

# Logics Meet 2-Way 1-clock Alternating Timed Automata

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## Abstract

In this paper, we study the extension of 1-clock Alternating Timed Automata (1-ATA) with ability to read in both forward and backward direction, the 2-Way 1-clock Alternating Timed Automata (2-Way 1-ATA). We show that subclass of 2-Way 1-ATA with reset free loops (2-Way 1-ATA-rfl) is expressively equivalent to  $\text{MSO}[\prec]$  extended with Guarded Metric Quantifiers (GQMSO). Emptiness Checking problem for 2-Way 1-ATA-rfl (and hence GQMSO) is undecidable, in general. We propose a “non punctuality” like restriction, called non adjacency, for 2-Way 1-ATA-rfl, and also for GQMSO, for which the emptiness (respectively, satisfiability) checking becomes decidable. Non-Adjacent 2-Way 1-ATA is the first such class of Timed Automata with alternations and 2-wayness for which the emptiness checking is decidable (and that too with elementary complexity). We also show that 2-Way 1-ATA-rfl, even with the non-adjacent restrictions, can express properties is not recognizable using 1-ATA.

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## 1 Introduction and Related Work

Exploring connections between different logics (e.g. the Kamp Theorem) and also between logics and automata (e.g. the Buchi Theorems) has been an active and influential area of work. Such connections often bring an ability to analyze logical questions algorithmically. Unfortunately, it has been challenging to find such tight connections between numerous timed logics and timed automata which have been proposed in the literature.

The 1-way 1-clock Alternating Timed Automata (1-ATA) were proposed as a Boolean closed model of timed languages with decidable emptiness. These were used to show the decidability of the future fragment of real-time logic  $\text{MTL}[\text{U}]$  (see [20] [17])<sup>1</sup>. However, the logic was not expressively complete for these automata. Exploring connections between real-time classical and temporal logics, Rabinovich [9] as well as Hunter [12] showed that logic  $\text{MITL}[\text{U}, \text{S}]$  extended with Pnueli modalities has the same expressive power as logic

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<sup>1</sup> These results are proved for automata and logics over finite timed words with point-wise interpretation. We shall also follow this in the current paper.



Q2MLO. The current authors [16] defined a more expressive and yet decidable extension of  $\text{MTL}[U]$  called  $\text{RatMTL}$ , and showed that this is expressively equivalent to the subclass of 1-ATA where all loops are reset free (1-ATA-rfl). Moreover these are expressively equivalent to a future time logic  $\text{QkMSO}$ .

The current paper explores a major extension of these results to logics and automata with both future and past. We show that 2-Way extension of 1-ATA-rfl (2-Way 1-ATA-rfl) is expressively equivalent to an extension of  $\text{MSO}[<]$  with Guarded Metric Quantifiers (GQMSO). The latter is a versatile and expressive logic, allowing properties of real-time systems to be defined conveniently. The use of Guarded Metric Quantifiers appeared in the pioneering formulations of logics QMLO and Q2MLO by Hirshfeld and Rabinovich [9] and it was further explored by Hunter [12]. We have generalized these to an anchored block of guarded quantifiers with arbitrary depth. This gives us the required power for obtaining expressive completeness.

To show the reduction from GQMSO to 2-Way 1-ATA-rfl (and vice versa), the proof factors via a recently proposed extension of MTL with “Pnueli-Automata” Modalities. This logic has been called Pnueli Extended Metric Temporal Logic (PnEMTL) [14]. Hence, as our first main result, we show the exact expressive equivalence  $\equiv$  of the following, by giving effective reductions between them. The readers may note the conceptual similarity of these results to the celebrated Kamp and Buchi Theorems.

$$\text{2-way 1-ATA-rfl} \equiv \text{PnEMTL} \equiv \text{GQMSO} \quad (1)$$

Unfortunately, the full PnEMTL being a syntactic extension of  $\text{MTL}[U, S]$ , is clearly undecidable. Hence, emptiness checking and satisfiability checking for both 2-Way 1-ATA-rfl and GQMSO are undecidable.

In [14], the authors have also proposed a novel generalization of the non punctuality condition of MITL to a non adjacency condition and shown that the non adjacent fragments of both PnEMTL as well as  $1\text{-TPTL}[U, S]$  have decidable satisfiability with  $\text{EXPSPACE}$ -complete complexity.

As our second contribution we define the non adjacency condition, suitably applied to 2-way 1-ATA automata and the logic GQMSO. We observe that the effective reductions between these formalisms and PnEMTL preserve this non adjacency. From the previously established  $\text{EXPSPACE}$ -complete decidability of non adjacent PnEMTL (see [14]), it follows that the emptiness of non adjacent 2-way 1-ATA-rfl as well as the satisfiability of non adjacent GQMSO are decidable. In fact, the former is  $\text{EXPSPACE}$ -complete. We also show that Non adjacent 2-Way 1-ATA-rfl can express properties that cannot be specified in 1-ATA, making their expressive powers incomparable.

To the best of our knowledge, this gives the first subclass of 2-way Alternating Timed Automata which has an elementary complexity for emptiness checking. In the past, Alur and Henzinger have explored 2-way deterministic timed automata with bounded reversals (Bounded 2DTA) and shown that their non emptiness is decidable with  $\text{PSPACE}$  complexity [3]. Ouaknine and Worrell [20] showed that emptiness checking of 1-ATA is decidable with non primitive recursive complexity and undecidable over infinite timed words. Abdulla *et al* [1] showed that generalizing 1-ATA, by allowing  $\epsilon$ -transitions, 2-wayness or omega words leads to undecidability of the emptiness checking problem. Thus, our model **non adjacent 2-Way 1-ATA with reset free loops**, is quite delicately poised. The expressively complete and decidable logic **Non adjacent GQMSO** can be seen as a powerful decidable generalization of Hirshfeld and Rabinovich’s Q2MLO [9] [10]. All our results, including decidability, hold for infinite timed words too.

## 2 Preliminaries

Let  $\Sigma$  be a finite set of propositions, and let  $\Gamma = 2^\Sigma \setminus \emptyset$ . A (finite) word over  $\Sigma$  is a (finite) sequence  $\sigma = \sigma_1 \sigma_2 \dots \sigma_n$ , where  $\sigma_i \in \Gamma$ . A (finite) timed word  $\rho$  over  $\Sigma$  is a (finite) sequence of pairs in  $\Gamma \times \mathbb{R}_{\geq 0}$ ;  $\rho = (\sigma_1, \tau_1) \dots (\sigma_n, \tau_n) \in (\Gamma \times \mathbb{R}_{\geq 0})^*$  where  $\tau_1 = 0$  and  $\tau_i \leq \tau_j$  for all  $1 \leq i \leq j \leq n$ . The  $\tau_i$  are called time stamps. For a timed or untimed word  $\rho$ , let  $\text{dom}(\rho) = \{i | 1 \leq i \leq |\rho|\}$ . The length of  $\rho$  is denoted  $|\rho|$ . Given a (timed) word  $\rho$  and  $i \in \text{dom}(\rho)$ , a pointed (timed) word is the pair  $\rho, i$ . The set of all timed words over  $\Gamma$  is denoted by  $T\Gamma^*$ . Let  $\mathcal{I}_+$  ( $\mathcal{I}_-$ ) be the set of open, half-open or closed time intervals containing real numbers, such that the end points of these intervals are in  $\mathbb{N} \cup \{0, \infty\}$  ( $\mathbb{Z} \cup \{-\infty\} \setminus \mathbb{N}$ ). Let  $\mathcal{I}_{+,-} = \mathcal{I}_+ \cup \mathcal{I}_-$ . For  $\tau \in \mathbb{R}$  and interval  $\langle a, b \rangle$ , with  $\langle \in \{ (, [ \}$  and  $\rangle \in \{ ], ) \}$ ,  $\tau + \langle a, b \rangle$  stands for the interval  $\langle \tau + a, \tau + b \rangle$ .

### 2.1 Anchored Interval Word Abstractions

Let  $I_\nu \subseteq \mathcal{I}_{+,-}$ . An  $I_\nu$ -interval word over  $\Sigma$  is a word  $\kappa$  of the form  $a_1 a_2 \dots a_n \in (2^{\Sigma \cup \{\text{anch}\}} \cup I_\nu)^*$ . There is a unique  $i \in \text{dom}(\kappa)$  called the *anchor* of  $\kappa$  such that  $\text{anch} \in a_i$ . Let  $J$  be any interval in  $I_\nu$ . We say that a point  $i \in \text{dom}(\kappa)$  is a  $J$ -time restricted point if and only if,  $J \in a_i$ .  $i$  is called time restricted point if and only if either  $i$  is  $J$ -time restricted for some interval  $J$  in  $I_\nu$  or  $\text{anch} \in a_i$ .

**From  $I_\nu$ -interval word to Timed Words.** Given an  $I_\nu$ -interval word  $\kappa = a_1 \dots a_n$  over  $\Sigma$  and a timed word  $\rho = (b_1, \tau_1) \dots (b_m, \tau_m)$ , the pointed timed word  $\rho, i$  is consistent with  $\kappa$  iff  $\text{dom}(\rho) = \text{dom}(\kappa)$ ,  $i = \text{anch}(\kappa)$ , for all  $j \in \text{dom}(\kappa)$ ,  $b_j = a_j \cap \Sigma$  and for  $j \neq i$ ,  $I \in a_j \cap I_\nu$  implies  $\tau_j - \tau_i \in I$ . Intuitively, each point  $j$  of  $\kappa$  does the following. (i) It stores the set of propositions that are true at point  $j$  of  $\rho$  and (ii) It also stores the set of intervals  $\mathcal{I} \subseteq I_\nu$  such that the time difference between point  $i$  and  $j$  of  $\rho$  lies within  $\bigcap \mathcal{I}$ , thus abstracting the time differences from the anchor point( $i$ ) using some set of intervals in  $I_\nu$ . We denote the set of all the pointed timed words consistent with a given interval word  $\kappa$  as  $\text{Time}(\kappa)$ . Similarly, given a set  $\Omega$  of  $I_\nu$  interval words,  $\text{Time}(\Omega) = \bigcup_{\kappa \in \Omega} (\text{Time}(\kappa))$ .

► **Example 1.** Let  $\kappa = \{a, b, (-1, 0)\} \{b, (-1, 0)\} \{a, \text{anch}\} \{b, [2, 3]\}$  be an interval word over the set of intervals  $\{(-1, 0), [2, 3]\}$ . Consider timed words  $\rho = (\{a, b\}, 0) (\{b\}, 0.5) (\{a\}, 0.95) (\{b\}, 3)$  and  $\rho' = (\{a, b\}, 0) (\{b\}, 0.8) (\{a\}, 0.9) (\{b\}, 2.9)$ . Then  $\rho, 3$  as well as  $\rho', 3$  are consistent with  $\kappa$  while  $\rho, 2$  is not. Likewise, for the timed word  $\rho'' = (\{a, b\}, 0) (\{b\}, 0.5) (\{a\}, 1.1) (\{b\}, 3)$ ,  $\rho'', 3$  is not consistent with  $\kappa$  as  $\tau_1 - \tau_3 \notin (-1, 0)$ , as also  $\tau_3 - \tau_2 \notin [2, 3]$ .

Note that the “consistency relation” is a many-to-many relation. For set of intervals  $I_\nu$ , a pointed timed word  $\rho, i$  can be consistent with more than one  $I_\nu$ -interval word and vice versa. Full technical details on interval words can be found in appendix B.

### 2.2 MSO with guarded metric quantifiers, GQMSO

We define a real-time logic GQMSO which is interpreted over timed words. It includes MSO[<] over words with respect to some alphabet  $\Sigma$ . This is extended with a notion of time constraint formula  $\psi(t)$ , where  $t$  is a free first order variable. All variables in our logic range over positions in the timed word and not over time stamps (unlike continuous interpretation of these logics). There are two sorts of formulae in GQMSO which are mutually recursively defined :  $\text{MSO}^{UT}$  and  $\text{MSO}^T$ .  $\text{MSO}^{UT}$  formulae  $\phi$  have no real-time constraints except the time constraint subformulae  $\psi(t_p) \in \text{MSO}^T$ . Formulae  $\psi(t_p)$  have only one free variable  $t_p$ , which is a first order variable.  $\psi(t_p)$  is defined as block of real-time constrained

quantification applied to a GQMSO formula with no free second order variables; it has the form  $\mathcal{Q}_1 t_1 \mathcal{Q}_2 t_2 \dots \mathcal{Q}_j t_j \phi(t_p, t_1, \dots, t_j)$  where  $\phi \in \text{MSO}^{UT}$ . All the metric quantifiers in the quantifier block constrain their variable relative to only the anchor  $t_p$ . The precise syntax follows below<sup>2</sup>.

This form of real time constraints in first order logic were pioneered by Hirshfeld and Rabinovich [9] in their logic Q2MLO (with only non punctual guards) and its punctual extension was later shown to be expressively complete to  $\text{FO}[\prec, +1]$  by Hunter [12] over signals. Here we extend the quantification to an **anchored block of quantifiers**.

Let  $t_0, t_1, \dots$  be first order variables and  $T_0, T_1, \dots$  the monadic second-order variables. We have a two sorted logic consisting of  $\text{MSO}^{UT}$  formulae  $\phi$  and time constrained formulae  $\psi$ . Let  $a \in \Sigma$ , and let  $t_i$  range over first order variables, while  $T_j$  range over second order variables. The syntax of  $\phi \in \text{MSO}^{UT}$  is given by:

$$t_p = t_q \mid t_p < t_q \mid Q_a(t_p) \mid T_j(t_i) \mid \phi \wedge \phi \mid \neg \phi \mid \exists t'. \phi \mid \exists T_i \phi \mid \psi(t_p).$$

Here,  $\psi(t_p) \in \text{MSO}^T$  is a time constraint formula whose syntax and semantics are given little later. A formula in  $\text{MSO}^{UT}$  with first order free variables  $t_0, t_1, \dots, t_k$  and second-order free variables  $T_1, \dots, T_m$  is denoted  $\phi(t_0, \dots, t_k, T_1, \dots, T_m)$ . The semantics of such formulae is as usual. Let  $\rho = (\sigma_1, \tau_1) \dots (\sigma_n, \tau_n)$  be a timed word over  $\Sigma$ . Given  $\rho$ , positions  $i_0, \dots, i_k$  in  $\text{dom}(\rho)$ , and sets of positions  $A_1, \dots, A_m$  with  $A_i \subseteq \text{dom}(\rho)$ , we define

$\rho, (i_0, i_1, \dots, i_k, A_1, \dots, A_m) \models \phi(t_0, t_1, \dots, t_k, T_1, \dots, T_m)$  inductively, as usual.

- $(\rho, i_0, \dots, i_k, A_1, \dots, A_m) \models t_x < t_y$  iff  $i_x < i_y$ ,
- $(\rho, i_0, \dots, i_k, A_1, \dots, A_m) \models Q_a(t_x)$  iff  $a \in \sigma_{i_x}$ ,
- $(\rho, i_0, \dots, i_k, A_1, \dots, A_m) \models T_j(t_x)$  iff  $a_x \in A_j$ ,
- $(\rho, i_0, \dots, i_k, A_1, \dots, A_m) \models \exists t'. \phi(t_0, \dots, t_k, t', T_1, \dots, T_m)$  iff  $(\rho, i_0, \dots, i_k, i', A_1, \dots, A_m) \models \phi(t_0, \dots, t_k, t', T_1, \dots, T_m)$  for some  $i' \in \text{dom}(\rho)$ .

The **time constraint** formulae  $\psi(t_p) \in \text{MSO}^T$  has the form  $\mathcal{Q}_1 t_1 \mathcal{Q}_2 t_2 \dots \mathcal{Q}_j t_j \phi(t_0, t_1, \dots, t_j)$  where  $\phi \in \text{MSO}^{UT}$  and  $j \in \mathbb{N}$ . Each quantifier  $\mathcal{Q}_x t_x$  has the form  $\exists t_x \in t_0 + I_x$  or  $\forall t_x \in t_0 + I_x$  for a time interval  $I_x \in \mathcal{I}_{+,-}$ .  $\mathcal{Q}_x$  is called a metric quantifier. Note that each metric quantifier constrains its variable only relative to the anchor variable  $t_0$ . The semantics of such an anchored metric quantifier is as follows.

$$\begin{aligned} (\rho, i_0, i_1, \dots, i_j) \models \exists t_1 \in t_0 + I. \phi(t_0, t_1, \dots, t_j) &\text{ iff } \left\{ \begin{array}{l} \text{there exist } i_1 \text{ such that } \tau_{i_1} \in \tau_{i_0} + I \text{ and,} \\ (\rho, i_0, i_1, \dots, i_j) \models \phi(t_0, t_1, \dots, t_j) \end{array} \right\}, \\ (\rho, i_0, i_1, \dots, i_j) \models \forall t_1 \in t_0 + I. \phi(t_0, t_1, \dots, t_j) &\text{ iff } \left\{ \begin{array}{l} \text{for all } i_1 \text{ such that } \tau_{i_1} \in \tau_{i_0} + I \text{ implies,} \\ (\rho, i_0, i_1, \dots, i_j) \models \phi(t_0, t_1, \dots, t_j) \end{array} \right\}. \end{aligned}$$

Note that metric quantifiers quantify over positions of the timed word and the metric constraint is applied on timestamp of the corresponding positions. Each time constraint formula in GQMSO has exactly one free variable; variables  $t_0, t_1, \dots, t_j$  are called time constrained in  $\psi(t_0)$ . Note that if we restrict the grammar of time constrained formulae  $\psi(t_0) \in \text{MSO}^T$  to contain only single metric quantifier (i.e.  $\mathcal{Q}_1 t_1 \phi(t_0, t_1)$ ), then you get **Q2MSO** of [16]. Moreover, if we disallow the usage of second order quantifiers altogether from Q2MSO formulae, we get the logic **Q2MLO** of [10].

Note that by definition, GQMSO is not closed under second order quantification: arbitrary use of second order quantification is not allowed, and its syntactic usage as explained above is restricted to prevent a second order free variable from occurring in the scope of the real-time constraint (similar to [21], [8] and [23]). For example,  $\exists X. \exists t. [X(t) \wedge \exists t' \in t + (1, 2) Q_a(t')]$  is a

<sup>2</sup> In [16], a similar logic called QkMSO was defined. QkMSO had yet another restriction that it can only quantify positions strictly in the future, and hence was not able to express past timed specifications.

well-formed GQMSO formula while,  $\exists X.\exists t.\exists t' \in t+(1,2)[Q_a(t') \wedge X(t)]$  is not, since  $X$  occurs free within the scope of the metric quantifier.

► **Example 2.** Let  $\rho=(\{a\},0) (\{b\},2.1) (\{b\},3.1)$  be a timed word. Consider the time constraint  $\psi(x) = \exists y \in x + (2, \infty) \exists z \in x + (3, \infty) (Q_b(y) \wedge Q_b(z) \wedge S(z, y))$ , where  $S(u, v)$  is the successor relation specifying that position  $(u = v + 1)$ . This relation is expressible in  $\text{FO}[<]$ . It can be seen that  $\rho, 1 \models Q_a(x) \wedge \psi(x)$  since taking position  $x = 1$ , we have  $a \in \sigma_1$ ,  $\tau_1 = 0$ ,  $\tau_2 \in \tau_1 + (2, \infty)$  with  $b \in \sigma_2$  and  $\tau_3 \in \tau_1 + (3, \infty)$  and  $b \in \sigma_4$ .

► **Example 3.** We define a language  $L_{\text{insterr}}$  over singleton alphabet  $\Sigma = \{b\}$  accepting words satisfying following conditions:

1. One  $b$  with timestamp 0 (first position).
2. Exactly two points in interval  $(0, 1)$  at positions  $x$  and  $y$  with timestamps  $\tau_x$  and  $\tau_y$ , respectively
3. Exactly one  $b$  in  $[\tau_x + 1, \tau_y + 1]$  at a position  $p$ . Other  $b$ 's can occur freely elsewhere.

