Chasing a Fast Robber on Planar Graphs and Random Graphs

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Received July 30, 2012; Revised December 26, 2013

Published online 13 March 2014 in Wiley Online Library (wileyonlinelibrary.com).
DOI 10.1002/jgt.21791

Abstract: We consider a variant of the Cops and Robber game, in which the robber has unbounded speed, that is, can take any path from her vertex in her turn, but she is not allowed to pass through a vertex occupied by a cop. Let $c_\infty(G)$ denote the number of cops needed to capture the robber in a graph G in this variant, and let $\mathrm{tw(G)}$ denote the treewidth of G. We show that if G is planar then $c_\infty(G) = \Theta(\mathrm{tw(G)})$, and there is a polynomial-time constant-factor approximation algorithm for computing $c_\infty(G)$. We also determine, up to constant factors, the value of $c_\infty(G)$ of the

Contract grant sponsor: ISF; Contract grant sponsor: USA-Israeli BSF; Contract grant sponsor: Israeli I-Core program.

Erdős–Rényi random graph $G = \mathcal{G}(n, p)$ for all admissible values of p, and show that when the average degree is $\omega(1)$, $c_\infty(G)$ is typically asymptotic to the domination number. J. Graph Theory 78: 81–96, 2015

Keywords: *cops and robber game; fast robber; planar graphs; treewidth; random graphs; domination number*

1. INTRODUCTION

The game of *Cops and Robber* is a perfect information game, played in a graph G. The players are a set of cops and a robber. Initially, the cops are placed at vertices of their choice in G (where more than one cop can be placed at a vertex). Then the robber, being fully aware of the cops' placement, positions herself at one of the vertices of G. Then the cops and the robber move in alternate rounds, with the cops moving first; however, players are permitted to remain stationary in their turn if they wish. The players use the edges of G to move from vertex to vertex. The cops win, and the game ends, if eventually a cop moves to the vertex currently occupied by the robber; otherwise, if the robber can elude the cops forever, the robber wins.

This game was defined (for one cop) by Nowakowski and Winkler [29] and Quilliot [33], and has been studied extensively, see Hahn [22] or Bonato and Nowakowski [8]. The best-known open question in this area is Meyniel's conjecture, published by Frankl [19], that states that for every connected graph on n vertices, $O(\sqrt{n})$ cops are sufficient to capture the robber. An intriguing fact about the Cops and Robber game is that although many scholars have studied the game it is not yet well understood. In particular, although the upper bound $O(\sqrt{n})$ was conjectured already in 1987, no upper bound better than $n^{1-o(1)}$ has been proved since then (see [20, 25, 34]).

One might try to change the rules of the game slightly in order to get a more approachable problem, and/or to understand what property of the original game causes the difficulty. Thus, various variations of the game have been studied [2, 7, 11, 20, 23]. The approach chosen by Fomin, Golovach, Kratochvíl, Nisse, and Suchan [18] is to allow the robber to move faster than the cops. Inspired by their work, in this article we let the robber take *any path* from her current position in her turn, but she is not allowed to pass through a vertex occupied by a cop. The parameter of interest is the *cop number* of G, which is defined as the minimum number of cops needed to ensure that the cops can win. We denote the cop number of G by $c_\infty(G)$, in which the ∞ at the subscript indicates that the robber has unbounded speed.

This variant was first studied by Fomin, Golovach, and Kratochvíl [17]. They proved that computing $c_\infty(G)$ is an NP-hard problem, even if G is a split graph. (A *split graph* is a graph whose vertex set can be partitioned into a clique and an independent set.) Next, Gavenčiak [21] gave a polynomial- time algorithm for interval graphs. This variant was further studied by Frieze, Krivelevich, and Loh [20], where the authors' approach is based on expansion. In [20], it is shown that for each n, there exists a connected graph on n vertices with cop number $\Theta(n)$. See the thesis of the second author [27] for more results on this variant, in particular about graphs with cop number one, interval graphs, chordal graphs, expander graphs, and Cartesian products of graphs.

We study this game on planar graphs and random graphs. Let tw(G) denote the treewidth of the graph G (see the next section for the formal definition). For planar graphs, we prove the following.

Theorem 1. *Let G be a connected planar graph on n vertices. Then we have $c_\infty(G) = \Theta(\mathrm{tw(G)}) = O(\sqrt{n})$, and there is a polynomial-time constant-factor approximation algorithm for computing $c_\infty(G)$.*

In fact, we show that the conclusion of Theorem 1 is true for graphs G that do not have a fixed apex graph as a minor. (An *apex graph* is a graph H that has a vertex v such that $H - v$ is planar.) Note that the $m \times m$ grid has cop number $\Omega(m)$ (see Theorem 3.3), so the bound $c_\infty(G) = O(\sqrt{n})$ in Theorem 1 is tight.

We denote the Erdős–Rényi random graph with parameters n and p by $\mathcal{G}(n, p)$. All asymptotics throughout are as $n \to \infty$. We say that an event in a probability space holds *asymptotically almost surely (a.a.s.)* if the probability that it holds approaches 1 as n goes to infinity. The second author [28] showed that if $np \geq 4.2 \log n$, then there are positive constants β_1, β_2 such that a.a.s.,

$$\beta_1/p \leq c_\infty\left(\mathcal{G}(n, p)\right) \leq \beta_2 \log(np)/p\,.$$

Let $\gamma(G)$ denote the domination number of the graph G (the formal definition appears in the next section). We prove the following theorem, tightening the result above and extending it to all admissible values of p.

