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# On $\omega$-Regular Sets

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The investigation of the acceptional power of finite automata with respect to several notions of acceptance for $\omega$-sets, done in the literature, has exhibited six subclasses of the class of $\omega$-regular sets. These classes are well characterized by means of topology and set representation. First we give an overview of these results. The main aim of this paper is the investigation of further natural subclasses of the class of $\omega$-regular sets. We define such subclasses by three methods: by the structural complexity of accepting automata, by the $m$-reducibility with finite automata, and by topological difficulty. It turns out that these classifications coincide or are at least comparable to each other.

## 1. INTRODUCTION

The study of $\omega$-regular sets was started in Büchi (1960). In this paper and in the following ones (cf. Muller, 1963; McNaughton, 1966) the problem of which sets of $\omega$-sequences are actually acceptable by finite automata in any natural sense was investigated. These sets of $\omega$-sequences were shown to coincide with the so-called $\omega$-regular sets defined by $\omega$-regular expressions (cf. Büchi, 1960; McNaughton, 1966). Though there are notions of acceptance by which all $\omega$-regular sets can be accepted with deterministic finite automata, there are on the other hand notions of acceptance by which even with nondeterministic finite automata not all $\omega$-regular sets can be accepted. Naturally, as a next step in the study of $\omega$-regular sets which $\omega$-regular sets can be accepted by several types of finite automata and several notions of acceptance were investigated (cf. Hartmanis and Stearns, 1967; Landweber, 1969; Trachtenbrot and Barsdin, 1970; Hossley, 1970; Johnson, 1970; Choueka, 1974; Staiger and Wagner, 1974; Wagner, 1977). It turned out that the corresponding subclasses of the class of $\omega$-regular sets coincide with certain levels in the Borel hierarchy (with respect to the product topology) and moreover that each of these subclasses can be characterized by a special language-theoretical representation. Further, all essential closure and decidability problems for these subclasses were solved. Thus the study of these subclasses can be considered as finished. The related results are given in Section 3.

As a further step in the study of $\omega$-regular sets in the present paper we classify these sets into a hierarchy. Evidently, three ways seem to be possible



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here: classification by the complexity of accepting automata, classification by $m$-reducibility with finite automata (degrees of reducibility), and classification by topological difficulty. We investigate all these possibilities and it turns out that the corresponding classifications are compatible mutually as well as with the former known subclasses. In Section 4 we introduce the basic structural complexity measures, and in Section 5 we investigate the complexity classes with respect to them. There is a hierarchy of these classes (see Fig. 6) and it turns out that the basic measures are independent of the special automaton accepting a given $\omega$-regular set. Thus, these measures give invariants of the $\omega$-regular sets.

The study of $m$-reducibility by finite automata is started in Section 6. We distinguish $m$-reducibility by finite synchronous automata ($DS$-reducibility) and $m$-reducibility by finite asynchronous automata ($DA$-reducibility). The coarse structure of the set of $DS$- as well as $DA$-degrees is shown to be identical with the structure of the above complexity classes (see Fig. 6). Further (Section 7) all these complexity classes can be characterized in terms of topology. Thus our hierarchy of complexity classes can be considered as a refinement of the low levels of the Borel hierarchy.

In Section 8 the relations between the basic measures and other measures partially based on other notions of acceptance are investigated. We suggest a classification of all measures into two categories: those which describe only structural, i.e., qualitative phenomena of $\omega$-regular sets and thus are compatible with our basic measures, and those which describe also quantitative phenomena and thus are not compatible with our basic measures. Examples for both categories are given. For instance, the so-called Rabin index which was already investigated by other authors belongs to the first category.

The continuation of the study of $DS$- as well as $DA$-reducibility in Section 9 leads us to a refinement of the hierarchy of complexity classes. The structure of all $DS$- ($DA$-) degrees is completely known. For their complexity we state the fact that maximal chains of $DS$- ($DA$-) degrees are of type $\omega^\omega$. Furthermore all $DS$- ($DA$-) degrees can be characterized by automata-theoretical properties. For this the construction of the so-called derivation of a given automaton is very important. By the way, this construction essentially does not depend on the special automaton accepting a given $\omega$-regular set, i.e., if two automata are equivalent (with respect to the accepted $\omega$-regular set) then their derivations are equivalent too.

At last, in Section 10 we show that $DA$-reducibility and $m$-reducibility by continuous functions are identical for $\omega$-regular sets. Consequently, membership to any $DA$-degree is a topological property. This result is a striking proof for the naturalness of this classification of $\omega$-regular sets.

## 2. Definitions

Let $N$ be the set of natural numbers. We use $\mathfrak{P}(M)$ to denote the set of all subsets of the set $M$. Set inclusion is indicated by $\subseteq$, proper inclusion by $\subset$, and card $M$ is the cardinality of $M$. For $T$, $T_1$, $T_2 \subseteq \mathfrak{P}(M)$ we define

$$\hat{T} \underset{\text{df}}{=} \{\bar{A}; A \in T\},$$

$$T_1 \triangle T_2 \underset{\text{df}}{=} \{A \cap B; A \in T_1 \wedge B \in T_2\},$$

$$T_1 \bigtriangledown T_2 \underset{\text{df}}{=} \{A \cup B; A \in T_1 \wedge B \in T_2\},$$

$$\underset{n}{\triangle} T \underset{\text{df}}{=} \underbrace{T \triangle T \triangle \cdots \triangle T}_{n \text{ times}} \qquad \text{for} \quad n \geqslant 1,$$

$$\underset{0}{\triangle} T \underset{\text{df}}{=} \{M\},$$

$$\underset{n}{\bigtriangledown} T \underset{\text{df}}{=} \underbrace{T \bigtriangledown T \bigtriangledown \cdots \bigtriangledown T}_{n \text{ times}} \qquad \text{for} \quad n \geqslant 1,$$

$$\underset{0}{\bigtriangledown} T \underset{\text{df}}{=} \{\varnothing\}.$$

Further let $\mathscr{B}(T)$ be the Boolean closure of $T$. The fact that $f$ is a function from (out) $M$ into $N$ is indicated by $f: M \mapsto N$ ($f: M \to N$).

For a nonempty finite alphabet $X$ we denote by $X^*$ the set of all finite sequences (words), by $X^\omega$ the set of all infinite sequences ($\omega$-sequences) and by $e$ the empty word. Because of card $X^\omega = 1$ for card $X = 1$ we propose card $X \geqslant 2$ and furthermore $X = \{0, 1, 2,..., \text{card } X - 1\}$.

If $w \in X^*$ and $p \in X^* \cup X^\omega$ then $w \cdot p$ or equivalently $wp$ is the concatenation of $w$ and $p$. For sets $W \subseteq X^*$ and $P \subseteq X^* \cup X^\omega$ let $W \cdot P =_{\text{df}} \{wp; w \in W \wedge p \in P\}$, $W^0 =_{\text{df}} \{e\}$, $W^{n+1} =_{\text{df}} W^n \cdot W$, $W^+ =_{\text{df}} \bigcup_{n=1}^{\infty} W^n$, $W =_{\text{df}} W^+ \cup W^0$ and, for $e \notin W$, $W^\omega =_{\text{df}} W \cdot W \cdot W \cdot \cdots$. For simplicity we sometimes omit the set braces in the case of singletons. For instance, instead of $(\{0\} \cup \{1\}^*)^\omega \cup \{0\}^\omega$ we write $(0 \cup 1^*)^\omega \cup 0^\omega$.

The initial word relation is indicated by $\sqsubseteq$, i.e., $w \sqsubseteq p \Leftrightarrow_{\text{df}} \exists q(q \in X^* \cup X^\omega \wedge w \cdot q = p)$ for $w \in X^*$ and $p \in X^* \cup X^\omega$, furthermore $w \sqsubset p \Leftrightarrow_{\text{df}} w \sqsubseteq p$ and $w \neq p$. The set of initial words is defined as $A(p) =_{\text{df}} \{w; w \sqsubset p\}$ for $p \in X^* \cup X^\omega$ and as $A(P) =_{\text{df}} \bigcup_{p \in P} A(p)$ for $P \subseteq X^* \cup X^\omega$.

The set of minimal words is defined as $\min W =_{\text{df}} \{w; A(w) \cap W = \{w\}\}$ for $W \subseteq X^*$. Furthermore, for $W \subseteq X^*$

$$\text{ls } W \underset{\text{df}}{=} \{\xi; \xi \in X^\omega \wedge A(\xi) \subseteq A(W)\},$$

$$\mathfrak{G}_\delta(W) \underset{\text{df}}{=} \{\xi; A(\xi) \cap W \text{ infinite}\}, \qquad \text{and}$$

$$\mathfrak{F}_\sigma(W) \underset{\text{df}}{=} \{\xi; A(\xi) \cap W \text{ finite}\}.$$

For $w \in X^*$ we denote by $|w|$ the length of $w$ and by $w(n)$ the $n$th symbol of $w$, i.e., $w = w(1)\,w(2) \cdots w(|w|)$. For $\xi \in X^\omega$ we denote by $\xi(n)$ the $n$th symbol of $\xi$, i.e., $\xi = \xi(1)\,\xi(2) \cdots$. Further let

$$\begin{aligned} \xi_n{}^m &\underset{\text{df}}{=} \xi(n+1)\,\xi(n+2) \cdots \xi(m) \qquad && \text{if} \quad n < m \\ &\underset{\text{df}}{=} e && \text{else} \end{aligned}$$

and

$$\xi_n{}^\infty \underset{\text{df}}{=} \xi(n+1)\,\xi(n+2) \cdots .$$

The set of all symbols which occur (infinitely often) in $\xi \in X^\omega$ is denoted by $E(\xi)$ $(U(\xi))$.

We shall use finite automata as acceptors as well as transducers. A nondeterministic partial finite $\omega$-acceptor ($\omega$-NPFA) is a system $\mathfrak{A} = [X, Z, f, z_0, \mathfrak{Z}]$, where $X$ is the input alphabet, $Z$ is a finite set of states with the initial state $z_0 \in Z$, $f\colon Z \times X \to \mathfrak{P}(Z)$ is the transition function, and $\mathfrak{Z} \subseteq \mathfrak{P}(Z)$ the system of final sets. $\mathfrak{A}$ is said to be full defined if $f\colon Z \times X \mapsto \mathfrak{P}(Z)$($\omega$-NFA) as well as deterministic if $f\colon Z \times X \to Z$($\omega$-DPFA, $\omega$-DFA). The transition function $f$ can be extended to $f\colon Z \times X^* \to \mathfrak{P}(Z)$ by $f(z, e) = \{z\}$, $f(z, wx) = \bigcup_{z' \in f(z,w)} f(z', x)$ for $w \in X^*$, $x \in X$.

For $\omega$-NPFA, $\mathfrak{A}$ and $\mathfrak{A}'$ means always $\mathfrak{A} = [X, Z, f, z_0, \mathfrak{Z}]$ and $\mathfrak{A}' = [X, Z', f', z_0', \mathfrak{Z}']$.

An $\omega$-NPFA $\mathfrak{A}$ generates a function $\Phi_{\mathfrak{A}}: X^\omega \to \mathfrak{P}(Z^\omega)$ as follows

$$\begin{aligned} \Phi_{\mathfrak{A}}(\xi) \underset{\text{df}}{=} \{\eta;\ \eta \in Z^\omega \wedge \eta(1) \in f(z_0, \xi(1)) \\ \wedge\ \forall n(n \geqslant 1 \to \eta(n+1) \in f(\eta(n), \xi(n+1)))\}. \end{aligned}$$

$\Phi_{\mathfrak{A}}(\xi)$ is the set of all possible state sequence if $\xi$ is put in. A sequence $\eta \in \Phi_{\mathfrak{A}}(\xi)$ is said to be a run of $\mathfrak{A}$ on $\xi$. If $\mathfrak{A}$ is an $\omega$-DFA then $\Phi_{\mathfrak{A}}(\xi) = \{\eta\}$ for a suitable $\eta \in Z^\omega$. In this case we write also $\Phi_{\mathfrak{A}}(\xi) = \eta$.

A deterministic asynchronous finite transducer is a system $\mathfrak{B} = [X, Z, f, g, z_0]$, where $X$ is the input alphabet, $Z$ is a finite set of states with the initial state $z_0 \in Z$, $f\colon Z \times X \mapsto Z$ the transition function and $g\colon Z \times X \mapsto X^*$ the output function. $\mathfrak{B}$ is said to be synchronous (DSFT) if $g\colon Z \times X \mapsto X$. The transition function $f$ can be extended to $f\colon Z \times X^* \mapsto Z$ by $f(z, e) = z$, $f(z, wx) = f(f(z, w), x)$ for $w \in X^*$, $x \in X$; and the output function $g$ can be extended to $g\colon Z \times X^* \mapsto X^*$ by $g(z, e) = e$, $g(z, wx) = g(z, w) \cdot g(f(z, w), x)$ for $w \in X^*$, $x \in X$. The global output function $\Phi_{\mathfrak{B}}\colon X^\omega \mapsto X^+ \cup X^\omega$ of $\mathfrak{B}$ is defined by

$$\Phi_{\mathfrak{B}}(\xi) \underset{\text{df}}{=} g(z_0, \xi(1)\,g(f(z_0, \xi_0{}^1), \xi(2))\,g(f(z_0, \xi_0{}^2), \xi(3)) \cdots .$$

In what follows only transducers $\mathfrak{B}$ with $\Phi_{\mathfrak{B}}\colon X^\omega \mapsto X^\omega$ are of interest. Therefore we introduce the abbrevation DAFT only for deterministic asynchronous finite transducers $\mathfrak{B}$ with $\Phi_{\mathfrak{B}}\colon X^\omega \mapsto X^\omega$. Clearly, every DSFT is a DAFT.

## 3. Basic Notions and Former Results

The acceptance of infinite sequences by finite automata has been studied first in Büchi (1960). He used the following notion of acceptance for $\omega$-NFA $\mathfrak{A}$ with a single final set $Z'$: A sequence $\xi \in X^\omega$ is said to be accepted by $\mathfrak{A}$ in the sense of Büchi iff at least one state of $Z'$ occurs infinitely many often in at least one run of $\mathfrak{A}$ on $\xi$, i.e., iff

$$\exists \eta(\eta \in \Phi_{\mathfrak{A}}(\xi) \wedge U(\eta) \cap Z' \neq \varnothing).$$

Büchi found out that in this sense exactly those sets of $\omega$-sequences are acceptable which can be represented as a finite union of sets $W \cdot V^\omega$, where $W, V \subseteq X^*$ are regular sets of words. For such sets the term $\omega$-regular set was introduced in McNaughton (1966). With $R$ we denote the class of all $\omega$-regular sets[1]. Thus, we have Büchi's result as

Theorem 1 (Büchi, 1960). *A set of $\omega$-sequences is acceptable by $\omega$-NFA in the sense of Büchi iff it is $\omega$-regular.*

Further the question was of interest whether, in conformity with the results for regular sets of words, all $\omega$-regular sets can be accepted by deterministic finite automata. The first step in this direction was done in Muller (1963). He considered the following notion of acceptance for $\omega$-DFA $\mathfrak{A}$: A sequence $\xi \in X^\omega$ is said to be accepted by $\mathfrak{A}$ in the sense of Muller iff the set of all states which occur infinitely often in the only run of $\mathfrak{A}$ on $\xi$ is a final set, i.e., iff $U(\Phi_{\mathfrak{A}}(\xi)) \in \mathfrak{z}$.

However, Muller's proof of the fact that in this manner exactly the $\omega$-regular sets can be accepted contains an error. In 1966 McNaughton gave a correct proof of this theorem.

Theorem 2 (McNaughton, 1966). *A set of $\omega$-sequences is acceptable by $\omega$-DFA in the sense of Muller iff it is $\omega$-regular.*

Naturally, now the following questions arose: What sets can be accepted in the sense of Büchi by $\omega$-DFA, in the sense of Muller by $\omega$-NFA and by other natural notions of acceptance and other types of automata, respectively.

Essentially, acceptance by the conditions "$U(\ ) \not\subseteq \bar{Z}'$" (this is equivalent to "$U(\ ) \cap Z' \neq \varnothing$"), "$U(\ ) \subseteq Z'$," "$U(\ ) = Z'$," "$E(\ ) \not\subseteq Z'$," "$E(\ ) \subseteq Z'$," and "$E(\ ) = Z'$," and by automata of the types $\omega$-DFA, $\omega$-NFA, $\omega$-DPFA, and $\omega$-NPFA has been investigated by several authors.

For $\alpha \in \{E, U\}$ and $\sigma \in \{\not\subseteq, \subseteq, =\}$ the set of $\omega$-sequences accepted by the acceptor $\mathfrak{A}$ in the sense $(\alpha, \sigma)$ is defined as

$$T_\sigma^{\alpha}(\mathfrak{A}) \underset{\mathrm{df}}{=} \{\xi; \exists Z'(Z' \in \mathfrak{Z} \wedge \alpha(\Phi_{\mathfrak{A}}(\xi))\, \sigma Z')\}$$

[1] Here and in the notation for other classes of sets of $\omega$-sequences we omit the reference to the alphabet $X$, which is fixed once and for all.

if $\mathfrak{A}$ is an $\omega$-DPFA and as

$$T_\sigma^\alpha(\mathfrak{A}) \underset{\text{df}}{=} \{\xi; \exists Z' \exists \eta (Z' \in \mathfrak{Z} \wedge \eta \in \Phi_{\mathfrak{A}}(\xi) \wedge \alpha(\eta)\, \sigma Z')\}$$

if $\mathfrak{A}$ is an $\omega$-NPFA. Since $(U, =)$-acceptance is the most important in this paper we shall use very often $T$ instead of $T_=^U$.

Clearly, $(U, =)$-acceptance is identical with acceptance in the sense of Muller and $(U, \not\subseteq)$ acceptance is identical with acceptance in the sense of Büchi because of

$$\exists Z'(Z' \in \mathfrak{Z} \wedge U(\ ) \not\subseteq Z') \Leftrightarrow \exists Z'(Z' \in \mathfrak{Z} \wedge U(\ ) \cap \overline{Z'} \neq \varnothing)$$
$$\Leftrightarrow U(\ ) \cap \bigcup_{Z' \in \mathfrak{Z}} \overline{Z'} \neq \varnothing .$$

It turned out that the accepting power of this notions of acceptance can be very well characterized in terms of topology. It is a well-known fact that the metric $\rho$,

$$\rho(\xi, \xi') \underset{\text{df}}{=} \frac{1}{\min\{n; \xi(n) \neq \xi'(n)\}} \qquad \text{if} \quad \xi \neq \xi'$$
$$\underset{\text{df}}{=} 0 \qquad \text{if} \quad \xi = \xi'$$

makes $X^\omega$ to a metric space which is homeomorphic to Cantor's discontinuum.

Evidently, continuous functions can be characterized as follows.

LEMMA 1. *A function $\Phi$: $X^\omega \mapsto X^\omega$ is continuous iff*

$$\forall n\, \exists k(\xi_0{}^k = \eta_0{}^k \rightarrow \Phi(\xi)_0^n = \Phi(\eta)_0^n)$$

*for any $\xi, \eta \in X^\omega$.*

Consequently $\Phi_{\mathfrak{A}}$ is continuous for a DAFT $\mathfrak{A}$:

LEMMA 2. *Let $\mathfrak{A}$ be a* DAFT.

(1) *$\Phi_{\mathfrak{A}}$ is a continuous function.*

(2) *If $A \subseteq X^\omega$ is an open (closed, $G_\delta$- , $F_\sigma$- , $\omega$-regular) set then $\Phi_{\mathfrak{A}}^{-1}(A)$ is an open (closed, $G_\delta$- , $F_\delta$- , $\omega$-regular) set also (for $\omega$-regularity see Rabin* (1969)).

For an $\omega$-NPFA $\mathfrak{A}$ the function $\Phi_{\mathfrak{A}}$ is not continuous in general. However, some topological properties are preserved by $\Phi_{\mathfrak{A}}^{-1}$.

LEMMA 3. *Let $\mathfrak{A}$ be an $\omega$*-NPFA.

*If $A \subseteq X^\omega$ is a closed ($F_\sigma$- , $\omega$-regular) set then $\Phi_{\mathfrak{A}}^{-1}(A)$ is a closed ($F_\sigma$- , $\omega$-regular) set also (cf. Staiger and Wagner*, 1974; *Wagner*, 1976).

In 1969 Büchi and Landweber showed that $\omega$-regular sets are in very low stages of the Borel hierarchy. Somewhat later Trachtenbrot made the same observation.

THEOREM 3 (Büchi and Landweber, 1969; Trachtenbrot, 1970). *Every ω-regular set is a $G_{\delta\sigma}$- as well as an $F_{\sigma\delta}$-set.*

That this result is the best possible has been shown in Landweber (1969) and independently in Thomas (1969).

By $G^R$, $F^R$, $G_\delta{}^R$, and $F_\sigma{}^R$ we denote the classes of ω-regular open sets, ω-regular closed sets, ω-regular $G_\delta$-sets, and ω-regular $F_\sigma$-sets, respectively.

The next theorem gives a characterization of the acceptational power of all notions of acceptance mentioned above.

THEOREM 4. *Table* I *is interpretable as follows: For instance, $G_\delta{}^R$ in the crossing of row* ω-DFA *and column* $(U, \not\subseteq)$ *means that a set of ω-sequences is acceptable by an* ω-DFA *in the sense* $(U, \not\subseteq)$ *iff it is an ω-regular $G_\delta$-set.*

TABLE I

| | $(E, \subseteq)$ | $(E, \not\subseteq)$ | $(E, =)$ | $(U, \subseteq)$ | $(U, \not\subseteq)$ | $(U, =)$ |
|---|---|---|---|---|---|---|
| ω-DFA | $F^R$ [a] | $G^R$ [a] | $G_\delta{}^R \cap F_\sigma{}^R$ [b] | $F_\sigma{}^R$ [a] | $G_\delta{}^R$ [a] | $R$ [c] |
| ω-DPFA | $F^R$ [d] | $G^R \triangle F^R$ [d] | $G_\delta{}^R \cap F_\sigma{}^R$ [d] | $F_\sigma{}^R$ [d] | $G_\delta{}^R$ [d] | $R$ [e] |
| ω-NFA | $F^R$ [f] | $G^R$ [f] | $F_\sigma{}^R$ [b] | $F_\sigma{}^R$ [f] | $R$ [g] | $R$ [e] |
| ω-NPFA | $F^R$ [d] | $F_\sigma{}^R$ [h] | $F_\sigma{}^R$ [d] | $F_\sigma{}^R$ [d] | $R$ [e] | $R$ [e] |

[a] Landweber (1969).
[b] Staiger and Wagner (1974).
[c] McNaughton (1966).
[d] Wagner (1976).
[e] Büchi (1960+).[2]
[f] Hossley (1970).
[g] Büchi (1960).
[h] Trachtenbrot and Barsdin (1970).

As in the case of Büchi-acceptance in several other cases a single final set is sufficient (see Staiger and Wagner, 1974).

*Remark* 1. (1) ω-DFA with a single final set are sufficient for $(E, \subseteq)$-acceptance of ω-regular closed sets.

(2) ω-DFA with a single final set which is a singleton are sufficient for $(E, \not\subseteq)$-acceptance of ω-regular open sets.

(3) ω-DFA with a single final set is sufficient for $(U, \subseteq)$-acceptance of ω-regular $F_\sigma$-sets.

[2] Buchi 60+ indicates that this result follows directly from Buchi (1960).

Now some results about the closure of above classes as well as the relations between them.

THEOREM 5. (1) *$G^R$, $F^R$, $G_\delta^R \cap F_\sigma^R$, $G_\delta^R$, $F_\sigma^R$, and $R$ are pairwise different* (*Landweber*, 1969).

(2) *$G^R$, $F^R$, $G_\delta^R \cap F_\sigma^R$, $G_\delta^R$, $F_\sigma^R$ and $R$ are closed under union and intersection* (*cf. Landweber*, 1969; *Johnson*, 1970; *Hossley*, 1970).

(3) *$G_\delta^R \cap F_\sigma^R$ and $R$ are closed under complementation* (*for $R$ see Büchi*, 1960). *$G^R$, $F^R$, $G_\delta^R$, and $F_\sigma^R$ are not* (*cf. Landweber*, 1969).

(4) $\widehat{G^R} = F^R$ *and* $\widehat{G_\sigma^R} = F_\sigma^R$.

(5) $\mathfrak{B}(G^R) = \mathfrak{B}(F^R) = G_\delta^R \cap F_\sigma^R$ (*cf. Staiger and Wagner*, 1974).

(6) $\mathfrak{B}(G_\delta^R) = \mathfrak{B}(F_\sigma^R) = R$ (*cf. Landweber*, 1969).

*Furthermore, each of the above classes has special set-theoretical representations.*

THEOREM 6. *Table* II *is interpretable as follows*: *For instance, the fourth row means that a set $A \subseteq X^\omega$ is an $\omega$-regular $G_\delta$-set iff there is a regular set $W \subseteq X^*$ such that $A = \mathfrak{G}_\delta(W)$.*

TABLE II

| Type | Representation | Conditions | References |
|---|---|---|---|
| $G^R$ | $W \cdot X^\omega$ | $W$ regular | Staiger (1977) |
| $F^R$ | $\operatorname{ls} W$ | $W$ regular | Staiger (1977) |
| $G^R \cap F^R$ | $\bigcup_{i=1}^{n} (\min W_i) \cdot \operatorname{ls} V_i$ | $W_i$, $V_i$ regular | Staiger and Wagner (1974) |
| $G_\delta^R$ | $\mathfrak{G}_\delta(W)$ | $W$ regular | Staiger and Wagner (1974) |
| $F_\sigma^R$ | $\mathfrak{F}_\sigma(W)$ | $W$ regular | Staiger and Wagner (1974) |
| $G_\delta^R$ | $\bigcup_{i=1}^{n} (\min W_i) \cdot (\min V_i)^\omega$ | $W_i$, $V_i$ regular | Choueka (1974), Staiger and Wagner (1974) |
| $F_\sigma^R$ | $\bigcup_{i=1}^{n} W_i \cdot \operatorname{ls} V_i$ | $W_i$, $V_i$ regular | Staiger and Wagner (1974) |
| $R$ | $\bigcup_{i=1}^{n} W_i \cdot (\min V_i)^\omega$ | $W_i$, $V_i$ regular | Choueka (1974) |
| $R$ | $\bigcup_{i=1}^{n} W_i \cdot V_i^\omega$ | $W_i$, $V_i$ regular | definition |

At last some decidability results.

