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A computable extension for D-finite functions: DD-finite functions ☆

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ABSTRACT

Differentiably finite (D-finite) formal power series form a large class of useful functions for which a variety of symbolic algorithms exists. Among these methods are several closure properties that can be carried out automatically. We introduce a natural extension of these functions to a larger class of computable objects for which we prove closure properties. These are again algorithmic. This extension can be iterated constructively preserving the closure properties.

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1. Introduction

During the past decades, the symbolic treatment of functions satisfying ordinary linear differential equations with polynomial coefficients has been widely investigated. A formal power series $\sum f_n x^n$ with this property is called holonomic or *differentiably finite* or in short D-finite (Kauers and Paule, 2011; Stanley, 1980, 1999). Such (formal) power series are interesting for different reasons: many special functions (Andrews et al., 1999; Rainville, 1971) are of this type as well as the generating functions of many combinatorial sequences arising in applications; it is well known that D-finite functions are closed under operations such as addition, multiplication, algebraic substitution, etc.; furthermore, for this large class several algorithms have been developed and implemented in various computer algebra systems (Chyzak, 1998; Koutschan, 2009; Kauers et al., 2014). For the latter, a necessary key feature is the finite representation of these objects. In order to specify a D-finite function only the

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order of the equation, the polynomial coefficients, and sufficiently many initial values need to be stored. Given functions in this form, a defining differential equation of the same type (plus initial values) for their sum, product, etc., can be computed from this input automatically. These algorithms can be used to prove identities for special functions or sequences in enumerative combinatorics (Kauers, 2013; Pillwein, 2008).

Even though many interesting objects are D-finite and can be covered by existing algorithms, there is more out there. Van Hoeij (1997) has considered the computation of formal solutions of linear differential equations with coefficients living in a differential field and the (approximate) factorization of differential operators of this type. Since general elements in this field consist of infinitely many terms, the factorization can only be determined up to some finite accuracy. The problem of determining regular solutions of linear differential systems of arbitrary order with power series coefficients has very recently been studied by Abramov et al. (Abramov and Khmelnov, 2014; Abramov et al., 2015). In our work we are not focused on finding closed form solutions, but we want to provide tools to carry out exact computations with functions satisfying linear differential equations, even if no closed form solution exists. One goal is to prove identities for these functions using closure properties as in the D-finite case.

The bigger class that we introduce are functions satisfying linear differential equations with D-finite coefficients and we refer to this class as *DD-finite functions*. As we stressed earlier, D-finite functions form a large class of useful functions - hence DD-finite functions are a considerable extension. In fact all examples given in the papers of van Hoeij and Abramov et al. are actually of this type. Also these functions can be represented exactly using a finite amount of data: the order of the equation, the finitely representable D-finite coefficients, and sufficiently many initial values. Simple examples of DD-finite but not D-finite functions are the double exponential $\exp(\exp(x))$, $\tan(x)$, and Mathieu's functions (DLMF, 2017). Outside this new class are functions such as the triple exponential or the Gamma function.

It turns out that we can carry over the classical closure properties to the setting of DD-finite functions and can even prove them in a more general setting that allows to iterate our construction. The proofs of these generalizations can be interpreted constructively and the closure properties can be implemented to be carried out entirely automatically. In this paper we first set up the definitions and basic properties of DD-finite functions. Then several closure properties are proven and examples are given. At this stage we do not yet discuss the issue of initial values much (as it also does not affect the results and computations). Initial values and uniqueness of the definition are addressed in a separate section. We provide some simple examples to illustrate the results.

2. Differentially definable functions

In the present section we develop the mathematical framework for DD-finite functions, i.e., functions satisfying linear differential equations with D-finite coefficients. Throughout this paper we fix the following notation: K is a field of characteristic zero, $K[[x]]$ denotes the ring of formal power series over K, ∂ the standard derivation in $K[[x]]$, and $\langle S\rangle_K$ the K-vector space generated by the set S.

Definition 1. Let R be a non-trivial differential subring of $K[[x]]$ and $R[\partial]$ the ring of linear differential operators over R. We call $f \in K[[x]]$ *differentially definable over* R if there is a non-zero operator $\mathcal{A} \in R[\partial]$ that annihilates f, i.e., $\mathcal{A}\cdot f = 0$. By $\mathrm{D}(R)$ we denote the set of all $f \in K[[x]]$ that are differentially definable over R. We define the *order of f w.r.t. R* as the minimal order of the operators that annihilate f (i.e., the minimal ∂-degree of $\mathcal{A} \in R[\partial]$ such that $\mathcal{A}\cdot f = 0$).

For non-trivial differential subrings of $K[[x]]$ the set of differentially definable functions is never empty. In fact, the set $\mathrm{D}(R)$ extends the base ring R.

Lemma 2. *Let R be a differential subring of $K[[x]]$. Then $R \subset \mathrm{D}(R)$.*

Proof. Let $f \in R$ and $\mathcal{A} = f\partial - f'$. Clearly, $\mathcal{A} \in R[\partial]$, and $\mathcal{A}\cdot f = f\partial(f) - f'f = ff' - f'f = 0$. $\square$

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For $R = K$, Definition 1 yields the ring of formal power series satisfying linear differential equations with *constant* coefficients, also referred to as C-finite functions. If R is the ring of polynomials in x over K, $K[x]$, then Definition 1 gives the classical D-finite functions. The new class of *DD-finite functions* can now be defined as follows.

Example 3 *(DD-finite).* Let R be the set of D-finite functions. It is well known (Kauers and Paule, 2011) that these functions are closed under addition, multiplication and derivation, i.e., they form a differential subring of $K[[x]]$. Then,

$$\mathrm{D}(R) = \{f \in K[[x]] : f \text{ is DD-finite}\} = \mathrm{D}(\mathrm{D}(K[x])) = \mathrm{D}^2(K[x]).$$

For classical D-finite functions there exist equivalent characterizations that help prove certain properties. These characterizations can be carried over immediately to differentially definable functions.

Theorem 4. *Let R be a differential subring of $K[[x]]$, $R[\partial]$ the ring of linear differential operators over R, and $F = Q(R)$ be the fraction field of R. Let $f \in K[[x]]$. Then the following are equivalent:*

1. *f is differentially definable over R.*
2. (Inhomogeneous) *There are $\mathcal{A} \in R[\partial]$ and $g \in \mathrm{D}(R)$ such that $\mathcal{A} \cdot f = g$.*
3. (Finite dimension) *The F-vector space generated by the set $\{f^{(i)} : i \in \mathbb{N}\}$ has finite dimension.*

In fact, f is differentially definable over R with order d if and only if the F-dimension of the vector space in (3) is exactly d.

