COMPACTNESS OF SYSTEMS OF EQUATIONS IN SEMIGROUPS*

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Communicated by D. Therien
Received 5 January 1996

AMS Mathematics Subject Classification: 20M05

We consider systems $u_i = v_i$ $(i \in I)$ of equations in semigroups over finite sets of variables. A semigroup (or a monoid) S is said to satisfy the compactness property (CP, for short), if each system of equations has an equivalent finite subsystem. We prove that all monoids in a variety $\mathcal{V}$ satisfy CP if and only if the finitely generated monoids in $\mathcal{V}$ satisfy the maximal condition on congruences. Consequently, all commutative monoids (and semigroups) satisfy CP. Also, if a finitely generated semigroup S satisfies CP, then S is necessarily hopfian and satisfies the chain condition on idempotents. It follows that the free inverse semigroups do not satisfy CP. Finally, we give two simple examples (the bicyclic monoid and the Baumslag-Solitar group) which do not satisfy CP, and show that the necessary conditions above are not sufficient.

1. Introduction

Let $X_n = \{x_1, x_2, \ldots, x_n\}$ be a finite set of variables. We denote by X_n^+ the free semigroup generated by the set X_n. An *equation* over X_n is a pair $(u,v) \in X_n^+ \times X_n^+$ of words, which is often written as $u = v$. A *solution* of an equation (u,v) *in a semigroup* S is a morphism $\alpha\colon X_n^+ \to S$ such that $\alpha(u) = \alpha(v)$, i.e., such that (u,v) is in the kernel of α, $(u,v) \in \ker(\alpha)$. Similarly, we let X_n^* and $X_n^{(*)}$ denote the free monoid and the free group generated by X_n, respectively. Equations $(u,v) \in X_n^* \times X_n^*$ $((u,v) \in X_n^{(*)} \times X_n^{(*)})$ and solutions $\alpha\colon X_n^* \to M$ $(\alpha\colon X_n^{(*)} \to G)$ in a monoid M and in a group G are defined similarly.

*A preliminary version of this paper was presented at the 22nd ICALP conference in Szeged Hungary 1995.
[†]Supported by Academy of Finland grant 4077.
[‡]Supported by the KBN grant 8 T11C 012 08.

International Journal of Algebra and Computation Vol. 7, No. 4 (1997) 457–470

Let S be a semigroup and let $L \subseteq X_n^+ \times X_n^+$ be a *system of equations* in X_n^+. We say that L is *equivalent to a subsystem* $L' \subseteq L$, if L and L' have the same set of solutions, i.e., if $\alpha: X_n^+ \to S$ is a solution of all $(u,v) \in L'$, then α is a solution of all $(u,v) \in L$. Further, S is said to satisfy the *compactness property* (*for systems of equations*), or *CP* for short, if for all $n \geq 1$ every system $L \subseteq X_n^+ \times X_n^+$ is equivalent to one of its *finite* subsystems $T \subseteq L$.

A class $\mathcal{C}$ of semigroups is said to satisfy the *compactness property*, if every semigroup $S \in \mathcal{C}$ satisfies it. We also say that a class $\mathcal{C}$ satisfies the compactness property *uniformly*, if for each system $L \in X_n^+ \times X_n^+$ of equations there exists a finite subsystem $T \subseteq L$ such that L is equivalent to T for all $S \in \mathcal{C}$.

Clearly, all finite semigroups satisfy CP, but as we will see in Sec. 2, the class of all finite semigroups does not satisfy CP uniformly.

As is easily seen the semigroups satisfying CP are closed under taking subsemigroups and finite direct products, but they are not closed under morphic images. In Sec. 2 we will prove that the semigroups satisfying CP are not closed under infinite direct products. This result follows from the existence of a residually finite semigroup that does not satisfy CP. As an example the monoid $\mathrm{Fin}(F_2)$ of all finite subsets of the 2-generator free monoid F_2 is residually finite, but as shown by Lawrence [15] it does not satisfy CP. Also, the free inverse semigroups are residually finite by Munn [20], but as we shall see in Sec. 4 they do not satisfy CP. From this result we deduce that the finite semigroups generated by two elements do not satisfy CP uniformly.

In Sec. 3, we generalize a group theoretical result of Albert and Lawrence [1] to varieties of monoids. We show that a variety $\mathcal{V}$ of monoids satisfies CP if and only if the monoids in $\mathcal{V}$ satisfy the maximal condition on congruences. In particular, by Redei's Theorem, the commutative semigroups and monoids satisfy the compactness property. It follows also that metabelian groups and nilpotent groups satisfy CP.

The above result for varieties does not hold for individual semigroups. Indeed, the free semigroups and the free groups satisfy CP (see Albert and Lawrence [2], Guba [19], de Luca and Restivo [16]), but they do not satisfy the maximal condition on congruences (see Sec. 4).

In Sec. 4 we prove some necessary conditions for the compactness property, and we give three simple examples (the bicyclic monoid, the free monogenic inverse semigroup and the Baumslag-Solitar group) showing that these conditions are not sufficient for the compactness property. These examples also show that the maximal condition on congruences does not imply CP.

We conclude this section with a few simple observations.

A system of equations $L \subseteq X_n^+ \times X_n^+$ is said to be *locally independent* in a semigroup S, if L is not equivalent to any of its finite proper subsystems. The following result was proved by Karhumäki and Plandowski [13].

Theorem 1.1. *A semigroup S satisfies the compactness property if and only if every locally independent system of equations is finite.*

The compactness property for systems of equations is also related to the following condition for morphisms. Let F_n be a free semigroup generated by n elements.

