WEAKLY DEFINABLE RELATIONS AND SPECIAL AUTOMATA*

BY

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In this paper we consider monadic second-order theories and study problems of definability. As a by-product we obtain certain decidability results. Let $\mathcal{N}_2 = \langle T, r_0, r_1 \rangle$ be the structure of two successor functions (see §1). Let L be the monadic second-order language appropriate for $\mathcal{N}_2$ which has individual variables $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \ldots$, ranging over elements of T, finite-set variables $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \ldots$, ranging over finite subsets of T, and set variables $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \ldots$, ranging over arbitrary subsets of T. A relation $H \subseteq P(T)^n$ between subsets of T is called *definable* in the second-order theory (language) of $\mathcal{N}_2$ if for some formula $F(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_n)$ of L

$$H = \{(A_1, \ldots, A_n) \,|\, (A_1, \ldots, A_n) \in P(T)^n, \mathcal{N}_2 \models F(A_1, \ldots, A_n)\}. \tag{1}$$

The relation H is *weakly-definable* if (1) holds for a formula $F(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_n)$ containing just individual and finite-set quantifiers.

In [6] we have characterized the definable relations by means of finite automata operating on infinite trees. This result was used to solve the decision problem of the second-order theory of $\mathcal{N}_2$. This in turn entailed the decidability of many theories.

Here we introduce the notion of a *special automaton* on infinite trees and use it to characterize the weakly definable sets. An automaton on infinite trees may be viewed as representing a relation $H \subseteq P(T)^n$ for some n. It turns out that a relation $H \subseteq P(T)^n$ is weakly definable if and only if both H and complement $P(T)^n - H$ are represented by appropriate special automata. On the other hand, not every relation H represented by a special automaton is weakly definable. Rather, a relation $H \subseteq P(T)^n$ is represented by a special automaton if and only if (1) holds with a formula $F(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_n)$ in prenex form which has only existential arbitrary-set quantifiers. This yields the following syntactical result. A formula $F(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_n)$ is equivalent (in $\mathcal{N}_2$) with some formula $G(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_n)$ containing only finite-set quantifiers, if and only if F is equivalent to some prenex formula $F_1(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_n)$ having only existential arbitrary-set quantifiers, and also to some prenex formula $F_2(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_n)$ having only universal arbitrary-set quantifiers.

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As a by-product of the characterization of weakly definbale relations we get the solution of certain decision problems. In [6] we have shown that the weak second-order theory of a unary function, and the weak second-order theory of linearly ordered sets (see [4]), are decidable. These results were actually corollaries of stronger theorems concerning the corresponding full monadic second-order theories. Here we deduce the same decidability results using the information concerning weakly definable relations and special automata. Also the many applications by D. M. Gabbay of [3] to the solution of the decision problem of various logical calculi follow already from Theorem 24 of the present paper.

1. Notations and basic standard definitions

We shall employ the standard notations and terminology concerning sets, mappings, structures, and logical calculi, used in [6].

As usual, each natural number n is the set of all smaller numbers. Thus $0 = \emptyset$, $1 = \{0\}$, $2 = \{0,1\}$, and $n = \{0,1,\dots,n-1\}$. An *n-termed sequence* is a mapping $x: n \to A$. The sequence x is also called a *word* on A. The ith coordinate of the sequence is $x(i)$, $0 \leqq i < n$, and will sometimes be denoted by x_i. The *length* $l(x)$ of x is $l(x) = n$. The sequence x will also be written as $(x_0, \dots, x_{n-1})$. If $x = (x_0, \dots, x_{n-1})$ and $y = (y_0, \dots, y_{m-1})$ then xy will denote the sequence $(x_0, \dots, x_{n-1}, y_0, \dots, y_{m-1})$. We have $l(xy) = l(x) + l(y)$. The sequence (x_0) of length one will also be written as x_0. Thus $x = x_0 x_1 \dots x_{n-1}$. The unique empty sequence of length 0 will be denoted by Λ.

For each $i < \omega$, the *projection* p_i is the function which is defined by $p_i(x) = x_i$ for $x = (x_0, \dots, x_{n-1})$, $i < n$.

The *infinite binary tree* is the set $T = \{0,1\}^*$ of all finite words on $\{0,1\}$. The elements $x \in T$ are the *nodes* of T. For $x \in T$, the nodes $x0, x1$ are called the *immediate successors* of x. The empty word Λ is called the *root* of T. Our language is suggested by the following picture. The lowest node of T is the root Λ. The root branches up to the (say) left into the node 0 and to the right into the node 1. The node 0 branches into 00 and 01; the node 1 branches into 10 and 11. And so on ad infinitum.

On T we define a partial-ordering by $x \leqq y$ (x is an *initial* of y) if and only if $\exists z[y = xz]$. If $x \leqq y$ and $x \neq y$ then we shall write $x < y$.

For $x \in T$, the *subtree* T_x with *root* x is defined by $T_x = \{y \mid y \in T, x \leqq y\}$. Thus $T_\Lambda = T$.

A *path* π of a tree T_x is a set $\pi \subset T_x$ satisfying 1) $x \in \pi$; 2) for $y \in \pi$, either $y0 \in \pi$ or $y1 \in \pi$, but not both; 3) π is the smallest subset of T_x satisfying 1–2.

Note that if $\pi \subset T$ is a path and $x, y \in \pi$, then $x \leqq y$ or $y \leqq x$.

A subset $F \subset T_x$ is called a *frontier* of T_x if for every path $\pi \subset T_x$ we have $c(\pi \cap F) = 1$. It is readily seen that if $F \subset T_x$ is a frontier then F is finite. If $F_1 \subset T_x$ and $F_2 \subset T_x$ are frontiers we shall say that F_2 *is bigger than* F_1 ($F_1 < F_2$) if for every $y \in F_2$ there exists a $x \in F_1$ such that $x < y$. For $S \subseteq T$ we have $c(S \cap \pi) = \omega$ for every path $\pi \subset T$, if and only if $S = \bigcup_{n<\omega} F_n$ where F_n is a frontier of T and $F_n < F_{n+1}$, $n < \omega$.

A *finite* (*frontiered*) *tree* is a set $E = \{x \mid x \leqq y \text{ for some } y \in F\}$ where F is a fixed frontier of T. For E as above, F is called the *frontier* of E and denoted by $Ft(E)$. By "finite tree" we shall always mean a finite frontiered tree.

For $a \in \{0,1\}$ define the (immediate) *successor function* $r_a: T \to T$ by $r_a(x) = xa$, $x \in T$. The *structure of two successor functions* is $\mathcal{N}_2 = \langle T, r_0, r_1 \rangle$.

With $\mathcal{N}_2$ we associate an appropriate (monadic) second-order language L_2. This L_2 has function-constants $\mathbf{r}_0, \mathbf{r}_1$, to denote r_0 and r_1; the usual logical connectives and quantifiers; the membership symbol $\in$; equality; individual variables $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \ldots$, ranging over elements of T; finite-set variables $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \ldots$, ranging over finite subsets of T; set variables $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \ldots$, ranging over arbitrary subsets of T. The atomic formulas of L_2 include formulas of the form $t \in V$ where t is a term of L_2 and V is a (finite or arbitrary) set variable. Quantification is possible over all the three sorts of variables.

The *second-order theory of two successor functions* (S2S) is the set of all sentences F of L_2 such that $\mathcal{N}_2 \models F$ (F is true in $\mathcal{N}_2$). The theory S2S was proved decidable in [6] by means of a theory of automata on infinite trees.

DEFINITION 1. An n-ary relation $R \subseteq P(T)^n$ between subsets of T is *definable* in L_2 (S2S) if there exists a formula $F(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_n)$ of L_2 such that

$$R = \{(A_1, \ldots, A_n) \mid \mathcal{N}_2 \models F(A_1, \ldots, A_n)\}. \tag{1}$$

The relation R is *weakly-definable* if (1) holds for a formula containing quantifiers only over individual and finite-set variables.

2. Special automata

As stated in the Introduction, our aim is to characterize the weakly-defined relations. To this end we develop a theory of special automata.

In the following, Σ denotes a finite set called the *alphabet.*

DEFINITION 2. A Σ-(*valued*)*tree* is a pair (v, T_x) such that $v: T_x \to \Sigma$. If (v, T) is a valued tree then (v, T_x) will denote the induced valued subtree $(v \mid T_x, T_x)$. The set of all Σ-trees (v, T_x), for a fixed $x \in T$, will be denoted by $V_{\Sigma,x}$. The set $\bigcup_{x \in T} V_{\Sigma,x}$ of all Σ-trees will be denoted by V_Σ.

DEFINITION 3. A *table* over Σ-trees is a pair $\langle S, M\rangle$ where S is a finite set, the *set of states*, and M is a function $M: S \times \Sigma \to P(S \times S)$, the (non-deterministic) *table of moves* ($P(A)$ denotes the set of all subsets of A).

A *special finite automaton* (s.f.a.) over Σ-trees (a *special Σ-automaton*) is a system $\mathfrak{A} = \langle S, M, S_0, F\rangle$ where $\langle S, M\rangle$ is as above, $S_0 \subseteq S$ is the set of *initial states*, $F \subseteq S$ is the set of *designated states*.

DEFINITION 4. A *run* of $\langle S, M\rangle$ on the Σ-tree $t = (v, T_x)$ is a mapping $r\colon T_x \to S$ such that for $y \in T_x$, $(r(y0), r(y1)) \in M(r(y), v(y))$. We also talk about a run of an automaton $\mathfrak{A}$ on a tree, meaning a run of the associated table. The set of all $\mathfrak{A}$-runs on t is denoted by $\mathrm{Rn}(\mathfrak{A}, t)$.

For a mapping $\phi: A \to B$ define $\mathrm{In}(\phi) = \{b \mid b \in B,\ c(\phi^{-1}(b)) \geqq \omega\}$.

DEFINITION 5. The special automaton $\mathfrak{A} = \langle S, M, S_0, F\rangle$ *accepts* (v, T_x) if there exists an *accepting* $\mathfrak{A}$-run r on (v, T_x) such that $r(x) \in S_0$ and for every path π of T_x, $\mathrm{In}(r \mid \pi) \cap F \neq \emptyset$. The set $T(\mathfrak{A})$ of Σ-trees *defined* by $\mathfrak{A}$ is

$$T(\mathfrak{A}) = \{(v, T_x) \mid x \in T,\ (v, T_x) \textit{ accepted by } \mathfrak{A}\}.$$

A set $A \subseteq V_\Sigma$ is *s.f.a. definable* if for some s.f.a. $\mathfrak{A}$, $A = T(\mathfrak{A})$.

REMARK. It is quite clear that the special automata introduced here are a weaker version of the automata defined in [6]. For every special automaton $\mathfrak{A}$ there exists an automaton $\mathfrak{B}$ in the sense of [6], such that $T(\mathfrak{A}) = T(\mathfrak{B})$. In fact, $\mathfrak{B}$ may be taken to have the same table and initial status as $\mathfrak{A}$. That the converse statement is not true is shown by the example in §3.

REMARK. A set $A \subset V_\Sigma$ is called *invariant* if for every Σ-tree $t = (v, T)$ and every $x \in T$, $t \in A$ if and only if the tree $t' = (v', T_x)$ defined by $v'(xy) = v(y)$, $y \in T$, is in A. The invariant subsets of V_Σ are a boolean algebra. It is clear from Definition 5 that every set $T(\mathfrak{A})$ is invariant. To prove that an invariant set A is s.f.a. definable, it suffices to construct an automaton $\mathfrak{A}$ such that $(v, T) \in T(\mathfrak{A})$ if and only if $(v, T) \in A$.

The following results are immediate.