Proof. First suppose that (ii) holds, i.e.,

$$\mathcal{A} \cdot f = g.$$

Then, there is a $\mathcal{B} \in R[\partial]$ that annihilates g, because $g \in \mathrm{D}(R)$. Hence,

$$\mathcal{B} \cdot (\mathcal{A} \cdot f - g) = (\mathcal{B}\mathcal{A}) \cdot f = 0.$$

Since $(\mathcal{B}\mathcal{A}) \in R[\partial]$, f is differentially definable over R, i.e., $(ii) \Rightarrow (i)$.

Suppose now that (i) holds for some operator $\mathcal{A}$ of order d. We are going to prove that for all $i \geq d$ $f^{(i)}(x)$ is an F-linear combination of the previous derivatives (iii).

Let $\mathcal{A} = r_0 + \cdots + r_d\partial^d$, for $r_i \in R$ and $r_d \neq 0$, and consider $\mathcal{A}_i = \partial^i \cdot \mathcal{A}$. Then for each i, $\mathcal{A}_i \in R[\partial]$ has order $d + i$ and leading coefficient r_d. Moreover,

$$\mathcal{A}_i \cdot f = (\partial^i \cdot \mathcal{A}) \cdot f = \partial^i \cdot (\mathcal{A} \cdot f) = \partial^i(0) = 0.$$

Thus, for any $i \in \mathbb{N}$, there are elements $s_0, \ldots, s_{d+i-1} \in R$ such that

$$s_0 f + \cdots + s_{d+i-1} f^{(d+i-1)} + r_d f^{(d+i)} = 0,$$

and then the following identity holds:

$$f^{(d+i)} = -\frac{s_0}{r_d} f - \cdots - \frac{s_{d+i-1}}{r_d} f^{(d+i-1)}.$$

Since all $-\frac{s_j}{r_d} \in F$, we have that $f^{(d+i)}$ is an F-linear combination of the first $d + i - 1$ derivatives of f for all $i \in \mathbb{N}$. Hence,

$$\langle f^{(i)} : i \in \mathbb{N}\rangle_F = \langle f, f', \ldots, f^{(d-1)}\rangle_F.$$

Finally, suppose now that (iii) holds and the F-vector space generated by the derivatives of f has finite dimension d. Then we have that the set $\{f, f', \ldots f^{(d)}\}$ is linearly dependent, i.e., for $i = 0, \ldots, d$ there are fractions $\frac{r_i}{s_i} \in F$ such that

$$\frac{r_0}{s_0} f + \cdots + \frac{r_d}{s_d} f^{(d)} = 0.$$

Let S be a common multiple of $\{s_0, \ldots, s_d\}$ and $p_i = S r_i / s_i \in R$. Then

$$p_0 f + \cdots + p_d f^{(d)} = 0.$$

For $\mathcal{A} = p_0 + \cdots + p_d \partial^d \in R[\partial]$ we have that $\mathcal{A} \cdot f = 0$, i.e., (ii) holds with $g = 0 \in \mathrm{D}(R)$. □

3. Closure properties

D-finite functions satisfy several closure properties (Kauers and Paule, 2011) and these are the key to proving identities for holonomic functions (Kauers, 2013). In this section we derive the analogous closure properties for differentially definable (and in particular DD-finite) functions. Furthermore it is easy to see that the construction of differentially definable functions can be iterated arbitrarily often. Starting from a fixed differential subring R of $K[[x]]$ a new differential subring is obtained. Iterating the procedure k times we obtain $\mathrm{D}^k(R)$, the set of differentially definable functions over $\mathrm{D}^{k-1}(R)$. The closure properties that we derive here hold in every layer of this hierarchy. We first present those for derivatives and antiderivatives.

Proposition 5 *(Differential closure properties). Let R be a differential subring of $K[[x]]$ and $f \in \mathrm{D}(R)$ with order d. Then:*

- *Any antiderivative g of f (i.e., $g' = f$) is in $\mathrm{D}(R)$ with order at most $d+1$.*
- *$f' \in \mathrm{D}(R)$ with order at most d.*

Proof. First, we prove that any antiderivative is in $\mathrm{D}(R)$. Let $\mathcal{A} \in R[\partial]$ with order d such that $\mathcal{A} \cdot f = 0$. Then we plug in the expression $g' = \partial \cdot g = f$, obtaining

$$0 = \mathcal{A} \cdot f = \mathcal{A} \cdot (\partial \cdot g) = (\mathcal{A}\partial) \cdot g.$$

Thus if f has order d, any antiderivate has at most order $d+1$.

To prove the closure property for the derivative we use a similar idea, but we need to distinguish two cases. Let $\mathcal{A} \in R[\partial]$ be an operator that annihilates f of the form

$$\mathcal{A} = r_d \partial^d + \cdots + r_1 \partial + r_0,$$

with $r_i \in R$ and $r_d \neq 0$.

If $r_0 = 0$ we can extract a right factor ∂ from $\mathcal{A}$ obtaining

$$0 = \mathcal{A} \cdot f = \left(r_d \partial^{d-1} + \cdots + r_1 \right) \cdot f'.$$

Hence $f' \in \mathrm{D}(R)$ with order at most $d-1$.

Now consider the case where $r_0 \neq 0$. Then it is easy to check that also the operator

$$\mathcal{B} = (r_0 \partial - r_0')\mathcal{A} = r_0(\partial \mathcal{A}) - r_0' \mathcal{A}$$

annihilates f. Indeed, we have that the coefficient of ∂^0 in $\mathcal{B}$ is

$$[\partial^0]\mathcal{B} = r_0 r_0' - r_0' r_0 = 0.$$

So, by the same reasoning as in the case $r_0 = 0$, we obtain an operator of order d annihilating f'. So $f' \in \mathrm{D}(R)$ with order at most d. □

Note that in the proof above only basic ring operations in $R[\partial]$ were needed. Next, we prove the closure properties for addition and Cauchy product.