A semigroup S is said to satisfy the *compactness condition for morphisms*, if for all subsets $A \subseteq F_n$ there exists a finite subset $B \subseteq A$ such that for any two morphisms $\alpha, \beta: F_n \to S$,

$$\alpha|B = \beta|B \iff \alpha|A = \beta|A\,,$$

where $\alpha|X$ is the restriction of α onto the subset X. The problem whether or not $S = F_n$ satisfies the compactness condition for morphisms is known as Ehrenfeucht's Conjecture, and it was proved to hold in 1985 by Albert and Lawrence [2] and independently by Guba, see [19]. The proof of Ehrenfeucht's Conjecture is based on Hilbert's Basis Theorem and on a result of Culik and Karhumäki [6] which states that for $S = F_n$ the compactness condition for morphisms is equivalent to the compactness property for systems of equations. The proof given in [6] generalizes immediately to all monoids and groups, and hence we have the following result.

Theorem 1.2. *For any semigroup S the compactness property of systems of equations is equivalent to the compactness condition for morphisms.*

2. Closure Properties

We consider only equations over a finite number of variables, and hence each system of equations $L \subseteq X_n^+ \times X_n^+$ is either finite or denumerable. Consequently L can be enumerated, $L = \{(u_i, v_i) | i = 1, 2, \ldots\}$. From this observation we obtain immediately the following lemma.

Lemma 2.1. *If a system of equations $L \subseteq X_n^* \times X_n^*$ does not have an equivalent finite subsystem in a semigroup S, then L has an infinite subsystem $L' = \{(r_i, t_i) | i = 1, 2, \ldots\}$ ordered in such a way that for each j there exists a solution of the system $r_i = t_i$ $(i = 1, 2, \ldots, j-1)$, which is not a solution of the equation $r_j = t_j$.*

Indeed, let $L_1 = \{(u_1, v_1)\}$ and define inductively

$$L_{i+1} = \begin{cases} L_i, & \text{if } L_{i+1} \text{ is equivalent to } L_i, \\ L_i \cup \{(u_{i+1}, v_{i+1})\}, & \text{if } L_{i+1} \text{ is not equivalent to } L_i, \end{cases}$$

and let $L' = \cup_{i \geq 1} L_i$.

The compactness property is preserved under some natural operations as shown in the next lemma, see [13].

Lemma 2.2.

(1) *If a semigroup S can be embedded into a semigroup that satisfies CP, then S satisfies CP.*
(2) *If a semigroup S satisfies CP, then the class of subsemigroups of S satisfies CP uniformly.*
(3) *If the semigroups S_1 and S_2 satisfy CP, then so does the direct product $S_1 \times S_2$.*

Consider a free semigroup F with an infinite number of generators. We observe that if $\alpha: X_n^+ \to F$ is a morphism, then the morphic image $\alpha(X_n^+)$ is a subsemigroup of a finitely generated free subsemigroup F_n of F. Since all n-generator free semigroups are isomorphic and they satisfy CP, it follows that F satisfies CP, and, moreover, each system $L \subseteq X_n^+ \times X_n^+$ of equations has a common equivalent finite subsystem for all free semigroups. The same conclusion is clearly true for all free monoids and free groups. Hence we have the following theorem.

Theorem 2.3. *The class of subsemigroups of free semigroups (submonoids of free monoids, and subgroups of free groups, respectively) satisfies CP uniformly.*

Notice also that a countably generated free semigroup F can be embedded into a 2-generator free semigroup F_2, and hence, by Lemma 2.2(1), if $T \subseteq L$ is an equivalent finite subsystem of $L \subseteq X_n^+ \times X_n^+$ in F_2, then T is an equivalent finite subsystem for all subsemigroups of free semigroups.

In above, the size of the finite subsystem T depends on L. Indeed, as shown in [1] for groups, for each $i \geq 1$ there exists a system $L_i \subseteq X_3^{(*)} \times X_3^{(*)}$ of equations over three variables such that the size of any equivalent subsystem $T_i \subseteq L_i$ is at least i. It is not known if the same result holds for semigroups and monoids, although for these similar but weaker lower bounds are presented in [13].

The compactness property is not necessarily inherited by morphic images (or by quotients), simply because the 2-generator free semigroup F_2 satisfies CP, and, as we shall see, its morphic image B (the bicyclic monoid) does not.

Theorem 2.4. *The semigroups satisfying CP are not closed under morphic images.*

Next we shall prove that the semigroups satisfying CP are not closed under infinite direct products. For convenience, this and some later proofs will be given for monoids only. The restriction to monoids is justified by the next result, where S^1 denotes the monoid, which is obtained from a given semigroup S without identity by adding an identity element 1 to S. If S is already a monoid, then we let $S^1 = S$.

Theorem 2.5. *A semigroup S satisfies CP if and only if the monoid S^1 satisfies CP.*

Proof. In one direction the claim follows from Lemma 2.2, since S is a subsemigroup of S^1.

For the proof of the converse statement, we notice first that if $u = v$ is an equation, where $v = 1$ or $u = 1$, then for any solution $\beta: X_n^* \to S^1$ of $u = v$, $\beta(x) = 1$ for all variables $x \in X_n$ that occur in u or v, because $\beta(w) \in S$ (and $1 \notin S$) for all $w \in X_n^+$.

For a subset $A \subseteq X_n$ let $\varepsilon_A: X_n^* \to X_n^*$ be the morphism that erases the variables in A: $\varepsilon_A(x) = 1$ for all $x \in A$, and $\varepsilon_A(x) = x$ for all $x \notin A$.

