Quantum Information Processing (2020) 19:423
https://doi.org/10.1007/s11128-020-02918-9

N-qudit SLOCC equivalent W states are determined by their bipartite reduced density matrices with tree form

Xia Wu[1,5] · Heng-Yue Jia[1,5] · Dan-Dan Li[2,3] · Ying-Hui Yang[4] · Fei Gao[5]

Received: 21 June 2020 / Accepted: 28 October 2020 / Published online: 21 November 2020

Abstract

It has been proved that N-qudit (i.e., d-level subsystems) generalized W states are determined by their bipartite reduced density matrices. In this paper, we prove that only $(N-1)$ of the bipartite reduced density matrices are sufficient. Furthermore, we find that N-qudit W states preserve their determinability under stochastic local operation and classical communication (SLOCC). That is, all multipartite pure states that are SLOCC equivalent to N-qudit W states can be uniquely determined (among pure, mixed states) by their $(N-1)$ of the bipartite reduced density matrices, if the $(N-1)$ pairs of qudits constitute a tree graph on N vertices, where each pair of qudits represents an edge.

Keywords Quantum state tomography · Generalized W states · Stochastic local operation and classical communication · Reduced density matrices

1 Introduction

As an important technique for quantum information processing, quantum state tomography provides a complete description of an arbitrary quantum state [1–6]. The objectives of quantum state tomography are to reconstruct the density matrix of a quantum state from a series of measurements obtained from experimental data and

✉ Heng-Yue Jia
jiahengyue@163.com

1 School of Information, Central University of Finance and Economics, Beijing 100081, China

2 School of Computer Science (National Pilot Software Engineering School), Beijing University of Posts and Telecommunications, Beijing 100876, China

3 State Key Laboratory of Cryptology, P.O. Box 5159, Beijing 100878, China

4 School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo 454000, China

5 State Key Laboratory of Networking and Switching Technology, Beijing University of Posts and Telecommunications, Beijing 100876, China

this is a major area of interest within the field of quantum communication and quantum cryptography [7,8]. In recent years, there has been an increasing interest in quantum state tomography via reduced density matrices(RDMs) [9–17]. One prerequisite for adopting this approach is that the original state must be uniquely determined among all states by its RDMs, which is called as "parts and whole problem".

A primary concern of "parts and whole problem" is whether the global state be the only state which is compatible with its RDMs. This research has been instrumental in our understanding of quantum correlations. This issue was first addressed by Linden et al. at 2002 in [9]. They proved that almost all pure states of three qubits are uniquely determined by their two particle RDMs, and the only states that do not have this property are those LU equivalent to the three-qubit GHZ state. Since then, a considerable amount of literature has been published on "parts and whole problem". In recent years, much effort has been spent on studying the determinability of quantum states with special forms, such as N-qubit stochastic local operations and classical communication (SLOCC) equivalent W states [18–21], Dicke states [22,23] and stabilizer states [24,25].

The research we have introduced above focus on the two-dimensional quantum systems. As is known to all, the high-dimensional quantum systems also play an important role in quantum information science [26–30]. Certain fundamental features of quantum mechanics can be tested with high-dimensional quantum systems only, such as quantum contextuality [27]. A larger amount of information can be encoded in single qudit than in qubit. Existing research recognizes the critical role played by high-dimensional quantum systems, so there is a growing body of literature focus on the study of the determinability of quantum state with high-dimensional systems [10–12,17,22].

Jones et al. showed that a generic N-qudit pure quantum state is uniquely determined among arbitrary states by its RDMS of $\left(\left\lceil \frac{N}{2} \right\rceil + 1\right)$-particle, and $\left\lfloor \frac{N}{2} \right\rfloor$-particle RDMs are proved to be an insufficient description of a generic N-qudit pure quantum state [12]. When N is odd, people wondered whether a generic pure state can be uniquely determined by its $\frac{N+1}{2}$-particle RDMs. In [17], Huang et al. proved that when N is odd, $\frac{N+1}{2}$-particle RDMs are sufficient to uniquely determine a generic pure state. When concerned about N-qudit generalized W states, Parashar et al. proved that the correlation in these states is reducible to the bipartite level [18,19]. As we can see, this bound on the size of the parts is much smaller than $\left(\left\lceil \frac{N}{2} \right\rceil + 1\right)$.

Previous research has established that any two states which can be obtained from each other by means of SLOCC are suited to implement the same task of quantum information processing [31,32]. But when considered about "parts and whole problem," people found that the determinability of a state from its RDMs cannot be preserved under SLOCC, and a typical instance is GHZ state [13]. To better understand the mechanisms of SLOCC and its effects on the determinability of a state, Rana et al. analyzed the determinability of N-qubit W type states. N-qubit W type states are those states that are SLOCC equivalent to the N-qubit standard W state, which is $\frac{1}{\sqrt{N}}\left(|1\cdots 00\rangle + |01\cdots 0\rangle + \cdots + |0\cdots 01\rangle\right)$. They proved in [19] that N-qubit standard W state and W type states are all uniquely determined by their $(N-1)$ of the bipartite RDMs. That is to say, the determinability of the N-qubit standard W state by its RDMs is preserved under SLOCC, which is different from the GHZ state.

What is not yet clear is the impact of SLOCC on determinability of high-dimensional quantum states. So it is worth to study the determinability of those states that are SLOCC equivalent to N-qudit W states, i.e., the N-qudit W type states. It has been proved in [22] that the N-qudit generalized W states are uniquely determined by their bipartite RDMs. Therefore, in view of the above, we can conjecture that the N-qudit W type states will preserve the determinability too.

In this paper, we will explore the determinability of all N-qudit SLOCC-equivalent W states. We find that the determinability of an N-qudit W states from its RDMs is preserved under SLOCC. That is, the multipartite pure quantum states which are SLOCC equivalent to N-qudit W states can be uniquely determined by their bipartite RDMs. We also prove that only $(N-1)$ of the bipartite RDMs are sufficient.

The paper is organized as follows. We begin by briefly describing a canonical form of N-qudit W type states. In Sec. 3, we show the bipartite RDMs of d-dimensional density matrix and N-qudit W type states. In Sec. 4 we prove that, among arbitrary states, all N-qudit W type states are uniquely determined by their $(N-1)$ of the bipartite RDMs. Finally, the paper ends with the conclusion in Sec. 5.