The above language was proposed by Lasota and Walukiewicz [18] (Theorem 2.8) as an example of language not recognizable by 1-ATA. This can be specified as GQMSO formula  $\exists t_0. \psi_1(t_0) \wedge \psi_2(t_0) \wedge \psi_3$ , with  $\psi_i(t_0)$  capturing the  $i^{\text{th}}$  condition above, as follows:

1.  $\psi_1(t_0) = \neg \exists w. S(t_0, w)$ . This holds only for  $t_0 = 1$ , the initial position.
2.  $\psi_2(t_0) = \exists x \in t_0 + (0, 1). \exists x \in t_0 + (0, 1). \forall z \in t_0 + (0, 1) [S(y, x) \wedge (z = x \vee z = y)]$
3.  $\psi_3 = \exists p. [\exists t \in p + [-1, 0) ISy(t) \wedge \forall t \in p + [-1, 0). \neg ISx(t)]$ , where  $ISy(t) = \exists t'. S(t, t') \wedge ISx(t')$ ,  $ISx(t) = \exists t'. S(t, t') \wedge \psi_1(t')$ . Note that in presence of constraints (1) and (2), the formula  $ISy(t)$  only holds for position  $t = 3$  with time stamp  $\tau_y$ . Similarly  $ISx(t)$  holds for  $t = 2$  with time stamp  $\tau_x$ .

**Metric Depth.** The *metric depth* of a formula  $\varphi$  denoted ( $\text{MtD}(\varphi)$ ) gives the nesting depth of time constraint constructs and is defined inductively: For atomic formulae  $\varphi$ ,  $\text{MtD}(\varphi) = 0$ .  $\text{MtD}[\varphi_1 \wedge \varphi_2] = \text{MtD}[\varphi_1 \vee \varphi_2] = \max(\text{MtD}[\varphi_1], \text{MtD}[\varphi_2])$  and  $\text{MtD}[\exists t. \varphi(t)] = \text{MtD}[\neg \varphi] = \text{MtD}(\varphi(t))$ .  $\text{MtD}[Q_1 t_1 Q_2 t_2 \dots Q_j t_j \phi] = \text{MtD}[\phi] + 1$ . For example, The sentence  $\forall t_3 \forall t_1 \in t_3 + (1, 2) \{Q_a(t_1) \rightarrow (\exists t_0 \in t_1 + [1, 1] Q_b(t_0))\}$  accepts all timed words such that for each  $a$  which is at distance  $(1, 2)$  from some time stamp  $t$ , there is a  $b$  at distance 1 from it. This sentence has metric depth two with time constrained variables  $t_0, t_1$ .

### 2.2.1 GQMSO with Alternation Free Metric Quantifiers(AF-GQMSO)

We define a syntactic fragment of GQMSO, called AF-GQMSO, where all the metric quantifiers in the outermost quantifier block of every  $\text{MSO}^T$  subformulae are existential metric quantifiers. More precisely, AF-GQMSO is a syntactic fragment of GQMSO where the **time constraint**  $\psi(t_0)$  has the form  $\exists t_1 \in t_0 + I_1 \exists t_2 \in t_0 + I_2 \dots \exists t_j \in t_0 + I_j \phi(t_0, t_1, \dots, t_j)$  with  $\phi \in \text{MSO}^{UT}$ . Hence, there is no alternation of metric quantifiers within a block of metric quantifier. Note that the negation of the timed subformulae is allowed in the syntax of GQMSO (and hence AF-GQMSO). Hence, alternation free  $\bar{\forall}^*$  formulae can also be expressed using AF-GQMSO. In later sections, we show that AF-GQMSO is as expressive as GQMSO.

### 2.3 Metric Temporal Logic (MTL)

MTL is a real-time extension of LTL where the modalities ( $\text{U}$  and  $\text{S}$ ) are guarded with intervals. Formulae of MTL are built from  $\Sigma$  using Boolean connectives and time constrained versions  $\text{U}_I$  and  $\text{S}_I$  of the standard  $\text{U}, \text{S}$  modalities, where  $I \in \mathcal{I}_+$ . Intervals of the form  $[x, x]$  are called punctual; a non punctual interval is one which is not punctual. Formulae in

MTL are defined as follows.  $\varphi ::= a \mid \top \mid \varphi \wedge \varphi \mid \neg \varphi \mid \varphi \mathbf{U}_I \varphi \mid \varphi \mathbf{S}_I \varphi$ , where  $a \in \Sigma$  and  $I \in \mathcal{I}_+$ . For a timed word  $\rho = (\sigma_1, \tau_1)(\sigma_2, \tau_2) \dots (\sigma_n, \tau_n) \in (\Gamma \times \mathbb{R}_{\geq 0})^*$ , with  $\Gamma = 2^\Sigma \setminus \emptyset$ , a position  $i \in \text{dom}(\rho)$ , an MTL formula  $\varphi$ , the satisfaction of  $\varphi$  at a position  $i$  of  $\rho$ , denoted  $\rho, i \models \varphi$ , is defined below. We discuss the time constrained modalities.

- $\rho, i \models \varphi_1 \mathbf{U}_I \varphi_2$  iff  $\exists j > i, \rho, j \models \varphi_2, \tau_j - \tau_i \in I$ , and  $\rho, k \models \varphi_1 \forall i < k < j$ ,
- $\rho, i \models \varphi_1 \mathbf{S}_I \varphi_2$  iff  $\exists j < i, \rho, j \models \varphi_2, \tau_i - \tau_j \in I$ , and  $\rho, k \models \varphi_1 \forall j < k < i$ .

The language of an MTL formula  $\varphi$  is defined as  $L(\varphi) = \{\rho \mid \rho, 1 \models \varphi\}$ . We say that a formula  $\varphi$  is **satisfiable** iff  $L(\varphi) \neq \emptyset$ . The subclass of MTL where punctual intervals are disallowed is called Metric Interval Temporal Logic MITL. Satisfiability checking for MTL[U, S] is undecidable [4] and for MITL it is EXPSPACE-complete[2].

### 2.3.1 MTL extended with Automata Modalities

There have been several attempts to extend logic MTL[U] with regular expression/automaton modalities [23, 15, 6, 11]. Among these, [23] was the first to extend the logic MITL with automata modalities, called Extended Metric Interval Temporal Logic (EMITL). In our very recent work [14], we used a generalization of these automata modalities to give the logic Pnueli Extended Metric Temporal Logic (PnEMTL). For any Non-Deterministic Finite Automaton (NFA)  $A$ , let  $L(A)$  denote the language of  $A$ .

For an alphabet  $\Sigma$ , the formulae of PnEMTL have the following syntax:

$$\varphi ::= a \mid \varphi \wedge \varphi \mid \neg \varphi \mid \mathcal{F}_{I_1, \dots, I_k}^k(A_1, \dots, A_{k+1})(S) \mid \mathcal{P}_{I_1, \dots, I_k}^k(A_1, \dots, A_{k+1})(S)$$

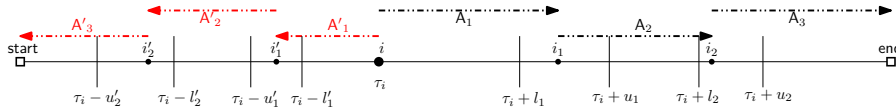
where  $a \in \Sigma$ ,  $I_1, I_2, \dots, I_k \in \mathcal{I}_+$  and  $A_1, \dots, A_{k+1}$  are automata over  $2^S$  where  $S$  is a set of formulae from PnEMTL.

Let  $\rho = (a_1, \tau_1) \dots (a_n, \tau_n) \in T\Gamma^*$ ,  $x, y \in \text{dom}(\rho)$ ,  $x \leq y$  and  $S = \{\varphi_1, \dots, \varphi_n\}$  be a given set of PnEMTL formulae. Let  $S_i$  be the exact subset of formulae from  $S$  evaluating to true at  $\rho, i$ , and let  $\text{Seg}^+(\rho, x, y, S)$  and  $\text{Seg}^-(\rho, y, x, S)$  be the untimed words  $S_x S_{x+1} \dots S_y$  and  $S_y S_{y-1} \dots S_x$  respectively. Then, the semantics for  $\rho, i_0$  satisfying a PnEMTL formula  $\varphi$  is defined recursively as :

- $\rho, i_0 \models \mathcal{F}_{I_1, \dots, I_k}^k(A_1, \dots, A_{k+1})(S)$  iff  $\exists i_0 < i_1 < i_2 \dots < i_k < n$  s.t.  

$$\bigwedge_{w=1}^k [(\tau_{i_w} - \tau_{i_0} \in I_w) \wedge \text{Seg}^+(\rho, i_{w-1} + 1, i_w, S) \in L(A_w)] \wedge \text{Seg}^+(\rho, i_k, n, S) \in L(A_{k+1})$$
- $\rho, i_0 \models \mathcal{P}_{I_1, \dots, I_k}^k(A_1, \dots, A_{k+1})(S)$  iff  $\exists i_0 > i_1 > i_2 \dots > i_k > 1$  s.t.  

$$\bigwedge_{w=1}^k [(\tau_{i_0} - \tau_{i_w} \in I_w) \wedge \text{Seg}^-(\rho, i_{w-1} - 1, i_w, S) \in L(A_w)] \wedge \text{Seg}^-(\rho, i_k, n, S) \in L(A_{k+1})$$
 <sup>3</sup>.



■ **Figure 1** Semantics of PnEMTL.  $\rho, i \models \mathcal{F}_{I_1, I_2}^2(A_1, A_2, A_3)$  &  $\rho, i \models \mathcal{P}_{J_1, J_2}^2(A'_1, A'_2, A'_3)$  where  $I_1 = \langle l_1, u_1 \rangle$ ,  $I_2 = \langle l_2, u_2 \rangle$ ,  $J_1 = \langle l'_1, u'_1 \rangle$ ,  $J_2 = \langle l'_2, u'_2 \rangle$

Language of any PnEMTL formulae  $\varphi$  is  $L(\varphi) = \{\rho \mid \rho, 1 \models \varphi\}$ . Given a PnEMTL formula  $\varphi$ , its arity is the maximum number of intervals appearing in any  $\mathcal{F}, \mathcal{P}$  modality of  $\varphi$ . For

<sup>3</sup> Unlike [14], we introduced the strict version of modalities, without loss of generality, for technical reasons. This doesn't affect the complexity of satisfiability checking for its non adjacent fragment.



example, the arity of  $\varphi = \mathcal{F}_{I_1, I_2}^2(A_1, A_2, A_3)(S_1) \wedge \mathcal{P}_{I_1}^1(A_1, A_2)(S_2)$  for some sets of formulae  $S_1, S_2$  is 2.

► **Example 4.** Consider the formulae  $\mathcal{F}_{(1,2)(2,3)}^2(a^*, b^*, a^*)(\{a, b\})$ . This formulae specifies, that there are sequence of points where  $a$  consecutively holds, followed by a sequence of  $b$ 's again followed by sequence of  $a$ 's. Moreover, the first sequence ends within time interval (1, 2) while the second sequence ends within interval (2, 3) from the present point.

**Modal Depth.** Modal Depth of a formula  $\varphi$ , denoted  $\text{MD}(\varphi)$ , is a measure of the nesting of its temporal modalities defined as recursively follows.  $\text{MD}(a)=0$  for any proposition  $a$ ,  $\text{MD}(\varphi \vee \psi) = \text{Max}(\text{MD}(\varphi), \text{MD}(\psi))$ ,  $\text{MD}(\neg\varphi) = \text{MD}(\varphi)$ ,  $\text{MD}(\mathcal{M}_{I_1, \dots, I_k}(A_1, \dots, A_{k+1})(S)) = \text{Max}_{\varphi \in S}(\text{MD}(\varphi)) + 1$ , where  $\mathcal{M} \in \{\mathcal{F}^k, \mathcal{P}^k\}$ .

## 2.4 Expressive Completeness and Strong Equivalence

. Given any specification (formulae or automata)  $X$  and  $Y$ ,  $X$  is *equivalent* to  $Y$  when for any pointed timed word  $\rho, i$ ,  $\rho, i \models X \iff \rho, i \models Y$ . We say that a formalism  $\mathcal{X}$  (logic or machine) is *expressively complete* to  $\mathcal{Y}$ , denoted by  $\mathcal{Y} \subseteq \mathcal{X}$ , if and only if, for any formulae/automata  $X \in \mathcal{X}$  there exists an equivalent  $Y \in \mathcal{Y}$ .  $\mathcal{X}$  is said to be *expressively equivalent* to  $\mathcal{Y}$ , denoted by  $\mathcal{X} \cong \mathcal{Y}$  when  $\mathcal{X} \subseteq \mathcal{Y}$  and  $\mathcal{Y} \subseteq \mathcal{X}$ .

### 3 Two Way 1-clock Alternating Timed Automata

We now define an extension of 1-ATA [20] [17], with “two wayness”. Let  $\Sigma$  be a finite alphabet. A 2-way 1-ATA is a 6 tuple  $\mathcal{A} = (\Sigma, Q^+, Q^-, \text{init}, \top, \perp, \delta, \mathcal{G})$ , where  $Q^+ \cap Q^- = \emptyset$ ,  $Q = Q^+ \cup Q^-$ , and  $Q^+$  and  $Q^-$  are finite sets of *forward* and *backward* moving locations, respectively.  $\text{init} \in (Q^+ \cup Q^- \cup \top \cup \perp)$  is the initial location,  $\top$  and  $\perp$  are accepting and rejecting location, respectively. Let  $x$  denote the clock variable as in 1-ATA, and  $x \in I$  denote a clock constraint where  $I \in \mathcal{I}_{+, -}$ . Then  $\mathcal{G}$  is a finite set of clock constraints. We say that a real number  $\nu$  satisfies a clock constraint  $x \in I$ , denoted by  $\nu \models x \in I$  if and only if  $\nu \in I$ .

Let  $\Sigma' = \Sigma \cup \{\vdash, \dashv\}$  where  $\vdash, \dashv$  are left and right end markers, respectively. Let  $\rho$  be any word over  $\Sigma$  with  $\tau_{\text{last}}$  being the timestamp of the last time point.

Let  $Q = Q^+ \cup Q^-$ . The transition function is defined as  $\delta : Q \times \Sigma' \times \mathcal{G} \rightarrow \Phi(Q \cup \{\top, \perp\})$  where  $\Phi(Q \cup \{\top, \perp\})$  is a set of formulae over  $Q \cup \{\top, \perp\}$  defined by the grammar as follows.  $\varphi ::= \psi \vee \varphi \mid \perp$ ,  $\psi ::= \psi \wedge \psi \mid q \mid x.q \mid \top$ , where  $q \in Q$  and  $x.q$  is a binding construct resetting clock  $x$  to 0. In other words,  $\Phi(Q)$ , is a positive boolean formulae in Disjunctive Normal Form (DNF) over literals from  $Q \cup \{x.q \mid q \in Q\}$ .

We denote by  $\text{free}(\varphi)$ , the set of all the locations in  $Q$  which does not appear within the scope of a binding construct in  $\varphi$ . Similarly, we denote by  $\text{bound}(\varphi)$ , the set of all the locations in  $Q$  which appear within the scope of a binding construct in  $\varphi$ . Note that  $\text{free}(\varphi)$  and  $\text{bound}(\varphi)$  are not necessarily disjoint sets as any location  $q$  can be both within and beyond the scope of binding construct. For example, in  $\varphi = q \wedge x.q$ ,  $\text{free}(\varphi) = \text{bound}(\varphi) = \{q\}$ . We define  $\delta_{\text{tr}}(q, a, \nu) = \bigvee_{g \in \mathcal{G} \wedge \nu \models g} \delta(q, a, g)$ <sup>4</sup>. Given any set of locations  $W$ , we denote by  $W_x$  the set  $\{x.q \mid q \in W\}$ . We apply following restrictions on transition functions to make sure that

<sup>4</sup> We define Alternating Finite Automata (AFA) in a similar way as a 7 tuple,  $\mathcal{A} = (\Sigma, Q^+, Q^-, \text{init}, \top, \perp, \delta)$ . The transition function is  $Q \times \Sigma \rightarrow \mathcal{B}(Q \cup \{\top, \perp\})$ , where  $\mathcal{B}(Q \cup \{\top, \perp\})$  is a Boolean formula (in DNF) over  $Q \cup \{\top, \perp\}$ , and,  $\delta_{\text{tr}}(q, a) = \delta(q, a)$ .

the automaton does not “fall off” the timed word.

- For any  $q \in Q^-$  and  $q' \in Q^+$ ,  $\delta(q, \vdash)$  and  $\delta(q', \dashv)$  is an expression of the form  $\Phi(Q^+ \cup \{\top, \perp\})$  and  $\Phi(Q^- \cup \{\top, \perp\})$ , respectively.

For any timed word  $\rho = (a_1, \tau_1), (a_2, \tau_2), \dots, (a_m, \tau_m)$ , we abuse the notation by assuming that  $\rho[0] = (\vdash, 0)$  and  $\rho[m+1] = (\dashv, \tau_m)$ . Let  $q \in Q$  and  $0 \leq \mathbf{h} \leq m+1$ . A state of a 2-way 1-ATA is either a  $\top$  (accepting state) or a  $\perp$  (reject state) or a tuple of the form  $(q, \nu, \mathbf{h})$  where  $q \in (Q^+ \cup Q^-)$ ,  $\nu$  is a clock valuation and  $\mathbf{h}$  is the head position. Formally, a state is an element of  $S = ((Q^+ \cup Q^-) \times \mathbb{R} \times (\{0\} \cup \mathbb{N})) \cup \{\top, \perp\}$ . A configuration is a set of states. For any 2-way 1-ATA, we define a function **Succ** (which depends solely on the transition function of the given 2-way 1-ATA) from a word  $\rho$  and a state  $s$  to a set of configurations,  $\text{Succ} : T\Sigma^* \times S \rightarrow 2^{2^S}$ , as follows:

- Let  $\rho = (a_1, \tau_1), (a_2, \tau_2), \dots, (a_m, \tau_m)$ . Let  $\tau_0=0$ ,  $\tau_{m+1}=\tau_m$ ,  $a_0=\vdash$  and  $a_{m+1}=\dashv$ .
- $\text{Succ}(\rho, \top) = \{\top\}$ ,  $\text{Succ}(\rho, \perp) = \{\perp\}$
- Let  $s = (q, \nu, \mathbf{h})$  be any state, where  $0 \leq \mathbf{h} \leq m+1$ . Let  $\mathbf{h}' = \mathbf{h} + 1$  if  $q \in Q^+$ . Otherwise,  $\mathbf{h}' = \mathbf{h} - 1$ . Let  $\nu' = \nu + \tau_{\mathbf{h}'} - \tau_{\mathbf{h}}$ . Let  $\delta_{\text{tr}}(q, a, \nu) = \bigvee_{i=1}^n (\varphi_i)$  where  $\varphi_i = \top$ ,  $\varphi_i = \perp$  or  $\varphi_i$  is of the form  $(\bigwedge Q_i \wedge x. \bigwedge Q'_i)$  where  $Q_i, Q'_i \subseteq Q$ . Any configuration  $\mathbb{C} \in \text{Succ}(\rho, s)$  if and only if there exists  $1 \leq i \leq n$ ,  $\mathbb{C} = \{(q', \nu', \mathbf{h}') \mid q' \in \text{free}(\varphi_i)\} \cup \{(q', 0, \mathbf{h}') \mid q' \in \text{bound}(\varphi_i)\}$ . Intuitively, if  $q$  is a forward (or backward) moving state then the  $\mathbf{h}$  is shifted forward (or backward, respectively) to  $\mathbf{h}'$ , the valuation of clock  $\nu$  is updated to  $\nu'$  by adding (or subtracting, respectively) the time delay incurred, the set of propositions at the new position  $\mathbf{h}'$  is read, and non deterministically, a conjunct (of the DNF) from an outgoing transition satisfied by the clock valuation  $\nu'$  is chosen. The state makes a transition to all the locations appearing in the chosen conjunct simultaneously with the clock valuation as  $\nu'$  if a location is free and 0 if a location is within the scope of a binding construct.