Suppose then that, as in Lemma 2.1, $L = \{(u_i, v_i) | i \geq 1\}$ is ordered such that $\beta_j: X_n^* \to S^1$, $j = 1, 2, \ldots$, is an infinite sequence of morphisms with $\beta_j(u_i) = \beta_j(v_i)$ for $i \leq j$, but $\beta_j(u_{j+1}) \neq \beta_j(v_{j+1})$. Write $B_j = \{x \in X_n | \beta_j(x) = 1\}$. Since X_n is finite, there is a subset $B \subseteq X_n$ and an infinite subsequence β_{m_j} such that $B_{m_j} = B$ for all $j \geq 1$. Now, by assumption, $\beta_{m_j}(u_{m_j+1}) \neq \beta_{m_j}(v_{m_j+1})$ for each $j \geq 1$, and hence u_{m_j+1}, $v_{m_j+1} \notin B^*$ for all $j \geq 1$. In particular, $\beta_{m_j}(\varepsilon_B(u_{m_i+1})) \neq 1$ and $\beta_{m_j}(\varepsilon_B(v_{m_i+1})) \neq 1$ for all i, j. Consider the system

$$R = \{(\varepsilon_B(u_{m_i+1}), \varepsilon_B(v_{m_i+1})) | i = 1, 2, \ldots\} \subseteq Y^+ \times Y^+ ,$$

of equations, where $Y = X_n \backslash B$. Define for each j a morphism $\alpha_j: Y^+ \to S$ by $\alpha_j(y) = \beta_{m_j}(y)$ for all $y \in Y$. Consequently, $\alpha_j(\varepsilon_B(u_{m_i+1})) = \alpha_j(\varepsilon_B(v_{m_i+1}))$ for all $i < j$, but $\alpha_j(\varepsilon_B(u_{m_j+1})) \neq \alpha_j(\varepsilon_B(v_{m_j+1}))$. This shows that the system R of equations does not have an equivalent finite subsystem for S, and the claim follows.

First we notice that the direct products are closely related to uniformity of CP.

Lemma 2.6. *A direct product $\Pi_{i\in I} S_i$ of monoids satisfies CP if and only if the monoids S_i $(i \in I)$ satisfy CP uniformly.*

Proof. Each monoid S_j, $j \in I$, has a natural embedding into $\Pi_{i\in I} S_i$, and hence, by Lemma 2.2, if $\Pi_{i\in I} S_i$ satisfies CP, then the monoids S_j satisfy CP uniformly.

In the other direction the claim follows from the observation that for a morphism $\alpha: X_n^* \to \Pi_{i\in I} S_i$, we have $\alpha(u) \neq \alpha(v)$ if and only if for some $j \in I$, $\pi_j\alpha(u) \neq \pi_j\alpha(v)$, where $\pi_j: \Pi_{i\in I} S_i \to S_j$ is the projection onto S_j.

Let $\Pi^\infty S$ denote the denumerably infinite direct product of S with itself.

Corollary 2.7. *A monoid S satisfies CP if and only if $\Pi^\infty S$ satisfies CP.*

Recall that a monoid S is *residually finite*, if for all distinct s, $r \in S$, there exists a finite monoid M and a morphism $\alpha: S \to M$ such that $\alpha(s) \neq \alpha(r)$. As the following result states, the free monoids are residually finite.

Lemma 2.8. *Let F_n be an n-generator free monoid. If $A \subseteq F_n$ is a finite subset of F_n, then there exists a morphism $\alpha: F_n \to S$ into a finite monoid S such that $\alpha(v) \neq \alpha(u)$ for all distinct u, $v \in A$.*

Proof. Let $|w|$ denote the length of the element $w \in F_n$. For each $k \geq 0$ define a relation θ_k by

$$(u, v) \in \theta_k \iff u = v, \quad \text{or } |u| > k \text{ and } |v| > k .$$

The relation θ_k is, as is easily seen, a congruence of F_n, and the monoid F_n/θ_k is finite. Let $\alpha_k: F_n \to F_n/\theta_k$ be the natural morphism. Now, if $A \subseteq F_n$ is a finite subset and $k = \max\{|w| \,|\, w \in A\}$, then clearly $\alpha_k(u) \neq \alpha_k(v)$ for all distinct u, $v \in A$, and the claim follows.

Let $\mathrm{Fin}(S)$ denote the monoid of all nonempty *finite subsets* of the semigroup S. It was shown by Lawrence [15] that the monoid $\mathrm{Fin}(F_2)$ does not satisfy CP. Indeed, the system L of equations

$$x_1 x_2^i x_1 = x_1 x_3^i x_1 \qquad (i \geq 1)$$

over three variables does not have an equivalent finite subsystem in $\mathrm{Fin}(F_2)$.

Theorem 2.9. *The monoid* $\mathrm{Fin}(F_2)$ *is a residually finite monoid that does not satisfy CP.*

Proof. That $\mathrm{Fin}(F_2)$ does not satisfy CP was shown in [15].

Denote $S = \mathrm{Fin}(F_2)$, for short. We observe first that each morphism $\alpha: F_2 \to M$ can be extended elementwise to a morphism $\alpha': S \to \mathrm{Fin}(M)$ by setting $\alpha'(U) = \{\alpha(u) | u \in U\}$ for all finite subsets U of F_2.

Now, let A, $B \in S$ be two distinct elements, and suppose $u \in A \setminus B$ (the case $u \in B \setminus A$ is symmetric). By Lemma 2.8, there exists a finite monoid S_{AB} and a morphism $\alpha_{AB}: S \to S_{AB}$ such that $\alpha_{AB}(u) \neq \alpha_{AB}(v)$ for all $u \in B$. Hence also $\alpha'_{AB}(A) \neq \alpha'_{AB}(B)$. Since $\alpha'_{AB}: S \to \mathrm{Fin}(S_{AB})$ is a morphism into a finite monoid, we conclude that S is residually finite.

Theorem 2.10.

(1) *The monoids satisfying CP are not closed under infinite direct products. In fact, there is a direct product of finite monoids, which does not satisfy CP.*
(2) *The class of all finite monoids does not satisfy CP uniformly.*

Proof. Let S be a residually finite monoid that does not satisfy CP. By Theorem 2.9, $\mathrm{Fin}(F_2)$ is an example of such a monoid. For any two distinct u, $v \in S$, let S_{uv} be a finite monoid such that there exists a morphism $\alpha_{uv}: S \to S_{uv}$ with $\alpha_{uv}(u) \neq \alpha_{uv}(v)$.

