The Quest for Strong Inapproximability Results with Perfect Completeness

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The Unique Games Conjecture has pinned down the approximability of all constraint satisfaction problems (CSPs), showing that a natural semidefinite programming relaxation offers the optimal worst-case approximation ratio for any CSP. This elegant picture, however, does not apply for CSP instances that are perfectly satisfiable, due to the imperfect completeness inherent in the Unique Games Conjecture.

This work is motivated by the pursuit of a better understanding of the approximability of perfectly satisfiable instances of CSPs. We prove that an "almost Unique" version of Label Cover can be approximated within a constant factor on satisfiable instances. Our main conceptual contribution is the formulation of a (hypergraph) version of Label Cover that we call *V Label Cover*. Assuming a conjecture concerning the inapproximability of V Label Cover on perfectly satisfiable instances, we prove the following implications:

- There is an absolute constant c_0 such that for $k \geq 3$, given a satisfiable instance of Boolean k-CSP, it is hard to find an assignment satisfying more than $c_0 k^2/2^k$ fraction of the constraints.
- Given a k-uniform hypergraph, $k \geq 2$, for all $\epsilon > 0$, it is hard to tell if it is q-strongly colorable or has no independent set with an ϵ fraction of vertices, where $q = \lceil k + \sqrt{k} - \frac{1}{2} \rceil$.
- Given a k-uniform hypergraph, $k \geq 3$, for all $\epsilon > 0$, it is hard to tell if it is $(k-1)$-rainbow colorable or has no independent set with an ϵ fraction of vertices.

CCS Concepts: • **Theory of computation → Problems, reductions and completeness**; *Approximation algorithms analysis;*

Additional Key Words and Phrases: Inapproximability, hardness of approximation, dictatorship testing, constraint satisfaction, hypergraph coloring

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1 INTRODUCTION

The sustained progress on approximation algorithms and inapproximability results for optimization problems since the early 1990s has been nothing short of extraordinary. This has led to a sharp understanding of the approximability threshold of many fundamental problems, alongside the development of a rich body of techniques on the algorithmic, hardness, and mathematical programming aspects of approximate optimization. Yet there also remain many problems that have resisted resolution, and for some there are in fact large gaps between the known algorithmic and hardness results. Examples include vertex cover, graph coloring, max-cut, feedback vertex set, undirected multicut, densest subgraph, and so on.

The **Unique Games Conjecture (UGC)** of Khot [44] postulates a strong inapproximability result for a particular class of arity 2 **constraint satisfaction problems (CSPs)**. This single assumption has a remarkable array of consequences, and implies *tight* inapproximability results for numerous problems including Vertex Cover [48], max-cut and indeed all CSPs [45, 56, 60], maximum acyclic subgraph and all ordering CSPs [25], scheduling problems [3, 4], graph pricing [51], and cut problems like directed multicut [52], to name a few. Furthermore, for CSPs, the UGC implies that a standard semidefinite programming relaxation gives the best approximation ratio [11, 60, 61].

Although the UGC has identified a common barrier against progress on a host of approximation problems, there are still several situations it does not apply to. Crucially, imperfect completeness, where Yes instances are only almost satisfiable, is inherent in the UGC, and this feature is inherited by the problems it reduces to. In particular, the UGC does not say *anything* about problems with *perfect completeness*, where Yes instances have a perfect solution obeying all of the constraints. Important classes of such problems include satisfiable instances of CSPs (which have a perfect satisfying assignment and the goal is maximize the number of satisfied constraints) and coloring graphs/hypergraphs with approximately optimal number of colors.

Our understanding of approximating satisfiable instances of CSPs still has many gaps. Håstad's tight hardness result for approximating Max 3-SAT on satisfiable instances was much harder to prove than the analogous result for near-satisfiable instances, and was an early sign of the subtleties of ensuring perfect completeness; albeit this proof was later simplified by Saket [64]. The approximability of satisfiable CSPs corresponds via a direct translation to the power of probabilistically checkable proof systems with perfect completeness—the best soundness error one can achieve with a k query (nonadaptive) probabilistically checkable proof is equal to the best inapproximability factor one can prove for a satisfiable arity k CSP. For $k = 3$, the best soundness is $5/8 + \epsilon$ for any $\epsilon > 0$, and this was established only recently via an intricate proof of the approximation resistance of satisfiable NTW (the arity 3 No-Two predicate that stipulates the number of true literals must be either $0, 1$, or 3) [41]. As a basic open question that still remains wide open, we do not know the approximability of satisfiable Max NAE-3-SAT (not-all-equal 3-SAT) under any plausible (or even not so plausible!) conjecture.

The preceding Unique Games hardness results consist of two components: (i) a dictatorship test that gives a way to test if a function is a dictator or is far from a dictator (e.g., has no influential coordinates), using constraints corresponding to the problem at hand (for NAE-3-SAT, this would be checking if certain triples of function values are not all equal), and (ii) a reduction from Unique Games via the dictatorship test that establishes inapproximability under the UGC. The second step is standard, and it gives a "free pass" from the world of combinatorics/analysis of Boolean functions to the complexity world. When we require perfect completeness, no such conjectured off-the-shelf compiler from dictatorship tests to hardness is known (and such a passage even appears unlikely). For instance, dictatorship tests with perfect completeness and optimal soundness are known for

Max k-CSP [68] (which was improved by Bhangale et al. [7]) and Max NAE-3-SAT (folklore, and this has connections to robust forms of Arrow's theorem from social choice theory, as established using Fourier analysis [42] and the work of O'Donnell [57, Section 4]). However, in both cases, we do not have matching inapproximability results under any plausible conjecture.

The closest to a UGC surrogate in the literature is the d-to-1 conjecture also made in the work of Khot [44]. The Unique Games problem is an arity 2 CSP whose constraints are bijections; the d-to-1 Label Cover is an arity 2 CSP whose constraints are d-to-1 functions. When $d \geq 2$, deciding satisfiability of a d-to-1 Label Cover instance is NP-complete, unlike Unique Games whose satisfiability is trivial to ascertain. Khot's d-to-1 conjecture states that d-to-1 Label Cover is also hard to approximate within any constant factor, even on satisfiable instances. Note that the UGC and d-to-1 conjecture are incomparable in strength; the UGC has simpler bijective constraints, but the d-to-1 conjecture asserts perfect completeness that the UGC cannot. Recently, the 2-to-1 conjecture without perfect completeness has been shown to be true [5, 16, 17, 46, 47], which gives further evidence that 2-to-1 with perfect completeness is likely to be true.

The d-to-1 conjecture has been used to show some strong inapproximability results. Such applications are, however, sporadic and also typically do not yield tight results. Some of these results are conditioned specifically on the 2-to-1 conjecture, such as a $\sqrt{2}-\epsilon$ inapproximability for vertex cover (mentioned in the work of Khot [44] and explicit in the work of Khot et al. [46]), which is now an NP-hardness due to the proof of the conjecture with imperfect completeness. Other implications of the 2-to-1 conjecture that are still open include max k-coloring with perfect completeness [33] and coloring 4-colorable graphs [18]. The d-to-1 conjecture, for any fixed d, has been used to show the approximation resistance of NTW [59] and a similar result for larger arity [38],[1] and finding independent sets in 2-colorable 3-uniform hypergraphs [50]. Yet, the implications of the d-to-1 conjecture are limited, and it has become apparent that it is not a versatile starting point for hardness results with perfect completeness.

1.1 Our Contributions

Given the preceding context, our work is motivated by the quest for a better starting point than 2-to-1 Label Cover for inapproximability results with perfect completeness, and that might be able to give striking consequences similar to the UGC.

Aggressive Unique Games variant. One version of Label Cover that is most similar to Unique Games, which we call (L, s)-*nearly unique Label Cover*, has constraint relations in[2] $[L]\times[L]$ consisting of a matching and s additional edges, for a small s that is a constant independent of L. For this version, it is NP-hard to decide satisfiability, and in fact one can give strong reductions matching the performance of dictatorship tests from it. However, this nearly unique form of Label Cover has a constant factor approximation algorithm with ratio depending only on s. We prove this result in Section 3.

V Label Cover. Our main conceptual contribution is the formulation of a (hypergraph) version of Label Cover that we call *V Label Cover*. This is a hypergraph variant[3] of 2-to-1 Label Cover. In 2-to-1 Label Cover, the constraint predicates are 2-to-1 maps from $[2L]$ to $[L]$, whose relation graph can be visualized as L disjoint "V's." In V Label Cover of arity k, we have "longer V's" where the two

[1] These were later improved to NP-hardness in the work of Håstad [41] and Wenner [72].
[2] We denote $[L] = \{1, \ldots, L\}$.
[3] That said, the "graph" variant of V Label Cover is not quite the same as 2-to-1 Label Cover. In particular, the arity of both sides of the predicate is the same, so there are an equal number of V's and Λ's in the relation graph.

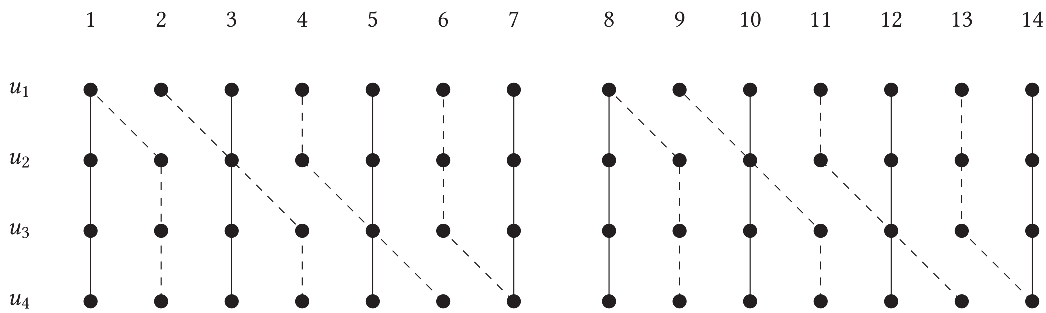

Fig. 1. A schematic diagram of the branches for an edge $e = (u_1, u_2, u_3, u_4)$ of V Label Cover instance Ψ with parameters $k = 4$ and $L = 2$. The ith row represents $\pi_i^{(e)}$, and the jth column represents the input j. The dashed and dotted lines indicate the two different branches with the same values with respect to $\pi^{(e)}$. For example, we may deduce from this diagram that $(10, 10, 10, 10)$ and $(9, 10, 11, 11)$ are two branches of e. In particular, we have that $\pi_1^{(e)}(9) = \pi_2^{(e)}(10) = \pi_3^{(e)}(11) = \pi_4^{(e)}(11)$. Note that $\psi_i^{(e)}(j) =\perp$ exactly when the node of the ith row and jth column is at the intersection of two branches. Compare with Figure 1 of Dinur et al. [18].

branches involve k variables that coincide in single variable.[4] This is best illustrated by Figure 1 in Section 4. We put forth the *V Label Cover conjecture*, which asserts a strong inapproximability result for this problem. For completeness, we want an assignment where for every constraint, the k variables involved get values in a single "V-branch." For soundness, we insist that no assignment even *weakly satisfies* more than a tiny fraction of constraints, where a constraint is weakly satisfied if two of its k variables get values in some V-branch.[5] For this to makesense, the "junction" of the V's cannot all be on the same variable (as in 2-to-1 Label Cover), as in that case we will have a Unique Label Cover constraint between the other $(k-1)$ variables, which we can perfectly satisfy. Therefore, in our V Label Cover constraints, we have V's with junctions at all k variables involved in the constraint. At a high level, this is similar to the correlation-breaking constraints of Chan [13].

Near-optimal inapproximability for Max k-CSP with perfect completeness. Assuming the V Label Cover conjecture, we prove a near-tight inapproximability result for approximating satisfiable Max k-CSP over any fixed domain.

Theorem 1.1. *Assume the V Label Cover conjecture. There is an absolute constant c_0 such that for $k \geq 3$, given a satisfiable instance of Boolean k-CSP, it is hard to find an assignment satisfying more than $c_0 k^2/2^k$ fraction of the constraints. For CSP over domain size $q \geq 3$, where q is a prime power, it is hard to satisfy more than $c_0 k^3 q^3/q^k$ of the constraints.*

The approximability of Max k-CSP has been the subject of many works in the past two decades since the advent of Håstad's optimal inapproximability results [35]; a partial list includes references [2, 13, 21, 22, 30, 36, 39, 65–67] on the hardness side and references [14, 30, 34, 53, 69, 70] on the algorithmic side.

The best known approximation guarantee for Max k-CSP over domain size q is $\Omega(kq/q^k)$ for $k \geq \Omega(\log q)$ and $0.62k/2^k$ for the Boolean case [53]. This is tight up to constant factors, due

[4]We should mention that our path to the formulation of V Label Cover was more circuitous and has its origins in attempts to define hypergraph versions of the "α Label Cover" problem of Dinur et al. [18].

[5]This stronger requirement in soundness is common in hypergraph versions of Label Cover. For general Label Cover, the stronger soundness guarantee can be ensured with a minor loss in parameters, but for V Label Cover we do not know such a reduction.

to Chan's inapproximability factor of $O(kq/q^k)$ [13]. However, this hardness does not apply for satisfiable instances. For satisfiable instances, the best hardness factor is $2^{O(k^{1/3})}/2^k$ for Boolean Max k-CSP [39] and $q^{O(\sqrt{k})}/q^k$ for Max k-CSP over domain size a prime q [36]. Note that our improved hardness factors (conditioned on the V Label Cover conjecture) from Theorem 1.1 are the first to get $\mathrm{poly}(k,q)/q^k$ type hardness for satisfiable instances (albeit only for prime powers) and are close to optimal. We note that satisfiable instances can be easier to approximate—Trevisan gave an elegant linear algebra based factor $(k+1)/2^k$ approximation algorithm for satisfiable Boolean Max k-CSP [70] long before Hast's $\Omega(k/2^k)$ algorithm for the general case [34].

Inapproximability for strong and rainbow colorable hypergraphs. Our other application of the V Label Cover conjecture is to hypergraph coloring, another fundamental problem where perfect completeness is crucial. We say a hypergraph is c-colorable if there is a coloring of its vertices with c colors so that no hyperedge is monochromatic. Given a 2-colorable k-uniform hypergraph for $k \geq 3$, strong inapproximability results that show the NP-hardness of coloring with any fixed ℓ number of colors are known [19, 26], and recent developments show hardness (for $k \geq 8$) even for $\ell = \exp((\log n)^{\Omega(1)})$, where n is the number of vertices [40, 49, 71]. However, these results do not apply when the hypergraph has some form of balanced coloring that is stronger than just being 2-colorable. Specifically, we consider the notions of strong and rainbow colorability in this work. A hypergraph is q-strongly colorable, $q \geq k$ (respectively, q-rainbow colorable, $q \leq k$), if it can be colored with q colors so that in every hyperedge, all vertices get distinct colors (respectively, all q colors are represented). We refer the reader to recent work [8, 9, 29, 32] for further context on these notions. When $k = q$, so that there is a perfectly balanced k-coloring where each hyperedge has exactly one vertex of each of the k colors, one can in polynomial time find a 2-coloring without any monochromatic hyperedge [54]. Here we prove a strong hardness result for coloring hypergraphs (in fact for finding sizable independent sets) when this perfect balance condition is relaxed even slightly (specifically, $q = k - 1$ for rainbow coloring, and $q = k + o(k)$ for strong coloring).