THEOREM 7. *For any type of $\omega$-acceptors, for any type* $(\alpha, \sigma)$ *of acceptance, and for any* $P \in \{G^R, F^R, G_\sigma{}^R \cap F_\sigma{}^R, G_\delta{}^R, F_\sigma{}^R\}$ *there is an algorithm for deciding whether the set accepted by an $\omega$-acceptor of this type in the sense* $(\alpha, \sigma)$ *is in* $P$ *(cf. Landweber,* 1969).

## 4. THE MEASURES $m^+$, $m^-$, $n^+$, $n^-$ OF STRUCTURAL COMPLEXITY

In this section we introduce four new measures of structural complexity of $\omega$-DFAs, give examples for their computation, and state some simple properties of them. Later it becomes clear that these structural measures do not depend on the special automaton accepting a given set $A$ of $\omega$-sequences, but they depend only on the set $A$ itself. Thus these measures give us invariants of the $\omega$-regular sets.

First we define for an $\omega$-DFA $\mathfrak{A}$ the set of "essential sets"

$$M(\mathfrak{A}) \underset{\mathrm{df}}{=} \{Z'; T([X, Z, f, z_0, \{Z'\}]) \neq \varnothing\}$$

which can be divided into "accepting" and "rejecting" ones:

$$M^+(\mathfrak{A}) \underset{\mathrm{df}}{=} M(\mathfrak{A}) \cap \mathfrak{Z}, \qquad M^-(\mathfrak{A}) \underset{\mathrm{df}}{=} M(\mathfrak{A}) \cap \overline{\mathfrak{Z}}.$$

LEMMA 4. *The following conditions are equivalent for* $Z' \subseteq Z$

(1) *$Z'$ is essential.*

(2) *There are $z \in Z'$ and $w_1, w_2 \in X^*$ such that $f(z_0, w_1) = f(z, w_2) = z$ and $\{f(z, w); w \subseteq w_2\} = Z'$.*

(3) *For any $z \in Z'$ there are $w_1, w_2 \in X^*$ such that $f(z_0, w_1) = f(z, w_2) = z$ and $\{f(z, w), w \subseteq w_2\} = Z'$.*

Now we look for alternating chains of accepting and rejecting sets with respect to set inclusion (for short: chains). We distinguish chains beginning with an accepting set (+chains) and chains beginning with a rejecting set (−chains)

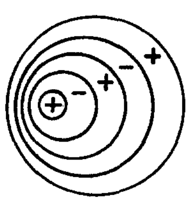


FIG. 1. A +chain.

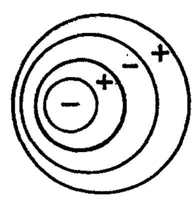


FIG. 2. A −chain.

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(see Figs. 1 and 2 in which + denotes an accepting set and — denotes a rejecting one).

For $m \geqslant 1$ and $Z' \subseteq Z$ let $Z'$ be in $M_m{}^+(\mathfrak{A})$ ($M_m{}^-(\mathfrak{A})$) iff $Z'$ is the last set of a +chain (—chain) of length $m$. More formally,

$$M_1{}^+(\mathfrak{A}) \underset{\mathrm{df}}{=} M^+(\mathfrak{A}), \qquad M_1{}^-(\mathfrak{A}) \underset{\mathrm{df}}{=} M^-(\mathfrak{A}),$$

$$M_{2m}^+(\mathfrak{A}) \underset{\mathrm{df}}{=} \{Z'; Z' \in M^-(\mathfrak{A}) \wedge \exists Z''(Z'' \in M_{2m-1}^+ \wedge Z'' \subset Z')\},$$

$$M_{2m}^-(\mathfrak{A}) \underset{\mathrm{df}}{=} \{Z'; Z' \in M^+(\mathfrak{A}) \wedge \exists Z''(Z'' \in M_{2m-1}^- \wedge Z'' \subset Z')\},$$

$$M_{2m+1}^+(\mathfrak{A}) \underset{\mathrm{df}}{=} \{Z'; Z' \in M^+(\mathfrak{A}) \wedge \exists Z''(Z'' \in M_{2m}^+(\mathfrak{A}) \wedge Z'' \subset Z')\},$$

$$M_{2m+1}^-(\mathfrak{A}) \underset{\mathrm{df}}{=} \{Z'; Z' \in M^-(\mathfrak{A}) \wedge \exists Z''(Z'' \in M_{2m}^-(\mathfrak{A}) \wedge Z'' \subset Z')\}.$$

The length of maximal +chains (—chains) is denoted by $m^+(\mathfrak{A})$ ($m^-(\mathfrak{A})$), i.e.,

$$m^+(\mathfrak{A}) \underset{\mathrm{df}}{=} \max(\{0\} \cup \{m; M_m{}^+(\mathfrak{A}) \neq \varnothing\}), \qquad \text{and}$$

$$m^-(\mathfrak{A}) \underset{\mathrm{df}}{=} \max(\{0\} \cup \{m; M_m{}^-(\mathfrak{A}) \neq \varnothing\}).$$

The numbers $m^+(\mathfrak{A})$ and $m^-(\mathfrak{A})$ are characteristics of the structure of the automaton $\mathfrak{A}$.

Now we state some simple properties of the sets $M_m{}^+(\mathfrak{A})$ and $M_m{}^-(\mathfrak{A})$ as well as the measures $m^+$ and $m^-$.

LEMMA 5. (1) *$Z' \in M_m{}^+ \Leftrightarrow$ there exist $Z_1, ..., Z_m \subseteq Z$ such that $Z_m = Z'$, $Z_i \subset Z_{i+1}$, and*

$$Z_i \begin{cases} \in M^+, & \textit{if } i \textit{ odd},[3] \\ \in M^-, & \textit{if } i \textit{ even}. \end{cases}$$

(2) *$Z' \in M_m{}^- \Leftrightarrow$ there exist $Z_1, ..., Z_m \subseteq Z$ such that $Z_m = Z'$, $Z_i \subset Z_{i+1}$, and*

$$Z_i \begin{cases} \in M^+, & \textit{if } i \textit{ even}. \\ \in M^-, & \textit{if } i \textit{ odd}. \end{cases}$$

(3) $M_{2m}^+ \subseteq M^-$, $M_{2m+1}^+ \subseteq M^+$.

(4) $M_{2m}^- \subseteq M^+$, $M_{2m+1}^- \subseteq M^-$.

(5) $M_{m+2}^+ \subseteq M_{m+1}^- \subseteq M_m{}^+$.

(6) $M_{m+2}^- \subseteq M_{m+1}^+ \subseteq M_m{}^-$.

[3] If no confusion is possible we omit the reference to the automaton $\mathfrak{A}$.

(7) $M_m^{\ +} \cap M_m^{\ -} = \varnothing$, or more generally,

(8) $M_{m_1}^+ \cap M_{m_2}^- = \varnothing \Leftrightarrow m_1 + m_2$ even,

(9) $M_{m_1}^+ \cap M_{m_2}^+ = \varnothing \Leftrightarrow M_{m_1}^- \cap M_{m_2}^- = \varnothing \Leftrightarrow m_1 + m_2$ odd,

(10) $M_m^{\ +} = \varnothing$ implies $M_{m+1}^+ = \varnothing$.

(11) $M_m^{\ -} = \varnothing$ implies $M_{m+1}^- = \varnothing$.

(12) $M_m^{\ +}([X, Z, f, z_0, \mathfrak{P}(Z)\backslash \mathfrak{Z}]) = M_m^{\ -}(\mathfrak{A})$.

(13) $M_m^{\ -}([X, Z, f, z_0, \mathfrak{P}(Z)\backslash \mathfrak{Z}]) = M_m^{\ +}(\mathfrak{A})$.

(14) *If* $Z' \subseteq Z''$ *and* $Z'' \in M$ *then*
$\max(\{0\} \cup \{m; Z' \in M_m^{\ +}\}) \leqslant \max(\{0\} \cup \{m; Z'' \in M_m^{\ +}\})$.

(15) *If* $Z' \subseteq Z''$ *and* $Z'' \in M$ *then*
$\max(\{0\} \cup \{m; Z' \in M_m^{\ -}\}) \leqslant \max(\{0\} \cup \{m; Z'' \in M_m^{\ -}\})$.

(16) $|\, m^+(\mathfrak{A}) - m^-(\mathfrak{A})| \leqslant 1$.

(17) $m^+([X, Z, f, z_0, \mathfrak{P}(Z)\backslash \mathfrak{Z}]) = m^-(\mathfrak{A})$.

(18) $m^-([X, Z, f, z_0, \mathfrak{P}(Z)\backslash \mathfrak{Z}]) = m^+(\mathfrak{A})$.

(19) $m^+(\mathfrak{A})$ *and* $m^-(\mathfrak{A})$ *are finite numbers.*

(20) *There exist algorithms for computing* $m^+(\mathfrak{A})$ *and* $m^-(\mathfrak{A})$ *for given* $\mathfrak{A}$.

The following example for the above definitions is useful also for our further investigations.

EXAMPLE 1. For $i = 1, 2, 3$, $m \geqslant 1$, $n \geqslant 1$, $k \geqslant 0$, $l \geqslant 0$ let $\mathfrak{A}_i(m, n, k, l)$ be that $\omega$-DFA with the transition graph represented in Fig. 2 (all symbols $\rho \geqslant 1$ are thought to be treated as 1), the initial state $s_0$ and the set system $\mathfrak{Z}_i(m, n)$, where

$$\mathfrak{Z}_1(m, n) \underset{\mathrm{df}}{=} \{\{z_0^{\nu}, z_1^{\nu}, \ldots, z_\mu^{\nu}\}; 1 \leqslant \mu \leqslant m \wedge \nu \leqslant n \wedge \nu + \mu \text{ odd}\},$$

$$\mathfrak{Z}_2(m, n) \underset{\mathrm{df}}{=} \{\{\hat{z}_0^{\nu}, \hat{z}_1^{\nu}, \ldots, \hat{z}_\mu^{\nu}\}; 1 \leqslant \mu \leqslant m \wedge \nu \leqslant n \wedge \nu + \mu \text{ even}\},$$

$$\mathfrak{Z}_3(m, n) \underset{\mathrm{df}}{=} \mathfrak{Z}_1(m, n) \cup \mathfrak{Z}_2(m, n).$$

As the essential sets we have

$$M(\mathfrak{A}_1(m, n, k, l)) = \{\{z_0^{\nu}, z_1^{\nu}, \ldots, z_\mu^{\nu}\}; 1 \leqslant \mu \leqslant m \wedge \nu \leqslant n\}$$
$$\cup \{\{\hat{z}_0^{\nu}, \hat{z}_1^{\nu}, \ldots, \hat{z}_\mu^{\nu}\}; 1 \leqslant \mu \leqslant m \wedge \nu \leqslant n\} \cup \{\{s\}\}$$

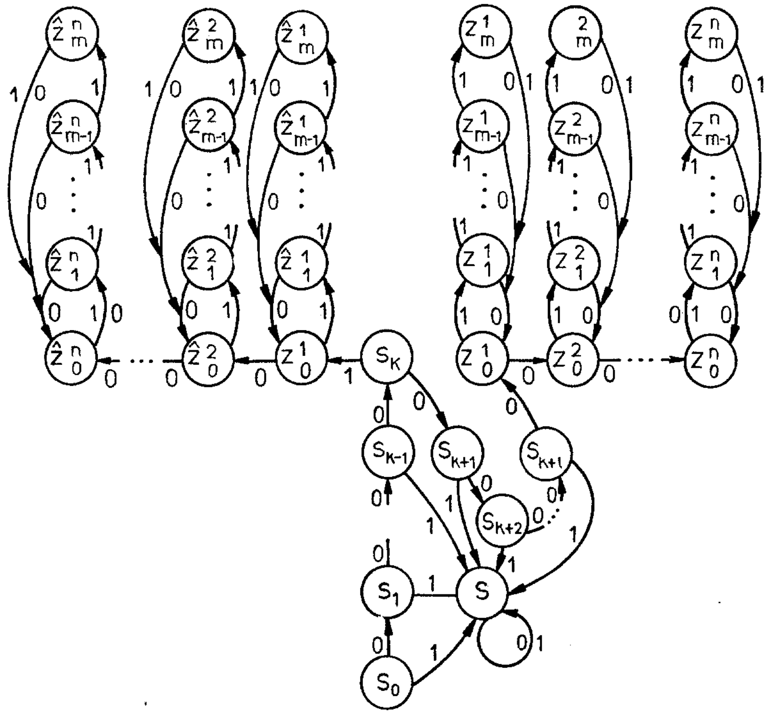

FIG. 3. The transition graph of $\mathfrak{A}_i(m, n, k, l)$.

and further for $i = 1$

$$M_1^+(\mathfrak{A}_1(m, n, k, l)) = M^+(\mathfrak{A}_1(m, n, k, l)) = \mathfrak{Z}_1(m, n),$$

$$M_1^-(\mathfrak{A}_1(m, n, k, l)) = M^-(\mathfrak{A}_1(m, n, k, l)) = M(\mathfrak{A}_1(m, n, k, l))\backslash\mathfrak{Z}_1(m, n),$$

$$M_2^-(\mathfrak{A}_1(m, n, k, l)) = \{\{z_0^\nu, \ldots, z_\mu^\nu\};\ 2 \leqslant \mu \leqslant m \wedge \nu \leqslant n \wedge \mu + \nu \text{ odd}\},$$

$$M_2^+(\mathfrak{A}_1(m, n, k, l)) = \{\{z_0^\nu, \ldots, z_\mu^\nu\};\ 2 \leqslant \mu \leqslant m \wedge \nu \leqslant n \wedge \mu + \nu \text{ even}\},$$

$$M_3^+(\mathfrak{A}_1(m, n, k, l)) = \{\{z_0^\nu, \ldots, z_\mu^\nu\};\ 3 \leqslant \mu \leqslant m \wedge \nu \leqslant n \wedge \mu + \nu \text{ even}\},$$

$$M_3^-(\mathfrak{A}_1(m, n, k, l)) = \{\{z_0^\nu, \ldots, z_\mu^\nu\};\ 3 \leqslant \mu \leqslant m \wedge \nu \leqslant n \wedge \mu + \nu \text{ even}\},$$

$$\vdots$$

$$M_m^+(\mathfrak{A}_1(m, n, k, l)) = \{\{z_0^\nu, \ldots, z_m^\nu\};\ \nu \leqslant n \wedge \nu \text{ even}\},$$

$$M_m^-(\mathfrak{A}_1(m, n, k, l)) = \{\{z_0^\nu, \ldots, z_m^\nu\};\ \nu \leqslant n \wedge \nu \text{ odd}\},$$

$$M_{m+1}^+(\mathfrak{A}_1(m, n, k, l)) = M_{m+1}^-(\mathfrak{A}_1(m, n, k, l)) = \varnothing .$$

Therefore,

$$\begin{aligned} m^+(\mathfrak{A}_1(m, n, k, l)) &= m, && \text{if} \quad n \geqslant 2, \\ &= m - 1, && \text{if} \quad n = 1, \\ m^-(\mathfrak{A}_1(m, n, k, l)) &= m. \end{aligned}$$

The reader is invited to experiment with the automata $\mathfrak{A}_2(m, n, k, l)$ and $\mathfrak{A}_3(m, n, k, l)$ for seeing that

$$\begin{aligned} m^+(\mathfrak{A}_2(m, n, k, l)) &= m, \\ m^-(\mathfrak{A}_2(m, n, k, l)) &= m, && \text{if} \quad n \geqslant 2, \\ &= \max\{1, m - 1\}, && \text{if} \quad n = 1, \\ m^+(\mathfrak{A}_3(m, n, k, l)) &= m^-(\mathfrak{A}_3(m, n, k, l)) = m. \end{aligned}$$

The measures $n^+$ and $n^-$ of structural complexity are defined in a similar manner based on maximal chains and the relation of reachability.

The state $z_2 \in Z$ is said to be reachable from $z_1 \in Z$ iff there exists a word $w \in X^*$ such that $f(z_1, w) = z_2$. The set $Z_2 \subseteq Z$ is said to be reachable from $Z_1 \subseteq Z$) ($Z_1 \vdash_{\mathfrak{A}} Z_2$ for short) iff some state of $Z_2$ is reachable from some state of $Z_1$. Note that for essential sets $Z_1$, $Z_2$ both occurrences of the word "some" in this definition could be replaced equivalently by the word "any." Let

$$m \underset{\mathrm{df}}{=} \max\{m^+(\mathfrak{A}), m^-(\mathfrak{A})\}$$

the length of maximal chains. We are interested in alternating chains of +chains and −chains of length $m$ with respect to reachability (for short: superchains). We distinguish superchains beginning with a +chain (+superchain) and superchains beginning with a −chain (−superchain) (see Figs. 4 and 5 in which the arrows stand for the reachability between the corresponding sets).

For $n \geqslant 1$ and $Z' \subseteq Z$ let $Z'$ be in $N_n^+(\mathfrak{A})$ ($N_n^-(\mathfrak{A})$) iff $Z'$ is the last set of the last chain of a +superchain (−superchain) of length $n$.

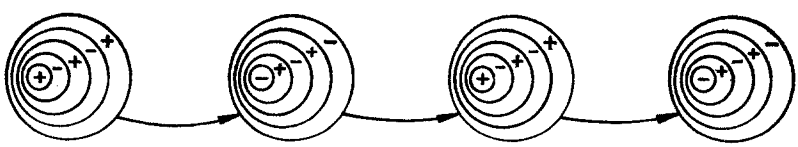

FIG. 4. A +superchain of length 4, where $m^+(\mathfrak{A}) = m^-(\mathfrak{A}) = 5$.

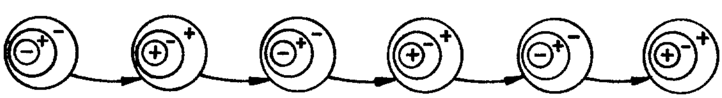

FIG. 5. A −superchain of length 6, where $m^+(\mathfrak{A}) = m^-(\mathfrak{A}) = 3$.

More formally,

$$N_1^+(\mathfrak{A}) \underset{\text{df}}{=} M_m^+(\mathfrak{A}), \qquad N_1^-(\mathfrak{A}) \underset{\text{df}}{=} M_m^-(\mathfrak{A}),$$

$$N_{2n}^+(\mathfrak{A}) \underset{\text{df}}{=} \{Z'; Z' \in M_m^-(\mathfrak{A}) \wedge \exists Z''(Z'' \in N_{2n-1}^+(\mathfrak{A}) \wedge Z'' \underset{\mathfrak{A}}{\vdash} Z')\},$$

$$N_{2n}^-(\mathfrak{A}) \underset{\text{df}}{=} \{Z'; Z' \in M_m^+(\mathfrak{A}) \wedge \exists Z''(Z'' \in N_{2n-1}^-(\mathfrak{A}) \wedge Z'' \underset{\mathfrak{A}}{\vdash} Z')\},$$

$$N_{2n+1}^+(\mathfrak{A}) \underset{\text{df}}{=} \{Z'; Z' \in M_m^+(\mathfrak{A}) \wedge \exists Z''(Z'' \in N_{2n}^+(\mathfrak{A}) \wedge Z'' \underset{\mathfrak{A}}{\vdash} Z')\},$$

$$N_{2n+1}^-(\mathfrak{A}) \underset{\text{df}}{=} \{Z'; Z' \in M_m^-(\mathfrak{A}) \wedge \exists Z''(Z'' \in N_{2n}^-(\mathfrak{A}) \wedge Z'' \underset{\mathfrak{A}}{\vdash} Z')\},$$

The length of maximal +superchains (−superchains) is denoted by $n^+(\mathfrak{A})$ ($n^-(\mathfrak{A})$), i.e.,

$$n^+(\mathfrak{A}) \underset{\text{df}}{=} \max(\{0\} \cup \{n; N_n^+(\mathfrak{A}) \neq \varnothing\})$$

and

$$n^-(\mathfrak{A}) \underset{\text{df}}{=} \max(\{0\} \cup \{n; N_n^-(\mathfrak{A}) \neq \varnothing\}).$$

The numbers $n^+(\mathfrak{A})$ and $n^-(\mathfrak{A})$ are also characteristic of the structure of the $\omega$-DFA $\mathfrak{A}$.

Now we state some simple properties of the sets $N_n^-(\mathfrak{A})$ and $N_n^+(\mathfrak{A})$ as well as the measures $n^+$ and $n^-$.

LEMMA 6. (1) *$Z' \in N_n^+ \Leftrightarrow$ there exist $Z_1, \ldots, Z_n$ such that $Z_n = Z'$, $Z_i \vdash_{\mathfrak{A}} Z_{i+1}$ and*

$$Z_i \in M_m^+, \quad \textit{if } i \textit{ odd}, \qquad \in M_m^-, \quad \textit{if } i \textit{ even}.$$

(2) *$Z' \in N_n^- \Leftrightarrow$ there exist $Z_1, \ldots, Z_n$ such that $Z_n = Z'$, $Z_i \vdash_{\mathfrak{A}} Z_{i+1}$ and*

$$Z_i \in M_m^+, \quad \textit{if } i \textit{ even}, \qquad \in M_m^-, \quad \textit{if } i \textit{ odd}.$$

(3) $N_{2n}^+ \subseteq M_m^-$, $N_{2n+1}^+ \subseteq M_m^+$.

(4) $N_{2n}^- \subseteq M_m^+$, $N_{2n+1}^- \subseteq M_m^-$.

(5) $$N_n^+ \subseteq M^+, \quad \textit{if } n+m \textit{ even}, \qquad \subseteq M^-, \quad \textit{if } n+m \textit{ odd}.$$

(6) $$N_n^- \subseteq M^+, \quad \textit{if } n+m \textit{ odd}, \qquad \subseteq M^-, \quad \textit{if } n+m \textit{ even}.$$

(7) $N_{n+2}^+ \subseteq N_{n+1}^- \subseteq N_n^+$.

(8) $N_{n+2}^- \subseteq N_{n+1}^+ \subseteq N_n^-$.

(9) $N_n^+ \cap N_n^- = \varnothing$, *or more generally*

(10) $N_{n_1}^+ \cap N_{n_2}^- = \varnothing \Leftrightarrow n_1 + n_2$ *even,*

(11) $N_{n_1}^+ \cap N_{n_2}^+ = \varnothing \Leftrightarrow N_{n_1}^- \cap N_{n_2}^- = \varnothing \Leftrightarrow n_1 + n_2$ *odd,*

(12) $N_n^+ = \varnothing$ *implies* $N_{n+1}^+ = \varnothing$.

(13) $N_n^- = \varnothing$ *implies* $N_{n+1}^- = \varnothing$.

(14) $N_n^+([X, Z, f, z_0, \mathfrak{P}(Z)\backslash\mathfrak{Z}]) = N_n^-(\mathfrak{A})$.

(15) $N_n^-([X, Z, f, z_0, \mathfrak{P}(Z)\backslash\mathfrak{Z}]) = N_n^+(\mathfrak{A})$.

(16) $| n^+(\mathfrak{A}) - n^-(\mathfrak{A})| \leqslant 1$.

(17) $n^+([X, Z, f, z_0, \mathfrak{P}(Z)\backslash\mathfrak{Z}]) = n^-(\mathfrak{A})$.

(18) $n^-([X, Z, f, z_0, \mathfrak{P}(Z)\backslash\mathfrak{Z}]) = n^+(\mathfrak{A})$.

(19) $m^+(\mathfrak{A}) = m^-(\mathfrak{A}) + 1$ *iff* $n^+(\mathfrak{A}) = 1$ *and* $n^-(\mathfrak{A}) = 0$.

(20) $m^+(\mathfrak{A}) + 1 = m^-(\mathfrak{A})$ *iff* $n^+(\mathfrak{A}) = 0$ *and* $n^-(\mathfrak{A}) = 1$.

(21) $m^+(\mathfrak{A}) = m^-(\mathfrak{A})$ *iff* $n^+(\mathfrak{A}) \geqslant 1$ *and* $n^-(\mathfrak{A}) \geqslant 1$.

(22) *There exist algorithms for computing* $n^+(\mathfrak{A})$ *and* $n^-(\mathfrak{A})$ *for given* $\mathfrak{A}$.

EXAMPLE 2. We consider the automata $\mathfrak{A}_i(m, n, k, l)$ defined in Example 1. In the right upper part of the transition graph of $\mathfrak{A}_1(m, n, k, l)$ (see Fig. 3) we have a-superchain of length $n$ and hence a $+$superchain of length $n - 1$. Because there are no greater $-$superchains and $+$superchains, respectively, we have $n^-(\mathfrak{A}_1(m, n, k, m)) = n$ and $n^+(\mathfrak{A}_1(m, n, k, l)) = n - 1$. Step by step we get

$$\max\{m^+(\mathfrak{A}_1(m, n, k, l)), m^-(\mathfrak{A}_1(m, n, k, l))\} = m \qquad \text{(Example 1)},$$

$$N_1^+(\mathfrak{A}_1(m, n, k, l)) = \{\{z_0^\nu, \ldots, z_m^\nu\}; \nu \text{ even} \wedge 1 \leqslant \nu \leqslant n\}$$

$$\begin{aligned} N_1^-(\mathfrak{A}_1(m, n, k, l)) = \{\{z_0^\nu, \ldots, z_m^\nu\}; \nu \text{ odd} \wedge 1 \leqslant \nu \leqslant n\} \\ \cup \{\{\hat{z}_0^\nu, \hat{z}_1^\nu\}, 1 \leqslant \nu \leqslant n\} \cup \{\{s\}\}, \qquad \text{if} \quad m = 1, \\ \cup\, \varnothing, \qquad \text{else}, \end{aligned}$$

$$N_2^+(\mathfrak{A}_1(m, n, k, l)) = \{\{z_0^\nu, \ldots, z_m^\nu\}; \nu \text{ odd} \wedge 2 \leqslant \nu \leqslant n\},$$

$$N_2^-(\mathfrak{A}_1(m, n, k, l)) = \{\{z_0^\nu, \ldots, z_m^\nu\}; \nu \text{ even} \wedge 2 \leqslant \nu \leqslant n\},$$

$$\vdots$$

$$N_n^+(\mathfrak{A}_1(m, n, k, l)) = \varnothing,$$

$$N_n^-(\mathfrak{A}_1(m, n, k, l)) = \{\{z_1^n, \ldots, z_m^n\}\} \qquad (\text{for } n > 1),$$

$$N_{n+1}^+(\mathfrak{A}_1(m, n, k, l)) = N_{n+1}^-(\mathfrak{A}_1(m, n, k, l)) = \varnothing,$$

and consequently

$$n^+(\mathfrak{A}_1(m, n, k, l)) = n - 1,$$
$$n^-(\mathfrak{A}_1(m, n, k, l)) = n.$$

The reader can do the same with the automata $\mathfrak{A}_2(m, n, k, l)$ and $\mathfrak{A}_3(m, n, k, l)$ and find out that

$$\begin{aligned} n^+(\mathfrak{A}_2(m, n, k, l)) &= n, \\ n^-(\mathfrak{A}_2(m, n, k, l)) &= n - 1, \qquad \text{if} \quad m + n > 2, \\ &= 1, \qquad\quad\; \text{if} \quad m = n = 1, \\ n^+(\mathfrak{A}_3(m, n, k, l)) &= n^-(\mathfrak{A}_3(m, n, k, l)) = n. \end{aligned}$$

Contrary to the case of $m^+(\mathfrak{A})$ and $m^-(\mathfrak{A})$ (see Lemma 5(19)) it is not immediately clear whether $n^+(\mathfrak{A})$ and $n^-(\mathfrak{A})$ are finite. The following two lemmas show us that $n^+(\mathfrak{A})$ and $n^-(\mathfrak{A})$ are really finite.

