μ-definable sets of integers

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ABSTRACT

The μ-calculus is a language consisting of standard first-order finitary logic with a least fixed point operator applicable to positive inductive definitions. The main theorem of this paper is a set-theoretic characterization of the sets of integers definable in the μ-calculus. Another theorem used but not proven here is a prenex normal form theorem for the μ-calculus.

1. INTRODUCTION

Inductive definability has been studied for some time already. (See [B], [M], and [RA] for material closely related to that here.) Nonetheless, there are some simple questions that have been overlooked. In particular, there is the problem of the expressibility of the μ-calculus.

The μ-calculus (developed by [SD], [HP], [P], [K], and others) is a language for including inductive definitions with first order logic. One can think of a first order logic formula as defining a subset of the universe, the set of elements that make it true. Then "and" corresponds to intersection, "or" to union, and "not" to complementation. Viewing the standard connectives as operations on sets, there is no reason not to include one more, least fixed point.

There are some interesting features of the μ-calculus coming from its being a language. A natural class of inductive definitions are those that are monotone: if $X \supset Y$ then $\Gamma(X) \supset \Gamma(Y)$ (where $\Gamma(X)$ is the result of one application of the operator to the set X). When studying monotonic operations in the context of a language, one would need a syntactic guarantor of monotonicity. This is provided by positivity. An occurrence of a set variable S is positive if that occurrence is in the scopes of exactly an even number of negations. S is positive in a formula if each occurrence of S is positive. Intuitively, the formula can ask whether $x \in S$, but not whether $x \notin S$. Such a ϕ can be considered an inductive definition: $\Gamma(X) = \{x \mid \phi(x)$, where the variable S is interpreted as $X\}$, and is monotone. Therefore, in the μ-calculus, a formula is well-formed only if all of its inductive definitions are positive.

A second aspect of the μ-calculus natural enough from its being a language is closure under the primitive constructs. There can be nesting: a least fixed point S can be a parameter, even negatively, to another fixed point T. Better yet, there can even be feedback: the definition of S can itself refer to T. This phenomenon happens naturally, as the simple-minded generation of the μ-formulas allows for it. It provides the major difficulty in understanding the expressive power of the μ-calculus.

The goal of this paper is to determine which sets of integers are definable in the

μ-calculus. The idea is a two-pronged attack, each of which involves working in L, the hierarchy of constructible sets. An upper bound is found by viewing an inductive construction as working its way up the L-hierarchy, each step corresponding to an increase of 1 in the ordinals. Using ideas from the fine structure of L, we can describe where such a construction must close off. Then we show that this upper bound can be obtained, by describing how to code the (inductive) construction of L into some manipulations on the integers, all within the bounds of the expressibility of the μ-calculus.

To be precise about the upper bound we need to examine reflection properties. For Γ a collection of formulas and X a class of ordinals, α is Γ-reflecting on X if, for all $\phi \in \Gamma$ (with parameters from L_α), if $L_\alpha \models \phi$ then $L_\beta \models \phi$ for some $\beta \in \alpha \cap X$. The μ-calculus calls for a tad more. Rather than reflecting just an ordinal, it wants to reflect a gap: an X-sized gap. So for $\alpha \in X$, let α^{+X} be the next member of X beyond α. α is Γ gap-reflecting on X if, for all $\phi \in \Gamma$ (with parameters from L_α <u>and constant symbol</u> $\mathcal{L}_\alpha$), if $\langle L_{\alpha+X}, L_\alpha \rangle \models \phi$, then for some $\beta < \alpha$, $\langle L_{\beta+X}, L_\beta \rangle \models \phi$. (Here, $\langle Y, Z \rangle \models \phi$ means that $Y \models \phi$ where the symbol $\mathcal{L}_\alpha$ in ϕ is interpreted as Z.) This possibly strange-sounding definition is actually suggested by various propositions in [RA].

Let α be a 1-reflecting admissible iff α is admissible, and an n+1-reflecting admissible iff α is Π_1 gap-reflecting on the set of n-reflecting admissibles. For notational convenience, the least n-reflecting admissible beyond α will be called α^{+n}, and α^{+1} will be α^+.

The notion of Σ_n definability in the μ-calculus measures the alternation of the higher-order quantifiers, the Σ side being μ and the Π side being ν. By the normal form theorem, every term and formula is equivalent to one in prenex form (i.e. all the quantifiers are up front) of the same quantifier complexity (suitably defined).

