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A GROUP OF PATHS IN $\mathbb{R}^2$

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Abstract. We define a group structure on the set of compact "minimal" paths in $\mathbb{R}^2$. We classify all finitely generated subgroups of this group G: they are free products of free abelian groups and surface groups. Moreover, each such group occurs in G.

The subgroups of G isomorphic to surface groups arise from certain topological 1-forms on the corresponding surfaces. We construct examples of such 1-forms for cohomology classes corresponding to certain eigenvectors for the action on cohomology of a pseudo-Anosov diffeomorphism.

Using G we construct a non-polygonal tiling problem in $\mathbb{R}^2$, that is, a finite set of tiles whose corresponding tilings are not equivalent to those of any set of polygonal tiles.

The group G has applications to combinatorial tiling problems of the type: given a set of tiles P and a region R, can R be tiled by translated copies of tiles in P?

1. Introduction

In [5], Conway and Lagarias introduced the beautiful notion of the "tiling group" of a set of polyominoes. Given a finite set T of polyominoes (i.e. polygons with edges parallel to the axes and edges of integral lengths), they define a finitely presented group H and for each polyomino R an element $b(R) \in H$ such that, if R is tilable by translates of the tiles in T, then $b(R)$ is the identity in H. Using this construction Thurston [20], and later Kenyon and Kenyon [15] were able for many simple sets of tiles T to give algorithms for determining the existence of a tiling of a given region R.

Our motivation for defining a group of paths in $\mathbb{R}^2$ in the present paper is to be able to generalize Conway and Lagarias' ideas to sets of tiles which are not polyominoes, or not even polygonal. For one successful result in this direction see [13].

Other motivation for studying this group of paths G comes from several sources. The first is the work of Dekking [6], who defines some "self-similar" curves in the plane by using free-group endomorphisms. (We'll use a construction related to his in section 4.3.) Secondly the work of the author on rigidity of tilings [14] led him to consider a generalization of interval exchanges which might be called "curve exchanges". These provided the first examples of interesting subgroups of G. We would in addition like to mention the article of Girault-Beauquier and Nivat [10]

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who also needed to consider group-theoretic properties of paths and in which a predecessor of Proposition 6 appears.

Another motivation for this work is that it provides one of the first applications of Rips' proof [9, 3] of Morgan and Shalen's conjecture [19] characterizing the finitely generated groups which act freely by isometries on $\mathbb{R}$-trees. Such groups are free products of free abelian groups and non-exceptional surface groups (fundamental groups of closed surfaces except for the three nonorientable surfaces of Euler characteristic ≥ -1).

Indeed we will prove

Theorem 1. *The finitely generated subgroups of G are free products of free abelian groups and non-exceptional surface groups.*

For any finitely generated subgroup H of G we construct an $\mathbb{R}$-tree on which H acts freely by isometries, and then apply Rips' theorem. In section 4.3 we construct examples of subgroups of G, giving a converse to Theorem 1:

Theorem 2. *There are subgroups of G isomorphic to $\mathbb{Z}^n$ and to the fundamental group of any closed surface of Euler characteristic ≤ -2. For any two finitely generated subgroups G_1, G_2 of G, there is a subgroup of G isomorphic to $G_1 * G_2$.*

Given a surface Σ with a singular foliation L (having "prong-type" singularities only), we define (following Fathi [7]) a *topological* 1*-form* transverse to L. For a foliation having two independent (i.e. non-cohomologous) topological 1-forms, at least one of which is "minimal" in a sense we describe, we construct a surface subgroup of G, and conversely we show that every surface subgroup of G occurs in this way:

Theorem 3. *If H is a subgroup of G, and H is isomorphic to the fundamental group of a surface Σ, then there is a singular foliation of Σ with two independent topological* 1*-forms giving rise to H.*

Our construction indicates to a certain extent how to construct foliations having more than one independent topological 1-form (each transversely-oriented measured foliation has at least one arising from its measure). In particular for the foliation arising from a pseudo-Anosov diffeomorphism there are at least as many non-cohomologous topological 1-forms as there are eigenvalues of modulus > 1 for the action on cohomology. The property of "minimality" of the resulting 1-forms seems at present harder to pin down.

The techniques of Keane [12] to construct non-uniquely ergodic measured foliations can be applied to make more general examples of topological 1-forms.

