THE COMPACTNESS THEOREM IN MATHEMATICAL LOGIC

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A topological space is said to be *compact* if every open covering contains a finite subcovering—or equivalently, if an intersection of closed sets is empty, then there must have been some finite intersection which was already empty. In logic the compactness theorem is sometimes stated in the following form: *If a (mathematical) theory is inconsistent, some finite subtheory must be inconsistent.* Because a proof is finite in length, the theorem follows from the fact that if a contradiction can be derived in a theory, there must be a finite set of axioms which imply both A and not A.

If, in addition, we restrict ourselves to those theories which are called *first order* or *elementary* (these terms are explained below) and assume a form of Gödel's completeness theorem—*A first order theory is consistent if and only if it has a model*—then the Compactness Theorem can be stated in the following more useful form:

A first order theory has a model if each of its finite subtheories has a model.

To fully understand this statement it is necessary to understand the notions of a *first order (elementary) theory*, a *finite subtheory*, and a *model*. Any mathematical theory is constructed within a *formal language*. The symbols of the language include logical symbols (individual variables, logical connectives, predicates, and quantifiers), nonlogical symbols (individual constants, functions and predicates) which give the language its content, and parentheses to group symbols. Thus, for example, we may use the following symbols:

individual variables: $x, y, z, \cdots$.
logical connectives: $\neg$ (not), $\vee$ (disjunction), $\wedge$ (conjunction), $\rightarrow$ (implication), $\leftrightarrow$ (biconditional).
logical quantifiers: $\forall$ (for all), $\exists$ (there exists).
logical predicate: $=$.
parentheses: (,), [,].
individual constants: $0, 1, \cdots$.
functions: $+, \cdot, f, g, \cdots$.
predicates: $<$, between, is a natural number, $P, Q, R, \cdots$.

A *term* in the language is defined as follows:

(1) each individual variable and constant is a term; (2) if f is a function of n arguments and $t_1, t_2, \cdots, t_n$ are n terms then $f(t_1, t_2, \cdots, t_n)$ is a term; (3) an expression is a term if and only if it satisfies (1) and (2).

Frequently, functions of two arguments like "$+$" and "$\cdot$" are called *operations* and are written between their arguments rather than in front of them.

If P is an n-place predicate and $t_1, t_2, \cdots, t_n$ are n terms then $P(t_1, t_2, \cdots, t_n)$ is called an *atomic formula*. It is customary to write binary predicates between two terms rather than in front of them. So "$x = y$" is an example of an atomic formula and if "B" is the predicate "between" then the expression "$x < y < z$" can be represented by the atomic formula "$B(x, y, z)$".

The *formulas* of the language are just those expressions which are built up from atomic formulas using logical connectives and quantifiers. For example, the following is a formula which expresses the fact that every nonzero element has a reciprocal:

$$(\forall x)\ [x \neq 0 \rightarrow (\exists y)\ (x \cdot y = 1)].$$

(We write "$x \neq 0$" instead of "$\neg (x = 0)$".)

A language is called a *first-order* language if quantification is permitted only on individual variables. The functions and predicates are constants. If there were function or predicate variables and quantification on these variables, the language would be called a *second-order* language.

New symbols can be introduced by definitions. For example, the symbol "2" is an abbreviation for "$1 + 1$", "$x \leqq y$" is an abbreviation for "$x < y \vee x = y$", and subtraction could be defined by the equivalence "$x - y = z \leftrightarrow x = y + z$". Defined terms are always replaceable by undefined terms in all formulas in which they occur. Also, it is convenient to use different types of letters for different types of variables. For example, if "$N(x)$" means "x is a natural number" and if the symbols "n", "m", $\cdots$ are used for variables which range over the set of natural numbers, then "$(\forall n)(\cdots)$" is an abreviation for "$(\forall x)\ (N(x) \rightarrow \cdots)$", and "$(\exists n)\ (\cdots)$" is an abreviation for "$(\exists x)\ (N(x) \wedge \cdots)$".

