PARTIALLY ORDERED SETS WITH TRANSITIVE AUTOMORPHISM GROUPS

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ABSTRACT

In this paper, we study the structure of infinite partially ordered sets $(\Omega, \leqslant)$ under suitable transitivity assumptions on their group $A(\Omega)$ of all order-automorphisms of $(\Omega, \leqslant)$. Let us call $A(\Omega)$ k-transitive (k-homogeneous) if whenever A, B are two isomorphic subsets of Ω each with k elements, then some (any) isomorphism from $(A, \leqslant)$ onto $(B, \leqslant)$ extends to an automorphism of Ω, respectively. We show that if $k \geqslant 4$ $(k = 3)$, there are precisely k (5) non-isomorphic countable partially ordered sets $(\Omega, \leqslant)$ not containing the pentagon such that $A(\Omega)$ is k-transitive but not k-homogeneous; if $k = 2$, there are a unique countable, and many different uncountable sets $(\Omega, \leqslant)$ of this type. We also give necessary and sufficient conditions for two partially ordered sets $(\Omega, \leqslant)$ not containing the pentagon and with k-transitive automorphism group $(k \geqslant 2)$ to be $L_{\infty\omega}$-equivalent.

1. *Introduction and results*

In this paper let $(\Omega, \leqslant)$ always be an infinite partially ordered set (p.o. set) and $A(\Omega) = \mathrm{Aut}((\Omega, \leqslant))$ be the group of all order-automorphisms of Ω. Let $k \in \mathbb{N}$. We call $A(\Omega)$ *k-transitive* (*k-homogeneous*), if whenever $A, B \subseteq \Omega$ each have k elements and $\varphi\colon A \to B$ is an isomorphism, then there exists $\alpha \in A(\Omega)$ with $A^{\alpha} = B$ $(\alpha|_A = \varphi)$, respectively. Further $A(\Omega)$ is *ω-transitive* (*ω-homogeneous*), if $A(\Omega)$ is k-transitive (k-homogeneous) for each $k \in \mathbb{N}$, respectively. Trivially, k-homogeneity implies k-transitivity of $A(\Omega)$ for each $k \in \mathbb{N}$, and ω-homogeneity implies ω-transitivity.

Linearly ordered sets (chains) $(\Omega, \leqslant)$ with 2-transitive automorphism groups have been extensively studied. Their automorphism groups $A(\Omega)$ have been used for example for the construction of certain infinite simple torsion-free groups (Higman [**23**]), or, in the theory of lattice-ordered groups, in dealing with embeddings of arbitrary lattice-ordered groups into simple divisible ones (Holland [**26**]). The interplay between the structure of these chains and the structure of the normal subgroup lattices of their automorphism groups was studied in [**1**, **2**, **11**, **13**, **15**, **26**]. Obviously, all linearly ordered fields are examples of such chains. For a variety of further results in this area see Glass [**20**]. Henson [**22**] showed that there are $2^{\aleph_0}$ non-isomorphic countable binary relational structures with ω-homogeneous automorphism group; precisely countably many of these are graphs (Lachlan and Woodrow [**29**]), see also [**17–19**, **28**, **34**]. Schmerl [**35**] characterized all countable p.o. sets with ω-homogeneous automorphism group. Further related work is listed in the references.

In this paper we examine the structure of infinite p.o. sets $(\Omega, \leqslant)$ of arbitrary cardinality under the assumption that $A(\Omega)$ is k-transitive for some $2 \leqslant k \in \mathbb{N}$. A study of this kind was proposed by Wielandt [**39**] and begun in [**11**, **12**], where we obtained a classification of these partial orders and in almost all cases a characterization of the condition that $A(\Omega)$ is k-transitive for some $k \geqslant 2$ by the

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structure of the p.o. set $(\Omega, \leqslant)$. A well-known model-theoretic technique in this context is to describe which p.o. sets can be embedded into $(\Omega, \leqslant)$. If $\kappa \geqslant 2$ is any finite or infinite cardinal number, let F_κ denote the p.o. set which is the smallest lattice containing an antichain A, that is, a set A consisting only of pairwise incomparable elements, with κ elements; thus $|F_\kappa \setminus A| = 2$. If $a, b \in \Omega$ satisfy $a < b$, we let $(a, b) = \{x \in \Omega\colon\ a < x < b\}$, an interval in Ω. First we show:

THEOREM 1. *Let $(\Omega, \leqslant)$ be an infinite p.o. set such that $A(\Omega)$ is k-transitive for some $k \geqslant 3$ and $(\Omega, \leqslant)$ contains copies of F_n for arbitrarily large $n \in \mathbb{N}$. Then $(\Omega, \leqslant)$ also contains a copy of $F_{\aleph_0}$. Moreover, if Ω also contains a chain with at least four elements, then $F_{\aleph_0}$ can in fact be embedded into any interval (a, b) of Ω.*

In **[12]**, we showed that for each $k \geqslant 2$, k-transitivity of $A(\Omega)$ is a strictly weaker assumption than k-homogeneity, but for any infinite p.o. set $(\Omega, \leqslant)$, ω-transitivity of $A(\Omega)$ is equivalent to ω-homogeneity. We will classify all partial orders $(\Omega, \leqslant)$ such that $A(\Omega)$ is k-transitive but not k-homogeneous for some $k \geqslant 2$; see Theorem 2.7. As a consequence we will obtain

THEOREM 2. (a) *Up to isomorphism, there exist for each $k \geqslant 4$ precisely k countable p.o. sets $(\Omega, \leqslant)$ such that $A(\Omega)$ is k-transitive but not k-homogeneous and such that $F_{\aleph_0}$ cannot be embedded into $(\Omega, \leqslant)$. Only one of these sets does not contain a copy of F_2.*

(b) *Up to isomorphism, there are precisely five countable p.o. sets $(\Omega, \leqslant)$ such that $A(\Omega)$ is 3-transitive but not 3-homogeneous and such that $F_{\aleph_0}$ cannot be embedded into $(\Omega, \leqslant)$. None of these p.o. sets contains a copy of F_3, and three of them do not contain F_2.*

It remains to examine the case that $A(\Omega)$ is 2-transitive but not 2-homogeneous. In this case $(\Omega, \leqslant)$ has either a trivial structure or a highly complicated one. If $x \in \Omega$ and $A \subseteq \Omega$, we write $x \parallel A$ if x is incomparable with each element of A. As a consequence of results of **[12]** we have

THEOREM (+) **[12]**. *Let $(\Omega, \leqslant)$ be an infinite p.o. set such that $A(\Omega)$ is 2-transitive. The following are equivalent:*

(1) *$A(\Omega)$ is not 2-homogeneous, and $(\Omega, \leqslant)$ does not contain a copy of each F_n $(2 \leqslant n \in \mathbb{N})$;*

(2) *$(\Omega, \leqslant)$ satisfies one of the following two conditions:*

  (i) *$\Omega = C \mathbin{\dot\cup} \{a\}$ where $a \parallel C$ and C is an infinite chain in Ω with 2-transitive automorphism group $A(C)$;*

  (ii) *either $(\Omega, \leqslant)$ or Ω with the inverse ordering is an uncountable tree with ramification order 2 such that $\operatorname{ram}(\Omega) = \Omega^+ \setminus \Omega$ and each point $a \in \operatorname{ram}(\Omega)$ has precisely two non-isomorphic cones of which at most one has countable coinitiality.*

Here a tree is a p.o. set $(\Omega, \leqslant)$ in which, in particular, for any two elements $a, b \in \Omega$ there is $c \in \Omega$ with $c < a$ and $c < b$, and for each $a \in \Omega$ the set $\{x \in \Omega\colon\ x < a\}$ is linearly ordered. For the convenience of the reader, the precise definitions of the notions of Condition (2ii) of Theorem (+), which were developed in [**12**], are summarized in § 3. Trees are natural examples of p.o. sets $(\Omega, \leqslant)$ for which $A(\Omega)$ can be k-transitive (m-homogeneous) only if $k \leqslant 3$ ($m \leqslant 2$), respectively [**12**, Theorem 5.8]. They are also relevant to the logical study of time (cf. van Benthem [**37**, **38**]), as we will show in § 4. Now we show:

THEOREM 3. *Let λ be an uncountable cardinal. If λ is regular, there are precisely 2^λ pairwise non-isomorphic trees $(\Omega, \leqslant)$ of cardinality λ such that $A(\Omega)$ is 2-transitive but not 2-homogeneous. If λ is singular, there are at least $2^{<\lambda} = \sum_{\rho<\lambda} 2^\rho$ such trees $(\Omega, \leqslant)$.*

This result answers an open question in [**12**, p. 97] and shows that there are indeed many p.o. sets $(\Omega, \leqslant)$ of the highly non-trivial structure described in Condition (2ii) of Theorem (+) for which $A(\Omega)$ is 2-transitive but not 2-homogeneous. The trees of Theorem 3 also provide first examples of p.o. sets $(\Omega, \leqslant)$ for which $A(\Omega)$ is 2-transitive but not 2-homogeneous and not 3-transitive.

Related to the question of whether $(\Omega, \leqslant)$ contains each lattice F_n is, in our context, the problem of whether $(\Omega, \leqslant)$ contains the *pentagon* $(P, \leqslant)$; here $P = \{a, b, c, d, e\}$ with $a < b < c < e$, $a < d < e$, and $d \parallel \{b, c\}$. Now let $L = \{\leqslant\}$ be the first order language of predicate calculus for p.o. sets in the following. Recall that the infinitary language $L_{\infty\omega}$ contains all atomic formulas of L and is closed under negation, quantification over finitely many variables, and conjunction and disjunction of arbitrary sets of formulas which are supposed to have altogether only a finite number of free variables. Two models $\mathfrak{A}$ and $\mathfrak{B}$ of L are called elementarily equivalent ($L_{\infty\omega}$-equivalent) if they satisfy the same sentences of L ($L_{\infty\omega}$), respectively. Obviously, $L_{\infty\omega}$-equivalence implies elementary equivalence. The theory of $\mathfrak{A}$, denoted by $\mathrm{Th}(\mathfrak{A})$, is the set of all sentences of L true in $\mathfrak{A}$; it is called $\aleph_0$-categorical if, up to isomorphism, there is a unique countably-infinite model of L with the same theory as $\mathfrak{A}$. We will show:

THEOREM 4. *Let $(\Omega, \leqslant)$ be an infinite p.o. set not containing the pentagon such that $A(\Omega)$ is k-transitive for some $k \geqslant 2$.*

(a) *Let $(\Omega', \leqslant)$ be another p.o. set. If $(\Omega, \leqslant)$ and $(\Omega', \leqslant)$ are elementarily equivalent, then they are also $L_{\infty\omega}$-equivalent.*
(b) $\mathrm{Th}(\Omega, \leqslant)$ *is $\aleph_0$-categorical.*

Here (b) is immediate from (a) since any two countable models which are $L_{\infty\omega}$-equivalent are, in fact, isomorphic. We will associate with each p.o. set $(\Omega, \leqslant)$ not containing the pentagon such that $A(\Omega)$ is k-transitive for some $k \geqslant 2$ a triple of cardinal invariants belonging to $\mathbb{N} \cup \{0, \aleph_0\}$. We let

$$\mathrm{Trans}(\Omega) = \{k \in \mathbb{N}\colon\ A(\Omega) \text{ is } k\text{-transitive}\}$$

and

$$\mathrm{Hom}(\Omega) = \{m \in \mathbb{N}\colon\ A(\Omega) \text{ is } m\text{-homogeneous}\}.$$

As a consequence of our results we will obtain

COROLLARY 5. *Let $(\Omega_i, \leqslant)$ be infinite p.o. sets not containing the pentagon such that $A(\Omega_i)$ is k_i-transitive for some $k_i \geqslant 2$ $(i = 1, 2)$. Then the following are* *equivalent:*

(1) $(\Omega_1, \leqslant)$ *and* $(\Omega_2, \leqslant)$ *have the same triple of cardinal invariants;*

(2) $(\Omega_1, \leqslant)$ *and* $(\Omega_2, \leqslant)$ *are elementarily equivalent.*

If one of these conditions (1), (2) *holds, then* $\mathrm{Trans}(\Omega_1) = \mathrm{Trans}(\Omega_2)$ *and* $\mathrm{Hom}(\Omega_1) = \mathrm{Hom}(\Omega_2)$, *except when* Ω_1, Ω_2 *or* Ω_1, Ω_2 *with the inverse ordering are trees such that* $A(\Omega_1)$ *is 2-transitive but not 2-homogeneous and* $A(\Omega_2)$ *is 3-transitive, or vice-versa. In this exceptional case,* $(\Omega_1, \leqslant)$ *and* $(\Omega_2, \leqslant)$ *are* $L_{\infty\omega}$*-equivalent, but* $\mathrm{Trans}(\Omega_1) \neq \mathrm{Trans}(\Omega_2)$ *and* $\mathrm{Hom}(\Omega_1) \neq \mathrm{Hom}(\Omega_2)$.

The exceptional case in Corollary 5 really occurs, as will be shown as a consequence of Theorem 3. As an immediate consequence of Theorem 4 and Corollary 5 we obtain a new proof for

COROLLARY 6 [**12**]. *Up to isomorphism, there are precisely countably many countable p.o. sets* $(\Omega, \leqslant)$ *not containing the pentagon such that* $A(\Omega)$ *is* k*-transitive for some* $k \geqslant 2$.

It is easy to see that if two relational structures $\mathfrak{A}$ and $\mathfrak{B}$ have, up to isomorphism, the same finite substructures and $\mathfrak{A}$ and $\mathfrak{B}$ each have an ω-homogeneous automorphism group, then they are $L_{\infty\omega}$-equivalent (since they are partially isomorphic). For p.o. sets, we generalize this remark:

COROLLARY 7. *Let* $(\Omega_1, \leqslant)$, $(\Omega_2, \leqslant)$ *be two infinite p.o. sets such that* $A(\Omega_1)$ *and* $A(\Omega_2)$ *are each* k*-transitive for infinitely many* $k \in \mathbb{N}$, *and* $(\Omega_1, \leqslant)$ *and* $(\Omega_2, \leqslant)$ *embed the same finite p.o. sets. Then* $(\Omega_1, \leqslant)$ *and* $(\Omega_2, \leqslant)$ *are* $L_{\infty\omega}$*-equivalent.*

The proofs of Theorems 1 and 2, 3, 4 are contained in §§ 2, 3, 4, respectively. Further consequences are also contained in § 4.