The paper is organized as follows. In section 2 we define the group G, prove that it is a group and indicate some of its properties. In section 3 we prove Theorem 1. In section 4 we discuss topological 1-forms and use them to construct surface subgroups of G. In section 5 we give an example of a non-polygonalizable set of tiles as an application of the group G. Finally, in section 6 we provide some generalizations of G and discuss their properties.

I would like to thank F. Paulin for introducing me to the theory of $\mathbb{R}$-trees and discussing many of the ideas in this paper.

2. The group of paths

2.1. **The group structure.** Let $\mathcal{P} = \{f\colon [0, a_f] \to \mathbb{R}^2 \mid f(0) = (0,0)\}$ be the set of continuous maps of a closed interval into the plane (where the domain $[0, a_f]$ depends on f) which begin at $(0,0)$.

Notationally, if $f \in \mathcal{P}$ has domain $[0,a]$ and $[b,c] \subset [0,a]$ then by $f([b,c])$ we will mean the path with domain $[0, c-b]$ defined by $t \mapsto f(b+t) - f(b)$. (This is just a subpath of f, translated to begin at the origin.) Also, a *prefix* of f is a path $f([0,b])$ for some $b \le a$, and a *suffix* of f is a path $f([c,a])$ for some $c \le a$.

We define a product on $\mathcal{P}$, represented by the symbol $*$, as follows: if f and g have domains $[0,a]$ and $[0,b]$ respectively, then $f * g$ has domain $[0, a+b]$ and

$$f * g(t) = \begin{cases} f(t), & 0 \le t \le a, \\ g(t-a) + f(a), & a \le t \le a+b. \end{cases}$$

Thus the product $f * g$ is just the concatenation of the two paths, with g translated to the end of f. We also define an involution $f \to f^{-1}$ on $\mathcal{P}$: if f has domain $[0,a]$, then so does f^{-1}, and

$$f^{-1}(t) = f(a-t) - f(a),$$

thus f^{-1} is f traced in the reverse direction. The unique path in $\mathcal{P}$ with domain $[0,0]$ is denoted e, the *constant path*.

A path f is said to be *minimal* if no subpath $f([x,y])$ is of the form $g * g^{-1}$ for some path $g \ne e$. Thus a minimal path has no "backtrackings". Let $G \subset \mathcal{P}$ be the set of minimal paths starting at $(0,0)$. We define a new product on G, denoted with a $\cdot$ or (usually) no symbol at all, as follows. Let $f, g \in G$ with domains $[0,a]$ and $[0,b]$ respectively. Let $h = f([c,a])$ be the largest suffix of f such that h^{-1} is a prefix of g, that is, $g([0, a-c]) = f([c,a])^{-1}$. Then define fg to be the path

$$fg(t) = f([0,c]) * g([a-c, b]).$$

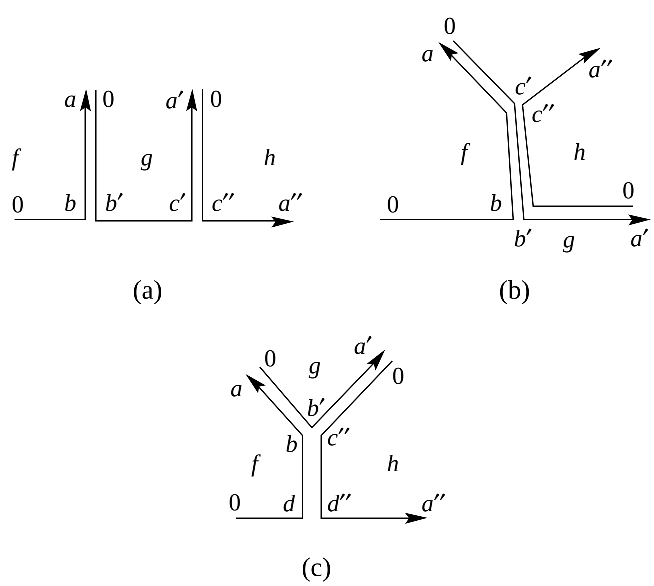

Figure 1. Associativity of the product

Thus fg is the minimal path obtained from the concatenation of f with g by cancellation of any common part. (It is important to distinguish between the two products $*$ and $\cdot$ in what follows.)

