

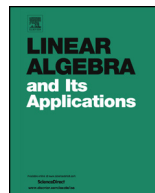


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Wildness for tensors



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ABSTRACT

In representation theory, a classification problem is called wild if it contains the problem of classifying matrix pairs up to simultaneous similarity. The latter problem is considered hopeless; it contains the problem of classifying an arbitrary finite system of vector spaces and linear mappings between them. We prove that an analogous “universal” problem in the theory of tensors of order at most 3 over an arbitrary field is the problem of classifying three-dimensional arrays up to equivalence transformations

$$[a_{ijk}]_{i=1}^m [u_{ii'}]_{i=1}^n [v_{jj'}]_{j=1}^t [w_{kk'}]_{k=1}^t \mapsto \left[\sum_{i,j,k} a_{ijk} u_{ii'} v_{jj'} w_{kk'} \right]_{i'=1}^m [u_{ii'}]_{i=1}^n [v_{jj'}]_{j=1}^t [w_{kk'}]_{k=1}^t$$

in which $[u_{ii'}]$, $[v_{jj'}]$, $[w_{kk'}]$ are nonsingular matrices: this problem contains the problem of classifying an arbitrary system of tensors of order at most three.

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1. Introduction and main result

We prove that

the problem of classifying three-dimensional arrays up to equivalence transformations

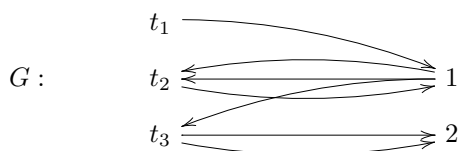
$$[a_{ijk}]_{i=1}^m [v_{jj'}]_{j=1}^n [w_{kk'}]_{k=1}^t \mapsto \left[\sum_{i,j,k} a_{ijk} u_{ii'} v_{jj'} w_{kk'} \right]_{i'=1}^m [v_{jj'}]_{j=1}^n [w_{kk'}]_{k=1}^t \quad (1)$$

in which $[u_{ii'}]$, $[v_{jj'}]$, $[w_{kk'}]$ are nonsingular matrices

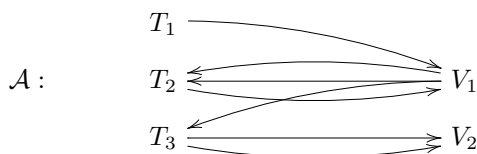
“contains” the problem of classifying an arbitrary system of tensors of order at most three; which means that the solution of the second problem can be derived from the solution of the first (see Definition 1.2 of the notion “contains”).

We are essentially concerned with systems of arrays. However, the main theorem in Section 1.3 is formulated in terms of systems of tensors that are considered as *representations of directed bipartite graphs* (i.e., directed graphs in which the set of vertices is partitioned into two subsets and all the arrows are between these subsets). The vertices t_1, \dots, t_p on the left represent tensors and the vertices $1, \dots, q$ on the right represent vector spaces.

For example, a representation of the graph



is a system



consisting of vector spaces V_1 and V_2 over a field \mathbb{F} and tensors

$$T_1 \in V_1^*, \quad T_2 \in V_1 \otimes V_1 \otimes V_1^*, \quad T_3 \in V_1 \otimes V_2^* \otimes V_2^*,$$

in which V^* denotes the dual space of V .

The *dimension* of a representation is the vector $(\dim V_1, \dots, \dim V_q)$. Two representations \mathcal{A} and \mathcal{A}' of the same dimension are *isomorphic* if \mathcal{A} is transformed to \mathcal{A}' by a system of linear bijections $V_1 \rightarrow V'_1, \dots, V_q \rightarrow V'_q$. All representations of dimension $\underline{n} := (n_1, \dots, n_q)$ are isomorphic to representations with $V_1 = \mathbb{F}^{n_1}, \dots, V_q = \mathbb{F}^{n_q}$, whose

sequences (T_1, \dots, T_p) of tensors are given by sequences $A = (A_1, \dots, A_p)$ of arrays over \mathbb{F} . These sequences form the vector space, which we denote by $\mathcal{W}_{\underline{n}}(G)$.

Let G be a directed bipartite graph, in which each vertex has at most 3 arrows (and so each representation of G consists of tensors of order at most 3). The aim of this paper is to construct an affine injection $F : \mathcal{W}_{\underline{n}}(G) \rightarrow \mathbb{F}^{m_1 \times m_2 \times m_3}$ with the following property: *two representations $A, A' \in \mathcal{W}_{\underline{n}}(G)$ are isomorphic if and only if the array $F(A)$ is transformed to $F(A')$ by equivalence transformations (1) (see Theorem 1.1).*

Note that the problem of classifying tensors of order 3 is motivated from seemingly independent questions in mathematics, physics, and computational complexity. Each finite dimensional algebra is given by a $(1, 2)$ -tensor; see Example 1.3. In computer science, this problem plays a role in algorithms for testing isomorphism of finite groups [3], algorithms for testing polynomial identities [1, 16, 15], and understanding the boundary of the determinant orbit closure [19, 11]. This problem also arises in the classification of quantum entangled states, which has applications in physics and quantum computing [17, 18, 27].

All arrays and tensors that we consider are over an arbitrary field \mathbb{F} .

1.1. Wild problems

Our paper was motivated by the theory of wild matrix problems; in this section we recall some known facts.

A classification problem over a field \mathbb{F} is called *wild* if it contains

the problem of classifying pairs (A, B) of square matrices of the same size over \mathbb{F} up to transformations of simultaneous similarity (2)
 $(S^{-1}AS, S^{-1}BS)$, in which S is a nonsingular matrix;

see formal definitions in [6], [8], and [9, Section 14.10].

Gelfand and Ponomarev [10] proved that the problem (2) (and even the problem of classifying pairs (A, B) of commuting nilpotent matrices up to simultaneous similarity) contains the problem of classifying t -tuples of square matrices of the same size up to transformations of simultaneous similarity

$$(M_1, \dots, M_t) \mapsto (C^{-1}M_1C, \dots, C^{-1}M_tC), \quad C \text{ is nonsingular.}$$

Example 1.1. Gelfand and Ponomarev's statement, but without the condition of commutativity of A and B , is easily proved. For each t -tuple $\mathcal{M} = (M_1, \dots, M_t)$ of $m \times m$ matrices, we define two $(t+1)m \times (t+1)m$ nilpotent matrices

$$A := \begin{bmatrix} 0 & I_m & & 0 \\ & 0 & \ddots & \\ & & \ddots & I_m \\ 0 & & & 0 \end{bmatrix}, \quad B(\mathcal{M}) := \begin{bmatrix} 0 & M_1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & M_t \\ 0 & & & 0 \end{bmatrix}.$$

Let $\mathcal{N} = (N_1, \dots, N_t)$ be another t -tuple of $m \times m$ matrices. Then \mathcal{M} and \mathcal{N} are similar if and only if $(A, B(\mathcal{M}))$ and $(A, B(\mathcal{N}))$ are similar. Indeed, let $(A, B(\mathcal{M}))$ and $(A, B(\mathcal{N}))$ be similar; that is,

$$AS = SA, \quad B(\mathcal{M})S = SB(\mathcal{N}) \quad (3)$$

for a nonsingular S . The first equality in (3) implies that S has an upper block-triangular form

$$S = \begin{bmatrix} C & & * \\ & C & \\ & & \ddots \\ 0 & & & C \end{bmatrix}, \quad C \text{ is } m \times m.$$

Then the second equality in (3) implies that $M_1C = CN_1, \dots, M_tC = CN_t$. Therefore, \mathcal{M} is similar to \mathcal{N} . Conversely, if \mathcal{M} is similar to \mathcal{N} via C , then $(A, B(\mathcal{M}))$ is similar to $(A, B(\mathcal{N}))$ via $\text{diag}(C, \dots, C)$.

Example 1.2. The problem of classifying pairs (M, N) of $m \times n$ and $m \times m$ matrices up to transformations

$$(M, N) \mapsto (C^{-1}MR, C^{-1}NC), \quad C \text{ and } R \text{ are nonsingular}, \quad (4)$$

looks simpler than the problem of classifying matrix pairs up to similarity since (4) has additional admissible transformations. However, these problems have the same complexity since for each two pairs (A, B) and (A', B') of $n \times n$ matrices the pair

$$\left(\begin{bmatrix} I_n \\ 0_n \end{bmatrix}, \begin{bmatrix} 0_n & A \\ I_n & B \end{bmatrix} \right) \text{ is reduced to } \left(\begin{bmatrix} I_n \\ 0_n \end{bmatrix}, \begin{bmatrix} 0_n & A' \\ I_n & B' \end{bmatrix} \right)$$

by transformations (4) if and only if (A, B) is similar to (A', B') .

Moreover, by [2] the problem (2) contains the problem of classifying representations of an arbitrary quiver over a field \mathbb{F} (i.e., of an arbitrary finite set of vector spaces over \mathbb{F} and linear mappings between them) and the problem of classifying representations of an arbitrary partially ordered set. Analogously, by [5] the problem of classifying pairs (A, B) of commuting complex matrices of the same size up to transformations of simultaneous consimilarity $(\bar{S}^{-1}AS, \bar{S}^{-1}BS)$, in which S is nonsingular, contains the problem of classifying an arbitrary finite set of complex vector spaces and linear or semilinear mappings between them.

Thus, all wild classification problems for systems of linear mappings have the same complexity and a solution of any one of them would imply a solution of every wild or non-wild problem.

The universal role of the problem (2) is not extended to systems of tensors: Belitskii and Sergeichuk [2] proved that the problem (2) is contained in the problem of classifying three-dimensional arrays up to equivalence but does not contain it.

1.2. Organization of the paper

The main theorem is formulated in Section 1.3. Its proof is given in Sections 2–4, in which we successively prove special cases of the main theorem. We describe them in this section.

Definition 1.1. An array of size $d_1 \times \cdots \times d_r$ over a field \mathbb{F} is an indexed collection $\underline{A} = [a_{i_1 \dots i_r}]_{i_1=1}^{d_1} \cdots [i_r=1]^{d_r}$ of elements of \mathbb{F} . (We denote arrays by underlined capital letters.) Let $\underline{A} = [a_{i_1 \dots i_r}]$ and $\underline{B} = [b_{i_1 \dots i_r}]$ be two arrays of size $d_1 \times \cdots \times d_r$ over a field \mathbb{F} . If there exist nonsingular matrices $S_1 = [s_{1ij}] \in \mathbb{F}^{d_1 \times d_1}, \dots, S_r = [s_{rij}] \in \mathbb{F}^{d_r \times d_r}$ such that

$$b_{j_1 \dots j_r} = \sum_{i_1, \dots, i_r} a_{i_1 \dots i_r} s_{1i_1 j_1} \cdots s_{ri_r j_r} \quad (5)$$

for all j_1, \dots, j_r , then we say that \underline{A} and \underline{B} are *equivalent* and write

$$(S_1, \dots, S_r) : \underline{A} \xrightarrow{\sim} \underline{B}. \quad (6)$$

We define partitioned three-dimensional arrays by analogy with block matrices as follows. Let $\underline{A} = [a_{ijk}]_{i=1}^m [j=1]_n^t [k=1]$ be an array of size $m \times n \times t$. Each partition of its index sets

$$\left. \begin{aligned} \{1, \dots, m\} &= \{1, \dots, i_1 \mid i_1 + 1, \dots, i_2 \mid \dots \mid i_{\bar{m}-1} + 1, \dots, m\} \\ \{1, \dots, n\} &= \{1, \dots, j_1 \mid j_1 + 1, \dots, j_2 \mid \dots \mid j_{\bar{n}-1} + 1, \dots, n\} \\ \{1, \dots, t\} &= \{1, \dots, k_1 \mid k_1 + 1, \dots, k_2 \mid \dots \mid k_{\bar{t}-1} + 1, \dots, t\} \end{aligned} \right\} \quad (7)$$

(we set $i_0 = j_0 = k_0 := 0$, $i_{\bar{m}} := m$, $j_{\bar{n}} := n$, and $k_{\bar{t}} := t$) defines the *partitioned array*

$$\underline{A} = [\underline{A}_{\alpha\beta\gamma}]_{\alpha=1}^{\bar{m}} [\beta=1]_{\bar{n}}^{\bar{t}} [\gamma=1]$$

with $\bar{m} \cdot \bar{n} \cdot \bar{t}$ spatial blocks

$$\underline{A}_{\alpha\beta\gamma} := [a_{ijk}]_{i=i_{\alpha-1}+1}^{i_{\alpha}} [j=j_{\beta-1}+1]^{j_{\beta}} [k=k_{\gamma-1}+1]^{k_{\gamma}}.$$

Thus, \underline{A} is partitioned into spatial blocks $\underline{A}_{\alpha\beta\gamma}$ by frontal, lateral, and horizontal planes.

