Machine Learning using Descriptive Complexity and Propositional Solvers

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Highlights '13

Why Logic in Machine Learning?

What is machine learning?

Given $(x^{(i)}, y^{(i)})_{i=1...m}$ **find** $h \in H$ **such that** $h(x^{(i)}) \approx y^{(i)}$

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Issues

- Is H **abstract** (e.g. PTIME) or **concrete** (e.g. $\{\theta \cdot \text{input} \mid \theta\}$)?
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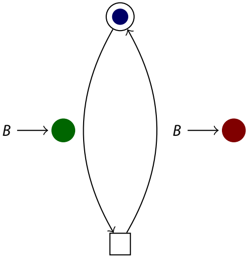
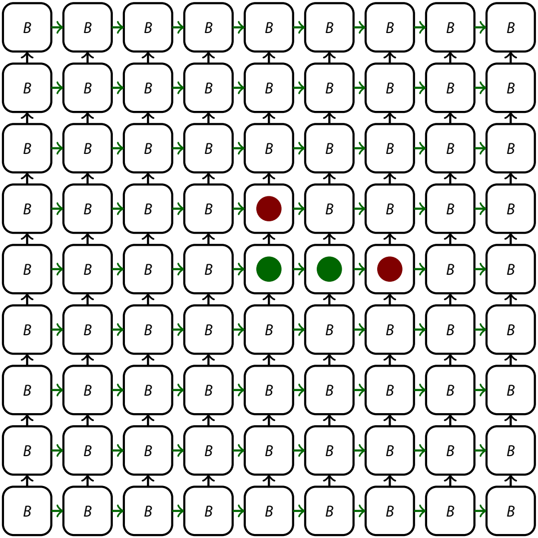
How can we use logic?

- Idea: H are **parametrized formulas**
- **Descriptive complexity** gives **theoretical guarantees**
- **Propositional solvers** used for **efficient learning**

Learning Board Game Rules

Representing Board Games

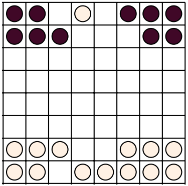




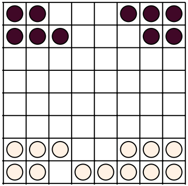
$$\exists x_1 \ldots x_5 \Big(\bigwedge_{1 \le i \le 5} G(x_i) \wedge \Big(\bigwedge_{1 \le i \le 5} R(x_i, x_{i+1}) \vee \bigwedge_{1 \le i \le 5} C(x_i, x_{i+1}) \vee$$

$$\bigwedge_{1 \le i \le 5} \exists y (R(x_i, y) \wedge C(y, x_{i+1})) \vee \bigwedge_{1 \le i \le 5} \exists y (R(x_i, y) \wedge C(x_{i+1}, y)) \Big) \Big)$$

Learning Winning Conditions



Positive Example $\mathfrak{A}$



Negative Example $\mathfrak{B}$

Find **minimal** φ such that $\mathfrak{A} \models \varphi$, $\mathfrak{B} \models \neg\varphi$

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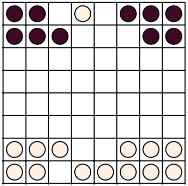

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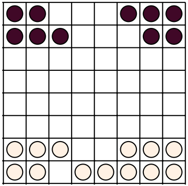

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Which logic and minimality?

- Full FO, minimal quantifier rank: **PSPACE-complete (Pezzoli ’98)**
- FO^k+C, minimal quantifier rank: **PTIME (Grohe ’99)**
- $k = 16$ and $\log(n)$ quantifiers suffice for …**(Pikhurko, Verbitsky ’10)**

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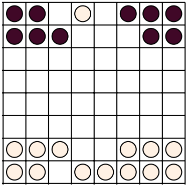

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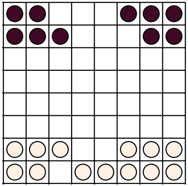

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Extensions: TC^m and guarded formulas, greedy shortening, …

computed formula: $\exists x \, (\mathbf{W}(x) \, \wedge \, \forall y \, \neg \mathbf{C}(x, y))$

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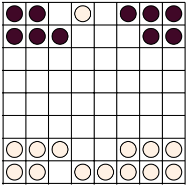