The *axioms* of our theory are a subset of the set of formulas—intuitively, those formulas which we decide in advance are to be true. For example, the following is one of the Peano axioms in elementary number theory:

$$(\forall n)\ (\forall m)\ (n + 1 = m + 1 \rightarrow n = m).$$

Sometimes a set of axioms is represented by an *axiom schema*—an expression which replaces an infinite number of axioms. An example of such a schema is the principle of mathematical induction: If $\phi(x)$ is any formula then the following is an axiom

$$[\phi(0) \wedge (\forall n)\ (\phi(n) \rightarrow \phi(n + 1)] \rightarrow (\forall n)\ \phi(n).$$

With this example we can see the restrictive nature of a first-order theory. The formula "$\phi(x)$" could be interpreted as "x has the property ϕ" or "x belongs to the set ϕ". (A property is identified with the set of all elements which have the property.) The principle of mathematical induction can then be interpreted as:

If ϕ is a set of natural numbers such that 0 belongs to ϕ and whenever n belongs to ϕ, the successor of n, $n + 1$ belongs to ϕ, then every natural number belongs to ϕ.

But in a first order theory the sets to which the principle of mathematical induction applies are restricted to those sets which are definable by a formula in a first order language (a first order formula). And, as we shall see later, not all sets are definable by such formulas.

Another example is the completeness property of the real numbers—every set of real numbers which is bounded above has a least upper bound. In a first order theory of the real numbers the completeness property only applies to those sets of real numbers which are definable by a first order formula.

Thus, a first order or elementary theory includes a first order language, a set of axioms, and all the first order formulas which can be deduced from the axioms using the rules of logic. A *subtheory* is obtained by omitting some axioms and the subtheory is called *finite* if it only has a finite number of axioms.

A *model* of a first order theory consists of a set of elements (the range of each individual variable) along with interpretations for each of the individual constants, functions and predicates, such that all the axioms of the theory have true interpretations. It should be noted that the interpretation of each individual constant is an element of the set of elements of a model and therefore is included in the range of each individual variable. As an example, let's look at elementary group theory. The nonlogical symbols are e, an individual constant; $\circ$, a function of two arguments; and I, a function of one argument. The axioms are

$$G_1. \quad (\forall x)(\forall y)(\forall z)\,[(x\circ y)\circ z = x\circ(y\circ z)].$$

$$G_2. \quad (\forall x)\,(x\circ e = x).$$

$$G_3. \quad (\forall x)\,(x\circ I(x) = e).$$

G_1 is the associative law for $\circ$, G_2 is the defining property of the identity element e, and G_3 is the defining property of the inverse. ("x^{-1}" is frequently used instead of "$I(x)$".) As a model for this theory we could take the set of natural numbers where "e" is interpreted as "0", "$\circ$" is interpreted as "+", and "I" is interpreted as "$-$", the additive inverse. It is well known that the associative law for addition of natural numbers holds, that $x + 0 = x$ for all natural numbers x, and that $x + (-x) = 0$ for all natural numbers x. Since all the axioms have true interpretations, it follows that we have defined a model of elementary group theory.

At this point the reader should understand the meaning of the compactness theorem; in what follows we give a few of its applications. First, suppose T is a first order theory which has arbitrarily large finite models. For example, the first order theory of fields is such a theory because for each prime p there is a field with p elements. We claim such a theory must have an infinite model. (In particular, there must be a field with an infinite number of elements.) To show that T has an infinite model, for each positive integer n let A_n be the statement that "there exist at least n elements". Thus, for example, A_3 is "$(\exists x)(\exists y)(\exists z)\,(x \neq y \wedge x \neq z \wedge y \neq z)$". Let T' be the theory obtained from T by adding the infinite set of axioms $A_1, A_2, \cdots$. We write $T' = T + \{A_1, A_2, \cdots\}$. Every finite subtheory of T' has a model, for if the subtheory contains the axioms $A_{i_1}, A_{i_2}, \cdots, A_{i_k}$, $i_1 < i_2 < \cdots < i_k$, then any model of T which has more than i_k elements is a model of the subtheory. Consequently, T' satisfies the hypothesis of the compactness theorem and must therefore have a model. We see that any model of T' is a model of T and also must be infinite.

For our next example, suppose T is the first-order theory of a well-ordering relation. That is, T contains only one nonlogical symbol, a binary predicate, R, and the following axioms:

A_1. $(\forall x)xRx$

A_2. $(\forall x)(\forall y)(xRy \wedge yRx \rightarrow x = y)$

A_3. $(\forall x)(\forall y)(\forall z)(xRy \wedge yRz \rightarrow xRz)$

A_4. $(\forall x)(\forall y)(xRy \vee yRx)$

A_5. If $\phi(x)$ is any formula then the following is an axiom:

$$(\exists x)\phi(x) \rightarrow (\exists x)\,[\phi(x) \wedge (\forall y)\,(\phi(y) \rightarrow xRy)].$$

(R is reflexive, antisymmetric, transitive, and connected, and every nonempty set has an R-smallest element).