2 Canonical form of *N*-qudit *W* type states

It is known to all that the N-qudit standard W state is denoted as

$$\left|W_N^d\right\rangle = \frac{1}{\sqrt{N(d-1)}} \sum_{i=1}^{d-1} (|i0\cdots 00\rangle + \cdots + |00\cdots 0i\rangle). \tag{1}$$

In [22], Parashar et al. have considered the following class of states as "N-qudit generalized W state"

$$\left|GW_N^d\right\rangle = \sum_{i=1}^{d-1} (a_{1i}\,|i0\cdots 00\rangle + \cdots + a_{Ni}\,|00\cdots 0i\rangle). \tag{2}$$

They showed that these states are uniquely determined by their bipartite RDMs. That is, there does not exist any other N-qudit density matrix having the same bipartite RDMs with the N-qudit generalized W state.

It is easy to check that N-qudit standard W state and generalized W state are SLOCC equivalent because there exists the invertible local operator $\otimes_{k=1}^{N} A_k$ that transforms the state $\left|GW_N^d\right\rangle$ into standard W state $\left|W_N^d\right\rangle$, where

$$A_k = |0\rangle\langle 0| + \sum_{i=1}^{d-1} \frac{1}{a_{ki}\sqrt{N(d-1)}} |i\rangle\langle i|.$$

In this paper, we will consider all possible N-qudit W type states, which is all N-qudit pure states that are SLOCC equivalent to $\left|W_N^d\right\rangle$ state. So we will introduce a convenient canonical form for all such states as [19] do. It is known two N-qudit pure quantum states $|\psi\rangle$, $\left|W_N^d\right\rangle$ are called equivalent under SLOCC if there are N invertible matrices $\{A_k|\det(A_k)\neq 0\}$ such that

$$|\psi\rangle = \bigotimes_{k=1}^{N} A_k \left|W_N^d\right\rangle. \tag{3}$$

Any invertible matrix A_k can be denoted as follows

$$A_k = \begin{bmatrix} \alpha_{00}^k & \alpha_{01}^k & \cdots & \alpha_{0(d-1)}^k \\ \alpha_{10}^k & \alpha_{11}^k & \cdots & \alpha_{1(d-1)}^k \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{(d-1)0}^k & \alpha_{(d-1)1}^k & \cdots & \alpha_{(d-1)(d-1)}^k \end{bmatrix}.$$

For brevity, we will rewrite A_k as follows

$$A_k = \left[\left|\alpha_0^k\right\rangle \left|\alpha_1^k\right\rangle \cdots \left|\alpha_{d-1}^k\right\rangle\right].$$

where $\left|\alpha_i^k\right\rangle$ $\{i = 0, 1, \ldots, d-1\}$ is the $(i+1)$th column of matrix A_k , i.e., $\left|\alpha_i^k\right\rangle = (\alpha_{0i}^k\ \alpha_{1i}^k\ \cdots\ \alpha_{(d-1)i}^k)^T$. Then we can get that A_k transforms $|i\rangle_k$ to $\left|\alpha_i^k\right\rangle$, that is $A_k\,|i\rangle = \left|\alpha_i^k\right\rangle$. From Eq. (3) , we can get

$$|\psi\rangle = \frac{1}{\sqrt{N(d-1)}} \sum_{i=1}^{d-1} (\left|\alpha_i^1\alpha_0^2\cdots\alpha_0^N\right\rangle + \left|\alpha_0^1\alpha_i^2\cdots\alpha_0^N\right\rangle + \cdots + \left|\alpha_0^1\alpha_0^2\cdots\alpha_i^N\right\rangle). \tag{4}$$

Since A_k is an invertible matrix, this implies $\left|\alpha_0^k\right\rangle, \left|\alpha_1^k\right\rangle, \ldots, \left|\alpha_{d-1}^k\right\rangle$ are independent. Hence, there is a useful method, the Gram–Schmidt procedure, which can be used to produce an orthogonal basis set $\left\{\left|p_0^k\right\rangle, \left|p_1^k\right\rangle, \ldots, \left|p_{d-1}^k\right\rangle\right\}$ for the d-dimensional local Hilbert space from the independent set $\left\{\left|\alpha_0^k\right\rangle, \left|\alpha_1^k\right\rangle, \ldots, \left|\alpha_{d-1}^k\right\rangle\right\}$. Thus, setting $\left|p_0^k\right\rangle$ parallel to $\left|\alpha_0^k\right\rangle$ and $\left|p_i^k\right\rangle$ orthogonal to $\left|p_o^k\right\rangle, \left|p_1^k\right\rangle, \ldots, \left|p_{i-1}^k\right\rangle$ by

$$\begin{aligned} \left|p_0^k\right\rangle &= \mu_{00}^k \left|\alpha_0^k\right\rangle \\ \left|p_1^k\right\rangle &= \mu_{01}^k \left|\alpha_0^k\right\rangle + \mu_{11}^k \left|\alpha_1^k\right\rangle \\ &\vdots \\ \left|p_i^k\right\rangle &= \sum_{j=0}^{i} \mu_{ji}^k \left|\alpha_j^k\right\rangle \\ &\vdots \end{aligned}$$

where $i = 0, 1, \ldots, d-1$. Therefore, the equations above can be compressed into a matrix expression as follows

$$\left[\left|p_0^k\right\rangle \left|p_1^k\right\rangle \cdots \left|p_{d-1}^k\right\rangle\right] = \left[\left|\alpha_0^k\right\rangle \left|\alpha_1^k\right\rangle \cdots \left|\alpha_{d-1}^k\right\rangle\right] U_k. \tag{5}$$

where

$$U_k = \begin{bmatrix} \mu_{00}^k & \mu_{01}^k & \cdots & \mu_{0(d-1)}^k \\ 0 & \mu_{11}^k & \cdots & \mu_{1(d-1)}^k \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu_{(d-1)(d-1)}^k \end{bmatrix}.$$

is an invertible upper triangular matrix. It is known that the inverse of an upper triangular matrix is triangular of the same kind, i.e., U_k^{-1} is also an upper triangular matrix. Thus we can get the following equation from Eq. (5) that

$$\left[\left|\alpha_0^k\right\rangle \left|\alpha_1^k\right\rangle \cdots \left|\alpha_{d-1}^k\right\rangle\right] = \left[\left|p_0^k\right\rangle \left|p_1^k\right\rangle \cdots \left|p_{d-1}^k\right\rangle\right] U_k^{-1}. \tag{6}$$

Without loss of generality, we suppose

$$U_k^{-1} = \begin{bmatrix} \nu_{00}^k & \nu_{01}^k & \cdots & \nu_{0(d-1)}^k \\ 0 & \nu_{11}^k & \cdots & \nu_{1(d-1)}^k \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \nu_{(d-1)(d-1)}^k \end{bmatrix}. \tag{7}$$