LEMMA 7. *Let* $m = \max(m^+(\mathfrak{A}), m^-(\mathfrak{A}))$, $Z_2 \in M_m^+(M_m^-)$ *and* $Z_1, Z_3 \in M_m^-(M_m^+)$. *Then* $Z_1 \vdash_{\mathfrak{A}} Z_2 \vdash_{\mathfrak{A}} Z_3$ *implies* $Z_1 \cap Z_3 = \varnothing$.

*Proof.* Assume that $m = \max(m^+(\mathfrak{A}), m^-(\mathfrak{A}))$, $Z_2 \in M_m^+(M_m^-)$, $Z_1$, $Z_3 \in M_m^-(M_m^+)$, $Z_1 \vdash_{\mathfrak{A}} Z_2 \vdash_{\mathfrak{A}} Z_3$, and $z_1 \in Z_1 \cap Z_3$. Hence there are $z_2 \in Z_2$ and $w_1, w_2 \in X^*$ such that $f(z_1, w_1) = z_2$ and $f(z_2, w_2) = z_1$. Since $Z_1$ and $Z_2$ are essential there are $w_3, w_4 \in X^*$ such that $f(z_1, w_3) = z_1$, $f(z_2, w_4) = z_2$, $\{f(z_1, w); w \subseteq w_3\} = Z_1$, and $\{f(z_2, w); w \subseteq w_4\} = Z_2$ (Lemma 4). Consequently $Z_4 =_{\mathrm{df}} \{f(z_1, w); w \subseteq w_1w_4w_2w_3\}$ is essential and satisfies $Z_4 \supseteq Z_1 \cup Z_2$. But because of $Z_1 \in M_n^-(M_m^+)$ and $Z_2 \in M_m^+(M_m^-)$ the set $Z_4$ is in $M_{m+1}^-$ or in $M_{m+1}^-$ which contradicts our supposition $m = \max\{m^+(\mathfrak{A}), m^-(\mathfrak{A})\}$. ∎

LEMMA 8. $n^+(\mathfrak{A})$ *and* $n^-(\mathfrak{A})$ *are finite numbers for any* $\omega$-DFA $\mathfrak{A}$.

*Proof.* Because of Lemma 6(16). It is sufficient to show that $n^+(\mathfrak{A})$ is finite. We assume the opposite. Consequently there are infinitely many $n$ such that $N_n^+(\mathfrak{A}) \neq \varnothing$ and by Lemma 6(12), 6(13) all $n$ have this property. Let $r =_{\mathrm{df}}$ card $\mathfrak{P}(Z) + 1$. Then there $Z_1, ..., Z_r$ such that $Z_1 \vdash_{\mathfrak{A}} Z_2 \vdash_{\mathfrak{A}} \cdots \vdash_{\mathfrak{A}} Z_r$ and $Z_\rho \in N_\rho^+(\mathfrak{A})$ for all $\rho = 1, ..., r$. Thus there exist $\rho_1, \rho_2$ which satisfy $1 \leqslant \rho_1 < \rho_2 \leqslant r$ and $Z_{\rho_1} = Z_{\rho_2}$ and therefore $Z_{\rho_1} \vdash_{\mathfrak{A}} Z_{\rho_1+1} \vdash_{\mathfrak{A}} Z_{\rho_2} = Z_{\rho_1}$. But this contradicts Lemma 7 because either $Z_{\rho_1} \in M_m^+(\mathfrak{A})$ and $Z_{\rho_1+1} \in M_m^-(\mathfrak{A})$ or $Z_{\rho_1} \in M_m^-(\mathfrak{A})$ and $Z_{\rho_1+1} \in M_m^+(\mathfrak{A})$ (where $m = \max m^+(\mathfrak{A}), m^-(\mathfrak{A})\}$). ∎

## 5. CLASSES OF STRUCTURAL COMPLEXITY

We start now our investigations of structural complexity classes of $\omega$-regular sets with respect to the measures $m^+$, $m^-$, $n^+$, and $n^-$. After stating some simple properties of them we establish a hierarchy of complexity classes and show that the numbers $m^+(\mathfrak{A})$, $m^-(\mathfrak{A})$, $n^+(\mathfrak{A})$, and $n^-(\mathfrak{A})$ are invariants of the $\omega$-regular set $T(\mathfrak{A})$.

At first we compare $\omega$-regular sets with respect to their complexity. Corresponding to the definition of our measures we have to compare first of all the length of maximal chains of $\omega$-DFA accepting these sets and then the length of maximal superchains of these automata. More formally, for $A, B \subseteq X^\omega$, $A \leqslant B \Leftrightarrow$ there are $\omega$-DFA $\mathfrak{A}$, $\mathfrak{B}$ such that $A = T(\mathfrak{A})$, $B = T(\mathfrak{B})$ and

$$\max(m^+(\mathfrak{A}), m^-(\mathfrak{A}) < \max(m^+(\mathfrak{B}), m^-(\mathfrak{B})) \text{ or}$$

$$\max(m^+(\mathfrak{A}), m^-(\mathfrak{A})) = \max(m^+(\mathfrak{B}), m^-(\mathfrak{B})), n^+(\mathfrak{A}) \leqslant n^+(\mathfrak{B}) \text{ and } n^-(\mathfrak{A}) \leqslant n^-(\mathfrak{B})$$

and furthermore $A \equiv B \Leftrightarrow_{\mathrm{df}} A \leqslant B$ and $B \leqslant A$.

As usual we transfer the relation $\leqslant$ to the set $R/\equiv$ of all $\equiv$-equivalence classes

LEMMA 9. (1) *The relation $\leqslant$ is a reflexive and transitive one.*

(2) $A \leqslant B \Leftrightarrow \bar{A} \leqslant \bar{B}$.

(3) $\equiv$ *is an equivalence relation.*

(4) *The equivalence classes of $\equiv$ are the classes $C_m{}^n$, $D_m{}^n$ and $E_m{}^n (m, n \geqslant 1)$ defined below.*

(5) *The relation $\leqslant$ is a partial ordering of the set of all $\equiv$-equivalence classes.*

*Hint.* For (2) see Lemmas 5(17), 5(18) and 6(17), 6(18).

$$C_m{}^n \underset{\mathrm{df}}{=} \{T(\mathfrak{A});\ \max(m^+(\mathfrak{A}), m^-(\mathfrak{A})) = m \wedge n^+(\mathfrak{A}) = n - 1 \wedge n^-(\mathfrak{A}) = n\},$$

$$D_m{}^n \underset{\mathrm{df}}{=} \{T(\mathfrak{A});\ \max(m^+(\mathfrak{A}), m^-(\mathfrak{A})) = m \wedge n^+(\mathfrak{A}) = n \wedge n^-(\mathfrak{A}) = n - 1\},$$

$$E_m{}^n \underset{\mathrm{df}}{=} \{T(\mathfrak{A});\ \max(m^+(\mathfrak{A}), m^-(\mathfrak{A})) = m \wedge n^+(\mathfrak{A}) = n^-(\mathfrak{A}) = n\}.$$

However, it is not clear now whether some of these classes coincide. Later we shall see that this is not possible.

The classes $C_m{}^n$, $D_m{}^n$, and $E_m{}^n$ can be considered as "exact" complexity classes. We also introduce the "downward" complexity classes

$$\begin{aligned}\hat{C}_m{}^n \underset{\mathrm{df}}{=} \{T(\mathfrak{A});\ &\max(m^+(\mathfrak{A}), m^-(\mathfrak{A})) < m \vee (\max(m^+(\mathfrak{A}), m^-(\mathfrak{A})) \\ &= m \wedge n^+(\mathfrak{A}) \leqslant n - 1)\},\end{aligned}$$

$$\begin{aligned}\hat{D}_m{}^n \underset{\mathrm{df}}{=} \{T(\mathfrak{A});\ &\max(m^+(\mathfrak{A}), m^-(\mathfrak{A})) < m \vee (\max(m^+(\mathfrak{A}), m^-(\mathfrak{A})) \\ &= m \wedge n^-(\mathfrak{A}) \leqslant n - 1)\},\end{aligned}$$

$$\begin{aligned}\hat{E}_m{}^n \underset{\mathrm{df}}{=} \{T(\mathfrak{A});\ &\max(m^+(\mathfrak{A}), m^-(\mathfrak{A})) < m \vee (\max(m^+(\mathfrak{A}), m^-(\mathfrak{A})) \\ &= m \wedge n^+(\mathfrak{A}) \leqslant n \wedge n^-(\mathfrak{A}) \leqslant n)\},\end{aligned}$$

as well as the "upward" complexity classes,

$$\check{C}_m{}^n \underset{\mathrm{df}}{=} \{T(\mathfrak{A});\ \max(m^+(\mathfrak{A}), m^-(\mathfrak{A})) > m \vee (\max(m^+(\mathfrak{A}), m^-(\mathfrak{A})) = m \wedge n^-(\mathfrak{A}) \geqslant n)\},$$

$$\check{D}_m{}^n \underset{\mathrm{df}}{=} \{T(\mathfrak{A});\ \max(m^+(\mathfrak{A}), m^-(\mathfrak{A})) > m \vee (\max(m^+(\mathfrak{A}), m^-(\mathfrak{A})) = m \wedge n^+(\mathfrak{A}) \geqslant n)\},$$

$$\check{E}_m{}^n \underset{\mathrm{df}}{=} \{T(\mathfrak{A});\ \max(m^+(\mathfrak{A}), m^-(\mathfrak{A})) > m \vee (\max(m^+(\mathfrak{A}), m^-(\mathfrak{A})) = m \wedge n^+(\mathfrak{A}) \geqslant n \wedge n^-(\mathfrak{A}) \geqslant n).$$

The following properties are evident.

LEMMA 10. (1) $R = \bigcup_{m,n\geqslant 1} (C_m{}^n \cup D_m{}^n \cup Z_m{}^n)$.

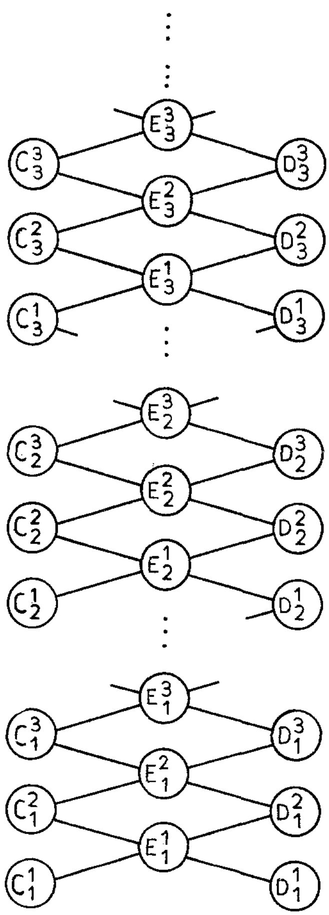


FIG. 6. The structure of the set of all exact complexity classes with respect to

(2) *The structure of the set of all exact complexity classes with respect to the partial ordering $\leqslant$ is represented in Fig. 6 in the following sense: if the classes P and Q are connected and P is not higher than Q then $P \leqslant Q$ (the converse is not clear yet as mentioned above).*

(3) $A \in C_m{}^n(\hat{C}_m{}^n, \check{C}_m{}^n) \Leftrightarrow \bar{A} \in D_m{}^n(\hat{D}_m{}^n, \check{D}_m{}^n)$.

(4) $A \in E_m{}^n(\hat{E}_m{}^n, \check{E}_m{}^n) \Leftrightarrow \bar{A} \in E_m{}^n(\hat{E}_m{}^n, \check{E}_m{}^n)$.

(5) $\hat{P}_m{}^n = \{A; \exists B(B \in P_m{}^n \wedge A \leqslant B)\}$ *for* $P = C, D, E$.

(6) $\hat{P}_m{}^n = \bigcup \{Q; Q \in R/\!\equiv \wedge\, Q \leqslant P_m{}^n\}$ *for* $P = C, D, E$.

(7) $\check{P}_m{}^n = \{A; \exists B(B \in P_m{}^n \wedge B \leqslant A)\}$ *for* $P = C, D, E$.

(8) $\check{P}_m{}^n = \bigcup \{Q; Q \in R/\!\equiv \wedge\, P_m{}^n \leqslant Q\}$ *for* $P = C, D, E$.

*Hint.* For (3) and (4) see Lemmas 5(17), 5(18) and 6(17), 6(18). At first it is necessary to show that these classes are not empty. Using the automata $\mathfrak{A}_i(m, n, k, l)$ defined in Example 1 this can easily be seen.

THEOREM 8. (1) card $C_m{}^n$ = card $D_m{}^n = \aleph_0$, *for* $m + n > 2$.

(2) card $E_m{}^n = \aleph_0$, *for* $m, n \geqslant 1$.

(3) $C_1{}^1 = \{\varnothing\}$ *and* $D_1{}^1 = \{X^\omega\}$.

*Proof.* By Examples 1 and 2 we have

$$\begin{aligned} &T(\mathfrak{A}_1(m, n, k, l)) \in C_m{}^n, && \text{for} \quad m, n \geqslant 2,\\ &T(\mathfrak{A}_2(m, n, k, l)) \in D_m{}^n, && \text{for} \quad m + n > 2,\\ &T(\mathfrak{A}_3(m, n, k, l)) \in E_m{}^n, && \text{for} \quad m, n \geqslant 1. \end{aligned}$$

Evidently, $T(\mathfrak{A}_2(m, n, k, l)) \neq \varnothing$ for $m + n > 2$ and all sequences of this set begin with $0^k1$. Thus $T(\mathfrak{A}_2(m, n, k_1, l)) \neq T(\mathfrak{A}_2(m, n, k_2, l))$ for $k_1 \neq k_2$. The hint of Lemma 10(3) completes the proof of (1). Similarly, $T(\mathfrak{A}_3(m, n, k, l)) \neq \varnothing$ for $m, n \geqslant 1$ and all sequences of this set begin with $0^k$. However, there is a sequence in $T(\mathfrak{A}_3(m, n, k, l))$ which begins with $0^k1$. This implies $T(\mathfrak{A}_3(m, n, k_1, l)) \neq T(\mathfrak{A}_3(m, n, k_2, l))$ for $k_1 \neq k_2$. At last, if $T(\mathfrak{A}) \in C_1{}^1$ then $\mathfrak{A}$ has no accepting set and, consequently, $T(\mathfrak{A}) \neq \varnothing$. By Lemma 10(3) we have $D_1{}^1 = \{X^\omega\}$. ▮

Next, we show that any two exact complexity classes which are different in name are in fact disjoint. It will turn out that for this end the proof of the disjointness of $C_m{}^n$ and $D_m{}^n$, for any $m, n \geqslant 1$, is sufficient. However, $C_m{}^n$ and $D_m{}^n$ are equal or disjoint because they are equivalence classes. Thus we shall prove that there is a set in $C_m{}^n \backslash D_m{}^n$.

For this reason we define

$$c_m^n \underset{\text{df}}{=} \bigcup_{\substack{0\leqslant\nu<n\\ 1\leqslant\mu\leqslant m\\ \nu+\mu\ \text{even}}} (\tilde{m}^*0)^\nu\, \tilde{m}^*(\widetilde{\mu-1}\cdot\bar{\mu})^\omega \quad\text{and}\quad d_m^n = \overline{c_m^n}, \qquad \text{for odd } n,$$

$$d_m^n \underset{\text{df}}{=} \bigcup_{\substack{0\leqslant\nu<n\\ 1\leqslant\mu\leqslant m\\ \nu+\mu\ \text{odd}}} (\tilde{m}^*0)^\nu\, \tilde{m}^*(\widetilde{\mu-1}\cdot\bar{\mu})^\omega \quad\text{and}\quad c_m^n \underset{\text{df}}{=} \overline{d_m^n}, \qquad \text{for even } n,$$

where $\bar{k} =_{\text{df}} 1^k0$ and $\tilde{k} =_{\text{df}} (\bar{1} \cup \bar{2} \cup \cdots \cup \bar{k})$.

LEMMA 11. (1) $c_m^n \in C_m^n \Leftrightarrow d_m^n \in D_m^n$.

(2) $c_m^n \in C_m^n$.

(3) $d_m^n \in D_m^n$.

*Proof.* Lemma 10(3) directly implies (1). Further we show (2) for odd $n$. By (1) we have (3) for odd $n$. The other case can be treated similarly. For odd $n$ the set $c_m^n$ can be accepted by the $\omega$-DFA with the transition graph represented in Fig. 7, the initial state $z_0^1$ and the system $\{\{z_0^\nu,\ldots, z_\mu^\nu\};\ 1 \leqslant \mu \leqslant m \wedge 1 \leqslant \nu \leqslant n \wedge \mu + \nu \text{ odd}\}$ of final sets. ∎

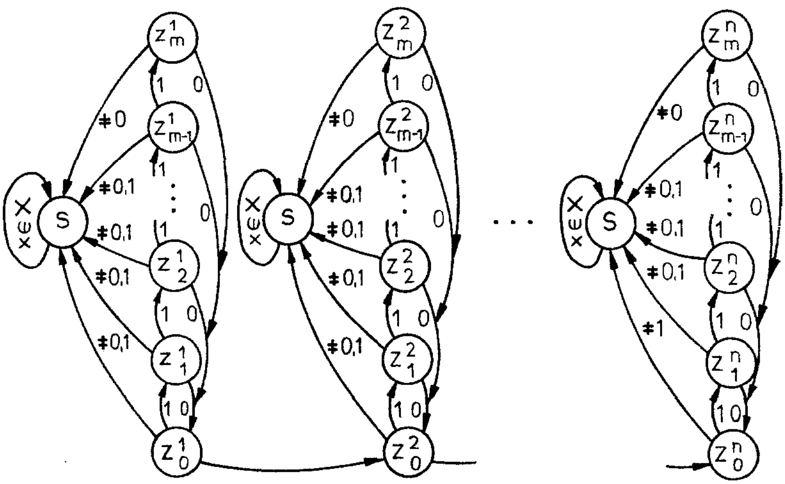


FIG. 7. The transition graph of an $\omega$-automaton accepting $c_m^n$.

LEMMA 12. (1) $c_m^n \notin D_m^n$.

(2) $d_m^n \notin C_m^n$.

*Proof.* By Lemma 11(1) it is sufficient to show $c_m^n \notin D_m^n$ for odd $n$ and $d_m^n \notin C_m^n$ for even $n$. We do the first. The latter can be done similarly. Let $\mathfrak{A}$

be a $\omega$-DFA such that $T(\mathfrak{A}) = c_m^{\ n}$. We show that either $\max(m^+(\mathfrak{A}), m^-(\mathfrak{A})) > m$ or $\max(m^+(\mathfrak{A}), m^-(\mathfrak{A})) = m$ and $n^-(\mathfrak{A}) \geqslant n$.

Let $s = \text{card } Z$,

$$
\begin{aligned}
w_1 &\underset{\text{df}}{=} \bar{1}^s, \\
w_2 &\underset{\text{df}}{=} (\bar{2}w_1)^s, \\
&\vdots \\
w_m &\underset{\text{df}}{=} (\bar{m}w_{m-1})^s,
\end{aligned}
$$

and $z_1 =_{\text{df}} f(z_0, w_m)$. We suppose that $m$ is odd.

Since $w_m^{\ \omega} = (\bar{m}w_{m-1})^\omega \notin c_m^{\ n} = T(\mathfrak{A})$ there is a $Z_m^{\ 1} \notin \mathfrak{Z}$ such that $w_m^{\ \omega} \in T([X, Z, f, z_0, \{Z_m^{\ 1}\}])$. And because of $f(z_0, (\bar{m}w_{m-1})^s) = z_1$ there is a $s_m < s$ such that $f(z_0, (\bar{m}w_{m-1})^{s_m}) = f(z_1, (\bar{m}w_{m-1})^{s-s_m}) = z_1$ and $\{f(z_1, w);\ w \subseteq (\bar{m}w_{m-1})^{s-s_m}\} = Z_m^{\ 1}$.

However,

$$w_m = (\bar{m} \cdot w_{m-1})^s = \underbrace{(\bar{m} \cdot w_{m-1})^{s-1} \cdot \bar{m}}_{v_m} \cdot (\overline{m-1} \cdot w_{m-2})^s$$

and we can argue in the same spirit:

Since $v_m \cdot w_{m-1}^{\omega} = v_m \cdot (\overline{m-1}\ w_{m-2})^\omega \in c_m^{\ n} = T(\mathfrak{A})$ there is a $Z_{m-1}^1 \in \mathfrak{Z}$ such that $v_m w_{m-1}^{\omega} \in T([X, Z, f, z_0, \{Z_{m-1}^1\}])$. And because of $f(z_0, w_m \cdot (\overline{m-1}\ w_{m-2})^s) = z_1$ there is an $s_{m-1} < s$ such that

$$f(z_0, v_m(\overline{m-1} \cdot w_{m-2})^{s_m-1}) = f(z_1, (\overline{m-1} \cdot w_{m-2})^{s-s_m-1}) = z_1$$

and

$$\{f(z_1, w);\ w \equiv (\overline{m-1} \cdot w_{m-2})^{s-s_m-1}\} = Z_{m-1}^1 .$$

Thus we have $z_1 \in Z_{m-1}^1 \subset Z_m^{\ 1}$.

We continue in this manner and at last we get a $-$chain $Z_1^{\ 1} \subset Z_2^{\ 1} \subset \cdots \subset Z_{m-1}^1 \subset Z_m^{\ 1}$ with $z_1 \in Z_1^{\ 1}$. The case "$m$ even" can be treated analogously with the same result. Consequently, $\max(m^+(\mathfrak{A}), m^-(\mathfrak{A})) \geqslant m$. In the case $\max(m^+(\mathfrak{A}), m^-(\mathfrak{A})) > m$ the proof would be finished. Thus we suppose $\max(m^+(\mathfrak{A}), m^-(\mathfrak{A})) = m$. Let $z_2 =_{\text{df}} f(z_0, w_m 0 w_m)$. As above we get a $+$chain $Z_1^{\ 2} \subset Z_2^{\ 2} \subset \cdots \subset Z_m^{\ 2}$ with $z_2 \in Z_1^{\ 2}$.

In the same manner we continue and get chain $Z_1^{\ \nu} \subset Z_2^{\ \nu} \subset \cdots \subset Z_m^{\ \nu}$ with $z_\nu =_{\text{df}} f(z_0, (w_m 0)^{\nu-1} w_m) \in Z_1^{\ \nu}$ where the chain with index $\nu$ is a $-$chain iff $\nu$ is odd $(1 \leqslant \nu \leqslant n)$.

Thus we have a $-$superchain of length $n$, i.e., $n^-(\mathfrak{A}) \geqslant n$. ∎

Since exact complexity classes are either disjoint or equal we have

Corollary 1. *$C_m^{\ n} \cap D_m^{\ n} = \varnothing$ for any $m, n \geqslant 1$.*

This holds even for all exact complexity classes.

THEOREM 9. *Any two exact complexity classes are disjoint if they are different by name.*

*Proof.* It is sufficient to show that any two complexity classes are unequal if they are different by name.

(1) Assume that $C_m{}^n = D_m{}^n$. This contradicts Corollary 1.

(2) Assume that $C_m{}^n = E_m{}^n$ or $D_m{}^n = E_m{}^n$. By Lemma 10(3), 10(4) we have $C_m{}^n = E_m{}^n$ and $D_m{}^n = E_m{}^n$ and this contradicts (1).

(3) Assume that $C_m^{n+1} = E_m{}^n$ or $D_m^{n+1} = E_m{}^n$. We get the same contradiction as in (2).

(4) Assume that $P_{m_1}^{n_1} = Q_{m_2}^{n_2}$, where $P \in \{C, D, E\}$ and $P_{m_1}^{n_1}$ and $Q_{m_2}^{n_2}$ are different by name (general case). If $P_{m_1}^{n_1}$ and $Q_{m_2}^{n_2}$ are not comparable in Fig. 6 (i.e., they are the same height) then this is a contradiction to (1). Hence, without lost of generality we can assume that $Q_{m_2}^{n_2}$ is higher than $P_{m_1}^{n_1}$ in Fig. 6.

By Lemma 10(2) we have $P_{m_1}^{n_1} \leqslant Q_{m_2}^{n_2}$. If $P = C$ or $P = D$ then $P_{m_1}^{n_1} \leqslant E_{m_1}^{n_1} \leqslant Q_{m_2}^{n_2} = P_{m_1}^{n_1}$ and this contradicts (2). If $P = E$ then $E_{m_1}^{n_1} \leqslant C_{m_1}^{n_1+1} \leqslant Q_{m_2}^{n_2} = E_{m_1}^{n_1}$ or $E_{m_1}^{n_1} \leqslant D_{m_1}^{n_1+1} \leqslant Q_{m_2}^{n_2} = E_{m_1}^{n_1}$ and this contradicts (3). ∎

By Theorem 9 we know both the structure of the set of all exact complexity classes with respect to $\leqslant$ and the structure of the set of downward (upward) complexity classes with respect to set inclusion.

COROLLARY 2. (1) *Figure* 6 *represents the structure of the set of all exact complexity classes*: $P \leqslant Q$ iff $P$ is connected with $Q$ and $P$ is not above $Q$.

(2) *Figure* 6 *represents the structure of the set of all downward* (*upward*) *complexity classes*: $\hat{P} \subseteq \hat{Q}$ ($\check{P} \subseteq \check{Q}$) iff $P$ is connected with $Q$ and $P$ is not above $Q$.