<u>Theorem</u> A set of integers is Σ_n definable in the μ-calculus iff it is Σ_1 definable over L_α, where α is the least n-reflecting admissible ordinal.

This paper continues with the ordinals involved, and the syntax and semantics of the μ-calculus, stating the relevant definitions and theorems. Following this are sketches of the proofs that the bound claimed in the main theorem is an upper bound, and that this upper bound can be attained.

I would like to thank Prof. Dexter Kozen for bringing this subject to my attention and providing me with valuable discussions about it.

2. STABILITY AND REFLECTION

Recall that α is β-stable if $L_\alpha \quad \Sigma^0{}_1 \; L_\beta$, for $\alpha < \beta$.

<u>Theorem</u>(Richter-Aczel) (α countable) α is $\Pi^1{}_1$-reflecting iff α is α^+-stable. (α^+ is the least admissible beyond α.)

<u>Prop</u> α *is $\Sigma^1{}_1$-reflecting iff α is 2-*reflecting admissible.
The proof of this is essentially the dual of the proof of the Richter-Aczel theorem.

<u>Cor</u> The least 2-reflecting admissible > the least α which is α^+-stable.
<u>Proof</u> By the proposition, the least 2-reflecting admissible is $\sigma^1{}_1$ from [RA].

Similarly, the least α which is α^+-stable is π^1_1. Their Theorem B is that $\pi^1_1 = |\Pi^1_1|$ and $\sigma^1_1 = |\Sigma^1_1|$, where, for a pointclass Λ, $|\Lambda|$ is the supremum of the closure ordinals of Λ-definable inductive definitions. By [A], $|\Pi^1_1| < |\Sigma^1_1|$. QED

Prop Suppose $n > m > 0$.

(1) $\{\beta < \alpha \mid \beta$ is an m-reflecting admissible$\}$ is Δ_1 uniformly in those α closed under taking the next m-1-reflecting admissible (that is, those α such that if $\beta < \alpha$ then $\beta^{+(m-1)} < \alpha$). (For $m = 1$, drop the condition on α.)

(2) If α is an n-reflecting admissible then α is an m-reflecting admissible.

(3) For all α, $\alpha^{+n} > \alpha^{+m}$.

For the case $m = 1$, something much stronger than (2) above is proved in [RA] (see their lemma 6.1, p. 333).

Cor For $n > m$, if α is n-reflecting, then α is a limit of m-reflecting ordinals.

Once the main theorem is proven, it can be combined with the previous proposition to show this

Cor The μ-definable sets of integers are a proper subset of the Δ^1_2 sets of integers. (See [B], ch. V.8, for background on Δ^1_2.)

3. SYNTAX, SEMANTICS, AND NORMAL FORM THEOREMS

The μ-calculus is an extension of $\mathcal{L}_{\omega\omega}$ which includes set variables and the symbols μ and ϵ. Terms and formulas are defined by a mutual induction. All terms of $\mathcal{L}_{\omega\omega}$ are individual terms, and all formula constructors of $\mathcal{L}_{\omega\omega}$ (i.e. $\wedge$, $\sim$, and $\exists$) are formula constructors. All set variables are set terms, and if ϕ is a formula in which the set variable S is positive, then $\mu S\ \phi$ is a set term. (The μ should bind not only S but also some individual variable x (see below), so the notation above is ambiguous. The default assumption is that the choice of x is clear.) If τ is an individual term and $\mathcal{S}$ a set term, then $\tau \in \mathcal{S}$ is a formula.

A structure for such a language is any structure $\mathfrak{M}$ for its first-order part. $\mathfrak{M}$ induces an interpretation of the higher-order part, with μ being the least-fixed-point operator and ϵ being set membership. More formally for $\mu S\ \phi$, define S^α, α an ordinal, inductively on α. S^0 is the empty set. For λ a limit, $S^\lambda = \bigcup\{S^\alpha \mid \alpha < \lambda\}$. $S^{\alpha+1} = \{m \mid \phi(m, S^\alpha)$ holds$\}$, where $\phi(m, S^\alpha)$ is ϕ with S interpreted as S^α and the x bound by the μ interpreted as m. If $\alpha > \beta$ then $S^\alpha \supset S^\beta$ since all occurrences of S in ϕ were positive. Then $\mu S\ \phi = \bigcup\{S^\alpha \mid \alpha \in \mathrm{ORD}\} = S^\gamma$ where γ is the least ordinal such that $S^\gamma = S^{\gamma+1}$.