LEMMA 1. *If $A \subseteq V_\Sigma$ is s.f.a. definable, then there exists an automaton $\mathfrak{A} = \langle S, M, S_0, F\rangle$ such that $S_0 = \{s_0\}$, $s_0 \in S$, and $T(\mathfrak{A}) = A$. This $\mathfrak{A}$ may be chosen so that $(s_1, s_2) \in M(s, \sigma)$ implies $s_1 \neq s_0$, and $s_2 \neq s_0$.*

THEOREM 2. *If $A, B \subseteq V_\Sigma$ are s.f.a. definable, then so are $A \cup B$ and $A \cap B$.*

Proof. Let $A = T(\mathfrak{A})$, $B = T(\mathfrak{B})$ where $\mathfrak{A} = \langle S, M, s_0, F\rangle$, $\mathfrak{B} = \langle S', M', s'_0, F'\rangle$; we assume that $S \cap S' = \emptyset$. Construct the automaton

$$\mathfrak{A} \cup \mathfrak{B} = \langle S \cup S',\ M \cup M',\ \{s_0, s_0'\},\ F \cup F'\rangle.$$

Clearly, $T(\mathfrak{A} \cup \mathfrak{B}) = A \cup B$.

With the above notations, define $\mathfrak{A} \times \mathfrak{B} = \langle S \times S' \times \{0,1,2\}, \bar{M}, (s_0, s_0', 0), \bar{F}\rangle$ as follows. $((s_1, s_1', b), (s_2, s_2', b)) \in \bar{M}((s, s', a), \sigma)$ if and only if $(s_1, s_2) \in M(s, \sigma)$, $(s_1', s_2') \in M'(s', \sigma)$; $b = 1$ if and only if $a = 0$ and $s \in F$, or $a = 1$ and $s' \notin F'$; $b = 2$ if and only if $a = 1$ and $s' \in F'$; $b = 0$ if and only if $a = 2$, or $a = 0$ and $s \notin F$. Put $\bar{F} = S \times S' \times \{2\}$. We have $T(\mathfrak{A} \times \mathfrak{B}) = A \cap B$.

DEFINITION 6. Let $t = (v, T)$ be a $\Sigma_1 \times \Sigma_2$-tree and let p_0 be the projection $p_0(x, y) = x$. The *projection* $p_0(t)$, by definition, is the Σ_1-tree $(p_0 v, T)$.

The *projection* $p_0(A)$ of a set $A \subseteq V_{\Sigma_1 \times \Sigma_2}$, is $p_0(A) = \{p_0(t) \mid t \in A\}$. The *$\Sigma_2$-cylindrification* of a set $B \subseteq V_{\Sigma_1}$ is the largest set $A \subset V_{\Sigma_1 \times \Sigma_2}$ such that $p_0(A) = B$.

THEOREM 3. *If $A \subseteq V_{\Sigma_1 \times \Sigma_2}$ is a s.f.a. definable set, then $p_0(A) \subseteq V_{\Sigma_1}$ is a s.f.a. definable set. If $B \subseteq V_{\Sigma_1}$ is s.f.a. definable, so is its Σ_2-cylindrification $A \subseteq V_{\Sigma_1 \times \Sigma_2}$.*

Proof. Let $\mathfrak{A} = \langle S, M, s_0, F\rangle$ be a $\Sigma_1 \times \Sigma_2$-automaton with $T(\mathfrak{A}) = A$. Define a Σ_1-automaton by $\mathfrak{A}_1 = \langle S, M_1, s_0, F\rangle$, where

$$M_1(s, \sigma_1) = \bigcup_{\sigma_2 \in \Sigma_2} M(s, (\sigma_1, \sigma_2)), \quad \sigma_1 \in \Sigma_1,\ s \in S.$$

One can check that $T(\mathfrak{A}_1) = p_0(A)$.

The proof concerning cylindrification is left to the reader.

3. A counterexample

We wish to show that the class of s.f.a. definable sets is not closed under complementation. We shall exhibit a s.f.a. definable set $B \subseteq V_\Sigma$ such that $A = V_\Sigma - B$ is not s.f.a. definable.

Let $\Sigma = \{0, 1\}$ and let B be the set of all Σ-trees (v, T_x) such that for some path $\pi \subset T_x$ we have: $1 \in \mathrm{In}(v \mid \pi)$. It is readily seen that B is s.f.a. definable.

The set $A = V_\Sigma - B$ consists of all Σ-trees (v, T_x) such that for every path $\pi \subset T_x$, $1 \notin \mathrm{In}(v \mid \pi)$. We claim that A is not s.f.a. definable. To prove this, let us assume that $\mathfrak{A} = \langle S, M, s_0, F\rangle$ is a special automaton such that $T(\mathfrak{A}) = A$ and derive a contradiction Throughout this section, unless otherwise specified, $\mathfrak{A}$ will denote this particular automaton. We shall need the following construction.

DEFINITION 7. Let $t = (v, T)$ and $t_1 = (v_1, T_x)$, $x \in T$, be Σ'-trees. The result of *grafting* the tree t_1 on t at $y \in T$ is the tree (v_2, T) such that $v_2(z) = v(z)$ for $x \notin T_y$, and $v_2(yz) = v_1(xz)$ for $z \in T$. (Note that $T_y = \{yz \mid z \in T\}$, and similarly for T_x.)

DEFINITION 8. Let $t_n = (v_n, T)$, $n < \omega$, be Σ'-trees. We shall say that $\lim_{n\to\infty} t_n = (v, T)$ if there exists an integral valued function $N(x)$, $x \in T$, such that $N(x) \leqq n$ implies $v_n(x) = v(x)$.

LEMMA 4. *Let $t = (v, T) \in T(\mathfrak{A})$ and let $r \in \mathrm{Rn}(\mathfrak{A}, t)$ be an accepting run If there exist nodes $x < z < y$ such that $r(x) = r(y) = s$, $s \in F$, and $v(z) = 1$, then there exists a tree $t' \notin A$ which is accepted by $\mathfrak{A}$.*

Proof. Assume that $y = xu$. Let $\Sigma' = S \times \Sigma$, and let (ϕ, T) be the Σ'-tree such that $\phi(z) = (r(z), v(z))$, $z \in T$. Graft $t_x = (\phi, T_x)$ on (ϕ, T) at the node $y = xu$ and call the resulting tree $t_1 = (\phi_1, T)$. Since $r(x) = r(y)$, we have that $p_0\phi_1$ is an $\mathfrak{A}$-run on $(p_1\phi_1, T)$.

Note that $p_0\phi_1(xu^2) = s$. Graft t_x on t_1 at xu^2 to obtain $t_2 = (\phi_2, T)$. Again $p_0\phi_2$ is an $\mathfrak{A}$-run on $(p_1\phi_2, T)$ and $p_0\phi_2(xu^3) = s$. Continue this process inductively for every $n < \omega$, where at the nth step we graft t_x on t_{n-1} at xu^n to obtain $t_n = (\phi_n, T)$. Let $\lim_{n\to\infty} t_n = \bar{t} = (\bar{\phi}, T)$, and $t' = (p_1\bar{\phi}, T)$. Since each $p_0\phi_n$ is an $\mathfrak{A}$-run on $(p_1\phi_n, T)$, $\bar{r} = p_0\bar{\phi}$ is an $\mathfrak{A}$-run on t'.

We claim that for every path $\pi \subset T$, $\mathrm{In}(\bar{r} \mid \pi) \cap F \neq \emptyset$ so that $t' \in T(\mathfrak{A})$. Namely, if $xu^n \in \pi$ for infinitely many (and hence all) $n < \omega$ then $\bar{r}(xu^n) = s$ and $s \in \mathrm{In}(\bar{r} \mid \pi) \cap F$. Otherwise, two cases may occur. Either π contains no xu^n, then $\bar{r} \mid \pi = r \mid \pi$ and $\mathrm{In}(\bar{r} \mid \pi) \cap F = \mathrm{In}(r \mid \pi) \cap F \neq \emptyset$. Or else there is an $n < \omega$ such that $xu^n \in \pi$, $xu^{n+1} \notin \pi$. In this case there exists a path $\pi' \subset T_x$ such that $\bar{r}(xu^n v) = r(xv)$ for all $xv \in \pi$. Thus $\mathrm{In}(\bar{r} \mid \pi) \cap F = \mathrm{In}(r \mid \pi') \cap F \neq \emptyset$.

Now, if $z = xw$ then $p_1\bar{\phi}(xu^n w) = 1$, $n < \omega$; hence $t' \notin A$.

Let $t = (v, T)$ be a $\{0,1\}$-tree. We define, by induction on n, subsets $D_n(t) \subseteq T$. $D_0(t) = \{x \mid v(x) = 1\}$; $D_{n+1}(t) = D_n(t) \cap \{x \mid c(D_n(t) \cap T_x) = \omega\}$.

LEMMA 5. *Let $\mathfrak{A} = \langle S, M, s_0, F\rangle$ be a special $\{0,1\}$-automaton; $t = (v, T) \in V_{\{0,1\}}$, $\Lambda \in D_{n+1}(t)$. If $r \in \mathrm{Rn}(\mathfrak{A}, t)$ is an accepting run such that $c(r(T - \{\Lambda\})) \leqq n$, then for some $x < z < y$, $r(x) = r(y) = s$, $s \in F$, and $r(z) = 1$.*

Proof. By induction on n. For every path π, let $x(\pi)$ be the first $\Lambda < x \in \pi$ such that $r(x) \in F$. The set $\{x(\pi) \mid \pi \subset T\}$ is a frontier and hence $E = \{y \mid y \leqq x(\pi),\ \textit{for some}\ \pi \subset T\}$, is finite. Since $c(D_n(t) \cap T) = \omega$, there must be a point $z \in D_n(t)$ such that $z \notin E$ and hence for some $x = x(\pi)$, $x < z$. Now $r(x) = s$, $s \in F$ and $v(z) = 1$. If for some $z < y$, $r(y) = s$.

then we are through. Otherwise $s \notin r(T_z - \{z\})$ and, since $s \in r(T - \{\Lambda\})$, $c(r(T_z - \{z\})) \leqq n - 1$. Since $z \in D_n(t)$, and $r \mid T_z$ is an accepting run of $\langle S, M, r(z), F \rangle$ on (v, T_z), the proof is completed by induction.

LEMMA 6. *For every* $1 \leqq n$ *there exists a tree* $t_n \in A$ *such that* $\Lambda \in D_{n-1}(t_n)$.

Proof. Let $t_1 = (v_1, T)$, $v_1(\Lambda) = 1$, $v_1(x) = 0$, $\Lambda < x$. The tree t_2 is obtained by grafting t_1 on t_1 at each of the nodes $1^k 0$, $1 \leqq k < \omega$. In general, t_n is obtained by grafting t_{n-1} on t_1 at each of the above mentioned nodes.

LEMMA 7. *The set* A *is not s.f.a.-definable.*

Proof. Assume $A = T(\mathfrak{A})$ for $\mathfrak{A} = \langle S, M, s_0, F \rangle$ where $c(S) = n$. Then t_{n+2} is accepted by $\mathfrak{A}$. By Lemma 5, there exist nodes $x < z < y$ satisfying the conditions of Lemma 4. Hence there exists a $t' \notin A$ which is accepted by $\mathfrak{A}$, a contradiction.

COROLLARY 8. *Not every set of valued trees definable by an automaton (in the sense of* [6]*) is definable by a special automaton. The set* A *is an example for this.*

4. A closure result

THEOREM 9. *Let* $\mathfrak{A} = \langle S, M, s_0, F \rangle$ *be a special* Σ*-automaton, and let* $\tau \in \Sigma$. *Define* D *to be the set of all* Σ*-trees* $t = (v, T)$ *such that* $t \in D$ *if and only if for every* $x \in T$, $v(x) = \tau$ *implies* $(v, T_x) \in T(\mathfrak{A})$. *The set* D *is s.f.a. definable.*

We need some definitions and lemmata to prove this result. The intuitive approach to the task of recognizing whether $t \in D$, is to start a copy of $\mathfrak{A}$ at each node x such that $v(x) = \tau$, and check whether (v, T_x) is accepted by $\mathfrak{A}$.

This process, however, cannot be accomplished by a finite automaton. The crucial observation is that for any $y \in T$, even though many copies of an $\mathfrak{A}$ may have been activated at various $x < y$, at y the number of *different* states of $\mathfrak{A}$ which appear is still bounded by the cardinality of the set S. Thus, all the copies of $\mathfrak{A}$ reaching y in the same state s can be replaced by just one of these copies. In this way, we have, at any node y, just a bounded number of copies of $\mathfrak{A}$, and this can be described by a finite Σ-table. In addition to having copies of $\mathfrak{A}$ move on (v, T), we will also need to record which copies merged when reaching the same state. The above considerations motivate the following formal definition of a Σ-table.