Two partitioned arrays

$$\underline{A} = [\underline{A}_{\alpha\beta\gamma}]_{\alpha=1}^{\bar{m}} [\beta=1]_{\bar{n}}^{\bar{t}} [\gamma=1] \quad \text{and} \quad \underline{B} = [\underline{B}_{\alpha\beta\gamma}]_{\alpha=1}^{\bar{m}} [\beta=1]_{\bar{n}}^{\bar{t}} [\gamma=1] \quad (8)$$

of the same size are *conformally partitioned* if the sizes of the space blocks $\underline{A}_{\alpha\beta\gamma}$ and $\underline{B}_{\alpha\beta\gamma}$ are equal for each α, β, γ .

Two conformally partitioned three-dimensional arrays (8) whose partition is given by (7), are *block-equivalent* if there exists an equivalence $(S_1, S_2, S_3) : \underline{A} \xrightarrow{\sim} \underline{B}$ (see (6)) in which

$$S = S_{11} \oplus \cdots \oplus S_{1\bar{m}}, \quad S_2 = S_{21} \oplus \cdots \oplus S_{2\bar{n}}, \quad S_3 = S_{31} \oplus \cdots \oplus S_{3\bar{l}} \quad (9)$$

and the sizes of diagonal blocks in (9) are given by (7).

In Section 2 we prove Theorem 2.1, which implies that

$$\text{the problem (1) contains the problem of classifying partitioned three-dimensional arrays up to block-equivalence.} \quad (10)$$

Theorem 2.1 is our main tool in the proof of Theorem 1.1.

In Section 3, we prove Corollary 3.2, which implies that

$$\text{for an arbitrary } t, \text{ the problem (1) contains the problem of classifying } t\text{-tuples of three-dimensional arrays up to simultaneous equivalence,} \quad (11)$$

which is a three-dimensional analogue of Gelfand and Ponomarev's statement from [10] about the problem (2).

In Section 4, we consider *linked block-equivalence transformations* of three-dimensional arrays; that is, block-equivalence transformations $(S_1, S_2, S_3) : \underline{A} \xrightarrow{\sim} \underline{B}$ of the form (9), in which some of the diagonal blocks are claimed to be equal ($S_{ij} = S_{i'j'}$) and some of the diagonal blocks are claimed to be mutually contragredient ($S_{ij} = (S_{i'j'}^{-1})^T$). We prove Theorem 4.1, which implies that

$$\text{the problem (1) contains the problem of classifying partitioned three-dimensional arrays up to linked block-equivalence.} \quad (12)$$

The main result of the article is Theorem 1.1, which generalizes (10)–(12) and implies that

$$\text{the problem (1) contains the problem of classifying an arbitrary system of tensors of order at most 3.} \quad (13)$$

Note that the second problem in (13) contains both the problem of classifying systems of linear mappings and bilinear forms (i.e., representations of mixed graphs) and the problem of classifying finite dimensional algebras; see [14,21] and Example 6.2.

Remark 1.1. Because of the potential applications in computational complexity, we remark that all of the containments we construct are easily seen to be uniform p -projections in the sense of Valiant [26]. In this way, our containments not only show that mathematically one classification problem contains another, but also that this holds in an effective,

computational sense. In particular, a polynomial-time algorithm for testing equivalence of three-dimensional arrays would yield a polynomial-time algorithm for all the other problems considered in this paper. (Perhaps the only caveat to be aware of is that for partitioned arrays with t parts, the reduction is polynomial in the size of the array and 2^t .)

1.3. Main theorem

All systems of tensors of fixed orders ≤ 3 form a vector space. We construct in Theorem 1.1 an embedding of this vector space into the vector space of three-dimensional arrays of a fixed size (the image of this embedding is an affine space) in such a way that two systems of tensors are isomorphic if and only if their images are equivalent arrays; see Remark 1.2.

An array $\underline{A} = [a_{i_1 \dots i_r}]_{i_1=1}^{d_1} \dots [a_{i_r=1}^{d_r}]$ is a *subarray* of an array $\underline{B} = [b_{j_1 \dots j_\rho}]_{j_1=1}^{\delta_1} \dots [b_{j_\rho=1}^{\delta_\rho}]$ if $r \leq \rho$ and there are nonempty (possible, one-element) subsets

$$J_1 \subset \{1, \dots, \delta_1\}, \dots, J_\rho \subset \{1, \dots, \delta_\rho\}$$

such that \underline{A} coincides with $[b_{j_1 \dots j_\rho}]_{j_1 \in J_1, \dots, j_\rho \in J_\rho}$ up to deleting the indices j_k with one-element J_k . The *size* of a p -tuple $\mathcal{A} = (\underline{A}_1, \dots, \underline{A}_p)$ of arrays is the sequence $d := (d_1, \dots, d_p)$ of their sizes. Each array classification problem \mathcal{C}_d that we consider is given by a set of \mathcal{C}_d -admissible transformations on the set of array p -tuples of size d .

We use the following definition of embedding of one classification problem about systems of aggregates to another, which generalizes the constructions from Examples 1.1 and 1.2. This definition is general; its concrete realization is given in Theorem 1.1.

Definition 1.2. Let $\mathcal{X} = (\underline{X}_1, \dots, \underline{X}_p)$ be a variable array p -tuple of size d , in which the entries of $\underline{X}_1, \dots, \underline{X}_p$ are independent variables (without repetition). We say that an array classification problem \mathcal{C}_d is contained in an array classification problem \mathcal{D}_δ if there is a variable array π -tuple $\mathcal{F}(\mathcal{X}) = (\underline{F}_1, \dots, \underline{F}_\pi)$ of size δ , in which every \underline{X}_i is a subarray of some \underline{F}_j and each entry of $\underline{F}_1, \dots, \underline{F}_\pi$ outside of $\underline{X}_1, \dots, \underline{X}_p$ is 0 or 1, such that

\mathcal{A} is reduced to \mathcal{B} by \mathcal{C}_d -admissible transformations if and only if $\mathcal{F}(\mathcal{A})$ is reduced to $\mathcal{F}(\mathcal{B})$ by \mathcal{D}_δ -admissible transformations

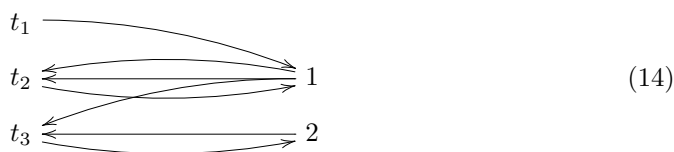
for all array p -tuples \mathcal{A} and \mathcal{B} of size d .

Note that $\mathcal{F}(\mathcal{X})$ defines an affine map $\mathcal{A} \mapsto \mathcal{F}(\mathcal{A})$ of the vector space of all array p -tuples of size d to the vector space of all array π -tuples of size δ .

Gabriel [7] (see also [9,14]) suggested to consider systems of vector spaces and linear mappings as representations of quivers: a quiver is a directed graph; its representation

is given by assigning a vector space to each vertex and a linear mapping of the corresponding vector spaces to each arrow. Generalizing this notion, Sergeichuk [23] suggested to study systems of tensors as representations of directed bipartite graphs. These representations are another form of *Penrose's tensor diagrams* [20], which are studied in [4,25].

A *directed bipartite graph* G is a directed graph in which the set of vertices is partitioned into two subsets and all the arrows are between these subsets. We denote these subsets by \mathbb{T} and \mathbb{V} , and write the vertices from \mathbb{T} on the left and the vertices from \mathbb{V} on the right. For example,

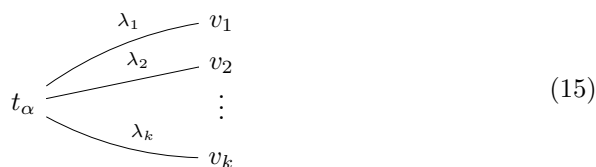


is a directed bipartite graph, in which $\mathbb{T} = \{t_1, t_2, t_3\}$ and $\mathbb{V} = \{1, 2\}$.

The following definition of representations of directed bipartite graphs is given in terms of arrays. We show in Section 6 that it is equivalent to the definition from [23] given in terms of tensors that are considered as elements of tensor products.

Definition 1.3. Let G be a directed bipartite graph with $\mathbb{T} = \{t_1, \dots, t_p\}$ and $\mathbb{V} = \{1, \dots, q\}$.

- An *array-representation* \mathcal{A} of G is given by assigning
 - a nonnegative integer number d_v to each $v \in \mathbb{V}$, and
 - an array \underline{A}_α of size $d_{v_1} \times \dots \times d_{v_k}$ ¹ to each $t_\alpha \in \mathbb{T}$ with arrows $\lambda_1, \dots, \lambda_k$ of the form



(each line $\xrightarrow{\quad}$ is \rightarrow or \leftarrow and some of v_1, \dots, v_k may coincide).

The vector $d = (d_1, \dots, d_q)$ is the *dimension* of \mathcal{A} . (For example, an array-representation \mathcal{A} of (14) of dimension $d = (d_1, d_2)$ is given by three arrays \underline{A}_1 , \underline{A}_2 , and \underline{A}_3 of sizes d_1 , $d_1 \times d_1 \times d_1$, and $d_1 \times d_2 \times d_2$.)

¹ For each sequence of nonnegative integers d_1, \dots, d_q with $\min\{d_1, \dots, d_q\} = 0$, there is exactly one array of size $d_1 \times \dots \times d_q$. In particular, the “empty” matrices of sizes $0 \times n$ and $m \times 0$ give the linear mappings $\mathbb{F}^m \rightarrow 0$ and $0 \rightarrow \mathbb{F}^m$.

- We say that two array-representations $\mathcal{A} = (\underline{A}_1, \dots, \underline{A}_p)$ and $\mathcal{B} = (\underline{B}_1, \dots, \underline{B}_p)$ of G of the same dimension $d = (d_1, \dots, d_q)$ are *isomorphic* and write $\mathcal{S} : \mathcal{A} \xrightarrow{\sim} \mathcal{B}$ (or $\mathcal{A} \simeq \mathcal{B}$ for short) if there exists a sequence $\mathcal{S} := (S_1, \dots, S_q)$ of nonsingular matrices of sizes $d_1 \times d_1, \dots, d_q \times d_q$ such that

$$(S_{v_1}^{T_1}, \dots, S_{v_k}^{T_k}) : \underline{A}_\alpha \xrightarrow{\sim} \underline{B}_\alpha \quad (16)$$

(see (6)) for each $t_\alpha \in \mathbb{T}$ with arrows (15), where

$$\tau_i := \begin{cases} 1 & \text{if } \lambda_i : t_\alpha \longleftarrow v_i \\ -T & \text{if } \lambda_i : t_\alpha \longrightarrow v_i \end{cases} \quad \text{for all } i = 1, \dots, k,$$

and $S^{-T} := (S^{-1})^T$ (which is called the *contragredient matrix* of S).

Example 1.3.