Theorem 2.

(a) *If $27 \leq np = O(1)$, then there exist positive constants η_1, η_2 such that a.a.s.*

$$\eta_1 \log(np)/p \leq c_\infty(G) \leq \gamma(G) \leq \eta_2 \log(np)/p\,.$$

(b) *If $np = \omega(1)$ and $p = 1 - \Omega(1)$, then a.a.s.*

$$c_\infty(G) = (1 + o(1))\frac{\log(np)}{-\log(1-p)}\,.$$

(c) *If $np = \omega(1)$, then a.a.s.*

$$c_\infty(G) = (1 + o(1))\gamma(G)\,.$$

Note that if $np < 27$ then a.a.s. the graph has $\Omega(n)$ isolated vertices, and hence in this case $c_\infty(G) = \Theta(n)$. Therefore, the above theorem and the fact that the proof of its last part presented in Section 4 shows that for all $\Omega(1) \leq p \leq 1$, $c_\infty(G) = \gamma(G)$ a.a.s. determines the typical asymptotic behavior of $c_\infty(\mathcal{G}(n, p))$ for all admissible values of p.

The *lollipop graph* L_n is obtained from a complete graph on n vertices and a path on $n + 1$ vertices by identifying some vertex of the complete graph with an end vertex of the path. Notice that $c_\infty(H) \leq \gamma(H)$ is true for any graph H, since if the cops start from a dominating set, they will capture the robber in the first round. Also, it is not hard to see that $c_\infty(H) \leq \mathrm{tw(H)} + 1$ is true for any graph H (see Theorem 3.3). These upper bounds are far from being tight for general graphs, as L_n has cop number one but domination number and treewidth $\Theta(n)$. Theorems 1 and 2 state that the two crude upper bounds are actually tight up to constant factors for two important classes of graphs.

2. PRELIMINARIES

Let G be the graph in which the game is played. In the following G is always finite, and n always denotes the number of vertices of G. We will assume that G is simple, because deleting multiple edges or loops does not affect the set of possible moves of the players. Note that the cop number of a disconnected graph equals the sum of the cop numbers of its connected components, and hence it suffices to understand the behavior of this parameter for connected graphs. As we are only interested in studying the cop number (and not the number of rounds in the game), we may assume without loss of generality that the cops choose vertices of our choice in the beginning, since they can move to the vertices of their choice later.

For a subset A of vertices, the *neighborhood* of A, denoted by $N(A)$, is the set of vertices in $V(G) \setminus A$ that have a neighbor in A, and the *closed neighborhood* of A, written $\overline{N}(A)$, is the union $A \cup N(A)$. If $A = \{v\}$ then we may write $N(v)$ and $\overline{N}(v)$ instead of $N(A)$ and $\overline{N}(A)$, respectively. A *dominating set* is a subset A of vertices with $V(G) = \overline{N}(A)$, and the *domination number* of G, written $\gamma(G)$, is the minimum size of a dominating set of G. The subgraph induced by A is written $G[A]$, and the subgraph induced by $V(G) \setminus A$ is written $G - A$. All logarithms are in the natural base. Write $\Delta = \Delta(G)$ for the maximum degree in G.

A *tree decomposition* of a graph G is a pair (T, W), where T is a tree and $W = (W_t : t \in V(T))$ is a family of subsets of $V(G)$ such that

(i) $\bigcup_{t\in V(T)} W_t = V(G)$,
(ii) Every edge of G has both endpoints in some W_t, and
(iii) For every $v \in V(G)$, the set $\{t : v \in W_t\}$ induces a subtree of T.

The *width* of (T, W) is

$$\max\{|W_t| - 1 : t \in V(T)\},$$

and the *treewidth* of G, written tw(G), is the minimum width of a tree decomposition of G.

We will use the following facts about tree decompositions, whose proofs can be found in Section 12.3 of [15].

Proposition 2.1. *Let (T, W) be a tree decomposition of a graph G.*

(a) *Let A be the vertex set of a clique in G. Then, there is a $t \in V(T)$ with $A \subseteq W_t$.*
(b) *Let t_1t_2 be an edge of T, and let T_1 and T_2 be the components of $T - t_1t_2$, with $t_1 \in T_1$ and $t_2 \in T_2$. Define $X = W_{t_1} \cap W_{t_2}$, $U_1 = \cup_{t\in T_1} W_t$, and $U_2 = \cup_{t\in T_2} W_t$. Then, X is a cut-set in G, and there is no edge between $U_1 \setminus X$ and $U_2 \setminus X$.*

We will use the following large deviations inequalities (see, e.g., Appendix A of [4]).

