# Solving Parity Games on Integer Vectors

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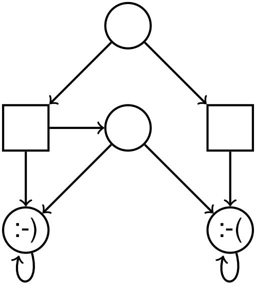
**Highlights - 21st september 2013**

# Finite-state parity games

Player 0

Player 1




- Finite number of control states
- Every state belongs to Player 0 or Player 1
- Colors (in $\{1, \ldots, k\}$) associated to each state
- **Parity winning condition:** Player 0 wins iff the highest color seen infinitely often is even

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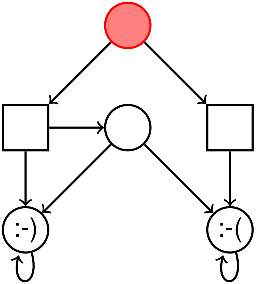


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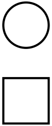

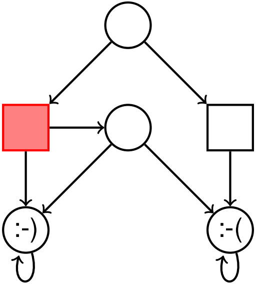


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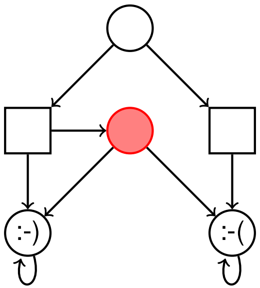


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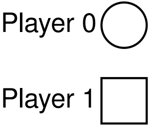


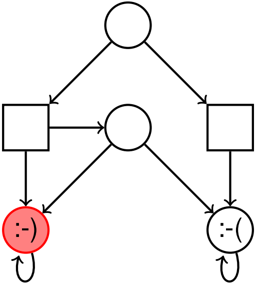


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# Integer vector games

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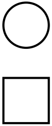
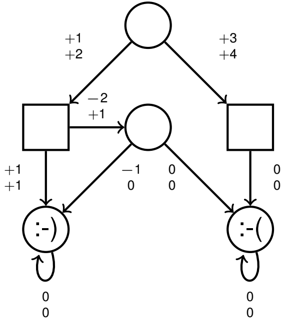
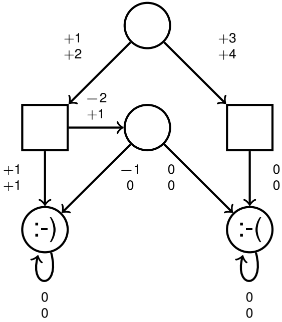





- Adding counters $C_1, \ldots, C_n$ to the game
- Transitions can decrement and increment the counter values
- Configurations are pairs $(q, \mathbf{v})$ with:
  - $q$ : control state
  - $\mathbf{v} \in \mathbb{Z}^n$ : values for the counters

**Which role play the counters in the winning condition and in the enabledness of transitions ?**

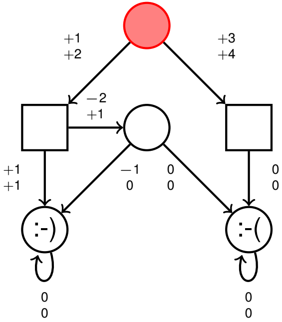
# Energy semantics




- Transitions are always enabled regardless of the counter values
- If the value of one counter drops below 0, Player 1 wins
- Player 0 wins if no counter drops below 0 and if the parity condition is respected
- Higher value of the counters are always better for Player 0

# Energy semantics

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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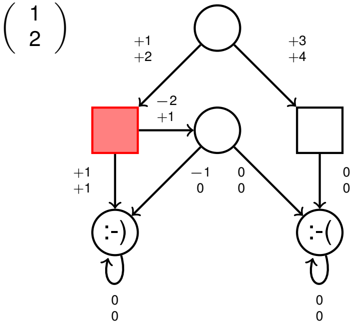