- Each array-representation of dimension $d = (d_1)$ of

$$t \begin{array}{c} \longleftarrow \\ \longrightarrow \end{array} 1 \quad \text{or} \quad t \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} 1 \quad (17)$$

is a $d_1 \times d_1$ matrix $A = [a_{ij}]$, which is isomorphic to an array-representation $B = [b_{ij}]$ if and only if there exists a $d_1 \times d_1$ nonsingular matrix $S = [s_{ij}]$ such that

$$b_{i'j'} = \sum_{i,j} a_{ij} s_{ii'} s_{jj'} \quad \text{or} \quad b_{i'j'} = \sum_{i,j} a_{ij} r_{ii'} s_{jj'},$$

respectively, where $[r_{ii'}] := S^{-T}$. Thus,

$$B = S^T A S \quad \text{or} \quad B = S^{-1} A S,$$

and so we can consider each array-representation of (17) as the matrix of a bilinear form or linear operator, respectively.

- Each array-representation of dimension $d = (d_1)$ of

$$t \begin{array}{c} \longleftarrow \\ \longrightarrow \\ \longrightarrow \end{array} 1 \quad (18)$$

is a $d_1 \times d_1 \times d_1$ array $\underline{A} = [a_{ijk}]$. It is reduced by transformations

$$(S, S, S^{-T}) : \underline{A} \mapsto \left[\sum_{i,j,k} a_{ijk} s_{ii'} s_{jj'} r_{kk'} \right]_{i'j'k'}, \quad [r_{kk'}] := S^{-T},$$

in which $S = [s_{ij}]$ is a $d_1 \times d_1$ nonsingular matrix.

- By (16), each array-representation of (15) defines an (m, n) -tensor (i.e., an m times contravariant and n times covariant tensor), where m is the number of arrows \longrightarrow and n is the number of arrows \longleftarrow ; see Section 6. In particular, each array-representation of (18) defines a $(1, 2)$ -tensor.

In Section 3 we show that the problem (1) contains the problems of classifying $(1, 2)$ -tensors and $(0, 3)$ -tensors.

Our main result is the following theorem (which ensures the statement (13)); the other theorems are its special cases.

Theorem 1.1. *Let G be a directed bipartite graph with the set $\mathbb{T} = \{t_1, \dots, t_p\}$ of left vertices and the set $\mathbb{V} = \{1, \dots, q\}$ of right vertices, in which each left vertex has at most three arrows. Let $d = (d_1, \dots, d_q)$ be an arbitrary sequence of nonnegative integers, and let $\mathcal{X} = (\underline{X}_1, \dots, \underline{X}_p)$ be a variable array-representation of G of dimension d , in which the entries of arrays $\underline{X}_1, \dots, \underline{X}_p$ are independent variables.*

Then there exists a partitioned three-dimensional variable array $\underline{F}(\mathcal{X})$ in which

- *p spatial blocks are $\underline{X}_1, \dots, \underline{X}_p$, and*
- *each entry of the other spatial blocks is 0 or 1,*

such that two array-representations \mathcal{A} and \mathcal{B} of dimension d of the graph G over a field are isomorphic if and only if

$$\underline{F}(\mathcal{A}) \text{ and } \underline{F}(\mathcal{B}) \text{ are equivalent as unpartitioned arrays.} \quad (19)$$

Remark 1.2. The variable array $\underline{F}(\mathcal{X})$ defines an embedding of the vector space of array-representations of dimension d of the graph G into the vector space of three-dimensional arrays of some fixed size. This embedding satisfies Definition 1.2 and its image (which consists of all $\underline{F}(\mathcal{A})$ with \mathcal{A} of dimension d) is an affine subspace. *Two representations are isomorphic if and only if their images are equivalent.*

2. Proof of the statement (10)

The statement (10) is proved in the following theorem.

Theorem 2.1. *For each partition (7), there exists a partitioned three-dimensional variable array $\underline{F}(\underline{X})$ in which*

- one spatial block is an $m \times n \times t$ variable array \underline{X} whose entries are independent variables, and*
- each entry of the other spatial blocks is 0 or 1,*

such that two $m \times n \times t$ arrays \underline{A} and \underline{B} partitioned into $\bar{m} \cdot \bar{n} \cdot \bar{t}$ spatial blocks are block-equivalent if and only if $\underline{F}(\underline{A})$ and $\underline{F}(\underline{B})$ are equivalent.

2.1. Slices and strata of three-dimensional arrays

We give a three-dimensional array $\underline{A} = [a_{ijk}]_{i=1}^m [j=1]^n [k=1]^t$ by the sequence of matrices

$$\underline{A} := (A_1, A_2, \dots, A_t), \quad A_k := [a_{ijk}]_{ij}, \quad (20)$$

which are the *frontal slices* of \underline{A} . For example, a $3 \times 3 \times 3$ array $\underline{A} = [a_{ijk}]_{i=1}^3 [j=1]^3 [k=1]^3$ can be given by its frontal slices

$$\begin{array}{c} \begin{array}{c} A_3 \\ A_2 \\ A_1 \end{array} \begin{array}{|c|c|c|} \hline a_{113} & a_{123} & a_{133} \\ \hline a_{213} & a_{223} & a_{233} \\ \hline a_{313} & a_{323} & a_{333} \\ \hline \end{array} \end{array} \quad (21)$$

An array $\underline{A} = [a_{ijk}]_{i=1}^m [j=1]^n [k=1]^t$ can be also given by the sequence of *lateral slices* $[a_{i1k}]_{ik}, \dots, [a_{ink}]_{ik}$, and by the sequence of *horizontal slices* $[a_{1jk}]_{jk}, \dots, [a_{mjk}]_{jk}$.

A *linear reconstruction* of a sequence (A_1, \dots, A_t) of matrices of the same size given by a nonsingular matrix $U = [u_{ij}]$ is the transformation

$$(A_1, \dots, A_t) \circ U := (A_1 u_{11} + \dots + A_t u_{t1}, \dots, A_1 u_{1t} + \dots + A_t u_{tt}). \quad (22)$$

Clearly, every linear reconstruction of (A_1, \dots, A_t) is a sequence of the following elementary linear reconstructions:

- (a) interchange of two matrices,
- (b) multiply any matrix by a nonzero element of \mathbb{F} ,
- (c) add a matrix multiplied by an element of \mathbb{F} to another matrix.

The following lemma is obvious.

Lemma 2.1. *Given two three-dimensional arrays \underline{A} and \underline{B} of the same size.*

- (a) \underline{A} and \underline{B} are equivalent if and only if \underline{B} can be obtained from \underline{A} by linear reconstructions of frontal slices, then of lateral slices, and finally of horizontal slices.
- (b) $(R, S, U) : \underline{A} \rightsquigarrow \underline{B}$ if and only if

$$\underline{B} = (R^T A_1 S, R^T A_2 S, \dots, R^T A_t S) \circ U. \quad (23)$$

Let $\underline{A} = [a_{ijk}]_{i=1}^m \substack{n \\ j=1} \substack{t \\ k=1}$ be a three-dimensional array, whose partition $\underline{A} = [\underline{A}_{\alpha\beta\gamma}]_{\alpha=1}^{\bar{m}} \substack{\bar{n} \\ \beta=1} \substack{\bar{t} \\ \gamma=1}$ into $\bar{m} \cdot \bar{n} \cdot \bar{t}$ spatial blocks is defined by partitions (7) of its index sets. The partition of $\{1, \dots, t\}$ in (7) into \bar{t} disjoint subsets defines also the division of \underline{A} by frontal planes into \bar{t} *frontal strata*

$$[a_{ijk}]_{i=1}^m \substack{n \\ j=1} \substack{k_1 \\ k=1}, [a_{ijk}]_{i=1}^m \substack{n \\ j=1} \substack{k_2 \\ k=k_1+1}, \dots, [a_{ijk}]_{i=1}^m \substack{n \\ j=1} \substack{k_{\bar{t}} \\ k=k_{\bar{t}-1}+1};$$

each frontal stratum is the union of frontal slices corresponding to the same subset. In the same way, \underline{A} is divided into \bar{n} *lateral strata* and into \bar{m} *horizontal strata*.

Two partitioned three-dimensional arrays \underline{A} and \underline{B} are block-equivalent if and only if \underline{B} can be obtained from \underline{A} by linear reconstructions of frontal strata, of lateral strata, and of horizontal strata.

2.2. The lemma that implies Theorem 2.1

Lemma 2.2. *For each partition (7), there exists a three-dimensional variable array $\underline{G}(\underline{X})$ partitioned into $\bar{m}' \cdot \bar{n}' \cdot \bar{t}'$ spatial blocks and satisfying (i) and (ii) from Theorem 2.1 such that two $m \times n \times t$ arrays \underline{A} and \underline{B} partitioned into $\bar{m} \cdot \bar{n} \cdot \bar{t}$ spatial blocks are block-equivalent if and only if $\underline{G}(\underline{A})$ and $\underline{G}(\underline{B})$ are block-equivalent with respect to partition into $\bar{m}' \cdot \bar{n}' \cdot \bar{t}'$ spatial blocks (i.e., we delete the horizontal partition).*

This lemma implies Theorem 2.1 since we can delete the horizontal partition, then the lateral partition, and finally the frontal partition in the same way.

2.3. Proof of Lemma 2.2 for a partitioned array of size $m \times n \times 6$

In order to make the proof of Lemma 2.2 clearer, we first prove it for arrays of size $m \times n \times 6$ partitioned by a frontal plane into two spatial blocks of size $m \times n \times 3$.

Such an array \underline{A} can be given by the sequence

$$\underline{A} = (A_1, A_2, A_3 \mid A_4, A_5, A_6)$$

of its frontal slices, which are $m \times n$ matrices (see (20)). Its two spatial blocks are given by the sequences (A_1, A_2, A_3) and (A_4, A_5, A_6) . Construct by \underline{A} the unpartitioned array \underline{M}^A given by the sequence of frontal slices

$$\begin{aligned} \underline{M}^A &= (M_1^A, \dots, M_6^A) \\ &:= \left(\left[\begin{array}{ccc|ccc} I_r & 0 & 0 & & & 0 \\ & & & 0 & 0 & 0 \\ & & & & & \end{array} \right] A_1, \left[\begin{array}{ccc|ccc} 0 & I_r & 0 & & & 0 \\ & & & 0 & 0 & 0 \\ & & & & & \end{array} \right] A_2, \left[\begin{array}{ccc|ccc} 0 & 0 & I_r & & & 0 \\ & & & 0 & 0 & 0 \\ & & & & & \end{array} \right] A_3, \right. \\ &\quad \left. \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & I_{2r} & 0 & 0 \\ & & & & & \end{array} \right] A_4, \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & I_{2r} & 0 \\ & & & & & \end{array} \right] A_5, \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & I_{2r} \\ & & & & & \end{array} \right] A_6 \right), \end{aligned} \quad (24)$$

in which

$$r := \min\{m, n\} + 1. \quad (25)$$

Let $\underline{B} := (B_1, B_2, B_3 \mid B_4, B_5, B_6)$ give another array \underline{B} of the same size as \underline{A} , which is partitioned by the frontal plane conformally to \underline{A} . Let us prove that

$$\begin{aligned} & \underline{A} \text{ and } \underline{B} \text{ are block-equivalent} \\ \iff & \underline{M}^A \text{ and } \underline{M}^B \text{ are equivalent.} \end{aligned} \quad (26)$$

\implies . In view of Lemma 2.1(b), \underline{B} can be obtained from \underline{A} by a sequence of the following transformations:

- (i) simultaneous equivalence transformations of A_1, \dots, A_6 ,
- (ii) linear reconstructions (a)–(c) (from Section 2.1) of (A_1, A_2, A_3) ,
- (iii) linear reconstructions (a)–(c) of (A_4, A_5, A_6) .

It suffices to consider the case when \underline{B} is obtained from \underline{A} by one of the transformations (i)–(iii).

If this transformation is (i), then \underline{M}^B can be obtained from \underline{M}^A by a simultaneous equivalence transformation of its frontal slices M_1^A, \dots, M_6^A . Hence, \underline{M}^A and \underline{M}^B are equivalent.

If this transformation is (ii), then we make a linear reconstruction of (M_1^A, M_2^A, M_3^A) . It spoils the blocks $[I_r \ 0 \ 0]$, $[0 \ I_r \ 0]$, $[0 \ 0 \ I_r]$ in (24), which can be restored by simultaneous elementary transformations of M_1^A, \dots, M_6^A that do not change the new A_1, \dots, A_6 . Hence \underline{M}^A and \underline{M}^B are equivalent.

