

Parametric updates in parametric timed automata[★]

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1 Introduction

Timed automata (TAs) are a powerful formalism to model and verify timed concurrent systems, both expressive enough to model many interesting systems and enjoying several decidability properties. In particular, the reachability of a discrete state is decidable and PSPACE-complete [AD94]. In TAs, clocks can be compared with constants in guards, and can be updated to 0 (“reset”) along edges.

Timed automata may turn insufficient to verify systems where the timing constants themselves are subject to some uncertainty, or when they are simply not known at the early design stage. Parametric timed automata (PTAs) [AHV93] address this drawback by allowing parameters (unknown constants) in the timing constraints; this high expressive power comes at the cost of the undecidability of most interesting problems (see *e. g.*, [And19]). In particular, the basic problem of EF-emptiness (“is the set of valuations for which a given location is reachable in the instantiated timed automaton empty?”) is “robustly” undecidable: even for a single rational-valued [Mil00] or integer-valued parameter [AHV93; Ben+15], or when only strict constraints are used [Doy07]. A famous syntactic subclass of PTAs that enjoys limited decidability is L/U-PTAs [Hun+02], where the parameters set is partitioned into lower-bound and upper-bound parameters, *i. e.*, parameters that can only be compared to a clock as a lower-bound (resp. upper-bound). The EF-emptiness problem is decidable for L/U-PTAs [Hun+02; BL09] and for PTAs under several restrictions [BO14]; however, most other problems are undecidable (*e. g.*, [BL09; Qua14; JLR15; AM15; ALR16; AL17]) (see [And19] for a survey).

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1.1 Contribution

We investigate parametric updates, which can model an unknown timing configuration in a system where processes need to synchronise together on common events, as in *e. g.*, programmable controller logic programs with concurrent tasks execution. We show that the EF-emptiness problem is decidable for PTAs augmented with parametric updates (*i. e.*, U2P-PTA), with the additional condition that whenever a clock is compared to a parameter in a guard or updated to a parameter, all clocks must be updated (possibly to parameters)—this gives R-U2P-PTA. This result holds when the parameters are *bounded rationals in guards*, and possibly *unbounded rationals in updates*. Non-trivial decidable subclasses of PTAs are a rarity (to the best of our knowledge, only L/U-PTAs [Hun+02] and integer-points (IP-)PTAs [ALR16]); this makes our positive result very welcome. In addition, not only the emptiness is decidable, but *exact synthesis* for bounded rational-valued parameters can be performed—which contrasts with L/U-PTAs and IP-PTAs as synthesis was shown intractable [JLR15; ALR16].

About this manuscript This is the extended version of [ALR19]. In addition to additional explanations and all proofs of our results, we added the whole new Section 7 adding stopwatches to our formalism.

1.2 Related Work

Our construction is reminiscent of the parametric difference bound matrices (PDBMs) defined in [QSW17, section III.C] where the author revisits the result of the binary reachability relation over both locations and clock valuations in TAs; however, parameters of [QSW17] are used to bound in time a run that reaches a given location, while we use parameters directly in guards and resets along the run, which make them active components of the run specifically for intersection with parametric guards, key point not tackled in [QSW17]. Related DBMs with an additional parameter are studied such as shrunk DBMs [SBM14; Bus+19] and infinitesimally enlarged DBMs [San15].

Allowing parameters in clock updates is inspired by the updatable TA formalism defined in [Bou+04] where clocks can be updated not only to 0 (“reset”) but also to rational constants (“update”). In [ALR18], we extended the result of [Bou+04] by allowing parametric updates (and no parameter elsewhere, *e. g.*, in guards): the EF-emptiness is undecidable even in the restricted setting of bounded rational-valued parameters, but becomes decidable when parameters are restricted to (unbounded) integers.

Synthesis is obviously harder than EF-emptiness: only three results have been proposed to synthesize the exact set of valuations for subclasses of PTAs, but they are all concerned with *integer*-valued parameters [BL09; JLR15; ALR18]. More precisely, it is possible to synthesize unbounded integers for L- or U-PTAs (L/U-PTAs with only lower-bound, or only upper-bound, parameters) [BL09]; bounded integers for PTAs [JLR15] unbounded integers for timed automata with parametric updates [ALR18].

In contrast, we deal here with (bounded) rational-valued parameters—which makes this result the first of its kind. The idea of updating all clocks when compared to parameters comes from our class of *reset-PTAs* briefly mentioned in [ALR16], but not thoroughly studied.

Finally, updating clocks on each transition in which a parameter appears is reminiscent of the initialized rectangular hybrid automata formalism defined in [Hen+98], which remains one of the few decidable subclasses of hybrid automata. Indeed, timed automata can be defined as a subclass of initialized rectangular hybrid automata where clocks evolve at the same fixed rate, in which diagonal constraints are not allowed for the purpose of leading to decidability of the reachability emptiness problem.

1.3 Outline

Section 2 recalls preliminaries. Section 3 presents R-U2P-PTA along with our decidability result. Sections 4 and 5 introduce operations on our \mathbf{p} -PDBMs and our extended region automaton. Section 6 proves the main decidability result. Section 7 extends our results to stopwatches. Section 8 gives a concrete application of our result. Section 9 concludes the paper.

2 Preliminaries

Let \mathbb{N} , \mathbb{Z} , \mathbb{Q}_+ and \mathbb{R}_+ denote the sets of non-negative integers, integers, non-negative rational numbers and non-negative real numbers respectively.

Throughout this paper, we assume a set $\mathbb{X} = \{x_1, \dots, x_H\}$ of *clocks*, *i. e.*, real-valued variables evolving at the same rate. A clock valuation is a function $w : \mathbb{X} \rightarrow \mathbb{R}_+$. We write $\mathbf{0}$ for the clock valuation that assigns 0 to all clocks. Given $d \in \mathbb{R}_+$, $w + d$ (resp. $w - d$) denotes the valuation such that $(w + d)(x) = w(x) + d$ (resp. $(w - d)(x) = w(x) - d$ if $w(x) - d > 0$, 0 otherwise), for all $x \in \mathbb{X}$. We assume a set $\mathbb{P} = \{p_1, \dots, p_M\}$ of *parameters*, *i. e.*, unknown constants. A parameter valuation v is a function $v : \mathbb{P} \rightarrow \mathbb{Q}_+$. We identify a valuation v with the point $(v(p_1), \dots, v(p_M))$ of \mathbb{Q}_+^M . Given $d \in \mathbb{N}$, $v + d$ (resp. $v - d$) denotes the valuation such that $(v + d)(p) = v(p) + d$ (resp. $(v - d)(p) = v(p) - d$ if $v(p) - d > 0$, 0 otherwise), for all $p \in \mathbb{P}$.

In the following, we assume $\triangleleft \in \{<, \leq\}$ and $\bowtie \in \{<, \leq, \geq, >\}$.

A *parametric guard* g is a constraint over $\mathbb{X} \cup \mathbb{P}$ defined as the conjunction of inequalities of the form $x \bowtie z$, where x is a clock and z is either a parameter or a constant in \mathbb{Z} . A *non-parametric guard* is a parametric guard without parameters (*i. e.*, over \mathbb{X}).

Given a parameter valuation v , $v(g)$ denotes the constraint over \mathbb{X} obtained by replacing in g each parameter p with $v(p)$. We extend this notation to an *expression*: a sum or difference of parameters and constants. Likewise, given a clock valuation w , $w(v(g))$ denotes the expression obtained by replacing in $v(g)$ each clock x with $w(x)$. A clock valuation w *satisfies* constraint $v(g)$ (denoted by $w \models v(g)$) if $w(v(g))$ evaluates to true. We say that v *satisfies* g , denoted by

$v \models g$, if the set of clock valuations satisfying $v(g)$ is nonempty. We say that g is *satisfiable* if $\exists w, v$ s.t. $w \models v(g)$.

A *parametric update* is a partial function $u : \mathbb{X} \rightarrow \mathbb{N} \cup \mathbb{P}$ which assigns to some of the clocks an integer constant or a parameter. For v a parameter valuation, we define a partial function $v(u) : \mathbb{X} \rightarrow \mathbb{Q}_+$ as follows: for each clock $x \in \mathbb{X}$, $v(u)(x) = k \in \mathbb{N}$ if $u(x) = k$ and $v(u)(x) = v(p) \in \mathbb{Q}_+$ if $u(x) = p$ a parameter. A non-parametric update is $u_{np} : \mathbb{X} \rightarrow \mathbb{N}$. The term *reset* has been used for clock updates to values different from 0 in [BY03]. For a clock valuation w and a parameter valuation v , we denote by $[w]_{v(u)}$ the clock valuation obtained after applying $v(u)$. We first define a new class of parametric timed automata in order to properly define further plain parametric timed automata and timed automata.

Definition 1. An *update-to-parameter PTA (U2P-PTA)* \mathcal{A} is a tuple $\mathcal{A} = (\Sigma, L, l_0, \mathbb{X}, \mathbb{P}, \zeta)$, where: *i)* Σ is a finite set of actions, *ii)* L is a finite set of locations, *iii)* $l_0 \in L$ is the initial location, *iv)* \mathbb{X} is a finite set of clocks, *v)* \mathbb{P} is a finite set of parameters, *vi)* ζ is a finite set of edges $e = \langle l, g, a, u, l' \rangle$ where $l, l' \in L$ are the source and target locations, g is a parametric guard, $a \in \Sigma$ and $u : \mathbb{X} \rightarrow \mathbb{N} \cup \mathbb{P}$ is a parametric update function.

An U2P-PTA is depicted in Figure 1. Note that all clocks are updated whenever there is a comparison with a parameter (as in `newBlock`) or a clock is updated to a parameter (as in `blockSolutionx`).

Given a parameter valuation v , we denote by $v(\mathcal{A})$ the structure where all occurrences of a parameter p_i have been replaced by $v(p_i)$. If $v(\mathcal{A})$ is such that all constants in guards and updates are rationals, then $v(\mathcal{A})$ is a *updatable timed automaton* [Bou+04] but will be called *timed automaton* (TA) for the sake of simplicity in this paper. In the following, we may denote as a timed automaton any structure $v(\mathcal{A})$, by assuming a rescaling of the constants: by multiplying all constants in $v(\mathcal{A})$ by their least common denominator, we obtain an equivalent timed automaton (with integer constants).

A *bounded U2P-PTA* is a U2P-PTA with a bounded parameter domain that assigns to each parameter a minimum integer bound and a maximum integer bound. That is, each parameter p_i ranges in an interval $[a_i, b_i]$, with $a_i, b_i \in \mathbb{N}$. Hence, a bounded parameter domain is a hyperrectangle of dimension M .

A parametric timed automaton (PTA) [AHV93] is a U2P-PTA where, for any edge $e = \langle l, g, a, u, l' \rangle \in \zeta$, $u : \mathbb{X} \rightarrow \{0\}$.

Definition 2 (Concrete semantics of a TA). Given a U2P-PTA $\mathcal{A} = (\Sigma, L, l_0, \mathbb{X}, \mathbb{P}, \zeta)$, and a parameter valuation v , the concrete semantics of $v(\mathcal{A})$ is given by the timed transition system (S, s_0, \rightarrow) , with

- $S = \{(l, w) \in L \times \mathbb{R}_+^H\}$, $s_0 = (l_0, \mathbf{0})$
- \rightarrow consists of the discrete and (continuous) delay transition relations:
 - *discrete transitions:* $(l, w) \xrightarrow{e} (l', w')$, if $(l, w), (l', w') \in S$, there exists $e = \langle l, g, a, u, l' \rangle \in \zeta$, $w' = [w]_{v(u)}$, and $w \models v(g)$.
 - *delay transitions:* $(l, w) \xrightarrow{d} (l, w + d)$, with $d \in \mathbb{R}_+$.

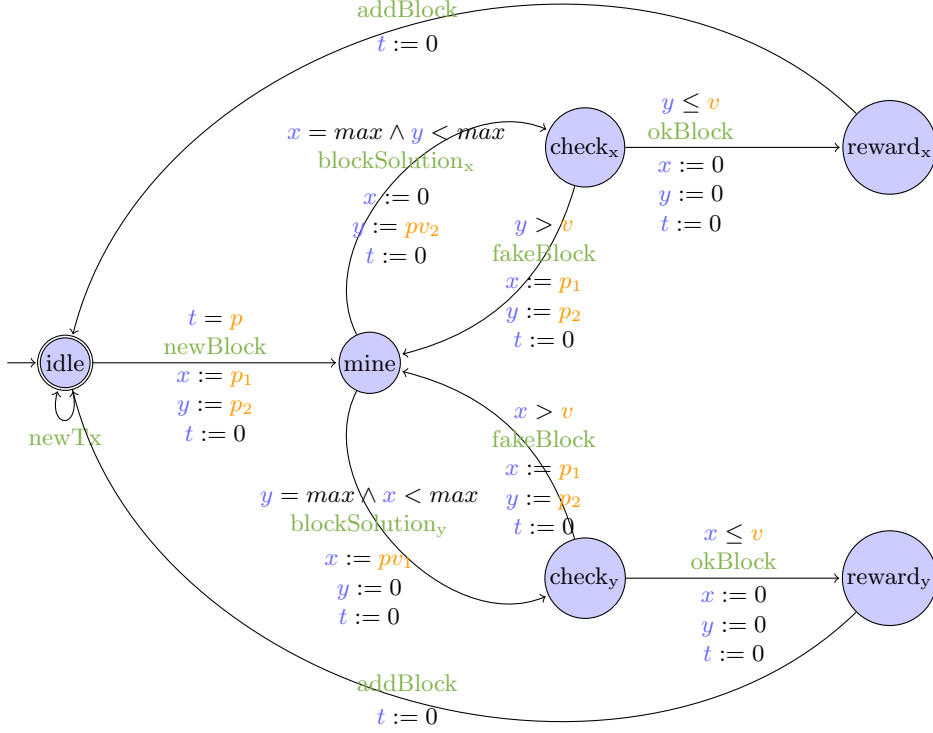


Fig. 1: A proof-of-work modeled with a bounded R-U2P-PTA.

Moreover we write $(l, w) \xrightarrow{e} (l', w')$ for a combination of a delay and discrete transitions where $((l, w), e, (l', w')) \in \rightarrow$ if $\exists d, w'' : (l, w) \xrightarrow{d} (l, w'') \xrightarrow{e} (l', w')$.

Given a TA $v(\mathcal{A})$ with concrete semantics (S, s_0, \rightarrow) , we refer to the states of S as the *concrete states* of $v(\mathcal{A})$. A (concrete) *run* of $v(\mathcal{A})$ is a possibly infinite alternating sequence of concrete states of $v(\mathcal{A})$ and edges starting from the initial concrete state s_0 of the form $s_0 \xrightarrow{e_0} s_1 \xrightarrow{e_1} \dots \xrightarrow{e_{m-1}} s_m \xrightarrow{e_m} \dots$, such that for all $i = 0, 1, \dots, e_i \in \zeta$, and $(s_i, e_i, s_{i+1}) \in \rightarrow$. Given a state $s = (l, w)$, we say that s is reachable (or that $v(\mathcal{A})$ reaches s) if s belongs to a run of $v(\mathcal{A})$. By extension, we say that l is reachable in $v(\mathcal{A})$, if there exists a state (l, w) that is reachable.

Throughout this paper, let K denotes the largest constant in a given U2P-PTA, *i. e.*, the maximum of the largest constant compared to a clock in a guard and the largest upper bound of a parameter (if the U2P-PTA is bounded).

Let us recall the notion of clock region [AD94]. Given a clock x and a clock valuation w , recall that $\lfloor w(x) \rfloor$ denotes the integer part of $w(x)$ while $\text{frac}(w(x))$ denotes its fractional part. We define the same notation for parameter valuations.

Definition 3 (clock region). For two clock valuations w and w' , \sim is an equivalence relation defined by: $w \sim w'$ iff

1. for all clocks x , either $\lfloor w(x) \rfloor = \lfloor w'(x) \rfloor$ or $w(x), w'(x) > K$;
2. for all clocks x, y with $w(x), w(y) \leq K$, $\text{frac}(w(x)) \leq \text{frac}(w(y))$ iff $\text{frac}(w'(x)) \leq \text{frac}(w'(y))$;
3. for all clocks x with $w(x) \leq K$, $\text{frac}(w(x)) = 0$ iff $\text{frac}(w'(x)) = 0$.

A clock region R_c is an equivalence class of \sim .

Two clock valuations in the same clock region (cf. Definition 3) and reach the same regions by time elapsing, satisfy the same guards and can take the same transitions [AD94]. In this paper, we address the EF-emptiness problem.

3 A decidable subclass of U2P-PTAs

We now impose that, whenever a guard or an update along an edge contains parameters, then all clocks must be updated (to constants or parameters). Our main contribution is to prove that this restriction makes EF-emptiness decidable.

Definition 4. An R-U2P-PTA is a U2P-PTA where for any edge $\langle l, g, a, u, l' \rangle \in \zeta$, u is a total function whenever:⁵

1. g is a parametric guard, or
2. $u(x) \in \mathbb{P}$ for some $x \in \mathbb{X}$.

The main idea for proving decidability is the following: given an R-U2P-PTA \mathcal{A} we will construct a finite region automaton that bisimulates \mathcal{A} , as in TA [AD94]. Our regions will contain both clocks and parameters, and will be a finite number. Since parameters are allowed in guards, we need to construct parameter regions and more restricted clock regions.

We will define a form of Parametric Difference Bound Matrices (viz., **p**-PDBMs for precise PDBMs, inspired by [Hum+02]) in which, once valuated by a parameter valuation, two clock valuations have the same discrete behavior and satisfy the same non-parametric guards. A **p**-PDBM will define the *set of clocks and parameter valuations* that satisfies it, while once valuated by a parameter valuation, a valuated **p**-PDBM will define the *set of clock valuations* that satisfies it. A key point is that in our **p**-PDBMs the parametric constraints used in the matrix will be defined from a *finite* set of predefined expressions involving parameters and constants, and we will prove that this defines a finite number of **p**-PDBMs. Decidability will come from this fact. We define this set ($\mathcal{P}\mathcal{L}\mathcal{T}$ for parametric linear term) as follows: $\mathcal{P}\mathcal{L}\mathcal{T} = \{\text{frac}(p_i), 1 - \text{frac}(p_i), \text{frac}(p_i) -$

⁵ In the following we only consider either non-parametric, or (necessarily total) fully parametric update functions. A total update function which is not fully parametric (i. e., an update of some clocks to parameters and all others to constants) can be encoded as a total fully parametric update immediately followed by a (partial) non-parametric update function.

$\text{frac}(p_j), \text{frac}(p_j)+1-\text{frac}(p_i), 1, 0, \text{frac}(p_i)-1-\text{frac}(p_j), -\text{frac}(p_i), \text{frac}(p_i)-1\}$, for all $1 \leq i, j \leq M$. Given a parameter valuation v and $d \in \mathcal{P}\mathcal{L}\mathcal{T}$, we denote by $v(d)$ the term obtained by replacing in d each parameter p by $v(p)$. Let us now define an equivalence relation between parameter valuations v and v' .

Definition 5 (regions of parameters). We write that $v \sim v'$ if

1. for all parameter p , $\lfloor v(p) \rfloor = \lfloor v'(p) \rfloor$;
2. for all $d_1, d_2, d_3 \in \mathcal{P}\mathcal{L}\mathcal{T}$, $v(d_1) \leq v(d_2) + v(d_3)$ iff $v'(d_1) \leq v'(d_2) + v'(d_3)$;

Parameter regions are defined as the equivalence classes of \sim , and we will use the notation R_p for parameter regions. The set of all parameter regions is denoted by \mathcal{R}_p . The definition is in a way similar to [Definition 3](#) but also involves comparisons of sums of elements of $\mathcal{P}\mathcal{L}\mathcal{T}$. In fact, we will need this kind of comparisons to define our p-PDBMs. Nonetheless we do not need more complicated comparisons as in R-U2P-PTA whenever a parametric guard or updated is met the update is a total function: this preserves us from the parameter accumulation, *e. g.*, obtaining expressions of the form $5\text{frac}(p_i) - 1 - 3\text{frac}(p_j)$ (that may occur in usual PTAs).

In the following, our p-PDBMs will contain pairs of the form $D = (d, \triangleleft)$, where $d \in \mathcal{P}\mathcal{L}\mathcal{T}$. We therefore need to define comparisons on these pairs.

We define an associative and commutative operator \oplus as $\triangleleft_1 \oplus \triangleleft_2 = <$ if $\triangleleft_1 \neq \triangleleft_2$, or \triangleleft_1 if $\triangleleft_1 = \triangleleft_2$. We define $D_1 + D_2 = (d_1 + d_2, \triangleleft_1 \oplus \triangleleft_2)$. Following the idea of parameter regions, we define the *validity* of a comparison between pairs of the form (d_i, \triangleleft_i) within a given parameter region, *i. e.*, whether the comparison is true for all parameter valuations v in the parameter region R_p .

Definition 6 (validity of comparison). Let R_p be a parameter region. Given any two linear terms d_1, d_2 over \mathbb{P} (*i. e.*, of the form $\sum_i \alpha_i p_i + d$ with $\alpha_i, d \in \mathbb{Z}$), the comparison $(d_1, \triangleleft_1) \triangleleft (d_2, \triangleleft_2)$ is valid for R_p if:

1. $\triangleleft = <$, and either
 - (a) for all $v \in R_p$, $v(d_1) < v(d_2)$ evaluates to true regardless of $\triangleleft_1, \triangleleft_2$, or
 - (b) for all $v \in R_p$, $v(d_1) \leq v(d_2)$ evaluates to true, $\triangleleft_1 = <$ and $\triangleleft_2 = \leq$;
2. $\triangleleft = \leq$, and either
 - (a) for all $v \in R_p$, $v(d_1) < v(d_2)$ evaluates to true regardless of $\triangleleft_1, \triangleleft_2$, or
 - (b) for all $v \in R_p$, $v(d_1) \leq v(d_2)$ evaluates to true, and $\triangleleft_1 = \triangleleft_2$, or $\triangleleft_1 = <$.

Transitivity is immediate from the definition: if $D_1 \triangleleft_1 D_2$ and $D_2 \triangleleft_2 D_3$ are valid for R_p , $D_1(\triangleleft_1 \oplus \triangleleft_2)D_3$ is valid for R_p .

The following lemma derives from [Definition 6](#):

Lemma 1 (validity of addition). Let $d_1, d_2, d_3, d_4 \in \mathcal{P}\mathcal{L}\mathcal{T}$. Let R_p be a parameter region. If $(d_1, \triangleleft_1) \leq (d_2, \triangleleft_2)$ and $(d_3, \triangleleft_3) \leq (d_4, \triangleleft_4)$ are valid for R_p then $(d_1, \triangleleft_1) + (d_3, \triangleleft_3) \leq (d_2, \triangleleft_2) + (d_4, \triangleleft_4)$ is valid for R_p .

Proof. See [Appendix A](#).

We can now define our data structure, namely **p-PDBMs** (for *precise Parametric Difference Bound Matrices*), inspired by the PDBMs of [Hun+02]; PDBMs were themselves inspired by the DBMs of [Dil90]. However, our **p-PDBM** compare differences of *fractional parts* of clocks, instead of clocks as in classical DBMs; therefore, our **p-PDBMs** are closer to clock regions than to DBMs and *fully contained* into clock regions of [AD94]. A **p-PDBM** is a pair made of an integer vector (encoding the clocks integer part), and a matrix (encoding the parametric differences between any two clock fractional parts). Their interpretation also follows that of PDBMs and DBMs: for $i \neq 0$, the matrix cell $D_{i,0} = (d_{i,0}, \triangleleft_{i0})$ is interpreted as the constraint $\text{frac}(x_i) \triangleleft_{i0} d_{i,0}$, and $D_{0,i} = (d_{0,i}, \triangleleft_{0i})$ as the constraint $-\text{frac}(x_i) \triangleleft_{0i} d_{0,i}$. For $i \neq 0$ and $j \neq 0$, the matrix cell $D_{i,j} = (d_{i,j}, \triangleleft_{ij})$ is interpreted as $\text{frac}(x_i) - \text{frac}(x_j) \triangleleft_{ij} d_{i,j}$. Finally for all i , $D_{i,i} = (0, \leq)$.

Our **p-PDBMs** are partitioned into two types: **open-p-PDBMs** and **point-p-PDBMs**. Different additional rules are applied on their matrix cells so we can precisely differentiate them. A **point-p-PDBM** is a clock region defined by only parameters which contains only one clock valuation; that is, it corresponds to a set of inequalities of the form $x_i = p_j$. In contrast, an **open-p-PDBM** is a clock region which can contain several clock valuations satisfying some possibly parametric constraints, or contain at least one clock valuation satisfying non-parametric constraints (as the corner-point of [AD94]). In particular, the initial clock region $\{0^H\}$ and any clock region $\{E_i^H\}$ where E_i is an integer for all clock x_i , is an **open-p-PDBM**.

Basically, only the first **p-PDBM** after a (necessarily total) parametric clock update will be a **point-p-PDBM**; any following **p-PDBM** will be an **open-p-PDBM** until the next (total) parametric update.

Definition 7 (open-p-PDBM). Let R_p be a parameter region. An **open-p-PDBM** for R_p is a pair (E, D) with $E = (E_1, \dots, E_H)$ a vector of H integers (or ∞) which is the integer part of each clock, and D is an $(H+1)^2$ matrix where each element $D_{i,j}$ is a pair $(d_{i,j}, \triangleleft_{ij})$ for all $0 \leq i, j \leq H$, where $d_{i,j} \in \mathcal{PCT}$. Moreover, for all $0 \leq i \leq H$, $D_{i,i} = (0, \leq)$. In addition:

1. For all i , $(-1, <) \leq D_{0,i} \leq (0, \leq)$ and $(0, \leq) \leq D_{i,0} \leq (1, <)$ are valid for R_p ,
2. For all $i \neq 0, j \neq 0$, either $(0, \leq) \leq D_{i,j} \leq (1, <)$ is valid for R_p and $(-1, <) \leq D_{j,i} \leq (0, \leq)$ is valid for R_p or $(0, \leq) \leq D_{j,i} \leq (1, <)$ is valid for R_p and $(-1, <) \leq D_{i,j} \leq (0, \leq)$ is valid for R_p .
3. For all i, j , if $d_{i,j} = -d_{j,i}$ and is different from 1 then $\triangleleft_{ij} = \triangleleft_{ji} = \leq$, else $\triangleleft_{ij} = \triangleleft_{ji} = <$,
4. For all i, j, k , $D_{i,j} \leq D_{i,k} + D_{k,j}$ is valid for R_p (canonical form), and
5. (a) There is at least one i s.t. $D_{i,0} = D_{0,i} = (0, \leq)$, or
(b) there is at least one i s.t. $D_{i,0} = (1, <)$ and for all j s.t. $D_{0,j} = (0, \triangleleft_{0j})$, then we have $\triangleleft_{0j} = <$.

An **open-p-PDBM** satisfying condition 5a can be seen as a subregion of an open line segment or a corner point region of [AD94, fig. 9 example 4.4] (it can

be seen as a *border* region) and one satisfying condition 5b can be seen as a subregion of an open region of [AD94, fig. 9 example 4.4] (it can be seen as a *center* region). Remark that sets of the form $\{\text{frac}(w(x)) \mid 0 \leq \text{frac}(w(x)) \leq 1\}$ are forbidden by Definition 7 (3), as in the regions of [AD94].

Let R_p be a parameter region. In the following, $p\text{-PDBM}_{\blacksquare}(R_p)$ is the set of all possible **open-p-PDBMs** (E, D) for R_p . This definition is similar to that of [Hun+02, def. 3.1].

The second type is the **point-p-PDBM**. It represents the unique clock valuation (for a given parameter valuation) obtained after a total parametric update in an U2P-PTA.

Definition 8 (point-p-PDBM). Let R_p be a parameter region. A **point-p-PDBM** for R_p is a pair (E, D) where D is an $(H+1)^2$ matrix where each element $D_{i,j}$ is a pair $(d_{i,j}, \leq)$ and for all $0 \leq i, j \leq H$, $d_{i,0} = \text{frac}(p_1) = -d_{0,i}$, and $d_{i,j} = \text{frac}(p_1) - \text{frac}(p_2) = -d_{j,i}$, for any $p_1, p_2 \in \mathbb{P}$. and for all $1 \leq i \leq H$, $E_i = \lfloor p_k \rfloor$ if $d_{i,0} = \text{frac}(p_k)$, for $1 \leq k \leq M$. In addition:

1. For all i , $(-1, <) \leq D_{0,i} \leq (0, \leq)$ and $(0, \leq) \leq D_{i,0} \leq (1, <)$ are valid for R_p ,
2. For all i, j, k , $D_{i,j} \leq D_{i,k} + D_{k,j}$ is valid for R_p (canonical form).

The fact that D is antisymmetric *i. e.*, for all i, j , $D_{i,j} = -D_{j,i}$, means that each clock is valued to a parameter and each difference of clocks is valued to a difference of parameters.

The set of all **point-p-PDBM** for R_p is denoted by $p\text{-PDBM}_{\odot}(R_p)$, and the set of all **p-PDBMs** for R_p is denoted by $p\text{-PDBM}(R_p)$ (hence $p\text{-PDBM}(R_p) = p\text{-PDBM}_{\blacksquare}(R_p) \cup p\text{-PDBM}_{\odot}(R_p)$).

The use of *validity* ensures the consistency of the **p-PDBM**. We denote the set of all **p-PDBMs** that are *valid* for R_p by $p\text{-PDBM}(R_p)$. Given a **p-PDBM** (E, D) , it defines the subset of $\mathbb{R}^H \cup \mathbb{Q}^M$ satisfying the constraints $\bigwedge_{i,j \in [0,H]} \text{frac}(x_i) - \text{frac}(x_j) \triangleleft_{i,j} d_{i,j} \wedge \bigwedge_{i \in [1,H]} \lfloor x_i \rfloor = E_i$. Given a **p-PDBM** (E, D) and a parameter valuation v , we denote by $(E, v(D))$ the *valuated p-PDBM*, *i. e.*, the set of clock valuations defined by the inequalities:

$$\bigwedge_{i,j \in [0,H]} \text{frac}(x_i) - \text{frac}(x_j) \triangleleft_{i,j} v(d_{i,j}) \wedge \bigwedge_{i \in [1,H]} \lfloor x_i \rfloor = E_i.$$

For a clock valuation w , we write $w \in (E, v(D))$ if it satisfies all constraints of $(E, v(D))$ ⁶.

The following two lemmas derive from the above definitions of **point-p-PDBM** and **p-PDBMs**:

Lemma 2 (positivity of reflexivity). Let R_p be a parameter region and (E, D) be a **p-PDBM** for R_p . For all clocks i, j , $(0, \leq) \leq D_{i,j} + D_{j,i}$ is valid for R_p .

⁶ If v is a valuation assigning an *integer* to each parameter, then $(E, v(D))$ is DBM as defined in [BY03].

Proof. See Appendix B.

Lemma 3 (neutral element of the set of cells). *Let R_p be a parameter region and (E, D) be a \mathbf{p} -PDBM for R_p . For all clocks i, j , $D_{i,j} \leq D_{i,j} + D_{j,j}$ and $D_{i,j} \leq D_{i,i} + D_{i,j}$ are valid for R_p .*

Proof. See [Appendix C](#).

But let us first clarify our needs graphically. Intuitively, our \mathbf{p} -PDBMs are partitioned into three types.

(1) The **point-p** PDBM is a clock region defined by *only parameters* which contains only one clock valuation; it represents the unique clock valuation (for a given parameter valuation) obtained after a total parametric update in an U2P-PTA. Each clock is valued to a parameter and each difference of clocks is valued to a difference of parameters (it corresponds to constraints of the form $x = p$ and $x - y = p_i - p_j$).

Let v be a parameter valuation. We assume $\lfloor v(p_2) \rfloor = \lfloor v(p_1) \rfloor = k \in \mathbb{N}$ and $\text{frac}(v(p_1)) > \text{frac}(v(p_2))$. The \mathbf{p} -PDBM obtained after an update $u(x) = v(p_2)$ and $u(y) = v(p_1)$ is represented using the following pair (where the indices $\mathbf{0}, \mathbf{x}, \mathbf{y}$ are shown for the sake of comprehension)

$$(E, D) = \left(\binom{k}{k}, \begin{pmatrix} \mathbf{0} & \mathbf{x} & \mathbf{y} \\ \mathbf{0} \ (0, \leq) & (-\text{frac}(p_2), \leq) & (-\text{frac}(p_1), \leq) \\ \mathbf{x} \ (\text{frac}(p_2), \leq) & (0, \leq) & (\text{frac}(p_2) - \text{frac}(p_1), \leq) \\ \mathbf{y} \ (\text{frac}(p_1), \leq) & (\text{frac}(p_1) - \text{frac}(p_2), \leq) & (0, \leq) \end{pmatrix} \right)$$

Once valuated with v , it contains a unique clock valuation. We represent it as the black dot in Figure 2.

