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Quickly Excluding a Forest

DAN BIENSTOCK*

Bell Communications Research,
445 South Street, Morristown, New Jersey 07960

NEIL ROBERTSON†

Department of Mathematics, Ohio State University,
231 W. 18th Avenue, Columbus, Ohio 43210

PAUL SEYMOUR

Bell Communications Research,
445 South Street, Morristown, New Jersey 07960

AND

ROBIN THOMAS‡

DIMACS Center, Hill Center, Busch Campus, Rutgers University,
New Brunswick, New Jersey 08903

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We prove that for every forest F, every graph with no minor isomorphic to F has path-width at most $|V(F)|-2$. © 1991 Academic Press, Inc.

1. INTRODUCTION

A *path-decomposition* of a graph G is a sequence $(W_1, ..., W_m)$ of subsets of $V(G)$ (the vertex set of G) such that

* Current address: Department of Industrial Engineering and Operations Research, Columbia University, New York, NY 10027.

† This research was performed under a consulting agreement with Bellcore.

‡ Current address: School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332.

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(i) $W_1 \cup \cdots \cup W_m = V(G)$, and for every $e \in E(G)$ there exists i with $1 \leqslant i \leqslant m$ such that W_i contains both ends of e

(ii) for $1 \leqslant i \leqslant j \leqslant k \leqslant m$, $W_i \cap W_k \subseteq W_j$.

(Graphs in this paper are finite and may have loops or multiple edges; $E(G)$ denotes the edge set of G.) The *path-width* of G is the minimum of $\max(|W_i| - 1 : 1 \leqslant i \leqslant m)$, taken over all path-decompositions $(W_1, ..., W_m)$ of G. (The null graph has path-width -1.)

In the first paper of the Graph Minors series [5], two of us proved that for every forest F, there is a number $N(F)$ such that every graph with path-width $\geqslant N(F)$ has a minor isomorphic to F. (A graph H is a *minor* of G if H can be obtained from a subgraph of G by contracting edges.) The proof was elaborate, and gave an upper bound on $N(F)$ which was enormous as a function of $|V(F)|$. In this paper we give a simple proof of a rather dramatic improvement, the following.

(1.1) *For every forest F, every graph with path-width $\geqslant |V(F)| - 1$ has a minor isomorphic to F.*

One reason that this simple proof was overlooked in [5] was that at that time, the concept of a "tangle" had not been developed, and that idea seems to be crucial. Tangles are obstructions to having small "tree-width" (see [6, 7]), but in Section 2 we introduce variants of tangles called "blockages," which are obstructions to having small path-width. In Section 3 we exploit properties of blockages to prove the theorem. Section 4 contains a discussion of the relationship between blockages and tangles, and finally in Section 5 we give a second application of our theorems about blockages, obtaining a short proof of a theorem of Kirousis and Papadimitriou about graph searching.

We remark that (1.1) is best possible in two senses. First, for every forest F with $V(F) \neq \varnothing$, the bound $|V(F)| - 1$ is best possible, for if $|V(F)| = n \geqslant 1$ then the complete graph K_{n-1} has path-width $|V(F)| - 2$ and has no minor isomorphic to F. Second, as was observed in [5], if F is not a forest then there is no number $N(F)$ such that all graphs with path-width $\geqslant N(F)$ have a minor isomorphic to F, for trees have arbitrarily high path-width and all their minors are forests.

2. Blockages

Let G be a graph, and let $V(G) = V$. For $X \subseteq V$ we denote by att(X) the set of all $v \in X$ with a neighbour in $V - X$ (att stands for "attachments"), and we write $\alpha(X) = |\text{att}(X)|$. Two subsets $X_1, X_2 \subseteq V$ are *complementary* if $X_1 \cup X_2 = V(G)$ and $\text{att}(X_1) \subseteq X_2$ (or equivalently, $\text{att}(X_2) \subseteq X_1$).