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Proposition 6 *(Arithmetic closure properties). Let R be a differential subring of $K[[x]]$. Let $f, g \in \mathrm{D}(R)$ with orders d_1, d_2, respectively. Then:*

- *$h = f + g$ is in $\mathrm{D}(R)$ with order at most $d_1 + d_2$.*
- *$h = fg$ is in $\mathrm{D}(R)$ with order at most $d_1 d_2$.*

Proof. Let F be the field of fractions of R. Define for $f(x) \in K[[x]]$ the F-vector space

$$V_F(f) = \langle f(x), f'(x), \ldots \rangle_F = \langle f^{(i)}(x) : i \in \mathbb{N} \rangle_F.$$

Then, using Theorem 4, we obtain

$$\dim(V_F(f)) = d_1, \ \dim(V_F(g)) = d_2.$$

On the other hand, if we compute the kth derivative of the addition and product of f and g we have:

- $(f+g)^{(k)} = f^{(k)} + g^{(k)} \in V_F(f) + V_F(g)$.
- $(fg)^{(k)} = \sum_{i=0}^{k} \binom{k}{i} f^{(i)} g^{(k-i)} \in V_F(f) \otimes V_F(g)$.

Thus we have that $V_F(f+g) \subset V_F(f) + V_F(g)$ and that $V_F(fg) \subset V_F(f) \otimes V_F(g)$, and so:

$$\dim(V_F(f+g)) \leq \dim(V_F(f)) + \dim(V_F(g)) = d_1 + d_2 < \infty,$$
$$\dim(V_F(fg)) \leq \dim(V_F(f)) \dim(V_F(g)) = d_1 d_2 < \infty.$$

Hence both $f+g$ and fg are in $\mathrm{D}(R)$ with orders at most $d_1 + d_2$ and $d_1 d_2$, respectively. $\square$

Although the proof presented here is not constructive, still a usual algorithm via ansatz can be derived to effectively compute an operator annihilating the addition or product of two functions, given their annihilating operators. The bound for the dimension of the F-vector space is the key to find a non-trivial operator. Let h be the function that we want to compute and d be the bound for its order. Then the algorithm proceeds as follows:

1. For $i = 0, \ldots, d$ express $h^{(i)}$ in the basis of $V_F(f) + V_F(g)$ or $V_F(f) \otimes V_F(g)$, respectively.
2. Set up an ansatz with undetermined coefficients of the form,

$$\alpha_d h^{(d)} + \cdots + \alpha_0 h = 0.$$

3. Obtain a linear system for the α_j by equating like coefficients of the basis elements to zero.

Since there are $d+1$ variables and at most d equations to set up the linear system it has a non-trivial solution. Note that in order to implement this algorithm, the nullspace must be effectively computable in R. For more details we refer to Jiménez-Pastor and Pillwein (2018).

An immediate consequence of the closure properties of Propositions 5 and 6 is that $\mathrm{D}(R)$ is a differential subring of $K[[x]]$.

Theorem 7. *Let R be a differential subring of $K[[x]]$. Then $\mathrm{D}(R)$ is a differential subring of $K[[x]]$.*

Hence, starting from any differential subring R of $K[[x]]$, the process of constructing the differentially definable functions over R can be iterated and, for each layer of $\mathrm{D}^k(R)$, the closure properties hold. This in particular also includes the following two results.

Proposition 8 *(Algebraic Closure). Let R be a differential subring of $K[[x]]$ and F its field of fractions. If $f \in K[[x]]$ is algebraic over F then $f \in \mathrm{D}(R)$.*

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Proof. Suppose that f is algebraic over F and let $m(y) \in F[y]$ be its minimal polynomial. Let κ_∂ denote the coefficient-wise derivation on $F[y]$ induced by the given derivation ∂ on F and ∂_y the usual derivation on $F[y]$ (i.e., $\partial_y y = 1$ and $\partial_y F = 0$). Then differentiating the equation $m(f) = 0$, we get that

$$(\partial_y m(f))f' + \kappa_\partial(m)(f) = 0. \tag{1}$$

As $m(y)$ is irreducible and $\deg_y(m) > \deg_y(\partial_y m)$, they are coprime polynomials in $F[y]$. Hence, there are $s(y), r(y) \in F[y]$ such that

$$r(y)m(y) + s(y)\partial_y m(y) = 1,$$

and substituting $y = f$ we obtain

$$s(f)\partial_y m(f) = 1.$$

Using this equality in (1), we have that

$$f' = -s(f)\kappa_\partial(m)(f) = \tilde{m}(f).$$

Now for any g with $g = p(f)$ for some $p(y) \in F[y]$,

$$g' = \partial_y p(f(x))f' + \kappa_\partial p(f) = (\partial_y p(y)\tilde{m}(y) + \kappa_\partial(p)(y))(f).$$

In other words, all the derivatives of any g that can be written as a polynomial in f with coefficients in F can be expressed in the same way. Summarizing we have

$$\langle f^{(i)} : i \in \mathbb{N}\rangle_F \subset \langle f^i : i \in \mathbb{N}\rangle_F.$$

Hence the F-vector space generated by f and its derivatives has dimension at most $\deg_y(m)$, so by Theorem 4, $f \in \mathrm{D}(R)$ with order at most $\deg_y(m)$. □

Lemma 9. *Let R be a differential subring of $K[[x]]$, $r \in R$ and $f \in \mathrm{D}(R)$.*

1. *If $r(0) \neq 0$ then its multiplicative inverse $1/r$ in the ring of formal power series is in $\mathrm{D}(R)$ with order 1.*
2. *If $f(0) \neq 0$ and f has* order 1 *then its multiplicative inverse $1/f$ in the ring of formal power series is in $\mathrm{D}(R)$ with order 1.*
3. *If there is $g \in K[[x]]$, $r \in R$ and $n \in \mathbb{N}$ such that $f = gr^n$, then $g \in \mathrm{D}(R)$ with order at most the order of f.*

Proof. Since, by Lemma 2, $R \subset \mathrm{D}(R)$ and all elements of R are of order 1, (i) is a consequence of (ii). For proving (ii), consider $\mathcal{A} \in R[\partial]$ that annihilates f of the form

$$\mathcal{A} = a\partial + b.$$

As $f(0) \neq 0$, it follows that $f^{-1} \in K[[x]]$. Let $\widetilde{\mathcal{A}} = a\partial - b$. Then,

$$\widetilde{\mathcal{A}} \cdot f^{-1} = -\frac{af'}{f^2} - \frac{b}{f} = -\frac{af' + bf}{f^2} = -\frac{\mathcal{A} \cdot f}{f^2} = 0,$$

hence $f^{-1} \in \mathrm{D}(R)$ with order 1.