The case of transformation (iii) is considered analogously.

\impliedby . Let \underline{M}^A and \underline{M}^B be equivalent. By Lemma 2.1(b), there exists a matrix sequence $\underline{N} = (N_1, \dots, N_6)$ such that the matrices in \underline{M}^A and \underline{N} are simultaneously equivalent and \underline{N} is reduced to \underline{M}^B by some linear reconstruction $\underline{N} \circ S^{-1} = \underline{M}^B$ given by a nonsingular 6×6 matrix $S = [s_{jk}]$ (see (22)). Then

$$(N_1, \dots, N_6) = (M_1^B, \dots, M_6^B) \circ S$$

and so

$$N_k = \left[\begin{array}{ccc|c} I_r s_{1k} & I_r s_{2k} & I_r s_{3k} & 0 \\ \hline I_{2r} s_{4k} & I_{2r} s_{5k} & I_{2r} s_{6k} & \\ \hline 0 & & & C_k \end{array} \right], \quad k = 1, \dots, 6, \quad (27)$$

where

$$(C_1, \dots, C_6) := (B_1, \dots, B_6) \circ S. \quad (28)$$

Since the matrices in $\overrightarrow{M^A}$ are simultaneously equivalent to the matrices in \overrightarrow{N} , $\text{rank } M_k^A = \text{rank } N_k$ for all k . Since A_k, B_k, C_k are $m \times n$, (25) shows that $\text{rank } A_k < r$ and $\text{rank } C_k < r$. If $k = 1, 2, 3$, then $\text{rank } N_k = \text{rank } M_k^A < 2r$, and so by (27) $s_{4k} = s_{5k} = s_{6k} = 0$. If $k = 4, 5, 6$, then $\text{rank } N_k = \text{rank } M_k^A \geq 2r$ and so $(s_{4k}, s_{5k}, s_{6k}) \neq (0, 0, 0)$; furthermore, $\text{rank } N_k = \text{rank } M_k^A < 3r$ and so $s_{1k} = s_{2k} = s_{3k} = 0$. Thus,

$$S = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} \oplus \begin{bmatrix} s_{44} & s_{45} & s_{46} \\ s_{54} & s_{55} & s_{56} \\ s_{64} & s_{65} & s_{66} \end{bmatrix}. \quad (29)$$

By (28) and (29), the array \underline{B} is block-equivalent to the conformally partitioned array \underline{C} given by $\underline{C} = (C_1, C_2, C_3 | C_4, C_5, C_6)$. It remains to prove that the arrays \underline{A} and \underline{C} are block-equivalent. By (27),

$$N_k = M_k^C Q, \quad k = 1, \dots, 6,$$

where

$$Q := \begin{bmatrix} s_{11}I_r & s_{21}I_r & s_{31}I_r \\ s_{12}I_r & s_{22}I_r & s_{32}I_r \\ s_{13}I_r & s_{23}I_r & s_{33}I_r \end{bmatrix} \oplus \begin{bmatrix} s_{44}I_{2r} & s_{54}I_{2r} & s_{64}I_{2r} \\ s_{45}I_{2r} & s_{55}I_{2r} & s_{65}I_{2r} \\ s_{46}I_{2r} & s_{56}I_{2r} & s_{66}I_{2r} \end{bmatrix} \oplus I_n.$$

Therefore, the matrices in \overrightarrow{N} and $\overrightarrow{M^C}$ are simultaneously equivalent. Since the matrices in $\overrightarrow{M^A}$ and \overrightarrow{N} are simultaneously equivalent, we have that the matrices in

$$\overrightarrow{M^A} = (M_1^A, \dots, M_6^A) \quad \text{and} \quad \overrightarrow{M^C} = (M_1^C, \dots, M_6^C) \quad (30)$$

are simultaneously equivalent. Hence, the representations

$$\begin{array}{ccc} & \begin{array}{c} \xrightarrow{M_1^A} \\ \xrightarrow{M_2^A} \\ \vdots \\ \xrightarrow{M_6^A} \end{array} & \\ \mathbb{F}^{9r+n} & \xrightarrow{\quad} & \mathbb{F}^{3r+m} \end{array} \quad \text{and} \quad \begin{array}{ccc} & \begin{array}{c} \xrightarrow{M_1^C} \\ \xrightarrow{M_2^C} \\ \vdots \\ \xrightarrow{M_6^C} \end{array} & \\ \mathbb{F}^{9r+n} & \xrightarrow{\quad} & \mathbb{F}^{3r+m} \end{array}$$

of the quiver

$$\bullet \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \vdots \\ \xrightarrow{\quad} \end{array} \bullet \quad (6 \text{ arrows}) \quad (31)$$

are isomorphic.

By (24), the sequences (30) have the form

$$(E_1, \dots, E_6) \oplus (A_1, \dots, A_6) \quad \text{and} \quad (E_1, \dots, E_6) \oplus (C_1, \dots, C_6).$$

We partition \underline{M}^A by lateral and horizontal planes that extend the partition of its spatial block \underline{A} , but we do not partition \underline{M}^A by frontal planes. Thus, each matrix M_k^A is partitioned as follows:

$$M_k^A = \left[\begin{array}{cccccc|c|c|c} \Delta_{k1} & \Delta_{k2} & \dots & 0 & 0 & 0 & 0 & \vdots & 0 \\ \dots & \dots & \dots & \dots & \dots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \Delta_{k,t-1} & \Delta_{kt} & 0 & 0 & \vdots & 0 \\ 0 & 0 & \dots & 0 & 0 & A_{11k} & A_{12k} & \dots & A_{1\bar{n}k} \\ \hline 0 & 0 & \dots & 0 & 0 & A_{21k} & A_{22k} & \dots & A_{2\bar{n}k} \\ \hline \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ \hline 0 & 0 & \dots & 0 & 0 & A_{\bar{m}1k} & A_{\bar{m}2k} & \dots & A_{\bar{m}\bar{n}k} \end{array} \right]. \quad (32)$$

Let $\underline{B} = [\underline{B}_{\alpha\beta\gamma}]_{\alpha=1}^{\bar{m}} [\beta=1]_{\bar{n}}^{\bar{t}} [\gamma=1]_{\bar{t}}$ be an array of the same size and with the same partition into spatial blocks as \underline{A} . Let us prove that

$$\begin{aligned} & \underline{A} \text{ and } \underline{B} \text{ are block-equivalent} \\ \iff & \underline{M}^A \text{ and } \underline{M}^B \text{ are block-equivalent.} \end{aligned} \quad (33)$$

\implies . It is proved as for (26).

\impliedby . Let \underline{M}^A and \underline{M}^B be block-equivalent. Then there exists a sequence $\underline{N} = (N_1, \dots, N_t)$ of matrices of the same size and with the same partition as (32) such that the matrices in \underline{M}^A and \underline{N} are simultaneously block-equivalent and $\underline{N} \circ S^{-1} = \underline{M}^B$ for some nonsingular matrix $S = [s_{jk}] \in \mathbb{F}^{t \times t}$. Thus,

$$(N_1, \dots, N_t) = (M_1^B, \dots, M_t^B) \circ S$$

and so

$$\begin{aligned} N_k &= [I_r s_{1k} \quad \dots \quad I_r s_{k_1 k}] \oplus [I_{2r} s_{k_1+1, k} \quad \dots \quad I_{2r} s_{k_2 k}] \\ &\quad \oplus \dots \oplus [I_{2\bar{t}-1r} s_{k_{\bar{t}-1}+1, k} \quad \dots \quad I_{2\bar{t}-1r} s_{k_{\bar{t}} k}] \oplus C_k, \end{aligned}$$

where

$$\underline{C} = (C_1, \dots, C_t) := (B_1, \dots, B_t) \circ S. \quad (34)$$

Denote by \underline{C} the array defined by (34) and partitioned conformally with the partitions of \underline{A} and \underline{B} .

Reasoning as in Section 2.3, we prove that

$$S = S_1 \oplus S_2 \oplus \dots \oplus S_{\bar{t}}, \quad S_1 \in \mathbb{F}^{k_1 \times k_1}, \quad S_2 \in \mathbb{F}^{(k_2-k_1) \times (k_2-k_1)}, \quad \dots$$

Thus, the arrays \underline{C} and \underline{B} are block-equivalent. It remains to prove that

$$\text{the arrays } \underline{A} \text{ and } \underline{C} \text{ are block-equivalent.} \quad (35)$$

We can reduce \underline{N} to $\underline{M}^C = (M_1^C, \dots, M_t^C)$ by simultaneous elementary transformations of columns $1, \dots, k_1$ of N_1, \dots, N_t , simultaneous elementary transformations of columns $k_1 + 1, \dots, k_2, \dots$, and simultaneous elementary transformations of columns $k_{t-1} + 1, \dots, k_t$ (these transformations do not change \underline{C}). Hence, the matrices in \underline{N} and \underline{M}^C are simultaneously block-equivalent. Since the matrices in \underline{M}^A and \underline{N} are simultaneously block-equivalent, we have that

$$\text{the matrices in } \underline{M}^A \text{ and } \underline{M}^C \text{ are simultaneously block-equivalent.} \quad (36)$$

By the *block-direct sum* of two block matrices $M = [M_{\alpha\beta}]_{\alpha=1}^p [\beta=1]^q$ and $N = [N_{\alpha\beta}]_{\alpha=1}^p [\beta=1]^q$, we mean the block matrix

$$M \boxplus N := [M_{\alpha\beta} \oplus N_{\alpha\beta}]_{\alpha=1}^p [\beta=1]^q.$$

This operation was studied in [22]; it is naturally extended to t -tuples of block matrices:

$$(M_1, \dots, M_t) \boxplus (N_1, \dots, N_t) := (M_1 \boxplus N_1, \dots, M_t \boxplus N_t).$$

By (32), $M_k^A = \Delta_k \boxplus A_k$ for $k = 1, \dots, t$, where

$$\Delta_k := \begin{array}{|cccc|cc|} \hline \Delta_{k1} & \Delta_{k1} & \dots & \dots & 0 & & \\ \hline \dots & \dots & \dots & \dots & \dots & & \\ \hline 0 & \dots & \dots & \Delta_{k,t-1} & \Delta_{kt} & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array}$$

(the empty strips do not have entries). Thus, $\underline{M}^A = \underline{\Delta} \boxplus \underline{A}$.

Each matrix M_k^C has the form (32) with C_{ijk} instead of A_{ijk} , and so $\underline{M}^C = \underline{\Delta} \boxplus \underline{C}$.

Let us prove that the matrices in \underline{A} and \underline{C} are simultaneously block-equivalent. Define the quiver Q with $\bar{m} + \bar{n}$ vertices $1, \dots, \bar{m}, 1', \dots, \bar{n}'$ and with $\bar{m}\bar{n}t$ arrows: with t arrows

$$\begin{array}{ccc} & \lambda_{\alpha\beta 1} & \\ & \lambda_{\alpha\beta 2} & \\ \beta' & \vdots & \alpha \\ & \lambda_{\alpha\beta t} & \end{array} \quad (37)$$

from each vertex $\beta' \in \{1', \dots, \bar{n}'\}$ to each vertex $\alpha \in \{1, \dots, \bar{m}\}$.

Let $\underline{K} = (K_1, \dots, K_t)$ be an arbitrary sequence of block matrices $K_k = [K_{\alpha\beta k}]_{\alpha=1}^{\bar{m}}_{\beta=1}^{\bar{n}}$, in which $K_{\alpha\beta k}$ is of size $m_\alpha \times n_\beta$. This sequence defines the array \underline{K} partitioned into $\bar{m} \cdot \bar{n} \cdot 1$ spatial blocks. Define the representation $R(\underline{K})$ of Q by assigning mappings to the arrows (37) as follows:

$$\begin{array}{ccc} & K_{\alpha\beta 1} & \\ & \curvearrowright & \\ & K_{\alpha\beta 2} & \\ \mathbb{R}^{n_\beta} & \xrightarrow{\quad} & \mathbb{R}^{m_\alpha} \\ & \vdots & \\ & \curvearrowleft & \\ & K_{\alpha\beta t} & \end{array}$$

Let $\underline{L} = (L_1, \dots, L_t)$ be another sequence of block matrices $L_k = [L_{\alpha\beta k}]_{\alpha=1}^{\bar{m}}_{\beta=1}^{\bar{n}}$, in which all matrices have the same size and the same partition into blocks as the matrices of \underline{K} . Clearly, the matrices in \underline{K} and \underline{L} are simultaneously block-equivalent if and only if the representations $R(\underline{K})$ and $R(\underline{L})$ are isomorphic.