Now, there exists a morphism $\alpha: S \to \Pi_{u \neq v} S_{uv}$ defined by its projections onto S_{uv}:

$$\pi_{uv}\alpha(s) = \alpha_{uv}(s) \qquad (s \in S)\,,$$

where π_{uv} is the projection of $\Pi_{u \neq v} S_{uv}$ onto S_{uv}. Since $\alpha_{uv}(u) \neq \alpha_{uv}(v)$, the morphism α is an embedding of S into $\Pi_{u \neq v} S_{uv}$. We conclude from Lemma 2.2 that the direct product $\Pi_{u \neq v} S_{uv}$ of finite monoids does not satisfy CP. Claim (2) follows now from Lemma 2.6.

3. Compactness Property for Varieties of Semigroups

A class $\mathcal{V}$ of monoids (resp., semigroups, groups) is a *variety*, if it is closed under taking submonoids (resp., subsemigroups, subgroups), morphic images, and arbitrary direct products. We refer to Cohn [5], Evans [8], or Neumann [21] for the theory of varieties. By Birkhoff's theorem, a variety of monoids becomes defined by a set of *identities* $u \equiv v$; these are equations $(u, v) \in X^*$ with a possibly infinite number of variables such that every morphism $\alpha: X^* \to S$ with $S \in \mathcal{V}$ is a solution of (u, v). As an example, the identity $x_1x_2 \equiv x_2x_1$ in X_2^* defines the variety of all commutative monoids.

A monoid S satisfies the *maximal condition on congruences*, if each set of congruences of S has a maximal element. The following general result is easy to prove (using Zorn's lemma).

Lemma 3.1. *The following conditions are equivalent for a monoid S.*

(1) *S satisfies the maximal condition on congruences.*
(2) *Each ascending chain $\theta_1 \subset \theta_2 \subset \ldots$ of congruences of S is finite.*
(3) *For each congruence θ of S generated by a subset $L \subseteq \theta$ there exists a finite subset $T \subseteq L$ such that T generates θ.*

Our main result of this section generalizes a group theoretical result of Albert and Lawrence [1] to varieties of monoids.

Theorem 3.2. *A variety $\mathcal{V}$ of monoids satisfies CP if and only if each finitely generated monoid $S \in \mathcal{V}$ satisfies the maximal condition on congruences.*

Proof. We recall first that for all $n \geq 1$, the variety $\mathcal{V}$ possesses a monoid V_n generated by an n-element subset B such that the following extension property holds: if $\gamma_B: B \to S$ is any mapping to $S \in \mathcal{V}$, then there exists a unique morphism $\gamma: V_n \to S$, which is an extension of γ_B, $\gamma|B = \gamma_B$. Such a monoid V_n is called a *free monoid of $\mathcal{V}$*. It follows from the extension property that each morphism $\alpha: X_n^* \to S$ (with $S \in \mathcal{V}$) can be factored as $\alpha = \beta\mu$, where $\mu: X_n^* \to V_n$ is the natural morphism onto V_n and $\beta: V_n \to S$ is a morphism.

Suppose first that each $S \in \mathcal{V}$ satisfies the maximal condition on congruences, and let $L = \{(u_i, v_i) | i \geq 1\} \subseteq X_n^* \times X_n^*$ be a system of equations. Further, let θ be the congruence of V_n generated by the relation $\mu(L) = \{(\mu(u_i), \mu(v_i)) | i \geq 1\}$, i.e., θ is the smallest congruence of V_n containing $\mu(L)$. By assumption and Lemma 3.1, θ is generated by a finite subset T' of $\mu(L)$. Clearly, $T' = \mu(T)$ for a finite subset T of L. Now, if $S \in \mathcal{V}$ and $\alpha = \beta\mu: X_n^* \to S$ is a solution to T, then $T \subseteq \ker(\alpha)$ and hence $T' = \mu(T) \subseteq \ker(\beta)$, which implies that $\theta \subseteq \ker(\beta)$. In particular, $\mu(L) \subseteq \ker(\beta)$, and, consequently, $L \subseteq \ker(\alpha)$. Hence S satisfies the compactness property.

To prove the converse claim, let $\mathcal{V}$ be a variety of monoids, and assume $S \in \mathcal{V}$ is a finitely generated monoid that does not satisfy the maximal condition on congruences. Let $\theta_1 \subset \theta_2 \subset \ldots$ be an ascending chain of congruences of S. Let $\alpha_i: S \to S_i \cong S/\theta_i$ be a surjective morphism with $\ker(\alpha_i) = \theta_i$. Each $S_i \in \mathcal{V}$ since $\mathcal{V}$ is a variety. Denote by $\mu: X_n^* \to S$ the natural morphism from the free monoid X_n^* onto S.

We observe that the congruences $\theta_i' = \ker(\alpha_i\mu)$ of X_n^* for $i \geq 1$ form a properly ascending chain. For each $i \geq 2$ we choose a pair $(u_i, v_i) \in \theta_i' \setminus \theta_{i-1}'$, and let $L = \{(u_i, v_i) | i \geq 2\}$.

Consider the direct product $\Pi_{i\geq 1}S_i$. Since $\mathcal{V}$ is a variety, also $\Pi_{i\geq 1}S_i \in \mathcal{V}$. Let $\beta_i: S_i \to \Pi_{i\geq 1}S_i$ be the natural embedding. Define $\gamma_i = \beta_i\alpha_i\mu: X_n^* \to \Pi_{i\geq 1}S_i$. Now, $\gamma_i(u_j) = \gamma_i(v_j)$ for all $j \leq i$, but $\gamma_i(u_{i+1}) \neq \gamma_i(v_{i+1})$, and thus the system L does not have an equivalent finite subsystem in $\Pi_{i\geq 1}S_i$.

