

# Verification of Continuous-time Markov Chains

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## Abstract

In this paper, we fill the long-standing gap in the field of the verification of continuous-time systems by proposing a linear-time temporal logic, named continuous linear logic (CLL) and the corresponding model checking algorithm for continuous-time Markov chains. The correctness of our model checking algorithm depends on Schanuel's conjecture, a central open question in transcendental number theory.

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# 1 Introduction

*Model checking* is a method for checking whether a finite-state model of a system meets a given specification, which is usually expressed by a temporal logic formula. Since the 1970s, the area of checking program has been promoted significantly through the discovery and development of the algorithmic methods for verifying discrete-time temporal logic properties of finite-state systems [Pnu77, CES86, LP85, QS82, VW86]. Regarding the nature of time, two dominant logics emerged. *Linear-time temporal logics (LTL)* was first proposed by Amir Pnueli in 1977 [Pnu77]. After that, in 1981, Clarke and Emerson introduced *computational tree logic (CTL)*, a branching-time logic [CE81]. Not surprisingly, LTL and CTL are expressively incomparable [EH86, CD88, Lam80]. CTL and LTL are used to specify qualitative properties for non-probabilistic model checking, for example, *safety* (nothing bad happens) and *liveness* (something good eventually happens). When considering probabilistic transition systems, in practice, discrete-time Markov chains with finite states are the most successful model for probabilistic dynamical systems. By allowing for probabilistic quantification on transition paths, Hansson and Jonsson introduced *probabilistic computational tree logic (PCTL)* in 1994 [HJ94]. The verification of the linear-time property of discrete-time Markov chains was achieved by Agrawal, Akshay, Genest, and Thiagarajan in [AAGT15]. They introduced *probabilistic linear-time logic (PLTL)* to reason the dynamics of discrete-time Markov chains by adding probabilistic quantifiers on the state space. The first three model-checking algorithms can be found in [BK08] and the last one is given in [AAGT15]. Recently, Yu provided *quantum temporal logic (QTL)*, a quantum generalization of LTL, mainly targeting on the verification of quantum programs in [Yu19].

Model checking continuous-time (real-time) systems is even nastier. For specifying the temporal properties of (non)deterministic systems, LTL and CTL both have been generalized from discrete-time domain to continuous-time domain in the 1990s, named *metric temporal logic (MTL)* [Koy90] and *timed computational temporal logic (TCTL)* [AH94], respectively. The syntaxes of the two continuous-time logics both use time-constrained temporal operators to replace the corresponding versions in LTL and CTL. The semantics are *dense*, i.e., the path is defined on infinitely countable time instants rather than the non-negative real numbers. Model checking for TCTL is decidable, while model checking for MTL is undecidable. However, model checking for *Metric Interval Temporal Logic (MITL)*, a fragment of MTL, is decidable. MITL was proposed by Alur et al. in [AFH96].

After that, temporal logics are also brought in the field of the verification of probabilistic continuous-time systems. *Continuous-time Markov chains (CTMCs)* form a fundamental class of probabilistic dynamical systems that have been widely studied theoretically and practically since Kolmogorov [Kol31]. Due to a broad range of applications, continuous-time model checking is invented for the verification of continuous-time Markov chains. Lots of efforts have been devoted in this field to constitute the underlying semantical model of a plethora of modeling formalisms for continuous-time probabilistic systems, in the last two decades, cf. the recent survey [Kat16]. In a majority of the verification related studies, the continuous-time Markov chain is viewed as a probabilistic interval transition system. The time is measured in discrete intervals, so at each time interval the system is in a particular state and the state will transfer to another state with an appropriate probability after the time interval. The paths of this transition system are viewed as computations and the goal is to use the branching-time probabilistic temporal logic, named *continuous stochastic logic (CSL)*, to reason about these computations [BHHK03]. Thus the semantic of CSL is also dense. Reusing the name presented in [ASSB00], CSL generalizes PCTL to fit for specifying properties of continuous-time Markov chains. Thanks to this time discretization method, efficient techniques

for analyzing discrete-time Markov chains have been developed for model checking continuous-time Markov chains. Furthermore, model-checking algorithms developed in [BHHK03] has been implemented in model checkers PRISM [KNP02] and MRMC [KZH<sup>+</sup>11]. As the semantics of all three continuous-time logics MITL, TCTL and CSL are dense, all model-checking algorithms for them are in general approximate. Table 1 overviews the current status of temporal logics.

	Linear-time	Branching-time
discrete-time (non)deterministic systems	LTL [Pnu77]	CTL [CE81]
continuous-time (non)deterministic systems	MTL [Koy90]	TCTL [AH94]
discrete-time probabilistic systems	PLTL [AAGT15]	PCTL [HJ94]
continuous-time probabilistic systems	?	CSL [ASSB00]

Table 1: Overview of Temporal Logics

There is an obvious gap in the above table.

**Problem 1** *Developing a linear-time temporal logic and the corresponding model-checking algorithm for continuous-time probabilistic systems?*

This is formulated as a challenging open problem mentioned in [CHKM11a]. We remark that linear-time temporal logics and branching-time temporal logics are naturally incomparable from many perspectives including the expressiveness. The techniques and results of branching-time temporal logic can not be used to solve the above problem.

## 1.1 Conceptual Contribution

In this paper, we introduce *continuous linear logics (CLL)* filling the mentioned gap in temporal logics.

We adopt an alternative approach to viewing the state space of the continuous-time Markov chain to be the set of probability distributions over the states of the chain, which was used in analyzing discrete-time Markov chains [AAGT15]. Thus the continuous-time Markov chain transforms initial probability distribution continuously. CLL studies the properties of the trajectory at all times generated from a given initial distribution. Many interesting dynamical properties can be formulated in this setting regarding the behaviors of the chain. For instance, in some time interval, no time will it be the case that the probability of being in the state  $s_i$  and the probability of being in the state  $s_j$  are both low, or starting from some time the system is most likely to be in state  $s_i$  or state  $s_j$ . To our best knowledge, such temporal properties have not been discussed in previous literature on the verification of continuous-time Markov chains.

We apply the method of symbolic dynamics on the distributions of continuous-time Markov chains. To be specified in our setting, we symbolize the probability value space  $[0, 1]$  into a finite set of intervals  $\mathcal{I} = \{\mathcal{I}_k\}_{k=1}^m$ . A probability distribution  $\mu$  over its set of states  $S = \{s_1, s_2, \dots, s_n\}$  is then represented symbolically as a set of symbols  $Sym = \{s := (j, \mathcal{I}_k) \in S \times \mathcal{I} : \mu_j \in \mathcal{I}_k\}$  which asserting that the  $j$ -th element of  $\mu$  falls in interval  $\mathcal{I}_k$ . For example, a special case of symbolization of distributions has been considered in [AAGT15]: choosing a disjoint cover of  $[0, 1]$ :  $\mathcal{I} = \{[0, p_1), [p_1, p_2), \dots, [p_n, 1]\}$ .  $Sym$  is regarded as the set of atomic propositions. It is direct to define if a probability distribution satisfies  $s \in Sym$ . A crucial fact is that the set of symbols  $Sym$  is finite. Consequently, the path over real numbers generated by an initial probability distribution

$\mu$  will induce a sequence over the finite alphabet  $Sym$ . Hence, given an initial distribution, the symbolic dynamics of continuous-time Markov chains can be studied in terms of a language over the alphabet  $Sym$ . Our focus here is on continuous behaviors over real numbers.

Our main motivation for studying continuous-time Markov chains in this fashion is that in many practical applications such as biochemical networks and queuing systems, obtaining exact estimations of the probability distributions (including the initial distribution) may be neither feasible nor necessary. Actually, interesting properties are stated in terms of probability ranges, such as “low, medium, or high” or “above the threshold 0.9” rather than exact probabilities.

We formulate the CLL in which an atomic proposition will assert that “the current probability of the state  $s_k$  lies in some interval”. Different from temporal logic in the discrete-time domain, CLL has two types of formulas: states and paths. The state type formulas are constructed under propositional connectives. The path type formulas are obtained under propositional connectives and the temporal model timed until in the usual way. The timed until is bounded. The usual next-step temporal operator is not meaningful in our logic. This is not surprised as in continuous-time, the steps of time can not be defined because the time domain (real numbers) is uncountable. On the other hand, keeping the time continuous instead of discretizing time, our logics expressive power is incomparable with logics such as CSL interpreted over the states transitions of the continuous-time Markov chain.

## 1.2 Technical contribution

Based on Schanuel’s conjecture, we develop an algorithm to model checking continuous-time Markov chains against CLL formulas.

We reduce the model checking problem to the real root isolation problem of real polynomial-exponential functions (PEFs) over the field of the algebraic number, a widely studied problem in recent symbolic and algebraic computation community [GCL<sup>+</sup>17, AMW08, LHXL16]. We resolve the latter problem under the assumption of the validity of Schanuel’s conjecture, a central open question in transcendental number theory [Lan66].

## 1.3 Comparison with Related Work

Symbolic dynamics is a classical topic in the theory of dynamical systems [MH38] with data storage, transmission and coding being the major application areas [LMDB95]. The basic idea is to partition the “smooth” state space into a finite set of blocks and represent a trajectory as a sequence of such blocks. The symbolic dynamics of discrete-time Markov chains was verified in [AAGT15]. This idea regards distributions as the state space instead of the states of discrete-time Markov chains and partitioning the probability value space  $[0, 1]$  into finite disjoint covers (intervals). Therefore, for a given initial distribution, there is only one single path, which PLTL can be introduced to reason the behavior of the dynamics of distributions. In our setting, we follow this view and symbolizing the probability value space  $[0, 1]$  into finite intervals, not needed to be a partition. This generalization endows CLL’s stronger power of expressiveness.

The verification of continuous-time Markov chains was studied in [ASSB00] using CSL, a branching-time logic, i.e., asserting the exact temporal properties with time continuous. The essential feature of CSL is that the path formula is the form of nesting of bounded timed until operators only reasoning the absolutely temporal properties (all time instants basing on one starting time).

In our CLL logic, the nesting of bounded timed until operator can also specify relatively temporal properties. This setting significantly enriches the expressiveness of temporal properties.

CSL was extended by adding next-step operator in [BHHK03]. For introducing the operator, they make time discretization such that the time is measured in discrete intervals. So at each time interval, the system is in a particular state. Each state is associated with a set of atomic propositions and the evolution of the system is described by an infinite sequence of symbols, which is represented as strings. The distinct feature in our CLL setting is that the discretization is applied on the distributions over states and we leave the time continuous. The set of all distributions is inherently not discrete. We fix finite intervals of  $[0, 1]$  representing a distribution by satisfying these intervals, to get a coarse-grained description of the system. This method ensures the time is real continuous such that we can exactly model checking continuous-time Markov chains instead of approximate verification by CSL. For more comparison, please see Table 1.3.

CLL	CSL
Linear-time	Branching-time
Distributions Discretization	Time Discretization
Labeling Distributions	Labeling States
Continuous Semantics	Dense Semantics

Table 2: CLL v.s. CSL

Linear-time model checking for continuous-time Markov chains has also been studied through time discretization in [CHKM11a]. The specification is given by deterministic timed automaton (DTA) with finite and Muller acceptance criteria. The central question they addressed is: what is the probability of the set of paths of continuous-time Markov chains that are accepted by a DTA. The model checker was also developed in [BCH<sup>+</sup>11]. Furthermore, the verification of the continuous-time Markov decision process (CTMDP) against DTA has been studied in [CHKM11b]. A DTA can only express a property. The logic system CLL provides the opportunity of covering many more properties we are interested in.