So we can get $\left|\alpha_i^k\right\rangle = \sum_{j=0}^{i} \nu_{ji}^k \left|p_j^k\right\rangle$ from Eqs. (6) and (7). Then take this equation to Eq. (4), we can get

$$|\psi\rangle = z_0 \left|p_0^1 p_0^2 \cdots p_0^d\right\rangle + \sum_{i=1}^{d-1} \left(z_{1i} \left|p_i^1 p_0^2 \cdots p_0^d\right\rangle + z_{2i} \left|p_0^1 p_i^2 \cdots p_0^d\right\rangle + \cdots + z_{Ni} \left|p_0^1 p_0^2 \cdots p_i^d\right\rangle\right) \tag{8}$$

where for any $k \in \{1, 2, \ldots, N\}$, $\left\{\left|p_0^k\right\rangle, \left|p_1^k\right\rangle, \ldots, \left|p_{d-1}^k\right\rangle\right\}$ is an othonormal basis set. Clearly the phases in the complex coefficient of the above equation can be absorbed by the bases, so Eq. (8) is written as

$$\left|SW_N^d\right\rangle = c_{00} \left|00\cdots 00\right\rangle + \sum_{i=1}^{d-1} (c_{1i} \left|i0\cdots 00\right\rangle + \cdots + c_{Ni} \left|00\cdots 0i\right\rangle), \tag{9}$$

$c_{ij} \geq 0$ and $c_{00}^2 + \sum_{i=1}^{d-1} \sum_{j=1}^{N} c_{ji}^2 = 1$ for normalization.

Thus, any SLOCC equivalent $\left|W_N^d\right\rangle$ state can be expressed as Eq. (9). In this paper, we will enlarge the $\left|SW_N^d\right\rangle$ states by not restricting the coefficients to be real, rather we will assume them as arbitrary nonzero complex numbers, where the sum of modulus square is 1.

3 Bipartite RDM of *d*-dimensional density matrix

The basis of our analysis lies in the concept of the graph, which has been introduced in [21]. Also a one-to-one correspondence between bipartite RDMs and graph has been established in [21], which is for a given N-qudit state ρ_M (pure or mixed) and arbitrary graph $G = (V, E)$ on N vertices, we define a set S which is composed by the bipartite RDMs of ρ_M as

$$S = \{\rho_M^{st} | (s, t) \in E; s, t = 1, 2, \dots, N\}.$$

So we have constructed a one-to-one correspondence between bipartite RDMs of the N-qudit quantum state and the graph, and the number of bipartite RDMs in S is equal to the number of edges with graph $G = (V, E)$.

Next, we will calculate the bipartite RDM ρ_M^{st} of an N-qudit mixed states ρ_M, which can be expressed as

$$\rho_M = \sum_{i_1,\dots,j_N=0}^{d-1} r_{(i_1 i_2 \cdots i_N)(j_1 j_2 \cdots j_N)} \left|i_1 i_2 \cdots i_N\right\rangle \left\langle j_1 j_2 \cdots j_N\right|. \tag{10}$$

So we can write

$$\rho_M^{st} = \sum_{i_s, i_t, j_s, j_t=0}^{d-1} b_{(i_s i_t)(j_s j_t)} \left|i_s i_t\right\rangle \left\langle j_s j_t\right|.$$

The bipartite RDM of an N-qudit quantum state is a matrix of order $d^2 \times d^2$. Since the RDM is obtained by tracing over $(N-2)$ parties, each $b_{(i_s i_t)(j_s j_t)}$ will be a sum of d^{N-2} number of terms with the form $r_{(i_1 \cdots i_{s-1} i_s i_{s+1} \cdots i_{t-1} i_t i_{t+1} \cdots i_N)(i_1 \cdots i_{s-1} j_s i_{s+1} \cdots i_{t-1} j_t i_{t+1} \cdots i_N)}$. We also call $b_{(i_s i_t)(j_s j_t)}$ as the coefficient of $|i_s i_t\rangle\langle j_s j_t|$ which is corresponds to the element at position $(di_s + i_t + 1, dj_s + j_t + 1)$ of the $d^2 \times d^2$ matrix. Then according to the above analysis, we can get the coefficient of $|i_s i_t\rangle\langle j_s j_t|$ is

$$b_{(i_s i_t)(j_s j_t)} = \sum_{i_{V\setminus\{s,t\}}=0}^{d-1} r_{(i_1 \cdots i_{s-1} i_s i_{s+1} \cdots i_{t-1} i_t i_{t+1} \cdots i_N)(i_1 \cdots i_{s-1} j_s i_{s+1} \cdots i_{t-1} j_t i_{t+1} \cdots i_N)}. \tag{11}$$

where $V = \{1, 2, \dots, N\}$, and $i_{V\setminus\{s,t\}}$ stands for $\{i_1, i_2, \dots, i_N\}$ except $\{i_s, i_t\}$.

A one-to-one correspondence can be observed between the sum terms in the expression of any two elements of matrix ρ_M^{st}. That is for any two elements of ρ_M^{st}, such as $b_{(i_s i_t)(i_s i_t)}$ and $b_{(i'_s i'_t)(j'_s j'_t)}$, any of the sum term in the expression of $b_{(i_s i_t)(i_s i_t)}$ can be written as $r_{(i_1 \cdots i_{s-1} i_s i_{s+1} \cdots i_{t-1} i_t i_{t+1} \cdots i_N)(i_1 \cdots i_{s-1} j_s i_{s+1} \cdots i_{t-1} j_t i_{t+1} \cdots i_N)}$. We will define

$$r_{(i_1 \cdots i_{s-1} i'_s i_{s+1} \cdots i_{t-1} i'_t i_{t+1} \cdots i_N)\left(i_1 \cdots i_{s-1} j'_s i_{s+1} \cdots i_{t-1} j'_t i_{t+1} \cdots i_N\right)}$$

in the expression of$b_{(i'_s i'_t)(j'_s j'_t)}$ as the unique corresponding term of $r_{(i_1 \cdots i_{s-1} i_s i_{s+1} \cdots i_{t-1} i_t i_{t+1} \cdots i_N)(i_1 \cdots i_{s-1} j_s i_{s+1} \cdots i_{t-1} j_t i_{t+1} \cdots i_N)}$.