Proposition 2.2. *Let $X = X_1 + X_2 + \cdots + X_m$, where the X_i are independent random variables taking values in $\{0, 1\}$. We have the following inequalities.*

(a) $\mathbb{P}[X \geq \mathbb{E}[X] + t] \leq \exp(-2t^2/m) \qquad \forall\, t \geq 0.$
(b) $\mathbb{P}[X \leq (1 - \epsilon)\mathbb{E}[X]] \leq \exp(-\epsilon^2\mathbb{E}[X]/2) \qquad \forall\, \epsilon \geq 0.$

3. PLANAR GRAPHS

In one of the first papers on the original Cops and Robber game, Aigner and Fromme [1] proved that three cops can capture the robber in any planar graph. In this section, we prove Theorem 1 that deals with the case of a fast robber in a planar graph.

Here is a high-level sketch of the proof. First, by relating our Cops and Robber game with the so-called Helicopter Cops and Robber game of Seymour and Thomas [35], we show that for any graph G,

$$\frac{\mathrm{tw(G)}+1}{\Delta(G)+1} \leq c_\infty(G) \leq \mathrm{tw(G)}+1.$$

Next, since the cop number cannot increase by contracting the edges, we may use a theorem from the bidimensionality theory of Demaine and Hajiaghayi [13] to infer that $c_\infty(G) = \Omega(\mathrm{tw(G)})$.

The Helicopter Cops and Robber game has two versions, and the one we define here is called jump-searching.

Definition. Helicopter Cops and Robber game (the jump-searching version). *For $X \subseteq V(G)$, an X-flap is the vertex set of a connected component of $G - X$. Two subsets $X, Y \subseteq V(G)$ touch if $\overline{N}(X) \cap Y \neq \emptyset$. A position is a pair (X, R), where $X \subseteq V(G)$ and R is an X-flap. (In the game X is the set of vertices currently occupied by the cops and R tells us where the robber is — since she can run arbitrarily fast, all that matters is which component of $G - X$ contains her.) At the start, the cops choose a subset X_0, and the robber chooses an X_0-flap R_0. Note that if there are k cops in the game, then $|X_0| \leq k$. At the start of round i, we have some position (X_{i-1}, R_{i-1}). The cops choose a new set $X_i \subseteq V(G)$ with $|X_i| \leq k$ (and no other restriction), and announce it. Then the robber, knowing X_i, chooses an X_i-flap R_i which touches R_{i-1}. If this is not possible then the cops have won. Otherwise, that is, if the robber never runs out of valid moves, the robber wins.*

The following lemma establishes a link between the two games.

Lemma 3.1. *Let G be a graph. If k cops can capture a robber with unbounded speed in the Cops and Robber game in G, then $k(\Delta + 1)$ cops can capture the robber in the Helicopter Cops and Robber game in G.*

Proof. We consider two games played in two copies of G: the first one, which we call the *real game*, is a game of Helicopter Cops and Robber with $k(\Delta + 1)$ cops; and the second one, the *virtual game*, is the usual Cops and Robber game with k cops and a robber with unbounded speed. Given a winning strategy for the cops in the virtual game, we need to give a capturing strategy for the cops in the real game. We translate the moves of the cops from the virtual game to the real game, and translate the moves of the robber from the real game to the virtual game, in such a way that all the translated moves are valid, and if the robber is captured in the virtual game, then she is captured in the real game as well. Hence, as the cops have a winning strategy in the virtual game, they have a winning strategy in the real game, too.

In the virtual game, initially the cops choose a subset C_0 of vertices. More precisely, C_0 is the set of vertices that accommodate at least one virtual cop. Then the real cops choose $X_0 = \overline{N}(C_0)$. Recall that $|C_0| \leq k$, so $|X_0| \leq k(\Delta + 1)$. The real robber chooses R_0, which is an X_0-flap, and the virtual robber chooses an arbitrary vertex $r_0 \in R_0$. In general, at the end of round $i - 1$ we have $X_{i-1} = \overline{N}(C_{i-1})$ and $r_{i-1} \in R_{i-1}$.

Suppose the virtual robber has not been captured at the end of round i. In round i, first the virtual cops move to a new set C_i. More precisely, C_i is the set of vertices that accommodate at least one virtual cop after the virtual cops' move. Each cop either stays still or moves to a neighbor, thus $C_i \subseteq \overline{N}(C_{i-1}) = X_{i-1}$ and since R_{i-1} was an X_{i-1}-flap, $C_i \cap R_{i-1} = \emptyset$. The real cops choose $X_i = \overline{N}(C_i)$ and announce it. The real robber, knowing X_i, chooses an X_i-flap R_i that touches R_{i-1}. If she cannot find a valid move, then she is captured and the lemma is proved. Otherwise, note that by definition $C_i \cap R_i = \emptyset$. Let r_i be an arbitrary vertex of R_i. The virtual robber moves from r_{i-1} to r_i. Since R_{i-1} and R_i touch, and both of them are connected, $R_{i-1} \cup R_i$ is connected. Moreover, C_i does not intersect $R_{i-1} \cup R_i$, so this is a valid move in the virtual game.

Now, suppose the virtual robber has not been captured at the end of round $i - 1$, but has been captured by the end of round i. We claim that if this happens then the real robber has already been captured in one of the previous rounds. If this is not the case, then in round i, the virtual cops move to a new set C_i such that $r_{i-1} \in C_i$. Each cop either stays still or moves to a neighbor, thus $C_i \subseteq \overline{N}(C_{i-1}) = X_{i-1}$ and since R_{i-1} was an X_{i-1}-flap, $C_i \cap R_{i-1} = \emptyset$. But $r_{i-1} \in C_i$ because the virtual robber has been captured in round i, and $r_{i-1} \in R_{i-1}$, thus $r_{i-1} \in C_i \cap R_{i-1}$, which is a contradiction. This shows that the real robber will be captured even before the virtual robber, and the proof is complete. ■

Seymour and Thomas [35] proved the following theorem.

