

A program for the full axiom of choice

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Introduction

The Curry-Howard correspondence enables to associate a program with each proof in classical natural deduction. But mathematical proofs not only use the rules of natural deduction, but also *axioms*, essentially those of Zermelo-Frænkel set theory with axiom of choice. To transform these proofs into programs, you must therefore associate with each of these axioms suitable instructions, which is far from obvious.

The theory of *classical realizability* (*c.r.*) solves this problem for all axioms of ZF, by means of a very rudimentary programming language : the λ_c -calculus, that is to say the λ -calculus with a control instruction.

The programs obtained in this way can therefore be written in practically any programming language. They are said to *realize* the axioms of ZF.

But, almost all the applications of mathematics in physics, probability, statistics, etc. use *Analysis*, that is to say *the axiom of dependent choice*. The first program for this axiom, known since 1998 [1], is a pure λ -term called *bar recursion*. In fact, c.r. shows that it provides also a program for the *continuum hypothesis* [13].

Nevertheless, this solution requires the programming language to be limited to λ_c -calculus, prohibiting any other instruction, which is a severe restriction.

Classical realizability provides other programs for this axiom [9, 10, 11], which require an additional instruction (clock, signature, ... , or as in this paper, introduction of “ fresh variables ”). On the other hand, this solution is very flexible regarding the programming language.

There remained, however, the problem of the *full* axiom of choice. It is solved here, by means of new instructions which allow *branching* or *parallelism*. The program obtained in this way is rather complicated and we may hope for new simpler solutions. However, it already shows that we can turn all proofs of ZFC into programs.

The framework of this article is therefore the *theory of classical realizability*, which is explained in detail in [9, 10, 11]. In order to simplify this article, we will often refer to these papers for definitions and standard notations.

Outline of the paper.

Section 1. Axioms and properties of *realizability algebras (r.a.)* which are first order structures, a generalization of both combinatory algebras and ordered sets of forcing conditions. Intuitively, their elements are programs.

Section 2. Each r.a. is associated with a *realizability model (r.m.)*, a generalization of the generic model in the theory of forcing. This model satisfies a set theory ZF_ε which is a conservative extension of ZF, with a strong non extensional membership relation ε .

Section 3. Generic extensions of realizability algebras and models.

Section 4. Definition and properties of a particular realizability algebra \mathfrak{A}_0 . It contains an *instruction of parallelism* but the programming language allows the introduction of many other instructions. This is therefore, in fact, a class of algebras.

At this stage, we obtain a program for the *axiom of well ordered choice (WOC)* : *the product of a family of non-empty sets whose index set is well ordered is non-empty*.

We get, at the same time, a new proof that WOC is weaker than AC (joint work with Laura Fontanella ; cf. [6] for the usual proof).

Section 5. Construction of a generic extension \mathfrak{A}_1 of the algebra \mathfrak{A}_0 which allows to *realize the axiom of choice : every set can be well ordered*.

Sections 3 and 4 are independent.

1 Realizability algebras (r.a.)

It is a first order structure, which is defined in [10]. We recall here briefly this definition and some essential properties :

A *realizability algebra* \mathcal{A} is made up of three sets : Λ (the set of *terms*), Π (the set of *stacks*), $\Lambda \star \Pi$ (the set of *processes*) with the following operations :

- $(\xi, \eta) \mapsto (\xi)\eta$ from Λ^2 into Λ (*application*) ;
- $(\xi, \pi) \mapsto \xi \bullet \pi$ from $\Lambda \times \Pi$ into Π (*push*) ;
- $(\xi, \pi) \mapsto \xi \star \pi$ from $\Lambda \times \Pi$ into $\Lambda \star \Pi$ (*process*) ;
- $\pi \mapsto \mathbf{k}_\pi$ from Π into Λ (*continuation*).

There are, in Λ , distinguished elements $\mathbf{B}, \mathbf{C}, \mathbf{I}, \mathbf{K}, \mathbf{W}, \mathbf{cc}$, called *elementary combinators* or *instructions*.

Notation. The term $(\dots(((\xi)\eta_1)\eta_2)\dots)\eta_n$ will be also written $(\xi)\eta_1\eta_2\dots\eta_n$ or even $\xi\eta_1\eta_2\dots\eta_n$. For instance : $\xi\eta\zeta = (\xi)\eta\zeta = (\xi\eta)\zeta = ((\xi)\eta)\zeta$.

We define a preorder on $\Lambda \star \Pi$, denoted by \succ , which is called *execution* ;

$\xi \star \pi \succ \xi' \star \pi'$ is read as : *the process $\xi \star \pi$ reduces to $\xi' \star \pi'$* .

It is the smallest reflexive and transitive binary relation, such that, for any $\xi, \eta, \zeta \in \Lambda$ and $\pi, \varpi \in \Pi$, we have :

- $(\xi)\eta \star \pi \succ \xi \star \eta \bullet \pi$ (*push*).
- $\mathbf{I} \star \xi \bullet \pi \succ \xi \star \pi$ (*no operation*).
- $\mathbf{K} \star \xi \bullet \eta \bullet \pi \succ \xi \star \pi$ (*delete*).
- $\mathbf{W} \star \xi \bullet \eta \bullet \pi \succ \xi \star \eta \bullet \eta \bullet \pi$ (*copy*).
- $\mathbf{C} \star \xi \bullet \eta \bullet \zeta \bullet \pi \succ \xi \star \zeta \bullet \eta \bullet \pi$ (*switch*).
- $\mathbf{B} \star \xi \bullet \eta \bullet \zeta \bullet \pi \succ \xi \star (\eta)\zeta \bullet \pi$ (*apply*).

$\text{cc} \star \xi \bullet \pi \succ \xi \star \text{k}_\pi \bullet \pi$ (*save the stack*).

$\text{k}_\pi \star \xi \bullet \varpi \succ \xi \star \pi$ (*restore the stack*).

We are also given a subset $\perp\!\!\!\perp$ of $\Lambda \star \Pi$, called "the pole", such that :

$$\xi \star \pi \succ \xi' \star \pi', \xi' \star \pi' \in \perp\!\!\!\perp \Rightarrow \xi \star \pi \in \perp\!\!\!\perp.$$

Given two processes $\xi \star \pi, \xi' \star \pi'$, the notation $\xi \star \pi \succ \xi' \star \pi'$ means :

$$\xi \star \pi \notin \perp\!\!\!\perp \Rightarrow \xi' \star \pi' \notin \perp\!\!\!\perp.$$

Therefore, obviously, $\xi \star \pi \succ \xi' \star \pi' \Rightarrow \xi \star \pi \succ \xi' \star \pi'$.

Finally, we choose a set of terms $\text{PL}_{\mathcal{A}} \subset \Lambda$, containing the elementary combinators :

$\text{B}, \text{C}, \text{I}, \text{K}, \text{W}, \text{cc}$ and closed by application. They are called the *proof-like terms of the algebra* \mathcal{A} . We write also PL instead of $\text{PL}_{\mathcal{A}}$ if there is no ambiguity about \mathcal{A} .

The algebra \mathcal{A} is called *coherent* if, for every proof-like term $\theta \in \text{PL}_{\mathcal{A}}$, there exists a stack π such that $\theta \star \pi \notin \perp\!\!\!\perp$.

Remark. A *set of forcing conditions* can be considered as a degenerate case of realizability algebras, if we present it in the following way : an inf-semi-lattice P , with a greatest element $\mathbf{1}$ and an initial segment $\perp\!\!\!\perp$ of P (the set of *false* conditions). Two conditions $p, q \in P$ are called *compatible* if their g.l.b. $p \wedge q$ is not in $\perp\!\!\!\perp$.

We get a realizability algebra if we set $\Lambda = \Pi = \Lambda \star \Pi = P$; $\text{B} = \text{C} = \text{I} = \text{K} = \text{W} = \text{cc} = \mathbf{1}$ and $\text{PL} = \{\mathbf{1}\}$; $(p)q = p \bullet q = p \star q = p \wedge q$ and $\text{k}_p = p$. The preorder $p \succ q$ is defined as $p \leq q$, i.e. $p \wedge q = p$. The condition of coherence is $\mathbf{1} \notin \perp\!\!\!\perp$.

We can translate λ -terms into terms of Λ built with the combinators $\text{B}, \text{C}, \text{I}, \text{K}, \text{W}$ in such a way that weak head reduction is valid :

$$\lambda x t[x] \star u \bullet \pi \succ t[u/x] \star \pi$$

where $\lambda x t[x], u$ are terms and π is a stack.

This is done in [10]. Note that the usual (K, S) -translation does not work.

λ -calculus is much more intuitive than combinatory algebra in order to write programs, so that we use it extensively in the following. But combinatory algebra is better for theory, in particular because it a first order structure.

2 Realizability models (r.m.)

The framework is very similar to that of forcing, which is anyway a particular case.

We use a first order language with three binary symbols $\not\in, \in, \subset$ (ε is intended to be a strong, non extensional membership relation ; \in and \subset have their usual meaning in ZF).

First order formulas are written with the only logical symbols $\rightarrow, \forall, \perp$.

The symbols $\neg, \wedge, \vee, \leftrightarrow, \exists$ are defined with them in the usual way.

Given a realizability algebra, we get a *realizability model (r.m.)* as follows :

We start with a model \mathcal{M} of ZFC (or even ZFL) called the *ground model*. The axioms of ZFL are written with the sublanguage $\{\not\in, \subset\}$.

We build a model \mathcal{N} of a new set theory ZF_ε , in the language $\{\not\in, \in, \subset\}$, the axioms of which are given in [9]. We recall them below, using the following rather standard abbreviations :

$F_1 \rightarrow (F_2 \rightarrow \dots (F_n \rightarrow G) \dots)$ is written $F_1, F_2, \dots, F_n \rightarrow G$ or even $\vec{F} \rightarrow G$.

We use the notation $\exists x\{F_1, F_2, \dots, F_n\}$ for $\forall x(F_1 \rightarrow (F_2 \rightarrow \dots \rightarrow (F_n \rightarrow \perp) \dots)) \rightarrow \perp$. Of course, $x \varepsilon y$ and $x \in y$ are the formulas $x \notin y \rightarrow \perp$ and $x \notin y \rightarrow \perp$. The notation $x =_{\varepsilon} y \rightarrow F$ means $x \subset y, y \subset x \rightarrow F$. Thus $x =_{\varepsilon} y$, which represents the usual (extensional) equality of sets, is the pair of formulas $\{x \subset y, y \subset x\}$. We use the notations $(\forall x \varepsilon a)F(x)$ for $\forall x(\neg F(x) \rightarrow x \notin a)$ and $(\exists x \varepsilon a)\vec{F}(x)$ for $\neg \forall x(\vec{F}(x) \rightarrow x \notin a)$. For instance, $(\exists x \varepsilon y)(t =_{\varepsilon} u)$ is the formula $\neg \forall x(t \subset u, u \subset t \rightarrow x \notin y)$.

The axioms of ZF_{ε} are the following :

0. Extensionality axioms.

$$\forall x \forall y (x \in y \leftrightarrow (\exists z \varepsilon y) x =_{\varepsilon} z) ; \forall x \forall y (x \subset y \leftrightarrow (\forall z \varepsilon x) z \in y).$$

1. Foundation scheme.

$$\forall \vec{a} (\forall x ((\forall y \varepsilon x) F(y, \vec{a}) \rightarrow F(x, \vec{a})) \rightarrow \forall x F(x, \vec{a})) \text{ for every formula } F(x, a_1, \dots, a_n).$$

The intuitive meaning of these axioms is that ε is a well founded relation and the relation \in is obtained by “collapsing” ε into an extensional relation.

The following axioms essentially express that the relation ε satisfies the Zermelo-Fraenkel axioms *except extensionality*.

2. Comprehension scheme.

$$\forall \vec{a} \forall x \exists y \forall z (z \varepsilon y \leftrightarrow (z \varepsilon x \wedge F(z, \vec{a}))) \text{ for every formula } F(z, \vec{a}).$$

3. Pairing axiom.

$$\forall a \forall b \exists x \{a \varepsilon x, b \varepsilon x\}.$$

4. Union axiom.

$$\forall a \exists b (\forall x \varepsilon a) (\forall y \varepsilon x) y \varepsilon b.$$

5. Power set axiom.

$$\forall a \exists b \forall x (\exists y \varepsilon b) \forall z (z \varepsilon y \leftrightarrow (z \varepsilon a \wedge z \varepsilon x)).$$

6. Collection scheme.

$$\forall \vec{a} \forall x \exists y (\forall u \varepsilon x) (\exists v F(u, v, \vec{a}) \rightarrow (\exists v \varepsilon y) F(u, v, \vec{a})) \text{ for every formula } F(u, v, \vec{a}).$$

7. Infinity scheme.

$$\forall \vec{a} \forall x \exists y \{x \varepsilon y, (\forall u \varepsilon y) (\exists v F(u, v, \vec{a}) \rightarrow (\exists v \varepsilon y) F(u, v, \vec{a}))\} \text{ for every formula } F(u, v, \vec{a}).$$

It is shown in [9] that ZF_{ε} is a *conservative extension* of ZF .