By (36), the matrices in $\underline{M}^A = \underline{\Delta} \boxplus \underline{A}$ and $\underline{M}^C = \underline{\Delta} \boxplus \underline{C}$ are simultaneously block-equivalent. Hence, the representations $R(\underline{M}^A) = R(\underline{\Delta}) \oplus R(\underline{A})$ and $R(\underline{M}^C) = R(\underline{\Delta}) \oplus R(\underline{C})$ of Q are isomorphic. By the Krull–Schmidt theorem for quiver representations, $R(\underline{A})$ and $R(\underline{C})$ are isomorphic too. Thus, the matrices in \underline{A} and \underline{C} are simultaneously block-equivalent, which proves (35).

We have proved (33), which finishes the proof of Lemma 2.2 and hence the proof of Theorem 2.1.

3. Proof of the statement (11)

In this section, we prove several corollaries of Theorem 2.1.

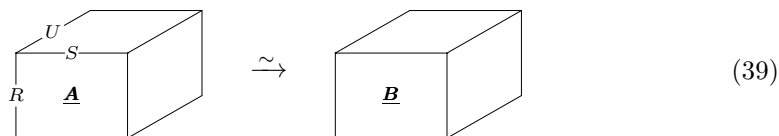
Corollary 3.1. *Theorem 1.1 holds if and only if it holds with the condition*

$$\underline{F}(\mathcal{A}) \text{ and } \underline{F}(\mathcal{B}) \text{ are block-equivalent} \quad (38)$$

instead of (19).

Proof. Suppose Theorem 1.1 with (38) instead of (19) holds for some partitioned three-dimensional variable array $\underline{F}(\mathcal{X})$. Reasoning as in Section 2.4, we first construct an array $\underline{F}_1(\mathcal{X}) := \underline{M}^{F(\mathcal{X})}$ that is not partitioned by frontal planes and satisfies the conditions of Theorem 1.1 with (38) instead of (19). Then we apply an analogous construction to $\underline{F}_1(\mathcal{X})$ and obtain an array $\underline{F}_2(\mathcal{X})$ that is not partitioned by frontal and lateral planes and satisfies the conditions of Theorem 1.1 with (38) instead of (19). At last, we construct an unpartitioned array $\underline{F}_3(\mathcal{X})$ that satisfies the conditions of Theorem 1.1. \square

We draw



$$\begin{array}{c} \text{U} \\ \text{S} \\ \text{R} \quad \underline{\mathbf{A}} \end{array} \quad \rightsquigarrow \quad \underline{\mathbf{B}} \quad (39)$$

if $(R, S, U) : \underline{\mathbf{A}} \rightsquigarrow \underline{\mathbf{B}}$. We denote arrays in illustrations by bold letters.

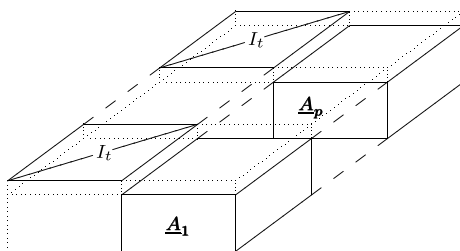
Gelfand and Ponomarev [10] proved that the problem of classifying pairs of matrices up to simultaneous similarity contains the problem of classifying p -tuples of matrices up to simultaneous similarity for an arbitrary p . A three-dimensional analogue of their statement is the following corollary, in which we use the notion “contains” in the sense of Definition 1.2.

Corollary 3.2. *The problem (1) contains the problem of classifying p -tuples of three-dimensional arrays up to simultaneous equivalence for an arbitrary p .*

Proof. Due to Theorem 2.1, it suffices to prove that the second problem is contained in the problem of classifying partitioned three-dimensional arrays up to block-equivalence.

Let $\mathcal{A} = (\underline{\mathbf{A}}_1, \dots, \underline{\mathbf{A}}_p)$ be a sequence of unpartitioned arrays of size $m \times n \times t$. Define the partitioned array

$$\underline{\mathbf{N}}^{\mathcal{A}} = [\underline{\mathbf{N}}_{\alpha\beta\gamma}^{\mathcal{A}}]_{\alpha=1}^2 [\beta=1]_2 [\gamma=1]^p =$$



in which

$$\underline{\mathbf{N}}_{111}^{\mathcal{A}} = \dots = \underline{\mathbf{N}}_{11p}^{\mathcal{A}} = I_t \text{ of size } 1 \times t \times t, \quad (\underline{\mathbf{N}}_{221}^{\mathcal{A}}, \dots, \underline{\mathbf{N}}_{22p}^{\mathcal{A}}) = \mathcal{A},$$

and the other spatial blocks are zero (the spatial blocks are indexed as the entries in (21); the diagonal lines in $\underline{\mathbf{N}}_{111}^{\mathcal{A}}$ and $\underline{\mathbf{N}}_{11p}^{\mathcal{A}}$ denote the main diagonal of I_n consisting of units).

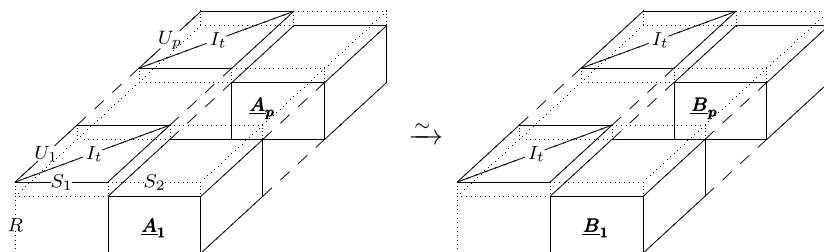
Let $\mathcal{B} = (\underline{\mathbf{B}}_1, \dots, \underline{\mathbf{B}}_p)$ be another sequence of unpartitioned arrays of the same size $m \times n \times t$. Let us prove that the arrays in \mathcal{A} and \mathcal{B} are simultaneously equivalent if and only if $\underline{\mathbf{N}}^{\mathcal{A}}$ and $\underline{\mathbf{N}}^{\mathcal{B}}$ are block-equivalent.

\implies . It is obvious.

\Leftarrow . Let $\underline{N}^{\mathcal{A}}$ and $\underline{N}^{\mathcal{B}}$ be block-equivalent; that is, there exists

$$([r] \oplus R, S_1 \oplus S_2, U_1 \oplus \cdots \oplus U_p) : \underline{N}^{\mathcal{A}} \xrightarrow{\sim} \underline{N}^{\mathcal{B}}.$$

In the notation (39),



Equating the spatial blocks $(1, 1, 1), \dots, (1, 1, p)$, we get $S_1^T I_t U_1 r = I_t, \dots, S_1^T I_t U_p r = I_t$ (which follows from (5) in analogy to (23)). Hence, $U_1 = \cdots = U_p$ and so $(R, S_2, U_1) : \underline{A}_1 \xrightarrow{\sim} \underline{B}_1, \dots, \underline{A}_p \xrightarrow{\sim} \underline{B}_p$. \square

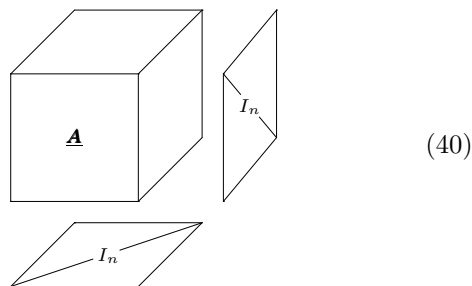
In the next two corollaries, we consider two important special cases of Theorem 1.1. Their proofs may help to understand the proof of Theorem 1.1. If $(R, S, U) : \underline{A} \xrightarrow{\sim} \underline{B}$, then we write that \underline{A} is reduced by (R, S, U) -transformations. Recall that the problems of classifying $(1, 2)$ -tensors and $(0, 3)$ -tensors are the problems of classifying 3-dimensional arrays up to (S, S, S^{-T}) -transformations and (S, S, S) -transformations, respectively. Recall also that each partitioned three-dimensional array is partitioned into strata by frontal, lateral, and horizontal planes, and each stratum consists of slices; see Section 2.1. By a plane stratum, we mean a stratum consisting of one slice.

Corollary 3.3. *The problem (1) contains the problem of classifying $(1, 2)$ -tensors.*

Proof. Due to Theorem 2.1, it suffices to prove that the second problem is contained in the problem of classifying partitioned three-dimensional arrays up to block-equivalence.

For each unpartitioned array \underline{A} of size $n \times n \times n$, define the partitioned array

$$\underline{K}^{\mathcal{A}} = [\underline{K}_{\alpha\beta\gamma}^{\mathcal{A}}]_{\alpha=1}^2 [\beta=1]_2 [\gamma=1]^1 :=$$



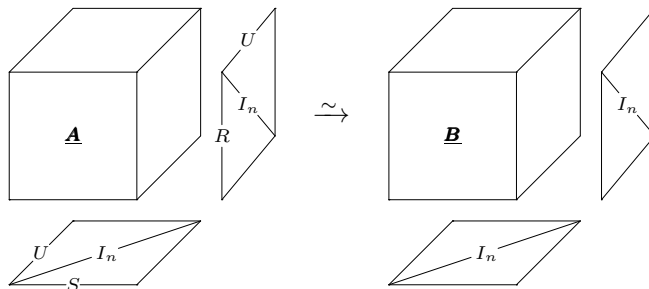
of size $(n+1) \times (n+1) \times n$. It is obtained from \underline{A} by attaching under it and on the right of it the plane strata that are the identity matrices:

$$\underline{K}_{111}^A = \underline{A}, \quad \underline{K}_{121}^A = I_n, \quad \underline{K}_{211}^A = I_n, \quad \underline{K}_{221}^A = 0$$

(the diagonal lines in \underline{K}_{121}^A and \underline{K}_{211}^A denote the main diagonal of I_n consisting of units). Let \underline{B} be another unpartitioned array of size $n \times n \times n$, and let \underline{K}^A and \underline{K}^B be block-equivalent. This means that there exists

$$(R \oplus [r], S \oplus [s], U) : \underline{K}^A \xrightarrow{\sim} \underline{K}^B;$$

that is,



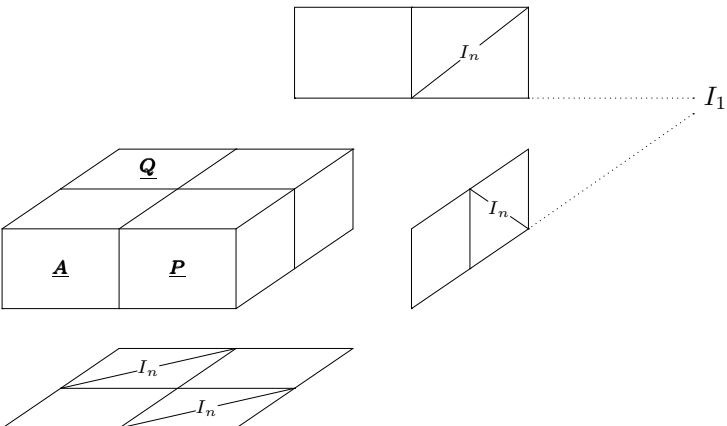
Equating the spatial blocks $(1, 2, 1)$ and $(2, 1, 1)$, we get $R^T I_n U s = I_n$ and $S^T I_n U r = I_n$. Hence $U^{-T} = R s = S r$, and so $(R, S, U) = (R, R s r^{-1}, R^{-T} s^{-1}) : \underline{A} \xrightarrow{\sim} \underline{B}$. Thus, $(R, R r^{-1}, R^{-T}) : \underline{A} \xrightarrow{\sim} \underline{B}$, and so $(R r^{-1}, R r^{-1}, (R r^{-1})^{-T}) : \underline{A} \xrightarrow{\sim} \underline{B}$. \square

Corollary 3.4. *The problem (1) contains the problem of classifying $(0, 3)$ -tensors.*

Proof. For each unpartitioned array \underline{A} of size $n \times n \times n$, define the array

$$\underline{L}^A := \begin{array}{c} \underline{Q} \\ \hline \begin{array}{cc} \underline{A} & \underline{P} \end{array} \end{array} \quad \text{with } \underline{P} = \underline{Q} = 0$$

of size $n \times 2n \times 2n$ partitioned into $1 \cdot 2 \cdot 2$ spatial blocks. We attach the plane strata to the right of \underline{L}^A , under it, and behind it. All blocks of the new plane strata are zero except for the identity matrices I_n under and directly behind of \underline{P} and under and on the right of \underline{Q} , and also except for the matrix I_1 at the intersection of these planes:

$$\underline{N}^A :=$$

(41)

The obtained array \underline{N}^A is partitioned into $2 \cdot 3 \cdot 3$ spatial blocks.

