SIMULTANEOUS LINEAR EQUATIONS OVER A DIVISION ALGEBRA

By A. R. RICHARDSON.

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PREVIOUS work on the theory of equations and determinants in non-commutative algebras suffers from a lack of generality owing to an assumption of one-sidedness*. Moreover, information about such matters as the theory of equations and invariants, when this assumption of one-sidedness is adopted, is obtained only by means of a number of further artificial restrictions†, for the product of two one-sided expressions is not, in general, one-sided. In this paper I overcome these difficulties, and give a satisfactory definition of determinants which can be used almost as freely as ordinary determinants, which are, indeed, only particular examples.

My method consists in defining hypercomplex determinants, by means of which I give an explicit form to the solution of the general linear equation

$$\Sigma a_s x b_s = a,$$

in all cases including the singular. I then generalize the definition of determinant, and show how to obtain the solution of the most general system of simultaneous linear equations.

PART I. ONE-SIDED LINEAR EQUATIONS AND THEIR DETERMINANTS.

The following properties of the elements of a division algebra will be assumed‡ :

(*a*) There are no divisors of zero.

(*b*) There is a modulus which will be denoted by 1.

(*c*) Every element a, except zero, has a unique inverse a^{-1} such that

$$a a^{-1} = a^{-1} a = 1.$$

* E. Study, *Acta Math.*, 42 (1918), 1–61 (for the reference to this important paper, I am indebted to a referee); A. R. Richardson, *Messenger of Math.*, 55 (1926), 145–152.

† A. R. Richardson, *Messenger of Math.*, 55 (1926), 175–182.

‡ L. E. Dickson, *Algebras and their arithmetics* (1923), 61.

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Definition of determinants.

The left-hand determinant

$$\Delta^{12}_{12} = \begin{vmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{vmatrix}$$

is defined as follows:

$$\Delta^{12}_{12} = a_{22}a_{11} - a_{22}a_{12}a_{22}^{-1}a_{21} \quad (a_{22} \neq 0), \tag{1}$$

$$\Delta^{12}_{12} = -a_{12}a_{21}. \quad (a_{22} = 0). \tag{2}$$

The right-hand determinant

$$\nabla^{12}_{12} = \begin{vmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{vmatrix}$$

is defined as follows:

$$\nabla^{12}_{12} = a_{11}a_{22} - a_{21}a_{22}^{-1}a_{12}a_{22} \quad (a_{22} \neq 0), \tag{3}$$

$$\nabla^{12}_{12} = -a_{21}a_{12} \quad (a_{22} = 0). \tag{4}$$

If the last column is zero, the determinant is defined to be zero.

In quaternions the conjugate of the left-hand determinant is seen to be the right-hand determinant of the conjugate elements.

The following results can be established by simple algebra.

Interchange of rows.

$$\left.\begin{aligned} \Delta^{12}_{12} &= -a_{22}a_{12}a_{22}^{-1}a_{12}^{-1}\Delta^{12}_{21} && (a_{22} \neq 0,\ a_{12} \neq 0) \\ &= -\Delta^{12}_{21} && (a_{22} = 0,\ a_{12} \neq 0) \\ &= -\Delta^{12}_{21} && (a_{22} \neq 0,\ a_{12} = 0) \end{aligned}\right\}. \tag{5}$$

If two rows are identical, the determinant vanishes. Notice that the last two expressions can be obtained by striking out the zero elements in the first.

Interchange of columns.

$$\left.\begin{aligned} \Delta^{12}_{12} &= -a_{22}a_{21}^{-1}\Delta^{21}_{12}a_{22}^{-1}a_{21} && (a_{22} \neq 0,\ a_{21} \neq 0) \\ &= -a_{21}^{-1}\Delta^{21}_{12}a_{21} && (a_{22} = 0,\ a_{21} \neq 0) \\ &= -a_{22}\Delta^{21}_{12}a_{22}^{-1} && (a_{22} \neq 0,\ a_{21} = 0) \end{aligned}\right\}. \tag{6}$$

If two columns are identical, the determinant vanishes. Notice that the last two expressions can be obtained by striking out the zero elements in the first.

Interchange of rows with columns.

$$\left.\begin{aligned} a_{22}^{-1}\begin{vmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{vmatrix} &= \begin{vmatrix} a_{11}, & a_{21} \\ a_{12}, & a_{22} \end{vmatrix} a_{22}^{-1} \qquad (a_{22} \neq 0) \\ \begin{vmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{vmatrix} &= \begin{vmatrix} a_{11}, & a_{21} \\ a_{12}, & a_{22} \end{vmatrix} \qquad\qquad (a_{22} = 0) \end{aligned}\right\}. \tag{7}$$

Multiplication of row or column.

$$\Delta_{(\lambda 1)2}^{12} = \begin{vmatrix} \lambda a_{11}, & \lambda a_{12} \\ a_{21}, & a_{22} \end{vmatrix} \begin{aligned} &= a_{22}\lambda a_{22}^{-1}\Delta_{12}^{12} \quad (a_{22} \neq 0) \\ &= \lambda\Delta_{12}^{12} \qquad\qquad\; (a_{22} = 0) \end{aligned}\Bigg\}, \tag{8}$$

$$\Delta_{1(\lambda 2)}^{12} = \begin{vmatrix} a_{11}, & a_{12} \\ \lambda a_{21}, & \lambda a_{22} \end{vmatrix} \begin{aligned} &= \lambda\Delta_{12}^{12} \qquad\qquad\; (a_{22} \neq 0) \\ &= a_{12}\lambda a_{12}^{-1}\Delta_{12}^{12} \quad (a_{22} = 0) \end{aligned}\Bigg\}, \tag{9}$$

$$\Delta_{12}^{(1\lambda)2} = \begin{vmatrix} a_{11}\lambda, & a_{12} \\ a_{21}\lambda, & a_{22} \end{vmatrix} = \Delta_{12}^{12}\lambda \quad \text{(in all cases)}, \tag{10}$$

$$\Delta_{12}^{1(2\lambda)} = \begin{vmatrix} a_{11}, & a_{12}\lambda \\ a_{21}, & a_{22}\lambda \end{vmatrix} \begin{aligned} &= a_{22}\lambda a_{22}^{-1}\Delta_{12}^{12} \quad (a_{22} \neq 0) \\ &= a_{12}\lambda a_{12}^{-1}\Delta_{12}^{12} \quad (a_{22} = 0) \end{aligned}\Bigg\}. \tag{11}$$

Definition. Expressions such as $a_{22}a_{12}a_{22}^{-1}a_{12}^{-1}$ and $a_{22}a_{21}^{-1}(\;)a_{22}^{-1}a_{21}$, in which each element is balanced by its inverse, will be termed *Permutators*. They are always of unit norm.

Addition of determinants.

$$\left.\begin{aligned} \begin{vmatrix} a_{11}+a_{11}', & a_{12}+a_{12}' \\ a_{21}, & a_{22} \end{vmatrix} &= \begin{vmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11}', & a_{12}' \\ a_{21}, & a_{22} \end{vmatrix} \\ \begin{vmatrix} a_{11}+a_{11}', & a_{12} \\ a_{21}+a_{21}', & a_{22} \end{vmatrix} &= \begin{vmatrix} a_{11}, & a_{12} \\ a_{12}, & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11}', & a_{12} \\ a_{21}', & a_{22} \end{vmatrix} \end{aligned}\right\}. \tag{12}$$

Hence we may add a left (right) multiple of the elements of the second row (column) into the first row (column) without altering the determinant.

If we add a multiple of the first row (or column) into the second row (or column), the determinant is multiplied by a *permutator*; for example,

$$\begin{vmatrix} a_{11}, & a_{12} \\ a_{21}+a'_{21}, & a_{22}+a'_{22} \end{vmatrix} = -(a_{22}+a'_{22})\,a_{12}(a_{22}+a'_{22})^{-1}\,a_{12}^{-1}(\Delta_{21}^{12}+\Delta'^{12}_{21}), \quad (13)$$

provided that $a_{22}+a'_{22} \neq 0$ and $a_{12} \neq 0$.

Solution of two simultaneous equations in two unknowns.

If we take the equations to be

$$\begin{cases} a_{11}x_1+a_{12}x_2 = a_1, \\ a_{21}x_1+a_{22}x_2 = a_2, \end{cases}$$

the necessary and sufficient condition for a unique solution is

$$\Delta_{12}^{12} \neq 0,$$

and this solution is given by

$$x_1 = (\Delta_{12}^{12})^{-1}\begin{vmatrix} a_1, & a_{12} \\ a_2, & a_{22} \end{vmatrix}, \qquad x_2 = (\Delta_{12}^{21})^{-1}\begin{vmatrix} a_1, & a_{11} \\ a_2, & a_{21} \end{vmatrix}.$$

If $\Delta_{12}^{12} = 0$, but both the determinants

$$\begin{vmatrix} a_1, & a_{12} \\ a_2, & a_{22} \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a_1, & a_{11} \\ a_2, & a_{21} \end{vmatrix}$$

are not zero, the equations are inconsistent. If both of these determinants are zero, the equations are not independent and the most general solution for x_1 is

$$x_1 = a_{11}^{-1}a_1+(-a_{11}^{-1}a_{12})\,Y,$$

where Y is arbitrary*.

The determinant of order n.