We lift the definition of **Succ** to configurations,  $\text{Successor} : T\Sigma^* \times 2^S \rightarrow 2^{2^S}$ . Given any two configurations,  $\mathbb{C}, \mathbb{C}'$ ,  $\mathbb{C}' \in \text{Successor}(\rho, \mathbb{C})$  if and only if  $\mathbb{C} = \{s_1, \dots, s_m\}$  and  $\mathbb{C}' = \mathbb{C}_1 \cup \dots \cup \mathbb{C}_m$  such that for every  $1 \leq i \leq m$ ,  $\mathbb{C}_i \in \text{Succ}(\rho, s_i)$ . Let  $\mathbb{C}' \in \text{Successor}^0(\rho, \mathbb{C})$  iff  $\mathbb{C} = \mathbb{C}'$ . Then we define a function  $\text{Successor}^i$ ,  $\text{Successor}^i : \rho \times 2^S \rightarrow 2^{2^S}$  such that  $\mathbb{C}' \in \text{Successor}^i(\rho, \mathbb{C})$  iff there exists a  $\mathbb{C}'' \in \text{Successor}^{i-1}(\rho, \mathbb{C})$  such that  $\mathbb{C}' \in \text{Successor}(\rho, \mathbb{C}'')$ . A configuration  $\mathbb{C}$  is accepting if and only if  $\mathbb{C} = \{\top\}$ . A configuration  $\mathbb{C}$  is a rejecting if and only if  $\perp \in \mathbb{C}$ . Given  $\rho$ , We say that a configuration  $\mathbb{C}$  is  $i^{\text{th}}$  successor of a configuration  $\mathbb{C}'$  with respect to  $\rho$  if and only if  $\mathbb{C} \in \text{Successor}^i(\rho, \mathbb{C}')$ . A configuration  $\mathbb{C}$  is eventually accepting on  $\rho$  iff there exists a non negative integer  $n$  such that  $\text{Successor}^n(\rho, \mathbb{C}) = \{\top\}$ .

We say that a pointed timed word  $\rho, i \models_{\mathcal{A}} (q, \nu)$ , iff  $\{(q, i, \nu)\}$  is eventually accepting on  $\rho$ . We say that a pointed timed word  $\rho, i$  is accepted by automata  $\mathcal{A}$  if and only if  $\{(init, 0, i)\}$  is eventually accepting on  $\rho$ . Similarly, a timed word  $\rho$  is accepted by automata  $\mathcal{A}$  if and only if  $\rho, 0$  is accepted by  $\mathcal{A}$ . The language of  $\mathcal{A}$ , denoted by  $L(\mathcal{A})$ , is the set of all timed words accepted by  $\mathcal{A}$ . To check whether language of a given automaton is empty is called **emptiness checking**.

► **Example 5.** Consider a 2-Way 1-ATA  $A = (\Sigma, Q^+, Q^-, q_0, \top, \perp, \delta, \mathcal{G})$  where  $\Sigma = \{a, b\}$ ,  $Q^+ = \{q_0, q_1\}$ ,  $Q^- = \{p_1\}$ ,  $\mathcal{G} = \{x \in (1, 2), x \in (0, 1)\}$  and transition relation is defined as follows.

- $\delta(q_0, a, x \in (0, 1)) = x.q_0 \wedge q_1$ ,  $\delta(q_1, a, x \neq 1) = \delta(q_1, b, x \neq 1) = q_1$ ,  $\delta(q_1, b, x = 1) = \top$ . These transitions only allow behaviours where every  $a$  within time interval  $(0, 1)$  has a  $b$  exactly after 1 time units.
- $\delta(q_0, \dashv) = \top$ ,  $\delta(q_0, b, x \in (1, 2)) = x.p_1$ ,  $\delta(p_1, a, x \neq 1) = \delta(p_1, b, x \neq 1) = p_1$ ,  $\delta(p_1, a, x = 1) = \top$ . These transitions only allow behaviours where every  $a$  within time interval  $(0, 1)$



has a  $b$  exactly after 1 time units.

- Moreover the transitions outgoing from  $q_0$  make sure that all the  $a$ 's and  $b$ 's occur with timestamps within  $(0, 1)$  and  $(1, 2)$ , respectively.

Note that the untimed behavior of the words expected by the above example is of the form  $a^n b^n$  for any  $n \in \mathbb{N}$ . This specification can also be expressed without the 2-Way extension used here.

### 3.1 Island Normal Form

We define a normal form for 2-way 1-ATA similar to the normal form of 1-ATA defined in [16]. A 2-way 1-ATA  $A = (\Sigma, Q, i, \top, \perp, \delta, \mathcal{G})$  is said to be in *Island Normal Form* iff  $Q$  can be partitioned into  $Q_1, \dots, Q_n$  and each  $Q_i$  has a location called the *header location*  $q_{i,r}$  such that:

- For every  $a \in \Sigma$  and  $q \in Q_i$ ,  $\text{free}(\delta(q, a)) \subseteq Q_i \setminus \{q_{i,r}\}$ . Hence, all non reset transitions outgoing from any location  $q \in Q_i$  leads to a non header location within  $Q_i$ .
- For any location  $q \in Q$  and  $a \in \Sigma$ ,  $\text{bound}(\delta(q, a)) \subseteq \{q_{1,r}, \dots, q_{n,r}\}$ .

We call such partitions as islands. Thus, any transition on which a clock variable is reset, can only lead to the header location of one of the islands. Therefore, once we enter an island, the only way to leave the island is via a reset transition. Moreover, entry to any island is via reset transition to its header location. Note that as opposed to the normal form of [16] for 1-ATA, each island here is a reset-free 2-way 1-ATA.

► **Lemma 6.** *Any 2-way 1-ATA  $A$  can be reduced to an equivalent automata in island normal form.*

The proof is identical to the normalization of 1-ATA described in [16] and [19]. Hence, without loss of generality we can assume that a given 2-way 1-ATA is in island normal form. Consider a 2-way 1-ATA  $\mathcal{A} = (\Sigma, Q, \text{init}, \top, \perp, \delta)$  with islands  $Q_1, \dots, Q_n$  and each  $Q_i$  has a header location  $q_{i,r}$ . Let  $Q_r = \{q_{1,r}, \dots, q_{n,r}\}$ .

### 3.2 2-way 1-ATA-rfl

$\mathcal{A}$  is a 2-way 1-ATA-rfl if and only if it satisfies the following: There is a partial order  $(Q_r, \preceq)$  on the header locations (equivalently, on islands  $Q_1, \dots, Q_n$ ). Moreover, for any location  $p \in Q_i$  and a location  $q$ , if  $x.q$  occurs in  $\delta(p, a)$  for any  $a$  (hence  $q = q_j^r$ ) then  $q_j^r \prec q_i^r$  ( $Q_j \prec Q_i$ ). Thus, islands (which are only connected by reset transitions) form a DAG, and every reset transition goes to a lower level island. Moreover, all transitions within an island are reset-free, but can form cycles. Hence, a cycle can never contain a transition with clock reset. An island  $Q_i$  is a *terminal* island if there is no reset outgoing from any of its states. Hence, all terminal islands are essentially reset free 2-way 1-ATA. Similarly, an island  $Q_j$  is said to be *initial* if its header state,  $q_j^r$ , is the initial state of  $\mathcal{A}$ . Note that terminal islands are minimal elements of  $\prec$ , while the initial island is the maximal element of  $\prec$ .

**Reset Depth** of any 2-way 1-ATA-rfl  $\mathcal{A}$  is the maximum number of reset transitions required to reach a terminal island from the initial island. Hence, the reset depth of a reset free automaton is 0. Similarly, the reset depth of a 2-way 1-ATA-rfl containing only 2 islands is 1.

**Boolean Closure of 2-way 1-ATA-rfl** 2-way 1-ATA (rfl) are closed under intersection, union and complementation. The proof of this statement is identical to the case of 1-ATA (proposition 4 [17] or proposition 7,8 of [20]).

#### 4 Non Adjacent 2-Way 1-ATA-rfl and GQMSO

Recently, a generalization of non punctuality restriction called, **non adjacency**, was explored in the context of logics TPTL and PnEMTL [14] to gain decidability. We propose similar non adjacent subclasses of 2-Way 1-ATA and GQMSO in this section and show the decidability for these fragments in section 5. Any set of intervals  $\mathcal{I}$  is said to be non adjacent iff for any  $l_1, l_2 \in \mathcal{I}$ ,  $\inf(l_1) \neq \sup(l_2)$ . For example,  $\{(2, 3), (4, 5), (2, 5)\}$  is non adjacent but  $\{(0, 1)(1, 2)\}$  and  $\{[1, 1]\}$  are adjacent. Note that  $[1, 1]$  is adjacent to itself and hence fails the test. Hence, non adjacency is a generalization of non punctual restriction of MITL.

**Non Adjacent PnEMTL (NAPnEMTL)** [14] is defined as a subclass of PnEMTL where every modality  $\mathcal{F}_{l_1, \dots, l_k}^k$  and  $\mathcal{P}_{l_1, \dots, l_k}^k$  is such that  $\{l_1, \dots, l_k\}$  is a non adjacent set of intervals. (Note that the same interval can appear several times in the list.)

##### 4.1 Non Adjacent 2-way 1-ATA-rfl (NA 2-way-1-ATA-rfl)

. Any 2-way-1-ATA  $\mathcal{A} = (\Sigma, Q, \text{init}, \top, \perp, \delta)$  is non adjacent iff the set of all the intervals,  $\mathcal{I}_i$ , appearing in the outgoing transitions from any location in  $Q_i$  is non adjacent. While this class of automata appears to be very restrictive, it can be shown that it can express properties which are not expressible using 1-Way 1-ATA.

► **Theorem 7.** *NA 2-Way-1-ATA-rfl  $\not\subseteq$  1-ATA.*

**Proof.** Theorem 2.8 [18] shows that language  $L_{\text{insterr}}$  presented in example 3 is not recognizable by 1-ATA. Hence, it suffices to show that the same can be expressed by NA 2-Way 1-ATA-rfl  $A$  with islands  $Q_0, \dots, Q_5$  with header states  $q_0, q_1, q_2^-, q_3^-, q_4^-, q_5^-$ , respectively, and transition function  $\delta$  detailed as follows. Backward moving locations are superscripted with  $-$  sign in the following. Let  $\rho = (b, 0)(b, \tau_2)(b, \tau_3) \dots (b, \tau_n)$  be any timed word.

- $Q_5 = \{q_5^-, q_{5,2}^-, q_{5,3}^-, q_{5,4}^-, q_{5,5}^-\}$ ,  $\delta(q_5^-, b) = q_5^- \vee q_{5,2}^-$ ,  $\delta(q_{5,2}^-, b, x \in (1, \infty)) = q_{5,3}^-$ ,  $\delta(q_{5,3}^-, b) = q_{5,4}^-$ ,  $\delta(q_{5,4}^-, b) = q_{5,5}^-$ ,  $\delta(q_{5,5}^-, \top) = \top$ . When called from any point  $i$  of  $\rho$ , this island makes sure that  $\tau_i > \tau_3 + 1$ .
- $Q_4 = \{q_4^-, q_{4,2}^-, q_{4,3}^-, q_{4,4}^-\}$ ,  $\delta(q_4^-, b) = q_4^- \vee q_{4,2}^-$ ,  $\delta(q_{4,2}^-, b, x \in [0, 1]) = q_{4,3}^-$ ,  $\delta(q_{4,3}^-, b) = q_{4,4}^-$ ,  $\delta(q_{4,4}^-, \top) = \top$ . When called from any point  $i$  of  $\rho$ , this island makes sure that  $\tau_i \leq \tau_2 + 1$ .
- $Q_3 = \{q_3^-, q_{3,2}^-, q_{3,3}^-, q_{3,4}^-\}$ ,  $\delta(q_3^-, b) = q_3^- \vee q_{3,2}^-$ ,  $\delta(q_{3,2}^-, b, x \in (1, \infty)) = q_{3,3}^-$ ,  $\delta(q_{3,3}^-, b) = q_{3,4}^-$ ,  $\delta(q_{3,4}^-, \top) = \top$ . When this island is called from any point  $i$  of  $\rho$ , it makes sure that  $\tau_i > \tau_2 + 1$ .
- $Q_2 = \{q_2^-, q_{2,2}^-, q_{2,3}^-, q_{2,4}^-, q_{2,5}^-\}$ ,  $\delta(q_2^-, b) = q_2^- \vee q_{2,2}^-$ ,  $\delta(q_{2,2}^-, b, x \in [0, 1]) = q_{2,3}^-$ ,  $\delta(q_{2,3}^-, b) = q_{2,4}^-$ ,  $\delta(q_{2,4}^-, b) = q_{2,5}^-$ ,  $\delta(q_{2,5}^-, \top) = \top$ . When called from any point  $i$  of  $\rho$ , this island makes sure that  $\tau_i \leq \tau_3 + 1$ .
- $Q_1 = \{q_1, q_{1,2}, q_{1,3}, q_{1,4}\}$ ,  $\delta(q_1, b) = q_{1,2}$ ,  $\delta(q_{1,2}, b) = q_{1,3}$ ,  $\delta(q_{1,3}, b) = q_{1,4}$ ,  $\delta(q_{1,4}, b, x \in (1, 2)) = \top$ . If this island is called from the first position, then it makes sure that  $\tau_4 \in (1, 2)$ .
- $Q_0 = \{q_0, q_{0,2}, q_{0,3}, q_{0,4}, q_{0,5}, q_{0,6}^-\}$ .  $q_0$  is the initial location of the automata  $A$ .  $\delta(q_0, b) = q_{0,2} \wedge x.q_1$ . On reading the first symbol, this transition moves to a location  $q_{0,2}$  and simultaneously calls island  $Q_1$  from the position 1 (and timestamp 0). Moreover from  $q_0$ , there are two consecutive  $b$  within interval  $(0, 1)$ .  $\delta(q_{0,2}, b, x \in (0, 1)) = q_{0,3}$ ,  $\delta(q_{0,3}, b, x \in (0, 1)) = q_{0,4}$ . Hence,  $\rho$  is accepted by  $A$  only if it has at least 4 points where  $b$  holds. Moreover, the second and third points are within interval  $(0, 1)$  and the fourth point is within interval  $(1, 2)$ .  $\delta(q_{0,4}, b) = q_{0,4} \vee (x.q_2^- \wedge x.p_3 \wedge q_{0,5} \wedge q_{0,6}^-)$ ,  $\delta(q_{0,6}^-, b) = x.q_4^-$ ,  $\delta(q_{0,5}, b) = x.q^-5$ ,  $\delta(q_{0,5}, \top) = x.q^-5$ . Location  $q_{0,4}$  loops on  $b$  and non deterministically chooses a position  $i$  of  $\rho$  from where it calls islands  $Q_2$  and  $Q_3$  simultaneously. Moreover,

island  $Q_4$  is called from the position  $i - 1$  and  $Q_5$  from  $i + 1$ . This implies that  $\rho$  is accepted by  $A$  iff  $\tau_2, \tau_3 \in (0, 1)$  and there exists exactly one point  $i \in \text{dom}(\rho)$  such that  $\tau_i \in [\tau_2 + 1, \tau_3 + 1]$  (as  $\tau_{i-1} \in [0, \tau_2 + 1)$ ,  $\tau_i + 1 \in (\tau_3 + 1, \infty)$  or  $\tau_i$  is the last position of  $\rho$ ).

◀

## 4.2 Non Adjacent GQMSO (NA-GQMSO)

. Any AF-GQMSO formula  $\varphi$  is said to be **non adjacent** if and only if for every subformula  $\psi$  of  $\varphi$  of the form  $\exists t_1 \in t + I_1 \dots \exists t_j \in t + I_j \Phi(t, t_1, \dots, t_j)$ , the set of intervals  $\{I_1, \dots, I_n\}$  is non adjacent. For example,  $\exists t_1 \in t_0 + (2, 3) \exists t_2 \in t_0 + (3, 4) [\exists t < t_0 \wedge \exists t_3 \in t_0 + (4, 5)]$  is not non adjacent as intervals  $(2, 3)$  and  $(3, 4)$  appear within the same metric quantifier block and are adjacent. On the other hand,  $\exists t_1 \in t_0 + (2, 3) \exists t_2 \in t_0 + (4, 5) [\exists t < t_0 \wedge \exists t_3 \in t_0 + (3, 4)]$  is non adjacent as  $\{(1, 2), (4, 5)\}$  is non adjacent and  $\{(2, 3)\}$  is non punctual (and hence non adjacent to itself). Language  $L_{\text{insterr}}$  of example 3 can be specified using NA-GQMSO. This formulae can be found in Appendix C

## 5 2-Way1-ATA-rfl $\equiv$ PnEMTL $\equiv$ GQMSO

This section is dedicated to prove the following theorem.

► **Theorem 8.**  $(NA)2\text{-way } 1\text{-ATA-rfl} \equiv (NA)PnEMTL \equiv (NA)GQMSO$ .

On closer examination of the reductions from 2-way 1-ATA-rfl to equivalent NAPnEMTL here, and from NAPnEMTL to EMITL $_{0,\infty}$  in [14] we get:

► **Theorem 9.** *Emptiness Checking for NA2-way 1-ATA-rfl is decidable and EXPSPACE-complete. Satisfiability checking for NA-GQMSO is decidable and non primitive recursive.*

► **Lemma 10**  $(PnEMTL \subseteq 2\text{-way } 1\text{-ATA-rfl})$ . *Given any  $(NA)PnEMTL$  formula  $\varphi$ , we can construct an equivalent  $(NA)2\text{-way } 1\text{-ATA-rfl}$   $\mathcal{A}$ .*

*Proof Sketch.* We apply induction on the modal depth of the formula  $\varphi$ . For modal depth 0,  $\varphi$  is a propositional logic formula. Hence, the lemma trivially holds. For modal depth 1, let  $\varphi$  be of the form  $\mathcal{F}_{I_1, \dots, I_k}^k(A_1, \dots, A_{k+1})(\Sigma)$ . In the case of  $\mathcal{P}^k$  modality, symmetrical construction applies. Moreover, dealing with Boolean operators is trivial as 2-way 1-ATA-rfl is closed under Boolean operations. Let  $A_j = (2^\Sigma, Q_j, \text{init}_j, F_j, \delta_j)$ . For  $a \in \Sigma$  let  $\text{Pre}(a, F_j) = \{q \mid q \in Q_j \wedge F_j \cap \delta_j(q, a) \neq \emptyset\}$ . Hence  $\text{Pre}(a, F_j)$  is the set of all the locations in  $A_j$  that has a transition to an accepting state on reading  $a$ . By semantics, for any timed word  $\rho = (a_1, \tau_1) \dots (a_m, \tau_m)$  and  $i_0 \in \text{dom}(\rho)$ ,  $\rho, i_0 \models \varphi$  iff there exists a sequence of points  $i_1, \dots, i_k, i_{k+1}$  lying in the strict future of  $i_0$  where  $i_{k+1} = m$  such that the behaviour of propositions in  $\Sigma$  between the segment from  $i_j$  to  $i_{j+1}$  (excluding  $i_j$  and including  $i_{j+1}$ ) is given by automata  $A_{j+1}$  for any  $0 \leq j \leq k + 1$ . This specification can be expressed using 1-clock non deterministic timed automata (NTA)  $\mathcal{A} = (2^\Sigma, Q, \text{init}, F, \delta, \mathcal{G})$ , constructed as follows.  $Q = Q_1 \cup \dots \cup Q_{k+1}$ .  $\text{init} = \text{init}_1$ ,  $F = F_{k+1}$ ,  $\mathcal{G} = \{x \in I_1, \dots, x \in I_k\}$ , for any  $1 \leq j \leq k + 1$   $q \in Q_j$ ,  $\delta(q, a) = \delta_j(q, a)$ , for any  $1 \leq j \leq k$   $q \in \text{Pre}(a, F_i)$ ,  $\delta(q, a, x \in I_i) = \text{init}_{j+1}$ . By semantics of NTA  $\rho, i \models \varphi$  iff,  $\mathcal{A}$  reaches accepting state on reading  $\rho$  starting from position  $i$ . Note that the NTA we constructed is a reset free NTA. In case of  $\mathcal{P}$  modality we would have a backward moving reset free NTA. Hence, the NTA constructed are “2-Way 1-ATA-rfl” with single island. Moreover,  $\mathcal{A}$  uses the same set of intervals as  $\varphi$ . Hence, if  $\varphi$  is in NAPnEMTL, then  $\mathcal{A}$  is in NA2-way 1-ATA-rfl. The induction part is identical to proof of Theorem 13(2) [16] and appears in appendix D.1.