Let \underline{B} be another unpartitioned array of size $n \times n \times n$. Let \underline{N}^A and \underline{N}^B be block-equivalent:

$$(R \oplus [r], S_1 \oplus S_2 \oplus [s], U_1 \oplus U_2 \oplus [u]) : \underline{N}^A \rightsquigarrow \underline{N}^B.$$

Equating the spatial blocks with I_n and I_1 , we get

$$S_1^T U_2 r = I_n, \quad S_2^T U_1 r = I_n, \quad R^T U_2 s = I_n, \quad R^T S_2 u = I_n, \quad rsu = 1.$$

Hence

$$S_1 = U_2^{-T} r^{-1} = R r^{-1} s, \quad U_1 = S_2^{-T} r^{-1} = R r^{-1} u$$

and we have

$$(R, S_1, U_1) = (R r^{-1} r, R r^{-1} s, R r^{-1} u) : \underline{A} \rightsquigarrow \underline{B}.$$

Since $rsu = 1$, $(R r^{-1}, R r^{-1}, R r^{-1}) : \underline{A} \rightsquigarrow \underline{B}$. \square

4. Proof of the statement (12)

Each block-equivalence transformation of a partitioned three-dimensional array \underline{A} has the form

$$(R, S, U) = (R_1 \oplus \cdots \oplus R_{\bar{m}}, S_1 \oplus \cdots \oplus S_{\bar{n}}, U_1 \oplus \cdots \oplus U_{\bar{t}}) : \underline{A} \rightsquigarrow \underline{B}. \quad (42)$$

In this section, we consider a special case of these transformations: some of the diagonal blocks in R, S, U are claimed to be equal or mutually contragredient.

Let us give formal definitions. Consider a finite set \mathcal{P} with two relations \sim and \bowtie satisfying the following conditions:

- (i) \sim is an equivalence relation,
- (ii) if $a \bowtie b$, then $a \approx b$,
- (iii) if $a \bowtie b$, then $b \bowtie c$ if and only if $a \sim c$.

Taking $a = c$ in (iii), we obtain that $a \bowtie b$ implies $b \bowtie a$.

It is clear that the relation \bowtie can be extended to the set \mathcal{P}/\sim of equivalence classes such that $a \bowtie b$ if and only if $[a] \bowtie [b]$, where $[a]$ and $[b]$ are the equivalence classes of a and b . Moreover, if an equivalence relation \sim on \mathcal{P} is fixed and $*$ is any involutive mapping on \mathcal{P}/\sim (i.e., $[a]** = [a]$ for each $[a] \in \mathcal{P}/\sim$), then the relation \bowtie defined on \mathcal{P} as follows:

$$a \bowtie b \iff [a] \neq [a]^* = [b]$$

satisfies (ii) and (iii), and each relation \bowtie satisfying (ii) and (iii) can be so obtained.

Let

$$\mathcal{P} := \{1, \dots, \bar{m}; 1', \dots, \bar{n}'; 1'', \dots, \bar{t}''\} \quad (43)$$

be the disjoint union of the set of first indices, the set of second indices, and the set of third indices of $\underline{A} = [\underline{A}_{\alpha\beta\gamma}]_{\alpha=1}^{\bar{m}}_{\beta=1}^{\bar{n}}_{\gamma=1}^{\bar{t}}$. Since these sets correspond to nonintersecting subsets of \mathcal{P} , we can denote all transforming matrices in (42) by the same letter:

$$(S_1 \oplus \dots \oplus S_{\bar{m}}, S_{1'} \oplus \dots \oplus S_{\bar{n}'}, S_{1''} \oplus \dots \oplus S_{\bar{t}''}) : \underline{A} \xrightarrow{\sim} \underline{B},$$

and give the partition of \underline{A} by the sequence

$$\underline{d} := (d_1, \dots, d_{\bar{m}}; d_{1'}, \dots, d_{\bar{n}'}; d_{1''}, \dots, d_{\bar{t}''}) \quad (44)$$

(in which the semicolons separate the sets of sizes of frontal, lateral, and horizontal strata) such that the size of each \underline{A}_{ijk} is $d_i \times d_{j'} \times d_{k''}$.

Let \sim and \bowtie be binary relations on (43) satisfying (i)–(iii). Let the partition (44) of $\underline{A} = [\underline{A}_{\alpha\beta\gamma}]_{\alpha=1}^{\bar{m}}_{\beta=1}^{\bar{n}}_{\gamma=1}^{\bar{t}}$ satisfies the condition:

$$d_\alpha = d_\beta \text{ if } \alpha \sim \beta \text{ or } \alpha \bowtie \beta \quad \text{for all } \alpha, \beta \in \mathcal{P}. \quad (45)$$

We say that (43) is a *linked block-equivalence transformation* if the following two conditions hold for all $\alpha, \beta \in \mathcal{P}$:

$$S_\alpha = S_\beta \text{ if } \alpha \sim \beta, \quad S_\alpha = S_\beta^{-T} \text{ if } \alpha \bowtie \beta. \quad (46)$$

It is convenient to give the relations \sim and \bowtie on \mathcal{P} by

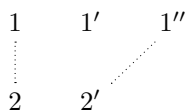
the graph Q with the set of vertices \mathcal{P} and with two types of arrows:
 two vertices α and β are linked by a solid line if $\alpha \sim \beta$ and $\alpha \neq \beta$, and
 by a dotted line if $\alpha \bowtie \beta$. (47)

Example 4.1. The graphs

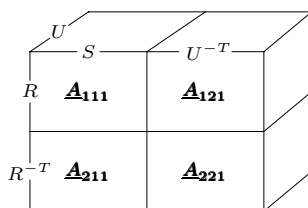


give the problems of classifying partitioned arrays consisting of a single spatial block \underline{A}_{111} up to (S, S, S^{-T}) -transformations and up to (S, S, S) -transformations, respectively; that is, the problems of classifying $(1, 2)$ -tensors and $(0, 3)$ -tensors.

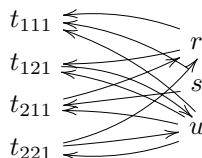
Example 4.2. The graph



gives the problem of classifying partitioned arrays $\underline{A} = [\underline{A}_{\alpha\beta 1}]$ consisting of $2 \cdot 2 \cdot 1$ spatial blocks up to $(R \oplus R^{-T}, S \oplus U^{-T}, U)$ -transformations; that is, up to linked block-equivalence transformations



This problem can be considered as the problem of classifying array-representations $\mathcal{A} = (\underline{A}_{111}, \underline{A}_{121}, \underline{A}_{211}, \underline{A}_{221})$ of the directed bipartite graph



We prove the statement (12) in the following theorem. Its array $\underline{H}(\underline{X})$ generalizes the arrays (40) and (41).

Theorem 4.1. Let $\mathcal{P} := \{1, \dots, \bar{m}; 1', \dots, \bar{n}'; 1'', \dots, \bar{t}''\}$ be a set with binary relations \sim and \bowtie satisfying the conditions (i)–(iii) from the beginning of Section 4. Let $\underline{X} = [\underline{X}_{\alpha\beta\gamma}]_{\alpha=1}^{\bar{m}} [\beta=1]_{\bar{n}}^{\bar{t}} [\gamma=1]_{\bar{t}}^{\bar{t}}$ be a variable array whose entries are independent parameters and whose partition into spatial blocks is given by some sequence (44) satisfying (45).

Then there exists a partitioned array $\underline{H}(\underline{X}) = [\underline{H}_{\alpha\beta\gamma}]_{\alpha=1}^{\bar{m}} [\beta=1]^{\bar{n}} [\gamma=1]^{\bar{t}}$ in which $\bar{m} < \bar{\bar{m}}$, $\bar{n} < \bar{\bar{n}}$, $\bar{t} < \bar{\bar{t}}$;

- (a) \underline{X} is the subarray of $\underline{H}(\underline{X})$ located at the first $\bar{m} \cdot \bar{n} \cdot \bar{t}$ spatial blocks (i.e. $\underline{H}_{\alpha\beta\gamma} = \underline{X}_{\alpha\beta\gamma}$ if $\alpha \leq \bar{m}$, $\beta \leq \bar{n}$, and $\gamma \leq \bar{t}$);
- (b) $\underline{H}_{\bar{m}\bar{n}\bar{t}} = [1]$ is of size $1 \times 1 \times 1$, and the other spatial blocks outside of \underline{X} are zero except for some of $\underline{H}_{\bar{m}\beta\gamma}$, $\underline{H}_{\alpha\bar{n}\gamma}$, $\underline{H}_{\alpha\beta\bar{t}}$ that are the identity matrices

such that two three-dimensional arrays \underline{A} and \underline{B} over a field, partitioned conformally to \underline{X} , are linked block-equivalent if and only if $\underline{H}(\underline{A})$ and $\underline{H}(\underline{B})$ are block-equivalent. Note that the disposition of the identity matrices outside of \underline{X} (see (b)) depends only on $(\mathcal{P}, \sim, \bowtie)$.

Proof. Let $\bar{\bar{m}} > \bar{m}$, $\bar{\bar{n}} > \bar{n}$, and $\bar{\bar{t}} > \bar{t}$ be large enough in order to make possible the further arguments. Let \underline{X} be the variable array from the theorem. Denote by $\underline{H}^0(\underline{X}) = [\underline{H}_{\alpha\beta\gamma}^0]_{\alpha=1}^{\bar{m}} [\beta=1]^{\bar{n}} [\gamma=1]^{\bar{t}}$ the array that satisfies (a) and in which all the spatial blocks outside of \underline{X} are zero except for $\underline{H}_{\bar{m}\bar{n}\bar{t}}^0 = [1]$ of size $1 \times 1 \times 1$.

Let Q be the graph defined in (47) that gives the relations \sim and \bowtie from the theorem. Consider the graph

$$Q^0 : \begin{array}{ccc} 1 & 1' & 1'' \\ 2 & 2' & 2'' \\ \vdots & \vdots & \vdots \\ \bar{m} & \bar{n} & \bar{t} \end{array} \quad (48)$$

without edges, whose vertices correspond to the indices of $\underline{H}^0(\underline{X})$. Each vertex of Q is the vertex of Q^0 . We will consecutively join the edges of Q to Q^0 and respectively modify $\underline{H}^0(\underline{X})$ until obtain $Q^r \supset Q$ and $\underline{H}(\underline{X}) := \underline{H}^r(\underline{X})$ satisfying Theorem 4.1.