The determinant

$$\Delta_{123\ldots n}^{123\ldots n} = \begin{vmatrix} a_{11}, & a_{12}, & \ldots, & a_{1n} \\ a_{21}, & a_{22}, & \ldots, & a_{2n} \\ \ldots & \ldots & & \ldots \\ a_{n1}, & a_{n2}, & \ldots, & a_{nn} \end{vmatrix}$$

* The Nabla function of Study is the norm of Δ_{12}^{12}.

is defined* to be equal to the following determinant of order $n-1$:

$$(-1)^{n-s}\begin{vmatrix} a_{sn}^{-1}\Delta_{1s}^{1n}, & \dots, & a_{sn}^{-1}\Delta_{1s}^{rn}, & \dots, & a_{sn}^{-1}\Delta_{1s}^{(n-1)n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{sn}^{-1}\Delta_{(s-1)s}^{1n}, & \dots, & a_{sn}^{-1}\Delta_{(s-1)s}^{rn}, & \dots, & a_{sn}^{-1}\Delta_{(s-1)s}^{(n-1)n} \\ a_{(s+1)1}, & \dots, & a_{(s+1)r}, & \dots, & a_{(s+1)(n-1)} \\ \dots & \dots & \dots & \dots & \dots \\ a_{(n-1)1}, & \dots, & a_{(n-1)r}, & \dots, & a_{(n-1)(n-1)} \\ a_{sn}a_{n1}, & \dots, & a_{sn}a_{nr}, & \dots, & a_{sn}a_{n(n-1)} \end{vmatrix} \tag{14}$$

where

$$\Delta_{rs}^{pq} = \begin{vmatrix} a_{rp}, & a_{rq} \\ a_{sp}, & a_{sq} \end{vmatrix}, \tag{15}$$

and s is defined by the conditions :

$$a_{nn} = a_{(n-1)n} = \dots = a_{(s+1)n} = 0, \quad a_{sn} \neq 0. \tag{16}$$

If all the elements of the last column are zero, the value of the determinant is defined to be zero.

Right-hand determinants are defined in a similar manner.

THEOREM 1. *In the successive reduction of the order of the determinant, the elements of a given row (say the t-th) remain in the t-th row, unless the diagonal minor of order $n-t$ in the lower right-hand corner contains a column of zeros in the last column at some stage in the reduction.*

For the elements in any column are never moved to a column on the right of the given one, and the elements of a given row are never moved from that row unless, at some stage in the process of reducing the order of the determinant, a zero appears as the right-hand element of every row below it. In symbols the elements of the t-th row remain in the t-th row unless

$$\Delta_{(t+1)\dots n}^{(t+1)\dots n} = 0.$$

The case in most frequent use is when $t = 1$. Hence

COROLLARY 1. *The elements of the first column always remain in the first column.*

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* This modification of the definition given in the *Messenger of Math.* (*loc. cit.*, 146) is due to C. Slocombe, *University of Wales Thesis.*

The elements of the first row always remain in the first row, unless

$$\Delta_{23\dots n}^{23\dots n} = 0.$$

Hence, if $\Delta_{34\dots n}^{34\dots n} \neq 0$,

$$\Delta_{123\dots n}^{123\dots n} = \begin{vmatrix} (\Delta_{34\dots n}^{34\dots n})^{-1}\,\Delta_{134\dots n}^{134\dots n}, & (\Delta_{34\dots n}^{34\dots n})^{-1}\,\Delta_{134\dots n}^{234\dots n} \\ \Delta_{234\dots n}^{134\dots n}, & \Delta_{234\dots n}^{234\dots n} \end{vmatrix}, \tag{17}$$

and, if $\Delta_{r(r+1)\dots n}^{r(r+1)\dots n} \neq 0$,

$$\Delta_{123\dots n}^{123\dots n} = \begin{vmatrix} (\Delta_{r(r+1)\dots n}^{r(r+1)\dots n})^{-1}\,\Delta_{1r(r+1)\dots n}^{1r(r+1)\dots n}, & \dots, & (\Delta_{r(r+1)\dots n}^{r(r+1)\dots n})^{-1}\,\Delta_{1r(r+1)\dots n}^{(r-1)r(r+1))\dots n} \\ \dots & \dots & \dots \\ \Delta_{(r-1)r(r+1)\dots n}^{1r(r+1)\dots n}, & \dots, & \Delta_{(r-1)r(r+1)\dots n}^{(r-1)r(r+1)\dots n} \end{vmatrix}. \tag{18}$$

THEOREM 2. *The coefficient of* a_{11} *is* $\Delta_{23\dots n}^{23\dots n}$.

For, from (16) and Cor. 1, a_{11} always remains in the leading position without multipliers, so that the determinant is certainly

$$\begin{vmatrix} a_{11}, & \dots \\ \dots, & \Delta_{23\dots n}^{23\dots n} \end{vmatrix}$$

if $\Delta_{23\dots n}^{23\dots n} \neq 0$. Hence the development begins with $\Delta_{23\dots n}^{23\dots n} a_{11} + \dots$.

If $\Delta_{23\dots n}^{23\dots n} = 0$, the term a_{11} does not occur in the development.

Hence, in all cases,

$$\Delta_{123\dots n}^{123\dots n} = \Delta_{23\dots n}^{23\dots n} a_{11} + \dots. \tag{19}$$

The following results may all be established by induction from the case $n = 2$.

THEOREM 3. *If all the elements of a row or column are zero, the determinant is equal to zero.*

THEOREM 4. *The determinant is equal to zero if two rows or two columns are identical.*

THEOREM 5. *The determinant is unaltered by adding to the elements of a row (or column) a left (right) multiple of the elements of any row (column) below (to the right of) it.*

THEOREM 6.

$$\left.\begin{aligned} \Delta_{12\dots n}^{12\dots(r\lambda)\dots n} &= P\lambda P^{-1}\,\Delta_{123\dots n}^{123\dots n} \\ \Delta_{12\dots n}^{(1\lambda)2\dots n} &= \Delta_{12\dots n}^{12\dots n}\lambda \end{aligned}\right\} \tag{20}$$

where $P(\ )P^{-1}$ denotes a certain permutator, independent of λ and of the elements of the first column.

In particular, if $\Delta^{r(r+1)\dots n}_{r(r+1)\dots n} \neq 0$,

$$P = \Delta^{r(r+1)\dots n}_{r(r+1)\dots n}. \tag{21}$$

THEOREM 7.

$$\left.\begin{aligned} \Delta^{12\dots n}_{12\dots(\lambda r)\dots n} &= P\lambda P^{-1}\Delta^{12\dots n}_{12\dots n} \\ \Delta^{12\dots n}_{(\lambda 1)2\dots n} &= \lambda\Delta^{12\dots n}_{12\dots n} \end{aligned}\right\}. \tag{22}$$

where $P(\ )P^{-1}$ denotes a certain permutator, independent of λ and of the elements of the first row.

In particular, if $\Delta^{r(r+1)\dots n}_{r(r+1)\dots n} \neq 0$,

$$P = \Delta^{r(r+1)\dots n}_{r(r+1)\dots n}. \tag{23}$$

THEOREM 8. *If $\Delta^{23\dots n}_{23\dots n} \neq 0$, then*

$$(\Delta^{23\dots n}_{23\dots(\lambda r)\dots n})^{-1}\Delta^{12\dots n}_{12\dots(\lambda r)\dots n} = (\Delta^{23\dots n}_{23\dots n})^{-1}\Delta^{123\dots n}_{123\dots n}. \tag{24}$$

THEOREM 9. *If $ijk\dots q$ is any permutation of $123\dots n$, and κ is the number of interchanges necessary to put the integers $i, j, k, \dots, q$ into their natural order, then*

$$\Delta^{123\dots n}_{123\dots n} = (-1)^{\kappa} P\Delta^{123\dots n}_{ijk\dots q}, \tag{25}$$

where P is a certain permutator, independent of the elements of the first column.

COROLLARY 2. *If $\Delta^{123\dots n}_{123\dots n} = 0$, then*

$$\Delta^{123\dots n}_{ijk\dots q} = 0. \tag{26}$$

COROLLARY 3. *If $\Delta^{23\dots n}_{23\dots n} \neq 0$, then*

$$\Delta^{123\dots n}_{123\dots n} = \Delta^{23\dots n}_{23\dots n}(\Delta^{23\dots n}_{jk\dots q})^{-1}\Delta^{123\dots n}_{ijk\dots q}. \tag{27}$$

COROLLARY 4. *If $a_{nn} = a_{n-1,n} = \dots = a_{s+1,n} = 0$, $a_{sn} \neq 0$, then*

$$\Delta^{12\dots n}_{12\dots s\dots t\dots n} = (-1)^{t-s}\Delta^{12\dots n}_{12\dots t\dots s\dots n}. \tag{28}$$

THEOREM 10. *Interchange of columns.*

$$\Delta^{123\dots n}_{123\dots n} = (-1)^{\kappa} P\Delta^{ijk\dots q}_{123\dots n} P', \tag{29}$$

where P and P' are certain permutators independent of the elements of the first row.