Furthermore Theorem 9 shows that the numbers $m^+(\mathfrak{A})$, $m^-(\mathfrak{A})$, $n^+(\mathfrak{A})$, and $n^-(\mathfrak{A})$ are invariants of the $\omega$-regular set $T(\mathfrak{A})$.

COROLLARY 3. *If* $T(\mathfrak{A}) = T(\mathfrak{B})$ *for any two* $\omega$-DFA $\mathfrak{A}$ *and* $\mathfrak{B}$ *then* $m^+(\mathfrak{A}) = m^+(\mathfrak{B})$, $m^-(\mathfrak{A}) = m^-(\mathfrak{B})$, $n^+(\mathfrak{A}) = n^+(\mathfrak{B})$, *and* $n^-(\mathfrak{A}) = n^-(\mathfrak{B})$.

Consequently, it makes sense to define for any $\omega$-regular set $A$ $m^+(A) =_{\mathrm{df}} m^+(\mathfrak{A})$, $m^-(A) =_{\mathrm{df}} m^-(\mathfrak{A})$, $n^+(A) =_{\mathrm{df}} n^+(\mathfrak{A})$, $n^-(A) =_{\mathrm{df}} n^-(\mathfrak{A})$, where $\mathfrak{A}$ is any $\omega$-DFA such that $T(\mathfrak{A}) = A$.

As we shall see in the next sections these invariants are of great importance for the investigation of the automata-theoretic and topological properties of $\omega$-regular sets.

## 6. Reducibilities by Finite Automata

In this section we introduce two notions of reducibility for sets of $\omega$-sequences: $m$-reducibility by deterministic finite synchronous automata ($DS$) and such by deterministic finite asynchronous automata ($DA$). These notions of reducibility seem to be the most natural ones for studying the $\omega$-regular sets. Evidently, $DS$-reducibility implies $DA$-reducibility, and it turns out that $DA$-reducibility implies comparability with respect to the relation $\leqslant$ studied in the preceding section. Therefore the decomposition of $R$ in $DA$-degrees is a refinement of the decomposition of $R$ generated by $\leqslant$. However, the coarse structure of the set of all $DS$-degrees is the same as the structure of the set of all exact complexity classes (see Fig. 6).

Let $A, B \subseteq X^\omega$. $A$ is said to be $DS$-reducible ($DA$-reducible) to $B$, for short $A \leqslant_{DS} B$ ($A \leqslant_{DA} B$), iff there is a $DSFA$ ($DAFA$)$\mathfrak{B}$ such that $\xi \in A \Leftrightarrow \Phi_{\mathfrak{B}}(\xi) \in B$, for any $\xi \in X^\omega$. Further $A \equiv_{DS} B$ iff $A \leqslant_{DS} B$ and $B \leqslant_{DS} A$ as well as $A \equiv_{DA} B$ iff $A \leqslant_{DA} B$ and $B \leqslant_{DA} A$. The $\equiv_{DS}$ ($\equiv_{DA}$) equivalence classes are said to be $DS$- ($DA$-) degrees. As usual we transfer the relation $\leqslant_{DS}$ ($\leqslant_{DA}$) to the set $R/{\equiv_{DS}}$ ($R/{\equiv_{DA}}$) of all $DS$- ($DA$-) degrees. Further let $\mathscr{D}\mathscr{S}(R) =_{\mathrm{df}} [R/{\equiv_{DS}}, \leqslant_{DS}]$ and $\mathscr{D}\mathscr{A}(R) =_{\mathrm{df}} [R/{\equiv_{DA}}, \leqslant_{DA}]$.

Lemma 13. (1) *$A \leqslant_{DS} B$ implies $A \leqslant_{DA} B$.*

(2) *$A \leqslant_{DS} B$ implies $\bar{A} \leqslant_{DS} \bar{B}$.*

(3) *$A \leqslant_{DA} B$ implies $\bar{A} \leqslant_{DA} \bar{B}$.*

(4) *$R/{\equiv_{DS}}$ is a refinement of $R/{\equiv_{DA}}$ (not necessary proper).*

(5) *$A \leqslant_{DS} B$ iff there is a DSFA $\mathfrak{B}$ such that $A = \Phi_{\mathfrak{B}}^{-1}(B)$.*

(6) *$A \leqslant_{DA} B$ iff there is a DAFA $\mathfrak{B}$ such that $A = \Phi_{\mathfrak{B}}^{-1}(B)$.*

We show now that $\leqslant$ is not weaker than $\leqslant_{DA}$.

Theorem 10. *$A \leqslant_{DA} B$ implies $A \leqslant B$.*

*Proof.* Let $A \leqslant_{DA} B$ via $\mathfrak{B}$ and let further $\mathfrak{A}$ be an $\omega$-DFA such that $T(\mathfrak{A}) = B$. The idea is to composite $\mathfrak{B}$ and $\mathfrak{A}$ in a suitable manner to an $\omega$-DFA$\mathfrak{C}$ such that $T(\mathfrak{C}) = A$ and $m^+(\mathfrak{C})$, $m^-(\mathfrak{C})$, $n^+(\mathfrak{C})$ and $n^-(\mathfrak{C})$ have the desired properties.

Let $\mathfrak{A} = [X, Z, f, z_0, \mathfrak{Z}]$ and $\mathfrak{B} = [X, Z', f', g', z_0']$. We define $\mathfrak{C} =_{\mathrm{df}} [X, Z'', f'', z_0'', \mathfrak{Z}'']$, where

$$Z'' \underset{\mathrm{df}}{=} Z' \times \bigcup_{k=0}^{k} Z^k, \qquad k \underset{\mathrm{df}}{=} \max\{|g'(z, x)|; z \in Z' \wedge x \in X\},$$

$$z_0'' \underset{\mathrm{df}}{=} [z_0', z_0],$$

$$f''([z', z_1 \cdots z_k], x) \underset{\mathrm{df}}{=} [f'(z', x), \hat{z}_1 \cdots \hat{z}_l] \text{ where } \hat{z}_1 \underset{\mathrm{df}}{=} f(z_k, x_1), \hat{z}_1 \underset{\mathrm{df}}{=} f(\hat{z}_1, x_2), \ldots, \hat{z}_l = f(\hat{z}_{l-1}, x_l) \text{ and } x_1 x_2 \cdots x_l = g'(z', x).$$

$$\mathfrak{Z}'' = \{Z'; \langle Z' \rangle \in \mathfrak{Z}\}, \qquad \langle Z' \rangle \underset{\mathrm{df}}{=} \bigcup \{\{z_1, \ldots, z_k\}; \exists z([z, z_1 \cdots z_k] \in Z')\}.$$

We explain this definition by observing an input sequence $\xi = x_1 x_2 x_3 \cdots \in X^\omega$: $z_1' z_2' z_3' \cdots$ is the corresponding state sequence in $\mathfrak{B}$, $w_1 w_2 w_3 \cdots = \Phi_{\mathfrak{B}}(\xi)$ is the corresponding output sequences of $\mathfrak{B}$, i.e., $w_i = g'(z_{i-1}, x_i)$. $z_1{}^1 z_2{}^1 \cdots z_{r_1}^1 z_1{}^2 z_2{}^2 \cdots z_{r_2}^2 z_1{}^3 z_2{}^3 \cdots z_{r_3}^3 \cdots = \Phi_{\mathfrak{A}}(\Phi_{\mathfrak{B}}(\xi))$ is the state sequence in $\mathfrak{A}$ corresponding to the input $\Phi_{\mathfrak{B}}(\xi)$, i.e., $r_i = |w_i|$, and $z_1{}^i z_2{}^i \cdots z_{r_i}^i = f(z_{r_{i-1}}^{i-1}, w_i(1)) f(z_1{}^i, w_i(2)) \cdots f(z_{r_i-1}^i, w_i(r_i))$, where $z_{r_0}^0 =_{\mathrm{df}} z_0$. Then

$$[z_1', z_1{}^1 z_2{}^1 \cdots z_{r_1}^1], [z_2', z_1{}^2 z_2{}^2 \cdots z_{r_2}^2], [z_3', z_1{}^3 z_2{}^3 \cdots z_{r_3}^3] \cdots = \Phi_{\mathfrak{C}}(\xi)$$

is the state sequence in $\mathfrak{C}$ corresponding to the input $\xi$.

Now it is evident that $\langle U(\Phi_{\mathfrak{C}}(\xi)) \rangle = U(\Phi_{\mathfrak{A}}(\Phi_{\mathfrak{B}}(\xi)))$ for any $\xi \in X^\omega$ and we can conclude

$$\xi \in A \Leftrightarrow \Phi_{\mathfrak{B}}(\xi) \in B \Leftrightarrow U(\Phi_{\mathfrak{A}}(\Phi_{\mathfrak{B}}(\xi))) \in \mathfrak{Z} \Leftrightarrow \langle U(\Phi_{\mathfrak{C}}(\xi)) \rangle \in \mathfrak{Z} \Leftrightarrow U_{\mathfrak{C}}(\Phi(\xi)) \in \mathfrak{Z}''.$$

Consequently, $A = T(\mathfrak{C})$.

Further simple observations are

(1) If $Z'$ is an accepting (rejecting) set of $\mathfrak{C}$ then $\langle Z' \rangle$ is an accepting (rejecting) set of $\mathfrak{A}$.

(2) If $Z' \subseteq Z''$ then $\langle Z' \rangle \subseteq \langle Z'' \rangle$.

(3) If $Z' \vdash_{\mathfrak{C}} Z''$ then $\langle Z' \rangle \vdash_{\mathfrak{A}} \langle Z'' \rangle$.

By (1) and (2), we have $m^+(\mathfrak{C}) \leqslant m^+(\mathfrak{A})$ and $m^-(\mathfrak{C}) \leqslant m^-(\mathfrak{A})$.

*Case* 1. $\max(m^+(\mathfrak{C}), m^-(\mathfrak{C})) < \max(m^+(\mathfrak{A}), m^-(\mathfrak{A}))$. Then $A = T(\mathfrak{C}) \leqslant T(\mathfrak{A}) = B$.

*Case* 2. $\max(m^+(\mathfrak{C}), m^-(\mathfrak{C})) = \max(m^+(\mathfrak{A}), m^-(\mathfrak{A})) = m$. Hence, if $Z' \in M_m{}^+(\mathfrak{C}(M_m{}^-(\mathfrak{C}))$ then $\langle Z' \rangle \in M_m{}^+(\mathfrak{A})(M_m{}^-(\mathfrak{A}))$. Taking 3 into consideration this means $n^+(\mathfrak{C}) \leqslant n^+(\mathfrak{A})$ and $n^-(\mathfrak{C}) \leqslant n^-(\mathfrak{A})$. Consequently, $A = T(\mathfrak{C}) \leqslant T(\mathfrak{A}) = B$. ∎

The converse of Theorem 10 is also true for sets $A$, $B$ which are not in the same class $E_m{}^n$. To show this we start with the following lemma

LEMMA 14. *For any* $A \in \check{C}_m{}^n(\check{D}_m{}^n)$ *there are* $v_\nu$, $w_\nu{}^\mu \in X^+$ $(\nu = 1, \ldots, n;$ $\mu = 1, \ldots, m)$ *such that*

$$\bigcup_{\substack{1 \leqslant \nu \leqslant n \\ 1 \leqslant \mu \leqslant m \\ \mu+\nu \text{ odd}}} \prod_{i=1}^{\nu} v_i \cdot (w_i{}^1 \cup \cdots \cup w_i{}^m)^* \cdot ((w_\nu{}^1 \cup \cdots \cup w_\nu^{\mu-1})^* w_\nu{}^\mu)^\omega \subseteq A(\bar{A})$$

*and*

$$\bigcup_{\substack{1\leqslant\nu\leqslant n\\1\leqslant\mu\leqslant m\\\mu+\nu\,\text{even}}}\prod_{i=1}^{\nu} v_i(w_i^1\cup\cdots\cup w_i^m)^*\cdot((w_\nu^1\cup\cdots\cup w_\nu^{\mu-1})^*\,w_\nu^\mu)^\omega\subseteq \bar{A}(A)$$

*Proof.* We prove the lemma for the case $A\in\check{C}_m{}^n$. Then we get the case $A\in\check{D}_m{}^n$ by Lemma 10(3).

Given an $\omega$-DFA $\mathfrak{A}$ such that $T(\mathfrak{A})=A\in\check{C}_m{}^n$.

*Case* 1. Let $\max(m^+(\mathfrak{A}),m^-(\mathfrak{A}))>m$ and without lost of generality $m^-(\mathfrak{A})>m$. Then there is a $+$chain $Z_1\subset Z_2\subset\cdots\subset Z_m\subset Z_{m+1}$ and hence there are a state $z\in Z_1$ and words $v,u_1,\ldots,u_{m+1}\in X^+$ such that $f(z_0,v)=f(z,u_\mu)=z$ and $\{f(z,u);\ u\sqsubseteq u_\mu\}=Z_\mu$ for $\mu=1,\ldots,m+1$. Then for odd (even) $\mu$ the set $Z_\mu$ is accepting (rejecting) and therefore $v\cdot(u_1\cup\cdots\cup u_{m+1})^*\cdot((u_1\cup\cdots\cup u_{\mu-1})^*u_\mu)^\omega\subseteq A(\bar{A})$.

Consequently, $v_1=_{\text{df}}v$, $v_\nu=_{\text{df}}u_1$ $(\nu=2,\ldots,n)$, and

$$\begin{aligned} w^\mu &\underset{\text{df}}{=} u_\mu && \text{for even } \nu\\ &\underset{\text{df}}{=} u_{\mu+1} && \text{for odd } \nu \qquad (\mu=1,\ldots,m;\ \nu=1,\ldots,n).\end{aligned}$$

fulfill the lemma.

*Case* 2. Let $m=\max(m^+(\mathfrak{A}),m^-(\mathfrak{A}))$ and $n^-(\mathfrak{A})\geqslant n$. Then there is a $-$superchain $Z_1{}^m\vdash_{\mathfrak{A}}Z_2{}^m\vdash_{\mathfrak{A}}\cdots\vdash_{\mathfrak{A}}Z_n{}^m$ and hence there are chains $Z_\nu^1\subset Z_\nu^2\subset\cdots\subset Z_\nu^m$ where this is a $-$chain iff $\nu$ is odd. Furthermore there are states $z_\nu\in Z_\nu^1$ and words $v_\nu,\ w_\nu{}^\mu\in X^+$ such that $f(z_{\nu-1},v_\nu)=f(z_\nu,w_\nu{}^\mu)=z_\nu$ and $\{f(z_\nu,w);\ w\sqsubseteq w_\nu{}^\mu\}=Z_\nu{}^\mu$ for $\nu=1,\ldots,n$ and $\mu=1,\ldots,m$. Then for odd (even) $\nu+\mu$ the set $Z_\nu{}^\mu$ is accepting (rejecting) and therefore

$$\left(\prod_{i=1}^{\nu} v_i\cdot(w_2^1\cup\cdots\cup w_i^m)^*\right)\cdot((w_\nu^1\cup\cdots\cup w_\nu^{\mu-1})^*\cdot w_\nu^\mu)^\omega\subseteq A(\bar{A}).$$

Thus $v_\nu$ and $w_\nu{}^\mu$ fulfill the lemma. ∎

THEOREM 11. (1) *If $A\in\hat{C}_m{}^n$ and $B\in\check{C}_m{}^n$ then $A\leqslant_{DS}B$.*

(2) *If $A\in\hat{D}_m{}^n$ and $B\in\check{D}_m{}^n$ then $A\leqslant_{DS}B$.*

*Proof.* We prove (1). Statement 2 can be proved similarly. Let $A\in\hat{C}_m{}^n$, $B\in\check{C}_m{}^n$ and let $\mathfrak{A}$ be an $\omega$-DFA such that $T(\mathfrak{A})=A$. By Lemma 14 there are $v_\nu$, $w_\nu{}^\mu$ $(\mu=1,\ldots,m;\ \nu=1,\ldots,n)$ such that

$$\bigcup_{\substack{1\leqslant\nu\leqslant n\\1\leqslant\mu\leqslant m\\\mu+\nu\,\text{odd}}}\left(\prod_{i=1}^{\nu} v_i(w_1^1\cup\cdots\cup w_1^m)^*\right)((w_\nu^1\cup\cdots\cup w_\nu^{\mu-1})\,w_\nu^\mu)^\omega\subseteq B$$

and

$$\bigcup_{\substack{1-\nu-n\\1-\mu-m\\\mu+\nu\text{ even}}} \left(\prod_{i=1}^{\nu} v_i(w_i^1 \cup \cdots \cup w_i^m)^*\right) \cdot ((w_\nu^1 \cup \cdots \cup w_\nu^{\mu-1})\, w_\nu^\mu)^\omega \subseteq B.$$

We now describe a *DSFA* $\mathfrak{B}$ which reduces $A$ to $B$.

The automaton $\mathfrak{B}$ gives out only words $w_\nu^\mu$ ($\mu = 0,..., m$; $\nu = 1,..., n$; $w_\nu^0 =_{\text{df}} v_\nu$). Since $\mathfrak{B}$ has to work synchronously, the output of a word $w_\nu^\mu$ possibly must be done in several steps. Here the phrase "$\mathfrak{B}$ gives out $w_\nu^\mu$" stands for the fact that $\mathfrak{B}$ gives out $w_\nu^\mu$ in $| w_\nu^\mu |$ steps. The automaton $\mathfrak{B}$ works in at most $n$ stages. In the $\nu$th stage $\mathfrak{B}$ gives out $v_\nu$ and then words $w_\nu^\mu$ ($\mu = 1,..., m$) while the state sequence of $\mathfrak{A}$ has not reached a $+$chain of length $m$ (if $\nu$ is odd) or a $-$chain of length $m$ (if $\nu$ is even), respectively. Thus $\mathfrak{B}$ gives out in the $\nu$th stage altogether a word from $v_\nu(w_\nu^1 \cup \cdots \cup w_\nu^m)^*$ if this stage is not the last one and a sequence from $v_\nu(w_\nu^1 \cup \cdots \cup w_\nu^m)^\omega$ if this stage is the last one.

First we have to show that $\mathfrak{B}$ cannot leave the $n$-th stage.

*Case* 1. $\max(m^+(\mathfrak{A}), m^-(\mathfrak{A})) < m$. Then we have no chain of length $m$ and cannot leave the first stage.

*Case* 2. $\max(m^+(\mathfrak{A}), m^-(\mathfrak{A})) = m$. Assume that $\mathfrak{B}$ can leave the $n$th stage. Then the state sequence of $\mathfrak{A}$ has at first reached a $+$chain of length $m$ (transition to the second stage), then it has reached a $-$chain of length $m$ (transition to the third stage) and so on up to the transition to the $(n + 1)$th stage. However this means that $\mathfrak{A}$ has a $+$superchain of length $n$ and therefore $n^+(\mathfrak{A}) \geqslant n$.

This contradicts our supposition $T(\mathfrak{A}) = A \in \hat{C}_m^n$.

Now we describe the working mode of $\mathfrak{B}$ in the $\nu$th stage by an algorithm. Let $Z_1 ,..., Z_r$ be the essential sets of $\mathfrak{A}$. Then we need special "testing sets" $S_1 ,..., S_r$ . During the whole stage the testing set $S_\rho$ ($\rho = 1,..., r$) will be filled up by the actual states of $\mathfrak{A}$ if these states are in $Z_\rho$ . If the actual state of $\mathfrak{A}$ is not in $Z_\rho$ then $S_\rho$ will be emptied and can be filled up again. The same happens if $S_\rho$ is "full," i.e., $S_\rho = Z_\rho$ . Parallel to this activity $\mathfrak{B}$ will do the following if $\nu$ is odd (for even $\nu$ see in the parentheses)

(1) $\mathfrak{B}$ gives out $v_\nu$ .

(2) If the state sequence of $\mathfrak{A}$ has reached a $+$chain ($-$chain) of length $m$ during the output of the last word given out then transition to the $(n + 1)$th stage, else

(3) If no testing set has been filled during the output of the last word given out, i.e., there was no $\rho = 1,..., r$ such that $S_\rho = Z_\rho$ during this time, then the next word given out is $w_\nu^1$ and go to (2), else

(4) If the testing sets $S_{\rho_1} ,..., S_{\rho_h}$ have been filled during the output of the last word given out then let

$$\mu \underset{\text{df}}{=} \max(\{\mu'; \exists i(i \in \{1,\dots,h\} \wedge Z_{\rho_i} \in M_{\mu'}^-)\}$$

$$\cup \{\mu' + 1; \exists i(i \in \{1,\dots,h\} \wedge Z_{\rho_i} \in M_{\mu'}^+)\}),$$

$$(\mu \underset{\text{df}}{=} \max(\{\mu'; \exists i(i \in \{1,\dots,h\} \wedge Z_{\rho_i} \in M_{\mu'}^+)\}$$

$$\cup \{\mu' + 1; \exists i(i \in \{1,\dots,h\} \wedge Z_{\rho_i} \in M_{\mu'}^-)\})$$

and the next word given out is $w_\nu{}^\mu$. Go to (2).

Note that $\mu \leqslant m$ because $m^-(\mathfrak{A}) \leqslant m$, $m^+(\mathfrak{A}) \leqslant m$ and a $+$chain ($-$chain) of length $m$ was not reached by the automaton $\mathfrak{A}$ in this stage. Next, we have to show that a *DSFA* $\mathfrak{B}$ which works in the described manner reduces $A$ to $B$.

Let $\xi \in A$. Then there is an essential set $Z_\rho$ ($\rho \in \{1,\dots, r\}$) such that $\xi \in T([X, Z, f, z_0, \{Z_\rho\}])$. Let the $\nu$th stage be the last stage of the work of $\mathfrak{B}$ if $\xi$ is put in. In the first $\nu - 1$ stage $\mathfrak{B}$ gives out a word of $\prod_{i=1}^{\nu-1} v_i(w_i{}^1 \cup \cdots \cup w_i{}^m)^*$. We restrict ourselves to the case that $\nu$ is odd. If $\nu$ is even one can conclude similarily.

Evidently, $S_\rho$ is infinitely often full in the $\nu$th stage, i.e., infinitely often occurs $S_\rho = Z_\rho$. And for any $\rho' \in \{1,\dots, r\}$ such that $S_{\rho'}$ was infinitely often full $Z_{\rho'} \subseteq Z_\rho$ holds. Then by Lemmas 5(14), 5(15), $\mu =_{\text{df}} \max(\{\mu'; Z_\rho \in M_{\mu'}^-\} \cup \{\mu' + 1; Z_\rho \in M_{\mu'}^+\})$ is the greatest number which is determined infinitely often by point (4) of our algorithm. Therefore, $\mu$ is the greatest of all numbers $\mu'$ such that $\mathfrak{B}$ gives out $w_\nu^{\mu'}$ infinitely often. Thus $\Phi_{\mathfrak{B}}(\xi) \in (\prod_{i=1}^{\nu} v_i(w_i{}^1 \cup \cdots \cup w_i{}^m)^*) \cdot ((w_\nu{}^1 \cup \cdots \cup w_\nu^{\mu-1})^* w_\nu{}^\mu)^\omega$.

Now we can conclude

$$\begin{aligned} \xi \in A &\Leftrightarrow Z_\rho \text{ is an accepting set} \\ &\Leftrightarrow \mu \text{ is even} \\ &\Leftrightarrow \mu + \nu \text{ is odd} \\ &\Leftrightarrow \Phi_{\mathfrak{B}}(\xi) \in B. \end{aligned}$$

Finally it is not hard to see that the DSFT $\mathfrak{B} = [X, Z', f', g', z_0']$ defined below realizes the algorithm described above, where $\sigma$ indicates whether $\mathfrak{A}$ has reached the next $+$chain or $-$chain, respectively, of length $m$; $\mu$, $\nu$ are the indexes of the next word $w_\mu{}^\nu$ given out by $\mathfrak{B}$; $\kappa$, $\lambda$ are the indexes of the word $w_\lambda{}^\kappa$ given out by $\mathfrak{B}$ at present, $t$ is the number of that letter of $w_\lambda{}^\kappa$ which $\mathfrak{B}$ will give out in the next step

$$Z' \underset{\text{df}}{=} Z \times \mathfrak{P}(Z)^r \times \{+, -\} \times \{1,\dots, n\} \times \{1,\dots, m\} \times \{1,\dots, n\} \times \{0, 1,\dots, m\}$$
$$\times\, x\{1,\dots, \max_{\substack{1 \leqslant \nu \leqslant n \\ 0 \leqslant \mu \leqslant m}} | w_\mu{}^\nu |\},$$

$$f'([z, S_1,\dots, S_r, \sigma, \nu, \mu, \lambda, \kappa, \tau], x) \underset{\text{df}}{=} [f(z, x), S_1',\dots, S_r', \sigma', \nu', \mu', \lambda', \kappa', \tau'],$$

where

$$\begin{aligned}
S'_\rho &\underset{\mathrm{df}}{=} S_\rho \cup \{z\}, && \text{if} \quad S_\rho \subset Z_\rho \text{ and } z \in Z_\rho, \\
&\underset{\mathrm{df}}{=} \varnothing, && \text{if} \quad S_\rho = Z_\rho \text{ or } z \notin Z_\rho,
\end{aligned}$$

$$\begin{aligned}
\sigma' &\underset{\mathrm{df}}{=} -, && \text{if} \quad \sigma = - \text{ and } z \notin \bigcup M_m^+(\mathfrak{A}), \\
&\underset{\mathrm{df}}{=} +, && \text{if} \quad \sigma = - \text{ and } z \in \bigcup M_m^+(\mathfrak{A}), \\
&\underset{\mathrm{df}}{=} +, && \text{if} \quad \sigma = + \text{ and } z \notin \bigcup M_m^-(\mathfrak{A}), \\
&\underset{\mathrm{df}}{=} -, && \text{if} \quad \sigma = + \text{ and } z \in \bigcup M_m^-(\mathfrak{A}),
\end{aligned}$$

$$\begin{aligned}
\nu' &\underset{\mathrm{df}}{=} \lambda, && \text{if} \quad \sigma = \sigma', \\
&\underset{\mathrm{df}}{=} \lambda + 1, && \text{if} \quad \sigma \neq \sigma',
\end{aligned}$$

$$\begin{aligned}
\mu' &\underset{\mathrm{df}}{=} 1, \quad \text{if} \quad \sigma \neq \sigma', \\
&\underset{\mathrm{df}}{=} \max(\{1\} \cup \{\mu''; \exists\rho(Z_\rho \in M_{\mu''}^- \wedge S_\rho = Z_\rho)\} \\
&\qquad \cup \{\mu'' + 1; \exists\rho(Z_\rho \in M_{\mu''}^+ \wedge S_\rho = Z_\rho)\}), \\
&\qquad\quad \text{if} \quad \sigma = \sigma' = - \text{ and } \tau = |\, w_\lambda^\kappa \,|, \\
&\underset{\mathrm{df}}{=} \max(\{1\} \cup \{\mu''; \exists\rho(Z_\rho \in M_{\mu''}^+ \wedge S_\rho = Z_\rho)\} \\
&\qquad \cup \{\mu'' + 1; \exists\rho(Z_\rho \in M_{\mu''}^- \wedge S_\rho = Z_\rho)\}), \\
&\qquad\quad \text{if} \quad \sigma = \sigma' = + \text{ and } \tau = |\, w_\lambda^\kappa \,|, \\
&\underset{\mathrm{df}}{=} \max(\{\mu\} \cup \{\mu''; \exists\rho(Z_\rho \in M_{\mu''}^- \wedge S_\rho = Z_\rho)\} \\
&\qquad \cup \{\mu'' + 1; \exists\rho(Z_\rho \in M_{\mu''}^+ \wedge S_\rho = Z_\rho)\}), \\
&\qquad\quad \text{if} \quad \sigma = \sigma' = - \text{ and } \tau < |\, w_\lambda^\kappa \,|, \\
&\underset{\mathrm{df}}{=} \max(\{\mu\} \cup \{\mu''; \exists\rho(Z_\rho \in M_{\mu''}^+ \wedge S_\rho = Z_\rho)\} \\
&\qquad \cup \{\mu'' + 1; \exists\rho(Z_\rho \in M_{\mu''}^- \wedge S_\rho = Z_\rho)\}), \\
&\qquad\quad \text{if} \quad \sigma = \sigma' = + \text{ and } \tau < |\, w_\lambda^\kappa \,|,
\end{aligned}$$

$$\begin{aligned}
\lambda' &\underset{\mathrm{df}}{=} \lambda, && \text{if} \quad \tau < |\, w_\lambda^\kappa \,|, \\
&\underset{\mathrm{df}}{=} \lambda, && \text{if} \quad \tau = |\, w_\lambda^\kappa \,| \text{ and } \lambda = \nu, \\
&\underset{\mathrm{df}}{=} \lambda + 1, && \text{if} \quad \tau = |\, w_\lambda^\kappa \,| \text{ and } \lambda \neq \nu,
\end{aligned}$$

$$\begin{aligned}
\kappa' &\underset{\mathrm{df}}{=} \kappa, && \text{if} \quad \tau < | w_\lambda^\kappa |, \\
&\underset{\mathrm{df}}{=} 0, && \text{if} \quad \tau = | w_\lambda^\kappa | \text{ and } \lambda \neq \nu, \\
&\underset{\mathrm{df}}{=} \mu, && \text{else,}
\end{aligned}$$

$$\begin{aligned}
\tau' &\underset{\mathrm{df}}{=} \tau + 1, && \text{if} \quad \tau < | w_\lambda^\kappa |, \\
&\underset{\mathrm{df}}{=} 1, && \text{else.}
\end{aligned}$$

Further $g'([z, S_1, ..., S_r, \sigma, \nu, \mu, \lambda, \kappa, \tau], x) =_{\mathrm{df}} w_\lambda^\kappa(\tau)$ and $z_0' =_{\mathrm{df}} [z_0, \phi, ..., \phi, -, 1, 1, 1, 0, 1]$. ▮

By this theorem and Lemma 13 we have

COROLLARY 4. (1) *$C_m^n$, $D_m^n$ are DS degrees as well as DA degrees* $(m, n \geqslant 1)$.

