Nash equilibrium in Infinite extensive-form games

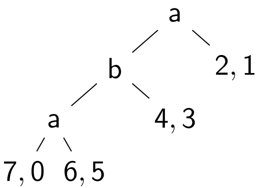
Stéphane Le Roux (TU Darmstadt)

HIGHLIGHTS 2013

September 17, 2013

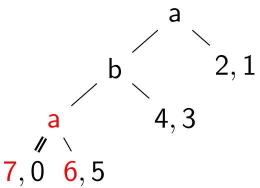
Nash equilibrium in finite extensive-form games

NE are traditionally obtained by **backward induction**:



Nash equilibrium in finite extensive-form games

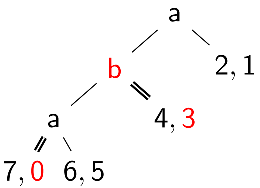
NE are traditionally obtained by **backward induction**:



Double lines represent strategical choices.

Nash equilibrium in finite extensive-form games

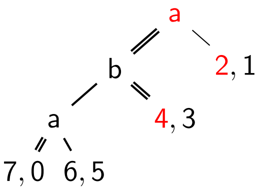
NE are traditionally obtained by **backward induction**:



Double lines represent strategical choices.

Nash equilibrium in finite extensive-form games

NE are traditionally obtained by **backward induction**:



Double lines represent strategical choices.

Gale-Stewart games

Let C be a non-empty set.

Player **a**	$p_0 \in C$		$p_2 \in C$		$\ldots$	
Player **b**		$p_1 \in C$		$p_3 \in C$		$\ldots$

Gale-Stewart games

Let C be a non-empty set.

Player **a**	$p_0 \in C$		$p_2 \in C$		$\dots$	
Player **b**		$p_1 \in C$		$p_3 \in C$		$\dots$

Let $W \subseteq C^\omega$, player **a** wins iff $p_0 p_1 p_2 \dots \in W$.

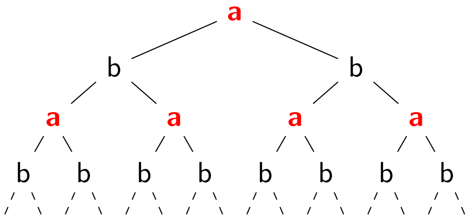
Gale-Stewart games

Let C be a non-empty set.

Player **a**	$p_0 \in C$		$p_2 \in C$		$\ldots$
Player **b**		$p_1 \in C$		$p_3 \in C$	$\ldots$

Let $W \subseteq C^\omega$, player **a** wins iff $p_0 p_1 p_2 \cdots \in W$.

Example with $C = \{\text{left}, \text{right}\}$



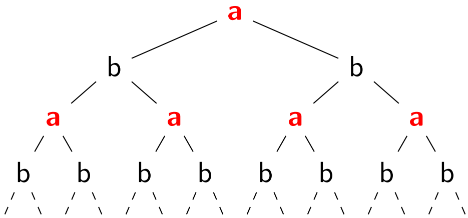
Gale-Stewart games

Let C be a non-empty set.

Player **a**	$p_0 \in C$		$p_2 \in C$		$\dots$	
Player **b**		$p_1 \in C$		$p_3 \in C$		$\dots$

Let $W \subseteq C^\omega$, player **a** wins iff $p_0 p_1 p_2 \dots \in W$.

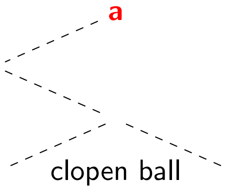
Example with $C = \{\text{left}, \text{right}\}$



Leaf-free infinite extensive-form games: no backward induction!!!

Quasi-Borel determinacy

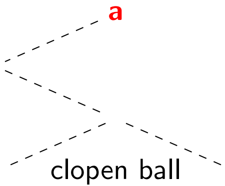
Discrete topology on C and product topology on C^ω.



clopen balls $= \{uC^\omega \mid u \in C^*\}$

Quasi-Borel determinacy

Discrete topology on C and product topology on C^ω.



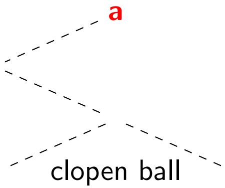
clopen balls $= \{uC^\omega \mid u \in C^*\}$

Borel sets include the open sets of C^ω and are close under:

1. complementation;
2. countable union;

Quasi-Borel determinacy

Discrete topology on C and product topology on C^ω.