COROLLARY 5. *If* $\Delta_{123\dots n}^{123\dots n} = 0$, *then*

$$\Delta_{123\dots n}^{ijk\dots q} = 0. \tag{30}$$

COROLLARY 6. *If* $\Delta_{23\dots n}^{23\dots n} \neq 0$, *then*

$$\Delta_{123\dots n}^{123\dots n} = -\Delta_{23\dots n}^{23\dots n}(\Delta_{23\dots n}^{jk\dots q})^{-1}\Delta_{123\dots n}^{ijk\dots q}(\Delta_{23\dots n}^{i2(i-1)(i+1)\dots n})^{-1}\Delta_{23\dots n}^{12\dots(i-1)(i+1)\dots n}. \tag{31}$$

COROLLARY 7. *If the first column is not involved in the interchange, then*

$$\Delta_{123\dots n}^{123\dots n} = \Delta_{23\dots n}^{23\dots n}(\Delta_{23\dots n}^{jk\dots q})^{-1}\Delta_{123\dots n}^{1jk\dots q}. \tag{32}$$

THEOREM 11. *Interchange of rows with columns. A left-hand determinant is the multiple by a permutator of the corresponding right-hand determinant in which rows have been interchanged with columns.*

If $\Delta_{23\dots n}^{23\dots n} \neq 0$, we have the important result that

$$\begin{vmatrix} a_{22}, & \dots, & a_{2n} \\ \dots & & \dots \\ a_{n2}, & \dots, & a_{nn} \end{vmatrix}^{-1} \times \begin{vmatrix} a_{11}, & a_{12}, & \dots, & a_{1n} \\ a_{21}, & a_{22}, & \dots, & a_{2n} \\ \dots & \dots & & \dots \\ a_{n1}, & a_{n2}, & \dots, & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11}, & a_{21}, & \dots, & a_{n1} \\ a_{12}, & a_{22}, & \dots, & a_{n2} \\ \dots & \dots & & \dots \\ a_{1n}, & a_{2n}, & \dots, & a_{nn} \end{vmatrix} \times \begin{vmatrix} a_{22}, & \dots, & a_{n2} \\ \dots & & \dots \\ a_{2n}, & \dots, & a_{nn} \end{vmatrix}^{-1}. \tag{33}$$

THEOREM 12. *The determinant, in which each element of a row (or column) is the sum of two numbers, is a permutator of the sum of the determinants in each of which only one of the two numbers appears in the corresponding row (or column).*

This follows from Theorems 6, 7, 9, and 10.

COROLLARY 8. *It follows from Theorems* 6, 7, 9, 10, *and* 12 *that the determinant resulting from the addition of a left (right) multiple of a row (column) to any other row (column) is a permutator of the original determinant.*

Theorem 5 should be compared with this result.

THEOREM 13. *Expansion in terms of the elements of the first row.*

It is evident from the work leading to (17) that, if $\Delta_{23\dots n}^{23\dots n} \neq 0$,

$$\Delta_{123\dots n}^{123\dots n} = \begin{vmatrix} (a_{11} - \dots), & (a_{12} - \dots) \\ \Delta_{234\dots n}^{134\dots n}, & \Delta_{234\dots n}^{234\dots n} \end{vmatrix} = \Delta_{23\dots n}^{23\dots n} a_{11} + \Delta_{23\dots n}^{23\dots n} a_{1s} B + \dots, \tag{34}$$

for, since $\Delta_{23\ldots n}^{23\ldots n} \neq 0$, neither the first row nor the first column is moved. Similarly

$$\Delta_{123\ldots s\ldots n}^{1s3\ldots 2\ldots n} = \begin{vmatrix} (a_{11}-\ldots), & (a_{1s}-\ldots) \\ \Delta_{234\ldots n}^{1s4\ldots 2\ldots n}, & \Delta_{234\ldots n}^{s34\ldots 2\ldots n} \end{vmatrix}$$

$$= \Delta_{23\ldots n}^{s3\ldots n} a_{11} - \Delta_{23\ldots s\ldots n}^{s3\ldots 2\ldots n} a_{1s} (\Delta_{23\ldots s\ldots n}^{s3\ldots 2\ldots n})^{-1} \Delta_{234\ldots s\ldots n}^{1s4\ldots 2\ldots n} + \ldots . \tag{35}$$

The second expression is a permutator of the first, and consequently

$$B = (\Delta_{23\ldots s\ldots n}^{s3\ldots 2\ldots n})^{-1} \Delta_{234\ldots s\ldots n}^{1s4\ldots 2\ldots n}. \tag{36}$$

Hence, if $\Delta_{23\ldots n}^{23\ldots n} \neq 0$,

$$\Delta_{123\ldots n}^{123\ldots n} = \Delta_{23\ldots n}^{23\ldots n} \left[a_{11} - \sum_{s=2}^{n} a_{1s} (\Delta_{23\ldots n}^{s3\ldots (s-1)2(s+1)\ldots n})^{-1} \Delta_{234\ldots n}^{1s4\ldots (s-1)2(s+1)\ldots n} \right]. \tag{37}$$

COROLLARY 9. *If, in the development* (37), *we substitute for the* a_{1s} *the corresponding elements of any other row, the resulting expression vanishes.*

THEOREM 14. *Expansion in terms of the elements of the first column.*

We do not require the form of the coefficients in this expansion; Cor. 1 shows that the elements of the first column always remain in the first column, so that

$$\Delta_{123\ldots n}^{123\ldots n} = \sum_{s=1}^{\infty} A_{s1} \, . \, a_{s1}. \tag{38}$$

When every $\Delta_{23\ldots (s-1)(s+1)\ldots n}^{34\ldots n} \neq 0$ and $\Delta_{23\ldots n}^{23\ldots n} \neq 0$, the actual expansion may be written out from considerations similar to those used in Theorem 13; the expansion is

$$\Delta_{123\ldots n}^{123\ldots n} = \Delta_{23\ldots n}^{23\ldots n} \left[a_{11} - \sum_{s=2}^{n} (\Delta_{23\ldots (s-1)(s+1)\ldots n}^{34\ldots n})^{-1} \Delta_{12\ldots (s-1)(s+1)\ldots n}^{23\ldots n} \right.$$
$$\left. \times (\Delta_{s2\ldots (s-1)(s+1)\ldots n}^{23\ldots n})^{-1} \Delta_{23\ldots (s-1)(s+1)\ldots n}^{34\ldots n} a_{s1} \right]. \tag{39}$$

In particular,

$$\Delta_{123}^{123} = \Delta_{23}^{23} [a_{11} - a_{33}^{-1} \Delta_{13}^{23} (\Delta_{23}^{23})^{-1} a_{33} a_{21} - a_{23}^{-1} \Delta_{12}^{23} (\Delta_{32}^{23})^{-1} a_{23} a_{31}], \tag{40}$$

except when $a_{33} = a_{23} = 0$; in other cases, where any of the symbols vanish they may be struck out.

When $a_{33} = a_{23} = 0$ and $a_{13} \neq 0$,

$$\Delta_{123}^{123} = a_{13} \Delta_{23}^{12}. \tag{41}$$

COROLLARY 10. *If, in the development* (38), *we substitute for the* a_{s1} *the corresponding elements of any other column, the resulting expression vanishes.*

One-sided linear equations.

Consider the system

$$\sum_{s=1}^{n} a_{rs} x_s = a_r \quad (r = 1, 2, 3, \ldots, n). \tag{42}$$

If $\Delta_{123\ldots n}^{123\ldots n} \neq 0$, there is one, and only one, solution, namely,

$$x_s = (\Delta_{123\ldots n}^{s12\ldots(s-1)(s+1)\ldots n})^{-1} \Delta_{123\ldots n}^{a12\ldots(s-1)(s+1)\ldots n}. \tag{43}$$

For consider

$$\begin{vmatrix} \beta_0, & \beta_1, & \ldots, & \beta_n \\ a_1, & a_{11}, & \ldots, & a_{1n} \\ a_2, & a_{21}, & \ldots, & a_{2n} \\ \ldots & \ldots & & \ldots \\ a_n, & a_{n1}, & \ldots, & a_{nn} \end{vmatrix}$$

Since $\Delta_{123\ldots n}^{123\ldots n} \neq 0$, it follows from Theorem 13 and Corollary 9 that the product of the elements of the r-th row by the algebraic complements of the first is zero.

Hence, from (37),

$$\Delta_{123\ldots n}^{123\ldots n} \left[a_r - \sum_{s=1}^{n} a_{rs} (\Delta_{123\ldots n}^{s12\ldots(s-1)(s+1)\ldots n} \Delta_{123\ldots n}^{a12\ldots(s-1)(s+1)\ldots n} \right] = 0,$$

so that
$$a_r = \sum_{s=1}^{n} a_{rs} (\Delta_{123\ldots n}^{s12\ldots(s-1)(s+1)\ldots n})^{-1} \Delta_{123\ldots n}^{a12\ldots(s-1)(s+1)\ldots n}],$$

and consequently x_s, given by (43), is a solution of the equations (42).

Again, multiply the left of the r-th row of (42) by the algebraic complements of a_{rs}, and sum for $r = 1, 2, 3, \ldots, n$.

Hence
$$\sum_{r=1}^{n} A_{rs} \sum_{s=1}^{n} a_{rs} x_s = \sum_{r=1}^{n} A_{rs} a_r,$$

that is to say

$$\Delta_{123\ldots n}^{s12\ldots(s-1)(s+1)\ldots n} x_s = \Delta_{123\ldots n}^{a12\ldots(s-1)(s+1)\ldots n},$$

so that the x_s of (43) must be the only solution. We thus obtain

THEOREM 15. *The system* (42) *has a unique solution, given by* (43), *if**

$$\Delta_{123\ldots n}^{123\ldots n} \neq 0.$$

* In the case of quaternions, the Nabla function of Study (*loc. cit.*) is the norm of $\Delta_{123\ldots n}^{123\ldots n}$.