► **Lemma 11 (2-way 1-ATA-rfl  $\subseteq$  PnEMTL).** *Given any (NA) 2-way 1-ATA-rfl  $\mathcal{A}$ , we can construct an equivalent (NA) PnEMTL formula  $\varphi$ .*

### Proof of lemma 11

Proof is via induction on reset depth of  $\mathcal{A}$ . We give the flow of the construction.

1. **Base Case:** For reset depth 0,  $\mathcal{A}$  is a reset free 2-way 1-ATA-rfl. Reduce  $\mathcal{A}$  to an untimed 2-way AFA over  $\mathcal{I}$  interval words called  $ABS(\mathcal{A})$  by treating guards as symbolic letters, such that  $\rho, i$  is accepted by  $\mathcal{A}$  iff  $\rho, i \in Time(L(ABS(\mathcal{A})))$ .
2. Reduce the 2-way AFA,  $ABS(\mathcal{A})$ , to NFA  $A$  over  $\mathcal{I}$  interval words using [7].
3. Give reduction from NFA  $A$  over  $\mathcal{I}$  interval words to PnEMTL formula  $\varphi$  such that  $\rho, i \in Time(L(A)) \iff \rho, i \models \varphi$ . Hence,  $\rho, i \models \varphi \iff \rho, i \models_{\mathcal{A}}(init, 0)$ . Moreover, if  $\mathcal{I}$  is non adjacent then  $\varphi$  is a non adjacent PnEMTL formula. This step is due to lemma 4 of [14] and appears in Appendix D.2.4.
4. **Induction:** Replace all the lower level islands by witness propositions. Then apply the reduction as in base case. Finally, the witnesses are replaced by subformula equivalent to the corresponding automata. This step is similar to Theorem 13(1) of [16] and appears in Appendix D.2.5.

We show step 1 here. Rest of the steps rely on [7] [14] [16] and appears in Appendix D.2.5.

**Construction of  $ABS(\mathcal{A})$ :** Let  $\mathcal{A} = (2^{\Sigma \cup \{\text{anch}\}}, Q^+ \cup \{\text{check}, \text{init}'\}, Q^-, \text{init}', \top, \perp, \Delta, \mathcal{G})$ . Let  $\mathcal{I}$  be the set of intervals appearing in the clock constraints of  $\mathcal{G}$ .  $ABS(\mathcal{A}) = (2^{\Sigma \cup \{\text{anch}\} \cup \mathcal{I}}, Q^+ \cup \{\text{check}, \text{init}'\}, Q^-, \text{init}', \top, \perp, \delta)$  such that for any  $q \in Q^+ \cup Q^-$ ,  $a \in \Gamma \cup \{\vdash, \dashv\}$ ,  $\mathcal{J} \subseteq \mathcal{I}$ ,  $\delta(q, a \cup \mathcal{J}) = \bigvee_{I \in \mathcal{J}} \Delta(q, a, x \in I)$ ,  $\delta(q, a \cup \{\text{anch}\}) = \bigvee_{0 \in I'} \Delta(q, a, x \in I')$ . That is, for every conjunction of outgoing edges from a location  $q$  to a set of locations  $Q'$  on reading  $a \in \Gamma$  with guard  $x \in I$  in  $\mathcal{A}$ , there is a conjunction of outgoing edges from state  $q$  to  $Q'$  on reading symbol  $a \cup \mathcal{J}$  for any  $\mathcal{J} \subseteq \mathcal{I}$  and  $I \in \mathcal{J}$  or on reading a symbol  $a \cup \text{anch}$  if  $0 \in I$ . Moreover, for any  $a \in \Gamma$  and  $I \in \mathcal{I}$ ,

- $\delta(\text{init}', a \cup \mathcal{J}) = \text{init}'$ ,  $\delta(\text{init}', a \cup \{\text{anch}\}) = \text{check} \wedge \text{init}'$  : Continue to loop till the anchor point is encountered. After reading anchor point, spawn two locations,  $\text{check}$  and  $\text{init}'$ , simultaneously.
- $\delta(\text{init}', \dashv) = \perp$  : If no anchor point is encountered before the head reaches the right end marker, reject the word.
- $\delta(\text{check}, a \cup \mathcal{J}) = \text{check}$ ,  $\delta(\text{check}, \dashv) = \top$ ,  $\delta(\text{check}, a \cup \{\text{anch}\}) = (\perp, r)$ ,  $\delta(\text{init}', \dashv) = \perp$  : Continue to loop on check after the first encounter of an anchor point. If another anchor point is encountered, reject the word.
- The above conditions will makes sure that a word is accepted only if it has exactly one anchor point and thus is a valid  $\mathcal{I}$ -interval word.

Proof of correctness for the above construction (lemma 13) requires the following proposition. Let  $\text{Succ}_{\Delta}$  be the successor relation of  $\mathcal{A}$ . The

► **Proposition 12.** *For any timed word  $\rho = (a_1, \tau_1) \dots (a_m, \tau_m)$ , any point  $i \in \text{dom}(\rho)$  and any non-negative integer  $g$ ,  $\mathcal{C} \in \text{Succ}_{\Delta}^g(\rho, \{(init, 0, i)\})$  implies that for all  $(q, \nu, \mathbf{h}) \in \mathcal{C}$ ,  $\nu = \tau_{\mathbf{h}} - \tau_i$ .*

The above proposition can be proved easily by applying induction on  $g$ . Intuitively, as there is no reset construct, valuation of the clock for any state reachable from the initial state will be equal to the delay from the point where the  $\mathcal{A}$  was started. Hence, the clock valuation of

all the reachable states will be  $\tau_{\mathbf{h}} - \tau_i$  where  $\mathbf{h}$  is a the header position of the state. The following lemma proves the language of  $ABS(\mathcal{A})$  is a set of interval abstractions of the words accepted by  $\mathcal{A}$ . Moreover, “concretizing” the language of  $ABS(\mathcal{A})$  gives back the language of  $\mathcal{A}$ .

► **Lemma 13.** *Any pointed timed word  $\rho, i$  is accepted by  $\mathcal{A}$  iff  $\rho, i \in \text{Time}(L(ABS(\mathcal{A})))$ .*

**Proof.** Intuitively, the state  $init'$  loops over itself and moves the read header left to right until the head reaches an anchor point. After which it spawns two states simultaneously,  $check, init$ . The  $check$  location checks that there is no other anchor point in the future and thus ensures the uniqueness of the anchor point. On the other hand,  $init$  starts imitating the transitions of automata  $\mathcal{A}$  in such a way that it precisely accepts interval abstractions of the set of pointed timed words accepted by  $\mathcal{A}$ . We say that states  $(q, \mathbf{h}, \nu)$  of  $\mathcal{A}$  and  $(q', \mathbf{h}')$  of  $ABS(\mathcal{A})$  are equivalent to each other iff  $q' = q$ , and  $\mathbf{h}' = \mathbf{h}$ . By construction of  $ABS(\mathcal{A})$ , for any word  $w$ ,  $w, 0 \models_{ABS(\mathcal{A})} init'$  if and only if  $w, i \models_{ABS(\mathcal{A})} init$  and  $w, i \models_{ABS(\mathcal{A})} check$  (i.e.  $w$  is a valid  $\mathcal{I}$ -interval word). Moreover, any word accepted by  $ABS(\mathcal{A})$  is such that all its point are time restricted points. Rephrasing the lemma as follows:

[ $\Rightarrow$ ] For any  $\mathcal{I}$ -interval word  $w$  and  $\text{anch}(w) = i$ , if  $w, i \models_{ABS(\mathcal{A})} init$  then for any  $\rho, i \in \text{Time}(w)$ ,  $\rho, i \models_{\mathcal{A}} (init, 0)$ .

[ $\Leftarrow$ ] For any  $\rho, i \models_{\mathcal{A}} (init, 0, 0)$ , there exists an  $\mathcal{I}$ -interval word  $w$  such that  $\rho, i \in \text{Time}(w)$  and  $w, i \models_{ABS(\mathcal{A})} init$ .

We prove [ $\Rightarrow$ ], for the converse [ $\Leftarrow$ ] refer Appendix D.2.2. Consider any arbitrary  $\mathcal{I}$ -interval word  $w = a'_1 \dots a'_m$ , where for some  $i \in \text{dom}(\rho)$ ,  $a'_i = a_i \cup \{\text{anch}\}$  and for all  $j \in \text{dom}(w)$ ,  $j \neq i$ ,  $a'_j = a_j \cup \mathcal{J}_j$  for some  $\mathcal{J}_j \subseteq \mathcal{I}$  such that  $T_j \in \bigcap \mathcal{J}$ . Let  $a'_0 = \{\vdash, \mathcal{J}_1\}$  and  $a'_{m+1} = \{\dashv, \mathcal{J}_m\}$ . Let  $\rho, i$  be any pointed timed word in  $\text{Time}(w)$ . Let  $a_0 = \vdash$ ,  $a_{m+1} = \dashv$ ,  $\tau_0 = 0$  and  $\tau_{m+1} = \tau_m$  and for any  $0 \leq j \leq m+1$  let  $T_j = \tau_j - \tau_i$ . Hence,  $\rho = (a_1, \tau_1) \dots (a_m, \tau_m)$  and for  $0 \leq j \leq m+1$   $T_j \in \mathcal{J}_j$ .

Let  $\text{Succ}_{\Delta}$  be the successor relation for  $\mathcal{A}$  and  $\text{Succ}_{\delta}$  be that of  $ABS(\mathcal{A})$ . By proposition 12, only states of the form  $(q, T_{\mathbf{h}}, \mathbf{h})$  are reachable from state  $(init, 0, i)$ . We say that a configuration  $C$  of  $ABS(\mathcal{A})$  is equivalent to a  $\mathbb{C} = \{(q_1, T_{\mathbf{h}_1}, \mathbf{h}_1), (q_2, T_{\mathbf{h}_2}, \mathbf{h}_2), \dots, (q_n, T_{\mathbf{h}_n}, \mathbf{h}_n)\}$  of  $\mathcal{A}$  iff  $C = \{(q_1, \mathbf{h}_1), (q_2, \mathbf{h}_2), \dots, (q_n, \mathbf{h}_n)\}$ . Let  $s = (q, \mathbf{h})$  be any state of  $ABS(\mathcal{A})$  such that  $q \in Q$ . Let  $s' = (q, T_{\mathbf{h}}, \mathbf{h})$  be any state of  $\mathcal{A}$ . Let  $\mathbf{h}' = \mathbf{h} + 1$  if  $q \in Q^+$  else  $\mathbf{h}' = \mathbf{h} - 1$ . Let  $\mathcal{J}'_{\mathbf{h}} = \{J' | J' \in \mathcal{I} \wedge T_{\mathbf{h}} \in J'\}$ . In other words,  $\mathcal{J}'_{\mathbf{h}}$  be the maximal subset of  $\mathcal{I}$  such that  $T_{\mathbf{h}} \in \bigcap \mathcal{J}'_{\mathbf{h}}$ . By construction of  $ABS(\mathcal{A})$ ,  $\delta(q, a_{\mathbf{h}} \cup \mathcal{J}_{\mathbf{h}}) = \bigvee_{I \in \mathcal{J}_{\mathbf{h}}} \Delta(q, a_{\mathbf{h}}, x \in I)$ .  $\delta_{\text{tr}}(q, a_{\mathbf{h}}, T_{\mathbf{h}}) = \bigvee_{I \in \mathcal{J}'_{\mathbf{h}}} \Delta(q, a_{\mathbf{h}}, x \in I)$ . As  $T_{\mathbf{h}} \in \bigcap \mathcal{J}_{\mathbf{h}}$ ,  $\mathcal{J}_{\mathbf{h}} \subseteq \mathcal{J}'_{\mathbf{h}}$ . Hence, any disjunct

of the form  $\bigwedge Q'$ , where  $Q' \subseteq Q^+ \cup Q^- \cup \{\top\}$ , that appears in  $\delta(q, a_{\mathbf{h}} \cup \mathcal{J}_{\mathbf{h}})$  also appears in  $\delta_{\text{tr}}(q, a_{\mathbf{h}}, T_{\mathbf{h}})$ . Hence, for every configuration  $C \in \text{Succ}_{\delta}(w, s)$  there exists a  $\mathbb{C} \in \text{Succ}_{\Delta}(\rho, s')$  such that  $C$  is equivalent to  $\mathbb{C}$  (obs 1). We show that for any  $g \geq 0$  and for any  $C \in \text{Successor}_{ABS(\mathcal{A})}^g(w, \{(init, i)\})$ , there exists  $\mathbb{C} \in \text{Successor}_{\mathcal{A}}^g(\rho, \{(init, 0, i)\})$  such that  $C$  is equivalent to  $\mathbb{C}$ . For  $g = 0$  the above statement is trivially true. Assume for  $g = k$  the statement is true. Let  $C' = \{(q_1, \mathbf{h}_1), \dots, (q_n, \mathbf{h}_n)\}$  be any configuration of  $ABS(\mathcal{A})$  such that  $C' \in \text{Successor}_{ABS(\mathcal{A})}^k(w, \{(init, i)\})$ . Then, by induction hypothesis, there exists a configuration  $\mathbb{C}' = \{(q_1, \mathbf{h}_1, \nu_1), \dots, (q_n, \mathbf{h}_n, \nu_n)\}$  of  $\mathcal{A}$  such that  $\mathbb{C}' \in \text{Successor}_{\mathcal{A}}^k(\rho, \{(init, 0, i)\})$ . Note that for every  $1 \leq j \leq n$ , by proposition 12,  $\nu_j = T_{\mathbf{h}_j}$ . Any configuration  $C'' \in \text{Successor}_{\delta}(w, C')$  if and only if  $C'' = C'_1 \cup \dots \cup C'_n$ , where for any  $1 \leq j \leq n$ ,  $C'_j \in \text{Succ}_{\delta}(q_j, \mathbf{h}_j)$ . Let  $C'' = \mathbb{C}'_1 \cup \mathbb{C}'_n$  such that  $C'_j$  is equivalent to  $\mathbb{C}'_j$  for all  $1 \leq j \leq n$ . By (obs 1), for any  $1 \leq j \leq n$ ,  $\mathbb{C}'_j \in \text{Succ}_{\Delta}(q_j, \nu_j, \mathbf{h}_j)$ . As a result,  $\mathbb{C}'' \in \text{Successor}_{\Delta}(\rho, \mathbb{C}')$ . Hence, for any configuration  $C'' \in \text{Successor}_{\delta}^{k+1}(w, \{(init, i)\})$  there

exists a configuration  $\mathbb{C}'' \in \text{Successor}_{\Delta}^{k+1}(\rho, \{(init, 0, i)\})$  such that  $C''$  is equivalent to  $\mathbb{C}''$ . Hence, if  $w, i \models_{ABS(\mathcal{A})} init$  then  $\rho, i \models_{\mathcal{A}} (init, 0)$ .  $\blacktriangleleft$

Lemma 11 and 10 imply that (NA) PnEMTL is equivalent to (NA)2-way 1-ATA-rfl.

► **Lemma 14 (PnEMTL  $\subseteq$  GQMSO).** *Given any (NA)PnEMTL formula  $\varphi$ , we can construct an equivalent (NA)GQMSO formula  $\psi$ .*

The key observation is that conditions of the form  $\text{Seg}(i, j, \rho, S) \in L(\mathcal{A})$  can be equivalently expressed as MSO[<] formulae  $\psi_{\mathcal{A}}(i, j)$  using Büchi Elgot Trakhtenbrot (BET) Theorem [13][5][22]. Replacing the former with latter we get an equivalent GQMSO formula. See Appendix D.3 for detailed proof. We first prove the following theorem which will be essential for proving the converse. Recall the fragment AF-GQMSO in section 2.2.1.

► **Theorem 15.** *Subset AF-GQMSO is expressively equivalent to GQMSO.*

**Proof.** Given formula  $\psi(t_0)$ , we first eliminate the outermost universal metric quantifier (shown underlined), using four additional existential quantifiers and some non metric universal quantifiers. Let

$$\psi(t_0) = \exists t_1 \in t_0 + I_1 \dots \exists t_{x-1} \in t_0 + I_{x-1} \underline{\forall t_x \in t_0 + [l, u]} \mathcal{Q}_{x+1} t_{x+1} \dots \mathcal{Q}_j t_j \varphi(t_0, t_1, \dots, t_j).$$

We eliminate  $\underline{\forall t_x \in t_0 + [l, u]}$  as follows. We assume  $l > 0$ . The reduction can be generalized to other type of intervals too. There are 3 possible cases:

1. There is no point within  $[l, u]$  of  $t_0$ . In this case, formula  $\psi$  will be vacuously true,  $C_1 = \neg \exists t \in t_0 + [l, u]$ .
2. There exists a point within  $[l, u]$  and a point in  $(u, \infty)$  from  $t_0$ . In this case, we replace the universal quantifier, with 4 existential metric quantifiers (shown underlined) and a non metric universal quantifier (also underlined) as follows.

$$C_2 = \frac{\exists t_1 \in t_0 + I_1 \dots \exists t'_x \in t_0 + [0, l]}{\frac{\exists t_x^- \in t_0 + [l, u] \exists t_x^+ \in t_0 + [l, u]}{\exists t_x'' \in t_0 + [u, \infty)} \dots \mathcal{Q}_j t_j} \left\{ \frac{S(t'_x, t_x^-) \wedge (\forall t_x t_x^- \leq t_x \leq t_x^+) \rightarrow \varphi(t_0, \dots, t_j) \wedge}{S(t_x^+, t_x'')} \right\},$$

where  $S$  is the successor relation definable in MSO[<]. The formula states that there exists a point  $t_x^-$  and  $t_x^+$  (not necessarily distinct) within  $[l, u]$  of  $t_0$  such that the previous of  $t_x^-$  is in  $[0, l]$  and next of  $t_x^+$  is in  $[u, \infty)$ . This makes  $t_x^-$  and  $t_x^+$  as first and last point in interval  $[l, u]$ , respectively. This implies,  $\forall t_x. t_x^- \leq t_x \leq t_x^+$  is equivalent to  $\forall t_x \in t_0 + [l, u]$ .