For each formula $F(\vec{a})$ of ZF_{ε} with parameters \vec{a} in the ground model \mathcal{M} we define, in \mathcal{M} , a *falsity value* $\|F(\vec{a})\|$ which is a subset of Π and a *truth value* $|F(\vec{a})|$ which is a subset of Λ . The notation $t \Vdash F(\vec{a})$ (read “ t realizes $F(\vec{a})$ ” or “ t forces $F(\vec{a})$ ” in the particular case of forcing) means $t \in |F(\vec{a})|$.

We set first $|F(\vec{a})| = \{t \in \Lambda ; (\forall \pi \in \|F(\vec{a})\|)(t \star \pi \in \perp)\}$ so that we only need to define $\|F(\vec{a})\|$, which we do by induction on F :

$$\|a \notin b\| = \{\pi \in \Pi ; (a, \pi) \in b\} ; \|\perp\| = \Pi ; \|\top\| = \emptyset ;$$

Definition by induction on $(\text{rk}(a) \cup \text{rk}(b), \text{rk}(a) \cap \text{rk}(b))$:

$$\|a \subset b\| = \bigcup_c \{t \bullet \pi \in \Pi ; t \Vdash c \notin b, (c, \pi) \varepsilon a\} ;$$

$$\|a \not\subset b\| = \bigcup_c \{t \bullet u \bullet \pi \in \Pi ; t \Vdash c \subset a, u \Vdash a \subset c, (c, \pi) \varepsilon b\}.$$

Definition by induction on the length of F :

$$\|F \rightarrow F'\| = \{t \bullet \pi \in \Pi ; t \Vdash F, \pi \in \|F'\|\} ;$$

$$\|\forall x F(x)\| = \bigcup_a \|F(a)\|.$$

We define the strong (Leibnitz) equality $a = b$ by $\forall z(a \notin z \rightarrow b \notin z)$. It is trivially transitive and it is symmetric by comprehension. This equality satisfy the first order axioms of equality $\forall x \forall y (x = y \rightarrow (F(x) \rightarrow F(y)))$ (by comprehension scheme of ZF_ε) and is therefore the equality in the r.m. \mathcal{N} .

Lemma 1. $\|a = b\| = \|\top \rightarrow \perp\| = \{\xi \bullet \pi ; \xi \in \Lambda, \pi \in \Pi\}$ if $a \neq b$;
 $\|a = a\| = \|\perp \rightarrow \perp\| = \{\xi \bullet \pi ; \xi \in \Lambda, \pi \in \Pi, \xi \star \pi \in \perp\}$.

Let $z = \{b\} \times \Pi$ so that $\|b \notin z\| = \Pi$. If $a \neq b$ then $\|a \notin z\| = \emptyset$ and therefore :
 $\|a \notin z \rightarrow b \notin z\| = \|\top \rightarrow \perp\|$.

If $a = b$, then $\|a \notin z\| = \Pi$ and therefore $\|a = b\| = \|\perp \rightarrow \perp\|$.

Q.E.D.

Finally, it is convenient to define first \neq by $\|a \neq a\| = \|\perp\| = \Pi$; $\|a \neq b\| = \|\top\| = \emptyset$ if $a \neq b$; and to define $a = b$ as $a \neq b \rightarrow \perp$.

This notion of realizability has two essential properties given by theorems 2 and 3 below. They are proved in [9].

Theorem 2 (Adequation lemma).

\Vdash is compatible with classical natural deduction, i.e. :

If t_1, \dots, t_n, t are λ_c -terms such that $t_1 : F_1, \dots, t_n : F_n \vdash t : F$ in classical natural deduction, then $t_1 \Vdash F_1, \dots, t_n \Vdash F_n \Rightarrow t \Vdash F$.

In particular, any valid formula is realized by a proof-like term.

Theorem 3. The axioms of ZF_ε are realized by proof-like terms.

It follows that every closed formula which is consequence of ZF_ε and, in particular, every consequence of ZF , is realized by a proof-like term.

In the following, we shall simply say “the formula F is realized” instead of “realized by a proof-like term” and use the notation $\Vdash F$.

Theorem 3 is valid for every r.a. The aim of this paper is to realize the full axiom of choice AC in some particular r.a. suitable for programming.

Remark. Note that AC is realized for any r.a. associated with a set of forcing conditions (generic extension of \mathcal{M}). But in this case, there is only one proof-like term which is the greatest element $\mathbf{1}$.

We define a preorder \leq on the set $\mathcal{P}(\Pi)$ of “falsity values” by setting :

$X \leq Y \Leftrightarrow$ there exists a proof-like term $\theta \Vdash X \rightarrow Y$. By theorem 2, we easily see [9] that $(\mathcal{P}(\Pi), \leq)$ is a Boolean algebra $\mathfrak{B}_\mathcal{A}$ if the r.a. \mathcal{A} is coherent. Every formula $F(\vec{a})$ of ZF_ε with parameters in the ground model \mathcal{M} has a value $\|F(\vec{a})\|$ in this Boolean algebra.

By means of any ultrafilter on $\mathfrak{B}_\mathcal{A}$, we thus obtain a complete consistent theory in the language $\{\notin, \subset\}$ with parameters in \mathcal{M} . We take any model \mathcal{N} of this theory and call it the realizability model (r.m.) of the realizability algebra \mathcal{A} .

Therefore, \mathcal{N} is a model of ZF_ε , and in particular, a model of ZF , that we will call \mathcal{N}_ε .

Thus \mathcal{N}_ε is simply the model \mathcal{N} restricted to the language $\{\notin, \subset\}$.

Remark. The ground model \mathcal{M} is contained in \mathcal{N} since every element of it is a symbol of constant. But \mathcal{M} is not a submodel of \mathcal{N} for the common language $\{\notin, \subset\}$; and, except in the case of forcing, not every element of \mathcal{N} “has a name” in \mathcal{M} .

Functionals. A functional relation defined in \mathcal{N} is given by a formula $F(x, y)$ of ZF_ε such that $\mathcal{N} \models \forall x \forall y \forall y' (F(x, y), F(x, y') \rightarrow y = y')$.

A *function* is a functional relation which is a set.

We define now some special functional relations on \mathcal{N} which we call *functionals defined in \mathcal{M}* or *functional symbols* :

For each functional relation $f : \mathcal{M}^k \rightarrow \mathcal{M}$ defined in the ground model \mathcal{M} , we add the functional symbol f to the language of ZF_ε . The application of f to an argument a will be denoted $f[a]$. Therefore f is also defined in \mathcal{N} .

We call this (trivial) operation *the extension to \mathcal{N} of the functional f* defined in the ground model. It is a fundamental tool in all that follows.

Theorem 4.

Let $t_1, u_1, \dots, t_n, u_n, t, u$ be k -ary terms built with functional symbols, such that :

$\mathcal{M} \models \forall \vec{x} (t_1[\vec{x}] = u_1[\vec{x}], \dots, t_n[\vec{x}] = u_n[\vec{x}] \rightarrow t[\vec{x}] = u[\vec{x}])$.

Then $\lambda x_1 \dots \lambda x_n \lambda x (x_1) \dots (x_n) x \Vdash \forall \vec{x} (t_1[\vec{x}] = u_1[\vec{x}], \dots, t_n[\vec{x}] = u_n[\vec{x}] \rightarrow t[\vec{x}] = u[\vec{x}])$.

This easily follows from the definition above of $\|a = b\|$.

Q.E.D.

As a first example let the unary functional $\Phi_F[X]$ be defined in \mathcal{M} by :

$$\Phi_F[X] = \{(x, \xi \bullet \pi) ; \xi \Vdash F(x), (x, \pi) \in X\}.$$

We shall denote it by $\{x \varepsilon X ; F(x)\}$ (in this notation, x is a bound variable) because it corresponds to the comprehension scheme in the model \mathcal{N} . Note that the use of ε reminds that this expression must be interpreted in \mathcal{N} .

We define now in \mathcal{M} the unary functional $\mathbf{J}X = X \times \Pi$, so that we have :

$\|x \notin \mathbf{J}X\| = \Pi$ if $x \in X$ and $\|x \notin \mathbf{J}X\| = \emptyset$ if $x \notin X$.

We define the quantifier $\forall x^{\mathbf{J}X}$ by setting $\|\forall x^{\mathbf{J}X} F(x)\| = \bigcup_{x \in X} \|F(x)\|$ so that :

Lemma 5. $\Vdash \forall x^{\mathbf{J}X} F(x) \leftrightarrow \forall x (x \varepsilon \mathbf{J}X \rightarrow F(x))$.

In fact, we have $\|\forall x (\neg F(x) \rightarrow x \notin \mathbf{J}X)\| = \|\forall x^{\mathbf{J}X} \neg F(x)\|$.

Now we have trivially $\lambda x(x) \Vdash \forall x^{\mathbf{J}X} F(x) \rightarrow \forall x^{\mathbf{J}X} \neg F(x)$

and $\text{cc} \Vdash \forall x^{\mathbf{J}X} \neg F(x) \rightarrow \forall x^{\mathbf{J}X} F(x)$.

Q.E.D.

Lemma 6. Let f be a functional k -ary symbol defined in \mathcal{M} such that $f : X_1 \times \dots \times X_k \rightarrow X$. Then its extension to \mathcal{N} is such that $f : \mathbf{J}X_1 \times \dots \times \mathbf{J}X_k \rightarrow \mathbf{J}X$.

Trivial.

Q.E.D.

By theorem 4 and lemma 6, the algebra operations on the Boolean algebra $2 = \{0, 1\}$ extended to $\mathbf{J}2$, turn it into a Boolean algebra which we call *the characteristic Boolean algebra* of the r.m. \mathcal{N} . In the ground model \mathcal{M} , we define the functional $(a, x) \mapsto ax$ from $2 \times \mathcal{M}$ into \mathcal{M} by $0x = \emptyset$ and $1x = x$. It extends to \mathcal{N} into a functional $\mathbf{J}2 \times \mathcal{N} \rightarrow \mathcal{N}$ such that $(ab)x = a(bx)$ for $a, b \varepsilon \mathbf{J}2$ and every x in \mathcal{N} .

Lemma 7.

$\Vdash \forall \vec{x} \forall \vec{y} \forall a^{\mathbf{J}2} (af[\vec{x}, \vec{y}] = af[a\vec{x}, \vec{y}])$ for every functional symbol f defined in \mathcal{M} .

Trivial.

Q.E.D.

For any formula $F(\vec{x})$ of ZF, we define, in \mathcal{M} , a functional $\langle F(\vec{x}) \rangle$ with value in $\{0, 1\}$ which is the truth value of this formula in \mathcal{M} . The extension of this functional to the model \mathcal{N} has its values in the Boolean algebra $\mathfrak{J}2$ (cf. [12]).

The binary functionals $\langle x \notin y \rangle$ and $\langle x \subset y \rangle$ define on the r.m. \mathcal{N} a structure of *Boolean model* on the Boolean algebra $\mathfrak{J}2$, that we denote by $\mathcal{M}_{\mathfrak{J}2}$. It is an elementary extension of \mathcal{M} since the truth value of every closed formula of ZF with parameters in \mathcal{M} is the same in \mathcal{M} and $\mathcal{M}_{\mathfrak{J}2}$.

Any ultrafilter \mathcal{U} on $\mathfrak{J}2$ would therefore give a (two-valued) model $\mathcal{M}_{\mathfrak{J}2}/\mathcal{U}$ which is an elementary extension of \mathcal{M} . In [12], it is shown that there exists one and only one ultrafilter \mathcal{D} on $\mathfrak{J}2$ such that the model $\mathcal{M}_{\mathfrak{J}2}/\mathcal{D}$, which we shall denote as $\mathcal{M}_{\mathcal{D}}$, is well founded (in \mathcal{N}). The binary relations \notin, \subset of $\mathcal{M}_{\mathcal{D}}$ are thus defined by $\langle x \notin y \rangle \varepsilon \mathcal{D}$ and $\langle x \subset y \rangle \varepsilon \mathcal{D}$.