A q-strong coloring of a hypergraph is also a legal q-coloring of the graph obtained by converting each of its hyperedges into a clique. For this reason, our hardness result for strongly colorable hypergraphs also implies hardness results in the more elementary setting of *approximate graph coloring*. There are several "pure" NP-hardness results known for graph coloring (e.g., the best known results in different regimes are presented elsewhere [9, 27, 40, 43]), but there is a gigantic gap between these results and the known algorithms. Dinur et al. [18] establish much improved results, assuming variants of both the 2-to-1 conjecture as well as a new variant known as *alpha label cover*. Their main result is that for all $\epsilon > 0$, given a 3−colorable graph G, under these assumptions, it is NP−hard to locate an independent set with $|G|\epsilon$ vertices. In this work, assuming the V Label Cover−conjecture, we give a substantial generalization of this hardness.

Theorem 1.2. *Assume the V Label Cover conjecture.*[6]

- *Given a k-uniform hypergraph, $k \geq 2$, for all $\epsilon > 0$, it is hard to tell if it is q-strongly colorable or has no independent set with an ϵ fraction of vertices, where $q = \lceil k + \sqrt{k} - \frac{1}{2} \rceil$.*
- *Given a k-uniform hypergraph, $k \geq 3$, for all $\epsilon > 0$, it is hard to tell if it is $(k-1)$-rainbow colorable or has no independent set with an ϵ fraction of vertices.*

Guruswami and Lee [29] showed that for any $\epsilon > 0$, it is NP-hard to distinguish if a k-uniform hypergraph (k even) is a $k/2$-rainbow colorable or does not have a independent set with ϵ fraction of the vertices. In [9], the authors present results for strong coloring, but they only apply when

[6]Technically, we need an "induced" version of the V Label Cover conjecture for this result.

$k = 2$ or when the weak coloring has only two colors. Thus, modulo the V Label Cover–conjecture, our results improve on those in the literature.

1.2 Proof Overview

We now briefly describe the steps needed to prove Theorem 1.1 and Theorem 1.2.

In each case, we reduce from a V Label Cover instance to a CSP (with weighted constraints). In Section 4.3, we detail this reduction. The structure of the reduction has the same standard form as many other inapproximability results. Each vertex of the V Label Cover instance is replaced by a constellation of variables, known as a long code. Each hyperedge of the V Label Cover instance is replaced by a probability distribution of constraints between the variables in the correspond long codes. This is done carefully to ensure that perfectly strongly satisfiable V Label Cover instances map to perfectly satisfiable CSPs.

For each problem type (Max-k-CSP, strong coloring, rainbow coloring), we craft a probability distribution that exploits its underlying structure. The probability distributions need to have a special correlation structure to be compatible with the V Label Cover constraints. We abstract a general notion termed *V Label Cover–compatibility* (Definition 4.1) that captures the properties common to these distributions. For example, we dictate that each vertex of each long code is sampled uniformly at random. Then, for each application, we outline the additional properties of our probability distributions for the reductions to have the proper soundness (Definitions 5.1 and 6.4).

For the soundness analysis, given a good approximation to the resulting CSP, we seek to find an approximate weak labeling of the original V Label Cover instance. To do that, we attempt to decode each long code by finding one (or many) low-degree influential coordinates; these coordinates can be viewed as candidate labels for the associated vertex. We then argue that for a sizable fraction of constraints, two of the decoded labels will belong to the a single V-branch in the constraint. We can then label our V Label Cover instance by assigning each vertex a label selected at random from among its decoded labels, which in expectation finds a good approximate weak labeling.

To guarantee these influential coordinates, we invoke a couple of invariance principles. For Max-k-CSP, we directly invoke a result due to Mossel (Theorem 2.5) on pairwise independent probability distributions. This version guarantees a common influential coordinate between *three* functions that belongs to a common "V." A pigeonhole principle then implies that two of these labels must be in the same branch. For the hypergraph coloring problems, where we do not have pairwise independence of the distributions, we generalize the invariance principles of Mossel (see Theorem 2.6) and Dinur et al. [18, Thm. 3.11] to yield a common influential coordinate for two functions that further lie on the same V-branch. This result, Theorem 2.7, is a key technical component of our reduction, which we hope will find other uses in the future.

1.3 A Path to NP-Hardness Results?

In several cases, the UGC conditioned hardness results were later replaced by NP-hardness results. Examples include some geometric inapproximability results [31], hardness of Unique Coverage [28], inapproximability results for agnostic learning [23], tight hardness results for scheduling [63], Chan's breakthrough showing an asymptotically tight inapproximability result for (near-satisfiable) Max k-CSP [13], and so on. In addition, the very recent breakthrough on 2-to-1 games without perfect completeness has led to a number of implications [5, 6, 16, 17, 46, 47], including approximating vertex cover and bounded-degree independence set.

We hope that establishing a similar body of conditional results for perfect completeness, based on the V Label Cover conjecture or related variants, will point to strong inapproximability results and spur unconditional results in this domain.

The recent proof of the 2-to-1 conjecture without perfect completeness and the accompanying implicit construction of strong SDP gaps (for the sum-of-squares hierarchy) for 2-to-1 Label Cover with perfect completeness raise similar questions about V Label Cover. Such a quest could be a good intermediate goal toward establishing hardness results or gaps for Unique Games.

1.4 Organization

In Section 2, we outline the necessary background on CSPs and probability spaces. In Section 3, we show that (L, s)-nearly unique Label Cover has a polynomial time approximation algorithm. In Section 4, we motivate and detail the V Label Cover–conjecture. In Section 5, we apply V Label Cover to the Max-k-CSP problem. In Section 6, we apply V Label Cover to the strong and rainbow hypergraph coloring problems. In Appendix A, we prove Theorem 2.7.

2 PRELIMINARIES

2.1 Probability Distributions

As is now commonplace in hardness of approximation reductions (e.g., [2, 13, 18, 55]), we utilize the following results on correlated probability spaces.

Definition 2.1 ([24, 37, 62][7]). Let $X \times Y$ be a finite joint probability space with a probability measure μ. The *correlation* between X and Y, denoted $\rho(X, Y)$, is defined to be

$$\rho(X, Y) = \sup_{\substack{f: X \to \mathbb{R}, g: Y \to \mathbb{R} \\ \mathbb{E}[f]=\mathbb{E}[g]=0,\ \mathrm{Var}[f]=\mathrm{Var}[g]=1}} \left[\mathop{\mathbb{E}}_{(x,y)\sim\mu} [f(x)g(y)] \right].$$

This is then easily extended to the correlation of $n \geq 3$ spaces.

Definition 2.2 (Definition 1.9 of Mossel [55]). Let $X_1 \times X_2 \times \cdots \times X_n$ be a finite joint probability space. Let $Z_i = X_1 \times X_2 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_n$. Then we define the correlation of $X_1, \ldots, X_n$ to be

$$\rho(X_1, X_2, \ldots, X_n) = \max_{1 \leq i \leq n} \rho(X_i, Z_i).$$

When a probability space can be decomposed into the product of independent subspaces, then the correlation behaves elegantly.

Lemma 2.1 (Theorem 1 of Witsenhausen [73]). *For all $i \in [n] := \{1, 2, \ldots, n\}$, let $X_i \times Y_i$ be a probability space with measure μ_i. Assume that $\mu_1, \ldots, \mu_n$ are independent. Then,*

$$\rho(X_1 \times X_2 \times \cdots \times X_n, Y_1 \times Y_2 \times \cdots \times Y_n) = \max_{1 \leq i \leq n} \rho(X_i, Y_i).$$

Often it can be difficult to bound the correlation of a distribution away from 1. The following result is key in reducing these complex correlation problems into rather elementary graph connectivity problems.

Lemma 2.2 (Lemma 2.9 of Mossel [55]). *Let $X \times Y$ be a finite joint probability space with measure μ. Let G be the bipartite graph on $X \cup Y$ such that $(x, y) \in X \times Y$ is an edge if and only if $\Pr[x, y] > 0$ with respect to μ. Assume that G is connected, and let δ be the minimum nonzero probability in the joint distribution. Then, we have that*

$$\rho(X, Y) \leq 1 - \delta^2/2.$$

[7] See the work of Anantharam et al. [1] for a history of this definition.

2.2 Influences

Recall the influence of a function over a probability space.

Definition 2.3. Let $X_1, \ldots, X_n$ be finite independent probability spaces, and let $f : X_1 \times \cdots \times X_n \to \mathbb{R}$ be a function. Let $Y_i = X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_n$. The influence is

$$\mathrm{Inf}_i(f) = \mathop{\mathbb{E}}_{x \in Y_i}[\mathrm{Var}_{z \in X_i} f(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n)].$$

Likewise, we need the notion of low-degree influences. We use the multilinear-polynomial definition used many times previously (e.g., [18, 55, 56]).

Definition 2.4 (e.g., Definition 3.4, 3.7 of Mossel et al. [56]). Let $X_1, \ldots, X_n$ be finite independent probability spaces, and let $f : X_1 \times \cdots \times X_n \to \mathbb{R}$ be a function. For each $i \in [n]$, let q_i be the cardinality of the support of X_i. Let $\alpha_1^{(i)}, \ldots, \alpha_{q_i}^{(i)} : X_i \to \mathbb{R}$ be an orthonormal basis of functions such that $\alpha_1^{(i)} \equiv 1$. Let $\Sigma = [q_1] \times \cdots [q_n]$. Now, f can be uniquely expressed as

$$f = \sum_{\sigma \in \Sigma} c_\sigma \prod_{i=1}^{n} \alpha_{\sigma_i}^{(i)},$$

for $c_\sigma \in \mathbb{R}$, which we call the Fourier coefficients. For $\sigma \in Q$, let $|\sigma| = |\{i \in [n] \mid \sigma_i \neq 1\}|$. The *low-degree influence* for $d \in [n]$ is

$$\mathrm{Inf}_i^{\leq d} f = \sum_{\sigma \in \Sigma,\, |\sigma| \leq d,\, \sigma_i \neq 1} c_\sigma^2.$$

The following is a key elementary fact concerning influences.

Lemma 2.3 (e.g., Proposition 3.8 of Mossel et al. [56]). *Consider $f : X_1 \times \cdots \times X_n \to \mathbb{R}$. For all integers $d \geq 1$,*

$$\sum_{i=1}^{n} \mathrm{Inf}_i^{\leq d} f \leq d \,\mathrm{Var}\, f.$$

In particular, for all $\tau > 0$, $|\{i \in [n] \mid \mathrm{Inf}_i^{\leq d} f \geq \tau\}| \leq \frac{d \,\mathrm{Var}\, f}{\tau}$.

Sometimes, we look at f from the perspective of different marginal distributions. Consider $f : X_1 \times \cdots X_n \to \mathbb{R}$ where the X_i's are independent. Furthermore, assume that each X_i can be written as $X_i = Y_{i,1} \times \cdots Y_{i,\ell_i}$, where these $Y_{i,j}$'s are independent. Then, we let $\mathrm{Inf}_{X_i}^{\leq d} f$ denote the low-degree influence of f in the ith coordinate with respect to the X_i's. Likewise, we let $\mathrm{Inf}_{Y_{i,j}}^{\leq d} f$ be the influence of the (i,j)th coordinate when viewed from the perspective of $f : Y_{1,1} \times \cdots \times Y_{n,\ell_n} \to \mathbb{R}$.

For each (i,j), let $\beta_1^{(i,j)}, \ldots, \beta_{q_{i,j}}^{(i,j)} : Y_{i,j} \to \mathbb{R}$ be an orthonormal basis of functions such that $\beta_1^{(i,j)} \equiv 1$. Note that $q_i = \prod_{j=1}^{\ell_i} q_{i,j}$. Let $\Sigma' = [q_{1,1}] \times \cdots [q_{n,\ell_n}]$. Then, we have that there exist c_σ's such that $f = \sum_{\sigma \in \Sigma'} c'_\sigma \prod_{i=1}^{n} \alpha_{\sigma_i}^{(i)}$. If $\ell_i \leq D$ for all i, then we have the following result.

Lemma 2.4 (e.g., Claim 2.7 of Dinur et al. [18]). *If $\ell_i \leq D$ for all $i \in [n]$, then we have for all $i, d \in [n]$ that*

$$\mathrm{Inf}_{X_i}^{\leq d} f \leq \sum_{k=1}^{\ell_i} \mathrm{Inf}_{Y_{i,k}}^{\leq Dd} f.$$

Thus, there exists $k \in [\ell_i]$ such that

$$\frac{1}{D} \mathrm{Inf}_{X_i}^{\leq d} f \leq \mathrm{Inf}_{Y_{i,k}}^{\leq Dd} f.$$

Proof. The proof is a straightforward adaptation of the proof of Claim 2.7 in the work of Dinur et al. [18]. □

For our applications, we only need the case $D = 2$.

2.3 Invariance Principles

Like Austrin and Mossel [2], we use the following result on pairwise independent probability spaces.

Theorem 2.5 (Lemmas 6.6 and 6.9 of Mossel [55]). *Fix $k \geq 3$. For $1 \leq i \leq n$, let $\Omega_i = X_i^{(1)} \times \cdots \times X_i^{(k)}$ be finite pairwise independent probability spaces with probability measure μ_i such that the probability measures corresponding to $\mu_1, \ldots, \mu_n$ are independent. Let δ be the minimum positive probability among all of the μ_i. Let*

$$\rho = \max_{1 \leq i \leq n} \rho(X_i^{(1)}, \ldots, X_i^{(k)})$$

and assume that $\rho < 1$. For every $\epsilon > 0$, there exists $\tau(\delta, \epsilon, \rho), d(\delta, \epsilon, \rho) > 0$ such that for any functions $f_1, \ldots, f_k$ where $f_i : X_1^{(i)} \times \cdots \times X_n^{(i)} \to [0, 1]$ if

$$\forall \ell \in [n], |\{i \mid \mathrm{Inf}^{\leq d}_{X_\ell^{(i)}} f_i > \tau\}| \leq 2$$

then

$$\left| \prod_{i=1}^{k} \mathbb{E}[f_i] - \mathbb{E}\left[\prod_{i=1}^{k} f_i\right] \right| \leq \epsilon.$$

In other words, if the product of the expected values and the expected value of the product significantly differ, then there must exist three functions with a common high low-degree influence coordinate. Note that the number "three" is crucially used in our reduction in Section 5.

As we cannot always obtain pairwise independent probability distributions (such as with our reduction to hypergraph coloring), we also need the following result on correlated probability spaces.