The case $\Delta_{123\dots n}^{123\dots n}=0$.

Definition. The matrix

$$\begin{pmatrix} a_{11}, & a_{12}, & \dots, & a_{1n} \\ a_{21}, & a_{22}, & \dots, & a_{2n} \\ \dots & \dots & & \dots \\ a_{p1}, & a_{p2}, & \dots, & a_{pn} \end{pmatrix}$$

is said to be of *left* (*right*) *rank* r, if all $(r+1)$-rowed left (right) hand minor determinants of the matrix vanish, but at least one r-rowed left (right) hand minor determinant of the matrix does not vanish.

Evidently the rank is unaltered by an interchange of rows or columns, and by the addition of a left (right) multiple of the elements of any row (column) to the corresponding elements of any other row (column).

Definition. The p systems of numbers in the p rows of the matrix are said to be *left-linearly independent* if the n equations

$$\sum_{s=1}^{p} \lambda_s a_{sr} = 0 \quad (r = 1, 2, 3, \dots, n)$$

are valid only for $\lambda_1 = \lambda_2 = \dots = \lambda_p = 0$.

It is easily proved that a system of p left-hand linear expressions in n-variables are left-linearly independent if, and only if, the left rank of their matrix is p. Further, if the left rank is r ($\leqslant p$), we may choose exactly r left-linearly independent expressions, the others being left-linear combinations of them.

If, therefore, the left rank of the system of homogeneous left-hand equations is r, we may write them as

$$\left.\begin{array}{l} a_{11}x_1+a_{12}x_2+\dots+a_{1r}x_r = -(a_{1(r+1)}x_{r+1}+a_{1(r+2)}x_{r+2}+\dots+a_{1n}x_n) \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ a_{r1}x_1+a_{r2}x_2+\dots+a_{rr}x_r = -(a_{r(r+1)}x_{r+1}+a_{r(r+2)}x_{r+2}+\dots+a_{rn}x_n) \end{array}\right\}, \tag{44}$$

and all others are linear combinations of them if

$$\Delta_{123\dots r}^{123\dots r} \neq 0. \tag{45}$$

The previous result (43) shows that

$$x_s = -\sum_{t=r+1}^{n} (\Delta_{12\dots r}^{s1\dots(s-1)(s+1)\dots r})^{-1}\Delta_{12\dots r}^{t1\dots(s-1)(s+1)\dots r}\, x_t. \tag{46}$$

For convenience, write (46) in the form

$$x_s = \sum_{t=r+1}^{n} B_s^{-1} A_{ts}. \tag{47}$$

Hence every solution is a right-linear combination of

$$\left.\begin{array}{llll|llll} x_1, & x_2, & \ldots, & x_r; & x_{r+1}, & x_{r+2}, & \ldots, & x_n \\ B_1^{-1}A_{(r+1)1}, & B_2^{-1}A_{(r+1)2}, & \ldots, & B_r^{-1}A_{(r+1)r}; & 1, & 0, & \ldots, & 0 \\ B_1^{-1}A_{(r+2)1}, & B_2^{-1}A_{(r+2)2}, & \ldots, & B_r^{-1}A_{(r+2)r}; & 0, & 1, & \ldots, & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ B_1^{-1}A_{n1}, & B_2^{-1}A_{n2}, & \ldots, & B_r^{-1}A_{nr}; & 0, & 0, & \ldots, & 1 \end{array}\right\}, \tag{48}$$

Conversely every right-linear combination of these is a solution. All this is strictly analogous to the ordinary case over commutative systems.

The system $$\sum_{s=1}^{n} a_{rs} x_r = a_r \quad (r = 1, 2, 3, \ldots, p),$$

evidently has a solution only if the matrices

$$\begin{pmatrix} a_{11}, & a_{12}, & \ldots, & a_{1n} \\ a_{21}, & a_{22}, & \ldots, & a_{2n} \\ \ldots & & \ldots & \ldots \\ a_{p1}, & a_{p2}, & \ldots, & a_{pn} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_1, & a_{11}, & a_{12}, & \ldots, & a_{1n} \\ a_2, & a_{21}, & a_{22}, & \ldots, & a_{2n} \\ \ldots & & \ldots & & \ldots \\ a_p, & a_{p1}, & a_{p2}, & \ldots, & a_{pn} \end{pmatrix}$$

have the same left rank. In such a case a solution is

$$x_t = x_t' = (\Delta_{12\ldots r}^{t1\ldots(t-1)(t+1)\ldots r})^{-1} \begin{vmatrix} a_1, & a_{11}, & \ldots, & a_{1(t-1)}, & a_{1(t+1)}, & \ldots, & a_{1r} \\ a_2, & a_{21}, & \ldots, & a_{2(t-1)}, & a_{2(t+1)}, & \ldots, & a_{2r} \\ \ldots & & \ldots & \ldots & \ldots & & \ldots \\ a_r, & a_{r1}, & \ldots, & a_{r(t-1)}, & a_{r(t+1)}, & \ldots, & a_{rr} \end{vmatrix} \tag{49}$$

by (43); and the most general solution is the sum of (49) and the general right-linear combinations of (48).

Multiplication of determinants. It is easily proved* that the left-hand determinant in which the element in the r-th row and s-th column is

$$\sum_{j=1}^{n} l_{sj} a_{rj}$$

is a permutator of the product of the determinants $|l_{rs}|$ and $|a_{rs}|$ in that order.

* *Messenger of Math.*, 55 (1926), 145–152 (151).

We require the following theorem :

THEOREM 16. *The expression B defined by the equation*

$$B = \begin{vmatrix} b_{11}, & b_{12}, & \dots, & b_{1n} \\ b_{21}, & b_{22}, & \dots, & b_{2n} \\ \dots & \dots & & \dots \\ b_{n1}, & b_{n2}, & \dots, & b_{nn} \end{vmatrix}^{-1} \times \begin{vmatrix} 1, & 0, & \dots, & 0 \\ 0, & b_{22}, & \dots, & b_{2n} \\ \dots & \dots & & \dots \\ 0, & b_{n2}, & \dots, & b_{nn} \end{vmatrix} \tag{50}$$

is unaltered if each of its constituent determinants is multiplied, row into row, by the non-vanishing scalar determinant

$$\delta = \begin{vmatrix} l_{11}, & l_{12}, & \dots, & l_{1n} \\ 0, & l_{22}, & \dots, & l_{2n} \\ \dots & \dots & & \dots \\ 0, & l_{n2}, & \dots, & l_{nn} \end{vmatrix}$$

For B is the value of x_1 which satisfies the system

$$\sum_{s=1}^{n} b_{rs} x_s = a_r \quad (r = 1, 2, 3, \dots, n), \tag{51}$$

with
$$a_1 = 1, \quad a_r = 0 \quad (r \neq 1).$$

Hence B is unaltered by all non-singular substitutions carried out on $x_2, x_3, \dots, x_n$, and, in particular, by a linear substitution of the type specified by δ.

A direct proof is as follows :

The expression B is unaltered by any interchange of columns not involving the first columns, and it is also unaltered by the multiplication of the elements of a column by a scalar, the same operations being carried out on the two constituent determinants of B. Hence the following operations leave B unaltered, where, for brevity, the steps are specified for one of the two constituents of B only.

Consider

$$B' = \begin{vmatrix} \sum_{s=1}^{n} l_{1s} b_{1s}, & \sum_{s=2}^{n} l_{2s} b_{1s}, & \dots, & \sum_{s=2}^{n} l_{ns} b_{1s} \\ \sum_{s=1}^{n} l_{1s} b_{2s}, & \sum_{s=2}^{n} l_{2s} b_{2s}, & \dots, & \sum_{s=2}^{n} l_{ns} b_{2s} \\ \dots & \dots & \dots & \dots \\ \sum_{s=1}^{n} l_{1s} b_{ns}, & \sum_{s=2}^{n} l_{2s} b_{ns}, & \dots, & \sum_{s=2}^{n} l_{ns} b_{ns} \end{vmatrix}$$

Multiply the r-th column ($r = 2, 3, \ldots, n$) by the algebraic complement L_{r1} of l_{r1} in δ, and add the results to the first column; B' becomes

$$\begin{vmatrix} \delta b_{11}, & \sum\limits_{s=2}^{n} l_{2s} b_{1s}, & \ldots, & \sum\limits_{s=2}^{n} l_{ns} b_{1s} \\ \delta b_{21}, & \sum\limits_{s=2}^{n} l_{2s} b_{2s}, & \ldots, & \sum\limits_{s=2}^{n} l_{ns} b_{2s} \\ \ldots & \ldots & \ldots & \ldots \\ \delta b_{n1}, & \sum\limits_{s=2}^{n} l_{2s} b_{ns}, & \ldots, & \sum\limits_{s=2}^{n} l_{ns} b_{ns} \end{vmatrix}.$$

Since
$$L_{11} = \begin{vmatrix} l_{22}, & \ldots, & l_{2n} \\ \ldots & \ldots & \ldots \\ l_{n2}, & \ldots, & l_{nn} \end{vmatrix} \neq 0,$$

at least one of the minor determinants of the elements of the first row is non-zero; let it be L_{2r}. Multiply the second, third, ..., n-th columns respectively by the algebraic complements $L_{2r}, L_{3r}, \ldots, L_{nr}$, and add into the second column; B' becomes

$$\begin{vmatrix} \delta b_{11}, & L_{11} b_{1r}, & \sum\limits_{s=2}^{n} l_{3s} b_{1s}, & \ldots, & \sum\limits_{s=2}^{n} l_{ns} b_{1s} \\ \delta b_{21}, & L_{11} b_{2r}, & \sum\limits_{s=2}^{n} l_{3s} b_{2s}, & \ldots, & \sum\limits_{s=2}^{n} l_{ns} b_{2s} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \delta b_{n1}, & L_{11} b_{nr}, & \sum\limits_{s=2}^{n} l_{3s} b_{ns}, & \ldots, & \sum\limits_{s=2}^{n} l_{ns} b_{ns} \end{vmatrix},$$

where the elements b_{qr} have been removed from all columns after the second by interchanges and subtractions not involving the first column; B is unaltered by these operations. If we proceed in this way, it is evident that B' ultimately becomes

$$\begin{vmatrix} b_{11}, & b_{1r}, & \ldots, & b_{1t} \\ b_{21}, & b_{2r}, & \ldots, & b_{2t} \\ \ldots & \ldots & & \ldots \\ b_{n1}, & b_{nr}, & \ldots, & b_{nt} \end{vmatrix},$$

where $1, r, \ldots, t$ is some rearrangement of $1, 2, \ldots, n$; and scalar numbers δ, L_{11}, etc. have been removed.