## 1.4 Organization of this paper

In the next section, we give the mathematical preliminaries used in this paper. In Section 3, we introduce the symbolic dynamics of continuous-time Markov chains by symbolizing distributions of them. In the subsequent section, we present our continuous-time temporal logic CLL and illustrate the model checking problem. In Section 5, with the help of the real root isolation of polynomial exponential functions, we develop an algorithm to solve the model checking problem. In Section 6, we illustrate the real root isolation of polynomial exponential functions over algebraic numbers and prove the correctness basing on Schanuel’s conjecture. In the concluding section, we summarize our results and point to future research directions.

## 2 Preliminaries

We use the set  $\mathbb{R}^+$  of the nonnegative real numbers (including 0) to denote the time domain. A bounded (time) *interval*  $\mathcal{T}$  is a subset of  $\mathbb{R}^+$ . Intervals may be open, half-open, or closed. Each

interval has one of the following forms:

$$[t_1, t_2], [t_1, t_2), (t_1, t_2], (t_1, t_2),$$

where  $t_2 \geq t_1 \geq 0$ . For an interval of the above forms,  $t_1$  is the left endpoint of  $\mathcal{T}$ , and  $t_2$  is the right endpoint of  $\mathcal{T}$ . The left and right endpoints of  $\mathcal{T}$  are denoted by  $\inf \mathcal{T}$  and  $\sup \mathcal{T}$ , respectively.

The expression  $t + \mathcal{T}$ , for  $t \in \mathbb{R}^+$ , denotes the interval  $\{t + t' : t' \in \mathcal{T}\}$ . Similarly,  $\mathcal{T} - t$  stands for the intervals  $\{-t + t' : t' \in \mathcal{T}\}$ . Two intervals  $\mathcal{T}_1$  and  $\mathcal{T}_2$  are *disjoint* if their intersection is an empty set, i.e.,  $\mathcal{T}_1 \cap \mathcal{T}_2 = \emptyset$ .

Throughout this paper, we write  $\mathbb{C}, \mathbb{R}, \mathbb{Q}$  and  $\mathbb{A}$  for the fields of complex, real, rational and algebraic numbers, respectively.

**Definition 1** *An algebraic number is any complex number that is a root of a non-zero polynomial in one variable with rational coefficients (or equivalently with integer coefficients, by eliminating denominators). An algebraic number  $\alpha$  is represented by  $(P, (a, b), \varepsilon)$  where  $P$  is the minimal polynomial of  $\alpha$ ,  $a, b \in \mathbb{Q}$  and  $a + bi$  is an approximation of  $\alpha$  such that  $|\alpha - (a + bi)| < \varepsilon$  and  $\alpha$  is the only root of  $P$  in the open ball  $B(a + bi, \varepsilon)$ . The minimal polynomial  $P$  of  $\alpha$  is the polynomial with the lowest degree in  $\mathbb{Q}[t]$  such that  $\alpha$  is a root of the polynomial.*

Furthermore, for any field  $\mathbb{F} \in \{\mathbb{R}, \mathbb{Q}, \mathbb{A}\}$ , we use  $\mathbb{F}^+$  to denote the set of positive elements (including 0) of  $\mathbb{F}$  and  $\mathbb{F}[t]$  to denote the set of polynomials in  $t$  with coefficients in  $\mathbb{F}$ ; let  $\mathbb{F}^{n \times m}$  be  $n$ -by- $m$  matrix with every entry on the field of  $\mathbb{F}$ . For any complex number  $z = a + bi$  where  $a, b \in \mathbb{R}$  and  $i$  is the imaginary unit, we denote the real part and the imaginary part of  $z$  by  $R(z) = a$  and  $I(z) = b$ , respectively. It is well known that a root of  $f(t) \in \mathbb{A}[t]$  is also algebraic. Moreover, given the representations of  $a, b \in \mathbb{A}$ , the representations of  $a \pm b$ ,  $\frac{a}{b}$  and  $a \cdot b$  can be computed in polynomial time, so is the equality checking [Coh13].

**Definition 2**  *$t^*$  is called a root of a function  $f(t)$  if  $f(t^*) = 0$ . Furthermore, the multiplicity of  $t^*$  is the maximum number  $m$  such that  $(t - t^*)^m$  is a factor of  $f(t)$ , i.e., there exists a function  $g(t)$  such that  $f(t) = g(t)(t - t^*)^m$ . Furthermore, if  $m = 1$ , then we call that  $t^*$  is a single root; otherwise  $t^*$  is a multiple root.*

## 2.1 The Real Root Isolation of Polynomial Exponential Functions

**Definition 3** *A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a polynomial-exponential function (PEF) if  $f$  has the following form:*

$$f(t) = \sum_{k=0}^K f_k(t) e^{\lambda_k t} \quad (1)$$

where for all  $0 \leq k \leq K < \infty$ ,  $f_k(t) \in \mathbb{F}_1[t]$ ,  $\lambda_k \in \mathbb{F}_2$  and  $\mathbb{F}_1, \mathbb{F}_2$  are fields. Furthermore,  $\text{Power}(f)$  denotes the set of power factors of  $f(t)$ , i.e.,  $\text{Power}(f) = \{\lambda_k\}_{k=0}^K$ .

We define the degree of  $f(t)$  be the maximum degree of  $f_k(t)$ s.

We usually write this PEF as  $f(t, e^{\lambda_0 t}, e^{\lambda_1 t}, \dots, e^{\lambda_K t})$ .

Generally, if the range of  $f(t)$  is in complex numbers  $\mathbb{C}$ , then let  $f'(t) = f(t) + f^*(t)$  is a PEF with the range in real numbers  $\mathbb{R}$ , where  $f^*(t)$  is the complex conjugate of  $f(t)$ . The factor  $t$  is omitted whenever convenient, i.e.,  $f = f(t)$ .

PEFs often appear in transcendental number theory as auxiliary functions in proofs involving the exponential function [Bak90].

**Definition 4** A (real) root isolation of function  $f(t)$  in interval  $\mathcal{T}$  is a set of mutually disjoint intervals, denoted by  $\text{Iso}(f)_{\mathcal{T}} = \{(a_j, b_j) \subseteq \mathcal{T}\}$  for  $a_j, b_j \in \mathbb{Q}$  such that

- for any  $j$ , there is at least and only one root of  $f(t)$  in  $(a_j, b_j)$ ;
- for any root  $\lambda$  of  $f(t)$ ,  $\lambda \in (a_j, b_j)$  for some  $j$ .

Furthermore, if  $f$  has no any root in  $\mathcal{T}$ , then  $\text{Iso}(f)_{\mathcal{T}} = \emptyset$ .

Although there are infinite kinds of real root isolations of  $f(t)$  in  $\mathcal{T}$ , the number of isolation intervals equals to the number of distinct roots of  $f(t)$  in  $\mathcal{T}$ .

Finding a real root isolation of PEFs is a long-standing problem and can at least backtrack to Ritt's paper [Rit29] in 1929. After that, some following results were obtained since the last century (e.g. [AH80, Tij71]). This problem plays an essential role in the reachability analysis of dynamical systems, one active field of symbolic and algebraic computation. In the case of  $\mathbb{F}_1 = \mathbb{Q}$  and  $\mathbb{F}_2 = \mathbb{N}^+$  in [AMW08], an algorithm named ISOL was proposed to isolate all real roots of  $f(t)$ . Later, this algorithm has been extended to the case of  $\mathbb{F}_1 = \mathbb{Q}$  and  $\mathbb{F}_2 = \mathbb{R}$  [GCL<sup>+</sup>17]. A variant of the problem has also been studied in [LHXL16]. The correctness of all these algorithms is based on very famous Schanuel's conjecture from transcendental number theory.

We will also pursue this problem in the context of continuous-time Markov chains. The distinct feature of solving the problem in our paper is to deal with complex numbers  $\mathbb{C}$ , more specifically algebraic numbers  $\mathbb{A}$ , i.e.,  $\mathbb{F}_1 = \mathbb{F}_2 = \mathbb{A}$ , while all the previous works can only handle the case over  $\mathbb{R}$ . Up to our best knowledge, finding a real root isolation of PEFs over  $\mathbb{A}$  has not been solved. From now on, we always assume that PEFs are over  $\mathbb{A}$ , i.e.,  $\mathbb{F}_1 = \mathbb{F}_2 = \mathbb{A}$ .

We remark that the algorithms for finding real root isolations of PEFs over  $\mathbb{R}$  can not be directly generalized to the case over  $\mathbb{C}$ . The main reason is that there are only finitely many real roots of PEF  $f(t)$  over  $\mathbb{R}$  in any interval  $\mathcal{T}$  [GCL<sup>+</sup>17], while there are infinitely many real roots over  $\mathbb{C}$  or  $\mathbb{A}$ , for example

$$f(t) = e^{i\theta t} + e^{-i\theta t} = 2\cos(\theta t) \quad \forall \theta \in \mathbb{R} \text{ and } \theta \neq 0$$

the real roots of  $f(t)$  is  $\{t = \frac{2k\pi + \pi}{2\theta}\}_{k \in \mathbb{N}}$ , i.e., infinitely many.

## 2.2 Existence Checking of Real Roots

Usually, finding a (real) root isolation of a function  $f$  bases on the decision problem: checking whether or not there is a root of  $f$  in some given interval  $\mathcal{T}$ . This can be done by the following proposition in [COW15].

**Proposition 1 ([COW15])** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a function defined on a closed interval of reals with endpoints  $a, b \in \mathbb{Q}$ . There is a procedure to decide the existence of the roots of  $f$  if  $f$  satisfies the following conditions:*

- 1 *there exists  $M > 0$  such that  $f$  is  $M$ -Lipschitz, i.e.,  $|f(s) - f(t)| \leq M|s - t|$  for all  $s, t \in [a, b]$ ;*
- 2 *given  $t \in [a, b] \cap \mathbb{Q}$  and positive error bound  $0 < \varepsilon \in \mathbb{Q}$ , we can compute  $q \in \mathbb{Q}$  such that  $|f(t) - q| < \varepsilon$ ;*
- 3  *$f(a) \neq 0$  and  $f(b) \neq 0$ ;*

4 for any  $t \in (a, b)$  such that  $f(t) = 0$ ,  $f'(t)$  exists and is non-zero, i.e.,  $f$  has no tangential zeros.

For our use in the latter discuss, we formate the procedure presented in [COW15] as Algorithm 1—*Exist()*.

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**Algorithm 1** *Exist*( $f, [a, b], \delta$ )

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**Require:** A  $f(t)$  satisfying all conditions in Proposition 1, a closed interval  $[a, b]$  and a positive number  $\delta > 0$

**Ensure:** **true** and **false** indicate whether or not there is a root of  $f(t)$  in closed interval  $[a, b]$

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1:  $M \leftarrow$  the Lipschitz constant
2:  $N \leftarrow \lceil \frac{4(b-a)M}{\delta} \rceil$ 
3: Let  $s_j = a + \frac{(b-a)j}{N}$  for  $j = 0, \dots, N$ .
4: For each sample point  $s_j$  compute  $q_j \in \mathbb{Q}$  such that  $|q_j - f(s_j)| < \delta/4$ .
5: Let  $f^-(s_j) = q_j - \delta/2$  and  $f^+(s_j) = q_j + \delta/2$ , and extend  $f^-$  and  $f^+$  linearly between sample points.
6: if  $f^+(t) < 0$  for all  $t \in [a, b]$  or  $f^-(t) > 0$  for all  $t \in [a, b]$  then
7:   return false
8: else if  $f^+(s) < 0$  and  $f^-(t) > 0$  for some  $s, t \in [a, b]$  then
9:   return true
10: else
11:   return Exist( $f, [a, b], \delta/2$ )
12: end if
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### 2.3 Jordan Decomposition

To reduce our continuous-time model checking problem to the real root isolation of PEFs, we need to use the *Jordan decomposition*.