Theorem 3.2 ([35]). *The minimum number of cops needed to capture a robber in the Helicopter Cops and Robber game is equal to the treewidth of the graph plus one.*

Using this, we have the following.

Theorem 3.3. *For every graph G, we have*

$$\frac{\text{tw(G)} + 1}{\Delta(G) + 1} \leq c_\infty(G) \leq \text{tw(G)} + 1 \ .$$

Proof. The lower bound follows from Lemma 3.1 and Theorem 3.2. To prove the upper bound, consider a tree decomposition (T, W) of G having minimum width. Assume that there are tw(G) + 1 cops in the game, so for every $t \in V(T)$, there are at least $|W_t|$ cops in the game. The cops start at W_{t_1} for some arbitrary $t_1 \in V(T)$. Assume that the robber starts at r_0, and let t be such that $r_0 \in W_t$. Let t_2 be the neighbor of t_1 in the unique (t_1, t)-path in T. Let T_1 and T_2 be the components of $T - t_1t_2$, with $t_1 \in T_1$ and $t_2 \in T_2$. Define $X = W_{t_1} \cap W_{t_2}$, $U_1 = \cup_{t \in T_1} W_t$, and $U_2 = \cup_{t \in T_2} W_t$. So, the cops are all in U_1 and the robber is at a vertex in $U_2 \setminus X$. Note that the number of cops is at least $|W_{t_2}|$. Now the cops move in order to occupy W_{t_2}, in such a way that the cops in X stay still. After some rounds, the cops will be located at W_{t_2}, and during those rounds the robber could not escape from $U_2 \setminus X$, because by part (b) of Proposition 2.2, there is no edge between $U_1 \setminus X$ and $U_2 \setminus X$. When the cops have covered W_{t_2}, the total space available to the robber has been decreased. Continuing similarly the cops will eventually capture the robber. ■

Remark. The complete graph on n vertices shows that the lower bound is tight. The upper bound is also tight: start with $m \geq 4$ vertices, and add m disjoint paths of length 3 between any two of them. This graph has treewidth $m - 1$ and cop number m. The details can be found in Theorem 4.5 of [27].

Recall that an apex graph is a graph H that has a vertex v such that $H - v$ is planar. The following theorem was proved in a weaker form by Demaine, Fomin, Hajiaghayi, and Thilikos [12], and then in its current form by Demaine and Hajiaghayi [14].

Theorem 3.4 ([12, 14]). *Let H be a fixed apex graph. There is a constant C_H such that the following holds. Let $g : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function, and $P(G)$ be a graph parameter with the following two properties.*

1. *If G is the $r \times r$ grid augmented with additional edges such that each vertex is incident to C_H edges connected to nonboundary vertices of the grid, then $P(G) \geq g(r)$.*
2. *$P(G)$ does not increase by contracting an edge of G.*

Then, for any graph G that does not contain H as a minor, the treewidth of G is $O\left(g^{-1}(P(G))\right)$.

Now, we prove Theorem 1.

Proof of Theorem 1. We show that the parameter $c_\infty(G)$ satisfies the two properties given in Theorem 3.4, with $g(r) = (r+1)/(5+C_H)$. First, an augmented $r \times r$ grid has treewidth r and maximum degree at most $4 + C_H$, so by Theorem 3.3 its cop number is at least $(r+1)/(5+C_H)$. Second, we need to show that the cop number does not increase by contracting an edge. It is easy to see that contracting an edge does not help the robber, since she has unbounded speed, and it does not hurt the cops. Therefore, contracting an edge does not increase the cop number.

Let H be the complete graph on five vertices. Since G is planar, it does not contain H as a minor. Therefore, by Theorem 3.4, we have $\mathrm{tw(G)} = O(c_\infty(G))$. By Theorem 3.3, $c_\infty(G) \leq \mathrm{tw(G)} + 1$, so we have $c_\infty(G) = \Theta(\mathrm{tw(G)})$. Moreover, it is known (see, e.g., [3]) that any G that does not have H as a minor has $\mathrm{tw(G)} = O(\sqrt{n})$. Finally, Feige, Hajiaghayi, and Lee [16] have developed a polynomial-time $O(1)$-approximation algorithm for finding the treewidth of a graph that does not contain H as a minor. ■

4. RANDOM GRAPHS

The original Cops and Robber game for Erdős–Rényi random graphs has been studied by several authors [6,9,26,31]. In particular, Prałat and Wormald [32] proved that Meyniel's conjecture holds for random graphs. In this section, we prove Theorem 2 that determines the typical asymptotic behavior of $c_\infty(G)$ for the Erdős–Rényi random graph.