Theorem 4. *G is a group.*

Proof. There is an identity element e. It is trivially true that $fg \in G$ if $f, g \in G$ and $ff^{-1} = f^{-1}f = e$. It remains to show that the product is associative. Let $f, g, h \in G$ with domains $[0,a], [0,a'], [0,a'']$ respectively. Let $[b,a]$ and $[0,b']$ denote the largest common portions of f and g (i.e. $f([b,a]) = g([0,b'])^{-1}$), and $[c',a']$ and $[0,c'']$ be the largest common portions of g and h. There are three cases to consider (see Figure 1). If $c' > b'$, then (Figure 1(a) $(fg)h = f(gh) = f([0,b]) * g([b',c']) * h([c'',a''])$. If $c' < b'$ then (Figure 1(b) $(fg)h = f(gh) = f([0,b]) * g([c',b'])^{-1} * h([c'',a''])$. Finally if $c' = b'$ let $[d,b]$ and $[c'',d'']$ be the common portions of $f([0,b])$ and $h([c'',a''])$; then (Figure 1(c) $(fg)h = f(gh) = f([0,d]) * h([d'',a''])$. •

3. Finitely generated subgroups of G

An $\mathbb{R}$-*tree* is a metric space in which any two points have a unique arc joining them and this arc is isometric to an interval.

By a *surface group* we will mean the fundamental group of a closed surface, *except* for the three non-orientable surfaces of Euler characteristic ≥ -1. (These are called exceptional surfaces by Morgan and Shalen [18]; along with the sphere, they are the surfaces which do not have pseudo-Anosov diffeomorphisms.)

Proof of Theorem 1. By a theorem of Rips [9, 3], a group which acts freely by isometries on an $\mathbb{R}$-tree is a free product of free abelian groups and surface groups.

Let H be a finitely generated subgroup of G, with generating paths $\{h_i\}_{i=1,\dots,k}$ with h_i parametrized by the interval $[0,s_i]$. For convenience in this proof we include the identity e as a generating path. We will construct an $\mathbb{R}$-tree on which H acts freely by isometries.

For each integer $m \geq 0$, let T_m be the space constructed as follows. Take a copy of the domain of the element $h_{i_1}^{\epsilon_1} h_{i_2}^{\epsilon_2} \dots h_{i_m}^{\epsilon_m}$ (where the $\epsilon_i = \pm 1$), for each word of length m in the generators of H. Glue these domains together by gluing every pair of domains along the largest common prefix of their associated paths (these prefixes may be reduced to the single point 0). Let T_m be the resulting space.

These gluings are isometric gluings of closed intervals along closed subintervals, each containing 0. So T_m is a tree. It is Hausdorff since the intervals along which the gluings are performed are closed, by continuity of the h_i. The tree has a well-defined metric since the gluings are isometric, making it an $\mathbb{R}$-tree.

For each m, we claim that $T_m \subset T_{m+1}$. The domains of paths of length $m+1$ contain the domains of paths of length m since e was included in the h_i. Also, two points which are not identified in T_m will not be identified in T_{m+1} either, since identifications are only along prefixes. This completes the claim. Furthermore the inclusion $T_m \subset T_{m+1}$ is isometric.

Define $T = \bigcup_{m \geq 0} T_m$. Then T is a tree: any loop in T would arise from a loop in T_m from some m. Furthermore, since the inclusions $T_m \to T_{m+1}$ are isometric, there is an induced metric on T making T an $\mathbb{R}$-tree.

Denote the origin $e \in T$ by simply e again. For $x \in T$ we let $\langle e, x \rangle$ denote the arc in T from e to x.

Let $\pi\colon T \to \mathbb{R}^2$ be the map sending $x \in T$ to the endpoint of the path in G whose domain is $\langle e, x\rangle$. This is a path in H, multiplied by a prefix of some h_i.

Lemma 5. *H acts freely on T.*

Proof. For $x \in T$, let $\gamma_x \in G$ denote the path whose domain is $\langle e, x\rangle$.

For $\alpha \in H$, α acts on T by sending $x \in T$ to $\alpha(x)$, the endpoint of the domain of the path $\alpha\gamma_x$. Since $\langle e, x\rangle$ is the domain of a path in H, followed by a prefix of some h_i, when we premultiply by α we again have a domain of the same form. It follows that $\alpha(x) \in T$.