Let $u \in S^*$ be a finite sequence on S. For $i < m = l(u)$, put $u(i)' = u(i)$ if $u(i) \neq u(j)$ for $j < i$, $u(i)' = \Lambda$ otherwise. Define $C(u)$, *the contraction of* u, by $C(u) = u(0)'u(1)' \dots u(m-1)'$.

Define the *contracting* function ϕ_u: $l(u) \to l(C(u))$ by $\phi_u(i) = j$ if and only if $u(i) = C(u)(j)$. Clearly ϕ_u maps $l(u)$ onto $l(C(u))$. Furthermore, $\phi_u(i) \leqq i$ for $i < l(u)$; and if $\phi_u(i) < i$ then $\phi_u(j) < j$ for $i \leqq j < l(u)$.

A word $u \in S^*$ is called *contracted* if $i < j < l(u)$ implies $u(i) \neq u(j)$. The word u is contracted if and only if $C(u) = u$. The word $C(u)$ is contracted for all $u \in S^*$. If u is contracted then $l(u) \leqq c(S)$.

Let $\mathfrak{A} = \langle S, M, s_0, F\rangle$ be a special Σ-automaton. Assume $c(S) = n$. Without loss of generality, we assume that $\mathfrak{A}$ is such that $(s_1, s_2) \in M(s, \sigma)$ implies $s_1 \neq s_0$, $s_2 \neq s_0$ (see Lemma 1).

If $f: A \to B$ then $D(f)$ and $R(f)$ will denote, respectively, the domain A and range $f(A)$ of f.

Definition 9. For $\mathfrak{A}$ and $\tau \in \Sigma$ as above, define the Σ-table $\mathfrak{B} = \langle S^{\mathfrak{B}}, M^{\mathfrak{B}}\rangle$ as follows:

Set $U = \{u \mid u \in S^*,\ \ C(u) = u,\ \ u(i) = s_0 \ \ \textit{implies} \ \ i = l(u) - 1\}$; $H = \{\phi \mid \phi\colon m \to k,\ R(\phi) = k,\ k \leqq m < n\}$. Define $S^{\mathfrak{B}} = (U \times H) \cup \{d\}$, where d is a *dump state.*

Define $M^{\mathfrak{B}}$ by cases. $M^{\mathfrak{B}}(d, \sigma) = \{(d,d)\}$ for $\sigma \in \Sigma$. Let $(u, \phi) \in S^{\mathfrak{B}}$, $l(u) = k$, and $\sigma \in \Sigma$. $M^{\mathfrak{B}}(u, \phi), \sigma) = \{(d,d)\}$ if $u(k-1) = s_0$ and $\sigma \neq \tau$, or $u(k-1) \neq s_0$ and $\sigma = \tau$.

In all other cases, $((u_1, \phi_1),\ (u_2, \phi_2)) \in M^{\mathfrak{B}}((u, \phi), \sigma)$ if and only if for some $w_1, w_2 \in S^k$ we have $(w_1(i), w_2(i)) \in M(u(i), \sigma)$ for $i < k$; $u_j = C(w_j)a_i$ where $a_j \in \{\Lambda, s_0\}$, $j = 1,2$; $\phi_j = \phi_{w_j}$, $j = 1,2$.

Remark. It is readily seen that, with the above notations,

$$(u_1(\phi_1(i)),\ u_2(\phi_2(i))) \in M(u(i), \sigma),\ \ i < l(u).$$

The idea behind this definition is as follows. There are up to n copies of $\mathfrak{A}$ scanning each node x of T. If $(u, \phi) \in S^{\mathfrak{B}}$, is the state at $x \in T$, $l(u) = k$, then copies $0, 1, \ldots, k-1$ are active at x and in states $u(0), u(1), \ldots, u(k-1)$. If $v(x) = \sigma$ then, unless $u(k) = s_0$ if and only if $\sigma = \tau$, $\mathfrak{B}$ moves into a dump state d, i.e. $M^{\mathfrak{B}}((u, \phi), \sigma) = \{(d,d)\}$. Thus for $\mathfrak{B}$ to avoid the state d, a copy of $\mathfrak{A}$ must be activated in state s_0 precisely at the nodes with $v(x) = \tau$, this copy will always have the highest number. Recall that we assume $\mathfrak{A}$ to be such that s_0 can appear only once in an $\mathfrak{A}$-run. If d is avoided, then each active copy i, $i < k$, of $\mathfrak{A}$ independently moves into the state $w_1(i)$ at $x0$ and $w_2(i)$ at $x1$ where $(w_1(i), w_2(i)) \in M(u(i), \sigma)$.

Now let $w_1(i_1), \ldots, w_1(i_{p-1})$, $i_1 < i_2 < \ldots < i_{p-1}$ be all the pairwise different states in $w_1(0), \ldots, w_1(k-1)$, where we define $i_j = \min(i \mid w_1(i) = w_1(i_j))$. If (u_1, ϕ_1) is the state of $\mathfrak{B}$ at $x0$, then $l(u_1) = p$ or $l(u_1) = p + 1$ (in which case $u_1(p) = s_0$), and $u_1(j) = w_1(i_j)$, $j < p$. Also, $\phi_1 : k \to p$ and $\phi_1(i) = j$ if and only if $u_1(i) = w_1(i_j)$. Similar statements hold for the state $(u_2, \phi_2) \in S$ at $x1$.

We introduce a set $S_0^{\mathfrak{B}} = \{(\Lambda,\emptyset),(s_0,\emptyset)\}$ of initial states of $\mathfrak{B}$.

DEFINITION 10. Let $t = (v,T) \in V_\Sigma$, and $r \in \mathrm{Rn}(\mathfrak{B},t)$ be such that $r(\Lambda) \in S_0^{\mathfrak{B}}$ and $r(x) \neq d$, $x \in T$. Denote $r(x) = (u^x, \phi^x)$, $x \in T$. If $l(u_x) = k$, $m < k$, then we shall say that m is *active at* x; if $u^x(k-1) = s_0$ then $k-1$ is *activated at* x. (Note that $k-1$ is activated at x if and only if $v(x) = \tau$.)

Let m be active at x and $y = x0$ or $y = x1$. We say that *m at x is replaced by m_1 at y* ($(m,x) \to (m_1,y)$) if $m_1 = \phi^y(m)$. The notion of replacement is extended by passing to the transitive closure. Thus assume $x < y$, $x = x_0$, $x_{i+1} = x_i\varepsilon_i$, $\varepsilon_i \in \{0,1\}$, $0 \leqq i \leqq k-1$, and $y = x_k$. We shall say that *m at x is replaced by m' at y* ($(mx) \to (m',y)$), if for a sequence $m_i \in n$, $0 \leqq i \leqq k$, $m_0 = m$, $m_k = m'$, and $(m_i,x_i) \to (m_{i+1},x_{i+1})$, $0 \leqq i \leqq k-1$.

Henceforth, to the end of this section, t, r, and (u^x,ϕ^x), retain their above meanings.

LEMMA 10. *Let k be activated at x. Define $r_x\colon T_x \to S$ by* $r_x(x) = s_0 = u^x(k)$, $r_x(y) = u^v(m)$ if $x < y$ and $(k,x) \to (m,y)$. For all x such that $v(x) = \tau$, $r_x \in \mathrm{Rn}(\mathfrak{A},(v,T_x))$. The proof is obvious.

Let $\pi \subset T$ be a path, and let $x \in \pi$. We say that m is *stabilized at x* (along π) if m is active at x and $(m,x) \to (m,y)$ for all $x < y \in \pi$.

LEMMA 11. *Let k be active at $z \in \pi$, there exists a $z \leqq x \in \pi$ and an m such that $(k,z) \to (m,x)$ and m is stabilized at x along π. If m is stabilized at $x \in \pi$ along π and $m_1 \leqq m$ then m_1 is stabilized at x along π. For every path $\pi \subset T$ for which $r(\pi) \neq \{(\Lambda,\emptyset)\}$ (i.e. $\tau \in v(\pi)$) there exists a $x_0 \in \pi$ and an m_0 such that m_0 is stabilized at x_0 and every $m > m_0$ is not stabilized at any $x \in \pi$.*

Proof. The statements follow at once from the properties of the contracting function mentioned just before Definition 9.

LEMMA 12. *$t \in D$ if and only is there exists a run $r \in \mathrm{Rn}(\mathfrak{B},t)$ such that $r(\Lambda) \in S_0^{\mathfrak{B}}$, $d \notin r(T)$, and for every $\pi \subset T$ if m is stabilized at $z \in \pi$ along π, then $c(\{y \mid z \leqq y \in \pi,\ u^y(m) \in F\}) = \omega$.*

Proof. Assume the existence of such a run r. Since $d \notin r(T)$, a $k < n$ is activated at precisely thoseɪx for which $v(x) = \tau$. Now $r_x \in \mathrm{Rn}(\mathfrak{A},(v,T_x))$, by Lemma 10, and $r_x(x) = s_0$. Let $\pi \subset T_x$ be a path. There exists a $x < z \in \pi$, a $m \leqq k$ such that $(k,x) \to (m,z)$ and m is stabilized at z along π. For $z \leqq y \in \pi$ we have $r_x(y) = u^y(m)$. Hence $\mathrm{In}(r_x \mid \pi) \cap F \neq \emptyset$. This implies $(v,T_x) \in T(\mathfrak{A})$. Thus $t \in D$.

Assume now $t \in D$. For every x such that $v(x) = \tau$ we have $(v,T_x) \in T(\mathfrak{A})$. Let $\bar{r}_x \in \mathrm{Rn}\,\mathfrak{A},(v,T_x))$ be an accepting run.

Let $r: T \to S^{\mathfrak{B}}$ be a $\mathfrak{B}$-run on t. For m active at $x \in T$, define $\rho(m,x)$-the *origin* of m at x, by: $\rho(m,x) = x$ if m is activated at x; and $\rho(m,x) = y$ if $y < x$ and for some k, $(k,y) \to (m,x)$, and no $z < y$ has this property. It is readily verified that $v(\rho(m,x)) = \tau$ holds for every $x \in T$ and m active at x. Note that $\rho(m,x)$ is completely determined by $r \big| \{y \,|\, y \leqq x\}$.

We shall define by induction on $l(x)$, $x \in T$, a run $r \in \mathrm{Rn}(\mathfrak{B},t)$. Put $r(\Lambda) = (s_0, \emptyset)$ if $v(\Lambda) = \tau$; $r(\Lambda) = (\Lambda, \emptyset)$ otherwise. Assume that $r(x) = (u^x, \phi^x)$ has been defined for all x such that $l(x) \leqq h$ and that $u^x(m) = \bar{r}_{\rho(m,x)}(x)$, $m < l(u^x)$, for all these x. Let $l(x) = h$, $l(u^x) = k$. Put $w_{a+1}(m) = \bar{r}_{\rho(m,x)}(xa)$, $m < k$, $a = 0,1$. Since $\bar{r}_{\rho(m,x)}$ is an $\mathfrak{A}$-run on $(v, T_{\rho(m,x)})$, and $u^x(m) = \bar{r}_{\rho(m,x)}(x)$, it follows that $(w_1(m), w_2(m)) \in M(u^x(m), v(x))$.

Let $y = x0$ (the case $y = x1$ is treated in the same way). Define $u^y = C(w_1(0) \dots w_1(k-1))$ if $v(y) \neq \tau$, and let ϕ^y be the corresponding contracting function. Define $u^y = C(w_1(0) \dots w_1(k-1))s_0$ if $v(y) = \tau$. Obviously $((u^{x0}, \phi^{x0}), (u^{x1}, \phi^{x1})) \in M^{\mathfrak{B}}((u^x, \phi^x), v(x))$. If $m = \min(m' \,|\, \phi^y(m') = i)$ then $\rho(i,y) = \rho(m,x)$. Thus $u^y(i) = w_1(m) = \bar{r}_{\rho(m,x)}(y) = \bar{r}_{\rho(i,y)}(y)$. Thus r was extended to a run $r: \{x \,|\, l(x) \leqq h + 1\} \to S^{\mathfrak{B}}$ with the same properties.

