Recognizability of languages via deterministic finite automata with values on a monoid: General Myhill-Nerode Theorem

José R. González de Mendívil, Federico Fariña Figueredo

Departamento de Estadística, Informática y Matemáticas

Universidad Pública de Navarra

31006 Pamplona (Spain)

Abstract

This paper deals with the problem of recognizability of functions $\ell: \Sigma^* \to M$ that map words to values in the support set M of a monoid $(M, \cdot, 1)$. These functions are called M-languages. M-languages are studied from the aspect of their recognition by deterministic finite automata whose components take values on M (M-DFAs). The characterization of an M-language ℓ is based on providing a right congruence on Σ^* that is defined through ℓ and a factor*ization* on the set of all M-languages, $L(\Sigma^*, M)$ (in short L). A factorization on L is a pair of functions (q, f) such that, for each $\ell \in L$, $q(\ell) \cdot f(\ell) = \ell$, where $q(\ell) \in M$ and $f(\ell) \in L$. In essence, a factorization is a form of common factor extraction. In this way, a general Myhill-Nerode theorem, which is valid for any $L(\Sigma^*, M)$, is provided. Basically, $\ell \in L$ is recognized by an M-DFA if and only if there exists a factorization on L, (g, f), such that the right congruence on Σ^* induced by the factorization (g, f) and $f(\ell) \in L$, has finite index. This paper shows that the existence of M-DFAs guarantees the existence of natural non-trivial factorizations on L without taking account any additional property on the monoid. In addition, the composition of factorizations is also a new factorization, and the composition of natural factorizations preserves the recognition capability of each individual natural factorization.

Keywords: Recognizability, languages, Deterministic finite automata, Myhill-Nerode Theorem, monoid, factorization, right congruence.

Email addresses: mendivil@unavarra.es (José R. González de Mendívil), fitxi@unavarra.es (Federico Fariña Figueredo)

1. Introduction

In formal languages and automata [20], the *Myhill-Nerode Theorem* [31] [32] provides necessary and sufficient conditions for a language to be recognized by a deterministic finite automaton (DFA). In that case, the language is said to be regular. Specifically, given a language $\ell \subseteq \Sigma^*$ (where Σ^* denotes the set of all finite words on an alphabet Σ) a right congruence relation \equiv_{ℓ} on Σ^* is defined in terms of the language ℓ , but with no regard to its representation. The recognizability of a language ℓ by a DFA is established by proving that: ℓ is a regular language if and only if \equiv_{ℓ} has finite index. It is worth of mention that the states of the DFA based on \equiv_{ℓ} , that recognizes ℓ , are the equivalence classes of that congruence. Furthermore, this DFA is minimal what makes Myhill-Nerode theorem to be considered in the topic of automata minimization.

In order to cope with different domains of practical applications, researchers have presented in the literature effective generalizations of automata using a wide diversity of algebraic structures. Fuzzy automata and weighted automata are ones of the best-known studied generalizations of automata [30][5]. For weighted automata, values on the transitions of those automata are usually taken from semirings [28], hemirings [6], or strong bimonoids [4][7]. For fuzzy automata, values on transitions are taken from certain ordered structures like lattice-ordered monoids [25], lattice-ordered structures [27], complete distributive lattices [1], general lattices [27], or complete residuated lattices [21][34][35].

Weighted deterministic finite state automata are called (sub)sequential transducers [8][37]. These transducers recognize languages called (sub) sequential rational functions. Characterization of the (sub)sequential rational functions in terms of a congruence relation in the flavour of Myhill-Nerode theorem has been studied for different special cases of monoids like free monoids [37], ($\mathbb{R}_0^+, +, 0$) [29], gcd monoids [39], or monoids based on sequentiable structures [10].

It is remarkable that Gerdjikov has provided five algebraic axioms [11][12], based on the relation divisor of 1 on the support set M of the monoid, for characterizing a wide class of monoids. Those axioms are satisfied by groups, free monoids, sequentiable structures, tropical monoids (including

Given a monoid $(M, \cdot, 1)$. For $a, b \in M$, a is divisor of b if there is $c \in M$ such that $b = a \cdot c$.

 $(\mathbb{Q}_0^+,+,0))$, and gcd monoids [12]. However, the axioms *left cancellation* and *right cancellation* considered in [12] avoid the existence of a *zero* element in M.

In fuzzy languages and automata [42], Ignjatović et al. [22] have proposed a Myhill-Nerode type theory for fuzzy languages with membership values in an arbitrary set with two distinguished elements 0 and 1, (M, 0, 1), which are needed to take common (crisp) languages in consideration. These fuzzy languages are studied in [22] from the aspect of their recognition by crisp determinist fuzzy automata. A crisp deterministic fuzzy automaton is simply an ordinary deterministic automaton equipped with a fuzzy subset of final states [1]. It is worth of mention, that the Myhill-Nerode theorem for fuzzy languages provided by Ignjatović [22] includes the recognizability of fuzzy languages with membership values on some well-known structures: Gödel structure [38] [30](Chap.7) [33]; distributive lattices [36]; finite monoids [2]; and general lattices [27]. However, all these previous works in fuzzy languages have a severe limitation: they only consider recognizability of fuzzy languages of finite rank. In fact, a fuzzy language is recognized by a crisp deterministic fuzzy automaton if and only if it has a finite rank and all its kernel languages are recognizable (see Theorem 4.3 in [22]).

In order to circumvent this restriction, a generalization of Myhill-Nerode theorem for fuzzy languages, having finite or infinite rank, is introduced in [16] for fuzzy languages based on continuous triangular norms (t-norms)[24], ([0,1], \otimes ,1). In that paper, the characterization of fuzzy languages is based on a right congruence defined by using the notion of factorization, in particular maximal factorization.

In essence, a factorization is a form of common factor extraction. Specifically, a factorization is a pair of functions (g,f) such that they satisfy $\ell=g(\ell)\otimes f(\ell)$ for any fuzzy language ℓ . This notion was initially introduced in weighted automata by Kirsten and Mäurer [23] and applied to the area of fuzzy automata in order to develop efficient constructions for determinization and minimization of fuzzy automata. In fact, factorizations have allowed researchers to provide minimization algorithms [28][9] and determinization methods for weighted automata [23]. In the context of fuzzy automata, factorizations have been studied to obtain determinization methods [14][15][40], and minimization algorithms for fuzzy automata [17][18][41].

Although [16] deals with characterizing fuzzy languages with infinite rank, that work is somewhat restrictive because it assumes *zero-divisor-free* t-norm based monoids and maximal factorizations. In general, that property on the monoid is a necessary condition for the existence of maximal factorizations as it has been recently proved in [13].

The motivation to write this paper is to study the recognizability of functions $\ell: \Sigma^* \to M$ defined for any arbitrary monoid $(M,\cdot,1)$. In other words, we do not consider any additional property on the monoid when we address the recognizability problem unlike other previous research works that have been referenced in this introductory section. In order to get a compact presentation of our results, we assume that each $\ell: \Sigma^* \to M$ is a total function. We call these functions M-languages. As in previous works, recognizability is based on deterministic finite automata, but in this case, their components take values on M (M-DFAs). Let $L(\Sigma^*, M)$ be the set of all M-languages. Our main objective is to provide a general Myhill-Nerode theorem valid for any $L(\Sigma^*, M)$. We use factorizations on $L(\Sigma^*, M)$ in order to get a characterization of M-languages. The characterization of an M-language $\ell \in L(\Sigma^*, M)$ is based on a right congruence on Σ^* induced by a factorization on $L(\Sigma^*, M)$, (g, f), and ℓ . That congruence is denoted by $\equiv_{\ell}^{(g,f)}$ in the paper.

The general Myhill-Nerode theorem presented in this paper establishes that $\ell \in L$ is recognized by an M-DFA if and only if there exists a factorization on L, (g,f), such that the right congruence on Σ^* induced by the factorization (g,f) and $f(\ell) \in L$, has finite index.

The proof is made by construction. In this way, we prove that if $\equiv_{f(\ell)}^{(g,f)}$ has finite index then there exists an M-DFA that recognizes the M-language $f(\ell)$, and, as (g, f) is a factorization, $g(\ell) \cdot f(\ell) = \ell$, which implies that ℓ is also recognized by such an M-DFA. In the other direction, if an M-language ℓ is recognized by some M-DFA, A, then there exists a natural factorization on $L(\Sigma^*, M)$ induced by A, (g_A, f_A) , such that $\equiv_{f_A(\ell)}^{(g_A, f_A)}$ has finite index. The construction of a factorization induced by an M-DFA is not trivial and requires the notion of transition-equalized automata. However, it is not necessary to construct explicitly such kind of M-DFAs to get their factorizations [19].

Each possible factorization on $L(\Sigma^*, M)$ has its own recognition capability. Thus, we study the recognition capability of three particular cases: trivial factorization, maximal factorizations and natural factorizations. The formulation of factorizations on $L(\Sigma^*, M)$ allows us to define the composition of factorizations to form new factorizations. In this way, we prove that the recognition capability of the composition of natural factorizations preserves the recognition capability of each individual natural factorization.

The rest of the paper is organized as follows. Section 2 and section 3 present a formal framework to define the operations and main properties of factorizations. Section 4 is a short introduction to M-DFAs. In section

5, we introduce the sufficient condition for recognizability of M-languages based on factorizations and the definition of right congruence based on factorizations. This section presents the main properties of the M-DFA based on this kind of congruences. Section 6 is devoted to natural factorizations, i.e., the factorizations induced via M-DFAs. The properties of the M-DFA constructed under a natural factorization are also included in this section. The general Myhill-Nerode theorem is proved in section 7. The recognition capability of a factorization is defined in section 8 and three cases of study are considered: trivial factorization, maximal factorizations and composition of natural factorizations. Finally, some concluding remarks end the paper.

2. Preliminaries

Let $f_1: A \to B$ and $f_2: B \to C$ be two well defined functions. In this paper, the composition of the functions f_1 and f_2 is denoted by $f_2 \circ f_1$ and defined by $(f_2 \circ f_1)(a) = f_2(f_1(a))$ for any $a \in A$. We will assume that every function is a total function.

Let $(M, \cdot, 1)$ be an arbitrary monoid where M, \cdot and 1 represent the support set, the multiplication operation, and the identity element of the monoid respectively. In general, we identify each monoid with the name of its support set.

Let Σ be a finite alphabet of symbols. The set Σ^* denotes the set of all finite words over Σ . We use ε to represent the empty word. Then, Σ^* is the free monoid generated by Σ under the operation of concatenation. Let us consider functions from Σ^* to M, i.e., $\ell:\Sigma^*\to M$. The set $L(\Sigma^*,M)=\{\ell\mid\ell:\Sigma^*\to M\}$ is the set containing all those kind of functions. We do not write (Σ^*,M) when it is clear in the context of discussion. Thus, $L(\Sigma^*,M)$ is simply denoted by L. We consider that L represents the set of all possible M-languages on Σ . In fact, if the monoid M is the elemental Boolean monoid then, any $\ell\in L$ may be interpreted as the characteristic function of an ordinary language, i.e., a subset of Σ^* .

In the following, we extend the multiplication operation \cdot of the monoid M to different contexts. The context for the symbol \cdot will clarify its interpretation. Given $\ell \in L$ and $m \in M$, $m \cdot \ell \in L$ is defined as $(m \cdot \ell)(\gamma) = m \cdot \ell(\gamma)$ for every $\gamma \in \Sigma^*$. Obviously, $(n \cdot m) \cdot \ell = n \cdot (m \cdot \ell)$ for any $n, m \in M$. For any function $r : \Sigma^* \to \Sigma^*$ (a word-transformation), any $\ell \in L$, and any $m \in M$, the next property holds:

$$m \cdot (\ell \circ r) = (m \cdot \ell) \circ r \tag{1}$$

Let us consider functions of the form $f:L\to L$ and $g:L\to M$. We define the sets $F(L)=\{f|f:L\to L\}$ and $G(L,M)=\{g|g:L\to M\}$ which are simply denoted by F and G respectively. In F, f_e denotes the identity function, i.e., $f_e(\ell)=\ell$ for any $\ell\in L$. In G, g_e denotes the constant function $g_e(\ell)=1$ for every $\ell\in L$. Since composition of functions is associative then (F,\circ,f_e) is a monoid. As the image of any function in G is a subset of M, we also extend \cdot in the following way: for any $g,g'\in G$, $g\cdot g'$ is defined as $(g\cdot g')(\ell)=g(\ell)\cdot g'(\ell)$ for any $\ell\in L$. Thus, $g\cdot g'$ is a well defined function of G. Clearly, (G,\cdot,g_e) is a monoid.