Redei's Theorem [23] states that the finitely generated commutative semigroups satisfy the maximal condition on congruences. For a short proof of this result, we refer to Freyd [9] or Grillet [10]. Hence we have the following corollary of Theorem 3.2 and Theorem 2.5.

Corollary 3.3. *Every commutative monoid or semigroup satisfies CP.*

In particular, by Corollary 3.3, the free commutative monoids satisfy CP. This result was improved by de Luca and Restivo [16] to trace monoids. Here a *trace monoid* is a monoid having a presentation $\langle A | ab = ba \ ((a,b) \in R)\rangle$, where R is an equivalence relation on the set A of generators. The proof of Theorem 3.4 relies again on Hilbert's Basis Theorem.

Theorem 3.4. *The finitely generated trace monoids satisfy CP.*

It is worth noting that in Theorem 3.2 and its corollary the compactness property holds not only for finitely generated monoids, but for infinitely generated monoids as well.

Theorem 3.2 holds also for varieties of groups. In group theory congruences correspond to normal subgroups, and hence as a corollary we have the following result of Albert and Lawrence [1].

Theorem 3.5. *A variety $\mathcal{V}$ of groups satisfies CP if and only if each finitely generated group of $\mathcal{V}$ satisfies the maximal condition on normal subgroups.*

For groups we have a stronger version of Corollary 3.3. Let $[a, b] = a^{-1}b^{-1}ab$ be the *commutator* of the elements a, b of a group. The *metabelian groups* form a (solvable) variety defined by a single identity $[[x_1, x_2], [x_3, x_4]] \equiv 1$, i.e., a group G is metabelian if and only if its second derived group is trivial. (Indeed, in [2] the free metabelian groups were used in showing that the free monoids satisfy Ehrenfeucht's Conjecture). Clearly, every abelian group is metabelian. Moreover, every finitely generated metabelian group satisfies the maximal condition on normal subgroups. We refer to Hall [11] for these and related results.

We also recall that a group G is *nilpotent of class* n, if in the lower central series of G we have $\gamma_{n+1}G = 1$. Hence a nontrivial abelian group is nilpotent of class one. The nilpotent groups of class at most n form a variety, since they are exactly the groups that satisfy the identity $[x_1, x_2, \ldots, x_n] \equiv 1$ for the generalized commutator over X_n. Moreover, a nilpotent group G satisfies the maximal condition on subgroups if and only if G is finitely generated, see Hall [11] or Schenkman [24, p. 200].

Corollary 3.6. *The metabelian groups and the nilpotent groups satisfy CP.*

Notice that the class of all nilpotent groups is not a variety, since it is not closed under infinite direct products. Indeed, the smallest variety that contains all nilpotent groups consists of all groups, see [24]. However, every nilpotent group belongs to a variety (of nilpotent groups of class n for some n) that satisfies CP. As will be seen in the next section, Corollary 3.6 does not extend to solvable groups.

We shall strengthen Theorem 3.2 by showing that in a variety satisfying the compactness property a system of equations $L \subseteq X_n^* \times X_n^*$ has an equivalent finite subsystem $T \subseteq L$ common to all $S \in \mathcal{V}$.

Theorem 3.7. *If a variety $\mathcal{V}$ satisfies CP, then it satisfies CP uniformly.*

Proof. Assume $\mathcal{V}$ satisfies CP, and let $L = \{(u_i, v_i) | i \geq 1\} \subseteq X_n^* \times X_n^*$ be a system of equations. For each $S \in \mathcal{V}$ there exists an equivalent finite subsystem $T_S \subset L$. We may assume that T_S is a *minimal prefix set* of L, that is, $T_S = T(k) = \{(u_i, v_i) | i = 1, 2, \ldots, k\}$ for some k such that $T(k-1)$ is not equivalent to the system L in S, see Lemma 2.1.

Suppose contrary to the claim that L has no common finite equivalent subsystem. There exists then an infinite sequence S_i $(i = 1, 2, \ldots)$ of monoids in $\mathcal{V}$ with minimal prefix sets $T(r_i)$ such that $r_1 < r_2 < \ldots$ forms an infinite increasing sequence, and therefore $L = \bigcup_{i \geq 1} T(r_i)$. Let $S = \Pi_{i \geq 1} S_i$ be the direct product of the monoids S_i. By assumption, there is an equivalent finite subsystem $T \subseteq L$ in S. Clearly, $T \subseteq T(r)$ for some $r \geq 1$, and hence we have a contradiction by Lemma 2.6.

We close this section with an application for groups. Denote by G' the *derived group* of a group G, i.e., G' is the subgroup of G generated by all the commutators $[a, b] = a^{-1}b^{-1}ab$ of G. The derived group G' is a normal subgroup of G, and the quotient G/G' is an abelian group, see Magnus, Karrass and Solitar [17].

Theorem 3.8. *Let A be a subset of the free group $X_n^{(*)}$. There is a finite subset $B \subseteq A$ such that for every group G and every morphism $\alpha: X_n^{(*)} \to G$, if $\alpha(B) \subseteq G'$, then $\alpha(A) \subseteq G'$.*

Proof. Let us identify the subset A with the system of equations $u = 1$, $u \in A$. By Theorem 3.7 (for groups), there is a finite subset $B \subseteq A$ such that A is equivalent to B for all abelian groups. Let G be a group and $\alpha: X_n^{(*)} \to G$ a morphism. Further, let $\mu: G \to G/G'$ be the natural morphism onto the abelian quotient G/G'. Now, $\alpha(u) \in G'$ if and only if $\mu\alpha(u) = 1_{G/G'}$, and hence $\alpha(u) \in G'$ for all $u \in A$ if and only if $\alpha(u) \in G'$ for all $u \in B$. This proves the claim.