Let $n \geqslant 0$ be an integer. A *blockage* (*in G, of order n*) is a set $\mathscr{B}$ such that

(i) each $X \in \mathscr{B}$ is a subset of V with $\alpha(X) \leqslant n$

(ii) if $X \in \mathscr{B}$ and $Y \subseteq X$ and $\alpha(Y) \leqslant n$, then $Y \in \mathscr{B}$

(iii) if X_1, X_2 are complementary and $|X_1 \cap X_2| \leqslant n$, then $\mathscr{B}$ contains exactly one of X_1, X_2.

We call these the *blockage axioms*. The main result of this section is the following, which may be regarded as a kind of minimax theorem for path-width.

(2.1) *There is a blockage of order n if and only if G has path-width $\geqslant n$.*

In the next section we show that if G has a blockage of order n then it has a minor isomorphic to each forest with $n+1$ vertices. Our proof of (2.1) closely resembles the proof of Theorem (4.3) of [6] (see also [1, 7] for related results). We need a series of lemmas. If $X \subseteq V$, we define $X^c = (V - X) \cup \operatorname{att}(X)$. (Thus $(X^c)^c \subseteq X$, but equality need not hold.)

(2.2) *Let $\mathscr{B}$ be a blockage of order n in G, let $X \in \mathscr{B}$, and let $Y \subseteq V$ with $\alpha(Y) \leqslant n$ and $|(Y - X) \cup \operatorname{att}(X)| \leqslant n$. Then $Y \in \mathscr{B}$.*

Proof. Since X, X^c are complementary, $|X \cap X^c| \leqslant \alpha(X) \leqslant n$, and $X \in \mathscr{B}$, it follows from the third blockage axiom that $X^c \notin \mathscr{B}$. Since $X \cup Y$, X^c are complementary and

$$|(X \cup Y) \cap X^c| = |(Y - X) \cup \operatorname{att}(X)| \leqslant n,$$

it follows from the same axiom that $X \cup Y \in \mathscr{B}$. Now $\alpha(Y) \leqslant n$, and so $Y \in \mathscr{B}$ from the second axiom. ∎

Let $\mathscr{B}$ satisfy the first two blockage axioms for some given n. A *fracture* in $\mathscr{B}$ is a sequence $(X_1, ..., X_m)$ with $m \geqslant 1$ such that

(i) for $1 \leqslant i \leqslant m$, $X_i \subseteq V$ and $\alpha(X_i) \leqslant n$

(ii) $X_1, X_m^c \in \mathscr{B}$

(iii) for $1 \leqslant i < m$, $|(X_{i+1} - X_i) \cup \operatorname{att}(X_i)| \leqslant n$.

A fracture $(X_1, ..., X_m)$ is *simple* if in addition $X_1 \subseteq X_2 \subseteq \cdots \subseteq X_m$.

(2.3) *Let $\mathscr{B}$ satisfy the first two blockage axioms. If there is a fracture in $\mathscr{B}$ then there is a simple fracture.*

Proof. Let us choose a fracture $(X_1, ..., X_m)$ such that

(a) $\sum_i \alpha(X_i)$ is minimum

(b) subject to (a), $\sum_i |X_i|$ is minimum.

We show that $(X_1, ..., X_m)$ is simple. For suppose that $X_i \nsubseteq X_{i+1}$ for some i with $1 \leqslant i \leqslant m-1$.

(1) $\alpha(X_i \cap X_{i+1}) > \alpha(X_i)$.

Let $X_i' = X_i \cap X_{i+1}$, and suppose that $\alpha(X_i') \leqslant \alpha(X_i)$. Then $(X_1, ..., X_{i-1}, X_i', X_{i+1}, ..., X_m)$ is a fracture, for

$$|X_{i+1} - X_i'| = |X_{i+1} - X_i| \leqslant n - \alpha(X_i) \leqslant n - \alpha(X_i')$$

and $|X_i' - X_{i-1}| \leqslant |X_i - X_{i-1}| \leqslant n - \alpha(X_{i-1})$ (if $i = 1$ we observe, instead, that $X_1' \in \mathscr{B}$, since $X_1' \subseteq X_1 \in \mathscr{B}$ and $\alpha(X_1') \leqslant \alpha(X_1) \leqslant n$). But this contradicts either (a) or (b).