Finally, for proving (iii) is enough to plug the formula,

$$f^{(k)} = (gr^n)^{(k)} = \sum_{i=0}^{k} \binom{k}{i} g^{(i)}(r^n)^{(k-i)},$$

into the equation $\mathcal{A} \cdot f = 0$ and rearrange terms to obtain the operator annihilating g. The resulting operator only depends on the coefficients of $\mathcal{A}$, binomial coefficients, and powers of r. As R is a differential subring of $K[[x]]$, provided that $f \in \mathrm{D}(R)$, this operator is in $R[\partial]$. These calculations do not affect the order of the operator, hence g has at most the same order as f. □

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Once more, note that, for any (non-trivial) differential subring of $K[[x]]$, $\mathrm{D}(R)$ is again a differential subring of $K[[x]]$. Hence, the construction can be iterated and an increasing chain of differential subrings of $K[[x]]$,

$$R \subset \mathrm{D}(R) \subset \mathrm{D}^2(R) \subset \ldots \subset \mathrm{D}^k(R) \subset \ldots \subset K[[x]],$$

can be built. A natural question is, what happens in the limit case when k tends to infinity. This, however, is beyond the scope of this paper.

4. A concrete example: the tangent

In order to illustrate the execution of closure properties, we consider the simple example of the tangent. It is one of the basic examples of a function that is *not* D-finite (Mallinger, 1996; Flajolet et al., 2005), but it is DD-finite. The sine ($\sin(x)$) and cosine ($\cos(x)$) are D-finite as they both satisfy the linear differential equation with polynomial coefficients $f''(x) + f(x) = 0$. Hence by Lemma 2 both are DD-finite and by Lemma 9 the multiplicative inverse of the cosine is DD-finite. Applying the closure property for multiplication from Proposition 6, we can conclude that the tangent $\tan(x) = \sin(x)/\cos(x)$ is DD-finite.

As an input for computing the annihilating operator for the tangent, we need operators annihilating $\sin(x)$ and $1/\cos(x)$, respectively. For the sine we may either use the above-mentioned operator

$$\mathcal{A} = \partial^2 + 1,$$

or, following the argument in the proof of Lemma 2, we also have that

$$\widetilde{\mathcal{A}} = \sin(x)\partial - \partial(\sin(x)) = \sin(x)\partial - \cos(x)$$

annihilates the sine. From Lemma 9 we obtain the following annihilating operator for $1/\cos(x)$

$$\mathcal{B} = \cos(x)\partial - \sin(x).$$

Next we derive annihilating operators with D-finite coefficients for the tangent using either of the operators $\mathcal{A}$ and $\widetilde{\mathcal{A}}$, yielding operators of order one and two, respectively. We show these computations using sine and cosine explicitly. Note that in general, we represent all coefficients in terms of their defining equations with polynomial coefficients using closure properties in the respective ring of D^k-finite functions. Furthermore we do not operate division-free in this example in order to obtain more easily readable results. For details on the actual implementation (features and issues) we refer to Jiménez-Pastor and Pillwein (2018).

Example 10 *(Tangent with order 1).* Given the DD-finite functions $f(x) = \sin(x)$ and $g(x) = 1/\cos(x)$ by their annihilating operators $\widetilde{\mathcal{A}}$ and $\mathcal{B}$ of orders $d_1 = d_2 = 1$, we seek to compute an annihilating operator for $h(x) = f(x)g(x)$. By Proposition 6 we have that the bound for the order of such an operator is $d = d_1 d_2 = 1$. In order to compute the coefficients of this operator, we need to express $h(x)$ and $h'(x)$ in terms of the basis $\{f(x)g(x)\}$ of the tensor product space:

$$\begin{aligned} h(x) &= f(x)g(x), \\ h'(x) &= f'(x)g(x) + f(x)g'(x). \end{aligned}$$

Using their defining operators, the derivatives f' and g' can be represented in terms of f and g as

$$f'(x) = \cos(x)/\sin(x) f(x), \quad \text{and} \quad g'(x) = \sin(x)/\cos(x) g(x).$$

Hence we can write

$$h'(x) = \left(\frac{\cos(x)}{\sin(x)} + \frac{\sin(x)}{\cos(x)}\right) f(x)g(x).$$

For the sought annihilating operator we set up an ansatz with undetermined coefficients

$$\alpha_0 h(x) + \alpha_1 h'(x) = 0,$$

plug in the representations for h and h' and equate the coefficients of basis elements to zero (which is in this case trivial). Thus we obtain that

$$\alpha_0 = -\alpha_1 \left(\frac{\cos(x)}{\sin(x)} + \frac{\sin(x)}{\cos(x)} \right).$$

With a little simplification we can write the resulting operator as

$$\widetilde{\mathcal{C}} = -\sin(x)\cos(x)\partial + 1 \in \mathrm{D}(K[x])[\partial],$$

and indeed $\widetilde{\mathcal{C}} \cdot \tan(x) = 0$.

As second example we compute again an annihilating operator for the tangent, but this time starting from the operators $\mathcal{A}$ and $\mathcal{B}$.

Example 11 *(Tangent with order 2).* Given f and g by the defining differential equations $f''(x) + f(x) = 0$ and $\cos(x)g'(x) - \sin(x)g(x) = 0$ we conclude using Proposition 6 that $h(x) = f(x)g(x)$ satisfies a linear differential equation of order at most two. As a basis for the tensor product space we have $\{f(x)g(x), f'(x)g(x)\}$. We need to express h, h', and h'' in terms of these basis elements and start by computing

$$\begin{aligned} h(x) &= f(x)g(x), \\ h'(x) &= f'(x)g(x) + f(x)g'(x), \\ h''(x) &= f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x). \end{aligned}$$

Using $\mathcal{A}$ and $\mathcal{B}$ we can replace f'', g', g'' above and obtain

$$\begin{aligned} h(x) &= f(x)g(x), \\ h'(x) &= \frac{\sin(x)}{\cos(x)} f(x)g(x) + f'(x)g(x), \\ h''(x) &= 2\frac{\sin^2(x)}{\cos^2(x)} f(x)g(x) + 2\frac{\sin(x)}{\cos(x)} f'(x)g(x). \end{aligned}$$

We plug these expressions into the ansatz $\alpha_0 h(x) + \alpha_1 h'(x) + \alpha_2 h''(x) = 0$ for a second order differential equation. Equating the coefficients of the basis elements to zero leads to the matrix

$$\begin{pmatrix} 1 & \sin(x)/\cos(x) & 2\sin^2(x)/\cos^2(x) \\ 0 & 1 & 2\sin(x)/\cos(x) \end{pmatrix},$$

whose nullspace has dimension one and is generated by the vector $(0, -2\sin(x), \cos(x))^T$. From this we obtain another annihilating operator for the tangent, namely,

$$\mathcal{C} = \cos(x)\partial^2 - 2\sin(x)\partial \in \mathrm{D}(K[x])[\partial].$$

Obviously, the differential equations given by the annihilators $\mathcal{A}, \mathcal{B}$, etc. do not yet define uniquely an element in the ring of formal power series. For this we need to also take into account certain initial values. This topic is addressed next.