Theorem 2.6 (Theorem 1.14 of Mossel [55]). *Fix $k \geq 2$. For $1 \leq i \leq n$, let $\Omega_i = X_i^{(1)} \times \cdots \times X_i^{(k)}$ be a finite probability spaces with measures μ_i such that $\mu_1, \ldots, \mu_n$ are independent. Let δ be the minimum positive probability among all of the μ_i. Let*

$$\rho = \max \left\{ \max_{1 \leq i \leq n} \rho(X_i^{(1)}, \ldots, X_i^{(k)}), \max_{\substack{1 \leq i \leq n \\ 1 \leq j < k}} \rho\left(\prod_{\ell=1}^{j} X_\ell^{(i)}, \prod_{\ell=j+1}^{k} X_\ell^{(i)} \right) \right\},$$

and assume that $\rho < 1$. For every $\epsilon > 0$, there exists $\epsilon'(\delta, \epsilon, \rho), \tau(\delta, \epsilon, \rho) > 0$ such that for any functions $f_1, \ldots, f_k$ where $f_i : X_1^{(i)} \times \cdots \times X_n^{(i)} \to [0, 1]$ and $\mathbb{E}[f_i] \geq \epsilon$ if

$$\forall \ell \in [n], \forall i \in [k], \mathrm{Inf}_{X_\ell^{(i)}} f_i < \tau$$

then

$$\mathbb{E}\left[\prod_{i=1}^{k} f_i\right] \geq \epsilon'.$$

We need a stronger version of this theorem for our applications. We prove Theorem 2.7 in Appendix A.

Theorem 2.7. *Fix $k \geq 2$. For $1 \leq \ell \leq n$, let $\Omega_\ell = X_\ell^{(1)} \times \cdots \times X_\ell^{(k)}$ be a finite probability space with distributions μ_ℓ such that the μ_ℓ's are independent. In addition, assume that for each $\ell \in [n]$ and $i \in [k]$, $X_\ell^{(i)} = \prod_{s=1}^{s_\ell^{(i)}} Y_{\ell,s}^{(i)}$, where the product is of otherwise independent distributions and $s_\ell^{(i)} \leq 2$ for all $i \in [k]$ and $\ell \in [n]$. Assume that we also have the following key property:*

- *If for distinct $i_1, i_2 \in [k]$ we have that $s_\ell^{(i_1)} = s_\ell^{(i_2)} = 2$, then $Y_{\ell,1}^{(i_1)}$ is independent of $Y_{\ell,2}^{(i_2)}$ (and $Y_{\ell,2}^{(i_2)}$ is independent of $Y_{\ell,1}^{(i_1)}$ by symmetry).*

For convenience of notation, if $s_\ell^{(i)} = 1$, let $Y_{\ell,2}^{(i)} := Y_{\ell,1}^{(i)}$. Let δ be the minimum positive probability among all the μ_ℓ's, $\ell \in [n]$. Let

$$\rho = \max\left\{ \max_{1\leq \ell \leq n} \rho(X_\ell^{(1)}, \ldots, X_\ell^{(k)}),\ \max_{\substack{1\leq \ell\leq n\\ 1\leq j<k}} \rho\left(\prod_{\ell=1}^{j} X_\ell^{(i)}, \prod_{\ell=j+1}^{k} X_\ell^{(i)}\right)\right\},$$

and assume that $\rho < 1$. For every $\epsilon > 0$, there exists $\epsilon'(\delta,\epsilon,\rho), \tau(\delta,\epsilon,\rho), d(\delta,\epsilon,\rho) > 0$ such that for any functions $f_1, \ldots, f_k$ where $f_i : X_1^{(i)} \times \cdots \times X_n^{(i)} \to [0,1]$ and $\mathbb{E}[f_i] \geq \epsilon$ if

$$\forall \ell \in [n], \forall s \in \{1,2\}, |\{i \mid \mathrm{Inf}^{\leq d}_{Y_{\ell,s}^{(i)}} f_i \geq \tau\}| \leq 1$$

then

$$\mathbb{E}\left[\prod_{i=1}^{k} f_i\right] \geq \epsilon'.$$

3 (L,S)-NEARLY 1–TO–1 LABEL COVER

Consider the following variant of the classic Label Cover problem.

Definition 3.1. Let L be a positive integer and $s \in \{0, \ldots, L^2\}$. An instance of (L,s)*-nearly 1-to-1 Label Cover* consists of $\Psi = (V, E, \{S_e\}_{e\in E}, \{\pi_{e,u}\}_{e\in E, u\in e})$, where (V,E) is a regular graph,[8] the $S_e \subseteq [L]\times[L]$ have size s,[9] and the maps $\pi_{e,u} : [L] \to [L]$ are permutations. A labeling is a function $\sigma : V \to [L]$. An edge $e \in E$ is satisfied if $(\pi_{e,u}(\sigma(u)), \pi_{e,v}(\sigma(v))) \in \{(\ell,\ell) : \ell \in [L]\} \cup S_e$.

Assume $s \geq 1$ (as the case $s = 0$ is Unique Games with perfect completeness). We show that when s is a constant relative to L, the (L,s)-nearly 1-to-1 Label Cover problem is efficiently approximable.

Theorem 3.1. *There exists a function $\eta : \mathbb{N} \to (0,1]$ (presumably decreasing) such that there is a randomized polynomial time algorithm that with high probability distinguishes the following two types of instances $\Psi = (V, E, \{\pi_{e,u}\}_{e\in E, u\in e})$ of (L,s)-nearly 1-to-1 Label Cover:*

- *Accept: Ψ is perfectly satisfiable.*
- *Reject: Every labeling of Ψ satisfies strictly less than $\eta(s)$ fraction of the edges.*

In fact, one may take $\eta(s) = \frac{1}{1024s^2}$.

For each $e \in E$, let $T_e = \{x : (x,y) \in S_e\} \cup \{y : (x,y) \in S_e\}$. Note that $|T_e| \leq 2s$.

[8] We assume that (V, E) is a regular graph for simplicity of presentation. The authors believe that the same result should hold for general graphs.

[9] If S_e is not symmetric, then the edge e is technically directed, but it is fine to assume that (V, E) is undirected for most of our analysis.

Assume that a perfect labeling exists for Ψ, and let $\Sigma : V \to [L]$ be such a labeling. We show that we can efficiently construct a labeling $\sigma : V \to [L]$ that satisfies at least $\eta(s)$ fraction of the edges. Such an algorithm will suffice to distinguish the two cases specified in the theorem statement.

For each $e \in E$, we say that a satisfied edge $e = (u, v)$ is *type-1-satisfied* by Σ if $\pi_{e,u}(\Sigma(u)) \notin T_e$ and $\pi_{e,v}(\Sigma(v)) \notin T_e$. Otherwise, we say that a satisfied edge e is *type-2-satisfied* by Σ. Let $E_1 \subseteq E$ be the type-1-satisfied edges, and let E_2 be the type-2-satisfied. Let D be the degree of each vertex of (V, E). Let $d_i(v)$ be the number of edges incident to vertex v that are type-i satisfied by σ.

First, we use a standard DFS algorithm to construct partial but perfect labelings of Ψ.

Lemma 3.2. *Given $v_0 \in V$ and $\ell \in L$, there is a polynomial time algorithm that outputs a subset $W \subseteq V$ and a partial labeling $\sigma : W \to [L]$ with the following properties:*

- *$v_0 \in W$ and $\sigma(v_0) = \ell$.*
- *Every $e \in E \cap W \times W$ is satisfied by σ.*
- *For every $\sigma' : V \to [L]$ that extends σ (i.e., $\sigma'(v) = \sigma(v)$ for all $v \in W$) which perfectly satisfies Ψ, every edge in the cut $E \cap W \times (V \setminus W)$ must be type-2-satisfied.*

If there is no satisfying assignment $\sigma : V \to [L]$ to Ψ with $\sigma(v_0) = \ell$, then the algorithm returns $\perp$.

Informally, the last condition means that the partial labeling cannot be extended any further by type-1 satisfying edges.

Proof. Consider the DFS/BFS-like Algorithm 1.

ALGORITHM 1: Finding a partial solution using type-1-satisfied edges.

```
Function Partial-Type-1-Labeling(Ψ, W, σ) do
    Data: (L, s)-nearly 1-to-1 Label Cover instance Ψ = (V, E, {S_e}_{e∈E}, {π_{e,u}}_{e∈E,u∈e}), W ⊆ V,
          σ : W → [L]
    Result: Either ⊥ or a pair (W′, σ′) where W′ ⊆ V and σ′ : V → [L].
    for v ∈ W do
        for e ∈ E where v ∈ e do
            Set u to other vertex of e
            if u ∈ W then
                if σ does not satisfy e then return ⊥
            end
            else if π_{e,v}(σ(v)) ∉ T_e then
                Set W′ = W ∪ {u}
                Set σ′ ↾ W = σ
                Set σ′(u) = (π_{e,u}^{-1} ∘ π_{e,v})(σ(v))
                return Partial-Type-1-Labeling(Ψ, W′, σ′)
            end
        end
    end
    return (W, σ)
end
```

We claim that calling Partial-Type-1-Labeling(Ψ, $\{v_0\}$, $v_0 \mapsto \ell$)[10] is the correct procedure. To prove efficiency, it is easy to see that during each recursive call, W will grow by at least one element or the procedure will terminate. Hence, there can be at most $|V|$ recursive calls (including

[10] $v_0 \mapsto \ell$ is shorthand for the function $\sigma : \{v_0\} \to [L]$ such that $\sigma(v_0) = \ell$.

the initial call). Furthermore, within one recursive call only a polynomial amount of work is done. Thus, the procedure runs in polynomial time.

To prove correctness, note that the final recursive call will verify that every edge inside the vertices of W is correctly labeled and every edge between W and $V \setminus W$ must be type-2-satisfied. Thus, if the algorithm outputs (W, σ), we know that W and σ will have the required properties. Furthermore, observe that the algorithm adds a new vertex to W only when the label of that vertex is forced. Thus, any contradiction found is proof that there is no fully satisfiable way to extend the initial choice that $\sigma(v_0) = \ell$.

Thus, the algorithm is correct and efficient. □

Note that the preceding algorithm will do quite well when Σ type-1-satisfies most of the edges. The following algorithm deals with the case in which most of the edges are type-2-satisfied. Let $\delta = |E_2|/|E|$.

Lemma 3.3. *Assume that Ψ is satisfiable and $\delta \geq 1/2$. Then there is a randomized polynomial time algorithm that finds a labeling $\sigma : V \to [L]$ which satisfies at least $f(s) = \frac{1}{1024s^2}$ of the constraints of Ψ with probability $1 - \frac{1}{2^{poly(|V|)}}$.*

Remark. We set $\eta(s) = f(s)$.

ALGORITHM 2: Finding a good approximate solution when there are many type-2 edges.

```
Function Approx-Type-2-Labeling(Ψ) do
    Data: (L, s)-nearly 1-to-1 Label Cover instance Ψ = (V, E, {S_e}_{e∈E}, {π_{e,u}}_{e∈E,u∈e})
    Result: An approximately satisfying labeling σ : V → [L].
    for v ∈ V do
        Pick e ∈ E uniformly at random such that v ∈ e.
        Pick ℓ ∈ T_e uniformly at random.
        Set σ(v) = π^{-1}_{e,v}(ℓ)
    end
    return σ
end
```

Proof. Consider Algorithm 2. Clearly the algorithm runs in polynomial time. It suffices to show that the preceding algorithm succeeds in finding an $\eta(s)$ approximation with constant probability, as one may repeat the subroutine polynomially many times and take the best solution. The first step of our analysis is the following simple claim.

Claim 3.4. For each $v \in V$, with probability at least $\frac{d_2(v)}{2sD}$, $\sigma(v) = \Sigma(v)$.

Proof. With probability $\frac{d_2(v)}{D}$, we pick an edge e that is type-2-satisfied by Σ. With probability $\frac{1}{|T_e|} \geq \frac{1}{2s}$, we then subsequently pick $\Sigma(v)$ since $\pi_{e,v}(\Sigma(v)) \in T_e$. □

Define a vertex $v \in V$ to be *good* if $d_2(v) \geq |D|/4$. Let $V' \subseteq V$ be the set of good vertices. By Markov's inequality and the fact that V is regular, $|V'| \geq |V|/4$. Define an edge $(u, v) \in E$ to be good if $u, v \in V'$. Let $E' \subseteq E$ be the set of good edges.

Claim 3.5. At least 1/8 fraction of the edges are good.

Proof. Let $\lambda = |E'|/|E_2|$. Pick a uniformly random edge $e \in E_2$, and pick a uniformly random vertex u of e. The probability that u is not good is at least $(1 - \lambda)/2$. Note that the probability that any particular u is picked is $\frac{d_2(u)}{2|E_2|}$. Thus,

$$\begin{aligned}\frac{1-\lambda}{2} &\leq \sum_{v\in V\setminus V'} \frac{d_2(u)}{2|E_2|}\\ &\leq |V\setminus V'|\frac{D}{8|E_2|}\\ &\leq \frac{3|V|D}{32|E_2|}\\ &= \frac{3|E|}{16|E_2|}\\ &\leq \frac{3}{8}.\end{aligned}$$

Thus, $\lambda \geq \frac{1}{4}$, so $|E'|/|E| \geq \lambda\delta \geq \frac{1}{8}$. □

The expected fraction of edges satisfied is then

$$\begin{aligned}\mathbb{E}\left[\frac{1}{|E|}\sum_{e\in E}\mathbf{1}[\sigma \text{ satisfies } e]\right] &\geq \frac{1}{|E|}\sum_{(u,v)\in E}\mathbb{E}[\mathbf{1}[(\sigma(u),\sigma(v)) = (\Sigma(u),\Sigma(v))]]\\ &\geq \frac{1}{|E|}\sum_{(u,v)\in E}\frac{d_2(u)d_2(v)}{(2s)^2D^2} \text{ (Claim 3.4 and independence)}.\\ &\geq \frac{1}{(2s)^2|E|}\sum_{(u,v)\in E'}\frac{d_2(u)d_2(v)}{D^2}.\\ &\geq \frac{|E'|}{64s^2|E|}\\ &\geq \frac{1}{512s^2} \text{ (Claim 3.5)}.\end{aligned}$$

Thus, by Markov's inequality, the preceding algorithm will find a solution satisfying at least $\frac{1}{1024s^2}$ fraction of the edges with probability at least $\frac{1}{1024s^2}$. □

Consider the case $\delta \leq 1/2$, Thus, most of the edges of E are type-1-satisfied by σ. Assume that (V, E_1) has k connected components $V_1 \cup \cdots \cup V_k = V$. We would like to show that one of these sets $V_1, \ldots, V_k$ has small *edge expansion*. First, recall the definition of edge expansion.

Definition 3.2. The *edge expansion* of a subset $V' \subseteq V$ of an undirected D-regular graph (V, E) is

$$\Phi(V') = \frac{|E \cap (V' \times (V \setminus V'))|}{D|V'|}.$$

Intuitively, if we can perfectly label the induced edges of a connected component V_i with poor edge expansion, we have made good progress toward a labeling satisfying a constant fraction of the edges, as we can recursively apply our algorithm to find an approximate labeling of $V \setminus V_i$ and union it with our labeling of V_i. The following lemma shows that such a V_i always exists.

Lemma 3.6. *Let (V, E) be an undirected D-regular graph, and let $V_1, \ldots, V_k$ be a partition of the vertices. Assume that at most δ fraction of the edges of E are between different V_i. Then there exists an $i \in [k]$ such that $\Phi(V_i) \leq \delta$.*

Proof. Let E' be the set of edges between the $V_1, V_2, \ldots, V_k$. Note that

$$\begin{aligned} 2|E'| &= \sum_{i \in [k]} |E \cap (V_i \times (V \setminus V_i))| \text{ (each edge of } E' \text{ is between two of the } V_i\text{'s)} \\ &= \sum_{i \in [k]} D|V_i|\Phi(V_i) \\ &\geq D|V| \min_i \Phi(V_i). \end{aligned}$$

Since $2|E'| = \delta(2|E|) = D|V|\delta$, we have that $\delta \geq \min_i \Phi(V_i)$. □

With this lemma proven, we may now state the final algorithm (3).