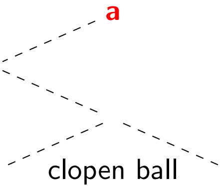
clopen balls $= \{uC^\omega \mid u \in C^*\}$

Quasi-Borel sets include the open sets of C^ω and are close under:

1. complementation;
2. countable union;
3. open-separated union. (For uncountable C.)

Quasi-Borel determinacy

Discrete topology on C and product topology on C^ω.



clopen balls $= \{uC^\omega \mid u \in C^*\}$

Quasi-Borel sets include the open sets of C^ω and are close under:

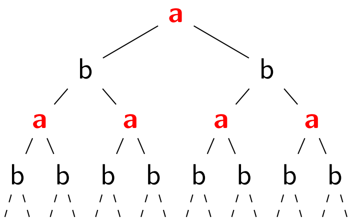
1. complementation;
2. countable union;
3. open-separated union. (For uncountable C.)

Theorem (Martin 1975, 1990)

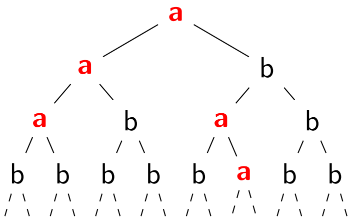
If W is (quasi)-Borel, one player has a winning strategy.

Messy Gale-Stewart games

Gale-Stewart games,
players play alternately:



Messy Gale-Stewart games,
arbitrary order:



First extension of Martin's theorem

Lemma

Let W be the winning set of a **messy** *Gale-Stewart game. If W is quasi-Borel, one player has a winning strategy.*

Proof.

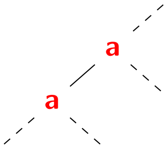
$\square$

First extension of Martin's theorem

Lemma

Let W be the winning set of a **messy** *Gale-Stewart game. If W is quasi-Borel, one player has a winning strategy.*

Proof.



□

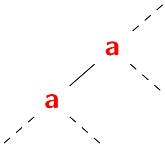
First extension of Martin's theorem

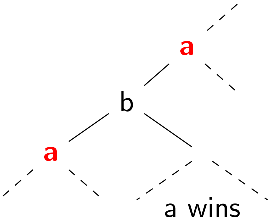
Lemma

Let W be the winning set of a **messy** *Gale-Stewart game.*
If W is quasi-Borel, one player has a winning strategy.

Proof.

Folklore transformation: insertion of dummy nodes.





□

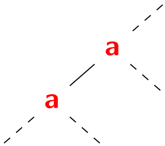
First extension of Martin's theorem

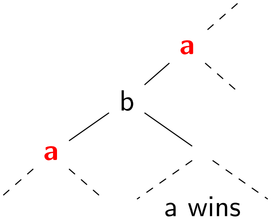
Lemma

Let W be the winning set of a **messy** *Gale-Stewart game.*
If W is quasi-Borel, one player has a winning strategy.

Proof.

Folklore transformation: insertion of dummy nodes.





If winning set W is quasi-Borel, modified W is quasi-Borel

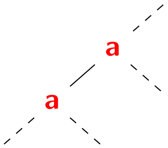
First extension of Martin's theorem

Lemma

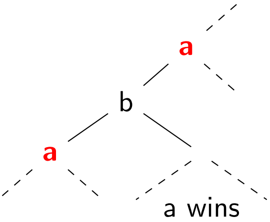
Let W be the winning set of a **messy** *Gale-Stewart game. If W is quasi-Borel, one player has a winning strategy.*

Proof.

Folklore transformation: insertion of dummy nodes.



If winning set W is quasi-Borel,

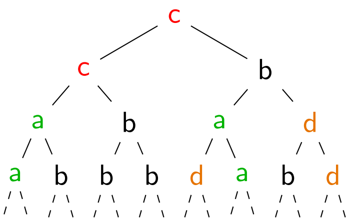


modified W is quasi-Borel

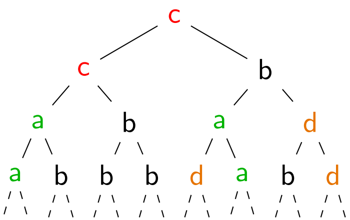
Win. strat. after dummy insertion translate back to win. strat.