In this way we obtain a run $r \in \mathrm{Rn}(\mathfrak{B},t)$ such that $d \notin r(T)$ and $u^y(m) = \bar{r}_{\rho(m,y)}(y)$ for $y \in T$, and m active at y. If $\pi \subset T$ and m is stabilized at $z \in \pi$, then $\rho(m,z) = x \in \pi$ and, for $z \leqq y$, $\rho(m,z) = \rho(m,y) = x$. Hence $u^y(m) = \bar{r}_x(y)$ for $z \leqq y$. Since $\mathrm{In}(\bar{r}_x \,|\, \pi) \cap F \neq \emptyset$, also $c(\{y \,|\, u^y(m) \in F,\ z \leqq y \in \pi\}) = \omega$.

The proof of Theorem 9 will result from the following.

LEMMA 13. *Let R be the set of $S^{\mathfrak{B}}$-valued trees (r,T) such that* a) $\exists t[r \in \mathrm{Rn}(\mathfrak{B}, t) \wedge t \in V_\Sigma]$; b) $d \notin r(T)$, $r(\Lambda) \in S_0^{\mathfrak{B}}$; c) *for every path $\pi \subset T$, $x \in \pi$ and m which is stabilized at x, $c(\{y \,|\, x < y \in \pi, u^y(m) \in F\}) = \omega$. The set R is s.f.a. definable.*

Proof. The sets R_a and R_b of $S^{\mathfrak{B}}$-trees satisfying conditions a) and b) are certainly definable by s.f.a. Thus it will be enough to construct an automaton $\mathfrak{C}$ accepting a $(r,T) \in R_a \cap R_b$ if and only if it satisfies c). The set of states of $\mathfrak{C}$ will be $n + 1 = \{0, 1, \dots, n\}$. Define $\bar{M}(n,s) = (0,0)$, $s \in S^{\mathfrak{B}}$; $\bar{M}(m,(u,\phi)) = (m+1, m+1)$, for $m < n$, $(u,\phi) \in S^{\mathfrak{B}}$, if $l(u) \leqq m$ or $\phi(m) < m$ or $u(m) \in F$; $\bar{M}(m,u,\phi)) = m$ otherwise. Put $\mathfrak{C} = \langle n+1, \bar{M}, 1, \{n\} \rangle$. If $h: T \to n + 1$, $h(\Lambda) = 0$ is the $\mathfrak{C}$-run on a tree $(r,T) \in R_a \cap R_b$ and $\pi \subset T$ then, by use of Lemma 11, we see that $n \notin \mathrm{In}(h \,|\, \pi)$ if and only if for some $x \in \pi$ and m stabilized at x, $c(\{y \,|\, x < y \in \pi,\ u^y(m) \in F\}) < \omega$. Thus $R = R_a \cap R_b \cap T(\mathfrak{C})$.

Proof of Theorem 9. By Lemma 12, $t = (v,T) \in D$ if and only if there exists a run $r \in \mathrm{Rn}(\mathfrak{B},t)$ such that $(r,T) \in R$. Consider the alphabet

$\bar{\Sigma} = S^{\mathfrak{B}} \times \Sigma$. The set A of $\bar{\Sigma}$-trees $(\bar{v}, T)$ such that $p_0\bar{v} \in \mathrm{Rn}(\mathfrak{B},(p_1\bar{v},T))$, is s.f.a. definable. The set $B = \{(\bar{v},T) \mid (p_0\bar{v}\,T) \in R\}$ is s.f.a. definable by Lemma 13 and Theorem 3. Now $D = p_1(A \cap B)$.

5. Automata on finite trees

We shall need some notions concerning finite Σ-trees and the definition of sets of such trees by automata. A *finite* Σ-tree is a pair (v,E) where $E \subset T$ is a finite frontiered tree and $v: E - Ft(E) \to \Sigma$. An automaton on finite Σ-trees is a system $\mathfrak{A} = \langle S, M, s_0, f\rangle$ where $M:(S - \{f\}) \times \Sigma \to P(S \times S)$, and $s_0 \in S$. The notion of an $\mathfrak{A}$-run on a finite Σ-tree is defined in the obvious way. An automaton $\mathfrak{A} = \langle S, M, s_0, f\rangle$ *accepts* $e = (v,E)$ if there exist an accepting $\mathfrak{A}$-run r on e such that $r(\Lambda) = s_0$ and $r(Ft(E)) = \{f\}$. The set of all the e accepted by $\mathfrak{A}$ is denoted by $T_f(\mathfrak{A})$. In [1, 5, 7] the notion of automata on finite trees is defined in a different way. It is, however, not hard to show the equivalence of the various definitions.

THEOREM 14. *Let $\mathfrak{A} = \langle S, M, s_0, f\rangle$ be a Σ-automaton. Define a set $B \subseteq V_\Sigma$ by: $(v,T) \in B$ if and only if there exists a sequence $(E_n)_{n<\omega}$ of frontiers of T such that $E_n < E_{n+1}$ and $(v, G_n) \in T_f(\mathfrak{A})$, $n < \omega$, where $G_n = \{x \mid x \leqq y$ for some $y \in E_n\}$. The set B is s.f.a. definable.*

Proof. Define a s.f.a. $\mathfrak{B} = \langle \bar{S}, \bar{M}, \bar{s}_0, \bar{F}\rangle$ as follows. Put $\bar{S} = S \times S$; $\bar{s}_0 = (s_0, f)$; $\bar{F} = S \times \{f\}$. The table $\bar{M}$ is defined by $((s_1,s_1'),(s_2,s_2')) \in \bar{M}((s,s'),\sigma)$ if and only if $(s_1,s_2) \in M(s,\sigma)$, $(s_1',s_2') \in M(s',\sigma)$ for $s' \neq f$, and $(s_1',s_2') \in M(s,\sigma)$ for $s' = f$. We claim $T(\mathfrak{B}) = B$.

Let $t = (v,T) \in T(\mathfrak{B})$ and let $\bar{r}: T \to \bar{S}$ be an accepting $\mathfrak{B}$-run. There exists a sequence $(E_n)_{n<\omega}$ of frontiers of T such that $\bar{r}(E_n) \subseteq \bar{F}$, $E_n < E_{n+1}$, $n < \omega$. Let $G_n = \{x \mid x \leqq y$ *for some* $y \in E_n\}$, $n < \omega$. Consider the mapping $r': G_{n+1} \to S$ defined by: $r'(x) = p_0(\bar{r}(x))$ for $x \in G_n$, and $r'(x) = p_1(\bar{r}(x))$ for $x \in G_{n+1} - G_n$. Clearly r' is an $\mathfrak{A}$-run on (v, G_{n+1}) and $r'(E_{n+1}) = \{f\}$. Thus $(v, G_{n+1}) \in T_f(\mathfrak{A})$, $n < \omega$. Hence $(v,T) \in B$.

Assume $(v,T) \in B$. Let $(E_n)_{n<\omega}$ and $(v,G_n)_{n<\omega}$ be as in the statement of the theorem. Let $r_n: G_n \to S$, $n < \omega$, be an $\mathfrak{A}$-run on (v, G_n) so that $r_n(E_n) = \{f\}$. Define $T_n = \{x \mid l(x) = n\}$. By König's Lemma, there exists a sequence $(n(i))_{i<\omega}$ of integers such that $n(i) < n(i+1)$, $i < \omega$, and $r_{n(i)} \mid T_i = r_{n(j)} \mid T_i$ for $i \leqq j < \omega$. Denote $r_i' = r_{n(i)}, E_i' = E_{n(i)}, G_i' = G_{n(i)}$, $i < \omega$. Let $r: T \to S$ be the limiting mapping of the sequence $(r_i')_{i<\omega}$, i.e. $r(x) = r_i'(x)$ for x such that $l(x) \leqq i$. r is an $\mathfrak{A}$-run satisfying $r(\Lambda) = s_0$.

We shall define inductively a sequence $m(k)$, $k < \omega$, of integers and $\mathfrak{B}$-runs $\bar{r}_k$: $G'_{m(k)} \to \bar{S}$. Define $m(0) = 0$, $p_0\bar{r}_0 = r \mid G_0'$, $p_1\bar{r}_0(x) = r_0'(x)$

for $\Lambda < x \in G_0'$, and $\bar{r}_0(\Lambda) = (s_0, f) = \bar{s}_0$. Assume $(m(i))_{i \leq k}$ and $(\bar{r}_i)_{i \leq k}$ to be defined. Let $j = \max(l(x) \mid x \in E'_{m(k)})$. Then $E'_{m(k)} < E'_j$ and $r'_j \mid G'_{m(k)} = r \mid G'_{m(k)}$. Define $m(k+1) = j$; $p_0 \bar{r}_{k+1} = r \mid G'_j$; $p_1 \bar{r}_{k+1}(x) = r'_j(x)$ for $x \in G'_j - G'_{m(k)}$. The sequence $(\bar{r}_k)_{k<\omega}$ defined in this way has the properties that $\bar{r}_{k+1} \mid G'_{m(k)} = \bar{r}_k$, and $p_1 \bar{r}_k(E_{m(k)}) = \{f\}$, $k < \omega$. The run $\bar{r} = \lim_{k \to \infty} \bar{r}_k$ is an accepting $\mathfrak{B}$-run of (v, T), hence $(v, T) \in T(\mathfrak{B})$. Thus $B = T(\mathfrak{B})$.

6. Weak definability and s.f.a.

In this section let $\Sigma = \{0, 1\}$. χ_A will denote the characteristic function of A. Let $A = (A_1, \ldots, A_n)$ be an n-tuple of subsets of T. With A we associate the Σ^n-tree $\tau(A) = (v_A, T)$ where $v_A(x) = (\chi_{A_1}(x), \ldots, \chi_{A_n}(x))$, $x \in T$. The mapping $\tau: P(T)^n \to V_{\Sigma^n, \Lambda}$ is clearly a one-to-one correspondence.

Definition 11. Let $R \subseteq P(T)^n$. We say that the Σ^n-automaton $\mathfrak{A}$ *represents* R if

$$\tau(R) = T(\mathfrak{A}) \cap V_{\Sigma_n, \Lambda}. \tag{2}$$

Lemma 15. *Let $K_1 \subseteq P(T)^n$ be the set of all $A = (A_1, \ldots, A_n)$ such that $c(A_1) = 1$; and let $K_2 \subseteq P(T)^n$ be the set of all A such that $c(A_1) < \omega$. The sets K_1 and K_2 are representable by special automata.*

Proof. Since $\Sigma = \Sigma \times \Sigma^{n-1}$ it suffices, by Theorem 3, to consider the case $n = 1$.

Let $\mathfrak{A}_1 = \langle \{s_0, s_1, s_2\}, M_1, s_0, \{s_1\} \rangle$, where $M_1(s_0, 0) = \{(s_0, s_1), (s_1, s_0)\}$, $M_1(s_0, 1) = M_1(s_1, 0) = \{(s_1, s_1)\}$, and $M_1(s_1, 1) = M_1(s_2, 0) = M_1(s_1, 1) = \{(s_2, s_2)\}$.

Let $\mathfrak{A}_2 = \langle \{s_0, s_1, s_2\}, M_2, s_0, \{s_1\} \rangle$, where $M_2(s_0, 1) = M_2(s_0, 1) = \{(s_0, s_0), (s_1, s_1)\}$, $M_2(s_1, 0) = \{(s_1, s_1)\}$, $M_2(s_1, 1) = M_2(s_2, 0) = M_2(s_2, 1) = \{(s_2, s_2)\}$.

It can easily be verified that K_i is represented by $\mathfrak{A}_i$, $i = 1, 2$.

Definition 12. Let $A = (x_1, \ldots, x_m, \alpha_{m+1}, \ldots, \alpha_{m+n}, A_{m+n+1}, \ldots, A_{m+n+k})$, where $x_i \in T$ $1 \leqq i \leqq m$, $\alpha_i \subseteq T$ is finite, $m + 1 \leqq i \leqq m + n$, and $A_i \subseteq T$ $m + n + 1 \leqq i \leqq m + n + k$. We shall say that A is of *type* (m, n, k). Note that some of the A_i may also be finite but that we disregard this in defining the type. We shall represent A by $\sigma(A) = (A_i)_{1 \leqq i \leqq m+n+k}$ where $A_i = \{x_i\}$, $1 \leqq i \leqq m$, $A_i = \alpha_i$, $m + 1 \leqq i \leqq m + n$, and the last k terms of $\sigma(A)$ coincides with those of A.