3. There exists a point within  $[l, u]$  and no point within  $(u, \infty)$  from  $t_0$ . This case is similar to the previous ones. We just need to assert that  $t_x^+$  is the last point of the timed word.

$$C_3 = \frac{\exists t_1 \in t_0 + I_1 \dots \exists t'_x \in t_0 + [0, l]}{\frac{\exists t_x^- \in t_0 + [l, u] \exists t_x^+ \in t_0 + [l, u]}{\dots \mathcal{Q}_j t_j} \left\{ \frac{S(t'_x, t_x^-) \wedge (\forall t_x t_x^- \leq t_x \leq t_x^+) \rightarrow \varphi(t_0, \dots, t_j) \wedge}{\neg \exists t > t_x^+} \right\}.$$

$C_1 \vee C_2 \vee C_3$  is the required formula.  $\blacktriangleleft$

► **Lemma 16 (GQMSO  $\subseteq$  PnEMTL).** *Given any (NA)GQMSO formula  $\varphi$ , we can construct an equivalent (NA)PnEMTL formula  $\psi$ .*

**Proof.** It suffices to show **AF-GQMSO  $\subseteq$  PnEMTL**(thanks to theorem 15). Proof is via induction on metric depth. Let  $\psi(t_0) = \exists t_1 \in t_0 + I_1 \dots \exists t_j \in t_0 + I_j \varphi(t_0, t_1, \dots, t_j)$  be any AF-GQMSO formula of metric depth 1. 1) By semantics of GQMSO, any pointed word  $\rho, i \models \exists t_1 \in t_0 + I_1 \dots \exists t_j \in t_0 + I_j \varphi(t_0, t_1, \dots, t_j)$  iff  $\exists i_1, i_2, \dots, i_j$  such that  $\tau_i - \tau_{i_1} \in I_1 \wedge \dots \wedge \tau_i - \tau_{i_j} \in I_j$  and the untimed behaviour of propositions in  $\Sigma$  is given by the MSO[<] formulae  $\varphi(t_0 = i, t_1 = i_1, \dots, t_j = i_j)$ . We add extra monadic predicates from  $\mathcal{I} \cup \{\text{anch}\}$  to get an interval word encoding the timed behaviour of  $\varphi$ . By definition of “consistency



relation” for interval words, any pointed timed word  $\rho, i \models \psi(t_0)$  iff there exists an  $\mathcal{I} = \{I_1, \dots, I_j\}$  interval word  $w$  such that  $w \models \psi_{ut}$  and  $\rho, i \in \text{Time}(w)$ , where  $\psi_{ut} = \exists t_0. [\text{anch}(t_0) \wedge \exists t_1 I_1(t_1) \dots \exists t_j I_j(t_j) \varphi(t_0, \dots, t_j) \wedge \forall t. (\text{anch}(t) \rightarrow (t = t_0))]$ . Hence,  $\rho, i \in \text{Time}(L(\psi_{ut}))$  iff  $\rho, i \models \psi(t_0)$ . Note that  $\psi_{ut}$  only accepts valid interval words. Recall that for any  $\mathcal{I}$  interval word  $w$ , for any point  $i' \in \text{dom}(w)$ , the truth of the predicate  $I(i')$  implies that  $\tau_{i'} - \tau_i \in I$  for some  $I_x \in \mathcal{I}$ . But  $\tau_{i'} - \tau_i \in I$  doesn't necessarily imply that  $I(i')$  is true. Hence, these monadic predicates only witness positive satisfaction of the timing condition. This is the key reason why we had to get rid of universal metric quantifier. For example, if  $\psi(t_0) = \forall t \in t_0 + I a(t)$ , then corresponding  $\psi_{ut} = \forall t \in t_0 + I \varphi(t_0, t)$ . But  $\psi_{ut}$  vacuously accepts all the interval words where  $I$  doesn't appear at all. Hence,  $\text{Time}(\psi_{ut})$  accepts all the pointed timed words and fails to encode the behaviour of  $\psi$  as  $\psi$  only accepts words  $\rho, i$  such that  $a$  holds at all the points within interval  $I$  from  $i$ . 2) By Büchi Elgot Trakhtenbrot Theorem [13] [5] [22],  $\text{MSO}[<]$  sentence  $\psi_{ut}$  can be reduced to an equivalent NFA  $A = (2^{\Sigma \cup \mathcal{I} \cup \{\text{anch}\}}, Q, \text{init}, F, \delta)$  over  $\mathcal{I}$  interval words. By lemma 22(lemma 4 [14]), for any NFA over  $\mathcal{I}$  interval words we can construct a PnEMTL formulae  $\phi$  such that for any pointed timed word  $\rho, i$ ,  $\rho, i \in \text{Time}(L(A))$  iff  $\rho, i \models \phi$ . Hence,  $\rho, i \models \psi(t_0)$  iff there exists  $\mathcal{I}$  interval word  $w \in L(A)$  such that  $\rho, i \in \text{Time}(w)$  iff  $\rho, i \in \text{Time}(L(A))$  iff  $\rho, i \models \phi$ . Moreover, if  $\psi$  is non adjacent then,  $\mathcal{I}$  is non adjacent and thus  $\varphi$  is in NAPnEMTL.

Assume that the lemma holds for all formula of depth less than  $n$ . Let  $\psi(t_0)$  be any AF-GQMSO formula of metric depth  $n$ . With every timed subformulae  $\psi_i(t)$  of  $\psi$ , we associate a witness proposition  $b_i$  such that  $b_i$  holds iff  $\psi_i$  holds. Let  $W$  be the set of witnesses. We replace each subformula  $\psi_i(t)$  of type  $\text{MSO}^T$  with its corresponding witness getting a formula  $\psi'(t_0)$ . As  $\psi'(t_0)$  doesn't contain any subformula of the form  $\text{MSO}^T$ , its metric depth is 1. As shown in the base case, we can construct a PnEMTL formula  $\varphi'$  equivalent to  $\psi'(t_0)$  containing symbols from  $\Sigma \cup W$ . Note that all subformulae  $\psi_i(t_0)$  of  $\psi$  are of metric depth less than  $n$ . Hence, by induction hypothesis, we can construct a PnEMTL formula  $\varphi_i$  equivalent to  $\psi_i(t_0)$ . Hence, the witnesses for  $\psi_i$  are also that for  $\varphi_i$ . Replacing the witnesses  $b_i$  with its corresponding PnEMTL formulae  $\varphi_i$ , we get the required PnEMTL formulae  $\varphi$ . Also note that if  $\psi$  is non adjacent then all its subformulae  $\psi_i$  and formula  $\psi'$  are non adjacent too. This implies that formulae  $\varphi_i$ ,  $\varphi'$  and, hence  $\varphi$  are NAPnEMTL formulae. ◀

## 6 Discussion

We established the expressive equivalences between timed logics and automata as given in Equation (1) in the introduction. Thus, we have extended the results of [16] to logics and automata with both future and past. Doing this requires new techniques of abstracting timed words by symbolic anchored interval words, and leveraging the results on untimed logics and automata. Moreover, We have applied the newly proposed non adjacency restriction from [14] to the three formalisms of Equation (1) and shown that this makes them all decidable. We conclude the paper by posing two open problems:

- i) Non punctual Q2MLO[9], most expressive known decidable fragment of  $\text{FO}[<, +n]$ , is a syntactic subclass of GQMLO (first order fragment of GQMSO). Is NA-GQMLO strictly more expressive than non punctual Q2MLO? A positive answer would make NA-GQMLO the most expressive decidable fragment of  $\text{FO}[<, +n]$ . A negative answer would imply that non punctual Q2MLO is expressively equivalent to non adjacent Very Weak 2-Way 1-ATA.
- ii) Is non adjacent PnEMTL strictly more expressive than EMITL [23]? A negative answer implies a tight automata and MSO logic characterization of EMITL.

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## A Useful Notations for the rest of the Appendix

We give some useful notations that will be used repeatedly in the following proofs follows.

1. For any set  $S$  containing propositions or formulae, let  $\bigvee S$  denote  $\bigvee_{s \in S} (s)$ . Similarly, let  $A = \{I_1, \dots, I_n\}$  be any set of intervals.  $\bigcap A = I_1 \cap \dots \cap I_n$ ,  $\bigcup A = I_1 \cup \dots \cup I_n$ . For any automaton  $A$  let  $L(A)$  denote the language of  $A$ .
2. For any NFA  $A = (Q, \Sigma, i, F, \delta)$ , for any  $q \in Q$  and  $F' \subseteq Q$ .  $A[q, F'] = (Q, \Sigma, q, F', \delta)$ . In other words,  $A[q, F']$  is the automaton where the set of states and transition relation are identical to  $A$ , but the initial state is  $q$  and the set of final states is  $F'$ . For the sake of brevity, we denote  $A[q, \{q'\}]$  as  $A[q, q']$ . Let  $\text{Rev}(A) = (Q \cup \{f\}, \Sigma, f, \{i\}, \delta')$ , where  $\delta'(f, \epsilon) = F$ , for any  $a \in \Sigma, q \in Q, (q, a, q') \in \delta'$  iff  $(q', a, q) \in \delta$ . In other words,  $\text{Rev}(A)$  is an automata that accepts the reverse of the words accepted by  $A$ .
3. Given any sequence  $\text{Str}$ , let  $|\text{Str}|$  denote length of the sequence  $\text{Str}$ .  $\text{Str}[x]$  denotes  $x^{\text{th}}$  letter of the sequence if  $x \leq |\text{Str}|$ .  $\text{Str}[1...x]$  denotes prefix of String  $\text{Str}$  ending at position  $x$ . Similarly,  $\text{Str}[x...]$  denotes suffix of string starting from  $x$  position. Let  $S_1, \dots, S_k$  be sets. Then, for any  $t \in S_1 \times \dots \times S_k$  if  $t = (x_1, x_2, \dots, x_k)$ .  $t(j)$ , for any  $j < k$ , denotes  $x_j$ .
4. For a timed word  $\rho$ ,  $\rho[i](1)$  gives the set of propositions true at point  $i$ .  $\rho[i](2)$  gives the timestamp of the point  $i$ .

## B Interval Word Abstraction

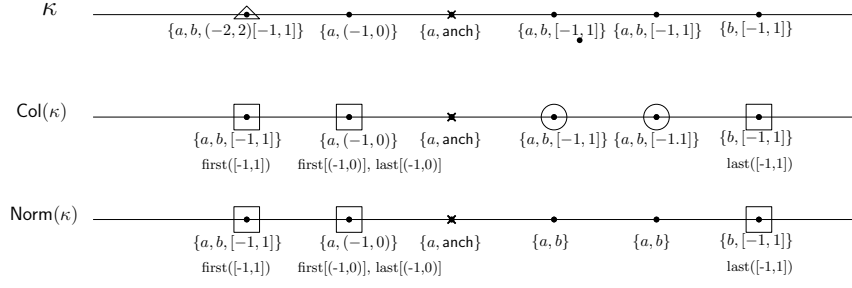
Let  $I_\nu \subseteq \mathcal{I}_{+,-}$ . An  $I_\nu$ -interval word over  $\Sigma$  is a word  $\kappa$  of the form  $a_1 a_2 \dots a_n \in (2^{\Sigma \cup \{\text{anch}\} \cup I_\nu})^*$ . There is a unique  $i \in \text{dom}(\kappa)$  called the *anchor* of  $\kappa$ . At the anchor position  $i$ ,  $a_i \subseteq \Sigma \cup \{\text{anch}\}$ , and  $\text{anch} \in a_i$ . Let  $J$  be any interval in  $I_\nu$ . We say that a point  $i \in \text{dom}(\kappa)$  is a  $J$ -time restricted point if and only if,  $J \in a_i$ .  $i$  is called time restricted point if and only if either  $i$  is  $J$ -time restricted for some interval  $J$  in  $I_\nu$  or  $\text{anch} \in a_i$ .

**From  $I_\nu$ -interval word to Timed Words :** Given a  $I_\nu$ -interval word  $\kappa = a_1 \dots a_n$  over  $\Sigma$  and a timed word  $\rho = (b_1, \tau_1) \dots (b_m, \tau_m)$ , the pointed timed word  $\rho, i = (b_i, \tau_i) \dots (b_m, \tau_m)$  is consistent with  $\kappa$  iff  $\text{dom}(\rho) = \text{dom}(\kappa)$ ,  $i = \text{anch}(\kappa)$ , for all  $j \in \text{dom}(\kappa)$ ,  $b_j = a_j \cap \Sigma$  and for  $j \neq i$ ,  $I \in a_j \cap I_\nu$  implies  $\tau_j - \tau_i \in I$ . Intuitively, each point  $j$  of  $\kappa$  does the following. (i) It stores the set of propositions that are true at point  $j$  of  $\rho$  and (ii) It also stores the set of intervals  $\mathcal{I} \subseteq I_\nu$  such that the time difference between point  $i$  and  $j$  of  $\rho$  lies within  $\bigcap \mathcal{I}$ , thus abstracting the time differences from the anchor point( $i$ ) using some set of intervals in  $I_\nu$ . We denote the set of all the pointed timed words consistent with a given interval word  $\kappa$  as  $\text{Time}(\kappa)$ . Similarly, given a set  $\Omega$  of  $I_\nu$  interval words,  $\text{Time}(\Omega) = \bigcup_{\kappa \in \Omega} (\text{Time}(\kappa))$ .

*Example.* Let  $\kappa = \{a, b, (-1, 0)\} \{b, (-1, 0)\} \{a, \text{anch}\} \{b, [2, 3]\}$  be an interval word over the set of intervals  $\{(-1, 0), [2, 3]\}$ . Consider timed words  $\rho$  and  $\rho'$  s.t.

$\rho = (\{a, b\}, 0)(\{b\}, 0.5)(\{a\}, 0.95)(\{b\}, 3)$ ,  $\rho' = (\{a, b\}, 0)(\{b\}, 0.8)(\{a\}, 0.9)(\{b\}, 2.9)$ . Then  $\rho, 3$  as well as  $\rho', 3$  are consistent with  $\kappa$  while  $\rho, 2$  is not. Likewise, for the timed word  $\rho'' = (\{a, b\}, 0)(\{b\}, 0.5)(\{a\}, 1.1)(\{b\}, 3)$ ,  $\rho'', 3$  is not consistent with  $\kappa$  as  $\tau_1 - \tau_3 \notin (-1, 0)$ , as also  $\tau_3 - \tau_2 \notin [2, 3]$ .

Note that the “consistency relation” is a many-to-many relation. For set of intervals  $I_\nu$ , a pointed timed word  $\rho, i$  can be consistent with more than one  $I_\nu$ -interval word and vice versa. Let  $I_\nu, I'_\nu \subseteq \mathcal{I}_{+,-}$ . Let  $\kappa = a_1 \dots a_n$  and  $\kappa' = b_1 \dots b_m$  be  $I_\nu$  and  $I'_\nu$  interval words, respectively.  $\kappa$  is *similar* to  $\kappa'$ , denoted by  $\kappa \sim \kappa'$  if and only if, (i)  $\text{dom}(\kappa) = \text{dom}(\kappa')$ , (ii) for all  $i \in \text{dom}(\kappa)$ ,  $a_i \cap \Sigma = b_i \cap \Sigma$ , and (iii)  $\text{anch}(\kappa) = \text{anch}(\kappa')$ .  $\kappa$  is *congruent* to  $\kappa'$ , denoted



■ **Figure 2** Point within the triangle has more than one interval. The encircled points are intermediate points and carry redundant information. The required timing constraint is encoded by first and last time restricted points of all the intervals (within boxes).

by  $\kappa \cong \kappa'$ , iff  $\text{Time}(\kappa) = \text{Time}(\kappa')$ . In other words,  $\kappa$  and  $\kappa'$  abstract the same set of pointed timed words. Note that  $\kappa \cong \kappa'$  implies  $\kappa \sim \kappa'$ .

**Boundary Points:** For any  $I \in I_\nu$ ,  $\text{first}(\kappa, I)$  and  $\text{last}(\kappa, I)$  respectively denote the first and last  $I$ -time restricted points in  $\kappa$ . If  $\kappa$  does not contain any  $I$ -time restricted point, then both  $\text{first}(\kappa, I) = \text{last}(\kappa, I) = \perp$ . We define,  $\text{Boundary}(\kappa) = \{i \mid i \in \text{dom}(\kappa) \wedge \exists I \in I_\nu \text{ s.t. } (i = \text{first}(\kappa, I) \vee i = \text{last}(\kappa, I) \vee i = \text{anch}(\kappa))\}$ .

**Collapsed Interval Words.** Given an  $I_\nu$  interval word  $\kappa = a_1 \dots a_n$ , let  $\mathcal{I}_j$  denote the largest set of intervals from  $I_\nu$  contained in  $a_j$ . Let  $\kappa' = \text{Collapse}(\kappa)$  be the word obtained by replacing  $\mathcal{I}_j \subseteq a_j$  with  $\bigcap_{I \in \mathcal{I}_j} I$  in  $a_j$ , for all  $j \in \text{dom}(\kappa)$ . It is clear that  $\text{Time}(\kappa) = \text{Time}(\kappa')$ .  $\kappa'$  is a  $\text{CL}(I_\nu)$  interval word, where

$\text{CL}(I_\nu) = \{I \mid I = \bigcap I', I' \subseteq I_\nu\}$ . An interval word  $\kappa$  is called *collapsed* iff  $\kappa = \text{Collapse}(\kappa)$ .

**Normalization of Interval Words.** Given an  $I_\nu$  interval word  $\kappa = a_1 \dots a_n$ , we define the normalized word corresponding to  $\kappa$ , denoted  $\text{Norm}(\kappa)$  as a  $\text{CL}(I_\nu)$  interval word  $\kappa_{\text{nor}} = b_1 \dots b_m$ , such that (i)  $\kappa_{\text{nor}} \sim \text{Collapse}(\kappa)$ , (ii) for all  $I \in \text{CL}(I_\nu)$ ,  $\text{first}(\kappa, I) = \text{first}(\kappa_{\text{nor}}, I)$ ,  $\text{last}(\kappa, I) = \text{last}(\kappa_{\text{nor}}, I)$ , and for all points  $j \in \text{dom}(\kappa_{\text{nor}})$  s.t.  $\text{first}(\kappa, I) < j < \text{last}(\kappa, I)$ ,  $j$  is not a  $I$ -time constrained point. Thus,  $\text{Norm}(\kappa)$  is an  $I_\nu$  interval word *similar* to  $\kappa$ , has identical first and last  $I$ -time restricted points and has no intermediate  $I$ -time restricted points, for any  $I \in I_\nu$ . An  $I_\nu$  interval word  $\kappa$  is normalized iff  $\text{Norm}(\kappa) = \kappa$ . Hence, a normalized word is a collapsed word where for any  $J \in I_\nu$  there are at most 2  $J$ -time restricted points. Refer Figure 2 for example.

► **Lemma 17.**  $\kappa \cong \text{Norm}(\kappa)$ . Hence, any  $I_\nu$  interval word,  $\kappa$ , can be reduced to a congruent word  $\kappa'$  such that  $\kappa'$  has at most  $2 \times |I_\nu|^2 + 1$  time restricted points.

We split the proof of Lemma 17 into two parts. First, Lemma 18 shows  $\kappa \cong \text{Collapse}(\kappa)$ . Lemma 19 implies that  $\text{Collapse}(\kappa) \cong \text{Norm}(\kappa)$ . Hence, both Lemma 18, 19 together imply Lemma 17.

► **Lemma 18.** Let  $\kappa$  be a  $I_\nu$  interval word and. Then  $\kappa \cong \text{Collapse}(\kappa)$ .

**Proof.** A pointed word  $\rho, i$  is consistent with  $\kappa$  iff

- (i)  $\text{dom}(\rho) = \text{dom}(\kappa)$ ,
- (ii)  $i = \text{anch}(\kappa)$ ,
- (iii) for all  $j \in \text{dom}(\kappa)$ ,  $\rho[j](1) = \kappa[j] \cap \Sigma$  and
- (iv) for all  $j \neq i$ ,  $I \in a_j \cap I_\nu$  implies  $\rho[j](2) - \rho[i](2) \in I$ .
- (v)  $\kappa \sim \text{Collapse}(\kappa)$ , by definition of  $\text{Collapse}$ .