(2) In contrast, an **open-p-PDBM** satisfying condition (5a) is a clock region which can contain several clock valuations satisfying some possibly parametric constraints, or contain at least one clock valuation satisfying non-parametric constraints (as the corner-point region of [AD94]). In particular, the initial clock region $\{0^H\}$ and any clock region that is a single integer clock valuation is a **p-PDBM**. An **open-p-PDBM** satisfying condition 5a is characterized by a $(0, \leq)$ and can be seen as a subregion of an region of [AD94, fig. 9 example 4.4]. After above **p-PDBM** (E, D) becomes

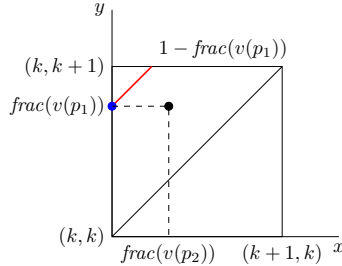


Fig. 2: Graphical representations of p-PDBMs and [AD94] regions

$$(E, D) = \left(\binom{k}{k}, \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{x} & \mathbf{y} \\ \mathbf{0} & (0, \leq) & (0, \leq) & (-\text{frac}(p_1), \leq) \\ \mathbf{x} & (0, \leq) & (0, \leq) & (-\text{frac}(p_1), \leq) \\ \mathbf{y} & (\text{frac}(p_1), \leq) & (\text{frac}(p_1), \leq) & (0, \leq) \end{pmatrix} \right)$$

We represent it once valuated with v as the blue dot in Figure 2. The open line segment of [AD94, fig. 9 example 4.4] can be represented as

$$\left(\binom{k}{k}, \begin{pmatrix} \mathbf{0} & \mathbf{x} & \mathbf{y} \\ \mathbf{0} & (0, \leq) & (0, \leq) & (0, <) \\ \mathbf{x} & (0, \leq) & (0, \leq) & (0, <) \\ \mathbf{y} & (1, <) & (1, <) & (0, \leq) \end{pmatrix} \right)$$

and is depicted as the vertical left black line in [Figure 2](#).

(3) An **open-p-PDBM** satisfying condition (5b) is a clock region which can contain several clock valuations satisfying some possibly parametric constraints (as the open region of [\[AD94\]](#)). An **open-p-PDBM** satisfying condition (5b) is characterized by at least one clock y s.t. $D_{y,0} = (1, <)$ and for all x s.t. $D_{0,x} = (0, \triangleleft_{ox})$, then we have $\triangleleft_{ox} = <$ and can be seen as a subregion of an open region of [\[AD94, fig. 9 example 4.4\]](#). After some time elapsing, and *before* any clock valuation reaches the next integer $k+1$ —therefore the next **open-p-PDBM** satisfying condition 5a—, the above **p-PDBM** (E, D) becomes

$$(E, D) = \left(\binom{k}{k}, \begin{pmatrix} \mathbf{0} & \mathbf{x} & \mathbf{y} \\ \mathbf{0} & (0, \leq) & (0, <) & (-\text{frac}(p_1), <) \\ \mathbf{x} & (1 - \text{frac}(p_1), <) & (0, \leq) & (-\text{frac}(p_1), \leq) \\ \mathbf{y} & (1, <) & (\text{frac}(p_1), \leq) & (0, \leq) \end{pmatrix} \right)$$

We represent it once valuated with v as the red line in [Figure 2](#). The open region of [\[AD94, fig. 9 example 4.4\]](#) can be represented as

$$\left(\binom{k}{k}, \begin{pmatrix} \mathbf{0} & \mathbf{x} & \mathbf{y} \\ \mathbf{0} & (0, \leq) & (0, <) & (0, <) \\ \mathbf{x} & (1, <) & (0, \leq) & (0, <) \\ \mathbf{y} & (1, <) & (1, <) & (0, \leq) \end{pmatrix} \right)$$

and is depicted as the top left black triangle in [Figure 2](#).

Remark that sets of the form $\{\text{frac}(w(x)) \mid 0 \leq \text{frac}(w(x)) \leq 1\}$ are in contradiction with [Definition 7 \(3\)](#) and therefore cannot be part of a **p-PDBM**, as in the regions of [\[AD94\]](#). Basically, only the first **p-PDBM** after a (necessarily total) parametric clock update will be a **point-p-PDBM**; any following **p-PDBM** will be a **open-p-PDBM** satisfying condition 5a or 5b until the next (total) parametric update.

The differentiation made in the previous paragraph between **open-p-PDBMs** satisfying condition 5a and 5b is intended to give an intuition to the reader about the inclusion of **p-PDBMs** into [\[AD94\]](#) clock regions. Technical details are given in the following [Section 4](#). In the following subsections [Sections 4.1 to 4.5](#), we are going to define operations on **p-PDBMs** (*i. e.*, update of clocks, time elapsing and guards satisfaction), and will show that the set of **p-PDBMs** is stable under these operations.

4 Operations on p-PDBMs

4.1 Non-parametric update

To apply a non-parametric update on a **p-PDBM**, following classical algorithms for DBMs [\[BY03\]](#), we define an update operator, given in [Algorithm 1](#). Given a **p-PDBM** (E, D) and u_{np} a non-parametric update function that updates a clock x to $k \in \mathbb{N}$, $\text{update}((E, D), u_{np})$ defines a new **p-PDBM** by

1. updating E_x to k ;
2. setting the fractional part of x to 0: $D_{x,0} := D_{0,x} := (0, \leq)$;

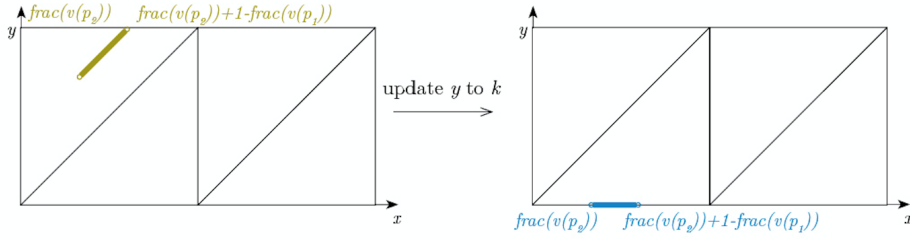
3. updating the new difference between fractional parts with all other clocks i , which is the range of values i can currently take: $D_{x,i} := D_{0,i}$ and $D_{i,x} := D_{i,0}$.

Example 1. Here is an open-p-PDBM satisfying condition 5b on the left of the figure below. Formally, it is written:

$$(E, D) = \left(\begin{pmatrix} k \\ k \end{pmatrix}, \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{x} & \mathbf{y} \\ \mathbf{x} & \begin{pmatrix} (0, \leq) \\ \text{frac}(p_2) + 1 - \text{frac}(p_1), < \end{pmatrix} & \begin{pmatrix} (-\text{frac}(p_2), <) \\ (0, \leq) \end{pmatrix} & \begin{pmatrix} (-\text{frac}(p_1), <) \\ (-\text{frac}(p_1) + \text{frac}(p_2), \leq) \end{pmatrix} \\ \mathbf{y} & (1, <) & (\text{frac}(p_1) - \text{frac}(p_2), \leq) & (0, \leq) \end{pmatrix} \right)$$

After an update of y to k prior to reaching $k + 1$, here is the open-p-PDBM satisfying condition 5a obtained, on the right of the figure below. Formally, it is written:

$$(E, D) = \left(\begin{pmatrix} k \\ k \end{pmatrix}, \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{x} & \mathbf{y} \\ \mathbf{x} & \begin{pmatrix} (0, \leq) \\ \text{frac}(p_2) + 1 - \text{frac}(p_1), < \end{pmatrix} & \begin{pmatrix} (-\text{frac}(p_2), <) \\ (0, \leq) \end{pmatrix} & \begin{pmatrix} (0, \leq) \\ \text{frac}(p_2) + 1 - \text{frac}(p_1), < \end{pmatrix} \\ \mathbf{y} & (0, \leq) & (-\text{frac}(p_2), <) & (0, \leq) \end{pmatrix} \right)$$



Algorithm 1: $\text{update}(D, u_{np})$: for all clock x_i where u_{np} is defined, update $\text{frac}(x_i) := 0$

```

1 foreach  $x_i$  where  $u_{np}(x_i)$  is defined do
2    $D_{i,0} := D_{0,i} = (0, \leq)$ 
3   for  $j$  from 1 to  $H$  do
4      $D_{i,j} = D_{0,j}$ 
5      $D_{j,i} = D_{j,0}$ 
6   end
7 end

```

Definition 9 (update of a p-PDBM). Let u_{np} be a non-parametric update function. Given $(E, D) \in p\text{-PDBM}(R_p)$, we define the update of (E, D) , denoted by $(E', D') = \text{update}((E, D), u_{np})$ as: D' is the result of Algorithm 1 and for each clock x if $u_{np}(x)$ is defined $E'_x := u_{np}(x)$, $E'_x := E_x$ otherwise.

Lemma 4 (stability under update). *Let R_p be a parameter region and $(E, D) \in p\text{-PDBM}(R_p)$. Let u_{np} be a non-trivial non-parametric update. Then $\text{update}((E, D), u_{np}) \in p\text{-PDBM}_{\blacksquare}(R_p)$.*

Proof. Intuitively, we update in (E, D) the lower and upper bounds of some clocks to $(0, \leq)$ and the difference between two clocks $D_{i,j}$ to $D_{0,j}$ if x_i is updated: that is, the new difference between two clocks if one has been updated is just the lower/upper bound of the one that is not updated. This allows us to conserve the canonical form as we only “moved” some cells in D that already verified the canonical form. Therefore $\text{update}((E, D), u_{np})$ is a p-PDBM. See [Appendix D](#) for details.

Applying a non-parametric *update* on any point-p-PDBM transforms it into an open-p-PDBM, and open-p-PDBMs are stable under *update*. It can seem a paradox that the (non-parametric) update of a point-p-PDBM becomes an open-p-PDBM; in fact, it remains geometrically speaking a point, *i. e.*, a singleton containing one clock valuation. Recall that our open-p-PDBMs include p-PDBMs geometrically corresponding to a point for each valuation. In contrast, point-p-PDBMs are also punctual (for each valuation), but are fully parametric.

The following lemma states that the update operator behaves as expected.

Lemma 5 (semantics of update on $p\text{-PDBM}(R_p)$). *Let R_p be a parameter region and $(E, D) \in p\text{-PDBM}(R_p)$. Let $v \in R_p$. Let u_{np} be a non-parametric update. For all w , $[w]_{u_{np}} \in \text{update}((E, v(D)), u_{np})$ iff $w \in (E, v(D))$.*

Proof. The technical part is (\Rightarrow) . The idea is to prove that, given $w' \in \text{update}((E, v(D)), u_{np})$ there is a non-empty set of clock valuations w s.t. $w' = [w]_{u_{np}}$ that is precisely defined by the constraints in $(E, v(D))$. See [Appendix E](#) for details.

4.2 Parametric update

Given $(E, D) \in p\text{-PDBM}(R_p)$ we write $\overline{\text{update}}((E, D), u)$ to denote the update of (E, D) by u , when u is a total parametric update function, *i. e.*, updating the set of clocks exclusively to parameters. We therefore obtain a point-p-PDBM, containing the parametric set of constraints defining a unique clock valuation. The semantics is straightforward. Recall that a total update function which is not fully parametric (*i. e.*, an update of some clocks to parameters and some others to constants) can be encoded as a total parametric update immediately followed by a partial non-parametric update function.

4.3 Time elapsing

Given a parameter region R_p , recall that constraints satisfied by parameters are known, and we can order elements of $\mathcal{P}\mathcal{L}\mathcal{T}$. Thanks to this order, within a p-PDBM (E, D) the clocks with the (possibly parametric) largest fractional part *i. e.*, the clocks that have a larger fractional part than any other clock, can

always be identified by their bounds in D . For a \mathbf{p} -PDBM (E, D) , we define the set of clocks with the largest fractional part (LFP) as $\text{LFP}_{R_p}(D) = \{x \in [1, H] \mid 0 \leq D_{x,i} \text{ is valid for } R_p, \text{ for all } 0 \leq i \leq H\}$. Clocks belonging to LFP are the first to reach the upper bound 1 by letting time elapse.

Definition 10 (clocks with the largest fractional part in a \mathbf{p} -PDBM). *Let R_p be a parameter region and $(E, D) \in \mathbf{p}\text{-PDBM}(R_p)$. A clock with the (possibly parametric) largest fractional part is a clock x s.t. for all $0 \leq i \leq H$, $(0, \leq) \leq D_{x,i}$ is valid for R_p .*

There is at least one clock with the (possibly parametric) largest fractional part:

Lemma 6 (existence of a clock with the largest fractional part). *Let R_p be a parameter region and $(E, D) \in \mathbf{p}\text{-PDBM}(R_p)$. There is at least one clock x s.t. for all $0 \leq i \leq H$, $(0, \leq) \leq D_{x,i}$ is valid for R_p .*

Proof. See [Appendix F](#).

Note that several clocks may have the largest fractional parts (up to some syntactic replacements⁷, in that case they satisfy the same constraints in (E, D)).

Suppose, given a parameter valuation $v \in R_p$, we have two different syntactic expressions that are equal once valuated (such as, given p , $v(p) = 1 - v(p)$ and by [Definition 5](#) if it is for v , it is for any $v' \in R_p$). Then we choose one and replace the second in every constraint where it appears (*e. g.*, replace $1 - v(p)$ by $v(p)$ everywhere). For a \mathbf{p} -PDBM (E, D) , we define the set of clocks with the largest fractional part (LFP) as $\text{LFP}_{R_p}(D) = \{x \in \mathbb{X} \mid 0 \leq D_{x,i} \text{ is valid for } R_p, \text{ for all } 0 \leq i \leq H\}$.

As we are able, thanks to the parameter regions, to order our parameter valuations (*i. e.*, whether one is greater or less than another one), we can define LFP from the constraints defined in the point- \mathbf{p} -PDBM. We will define and apply successively two time-elapsing algorithms: the first one starts from a point- \mathbf{p} -PDBM or an open- \mathbf{p} -PDBM respecting condition [Definition 7 \(5a\)](#). We will prove that we obtain an open- \mathbf{p} -PDBM respecting condition [Definition 7 \(5b\)](#). The second one, starts from an open- \mathbf{p} -PDBM respecting condition [Definition 7 \(5b\)](#) and will define the set of constraints defining the possible clocks valuations exactly when any clock of LFP has reached its upper bound 1. We will prove that we obtain an open- \mathbf{p} -PDBM respecting condition [Definition 7 \(5a\)](#). As we will obtain at each iteration of the algorithm an open- \mathbf{p} -PDBM respecting either condition [Definition 7 \(5a\)](#) or (5b), this will prove we have a stable set of open- \mathbf{p} -PDBMs. Now we explain our algorithms more precisely.

Clocks belonging to LFP are the first to reach the upper bound 1 by letting time elapse. Since LFP can contain multiple clocks and they have the same fractional part, we can consider any $x \in \text{LFP}$.

⁷ Let $v \in R_p$ and suppose, we have two different syntactic expressions, such as $p, 1 - p$ that are equal once valuated *i. e.*, $v(p) = 1 - v(p)$. From [Definition 5](#) remark that if it is for v , it is for any $v' \in R_p$. We choose one *e. g.*, $1 - v(p)$ and replace the second, $v(p)$, everywhere it appears.

Let $(E, D) \in p\text{-PDBM}(R_p)$ and $x \in \text{LFP}_{R_p}(D)$. To formalize time elapsing until the largest fractional part $\text{frac}(x)$ reaches 1, we define a time elapsing operator that will decline in two variants depending on the input: **open-p-PDBM** (Definition 7) satisfying condition (5a) and **point-p-PDBM** (Definition 8) or **open-p-PDBM** (Definition 7) satisfying condition (5b)).

Given an **open-p-PDBM** satisfying condition 5a or a **point-p-PDBM** (E, D) with $E_x = k$, $TE((E, D))$ described in Algorithm 2 and named $TE_{<}$, defines a new **open-p-PDBM** satisfying condition 5b by

1. setting $D_{x,0} := (1, <)$ as x is the first one that will reach $k + 1$;
2. updating the upper bound of all other clocks i , which has increased: $D_{i,0} := D_{i,x} + (1, <)$;
3. updating all lower bounds as they have to leave the *border*: $D_{0,i} := D_{0,i} + (0, <)$ (x included).

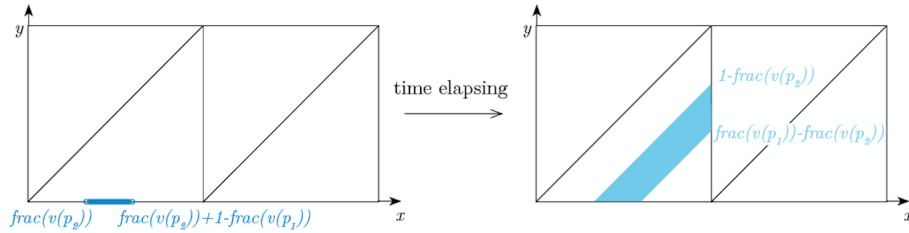
This gives the range of possible clock valuations *before* $\text{frac}(x)$ reaches 1. Intuitively it represents the transformation from an open line segment or the corner-point region of [AD94] into an open region of [AD94].

Example 2. Here is an **open-p-PDBM** satisfying condition 5a, on the left of the figure below. Formally, it is written:

$$(E, D) = \left(\begin{pmatrix} k \\ k \end{pmatrix}, \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{x} & \mathbf{y} \\ \mathbf{0} & (0, \leq) & (-\text{frac}(p_2), <) & (0, \leq) \\ \mathbf{x} & (\text{frac}(p_2) + 1 - \text{frac}(p_1), <) & (0, \leq) & (\text{frac}(p_2) + 1 - \text{frac}(p_1), <) \\ \mathbf{y} & (0, \leq) & (-\text{frac}(p_2), <) & (0, \leq) \end{pmatrix} \right)$$

Time elapsing before $x \in \text{LFP}$ reaches the next integer gives the following **open-p-PDBM** satisfying condition 5b, on the right of the figure below. Formally, it is written:

$$(E, D) = \left(\begin{pmatrix} k \\ k \end{pmatrix}, \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{x} & \mathbf{y} \\ \mathbf{0} & (0, \leq) & (-\text{frac}(p_2), <) & (0, <) \\ \mathbf{x} & (1, <) & (0, \leq) & (\text{frac}(p_2) + 1 - \text{frac}(p_1), <) \\ \mathbf{y} & (1 - \text{frac}(p_2), <) & (-\text{frac}(p_2), <) & (0, \leq) \end{pmatrix} \right)$$



Algorithm 2: $TE_{<}((E, D))$: set upper bound of all $frac(x_i) \in \text{LFP}_{R_p}(D)$ to 1

```

1 pick  $x_i \in \text{LFP}_{R_p}(D)$ 
2 for  $j$  from 1 to  $H$  do
3   if  $j \in \text{LFP}_{R_p}(D)$  then
4      $D_{j,0} := (1, <)$ 
5   else
6      $D_{j,0} := D_{j,i} + (1, <)$ 
7   end
8    $D_{0,j} := D_{0,j} + (0, <)$ 
9 end

```

$TE_{<}$ is applied to **point-p-PDBMs** and **open-p-PDBMs** respecting condition 5a; it sets $D_{x,0} := (1, <)$ and $D_{0,x} := D_{0,x} + (0, <)$ for all $x \in \text{LFP}_{R_p}(D)$. Then, for all clocks $1 \leq j \leq H$ not in LFP sets $D_{j,0} := D_{j,i} + (1, <)$ and $D_{0,j} := D_{0,j} + (0, <)$. This gives the range of possible clock valuations before $frac(x_i)$ reaches 1. The obtained result is denoted by $TE_{<}((E, D))$, and it leaves E unchanged.

The time elapsing operator also operates the transformation from an open region of [AD94] to the upper open line segment or the corner-point region of [AD94], given in Algorithm 3 as $TE_{=}$. Given an **open-p-PDBM** satisfying condition 5b (E, D) where $E_x = k$, $TE_{=}(E, D)$ defines a new **open-p-PDBM** satisfying condition 5a by

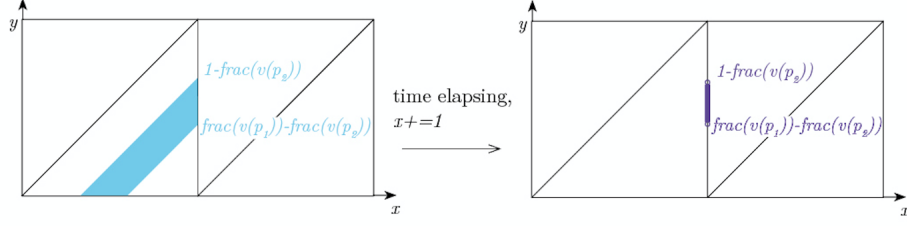
1. setting $D_{x,0} := D_{0,x} := (0, \leq)$ (intuitively both became $(1, \leq)$) and $E_x = k + 1$ (if $E_x \leq K + 1$), as x is now in the upper *border*;
2. updating the upper/lower bounds of all other clocks i : $D_{i,0} := D_{i,x} + (1, \leq)$ and $D_{0,i} := D_{x,i} + (-1, \leq)$;
3. updating the new difference between fractional parts with all other clocks i , which is the range of values i can currently take (as in the update operator): $D_{x,i} := D_{0,i}$ and $D_{i,x} := D_{i,0}$.

Example 3. Here is an **open-p-PDBM** satisfying condition 5b, on the left of the figure below. Formally, it is written:

$$(E, D) = \left(\binom{k}{k}, \begin{pmatrix} \mathbf{0} & \mathbf{x} & \mathbf{y} \\ \mathbf{x} & \begin{pmatrix} (0, \leq) & (-frac(p_2), <) \\ (1, <) & (0, \leq) \end{pmatrix} & \begin{pmatrix} (0, <) \\ frac(p_2) + 1 - frac(p_1), < \end{pmatrix} \\ \mathbf{y} & \begin{pmatrix} (1 - frac(p_2), <) & (-frac(p_2), <) \end{pmatrix} & \begin{pmatrix} (0, \leq) \end{pmatrix} \end{pmatrix} \right)$$

When $x \in \text{LFP}$ reaches $k + 1$, the **open-p-PDBM** satisfying condition 5a obtained is given on the right of the figure below. Formally, it is written:

$$(E, D) = \left(\binom{k+1}{k}, \begin{pmatrix} \mathbf{0} & \mathbf{x} & \mathbf{y} \\ \mathbf{x} & \begin{pmatrix} (0, \leq) & (0, \leq) \\ (0, \leq) & (0, \leq) \end{pmatrix} & \begin{pmatrix} (-frac(p_1) + frac(p_2), <) \\ (-frac(p_1) + frac(p_2), <) \end{pmatrix} \\ \mathbf{y} & \begin{pmatrix} (1 - frac(p_2), <) & (1 - frac(p_2), <) \end{pmatrix} & \begin{pmatrix} (0, \leq) \end{pmatrix} \end{pmatrix} \right)$$



Algorithm 3: $TE_{=}(E, D)$: set upper and lower bound of all $frac(x_i) \in \text{LFP}_{R_p}(D)$ to 1

```

1  pick  $x_i \in \text{LFP}_{R_p}(D)$ 
2  for  $j$  from 1 to  $H$  do
3      if  $j \in \text{LFP}_{R_p}(D)$  then
4           $D_{j,0} := (0, \leq)$ 
5           $D_{0,j} := (0, \leq)$ 
6           $E_j := E_j + 1$ 
7      else
8           $D_{j,0} := D_{j,i} + (1, \leq)$ 
9           $D_{0,j} := D_{i,j} + (-1, \leq)$ 
10     end
11 end
12 for  $j$  from 1 to  $H$  do
13      $D_{j,i} := D_{j,0}$ 
14      $D_{i,j} := D_{0,j}$ 
15 end

```

$TE_{=}$ is applied to open-p-PDBMs respecting condition 5b and sets $D_{x,0} := (0, \leq)$ and $D_{0,x} := (0, \leq)$ for all $x \in \text{LFP}_{R_p}(D)$. Then, for all clocks $x_j \in H \setminus \text{LFP}_{R_p}(D)$ sets $D_{0,j} := (-1, \leq) + D_{i,j}$ and $D_{j,0} := D_{j,i} + (1, \leq)$; it gives the range of clock valuations when $frac(x)$ reaches 1, and increments E_x , for $x \in \text{LFP}_{R_p}(D)$ if E_x is not greater than $K+1$. It then sets, regardless of whether $x_j \in \text{LFP}_{R_p}(D)$ $D_{i,j} := D_{0,j}$ and $D_{j,i} := D_{j,0}$. Finally, for $x \in \text{LFP}_{R_p}(D)$ it sets $E_x := E_x + 1$. The obtained result is denoted by $TE_{=}(E, D)$.

Definition 11 (time elapsing in a p-PDBM). Let R_p be a parameter region and $(E, D) \in p\text{-PDBM}_{\odot}(R_p) \cup p\text{-PDBM}_{\blacksquare}(R_p)$. We define $(E', D') = TE((E, D))$ as applying either $TE_{<}$ if (E, D) respects condition 5a or $(E, D) \in p\text{-PDBM}_{\odot}(R_p)$, or $TE_{=}$ if (E, D) respects condition 5b.

Lemma 7 (stability under time elapsing). Let R_p be a parameter region. Let $(E, D) \in p\text{-PDBM}(R_p)$. Then $TE((E, D)) \in p\text{-PDBM}(R_p)$.

Proof. Although we perform some additions such as $D_{j,i} + (1, <)$, we do not create new expressions that are not in \mathcal{PLT} . In fact, this addition is performed on a negative term (e. g., $frac(p) - 1$), as x_i is a clock with the largest fractional

part and adding 1 transforms it into another term of \mathcal{PCT} . The intuition is similar when performing additions such as $D_{i,j} + (-1, \leq)$: as x_i is a clock with the largest fractional part, $d_{i,j}$ is a positive term. The canonical form is also preserved by the last setting operations of the algorithm, as in the update operator. Therefore $TE((E, D))$ is a \mathbf{p} -PDBM. See [Appendix G](#) for details.

Note that, by [Lemma 7](#) (E', D') is a \mathbf{p} -PDBM. $\mathbf{open-p}$ -PDBMs are stable under $TE_{<}$ and $TE_{=}$, switching the condition they respect ([5a](#), [5b](#)). Applying $TE_{<}$ on a $\mathbf{point-p}$ -PDBM transforms it into an $\mathbf{open-p}$ -PDBM.

The following proposition proves that time elapsing behaves as we expect.

Proposition 1 (semantics of \mathbf{p} -PDBM under TE). *Let R_p be a parameter region and $(E, D) \in \mathbf{p-PDBM}(R_p)$. Let $v \in R_p$. There exists $w' \in TE((E, v(D)))$ iff there exist $w \in (E, v(D))$ and a delay δ s.t. $w' = w + \delta$.*

Proof. This proof is quite technical. Intuitively, we bound the difference of each upper bound $v(d_{i,0})$ and $w(x_i)$ and each lower bound $v(d_{0,i})$ and $w(x_i)$. This allows us to take a delay δ inside these bounds that allows us to reach the next \mathbf{p} -PDBM. See [Appendix H](#) for details.

Running example: [Figure 3](#) represents graphically different \mathbf{p} -PDBMs obtained after an update $u(x) = v(p_2)$ and $u(y) = v(p_1)$ ([figure 1](#)). Time elapsing before $y \in \mathbf{LFP}$ reaches the next integer gives the $\mathbf{open-p}$ -PDBM satisfying condition [5b](#) ([figure 2](#))

$$(E, D) = \left(\binom{k}{k}, \begin{pmatrix} \mathbf{0} & \mathbf{x} & \mathbf{y} \\ \mathbf{0} & (-\text{frac}(p_2), <) & (-\text{frac}(p_1), <) \\ \mathbf{x} & (\text{frac}(p_2) + 1 - \text{frac}(p_1), <) & (-\text{frac}(p_1) + \text{frac}(p_2), \leq) \\ \mathbf{y} & (1, <) & (0, \leq) \end{pmatrix} \right)$$

After an update of y to k prior to reaching $k+1$, the $\mathbf{open-p}$ -PDBM satisfying condition [5a](#) obtained is ([figure 3](#))

$$(E, D) = \left(\binom{k}{k}, \begin{pmatrix} \mathbf{0} & \mathbf{x} & \mathbf{y} \\ \mathbf{0} & (-\text{frac}(p_2), <) & (0, \leq) \\ \mathbf{x} & (\text{frac}(p_2) + 1 - \text{frac}(p_1), <) & (0, \leq) \\ \mathbf{y} & (0, \leq) & (\text{frac}(p_2) + 1 - \text{frac}(p_1), <) \end{pmatrix} \right)$$

Time elapsing before $x \in \mathbf{LFP}$ reaches the next integer gives the $\mathbf{open-p}$ -PDBM satisfying condition [5b](#) ([figure 4](#))

$$(E, D) = \left(\binom{k}{k}, \begin{pmatrix} \mathbf{0} & \mathbf{x} & \mathbf{y} \\ \mathbf{0} & (-\text{frac}(p_2), <) & (0, <) \\ \mathbf{x} & (1, <) & (0, \leq) \\ \mathbf{y} & (1 - \text{frac}(p_2), <) & (\text{frac}(p_2) + 1 - \text{frac}(p_1), <) \end{pmatrix} \right)$$

When $x \in \mathbf{LFP}$ reaches $k+1$, the $\mathbf{open-p}$ -PDBM satisfying condition [5a](#) obtained is ([figure 5](#))

$$(E, D) = \left(\binom{k+1}{k}, \begin{pmatrix} \mathbf{0} & \mathbf{x} & \mathbf{y} \\ \mathbf{0} & (0, \leq) & (-\text{frac}(p_1) + \text{frac}(p_2), <) \\ \mathbf{x} & (0, \leq) & (-\text{frac}(p_1) + \text{frac}(p_2), <) \\ \mathbf{y} & (1 - \text{frac}(p_2), <) & (1 - \text{frac}(p_2), <) \end{pmatrix} \right)$$

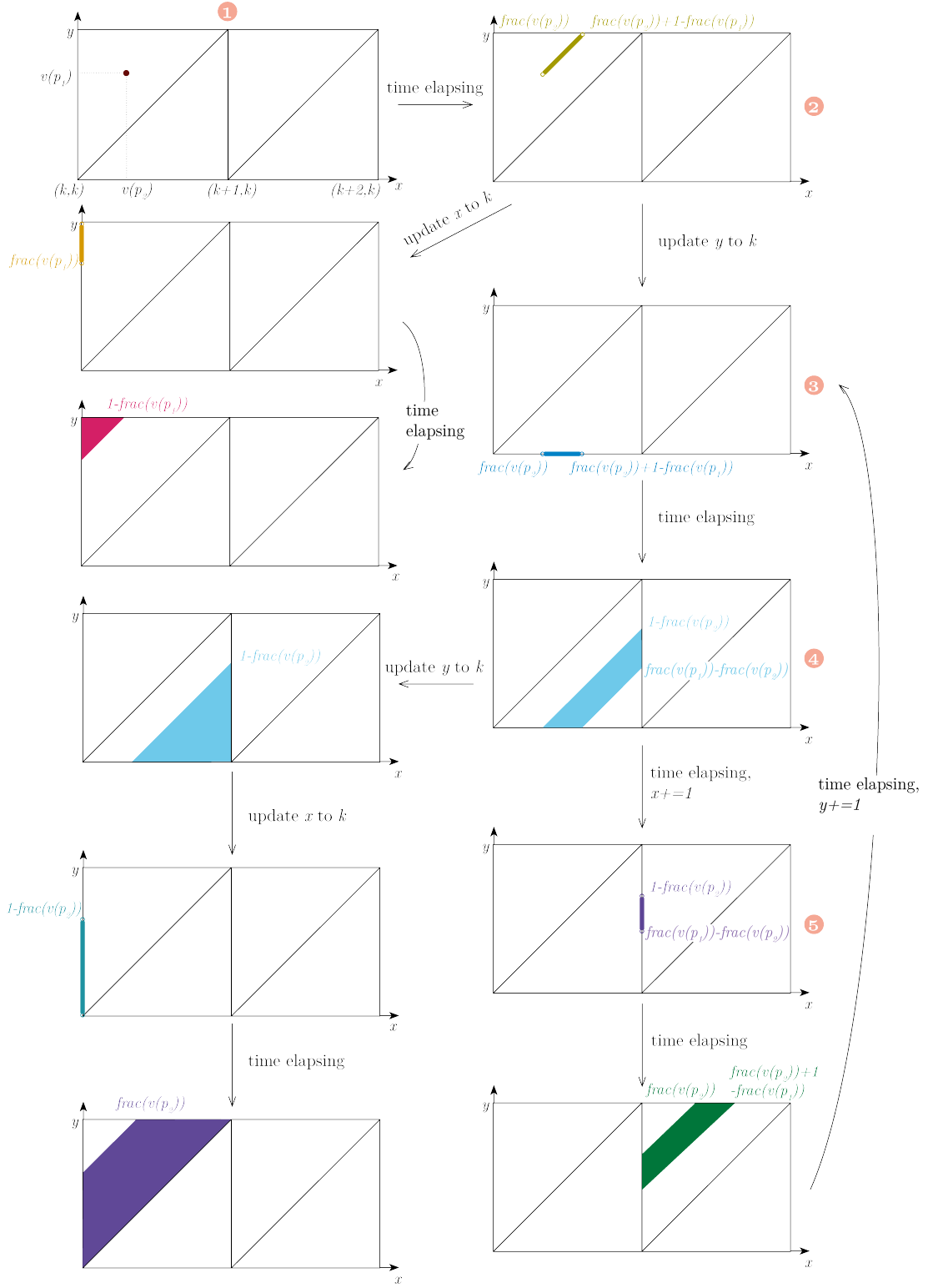


Fig. 3: Representation of p-PDBMs in two dimensions with two clocks x, y , two parameters p_1, p_2 and v s.t. $\lfloor v(p_1) \rfloor = \lfloor v(p_2) \rfloor$ and $\text{frac}(v(p_1)) > \text{frac}(v(p_2))$.

4.4 Non-parametric guard

From [AD94, Section 4.2] we have that either every clock valuation of a clock region satisfies a guard, or none of them does. Note that a \mathbf{p} -PDBM for R_p is contained into a clock region of Definition 3, therefore we have that if $w \in (E, v(D))$ satisfies a non-parametric guard g , then for all $w' \in (E, v(D))$ we also have w' satisfies g .