Corollary 16. *Let $K(m, n, k) = \{\sigma(A) \mid A$ is of type $(m, n, k)\}$. The set $K(m, n, k)$ is representable by a special automaton.*

Proof. This follows from Lemma 15 and from the closure of s.f.a. definable sets under intersections (Theorem 2).

THEOREM 17. *Let $R \subseteq P(T)^n$ and $Q \subseteq P(T)^n$ be, respectively, represented by the special automata $\mathfrak{A}$ and $\mathfrak{B}$. The following sets are representable by special automata.*

(a) $R \cup Q$;

(b) $R \cap Q$;

(c) $R_1 = \{(A_1, \dots, A_{n-1}) \mid \exists A_n[(A_1, \dots, A_{n-1}) \in R]\}$;

(d) $R_2 = \{(A_1, \dots, A_{n-1}) \mid \forall \alpha_n[(A_1, \dots, A_{n-1}, \alpha_n) \in R]\}$,

here α_n ranges over all finite subsets of T.

Proof. Since $\tau(R \cup Q) = \tau(R) \cup \tau(Q)$, $\tau(R \cap Q) = \tau(R) \cap \tau(Q)$, (a) and (b) follow from Theorem 2.

Let p be the mapping $(x_1, \dots, x_{n-1}, x_n) \to (x_1, \dots, x_{n-1})$. We have $R_1 = pR$ Now $\Sigma^n = \Sigma^{n-1} \times \Sigma$ and p, which may also be considered as inducing the projection $p_0 : \Sigma^{n-1} \times \Sigma \to \Sigma^{n-1}$, commutes with τ. I.e. for $A \in P(T)^n$, $\tau(p(A)) = p(\tau(A))$. Thus

$$\tau(R_1) = \tau(p(R)) = p(\tau(R)) = p(T(\mathfrak{A}) \cap V_{\Sigma_n, \Lambda});$$

and R_1 is representable by Theorem 3.

Let (v, T) be a Σ^{n-1}-tree and $\chi : T \to \{0,1\}$. By $(v \times \chi, T)$ we shall denote the $\Sigma^n = \Sigma^{n-1} \times \Sigma$-tree such that $(v \times \chi)(x) = v(x)\chi(x)$, $x \in T$. A similar notation will be used for finite trees (v, E).

To prove (d), let us look at $\bar{\Sigma} = \Sigma^{n-1} \times (P(S) - \{\emptyset\})$-trees, where S is the set of states of $\mathfrak{A} = \langle S, M, s_0, F \rangle$. Let η be the mapping $\eta : T \to \{0\}$. Define $\tilde{A} \subseteq V_{\bar{\Sigma}}$ to be the set of trees $(\bar{v}, T_x)$ so that for each $s \in p_1 \bar{v}(x)$, $((p_0 \bar{v}) \times \eta, T_x)$ is accepted by $\langle S, M, s, F \rangle$. The set A is obviously s.f.a. definable.

Define now an invariant set $A \subseteq V_{\bar{\Sigma}}$ by $(\bar{v}, T) \in A$ if and only if $(\bar{v}, T_x) \in \tilde{A}$ for *all* $x \in T$. It follows at once from Theorem 9 that A is s.f.a. definable.

Consider the following set P of finite $\bar{\Sigma}$-trees. $(\bar{v}, G) \in P$ if and only if there exists a frontiered tree $H \subset G$ such that 1) for every $y \in Ft(G)$ there exists an $x \in Ft(H)$ such that $y = x0$ or $y = x1$; 2) for every $\chi : H \to \{0,1\}$ such that $\chi(Ft(H)) = \{0\}$, there exists an $\mathfrak{A}$-run $r : H \to S$ on $((p_0 \bar{v}) \times \chi, H)$ such that $r(\Lambda) = s_0$ and $r(x) \in p_1 \bar{v}(x)$ for all $x \in Ft(H)$. The above conditions for $(\bar{v}, G) \in P$ can be expressed in the weak second-order theory of two successor functions. Hence $P = T_f(\mathfrak{C})$ for some finite automaton $\mathfrak{C}$ (see [1,7]).

Thus, by Theorem 14, the invariant set $B \subset V_{\bar{\Sigma}}$ defined by $(\bar{v}, T) \in B$ if and only if there exists a sequence $(G_n)_{n<\omega}$ of finite subtrees of T such that $Ft(G_n) < Ft(G_{n+1})$ and $(\bar{v}, G_n) \in P$, $n < \omega$, is definable by a special automaton.

If we shall show that $p_0(A \cap B) \cap V_{\Sigma^{n-1}, \Lambda} = \tau(R_2)$, then the proof of (d) will be completed by Theorem 3.

Let $t = (v, T)$ be a Σ^{n-1}-tree such that $t = \tau((A_1, \ldots, A_{n-1}))$. Assume $t \in p_0(A \cap B)$. Then there exists a $\bar{\Sigma} = \Sigma^{n-1} \times (P(S) - \emptyset)$-tree $\bar{t} = (\bar{v}, T)$ such that $p_0 \bar{v} = v$, and $\bar{t} \in A \cap B$. Let $\alpha \subset T$ be a finite set. $\tau((A_1, \ldots, A_{n-1}, \alpha)) = (v \times \chi_\alpha, T)$. Now $(\bar{v}, T) \in B$, hence there exist finite trees $H \subset G$ such that conditions 1–2 of the definition of P are satisfied, and for $x \in (T - H) \cup Ft(H)$, $\chi_\alpha(x) = 0$. Namely, with the notations of the previous paragraph, $G = G_n$ for some large enough $n < \omega$. Thus there exists a $r \in \mathrm{Rn}(\mathfrak{A}, (v \times \chi_\alpha, H))$ such that $r(\Lambda) = s_0$ and $r(x) \in p_1 \bar{v}(x)$ for $x \in Ft(H)$. Since $(\bar{v}, T) \in A$, and $\chi_\alpha(y) = 0$ for $y \in T_x$ and $x \in Ft(H)$, $r(x) \in p_1 \bar{v}(x)$ implies that $(v \times \chi_\alpha, T_x)$ is accepted by $\langle S, M, r(x), F \rangle$ for $x \in Ft(H)$; let $r_x \colon T_x \to S$ be an accepting run. Define $r' \colon T \to S$ by $r'(y) = r(y)$ for $y \in H$, $r'(y) = r_x(y)$ for $y \in T_x$, $x \in Ft(H)$. Since $r_x(x) = r(x)$ for $x \in Ft(H)$, the mapping r' is well-defined. It is readily seen that r' is an accepting run of $\mathfrak{A}$ on $(v \times \chi_A, T)$. Hence $\tau((A_1, \ldots, A_{n-1}, \alpha)) \in T(\mathfrak{A})$ for every finite α. Thus $(A_1, \ldots, A_{n-1}) \in R_2$.

Assume now that $t = (v, T) = \tau((A_1, \ldots, A_{n-1}))$, where $(A_1, \ldots, A_{n-1}) \in R_2$. Let $\eta \colon T \to \{0, 1\}$, $\eta(T) = \{0\}$. For every $x \in T$ define $s(x) = \{s \mid s \in S,\ (v \times \eta, T_x) \in T(\langle S, M, s, F \rangle)\}$. Since $\eta = \chi_\emptyset$ and $(A_1, \ldots, A_{n-1}, \emptyset) \in R$, it follows that $s(x) \neq \emptyset$ for $x \in T$. Let $\bar{t} = (\bar{v}, T)$ be the $\bar{\Sigma}$-tree such that $p_0 \bar{v} = v$ and $p_1 \bar{v}(x) = s(x)$ for all $x \in T$. We want to show that $\bar{t} \in A \cap B$ which will entail $t \in p_0(A \cap B)$ and hence $\tau(R_2) = p_0(A \cap B) \cap V_{\Sigma^{n-1}, \Lambda}$.

That $\bar{t} \in A$ follows at once from the definition of $s(x)$ and $\bar{v}$. Now let $H \subset T$ be any finite tree and $G = H \cup \{xa \mid x \in Ft(H),\ a = 0, 1\}$. Let $\chi \colon H \to \{0, 1\}$ be such that $\chi(Ft(H)) = \{0\}$, and let $\alpha \subset T$ be the finite set such that $\chi_\alpha \mid H = \chi$, $\chi_\alpha(y) = 0$ for $y \notin H$. Since $(A_1, \ldots, A_{n-1}, \alpha) \in R$, it follows that $(v \times \chi_\alpha, T) \in T(\mathfrak{A})$. Let $r \in \mathrm{Rn}(\mathfrak{A}, (v \times \chi_\alpha, T))$ be an accepting run. Then $r \in \mathrm{Rn}(\mathfrak{A}, (v \times \chi, H))$, $r(\Lambda) = s_0$. Now $(v \times \chi_\alpha, T_x)$ is accepted by $\langle S, M, r(x), F \rangle$, $x \in Ft(H)$. Since $\chi_\alpha(y) = 0$ for $y \in T_x$, $x \in Ft(H)$, we have $r(x) \in s(x)$. Thus $(\bar{v}, G) \in P$. Chose now $G_n = \{x \mid l(x) \leqq n + 1\}$, $n < \omega$, then $(\bar{v}, G_n) \in P$. Hence $\bar{t} \in B$.

LEMMA 18. *Let $w \in \{0, 1\}^*$ and $P \subseteq P(T)^n$ be the set of n-tuples $(A_1, \ldots, A_n)$ such that $A_i = \{x\}$ for some $x \in T$ and $xw \in A_j$. Let Q be similarly defined with $xw \notin A_j$. The relations P and Q are representable by special automata.*

Proof. Because of Theorem 3 it suffices to consider the case $n = 2$, $i = 1$, $j = 2$. We shall construct a Σ^2-automaton. Let $w = \varepsilon_1 \ldots \varepsilon_k$. Put $S = \{s_0, s_1, \ldots, s_k, d\}$, (if $w = \Lambda$ then $S = \{s_0, d\}$).

Let $\delta = 0, 1$, $\sigma \in \Sigma^2$. Define $M(s_0, (0, \delta)) = \{(s_0, s_0)\}$, $M(s_0, (1, \delta)) = \{(s_1, s_0)\}$ if $\varepsilon_1 = 0$, and $M(s_0, (1\ \delta)) = \{(s_0, s_1)\}$ if $\varepsilon_1 = 1$. For $1 \leqq i < k$, define $M(s_i, \sigma) = \{(s_{i+1}, s_0)\}$ if $\varepsilon_\iota = 0$, $M(s_i, \sigma) = \{(s_0, s_{i+1})\}$ if $\varepsilon_{i+1} = 1$.

Finally, $M(s_k,(\delta,1)) = \{(s_0,s_0)\}$, $M(s_k,(\delta,0)) = M(d,\sigma) = \{(d,d)\}$. It is easily verified that $V_{\Sigma^2,\Lambda} \cap T(\langle S, M, s_0, \{s_0\}\rangle) \cap K(1,0,1) = \tau(P)$. It follows from Corollary 16 that P is representable by a s.f.a.

The proof for Q is as above with an appropriate change in the definition of $M(s_k,\sigma)$.