Given the monoids G and F introduced above, we put our attention to the cartesian product $G \times F$ and pairs of functions $(g,f) \in G \times F$. We define the binary operation $\bullet: G \times F \to F$ as follows: given $g \in G$ and $f \in F$, $(g \bullet f)(\ell) = g(\ell) \cdot f(\ell)$ for any $\ell \in L$. Thus, $(g \bullet f)$ is a well defined function of F. Furthermore, for any $g, g' \in G$ and $f, f' \in F$, it is simple to prove that,

$$(g \cdot g') \bullet f = g \bullet (g' \bullet f) \tag{2}$$

$$(g \circ f') \bullet (f \circ f') = (g \bullet f) \circ f' \tag{3}$$

We provide a product operation for elements of $G \times F$. The binary operation $*: (G \times F)^2 \to G \times F$ is defined as follows:

$$(g_1, f_1) * (g_2, f_2) = (g_1 \cdot (g_2 \circ f_1), f_2 \circ f_1)$$
 (4)

for any (g_1, f_1) , $(g_2, f_2) \in G \times F$. Obviously, $(g_1, f_1) * (g_2, f_2) \in G \times F$. It is not difficult to prove that * is associative. In addition, the pair (g_e, f_e) is the identity element for *. Therefore, $(G \times F, *, (g_e, f_e))$ is a monoid. We abuse of the confidence of the reader by using the term *composition* instead of * when this operation is evident from its context of application.

Given a finite family $\{(g_i, f_i) \in G \times F\}_{i:1..n}$ with $n \geq 1$, the composition $(g_1, f_1) * (g_2, f_2) * ... * (g_n, f_n)$, is denoted by $*|_{i=1}^n (g_i, f_i)$. By using (4) successively, the result can be expressed in the form:

$$*|_{i=1}^{n}(g_{i}, f_{i}) = (\prod_{i=1}^{n} g_{i} \circ (\circ|_{j=i-1}^{1} f_{j}), \circ|_{i=n}^{1} f_{i})$$
(5)

where \circ and \prod represent the quantifiers of \circ and \cdot respectively. By convention, $\circ|_{\emptyset}(.) = f_e$ and $\prod_{\emptyset}(.) = g_e$. Thus, (5) is also applicable when n = 0. In that case, $*|_{\emptyset}(f_i, g_i) = (f_e, g_e)$.

Example 1. As an example of (5), let us consider n = 3:

$$*|_{i=1}^{3}(g_{i}, f_{i}) = (g_{1}, f_{1}) * (g_{2}, f_{2}) * (g_{3}, f_{3}) = ((g_{1} \circ f_{e}) \cdot (g_{2} \circ f_{1}) \cdot (g_{3} \circ (f_{2} \circ f_{1})), f_{3} \circ f_{2} \circ f_{1}) = (g_{1} \cdot (g_{2} \circ f_{1}) \cdot (g_{3} \circ (f_{2} \circ f_{1})), f_{3} \circ f_{2} \circ f_{1})$$

In order to obtain a more compact notation, we will write equation (5) in the form

$$[(g_i, f_i)]_1^n = ([g_i \circ [f_j]_{i-1}^1]_1^n, [f_i]_n^1)$$
(6)

where $[(g_i, f_i)]_1^n = *|_{i=1}^n (g_i, f_i), [f_i]_n^1 = \circ|_{i=n}^1 f_i, \text{ and } [g_i \circ [f_j]_{i=1}^1]_1^n = \prod_{i=1}^n g_i \circ (\circ|_{i=i-1}^1 f_j).$ By using this notation, we obtain, by (4), that

$$[(g_i, f_i)]_1^{n+1} = [(g_i, f_i)]_1^n * (g_{n+1}, f_{n+1}) = ([g_i \circ [f_j]_{i-1}]_1^n \cdot (g_{n+1} \circ [f_j]_n^1), f_{n+1} \circ [f_i]_n^1)$$
(7)

for the composition of a family of n+1 pairs of functions in $G \times F$, with $n \ge 0$.

Remark 1. Let us observe that, by (6), $[(g_i, f_i)]_1^{n+1} = ([g_i \circ [f_j]_{i-1}^1]_1^{n+1}, [f_i]_{n+1}^1)$. In some proofs, we are interested in the operation $[g_i \circ [f_j]_{i-1}^1]_1^{n+1} \bullet [f_i]_{n+1}^1$.

$$[g_{i} \circ [f_{j}]_{i-1}^{1}]_{1}^{n+1} \bullet [f_{i}]_{n+1}^{1} = (\text{by } (7)) = ([g_{i} \circ [f_{j}]_{i-1}^{1}]_{1}^{n} \cdot (g_{n+1} \circ [f_{j}]_{n}^{1})) \bullet (f_{n+1} \circ [f_{i}]_{n}^{1}) = (\text{by } (2)) = ([g_{i} \circ [f_{j}]_{i-1}^{1}]_{1}^{n}) \bullet ((g_{n+1} \circ [f_{j}]_{n}^{1}) \bullet (f_{n+1} \circ [f_{i}]_{n}^{1})) = (\text{by } (3)) = ([g_{i} \circ [f_{j}]_{i-1}^{1}]_{1}^{n}) \bullet ((g_{n+1} \bullet f_{n+1}) \circ [f_{i}]_{n}^{1})$$

In conclusion,

$$[g_i \circ [f_j]_{i-1}^1]_1^{n+1} \bullet [f_i]_{n+1}^1 = [g_i \circ [f_j]_{i-1}^1]_1^n \bullet ((g_{n+1} \bullet f_{n+1}) \circ [f_i]_n^1)$$
 (8)

For each word-transformation $r: \Sigma^* \to \Sigma^*$, we define the function $\partial_r \in F$, $\partial_r: L \to L$, as $\partial_r(\ell) = \ell \circ r$ for any $\ell \in L$. We say that $\partial_r(\ell)$ is the derivative of ℓ by the word-transformation r.

Example 2. Let us consider, for each $\alpha \in \Sigma^*$, the transformation $\underline{\alpha} : \Sigma^* \to \Sigma^*$ defined by $\underline{\alpha}(\gamma) = \alpha \gamma$ for any $\gamma \in \Sigma^*$. Thus, the derivative $\partial_{\underline{\alpha}}(\ell)$, satisfies that $\partial_{\underline{\alpha}}(\ell) = \ell \circ \underline{\alpha}$ by the definition given above. Then, $\partial_{\underline{\alpha}}(\ell)(\gamma) = \ell(\underline{\alpha}(\gamma)) = \ell(\alpha \gamma)$ for each word γ . Let us observe that $\partial_{\underline{\alpha}}(\ell)$ may be viewed

as a generalization of the Brzozowski derivative of an ordinary language by a word [3]. In addition, $\partial_{\underline{\beta}} \circ \partial_{\underline{\alpha}} = \partial_{\underline{\alpha} \circ \underline{\beta}} = \partial_{\underline{\alpha} \beta}$ holds for any words α and β . Clearly, $\partial_{\underline{\varepsilon}} = f_e$. In the rest of this paper, $\overline{\partial}_{\underline{\alpha}}$ is simply denoted by ∂_{α} for any word α .

Given a word-transformation $r: \Sigma^* \to \Sigma^*$ and a pair $(g, f) \in G \times F$, the next equation holds:

$$g \bullet (\partial_r \circ f) = \partial_r \circ (g \bullet f) \tag{9}$$

Proof: For any $\ell \in L$,

$$(g \bullet (\partial_r \circ f))(\ell) = g(\ell) \cdot (\partial_r (f(\ell))) = g(\ell) \cdot (f(\ell) \circ r) = (\text{ by } (1)) = (g(\ell) \cdot f(\ell)) \circ r = (g \bullet f)(\ell) \circ r = \partial_r ((g \bullet f)(\ell)) = (\partial_r \circ (g \bullet f))(\ell)$$

3. Factorizations on $L(\Sigma^*, M)$

Let us observe that the identity element $(g_e, f_e) \in G \times F$ satisfies that $g_e \bullet f_e = f_e$ since, for any $\ell \in L$, $(g_e \bullet f_e)(\ell) = g_e(\ell) \cdot f_e(\ell) = 1 \cdot \ell = \ell$. We assume the hypothesis that there exist other pairs $(g, f) \in G \times F$ with the same property, i.e., $(g \bullet f)(\ell) = g(\ell) \cdot f(\ell) = \ell$ for any $\ell \in L$. In that case, $g(\ell) \in M$ divides each value $\ell(\alpha) \in M$ for any word α , i.e., it is a common factor for ℓ . Thus, we may say that the pair (g, f) factorizes L. In general,

Definition 1. A pair of functions $(g, f) \in G \times F$ is a factorization on L if (g, f) satisfies that

$$g \bullet f = f_e \tag{10}$$

The identity element (g_e, f_e) is called the *trivial factorization* on L. We will study the properties obtained from the definition of factorization without considering additional properties about the monoid or the functions involved in the factorization. If (g, f) is a factorization on L then

$$\partial_r = \partial_r \circ (g \bullet f) = g \bullet (\partial_r \circ f) \tag{11}$$

$$\partial_r = (g \bullet f) \circ \partial_r = (g \circ \partial_r) \bullet (f \circ \partial_r) \tag{12}$$

for any word-transformation $r: \Sigma^* \to \Sigma^*$. Those results are consequence of (9), (3), and Definition 1.

Lemma 1. Let $\{(g_i, f_i) \in G \times F\}_{i:1..n}$ be an arbitrary finite family of $n \geq 0$ factorizations on L. The composition $[(g_i, f_i)]_1^n$ is a factorization on L.

Proof: By (6), $[(g_i, f_i)]_1^n = ([g_i \circ [f_i]_{i-1}^1]_1^n, [f_i]_n^1)$. By induction on n:

- Basis. if n=0 then $[(g_i,f_i)]_{\emptyset}=(g_e,f_e)$, the trivial factorization.
- Hypothesis. Let us assume that $[(g_i, f_i)]_1^n$ is a factorization on L for some arbitrary $n \geq 0$, i.e., $[g_i \circ [f_j]_{i-1}^1]_1^n \bullet [f_i]_n^1 = f_e$ (Definition 1).
- Induction Step. Let us consider the composition $[(g_i, f_i)]_1^{n+1}$ where (g_{n+1}, f_{n+1}) is a factorization on L. By (8), the fact that $g_{n+1} \bullet f_{n+1} = f_e$, and induction Hypothesis, $[g_i \circ [f_j]_{i-1}^1]_1^{n+1} \bullet [f_i]_{n+1}^1 = f_e$. Therefore, $[(g_i, f_i)]_1^{n+1}$ is a factorization on L.

Lemma 2. Let $\{(g'_i, f'_i) \in G \times F\}_{i:1..n}$ be a finite family of $n \geq 0$ factorizations on L. Let $\{r_i: \Sigma^* \to \Sigma^*\}_{i:1..n}$ be a finite family of $n \geq 0$ wordtransformations. For each i:1..n, the pair $(g_i,f_i)\in G\times F$ is defined as $(g_i, f_i) = (g'_i \circ \partial_{r_i}, f'_i \circ \partial_{r_i})$ where ∂_{r_i} is the derivative operator by the wordtransformation r_i . The composition of the family $\{(g_i, f_i) \in G \times F\}_{i:1..n}$, $[(g_i, f_i)]_1^n = ([g_i \circ [f_j]_{i-1}^1]_1^n, [f_i]_n^1), \text{ satisfies that}$

$$[g_i \circ [f_j]_{i-1}^1]_1^n \bullet [f_i]_n^1 = \partial_{\circ|_{i=1}^n r_i}$$
(13)

Proof: Let us recall that $\partial_{r_i} \in F$ and $\partial_{r_i}(\ell) = \ell \circ r_i$ for any $\ell \in L$. By induction on n:

- Basis. if n=0 then $[(g_i,f_i)]_{\emptyset}=(g_e,f_e)$. By convention $\circ|_{\emptyset}r_i$ is the identity word-transformation. Thus, $\partial_{\circ|_{\emptyset}r_i} = f_e$. Therefore, $g_e \bullet f_e = f_e$ since (g_e, f_e) is the trivial factorization on L.
- Hypothesis. Let us assume that (13) holds for an arbitrary $n \geq 0$.
- Induction Step. Let us consider the composition $[(g_i, f_i)]_1^{n+1}$ where each (g_i, f_i) is defined in Lemma 2. The new pair (g_{n+1}, f_{n+1}) is also defined as $(g_{n+1}, f_{n+1}) = (g'_{n+1} \circ \partial_{r_{n+1}}, f'_{n+1} \circ \partial_{r_{n+1}})$ for a given factorization on L, (g'_{n+1}, f'_{n+1}) , and a word-transformation r_{n+1} .

By definition of $(g_{n+1}, f_{n+1}), g_{n+1} \bullet f_{n+1} = (g'_{n+1} \circ \partial_{r_{n+1}}) \bullet (f'_{n+1} \circ \partial_{r_{n+1}}).$ Then, by (12), $g_{n+1} \bullet f_{n+1} = \partial_{r_{n+1}}$.

By (8), and substitution,

$$\begin{split} &[g_i \circ [f_j]_{i-1}^1]_1^{n+1} \bullet [f_i]_{n+1}^1 = [g_i \circ [f_j]_{i-1}^1]_1^n \bullet (\partial_{r_{n+1}} \circ [f_i]_n^1). \\ &\text{By (9), } [g_i \circ [f_j]_{i-1}^1]_1^{n+1} \bullet [f_i]_{n+1}^1 = \partial_{r_{n+1}} \circ ([g_i \circ [f_j]_{i-1}^1]_1^n \bullet [f_i]_n^1). \\ &\text{By Hypothesis and definition of } \partial_{r_{n+1}}, \end{split}$$

 $[g_i \circ [f_j]_{i-1}^1]_1^{n+1} \bullet [f_i]_{n+1}^1 = \partial_{r_{n+1}} \circ \partial_{\circ|_{i=1}^n r_i}^{n+1} = \partial_{\circ|_{i-1}^{n+1} r_i}. \text{ Therefore, the Lemma}$ holds.