Let $v \in R_p$. We define $v \in \text{guard}_\forall(g, E, D)$ iff for all $w \in (E, v(D))$, $w \models g$. As any two $v, v' \in R_p$ satisfy the same constraints, the following lemma is straightforward

Lemma 8. *Let (E, D) be a \mathbf{p} -PDBM for R_p and $v \in R_p$. Let g be a non-parametric guard. If $v \in \text{guard}_\forall(g, E, D)$, then for all $v' \in R_p$, $v' \in \text{guard}_\forall(g, E, D)$.*

Proof. See Appendix I.

4.5 Parametric guard

As for the previous result, using a projection on parameters *i. e.*, eliminating clocks, does not create new constraints on parameters that are not already in a parameter region R_p . Indeed, a parametric guard g only adds new constraints of the form $x \bowtie p$ which gives, when eliminating clocks in both a \mathbf{p} -PDBM (E, D) and a parametric guard, again a comparison between elements of \mathcal{PCT} . Therefore, these new constraints already belong to \mathcal{PCT} and we can decide whether the set of clock valuations satisfying this set of constraints is non-empty *i. e.*, given $v \in R_p$, $v(g)$ is satisfied by some clock valuation $w \in (E, v(D))$. This is a key point in the overall process of proving the decidability of our R-U2P-PTAs.

Note that there will also be additional constraints involving clocks (with other clocks, constants or parameters), but they will not be relevant as we immediately update all clocks, therefore replacing these constraints with new constraints encoding the clock updates.

Let $v \in R_p$. We define $v \in \text{p-guard}_\exists(g, E, D)$ iff there is a $w \in (E, v(D))$ s.t. $w \models v(g)$.⁸ Again, as any two $v, v' \in R_p$ satisfy the same constraints, the following lemma is straightforward

Lemma 9. *Let (E, D) be a \mathbf{p} -PDBM for R_p and $v \in R_p$. Let g be a parametric guard. If $v \in \text{p-guard}_\exists(g, E, D)$, then for all $v' \in R_p$, $v' \in \text{p-guard}_\exists(g, E, D)$.*

Proof. See Appendix J.

Now that we have defined useful operations on \mathbf{p} -PDBMs, we are going, given a parameter region R_p , to construct a finite region automaton in which for any run, there is an equivalent concrete run in the R-U2P-PTA.

⁸ Remark that here is why our construction works for EF-emptiness, but cannot be used for, *e. g.*, AF-emptiness (“is there a parameter valuation such that all runs reach a goal location l ”): unlike $\text{guard}_\forall(g, E, D)$, not all clock valuations in a \mathbf{p} -PDBM $(E, v(D))$ can satisfy a parametric guard if $v \in \text{p-guard}_\exists(g, E, D)$.

5 Parametric region automaton

Let $(E, D) \in p\text{-PDBM}(R_p)$, we say $(E', D') \in \text{Succ}((E, D)) \Leftrightarrow \exists i \geq 0$ s.t. $(E', v(D')) = TE^i((E, D))$. In other words, (E', D') is obtained after applying $TE((E, D))$ a finite number of times. $\text{Succ}((E, D))$ is also called the *time successors* of (E, D) .

In order to finitely simulate an R-U2P-PTA, we create a parametric region automaton.

Definition 12 (Parametric region automaton). Let R_p be a parameter region. For an R-U2P-PTA $\mathcal{A} = (\Sigma, L, l_0, \mathbb{X}, \mathbb{P}, \zeta)$, given (E_0, D_0) the initial $p\text{-PDBM}$ where all clocks are 0, the parametric region automaton $\mathcal{R}(\mathcal{A})$ over R_p is the tuple $(L', \Sigma, L'_0, \zeta')$ where:

1. $L' = L \times p\text{-PDBM}(R_p)$
2. $L'_0 = (l_0, (E_0, D_0))$
3. $\zeta' = \{((l, (E, D)), a, (l', (E', D'))) \in L' \times \Sigma \times L' \mid \text{either } \exists e = \langle l, g, a, u_{np}, l' \rangle \in \zeta, g \text{ is a non-parametric guard, } \exists (E'', D'') \in \text{Succ}((E, D)), R_p \subseteq \text{guard}_\forall(g, (E'', D'')) \text{ and } (E', D') = \text{update}(E'', D'', u_{np}) \text{ is an open-}p\text{-PDBM, or } \exists e = \langle l, g, a, u, l' \rangle \in \zeta, g \text{ is a parametric guard, } \exists (E'', D'') \in \text{Succ}((E, D)), R_p \subseteq p\text{-guard}_\exists(g, (E'', D'')) \text{ and } (E', D') = \text{update}(E'', D'', u) \text{ is a point-}p\text{-PDBM.}\}$

Let R_p be a parameter region, \mathcal{A} be an R-U2P-PTA and $\mathcal{R}(\mathcal{A}) = (L', \Sigma, L'_0, \zeta')$ its parametric region automaton over R_p . A run in $\mathcal{R}(\mathcal{A})$ is an untimed sequence

$\sigma : (l_0, (E_0, D_0))e_0(l_1, (E_1, D_1))e_1 \cdots (l_i, (E_i, D_i))e_i(l_{i+1}, (E_{i+1}, D_{i+1}))e_{i+1} \cdots$ such that for all i we have $((l_i, (E_i, D_i)), a_i, (l_{i+1}, (E_{i+1}, D_{i+1}))) \in \zeta'$, which we also write $(l_i, (E_i, D_i)) \xrightarrow{e_i} (l_{i+1}, (E_{i+1}, D_{i+1}))$. Note that we label our transitions with the edges of the R-U2P-PTA.

6 Decidability of EF-emptiness and synthesis

Using our construction of the parametric region automaton $\mathcal{R}(\mathcal{A})$ for a given R-U2P-PTA \mathcal{A} , we state the next proposition.

Proposition 2. Let R_p be a parameter region. Let \mathcal{A} be an R-U2P-PTA and $\mathcal{R}(\mathcal{A})$ its parametric region automaton over R_p . There is a run $\sigma : (l_0, (E_0, D_0)) \xrightarrow{e_0} (l_1, (E_1, D_1)) \xrightarrow{e_1} \cdots (l_{f-1}, (E_{f-1}, D_{f-1})) \xrightarrow{e_{f-1}} (l_f, (E_f, D_f))$ in $\mathcal{R}(\mathcal{A})$ iff for all $v \in R_p$ there is a run $\rho : (l_0, w_0) \xrightarrow{e_0} (l_1, w_1) \xrightarrow{e_1} \cdots (l_{f-1}, w_{f-1}) \xrightarrow{e_{f-1}} (l_f, w_f)$ in $v(\mathcal{A})$ s.t. for all $0 \leq i \leq f$, $w_i \in (E_i, v(D_i))$.

Proof. We prove this result by induction on the length of the run. It is quite direct as we construct runs without parametric guards. See [Appendix K](#) for details.

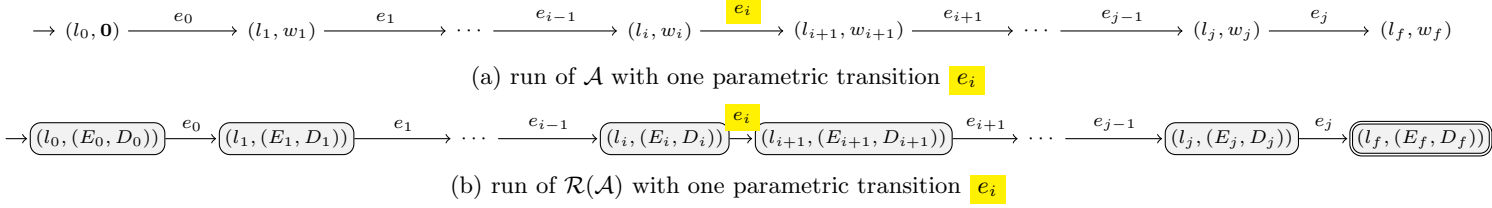


Fig. 4: A run in an R-U2P-PTA \mathcal{A} (above) and its equivalent run in $\mathcal{R}(\mathcal{A})$ (below)

Example 4. Consider Figure 4. Let \mathcal{A} be an R-U2P-PTA, R_p a parameter region and $v \in R_p$. Suppose there is a run in \mathcal{A} , starting from the initial location $(l_0, \mathbf{0})$ reaching a goal location (l_f, w_f) . Along this run, all edges are non-parametric transitions but $e_i = \langle l_i, g, a_i, u, l_{i+1} \rangle$. That is, u is a total parametric update, and g is a possibly parametric guard.

The first part of this run, from $(l_0, \mathbf{0})$ to (l_i, w_i) starts from $(l_0, (E_0, D_0))$ where (E_0, D_0) is the \mathbf{p} -PDBM of the initial clock region $\{\mathbf{0}\}$, and ends in $(l_i, (E_i, D_i))$. The second part of this run, from (l_{i+1}, w_{i+1}) to (l_f, w_f) starts from $(l_{i+1}, (E_{i+1}, D_{i+1}))$ where (E_{i+1}, D_{i+1}) is a $\mathbf{point-p}$ -PDBM, and can reach $(l_f, (E_f, D_f))$ and further ends in $(l_s, (E_s, D_s))$.

These runs contain only non-parametric transitions, and as there is an edge in \mathcal{A} from (l_i, w_i) to (l_{i+1}, w_{i+1}) , we have to bisimulate this run in $\mathcal{R}(\mathcal{A})$ containing the parametric transition e_i , where $\overline{\text{update}}((E_i, D_i), u)$ gives (E_{i+1}, D_{i+1}) .

From Proposition 2, we deduce that if there is a run reaching a goal location in an instantiated R-U2P-PTA, then for another parameter valuation in the same parameter region there is a run in the instantiated R-U2P-PTA with the same locations and transitions (but possibly different delays), reaching the same location.

Theorem 1. *Let \mathcal{A} be an R-U2P-PTA. Let R_p be a parameter region and $v \in R_p$. If there is a run $\rho = (l_0, w_0) \xrightarrow{e_0} \dots \xrightarrow{e_{i-1}} (l_i, w_i)$ in $v(\mathcal{A})$, then for all $v' \in R_p$ there is a run $\rho' = (l_0, w'_0) \xrightarrow{e_0} \dots \xrightarrow{e_{i-1}} (l_i, w'_i)$ in $v'(\mathcal{A})$ such that for all i , $(w_i, v) \simeq (w'_i, v')$.*

Proof. Let $v \in R_p$ and ρ a run of $v(\mathcal{A})$ reaching (l_i, w_i) . From Proposition 2, there is a run σ in $\mathcal{R}(\mathcal{A})$ s.t. each clock valuation at a location in ρ is in the \mathbf{p} -PDBM at the same location in σ . Still from Proposition 2, for all $v' \in R_p$ there is a run ρ' in $v'(\mathcal{A})$ reaching (l_i, w'_i) s.t. each clock valuation at a location in ρ' is in the \mathbf{p} -PDBM at the same location in σ (note that possibly $v = v'$). Therefore, we have for all $0 \leq j \leq i$ that $(w_j, v) \simeq (w'_j, v')$ and the expected result.

Note that there is a finite number of \mathbf{p} -PDBMs for each parameter region R_p . Let $(E, D) \in \mathbf{p-PDBM}(R_p)$ and consider $\mathcal{P}\mathcal{L}\mathcal{T}$: D is an $(H+1)^2$ matrix made of pairs (d, \triangleleft) where $d \in \mathcal{P}\mathcal{L}\mathcal{T}$ and $\triangleleft \in \{\leq, <\}$. Therefore the number of possible

D is bounded by $(2 \times (2 + 3 \times \binom{M}{2}) + 4 \times M)^{(H+1)^2}$. Moreover the number of E is unbounded, but only a finite subset of all values needs to be explored, *i. e.*, those smaller than $K + 1$: indeed, following classical works on timed automata [AD94; Bou+04], (integer) values exceeding the largest constant used in the guards or the parameter bounds are equivalent.

To test EF-emptiness given an R-U2P-PTA \mathcal{A} and a goal location l , we first enumerate all parameter regions (which are a finite number), and apply for each R_p the following process: we pick $v \in R_p$ (*e. g.*, using a linear programming algorithm [Kar84]). Then, we consider $v(\mathcal{A})$ which is an updatable timed automaton and test the reachability of l in $v(\mathcal{A})$ [Bou+04]. Then EF-emptiness is false if and only if there is v and a run in $v(\mathcal{A})$ reaching l .

Theorem 2. *The EF-emptiness problem is PSPACE-complete for bounded R-U2P-PTAs.*

Proof. Since a TA is a special case of R-U2P-PTA we have the PSPACE-hardness [AD94]. Now, let G be a set of goal locations of \mathcal{A} . We build a non-deterministic Turing machine that:

1. takes \mathcal{A} , G and K as input
2. non-deterministically “guesses” a parameter region R_p
3. takes $v \in R_p$ and writes it to the tape
4. overwrites on the tape each parameter p by $v(p)$, giving the updatable TA $v(\mathcal{A})$
5. solves reachability in $v(\mathcal{A})$ for G
6. accepts iff the result of the previous step is “yes”.

The machine accepts iff there is an integer valuation v bounded by K and a run in $v(\mathcal{A})$ reaching a location $l \in G$.

The size of the input is $|\mathcal{A}| + |G| + |K|$, using $|\cdot|$ to denote the size in bits of the different objects. Moreover, the number of parameter regions is bounded (M is the number of parameters in \mathcal{A}) by $(M! \times 2^M \times \prod_{p \in \mathbb{P}} (2M + 2)) \times (2 \times (2 + M(3\frac{M-1}{2} + 4))^3)$ since they are constructed as the clock regions of [AD94], the second part being the maximal number of constraints in a parameter region. Picking v at step *iii*) uses a PSPACE linear programming algorithm (*e. g.*, [Kar84]). Storing the valuation at step *iv*) uses at most $M \times |K|$ additional bits, which is polynomial w.r.t. the size of the input. Step *v*) also needs polynomial space from [Bou+04]. So globally this non-deterministic machine runs in polynomial space. Finally, by Savitch’s theorem we have PSPACE = NPSPACE [Sav70], and the expected result.

Given a goal location l and a bounded R-U2P-PTA \mathcal{A} , we can exactly synthesize the parameter valuations v s.t. there is a run in $v(\mathcal{A})$ reaching l by enumerating each parameter region (of which there is a finite number) and test if l is reachable for one of its parameter valuations. The result of the synthesis is the union of the parameter regions for which one valuation (and, from our results, all valuations in that region) indeed reaches the goal location in the instantiated TA.

Corollary 1. *Given a bounded R-U2P-PTA \mathcal{A} and a goal location l we can effectively compute the set of parameter valuations v s.t. there is a run in $v(\mathcal{A})$ reaching l .*

Proof. The procedure to obtain synthesis is as follows. We assume an R-U2P-PTA \mathcal{A} and a goal location l .

1. enumerate all parameter regions (of which there is a finite number)
2. for each R_p , pick a parameter valuation we pick $v \in R_p$ (e. g., using a linear programming algorithm [Kar84])
3. test the reachability of l in the updatable timed automaton $v(\mathcal{A})$, which is decidable [Bou+04]
4. if l is reachable in $v(\mathcal{A})$, add R_p to the list of synthesized regions

We finally return the union of all regions R_p that reach l .

The correctness immediately comes from Theorems 1 and 2.

Remark 1. By bounding parameter valuations in guards but not those used in updates, we still have a finite number of parameter regions. Indeed, an integer vector E with components E_x greater than $\lfloor K \rfloor + 1$ is equivalent to an integer vector E' with $E'_x = E_x$ if $E_x < \lfloor K \rfloor + 1$ and $E'_x = \lfloor K \rfloor + 1$ if $E_x \geq \lfloor K \rfloor + 1$. Moreover for all p , we have to replace each parameter valuation v used in an update by $v(p) = v'(p)$ if $v(p) \leq K$ and $v'(p) = K + 1$ if $v(p) > K$.

7 Parametric updates and stopwatches

We define \mathcal{A} an R-U2P-PTA and its clocks as stopwatches [CL00]. Clocks can be stopped at each full update function, and start again at the next full update function.

Definition 13. *A stopwatch reset update-to-parameter PTA (S-R-U2P-PTAs) \mathcal{A} is a tuple $\mathcal{A} = (\Sigma, L, l_0, \mathbb{X}, \mathbb{P}, \zeta, \text{stop})$, where: i) Σ is a finite set of actions, ii) L is a finite set of locations, iii) $l_0 \in L$ is the initial location, iv) \mathbb{X} is a finite set of clocks, v) \mathbb{P} is a finite set of parameters, vi) ζ is a finite set of edges $e = \langle l, g, a, u, l' \rangle$ where $l, l' \in L$ are the source and target locations, g is a parametric guard, $a \in \Sigma$ and $u : \mathbb{X} \rightarrow \mathbb{N} \cup \mathbb{P}$ is a parametric update function. vii) $\text{stop} : L \rightarrow 2^{\mathbb{X}}$ assigns to each location a set of clocks that are stopped at this location. Moreover, u is a total function whenever: i) g is a parametric guard, ii) $u(x) \in \mathbb{P}$ for some $x \in \mathbb{X}$, or iii) $\text{stop}(l) \neq \text{stop}(l')$ for $e = \langle l, g, a, u, l' \rangle$.*

The semantic is defined in a straightforward manner.

Theorem 3. *The EF-emptiness problem is PSPACE-complete for bounded S-R-U2P-PTAs.*

Proof. To prove this result, we first remove stopped clocks from p-PDBMs, and show that we can still reason as in Theorem 2.

Note that after a full update, stopped clocks satisfy a set of constraints defined by a $p\text{-PDBM}_{\odot}(R_p)$. These constraints, defined in Definition 8, are

- (1) For all i , $(-1, <) \leq D_{0,i} \leq (0, \leq)$ and $(0, \leq) \leq D_{i,0} \leq (1, <)$ are valid for R_p ,
- (2) For all i, j, k , $D_{i,j} \leq D_{i,k} + D_{k,j}$ is valid for R_p (canonical form).

Let x be a stopped clock, we remove the column and the row corresponding to x in the **point-p-PDBM**, and we put these constraints aside.

(1) is still satisfied for clocks different from x as nothing changed. As both the column and the row are removed for x , constraints of the form $D_{x,j} \leq D_{x,k} + D_{k,j}$ and $D_{i,x} \leq D_{i,k} + D_{k,x}$ are removed as well. With the same argument, constraints of the form $D_{i,j} \leq D_{i,x} + D_{x,j}$ are also removed. Only constraints of the form $D_{i,j} \leq D_{i,k} + D_{k,j}$ where i, j, k are different from x remain, and therefore (2) is still satisfied. We apply the same reasoning for all stop clocks.

After this modification, we still have a **point-p-PDBM**. Lemmas [Lemmas 4, 5 and 7](#) and [Proposition 1](#) are still applicable in this context.

Remains the comparison with non parametric guards [Lemma 8](#) and parametric guards [Lemma 9](#).

As stated, constraints on stopped clocks are put aside, call this set of constraints C_{stop} ; as these constraints are the result of a full update it means there is one clock valuation that satisfies these constraints (it is not an interval, or a set of clock valuations). This argument is crucial in the following: we do not need to be able to bound difference of a stopped clock and a running clock (*i. e.*, $D_{x,y}$ in the **p-PDBM** for some stopped x and running y , or conversely) to compare separately the **p-PDBM** with a guard, parametric or not.

First, we compare the **p-PDBM** with a possibly parametric guard. Let R_p a parameter region and $v \in R_p$. Separately we compare the guard g with the constraints on stopped clock C_{stop} :

- if g is a non parametric guard, let x be a stopped clock appearing in g .
 - If x has been reset to a constant, we can test whether g is satisfied by testing the conjunction of the constraints $v(C_{\text{stop}})$ projected on x and g projected on x ;
 - if x has been reset to a parameter, we can test the same conjunction as the order between parameters is defined in R_p .
- If g is a parametric guard, let x be a stopped clock appearing in g . In both cases (x reset to a parameter or a constant) we can test whether $v(g)$ is satisfied by testing the conjunction of the constraints $v(C_{\text{stop}})$ projected on x and $v(g)$ projected on x as the order between parameters and constants is defined in R_p .

Therefore, [Lemma 8](#) and parametric guards [Lemma 9](#) are still applicable in this context.

With our modified structure of **p-PDBM** with stopwatches, the core operators for clock updates, time elapsing and comparison with guards are still applicable. We obtain our result by a similar reasoning as [Theorem 2](#).

Similarly to [Corollary 1](#), we obtain the following result:

Corollary 2. *Given a bounded S-R-U2P-PTA \mathcal{A} and a goal location l we can effectively compute the set of parameter valuations v s.t. there is a run in $v(\mathcal{A})$ reaching l .*

8 Case study

We implemented EFsynth for R-U2P-PTAs in IMITATOR, a parametric model checker for (extensions of) PTAs [And+12].

Our class is the first for which synthesis is possible over bounded rational parameters. We believe our formalism is useful to model several categories of case studies, notably distributed systems with a periodic (global) behavior for which the period is unknown: this can be encoded using a parametric guard while resetting all clocks—possibly to other parameters.

Consider the R-U2P-PTA in Figure 1 with six locations, three clocks compared to parameters (x, y, t) , one constant (max) and six parameters $(p, p_1, p_2, v, pv_1, pv_2)$.

We consider the case of a network of peers exchanging transactions grouped by blocks, *e. g.*, a blockchain, using the Proof-of-Work as a mean to validate new blocks to add. In this simplified example, we consider a set of two peers (represented by x, y) which have different computation power (represented by p_1, p_2). Peers write new transactions on the current block ($newTx$). If it is full ($t = p$), both peers try to add a new block ($newBlock$) to write the transaction on it. We update x to p_1 , y to p_2 , and t to 0 as the peers have a different computation power, and they start “mining” the block (find a solution to a computation problem). Either x or y will eventually offer a solution to the problem ($blockSolution_x$ if $x = max$ or $blockSolution_y$ if $y = max$). If y offers a solution, x will check whether the solution is correct: x is updated to pv_1 to represent its rapidity to verify an offer. x can refuse the offer if the verification is too long ($fakeBlock$ if $x > v$) therefore the mining step restarts. x can approve the offer ($okBlock$ if $x \leq v$), y is rewarded and the block is added to the blockchain ($addBlock$).

We are interested in a malicious peer x that wants to avoid y to be rewarded for every new block. Therefore x asks: “*what are the possible computation power configurations and verification rapidity so that y is eventually rewarded*” ($EF(\text{reward}_y)$ -synthesis), considered as a bug state in the automaton.

We run this R-U2P-PTA using IMITATOR [And+12]⁹. We set $max = 30$ units of time and also the upper bound of p and $1 \geq v > 0$ unit of time. IMITATOR computes a disjunction of constraints so that reward_y is unreachable: we keep two relevant ones;

1. $p_1 \geq p_2$: x has strictly more computation power than y in which case x always offers a block solution, or has the same computation power than y in which case the systems blocks. x should invest heavily into hardware to keep its computation power high;

⁹ Experiments were conducted with IMITATOR 2.10.4 “Butter Jellyfish” on a 2.4 GHz Intel Core i5 processor with 2 GiB memory. Computation time is less than 1 second. Sources, binaries, models and results are available at imitator.fr/static/FORTE19/

2. $pv_1 > v$: the malicious peer x is always faster to verify the solution offered by y and refuses it. The blockchain is probably compromised.

Using a parameter valuation respecting one of the previous constraints guarantees that y is never rewarded.

9 Conclusion and perspectives

Our class of R-U2P-PTAs is one of the few subclasses of PTAs (actually even extended with parametric updates) to enjoy decidability of EF-emptiness. In addition, R-U2P-PTAs are the first “subclass” of PTAs to allow exact synthesis of bounded *rational*-valued parameters.

In terms of future works, beyond reachability emptiness, we aim at studying unavailability-emptiness and language preservation emptiness (“given a reference parameter valuation, does there exist another parameter valuation with the same untimed language” [AM15]), as well as their synthesis.

We would also study the possibility to stop and start stopwatches without full updates; it would change the \mathbf{p} -PDBM structure by creating new clock constraints, but seems promising and the reachability emptiness problem might be decidable as well.

Time bounded reachability is also interesting to study: given an R-U2P-PTA \mathcal{A} , a parameter \max and a goal location l , is there a parameter valuation v and a run in $v(\mathcal{A})$ reaching l in less than $v(\max)$ time?

Finally, we would like to investigate whether our parametric updates can be applied to decidable hybrid extensions of TAs [Hen+98; Bri+13].

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Appendix

A Proof of Lemma 1

Lemma 1 (recalled). *Let $d_1, d_2, d_3, d_4 \in \mathcal{PLT}$. Let R_p be a parameter region. If $(d_1, \triangleleft_1) \leq (d_2, \triangleleft_2)$ and $(d_3, \triangleleft_3) \leq (d_4, \triangleleft_4)$ are valid for R_p then $(d_1, \triangleleft_1) + (d_3, \triangleleft_3) \leq (d_2, \triangleleft_2) + (d_4, \triangleleft_4)$ is valid for R_p .*

Proof. Four cases show up: for all $v \in R_p$,

- $v(d_1) < v(d_2)$ and $v(d_3) < v(d_4)$, then clearly $v(d_1) + v(d_3) < v(d_2) + v(d_4)$ and we have our result from Definition 6 (2a).
- $v(d_1) < v(d_2)$ and $v(d_3) \leq v(d_4)$, then $v(d_1) + v(d_3) < v(d_2) + v(d_4)$ and we have our result from Definition 6 (2a).
- $v(d_1) \leq v(d_2)$ and $v(d_3) < v(d_4)$, then $v(d_1) + v(d_3) < v(d_2) + v(d_4)$ and we have our result from Definition 6 (2a).
- $v(d_1) \leq v(d_2)$ and $v(d_3) \leq v(d_4)$, then $v(d_1) + v(d_3) \leq v(d_2) + v(d_4)$ and
 1. if $\triangleleft_1 = \triangleleft_2$ and $\triangleleft_3 = \triangleleft_4$ then $\triangleleft_1 \oplus \triangleleft_3 = \triangleleft_2 \oplus \triangleleft_4$ and we have our result from Definition 6 (2b).
 2. if $\triangleleft_1 = \triangleleft_2$ and $\triangleleft_3 = <$, $\triangleleft_4 = \leq$ then $\triangleleft_1 \oplus \triangleleft_3 = <$ and $\triangleleft_2 \oplus \triangleleft_4$ is either $<$ or \leq and we have our result from Definition 6 (2b).
 3. if $\triangleleft_1 = <$, $\triangleleft_2 = \leq$ and $\triangleleft_3 = \triangleleft_4$ then $\triangleleft_1 \oplus \triangleleft_3 = <$ and $\triangleleft_2 \oplus \triangleleft_4$ is either $<$ or \leq and we have our result from Definition 6 (2b).
 4. if $\triangleleft_1 = \triangleleft_3 = <$ and $\triangleleft_2 = \triangleleft_4 = \leq$ then $\triangleleft_1 \oplus \triangleleft_3 = <$ and $\triangleleft_2 \oplus \triangleleft_4 = \leq$ and we have our result from Definition 6 (2b).

From Definition 6 (2a, 2b) we have that $(d_1, \triangleleft_1) + (d_3, \triangleleft_3) \leq (d_2, \triangleleft_2) + (d_4, \triangleleft_4)$ is valid for R_p .

B Proof of Lemma 2

Lemma 2 (recalled). *Let R_p be a parameter region and (E, D) be a p-PDBM for R_p . For all clocks i, j , $(0, \leq) \leq D_{i,j} + D_{j,i}$ is valid for R_p .*

Proof. By condition (4) in Definition 7 and Definition 8 (2), we have that $D_{i,i} \leq D_{i,j} + D_{j,i}$ is valid for R_p ; the result follows from the fact that $D_{i,i} = (0, \leq)$ (again from Definition 7 and Definition 8).

C Proof of Lemma 3

Lemma 3 (recalled). *Let R_p be a parameter region and (E, D) be a p-PDBM for R_p . For all clocks i, j , $D_{i,j} \leq D_{i,j} + D_{j,j}$ and $D_{i,j} \leq D_{i,i} + D_{i,j}$ are valid for R_p .*

Proof. Let R_p be a parameter region and (E, D) be a p -PDBM for R_p . Let $D_{i,j} = (d_{i,j}, \triangleleft_{ij})$ with $d_{i,j} \in \mathcal{P}\mathcal{L}\mathcal{T}$. By Definition 7 and Definition 8 for all clock i , $D_{i,i} = (0, \leq)$. We have $D_{j,i} + D_{i,i} = (d_{j,i} + 0, \triangleleft_{ij} \oplus \leq) = D_{j,i}$. Moreover from Definition 6 (2b) $D_{i,j} \leq D_{i,i}$ is valid for R_p . Hence $D_{i,j} \leq D_{i,i} + D_{i,j}$ is valid for R_p . The same way we prove $D_{i,j} \leq D_{i,j} + D_{j,j}$ is valid for R_p .

D Proof of Lemma 4

Lemma 4 (recalled). *Let R_p be a parameter region and $(E, D) \in p\text{-PDBM}(R_p)$. Let u_{np} be a non-trivial non-parametric update. Then $\text{update}((E, D), u_{np}) \in p\text{-PDBM}_{\blacksquare}(R_p)$.*

Proof. We split this proof in two parts: the first one treats the case of point- p -PDBMs and the second one of open- p -PDBMs.

The following lemma shows that applying a *update* on any point- p -PDBM transforms it into an open- p -PDBM.

Lemma 10 (*$p\text{-PDBM}_{\odot}(R_p)$ becomes $p\text{-PDBM}_{\blacksquare}(R_p)$ after update*). *Let R_p be a parameter region and $(E, D) \in p\text{-PDBM}_{\odot}(R_p)$. Let u_{np} be a non-parametric update. Then $\text{update}((E, D), u_{np}) \in p\text{-PDBM}_{\blacksquare}(R_p)$.*

Proof. Let R_p be a parameter region and $(E, D) \in p\text{-PDBM}_{\odot}(R_p)$. Consider $(E', D') = \text{update}((E, D), u_{np})$. After applying Algorithm 1, Algorithm 1 for all clock x_i of (E, D) where u_{np} is defined, $E'_i = u_{np}(x_i)$; moreover for all clock j , $D'_{i,j} = D_{0,j}$ and $D'_{j,i} = D_{j,0}$. First note that if x_i, x_j have been updated, $D'_{i,j} = D'_{j,i} = D'_{0,j} = D'_{j,0} = D'_{0,i} = D'_{i,0} = (0, \leq) = D_{0,0}$. For all clocks i, j, k , the following inequalities are valid for R_p :

1. (a) if x_i is updated: $D'_{i,0} = (0, \leq) = D'_{0,i}$ and therefore trivially it holds that $-1 \leq D'_{0,i} \leq 0$ and $0 \leq D'_{i,0} \leq 1$ are valid for R_p ;
 (b) if x_i is not updated: $D'_{i,0} = D_{i,0}$ and therefore $-1 \leq D'_{0,i} \leq 0$ and $0 \leq D'_{i,0} \leq 1$ are valid for R_p because these constraints were already satisfied in (E, D) .
2. For all x_i, x_j , if neither x_i nor x_j is updated, $D_{i,j}$ and $D_{j,i}$ are not modified so condition Definition 7 (2) still holds. If either x_i is updated, as $D'_{i,j} = D_{0,j}$ and $D'_{j,i} = D_{j,0}$ condition Definition 7 (2) still holds as it holds for $D_{0,j}$ and $D_{j,0}$ and we apply the same reasoning if x_j is updated. If both x_i, x_j are updated, condition Definition 7 (2) trivially holds.
3. For all x_i , if it is updated then $D'_{0,i} = D'_{i,0} = (0, \leq)$, hence $d_{0,i} = -d_{i,0} = 0$ and $\triangleleft_{0i} = \triangleleft_{i0} = \leq$; condition Definition 7 (3) holds. For all x_i, x_j , if neither x_i nor x_j is updated, $D'_{i,j} = D_{i,j}$ and $D'_{j,i} = D_{j,i}$ so condition Definition 7 (3) holds as it holds for $D_{i,j}$ and $D_{j,i}$. If either x_i is updated, as $D'_{i,j} = D_{0,j}$ and $D'_{j,i} = D_{j,0}$, condition Definition 7 (3) holds as it holds for $D_{0,j}$ and $D_{j,0}$. We treat the case where x_j is updated similarly. If both x_i, x_j are updated, condition Definition 7 (3) trivially holds.
4. Canonical form is preserved:

- (a) if x_i, x_j, x_k are not updated: since no clock is updated we have $D'_{i,j} = D_{i,j}$, $D'_{j,k} = D_{j,k}$ and $D'_{i,k} = D_{i,k}$ since $(E, D) \in p\text{-}\mathcal{PDBM}_\odot(R_p)$ from [Definition 8 \(2\)](#), we know that $D_{i,k} \leq D_{i,j} + D_{j,k}$ is valid for R_p ; therefore it remains valid.
 - (b) if x_k is updated and x_i, x_j are not updated: $D'_{i,j} = D_{i,j}$ and $D'_{j,k} = D_{j,0}$, $D'_{i,k} = D_{i,0}$ because x_k is updated. Since $(E, D) \in p\text{-}\mathcal{PDBM}_\odot(R_p)$ from [Definition 8 \(2\)](#), we know that $D_{i,0} \leq D_{i,j} + D_{j,0}$ is valid for R_p ; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p .
 - (c) if x_j is updated and x_i, x_k are not updated: then $D'_{i,k} = D_{i,k}$ because neither x_i nor x_k are updated; since x_k is updated we have $D'_{j,k} = D_{0,k}$ and $D'_{i,j} = D_{i,0}$; since $(E, D) \in p\text{-}\mathcal{PDBM}_\odot(R_p)$ from [Definition 8 \(2\)](#), we know that $D_{i,k} \leq D_{i,0} + D_{0,k}$ is valid for R_p ; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p .
 - (d) if x_j, x_k are updated and x_i is not updated: then $D'_{i,k} = D_{i,0}$ because x_k is updated; since x_j is updated we have $D'_{i,j} = D_{i,0}$ and $D'_{j,k} = D_{0,0}$; since $(E, D) \in p\text{-}\mathcal{PDBM}_\odot(R_p)$ from [Definition 8 \(2\)](#) and [Lemma 3](#), we know that $D_{i,0} \leq D_{i,0} + D_{0,0}$ is valid for R_p ; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p .
 - (e) if x_i is updated and x_j, x_k are not updated: then $D'_{i,k} = D_{0,k}$, $D'_{i,j} = D_{0,j}$ because x_i is updated; since x_j, x_k are not updated, we have $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{-}\mathcal{PDBM}_\odot(R_p)$ from [Definition 8 \(2\)](#), we know that $D_{0,k} \leq D_{0,j} + D_{j,k}$ is valid for R_p ; therefore $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p .
 - (f) if x_i, x_k are updated and x_j is not updated: we have $D'_{i,k} = (0, \leq) = D_{0,0}$, $D'_{i,j} = D_{0,j}$ and $D'_{j,k} = D_{j,0}$ because x_i, x_k are updated. Since $(E, D) \in p\text{-}\mathcal{PDBM}_\odot(R_p)$ from [Definition 8 \(2\)](#), we know that $D_{0,0} \leq D_{0,j} + D_{j,0}$ is valid for R_p ; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p .
 - (g) if x_i, x_j are updated and x_k is not updated: we have $D'_{i,k} = D_{0,k}$, $D'_{i,j} = (0 \leq) = D_{0,0}$ and $D'_{j,k} = D_{0,k}$ because x_i, x_j are updated. Since $(E, D) \in p\text{-}\mathcal{PDBM}_\odot(R_p)$ from [Definition 8 \(2\)](#) and [Lemma 3](#), we know that $D_{0,k} \leq D_{0,0} + D_{0,k}$ is valid for R_p ; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p .
 - (h) if x_i, x_j, x_k are updated: we have $D'_{i,k} = D_{0,0}$, $D'_{i,j} = D_{0,0}$ and $D'_{j,k} = D_{0,0}$ because x_i, x_j, x_k are updated. Since $(E, D) \in p\text{-}\mathcal{PDBM}_\odot(R_p)$ from [Definition 8 \(2\)](#) and [Lemma 3](#), we know that $D_{0,0} \leq D_{0,0} + D_{0,0}$ is valid for R_p ; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p .
5. there is at least one clock x s.t. $D'_{x,0} = D'_{0,x} = (0, \leq)$.