(2) *If $A, B \in R \backslash E_m^n$ for any $m$, $n$ then*

$$A \leqslant_{DS} B \Leftrightarrow A \leqslant_{DA} B \Leftrightarrow A \leqslant B.$$

Corollary 4.2 says that $\leqslant_{DS}$, $\leqslant_{DA}$, and $\leqslant$ can differ at most in the classes $E_m^n$. Later we shall see that they actually do so.

## 7. TOPOLOGICAL CHARACTERIZATIONS OF THE DOWNWARD COMPLEXITY CLASSES

As mentioned in Section 3 topological properties play an important role in the study of $\omega$-regular sets. Very often topological properties are more transparent and allow deeper insights than automaton-theoretical ones. Thus it would be desirable to have topological characterizations of the classes studied in the above sections. As shown in Wagner (1976, 1977) such topological characterizations for the classes $C_m^n$, $D_m^n$, and $E_m^n$ are possible, and we state here these results without proofs.

As a first step it can be shown that the simple topological properties "open," "closed," "to be a $G_\delta$-set," and "to be a $F_\sigma$-set" correspond to several classes of our hierarchy.

THEOREM 12. (1) $G^R = \hat{C}_1^2$.

(2) $F^R = \hat{D}_1^2$.

(3) $G_\delta^R = \hat{C}_2^1$.

(4) $F_\sigma^R = \hat{D}_2^1$.

(5) $G_\sigma^R \cap F_\sigma^R = \bigcup_{n=1}^\infty (\hat{C}_1^n \cup \hat{D}_1^n)$.

Hence by Theorem 5 we know that $\bigcup_{n=1}^\infty (\hat{C}_1^n \cup \hat{D}_1^n) = \mathscr{B}(G^R) = \mathscr{B}(F^R)$. Taking this fact into consideration one can get the idea that the classes $C_1^n$ and

$D_1^n$ might have something to do with certain set-theoretical subclasses of the Boolean closure of $G^R$ or $F^R$, respectively.

We define inductively the class $\mathcal{T}(M)$ of all stages of Boolean generation with respect to a set system $M$.

(1) $M \in \mathcal{T}(M)$.

(2) If $T_1, T_2 \in \mathcal{T}(M)$ then $T_1 \triangledown T_2$, $T_1 \triangle T_2 \in (M)$.

(3) If $T \in \mathcal{T}(M)$ then $\hat{T} \in \mathcal{T}(M)$.

(4) Further set systems do not belong to $\mathcal{T}(M)$.

Indeed, the stages of Boolean generation with respect to $G^R$ and $G_\delta{}^R$ correspond to topological properties.

Lemma 15. *If* $T \in \mathcal{T}(G^R)$ *or* $T \in \mathcal{T}(G_\delta{}^R)$ *then* "$A \in T$" *is a topological property.*[4]

Further, under certain assumptions any stage of Boolean generation coincides with such a stage of normal form.

Lemma 16. *If $M$ is closed under union and intersection then*

$$\mathcal{T}(M) = \left\{\underset{n+1}{\triangledown} (M \triangle \hat{M}); n \geqslant 0\right\} \cup \left\{M \triangledown \underset{n}{\triangledown} (M \triangle \hat{M}); n \geqslant 0\right\}$$
$$\cup \left\{\hat{M} \triangledown \underset{n}{\triangledown} (M \triangle \hat{M}); n \geqslant 0\right\} \cup \left\{M \triangledown \check{M} \triangledown \underset{n}{\triangledown} (M \triangle \hat{M}); n \geqslant 0\right\}.$$

Note that $G^R$ and $G_\delta{}^R$ are closed under union and intersection. Thus the stages of Boolean generation with respect to $G^R$ form a "ladder" with respect to set inclusion which has the same structure as the set of all $C_1{}^n$ and $D_1{}^n$ with respect to set inclusion. The reason for this can be found in the next theorem

Theorem 13. *For $n \geqslant 1$ there hold*

(1) $\hat{C}_1^{2n} = G^R \triangledown \triangledown_{n-1} (G^R \triangle F^R)$.

(2) $\hat{D}_1^{2n} = F^R \triangledown \triangledown_{n-1} (G^R \triangle F^R)$.

(3) $\hat{C}_1^{2n+1} = \triangledown_n (G^R \triangle F^R)$.

(4) $\hat{D}_1^{2n+1} = G^R \triangledown F^R \triangledown \triangledown_{n-1} (G^R \triangle F^R)$.

A similar situation exists in the case of $\mathcal{T}(B_\delta{}^R)$ and the sets $\hat{C}_m{}^1$ and $\hat{D}_m{}^1$

[4] A property is said to be topological iff inverse continuous functions preserve this property.

THEOREM 14. *For $m \geqslant 1$ there holds*

(1) $\hat{C}^1_{2m} = G_\delta{}^R \triangledown \triangledown_{m-1}(G_\delta{}^R \triangledown F_\sigma{}^R)$.

(2) $\hat{D}^1_{2m} = F_\sigma{}^R \triangledown \triangledown_{m-1}(G_\delta{}^R \triangledown F_\sigma{}^R)$.

(3) $\hat{C}^1_{2m+1} = \triangledown_m(G_\delta{}^R \vartriangle F_\sigma{}^R)$.

(4) $\hat{D}^1_{2m+1} = G_\delta{}^R \triangledown F_\sigma{}^R \triangledown \triangledown_{m-1}(G_\delta{}^R \vartriangle F_\sigma{}^R)$.

Thus it remains to speak about the classes $\hat{C}_m{}^n$ and $\hat{D}_m{}^n$ for $m, n \geqslant 2$ (because of $\hat{E}_m{}^n = \hat{C}_m^{n+1} \cap \hat{D}_m^{n+1}$ it is not necessary to speak about $E_m{}^n$ separately).

Let $[A]$ be the topological closure of $A \subseteq X^\omega$ and define

$$A \prec B \qquad \text{iff} \quad [A] \cap B = \varnothing \quad \text{and} \quad [A] \subseteq [B]$$

for $A, B \subseteq X^\omega$.

THEOREM 15. *Let $m \geqslant 2$ and $n \geqslant 1$.*

(1) $A \in \hat{C}_m{}^n$ *iff there are* $A_1, \ldots, A_n \subseteq X^\omega$ *such that* $A_1 \prec A_2 \prec \cdots \prec A_n$,

$$A_\nu \in \hat{C}_m{}^1(\hat{D}_m{}^1) \text{ for odd (even) } \nu \text{ and } A = \bigcup_{\nu=1}^{n} A_\nu . \tag{$*$}$$

(2) $A \in \hat{D}_m{}^n$ *iff there are* $A_1, \ldots, A_n \subseteq X^\omega$ *such that* $A_1 \prec A_2 \prec \cdots \prec A_n$,

$$A_\nu \in \hat{D}_m{}^1(\hat{C}_m{}^1) \text{ for odd (even) } \nu \text{ and } A = \bigcup_{\nu=1}^{n} A_\nu . \tag{$**$}$$

Thus we have characterization of all classes $\hat{C}_m{}^n$, $\hat{D}_m{}^n$, and $\hat{E}_m{}^n$ by topological properties since

LEMMA 17. *The properties* $(*)$ *and* $(**)$ *are topological ones.*

In other words

COROLLARY 5. *If $\Phi: X^\omega \mapsto X^\omega$ is a continuous function and $A \in \hat{C}_m{}^n(\hat{D}_m{}^n, \hat{E}_m{}^n)$ then $\Phi^{-1}(A) \in \hat{C}_m{}^n(\hat{D}_m{}^n, \hat{E}_m{}^n)$.*

This result stimulates us to consider "continuous reducibility." For $A, B \subseteq X^\omega$ let $A \leqslant_{CA} B(A \leqslant_{CS} B)$ iff there is a (synchronous) continuous function $\Phi$: $X^\omega \mapsto X^\omega$ such that $\xi \in A \Leftrightarrow \Phi(\xi) \in B$ for any $\xi \in X^\omega$. There a continuous function $\Phi$: $X^\omega \mapsto X^\omega$ is said to be synchronous iff there is a function $\varphi$: $X^* \mapsto X$ such that $\Phi(\xi)(n) = \varphi(\xi_0{}^n)$ for $\xi \in X^\omega$ and $n \in \mathrm{m}$. Evidently there holds

LEMMA 18. (1) *$A \leqslant_{CA} B$ implies $\bar{A} \leqslant_{CA} \bar{B}$ and $A \leqslant_{CS} B$ implies $\bar{A} \leqslant_{CS} \bar{B}$.*

(2) *$A \leqslant_{DA} B$ implies $A \leqslant_{CA} B$ and $A \leqslant_{DS} B$ implies $A \leqslant_{CS} B$.*

(3) *$A \leqslant_{CA} B$ ($A \leqslant_{CS} B$) iff there is a (synchronous) continuous function $\Phi: X^\omega \mapsto X^\omega$ such that $A = \Phi^{-1}(B)$.*

There are two other formulations of Corollary 5

COROLLARY 6. (1) *$A \leqslant_C B$ implies $A \leqslant B$.*

(2) *If $B \in \hat{C}_m{}^n(\hat{D}_m{}^n, \hat{E}_m{}^n)$ and $A \leqslant_C B$ then $A \in \hat{C}_m{}^n(\hat{D}_m{}^n, \hat{E}_m{}^n)$.*

## 8. COMPARISON WITH OTHER MEASURES

At this point of our investigation we are entitled to be thoroughly convinced that the basic measures $m^+$, $m^-$, $n^+$, and $n^-$ together describe exactly the essential structure of an $\omega$-regular set, i.e., that they describe the qualitative phenomena and neglect all quantitative ones. Naturally the question arises whether other measures of $\omega$-regular sets are of the same nature and consequently are compatible with our basic measures, or describe quantitative aspects, too, and consequently are not compatible with these measures.

As examples we consider two measures of the first kind and one of the second one.

1. First we investigate a pair of measures based on the $T_=^E$-acceptance. Analogously to the case of $T_=^U$-acceptance we define for any $\omega$-DFA $\mathfrak{A}$ the following:

$Z' \subseteq Z$ is said to be $E$-essential iff $T_=^E([X, Z, f, z_0, \{Z'\}]) \neq \varnothing$ and $Z' \subseteq Z$ is said to be $E$-accepting (-rejecting) iff $Z'$ is $E$-essential and $Z' \in \mathfrak{Z}$ ($Z' \notin \mathfrak{Z}$).

Let further $k^+(\mathfrak{A})$ ($k^-(\mathfrak{A})$) be the length of maximal alternating inclusion chains of $E$-accepting and $E$-rejecting sets beginning with an $E$-accepting (-rejecting) set.

Clearly, these measures correspond to the measures $m^+$ and $m^-$ based on $T_=^U$-acceptance. However, as we shall see below, contrary to $m^+$ and $m^-$ the measures $k^+$ and $k^-$ depend on the special automaton accepting a given $\omega$-regular set. Therefore it is useful to define for any $\omega$-regular set $A \in G_\delta{}^R \cap F_\sigma{}^R$ (by Theorem 4 only $G_\delta{}^R \cap F_\sigma{}^R$ sets can be accepted with $T_=^E$-acceptance)

$$k^+(A) \underset{\mathrm{df}}{=} \min\{k^+(\mathfrak{A});\ T_=^E(\mathfrak{A}) = A\},$$

$$k^-(A) \underset{\mathrm{df}}{=} \min\{k^-(\mathfrak{A});\ T_=^E(\mathfrak{A}) = A\}.$$

Evidently we have

LEMMA 19. *For any $A \in G_\delta{}^R \cap F_\sigma{}^R$*

(1) $|\,k^+(A) - k^-(A)| \leqslant 1.$

(2) $k^+(\bar{A}) = k^-(A).$

The measures describe only structural aspects of $G_\delta^R \cap F_\sigma^R$ sets and therefore they are compatible with the measures $m^+$, $m^-$, $n^+$, and $n^-$. We remember that $G_\delta^R \cap F_\sigma^R = \bigcup_{n=1} (\hat{C}_1^n \cup \hat{D}_1^n \cup \hat{E}_1^n)$ by Theorem 12(5).

THEOREM 16. *For any* $A \in G_\delta^R \cap F_\sigma^R$

(1) $A \in C_1^n \Leftrightarrow k^+(A) = n - 1$ *and* $k^-(A) = n$.

(2) $A \in D_1^n \Leftrightarrow k^+(A) = n$ *and* $k^-(A) = n - 1$.

(3) $A \in E_1^n \Leftrightarrow k^+(A) = n$ *and* $k^-(A) = n$.

(4) $A \in C_1^n \Leftrightarrow k^+(A) \leqslant n - 1$.

(5) $A \in D_1^n \Leftrightarrow k^-(A) \leqslant n - 1$.

(6) $A \in E_1^n \Leftrightarrow k^+(A) \leqslant n$ *and* $k^-(A) \leqslant n$.

*Proof.* We prove (4). By transition to the complement we then have (5) (Lemmas 19(2), 10(3)). Then the other statements are evident.

First let $A \in \hat{C}_1^n$. Hence there is an $\omega$-DFA $\mathfrak{A}$ such that $T(\mathfrak{A}) = A$, $\max(m^+(\mathfrak{A}), m^-(\mathfrak{A})) = 1$, and $n^+(\mathfrak{A}) \leqslant n - 1$. With another system of final sets the same automaton can be used to accept $A$ with $T_{=}^{E}$-acceptance. The new system $\mathfrak{Z}$ consists exactly of the $E$-essential sets which arise from accepting sets (in the sense of $T_{=}^{U}$-acceptance), i.e.,

$$\hat{\mathfrak{Z}} \underset{\text{df}}{=} \{Z' \cup \{f(z_0, u); u \sqsubseteq w\}; Z' \text{ accepting} \wedge f(z_0, w) \in Z'\}$$

and

$$\hat{\mathfrak{A}} \underset{\text{df}}{=} [X, Z, f, z_0, \hat{\mathfrak{Z}}].$$

Evidently, if $\xi \in A$ then $E(\Phi_{\hat{\mathfrak{A}}}(\xi)) \in \hat{\mathfrak{Z}}$. Assume that $\xi \notin A$ and $Z_1 =_{\text{df}} E(\Phi_{\hat{\mathfrak{A}}}(\xi)) \in \hat{\mathfrak{Z}}$. Then we have not only $Z_1 = Z' \cup \{f(z_0, u); u \sqsubseteq w'\}$ for some accepting $Z'$ and $w'$ such that $f(z_0, w') \in Z'$ but also $Z_1 = Z'' \cup \{f(z_0, u); u \sqsubseteq w''\}$ for some rejecting $Z''$ and $w''$ such that $f(z_0, w'') \in Z''$. Consequently there holds $Z' \vdash_{\mathfrak{A}} Z'' \vdash_{\mathfrak{A}} Z'$ which contradicts Lemma 7. Hence we have $A = T_{=}^{E}(\hat{\mathfrak{A}})$.

We show now $k^+(\hat{\mathfrak{A}}) \leqslant n - 1$. Assume that $k^+(\hat{\mathfrak{A}}) \geqslant n$. Then there are $E$-essential sets $Z_1, \ldots, Z_n$ such that $Z_1 \subset Z_2 \subset \cdots \subset Z_n$ where $Z_\nu$ is $E$-accepting iff $\nu$ is odd. Consequently there are essential sets $Z_1', \ldots, Z_n'$ and words $w_1, \ldots, w_n$ such that $Z_\nu = Z_\nu' \cup \{f(z_0, u); u \sqsubseteq w_\nu\}$ where $Z_\nu'$ is accepting iff $\nu$ is odd.

Because of $Z_{\nu-1}' \subseteq Z_\nu'$ we have $Z_{\nu-1}' \vdash_{\mathfrak{A}} Z_\nu'$ for $\nu = 2, \ldots, n$. This implies $n^+(\mathfrak{A}) \geqslant n$ which contradicts our assumption $A \in \hat{C}_1^n$.

Now we suppose $A \in (G_\delta^R \cap R_\sigma^R) \backslash \hat{C}_1^n$. Let $\mathfrak{A}$ be any $\omega$-DFA such that $T_{=}^{E}(\mathfrak{A}) = A$. We construct an $\omega$-DFA $\hat{\mathfrak{A}} = [X, \tilde{Z}, \tilde{f}, \tilde{z}_0, \tilde{\mathfrak{Z}}]$ which accepts $A$ with $T_{=}^{U}$-acceptance. The idea is that $\hat{\mathfrak{A}}$ works as the automaton $\mathfrak{A}$ and additionally collects the states which have occured up to the corresponding moment:

$$\tilde{Z} =_{\text{df}} \mathfrak{P}(Z) \times Z$$

$$\tilde{f}([Z', z], x) \underset{\mathrm{df}}{=} [Z' \cup \{f(z, x)\}, f(z, x)],$$

$$\tilde{z}_0 \underset{\mathrm{df}}{=} [\{z_0\}, z_0].$$

Evidently there holds $U(\Phi_{\tilde{\mathfrak{A}}}(\xi)) = \{E(\Phi_{\mathfrak{A}}(\xi))\} \times Z''$ for a suitable $Z'' \subseteq Z$.

Consequently, defining $\tilde{\mathfrak{Z}} =_{\mathrm{df}} \{\{Z'\} \times Z''; Z' \in \mathfrak{Z} \wedge Z'' \subseteq Z\}$ we have $T^U_{=}(\tilde{\mathfrak{A}}) = A$.

Note that $\{Z'\} \times Z''$ is accepting (rejecting) with respect to $\tilde{\mathfrak{A}}$ iff (*) $Z'$ is $E$-accepting (-rejecting) with respect to $\mathfrak{A}$.

Because of $A \in (G_\delta^R \cap F_\sigma^R) \backslash \hat{C}_1^n$ there are $\tilde{Z}_1, ..., \tilde{Z}_n \subseteq \tilde{Z}$ such that $\tilde{Z}_1 \vdash_{\tilde{\mathfrak{A}}} \tilde{Z}_2 \vdash_{\tilde{\mathfrak{A}}} \cdots \vdash_{\tilde{\mathfrak{A}}} \tilde{Z}_n$, where $\tilde{Z}_\nu$ is accepting (rejecting) iff is odd (even). Hence there are $Z_1', ..., Z_n', Z_1'', ..., Z_n'' \subseteq Z$ such that $\tilde{Z}_\nu = \{Z_\nu'\} \times Z''$. By (*) $Z_\nu'$ is $E$-accepting (-rejecting) iff $\nu$ is odd (even).

But $\{Z_\nu'\} \times Z_\nu'' \vdash_{\tilde{\mathfrak{A}}} \{Z_{\nu+1}'\} \times Z_{\nu+1}'$ implies $Z_\nu' \subseteq Z_{\nu+1}'$. Consequently we have $k^+(\mathfrak{A}) \geqslant n$. Since $\mathfrak{A}$ was any $\omega$-DFA with $T^E_{=}(\mathfrak{A}) = A$ we have also $k^+(A) \geqslant n$. ∎

*Remark* 2. As mentioned above, contrary to $m^+$ and $m^-$ the measures $k^+$ and $k^-$ depend on the special automaton which accepts a given set $A$. For example we consider $A = 0 \cdot X^\omega$ and the $\omega$-DFA

$$\mathfrak{A}_1 = [X, \{A, B, C\}, f_1, A, \{\{A, B\}\}]$$

and

$$\mathfrak{A}_2 = [X, \{A, B, C\}, f_2, A, \{\{A, B\}\}].$$

The transition graphs of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are shown in Figs. 8 and 9, respectively. Evidently there holds $T^E_{=}(\mathfrak{A}_1) = T^E_{=}(\mathfrak{A}_2) = A$ but $k^+(\mathfrak{A}_1) = 1$ and $k^+(\mathfrak{A}_2) = 2$.

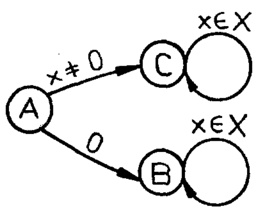


FIG. 8. The transition graph of $\mathfrak{A}_1$.

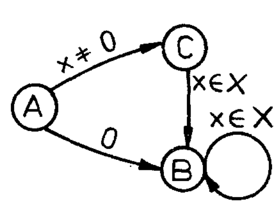


FIG. 9. The transition graph of $\mathfrak{A}_2$.

2. As a second example for measures which are compatible with $m^+$, $m^-$, $n^+$, and $n^-$ we consider the so-called Rabin index $I_R$. This measure is based on a notion of acceptance first studied by Rabin (1969). The automata used here differ from the $\omega$-DFA only by the condition $\mathfrak{Z} \subseteq \mathfrak{P}(Z) \times \mathfrak{P}(Z)$ (shortly: modified $\omega$-DFA). A set $A \subseteq X^\omega$ is said to be acceptable in the sense of Rabin iff there is a modified $\omega$-DFA $\mathfrak{A} = [X, Z, f, z_0, \mathfrak{Z}]$ such that

$$\xi \in A \Leftrightarrow \exists Z_1 \, \exists Z_2((Z_1, Z_2) \in \mathfrak{Z} \wedge U(\Phi_{\mathfrak{A}}(\xi)) \cap Z_1 \neq \varnothing \wedge U(\Phi_{\mathfrak{A}}(\xi)) \cap Z_2 = \varnothing).$$

It is known and can easily be seen that exactly the $\omega$-regular sets can be accepted in the sense of Rabin.

The Rabin index of an $\omega$-regular set $A$ is defined as the minimal number of pairs of final sets sufficient for it's acceptance in the sense of Rabin, i.e.,

$$I_R(A) \underset{\mathrm{df}}{=} \min\{m;\ \exists\mathfrak{A};\ \mathfrak{A} = [X, Z, f, z_0, \mathfrak{Z}] \text{ accepts } A \text{ in the sense of Rabin card } \mathfrak{Z} = m)\}.$$

The proof of the following theorem uses the topological results of Section 7 and can be found in Wagner (1977).

THEOREM 17. $A \in \hat{C}^1_{2m+1} \Leftrightarrow I_R(A) < m$, *for* $m \geqslant 0$.

And by Lemma 6(20) we have the following connection with the measure $m^+$.

COROLLARY 7.

$$I_R(A) = \left[\frac{m^+(A) + 1}{\cdot 2}\right]^5.$$

Consequently by Lemma 5(20) we have

COROLLARY 8. *The Rabin index of an $\omega$-regular set* (*given by an* $\omega$-DFA *accepting it*) *is computable.*

We mention that Corollary 8 remains true if the $\omega$-regular set is given by a modified $\omega$-DFA or by an $\omega$-regular expression because the transition to a corresponding $\omega$-DFA is known as computable. We are convinced that it would not be easy to get Corollary 8 directly. At least an algorithm by successive trial is not seen to be possible because there are infinitely many modified $\omega$-DFA with the same number of pairs of final sets.

3. Analogously we can define the Muller index of an $\omega$-regular set $A$ as the minimal number of final sets sufficient for it's acceptance in the sense of Muller (deterministic $T^U_=$-acceptance), i.e.

$$I_M(A) \underset{\mathrm{df}}{=} \min\{m;\ \exists\mathfrak{A}(T^U_=(\mathfrak{A}) = A_1 \text{ card } \mathfrak{Z} = m)\}$$

This measure is not a pure structural one, but it measures also quantitative phenomena.

