

Non-Holonomic Sequences and Functions

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May 16, 2006

Holonomic Sequences and Functions

- ▶ A function $f(z)$ is holonomic if it satisfies an LODE

$$p_0(z)f^{(0)}(z) + \cdots + p_d(z)f^{(d)}(z) = 0$$

with polynomial coefficients.

- ▶ A sequence (a_n) is holonomic if it satisfies an LORE

$$p_0(n)a_n + \cdots + p_d(n)a_{n+d} = 0$$

with polynomial coefficients.

- ▶ Rather general way to specify concrete functions/sequences in finite terms
- ▶ Several algorithms are available for the symbolic manipulation of holonomic functions and sequences.

Examples and Basic Properties

- ▶ Examples of holonomic functions: rational functions, exp, sin, cos, Bessel, etc.
- ▶ Examples of holonomic sequences: α^n , $n!$, $\sum_{k=1}^n 1/k$, $\sharp(\text{involutions})$, etc.
- ▶ Sum and Product of two holonomic sequences (functions) are holonomic
- ▶ a_n is holonomic iff $\sum_{n \geq 0} a_n z^n$ is holonomic
- ▶ Definition can be extended to several variables $(n_1, \dots, n_r, z_1, \dots, z_s)$

Proving *Non*-Holonomicity

- ▶ Holonomic sequences \approx algebraic numbers
- ▶ Annihilated by operators instead of polynomials
- ▶ If a sequence (function) is not obviously holonomic, it is usually not holonomic
- ▶ But how to come up with a rigorous proof?

Why?

- ▶ Lower bound on the algorithmic complexity of a sequence (function)
- ▶ Relations to many areas of mathematics:
- ▶ Analytic combinatorics, complex analysis, algebraic geometry, number theory
- ▶ Many ways to define sequences (functions) \implies ample opportunities for research

Two Easy arguments

- ▶ For each holonomic sequence a_n there is α such that $a_n = O(n!^\alpha)$
- ▶ Hence 2^{2^n} and 2^{n^2} are not holonomic (What about n^n ?).
- ▶ Other argument: Singularities of a holonomic function $f(z)$ are roots of $p_d(z)$.
- ▶ Hence $\tan z$, $z/(e^z - 1)$, and $\prod(1 - z^n)^{-1}$ are not holonomic.

Infinitely Many Singularities

- ▶ We show that the sequence $(\zeta(n))_{n \geq 2}$ of values of the Riemann zeta function is not holonomic.



$$f_{\text{even}}(z) := \sum_{n \geq 1} \zeta(2n) z^n, \quad f_{\text{odd}}(z) := \sum_{n \geq 1} \zeta(2n + 1) z^n$$

- ▶ Euler:

$$f_{\text{even}}(z) = \frac{1}{2} - \frac{\pi}{2} z \cot \pi z$$

- ▶ Molteni (2001):

$$f_{\text{even}}(z)/z - f_{\text{odd}}(z) = \gamma - \psi(1 + z),$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function.

Known Non-Trivial Results

- ▶ Harris, Sibuya (1985): The reciprocal $1/f$ of a holonomic function is holonomic iff f'/f is algebraic.
- ▶ Singer (1986): Similar results about exponentiation and composition
- ▶ Van der Put, Singer (1997): The reciprocal $1/a_n$ of a holonomic sequence is holonomic iff a_n is an interlacement of hypergeometric sequences.
- ▶ Pólya-Carlson (1921): If $f(z)$ has integer coefficients and radius of convergence 1, then $f(z)$ is rational or has the unit circle for its natural boundary.

Proofs by Number Theory (SG, 2004)

- ▶ W.l.o.g. the polynomials $p_k(n)$ have coefficients in $\mathbb{Q}(a_j : j \geq 0)$.
- ▶ $a_n = \sqrt{n}$ does not satisfy a recurrence

$$p_0(n)a_n + \cdots + p_d(n)a_{n+d} = 0$$

with coefficients in $\mathbb{Q}(\sqrt{j} : j \geq 0)[n]$, since

$$[\mathbb{Q}(\sqrt{\rho_1}, \dots, \sqrt{\rho_s}) : \mathbb{Q}] = 2^s$$

for distinct primes ρ_1, \dots, ρ_s .

- ▶ Other argument: transcendence of e implies non-holonomicity of n^n .

Proofs by Asymptotics (P. Flajolet, SG, B. Salvy, 2005)

- Fuchs-Frobenius theory: Asymptotic expansion of holonomic functions as $|z| \rightarrow \infty$ must be linear combination of series of the form

$$e^{P(z^{1/r})} z^\alpha \sum_{j \geq 0} Q_j(\log z) z^{-js},$$

where P and Q_j are polynomials, the Q_j have bounded degree, $r \in \mathbb{N}$, $\alpha \in \mathbb{C}$, $0 < s \in \mathbb{Q}$.