Finally, we remark that in the literature the definitions of the concepts of transitivity and homogeneity are slightly inconsistent, unfortunately. We have chosen our present usage to coincide with the one of [**12**], since this is our main reference.

2. *Embedding the lattice* $F_{\aleph_0}$

This section is devoted to the proof of Theorems 1 and 2. For the convenience of the reader, we also summarize the basic notation and those results of [**12**] which we are going to use throughout this paper. For details, we refer the reader to [**12**].

Let $(\Omega, \leqslant)$ be a p.o. set and $A \subseteq \Omega$. We say that A is *dense* in Ω if whenever $a, b \in \Omega$ with $a < b$, there is some $x \in A$ with $a < x < b$. We call A *bounded above* (*below*) in Ω if there exists $x \in \Omega$ with $a \leqslant x$ ($x \leqslant a$) for all $a \in A$; A is *unbounded above* (*below*) in Ω if it is not bounded above (below) in Ω, respectively; A is *unbounded* in Ω if it is unbounded both above and below in Ω. The set $(A, \leqslant)$ is called dense (unbounded) if $(A, \leqslant)$ is dense (unbounded) in itself. In particular, a chain $(A, \leqslant)$ is unbounded if and only if A contains neither a greatest nor a smallest element.

For elements $a, b \in \Omega$ we write $a \parallel b$ if a and b are incomparable, i.e. if neither $a \leqslant b$ nor $b \leqslant a$ holds. For subsets $A, B \subseteq \Omega$ let $A \parallel B$ ($A < B$) denote that $a \parallel b$ ($a < b$) for all $a \in A$, $b \in B$, respectively. In particular, $A \parallel B$ and $A < B$ each imply that A and B are disjoint. We also write $a \parallel B$ ($a < B$) for $\{a\} \parallel B$ ($\{a\} < B$).

Next let C be an infinite chain. As is well-known, the following three conditions are equivalent:

(1) $A(C)$ is ω-transitive;

(2) $A(C)$ is k-transitive for some $k \geqslant 2$;

(3) C is unbounded and any two closed intervals $[a, b]_C$ and $[c, d]_C$ ($a, b, c, d \in C$, $a < b$, $c < d$) of C are isomorphic.

If one of these conditions holds, C is also called *doubly homogeneous*, and such chains are considered basic for our study. Now let $r \geqslant 1$ be an arbitrary cardinal. Identify r with some fixed set of cardinality r. We let $C \vec{\times} r$ be the set $C \times r$ with a partial ordering defined by $(c_1, r_1) < (c_2, r_2)$ if and only if $c_1 < c_2$ ($c_1, c_2 \in C$, $r_1, r_2 \in r$). Clearly, $C \vec{\times} r$ is a chain if and only if $r = 1$. For any $r \geqslant 1$ and $2 \leqslant k \in \mathbb{N}$, the following are equivalent:

(1) $A(C \vec{\times} r)$ is ω-homogeneous;

(2) $A(C \vec{\times} r)$ is k-transitive;

(3) C is doubly homogeneous.

We have the following classification of all partial orders $(\Omega, \leqslant)$ with k-transitive automorphism group $A(\Omega)$:

THEOREM 2.1 [**12**, Theorems 4.14–4.16, 5.8, 7.2(b)]. *Let $(\Omega, \leqslant)$ be an infinite p.o. set and $2 \leqslant k \in \mathbb{N}$. If $A(\Omega)$ is k-transitive, then $(\Omega, \leqslant)$ satisfies one of the following conditions.*

(0) *The order on Ω is trivial.*

(1) *$\Omega = A \,\dot\cup\, B \,\dot\cup\, C$ with antichains A, B, C in Ω such that $A < B < C$ and $1 \leqslant |A| + |C| \leqslant k - 1$.*

(2) *$\Omega = \dot\bigcup_{i \in I} C_i$ where $1 \leqslant |I| \leqslant |\Omega|$ is an arbitrary cardinal such that the following conditions are satisfied:*

(i) *for each $i \in I$, C_i is a doubly homogeneous chain;*

(ii) *whenever $i, j \in I$ with $i \neq j$, then $C_i \cong C_j$ and $C_i \parallel C_j$.*

(3) *There exists a doubly homogeneous chain C such that precisely one of the following three conditions holds:*

(a) *$\Omega \cong C \vec{\times} r$ for some finite or infinite cardinal $r \geqslant 2$;*

(b) *$\Omega = \Omega_1 \,\dot\cup\, \Omega_2$ such that $\Omega_1 \parallel \Omega_2$ and $\Omega_1 \cong \Omega_2 \cong C \vec{\times} r$ for some $r \in \mathbb{N}$ with $r \geqslant 2$ and $2r - 1 \leqslant k$;*

(c) *$\Omega = \Omega' \,\dot\cup\, A$ such that A is a non-empty antichain in Ω, $A \parallel \Omega'$, and $\Omega' \cong C \vec{\times} r$ for some $r \in \mathbb{N}$ with $r + |A| \leqslant k$.*

(4) *Ω contains a set $\{x, y, z\} \subseteq \Omega$ with $x < y$ and $z \parallel \{x, y\}$. Moreover, precisely one of the following conditions is satisfied:*

(a) *any finite subset of Ω is bounded below in Ω, and for each $a \in \Omega$ the set $\{x \in \Omega\colon\ x < a\}$ is non-empty and a chain;*

(b) *any finite subset of Ω is bounded above in Ω, and for each $a \in \Omega$ the set $\{x \in \Omega\colon\ a < x\}$ is non-empty and a chain;*

(c) *any finite subset of* Ω *is bounded below and above in* Ω.
In the case of (a) *or* (b) *we have* $k \in \{2, 3\}$.

Conversely, if $(\Omega, \leqslant)$ *satisfies one of the conditions* (0)–(3), *then* $A(\Omega)$ *is* k*-transitive.*

In the following we will say that $(\Omega, \leqslant)$ or Ω is of Type (2), (3c) etc., if $(\Omega, \leqslant)$ satisfies Condition (2), (3c), respectively, of Theorem 2.1. Sets of Type (4a) with k-transitive automorphism group are trees; see § 3. First we examine sets of Type (4c). Their basic properties are summarized in

THEOREM 2.2 [**12**, Theorems 7.2, 7.3]. *Let* $(\Omega, \leqslant)$ *be of Type* (4c) *and* $A(\Omega)$ *be* k*-transitive for some* $k \geqslant 2$.

(a) *No element of* Ω *is maximal or minimal in* Ω.

(b) *For each finite subset* $A \subseteq \Omega$ *there is an infinite antichain* $B \subseteq \Omega$ *with* $A \parallel B$.

(c) $A(\Omega)$ *is* m*-transitive (and* m*-homogeneous if* $A(\Omega)$ *is also* k*-homogeneous) for each* $m \leqslant k$.

(d) *If* $A(\Omega)$ *is* 3*-transitive, then* $A(\Omega)$ *is also* 2*-homogeneous.*

(e) *Let* $a, b \in \Omega$ *with* $a < b$, *and* $n \in \mathbb{N}$. *There exists a finite antichain* $A \subseteq \Omega$ *such that* $a < A < b$ *and* $|A| = n$.

Proof. Only (e) remains to be proved. By (b), there is an antichain $B \subseteq \Omega$ with $|B| = n$. Choose $c, d \in \Omega$ with $c < B < d$. Now by (c) we have $\{c, d\}^\alpha = \{a, b\}$ for some $\alpha \in A(\Omega)$. Then $A = B^\alpha$ is an antichain with $|A| = n$ and $a < A < b$.

As a consequence of Theorems 2.1 and 2.2 we have

COROLLARY 2.3. *Let* $(\Omega, \leqslant)$ *be an infinite p.o. set such that* $A(\Omega)$ *is* k*-transitive for some* $k \geqslant 2$. *The following are equivalent:*

(1) $(\Omega, \leqslant)$ *is of Type* (4c);

(2) $(\Omega, \leqslant)$ *contains a subset* $\{x, y, z\} \subseteq \Omega$ *with* $x < y$ *and* $z \parallel \{x, y\}$, *and also a copy of each* F_n $(2 \leqslant n \in \mathbb{N})$;

(3) $(\Omega, \leqslant)$ *contains a pentagon.*

Proof. (1)$\Rightarrow$(2). This is immediate by Theorem 2.2(e).

(1)$\Rightarrow$(3). By assumption, Ω contains a set $\{b, c, d\}$ with $b < c$ and $d \parallel \{b, c\}$, and we can choose $a, e \in \Omega$ such that $a < \{b, c, d\} < e$. Then $P = \{a, b, c, d, e\}$ is a pentagon in Ω.

(2)$\Rightarrow$(1) and (3)$\Rightarrow$(1). This is straightforward by Theorem 2.1.

Next we note

PROPOSITION 2.4 [**12**, Lemmas 7.4, 7.5]. *Let* $(\Omega, \leqslant)$ *be of Type* (4c).

(a) *Let* $A(\Omega)$ *be* 2*-homogeneous and* $a, b \in \Omega$ *with* $a \parallel b$. *Then there is* $c \in \Omega$ *with* $a < c$ *and* $c \parallel b$.

(b) *Let* $A(\Omega)$ *be* 3*-transitive. Whenever* $a \in \Omega$ *and* $A \subseteq \Omega$ *is finite with* $a < A$, *there exists* $b \in \Omega$ *with* $a < b < A$. *In particular,* $(\Omega, \leqslant)$ *is not a lattice.*

After these preparations, we now turn to the proof of Theorem 1.

LEMMA 2.5. *Let* $(\Omega, \leqslant)$ *be of Type* (4c) *and* $A(\Omega)$ *be* 3-*transitive. Let* $a, b \in \Omega$ *with* $a < b$, *and* $A = \{a_1, \ldots, a_n\} \subseteq \Omega$ $(2 \leqslant n \in \mathbb{N})$ *be an antichain with* $a < A < b$. *Then there exists an infinite antichain* $B \subseteq \Omega$ *such that* $a < B < b$ *and* $\{a_1, \ldots, a_{n-1}\} \subseteq B$.

Proof. We first show that there are two elements $b_1, b_2 \in \Omega$ such that $a < \{b_1, b_2\} < b$ and $\{a_1, \ldots, a_{n-1}, b_1, b_2\}$ is an antichain. By Theorem 2.2(d) and Proposition 2.4(a), for each $i \in \{1, \ldots, n-1\}$ there exists an element $d_i \in \Omega$ such that $a_n < d_i$ and $d_i \parallel a_i$. By Proposition 2.4(b), there is $c \in \Omega$ with

$$a_n < c < \{b, d_1, \ldots, d_{n-1}\}.$$

Then $c \parallel \{a_1, \ldots, a_{n-1}\}$. By Theorem 2.2(e), there are $b_1, b_2 \in \Omega$ with

$$a < a_n < \{b_1, b_2\} < c < b$$

and $b_1 \parallel b_2$. It follows that $A' = \{a_1, \ldots, a_{n-1}, b_1, b_2\}$ is an antichain with $a < A' < b$.

Now induction immediately implies the result.

Our main new result for sets $(\Omega, \leqslant)$ of Type (4c) is the subsequent

THEOREM 2.6. *Let* $(\Omega, \leqslant)$ *be of Type* (4c) *and* $A(\Omega)$ *be* 3-*transitive.*

(a) *Whenever* $a, b \in \Omega$ *with* $a < b$, *there exists an infinite antichain* $B \subseteq \Omega$ *with* $a < B < b$.

(b) *Let* $A \subseteq \Omega$ *be a finite antichain. Then* A *is contained in an infinite antichain* $B \subseteq \Omega$ *which is bounded both above and below in* Ω. *Moreover, if* $A(\Omega)$ *is* 4-*transitive and* $a, b \in \Omega$ *satisfy* $a < A < b$, *then* B *can be chosen such that* $a < B < b$ *also.*

Proof. (a) By Theorem 2.2(e) there are $a_1, a_2 \in \Omega$ with $a_1 \parallel a_2$ and $a < \{a_1, a_2\} < b$. Now apply Lemma 2.5.

(b) Let $A \subseteq \Omega$ be a finite antichain. First choose by Theorem 2.2(b) an element $x \in \Omega$ such that $x \parallel A$ and then $y, z \in \Omega$ with $y < A \mathbin{\dot\cup} \{x\} < z$. Now by Lemma 2.5 there is an infinite antichain $B \subseteq \Omega$ such that $y < B < z$ and $A \subseteq B$.

Now assume $A(\Omega)$ is also 4-transitive and $a, b \in \Omega$ satisfy $a < A < b$. By Theorem 2.2(c) and Proposition 2.4(b) there are $c, d \in \Omega$ with $a < c < A < d < b$. By Corollary 2.3 choose a pentagon P in Ω, that is, a set $P = \{a', b', c', d', e'\} \subseteq \Omega$ such that $a' < c' < d' < b'$, $a' < e' < b'$, and $e' \parallel \{c', d'\}$. Then $\{a, c, d, b\} = \{a', c', d', b'\}^{\alpha}$ for some $\alpha \in A(\Omega)$. Let $e = e'^{\alpha}$. We obtain $e \parallel \{c, d\}$ and $a < e < b$. So $e \parallel A$, and $A' = A \mathbin{\dot\cup} \{e\}$ is an antichain in Ω with $a < A' < b$. Now the result follows by induction.

Now we are ready for the

Proof of Theorem 1. It follows from the assumptions and Theorem 2.1 that either Ω is of Type (1) and each maximal chain in Ω has precisely three elements, or Ω is of Type (3a) or (4c). In the first case, the result is clear. If Ω is of Type (3a), we can assume that $\Omega = C \vec{\times} r$ for some doubly homogeneous chain C and some infinite cardinal r. Now let $a, b \in \Omega$ with $a < b$. Since C is dense in itself, so is $(\Omega, \leqslant)$, and there are $c, d, e \in \Omega$ with $a < c < d < e < b$. Assume $d = (x, q)$ with $x \in C$, $q \in r$. Then $A = \{x\} \times r$ is an antichain in Ω with $|A| = r$ and $c < A < e$, and hence $A \cup \{c, e\}$ is a copy of the lattice F_r in (a, b). Finally, if Ω is of Type (4c), the result follows from Theorems 2.2(c) and 2.6(a).

