Well-Quasi-Ordering, Overview and its Applications

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This talk shows theorems, techniques, and applications of *well-quasi-ordering* (WQO).

First, we introduce what is WQO, and Kruskal-type theorems on various data structures [31, 15, 16, 23, 34, 33]. The basic proof technique, *minimal bad sequence* (MBS), will be introduced for the simplest case, *Higman's Lemma*. Then, WQO will be extended to *better-quasi-ordering* (BQO) for infinite data structures [31, 25, 24, 26, 18, 36, 39]. The variation with the *gap-condition* will be also mentioned [37, 17, 12, 13, 40].

Second, the relation between WQO and regularity will be discussed. The basic fact is that a set of finite words is regular if, and only if, it is upward-closed wrt some WQO on finite words [8]. Its variations are also mentioned [14, 5, 28].

Based on these theoretical results, we overview applications of WQO techniques:

1. simple termination in rewriting [6], its extensions [30, 19], and automation [7],
2. upper bound of complexity [9, 4], including some instance of algorithm generation [27],
3. well structured transition system [1, 10, 3, 2].

Simple termination is one of well-known applications of WQO. It reduces termination to syntactical conditions, and validates well-foundedness of the family of path orderings (such as lexicographical path ordering). This also gives a reasonably large class in which termination can be automatically detected. For instance, termination of Ackerman function can be automatically proved. These techniques are also used to show termination of partial evaluation [38, 20, 21].

For a quite broad class of problems, the upper bound of complexity can be estimated by finiteness of *forbidden minors*. For instance, a known algorithm of the k-searcher problem remains at $O(n^{2k^2+4k+8})$, whereas WQO techniques shows the existence of a linear time algorithm [35, 29]. Note that this *does not* give any sight on how to construct such a linear time algorithm. This strange phenomena occurs from highly non-constructive nature of MBS. When a Kruskal-type theorem has a constructive proof, one may be able to construct an efficient algorithm. For instance, constructive proofs of Higman's Lemma are quite well-investigated [22, 32, 11] (supplied even in Coq theorem prover library!), and an open problem in relational database is solved by automatic generation [28].

Model checking is one of recent hot topics. Usually, decidability of model checking relies on finiteness of a model space. WQO sometimes guarantees decidability of model checking on an infinite space. This is known as *well-structured transition system*, and we will conclude the talk with its overview.

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