Here is a high-level sketch of the proof. To give an upper bound for the cop number, we simply bound the domination number. For the lower bound, we give an escaping strategy for the robber. We first find a number s such that a.a.s. there is an edge between any two disjoint subsets of vertices of size s. It follows that a.a.s. any subset A of vertices of size at least $3s$ induces a connected component of size at least s. Call this component a *continent* of A. Now, if the number of cops is small enough such that when they are in a subset S of vertices, $\overline{N}(S)$ leaves out at least $3s$ vertices, then the robber always moves to a continent R of $V(G) \setminus \overline{N}(S)$. Assume that the cops move to a new subset S' in their turn, so $\overline{N}(S')$ also leaves out at least $3s$ vertices, hence there is a continent $R' \subseteq V(G) \setminus \overline{N}(S')$. Now, since R and R' have size at least s, if they do not intersect, there is an edge between them. In either case the robber can move to R', and thus she will never be captured.

Let $G = \mathcal{G}(n, p)$ and $d = d(n) := np$. As mentioned after the statement of Theorem 2, the fact that for, say, $d < 27$, $c_\infty(G) = \Theta(n)$ a.a.s. is very simple, and hence we may and will assume that $d \geq 27$. In particular, assume that $d > \exp(1)$, and that n is sufficiently large.

Lemma 4.1. *A.a.s. for any two disjoint subsets $A_1, A_2 \subseteq V(G)$ of size at least $en \log d/d$, there exists an edge between A_1 and A_2.*

Proof. By the union bound, the probability that there exist disjoint A_1, A_2 of size $en \log d/d$ with no edge between them is at most

$$\binom{n}{en \log d/d}^2 (1-p)^{(en \log d/d)^2} \leq \left(\frac{ed}{e \log d}\right)^{2en \log d/d} \exp(-pe^2 n^2 \log^2 d/d^2)$$
$$= \exp\left(\frac{2en \log d(\log d - \log\log d) - e^2 n \log^2 d}{d}\right) = o(1).$$

■

The following lemma can be easily proved (see, e.g., Lemma 7.3 in [27]).

Lemma 4.2. *Let $a_1, a_2, \ldots, a_m$ be positive integers such that each of them is at most $t/2$, and their sum is t. Then one can choose a subset of $\{a_1, \ldots, a_m\}$ whose sum is between $t/3$ and $t/2$ (inclusive).*

The following corollary follows from Lemmas 4.1 and 4.2.

Corollary 4.3. *A.a.s. for any $A \subseteq V(G)$ of size at least $3en \log d/d$, there exists a connected component of $G[A]$ of size at least $en \log d/d$.*

We are now ready to prove our main lower bound for the cop number.

Theorem 4.4. *Assume that $p = o(1)$, $d = np$. Let α, α' be fixed numbers with $0 < \alpha < \alpha' < 1$ and such that for all sufficiently large n,*

$$(3e + \alpha) \log d < d^{1-\alpha'}, \tag{1}$$

and

$$\left(1 - (3e + \alpha) d^{\alpha' - 1} \log d\right)^2 d^{1-\alpha'} > 2\alpha (\log d) \log\left(\frac{de^2}{\alpha \log d}\right). \tag{2}$$

Then a.a.s.

$$c_\infty(G) > \alpha n \log d/d.$$

Remark. For any fixed d with $d > 3e \log d$ there exist fixed $0 < \alpha < \alpha' < 1$ satisfying (1) and (2). The largest d for which such α, α' do not exist is around 26.82. Also, if $d = \omega(1)$, then any fixed $0 < \alpha < \alpha' < 1$ satisfy (1) and (2).

Proof.
Claim. A.a.s. for any subset $X \subseteq V(G)$ of size $\alpha n \log d/d$, we have

$$|V(G) \setminus N(X)| > (3e + \alpha) n \log d/d.$$

Proof of Claim. Let $y = \alpha' \log d$ and let X be a set of size $\alpha n \log d/d$. Then for any vertex $v \in V(G)$, since $p = o(1)$, for n large enough,

$$\mathbb{P}[v \notin N(X)] \geq (1-p)^{|X|} \geq \exp(-p\alpha'/\alpha)^{|X|} = e^{-y}.$$

Thus

$$\mathbb{E}[|V(G) \setminus N(X)|] \geq ne^{-y} = nd^{-\alpha'}.$$

Define

$$\varepsilon = 1 - (3e+\alpha)d^{\alpha'-1}\log d\,.$$

Note that $\varepsilon > 0$ by (1), so by Proposition 2.2(b),

$$\begin{aligned}
&\mathbb{P}\big[|V(G) \setminus N(X)| \leq (3e+\alpha)n\log d/d\big] \\
&\quad\leq \mathbb{P}\big[|V(G) \setminus N(X)| \leq (1-\varepsilon)\mathbb{E}[|V(G) \setminus N(X)|]\big] \\
&\quad\leq \exp\left(-\varepsilon^2 nd^{-\alpha'}/2\right).
\end{aligned}$$

The total number of such sets X is

$$\binom{n}{\alpha n \log d/d} \leq \left(\frac{ed}{\alpha \log d}\right)^{\alpha n \log d/d} = \exp\left(\frac{\alpha n \log d}{d} \log\left(\frac{ed}{\alpha \log d}\right)\right).$$

We have

$$\exp\left[n\left(\frac{\alpha \log d}{d}\log\left(\frac{ed}{\alpha\log d}\right) - \frac{\varepsilon^2 d^{-\alpha'}}{2}\right)\right] \leq \exp\left[-n\alpha\log d/d\right] = o(1),$$

where we have used (2). So by the union bound, a.a.s. for every X of size $\alpha n \log d/d$ we have

$$|V(G) \setminus N(X)| > (3e+\alpha)n\log d/d. \qquad \square$$

Assuming that there are $\alpha n \log d/d$ cops in the game, we give an escaping strategy for the robber. For a subset $S \subseteq V(G)$, denote by G/S the subgraph obtained by deleting the vertices in $\overline{N}(S)$. The following conditions are satisfied a.a.s.