THEOREM 19. *If* $F(\boldsymbol{x},\boldsymbol{\alpha},A)$, *where* $\boldsymbol{x} = (x_1,\ldots,x_n)$, $\boldsymbol{\alpha} = (\alpha_{m+1},\ldots,\alpha_{m+n})$, $A = (A_{m+n+1},\ldots,A_{m+n+k})$, *is a formula of S2S involving quantifiers over individual variables and finite-set variables only, and*

$$P = \{(x,\alpha,A)\,|\,\mathcal{N}_2 \models F(x,\alpha,A)\},$$

then $\sigma(P) = R(F)$ *(see Definition 12) is representable by a s.f.a.*

Proof. In F replace all occurrences of $t_1 = t_2$, where t_1,t_2 are terms, by $\forall\boldsymbol{\alpha}[t_1 \in \boldsymbol{\alpha} \rightarrow t_2 \in \boldsymbol{\alpha}]$. Replace all occurrences of $u = v$, where u and v are finite or arbitrary set variables, by $\forall\boldsymbol{x}[\boldsymbol{x} \in u \leftrightarrow \boldsymbol{x} \in v]$. Call a Boolean (propositional) combination of formulas *positive* if it involves only the use of $\vee$ and $\wedge$. Transform F into an equivalent formula by transporting the negation sign inside next to the atomic formulas.(1)

This is done by use of the rules $\sim [F \vee G] \equiv [\sim F \wedge \sim G]$, $\sim [F \wedge G] \equiv [\sim F \vee \sim G]$, $\sim \forall vF \equiv \exists v \sim F$, $\sim \exists vF \equiv \forall v \sim F$, where v is an individual or finite-set variable. Thus we may assume that F is obtained from atomic formulas $t \in u$ and $\sim t \in u$, where t is a term and u a set variable, by forming positive boolean combinations and using quantifiers over individual and finite-set variables.

The proof is by induction on the structure of F. If F is in one of the forms $x_i w \in A_j$, $x_i w \in \alpha_j$, where $w \in T$, or a negation of such a formula, then $R(F)$ is representable by Lemma 18, Corollary 16, and Theorem 3.

If $R(F_1)$ and $R(F_2)$ are representable then, since $R(F_1 \vee F_2) = R(F_1) \cup R(F_2)$ and $R(F_1 \wedge F_2) = R(F_1) \cap R(F_2)$, so are $R(F_1 \vee F_2)$ and $R(F_1 \wedge F_2)$ by Theorem 17(a), (b).

Let $F(\boldsymbol{x},\boldsymbol{\alpha},A)$ be a formula of the form in the statement of the theorem, then $R(F)$ is of type (m,n,k). Assume that $F(F)$ is representable and let v_i be one of the individual or finite-set variables of F; i.e. $i \leqq m + n$. We have

$$(3)\quad R(\exists v_i F) = \{(A_1,\ldots,A_{i-1},A_{i+1},\ldots,A_{m+n+k})\,|\,\exists A_i[(A_1,\ldots,A_i,\ldots,A_{n+m+k}) \in R(F)]\}$$

$$(4)\quad R(\forall v_i F) = \{(A_1,\ldots,A_{i-1},A_{i+1},\ldots,A_{m+n+k})\,|\, \forall\alpha_i[(A_1,\ldots,A_{i-1},\alpha_i,A_{i+1},\ldots,A_{m+n+k}) \in R(F)\},$$

(1) The idea of pushing negation signs next to the atomic formulas, was suggested to me by M. Magidor.

here $\alpha \subset T$ ranges over finite sets. These relations (3) and (4) hold because $R(F) \subseteq K(m,n,k)$, so that in (3) if $i \leqq m$, A_i will automatically range over singletons and if $m < i \leqq m+n$ then A_i will range over finite sets; in (4) for $i \leqq m+n$, any A_i for which $(A_1, \cdots, A_i, \cdots, A_{m+n+k}) \in R(F)$ is finite so that quantifying over just finite sets is no restriction. Now $R(\exists v_i F)$ and $R(\forall v F)$ are representable by Theorem 17(c), (d).

Thus our proof is completed by induction on formulas.

LEMMA 20. *The partial order $x \leqq y$ on T, and the lexicographic order $x \preccurlyeq y$ defined by $x \preccurlyeq y \equiv x \leqq y \vee \exists z[z0 \leqq x \wedge z1 \leqq y]$, are weakly definable.*

Proof. The formula

$$\forall \boldsymbol{\alpha}[\forall \boldsymbol{u} \forall \boldsymbol{v}[\boldsymbol{u} \in \boldsymbol{\alpha} \wedge [\boldsymbol{v}0 = \boldsymbol{u} \vee \boldsymbol{v}1 = \boldsymbol{u}] \rightarrow \boldsymbol{v} \in \boldsymbol{\alpha}] \wedge \boldsymbol{y} \in \boldsymbol{\alpha} \rightarrow \boldsymbol{x} \in \boldsymbol{\alpha}],$$

defines $x \leqq y$. That $x \preccurlyeq y$ is definable, follows from the weak definability of $x \leqq y$.

LEMMA 21. *The relations $\alpha \subset T_x$ is a frontier of T_x; $\beta \subset T$ is a finite frontiered tree; and $\alpha = Ft(\beta)$; are weakly definable.*

Proof. The formula

$$\forall \boldsymbol{y}[\boldsymbol{y} \in \boldsymbol{\alpha} \rightarrow \boldsymbol{x} \leqq \boldsymbol{y}] \wedge \forall \boldsymbol{y} \forall \boldsymbol{z}[\boldsymbol{y} \in \boldsymbol{\alpha} \wedge \boldsymbol{z} \in \boldsymbol{\alpha} \wedge \boldsymbol{y} \leqq \boldsymbol{z} \rightarrow \boldsymbol{y} = \boldsymbol{z}] \wedge$$
$$\forall \boldsymbol{y} \exists \boldsymbol{z}[\boldsymbol{x} \leqq \boldsymbol{y} \rightarrow \boldsymbol{z} \in \boldsymbol{\alpha} \wedge [\boldsymbol{y} \leqq \boldsymbol{z} \wedge \boldsymbol{z} \leqq \boldsymbol{y}]],$$

defines the relation: $\alpha \subset T_x$ and α a frontier of T_x. The other statements are immediate.

THEOREM 22. *A relation $R \subseteq P(T)^n$ is representable by a s.f.a. if and only if for some m there exists a weakly definable relation $P \subseteq P(T)^{n+m}$ such that*

$$R = \{(A_1, \ldots, A_n) \mid \exists B_1 \ldots \exists B_m[(A_1, \ldots, A_n, B_1, \ldots, B_m) \in P]\}. \tag{5}$$

Proof. That a relation R satisfying (5) for a weakly definable P is representable, follows from Theorem 19 and Theorem 17(c).

Assume $\tau(R) = T(\mathfrak{A}) \cap V_{\Sigma^n, \Lambda}$, where $\mathfrak{A} = \langle S, M, s_0, F \rangle$. By possibly adding states to S, we may assume $S = \{0,1\}^m$ for some $m < \omega$. Thus every $r = (B_1, \ldots, B_m) \in P(T)^m$ may be viewed as the run $r: T \rightarrow S$ such that $r(x) = (\chi_{B_1}(x), \ldots, \chi_{Bm}(x))$, $x \in T$. This sets a 1-1 correspondence between $P(T)^m$ and the set S^T.

It is readily checked that the relations $r(x) = s$, where $s \in S$, $x \in T$, and $r(x) \in F \subseteq S$, are weakly definable. Hence $r \in \mathrm{Rn}(\mathfrak{A}, \tau(A_1, \ldots, A_n)) \wedge r(\Lambda) = s_0$, where $r = (B_1, \ldots, B_m)$, is weakly definable (by a formula H involving quantification over just individual variables). The relation $\forall\pi[\mathrm{In}(r|\pi)\cap F \neq \phi]$ is shown to be weakly definable by transforming

$$\forall x \exists \alpha [\alpha \textit{ frontier of } T_x \textit{ and } \forall y [y \in \alpha \rightarrow r(y) \in F]]$$

into a formula $G(\boldsymbol{B}_1, \ldots, \boldsymbol{B}_m)$ of S2S which involves quantification over just individual variables and the quantifier $\exists\boldsymbol{\alpha}$. Now $\tau(A_1, \ldots, A_n) \in T(\mathfrak{A})$, which is equivalent to $(A_1, \ldots, A_n) \in \mathrm{R}$, defined in S2S by

$$\exists \boldsymbol{B}_1 \ldots \exists \boldsymbol{B}_m [H(A_1, \ldots, A_n, \boldsymbol{B}_1, \ldots, \boldsymbol{B}_m) \wedge G(\boldsymbol{B}_1, \ldots, \boldsymbol{B}_m)].$$

7. Applications to decidability and definability

The decidability results in this section were obtained in an even stronger form in [6] by use of the general theory of automata on infinite trees. We wish to show that for some of the results of [6], the simpler theory of special automata is sufficient. Also, the decision procedures involving special automata require fewer computational steps than the procedures of [6].

THEOREM 23. *There exists an effective procedure of deciding for a special Σ-automaton $\mathfrak{A} = \langle S, M, s_0, F \rangle$ whether $T(\mathfrak{A}) \neq \emptyset$. If $c(S) = n$, then this procedure requires n^4 computational steps.*

Proof. By forming the $\{\alpha\}$-table $M'(s,\alpha) = \bigcup_{\sigma \in \Sigma} M(s,\sigma)$, we pass to the case of a single letter alphabet $\{\alpha\}$. Clearly $T(\mathfrak{A}) \neq \emptyset$ if and only if $T(\langle S, M', s_0, F \rangle) \neq \emptyset$. Denote $M'(s,\alpha) = M(s)$. Since every tree has just one $\{\alpha\}$-valuation, we shall just talk about the $\{\alpha\}$-tree T or the finite $\{\alpha\}$-tree $E \subset T$. Thus assume $\mathfrak{A}$ to be an $\{\alpha\}$-automaton. Let $H \subseteq S$, denote by $R(H)$ the set of all $s \in S$ such that there exists a finite tree $E \neq \{\Lambda\}$ and an $\mathfrak{A}$-run $r: E \rightarrow S$ such that $r(\Lambda) = s$ and $r(Ft(E)) \subseteq H$.

An algorithm to compute $R(H)$ will proceed as follows. Put $H_0 = \emptyset$, and inductively for $i < \omega$,

$$H_{i+1} = H_i \cup \{s \,|\, \exists s_1 \exists s_2 [(s_1, s_2) \in M(s), \{s_1, s_2\} \subseteq H \cup H_i]\}.$$

Now, $H_i \subseteq H_{i+1}$ for $i < \omega$, and if $H_i = H_{i+1}$ then $H_i = H_{i+k} = R(H)$, $k < \omega$. Since $H_i \subseteq S$, we certainly have $H_n = H_{n+1} = R(H)$. Given H_i, the calculation of H_{i+1} requires at most n^2 steps since $c(H_i) \leqq n$. Because $H_n = R(H)$, the calculation of $R(H)$ requires at most n^3 steps.

Define $F_0 = F$, and inductively for $i < \omega$, $F_{i+1} = F_i \cap R(F_i)$. Thus $F_{i+1} \subseteq F_i$ for $i < \omega$. Also, $F_i = F_{i+1}$ implies $F_i = F_{i+k}$ for $k < \omega$. Thus certainly $F_n = F_{n+1}$; put $F_n = G$. Note that the calculation of G requires at most $c(F) \leqq n$ calculations of sets $R(H)$, i.e. at most n^4 steps.

We claim that $T(\mathfrak{A}) \neq \emptyset$ if and only if $s_0 \in R(G)$. That $T \in T(\mathfrak{A})$ implies $s_0 \in R(G)$ is proved by methods similar to those of the proof of Theorem 27 and is left to the reader.

Assume $s_0 \in R(G)$ and let us construct an accepting $\mathfrak{A}$-run $r: T \to S$, thereby showing $T \in T(\mathfrak{A})$. Let $t_0 = (r_0, E_0)$ be a non-trivial finite S-tree such that $r_0 \in \mathrm{Rn}(\mathfrak{A}, E_0)$, $r_0(\Lambda) = s_0$, and $r_0(Ft(E_0)) \subseteq G$. For each $s \in G$ let $t_s = (r_s, E_s)$ be a non-trivial finite S-tree such that $r_s \in \mathrm{Rn}(\mathfrak{A}, E_s)$, $r_s(\Lambda) = s$, and $r_s(Ft(E_s)) \subseteq G$. Graft t_0 onto T at Λ; call the resulting (partial) S-tree (r_0, T). For each $x \in Ft(E_0)$, graft $t_{r_0(x)}$ onto (r_0, T) at x; call the resulting tree (r_1, T). Let $E_1 \subset T$ be the subtree of T on which r_1 is defined. We have $r_1(Ft(E_1)) \subseteq G$. For each $x \in Ft(E_1)$, graft $t_{r_1(x)}$ onto (r_1, T) at x; call the resulting tree—(r_2, T). Continuing in this manner, we get a sequence of partial S-trees (r_i, T), $i < \omega$ and finite trees $E_i = D(r_i)$, $i < \omega$. Because $E_s \neq \{\Lambda\}$ for $s \in G$, we have $Ft(E_i) < Ft(E_{i+1})$, $i < \omega$. Thus $T = \bigcup_{i<\omega} E_i$. Let $r = \lim_{i\to\infty} r_i$. Then $r: T \to S$ is an $\mathfrak{A}$-run on T, $r(\Lambda) = r_0(\Lambda) = s_0$, and $r(Ft(E_i)) = r_i(Ft(E_i)) \subseteq G \subseteq F$, $i < \omega$. Hence, $T \in T(\mathfrak{A}) \neq \emptyset$.