**Definition 5** A Jordan block is a square matrix of the following form

$$\begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}.$$

A square matrix  $J$  is in Jordan norm form if

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_n \end{bmatrix},$$

where  $J_k$  is a Jordan blocks for each  $1 \leq k \leq n$ .



Because  $\mathbb{A}$  is algebraic closed, we know that

**Proposition 2** *Any matrix  $A \in \mathbb{Q}^{n \times n}$  is algebraically similar to a matrix in Jordan normal form over algebraic number field  $\mathbb{A}$ . Namely, there exists some invertible  $P \in \mathbb{A}^{n \times n}$  and  $J \in \mathbb{A}^{n \times n}$  in Jordan form such that  $A = P^{-1}JP$ , where  $\mathbb{A}^{n \times n}$  is the set of all  $n$ -by- $n$  matrices with every entry being algebraic number.*

## 2.4 Transcendental Number Theory

In the latter discussion, we will see that transcendental number theory can be applied to compare time instants when the values are *transcendental numbers*. A transcendental number is a complex number that is not an algebraic number, such as  $\pi, e^\pi$ .

In general, it is extremely difficult to verify relationships between two transcendental numbers [Ric97]. But for some transcendental numbers represented by PEFs can be compared with the help of Lindemann-Weierstrass theorem.

**Theorem 1 (Lindemann-Weierstrass theorem)** *Let  $\eta_1, \dots, \eta_n$  be pairwise distinct algebraic complex numbers. Then there exists no equation  $\sum_k \lambda_k e^{\eta_k} = 0$  in which  $\lambda_1, \dots, \lambda_n$  are algebraic numbers and are not all zero.*

The following fact was observed in [ASSB00] by Lindemann-Weierstrass theorem.

**Observation 1** *Given a real number  $r \in \mathbb{R}$  of the form  $\sum_k \mu_k e^{\eta_k}$  where  $\mu_k$ 's and  $\eta_k$ 's are algebraic complex numbers, and the  $\eta_k$ 's are pairwise distinct, there is an effective procedure to compare the values of  $r$  and  $c$  for any  $c \in \mathbb{Q}$ .*

First by Lindemann-Weierstrass theorem, we know that  $r = c$  if and only if  $r = c = 0$  or  $c = -\lambda_k$  for some  $k$  with  $\eta_k = 0$ . Then the main idea of the above observation is for each  $k$ ,  $e^{\eta_k}$  can be approximated with an error of less than  $\varepsilon$  (when  $\varepsilon < 1$ ) by taking the first  $\lceil 3|\eta_k|^2/\varepsilon \rceil + 1$  terms of the Maclaurin expansion for  $e^{\eta_k}$ . This can be extended to obtain an upper bound on the number of terms needed to approximate  $r$  to within  $\varepsilon$ . Since the individual terms in the Maclaurin expansion are algebraic functions of the  $\eta_k$ 's, it follows that the approximations are algebraic. Then we can check if  $r > c$  by the comparison between the approximations and  $c$ .

For finding a real root isolation of PEFs, we also need the famous open problem Schanuel's conjecture to factoring a PEF into finite *irreducible* PEFs. Before introducing the conjecture, we need more concepts.

**Definition 6 (Algebraic independence)** *A set of complex numbers  $S = \{a_1, \dots, a_n\}$  is algebraically independent over  $\mathbb{Q}$  if the elements of  $S$  do not satisfy any nontrivial (non-constant) polynomial equation with coefficients in  $\mathbb{Q}$ .*

By the above definition, for any transcendental number  $u$ ,  $\{u\}$  is algebraically independent over  $\mathbb{Q}$ ,  $\{a\}$  for any algebraic number  $a \in \mathbb{A}$  is not. Thus, a number in an algebraically independent set over  $\mathbb{Q}$  must be a transcendental number. Excluding singlet sets,  $\{\pi, e^{\pi\sqrt{n}}\}$  is also algebraically independent over  $\mathbb{Q}$  for any positive integer  $n$  [Nes96]. Checking the algebraic independence is a challenging problem, and there are many open problems. It is still not known whether  $\{e, \pi\}$  is algebraically independent over  $\mathbb{Q}$ .

**Definition 7 (Transcendence degree)** *Let  $L$  be a field extension of  $\mathbb{Q}$ , the transcendence degree of  $L$  over  $\mathbb{Q}$  is defined as the largest cardinality of an algebraically independent subset of  $L$  over  $\mathbb{Q}$ .*

For instance, let  $\mathbb{Q}(e) = \{a + be \mid a, b \in \mathbb{Q}\}$  and  $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$  be two field extensions of  $\mathbb{Q}$ . Then the transcendence degree of them are 1 and 0, respectively, by noting that  $e$  is a transcendental number and  $\sqrt{2}$  is an algebraic number.

**Conjecture 2 (Schanuel's conjecture)** *Given any complex numbers  $z_1, \dots, z_n$  that are linearly independent over  $\mathbb{Q}$ , the extension field  $\mathbb{Q}(z_1, \dots, z_n, e^{z_1}, \dots, e^{z_n})$  has transcendence degree of at least  $n$  over  $\mathbb{Q}$ .*

This conjecture was proposed by Stephen Schanuel during a course given by Serge Lang at Columbia in the 1960s [Lan66]. Schanuel's conjecture concerning the transcendence degree of certain field extensions of the rational numbers. The conjecture, if proven, would generalize the most known results in transcendental number theory [Ter08, MW96]. For example,  $\{e, \pi\}$  is algebraically independent simply by setting  $z_1 = 1$  and  $z_2 = \pi i$ , and using Euler's identity  $e^{\pi i} + 1 = 0$ .

**Corollary 1** [GCL<sup>+</sup>17, Corollary 3] *Let  $a_1, \dots, a_n$  be algebraic numbers that are linearly independent over  $\mathbb{Q}$ . Based on Schanuel's conjecture, the transcendence degree of the field extension  $\mathbb{Q}(t, e^{a_1 t}, \dots, e^{a_n t})$  is at least  $n$  if  $t \neq 0$ .*

## 2.5 Factoring Multivariate Polynomials over Algebraic Number Fields

Recall that PEF  $f(t)$  in Eq.(1) over algebraic number field has the form:

$$f(t) = \sum_{k=0}^K f_k(t) e^{\lambda_k t} \quad (2)$$

where  $f_k(t) \in \mathbb{A}[t]$  and  $\lambda_k \in \mathbb{A}$  for all  $0 \leq k \leq K$ .

$f(t)$  is called *exponential* if there exists some  $a, \eta \in \mathbb{C}$  such that

$$f(t) \equiv a \cdot e^{\eta t}.$$

$f(t, e^{a_1 t}, \dots, e^{a_n t})$  is called *irreducible* in  $\mathbb{A}$  if it has only trivial factoring as follows

$$f(t, e^{a_1 t}, \dots, e^{a_n t}) = e^{\mu t} \cdot g(t, e^{b_1 t}, \dots, e^{b_n t}),$$

for some  $\mu, b_i \in \mathbb{A}$  and a PEF  $g(t)$ .

We are interested in factoring a non exponential PEF into product of PEFs, i.e.,

$$f(t, e^{a_1 t}, \dots, e^{a_n t}) = e^{\mu t} \cdot f_1^{m_1}(t, e^{b_{1,1} t}, \dots, e^{b_{1,n_1} t}) \cdots f_s^{m_s}(t, e^{b_{s,1} t}, \dots, e^{b_{s,n_s} t}),$$

for some  $\mu, b_{i,j} \in \mathbb{C}$ , and non-exponential  $f_i(t, e^{b_{i,1} t}, \dots, e^{b_{i,n_i} t})$ .

**Definition 8** *A PEF  $f(t)$  is square free if*

$$f(t) \not\equiv f_1^2(t) \cdots f_s(t)$$

*where  $f_i(t)$ s are PEFs and  $f_1(t)$  is not exponential.*

There is an efficient algorithm to factor multivariate polynomials over  $\mathbb{A}$  with polynomial-time complexity in the degrees of the polynomial to be factored [Len87]. The main idea is that the multivariate polynomial  $f(x_1, \dots, x_n)$  is first reduced to a polynomial in just one variable by substituting properly selected integers (e.g.,  $x_2 = b_1, \dots, x_n = b_n$ ) for all but one variable  $x_1$ . The resulting univariate polynomial  $f(x_1, b_1, \dots, b_n)$  is then factored over  $\mathbb{A}$ . Then the well-known algorithm for factoring univariate polynomial over rational numbers [LLL<sup>+</sup>82] can be generalized to factor  $f(x_1, b_1, \dots, b_n)$ .

We can employ this algorithm for factoring multivariate polynomials into the factoring of PEFs, an important step to get real root isolation of PEF  $f(t)$  over algebraic number field  $\mathbb{A}$ .

If  $f(t)$  is exponential, the factoring is trivial. Assume  $f(t)$  is not exponential, and we have the following factoring,

$$f(t) = \sum_{k=0}^K f_k(t) e^{\lambda_k t} = f_1^{m_1}(t, e^{b_{1,1}t}, \dots, e^{b_{1,n_1}t}) \cdots f_s^{m_s}(t, e^{b_{s,1}t}, \dots, e^{b_{s,n_s}t}),$$

where  $f_i(t)$ s are not exponential.

We know that  $\lambda_k$  can be written as some natural number combinations of  $b_{i,j}$ 's and this is sufficient requirement for  $b_{i,j}$ 's by expanding this equation observing that  $\sum_{k=0}^K f_k(t) e^{\lambda_k t} \equiv 0$  if and only if  $f_k(t) \equiv 0$  for all  $k$ .

We first compute an integral basis of  $\lambda_0, \dots, \lambda_K$ , says  $a_1, \dots, a_n$ . Then  $a_1, \dots, a_n$  are linearly independent over  $\mathbb{Q}$ , and  $f(t)$  is a *multivariate polynomial* of  $t, e^{a_1 t}, \dots, e^{a_n t}, e^{-a_1 t}, \dots, e^{-a_n t}$ , denoted by  $P(t, e^{a_1 t}, \dots, e^{a_n t}, e^{-a_1 t}, \dots, e^{-a_n t})$ . We can always choose  $b_{i,j}$  to be some natural number combination of  $a_1, \dots, a_n, -a_1, \dots, -a_n$ , thus,  $f_i(t)$  are all *multivariate polynomials* of  $t, e^{a_1 t}, \dots, e^{a_n t}, e^{-a_1 t}, \dots, e^{-a_n t}$ , denoted by  $P_i(t, e^{a_1 t}, \dots, e^{a_n t}, e^{-a_1 t}, \dots, e^{-a_n t})$ . Therefore,

$$P(t, e^{a_1 t}, \dots, e^{a_n t}, e^{-a_1 t}, \dots, e^{-a_n t}) = \prod_{i=1}^s P_i^{m_i}(t, e^{a_1 t}, \dots, e^{a_n t}, e^{-a_1 t}, \dots, e^{-a_n t})$$

Comparing the polyomial degrees of both sides. The degree of the left hand side is the maximum degree of  $f_k(t)$ s, says  $d$ . The right hand side is at least  $\sum_{i=1}^s m_i$  because the degree of PEFs is additive and each non-exponential PEF has degree at least 1.