Hence B is unaltered by the specified multiplication.

COROLLARY 11. *Similarly B is unaltered by multiplying rows of δ into columns of B, provided that the elements of B which are in a given column are kept in that column. The process is evidently equivalent to that of forming suitable linear combinations of the n equations* (51).

PART II. THE MOST GENERAL LINEAR SYSTEM. : two-sided

We now proceed to generalize the preceding results by removing the restriction of one-sidedness from our equations and determinants.

Consider the single linear equation*

$$\sum_s b_s x b'_s = a.$$

It may be written

$$a_1 x + a_2 x e_2 + a_3 x e_3 + \ldots + a_n x e_n = a, \tag{52}$$

where $1, e_2, e_3, \ldots, e_n$ is a suitably chosen basis of the division algebra, that is to say the elements are linearly independent over the field F over which the algebra is defined†, and

$$e_r . e_s = \sum_{t=1}^{n} \gamma_{rst} e_t, \tag{53}$$

where the γ_{rst} are numbers of F, and consequently are commutative with all numbers of the algebra.

Let ϕ denote the operator $a_1(x) + a_2(x)e_2 + a_3(x)e_3 + \ldots + a_n(x)e_n$, so that (52) may be written

$$\phi(x) = a. \tag{54}$$

To solve (52) multiply on the right in succession by $e_2, e_3, \ldots, e_n$, use (53), carry the scalar elements γ_{rst} through the x's (with which they are commutative) into the a's, and again write the expressions in the form (52).

Since $$e_r . 1 = 1 . e_r = e_r,$$

we have

$$\gamma_{r1t} = \gamma_{1rt} = 0 \quad (r \neq t), \qquad \gamma_{r1r} = \gamma_{1rr} = 1, \qquad \gamma_{111} = 1,$$

* Sylvester, *Comptes rendus*, 99 (1884), 117–118, 473–476, 502–505: 409–412, 432–436, 527–529. [*Mathematical Papers*, 4 (1912), 181–187; 199–207.]

† Dickson, *Algebras and their arithmetics* (1923), 17.

and hence we get

$$\left.\begin{array}{l}
\sum_{t=1}^{n} \gamma_{t11} a_t . x + \sum_{t=1}^{n} \gamma_{t12} a_t . xe_2 + \sum_{t=1}^{n} \gamma_{t13} a_t . xe_3 + \ldots + \sum_{t=1}^{n} \gamma_{t1n} a_t . xe_n = a \\
\sum_{t=1}^{n} \gamma_{t21} a_t . x + \sum_{t=1}^{n} \gamma_{t22} a_t . xe_2 + \sum_{t=1}^{n} \gamma_{t23} a_t . xe_3 + \ldots + \sum_{t=1}^{n} \gamma_{t2n} a_t . xe_n = ae_2 \\
\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
\sum_{t=1}^{n} \gamma_{ts1} a_t . x + \sum_{t=1}^{n} \gamma_{ts2} a_t . xe_2 + \sum_{t=1}^{n} \gamma_{ts3} a_t . xe_3 + \ldots + \sum_{t=1}^{n} \gamma_{tsn} a_t . xe_n = ae_s \\
\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
\sum_{t=1}^{n} \gamma_{tn1} a_t . x + \sum_{t=1}^{n} \gamma_{tn2} a_t . xe_2 + \sum_{t=1}^{n} \gamma_{tn3} a_t . xe_3 + \ldots + \sum_{t=1}^{n} \gamma_{tnn} a_t . xe_n = ae_n
\end{array}\right\} . \quad (55)$$

Multiply, in succession, on the left by the algebraic complements A of the elements of the first column in

$$\Delta = \begin{vmatrix}
\sum_{t=1}^{n} \gamma_{t11} a_t, & \sum_{t=1}^{n} \gamma_{t12} a_t, & \sum_{t=1}^{n} \gamma_{t13} a_t, & \ldots, & \sum_{t=1}^{n} \gamma_{t1n} a_t \\
\sum_{t=1}^{n} \gamma_{t21} a_t, & \sum_{t=1}^{n} \gamma_{t22} a_t, & \sum_{t=1}^{n} \gamma_{t23} a_t, & \ldots, & \sum_{t=1}^{n} \gamma_{t2n} a_t \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\sum_{t=1}^{n} \gamma_{ts1} a_t, & \sum_{t=1}^{n} \gamma_{ts2} a_t, & \sum_{t=1}^{n} \gamma_{ts3} a_t, & \ldots, & \sum_{t=1}^{n} \gamma_{tsn} a_t \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\sum_{t=1}^{n} \gamma_{tn1} a_t, & \sum_{t=1}^{n} \gamma_{tn2} a_t, & \sum_{t=1}^{n} \gamma_{tn3} a_t, & \ldots, & \sum_{t=1}^{n} \gamma_{tnn} a_t
\end{vmatrix}, \quad (56)$$

and add : we get

$$\left.\begin{array}{l}
\Delta x = \sum_{s=1}^{n} A_s a e_s, \\
\Delta = \sum_{s=1}^{n} A_s . \sum_{t=1}^{n} \gamma_{ts1} a_t, \\
0 = \sum_{s=1}^{n} A_s \sum_{t=1}^{n} \gamma_{tsr} a_t \quad (r \neq 1)
\end{array}\right\} . \quad (57)$$

and (the last line above).

We write
$$\phi^{-1} = \sum_{s=1}^{n} \Delta^{-1} A_s(\quad) e_s,$$

when $\Delta \neq 0$, or, more conveniently,

$$\phi^{-1} = \sum_{s=1}^{n} H_s(\quad) e_s, \quad H_s = \Delta^{-1} A_s.$$

We shall now prove that

$$x = \phi^{-1}(a)$$

satisfies the equation (52).

Substituting on the left of (52) we get

$$a_1 \sum_{s=1}^{n} H_s a e_s + a_2 \sum_{s=1}^{n} H_s a e_s \,.\, e_2 + a_3 \sum_{s=1}^{n} H_s a e_s \,.\, e_3 + \ldots + a_n \sum_{s=1}^{n} H_s a e_s \,.\, e_n.$$

If we use the multiplication table (53) and collect corresponding terms, this becomes

$$\left.\begin{array}{l}
\left[\sum_{t=1}^{n} \gamma_{1t1} a_t \,.\, H_1 + \sum_{t=1}^{n} \gamma_{2t1} a_t \,.\, H_2 + \sum_{t=1}^{n} \gamma_{3t1} a_t \,.\, H_3 + \ldots + \sum_{t=1}^{n} \gamma_{nt1} a_t \,.\, H_n\right] a \\
+\left[\sum_{t=1}^{n} \gamma_{1t2} a_t \,.\, H_1 + \sum_{t=1}^{n} \gamma_{2t2} a_t \,.\, H_2 + \sum_{t=1}^{n} \gamma_{3t2} a_t \,.\, H_3 + \ldots + \sum_{t=1}^{n} \gamma_{nt2} a_t \,.\, H_n\right] a e_2 \\
+\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
+\left[\sum_{t=1}^{n} \gamma_{1tr} a_t \,.\, H_1 + \sum_{t=1}^{n} \gamma_{2tr} a_t \,.\, H_2 + \sum_{t=1}^{n} \gamma_{3tr} a_t \,.\, H_3 + \ldots + \sum_{t=1}^{n} \gamma_{ntr} a_t \,.\, H_n\right] a e_r \\
+\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
+\left[\sum_{t=1}^{n} \gamma_{1tn} a_t \,.\, H_1 + \sum_{t=1}^{n} \gamma_{2tn} a_t \,.\, H_2 + \sum_{t=1}^{n} \gamma_{3tn} a_t \,.\, H_3 + \ldots + \sum_{t=1}^{n} \gamma_{ntn} a_t \,.\, H_n\right] a e_n
\end{array}\right\}, \tag{58}$$

We shall prove that this expression is equal to a by showing that the equations in $H_1, H_2, \ldots, H_n$ obtained from (58) by direct comparison of (58) with a have the same values for $H_1, H_2, \ldots, H_n$ as equations (57).