If $\alpha, \beta \in H$, then $(\alpha\beta)(x)$ is the endpoint of the domain of $(\alpha\beta)\gamma_x = \alpha(\beta\gamma_x))$ by associativity of the product in G, which in turn equals the endpoint of the domain of $\alpha\gamma_{\beta(x)}$. Thus $\alpha\beta(x) = \alpha(\beta(x))$, and the action is well-defined. It is also clearly isometric.

Now H acts freely on T if no element but the identity fixes any point of T. This follows from the fact that $\alpha\gamma_x \neq \gamma_x$ in G unless $\alpha = e$. •

So H acts freely by isometries on the $\mathbb{R}$-tree T. By the theorem of Rips, H is a free product of free abelian groups and surface groups. This completes the proof of Theorem 1. •

4. Surface subgroups of G

Proof of Theorem 2. To construct a subgroup of G isomorphic to $\mathbb{Z}^n$, take as generators the paths $h_i(t) = (t, 0)$ with domains $[0, a_i]$, where the a_i are rationally independent. Any two generators commute, and rational independence guarantees that the map $\mathbb{Z}^n \to G$ is injective. Thus this is a subgroup isomorphic to $\mathbb{Z}^n$.

For the construction of a surface group, see section 4.3. To construct a subgroup of G isomorphic to the free product of two finitely generated subgroups G_1, G_2 of G, let $\{g_i\}$ be a finite set of generators for G_1 and $\{g_i'\}$ be likewise for G_2. Define α_s for $s \in [0, 2\pi)$ to be the rotation $\alpha_s(z) = e^{is}z$. Note that for a fixed s, $\{\alpha_s g_i'\}$ generates a subgroup isomorphic to G_2 (in fact α_s is an automorphism of G).

Consider the subgroup H_s of G generated by $\{g_i\}\cup\{\alpha_s g_i'\}$. There are a countable number of possible relations between the $\{g_i\}$ and the $\{\alpha_s g_i'\}$, and we claim that each such relation can occur for at most a finite set of values of s. Thus there is some value of $s \in [0, 2\pi)$ for which no relation occurs, i.e. H_s is the free product $G_1 * G_2$.

To prove the claim, let $u_1v_1 \dots u_kv_k = e$ be a relation, with for all i, $u_i \in G_1$ and $v_i \in \alpha_s G_2$ (and $u_i, v_i \neq e$). Suppose for some s' close to s the same relation with the v_i replaced with $\alpha_{s'-s}v_i$ holds.

For some i the path $u_i * v_i$ or $v_i * u_{i+1}$ is not minimal in order for the product $u_1v_1 \dots u_kv_k$ to be the identity. If u_i*v_i is minimal then for ϵ small the path $u_i*\alpha_\epsilon v_i$ will still be minimal. So there exists some $u_i * v_i$ (say) which is not minimal, and for which $u_i * \alpha_\epsilon v_i$ is also not minimal. This is impossible, since in this path points of $\alpha_\epsilon v_i$ are rotated around the endpoint of u_i. •

4.1. **Abelian subgroups.**

Proposition 6. *The paths $a, b \in G$ commute iff either there is a $u \in G$ such that $a = u^k$, $b = u^l$ for some integers k, l, or there is an s such that $a = sa's^{-1}$, $b = sb's^{-1}$ where a' and b' are parallel line segments parametrized proportionally to arc length.*

Proof. Assume paths a and b commute in G and are not equal to the identity (if a or b is the identity the lemma is true with k or $l = 0$).

Suppose first that none of the paths $a * b, b * a^{-1}, a^{-1} * b^{-1}, b^{-1} * a$ are minimal. Then since the path $a * b$ is not minimal, there is a maximal path c_1 such that $a = a_1 * c_1^{-1}$ and $b = c_1 * b_1$. Similarly since $b * a^{-1}, a^{-1} * b^{-1}$, and $b^{-1} * a$ are not minimal, there are maximal paths c_2, c_3, c_4 respectively such that $b = b_2 * c_2^{-1}$ and $a = a_2 * c_2^{-1}$, $a = c_3 * a_3$ and $b = b_3 * c_3^{-1}$, $a = c_4 * a_4$ and $b = c_4 * b_4$.