Next we examine p.o. sets $(\Omega, \leqslant)$ for which $A(\Omega)$ is k-transitive but not k-homogeneous, for some $k \geqslant 2$. A description of their structure is contained in

THEOREM 2.7. *Let $(\Omega, \leqslant)$ be an infinite p.o. set such that $A(\Omega)$ is k-transitive for some $k \geqslant 2$. If $A(\Omega)$ is not k-homogeneous, then $(\Omega, \leqslant)$ satisfies one of the following five mutually exclusive conditions:*

(1) *there exist a doubly homogeneous chain C and an integer $r \geqslant 2$ such that $\Omega = \Omega_1 \mathbin{\dot\cup} \Omega_2$ with $\Omega_1 \parallel \Omega_2$, $\Omega_1 \cong \Omega_2 \cong C \vec{\times} r$, and $k \in \{2r-1, 2r\}$; in particular, $k \geqslant 3$;*

(2) *there exist a doubly homogeneous chain C, a non-empty antichain $A \subseteq \Omega$, and an integer $r \in \mathbb{N}$ such that $\Omega = \Omega' \mathbin{\dot\cup} A$ with $\Omega' \parallel A$, $\Omega' \cong C \vec{\times} r$, and $r + |A| = k$;*

(3) *Ω is of Type* (4a) *or* (4b), *and $k = 3$;*

(4) *Ω is of Type* (4a) *or* (4b), *and $k = 2$;*

(5) *Ω is of Type* (4c).

Conversely, if Ω satisfies one of the Conditions (1)–(3), *then $A(\Omega)$ is not k-homogeneous.*

Proof. The result follows by a case-by-case-examination from Theorem 2.1 and [**12**, Theorems 3.2.7, 4.15, 4.16, 5.8].

A characterization of all p.o. sets $(\Omega, \leqslant)$ of Type (4a) or (4b) for which $A(\Omega)$ is 2-transitive but not 2-homogeneous is contained in § 3. Next we apply Theorems 2.6, 2.7, and a result of [**12**] on p.o. sets of Type (4a, b) to prove Theorem 2. Notice that each countable doubly homogeneous chain C is isomorphic to $(\mathbb{Q}, \leqslant)$, since it is dense in itself and unbounded.

Proof of Theorem 2. (a) By Theorems 2.2(c), 2.6(a), and (2.7) it follows that each of the countable p.o. sets $(\Omega, \leqslant)$ in question satisfies Condition (1) or (2) of Theorem 2.7, where the chain C can be replaced by $\mathbb{Q}$. In the case of Condition (1), the integer r is uniquely determined by k. Hence there are precisely $1 + (k-1) = k$ such countable p.o. sets $(\Omega, \leqslant)$. In each of these cases, F_2 can be embedded into $(\Omega, \leqslant)$ unless $(\Omega, \leqslant)$ satisfies Condition (2) of Theorem 2.7 with $r = 1$.

(b) Again by Theorems 2.2(c), 2.6(a), and 2.7 the countable p.o. sets in question satisfy one of the Conditions (1)–(3) of Theorem 2.7; hence none of these sets contains a copy of F_3. As shown above, there are precisely three countable sets satisfying Condition (2.7)(1) or (2). By [**12**, Theorem 6.22] (cf. the subsequent Corollary 4.7(b)) there is a unique countable p.o. set $(\Omega, \leqslant)$ of Type (4a) and another one (the anti-isomorphic copy) of Type (4b) for which $A(\Omega)$ is 3-transitive and hence not 3-homogeneous by Theorem 2.7. This implies the result.

Finally we note that the statements of Theorem 2 remain true if we replace the lattice $F_{\aleph_0}$ by the pentagon $(P, \leqslant)$; this is immediate by

COROLLARY 2.8. *Let $(\Omega, \leqslant)$ be an infinite p.o. set and $A(\Omega)$ be k-transitive but* *not k-homogeneous for some $k \geqslant 3$. The following are equivalent:*

(1) $(\Omega, \leqslant)$ *contains a copy of* $F_{\aleph_0}$;

(2) $(\Omega, \leqslant)$ *contains a copy of the pentagon* P;

(3) $(\Omega, \leqslant)$ *is of Type* (4c).

Proof. (1)$\Leftrightarrow$(3). This is immediate by Theorems 2.6 and 2.7.
(2)$\Leftrightarrow$(3). See Corollary 2.3.

3. *Trees*

In this section we examine p.o. sets $(\Omega, \leqslant)$ of Type (4a) or (4b) of Theorem 2.1, and we will prove Theorem 3. Because of symmetry, we can restrict ourselves to sets $(\Omega, \leqslant)$ of Type (4a). Let us start with

DEFINITION 3.1 [12]. Let $(\Omega, \leqslant)$ be a p.o. set with the following properties:
(i) any finite subset of Ω has a lower bound in Ω;
(ii) for each $a \in \Omega$, the set $\{x \in \Omega\colon x < a\}$ is a chain;
(iii) there are $a, b \in \Omega$ with $a \parallel b$;
(iv) whenever $a, b \in \Omega$ satisfy $a \parallel b$, there exists $c \in \Omega$ with $c < a$ and $c \parallel b$;
(v) Ω has no maximal or minimal elements.
Then $(\Omega, \leqslant)$ is called a *tree.* If, in addition, $(\Omega, \leqslant)$ is dense, then $(\Omega, \leqslant)$ is called a *dense tree.*

Let us summarize those results of [12, § 5] which are relevant for us here. If $(\Omega, \leqslant)$ is a p.o. set and $A(\Omega)$ is k-transitive for some $k \geqslant 2$, then $(\Omega, \leqslant)$ is a dense tree if and only if $(\Omega, \leqslant)$ is of Type (4a). For the rest of this section, let $(\Omega, \leqslant)$ always be a dense tree. Up to isomorphism, there exists a unique tree $(\Omega^+, \leqslant)$ such that $(\Omega^+, \leqslant)$ is a meet-semilattice, $\Omega \subseteq \Omega^+$, the order of Ω^+ extends the order of Ω, Ω is dense in Ω^+, and, moreover, no proper subset of Ω^+ has these properties. This final minimality condition is equivalent to $\Omega^+ = \Omega \cup \mathrm{ram}(\Omega)$, where

$$\mathrm{ram}(\Omega) = \{a \in \Omega^+\colon \exists\, b, c \in \Omega \text{ such that } b \parallel c \text{ and } a = \inf\{b, c\}\},$$

the set of all *ramification points* of Ω. Here and in the following, all infima of subsets of Ω or Ω^+ are to be taken in Ω^+. Let $a \in \mathrm{ram}(\Omega)$. The maximal subsets A of $\{x \in \Omega\colon a < x\}$ with the property that $a < \inf\{x, y\}$ for any $x, y \in A$ are called the *cones* of a. Let $C(a)$ be the set of all cones of a. Clearly, $A \parallel B$ for any two different cones A, B of a. If $|C(a)| = \kappa$ for all $a \in \mathrm{ram}(\Omega)$, then κ is called the *ramification order* of Ω. If A is a cone of a, the cardinal number $\mathrm{coi}(A) = \inf\{|T|\colon T \subseteq A,\ a = \inf T\}$ is called the *coinitiality* of A. For all $a, b \in \Omega^+$ with $a < b$ we let

$$\begin{aligned}\langle a, b\rangle &= (\{x \in \Omega\colon a \leqslant x\} \setminus \{x \in \Omega\colon b < x\}) \cup \{a, b\}\\ &= \{x \in \Omega\colon a < x < b \text{ or } a < x,\ x \parallel b\} \,\dot{\cup}\, \{a, b\}.\end{aligned}$$

Then a (b) is the smallest (unique maximal) element of the set $\langle a, b\rangle$, respectively.

As follows from Theorems 2.1 and 2.7, $A(\Omega)$ can be k-transitive or m-homogeneous only if $k \leqslant 3$ ($m \leqslant 2$), respectively, and we have the following implications for $A(\Omega)$:

$$\text{3-transitive} \Rightarrow \text{2-homogeneous} \Rightarrow \text{2-transitive} \quad (\Rightarrow \text{1-transitive}).$$

In **[12]** we characterized the conditions under which the first implication can be reversed, and we have the following characterization of the structure of dense trees $(\Omega, \leqslant)$ whose automorphism group is 2-transitive but not 2-homogeneous:

THEOREM 3.2 [**12**, Theorem 5.17, Corollaries 5.36, 5.37]. *Let $(\Omega, \leqslant)$ be a dense tree. Then the following are equivalent:*

(1) *$A(\Omega)$ is 2-transitive but not 2-homogeneous;*

(2) *$(\Omega, \leqslant)$ has the following properties:*

  (i) *whenever $a, c \in \Omega$ and either $b, d \in \Omega$ or $b, d \in \mathrm{ram}(\Omega)$ with $a < b$ and $c < d$, then $\langle a, b\rangle \cong \langle c, d\rangle$;*

  (ii) $\mathrm{ram}(\Omega) = \Omega^{+} \setminus \Omega$;

  (iii) *whenever $a, b \in \mathrm{ram}(\Omega)$, then $C(a) = \{A, B\}$ and $C(b) = \{C, D\}$ such that $A \cong C$, $B \cong D$, and $A \not\cong B$.*

In particular, if one of these conditions holds, then for each $a \in \mathrm{ram}(\Omega)$ at most one cone of a has countable coinitiality, and Ω is uncountable.

Next we give the

Proof of Theorem (+). (1)$\Rightarrow$(2). By Theorem 2.2(e), $(\Omega, \leqslant)$ is not of Type (4c). Hence $(\Omega, \leqslant)$ satisfies either Condition (2) of Theorem 2.7, which implies (2i), or Condition (2.7)(4), which yields (2ii) by Theorem 3.2.

(2)$\Rightarrow$(1). This is immediate by Theorems 2.7 and 3.2.

In order to prove Theorem 3, we will now construct dense trees Ω in which each point $a \in \mathrm{ram}(\Omega)$ has two cones A, B with different coinitialities; then A and B cannot be isomorphic and $A(\Omega)$ is not 2-homogeneous. A construction similar to the following is contained in [**12**, pp. 56–61]; however, the present situation is much more complicated. Before introducing the explicit construction, let us first give an example which shows how to visualize these trees. If C is a dense unbounded chain, we let $\bar{C}$ always denote its Dedekind-completion. Thus $\bar{C}$ is an unbounded chain containing C as a dense subset such that any bounded subset of $\bar{C}$ has both a supremum and an infimum in $\bar{C}$.

EXAMPLE 3.3. Let C be a dense unbounded chain and $C_1, C_2 \subseteq \bar{C} \setminus C$ two disjoint subsets, each of which is dense in $\bar{C}$. Fix $x \in C_1$, $y \in C_2$. Let $C^{+} = C \,\dot{\cup}\, C_1 \,\dot{\cup}\, C_2$ and $C_x^{+} = \{z \in C^{+}\colon\ x < z\}$, $C_y^{+} = \{z \in C^{+}\colon\ y < z\}$. We define $(\Omega^{+}, \leqslant)$ in the natural way as the union of a countable chain of p.o. sets $(A_i, \leqslant)$ ($i \in \mathbb{N}$) where, for each $i \in \mathbb{N}$, $A_i \subseteq A_{i+1}$ and the order of A_{i+1} extends the order of A_i. First, let $A_1 = C^{+}$. Now define $(A_2, \leqslant)$ by adjoining at each $z \in C_1 \,\dot{\cup}\, C_2$ a chain $(C_z, \leqslant)$ to $(A_1, \leqslant)$ such that $z < C_z$ in $(A_2, \leqslant)$, where C_z is a copy of C_x^{+} (C_y^{+}) if $z \in C_2$ ($z \in C_1$), respectively. That is, we put $A_2 = A_1 \,\dot{\cup}\, \dot{\bigcup} \{C_z\colon\ z \in C_1 \cup C_2\}$ and let the order of A_2 extend the order of A_1 and of each C_z such that $a < C_z$ for each $a \in A_1$ with $a \leqslant z$ (and no other relations hold). Next define $(A_3, \leqslant)$ by adjoining to $(A_2, \leqslant)$ for each $z \in C_1 \cup C_2$ at each point $w \in C_z$ which is in the copy C_z an image of an element of C_1 (C_2) a copy C_w of C_y^+ (C_x^+), respectively, such that $w < C_w$ in $(A_3, \leqslant)$. Continue inductively to obtain the dense tree $(\Omega^+, \leqslant)$, a meet-semilattice, and let $\Omega = \Omega^+ \setminus \mathrm{ram}(\Omega^+)$. Then Ω is a dense tree with ramification order 2 and $\mathrm{ram}(\Omega) = \mathrm{ram}(\Omega^+) = \Omega^+ \setminus \Omega$.

Now let us give the precise numerical definition of a large class of trees which we are going to consider.

CONSTRUCTION 3.4. Let C be a dense unbounded chain, $\bar{C}$ its Dedekind-completion, and $C_1, C_2 \subseteq \bar{C} \setminus C$ two non-empty disjoint subsets. Fix $x \in C_1$, $y \in C_2$. We put

$$\begin{aligned} S &= S(C, C_1, C_2, x, y) \\ &= \{(a_1, \ldots, a_k):\ k \in \mathbb{N},\ a_i \in C_1 \cup C_2 \text{ for each } 1 \leqslant i \leqslant k,\ x < a_{i+1} \\ &\qquad \text{if } a_i \in C_2, \text{ and } y < a_{i+1} \text{ if } a_i \in C_1, \text{ for each } 1 \leqslant i \leqslant k-1\}. \end{aligned}$$

For each $A \subseteq \bar{C}$ we let

$$\begin{aligned} T(A, S) = \{(a):\ a \in A\} \cup \{(s\,; a):\ & s = (a_1, \ldots, a_k) \in S,\ a \in A,\ x < a \text{ if} \\ & a_k \in C_2, \text{ and } y < a \text{ if } a_k \in C_1\}. \end{aligned}$$

For $s = (a_1, \ldots, a_k) \in S$ and $a \in A$ we also formally write

$$(s\,; a) = (a_1, \ldots, a_k\,; a),$$

and for $k = 0$ and $a \in A$,

$$(a) = (a_1, \ldots, a_k\,; a).$$

Define a partial order on $T(A, S)$ as follows. Let

$$a' = (a_1, \ldots, a_k\,; a), \quad b' = (b_1, \ldots, b_j\,; b), \quad a', b' \in T(A, S).$$

We put $a' \leqslant b'$ if and only if

(i) $k \leqslant j$ and $a_i = b_i$ for all $i = 1, \ldots, k$, and
(ii) $a \leqslant b$ if $k = j$, and $a \leqslant b_{k+1}$ if $k < j$.