THEOREM 18. (1) $A \in C_m{}^n$ *implies* $I_M(A) \geqslant [(m \cdot n)/2]$.

(2) $A \in D_m{}^n$ *implies* $I_M(A) \geqslant [(m \cdot n + 1)/2]$.

(3) $A \in E_m{}^n$ *implies* $I_M(A) \geqslant m \cdot n$.

(4) *The lower bounds in* (1), (2), *and* (3) *cannot be improved.*

[5] For a real number $\alpha$ let $[\alpha]$ be the greatest integer not greater than $\alpha$.

(5) *There are no upper bounds for $I_M(A)$ depending on $m$ and $n$ for $A \in C_m{}^n(D_m{}^n, E_m{}^n) \geqslant D_1{}^2$. For $A \in C_1{}^1$ we have $I_M(A) = 0$ and for $A \in D_1{}^1 \cup E_1{}^1 \cup C_1{}^2$ we have $I_M(A) = 1$.*

*Proof.* Statements (1), (2), and (3) can easily be seen by counting all accepting sets necessary for the acceptance of any $A \in C_m{}^n(D_m{}^n, E_m{}^n)$ by the definition of these classes. Thereby it is clear that sets $A$ exist for which the correspondend equality holds. (For this one can take for instance automata similar to $\mathfrak{A}_i(m, n, k, l)$, $i = 1, 2, 3$. Only the state $s$ must be omitted.)

Now we prove (5) $A \in C_1{}^1$ implies $A = \varnothing$ and therefore $I_M(A) = 0$. Further, by Remark 1 any regular open set can be accepted with one final set only. Hence $I_M(A) \leqslant 1$ for $A \in D_1{}^1 \cup E_1{}^1 \cup C_1{}^2$ (see Theorem 12(1)). Together with the statements (1), (2), and (3) we have $I_M(A) = 1$ for $A \in D_1{}^1 \cup E_1{}^1 \cup C_1{}^2$.

Evidently, the set $F_k =_{\mathrm{df}} (01)^\omega \cup (0^21)^\omega \cup \cdots \cup (0^k1)^\omega$ is in $D_1{}^2$. Assume that $l_1 \neq l_2$ and $(0^{l_1} \cdot 1)^\omega$ as well as $(0^{l_2} \cdot 1)^\omega$ can be accepted with the same final set by the $\omega$-DFA $\mathfrak{A}$, i.e., $U(\Phi_{\mathfrak{A}}((0^{l_1}1)^\omega) = U(\Phi_{\mathfrak{A}}((0^{l_2}1)^\omega))_{\mathrm{df}} = Z'$. Then there is a $w \in X^*$ such that $U(\Phi_{\mathfrak{A}}(0^{l_1}1 \cdot w \cdot (0^{l_2} \cdot 1)^\omega)) = Z'$. Because of $0^{l_1}1 \cdot w(0^{l_2}1)^\omega \notin F_k$ we have $I_M(F_k) \geqslant k$. ∎

Now let $A \in P \geqslant D_1{}^2$. Then $1 \cdot F_k \cup 0 \cdot A \in P$ and we can conclude in the same manner that $I_M(1 \cdot F_k \cup 0 \cdot A) \geqslant k$.

The proof of statement (5) shows us why the Muller index measures not only structural complexities of an $\omega$-regular sets. The sets $(01)^\omega, (0^21)^\omega, (0^31)^\omega, \ldots$, are of the same structure but they cannot be accepted by the same final set. Therefore $I_M$ counts out not only how complicated the structure of an $\omega$-regular set is, but also how many times subsets of the same structure exist.

In the same spirit one can investigate the connections of many other measures for $\omega$-regular sets to our basic measures $m^+$, $m^-$, $n^+$, and $n^-$. Especially we think of measures counting the number of states necessary for the $T_\sigma{}^\alpha$-acceptance ($\alpha \in \{U, E\}$; $\sigma \in \{\subseteq, \nsubseteq, =\}$) of a $\omega$-regular set, or of measures indicating the structure of the corresponding $\omega$-regular expressions. However, we believe that these measures belong to the second category (i.e., they are not compatible with the basic measures).

## 9. $DS$- and $DA$-Degrees in the Classes $E_m{}^n$

By Corollaries 2 and 4 we know the coarse structure of $\mathscr{DS}(R)$ and $\mathscr{DA}(R)$, i.e., we know that $C_m{}^n$ and $D_m{}^n$ are $DS$- ($DA$-) degrees and that $E_m{}^n$ is the union of some $DS$- ($DA$-) degrees. The structure of the classes $C_m{}^n$, $D_m{}^n$, and $E_m{}^n$ with respect to $\leqslant_{DS}$ ($\leqslant_{DA}$) is represented by Fig. 6. For knowing the exact structure of $\mathscr{DS}(R)$ and $\mathscr{DA}(R)$ we have to study the following two questions

1. Which $DS$- ($DA$-) degrees contains $E_m{}^n$?
2. What is the structure of the set of these degrees?

In this section we shall answer these questions.

The "most difficult parts" of two sets $A, B \in E_m{}^n$ are essentially the same, i.e., both sets have $+$superchains as well as $-$superchains of length $n$. As we have seen in Theorem 11 all $\omega$-regular sets accepted by the final sets of a $+$superchain ($-$superchain) of length $n$ can be $DS$-reduced to each other. Therefore, if $A$ cannot be reduced to $B$ then those parts of $A$ and $B$ not accepted by the final sets of the superchains of length $n$ are responsible for it. For instance, if the $\omega$-DFA $\mathfrak{A}$ has a structure as shown in Fig. 10 and the $\omega$-DFA $\mathfrak{B}$ has a structure as shown in Fig. 11 then we have $T(\mathfrak{A})$, $T(\mathfrak{B}) \in E_3{}^2$, but $T(\mathfrak{A})$ cannot be reduced to $T(\mathfrak{B})$ because the structure of $\mathfrak{A}$ not belonging to the superchains is more complicated than that of $\mathfrak{B}$.

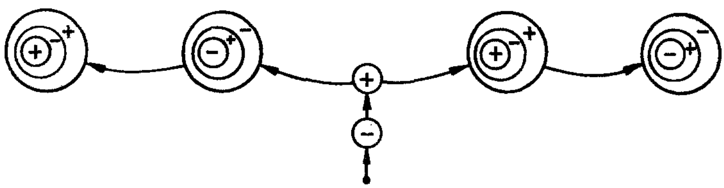

FIG. 10. The transition graph of $\mathfrak{A}$.

FIG. 11. The transition graph of %.

Now the idea is the following: Cutting off the superchains of length $n$ of $\mathfrak{A}$ and $\mathfrak{B}$ we get new automata $\partial\mathfrak{A}$ and $\partial\mathfrak{B}$, the so-called derivations of $\mathfrak{A}$ and $\mathfrak{B}$, and we now compare the structure of the derivations. These structure are simpler and as we shall see in Theorem 21:

$$\mathfrak{A} \text{ is reducible to } \mathfrak{B} \text{ iff } \partial\mathfrak{A} \text{ is reducible to } \partial\mathfrak{B}.$$

Applying this trick several times we can trace back the problem "$T(\mathfrak{A}) \leqslant_{DS} T(\mathfrak{B})$ for $T(\mathfrak{A})$, $T(\mathfrak{B}) \in E_m{}^n$" to a reducibility problem which can be answered easily.

We define now the notion of the derivation of an $\omega$-DFA. Let $m =_{\text{df}} \max(m^+(\mathfrak{A}), m^-(\mathfrak{A}))$ and $n =_{\text{df}} \max(n^+(\mathfrak{A}), n^-(\mathfrak{A}))$. The $\omega$-DFA $\partial\mathfrak{A} =_{\text{df}} [X, \partial Z, \partial f, \partial z_0, \partial\mathfrak{Z}]$ is said to be the first derivation of $\mathfrak{A}$ where

$$\begin{aligned} \partial Z &\underset{\text{df}}{=} (\partial^+ Z \cap \partial^- Z) \cup \{s^+, s^-\} && \text{if } \partial^+ Z \cap \partial^- Z \neq \varnothing \\ &\underset{\text{df}}{=} \{s^+\} && \text{if } \partial^- Z = \varnothing \\ &\underset{\text{df}}{=} \{s^-\} && \text{if } \partial^+ Z = \varnothing, \end{aligned}$$

$s^+$, $s^-$ are new states not in $Z$,

$$\partial^+ Z \underset{\text{df}}{=} \{z;\, z \in Z \wedge \exists(Z_1, \dots, Z_n)(Z_1 \in N_1^+(\mathfrak{A}) \wedge \cdots \wedge Z_n \in N_n^+(\mathfrak{A}) \wedge \{z_0\} \underset{\mathfrak{A}}{\vdash} \{z\} \underset{\mathfrak{A}}{\vdash} Z_1 \underset{\mathfrak{A}}{\vdash} \cdots \underset{\mathfrak{A}}{\vdash} Z_n)\}$$

is the set of all states from which $+$superchains of length $n$ are reachable,

$$\partial^- Z \underset{\text{df}}{=} \{z;\, z \in Z \wedge \exists(Z_1, \dots, Z_n)(Z_1 \in N_1^-(\mathfrak{A}) \wedge \cdots \wedge Z_n \in N_n^-(\mathfrak{A}) \wedge \{z_0\} \underset{\mathfrak{A}}{\vdash} \{z\} \underset{\mathfrak{A}}{\vdash} Z_1 \underset{\mathfrak{A}}{\vdash} \cdots \underset{\mathfrak{A}}{\vdash} Z_n)\}$$

is the set of all states from which $-$superchains of length $n$ are reachable,

$$\begin{aligned} \partial z_0 &\underset{\text{df}}{=} z_0 && \text{if } \partial^+ Z \cap \partial^- Z \neq \varnothing \\ &\underset{\text{df}}{=} s^+ && \text{if } \partial^- Z = \varnothing \\ &\underset{\text{df}}{=} s^- && \text{if } \partial^+ Z = \varnothing, \end{aligned}$$

$$\begin{aligned} \partial f(z, x) &\underset{\text{df}}{=} f(z, x) && \text{if } z \in \partial^+ Z \cap \partial^- Z \text{ and } f(z, x) \in \partial^+ Z \cap \partial^- Z \\ &\underset{\text{df}}{=} s^+ && \text{if } z \in \partial^+ Z \cap \partial^- Z \text{ and } f(z, x) \in \partial^+ Z \setminus \partial^- Z \\ &\underset{\text{df}}{=} s^- && \text{if } z \in \partial^+ Z \cap \partial^- Z \text{ and } f(z, x) \notin \partial^+ Z \\ &\underset{\text{df}}{=} s^+ && \text{if } z = s^+ \\ &\underset{\text{df}}{=} s^- && \text{if } z = s^-, \end{aligned}$$

and

$$\partial \mathfrak{Z} \underset{\text{df}}{=} \mathfrak{Z} \cup \{\{s^+\}\}.$$

Further $\partial^0 \mathfrak{A} =_{\text{df}} \mathfrak{A}$ and $\partial^{r+1} \mathfrak{A} =_{\text{df}} \partial \partial^r \mathfrak{A}$.

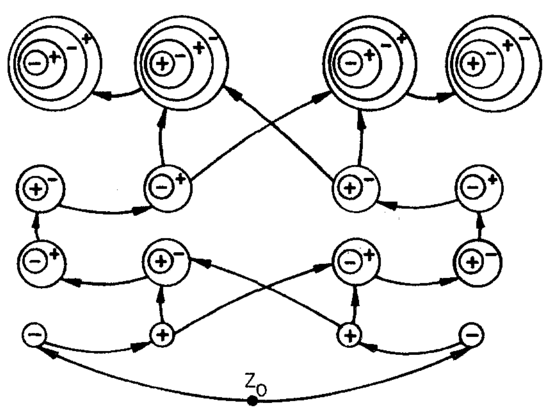


FIG. 12. The structure of $\mathfrak{A}$.

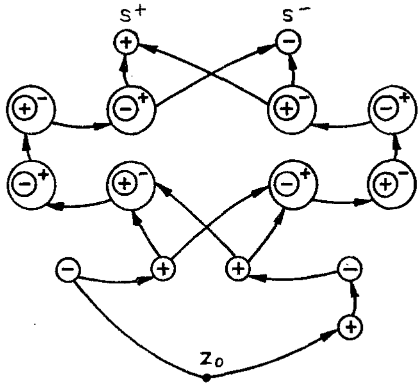


FIG. 13. The structure of $\partial\mathfrak{A}$.

EXAMPLE 3. Let $\mathfrak{A}$ be an $\omega$-DFA with the structure shown in Fig. 12. Then $T(\mathfrak{A}) \in E_4^2$ and $\partial\mathfrak{A}$ has the structure shown in Fig. 13. Further, $T(\partial\mathfrak{A}) \in E_2^4$ and $\partial^2\mathfrak{A}$ has the structure shown in Fig. 14. At last, $T(\partial^2\mathfrak{A}) \in D_1^4$ and $\partial^3\mathfrak{A}$ as well as every higher derivation has the structure shown in Fig. 15.

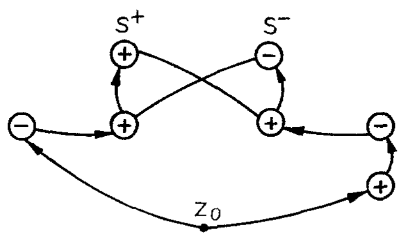


FIG. 14. The structure of $\partial^2\mathfrak{A}$.

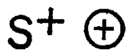


FIG. 15. The structure of $\partial^2\mathfrak{A}$.

Now we make some observations helpful for the further investigations.

LEMMA 20. (1) *$E(\Phi_{\mathfrak{A}}(\xi)) \subseteq \partial^+Z \cap \partial^-Z$ iff $E(\Phi_{\partial\mathfrak{A}}(\xi)) \subseteq \partial^+Z \cap \partial^-Z$*

(2) *$E(\Phi_{\mathfrak{A}}(\xi)) \subseteq \partial^+Z \cap \partial^-Z$ implies*

(a) $\Phi_{\mathfrak{A}}(\xi) = \Phi_{\partial\mathfrak{A}}(\xi)$,

(b) $\xi \in T(\mathfrak{A}) \Leftrightarrow \xi \in T(\partial\mathfrak{A})$.

(3) *If $\Phi_{\mathfrak{A}}(\xi)$ leaves $\partial^-Z$ earlier than $\partial^+Z$ then*

(a) $U(\Phi_{\partial\mathfrak{A}}(\xi)) = \{s^+\}$,

(b) $\xi \in T(\partial\mathfrak{A})$.

(4) *If $\Phi_{\mathfrak{A}}(\xi)$ leaves $\partial^+Z$ but not later than $\partial^-Z$ then*

(a) $U(\Phi_{\partial\mathfrak{A}}(\xi)) = \{s^-\}$,

(b) $\xi \notin T(\partial\mathfrak{A})$.

(5) *If $m = \max(m^+(\mathfrak{A}), m^-(\mathfrak{A}))$ then*

(a) *$Z' \in M_m^+$ implies $Z' \subseteq \overline{\partial^-Z}$,*

(b) *$Z' \in M_m^-$ implies $Z' \subseteq \overline{\partial^+Z}$.*

(6) *From any $z \in \partial^+Z \cap \partial^-Z$ the state $s^+$ as well as $s^-$ can be reached in $\partial\mathfrak{A}$.*

(7) *There is an algorithm for constructing $\partial\mathfrak{A}$ for given $\mathfrak{A}$.*

*Proof.* Statements (1), (2), (3), and (4) are evident

(5) Assume that $Z' \in M_m^+$ and there is a $z \in Z' \cap \partial^-Z$. Let $n = \max(n^+(\mathfrak{A}), n^-(\mathfrak{A}))$. Because of $z \in \partial^-Z$ a —superchain of length $n$ is reachable from $z$ and therefore also from $Z'$. However, $Z'$ is a maximal $+$chain and hence there is a $+$superchain of length $n+1$ which contradicts $n = \max(n^+(\mathfrak{A}), n^-(\mathfrak{A}))$. Statement (b) can be proved analogously.

(6) Let $z \in \partial^+Z \cap \partial^-Z$. Then a maximal $+$superchain can be reached from $z$. Let $Z'$ be the last member of the first chain of this maximal $+$superchain. Consequently, $Z' \subseteq \partial^+Z$ and $\{z\} \vdash_{\mathfrak{A}} Z'$. By statement (5a) of this lemma we have $Z' \subseteq \overline{\partial^-Z}$. Hence $\{z\} \vdash_{\mathfrak{A}} Z' \subseteq \partial^+Z \backslash \partial^-Z$ and $\{z\} \vdash_{\partial\mathfrak{A}} \{s^+\}$ by definition of $\partial\mathfrak{A}$. Analogously we have $\{z\} \vdash_{\partial\mathfrak{A}} \{s^-\}$.

(7) The definition of $\partial\mathfrak{A}$ can be considered as such an algorithm. ▮

The following theorem describes the complexity of the derivation of an $\omega$-DFA.

THEOREM 19. (1) *$T(\mathfrak{A}) \in C_m^n$ implies $T(\partial\mathfrak{A}) \in C_1^1$.*

(2) *$T(\mathfrak{A}) \in D_m^n$ implies $T(\partial\mathfrak{A}) \in D_1^1$.*

(3) *$T(\mathfrak{A}) \in E_m^n$ implies $T(\partial\mathfrak{A}) \in \hat{C}_m^1 \cap \hat{D}_m^1$ for $m \geq 2$.*

(4) *$T(\mathfrak{A}) \in E_1^n$ implies $T(\partial\mathfrak{A}) \in E_1^1$.*

*Proof.* (1) If $T(\mathfrak{A}) \in C_m^n$ then $\partial^+Z = \varnothing$, $\partial\mathfrak{A} = [X, \{s^-\}, s^-, s^-, \mathfrak{Z} \cup \{\{s^+\}\}]$ and hence $T(\partial\mathfrak{A}) = \varnothing \in C_1^1$.

(2) If $T(\mathfrak{A}) \in D_m^n$ then $\partial^-Z = \varnothing$, $\partial\mathfrak{A} = [X, \{s^+\}, s^+, s^+, \mathfrak{Z} \cup \{\{s^+\}\}]$ and hence $T(\partial\mathfrak{A}) = X^\omega \in D_1^1$.

(3) Assume that $T(\mathfrak{A}) \in E_m^n$, $m \geq 2$ and $T(\partial\mathfrak{A}) \notin \hat{C}_m^1 \cap \hat{D}_m^1$. Hence $m_0 =_{\text{df}} \max(m^+(\partial\mathfrak{A}), m^-(\partial\mathfrak{A})) \geq m$. Then there is a chain of length $m_0$ in $\partial\mathfrak{A}$.

Since $\{s^+\}$ and $\{s^-\}$ cannot contribute to a chain of length $>1$ this chain consists of subsets of $\partial^+Z \cap \partial^-Z$ and hence this chain of length $m_0$ is also a chain in $\mathfrak{A}$, i.e., there is a set $Z' \subseteq \partial^+Z \cap \partial^-Z$ such that $Z' \subseteq M_{m_0}^+ \cup M_{m_0}^-$. Because of $T(\mathfrak{A}) \in E_m{}^n$ the sets $M_{m_0}^+$ and $M_{m_0}^-$ are empty for $m_0 > m$. Consequently $m_0 = m$ and $Z' \subseteq M_m{}^+ \cup M_m{}^-$. This contradicts Lemma 20(5).

(4) Let $T(\mathfrak{A}) \in E_1{}^n$. Hence $\max(m^+(\mathfrak{A}), m^-(\mathfrak{A})) = 1$. By Lemma 20(5) any essential set of $\mathfrak{A}$ is a subset of $\overline{\partial^+Z \cup \partial^-Z}$. Therefore the only essential sets of $\partial\mathfrak{A}$ are $\{s^+\}$ and $\{s^-\}$. Because neither $\{s^+\} \vdash_{\partial\mathfrak{A}} \{s^-\}$ nor $\{s^-\} \vdash_{\partial\mathfrak{A}} \{s^+\}$ we have $m^+(\partial\mathfrak{A}) = m^-(\partial\mathfrak{A}) = n^+(\partial\mathfrak{A}) = n^-(\partial\mathfrak{A}) = 1$ and hence $T(\partial\mathfrak{A}) \in E_1{}^1$. ∎

In the following, together with an $\omega$-DFA $\mathfrak{A} = [X, Z, f, z_0, \mathfrak{Z}]$, the $\omega$-DFA $\mathfrak{A}_z =_{\text{df}} [X, Z, f, z, \mathfrak{Z}]$ for any $z \in Z$ is of interest. Here are some easy properties of $\mathfrak{A}_z$.

Lemma 21. (1) *$z_0 \vdash_{\mathfrak{A}} z$ implies $m^+(\mathfrak{A}_z) \leqslant m^+(\mathfrak{A})$ and $m^-(\mathfrak{A}_z) \leqslant m^-(\mathfrak{A})$.*

(2) *If $z_0 \vdash_{\mathfrak{A}} z$ and $\max(m^+(\mathfrak{A}_z), m^-(\mathfrak{A}_z)) = \max(m^+(\mathfrak{A}), m^-(\mathfrak{A}))$ then $n^+(\mathfrak{A}_z) \leqslant n^+(\mathfrak{A})$ and $n^-(\mathfrak{A}_z) \leqslant n^-(\mathfrak{A})$.*

(3) *If $z_0 \vdash_{\mathfrak{A}} z$ and $T(\mathfrak{A}) \in E_m{}^n$ then*

(a) *$z \in \partial^+Z$ implies $T(\mathfrak{A}_z) \in \check{D}_m{}^n$,*

(b) *$z \in \partial^-Z$ implies $T(\mathfrak{A}_z) \in \check{C}_m{}^n$,*

(c) *$z \notin \partial^+Z$ implies $T(\mathfrak{A}_z) \in \hat{C}_m{}^n$,*

(d) *$z \notin \partial^-Z$ implies $T(\mathfrak{A}_z) \in \hat{D}_m{}^n$.*

(4) *$T(\mathfrak{A}) = T(\mathfrak{A}')$ implies $T(\mathfrak{A}_{f(z_0, w)}) = T(\mathfrak{A}'_{f'(z_0', w)})$ for any $w \in X^*$.*

It turns out that the operation of derivation is actually an operation on $\omega$-regular sets independent on the special $\omega$-DFA accepting them.

Theorem 20. *$T(\mathfrak{A}) = T(\mathfrak{A}')$ implies $T(\partial\mathfrak{A}) = T(\partial\mathfrak{A}')$.*

*Proof.* First let $T(\mathfrak{A}) = T(\mathfrak{A}') \in C_m{}^n(D_m{}^n)$. Then by Theorem 19(1) (19(2)) we have $T(\partial\mathfrak{A}), T(\partial\mathfrak{A}') \in C_1{}^1(D_1{}^1)$ and consequently $T(\partial\mathfrak{A}) = T(\partial\mathfrak{A}') = \varnothing\ (X^\omega)$ by Theorem 8(3).

Now the case $T(\mathfrak{A}) = T(\mathfrak{A}') \in E_m{}^n$. First of all we prove that for any $w \in X^*$

$$f(z_0, w) \in \partial^+Z \Leftrightarrow f'(z_0', w) \in \partial^+Z' \tag{$*$}$$

and

$$f(z_0, w) \in \partial^-Z \Leftrightarrow f'(z_0', w) \in \partial^-Z' \tag{$**$}$$

It suffices to show (*); (**) can be proved similarly. Assume that there is a $w \in X^*$ such that $f(z_0, w) \in \partial^+Z$ and $f'(z_0', w) \notin \partial^+Z$. By Lemma 21(3a) and (3b) we have $T(\mathfrak{A}_{f(z_0, w)}) \in \check{D}_m{}^n$ and $T(\mathfrak{A}_{f'(z_0', w)}) \in \hat{C}_m{}^n$. However, $\hat{C}_m{}^n \cap \check{D}_m{}^n$

$= \varnothing$ implies $T(\mathfrak{A}_{f(z_0, w)}) \neq T(\mathfrak{A}'_{f'(z_0', w)})$ contradictory to Lemma 21(4). Thus (*) is proved.

We distinguish 3 cases

*Case* 1. $\Phi_{\mathfrak{A}}(\xi)$ does not leave $\partial^+ Z \cap \partial^- Z$, i.e., $E(\Phi_{\mathfrak{A}}(\xi)) \subseteq \partial^+ Z \cap \partial^- Z$. By (*) and (**), $\Phi_{\mathfrak{A}'}(\xi)$ does not leave $\partial^+ Z' \cap \partial^- Z'$ and by Lemma 20(2b) we have

$$\xi \in T(\partial\mathfrak{A}) \Leftrightarrow \xi \in T(\mathfrak{A}) = T(\mathfrak{A}') \Leftrightarrow \xi \in T(\partial\mathfrak{A}').$$

*Case* 2. $\Phi_{\mathfrak{A}}(\xi)$ leaves $\partial^- Z$ earlier than $\partial^+ Z$. By (*) and (**), $\Phi_{\mathfrak{A}'}(\xi)$ leaves $\partial^- Z'$ earlier than $\partial^+ Z'$ and by Lemma 20(3b) we have

$$\xi \in T(\partial\mathfrak{A}) \qquad \text{and} \qquad \xi \in T(\partial\mathfrak{A}').$$

*Case* 3. $\Phi_{\mathfrak{A}'}(\xi)$ leaves $\partial^+ Z$ but not later than $\partial^- Z$. By (*) and (**), $\Phi_{\mathfrak{A}'}(\xi)$ leaves $\partial^+ Z'$ but not later than $\partial^- Z'$ and by Lemma 20(4b) we have

$$\xi \notin T(\partial\mathfrak{A}) \qquad \text{and} \qquad \xi \notin T(\partial\mathfrak{A}'). \quad \blacksquare$$

Theorem 20 justifies the following definition.

For $A \in X^\omega$ and any $\omega$-DFA $\mathfrak{A}$ with $T(\mathfrak{A}) = A$ we define

$$\partial A \underset{\text{df}}{=} T(\partial\mathfrak{A})$$

and

$$\partial^0 A \underset{\text{df}}{=} A, \qquad \partial^{r+1} A \underset{\text{df}}{=} \partial\partial^r A.$$

Thus Theorem 19 can be formulated as follows

COROLLARY 9. (1) *$A \in C_m{}^n$ implies $\partial A \in C_1{}^1$.*

(2) *$A \in D_m{}^n$ implies $\partial A \in D_1{}^1$.*

(3) *$A \in E_m{}^n$ implies $\partial A \in \hat{C}_m{}^1 \cap \hat{D}_m{}^1$ for $m \geqslant 2$.*

(4) *$A \in E_1{}^n$ implies $\partial A \in E_1{}^1$.*

For the proof of the basic theorem about the connections between $\omega$-DFA and their derivations the following lemma is helpful.