Since c_4 and c_3 are both prefixes of the minimal path a, one is a prefix of the other. Since c_3^{-1} and c_2^{-1} are both suffixes of the minimal path b, one is the suffix of the other, that is, one of c_2 and c_3 is a prefix of the other. Since c_1^{-1} and c_2^{-1} are suffixes of a, one of c_1 and c_2 is a prefix of the other. Define c to be the largest common prefix of c_1, c_2, c_3, c_4.

So a and b are of the form $a = c * A * c^{-1},\ b = c * B * c^{-1}$ for some minimal paths A, B. At least one of the four paths $A * B, B * A^{-1}, A^{-1} * B^{-1}, B^{-1} * A$ will be minimal by maximality of c. Let us assume without loss of generality (up to exchanging a and b or taking inverses) that it is $A * B$. Then $B * A$ is also minimal, since $AB = BA$ and the domain of BA has the same length as the domain of AB.

Thus $A * B$ and $B * A$ are the same minimal path. Let $[0, s]$ and $[0, t]$ be the domains of A and B, respectively. For a point x in the image of A, $x + B(t)$ is in the image of $AB = BA$. Likewise, for a point x in the image of the suffix B of the path AB, the point $x - A(s)$ is in the image of AB. Thus for a point x in the image of AB we can repeatedly add either $B(t)$ or $-A(s)$ depending on where the image of x falls, and still have the new point in the image of AB. If $B(t)$ and $-A(s)$ as vectors in $\mathbb{R}^2$ do not point in opposite directions, the orbit of a point x under this mapping tends to infinity, a contradiction since the image of AB is bounded. Let ℓ denote the line joining 0 and $A(s)$.

If $s = t$ then we're done since $A = B$ in that case. Assume without loss of generality that $s < t$; then A is a prefix of B. Write $B = A * B'$; then $AB = AAB'$ and $BA = AB'A$; so $AB' = B'A$. Let $A' = A$. The domains of A', B' are $[0, s]$ and $[0, t - s]$ respectively, and $A' * B' = B' * A'$, both paths being minimal. The endpoints of A', B' are $A(s), B(t) - A(s)$ respectively.

Repeat this argument starting with A' and B' (either A' is a prefix of B', or vice versa; define $A^{(2)}, B^{(2)}$, and so on). If the domains of some pair $A^{(n)}, B^{(n)}$ have equal length, working backwards we see that this implies $A \sim u^k, B \sim u^m$ for some path u and integers k, m. If not, the domains of $A^{(n)}$ and $B^{(n)}$ decrease to 0, so they converge to the constant path. Furthermore the endpoints of $A^{(n)}$ and $B^{(n)}$ lie on ℓ. Since A and B are products of $A^{(n)}$ and $B^{(n)}$ for each n, A and B are segments. Since reversing the domains doesn't change the parametrization, the derivative of the parametrization gives a measure on the circle $\mathbb{R}/(s+t)\mathbb{Z}$ invariant under an irrational rotation $x \mapsto x+s$. The only such measure is Lebesgue measure, implying that the parametrization is a multiple of arclength. •

4.2. **Topological 1-forms.** Let Σ be a surface with a (singular) foliation L. Let $H = \pi_1(\Sigma)$ be the fundamental group.

An $\mathbb{R}$-valued *topological* 1-*form* (see Fathi [7]) is an Alexander 1-cocycle ω which assigns to each path α in Σ a real number $\omega(\alpha)$ (which depends only on the homology class of α between its endpoints), which is invariant under holonomy along leaves of L and continuous as a function of the endpoints of α.

Equivalently, we can define a topological 1-form ω as follows. Let $\mathcal{T}$ be a continuous map from the universal cover $\tilde{\Sigma}$ of Σ to $\mathbb{R}$, which is constant on leaves of the lift of L, and satisfies: for $\gamma \in H$ and $x \in \tilde{\Sigma}$,

$$\mathcal{T}(\gamma x) = \mathcal{T}(x) + \rho(\gamma) \tag{1}$$

for some homomorphism $\rho\colon H \to \mathbb{R}$. For a path α_{xy} on Σ from x to y , define $\omega(\alpha_{xy}) = \mathcal{T}(y') - \mathcal{T}(x')$ where x', y' are the endpoints of some lift of α_{xy} to $\tilde{\Sigma}$. By (1) this is independent of the lift.