PROPOSITION 3.5. *Under the assumptions of Construction* 3.4, $\Omega = T(C, S)$ *is a dense tree with ramification order* 2 *such that* $\Omega^+ = T(C^+, S)$, *where* $C^+ = C \,\dot{\cup}\, C_1 \,\dot{\cup}\, C_2$, *and*

$$\mathrm{ram}(\Omega) = \Omega^+ \setminus \Omega = T(C_1 \cup C_2, S).$$

Furthermore, $|\Omega| = |\Omega^+| = |C^+|$.

Proof. We prove this proposition by checking the definitions. For brevity let $\Omega^* = T(C^+, S)$. The construction immediately implies that the p.o. sets Ω and Ω^* satisfy Conditions (3.1)(i), (iii)–(v), and Ω is dense in Ω^*. Thus, in particular, Ω is dense. In order to show that Condition (3.1)(ii) holds, let

$$a' = (a_1, \ldots, a_k\,; a_{k+1}) \in \Omega^* \quad (k \in \mathbb{N} \cup \{0\})$$

and

$$A' = \{z' \in \Omega^*:\ z' \leqslant a'\}.$$

We put

$$A_1 = \{(z):\ z \in C^+,\ z \leqslant a_1\}$$

and

$$A_{i+1} = \{(a_1, \ldots, a_i\,; z):\ z \in C^+,\ z \leqslant a_{i+1}, x < z \text{ if } a_i \in C_2, \text{ and } y < z \text{ if } a_i \in C_1\}$$

for $1 \leqslant i \leqslant k$. Then $A' = A_1 \dot\cup \dot\bigcup_{i=1}^{k} A_{i+1}$ and $A_i < A_{i+1}$ in Ω^* for each $1 \leqslant i \leqslant k$. Thus A' is a chain, and hence Ω and Ω^* are dense trees.

Trivially, $\Omega^* \setminus \Omega = T(C_1 \cup C_2, S)$. Next we show that Ω^* is a meet-semilattice and that $\mathrm{ram}(\Omega) \subseteq \Omega^* \setminus \Omega$. For this it suffices to check that whenever $b', c' \in \Omega^*$ satisfy $b' \parallel c'$, there exists $a' \in \Omega^* \setminus \Omega$ such that $a' = \inf\{b', c'\}$ in $(\Omega^*, \leqslant)$. Let $b' = (b_1, \ldots, b_k\,; b)$ and $c' = (c_1, \ldots, c_j\,; c)$ with $j, k \in \mathbb{N} \cup \{0\}$. We may assume $k \leqslant j$. If $b_i = c_i$ for all $1 \leqslant i \leqslant k$ (if $k = 0$, this is trivially satisfied), then $b' \parallel c'$ implies $k < j$ and $c_{k+1} < b$; in this case put $a' = (c_1, \ldots, c_k\,; c_{k+1})$. On the other hand, if $m = \min\{i:\ b_i \neq c_i\} \in \mathbb{N}$ exists, we have either $b_m < c_m$ or $c_m < b_m$, and we put $a' = (c_1, \ldots, c_{m-1}\,; \min\{b_m, c_m\})$. In both cases we have $a' \in T(C_1 \cup C_2, S)$ and $a' = \inf\{b', c'\}$ in Ω^*.

Next we prove that $\Omega^* \setminus \Omega \subseteq \mathrm{ram}(\Omega)$ and Ω has ramification order 2. Choose any $a' = (s\,; a) \in T(C_1 \cup C_2, S)$. We may assume $a \in C_1$. Choose $b, c \in C$ such that $a < b$ and $y < c$. Then $b' = (s\,; b)$ and $c' = (s, a\,; c)$ satisfy $b', c' \in \Omega$, $b' \parallel c'$, and $a' = \inf\{b', c'\}$, showing that $a' \in \mathrm{ram}(\Omega)$ and also $|C(a')| \geqslant 2$. Now let $d' \in \Omega$ with $a' < d'$. Then either

(1) $d' = (s\,; d)$ or $d' = (s, d, \ldots)$ for some $d \in C^+$ with $a < d$, or
(2) $d' = (s, a\,; d)$ or $d' = (s, a, d, \ldots)$ for some $d \in C^+$ with $y < d$.

In the first case, $e' = (s\,; \min\{b, d\}) \in \Omega^*$ satisfies $a' < e' = \inf\{b', d'\}$, so d' and b' belong to the same cone. In the second case, $e' = (s, a\,; \min\{c, d\}) \in \Omega^*$ satisfies $a' < e' = \inf\{c', d'\}$, and thus d' and c' belong to the same cone. Hence $|C(a')| = 2$.

Since $\Omega^* = \Omega \dot\cup \mathrm{ram}(\Omega)$ is a meet-semilattice and Ω is dense in Ω^*, we can conclude that $\Omega^+ = \Omega^* = T(C^+, S)$. The fact that $|\Omega| = |\Omega^+| = |C^+|$ is immediate by standard cardinal arithmetic.

With similar methods it can be shown that the p.o. set $T(\bar{C}, S)$ is the Dedekind-completion of $T(C, S)$ (cf. [**12**, §§ 5, 6]). Next we will impose on the chains C, C_1, C_2 certain homogeneity assumptions (in particular, C will be doubly homogeneous) which will imply homogeneity properties of the trees $T(C^+, S)$. The following notation will be useful.

Definition 3.6. Let $(A, \leqslant)$, $(B, \leqslant)$ be p.o. sets and $(Z_i)_{i\in I}$ a sequence of arbitrarily many sets. A mapping φ: $A \to B$ is called a *$(Z_i)_{i\in I}$-isomorphism from A onto B* if φ is an isomorphism from A onto B such that $(A \cap Z_i)^\varphi = B \cap Z_i$ for each $i \in I$. We say that $(A, \leqslant)$ and $(B, \leqslant)$ are *$(Z_i)_{i\in I}$-isomorphic* if there exists a $(Z_i)_{i\in I}$-isomorphism φ: $A \to B$.

In the proof of Theorem 3.8 we will need the following

Lemma 3.7. *Let Ω be a dense tree, $k \in \mathbb{N}$, and $a_i, b_i \in \Omega^+$ for each $i = 1, \ldots, k+1$ such that $a_i \in \Omega$ if and only if $b_i \in \Omega$ for each $2 \leqslant i \leqslant k$. For each*

$1 \leq i \leq k$ *assume* $a_i < a_{i+1}$, $b_i < b_{i+1}$, *and* $\langle a_i, a_{i+1}\rangle \cong \langle b_i, b_{i+1}\rangle$. *Then* $\langle a_1, a_{k+1}\rangle \cong \langle b_1, b_{k+1}\rangle$.

Proof. Let $A_i = \langle a_i, a_{i+1}\rangle$, $B_i = \langle b_i, b_{i+1}\rangle$, φ_i: $A_i \to B_i$ be an isomorphism $(1 \leq i \leq k)$, and $A = \langle a_1, a_{k+1}\rangle$, $B = \langle b_1, b_{k+1}\rangle$. Observe that

$$A = \dot{\bigcup_{i=1}^{k}} ((A_i \setminus \{a_{i+1}\}) \cap \Omega) \cup \{a_1, a_{k+1}\},$$

$$B = \dot{\bigcup_{i=1}^{k}} ((B_i \setminus \{b_{i+1}\}) \cap \Omega) \cup \{b_1, b_{k+1}\},$$

and $a_1^{\varphi_1} = b_1$, $a_i^{\varphi_{i-1}} = b_i = a_i^{\varphi_i}$, $a_i \in A$ if and only if $b_i \in B$, for each $2 \leq i \leq k$. Hence we can define a bijection φ: $A \to B$ by letting $a_1^{\varphi} = b_1$, $a_{k+1}^{\varphi} = b_{k+1}$, and $\varphi|_{A_i} = \varphi_i$ for each $i = 1, \ldots, k+1$. We claim that φ is an isomorphism.

Let $a, b \in A$. We have to show that $a < b$ if and only if $a^{\varphi} < b^{\varphi}$. Since $\varphi|_{A_i} = \varphi_i$ is an isomorphism, we can assume $a \in (A_i \setminus \{a_{i+1}\}) \cap \Omega$ and $b \in (A_j \setminus \{a_{j+1}\}) \cap \Omega$ with $i < j$. Hence $a_{i+1} \leq a_j \leq b$. Clearly

$$a^{\varphi} \in (B_i \setminus \{b_{i+1}\}) \cap \Omega, \quad b^{\varphi} \in (B_j \setminus \{b_{j+1}\}) \cap \Omega.$$

We have either $a_i \leq a < a_{i+1} \leq b$ or $a_i < a$, $a \parallel a_{i+1}$ and hence $a \parallel b$. In the first case, we get

$$b_i \leq a^{\varphi_i} = a^{\varphi} < b_{i+1} \leq b_j \leq b^{\varphi_j} = b^{\varphi}.$$

In the second case, we have

$$b_i < a^{\varphi_i} = a^{\varphi}, \quad a^{\varphi} \parallel b_{i+1}, \quad b_{i+1} \leq b^{\varphi}$$

and thus $a^{\varphi} \parallel b^{\varphi}$. Hence φ is an isomorphism.

If Z is any chain and $a, b \in Z$ with $a < b$, let

$$[a, b]_Z = \{z \in Z\colon\ a \leq z \leq b\} \quad \text{and} \quad (-\infty, a)_Z = \{z \in Z\colon\ z < a\};$$

the sets $(a, b)_Z$, $(a, \infty)_Z$, etc. are defined similarly. Now we show:

THEOREM 3.8. *Let C be a dense unbounded chain with Dedekind-completion $\bar{C}$, let $C_1, C_2 \subseteq \bar{C} \setminus C$ be two non-empty disjoint subsets, and let $C^+ = C \mathbin{\dot\cup} C_1 \mathbin{\dot\cup} C_2$ be such that the following properties are satisfied:*

(i) *whenever $a, b, c, d \in C^+$ all belong to the same one of the chains C, C_1, C_2 such that $a < b$ and $c < d$, then the intervals $[a, b]_{C^+}$ and $[c, d]_{C^+}$ are (C, C_1, C_2)-isomorphic;*
(ii) *for some $a \in C_1$, $b \in C_2$, the sets $(-\infty, a)_{C^+}$ and $(-\infty, b)_{C^+}$ are (C, C_1, C_2)-isomorphic, but $(a, \infty)_C$ and $(b, \infty)_C$ are not isomorphic.*

Choose any $x \in C_1$, $y \in C_2$ and put $S = S(C, C_1, C_2, x, y)$. Then $\Omega = T(C, S)$ is a dense tree of cardinality $|\Omega| = |C^+|$ such that $A(\Omega)$ is 2-transitive but not 2-homogeneous.

Proof. First note that assumption (i) yields that C_1 and C_2 are dense and unbounded in $\bar{C}$. Next we show:

(1) For any $c, d \in C$ or $c, d \in C_1 \cup C_2$, the sets $(-\infty, c)_{C^+}$ and $(-\infty, d)_{C^+}$ are (C, C_1, C_2)-isomorphic.

Indeed, if $c, d \in C$ ($c, d \in C_1$, $c, d \in C_2$), choose $a \in C$ (C_1, C_2 respectively) with $a < \{c, d\}$. By (i), there is a (C, C_1, C_2)-isomorphism φ from $[a, c]_{C^+}$ onto $[a, d]_{C^+}$. Now by extending φ trivially, we obtain a (C, C_1, C_2)-isomorphism from $(-\infty, c]_{C^+}$ onto $(-\infty, d]_{C^+}$. If $c \in C_1$, $d \in C_2$, then the result already proved and assumption (ii) imply that $(-\infty, c)_{C^+}$ is (C, C_1, C_2)-isomorphic to $(-\infty, d)_{C^+}$.

By an argument symmetrical to the one just applied, we also obtain

(2) For any $c, d \in C_1$ or $c, d \in C_2$, the sets $(c, \infty)_{C^+}$ and $(d, \infty)_{C^+}$ are (C, C_1, C_2)-isomorphic.

Next we show:

(3) Whenever $a, b, c, d \in C^+$ with $a < b$, $c < d$ such that a, c belong to the same chain, C, C_1, or C_2, and $b \in C$ if and only if $d \in C$, then $(a, b)_{C^+}$ and $(c, d)_{C^+}$ are (C, C_1, C_2)-isomorphic.

Let us first assume that $b = d$. Choose $e \in C^+$ such that $\{a, c\} < e < b$ and a, c, e all belong to the same one of the chains C, C_1, C_2. By (i) there is a (C, C_1, C_2)-isomorphism φ from $[a, e]_{C^+}$ onto $[c, e]_{C^+}$. By extending φ trivially, we obtain a (C, C_1, C_2)-isomorphism from $[a, b]_{C^+}$ onto $[c, d]_{C^+}$.

Now we consider the general case. By (1), there is a (C, C_1, C_2)-isomorphism ψ from $(-\infty, b)_{C^+}$ onto $(-\infty, d)_{C^+}$. As just shown, there exists a (C, C_1, C_2)-isomorphism π from $(a^\psi, d)_{C^+}$ onto $(c, d)_{C^+}$. Then $\psi \cdot \pi$ maps $(a, b)_{C^+}$ (C, C_1, C_2)-isomorphically onto $(c, d)_{C^+}$, establishing (3).

After these preparations we wish to apply Theorem 3.2 to prove the result of the theorem. By Proposition 3.5, Ω is a dense tree with $\mathrm{ram}(\Omega) = \Omega^+ \setminus \Omega$, ramification order 2, and cardinality $|C^+|$.