Previous results have been provided for arbitrary factorizations, wordtransformations and their derivatives. The importance of factorizations on L and the equation (13) is that they provide a reasonable theoretical basis for recognizability of M-languages via deterministic finite automata with values on a monoid M as we will show in the next sections.

4. Deterministic finite automata with values on a monoid

We present a short introduction to deterministic finite automata with values on a monoid.

Definition 2. Let $(M, \cdot, 1)$ be a monoid. A Deterministic Finite Automaton with values on the monoid M, (M-DFA in short), is a tuple $A = (Q, \Sigma, u, i_u, \delta, w, \rho)$ where

- Q is a finite nonempty set of states;
- Σ is a finite alphabet;
- $u \in Q$ is the unique initial state;
- $i_u \in M$ is the initial value assigned to the initial state $u \in Q$;
- $\delta: Q \times \Sigma \to Q$ is the state-transition function;
- $w: Q \times \Sigma \to M$ is the monoid-transition function that assigns values from M to each transition; and
- $\rho: Q \to M$ is the final-function that assigns final values from M to each state in Q.

The state-transition function δ is extended to Σ^* . The extended function $\delta^*: Q \times \Sigma^* \to Q$ is defined as

(i)
$$\delta^*(q,\varepsilon) = q$$
; and, (ii) $\delta^*(q,\alpha\sigma) = \delta(\delta^*(q,\alpha),\sigma)$ (14)

for any $q \in Q$, $\alpha \in \Sigma^*$, and $\sigma \in \Sigma$. As δ is a total function then $\delta^*(q,\alpha) \in Q$ for every word α . Due this fact, it is common to say that A is *complete*. In the rest of this paper, $\delta^*(q,\alpha)$ is simply denoted $q\alpha$ for each state q of A, and word α . As A is a deterministic automaton, $q\alpha$ is the unique reachable state from q by the word α . In other words, $q\alpha$ is an accessible state from q (by the word α). An M-DFA A is accessible if $Q = \{u\alpha | \alpha \in \Sigma^*\}$, i.e., any state in Q is accessible from the initial state.

The monoid-transition function w is also extended to Σ^* . The extended function $w^*: Q \times \Sigma^* \to M$ is defined as

(i)
$$w^*(q,\varepsilon) = 1$$
; and, (ii) $w^*(q,\alpha\sigma) = w^*(q,\alpha) \cdot w(q\alpha,\sigma)$ (15)

for any $q \in Q$, $\alpha \in \Sigma^*$, and $\sigma \in \Sigma$. Let us observe that the definition of w^* uses the extended function δ^* .

Let $\alpha \in \Sigma^*$ be a word. The length of α is denoted by $|\alpha|$. The k-th prefix of α is $\alpha[k]$ where $0 \le k \le |\alpha|$. By convention, $\alpha[0] = \varepsilon$. In addition, the k-th symbol in α is denoted by $\alpha(k)$ where, by convention, $\alpha(k) = \varepsilon$ when k < 1 or $k > |\alpha|$. Taken into account such a notation, by (15), we can expand $w^*(q, \alpha)$ as follows:

$$w^*(q,\alpha) = w(q,\alpha(1)) \cdot \dots w(q\alpha[i-1],\alpha(i)) \dots \cdot w(q\alpha[n-1],\alpha(n))$$

$$= \prod_{i=1}^{n} w(q\alpha[i-1], \alpha(i))$$
(16)

for any word α of length $n \geq 0$ and $q \in Q$. It is also clear that, for any two words α and β , and $q \in Q$,

$$w^*(q,\alpha\beta) = w^*(q,\alpha) \cdot w^*(q\alpha,\beta) \tag{17}$$

Given those previous definitions and extended functions, it is possible to define the M-language recognized (or generated) by an M-DFA. Let $A = (Q, \Sigma, u, i_u, \delta, w, \rho)$ be an M-DFA. The M-language recognized by A, denoted A, $A \in L$, is defined by

$$\mathcal{A}(\alpha) = i_u \cdot w^*(u, \alpha) \cdot \rho(u\alpha) \tag{18}$$

for any $\alpha \in \Sigma^*$. In addition, for each state q of A, the M-language $\mathcal{A}_q \in L$ is defined by

$$\mathcal{A}_q(\alpha) = w^*(q, \alpha) \cdot \rho(q\alpha) \tag{19}$$

for any $\alpha \in \Sigma^*$. For each word $\alpha \in \Sigma^*$ and its derivative $\partial_{\alpha} \in F$ (see Example 2), it is simple to prove that

$$\partial_{\alpha}(\mathcal{A}_q) = w^*(q,\alpha) \cdot \mathcal{A}_{q\alpha} \tag{20}$$

for any $q \in Q$. Finally, $\partial_{\alpha}(A) = i_u \cdot \partial_{\alpha}(A_u)$.

Definition 3. An M-language $\ell \in L$, is a recognizable M-language, or simply recognizable, if ℓ is recognized by some M-DFA.

Two trivial consequences of Definition 3 are:

(**Rcg1**) If $\ell \in L$ is recognizable then, for any $m \in M$, $m \cdot \ell$ is recognizable; and

(**Rcg2**) if $\ell \in L$ is recognizable then, there exists $\ell' \in L$ such that ℓ' is recognizable and $\ell = m \cdot \ell'$ for some $m \in M$.

If $\ell \in L$ is recognized by $A = (Q, \Sigma, u, i_u, \delta, w, \rho)$, i.e., $\mathcal{A} = \ell$, then, by (18), (**Rcg1**) $m \cdot \ell$ is recognized by the automaton $A' = (Q, \Sigma, u, m \cdot i_u, \delta, w, \rho)$. For (**Rcg2**), let us consider $A_u = (Q, \Sigma, u, 1, \delta, w, \rho)$. Then, by (18), $\ell' = \mathcal{A}_u$ and $m = i_u$.

Given two M-DFAs, A and B, we say that A is (language) equivalent to B when they recognize the same M-language, i.e., A = B.

Among the equivalent automata to A, we may find a *minimal* one. The minimal automaton has the minimal number of states for recognizing the same M-language.

Property 1. Let $A = (Q, \Sigma, u, i_u, \delta, w, \rho)$ be an M-DFA. If A is minimal then A satisfies the next conditions:

- 1. A is an accessible M-DFA.
- 2. (NcndS) For all states $p, q \in Q$, $(\exists m \in M : m \cdot A_p = A_q \vee A_p = m \cdot A_q) \Rightarrow p = q$
- 3. (NcndW) For all states $p, q \in Q$, $A_p = A_q \Rightarrow p = q$

Proof: Let $A = (Q, \Sigma, u, i_u, \delta, w, \rho)$ be a minimal M-DFA.

- 1. If A is not accessible then there is an equivalent M-DFA A' with a lesser number of states than A. A' is built by removing the unaccessible states of A. This is in contradiction with the initial hypothesis.
- 2. Let us consider that for two states $p \neq q$, $m \cdot \mathcal{A}_p = \mathcal{A}_q$ for some $m \in M$. Let us define the M-DFA $A' = (Q, \Sigma, u, i_u, \delta', w', \rho)$. A' is exactly equal to A excepting that, for any $s \in Q$ and $\sigma \in \Sigma$, if $\delta(s, \sigma) = q$ then $\delta'(s, \sigma) = p$ and $w'(s, \sigma) = w(s, \sigma) \cdot m$. That is, every transition (s, σ) ending in q in A ends in p in A'. It is easily to prove that A' is equivalent to A. However, in A', q is not an accessible state. Removing q in A', constructs an equivalent automaton to A but with less states. Again, this is a contradiction.

3. NcndW is consequence of NcndS when
$$m=1$$
.

Those previous conditions are *necessary* conditions for an M-DFA to be a minimal one. Let us observe that (**NcndS**) is stronger than (**NcndW**). It is also possible to provide a *sufficient* condition for minimality.

Property 2. Let $A = (Q, \Sigma, u, i_u, \delta, w, \rho)$ be an accessible M-DFA, then

$$\neg(\exists \alpha, \beta \in \Sigma^*, \ell \in L, m, m' \in M :
 u\alpha \neq u\beta \land \partial_{\alpha}(A) = m \cdot \ell \land \partial_{\beta}(A) = m' \cdot \ell)
 \Rightarrow
 A is minimal$$
(21)

Proof: As A is accessible, $Q = \{u\alpha \mid \alpha \in \Sigma^*\}$. Assume that $\|Q\| = n$. As A is a deterministic automaton, there exists n different words, $\alpha_1...\alpha_n$, such that $Q = \{u\alpha_i \mid i : 1..n\}$. Suppose that A is not minimal. Then, there exists a minimal M-DFA $A' = (Q', \Sigma, u', i_{u'}, \delta', w', \rho')$ equivalent to A with $\|Q'\| < \|Q\|$. As A' is complete $u'\alpha_i \in Q'$ for each i : 1..n. By the Pigeonhole principle, there are at least two different words α_k and $\alpha_{k'}$, with $1 \le k < k' \le n$, such that $u_{\alpha_k} \ne u_{\alpha_{k'}}$ and $u'_{\alpha_k} = u'_{\alpha_{k'}}$. Call α_k and $\alpha_{k'}$ by α and β respectively. Thus, $\partial_{\alpha}(A) = \partial_{\alpha}(A')$ and $\partial_{\beta}(A) = \partial_{\beta}(A')$ since A is equivalent to A'. By (20), $\partial_{\alpha}(A') = i_{u'} \cdot w'^*(u', \alpha) \cdot A'_{u'\alpha}$ and $\partial_{\beta}(A') = i_{u'} \cdot w'^*(u', \beta) \cdot A'_{u'\beta}$. As $u'_{\alpha} = u'_{\beta}$, then $A'_{u'\alpha} = A'_{u'\beta} = \ell$. In conclusion, $u\alpha \ne u\beta \wedge \partial_{\alpha}(A) = m \cdot \ell \wedge \partial_{\beta}(A) = m' \cdot \ell$. A contradiction happens and the property holds.

In general, we say that an M-DFA $A = (Q, \Sigma, u, i_u, \delta, \omega, \rho)$ is transition-equalized if for any $p, q \in Q$ and $\sigma, \tau \in \Sigma$,

$$\partial_{\sigma}(\mathcal{A}_q) = \partial_{\tau}(\mathcal{A}_p) \Rightarrow \delta(q, \sigma) = \delta(p, \tau) \wedge w(q, \sigma) = w(p, \tau)$$
 (22)

This property has an important relation with the construction of factorizations induced by an M-DFA (see section 6).

5. Recognizability via factorizations on $L(\Sigma^*, M)$

In this section, our main result states a sufficient condition for the recognizability of M-languages via factorizations on L. We recall that, for a word $\alpha \in \Sigma^*$, $\alpha(i)$ denotes the i-th symbol in α and $\alpha[i]$ denotes the i-th prefix of α . Given a factorization on L, (g, f), and a word α , let us define the pairs $(g \circ \partial_{\alpha(i)}, f \circ \partial_{\alpha(i)}) \in G \times F$ with $i : 1..|\alpha|$, and the composition $[(g \circ \partial_{\alpha(i)}, f \circ \partial_{\alpha(i)})]_1^{|\alpha|}$. By (5) and the notation given in (6), that composition returns the pair $(W_{\alpha}^{(g,f)}, S_{\alpha}^{(g,f)}) \in G \times F$, where

$$S_{\alpha}^{(g,f)} = [f \circ \partial_{\alpha(i)}]^{1}_{|\alpha|} \tag{23}$$

$$W_{\alpha}^{(g,f)} = [g \circ \partial_{\alpha(i)} \circ S_{\alpha[i-1]}^{(g,f)}]_{1}^{|\alpha|}$$

$$(24)$$

We write $S_{\alpha}(\ell)$ and $W_{\alpha}(\ell)$ instead of $S_{\alpha}^{(g,f)}$ and $W_{\alpha}^{(g,f)}$ in many part of the text when the factorization (g,f) is clear in the context of discussion. We recognize that it is not simple to provide an interpretations of these functions. This is the reason for providing their main properties:

$$S_{\varepsilon} = f_e \text{ and } W_{\varepsilon} = g_e$$
 (25)

$$S_{\sigma} = f \circ \partial_{\sigma} \text{ and } W_{\sigma} = g \circ \partial_{\sigma}$$
 (26)

$$S_{\beta} \circ S_{\alpha} = S_{\alpha\beta} \tag{27}$$

$$W_{\alpha} \cdot (W_{\beta} \circ S_{\alpha}) = W_{\alpha\beta} \tag{28}$$

$$W_{\alpha} \bullet S_{\alpha} = \partial_{\alpha} \tag{29}$$

$$(W_{\beta} \circ S_{\alpha}) \bullet (S_{\beta} \circ S_{\alpha}) = \partial_{\beta} \circ S_{\alpha} \tag{30}$$

for any $\sigma \in \Sigma$, and $\alpha, \beta \in \Sigma^*$.

Equations (25), (26), (27) and (28) are directly obtained from the definition of S_{α} (23) and W_{α} (24). Equation (29) is a consequence of the fact that (W_{α}, S_{α}) is an element of $G \times F$ that satisfies (13) in Lemma 2. The last equation (30) is obtained by (3) and the previous one (29).