On each step k , we construct Q^k and $\underline{H}^k(\underline{X}) = [\underline{H}_{\alpha\beta\gamma}^k]$ such that

- (1°) the conditions (a) and (b) hold with $\underline{H}^k(\underline{X})$ instead of $\underline{H}(\underline{X})$, and
- (2°) for every two arrays \underline{A} and \underline{B} partitioned conformally to \underline{X} , each block-equivalence

$$(S_1 \oplus \cdots \oplus S_{\bar{m}}, S_{1'} \oplus \cdots \oplus S_{\bar{n}'}, S_{1''} \oplus \cdots \oplus S_{\bar{t}''}) : \underline{H}^k(\underline{A}) \xrightarrow{\sim} \underline{H}^k(\underline{B}) \quad (49)$$

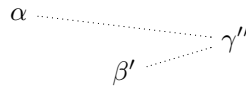
with $S_{\bar{m}} = S_{\bar{n}'} = S_{\bar{t}''} = [1]$ satisfies (46) in which the relations \sim and \bowtie are given by the graph Q^k .

If $k = 0$, then (1°) and (2°) hold since the block-equivalence coincides with the linked block-equivalence with respect to the relations \sim and \bowtie given by Q^0 .

Reasoning by induction, we assume that Q^k and $\underline{H}^k(\underline{X})$ satisfying (1°) and (2°) have been constructed. We construct Q^{k+1} and $\underline{H}^{k+1}(\underline{X}) = [\underline{H}_{\alpha\beta\gamma}^{k+1}]$ as follows. Let λ be an edge of Q that does not belong to Q^k . Denote by Q^{k+1} the graph obtained from Q^k by joining λ and all the edges that appear automatically due to the transitivity of \sim and the condition (iii) from the beginning of this section.

For definiteness, we suppose that λ connects a vertex from the first column and a vertex from the first or second column in (48). The following cases are possible.

- Case 1:** λ is dotted and connects a vertex from the first column and a vertex from the second column in (48). Let $\lambda : \alpha \cdots \beta'$. We replace the spatial block $\underline{H}_{\alpha\beta\bar{t}}^k = 0$ of size $d_\alpha \times d_{\beta'} \times 1$ by the identity matrix: $\underline{H}_{\alpha\beta\bar{t}}^{k+1} = I$ ($d_\alpha = d_{\beta'}$ by (45)); the other spatial blocks of \underline{H}^k and \underline{H}^{k+1} coincide. Since $\underline{H}_{\alpha\beta\bar{t}}^{k+1} = I$ and $S_{\bar{t}''} = [1]$, we have $S_\alpha^T I S_{\beta'} = I$ in (49). Hence, $S_\alpha^{-T} = S_{\beta'}$, which ensures the conditions (1°) and (2°) with $k+1$ instead of k .
- Case 2:** λ is solid and connects a vertex from the first column and a vertex from the second column in (48). Let $\lambda : \alpha \text{---} \beta'$ and let $\gamma'' \in \{\bar{t}+1, \dots, \bar{t}-1\}$ be a vertex from the third column in (48) that does not have arrows (it exists since we have supposed that $\bar{m}, \bar{n}, \bar{t}$ are large enough). Reasoning as in Case 1, we join the following two dotted arrows to Q^k :

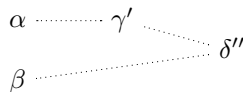


Then the solid arrow $\lambda : \alpha \text{---} \beta'$ is joined automatically by (iii).

- Case 3:** λ is solid and connects two vertices from the first column. Let $\lambda : \alpha \text{---} \beta$. Reasoning as in Case 2, we join the following two dotted arrows to Q^k :



- Case 4:** λ is dotted and connects two vertices from the first column. Let $\lambda : \alpha \cdots \beta$. Let $\gamma' \in \{\bar{n}+1, \dots, \bar{n}-1\}$ and $\delta'' \in \{\bar{t}+1, \dots, \bar{t}-1\}$ be vertices from the second and third columns in (48) that do not have arrows. We join the following three dotted arrows to Q^k :



Then the dotted arrow $\lambda : \alpha \cdots \beta$ is attached automatically by (iii).

The conditions (1°) and (2°) with $k + 1$ instead of k hold in all the cases. We repeat this construction until obtain $Q^r \supset Q$.

Let \underline{A} and \underline{B} be three-dimensional arrays partitioned conformally to \underline{X} such that $\underline{H}^r(\underline{A})$ and $\underline{H}^r(\underline{B})$ are block-equivalent; that is, there exists

$$(R_1 \oplus \cdots \oplus R_{\bar{m}}, R_{1'} \oplus \cdots \oplus R_{\bar{n}'}, R_{1''} \oplus \cdots \oplus R_{\bar{t}''}) : \underline{H}^r(\underline{A}) \xrightarrow{\sim} \underline{H}^r(\underline{B}).$$

Since $\underline{H}_{\bar{m}\bar{n}\bar{t}}^r$ is of size $1 \times 1 \times 1$, the summands $R_{\bar{m}}, R_{\bar{n}'}, R_{\bar{t}''}$ are 1×1 . Let $R_{\bar{m}} = [a]$, $R_{\bar{n}'} = [b]$, and $R_{\bar{t}''} = [c]$. Since $\underline{H}_{\bar{m}\bar{n}\bar{t}}^r = [1]$, $abc = 1$. Hence,

$$(a^{-1}(R_1 \oplus \cdots \oplus R_{\bar{m}}), b^{-1}(R_{1'} \oplus \cdots \oplus R_{\bar{n}'}), c^{-1}(R_{1''} \oplus \cdots \oplus R_{\bar{t}''})) : \underline{H}^r(\underline{A}) \xrightarrow{\sim} \underline{H}^r(\underline{B}).$$

Since $a^{-1}R_{\bar{m}} = b^{-1}R_{\bar{n}'} = c^{-1}R_{\bar{t}''} = [1]$, we use (1°) and (2°) with $k = r$ and get that

$$(a^{-1}(R_1 \oplus \cdots \oplus R_{\bar{m}}), b^{-1}(R_{1'} \oplus \cdots \oplus R_{\bar{n}'}), c^{-1}(R_{1''} \oplus \cdots \oplus R_{\bar{t}''})) : \underline{A} \xrightarrow{\sim} \underline{B}$$

is a linked block-equivalence with respect to the relations \sim and \boxtimes from Theorem 4.1. \square

5. Proof of Theorem 1.1

In this section, we prove Theorem 1.1, which ensures the statement (13).

Lemma 5.1. *It suffices to prove Theorem 1.1 for all graphs G in which each left vertex has three arrows.*

Proof. Let G be a directed bipartite graph with $\mathbb{T} = \{t_1, \dots, t_p\}$ and $\mathbb{V} = \{1, \dots, q\}$.

(1°) Let G have a left vertex t with exactly two arrows:

$$t \begin{array}{c} \nearrow u \\ \longrightarrow v \end{array} \quad \text{or} \quad t \begin{array}{c} \frown \\ \longrightarrow v \end{array} \quad (50)$$

where each line is \longrightarrow or \longleftarrow . Denote by G' the graph obtained from G by replacing (50) with

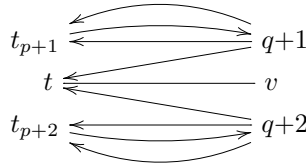
$$t \begin{array}{c} \nearrow u \\ \longrightarrow v \\ \longrightarrow q+1 \end{array} \quad \text{or} \quad t \begin{array}{c} \frown \\ \longrightarrow v \\ \longrightarrow q+1 \end{array} \quad (51)$$

respectively (thus, $\mathbb{T}' = \{t_1, t_2, \dots, t_{p+1}\}$ and $\mathbb{V}' = \{1, 2, \dots, q+1\}$). Let us extend each array-representation \mathcal{A} of G to the array-representation \mathcal{A}' of G' by assigning

the $1 \times 1 \times 1$ array $\underline{A}'_{t_{p+1}} := [1]$ to the new vertex t_{p+1} .

Let us prove that $\mathcal{A} \simeq \mathcal{B}$ if and only if $\mathcal{A}' \simeq \mathcal{B}'$. If $(S_1, \dots, S_q) : \mathcal{A} \xrightarrow{\sim} \mathcal{B}$, then $(S_1, \dots, S_q, [1]) : \mathcal{A}' \xrightarrow{\sim} \mathcal{B}'$. Conversely, let $(S_1, \dots, S_q, [a]) : \mathcal{A}' \xrightarrow{\sim} \mathcal{B}'$. Then $([a], [a], [a]^{-T}) : \underline{A}_{t_{p+1}} \xrightarrow{\sim} \underline{B}_{t_{p+1}}$. Since $\underline{A}_{t_{p+1}} = \underline{B}_{t_{p+1}} = [1]$, we have $a = 1$, and so $(S_1, \dots, S_q) : \mathcal{A} \xrightarrow{\sim} \mathcal{B}$.

(2°) Let G have a left vertex t with exactly one arrow: $t \rightarrow v$ or $t \leftarrow v$. Then by analogy with (51) we replace it with



in which $t \text{ --- } v$ is $t \rightarrow v$ or $t \leftarrow v$.

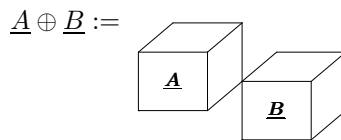
We repeat (1°) and (2°) until extend G to a graph \tilde{G} with the set of left vertices $\tilde{\mathbb{T}} = \{t_1, t_2, \dots, t_{p+p'}\}$ in which each left vertex has exactly three arrows. For array-representations $\mathcal{A} = (\underline{A}_1, \dots, \underline{A}_p)$ and $\mathcal{B} = (\underline{B}_1, \dots, \underline{B}_p)$ of G , define the array-representations

$$\tilde{\mathcal{A}} := (\underline{A}_1, \dots, \underline{A}_p, [1], \dots, [1]), \quad \tilde{\mathcal{B}} := (\underline{B}_1, \dots, \underline{B}_p, [1], \dots, [1]) \quad (52)$$

of \tilde{G} and obtain that $\mathcal{A} \simeq \mathcal{B}$ if and only if $\tilde{\mathcal{A}} \simeq \tilde{\mathcal{B}}$.

Suppose that Theorem 1.1 holds for \tilde{G} and some array $\tilde{F}(\tilde{\mathcal{X}})$, in which $\tilde{\mathcal{X}} = (\underline{X}_1, \dots, \underline{X}_p, [x_{p+1}], \dots, [x_{p+p'}])$ is a variable array-representation of \tilde{G} . Substituting the array-representations (52), we find that $\tilde{F}(\tilde{\mathcal{A}})$ is equivalent to $\tilde{F}(\tilde{\mathcal{B}})$ if and only if $\tilde{\mathcal{A}} \simeq \tilde{\mathcal{B}}$, if and only if $\mathcal{A} \simeq \mathcal{B}$. Hence, Theorem 1.1 holds for G and for the array $\underline{F}(\mathcal{X}) := \tilde{F}(\underline{X}_1, \dots, \underline{X}_p, [1], \dots, [1])$ with $\mathcal{X} := (\underline{X}_1, \dots, \underline{X}_p)$. \square

The *direct sum* $\underline{A} \oplus \underline{B}$ of three-dimensional arrays \underline{A} and \underline{B} is the partitioned array $[\underline{C}_{\alpha\beta}]_{\alpha,\beta=1}^2$, in which $\underline{C}_{11} := \underline{A}$, $\underline{C}_{22} := \underline{B}$, and the other $\underline{C}_{\alpha\beta} := 0$:



Proof of Theorem 1.1. Let G be a directed bipartite graph with left vertices t_1, \dots, t_p and right vertices $1, \dots, q$. Due to Lemma 5.1, we can suppose that each left vertex of G has exactly three arrows:



Let

$$\mathcal{A} = (\underline{A}_1, \dots, \underline{A}_p), \quad \mathcal{B} = (\underline{B}_1, \dots, \underline{B}_p) \quad (53)$$

be two array-representations of G of the same size. By Definition 1.3, each isomorphism $\mathcal{S} : \mathcal{A} \xrightarrow{\sim} \mathcal{B}$ is given by a sequence

$$\mathcal{S} := (S_1, \dots, S_q) \quad (54)$$

of nonsingular matrices such that

$$(S_{v_i}^{\tau_i}, S_{v_{i'}}^{\tau_{i'}}, S_{v_{i''}}^{\tau_{i''}}) : \underline{A}_i \xrightarrow{\sim} \underline{B}_i, \quad i = 1, \dots, p, \quad (55)$$

where

$$\tau_{i^{(\varepsilon)}} := \begin{cases} 1 & \text{if } \lambda_{i^{(\varepsilon)}} : t_i \longleftarrow v_{i^{(\varepsilon)}} \\ -T & \text{if } \lambda_{i^{(\varepsilon)}} : t_i \longrightarrow v_{i^{(\varepsilon)}} \end{cases} \quad \text{for } i = 1, \dots, p \text{ and } \varepsilon = 0, 1, 2,$$

in which $i^{(0)} := i$, $i^{(1)} := i'$, and $i^{(2)} := i''$.