Moreover, $\mathcal{M}_{\mathcal{D}}$ is isomorphic with a transitive submodel of \mathcal{N}_{\in} with the same ordinals. In fact, if we start with a ground model \mathcal{M} which satisfies $V = L$, then $\mathcal{M}_{\mathcal{D}}$ is isomorphic with the constructible class of \mathcal{N}_{\in} .

Remark. We have defined many first order structures on the model \mathcal{N} :

- The realizability model \mathcal{N} itself uses the language $\{\notin, \subset, \subset\}$ of ZF_{ε} ; the equality on \mathcal{N} is the Leibnitz equality $=$, which is the strongest possible.
- The model \mathcal{N}_{\in} of ZF is restricted to the language $\{\notin, \subset\}$; the equality on \mathcal{N}_{\in} is the extensional equality $=_{\in}$.
- The Boolean model $\mathcal{M}_{\mathfrak{J}2}$ with the language $\{\notin, \subset\}$ of ZF and with truth values in $\mathfrak{J}2$; it is an elementary extension of the ground model \mathcal{M} . The equality on $\mathcal{M}_{\mathfrak{J}2}$ is $\langle x = y \rangle = 1$ i.e. the Leibnitz equality.
- The model $\mathcal{M}_{\mathcal{D}}$ with the same language, also an elementary extension of \mathcal{M} ; if $F(\vec{a})$ is a closed formula of ZF with parameters (in \mathcal{N}), then $\mathcal{M}_{\mathcal{D}} \models F(\vec{a})$ iff $\mathcal{N} \models \langle F(\vec{a}) \rangle \varepsilon \mathcal{D}$.

The equality on $\mathcal{M}_{\mathcal{D}}$ is given by $\langle x = y \rangle \varepsilon \mathcal{D}$.

The proof of existence of the ultrafilter \mathcal{D} in [12] is not so simple. But it is useless in the present paper, because $\mathfrak{J}2$ will be the four elements algebra, with two atoms a_0, a_1 which give the two trivial ultrafilters on $\mathfrak{J}2$. It is easily seen that one of them, say a_0 gives a well founded model denoted by \mathcal{M}_{a_0} which is the class $a_0\mathcal{N} = \mathcal{M}_{\mathcal{D}}$. The class $\mathcal{M}_{a_1} = a_1\mathcal{N}$ is also an elementary extension of \mathcal{M} (but not well founded, cf. the remark after lemma 25). Finally we have $\mathcal{M}_{\mathfrak{J}2} = \mathcal{N} = a_0\mathcal{N} \times a_1\mathcal{N}$ since the Boolean model is simply a product in this case.

Some useful notations.

For every set of terms $X \subset \Lambda$ and every closed formula F we can define an “extended formula” $X \rightarrow F$ by setting $\|X \rightarrow F\| = \{\xi \bullet \pi ; \xi \in X, \pi \in \|F\|\}$.

For instance, for every formula F , we define $\neg F = \{\mathbf{k}_{\pi} ; \pi \in \|F\|\}$. It is a useful equivalent of $\neg F$ by the following :

Lemma 8. $\Vdash \neg F \leftrightarrow \neg F$.

If $\pi \in \|F\|$, then $\mathbf{k}_{\pi} \Vdash \neg F$ and therefore $\mathbf{l} \Vdash \neg F \rightarrow \neg F$.

Conversely, if $\xi \Vdash \neg F \rightarrow \perp$, then $\xi \star \mathbf{k}_{\pi} \bullet \pi \in \perp$ for every $\pi \in \|F\|$; thus $\text{cc} \Vdash \neg \neg F \rightarrow F$.

Q.E.D.

If t, u are terms of the language of ZF, built with functionals in \mathcal{M} , we define another “extended formula” $t = u \hookrightarrow F$ by setting :

$$\|t = u \hookrightarrow F\| = \emptyset \text{ if } t \neq u ; \|t = u \hookrightarrow F\| = \|F\| \text{ if } t = u.$$

Lemma 9. $\Vdash ((t = u \hookrightarrow F) \leftrightarrow (t = u \rightarrow F)).$

We have immediately $\Vdash \neg F, (t = u \hookrightarrow F) \rightarrow t \neq u.$

Conversely $\lambda x(x) \Vdash (t = u \rightarrow F) \rightarrow (t = u \hookrightarrow F).$

Q.E.D.

We define the quantifier $\forall n^{\text{int}}$ by setting $\|\forall n^{\text{int}} F(n)\| = \{\underline{n} \bullet \pi ; n \in \mathbb{N}, \pi \in \|F(n)\|\}$ where \underline{n} is the λ -term $s^n \underline{0}$ (note that the Church integers do not work).

It is shown in [9] that $\Vdash \forall n^{\text{int}} F(n) \leftrightarrow (\forall n \in \tilde{\mathbb{N}}) F(n)$ where $\tilde{\mathbb{N}}$ is the set of integers of the model \mathcal{N} .

Lemmas 10, 11 and theorem 12 below will be used in the following sections.

Lemma 10. $\Vdash \forall x \forall y (x \varepsilon y \rightarrow \langle x \in \text{Cl}[y] \rangle = 1).$

In the model \mathcal{M} , the unary functional symbol Cl denotes the *transitive closure*.

We show $\Vdash \forall x \forall y (\langle x \in \text{Cl}[y] \rangle \neq 1 \rightarrow x \notin y)$: let $\xi \Vdash \langle x \in \text{Cl}[y] \rangle \neq 1$ and $\pi \in \|x \notin y\|$.

Then $(x, \pi) \in y$, therefore $x \in \text{Cl}(y)$. It follows that $\xi \Vdash \perp$.

Q.E.D.

Lemma 11. Let $F(x, \vec{y})$ be a formula in ZF_ε . Then :

$$\Vdash \forall \vec{y} (\forall \varpi^{\text{III}} F(f[\varpi, \vec{y}], \vec{y}) \rightarrow \forall x F(x, \vec{y}))$$

where f is a functional symbol defined in \mathcal{M} .

Let $\vec{a} = (a_1, \dots, a_k)$ in \mathcal{M} . Then, we have :

$$\pi \in \|\forall x F(x, \vec{a})\| \Leftrightarrow \mathcal{M} \models \exists x (\pi \in \|F(x, \vec{a})\|) \Leftrightarrow \mathcal{M} \models \pi \in \|F(f[\pi, \vec{a}], \vec{a})\|$$

where f is a functional defined in \mathcal{M} , (choice principle in \mathcal{M}).

Thus we have $\|\forall x F(x, \vec{a})\| \subset \bigcup_{\varpi \in \Pi} \|F(f[\varpi, \vec{a}], \vec{a})\|$ hence the result.

Q.E.D.

Theorem 12.

Let $\mathcal{L}_{\mathcal{M}}$ be the language $\{\varepsilon, \in, \subset\}$ of ZF_ε , with a symbol for each functional definable in \mathcal{M} . Then, there exists an ε -transitive $\mathcal{L}_{\mathcal{M}}$ -elementary substructure $\tilde{\mathcal{N}}$ of \mathcal{N} such that : For all a in $\tilde{\mathcal{N}}$, there is an ordinal α of \mathcal{M} such that $\tilde{\mathcal{N}} \models a \varepsilon \mathfrak{I}V_\alpha$.

$\tilde{\mathcal{N}}$ is made up of the elements a of \mathcal{N} such that $a \varepsilon \mathfrak{I}V_\alpha$ for an ordinal α of \mathcal{M} (note that it is not a class defined in \mathcal{N}).

Let $F(x, \vec{y})$ be a formula of ZF_ε and $\vec{b} = (b_1, \dots, b_k)$ be elements of \mathcal{M} .

Suppose that $\tilde{\mathcal{N}} \models \forall x F(x, \vec{b})$; we show that $\mathcal{N} \models \forall x F(x, \vec{b})$ by induction on F .

By lemma 11, it suffices to show that $\mathcal{N} \models \forall \varpi^{\text{III}} F(f(\varpi, \vec{b}), \vec{b})$. Thus let $\pi \varepsilon \mathfrak{I}\Pi$; thus π is in $\tilde{\mathcal{N}}$. We have $\tilde{\mathcal{N}} \models F(f(\pi, \vec{b}), \vec{b})$, hence $\mathcal{N} \models F(f(\pi, \vec{b}), \vec{b})$ by the recurrence hypothesis.

Q.E.D.

Replacing \mathcal{N} by this elementary substructure, we shall suppose in the following :

① For all a in \mathcal{N} , there is an ordinal α of \mathcal{M} such that $\mathcal{N} \models a \varepsilon \mathfrak{I}V_\alpha$.

3 Extensional generic extensions

In this section we build some tools in order to manage generic extensions $\mathcal{N}_\varepsilon[G]$ of the *extensional* model \mathcal{N}_ε . We define a new r.a. and give, in this r.a., a new way to compute the truth value of ZF-formulas in $\mathcal{N}_\varepsilon[G]$.

Let $\mathbf{V} = V_\alpha$ be fixed in \mathcal{M} . We have :

$$\|x \notin \mathbf{V}\| = \|\langle x \in \mathbf{V} \rangle \neq 1\|, \text{ thus } \Vdash \forall x(x \varepsilon \mathbf{V} \leftrightarrow \langle x \in \mathbf{V} \rangle = 1).$$

Remember also the important (and obvious) equivalence $\Vdash \forall x \forall y (x = y \leftrightarrow \langle x = y \rangle = 1)$ which follows from $\|x \neq y\| = \|\langle x = y \rangle \neq 1\|$ and which identifies the r.m. \mathcal{N} with the boolean model $\mathcal{M}_{\mathbf{J}_2}$.

$\mathbf{J}\mathbf{V}$ is ε -transitive, by the :

Lemma 13.

If $X \in \mathcal{M}$ is transitive, then $\mathbf{J}X$ is ε -transitive, i.e. $K \Vdash \forall x \forall y (y \varepsilon x, y \notin \mathbf{J}X \rightarrow x \notin \mathbf{J}X)$.

We write $y \varepsilon x \equiv \neg(y \notin x)$. Let $\varpi \in \Pi, \xi \in \mathbf{\Lambda}$ be such that $(y, \varpi) \in x$ and $\xi \Vdash y \notin \mathbf{J}X$.

We suppose $\|x \notin \mathbf{J}X\| \neq \emptyset$ and therefore $x \in X$. Thus $y \in X$ and $\xi \Vdash \perp$.

Q.E.D.

Theorem 14. $\forall a \forall n^{int} (a \varepsilon (\mathbf{J}\mathbf{V})^n \rightarrow \exists ! b \{b \varepsilon \mathbf{J}(\mathbf{V}^n), \forall i^{int} (i < n \rightarrow a(i) = b[i])\})$.

In other words, each finite sequence of $\mathbf{J}\mathbf{V}$ in \mathcal{N} is represented by a unique finite sequence of \mathbf{V} in the boolean model $\mathcal{M}_{\mathbf{J}_2}$.

Unicity. Note first that, since there is no extensionality in \mathcal{N} , you may have two sequences $a \neq a' \varepsilon (\mathbf{J}\mathbf{V})^n$ such that $a(i) = a'(i)$ for $i < n$. Now suppose $b, b' \varepsilon \mathbf{J}(\mathbf{V}^n)$ be such that $b[i] = b'[i]$ for $i < n$. Then, we have $\langle b, b' \in \mathbf{V}^n \rangle = 1$ and $\langle (\forall i < n) (b[i] = b'[i]) \rangle = 1$. Since the boolean model $\mathcal{M}_{\mathbf{J}_2}$ satisfies extensionality, we get $\langle b = b' \rangle = 1$ that is $b = b'$.

Existence. Proof by induction on n . This is trivial if $n = 0$: take $b = \emptyset$.

Now, let $a \varepsilon (\mathbf{J}\mathbf{V})^{n+1}$ and a' be a restriction of a to n . Let $b' \varepsilon \mathbf{J}(\mathbf{V}^n)$ such that $a'(i) = b'[i]$ for $i < n$ (induction hypothesis).

In the ground model \mathcal{M} , we define the binary functional $+$ as follows :

if u is a finite sequence (u_0, \dots, u_{n-1}) , then $u + v$ is the sequence (u_0, \dots, u_{n-1}, v) .