LEMMA 22. *Let $T(\mathfrak{A})$, $T(\mathfrak{A}') \in E_m{}^n$.*

(1) *If $T(\mathfrak{A}) \leqslant_{DS} T(\mathfrak{A}')$ by $\mathfrak{B} = [X, X, \tilde{Z}, \tilde{f}, \tilde{g}, \tilde{z}_0]$ then for any $w \in X^*$*

$$T(\mathfrak{A}_{f(z_0, w)}) \leqslant_{DS} T(\mathfrak{A}'_{f'(z_0', \tilde{g}(\tilde{z}_0, w))}).$$

(2) *If $T(\mathfrak{A}) \leqslant_{DA} T(\mathfrak{A}')$ by $\mathfrak{B} = [X, X, \tilde{Z}, \tilde{f}, \tilde{g}, \tilde{z}_0]$ then for any $w \in X^*$*

$$T(\mathfrak{A}_{f(z_0, w)}) \leqslant_{DA} T(\mathfrak{A}'_{f'(z_0', \tilde{g}(\tilde{z}_0, w))}).$$

(3) *If* $T(\mathfrak{A}) \leqslant_{DA} T(\mathfrak{A}')$ *by* $\mathfrak{B} = [X, X, \tilde{Z}, \tilde{f}, \tilde{g}, \tilde{z}_0]$ *then for any* $w \in X^*$

(a) $f(z_0, w) \in \partial^+ Z$ *implies* $f'(z'_0, \tilde{g}(\tilde{z}_0, w)) \in \partial^+ Z'$,

(b) $f(z_0, w) \in \partial^- Z$ *implies* $f'(z'_0, \tilde{g}(\tilde{z}_0, w)) \in \partial^- Z'$.

(4) *If* $T(\partial\mathfrak{A}) \leqslant_{DA} T(\partial\mathfrak{A}')$ *by* $\mathfrak{B} = [X, X, \tilde{Z}, \tilde{f}, \tilde{g}, \tilde{z}_0]$ *then for any* $w \in X^*$

(a) $\partial f(\partial z_0, w) \neq s^+$ *implies* $\partial f'(\partial z'_0, \tilde{g}(\tilde{z}_0, w)) \neq s^+$,

(b) $\partial f(\partial z_0, w) \neq s^-$ *implies* $\partial f'(\partial z'_0, \tilde{g}(\tilde{z}_0, w)) \neq s^-$.

*Proof.* (1) The asked reduction can be made by the DSFA

$$\mathfrak{B}_{\tilde{f}(\tilde{z}_0,w)} = [X, X, \tilde{Z}, \tilde{f}, \tilde{g}, \tilde{f}(\tilde{z}_0, w)],$$

because of

$$\begin{aligned}
\xi \in T(\mathfrak{A}_{f(z_0,w)}) &\Leftrightarrow U(\Phi_{\mathfrak{A}_{f(z_0,w)}}(\xi)) \in \mathfrak{Z} \\
&\Leftrightarrow U(\Phi_{\mathfrak{A}}(w\xi)) \in \mathfrak{Z} \\
&\Leftrightarrow w\xi \in T(\mathfrak{A}) \\
&\Leftrightarrow \Phi_{\mathfrak{B}}(w\xi) \in T(\mathfrak{A}') \\
&\Leftrightarrow U(\Phi_{\mathfrak{A}'}(\Phi_{\mathfrak{B}}(w\xi))) \in \mathfrak{Z}' \\
&\Leftrightarrow U(\Phi_{\mathfrak{A}'}(\tilde{g}(\tilde{z}_0 w)\, \Phi_{\mathfrak{B}_{\tilde{f}(\tilde{z}_0,w)}}(\xi))) \in \mathfrak{Z}' \\
&\Leftrightarrow U(\Phi_{\mathfrak{A}'_{f'(z'_0,\tilde{f}(\tilde{z}_0,w))}}(\Phi_{\mathfrak{B}_{\tilde{f}(\tilde{z}_0,w)}}(\xi))) \in \mathfrak{Z}' \\
&\Leftrightarrow \Phi_{\mathfrak{B}_{\tilde{f}(\tilde{z}_0,w)}}(\xi) \in T(\mathfrak{A}'_{f'(z'_0,\tilde{g}(\tilde{z}_0,w))}).
\end{aligned}$$

(2) Can be proved analogously.

(3) Assume that $f(z_0, w) \in \partial^+ Z$ and $f'(z'_0, \tilde{g}(\tilde{z}_0, w)) \notin \partial^+ Z'$. By Lemma 21(1) we have $T(\mathfrak{A}_{f(z_0,w)}) \in \check{D}_m{}^n$ and $T(\mathfrak{A}'_{f'(z'_0,\tilde{g}(\tilde{z}_0,w))}) \in \hat{C}_m{}^n$. Because of statement (2) of this lemma $T(\mathfrak{A}_{f(z_0,w)})$ is $DA$-reducible to $T(\mathfrak{A}'_{f'(z'_0,\tilde{g}(\tilde{z}_0,w))})$ and therefore $T(\mathfrak{A}_{f(z_0,w)})$ is also in $\hat{C}_m{}^n$ (Theorem 10) contradictory to $\hat{C}_m{}^n \cap \check{D}_m{}^n = \varnothing$. Statement (b) can be proved similarly.

(4) Assume that $\partial f(\partial z_0, w) \neq s^+$ and $\partial f'(\partial z'_0, \tilde{g}(\tilde{z}_0, w)) = s^+$. Consequently $\tilde{g}(\tilde{z}_0, w)\xi \in T(\partial\mathfrak{A}')$ for any $\xi \in X^\omega$. (*)

Because of $\partial f(\partial z_0, w) \neq s^+$ and Lemma 20(6) the state $s^-$ can be reached from $\partial f(\partial z_0, w)$ in $\partial\mathfrak{A}$, for instance, with $v \in X^*$, i.e., $\partial f(\partial z_0, w \cdot v) = \partial f(\partial f(\partial z_0, w), v) = s^-$. Consequently $w \cdot v \cdot \xi \notin T(\partial\mathfrak{A})$ for any $\xi \in X^\omega$. Since $T(\partial\mathfrak{A}) \leqslant_{DA} T(\partial\mathfrak{A}')$ by $\mathfrak{B}$ we have $\tilde{g}(\tilde{z}_0, w)\Phi_{\mathfrak{B}_{\tilde{f}(\tilde{z}_0,w)}}(v \cdot \xi) \notin T(\partial\mathfrak{A}')$ for any $\xi \in X^\omega$ contradictory to (*).

Statement (b) can be proved similarly. ∎

We prove now the fact which is very important for the main problem of this section that for any $A, B \in E_m{}^n$ the reducibility of $\partial A$ to $\partial B$ is necessary and sufficient for the reducibility of $A$ to $B$.

THEOREM 21. *For any* $A, B \in E_m{}^n$

(1) $A \leqslant_{DS} B$ *iff* $\partial A \leqslant_{DS} \partial B$,

(2) $A \leqslant_{DA} B$ *iff* $\partial A \leqslant_{DA} \partial B$.

*Proof.* We show (1), statement (2) can be proved analogously.

Let $\mathfrak{A}, \mathfrak{A}'$ be $\omega$-DFA such that $T(\mathfrak{A}), T(\mathfrak{A}') \in E_m{}^n$ and $T(\mathfrak{A}) \leqslant_{DS} T(\mathfrak{A}')$ by a DSFA $\mathfrak{B} = [X, X, \tilde{Z}, \tilde{f}, \tilde{g}, \tilde{z}_0]$. We describe a DSFA $\mathfrak{B}'$ which reduces $T(\mathfrak{A})$ to $T(\partial\mathfrak{A}')$. Let $\xi \in X^\omega$ be an input sequence.

As long as the automaton $\partial\mathfrak{A}$ does not leave the set $\partial^+Z \cap \partial^-Z$, the DSFA $\mathfrak{B}'$ works as $\mathfrak{B}$. Hence, by Lemmas 20(1) and 22(3), in the same time the automaton $\partial\mathfrak{A}'$ does not leave the set $\partial^+Z' \cap \partial^-Z'$ on the input sequence $\Phi_{\mathfrak{B}}(\xi)$. If $\partial\mathfrak{A}$ reaches the state $s^+(s^-)$ then letter by letter $\mathfrak{B}'$ puts out a word which leads $\partial\mathfrak{A}'$ to the state $s^+$ $(s^-)$ also. By Lemma 20(6) this is possible. Further in each case $\mathfrak{B}'$ puts out the input symbol. We check now whether $\mathfrak{B}'$ works in the desired manner.

*Case* 1. Let $E(\Phi_{\mathfrak{A}}(\xi)) \subseteq \partial^+Z \cap \partial^-Z$. Hence $E(\Phi_{\mathfrak{A}'}(\Phi_{\mathfrak{B}}(\xi))) \subseteq \partial^+Z' \cap \partial^-Z'$ by Lemma 22(3), $\xi \in T(\mathfrak{A}) \Leftrightarrow \xi \in T(\partial\mathfrak{A})$ as well as $\Phi_{\mathfrak{B}}(\xi) \in T(\mathfrak{A}') \Leftrightarrow \Phi_{\mathfrak{B}}(\xi) \in T(\partial\mathfrak{A}')$ by Lemma 20(2b) and $\Phi_{\mathfrak{B}}(\xi) = \Phi_{\mathfrak{B}'}(\xi)$ by the construction on $\mathfrak{B}'$. Consequently $\xi \in T(\partial\mathfrak{A}) \Leftrightarrow \xi \in T(\mathfrak{A}) \Leftrightarrow \Phi_{\mathfrak{B}}(\xi) \in T(\mathfrak{A}') \Leftrightarrow \Phi_{\mathfrak{B}}(\xi) \in T(\partial\mathfrak{A}') \Leftrightarrow \Phi_{\mathfrak{B}'}(\xi) \in T(\partial\mathfrak{A}')$.

*Case* 2. $\Phi_{\mathfrak{A}}(\xi)$ leaves $\partial^-Z$ earlier than $\partial^+Z$. Hence $\Phi_{\partial\mathfrak{A}}(\xi)$ reaches $s^+$ by Lemma 20(3a) and $\Phi_{\partial\mathfrak{A}'}(\Phi_{\mathfrak{B}'}(\xi))$ reaches $s^+$ by the construction on $\mathfrak{B}'$. Consequently $\xi \in T(\partial\mathfrak{A})$ and $\Phi_{\mathfrak{B}'}(\xi) \in T(\partial\mathfrak{A}')$.

*Case* 3. $\Phi_{\mathfrak{A}}(\xi)$ leaves $\partial^+Z$ but not later than $\partial^-Z$. Hence $\Phi_{\partial\mathfrak{A}}(\xi)$ reaches $s^-$ by Lemma 20(4a) and $\Phi_{\partial\mathfrak{A}'}(\Phi_{\mathfrak{B}'}(\xi))$ reaches $s^-$ by the construction of $\mathfrak{B}'$. Consequently $\xi \notin T(\partial\mathfrak{A})$ and $\Phi_{\mathfrak{B}'}(\xi) \notin T(\partial\mathfrak{A}')$.

Now let $\mathfrak{A}, \mathfrak{A}'$ be $\omega$-DFA such that $T(\mathfrak{A}), T(\mathfrak{A}') \in E_m{}^n$ and $T(\partial\mathfrak{A}) \leqslant_{DS} T(\partial\mathfrak{A}')$ by a DSFA $\mathfrak{B} = [X, X, \tilde{Z}, \tilde{f}, \tilde{g}, \tilde{z}_0]$. We describe a DSFA $\mathfrak{B}'$ which reduces $T(\mathfrak{A})$ to $T(\mathfrak{A}')$. Let $\xi \in X^\omega$ be an input sequences.

(a) While the automaton $\mathfrak{A}$ does not leave the set $\partial^+Z \cap \partial^-Z$, the DSFA $\mathfrak{B}'$ works as $\mathfrak{B}$. Hence, by Lemma 22(3) in the same time the automaton $\mathfrak{A}'$ does not leave the set $\partial^+Z' \cap \partial^-Z'$ on the input sequence $\Phi_{\mathfrak{B}}(\xi)$.

(b) If $\mathfrak{A}$ leaves in the $k$th step $\partial^-Z$ but remains in $\partial^+Z$ then $\mathfrak{B}'$ puts out in this step such a symbol $x \in X$ that $f'(z_0', \Phi_{\mathfrak{B}'}(\xi)_0^k) = f'(z_0', \tilde{g}(\tilde{z}_0, \xi_0^{k-1})x) \in \partial^+Z$. Further the automaton $\mathfrak{B}'$ works as a DSFA which reduces $T(\mathfrak{A}_{f(z_0, \xi_0{}^k)})$ to $T(\mathfrak{A}'_{f'(z_0', \tilde{g}(\tilde{z}_0, \xi_0^{k-1})x)})$. The latter is possible because $T(\mathfrak{A}_{f(z_0, \xi_0{}^k)}) \in \hat{D}_m{}^n$ (Lemma 21(3d)) and $T(\mathfrak{A}'_{f'(z_0', \tilde{g}(\tilde{z}_0, \xi_0^{k-1})x)}) \in \check{D}_m{}^n$ (Lemma 21(3a)).

(c) If $\mathfrak{A}$ has been in the $(k-1)$th step in $\partial^+Z \cap \partial^-Z$ and leaves in the $k$th step $\partial^+Z$ then in this step $\mathfrak{B}'$ puts out such a symbol $x \in X$ that $f'(z_0', \Phi_{\mathfrak{B}'}(\xi)_0^k) = f'(z_0', \tilde{g}(\tilde{z}_0, \xi_0^{k-1})x) \in \partial^-Z$. Further the automaton $\mathfrak{B}'$ works as a DSFA which reduces $T(\mathfrak{A}_{f(z_0,\xi_0^k)})$ to $T(\mathfrak{A}'_{f'(z_0,\tilde{g}(\tilde{z}_0,\xi_0^{k-1})x)})$. The latter is possible because $T(\mathfrak{A}_{f(z_0,\xi_0^k)}) \in \hat{C}_m{}^n$ (Lemma 21(3c)) and $T(\mathfrak{A}'_{f'(z_0,\tilde{g}(\tilde{z}_0,\xi_0^{k-1})x)}) \in \check{C}_m{}^n$ (Lemma 21(3b)).

We check now whether $\mathfrak{B}'$ works in the desired manner.

*Case* 1. Let $E(\Phi_{\mathfrak{A}}(\xi)) \subseteq \partial^+Z \cap \partial^-Z$. Hence $E(\Phi_{\partial\mathfrak{A}}(\xi)) \subseteq \partial^+Z \cap \partial^-Z$ by Lemma 20(1) and $E(\Phi_{\partial\mathfrak{A}'}(\Phi_{\mathfrak{B}}(\xi))) \subseteq \partial^+Z' \cap \partial^-Z'$ by Lemma 22(4). Consequently

$$
\begin{aligned}
\xi \in T(\mathfrak{A}) &\Leftrightarrow \xi \in T(\partial\mathfrak{A}) && \text{(Lemma 20(2b))}\\
&\Leftrightarrow \Phi_{\mathfrak{B}}(\xi) \in T(\partial\mathfrak{A}') &&\\
&\Leftrightarrow \Phi_{\mathfrak{B}}(\xi) \in T(\mathfrak{A}') && \text{(Lemma 20(1) and (2b))}\\
&\Leftrightarrow \Phi_{\mathfrak{B}'}(\xi) \in T(\mathfrak{A}') && \text{(see item (a) of the construction of } \mathfrak{B}').
\end{aligned}
$$

*Case* 2. Let $f(z_0, \xi_0^{k-1}) \in \partial^+Z \cap \partial^-Z$ and $f(z_0, \xi_0^k) \in \partial^+Z \cap \partial^-Z$. By item (b) of the construction of $\mathfrak{B}'$ we have $\Phi_{\mathfrak{B}'}(\xi) = \tilde{g}(\tilde{z}_0, \xi_0^{k-1}) \cdot x \cdot \eta$, where $\xi_k{}^\omega \in T(\mathfrak{A}_{f(z_0,\xi_0^k)}) \Leftrightarrow \eta \in T(\mathfrak{A}'_{f'(z_0',\tilde{g}(\tilde{z}_0,\xi_0^{k-1})x)})$. Therefore

$$
\begin{aligned}
\xi \in T(\mathfrak{A}) &\Leftrightarrow \xi_k{}^\omega \in T(\mathfrak{A}_{f(z_0,\xi_0^k)})\\
&\Leftrightarrow \eta \in T(\mathfrak{A}'_{f'(z_0',\tilde{g}(\tilde{z}_0,\xi_0^{k-1})x)})\\
&\Leftrightarrow \tilde{g}(\tilde{z}_0, \xi_0^{k-1})x\eta \in T(\mathfrak{A}')\\
&\Leftrightarrow \Phi_{\mathfrak{B}'}(\xi) \in T(\mathfrak{A}').
\end{aligned}
$$

*Case* 3. Let $f(z_0, \xi_0^{k-1}) \in \partial^+Z \cap \partial^-Z$ and $f(z_0, \xi_0^k) \notin \partial^+Z$. This case can be treated analogously. ∎

Taking Corollaries 4 and 9 into consideration we get

COROLLARY 10. *Let* $A, B \in R$.

(1) $A \equiv_{DS(DA)} B \Leftrightarrow$ *there is a number* $r \geqslant 0$ *such that*

(a) $\partial^\rho A, \partial^\rho B \in E_{m_\rho}^{n_\rho}$ *for* $\rho = 0, \ldots, r-1$ *and suitable* $m_\rho, n_\rho$

(b) $\partial^r A, \partial^r B \in C_{m_r}^{n_r}$ *for suitable* $m_r, n_r$ *or*

$\partial^r A, \partial^r B \in D_{m_r}^{n_r}$ *for suitable* $m_r, n_r$ *or*

$\partial^r A, \partial^r B \in E_1{}^1$ *and* $\partial^r A \equiv_{DS(DA)} \partial^r B$.

(2) $A \leqslant_{DS(DA)} B \Leftrightarrow$ *there is a number* $r \geqslant 0$ *such that*

(a) $\partial^\rho A, \partial^\rho B \in E_{m_\rho}^{n_\rho}$ *for* $\rho = 0,\ldots, r-1$ *and suitable* $m_\rho, n_\rho$

(b) $\partial^r A \in P, \partial^r B \in Q$ *and* $P < Q$ *for suitable exact complexity classes* $P$ *and* $Q$ *or*

$\partial^r A, \partial^r B \in C_{m_r}^{n_r}$ *for suitable* $m_r, n_r$ *or*

$\partial^r A, \partial^r B \in D_{m_r}^{n_r}$ *for suitable* $m_r, n_r$ *or*

$\partial^r A, \partial^r B \in E_1^{\,1}$ *and* $\partial^r A \leqslant_{DS(DA)} \partial^r B$.

Consequently we know all $DS$- ($DA$-) degrees as well as the structure of $\mathscr{DS}(R)$ ($\mathscr{DA}(R)$) if we know all $DS$- ($DA$-) degrees in $E_1^{\,1}$ and the structure of the set of all $DS$- ($DA$-) degrees in $E_1^{\,1}$. We shall investigate the latter now.

First we note a lemma about $E_1^{\,1}$-sets.

LEMMA 23. *Let* $A \in E_1^{\,1}$.

(1) *If* $T(\mathfrak{A}) = A$ *for any* $\omega$-DFA $\mathfrak{A}$ *then* $\max\{| w |; f(z_0, w) \in \partial^+ Z \cap \partial^+ Z\}$ *is finite.*

(2) *There is a finite set* $W \subseteq X^*$ *such that* $A = W \cdot X^\omega$.

*Proof.* (1) Assume that $\max\{| w |; f(z_0, w) \in \partial^+ Z \cap \partial^- Z\} = \infty$. Then there is a essential set $Z' \subseteq \partial^+ Z \cap \partial^- Z$. This contradicts Lemma 20(5).

(2) Let $T(\mathfrak{A}) = A$ and $l =_{\mathrm{df}} \max\{| w |; f(z_0, w) \in \partial^+ Z \cap \partial^- Z\}$. We define $W^+ =_{\mathrm{df}} \{w; | w | = l + 1 \wedge f(z_0, w) \in \partial^+ Z\}$. Since maximal $+$ $(-)$ superchains consists of $+$ $(-)$ chains of length 1, i.e., of accepting (rejecting) sets we have for $z \in Z$

$$z \in \partial^+ Z \Leftrightarrow \text{an accepting set is reachable from } z,$$

$$z \in \partial^- Z \Leftrightarrow \text{an rejecting set is reachable from } z.$$

Consequently,

$$\begin{aligned} \xi \in A &\Leftrightarrow U(\Phi_{\mathfrak{A}}(\xi)) \text{ is an accepting set} \\ &\Leftrightarrow \text{for any } k \text{ an accepting set is reachable from } f(z_0, \xi_0^{\,k}) \\ &\Leftrightarrow f(z_0, \xi_0^{\,k}) \in \partial^+ Z \text{ for any } k \\ &\Leftrightarrow f(z_0, \xi_0^{l+1}) \in \partial^+ Z \\ &\Leftrightarrow \xi_0^{l+1} \in W^+ \\ &\Leftrightarrow \xi \in W^+ X^\omega. \quad \blacksquare \end{aligned}$$

Now question (1) can be answered for the $DA$-degrees.

THEOREM 22. *$E_1^1$ is a DA-degree.*

*Proof.* Because of Lemma 13(4) the class $E_1^1$ is a union of $DA$-degrees. It remains to prove that $A \leqslant_{DA} B$ for any $A, B \in E_1^1$.

Let $A, B \in E_1^1$. By Lemma 23(2) there are finite sets $W_1$, $W_2$ such that $A = W_1 \cdot X^\omega$ and $B = W_2 \cdot X^\omega$. The proof of Lemma 23(2) shows that $W_1$ and $W_2$ can be chosen in such a manner that $W_1 \subseteq X^{l_1}$ and $W_2 \subseteq X^{l_2}$ for suitable $l_1, l_2 > 0$. Because of $B \neq \varnothing \in C_1^1$ and $B \neq X^\omega \in D_1^1$ there are words $w_1 \in W_2$ and $w_2 \in X^{l_2} \backslash W_2$. We describe now a DAFA $\mathfrak{B}$ which reduces $A$ to $B$. Let $\xi \in X^\omega$. In each case in the first $l_1 - 1$ steps $\mathfrak{B}$ puts out the empty word. In the $l_1$th step $\mathfrak{B}$ puts out $w_1$ if $\xi^{l_1} \in W_1$ and $w_2$ if $\xi^{l_1} \notin W_1$. Further in each case $\mathfrak{B}$ puts out the input symbol. Thus

$$\begin{aligned} \xi \in A &\Leftrightarrow \xi_0^{l_1} \in W \\ &\Leftrightarrow \Phi_{\mathfrak{B}}(\xi)_0^{l_1} = W_1 \\ &\Leftrightarrow \Phi_{\mathfrak{B}}(\xi) \in W_2 X^\omega \\ &\Leftrightarrow \Phi_{\mathfrak{B}}(\xi) \in B. \quad \blacksquare \end{aligned}$$

As a direct consequence of this theorem, Corollary 9, and Theorem 21 we have

COROLLARY 11. *$E_1^n$ is a DA-degree for any $n \geqslant 1$.*

However, this result cannot be transferred to the case of $DS$-reducibility. The reason for this is very transparent: If the automaton $\mathfrak{B}$ which reduces $T(\mathfrak{A})$ to $T(\mathfrak{A}')$ must be synchronous then we have by Lemma 22(3)

$$\max\{|w|; f(z_0, w) \in \partial^+ Z \cap \partial^- Z\} \leqslant \max\{|w|; f'(z_0', w) \in \partial^+ Z' \cap \partial^- Z'\},$$

which cannot be for any two sets of $E_1^1$.

Nevertheless, this leads us to the idea that only the number $\max\{|w|; f(z_0, w) \in \partial^+ Z \cap \partial^- Z\}$ is responsible for the $DS$-reducibility of a set $T(\mathfrak{A}) \in E_1^1$ to another one. We define for any $\mathfrak{A}$ with $T(\mathfrak{A}) \in E_1^1$

$$l(\mathfrak{A}) \underset{\mathrm{df}}{=} \max\{|w|; f(z_0, w) \in \partial^+ Z \cap \partial^- Z\}.$$

By Lemma 23(1) this maximum is finite.

EXAMPLE 4. For the automaton $\mathfrak{A}_3(1, 1, k, l)$ defined in Example 1 evidently we have $l(\mathfrak{A}_3(1, 1, k, l)) = k$.

Now we state some properties of the measure 1

LEMMA 24. *Let $T(\mathfrak{A}) \in E_1^1$.*

(1) $T(\mathfrak{A}) \leqslant_{DS} T(\mathfrak{A}')$ *implies* $l(\mathfrak{A}) \leqslant l(\mathfrak{A}')$.

(2) $l(\mathfrak{A}) = \max\{| w |; \exists\xi(w \cdot \xi \in T(\mathfrak{A})) \wedge \exists\xi(w \cdot \xi \notin T(\mathfrak{A}))\}$.

(3) *There is an algorithm for computing* $l(\mathfrak{A})$ *for given* $\mathfrak{A}$.

*Proof.* (1) See the argument before the definition of (1).

(2) Here the hint that $f(z_0, w) \in \partial^+ Z \Leftrightarrow \exists\xi(w \cdot \xi \in T(\mathfrak{A}))$ and $f(z_0, w) \in \partial^- Z \Leftrightarrow \exists\xi(w \cdot \xi \notin T(\mathfrak{A}))$ suffices.

(3) To compute $l(\mathfrak{A})$ first of all we determine the set $\partial^+ Z \cap \partial Z$. Because no essential set is included in $\partial^+ Z \cap \partial^- Z$ (Lemma 20(5)) there is a largest chain $z_0 \vdash_{\mathfrak{A}} z_1 \vdash_{\mathfrak{A}} \cdots \vdash_{\mathfrak{A}} z_k$ such that $z_i \in \partial^+ Z \cap \partial^- Z$ for $i = o,\ldots, k$. Consequently, $l(\mathfrak{A}) = k$. ▌

Because of Lemma 24(2) the measure l depends only on the set $A \in E_1^1$ itself and not on the special automaton accepting $A$. This justifies the following definition.