The array-representations (53) define the partitioned arrays

$$\underline{A} := \underline{A}_1 \oplus \dots \oplus \underline{A}_p, \quad \underline{B} := \underline{B}_1 \oplus \dots \oplus \underline{B}_p.$$

The sequence (54) defines the isomorphism $\mathcal{S} : \mathcal{A} \xrightarrow{\sim} \mathcal{B}$ if and only if (55) holds if and only if

$$(S_{v_1}^{\tau_1} \oplus \dots \oplus S_{v_p}^{\tau_p}, S_{v_{1'}}^{\tau_{1'}} \oplus \dots \oplus S_{v_{p'}}^{\tau_{p'}}, S_{v_{1''}}^{\tau_{1''}} \oplus \dots \oplus S_{v_{p''}}^{\tau_{p''}}) : \underline{A} \xrightarrow{\sim} \underline{B}. \quad (56)$$

An arbitrary block-equivalence

$$(R_1 \oplus \dots \oplus R_p, R_{1'} \oplus \dots \oplus R_{p'}, R_{1''} \oplus \dots \oplus R_{p''},) : \underline{A} \xrightarrow{\sim} \underline{B} \quad (57)$$

has the form (56) if and only if the following two conditions hold:

- $R_{i^{(\varepsilon)}} = R_{j^{(\delta)}}$ if $v_{i^{(\varepsilon)}} = v_{j^{(\delta)}}$ and $\tau_{i^{(\varepsilon)}} = \tau_{j^{(\delta)}}$,
- $R_{i^{(\varepsilon)}} = R_{j^{(\delta)}}^{-T}$ if $v_{i^{(\varepsilon)}} = v_{j^{(\delta)}}$ and $\tau_{i^{(\varepsilon)}} \neq \tau_{j^{(\delta)}}$;

that is, if and only if (57) is a linked block-equivalence with respect to the relations \sim and \boxtimes on

$$\mathcal{P} := \{1, \dots, p; 1', \dots, p'; 1'', \dots, p''\}$$

that are defined as follows:

- $i^{(\varepsilon)} \sim j^{(\delta)}$ if $v_{i^{(\varepsilon)}} = v_{j^{(\delta)}}$ and $\tau_{i^{(\varepsilon)}} = \tau_{j^{(\delta)}}$,
- $i^{(\varepsilon)} \bowtie j^{(\delta)}$ if $v_{i^{(\varepsilon)}} = v_{j^{(\delta)}}$ and $\tau_{i^{(\varepsilon)}} \neq \tau_{j^{(\delta)}}$.

Thus, $\mathcal{A} \simeq \mathcal{B}$ if and only if \underline{A} and \underline{B} are linked block-equivalent.

Let $\mathcal{X} = (\underline{X}_1, \dots, \underline{X}_p)$ be a variable array-representation of G of dimension $d = (d_1, \dots, d_q)$. Define the array $\underline{K}(\mathcal{X}) := \underline{X}_1 \oplus \dots \oplus \underline{X}_p$. Two array-representations \mathcal{A} and \mathcal{B} of G of dimension d are isomorphic if and only if $\underline{K}(\mathcal{A})$ and $\underline{K}(\mathcal{B})$ are linked block-equivalent. This proves Theorem 1.1 due to Corollary 3.1 and Theorem 4.1. \square

Note that the array $\underline{K}(\mathcal{X})$ in the proof of Theorem 1.1 is somewhat large. One can construct smaller arrays in most specific cases (as in Example 4.2).

6. Tensor-representations of directed bipartite graphs

Definition 1.3 of representations of directed bipartite graphs is given in terms of arrays. In this section, we give an equivalent definition in terms of tensors that are considered as elements of tensor products, which may extend the range of validity of Theorem 1.1.

Recall that a *tensor on vector spaces* V_1, \dots, V_{m+n} over a field \mathbb{F} (some of them can be equal) is an element of the tensor product

$$\mathcal{T} \in V_1 \otimes \dots \otimes V_m \otimes V_{m+1}^* \otimes \dots \otimes V_{m+n}^*, \quad (58)$$

in which V_i^* denotes the dual space consisting of all linear forms $V_i \rightarrow \mathbb{F}$. The tensor \mathcal{T} is called *m times contravariant and n times covariant*, or an (m, n) -*tensor* for short.

Choose a basis $f_{\alpha 1}, \dots, f_{\alpha d_\alpha}$ in each space V_α ($\alpha = 1, \dots, m+n$) and take the *dual basis* $f_{\alpha 1}^*, \dots, f_{\alpha d_\alpha}^*$ in V_α^* , where $f_{\alpha i}^* : V_i \rightarrow \mathbb{F}$ is defined by $f_{\alpha i}^*(f_{\alpha i}) := 1$ and $f_{\alpha i}^*(f_{\alpha j}) := 0$ if $i \neq j$. The tensor product in (58) is the vector space over \mathbb{F} with the basis formed by

$$f_{1i_1} \otimes \dots \otimes f_{mi_m} \otimes f_{m+1, i_{m+1}}^* \otimes \dots \otimes f_{m+n, i_{m+n}}^*,$$

and so each tensor (58) is uniquely represented in the form

$$\mathcal{T} = \sum_{i_1, \dots, i_{m+n}} a_{i_1 \dots i_{m+n}} f_{1i_1} \otimes \dots \otimes f_{mi_m} \otimes f_{m+1, i_{m+1}}^* \otimes \dots \otimes f_{m+n, i_{m+n}}^* \quad (59)$$

and can be given by *its array* $\underline{A} = [a_{i_1 \dots i_{m+n}}]$ over \mathbb{F} .

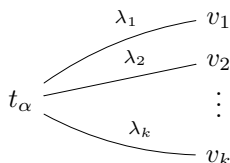
Choose a new basis $g_{\alpha 1}, \dots, g_{\alpha d_\alpha}$ in each V_α , and let $S_\alpha := [s_{\alpha ij}]$ be the change of basis matrix (i.e., $g_{\alpha j} = \sum_{i,j} s_{\alpha ij} f_{\alpha i}$) for each $\alpha = 1, \dots, m+n$. Denote by \underline{B} the array of \mathcal{T} in the new set of bases. Then

$$(S_1^{-T}, \dots, S_m^{-T}, S_{m+1}, \dots, S_{m+n}) : \underline{A} \xrightarrow{\sim} \underline{B},$$

which allows us to reformulate Definition 1.3 as follows.

Definition 6.1 (see [23, Sec. 4]). Let G be a directed bipartite graph with $\mathbb{T} = \{t_1, \dots, t_p\}$ and $\mathbb{V} = \{1, \dots, q\}$.

- A *representation* (\mathcal{T}, V) of G is given by assigning a finite dimensional vector space V_v over \mathbb{F} to each $v \in \mathbb{V}$ and a tensor \mathcal{T}_α to each $t_\alpha \in \mathbb{T}$ so that if



are all arrows in a vertex t_α (each line --- is \longrightarrow or \longleftarrow ; some of $v_1, \dots, v_k \in \mathbb{T}$ may coincide), then

$$\mathcal{T}_\alpha \in V_{v_1}^{\varepsilon_1} \otimes \cdots \otimes V_{v_p}^{\varepsilon_p}, \quad \varepsilon_i := \begin{cases} 1 & \text{if } \lambda_i : t_\alpha \longleftarrow v_i, \\ * & \text{if } \lambda_i : t_\alpha \longrightarrow v_i. \end{cases}$$

- Two representations (\mathcal{T}, V) and (\mathcal{T}', V') of G are *isomorphic* if there exists a system of linear bijections $\varphi_v : V_v \rightarrow V'_v$ ($v \in \mathbb{V}$) that together with the contragredient bijections $\varphi_v^{*-1} : V_v^* \rightarrow V'^*_v$ transform $\mathcal{T}_1, \dots, \mathcal{T}_p$ to $\mathcal{T}'_1, \dots, \mathcal{T}'_p$.

Example 6.1. A special case of (59) is a $(0, 2)$ -tensor $\mathcal{T} = \sum_{i,j} a_{ij} f_{1i}^* \otimes f_{2j}^* \in V_1^* \otimes V_2^*$. It can be identified with the bilinear form

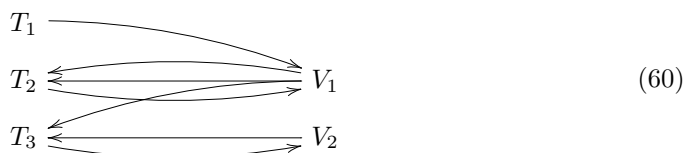
$$\mathcal{T} : V_1 \times V_2 \rightarrow \mathbb{F}, \quad (v_1, v_2) \mapsto \sum_{i,j} a_{ij} f_{1i}^*(v_1) f_{2j}^*(v_2).$$

A $(1, 1)$ -tensor $\mathcal{T} = \sum_{i,j} a_{ij} f_{1i} \otimes f_{2j}^* \in V_1 \otimes V_2^*$ can be identified with the linear mapping

$$\mathcal{T} : V_2 \rightarrow V_1, \quad v_2 \mapsto \sum_{i,j} a_{ij} f_{1i} f_{2j}^*(v_2).$$

The arrays $[a_{ij}]$ of these tensors are the matrices of these bilinear forms and linear mappings. The systems of tensors of order 2 are studied in [13,14,21,23,24] as representations of graphs with undirected, directed, and double directed (\longleftrightarrow) edges that are assigned, respectively, by $(0, 2)$ -, $(1, 1)$ -, and $(2, 0)$ -tensors on the vector spaces assigned to the corresponding vertices. The problem of classifying such representations (i.e., of arbitrary systems of bilinear forms, linear mappings, and bilinear forms on dual spaces) was reduced in [21] to the problem of classifying representations of quivers.

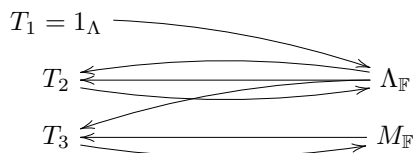
Example 6.2 (see [23, Sec. 4]). Each representation



of the bipartite directed graph (14) consists of vector spaces V_1 and V_2 and tensors

$$T_1 \in V_1, \quad T_2 \in V_1^* \otimes V_1^* \otimes V_1, \quad T_3 \in V_1^* \otimes V_2^* \otimes V_2.$$

Each pair (Λ, M) , consisting of a finite dimensional algebra Λ with unit 1_Λ over a field \mathbb{F} and a Λ -module M , defines the representation



of (14), in which

$$T_2 = \sum_i a_{i1}^* \otimes a_{i2}^* \otimes a_{i3}, \quad T_3 = \sum_i a_i^* \otimes m_{i1}^* \otimes m_{i2}$$

(all $a_{ij} \in \Lambda$ and $m_{ij} \in M$) are the tensors that define the multiplications in Λ and M :

$$(a', a'') \mapsto \sum_i a_{i1}^*(a') a_{i2}^*(a'') a_{i3}, \quad (a, m) \mapsto \sum_i a_{i1}^*(a) m_{i1}^*(m) m_{i2}.$$

Note that the identities (additivity, distributivity, ...) that must satisfy the operations in Λ and M can be written via tensor contractions in such a way that each representation (60) satisfying these relations defines a pair consisting of a finite dimensional algebra and its module. This leads to the theory of representations of bipartite directed graphs with relations that generalizes the theory of representations of quivers with relations.

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