Therefore,

$$\sum_{i=1}^s m_i \leq d.$$

There exists a positive integer  $r$  such that

$$\begin{aligned} P(t, e^{a_1 t}, \dots, e^{a_n t}, e^{-a_1 t}, \dots, e^{-a_n t}) \cdot e^{\sum_{i=1}^n r d a_i t} &= Q(t, e^{a_1 t}, \dots, e^{a_n t}) \\ P_i(t, e^{a_1 t}, \dots, e^{a_n t}, e^{-a_1 t}, \dots, e^{-a_n t}) \cdot e^{\sum_{i=1}^n r m_i a_i t} &= Q_i(t, e^{a_1 t}, \dots, e^{a_n t}) \end{aligned}$$

for some multivariate polynomials  $Q$  and  $Q_i$ s.

Now we have

$$Q(t, e^{a_1 t}, \dots, e^{a_n t}) = \prod_{i=1}^s Q_i(t, e^{a_1 t}, \dots, e^{a_n t}) \cdot e^{\sum_{i=1}^n h a_i t},$$

for some integer  $h$ .

One can use the factoring algorithm for multivariate polynomials by regarding  $h$  and  $h_i$ 's as multivariate polynomial on  $t, e^{a_1 t}, \dots, e^{a_n t}$ .

In Section 6, we will prove that assuming Schanuel's conjecture,  $f(t)$  has only single roots excepting 0 if  $f(t)$  is square free. Thus the algorithm of factoring multivariate polynomials over  $\mathbb{A}$  can be used to getting  $f'(t)$  from PEF  $f(t)$  such that  $f'(t)$  inherits all roots of  $f(t)$  and each root is single excepting 0. As we will see latter, this simplification plays essential role in finding a root isolation of  $f(t)$ .

### 3 Symbolic Dynamics of Continuous-time Markov Chains

We begin with *continuous-time Markov chains*. A continuous-time Markov chain is a Markovian (i.e. memoryless) stochastic process that takes values on a finite state set  $S$  ( $|S| = d < \infty$ ) and evolves in continuous-time  $t \in \mathbb{R}^+$ . More formally:

**Definition 9** A stochastic process  $\{X(t) : t \in \mathbb{R}^+\}$  with finite state set  $S$  is a continuous-time Markov chain if it satisfies the Markov property, i.e. for any  $k \in \mathbb{R}^+$ :

$$p(X_{t+k} = j | X_t = i, \{X_h : 0 \leq h < t\}) = p(X_{t+k} = j | X_t = i),$$

where  $i, j \in S$  and  $p(\cdot | \cdot)$  is the conditional probability of events. The number of states  $d = |S|$  is the dimension of the chain.

It turns out that a continuous-time Markov chain is fully characterized by a *transition rate matrix*  $Q$  [GS07].  $Q$  is  $d$ -by- $d$  matrix and the off-diagonal entries  $\{Q_{i,j}\}_{i \neq j}$  are nonnegative rational numbers, representing the transition rate from state  $i$  to state  $j$ , while the diagonal entries  $\{Q_{j,j}\}$  are constrained to be  $-\sum_{i \neq j} Q_{i,j}$  for all  $1 \leq j \leq d$ . Consequently, the row summations of  $Q$  are all zero. The dynamic of a continuous-time Markov chain is fully described by a master equation, which is a system of coupled ordinary differential equations that describe how the probability distribution changes over time for each of the states of the system. Specifically, the master equation is:

$$dP(t) = P(t)Qdt,$$

where  $P(t)$  is the  $d$ -by- $d$  transition probability matrix at time  $t$ ; the quantity  $P_{i,j}(t) = p(X_t = j | X_0 = i)$  denotes the probability from state  $i$  at time 0 to state  $j$  at time  $t \in \mathbb{R}^+$ . The master equation is solved subject to initial conditions  $P(0) = I$ , the identity matrix:

$$P(t) = e^{Qt}.$$

Thus, given an initial distribution  $\mu_0$ , the distribution at time  $t \in \mathbb{R}^+$  is:

$$\mu_t = P(t)\mu_0 = e^{Qt}\mu_0.$$

Therefore, we have the brief definition of continuous-time Markov chains:

**Definition 10** A continuous-time Markov chain is a pair  $\mathcal{M} = (S, Q)$ , where  $S$  ( $|S| = d$ ) is a finite state set,  $Q$  a  $d$ -by- $d$  transition rate matrix.

W.l.o.g, we always denote  $S = \{s_1, s_2, \dots, s_d\}$  in this paper.

**Example 1** Consider continuous-time Markov chain  $\mathcal{M} = (S, Q)$ , where  $|S| = 3$ ,

$$Q = \begin{pmatrix} -0.025 & 0.02 & 0.005 \\ 0.3 & -0.5 & 0.2 \\ 0.02 & 0.4 & -0.42 \end{pmatrix}$$

Given continuous-time Markov chain  $\mathcal{M} = (S, Q)$ , we denote  $\mathcal{D}(S)$  the set of all distributions over  $S$ , which is called the *state space* of continuous-time Markov chains, while in previous work (e.g. [ASSB00, CHKM11a, BHHK03]),  $S$  is regarded as the state space of the chain. The path of  $\mathcal{M}$  is a continuous function indexed by an initial distribution  $\mu \in \mathcal{D}(S)$ :

$$\sigma_\mu: \mathbb{R}^+ \rightarrow \mathcal{D}(S), \quad \sigma_\mu(t) = e^{Qt}\mu. \quad (3)$$

We remark that defining path  $\sigma_\mu$  over distributions transitions is different to the usual way over the transitions of states set  $S$ , i.e., functions  $\omega_s: \mathbb{R}^+ \rightarrow S$  with the starting state  $s$ . Furthermore,  $\omega_s$  must have *finite variability*, i.e., its set of discontinuities has no accumulation points (in other words, on any finite interval the value of  $\omega_s$  can only change a finite number of times). This restriction is for applying analyzing methods of discrete-time Markov chains on the continuous-time counterpart as the domain  $\omega_s$  is actually  $\mathbb{N}^+$  rather than  $\mathbb{R}^+$ . However, we do not put the constrain on the path  $\sigma_\mu$  such that we track all time instants not a part domain of continuous-time Markov chains. Thus in any time interval, the number of  $\sigma_\mu$  changing may be infinite.

We move to the *symbolic dynamic (path)* of continuous-time Markov chains. The *symbolization of distributions* is a generalization of the discretization of distributions in [AAGT15]. First, we fix a finite set of intervals  $\mathcal{J} = \{\mathcal{I}_k \subseteq [0, 1]\}_{k \in K}$ . With the states  $S = \{s_j\}_{j=1}^d$ , we define the symbolization of distributions as a function:

$$\mathbb{S}: \mathcal{D}(S) \rightarrow 2^{S \times \mathcal{J}} \quad \mathbb{S}(\mu) = \{\langle j, \mathcal{I}_k \rangle : \mu(j) \in \mathcal{I}_k\}. \quad (4)$$

where  $\times$  denotes the Cartesian product, and  $2^{S \times \mathcal{J}}$  is the set of all subsets of  $S \times \mathcal{J}$ .

$\langle j, \mathcal{I}_k \rangle \in \mathbb{S}(\mu)$  asserts that the  $j$ -th element of  $\mu$  is in the interval  $\mathcal{I}_k$ . Specially, in [AAGT15],  $\mathcal{J}$  must be fixed as a partition of  $[0, 1]$ , i.e.,  $\mathcal{I}_k \cap \mathcal{I}_m = \emptyset$  for all  $k \neq m$ . Thus for any  $\mu \in \mathcal{D}(S)$ , we can label it by finite symbols from  $S \times \mathcal{J}$ .

**Example 2** *Considering distributions*

$$\mu_0 = \begin{pmatrix} 0.3 \\ 0.3 \\ 0.4 \end{pmatrix}, \mu_1 = \begin{pmatrix} 0 \\ 0.7 \\ 0.3 \end{pmatrix}$$

suppose

$$\mathcal{J} = \{[0, 0.2], [0.4, 0.4], (0.5, 0.7], (0.3, 1]\}.$$

Then,  $\mathbb{S}(\mu_0) = \{\langle 3, [0.4, 0.4] \rangle, \langle 3, (0.3, 1] \rangle\}$ ,  $\mathbb{S}(\mu_1) = \{\langle 1, [0, 0.2] \rangle, \langle 2, (0.5, 0.7] \rangle, \langle 2, (0.3, 1] \rangle\}$ .

The benefits of the symbolization are that in practice, we do not care the exact probability of some state but the range of the probability, which can be represented by intervals. In the next section, we will further explain this with our temporal logic CLL.

With the symbolization, we have

**Definition 11** A symbolized continuous-time Markov chain is a tuple  $\mathcal{SM} = (S, Q, \mathcal{J})$ , where  $(S, Q)$  is a continuous-time Markov chain and  $\mathcal{J}$  is a finite set of intervals in  $[0, 1]$ .

Throughout this paper, we always assume that  $\mathcal{SM}$  is rational, i.e., all elements of  $Q$ , and for all  $\mathcal{I} \in \mathcal{J}$  is a rational interval, i.e., the endpoints must be rational numbers. Thus  $(a, b)$ ,  $(a, b]$ ,  $[a, b)$  and  $[a, b]$  are all valid rational intervals for all  $a, b \in \mathbb{Q}$ . Furthermore, the elements of all the distributions are also rational.

Next, we extend this symbolization to the path  $\sigma_\mu$ :

$$\mathbb{S} \circ \sigma_\mu: \mathbb{R}^+ \rightarrow 2^{S \times \mathcal{J}}. \quad (5)$$

**Definition 12** Given a symbolized continuous-time Markov chain  $\mathcal{SM}$ ,  $\mathbb{S} \circ \sigma_\mu$  is a symbolic dynamic (path) of  $\mathcal{M}$ .

In the end of this section, we prove that the path (continuous function)  $\sigma_\mu = e^{Qt}\mu$  is a system of PEFs over algebraic number  $\mathbb{A}$ .

**Lemma 1** Given a continuous-time Markov chain  $\mathcal{M} = (S, Q)$  with an initial distribution  $\mu$ , for any  $1 \leq i \leq d$  ( $|S| = d$ ),  $e^{Qt}\mu(i)$ , the  $i$ -th element of  $e^{Qt}\mu$ , can be expressed as a PEF  $f : \mathbb{R}^+ \rightarrow [0, 1]$  of  $f(t) = \sum_{k=0}^K f_k(t)e^{\lambda_k t}$  over  $\mathbb{A}$ .

*Proof.* As the elements of  $\mu$  are rational, we only need to prove that any entry of  $e^{Qt}$  can be expressed as a finite sum of  $\sum_k f_k(t)e^{\eta_k t}$  for  $\eta_k \in \mathbb{A}$  and  $f_k(t) \in \mathbb{A}[t]$ .

By Proposition 2, we have that there is a  $P \in \mathbb{A}^{n \times n}$  such that  $Q = P^{-1}(\oplus_k J_k)P$  such that

$$J_k = \begin{bmatrix} \lambda_k & 1 & 0 & \cdots & 0 \\ 0 & \lambda_k & 1 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & \lambda_k \end{bmatrix}$$

Note that  $\lambda_k$  is an eigenvalue of  $Q$  and  $Q \in \mathbb{Q}^{d \times d}$ , so  $\lambda_k$  is algebraic. Furthermore,  $J_k \in \mathbb{A}^{n_k \times n_k}$ , where  $n_k$  is the dimension of  $J_k$ .