An Alexander 1-cocycle defines an element of Alexander cohomology, which in this case agrees with singular cohomology (with real coefficients). In the present case the cohomology element of ω is defined by ρ in the above formula (1). From a topological 1-form ω, $\mathcal{T}_\omega$ satisfying (1) can be defined up to an additive constant by integration.

Topological 1-forms on measured foliations are also special cases of the "transverse Hölder distributions" of Bonahon [4]. In fact they are precisely the *continuous* transverse Hölder distributions. In particular the topological 1-forms on a given foliation, like transverse Hölder distributions, form a finite-dimensional vector space [4].

One example of a topological 1-form is a differential 1-form on a surface. This is a topological 1-form with respect to its associated foliation (the foliation on which the form vanishes). For a differential 1-form ω, the value $\omega(\alpha)$ for a path α on the surface is just $\int_\alpha \omega$.

In particular given a measured foliation on Σ, with a transverse orientation for the leaves, then there is a natural 1-form associated to the measure: the value of this form on a short transversal is the measure of the transversal if the orientation of the transversal agrees with the transverse orientation, and the negative of this if they disagree.

4.3. **Construction of surface subgroups of G.** Let Σ be a surface of Euler characteristic $\chi \leq -2$. Let $\ell = 2 - \chi$ be the minimal number of generators for the fundamental group. Let k be the rank of the integer 1-homology, $k = \ell$ if the surface is orientable, $k = \ell - 1$ if not. Let ψ be a pseudo-Anosov diffeomorphism of Σ whose associated stable foliation has a transverse orientation (i.e. the unstable foliation has oriented leaves). Suppose that ψ^*, the action of ψ on the free part of $H^1(\Sigma, \mathbb{Z})$, is irreducible (has irreducible characteristic polynomial) and has at least two eigenvalues λ_1, λ_2 of modulus larger than 1.

Lemma 7. *Such a ψ exists.*

For the proof, see the appendix.

Since the stable foliation is transversely oriented we can suppose λ_1 is the growth rate of ψ, that is, ψ multiplies the transverse measure by λ_1 (see e.g. [8]).

Let $\eta_\lambda \in H^1(\Sigma, \mathbb{R})$ be the eigenvector for ψ^* corresponding to an eigenvalue λ satisfying $|\lambda| > 1$ (we assume $\lambda \in \mathbb{R}$: the complex case is similar but will give a $\mathbb{C}$-valued topological 1-form, or two $\mathbb{R}$-valued forms by taking the real and imaginary parts). We construct a topological 1-form ω_λ on Σ transverse to the stable foliation, whose cohomology class is $[\omega_\lambda] = \eta_\lambda$.

Let $x, y \in \Sigma$ and α be a path from x to y. For each n complete the path $\alpha_n = \psi^n(\alpha)$ to a closed path α'_n by adding a path of bounded length between the endpoints $\psi^n(x)$ and $\psi^n(y)$. Then α'_n defines a homology element $[\alpha'_n]$. Let

r_n be the real number (or complex number if λ is nonreal) $r_n = \lambda^{-n}\eta_\lambda[\alpha'_n]$, i.e. the component of $[\alpha'_n]$ in the eigendirection for λ scaled by λ^{-n}. By definition, $\eta_\lambda([\psi\alpha'_n]-\lambda[\alpha'_n])=0$, and $|\eta_\lambda([\alpha'_{n+1}]-[\psi\alpha'_n])|\leq C$ for some constant C determined by the choices of closing path. So

$$|\lambda^{-n-1}\eta_\lambda([\alpha'_{n+1}]-\lambda[\alpha'_n])|\leq C|\lambda^{-n-1}|,$$

and since $|\lambda|>1$, this implies that r_n converges. Define $\omega_\lambda(\alpha)=\lim_{n\to\infty} r_n$.

Now $\omega_\lambda(\alpha)$ is easily seen to be a continuous function of the endpoints x and y, and is invariant under holonomy along leaves of the stable foliation (the leaves of the stable foliation collapse under ψ^n). For a closed path γ we have

$$\omega_\lambda(\gamma)=\lim_{n\to\infty}\lambda^{-n}\eta_\lambda([\psi^n\gamma])=\lim_{n\to\infty}\lambda^{-n}(\lambda^n\eta_\lambda([\gamma]))=\eta_\lambda([\gamma]).$$

So ω_λ defines a topological 1-form whose cohomology is η_λ.