We extend it to \mathcal{N} and we set $b = b' + a(n)$ i.e. $\langle b = b' + a(n) \rangle = 1$. Therefore $\langle b[i] = b'[i] \rangle = 1$ for $i < n$ and $\langle b[n] = a(n) \rangle = 1$, i.e. $a(i) = b[i]$ for $i < n$ and $a(n) = b[n]$.

Q.E.D.

Consider an arbitrary ordered set (C, \leq) in the model \mathcal{N} . By theorem 12 and property ① (section 2), we may suppose that $(C, \leq) \varepsilon \mathbf{J}\mathbf{V}$. As a set of forcing conditions, C is equivalent to the set \mathfrak{C} of finite subsets X of $\mathbf{J}\mathbf{V}$ such that $X \cap C$ has a lower bound in C , \mathfrak{C} being ordered by inclusion.

Thus, we can define \mathfrak{C} by the following formula of ZF_ε :

$$\mathfrak{C}(u) \equiv u \varepsilon (\mathbf{J}\mathbf{V})^{<\omega} \wedge (\text{Im}(u) \cap C) \text{ has a lower bound in } C$$

where $\text{Im}(u) \subseteq \mathbf{J}\mathbf{V}$ is the (finite) image of the finite sequence u .

We have $\mathcal{N} \models \mathfrak{C}(u) \rightarrow u \varepsilon (\mathbf{J}\mathbf{V})^{<\omega}$ and therefore $\mathcal{N} \models \mathfrak{C}(u) \rightarrow u \varepsilon (\mathbf{J}\mathbf{V}^{<\omega})$ by theorem 14.

Moreover, we have $\mathcal{N} \models \mathfrak{C}(\emptyset)$.

Remark. When we write $\mathcal{N} \models \Phi$ where Φ is a formula of ZF_ε , we really mean $\Vdash \Phi$ because we mean that Φ is true in *any* r.m. \mathcal{N} of the algebra \mathcal{A} .

In \mathcal{M} , the function $(u, v) \mapsto uv$, from $(\mathbf{V}^{<\omega})^2$ into $\mathbf{V}^{<\omega}$, which is the concatenation of sequences, is associative with \emptyset as neutral element, also denoted by $\mathbf{1}$ (monoid).

This function extends to \mathcal{N} into an application of $\mathfrak{J}(\mathbf{V}^{<\omega})^2$ into $\mathfrak{J}(\mathbf{V}^{<\omega})$ with the same properties. Thus, we write uvw for $u(vw)$, $(uv)w$, etc.

The formula $\mathfrak{C}(uv)$ of ZF_ε means that u, v are two compatible finite sequences of elements of C , i.e. the union of their images has a lower bound in C . Thus \mathfrak{C} becomes a set of forcing conditions equivalent to C by means of this compatibility relation.

This formula has the following properties :

$$\Vdash \mathfrak{C}(puvq) \rightarrow \mathfrak{C}(pvuq), \Vdash \mathfrak{C}(puvq) \rightarrow \mathfrak{C}(puq), \Vdash \mathfrak{C}(puq) \rightarrow \mathfrak{C}(puuv).$$

It will be convenient to have only one formula and to use simply the following consequence :

$$\textcircled{2} \quad \text{There exists a proof-like term } \mathfrak{c} \text{ such that } \mathfrak{c} \Vdash \mathfrak{C}(pqrtuvw) \rightarrow \mathfrak{C}(ptruuv).$$

Consider now, in the ground model \mathcal{M} , a r.a. \mathcal{A}_0 which gives the r.m. \mathcal{N} .

We suppose to have an operation $(\pi, \tau) \mapsto \pi^\tau$ from $\Pi \times \Lambda$ into Π such that :

$$(\xi \cdot \pi)^\tau = \xi \cdot \pi^\tau \text{ for every } \xi, \tau \in \Lambda \text{ and } \pi \in \Pi.$$

and two new combinators χ (*read*) and χ' (*write*) such that :

$$\begin{aligned} \chi \star \xi \cdot \pi^\tau &\succ \xi \star \tau \cdot \pi \text{ (i.e. } \xi \star \tau \cdot \pi \in \mathbb{L} \Rightarrow \chi \star \xi \cdot \pi^\tau \in \mathbb{L}) \\ \chi' \star \tau \cdot \xi \cdot \pi &\succ \xi \star \pi^\tau \text{ (i.e. } \xi \star \pi^\tau \in \mathbb{L} \Rightarrow \chi' \star \tau \cdot \xi \cdot \pi \in \mathbb{L}). \end{aligned}$$

Intuitively, π^τ is obtained by putting the term τ *at the end* of the stack π , in the same way that $\tau \cdot \pi$ is obtained by putting τ *at the top* of π .

We define now, in the ground model \mathcal{M} , a new r.a. \mathcal{A}_1 ; its r.m. will be called *the extension of \mathcal{N} by a \mathfrak{C} -generic (or a C -generic)*.

We define the terms $\mathbf{B}^*, \mathbf{C}^*, \mathbf{I}^*, \mathbf{K}^*, \mathbf{W}^*, \mathbf{cc}^*$ and \mathbf{k}_π^* by the conditions :

$$\begin{aligned} \textcircled{3} \quad &\mathbf{B}^* = \mathbf{B} ; \mathbf{I}^* = \mathbf{I} ; \\ &\mathbf{C}^* \star \xi \cdot \eta \cdot \zeta \cdot \pi^\tau \succ \xi \star \zeta \cdot \eta \cdot \pi^{\tau\tau} ; \text{i.e. } \mathbf{C}^* = \lambda x \lambda y \lambda z (\chi) \lambda t ((\chi')(\mathfrak{c})t) xzy ; \\ &\mathbf{K}^* \star \xi \cdot \eta \cdot \pi^\tau \succ \xi \star \pi^{\tau\tau} ; \text{i.e. } \mathbf{K}^* = \lambda x \lambda y (\chi) \lambda t ((\chi')(\mathfrak{c})t) x ; \\ &\mathbf{W}^* \star \xi \cdot \eta \cdot \pi^\tau \succ \xi \star \eta \cdot \eta \cdot \pi^{\tau\tau} ; \text{i.e. } \mathbf{W}^* = \lambda x \lambda y (\chi) \lambda t ((\chi')(\mathfrak{c})t) xyy ; \\ &\mathbf{k}_\pi^* \star \xi \cdot \varpi^\tau \succ \xi \star \pi^{\tau\tau} ; \text{i.e. } \mathbf{k}_\pi^* = \lambda x (\chi) \lambda t (\mathbf{k}_\pi) ((\chi')(\mathfrak{c})t) x ; \\ &\mathbf{cc}^* \star \xi \cdot \pi^\tau \succ \xi \star \mathbf{k}_\pi^* \cdot \pi^{\tau\tau} ; \text{i.e.} \\ &\mathbf{cc}^* = \lambda x (\chi) \lambda t (\mathbf{cc}) \lambda k (((\chi')(\mathfrak{c})t)x) \lambda x' (\chi) \lambda t' (k) (((\chi')(\mathfrak{c})t')x'). \end{aligned}$$

When checking below the axioms of r.a., the property needed for each combinator is :

$$\begin{aligned} \textcircled{4} \quad &\text{for } \mathbf{C}^* : \mathfrak{c} \Vdash \mathfrak{C}(prtv) \rightarrow \mathfrak{C}(ptrv) ; \text{ for } \mathbf{K}^* : \mathfrak{c} \Vdash \mathfrak{C}(pqr) \rightarrow \mathfrak{C}(pr) ; \\ &\text{for } \mathbf{W}^* : \mathfrak{c} \Vdash \mathfrak{C}(puv) \rightarrow \mathfrak{C}(puuv) ; \text{ for } \mathbf{cc}^* : \mathfrak{c} \Vdash \mathfrak{C}(pu) \rightarrow \mathfrak{C}(puu) ; \\ &\text{for } \mathbf{k}_\pi^* : \mathfrak{c} \Vdash \mathfrak{C}(rtw) \rightarrow \mathfrak{C}(tr). \end{aligned}$$

We get them replacing by $\mathbf{1}$ some of the variables p, q, r, t, u, v, w in the definition $\textcircled{2}$ of \mathfrak{c} .

We define the r.a. \mathcal{A}_1 by setting $\Lambda = \Lambda \times \mathbf{V}^{<\omega} ; \Pi = \Pi \times \mathbf{V}^{<\omega} ; \Lambda \star \Pi = (\Lambda \star \Pi) \times \mathbf{V}^{<\omega}$.

$$(\xi, u) \cdot (\pi, v) = (\xi \cdot \pi, uv) ;$$

$$(\xi, u) \star (\pi, v) = (\xi \star \pi, uv) ;$$

$$(\xi, u)(\eta, v) = (\xi\eta, uv).$$

The pole \mathbb{L} of \mathcal{A}_1 is defined by :

$$(\xi \star \pi, u) \in \mathbb{L} \Leftrightarrow (\forall \tau \in \Lambda) (\tau \Vdash \mathfrak{C}(u) \Rightarrow \xi \star \pi^\tau \in \mathbb{L}).$$

The combinators are :

$$\mathbf{B} = (\mathbf{B}, \mathbf{1}), \mathbf{C} = (\mathbf{C}^*, \mathbf{1}), \mathbf{I} = (\mathbf{I}, \mathbf{1}), \mathbf{K} = (\mathbf{K}^*, \mathbf{1}), \mathbf{W} = (\mathbf{W}^*, \mathbf{1}), \mathbf{cc} = (\mathbf{cc}^*, \mathbf{1}) ;$$

$$\mathbf{k}_{(\pi, u)} = (\mathbf{k}_\pi^*, u) ; \text{PL}_{\mathcal{A}_1} \text{ is } \{(\theta, \mathbf{1}) ; \theta \in \text{PL}_{\mathcal{A}_0}\}.$$

Theorem 15. \mathcal{A}_1 is a coherent r.a..

Let us prove first that \mathcal{A}_1 is coherent : let $\theta \in \text{PL}_{\mathcal{A}_0}$; by hypothesis on the formula \mathfrak{C} , there exists $\tau_0 \in \text{PL}$ such that $\tau_0 \Vdash \mathfrak{C}(\mathbf{1})$.

Since $\chi'\theta\tau_0 \in \text{PL}_{\mathcal{A}_0}$, there exists $\pi \in \Pi$ such that $\chi'\theta\tau_0 \star \pi \notin \perp$, thus $\theta \star \pi^{\tau_0} \notin \perp$, therefore $(\theta, \mathbf{1}) \star (\pi, \mathbf{1}) \notin \perp$.

We check now that \mathcal{A}_1 is a r.a. :

- $(\xi, u) \star (\eta, v) \cdot (\pi, w) \in \perp \Rightarrow (\xi, u)(\eta, v) \star (\pi, w) \in \perp$.

By hypothesis, we have $(\xi \star \eta \cdot \pi, uvw) \in \perp$, therefore :

$(\forall \tau \in \Lambda)(\tau \Vdash \mathfrak{C}(uvw) \rightarrow \xi \star \eta \cdot \pi^\tau \in \perp)$ and therefore :

$(\forall \tau \in \Lambda)(\tau \Vdash \mathfrak{C}(uvw) \rightarrow \xi \eta \star \pi^\tau \in \perp)$ hence the result.

- $(\xi, u) \star (\eta, v)(\zeta, w) \cdot (\pi, z) \in \perp \Rightarrow (\mathbf{B}, \mathbf{1}) \star (\xi, u) \cdot (\eta, v) \cdot (\zeta, w) \cdot (\pi, z) \in \perp$.

By hypothesis, we have $(\xi \star \eta \zeta \cdot \pi, uvwz) \in \perp$ therefore :

$(\forall \tau \in \Lambda)(\tau \Vdash \mathfrak{C}(uvwz) \rightarrow \xi \star \eta \zeta \cdot \pi^\tau \in \perp)$ thus :

$(\forall \tau \in \Lambda)(\tau \Vdash \mathfrak{C}(uvwz) \rightarrow \mathbf{B} \star \xi \cdot \eta \cdot \zeta \cdot \pi^\tau \in \perp)$.

- $(\xi, u) \star (\zeta, w) \cdot (\eta, v) \cdot (\pi, z) \in \perp \Rightarrow (\mathbf{C}^*, \mathbf{1}) \star (\xi, u) \cdot (\eta, v) \cdot (\zeta, w) \cdot (\pi, z) \in \perp$.