(2) $\alpha(X_i \cup X_{i+1}) \geqslant \alpha(X_{i+1})$.

Let $X_{i+1}' = X_i \cup X_{i+1}$, and suppose that $\alpha(X_{i+1}') < \alpha(X_{i+1})$. Then $(X_1, ..., X_i, X_{i+1}', X_{i+2}, ..., X_m)$ is a fracture, for $|X_{i+1}' - X_i| = |X_{i+1} - X_i| \leqslant n - \alpha(X_i)$ and

$$|X_{i+2} - X_{i+1}'| \leqslant |X_{i+2} - X_{i+1}| \leqslant n - \alpha(X_{i+1}) \leqslant n - \alpha(X_{i+1}')$$

if $i+1 < m$ (if $i+1 = m$ we observe, instead, that $X_m'^{c} \in \mathscr{B}$, since $X_m'^{c} \subseteq X_m^{c} \in \mathscr{B}$ and $\alpha(X_m'^{c}) \leqslant \alpha(X_m') \leqslant \alpha(X_m) \leqslant n$). But this contradicts (a).

Now it is easily verified that

$$\operatorname{att}(X_i \cap X_{i+1}) \cup \operatorname{att}(X_i \cup X_{i+1}) \subseteq \operatorname{att}(X_i) \cup \operatorname{att}(X_{i+1}),$$

$$\operatorname{att}(X_i \cap X_{i+1}) \cap \operatorname{att}(X_i \cup X_{i+1}) \subseteq \operatorname{att}(X_i) \cap \operatorname{att}(X_{i+1}).$$

It follows that $\alpha(X_i \cap X_{i+1}) + \alpha(X_i \cup X_{i+1}) \leqslant \alpha(X_i) + \alpha(X_{i+1})$, contrary to (1) and (2). Thus there is no such i, and so $(X_1, ..., X_m)$ is simple, as required. ∎

(2.4) *If* $X \subseteq V$ *then* $X - \operatorname{att}(X) \subseteq Y - \operatorname{att}(Y)$ *and hence* $Y^c \subseteq X^c$ *and* $(X^c - Y^c) \cup \operatorname{att}(Y^c) \subseteq (Y - X) \cup \operatorname{att}(X)$.

Proof. Certainly $X - \operatorname{att}(X) \subseteq Y - \operatorname{att}(Y)$, for if X includes all neighbours of $v \in V$ then so does Y. Since $X^c = V - (X - \operatorname{att}(X))$ and $Y^c = V - (Y - \operatorname{att}(Y))$, it follows that $Y^c \subseteq X^c$. Moreover,

$$\operatorname{att}(Y^c) \subseteq \operatorname{att}(Y) \subseteq (Y - X) \cup \operatorname{att}(X)$$

and

$$\begin{aligned} X^c - Y^c &= (V - (X - \operatorname{att}(X))) - (V - (Y - \operatorname{att}(Y))) \\ &= (Y - \operatorname{att}(Y)) - (X - \operatorname{att}(X)) \subseteq (Y - X) \cup \operatorname{att}(X). \end{aligned}$$

The lemma follows. ∎

For a given integer $n \geqslant 0$, a *partial blockage* is a set $\mathscr{B}$ satisfying the first two blockage axioms, with $\varnothing \in \mathscr{B}$, such that there is no fracture in $\mathscr{B}$.

(2.5) *Given $n \geqslant 0$, every partial blockage is a subset of a blockage of order n.*

Proof. Let $\mathscr{B}$ be a partial blockage. We assume the result is true for every partial blockage $\mathscr{B}'$ with $|\mathscr{B}'| > |\mathscr{B}|$. We may assume that $\mathscr{B}$ is not a blockage of order n.

(1) *There exist complementary sets B_1, $B_2 \subseteq V$ with $|B_1 \cap B_2| \leqslant n$, such that $B_1, B_2 \notin \mathscr{B}$.*

Since $\mathscr{B}$ is not a blockage of order n and satisfies the first two blockage axioms, it does not satisfy the third. Thus there exist complementary $B_1, B_2 \subseteq V$ with $|B_1 \cap B_2| \leqslant n$ such that $\mathscr{B}$ contains both or neither B_1, B_2. If it contains both, then the 1-term sequence (B_1) is a fracture in $\mathscr{B}$ (for $B_1^c \subseteq B_2$ and hence $B_1^c \in \mathscr{B}$, since $\mathscr{B}$ satisfies the second blockage axiom), which is impossible. Thus $B_1, B_2 \notin \mathscr{B}$, as required.