$\vee$, $\forall$, and ν are the duals of $\wedge$, $\exists$, and μ respectively, which we use freely without comment. ν is the greatest fixed point operator. That is, to calculate $\nu S\ \phi(S)$, start with S being the whole universe of $\mathfrak{M}$, and use ϕ to throw things out inductively until the procedure stabilizes, which is at the greatest fixed point. Observe that $\nu S\ \phi(S)$ is the complement of $\mu S\ {\sim}\phi({\sim}S)$. (Notice that it is necessary syntactically to negate the S internally in order to keep all occurrences of S positive before binding with the μ.) In the cases of all three duals, by the double negations used in their definitions, their occurrence does not affect the positivity or negativity of any free set variable.

Definitions

A set term is in μ-normal form if it is a set variable or if it is of the form $\mu S\ \phi$ or $\nu S\ \phi$, ϕ a formula in μ-normal form.

A formula is quantifier-free if it contains

no occurrences of $\forall, \exists, \mu$, or ν.
A formula is in μ-normal form if it is quantifier-free, or of the form $\tau \in \mathcal{S}$ with τ an individual term and $\mathcal{S}$ a set term in μ-normal form, or of the form $\exists x\ \phi$ or $\forall x\ \phi$ with ϕ a formula in μ-normal form.

Theorem Every formula is equivalent to one in μ-normal form, and every set term is equivalent to one in μ-normal form.
The proof is by a mutual induction.

Suppose there were a pairing function $\langle\ ,\ \rangle$ with projections $(\ )_0$ and $(\ )_1$. Then as usual like first-order quantifiers could be collapsed when they occur consecutively ($\exists x \exists y\ \phi(x, y)$ is equivalent to $\exists z\ \phi((z)_0, (z)_1)$). If in addition there were at least two elements in the universe, then we could do much more.

Definitions The μ-calculus with pairing is the μ-calculus with distinguished function symbols 0 and 1 (0-ary), $(\ )_0$ and $(\ )_1$ (unary), and $\langle\ ,\ \rangle$ (binary). A model for such a language is a model for the μ-calculus in which 0 and 1 are interpreted as distinct objects, and $\langle\ ,\ \rangle$ is a pairing function with projections $(\ )_0$ and $(\ )_1$.

A formula is in pair normal form if it is a Boolean combination of first-order Π_1 and Σ_1 formulas, or of the form $\tau \in \mathcal{S}$ with τ an individual term and $\mathcal{S}$ a set term in pair normal form. A set term is in pair normal form if it is a set variable, or of the form $\mu S\ \phi$ or $\nu S\ \phi$, with ϕ a pair normal formula of the opposite type of S. "Of the opposite type" means in the former case that ϕ has the form $\tau \in \nu T\ \phi'$ (or is first-order), and dually in the latter. So a pair normal term or formula can be described as a string of second-order quantifiers alternating in type followed by a Boolean combination of first-order Π_1 and Σ_1 formulas.

The 0th slice of a set S is $\{x \mid \langle 0,x\rangle \in S\}$. The 1st slice is defined similarly.

Theorem In the μ-calculus with pairing, every formula is equivalent to one in pair normal form, and every set term is equivalent to the 0th slice of one in pair normal form.

The proof once again is by a mutual induction.

A closer examination of the proofs shows that each formula or set term is equivalent to a normal one of the same quantifier complexity, where the complexity counts alternations of higher-order quantifiers, the lowest level being Δ_1.

4. THE LIMIT OF EXPRESSIBILITY

The goal of this section is to prove one half of the main theorem: if a set of integers is Σ_n definable then it is Σ_1 over the least n-reflecting admissible ordinal.

$\mathbb{N}$ can be construed as a model of the μ-calculus with pairing, using any recursive pairing function. Hence it suffices to consider terms and formulas in pair normal form.

Definition X is Σ_n (definable over $\mathfrak{M}$) if X is the 0th slice of a set definable by a Σ_n formula.
For the purposes of this section, allowing the slice of a set does not make a difference: if a set if Σ_1 over some ordinal so are its projections. Hence the proof given here is only for those sets which are the extensions are Σ_n formulas. It is in the proof of the converse (section 5) that this more general definition of Σ_n is used.