By hypothesis, we have $(\xi \star \zeta \cdot \eta \cdot \pi, uvwz) \in \perp$, therefore :

$(\forall \tau \in \Lambda)(\tau \Vdash \mathfrak{C}(uvwz) \rightarrow \xi \star \zeta \cdot \eta \cdot \pi^\tau \in \perp)$ therefore, by definition ③ of \mathbf{C}^* :

$(\forall \tau \in \Lambda)(\tau \Vdash \mathfrak{C}(uvwz) \rightarrow \mathbf{C}^* \star \xi \cdot \eta \cdot \zeta \cdot \pi^\tau \in \perp)$. But, by the property ④ of \mathbf{c} , we have :

$\tau \Vdash \mathfrak{C}(uvwz) \rightarrow \mathbf{c}\tau \Vdash \mathfrak{C}(uvwz)$, hence the result.

- $(\xi, u) \star (\pi, v) \in \perp \Rightarrow (\mathbf{I}, \mathbf{1}) \star (\xi, u) \cdot (\pi, v) \in \perp$.

Immediate.

- $(\xi, u) \star (\pi, w) \in \perp \Rightarrow (\mathbf{K}^*, \mathbf{1}) \star (\xi, u) \cdot (\eta, v) \cdot (\pi, w) \in \perp$.

By hypothesis, we have $(\xi \star \pi, uw) \in \perp$, therefore :

$(\forall \tau \in \Lambda)(\tau \Vdash \mathfrak{C}(uw) \rightarrow \xi \star \pi^{\tau} \in \perp)$, therefore by the definition ③ of \mathbf{K}^* :

$(\forall \tau \in \Lambda)(\tau \Vdash \mathfrak{C}(uw) \rightarrow \mathbf{K}^* \star \xi \cdot \eta \cdot \pi^\tau \in \perp)$. But, by ④, we have :

$\tau \Vdash \mathfrak{C}(uw) \rightarrow \mathbf{c}\tau \Vdash \mathfrak{C}(uw)$, hence the result.

- $(\xi, u) \star (\eta, v) \cdot (\eta, v) \cdot (\pi, w) \in \perp \Rightarrow (\mathbf{W}^*, \mathbf{1}) \star (\xi, u) \cdot (\eta, v) \cdot (\pi, w) \in \perp$.

By hypothesis, we have $(\xi \star \eta \cdot \eta \cdot \pi, uvvw) \in \perp$, therefore :

$(\forall \tau \in \Lambda)(\tau \Vdash \mathfrak{C}(uvvw) \rightarrow \xi \star \eta \cdot \eta \cdot \pi^\tau \in \perp)$, thus, by the definition ③ of \mathbf{W}^* :

$(\forall \tau \in \Lambda)(\tau \Vdash \mathfrak{C}(uvvw) \rightarrow \mathbf{W}^* \star \xi \cdot \eta \cdot \pi^\tau \in \perp)$. Now, by ④, we have :

$\tau \Vdash \mathfrak{C}(uvw) \rightarrow \mathbf{c}\tau \Vdash \mathfrak{C}(uvvw)$, hence the result : $\tau \Vdash \mathfrak{C}(uvw) \rightarrow \mathbf{W}^* \star \xi \cdot \eta \cdot \pi^\tau \in \perp$.

- $(\xi, v) \star (\pi, u) \in \perp \Rightarrow (\mathbf{k}_\pi^*, u) \star (\xi, v) \cdot (\varpi, w) \in \perp$.

By hypothesis, we have $(\xi \star \pi, vu) \in \perp$, therefore :

$(\forall \tau \in \Lambda)(\tau \Vdash \mathfrak{C}(vu) \rightarrow \xi \star \pi^{\tau} \in \perp)$, thus, by the definition ③ of \mathbf{k}_π^* :

$(\forall \tau \in \Lambda)(\tau \Vdash \mathfrak{C}(vu) \rightarrow \mathbf{k}_\pi^* \star \xi \cdot \varpi^\tau \in \perp)$. Now, by ④, we have :

$\tau \Vdash \mathfrak{C}(uvw) \rightarrow \mathbf{c}\tau \Vdash \mathfrak{C}(vu)$, hence the result : $\tau \Vdash \mathfrak{C}(uvw) \rightarrow \mathbf{k}_\pi^* \star \xi \cdot \varpi^\tau \in \perp$.

- $(\xi, u) \star (\mathbf{k}_\pi^*, v) \cdot (\pi, v) \in \perp \Rightarrow (\mathbf{cc}^*, \mathbf{1}) \star (\xi, u) \cdot (\pi, v) \in \perp$.

By hypothesis, we have $(\xi \star \mathbf{k}_\pi^* \cdot \pi, uvv) \in \perp$, therefore :

$(\forall \tau \in \Lambda)(\tau \Vdash \mathfrak{C}(uvv) \rightarrow \xi \star \mathbf{k}_\pi^* \cdot \pi^\tau \in \perp)$, thus, by the definition ③ of \mathbf{cc}^* :

$(\forall \tau \in \Lambda)(\tau \Vdash \mathfrak{C}(uvv) \rightarrow \mathbf{cc}^* \star \xi \cdot \pi^\tau \in \perp)$. But, by ④, we have :

$\tau \Vdash \mathfrak{C}(uv) \rightarrow \mathbf{c}\tau \Vdash \mathfrak{C}(uvv)$, hence the result : $\tau \Vdash \mathfrak{C}(uv) \rightarrow \mathbf{cc}^* \star \xi \cdot \pi^\tau \in \perp$.

Q.E.D.

The C-forcing defined in \mathcal{N}

We suppose now that the set of conditions (C, \leq) is in \mathcal{N}_ϵ , more precisely that it is saturated in $\mathbf{J}\mathbf{V}$ for the equivalence extensionality relation $=_\epsilon$:

$$\mathcal{N} \models \forall x \mathbf{J}\mathbf{V} \forall y \mathbf{J}\mathbf{V} \forall x' \mathbf{J}\mathbf{V} \forall y' \mathbf{J}\mathbf{V} (x =_\epsilon x', y =_\epsilon y', x \leq y \rightarrow x' \leq y').$$

We define the formula $p \Vdash F(\vec{a})$, where $F(\vec{a})$ is a formula of the language of ZF, *as the formula of the language of ZF which expresses the C-forcing on \mathcal{N}_ϵ* . The equivalences ⑤.1 and ⑤.2 below are therefore consequences of ZF_ϵ and therefore realized by proof-like terms $\mathbf{p}_1, \mathbf{p}'_1, \mathbf{p}_2, \mathbf{p}'_2$ of \mathcal{A} :

1. $(\mathbf{p}_1, \mathbf{p}'_1) \Vdash (p \Vdash a \notin b \leftrightarrow \forall q \forall r \forall c (\mathfrak{C}(pqr), q \Vdash c \subset a, r \Vdash a \subset c \rightarrow (c, \text{Im}(qr)) \notin b))$.
- ⑤ 2. $(\mathbf{p}_2, \mathbf{p}'_2) \Vdash (p \Vdash a \subset b \leftrightarrow \forall q \forall c (\mathfrak{C}(pq), q \Vdash c \notin b \rightarrow (c, \text{Im}(q)) \notin a))$.
3. $p \Vdash F \rightarrow F' \equiv \forall q (q \Vdash F \rightarrow pq \Vdash F')$.
4. $p \Vdash \forall x F \equiv \forall x (p \Vdash F)$.

Remarks.

We write “ $(\mathbf{p}, \mathbf{p}') \Vdash A \leftrightarrow B$ ” for “ $\mathbf{p} \Vdash A \rightarrow B$ and $\mathbf{p}' \Vdash B \rightarrow A$ ”.

$\text{Im}(p)$ is the finite subset of C which is the image of the sequence $p \in \mathfrak{C}$.

We have $\mathcal{N} \models \neg \mathfrak{C}(p) \rightarrow p \Vdash F$: if p is not a condition, then p forces everything.

In the general theory of classical realizability, we define a truth value for the formulas of ZF_ϵ and therefore, in particular, for the formulas of ZF. We will define here directly a new truth value $\|F(a_1, \dots, a_n)\|$ for a formula of ZF with parameters in \mathcal{M} for the r.a. \mathcal{A}_1 . To this aim, we first define the truth values $\|a \notin b\|, \|a \subset b\|$ of the atomic formulas of ZF ; then that of $F(a_1, \dots, a_n)$, by induction on the length of the formula :

$$\begin{aligned} \|a \notin b\| &= \{(\eta \cdot \zeta \cdot \pi, qr) ; q, r \in \mathbf{V}^{<\omega}, \eta, \zeta \in \Lambda, \\ &\quad \eta \Vdash (q \Vdash c \subset a), \zeta \Vdash (r \Vdash a \subset c), \pi \in \|(c, \text{Im}(qr)) \notin b\|\} ; \\ \|a \subset b\| &= \{(\eta \cdot \pi, q) ; q \in \mathbf{V}^{<\omega}, \eta \in \Lambda, \eta \Vdash (q \Vdash c \notin b), \pi \in \|(c, \text{Im}(q)) \notin a\|\} ; \\ \|F \rightarrow F'\| &= \{(\xi \cdot \pi, pq) ; (\xi, p) \Vdash F, (\pi, q) \in \|F'\|\} ; \\ \|\forall x F(x, a_1, \dots, a_n)\| &= \bigcup_a \|F(a, a_1, \dots, a_n)\|. \end{aligned}$$

Remark. Be careful, as we said before, these are not the truth values, in the r.a. \mathcal{A}_1 , of $a \notin b$ and $a \subset b$ considered as formulas of ZF_ϵ . We seek to define here directly the C-generic model on \mathcal{N}_ϵ , without going through a model of ZF_ϵ .

Lemma 16. Let q_1, \dots, q_n range in $\mathbf{V}^{<\omega}$; denote $\vec{q} = (q_1, \dots, q_n)$ and $|\vec{q}| = q_1 \cdots q_n$. Let $\Phi = \bigcup_{\vec{q}} \|F(\vec{q})\| \times \{|\vec{q}|\}$ where $F(\vec{q})$ is a formula of ZF_ϵ . Then :

- a) $\xi \Vdash \forall \vec{q} (\mathfrak{C}(p|\vec{q}|) \rightarrow F(\vec{q})) \Rightarrow (\chi\xi, p) \Vdash \Phi$.
- b) $(\xi, p) \Vdash \Phi \Rightarrow \chi'\xi \Vdash \forall \vec{q} (\mathfrak{C}(p|\vec{q}|) \rightarrow F(\vec{q}))$.

a) Let $\pi \in \|F(\vec{q})\|$ and $\tau \Vdash \mathfrak{C}(p|\vec{q}|)$; by hypothesis, we have $\xi \star \tau \cdot \pi \in \perp$ and therefore $\chi\xi \star \pi^\tau \in \perp$. Thus, we have shown $\forall \tau (\tau \Vdash \mathfrak{C}(p|\vec{q}|) \rightarrow \chi\xi \star \pi^\tau \in \perp)$; therefore $(\chi\xi, p) \star (\pi, |\vec{q}|) \in \perp$ for every $(\pi, q) \in \|\Phi\|$.

b) Conversely, let $(\pi, |\vec{q}|) \in \|\Phi\|$ so that $\pi \in \|F(\vec{q})\|$.

By hypothesis, we have $(\xi, p) \star (\pi, |\vec{q}|) \in \perp$, i.e. $\forall \tau (\tau \Vdash \mathfrak{C}(p|\vec{q}|) \rightarrow \xi \star \pi^\tau \in \perp)$.

Therefore we have $\forall \tau (\tau \Vdash \mathfrak{C}(p|\vec{q}|) \rightarrow \chi'\xi \star \tau \cdot \pi \in \perp)$ which is the desired result.

Q.E.D.

Lemma 17. *There exist four proof-like terms $\chi_\epsilon, \chi'_\epsilon, \chi_\subset, \chi'_\subset$ such that :*

- 1a. $\xi \Vdash (p \Vdash a \notin b) \Rightarrow (\chi_\epsilon \xi, p) \Vdash a \notin b$.
- 1b. $(\xi, p) \Vdash a \notin b \Rightarrow \chi'_\epsilon \xi \Vdash (p \Vdash a \notin b)$.
- 2a. $\xi \Vdash (p \Vdash a \subset b) \Rightarrow (\chi_\subset \xi, p) \Vdash a \subset b$.
- 2b. $(\xi, p) \Vdash a \subset b \Rightarrow \chi'_\subset \xi \Vdash (p \Vdash a \subset b)$.