Therefore,  $e^{Qt} = P^{-1}e^{\oplus_k J_k t}P = P^{-1}(\oplus_k e^{J_k t})P$ . So we complete the proof by proving that for each  $k$ , any entry of  $e^{J_k t}$  can be expressed as a finite sum of  $\sum_k f_k(t)e^{\eta_k t}$  for  $\eta_k \in \mathbb{A}$  and  $f_k(t) \in \mathbb{A}[t]$ . Computing  $e^{J_k t}$ , we obtain that

$$e^{J_k t} = \begin{bmatrix} e^{\lambda_k t} & te^{\lambda_k t} & t^2 e^{\lambda_k t}/2! & \cdots & t_k^n e^{\lambda_k t}/n_k! \\ 0 & e^{\lambda_k t} & te^{\lambda_k t} & \cdots & t^{n_k-1} e^{\lambda_k t}/(n_k-1)! \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & e^{\lambda_k t} \end{bmatrix}.$$

□

By the above lemma, analyzing dynamics  $\sigma_\mu$  of continuous-time Markov chains is equivalent to studying corresponding PEFs.

## 4 Continuous Linear Logic

In this section, we introduce *continuous linear logic (CLL)* to specify the temporal properties of symbolized continuous-time Markov chain  $\mathcal{SM} = (S, Q, \mathcal{J})$ . In summary, CLL is a linear-time temporal logic. Unlike LTL, CLL has the path and state formulas expressing temporal properties of  $\mathcal{SM}$ . However, the path formulas in CLL are simpler because the only applied temporal operator is a timed version of until operator. Furthermore, the nesting of until temporal operator is allowed, which does not appear in the previous work [BHHK03]. More importantly, CLL formulas can express not only absolutely temporal properties (all-time instants basing on one starting time) but also relative versions. Up to our best knowledge, this is the first logic with this distinctive feature in the context of the verification of continuous-time Markov chains.

**Remark:** The next-step operator is important in the expressiveness of LTL and CSL with time-domain non-negative positive numbers  $\mathbb{N}^+$ . However, the time domain of CLL is non-negative real numbers  $\mathbb{R}^+$ , so “next-step” is not meaningful.

**Definition 13** *CLL path formulas are formed according to the following grammar:*

$$\varphi := \mathbf{true} \mid \Phi_0 U^{\mathcal{T}_1} \Phi_1 U^{\mathcal{T}_2} \Phi_2 \dots U^{\mathcal{T}_n} \Phi_n \mid \neg \varphi \mid \varphi_1 \wedge \varphi_2$$

where  $n \in \mathbb{N}^+$  is a positive integer, for all  $0 \leq k \leq n$ ,  $\Phi_k$  is a state formula, and  $\mathcal{T}_k$ 's are arbitrary rational time intervals (with rational endpoints) in  $\mathbb{R}^+$ , i.e.,  $\mathcal{T}_k$  is one of the forms:

$$(a, b), [a, b], (a, b], [a, b) \quad \forall a, b \in \mathbb{Q}^+.$$

The syntax of CLL state formulas is described by the following grammar:

$$\Phi := \mathbf{true} \mid a \in AP \mid \neg \Phi \mid \Phi_1 \wedge \Phi_2$$

where  $AP$  denotes the set of atomic propositions.

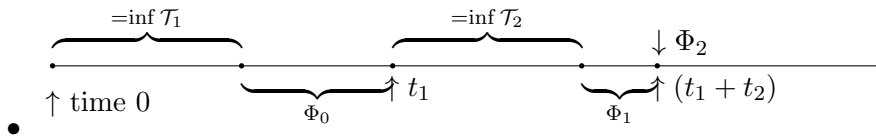
For  $n = 1$ , the until operator  $\Phi_0 U^{\mathcal{T}_1} \Phi_1$  is a timed variant of the until operator of LTL; the path formula  $\Phi_0 U^{\mathcal{T}_1} \Phi_1$  asserts that  $\Phi_1$  is satisfied at some time instant in the interval  $\mathcal{T}_1$  and that at all preceding time instants holds  $\Phi_0$ . This can be extended to arbitrary  $n$  by noting that  $U^{\mathcal{T}}$  is right-associative, e.g.,  $\Phi_0 U^{\mathcal{T}_1} \Phi_1 U^{\mathcal{T}_2} \Phi_2$  stands for  $\Phi_0 U^{\mathcal{T}_1} (\Phi_1 U^{\mathcal{T}_2} \Phi_2)$ . One important point of CLL is that we consider the relative time, i.e.,  $\mathcal{T}_1$  and  $\mathcal{T}_2$  do not have to be disjoint, and the starting time point 0 of  $\mathcal{T}_2$  is based on some time event in  $\mathcal{T}_1$ . For instance  $\Phi_0 U^{[3,9]} \Phi_1 U^{(2,7)} \Phi_2$  is a valid path formula, which is different to CLL path formula  $\Phi_0 U^{[3,9]} \Phi_1 \wedge \Phi_1 U^{(2,7)} \Phi_2$ . This picture will be more clear by the following semantics of CLL.

Given a symbolized continuous-time Markov chain  $\mathcal{SM} = (S, Q, \mathcal{J})$ , the semantics of CLL path formulas is defined on paths  $\{\sigma_\mu\}_{\mu \in \mathcal{D}(S)}$ .

- $\sigma_\mu \models \mathbf{true}$  for all distributions  $\mu \in \mathcal{D}(S)$ ;
- $\sigma_\mu \models \Phi_0 U^{\mathcal{T}_1} \Phi_1 U^{\mathcal{T}_2} \Phi_2 \dots U^{\mathcal{T}_n} \Phi_n$  iff there is a time  $t \in \mathcal{T}_1$  such that  $\sigma_{\mu_t} \models \Phi_1 U^{\mathcal{T}_2} \Phi_2 \dots U^{\mathcal{T}_n} \Phi_n$ , and for any  $t' \in [0, t) \cap \mathcal{T}_1$ ,  $\mu_{t'} \models \Phi_0$ , where  $\sigma_\mu \models \Phi$  iff  $\mu \models \Phi$ ,  $\mu_t = e^{Qt} \mu \forall t \in \mathbb{R}^+$ ;
- $\sigma_\mu \models \neg \varphi$  iff it is not the case that  $\sigma_\mu \models \varphi$  (written  $\sigma_\mu \not\models \varphi$ );
- $\sigma_\mu \models \varphi_1 \wedge \varphi_2$  iff  $\sigma_\mu \models \varphi_1$  and  $\sigma_\mu \models \varphi_2$ .

As we see, the CLL path formula  $\sigma_\mu \models \Phi_0 U^{\mathcal{T}_1} \Phi_1 U^{\mathcal{T}_2} \Phi_2 \dots U^{\mathcal{T}_n} \Phi_n$  is explained over the induction on  $n$ . This makes time instants relative. This can be further explained by comparing  $\Phi_0 U^{\mathcal{T}_1} \Phi_1 U^{\mathcal{T}_2} \Phi_2$  and  $\Phi_0 U^{\mathcal{T}_1} \Phi_1 \wedge \Phi_1 U^{\mathcal{T}_2} \Phi_2$ .

- $\sigma_\mu \models \Phi_0 U^{\mathcal{T}_1} \Phi_1 U^{\mathcal{T}_2} \Phi_2$  asserts that there are time instants  $t_1 \in \mathcal{T}_1, t_2 \in \mathcal{T}_2$  such that  $\mu_{t_1+t_2} \models \Phi_2$  and for any  $t'_1 \in \mathcal{T}_1 \cap [0, t_1)$  and  $t'_2 \in \mathcal{T}_2 \cap [0, t_2)$ ,  $\mu_{t'_1} \models \Phi_0$  and  $\mu_{t_1+t'_2} \models \Phi_1$ , where  $\mu_t = e^{Qt} \mu \forall t \in \mathbb{R}^+$ . This is more clear in the following picture.



- $\sigma_\mu \models \Phi_0 U^{\mathcal{T}_1} \Phi_1 \wedge \Phi_1 U^{\mathcal{T}_2} \Phi_2$  asserts that there are time instants  $t_1 \in \mathcal{T}_1, t_2 \in \mathcal{T}_2$  such that  $\mu_{t_1} \models \Phi_1$  and  $\mu_{t_2} \models \Phi_2$ , and for any  $t'_1 \in \mathcal{T}_1 \cap [0, t_1)$  and  $t'_2 \in \mathcal{T}_2 \cap [0, t_2)$ ,  $\mu_{t'_1} \models \Phi_0$  and  $\mu_{t'_2} \models \Phi_1$ , where  $\mu_t = e^{Qt} \mu \forall t \in \mathbb{R}^+$ .

The semantics of CLL state formula are defined on the set  $\mathcal{D}(S)$  of distributions over  $S$  with the symbolized function  $\mathbb{S}$  in Eq.(4).

- $\mu \models \mathbf{true}$  for all distributions  $\mu \in \mathcal{D}(S)$ ;
- $\mu \models a$  iff  $a \in \mathbb{S}(\mu)$ ;
- $\mu \models \neg\Phi$  iff it is not the case that  $\mu \models \Phi$  (written  $\mu \not\models \Phi$ );
- $\mu \models \Phi_1 \wedge \Phi_2$  iff  $\mu \models \Phi_1$  and  $\mu \models \Phi_2$ .

Other Boolean connectives are derived in the standard way, i.e.,  $\mathbf{false} = \neg\mathbf{true}$ ,  $\Phi_1 \vee \Phi_2 = \neg(\neg\Phi_1 \wedge \neg\Phi_2)$  and  $\Phi_1 \rightarrow \Phi_2 = \neg\Phi_1 \vee \Phi_2$ , and the path formulas  $\varphi$  follows the same way.

Temporal operators like  $\diamond$  (eventually), its timed variant  $\diamond^T$  is derived as follows:

$$\diamond^T \Phi = \mathbf{true} U^T \Phi.$$

For  $\square$  (always), its timed variant  $\square^T$ , we have:

$$\square^T \Phi = \neg \diamond^T \neg \Phi.$$

In this paper, we aim to solve the decidability of model checking continuous-time Markov chains against CLL formulas.

**Problem 2 (CLL Model-checking Problem)** *Given a symbolized continuous-time Markov chain  $\mathcal{SM} = (S, Q, \mathcal{J})$  with an initial distribution  $\mu$  and a CLL path formula  $\varphi$  on  $AP = S \times \mathcal{J}$ , the goal is to decide whether  $\sigma_\mu \models \varphi$ .*

Before proceeding to solve this model-checking problem, we shall first consider what can be specified in our logic CLL.

Given a continuous-time Markov chain  $(S, Q)$  with initial distribution  $\mu$ , the liveness property  $\diamond^{[0,1000]} \langle k, [1, 1] \rangle$  expresses state  $s_k \in S$  is eventually reached with probability one before time instants 1000. The safety property  $\square^{[100,1000]} \langle k, [0, 0] \rangle$  expresses that state  $s_k \in S$  is always not reached (with probability zero) in time interval  $[100, 1000]$ . Furthermore, setting the intervals non-trivial (not  $[0, 0]$  or  $[1, 1]$ ), liveness and safety can be asserted with probabilities, such as  $\diamond^{[0,1000]} \langle k, [0.5, 1] \rangle$  and  $\square^{[100,1000]} \langle k, [0, 0.5] \rangle$ . Further,

$$\langle k, [0.7, 1] \rangle U^{[2,3]} \langle k, [0.7, 1] \rangle \dots U^{[2,3]} \langle k, [0.7, 1] \rangle,$$

where the number of  $U^{[2,3]}$  is 100, asserts that the probability of  $s_k$  is beyond 0.7 in every time instant 2 to 3, and this happens at least 100 times.