The proof is by induction on n. To account for parameters, it uses for a stronger inductive hypothesis the theorem relativized to a set of integers **R**. Notice that the definition of n-reflecting admissible, as well as the statement of the theorem, relativize straightfowardly. We ambiguously use the same notation — e.g. α^{+n} is the least n-**R**-reflecting admissible beyond α -- and hope that the context makes the unstated choice of **R** clear.

<u>n=1</u>

Suppose $\mathcal{S}$ is Σ_1: $\mathcal{S} = \mu S\ \phi$, ϕ first order with parameter T. Work in $L_\alpha[T]$, where α is the least **T**-admissible. The inductive generation of $\mathcal{S}$'s interpretation **S** can be done in $L_\alpha[T]$, at least for α-many steps. More particularly, consider the inductive step $\mathbf{S}^{\beta+1} = \{x \mid \phi(x, \mathbf{S}^\beta)\}$. Since ϕ is first-order, $\mathbf{S}^{\beta+1}$ is definable over $L_\omega[\mathbf{S}^\beta, \mathbf{T}]$. Inductively, $\mathbf{S}^\beta$ is definable over $L_{\omega+\beta}[\mathbf{T}]$, so $\mathbf{S}^{\beta+1}$ is definable over $L_{\omega+\beta+1}[T]$. (It's true that these ordinals can be decreased, but this does not help the argument.) By the uniformity in the generation of S, the same holds for limits: $\mathbf{S}^\lambda$ is definable over $L_{\omega+\lambda}[T]$. The base case of $\mathbf{S}^0$ = the empty set is trivial. Thus $\mathbf{S}^\alpha$ is Σ_1 over $L_\alpha[T]$: $n \in \mathbf{S}^\alpha$ iff $\exists f$ (f is a function with domain some ordinal, and f enumerates the generation of S, and $n \in f(\beta)$ for some β). Furthermore, $\mathbf{S}^\alpha = \mathbf{S}$, by the following argument using admissibility. $n \in \mathbf{S}^{\alpha+1}$ iff $\phi(n, \mathbf{S}^\alpha)$. Replace "$\xi \in \mathbf{S}^\alpha$" by "$\exists\beta\ \xi \in \mathbf{S}^\beta$" in ϕ. All of the syntactically first-order quantifiers in ϕ are bounded quantifiers in $L_\alpha[T]$, so by admissibility the "$\exists\beta$"s can all be pulled out in front: *$n \in \mathbf{S}^{\alpha+1}$ iff $\exists\beta\ \phi(n,$ $\mathbf{S}^\beta)$*. This happens iff $n \in \mathbf{S}^\alpha$, which suffices.

<u>n>1</u>

For the induction step, suppose that $\mathcal{S}$ is Σ_n, n > 1. For the sake of notational convenience assume that n = 2; the argument is essentially unchanged. So $\mathcal{S} = \mu S\ \sigma \in \nu T\ \phi$, where μS binds x, νT binds y, and ϕ is first order with parameter **U**. Let α be the least 2-**U**-reflecting admissible. Recall that α is **U**-admissible, by section 2.

As above, the generation of $\mathcal{S}$'s interpretation **S** can be started in $L_\alpha[\mathbf{U}]$, with some care paid to the ordinals. $\mathbf{S}^0 = \emptyset$ is definable over L_ω, and induces $\mathbf{T}_0 = \nu T\ \phi(y, \mathbf{S}^0, T)$. Recall that since ϕ might have x free, which has been suppressed in the notation, $\mathbf{T}_0$ is to be thought of as an $\mathbb{N}$-indexed sequence of sets, one for each x in $\mathbb{N}$. $\mathbf{T}_0$, being Π_1, is Π_1 over the least **U**-admissible, inductively. Using $\mathbf{T}_0$, the next step in the construction of **S**, $\mathbf{S}^1$, is $\{x \mid \langle x, \sigma\rangle \in \mathbf{T}_0\}$, and is definable over the least **U**-admissible. Now $\mathbf{S}^1$ induces $\mathbf{T}_1 = \nu T\ \phi(y, \mathbf{S}^1, T)$. $\mathbf{T}_1$ is Π_1 over the least $\langle \mathbf{S}^1, \mathbf{U}\rangle$-admissible, which is at most the second **U**-admissible. Again, $\mathbf{S}^2 = \{x \mid \langle x, \sigma\rangle \in \mathbf{T}_1\}$ is therefore definable over the second **U**-admissible. Other successors $\mathbf{S}^{\beta+1}$ are handled similarly. For limits λ, by the uniformity of the definition of $\mathbf{S}^\beta$, $\mathbf{S}^\lambda$ is definable over the limit of the first λ-many **U**-admissible ordinals. Thus we see that $\mathbf{S}^\beta$ is definable over the βth ordinal in the (topological) closure of the set of **U**-admissible ordinals. Since there are α-many such ordinals in $L_\alpha[\mathbf{U}]$, $\mathbf{S}^\alpha$ is Σ_1 over $L_\alpha[\mathbf{U}]$, as $\{x \mid \exists f$ f is a function which enumerates the inductive construction of S, and for some β $x \in f(\beta)\}$.