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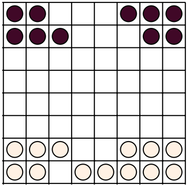

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Learning game rules from videos (K., AAAI-12)

`http://toss.sf.net/learn.html`

Learning Reductions

Formula Outlines

Conjunction outline (conjunction with Boolean guards on all atoms)

$$X_1\mathbf{E}(x_1,x_1) \quad\wedge\quad X_2\mathbf{E}(x_1,x_2) \quad\wedge\quad X_3\mathbf{E}(x_2,x_1) \quad\wedge\quad X_4\mathbf{E}(x_2,x_2) \quad\wedge$$
$$X_5\neg\mathbf{E}(x_1,x_1) \quad\wedge\quad X_6\neg\mathbf{E}(x_1,x_2) \quad\wedge\quad X_7\neg\mathbf{E}(x_2,x_1) \quad\wedge\quad X_8\neg\mathbf{E}(x_2,x_2)$$

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l-DNF outline (all quantifier-free formulas)

$$C_1 \vee C_2 \vee \cdots \vee C_l$$

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Extensions

- ***k*-Variable ∃ *l*-DNF outline**

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- ***m*-Predicate *k*-Variable $\exists$ *l*-DNF outline**

$$P_i(x) = \exists x_1 \dots x_k \, (C_1 \vee C_2 \vee \cdots \vee C_l), \quad i = 1 \dots m$$

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- **n-Layer m-Predicate k-Variable $\exists$ l-DNF outline**

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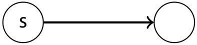
Automatic Reduction Finding

Representing reductions by **k-dimensional quantifier-free queries**

$\left(k = 2,\ \varphi_U = \top,\ \psi_E(x_1, x_2, y_1, y_2) = \mathbf{E}(x_1, y_1) \wedge (x_2 = y_2 \vee y_2 = \mathbf{s})\right)$



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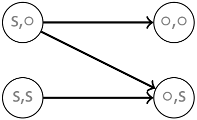

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Finding reductions by CEGAR and SAT-solvers

- Find a l-DNF reduction θ_i good on counter-examples $\mathfrak{E}_0, \ldots, \mathfrak{E}_i$
- Find a counter-example $\mathfrak{E}_{i+1}$ to θ_i, iterate

(**Jordan, K., SAT '13** improving on Crouch, Immerman, Moss '10)

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Easy example: s-t reachability to strongly connected (both NL-complete)

$$\text{Reach} = [\text{tc}_{x,y}\ \mathbf{E}(x, y)](.s, .t) \qquad \text{SC} := \forall x, y(\text{tc}_{x,y}\ \mathbf{E}(x, y))$$

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`http://toss.sf.net/reduct.html`

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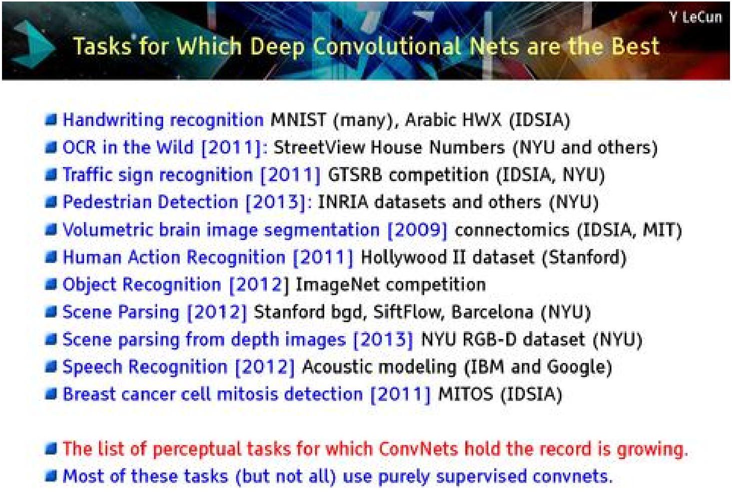
Other applications: game rule learning, program synthesis, …

Looking Forward

Machine Learning Motivation

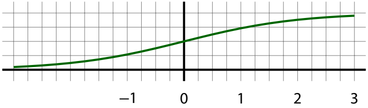


It's hard to prove anything about deep learning systems

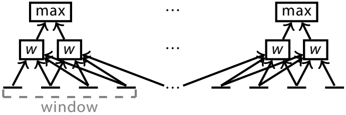
Y. LeCun, COLT '13

What are Convolutional Networks?