4. Semigroups Without Compactness Property

As shown by Lawrence [15], the monoid $\mathrm{Fin}(F_2)$ of all finite subsets of a free monoid F_2 does not satisfy CP. In this section we show that there are even simpler monoids, semigroups and groups that do not possess CP either.

We notice that the free monoid F_n does not satisfy the maximal condition on congruences. Indeed, as an example, consider the submonoid S of the free monoid F_2 generated by the elements a, aba, $baba$, $baab$, where a and b are the generators of F_2. By Markov [18], S has no finite presentation. Let $S = \langle X_4 | u_i = v_i\ (i \in I)\rangle$ be any presentation of S, and let θ_k be the congruence of X_4^* generated by $\{(u_i, v_i) | i = 1, 2, \ldots, k\}$. It follows that $\theta_1 \subset \theta_2 \subset \ldots$ is an infinite ascending chain of congruences of X_4^*.

However, all free monoids satisfy CP. Furthermore, as shown below, the bicyclic monoid satisfies the maximal condition on congruences, but does not satisfy CP. Consequently the notions 'compactness property' and 'maximality condition on congruences' (on finitely generated subsemigroups) are incomparable in general, although they coincide on varieties.

The *bicyclic monoid* B is a 2-generator and 1-relator semigroup with the presentation $\langle a, b | ab = 1\rangle$. B is isomorphic to the submonoid of the transformation semigroup $T_{\mathbb{N}}$ generated by the functions α, β: $\mathbb{N} \to \mathbb{N}$:

$$\alpha(n) = \max\{0, n-1\}, \qquad \beta(n) = n + 1\,,$$

see Clifford and Preston [4]. Here $\alpha\beta = 1$, the identity transformation on $\mathbb{N}$, but $\beta\alpha \neq 1$. Define $\gamma_i = \beta^i\alpha^i$, for $i \geq 0$. Hence

$$\gamma_i(n) = \begin{cases} i & \text{if } n \leq i\,, \\ n & \text{if } n > i\,. \end{cases}$$

Next we observe that $\gamma_i\gamma_j = \gamma_{\max\{i,j\}}$. In particular, each γ_i is an idempotent of B, i.e., $\gamma_i^2 = \gamma_i$. Consider then the system $L \subseteq X_3^+$ consisting of the equations

$$x_1^i x_2^i x_3 = x_3 \qquad (i = 1, 2, \ldots)\,.$$

We conclude that the morphism δ_j defined by $\delta_j(x_1) = \beta$, $\delta_j(x_2) = \alpha$ and $\delta_j(x_3) = \gamma_j$, is a solution of $x_1^i x_2^i x_3 = x_3$ for all $i \leq j$, but δ_j is not a solution of $x_1^{j+1} x_2^{j+1} x_3 = x_3$. Hence the system L does not have an equivalent finite subsystem, and therefore the bicyclic monoid does not satisfy CP.

The bicyclic monoid B is an inverse semigroup, and it is simple, i.e., it has no nontrivial ideals. Furthermore, for every nontrivial congruence θ, the quotient B/θ is a cyclic group, see [4] or Lallement [14]. In particular, B satisfies the maximal condition on congruences by Redei's Theorem. Finally, we notice that B is a monoid that does not satisfy CP, but all the proper quotients of which do satisfy CP.

If we consider the subsemigroup B_1 of B generated by the two elements $\rho = \beta^2\alpha$ and α, we obtain a semigroup without the identity element, which does not satisfy CP. Indeed, in B_1 we have, in the above notations, $\gamma_i = \rho^i\alpha^{i-1}$ for $i \geq 2$, and the claim follows when we consider the system of equations $x_1^i x_2^{i-1} x_3 = x_3$ for $i \geq 2$.

The set $E(S) = \{e \in S | e^2 = e\}$ of idempotents of a semigroup S can be partially ordered as follows: if $fe = e = ef$, then $e \leq f$, see [4]. We say that two elements a and b of a semigroup S form an *inverse pair*, if $a = aba$ and $b = bab$. In this case the elements ab and ba are idempotents of S. For the first claim of the next theorem we refer to Petrich [22, p. 432].

Theorem 4.1. *Let S be a semigroup which contains an inverse pair a, b such that $ba < ab$. Then the subsemigroup of S generated by a and b is a bicyclic monoid with identity ab. In particular, S does not satisfy CP.*

We say that S satisfies the *chain condition on idempotents*, if each subset E_1 of $E(S)$ contains a maximal and a minimal element, i.e., each chain $e_i < e_{i+1}$ $(i \in \mathbb{Z})$ of idempotents is finite.

Theorem 4.2. *If a finitely generated semigroup S satisfies CP, then S satisfies the chain condition on idempotents.*

Proof. Let S be generated by n elements, and let $\mu: X_n^+ \to S$ be the natural morphism onto S. Suppose first that $e_1 > e_2 > \dots$ is an infinite descending chain of idempotents in $E(S)$. Hence $e_ie_j = e_{\max\{i,j\}}$. Let then $w_i \in X_n^+$ be an element such that $\mu(w_i) = e_i$ for each $i \geq 1$, and let $X = X_n \cup \{y\}$. Consider the system L of equations $w_iy = y$ $(i = 1, 2, \dots)$ over X. For each $j \geq 1$ define a morphism $\alpha_j: X^+ \to S$ by $\alpha_j(x) = \mu(x)$ for $x \in X$ and $\alpha_j(y) = e_j$. Now, $\alpha_j(w_iy) = e_ie_j$ for all i and j. Consequently, α_j is a solution of the equations $w_iy = y$ for all i with $i \leq j$, but α_j is not a solution to $w_{j+1}y = y$. We conclude that the system L of equations does not have an equivalent finite subsystem for the semigroup S.