ALGORITHM 3: The full algorithm.

Function `Approximate-Labeling`(Ψ) **do**
  Data: (L, s)-nearly 1-to-1 Label Cover instance $\Psi = (V, E, \{\pi_{e,u}\}_{e \in E, u \in e})$
  Result: Either $\perp$ or an approximately-satisfying labeling $\sigma : V \to [L]$.
  Set σ_1 = `Approx-Type-2-Labeling` (Ψ)
  if σ_1 *satisfies* $\eta(s)$ *fraction of the edges of* Ψ **then**
    return σ_1
  end
  for $v \in V, \ell \in [L]$ **do**
    Set $\sigma_2(v) = \ell$.
    Set τ = `Partial-Type-1-Labeling`(Ψ, v, σ_2)
    if $\tau = (W, \sigma_2')$ ***and*** $\Phi(W) \leq 1/2$ **then**
      Set $V' = V \setminus W$
      Set $E' = E \cap (V')^2$
      Set $\Psi' = (V', E', \{\pi_{e,u}\}_{e \in E', u \in e})$
      Set σ_3 = `Approximate-Labeling`(Ψ')
      if $\sigma_3 \neq \perp$ **then return** $\sigma_2' \cup \sigma_3$
    end
  end
  return $\perp$
end

Proof of Theorem 3.1. We prove that the algorithm works by strong induction on $|V|$.

Assume that Ψ is perfectly satisfiable. If $\delta = |E_2'|/|E| \geq 1/2$, then Lemma 3.3 guarantees that we will find an $\eta(s)$ approximation with high probability. Otherwise, if $\delta \leq 1/2$, we know by Lemma 3.6 that there exists $W \subseteq V$ and a perfect partial labeling $\sigma_2' : W \to [L]$ such that W is connected by edges type-1-satisfied by σ and $\Phi(W) \leq 1/2$. By Lemma 3.2, the above for loop will succeed in finding some (W, σ_2') with these properties in polynomial time. By the strong induction hypothesis, we can with high probability find a $\eta(s)$-approximate labeling σ_3 to the instance Ψ' induced by $V \setminus W$. Thus, the labeling $\sigma_2' \cup \sigma_3$ satisfies at least 1/2 fraction of the edges incident with at least one vertex of W (since $\Phi(W) \leq 1/2$ and the edges inside of W are perfectly satisfied) and at least $\eta(s)$ of the edges not incident with W. Thus, we have efficiently found a $\min(1/2, \eta(s)) = \eta(s)$ approximation for Ψ. If the algorithm does not succeed, then Ψ is not perfectly satisfiable. □

4 V LABEL COVER

In this section, we propose a variant of hypergraph Label Cover that seems to plausibly have perfect completeness while also allowing for new hardness reductions. It can be thought of as a generalization of 2-to-1 Label Cover.

4.1 Definition

Let $k \geq 2$ and $L \geq 1$ be positive integers. An instance of k-uniform V-Label Cover is a k-uniform hypergraph on vertex set U. The constraints are on k-tuples $E \subseteq U^k$. Each edge $e = (u_1, \ldots, u_k)$ also has projection maps $\pi_1^{(e)}, \ldots, \pi_k^{(e)} : [(2k-1)L] \to [kL]$ with the following special property:

- The maps are surjective, particularly for all $i \in [k]$ and $j \in [kL]$,

$$|(\pi_i^{(e)})^{-1}(j)| = \begin{cases} 1 & i \equiv j \mod k \\ 2 & \text{otherwise.} \end{cases}$$

In addition, we would like to be able to distinguish between the two labels that map to a common value. To do this, we supplement the projection maps with *distinguishing functions* $\psi_1, \ldots, \psi_k : [(2k-1)L] \to \{0, 1, \perp\}$ such that for all $i \in [k]$, the map $x \mapsto (\pi_i^{(e)}(x), \psi_i^{(e)}(x))$ is injective. Furthermore, if $|(\pi_i^{(e)})^{-1}(\pi_i^{(e)}(x))| = 1$, then we define $\psi_i^{(e)}(x) = \perp$, and otherwise $\psi_i^{(e)}(x) \in \{0, 1\}$. We say that $(t_1, \ldots, t_k) \in [(2k-1)L]^k$ is a *branch* of e if there is $\ell \in [kL]$ and $b \in \{0, 1\}$ such that for all i, $(\pi_i^{(e)}(t_i), \psi_i^{(e)}(t_i))$ equals (ℓ, b) or $(\ell, \perp)$. Note that for each branch, there is exactly one $j \in [k]$ such that $\psi_j^{(e)}(t_j) = \perp$. In fact, such an index satisfies $j \equiv \pi_i^{(e)}(t_i) \mod k$ for all i. We say that j is the *junction* of the branch. To better understand the setup, see Figure 1.

The goal of V-Label Cover is to produce a labeling of the vertices $\sigma : U \to [(2k-1)L]$. We say that a hyperedge $e = (u_1, \ldots, u_k)$ is *strongly satisfied* if $(\sigma(u_1), \ldots, \sigma(u_k))$ is a branch. In other words, for all $i, j \in [k]$, $\pi_i^{(e)}(\sigma(u_i)) = \pi_j^{(e)}(\sigma(u_j))$ *and* either $\psi_i^{(e)}(\sigma(u_i)) = \psi_j^{(e)}(\sigma(u_j)) \neq \perp$ or exactly one of $\psi_i^{(e)}(\sigma(u_i)), \psi_j^{(e)}(\sigma(u_j))$ is $\perp$. Another way to express this is that $(\pi_i^{(e)}(\sigma(u_i)), \psi_i^{(e)}(\sigma(u_i)))$ is uniform except for one i for which $\psi_i^{(e)}(\sigma(u_i)) = \perp$ (the meeting point in the "V" of the two branches).

We say the hyperedge is *weakly satisfied* if for some distinct $i, j \in [k]$, $\pi_i^{(e)}(\sigma(u_i)) = \pi_j^{(e)}(\sigma(u_j))$ *and $\sigma(u_i)$ and $\sigma(u_j)$ are in the same branch.*

We now formally state our conjectured intractability of approximating V Label Cover. In the following, we state an "induced" version where in the soundness guarantee, for every labeling, most of the hyperedges within any subset of vertices of density ϵ fail to be weakly satisfied. The induced version is needed for our reduction to hypergraph coloring (this is similar to the α conjecture of Dinur et al. [18] that was also defined in an induced form). For our Max k-CSP result, it suffices to assume the soundness condition that at most ϵ fraction of edges are weakly satisfiable. For simplicity, we only state the stronger induced version in the following.

Conjecture 4.1 (V Label Cover–conjecture, Induced Version). *For all $k \geq 2$ and $\epsilon > 0$, there exists an $L \geq 1$ such that for any k-uniform V Label Cover instance Ψ on label set L and vertex set U and hyperedge set E, it is NP-hard to distinguish between*

- *YES: There exists a labeling for which every hyperedge is strongly satisfied.*
- *NO: For every labeling and every subset $U' \subset U$ with $|U'| \geq |U|\epsilon$, less than ϵ fraction of the edges in $(U')^k \cap E$ are weakly satisfied by the labeling.*

4.2 Compatibility

Consider a domain size $q \geq 2$, an arity $k \geq 2$, and a predicate $P \subseteq [q]^k$. To understand the "V Label Cover–hardness" of this predicate P, for each edge $e = (u_1, \ldots, u_k)$ of our V Label Cover instance we seek to construct probability distributions on $[q]^{k \times (2k-1)L}$ such that the marginal distribution of each branch of e is supported by P. We define the notion of *V Label Cover–compatibility* to capture exactly what we need.

Definition 4.1. For a predicate $P \subseteq [q]^k$, consider $\mu_1, \ldots, \mu_k$ supported on P^2. For $i, j \in [k]$, let $X_{i,j} \sim [q]^2$ be the marginal distribution of μ_i on the jth coordinates. In other words, for all $(a, b) \in [q]^2$,

$$\Pr_{(x', y') \sim X_{i,j}}[(x', y') = (a, b)] = \Pr_{(x,y) \sim \mu_i}[(x_j, y_j) = (a, b)].$$

We call the distributions $\mu_1, \ldots, \mu_k$ a *V Label Cover–compatible family* if they satisfy the following properties:

(1) For all $i \in [k]$, $X_{i,i}$ is uniform on $\{(a, a) \mid a \in [q]\}$.
(2) For all $i, j \in [k]$ with $i \neq j$ and $X_{i,j}$ is uniform on $[q]^2$.
(3) For all $i \in [k]$, $\rho(\mu_i) < 1$, which we define to be

$$\rho(\mu_i) := \rho(X_{i,1}, \ldots, X_{i,k}).$$

We say that P is *V Label Cover–compatible* if a V Label Cover–compatible family $\mu_1, \ldots, \mu_k$ exists.

The reason we have k different distributions is because the two connected branches can intersect in k different rows (see Figure 1).

Property (3) of Definition 4.1 precludes any algebraic structure in our predicate that would permit a polynomial time algorithm. For example, the uniform distribution on the predicate $\{x \in \mathbb{Z}_2^n \mid x_1 + \cdots + x_n = 0\}$ has correlation 1 and allows for Gaussian elimination to solve exactly.

4.3 Reduction from V Label Cover to P-CSP

Let $P \subseteq [q]^k$ be a predicate for $q, k \geq 2$ that is V Label Cover–compatible with distributions $\mu_1, \ldots, \mu_k$. In this section, we show how to reduce an arbitrary instance of V Label Cover into an instance of P-CSP, the CSP where all clauses are of the form $(x_{i_1}, \ldots, x_{i_k}) \in P$. Furthermore, we assign weights to the clauses of this CSP, in which the weights are determined by these distributions μ_i. This reduction is the starting point for showing the conditional NP-hardness results in Sections 5 and 6.

Let $\Psi = (U, E, L, \{\pi_i^{(e)}\}_{e \in E, i \in [k]}, \{\psi_i^{(e)}\}_{e \in E, i \in [k]})$ be our instance of k-uniform V Label Cover. For each $u \in U$, we construct $q^{(2k-1)L}$ variables $x_s^{(u)}$, where $s \in [q]^{(2k-1)L}$. Now, for every edge $e = (u_1, \ldots, u_k) \in E$ and every $s^{(1)}, \ldots, s^{(k)} \in [q]^{(2k-1)L}$ with the following property:

- For any $t_1, \ldots, t_k \in [(2k-1)L]$ such that $(t_1, \ldots, t_k)$ is a branch of e, we have $(s_{t_1}^{(1)}, \ldots, s_{t_k}^{(k)}) \in P$,

we add the constraint $(x_{s^{(1)}}^{(u_1)}, \ldots, x_{s^{(k)}}^{(u_k)}) \in P$. Looking back at Figure 1, we have that any assignment of values from $[q]$ to the nodes of the schematic such that each branch is an element of P corresponds to some choice $(s^{(1)}, \ldots, s^{(k)})$.

Let Φ be the resulting instance. Although we have described the clauses, we have not yet determined the relative weights of the clauses.

Claim 4.2. If Ψ has a labeling $\sigma : U \to [(2k-1)L]$ that strongly satisfies every hyperedge, then we have that Φ has a perfect satisfying assignment. In other words, this reduction has perfect completeness.

Proof. For each $u \in U$, and $s \in [q]^{(2k-1)L}$, we let $x_s^{(u)} = s_{\sigma(u)}$. One can verify that this assignment satisfies Φ. □

Now, fix $e = (u_1, \ldots, u_k) \in E$. For each $\ell \in [kL]$, let $(a_1, \ldots, a_k), (b_1, \ldots, b_k)$ be the two branches of e such that $\pi_i^{(e)}(a_i) = \pi_i^{(e)}(b_i) = \ell$ for all i. Let $j \in [k]$ be the unique index for which $a_j = b_j$, (i.e., j is the junction). Let I be the index set $I := \{(i, a_i) \mid i \in [k]\} \cup \{(i, b_i) \mid i \in [k]\}$; note that $|I| = 2k-1$. Let $\Omega_\ell^{(e)} \sim [q]^I$ be the probability distribution isomorphic to μ_j such that the marginals $x_1, \ldots, x_k, y_1, \ldots, y_k$ of μ_j correspond to the marginals indexed by $(1, a_1), \ldots, (k, a_k), (1, b_1), \ldots, (k, b_k)$ of $\Omega_\ell^{(e)}$.

Let

$$\nu^{(e)} := \prod_{\ell \in [kL]} \Omega_\ell^{(e)},$$

where the product is over independent distributions. Note that the support of $\nu^{(e)}$ can be identified with $[q]^{[k]\times[(2k-1)L]}$ since each $(i, a_i) \in [k] \times [(2k-1)L]$ is accounted for in some branch. We let $Y_j^{(e,i)}$ be the marginal distribution of coordinate $(i, j) \in [k]\times[(2k-1)L]$ of $\nu^{(e)}$. For any $i \in [k]$ and $\ell \in [kL]$, we let $X_{i,\ell}^{(e)}$ be the marginal distribution on the indices $\{(i, t) \mid \pi_i^{(e)}(t) = \ell\}$. In particular, if i is a junction, the meeting point of the branches, then $Y_t^{(e,i)} = X_{i,\ell}^{(e)}$. Otherwise, $X_{i,\ell}^{(e)}$ is the product of two Y's:

$$X_{i,\ell}^{(e)} = \prod_{t \in (\pi_i^{(e)})^{-1}(\ell)} Y_t^{(e,i)}.$$

This distribution $\nu^{(e)}$ specifies the probability distribution of the clauses corresponding to a particular edge of the Label Cover instance. These probabilities are the relative weights of the clauses in the instance.

5 PERFECT-COMPLETENESS APPROXIMATION RESISTANCE AND MAX-K-CSP$_Q$

A natural question to ask concerning V Label Cover is if it reduces to natural families of predicates that are hard to approximate, even when guaranteed perfect completeness. In the case of imperfect completeness, Austrin and Mossel [2] showed assuming the UGC that if a predicate $P \subseteq [q]^k$, for some finite domain size q, supports a balanced pairwise independent distribution, then P is *approximation resistant.* In other words, for all $\epsilon > 0$, it is NP-hard to distinguish between $1-\epsilon$-satisfiable and $\frac{|P|}{q^k} + \epsilon$-satisfiable P-CSPs. Only a few years later, in a breakthrough by Chan [12], unconditional approximation resistance was shown for any P that supports a balanced pairwise independent subgroup. We hope that establishing a similar conditional results for perfect completeness will spur unconditional results in this domain.

To reduce from V Label Cover, we need a more stringent criteria than merely supporting a balanced pairwise independent distribution. We call these more structured distributions *pairwise independent V Label Cover–compatible.*

Definition 5.1. Let $q \geq 2, k \geq 3$ be parameters. Let $P \subseteq [q]^k$ be a predicate. We say that P is *pairwise independent V Label Cover–compatible* if there exists a V Label Cover–compatible family $\mu_1, \ldots, \mu_k$ supported on P^2 (with marginals $X_{i,j}$, $i, j \in [k]$) with the additional property that

(4) For all $i \in [k]$ and $j \neq j' \in [k]$, we have that $X_{i,j}$ and $X_{i,j'}$ are pairwise independent.