The main justification for introducing such a pair (W_{α}, S_{α}) is that, for any $\ell \in L$ and $\alpha \in \Sigma^*$,

$$(W_{\alpha} \bullet S_{\alpha})(\ell) = \partial_{\alpha}(\ell)$$

$$W_{\alpha}(\ell) \cdot S_{\alpha}(\ell) = \ell \circ \underline{\alpha}$$

$$(W_{\alpha}(\ell) \cdot S_{\alpha}(\ell))(\varepsilon) = (\ell \circ \underline{\alpha})(\varepsilon)$$

$$W_{\alpha}(\ell) \cdot (S_{\alpha}(\ell))(\varepsilon) = \ell(\alpha)$$

In conclusion, for any factorization on L, (g, f), $\ell \in L$ and $\alpha \in \Sigma^*$:

$$W_{\alpha}^{(g,f)}(\ell) \cdot (S_{\alpha}^{(g,f)}(\ell))(\varepsilon) = \ell(\alpha)$$
(31)

This last equation suggests us a way for constructing the M-language ℓ taken into account that, by (28), $W_{\alpha}^{(g,f)}(\ell)$, with $|\alpha| = n$, can be expanded as

$$W_{\alpha}(\ell) = W_{\alpha(1)}(\ell) \cdot (W_{\alpha(2)} \circ S_{\alpha[1]})(\ell) \cdot \dots \cdot (W_{\alpha(n)} \circ S_{\alpha[n-1]})(\ell) W_{\alpha}(\ell) = \prod_{i=1}^{|\alpha|} (W_{\alpha(i)} \circ S_{\alpha[i-1]})(\ell)$$
(32)

By (31) and the considered expansion, each value $\ell(\alpha) \in M$, is the product of $|\alpha| + 1$ values from M for any factorization. The reader may compare the language accepted by an M-DFA (18) with (31), and (16) with (32). The main question is how to ensure that every $\ell(\alpha)$ is finitely generated by some automaton given a factorization. This question is solved in the following lemma for recognizability.

Lemma 3. Let $\ell \in L$ be an M-language. If there exists a factorization on L, (g, f), such that the set $\{S_{\alpha}^{(g,f)}(\ell) | \alpha \in \Sigma^*\}$ is finite, then ℓ is a recognizable M-language.

Proof: We construct an M-DFA that recognizes $\ell \in L$. By assumption $\{S_{\alpha}^{(g,f)}(\ell) | \alpha \in \Sigma^*\}$ is finite. It is a nonempty set since ℓ is in that set. In the rest of the proof, we omit the superscript (g,f) excepting for some definitions. Let us consider the following automaton:

Definition 4. $N_{\ell}^{(g,f)} = (Q, \Sigma, S_{\varepsilon}(\ell), 1, \delta, w, \rho)$:

- $Q = \{S_{\alpha}(\ell) | \alpha \in \Sigma^*\}$ is the set of sates, each state is an M-language;
- the initial state is $S_{\varepsilon}(\ell) \in Q$, by (25), $S_{\varepsilon}(\ell) = \ell$;
- the initial value of $S_{\varepsilon}(\ell)$ is 1;
- the state-transition function δ is defined as $\delta(S_{\alpha}(\ell), \sigma) = S_{\alpha\sigma}(\ell)$ for any $\alpha \in \Sigma^*$ and $\sigma \in \Sigma$;
- the monoid-transition function w is defined as $w(S_{\alpha}(\ell), \sigma) = (W_{\sigma} \circ S_{\alpha})(\ell)$ for any $\alpha \in \Sigma^*$ and $\sigma \in \Sigma$; and
- for each state, its final value is $\rho(S_{\alpha}(\ell)) = (S_{\alpha}(\ell))(\varepsilon)$ for any $\alpha \in \Sigma^*$.

Both $\delta: Q \times \Sigma \to Q$ and $w: Q \times \Sigma \to M$ are well defined functions. Given two words α and β , if $S_{\alpha}(\ell) = S_{\beta}(\ell)$ then $S_{\alpha\sigma}(\ell) = (S_{\sigma} \circ S_{\alpha})(\ell) = (S_{\sigma} \circ S_{\beta})(\ell) = S_{\beta\sigma}(\ell)$; and, trivially, $(W_{\sigma} \circ S_{\alpha})(\ell) = (W_{\sigma} \circ S_{\beta})(\ell)$. Therefore, the automaton $N_{\ell}^{(g,f)}$ is a well defined M-DFA since Q is a nonempty finite set by the conditions of the lemma.

It is simple to show that $\delta^*(S_{\alpha}(\ell), \beta) = S_{\alpha\beta}(\ell)$ for any words α and β . By notation, $\delta^*(S_{\alpha}(\ell), \beta)$ is represented as $S_{\alpha}(\ell)\beta$ (as in section 4). Thus, $S_{\alpha}(\ell)\beta = S_{\alpha\beta}(\ell)$.

In the following, we prove that the M-language recognized by $N_{\ell}^{(g,f)}$, in notation $\mathcal{N}_{\ell}^{(g,f)}$, satisfies $\mathcal{N}_{\ell}^{(g,f)} = \ell$.

For any $\alpha \in \Sigma^*$: By (18), $\mathcal{N}_{\ell}^{(g,f)}(\alpha) = 1 \cdot w^*(S_{\varepsilon}(\ell), \alpha) \cdot \rho(S_{\varepsilon}(\ell)\alpha)$. Since, $S_{\varepsilon}(\ell)\alpha = S_{\alpha}(\ell)$, then $\mathcal{N}_{\ell}^{(g,f)}(\alpha) = w^*(S_{\varepsilon}(\ell), \alpha) \cdot \rho(S_{\alpha}(\ell))$, By (16), $\mathcal{N}_{\ell}^{(g,f)}(\alpha) = (\prod_{i=1}^{|\alpha|} w(S_{\varepsilon}(\ell)\alpha[i-1], \alpha(i))) \cdot \rho(S_{\alpha}(\ell))$, and again, $S_{\varepsilon}(\ell)\alpha[i-1] = S_{\alpha[i-1]}(\ell)$, then, $\mathcal{N}_{\ell}^{(g,f)}(\alpha) = (\prod_{i=1}^{|\alpha|} w(S_{\alpha[i-1]}(\ell), \alpha(i))) \cdot \rho(S_{\alpha}(\ell))$ By definition of w() and $\rho()$ given for the automaton $\mathcal{N}_{\ell}^{(g,f)}$,
$$\begin{split} \mathcal{N}_{\ell}^{(g,f)}(\alpha) &= (\prod_{i=1}^{|\alpha|} (W_{\alpha(i)} \circ S_{\alpha[i-1]})(\ell)) \cdot (S_{\alpha}(\ell))(\varepsilon) \\ \text{By the considered expansion of } W_{\alpha}(\ell) \text{ given in (32)}, \\ \mathcal{N}_{\ell}^{(g,f)}(\alpha) &= W_{\alpha}(\ell) \cdot (S_{\alpha}(\ell))(\varepsilon). \\ \text{Finally, by (31)}, \, \mathcal{N}_{\ell}^{(g,f)}(\alpha) &= \ell(\alpha). \end{split}$$

Therefore, $\mathcal{N}_{\ell}^{(g,f)} = \ell$. This fact concludes that ℓ is a recognizable M-language via the factorization (g,f).

As Lemma 3 indicates, $\ell \in L$ is recognizable if the automaton $N_{\ell}^{(g,f)}$ is an M-DFA for some factorization on L, (g,f). In the construction provided in Definition 4, $N_{\ell}^{(g,f)}$ depends on ℓ and the factorization (g,f). The initial value of this automaton may be also changed without modifying the rest of components. In that case, we write $N^{(g,f)}(\ell,1)$ for the original automaton and $N^{(g,f)}(\ell,m)$ for the same automaton but with initial value $m \in M$.

Let us observe that the finiteness property of the set of states $Q_{\ell}^{(g,f)} = \{S_{\alpha}(\ell) | \alpha \in \Sigma^*\}$ of $N^{(g,f)}(\ell,1)$, is due to the fact that infinitely many words satisfy $S_{\alpha}(\ell) = S_{\beta}(\ell)$. This argument allows us to propose a right congruence on Σ^* .

Definition 5. Let (g, f) be a factorization on L and let ℓ be an M-language. The binary relation on Σ^* denoted by $\equiv_{\ell}^{(g,f)}$ is defined as follows:

$$\alpha \equiv_{\ell}^{(g,f)} \beta \Leftrightarrow S_{\alpha}^{(g,f)}(\ell) = S_{\beta}^{(g,f)}(\ell) \tag{33}$$

for any α , $\beta \in \Sigma^*$.

Clearly, $\equiv_{\ell}^{(g,f)}$ is a congruence on Σ^* . In addition, $\equiv_{\ell}^{(g,f)}$ satisfies

$$\alpha \equiv_{\ell}^{(g,f)} \beta \Rightarrow \alpha \gamma \equiv_{\ell}^{(g,f)} \beta \gamma \tag{34}$$

for any $\alpha, \beta, \gamma \in \Sigma^*$.

Proof: By definition of the congruence $S_{\alpha}^{(g,f)}(\ell) = S_{\beta}^{(g,f)}(\ell)$. Then, by (27), $S_{\alpha\gamma}(\ell) = (S_{\gamma} \circ S_{\alpha})(\ell) = (S_{\gamma} \circ S_{\beta})(\ell) = S_{\beta\gamma}(\ell)$. Thus, $\alpha\gamma \equiv_{\ell}^{(g,f)} \beta\gamma$ for every $\gamma \in \Sigma^*$.

It concludes that $\equiv_{\ell}^{(g,f)}$ is a right congruence on Σ^* . Furthermore, if the quotient set $\Sigma^*/\equiv_{\ell}^{(g,f)}$ is finite, i.e., $\equiv_{\ell}^{(g,f)}$ has finite index, then the set $Q_{\ell}^{(g,f)}$ is also finite (and vice versa):

$$\equiv_{\ell}^{(g,f)}$$
 has finite index $\Leftrightarrow Q_{\ell}^{(g,f)} = \{S_{\alpha}(\ell) | \alpha \in \Sigma^*\}$ is finite (35)

Corollary 1. Let $\ell \in L$ be an M-language. If there exists a factorization on L, (g, f), such that the right congruence on Σ^* induced by the factorization, $\equiv_{\ell}^{(g,f)}$, has finite index, then ℓ is a recognizable M-language.

Proof: By (35), $Q_{\ell}^{(g,f)}$ is a finite set. Therefore, by Lemma 3, ℓ is recognized by the M-DFA $N^{(g,f)}(\ell,1)$.

We end this section by providing the main properties of the automaton $N^{(g,f)}(\ell,m)$.

Property 3. Let $\ell \in L$ be an M-language. Let (g, f) be a factorization on L. If the automaton $N^{(g,f)}(\ell,1)$ is an M-DFA then, for any $m \in M$, the M-DFA $N^{(g,f)}(\ell,m)$ satisfies the properties:

- 1. $\mathcal{N}^{(g,f)}(\ell,m) = m \cdot \ell$
- 2. $N^{(g,f)}(\ell,m)$ is an accessible M-DFA.
- 3. Each state $S_{\alpha}(\ell) \in Q_{\ell}^{(g,f)}$ is a recognizable M-language.
- 4. For each state $S_{\alpha}(\ell) \in Q_{\ell}^{(g,f)}$, $(\mathcal{N}^{(g,f)}(\ell,m))_{S_{\alpha}(\ell)} = S_{\alpha}(\ell)$.
- 5. $N^{(g,f)}(\ell,m)$ satisfies the necessary condition of minimality (NcdW).