Choose B_1, B_2 as in (1) with B_1 minimal.

(2) *If $X \subseteq B_1$ and $\alpha(X) \leqslant n$, then either $X \in \mathscr{B}$, or $X = B_1$, or $B_1^c \in \mathscr{B}$.*

Suppose that $X \notin \mathscr{B}$ and $X \neq B_1$. From the minimality of B_1, it follows that $X^c \in \mathscr{B}$. But $B_1^c \subseteq X^c$ by (2.4), since $X \subseteq B_1$, and so $B_1^c \in \mathscr{B}$, as required.

For $i = 1, 2$, let $\mathscr{B}_i$ be the set of all $X \subseteq B_i$ with $\alpha(X) \leqslant n$. We suppose, for a contradiction, that neither $\mathscr{B} \cup \mathscr{B}_1$ nor $\mathscr{B} \cup \mathscr{B}_2$ is a subset of a blockage of order n. Since $|\mathscr{B} \cup \mathscr{B}_i| > |\mathscr{B}|$ (because $B_i \in \mathscr{B}_i - \mathscr{B}$) and every partial blockage of greater cardinality than $\mathscr{B}$ is a subset of a blockage of order n, it follows that $\mathscr{B} \cup \mathscr{B}_i$ is not a partial blockage, and hence there is a fracture in $\mathscr{B} \cup \mathscr{B}_i$. From (2.3) we deduce that

(3) *For $i = 1, 2$ there is a simple fracture in $\mathscr{B} \cup \mathscr{B}_i$.*

We claim

(4) *There is a simple fracture $(X_1, \ldots, X_r)$ in $\mathscr{B} \cup \mathscr{B}_1$ with $X_1 = B_1$ and $X_r^c \in \mathscr{B}$.*

If $B_1^c \in \mathscr{B}$ we may set $r = 1$ and $X_1 = B_1$, and so we may assume that $B_1^c \notin \mathscr{B}$. Let $(X_1, \ldots, X_r)$ be a simple fracture in $\mathscr{B} \cup \mathscr{B}_1$. By (2.4), $(X_r^c, X_{r-1}^c, \ldots, X_1^c)$ is also a simple fracture in $\mathscr{B} \cup \mathscr{B}_1$, for $(X_1^c)^c \in \mathscr{B} \cup \mathscr{B}_1$, since $(X_1^c)^c \subseteq X_1 \in \mathscr{B} \cup \mathscr{B}_1$. Since $(X_1, \ldots, X_r)$ is not a fracture in $\mathscr{B}$ it follows that one of X_1, X_r^c is not in $\mathscr{B}$, and by replacing $(X_1, \ldots, X_r)$ by $(X_r^c, \ldots, X_1^c)$ if necessary we may assume that $X_1 \notin \mathscr{B}$. Hence $X_1 \in \mathscr{B}_1$, and so $X_1 \subseteq B_1$, and by (2), $X_1 = B_1$. Suppose that $X_r^c \notin \mathscr{B}$. Then $X_r^c \neq \varnothing$, since $\varnothing \in \mathscr{B}$, and so $X_r^c \not\subseteq X_r$. Now $B_1 = X_1 \subseteq X_r$, since $(X_1, ..., X_r)$ is a simple fracture, and so $X_r^c \not\subseteq B_1$. Hence $X_r^c \notin \mathscr{B} \cup \mathscr{B}_1$, a contradiction. We deduce that $X_r^c \in \mathscr{B}$, as required.

Let $(Y_1, ..., Y_s)$ be a simple fracture in $\mathscr{B} \cup \mathscr{B}_2$. Since there is no fracture in $\mathscr{B}$ it follows that one of Y_1, Y_s^c does not belong to $\mathscr{B}$, and by replacing $(Y_1, ..., Y_s)$ by $(Y_s^c, ..., Y_1^c)$ if necessary, we may assume that $Y_1 \notin \mathscr{B}$.