Similarly, if $e_1 < e_2 < \dots$ is an infinite ascending chain of idempotents, then $e_ie_j = e_{\min\{i,j\}}$. In this case, consider the system L of equations $w_iy = w_i$, and define w_i and α_j analogously to the above. We conclude that α_j is a solution of all $w_iy = w_i$ for $i \leq j$, but not to $w_{j+1}y = w_{j+1}$. Hence L has no equivalent finite subsystem for S, and the claim follows.

The free inverse semigroups do not satisfy the chain condition on idempotents. Indeed, the *free monogenic inverse semigroup*, which is generated by one element as an inverse semigroup, has a semigroup presentation

$$FI_1 = \langle a, b | a = aba,\ b = bab,\ a^mb^{m+n}a^n = b^na^{n+m}b^n \quad (n, m \geq 1)\rangle\,,$$

see Petrich [22, p. 427]. Here a^nb^n is an idempotent for each $n \geq 1$, and $a^nb^n \cdot a^mb^m = a^nb^n = a^mb^m \cdot a^nb^n$, i.e., $a^nb^n \leq a^mb^m$, for all $n \geq m$. By Theorem 4.2 we have the following result.

Theorem 4.3. *The free inverse semigroups do not satisfy CP.*

We notice also that the free inverse semigroups do not contain the bicyclic monoid.

The finitely generated free inverse semigroups are residually finite by Munn [20]. Since these semigroups do not satisfy CP, they can be used to prove Theorem 2.10 instead of the monoid $\mathrm{Fin}(F_2)$. In particular, the free monogenic inverse semigroup FI_1 is a 2-generator semigroup, and therefore we have the following corollary.

Corollary 4.4. *The 2-generator finite (inverse) semigroups do not satisfy CP uniformly.*

Below we prove that a semigroup with the compactness property satisfies a maximal condition on restricted congruences, namely on nuclear congruences.

Following Dubreil [7] we say that a congruence θ of a semigroup S is *nuclear*, if it is induced by an endomorphism, i.e., $\theta = \ker(\alpha)$ for a morphism $\alpha: S \to S$. Hence a congruence θ of S is nuclear, if the quotient S/θ is isomorphic to a subsemigroup of S.

Lemma 4.5. *If a semigroup S satisfies CP then each sequence $\alpha_i: X_n^+ \to S$ of morphisms with $\ker(\alpha_i) \subset \ker(\alpha_{i+1})$ $(i = 1, 2, \ldots)$ is finite for all n.*

Proof. Suppose α_i $(i = 1, 2, \ldots)$ is an infinite sequence such that $\ker(\alpha_i) \subset \ker(\alpha_{i+1})$. Consider a system $L = \{(u_i, v_i) | i = 1, 2, \ldots\}$ of equations, where $(u_i, v_i) \in \ker(\alpha_{i+1}) \setminus \ker(\alpha_i)$ for each $i = 1, 2, \ldots$. Clearly, L has no equivalent finite subsystem w.r.t. to S. It follows that S does not satisfy CP.

As a corollary to Ehrenfeucht's Conjecture it was shown by Harju and Karhumäki [12] that the finitely generated subsemigroups of free semigroups satisfy the maximal condition on nuclear congruences. The next theorem generalizes this result for semigroups satisfying the compactness property.

Theorem 4.6. *If a semigroup S satisfies CP then the finitely generated subsemigroups of S satisfy the maximal condition on nuclear congruences.*

Proof. Assume that S_0 is a finitely generated subsemigroup of S that has an infinite sequence $\alpha_i: S_0 \to S_0$ $(i = 1, 2, \ldots)$ of endomorphisms such that $\ker(\alpha_i) \subset \ker(\alpha_{i+1})$ for all $i \geq 1$. Let $\mu: X_n^+ \to S_0$ be the natural morphism onto S_0. Consequently, $\ker(\alpha_i\mu) \subset \ker(\alpha_{i+1}\mu)$ for all $i \geq 1$, and the claim follows from Lemma 4.5.

We can strengthen Lemma 4.5 (and Corollary 2.7) as follows.

Theorem 4.7. *A monoid S satisfies CP if and only if each sequence $\alpha_i: X_n^* \to \Pi^\infty S$ of morphisms with $\ker(\alpha_i) \subset \ker(\alpha_{i+1})$ is finite for each $n \geq 1$.*

Proof. If S satisfies CP, then by Corollary 2.7, so does the direct product $\Pi^\infty S$. The claim from left to right follows now from Lemma 4.5.

Suppose then that S does not satisfy CP, and let $L = \{(u_i, v_i) | i \geq 1\} \subseteq X_n^* \times X_n^*$ be a system of equations, for which there is infinite sequence of morphisms $\beta_j: X_n^* \to S$ $(j \geq 1)$ such that $\beta_j(u_i) = \beta_j(v_i)$ for all $i < j$, but $\beta_j(u_j) \neq \beta_j(v_j)$ (see Lemma 2.1.). For each $i \geq 1$ define a morphism $\alpha_i: X_n^* \to \Pi^\infty S$ by its projections

$$\pi_j \alpha_i(x) = \begin{cases} 1, & \text{if } j < i \\ \beta_j(x), & \text{if } j \geq i \end{cases}$$

for all $x \in X_n$. Now, $\ker(\alpha_i) \subset \ker(\alpha_{i+1})$ for all $i \geq 1$, and the claim follows from this.

We conclude with one more example when the compactness property does not hold.

A semigroup S is said to be ***hopfian***, if S is not isomorphic to a quotient S/θ for any nontrivial congruence θ of S, see Evans [8] or Magnus, Karrass and Solitar [17]. Equivalently, S is hopfian, if every surjective endomorphism $\alpha: S \to S$ is an automorphism.

Theorem 4.8. *If S satisfies CP then every finitely generated subsemigroup of S is hopfian.*

Proof. Assume S_0 is a nonhopfian finitely generated subsemigroup of S, and let $\alpha: S_0 \to S_0$ be a surjective endomorphism which is not injective. The nuclear congruences $\theta_i = \ker(\alpha^i)$, $i = 1, 2, \ldots$, form a properly ascending chain, and hence the claim follows by Theorem 4.6.