Next, we can classify members of  $\mathcal{J}$  as representing “low” and “high” probabilities. For example, if  $\mathcal{J}$  contains 3 intervals  $\{[0, 0.2], (0.2, 0.8), [0.8, 1]\}$ , we can declare the first interval as “low” and the last interval as “high”. In this case  $\square^{[10,1000]}(\langle i, [0, 0.2] \rangle \rightarrow \langle j, [0.8, 1] \rangle)$  will say that whenever the probability of  $s_i$  is low, the probability of  $s_j$  will be high in time interval  $[10, 1000]$ .



## 5 CLL Model Checking

In this section, we establish that there is an algorithm to model checking continuous-time Markov chains against CLL formulas, i.e., the model-checking problem in Problem 2 is decidable. This result mainly depends on the following theorem about the root isolation of PEFs. Thus the correctness of our model checking algorithm is based on Schanuel's conjecture.

**Theorem 3** *Based on Schanuel's conjecture, given a PEF  $f(t)$  and a bounded time interval  $\mathcal{T}$ , there is an algorithm (Algorithm 2) to find a real root isolation  $\text{Iso}(f)_{\mathcal{T}}$  of  $f(t)$ . Furthermore, the*

*number of real roots is finite, i.e.,  $|\text{Iso}(f)| < \infty$ . Does this depend on Schanuel's conjecture?*

For the readability, the proof of the above theorem will be postponed in the next section.

**Theorem 4** *Based on Schanuel's conjecture, the CLL model checking problem in Problem 2 is decidable.*

*Proof.* The nontrivial step is to model check formulas of the forms

$$\Phi_0 U^{\mathcal{T}_1} \Phi_1 U^{\mathcal{T}_2} \Phi_2 \dots U^{\mathcal{T}_n} \Phi_n,$$

where  $\{\mathcal{T}_j\}_{j=1}^n$  is a set of bounded rational intervals in  $\mathbb{R}^+$ , and for  $0 \leq k \leq n+1$ ,  $\Phi_k$  is a state formula by the following grammar:

$$\Phi := \mathbf{true} \mid a \in AP \mid \neg\Phi \mid \Phi_1 \wedge \Phi_2.$$

Thus  $\Phi_k$  is actually the same to the formula of propositional logics. So  $\Phi$  admits *conjunctive normal form (CNF)*, i.e.,

$$\Phi_k = \bigwedge_{j \in J_k} \bigvee_{l \in L_k} \text{lit}_{j,l},$$

where  $\text{lit}_{j,l}$  is a *literal* of  $a \in AP$  or  $\neg a$ , and  $J_k$  and  $L_k$  are finite sets. Furthermore, we observe that  $\neg a$  can be represented by the disjunction of (at most) two extra atomic propositions. For example, if  $a = \langle k, [0.1, 0.9] \rangle$ , then  $\neg a = a_1 \vee a_2$ , where  $a_1 = \langle k, [0, 0.1] \rangle$  and  $a_2 = \langle k, (0.9, 1] \rangle$ . Therefore, the left problem is to model check

$$\sigma_\mu \models \Phi_0 U^{\mathcal{T}_1} \Phi_1 U^{\mathcal{T}_2} \Phi_2 \dots U^{\mathcal{T}_n} \Phi_n, \quad (6)$$

where  $\Phi_k = \bigwedge_{j \in J_k} \bigvee_{l \in L_k} a_{j,l}$  and  $a_{j,l} \in AP$  for all  $j, l$ .

In the following, we always assume that each  $a \in AP$  contains closed interval, i.e.,  $a = \langle k, [a, b] \rangle$  for  $a, b \in \mathbb{Q}$ , and  $\forall 1 \leq j \leq n$ ,  $\mathcal{T}_j$  is left closed, i.e.  $\mathcal{T}_j = [t_j, t'_j)$  for  $t_j, t'_j \in \mathbb{Q}^+$ . Other situations can also be handled in the similar way. First of all, we claim that the relative time relationship can be rewritten as absolute versions.

**Lemma 2**  $\sigma_\mu \models \Phi_0 U^{\mathcal{T}_1} \Phi_1 U^{\mathcal{T}_2} \Phi_2 \dots U^{\mathcal{T}_n} \Phi_n$  if and only if there exists time intervals  $\{\mathcal{I}_k \subseteq \mathbb{R}^+\}_{k=0}^n$  with  $\mathcal{I}_0 = [0, 0]$  such that

- The satisfaction of intervals: for all  $1 \leq k \leq n$ ,  $\mu_t \models \Phi_{k-1}$  for all  $t \in \mathcal{I}_k$ , and  $\mu_{t^*} \models \Phi_n$ , where  $t^* = \sup \mathcal{I}_n$  and  $\mu_t = e^{Qt} \mu \forall t \in \mathbb{R}^+$ ;
- The order of intervals: for all  $1 \leq k \leq n$ ,  $\mathcal{I}_k \subseteq \sup \mathcal{I}_{k-1} + \mathcal{T}_k$  and  $\inf \mathcal{I}_k = \sup \mathcal{I}_{k-1} + \inf \mathcal{T}_k$ .

*Proof.* We first prove the sufficient direction. Let  $\varphi_1 = \Phi_1 U^{\mathcal{T}_2} \Phi_2 \dots U^{\mathcal{T}_n} \Phi_n$ . Then the above formula is  $\Phi_0 U^{\mathcal{T}_1} \varphi_1$ . By the semantic of CLL, we have that there is a time  $t_1 \in \mathcal{T}_1$  such that  $\sigma_{\mu_{t_1}} \models \varphi_1$ , and for any  $t'_1 \in [0, t) \cap \mathcal{T}_1$ ,  $\mu_{t'_1} \models \Phi_0$ . Then let  $\varphi_2 = \Phi_2 U^{\mathcal{T}_3} \Phi_3 \dots U^{\mathcal{T}_n} \Phi_n$  and we get  $\varphi_1 = \Phi_1 U^{\mathcal{T}_2} \varphi_2$ . In the similar way, we have that there is a time  $t_2 \in \mathcal{T}_2$  such that  $\sigma_{\mu_{t_1+t_2}} \models \varphi_2$ , and for any  $t'_2 \in [0, t_2) \cap \mathcal{T}_2$ ,  $\mu_{t_1+t'_2} \models \Phi_1$ . Iteratively, we get a set of time instants  $\{t_k\}_{k=0}^n$  with  $t_0 = 0$ . Let  $\mathcal{I}_k = \sum_{j=0}^{k-1} t_j + [0, t_k) \cap \mathcal{T}_k$  for all  $1 \leq k \leq n$ . Then it is easy to check that  $\{\mathcal{I}_k \in \mathbb{R}^+\}_{k=0}^n$  with  $\mathcal{I}_0 = [0, 0]$  are the desired intervals satisfying the above two conditions.

Considering the necessary direction, by the above proof, we only need to identify  $\{t_k\}_{k=1}^n$  throughout intervals  $\{\mathcal{I}_k\}_{k=0}^n$ . Let  $t_k = \sup \mathcal{I}_k - \sup \mathcal{I}_{k-1}$  for all  $1 \leq k \leq n$ .  $\square$

Thus from the above lemma, for model checking the formula in Eq.(6), we only need to check the existence of time intervals  $\{\mathcal{I}_k\}_{k=0}^n$ . The following procedure can construct such set of intervals if it exists:

- (1) Let  $\mathcal{S}_0 = \{\mathcal{I}_0 = [0, 0]\}$  ;
- (2) For each  $1 \leq k \leq n$ , compute the set  $\mathcal{S}_k$  in  $[0, \sum_{j=1}^k \sup \mathcal{T}_j]$  of all maximum intervals such that  $\mu_t \models \Phi_{k-1}$  for all  $t \in \mathcal{I}$  of  $\mathcal{I} \in \mathcal{S}$ , where  $\mu_t = e^{Q_t} \mu$ , and an interval  $\mathcal{I}$  satisfying some property is maximum if there is no sub-interval  $\mathcal{I}' \subseteq \mathcal{I}$  satisfying the same property; Noting that  $\mathcal{S}_k$  can be the empty set, i.e.,  $\mathcal{S}_k = \emptyset$ ;
- (3) Let  $k$  from 1 to  $n+1$ . Updating  $\mathcal{S}_k$  by

$$\{\mathcal{I} \cap (\sup \mathcal{I}' + \mathcal{T}_k) : \inf \mathcal{I} \leq \sup \mathcal{I}' + \inf \mathcal{T}_k, \mathcal{I} \in \mathcal{S}_k \text{ and } \mathcal{I}' \in \mathcal{S}_{k-1}\}.$$

Deleting the element of  $\emptyset$  in  $\mathcal{S}_k$  if there exists one in  $\mathcal{S}_k$ . If  $\mathcal{S}_k = \emptyset$ , then the formula is not satisfied;

- (4) For each  $\mathcal{I} \in \mathcal{S}_n$  Updating  $\mathcal{S}_n$  by

$$\{\mathcal{I} : \mu_{\sup \mathcal{I}} \models \Phi_n, \mathcal{I} \in \mathcal{S}_k\},$$

where  $\mu_{\sup \mathcal{I}} = e^{Q_{\sup \mathcal{I}}} \mu$ . If  $\mathcal{S}_n = \emptyset$ , then the formula is not satisfied.

Thus after the above procedure, we have non-empty sets  $\{\mathcal{S}_k\}_{k=0}^n$  with the following properties.

- for each  $1 \leq k \leq n$ ,  $\mu_t \models \Phi_{k-1}$  for all  $t \in \mathcal{I}_k$  and  $\mathcal{I}_k \in \mathcal{S}_k$ , and  $\mu_{t^*} \models \Phi_n$ , where  $t^* = \sup \mathcal{I}_n$ ;
- for each  $1 \leq k \leq n$ ,  $\mathcal{I} \in \mathcal{S}_k$ , there exists at least one  $\mathcal{I}' \in \mathcal{S}_{k-1}$  such that  $\mathcal{I} \subseteq \sup \mathcal{I}' + \mathcal{T}_k$  and  $\inf \mathcal{I} = \sup \mathcal{I}' + \inf \mathcal{T}_k$ .

Therefore, we can get a set of intervals  $\{\mathcal{I}_k\}_{k=0}^n$  satisfying the two conditions in Lemma 2 if it exists. On the other hand, it is easy to check that all such  $\{\mathcal{I}_k\}_{k=0}^n$  must be in  $\{\mathcal{S}_k\}_{k=0}^n$ , i.e., for each  $k$ ,  $\mathcal{I}_k \subseteq \mathcal{I}$  for some  $\mathcal{I} \in \mathcal{S}_k$ . This ensures the correctness of the above procedure.

Finally, we are going to give the specific steps to implement the above procedure by transcendental number theory and real root isolation of PEFs.