(5) $|(Y_1 - X_1^c) \cup \operatorname{att}(X_1^c)| \leqslant n.$

For $(Y_1 - X_1^c) \cup \operatorname{att}(X_1^c) \subseteq (B_2 - B_1^c) \cup \operatorname{att}(B_1) \subseteq B_1 \cap B_2$, and $|B_1 \cap B_2| \leqslant n$.

Now $(X_r^c, X_{r-1}^c, ..., X_1^c, Y_1, ..., Y_s)$ is not a fracture in $\mathscr{B}$, because there is none. Yet $X_r^c \in \mathscr{B}$, and for $1 \leqslant i < r$, $|(X_i^c - X_{i+1}^c) \cup \operatorname{att}(X_{i+1}^c)| \leqslant n$, by (2.4). Thus, by (5), it follows that $Y_s^c \notin \mathscr{B}$, and so $Y_s^c \subseteq B_2$. Now

$$(X_r^c, X_{r-1}^c, ..., X_1^c, Y_1, ..., Y_s, B_2^c, X_1, ..., X_r)$$

is not a fracture in $\mathscr{B}$. Yet $X_r^c \in \mathscr{B}$, $B_2^c \subseteq Y_s$ (because $Y_s^c \subseteq B_2$), and

$$(X_1 - B_2^c) \cup \operatorname{att}(B_2^c) = (B_1 \cap (B_2 - \operatorname{att}(B_2))) \cup \operatorname{att}(B_2^c) \subseteq B_1 \cap B_2,$$

and so $|(X_1 - B_2^c) \cup \operatorname{att}(B_2^c)| \leqslant n$, a contradiction.

We deduce that one of $\mathscr{B} \cup \mathscr{B}_1$, $\mathscr{B} \cup \mathscr{B}_2$ is a subset of a blockage of order n, and hence so is $\mathscr{B}$, as required. ∎

Finally, we can prove (2.1).

Proof of (2.1). Suppose that G has path-width $< n$, and $\mathscr{B}$ is a blockage of order n. Let $(W_1, ..., W_m)$ be a path-decomposition, where each $|W_i| \leqslant n$. Since $\varnothing$, V are complementary and $\varnothing \subseteq V$, it follows from the second and third blockage axioms that $\varnothing \in \mathscr{B}$. From (2.2), $W_1 \in \mathscr{B}$. For $1 \leqslant i \leqslant n$, let $X_i = W_1 \cup \cdots \cup W_i$, and choose i maximum with $X_i \in \mathscr{B}$. Now $i \neq m$, because $V \notin \mathscr{B}$ (for $\varnothing \in \mathscr{B}$, and $\varnothing$, V are complementary). Moreover, $\operatorname{att}(X_i) \subseteq W_{i+1}$, and so $|(X_{i+1} - X_i) \cup \operatorname{att}(X_i)| \leqslant |W_{i+1}| \leqslant n$. By (2.2), $X_{i+1} \in \mathscr{B}$, contrary to the maximality of i. This proves the "easy" half of (2.1).

For the converse, suppose that there is no blockage of order n. By (2.5), $\{\varnothing\}$ is not a partial blockage, and so there is a simple fracture $(X_1, ..., X_m)$ in $\{\varnothing\}$. Hence $X_1, X_m^c = \varnothing$. For $1 \leqslant i \leqslant m-1$, let $W_i = (X_{i+1} - X_i) \cup \operatorname{att}(X_i)$. Then each $|W_i| \leqslant n$.