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5. Initial values and uniqueness

Naturally when working with differential equations, initial values need to be given to characterize one solution to the given equation uniquely. Under certain regularity assumptions on the coefficients, existence and uniqueness of a solution can be proven (Rainville and Bedient, 1969). This solution is formed as a linear combination of d linearly independent solutions, where d is the order of the differential equation. The coefficients in this linear combination are given by d initial conditions.

In the case of formal power series $f(x) = \sum_{n\geq 0} f_n x^n$, the only evaluation possible is at the origin with $f(0)$ equal to the constant coefficient in this series expansion. Hence, we define the *solution space* of a given differential operator $\mathcal{A}$ as the K-vector space of formal power series annihilated by $\mathcal{A}$.

In this section we address the issue of initial values $(f(0), f'(0), f''(0), \dots)$ to define the solution within $K[[x]]$ of the given linear differential equation uniquely. This is in particular of importance, if the regularity assumptions on the leading coefficient do not hold.

As a first motivating example consider the simple differential operators $\mathcal{B}_m = x\partial - m$ for $m \in \mathbb{N} \setminus \{0\}$. The solutions are of the form Cx^m, for some $C \in K$. It is easy to see that these equations fix the first m initial values to be 0 and the $(m+1)$th initial value determines C (indeed, $f^{(m)}(0) = m!C$).

Another example is given by the operator $\mathcal{C} = x\partial^2 + \partial + x$. This operator annihilates the Bessel functions of first $(J_0(x))$ and second $(Y_0(x))$ kind (DLMF, 2017). The latter function $Y_0(x)$ has a pole at the origin and is not in $K[[x]]$. In fact, from the equation it follows that $f'(0)$ has to vanish and the value of f at zero defines uniquely an element of $K[[x]]$.

Similar to the case of D-finite functions we transfer the problem of computing initial values to the level of the coefficient sequences (Kauers and Paule, 2011). We provide a method to compute the dimension of the solution space for a linear differential equation and divide this process into three steps:

1. Set up an infinite linear system for the coefficient sequence from the differential equation.
2. Examine the structure of this linear system.
3. Relate the dimension of the solution space to the rank of a finite matrix.

In the following we denote the coefficient sequence of an element $f(x) = \sum_{n\geq 0} f_n x^n \in K[[x]]$ by $\mathbf{f} = (f_n)_{n\geq 0}$. Let R be a differential subring of $K[[x]]$ and let us consider an operator $\mathcal{A} \in R[\partial]$ with

$$\mathcal{A} = r_d\partial^d + \cdots + r_1\partial + r_0$$

for some $r_j(x) = \sum_{n\geq 0} r_{j;n}x^n \in R$. Now $\mathcal{A} \cdot f = 0$ if and only if all its coefficients vanish, i.e.,

$$\sum_{i=0}^{d}\sum_{k=0}^{n} r_{i;n-k}\,(k+i)^{\underline{i}}\, f_{k+i} = 0, \quad \text{for all } n \in \mathbb{N},$$

where $(a)^{\underline{m}} = a\cdot(a-1)\cdots(a-m+1)$ denotes the falling factorial. Reorganizing this sum, one can extract the coefficients f_k from the innermost sum and we obtain

$$\sum_{k=0}^{n+d}\left(\sum_{l=\max\{0,k-n\}}^{\min\{d,k\}} (k)^{\underline{l}}\, r_{l;n-k+l}\right) f_k = 0, \quad \text{for all } n \in \mathbb{N},$$

as the condition for $\mathcal{A}$ annihilating f on the sequence level. Now let's define the infinite dimensional matrix $M^{\mathcal{A}} = (m_{n,k})_{n,k\geq 0}$ with

$$m_{n,k} = \begin{cases} \displaystyle\sum_{l=\max\{0,k-n\}}^{\min\{d,k\}} (k)^{\underline{l}}\, r_{l;n-k+l}, & k \leq n+d, \\ 0, & k > n+d. \end{cases}$$

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With this definition we have that $\mathcal{A}\cdot f=0$ if and only if $M^{\mathcal{A}}\mathbf{f}=\mathbf{0}$. These matrices have almost a lower left triangular structure since $m_{n,k}=0$ for $k>n+d$. If we consider the strip $n\leq k\leq n+d$, then these matrix entries can be computed as follows for each $n\geq d$:

$$m_{n,k}=\sum_{l=\max\{0,k-n\}}^{\min\{d,k\}}(k)^{\underline{l}}\,r_{l;n-k+l}=\sum_{l=k-n}^{d}(k)^{\underline{l}}\,r_{l;n-k+l}.$$

Rewriting $k=n+i$, for $i=0,\ldots,d$, we obtain for each i the following polynomials in n of degree at most d,

$$m_{n,n+i}=\sum_{l=i}^{d}(n+i)^{\underline{l}}r_{l;l-i}=:\varphi_i(n).$$

We single out these polynomials as they play an important role in understanding the matrix $M^{\mathcal{A}}$.

For computing the number of initial values necessary to define a formal power series uniquely with a given operator, it plays an essential role whether or not the leading coefficient vanishes at $x=0$. Moreover, it may happen during computations (e.g., execution of closure properties) that *all* coefficients vanish at zero. We want to exclude that case and below we show that for any DD-finite function an operator with this property exists.