The claim is that $\mathbf{S}^\alpha = \mathbf{S}$. So consider the definition of $\mathbf{S}^{\alpha+1}$. In order to evaluate it we must consider $\mathbf{T}_\alpha = \nu T\ \phi(y, \mathbf{S}^\alpha, T)$. $\mathbf{T}_\alpha$ itself requires generation.

$$(\mathbf{T}_\alpha)^0 = \omega \times \omega,$$

$$(\mathbf{T}_\alpha)^{\beta+1} = \{\langle x, y\rangle \mid \phi(y, \mathbf{S}^\alpha, (\mathbf{T}_\alpha)^\beta)\}, \text{ and}$$

$$(\mathbf{T}_\alpha)^\lambda = \cap_{\beta<\lambda}(\mathbf{T}_\alpha)^\beta.$$

Inductively, the generation of $\mathbf{T}$ closes off by the next $\mathbf{U}$-admissible: $\mathbf{T}_\alpha = \bigcap_{\beta<\alpha^+} (\mathbf{T}_\alpha)^\beta$. Thus,

$n \in \mathbf{T}_\alpha$ iff $L_{\alpha^+} \models$ "$\forall f$ (if f is a function with domain an ordinal γ,

$f(0) = \omega \times \omega$,

$f(\beta+1) = \{\langle x, y\rangle \mid \phi(y, S^\alpha, f(\beta))\}$, and

$f(\lambda) = \bigcap_{\beta<\lambda} f(\beta)$,

then for all $\beta<\gamma$ $n \in f(\beta)$)".

The formula being evaluated in L_{α^+} is Π_1, with parameter L_α (used in defining S^α). By the choice of α, it is true in some L_{β^+}, $\beta<\alpha$, with S^α replaced by S^β (by uniformity). So $n \in T_\beta$.

Conversely, suppose $n \in \mathbf{T}_\beta$, $\beta<\alpha$. $\mathbf{T}_\alpha$ is defined just as $\mathbf{T}_\beta$ was, only the parameter $\mathbf{S}^\alpha$ is used instead of its subset $\mathbf{S}^\beta$. By the positivity of S in ϕ, $\mathbf{T}_\alpha \supset \mathbf{T}_\beta$. Therefore, $n \in T_\alpha$.

So $\mathbf{T}_\alpha = \bigcup_{\beta<\alpha} \mathbf{T}_\beta$. Finally, $x \in \mathbf{S}^{\alpha+1}$ iff $\langle x, \sigma\rangle \in T_\alpha$ iff $\langle x, \sigma\rangle \in \mathbf{T}_\beta$ for some $\beta<\alpha$ iff $x \in \mathbf{S}^{\beta+1}$ for some $\beta<\alpha$ iff $x \in \mathbf{S}^\alpha$. Thus $\mathbf{S} = \mathbf{S}^\alpha$, which is Σ_1 over $L_\alpha[\mathbf{U}]$, as desired.

5. REACHING THE LIMIT

In this section we develop formulas with the maximal degrees of complexity. The idea is to code the inductive generation of L and assertions about it into $\mathbb{N}$, the ur-elements.

Clearly we will rely heavily upon Goedel coding. How to do this is so standard that we will freely introduce new types of assertions in ordinary English, the myth being that we have merely suppressed the list at beginning of the paper of all pieces of syntax and their corresponding integer codes, as well as the recursive rules for manipulating them.

The construction of the formulas is by induction on n. The result for the base case n=1 is nothing new (see [B] ch.6 for instance), and so we include only a brief sketch. For the inductive step, we do n=2 as a typical case, and mention the general argument.