(1) *$W_1 \cup \cdots \cup W_{m-1} = V$, and for every edge of G, one of $W_1, ..., W_{m-1}$ contains both its ends.*

If $v \in V$, choose $i \geqslant 1$ maximum such that $v \notin X_i$. (This is possible since $X_1 = \varnothing$.) (Since $X_m^c = \varnothing$ and hence $X_m = V$, it follows that $i < m$; hence $v \in X_{i+1}$ and so $v \in W_i$. Thus $W_1 \cup \cdots \cup W_{m-1} = V$. Now let $e \in E(G)$ with ends u, v, and choose $i \geqslant 1$ maximum such that $\{u, v\} \nsubseteq X_i$. Again, $i < m$, and we may assume that $v \in X_{i+1} - X_i$. Hence either $u \in X_{i+1} - X_i$ or $u \in \mathrm{att}(X_i)$, and in either case $\{u, v\} \subseteq W_i$, as required.

(2) *For $1 \leqslant i \leqslant j \leqslant k \leqslant m$, $W_i \cap W_k \subseteq W_j$.*

$W_k \cap (X_k - \mathrm{att}(X_k)) = \varnothing$, and so by (2.4), $W_k \cap (X_j - \mathrm{att}(X_j)) = \varnothing$, because $X_j \subseteq X_k$. Moreover, $W_i \subseteq X_{i+1} \subseteq X_{j+1}$, and so

$$W_i \cap W_k \subseteq X_{j+1} - (X_j - \mathrm{att}(X_j)) = W_j,$$

as required.

From (1) and (2) we see that $(W_1, ..., W_{m-1})$ is a path-decomposition of G of width $\leqslant n-1$, as required. ∎

3. Minors

In view of (2.1), to prove (1.1) it suffices to prove the following.

(3.1) *Let $\mathscr{B}$ be a blockage of order n in G, and let F be a forest with $|V(F)| = n+1$. Then G has a minor isomorphic to F.*

Proof. We may assume (by adding edges to F) that F is a tree. Let the vertices of F be $v_1, ..., v_{n+1}$, numbered so that for $1 \leqslant i \leqslant n$, exactly one of $v_1, ..., v_i$ is adjacent to v_{i+1}. We say that $X \in \mathscr{B}$ is *useful* if, writing $k = \alpha(X)$,

(i) there is no $Y \in \mathscr{B}$ with $\alpha(Y) < \alpha(X)$ and $X \subseteq Y$

(ii) there are vertex-disjoint subgraphs $C_1, ..., C_k$ of G, each connected and with $V(C_i) \subseteq X$ and $V(C_i) \cap \mathrm{att}(X) \neq \varnothing$, such that for $1 \leqslant i < j \leqslant k$, if v_i and v_j are adjacent in F then some edge of G has one end in $V(C_i)$ and the other in $V(C_j)$.

Certainly $\varnothing$ is useful, and so we may choose k maximum such that there is a useful $X \in \mathscr{B}$ with $\alpha(X) = k$. Choose such a set X, maximal, and let $C_1, ..., C_k$ be subgraphs as in (ii). Since each C_i meets $\mathrm{att}(X)$ and $\alpha(X) = k$, it follows that $\mathrm{att}(X) \subseteq V(C_1) \cup \cdots \cup V(C_k)$, and each C_i contains exactly one vertex of $\mathrm{att}(X)$.

(1) *There is no $Y \in \mathscr{B}$ with $\alpha(Y) = k$ and $X \subset Y$.*

Suppose that there is such a Y. We claim that there are k mutually vertex-disjoint paths of G from X to Y^c. For otherwise, by Menger's theorem, there exist complementary sets $A, B \subseteq V(G)$ with $X \subseteq A$, $Y^c \subseteq B$, and $|A \cap B| < k$. Now $Y \in \mathscr{B}$, and so $Y^c \notin \mathscr{B}$, from the third blockage axiom.

Hence $B \notin \mathscr{B}$, from the second axiom, and so $A \in \mathscr{B}$, from the third. But $\text{att}(A) \subseteq A \cap B$, and so $\alpha(A) < k$, contrary to the fact that X is useful. Thus the k disjoint paths exist. Let us choose them minimal. Since each meets X and $Y^c \subseteq X^c$, it follows that each meets $\text{att}(X)$, and similarly each meets $\text{att}(Y)$. From the minimality of the paths, we deduce that each path has its first vertex in $\text{att}(X)$ and no other vertex in X, its last vertex in $\text{att}(Y)$, and all its vertices in $(Y - X) \cup \text{att}(X)$. Let the paths be $P_1, ..., P_k$, where the first vertex of P_i belongs to $V(C_i)$ $(1 \leqslant i \leqslant k)$. Now