Step I. We check Condition (2i) of Theorem 3.2.

Let $a' \in \Omega$, $b' \in \Omega^+$ with $a' < b'$. Choose any $c \in C$, $d \in C^+$ with $c < d$ and $d \in C$ if and only if $b' \in \Omega$. Put $c' = (c)$, $d' = (d)$. Clearly it suffices to show that $\langle a', b' \rangle \cong \langle c', d' \rangle$. Let a' have the form $a' = (s\,; a)$ with $a \in C$. We distinguish between two cases.

Case 1. Assume $b' > a'$ is of the form $b' = (s\,; b)$ with $a < b \in C^+$ and $b \in C$ if and only if $b' \in \Omega$. Hence $b \in C$ if and only if $d \in C$. We have

$$\langle c', d' \rangle = \{c', d'\} \,\dot\cup\, \{(z)\colon\ c < z < d,\ z \in C\} \,\dot\cup\, \{(z, \ldots)\colon\ c < z < d,\ z \in C_1 \cup C_2\}$$

and

$$\langle a', b' \rangle = \{a', b'\} \,\dot\cup\, \{(s\,; z)\colon\ a < z < b,\ z \in C\}$$
$$\dot\cup\, \{(s, z, \ldots)\colon\ a < z < b,\ z \in C_1 \cup C_2\}.$$

According to (3), there exists a (C, C_1, C_2)-isomorphism φ from $(a, b)_{C^+}$ onto $(c, d)_{C^+}$. Now define $\psi\colon \langle a', b' \rangle \to \langle c', d' \rangle$ canonically by letting $a'^\psi = c'$, $b'^\psi = d'$, $(s\,; z)^\psi = (z^\varphi)$ for each $z \in C$ with $a < z < b$, and $(s, z, v)^\psi = (z^\varphi, v)$ for each $z \in C_1 \cup C_2$ with $a < z < b$ and each v which represents the other variables. Then ψ is well-defined since φ is a (C, C_1, C_2)-isomorphism, and it is easy to check that ψ is an isomorphism.

Case 2. Assume $b' > a'$ is of the form $b' = (s, b_1, \ldots, b_k\,; b_{k+1})$ with $k \in \mathbb{N}$, $a < b_1$, $b_{k+1} \in C^+$, and $b_{k+1} \in C$ if and only if $b' \in \Omega$ which holds if and only if $d \in C$.

Note that $b_i \in C_1 \cup C_2$, $x < b_{i+1}$ if $b_i \in C_2$, and $y < b_{i+1}$ if $b_i \in C_1$, for each $1 \leq i \leq k$. We let $b_i' = (s, b_1, \ldots, b_{i-1}\,; b_i)$ for $i = 1, \ldots, k+1$. Then $a' < b_1' < \ldots <$

$b'_{k+1}=b'$. Next we choose elements $d_i \in C_1 \cup C_2$ with $d_i \in C_2$ if and only if $b_i \in C_1$ $(i=1, \ldots, k)$ such that $c<d_1<\ldots<d_k<d_{k+1}:=d$. Let $d'_i=(d_i)$ for $i=1, \ldots, k+1$. Then $c'<d'_1<\ldots<d'_k<d'_{k+1}=d'$. From Case 1 it follows that $\langle a', b'_1\rangle \cong \langle c', d'_1\rangle$. Let $i \in \{1, \ldots, k\}$. We claim that $\langle b'_i, b'_{i+1}\rangle \cong \langle d'_i, d'_{i+1}\rangle$. We may assume $b_i \in C_1$, and thus $d_i \in C_2$. Put $t=(s, b_1, \ldots, b_{i-1})$. We have

$$\begin{aligned}\langle b'_i, b'_{i+1}\rangle = \{b'_i, b'_{i+1}\} &\,\dot\cup\, \{(t\,; z)\colon\ y<z<b_{i+1},\ z \in C\}\\ &\,\dot\cup\, \{(t, z, \ldots)\colon\ y<z<b_{i+1},\ z \in C_1 \cup C_2\}\end{aligned}$$

and

$$\begin{aligned}\langle d'_i, d'_{i+1}\rangle = \{d'_i, d'_{i+1}\} &\,\dot\cup\, \{(z)\colon\ d_i<z<d_{i+1},\ z \in C\}\\ &\,\dot\cup\, \{(z, \ldots)\colon\ d_i<z<d_{i+1},\ z \in C_1 \cup C_2\}.\end{aligned}$$

Since $y, d_i \in C_2$, by (3) there is a (C, C_1, C_2)-isomorphism φ from $(y, b_{i+1})_{C^+}$ onto $(d_i, d_{i+1})_{C^+}$. Now we define $\psi\colon \langle b'_i, b'_{i+1}\rangle \to \langle d'_i, d'_{i+1}\rangle$ by $b_i'^{\psi}=d'_i$, $b_{i+1}'^{\psi}=d'_{i+1}$, $(t, z)^{\psi}=(z^{\varphi})$ for each $z \in C$ with $y<z<b_{i+1}$, and $(t, z, v)^{\psi}=(z^{\varphi}, v)$ for each $z \in C_1 \cup C_2$ with $y<z<b_{i+1}$ and each v representing the other variables. Then ψ is an isomorphism, establishing our claim.

Note that $b'_i, d'_i \in \Omega^+\backslash\Omega$ for each $1 \leqslant i \leqslant k$. Hence $\langle a', b'\rangle \cong \langle c', d'\rangle$ by Lemma 3.7.

Step II. We check Condition (2iii) of Theorem 3.2.

It suffices to check this condition for any $a', b' \in \operatorname{ram}(\Omega)$ where $b'=(b)$ for some $b \in C_1 \cup C_2$. Let a' be of the form $a'=(s\,; a)$ with $a \in C_1 \cup C_2$. We may assume $a \in C_1$. Then $C(a')=\{A, B\}$ where

$$\begin{aligned}A &= \{(s\,; c)\colon\ a<c \in C\} \,\dot\cup\, \{(s, d, \ldots)\colon\ a<d \in C_1 \cup C_2\},\\ B &= \{(s, a\,; c)\colon\ y<c \in C\} \,\dot\cup\, \{(s, a, d, \ldots)\colon\ y<d \in C_1 \cup C_2\}.\end{aligned}$$

Now we distinguish between two cases.

Case 1. Assume $b \in C_1$. Then $C(b')=\{A', B'\}$ where

$$\begin{aligned}A' &= \{(c)\colon\ b<c \in C\} \,\dot\cup\, \{(d, \ldots)\colon\ b<d \in C_1 \cup C_2\},\\ B' &= \{(b\,; c)\colon\ y<c \in C\} \,\dot\cup\, \{(b, d, \ldots)\colon\ y<d \in C_1 \cup C_2\}.\end{aligned}$$

By (2), there is a (C, C_1, C_2)-isomorphism φ from $(a, \infty)_{C^+}$ onto $(b, \infty)_{C^+}$. Define $\psi\colon A \to A'$ canonically by letting $(s\,; c)^{\psi}=(c^{\varphi})$ for each $a<c \in C$, and $(s, d, v)^{\psi}=(d^{\varphi}, v)$ for each $a<d \in C_1 \cup C_2$ and each v which represents the other variables. Next define $\pi\colon B \to B'$ (even easier) by putting $(s, a\,; c)^{\pi}=(b\,; c)$ for each $y<c \in C$ and $(s, a, d, v)^{\pi}=(b, d, v)$ for each $y<d \in C_1 \cup C_2$ and each v representing the other variables. It is straightforward to check that ψ and π are isomorphisms.

Case 2. Assume $b \in C_2$. Then $C(b')=\{A^*, B^*\}$ where

$$\begin{aligned}A^* &= \{(c)\colon\ b<c \in C\} \,\dot\cup\, \{(d, \ldots)\colon\ b<d \in C_1 \cup C_2\},\\ B^* &= \{(b\,; c)\colon\ x<c \in C\} \,\dot\cup\, \{(b, d, \ldots)\colon\ x<d \in C_1 \cup C_2\}.\end{aligned}$$

Here let φ be a (C, C_1, C_2)-isomorphism from $(a, \infty)_{C^+}$ onto $(x, \infty)_{C^+}$. Define $\psi\colon A \to B^*$ by $(s\,; c)^{\psi}=(b\,; c^{\varphi})$ for each $a<c \in C$, and $(s, d, v)^{\psi}=(b, d^{\varphi}, v)$ for each $a<d \in C_1 \cup C_2$ and each v representing the other variables. Similarly, let φ' be a (C, C_1, C_2)-isomorphism from $(y, \infty)_{C^+}$ onto $(b, \infty)_{C^+}$. We define $\pi\colon B \to A^*$ by $(s, a\,; c)^{\pi}=(c^{\varphi'})$ for each $y<c \in C$, and $(s, a, d, v)^{\pi}=(d^{\varphi'}, v)$ for each

$y<d\in C_1\cup C_2$ and each v which represents the other variables. Again, ψ and π are isomorphisms.

To finish the verification of Condition (3.2)(2iii), it remains to check that the cones A and B are not isomorphic. Assume $\varphi\colon A\to B$ is an isomorphism. Choose $a<c\in C$ and let $c'=(s\,;c)\in A$. Then $c'^{\varphi}=d'\in B$ and either $d'=(s,a\,;d)$ or $d'=(s,a,d,\ldots)$ for some $y<d\in C^+$. Choose $f\in C$ with $y<f<d$ and put $f'=(s,a\,;f)\in B$. There is $e'\in A$ with $e'^{\varphi}=f'$. Since $f'<d'$, we have $e'<c'$ and hence $e'=(s\,;e)$ for some $e\in C$ with $a<e<c$. Thus φ maps $A_1=\{z'\in A\colon z'\leqslant e'\}$ onto $B_1=\{z'\in B\colon z'\leqslant f'\}$. Clearly $A_1=\{(s\,;z)\colon a<z\leqslant e,\ z\in C\}$ is isomorphic to the interval $(a,e]_C$ in C, and B_1 is isomorphic to $(y,f]_C$. Choose $g\in C$ with $\{e,f\}<g$. Since $e,f\in C$, we obtain

$$(a,g]_C\cong(a,e]_C\cong A_1\cong B_1\cong(y,f]_C\cong(y,g]_C$$

by (3), and so $(a,\infty)_C\cong(y,\infty)_C$. Since $a\in C_1$, $y\in C_2$, this implies by (2) a contradiction to (ii).

Now Theorem 3.2 shows that $A(\Omega)$ is 2-transitive but not 2-homogeneous.

It remains to construct chains C, C_1, C_2 satisfying the assumptions of Theorem 3.8. This we will now do in a series of steps. The following theorem and its corollary are Löwenheim–Skolem-type results for doubly homogeneous chains.

THEOREM 3.9. *Let C be an infinite chain, I an index set, and $C=\bigcup_{i\in I}C_i$ such that whenever $j\in I$ and $a,b,c,d\in C_j$ with $a<b$ and $c<d$, then the intervals $[a,b]_C$ and $[c,d]_C$ are $(C_i)_{i\in I}$-isomorphic. Let $A_i\subseteq C_i$ for each $i\in I$, let $A=\bigcup_{i\in I}A_i$, and let λ be an infinite cardinal with $|A|\leqslant\lambda\leqslant|C|$. Then there are sets B_i with $A_i\subseteq B_i\subseteq C_i$ for each $i\in I$ such that $B=\bigcup_{i\in I}B_i$ has cardinality λ and whenever $j\in I$ and $a,b,c,d\in B_j$ with $a<b$ and $c<d$, then the intervals $[a,b]_B$ and $[c,d]_B$ are $(B_i)_{i\in I}$-isomorphic.*

Proof. We may assume that $A_i\neq\varnothing$ for each $i\in I$. Hence $|I|\leqslant\lambda$. For each $i\in I$, we identify the subsets A_i, C_i of C with unary predicates of C. By assumption, there is a function $f\colon C^5\to C$ such that for all $j\in I$ and $a,b,c,d\in C_j$ with $a<b$ and $c<d$, the function $f(a,b,c,d,\cdot)$ maps the interval $[a,b]_C$ $(C_i)_{i\in I}$-isomorphically onto $[c,d]_C$. By the downward Löwenheim–Skolem theorem, the model $\mathfrak{C}=\langle C,<,f,(C_i)_{i\in I},(A_i)_{i\in I}\rangle$ has an elementary submodel $\mathfrak{B}\prec\mathfrak{C}$ whose domain B satisfies $A\subseteq B\subseteq C$ and $|B|=\lambda$. Putting $B_i=B\cap C_i$ for each $i\in I$, we obtain the result.

Of course, there also exists a direct construction of the chain B of Theorem 3.9, but this is technically more complicated. Recall that a chain C is doubly homogeneous if and only if C is unbounded and any two closed intervals of C are isomorphic. Any such chain is dense. As a consequence of Theorem 3.9 we have

COROLLARY 3.10. *Let C be a doubly homogeneous chain, λ an infinite cardinal, and $A\subseteq C$ such that $|A|\leqslant\lambda\leqslant|C|$. Then there is a doubly homogeneous chain B such that $A\subseteq B\subseteq C$ and $|B|=\lambda$.*

Proof. Since C is unbounded, we may assume that A is also unbounded (otherwise enlarge A correspondingly). By Theorem 3.9, there is a chain B' with $A\subseteq B'\subseteq C$ and $|B'|=\lambda$ in which any two closed intervals of B' are isomorphic.

Hence, by deleting the smallest or greatest elements of B' (if such exists), we obtain a doubly homogeneous chain B with the required properties.

Let C be a dense unbounded chain and $\bar{C}$ its Dedekind-completion. Each automorphism α of C has a unique extension to an automorphism of $\bar{C}$ which we will also denote by α. If $c \in \bar{C}$, the set $\{c^\alpha\colon\ \alpha \in A(C)\}$ is called the $A(C)$-*orbit* of c in $\bar{C}$. A set $A \subseteq \bar{C}$ is called an orbit of $A(C)$ in $\bar{C} \setminus C$ if it is the $A(C)$-orbit of some $c \in \bar{C} \setminus C$ in $\bar{C}$.