The following result follows from the decidability of S2S proved in [6]. Here we deduce it from the theory of special automata.

THEOREM 24. *There exists an effective procedure of deciding for every sentence G of the form $\exists A_1 \ldots \exists A_n F(A_1, \ldots, A_n)$* or *$\forall A_1 \ldots \forall A_n F(A_1, \ldots, A_n)$, where F is a formula with quantification over just individual of finite-set variables, whether $\mathcal{N}_2 \models G$.*

Proof. It is enough to consider the case of sentences G with existential quantifiers, If G is universal, then $\sim G$ is existential of the above form and $\mathcal{N}_2 \models G$ if and only if *not* $\mathcal{N}_2 \models \sim G$.

Assume that $G = \exists A_1 \ldots \exists A_n F(A_1, \ldots, A_n)$, where F is as above. To decide whether $\mathcal{N}_2 \models G$, construct by the procedure given in the proof of Theorem 19 a special $\{0,1\}^n$-automation $\mathfrak{A}$ which represents the relation $\{(A_1, \ldots, A_n) \mid \mathcal{N}_2 \models F(A_1, \ldots, A_n)\}$. Now $\mathcal{N}_2 \models G$ if and only if $T(\mathfrak{A}) \neq \emptyset$ and the latter question is decidable by Theorem 23.

The following theorem is due to Läuchli [4]. A stronger form is found in [6].

THEOREM 25. *The weak second-order theory WTO of linearly ordered sets is decidable.*

Proof. It is a well known result of Tarski that the downward Skolem-Löwenheim theorem holds for weak second-order languages. This implies that WTO is the same as the weak second-order theory of countable linearly ordered sets. Thus if G is a sentence of the weak second-order language of linear order, then $G \in WTO$ if and only if $P = \langle \bar{A}, \leqq \rangle \models G$ for every linearly ordered system P with $c(\bar{A}) \leqq \omega$.

In [6] we have shown that there exists a subset $B \subseteq T$ such that $\langle B, \preccurlyeq \rangle$

(see Lemma 20) has the order type of the rationals. This implies that for every countable linearly ordered system P there exists a subset $A \subseteq T$ such that $P \simeq \langle A, \preccurlyeq \rangle$.

Let G be a sentence as above. From the formula $G(A)$ of S2S by replacing all occurrences of $x \leqq y$ by $x \preccurlyeq y$, and relativizing all individual and finite-set quantifiers to A. The formula $G(A)$ does not contain arbitrary set quantifiers. If $A \subseteq T$ then $\langle A, \preccurlyeq \rangle \models G$ if and only if $\mathcal{N}_2 \models G(A)$. Hence $G \in WTO$ if and only if $\mathcal{N}_2 \models \forall A G(A)$. The last question is decidable by Theorem 24.

As in [6], the decision procedure given here is primitive recursive (in fact, even elementary recursive) which improves upon Läuchli's corresponding result [4].

The following theorem is also a special case of a result in [6]. It is an improvement of Ehrenfeucht's result [2] both in extending his result to weak second-order logic, and in yielding a primitive recursive decision procedure.

THEOREM 26. *The weak second-order theory WSU of a unary function is decidable.*

Proof. In the proof of Theorem 2.4 of [6], we constructed two formulas $F(x, y, C)$ and $Al(A,C)$ with the following properties. For $x \in T$ and $C \subseteq T$ there exists at most one $y \in T$ such that $\mathcal{N}_2 \models F(x, y, C)$. Thus for a fixed $C \subseteq T$, $F(x, y, C)$ defines a binary relation $f \subseteq T \times T$ which is a mapping from $D(f) = \{x \mid \mathcal{N}_2 \models F(x, y, C)\}$ into T. For $A \subseteq T, C \subseteq T$, $\mathcal{N}_2 \models Al(A, C)$ holds if and only if $\langle A, f \mid A \rangle$ is an algebra, i.e. $f(A) \subseteq A$. For every countable unary algebra $P = \langle B, g \rangle$ there exist sets $A \subseteq T$, $C \subseteq T$ such that with the above notations, $\mathcal{N}_2 \models Al(A,C)$ and $P \simeq \langle A, f \mid A \rangle$. Examination of the construction of $F(x,y,C)$ and $Al(A,C)$ will show that these formulas do not involve quantification over arbitrary set variables.

An argument similar to the one in the previous proof shows that for every sentence G of the weak second-order theory of a unary function there exists a sentence $\bar{G} = \forall A \forall A G(A,C)$ of S2S, where G has no arbitrary set quantifiers, such that $G \in WSU$ if and only if $\mathcal{N}_2 \models \bar{G}$. This shows the decidability of WSU.

8. Characterization of weakly definable relations

We wish to show that a relation $R \subseteq P(T)^n$ is weakly-definable if and only if both R and its complement $P(T)^n - R$ are representable by special automata. To this end we study, for special automata, the question when is $T(\mathfrak{A}) \cap T(\mathfrak{B}) \neq \emptyset$.

Let $\mathfrak{A} = \langle S, M, s_0, F \rangle, \mathfrak{B} = \langle S', M', s_0', F' \rangle$. If $(v, T) = t \in T(\mathfrak{A}) \cap T(\mathfrak{B})$, then there are two accepting runs $r \in \mathrm{Rn}(\mathfrak{A}, t)$ and $r' \in \mathrm{Rn}(\mathfrak{B}, t)$. Hence there exists a finite subtree $E \subset T$ and two frontiers (of T) G and G' such that $G < Ft(E)$, $G' < Ft(E)$, $r(G) \subseteq F$ and $r'(G') \subseteq F'$. For every $x \in Ft(E)$ there exists a finite subtree $E_1 \subseteq T_x$, frontiers G_1 and G_1' (of T_x) such that $G_1 < Ft(E_1)$, $G_1' < Ft(E_1)$, $r(G_1) \subseteq F$, and $r'(G_1') \subseteq F'$. And so on, for the nodes $x \in Ft(E_1)$. These considerations motivate the following construction of a sequence of subsets of $S \times S'$.

Define $H_0 = S \times S'$. Define H_{i+1} inductively on i by: $(s, s') \in H_{i+1}$ if and only if $(s, s') \in H_i$ and there exists a finite Σ-tree $e = (v, E)$, where $E \neq \{\Lambda\}$, frontiers $G < Ft(E)$ and $G' < Ft(E)$, and runs $r \in \mathrm{Rn}(\mathfrak{A}, e)$, $r' \in \mathrm{Rn}(\mathfrak{B}, e)$ such that: (1) $r(\Lambda) = s$, $r'(\Lambda) = s'$; (2) $r(G) \subseteq F$, $r'(G') \subseteq F'$; (3) for every $x \in Ft(E)$ we have $(r(x), r'(x)) \in H_i$. Obviously, $H_0 \supseteq H_1 \supseteq \ldots$. Also, if $H_i = H_{i+1}$, then $H_i = H_{i+k}$ for every $k < \omega$. Thus, if $c(S)c(S') = m$, then certainly $H_m = H_{m+k}$ for $k < \omega$. With the above notations we have the following.

THEOREM 27. *$T(\mathfrak{A}) \cap T(\mathfrak{B}) \neq \emptyset$ if and only if $(s_0 s_0') \in H_m$.*

Proof. Assume $(v, T) = t \in T(\mathfrak{A}) \cap T(\mathfrak{B})$. There exist runs $r \in \mathrm{Rn}(\mathfrak{A}, t)$ and $r' \in \mathrm{Rn}(\mathfrak{B}, t)$ such that $r(\Lambda) = s_0$, $r'(\Lambda) = s_0'$ and for every path $\pi \subset T$, $\mathrm{In}(r | \pi) \cap F \neq \emptyset$, $\mathrm{In}(r' | \pi) \cap F' \neq \emptyset$. This implies the existence of a strictly increasing sequence $(E_i)_{i \leqq m}$ of finite subtrees $E_i \subset E_{i+1} \subset T$, $i < m$, such that $E_0 = \{\Lambda\}$ and, for each $i < m$, there are two frontiers (of T) G_i, G_i' satisfying $Ft(E_i) \leqq G_i < Ft(E_{i+1})$, $r(G_i) \subseteq F$, $Ft(E_i) \leqq G_i' < Ft(E_{i+1})$, $r'(G_i) \subseteq F'$.

Let $x \in Ft(E_i)$, $i \leqq m-1$ and consider the finite tree (with root x) $E = T_x \cap E_{i+1}$. Note that $Ft(E) = T_x \cap Ft(E_{i+1})$ and that $G_i \cap T_x$ and $G_i' \cap T_x$ are frontiers of E (and of T_x). Now, $r | E$ and $r' | E$ are, respectively, $\mathfrak{A}$-and $\mathfrak{B}$-runs on (v, E). Also $r(G_i \cap T_x) \subseteq F$ and $r'(G_i \cap T_x) \subseteq F'$. Hence $(r(x), r'(x)) \in H_1$.

Assume now that $i \leqq m-2$, then $i+1 \leqq m-1$ and by the previous result $(r(y), r'(y)) \in H_1$ for every $y \in Ft(E_{i+1})$. Thus, by considering the same tree (v, E), we have $(r(x), r'(x)) \in H_2$.

Using induction, we get $(r(x), r'(x)) \in H_k$ for $x \in Ft(E)$, $i \leqq m-k$. Since $\Lambda \in Ft(E_0)$ we have, in particular, $(s_0, s_0') = (r(\Lambda), r'(\Lambda)) \in H_m$.

To prove the converse assertion, let $(s_0, s_0') \in H_m$ Because $H_m = H_{m+1}$, we have for every $(s, s') \in H_m$ a finite Σ-tree $e(s, s') = (v(s, s'), E(s, s'))$ where $E(s, s') \neq \{\Lambda\}$, runs $r(s, s') = r \in \mathrm{Rn}(\mathfrak{A}, e(s, s'))$, $r'(s, s') = r' \in \mathrm{Rn}(\mathfrak{B}, e(s, s'))$, and frontiers $G < Ft(E(s, s'))$, $G' < Ft(E(s, s'))$ satisfying (1) $r(\Lambda) = s$, $r'(\Lambda) = s'$; (2) $r(G) \subseteq F$, $r'(G') \subseteq F'$; (3) $x \in Ft(E(s, s'))$ implies $(r(x), r'(x)) \in H_m$.

The notion of grafting a subtree t onto a tree (v, T) at $x \in T$, can be extended to the case that $t = (u, E)$ is a finite tree and the valuation is not completely defined. Namely, the result of this graft is defined to be the tree (w, T) such that $w(y) = v(y)$ if $y \notin xE$ and $w(xz) = u(z)$ for $z \in E$.

Let t_1 be the result of grafting $e(s_0, s_0')$ onto $(\emptyset, T)$ at Λ. On $t_1 = (w_1, T)$ define a partial $\mathfrak{A}$-run r_1 by $r_1(x) = r(s, s_0')(x)$ for $x \in E(s_0, s_0')$, and a partial $\mathfrak{B}$-run r_1' by $r_1'(x) = r'(s_0, s_0')(x)$, $x \in E(s_0, s_0')$. Note that for $x \in Ft(E(s_0, s_0'))$, $(r_1(x), r_1'(x)) \in H_m$.