$$V(C_i \cap P_j) \subseteq X \cap V(P_j) \subseteq V(C_j),$$

and so $C_i \cap P_j$ is null unless $i = j$. Thus, the connected subgraphs $C_i \cup P_i$ $(1 \leqslant i \leqslant k)$ are mutually disjoint. Moreover, there is no $Z \in \mathscr{B}$ with $\alpha(Z) < k$ and $Y \subseteq Z$, for $X \subseteq Y$ and X is useful. It follows that Y is useful, contrary to the maximality of X. This proves (1).

Now $k = \alpha(X) \leqslant n$ since $X \in \mathscr{B}$; choose i with $1 \leqslant i \leqslant k$ such that v_i and v_{k+1} are adjacent in F. Let $u \in V(C_i) \cap \text{att}(X)$, and let $v \in V(G) - X$ be a neighbour of u. Let C_{k+1} be the subgraph with $V(C_{k+1}) = \{v\}$, $E(C_{k+1}) = \emptyset$, and let $X' = X \cup \{v\}$. Suppose that $X' \in \mathscr{B}$. Since $\alpha(X') \leqslant k+1$, we deduce from (1) that $\alpha(X') = k+1$ and hence $\text{att}(X') = \text{att}(X) \cup \{v\}$. Moreover, from (1) again, there is no $Y \in \mathscr{B}$ with $X' \subseteq Y$ and $\alpha(Y) < \alpha(X')$. The existence of $C_1, ..., C_{k+1}$ implies that X' is useful, contrary to the maximality of k. Hence $X' \notin \mathscr{B}$. From (2.2), it follows that $\alpha(X) = n$, and so $k = n$. But then G has a minor isomorphic to F, as required. ∎

4. Blockages and Tangles

In this section we mention a connection between blockages and tangles, introduced in [6]. A *cut* of a graph G is a pair (A, B) of complementary subsets of $V(G)$, that is, such that $A \cup B = V(G)$ and for every $e \in E(G)$, one of A, B contains both ends of e. The *order* of a cut (A, B) is $|A \cap B|$. Let $\mathscr{Y}$ be a set of cuts of G, all of order $\leqslant n$, such that

(i) if (A, B) is a cut of G of order $\leqslant n$, then $\mathscr{Y}$ contains one of (A, B), (B, A)

(ii) if (A_1, B_1), $(A_2, B_2) \in \mathscr{Y}$, then $(G|A_1) \cup (G|A_2) \neq G$ (where $G|A$ is the restriction of G to A).

Let us call such a set $\mathscr{Y}$ a *stoppage* of order n. We leave the proof of the following to the reader.

(4.1) *If $\mathscr{Y}$ is a stoppage of order n, then $\{A\colon (A, B) \in \mathscr{Y}\}$ is a blockage of order n. Conversely, if $\mathscr{B}$ is a blockage of order n, then the set of all cuts (A, B) of order $\leqslant n$ with $A \in \mathscr{B}$ is a stoppage of order n.*

If we replace condition (ii) by the stronger requirement that if (A_1, B_1), (A_2, B_2), $(A_3, B_3) \in \mathscr{Y}$ then $(G | A_1) \cup (G | A_2) \cup (G | A_3) \neq G$, the new conditions we produce are equivalent to the "tangle axioms" of [6] (again, we leave verifying this to the reader). We hope that this helps clarify the relationship between these concepts.

5. Graph Searching

It seems remarkable to us that a minimax formula like (2.1) for an NP-hard invariant like path-width [3] could be of any real use. But we have shown one application of (2.1), and to confirm its nontriviality we show another in this section, to the problem of searching a graph.

Suppose we have a system of tunnels which we wish to search for some hapless explorer who has become lost in them and who is now wandering around unpredictably. We have a map of the tunnels, which is a graph, but we only have a limited number of searchers at our disposal. We wish to devise a search plan by which rescue can be guaranteed. This is easy, even with one searcher, if only the victim would keep still, but we cannot be sure of this, and we must take care that he does not accidentally avoid us.