By the definition above of $\Vdash a \notin b$ and $\Vdash a \subset b$, we have :

$$((\xi, p) \Vdash a \notin b) \equiv \forall \pi \forall \eta \forall \zeta \forall q \forall r ((\eta \cdot \zeta \cdot \pi, qr) \in \Vdash a \notin b \rightarrow (\xi \star \eta \cdot \zeta \cdot \pi, pqr) \in \mathbb{L}).$$

$$\equiv \forall c \forall \pi \forall \eta \forall \zeta \forall q \forall r \forall \tau$$

$$(\eta \Vdash (q \Vdash c \subset a), \zeta \Vdash (r \Vdash a \subset c), \pi \in \|(c, \text{Im}(qr)) \notin b\|, \tau \Vdash \mathfrak{C}(pqr) \rightarrow \xi \star \eta \cdot \zeta \cdot \pi^\tau \in \mathbb{L}).$$

In the same way, we get :

$$((\xi, p) \Vdash a \subset b) \equiv \forall c \forall \pi \forall \eta \forall q \forall \tau (\eta \Vdash (q \Vdash c \notin b), \pi \in \|(c, \text{Im}(q)) \notin a\|, \tau \Vdash \mathfrak{C}(pq) \rightarrow \xi \star \eta \cdot \pi^\tau \in \mathbb{L}).$$

Proof of (2a, 2b). Let $F(q) \equiv \forall c (q \Vdash c \notin b \rightarrow (c, \text{Im}(q)) \notin a)$.

Let $\xi \Vdash (p \Vdash a \subset b)$; by ⑤.2, we have :

$$\mathfrak{p}_2 \xi \Vdash \forall q \forall c (\mathfrak{C}(pq), q \Vdash c \notin b \rightarrow (c, \text{Im}(q)) \notin a) \text{ i.e. } \mathfrak{p}_2 \xi \Vdash \forall q (\mathfrak{C}(pq) \rightarrow F(q)).$$

By lemma 16.a, we get $((\chi)(\mathfrak{p}_2)\xi, p) \Vdash \Phi$ with $\|\Phi\| = \bigcup_q \|F(q)\| \times \{q\} = \Vdash a \subset b$.

This gives $\chi_\subset = \lambda x (\chi)(\mathfrak{p}_2)x$.

Conversely, if $(\xi, p) \Vdash a \subset b$, we have $(\xi, p) \Vdash \Phi$. By lemma 16.b, we get :

$$\chi'_\subset \xi \Vdash \forall q (\mathfrak{C}(pq) \rightarrow F(q)) \text{ i.e. } \chi'_\subset \xi \Vdash \forall q \forall c (\mathfrak{C}(pq), q \Vdash c \notin b \rightarrow (c, \text{Im}(q)) \notin a).$$

By ⑤.2, we obtain $(\mathfrak{p}'_2)(\chi'_\subset)\xi \Vdash (p \Vdash a \subset b)$ and this gives $\chi'_\subset = \lambda x (\mathfrak{p}'_2)(\chi'_\subset)x$.

Proof of (1a, 1b). The same, with $F(q, r) \equiv \forall c (q \Vdash c \subset a, r \Vdash a \subset c \rightarrow (c, \text{Im}(qr)) \notin b)$.

We get $\chi_\epsilon = \lambda x (\chi)(\mathfrak{p}_1)x$ and $\chi'_\epsilon = \lambda x (\mathfrak{p}'_1)(\chi')x$.

Q.E.D.

Theorem 18 below may be considered as a generalization of the well known result about iteration of forcing : the r.a. \mathcal{A}_1 , which is a kind of product of \mathcal{A}_0 by \mathfrak{C} , gives the same r.m. as the \mathfrak{C} -generic extension of \mathcal{N}_ϵ .

Theorem 18.

For each closed formula F of ZF with parameters in the model \mathcal{N} , there exists two proof-like terms χ_F, χ'_F , which only depend on the propositional structure of F , such that we have :

$$\xi \Vdash (p \Vdash F) \Rightarrow (\chi_F \xi, p) \Vdash F \text{ and } (\xi, p) \Vdash F \Rightarrow \chi'_F \xi \Vdash (p \Vdash F)$$

for every $\xi \in \Lambda$ and $p \in \mathbf{V}^{<\omega}$.

The *propositional structure* of F is the propositional formula built with the connective \rightarrow and only one atom O , which is obtained from F by deleting all quantifiers and by identifying all atomic formulas with O .

For instance, the propositional structure of the formula :

$$\forall X (\forall x (\forall y ((x, y) \in Y \rightarrow y \notin X) \rightarrow x \notin X) \rightarrow \forall x (x \notin X)) \text{ is } ((O \rightarrow O) \rightarrow O) \rightarrow O.$$

Proof by recurrence on the length of F .

- If F is atomic, we have $F \equiv a \notin b$ or $a \subset b$. Apply lemma 17.
- If $F \equiv \forall x F'$, then $p \Vdash F \equiv \forall x (p \Vdash F')$. Therefore $\xi \Vdash p \Vdash F \equiv \forall x (\xi \Vdash (p \Vdash F'))$. Moreover, $(\xi, p) \Vdash F \equiv \forall x ((\xi, p) \Vdash F')$.

The result is immediate, from the recurrence hypothesis.

• If $F \equiv F' \rightarrow F''$, we have $p \Vdash F \equiv \forall q(q \Vdash F' \rightarrow pq \Vdash F'')$ and therefore :

⑧ $\xi \Vdash (p \Vdash F) \Rightarrow \forall \eta \forall q(\eta \Vdash (q \Vdash F') \rightarrow \xi \eta \Vdash (pq \Vdash F''))$.

Suppose that $\xi \Vdash (p \Vdash F)$ and set $\chi_F = \lambda x \lambda y(\chi_{F''})(x)(\chi'_{F'}y)$.

We must show $(\chi_F \xi, p) \Vdash F' \rightarrow F''$; thus, let $(\eta, q) \Vdash F'$ and $(\pi, r) \in \Vdash F'' \Vdash$.

We must show $(\chi_F \xi, p) \star (\eta, q) \bullet (\pi, r) \in \Vdash$ that is $(\chi_F \xi \star \eta \bullet \pi, pqr) \in \Vdash$.

Thus, let $\tau \Vdash \mathfrak{C}(pqr)$; we must show $\chi_F \xi \star \eta \bullet \pi^\tau \in \Vdash$ or else $\chi_F \star \xi \bullet \eta \bullet \pi^\tau \in \Vdash$.

From the recurrence hypothesis applied to $(\eta, q) \Vdash F'$, we have $\chi'_{F'} \eta \Vdash (q \Vdash F')$.

From ⑧, we have therefore $(\xi)(\chi'_{F'} \eta) \Vdash (pq \Vdash F'')$.

Applying again the recurrence hypothesis, we get :

$((\chi_{F''})(\xi)(\chi'_{F'} \eta), pq) \Vdash F''$. But since $(\pi, r) \in \Vdash F'' \Vdash$, we have :

$((\chi_{F''})(\xi)(\chi'_{F'} \eta), pq) \star (\pi, r) \in \Vdash$, that is $((\chi_{F''})(\xi)(\chi'_{F'} \eta) \star \pi, pqr) \in \Vdash$.

Since $\tau \Vdash \mathfrak{C}(pqr)$, we have $(\chi_{F''})(\xi)(\chi'_{F'} \eta) \star \pi^\tau \in \Vdash$.

But, by definition of χ_F , we have $\chi_F \star \xi \bullet \eta \bullet \pi^\tau \succ (\chi_{F''})(\xi)(\chi'_{F'} \eta) \star \pi^\tau$ which gives the desired result : $\chi_F \star \xi \bullet \eta \bullet \pi^\tau \in \Vdash$.

Suppose now that $(\xi, p) \Vdash F' \rightarrow F''$; we set $\chi'_F = \lambda x \lambda y(\chi'_{F''})(x)(\chi_{F'}y)$.

We must show $\chi'_F \xi \Vdash (p \Vdash F' \rightarrow F'')$ that is $\forall q(\chi'_F \xi \Vdash (q \Vdash F' \rightarrow pq \Vdash F''))$.

Thus, let $\eta \Vdash q \Vdash F'$ and $\pi \in \Vdash pq \Vdash F'' \Vdash$; we must show $\chi'_F \xi \star \eta \bullet \pi \in \Vdash$.

By the recurrence hypothesis, we have $(\chi_{F'} \eta, q) \Vdash F'$, therefore $(\xi, p)(\chi_{F'} \eta, q) \Vdash F''$ or else, by definition of the algebra \mathcal{A}_1 : $((\xi)(\chi_{F'} \eta), pq) \Vdash F''$.

Applying again the recurrence hypothesis, we have $(\chi'_{F''})(\xi)(\chi_{F'} \eta) \Vdash (pq \Vdash F'')$ and therefore $(\chi'_{F''})(\xi)(\chi_{F'} \eta) \star \pi \in \Vdash$. But we have, by definition of χ'_F :

$\chi'_F \xi \star \eta \bullet \pi \succ \chi'_{F''} \star \xi \bullet \eta \bullet \pi \succ (\chi'_{F''})(\xi)(\chi_{F'} \eta) \star \pi$; the desired result $\chi'_F \xi \star \eta \bullet \pi \in \Vdash$ follows.

Q.E.D.

Theorem 19. *For each axiom A of ZF, there exists a proof like term Θ_A of the r.a. \mathcal{A}_0 such that $(\Theta_A, \mathbf{1}) \Vdash A$.*

Indeed, if we denote by $\mathcal{N}_\infty[G]$ the C -generic model over \mathcal{N}_∞ , with $G \subseteq C$ being the generic set, we have $\mathcal{N}_\infty[G] \models \text{ZF}$. Therefore, $\mathcal{N} \models (\mathbf{1} \Vdash A)$, which means that there is a proof-like term Θ'_A such that $\Theta'_A \Vdash (\mathbf{1} \Vdash A)$. By theorem 18, we can take $\Theta_A = \chi_A \Theta'_A$.

Q.E.D.

4 The algebra \mathfrak{A}_0

We define a r.a. \mathfrak{A}_0 which gives a very interesting r.m. \mathcal{N} . In the following, we use only this r.a. and a generic extension \mathfrak{A}_1 .

The terms of \mathfrak{A}_0 are finite sequences of symbols :

$$), (, \mathbf{B}, \mathbf{C}, \mathbf{I}, \mathbf{K}, \mathbf{W}, \mathbf{cc}, \mathbf{a}, \mathbf{p}, \gamma, \kappa, \mathbf{e}, \chi, \chi', h_0, h_1, \dots, h_i, \dots$$

Λ is the least set which contains these symbols (except parentheses) and is such that :

$t, u \in \Lambda \Rightarrow (t)u \in \Lambda$.

PL is the set of terms which do not contain neither \mathbf{p} nor any h_i .

A stack is a finite sequence of terms, separated by the symbol \bullet and terminated by the symbol π_0 (*the empty stack*).

Π is therefore the least set such that $\pi_0 \in \Pi$ and $t \in \Lambda, \pi \in \Pi \Rightarrow t \bullet \pi \in \Pi$.

\mathbf{k}_π is defined by recurrence : $\mathbf{k}_{\pi_0} = \mathbf{a}$; $\mathbf{k}_{t.\pi} = \lambda x(\mathbf{k}_\pi)(x)t$.

The application $(\tau, \pi) \mapsto \pi^\tau$ from $\Lambda \times \Pi$ into Π consists in replacing π_0 by $\tau \cdot \pi_0$. It is therefore recursively defined by : $\pi_0^\tau = \tau \cdot \pi_0$; $(t \cdot \pi)^\tau = t \cdot \pi^\tau$.

$\Lambda \star \Pi$ is $\Lambda \times \Pi$.