Lemma 3.11 (McCleary [**31**, Proposition 2]). *Let C be a doubly homogeneous chain, C_1, C_2 two different orbits of $A(C)$ in $\bar{C} \setminus C$, and let $C^+ = C \cup C_1 \cup C_2$. Let $a, b, c, d \in C^+$ all belong to the same one of the chains C, C_1, C_2 such that $a < b$ and $c < d$. Then there is $\alpha \in A(C)$ with $a^\alpha = c$ and $b^\alpha = d$. In particular, $[a, b]_{C^+}$ is (C, C_1, C_2)-isomorphic to $[c, d]_{C^+}$.*

Again let C be a dense unbounded chain and $a \in \bar{C}$. Then

$$\operatorname{cof}(a) = \min\{|A|\colon\ A \subseteq C, A < a, a = \sup A \text{ in } (\bar{C}, \leqslant)\},$$

the *cofinality* of a, and

$$\operatorname{coi}(a) = \min\{|A|\colon\ A \subseteq C, a < A, a = \inf A \text{ in } (\bar{C}, \leqslant)\},$$

the *coinitiality* of a in $\bar{C}$. As a consequence of the preceding results, we show now:

Corollary 3.12. *Let C be a doubly homogeneous chain for which there are $a, b \in \bar{C} \setminus C$ with $\operatorname{cof}(a) = \operatorname{coi}(a) = \operatorname{cof}(b) = \aleph_0$ and $\operatorname{coi}(b) \neq \aleph_0$. Then there are two non-empty disjoint sets $C_1, C_2 \subseteq \bar{C} \setminus C$ such that C, C_1, C_2 satisfy the assumptions of Theorem* 3.8 $|C_1| + |C_2| \leqslant |C|$.

Proof. Let D_1 (D_2) be the $A(C)$-orbit of a (b) in $\bar{C} \setminus C$, respectively. Clearly, D_1 and D_2 are disjoint. By Lemma 3.11 and Theorem 3.9 there are sets $C_1 \subseteq D_1$, $C_2 \subseteq D_2$ such that $a \in C_1$, $b \in C_2$, $C^+ = C \cup C_1 \cup C_2$ has cardinality $|C|$, and Condition (i) of Theorem 3.8 is satisfied. It only remains to check Condition (3.8)(ii). Clearly $(a, \infty)_C$ and $(b, \infty)_C$ are not isomorphic since $\operatorname{coi}(a) \neq \operatorname{coi}(b)$. Now choose two sequences of elements a_i, $b_i \in C$ such that $a_i < a_{i+1} < a$, $b_i < b_{i+1} < b$ for each $i \in \mathbb{N}$, $a_1 = b_1$, and $a = \sup\{a_i\colon\ i \in \mathbb{N}\}$, $b = \sup\{b_i\colon\ i \in \mathbb{N}\}$ in $\bar{C}$. By Condition (3.8)(i), for each $i \in \mathbb{N}$ there is a (C, C_1, C_2)-isomorphism φ_i from $[a_i, a_{i+1}]_{C^+}$ onto $[b_i, b_{i+1}]_{C^+}$. Define $\varphi\colon (-\infty, a)_{C^+} \to (-\infty, b)_{C^+}$ such that φ extends each φ_i ($i \in \mathbb{N}$) and is the identity on $(-\infty, a_1]_{C^+}$. Then φ is a (C, C_1, C_2)-isomorphism.

If $(C, \leqslant)$ is a chain and $c \in C$, the set $(-\infty, c)_C$ will be called an *initial segment* of C. If C is doubly homogeneous, clearly all initial segments of C are isomorphic. Next we note that for each cardinal λ there are 'many' chains C of cardinality λ satisfying the hypotheses of Corollary 3.12:

Proposition 3.13 [**14**, Corollary 3.3]. *Let λ be an uncountable cardinal. If λ is regular, there are 2^λ doubly homogeneous chains $(C_i, \leqslant)$ ($i < 2^\lambda$) of cardinality λ such that the following two conditions hold for all $i, j < 2^\lambda$ with $i \neq j$:*

(1) *there are $a, b \in \overline{C_i} \setminus C_i$ with $\operatorname{cof}(a) = \operatorname{coi}(a) = \operatorname{cof}(b) = \aleph_0$ and $\operatorname{coi}(b) = \aleph_1$;*

(2) *whenever* $c \in C_i$, $d \in C_j$, *then the initial segments* $(-\infty, c)_{C_i}$, $(-\infty, d)_{C_j}$ *are not isomorphic.*

If λ *is singular, there are at least* $2^{<\lambda} = \sum_{\rho<\lambda} 2^{\rho}$ *such chains* $(C_i, \leqslant)$.

Finally, we prove that if we have 'sufficiently different' chains C in the situation of Theorem 3.8, then the corresponding trees $\Omega = T(C, S)$ are also non-isomorphic.

LEMMA 3.14. *Let* C, C_1, C_2, x, y *and* C', C_1', C_2', x',y' *be two quintuples of chains and elements each satisfying the assumptions of Theorem* 3.8. *Let* $S = S(C, C_1, C_2, x, y)$ *and* $S' = S(C', C_1', C_2', x', y')$. *Assume that for some* $c \in C$, $c' \in C'$, *the initial segments* $(-\infty, c)_C$ *and* $(-\infty, c')_{C'}$ *are not isomorphic. Then the trees* $\Omega = T(C, S)$ *and* $\Omega' = T(C', S')$ *are not isomorphic.*

Proof. Suppose there is an isomorphism φ: $\Omega \to \Omega'$. Then $(c) \in \Omega$ and $(c)^{\varphi}$, $(c') \in \Omega'$. By Theorem 3.8, there is $\beta \in A(\Omega')$ with $((c)^{\varphi})^{\beta} = (c')$. Thus $\varphi \cdot \beta$ maps $A = \{(z)\colon z \in C, z \leqslant c\}$ isomorphically onto $A' = \{(z')\colon z' \in C', z' \leqslant c'\}$. Hence

$$(-\infty, c)_C \cong A \cong A' \cong (-\infty, c')_{C'},$$

a contradiction.

Finally, the *Proof of Theorem* 3 is straightforward by Proposition 3.13, Corollary 3.12, Theorem 3.8, and Lemma 3.14; clearly there cannot be more than 2^{λ} pairwise non-isomorphic trees of cardinality λ.

It remains an open problem to construct *countable* p.o. sets $(\Omega, \leqslant)$ whose automorphism group is 2-transitive but not 2-homogeneous, other than the one example of Theorem (+)(2i) (with $C = \mathbb{Q}$). By Theorems 2.7 and 3.2, any p.o. set $(\Omega, \leqslant)$ of this kind is of Type (4c). Since the example of Theorem (+)(2i) has a 3-transitive automorphism group (by Theorem 2.1), by Theorem 2.2(d) it is an equivalent version of this problem to construct countable p.o. sets $(\Omega, \leqslant)$ such that $A(\Omega)$ is 2-transitive, but neither 2-homogeneous nor 3-transitive.

As a further consequence of our considerations we have the following result which will also be useful to us in § 4 (see Corollary 4.8).

COROLLARY 3.15. *Let* λ *be any infinite cardinal and* κ *any (finite or infinite) cardinal with* $2 \leqslant \kappa \leqslant \lambda$. *Then there are two trees* $(\Omega_1, \leqslant)$, $(\Omega_2, \leqslant)$ *of cardinality* λ *and ramification order* κ *such that* $A(\Omega_1)$ *and* $A(\Omega_2)$ *are 2-homogeneous,* $(\Omega_1, \leqslant)$ *is a meet-semilattice, and* $(\Omega_2, \leqslant)$ *is not a meet-semilattice.*

Proof. We claim that there are a dense unbounded chain C and a non-empty subset $D \subseteq \bar{C} \setminus C$ such that $|D| \leqslant |C| = \lambda$ and whenever $a, b, c, d \in C$ or $a, b, c, d \in D$ with $a < b$ and $c < d$, then the intervals $[a, b]_{C \cup D}$ and $[c, d]_{C \cup D}$ are (C, D)-isomorphic. Indeed, if λ is uncountable, this claim is immediate by Proposition 3.13 and Corollary 3.12; otherwise let $C = \mathbb{Q}$ and $D = \{q + \sqrt{2}\colon q \in \mathbb{Q}\}$. Now the result follows directly from [**12**, Theorem 6.8] (put $\Omega_1 = T_{\kappa}(C, C)$ and $\Omega_2 = T_{\kappa}(C, D)$).

4. *Partial isomorphisms and $L_{\infty\omega}$-equivalence*

This section is devoted to the proof of Theorem 4 and its consequences. We first need some model-theoretic preparations. The following remarks hold in a more general context, but we formulate them for our purposes only for p.o. sets. If $(\Omega_1, \leqslant)$, $(\Omega_2, \leqslant)$ are two p.o. sets, $A \subseteq \Omega_1$, $B \subseteq \Omega_2$, and φ: $A \to B$ is an isomorphism, then φ is called a *partial isomorphism* from Ω_1 into Ω_2. We say that $(\Omega_1, \leqslant)$ and $(\Omega_2, \leqslant)$ are *partially isomorphic* if there exists a system S of partial isomorphisms from Ω_1 into Ω_2 such that whenever $\varphi \in S$ and $a \in \Omega_1$ ($b \in \Omega_2$), there exists $\psi \in S$ which extends φ such that the domain (range) of ψ contains a (b), respectively; in this case we say that $(\Omega_1, \leqslant)$ and $(\Omega_2, \leqslant)$ are *partially isomorphic by S.* Now we have:

PROPOSITION 4.1 (Karp [27]). *Let $(\Omega_1, \leqslant)$ and $(\Omega_2, \leqslant)$ be two p.o. sets and $L = \{\leqslant\}$ the first order language for p.o. sets.*

(a) *$(\Omega_1, \leqslant)$ and $(\Omega_2, \leqslant)$ are partially isomorphic if and only if they are $L_{\infty\omega}$-equivalent.*

(b) *Let $(\Omega_1, \leqslant)$ and $(\Omega_2, \leqslant)$ be countable and partially isomorphic by a system S of partial isomorphisms from Ω_1 into Ω_2. Then any $\varphi \in S$ extends to an isomorphism ψ: $\Omega_1 \to \Omega_2$. In particular, $(\Omega_1, \leqslant)$ and $(\Omega_2, \leqslant)$ are isomorphic.*

First let us study trees. A tree $(\Omega, \leqslant)$ is called *almost normal,* if $(\Omega, \leqslant)$ is dense, either $\Omega^+ = \Omega$ or $\mathrm{ram}(\Omega) = \Omega^+ \setminus \Omega$, $\mathrm{ram}(\Omega)$ is dense in Ω^+, and either $|C(a)| = |C(b)| < \infty$ for all $a, b \in \mathrm{ram}(\Omega)$ or $|C(a)|$ is infinite for all $a \in \mathrm{ram}(\Omega)$. A tree $(\Omega, \leqslant)$ is *normal* if it is almost normal and $|C(a)| = |C(b)|$ for all $a, b \in \mathrm{ram}(\Omega)$, that is, the ramification order of Ω exists. The importance of normal trees is clear by

PROPOSITION 4.2 [**12**, 6.2]. *Let $(\Omega, \leqslant)$ be a tree. If $A(\Omega)$ is 2-transitive, then $(\Omega, \leqslant)$ is normal.*

Next we note that the class of p.o. sets which are almost normal trees is closed under elementary equivalence.

PROPOSITION 4.3. *Let $(\Omega_1, \leqslant)$ and $(\Omega_2, \leqslant)$ be two elementarily equivalent p.o. sets such that $(\Omega_1, \leqslant)$ is an almost normal tree. Then $(\Omega_2, \leqslant)$ is also an almost normal tree. Moreover, $(\Omega_2, \leqslant)$ is a meet-semilattice if and only if $(\Omega_1, \leqslant)$ is too, and for each $n \in \mathbb{N}$ and all $a_i \in \mathrm{ram}(\Omega_i)$ $(i = 1, 2)$, we have $|C(a_1)| = n$ if and only if $|C(a_2)| = n$.*

Proof. Clearly $(\Omega_2, \leqslant)$ is a dense tree. The fact that $\mathrm{ram}(\Omega_i)$ is dense in Ω_i^+ $(i = 1, 2)$, can be expressed by the first order sentence

$$\sigma \equiv (\forall x, y)[x < y \Rightarrow (\exists v, w)(x < v < \{y, w\} \wedge y \parallel w)],$$

using our standard abbreviations. Since either all pairs of elements $a, b \in \Omega_1$ or no two such elements have an infimum in Ω_1, the same applies to Ω_2. Hence $\mathrm{ram}(\Omega_2)$ is dense in Ω_2^+, and either $\mathrm{ram}(\Omega_2) = \Omega_2$ or $\mathrm{ram}(\Omega_2) = \Omega_2^+ \setminus \Omega_2$. Now if

$x_1, \ldots, x_n$ are variables ($3 \leqslant n \in \mathbb{N}$), we put

$$\psi_n(x_1, \ldots, x_n) \equiv \bigwedge_{\substack{i \neq j \\ k \neq m}} [(x_i \parallel x_j) \wedge (x_k \parallel x_m) \wedge (\forall z)(z < \{x_i, x_j\} \Leftrightarrow z < \{x_k, x_m\})],$$

where the conjunction is taken over all $i, j, k, m \in \{1, \ldots, n\}$ with $i \neq j$ and $k \neq m$. Now, for each $3 \leqslant n \in \mathbb{N}$, the sentence

$$\sigma_n \equiv (\forall x_1, x_2)[x_1 \parallel x_2 \Rightarrow (\exists x_3, \ldots, x_n)\psi(x_1, x_2, \ldots, x_n)]$$

holds in $(\Omega_1, \leqslant)$ if and only if $|C(a)| \geqslant n$ for each $a \in \operatorname{ram}(\Omega_1)$. Now it is straightforward to conclude that $(\Omega_2, \leqslant)$ is also an almost normal tree, and the result follows.