To motivate the definition, one way to view property (4), when combined with properties (1) and (2) of Definition 4.1, is that P does not just support a pairwise independent distribution, but that the distribution can preserve pairwise independence even when conditioning on the value of a coordinate.[11] Assuming the V Label Cover conjecture, this property suffices to establish perfect-completeness approximation resistance if we allow what are known as *folded* predicates.[12] Assume that $[q]$ has a + operator (e.g., addition modulo q). We specify that we may use folded versions of our predicate P to be the predicates

$$a \in [q]^k,\ P^{(a)} := \{(x_1 + a_1, \dots, x_k + a_k) \mid (x_1, \dots, x_k) \in P\}.$$

Each $P^{(a)}$ has the same cardinality, so incorporating these extra predicates can only increase the severity of the hardness of approximation because a lower bound argument can choose to ignore these additional predicates. Thus, more precisely we say that the *family* of predicates $\{P^{(a)} \mid a \in [q]^k\}$ is perfect-completeness approximation resistant. In other words, for every $\epsilon > 0$, it is NP-hard to distinguish whether a CSP with predicates from $\{P^{(a)} \mid a \in [q]^k\}$ is perfectly satisfiable or is $\frac{|P|}{q^k} + \epsilon$ satisfiable.

Theorem 5.1. *Let $P \subseteq [q]^k$ be a predicate that supports a pairwise independent V Label Cover–compatible distribution. Then, assuming the V Label Cover conjecture, we have that the collection of predicates $\{P^{(a)} \mid a \in [q]^k\}$ is perfect-completeness approximation resistant.*

Proof. The high-level structure of our proof is analogous to that of Austrin and Mossel [2]. The proof proceeds in a couple of stages. First, we describe the reduction from a V Label Cover instance to an instance of P-CSP and note that such a reduction preserves perfect completeness. Second, we analyze the soundness of our reduction using Theorem 2.5 to show that if our P-CSP can be well approximated, then our original V Label Cover instance also admits an approximation.

Reduction. Let $\Psi = (U, E, L, \{\pi_i^{(e)}\}_{e \in E, i \in [k]}, \{\psi_i^{(e)}\}_{e \in E, i \in [k]})$ be our instance of k-uniform V Label Cover. Let Φ be the instance of P-CSP guaranteed by the construction in Section 4.3. Let $\nu^{(e)} \in [q]^{[k] \times [(2k-1)L]}$ be the weighting distributions on the clauses corresponding to the hyperedges. Let $\Omega_\ell^{(e)}, X_{i,j}^{(e)}, Y_j^{(e,i)}$ be the marginal distributions described in Section 4.3. By Claim 4.2, our reduction has perfect completeness.

We now modify the CSP Φ into a new CSP Φ' that incorporates folding. For each constraint

$$\left(x_{s^{(1)}}^{(u_1)}, \dots, x_{s^{(k)}}^{(u_k)}\right) \in P$$

and for each $i \in [k]$, let $(s^{(i)})' = s^{(i)} - s_1^{(i)}$ (i.e., subtract $s_1^{(i)}$ from every coordinate). Then, we specify that

$$\left(x_{(s^{(1)})'}^{(u_1)}, \dots, x_{(s^{(k)})'}^{(u_k)}\right) \in P^{(s_1^{(1)}, \dots, s_1^{(k)})}.$$

One may check that this modification preserves perfect completeness.

Soundness. We view an assignment to Φ' as a collection of functions $\mathcal{F} = \{f_u : [q]^{(2k-1)L} \to [q] \mid u \in U\}$, where $f_u(s)$ is the assigned value for x_s^u. Because of our modification to the CSP, we only specify constraints for $f_u(s)$ when $s_1 = q$. Thus, we may assume that each f_u is *folded*. In other words, $f_u(s) + a \equiv f_u(s + (a, \dots, a)) \mod q$ for all $a \in [q]$. One may check that the f_u's satisfy a clause in Φ' if and only if they satisfy the corresponding clause in Φ. Thus, it is equivalent to focus on the f_u's satisfaction of Φ.

[11] The definition permits a slightly broader class of P (i.e., the distribution can change depending on which coordinate is conditioned on), but our applications will construct P of the type specified in the motivation.
[12] This is a standard assumption in the CSP literature (e.g., [2]).

For $a \in [q]$, we let

$$f_u^{(a)}(x) = \begin{cases} 1 & f_u(x) = a \\ 0 & \text{otherwise.} \end{cases}$$

We define the influences and low-degree influences (Definitions 2.3 and 2.4) of the $f_u^{(a)}$'s to be with respect to the uniform distribution.

Let $\Phi(\mathcal{F})$ be the fraction of constraints of Φ satisfied by $\mathcal{F}$, using the weights specified by the $\nu^{(e)}$ distributions. We seek to show for any $\epsilon > 0$ if there exists a $\mathcal{F}$ such that $\Phi(\mathcal{F}) > \frac{|P|}{q^k} + \epsilon$, then there exists $\delta > 0$ and $\sigma : U \to [(2k-1)L]$ such that σ weakly satisfies δ fraction of the constraints of Ψ.

It is evident from the construction that a group of constraints are associated with each $e \in E$. Let $e(\mathcal{F})$ be the fraction of constraints corresponding to ϕ satisfied by $\mathcal{F}$ (that is the measure with respect to $\nu^{(e)}$ of the clauses satisfied by $\mathcal{F}$). We have that

$$\Phi(\mathcal{F}) = \frac{1}{|E|} \sum_{e \in E} e(\mathcal{F}).$$

Thus, if $\Phi(\mathcal{F}) > \frac{|P|}{q^k} + \epsilon$, there exists a subset $E' \subseteq E$ such that $|E'| > (\epsilon/2)|E|$ and $e(\mathcal{F}) \geq \frac{|P|}{q^k} + \epsilon/2$ for all $e \in E'$, as otherwise $\Phi(\mathcal{F}) \leq \epsilon/2 \cdot 1 + (1 - \epsilon/2) \cdot (\frac{|P|}{q^k} + \epsilon/2) < \frac{|P|}{q^k} + \epsilon$.

Fix $e = (u_1, \ldots, u_k) \in E'$. Note that

$$\begin{aligned} e(\mathcal{F}) &= \mathop{\mathbb{E}}_{(s_1,\ldots,s_k)\sim\nu^{(e)}} [(f_{(u_1)}(s_1), \ldots, f_{(u_k)}(s_k)) \in P] \\ &= \sum_{r \in P} \mathop{\mathbb{E}}_{(s_1,\ldots,s_k)\sim\nu^{(e)}} \left[f_{u_1}^{(r_1)}(s_1) \cdots f_{u_k}^{r_k}(s_k) \right]. \end{aligned}$$

Thus, for some $r \in P$, we have that

$$\mathop{\mathbb{E}}_{(s_1,\ldots,s_k)\sim\nu^{(e)}} \left[f_{u_1}^{(r_1)}(s_1) \cdots f_{u_k}^{r_k}(s_k) \right] > \frac{1}{q^k} + \frac{\epsilon}{2|P|}.$$

Let $\epsilon' = \epsilon/(2|P|) > 0$. In addition, for all $i \in [k]$, let $\Pi_i^{(e)} = \prod_{\ell=1}^{kL} X_{i,\ell}^{(e)}$. Since each $\Pi_i^{(e)}$ is uniform and f_{u_i} is folded, we have that

$$\mathop{\mathbb{E}}_{s_i \sim \Pi_i^{(e)}} \left[f_{u_i}^{(r_i)}(s_i) \right] = \frac{1}{q}.$$

In particular, this implies that

$$\left| \prod_{i=1}^{\ell} \mathbb{E}[f_{u_i}^{(r_i)}(s_i)] - \mathbb{E}\left[\prod_{i=1}^{\ell} f_{u_i}^{(r_i)}(s_i) \right] \right| > \epsilon'.$$

Note that $\nu^{(e)} = \Omega_1^{(e)} \times \cdots \times \Omega_{kL}^{(e)}$ meets the requirements of Theorem 2.5. Thus, there exists $\tau, d > 0$, which are functions of only ϵ' and parameters of $|P|$, such that

$$\exists \ell \in [kL], |\left\{ i : \mathrm{Inf}_{X_{i,\ell}^{(e)}}^{\leq d} f_{u_i}^{(r_i)} > \tau \right\}| \geq 3.$$

Let $i_1, i_2, i_3 \in [k]$ be three of these coordinates, and let $\ell \in [kL]$ be the guaranteed value of ℓ. Observe that we can also write $\Pi_{i_a}^{(e)}$ as

$$\Pi_{i_a}^{(e)} = \prod_{t \in [(2k-1)L]} Y_t^{(e, i_a)}.$$

Note that each $X^{(e)}_{i_a,\ell}$ can be written as the product distribution of at most 2 $Y^{(e,i_a)}_t$'s, where $\pi^{(e)}_{i_a}(t) = \ell$. By invoking Lemma 2.4 with $D = 2$, we have that there exists t_1, t_2, t_3 such that $\pi^{(e)}_{i_a}(t_a) = \ell$ for all $a \in \{1,2,3\}$ and

$$\mathrm{Inf}^{\le 2d}_{Y^{(e,i_a)}_{t_a}} f^{(r_i)}_{u_{i_a}} = \mathrm{Inf}^{\le 2d}_{t_a} > \frac{\tau}{2},$$

where the equality is due to the fact that the $Y^{(e,i_a)}_{t_a}$ distributions are uniform distributions on $[q]$.

Note that since each "component" of (e) has two branches, by the Pigeonhole principle, some two of $\{t_1, t_2, t_3\}$ are in the same branch. Thus, any assignment σ for which $\sigma(u_{i_a}) = t_a$ for all $a \in \{1,2,3\}$ weakly satisfies e.

For each $u \in U$. Let $S_u \subseteq [(2k-1)L]$ be the set of labels j for which $\mathrm{Inf}^{\le 2d}_j f^{(a)}_u > \tau/2$ for some $a \in [q]$. Since $\mathrm{Var}\, f^{(a)}_u \le \max(f^{(a)}_u)^2 = 1$, we have by Lemma 2.3 that $|S_u| \le 4dq/\tau$, which is independent of L. Construct a random labeling $\sigma : U \to [(2k-1)L]$ by sampling each $\sigma(u)$ from S_u independently and uniformly at random (if S_u is empty, let $\sigma(u) = 1$). For each $e \in E'$, we established that there exists $i, i' \in [k]$ and $\ell \in S_{u_i}$ and $\ell' \in S_{u_{i'}}$ such that setting $\sigma(u_i) = \ell$ and $\sigma(u_{i'}) = \ell'$ weakly satisfies e. Thus, in expectation at least

$$\frac{|E'|}{|E|} \cdot \frac{1}{(\max |S_u|)^2} = \frac{\tau^2 \epsilon}{16 d^2 q^2} > 0$$

of the edges are weakly satisfied. Note that this expression is independent of L and the size of Ψ, as desired. □

We use this theorem to obtain hardness of approximation results for Max-k-CSP$_q$ when $q \ge 2$ is a prime power. We start with the following combinatorial lemma.

Claim 5.2. Let $q \ge 2$ be a prime power, and let $\ell \ge 1$ be odd. There exists $S \subset \mathbb{F}_q^\ell$ with $|S| = q^{(\ell-1)/2}$ such that S is 3-wise linearly independent over $\mathbb{F}_q$. In other words, each three-element subset of S is linearly independent.

Remark. The proof is substantially simplified from the conference version [10], based on suggestions by Michael Forbes and Sergey Yekhanin.

Proof. Let $\ell' = (\ell-1)/2$. Consider

$$S = \{(1, x, x^2) : x \in \mathbb{F}_{q^{\ell'}}\}.$$

Here, $\mathbb{F}_{q^{\ell'}}$ is identified with $\mathbb{F}_q^{\ell'}$ in the canonical way. Any three distinct vectors are linearly independent because the Vandermonde determinant is nonzero.

$$\det \begin{pmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{pmatrix} = (x-y)(y-z)(z-x).$$ □

Lemma 5.3. *For all $q \ge 2$ a prime power and $k \ge 2$, there exists $P \subseteq [q]^k$ that is pairwise independent V Label Cover–compatible with $|P| = 2k^3q^3$.*

Remark. Because of the recent breakthrough that subsets of $\mathbb{Z}_q^n$ that do not have an arithmetic progress of length three have size at most q^{cn} for some $c < 1$, [15, 20], it is impossible to improve that factor of 1/2 in the exponent of Claim 5.2 to 1 when $q \ge 3$. In particular, Lemma 5.3 can at best be improved to $O_q(k^{2+\gamma})$ for some $\gamma > 0$ (where the O_q notation hides the dependence of q).

Proof. We use a modification of the constructions of Austrin and Mossel [2] and Tamaki and Yoshida [68]. Let $\ell \geq 3$ be the least odd integer such that $q^{(\ell-1)/2} \geq k$. Thus, $q^\ell \leq k^2 q^3$. View $\mathbb{F}_q^\ell$ as a vector space over $\mathbb{F}_q$. By Claim 5.2, there exists $S \subset \mathbb{F}_q^\ell$ with $|S| \geq q^{(\ell-1)/2} \geq k$ such that S is 3-wise linearly independent (i.e., every 3-element subset is linearly independent). Let $v^{(1)}, \ldots, v^{(k)} \in S$ be k distinct elements from this set. Define $\langle \cdot, \cdot \rangle$ to be the canonical bilinear form on $\mathbb{F}_q^\ell$. In other words, $\langle x, y \rangle = \sum_{i=1}^{\ell} x_i y_i$.

We give an initial attempt to construct our predicate. Let[13]

$$P_0 = \{(\langle v^{(1)}, X \rangle, \ldots, \langle v^{(k)}, X \rangle) : X \in \mathbb{F}_q^\ell\}.$$

We have that $|P_0| \leq q^\ell \leq k^2 q^3$. We show that P_0 satisfies properties (1), (2), and (4) of Definitions 4.1 and 5.1. Note that the definition of P_0 defined a natural probability distribution μ. It is clear that μ has uniform marginal distributions (since each $v^{(i)}$ is nonzero and X is uniform). Furthermore, the marginal distributions are 3-wise independent (and thus 3-wise uniform) since the $v^{(i)}$'s are 3-wise linearly independent. (We omit the proof, a similar result for pairwise independence is Lemma 4.2 of Austrin and Mossel [2].)

Now, fix $i \in [k]$, define μ_i to be

$$\mu_i := \{x, y \sim \mu \text{ independent} : x_i = y_i\}.$$

Let $X_{i,j}$ with $j \in [k]$ be the marginals of μ_i. We seek to show that μ_i satisfies properties (1), (2), and (4) of Definitions 4.1 and 5.1. Property (1) follows immediately from the uniform marginals of μ. Now, fix $j \neq i$, since (x_i, x_j) and $(x_i = y_i, y_j)$ are uniform distributions and x_j and y_j are conditionally independent given x_i, we have that

$$\Pr[x_i \wedge x_j \wedge y_j] = \Pr[x_j \wedge y_j | x_i] \Pr[x_i] = \Pr[x_j | x_i] \Pr[y_j | x_i] \Pr[x_i] = \Pr[x_j] \Pr[y_j] \Pr[x_i].$$

Therefore, (x_i, x_j, y_j) is uniform on $\mathbb{F}_q^\ell$. Thus, property (2) and the case $j' = i$ of property (4) follow.