Proof:

- 1. We recall that $m \in M$ is the initial value in the automaton $N^{(g,f)}(\ell,m)$ that replaces the original initial value 1 in $N^{(g,f)}(\ell,1)$. Both automata are exactly the same by excluding those initial values. By condition (**Rcg1**) (see section 4), as $N^{(g,f)}(\ell,1)$ is an M-DFA that recognizes ℓ (proof of Lemma 3) then, $N^{(g,f)}(\ell,m)$ is an M-DFA that recognizes $m \cdot \ell \in L$ for any $m \in M$.
- 2. By Definition 4 in Lemma 3, the set $Q_{\ell}^{(g,f)} = \{S_{\alpha}(\ell) | \alpha \in \Sigma^*\}$ contains all accessible states from the initial state $S_{\varepsilon}(\ell) = \ell$ (see the proof below Definition 4 in Lemma 3).
- 3. As $Q_{\ell}^{(g,f)}$ is finite then, for any $S_{\alpha}(\ell) \in Q_{\ell}^{(g,f)}$, the set $Q_{S_{\alpha}(\ell)}^{(g,f)}$, that is obtained by replacing ℓ by $S_{\alpha}(\ell)$, is also finite and; thus, each state $S_{\alpha}(\ell)$ is a recognizable M-language by Lemma 3. In fact, $Q_{S_{\alpha}(\ell)}^{(g,f)} \subseteq Q_{\ell}^{(g,f)}$ because, by (27), for any word β , $S_{\alpha\beta}(\ell) = (S_{\beta} \circ S_{\alpha})(\ell) = S_{\beta}(S_{\alpha}(\ell)) \in Q_{\ell}^{(g,f)}$.
- 4. As $S_{\alpha}(\ell)$ is a state of $N^{(g,f)}(\ell,m)$ then, by (19), $(\mathcal{N}^{(g,f)}(\ell,m))_{S_{\alpha}(\ell)}$ is given by $(\mathcal{N}^{(g,f)}(\ell,m))_{S_{\alpha}(\ell)}(\beta) = w^*(S_{\alpha}(\ell),\beta) \cdot \rho(S_{\alpha}(\ell)\beta)$ for any $\beta \in \Sigma^*$, where $w^*()$ and $\rho()$ are the extended monoid-transition function and final

function provided in (15) and Definition 4. Following a similar reasoning as in the proof of Lemma 3, we have that

$$w^*(S_{\alpha}(\ell),\beta) = \prod_{i=1}^{|\beta|} (W_{\beta(i)} \circ S_{\beta[i-1]})(S_{\alpha}(\ell)) = W_{\beta}(S_{\alpha}(\ell)) = (W_{\beta} \circ S_{\alpha})(\ell),$$
 and $\rho(S_{\alpha}(\ell)\beta) = \rho(S_{\alpha\beta}(\ell)) = (S_{\alpha\beta}(\ell))(\varepsilon) = ((S_{\beta} \circ S_{\alpha})(\ell))(\varepsilon)$
Thus, for any word β ,

$$(\mathcal{N}^{(g,f)}(\ell,m))_{S_{\alpha}(\ell)}(\beta) = (W_{\beta} \circ S_{\alpha})(\ell) \cdot ((S_{\beta} \circ S_{\alpha})(\ell))(\varepsilon) = (((W_{\beta} \circ S_{\alpha}) \bullet (S_{\beta} \circ S_{\alpha}))(\ell))(\varepsilon) = \text{by } (30) ((\partial_{\beta} \circ S_{\alpha})(\ell))(\varepsilon) = (\partial_{\beta}(S_{\alpha}(\ell)))(\varepsilon) = S_{\alpha}(\beta)$$

In conclusion, $(\mathcal{N}^{(g,f)}(\ell,m))_{S_{\alpha}(\ell)} = S_{\alpha}(\ell)$.

5. The necessary condition for minimality (**NcndW**) (see Property 1.3) is fulfilled in the M-DFA $N^{(g,f)}(\ell,m)$ because for all states $S_{\alpha}(\ell)$ and $S_{\beta}(\ell)$ of that automaton, if $(\mathcal{N}^{(g,f)}(\ell,m))_{S_{\alpha}(\ell)} = (\mathcal{N}^{(g,f)}(\ell,m))_{S_{\beta}(\ell)}$ then, by the previous Property 3.4, $S_{\alpha}(\ell) = S_{\beta}(\ell)$.

6. Natural factorizations on $L(\Sigma^*, M)$

Recognizability of M-languages provided in section 5 has been based on the hypothesis that, for arbitrary monoids, there exist general factorizations on $L(\Sigma^*, M)$. In this section, we show that each M-DFA is able to define a natural factorization on L. The construction of a factorization on L induced by an M-DFA is based on equalization of transitions (see (22)).

Let $A = (Q, \Sigma, u, i_u, \delta, w, \rho)$ be an M-DFA (Definition 2). Recall that the M-language recognized by A is denoted by \mathcal{A} (see (18)), and that \mathcal{A}_q (see 19) is the M-language given for each state q of A.

We define the set $P_A = \{((\sigma, q), \partial_{\sigma}(\mathcal{A}_q)), ((\varepsilon, u), \mathcal{A}) | \sigma \in \Sigma, q \in Q\}$ and the following relation on P_A , denoted by \approx_A :

$$((\sigma, q), \ell) \approx_A ((\tau, p), \ell') \Leftrightarrow \ell = \ell'$$
(36)

for every $((\sigma,q),\ell)$, $((\tau,p),\ell') \in P_A$.

By the given definition, \approx_A is an equivalence relation on P_A . This equivalence relation has been constructed taken into account the antecedent of the implication given in (22), and it will allow us to define a factorization on L. Clearly, the quotient set P_A/\approx_A contains a finite number of equivalence classes. Let π be a function for selecting a unique representative element for each class in P_A/\approx_A . Under a given selection π , each class $C_A \in P_A/\approx_A$,

is denoted by $C_A^{\pi}((\sigma, q), \partial_{\sigma}(\mathcal{A}_q))$. This notation indicates explicitly the representative element of the class. The given selection π is defined is such a way that it always provides the class $C_A^{\pi}((\varepsilon, u), \mathcal{A})$.

Definition 6. Let $A = (Q, \Sigma, u, i_u, \delta, w, \rho)$ be an M-DFA, let P_A / \approx_A be the quotient set of the equivalence relation \approx_A (36), and let π be a selection function of representative elements for the classes in P_A / \approx_A . The functions $f_A^{\pi}: L \to L$ and $g_A^{\pi}: L \to M$ are defined as follows:

For each $\ell \in L$

$$f_A^{\pi}(\ell) = \mathcal{A}_u \quad \text{if } \ell = \mathcal{A} \ \land \ C_A^{\pi}((\varepsilon, u), \mathcal{A}) \in P_A / \approx_A$$

$$f_A^{\pi}(\ell) = \mathcal{A}_{q\sigma} \quad \text{if } \ell = \partial_{\sigma}(\mathcal{A}_q) \ \land \ C_A^{\pi}((\sigma, q), \partial_{\sigma}(\mathcal{A}_q)) \in P_A / \approx_A$$

$$f_A^{\pi}(\ell) = \ell \quad \text{otherwise}$$

$$(37)$$

$$g_A^{\pi}(\ell) = i_u \qquad if \ \ell = \mathcal{A} \ \land \ C_A^{\pi}((\varepsilon, u), \mathcal{A}) \in P_A / \approx_A$$

$$g_A^{\pi}(\ell) = w(q, \sigma) \quad if \ \ell = \partial_{\sigma}(\mathcal{A}_q) \ \land \ C_A^{\pi}((\sigma, q), \partial_{\sigma}(\mathcal{A}_q)) \in P_A / \approx_A$$

$$g_A^{\pi}(\ell) = 1 \qquad otherwise$$

$$(38)$$

By the given definition, f_A^{π} and g_A^{π} are well defined functions of F and G respectively. We prove that such pair of functions is a factorization on L.

Lemma 4. Let $A = (Q, \Sigma, u, i_u, \delta, w, \rho)$ be an M-DFA. The pair $(g_A^{\pi}, f_A^{\pi}) \in G \times F$ (Definition 6) is a factorization on L induced by the automaton A.

Proof: For any $\ell \in L$. By Definition 6,

- $\ell = \mathcal{A}$. Then, $g_A^{\pi}(\ell) \cdot f_A^{\pi}(\ell) = i_u \cdot \mathcal{A}_u = \mathcal{A}$ (by (19) and (18)).
- $\ell = \partial_{\tau}(\mathcal{A}_p)$ for some $\tau \in \Sigma$ and $p \in Q$, then there exists some class $C_A^{\pi} \in P_A / \approx_A$ such that $\ell = \partial_{\sigma}(\mathcal{A}_q)$ or $\ell = \mathcal{A}$. This last case is the same as the given above. For the former case, assume that $((\sigma, q), \partial_{\sigma}(\mathcal{A}_q))$ is the given representative by π . Then, $g_A^{\pi}(\ell) \cdot f_A^{\pi}(\ell) = w(q, \sigma) \cdot \mathcal{A}_{q\sigma} = \partial_{\sigma}(\mathcal{A}_q) = \ell$ (by (20)).
- Otherwise, $g_A^{\pi}(\ell) \cdot f_A^{\pi}(\ell) = 1 \cdot \ell$.

Therefore, $g_A^{\pi} \bullet f_A^{\pi} = f_e$. By Definition 1, the pair $(g_A^{\pi}, f_A^{\pi}) \in G \times F$ is a factorization on L.

Let us observe that the pair (g_A^{π}, f_A^{π}) depends on the selection function π , and that Lemma 4 holds for any selection π . Given an M-DFA A =

 $(Q, \Sigma, u, i_u, \delta, w, \rho)$, define the sets $\widehat{P}_A = \{A, \partial_{\sigma}(A_q) | \sigma \in \Sigma, q \in Q\}$ and $\widehat{Q}_A = \{A_q | q \in Q\}$. Any factorization on L induced by A, (g_A^{π}, f_A^{π}) , satisfies that

$$f_A^{\pi}(\ell) \in \widehat{Q}_A \quad \text{if } \ell \in \widehat{P}_A$$
 $f_A^{\pi}(\ell) = \ell \quad \text{otherwise}$ (39)

for any $\ell \in L$.

Lemma 5. Let $A = (Q, \Sigma, u, i_u, \delta, w, \rho)$ be an M-DFA. For any factorization on L induced by A, (g_A^{π}, f_A^{π}) , the right congruence $\equiv_{f_A^{\pi}(A)}^{(g_A^{\pi}, f_A^{\pi})}$ has finite index.

Proof: Recall that \mathcal{A} is the M-language recognized by A. As A is an M-DFA, the set of states Q is finite, and the sets of M-languages $\widehat{Q}_A = \{\mathcal{A}_q | q \in Q\}$ and $\widehat{P}_A = \{\mathcal{A}, \partial_{\sigma}(\mathcal{A}_q) | \sigma \in \Sigma, q \in Q\}$ are also finite. We prove that for any $\alpha \in \Sigma^*$,

$$S_{\alpha}^{(g_A^{\pi}, f_A^{\pi})}(f_A^{\pi}(\mathcal{A})) \in \widehat{Q}_A \tag{40}$$

For convenience, we omit the superscript (g_A^{π}, f_A^{π}) in $S_{\alpha}^{(g_A^{\pi}, f_A^{\pi})}$. By induction on the prefixes of α :

- Basis. By (25), $S_{\varepsilon}(f_A^{\pi}(\mathcal{A})) = f_A^{\pi}(\mathcal{A})$. As $\mathcal{A} \in \widehat{P}_A$, then, by (39), $S_{\varepsilon}(f_A^{\pi}(\mathcal{A})) \in \widehat{Q}_A$.
- Hypothesis. Let us assume that $S_{\alpha'}(f_A^{\pi}(\mathcal{A})) \in \widehat{Q}_A$ where α' is a prefix of α .
- Induction step. Let us consider $\alpha'\sigma$ be a prefix of α with $\sigma \in \Sigma$. By (27) and (26), $S_{\alpha'\sigma}(f_A^{\pi}(\mathcal{A})) = (S_{\sigma} \circ S_{\alpha'})(f_A^{\pi}(\mathcal{A})) = f_A^{\pi}(\partial_{\sigma}(S_{\alpha'}(f_A^{\pi}(\mathcal{A}))))$. By induction hypothesis, $S_{\alpha'}(f_A^{\pi}(\mathcal{A})) \in \widehat{Q}_A$, i.e., there exists some $q \in Q$ such that $S_{\alpha'}(f_A^{\pi}(\mathcal{A})) = \mathcal{A}_q$. By substitution, $S_{\alpha'\sigma}(f_A^{\pi}(\mathcal{A})) = f_A^{\pi}(\partial_{\sigma}(\mathcal{A}_q))$. As $\partial_{\sigma}(\mathcal{A}_q) \in \widehat{P}_A$, then, by (39), $f_A^{\pi}(\partial_{\sigma}(\mathcal{A}_q)) \in \widehat{Q}_A$. Therefore, $S_{\alpha'\sigma}(f_A^{\pi}(\mathcal{A})) \in \widehat{Q}_A$.

This fact proves that the set $Q_{f_A^{\pi}(\mathcal{A})}^{(g_A^{\pi}, f_A^{\pi})} = \{S_{\alpha}(f_A^{\pi}(\mathcal{A})) | \alpha \in \Sigma^* \}$ is finite because $Q_{f_A^{\pi}(\mathcal{A})}^{(g_A^{\pi}, f_A^{\pi})} \subseteq \widehat{Q}_A$. Therefore, by (35), $\equiv_{f_A^{\pi}(\mathcal{A})}^{(g_A^{\pi}, f_A^{\pi})}$ has finite index. \square

Remark 2. Given an M-DFA $A = (Q, \Sigma, u, i_u, \delta, w, \rho)$, Lemma 5 states that $\equiv_{f_A^{\pi}(\mathcal{A})}^{(g_A^{\pi}, f_A^{\pi})}$ has finite index. By Corollary 1 and Lemma 3, the M-DFA $N^{(g_A^{\pi}, f_A^{\pi})}(f_A^{\pi}(\mathcal{A}), 1)$ recognizes the M-language $f_A^{\pi}(\mathcal{A})$.

By Property 3.1, $N^{(g_A^{\pi}, f_A^{\pi})}(f_A^{\pi}(\mathcal{A}), g_A^{\pi}(\mathcal{A}))$ recognizes $g_A^{\pi}(\mathcal{A}) \cdot f_A^{\pi}(\mathcal{A})$. By Definition 6 (or Lemma 4), $g_A^{\pi}(\mathcal{A}) \cdot f_A^{\pi}(\mathcal{A}) = i_u \cdot \mathcal{A}_u = \mathcal{A}$. Therefore, $N^{(g_A^{\pi}, f_A^{\pi})}(f_A^{\pi}(\mathcal{A}), g_A^{\pi}(\mathcal{A}))$ is equivalent to A.