It should be observed that we're actually defining in the μ-calculus not an arbitrary $\Sigma_1(L_\alpha)$ set, but only a universal one, which suffices.

Base case: n=1

This case proceeds by inductively defining integer notation for all ordinals through ω^1_{CK} and hence for all sets in $L_{\omega^1_{CK}}$. An ordinal α is represented by the Goedel code of a formula ϕ if ϕ represents a Σ_1 function from ω cofinally through α. Not only are such names carried along, but so are all Σ_1 assertions about them provable from ZF and previously proven statements. Hence the final fixed point S includes (integer codes for) statements of the form "ϕ is an ordinal name", "τ is a term", and ϕ, ϕ a $\mathbf{\Sigma_1}$ assertion over $L_{\omega^1_{CK}}$. Furthermore, S is closed under deducibility from ZF and S, using the ω-rule.

Dually, we can develop T, the set of true Π_1 sentences, as a greatest fixed point. The idea is to begin with everything, including a lot of syntactically ill-formed stuff as well as simple falsehoods, and throw out garbage. T will contain assertions like "ϕ is not an ordinal name", "τ is not a term", and ϕ, ϕ $\mathbf{\Pi_1}$ over $L_{\omega^1_{CK}}$. Again, T is closed under deducibility, as with S. A typical step is that "$\forall x\ \phi$" will get thrown out just after both "$\phi(\sigma)$" and "σ is not a term" got thrown out. Thus after ω^1_{CK}-many steps we will have thrown away all false Π_1 statements, but none of the true ones since there was no reason to.

The Case n=2

To understand this better, we dwell for a bit on the first step: embedding S and T (suitably re-defined) into another μ-formula, which goes up through the second admissible.

After generating the true Σ_1 and Π_1 formulas of ω^1_{CK}, gather them together into one set U. So "ϕ" $\in$ U if

1) "ϕ" $\in$ S, or

2) "ϕ" $\in$ T, ϕ is Π_1 over $L_{\omega^1_{CK}}$, and if c is a parameter of ϕ then "c is a term" $\in$ S.

Clearly, U is as desired.

Allowing S and T also to depend on U only positively, this construction clearly can continue. Introduce into the definition of S one new clause:

"ϕ" $\in$ S if "ϕ" $\in$ U.

Thus using all of the information in U, S will now generate terms for all of the sets up through the second admissible ordinal, and all true Σ_1 statements about them. There is a corresponding clause for T, which acts as an elimination clause:

"ϕ" $\in$ T if every Σ_1 and Π_1 consequence of ϕ about $L_{\omega^1_{CK}}$ from ZF is in U.

This clause eliminates all false Σ_1 information from T: if ϕ Σ_1 is false, then $\phi \notin$ U, so eliminate ϕ from T. Continuing in such a fashion, the generation of T will terminate at the second admissible, with all and only the true Π_1 statements over this ordinal.

This suggests a twofold task. One is to develop a system of notation for admissible ordinals less than the least ordinal Π_1 gap-reflecting on admissibles. The other is to rework the definitions of S and T accordingly, to develop the proper control structure. Even if it has proper notation for α, the procedure still must know exactly when to work on α.

If α is not Π_1 gap-reflecting on admissibles, then it has a witness ψ to such. Recall that ψ must be a Π_1 formula from the language of set theory (extended by names for members of L_α and the constant symbol $\mathcal{L}_\alpha$) true in $\langle L_{\alpha^+}, L_\alpha\rangle$ but not reflecting down to $\langle L_{\beta^+}, L_\beta\rangle$. Any such formula itself provides adequate notation for α. Thus α will be called ω^Ψ.

T will work on all possible ω^Ψs at all times, since it may not be able to tell just from U when it is time to work on a given ω^Ψ. Such work for different ω^Ψs must be kept separate from one another. So T will identify ω with $\omega \times \omega$ in some recursive fashion. The nth slice of ω is dedicated to working on ω^Ψ, where n is the Goedel number of $\nu\psi$. To save on notation, <"ω^Ψ", "ϕ"> will be abbreviated by "ϕ" when the choice of ψ is obvious.

The appropriate time to work on ω^Ψ is just after all bounded information about it has been accumulated. So assume inductively that U contains notation for, and all bounded information about, all members of L_α. (Of course, we will check later that the ω^Ψth slice is not messed up at other stages.) Thus two related conditions that must be met to get into the ω^Ψth slice are:

a) for all terms σ of ψ, "σ is a term" $\in$ U;

b) "$\sigma \in L_\alpha$" $\in$ T only if "σ is a term" $\in$ U.