In particular, the finitely generated commutative semigroups are hopfian and they satisfy the chain condition on idempotents.

We notice that if a semigroup S satisfies CP, then S itself need not be hopfian (unless S is finitely generated). Indeed, the countably generated free semigroup F_∞ satisfies CP, but, as is easily seen, it is nonhopfian. For another example, consider the multiplicative group $\mathbb{C}^*$ of the nonzero complex numbers. Then α: $\mathbb{C}^* \to \mathbb{C}^*$, $\alpha(c) = c^n$ $(n \geq 2)$, is a surjective endomorphism of $\mathbb{C}^*$, for which $\alpha(r) = 1$ for all nth roots of unity. In particular, α is not injective, and hence $\mathbb{C}^*$ is a nonhopfian abelian group. However, as an abelian group $\mathbb{C}^*$ satisfies CP.

We notice also that the bicyclic B monoid is hopfian, since every nontrivial quotient B/θ is a cyclic group. However, as shown above, B does not satisfy CP.

By Theorem 4.8 a finitely generated nonhopfian semigroup does not satisfy CP. Of course, the same holds for groups. Possibly the simplest nonhopfian group is the *Baumslag-Solitar group* [3], which has a group presentation $G_{BS} = \langle a, b | b^2 a = ab^3 \rangle$, i.e., with two generators and one defining relation. Hence G_{BS} does not satisfy CP.

Indeed, the morphism $\alpha: G_{BS} \to G_{BS}$ defined by $\alpha(a) = a$ and $\alpha(b) = b^2$ is surjective, because $\alpha([a, b^{-1}]) = b$. However, as can be shown, $\alpha(u) = 1$ for $u = [a^{-1}ba, b] \neq 1$, and therefore α is not injective. Let $\beta: G_{BS} \to G_{BS}$ be defined by $\beta(a) = a$ and $\beta(b) = [a, b^{-1}]$. When we consider a and b as variables, we obtain a system of equations, $L = \{\beta^i(u) | i = 1, 2, \ldots\}$, which has no equivalent finite subsystem, because $\alpha^i(\beta^j(u)) = 1$ for $j < i$, but $\alpha^i(\beta^i(u)) \neq 1$.

One should notice that G_{BS} is a solvable group [3], which does not satisfy CP. Moreover, since the factors of the derived series (of length three) are abelian groups, it follows that the compactness property is not closed under group extensions.

References

1. M. H. Albert and J. Lawrence, *The descending chain condition on solution sets for systems of equations in groups*, Proc. Edinburgh Math. Soc. **29** (1985), 69–73.
2. M. H. Albert and J. Lawrence, *A proof of Ehrenfeucht's Conjecture*, Theoret. Comput. Sci. **41** (1985), 121–123.
3. G. Baumslag and D. Solitar, *Some two-generator one-relator nonhopfian groups*, Bull. Amer. Math. Soc. **68** (1962), 199–201.
4. A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups Vol I*, Math. Surveys of the American Math. Soc. **7**, Providence, R. I., 1961.
5. P. M. Cohn, *Universal Algebra*, D. Reidel Publ. Co., Dordrecht, 1981.
6. K. Culik II and J. Karhumäki, *Systems of equations over a free monoid and Ehrenfeucht's Conjecture*, Discrete Math. **43** (1983), 139–153.
7. P. Dubreil, *Sur le demi-groupe des endomorhismes d'une algébre abstraite*, Lincei-Rend. Sc. fis. mat. e nat. **XLVI** (1969), 149–153.
8. T. Evans, *The lattice of semigroup varieties*, Semigroup Forum **2** (1971), 1–43.
9. P. Freyd, *Redei's finiteness theorem for commutative semigroups*, Proc. Amer. Math. Soc. **19** (1968), 1003.
10. P. A. Grillet, *A short proof of Redei's Theorem*, Semigroup Forum **46** (1993), 126–127.
11. P. Hall, *Finiteness conditions for soluble groups*, Proc. London Math. Soc. **4** (1954), 419–436.
12. T. Harju and J. Karhumäki, *On the defect theorem and simplifiability*, Semigroup Forum **33** (1986), 199–217.
13. J. Karhumäki and W. Plandowski, *On the size of independent systems of equations in semigroups*, Lecture Notes in Comput. Sci. **841** (1994), 443–452.
14. G. Lallement, *Semigroups and Combinatorial Applications*, John Wiley & Sons, New York, 1979.
15. J. Lawrence, *The nonexistence of finite test set for set-equivalence of finite substitutions*, Bull. of the EATCS **28** (1986), 34–37.
16. A. de Luca and A. Restivo, *On a generalization of a conjecture of Ehrenfeucht*, Bulletin of the EATCS **30** (1986), 84–90.
17. W. Magnus, A. Karrass and D. Solitar, *Combinatorial Group Theory*, Dover Publ., New York, 1976.
18. Al. A. Markov, *On finitely generated subsemigroups of a free semigroup*, Semigroup Forum **3** (1971), 251–258.
19. A. A. Muchnik and A. L. Semenov, *Jewels of Formal Languages* (Russian translation of a book by A. Salomaa), Mir, Moscow, 1986.
20. W. D. Munn, *Free inverse semigroups*, Proc. London Math. Soc. **29** (1974), 385–404.
21. H. Neumann, *Varieties of Groups*, Springer-Verlag, Berlin, 1967.
22. M. Petrich, *Inverse Semigroups*, John Wiley & Sons, New York, 1984.
23. L. Redei, *The Theory of Finitely Generated Commutative Semi-Groups*, Pergamon Press, Oxford, 1965.
24. E. Schenkman, *Group Theory*, D. van Nostrand Co., Princeton, 1965.