Let ω_{λ_1} and ω_{λ_2} be the topological 1-forms constructed in this way for eigenvalues λ_1,λ_2. The form ω_{λ_1} constructed from λ_1, the growth rate of ψ, is the same as the 1-form arising from the transverse measure.

Let $\{a_1,\dots,a_\ell\}$ be a set of loops on Σ *transverse* to the stable foliation, and generating the fundamental group. Define paths β_i in $\mathbb{R}^2$ by

$$\beta_i(t)=(\omega_{\lambda_1}(a_i[0,t]),\omega_{\lambda_2}(a_i[0,t])),$$

where $a_i[0,t]$ is the initial segment of a_i of length t with respect to the invariant transverse measure. Each β_i is parametrized by $[0,|a_i|]$, where $|a_i|$ is the length of a_i in the transverse measure. Furthermore since the paths a_i were chosen transversely to the foliation, the first coordinates of β_i are monotone functions. Thus the β_i are *minimal* paths, that is, elements of G.

Let $\mathcal{H}$ be the subgroup of G generated by $\{\beta_1,\dots,\beta_\ell\}$. We will show that $\mathcal{H}$ is isomorphic to H.

Let D be a 2ℓ-gon in the universal cover $\tilde{\Sigma}$, which is a fundamental domain for Σ, so that it spans the relator $R=a_{i_1}a_{i_2}\dots a_{i_{2\ell}}$ defining H. (Here the a_i's are the loops which we used to define the β_i, and the jth edge of ∂D is the domain of a_{i_j}.)

Lift the stable foliation to D; by our choice of a_i, it is transverse to the boundary except possibly at the vertices of D. The map $(\mathcal{T}_{\lambda_1},\mathcal{T}_{\lambda_2})$ from D to $\mathbb{R}^2$ maps the boundary of D to the path $\beta_{i_1}*\beta_{i_2}*\dots*\beta_{i_{2\ell}}$. This map is constant on the leaves of the stable foliation, and in fact the leaves describe precisely the cancellations in going from $\beta_{i_1}*\beta_{i_2}*\dots*\beta_{i_{2\ell}}$ to the path $\beta_{i_1}\beta_{i_2}\dots\beta_{i_{2\ell}}=e$ (this is described in section 4.5). So the paths β_i considered as elements of G satisfy the relation defining the fundamental group of Σ.

Theorem 8. *The paths $\{\beta_i\}$ generate a subgroup $\mathcal{H}$ of G isomorphic to H.*

Proof. We just showed that $\mathcal{H}$ is a quotient of H. We just need to show that there are no more relations.

The endpoint map is a surjective homomorphism of $\mathcal{H}$ to $\mathbb{Z}^k$: the x-coordinates of the endpoints of the β_i satisfy no rational relation in the orientable case and exactly one relation in the nonorientable case since we assumed ψ^* was irreducible (the endpoints are components of eigenvectors for λ_1,λ_2 of ψ^*).

Denote by the same symbol ψ the automorphism of $\mathcal{H}$ induced by ψ on H. Since $\mathcal{H}$ is a finitely generated subgroup of G, it acts freely on an $\mathbb{R}$-tree, and so it is a free product of free abelian groups and surface groups. Consider the three possibilities: $\mathcal{H}$ is free, $\mathcal{H}$ is a non-trivial free product, $\mathcal{H}$ is $\mathbb{Z}^k$. We will rule out each of these.

By Lemma 10 below using $k = -\chi + 2$ or $-\chi + 1$, and the fact that the rank of the abelianization of $\mathcal{H}$ is k we see that $\mathcal{H}$ is not free.

Lemma 9 below shows that if $\mathcal{H}$ were a non-trivial free product not equal to the free group, then ψ^* would not be irreducible (the subspaces of $\mathbb{Z}^k$ corresponding to the factors in the free product representing $\mathcal{H}$ would be invariant under ψ^*).

By Proposition 6, we know $\mathcal{H}$ is not abelian since the eigenvectors for λ_1 and λ_2 are not proportional.