Neuron: σ(weighted sum)



Network with **shared weights**



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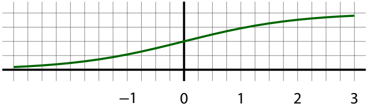

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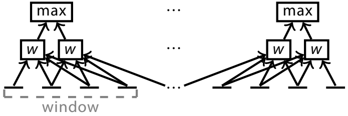
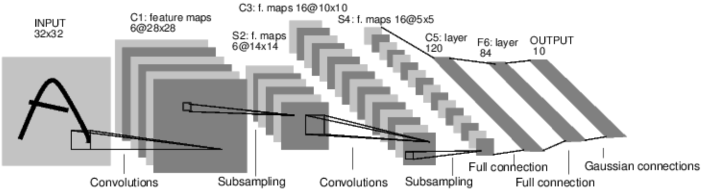
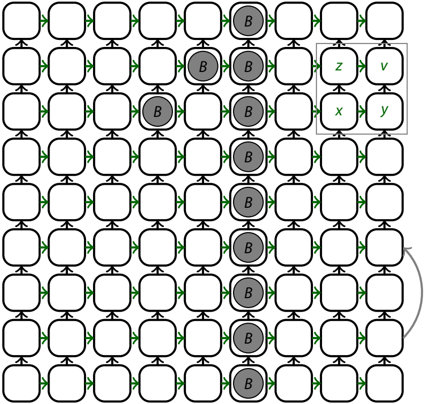


Example (LeNet-5, 0.95% MNIST error rate)



(credit: EBLearn (`eblearn.cs.nyu.edu`))

Threshold Convolutional Formulas



Convolution

$$P^1(x) = \exists y, z, v.(R(x,y) \wedge C(x,z) \wedge R(z,v))\varphi$$

with $\varphi = w_1\chi[B(x)] + \cdots + w_4\chi[B(v)] \geq t$

Subsampling (max-pooling)

$$P^2(x) = \exists y, z, v.(R_2(x,y) \wedge C_2(x,z) \wedge \ldots)\psi$$

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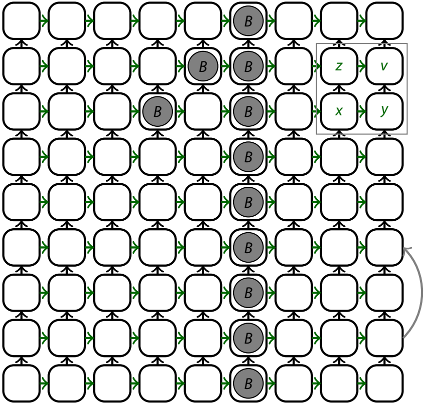

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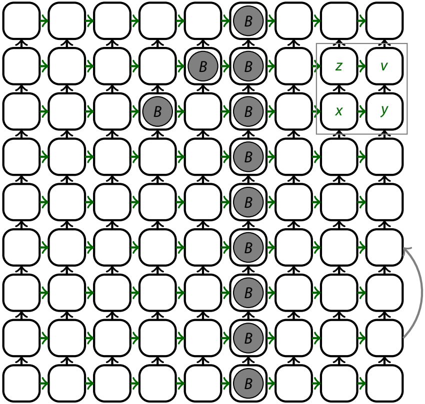

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Inspiring goal: uniform learning platform and theory

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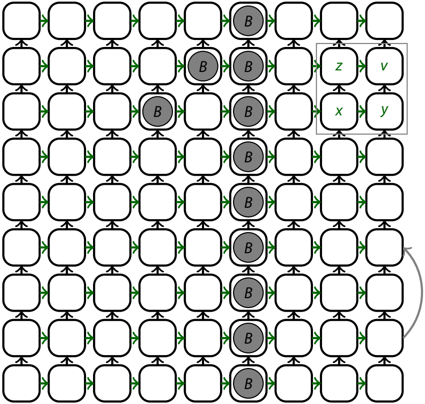

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Inspiring goal: uniform learning platform and theory **Thank You!**