To finish establishing property (4), consider $j \neq j' \in [k] \setminus \{i\}$. We seek to show that $(x_j, x_{j'}, y_j, y_{j'})$ is uniform for which it suffices to show that $(x_i, x_j, x_{j'}, y_j, y_{j'})$ is uniform. Like before,

$$\begin{aligned}\Pr[x_i \wedge x_j \wedge x_{j'} \wedge y_j \wedge y_{j'}] &= \Pr[x_j \wedge x_{j'} | x_i] \Pr[y_j \wedge y_{j'} | x_i] \Pr[x_i] \\ &= \Pr[x_j] \Pr[x_{j'}] \Pr[y_j] \Pr[y_{j'}] \Pr[x_i] \text{ (3-wise independence of } \mu).\end{aligned}$$

Thus, the μ_i's satisfy properties (1), (2), and (4) of Definitions 4.1 and 5.1. Sadly, due to the nice algebraic structure of P_0, we have that $\rho(\mu_i) = 1$ for all i. To rectify this, we create a "noisy" version of P_0. For $x \in \mathbb{F}_q^k$, let $|x|$ be the number of nonzero coordinates of x. Then, we define P to be

$$P := \{x \in \mathbb{F}_q^k \mid \exists y \in P_0, |x - y| \leq 1\}.$$

Note that $|P| \leq (k+1)|P_0| \leq 2k^3 q^3$. Now, modify the μ_i's to get μ_i''s by the following procedure:

(1) Sample $(x, y) \in \mu_i$.
(2) Sample $j \in [k]$ and $a, b \in \mathbb{F}_q$ uniformly.
(3) If $i = j$, set $x_i = y_j = a$. Otherwise, set $x_i = a$ and $y_j = b$.

Clearly the support of μ_i' is P^2. Also μ_i' preserves properties (1), (2), and (4) of Definitions 4.1 and 5.1 of being V Label Cover–compatible since re-randomizing coordinates can only assist in maintaining pairwise independent distributions.

[13] Note that we identify $[q]$ with $\mathbb{F}_q$ in some canonical way.

It remains to show that μ_i' satisfies property (3) of Definition 4.1. The proof of this is similar to that of Lemma 4.6 of Tamaki and Yoshida [68]. Let

$$Z_{i,j} := \prod_{j=1, j\neq i}^{k} X_{i,j}.$$

It suffices to show that $\rho(X_{i,j}, Z_{i,j}) < 1$. To do that, it suffices to show by Lemma 2.2 that the bipartite graph whose edges are the support of $X_{i,j} \times Z_{i,j}$ is connected. For any $(\alpha, \beta) \in X_{i,j} \times Z_{i,j}$, since with nonzero probability the jth coordinate is re-randomized, we have that $(\alpha', \beta) \in X_{i,j} \times Z_{i,j}$ for all α' in the support of $X_{i,j}$. From this connectivity immediately follows.

Therefore, P has the desired properties. □

Using the same proof techniques, we have the following corollary.

Corollary 5.4. *For $q = 2$ and all $k \geq 2$, there exists $P \subseteq [2]^k$ that is pairwise independent V Label Cover–compatible and $|P| = O(k^2)$.*

Proof. Repeat the proof of Lemma 5.3, but note that $S = \{x \in \mathbb{F}_2^\ell : \sum_{i=1}^{\ell} x_i = 1\}$ is a 3-wise-independent subset of size $2^{\ell-1}$. □

Now we may obtain Theorem 1.1.

Proof of Theorem 1.1. The case $q = 2$ follows immediately from Corollary 5.4 and Theorem 5.1. Similarly, if $q \geq 3$ is a prime power, then the result follows from Lemma 5.3 and Theorem 5.1. □

Remark. If q is not a prime power, we cannot invoke the monotonicity result of Austrin and Mossel [2, Cor. B.1], since they crucially assume a lack of perfect completeness. In fact, their reduction does not even produce instances that are near-perfectly satisfiable. If for a general q, we can find a distribution $\mu \sim [q]^k$ whose support is of size poly(q, k), has uniform marginals, and has 3-wise independence, then by Theorem 5.1 we can extend our result to Max-k-CSP$_q$.

This is the first conditional NP-hardness reduction that obtains a soundness of $\frac{\text{poly}(q,k)}{q^k}$ for even one fixed q. Previously, a long code test due to Tamaki and Yoshida [68] obtained $\frac{O(k)}{2^k}$ for when $q = 2$. The currently best known unconditional result for Max-k-CSP$_2$ is $\frac{2^{O(k^{1/3})}}{2^k}$ due to Huang [39]. For $q \geq 3$, the best known result is that of Håstad and Khot [36] and [53] Makarychev and Makarychev.

Remark. Using a modification of the predicate of Tamaki and Yoshida [68], we speculate that it is possible to improve the hardness factor for Boolean Max-k-CSP to $O(k/2^k)$.

6 REDUCTION TO STRONG/RAINBOW HYPERGRAPH COLORING

Recall the notions of strong and rainbow graph coloring [8, 9, 29].

Definition 6.1. Let $H = (V, E)$ be a hypergraph of uniformity $k \geq 2$. Let $q \geq k$ be a positive integer. A function $\chi : V \to [q]$ is a (k, q)*-strong coloring* of H if for all $e \in E$, $\chi \restriction e$ is an injection. In other words, no two vertices in the same hyperedge receive the same color.

Definition 6.2. Let $H = (V, E)$ be a hypergraph of uniformity $k \geq 2$. Let $q \leq k$ be a positive integer. A function $\chi : V \to [q]$ is a (k, q)*-rainbow coloring* of H if for all $e \in E$, $\chi \restriction e$ is a surjection. In other words, for all $e \in E$ and $c \in [q]$, there is $v \in e$ such that $\chi(v) = c$.

Note that the notions of strong and rainbow coloring coincide when $k = q$. In these hypergraphs, we would like to know if we can tractably identify large *weak independent sets.*

Definition 6.3. Let $H = (V, E)$ be a hypergraph. A subset $W \subseteq V$ is a *weak independent set* if for all $e \in E$, $e \cap W \neq e$.

Theorem 6.1. *Assume the induced version of the V Label Cover conjecture (Conjecture 4.1). For all $k \geq 2$, $q > k + \sqrt{k} - \frac{1}{2}$ and $\epsilon > 0$, given a k-uniform hypergraph $H = (V, E)$, it is NP-hard to distinguish between the following two settings:*

- *YES: H admits a (k, q)-strong coloring.*
- *NO: H does not have a weak independent set of density ϵ ($|V|\epsilon$ vertices).*

Theorem 6.2. *Assume the induced version of the V Label Cover conjecture (Conjecture 4.1). For all $k > q \geq 2$ and $\epsilon > 0$, given a k-uniform hypergraph $H = (V, E)$, it is NP-hard to distinguish between the following two settings:*

- *YES: H admits a (k, q)-rainbow coloring.*
- *NO: H does not have a weak independent set of density ϵ ($|V|\epsilon$ vertices).*

We can view strong and rainbow hypergraph coloring as CSPs. In particular, let

$$S_{k,q} = \{(c_1, \ldots, c_k) \in [q]^k \mid \forall i, j \in [k], \text{ if } i \neq j \text{ then } c_i \neq c_j\}$$

be the strong coloring predicate, and let

$$R_{k,q} = \{(c_1, \ldots, c_k) \in [q]^k \mid \forall c \in [q], \exists i \in [k], c = c_i\}$$

be the rainbow coloring predicate.

These predicates have structure that we call *unpredictable*.

6.1 Unpredictable Predicates

In this section, we supplement Definition 4.1 to give our distributions additional properties that we need for our hardness reduction.

Definition 6.4. Let $q, k \geq 2$ be parameters. Let $P \subseteq [q]^k$ be a predicate. We say that P is *unpredictably V Label Cover–compatible* if there exists a V Label Cover–compatible family $\mu_1, \ldots, \mu_k$ supported on P^2 (with marginals $X_{i,j}$, $i, j \in [k]$) with the additional properties that

(4) For all $i \in [k]$ and $1 \leq j < k$, we have that

$$\rho\left(\prod_{\ell=1}^{j} X_{i,\ell}, \prod_{\ell=j+1}^{k} X_{i,\ell}\right) < 1.$$

(5) Each $i \in [k]$ and $j_1, j_2 \in [k] \setminus \{i\}$ with $j_1 \neq j_2$, we have that the marginal distribution of (x_{j_1}, y_{j_2}) in μ_i (recall that x_{j_1} and y_{j_2} are in separate "branches" of μ_i) is uniform over $[q]^2$.

As the properties are rather technical, the following definition helps to streamline our understanding.

Definition 6.5 (c.f., Section 1.4 of Mossel [55]). Let $\Omega = X^{(1)} \times \cdots \times X^{(k)}$ be a probability space. We say that Ω is *connected* if for all atoms (elements with nonzero probability) $x, y \in \Omega$, there exists a sequence $z_0, \ldots, z_n \in \Omega$ of atoms such that $x = z_0, y = z_n$, and z_i and z_{i-1} differ in exactly one of the k coordinates for all $i \in [n]$.

The following lemma demonstrates the utility of connected predicates.

Lemma 6.3. *If P admits a family $\mu_1, \ldots, \mu_k$ (with marginals $X_{i,j}$, $i, j \in [k]$) of probability distributions such that they are connected. Then P satisfies property (3) of Definition 4.1 and property (4) of Definition 6.4.*

Proof. First we verify property (4) of Definition 6.4. Fix $i \in [k]$. It suffices to check for all $1 \le j < k$ that

$$\rho\left(\prod_{\ell=1}^{j} X_{i,\ell}, \prod_{\ell=j+1}^{k} X_{i,\ell}\right) < 1.$$

By Lemma 2.2, it suffices to check that the bipartite graph $G_{i,j} := \prod_{\ell=1}^{j} X_{i,\ell} \times \prod_{\ell=j+1}^{k} X_{i,\ell}$ corresponding to nonzero probability events is connected. Consider any atom $x \in \prod_{\ell=1}^{j} X_{i,\ell}$ and $y \in \prod_{\ell=1}^{j} X_{i,\ell}$. Since x and y are marginals with nonzero probability, there exist atoms $x', y' \in \mu_i$ such that x is a prefix of x' and y is a suffix of y'. Since μ_i is connected, there exists $z_0, \dots, z_n$ such that $z_0 = x', z_n = y'$ and z_i and z_{i-1} differ in exactly one coordinate for all $i \in [n]$. In particular, this implies that each z_i corresponds to an edge of $G_{i,j}$ and consecutive edges share a vertex. Thus, x and y are connected; therefore $G_{i,j}$ is connected. Hence, the μ_i's satisfy property (4) of Definition 6.4.

By essentially the same argument, we can see that the μ_i's satisfy property (3) of Definition 4.1. □

We can apply this lemma to obtain results about the CSPs corresponding to strong and rainbow hypergraph coloring.

Lemma 6.4. *For all $k \ge 2$ and $q > k+\sqrt{k}-\frac{1}{2}$, $S_{k,q}$ is unpredictably V Label Cover–compatible.*

Proof. Since $S_{k,q}$ is a symmetric predicate, it suffices without loss of generality to construct the distribution μ_1. The distribution μ_1 corresponds to the following algorithm:

(1) Pick $m \in \{2k-q-1, \dots, k-1\}$ according to a distribution Ω to be specified.
(2) Pick uniformly at random a partial matching $(a_1, b_1), \dots, (a_m, b_m) \in \{2, \dots, k\}^2$ such that $a_i \neq a_j$ and $b_i \neq b_j$ for all distinct $i, j \in [m]$.
(3) Define $S' \subset S_{k,q}$ to be

$$\begin{aligned} S' := \{((x_1, \dots, x_k), (y_1, \dots, y_k)) \in S_{k,q} \mid x_1 = y_1, \forall a, b \in \{2, \dots, k\},\\ y_a = x_b \text{ iff } \exists i \in [m], (a,b) = (a_i, b_i)\}. \end{aligned}$$

Pick $((x_1, \dots, x_k), (y_1, \dots, y_k)) \sim S'$ uniformly at random.

Our sample from μ_1 is then $((x_1, \dots, x_k), (y_1, \dots, y_k))$. For this to be sensible, we need to verify the following claim.

Claim 6.5. For any choice of $m \in \{2k-q-1, \dots, k-1\}$ and $(a_1, b_1), \dots, (a_m, b_m)$, we have that S' is nonempty.

Proof. By symmetry, we may assume without loss of generality that $a_i = b_i = i+1$ for all $i \in [m]$. Let $x_i = i$ for all $i \in [k]$. Let $y_i = i$ for all $i \in [m]$. For $i \in \{m+1, \dots, k\}$, let $y_i = k+i-m \le k+k-(2k-q) = q$. Thus, $((x_1, \dots, x_n), (y_1, \dots, y_n)) \in S'$, as desired. □

Now, we pick our distribution Ω to satisfy the property guaranteed by the following claim.

Claim 6.6. There is a distribution Ω supported on $\{2k-q+1, \dots, k-1\}$ such that each element of the set has nonzero probability and

$$\mathbb{E}[\Omega] = \frac{(k-1)^2}{q}.$$

Proof. Since $q > k + \sqrt{k} - \frac{1}{2}$, we have that

$$2k - q + 1 < \frac{(k-1)^2}{q} < k - 1. \tag{1}$$

Let X be the probability distribution on Ω that samples $2k - q + 1$ with probability 1. Let Y be the probability distribution on Ω that samples $k-1$ with probability 1. Let Z be the uniform distribution on Ω. By (1), there exists $\epsilon > 0$ such that the mixture $(1-\epsilon)X + \epsilon Z$ (i.e., the probability distribution that samples X with probaiblity $1-\epsilon$ and from Z with probability ϵ) has mean less than $\frac{(k-1)^2}{q}$ and the mixture $(1-\epsilon)Y + \epsilon Z$ has mean greater than $\frac{(k-1)^2}{q}$. Note that both of these distributions have full support.

By an application of the intermediate value theorem, there must be some a mixture of $(1-\epsilon)X + \epsilon Z$ and $(1-\epsilon)Y + \epsilon Z$ on Ω with mean $\frac{(k-1)^2}{q}$ that gives every $m \in \{2k - q + 1, \ldots, k - 1\}$ nonzero probability. □

Since the algorithm is symmetric with respect to the colors, we have that x_1 (and thus also y_1) is chosen uniformly at random. Therefore, μ_1 has property (1) of Definition 4.1. Fix $i, j \in \{2, \ldots, k\}$ (not necessarily distinct). Since our algorithm is symmetric with respect to these pairs (i, j), Claim 6.6 guarantees that $x_j = y_j$ with probability $1/q$. Thus, $x_j \neq y_j$ with probability $(q-1)/q$. These probabilities are consistent with the uniform distribution on $[q]^2$. By the symmetry of the algorithm, we have that once we decide whether $x_j = y_j$ or $x_j \neq y_j$, the coloring is chosen uniformly from the valid options. Thus, (x_j, y_j) is a uniform distribution on $[q]^2$, so μ_1 has property (2) of Definition 4.1 and property (5) of Definition 6.4.