Therefore, $(E', D') \in p\text{-}\mathcal{PDBM}_\blacksquare(R_p)$.

The following lemma shows that applying a *update* on any open-p-PDBM transforms it into an open-p-PDBM respecting [Definition 7 \(2\)](#).

Lemma 11 (stability of $p\text{-}\mathcal{PDBM}_\blacksquare(R_p)$ under update). *Let R_p be a parameter region and $(E, D) \in p\text{-}\mathcal{PDBM}_\blacksquare(R_p)$. Let u_{np} be a non-parametric update. Then $\text{update}((E, D), u_{np}) \in p\text{-}\mathcal{PDBM}_\blacksquare(R_p)$.*

Proof. Most cases are similar to the proof of [Lemma 10](#).

The remaining cases to treat are the cases of [Definition 7 \(2\)](#). If i, j are different from 0, and

1. if i, j are not updated then $D'_{i,j} = D_{i,j}$ and since it is the case in (E, D) , condition [Definition 7 \(2\)](#) holds.
2. if j is updated and i is not updated then $D'_{i,j} = D_{i,0}$ and $D'_{j,i} = D_{0,i}$ and as condition [Definition 8 \(1\)](#) holds for $D_{i,0}$ and $D_{0,i}$ in (E, D) , condition [Definition 7 \(2\)](#) holds in (E', D') .
3. if i is updated and j is not updated then $D'_{i,j} = D_{0,j}$ and $D'_{j,i} = D_{j,0}$ and as condition [Definition 8 \(1\)](#) holds for $D_{j,0}$ and $D_{0,j}$ in (E, D) , condition [Definition 7 \(2\)](#) holds in (E', D') .
4. if i, j are updated then trivially $D'_{i,j} = D'_{j,i} = (0, \leq)$ and condition [Definition 7 \(2\)](#) holds.

E Proof of [Lemma 5](#)

Lemma 5 (recalled). *Let R_p be a parameter region and $(E, D) \in p\text{-}\mathcal{PDBM}(R_p)$. Let $v \in R_p$. Let u_{np} be a non-parametric update. For all w , $[w]_{u_{np}} \in \text{update}((E, v(D)), u_{np})$ iff $w \in (E, v(D))$.*

Proof. We first treat the case of the $p\text{-}\mathcal{PDBM}_{\blacksquare}(R_p)$ (the case of the $p\text{-}\mathcal{PDBM}_{\odot}(R_p)$ will be handled similarly at the end). We also prove this lemma for a singleton update (only one clock, say x_i) since updating several clocks can be done by applying several singleton updates in a 0 delay.

$(E, D) \in p\text{-}\mathcal{PDBM}_{\blacksquare}(R_p)$, (\Rightarrow) Let R_p be a parameter region and $(E, D) \in p\text{-}\mathcal{PDBM}_{\blacksquare}(R_p)$. Let $v \in R_p$. Let u_{np} be a non-parametric update which updates x_i to an integer n and lets the value of other clocks unchanged. Consider $(E', D') = \text{update}((E, v(D)), u_{np})$ and suppose $w' \in (E', D')$. We want to construct a valuation $w \in (E, v(D))$ s.t. $w' = u_{np}(w)$.

Let w be a clock valuation s.t. for all clock x_j where $i \neq j$, $w(x_j) = w'(x_j)$. That means that for all $j \neq i$,

$$\text{frac}(w(x_j)) \triangleleft_{j0} v(d_{j,0}), \quad -\text{frac}(w(x_j)) \triangleleft_{0j} v(d_{0,j}) \quad \text{and} \quad \lfloor w(x_j) \rfloor = E_j$$

hold from [Definition 9](#) since it is the case in (E', D') and these values are left untouched by the update. Moreover for all $j \neq i$, $k \neq i$,

$$\text{frac}(w(x_j)) - \text{frac}(w(x_k)) \triangleleft_{jk} v(d_{j,k}) \quad \text{and} \quad \text{frac}(w(x_k)) - \text{frac}(w(x_j)) \triangleleft_{kj} v(d_{k,j})$$

again hold from [Definition 9](#) since it is the case in (E', D') and these values are left untouched by the update.

We want a valuation for $w(x_i)$ s.t.

$$\text{frac}(w(x_i)) \triangleleft_{i0} v(d_{i,0}) \quad -\text{frac}(w(x_i)) \triangleleft_{0i} v(d_{0,i}) \quad \text{and} \quad \lfloor w(x_i) \rfloor = E_i$$

hold, and for all $j \neq i, k \neq i$,

$$\text{frac}(w(x_j)) - \text{frac}(w(x_k)) \triangleleft_{ij} v(d_{i,j}) \quad \text{and} \quad \text{frac}(w(x_k)) - \text{frac}(w(x_i)) \triangleleft_{ki} v(d_{k,i}) \quad (1)$$

hold. Let us prove that such a valuation w exists. We set $\lfloor w(x_i) \rfloor = E_i$.

The following lemma proves transitivity of constraints on clocks with respect to constraints in a \mathbf{p} -PDBM.

Lemma 12. *Let R_p be a parameter region and $(E, D) \in p\text{-PDBM}(R_p)$. Let $v \in R_p$. Let $w \in (E, v(D))$. For all clocks i, j, k , $\text{frac}(w(x_j)) - \text{frac}(w(x_k)) (\triangleleft_{ji} \oplus \triangleleft_{ik}) v(d_{j,i}) + v(d_{i,k})$.*

Proof. Let R_p be a parameter region and $(E, D) \in p\text{-PDBM}(R_p)$. Let $v \in R_p$. Let $w \in (E, v(D))$.

Since $(E, D) \in p\text{-PDBM}(R_p)$, for all i, j, k we have from [Definition 7 \(4\)](#),

$$D_{j,k} \leq D_{j,i} + D_{i,k}$$

is valid for R_p hence since $v \in R_p$, we have $v(D_{j,k}) \leq v(D_{j,i}) + v(D_{i,k})$. Precisely that is $(v(d_{j,k}), \triangleleft_{jk}) \leq (v(d_{j,i}), \triangleleft_{ji}) + (v(d_{i,k}), \triangleleft_{ik})$ i. e.,

$$(v(d_{j,k}), \triangleleft_{jk}) \leq (v(d_{j,i}) + v(d_{i,k}), \triangleleft_{ji} \oplus \triangleleft_{ik}).$$

For all clocks j, k satisfying constraints of (E, D) ,

$$\text{frac}(w(x_j)) - \text{frac}(w(x_k)) \triangleleft_{jk} v(d_{j,k}).$$

Then for all i, j, k , either:

- from [Definition 6 \(2a\)](#): $v(d_{j,k}) < v(d_{j,i}) + v(d_{i,k})$ and then, regardless of \triangleleft_{jk} and $\triangleleft_{ji} \oplus \triangleleft_{ik}$ we have $\text{frac}(w(x_j)) - \text{frac}(w(x_k)) (\triangleleft_{ji} \oplus \triangleleft_{ik}) v(d_{j,i}) + v(d_{i,k})$, or
- from [Definition 6 \(2b\)](#):
 - $v(d_{j,k}) \leq v(d_{j,i}) + v(d_{i,k})$ and $\triangleleft_{jk} = <, \triangleleft_{ji} \oplus \triangleleft_{ik} = \leq$ and then we have $\text{frac}(w(x_j)) - \text{frac}(w(x_k)) (\triangleleft_{ji} \oplus \triangleleft_{ik}) v(d_{j,i}) + v(d_{i,k})$, or
 - $v(d_{j,k}) \leq v(d_{j,i}) + v(d_{i,k})$ and $\triangleleft_{jk} = \triangleleft_{ji} \oplus \triangleleft_{ik}$ and then we have $\text{frac}(w(x_j)) - \text{frac}(w(x_k)) (\triangleleft_{ji} \oplus \triangleleft_{ik}) v(d_{j,i}) + v(d_{i,k})$ which completes the proof.

This completes the proof of [Lemma 12](#).

For all $j \neq i$ and $k \neq i$, since $v(D_{j,k}) \leq v(D_{j,i}) + v(D_{i,k})$ from [Definition 7 \(4\)](#), we have $\text{frac}(w(x_j)) - \text{frac}(w(x_k)) \triangleleft_{jk} v(d_{j,k})$ and

$$\text{frac}(w(x_j)) - \text{frac}(w(x_k)) (\triangleleft_{ji} \oplus \triangleleft_{ik}) v(d_{j,i}) + v(d_{i,k})$$

holds from [Lemma 12](#). Hence

$$\text{frac}(w(x_j)) - v(d_{j,i}) (\triangleleft_{ji} \oplus \triangleleft_{ik}) \text{frac}(w(x_k)) + v(d_{i,k}) \quad (2)$$

holds. Note that $\triangleleft_{ji} \oplus \triangleleft_{ik}$ is either \leq or $<$. Note the following trick is inspired by [Hun+02, Proof of Lemma 3.5] and [Hun+02, Proof of Lemma 3.13]. Hence

$$I = \{t \in \mathbb{R}_+ \mid \text{frac}(w(x_j)) - v(d_{j,i}) \leq t \leq \text{frac}(w(x_k)) + v(d_{i,k}) \text{ for all clocks } j, k\}$$

is a non empty set. That means that choosing a $\text{frac}(w(x_i))$ with respect to constraints (1), recall that they are

$$\text{frac}(w(x_j)) - \text{frac}(w(x_i)) \triangleleft_{ji} v(d_{j,i}) \quad \text{and} \quad \text{frac}(w(x_i)) - \text{frac}(w(x_k)) \triangleleft_{ik} v(d_{i,k})$$

is equivalent to choose a $\text{frac}(w(x_i))$ s.t.

$$\text{frac}(w(x_j)) - v(d_{j,i}) \triangleleft_{ji} \text{frac}(w(x_i)) \quad \text{and} \quad \text{frac}(w(x_i)) \triangleleft_{ik} \text{frac}(w(x_k)) + v(d_{i,k})$$

which is a nonempty set from formula (2). Finally we choose a $\text{frac}(w(x_i)) \in I$, then $w \in (E, v(D))$ and it completes the proof.

$(E, D) \in p\text{-}\mathcal{PDBM}_{\blacksquare}(R_p)$, (\Leftarrow) Let R_p be a parameter region and $(E, D) \in p\text{-}\mathcal{PDBM}_{\blacksquare}(R_p)$. Let $v \in R_p$. Let u_{np} be a non-parametric update which updates x_i to an integer n and lets the value of other clocks unchanged. Consider $(E', D') = \text{update}((E, v(D)), u_{np})$. Now suppose $w \in (E, v(D))$ and let $w' = [w]_{u_{np}}$.

- for x_i , since u_{np} is defined, $w'(x_i) = u_{np}(x_i) = E'_{x_i}$ (i. e., $\text{frac}(w'(x_i)) = 0$) by applying update as defined in Definition 9. By applying update as defined in Definition 9, $D'_{i,0} = D'_{0,i} = (0, \leq)$, hence

$$-\text{frac}(w'(x_i)) \triangleleft_{0i} v(d'_{0,i}) \quad \text{and} \quad \text{frac}(w'(x_i)) \triangleleft_{i0} v(d'_{i,0})$$

hold from Definition 9 and Lemma 10. Moreover we know that for all $j \neq i$

$$-v(D'_{i,j}) = -v(D'_{0,j}) \quad \text{and} \quad v(D'_{j,i}) = v(D'_{j,0}) \quad (3)$$

holds from Definition 9, and we also know that

$$\text{frac}(w'(x_j)) - \text{frac}(w'(x_i)) = \text{frac}(w'(x_j)) \quad (4)$$

since $\text{frac}(w'(x_i)) = 0$. Hence, combining (3) and (4), clearly since

$$-\text{frac}(w'(x_j)) \triangleleft_{0j} v(d'_{0,j}) \quad \text{and} \quad \text{frac}(w'(x_j)) \triangleleft_{j0} v(d'_{j,0})$$

hold in (E', D') ,

$$\text{frac}(w'(x_j)) - \text{frac}(w'(x_i)) \triangleleft_{ji} v(d'_{j,i}) \quad \text{and} \quad \text{frac}(w'(x_i)) - \text{frac}(w'(x_j)) \triangleleft_{ij} v(d'_{i,j})$$

hold.

- for any two clocks x_j, x_k where u_{np} is not defined, $w(x_j) = w'(x_j)$ and $w(x_k) = w'(x_k)$. Hence

$$-v(D'_{0,j}) \triangleleft_{0j} \text{frac}(w'(x_j)) \triangleleft_{j0} v(D'_{j,0})$$

and

$$-v(D'_{k,j}) \triangleleft_{kj} \text{frac}(w'(x_j)) - \text{frac}(w'(x_k)) \triangleleft_{jk} v(D'_{j,k})$$

hold from Definition 9 and Lemma 10 since bounds remain unchanged.

Then $w' \in \text{update}((E, v(D)), u_{np})$.

This concludes the case $(E, D) \in p\text{-PDBM}_{\blacksquare}(R_p)$.

Let us now treat the case $(E, D) \in p\text{-PDBM}_{\odot}(R_p)$.

$(E, D) \in p\text{-PDBM}_{\odot}(R_p)$, (\Rightarrow) Let R_p be a parameter region and $(E, D) \in p\text{-PDBM}_{\odot}(R_p)$. Let $v \in R_p$. Let u_{np} be a non-parametric update which updates x_i to an integer n and lets the value of other clocks unchanged. Consider $(E', D') = \text{update}((E, v(D)), u_{np})$ and suppose $w' \in (E', D')$. We want to construct a valuation

$$w \in (E, v(D)) \quad \text{s.t.} \quad w' = u_{np}(w)$$

Let w be a clock valuation s.t. for all clock x_j where $j \neq i$, $w(x_j) = w'(x_j)$. That means for all $j \neq i$,

$$\text{frac}(w(x_j)) \triangleleft_{j0} v(d_{j,0}), \quad -\text{frac}(w(x_j)) \triangleleft_{0j} v(d_{0,j}) \quad \text{and} \quad \lfloor w(x_j) \rfloor = E_j$$

hold from [Definition 9](#) since it is the case in (E', D') and bounds remain unchanged i. e., $D_{0,j} = D'_{0,j}$ and $D_{j,0} = D'_{j,0}$. Moreover for all $k \neq i$ and $k \neq j$,

$$\text{frac}(w(x_j)) - \text{frac}(w(x_k)) \triangleleft_{jk} v(d_{j,k}) \quad \text{and} \quad \text{frac}(w(x_k)) - \text{frac}(w(x_j)) \triangleleft_{kj} v(d_{k,j})$$

also hold from [Definition 9](#) since it is the case in (E', D') and bounds remain unchanged i. e., $D_{k,j} = D'_{k,j}$ and $D_{j,k} = D'_{j,k}$.

Recall that (E, D) contains only one clock valuation for each parameter valuation $v \in R_p$.

Let $\text{frac}(w(x_i)) = v(d_{i,0})$ (or equivalently $\text{frac}(w(x_i)) = -v(d_{0,i})$ since by [Definition 8](#) we have $(d_{i,0}, \triangleleft_{i0}) = (-d_{0,i}, \triangleleft_{0i})$). Then, as it is the case in (E, D) ,

$$\text{frac}(w(x_i)) \triangleleft_{i0} v(d_{i,0}), \quad -\text{frac}(w(x_i)) \triangleleft_{0i} v(d_{0,i}) \quad \text{and} \quad \lfloor w(x_i) \rfloor = E_i$$

hold, and for all $j \neq i$, $k \neq i$,

$$\text{frac}(w(x_i)) - \text{frac}(w(x_j)) \triangleleft_{ij} v(d_{i,j}) \quad \text{and} \quad \text{frac}(w(x_k)) - \text{frac}(w(x_i)) \triangleleft_{ki} v(d_{k,i})$$

hold, which completes the proof, as $w \in (E, v(D))$ and $w' = u_{np}(w)$.

$(E, D) \in p\text{-PDBM}_{\odot}(R_p)$, (\Leftarrow) This case is straightforward and similar to the case (\Leftarrow) above of open-p-PDBMs .

F Proof of [Lemma 6](#)

Lemma 6 (recalled). *Let R_p be a parameter region and $(E, D) \in p\text{-PDBM}(R_p)$. There is at least one clock x s.t. for all $0 \leq i \leq H$, $(0, \leq) \leq D_{x,i}$ is valid for R_p .*

Proof. Reductio ad absurdum: Let R_p be a parameter region and $(E, D) \in p\text{-PDBM}_{\blacksquare}(R_p)$ with at least 2 clocks i, j . Suppose for all clock x_i there is another clock x_j s.t. $D_{i,j} < 0$ is valid for R_p . Let $v \in R_p$. Then $v(D_{i,j}) < 0$.

- Suppose for x_j , x_i is the clock s.t. $D_{j,i} < 0$ is valid for R_p . Then $v(D_{j,i}) < 0$. We have $v(D_{i,j}) + v(D_{j,i}) < 0$ holds, therefore $0 \leq v(D_{i,j}) + v(D_{j,i})$ does not hold, and hence $0 \leq D_{i,j} + D_{j,i}$ is not valid for R_p . Then (E, D) does not respect [Lemma 2](#) and violates condition (4) of [Definition 7](#). So $(E, D) \notin p\text{-PDBM}_{\blacksquare}(R_p)$.
- Suppose for x_j , a third clock x_k is the clock s.t. $D_{j,k} < 0$ is valid for R_p . Then $v(D_{j,k}) < 0$. Suppose we have only three clocks. Then for x_k , either x_i or x_j is the clock s.t. $D_{k,i} < 0$ is valid for R_p .
 - Assume this is x_i . Then $v(D_{k,i}) < 0$. We have $v(D_{k,i}) + v(D_{i,j}) < 0$ and $v(D_{k,j}) \leq v(D_{k,i}) + v(D_{i,j})$ by [Definition 7](#) (4). Follows that $v(D_{k,j}) + v(D_{j,k}) < 0$ and $0 \leq D_{k,j} + D_{j,k}$ is not valid for R_p . Then (E, D) does not respect [Lemma 2](#) and violates condition (4) of [Definition 7](#). So $(E, D) \notin p\text{-PDBM}_{\blacksquare}(R_p)$.
 - Assume this is x_j . This case is similar (and simpler).

We apply the same reasoning for more than 3 clocks. Now suppose $(E, D) \in p\text{-PDBM}_{\odot}(R_p)$. We apply the same reasoning, replacing the argument of condition (4) of [Definition 7](#) by the fact from [Definition 8](#) that D is antisymmetric.

G Proof of [Lemma 7](#)

Lemma 7 (recalled). *Let R_p be a parameter region. Let $(E, D) \in p\text{-PDBM}(R_p)$. Then $TE((E, D)) \in p\text{-PDBM}(R_p)$.*

Proof. We prove our lemma for the two types of open-p-PDBMs and for point-p-PDBMs, and split this proof in three lemmas.

[Definition 7](#) type (5a) to (5b)

Lemma 13 (modification of an open-p-PDBM respecting condition 5a under $TE_{<}$). *Let R_p be a parameter region and $(E, D) \in p\text{-PDBM}_{\blacksquare}(R_p)$ respecting condition 5a, then $TE_{<}((E, D)) \in p\text{-PDBM}_{\blacksquare}(R_p)$ respecting condition 5b.*

Proof (Proof of [Lemma 13](#)).

Suppose $(E, D) \in p\text{-PDBM}_{\blacksquare}(R_p)$ respects condition (5a) of [Definition 7](#), i. e., we have at least an x s.t. $D_{x,0} = D_{0,x} = (0, \leq)$. Since, in R_p , we know which parameters have the largest fractional part, we can determine $\text{LFP}_{R_p}(D)$ from [Lemma 6](#). If more than one clock belong to $\text{LFP}_{R_p}(D)$ then their valuations have the same fractional part. Indeed, from [Definition 10](#) if $x_i, x_j \in \text{LFP}_{R_p}(D)$ then both $(0, \leq) \leq D_{i,j}$ and $(0, \leq) \leq D_{j,i}$ are valid for R_p , and from [Definition 7](#) (2) we must have $D_{i,j} = D_{j,i} = (0, \leq)(\star)$.

Let $v \in R_p$. Assume $x_i \in \text{LFP}_{R_p}(D)$ and $w \in (E, v(D))$, by letting time elapse, $\text{frac}(w(x_i))$ is the first that might reach 1. Moreover, for all $x_j \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$, $\text{frac}(w(x_j))$ cannot reach 1 before $\text{frac}(w(x_i))$. We are going to construct a new $(E', D') = TE_{<}((E, D))$, which will be an open-p-PDBM respecting condition 5b of Definition 7. While detailing the procedure of $TE_{<}$, we are going to prove that Definition 7 (1) and (2) hold for (E', D') . Further we will prove that (4) and (5b) also hold.

proof that Definition 7 (1) holds According to the definition of $TE_{<}$ (Algorithm 2) the first step is to set a new upper bound

$$D'_{i,0} = (1, <) \quad \text{for all } x_i \in \text{LFP}_{R_p}(D)$$

and obviously $(0, \leq) \leq D'_{i,0} \leq (1, \leq)$ is valid for R_p . Then we set new upper bounds for all other clock $x_j \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$ by setting

$$D'_{j,0} = D_{j,i} + (1, <).$$

Indeed, $D_{j,i}$ is the constraint on the lower bound of $\text{frac}(w(x_j)) - \text{frac}(w(x_i))$ and since the upper bound of x_i has increased, this gives the new upper bound of x_j . Note that since $x_i \in \text{LFP}_{R_p}(D)$, from Definition 10 and Definition 7 (2) we have that $-1 \leq D_{j,i} \leq 0$ is valid for R_p for all clock x_j . Precisely, $d_{j,i} \in \{0, -p_1, p_2 - p_1, p_1 - 1 - p_2, p_1 - 1\}$ for some $p_1, p_2 \in \mathbb{P}$ where $p_2 \leq p_1$ is valid for R_p . Hence as $d_{j,i} + 1 \in \{1, 1 - p_1, p_2 + 1 - p_1, p_1 - p_2, p_1\}$, we have that $d'_{j,0} \in \mathcal{PLT}$, $\triangleleft_{ji'} = \triangleleft_{ji} \oplus < = <$ so $(0, \leq) \leq D'_{j,0} \leq (1, <)$ is valid for R_p .

Note that we cannot have $(d_{j,i}, \triangleleft_{ji}) = (-1, <)$ because even if $(d_{i,j}, \triangleleft_{ij}) = (1, <)$, since $(E, D) \in p\text{-PDBM}_{\blacksquare}(R_p)$ we do not have have $0 \leq D_{j,i} + D_{i,j}$ is valid for R_p from Definition 7 (4) and Lemma 2.

Secondary we set for all clock x regardless of whether they are in $\text{LFP}_{R_p}(D)$

$$D'_{0,x} = D_{0,x} + (0, <).$$

Since some time elapsed, lower bounds of all clocks are increased. Moreover, as $(-1, <) \leq D_{0,x} \leq (0, \leq)$ is valid for R_p from Definition 7 (1), $(-1, \leq) \leq D'_{0,x} \leq (0, \leq)$ is also valid for R_p .

Therefore, Definition 7 (1) holds.

proof that Definition 7 (2) holds Third we set for all clocks x, y regardless of whether they are in $\text{LFP}_{R_p}(D)$

$$D'_{x,y} = D_{x,y}$$

so as Definition 7 (2) holds in (E, D) , it still does. More intuitively since no fractional part has reached 1, constraints on differences of clocks and integer parts remain unchanged.

proof that [Definition 7 \(3\)](#) holds For all x_i :

- if $x_i \in \text{LFP}_{R_p}(D)$, $D'_{i,0} = (1, <)$, $D'_{0,i} = D_{0,i} + (0, <)$ hence $d'_{i,0} \neq d'_{0,i}$ and $\triangleleft_{i0'} = \triangleleft_{0i'} = <$, condition [Definition 7 \(3\)](#) holds;
- if $x_i \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$, $x \in \text{LFP}_{R_p}(D)$, $D'_{i,0} = D_{i,x} + (1, <)$, $D'_{0,i} = D_{0,i} + (0, <)$ hence as $(0, \leq) \leq D'_{i,0}$ is valid for R_p and $D'_{0,i} \leq (0, \leq)$ is valid for R_p , we have $d'_{i,0} \neq d'_{0,i}$ and $\triangleleft_{i0'} = \triangleleft_{0i'} = <$ and condition [Definition 7 \(3\)](#) holds.

For all x_i, x_j :

- if $x_i, x_j \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$, $D'_{i,j} = D_{i,j}$ and $D'_{j,i} = D_{j,i}$, condition [Definition 7 \(3\)](#) holds as it holds for $D_{i,j}$ and $D_{j,i}$.
- if $x_i \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$, $x_j \in \text{LFP}_{R_p}(D)$, $D'_{i,0} = D_{i,j} + (1, <)$, $D'_{0,i} = D_{0,i} + (0, <)$ hence as $(0, \leq) \leq D'_{i,0}$ is valid for R_p and $D'_{0,i} \leq (0, \leq)$ is valid for R_p , we have $d'_{i,0} \neq d'_{0,i}$ and $\triangleleft_{i0'} = \triangleleft_{0i'} = <$, condition [Definition 7 \(3\)](#) holds. The case $x_j \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$, $x_i \in \text{LFP}_{R_p}(D)$ is treated similarly.
- if $x_i, x_j \in \text{LFP}_{R_p}(D)$, $D'_{i,j} = D'_{j,i} = (0, \leq)$, hence $d'_{i,j} = -d'_{j,i} = 0$ and $\triangleleft_{ij'} = \triangleleft_{ji'} = \leq$ and condition [Definition 7 \(3\)](#) holds.

proof that [Definition 7 \(4\)](#) holds Now we prove that [Definition 7 \(4\)](#) holds, i. e., for all clocks x_i, x_j, x_k , valid conditions such as $D'_{i,j} \leq D'_{i,k} + D'_{k,j}$ remain valid in R_p . Indeed, when time elapses, all clocks have the same behavior, hence the difference between two clocks does not change without an update. Precisely, for all clocks x_i, x_j, x_k , are valid for R_p :

1. if $x_i, x_j, x_k \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$: let $x \in \text{LFP}_{R_p}(D)$ and
 - if i, j, k are different from 0, we have $D'_{i,k} = D_{i,k}$, $D'_{i,j} = D_{i,j}$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{-PDBM}_{\blacksquare}(R_p)$ from [Definition 7 \(4\)](#), we know that $D_{i,k} \leq D_{i,j} + D_{j,k}$ is valid for R_p ; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p .
 - if i, j are different from 0, $k = 0$, we have $D'_{i,0} = D_{i,x} + (1, <)$, $D'_{i,j} = D_{i,j}$ and $D'_{j,0} = D_{j,x} + (1, <)$; since $(E, D) \in p\text{-PDBM}_{\blacksquare}(R_p)$ from [Definition 7 \(4\)](#), we know that $D_{i,x} \leq D_{i,j} + D_{j,x}$ is valid for R_p ; then $D_{i,x} + (1, <) \leq D_{i,j} + D_{j,x} + (1, <)$ is valid for R_p from [Lemma 1](#) and therefore, $D'_{i,0} \leq D'_{i,j} + D'_{j,0}$ is valid for R_p .
 - if i, k are different from 0, $j = 0$, we have $D'_{i,k} = D_{i,k}$, $D'_{i,0} = D_{i,x} + (1, <)$ and $D'_{0,k} = D_{0,k} + (0, <)$; we claim that

$$D_{i,k} \leq D_{i,x} + (1, <) + D_{0,k} + (0, <) \quad (5)$$

is valid for R_p , which is equivalent to $D'_{i,k} \leq D'_{i,0} + D'_{0,k}$ is valid for R_p . Since $(E, D) \in p\text{-PDBM}_{\blacksquare}(R_p)$ from [Definition 7 \(1\)](#), we know that

$$D_{x,0} \leq (1, <); \quad (6)$$

moreover we have

$$(1, <) + (0, <) = (1 + 0, < \oplus <) = (1, <) \quad (7)$$

Since $(E, D) \in p\text{-}\mathcal{PDBM}_{\blacksquare}(R_p)$ from [Definition 7 \(4\)](#), we know that $D_{x,k} \leq D_{x,0} + D_{0,k}$ is valid for R_p ; combining with [\(6\)](#) and [\(7\)](#) we obtain

$$D_{x,k} \leq (1, <) + D_{0,k} + (0, <). \quad (8)$$

Now, since $(E, D) \in p\text{-}\mathcal{PDBM}_{\blacksquare}(R_p)$ from [Definition 7 \(4\)](#), we know that $D_{i,k} \leq D_{i,x} + D_{x,k}$ is valid for R_p and combining with [\(8\)](#) we obtain [\(5\)](#) and therefore our result.

- if i is different from 0, $j = k = 0$, we have $D'_{i,0} = D_{i,x} + (1, <)$; from [Definition 6 \(2b\)](#) we have that

$$D_{i,x} + (1, <) \leq D_{i,x} + (1, <)$$

is valid for R_p . Hence from [Lemma 3](#)

$$D'_{i,0} \leq D'_{i,0} + D'_{0,0}$$

is valid for R_p .

- if j, k are different from 0, $i = 0$, we have $D'_{0,k} = D_{0,k} + (0, <)$, $D'_{0,j} = D_{0,j} + (0, <)$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{-}\mathcal{PDBM}_{\blacksquare}(R_p)$, from [Definition 7 \(4\)](#) we know that $D_{0,k} \leq D_{0,j} + D_{j,k}$ is valid for R_p . Moreover we have that

$$D_{0,k} + (0, <) = (d_{0,k}, <) \quad \text{and} \quad D_{0,j} + (0, <) + D_{j,k} = (d_{0,j} + d_{j,k}, <)$$

so we have from [Definition 6 \(2b\)](#)

$$D_{0,k} + (0, <) \leq D_{0,j} + (0, <) + D_{j,k}$$

is valid for R_p . Hence $D'_{0,k} \leq D'_{0,j} + D'_{j,k}$ is valid for R_p .

- if j is different from 0, $i = k = 0$, we have $D'_{0,0} = (0, \leq)$, $D'_{0,j} = D_{0,j} + (0, <)$ and $D'_{j,0} = D_{j,x} + (1, <)$; since $(E, D) \in p\text{-}\mathcal{PDBM}_{\blacksquare}(R_p)$, from [Definition 7 \(4\)](#) we know that $D_{0,x} \leq D_{0,j} + D_{j,x}$ is valid for R_p ; moreover, from [Definition 6 \(2b\)](#) and [Lemma 1](#),

$$D_{0,x} + (0, <) \leq D_{0,j} + (0, <) + D_{j,x}$$

is valid for R_p . Recall that from [Lemma 2](#) $(0, \leq) \leq D_{0,x} + D_{x,0}$ is valid for R_p and since $D_{x,0} \leq (1, <)$ from [Definition 7 \(1\)](#), we have

$$(0, \leq) \leq D_{0,x} + (1, <)$$

is valid for R_p . As we have $(1, <) + (0, <) = (1 + 0, < \oplus <) = (1, <)$, we obtain that

$$D_{0,x} + (1, <) \leq D_{0,j} + D_{j,x} + (1, <)$$

is valid for R_p and therefore $D'_{0,0} \leq D'_{0,j} + D'_{j,0}$ is valid for R_p .