□

Infinite game in extensive form

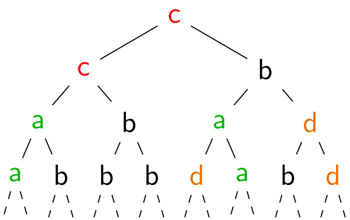


Infinite game in extensive form



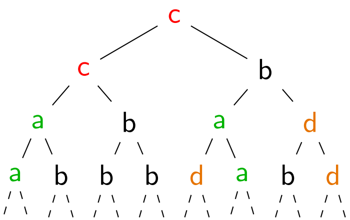
- $v : C^\omega \to O$ from plays to outcomes.

Infinite game in extensive form



- $v : C^\omega \to O$ from plays to outcomes.
- $\prec_a$ is a binary relation over O, the preference of a.

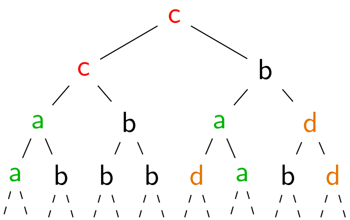
Infinite game in extensive form



- $v : C^\omega \to O$ from plays to outcomes.
- $\prec_a$ is a binary relation over O, the preference of a.

$$\begin{array}{cccccccccccc} \text{start} \longrightarrow & a & \to & a & \to & a & \text{- - - - -} & a & \to & a & \text{- - - - -} & 0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ & 1 & & 2 & & 3 & & n & & n+1 & & \end{array}$$

Infinite game in extensive form



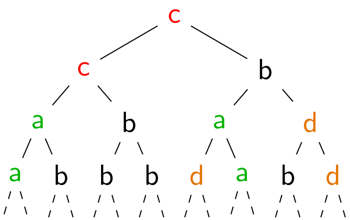
- $v : C^\omega \to O$ from plays to outcomes.
- $\prec_a$ is a binary relation over O, the preference of a.

$$
\begin{array}{ccccccccccccc}
\text{start} & \longrightarrow & a & \to & a & \to & a & \text{- - - - - -} & a & \to & a & \text{- - - - - -} & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 2 & & 3 & & n & & n+1 & &
\end{array}
$$

Theorem

The game has an NE, if the $\prec_a^{-1}$ are strictly well-founded,

Infinite game in extensive form



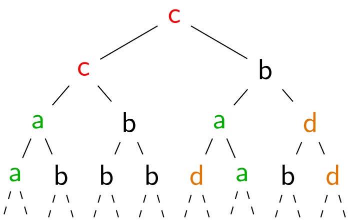
- $v : C^\omega \to O$ from plays to outcomes.
- $\prec_a$ is a binary relation over O, the preference of a.

$$\begin{array}{ccccccccccc} \text{start} \longrightarrow & a & \to & a & \to & a & \text{- - - - - -} & a & \to & a & \text{- - - - - -}\ 0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ & 1 & & 2 & & 3 & & n & & n{+}1 & \end{array}$$

Theorem

The game has an NE, if the $\prec_a^{-1}$ are strictly well-founded, $v^{-1}(o)$ is quasi-Borel for all o

Infinite game in extensive form



- $v : C^\omega \to O$ from plays to outcomes.
- $\prec_a$ is a binary relation over O, the preference of a.

$$\begin{tikzcd}[column sep=small]
\text{start} \arrow[r] & a \arrow[r] \arrow[d] & a \arrow[r] \arrow[d] & a \arrow[r, dash, dashed] \arrow[d] & a \arrow[r] \arrow[d] & a \arrow[r, dash, dashed] \arrow[d] & 0 \\
& 1 & 2 & 3 & n & n+1 &
\end{tikzcd}$$

Theorem

The game has an NE, if the $\prec_a^{-1}$ are strictly well-founded, $v^{-1}(o)$ is quasi-Borel for all o in countable O.

Idea of the proof for finite games

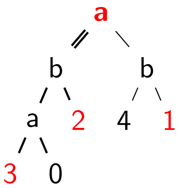
Both players get the same payoffs.

```
        a
      /   \
    b       b
   / \     / \
  a   2   4   1
 / \
3   0
```

- At the root, *a* seeks the best guaranteed payoff.

Idea of the proof for finite games

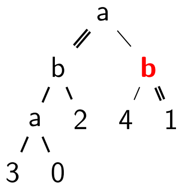
Both players get the same payoffs.



- At the root, *a* seeks the best guaranteed payoff.