\perp is the least subset of $\Lambda \star \Pi$ satisfying the conditions :

1. $\mathbf{p} \star \pi \in \perp$ for every stack $\pi \in \Pi$ (*stop*) ;
 2. $\xi \star \pi_0 \in \perp \Rightarrow \mathbf{a} \star \xi \cdot \pi \in \perp$ for every $\xi \in \Lambda, \pi \in \Pi$ (*abort*) ;
 3. If at least two out of $\xi \star \pi, \eta \star \pi, \zeta \star \pi$ are in \perp , then $\gamma \star \xi \cdot \eta \cdot \zeta \cdot \pi \in \perp$ (*fork*) ;
 4. $\xi \star \pi \in \perp \Rightarrow \mathbf{e} \star h_i \cdot h_i \cdot \eta \cdot \xi \cdot \pi \in \perp$ for every $\xi, \eta \in \Lambda$ and $i \in \mathbb{N}$ (*elimination of constants*) ;
 5. $\xi \star \pi \in \perp \Rightarrow \mathbf{e} \star h_i \cdot h_j \cdot \xi \cdot \eta \cdot \pi \in \perp$ for every $\xi, \eta \in \Lambda$ and $i, j \in \mathbb{N}, i \neq j$; (*elimination of constants*)
 6. $\xi \star h_n \cdot \pi \in \perp \Rightarrow \kappa \star \xi \cdot \pi \in \perp$ if h_n does not appear in ξ, π ; (*introduction of constants*)
- and also the general axiomatic conditions for $\mathbf{B}, \mathbf{C}, \mathbf{I}, \mathbf{K}, \mathbf{W}, \mathbf{cc}, \chi, \chi'$ and the application :
7. $\xi \star \eta \cdot \pi \in \perp \Rightarrow (\xi)\eta \star \pi \in \perp$ (*push*) ;
 8. $\xi \star \pi \in \perp \Rightarrow \mathbf{I} \star \xi \cdot \pi \in \perp$ (*no operation*) ;
 9. $\xi \star \pi \in \perp \Rightarrow \mathbf{K} \star \xi \cdot \eta \cdot \pi \in \perp$ (*delete*) ;
 10. $\xi \star \eta \cdot \eta \cdot \pi \in \perp \Rightarrow \mathbf{W} \star \xi \cdot \eta \cdot \pi \in \perp$ (*copy*) ;
 11. $\xi \star \zeta \cdot \eta \cdot \pi \in \perp \Rightarrow \mathbf{C} \star \xi \cdot \eta \cdot \zeta \cdot \pi \in \perp$ (*switch*) ;
 12. $\xi \star (\eta)\zeta \cdot \pi \in \perp \Rightarrow \mathbf{B} \star \xi \cdot \eta \cdot \zeta \cdot \pi \in \perp$ (*apply*) ;
 13. $\xi \star \mathbf{k}_\pi \cdot \pi \in \perp \Rightarrow \mathbf{cc} \star \xi \cdot \pi \in \perp$ (*save the stack*)
 14. $\xi \star \tau \cdot \pi \in \perp \Rightarrow \chi \star \xi \cdot \pi^\tau \in \perp$ (*read the end of the stack*).
 15. $\xi \star \pi^\tau \in \perp \Rightarrow \chi' \star \tau \cdot \xi \cdot \pi \in \perp$ (*write at the end of the stack*).

The property :

$\xi \star \pi \in \perp \Rightarrow \mathbf{k}_\pi \star \xi \cdot \varpi \in \perp$ (*restore the stack*)

now follows easily from the definition of \mathbf{k}_π .

Theorem 20. $\mathfrak{J}2$ has at most 4 elements.

Indeed, we have immediately $\gamma \Vdash \forall x^{\mathfrak{J}2} \forall y^{\mathfrak{J}2} (\langle x \leq y \rangle = 1 \hookrightarrow (x \neq 0, y \neq 1, x \neq y \rightarrow \perp))$.
Q.E.D.

If $\mathfrak{J}2$ is trivial, everything in the following is also. Therefore, we now assume that $\mathfrak{J}2$ has 4 elements.

Let a_0, a_1 be the atoms of $\mathfrak{J}2$. Then $\mathcal{M}_{\mathcal{D}} = \mathcal{M}_{a_0} = a_0 \mathcal{N}$ and $\mathcal{M}_{a_1} = a_1 \mathcal{N}$ are classes in the r.m. \mathcal{N} respectively defined by the formulas $x = a_0 x$ and $x = a_1 x$.

We define the binary functional \sqcup in \mathcal{N} as the extension of the functional $(x, y) \mapsto x \cup y$ on \mathcal{M} . We don't use the symbol \cup , because it already denotes the union. For instance, we have $a_0 \sqcup a_1 = 1$ but $a_0 \cup a_1$ is empty, since a_0, a_1 are.

The identity $x = a_0 x \sqcup a_1 x$ gives a bijection from \mathcal{N} onto $\mathcal{M}_{a_0} \times \mathcal{M}_{a_1}$.

We have $\mathcal{M} \prec \mathcal{M}_{a_0}, \mathcal{M}_{a_1}$. Moreover, $\mathcal{M}_{a_0} = \mathcal{M}_{\mathcal{D}}$ is well founded in \mathcal{N} and therefore :

The class of ordinals On is defined in \mathcal{N} .

Remark. If the ground model \mathcal{M} satisfies $V = L$, then \mathcal{M}_{a_0} is isomorphic to $L^{\mathcal{N}_{\in}}$, the class of constructible sets of \mathcal{N}_{\in} .

Execution of processes

If a given process $\xi \star \pi$ is in \perp , it is obtained by applying precisely one of the rules of definition of \perp : if ξ is an application, it is rule 7, else ξ is an instruction and there is one and only one corresponding rule.

In this way, we get a finite tree, which is the proof that $\xi \star \pi \in \perp$; it is linear, except in the case of rule 3 (instruction γ) where there is a triple branch. This tree is called *the execution of the process $\xi \star \pi$* .

We can, of course, build this tree for any process ; it may then be infinite.

Computing with the instruction γ

Let us consider, in the ground model \mathcal{M} , some functions $f_i : \mathbb{N}^{k_i} \rightarrow \mathbb{N}$, satisfying a set of axioms of the form : $(\forall \vec{x} \in \mathbb{N}^k)(t_0[\vec{x}] = u_0[\vec{x}], \dots, t_{n-1}[\vec{x}] = u_{n-1}[\vec{x}] \rightarrow t[\vec{x}] = u[\vec{x}])$

where t_i, u_i are terms built with the symbols f_i and the variables \vec{x} .

These function symbols are also interpreted in \mathcal{N} where we have $f_i : \mathbb{N}^{k_i} \rightarrow \mathbb{N}$ and in this way, we have a set \mathcal{E} of realized axioms :

$\lambda f_0 \dots \lambda f_{n-1} \lambda x (f_0) \dots (f_{n-1}) x \Vdash \forall \vec{x}^{\mathbb{N}^k} (t_0[\vec{x}] = u_0[\vec{x}], \dots, t_{n-1}[\vec{x}] = u_{n-1}[\vec{x}] \rightarrow t[\vec{x}] = u[\vec{x}])$.

Now suppose that, with the axioms \mathcal{E} and some other axioms realized in \mathcal{N} like, for instance :

(*) $\text{ZF}_\varepsilon + \mathcal{E} + \mathfrak{I}2$ has at most 4 elements + \mathbb{N} is countable

we can prove $\exists n^{\text{int}} (f[n] = 0)$. In this way, we get a proof-like term :

$\theta \Vdash \forall n^{\text{int}} (f[n] \neq 0) \rightarrow \perp$ and θ may contain the instructions \mathbf{e}, κ and, above all, γ .

We shall show how θ allows to compute a solution of the equation $f[n] = 0$.

Note that *we do not assume that the f_i (in particular f) are recursive*. On the other hand, we may add symbols for all recursive functions, since they are defined by axiom systems of this form.

First, we replace f by the function f' always equal to 1, except that $f'[n_0] = 0$ for the first zero n_0 of f if there exists one.

To this aim, we define a function g by $g[0] = \inf[1, f[0]]$; $g[n+1] = \inf[g[n], f[n+1]]$; then a function f' by $f'[0] = g[0]$; $f'[n+1] = 1 - g[n] + g[n+1]$.

With these equations, it easy to show, in arithmetic that :

$\forall n^{\text{int}} (f'[n] = 0 \rightarrow f[n] = 0)$; $\forall m^{\text{int}} \forall n^{\text{int}} (f'[m] = 0, f'[n] = 0 \rightarrow m = n)$;

$\forall n^{\text{int}} (f'[n] \neq 0) \rightarrow \forall n^{\text{int}} (f[n] \neq 0)$.

Thus, there is a proof-like term $\theta' \Vdash \forall n^{\text{int}} (f'[n] \neq 0) \rightarrow \perp$. We shall see that θ' allows to compute n_0 . To this aim, we add a term constant δ and the rule $\delta \star \underline{n}_0 \cdot \pi \in \perp$ in the definition of \perp . Thus, we have $\delta \Vdash \forall n^{\text{int}} (f'[n] \neq 0)$ and therefore $\theta' \star \delta \cdot \pi_0 \in \perp$.

But any process $\xi \star \pi \in \perp$ which does not contain \mathbf{p} (but possibly containing δ) computes n_0 . We show this by recurrence on the number of applications of rules for \perp used to build it :

If this number is 1, the process is $\delta \star \underline{n}_0 \cdot \pi$. Else, the only non trivial case is when the process is $\gamma \star \xi \cdot \eta \cdot \zeta \cdot \pi$ and when two out of the three processes $\xi \star \pi, \eta \star \pi, \zeta \star \pi$ are in \perp (but we don't know which). By the recurrence hypothesis, at least two of them will give the integer n_0 which is therefore determined as the only integer obtained at least two times (example : $\gamma \star \delta \underline{n}_0 \cdot \tau \underline{n}_0 \cdot \delta \underline{n} \cdot \pi$).

\mathbb{N} is countable

We define two sets, in the ground model \mathcal{M} :

$$\mathbb{H} = \{h_i ; i \in \mathbb{N}\}, \mathbf{H} = \{(h_i, h_i \cdot \pi) ; i \in \mathbb{N}, \pi \in \Pi\}$$

and also the bijection $\mathbf{h} : \mathbb{N} \rightarrow \mathbb{H}$ such that $\mathbf{h}[i] = h_i$ for every $i \in \mathbb{N}$.

This bijection extends to the model \mathcal{N} into a bijection $\mathbf{h} : \mathbb{N} \rightarrow \mathbb{H}$. Moreover, we have trivially $\Vdash \mathbf{H} \subseteq \mathbb{H}$; in fact $\Vdash \forall x (x \notin \mathbb{H} \rightarrow x \notin \mathbf{H})$.

Lemma 21. $\mathbf{e} \Vdash \forall i \in \mathbb{N} \forall j \in \mathbb{N} (h[i] \in \mathbf{H}, h[j] \in \mathbf{H}, \langle i = j \rangle \neq 0 \rightarrow i = j)$.

Note that we have $|\mathbf{h}[i] \in \mathbf{H}| = \{h_i\}$. Let $t \Vdash \langle i = j \rangle \neq 0$ and $u \Vdash i \neq j$. We must show that $\mathbf{e} \star h_i \cdot h_j \cdot t \cdot u \cdot \pi \in \perp$, which follows immediately from the execution rule of \mathbf{e} .

Q.E.D.

Thus we have $\mathcal{N} \models \forall i \in \mathbb{N} \forall j \in \mathbb{N} (h[i] \in \mathbf{H}, h[j] \in \mathbf{H}, ia_0 = ja_0 \rightarrow i = j)$. It follows that :

$$\mathcal{N} \models (\mathbf{H} \text{ is countable}).$$

Indeed, if $i \in \mathbb{N}$ and $\mathbf{h}[i] \in \mathbf{H}$, then i is determined by ia_0 , which is an integer of \mathcal{M}_{a_0} and therefore an integer of \mathcal{N} .

Define the function symbols $\text{pr}_0, \text{pr}_1 : \mathbb{N} \rightarrow \mathbb{N}$ by :

$$n = \text{pr}_1[n] + \frac{1}{2}(\text{pr}_0[n] + \text{pr}_1[n])(\text{pr}_0[n] + \text{pr}_1[n] + 1) \text{ (bijection from } \mathbb{N}^2 \text{ onto } \mathbb{N}).$$

Theorem 22. $\kappa \Vdash \forall \nu \in \mathbb{N} \exists n \in \mathbb{N} \{h[n] \in \mathbf{H}, \nu = \text{pr}_1[n]\}$.

$\Vdash \mathbb{N} \text{ is countable}$.

Let $\nu \in \mathbb{N}, \pi \in \Pi$ and $\xi \Vdash \forall n \in \mathbb{N} \{\nu = \text{pr}_1[n] \leftrightarrow \mathbf{h}[n] \notin \mathbf{H}\}$. Thus, we have $\xi \star h_n \cdot \pi \in \perp$ for all $n \in \mathbb{N}$ such that $\nu = \text{pr}_1[n]$. There is an infinity of such n , so that we can choose one such that h_n does not appear in ξ, π . It follows that $\kappa \star \xi \cdot \pi \in \perp$.