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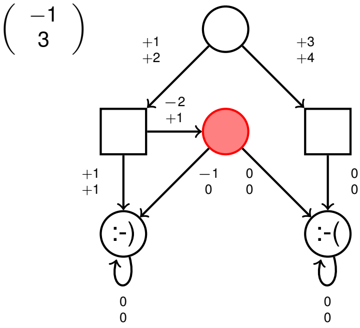


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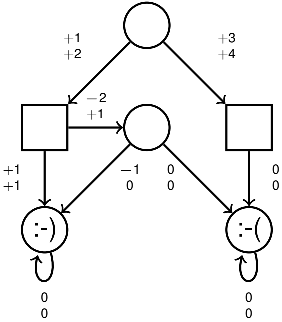


- Configurations : $(q, \mathbf{v})$ with $\mathbf{v} \in \mathbb{N}^n$
- Transitions that make a counter drop below 0 are disabled
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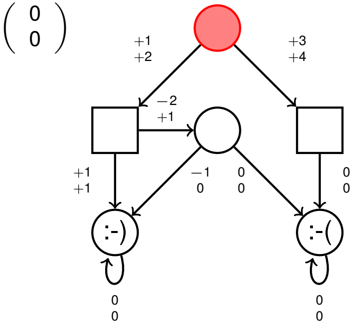


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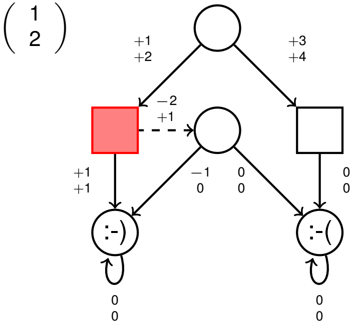


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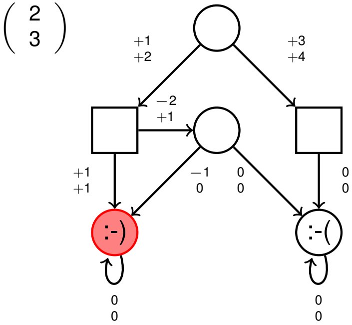


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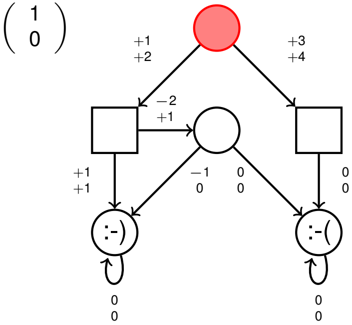


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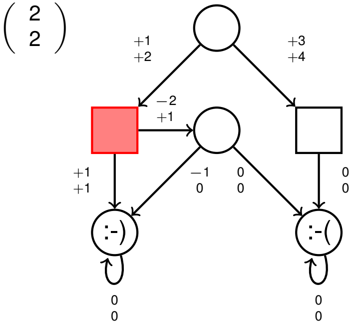


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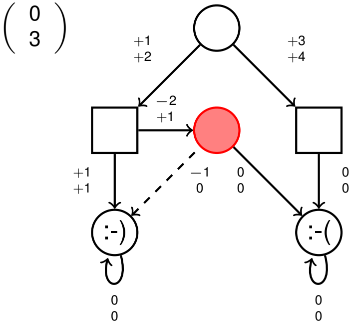


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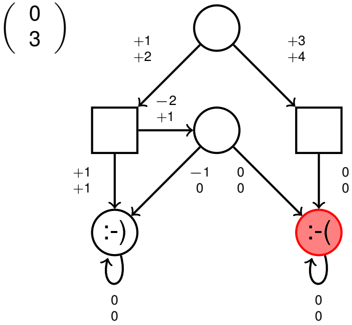


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# Problems

For $\mathtt{I} \in \{\mathtt{Energy}, \mathtt{VASS}\}$ and a game $\mathcal{G}$:

- $\mathtt{Win}(\mathcal{G}, \mathtt{I}) = \{(q, \mathbf{v}) \in Q \times \mathbb{N}^n \mid$ Player 0 has a winning strategy from $(q, \mathbf{v})\}$

## Unknown initial credit problem

- **Input:** A game $\mathcal{G}$ and a semantic $\mathtt{I} \in \{\mathtt{Energy}, \mathtt{VASS}\}$
- **Output:** Is $\mathtt{Win}(\mathcal{G}, \mathtt{I})$ not empty ?