Lemma 12. *Let R be a differential subring of $K[[x]]$. Then $f\in\mathrm{D}(R)$ if and only if there exists an operator $\mathcal{A}=r_d\partial^d+\cdots+r_0\in R[\partial]$ annihilating f with the property $\exists\, j\in\{0,\ldots,d\}: r_j(0)\neq 0$.*

Proof. Let $f\in\mathrm{D}(R)$ with annihilating operator $\mathcal{A}=r_d\partial^d+\cdots+r_0\in R[\partial]$. Assume that $r_j(0)=0$ for all $j=0,\ldots,d$ (else there is nothing to prove). Then there exists a maximal index k such that $r_{j;l}=0$ for all $j=0,\ldots,d$ and for all $l=0,\ldots,k$. Thus the factor x^k can be pulled out of each coefficient $r_j(x)$. The operator $\widetilde{\mathcal{A}}=x^{-k}\mathcal{A}$ is in $R[\partial]$, annihilates f, and not all its coefficients vanish at $x=0$. □

By Lemma 13 below, it follows that for all operators with at least one coefficient function *not* vanishing at zero, also not all polynomials $\varphi_i(n)$ vanish identically.

Lemma 13. *Let R be a differential subring of $K[[x]]$ and $\mathcal{A}=r_d\partial^d+\cdots+r_0\in R[\partial]$. If there exists an index $i\in\{0,\ldots,d\}$ s.t. $r_i(0)\neq 0$, then $\varphi_i(n)\neq 0$.*

Proof. Let i be s.t. $r_i(0)=r_{i;0}\neq 0$. Now $\varphi_i(n)=0$, iff all its coefficients vanish. But

$$\varphi_i(n)=\sum_{l=i}^{d}(n+i)^{\underline{l}}r_{l;l-i}=(n+i)^{\underline{i}}r_{i;0}+\sum_{l=i+1}^{d}(n+i)^{\underline{l}}r_{l;l-i},$$

so at least one coefficient is non-zero. □

Operators for which at least one of the coefficient functions does *not* vanish at zero, we call *zero-reduced* and the maximal index i, $0\leq i\leq d$, for which $\varphi_i(n)\neq 0$ we denote by $\nu(\mathcal{A})$.

Corollary 14. *Let R be a differential subring of $K[[x]]$ and $\mathcal{A}$ a zero-reduced operator of order d. Then for $M^{\mathcal{A}}=(m_{n,k})_{n,k\geq 0}$ and $n\geq d-\nu(\mathcal{A})$,*

$$\begin{cases} m_{n,k}=0 & \text{for all } k>n+\nu(\mathcal{A}),\\ m_{n,n+\nu(\mathcal{A})}=\varphi_{\nu(\mathcal{A})}(n) \end{cases}$$

Proof. Once again, writing $k=n+i$, $i\geq\nu(\mathcal{A})$, we have $\varphi_i(n)=m_{n,n+i}$ for $n\geq d-\nu(\mathcal{A})$. Then the statement follows for $n+\nu(\mathcal{A})\leq k\leq n+d$ by definition of $\nu(\mathcal{A})$ and for $k>n+d$ by definition of $M^{\mathcal{A}}$. □

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In order to discuss which initial values are needed to uniquely define a formal power series annihilated by a zero-reduced operator $\mathcal{A}$, we define finite submatrices of $M^{\mathcal{A}}$ and an auxiliary index.

Definition 15. Let R be a differential subring of $K[[x]]$ and $\mathcal{A} \in R[\partial]$ a zero-reduced operator of order d.

- We define the *pth recursion matrix* as $M_p^{\mathcal{A}} = (m_{n,k})_{0\le n\le p}^{0\le k\le p+\nu(\mathcal{A})}$, i.e., the upper left submatrix of $M^{\mathcal{A}}$ of dimension $(p+1)\times(p+\nu(\mathcal{A})+1)$.
- Let $\varphi(n) = \varphi_{\nu(\mathcal{A})}(n)$ and μ be the maximal integer root of $\varphi(n)$ or $-\infty$, if there are no integer roots. Then we define $\lambda(\mathcal{A}) := \max\{d-\nu(\mathcal{A}), \mu\}$ and call it *singular value*.

With these notations at hand, we are in the position to state our result about the dimension of the solution space of a zero-reduced operator. First note that for any $p \ge \lambda(\mathcal{A})$, the dimension of the nullspace of $M_p^{\mathcal{A}}$ does not change any more.

Proposition 16. *Let R be a differential subring of $K[[x]]$ and $\mathcal{A} \in R[\partial]$ a zero-reduced operator of order d. Let $p > \lambda(\mathcal{A})$. Then:*

$$\mathrm{rk}(M_p^{\mathcal{A}}) = 1 + \mathrm{rk}(M_{p-1}^{\mathcal{A}}).$$

Proof. Using Corollary 14 we have, for $p \ge d-\nu(\mathcal{A})$ that the matrix $M_p^{\mathcal{A}}$ has the following recursive shape:

$$M_p^{\mathcal{A}} = \begin{pmatrix} M_{p-1}^{\mathcal{A}} & 0 \\ * & \varphi(p) \end{pmatrix}.$$

Now, as we have $p > \lambda(\mathcal{A})$, we know that $p > d-\nu(\mathcal{A})$ and also that $\varphi(p) \neq 0$. Hence, the last row of the matrix is always linearly independent of the others, which makes that

$$\mathrm{rk}(M_p^{\mathcal{A}}) = 1 + \mathrm{rk}(M_{p-1}^{\mathcal{A}}). \quad \square$$

Proposition 17. *Let R be a differential subring of $K[[x]]$ and $\mathcal{A} \in R[\partial]$ a zero-reduced operator of order d. Let v be a vector in the right nullspace of $M_{\lambda(\mathcal{A})}^{\mathcal{A}}$. Then there is a unique sequence $\mathbf{f} = (f_n)_{n\ge 0}$ such that $f_i = v_i$ for $i = 0, \dots, \lambda(\mathcal{A})+\nu(\mathcal{A})$ and, for all $p \in \mathbb{N}$,*

$$M_p^{\mathcal{A}}\mathbf{f}_p = 0,$$

where $\mathbf{f}_p = (f_0, \dots, f_{p+\nu(\mathcal{A})})$.