- if k is different from 0, $i = j = 0$, we have $D'_{0,k} = D_{0,k} + (0, <)$; From [Definition 6 \(2b\)](#) and [Lemma 1](#) we have that

$$D_{0,k} + (0, <) \leq D_{0,k} + (0, <)$$

is valid for R_p . Hence from [Lemma 3](#)

$$D'_{0,k} \leq D'_{0,0} + D'_{0,k}$$

is valid for R_p .

- if $i = j = k = 0$, from [Definition 7 \(4\)](#) and [Lemma 3](#) we trivially have

$$D'_{0,0} \leq D'_{0,0} + D'_{0,0}$$

is valid for R_p .

2. if $x_k \in \text{LFP}_{R_p}(D)$ and $x_i, x_j \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$: $k \neq 0$ and

- if i, j are different from 0, we have $D'_{i,k} = D_{i,k}$, $D'_{i,j} = D_{i,j}$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{-}\mathcal{PDBM}_{\blacksquare}(R_p)$ from [Definition 7 \(4\)](#), we know that $D_{i,k} \leq D_{i,j} + D_{j,k}$; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$.
- if $i \neq 0, j = 0$, we have $D'_{i,k} = D_{i,k}$, $D'_{i,0} = D_{i,k} + (1, <)$ and $D'_{0,k} = D_{0,k} + (0, <)$; we claim that $D_{i,k} \leq D_{i,k} + (1, <) + D_{0,k} + (0, <)$ is valid for R_p , i. e.,

$$(0, \leq) \leq (1, <) + D_{0,k} + (0, <) \quad (9)$$

is valid for R_p , which is equivalent to $D'_{i,k} \leq D'_{i,0} + D'_{0,k}$ is valid for R_p . We have

$$(1, <) + (0, <) = (1 + 0, < \oplus <) = (1, <). \quad (10)$$

Since $(E, D) \in p\text{-}\mathcal{PDBM}_{\blacksquare}(R_p)$, from [Definition 7 \(4\)](#) we know that $(0, \leq) \leq D_{0,k} + D_{k,0}$ is valid for R_p and from [Definition 7 \(1\)](#) that $D_{k,0} \leq (1, <)$ is valid for R_p ; combining with (9) and (10) we obtain our result.

- if $i = 0, j \neq 0$, we have $D'_{0,k} = D_{0,k} + (0, <)$, $D'_{0,j} = D_{0,j} + (0, <)$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{-}\mathcal{PDBM}_{\blacksquare}(R_p)$, from [Definition 7 \(4\)](#) we know that $D_{0,k} \leq D_{0,j} + D_{j,k}$. Moreover we have that

$$D_{0,k} + (0, <) = (d_{0,k}, <) \quad \text{and} \quad D_{0,j} + (0, <) + D_{j,k} = (d_{0,j} + d_{j,k}, <)$$

so we have from [Definition 6 \(2b\)](#)

$$D_{0,k} + (0, <) \leq D_{0,j} + (0, <) + D_{j,k}$$

is valid for R_p . Hence $D'_{0,k} \leq D'_{0,j} + D'_{j,k}$ is valid for R_p .

- if $i = j = 0$, from [Definition 7 \(4\)](#) and [Lemma 3](#) we trivially have

$$D'_{0,k} \leq D'_{0,0} + D'_{0,k}$$

is valid for R_p .

3. if $x_j \in \text{LFP}_{R_p}(D)$ and $x_i, x_k \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$: $j \neq 0$ and

- if i, k are different from 0, we have $D'_{i,k} = D_{i,k}$, $D'_{i,j} = D_{i,j}$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{-}\mathcal{PDBM}_{\blacksquare}(R_p)$, from [Definition 7 \(4\)](#) we know that $D_{i,k} \leq D_{i,j} + D_{j,k}$ is valid for R_p ; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p .
- if $i \neq 0, k = 0$, we have $D'_{i,0} = D_{i,j} + (1, <)$, $D'_{i,j} = D_{i,j}$ and $D'_{j,0} = (1, <)$; From [Definition 6 \(2b\)](#) we trivially have that $D_{i,j} + (1, <) \leq D_{i,j} + (1, <)$ is valid for R_p and therefore, $D'_{i,0} \leq D'_{i,j} + D'_{j,0}$ is valid for R_p .
- if $i = 0, k \neq 0$, we have $D'_{0,k} = D_{0,k} + (0, <)$, $D'_{0,j} = D_{0,j} + (0, <)$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{-}\mathcal{PDBM}_{\blacksquare}(R_p)$, from [Definition 7 \(4\)](#) we know that $D_{0,k} \leq D_{0,j} + D_{j,k}$ is valid for R_p . Moreover we have that

$$D_{0,k} + (0, <) = (d_{0,k}, <) \quad \text{and} \quad D_{0,j} + (0, <) + D_{j,k} = (d_{0,j} + d_{j,k}, <)$$

so we have from [Definition 6 \(2b\)](#) and [Lemma 1](#)

$$D_{0,k} + (0, <) \leq D_{0,j} + (0, <) + D_{j,k}$$

is valid for R_p . Hence $D'_{0,k} \leq D'_{0,j} + D'_{j,k}$ is valid for R_p .

- if $i = k = 0$, we have $D'_{0,0} = (0, \leq)$, $D'_{0,j} = D_{0,j} + (0, <)$ and $D'_{j,0} = (1, <)$; since $(E, D) \in p\text{-}\mathcal{PDBM}_{\blacksquare}(R_p)$, from [Lemma 2](#) we know that $(0, \leq) \leq D_{0,j} + D_{j,0}$ is valid for R_p , and since from [Definition 7 \(1\)](#) $D_{j,0} \leq (1, \leq)$ is valid for R_p , that means $(0, \leq) \leq D_{0,j} + (1, <)$ is valid for R_p . As we have

$$(1, <) + (0, <) = (1 + 0, < \oplus <) = (1, <)$$

we obtain that

$$(0, \leq) \leq D_{0,j} + (0, <) + (1, <)$$

is valid for R_p and therefore $D'_{0,0} \leq D'_{0,j} + D'_{j,0}$ is valid for R_p .

- if $x_j, x_k \in \text{LFP}_{R_p}(D)$ and $x_i \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$: $j \neq 0, k \neq 0$ and
 - if i is different from 0, we have $D'_{i,k} = D_{i,k}$, $D'_{i,j} = D_{i,j}$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{-}\mathcal{PDBM}_{\blacksquare}(R_p)$, from [Definition 7 \(4\)](#) we know that $D_{i,k} \leq D_{i,j} + D_{j,k}$; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$.
 - if $i = 0$, we have $D'_{0,k} = D_{0,k} + (0, <)$, $D'_{0,j} = D_{0,j} + (0, <)$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{-}\mathcal{PDBM}_{\blacksquare}(R_p)$, from [Definition 7 \(4\)](#) we know that $D_{0,k} \leq D_{0,j} + D_{j,k}$. Moreover we have that

$$D_{0,k} + (0, <) = (d_{0,k}, <) \quad \text{and} \quad D_{0,j} + (0, <) + D_{j,k} = (d_{0,j} + d_{j,k}, <)$$

so we have from [Definition 6 \(2b\)](#) and [Lemma 1](#)

$$D_{0,k} + (0, <) \leq D_{0,j} + (0, <) + D_{j,k}$$

is valid for R_p . Hence $D'_{0,k} \leq D'_{0,j} + D'_{j,k}$ is valid for R_p .

- if $x_i \in \text{LFP}_{R_p}(D)$ and $x_j, x_k \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$: $i \neq 0$ and
 - if j, k are different from 0, we have $D'_{i,k} = D_{i,k}$, $D'_{i,j} = D_{i,j}$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{-}\mathcal{PDBM}_{\blacksquare}(R_p)$, from [Definition 7 \(4\)](#) we know that $D_{i,k} \leq D_{i,j} + D_{j,k}$; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$.

- if $j \neq 0, k = 0$, we have $D'_{i,0} = (1, <)$, $D'_{i,j} = D_{i,j}$ and $D'_{j,0} = D_{j,i} + (1, <)$; from [Definition 7 \(4\)](#) and [Lemma 2](#) we know that $(0, \leq) \leq D_{i,j} + D_{j,i}$ is valid for R_p . Since, from [Definition 6 \(2b\)](#) $(1, <) \leq (1, <)$ is valid for R_p , then from [Lemma 1](#)

$$(1, <) \leq D_{i,j} + D_{j,i} + (1, <)$$

- is valid for R_p and therefore, $D'_{i,0} \leq D'_{i,j} + D'_{j,0}$ is valid for R_p .
- if $j = 0, k \neq 0$, we have $D'_{i,k} = D_{i,k}$, $D'_{i,0} = (1, <)$ and $D'_{0,k} = D_{0,k} + (0, <)$; we claim that

$$D_{i,k} \leq (1, <) + D_{0,k} + (0, <)$$

is valid for R_p , which is equivalent to $D'_{i,k} \leq D'_{i,0} + D'_{0,k}$ is valid for R_p . Since $(E, D) \in p\text{-}\mathcal{PDBM}_{\blacksquare}(R_p)$ from [Definition 7 \(4\)](#), we know that $D_{i,k} \leq D_{i,0} + D_{0,k}$ is valid for R_p ; moreover, from [Definition 7 \(1\)](#), we know that $D_{i,0} \leq (1, <)$ is valid for R_p . We have

$$(1, <) + (0, <) = (1 + 0, < \oplus <) = (1, <)$$

so we obtain that

$$D_{i,k} \leq D_{i,0} + D_{0,k} \leq (1, <) + D_{0,k} = (1, <) + D_{0,k} + (0, <)$$

is valid for R_p and therefore our result.

- if i is different from 0, $j = k = 0$, we have $D'_{i,0} = (1, <)$, $D'_{0,0} = (0, \leq)$; from [Definition 6 \(2b\)](#) we have that

$$(1, <) \leq (1, <)$$

is valid for R_p . Hence from [Lemma 3](#)

$$D'_{i,0} \leq D'_{i,0} + D'_{0,0}$$

is valid for R_p .

6. if $x_i, x_k \in \text{LFP}_{R_p}(D)$ and $x_j \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$: $i \neq 0, k \neq 0$ and
 - if $j \neq 0$, we have $D'_{i,k} = D_{i,k}$, $D'_{i,j} = D_{i,j}$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{-}\mathcal{PDBM}_{\blacksquare}(R_p)$, from [Definition 7 \(4\)](#) we know that $D_{i,k} \leq D_{i,j} + D_{j,k}$; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p .
 - if $j = 0$, we have $D'_{i,k} = D_{i,k}$, $D'_{i,0} = (1, <)$ and $D'_{0,k} = D_{0,k} + (0, <)$; we claim that

$$D_{i,k} \leq (1, <) + D_{0,k} + (0, <)$$

is valid for R_p , which is equivalent to $D'_{i,k} \leq D'_{i,0} + D'_{0,k}$ is valid for R_p . Since $(E, D) \in p\text{-}\mathcal{PDBM}_{\blacksquare}(R_p)$ from [Definition 7 \(4\)](#), we know that $D_{i,k} \leq D_{i,0} + D_{0,k}$ is valid for R_p ; moreover, from [Definition 7 \(1\)](#), we know that $D_{i,0} \leq (1, <)$ is valid for R_p . We have

$$(1, <) + (0, <) = (1 + 0, < \oplus <) = (1, <)$$

so we obtain that

$$D_{i,k} \leq D_{i,0} + D_{0,k} \leq (1, <) + D_{0,k} = (1, <) + D_{0,k} + (0, <)$$

is valid for R_p and therefore our result.

7. if $x_i, x_j \in \text{LFP}_{R_p}(D)$ and $x_k \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$: $i \neq 0, j \neq 0$ and
 - if $k \neq 0$, we have $D'_{i,k} = D_{i,k}$, $D'_{i,j} = D_{i,j}$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{-PDBM}_{\blacksquare}(R_p)$, from [Definition 7 \(4\)](#) we know that $D_{i,k} \leq D_{i,j} + D_{j,k}$ is valid for R_p ; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p .
 - if $k = 0$, we have $D'_{i,0} = (1, <)$, $D'_{i,j} = D_{i,j} = (0, \leq)$ since both $x_i, x_j \in \text{LFP}_{R_p}(D)$ (cf. (\star)) and $D'_{j,0} = (1, <)$; then $(1, <) \leq (0, \leq) + (1, <)$ is valid for R_p and therefore, $D'_{i,0} \leq D'_{i,j} + D'_{j,0}$ is valid for R_p .
8. if $x_i, x_j, x_k \in \text{LFP}_{R_p}(D)$: i, j, k are different from 0, we have $D'_{i,k} = D_{i,k}$, $D'_{i,j} = D_{i,j}$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{-PDBM}_{\blacksquare}(R_p)$, from [Definition 7 \(4\)](#) we know that $D_{i,k} \leq D_{i,j} + D_{j,k}$ is valid for R_p ; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p .

proof that [Definition 7 \(5b\)](#) holds Finally, for $x_i \in \text{LFP}_{R_p}(D)$, $D'_{i,0} = (1, <)$ and for all clock j s.t. $D'_{0,j} = (0, \triangleleft_{0j'})$, then we have $\triangleleft_{0j'} = <$. Condition [Definition 7 \(5b\)](#) is satisfied.

We denote by (E, D') the obtained p-PDBM and $(E, D') \in p\text{-PDBM}_{\blacksquare}(R_p)$.

Definition 7 type 5b to (5a)

Lemma 14. *Let $(E, D) \in p\text{-PDBM}_{\blacksquare}(R_p)$; let $x_i \in \text{LFP}_{R_p}(D)$, $x_j \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$. If $(d_{i,j}, \triangleleft_{ij}) = (0, \triangleleft)$, then $\triangleleft = <$*

Proof. Let $x_i \in \text{LFP}_{R_p}(D)$, $x_j \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$. Suppose $(d_{i,j}, \triangleleft_{ij}) = (0, \leq)$. From [Definition 7 \(2\)](#) we should have that $(d_{j,i}, \triangleleft_{ji}) = (0, \leq)$ so [Lemma 2](#) is satisfied, and then $x_j \in \text{LFP}_{R_p}(D)$.

Lemma 15 (modification of an open-p-PDBM respecting condition 5b under $TE_{=}$). *Let R_p be a parameter region and $(E, D) \in p\text{-PDBM}_{\blacksquare}(R_p)$ respecting condition 5b, then $TE_{=}(E, D) \in p\text{-PDBM}_{\blacksquare}(R_p)$ respecting condition 5a.*

Proof. Suppose $(E, D) \in p\text{-PDBM}_{\blacksquare}(R_p)$ respects condition (5a) of [Definition 7](#) i. e., we have at least an x s.t. $D_{x,0} = (1, <)$ and for all other j s.t. $D_{0,j} = (0, \triangleleft_{0j})$, $\triangleleft_{0j} = <$. First we can determine $\text{LFP}_{R_p}(D)$. Let $x \in \text{LFP}_{R_p}(D)$. If more than one clock belong to $\text{LFP}_{R_p}(D)$ then their valuations have the same fractional part. Indeed, from [Definition 10](#) if $x_i, x_j \in \text{LFP}_{R_p}(D)$ then both $(0, \leq) \leq D_{i,j}$ and $(0, \leq) \leq D_{j,i}$ are valid for R_p , and from [Definition 7 \(2\)](#) we must have $D_{i,j} = D_{j,i} = (0, \leq)$.

Let $v \in R_p$. Let $x_i \in \text{LFP}_{R_p}(D)$ and $w \in (E, v(D))$. By letting time elapse, $\text{frac}(w(x))$ is the first to actually reach 1. Moreover, for all $x_j \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$, $\text{frac}(w(x_j))$ cannot reach 1 before $\text{frac}(w(x_i))$. We are going to construct a new $(E', D') = TE_{=}(E, D)$ which is an open-p-PDBM respecting condition 5b. While detailing the procedure of $TE_{=}$, we are going to prove that [Definition 7 \(1\)](#) and (2) hold for (E', D') . Further we will prove that (4) and (5a) also hold.

proof that [Definition 7 \(1\)](#) holds According to the definition of $TE_{=}$ Algorithm 3, the first step is to fix the value of $frac(x_i)$ to 0 by setting

$$D'_{i,0} = (0, \leq) \quad \text{and} \quad D'_{0,i} = (0, \leq) \quad \text{for all } x_i \in \text{LFP}_{R_p}(D).$$

Indeed, when $frac(x_i)$ reaches 1, in the constraints expressed by $(E, v(D))$ we have to increase the integer part by 1 and set the new constraints on the fractional part to 0.

Secondary we set new upper and lower bound for all other clock $x_j \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$

$$D'_{0,j} = D_{i,j} + (-1, \leq) \quad \text{and} \quad D'_{j,0} = D_{j,i} + (1, \leq).$$

We have to force now upper and lower bounds for other clocks since we know the interval of time that elapsed when x_i reached 1.

Note that since $x_i \in \text{LFP}_{R_p}(D)$, $x_j \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$ from [Definition 10](#) we have that $(0, \leq) \leq D_{i,j} \leq (1, <)$ is valid for R_p for all clock x_j . Nonetheless, since $x_j \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$, we even have $D_{i,j} \neq (0, \leq)$: suppose $(d_{i,j}, \triangleleft_{ij}) = (0, \leq)$: from [Definition 7 \(2\)](#) we should have that $(d_{j,i}, \triangleleft_{ji}) = (0, \leq)$ so [Lemma 2](#) is satisfied, and then $x_j \in \text{LFP}_{R_p}(D)$. The same reasoning leads to $D_{j,i} \neq (0, \leq)$.

Obviously, we have $D_{i,j} \neq (0, <)$: suppose $D_{i,j} = (0, <)$, since $x_i \in \text{LFP}_{R_p}(D)$ then from [Definition 10](#) $(0, \leq) \leq D_{i,j}$ should be valid for R_p , which is not from [Definition 6 \(2b\)](#).

Precisely, $d_{i,j} \in \{1, 1 - p_1, p_2 + 1 - p_1, p_1 - p_2, p_1\}$ for any two $p_1, p_2 \in \mathbb{P}$ where $p_2 \leq p_1$ is valid for R_p . Hence as $-1 + d_{i,j} \in \{0, -p_1, p_2 - p_1, p_1 - 1 - p_2, p_1 - 1\}$, we have that $D'_{0,j} \in \mathcal{P}\mathcal{L}\mathcal{T}$ and $(-1, <) \leq D'_{0,j} \leq (0, \leq)$ is valid for R_p from [Lemma 14](#).

Also note that since $x_i \in \text{LFP}_{R_p}(D)$, from [Definition 10](#) and [Definition 7 \(2\)](#) we have that $(-1, <) \leq D_{j,i} \leq (0, \leq)$ is valid for R_p for all clock x_j . Precisely, $d_{j,i} \in \{0, -p_1, p_2 - p_1, p_1 - 1 - p_2, p_1 - 1\}$ for some $p_1, p_2 \in \mathbb{P}$ where $p_2 \leq p_1$ is valid for R_p . Hence as $d_{j,i} + 1 \in \{1, 1 - p_1, p_2 + 1 - p_1, p_1 - p_2, p_1\}$, we have that $d'_{j,0} \in \mathcal{P}\mathcal{L}\mathcal{T}$ and $(0, \leq) \leq D'_{j,0} \leq (1, <)$ is valid for R_p .

Clearly [Definition 7 \(1\)](#) holds.

proof that [Definition 7 \(2\)](#) holds Third we set for all two clocks i, j where $x_i \in \text{LFP}_{R_p}(D)$, $x_j \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$

$$D'_{i,j} = D'_{0,j} \quad \text{and} \quad D'_{j,i} = D'_{j,0},$$

for all two clocks $x_j, x_k \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$

$$D'_{j,k} = D_{j,k}$$

and for all two clocks $x, y \in \text{LFP}_{R_p}(D)$

$$D'_{x,y} = D'_{y,x} = (0, \leq).$$

Here as we have already proven above that $(-1, <) \leq D'_{0,j} \leq (0, \leq)$ and $(0, \leq) \leq D'_{j,0} \leq (1, <)$ are valid for R_p , [Definition 7 \(2\)](#) holds.

proof that [Definition 7 \(3\)](#) holds For all x_i :

- if $x_i \in \text{LFP}_{R_p}(D)$, $D'_{i,0} = (0, \leq)$, $D'_{0,i} = (0, \leq)$ hence $d'_{i,0} = -d'_{0,i}$ and $\triangleleft_{i0'} \triangleleft_{0i'} = \leq$, condition [Definition 7 \(3\)](#) holds;
- if $x_i \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$, $x \in \text{LFP}_{R_p}(D)$, $D'_{i,0} = D_{i,x} + (1, \leq)$, $D'_{0,i} = D_{x,i} + (-1, \leq)$ as condition [Definition 7 \(3\)](#) holds for $D_{i,x}$ and $D_{x,i}$ and $\triangleleft_{ij} \oplus \leq = \triangleleft_{ij}$, $\triangleleft_{ji} \oplus \leq = \triangleleft_{ji}$, condition [Definition 7 \(3\)](#) holds for $D'_{i,0}$ and $D'_{0,i}$.

For all x_i, x_j :

- if $x_i, x_j \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$, $D'_{i,j} = D_{i,j}$ and $D'_{j,i} = D_{j,i}$, condition [Definition 7 \(3\)](#) holds as it holds for $D_{i,j}$ and $D_{j,i}$.
- if $x_i \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$, $x_j \in \text{LFP}_{R_p}(D)$, $D'_{i,j} = D_{i,j} + (1, \leq)$, $D'_{j,i} = D_{j,i} + (-1, \leq)$ condition [Definition 7 \(3\)](#) holds for $D_{i,j}$ and $D_{j,i}$ and $\triangleleft_{ij} \oplus \leq = \triangleleft_{ij}$, $\triangleleft_{ji} \oplus \leq = \triangleleft_{ji}$, condition [Definition 7 \(3\)](#) holds for $D'_{i,j}$ and $D'_{j,i}$. The case $x_j \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$, $x_i \in \text{LFP}_{R_p}(D)$ is treated similarly.
- if $x_i, x_j \in \text{LFP}_{R_p}(D)$, $D'_{i,j} = D'_{j,i} = (0, \leq)$, hence $d'_{i,j} = -d'_{j,i} = 0$ and $\triangleleft_{ij'} \triangleleft_{ji'} = \leq$ and condition [Definition 7 \(3\)](#) holds.

proof that [Definition 7 \(4\)](#) holds Now we prove that [Definition 7 \(4\)](#) holds, i. e., for all clocks x_i, x_j, x_k , valid conditions such as $D'_{i,j} \leq D'_{i,k} + D'_{k,j}$ remain valid in R_p . This is not trivial since, in this construction some clocks have been updated. Precisely, for all clocks x_i, x_j, x_k , are valid for R_p :

1. if $x_i, x_j, x_k \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$: let $x \in \text{LFP}_{R_p}(D)$ and
 - if i, j, k are different from 0, we have $D'_{i,k} = D_{i,k}$, $D'_{i,j} = D_{i,j}$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{-}\mathcal{PDBM}_{\blacksquare}(R_p)$, from [Definition 7 \(4\)](#) we know that $D_{i,k} \leq D_{i,j} + D_{j,k}$ is valid for R_p ; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p .
 - if i, j are different from 0, $k = 0$, we have $D'_{i,0} = D_{i,x} + (1, \leq)$, $D'_{i,j} = D_{i,j}$ and $D'_{j,0} = D_{j,x} + (1, \leq)$; since $(E, D) \in p\text{-}\mathcal{PDBM}_{\blacksquare}(R_p)$, from [Definition 7 \(4\)](#) we know that $D_{i,x} \leq D_{i,j} + D_{j,x}$ is valid for R_p ; then from [Lemma 1](#) $D_{i,x} + (1, \leq) \leq D_{i,j} + D_{j,x} + (1, \leq)$ is valid for R_p and therefore, $D'_{i,0} \leq D'_{i,j} + D'_{j,0}$ is valid for R_p .
 - if i, k are different from 0, $j = 0$, we have $D'_{i,k} = D_{i,k}$, $D'_{i,0} = D_{i,x} + (1, \leq)$ and $D'_{0,k} = D_{x,k} + (-1, \leq)$; we claim that

$$D_{i,k} \leq D_{i,x} + (1, \leq) + D_{x,k} + (-1, \leq) \quad (11)$$

is valid for R_p , which is equivalent to $D'_{i,k} \leq D'_{i,0} + D'_{0,k}$ is valid for R_p . We have

$$(1, \leq) + (-1, \leq) = (1 + -1, \leq \oplus \leq) = (0, \leq) \quad (12)$$

Since $(E, D) \in p\text{-}\mathcal{PDBM}_{\blacksquare}(R_p)$ from [Definition 7 \(4\)](#), we know that $D_{i,k} \leq D_{i,x} + D_{x,k}$ is valid for R_p ; combining with (12) and since $D_{x,k} + (0, \leq) = D_{x,k}$, we obtain (11) and therefore our result.

- if i is different from 0, $j = k = 0$, we have $D'_{i,0} = D_{i,x} + (1, \leq)$, $D'_{j,k} = D'_{0,0} = (0, \leq)$; we have from [Definition 6 \(2b\)](#) that

$$D_{i,x} + (1, \leq) \leq D_{i,x} + (1, \leq)$$

is valid for R_p . Hence [Lemma 3](#) gives that

$$D'_{i,0} \leq D'_{i,0} + D'_{0,0}$$

is valid for R_p .

- if j, k are different from 0, $i = 0$, we have $D'_{0,k} = D_{x,k} + (-1, \leq)$, $D'_{0,j} = D_{x,j} + (-1, \leq)$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{-}\mathcal{PDBM}_{\blacksquare}(R_p)$, from [Definition 7 \(4\)](#) we know that $D_{x,k} \leq D_{x,j} + D_{j,k}$ is valid for R_p . Moreover we have that

$$(-1, \leq) \leq (-1, \leq)$$

is valid for R_p so we have from [Definition 6 \(2b\)](#) and [Lemma 1](#)

$$D_{x,k} + (-1, \leq) \leq D_{x,j} + (-1, \leq) + D_{j,k}$$

is valid for R_p . Hence $D'_{0,k} \leq D'_{0,j} + D'_{j,k}$ is valid for R_p .

- if j is different from 0, $i = k = 0$, we have $D'_{0,j} = D_{x,j} + (-1, \leq)$ and $D'_{j,0} = D_{j,x} + (1, \leq)$; since $(E, D) \in p\text{-}\mathcal{PDBM}_{\blacksquare}(R_p)$, from [Lemma 2](#) we know that $(0, \leq) \leq D_{x,j} + D_{j,x}$ is valid for R_p ; moreover, we have that

$$(1, \leq) + (-1, \leq) = (1 + -1, \leq \oplus \leq) = (0, \leq)$$

and $D_{j,x} + (0, \leq) = D_{j,x}$. Then we have from [Lemma 1](#)

$$(0, \leq) \leq D_{x,j} + (-1, \leq) + D_{j,x} + (1, \leq)$$

is valid for R_p and therefore $D'_{0,0} \leq D'_{0,j} + D'_{j,0}$ is valid for R_p .

- if k is different from 0, $i = j = 0$, we have $D'_{0,k} = D_{x,k} + (-1, \leq)$, $D'_{i,j} = D'_{0,0} = (0, \leq)$; we have from [Definition 6 \(2b\)](#) that

$$D_{x,k} + (-1, \leq) \leq D_{x,k} + (-1, \leq)$$

is valid for R_p . Hence, as $D_{x,k} + (-1, \leq) + (0, \leq) = D_{x,k} + (-1, \leq)$ we have

$$D'_{0,k} \leq D'_{0,0} + D'_{0,k}$$

is valid for R_p .

- if $i = j = k = 0$, we trivially have from [Definition 7 \(4\)](#) and [Lemma 3](#)

$$D'_{0,0} \leq D'_{0,0} + D'_{0,0}$$

is valid for R_p .

2. if $x_k \in \text{LFP}_{R_p}(D)$ and $x_i, x_j \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$: $k \neq 0$ and

- if i, j are different from 0, we have $D'_{i,k} = D'_{i,0} = D_{i,k} + (1, \leq)$, $D'_{i,j} = D_{i,j}$ and $D'_{j,k} = D'_{j,0} = D_{j,k} + (1, \leq)$; since $(E, D) \in p\text{-}\mathcal{PDBM}_{\blacksquare}(R_p)$, from [Definition 7 \(4\)](#) we know that $D_{i,k} \leq D_{i,j} + D_{j,k}$ is valid for R_p ; moreover, since we have $(1, \leq) \leq (1, \leq)$ is valid for R_p then from [Lemma 1](#)

$$D_{i,k} + (1, \leq) \leq D_{i,j} + D_{j,k} + (1, \leq)$$

is valid for R_p , therefore we have $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p .

- if $i \neq 0, j = 0$, we have $D'_{i,k} = D'_{i,0} = D_{i,k} + (1, \leq)$, $D'_{i,0} = D_{i,k} + (1, \leq)$ and $D'_{0,k} = (0, \leq)$; clearly

$$(1, \leq) \leq (1, \leq) + (0, \leq)$$

and

$$D_{i,k} \leq D_{i,k}$$

are valid for R_p , then from [Lemma 1](#) we obtain $D'_{i,k} \leq D'_{i,0} + D'_{0,k}$ is valid for R_p .

- if $i = 0, j \neq 0$, we have $D'_{0,k} = (0, \leq)$, $D'_{0,j} = D_{k,j} + (-1, \leq)$ and $D'_{j,k} = D'_{j,0} = D_{j,k} + (1, \leq)$; since $(E, D) \in p\text{-}\mathcal{PDBM}_{\blacksquare}(R_p)$, from [Lemma 2](#) we know that $(0, \leq) \leq D_{k,j} + D_{j,k}$ is valid for R_p . Moreover we have that

$$(1, \leq) + (-1, \leq) = (1 + -1, \leq \oplus \leq) = (0, \leq)$$

so we have from [Lemma 1](#)

$$(0, \leq) \leq D_{k,j} + D_{j,k} + (0, \leq)$$

is valid for R_p . Hence $D'_{0,k} \leq D'_{0,j} + D'_{j,k}$ is valid for R_p .

- if $i = j = 0$, we trivially have from [Definition 7 \(4\)](#) and [Lemma 3](#)

$$D'_{0,k} \leq D'_{0,0} + D'_{0,k}$$

is valid for R_p .

3. if $x_j \in \text{LFP}_{R_p}(D)$ and $x_i, x_k \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$: $j \neq 0$ and
 - if i, k are different from 0, we have $D'_{i,k} = D_{i,k}$, $D'_{i,j} = D'_{i,0} = D_{i,j} + (1, \leq)$ and $D'_{j,k} = D'_{0,k} = D_{j,k} + (-1, \leq)$; since $(E, D) \in p\text{-}\mathcal{PDBM}_{\blacksquare}(R_p)$, from [Definition 7 \(4\)](#) we know that $D_{i,k} \leq D_{i,j} + D_{j,k}$ is valid for R_p ; moreover, since we have

$$(1, \leq) + (-1, \leq) = (1 + (-1), \leq \oplus \leq) = (0, \leq)$$

then as $D_{i,j} + D_{j,k} + (0, \leq) = D_{i,j} + D_{j,k}$, clearly $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p .

- if $i \neq 0, k = 0$, we have $D'_{i,0} = D_{i,j} + (1, \leq)$, $D'_{i,j} = D'_{i,0} = D_{i,j} + (1, \leq)$ and $D'_{j,0} = (0, \leq)$; From [Definition 6 \(2b\)](#) we trivially have that $D_{i,j} + (1, \leq) \leq D_{i,j} + (1, \leq)$ is valid for R_p and therefore, $D'_{i,0} \leq D'_{i,j} + D'_{j,0}$ is valid for R_p .

- if $i = 0, k \neq 0$, we have $D'_{0,k} = D_{j,k} + (-1, \leq)$, $D'_{0,j} = (0, \leq)$ and $D'_{j,k} = D'_{0,k} = D_{j,k} + (-1, \leq)$; since $(E, D) \in p\text{-}\mathcal{PDBM}_{\blacksquare}(R_p)$, from [Definition 7 \(4\)](#) we know that $D_{0,k} \leq D_{0,j} + D_{j,k}$ is valid for R_p . From [Definition 6 \(2b\)](#) we trivially have that $D_{j,k} + (-1, \leq) \leq D_{j,k} + (-1, \leq)$ is valid for R_p . As $(-1, \leq) + (0, \leq) = (-1, \leq)$, we have $D'_{0,k} \leq D'_{0,j} + D'_{j,k}$ is valid for R_p .
- if $i = k = 0$, we have $D'_{0,j} = (0, \leq)$ and $D'_{j,0} = (0, \leq)$; As we have

$$(0, \leq) + (0, \leq) = (0, \leq)$$

we clearly have that $D'_{0,0} \leq D'_{0,j} + D'_{j,0}$ is valid for R_p .

- if $x_j, x_k \in \text{LFP}_{R_p}(D)$ and $x_i \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$: $j \neq 0, k \neq 0$ and
 - if i is different from 0, we have $D'_{i,k} = D'_{i,0} = D_{i,k} + (-1, \leq)$, $D'_{i,j} = D'_{i,0} = D_{i,k} + (-1, \leq)$ and $D'_{j,k} = (0, \leq)$; we have that $(-1, \leq) + (0, \leq) = (-1, \leq)$ and

$$D_{i,k} + (-1, \leq) \leq D_{i,k} + (-1, \leq)$$

holds from [Definition 6 \(2b\)](#). Therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$.