1. For any subset $S \subseteq V(G)$ of size at most $\alpha n \log d/d$, G/S has at least $3en\log d/d$ vertices. This is true with probability $1 - o(1)$ by the claim.
2. For any subset $S \subseteq V(G)$ of size at most $\alpha n \log d/d$, the largest connected component of G/S has at least $en \log d/d$ vertices. This is true with probability $1 - o(1)$ by Corollary 4.3 and the previous condition.
3. For any two disjoint subsets $A_1, A_2 \subseteq V(G)$ of size at least $e\log(np)/p$, there is an edge between A_1 and A_2. This is true with probability $1 - o(1)$ by Lemma 4.1.

The robber plays in such a way that whenever the cops are in a subset S, she is in a component of G/S with at least $en \log d/d$ vertices. This ensures that she will never be captured. She can clearly position herself as required in the beginning. Assume that at the end of round i, the cops are in S_i, the robber is in some vertex in C_i, where C_i is a component of G/S_i with at least $en\log d/d$ vertices. In round $i+1$, the cops move to S_{i+1}. Note that S_{i+1} and C_i are disjoint. Let C_{i+1} be a component of G/S_{i+1} with at least $en \log d/d$ vertices. Either C_i and C_{i+1} have a vertex in common, or they are disjoint. In the latter case, since both have size at least $en \log d/d$, there is an edge between them. In either case the robber moves to a vertex in C_{i+1}, and this completes the proof. ■

Now we turn to proving upper bounds for $\gamma(G)$, which results in upper bounds for $c_\infty(G)$. The following lemma is an adaptation of Theorem 1.2.2 in [4].

Lemma 4.5. *Let d_0 be a positive integer and let H be a graph on n vertices, which has at most a vertices of degree less than $d_0 - 1$. Then we have*

$$\gamma(H) \le \frac{1 + \log d_0}{d_0} n + a.$$

Proof. Let $q = \log d_0/d_0$, and let A be the set of vertices of H with degree less than $d_0 - 1$. Form a random subset $X \subseteq V(H)$ by choosing each vertex independently with probability q. Let

$$Y_X = V(H) \setminus (\overline{N}(X) \cup A),$$

and note that $X \cup Y_X \cup A$ is a dominating set for H.

For every vertex v, the probability that $v \in Y_X$ is at most $(1-q)^{d_0} \le e^{-qd_0}$, since for this to happen v must have degree at least $d_0 - 1$, and none of v and its neighbors must have been chosen. We conclude that

$$\mathbb{E}[|X \cup Y_X \cup A|] \le \mathbb{E}[|X|] + \mathbb{E}[|Y_X|] + \mathbb{E}[|A|] \le nq + ne^{-qd_0} + a \le \frac{1 + \log d_0}{d_0} n + a.$$

Hence there is a choice of X for which

$$|X \cup Y_X \cup A| \le \frac{1 + \log d_0}{d_0} n + a.$$

■

By bounding the number of vertices having small degree, we get the following.

Theorem 4.6. *Let $\delta \in (0, 1)$ be fixed.*

(a) *If $d = \omega(1)$, then a.a.s.*

$$\gamma(G) \le \frac{1 + \log d}{(1-\delta)p} + dn \exp(-\delta^2 d/2).$$

(b) *If $d = O(1)$, then a.a.s.*

$$\gamma(G) \le \frac{1 + \log d}{(1-\delta)p} + (1 + o(1))n \exp(-\delta^2 d/8).$$

Proof. Let $d_0 = (1-\delta)d$. Say vertex v is *terrible* if its degree is less than $d_0 - 1$. We show that if $d = \omega(1)$, then the number of terrible vertices is a.a.s. at most $dn \exp(-\delta^2 d/2)$, and if $d = O(1)$ then the number of terrible vertices is a.a.s. at most $(1 + o(1))n \exp(-\delta^2 d/8)$. Then we are done by Lemma 4.5 since

$$\frac{1 + \log d_0}{d_0} n = \frac{1 + \log((1-\delta)d)}{(1-\delta)d} n \le \frac{1 + \log d}{(1-\delta)p}.$$

(a) For every vertex v, deg(v) is a binomial random variable with parameters $n-1$ and p, so by Proposition 2.2(b),

$$\mathbb{P}\Big[\deg(\mathrm{v}) \le d_0 - 1\Big] \le \mathbb{P}\Big[\deg(\mathrm{v}) \le (1-\delta)\mathbb{E}[\deg(\mathrm{v})]\Big] \le \exp(-\delta^2(n-1)p/2).$$

Thus the expected number of terrible vertices is at most $n\exp(-\delta^2(n-1)p/2)$. Since $d = \omega(1)$, by the Markov inequality the number of terrible vertices is a.a.s. at most $dn\exp(-\delta^2 d/2)$.