The last thing to verify is that μ_1 is a connected distribution. Let $S''_{k,q} := \{(x, y) \in S^2_{k,q} \mid x_1 = y_1\}$. Note that each element $(x, y) \in S''_{k,q}$ has nonzero probability in μ_1, since there is a nonzero probability that m is chosen and $\{(a_i, b_i) \mid i \in [m]\}$ are drawn in order to equal to $\{(i, j) \in \{2, \ldots, k\}^2 \mid x_i = y_j\}$. Then, since $(x, y) \in S'$, there is a nonzero probability (x, y) is drawn. Thus, to show that μ_1 is connected, it suffices to show that each $(x, y) \in S''_{k,q}$ can reach $((1, 2, \ldots, k), (1, 2, \ldots, k)) \in S''_{k,q}$ by changing pairs (x_i, y_i) while staying in $S''_{k,q}$. This can be done by Algorithm 4.

ALGORITHM 4: Algorithm demonstrating connectivity of $S''_{k,q}$.

> **for** $c \in [k]$ **do**
>   **for** $j \in [k] \setminus \{1\}$ ***with*** $x_j = c$ **do**
>   Set x_j to some color in $[q] \setminus \{x_1, \ldots, x_k\}$
>   **end**
>   **for** $j \in [k] \setminus \{1\}$ ***with*** $y_j = c$ **do**
>   Set y_j to some color in $[q] \setminus \{y_1, \ldots, y_k\}$
>   **end**
>   Set $x_c = y_c = c$
> **end**

In the two internal for loops, the modification is always legal, as we purposely select a color not among those used by the other x_k's. The last line is also legal for $c = 1$ since every other variable has value other than 1. The last line is also legal for $c > 1$ since $x_j, y_j \neq c$ for all $j \in \{2, \ldots, n\}$ and $x_1 = y_1 = 1 \neq c$. Thus, we have that μ_1 is connected.

Thus, by Lemma 6.3, we have that μ_1 is unpredictably V Label Cover–compatible. □

Lemma 6.7. *For all $k \geq 3$, $R_{k,k-1}$ is unpredictably V Label Cover–compatible*

Proof. Again, it suffices to construct μ_1 only. Consider the following distribution. Note that the support of this distribution is a strict subset of $R_{k,q}$, where $q = k - 1$:

(1) Let $(x_2, \ldots, x_k)$ and $(y_2, \ldots, y_k)$ be independently chosen uniformly random permutations of $(1, \ldots, q)$.
(2) Pick $b \in \{0, 1\}$ and $\ell \in \{2, \ldots, k\}$ uniformly at random.
(3) If $b = 0$, set $x_1 = y_1 = x_\ell$ and then recolor x_ℓ uniformly at random (possibly the same color). Otherwise, if $b = 1$, set $x_1 = y_1 = y_\ell$ and recolor y_ℓ uniformly at random.

Like usual, $((x_1, \ldots, x_k), (y_1, \ldots, y_k))$ is our sample from μ_1. It is straightforward to verify that this distribution μ_1 satisfies properties (1) and (2) of Definition 4.1 and property (5) of Definition 5.1. To verify the other properties, by Lemma 6.3, it suffices to show that the support of μ_1 is connected. We do this by demonstrating that everything connects to $\{(1, 1, 2, \ldots, q), (1, 1, 2, \ldots, q)\}$.

First, note that for any $(x, y) \in \mu_1$, we have that (x, y) is connected to $(x', y') \in \mu_1$ such that $(x'_2, \ldots, x'_k)$ and $(y'_2, \ldots, y'_k)$ are permutations of $(1, \ldots, q)$, because by Step (3) we can change the color of either x_ℓ or y_ℓ to make the permutations.

Second, observe that if $(x, y) \in \mu_1$ has the property that $(x_2, \ldots, x_k)$ and $(y_2, \ldots, y_k)$ are permutations of $(1, \ldots, q)$, then the modification (x', y') with $x'_1 = y'_1 = 1$, but otherwise equal to (x, y), is also in the support of μ_1.

Next, we show that if $(x, y) \in \mu_1$ has $x_1 = y_1 = 1$ and $(x_2, \ldots, x_k)$ and $(y_2, \ldots, y_k)$ are permutations of $(1, \ldots, q)$, then for any distinct $i, j \in \{2, \ldots, k\}$, $(x', y') \in \mu_1$ with $x'_j = x_i$ and $x'_i = x_j$, but otherwise equal to (x, y), is connected to (x, y). We do this as follows:

(1) Set $x_1 = y_1 = x_i$.
(2) Set $x_i = x_j$.
(3) Set $x_j = x_1$.
(4) Set $x_1 = 1$.

It is clear a similar result holds for transposing the elements of y instead of the elements of x.

Now, by applying a standard sorting algorithm, we can see that all $(x, y) \in \mu_1$ are connected to

$$((1, 1, 2, \ldots, q), (1, 1, 2, \ldots, q)),$$

as desired. Thus, $R_{k,k-1}$ is unpredictably V Label Cover–compatible. □

6.2 Hardness Results

Now that we know are predicates are unpredictably V Label Cover–compatible, we may proceed with establishing Theorems 6.1 and 6.2.

If Φ is a P-CSP, in which $P \subseteq [q]^k$, define the *underlying k-uniform hypergraph of* Φ to be the k-uniform hypergraph whose vertices are the variables of Φ and those hyperedges are the clauses of Φ.

Theorem 6.8. *Let $P \subseteq [q]^k$ $(q, k \geq 2)$ be a predicate that supports a unpredictably V Label Cover–compatible distribution. Then, assuming the induced version of the V Label Cover conjecture (Conjecture 4.1), for all $\epsilon > 0$, it is NP-hard to distinguish the following for a P-CSP Φ:*

- *YES: Φ is perfectly satisfiable.*
- *NO: The underlying k-uniform hypergraph of Φ does not have an ϵ-density weak independent set.*

Proof. The proof mirrors the structure of Theorem 5.1 and also incorporates some ideas from Dinur et al. [18]. First, we describe the reduction from a V Label Cover instance to an instance of P-CSP and note that such a reduction preserves perfect completeness. Second, we analyze the soundness of our reduction using Theorem 2.6 to show that if the underlying hypergraph of the P-CSP has a large weak independent set, then our original V Label Cover instance also admits an approximation.

Reduction. The reduction is exactly that specified in Section 4.3. This time, we make no modifications for folding. In particular by Claim 4.2, the reduction has perfect completeness.

Soundness. Assume that the underlying hypergraph $H_\Phi = (V_\Phi, E_\Phi)$ has a large weak independent set $I \subset V_\Phi$ with $|I| \geq \epsilon |V_\Phi|$. We view I as a collection of functions $\mathcal{F} = \{f_u : [q]^{(2k-1)L} \to \{0,1\} : u \in U\}$, where $f_u(s) = 1$ if and only if $x_s^{(u)} \in I$. From this, it is clear that

$$\frac{1}{|U|} \sum_{u \in U} \mathbb{E}[f_u] \geq \epsilon,$$

where the expectation is taken over the uniform distribution on $[q]^{(2k-1)L}$. We also define the influences and the low-degree influences of the f_u's with respect to the uniform distribution of $[q]^{(2k-1)L}$. Thus, there exists a subset $U' \subseteq U$ of size $|U'| > (\epsilon/2)|U|$ for which $\mathbb{E}[f_u] > \epsilon/2$ for all $u \in U'$. As otherwise,

$$\frac{1}{|U|} \sum_{u \in U} \mathbb{E}[f_u] \geq \epsilon < \frac{\epsilon}{2}(1) + \left(1 - \frac{\epsilon}{2}\right) < \epsilon.$$

For each $e = (u_1, \ldots, u_k) \in E \cap (U')^k$, since I is a weak independent set of H_Φ,

$$\begin{aligned} 0 &= \mathop{\mathbb{E}}_{(s_1,\ldots,s_k) \sim \nu^{(e)}} \left[x_{s_1}^{(u_1)} \in I \wedge \cdots \wedge x_{s_k}^{(u_k)} \in I\right] \\ &= \mathop{\mathbb{E}}_{(s_1,\ldots,s_k) \sim \nu^{(e)}} [f_{u_1}(s_1) \cdots f_{u_k}(s_k)]. \end{aligned}$$

For all $i \in [k]$, let $\Pi_i^{(e)} = \prod_{\ell=1}^{kL} X_{i,\ell}^{(e)}$. Since each $\Pi_i^{(e)}$ is uniform, $\mathbb{E}_{s_i \sim \Pi_i^{(e)}}[f_{u_i}] > (\epsilon/2)$.

Because P is unpredictably V Label Cover–compatible, $\nu^{(e)} = \Omega_1^{(e)} \times \cdots \times \Omega_{kL}^{(e)}$ meets the requirements of Theorem 2.7. Thus, there exists $\epsilon', \tau, d > 0$, which are functions of only $\epsilon/2$ and parameters of $|P|$, such that there are $i_1, i_2 \in [k]$ and $t_1, t_2 \in [(2k-1)L]$ such that (i_1, t_1) and (i_2, t_2) are in the same branch and

$$\mathrm{Inf}_{Y_{t_a}^{(e,i_a)}}^{\leq d} f_{u_i} = \mathrm{Inf}_{t_a}^{\leq d} f_{u_i} > \tau.$$

For each $u \in U'$. Let $S_u \subseteq [(2k-1)L]$ be the set of labels j for which $\mathrm{Inf}_j^{\leq d} f_u > \tau$. Since $\mathrm{Var}\, f_u \leq \max(f_u)^2 = 1$, we have by Lemma 2.3 that $|S_u| \leq d/\tau$, which is independent of L. Construct a random partial labeling $\sigma : U' \to [(2k-1)L]$ by sampling each $\sigma(u)$ from S_u independently and uniformly at random (if S_u is empty, let $\sigma(u) = 1$). For each $e \in E \cap (U')^k$, we established that there exists $i, i' \in [k]$ and $\ell \in S_{u_i}$ and $\ell' \in S_{u_{i'}}$ such that setting $\sigma(u_i) = \ell$ and $\sigma(u_{i'}) = \ell'$ weakly satisfies e. Thus, inside U' expectation at least

$$\frac{1}{(\max |S_u|)^2} = \frac{\tau^2}{d^2} > 0$$

of the edges are weakly satisfied. Note that this expression is independent of L and the size of Ψ, as desired. □

Note that Theorem 6.1 follows as a corollary of Theorem 6.8 combined with Lemma 6.4.

Proof of Theorem 6.2. Theorem 6.8 and Lemma 6.7 imply the case $q = k-1$. For $q < k-1$, one can see that a $(k, k-1)$-rainbow colorable hypergraph is also a (k, q)-rainbow colorable hypergraph since we can "merge" colors together while preserving the rainbow property. Therefore, since the V Label Cover–conjecture implies for $\epsilon > 0$, it is NP-hard to distinguish $(k, k-1)$-rainbow colorable hypergraphs from graphs without an ϵ-density independent set, then for any $q \leq k-1$ it must be NP-hard to distinguish (k, q)-rainbow colorable hypergraphs from graphs without an ϵ-density independent set. □

Theorems 6.1 and 6.2 together imply Theorem 1.2.

APPENDIX

A PROOF OF THEOREM 2.7

Recall the statement of Theorem 2.7.

Theorem A.1 (Theorem 2.7). *Fix $k \geq 2$. For $1 \leq \ell \leq n$, let $\Omega_\ell = X_\ell^{(1)} \times \cdots \times X_\ell^{(k)}$ be a finite probability space with distributions μ_ℓ such that the μ_ℓ's are independent. In addition, assume that for each $\ell \in [n]$ and $i \in [k]$, $X_\ell^{(i)} = \prod_{s=1}^{s_\ell^{(i)}} Y_{\ell,s}^{(i)}$, where the product is of otherwise independent distributions and $s_\ell^{(i)} \leq 2$ for all $i \in [k]$ and $\ell \in [n]$. Assume that we also have the following key property:*

- *If for distinct $i_1, i_2 \in [k]$ we have that $s_\ell^{(i_1)} = s_\ell^{(i_2)} = 2$, then $Y_{\ell,1}^{(i_1)}$ is independent of $Y_{\ell,2}^{(i_2)}$ (and $Y_{\ell,2}^{(i_2)}$ is independent of $Y_{\ell,1}^{(i_1)}$ by symmetry).*

For convenience of notation, if $s_\ell^{(i)} = 1$, let $Y_{\ell,2}^{(i)} := Y_{\ell,1}^{(i)}$. Let δ be the minimum positive probability among all the μ_ℓ's, $\ell \in [n]$. Let

$$\rho = \max\left\{\max_{1\leq \ell\leq n} \rho(X_\ell^{(1)}, \ldots, X_\ell^{(k)}),\ \max_{\substack{1\leq \ell\leq n\\ 1\leq j<k}} \rho\left(\prod_{\ell=1}^{j} X_\ell^{(i)}, \prod_{\ell=j+1}^{k} X_\ell^{(i)}\right)\right\},$$

and assume that $\rho < 1$. For every $\epsilon > 0$, there exists $\epsilon'(\delta, \epsilon, \rho), \tau(\delta, \epsilon, \rho), d(\delta, \epsilon, \rho) > 0$ such that for any functions $f_1, \ldots, f_k$ where $f_i : X_1^{(i)} \times \cdots \times X_n^{(i)} \to [0, 1]$ and $\mathbb{E}[f_i] \geq \epsilon$ if

$$\forall \ell \in [n], \forall s \in \{1, 2\}, |\{i \mid \mathrm{Inf}_{Y_{\ell,s}^{(i)}}^{\leq d} f_i \geq \tau\}| \leq 1$$

then

$$\mathbb{E}\left[\prod_{i=1}^{k} f_i\right] \geq \epsilon'.$$

Proof. The proof of this theorem follows a similar structure to the proof of Theorem 3.11 of Dinur et al. [18]. Let ϵ_0', τ_0 be the values guaranteed by Theorem 2.6 for parameters ρ, ϵ, δ. For each $i \in [k]$, we define

$$\Sigma^{(i)} = \left[|X_1^{(i)}|\right] \times \cdots \times \left[|X_n^{(i)}|\right]$$
$$(\Sigma^{(i)})' = \left[|Y_{1,1}^{(i)}|\right] \times \left[|Y_{1,2}^{(i)}|\right] \times \cdots \times \left[|Y_{n,2}^{(i)}|\right]$$
$$\forall \ell \in [n], \alpha_1^{(i,\ell)}, \ldots, \alpha_{|X_\ell^{(i)}|}^{(i,\ell)} : X_\ell^{(i)} \to \mathbb{R} \text{ orthonormal basis with } \alpha_1^{(i,\ell)} \equiv 1$$
$$\forall (\ell, s) \in [n] \times [2], \beta_1^{(i,\ell,s)}, \ldots, \beta_{|Y_{\ell,s}^{(i)}|}^{(i,\ell,s)} : Y_{\ell,s}^{(i)} \to \mathbb{R} \text{ orthonormal basis with } \beta_1^{(i,\ell,s)} \equiv 1.$$

We also require that α's and β's are consistent in the following sense. Since $X_\ell^{(i)} = Y_{\ell,1}^{(i)} \times \cdots \times Y_{\ell,s_\ell^{(i)}}^{(i)}$, we have that

$$\beta_{j_1}^{(i,\ell,1)} \cdots \beta_{j_k}^{(i,\ell,s_\ell^{(i)})}$$

is an orthonormal basis of the functions from $X_\ell^{(i)}$ to $\mathbb{R}$, where $(j_1, \ldots, j_{s_\ell^{(i)}}) \in [|Y_{\ell,1}^{(i)}|] \times \cdots \times [|Y_{\ell,s_\ell^{(i)}}^{(i)}|]$. Since $\beta_1^{(i,\ell,1)} \cdots \beta_1^{(i,\ell,s_\ell^{(i)})} = 1$, we may assume that the $\alpha_j^{(i,\ell)}$'s are some enumeration of this basis.