Next, we will calculate the bipartite RDM ρ_{SW}^{st} of $\left|SW_N^d\right\rangle$. And we also show only the upper-half entries, as density matrix is hermitian and the upper-half r_{ij}, $\forall i \leq j$ is sufficient to describe a hermitian matrix $\rho = (r_{ij})$. In order to get the bipartite RDM ρ_{SW}^{st} of $\left|SW_N^d\right\rangle$, we will calculate the coefficient of $|i_s i_t\rangle\langle j_s j_t|$, where $i_s, i_t, j_s, j_t \in \{0, 1, \ldots, d-1\}$.

Firstly, we will calculate the diagonal elements of the reduced density matrix ρ_{SW}^{st}. From Eq. (11) , we can get

(i) the coefficient of $|00\rangle\langle 00|$ is $1 - \sum_{j=1}^{d-1} \left(|c_{sj}|^2 + |c_{tj}|^2\right)$.
(ii) the coefficient of $|0j\rangle\langle 0j|$ is $|c_{tj}|^2$ and the coefficient of $|i0\rangle\langle i0|$ is $|c_{si}|^2$.
(iii) the remaining diagonal elements of ρ_{SW}^{st} are all equal to zero, that is the coefficient of $|ij\rangle\langle ij|$ is 0 if $1 \leq i \leq d-1$, $1 \leq j \leq d-1$, i.e., $i, j \neq 0$.

Next we can get the non diagonal elements of the matrix ρ_{SW}^{st} as follows

(iv) the coefficient of $|00\rangle\langle 0j|$ is $c_{00}\bar{c}_{tj}$ and the coefficient of $|00\rangle\langle i0|$ is $c_{00}\bar{c}_{si}$.
(v) the coefficient of $|0i\rangle\langle 0j|$ is $c_{ti}\bar{c}_{tj}$ and the coefficient of $|i0\rangle\langle j0|$ is $c_{si}\bar{c}_{sj}$, where $1 \leq i < j \leq d-1$.
(vi) the coefficient of $|0i\rangle\langle j0|$ is $c_{ti}\bar{c}_{sj}$, where $1 \leq i \leq d-1$, $1 \leq j \leq d-1$.
(vii) the remaining non diagonal elements are all equal to zero if $1 \leq i \leq d-1$, $1 \leq j \leq d-1$, that is the elements that at the $(di + j + 1)$ column and the elements at the $(di + j + 1)$ row of the matrix ρ_{SW}^{st} are all equal to zero if $i, j \neq 0$.

4 Determinability of *N*-qudit SLOCC *W* states from bipartite RDMs

It is known Parashar et al. have considered the determinability in N-qudit generalized W states $\left|GW_N^d\right\rangle$. They proved that $\left|GW_N^d\right\rangle$ states as shown in Eq. (2) are uniquely determined by their bipartite RDMs [22]. In this paper, we will consider the determinability in the class of sates Eq. (9), and the quantum state space in which $|SW_N^d\rangle$ states are located is much larger than the state space in which $\left|GW_N^d\right\rangle$ states are located. We will show that the class of states $\left|SW_N^d\right\rangle$ is uniquely determined, among arbitrary states, by its bipartite RDMs too, furthermore, only $(N-1)$ of the bipartite RDMs are sufficient.

Theorem Among arbitrary states, SLOCC equivalent $\left|W_N^d\right\rangle$ states are uniquely determined by their bipartite RDMs set $S = \{\rho_{SW}^{st} | (s, t) \in E; s, t = 1, 2, \ldots, N\}$, where $G = (V, E)$ is any tree graph on N vertices.

Before giving specific proof, we will first list some properties with the positive semidefinite matrix, which have presented in [19] . If a Hermitian matrix $\rho = (r_{ij})$ is positive semidefinite matrix, then

(i) $r_{ii} \geq 0, \forall i$.
(ii) if some $r_{kk} = 0$, then $r_{ik} = r_{kj} = 0, \forall i, j$.

(iii) $r_{ii}r_{jj} \geq \left|r_{ij}\right|^2, \forall i, j$.
(iv) all principle minors of ρ are nonnegative.

Proof (1). Without loss of generality, suppose another N-qudit matrix, possibly mixed, is ρ_M, which has been presented in Eq. (10). What we are going to prove is that, for any given tree graph $G = (V, E)$, if $\rho_M^{st} = \rho_{SW}^{st}$ with all of the $(s, t) \in E$, then $\rho_M = |SW_N^d\rangle\langle SW_N^d|$.

It can obviously be obtained from the property of density matrix that $\bar{r}_{(i_1i_2\cdots i_N)(j_1j_2\cdots j_N)} = r_{(j_1j_2\cdots j_N)(i_1i_2\cdots i_N)}$ and $\sum_{i_1,\ldots,i_N=0}^{d-1} r_{(i_1i_2\cdots i_N)(i_1i_2\cdots i_N)} = 1$.

(2). We have shown in (iii) of Sec. 3 that the coefficient of $|i_si_t\rangle\langle i_si_t|$ in ρ_{SW}^{st} is equal to 0, if $i_s, i_t \neq 0$. For $(s, t) \in E$, by comparing the coefficient of the same $|i_si_t\rangle\langle i_si_t|$ from ρ_M^{st} and ρ_{SW}^{st}, we can get

$$
\begin{aligned}
b_{(i_si_t)(i_si_t)} &= \sum_{i_{V\setminus\{s,t\}}=0}^{d-1} r_{(i_1\cdots i_s\cdots i_t\cdots i_N)(i_1\cdots i_s\cdots i_t\cdots i_N)} \\
&= 0,
\end{aligned}
$$

if $i_s, i_t \neq 0$. As all of the diagonal elements with hermitian positive semidefinite matrix are greater than or equal to 0, we have

$$
r_{(i_1\cdots i_s\cdots i_t\cdots i_N)(i_1\cdots i_s\cdots i_t\cdots i_N)} = 0,
$$

if $i_s, i_t \neq 0$ and s, t are adjacent vertices in G.

(3). For any $(s, t) \in E$ and $i_t, j_s \neq 0$, if one of the sum term in the expression of $b_{(0_si_t)(0_si_t)}$ (i.e., the coefficient of $|0_si_t\rangle\langle 0_si_t|$) with ρ_{SW}^{st}) is equal to zero, then we can get the corresponding sum term in the expression of $b_{(j_s0_t)(j_s0_t)}$ is equal to zero too, and vice versa. That is, for any $(s, t) \in E$ and $i_t, j_s \neq 0$, if

$$
r_{(i'_1\cdots i'_{s-1}0_si'_{s+1}\cdots i'_{t-1}i_ti'_{t+1}\cdots i'_N)(i'_1\cdots i'_{s-1}0_si'_{s+1}\cdots i'_{t-1}i_ti'_{t+1}\cdots i'_N)} = 0,
$$

we can get

$$
r_{(i'_1\cdots i'_{s-1}j_si'_{s+1}\cdots i'_{t-1}0_ti'_{t+1}\cdots i'_N)(i'_1\cdots i'_{s-1}j_si'_{s+1}\cdots i'_{t-1}0_ti'_{t+1}\cdots i'_N)} = 0.
$$

We relegate the proof to the appendix.