Previous Remark shows that $N^{(g_A^{\pi},f_A^{\pi})}(f_A^{\pi}(\mathcal{A}),g_A^{\pi}(\mathcal{A}))$ is equivalent to A. Those automata could be very different because the former one satisfies all the properties given in Property 3 but, however, these properties may be absent in the automaton A. The M-DFA $N^{(g_A^{\pi},f_A^{\pi})}(f_A^{\pi}(\mathcal{A}),g_A^{\pi}(\mathcal{A}))$ has their transitions equalized. This is derived by the way factorization (g_A^{π},f_A^{π}) is constructed. The next property collects the main properties involved any factorization on L induced by A.

Property 4. Let $A = (Q, \Sigma, u, i_u, \delta, w, \rho)$ be an M-DFA. For any factorization on L induced by $A, (g_A^{\pi}, f_A^{\pi})$, the following properties hold:

- 1. $N^{(g_A^{\pi}, f_A^{\pi})}(f_A^{\pi}(A), g_A^{\pi}(A))$ is equivalent to A.
- 2. $N^{(g_A^{\pi}, f_A^{\pi})}(f_A^{\pi}(\mathcal{A}), g_A^{\pi}(\mathcal{A}))$ is transition-equalized.
- 3. If A is minimal then $N^{(g_A^{\pi}, f_A^{\pi})}(f_A^{\pi}(\mathcal{A}), g_A^{\pi}(\mathcal{A}))$ is minimal.
- 4. There is a minimal and transition-equalized M-DFA equivalent to A.
- 5. If A is minimal then f_A^{π} is idempotent, i.e., $f_A^{\pi} \circ f_A^{\pi} = f_A^{\pi}$
- 6. If A is minimal and transition-equalized then $f_A^{\pi_i} = f_A^{\pi_j}$ for any selection functions π_i and π_j .

Proof:

- 1. The equivalence is shown in Remark 2.
- 2. Let $N^{(g_A^\pi,f_A^\pi)}(f_A^\pi(\mathcal{A}),g_A^\pi(\mathcal{A}))=(Q_{f_A^\pi(\mathcal{A})}^{(g_A^\pi,f_A^\pi)},\Sigma,f_A^\pi(\mathcal{A}),g_A^\pi(\mathcal{A}),\delta,w,\rho)$ be the structure of such automaton provided in Definition 4. Let us consider two states $S_\alpha(f_A^\pi(\mathcal{A}))$ and $S_\beta(f_A^\pi(\mathcal{A}))$ and two symbols $\sigma,\ \tau\in\Sigma$. By (40), $S_\alpha(f_A^\pi(\mathcal{A})),\ S_\beta(f_A^\pi(\mathcal{A}))\in\widehat{Q}_A$, i.e., there are two states $q,\ p\in Q$, such that $S_\alpha(f_A^\pi(\mathcal{A}))=\mathcal{A}_q$ and $S_\beta(f_A^\pi(\mathcal{A}))=\mathcal{A}_p$. Now consider (22):

$$\begin{split} \partial_{\sigma}(\mathcal{N}^{(g_{A}^{\pi},f_{A}^{\pi})}(f_{A}^{\pi}(\mathcal{A}),g_{A}^{\pi}(\mathcal{A}))_{S_{\alpha}(f_{A}^{\pi}(\mathcal{A}))}) &= \partial_{\tau}(\mathcal{N}^{(g_{A}^{\pi},f_{A}^{\pi})}(f_{A}^{\pi}(\mathcal{A}),g_{A}^{\pi}(\mathcal{A}))_{S_{\beta}(f_{A}^{\pi}(\mathcal{A}))}), \\ \text{by Property 3.4} \\ \partial_{\sigma}(S_{\alpha}(f_{A}^{\pi}(\mathcal{A}))) &= \partial_{\tau}(S_{\beta}(f_{A}^{\pi}(\mathcal{A}))) \\ \text{then,} \\ \partial_{\sigma}(\mathcal{A}_{a}) &= \partial_{\tau}(\mathcal{A}_{n}) \end{split}$$

This fact implies that $((\sigma,q),\partial_{\sigma}(\mathcal{A}_q))$ and $((\tau,p),\partial_{\tau}(\mathcal{A}_p)) \in C_A^{\pi}$, i.e, they are in the same class of P_A/\approx_A . Then, by Definition 6, $f_A^{\pi}(\partial_{\sigma}(\mathcal{A}_q)) =$

$$f_A^{\pi}(\partial_{\tau}(\mathcal{A}_p))$$
 and $g_A^{\pi}(\partial_{\sigma}(\mathcal{A}_q)) = g_A^{\pi}(\partial_{\tau}(\mathcal{A}_p)).$

Thus, $\delta(S_{\alpha}(f_A^{\pi}(\mathcal{A})), \sigma) = S_{\alpha\sigma}(f_A^{\pi}(\mathcal{A})) = (S_{\sigma} \circ S_{\alpha})(f_A^{\pi}(\mathcal{A})) = f_A^{\pi}(\partial_{\sigma}(\mathcal{A}_q)) =$ $f_A^{\pi}(\partial_{\tau}(\mathcal{A}_p)) = \delta(S_{\beta}(f_A^{\pi}(\mathcal{A})), \tau)$. In a similar way, $w(S_{\alpha}(f_A^{\pi}(\mathcal{A})), \sigma) = (W_{\sigma} \circ \mathcal{A}_{\sigma})$ $S_{\alpha}(f_A^{\pi}(\mathcal{A})) = g_A^{\pi}(\partial_{\sigma}(\mathcal{A}_q)) = g_A^{\pi}(\partial_{\tau}(\mathcal{A}_p)) = w(S_{\beta}(f_A^{\pi}(\mathcal{A})), \tau).$ In conclusion, $N^{(g_A^{\pi}, f_A^{\pi})}(f_A^{\pi}(\mathcal{A}), g_A^{\pi}(\mathcal{A}))$ is transition-equalized.

- 3. By the proof in Lemma 5, $Q_{f_A^{\pi}(\mathcal{A})}^{(g_A^{\pi}, f_A^{\pi})} \subseteq \widehat{Q}_A$. By definition $\|\widehat{Q}_A\| \leq \|Q\|$ where Q is the set of states of the minimal M-DFA A. Then, $\|Q_{f_{-}^{\pi}(A)}^{(g_{A}^{\pi},f_{A}^{\pi})}\| \leq$ ||Q||. Therefore, $N^{(g_A^{\pi}, f_A^{\pi})}(f_A^{\pi}(\mathcal{A}), g_A^{\pi}(\mathcal{A}))$ is a minimal M-DFA.
- 4. It is a consequence of the previous properties.
- 5. This property is proved by exhaustive case analysis and the application of the condition NcndS (Property 1.2) for a minimal M-DFA. We omit the proof by brevity.
- 6. By Definition 6 and (39):
 - If $\ell \notin \widehat{P}_A$, then $f_A^{\pi_i}(\ell) = f_A^{\pi_j}(\ell) = \ell$.

 - If $\ell \in \widehat{P}_A$, which $f_A(\ell) = f_A(\ell) = \ell$. If $\ell \in \widehat{P}_A$ and $\ell = \mathcal{A}$, then $f_A^{\pi_i}(\ell) = f_A^{\pi_j}(\ell) = \mathcal{A}_u$. If $\ell \in \widehat{P}_A$ and $\ell = \partial_{\sigma}(A_q)$ and $C_A^{\pi_i}((\sigma, q), \partial_{\sigma}(A_q)) \in P_A/\approx_A$, then $C_A^{\pi_i}((\sigma, q), \partial_{\sigma}(A_q)) = C_A^{\pi_j}((\tau, p), \partial_{\tau}(A_p))$ with $\partial_{\sigma}(A_q) = \partial_{\tau}(A_p)$. As A is transition-equalized, by (22), $q\sigma = p\tau$. Therefore, $f_A^{\pi_i}(\ell) = \mathcal{A}_{q\sigma} = \mathcal{A}_{p\tau} = f_A^{\pi_j}(\ell)$.

7. General Myhill-Nerode Theorem

Results presented in previous sections allow us to enunciate a general theorem for recognizability of M-languages.

Theorem 1. (General Myhill-Nerode Theorem) Let $\ell \in L$ be an Mlanguage. The following two conditions are equivalent:

- (i) ℓ is a recognizable M-language;
- (ii) there exists a factorization on L, (g, f), such that the right congruence $\equiv_{f(\ell)}^{(g,f)}$ has finite index.

Proof:

(i) \Rightarrow (ii). If ℓ is a recognizable M-language then there is an M-DFA $A = (Q, \Sigma, u, i_u, \delta, w, \rho)$ such that $\mathcal{A} = \ell$. By Lemma 4, the pair $(g_A^{\pi}, f_A^{\pi}) \in G \times F$ (Definition 6) is a factorization on L induced by the automaton A. By Lemma 5, the right congruence $\equiv_{f_A^{\pi}(\mathcal{A})}^{(g_A^{\pi}, f_A^{\pi})}$ has finite index. In addition, by Remark 2, the M-DFA $N^{(g_A^{\pi}, f_A^{\pi})}(f_A^{\pi}(\mathcal{A}), g_A^{\pi}(\mathcal{A}))$ recognizes $\mathcal{A} = \ell$.

(ii) \Rightarrow (i). If there exists a factorization on L, (g,f), such that the right congruence $\equiv_{f(\ell)}^{(g,f)}$ has finite index then, by Lemma 3, $f(\ell)$ is recognized by the M-DFA $N^{(g,f)}(f(\ell),1)$. By Property 3.1, the M-DFA $N^{(g,f)}(f(\ell),g(\ell))$ recognizes the M-language $g(\ell) \cdot f(\ell) = \ell$ since (g,f) is a factorization on L.

8. Recognition Capability of factorizations on $L(\Sigma^*, M)$

For a monoid M and an alphabet Σ , the set of all recognizable M-languages in $L(\Sigma^*, M)$ is denoted by \mathcal{R}_L . By following the previous general Myhill-Nerode Theorem, we can define the *recognition capability* of a factorization on L, (g, f), as the set

$$\operatorname{RcgCap}_{L}^{(g,f)} = \{ \ell \in L \mid N^{(g,f)}(f(\ell), 1) \text{ is an } M\text{-DFA} \}$$
 (41)

This set, $\operatorname{RcgCap}_L^{(g,f)}$, represents the set of all recognizable M-languages for a given factorization on L, i.e, the languages recognized by $N^{(g,f)}(f(\ell),m)$ for some $m \in M$. Obviously, $\operatorname{RcgCap}_L^{(g,f)} \subseteq \mathcal{R}_L$. In this section, we study the recognition capability of three types of factorizations: (a) trivial factorization; (b) maximal factorizations; and (c) composition of natural factorizations.

8.1. Case of study: Trivial factorization

For every monoid M, $L(\Sigma^*, M)$ admits the trivial factorization (g_e, f_e) , which is the identity element of $G \times F$ under composition of factorizations. By (23) and (24), for any $\ell \in L$ and $\alpha \in \Sigma^*$, $S_{\alpha}^{(g_e, f_e)}(\ell) = \partial_{\alpha}(\ell)$ and $W_{\alpha}^{(g_e, f_e)}(\ell) = 1$. In addition, $f_e(\ell) = \ell$.

Let us consider that $N^{(g_e,f_e)}(\ell,1)$ is an M-DFA for a language $\ell \in L$. By Definition 4, $Q_{\ell}^{(g_e,f_e)} = \{\partial_{\alpha}(\ell) | \alpha \in \Sigma^*\}$ is finite and its monoid-transition function satisfies that $w(\partial_{\alpha},\sigma) = 1$ for any state and symbol.

The M-DFA $N^{(g_e,f_e)}(\ell,1)$, which recognizes ℓ , is merely and ordinary DFA

equipped with a final-state function ρ . Furthermore, by (18), $(\mathcal{N}^{(g_e,f_e)}(\ell,1))(\beta) = \rho(\partial_{\beta}(\ell)) = (\partial_{\beta}(\ell))(\varepsilon) = \ell(\beta)$. Therefore, as $Q_{\ell}^{(g_e,f_e)}$ is finite, ℓ has finite rank. In conclusion, only M-languages of finite rank may be recognized by using the trivial factorization.

Previous discussion allows us to introduce the notion of a ρ -DFA, i.e., an ordinary DFA with a final-state function ρ . We can represent a ρ -DFA by $A = (Q, \Sigma, u, \delta, \rho)$ where (Q, Σ, u, δ) is a DFA and $\rho : Q \to M$. A ρ -DFA A, recognizes the M-language $A(\alpha) = \rho(u\alpha)$ for any word α . One can define the notion of minimal ρ -DFA, and by Property 1.3 and Property 2, it is concluded that the sufficient and necessary condition for an accessible ρ -DFA to be minimal is just the condition \mathbf{NcndW} , i.e., A is a minimal ρ -DFA if and only if for every $p, q \in Q$, $A_p = A_q \Rightarrow p = q$. Therefore, $N^{(g_e, f_e)}(\ell, 1)$, when it is finite, is the minimal ρ -DFA that recognizes ℓ since $N^{(g_e, f_e)}(\ell, 1)$ satisfies Property 1.3 (\mathbf{NcdW}). However, this fact does not prevent to find another factorization on L that provides an M-DFA with lesser states than a minimal ρ -DFA (see fig.2 and fig. 3 in [17]).