**Lemma 3** *Based on Schanuel's conjecture, given a bounded rational time interval  $\mathcal{T}$ , the set  $\mathcal{S}$  of all maximum intervals satisfying  $\mu_t \models \Phi$  can be computed, where  $\Phi = \bigwedge_{j \in J} \bigvee_{l \in L_j} a_{j,l}$  and  $a_{j,l} \in AP$  for all  $j, l$ . Furthermore,*

- The number of all intervals in  $\mathcal{I}$  is finite;
- The left and right endpoints of each interval in  $\mathcal{I}$  are roots of PEFs.

*Proof.* From  $\mu_t \models \bigwedge_{j \in J} \bigvee_{l \in L_j} a_{j,l}$ , by the semantic of CLL, we have that for all  $j \in J$ ,  $\mu_t \models a_{j,l}$  for some  $l \in L_j$ . As  $J$  and all  $L_j$  are finite sets, we can one-by-one check whether or not  $\mu_t \models a_{j,l}$ . Let  $\mathcal{I}_{j,l}$  be the set of all the maximum intervals such that for each  $\mathcal{I} \in \mathcal{I}_{j,l}$ ,  $\mu_t \models a_{j,l}$  for all  $t \in \mathcal{I}$ . Then  $\mathcal{I} = \bigcap_{j \in J} \mathcal{I}_{L_j}$ , where

$$\mathcal{I}_{L_j} = \{\mathcal{I}_1 \cup \mathcal{I}_2 : \mathcal{I}_1 \in \mathcal{I}_{l_1} \text{ and } \mathcal{I}_2 \in \mathcal{I}_{l_2} \text{ for all distinct } l_1, l_2 \in L_j\} \text{ for all } j \in J,$$

and

$$\mathcal{I}_{L_i} \cap \mathcal{I}_{L_j} = \{\mathcal{I}_1 \cap \mathcal{I}_2 : \mathcal{I}_1 \in \mathcal{I}_{L_i} \text{ and } \mathcal{I}_2 \in \mathcal{I}_{L_j}\}.$$

Next, we first show how to compute  $\mathcal{I}_{j,l}$  and then deal with  $\mathcal{I}_1 \cap \mathcal{I}_2$  and  $\mathcal{I}_1 \cup \mathcal{I}_2$ .

**Lemma 4** *Given a bounded time interval  $\mathcal{T}$ , a symbolized continuous-time Markov chain  $(S, Q, \mathcal{I})$  with initial distribution  $\mu$  and  $\langle k, [a, b] \rangle \in AP$ , Let  $\mathcal{I} \subseteq \mathcal{T}$  be an maximum interval in  $\mathcal{T}$  such that  $\mu_t \models \langle k, [a, b] \rangle$  for all  $t \in \mathcal{I}$ , where  $\mu_t = e^{Qt} \mu$  for all  $t \in \mathbb{R}^+$ . Then the left and right endpoints of  $\mathcal{I}$  must be  $\inf \mathcal{T}$ ,  $\sup \mathcal{T}$  or a root of  $f(t) = a$  or  $f(t) = b$ , where  $f(t) = e^{Qt} \mu(k)$ .*

*Proof.* Let  $\mathcal{I}$  be the set of intervals in  $\mathcal{T}$  such that  $\mu_t \models \langle k, [a, b] \rangle$  for all  $t \in \mathcal{I}$ , where  $\mu_t = e^{Qt} \mu$  for all  $t \in \mathbb{R}^+$ . W.l.o.g., we assume that  $\mathcal{I}$  is the first one, i.e.  $\inf \mathcal{I} < \inf \mathcal{I}'$  for any  $\mathcal{I}' \in \mathcal{I}$  and  $\mathcal{I}' \neq \mathcal{I}$ . By the semantics of CLL,  $\mu_t \models a$  means  $f(t) = e^{Qt} \mu(k) \in \mathcal{I}$ .

We first consider the left endpoint of  $\mathcal{I}$ , denoted by  $t^* = \inf \mathcal{I}$ , and the right endpoint follows the same way.

Let  $t_1 = \inf \mathcal{T}$ . We analyze the roots of two PEFs  $f(t) - a$  and  $f(t) - b$  case by case.

- If  $\text{Iso}(f(t) - a) = \text{Iso}(f(t) - b) = \emptyset$ , then by the continuity of  $f(t)$ , we have that

$$\begin{cases} t^* \text{ does not exist } (\mathcal{I} = \emptyset) & f(t_1) < a \text{ or } f(t_1) > b; \\ t^* = t_1 & \text{others} \end{cases}$$

.

- If  $\text{Iso}(f(t) - a) \neq \emptyset$  and  $\text{Iso}(f(t) - b) = \emptyset$ , then, by the continuity of  $f(t)$  again,

$$t^* = \begin{cases} t_1 & f(t_1) \geq a; \\ \min_{f(t)=a} t \in \mathcal{T} & \text{others} \end{cases}$$

.

- If  $\text{Iso}(f(t) - a) = \emptyset$  and  $\text{Iso}(f(t) - b) \neq \emptyset$ , then, by the continuity of  $f(t)$  again,

$$t^* = \begin{cases} t_1 & f(t_1) \leq b; \\ \min_{f(t)=b} t \in \mathcal{T} & \text{others} \end{cases}$$

.

- If  $\text{Iso}(f(t) - a) \neq \emptyset$  and  $\text{Iso}(f(t) - b) \neq \emptyset$ , then, by the continuity of  $f(t)$  again,

$$t^* = \begin{cases} \min_{f(t)=a} t \in \mathcal{T} & f(t_1) < a; \\ \min_{f(t)=b} t \in \mathcal{T} & f(t_1) > b; \\ t_1 & \text{others.} \end{cases}$$

where  $\min_{f(t)=a} t \in \mathcal{T}$  is the minimum  $t \in \mathcal{T}$  with  $t$  being a root of  $f(t) - a$ . Theorem 3 ensures the existence checking of PEFs. □

By the above lemma, the left and right endpoints of  $\mathcal{I} \in \mathcal{J}$  must be a root of a PEF  $f(t)$  by noting that  $\inf \mathcal{T}$  is a rational number, and can be a root of PEF  $f(t) = t - \inf \mathcal{T}$ . By Theorem 3, the number of roots of  $f(t)$  in  $\mathcal{T}$  is finite. Therefore, the number of all intervals in  $\mathcal{J}$  is finite.

The left problem is to handle  $\mathcal{I}_1 \cap \mathcal{I}_2$  and  $\mathcal{I}_1 \cup \mathcal{I}_2$ , where the left and right endpoints of  $\mathcal{I}_1$  and  $\mathcal{I}_2$  are the roots of PEFs. It is equivalent to compare the values of two real roots of two (different) PEFs.

**Lemma 5** *Let  $f_1(t)$  and  $f_2(t)$  two PEFs in  $\mathcal{T}_1$  and  $\mathcal{T}_2$ , respectively. For any roots  $t_1 \in \mathcal{T}_1$  and  $t_2 \in \mathcal{T}_2$  of  $f_1(t)$  and  $f_2(t)$ , respectively, based on Schanuel's conjecture, there is an efficient way to check whether or not  $t_1 - t_2 < g$  for any given rational number  $g \in \mathbb{Q}$ .*

*Proof.* First, by Theorem 3, isolating the real roots of  $f_1(t)$  and  $f_2(t)$ , we have  $t_1 \in (a_1, b_1)$  and  $t_2 \in (a_2, b_2)$  for  $a_1, a_2, b_1, b_2 \in \mathbb{Q}$ . Then we first check if  $t_1 - t_2 = g$ . Note that  $t_1 - t_2 = g$  if and only if  $f_1^2(t) + f_2^2(t + g) = 0$  has a root in  $(a_1, b_1) \cup (a_2, b_2)$ .  $f_1^2(t) + f_2^2(t + g)$  is still a PEF, then we can check whether or not there is a root of it in  $(a_1, b_1) \cup (a_2, b_2)$  by Algorithm 1.

If  $t_1 - t_2 \neq g$ , we answer whether or not  $t_1 - (t_2 - g) < 0$  by narrowing  $(a_1, b_1)$  and  $(a_2, b_2)$  and maintaining the roots of  $f_1(t)$  and  $f_2(t)$  in the intervals. As we can arbitrarily narrowing the intervals and there is a gap between  $t_1$  and  $t_2 - g$ , we can compare them by comparing  $a_1$  and  $b_2$  ( $a_2$  and  $b_1$ ). □

By the above lemma, back to the procedure Item (3), we can compute  $\mathcal{I} \cap (\sup \mathcal{I}' + \mathcal{T}_k)$  and check whether or not  $\inf \mathcal{I} \leq \sup \mathcal{I}' + \inf \mathcal{T}_k$ . The reason is that  $\mathcal{T}_k$  is a rational interval. □

Finally, for Item (4) in the procedure,  $\mu_{\sup \mathcal{I}} \models \Phi_n$  can be checked by Lemma 3. □

## 6 Real Root Isolation of PEFs

In this section, we give an algorithm to find a real root isolation of PEFs and finish the proof of Theorem 3.

From Rolle's theorem, we know there must exist at last one real root of  $f'(t)$  between every two distinct real roots of  $f(t)$ , if  $f(t)$  is continuous differentiable. In order to isolating the real roots of  $f(t)$ , we can try to get an real root isolation of  $f'(t)$  first. Likewise, in order to obtain the real roots of  $f'(t)$ , we can try to get the real roots of  $f''(t)$  first. We can repeat the above procedure until the real solutions of the  $i$ -th derivative of  $f(t)$  for some  $i$  can be achieved. Then, we lift the real solutions of the respective derivative in the inverse order until  $f(t)$  itself. Following the basic idea, we have

**Lemma 6** Let  $f(t)$  be a PEF,  $f'(t)$  the derivative of  $f(t)$  w.r.t.  $t$ , and  $\text{Iso}(f')_{\mathcal{T}} = \{\mathcal{T}_j = (a_j, b_j)\}_{j=1}^J$ , in which  $\mathcal{T} = [a, b]$  and  $a = b_0 < a_1 < b_1 < \dots < a_J < b_J < a_{J+1} = b$ . Furthermore,  $f(t)$  has no real roots in any closed interval  $[a_j, b_j]$ ,  $1 \leq j \leq J$ . Then,  $\text{Iso}(f)_{\mathcal{T}} = \{(b_j, a_{j+1}) | f(b_j)f(a_{j+1}) < 0, 0 \leq j \leq J\}$ .

Proof: Since  $f(t)$  has no real roots in any closed interval  $[a_j, b_j]$ ,  $1 \leq j \leq J$ , all real roots of  $f(t)$  are in  $\cup_{j=0}^J (b_j, a_{j+1})$  and  $f(b_j)f(a_{j+1}) \neq 0$ . Moreover,  $f(t)$  has at most one real root in each  $(b_j, a_{j+1})$ , otherwise, there must be at least one real root of  $f'(t) = 0$  on it by Rolle's theorem, which is a contradiction with  $\text{Iso}(f')_{\mathcal{T}} = \{\mathcal{T}_j = (a_j, b_j)\}_{j=1}^J$ . So, if  $f(b_j)f(a_{j+1}) < 0$ , then there exists only one real root of  $f(t)$  in  $(b_j, a_{j+1})$ , otherwise no real root of  $f(t)$  in  $(b_j, a_{j+1})$ .  $\square$

From the above lemma, if  $f(t)$  has only single roots, then such  $\text{Iso}(f') = \{\mathcal{T}_j = (a_j, b_j)\}_{j=1}^J$  always exists because there is always a gap between the roots of  $f(t)$  and  $f'(t)$ . Thus  $\text{Iso}(f')$  can be used to reason about  $\text{Iso}(f)$ . Otherwise,  $f(t)$  has one multiple root, then the root will be hidden by the root of  $f'(t)$ , leading that such  $\text{Iso}(f') = \{\mathcal{T}_j = (a_j, b_j)\}_{j=1}^J$  does not exist. Therefore, we need ensure that the roots of  $f(t)$  are all single. This can be done by factoring  $f(t)$  and replace  $f(t)$  by its square free part  $\hat{f}(t)$ . Then the following result guarantees that  $\hat{f}(t)$  is single.