It is therefore a question of identifying

$$H_s = \begin{vmatrix}
\Sigma\gamma_{st1}a_t, & \Sigma\gamma_{2t1}a_t, & \ldots, & \Sigma\gamma_{(s-1)t1}a_t. & \Sigma\gamma_{1t1}a_t, & \Sigma\gamma_{(s+1)t1}a_t, & \ldots, & \Sigma\gamma_{nt1}a_t \\
\Sigma\gamma_{st2}a_t, & \Sigma\gamma_{2t2}a_t, & \ldots, & \Sigma\gamma_{(s-1)t2}a_t, & \Sigma\gamma_{1t2}a_t, & \Sigma\gamma_{(s+1)t2}a_t, & \ldots, & \Sigma\gamma_{nt2}a_t \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\Sigma\gamma_{str}a_t, & \Sigma\gamma_{2tr}a_t, & \ldots, & \Sigma\gamma_{(s-1)tr}a_t, & \Sigma\gamma_{1tr}a_t, & \Sigma\gamma_{(s+1)tr}a_t, & \ldots, & \Sigma\gamma_{ntr}a_t \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\Sigma\gamma_{stn}a_t, & \Sigma\gamma_{2tn}a_t, & \ldots, & \Sigma\gamma_{(s-1)tn}a_t, & \Sigma\gamma_{1tn}a_t, & \Sigma\gamma_{(s+1)tn}a_t, & \ldots, & \Sigma\gamma_{ntn}a_t
\end{vmatrix}^{-1}$$

$$\times \text{(the minor outlined by dots).} \tag{59}$$

with

$H'_s =$ (the minor outlined by dots in the following determinant)

$$\times \begin{vmatrix} \Sigma\gamma_{ts1}a_t, & \Sigma\gamma_{t21}a_t, & \dots, & \Sigma\gamma_{t(s-1)1}a_t, & \Sigma\gamma_{t11}a_t, & \Sigma\gamma_{t(s+1)1}a_t, & \dots, & \Sigma\gamma_{tn1}a_t \\ \Sigma\gamma_{ts2}a_t, & \Sigma\gamma_{t22}a_t, & \dots, & \Sigma\gamma_{t(s-1)2}a_t, & \Sigma\gamma_{t12}a_t, & \Sigma\gamma_{t(s+1)2}a_t, & \dots, & \Sigma\gamma_{tn2}a_t \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \Sigma\gamma_{tsr}a_t, & \Sigma\gamma_{tsr}a_t, & \dots, & \Sigma\gamma_{t(s-1)r}a_t, & \Sigma\gamma_{t1r}a_t, & \Sigma\gamma_{t(s+1)r}a_t, & \dots, & \Sigma\gamma_{tnr}a_t \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \Sigma\gamma_{tsn}a_t, & \Sigma\gamma_{t2n}a_t, & \dots, & \Sigma\gamma_{t(s-1)n}a_t, & \Sigma\gamma_{t1n}a_t, & \Sigma\gamma_{t(s+1)n}a_t, & \dots, & \Sigma\gamma_{tnn}a_t \end{vmatrix}^{-1}$$

If $H'_s \neq 0$, we may interchange rows with columns in this expression, by Theorem 11, and get

$$H'_s = \begin{vmatrix} \Sigma\gamma_{ts1}a_t, & \Sigma\gamma_{ts2}a_t, & \dots, & \Sigma\gamma_{tsr}a_t, & \dots, & \Sigma\gamma_{tsn}a_t \\ \Sigma\gamma_{t21}a_t, & \Sigma\gamma_{t22}a_t, & \dots, & \Sigma\gamma_{t2r}a_t, & \dots, & \Sigma\gamma_{t2n}a_t \\ \dots \quad \dots & \dots & \dots & \dots & \dots & \dots \\ \Sigma\gamma_{t(s-1)1}a_t, & \Sigma\gamma_{t(s-1)2}a_t, & \dots, & \Sigma\gamma_{t(s-1)r}a_t, & \dots, & \Sigma\gamma_{t(s-1)n}a_t \\ \Sigma\gamma_{t11}a_t, & \Sigma\gamma_{t12}a_t & \dots, & \Sigma\gamma_{t1r}a_t, & \dots, & \Sigma\gamma_{t1n}a_t \\ \Sigma\gamma_{t(s+1)1}a_t, & \Sigma\gamma_{t(s+1)2}a_t, & \dots, & \Sigma\gamma_{t(s+1)r}a_t, & \dots, & \Sigma\gamma_{t(s+1)n}a_t \\ \dots \quad \dots & \dots & \dots & \dots & \dots & \dots \\ \Sigma\gamma_{tn1}a_t, & \Sigma\gamma_{tn2}a_t, & \dots, & \Sigma\gamma_{tnr}a_t, & \dots, & \Sigma\gamma_{tnn}a_t \end{vmatrix}^{-1}$$

$$\times \text{(the minor outlined by dots)}. \qquad (60)$$

Now, since the algebra is a division algebra, the equation $\beta x = e_s$ regarded as an equation in β has one and only one solution for every x which is not zero.

Consequently the equation

$$\sum_{r=1}^{n} \beta_r e_r \,.\, \sum_{t=1}^{n} x_t e_t = e_s,$$

that is to say

$$\sum_{r=1}^{n} \sum_{t=1}^{n} \beta_r x_t \,.\, e_r e_t = \sum_{r=1}^{n} \sum_{t=1}^{n} \beta_r x_t \sum_{k=1}^{n} \gamma_{trk} e_k = e_s$$

has a solution for all x_t which are not all zero.

Hence the determinant of which the element in the r-th row and t-th column is γ_{trs} is non-zero for any of the values $1, 2, \dots,$, n of s. By

Theorem (16) multiplication of the constituent determinants of H_s and H'_s by non-vanishing determinants which contain only one non-zero element in the first row and first column leaves them unaltered.

Hence H_s is unaltered if we multiply its constituent determinants (rows into columns) by

$$\begin{vmatrix} \gamma_{1ss}, & \gamma_{2ss}, & \ldots, & \gamma_{jss}, & \ldots, & \gamma_{nss} \\ \gamma_{12s}, & \gamma_{22s}, & \ldots, & \gamma_{j2s}, & \ldots, & \gamma_{n2s} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \gamma_{1(s-1)s}, & \gamma_{2(s-1)s}, & \ldots, & \gamma_{j(s-1)s}, & \ldots, & \gamma_{n(s-1)s} \\ \gamma_{11s}, & \gamma_{21s}, & \ldots, & \gamma_{j1s}, & \ldots, & \gamma_{n1s} \\ \gamma_{1(s+1)s}, & \gamma_{2(s+1)s}, & \ldots, & \gamma_{j(s+1)s}, & \ldots, & \gamma_{n(s+1)s} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \gamma_{1ns}, & \gamma_{2ns}, & \ldots, & \gamma_{jns}, & \ldots, & \gamma_{nns} \end{vmatrix},$$

a determinant which does not vanish and whose first column (except for the leading element) consists of zeros.

There results

$$H_s = \begin{vmatrix} \sum_{t=1}^{n} a_t \sum_{j=1}^{n} \gamma_{jss}\gamma_{stj}, & \sum_{t=1}^{n} a_t \sum_{j=1}^{n} \gamma_{jss}\gamma_{2tj}, & \ldots, & \sum_{t=1}^{n} a_t \sum_{j=1}^{n} \gamma_{jss}\gamma_{rtj}, & \ldots, & \sum_{t=1}^{n} a_t \sum_{j=1}^{n} \gamma_{jss}\gamma_{ntj} \\ \sum_{t=1}^{n} a_t \sum_{j=1}^{n} \gamma_{j2s}\gamma_{stj}, & \sum_{t=1}^{n} a_t \sum_{j=1}^{n} \gamma_{j2s}\gamma_{2tj}, & \ldots, & \sum_{t=1}^{n} a_t \sum_{j=1}^{n} \gamma_{j2s}\gamma_{rtj}, & \ldots, & \sum_{t=1}^{n} a_t \sum_{j=1}^{n} \gamma_{j2s}\gamma_{ntj} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \sum_{t=1}^{n} a_t \sum_{j=1}^{n} \gamma_{j1s}\gamma_{stj}, & \sum_{t=1}^{n} a_t \sum_{j=1}^{n} \gamma_{j1s}\gamma_{2tj}, & \ldots, & \sum_{t=1}^{n} a_t \sum_{j=1}^{n} \gamma_{j1s}\gamma_{rtj}, & \ldots, & \sum_{t=1}^{n} a_t \sum_{j=1}^{n} \gamma_{j1s}\gamma_{ntj} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \sum_{t=1}^{n} a_t \sum_{j=1}^{n} \gamma_{jns}\gamma_{stj}, & \sum_{t=1}^{n} a_t \sum_{j=1}^{n} \gamma_{jns}\gamma_{2tj}, & \ldots, & \sum_{t=1}^{n} a_t \sum_{j=1}^{n} \gamma_{jns}\gamma_{rtj}, & \ldots, & \sum_{t=1}^{n} a_t \sum_{j=1}^{n} \gamma_{jns}\gamma_{ntj} \end{vmatrix}^{-1}$$

$\times$ (the minor outlined by dots).