Our main new result for trees is contained in

THEOREM 4.4. *Let $(\Omega_1, \leqslant)$, $(\Omega_2, \leqslant)$ be two almost normal trees such that Ω_1 is a meet-semilattice if and only if Ω_2 is a meet-semilattice, and such that for all $a \in \operatorname{ram}(\Omega_1)$, $b \in \operatorname{ram}(\Omega_2)$ we have $|C(a)| = |C(b)|$ provided that one of these numbers is finite.*

(a) *$(\Omega_1^+, \leqslant)$ and $(\Omega_2^+, \leqslant)$ are partially isomorphic by the system S of all isomorphisms $\varphi\colon A \to B$ where $A \subseteq \Omega_1^+$, $B \subseteq \Omega_2^+$ are finite meet-semisublattices such that $(A \cap \Omega_1)^\varphi = B \cap \Omega_2$.*

(b) *$(\Omega_1, \leqslant)$ and $(\Omega_2, \leqslant)$ are partially isomorphic by the system of all isomorphisms $\varphi\colon A_1 \to A_2$ where $A_i \subseteq \Omega_i$ is finite such that if B_i is the smallest meet-semisublattice of Ω_i^+ containing A_i $(i = 1, 2)$, then φ extends to an isomorphism $\psi\colon B_1 \to B_2$.*

Proof. (a) Let $A \subseteq \Omega_1^+$, $B \subseteq \Omega_2^+$ be two finite meet-semisublattices, $\varphi\colon A \to B$ an isomorphism with $(A \cap \Omega_1)^\varphi = B \cap \Omega_2$, and $a \in \Omega_1^+ \setminus A$. We claim that there are two finite meet-semisublattices $C \subseteq \Omega_1^+$, $D \subseteq \Omega_2^+$ and an isomorphism $\psi\colon C \to D$ such that $A \cup \{a\} \subseteq C$, $B \subseteq D$, ψ extends φ, and $(C \cap \Omega_1)^\psi = D \cap \Omega_2$. Let C be the smallest meet-semisublattice of Ω_1^+ containing $A \cup \{a\}$. We distinguish between several cases. Note in the following that either $\Omega_i = \Omega_i^+$ for $i = 1, 2$, or $\operatorname{ram}(\Omega_i) = \Omega_i^+ \setminus \Omega_i$ for $i = 1, 2$, and that either $|C(a)| = |C(b)| < \infty$ for all $a \in \operatorname{ram}(\Omega_1)$, $b \in \operatorname{ram}(\Omega_2)$, or $|C(a)|$, $|C(b)|$ are infinite for all $a \in \operatorname{ram}(\Omega_1)$, $b \in \operatorname{ram}(\Omega_2)$.

If $a < A$, let $m = \inf A$. Then $m \in A$, and hence $a < m$ and $C = A \,\dot\cup\, \{a\}$. Choose $b \in \Omega_2^+$ with $b < \inf B$ and $b \in \Omega_2$ if and only if $a \in \Omega_1$. Let $D = B \,\dot\cup\, \{b\}$ and define $\psi\colon C \to D$ by $\psi|_A = \varphi$ and $a^\psi = b$. Clearly our claim follows.

If $A < a$, we proceed dually.

If $a \parallel A$, again let $m = \inf A$. Then $m \in A$, and hence $m \parallel a$. Let $x = \inf\{m, a\} \in \operatorname{ram}(\Omega_1)$. Then $C = A \,\dot\cup\, \{a, x\}$. Choose $y \in \operatorname{ram}(\Omega_2)$ with $y < \inf B$ and then $b \in \Omega_2^+$ such that $b \parallel \inf B$, $y = \inf\{b, \inf B\}$, and $b \in \Omega_2$ if and only if $a \in \Omega_1$. Put $D = B \,\dot\cup\, \{y, b\}$ and define $\psi\colon C \to D$ by $\psi|_A = \varphi$ and $x^\psi = y$, $a^\psi = b$. Again the claim follows.

Now assume that neither $a < A$ nor $a \parallel A$ nor $A < a$. Put $A_1 = \{w \in A\colon a < w\}$ and $A_2 = \{w \in A\colon w \parallel a\}$. We claim that there is some $v \in A$ with $v < a$. Indeed, if $A_1 = \emptyset$ or $A_2 = \emptyset$, this is clear by our assumption. Now let $A_1 \neq \emptyset$ and $A_2 \neq \emptyset$. So $u_i = \inf A_i \in A$ $(i = 1, 2)$ exist, and $v = \inf\{u_1, u_2\} \in A$ satisfies $v < u_1$ (otherwise $a \leqslant u_1 = v \leqslant u_2$, a contradiction), and hence $v < a$. Thus in any case $m = \max\{w \in A\colon w < a\} \in A$ exists.

If $A_1 \neq \emptyset$, then $u = \inf A_1 \in A$ and hence $a < u$. So $C = A \,\dot\cup\, \{a\}$, since for any

$w \in A_2$ we have $\inf\{w, a\} = \inf\{w, u\} \in A$. Choose $b \in \Omega_2^+$ with $m^\varphi < b < u^\varphi$ and $b \in \Omega_2$ if and only if $a \in \Omega_1$. Put $D = B \cup \{b\}$ and define ψ: $C \to D$ by $\psi|_A = \varphi$ and $a^\psi = b$. This implies our claim.

Therefore assume $A_1 = \varnothing$ now, and hence $A_2 \neq \varnothing$, since otherwise $A < a$. Let $x = \max\{z \in \mathrm{ram}(\Omega_1)$: $\exists\, v \in A_2$ such that $z = \inf\{v, a\}\}$. Then $x < a$. Let $A_3 = \{w \in A_2$: $x < w\}$. Thus $A_3 \neq \varnothing$ and $x = \inf\{w, a\}$ for each $w \in A_3$. For each $w \in A_2 \backslash A_3$ we have $w \parallel x$, and hence $\inf\{w, a\} = \inf\{w, x\} = \inf(\{w\} \cup A_3) \in A$. Now we distinguish between three cases.

If $x < m$, we get $x = \inf(A_3 \cup \{m\}) \in A$ and thus $C = A \cup \{a\}$. Choose $b \in \Omega_2^+$ such that $m^\varphi < b$ and $b \in \Omega_2$ if and only if $a \in \Omega_1$, and put $D = B \cup \{b\}$. Again ψ: $C \to D$, defined by $\psi|_A = \varphi$ and $a^\psi = b$, satisfies all our requirements.

If $m < x$, we have $x \notin A$ by definition of m. Hence $u = \inf A_3 \in A$ satisfies $x < u$ and $u \parallel a$. Clearly there is no $w \in A$ with $m < w < u$, since any such w would have to satisfy either $w \parallel a$ or $w < a$, a contradiction. Choose $y \in \mathrm{ram}(\Omega)$ with $m^\varphi < y < u^\varphi$; then $y \notin B$. Next choose $b \in \Omega_2^+$ with $b \parallel u^\varphi$, $y = \inf\{b, u^\varphi\}$, and $b \in \Omega_2$ if and only if $a \in \Omega_1$. Thus $b \notin B$. Let $D = B \cup \{y, b\}$ and define ψ: $C \to D$ by $\psi|_A = \varphi$, $x^\psi = y$, and $a^\psi = b$. Again our result follows.

Finally, assume $x = m$. Then $C = A \cup \{a\}$, and since $A_1 = \varnothing$ and by the definition of x and m, a belongs to a cone of x in Ω_1 which is disjoint from A. Let $T \subseteq A$ be a maximal subset such that $m < T$ and $m = \inf T'$ for all $T' \subseteq T$ with $|T'| = 2$. Trivially, $|T| \geqslant 1$ and for all $z \in \mathrm{ram}(\Omega_1)$ we have $|C(z)| \geqq |T| + 1$, since all elements of $T \cup \{a\}$ belong to different cones of $m = x$ in Ω_1. Hence for all $z \in \mathrm{ram}(\Omega_2)$ we have $|C(z)| > |T^\varphi|$, and we can find a cone Z of m^φ in Ω_2 which is disjoint from T^φ and hence satisfies $m^\varphi = \inf\{z, t^\varphi\}$ for each $z \in Z$, $t \in T$. Now the maximality of T implies that Z is disjoint from B. Choose any $z \in Z$ and then $b \in \Omega_2^+$ with $z \leqslant b$ and $b \in \Omega_2$ if and only if $a \in \Omega_1$. Then $b \parallel w$ and $m^\varphi = \inf\{b, w\}$ for each $w \in B$ with $m^\varphi < w$. For all $w \in B$ with $w \parallel b$ and not $m^\varphi < w$ we have $m^\varphi \parallel w$ and hence $\inf\{w, b\} = \inf\{w, m^\varphi\} \in B$. Put $D = B \cup \{b\}$ and define ψ: $C \to D$ again by $\psi|_A = \varphi$ and $a^\psi = b$. Then ψ satisfies the requirements, and our claim is proved.

Together with a symmetrical argument, this shows that $(\Omega_1^+, \leqslant)$ and $(\Omega_2^+, \leqslant)$ are partially isomorphic by the system S given in the theorem.

(b) Let $A_i \subseteq \Omega_i$, $B_i \subseteq \Omega_i^+$ $(i = 1, 2)$, and φ, ψ be given as stated in the theorem. Then $B_i = \{\inf T$: $T \subseteq A_i\}$ is finite $(i = 1, 2)$ and $(B_1 \cap \Omega_1)^\psi = B_2 \cap \Omega_2$, since if $\Omega_i \neq \Omega_i^+$, we have $B_i \cap \Omega_i = A_i$ $(i = 1, 2)$. Now the result follows from (a).

Now the use of Theorem 4.4 allows us to considerably shorten the argument of [**12**] for the two following important results of Corollary 4.5 and 4.7 for countable trees $(\Omega, \leqslant)$.

Corollary 4.5 [**12**, Theorem 6.21]. *Let* $(\Omega, \leqslant)$ *be a countable tree. The following are equivalent:*

(1) $(\Omega, \leqslant)$ *is normal;*

(2) $A(\Omega)$ *is 2-transitive;*

(3) $A(\Omega)$ *is 2-homogeneous.*

Proof. (1)$\Rightarrow$(3). Let $A_i \subseteq \Omega$ with $|A_i| = 2$ $(i = 1, 2)$ and φ: $A_1 \to A_2$ be an isomorphism. It is clear that φ belongs to the system of partial isomorphisms described in Theorem 4.4(b). Hence by Theorem 4.4(b) and Proposition 4.1(b), φ extends to an automorphism of Ω.

(3)$\Rightarrow$(2). This is trivial.

(2)$\Rightarrow$(1). This is immediate by Proposition 4.2.

We will also need the following result.

THEOREM 4.6 [**12**, Theorem 5.33]. *Let $(\Omega, \leqslant)$ be a dense tree. Then $A(\Omega)$ is 3-transitive if and only if $A(\Omega)$ is 2-homogeneous and Ω has ramification order 2 and is not a meet-semilattice.*

In this context we note that a short proof of Theorem 4.6 for *countable* trees also follows from Propositions 4.2, 4.1(b), Theorem 4.4(b), and the argument in the proof of [**12**, Theorem 5.33].

COROLLARY 4.7 [**12**, Theorems 6.21, 6.22]. (a) *For each $2 \leqslant n \in \mathbb{N} \cup \{\aleph_0\}$, there exist up to isomorphism precisely two countable trees $(\Omega, \leqslant)$ with 2-transitive automorphism group and ramification order n; one of these two trees is a meet-semilattice.*

(b) *Up to isomorphism, there exists a unique countable tree $(\Omega, \leqslant)$ with 3-transitive automorphism group.*

Proof. (a) The uniqueness part is immediate by Propositions 4.2, 4.1(b), and Theorem 4.4(b). The existence result follows from Corollary 4.5 and [**12**, Proposition 6.5].

(b) This is immediate by (a), Corollary 4.5, and Theorem 4.6.

Now we can give the

Proof of Theorem 4. Let $(\Omega', \leqslant)$ be a p.o. set wich is elementarily equivalent to $(\Omega, \leqslant)$. Clearly Ω' is infinite. According to Proposition 4.1(a), we have to show that $(\Omega, \leqslant)$ and $(\Omega', \leqslant)$ are partially isomorphic. First assume that Ω is of one of the Types (0)–(3) of Theorem 2.1. We use the notation of Theorem 2.1 in the following and distinguish between several cases to describe the structure of $(\Omega', \leqslant)$.

(0) If $(\Omega, \leqslant)$ is trivially ordered, so is $(\Omega', \leqslant)$.

(1) If $(\Omega, \leqslant)$ satisfies Condition (1) of Theorem 2.1, then $\Omega' = A' \,\dot\cup\, B' \,\dot\cup\, C'$ with antichains A', B', C' in Ω' such that $A' < B' < C'$, $|A'| = |A|$, $|C'| = |C|$, and B' is infinite.

(2) Assume $(\Omega, \leqslant)$ satisfies Condition (2.1)(2). Then $\Omega' = \dot\bigcup_{i \in I'} C_i'$ where either $|I'| = |I| < \infty$ or I' and I are both infinite sets such that the following two conditions are satisfied:

(i) for each $i \in I'$, C_i' is a dense unbounded chain;

(ii) whenever $i, j \in I'$ with $i \neq j$, then $C_i' \parallel C_j'$.

(3a) Let $(\Omega, \leqslant)$ satisfy Condition (2.1)(3a). There exists a dense unbounded chain $C' \subseteq \Omega'$ such that the following two conditions hold:

(i) if r is finite, then $\Omega' \cong C' \vec{\times} r$;

(ii) if r is infinite, then $(\Omega', \leqslant)$ is a weak order with infinite levels of pairwise incomparable elements such that each maximal subchain of Ω' is isomorphic to $(C', \leqslant)$; that is, for each $c \in C'$ there are infinite sets M_c such that $(\Omega', \leqslant) \cong (W, \leqslant)$ where $W = \dot{\bigcup}_{c \in C'} (\{c\} \vec{\times} M_c)$ and for any $c, d \in C'$, $x \in M_c$, and $y \in M_d$ we have $(c, x) < (d, y)$ in $(W, \leqslant)$ if and only if $c < d$.

(3b) Assume $(\Omega, \leqslant)$ satisfies Condition (2.1)(3b). Then there are two dense unbounded chains $C_1, C_2 \subseteq \Omega'$ such that $\Omega' = \Omega_1' \dot{\cup} \Omega_2'$ with $\Omega_1' \parallel \Omega_2'$ and $\Omega_i' \cong C_i \vec{\times} r$ $(i = 1, 2)$.