As a result, we can get the number of nonzero sum terms in the expression of $b_{(j_s0_t)(j_s0_t)}$ is equal to the number of nonzero sum terms in the expression of $b_{(0_si_t)(0_si_t)}$. In addition, by property (ii) of positive semidefinite matrix, we can get the number of nonzero sum terms in the expression of $b_{(0_si_t)(j_s0_t)}$ is also equal to the number of nonzero sum terms in the expression of $b_{(0_si_t)(0_si_t)}$, where $1 \leq i_t \leq d-1$, $1 \leq j_s \leq d-1$ and s, t are adjacent vertices.

(4). In this step, we will show that $r_{(i_1i_2\cdots i_N)(i_1i_2\cdots i_N)} = 0$ if two or more i_ks are not equal to zero. This is equivalent to proving $b_{(i_ui_v)(i_ui_v)} = 0$ (the coefficient of $|i_ui_v\rangle\langle i_ui_v|$) with ρ_M^{st}) for any $i_u, i_v \neq 0$.

Since any two vertices in G can be connected by a unique simple path if G is a tree graph, we can suppose the path between u and v is $V = \{u, k_1, k_2, \ldots, k_m, v\}$, $E = \{uk_1, k_1k_2, \ldots, k_m v\}$.

Therefore, from the analysis in step (2) and step (3), we can get the following equivalent equations

$$
\begin{aligned}
& b_{(i_u i_v)(i_u i_v)} = 0 \\
\Leftrightarrow & \sum_{i_{V\setminus\{u,v\}}=0}^{d-1} r_{(i_1\cdots i_u\cdots i_v\cdots i_N)(i_1\cdots i_u\cdots i_v\cdots i_N)} = 0 \\
\Leftrightarrow & \sum r_{\left(i_1\cdots i_u\cdots 0_{k_1}\cdots i_v\cdots i_N\right)\left(i_1\cdots i_u\cdots 0_{k_1}\cdots i_v\cdots i_N\right)} = 0 \\
\Leftrightarrow & \sum r_{\left(i_1\cdots 0_u\cdots 1_{k_1}\cdots i_v\cdots i_N\right)\left(i_1\cdots 0_u\cdots 1_{k_1}\cdots i_v\cdots i_N\right)} = 0 \\
\Leftrightarrow & \sum r_{\left(i_1\cdots 0_u\cdots 1_{k_1}\cdots 0_{k_2}\cdots i_v\cdots i_N\right)\left(i_1\cdots 0_u\cdots 1_{k_1}\cdots 0_{k_2}\cdots i_v\cdots i_N\right)} = 0 \\
\Leftrightarrow & \sum r_{\left(i_1\cdots 0_u\cdots 0_{k_1}\cdots 1_{k_2}\cdots i_v\cdots i_N\right)\left(i_1\cdots 0_u\cdots 0_{k_1}\cdots 1_{k_2}\cdots i_v\cdots i_N\right)} = 0 \\
& \vdots \\
\Leftrightarrow & \sum r_{\left(i_1\cdots 0_u\cdots 0_{k_1}\cdots 0_{k_2}\cdots 1_{k_m}\cdots i_v\cdots i_N\right)\left(i_1\cdots 0_u\cdots 0_{k_1}\cdots 0_{k_2}\cdots 1_{k_m}\cdots i_v\cdots i_N\right)} = 0.
\end{aligned}
$$

Next we only need to prove

$$
r_{\left(i_1\cdots 0_u\cdots 0_{k_1}\cdots 0_{k_2}\cdots 1_{k_m}\cdots i_v\cdots i_N\right)\left(i_1\cdots 0_u\cdots 0_{k_1}\cdots 0_{k_2}\cdots 1_{k_m}\cdots i_v\cdots i_N\right)}=0.
$$

It is obvious that this equation hold from the analysis in step(2), as k_m and v are adjacent vertices and $i_v \neq 0$. So we get $b_{(i_u i_v)(i_u i_v)} = 0$ hold. Therefore we get $r_{(i_1\cdots i_u\cdots i_v\cdots i_N)(i_1\cdots i_u\cdots i_v\cdots i_N)} = 0$ hold for any $i_u, i_v \neq 0$. Hence, by property (ii) of positive semidefinite matrix, we have $r_{(i_1 i_2\cdots i_N)(j_1 j_2\cdots j_N)} = 0$ if two or more i_ks are not equal to zero, or two or more j_ks are not equal to zero.

In addition, as a corollary, we can draw

$$
r_{(0\cdots 0 i_t 0\cdots 0)(0\cdots 0 i_t 0\cdots 0)}=|c_{ti}|^2,
$$

for any $1 \leq i_t \leq d-1$ and $t = 1, 2, \ldots, N$.

Furthermore, by normalization, we have

$$
r_{(0\cdots 0)(0\cdots 0)} = |c_{00}|^2.
$$

(5). For any $(s, t) \in E$, comparing the coefficients of $|0_s i_t\rangle\langle 0_s j_t|$ and $|i_s 0_t\rangle\langle j_s 0_t|$ from ρ_M^{st} and ρ_{SW}^{st}, we have

$$
r_{(0\cdots 0 i_t 0\cdots 0)(0\cdots 0 j_t 0\cdots 0)} = c_{ti}\bar{c}_{tj},
$$

where $t \in \{1, 2, \ldots, N\}$ and $i_t, j_t \in \{1, 2, \ldots, d-1\}$. Next, for any $(s, t) \in E$, comparing the coefficient of $|0_s i_t\rangle\langle j_s 0_t|$ from ρ_M^{st} and ρ_{SW}^{st}, we have

$$r_{(0\cdots0i_t0\cdots0)(0\cdots0j_s0\cdots0)} = c_{ti}\bar{c}_{sj},$$

where $i_t, j_s \in \{1, 2, \ldots, d-1\}$.