But, since the algebra is associative*,

$$\sum_{j=1}^{n} \gamma_{srj} \cdot \gamma_{ijk} = \sum_{j=1}^{n} \gamma_{isj}\gamma_{jrk} \quad (i, s, r, k = 1, 2, 3, \ldots, n),$$

* Dickson, *Algebras and their arithmetics* (1923), 92.

and hence

$$H_s = \begin{vmatrix} \sum\limits_{t,j} \gamma_{tsj}\,.\,\gamma_{sjs}\,a_t, & \sum\limits_{t,j} \gamma_{tsj}\,.\,\gamma_{2js}\,a_t, & \ldots, & \sum\limits_{t,j} \gamma_{tsj}\,.\,\gamma_{rjs}\,a_t, & \ldots, & \sum\limits_{t,j} \gamma_{tsj}\,.\,\gamma_{njs}\,a_t \\ \sum\limits_{t,j} \gamma_{t2j}\,.\,\gamma_{sjs}\,a_t, & \sum\limits_{t,j} \gamma_{t2j}\,.\,\gamma_{2js}\,a_t, & \ldots, & \sum\limits_{t,j} \gamma_{t2j}\,.\,\gamma_{rjs}\,a_t, & \ldots, & \sum\limits_{t,j} \gamma_{t2j}\,.\,\gamma_{njs}\,a_t \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \sum\limits_{t,j} \gamma_{t1j}\,.\,\gamma_{sjs}\,a_t, & \sum\limits_{t,j} \gamma_{t1j}\,.\,\gamma_{2js}\,a_t, & \ldots, & \sum\limits_{t,j} \gamma_{t1j}\,.\,\gamma_{rjs}\,a_t, & \ldots, & \sum\limits_{t,j} \gamma_{t1j}\,.\,\gamma_{njs}\,a_t \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \sum\limits_{t,j} \gamma_{tnj}\,.\,\gamma_{sjs}\,a_t, & \sum\limits_{t,j} \gamma_{tnj}\,.\,\gamma_{2js}\,a_t, & \ldots, & \sum\limits_{t,j} \gamma_{tnj}\,.\,\gamma_{rjs}\,a_t, & \ldots, & \sum\limits_{t,j} \gamma_{tnj}\,.\,\gamma_{njs}\,a_t \end{vmatrix}^{-1}$$

$$\times \text{(the minor outlined by dots).} \qquad (61)$$

But this is precisely the result of multiplying the rows in the constituent determinants of H_s' by the rows of

$$\begin{vmatrix} \gamma_{s1s}, & \gamma_{s2s}, & \ldots, & \gamma_{sjs}, & \ldots, & \gamma_{sns} \\ \gamma_{21s}, & \gamma_{22s}, & \ldots, & \gamma_{2js}, & \ldots, & \gamma_{2ns} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \gamma_{(s-1)1s}, & \gamma_{(s-1)2s}, & \ldots, & \gamma_{(s-1)js}, & \ldots, & \gamma_{(s-1)ns} \\ \gamma_{11s}, & \gamma_{12s}, & \ldots, & \gamma_{1js}, & \ldots, & \gamma_{1ns} \\ \gamma_{(s+1)1s}, & \gamma_{(s+1)2s}, & \ldots, & \gamma_{(s+1)js}, & \ldots, & \gamma_{(s+1)ns} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \gamma_{n1s}, & \gamma_{n2s}, & \ldots, & \gamma_{njs}, & \ldots, & \gamma_{nns} \end{vmatrix},$$

an operation which leaves H_s' unchanged.

Hence $$H_s = H_s',$$

provided that neither is zero.

If H_s is zero, the minor which is outlined in (59) vanishes, and hence, by the theorem on multiplication of determinants, the determinant shown in (61) also vanishes; and therefore H_s' also vanishes.

Hence, in all cases, $H_s = H_s'$ and the expression (58) is equal to α. Consequently

$$\phi\{\phi^{-1}(x)\} = x,$$

provided that $\Delta \neq 0$.

Therefore the necessary and sufficient condition for the equation (52) *to have one and only one solution is that* $\Delta \neq 0$, *and the solution is then given by* $x = \phi^{-1}(\alpha)$.

We shall now prove that

$$\phi^{-1}\{\phi(z)\} = z.$$

For

$$\phi^{-1}\{\phi(z)\} = \Delta^{-1} \begin{vmatrix} \phi(z), & \sum_{t=1}^{n} \gamma_{t12} a_t, & \ldots, & \sum_{t=1}^{n} \gamma_{t1j} a_t, & \ldots, & \sum_{t=1}^{n} \gamma_{t1n} a_t \\ \phi(z)\, e_2, & \sum_{t=1}^{n} \gamma_{t22} a_t, & \ldots, & \sum_{t=1}^{n} \gamma_{t2j} a_t, & \ldots, & \sum_{t=1}^{n} \gamma_{t2n} a \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \phi(z)\, e_s, & \sum_{t=1}^{n} \gamma_{ts2} a_t, & \ldots, & \sum_{t=1}^{n} \gamma_{tsj} a_t, & \ldots, & \sum_{t=1}^{n} \gamma_{tsn} a_t \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \phi(z)\, e_n, & \sum_{t=1}^{n} \gamma_{tn2} a_t, & \ldots, & \sum_{t=1}^{n} \gamma_{tnj} a_t, & \ldots, & \sum_{t=1}^{n} \gamma_{tnn} a_t \end{vmatrix}$$

$$= \Delta^{-1} \begin{vmatrix} \sum_{j,t} \gamma_{t1j} a_t z e_j, & \sum_{t=1}^{n} \gamma_{t12} a_t, & \ldots, & \sum_{t=1}^{n} \gamma_{t1j} a_t, & \ldots, & \sum_{t=1}^{n} \gamma_{t1n} a_t \\ \sum_{j,t} \gamma_{t2j} a_t z e_j, & \sum_{t=1}^{n} \gamma_{t22} a_t, & \ldots, & \sum_{t=1}^{n} \gamma_{t2j} a_t, & \ldots, & \sum_{t=1}^{n} \gamma_{t2n} a_t \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \sum_{j,t} \gamma_{tsj} a_t z e_j, & \sum_{t=1}^{n} \gamma_{ts2} a_t, & \ldots, & \sum_{t=1}^{n} \gamma_{tsj} a_t, & \ldots, & \sum_{t=1}^{n} \gamma_{tsn} a_t \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \sum_{j,t} \gamma_{tnj} a_t z e_j, & \sum_{t=1}^{n} \gamma_{tn2} a_t, & \ldots, & \sum_{t=1}^{n} \gamma_{tnj} a_t, & \ldots, & \sum_{t=1}^{n} \gamma_{tnn} a_t \end{vmatrix}$$

If we multiply the j-th column on the right by ze_j $(j = 2, 3, \ldots, n)$, and subtract the results from the first column, this becomes

$$\Delta^{-1} \begin{vmatrix} \sum_{t=1}^{n} \gamma_{t11} a_t \,.\, z, & \sum_{t=1}^{n} \gamma_{t12} a_t, & \ldots, & \sum_{t=1}^{n} \gamma_{t1j} a_t, & \ldots, & \sum_{t=1}^{n} \gamma_{t1n} a_t \\ \sum_{t=1}^{n} \gamma_{t21} a_t \,.\, z, & \sum_{t=1}^{n} \gamma_{t22} a_t, & \ldots, & \sum_{t=1}^{n} \gamma_{t2j} a_t, & \ldots, & \sum_{t=1}^{n} \gamma_{t2n} a_t \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \sum_{t=1}^{n} \gamma_{ts1} a_t \,.\, z, & \sum_{t=1}^{n} \gamma_{ts2} a_t, & \ldots, & \sum_{t=1}^{n} \gamma_{tsj} a_t, & \ldots, & \sum_{t=1}^{n} \gamma_{tsn} a_t \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \sum_{t=1}^{n} \gamma_{tn1} a_t \,.\, z, & \sum_{t=1}^{n} \gamma_{tn2} a_t, & \ldots, & \sum_{t=1}^{n} \gamma_{tnj} a_t, & \ldots, & \sum_{t=1}^{n} \gamma_{tnn} a_t \end{vmatrix}$$

$$= \Delta^{-1} \,.\, \Delta \,.\, z = z.$$

Hence $$\phi^{-1}\{\phi(z)\} = z,$$

so that ϕ is commutative with ϕ^{-1}.

Similarly, $$(\phi^{-1})^{-1}(z) = \phi(z),$$

so that $$(\phi^{-1})^{-1} = \phi.$$

We see, therefore, that ϕ may be treated as if it were a number; and provided that we always interpret $\phi_1\phi_2(z)$ to mean $\phi_1\{\phi_2(z)\}$, no difficulties arise. In fact, the ϕ's constitute an algebra with n^2 units, namely $\phi_{rs} = e_r(\ \ )e_s$, but this algebra is not, in general, a division algebra.

Singular linear operators. The case $\Delta = 0$ remains for consideration. Let equations (55) be of left rank r, so that at least one left-hand determinant of order r in Δ is non-zero. It follows from the investigation in Part I following on the definition of left-linear independence that the corresponding matrices which involve a must also have left rank r, or the equations are inconsistent.

This is the case with the equation

$$xa - ax = 1,$$

which arises in quantum algebras; this equation has no solution in any *finite* associative algebra, nor, indeed, in a very extensive class of infinite algebras.

If there is more than one solution, the difference of two solutions is a solution of (55) with the right-hand sides replaced by zeros; and when we multiply by the algebraic complements of the elements of a column in Δ and add, we get

$$\Delta(x'-x) = 0.$$

Since there are no divisors of zero in a division algebra we must have $\Delta = 0$.