Hence given (v), (i) iff (a) (ii) iff (b)(iii) iff (c) where:

(a)  $\text{dom}(\rho) = \text{dom}(\kappa) = \text{dom}(\text{Collapse}(\kappa))$ , (b)  $i = \text{anch}(\kappa) = \text{anch}(\text{Collapse}(\kappa))$ , (c) for all  $j \in \text{dom}(\kappa)$ ,  $\rho[j](1) = \kappa[j] \cap \Sigma = \text{Collapse}(\kappa)[j] \cap \Sigma$ . (iv) is equivalent to  $\rho[j](2) - \rho[i](2) \in \bigcap (\kappa[j] \cap I_\nu)$ , but  $\bigcap (\kappa[j] \cap I_\nu) = \text{Collapse}(\kappa)[j]$ . Hence, (iv) iff (d)  $\rho[j](2) - \rho[i](2) \in \text{Collapse}(\kappa)[j]$ . Hence, (i)(ii)(iii) and (iv) iff (a)(b)(c) and (d). Hence,  $\rho, i$  is consistent with  $\kappa$  iff it is consistent with  $\text{Collapse}(\kappa)$ .  $\blacktriangleleft$

► **Lemma 19.** *Let  $\kappa$  and  $\kappa'$  be  $I_\nu$  interval words such that  $\kappa \sim \kappa'$ . If for all  $I \in I_\nu$ ,  $\text{first}(\kappa, I) = \text{first}(\kappa', I)$  and  $\text{last}(\kappa, I) = \text{last}(\kappa', I)$ , then  $\kappa \cong \kappa'$ .*

**Proof.** The proof idea is the following:

- As  $\kappa \sim \kappa'$ , the set of timed words consistent with any of them will have identical untimed behaviour.
- As for the timed part, the intermediate  $I$ -time restricted points ( $I$ -time restricted points other than the first and the last) do not offer any extra information regarding the timing behaviour. In other words, the restriction from the first and last  $I$  restricted points will imply the restrictions offered by intermediate  $I$  restricted points.

Let  $\rho = (a_1, \tau_1), \dots (a_n, \tau_n)$  be any timed word.  $\rho, i$  is consistent with  $\kappa$  iff

- 1.(i)  $\text{dom}(\rho) = \text{dom}(\kappa)$ ,
- (ii)  $i = \text{anch}(\kappa)$ ,
- (iii) for all  $j \in \text{dom}(\rho)$ ,  $\kappa[j] \cap \Sigma = a_j$  and
- (iv) for all  $j \neq i \in \text{dom}(\rho)$ ,  $\tau_j - \tau_i \in \bigcap (I_\nu \cap \kappa[j])$ .

Similarly,  $\rho, i$  is consistent with  $\kappa'$  if and only if

- 2(a)  $\text{dom}(\rho) = \text{dom}(\kappa')$ ,
- (b)  $i = \text{anch}(\kappa')$ ,
- (c) for all  $j \in \text{dom}(\rho)$ , if  $\kappa'[j] \cap \Sigma = a_j$  and
- (d) for all  $j \neq i \in \text{dom}(\rho)$ ,  $\tau_j - \tau_i \in \bigcap (I_\nu \cap \kappa'[j])$ .

Note that as  $\kappa \sim \kappa'$ , we have,  $\text{dom}(\kappa) = \text{dom}(\kappa')$ ,  $\text{anch}(\kappa) = \text{anch}(\kappa')$ , for all  $j \in \text{dom}(\kappa)$ ,  $\kappa[j] \cap \Sigma = \kappa'[j] \cap \Sigma$ . Thus, 2(a)  $\equiv$  1(i), 2(b)  $\equiv$  1(ii) and 2(c)  $\equiv$  1(iii).

Suppose there exists a  $\rho, i$  consistent with  $\kappa$  but there exists  $j' \neq i \in \text{dom}(\rho)$ ,  $\tau_{j'} - \tau_i \notin I'$  for some  $I' \in \kappa[j']$ . By definition,  $\text{first}(\kappa', I') \leq j' \leq \text{last}(\kappa', I')$ . But  $\text{first}(\kappa', I') = \text{first}(\kappa, I')$ ,  $\text{last}(\kappa', I') = \text{last}(\kappa, I')$ . Hence,  $\text{first}(\kappa, I') \leq j' \leq \text{last}(\kappa, I')$ . As the time stamps of the timed word increases monotonically,  $x \leq y \leq z$  implies that  $\tau_x \leq \tau_y \leq \tau_z$  which implies that  $\tau_x - \tau_i \leq \tau_y - \tau_i \leq \tau_z - \tau_i$ . Hence,  $\tau_{\text{first}(\kappa, I')} - \tau_i \leq \tau_{j'} - \tau_i \leq \tau_{\text{last}(\kappa, I')} - \tau_i$ . But  $\tau_{\text{first}(\kappa, I')} - \tau_i \in I'$  and  $\tau_{\text{last}(\kappa, I')} - \tau_i \in I'$  because  $\rho$  is consistent with  $\kappa$ . This implies, that  $\tau_{j'} - \tau_i \in I'$  (as  $I'$  is a convex set) which is a contradiction. Hence, if  $\rho, i$  is consistent with  $\kappa$  then it is consistent with  $\kappa'$  too. By symmetry, if  $\rho, i$  is consistent with  $\kappa'$ , it is also consistent with  $\kappa$ . Hence  $\kappa \cong \kappa'$ .  $\blacktriangleleft$

## C Non Adjacent GQMSO - Example

► **Example 20.** We give a NA-GQMSO formulae equivalent to  $L_{\text{insterr}}$  given Example 3 as follows:

$$\begin{aligned} \exists t_0. & \left[ \begin{array}{l} \text{First}(t_0) \wedge \\ (\exists t_1 \in t_0 + (0, 1). \exists t_2 \in t_0 + (0, 1). ISx(t_1) \wedge ISy(t_2) \wedge \\ (\exists t_3 \in t_0 + (1, 2). (\exists t. S(t, t_0) \wedge ISy(t))) \end{array} \right] \wedge \\ \exists t_4. \exists t_5. & \left[ \begin{array}{l} \{S(t_5, t_4) \wedge \exists t \in t_4 - (0, 1). ISx(t)\} \wedge \\ \{\exists t' \in t_5 - (0, 1). ISy(t')\} \wedge \{\exists t \in t_5 - (1, \infty). ISx(t)\} \\ \wedge \{\exists t_6. S(t_6, t_5) \rightarrow (\exists t' \in t_6 - (1, \infty). ISy(t'))\} \end{array} \right] \end{aligned}$$



where  $ISx()$  and  $ISy()$  are defined in example 3 and  $First(t_0)$  is defined in the same example as  $\psi_1(t_0)$ .

## D Proofs for section 5

### D.1 Proof of Lemma 10

We apply induction on modal depth of the formulae  $\varphi$ . For modal depth 0,  $\varphi$  is a propositional formulae. Hence, the lemma trivially holds. For modal depth 1, let  $\varphi$  be of the form  $\varphi = \mathcal{F}_{I_1, \dots, I_k}^k(A_1, \dots, A_{k+1})(\Sigma)$ . In the case of  $\mathcal{P}^k$  modality, symmetrical construction applies. Moreover, dealing with boolean operators is trivial as the resulting 2-Way 1-ATA-rfl are closed under boolean operations. Let  $A_j = (2^\Sigma, Q_j, init_j, F_j, \delta_j)$ . For  $a \in \Sigma$  let  $\text{Pre}(a, F_j) = \{q | q \in Q_j \wedge F_j \cap \delta_j(q, a) \neq \emptyset\}$ . Hence  $\text{Pre}(a, F_j)$  denote set of all the locations in  $A_j$  that has a transition to an accepting state on reading  $a$ . By semantics, for any timed word  $\rho = (a_1, \tau_1) \dots (a_m, \tau_m)$  and  $i_0 \in \text{dom}(\rho')$ ,  $\rho, i_0 \models \varphi$  iff there exists a sequence of point  $i_1, \dots, i_k, i_{k+1}$  in strict future of  $i_0$  where  $i_{k+1} = m$  such that the behaviour of propositions in  $\Sigma$  between the segment from  $i_j + 1$  to  $i_{j+1}$  is given by automata  $A_{j+1}$  for any  $0 \leq j \leq k + 1$ . This specification can be expressed using 1-clock Non Deterministic Timed Automata,  $\mathcal{A} = (2^\Sigma, Q, init, F, \delta, \mathcal{G})$ , constructed as follows.  $Q = Q_1 \dots Q_{k+1}$ .  $init = init_1$ ,  $F = F_{k+1}$ ,  $\mathcal{G} = \{x \in I_1 \dots, x \in I_k\}$ , for any  $1 \leq j \leq k + 1$   $q \in Q_j$ ,  $\delta(q, a) = \delta_j(q, a)$ , for any  $1 \leq j \leq k$   $q \in \text{Pre}(a, F_i)$ ,  $\delta(q, a, x \in I_i) = init_{j+1}$ . By semantics of NTA  $\rho, i \models \varphi'$  if and only if,  $\mathcal{A}$  reaches accepting state on reading  $\rho$  starting from position  $i$ . Note that the NTA we constructed is a reset free NTA. In case, of  $\mathcal{P}$  modality we will have a backward moving reset free NTA. Hence, the NTA constructed are “2-Way 1-ATA-rfl” with single island.

Let us assume that the lemma holds for every PnEMTL formulae of depth less than  $n$ . Let  $\varphi$  be any PnEMTL formulae of modal depth  $n$  of the form  $\mathcal{F}_{I_1, \dots, I_k}^k(A_1, \dots, A_{k+1})(S)$ . As the  $\text{MD}(\varphi) = n$ , any formulae  $\phi_j \in S$  is s.t.  $\text{MD}(\phi_i) < n$ . We consider the set of timed words  $T$  over extended set of propositions  $\Sigma \cup W$ , where  $W$  is a set of propositions containing witness  $b_j$  for each formulae  $\phi_j \in S$  such that for any  $\rho' \in T$  and  $i \in \text{dom}(\rho')$ ,  $\rho', i \models \phi_i \iff \rho', i \models b_i$ . Let  $\varphi'$  be a formulae obtained from  $\varphi$  by replacing occurrence of every  $\phi_i \in S$  by its corresponding witness  $b_i$ . Given any word  $\rho$  over  $\Sigma$ , Let  $\rho' \Downarrow \Sigma$  denote a word  $\rho$  over  $\Sigma$  obtained from  $\rho'$  by hiding symbols from  $W$ . For any  $i \in \text{dom}(\rho')$ ,  $\rho', i \models \varphi' \iff \rho' \Downarrow \Sigma, i \models \varphi$ . Hence, any pointed word  $\rho, i$  satisfies  $\varphi$  if and only if it is a projection on  $\Sigma$  of a timed word  $\rho' \in T$  and  $\rho', i \models \varphi'$ . Note that  $\varphi'$  is a modal depth 1 formulae of the form  $\mathcal{F}_{I_1, \dots, I_k}^k(A'_1, \dots, A'_{k+1})(\Sigma \cup W)$ . Hence, we can construct a 2-Way 1-ATA-rfl  $\mathcal{A}'$  with only 1 island over  $\Sigma \cup W$ .

To get an automata  $\mathcal{A}$  equivalent to  $\varphi$ , we need to make sure that the it accepts all and only those words  $\rho, i$  where  $\rho$  is timed word over  $\Sigma$  which can be obtained from a word  $\rho'$  over  $\Sigma \cup W$  in  $T$  such that  $\rho', i$  is accepted by  $\mathcal{A}'$ . This can be done by as follows. By induction hypothesis, for any subformulae  $\phi_i \in S$ , we can construct a 2-Way 1-ATA-rfl  $\mathcal{A}_i = (2^\Sigma, Q_i^+, Q_i^-, q_i, F_i, \delta_i, \mathcal{G}_i)$  and  $\mathcal{A}_i^c = (2^\Sigma, Q_i^{+c}, Q_i^{-c}, q_i', F_i', \delta_i', \mathcal{G}_i')$  such that  $\mathcal{A}_i$  is equivalent to  $\phi_i$  and  $\mathcal{A}_i^c$  to  $\neg\phi_i$ . Let  $S = \{\phi_1, \dots, \phi_n\}$ ,  $\mathcal{Q}^\sim = Q^\sim \cup Q_1^\sim \cup \dots \cup Q_n^\sim \cup Q_1'^\sim \cup \dots \cup Q_n'^\sim$  for  $\sim \in \{+, -\}$ ,  $\mathcal{F} = F \cup F_1 \cup \dots \cup F_n \cup F_1' \cup \dots \cup F_n'$ ,  $\mathcal{G} = \mathcal{G}' \cup \mathcal{G}_1 \cup \dots \mathcal{G}_n \cup \dots \mathcal{G}_1' \cup \dots \mathcal{G}_n'$ . We now construct the required automata  $\mathcal{A}$ . Intuitively, every transition (not) labelled by  $b_i$  is conjuncted with a new transition to  $(q_j')$   $q_j$ , respectively.  $\mathcal{A} = (2^\Sigma, \mathcal{Q}^+, \mathcal{Q}^-, init, \mathcal{F}, \delta, \mathcal{G})$  where for any  $a \in \Sigma$ ,  $g \in \mathcal{G}'$ , if  $q \in Q$  then

$$\delta(q, a, g) = \bigvee_{W' \subseteq W} [\delta'(q, a \cup W', g) \wedge \bigwedge_{b_i \in W'} x.q_i \wedge \bigwedge_{b_i \notin W'} x.q_i'],$$

if  $q \in Q_i$  then  $\delta(q, a, g) = \delta_i(q, a, g)$ , if  $q \in Q_i'$  then  $\delta(q, a, g) = \delta_i'(q, a, g)$ .

Note that, by construction, each of  $Q, Q_1, \dots, Q_n, Q'_1, \dots, Q'_n$  for islands of  $\mathcal{A}$ . Moreover, if  $\varphi$  is non adjacent then island  $Q$  uses non adjacent set of intervals as all its outgoing transitions use the same set of intervals as used by  $\mathcal{A}^c$ . Also if  $\varphi$  is non adjacent then all the its subformulae in  $S$  are non adjacent. By inductive hypothesis islands  $Q_1, \dots, Q_n, Q'_1, \dots, Q'_n$  are also non adjacent. This proves the lemma. Note that if  $\varphi$  is a  $\mathcal{P}^k$  formula then the initial island would have been a backward moving island.

## D.2 Proof of Lemma 11

We apply induction on reset depth of  $\mathcal{A}$ . The key difference between reduction from 1-ATA-rfl to RatMTL in [16] is in the reduction of single island (or reset free) Automata to an equivalent formulae (base case). In [16], the reduction was via region abstraction of words. These abstractions do not preserve the non adjacency restriction. In this case, we use a coarser abstraction of interval words which helps us to preserve non-adjacency while reduction and hence get decidable fragments for 2-Way 1-ATA. We give the flow of the construction here.

We break the construction into following steps:

1. **Base Case:** For reset depth 0,  $\mathcal{A}$  is reset free 2 way 1-ATA-rfl. Reduce  $\mathcal{A}$  to AFA  $ABS(\mathcal{A})$  over  $\mathcal{I}$  intervals words such that  $\rho, i \models_{\mathcal{A}} init$  if and only if  $\rho, i \in Time(L(\mathcal{A}))$ .
2. Reduce 2-Way AFA  $ABS(\mathcal{A})$  to 1-way NFA  $A$  over  $\mathcal{I}$  interval words using result from [7].
3. Show that given any NFA  $A$  over  $\mathcal{I}$  interval words, one can construct a PnEMTL formula  $\varphi$  such that  $\rho, i \in Time(L(A)) \iff \rho, i \models \varphi$ . Hence,  $\rho, i \models \varphi \iff \rho, i \models_{\mathcal{A}} (init, 0)$ . Moreover, if  $\mathcal{I}$  is a non adjacent set of intervals then  $\varphi$  is a non adjacent PnEMTL formula.
4. **Induction:** Replace all the lower level islands by witness propositions. Then apply the reduction as in base case. Finally, the witnesses are replaced by subformula equivalent to the corresponding automata.

### D.2.1 From Reset Free 2-Way 1-ATA to AFA over interval words

We first give a construction of two way AFA over  $\mathcal{I}$  interval words constructed from  $\mathcal{A}$  as follows, denoted by  $ABS(\mathcal{A})$ , such that  $\rho, i \in Time(L(ABS(\mathcal{A}))) \iff \rho, i \models_{\mathcal{A}} (init, 0)$ .  $ABS(\mathcal{A}) = (2^{\Sigma \cup \{\text{anch}\} \cup I_\nu}, Q^+ \cup \{check, init'\}, Q^-, init', \top, \perp, \delta)$  such that for any  $q \in Q^+ \cup Q^-$ ,  $a \in \Gamma \cup \{\vdash, \dashv\}$ ,  $\mathcal{J} \subseteq \mathcal{I}$ ,  $\delta(q, a \cup \mathcal{J}) = \bigvee_{I \in \mathcal{J}} \Delta(q, a, x \in I)$ ,  $\delta(q, a \cup \{\text{anch}\}) = \bigvee_{0 \in I'} \Delta(q, a, x \in I')$ . That is, for every conjunction of outgoing edges from a location  $q$  to a set of locations  $Q'$  on reading  $a \in \Gamma$  with guard  $x \in I$  in  $\mathcal{A}$ , there is a conjunction of outgoing edges from state  $q$  to  $Q'$  on reading symbol  $a \cup \mathcal{J}$  for any  $\mathcal{J} \subseteq \mathcal{I}$  and  $I \in \mathcal{J}$  or on reading a symbol  $a \cup \text{anch}$  if  $0 \in I$ . Moreover, for any  $a \in \Gamma$  and  $I \in I_\nu$ ,

- $\delta(init', a \cup \mathcal{J}) = init'$ ,  $\delta(init', a \cup \{\text{anch}\}) = check \wedge init$  : Continue to loop till the anchor point is encountered. After reading anchor point, spawn two locations,  $check$  and  $init$ , simultaneously.
- $\delta(init', \dashv) = \perp$  : If no anchor point is encountered before the head reaches the right end marker, reject the word.
- $\delta(check, a \cup \mathcal{J}) = check$ ,  $\delta(check, \dashv) = \top$ ,  $\delta(check, a \cup \{\text{anch}\}) = (\perp, r)$ ,  $\delta(init', \dashv) = \perp$  : Continue to loop on check after the first encounter of an anchor point. If another anchor point is encountered, reject the word.
- The above conditions will makes sure that a word is accepted only if it has exactly one anchor point and thus is a valid  $I_\nu$ -interval word.

