathbb{D}$ maps a pair $(d, i) \in \mathbb{D} \times \{1, \dots, d\}$ to $(\theta(d), i)$; and maps a data vector $\mathbf{v} \in V$ to $\mathbf{v} \circ \theta^{-1}$. Similarly, another special case has been investigated in [11], where finiteness up to the natural action of automorphisms of the data domain $(\mathbb{D}, =)$ played a similar role. As another example, in [13] the orbit-finite solvability problem has been investigated (in the framework of CSP) for the same data domain $(\mathbb{D}, =)$, in the case where \mathbb{R} is a finite field.

It is an exciting research challenge to fully understand the complexity landscape of orbit-finite systems of linear equations, as a function of the choice of data domain. In this direction, the results of this paper are discouraging: the case of ordered data, compared to the case of unordered data investigated in [11], requires significantly new techniques and the complexity of the nonnegative integer solvability differs significantly too; thus it is expectable that different choices of data domain will require different approaches. Nevertheless, investigation of orbit-finite dimensional linear algebra seems to be a tempting continuation of our work.

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A Missing proofs

Proof of Lemma 4.5. The if direction is easy as sum of histograms is a histogram, and the degree is the sum of degrees.

We prove the only if direction by induction on the degree s of a histogram. For s=1 the claim is trivial, so assume s>1 and fix a $r\times c$ -histogram H of degree s. We are going to extract from H a simple histogram S in such a way that remaining matrix H-S is still a histogram, necessarily of degree s-1. Then one can use the induction assumption to deduce that H can be decomposed as a sum of simple histograms.

Consider a function a function $f:\{1\dots r\} \to \{1\dots c\}$ which maps i to the smallest j with H(i,j)>0. Let S be the simple $r\times c$ -histogram induces by f. We need to check that the matrix H-S is a histogram. As the first defining condition of histogram is obvious, we concentrate on the second one, i.e., for any $1 \le i < r$ and $0 \le j < c$ we aim at showing

$$\sum [H-S](i,1\ldots j) \geq \sum [H-S](i+1,1\ldots j+1).$$

We consider separately three cases:

(i)
$$f(i) < f(i+1) \le j+1$$
;
(ii) $f(i) \le j+1 < f(i+1)$; and
(iii) $j+1 < f(i) < f(i+1)$.

In the case (i) we have

$$\begin{split} &\sum \big[H-S\big](i,1\ldots j) \;=\; \sum H(i,1\ldots j)-1 \geq \\ &\sum H(i+1,1\ldots j+1)-1 \;=\; \sum \big[H-S\big](i+1,1\ldots j+1); \end{split}$$

the inequality holds as H is a histogram. In the case (ii) we have

$$\begin{split} \sum \left[H-S\right](i,1\ldots j) \; &=\; \sum H(i,1\ldots j)-1 \geq 0 \\ &=\; \sum \left[H-S\right](i+1,1\ldots j+1); \end{split}$$

the inequality holds due to definition of the function x. In the case (iii) we have

$$\sum \big[H-S\big](i,1\ldots j)=0\geq 0=\sum \big[H-S\big](i+1,1\ldots j+1).$$

Thus H - S is a histogram, which allows us to apply the induction assumption for s - 1.

Proof of Lemma 4.6. Simple $r \times c$ -histograms are in one-to-one correspondence with monotonic functions $\{1 \dots r\} \to \{1 \dots c\}$. Indeed, such a function f induces a simple histogram S with S(i,j) = 1 if f(i) = j, and S(i,j) = 0 otherwise; on the other hand a simple histogram S defines a function that maps i to the first (and the only) index j with S(i,j) = 1, and this function is necessarily monotonic.

Fix a matrix M of row dimension r and of column dimensions c. Consider a simple $c \times c'$ -histogram S and the corresponding monotonic function $f:\{1\dots c\} \to \{1\dots c'\}$. The multiplication $M \cdot S$ yields a $r \times c'$ -matrix whose f(i)-th column equals the i-th column of M, for $i=1\dots c$, and all other columns are zero ones. Thus $M \cdot S \in 0$ -ext(M). Moreover, every $r \times c'$ -matrix $N \in 0$ -ext(M) is obtained in the same way via some monotonic function $f:\{1\dots c\} \to \{1\dots c'\}$, and thus $N=M \cdot S$ where S is the corresponding simple $c \times c'$ -histogram.

Proof of Lemma 6.3. The \mathbb{Q}_+ -solution set of a homogenous system of equations is a finitely generated cone (cf. [6], Prop.3). Thus every transition in \mathcal{V} can be expressed as a \mathbb{Q}_+ -sum of a set of basic solutions, and in consequence as a sequence of transitions of the form $c_i \mathbf{v}_i$, for \mathbf{v}_i generators of the cone and $c_i \in \mathbb{Q}_+$, which are exactly transitions of a continuous PN.

In the opposite direction, every transition of the continuous PN, as described above, will be also a transition of the homogeneous PN $\hfill\Box$