Since $\mathbf{h} : \mathbb{N} \rightarrow \mathbb{H}$ is a bijection, we obtain a surjection from \mathbf{H} onto \mathbb{N} . It follows that :

$$\mathcal{N} \models (\mathbb{N} \text{ is countable}).$$

Q.E.D.

Theorem 23. $\mathcal{N} \models \text{NEPC}$ (the non extensional principle of choice).

This means that for any formula $R(x, y)$ of ZF_ε , there is a binary relation $\Phi(x, y)$ such that $\Vdash \forall x \forall y \forall y' (\Phi(x, y), \Phi(x, y') \rightarrow y = y')$ (functional relation) ;

$$\Vdash \forall x \forall y (R(x, y) \rightarrow \exists y' \{R(x, y'), \Phi(x, y')\}) \text{ (choice)}.$$

This does not give the usual principle of choice in the model \mathcal{N}_ε of ZF because, even if R is compatible with the extensional equivalence $=_\varepsilon$, Φ is not necessarily so.

By lemma 11, we have $\Vdash \forall x \forall y (R(x, y) \rightarrow \exists \varpi \in \mathbb{I} R(x, f[x, \varpi]))$ where f is a functional symbol defined in \mathcal{M} . Now, Π is countable in \mathcal{M} , thus \mathbb{I} is equipotent to \mathbb{N} and therefore countable by theorem 22. Therefore, we can define $\Phi(x, y)$ as “ $y = f[x, \varpi]$ for the first $\varpi \in \mathbb{I}$ such that $R(x, f[x, \varpi])$ ”.

Q.E.D.

By the results of [10], it follows also that $\mathcal{N} \models (\mathbb{R} \text{ is not well orderable})$.

Note that these results do not use the instruction γ and therefore are valid for any $\mathbb{I}2$.

On the other hand, the following result uses γ , i.e. the fact that $\mathbb{I}2$ is finite :

Theorem 24. $\mathcal{N} \models$ the axiom of well ordered choice (WOC) i.e. : The product of a family of non void sets indexed by a well ordered set is non void.

This follows immediately from NEPC and the fact that On is isomorphic to a class of \mathcal{N} .

Q.E.D.

Remark. In fact, since \mathcal{N} satisfies NEPC, it also satisfies the well ordered principle of choice (WOPC).

Theorem 24 has two interesting consequences :

1. There exists a proof-like term $\Theta_{WOC} \Vdash \text{WOC}$ which means that we now have a program for the axiom WOC, which is a λ -term with the instructions γ, κ, e .
2. This gives a new proof that AC is not a consequence of $ZF + \text{WOC}$ [6].

5 The algebra \mathfrak{A}_1 and the program for AC

Lemma 25. *Let $R(x, y)$ be a formula of ZF_ε such that $R(a_0x, a_1y)$ defines, in \mathcal{N} , a functional from \mathcal{M}_{a_0} into \mathcal{M}_{a_1} or from \mathcal{M}_{a_1} into \mathcal{M}_{a_0} . Then, this functional has a countable image.*

Remember that $\mathcal{M}_{a_i} (i = 0, 1)$ is the class defined by $a_ix = x$.

Suppose, for instance, that $R(x, y)$ defines a functional from \mathcal{M}_{a_0} into \mathcal{M}_{a_1} i.e. :

$$\mathcal{N} \models \forall x \forall y \forall y' (R(a_0x, a_1y), R(a_0x, a_1y') \rightarrow a_1y = a_1y').$$

Applying lemma 11 to the formula $\neg R(a_0x, a_1y)$, we obtain :

$$\vdash \Vdash \forall \varpi^{\mathbb{I}\Pi} \neg R(a_0x, a_1f[a_0x, \varpi]) \rightarrow \forall y \neg R(a_0x, a_1y)$$

for some functional $f : \mathcal{M} \times \Pi \rightarrow \mathcal{M}$ defined in \mathcal{M} .

By lemma 7, we have $a_1f[a_0x, \varpi] = a_1f[a_1a_0x, \varpi]$. Since $a_1a_0 = 0$, we get :

$$\vdash \Vdash \forall \varpi^{\mathbb{I}\Pi} \neg R(a_0x, a_1f[\emptyset, \varpi]) \rightarrow \forall y \neg R(a_0x, a_1y).$$

Now Π is countable (in \mathcal{M}), thus $\mathbb{I}\Pi$ is equipotent to $\mathbb{I}\mathbb{N}$; therefore $\mathbb{I}\Pi$ is countable (in \mathcal{N}) by theorem 22. Hence we have, for some bijection g from \mathbb{N} onto $\{f[\emptyset, \varpi] ; \varpi \varepsilon \mathbb{I}\Pi\}$:

$$\vdash \Vdash \forall n^{\text{int}} \neg R(a_0x, a_1g(n)) \rightarrow \forall y \neg R(a_0x, a_1y).$$

Thus, we have $\vdash \exists y R(a_0x, a_1y) \rightarrow \exists n^{\text{int}} R(a_0x, a_1g(n))$. It follows that g is a surjection from \mathbb{N} onto the image of R .

Q.E.D.

Remark. This shows that \mathcal{M}_{a_0} and \mathcal{M}_{a_1} cannot be both well founded : otherwise, their classes of ordinals would be isomorphic, which is excluded by Lemma 25.

Much more general results are given in [10, 12].

In order to simplify a little, we suppose that the ground model \mathcal{M} satisfy $V = L$ (this is not really necessary). The important point is the *principle of choice (PC)* : *there is a bijective functional between \mathcal{M} and On .*

Let On_{a_0} be the class of ordinals of \mathcal{M}_{a_0} , which is order isomorphic to the class $On^{\mathcal{N}_\varepsilon}$ of ordinals of \mathcal{N}_ε .

For each α in On_{a_0} , we can choose, by WOPC, an element W_α of the class of sets extensionally equivalent to V_α .

Lemma 26. $\mathcal{N} \models \exists X \forall x \exists y (\langle x \subset a_0y \sqcup a_1X \rangle = 1)$.

The functional $\alpha \mapsto a_1W_\alpha$ from On_{a_0} into \mathcal{M}_{a_1} has a countable image by lemma 25.

It follows that there exists α_0 such that $a_1W_\alpha = a_1W_{\alpha_0}$ for an unbounded class U of α in On_{a_0} . Therefore, we have $W_\alpha = a_0W_\alpha \sqcup a_1W_{\alpha_0}$ for α in U .

Now, we have $\forall x \exists \alpha \{U(\alpha), x \subset W_\alpha\}$ (in any model of ZF, every set is included in some V_α) and therefore, by lemma 10 : $\forall x \exists \alpha \{U(\alpha), \langle x \subset \text{Cl}[W_\alpha] \rangle = 1\}$. This gives finally : $\forall x \exists \alpha \{U(\alpha), \langle x \subset a_0 \text{Cl}[W_\alpha] \sqcup a_1 \text{Cl}[W_{\alpha_0}] \rangle = 1\}$. Set $X = a_1 \text{Cl}[W_{\alpha_0}]$.

Q.E.D.

Theorem 27. *There exists a generic extension $\mathcal{N}_\epsilon[G]$ of \mathcal{N}_ϵ which satisfies AC.*

By lemma 26, there is a surjective functional from $\mathcal{M}_{a_0} \times X$ onto the model \mathcal{N}_ϵ of ZF. Since $\mathcal{M}_{a_0} \models V = L$, there exists a surjective functional $\Psi : On \times X \rightarrow \mathcal{N}_\epsilon$ and therefore \mathcal{N}_ϵ is the union of the $Z_\alpha = \Psi(\{\alpha\} \times X)$ with α in On .

Let us consider, for each ordinal α of \mathcal{N}_ϵ , the equivalence relation on X associated with the restriction of $=_\epsilon$ to $\{\alpha\} \times X$. These equivalence relations form a set (included in $\mathcal{P}(X^2)$).

Thus, there exist an ordinal α_0 and for all β , a surjection $S_\beta : Z \rightarrow Z_\beta$ with $Z = \bigcup_{\alpha < \alpha_0} Z_\alpha$. Finally, using NECP (non extensional choice principle) we get a surjective functional from $On \times Z$ onto \mathcal{N}_ϵ (note that, by definition, Z is a set of \mathcal{N}_ϵ).

Destroying Z by means of a generic G on \mathcal{N}_ϵ , we see that $\mathcal{N}_\epsilon[G] \models AC$.

Q.E.D.

In the following, we take for (C, \leq) the set of conditions of \mathcal{N}_ϵ given by theorem 27. Applying the constructions of section 3, we obtain a r.a. \mathfrak{A}_1 and a generic model $\mathcal{N}_\epsilon[G]$ which satisfies AC by theorem 27. Therefore, we have $\mathcal{N}_\epsilon \models (\mathbf{1} \Vdash AC)$.

It follows that $\Vdash (\mathbf{1} \Vdash AC)$ and finally $\Vdash AC$ by theorem 18. Hence the :

Theorem 28. *There exists a proof like term Θ_{AC} of the r.a. \mathfrak{A}_0 such that $(\Theta_{AC}, \mathbf{1}) \Vdash AC$.*

More generally by theorem 19, it follows that for each axiom A of ZFC, there exists a proof-like term Θ_A of the r.a. \mathfrak{A}_0 such that $(\Theta_A, \mathbf{1}) \Vdash A$. Note that Θ_{AC} is the only one which contains the instructions γ, κ, e .

Example of computation with Θ_{AC}

Consider a function $f : \mathbb{N} \rightarrow 2$ such that we have a proof of $\exists n^{\text{int}}(f[n] = 0)$ in the theory $ZF + AC + \mathcal{E}$ where \mathcal{E} is a set of axioms of the form :

$$(\forall \vec{x} \in \mathbb{N}^k)(t_0[\vec{x}] = u_0[\vec{x}], \dots, t_{n-1}[\vec{x}] = u_{n-1}[\vec{x}] \rightarrow t[\vec{x}] = u[\vec{x}])$$

(cf. section 4, *Computing with the instruction γ*).

We denote by \mathcal{E}_ϵ the conjunction of the corresponding set, written in the language of \mathcal{N}_ϵ : $(\forall \vec{x} \in \mathbb{N}^k)(t_0[\vec{x}] =_\epsilon u_0[\vec{x}], \dots, t_{n-1}[\vec{x}] =_\epsilon u_{n-1}[\vec{x}] \rightarrow t[\vec{x}] =_\epsilon u[\vec{x}])$.

This proof gives a term Φ written with the only combinators B, C, I, K, W, cc such that : $\vdash \Phi : ZFC_0, \mathcal{E}_\epsilon \rightarrow (\exists n \in \mathbb{N})(f[n] =_\epsilon 0)$

for some finite conjunction ZFC_0 of axioms of ZFC.

Therefore, by theorems 19 and 28, we have in the r.a. \mathfrak{A}_1 :

$$(\Phi^* \Theta_{ZFC_0}, \mathbf{1}) \Vdash (\mathcal{E}_\epsilon \rightarrow (\exists n \in \mathbb{N})(f[n] =_\epsilon 0))$$

(remember that if $t \in \Lambda$, we obtain t^* replacing C, K, W, cc by C^*, K^*, W^*, cc^*).

Let $F \equiv \mathcal{E}_\epsilon \rightarrow (\exists n \in \mathbb{N})(f[n] =_\epsilon 0)$. By theorem 18, it follows that :

$$(\chi'_F)(\Phi^*) \Theta_{ZFC_0} \Vdash (\mathbf{1} \Vdash (\mathcal{E}_\epsilon \rightarrow (\exists n \in \mathbb{N})(f[n] =_\epsilon 0))).$$

Since \Vdash is a forcing on \mathcal{N}_ϵ , and F is arithmetical, we have :

$$ZF_\epsilon \vdash (\mathbf{1} \Vdash (\mathcal{E}_\epsilon \rightarrow (\exists n \in \mathbb{N})(f[n] =_\epsilon 0))) \rightarrow (\mathcal{E} \rightarrow \exists n^{\text{int}}(f[n] = 0)).$$

Hence, there is a proof-like term Ξ in the r.a. \mathfrak{A}_0 such that :

$(\Xi)(\chi'_F)(\Phi^*)\Theta_{ZFC_0} \Vdash \mathcal{E} \rightarrow \exists n^{\text{int}}(f[n] = 0)$.

Now we can apply the algorithm of section 4, *Computing with the instruction γ* .

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