One model of T is the set of natural numbers as ordered by $\leqq$. Now let us add to the language of T an infinite number of constants, $a_0, a_1, a_2, \cdots$, and an infinite number of axioms:

B_1. $a_1Ra_0 \wedge a_1 \neq a_0$

B_2. $a_2Ra_1 \wedge a_2 \neq a_1$

. . . . .

B_n $a_nRa_{n-1} \wedge a_n \neq a_{n-1}$

. . . . .

Call the extended system T'. Every finite subtheory of T' has a model because any infinite model of T can be expanded to form a model for any finite subtheory of T'. To see this, suppose the finite subtheory contains the axioms B_2, B_3, B_5, and B_9. (The proof is similar for any other finite set of axioms.) Suppose also that b_0, b_1, b_2, b_3, b_4, b_5, b_8, and b_9 are 8 distinct elements in an infinite model of T such that b_9Rb_8, b_5Rb_4, b_3Rb_2, and b_2Rb_1. (We use the same symbol "R" for both the predicate symbol in the formal language and its interpretation in the model.) Let b_i be the interpretation of a_i, for $i = 1,2,3,4,5,8,9$, and let b_0 be the interpretation of a_i otherwise. Then B_2, B_3, B_5 and B_9 are true and we have a model for this finite subtheory of T'.

Consequently, the compactness theorem implies that T' has a model. Let $b_0, b_1, b_2, \cdots$ be the interpretation of $a_0, a_1, a_2, \cdots$ respectively, in a model of T'. Then we obtain the apparent contradiction that the set $\{b_0, b_1, b_2, \cdots\}$ has no-R-smallest element. That is, we apparently have a model of T' in which one of the axioms of T', namely A_5, is not true. To see what happened let us return to our earlier statement about the restrictive nature of a first order theory. The only possible explanation is that the set $\{b_0, b_1, b_2, \cdots\}$ is not definable by a first-order formula. Thus, it just appears that A_5 is false in the model, while in fact it is not.

One final example. Let T be the first order theory of the real numbers. Add a new individual constant, say ω, to the language of T and add the following axioms to T: $\omega > 1$, $\omega > 2$, $\omega > 3, \cdots$. Then the compactness theorem implies that if T has a model, the extended theory also has a model. The reason for this is that if only a finite number of axioms of the form: $\omega > n_1, \omega > n_2, \cdots, \omega > n_k$ are added then in a model of T choose for ω any real number greater than $n_1, n_2, \cdots,$ and n_k and it is clear the axioms are satisfied. Therefore, any model of T serves as a model of each finite extension. Thus, non-standard analysis was born.

Since the axioms for the theory of the non-standard reals include all axioms for the standard reals, all theorems and axioms for the standard reals also hold for the non-standard reals. In addition, of course, there is an infinite non-standard real number since ω is greater than every standard natural number. In fact, there are infinitely many infinite numbers because $\omega, \omega + 1, \omega + 2, \cdots$ are all infinite numbers. Moreover, there must be infinitely small numbers (called *infinitesimals*), because the reciprocal of each number also exists.

Let N be the set of standard natural numbers. N is bounded above by ω, but N has no least upper bound. For any upper bound, b, of N must be infinite, $b - 1 < b$ and $b - 1$ is also an upper bound of N. It appears that the non-standard reals do not satisfy the completeness property. But, of course, this is not the case. The completeness property only applies to sets of real numbers definable by a first order formula. Consequently, the set of standard natural numbers is not definable by a first order formula. Similarly we see there are many other sets not definable by a first order formula, for example the set of standard real numbers, the set of standard rational numbers and the set of infinitesimals.

Perhaps the preceding examples give some idea of the strength and limitations of the compactness theorem and indicate some of its applications in logic.

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COVERING SETS OF CONGRUENCES

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1. Introduction. In a paper in 1952, Erdös [1] introduced the following concept:

DEFINITION 1. *Let* $1 < n_1 < n_2 < \cdots < n_k$, *each* $n_i \in Z$ (*the rational integers*). *Let* $0 \leqq b_i < n_i$ *for each value of* i. *The set of ordered pairs* $\{(b_1, n_1), \cdots, (b_k, n_k)\}$ *of*