Idea of the proof for finite games

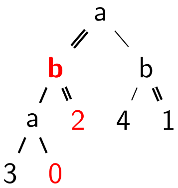
Both players get the same payoffs.



- At the root, *a* seeks the best guaranteed payoff.
- In the subgame ignored by *a*, player *b* threatens *a* (old idea).

Idea of the proof for finite games

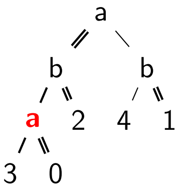
Both players get the same payoffs.



- At the root, *a* seeks the best guaranteed payoff.
- In the subgame ignored by *a*, player *b* threatens *a* (old idea).
- Repeat this in the subgame chosen by *a*.

Idea of the proof for finite games

Both players get the same payoffs.



- At the root, *a* seeks the best guaranteed payoff.
- In the subgame ignored by *a*, player *b* threatens *a* (old idea).
- Repeat this in the subgame chosen by *a*.

Idea of the proof for infinite games

$$
\begin{array}{ccccccccc}
\mathsf{a} & \rightarrow & \mathsf{a} & \rightarrow & \mathsf{a} & \rightarrow & \mathsf{a} & \text{- - - - - -} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & & 1 & & 1 & & 1 & &
\end{array}
$$

Idea of the proof for infinite games

$$\begin{array}{ccccccccc} \color{red}{\mathbf{a}} & \Rightarrow & a & \rightarrow & a & \rightarrow & a & \text{- - - - - -} & 0 \\ \downarrow & & \Downarrow & & \downarrow & & \downarrow & & \\ 1 & & 1 & & 1 & & 1 & & \end{array}$$

Idea of the proof for infinite games

$$
\begin{tikzcd}
a \arrow[r, very thick] \arrow[d] & \color{red}{\mathbf{a}} \arrow[r, very thick] \arrow[d] & a \arrow[r] \arrow[d, very thick] & a \arrow[r, dashed, dash] \arrow[d] & 0 \\
1 & 1 & 1 & 1 &
\end{tikzcd}
$$

Idea of the proof for infinite games

$$\begin{array}{ccccccccc} a & \rightarrow & a & \rightarrow & \color{red}{\mathbf{a}} & \rightarrow & a & \text{- - - - - -} & 0 \\ \downarrow & & \downarrow & & \downarrow & & \Downarrow & & \\ 1 & & 1 & & 1 & & 1 & & \end{array}$$

Idea of the proof for infinite games

$$
\begin{array}{ccccccccc}
a & \Rightarrow & a & \Rightarrow & \mathbf{a} & \Rightarrow & a & = = = = = & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & & 1 & & 1 & & 1 & &
\end{array}
$$

Idea of the proof for infinite games

$$\begin{array}{ccccccccc} a & \dashrightarrow & a & \dashrightarrow & \color{red}{\mathbf{a}} & \dashrightarrow & a & = = = = = & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & & 1 & & 1 & & 1 & & \end{array}$$

- To fix it, change subprofiles only if improving the guarantee.

Idea of the proof for infinite games

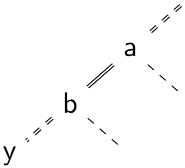
$$\begin{array}{lllllll} a & \rightarrow & a & \rightarrow & \mathbf{a} & \rightarrow & a = = = = = 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 1 & & 1 & & 1 & & 1 \end{array}$$

- To fix it, change subprofiles only if improving the guarantee.
- No infinite ascending preference chain, so each player improves guarantee only finitely many times.

Idea of the proof for infinite games

$$\begin{array}{ccccccccc} a & \rightarrow & a & \rightarrow & \color{red}{\mathbf{a}} & \rightarrow & a & ===== & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & & 1 & & 1 & & 1 & & \end{array}$$

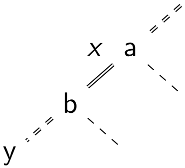
- To fix it, change subprofiles only if improving the guarantee.
- No infinite ascending preference chain, so each player improves guarantee only finitely many times.



Idea of the proof for infinite games

$$\begin{tikzcd} a \arrow[r] \arrow[d] & a \arrow[r] \arrow[d] & \color{red}{a} \arrow[r] \arrow[d] & a \arrow[r, dashed, no head, Rightarrow] \arrow[d] & 0 \\ 1 & 1 & 1 & 1 & \end{tikzcd}$$

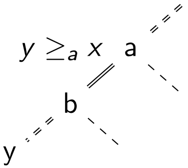
- To fix it, change subprofiles only if improving the guarantee.
- No infinite ascending preference chain, so each player improves guarantee only finitely many times.