- Hence $\log \log z$, $e^{e^z - 1}$, and Lambert W are not holonomic.

Proofs by Asymptotics

- **Basic Abelian theorem.** Let $\phi(x)$ be any of the functions

$$x^\alpha (\log x)^\beta (\log \log x)^\gamma, \quad \alpha \geq 0, \quad \beta, \gamma \in \mathbb{C}. \quad (1)$$

Let (a_n) be a sequence that satisfies the asymptotic estimate

$$a_n \underset{n \rightarrow \infty}{\sim} \phi(n).$$

Then the generating function $f(z) := \sum_{n \geq 0} a_n z^n$ satisfies the asymptotic estimate

$$f(z) \underset{z \rightarrow 1-}{\sim} \Gamma(\alpha + 1) \frac{1}{(1-z)} \phi\left(\frac{1}{1-z}\right). \quad (2)$$

Proofs by Asymptotics

- ▶ The sequence of prime numbers:

$$n\text{-th prime} = n \log n + n \log \log n + O(n),$$

hence

$$(n\text{-th prime})/n - H_n \sim \log \log n.$$

- ▶ The sequences of powers ($\alpha \in \mathbb{C} \setminus \mathbb{Z}$):

$$\sum_{k=1}^n \binom{n}{k} (-1)^k k^\alpha \sim \frac{(\log n)^{-\alpha}}{\Gamma(1-\alpha)}.$$

$e^{1/n}$ by Asymptotics (P. Flajolet, SG, B. Salvy, 2005)

- ▶ Lindelöf integral representation

$$\sum_{n \geq 1} e^{1/n} (-z)^n = -\frac{1}{2i} \int_{1/2-i\infty}^{1/2+i\infty} \frac{z^s e^{1/s}}{\sin \pi s} ds$$

- ▶ Asymptotics (saddle point method)

$$\sum_{n \geq 1} e^{1/n} (-z)^n \sim -\frac{e^{2\sqrt{\log z}}}{2\sqrt{\pi}(\log z)^{1/4}} \quad \text{as } |z| \rightarrow \infty.$$

- ▶ Hence $e^{1/n}$ is not holonomic.
- ▶ Work in progress: generalize asymptotics to α^{n^β} .

Closed-Form Sequences (J.P. Bell, SG, M. Klazar, F. Luca, 2006)

- ▶ For sequences like n^α , $e^{1/n}$, there is a more direct way to exploit the additional structure
- ▶ $a_n = f(n)$ for a smooth function f
- ▶ The recurrence means that

$$F(z; p_0, \dots, p_d) := \sum_{k=0}^d p_k(z) f(z+k)$$

vanishes at $z = 0, 1, 2, \dots$.

- ▶ Example: $f(z) = z^\alpha$. If F vanishes identically, the left hand side of

$$f(z) = -\frac{1}{p_0(z)} \sum_{k=1}^d p_k(z) f(z+k)$$

is meromorphic at $z = 0$, hence $\alpha \in \mathbb{Z}$.

Zeroes of Analytic and of Elementary Functions

- ▶ **Carlson's theorem** (1921). The function $\sin \pi z$ is the “smallest” function analytic for $\Re(z) \geq 0$ that vanishes at $z = 0, 1, 2, \dots$
- ▶ Shows non-holonomicity of n^α , $\log n$, $e^{1/n}$, \dots
- ▶ Khovanskii investigates the geometry of the zero set of elementary functions in his book “Fewnomials”.
- ▶ **Definition.** *Elementary functions* are built by composing rational functions, $\exp(x)$, $\log(x)$, $\sin(x)$, $\cos(x)$, $\tan x$, $\arcsin(x)$, $\arccos(x)$, and $\arctan(x)$. The domain of definition must be such that arguments of \sin and \cos are bounded.
- ▶ **Theorem** (Khovanskii). An elementary function has only finitely many simple zeros in its domain of definition.

Results proved using Carlson or Khovanskii

- ▶ For distinct complex u_1, \dots, u_s , the sequence $\Gamma(n - u_1)^{\alpha_1} \dots \Gamma(n - u_s)^{\alpha_s}$ is holonomic if and only if $\alpha_1, \dots, \alpha_s$ are integers.
- ▶ If a sequence from $\mathbb{R}(n, e^n)$ is holonomic, then the denominator has just one summand.
- ▶ If $(f(n))_{n \geq 1}$ is holonomic for an algebraic function $f :]1, \infty] \rightarrow \mathbb{R}$, then f is a rational function.

Conclusion

- ▶ “Negative” results, but “positive” ones obtained along the way
- ▶ Opportunity to apply methods and results from various areas
- ▶ There is a good chance that holonomicity of a sequence can be decided if it has (i) a closed form representation or (ii) a known asymptotic expansion.