A method of solution in the case of quaternions makes use of a formula given by Sylvester*, where the operator ϕ is shown to satisfy an equation of the fourth degree with real coefficients†

$$\phi^4 - A\phi^3 + B\phi^2 + C\phi - D = 0.$$

* *Comptes rendus*, 99 (1884), 504.

† He does not calculate the coefficients.

In our case, since ϕ is singular, $D = 0$; and, if we suppose that $C \neq 0$, we have

$$\phi(\phi^3 - A\phi^2 + B\phi + C) = 0.$$

Hence the general solution of $\phi(x) = a$ is

$$x = X + (\phi^3 - A\phi^2 + B\phi + C)Y,$$

where Y is an arbitrary quaternion and X is a particular solution of the equation.

A more direct method, based on the theory of determinants, is the following, which has the advantage of exhibiting the solution in an explicit form. We confine our attention to quaternions, since the work for the general algebra is much more involved and tedious.

Let our equations be first of rank three, and, for clearness, suppose that the non-vanishing minor of order three is in the top left-hand corner of Δ.

Then, if we multiply the first three equations of (55) by the algebraic complements of the first column in

$$\delta = \begin{vmatrix} a_1, & a_2, & a_3 \\ -a_2, & a_1, & a_4 \\ -a_3, & -a_4, & a_1 \end{vmatrix}$$

and add, we see that x must be a solution of

$$\delta \, . \, x = \begin{vmatrix} a - a_4 xk, & a_2, & a_3 \\ ai + a_3 xk, & a_1, & a_4 \\ aj - a_2 xk, & -a_4, & a_1 \end{vmatrix} = X'(a) + H' \, . \, xk,$$

X' being a linear operator on a, and H' being a left multiple of xk.

That is to say $$x = X(a) + H \, . \, xk, \qquad (62)$$

where $$H = \delta^{-1} H', \quad X = \delta^{-1} X',$$

since $\delta \neq 0$; and consequently

$$(1 + H^2)x = X(a) + H \, . \, X(a)\, k.$$

If* $1 + H^2 \neq 0$, the solution is obvious. In general, however, $1 + H^2 = 0$, so that

$$X(a) + HX(a)\, k = 0.$$

* This means that there is a unique solution in spite of Δ vanishing, and, from other considerations, this seems to be ruled out.

A particular solution of (62) is therefore $x = \frac{1}{2}X(a)$, and the most general solution is

$$x = \tfrac{1}{2}X(a)+(Y+HYk),$$

where Y is an arbitrary quaternion.

Δ *of rank two.* Again, for convenience, let the non-vanishing minor of order two be

$$d = \begin{vmatrix} a_1, & a_2 \\ -a_2, & a_1 \end{vmatrix},$$

so that, as above, x must be a solution of the equation

$$dx = \begin{vmatrix} a, & a_2 \\ ai, & a_1 \end{vmatrix} - \begin{vmatrix} a_3, & a_2 \\ a_4, & a_1 \end{vmatrix} xj + \begin{vmatrix} -a_4, & a_2 \\ a_3, & a_1 \end{vmatrix} xk,$$

that is, of
$$x = X(a)+Jxj+Kxk, \tag{63}$$

where $X(a)$ is a linear operator on a, and J and K are left-hand multiples of xj and xk respectively.

Hence

$$(1+J^2+K^2)\,x = X(a)+JX(a)\,j+KX(a)\,k+(KJ-JK)\,xi. \tag{64}$$

If both $\quad 1+J^2+K^2 = 0 \quad$ and $\quad KJ-JK = 0,$

then
$$X(a)+JX(a)\,j+KX(a)\,k = 0. \tag{65}$$

A particular solution is

$$x = \tfrac{1}{2}X(a).$$

The general solution is

$$x = \tfrac{1}{2}X(a)+(Y+JYj+KYk),$$

where Y is an arbitrary quaternion.

If one of $1+J^2+K^2$ and $KJ-JK$ is zero, and the other is not zero, there is only one solution, and it is obtained by direct division from (64).

There remains the case in which neither is zero; (64) may then be rewritten as

$$A'x+B'xi = \Pi'(a),$$

which is reducible to the form

$$x+Bxi = \Pi(a);$$

the general solution of this, as in (62), is

$$x = \tfrac{1}{2}\Pi(a)+(Y-BYi),$$

where Y is an arbitrary quaternion and

$$\Pi(a) = (1+J^2+K^2)^{-1}[X(a)+JX(a)j+KX(a)k],$$

$$B = (1+J^2+K^2)^{-1}(JK-KJ).$$

Hence, in all cases, we can write down explicit expressions for the solution of the equation $\phi(x) = a$ in terms of our determinants.

In the case of quaternions the method leading to (64) simplifies the actual calculations, for, from (64), the equation $\phi(x) = a$ is now

$$x+Bxi = \Pi(a),$$

that is
$$(1+B^2)x = \Pi(a)-B\Pi(a)i,$$

so that
$$x = (1+B^2)^{-1}[\Pi-B\Pi(a)i].$$

Solution of the general system of simultaneous linear equations in m unknowns.

Let ϕ_{rp} denote the linear operator

$$a_{rp}^{(1)}(\ )+a_{rp}^{(2)}(\ )e_2+\ldots+a_{rp}^{(n)}(\ )e_n.$$

The most general set of linear equations in m unknowns may be written

$$\left.\begin{array}{l} \phi_{11}(x_1)+\phi_{12}(x_2)+\ldots+\phi_{1m}(x_m) = a_1 \\ \phi_{21}(x_1)+\phi_{22}(x_2)+\ldots+\phi_{2m}(x_m) = a_2 \\ \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots \\ \phi_{p1}(x_1)+\phi_{p2}(x_2)+\ldots+\phi_{pm}(x_m) = a_p \end{array}\right\}. \tag{66}$$

Since the solution cannot depend on the order in which the equations and variables are written, if not all the linear operators are singular, we may suppose ϕ_{rp} to be non-singular, and use the theory of determinants set out in this paper, avoiding always divisions by singular or zero operators.

In particular, if the symbolic determinant

$$\Delta = \begin{vmatrix} \phi_{11}, & \phi_{12}, & \ldots, & \phi_{1m} \\ \phi_{21}, & \phi_{22}, & \ldots, & \phi_{2m} \\ \ldots & \ldots & & \ldots \\ \phi_{m1}, & \phi_{m2}, & \ldots, & \phi_{mm} \end{vmatrix} \tag{67}$$

is non-singular, there is one and only one solution, namely

$$x_s = \begin{vmatrix} \phi_{1s}, & \phi_{12}, & \dots, & \phi_{11}, & \dots, & \phi_{1m} \\ \phi_{2s}, & \phi_{22}, & \dots, & \phi_{21}, & \dots, & \phi_{2m} \\ \dots & \dots & & \dots & & \dots \\ \phi_{ms}, & \phi_{m2}, & \dots, & \phi_{m1}, & \dots, & \phi_{mm} \end{vmatrix}^{-1} \begin{vmatrix} a_1, & \phi_{12}, & \dots, & \phi_{11}, & \dots, & \phi_{1m} \\ a_2, & \phi_{22}, & \dots, & \phi_{21}, & \dots, & \phi_{2m} \\ \dots & \dots & & \dots & & \dots \\ a_m, & \phi_{m2}, & \dots, & \phi_{m1}, & \dots, & \phi_{mm} \end{vmatrix}$$

$$(s = 1, 2, 3, \dots, m).$$

If all the ϕ_{sm} are singular, it may still be possible to eliminate the other variables and get

$$\begin{vmatrix} \phi_{1m}, & \phi_{12}, & \dots, & \phi_{11} \\ \phi_{2m}, & \phi_{22}, & \dots, & \phi_{21} \\ \dots & \dots & & \dots \\ \phi_{mm}, & \phi_{m2}, & \dots, & \phi_{mm} \end{vmatrix} x_m = \begin{vmatrix} a_1, & \phi_{12}, & \dots, & \phi_{11} \\ a_2, & \phi_{22}, & \dots, & \phi_{21} \\ \dots & \dots & & \dots \\ a_m, & \phi_{m2}, & \dots, & \phi_{m1} \end{vmatrix},$$

the general solution of which may be obtained by the method just given. The substitution of these values in (66) now leaves a set of equations which can be solved by the above process.

The outstanding case is that in which all the ϕ_{rp} are singular. As an example, we take

$$\left.\begin{aligned} \phi_{11}(x_1)+\phi_{12}(x_2) &= a_1 \\ \phi_{21}(x_1)+\phi_{22}(x_2) &= a_2 \end{aligned}\right\}. \tag{68}$$

As on p. 410, operators Φ_{12} and Φ_{22} exist such that $\Phi_{12}\phi_{12}$ and $\Phi_{22}\phi_{22}$ are zero. Hence x_1 satisfies the two equations

$$\Phi_{12}\{\phi_{11}(x_1)\} = \Phi_{12}(a_1), \qquad \Phi_{22}\{\phi_{21}(x_1)\} = \Phi_{22}(a_2).$$

If these have a common solution, x_2 may be obtained from one of the set (68) by the method for solving equations in one unknown.

Generalization of determinants. Symbolic determinants such as (67) include both the right-hand and the left-hand determinants of Part I as special cases, and furnish their natural generalization. By using them we overcome the grave restrictions of one-sidedness in such matters as the theories of equations and of invariants over division algebras*.

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* *Messenger of Math.*, 55 (1926), 175.