Proof. We build the sequence $\mathbf{f}$ inductively and initialize $f_i = v_i$ for $i = 0, \dots, \lambda(\mathcal{A})+\nu(\mathcal{A})$. As v is in the right nullspace of $M_{\lambda(\mathcal{A})}^{\mathcal{A}}$ and because of the structure of the matrix (see Definition 15), for $0 \le p \le \lambda(\mathcal{A})$,

$$M_p^{\mathcal{A}}\mathbf{f}_p = 0.$$

Now suppose we have constructed $\mathbf{f}_n$ such that $M_p^{\mathcal{A}}\mathbf{f}_p = 0$ for all $p = 0, \dots, n$. Using again the block structure of the matrix it follows immediately that

$$M_{n+1}^{\mathcal{A}}\mathbf{f}_{n+1} = \begin{pmatrix} M_n^{\mathcal{A}} & \mathbf{0} \\ m_{n+1,0} \dots m_{n+1,n+\nu(\mathcal{A})} & \varphi(n+1) \end{pmatrix} \begin{pmatrix} \mathbf{f}_n \\ f_{n+\nu(\mathcal{A})+1} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ * \end{pmatrix}.$$

Note that $\varphi(n+1) \neq 0$, hence

$$f_{n+\nu(\mathcal{A})+1} = -\left(m_{n+1,0}f_0 + \dots + m_{n+1,n+\nu(\mathcal{A})}f_{n+\nu(\mathcal{A})}\right)/\varphi(n+1)$$

defines the unique extension. $\square$

The sequence $\mathbf{f}$ constructed in the previous proposition yields the formal power series annihilated by $\mathcal{A}$. Hence we can conclude with the explicit formula for the dimension of the solution space, which is equivalent to the number of initial values needed to define a unique solution of $\mathcal{A}$ (Horn and Johnson, 1985).

Theorem 18 *(Dimension of solution space). Let R be a differential subring of $K[[x]]$ and $\mathcal{A} \in R[\partial]$ a zero-reduced operator. Then the dimension of the solution space of $\mathcal{A}$ within $K[[x]]$ is the dimension of the right nullspace of $M^{\mathcal{A}}_{\lambda(\mathcal{A})}$, i.e.:*

$$\lambda(\mathcal{A}) + \nu(\mathcal{A}) - \mathrm{rk}(M^{\mathcal{A}}_{\lambda(\mathcal{A})}) + 1.$$

Let $\{v_1, \dots, v_r\}$ be a basis for the nullspace of $M^{\mathcal{A}}_{\lambda(\mathcal{A})}$, i.e., the dimension of the solution space of the operator $\mathcal{A}$ is r. Then the first $\lambda(\mathcal{A}) + \nu(\mathcal{A}) + 1$ values of the coefficient sequence $\mathbf{f}$ are given as $\mathbf{f}_{\lambda(\mathcal{A})} = \gamma_1 v_1 + \dots + \gamma_r v_r$. Obviously the elements f_m for which all components $v_{j;m} = 0$, $j = 1, \dots, r$, are fixed to be zero. Hence we know how many and which initial values are necessary to define a unique solution in the ring of formal power series.

This is a generalization of the classical result where the leading coefficient r_d does not vanish at zero. In that case, we have $\varphi_d(n) = r_d(0)\,(n+d)^{\underline{d}}$, hence $\nu(\mathcal{A}) = d$ and $\lambda(\mathcal{A}) = 0$. Then the dimension of the solution space is determined by the nullspace of the matrix $M^{\mathcal{A}}_0$ which is a row vector with $d+1$ entries with the last being $\varphi(0) \neq 0$. Hence the dimension is d and the initial values $f(0), \dots, f^{(d-1)}(0)$ are required to characterize a solution of $\mathcal{A}$.

Now we apply the results developed in this section to the previously presented examples.

Example 19. Let $\mathcal{C} = x\partial^2 + \partial + x$ be the annihilating second order operator for the Bessel functions mentioned at the beginning of this section. This operator is already zero-reduced since $[\partial^1]\mathcal{C} = 1$. Hence, we can compute the polynomials φ as described above yielding

$$\varphi_0(n) = 0, \quad \varphi_1(n) = (n+1)^2, \quad \varphi_2(n) = 0.$$

From this is easily obtained that $\nu(\mathcal{C}) = 1$ and that the singular value is $\max\{-1, (2-1)\} = 1$. Hence, to determine the dimension and structure of the solution space of $\mathcal{C}$ within $K[[x]]$ we need to consider the matrix

$$M^{\mathcal{C}}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 4 \end{pmatrix}.$$

This matrix has a one-dimensional right nullspace with basis $\{(1, 0, -1/4)^T\}$. Hence to define a unique solution in $K[[x]]$ of the equation given by $\mathcal{C}$ one initial value needs to specified (not two as the order would indicate). Furthermore we have that f_0 or f_2 need to be specified and that f_1 are fixed to be zero.

Example 20. Next consider the linear differential operator $\mathcal{B}_m = x\partial - m$ for some $m \in \mathbb{N} \setminus \{0\}$ with general solution Cx^m. This operator is also already zero-reduced and its order is one. We have that $\varphi_0(n) = n - m$ and $\varphi_1(n) = 0$. Hence, $\nu(\mathcal{B}_m) = 0$ for all $m \in \mathbb{N}$ and the singular value is m. Now we can set up the matrix $M^{\mathcal{B}_m}_m$ which has the following structure

$$M^{\mathcal{B}_m}_m = \begin{pmatrix} -m & 0 & 0 & \cdots & 0 & 0 \\ 0 & -m+1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -m+2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

For any m, the right nullspace of this matrix has dimension one and is generated by the vector $(0, \dots, 0, 1)^T$. Consequently f_m needs to be specified and all $f_j = 0$ for $j = 0, \dots, m-1$.

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In Examples 10 and 11, two different operators annihilating the tangent have been computed. Next we determine how many and which initial values are needed to uniquely define the tangent. Note that given $f(x) = \sin(x) = \sum_{n\geq 0} f_n x^n$ and $g(x) = 1/\cos(x) = \sum_{n\geq 0} g_n x^n$ and their initial values, the coefficients of the tangent $h(x) = f(x)g(x)$ can be computed using the Cauchy product $\sum_{n\geq 0}\left(\sum_{m=0}^{n} f_{n-m}g_m\right)x^n$.