(3c) If $(\Omega, \leqslant)$ satisfies Condition (2.1)(3c), there exist a dense unbounded chain $C' \subseteq \Omega'$ and an antichain $A' \subseteq \Omega'$ such that $\Omega' = \Omega^* \dot{\cup} A'$, $\Omega^* \parallel A'$, $\Omega^* \cong C' \vec{\times} r$, and $|A'| = |A|$.

Since any two dense unbounded chains are partially isomorphic, it is easy (and tedious) to check that in each of these cases, $(\Omega, \leqslant)$ and $(\Omega', \leqslant)$ are partially isomorphic by the system S of all isomorphisms φ: $A \to B$ where $A \subseteq \Omega_1$, $B \subseteq \Omega_2$ are finite subsets with $|A| = |B| \geqslant k + 2$.

Now let us assume that Ω is of Type (4a). By Proposition 4.2, $(\Omega, \leqslant)$ is a normal tree. Now Proposition 4.3 and Theorem 4.4(b) shows that $(\Omega, \leqslant)$ and $(\Omega', \leqslant)$ are partially isomorphic. If Ω is of Type (4b), then Ω with the inverse order is of Type (4a) and thus partially isomorphic to Ω' with the inverse ordering, which implies the result.

Now let $(\Omega, \leqslant)$ be an infinite p.o. set such that $A(\Omega)$ is k-transitive for some $k \geqslant 2$ and Ω does not contain the pentagon, that is, Ω is not of Type (4c) of Theorem 2.1. We wish to associate with each such p.o. set $(\Omega, \leqslant)$ a triple of cardinal invariants in $\mathbb{N} \cup \{0, \aleph_0\}$. For the Types (0)–(3), this is done in Table 1. In the left-hand column we assume that $(\Omega, \leqslant)$ is as described in the corresponding condition of Theorem 2.1. Then the right-hand column contains the associated triple of invariants.

TABLE 1

Condition satisfied by $(\Omega, \leqslant)$	Associated triple of invariants
(2.1)(0)	$(1, 0, 0)$
(2.1)(1)	$(1, \lvert A\rvert, \lvert C\rvert)$
(2.1)(2)	$(2, \min\{\lvert I\rvert, \aleph_0\}, 0)$
(2.1)(3a)	$(3, 1, \min\{r, \aleph_0\})$
(2.1)(3b)	$(3, 2, r)$
(2.1)(3c)	$(3, \lvert A\rvert + 2, r)$

Next we assume that Ω is of Type (4a), that is, Ω is a tree. If Ω is a meet-semilattice (not a meet-semilattice) with ramification order κ, the associated triple is $(4, 1, \min\{\kappa, \aleph_0\})$ $((4, 2, \min\{\kappa, \aleph_0\}))$, respectively. If Ω is of Type (4b), we proceed analogously for the set Ω with the inverse ordering, but letting the first number of the triple be 5.

Next we give the

Proof of Corollary 5. (1) $\Rightarrow$ (2). Proceeding as in the proof of Theorem 4, we obtain that $(\Omega_1, \leqslant)$ and $(\Omega_2, \leqslant)$ are even $L_{\infty\omega}$-equivalent.

(2)$\Rightarrow$(1). Since $A(\Omega_i)$ is k_i-transitive, it is easy to find sets of first order sentences characterizing which type and, moreover, which triple of cardinal invariants $(\Omega_i, \leqslant)$ has $(i = 1, 2)$. Since $(\Omega_1, \leqslant)$ and $(\Omega_2, \leqslant)$ are elementarily equivalent, it follows that they have the same triple of cardinal invariants.

Now assume one of the Conditions (1), (2) holds. If Ω_1, Ω_2 are of one of the types (0)–(3), the fact that $\mathrm{Trans}(\Omega_1) = \mathrm{Trans}(\Omega_2)$ and $\mathrm{Hom}(\Omega_1) = \mathrm{Hom}(\Omega_2)$ follows from [**12**, Theorems 3.2.7, 4.15, 4.16]. Hence we may assume now that Ω_1, Ω_2 are of Type (4a). Then for $i = 1, 2$, we have $k_i \leqslant 3$, $A(\Omega_i)$ is 1- and 2-transitive, and if $A(\Omega_i)$ is m-homogeneous, then $m \leqslant 2$, as noted at the beginning of § 3. If $A(\Omega_1)$ and $A(\Omega_2)$ are both 2-homogeneous, we see by Theorem 4.6 that $A(\Omega_1)$ is 3-transitive if and only if $A(\Omega_2)$ is too. If, however, $A(\Omega_1)$ and $A(\Omega_2)$ are not 2-homogeneous, then they are also both not 3-transitive by Theorem 4.6. In both of these cases, we therefore get $\mathrm{Trans}(\Omega_1) = \mathrm{Trans}(\Omega_2)$ and $\mathrm{Hom}(\Omega_1) = \mathrm{Hom}(\Omega_2)$. Now let us assume that $A(\Omega_1)$ is not 2-homogeneous, but $A(\Omega_2)$ is. Then Theorem 3.2 shows that $\mathrm{ram}(\Omega_i) = \Omega_i^+ \setminus \Omega_i$ and Ω_i has ramification order 2 for $i = 1$ and hence also for $i = 2$. Thus $A(\Omega_2)$ is 3-transitive by Theorem 4.6, and we are hence in the exceptional case.

Conversely, if Ω_1, Ω_2 or Ω_1, Ω_2 with the inverse orderings are trees, $A(\Omega_1)$ is 2-transitive but not 2-homogeneous, and $A(\Omega_2)$ is 3-transitive, then $A(\Omega_1)$ is not 3-transitive and $A(\Omega_2)$ is also 2-homogeneous, showing that $\mathrm{Trans}(\Omega_1) \neq \mathrm{Trans}(\Omega_2)$ and $\mathrm{Hom}(\Omega_1) \neq \mathrm{Hom}(\Omega_2)$. However, Theorems 3.2 and 4.6 show that $\mathrm{ram}(\Omega_i) = \Omega_i^+ \setminus \Omega_i$ and Ω_i has ramification order 2 $(i = 1, 2)$, whence Ω_1, Ω_2 have the same triple of cardinal invariants.

Concerning the existence of the exception mentioned in Corollary 5, we note

COROLLARY 4.8. *Let λ be an uncountable and μ any infinite cardinal, and $L = \{\leqslant\}$ a first order language for p.o. sets. Then there are two p.o. sets $(\Omega_1, \leqslant)$, $(\Omega_2, \leqslant)$ not containing the pentagon such that $A(\Omega_1)$, $A(\Omega_2)$ are both 2-transitive, $(\Omega_1, \leqslant)$ and $(\Omega_2, \leqslant)$ are $L_{\infty\omega}$-equivalent, $|\Omega_1| = \lambda$, $|\Omega_2| = \mu$, but* $\mathrm{Trans}(\Omega_1) \neq \mathrm{Trans}(\Omega_2)$ *and* $\mathrm{Hom}(\Omega_1) \neq \mathrm{Hom}(\Omega_2)$.

Proof. By Theorem 3, there exists a tree $(\Omega_1, \leqslant)$ of cardinality λ for which $A(\Omega_1)$ is 2-transitive but not 2-homogeneous. By Corollary 3.15 and Theorem 4.6 there is a tree $(\Omega_2, \leqslant)$ of cardinality μ for which $A(\Omega_2)$ is 3-transitive and hence also 2-transitive. Now the result follows from Corollary 5.

Next we give the

Proof of Corollary 6. The existence part of the corollary is immediate by Theorem 2.1. For the uniqueness part note that if $(\Omega_1, \leqslant)$, $(\Omega_2, \leqslant)$ are two countable p.o. sets not containing the pentagon such that $A(\Omega_i)$ is k_i-transitive for some $k_i \geqslant 2$ $(i = 1, 2)$, and if $(\Omega_1, \leqslant)$ and $(\Omega_2, \leqslant)$ have the same triple of cardinal invariants, then $(\Omega_1, \leqslant)$ and $(\Omega_2, \leqslant)$ are isomorphic by Corollary 5 and Theorem 4. The result follows.

Recall that if T is a theory and λ a cardinal, T is called λ-categorical if, up to isomorphism, T has exactly one model of cardinality λ. By a well-known theorem of Morley [**32**], if T is λ-categorical for some uncountable cardinal λ, then it is

λ-categorical for any uncountable cardinal λ. Therefore it suffices to study $\aleph_0$- and $\aleph_1$-categoricity. For p.o. sets $(\Omega, \leqslant)$ with k-transitive automorphism group ($k \geqslant 2$) we have the following characterization of $\aleph_1$-categoricity of $\mathrm{Th}(\Omega, \leqslant)$.

THEOREM 4.9. *Let $(\Omega, \leqslant)$ be an infinite p.o. set, and let $2 \leqslant k \in \mathbb{N}$. Then the following are equivalent:*

(1) *$A(\Omega)$ is k-transitive, and $\mathrm{Th}(\Omega, \leqslant)$ is $\aleph_1$-categorical;*

(2) *$\Omega = A \,\dot{\cup}\, B \,\dot{\cup}\, C$ with antichains A, B, C in Ω such that $A < B < C$ and $|A| + |B| \leqslant k-1$.*

Proof. (1)$\Rightarrow$(2). Since $\mathrm{Th}(\Omega, \leqslant)$ is $\aleph_1$-categorical, it is stable, and hence all chains in $(\Omega, \leqslant)$ are finite (cf., for example, [**9**, 7.1.4, 7.1.33]). By Theorems 2.1 and 2.2(a), $(\Omega, \leqslant)$ satisfies conditions (0) or (1) of Theorem 2.1.

(2)$\Rightarrow$(1). By Theorem 2.1, $A(\Omega)$ is k-transitive. It is straightforward to check that $\mathrm{Th}(\Omega, \leqslant)$ is $\aleph_1$-categorical.

In a study of tense logic and time, van Benthem [**38**] proposed to characterize all countable p.o. sets $(\Omega, \leqslant)$ such that $A(\Omega)$ is 1-transitive and the following property (*) (called 'indistinguishability') is satisfied:

(*) For every two finite sequences $(a_1, \ldots, a_n)$, $(b_1, \ldots, b_n)$ of elements of Ω ($n \in \mathbb{N}$) with the same type (i.e. verifying the same first order formulas), there exists $\alpha \in A(\Omega)$ such that $a_i^\alpha = b_i$ for all $i = 1, \ldots, n$.

Next we wish to show that the class of all such p.o. sets $(\Omega, \leqslant)$ is quite rich; it includes all trees with a 2-homogeneous automorphism group. We need the following result from [**12**].

PROPOSITION 4.10 [**12**, Corollary 5.32]. *Let $(\Omega, \leqslant)$ be a tree, $A(\Omega)$ 2-homogeneous, $A, B \subseteq \Omega$ finite subsets, and φ: $A \to B$ an isomorphism. Let $\bar{A}, \bar{B} \subseteq \Omega^+$ be the smallest (finite) meet-semisublattices of Ω^+ with $A \subseteq \bar{A}$, $B \subseteq \bar{B}$. Then φ extends to an automorphism of Ω if and only if φ extends to an isomorphism from $\bar{A}$ onto $\bar{B}$.*

We just remark that by this result and Theorem 3.2, if Ω is a tree and a meet-semilattice and $A(\Omega)$ is 2-transitive, then indeed any isomorphism between two finite meet-semisublattices of Ω extends to an automorphism of Ω. Now we show:

THEOREM 4.11. *Let $(\Omega, \leqslant)$ be a tree such that $A(\Omega)$ is 2-homogeneous. Then $(\Omega, \leqslant)$ satisfies Condition (*).*

Proof. Let $(a_1, \ldots, a_n)$, $(b_1, \ldots, b_n)$ be two n-tuples of elements of Ω satisfying the same first order formulas in Ω ($n \in \mathbb{N}$). Let $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$. Then the smallest meet-semisublattice of Ω^+ containing A (B) is $\bar{A} = \{\inf T\colon\ T \subseteq A, |T| \leqslant 2\}$ ($\bar{B} = \{\inf T\colon\ T \subseteq B, |T| \leqslant 2\}$), respectively. Clearly, the mapping φ: $A \to B$ with $a_i^\varphi = b_i$ ($i = 1, \ldots, n$) is an isomorphism. Next observe that for all $a \in \Omega$, $x \in \Omega^+$ we have $x \leqslant a$ if and only if $z \leqslant a$ for all $z \in \Omega$

with $z \leqslant x$. Assume $i, j, k \in \{1, \ldots, n\}$ are pairwise different. Since the formula

$$\psi(x_1, x_2, x_3) \equiv (\forall z)(z \leqslant \{x_1, x_2\} \Rightarrow z \leqslant x_3)$$

is satisfied by a_i, a_j, a_k in Ω if and only if it is satisfied by b_i, b_j, b_k, we obtain that

$$\inf\{a_i, a_j\} \leqslant a_k \quad \text{if and only if} \quad \inf\{b_i, b_j\} \leqslant b_k \quad \text{in } \Omega^+.$$

It follows that φ can be extended to an isomorphism from $\bar{A}$ onto $\bar{B}$ and therefore, by Proposition 4.10, also to an automorphism of Ω.

Using a result of [**12**], we finally give the

Proof of Corollary 7. By [**12**, Theorem 8.12(a)], there exists $m \in \mathbb{N}$ such that $A(\Omega_1)$, $A(\Omega_2)$ are both n-homogeneous for each $n \geqslant m$. It follows that $(\Omega_1, \leqslant)$ and $(\Omega_2, \leqslant)$ are partially isomorphic by the system S of all isomorphisms $\varphi: A \to B$ where $A \subseteq \Omega_1$, $B \subseteq \Omega_2$ are finite subsets with $|A| = |B| \geqslant m$. Now Proposition 4.1(a) implies the result.

Clearly, the assumption of Corollary 7 cannot be weakened to the extent that the automorphism groups $A(\Omega_1)$, $A(\Omega_2)$ are each k-transitive for *some* $k \geqslant 2$. For, as Theorem 4.4 and Corollary 4.7 show, there are up to $L_{\infty\omega}$-equivalence precisely countably many different trees $(\Omega, \leqslant)$ with 2-transitive automorphism groups $A(\Omega)$, and these trees clearly all embed the same finite p.o. sets.

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