(b) First, assume that $\sqrt{n}$ is an integer and let $k = \sqrt{n}$. Partition $V(G)$ arbitrarily into k parts $B_1, \dots, B_k$ of size k. For a vertex v in part B_i, say v is *bad* if

$$|N(v) \cap (V(G) \setminus B_i)| < (1-\delta)np - 1.$$

Clearly, any terrible vertex is bad. Observe that for any s vertices $v_1, \dots, v_s$ in the same part, the events $\{v_j \text{ is bad} : 1 \le j \le s\}$ are mutually independent. Fix an $\epsilon \in (0,1)$ and we will show that a.a.s. the number of bad vertices is at most $(1+\epsilon)n\exp(-\delta^2 np/8)$.

Say a part is *bad* if it has more than $(1+\epsilon)k\exp(-\delta^2 np/8)$ bad vertices. Consider a part $B_i = \{v_1, v_2, \dots, v_k\}$ and let X_j be the indicator variable for v_j being bad. Thus $X := X_1 + X_2 + \cdots + X_k$ is the number of bad vertices of B_i. For each j, notice that $\mathbb{E}[|N(v_j) \cap (V(G) \setminus B_i)|] = (n-k)p$ so by Proposition 2.2(b)

$$\begin{aligned}\mathbb{E}[X_j] &= \mathbb{P}\Big[|N(v_j) \cap (V(G)\setminus B_i)| < (1-\delta)np - 1\Big] \\ &\le \mathbb{P}\left[|N(v_j) \cap (V(G)\setminus B_i)| \le \left(1 - \frac{\delta}{\sqrt{2}}\right)(n-k)p\right] \\ &\le \exp(-\delta^2(n-k)p/4).\end{aligned}$$

Hence

$$\mathbb{E}[X] = \mathbb{E}[X_1 + X_2 + \cdots + X_k] \le k\exp(-\delta^2(n-k)p/4).$$

Let $\mu := k\exp(-\delta^2(n-k)p/4)$. The probability that B_i is bad is at most

$$\mathbb{P}[X > (1+\epsilon)k\exp(-\delta^2 np/8)] \le \mathbb{P}[X > (1+\epsilon)\mu] \le \exp(-2(\epsilon\mu)^2/k),$$

where we have used Proposition 2.2(a) and the fact that the X_is are mutually independent.

So by the union bound, the probability that there exists a bad part is at most

$$k\exp(-2(\epsilon\mu)^2/k) = k\exp\left[-2\epsilon^2 k\exp(-\delta^2(n-k)p/2)\right],$$

which is $o(1)$ as $np = O(1)$ and $k = \omega(1)$. Consequently, a.a.s. there is no bad part, and the total number of bad vertices is at most $(1+\epsilon)n\exp(-\delta^2 np/8)$.
Now, assume that $\sqrt{n}$ is not an integer, and write $V(G) = V_0 \cup A$, where $|V_0|$ is a square number, and $|A| = O(\sqrt{n})$. Partitioning V_0 and doing the same analysis as above shows that there are at most $(1+o(1))n\exp(-\delta^2 np/8)$ terrible vertices in V_0. Thus G has at most $(1+o(1))n\exp(-\delta^2 np/8) + |A| = (1+o(1))n\exp(-\delta^2 np/8)$ terrible vertices, and the proof is complete. ■

Finally, we prove Theorem 2 .

Proof of Theorem 2.

(a) Since $3e\log(27) < 27$, there exist fixed $0 < \alpha < \alpha' < 1$ satisfying the conditions of Theorem 4.4. Setting $\eta_1 = \alpha$ proves the lower bound.
For the upper bound, let $\delta = \sqrt{8/27}$ and let η_2 be the constant satisfying

$$\frac{1+\log 27}{1-\delta} + 2\times 27\times \exp(-\delta^2\times 27/8) = \eta_2 \log 27.$$

It can be verified, by differentiating both sides with respect to x, that for all $x \geq 27$ we have

$$\frac{1+\log x}{1-\delta} + 2x\exp(-\delta^2 x/8) \leq \eta_2 \log x.$$

Thus, by Theorem 4.6(b), a.a.s.

$$\gamma(G) \leq \frac{1+\log d}{(1-\delta)p} + (1+o(1))n\exp(-\delta^2 d/8) \leq \eta_2 n\log d/d.$$

(b) When $p = \Omega(1)$, the bound follows from part (c) of Theorem 4.6 in [28]. When $p = o(1)$, note that $p = (1+o(1))(-\log(1-p))$. Fix $\epsilon > 0$, and we will show that

$$(1-\epsilon)\frac{\log d}{p} \leq c_\infty(G) \leq \gamma(G) \leq (1+\epsilon)\frac{\log d}{p}.$$

First, since $d = \omega(1)$, the pair $(\alpha, \alpha') = (1-\epsilon, 1-\epsilon/2)$ satisfy the conditions of Theorem 4.4, thus by that theorem, a.a.s.