(6). Comparing the coefficient of $|00\rangle\langle 0i|$ and coefficient of $|00\rangle\langle j0|$ of ρ_M^{st} and ρ_{SW}^{st} for any $(s, t) \in E$, we have

$$r_{(00\cdots0)(0\cdots0i_t0\cdots0)} = c_{00}\bar{c}_{ti},$$

for $t = 1, 2, \ldots, N$ and $i_t = 1, 2, \ldots, d-1$.

(7). Collecting all the results it follows that the only remaining task is to prove

$$r_{(0\cdots0i_K0\cdots0)(0\cdots0j_L0\cdots0)} = c_{Ki}\bar{c}_{Lj}, \tag{12}$$

and

$$r_{(0\cdots0i_K0\cdots0)(0\cdots0j_K0\cdots0)} = c_{Ki}\bar{c}_{Kj}, \tag{13}$$

where $K, L \in \{1, 2, \ldots, N\}$, and $i_K, j_L, j_K \neq 0$.

In order to prove Eq. (12), we consider the following principal minor which consisting the rows $(00\cdots0)$, $(0\cdots0i_K0\cdots0)$ and $(0\cdots0j_L0\cdots0)$ in ρ_M. The analysis is similar to proof in [19], suppose $r_{(0\cdots0i_K0\cdots0)(0\cdots0j_L0\cdots0)} = r$, then the principal minor is

$$\begin{vmatrix} |c_{00}|^2 & c_{00}\bar{c}_{Ki} & c_{00}\bar{c}_{Lj} \\ \bar{c}_{00}c_{Ki} & |c_{Ki}|^2 & r \\ \bar{c}_{00}c_{Lj} & \bar{r} & |c_{Lj}|^2 \end{vmatrix}.$$

The value of this determinant is $-|c_{00}|^2|c_{Ki}|^2|c_{Lj}|^2\left|1 - \frac{r}{c_{Ki}\bar{c}_{Lj}}\right|^2$, which should be nonnegative according to (iv) property of positive semidefinite matrix, so $r_{(0\cdots0i_K0\cdots0)(0\cdots0j_L0\cdots0)} = c_{Ki}\bar{c}_{Lj}$. Using the same method, we can get Eq. (13) hold.

This end the proof. □

5 Conclusion

It has been shown that bipartite RDMs can uniquely determine the N-qudit generalized W state, which is shown as Eq. (2) [22]. Among arbitrary states(pure or mixed), in order to uniquely determine an N-qudit generalized W state, all of the $\frac{N(N-1)}{2}$ bipartite RDMs are needed. Moreover, when only pure states are considered, then $(N-1)$ of them having one party common to all are sufficient. In this paper, we considered determinability with the N-qudit SLOCC equivalent W state, which is shown as Eq. (9). We proved that, among arbitrary states(pure or mixed), this class of states are uniquely determined by their bipartite RDMs , and only $(N-1)$ number of them with tree form are sufficient. We hope our paper will help to handle the high dimensional quantum state tomography.

Acknowledgements We appreciate the anonymous reviewers for their valuable suggestions. This work is supported by National Natural Science Foundation of China (Grant Nos. 61701553, 61601171, 61772134, 61802023) and the Open Foundation of State key Laboratory of Networking and Switching Technology (Beijing University of Posts and Telecommunications) (SKLNST-2016-2-10, SKLNST-2018-1-03).

Appendix

In this section, we prove for any $(s,t)\in E$ and $i_t, j_s \neq 0$, if

$$r_{(i'_1\cdots i'_{s-1}0_s i'_{s+1}\cdots i'_{t-1}i_t i'_{t+1}\cdots i'_N)(i'_1\cdots i'_{s-1}0_s i'_{s+1}\cdots i'_{t-1}i_t i'_{t+1}\cdots i'_N)}=0,$$

then

$$r_{(i'_1\cdots i'_{s-1}j_s i'_{s+1}\cdots i'_{t-1}0_t i'_{t+1}\cdots i'_N)(i'_1\cdots i'_{s-1}j_s i'_{s+1}\cdots i'_{t-1}0_t i'_{t+1}\cdots i'_N)}=0.$$

Proof For any $(s,t)\in E$, comparing the coefficients of $|0_s i_t\rangle\langle 0_s i_t|$, $|j_s 0_t\rangle\langle j_s 0_t|$ and $|0_s i_t\rangle\langle j_s 0_t|$ from ρ_M^{st} and ρ_{SW}^{st}, since $i_t, j_s\neq 0$, we have

$$\sum_{i_{V\backslash\{s,t\}}=0}^{d-1} r_{(i_1\cdots 0_s\cdots i_t\cdots i_N)(i_1\cdots 0_s\cdots i_t\cdots i_N)}=|c_{ti}|^2, \tag{14}$$

$$\sum_{i_{V\backslash\{s,t\}}=0}^{d-1} r_{(i_1\cdots j_s\cdots 0_t\cdots i_N)(i_1\cdots j_s\cdots 0_t\cdots i_N)}=\left|c_{sj}\right|^2, \tag{15}$$

and

$$\sum_{i_{V\backslash\{s,t\}}=0}^{d-1} r_{(i_1\cdots 0_s\cdots i_t\cdots i_N)(i_1\cdots j_s\cdots 0_t\cdots i_N)}=c_{ti}\bar{c}_{sj}. \tag{16}$$

As $r_{(i'_1\cdots 0_s\cdots i_t\cdots i'_N)(i'_1\cdots 0_s\cdots i_t\cdots i'_N)} = 0$, by property (ii) of positive semidefinite matrices, we can get $r_{(i'_1\cdots 0_s\cdots i_t\cdots i'_N)(i'_1\cdots j_s\cdots 0_t\cdots i'_N)} = 0$ too. Then we can rewrite Eqs. (14), (15) and (16) as follows

$$\sum_{i_{V\backslash\{s,t\}}\in I} r_{(i_1\cdots 0_s\cdots i_t\cdots i_N)(i_1\cdots 0_s\cdots i_t\cdots i_N)}=|c_{ti}|^2, \tag{17}$$

$$\sum_{i_{V\backslash\{s,t\}}\in I} r_{(i_1\cdots j_s\cdots 0_t\cdots i_N)(i_1\cdots j_s\cdots 0_t\cdots i_N)}+r_{(i'_1\cdots j_s\cdots 0_t\cdots i'_N)(i'_1\cdots j_s\cdots 0_t\cdots i'_N)}=\left|c_{sj}\right|^2, \tag{18}$$

and

$$\sum_{i_{V\backslash\{s,t\}}\in I} r_{(i_1\cdots 0_s\cdots i_t\cdots i_N)(i_1\cdots j_s\cdots 0_t\cdots i_N)}=c_{ti}\bar{c}_{sj}, \tag{19}$$