Recall how these would work. T, as a greatest fixed point, begins with all of ω, including "$\sigma \in L_\alpha$" for all sorts of weird ψ and σ (ψ being a subterm of α). By a), T won't work on ω^Ψ if ψ is still ill-formed; by b), all the really bad σ get thrown out immediately.

Just as for n=1 T was closed under deducibility from ZF and T, so now T is closed under such deducibility including the axiom:

c) $\langle L_{\alpha^+}, L_\alpha\rangle$ is the least place where ψ holds.

Similarly, the meaning of α^+, another piece of syntax, must also be legislated:

d) L_{α^+} is the least admissible $> \alpha$.

Analogously to the situation with the second admissible, when T could contain ϕ only if all of ϕ's Σ_1 and Π_1 consequences about $L_{\omega_1^{CK}}$ were in U, we have:

e) $\phi \in$ T only if every consequence which is Σ_1 over L_α is in U.

It is well to consider now why this works. First assume that α is recursively inaccessible, and let ψ witness that it's not Π_1 gap-reflecting on admissibles. $U_\alpha =_{def} \bigcup_{\beta<\alpha} U_\beta$ is the Σ_1 diagram of L_α, inductively. T's ω^ψth slice, after one step, will have "$\sigma \in L_\alpha$" only if σ is legitimate notation, by clause b). At the next step T will lose all false Σ_1 assertions about L_α, as before. T will also lose all false Π_1 assertions about L_α, but that may take α-many steps. This is because the $\Pi_1(L_\alpha)$ statement $\forall^\alpha x\ \phi$ is in T if, for each term σ, either $\phi(\sigma) \in$ T or "σ is not a legitimate term of rank $< \alpha$" $\in$ T. To extract from U_α the comparison-with-α function even on legitimate terms would require a negative use of U. Therefore, to eliminate the false assertion $\forall^\alpha x\ \phi$ from T, wait until "$\sim rk(\sigma) < \alpha$" is eliminated from T, for σ a witness to $\sim\phi$. This will take $rk(\sigma)$-many steps, which is arbitrarily close to α. Only then (actually, at $\max(rk(\sigma), rk(\phi))$) will $\forall^\alpha x\ \phi$ be removed from T. With only true Σ_1 and Π_1 information about L_α still remaining, T will now produce the Π_1 truths about L_{α^+}, much as in the previous cases.

What happens to the ω^ψth slice at some stage less than α? If $\beta<\alpha$, then ψ is false in $\langle L_{\beta^+}, L_\beta\rangle$. T throws out all false Π_1 information. So eventually ψ will get thrown out of its own slice. Then all of the slice will be emptied. In more detail, T does not have "$\langle L_{\beta^+}, L_\beta\rangle$ is not the least place where ψ holds", by c). Nor does it have "$\langle L_{\beta^+}, L_\beta\rangle$ is the least place where ψ holds" once it loses ψ. The next step T will be missing something of the form $\phi \vee \sim\phi$, which is a consequence of everything, so then T will lose everything in the slice under consideration.

What happens at some stage $\beta>\alpha$? T begins with a larger set of legitimate terms; there are many more statements of the form "$\sigma \in L_\alpha$" in T. Still, there is no guarantee that all remain. Many engender a contradiction, such as "$L_\alpha \in L_\alpha$". These are thrown out of T. What remains might have names for and information about sets beyond L_{α^+}, which was not the case at stage α, because of the extra terms from the beginning. Nonetheless, the Π_1 information about L_{α^+} is the same.

The argument above assumed that α was recursively inaccessible. Nothing much different is going on at other α. The only place that recursive inaccessibility was used is that at the αth stage of the generation of U, U_α is the complete Σ_1 diagram of L_α. All that's really necessary is that U_α is the Σ_1 diagram for some L_γ. The demonstration of this depends upon having the right control structure. (See part 2 for the fact that inadmissible limits of admissibles and successor admissibles are not Π_1 gap-reflecting on admissibles.)