More precisely, let us say a *search in G* is a sequence $(X_1, ..., X_m)$ of subsets of $V(G)$, such that $X_1 = \emptyset$ and for $1 \leqslant i < m$, either $X_{i+1} \subseteq X_i$ or $X_i \subseteq X_{i+1}$. (These are the positions occupied by the searchers at each stage.) Let $B_1 = V(G)$, and inductively let B_i be the set of all vertices v of G such that there is a path P of G between v and some vertex of B_{i-1}, with $V(P) \cap X_i = \emptyset$. ($B_i$ represents the places where the person we are rescuing may currently be, if we have not found him yet.) The search is *successful* if $B_m = \emptyset$. We want to know if there is a successful search $(X_1, ..., X_m)$ with each $|X_i| \leqslant n$, for some given n.

A successful search is *monotone* if $B_1 \supseteq B_2 \supseteq \cdots \supseteq B_m = \emptyset$, in other words, if no part of the graph is searched twice. The equivalence of (i) and (iv) in the following theorem is due to Kirousis and Papadimitriou [2, 3], who deduced it from a difficult theorem of LaPaugh [4]. We show that it follows from (2.1).

(5.1) *For a graph G and integer $n \geqslant 0$, the following are equivalent*:

(i) *there is a successful search $(X_1, ..., X_m)$ in G with each $|X_i| \leqslant n$*

(ii) *there is no blockage in G of order n*

(iii) *G has path-width $\leqslant n-1$*

(iv) *there is a monotone successful search $(X_1, ..., X_m)$ in G with each $|X_i| \leqslant n$.*

Proof. First we show that (i) implies (ii). Let $(X_1, ..., X_m)$ be a successful search with each $|X_i| \leqslant n$, and let $B_1, ..., B_m$ be as before. Suppose that $\mathscr{B}$ is a blockage in G of order n. For each i, let $A_i = V(G) - B_i$; then $\text{att}(A_i) \subseteq X_i$, by definition of B_i. Now $A_1 = \varnothing \in \mathscr{B}$ and $A_m = V(G) \notin \mathscr{B}$, and so we may choose i with $1 \leqslant i < m$ such that $A_i \in \mathscr{B}$ and $A_{i+1} \notin \mathscr{B}$. Since $(A_{i+1} - A_i) \cup \text{att}(A_i) \subseteq X_i$ and $|X_i| \leqslant n$, this contradicts (2.2). Hence (i) implies (ii).

Now (ii) implies (iii), by (2.1). To show that (iii) implies (iv), let $(W_1, ..., W_m)$ be a path-decomposition with each $|W_i| \leqslant n$. Then $(\varnothing, W_1, W_1 \cap W_2, W_2, W_2 \cap W_3, W_3, ..., W_m)$ is a successful monotone search. Finally, that (iv) implies (i) is trivial. ∎

LaPaugh proved a similar result for a closely related kind of graph searching [4], and a similar short proof can be given of her result. See also [1].

References

1. D. Bienstock and P. D. Seymour, Monotonicity in graph searching, *J. Algorithms*, to appear.
2. L. M. Kirousis and C. H. Papadimitriou, Interval graphs and searching, *Discrete Math.* **55** (1985), 181–184.
3. L. M. Kirousis and C. H. Papadimitriou, Searching and pebbling, *J. Theoret. Comput. Sci.* **47** (1986), 205–218.
4. A. LaPaugh, "Recontamination Does Not Help Search a Graph," Technical Report 335, Department of Electrical Engineering and Computer Science, Princeton University, 1982.
5. Neil Robertson and P. D. Seymour, Graph minors. 1. Excluding a forest, *J. Combin. Theory Ser. B* **35** (1983), 39–61.
6. Neil Robertson and P. D. Seymour, Graph minors. X. Obstructions to tree-decomposition, *J. Comb. Theory Ser. B* **52** (1991), 153–190.
7. P. D. Seymour and Robin Thomas, Graph searching, and a minimax theorem for tree-width, submitted for publication.