- if $i = 0$, we have $D'_{0,k} = (0, \leq)$, $D'_{0,j} = (0, \leq)$ and $D'_{j,k} = (0, \leq)$; since $(E, D) \in p\text{-}\mathcal{PDBM}_{\blacksquare}(R_p)$ from [Definition 7 \(4\)](#), we know that $D_{0,k} \leq D_{0,j} + D_{j,k}$. As we have

$$(0, \leq) + (0, \leq) = (0, \leq)$$

we clearly have that $D'_{0,k} \leq D'_{0,j} + D'_{j,k}$ is valid for R_p .

- if $x_i \in \text{LFP}_{R_p}(D)$ and $x_j, x_k \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$: $i \neq 0$ and
 - if j, k are different from 0, we have $D'_{i,k} = D'_{0,k} = D_{i,k} + (-1, \leq)$, $D'_{i,j} = D'_{0,j} = D_{i,j} + (-1, \leq)$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{-}\mathcal{PDBM}_{\blacksquare}(R_p)$, from [Definition 7 \(4\)](#) we know that $D_{i,k} \leq D_{i,j} + D_{j,k}$ is valid for R_p ; moreover, since we have

$$(-1, \leq) \leq (-1, \leq)$$

is valid for R_p then from [Lemma 1](#) we have $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p .

- if $j \neq 0, k = 0$, we have $D'_{i,0} = (0, \leq)$, $D'_{i,j} = D'_{0,j} = D_{i,j} + (-1, \leq)$ and $D'_{j,0} = D_{j,i} + (1, \leq)$; from [Lemma 2](#) we know that $(0, \leq) \leq D_{i,j} + D_{j,i}$ is valid for R_p . Moreover, we have

$$(1, \leq) + (-1, \leq) = (1 + (-1), \leq \oplus \leq) = (0, \leq)$$

then

$$(0, \leq) \leq D_{i,j} + D_{j,i} + (0, \leq)$$

is valid for R_p and therefore, $D'_{i,0} \leq D'_{i,j} + D'_{j,0}$ is valid for R_p .

- if $j = 0, k \neq 0$, we have $D'_{i,k} = D_{i,k}$, $D'_{i,0} = (1, \leq)$ and $D'_{0,k} = D_{i,k} + (-1, \leq)$; we have that

$$(1, \leq) + (-1, \leq) = (1 + (-1), \leq \oplus \leq) = (0, \leq)$$

and from [Definition 6 \(2b\)](#) that

$$D_{i,k} \leq D_{i,k} + (0, \leq)$$

is valid for R_p , which gives us our result.

- if i is different from 0, $j = k = 0$, we have $D'_{i,0} = (0, \leq)$, $D'_{j,k} = D'_{0,0} = (0, \leq)$; we have from [Definition 6 \(2b\)](#) that

$$(0, \leq) \leq (0, \leq)$$

is valid for R_p . Hence

$$D'_{i,0} \leq D'_{i,0} + D'_{0,0}$$

is valid for R_p .

6. if $x_i, x_k \in \text{LFP}_{R_p}(D)$ and $x_j \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$: $i \neq 0, k \neq 0$ and
 - if $j \neq 0$, we have $D'_{i,k} = (0, \leq)$, $D'_{i,j} = D'_{0,j} = D_{i,j} + (-1, \leq)$ and $D'_{j,k} = D'_{j,0} = D_{j,i} + (1, \leq)$; since $(E, D) \in p\text{-PDBM}_{\blacksquare}(R_p)$, from [Lemma 2](#) we know that $(0, \leq) \leq D_{i,j} + D_{j,i}$ is valid for R_p ; we have

$$(1, \leq) + (-1, \leq) = (1 + (-1), \leq \oplus \leq) = (0, \leq)$$

and therefore from [Lemma 1](#), $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p .

- if $j = 0$, we have $D'_{i,k} = (0, \leq)$, $D'_{i,0} = (0, \leq)$ and $D'_{0,k} = (0, \leq)$; we have that $(0, \leq) + (0, \leq) = (0, \leq)$ and from [Definition 6 \(2b\)](#)

$$(0, \leq) \leq (0, \leq)$$

is valid for R_p . Therefore we obtain our result.

7. if $x_i, x_j \in \text{LFP}_{R_p}(D)$ and $x_k \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$: $i \neq 0, j \neq 0$ and
 - if $k \neq 0$, we have $D'_{i,k} = D'_{0,k} = D_{i,k} + (-1, \leq)$, $D'_{i,j} = (0, \leq)$ and $D'_{j,k} = D'_{0,k} = D_{i,k} + (-1, \leq)$; we have that

$$D_{i,k} \leq D_{i,k}$$

is valid for R_p and from [Lemma 1](#)

$$(-1, \leq) \leq (-1, \leq)$$

is valid for R_p . Therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p .

- if $k = 0$, we have $D'_{i,0} = (0, \leq)$, $D'_{i,j} = (0, \leq)$ and $D'_{j,0} = (0, \leq)$; we have that $(0, \leq) + (0, \leq) = (0, \leq)$ and from [Definition 6 \(2b\)](#)

$$(0, \leq) \leq (0, \leq)$$

is valid for R_p ; therefore $D'_{i,0} \leq D'_{i,j} + D'_{j,0}$ is valid for R_p .

8. if $x_i, x_j, x_k \in \text{LFP}_{R_p}(D)$: i, j, k are different from 0, we have $D'_{i,k} = (0, \leq)$, $D'_{i,j} = (0, \leq)$ and $D'_{j,k} = (0, \leq)$; we have that $(0, \leq) + (0, \leq) = (0, \leq)$ and from [Definition 6 \(2b\)](#)

$$(0, \leq) \leq (0, \leq)$$

is valid for R_p ; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p .

proof that [Definition 7 \(5a\)](#) holds. Finally, there is at least one clock $x_i \in \text{LFP}_{R_p}(D)$ s.t. $D_{0,i} = D_{i,0} = (0, \leq)$. Hence condition [Definition 7 \(5a\)](#) holds.

Finally, we set $E'_i = E_i + 1$ if $x_i \in \text{LFP}_{R_p}(D)$ and $E'_j = E_j$ if $x_j \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$. We denote by (E, D') the obtained p-PDBM and $(E', D') \in p\text{-PDBM}_{\blacksquare}(R_p)$.

Definition 8 to Definition 7 type (5a)

Lemma 16 ($p\text{-PDBM}_{\odot}(R_p)$ becomes $p\text{-PDBM}_{\blacksquare}(R_p)$ after $TE_{<}$). *Let R_p be a parameter region and $(E, D) \in p\text{-PDBM}_{\odot}(R_p)$, then $TE_{<}((E, D)) \in p\text{-PDBM}_{\blacksquare}(R_p)$ respecting condition 5b.*

Proof. Suppose $(E, D) \in p\text{-PDBM}_{\odot}(R_p)$. Since, in R_p , we know which parameters have the largest fractional part, we can determine $\text{LFP}_{R_p}(D)$ from [Lemma 6](#). If more than one clock belong to $\text{LFP}_{R_p}(D)$ then their valuations have the same fractional part.

Indeed, from [Definition 10](#) if $x_i, x_j \in \text{LFP}_{R_p}(D)$ then both $(0, \leq) \leq D_{i,j}$ and $(0, \leq) \leq D_{j,i}$ are valid for R_p , and from [Definition 7 \(2\)](#) we must have $D_{i,j} = D_{j,i} = (0, \leq)$.

Let $v \in R_p$. Let $x_i \in \text{LFP}_{R_p}(D)$ and $w \in (E, v(D))$. By letting time elapse, $\text{frac}(w(x_i))$ is the first that might reach 1. Moreover, for all $x_j \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$, $\text{frac}(w(x_j))$ cannot reach 1 before $\text{frac}(w(x_i))$. We are going to construct a new $(E', D') = TE_{<}(E, D)$ which is an open-p-PDBM respecting condition 5b. While detailing the procedure of $TE_{<}$, we are going to prove that [Definition 7 \(1\)](#) and [\(2\)](#) hold for (E', D') . Further we will prove that [\(4\)](#) and [\(5b\)](#) also hold.

proof that [Definition 7 \(1\)](#) holds. According to the definition of $TE_{<}$ ([Algorithm 2](#)) the first step is to set a new upper bound

$$D'_{i,0} = (1, <) \quad \text{for all } x_i \in \text{LFP}_{R_p}(D)$$

and obviously $(0, \leq) \leq D'_{i,0} \leq (1, <)$ is valid for R_p . Then we set new upper bounds for all other clock $x_j \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$ by setting

$$D'_{j,0} = D_{j,i} + (1, <).$$

Indeed, $D_{j,i}$ is the constraint on the lower bound of $w(x_j) - w(x_i)$ and since the upper bound of x_i has increased, this gives the new upper bound of x_j . Note that since $x_i \in \text{LFP}_{R_p}(D)$, from [Definition 10](#) we have for all clock x_j that $(-1, <) \leq D_{j,i} \leq (0, \leq)$ is valid for R_p . Precisely, $d_{j,i} \in \{0, -p_1, p_2 - p_1, p_1 - 1 - p_2, p_1 - 1\}$ for some $p_1, p_2 \in \mathbb{P}$ where $p_2 \leq p_1$ is valid for R_p . Hence as $d_{j,i} + 1 \in \{1, 1 - p_1, p_2 + 1 - p_1, p_1 - p_2, p_1\}$, we have that $d'_{j,0} \in \mathcal{PLT}$, $\triangleleft_{j,0'} = \triangleleft_{j,0} \oplus < = <$ and $(0, \leq) \leq D'_{j,0} \leq (1, <)$ is valid for R_p . Note that we cannot have $(d_{j,i}, \triangleleft_{ji}) = (-1, <)$ because even if $(d_{i,j}, \triangleleft_{ij}) = (1, <)$, since $(E, D) \in p\text{-PDBM}_{\blacksquare}(R_p)$ we do not have have $0 \leq D_{j,i} + D_{i,j}$ is valid for R_p from [Definition 7 \(4\)](#) and [Lemma 2](#).

Secondary we set for all clock x regardless of whether they are in $\text{LFP}_{R_p}(D)$

$$D'_{0,x} = D_{0,x} + (0, <).$$

Since some time elapsed, lower bounds of all clocks are increased. Moreover, from [Definition 8 \(1\)](#) as $(-1, <) \leq D_{0,x} \leq (0, \leq)$ is valid for R_p , $(-1, <) \leq D'_{0,x} \leq (0, \leq)$ is also valid for R_p .

proof that [Definition 7 \(2\)](#) holds Third we set for all clocks x, y regardless of whether they are in $\text{LFP}_{R_p}(D)$

$$D'_{x,y} = D_{x,y}$$

since no fractional part has reached 1, constraints on differences of clocks and integer parts remain unchanged. As it is the case in (E, D) , [Definition 7 \(2\)](#) holds.

proof that [Definition 7 \(3\)](#) holds For all x_i :

- if $x_i \in \text{LFP}_{R_p}(D)$, $D'_{i,0} = (1, <)$, $D'_{0,i} = D_{0,i} + (0, <)$ hence $d'_{i,0} \neq d'_{0,i}$ and $\triangleleft_{i0'} \triangleleft_{0i'} = <$, condition [Definition 7 \(3\)](#) holds;
- if $x_i \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$, $x \in \text{LFP}_{R_p}(D)$, $D'_{i,0} = D_{i,x} + (1, <)$, $D'_{0,i} = D_{0,i} + (0, <)$ hence as $(0, \leq) \leq D'_{i,0}$ is valid for R_p and $D'_{0,i} \leq (0, \leq)$ is valid for R_p , we have $d'_{i,0} \neq d'_{0,i}$ and $\triangleleft_{i0'} \triangleleft_{0i'} = <$ and condition [Definition 7 \(3\)](#) holds.

For all x_i, x_j :

- if $x_i, x_j \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$, $D'_{i,j} = D_{i,j}$ and $D'_{j,i} = D_{j,i}$, condition [Definition 7 \(3\)](#) holds as it holds for $D_{i,j}$ and $D_{j,i}$.
- if $x_i \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$, $x_j \in \text{LFP}_{R_p}(D)$, $D'_{i,0} = D_{i,j} + (1, <)$, $D'_{0,i} = D_{0,i} + (0, <)$ hence as $(0, \leq) \leq D'_{i,0}$ is valid for R_p and $D'_{0,i} \leq (0, \leq)$ is valid for R_p , we have $d'_{i,0} \neq d'_{0,i}$ and $\triangleleft_{i0'} \triangleleft_{0i'} = <$, condition [Definition 7 \(3\)](#) holds. The case $x_j \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$, $x_i \in \text{LFP}_{R_p}(D)$ is treated similarly.
- if $x_i, x_j \in \text{LFP}_{R_p}(D)$, $D'_{i,j} = D'_{j,i} = (0, \leq)$, hence $d'_{i,j} = -d'_{j,i} = 0$ and $\triangleleft_{ij'} \triangleleft_{ji'} = \leq$ and condition [Definition 7 \(3\)](#) holds.

proof that [Definition 7 \(4\)](#) holds Now we prove that [Definition 7 \(4\)](#) holds, i. e., for all clocks x_i, x_j, x_k valid conditions such as $D'_{i,j} \leq D'_{i,k} + D'_{k,j}$ remain valid in R_p . Indeed, when time elapses, all clocks have the same behavior, hence the difference between two clocks does not change without an update. Precisely, for all clocks x_i, x_j, x_k , are valid for R_p :

1. if $x_i, x_j, x_k \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$: let $x \in \text{LFP}_{R_p}(D)$ and
 - if i, j, k are different from 0, we have $D'_{i,k} = D_{i,k}$, $D'_{i,j} = D_{i,j}$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{-PDBM}_{\odot}(R_p)$ from [Definition 8 \(2\)](#), we know that $D_{i,k} \leq D_{i,j} + D_{j,k}$ is valid for R_p ; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p .

- if i, j are different from 0, $k = 0$, we have $D'_{i,0} = D_{i,x} + (1, <)$, $D'_{i,j} = D_{i,j}$ and $D'_{j,0} = D_{j,x} + (1, <)$; since $(E, D) \in p\text{-}\mathcal{PDBM}_\odot(R_p)$, from [Definition 8 \(2\)](#) we know that $D_{i,x} \leq D_{i,j} + D_{j,x}$ is valid for R_p ; then from [Lemma 1](#) $D_{i,x} + (1, <) \leq D_{i,j} + D_{j,x} + (1, <)$ is valid for R_p and therefore, $D'_{i,0} \leq D'_{i,j} + D'_{j,0}$ is valid for R_p .
- if i, k are different from 0, $j = 0$, we have $D'_{i,k} = D_{i,k}$, $D'_{i,0} = D_{i,x} + (1, <)$ and $D'_{0,k} = D_{0,k} + (0, <)$; we claim that

$$D_{i,k} \leq D_{i,x} + (1, <) + D_{0,k} + (0, <) \quad (13)$$

is valid for R_p , which is equivalent to $D'_{i,k} \leq D'_{i,0} + D'_{0,k}$ is valid for R_p . Since $(E, D) \in p\text{-}\mathcal{PDBM}_\odot(R_p)$, from [Definition 8 \(1\)](#) we know that

$$D_{x,0} \leq (1, <) \quad (14)$$

is valid for R_p ; moreover we have

$$(1, <) + (0, <) = (1 + 0, < \oplus <) = (1, <). \quad (15)$$

Since $(E, D) \in p\text{-}\mathcal{PDBM}_\odot(R_p)$, from [Definition 8 \(2\)](#) we know that $D_{x,k} \leq D_{x,0} + D_{0,k}$ is valid for R_p ; combining with (14) and (15) we obtain $D_{x,k} \leq (1, <) + D_{0,k} + (0, <)$ is valid for R_p . As $D_{i,x} \leq D_{i,x}$ is valid for R_p , using [Lemma 1](#) we obtain

$$D_{i,x} + D_{x,k} \leq D_{i,x} + (1, <) + D_{0,k} + (0, <) \quad (16)$$

is valid for R_p . Now, since $(E, D) \in p\text{-}\mathcal{PDBM}_\odot(R_p)$, from [Definition 8 \(2\)](#) we know that $D_{i,k} \leq D_{i,x} + D_{x,k}$ is valid for R_p and combining with (16) we obtain (13) and therefore our result.

- if i is different from 0, $j = k = 0$, we have $D'_{i,0} = D_{i,x} + (1, <)$, $D'_{j,k} = D'_{0,0} = (0, \leq)$; we have from [Definition 6 \(2b\)](#) that

$$D_{i,x} + (1, <) \leq D_{i,x} + (1, <)$$

is valid for R_p . Hence

$$D'_{i,0} \leq D'_{i,0} + D'_{0,0}$$

is valid for R_p .

- if j, k are different from 0, $i = 0$, we have $D'_{0,k} = D_{0,k} + (0, <)$, $D'_{0,j} = D_{0,j} + (0, <)$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{-}\mathcal{PDBM}_\odot(R_p)$, from [Definition 8 \(2\)](#) we know that $D_{0,k} \leq D_{0,j} + D_{j,k}$ is valid for R_p . Moreover we have that

$$D_{0,k} + (0, <) = (d_{0,k}, <) \quad \text{and} \quad D_{0,j} + (0, <) + D_{j,k} = (d_{0,j} + d_{j,k}, <)$$

so we have from [Lemma 1](#)

$$D_{0,k} + (0, <) \leq D_{0,j} + (0, <) + D_{j,k}$$

is valid for R_p . Hence $D'_{0,k} \leq D'_{0,j} + D'_{j,k}$ is valid for R_p .

- if j is different from 0, $i = k = 0$, we have $D'_{i,k} = D'_{0,0} = (0, \leq)$, $D'_{0,j} = D_{0,j} + (0, <)$ and $D'_{j,0} = D_{j,x} + (1, <)$; since $(E, D) \in p\text{-}\mathcal{PDBM}_\odot(R_p)$, from [Definition 8 \(2\)](#) we know that $D_{0,x} \leq D_{0,j} + D_{j,x}$ is valid for R_p ; moreover from [Lemma 1](#),

$$D_{0,x} + (0, <) \leq D_{0,j} + (0, <) + D_{j,x}$$

is valid for R_p . As we have

$$(1, <) + (0, <) = (1 + 0, < \oplus <) = (1, <)$$

we obtain from [Lemma 1](#) that

$$D_{0,x} + (1, <) \leq D_{0,j} + D_{j,x} + (1, <)$$

is valid for R_p . Recall that from [Lemma 2](#) $(0, \leq) \leq D_{0,x} + D_{x,0}$ is valid for R_p . Since from [Definition 8 \(1\)](#) $D_{x,0} \leq (1, <)$ is valid for R_p , we have $(0, \leq) \leq D_{0,x} + (1, <)$ is valid for R_p . Therefore $D'_{0,0} \leq D'_{0,j} + D'_{j,0}$ is valid for R_p .

- if k is different from 0, $i = j = 0$, we have $D'_{i,k} = D'_{j,k} = D'_{0,k} = D_{0,k} + (0, <)$, $D'_{i,j} = D'_{0,0} = (0, \leq)$; we have from [Definition 6 \(2b\)](#) that

$$D_{0,k} + (0, <) \leq D_{0,k} + (0, <)$$

is valid for R_p . Hence from [Lemma 3](#)

$$D'_{0,k} \leq D'_{0,0} + D'_{0,k}$$

is valid for R_p .

- if $i = j = k = 0$, we trivially have

$$D'_{0,0} \leq D'_{0,0} + D'_{0,0}$$

is valid for R_p .

- if $x_k \in \text{LFP}_{R_p}(D)$ and $x_i, x_j \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$: $k \neq 0$ and
 - if i, j are different from 0, we have $D'_{i,k} = D_{i,k}$, $D'_{i,j} = D_{i,j}$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{-}\mathcal{PDBM}_\odot(R_p)$, from [Definition 8 \(2\)](#) we know that $D_{i,k} \leq D_{i,j} + D_{j,k}$ is valid for R_p ; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p .
 - if $i \neq 0$, $j = 0$, we have $D'_{i,k} = D_{i,k}$, $D'_{i,0} = D_{i,k} + (1, <)$ and $D'_{0,k} = D_{0,k} + (0, <)$; we claim that $D_{i,k} \leq D_{i,k} + (1, <) + D_{0,k} + (0, <)$, *i. e.*,

$$0 \leq (1, <) + D_{0,k} + (0, <) \tag{17}$$

is valid for R_p , which is from [Lemma 1](#) equivalent to $D'_{i,k} \leq D'_{i,0} + D'_{0,k}$ is valid for R_p . We have

$$(1, <) + (0, <) = (1 + 0, < \oplus <) = (1, <). \tag{18}$$

Since $(E, D) \in p\text{-}\mathcal{PDBM}_\odot(R_p)$, from [Definition 8 \(2\)](#) we know that $0 \leq D_{0,k} + D_{k,0}$ is valid for R_p and from [Definition 8 \(1\)](#) that $D_{k,0} \leq (1, <)$ is valid for R_p ; combining with (18) we obtain (17) and therefore our result.

- if $i = 0, j \neq 0$, we have $D'_{0,k} = D_{0,k} + (0, <)$, $D'_{0,j} = D_{0,j} + (0, <)$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{-}\mathcal{PDBM}_\odot(R_p)$, from [Definition 8 \(2\)](#) we know that $D_{0,k} \leq D_{0,j} + D_{j,k}$ is valid for R_p . Moreover we have that $(0, <) \leq (0, <)$ is valid for R_p so we have from [Lemma 1](#)

$$D_{0,k} + (0, <) \leq D_{0,j} + (0, <) + D_{j,k}$$

is valid for R_p . Hence $D'_{0,k} \leq D'_{0,j} + D'_{j,k}$ is valid for R_p .

- if $i = j = 0$, from [Definition 8 \(2\)](#) we trivially have

$$D'_{0,k} \leq D'_{0,0} + D'_{0,k}$$

is valid for R_p .

- if $x_j \in \text{LFP}_{R_p}(D)$ and $x_i, x_k \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$: $j \neq 0$ and
 - if i, k are different from 0, we have $D'_{i,k} = D_{i,k}$, $D'_{i,j} = D_{i,j}$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{-}\mathcal{PDBM}_\odot(R_p)$, from [Definition 8 \(2\)](#) we know that $D_{i,k} \leq D_{i,j} + D_{j,k}$ is valid for R_p ; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p .
 - if $i \neq 0, k = 0$, we have $D'_{i,0} = D_{i,j} + (1, <)$, $D'_{i,j} = D_{i,j}$ and $D'_{j,0} = (1, <)$; from [Definition 6 \(2b\)](#) we trivially have that $D_{i,j} + (1, <) \leq D_{i,j} + (1, <)$ is valid for R_p and therefore, $D'_{i,0} \leq D'_{i,j} + D'_{j,0}$ is valid for R_p .
 - if $i = 0, k \neq 0$, we have $D'_{0,k} = D_{0,k} + (0, <)$, $D'_{0,j} = D_{0,j} + (0, <)$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{-}\mathcal{PDBM}_\odot(R_p)$, from [Definition 8 \(2\)](#) we know that $D_{0,k} \leq D_{0,j} + D_{j,k}$ is valid for R_p . Moreover we have that $(0, <) \leq (0, <)$ is valid for R_p so we have

$$D_{0,k} + (0, <) \leq D_{0,j} + (0, <) + D_{j,k}$$

holds from [Definition 6 \(2b\)](#). Hence $D'_{0,k} \leq D'_{0,j} + D'_{j,k}$ is valid for R_p .

- if $i = k = 0$, we have $D'_{0,j} = D_{0,j} + (0, <)$ and $D'_{j,0} = (1, <)$; since $(E, D) \in p\text{-}\mathcal{PDBM}_\odot(R_p)$, from [Lemma 2](#) we know that $0 \leq D_{0,j} + D_{j,0}$ is valid for R_p , from [Definition 8 \(1\)](#) we know that $D_{j,0} \leq 1$ is valid for R_p which means $0 \leq D_{0,j} + (1, <)$ is valid for R_p . As we have

$$(1, <) + (0, <) = (1 + 0, < \oplus <) = (1, <)$$

we obtain that

$$(0, \leq) \leq D_{0,j} + (0, <) + (1, <)$$

is valid for R_p and therefore $D'_{0,0} \leq D'_{0,j} + D'_{j,0}$ is valid for R_p .

- if $x_j, x_k \in \text{LFP}_{R_p}(D)$ and $x_i \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$: $j \neq 0, k \neq 0$ and
 - if i is different from 0, we have $D'_{i,k} = D_{i,k}$, $D'_{i,j} = D_{i,j}$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{-}\mathcal{PDBM}_\odot(R_p)$, from [Definition 8 \(2\)](#) we know that $D_{i,k} \leq D_{i,j} + D_{j,k}$ is valid for R_p ; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p .
 - if $i = 0$, we have $D'_{0,k} = D_{0,k} + (0, <)$, $D'_{0,j} = D_{0,j} + (0, <)$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{-}\mathcal{PDBM}_\odot(R_p)$, from [Definition 8 \(2\)](#) we know

that $D_{0,k} \leq D_{0,j} + D_{j,k}$ is valid for R_p . Moreover we have that $(0, <) \leq (0, <)$ is valid for R_p so we have from [Lemma 1](#)

$$D_{0,k} + (0, <) \leq D_{0,j} + (0, <) + D_{j,k}$$

is valid for R_p . Hence $D'_{0,k} \leq D'_{0,j} + D'_{j,k}$ is valid for R_p .

5. if $x_i \in \text{LFP}_{R_p}(D)$ and $x_j, x_k \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$: $i \neq 0$ and
- if j, k are different from 0, we have $D'_{i,k} = D_{i,k}$, $D'_{i,j} = D_{i,j}$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{-PDBM}_\odot(R_p)$, from [Definition 8 \(2\)](#) we know that $D_{i,k} \leq D_{i,j} + D_{j,k}$ is valid for R_p ; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p .
 - if $j \neq 0, k = 0$, we have $D'_{i,0} = (1, <)$, $D'_{i,j} = D_{i,j}$ and $D'_{j,0} = D_{j,i} + (1, <)$; since $(E, D) \in p\text{-PDBM}_\odot(R_p)$, from [Lemma 2](#) we know that $0 \leq D_{i,j} + D_{j,i}$. Then from [Lemma 1](#)

$$(1, <) \leq D_{i,j} + D_{j,i} + (1, <)$$

is valid for R_p and therefore, $D'_{i,0} \leq D'_{i,j} + D'_{j,0}$ is valid for R_p .

- if $j = 0, k \neq 0$, we have $D'_{i,k} = D_{i,k}$, $D'_{i,0} = (1, <)$ and $D'_{0,k} = D_{0,k} + (0, <)$; we claim that

$$D_{i,k} \leq (1, <) + D_{0,k} + (0, <)$$

is valid for R_p , which is equivalent to $D'_{i,k} \leq D'_{i,0} + D'_{0,k}$ is valid for R_p . Since $(E, D) \in p\text{-PDBM}_\odot(R_p)$ from [Definition 8 \(2\)](#), we know that $D_{i,k} \leq D_{i,0} + D_{0,k}$ is valid for R_p ; moreover, from [Definition 8 \(1\)](#), we know that $D_{i,0} \leq (1, <)$ is valid for R_p . We have

$$(1, <) + (0, <) = (1 + 0, < \oplus <) = (1, <)$$

We obtain that

$$D_{i,k} \leq D_{i,0} + D_{0,k} \leq (1, <) + D_{0,k} = (1, <) + D_{0,k} + (0, <)$$

is valid for R_p and therefore our result.

- if i is different from 0, $j = k = 0$, we have $D'_{i,0} = (1, <)$, $D'_{j,k} = D'_{0,0} = (0, \leq)$; from [Definition 6 \(2b\)](#) we have that

$$(1, <) \leq (1, <)$$

is valid for R_p . Hence

$$D'_{i,0} \leq D'_{i,0} + D'_{0,0}$$

is valid for R_p .

6. if $x_i, x_k \in \text{LFP}_{R_p}(D)$ and $x_j \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$: $i \neq 0, k \neq 0$ and
- if $j \neq 0$, we have $D'_{i,k} = D_{i,k}$, $D'_{i,j} = D_{i,j}$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{-PDBM}_\odot(R_p)$, from [Definition 8 \(2\)](#) we know that $D_{i,k} \leq D_{i,j} + D_{j,k}$ is valid for R_p ; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p .

- if $j = 0$, we have $D'_{i,k} = D_{i,k}$, $D'_{i,0} = (1, <)$ and $D'_{0,k} = D_{0,k} + (0, <)$; we claim that

$$D_{i,k} \leq (1, <) + D_{0,k} + (0, <)$$

is valid for R_p , which is equivalent to $D'_{i,k} \leq D'_{i,0} + D'_{0,k}$ is valid for R_p . Since $(E, D) \in p\text{-PDBM}_\odot(R_p)$ from [Definition 8 \(2\)](#), we know that $D_{i,k} \leq D_{i,0} + D_{0,k}$ is valid for R_p ; moreover, from [Definition 8 \(1\)](#), we know that $D_{i,0} \leq (1, <)$ is valid for R_p . We have

$$(1, <) + (0, <) = (1 + 0, < \oplus <) = (1, <)$$

We obtain that

$$D_{i,k} \leq D_{i,0} + D_{0,k} \leq (1, <) + D_{0,k} = (1, <) + D_{0,k} + (0, <)$$

is valid for R_p and therefore our result.

- if $x_i, x_j \in \text{LFP}_{R_p}(D)$ and $x_k \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$: $i \neq 0, j \neq 0$ and
 - if $k \neq 0$, we have $D'_{i,k} = D_{i,k}$, $D'_{i,j} = D_{i,j}$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{-PDBM}_\odot(R_p)$, from [Definition 8 \(2\)](#), we know that $D_{i,k} \leq D_{i,j} + D_{j,k}$ is valid for R_p ; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p .
 - if $k = 0$, since both $x_i, x_j \in \text{LFP}_{R_p}(D)$ we have $D'_{i,j} = D_{i,j} = (0, \leq)$, $D'_{i,0} = (1, <)$ and $D'_{j,0} = (1, <)$; trivially $(1, <) \leq (0, \leq) + (1, <)$ is valid for R_p and therefore, $D'_{i,0} \leq D'_{i,j} + D'_{j,0}$ is valid for R_p .
- if $x_i, x_j, x_k \in \text{LFP}_{R_p}(D)$: i, j, k are different from 0, we have $D'_{i,k} = D_{i,k}$, $D'_{i,j} = D_{i,j}$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{-PDBM}_\odot(R_p)$, from [Definition 8 \(2\)](#) we know that $D_{i,k} \leq D_{i,j} + D_{j,k}$ is valid for R_p ; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p .

proof that [Definition 7 \(5b\)](#) holds Finally, for $x_i \in \text{LFP}_{R_p}(D)$, $D'_{i,0} = (1, <)$ and for all clock j s.t. $D'_{0,j} = (0, \triangleleft)$, then we have $\triangleleft = <$. Condition [Definition 7 \(5b\)](#) is satisfied.

We set $E' = E$ and denote by (E, D') the obtained $p\text{-PDBM}$, which is $(E, D') \in p\text{-PDBM}_\blacksquare(R_p)$.

H Proof of [Proposition 1](#)

Proposition 1 (recalled). *Let R_p be a parameter region and $(E, D) \in p\text{-PDBM}(R_p)$. Let $v \in R_p$. There exists $w' \in TE((E, v(D)))$ iff there exist $w \in (E, v(D))$ and a delay δ s.t. $w' = w + \delta$.*

Proof. Note that this proof is inspired by [[Hun+02](#), Proof of Lemma 3.13]

Lemma 17. *Let $(E, D) \in p\text{-PDBM}_\blacksquare(R_p)$. If (E, D) satisfies condition [Definition 7 \(5b\)](#) it has been obtained after applying Algorithm 2 on another open- $p\text{-PDBM}$ satisfying condition [Definition 7 \(5a\)](#) or a point- $p\text{-PDBM}$.*

Let $(E, D) \in p\text{-PDBM}_{\blacksquare}(R_p)$. If (E, D) satisfies condition [Definition 7 \(5a\)](#) it has been obtained after applying Algorithm 3 on another open-p-PDBM satisfying condition [Definition 7 \(5b\)](#) or after a non-parametric update applied on another open-p-PDBM or a point-p-PDBM.

Proof. Let $(E, D) \in p\text{-PDBM}_{\blacksquare}(R_p)$ and suppose (E, D) satisfies condition [Definition 7 \(5b\)](#). Since for all y , if $d_{0,y} = 0$ we have $\triangleleft_{0y} = <$, from [Lemma 10](#) and [Lemma 11](#) it cannot be the result of a non-parametric update where there is at least a clock x update and $D_{x,0} = D_{0,x} = (0, \leq)$. From [Lemma 15](#) it cannot be the result of Algorithm 3, as there must be at least a clock x s.t. $D_{x,0} = D_{0,x} = (0, \leq)$. Then it is the result either from [Lemma 13](#) of Algorithm 2 applied on an open-p-PDBM satisfying condition [Definition 7 \(5a\)](#), or from [Lemma 16](#) of Algorithm 2 applied on a point-p-PDBM.

Let $(E, D) \in p\text{-PDBM}_{\blacksquare}(R_p)$ and suppose (E, D) satisfies condition [Definition 7 \(5a\)](#). Since there is at least a clock y s.t. $D_{y,0} = D_{0,y} = (0, \leq)$, from [Lemma 13](#) and [Lemma 16](#) it cannot be the result of Algorithm 2, as for all x , if $d_{0,x} = 0$ we must have $\triangleleft_{0x} = <$. Then it is the result of either from [Lemma 15](#) of Algorithm 3 applied on an open-p-PDBM satisfying condition [Definition 7 \(5b\)](#) or from [Lemma 10](#) and [Lemma 11](#) of Algorithm 1 applied on an open-p-PDBM or a point-p-PDBM.