Denote $E(s_0, s_0') = E_1$. Graft simultaneously onto t_1, at each $x \in Ft(E_1)$, the tree $e(r_1(x), r_1'(x))$ and call the resulting tree $t_2 = (w_2, T)$. The domain of definition of w_2 is a finite tree, call it E_2. Extend r_1 and r_1 to runs $r_2 \in \mathrm{Rn}(\mathfrak{A}, (w_2, E_2))$, $r_2' \in \mathrm{Rn}(\mathfrak{B}, (w_2, E_2))$ as follows. If $y \in (E_2 - E_1) \cup Ft(E_1)$ then there is a unique $x \in Ft(E_1)$ such that $y \in xE(s, s')$, where $r_1(x) = s$, and $r_1'(x) = s'$. Assume $y = xz$, $z \in E(s, s')$ and set $r_2(y) = r(s, s')(z)$, $r_2'(y) = r'(s, s')(z)$. Note that for $y \in Ft(E_1)$ we have $r_1(y) = r_2(y)$, $r_1'(y) = r_2'(y)$. Thus r_2 and r_2' indeed extend r_1 and r_1', respectively. Also, for $x \in Ft(E_2)$ we have $(r_2(x), r_2'(x)) \in H_m$. Thus the process can be continued by induction to obtain sequences of partial Σ-trees $(t_i = (w_i, T))_{i<\omega}$, of finite trees $((w, E_i))_{i<\omega}$, and of runs $(r_i)_{i<\omega}$ and $(r_i)_{i<\omega}$. Let $t = \lim_{i<\omega} t_i$; $r = \lim_{i<\omega} r_i$; and $r' = \lim_{i<\infty} r_i'$. Then, $r \in \mathrm{Rn}(\mathfrak{A}, t)$, $r' \in \mathrm{Rn}(\mathfrak{B}, t)$, $r(\Lambda) = s_0'$, $r'(\Lambda) = s_0'$. Our construction implies that for evevery $x \in Ft(E_i)$, $r \,|\, (T_x \cap E_{i+1})$ and $r' \,|\, (T_x \cap E_{i+1})$ coincide with some $r(s, s')$ and $r'(s, s')$, respectively. This entails the existence of two frontiers, G_i, G_i' of T, $Ft(E_i) \leqq G_i < Ft(E_{i+1})$ $Ft(E_i) \leqq G_i' < Ft(E_{i+1})$, such that $r(G_i) \subseteq F$, $r'(G_i) \subseteq F'$. Thus r is an accepting $\mathfrak{A}$-run for t and r' is an accepting $\mathfrak{B}$-run for t. Hence $t \in T(\mathfrak{A}) \cap T(\mathfrak{B})$.

Keeping the notations of the previour theorem we have

COROLLARY 28. *If there exists a Σ-tree $t = (v, T)$ and a strictly increasing sequence $(E_i)_{i \leqq m}$ of finite trees as in the first part of the previous proof, then $T(\mathfrak{A}) \cap T(\mathfrak{B}) \neq \emptyset$.*

Proof. It was shown before that our assumption implies $(s_0, s_0') \in H_m$. Hence $T(\mathfrak{A}) \cap T(\mathfrak{B}) \neq \emptyset$.

To simplify notations we shall now restrict ourselves to Σ^n $(= \{0,1\}^n)$-trees. We shall talk interchangeably about a sequence $A = (A_1, \ldots, A_n) \in P(T)^n$ and the corresponding Σ^n-tree $\tau(A)$.

Let $t = (v, T)$ be a tree accepted by $\mathfrak{A}$. We wish to approximate this fact by certain statements which are weakly definable. On t there exists a run $r \in \mathrm{Rn}(\mathfrak{A}, t)$ and an infinite strictly increasing sequence of finite subtrees $(G_i)_{i<\omega}$ such that $r(\Lambda) = s_0$, $r(Ft(G_i)) \subseteq F$, $i < \omega$. This implies that for every finite subtree $E \subset T$ there exists a subtree $E \subset G \subset T$ and a run $r' \in \mathrm{Rn}(\mathfrak{A}, (v, G))$ such that $r'(\Lambda) = s_0$, $r'(Ft(G)) \subseteq F$. Namely, $G = G_t$

for an appropriate i, and $r' = r|G$. This r' has the property that for every $x \in Ft(G)$ and every finite subtree $E' \subset T_x$ with root x, there exists a finite tree $E' \subset G' \subset T_x$ and a run $r'' \in \mathrm{Rn}(\mathfrak{A},(v,G'))$ such that $r''(x) = r'(x)$, $r''(Ft(G')) \subseteq F$. Namely, $G' = G_j \cap T_x$ for an appropriate $j > i$, and $r'' = r|G'$. And so on. These facts will now be formalized by an inductive definition.

Define $K_0 = S \times T \times V_{\Sigma,\Lambda}$, i.e. $K_0(s,x,t)$ is always true. Let $s \in S$, $x \in T$, (v,T) be a Σ^n-tree, and $i < \omega$; define

$$(6) \quad K_{i+1}(s,x,(v,T)) = \forall E \exists G \exists r[E \textit{ finite subtree of } T_x \rightarrow E \subseteq G \textit{ is a finite subtree} \text{ of } T_x \wedge r \in \mathrm{Rn}(\mathfrak{A},(v,G)) \wedge r(x) = s \wedge r(Ft(G)) \subseteq F \wedge \forall y[y \in Ft(G) \rightarrow K_i(r(y),y,(v,T))]].$$

For every fixed $i < \omega$ and fixed $s \in S$, K_i is a relation $K_i \subseteq T \times P(T)^n$. This relation is weakly definable in $\mathcal{N}_2$. Namely, let $\eta: S \rightarrow \Sigma^k$ be a fixed coding.

In (6), the finite trees G, E, are represented by finite sets α, β. The Σ^n-tree (v,T) is, of course, represented by an n-tuple $(A_1,\dots,A_n)$ of arbitrary sets. These set variables remain free. We recall that the relation $x \leqq y$ is weakly definable. Hence the relation (between $G \subset T$ and $x \in T$): G is a finite subtree of T_x with root x, is weakly definable. Now, a run $r: G \rightarrow S$ is represented, via η, by k finite sets $\alpha_1,\dots,\alpha_k$. A relation $r(x) = s$ is then weakly definable. Finally, $K_i(r(y),y,t)$ is replaced by $\bigwedge_{s \in S}[r(y) = s \rightarrow K_i(s,y,t)]$.

THEOREM 29. *If $R \subseteq P(T)^n$ and $Q = P(T)^n - R$ are representable by special automata, then R is weakly-definable. Conversely, if R is weakly definable, then R and Q are representable by special automata.* (2)

Proof. The latter part of the theorem is simply Theorem 19 and requires no proof.

Let R be represented by $\mathfrak{A} = \langle S, M, s_0, F \rangle$, and Q be represented by $\mathfrak{B} = \langle S', M', s'_0, F' \rangle$; let $c(S)c(S') = m$. Thus $T(\mathfrak{A}) \cap T(\mathfrak{B}) = \emptyset$. We claim that $t = (v,T) \in T(\mathfrak{A})$ if and only if $K_m(s_0,\Lambda,t)$ is true. Since this last relation is weakly definable, this will prove our theorem.

If $t \in T(\mathfrak{A})$ then $K_i(r(x),x,t)$ is true for every $i < \omega$, $x \in T$, by the remarks preceding the definition of $K_i(s,x,t)$. Hence $K_m(s_0,\Lambda,t)$ is true.

Assume by way of contradiction that $K_m(s_0,\Lambda,t)$ for $t = (v,T)$ is true but $t \notin T(\mathfrak{A})$, i.e. $t \in T(\mathfrak{B})$. Let $r' \in \mathrm{Rn}(\mathfrak{B},t)$ be an accepting run of $\mathfrak{B}$ on t. Define $E_0 = \{\Lambda\}$ and let G'_0 be a frontier of T such that $r'(G'_0) \subseteq F'$. Since $K_m(s_0,\Lambda,t)$ is true, there exists a finite tree $\bar{G}_0$ and an $\mathfrak{A}$-run $r_0: \bar{G}_0 \rightarrow S$ on $(v,\bar{G}_0)$ such that $r_0(\Lambda) = s_0$ and, for $G_0 = Ft(\bar{G}_0)$, (1) $r_0G(_0) \subseteq F$ and (2) $K_{m-1}(r_0(x),x,t)$ is true for all $x \in G_0$. Let $E_1 \subset T$ be a finite tree such

(2) This theorem was suggested by A. Tarski.

that $G_0, G_0' < Ft(E_1)$. There exists a frontier G_1' of T such that $Ft(E_1) \leqq G_1'$ and $r'(G_1') \subseteq F'$. Applying the above statement (2) to each $x \in G_0$ and the finite subtree $T_x \cap E_1$ of T_x, we get the existence of a finite tree $E_1 \subseteq \bar{G}_1$ and an extension $r_1 \colon \bar{G}_1 \to S$ of r_0 such that r_1 is an $\mathfrak{A}$-run on $(v, \bar{G}_1)$ and, for $G_1 = Ft(\bar{G}_1)$, (1) $r_1(G_1) \subseteq F$ and (2) $K_{m-2}(r_0(x), x, t)$ is true for all $x \in G_1$. Continue in this way up to a tree E_{m-1} and two frontiers $Ft(E_{m-1}) \leqq G_{m-1}$, G'_{m-1}. Extend the run $r_{m-1} \colon \bar{G}_{m-1} \to S$ in some way to a run $r \in \mathrm{Rn}(\mathfrak{A}, t)$. Then $(E_i)_{i<m}$, r and r' satisfy the conditions of Corollary 28. Hence $T(\mathfrak{A}) \cap T(\mathfrak{B}) \neq \emptyset$, a contradiction. Thus $K_m(s_0, \Lambda, t)$ is true if and only if $t \in T(\mathfrak{A})$ and the proof is completed.

THEOREM 30. *A formula $H(A_1, \dots, A_n)$ of* S2S *is equivalent in* S2S *with some formula $H_1(A_1, \dots, A_n)$ which has no arbitrary set quantifiers, if and only if there exist two formulas $G_1 = \forall B_1 \dots \forall B_p F_1(A_1, \dots, A_n, B_1, \dots B_p)$ and $G_2 = \exists B_1 \dots \exists B_q F_2(A_1, \dots, A_n, B_1, \dots, B_q)$, such that F_i has no arbitrary set quantifier and H is equivalent with G_i in* S2S *for $i = 1, 2$.*

Proof. This follows at once from Theorems 22 and 29.

We conclude our discussion with a decision problem suggested by H. Gaifman. Is it decidable to determine for every given formula $H(A_1, \dots, A_n)$ of S2S whether there exists a formula $H_1(A_1, \dots, A_n)$ without arbitrary set quantifiers which is equivalent to H in S2S?

A positive solution for this problem will have interesting consequences. We have seen in §7 that every countable linearly ordered set $P = \langle \bar{A}, \leqq \rangle$ is reproducible (up to isomorphism) as $\langle A, \preccurlyeq \rangle$ where $A \subseteq T$. Thus every question concerning weak definability of a relation definable in the (monadic) second-order theory of countable linear order can be converted into a corresponding question for S2S. If the latter question can be effectively answered so can the original question. Similar remarks apply to the second-order theory of a unary function and to other theories.

REFERENCES

[1] J. E. DONER, *Decidability of the weak second-order theory of two successors,* Notices Amer. Math. Soc., **12** (1965), 819.

[2] A. EHRENFEUCHT, *Decidability of the theory of one function,* Notices Amer. Math. Soc., **6** (1959), 268.

[3] D. M. GABBAY, *Decidability results in non-classical logics I,* Technical Report No. 29, Applied Logic Branch, Jerusalem, 1969, p. 42.

[4] H. LÄUCHLI, *A decision procedure for the weak second order theory of linear order, Contributions to mathematical logic,* K. Schütte, editor, North-Holland, Amsterdam, 1968, 189–197.

[5] M. O. RABIN, *Mathematical theory of automata,* Proc. Sympos. Appl. Math., Vol. 19, Amer. Math. Soc., Providence, R.I., 1968, 153–175.

[6] ———, *Decidability of second order theories and automata on infinite trees,* Trans. Amer. Math. Soc., **141** (1969), 1–35.

[7] J. W. THATCHER and J. B. WRIGHT, *Generalized finite automata,* Notices Amer. Math. Soc., **12** (1965), 820.