Idea of the proof for infinite games

$$\begin{array}{ccccccccc}
a & \nrightarrow & a & \nrightarrow & \color{red}{\mathbf{a}} & \nrightarrow & a & = = = = = = & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & & 1 & & 1 & & 1 & &
\end{array}$$

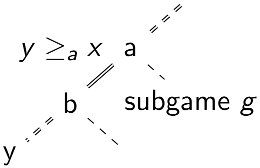
- To fix it, change subprofiles only if improving the guarantee.
- No infinite ascending preference chain, so each player improves guarantee only finitely many times.



Idea of the proof for infinite games

$$\begin{array}{ccccccccc} a & \rightarrow & a & \rightarrow & \color{red}{a} & \rightarrow & a & ====== & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & & 1 & & 1 & & 1 & & \end{array}$$

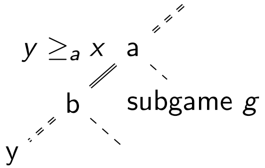
- To fix it, change subprofiles only if improving the guarantee.
- No infinite ascending preference chain, so each player improves guarantee only finitely many times.



Idea of the proof for infinite games

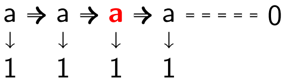
$$\begin{array}{ccccccccc} a & \rightarrow & a & \rightarrow & \color{red}{a} & \rightarrow & a & = = = = = & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & & 1 & & 1 & & 1 & & \end{array}$$

- To fix it, change subprofiles only if improving the guarantee.
- No infinite ascending preference chain, so each player improves guarantee only finitely many times.

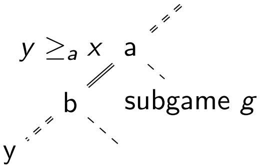


Transform g into a messy Gale-Stewart game:
a's opponents team up and win if a gets $x' \leq_a x$.

Idea of the proof for infinite games



- To fix it, change subprofiles only if improving the guarantee.
- No infinite ascending preference chain, so each player improves guarantee only finitely many times.

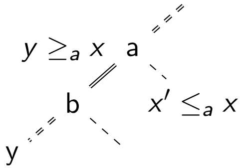


Transform g into a messy Gale-Stewart game:
a's opponents team up and win if a gets $x' \leq_a x$.
Player a has no winning strategy, by definition of the guarantee.
The winning set is quasi-Borel, so a's opponents have a win. strat.

Idea of the proof for infinite games

$$\begin{array}{ccccccccc} a & \nrightarrow & a & \nrightarrow & \color{red}{\mathbf{a}} & \nrightarrow & a & = = = = = = & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & & 1 & & 1 & & 1 & & \end{array}$$

- To fix it, change subprofiles only if improving the guarantee.
- No infinite ascending preference chain, so each player improves guarantee only finitely many times.



Transform g into a messy Gale-Stewart game:
a's opponents team up and win if a gets $x' \leq_a x$.
Player a has no winning strategy, by definition of the guarantee.
The winning set is quasi-Borel, so a's opponents have a win. strat.

Conclusion

The actual result works for more than just quasi-Borel sets:

- If a pointclass with reasonnable closure properties yields determined games, it yields also games with NE.

Conclusion

The actual result works for more than just quasi-Borel sets:

- If a pointclass with reasonnable closure properties yields determined games, it yields also games with NE.
- Axiom of determinacy $\Rightarrow$ every game has an NE (under preference condition).

Conclusion

The actual result works for more than just quasi-Borel sets:

- If a pointclass with reasonnable closure properties yields determined games, it yields also games with NE.
- Axiom of determinacy $\Rightarrow$ every game has an NE (under preference condition).

Questions:

- Several proofs in logic invoke Borel determinacy, can the results be extended by invoking existence of NE?

Conclusion

The actual result works for more than just quasi-Borel sets:

- If a pointclass with reasonnable closure properties yields determined games, it yields also games with NE.
- Axiom of determinacy $\Rightarrow$ every game has an NE (under preference condition).

Questions:

- Several proofs in logic invoke Borel determinacy, can the results be extended by invoking existence of NE?
- Is anyone familiar with the proof of Borel determinacy?