Let R_p be a parameter region and $(E, D) \in p\text{-PDBM}(R_p)$. We have to consider two different cases: $(E, D) \in p\text{-PDBM}_{\blacksquare}(R_p)$ and $(E, D) \in p\text{-PDBM}_{\odot}(R_p)$.

Lemma 18. *Let R_p be a parameter region and $(E, D) \in p\text{-PDBM}_{\blacksquare}(R_p)$. Let $v \in R_p$. There is $w' \in TE((E, v(D)))$ iff there is $w \in (E, v(D))$ and a delay δ s.t. $w' = w + \delta$.*

Proof. Let R_p a parameter region and $(E, D) \in p\text{-PDBM}_{\blacksquare}(R_p)$. Let $v \in R_p$.

\implies open-p-PDBM respecting [Definition 7 \(5a\)](#)

Let $v \in R_p$. Consider $(E', D') = TE((E, D))$ respecting condition [Definition 7 \(5a\)](#), i. e., suppose there is x_i s.t. $D'_{i,0} = -D'_{0,i} = (0, \leq)$. Let $w' \in (E', v(D'))$, for this x_i we have $w'(x_i) = 0$. We need to find a value δ s.t. $w' - \delta \in (E, v(D))$ which is equivalent to prove for all x_i, x_j

$$\text{frac}(w'(x_j)) - \delta - \text{frac}(w'(x_i)) + \delta \triangleleft_{ji} v(d_{j,i})$$

and

$$\text{frac}(w'(x_i)) - \delta - \text{frac}(w'(x_j)) + \delta \triangleleft_{ij} v(d_{i,j})$$

and

$$-\text{frac}(w'(x_j)) + \delta \triangleleft_{0j} v(d_{0,j}) \quad \text{and} \quad \text{frac}(w'(x_j)) - \delta \triangleleft_{j0} v(d_{j,0}).$$

In this proof we are going to define a δ which is different from 0, and give it an upper bound in order to show that constraints in (E, D) are satisfied while going backward of δ units of time from w' .

First we will prove that for all clock j , its constraints of lower bound $D_{0,j}$ and upper bound $D_{j,0}$ are satisfied. Second we will prove that for all i , bounds on their difference $D_{i,j}$ and $D_{j,i}$ are also satisfied.

We want to show that we have to go a little backward in time from w' to ensure the upper bounds $D_{j,0}$ of (E, D) hold. For this purpose, we are going to prove that for all x_j

$$D_{j,0} \leq D'_{j,0}$$

is valid for R_p . Intuitively this means upper bounds of clocks in (E', D') are greater than in (E, D) , which is consistent as time is elapsing.

As (E', D') respects [Definition 7 \(5a\)](#) and precisely $(E', D') = TE_=((E, D))$, we know (E, D) is respecting condition [Definition 7 \(5b\)](#) from [Lemma 13](#). As $\text{frac}(w'(x_i)) = 0$ it was in (E, D) a clock with the largest fractional part, *i. e.*, $x_i \in \text{LFP}_{R_p}(D)$ and $D_{i,0} = (1, <)$.

By definition of $TE_<$ (cf. Algorithm 2), in (E, D) which is the open-p-PDBM obtained after the application of $TE_<$ on another p-PDBM (see [Lemma 17](#)), for each $x_j \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$, $D_{j,0} = D_{j,i} + (1, <)$ and for all $x_j \in \mathbb{X}$, we have $D_{j,0}$ is of the form $(d_{j,0}, <)$ for some $d_{j,0}$.

By definition of $TE_ =$ applied to (E, D) (cf. Algorithm 3), in (E', D') , for each $x_j \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$, $D'_{j,0} = D_{j,i} + (1, \leq)$, *i. e.*, $d_{j,0} = d'_{j,0}$. Hence by [Definition 6 \(2b\)](#) and as $\triangleleft_{j0'}$ is either \leq or $<$, we have

$$(d_{j,0}, <) = D_{j,0} \leq D'_{j,0} = (d_{j,0}, \triangleleft_{j0'})$$

is valid for R_p . Next we define the largest amount of time so that all upper bounds of (E, D) are satisfied.

We claim that for all x_j , $\text{frac}(w'(x_j)) - v(d_{j,0}) \leq 0$. Indeed, remark that by applying Algorithm 2 then 3, constraints on upper bounds of clocks in (E, D) and (E', D') differ only by their \triangleleft . As for $i \in \text{LFP}_{R_p}(D)$ and $j \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$ it we have $D_{j,0} = D_{j,i} + (1, <)$ in (E, D) and $D'_{j,0} = D_{j,i} + (1, \leq)$ in (E', D') , so $d_{j,0} = d'_{j,0}$. Since for any x , its fractional part is less or equal to its upper bound in D and therefore in D' , any difference between a fractional part and its upper bound is either negative or null. For all x , since $\text{frac}(w'(x)) \triangleleft_{x0'} v(d'_{x,0})$ we have $\text{frac}(w'(x)) - v(d'_{x,0}) \triangleleft_{x0'} 0$. Since $v(d'_{x,0}) = v(d_{x,0})$, $\text{frac}(w'(x)) - v(d_{x,0}) \triangleleft_{x0'} 0$, therefore we have our result.

Now we claim that we have to go at least an $\epsilon > 0$ backward in time to ensure all bounds of (E, D) are met. Let $x_j \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$. As

$$\text{frac}(w'(x_j)) \triangleleft_{j0'} v(d_{j,0})$$

we have

- either $\triangleleft_{j0'} = <$ and we already have $\text{frac}(w'(x_j)) < v(d_{j,0})$,
- or $\triangleleft_{j0'} = \leq$ and for any $\epsilon > 0$ we have $\text{frac}(w'(x_j)) - \epsilon < v(d_{j,0})$.

It is also true for each $x_i \in \text{LFP}_{R_p}(D)$: after applying $TE_<$ recall that we have $D_{i,0} = (1, <)$. We can take $\epsilon > 0$ and define $\text{frac}(w(x_i)) = 1 - \epsilon$, so we have $\text{frac}(w(x_i)) < v(d_{i,0})$.

Now that we know we have to go a little backward in time (at least an $\epsilon > 0$) so upper bounds of (E, D) are satisfied, we are going to give an upper bound to ϵ so that all lower bounds $D_{0,j}$ of (E, D) are also satisfied.

Let

$$t_1 = \min_{x \in \mathbb{X}} \{ \text{frac}(w'(x)) + v(d_{0,x}) \}$$

We want to prove that $t_1 > 0$.

Let us prove that for all x_j , $D'_{0,j} \leq D_{0,j}$ is valid for R_p . Recall that for $x_i \in \text{LFP}_{R_p}(D)$, we have that $D_{i,0} = (1, <)$. Moreover, from [Definition 7 \(4\)](#) $D_{i,j} \leq D_{i,0} + D_{0,j}$ is valid for R_p , then we have

$$D_{i,j} \leq (1, <) + D_{0,j}$$

is valid for R_p . Recall that after applying Algorithm 3, $D'_{0,j} = D_{i,j} + (-1, \leq)$. By [Definition 6 \(2b\)](#) we have $(-1, \leq) \leq (-1, \leq)$. We invoke [Lemma 1](#) which gives

$$D_{i,j} + (-1, \leq) \leq (1, <) + D_{0,j} + (-1, \leq) = D_{0,j} + (0, <) \text{ is valid for } R_p. \quad (19)$$

As, from [Definition 6 \(2b\)](#) we have $D_{0,j} + (0, <) \leq D_{0,j}$ is valid for R_p , we infer (19) and it gives

$$D'_{0,j} \leq D_{0,j} \text{ is valid for } R_p.$$

Since $w' \in (E', v(D'))$ we have $-\text{frac}(w'(x_j)) \triangleleft_{0j'} v(d'_{0,j})$,

$$0 \triangleleft_{0j'} \text{frac}(w'(x_j)) + v(d'_{0,j}).$$

Then we have that

$$0 \triangleleft_{0j'} \text{frac}(w'(x_j)) + v(d'_{0,j}) \leq \text{frac}(w'(x_j)) + v(d_{0,j})$$

where ,

- either from [Definition 6 \(2a\)](#) $d'_{0,j} < d_{0,j}$;
- or from [Definition 6 \(2b\)](#), $d'_{0,j} \leq d_{0,j}$ and then $\triangleleft_{0j'} = \triangleleft_{0j} = <$. Indeed as $D'_{0,j} \leq D_{0,j}$ is valid for R_p , and since (E, D) is the **open-p**-PDBM obtained after the application of $TE_{<}$ (cf. Algorithm 2) on another **p**-PDBM (see [Lemma 17](#)), we have $\triangleleft_{0j} = <$.

To conclude we have that for all x_j either

$$0 \triangleleft_{0j'} \text{frac}(w'(x_j)) + v(d'_{0,j}) < \text{frac}(w'(x_j)) + v(d_{0,j})$$

or

$$0 < \text{frac}(w'(x_j)) + v(d'_{0,j}) \leq \text{frac}(w'(x_j)) + v(d_{0,j}).$$

As t_1 is by definition the minimum value of an expression $\text{frac}(w'(x_j)) + v(d_{0,j})$ for a given x_j , which as we just proved are all strictly positive, we have that for all x_j

$$0 < t_1 \leq \text{frac}(w'(x_j)) + v(d_{0,j}).$$

We proved that $t_1 > 0$, so we can set $\delta = \frac{t_1}{2}$ (therefore $\delta > 0$).
 More intuitively δ is the value right in the middle of the least and the largest amount of time s.t. we can go backward in time from w' and respect all constraints defined in $(E, v(D))$.
 Now we are going to prove that for any clock x_j , its constraints on lower and upper bounds are satisfied, *i. e.*,

$$-v(d_{0,j}) \triangleleft_{0j} \text{frac}(w'(x_j)) - \delta \triangleleft_{j0} v(d_{j,0}).$$

First as $\delta < t_1$, we have

$$-\text{frac}(w'(x_j)) + \delta < -\text{frac}(w'(x_j)) + t_1 \leq -\text{frac}(w'(x_j)) + \text{frac}(w'(x_j)) + v(d_{0,j}) = v(d_{0,j})$$

which is $-v(d_{0,j}) < \text{frac}(w'(x_j)) - \delta$. Since (E, D) is the **open-p-PDBM** obtained after the application of $TE_<$ (cf. Algorithm 3) on another **p-PDBM** (see Lemma 17), we have $\triangleleft_{0j} = <$ so $-v(d_{0,j}) \triangleleft_{0j} \text{frac}(w'(x_j)) - \delta$.
 Secondary as $0 < \delta$, we have

$$\text{frac}(w'(x_j)) - \delta < \text{frac}(w'(x_j)) - 0 \leq \text{frac}(w'(x_j)) - \text{frac}(w'(x_j)) + v(d_{j,0}) = v(d_{j,0})$$

which is $\text{frac}(w'(x_j)) - \delta < v(d_{j,0})$. Since (E, D) is the **open-p-PDBM** obtained after the application of $TE_<$ (cf. Algorithm 3) on another **p-PDBM** (see Lemma 17), we have $\triangleleft_{j0} = <$ so $\text{frac}(w'(x_j)) - \delta \triangleleft_{j0} v(d_{j,0})$.
 Now we prove that constraints defined in (E, D) on differences of clocks are also satisfied by going back of δ units of time from w' .
 Recall that in (E', D') we have for all clock x_j ,

$$D'_{j,i} = D'_{j,0} = D_{j,i} + 1 \quad \text{and} \quad D'_{i,j} = D'_{0,j} = -1 + D_{i,j}.$$

In addition by definition of $TE_<$, for $x_i \in \text{LFP}_{R_p}(D)$, $E_{x_i} = E'_{x_i} - 1$ and for $x_j \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$, $E_{x_j} = E'_{x_j}$.

We already treated the case whether i or j are 0, now suppose i, j are both different from 0.

- if $x_i, x_j \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$: let $x \in \text{LFP}_{R_p}(D)$ and recall that after applying Algorithm 3, $D'_{i,j} = D_{i,j}$, $D'_{j,i} = D_{j,i}$; we have that $\text{frac}(w'(x_j)) - \text{frac}(w'(x_i)) \triangleleft_{ij} d'_{j,i} = d_{j,i}$, and therefore $\text{frac}(w'(x_j)) - \delta - \text{frac}(w'(x_i)) + \delta \triangleleft_{ji} d_{j,i}$.

We also have that $\text{frac}(w'(x_i)) - \text{frac}(w'(x_j)) \triangleleft_{ij'} d'_{i,j} = d_{i,j}$, therefore $\text{frac}(w'(x_i)) - \delta - \text{frac}(w'(x_j)) + \delta \triangleleft_{ij} d_{i,j}$;

- if $x_i \in \text{LFP}_{R_p}(D)$ and $x_j \in \mathbb{X} \setminus \text{LFP}_{R_p}(D)$: recall that after applying Algorithm 3, $D'_{j,0} = D_{j,i} + (1, \leq)$, and $D'_{0,j} = D_{i,j} + (-1, \leq)$. Observe that as we added \leq which is the neutral element of the addition \oplus between two operators \triangleleft , we have $\triangleleft_{j0'} = \triangleleft_{ji}$ and $\triangleleft_{0j'} = \triangleleft_{ij}$. Note that as $x_i \in \text{LFP}_{R_p}(D)$, in (E', D') we have $D'_{0,i} = (0, \leq) = D'_{i,0}$ which means $\text{frac}(w'(x_i)) = 0$. Going backward in time of δ units of time from $w'(x_i)$ means that $\text{frac}(w'(x_i)) = 1 - \delta$.

We have that

$$\text{frac}(w'(x_j)) \triangleleft_{j0'} v(d'_{j,0}) = v(d_{j,i}) + 1$$

hence $\text{frac}(w'(x_j)) - 1 \triangleleft_{ji} v(d_{j,i})$ which is equivalent to

$$\text{frac}(w'(x_j)) - \delta - 1 + \delta \triangleleft_{ji} v(d_{j,i}).$$

The same way we have

$$-\text{frac}(w'(x_j)) \triangleleft_{0j'} v(d'_{0,j}) = v(d_{i,j}) - 1$$

hence $1 - \text{frac}(w'(x_j)) \triangleleft_{ij} v(d_{i,j})$ which is equivalent to

$$1 - \delta - \text{frac}(w'(x_j)) + \delta \triangleleft_{ij} v(d_{i,j}).$$

To conclude, we define for all x_j s.t. $D'_{0,j} \neq (0, \leq)$ and $D'_{j,0} \neq (0, \leq)$

$$w(x_j) = w'(x_j) - \delta$$

and for all x_i s.t. $D'_{0,i} = (0, \leq) = D'_{i,0}$

$$w(x_i) = (w'(x_i) - 1) + 1 - \delta$$

and clearly, $w \in (E, v(D))$.

\Rightarrow **open-p-PDBM respecting Definition 7 (5b)**

Let $v \in R_p$. Consider $(E', D') = TE((E, D))$ respecting condition Definition 7 (5b), i. e., suppose there is at least an x_i s.t. $D'_{i,0} = (1, <)$ and for all j s.t. $D_{0,j} = (0, \triangleleft_{0j})$, then we have $\triangleleft_{0j} = <$. Let $w' \in (E', v(D'))$.

We need to find a value δ s.t. $w' - \delta \in (E, v(D))$ which is equivalent to prove for all x_i, x_j

$$\text{frac}(w'(x_j)) - \delta - \text{frac}(w'(x_i)) + \delta \triangleleft_{ji} v(d_{j,i})$$

and

$$\text{frac}(w'(x_i)) - \delta - \text{frac}(w'(x_j)) + \delta \triangleleft_{ij} v(d_{i,j})$$

and

$$-\text{frac}(w'(x_j)) + \delta \triangleleft_{0j} v(d_{0,j}) \quad \text{and} \quad \text{frac}(w'(x_j)) - \delta \triangleleft_{j0} v(d_{j,0}).$$

As done previously we are going to define a δ which is different from 0 so we satisfy condition Definition 7 (5a), and show that constraints in (E, D) are satisfied while going backward of δ units of time from w' .

We define the largest and the least amount of time so that all upper bounds of (E, D) are satisfied. Let

$$t_0 = \max_{x \in \mathbb{X}} \{0, \text{frac}(w'(x)) - v(d_{x,0})\}$$

and

$$t_1 = \min_{x \in \mathbb{X}} \{\text{frac}(w'(x)) + v(d_{0,x})\}.$$

We want to prove that $t_0 = t_1 > 0$. For this purpose, let us first show that for all i, j we have $\text{frac}(w'(x_j)) - v(d'_{j,0}) \leq \text{frac}(w'(x_i)) + v(d'_{0,i})$, which is $t_0 \leq t_1$.

First note that for all i, j

$$\text{frac}(w'(x_j)) - \text{frac}(w'(x_i)) \triangleleft_{ji'} v(d'_{j,i}).$$

By applying $TE_<$ (Algorithm 2) to (E, D) , we have that $D'_{j,i} = D_{j,i}$, i. e., $(d_{i,j}, \triangleleft_{ij}) = (d'_{i,j}, \triangleleft_{ij'})$, and from Definition 7 (4) we have that $D_{j,i} \leq D_{j,0} + D_{0,i}$ is valid for R_p .

Hence, we have from Definition 6 (2b) that either $v(d_{j,i}) < v(d_{j,0}) + v(d_{0,i})$ or $v(d_{j,i}) \leq v(d_{j,0}) + v(d_{0,i})$ and $\triangleleft_{ji} = \triangleleft_{j0} \oplus \triangleleft_{0i}$ or $\triangleleft_{ji} = <$ and $\triangleleft_{j0} \oplus \triangleleft_{0i} = \leq$. We can then write that

$$\text{frac}(w'(x_j)) - \text{frac}(w'(x_i)) (\triangleleft_{j0} \oplus \triangleleft_{0i}) v(d_{j,0}) + v(d_{0,i})$$

which is equivalent to

$$\text{frac}(w'(x_j)) - v(d_{j,0}) (\triangleleft_{j0} \oplus \triangleleft_{0i}) \text{frac}(w'(x_i)) + v(d_{0,i})$$

so we obtain our result, as $(\triangleleft_{j0} \oplus \triangleleft_{0i})$ is either \leq or $<$.

Now, recall that (E, D) respects condition Definition 7 (5a) so we have at least an x s.t. $D_{x,0} = D_{0,x} = (0, \leq)$.

For this clock x we have that $\text{frac}(w'(x)) = \text{frac}(w'(x)) - v(d_{x,0}) \leq t_0$ and that $t_1 \leq \text{frac}(w'(x)) + v(d_{0,x}) = \text{frac}(w'(x))$.

Hence $t_0 = t_1 = \text{frac}(w'(x))$.

As $\triangleleft_{x0} = \leq$, we have $(\triangleleft_{x0} \oplus \triangleleft_{0i}) = \triangleleft_{0i}$ and $(\triangleleft_{j0} \oplus \triangleleft_{0x}) = \triangleleft_{j0}$, which gives

$$\text{frac}(w'(x)) = \text{frac}(w'(x)) - v(d_{x,0}) \triangleleft_{0i} \text{frac}(w'(x_i)) + v(d_{0,i})$$

and

$$\text{frac}(w'(x_j)) - v(d_{j,0}) \triangleleft_{j0} \text{frac}(w'(x)) + v(d_{0,x}) = \text{frac}(w'(x)).$$

Moreover in (E', D') we have that $\text{frac}(w'(x)) \triangleleft_{0x'} v(d'_{0,x})$. Since (E', D') respects condition Definition 7 (5b), if $D'_{0,x} = (0, \triangleleft_{0x'})$ then $\triangleleft_{0x'} = <$. Hence $0 < \text{frac}(w'(x))$ and

$$0 < t_0 = t_1.$$

Let $\delta = t_0 = t_1$. More intuitively δ is the value right in the middle of the least and the largest amount of time s.t. we can go backward in time from w' and respect all constraints defined in $(E, v(D))$.

First we have

$$-\text{frac}(w'(x_j)) + \delta \leq -\text{frac}(w'(x_j)) + t_1 \triangleleft_{j0} - \text{frac}(w'(x_j)) + \text{frac}(w'(x_j)) + v(d_{0,j}) = v(d_{0,j})$$

which is $-v(d_{0,j}) \triangleleft_{j0} \text{frac}(w'(x_j)) - \delta$.

Secondary we have

$$\text{frac}(w'(x_j)) - \delta \leq \text{frac}(w'(x_j)) - t_0 \triangleleft_{0j} \text{frac}(w'(x_j)) - \text{frac}(w'(x_j)) + v(d_{j,0}) = v(d_{j,0})$$

which is $\text{frac}(w'(x_j)) - \delta \triangleleft_{0j} v(d_{j,0})$.

Now we prove that constraints defined in (E, D) on differences of clocks are also satisfied by going back of δ units of time from w' . Recall that in (E', D') from the definition of Algorithm 2 we have for all clocks x_i, x_j ,

$$D'_{j,i} = D_{j,i} \quad \text{and} \quad D'_{i,j} = D_{i,j}.$$

Since we already treated the case whether i or j are 0, now suppose i, j are both different from 0. We have that $\text{frac}(w'(x_j)) - \text{frac}(w'(x_i)) \triangleleft_{ji'} v(d'_{j,i}) = v(d_{j,i})$, and therefore as $\triangleleft_{ji'} = \triangleleft_{ji}$,

$$\text{frac}(w'(x_j)) - \delta - \text{frac}(w'(x_i)) + \delta \triangleleft_{ji} v(d_{j,i}).$$

We also have that $\text{frac}(w'(x_i)) - \text{frac}(w'(x_j)) \triangleleft_{ij'} v(d'_{i,j}) = v(d_{i,j})$, therefore as $\triangleleft_{ij'} = \triangleleft_{ij}$,

$$\text{frac}(w'(x_i)) - \delta - \text{frac}(w'(x_j)) + \delta \triangleleft_{ij} v(d_{i,j}).$$

To conclude, we define for all x_j

$$w(x_j) = w'(x_j) - \delta$$

and clearly, $w \in (E, v(D))$.

Conversely, let $w \in (E, v(D))$,

\Leftarrow **open-p-PDBM respecting Definition 7 (5b)**

Suppose in (E, D) there is at least an x_i s.t. $D_{i,0} = (1, <)$ and for all j s.t. $D_{0,j} = (0, \triangleleft)$, we have $\triangleleft = <$. Let x_i be such a clock and $v \in R_p$. Now consider $(E', D') = TE((E, D))$. We need to find a value δ s.t. $w + \delta \in (E', v(D'))$. which is equivalent to prove for all x_i, x_j

$$\text{frac}(w(x_j)) + \delta - \text{frac}(w(x_i)) - \delta \triangleleft_{ji'} v(d'_{j,i})$$

and

$$\text{frac}(w(x_i)) + \delta - \text{frac}(w(x_j)) - \delta \triangleleft_{ij'} v(d'_{i,j})$$

and

$$-\text{frac}(w(x_j)) - \delta \triangleleft_{0j'} v(d'_{0,j}) \quad \text{and} \quad \text{frac}(w(x_j)) + \delta \triangleleft_{j0'} v(d'_{j,0}).$$

As done previously we are going to define a δ which is different from 0, and show that constraints in (E, D) are satisfied while going forward of δ units of time from w .

Recall that $x_i \in \text{LFP}_{R_p}(D)$ and let $\delta = 1 - \text{frac}(w(x_i))$ which we will prove is the exact amount of time so that all upper bounds of (E', D') are satisfied. Let

$$t_0 = \max_{x \in \mathbb{X}} \{-\text{frac}(w(x)) - \text{frac}(v(d'_{0,x}))\}$$

and

$$t_1 = \min_{x \in \mathbb{X}} \{\text{frac}(v(d'_{x,0})) - \text{frac}(w(x))\}.$$

Recall that since (E, D) respects condition [Definition 7 \(5b\)](#), for all j s.t. $D_{0,j} = (0, \triangleleft_{0j})$, we have $\triangleleft_{0j} = <$. Hence as $-frac(w(x_i)) < v(d_{0,j})$, $frac(w(x_i)) \neq 0$. Using the same reasoning as before, we are going to prove that $t_0 \leq \delta \leq t_1$.

First we will prove that $t_0 \leq \delta$. Consider $x_i \in \text{LFP}_{R_p}(D)$. For all clock x_j , since $w \in (E, v(D))$ we have $frac(w(x_i)) - frac(w(x_j)) \triangleleft_{ij} frac(v(d_{i,j}))$. From Algorithm 3 applied to (E, D) and since $x_i \in \text{LFP}_{R_p}(D)$ we obtain in (E', D') that $D'_{0,j} = D_{i,j} + (-1, \leq)$. Clearly we have $\triangleleft_{0j'} = \triangleleft_{ij} \oplus \leq = \triangleleft_{ij}$. It gives that

$$frac(w(x_i)) - frac(w(x_j)) - 1(\triangleleft_{ij} \oplus \leq) frac(v(d_{i,j})) - 1$$

which is equivalent to $frac(w(x_i)) - frac(w(x_j)) - 1 \triangleleft_{0j'} frac(v(d'_{0,j}))$ which is equivalent to

$$frac(w(x_i)) - 1 \triangleleft_{0j'} frac(v(d'_{0,j})) + frac(w(x_j)).$$

This gives us our first result.

Second we will prove that $\delta \leq t_1$. Consider $x_i \in \text{LFP}_{R_p}(D)$. For all clock x_j , from [Definition 7 \(4\)](#) we have $frac(w(x_j)) - frac(w(x_i)) \triangleleft_{ji} frac(v(d_{j,i}))$. We have

$$frac(w(x_j)) - frac(w(x_i)) + 1 \triangleleft_{ji} frac(v(d_{j,i})) + 1.$$

From Algorithm 3 applied to (E, D) and since $x_i \in \text{LFP}_{R_p}(D)$ we obtain in (E', D') that $D'_{j,0} = D_{j,i} + (1, \leq)$. Clearly we have $\triangleleft_{j0'} = \triangleleft_{ji} \oplus \leq = \triangleleft_{ji}$. Then we can write that $frac(w(x_j)) - frac(w(x_i)) + 1 \triangleleft_{j0'} frac(v(d'_{j,0}))$ which is equivalent to

$$1 - frac(w(x_i)) \triangleleft_{j0'} frac(v(d'_{j,0})) - frac(w(x_j)).$$

This gives us our second result.

Now for all clock x_j , we obtain two results. First we have

$$-frac(w(x_j)) - \delta \triangleleft_{0j'} - frac(w(x_j)) - t_1 \leq -frac(w(x_j)) + frac(w(x_j)) + v(d'_{0,j}) = v(d'_{0,j})$$

which is $-v(d'_{0,j}) \triangleleft_{0j'} frac(w(x_j)) + \delta$.

Secondary we have

$$frac(w(x_j)) + \delta \triangleleft_{j0'} frac(w(x_j)) + t_0 \leq frac(w(x_j)) - frac(w(x_j)) + v(d'_{j,0}) = v(d'_{j,0})$$

which is $frac(w(x_j)) + \delta \triangleleft_{j0'} v(d'_{j,0})$.

Since we already treated the case whether i or j are 0, now suppose i, j are both different from 0.

Note that if both $x_i, x_j \in \text{LFP}_{R_p}(D)$, as $frac(w(x_i)) = frac(w(x_j))$, $D_{i,j} = D'_{i,j} = (0, \leq)$ and $D_{j,i} = D'_{j,i} = (0, \leq)$ from [Definition 10](#). Hence $frac(w(x_i)) + \delta - frac(w(x_j)) - \delta \triangleleft_{ij'} frac(v(d'_{i,j}))$ and $frac(w(x_j)) + \delta - frac(w(x_j)) - \delta \triangleleft_{ji'} frac(v(d'_{j,i}))$.

The same way, if both $x_i, x_j \notin \text{LFP}_{R_p}(D)$ we have $D_{i,j} = D'_{i,j}$ and $D_{j,i} = D'_{j,i}$ and again our result. If either x_i or x_j is in $\text{LFP}_{R_p}(D)$, the case is similar to $D'_{0,j}$ or $D'_{i,0}$.

Finally, we define $w' = w + \delta$ and $w' \in (E', v(D'))$.

\Leftarrow open-p-PDBM respecting **Definition 7 (5a)**

Suppose in (E, D) there is at least an x_j s.t. $D_{j,0} = D_{0,j} = (0, \leq)$. Let $v \in R_p$, and $x_i \in \text{LFP}_{R_p}(D)$.

Now consider $(E', D') = TE((E, D))$. We need to find a value δ s.t. $w + \delta \in (E', v(D'))$. which is equivalent to prove for all x_i, x_j

$$\text{frac}(w(x_j)) + \delta - \text{frac}(w(x_i)) - \delta \triangleleft_{ji'} v(d'_{j,i})$$

and

$$\text{frac}(w(x_i)) + \delta - \text{frac}(w(x_j)) - \delta \triangleleft_{ji'} v(d'_{i,j})$$

and

$$-\text{frac}(w(x_j)) - \delta \triangleleft_{0j'} v(d'_{0,j}) \quad \text{and} \quad \text{frac}(w(x_j)) + \delta \triangleleft_{j0'} v(d'_{j,0}).$$

As done previously we are going to define a δ which is different from 0, and show that constraints in (E, D) are satisfied while going forward of δ units of time from w .

Let

$$t_0 = \max_{x \in \mathbb{X}} \{0, -\text{frac}(w(x)) - \text{frac}(v(d'_{0,x}))\}$$

and

$$t_1 = \min_{x \in \mathbb{X}} \{\text{frac}(v(d'_{x,0})) - \text{frac}(w(x))\}.$$

We want to prove that $t_0 \leq t_1$. For this purpose, we are going to prove for all clocks i, j that $-\text{frac}(w(x_j)) - v(d'_{j,0}) \leq v(d'_{0,i}) - \text{frac}(w(x_i))$.

First note that

$$\text{frac}(w(x_j)) - \text{frac}(w(x_i)) \triangleleft_{ji} v(d_{j,i})$$

By definition of $TE_{<}$ applied to (E, D) , we have that $D'_{j,i} = D_{j,i}$, and from **Definition 7 (4)** we have that $D'_{j,i} \leq D'_{j,0} + D'_{0,i}$.

Hence, we have from **Definition 6 (2b)** that either $d'_{j,i} < d'_{j,0} + d'_{0,i}$ or $d'_{j,i} = d'_{j,0} + d'_{0,i}$ and $\triangleleft_{ji'} = \triangleleft_{j0'} \oplus \triangleleft_{0i'}$ or $\triangleleft_{ji'} = <$ and $\triangleleft_{j0'} \oplus \triangleleft_{0i'} = \leq$.

We can then write that

$$\text{frac}(w(x_j)) - \text{frac}(w(x_i)) (\triangleleft_{j0'} \oplus \triangleleft_{0i'}) v(d'_{j,0}) + v(d'_{0,i})$$

which is equivalent to

$$-\text{frac}(w(x_i)) - v(d'_{0,i}) (\triangleleft_{j0'} \oplus \triangleleft_{0i'}) v(d'_{j,0}) - \text{frac}(w(x_j))$$

Now we prove that $t_0 = 0$. Clearly from **Definition 7** for any clock i we have that $-\text{frac}(w(x_i)) \triangleleft_{0i} v(d_{0,i})$ which is equivalent to $-\text{frac}(w(x_i)) - v(d_{0,i}) \triangleleft_{0i} 0$.

Hence if as (E, D) there is at least an x_j s.t. $D_{j,0} = D_{0,j} = (0, \leq)$, for this clock j we have $-\text{frac}(w(x_j)) - v(d_{0,j}) = 0$.

By definition of $TE_{<}$ applied to (E, D) , we have that $D'_{0,i} = D_{0,i} + (0, <)$. In order to respect the constraint $-\text{frac}(w(x_i)) - \delta \triangleleft_{0i'} v(d'_{0,i})$ which is,

as $\triangleleft_{0i'} = <$, $-frac(w(x_i)) - \delta < v(d'_{0,i})$ and especially for j where $v(d'_{0,j}) = 0$ we have to find a $\delta > 0$.

In order to find an upper bound for δ , we are going to prove that $t_1 > 0$. From Definition 7 (4) we have in (E, D) that for any clocks i, j $D_{j,0} \leq D_{j,i} + D_{i,0}$. Let $x_i \in \text{LFP}_{R_p}(D)$. From Definition 7 (1), we have that $D_{i,0} \leq (1, <)$. This gives that $D_{j,i} + D_{i,0} \leq D_{j,i} + (1, <)$.

By definition of $TE_{<}$ applied to (E, D) , we have that $D'_{j,0} = D_{j,i} + (1, <)$. Hence we have $D_{j,0} \leq D'_{j,0}$.

Now as $frac(w(x_i)) \triangleleft_{i0} v(d_{i,0})$ we can write $frac(w(x_i)) \triangleleft_{i0'} v(d'_{i,0})$ and then $0 \triangleleft_{i0'} v(d'_{i,0}) - frac(w(x_i))$ where $\triangleleft_{i0'} = <$, which prove our result.

We define $\delta = \frac{t_1}{2}$, therefore $t_0 < \delta < t_1$. Now for all clock x_j , we obtain two results. First we have

$$-frac(w(x_j)) - \delta < -frac(w(x_j)) - t_1 \triangleleft_{0j'} -frac(w(x_j)) + frac(w(x_j)) + v(d'_{0,j}) = v(d'_{0,j})$$

which is $-v(d'_{0,j}) \triangleleft_{0j} frac(w(x_j)) + \delta$ as $\triangleleft_{0j'} = <$.

Secondary we have

$$frac(w(x_j)) + \delta < frac(w(x_j)) + t_0 \triangleleft_{j0'} frac(w(x_j)) - frac(w(x_j)) + v(d'_{j,0}) = v(d'_{j,0})$$

which is $frac(w(x_j)) + \delta \triangleleft_{j0} v(d'_{j,0})$ as $\triangleleft_{j0'} = <$.