Let us observe that $N^{(g_e,f_e)}(\ell,m) = N^{(g_e,f_e)}(m \cdot \ell,1)$ since ℓ is of finite rank and $N^{(g_e,f_e)}(\ell,m)$ is a ρ -DFA. Therefore, for any monoid M, $\mathcal{R}_L \supseteq \operatorname{RcgCap}_L^{(g_e,f_e)} = \{\ell \in L | \ell \text{ is recognized by a } \rho\text{-DFA}\}$

In the context of fuzzy automata, ρ -DFAs are called crisp deterministic fuzzy automata. This kind of automata has been studied by Ignatović et al. [22] from the perspective of the Myhill-Nerode Theorem and minimization algorithms for crisp deterministic fuzzy automata.

8.2. Case of study: Maximal factorizations

Let us consider that $L(\Sigma^*, M)$ admits a maximal factorization. By the results in [13], if M contains a zero element then $(M, \cdot, 1)$ is zero-divisor-free. In addition, $(M \setminus \{0\}, \cdot, 1)$ is also a monoid. In particular, [13] studies mge-monoids² and their conditions to obtain maximal factorizations. In order to simplify this case of study, we consider that $(M, \cdot, 1)$ is a monoid without a zero element

A maximal factorization on L, $(g_h, h) \in G \times F$, is a factorization that satisfies $h(m \cdot \ell) = h(\ell)$ for any $\ell \in L$ and $m \in M$. This strong property implies that h is idempotent: $h(\ell) = h(g_h(\ell) \cdot h(\ell)) = h(h(\ell))$ for any $\ell \in L$. Let us consider that $N^{(g_h,h)}(h(\ell),1)$ is an M-DFA for a language $\ell \in L$.

²An *mge*-monoid satisfies left and right cancellation axioms and the right most general equalizer axiom [13].

By (23) for any $\ell \in L$ and $\alpha \in \Sigma^*$, $S_{\alpha}^{(g_h,h)}(h(\ell)) = h(\partial_{\alpha}(h(\ell)))$. The reader may prove the given expression by induction on $|\alpha|$ by using that h is idempotent and a maximal factorization. The hint is to apply equation (11) when proving the induction step. This expression simplifies the definition of $W_{\alpha}^{(g_h,h)}(h(\ell))$ in (24). Then, the set of states of $N^{(g_h,h)}(h(\ell),1)$ is the set $Q_{h(\ell)}^{(g_h,h)} = \{h(\partial_{\alpha}(h(\ell))) | \alpha \in \Sigma^*\}$.

Let us recall that by Property 3.1, $N^{(g_h,h)}(h(\ell),g(\ell))$ recognizes $\ell \in L$. The interesting aspect of this automata is that it satisfies the sufficient condition of minimality (Property 2).

Property 5. Let (g_h, h) be a maximal factorization on L. If $N^{(g_h, h)}(h(\ell), g(\ell))$ is an M-DFA then it is a minimal M-DFA that recognizes $\ell \in L$.

Proof: Let us consider an M-DFA $A=(Q,\Sigma,u,i_u,\delta,w,\rho)$ such that it is accessible, but it does not satisfy the sufficient condition for minimality (see Property 2): for two words $\alpha, \beta \in \Sigma^*$, two values $m, m' \in M$, and an M-language $\ell' \in L$, $u\alpha \neq u\beta \wedge \partial_{\alpha}(\mathcal{A}) = m \cdot \ell' \wedge \partial_{\beta}(\mathcal{A}) = m' \cdot \ell'$. As (g_h,h) is a maximal factorization, $h(\partial_{\alpha}(\mathcal{A})) = h(\partial_{\beta}(\mathcal{A}))$ because $h(m \cdot \ell') = h(m' \cdot \ell') = h(\ell')$. By (20), $\partial_{\alpha}(\mathcal{A}) = i_u \cdot w^*(u,\alpha) \cdot \mathcal{A}_{u\alpha}$. Then, $h(\partial_{\alpha}(\mathcal{A})) = h(\mathcal{A}_{u\alpha})$. Similarly, $h(\partial_{\beta}(\mathcal{A})) = h(\mathcal{A}_{u\beta})$. Therefore, $h(\mathcal{A}_{u\alpha}) = h(\mathcal{A}_{u\beta})$. By identifying, A with $N^{(g_h,h)}(h(\ell),g(\ell))$, which is accessible, then $u\alpha = S_{\alpha}(h(\ell)) = h(\partial_{\alpha}(h(\ell)))$ and $u\beta = h(\partial_{\beta}(h(\ell)))$. By Property 3.4, as $\mathcal{A}_{u\alpha} = u\alpha$ and $\mathcal{A}_{u\beta} = u\beta$, then $h(h(\partial_{\alpha}(h(\ell)))) = h(h(\partial_{\beta}(h(\ell))))$. As h() is idempotent, then $u\alpha = u\beta$ following our identification. This is a contradiction with the initial hypothesis. Therefore, $N^{(g_h,h)}(h(\ell),g(\ell))$ satisfies the sufficient condition for minimality.

A maximal factorization on $L(\Sigma^*, M)$ achieves the maximal recognition capability.

Lemma 6. Let (g_h, h) be a maximal factorization on L. Then, $\mathcal{R}_L = RcgCap_L^{(g_h, h)}$

Proof: Let $\ell \in \mathcal{R}_L$ be recognized by an M-DFA $A = (Q, \Sigma, u, i_u, \delta, w, \rho)$, i.e., $\mathcal{A} = \ell$. Without loss of generality, A is accessible, i.e, $Q = \{u\alpha \mid \alpha \in \Sigma^*\}$. The set $\widehat{Q}_A = \{\mathcal{A}_{u\alpha} \mid \alpha \in \Sigma^*\}$ is finite. Thus, $h(\widehat{Q}_A) = \{h(\mathcal{A}_{u\alpha}) \mid \alpha \in \Sigma^*\}$ is also finite. By (20), $\partial_{\alpha}(\mathcal{A}) = i_u \cdot w^*(u,\alpha) \cdot \mathcal{A}_{u\alpha}$. Then, as (g_h, h) is a maximal factorization, $h(\partial_{\alpha}(\mathcal{A})) = h(\mathcal{A}_{u\alpha})$. As $\mathcal{A} = g_h(\mathcal{A}) \cdot h(\mathcal{A})$; by (11), $h(\partial_{\alpha}(\mathcal{A})) = h(\partial_{\alpha}(g_h(\mathcal{A}) \cdot h(\mathcal{A}))) = h(g_h(\mathcal{A}) \cdot \partial_{\alpha}(h(\mathcal{A}))) = h(\partial_{\alpha}(h(\mathcal{A})))$. Therefore, the set $\{h(\partial_{\alpha}(h(\ell))) \mid \alpha \in \Sigma^*\}$ is finite. This set is the set of states of the M-DFA $N^{(g_h,h)}(h(\ell),g(\ell))$ which recognizes ℓ . By the definitions given

at the beginning of this section, $\ell \in \operatorname{RcgCap}_L^{(g_h,h)}$, i.e., $\mathcal{R}_L \subseteq \operatorname{RcgCap}_L^{(g_h,h)}$; thus, $\mathcal{R}_L = \operatorname{RcgCap}_L^{(g_h,h)}$.

When the maximal factorization has an explicit formulae, it is possible to construct determinization and minimization algorithms for automata. Maximal factorizations produce very efficient constructions. Kirsten and Mäurer [23] show that their determinization algorithm of weighted automata is optimal using maximal factorizations and the zero-divisor-free condition (Theorem 3.3 in [23]). This behaviour has been corroborated in some determinization methods for fuzzy automata [15][40]. The original Mohri's minimization algorithm for weighted automata over tropical semiring applies a maximal factorization [28]. Other examples of applications of maximal factorizations are in [9][17][41][16].

8.3. Case of study: Composition of Natural factorizations

Let us consider an arbitrary monoid M. Let A be an M-DFA. By Property 4.4, there exists a minimal and transition-equalized M-DFA equivalent to A. Thus, we consider that A is a minimal and transition-equalized M-DFA. By Property 4.6, the factorization on L induced by A is unique. That factorization is simply denoted (g_A, f_A) . By Lemma 5 and Remark 2, $\operatorname{RcgCap}_L^{(g_A, f_A)}$ is not empty since $f_A(A)$ is in this set. Let us recall that, by Lemma 1, the composition of factorizations on L is again a factorization on L. We study the composition of natural factorizations to provide the result that the composition preserves the recognition capability of each individual natural factorization.

Lemma 7. Let $\{Ak\}_{k=1..n}$ be a finite family of n minimal and transition-equalized M-DFAs; and, let $\{(g_{Ak}, f_{Ak})\}_{k=1..n}$ be the famility of the natural factorizations on L induced by those automata. The factorization on L, $(g, f) = [(g_{Ak}, f_{Ak})]_1^n$, obtained by the composition of the family $\{(g_{Ak}, f_{Ak})\}_{k=1..n}$, satisfies the next property,

$$RcgCap_L^{(g,f)} \supseteq \bigcup_{k=1}^n RcgCap_L^{(g_{Ak},f_{Ak})}$$
 (42)

Proof: For each M-DFA $Ak = (Q_{Ak}, \Sigma, uk, i_{uk}, \delta_k, w_k, \rho_k)$, with k : 1..n: Q_{Ak} is the set of states; $\widehat{Q}_{Ak} = \{Ak_q | q \in Q_{Ak}\}$; and $\widehat{P}_{Ak} = \{Ak, \partial_{\sigma}(Ak_q) | q \in Q_{Ak}, \sigma \in \Sigma\}$. By definition of composition of factorizations on $L, f = f_{An} \circ ... \circ f_{A1}$. By (39), it is simple to show that,

$$f(\ell) \in \bigcup_{k=1}^{n} \widehat{Q}_{Ak} \quad \text{if } \ell \in \bigcup_{k=1}^{n} \widehat{P}_{Ak}$$

$$f(\ell) = \ell \quad \text{otherwise}$$

$$(43)$$

for any $\ell \in L$. Let us observe that if $\ell \notin \bigcup_{k=1}^n \widehat{P}_{Ak}$ then $f(\ell) = \ell$. Let j, $n \geq j \geq 1$, be the first index such that $\ell \in \widehat{P}_{Aj}$, then $f(\ell) \in \bigcup_{k=j}^n \widehat{Q}_{Ak}$. This is so because, by (39), $(f_{Aj-1} \circ ... \circ f_{A1})(\ell) = \ell$, $f_{Aj}(\ell) \in \widehat{Q}_{Aj}$ and $f_{Aj}(\ell)$ may belong (or not) to any \widehat{P}_{Ak} with k: j+1..n.

Let us consider an arbitrary $\ell \in L$ such that, for some arbitrary Aj, with $1 \leq j \leq n$, $\ell \in \text{RcgCap}_L^{(g_{Aj},f_{Aj})}$. That is, the M-DFA $N^{(g_{Aj},f_{Aj})}(f_{Aj}(\ell),1)$ recognizes $f_{Aj}(\ell)$. The finite set of states of this automaton is $Q_{f_{Aj}(\ell)}^{(g_{Aj},f_{Aj})} = \{S_{\alpha}^{(g_{Aj},f_{Aj})}(f_{Aj}(\ell)) \mid \alpha \in \Sigma^*\}$. By (23) and (39), it is simple to show that $S_{\alpha}^{(g_{Aj},f_{Aj})}(f_{Aj}(\ell)) \in \widehat{Q}_{Aj}$ or $S_{\alpha}^{(g_{Aj},f_{Aj})}(f_{Aj}(\ell)) = \partial_{\alpha}(\ell)$. This last case happens when each language in the composition of $S_{\alpha}^{(g_{Aj},f_{Aj})}(f_{Aj}(\ell))$ does not belong to \widehat{P}_{Aj} and f_{Aj} behaves like the identity f_e . This structure of $Q_{f_{Aj}(\ell)}^{(g_{Aj},f_{Aj})}$ is important in the next step of the proof.

We claim that the following property holds for the composition and the language ℓ ,

$$S_{\alpha}^{(g,f)}(f(\ell)) \in (\bigcup_{k=1}^{n} \widehat{Q}_{Ak}) \cup Q_{f_{Aj}(\ell)}^{(g_{Aj},f_{Aj})}$$
 (44)

for any $\alpha \in \Sigma^*$.