where the sum $\sum\limits_{i_{V\backslash\{s,t\}}\in I}$ varies over $0,1,\ldots,d-1$ at the $1,\ldots,s-1,s+1,\ldots,t-1,t+1,\ldots,N$ parties, excepts for $i'_1\cdots i'_{s-1}i'_{s+1}\cdots i'_{t-1}i'_{t+1}\cdots i'_N$. Therefore, from the above three equations, we can get

$$
\begin{aligned}
&\left|\sum_{i_{V\setminus\{s,t\}}\in I} r_{(i_1\cdots 0_s\cdots i_t\cdots i_N)(i_1\cdots j_s\cdots 0_t\cdots i_N)}\right|^2\\
=&\Bigg(\sum_{i_{V\setminus\{s,t\}}\in I} r_{(i_1\cdots 0_s\cdots i_t\cdots i_N)(i_1\cdots 0_s\cdots i_t\cdots i_N)}\Bigg)\Bigg(\sum_{i_{V\setminus\{s,t\}}\in I} r_{(i_1\cdots j_s\cdots 0_t\cdots i_N)(i_1\cdots j_s\cdots 0_t\cdots i_N)}ht.\\
&+r_{(i'_1\cdots j_s\cdots 0_t\cdots i'_N)(i'_1\cdots j_s\cdots 0_t\cdots i'_N)}\Bigg)\\
=&\Bigg(\sum_{i_{V\setminus\{s,t\}}\in I} r_{(i_1\cdots 0_s\cdots i_t\cdots i_N)(i_1\cdots 0_s\cdots i_t\cdots i_N)}\Bigg)\Bigg(\sum_{i_{V\setminus\{s,t\}}\in I} r_{(i_1\cdots j_s\cdots 0_t\cdots i_N)(i_1\cdots j_s\cdots 0_t\cdots i_N)}\Bigg)\\
&+\Bigg(\sum_{i_{V\setminus\{s,t\}}\in I} r_{(i_1\cdots 0_s\cdots i_t\cdots i_N)(i_1\cdots 0_s\cdots i_t\cdots i_N)}\Bigg) r_{(i'_1\cdots j_s\cdots 0_t\cdots i'_N)(i'_1\cdots j_s\cdots 0_t\cdots i'_N)}.
\end{aligned}
\tag{20}
$$

By the property (iii) of positive semidefinite matrices, it follows that

$$
\begin{aligned}
&\left|r_{(i_1\cdots 0_s\cdots i_t\cdots i_N)(i_1\cdots j_s\cdots 0_t\cdots i_N)}\right|\\
\le&\sqrt{r_{(i_1\cdots 0_s\cdots i_t\cdots i_N)(i_1\cdots 0_s\cdots i_t\cdots i_N)}r_{(i_1\cdots j_s\cdots 0_t\cdots i_N)(i_1\cdots j_s\cdots 0_t\cdots i_N)}}
\end{aligned}
$$

So we can get

$$
\begin{aligned}
&\left|\sum_{i_{V\setminus\{s,t\}}\in I} r_{(i_1\cdots 0_s\cdots i_t\cdots i_N)(i_1\cdots j_s\cdots 0_t\cdots i_N)}\right|\\
\le&\sum_{i_{V\setminus\{s,t\}}\in I}\left|r_{(i_1\cdots 0_s\cdots i_t\cdots i_N)(i_1\cdots j_s\cdots 0_t\cdots i_N)}\right|\\
\le&\sum_{i_{V\setminus\{s,t\}}\in I}\sqrt{r_{(i_1\cdots 0_s\cdots i_t\cdots i_N)(i_1\cdots 0_s\cdots i_t\cdots i_N)}r_{(i_1\cdots j_s\cdots 0_t\cdots i_N)(i_1\cdots j_s\cdots 0_t\cdots i_N)}}\\
\le&\sqrt{\Bigg(\sum_{i_{V\setminus\{s,t\}}\in I} r_{(i_1\cdots 0_s\cdots i_t\cdots i_N)(i_1\cdots 0_s\cdots i_t\cdots i_N)})(\sum_{i_{V\setminus\{s,t\}}\in I} r_{(i_1\cdots j_s\cdots 0_t\cdots i_N)(i_1\cdots j_s\cdots 0_t\cdots i_N)}\Bigg)}.
\end{aligned}
\tag{21}
$$

After both sides of this Formula (21) is squared and subtracted, we obtain

$$\begin{aligned}&\left|\sum_{i_{V\setminus\{s,t\}}\in I} r_{(i_1\cdots 0_s\cdots i_t\cdots i_N)(i_1\cdots j_s\cdots 0_t\cdots i_N)}\right|^2\\&\quad-\left(\sum_{i_{V\setminus\{s,t\}}\in I} r_{(i_1\cdots 0_s\cdots i_t\cdots i_N)(i_1\cdots 0_s\cdots i_t\cdots i_N)}\right)\left(\sum_{i_{V\setminus\{s,t\}}\in I} r_{(i_1\cdots j_s\cdots 0_t\cdots i_N)(i_1\cdots j_s\cdots 0_t\cdots i_N)}\right)\\&\leq 0.\end{aligned}\tag{22}$$

But from Eq. (20), we obtain that the difference between the two expressions in left side of Formula (22) is $(\sum_{i_{V\setminus\{s,t\}}\in I} r_{(i_1\cdots 0_s\cdots i_t\cdots i_N)(i_1\cdots 0_s\cdots i_t\cdots i_N)})r_{(i'_1\cdots j_s\cdots 0_t\cdots i'_N)(i'_1\cdots j_s\cdots 0_t\cdots i'_N)}$, which is greater than or equal to 0, so we have

$$\begin{aligned}&\left(\sum_{i_{V\setminus\{s,t\}}\in I} r_{(i_1\cdots 0_s\cdots i_t\cdots i_N)(i_1\cdots 0_s\cdots i_t\cdots i_N)}\right) r_{(i'_1\cdots j_s\cdots 0_t\cdots i'_N)(i'_1\cdots j_s\cdots 0_t\cdots i'_N)}\\&=|c_{ti}|^2 r_{(i'_1\cdots j_s\cdots 0_t\cdots i'_N)(i'_1\cdots j_s\cdots 0_t\cdots i'_N)}\\&=0.\end{aligned}$$

As $|c_{ti}|^2\neq 0$, we can get $r_{(i'_1\cdots j_s\cdots 0_t\cdots i'_N)(i'_1\cdots j_s\cdots 0_t\cdots i'_N)}=0$.