$$c_\infty(G) \geq (1-\epsilon)n\log d/d = (1-\epsilon)\frac{\log d}{p}.$$

For the upper bound, pick a $\delta \in (0, 1/2)$ small enough so that $(1-\delta)(1+\epsilon/2) \geq 1$. By Theorem 4.6(a), a.a.s. we have

$$\begin{aligned}
\gamma(G) &\leq \frac{1+\log d}{(1-\delta)p} + dn\exp(-\delta^2 d/2)\\
&= \frac{1}{p}\left[\frac{1}{1-\delta} + \frac{\log d}{1-\delta} + d^2\exp\left(-\delta^2 d/2\right)\right]\\
&\leq \frac{1}{p}\left[2 + \left(1+\frac{\epsilon}{2}\right)\log d + o(1)\right]\\
&< (1+\epsilon)\log d/p.
\end{aligned}$$

(c) When $p = o(1)$, the proof of part (b) does the job. Now, assume that $p = \Omega(1)$. Note that for this value of p, G is a.a.s. $(\sqrt{n})$-connected, and a.a.s. $\gamma(G) < \sqrt{n}$. Indeed, a.a.s. any two vertices of G have $\Omega(n)$ common neighbors, and hence even after deleting $\sqrt{n}$ vertices the remaining graph still has diameter 2 (in fact, it is known that a.a.s. the connectivity of G is equal to its minimum degree-see, e.g., [5] Chapter VII.2, but we only need here the much weaker bound stated above). Conditioned on these two events, we show that $c_\infty(G) \geq \gamma(G)$. Indeed, if there are less than $\gamma(G)$ cops in the game, then there exists a nondominated vertex in every round, and since G is $\gamma(G)$-connected, there exists an unblocked path from the robber's vertex to that vertex; so the robber can move there, and will never be captured. ■

5. CONCLUDING REMARKS

Note that each cop has two functions: attacking and blocking. Consider a version in which each cop can just attack, or in other words, the robber can jump over the cops. Let $c_A(G)$ denote the cop number of G in this version. Now, consider another version in which each cop can just block, and say the robber is captured if she is in a vertex v such that the cops occupy $\overline{N}(v)$. Let $c_B(G)$ denote the cop number of G in the second version. Then it is clear that $c_\infty(G) \leq c_A(G) = \gamma(G)$, and that $c_\infty(G) \leq c_B(G) \leq \mathrm{tw(G)} + 1$.

Therefore, in this notation Theorem 1 asserts that if G is planar, then $c_\infty(G) = \Theta(c_B(G))$. In other words, *blocking* is the crucial function in planar graphs. On the other hand, Theorem 2 asserts that if G is random with average degree $\omega(1)$, then $c_\infty(G) = (1 + o(1))c_A(G)$, that is, in a random graph, the *attacking* function is the crucial one. This shows an interesting contrast between planar graphs and random graphs in the context of pursuit-evasion games.

In the proof of Theorem 1, we showed that the cop number is a bidimensional parameter, and used this to conclude that it is of the same order as the treewidth (in planar graphs). In almost all bidimensional parameters previously studied [13], the parameter of interest is of quadratic order in the treewidth (a typical example is the domination number). So, this article might be the first place where this theory is used to give a nontrivial result for a parameter that is linear in the treewidth.

We conclude with some open questions and directions for further research. As mentioned in the Introduction, treewidth and domination number are two easy upper bounds for the cop number, and Theorems 1 and 2 imply that they are tight up to constant factors for two well-studied graph classes. It would be interesting to find other natural graph classes for which these bounds are tight.

Let H be a fixed apex graph. It follows from the proof of Theorem 1 that if G does not have H as a minor, then $c_\infty(G) = \Theta(tw(G))$. It is natural to ask whether the conclusion is true when H is a general graph. Also, one can ask what is the largest constant κ such that any planar G has $\kappa\mathrm{tw(G)} \leq c_\infty(G)$.

It would be interesting to determine the cop number of the m-dimensional hypercube graph $\mathcal{H}_m$. Note that since $\mathcal{H}_m$ has maximum degree m, treewidth $\Theta\left(2^m/\sqrt{m}\right)$ (see [10]), and domination number $\Theta\left(2^m/m\right)$ (see [30], or note that this follows easily from the existence of Hamming codes), there exist positive constants ζ_1, ζ_2 such that

$$\frac{\zeta_1 2^m}{m\sqrt{m}} \leq c_\infty(\mathcal{H}_m) \leq \frac{\zeta_2 2^m}{m} .$$

Fomin et al. [17] proved that computing $c_\infty(G)$ is NP-hard, but it is actually not known if this problem is in NP (Kinnersley [24] has recently proved that it is EXPTIME-complete to decide if k cops can capture the robber in the original version of the game, with cops and robber of speed one). To show that this problem is in NP, one needs to prove that there is always an efficient way to describe the cops' strategy. This has been done for the Helicopter Cops and Robber game [35]. As another algorithmic question, it would be interesting to extend the constant-factor approximation algorithm of Theorem 1 for computing the cop number of planar graphs to other graph classes, and/or to prove hardness of approximation results for general graphs.

ACKNOWLEDGMENTS

The second author is grateful to Nick Wormald for continuous support and lots of fruitful discussions. We thank the anonymous referees for useful comments.

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