Now we prove that constraints defined in (E', D') on differences of clocks are also satisfied by going forward of δ units of time from w

Recall that in (E', D') from the definition of Algorithm 2 we have for all clock x_j ,

$$D'_{j,i} = D_{j,i} \quad \text{and} \quad D'_{i,j} = D_{i,j}.$$

Since we already treated the case whether i or j are 0, now suppose i, j are both different from 0. We have that $frac(w(x_j)) - frac(w(x_i)) \triangleleft_{ji} v(d_{j,i}) = v(d'_{j,i})$, and therefore as $\triangleleft_{ji'} = \triangleleft_{ji}$,

$$frac(w(x_j)) + \delta - frac(w(x_i)) - \delta \triangleleft_{ji'} v(d'_{j,i}).$$

We also have that $frac(w(x_i)) - frac(w(x_j)) \triangleleft_{ij} v(d_{i,j}) = v(d'_{i,j})$, therefore as $\triangleleft_{ij'} = \triangleleft_{ij}$,

$$frac(w(x_i)) + \delta - frac(w(x_j)) - \delta \triangleleft_{ij'} v(d'_{i,j}).$$

Finally, we define $w' = w + \delta$ and $w' \in (E', v(D'))$.

Lemma 19. *Let R_p be a parameter region and $(E, D) \in p\text{-PDBM}_{\odot}(R_p)$. Let $v \in R_p$. There is $w' \in TE((E, v(D)))$ iff there is $w \in (E, v(D))$ and a delay δ s.t. $w' = w + \delta$.*

Proof. $\Leftarrow p\text{-PDBM}_{\odot}(R_p)$

Let $v \in R_p$. Consider $(E', D') = TE((E, D))$ respecting condition Definition 7 (5b), i.e., suppose there is at least an x_i s.t. $D'_{i,0} = (1, <)$ and for all j s.t. $D_{0,j} = (0, \triangleleft_{0j})$, then we have $\triangleleft_{0j} = <$. Let $w' \in (E', v(D'))$.

We need to find a value δ s.t. $w' - \delta \in (E, v(D))$ which is equivalent to prove for all x_i, x_j

$$\text{frac}(w'(x_j)) - \delta - \text{frac}(w'(x_i)) + \delta \triangleleft_{ji} v(d_{j,i})$$

and

$$\text{frac}(w'(x_i)) - \delta - \text{frac}(w'(x_j)) + \delta \triangleleft_{ij} v(d_{i,j})$$

and

$$-\text{frac}(w'(x_j)) + \delta \triangleleft_{0j} v(d_{0,j}) \quad \text{and} \quad \text{frac}(w'(x_j)) - \delta \triangleleft_{j0} v(d_{j,0}).$$

As done previously we are going to define a δ which is different from 0, and show that constraints in (E, D) are satisfied while going backward of δ units of time from w' .

We define the largest and the least amount of time so that all upper bounds of (E, D) are satisfied. Let

$$t_0 = \max_{x \in \mathbb{X}} \{0, \text{frac}(w'(x)) - v(d_{x,0})\}$$

and

$$t_1 = \min_{x \in \mathbb{X}} \{\text{frac}(w'(x)) + v(d_{0,x})\}.$$

We want to prove that $t_0 = t_1 > 0$. For this purpose, let us first show that for all i, j we have $\text{frac}(w'(x_j)) - v(d'_{j,0}) \leq \text{frac}(w'(x_i)) + v(d'_{0,i})$, which is $t_0 \leq t_1$. First note that for all i, j

$$\text{frac}(w'(x_j)) - \text{frac}(w'(x_i)) \triangleleft_{ji'} v(d'_{j,i}).$$

By applying $TE_{<}$ (Algorithm 2) to (E, D) , we have that $D'_{j,i} = D_{j,i}$, i. e., $(d_{i,j}, \triangleleft_{ij}) = (d'_{i,j}, \triangleleft_{ij'})$, and from [Definition 8 \(2\)](#) we have that $D_{j,i} \leq D_{j,0} + D_{0,i}$ is valid for R_p .

Hence, we have from [Definition 6 \(2b\)](#) that either $v(d_{j,i}) < v(d_{j,0}) + v(d_{0,i})$ or $v(d_{j,i}) \leq v(d_{j,0}) + v(d_{0,i})$ and $\triangleleft_{ji} = \triangleleft_{j0} \oplus \triangleleft_{0i}$ or $\triangleleft_{ji} = <$ and $\triangleleft_{j0} \oplus \triangleleft_{0i} = \leq$. We can then write that

$$\text{frac}(w'(x_j)) - \text{frac}(w'(x_i)) (\triangleleft_{j0} \oplus \triangleleft_{0i}) v(d_{j,0}) + v(d_{0,i})$$

which is equivalent to

$$\text{frac}(w'(x_j)) - v(d_{j,0}) (\triangleleft_{j0} \oplus \triangleleft_{0i}) \text{frac}(w'(x_i)) + v(d_{0,i})$$

so we obtain our result, as $(\triangleleft_{j0} \oplus \triangleleft_{0i})$ is either \leq or $<$.

Now, recall that in (E, D) for all x we have $d_{0,x} = -d_{x,0}$ and $\triangleleft_{0x} = \triangleleft_{x0}$.

For any clock x we have that $\text{frac}(w'(x)) - v(d_{x,0}) \leq t_0$ and that $t_1 \leq \text{frac}(w'(x)) + v(d_{0,x}) = \text{frac}(w'(x)) - v(d_{x,0})$.

Hence $t_0 = t_1$.

As for all x , $\triangleleft_{x0} = \leq$, we have for all i, j that $(\triangleleft_{x0} \oplus \triangleleft_{0i}) = \triangleleft_{0i}$ and $(\triangleleft_{j0} \oplus \triangleleft_{0x}) = \triangleleft_{j0}$, which gives

$$t_1 \triangleleft_{0i} \text{frac}(w'(x_i)) + v(d_{0,i})$$

and

$$\text{frac}(w'(x_j)) - v(d_{j,0}) \triangleleft_{j0} t_0.$$

Moreover in (E', D') we have that $\text{frac}(w'(x)) \triangleleft_{0x'} v(d'_{0,x})$. From [Lemma 17](#), (E', D') is obtained after applying Algorithm 2 and therefore $\triangleleft_{0x'} = <$. Hence $0 < \text{frac}(w'(x))$ and

$$0 < t_0 = t_1.$$

Let $\delta = t_0 = t_1$. More intuitively δ is the value right in the middle of the least and the largest amount of time s.t. we can go backward in time from w' and respect all constraints defined in $(E, v(D))$.

First we have

$$-\text{frac}(w'(x_j)) + \delta \leq -\text{frac}(w'(x_j)) + t_1 \triangleleft_{j0} -\text{frac}(w'(x_j)) + \text{frac}(w'(x_j)) + v(d_{0,j}) = v(d_{0,j})$$

which is $-v(d_{0,j}) \triangleleft_{j0} \text{frac}(w'(x_j)) - \delta$.

Secondary we have

$$\text{frac}(w'(x_j)) - \delta \leq \text{frac}(w'(x_j)) - t_0 \triangleleft_{0j} \text{frac}(w'(x_j)) - \text{frac}(w'(x_j)) + v(d_{j,0}) = v(d_{j,0})$$

which is $\text{frac}(w'(x_j)) - \delta \triangleleft_{0j} v(d_{j,0})$.

Now we prove that constraints defined in (E, D) on differences of clocks are also satisfied by going back of δ units of time from w'

Recall that in (E', D') from the definition of Algorithm 2 we have for all clocks x_i, x_j ,

$$D'_{j,i} = D_{j,i} \quad \text{and} \quad D'_{i,j} = D_{i,j}.$$

Since we already treated the case whether i or j are 0, now suppose i, j are both different from 0. We have that $\text{frac}(w'(x_j)) - \text{frac}(w'(x_i)) \triangleleft_{ji'} v(d'_{j,i}) = v(d_{j,i})$, and therefore as $\triangleleft_{ji'} = \triangleleft_{ji}$,

$$\text{frac}(w'(x_j)) - \delta - \text{frac}(w'(x_i)) + \delta \triangleleft_{ji} v(d_{j,i}).$$

We also have that $\text{frac}(w'(x_i)) - \text{frac}(w'(x_j)) \triangleleft_{ij'} v(d'_{i,j}) = v(d_{i,j})$, therefore as $\triangleleft_{ij'} = \triangleleft_{ij}$,

$$\text{frac}(w'(x_i)) - \delta - \text{frac}(w'(x_j)) + \delta \triangleleft_{ij} v(d_{i,j}).$$

To conclude, we define for all x_j

$$w(x_j) = w'(x_j) - \delta$$

and clearly, $w \in (E, v(D))$.

$\implies p\text{-}\mathcal{PDBM}_\odot(R_p)$

Assume in $(E, D) \in p\text{-}\mathcal{PDBM}_\odot(R_p)$. Let $v \in R_p$, and $x_i \in \text{LFP}_{R_p}(D)$.

Now consider $(E', D') = TE((E, D))$. We need to find a value δ s.t. $w + \delta \in (E', v(D'))$. which is equivalent to prove for all x_i, x_j

$$\text{frac}(w(x_j)) + \delta - \text{frac}(w(x_i)) - \delta \triangleleft_{j,i'} v(d'_{j,i})$$

and

$$\text{frac}(w(x_i)) + \delta - \text{frac}(w(x_j)) - \delta \triangleleft_{i,j'} v(d'_{i,j})$$

and

$$-\text{frac}(w(x_j)) - \delta \triangleleft_{0,j'} v(d'_{0,j}) \quad \text{and} \quad \text{frac}(w(x_j)) + \delta \triangleleft_{j,0'} v(d'_{j,0}).$$

As done previously we are going to define a δ which is different from 0, and show that constraints in (E, D) are satisfied while going forward of δ units of time from w .

Let

$$t_0 = \max_{x \in \mathbb{X}} \{0, -\text{frac}(w(x)) - \text{frac}(v(d'_{0,x}))\}$$

and

$$t_1 = \min_{x \in \mathbb{X}} \{\text{frac}(v(d'_{x,0})) - \text{frac}(w(x))\}.$$

We prove that $t_1 \leq t_0$.

for any clock i we have that $D_{i,0} = (\text{frac}(p), \leq)$ and $D_{i,0} = (-\text{frac}(p), \leq)$ i. e., $d_{0,i} = -d_{i,0}$ for some p , hence $-\text{frac}(w(x_i)) - v(d_{0,i}) = -\text{frac}(w(x_i)) + v(d_{i,0})$.

By definition of $TE_<$ applied to (E, D) , we have that $D'_{0,i} = D_{0,i} + (0, <)$. In order to respect the constraint $-\text{frac}(w(x_i)) - \delta \triangleleft_{0,i'} v(d'_{0,i})$ which is, as $\triangleleft_{0,i'} = <$, $-\text{frac}(w(x_i)) - \delta < v(d'_{0,i})$, we have to find a $\delta > 0$.

In order to find an upper bound for δ , we are going to prove that $t_1 > 0$. From [Definition 8 \(2\)](#) we have in (E, D) that for any clocks i, j $D_{j,0} \leq D_{j,i} + D_{i,0}$. Let $x_i \in \text{LFP}_{R_p}(D)$. From [Definition 8 \(1\)](#), we have that $D_{i,0} \leq (1, <)$. This gives that $D_{j,i} + D_{i,0} \leq D_{j,i} + (1, <)$.

By definition of $TE_<$ applied to (E, D) , we have that $D'_{j,0} = D_{j,i} + (1, <)$. Hence we have $D_{j,0} \leq D'_{j,0}$.

Now as $\text{frac}(w(x_i)) \triangleleft_{i,0} v(d_{i,0})$ we can write $\text{frac}(w(x_i)) \triangleleft_{i,0'} v(d'_{i,0})$ and then $0 \triangleleft_{i,0'} v(d'_{i,0}) - \text{frac}(w(x_i))$ where $\triangleleft_{i,0'} = <$, which prove our result.

We define $\delta = \frac{t_1}{2}$, therefore $t_0 < \delta < t_1$. Now for all clock x_j , we obtain two results. First we have

$$-\text{frac}(w(x_j)) - \delta < -\text{frac}(w(x_j)) - t_1 \triangleleft_{0,j'} -\text{frac}(w(x_j)) + \text{frac}(w(x_j)) + v(d'_{0,j}) = v(d'_{0,j})$$

which is $-v(d'_{0,j}) \triangleleft_{0,j} \text{frac}(w(x_j)) + \delta$ as $\triangleleft_{0,j'} = <$.

Secondary we have

$$\text{frac}(w(x_j)) + \delta < \text{frac}(w(x_j)) + t_0 \triangleleft_{j,0'} \text{frac}(w(x_j)) - \text{frac}(w(x_j)) + v(d'_{j,0}) = v(d'_{j,0})$$

which is $\text{frac}(w(x_j)) + \delta \triangleleft_{j,0} v(d'_{j,0})$ as $\triangleleft_{j,0'} = <$.

Now we prove that constraints defined in (E', D') on differences of clocks are also satisfied by going forward of δ units of time from w . Recall that in (E', D') from the definition of Algorithm 2 we have for all clock x_j ,

$$D'_{j,i} = D_{j,i} \quad \text{and} \quad D'_{i,j} = D_{i,j}.$$

Since we already treated the case whether i or j are 0, now suppose i, j are both different from 0. We have that $\text{frac}(w(x_j)) - \text{frac}(w(x_i)) \triangleleft_{ji} v(d_{j,i}) = v(d'_{j,i})$, and therefore as $\triangleleft_{ji'} = \triangleleft_{ji}$,

$$\text{frac}(w(x_j)) + \delta - \text{frac}(w(x_i)) - \delta \triangleleft_{ji'} v(d'_{j,i}).$$

We also have that $\text{frac}(w(x_i)) - \text{frac}(w(x_j)) \triangleleft_{ij} v(d_{i,j}) = v(d'_{i,j})$, therefore as $\triangleleft_{ij'} = \triangleleft_{ij}$,

$$\text{frac}(w(x_i)) + \delta - \text{frac}(w(x_j)) - \delta \triangleleft_{ij'} v(d'_{i,j}).$$

Finally, we define $w' = w + \delta$ and $w' \in (E', v(D'))$.

I Proof of Lemma 8

Lemma 8 (recalled). *Let (E, D) be a p-PDBM for R_p and $v \in R_p$. Let g be a non-parametric guard. If $v \in \text{guard}_\forall(g, E, D)$, then for all $v' \in R_p$, $v' \in \text{guard}_\forall(g, E, D)$.*

Proof. Our idea is to define a clock region “larger” than our p-PDBM (following Definition 3) and show that, even for this (larger) clock region, either all clock valuations satisfy the guard—or none do.

Definition 14. *Let R_p be a parameter region, $v \in R_p$. Let (E, D) be a p-PDBM for R_p . We define the clock region containing $(E, v(D))$, denoted by $[(E, v(D))]_{R_c}$, as follows: for all $w \in [(E, v(D))]_{R_c}$, for all clocks x_i, x_j ,*

- if $E_{x_i} < K$, $\lfloor w(x_i) \rfloor = E_{x_i}$, else if $E_{x_i} = \infty$, $w(x_i) \geq K$
- if $(0, \leq) < D_{i,j}$ is valid for R_p and $E_{x_i} < K$, $\text{frac}(w(x_j)) < \text{frac}(w(x_i))$
- if $(0, \leq) = D_{i,j}$ is valid for R_p and $E_{x_i} < K$, $\text{frac}(w(x_j)) = \text{frac}(w(x_i))$
- if $D_{i,0} = D_{0,i} = (0, \leq)$ and $E_{x_i} < K$, $\text{frac}(w(x_i)) = 0$.
- if $D_{i,0} \neq (0, \leq)$, $D_{0,i} \neq (0, \leq)$ and $E_{x_i} < K$, $\text{frac}(w(x_i)) \neq 0$.

Lemma 20. *Let (E, D) be a p-PDBM for R_p and $v \in R_p$. We have $(E, v(D)) \subseteq [(E, v(D))]_{R_c}$.*

Proof. Clock regions of Definition 3 define constraints on clocks of the form $0 = \text{frac}(x)$, $0 < \text{frac}(x) < 1$, $0 = \text{frac}(x) - \text{frac}(y)$ and $0 < \text{frac}(x) - \text{frac}(y) < 1$ for some x, y , and $\lfloor x \rfloor = k$ for some integer k . Let (E, D) be a p-PDBM for R_p and $v \in R_p$. It defines a set of constraints

$$\bigwedge_{i,j \in [0,H]^2} \text{frac}(x_i) - \text{frac}(x_j) \triangleleft_{i,j} v(d_{i,j}) \quad \wedge \quad \bigwedge_{i \in [1,H]} \lfloor x_i \rfloor = E_i.$$

straints $\bigwedge_{i \in [1, H]} \lfloor x_i \rfloor = E_i$. Clearly, if $w \in (E, v(D))$ satisfies $\lfloor x_i \rfloor = E_i$ then it satisfies the same constraint defined in $[(E, v(D))]_{R_c}$.

Consider the constraints $\text{frac}(x_i) - \text{frac}(x_j) \triangleleft_{i,j} v(d_{i,j})$ and $\text{frac}(x_j) - \text{frac}(x_i) \triangleleft_{j,i} v(d_{j,i})$.

- If i, j are both different from 0. From [Definition 7 \(3\)](#) and [Definition 8](#), either $d_{i,j} = d_{j,i}$ and then $\triangleleft_{i,j} = \leq = \triangleleft_{j,i}$, then if $d_{i,j} = d_{j,i} = 0$ it satisfies the same constraint defined in $[(E, v(D))]_{R_c}$, or $d_{i,j}$ and $d_{j,i}$ are different from 0, as they are elements of \mathcal{PLT} which are strictly smaller than 1, it satisfies either $0 < \text{frac}(x_i) - \text{frac}(x_j) < 1$ or $0 = \text{frac}(x_i) - \text{frac}(x_j)$ in $[(E, v(D))]_{R_c}$. Finally if $d_{i,j} \neq d_{j,i}$, then $\triangleleft_{i,j} = < = \triangleleft_{j,i}$ and it satisfies $0 < \text{frac}(x_i) - \text{frac}(x_j) < 1$ in $[(E, v(D))]_{R_c}$.
- If i is different from 0 and $j = 0$. From [Definition 7 \(3\)](#) and [Definition 8](#), either $d_{i,0} = d_{0,i}$ and then $\triangleleft_{i,0} = \leq = \triangleleft_{0,i}$, then if $d_{i,0} = d_{0,i} = 0$ it satisfies the same constraint defined in $[(E, v(D))]_{R_c}$, or $d_{i,0}$ and $d_{0,i}$ are different from 0, as they are elements of \mathcal{PLT} which are strictly smaller than 1, it satisfies either $0 < \text{frac}(x_i) < 1$ or $0 = \text{frac}(x_i)$ in $[(E, v(D))]_{R_c}$. Finally if $d_{i,0} \neq d_{0,i}$, then $\triangleleft_{i,0} = < = \triangleleft_{0,i}$ and it satisfies $0 < \text{frac}(x_i) < 1$ in $[(E, v(D))]_{R_c}$.
- The case j is different from 0 and $i = 0$ is similar.
- If both i, j are 0, the constraint is not taken into account as we have not x_0 in $[(E, v(D))]_{R_c}$.

Finally, we have that if $w \in (E, v(D))$ then $w \in [(E, v(D))]_{R_c}$.

Now we come back to the proof of the main lemma:

Let (E, D) be a \mathbf{p} -PDBM for R_p and $v \in R_p$. It defines a set of constraints

$$\bigwedge_{i,j \in [0, H]^2} \text{frac}(x_i) - \text{frac}(x_j) \triangleleft_{i,j} v(d_{i,j}) \quad \wedge \quad \bigwedge_{i \in [1, H]} \lfloor x_i \rfloor = E_i.$$

Moreover, let g be a non-parametric guard. It defines a set of constraints for a finite number of integer constants k_i with $i \in I \subseteq [1, H]$

$$\bigwedge_{i \in I} \text{frac}(x_i) \leq 0 \quad \wedge \quad \bigwedge_{i \in I} -\text{frac}(x_i) \leq 0 \quad \wedge \quad \bigwedge_{i \in I} \lfloor x_i \rfloor \bowtie k_i.$$

The intersection between the two is given by the conjunction of those constraints. We project this intersection on parameter variables (by elimination of clock variables) and we prove that the intersection does not create new constraints on parameters different from those we already have in $(E, v(D))$ (and therefore in R_p). For some set of clocks $I \subseteq [1, H]$ and $i \in I$, suppose we have the constraints $\text{frac}(x_i) \leq 0$ and $-\text{frac}(x_i) \leq 0$ in g . When eliminating x_i in any constraint of the form $\text{frac}(x_i) - \text{frac}(x_j) \triangleleft_{i,j} v(d_{i,j})$, it is clear that we proceed on \mathcal{PLT} to the operation $(0, \leq) + (d_{i,j}, \triangleleft_{i,j}) = (0 + d_{i,j}, \leq \oplus \triangleleft_{i,j}) = (d_{i,j}, \triangleleft_{i,j})$. The same way on any constraint of the form $\text{frac}(x_i) \triangleleft_{i,0} v(d_{i,0})$, eliminating x_i gives the constraint $(0, \leq) + (d_{i,0}, \triangleleft_{i,0}) = (d_{i,0}, \triangleleft_{i,0})$. Hence it does not create new inequalities not belonging to R_p .

Now suppose $v \in \text{guard}_\forall(g, E, D)$. We have that all $w \in (E, v(D))$ satisfies g . As no new constraints not in PLT have been created, all $v' \in R_p$ respect the same constraints on their fractional part and integer part as v and therefore, $(E, v'(D))$ is contained in the same clock region as $(E, v(D))$ is, *i. e.*, $[(E, v(D))]_{R_c} = [(E, v'(D))]_{R_c}$. Finally, $v' \in \text{guard}_\forall(g, E, D)$.

J Proof of Lemma 9

Lemma 9 (recalled). *Let (E, D) be a \mathbf{p} -PDBM for R_p and $v \in R_p$. Let g be a parametric guard. If $v \in p\text{-guard}_\exists(g, E, D)$, then for all $v' \in R_p$, $v' \in p\text{-guard}_\exists(g, E, D)$.*

Proof. Let (E, D) be a \mathbf{p} -PDBM for R_p and $v \in R_p$. Let g be a parametric guard and suppose $v \in p\text{-guard}_\exists(g, E, D)$. After applying a projection on parameters, we obtain a set of constraints on elements of \mathcal{PLT} . By hypothesis, all these constraints are satisfied by v . Suppose $v' \in R_p$. By definition of our parameter regions, and since v and v' both belong to R_p , v' satisfies the same set of constraints on elements of \mathcal{PLT} . Therefore, the same set of constraints is satisfied by v' and $v' \in p\text{-guard}_\exists(g, E, D)$.

K Proof of Proposition 2

Proposition 2 (recalled). *Let R_p be a parameter region. Let \mathcal{A} be an R -U2P-PTA and $\mathcal{R}(\mathcal{A})$ its parametric region automaton over R_p . There is a run $\sigma : (l_0, (E_0, D_0)) \xrightarrow{e_0} (l_1, (E_1, D_1)) \xrightarrow{e_1} \dots (l_{f-1}, (E_{f-1}, D_{f-1})) \xrightarrow{e_{f-1}} (l_f, (E_f, D_f))$ in $\mathcal{R}(\mathcal{A})$ iff for all $v \in R_p$ there is a run $\rho : (l_0, w_0) \xrightarrow{e_0} (l_1, w_1) \xrightarrow{e_1} \dots (l_{f-1}, w_{f-1}) \xrightarrow{e_{f-1}} (l_f, w_f)$ in $v(\mathcal{A})$ s.t. for all $0 \leq i \leq f$, $w_i \in (E_i, v(D_i))$.*

Proof. \Leftarrow By induction on the length of the run.

Let $v \in R_p$. As the basis for the induction, in the initial location $(l_0, \{0\}^H)$ the only valuation is reachable by an empty run of $v(\mathcal{A})$. Moreover $\{0\}^H \in (E_0, v(D_0))$ the initial \mathbf{p} -PDBM containing only 0. Therefore the initial location $(l_0, (E_0, v(D_0)))$ is reachable by an empty run of $\mathcal{R}(\mathcal{A})$.

For the induction step, suppose for all v , there is run in $v(\mathcal{A})$ of length $f-1$ we have our result.

Let $v \in R_p$ and $\rho = (l_0, w_0) \xrightarrow{e_0} \dots \xrightarrow{e_{f-2}} (l_{f-1}, w_{f-1}) \xrightarrow{e_{f-1}} (l_f, w_f)$ be a run of $v(\mathcal{A})$ of length f . By induction hypothesis, there is a run $\sigma = (l_0, (E_0, D_0)) \xrightarrow{e_0} \dots \xrightarrow{e_{f-2}} (l_{f-1}, (E_{f-1}, D_{f-1}))$ in $\mathcal{R}(\mathcal{A})$ and for all $0 \leq i \leq f-1$, $w_i \in (E_i, v(D_i))$.

Consider e_{f-1} . By Definition 12 of the parametric region automaton, it is also in its set of edges ζ' . Three cases show up:

- If $e_{f-1} = \langle l_{f-1}, a, g, u_{np}, l_f \rangle$ contains no parametric guard nor parametric update. Using [Definition 2](#) there is a delay δ (possibly 0) s.t. $(l_{f-1}, w_{f-1}) \xrightarrow{\delta} (l_{f-1}, w'_{f-1}) \xrightarrow{e_{f-1}^{-1}} (l_f, w_f)$ where $w'_{f-1} \models g$ and $w_f = [w'_{f-1}]_{u_{np}}$. As $w_{f-1} \in (E_{f-1}, v(D_{f-1}))$ there is $(E'_{f-1}, D'_{f-1}) \in \text{Succ}((E_{f-1}, D_{f-1}))$ s.t. from [Proposition 1](#) we have $w'_{f-1} \in (E'_{f-1}, v(D'_{f-1}))$. As $w'_{f-1} \models g$ by construction of our p-PDBMs (see [Section 4.4](#)) any other clock valuation belonging to $(E'_{f-1}, v(D'_{f-1}))$ satisfies g . Therefore $v \in \text{guard}_\forall(g, E'_{f-1}, D'_{f-1})$ and from [Lemma 8](#), $R_p \subseteq \text{guard}_\forall(g, E'_{f-1}, D'_{f-1})$. Now, as $w_f = [w'_{f-1}]_{u_{np}}$ consider the open-p-PDBM $(E_f, D_f) = \text{update}((E'_{f-1}, D'_{f-1}), u_{np})$; from [Lemma 5](#) we have $w_f \in (E_f, v(D_f))$. Finally there is an edge $(l_{f-1}, (E_{f-1}, D_{f-1})) \xrightarrow{e_{f-1}^{-1}} (l_f, (E_f, D_f))$.
- If $e_{f-1} = \langle l_{f-1}, a, g, u, l_f \rangle$ contains a parametric guard and a parametric update. Using [Definition 2](#) there is a delay δ (possibly 0) s.t. $(l_{f-1}, w_{f-1}) \xrightarrow{\delta} (l_{f-1}, w'_{f-1}) \xrightarrow{e_{f-1}^{-1}} (l_f, w_f)$ where $w'_{f-1} \models v(g)$ and $w_f = [w'_{f-1}]_{v(u)}$. As $w_{f-1} \in (E_{f-1}, v(D_{f-1}))$ there is $(E'_{f-1}, D'_{f-1}) \in \text{Succ}((E_{f-1}, D_{f-1}))$ s.t. from [Proposition 1](#) we have $w'_{f-1} \in (E'_{f-1}, v(D'_{f-1}))$. As $w'_{f-1} \models v(g)$, $v \in p\text{-guard}_\exists(g, E'_{f-1}, D'_{f-1})$ and from [Lemma 9](#), $R_p \subseteq p\text{-guard}_\exists(g, E'_{f-1}, D'_{f-1})$. Now, as $w_f = [w'_{f-1}]_{v(u)}$ consider the point-p-PDBM $(E_f, D_f) = \overline{\text{update}}((E'_{f-1}, D'_{f-1}), u)$; $(E_f, v(D_f))$ contains only one clock valuation, precisely defined by the fully parametric update $v(u)$ so we have $w_f \in (E_f, v(D_f))$. Finally there is an edge $(l_{f-1}, (E_{f-1}, D_{f-1})) \xrightarrow{e_{f-1}^{-1}} (l_f, (E_f, D_f))$.
- The case where e_{f-1} contains a non parametric guard and a parametric update is similar to the previous one.

Finally, there is a run $\sigma' = \sigma \xrightarrow{e_{f-1}^{-1}} (l_f, (E_f, D_f))$ of length f in $\mathcal{R}(\mathcal{A})$ s.t. for all $0 \leq i \leq f$, $w_i \in (E_i, v(D_i))$.

\Rightarrow By induction on the length of the run.

Let $v \in R_p$. As the basis for the induction, the initial location $(l_0, (E_0, v(D_0)))$ is reachable by an empty run of $\mathcal{R}(\mathcal{A})$. Moreover, as $\{0\}^H \in (E_0, v(D_0))$, the initial location $(l_0, \{0\}^H)$ is reachable by an empty run of $v(\mathcal{A})$.

For the induction step, suppose it is true for all run in $\mathcal{R}(\mathcal{A})$ of length $f-1$.

Let $v \in R_p$ and $\sigma = (l_0, (E_0, D_0)) \xrightarrow{e_0} \dots \xrightarrow{e_{f-2}} (l_{f-1}, (E_{f-1}, D_{f-1})) \xrightarrow{e_{f-1}^{-1}} (l_f, (E_f, D_f))$ be a run of $\mathcal{R}(\mathcal{A})$ of length f . Consider e_{f-1} . By [Definition 12](#) of the parametric region automaton, it is also in the set of edges ζ of \mathcal{A} . Two cases show up:

- If $e_{f-1} = \langle l_{f-1}, a, g, u_{np}, l_f \rangle$ contains no parametric guard nor parametric update. By induction hypothesis, there is a run $\rho = (l_0, w_0) \xrightarrow{e_0} \dots \xrightarrow{e_{f-2}} (l_{f-1}, w_{f-1})$ of $v(\mathcal{A})$ of length $f-1$ s.t. for all $0 \leq i \leq f-1$, $w_i \in (E_i, v(D_i))$. Using [Definition 12](#) there is $(E'_{f-1}, D'_{f-1}) \in \text{Succ}((E_{f-1}, D_{f-1}))$, $R_p \subseteq \text{guard}_\forall(g, E'_{f-1}, D'_{f-1})$ and $(E_f, D_f) = \text{update}((E'_{f-1}, D'_{f-1}), u_{np})$. From [Proposition 1](#) we

have $w'_{f-1} \in (E'_{f-1}, v(D'_{f-1}))$ and a delay δ s.t. $w'_{f-1} = w_{f-1} + \delta$. As $R_p \subseteq \text{guard}_{\forall}(g, E'_{f-1}, D'_{f-1})$ from Lemma 8 we have $v \in \text{guard}_{\forall}(g, E'_{f-1}, D'_{f-1})$ and $w'_{f-1} \models g$. Moreover, since $(E_f, D_f) = \text{update}((E'_{f-1}, D'_{f-1}), u_{np})$, we define $w_f = [w'_{f-1}]_{u_{np}}$ and therefore from Lemma 5, $w_f \in (E_f, v(D_f))$. Finally there is an edge $(l_{f-1}, w_{f-1}) \xrightarrow{e_{f-1}} (l_f, w_f)$ and a run $\rho' = \rho \xrightarrow{e_{f-1}} (l_f, w_f)$ in $v(\mathcal{A})$ of length f s.t. for all $0 \leq i \leq f$, $w_i \in (E_i, v(D_i))$.

- If $e_{f-1} = \langle l_{f-1}, a, g, u, l_f \rangle$ contains a parametric guard and a parametric update. Using Definition 12 there is $(E'_{f-1}, D'_{f-1}) \in \text{Succ}((E_{f-1}, D_{f-1}))$, $R_p \subseteq p\text{-guard}_{\exists}(g, E'_{f-1}, D'_{f-1})$ and $(E_f, D_f) = \text{update}((E'_{f-1}, D'_{f-1}), u)$. From Lemma 9 we can take $w'_{f-1} \in (E'_{f-1}, v(D'_{f-1}))$ s.t. $w'_{f-1} \models v(g)$. Let $w_f = [w'_{f-1}]_{v(u)}$. Clearly, $(E_f, D_f) = \text{update}((E'_{f-1}, D'_{f-1}), u)$ is a point-p-PDBM; as $(E_f, v(D_f))$ contains only one clock valuation precisely defined by the fully parametric update $v(u)$, we have $w_f \in (E_f, v(D_f))$. From Proposition 1 as $w'_{f-1} \in (E'_{f-1}, v(D'_{f-1}))$ there is a delay δ and a $w_{f-1} \in (E_{f-1}, v(D_{f-1}))$ s.t. $w'_{f-1} = w_{f-1} + \delta$. Using the induction hypothesis, there is a run $\rho = (l_0, w_0) \xrightarrow{e_0} \dots \xrightarrow{e_{f-2}} (l_{f-1}, w_{f-1})$ of $v(\mathcal{A})$ of length $f-1$ s.t. for all $0 \leq i \leq f-1$, $w_i \in (E_i, v(D_i))$. Finally there is an edge $(l_{f-1}, w_{f-1}) \xrightarrow{e_{f-1}} (l_f, w_f)$ and a run $\rho' = \rho \xrightarrow{e_{f-1}} (l_f, w_f)$ in $v(\mathcal{A})$ of length f s.t. for all $0 \leq i \leq f$, $w_i \in (E_i, v(D_i))$.
- The case where e_{f-1} contains a non parametric guard and a parametric update is similar to the previous one.