Example 21 *(Continuation of Example 10).* Let $\widetilde{\mathcal{C}} = -\sin(x)\cos(x)\partial + 1$. This operator is zero-reduced and of order one, and we have that $\varphi_0(n) = 1 - n$ and $\varphi_1(n) = 0$. Hence, $\nu(\widetilde{\mathcal{C}}) = 0$ and, since $\varphi_0(1) = 0$, we have that $\lambda(\widetilde{\mathcal{C}}) = \max\{1, 1-0\} = 1$. The first recursion matrix of $\widetilde{\mathcal{C}}$ is

$$\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix},$$

and a basis for its one-dimensional nullspace is given by $(0, 1)^T$. From this we obtain that $h(0) = 0$ and the following exact representation of the tangent,

$$\begin{cases} -\sin(x)\cos(x)h'(x) + h(x) = 0 \\ h'(0) = 1 \end{cases}$$

Example 22 *(Continuation of Example 11).* Let $\mathcal{C} = \cos(x)\partial^2 - 2\sin(x)\partial$. This operator is zero-reduced and of order two, and we have that $\varphi_0(n) = -\frac{1}{2}n(n+3), \varphi_1(n) = 0, \varphi_2(n) = (n+1)(n+2)$.

Hence, $\nu(\mathcal{C}) = 2$ and we have that $\lambda(\mathcal{C}) = \max\{-1, 2-2\} = 0$. The recursion matrix of $M_0^{\mathcal{C}}$ is

$$\begin{pmatrix} 0 & 0 & 2 \end{pmatrix},$$

and a basis for its two-dimensional nullspace is given by $\{(1, 0, 0)^T, (0, 1, 0)^T\}$, which was the expected since $\cos(0) \neq 0$. From this we conclude that the first and second initial value need to be given and we obtain the following exact representation of the tangent,

$$\begin{cases} \cos(x)h''(x) - 2\sin(x)h'(x) = 0, \\ h(0) = 0, h'(0) = 1. \end{cases}$$

As a final example, we consider a slightly more intricate DD-finite function and sketch how to prove an identity by applying DD-finite closure properties. More details on this example, and in general on the implementation of DD-finite closure properties, can be found in Jiménez-Pastor and Pillwein (2018).

Example 23 *(Proving identities).* Mathieu's equation in its standard form is given by DLMF (2017); McLachlan (1964)

$$w'' + (a - 2q\cos(2x))w = 0, \tag{2}$$

for some parameters a and q. This differential equation has a pair of fundamental solutions (w_1, w_2) with initial values

$$w_1(0; a, q) = 1,\ w_1'(0; a, q) = 0, \text{ and}$$
$$w_2(0; a, q) = 0,\ w_2'(0; a, q) = 1.$$

$w_1(z; a, q)$ is even and $w_2(z; a, q)$ is odd and both are DD-finite functions. A well-known result is that the Wronskian of these two functions equals one, i.e.,

$$W(x; a, q) := w_1(x; a, q)w_2'(x; a, q) - w_1'(x; a, q)w_2(x; a, q) = 1.$$

As this is an identity between DD-finite functions, the closure properties derived in this paper can be applied to prove it.

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Let $\mathcal{A} = \partial^2 + (a - 2q\cos(2x))$ denote the annihilating operator for the Mathieu functions. In the first step, the equations for w_1' and w_2' have to be computed. Since w_1, w_2 satisfy the same differential equation, their derivatives are annihilated by the same operator as well. Applying Proposition 5, we compute the operator

$$\mathcal{B} = (a - 2q\cos(2x))\partial^2 - 4q\sin(2x)\partial + (a - 2q\cos(2x))^2$$

satisfying $\mathcal{B} \cdot w_1' = \mathcal{B} \cdot w_2' = 0$. Next, the closure property "Multiplication" can be applied to $\mathcal{A}$ and $\mathcal{B}$ to derive an annihilating operator for both $w_1 w_2'$ and $w_1' w_2$. Using Proposition 6, we obtain the differential operator

$$\mathcal{C} = \beta_4\partial^4 + \beta_3\partial^3 + \beta_2\partial^2 + \beta_1\partial,$$

with coefficients

$$\begin{aligned}
\beta_4 &= q\sin(2x)^2 + 2q - a\cos(2x),\\
\beta_3 &= -2\sin(2x)(a + 2q\cos(2x)),\\
\beta_2 &= 2q(-(2a^2+9)q\cos(2x) + (a+1)(\cos(4x)+7) + q\cos(6x)),\\
\beta_1 &= -4\sin(2x)(2a^2 + 10aq\cos(2x) + q^2\cos(4x) - 29q^2).
\end{aligned}$$

Since both $w_1 w_2'$ and $w_1' w_2$ are annihilated by the same operator $\mathcal{C}$, so is their difference W. It remains to verify that the constant function 1 is also annihilated by $\mathcal{C}$ and check that sufficiently many initial values agree. In this example, no more than the four initial values

$$W(0; a, q) = 1, \quad W'(0; a, q) = 0, \quad W''(0; a, q) = 0, \quad W'''(0; a, q) = 0,$$

are needed. This completes the proof.

6. Conclusions

In this paper we have introduced a generalization of the class of D-finite functions and derived several closure properties for this new class and discussed the issue of initial values in the ring of formal power series. The results presented here have been implemented in the open source mathematical software SAGE (Stein et al., 2017) and are described in Jiménez-Pastor and Pillwein (2018).

Among the known closure properties for D-finite functions a prominent one that has not been discussed here is the composition with algebraic functions. Also this closure property carries over to the setting of DD-finite functions. Furthermore, it can be shown that the composition of two D^k-finite functions is in a different layer. Also these closure properties have been implemented already and an article on these results is in preparation.

There are still many interesting open topics related to DD-finite functions, such as characterizing the different layers of $\mathrm{D}^k(R)$. We plan to apply our procedures to prove identities on DD-finite (but not D-finite) functions. In this context for D-finite functions a useful tool often is to apply first some guessing routine (Salvy and Zimmermann, 1994; Mallinger, 1996; Kauers, 2009) and then use, e.g., closure properties to prove the guessed identity. Among the open topics for DD-finite functions, the development of an efficient guessing tool is certainly one of the more interesting and pressing ones.

A natural question that has not been addressed in this paper, is the discrete analogon. Holonomic sequences are defined by linear recurrence equations with polynomial coefficients. This space is no longer an integral domain. Hence, the algebraic set-up has to be such as to avoid conflicts with zero divisors. Then the generalization of the closure properties derived in this paper to sequences satisfying linear recurrences with holonomic sequence coefficients should be immediate. Besides that, it will be interesting to study the structure of the coefficient sequences of DD-finite functions.

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