### D.2.2 Proof of lemma 13-Converse Direction [ $\Leftarrow$ ]

$\Leftarrow$  Let  $\rho = (a_1, \tau_1) \dots (a_m, \tau_m)$ . Let  $a_0 = \top$ ,  $a_{m+1} = \perp$ ,  $\tau_0 = 0$  and  $\tau_{m+1} = \tau_m$  and for any  $0 \leq j \leq m+1$  let  $T_j = \tau_j - \tau_i$ . Consider a word  $w = a_1 \cup \mathcal{I}_1 \dots a_m \cup \mathcal{I}_m$  where  $\mathcal{I}_i = \{anch\}$  and for all  $j \in \text{dom}(w)$  and  $j \neq i$ ,  $\mathcal{I}'_j = I | I \in \mathcal{I} \wedge T_j \in I$ . Hence,  $\mathcal{I}'_j$  is a maximal subset of  $\mathcal{I}$  such that for all intervals  $I \in \mathcal{I}'_j$ ,  $T_j \in I$ . Clearly,  $\rho, i \in \text{Time}(w)$ . Moreover,  $\delta(q, a_h \cup \mathcal{I}_h) = \delta_{tr}(q, a_h, T_h)$  for any  $0 \leq h \leq m+1$  and for any  $q \in Q$ . Let  $s$  ( $s'$ ) be any state of  $ABS(\mathcal{A})$  ( $\mathcal{A}$ ). Let  $\{s\}$  be equivalent to  $\{s'\}$ . Hence, for any configuration  $C \in \text{Succ}_\delta(w, s)$  there exists a  $\mathbb{C} \in \text{Succ}_\Delta(\rho, s')$  such that  $C$  is equivalent to  $\mathbb{C}$ . Moreover, for any configuration  $\mathbb{C} \in \text{Succ}_\Delta(\rho, s')$  there exists a configuration  $C' \in \text{Succ}_\delta(w, s)$  such that  $C'$  is equivalent to  $\mathbb{C}$ . For any  $g \geq 0$  and for any  $\mathbb{C} \in \text{Successor}_\Delta^g(\rho, \{(init, 0, i)\})$  there exists  $C \in \text{Successor}_\delta^g(w, \{(init, i)\})$  such that  $C$  is equivalent to  $\mathbb{C}(\text{obs } 2)$ . For  $g = 0$  the above statement is trivially true. Assume for  $g = k$  the statement is true. Let  $\mathbb{C}' = \{(q_1, \mathbf{h}_1, T_{\mathbf{h}_1}), \dots, (q_n, \mathbf{h}_n, T_{\mathbf{h}_n})\}$  of  $\mathcal{A}$  such that  $\mathbb{C}' \in \text{Successor}_\Delta^k(\rho, \{(init, 0, i)\})$ . Then by induction hypothesis,  $C' = \{(q_1, \mathbf{h}_1), \dots, (q_n, \mathbf{h}_n)\} \in \text{Successor}_\Delta^k(w, \{(init, i)\})$ . Any configuration  $\mathbb{C}'' \in \text{Successor}_\Delta(\rho, \mathbb{C}')$  if and only if  $\mathbb{C}'' = \mathbb{C}'_1 \cup \dots \cup \mathbb{C}'_n$ , where for any  $1 \leq j \leq n$ ,  $\mathbb{C}'_j \in \text{Succ}_{\mathcal{A}}(q_j, T_{\mathbf{h}_j}, \mathbf{h}_j)$ . Let  $C'' = C'_1 \cup C'_n$  such that  $C'_j$  is equivalent to  $\mathbb{C}'_j$  for all  $1 \leq j \leq n$ . By (obs 2), for any  $1 \leq j \leq n$ ,  $C'_j \in \text{Succ}_\delta(q_j, \mathbf{h}_j)$ . As a result,  $C'' \in \text{Successor}_\delta(\rho, \mathbb{C}')$ . Hence, for any configuration  $\mathbb{C}'' \in \text{Successor}_\Delta^{k+1}(\rho, \{(init, 0, i)\})$  there exists a configuration  $C'' \in \text{Successor}_\Delta^{k+1}(w, \{(init, i)\})$  such that  $C''$  is equivalent to  $\mathbb{C}''$ . Hence, if  $\rho, 0, i \models_{\mathcal{A}} \text{init}$  then there exists  $w'$  such that  $\rho, i \in \text{Time}(w')$  and  $w', i \models_{ABS(\mathcal{A})} \text{init}$ .

### D.2.3 2-Way AFA to NFA

► **Theorem 21** ([7]). *For any 2-Way Alternating finite Automata  $A$ , one can construct a 1 Way Non Deterministic Finite Automata (NFA)  $A'$  with at most exponential number of states.*

We use above theorem to construct 1-Way NFA  $A'$  equivalent to  $ABS(\mathcal{A})$ .

### D.2.4 From NFA over Interval words to PnEMTL

► **Lemma 22.** *Given any NFA  $A'$  over  $\mathcal{I}$  interval words, we can construct a PnEMTL formula  $\varphi$  such that  $\rho, i \models \varphi \iff \rho, i \in \text{Time}(L(A'))$ . Moreover, if  $\mathcal{I}$  is non adjacent then  $\varphi$  is a non adjacent PnEMTL formulae.*

We encourage readers to first read section B to for definition and notations not introduced in the main paper but is used in this proof.

**Automata over Collapsed Interval Word-** From  $A'$ , we construct an automaton  $A = (Q, \text{init}, 2^{\Sigma'}, \delta, F)$  s.t.  $L(A) = \text{Collapse}(L(A'))$ . For any  $q, q' \in Q$ ,  $S \subseteq \Sigma'$ ,  $\mathcal{I} \subseteq I_\nu$ ,  $(q, S \cup \mathcal{I}, q') \in \delta'$  iff  $(q, S \cup \{I\}, q') \in \delta$  where  $I = \bigcap \mathcal{I}$ ,  $(q, \{a, \text{anch}\}, q') \in \delta'$  iff  $(q, \{a, \text{anch}\}, q') \in \delta$ .  $A$  is obtained from  $A'$  by replacing  $\bigcap \mathcal{I}$  in place of  $\mathcal{I}$  on the transitions. This gives  $L(A) = \text{Collapse}(L(A'))$ .

**Partitioning Interval Words-** We discuss here how to partition  $W$ , the set of all  $I_\nu$  interval words using some finite sequences  $\text{seq}$  over  $I_\nu \cup \{\text{anch}\}$ . For any collapsed  $w \in W$ ,  $\text{seq}$  gives an ordering between  $\text{anch}(w)$ ,  $\text{first}(w, I)$  and  $\text{last}(w, I)$  for all  $I \in I_\nu$ , such that, any  $I \in I_\nu$  appears exactly twice and  $\text{anch}$  appears exactly once in  $\text{seq}$ . For instance,  $\text{seq} = I_1 I_1 \text{anch} I_2 I_2$  is a sequence different from  $\text{seq}' = I_1 I_2 \text{anch} I_2 I_1$  since the relative orderings between the first and last occurrences of  $I_1, I_2$  and  $\text{anch}$  differ in both. Let  $\mathcal{T}(I_\nu)$  be the set of all such sequences; by definition,  $\mathcal{T}(I_\nu)$  is finite. Given  $w \in W$ , let

$\text{Boundary}(w) = \{i_1, i_2, \dots, i_k\}$  be the positions of  $w$  which are either  $\text{first}(w, I)$  or  $\text{last}(w, I)$  for some  $I \in I_\nu$  or is  $\text{anch}(w)$ . Let  $w \downarrow_{\text{Boundary}(w)}$  be the subword of  $w$  obtained by projecting  $w$  to the positions in  $\text{Boundary}(w)$ , restricted to the sub alphabet  $2^{I_\nu} \cup \{\text{anch}\}$ . For example,  $w = \{a, I_1\}\{b, I_1\}\{c, I_2\}\{\text{anch}, a\}\{b, I_1\}\{b, I_2\}\{c, I_2\}$  gives  $w \downarrow_{\text{Boundary}(w)}$  as  $I_1 I_2 \text{anch} I_1 I_2$ . Then  $w$  is in the partition  $W_{\text{seq}}$  iff  $w \downarrow_{\text{Boundary}(w)} = \text{seq}$ . Clearly,  $W = \uplus_{\text{seq} \in \mathcal{T}(I_\nu)} W_{\text{seq}}$ . Continuing with the example above,  $w$  is a collapsed  $\{I_1, I_2\}$ -interval word over  $\{a, b, c\}$ , with  $\text{Boundary}(w) = \{1, 3, 4, 5, 7\}$ , and  $w \in W_{\text{seq}}$  for  $\text{seq} = I_1 I_2 \text{anch} I_1 I_2$ , while  $w \notin W_{\text{seq}'}$  for  $\text{seq}' = I_1 I_1 \text{anch} I_2 I_2$ . Finally, all the timed words abstracted by interval words in a partition  $W_{\text{seq}}$  for  $\text{seq} = I'_1 \dots I'_m \text{anch} I_1 \dots I_n$  is expressed using (disjunction of) formulae of the form  $\mathcal{F}_{I_1, \dots, I_n}^n(A_1, \dots, A_{n+1}) \wedge \mathcal{P}_{I'_1, \dots, I'_m}^m(A'_1, \dots, A'_{m+1})$ .

**Construction of NFA for each type-** Let  $\text{seq}$  be any sequence in  $\mathcal{T}(I_\nu)$ . In this section, given  $A = (Q, \text{init}, 2^{\Sigma'}, \delta', F)$  as constructed above, we construct an NFA  $A_{\text{seq}} = (Q \times \{1, 2, \dots, |\text{seq}| + 1\} \cup \{\perp\}, (\text{init}, 1), 2^{\Sigma'}, \delta_{\text{seq}}, F \times \{|\text{seq}| + 1\})$  such that  $L(A_{\text{seq}}) = \text{Norm}(L(A) \cap W_{\text{seq}})$ . Intuitively, the second element of the state indicates the next time restricted point expected to be read. More precisely, from any state  $(q, j)$  the automaton does not have any transition on a time restricted point labelled  $S$  if  $S \cap \text{seq}[j] = \emptyset$ . Moreover, from any state  $(q, j)$ , on reading an unrestricted point of the form  $S \subseteq \Sigma$ , it non deterministically proceeds to a state  $(q', j)$  if and only if, in automaton  $A$ , there is a transition of the form  $q \xrightarrow{S \cup J} q'$  where  $J = \emptyset$  or  $J$  is any interval in  $I_\nu$  such that  $\text{first}(J, w)$  has already been read and  $\text{last}(J, w)$  is yet to be read in the future.

Let  $\text{seq}$  be any sequence in  $\mathcal{T}(I_\nu)$ . Given  $A = (Q, \text{init}, 2^{\Sigma'}, \delta', F)$  over collapsed interval words from LTL formula  $\alpha$ . We construct an NFA  $A_{\text{seq}} = (Q \times \{1, 2, \dots, |\text{seq}| + 1\} \cup \{\perp\}, (\text{init}, 1), 2^{\Sigma'}, \delta_{\text{seq}}, F \times \{|\text{seq}| + 1\})$  such that  $L(A_{\text{seq}}) = \text{Norm}(L(A) \cap W_{\text{seq}})$ .

For any  $(q, i) \in Q \times \{1, \dots, |\text{seq}| + 1\}$ ,  $S \in 2^{\Sigma \cup I_\nu \cup \text{anch}}$  and  $I \in I_\nu \cup \{\text{anch}\}$  such that  $\text{seq}[i] = I$ ,  $\delta_{\text{seq}}$  is defined as follows:

- If  $1 \leq i \leq |\text{seq}|$ 
  - (i) If  $\text{seq}[i] \in S$ , then  $\delta_{\text{seq}}((q, i), S) = \delta(q, S) \times \{i + 1\}$
  - (ii) If  $\text{seq}[i] \notin S \wedge S \cap \Sigma \neq \emptyset$ , then  $\delta_{\text{seq}}((q, i), S) = \emptyset$
  - (iii) If  $S \cap \Sigma = \emptyset$ , then  $\delta_{\text{seq}}((q, i), S) = [\bigcup_{I' \in \mathcal{I}_i} \delta(q, S \cup \{I'\}) \cup \delta_{\text{seq}}(q, S)] \times \{i\}$  where  $\mathcal{I}_i = \{I' \mid I' \in I_\nu \wedge \exists i', i'' : i' < i \leq i'', \text{seq}[i'] = \text{seq}[i''] = I'\}$ .
- If  $i = |\text{seq}| + 1$ ,  $\delta_{\text{seq}}((q, i), S) = \emptyset$  if  $S \cap \Sigma \neq \emptyset$ ,  $\delta_{\text{seq}}((q, i), S) = \delta(q, S)$  if  $S \cap \Sigma = \emptyset$

Let  $W_{\text{seq}}$  be all the set of  $I_\nu$  intervals words over  $\Sigma$  of type  $\text{seq}$ .

► **Lemma 23.**  $L(A_{\text{seq}}) = \text{Norm}(L(A) \cap W_{\text{seq}})$ . Hence,  $\bigcup_{\text{seq} \in \mathcal{T}(I_\nu)} L(A_{\text{seq}}) = \text{Norm}(L(A))$ .

**Proof.** Let  $w$  be any collapsed timed word of type  $\text{seq}$  and  $w' = \text{Norm}(w)$ . Let  $\text{BSequence}(w) = \text{BSequence}(w') = i_1 i_2 \dots i_n$  be the boundary positions.

- (i) If a state  $q$  is reachable by  $A$  on reading first  $j$  letters of  $w$ , then  $(q, k)$  is reachable by  $A_{\text{seq}}$  on reading the corresponding first  $j$  letters of  $w'$  where  $i_{k-1} < j < i_k$ .
- (ii) If a state  $(q, k)$  is reachable by  $A_{\text{seq}}$  on reading first  $j$  letters of  $w'$ , then  $q$  is reachable by  $A_{\text{seq}}$  on reading the corresponding first  $j$  letters of  $w$  and  $i_{k-1} < j < i_k$ .

The above two statements imply that on reading any word  $w \in W_{\text{seq}}$ ,  $A$  reaches the final state if and only if  $A'$  reaches the final state on reading  $w' = \text{Norm}(w)$ . Statement (i) and (ii) are formally proved in Lemma 25 and Lemma 26, respectively. By (i) and (ii), we get  $L(A_{\text{seq}}) \cap W_{\text{seq}} \supseteq \text{Norm}(L(A) \cap W_{\text{seq}})$ .

(1). By Proposition 24 (below)  $L(A_{\text{seq}}) \subseteq \text{Norm}(W_{\text{seq}}) \subseteq W_{\text{seq}}$ , and (2).  $L(A_{\text{seq}}) \cap W_{\text{seq}} = L(A_{\text{seq}})$ . Hence, by (1) and (2),  $L(A_{\text{seq}}) \subseteq \text{Norm}(L(A) \cap W_{\text{seq}})$ . ◀

► **Proposition 24.**  $L(A_{\text{seq}}) \subseteq \text{Norm}(W_{\text{seq}})$

**Proof.** Let  $Q_i = Q \times \{i\}$ . By construction of  $A_{\text{seq}}$ , transition from a state in  $Q_i$  to  $Q_{i'}$ , where  $i \neq i'$  happens only on reading an interval  $I = \text{seq}[i]$ <sup>5</sup>. Moreover,  $i' = i + 1$ . Thus, any word  $w$  is accepted by  $A_{\text{seq}}$  only if there exists  $1 \leq i_1 < i_2 < \dots < i_{|\text{seq}|} \leq |w|$  such that  $w[i_k] \setminus \Sigma = \{\text{seq}[i_k]\}$  and all other points except  $\{i_1, \dots, i_k\}$  are unrestricted points. This implies,  $w \in L(A_{\text{seq}}) \rightarrow w \in \text{Norm}(W_{\text{seq}})$ . ◀

Let the set of the states reachable from initial state,  $\text{init}$ , of any NFA  $C$  on reading first  $j$  letters of a word  $w$  be denoted as  $C < w, j >$ . Hence,  $A < w, 0 > = \{\text{init}\}$  and  $A_{\text{seq}} < w, 0 > = \{(\text{init}, 1)\}$ .

► **Lemma 25.** Let  $w$  be any collapsed  $I_\nu$  interval word of type  $\text{seq}$  and  $\text{BSequence}(w) = i_1 i_2 \dots i_n$ . Let  $w' = \text{Norm}(w)$ . Hence,  $\text{BSequence}(w) = \text{BSequence}(w')$ . For any  $q \in Q$ ,  $q \in A < w, j >$  implies  $(q, k) \in A_{\text{seq}} < w', j >$  where  $i_{k-1} < j < i_k$ .

**Proof.** Recall that  $\text{BSequence}$  is the sequence of boundary points in order. We apply induction on the number of letters read,  $j$ . Note that for  $j = 0$ , by definition,  $A < w, 0 > = \{\text{init}\}$  and  $A_{\text{seq}}(\text{Norm}(w), 0) = \{(\text{init}, 1)\}$  the statement trivially holds as  $0 < i_1 \dots < i_n$ . Let us assume that for some  $m$ , for every state  $q \in A < w, m >$  there exists  $(q, k) \in A_{\text{seq}} < \text{Norm}(w), m >$  such that  $i_1 < \dots < i_{k-1} \leq m < i_k < \dots < i_n$ . Now let  $j = m + 1$ . Let us assume that  $q'$  is any state in  $A < w, m + 1 >$ . We just need to show that for some  $(q', k') \in A_{\text{seq}} < w', m + 1 >$  where  $k' = k + 1$  if  $m + 1 \in \text{Boundary}(w)$ . Else  $k' = k$ .

As  $q' \in A < w, m + 1 >$ , there exists a state  $q \in A < w, m >$  such that  $q' \in \delta(q, w[m + 1])$ . By induction hypothesis,  $(q, k) \in A_{\text{seq}} < \text{Norm}(w), m >$ . Note that  $(q', k') \in \delta_{\text{seq}}((q, k), w'[m + 1])$  implies  $(q', k') \in A_{\text{seq}} < w', m + 1 >$ . Let  $w[m + 1] = S_J$ , where  $S_J \subseteq \Sigma \cup I_\nu \cup \{\text{anch}\}$  and  $S_J \setminus \Sigma$  contains at most 1 element.

**Case 1:**  $m + 1 \in \text{Boundary}(w)$ . This implies that  $w'[m + 1] = w[m + 1]$ . As both  $w$  and  $w'$  are of type  $\text{seq}$ ,  $\{\text{seq}[k]\} = S_J \setminus \Sigma$  (by definition of  $\text{seq}$ ). Hence, by construction of  $A_{\text{seq}}$ ,  $\delta_{\text{seq}}((q, k), S_J) = \delta(q, S_J) \times \{k + 1\}$ . As  $q' \in \delta(q, S_J)$ ,  $(q', k + 1) \in \delta_{\text{seq}}((q, k), S_J)$ .

**Case 2:**  $m + 1 \notin \text{Boundary}(w)$ . This implies that  $w'[m + 1] = S = S_J \cap \Sigma$ .

*Case 2.1:*  $S = S_J$ . By construction of  $A_{\text{seq}}$ ,  $\delta((q, k), S) \supseteq \delta(q, S) \times k$ . Thus,  $(q', k) \in \delta_{\text{seq}}((q, k), S_J \cap \Sigma)$ .

*Case 2.2:*  $S \neq S_J$ . Let  $S_J \setminus \Sigma = \{J\}$  where  $J \in I_\nu \cup \{\text{anch}\}$ . Then  $m + 1$  is neither the first nor the last  $J$ -time restricted point nor the anchor point in  $w$ . Hence,  $\text{first}(J, w) < m + 1 < \text{last}(J, w)$ . By induction hypothesis,  $i_{k-1} \leq m < i_k$ . Note, as  $m + 1$  is not in  $\text{Boundary}(w)$ ,  $m + 1 \neq i_k$ . Hence,  $i_{k-1} \leq m < m + 1 < i_k$ . This implies,  $\text{first}(J, w) < i_k \leq \text{last}(J, w)$ . By definition of  $\text{seq}$ , there exists  $k'$  and  $k''$  such that  $k' < k \leq k''$  and  $\text{seq}[k'] = \text{seq}[k''] = J$ . Hence, by construction of  $\delta_{\text{seq}}$ ,  $\delta_{\text{seq}}((q, k), S) \supseteq \delta(a, S_J) \times \{k\}$ . Hence  $(q', k) \in \delta_{\text{seq}}((q, k), S_J)$ . ◀

► **Lemma 26.** Let  $w'$  be any normalized  $I_\nu$  interval word of type  $\text{seq}$  and  $\text{BSequence}(w') = i_1 i_2 \dots i_n$ . Let  $i_0 = 0$ . For any  $q \in Q$ ,  $(q, k) \in A_{\text{seq}} < w', j >$  implies there exists a collapsed  $I_\nu$  interval word  $w$ , such that  $\text{Norm}(w) = w'$ ,  $q \in A_{\text{seq}} < w, j >$  and  $i_{k-1} < j \leq i_k$

<sup>5</sup> Let  $I$  be any symbol in  $I_\nu \cup \{\text{anch}\}$ . By “reading of an interval  $I$ ” we mean “reading a symbol  $S$  containing interval  $I$ ”.

**Proof.** We apply induction on the value of  $j$  as in proof of Lemma 25. For  $j = 0$ , the statement trivially holds. Assume that for  $j = m$ , the statement holds and  $(q', k') \in A_{\text{seq}} < w', m + 1 >$  (Assumption 1). We need to show

- (i)  $i'_{k'-1} \leq m + 1 < i'_k$  and,
- (ii) there exists  $w$  such that  $\text{Norm}(w) = w'$  and  $q' \in A < w, m + 1 >$ .  $(q', k') \in A_{\text{seq}} < w', m + 1 >$  implies, there exists  $(q, k) \in A_{\text{seq}} < w', m >$  such that  $(q', k') \in \delta_{\text{seq}}((q, k), w'[m + 1])$ . By induction hypothesis,  $i_{k-1} \leq m < i_k$  [IH1] and there exists a word  $w''$  such that  $\text{Norm}(w'') = w'$  and  $q \in A < w'', m >$  [IH2].