$l(A) =_{\text{df}} l(\mathfrak{A})$ for any $A \in E_1^1$ and any $\omega$-DFA such that $T(\mathfrak{A}) = A$. Further we define for $k \geqslant 0$ the exact complexity class

$$E_k \underset{\text{df}}{=} \{A; l(A) = k\}$$

and the downward complexity class

$$\hat{E}_k \underset{\text{df}}{=} \{A; l(A) \leqslant k\}.$$

By Example 4 and Lemma 24 the classes $E_k$ and $\hat{E}_k$ have the following properties

LEMMA 25. (1) $E_k \neq \varnothing$ *for any* $k \geqslant 0$.

(2) $E_1^1 = \bigcup_{k\geqslant 0} E_k$.

(3) $A \in E_k$ *iff* $\max\{| w |; \exists\xi(w \cdot \xi \in A) \wedge \exists\xi(w \cdot \xi \notin \bar{A})\} = k$.

(4) $A \in \hat{E}_k$ *iff there is a* $W \subseteq X^{k+1}$ *such that* $A = W \cdot X^\omega$.

Now it is not hard to recognize the $DS$-degrees in $E_1^1$ and their structure with respect to $\leqslant_{DS}$.

THEOREM 23. *For any* $k \geqslant 0$

(1) $E_k$ *is a DS-degree.*

(2) $E_k \leqslant_{DS} E_{k+1}$.

(3) $\text{card}\, \hat{E}_k = 2^{(\text{card}\, X)^{k+1}}$.

*Proof.* By Lemma 24(1) we know that $A \equiv_{DS} B$ implies $l(A) = l(B)$ for $A, B \in E_1^1$ (i.e., $A, B \in E_k$ for a suitable $k \geqslant 0$). For statement (1) it remains to show that $l(A) \leqslant l(B)$ implies $A \leqslant_{DS} B$. This is sufficient also for statement (2).

Let $A, B \in E_1{}^1$ and $l(A) \leqslant l(B)$. By Lemma 25(4) we have sets $W_1 \subseteq X^{l(A)+1}$ and $W_2 \subseteq X^{l(B)+1}$ such that $A = W_1 \cdot X^\omega$ and $B = W_2 \cdot X^\omega$. Since $B \in E_1{}^1$ neither $B = \varnothing$ nor $B = X^\omega$ holds and consequently there is a $w \in X^{l(B)}$ and symbols $x_1, x_2 \in X$ such that $w \cdot x_1 \in W_2$ and $w \cdot x_2 \notin W_2$. We describe now a DSFA $\mathfrak{B}$ which reduces $A$ to $B$. Let $\xi \in X^\omega$. For $k = 1,\ldots, l(B)$ in the $k$th step $\mathfrak{B}$ puts out $w(k)$. In the $(l(B) + 1)$th step $\mathfrak{B}$ puts out $x_1$ if some initial word of $\xi_0^{l(B)+1}$ is in $W_1$ and $\mathfrak{B}$ puts out $x_2$ otherwise. Further $\mathfrak{B}$ puts out in each case the input symbol.

Thus

$$\begin{aligned} \xi \in A &\Leftrightarrow \xi_0^{l(A)+1} \in W_1 \\ &\Leftrightarrow \text{an initial word of } \xi_0^{l(B)+1} \text{ is in } W_1 \\ &\Leftrightarrow \Phi_{\mathfrak{B}}(\xi)_0^{l(B)+1} = wx_1 \\ &\Leftrightarrow \Phi_{\mathfrak{B}}(\xi) \in W_2 X^\omega \\ &\Leftrightarrow \Phi_{\mathfrak{B}}(\xi) \in A \end{aligned}$$

(3) By Lemma 25(4) we have for any $k \geqslant 0$

$$\operatorname{card} \hat{E}_k = \operatorname{card} \mathfrak{P}(X^{k+1}) = 2^{\operatorname{card} X^{k+1}} = 2^{(\operatorname{card} X)^{k+1}}. \quad \blacksquare$$

By Corollary 11 and Theorem 23 we can improve Corollary 10.

COROLLARY 12. *Let $A, B \in R$*

(1) $A \equiv_{DS(DA)} B \Leftrightarrow$ *there is a number $r \geqslant s$ such that*

(a) $\partial^\rho A, \partial^\rho B \in E_{m_\rho}^{n_\rho}$ *for $\rho = 0,\ldots, r - 1$ and suitable $m_\rho$, $n_\rho$,*

(b) $\partial^\rho A, \partial^\rho B \in C_{m_r}^{n_r}$ *for suitable $m_r$, $n_r$ or*

$\partial^\rho A, \partial^\rho B \in D_{m_r}^{n_r}$ *for suitable $m_r$, $n_r$ or*

$\partial^r A, \partial^r B \in E_k$ *for suitable $k$ ($\partial^r A, \partial^r B \in E_1^{n_r}$ for suitable $n_r$).*

(2) $A \leqslant_{DS(DA)} B \Leftrightarrow$

(a) $\partial^\rho A, \partial^\rho B \in E_{m_\rho}^{n_\rho}$ *for $\rho = 0,\ldots, r - 1$ and suitable $m_\rho$, $n_\rho$,*

(b) $\partial^r A \in P, \partial^r B \in Q$, *and $P < Q$ for suitable exact complexity classes $P$ and $Q$ or*

$\partial^r A, \partial^r B \in C_{m_r}^{n_r}$ *for suitable $m_r$, $n_r$ or*

$\partial^r A, \partial^r B \in D_{m_r}^{n_r}$ *for suitable $m_r$, $n_r$ or*

$$\partial^r A \in E_{k_1}, B \in E_{k_2}, \text{ and } k_1 \leqslant k_2 \qquad \text{for suitable } k_1, k_2$$

$$(\partial^r A, \partial^r B \in E_1^{n_r} \text{ for suitable } n_r)$$

Because of Corollary 12 we have a very strong connection between $\equiv_{DA}$ and $\equiv$ as well as $\leqslant_{DA}$ and $\leqslant$. Consequently, the relation $\leqslant_{DA}$ of $DA$-reducibility can be formulated in terms of the complexity-theoretic relation $\leqslant$.

COROLLARY 13. *Let $A, B \in R$.*

(1) *$A \equiv_{DA} B \Leftrightarrow \partial^r A \equiv \partial^r B$ for all $r \geqslant 0$.*

(2) *$A \leqslant_{DA} B \Leftrightarrow \partial^r A \equiv \partial^r B$, for all $r \geqslant 0$, or there is an $r \geqslant 0$ such that $\partial^\rho A \equiv \partial^\rho B$ for $\rho = 0, \ldots, r-1$ and $\partial^r A < \partial^r B$.*

Now we introduces a suggestive notation for the $DS$- as well as $DA$-degrees. We define

$$E_{m_0}^{n_0} E_{m_1}^{n_1} \cdots E_{m_{r-1}}^{n_{r-1}} C_{m_r}^{n_r} \underset{\text{df}}{=} \{A;\ \partial^\rho A \in E_{m_\rho}^{n_\rho} \text{ for } \rho = 0, \ldots, r-1 \text{ and } \partial^r A \in C_{m_r}^{n_r}\},$$

$$\begin{aligned} E_{m_0}^{n_0} E_{m_1}^{n_1} \cdots E_{m_{r-1}}^{n_{r-1}} D_{m_r}^{n_r} \underset{\text{df}}{=} \{A;\ &\partial^\rho A \in E_{m_\rho}^{n_\rho} \text{ for } \rho = 0, \ldots, r-1 \text{ and } \partial^r A \in D_{m_r}^{n_r}\} \\ \text{for} \quad &r \geqslant 0,\ m_0 > m_1 > \cdots > m_{r-1} > m_r \geqslant 1, \\ &n_0, n_1, \ldots, n_r \geqslant 1 \text{ and } m_r + n_r > 2, \end{aligned}$$

$$\begin{aligned} E_{m_0}^{n_0} E_{m_1}^{n_1} \cdots E_{m_{r-1}}^{n_{r-1}} E_k \underset{\text{df}}{=} \{A;\ &\partial^\rho A \in E_{m_\rho}^{n_\rho} \text{ for } \rho = 0, \ldots, r-1 \text{ and } \partial^r A \in E_k\} \\ \text{for} \quad &r \geqslant 0, m_0 > m_1 > \cdots > m_{r-1} \geqslant 1, m_{r-1} + n_{r-1} > 2, \\ &n_0, n_1, \ldots, n_{r-1} \geqslant 1 \text{ and } k \geqslant 0; \end{aligned}$$

$$\begin{aligned} E_{m_0}^{n_0} E_{m_1}^{n_1} \cdots E_{m_{r-1}}^{n_{r-1}} E_1^{n_r} \underset{\text{df}}{=} \{A;\ &\partial^\rho A \in E_{m_\rho}^{n_\rho} \text{ for } \rho = 0, \ldots, r-1 \text{ and } \partial^r A \in E_1^{n_r}\} \\ \text{for} \quad &r \geqslant 0,\ m_0 > m_1 > \cdots > m_{r-1} > 1 \text{ and} \\ &n_0, n_1, \ldots, n_r \geqslant 1. \end{aligned}$$

The next theorem completes and summarizes our knowledge about the $DS$- as well as $DA$-degrees and the structure of $\mathscr{DS}(R)$ and $\mathscr{DA}(R)$.

THEOREM 24. (1) *$C_1^1$ and $D_1^1$ are DS- as well as DA-degrees of cardinality 1.*

(2) *$E_k$ is a DS-degree of cardinality $2^{(\text{card}\, X)^{k+1}} - 2^{(\text{card}\, X)^k}$ for $k \geqslant 0$*

(3) *$E_{m_0}^{n_0} E_{m_1}^{n_1} \cdots E_{m_{r-1}}^{n_{r-1}} C_{m_r}^{n_r}$ is a DS- as well as a DA-degree of cardinality $\aleph_0$ for $r \geqslant 0$, $m_0 > m_1 > \cdots > m_r \geqslant 1$, $n_0, n_1, \ldots, n_r \geqslant 1$ and $m_r + n_r > 2$*

(4) $E^{n_0}_{m_0}E^{n_1}_{m_1} \cdots E^{n_{r-1}}_{m_{r-1}}D^{n_r}_{m_r}$ *is a DS- as well as a DA-degree of cardinality* $\aleph_0$ *for* $r \geqslant 0$, $m_0 > m_1 > \cdots > m_r \geqslant 1$, $n_0, n_1, \ldots, n_r \geqslant 1$ *and* $m_r + n_r > 2$

(5) $E^{n_0}_{m_0}E^{n_1}_{m_1} \cdots E^{n_{r-1}}_{m_{r-1}}E_k$ *is a DS-degree of cardinality* $\aleph_0$ *for* $r \geqslant 1$, $m_0 > m_1 > \cdots > m_{r-1} \geqslant 1$, $n_0, n_1, \ldots, n_{r-1} \geqslant 1$ *and* $m_{r-1} + n_{r-1} > 2$

(6) $E^{n_0}_{m_0}E^{n_1}_{m_1} \cdots E^{n_{r-1}}_{m_{r-1}}E^{n_r}_1$ *is a DA-degree of cardinality* $\aleph_0$ *for* $r \geqslant 0$, $m_0 > m_1 > \cdots > m_{r-1} > 1$ *and* $n_0, n_1, \ldots, n_r \geqslant 1$.

(7) *All DS- (DA-) degrees listed above are pairwise different.*

(8) *There are no other DS- (DA-) degrees.*

(9) *Let P, Q be DS-degrees.* $P \leqslant_{DS} Q \Leftrightarrow P = Q$ *or if* $P' \in \{C_m{}^n, D_m{}^n, E_m{}^n, E_k\}$ *and* $Q' \in \{C_m{}^n, D_m{}^n, E_m{}^n, E_k\}$ *are the first places in the names of P and Q, respectively, in which they are different, then* $P' < Q'$ *if* $P' \neq E_k$ *and* $Q' \neq E_k$, $E_1{}^1 < Q'$ *if* $P' = E_k$ *and* $Q' \neq E_k$, $P' < E_1{}^1$ *if* $P' \neq E_k$ *and* $Q' = E_k$, $k_1 < k_2$ *if* $P' = E_{k_1}$ *and* $Q' = E_{k_2}$.

(10) *Let P, Q be DA-degrees.* $P \leqslant_{DA} Q \Leftrightarrow P = Q$ *or if* $P' \in \{C_m{}^n, D_m{}^n, E_m{}^n\}$ *and* $Q' \in \{C_m{}^n, D_m{}^n, E_m{}^n\}$ *are the first places in the names of P and Q, respectively, in which they are different, then* $P' < Q'$.

*Proof.* That the classes listed in (1)–(6) are actually *DS*- and *DA*-degrees, respectively, is an immediate consequence of Corollary 12(1). Further, all names listed in (1)–(6) are pairwise different. The names of *DS*- (*DA*-) degrees are defined in such a way, that only the derivations up to the first one which is in $C_m{}^n$, $D_m{}^n$, or $E_k$ ($C_m{}^n$, $D_m{}^n$, $E_1{}^n$) (these classes are already known as *DS*- (*DA*-) degrees) are taken into consideration. Therefore, if the names are different then the degrees are different, too. And, because for every possible series of derivations of an $\omega$-regular set a corresponding class containing this set is defined, all *DS*- (*DA*-) degrees are listed in (1)–(6). Statements (9) and (10) are direct consequences of Corollary 12(2). Thus it remains to prove the cardinality statements.

For (1) and (2) they are already proved in Theorems 8(3) and 23(3), respectively.

(3) We use here the automata $\mathfrak{A}_i(m, n, k, l)$ defined in Example 1. Let $\mathfrak{A}_i{}^r(m, n, k, l)$ be that $\omega$-DFA which arises from $\mathfrak{A}_i(m, n, k, l)$ by renaming the states $s$ in $s(r)$, $s_\lambda$ is $s_\lambda(r)$, $\hat{z}_\mu{}^\nu$ in $\hat{z}_\mu{}^\nu(r)$, and $z_\mu{}^\nu$ in $z_\mu{}^\nu(z)$. The corresponding system of final sets is denoted by $\mathfrak{Z}_i{}^r(m, n)$.

Further let $\mathfrak{A}_3{}^0(m_0, n_0, k_0, l_0) \circ \mathfrak{A}_3{}^1(m_1, n_1, k_1, l_1) \circ \cdots \circ \mathfrak{A}_3^{r-1}(m_{r-1}, n_{r-1}, k_{r-1}, l_{r-1}) \circ \mathfrak{A}_1{}^r(m_r, n_r, k_r, l_r)$ be that $\omega$-DFA which is determined by the system $\bigcup_{\rho=0}^{r-1} \mathfrak{Z}_3{}^\rho(m_\rho, n_\rho) \cup \mathfrak{Z}_1{}^r(m_r, n_r)$ of final sets, the initial state $s_0(r)$, and a transition graph which originates from the transition graphs of the $\omega$-DFA

$\mathfrak{A}_3{}^0(m_0, n_0, k_0, l_0), \ldots, \mathfrak{A}_3^{r-1}(m_{r-1}, n_{r-1}, k_{r-1}, l_{r-1})$ and $\mathfrak{A}_1{}^r(m_r, n_r, k_r, l_r)$ in the following way:

The states $z_{m_\rho}^{n_\rho}(\rho)$ and $\hat{z}_{m_\rho}^{n_\rho}(\rho)$ are not removed in $z_0^{n_\rho}$ and $\hat{z}_0^{n_\rho}$, respectively, but in $s_0(\rho - 1)$ by input 1 for all $\rho = 1, \ldots, r + 1$. Evidently there holds

$$T(\mathfrak{A}_3{}^0(m_0, n_0, 0, 0) \circ \cdots \circ \mathfrak{A}_3^{r-1}(m_{r-1}, n_{r-1}, 0, 0) \circ \mathfrak{A}_1{}^r(m_r, n_r, 0, l))$$
$$\in E_{m_0}^{n_0} \cdots E_{m_{r-1}}^{n_{r-1}} C_{m_r}^{n_r}$$

for $m_r > 1$ and

$$T(\mathfrak{A}_3{}^0(m_0, n_0, 0, 0) \circ \cdots \circ \mathfrak{A}_3^{r-1}(m_{r-1}, n_{r-1}, 0, 0) \circ \mathfrak{A}_1{}^r(1, n_r - 1, 0, l))$$
$$\in E_{m_0}^{n_0} \cdots E_{m_{r-1}}^{n_{r-1}} C_1^{n_r}.$$

In both cases the accepted set has the form $A_l = 1 \cdot A_1 \cup 0^{l+1} A_2$, where $A_2 \neq \varnothing$ and $A_2 \neq 0^\omega$ and $A_1$, $A_2$ do not depend on $(l)$. Consequently $A_{l_1} \neq A_{l_2}$ for $l_1 \leqslant l_2$. (4)–(6) can be treated similarly, where in (5) the result of Example 4 must be taken into consideration. ∎

With Theorem 24 we know all $DS$- ($DA$-) degrees and the structures of $\mathscr{DS}(R)(\mathscr{DA}(R))$. We summarize once more: By Corollary 4 we know that $C_m{}^n$ and $D_m{}^n$ are $DS$- ($DA$-) degrees and that the coarse structure of $\mathscr{DS}(R)(\mathscr{DA}(R))$ is represented in Fig. 6.

It remains to speak about the $DS$- ($DA$-) degrees of $E_m{}^n$ and their structure. The class $E_1{}^1$ decomposes into the $DS$-degrees $E_0, E_1, \ldots$ with $E_0 <_{DS} E_1 <_{DS} E_2 < \cdots$, and the $DS$-degrees in $E_1{}^n$ have all the same structure. Further the classes $E_1{}^n$ are $DA$-degrees themselves. The set of all $DS$- ($DA$-) degrees in $E_2{}^n$ has the same structure as the set of all $DS$- ($DA$-) degrees in $\hat{C}_2{}^1 \cap \hat{D}_2{}^1 \backslash (C_1{}^1 \cup D_1{}^1)$. The set of all $DS$- ($DA$-) degrees in $E_3{}^n$ has the same structure as the set of all $DS$- ($DA$-) degrees in $\hat{C}_3{}^1 \cap \hat{D}_3{}^1 \backslash (C_1{}^1 \cup D_1{}^1) \cdots$. The set of all $DS$- ($DA$-) degrees in $E_m{}^n$ has the same structure as the set of all $DS$- ($DA$-) degrees in $\hat{C}_m{}^1 \cap \hat{D}_m{}^1 \backslash (C_1{}^1 \cup D_1{}^1)$ for $m > 1$. Thus one can form an idea of the structure of $\mathscr{DS}(R)$ and $\mathscr{DA}(R)$.

By the way, this implies directly

Corollary 14. *Maximal chains in $\mathscr{DS}(R)(\mathscr{DA}(R))$ are of type $\omega^\omega$.*

Further, because the measures $m^+$, $m^-$, $n^+$, $n^-$, and $l$ as well the derivation of an $\omega$-DFA is computable there holds.

Corollary 15. *There is an algorithm for computing the DA-degree of $T(\mathfrak{A})$ for given $\omega$-DFA $\mathfrak{A}$.*

## 10. THE TOPOLOGICAL NATURE OF THE REDUCIBILITIES

In Section 7 we have exhibited the topological nature of all complexity classes $\hat{C}_m{}^n$, $\hat{D}_m{}^n$, and $\hat{E}_m{}^n$, i.e., by Corollary 6(1), that $A \leqslant_C B$ implies $A \leqslant B$. However, $\leqslant_{DA}$ and $\leqslant_{DS}$ are weaker than $\leqslant$ and therefore the question arises whether $A \leqslant_{CA} B$ implies $A \leqslant_{DA} B$ or $A \leqslant_{CS} B$ implies $A \leqslant_{DS} B$ also. By Lemma 18(2) this would say that $\leqslant_{CA}$ and $\leqslant_{DA}$ as well as $\leqslant_{CS}$ and $\leqslant_{DS}$ coincide for $\omega$-regular sets. We show that this is actually the case.

We start with the following lemma

LEMMA 26. *Let $A, B \in E_m{}^n$.*

(1) *$A \leqslant_{CA} B$ implies $\partial A \leqslant_{CA} \partial B$,*

(2) *$A \leqslant_{CS} B$ implies $\partial A \leqslant_{CS} \partial B$.*

*Proof.* Let $A \leqslant_{CA} B$ ($A \leqslant_{CS} B$). We construct now a function reducing $\partial A$ to $\partial B$ in the same way as we have done it in Theorem 21 for showing $\partial A \leqslant_{DA} \partial B$ if $A \leqslant_{DA} B$ ($\partial A \leqslant_{DS} \partial B$ if $A \leqslant_{DS} B$). There we start from a function $\Phi_{\mathfrak{B}}$ generated by a DSFA $\mathfrak{B}$ and get such a function again. Here we start from a (synchronous) continous function and, because the construction in Theorem 21 preserves the sequential mode of working of the original function, the so-constructed reducing function must be synchronous and continous too (see Lemma 1(1)). ∎

THEOREM 25. *For any $\omega$-regular sets $A, B$ we have $A \leqslant_{DA} B$ iff $A \leqslant_{CA} B$, and $A \leqslant_{DS} B$ iff $A \leqslant_{CS} B$.*

*Proof.* Because of Lemma 18(2) we have only to show that $A \leqslant_{CA} B$ implies $A \leqslant_{DA} B$ ($A \leqslant_{CS} B$ implies $A \leqslant_{DS} B$). First of all $A \leqslant_{CA} B$ ($A \leqslant_{CS} B$) implies $A \leqslant B$ (Corollary 6(1)).

By Corollary 4(2) we have $A \leqslant_{DA} B$ ($A \leqslant_{DS} B$) or $A, B \in E_m{}^n$ for suitable $m, n \geqslant 1$. In the latter case there holds $\partial A \leqslant_{CA} \partial B$ ($\partial A \leqslant_{CS} \partial B$) by Lemma 26. This implies $\partial A \leqslant \partial B$ (Corollary 6(1)) and by Corollary 4(2) we have $\partial A \leqslant_{DA} \partial B$ ($\partial A \leqslant_{DS} \partial B$) or $\partial A, \partial B \in E_m{}^n$ for suitable $m, n \geqslant 1$. We continue in such a way and we get $\partial^r A \leqslant_{DA} \partial^r B$ ($\partial^r A \leqslant_{DS} \partial^r B$) and $\partial^\rho A$, $\partial^\rho B \in E_{m_\rho}^{n_\rho}$ for $r \geqslant 0$, $\rho = 0,\ldots, r-1$ and suitable $m_\rho$, $n_\rho \geqslant 1$, or $\partial^\rho A$, $\partial^\rho B \in E_{m_\rho}^{n_\rho}$ and $\partial^\rho A \leqslant_{CA} \partial^\rho B$ ($\partial^\rho A \leqslant_{DS} \partial^\rho B$) for all $\rho \geqslant 0$ and suitable $m_\rho$, $n_\rho \geqslant 1$. The first implies $A \leqslant_{CA} B$ ($A \leqslant_{DS} B$) by Theorem 21. The latter implies $\partial^r A$, $\partial^r B \in E_1{}^1$ for sufficient large $r$ (Corollary 9). By Theorem 22 we have $\partial^r A \leqslant_{DA} \partial^r B$ and this implies $A \leqslant_{DA} B$ by Theorem 21. (By Lemma 25.2 we have $\partial^r A \in E_{k_1}$ and $\partial^r B \in E_{k_2}$ for suitable $k_1$, $k_2 \geqslant 0$. Since $\partial^r A \leqslant_{CS} \partial^r B$ implies, for instance, $0^{k_1} \cdot 1 \cdot X^\omega \leqslant_{CS} 0^{k_2} \cdot 1 \cdot X^\omega$ we have $k_1 \leqslant k_2$ and therefore $\partial^r A \leqslant_{DS} \partial^r B$. Consequently, by Theorem 21 we have $A \leqslant_{DS} B$.) ∎

Thus, for any $DA$-degree ($DS$-degree) $P$ the class

$$\hat{P} \underset{\text{df}}{=} \{A; \exists B(B \in P \wedge A \leqslant_{DA} B)\} = \bigcup \{Q; Q\ DA\text{-degree} \wedge Q \leqslant_{DA} P\},$$

$$(\hat{P} \underset{\text{df}}{=} \{A; \exists B(B \in P \wedge A \leqslant_{DS} B)\} = \bigcup \{Q; Q\ DS\text{-degree} \wedge Q \leqslant_{DS} P\}$$

is closed under inverse (synchronous) continuous mappings, i.e.,

COROLLARY 16. (1) *Let $P$ be a DA-degree. Then $A \leqslant_{CA} B$ and $B \in \hat{P}$ implies $A \in \hat{P}$.*

(2) *Let $P$ be a DS-degree. Then $A \leqslant_{CS} B$ and $B \in \hat{P}$ implies $A \in \hat{P}$.*

Consequently, for any $DA$-degree $P$ the property "$A \in \hat{P}$" is a topological one. On the other hand other topological properties of $\omega$-regular sets cannot exist, because other topological properties would give rise to other "$C$-degrees" and hence to other $DA$-degrees, which cannot be. By the way, using the topological characterizations of $\hat{C}_m{}^n$, $\hat{D}_m{}^n$, and $\hat{E}_m{}^n$ one easily gets a characterization of all these properties "$A \in \hat{P}$" in terms of topology.

*Remark* 3. As J. R. Buchi pointed out[6] the result $A \leqslant_{DS} B \Leftrightarrow A \leqslant_{CS} B$ can be also derived from Büchi and Landweber (1969b). Since

$$\mathfrak{C}(\xi, \eta) \underset{\text{df}}{\equiv} \xi \in T(\mathfrak{A}) \Leftrightarrow \eta \in T(\mathfrak{A}')$$

for given $\omega$-DFA $\mathfrak{A}$, $\mathfrak{A}'$ is a sequential finite-state condition, we have (see Büchi and Landweber, 1969b):

If there is a synchronous continous solution $\Phi$ of $\mathfrak{C}$ (i.e., $\mathfrak{C}(\xi, \Phi(\xi))$ for all $\xi \in X^\omega$) then there is even a synchronous finite-state solution of $\mathfrak{C}$ (i.e., there is a DSFT $\mathfrak{B}$ such that $\mathfrak{C}(\xi, \Phi_{\mathfrak{B}}(\xi))$ for all $\xi \in X^\omega$). However, this means $A \leqslant_{CS} B \Leftrightarrow A \leqslant_{DS} B$.


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[6] Personal communication.

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