By induction on the length of $\alpha \in \Sigma^*$:

- Basis. Let $\alpha = \varepsilon$. By (25), $S_{\varepsilon}^{(g,f)}(f(\ell)) = f(\ell)$. By (43), $f(\ell) \in \bigcup_{k=1}^{n} \widehat{Q}_{Ak}$ or $f(\ell) = \ell$. In this last case, $\ell \notin \widehat{P}_{Aj}$. This implies that, by (25) and (39), $S_{\varepsilon}^{(g_{Aj},f_{Aj})}(f_{Aj}(\ell)) = S_{\varepsilon}^{(g_{Aj},f_{Aj})}(\ell) = \ell \in Q_{f_{Aj}(\ell)}^{(g_{Aj},f_{Aj})}$. The property holds.
- Hypothesis. Let us assume that (44) is valid for $\alpha \in \Sigma^*$.
- Induction Step. Let $\alpha' = \alpha \sigma$ with $\alpha \in \Sigma^*$ and $\sigma \in \Sigma$. By (26) and (27), $S_{\alpha\sigma}^{(g,f)}(f(\ell)) = (S_{\sigma}^{(g,f)} \circ S_{\alpha}^{(g,f)})(f(\ell)) = f(\partial_{\sigma}(S_{\alpha}^{(g,f)}(f(\ell))))$. By Hypothesis and (44), we have two main cases:

Case (a). $S_{\alpha}^{(g,f)}(f(\ell)) \in \widehat{Q}_{Ar}$ for some $1 \leq r \leq n$. Then, $S_{\alpha}^{(g,f)}(f(\ell)) = \mathcal{A}r_q$ for some $q \in Q_{Ar}$. In that case, $\partial_{\sigma}(\mathcal{A}r_q) \in \widehat{P}_{Ar}$. By (43), $f(\partial_{\sigma}(\mathcal{A}r_q)) \in \bigcup_{k=1}^{n} \widehat{Q}_{Ak}$. Thus, $S_{\alpha\sigma}^{(g,f)}(f(\ell)) \in \bigcup_{k=1}^{n} \widehat{Q}_{Ak}$, and the property holds.

 $\bigcup_{k=1}^{n} \widehat{Q}_{Ak}. \text{ Thus, } S_{\alpha\sigma}^{(g,f)}(f(\ell)) \in \bigcup_{k=1}^{n} \widehat{Q}_{Ak}, \text{ and the property holds.}$ Case (b). $S_{\alpha}^{(g,f)}(f(\ell)) \in Q_{f_{Aj}(\ell)}^{(g_{Aj},f_{Aj})}.$ By the structure of this set of states given above, we have two subcases:

Case (b1). $S_{\alpha}^{(g,f)}(f(\ell)) \in \widehat{Q}_{Aj}$. The proof is the same as in Case (a).

Case (b2). $S_{\alpha}^{(g,f)}(f(\ell)) = \partial_{\beta}(\ell)$ for some $\beta \in \Sigma^*$. Thus, $f(\partial_{\sigma}(\partial_{\beta}(\ell))) = f(\partial_{\beta\sigma}(\ell))$. Then, by (43), if $\partial_{\beta\sigma}(\ell) \notin \bigcup_{k=1}^n \widehat{P}_{Ak}$, then $f(\partial_{\beta\sigma}(\ell)) = \partial_{\beta\sigma}(\ell)$. As $\partial_{\beta\sigma}(\ell) \notin \widehat{P}_{Aj}$, then $\partial_{\beta\sigma}(\ell) = f_{Aj}(\partial_{\beta\sigma}(\ell)) = f_{Aj}(\partial_{\sigma}(\partial_{\beta}(\ell))) \in Q_{f_{Aj}(\ell)}^{(g_{Aj},f_{Aj})}$. Then, $S_{\alpha\sigma}^{(g,f)}(f(\ell)) \in Q_{f_{Aj}(\ell)}^{(g_{Aj},f_{Aj})}$ and the property holds. Finally, by (43), if $\partial_{\beta\sigma}(\ell) \in \bigcup_{k=1}^n \widehat{P}_{Ak}$, then $f(\partial_{\beta\sigma}(\ell)) \in \bigcup_{k=1}^n \widehat{Q}_{Ak}$. In this case, $S_{\alpha\sigma}^{(g,f)}(f(\ell)) \in \bigcup_{k=1}^n \widehat{Q}_{Ak}$ and the property holds again.

Therefore, $\{S_{\alpha}^{(g,f)}(f(\ell)) \mid \alpha \in \Sigma^*\} \subseteq (\bigcup_{k=1}^n \widehat{Q}_{Ak}) \cup Q_{f_{Aj}(\ell)}^{(g_{Aj},f_{Aj})}$. This implies that it is finite. By Lemma 3, $f(\ell) \in L$ is recognized by the M-DFA $N^{(g,f)}(f(\ell),1)$. In conclusion, $\ell \in \operatorname{RcgCap}_L^{(g_{Aj},f_{Aj})}$ implies $\ell \in \operatorname{RcgCap}_L^{(g,f)}$; and, as ℓ and Aj has been selected in an arbitrary way, then (42) holds. \square

9. Conclusions

The Myhill-Nerode theory studies formal languages and deterministic automata through right congruences and congruences on a free monoid. Let $(M,\cdot,1)$ be an arbitrary monoid. In this paper, we provide a general Myhill-Nerode theorem for M-languages, i.e., functions of the form $\ell: \Sigma^* \to M$. M-languages are studied from the aspect of their recognition by deterministic finite automata whose components take values on M (M-DFAs). Unlike other previous papers in the literature that deal with the problem of recognizability of languages on different algebraic structures, we do not assume any additional property on the monoid. We characterize an M-language ℓ by a right congruence on Σ^* that is defined through the language ℓ and a factorization (f,g) on the set of all M-languages, denoted $\equiv_{\ell}^{(g,f)}$. As (g,f) is a factorization then $\ell=g(\ell)\cdot f(\ell)$ where $g(\ell)\in M$ and $f(\ell)$ is an M-language. Then, ℓ is characterized equivalently by both $\equiv_{\ell}^{(g,f)}$ or $\equiv_{f(\ell)}^{(g,f)}$ congruences. The main result of the paper (Theorem 1) states the equivalence of the conditions:

- (i) ℓ is a recognizable M-language;
- (ii) there exists a factorization on L, (g, f), such that the right congruence $\equiv_{f(\ell)}^{(g,f)}$ has finite index.

The proof of the implication (ii) \Rightarrow (i) requires the explicit construction of an M-DFA that recognizes $f(\ell)$. The properties of this automata based on the congruence $\equiv_{f(\ell)}^{(g,f)}$ state that it is accessible and satisfies a weak necessary condition of minimality (Property 3). The proof of the implication

(i) \Rightarrow (ii) requires that any M-DFA induces a factorization on the set of all M-languages. That factorization is called natural factorization induced by an M-DFA. The construction and composition of this kind of factorizations is also studied in the paper. The provided formalism to study factorizations and their composition establishes that the composition of natural factorizations is also a factorization that preserves the recognition capability of each individual natural factorization (Lemma 7). In the particular case of the existence of a maximal factorization on the set of all M-languages, we obtain that the automata based on a maximal factorization is minimal (Property 5) and that the recognition capability is maximal (Lemma 6).

References

- [1] R. Bělohlávek. Determinism and fuzzy automata. Information Sciences 143 (2002) 205–209.
- [2] S. Bozapalidis, O. Louscou-Bozapalidou. On the recognizability of fuzzy languages II. Fuzzy Sets and Systems 159 (2008) 107–113.
- [3] J. A. Brzozowski. Derivatives of Regular Expressions. Journal of ACM. 11:4 (1964) 481-494.
- [4] M. Ćirić, M. Droste, J. Ignjatović, H. Vogler. Determinization of weighted finite automata over strong bimonoids. Information Sciences 180 (2010) 3497–3520.
- [5] M. Droste, W. Kuich, and H. Vogler, editors. Handbook of Weighted Automata. Springer-Verlag, Berlin, 2009.
- [6] M. Droste, W. Kuich, Weighted finite automata over hemirings, Theoretical Computer Science 485 (2013) 38–48.
- [7] M. Droste, T. Stüber, H. Vogler. Weighted finite automata over strong bimonoids. Information Sciences 180 (2010) 156–166.
- [8] S. Eilenberg. Automata, Languages and Machines. Academic Press. New York and London 1974.
- [9] J. Eisner. Simpler and more general minimization for weighted finite-state automata. In Proceedings of HLT-NAACL2003 conference, pp. 64–71, 2003.

- [10] S. Gerdjikov, S. Mihov. Myhill-Nerode Relation for Sequentiable Structures. ArXiv e-prints. https://arxiv.org/abs/1706.02910, (2017).
- [11] S. Gerdjikov. A general class of monoids supporting canonisation and minimisation of (sub)sequential transducers. Klein, S.T., Martín-Vide, C., Shapira, D. (eds.) Language and Automata Theory and Application, 12th International Conference LATA2018 (2018).
- [12] S. Gerdjikov. Characterisation of (sub)sequential rational function over a general class monoids. CoRR, abs/1801.10063, (2018).
- [13] S. Gerdjikov, J. R. González de Mendivil. Conditions for the existence of maximal factorizations. Fuzzy Sets and Systems. (on line) https://doi.org/10.1016/j.fss.2019.07.006 (2019).
- [14] J. R. Gonzalez de Mendivil, J. R. Garitagoitia. Fuzzy languages of infinite range: pumping lemmas and determinization procedure. Fuzzy Sets and Systems 249 (2014) 1–26.
- [15] J. R. Gonzalez de Mendivil, J. R. Garitagoitia. Determinization of fuzzy automata via factorization of fuzzy states. Information Sciences 283 (2014) 165–179.
- [16] J. R. González de Mendivil. A generalization of Myhill-Nerode theorem for fuzzy languages. Fuzzy Sets and Systems 301 (2016) 103–115.
- [17] J. R. González de Mendivil. Conditions for Minimal Fuzzy Deterministic Finite Automata via Brzozowski's Procedure. IEEE Transactions on Fuzzy Systems. 26 (4) (2018) 2409–2420.
- [18] J. R. González de Mendivil, F. Fariña. Canonization of max-min fuzzy automata. Fuzzy Sets and Systems. 376 (2019) 152–168.
- [19] J. R. González de Mendivil, F. Fariña. Recognizability of languages with values on a monoid. Report number: M02-2019-gsd. Universidad Pública de Navarra. 2019.
- [20] J. E. Hopcroft, R. Motwani, J. D. Ullman. Introduction to Automata Theory. 3rd Edition. Addison-Wesley, 2007.
- [21] J. Ignjatović, M. Čirić, S. Bogdanović. Determinization of fuzzy automata with membership values in complete residuated lattices. Information Sciences 178 (2008) 164–180.

- [22] J. Ignjatović, M. Ćirić, S. Bogdanović, T. Petković. Myhill-Nerode type theory for fuzzy languages and automata. Fuzzy Sets and Systems 161 (2010) 1288–1324.
- [23] D. Kirsten, L. Mäurer. On the determinization of weighted automata. Journal of Automata, Languages and Combinatorics 10 (2005) 287–312.
- [24] E.P. Klement, R. Mesiar and E. Pap. Triangular Norms, Kluwer Academic Publishers, Dordrecht, 2000.
- [25] Y. M. Li, W. Pedrycz, Fuzzy finite automata and fuzzy regular expressions with membership values in lattice ordered monoids. Fuzzy Sets and Systems 156 (2005) 68–92.
- [26] Y. M. Li, W. Pedrycz. Minimization of lattice finite automata and its application to the decomposition of lattice languages. Fuzzy Sets and Systems 158 (2007) 1423–1436.
- [27] Y. Li. Finite automata theory with membership values in lattices. Information Sciences 181 (2011) 1003–1017.
- [28] M. Mohri. Finite-state transducers in language and speech processing. Computing Linguistics 23(2) (1997) 269–311.
- [29] M. Mohri. Minimization algorithms for sequential transducers. Theoretical Computer Science 234 (2000) 177–201.
- [30] J. Mordeson, D. Malik. Fuzzy Automata and Languages: Theory and Applications. Chapman & Hall, CRC Press, London, Boca Raton, FL., 2002.
- [31] J. Myhill. Finite automata and the representation of events. WADD TR-57-624, Wright Patterson AFB, Ohio, pp. 112–137, 1957.
- [32] A. Nerode. Linear automata transformation. Proceedings of AMS 9, pp. 541–544, 1958.
- [33] T. Petković. Varietes of fuzzy languages. Proc. 1st Internat. Conf. on Algebraic Informatics, Aristotle University of Thessaloniki, Thessaloniki, pp. 197–205, 2005.
- [34] D.W. Qiu. Automata theory based on completed residuated lattice-valued logic (I). Science in China, Ser. F, 44 (6) (2001) 419–429.

- [35] D.W. Qiu. Automata theory based on completed residuated lattice-valued logic (II). Science in China, Ser. F, 45 (6) (2002) 442–452.
- [36] G. Rahonis. Fuzzy languages. Handbook of Weighted Automata. M. Droste, W. Kuich, H. Vogler (Eds.), Springer-Verlag, Berlin, 2009.
- [37] J. Sakarovitch. Elements of Automata Theory. Cambridge University Press. 2009.
- [38] J. Shen. Fuzzy language on free monoid. Information Sciences 88 (1996) 149–168.
- [39] R. Souza. Properties of some classes of rational relations (short version in English). Master's thesis, University of Sao Paulo, 2004.
- [40] S. Stanimirović et. al. Determinization of fuzzy automata by factorizations of fuzzy states and right invariant fuzzy quasi-orders. Information Sciences 469 (2018) 79–100.
- [41] S. Stanimirović et. al. A double reverse canonization method for fuzzy automata. Internal communication 2017.
- [42] L. A. Zadeh. Fuzzy Sets. Information and Control 8 (3) (1965) 338–353.