## Fixed initial credit problem

- **Input:** A game $\mathcal{G}$,a semantic $\mathtt{I} \in \{\mathtt{Energy}, \mathtt{VASS}\}$ and a configuration $(q, \mathbf{v})$
- **Output:** Do we have $(q, \mathbf{v}) \in \mathtt{Win}(\mathcal{G}, \mathtt{I})$?

## Computing the winning set

- **Input:** A game $\mathcal{G}$ and a semantic $\mathtt{I} \in \{\mathtt{Energy}, \mathtt{VASS}\}$
- **Output:** Can we compute (and represent finitely) $\mathtt{Win}(\mathcal{G}, \mathtt{I})$ ?

# Previous results

**Theorem** [Chaterjee et al., Concur'12]

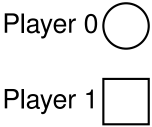
The unknown initial credit problem is $\mathtt{coNP}$-complete for energy games.

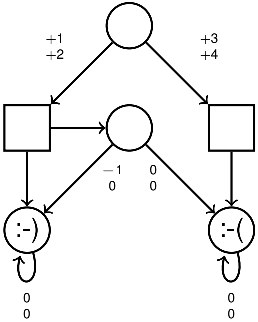
**Theorem** [Abdulla et al., CSL'03]

The fixed initial credit problem is undecidable for VASS games (even with reachability objectives).

# Single sided games

- Player 1 cannot change the counter values only the control states




**Theorem** [Raskin et al.,AVoCS'04]

The fixed initial credit problem is decidable for single-sided VASS games with reachability objectives.

# Upward-closed winning sets

## Upward-closed set

A set $S \in Q \times \mathbb{N}^n$ is upward-closed iff for $(q, \mathbf{v}) \in S$ and all $\mathbf{v}$' $\in \mathbb{N}^n$, $\mathbf{v} \leq \mathbf{v}'$ implies $(q, \mathbf{v}') \in S$.

## Dickson's Lemma

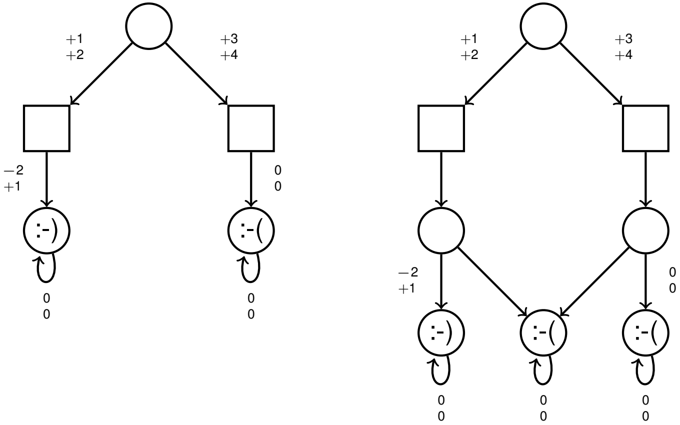
Upward-closed set of $Q \times \mathbb{N}^n$ have a finite number of minimal elements.

- To represent an upward-closed set it is hence enough to store its minimal elements

## Proposition

For energy games and single-sided VASS games, the winning sets are upward closed.

# From energy games to single-sided games




**Proposition**

Energy games and single-sided VASS games are PTIME inter-reducible.

# Results

**Theorem**

For single-sided VASS games, the minimal elements of the winning sets are computable.

**Corollary**

For energy games, the minimal elements of the winning sets are computable.

Hence, for single-sided VASS games and energy games we can solve:

- The unknown initial credit problem
- The fixed initial credit problem

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- This fragment of $\mu$-calculus is incomparable with EF,EG,CTL, and can express non-trivial properties.

# Conclusion

## What have we done ?

- The winning set of single-sided VASS games can be computed
- Energy games and single-sided VASS games are interreducible
- Our decidability result can be used for model-checking VASS

## What's next ?

- Use single-sided VASS to verify more complex systems
- Extend our result with other games (imperfect information for instance)
- Extend our result to stochastic games