This end the proof. □

References

1. Cramer, M., Plenio, M.B., Flammia, S.T., Somma, R., Gross, D., Bartlett, S.D., Landon-Cardinal, O., Poulin, D., Liu, Y.K.: Efficient quantum state tomography. Nat. Commun. **1**, 149 (2010)
2. Christandl, M., Renner, R.: Reliable quantum state tomography. Phys. Rev. Lett. **109**, 120403 (2012)
3. Ohliger, M., Nesme, V., Eisert, J.: Efficient and feasible state tomography of quantum many-body systems. New J. Phys. **15**, 015024 (2013)
4. Schwemmer, C., Knips, L., Richart, D., Weinfurter, H., Moroder, T., Kleinmann, M., Guhne, O.: Systematic errors in current quantum state tomography tools. Phys. Rev. Lett. **114**, 080403 (2015)
5. Lu, D.W., Xin, T., Yu, N.K., Ji, Z.F., Chen, J.X., Long, G.L., Baugh, J., Peng, X.H., Zeng, B., Laflamme, R.: Tomography is necessary for universal entanglement detection with single-copy observables. Phys. Rev. Lett. **116**, 230501 (2016)
6. Acharya, A., Kypraios, T., Guta, M.: A comparative study of estimation methods in quantum tomography. J. Phys. A: Math. Theor. **52**, 234001 (2019)
7. Gao, F., Qin, S.J., Huang, W., Wen, Q.Y.: Quantum private query: A new kind of practical quantum cryptographic protocol. Sci. China-Phys. Mech. Astron. **62**, 070301 (2019)
8. Liu, B., Gao, F., Huang, W.: QKD-based quantum private query without a failure probability. Sci. China-Phys. Mech. Astron. **58**, 100301 (2015)
9. Linden, N., Popescu, S., Wootters, W.K.: Almost every pure state of three qubits is completely determined by its two-particle reduced density matrices. Phys. Rev. Lett. **89**, 207901 (2002)
10. Linden, N., Wootters, W.K.: The parts determine the whole in a generic pure quantum state. Phys. Rev. Lett. **89**, 277906 (2002)

11. Diosi, L.: Three-party pure quantum states are determined by two two-party reduced states. Phys. Rev. A **70**, 010302(R) (2004)
12. Jones, N.S., Linden, N.: Parts of quantum states. Phys. Rev. A **71**, 012324 (2005)
13. Walck, S.N., Lyons, D.W.: Only n-qubit Greenberger-Horne-Zeilinger states are undetermined by their reduced density matrices. Phys. Rev. Lett. **100**, 050501 (2008)
14. Chen, J.X., Dawkins, H., Ji, Z.F., Johnston, N., Kribs, D., Shultz, F., Zeng, B.: Uniqueness of quantum states compatible with given measurement results. Phys. Rev. A **88**, 012109 (2013)
15. Xin, T., Lu, D.W., Klassen, J., Yu, N.K., Ji, Z.F., Chen, J.X., Ma, X., Long, G.L., Zeng, B., Laflamme, R.: Quantum state tomography via reduced density matrices. Phys. Rev. Lett. **118**, 020401 (2017)
16. Wyderka, N., Huber, F., Guhne, O.: Almost all four-particle pure states are determined by their two-body marginals. Phys. Rev. A **96**, 010102(R) (2017)
17. Huang, S.L., Chen, J.X., Li, Y.N., Zeng, B.: Quantum state tomography for generic pure states. Sci. China-Phys. Mech. Astron. **61**, 110311 (2018)
18. Parashar, P., Rana, S.: N-qubit W states are determined by their bipartite marginals. Phys. Rev. A **80**, 012319 (2009)
19. Rana, S., Parashar, P.: Optimal reducibility of all W states equivalent under stochastic local operations and classical communication. Phys. Rev. A **84**, 052331 (2011)
20. Yu, N.K.: Multipartite W-type state is determined by its single-particle reduced density matrices among all W-type states. Phys. Rev. A **87**, 052310 (2013)
21. Wu, X., Tian, G.J., Huang, W., Wen, Q.Y., Qin, S.J., Gao, F.: Determination of W states equivalent under stochastic local operations and classical communication by their bipartite reduced density matrices with tree form. Phys. Rev. A **90**, 012317 (2014)
22. Parashar, P., Rana, S.: Reducible correlations in Dicke states. J. Phys. A: Math. Theor. **42**, 462003 (2009)
23. Wu, X., Yang, Y.H., Wen, Q.Y., Qin, S.J., Gao, F.: Determination of Dicke states equivalent under stochastic local operations and classical communication. Phys. Rev. A **92**, 052338 (2015)
24. Zhou, D.L.: Irreducible multiparty correlations in quantum states without maximal rank. Phys. Rev. Lett. **101**, 180505 (2008)
25. Wu, X., Yang, Y.H., Wang, Y.K., Wen, Q.Y., Qin, S.J., Gao, F.: Determination of stabilizer states. Phys. Rev. A **92**, 012305 (2015)
26. Martinez, D., Solis-Prosser, M.A., Canas, G., Jimenez, O., Delgado, A., Lima, G.: Experimental quantum tomography assisted by multiply symmetric states in higher dimensions. Phys. Rev. A **99**, 012336 (2019)
27. Canas, G., Etcheverry, S., Gomez, E.S., Saavedra, C., Xavier, G.B., Lima, G., Cabello, A.: Experimental implementation of an eight-dimensional Kochen-Specker set and observation of its connection with the Greenberger-Horne-Zeilinger theorem. Phys. Rev. A **90**, 012119 (2014)
28. Collins, D., Gisin, N., Linden, N., Massar, S., Popescu, S.: Bell inequalities for arbitrarily high-dimensional systems. Phys. Rev. Lett. **88**, 040404 (2002)
29. Tonchev, H.S., Vitanov, N.V.: Quantum phase estimation and quantum counting with qudits. Phys. Rev. A **94**, 042307 (2016)
30. Nikolopoulos, G.M., Ranade, K.S., Alber, G.: Error tolerance of two-basis quantum-key-distribution protocols using qudits and two-way classical communication. Phys. Rev. A **73**, 032325 (2006)
31. Dur, W., Vidal, G., Cirac, J.I.: Three qubits can be entangled in two inequivalent ways. Phys. Rev. A **62**, 062314 (2000)
32. Horodecki, R., Horodecki, P., Horodecki, M., Horodecki, K.: Quantum entanglement. Rev. Mod. Phys. **81**, 865 (2009)