**Lemma 7** Let  $f(t) = \sum_{k=0}^K f_k(t)e^{\lambda_k t}$  be a square free PEF. Based on Schanuel's conjecture,  $f(t)$  has no multiple real root except 0.

*Proof.* The proof is similar to [GCL<sup>+</sup>17, Corollary 4] (PEFs over real numbers). For the completeness, we present the proof as follows.

Let  $\{a_j\}_{j=1}^n$  be an integral basis of  $\text{Power}(f) = \{\lambda_k\}_{k=0}^K$ . Since  $f(t)$  is square free, we may write

$$f(t, e^{a_1 t}, \dots, e^{a_n t}) = f_1(t, e^{a_1 t}, \dots, e^{a_n t}) \dots f_n(t, e^{a_1 t}, \dots, e^{a_n t})$$

where for any  $1 \leq i, j \leq n$ ,  $i \neq j$ ,  $f_i(t, e^{a_1 t}, \dots, e^{a_n t})$  is irreducible and  $f_i(t, e^{a_1 t}, \dots, e^{a_n t})$  and  $f_j(t, e^{a_1 t}, \dots, e^{a_n t})$  are coprime, i.e., they are not a factor of each other.

We first prove, by contradiction, that  $f_i(t, e^{a_1 t}, \dots, e^{a_n t})$  and  $f_j(t, e^{a_1 t}, \dots, e^{a_n t})$  have no nonzero common real root w.r.t.  $t$ . Suppose  $t_0 \neq 0$  is a common real root of  $f_i(t, e^{a_1 t}, \dots, e^{a_n t})$  and  $f_j(t, e^{a_1 t}, \dots, e^{a_n t})$ . By Corollary 1, we have that the transcendence degree of  $\mathbb{Q}(t_0, e^{a_1 t_0}, \dots, e^{a_n t_0})$  is at least  $n$ . Then, there must exist  $n$  elements in  $\{t_0, e^{a_1 t_0}, \dots, e^{a_n t_0}\}$  that are algebraically independent. Without loss of generality, let  $\{t_0, e^{a_1 t_0}, \dots, e^{a_{n-1} t_0}\}$  be the  $n$  elements that are algebraically independent. Besides, let  $g(t, e^{a_1 t}, \dots, e^{a_{n-1} t})$  be the resultant of  $f_i(t, e^{a_1 t}, \dots, e^{a_n t})$  and  $f_j(t, e^{a_1 t}, \dots, e^{a_n t})$  w.r.t.  $e^{a_n t}$ , then  $(t_0, e^{a_1 t_0}, \dots, e^{a_{n-1} t_0})$  is a real root of  $g(t, y_1, \dots, y_{n-1})$ . Further since  $f_i(t, e^{a_1 t}, \dots, e^{a_n t})$  and  $f_j(t, e^{a_1 t}, \dots, e^{a_n t})$  are coprime,  $g(t, y_1, \dots, y_{n-1})$  is nontrivial polynomial, indicating that  $(t_0, e^{a_1 t_0}, \dots, e^{a_{n-1} t_0})$  is a real root of some nontrivial polynomial. This contradicts that  $\{t_0, e^{a_1 t_0}, \dots, e^{a_{n-1} t_0}\}$  are algebraically independent.

Next, we prove that  $f_i(t, e^{a_1 t}, \dots, e^{a_n t})$  has no multiple real root. Suppose

$$f_i(t, e^{a_1 t}, \dots, e^{a_n t}) = h_0(t) + \sum_{k=1}^s h_k(t)(e^{a_1 t})^{b_{k1}} \dots (e^{a_n t})^{b_{kn}}$$

where  $h_0(t), \dots, h_n(t)$  are nontrivial polynomials and  $b_{jk} \in \mathbb{N}$  for  $1 \leq j \leq s$  and  $1 \leq k \leq n$ . Then we have

$$f'_i(t, e^{a_1 t}, \dots, e^{a_n t}) = h'_0(t) + \sum_{k=1}^s [h'_k(t) + h_k(t)(a_1 b_{k1} + \dots + a_n b_{kn})](e^{a_1 t})^{b_{k1}} \dots (e^{a_n t})^{b_{kn}}$$

Moreover

$$f_i(t, y_1, \dots, y_n) = h_0(t) + \sum_{k=1}^s h_k(t)(y_1)^{b_{k1}} \dots (y_n)^{b_{kn}}$$

$$f'_i(t, y_1, \dots, y_n) = h'_0(t) + \sum_{k=1}^s [h'_k(t) + h'_k(t)(a_1 b_{k1} + \dots + a_n b_{kn})](y_1)^{b_{k1}} \dots (y_n)^{b_{kn}}$$

Consider the degree and  $h_0(t)$  to be nontrivial, it is evident to see that  $f_i(t, y_1, \dots, y_n)$  is not a factor of  $f'_i(t, y_1, \dots, y_n)$ . Then,  $f_i(t, y_1, \dots, y_n)$  and  $f'_i(t, y_1, \dots, y_n)$  are coprime, since  $f_i(t, y_1, \dots, y_n)$  is irreducible. For the same reason as above  $f_i(t, e^{a_1 t}, \dots, e^{a_n t})$  and  $f'_i(t, e^{a_1 t}, \dots, e^{a_n t})$  have no common real roots. Therefore,  $f_i(t, e^{a_1 t}, \dots, e^{a_n t})$  has no multiple real root.  $\square$

Now we can present our algorithm to find a real root isolation of a given PEF  $f(t)$  in interval  $(B, C)$ . This procedure is implemented in Algorithm 2, whose main steps can be understood as follows:

- Step 1: From line 1 to 15, construct a sequence PEFs  $S_0(t), S_1(t), S_2(t), \dots, S_{r+1}(t)$ , where  $S_0(t)$  is the square free part of  $f(t)$ , for  $1 \leq k \leq r$ ,  $S_k(t)$  is square free which has the same real roots as the derivative of  $S_{k-1}$  and  $S_{r+1}(t) \in \mathbb{A}[t]$ , a polynomial in  $t$ .
- Step 2: In line 16 to 19, finding a all real roots of  $S_r(t)$  on  $(B, C)$ . If  $\text{Iso}S_r(t) = \emptyset$ , then let  $\text{Iso}S_r(t) = (B, C)$ . Note that the problem of isolating real roots of a univariate polynomial is well studied [CA76, Joh98].
- Step 3: In the rest part, for  $k = r$  down to 0, for each  $(u, l) \in \text{Iso}(S_k)$ , adjusting  $u$  and  $v$  such that  $(u, l)$  is still in one of sets of real root isolations of  $S_{k+1}$  and  $S_k$  has no any root in  $[u, l]$  by calling *Exist()* to satisfy the requirements of Lemma 6. Then by Lemma 6, finding a real root isolation of  $S_k$ .

From the above procedure, as we can always get a polynomial  $S_t(t)$  and the number of roots of  $S_t(t)$  in  $(B, C)$  is finite, we can claim that the number of roots of  $f(t)$  in  $(B, C)$  is finite. If not, we can use  $\text{Iso}(S_t(t))$  to reason  $\text{Iso}(f)$ , contradicting the fact in Lemma 6.

**Theorem 5 (Correctness of IsolatePEF)** *Based on Schanuel's conjecture, Algorithm 2 IsolatePEF always terminates and returns a real root isolation for a given PEF  $f(t)$ .*

*Proof.* Termination is immediately obtained from the correctness of Algorithm 1. The correctness of procedures are obtained by Lemmas 6 and 7.  $\square$

## 7 Conclusion

In this paper, we have initiated here the study of the symbolic dynamics of finite-state continuous-time Markov chains obtained by symbolizing the probability value space  $[0, 1]$  into a finite set of intervals in every dimension of the distributions. Then a new continuous-time logic, named continuous linear logic, was proposed to specify the properties of the symbolic dynamics of continuous-time Markov chains. The logic is an extension of linear-time temporal logic in the discrete-time domain. We have considered the model checking problem in this setting. Our main result is that based on Schanuel's conjecture, the problem is decidable for the full class of continuous-time Markov chains.

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**Algorithm 2** IsolatePEF( $f$ )

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**Require:** A PEF  $f(t)$  with bounded time interval  $(B, C)$

**Ensure:** Iso( $f$ ) a real root isolation of  $f(t)$  in  $\mathbb{R}^+$

```
1:  $r = 0$ 
2: while |Power( $f$ )|! = 1 do
3:    $f(t) = \hat{f}(t)$ , the square part of  $f(t)$ 
4:    $S_r(t) = f(t)$ 
5:    $r = r + 1$ 
6: while  $0 \in \text{Power}(f)$  do
7:    $f(t) = f'(t)$ 
8:    $f(t) = \hat{f}(t)$ , the square part of  $f(t)$ 
9:    $S_r(t) = f(t)$ 
10:   $r = r + 1$ 
11: end while
12:  $\lambda \leftarrow$  the first element of Power( $f$ )
13:  $f(t) = f(t)e^{-\lambda t}$ 
14: end while
15:  $S_{r+1}(t) = f(t)$ 
16: Iso( $S_{r+1}$ )  $\leftarrow$  a real root isolation of polynomial  $S_r(t)$  on  $(B, C)$ ;
17: if Iso( $S_{r+1}$ ) =  $\emptyset$  then
18:   Iso( $S_{r+1}$ ) =  $\{(B, C)\}$ 
19: end if
20: for each  $k = r; k \geq 0; k--$  do
21:   for each  $(u, l) \in \text{Iso}(S_{k+1})$  do
22:     while  $S_k(u) = 0$  or  $S_k(l) = 0$  or  $\text{Exist}(S_k, [u, l], 0.5) = \text{true}$  do
23:       if  $S_{k+1}(u)S_{k+1}(\frac{l+u}{2}) < 0$  then
24:          $l = \frac{l+u}{2}$ 
25:       else if  $S_{k+1}(\frac{l+u}{2}) = 0$  then
26:          $l = u = \frac{l+u}{2}$ 
27:       else
28:          $u = \frac{l+u}{2}$ 
29:       end if
30:     end while
31:     Iso( $S_k$ ) =  $\emptyset$ .
32:     Let Iso( $S_r$ ) =  $\{(a_k, b_k)\}_{k=1}^m$  with  $a_1 < b_1 < a_2 < b_2 < \dots < a_m < b_m$ 
33:     Let  $b_0 = B$  and  $a_{m+1} = C$ 
34:     for each  $j = 0; j < m + 1; j++$  do
35:       if  $S_k(b_j)S_k(a_{j+1}) < 0$  then
36:         Iso( $S_k$ )  $\leftarrow$  Iso( $S_k$ )  $\cup \{(b_j, a_{j+1})\}$ 
37:       end if
38:     end for
39:   end for
40: end for
41: return Iso( $S_0$ )
```

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As mentioned in the introduction, our main goal has been to prove decidability in as simple a fashion as possible without paying much attention to complexity issues. We are however confident that number theory and linear algebraic techniques can considerably lower the complexity of many of our constructions. This could lead to a significant improvement of our algorithm from a practical stand-point.

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