We define the Fourier coefficients of the f_i's (see Definition 2.4 for notation) to be

$$\begin{aligned} f_i &:= \sum_{\sigma \in \Sigma^{(i)}} c_\sigma^{(i)} \prod_{\ell=1}^n \alpha_{\sigma_\ell}^{(i,\ell)} && (X_\ell^{(i)} \text{ marginals}) \\ &= \sum_{\sigma' \in (\Sigma^{(i)})'} c_{\sigma'}^{(i)} \prod_{\ell=1}^n \prod_{s=1}^{s_\ell^{(i)}} \beta_{\sigma'_{\ell,s}}^{(i,\ell,s)} && (Y_{\ell,s}^{(i)} \text{ marginals}), \end{aligned}$$

where $c_{\sigma'}^{(i)} := c_\sigma^{(i)}$ if $\prod_{\ell=1}^n \alpha_{\sigma_\ell}^{(i,\ell)} = \prod_{\ell=1}^n \prod_{s=1}^{s_\ell^{(i)}} \beta_{\sigma'_{\ell,s}}^{(i,\ell,s)}$.

Denote $|\sigma| = \{\ell \in [n] \mid \sigma_\ell \neq 1\}$ for $\sigma \in \Sigma^{(i)}$. For $\sigma' \in (\Sigma^{(i)})'$, we denote $|\sigma'| = \{\ell \in [n] \mid s \in [s_\ell^i], \sigma_{i,s} \neq 1\}$. A key property is that if $\prod_{\ell=1}^n \alpha_{\sigma_\ell}^{(i,\ell)} = \prod_{\ell=1}^n \prod_{s=1}^{s_\ell^{(i)}} \beta_{\sigma'_{\ell,s}}^{(i,\ell,s)}$, then $|\sigma| = |\sigma'|$.

To not be concerned with low-degree influences, we first replace each f_i, with a noised version $T_\eta^X f_i$,[14] which is defined in terms of Fourier coefficients to be

$$T_\eta^X f_i := \sum_{\sigma \in \Sigma^{(i)}} \eta^{|\sigma|} c_\sigma^{(i)} \prod_{\ell=1}^n \alpha_{\sigma_\ell}^{(i,\ell)}.$$

Note that this noise operator is applied to the $X_\ell^{(i)}$ marginals. Rewriting this in terms of the $Y_{\ell,s}^{(i)}$ basis,

$$T_\eta^X f_i = \sum_{\sigma' \in (\Sigma^{(i)})'} \eta^{|\sigma'|} c_{\sigma'}^{(i)} \prod_{\ell=1}^n \prod_{s=1}^{s_\ell^{(i)}} \beta_{\sigma'_{\ell,s}}^{(i,\ell,s)}.$$

Since the range of each f_i is a subset of $[0,1]$, it is well known that $T_\eta^X f_i$'s range is also a subset of $[0,1]$ (e.g., Definition 8.28 of O'Donnell [58]).

Let $\epsilon_1 = \epsilon_0'/(4k) > 0$. Since ρ, our correlation, is bounded away from 1, by Lemma 6.2 of Mossel [55], there exists $\eta < 1$ such that

$$\begin{aligned} \left| \mathbb{E}\left[\prod_{i=1}^k f_i\right] - \mathbb{E}\left[\prod_{i=1}^k (T_\eta^X f_i)\right] \right| &\le \epsilon_1 \sum_{i=1}^k \sqrt{\mathrm{Var}[f_i]} \sqrt{\mathrm{Var}\left[\prod_{j<i} T_\eta^X f_j \prod_{j>i} f_j\right]} \\ &\le \epsilon_1 k = \frac{\epsilon_0'}{4}. \end{aligned} \tag{2}$$

The second inequality follows from the fact that range for both f_i and $\left(\prod_{j<i} T_\eta^X f_j \prod_{j>i} f_j\right)$ are inside $[0,1]$, so their variances are bounded by 1. Let $g_i := T_\eta^X f_i$ for all i. Note that $\mathbb{E}[g_i] = \mathbb{E}[f_i] \ge \epsilon$ and $\mathrm{Var}[g_i] \le \mathrm{Var}[f_i] \le 1$. From (2), it suffices to give a lower bound on $\mathbb{E}\left[\prod_{i=1}^k g_i\right]$.

Similar to Dinur et al. [18], we have $d \in \mathbb{N}$ such that $2^8(d+1)\eta^d < (\epsilon_0')^2 \tau_0$. Also fix

$$\tau = \frac{\tau_0 (\epsilon_0')^2}{2^8 d k^3} < \frac{\tau_0}{4}.$$

Thus, $\eta^d < \tau$. We need the following quantitative bound.

[14] We add a superscript X to signify that the noise operator is with respect to the $X_1^{(i)} \cdots \times \cdots X_n^{(i)}$ basis.

Claim A.2. For all $i \in [k]$ and $(\ell, s) \in [n] \times [2]$, we have that

$$\mathrm{Inf}_{Y^{(i)}_{\ell,s}} g_i \le \mathrm{Inf}^{\le d}_{Y^{(i)}_{\ell,s}} f_i + \tau.$$

Proof. We have that

$$\begin{aligned}
\mathrm{Inf}_{Y^{(i)}_{\ell,s}} g_i &= \sum_{\substack{\sigma' \in (\Sigma^{(i)})' \\ \sigma'_{\ell,s} \neq 1}} \eta^{2|\sigma'|} \left(c^{(i)}_{\sigma'}\right)^2 \\
&\le \sum_{\substack{\sigma' \in (\Sigma^{(i)})' \\ \sigma'_{\ell,s} \neq 1}} \eta^{|\{(\ell',s') | \sigma'_{\ell',s'} \neq 1\}|} \left(c^{(i)}_{\sigma'}\right)^2 \qquad \left(= \left\langle f_i, T^Y_\eta f_i \right\rangle\right) \\
&\le \mathrm{Inf}^{\le d}_{Y^{(i)}_{\ell,s}} f_i + \eta^d \,\mathrm{Var}[f_i] \\
&\le \mathrm{Inf}^{\le d}_{Y^{(i)}_{\ell,s}} f_i + \tau,
\end{aligned}$$

as desired. □

Now consider

$$B := \{\ell \in [n] \mid \exists s \in \{1,2\}, \exists i \in [k], \mathrm{Inf}_{Y^{(i)}_{\ell,s}} g_i \ge \tau_0/2\}.$$

(This is analogous to the "B" in the proof of Theorem 3.11 of Dinur et al. [18].) Then note that for all $\ell \in B$, there exists $s \in [2]$ and $i \in [k]$ such that

$$\frac{\tau_0}{2} \le \mathrm{Inf}_{Y^{(i)}_{\ell,s}} g_i \le \mathrm{Inf}^{\le d}_{Y^{(i)}_{\ell,s}} f_i + \tau,$$

where the inequality follows from Claim A.2. Since $\tau < \tau_0/4$, we have that $\mathrm{Inf}^{\le d}_{Y^{(i)}_{\ell,s}} f_i > \tau_0/4$. Therefore, by Lemma 2.3, we have that each $i \in [k]$ has at most $4d/\tau_0$ marginals $Y^{(i)}_{\ell,s}$ with $\mathrm{Inf}_{Y^{(i)}_{\ell,s}} g_i \ge \tau_0/2$. Thus, $|B| \le \frac{4kd}{\tau_0}$.

Now, we show that we can "smooth out" these high-influence coordinates without substantially changing the product of the g_i's. For each $i \in [k]$, define

$$\begin{aligned}
B^{\mathrm{lo}}_i &:= \{(\ell, s) \in B \times [2] \mid \mathrm{Inf}_{Y^{(i)}_{\ell,s}} g_i \le 2\tau\} \\
B^{\mathrm{hi}}_i &:= \{(\ell, s) \in B \times [2] \mid \mathrm{Inf}_{Y^{(i)}_{\ell,s}} g_i > 2\tau\}.
\end{aligned}$$

For each $i \in [k]$, we define $g'_i, g''_i : X^{(i)}_1 \times \cdots \times X^{(i)}_n$ to be

$$\begin{aligned}
g'_i &:= \mathop{\mathbb{E}}_{\prod_{(\ell,s)\in B^{\mathrm{lo}}_i} Y^{(i)}_{\ell,s}} [g_i] \\
g''_i &:= \mathop{\mathbb{E}}_{\prod_{(\ell,s)\in B^{\mathrm{hi}}_i} Y^{(i)}_{\ell,s}} [g'_i].
\end{aligned}$$

(c.f., the "averaging operator" in Section 2.1 of Dinur et al. [18]). In other words, in g'_i, we average out all of the low-influence marginals in the blocks $\Omega_\ell = X^{(1)}_\ell \times \cdots \times X^{(k)}_\ell$ that contain a high-influence marginal. In g''_i, we then average out the remaining marginals. For each $i \in [k]$, we average out at most $2|B| \le \frac{4kd}{\tau_0}$ coordinates, so we have that

$$\mathrm{Var}[g_i - g'_i] \le \sum_{(\ell,s)\in B^{\mathrm{lo}}_i} \mathrm{Inf}_{Y^{(i)}_{\ell,s}} g_i \le 2|B|(2\tau) = \frac{16kd\tau}{\tau_0} \le \frac{(\epsilon'_0)^2}{16k^2}.$$

Thus, by Cauchy-Schwartz,

$$\left|\mathbb{E}\left[\prod_{i=1}^{k} g_i - \prod_{i=1}^{k} g_i'\right]\right| \leq \sum_{i=1}^{k} \sqrt{\mathrm{Var}[g_i - g_i']}\sqrt{\mathrm{Var}\left[\prod_{j<i} g_j' \prod_{j>i} g_j\right]}$$
$$\leq \sum_{i=1}^{k} \frac{\epsilon_0'}{4k} = \frac{\epsilon_0'}{4}. \tag{3}$$

Now, we claim the (rather remarkable) fact that

$$\mathbb{E}\left[\prod_{i=1}^{k} g_i'\right] = \mathbb{E}\left[\prod_{i=1}^{k} g_i''\right]. \tag{4}$$

First recall the assumption that for all $(\ell, s) \in [n] \times [2]$, we have that

$$\left|\left\{i \mid \mathrm{Inf}^{\leq d}_{Y^{(i)}_{\ell,s}} f_i \geq \tau\right\}\right| \leq 1. \tag{5}$$

By Claim A.2, if $(\ell, s) \in B_i^{\mathrm{hi}}$ for some $i \in [k]$, then $\mathrm{Inf}^{\leq d}_{Y^{(i)}_{\ell,s}} f_i \geq \tau$. Thus, each $(\ell, s) \in [n] \times [2]$ is in at most one B_i^{hi}. Consider the set of marginal distributions

$$S := \left\{Y^{(i)}_{\ell,s} \mid i \in [k], (\ell, s) \in [n] \times [2] \text{ s.t. } (\ell, s) \in B_i^{\mathrm{hi}}\right\}.$$

We claim that this set of marginal distributions is independent. Since $\Omega_1, \ldots, \Omega_n$ are independent, it suffices to check independence of each subset

$$S \supset S_\ell := \{Y^{(i)}_{\ell,s} \mid i \in [k], s \in [2] \text{ s.t. } (\ell, s) \in B_i^{\mathrm{hi}}\}$$

for all $\ell \in [n]$. Note that $|S_\ell| \leq 2$ for all ℓ because of (5). If $|S_\ell| = 1$, we are immediately done. If $Y^{(i)}_{\ell,s} \in S_\ell$ in which $s^{(i)}_\ell = 1$, then recall that $Y^{(i)}_{\ell,1} = Y^{(i)}_{\ell,2}$, so $(\ell, 1), (\ell, 2) \in B_i^{\mathrm{hi}}$. Thus, there cannot be any more elements of S_ℓ besides $Y^{(i)}_{\ell,1}$. Thus, $|S_\ell| = 1$, so S_ℓ is vacuously a set of independent random variables. In the last case, we have $|S_\ell| = 2$ and $Y^{(i_1)}_{\ell,1}, Y^{(i_2)}_{\ell,2} \in S_\ell$ have the property that $s^{(i_1)}_\ell = s^{(i_2)}_\ell = 2$. Then, by our assumption on the marginal distributions, we have that $Y^{(i_1)}_{\ell,1}$ and $Y^{(i_2)}_{\ell,2}$ are independent. Thus, S_ℓ is independent for all $\ell \in [n]$. Therefore, S is independent.

Let $Z = \prod_{\ell \in [n] \setminus B} \Omega_\ell$. It is easy to see that $S \cup Z$ is independent. Note that $\prod_{i=1}^{k} g_i'$ is a function of only $Z \times \prod_{Y \in S} Y$ and that $\prod_{i=1}^{k} g_i''$ is a function of only Z. Thus,

$$\mathop{\mathbb{E}}_{Z \times \prod_{Y \in S} Y}\left[\prod_{i=1}^{k} g_i'\right] = \mathop{\mathbb{E}}_{Z}\left[\mathop{\mathbb{E}}_{\prod_{Y \in S} Y}\left[\prod_{i=1}^{k} g_i'\right]\right] \text{ (independence)}$$
$$= \mathop{\mathbb{E}}_{Z}\left[\left[\prod_{i=1}^{k} \mathop{\mathbb{E}}_{\prod_{Y^{(i)} \in S} Y^{(i)}} g_i'\right]\right] \text{ (magic: independence)}$$
$$= \mathop{\mathbb{E}}_{Z}\left[\prod_{i=1}^{k} g_i''\right].$$

The second equality follows from the fact that each $Y^{(i)}_{\ell,s} \in S$ only affects the value of g_i'. Thus, (4) holds. Now, note that for all $i \in [k]$ and $\ell \in [n]$, we have that

$$\mathrm{Inf}_{X^{(i)}_\ell} g_i'' < \tau_0,$$

as otherwise by Lemma 2.4 (where $d = n$), we would have that there exists $s \in [2]$ for which $\mathrm{Inf}_{Y^{(i)}_{\ell,s}} g_i'' \geq \tau_0/2 > 2\tau$, but such coordinates were averaged out from g_i from construction of g_i''. Thus, we may invoke Theorem 2.6 to obtain that

$$\mathbb{E}\left[\prod_{i=1}^{k} g_i''\right] \geq \epsilon_0'. \tag{6}$$

Thus, by combining (2–4, 6), we have that

$$\mathbb{E}\left[\prod_{i=1}^{k} f_i\right] \geq \frac{\epsilon_0'}{2}.$$

Therefore, we may set $\epsilon' = \epsilon_0'/2 > 0$. □

Remark. Unlike Dinur et al. [18], we do not require that the distributions are symmetric.

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