**Case 1**  $m + 1 \in \text{Boundary}(w')$ : This implies

- (a)  $m + 1 \in \{i_1, i_2, \dots, i_k\}$ .
- (b)  $k' = k + 1$  (by construction of  $\delta_{\text{seq}}$ ).
- (c)  $w''[m + 1] = w'[m + 1] = S \cup \{J\}$  such that  $S \subseteq \Sigma$  and  $J \in (I_\nu \cup \{\text{anch}\})$ . In other words,  $m + 1$  is either a time restricted point or an anchor point in both  $w''$  and  $w'$ .
- d)  $\text{seq}[i'_k] = J$ , otherwise  $\delta_{\text{seq}}((q, k), S \cup \{J\}) = \emptyset$  which contradicts Assumption 1.
- (i) IH1 and a) implies that  $m + 1 = i_k$ . This along with b) implies that  $m + 1 = i_{k'-1}$ . Hence proving (i) for Case 1.
- (ii) IH2 along with c) and d) implies that  $\delta_{\text{seq}}((q, k), w'[m + 1]) = \delta(q, w[m + 1]) \times \{k + 1\}$ . Hence, if  $(q', k') \in A_{\text{seq}} < w', m + 1 >$  then  $q' \in A < w'', m + 1 >$ . Hence, there exists a  $w = w''$  such that  $q' \in A < w, m + 1 >$ , proving (ii) for Case 1.

**Case 2**  $m + 1 \in \text{Boundary}(w')$  : This implies

- (1)  $m + 1 \notin \{i_1, i_2, \dots, i_k\}$ .
- (2)  $k' = k$  (by construction of  $\delta_{\text{seq}}$ ).
- (3)  $w''[m + 1] \subseteq \Sigma$ . In other words,  $m + 1$  is either an unrestricted point in both  $w''$ .

Now we have

- (i) IH1 implies  $i_{k-1} \leq m < m + 1 \leq i_k$ . This along with 1) and 2) implies  $i_{k'-1} \leq m < m + 1 < i_{k'}$ . Hence proving (i) for Case 2.
- (ii) IH2 along with 3) and the construction of  $\delta_{\text{seq}}$  implies  $\delta_{\text{seq}}((q, k), w'[m + 1]) = (\bigcup \delta_{\text{seq}}((q, k), w'[m + 1] \cup \{J\}) \cup \delta_{\text{seq}}((q, k), w'[m + 1])) \times k$  for  $J \in I_\nu$  such that there exists  $j < k' < l$  such that  $\text{seq}[j] = \text{seq}[l] = J$ .

Hence,  $J$  is an interval which appears twice in  $\text{seq}$  and only one of those  $J$ 's have been encountered within first  $m$  letters. Hence, the prefix  $w'[1..m + 1]$  and suffix  $w'[m + 2..]$  contains exactly one  $J$  time restricted point. This implies that

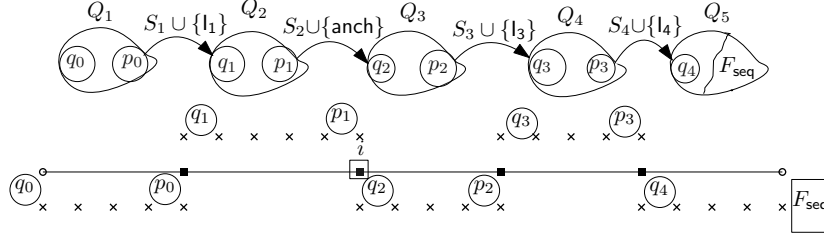
(Case A)  $q' \in \delta_{\text{seq}}((q, k), w'[m + 1] \cup \{J\})$  for some  $J$  such that  $w'[1..m]$  and  $w'[m + 2..]$  contains exactly one  $J$ - time restricted point or

(Case B)  $q' \in \delta_{\text{seq}}((q, k), w'[m + 1])$ .

As  $\text{Norm}(w'') = w'$ , first and last  $J$  time restricted points are the same in both  $w''$  and  $w'$ . Hence, first  $J$ -time restricted point in  $w''$  is within  $w''[1..m]$  and the last is within  $w''[m + 2..]$ . Consider a set of words  $W$  such that for any  $w \in W$ ,  $w[1..m] = w''[1..m]$ ,  $w[m + 2..] = w'[m + 2..]$  and either  $w[m + 1] = w'[m + 1]$  or  $w[m + 1] = w'[m + 1] \cup \{J\}$  where  $J \in I_\nu$  such that both  $w'[1..m]$  and  $w'[m + 2..]$  contains  $J$ -time restricted points. Notice that  $m + 1 \notin \text{Boundary}(w)$ . Hence, making it time unrestricted will still imply  $\text{Boundary}(w) = \text{Boundary}(w')$ .

When there exists a  $J$  restricted time point in prefix  $w[1..m]$  and suffix  $w[m + 2..]$  for  $J \in I_\nu$ , making point  $m + 1$  as  $J$  restricted time point will still imply  $\text{Boundary}(w) = \text{Boundary}(w')$ . Hence, this implies that  $\text{Norm}(w) = w'$  for any  $w \in W$ . Moreover, as for





■ **Figure 3** Figure representing set of runs  $A_{l_1 anch l_3 l_4}$  of type  $R_{Qseq}$  where  $Qseq = T_1 T_2 T_3 T_4$ , where for  $1 \leq i \leq 4$ ,  $T_i = (p_{i-1} \xrightarrow{S_i \cup \{I_i\}} q_i)$ ,  $I_2 = \{\text{anch}\}$

any  $w \in W$ ,  $w[1..m] = w''[1..m]$ ,  $A < w, m \rangle = A < w'', m \rangle$  and  $A_{seq} < w, m \rangle = A_{seq} < w'', m \rangle$ . Hence, for any  $q \in Q$  such that  $(q, k) \in A_{seq} < w', m \rangle$  implies for every  $w \in W$ ,  $q \in A < w, m \rangle$ .

It suffices to show that there exists a  $w \in W$  such that  $q' \in A < w, m+1 \rangle$ . In case of Case A, for any word  $w \in W$  such that  $w[m+1]$  is a  $J$ -time restricted point  $q' \in A < w, m+1 \rangle$ . Note that such a word exists as Case A implies that  $w'[1..m]$  and  $w'[m+2..]$  contains exactly one  $J$ -time restricted point. In case of B, for any word  $w \in W$  where  $w[m+1] \subseteq \Sigma$ ,  $q' \in A < w, m+1 \rangle$ . Hence, proving for Case 2. ◀

The words in  $L(A_{seq})$  are all normalized, and have at most  $2|I_\nu| + 1$  time restricted points. Thanks to this, its corresponding timed language can be expressed using PnEMTL formulae with arity at most  $2|I_\nu|$ .

**Reducing NFA of each type to PnEMTL:** Next, for each  $A_{seq}$  we construct PnEMTL formula  $\phi_{seq}$  such that, for a timed word  $\rho$  with  $i \in \text{dom}(\rho)$ ,  $\rho, i \models \phi_{seq}$  iff  $\rho, i \in \text{Time}(L(A_{seq}))$ . For any NFA  $N = (St, \Sigma, i, Fin, \Delta)$ ,  $q \in Q$ ,  $F' \subseteq Q$ , let  $N[q, F'] = (St, \Sigma, q, F', \Delta)$ . For the sake of brevity, we denote  $N[q, \{q'\}]$  as  $N[q, q']$ . We denote by  $\text{Rev}(N)$ , the NFA  $N'$  that accepts the reverse of  $L(N)$ . The right/left concatenation of  $a \in \Sigma$  with  $L(N)$  is denoted  $N \cdot a$  and  $a \cdot N$  respectively.

► **Lemma 27.** *We can construct a PnEMTL formulae  $\phi_{seq}$  with  $\text{Constraint}(\phi_{seq}) \subseteq I_\nu$  such that  $\rho, i \models \phi_{seq}$  iff  $\rho, i \in \text{Time}(L(A_{seq}))$ .*

**Proof.** Let  $seq = I_1 I_2 \dots I_n$ , and  $I_j = \text{anch}$  for some  $1 \leq j \leq n$ . Let  $\Gamma = 2^\Sigma$  and

$Qseq = T_1 T_2 \dots T_n$  be a sequence of transitions of  $A_{seq}$  where for any  $1 \leq i \leq n$ ,  $T_i = p_{i-1} \xrightarrow{S'_i} q_i$ ,  $S'_i = S_i \cup \{I_i\}$ ,  $S_i \subseteq \Sigma$ ,  $p_{i-1} \in Q \times \{i-1\}$ ,  $q_i \in Q \times \{i\}$ . Let  $q_0 = (\text{init}, 1)$ . We define  $R_{Qseq}$  as set of accepting runs containing transitions  $T_1 T_2 \dots T_n$ . Hence the runs in  $R_{Qseq}$  are of the following form:

$T_{0,1} T_{0,2} \dots T_{0,m_0} T_1 T_{1,1} \dots T_{1,m_1} T_2 \dots T_{n-1,1} T_{n-1,2} \dots T_n T_{n,1} \dots T_{n+1}$

where the source of the transition  $T_{0,1}$  is  $q_0$  and the target of the transition  $T_{n+1}$  is any accepting state of  $A_{seq}$ . Moreover, all the transitions  $T_{i,j}$  for  $0 \leq i \leq n$ ,  $1 \leq j \leq n_i$  are of the form  $(p' \xrightarrow{S_{i,j}} q')$  where  $S_{i,j} \subseteq \Sigma$  and  $p', q' \in Q_{i+1}$ . Hence, only  $T_1, T_2, \dots, T_n$  are labelled by any interval from  $I_\nu$ . Moreover, only on these transitions the the counter (second element of the state) increments. Let  $W_{Qseq}$  be set of words associated with any run in  $R_{Qseq}$ . Refer figure 3 for illustration.  $w \in W_{Qseq}$  if and only if  $w \in L(A_1).S'_1.L(A_2).S'_2 \dots L(A_n).S'_n.L(A_{n+1})$  where

$A_i = (Q_i, 2^\Sigma, q_{i-1}, \{p_{i-1}\}, \delta_{seq}) \equiv A_{seq}[q_{i-1}, p_{i-1}]$  for  $1 \leq i \leq n$  and

$A_{n+1} = (Q_{n+1}, 2^\Sigma, q_n, F_{seq}, \delta_{seq}) \equiv A[q_n, F]$ . Let  $A'_k = S_{k-1} \cdot A_k \cdot S_k$  for  $1 \leq k \leq n+1$ , with  $S_0 = S_{n+1} = \epsilon$ . Let  $\rho = (b_1, \tau_1) \dots (b_m, \tau_m)$  be a timed word over  $\Gamma$ . Then  $\rho, i_j \in \text{Time}(W_{Qseq})$  iff

$\exists 0 \leq i_1 \leq i_2 \leq \dots \leq i_{j-1} \leq i_j \leq i_{j+1} \leq \dots \leq i_n \leq m$  s.t.

$$\bigwedge_{k=1}^{j-1} [(\tau_{i_k} - \tau_{i_j} \in I_k) \wedge \text{Seg}^-(\rho, i_{k+1}, i_k, \Gamma) \in L(\text{Rev}(A'_k))] \wedge \bigwedge_{k=j}^n [(\tau_{i_k} - \tau_{i_j} \in I_k) \wedge \text{Seg}^+(\rho, i_k, i_{k+1}, \Gamma) \in L(A'_k)],$$

where  $i_0=0$  and  $i_{n+1}=m$ . Hence, by semantics of  $\mathcal{F}^k$  and  $\mathcal{P}^k$  modalities,  $\rho, i \in \text{Time}(\mathcal{W}_{Q_{\text{seq}}})$  if and only if  $\rho, i \models \phi_{\text{qseq}}$  where

$$\phi_{\text{qseq}} = \mathcal{P}_{I_{j-1}, \dots, I_1}^j(\text{Rev}(A'_1), \dots, \text{Rev}(A'_j))(\Gamma) \wedge \mathcal{F}_{I_{j+1}, \dots, I_n}^{n-j}(A'_{j+1}, \dots, A'_{n+1})(\Gamma).$$

Let  $\text{State-seq}$  be set of all possible sequences of the form  $\text{Qseq}$ . As  $A_{\text{seq}}$  accepts only words which has exactly  $n$  time restricted points, the number of possible sequences of the form  $\text{Qseq}$  is bounded by  $|Q|^n$ . Hence any word  $\rho, i \in \text{Time}(L(A_{\text{seq}}))$  iff  $\rho, i \models \phi_{\text{seq}}$  where  $\phi_{\text{seq}} = \bigvee_{\text{qseq} \in \text{State-seq}} \phi_{\text{qseq}}$ . Disjuncting over all possible sequences  $\text{seq} \in \mathcal{T}(I_\nu)$  we get

the required PnEMTL formula  $\phi$ . Moreover, the timing intervals appearing in all the  $\mathcal{F}^k$  subformulae of  $\phi$  are from  $I_\nu$ . Similarly, the timing intervals appearing in all the  $\mathcal{P}^k$  formulae are from  $I_\nu^- = \{\langle -l, -u \rangle \mid \langle l, u \rangle \in I_\nu \wedge l < u \leq 0\}$ . If  $\mathcal{I}$  is non adjacent, then its intersection closure  $I_\nu^-, I_\nu, I_\nu^+$  are also non adjacent. Hence, if  $\mathcal{I}$  is non adjacent then  $\phi$  is a non adjacent PnEMTL formula.  $\blacktriangleleft$

Disjuncting over all possible sequences  $\text{seq} \in \mathcal{T}(I_\nu)$  we get the following lemma 22.

As a consequence of 13, 21 and 22 any (Non Adjacent) 2-Way 1-ATA-rfl of reset depth 0 can be reduced to an equivalent (Non Adjacent) PnEMTL formulae.

## D.2.5 Induction

Assume that the lemma 11 holds if  $\mathcal{A}$  has reset depth less than  $n$ . Let reset depth of  $\mathcal{A} = (\Gamma, Q^+, Q^-, \text{init}, \top, \perp, \Delta, \mathcal{G})$  be  $n$ . Let  $Q_0, \dots, Q_m$  be the set of islands of  $\mathcal{A}$  with header locations  $q_0^r, \dots, q_m^r$ , respectively. Let  $Q_0$  be the initial island. Let  $\mathcal{A}[q_j^r]$  of  $\mathcal{A}$  is a 2-Way 1-ATA-rfl same as  $\mathcal{A}$  but with initial state  $q_j^r$  for any  $1 \leq j \leq m$ . Note that, for any  $1 \leq j \leq m$ , as  $q_j^r \prec \text{init}$ , all the states reachable from  $q$  are the states within island  $Q_j$  and all the islands lower than  $Q_j$ . Hence, the reset depth of any subautomata of  $\mathcal{A}$  is less than  $n$ . by induction hypothesis, we can construct a PnEMTL formulae  $\varphi_j$  equivalent to  $\mathcal{A}[q_j^r]$  for  $1 \leq j \leq m$ . Let  $W = \{b_1, \dots, b_m\}$  be witness variables for  $\mathcal{A}[q_j^r]$  and  $\varphi_j$ . We now construct an automata  $\mathcal{A}'$  from  $\mathcal{A}$  with transition function  $\delta'$ , set of locations in  $Q_0$  and over symbols in  $\Gamma \times \{0, 1\}^m$  where  $j^{\text{th}}$  component of the bit vector encodes the truth value of witness  $b_j$ . For any  $q \in Q_0, a \in \Sigma, g \in \mathcal{G}$ , let  $\delta'(q, a, g)$  is a boolean expression constructed from  $\delta(q, a, g)$  by replacing all the occurrences of  $x.q_j^r$  with truth value of  $b_j$ . Hence, whenever  $b_j$  is false the conjunction of transitions calling  $x.s_j^r$  is vanishes in  $\delta'$ . Note that automata  $\mathcal{A}'$  is a reset free automata (as all the literals reset construct are replaced with either 0 or 1). As shown for the base case, we can construct a PnEMTL formulae  $\varphi'$  equivalent to  $\mathcal{A}'$  over extended alphabets. For any  $a \in \Gamma$  and  $b \in \{0, 1\}^k$  we replace occurrence of  $(a, b)$  in  $\varphi'$  with  $a \wedge \bigwedge_{b(i)=1} \varphi_i \wedge \bigwedge_{b(j)=0} \neg \varphi_j$ . Hence, By replacing the witnesses with their corresponding

formulae we get the required formulae  $\varphi$  equivalent to  $\mathcal{A}$ . Moreover, note that if  $\mathcal{A}$  is non adjacent all its sub automata and  $\mathcal{A}'$  are non adjacent. Then, by induction hypothesis and by construction of PnEMTL for reset free automata  $\varphi$  is a non adjacent PnEMTL formula.

## D.3 Proof of Lemma 14

**Lemma 14.** (PnEMTL  $\subseteq$  GQMSO). Given any (NA)PnEMTL formula  $\varphi$ , we can construct an equivalent (NA)GQMSO formula  $\psi$ . **Proof.** We apply induction on modal depth of the given formula  $\varphi$ .

**Base Case:** For modal depth 0,  $\varphi$  is a propositional formula and hence it is trivially a AF-GQMSO formula. Let  $\varphi$  be a modal depth 1 formula of the form  $\mathcal{F}_{I_1, \dots, I_k}^k(A_1, \dots, A_{k+1})(\Sigma)$ . The reduction for  $\mathcal{P}^k$  modality is identical. Moreover, dealing with boolean operators is trivial as the AF-GQMSO are closed under boolean operations. Let  $A_j = (2^\Sigma, Q_j, \text{init}_j, F_j, \delta_j)$ . By semantics, for any timed word  $\rho = (a_1, \tau_1) \dots (a_m, \tau_m)$  and  $i_0 \in \text{dom}(\rho')$ ,  $\rho, i_0 \models \varphi$  iff  $\exists i_0 < i_1 < i_2 \dots < i_k < n$  s.t.

$\bigwedge_{w=1}^k [(\tau_{i_w} - \tau_{i_0} \in I_w) \wedge \text{Seg}^+(\rho, i_{w-1} + 1, i_w, \Sigma) \in L(A_w)]$ . By BET theorem, we can construct an MSO[<] formula  $\psi_w(i_{w-1}, i_w)$  equivalent to condition  $\text{Seg}^+(\rho, i_{w-1} + 1, i_w, \Sigma) \in L(A_w)$ . Note that replacing conditions  $\text{Seg}^+(\rho, i_{w-1} + 1, i_w, \Sigma) \in L(A_w)$  with  $\psi_i$  will result in a AF-GQMSO formula. Moreover if  $\varphi$  is non adjacent then the resulting AF-GQMSO formula is also non adjacent. We assume that the lemma holds for all the PnEMTL formulae of modal depth  $< n$ . Let  $\varphi = \mathcal{F}_{I_1, \dots, I_k}^k(A_1, \dots, A_{k+1})(\Sigma \cup S)$  of modal depth  $n$ . Therefore,  $S$  is a set of PnEMTL formula with modal depth  $< n$ . We replace all the subformulae in  $S$  by a witness propositions getting a formula  $\varphi'$  of modal depth 1. As with the base case, we can construct an AF-GQMSO formula  $\psi'$  equivalent to  $\varphi'$ . By inductive hypothesis every subformulae  $\varphi_i$  in  $S$  can be reduced to an equivalent AF-GQMSO formula  $\psi_i$ . We replace all the witnesses of  $\varphi_i$  by  $\psi_i$  getting an equivalent formulae  $\psi$  over  $\Sigma$ . Note that if formula  $\varphi_i$  in  $S$  are non adjacent then, by induction hypothesis, equivalent  $\psi_i$  are in NA-GQMSO formula. Similarly, if  $\varphi'$  is NAPnEMTL formula then  $\psi'_i$  is NA-GQMSO formula. Hence, if  $\varphi$  in non adjacent then equivalent formula  $\psi$  is non adjacent too.