To get the proper control structure, observe that if T's ω^{Ψ}th slice is non-empty, then its statements about $\mathcal{L}_\alpha$ are exactly the true ones about L_α, and its Π_1 statements about L_{α^+} are exactly the true ones. So clause 2) of U should be changed to:

2) <"ω^{Ψ}", "ϕ"> $\in$ U if <"ω^{Ψ}", "ϕ"> $\in$ T and ϕ is an assertion about L_α.

Now S should go work on L_{α^+}, and should be told as much:

3) "S should work on L_{α^+}" $\in$ U if $\exists\phi$ <"ω^{Ψ}", "ϕ"> $\in$ T.

S should now work on L_{α^+}, and using the information about L_α that it gets from U, it will generate all true Σ_1 information about L_{α^+}.

When S is done, U will extract such information from it, by clause 1) of U. U must also take the proper Π_1 information from T:

4) "ϕ" $\in$ U if ϕ is Π_1 over L_{α^+}, <"ω^{Ψ}", "ϕ"> $\in$ T, "S should work on L_{α^+}" $\in$ U, and if σ is a parameter of ϕ then "$\sigma \in L_{\alpha^+}$" $\in$ S.

Looking toward the future,

5) "S should work on L_{β^+}" $\in$ U if "S should work on L_β" $\in$ U.

Again, it is well to consider how this is working. At some recursive inaccessible α, U_α will be all Σ_1 information about L_α. Constructing $U_{\alpha+1}$ involves evaluating S and T with U_α. S will do nothing new, by the admissibility of α. T's various slices will either be empty, the same as in an earlier evaluation (at least when restricted to the relevant formulas), or, for the appropriate ψ, the Π_1 facts about L_{α^+}. U collects the new information from T, but only about L_α. In the next evaluations of S and T, using $U_{\alpha+1}$, S generates the Σ_1 facts about L_{α^+}. T, however, can do nothing new, since $U_{\alpha+1}$ has no assertions "σ is a term" that U_α didn't already have. $U_{\alpha+2}$ will then collect all Σ_1 and Π_1 information about L_{α^+} from S and T respectively, and also have "S should work on $L_{\alpha^{++}}$", by clause 5). Using $U_{\alpha+2}$, S and T will do $L_{\alpha^{++}}$, and U will extract that information from them, by clauses 1) and 4).

If α is an inadmissible limit of admissibles, say the first λ-many, the picture is slightly different. Using U_λ, not only will T generate the Π_1 facts about L_{α^+}, but S will generate its Σ_1 facts. This is because the mechanism in S that made it go beyond inadmissibles less that ω_1CK in the first place will now just as well make it go beyond α. $U_{\lambda+1}$ absorbs the $\Sigma_1(L_{\alpha^+})$ facts, which includes everything it gets from T via clause 2). Therefore the next evaluation of S, using $U_{\lambda+1}$, will be no different from that using U_λ. The corresponding evaluation of T will go through $L_{\alpha^{++}}$, because $U_{\lambda+1}$ does include a term for L_α. $U_{\lambda+2}$ than extracts from T only the Π_1 information about L_{α^+}. Using $U_{\lambda+2}$, S and T generate the Σ_1 and Π_1 facts about $L_{\alpha^{++}}$ respectively, and the situation continues as above.

<u>The Case n>2</u>

We have seen that by embedding a Σ_1 and a Π_1 formula inside of a μ formula, we can construct one that goes far past the limits of Σ_1 definitions. The obvious thing to do next is to mimic this, by embedding Σ_n and Π_n formulas within a μ, inductively. There are two vital properties that must be carried along for this to work. First, there is a universal Σ_n formula. Second, these operations relativize to a parameter of the right form, uniformly. In practice, the import of the latter is that after the μ extracts the correct Σ_1 and Π_1 information from the Σ_n and Π_n formulas, it then returns that information to the same formulas, which can then correctly go ahead to the next reflecting ordinal. Thus the outer μ keeps accumulating complete Σ_1 information about n-reflecting admissibles, which it then returns to the embedded formulas, which then go up to the next n-reflecting admissible. Notation and control strategies can be developed by analogy with those above, only instead of having depth 1 it would have depth n. The depth of the nesting refers to the size of the tuples we are pretending constitute ω: for depth n, identify ω with ω^{n+1}. Thus, so long as there is a formula not reflecting on n-reflecting admissibles, there would be enough notation available for the process to continue. As mentioned in the previous section, once the first n+1-reflecting admissible were reached, it would have to stop. As the ideas involved are not new, but the calculations are more elaborate, these details are left to the reader. QED

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