So $\mathcal{H}$ must be isomorphic to H. •

Lemma 9. *Let G be a free product of surface groups and free abelian groups, and ψ an automorphism of G. If ψ^*, the action of ψ on the free part of the abelianization of G, is irreducible, then G either consists of a single free factor, or is a free group.*

Proof. This follows from the "Kurosh subgroup theorem" [17] which states that every subgroup H of a free product $G_1 * G_2 * \cdots * G_k$ is itself a free product of a free group and the intersection of H with conjugates of the G_i.

For a free factor $G_i \ncong \mathbb{Z}$ of G, the subgroup $\psi(G_i) \cong G_i$ cannot be written as a non-trivial free product and so is a (conjugate of a) subgroup of some G_j; by taking G_i of minimal rank among the nonabelian free factors, or of maximal rank among the abelian free factors, this shows that G_i must be a conjugate of one of the G_j (and $G_j \cong G_i$).

Hence if one of the G_i is not $\mathbb{Z}$ the map ψ^* is reducible, ψ^* permutes the images of free factors isomorphic to G_i in the abelianization. •

Lemma 10. *The fundamental group G of a closed surface of Euler characteristic χ does not have a free quotient group of rank larger than $-\chi/2 + 1$.*

Note. The largest rank of a free quotient of a group G is called the *first nonabelian Betti number* of G, by Arnoux and Levitt [2]. Also, it is easy to see that the bound above is sharp.

Proof. Let Σ be a surface with fundamental group G. A homomorphism $G \to F_m$ can be realized by a continuous map $f \colon \Sigma \to \wedge_m S^1$, the wedge of m circles (since a wedge of m circles is a $K(F_m, 1)$). Pick a point c_i on each circle in the wedge, which is not the base point. By a theorem of Levitt [11] one can arrange the map f so that the preimage of the complement of this set of points is connected. Furthermore we can arrange so that the map f is transverse at each c_i.

Then the inverse images of the c_i are unions of closed loops on the surface Σ (*two-sided* loops if Σ is nonorientable; a perturbation of a one-sided loop is two-sided). For each i pick one simple closed loop in the preimage of c_i. These loops are then pairwise disjoint and the complement of the set is connected. Such a set of loops has cardinality at most $-\chi/2 + 1$. This can be proved by induction: let χ denote the Euler characteristic and b the number of boundary components of a surface; cutting along a two-sided loop increases $b + \chi$ by 2; but for a connected oriented surface $b + \chi \leq 2$, and for a connected non-orientable surface $b + \chi \leq 1$. •

4.4. **Examples.** Since non-uniquely ergodic (transversely oriented) foliations (see [12]) have at least two independent transverse measures, they can be used to construct surface subgroups of G. In this case the values $\omega_1(\alpha[0, t]), \omega_2(\alpha[0, t])$ are both monotone functions of t and so the resulting paths are minimal and even *rectifiable.*

Proposition 11. *There are surface subgroups of G consisting of rectifiable paths: the generating paths $f_i\colon [0, s_i] \to \mathbb{R}^2$ are monotone and Lipschitz.*

Many laminations on surfaces have several topological 1-forms, as the above examples show. The question arises as to how many exist on a given lamination, and what is the maximum number that a lamination can have. Using Lemma 6 one can show that a foliation of a torus with lines of irrational slope has only one topological 1-form. Similarly, the lift of this foliation to a branched cover of the torus gives a foliation on a higher-genus surface with only one topological 1-form. We suspect that there are foliations with no topological 1-forms at all.

In regards to the upper bound, we make the following conjecture.

Conjecture 1. *The number of independent topological 1-forms on a given lamination of a surface is at most $-\chi/2 + 1$, where χ is the Euler characteristic of the surface.*

Regarding minimality, it is not necessary in the construction of the previous section that the leaves of the unstable foliation be transversely oriented; we just used that to prove that the resulting paths were minimal. In some cases one can prove minimality by different means. In the following example, one sees "just by looking" that the paths are minimal.

Let H be the group $\langle a, b, c, d \mid abcd = bdac\rangle$, the fundamental group of the orientable surface of genus 2. The automorphism we consider is a product of four Dehn twists along annuli freely homotopic to curves bd, ac, a and d. For these generators, the twist around d is the automorphism tw_d which fixes all generators but c, with $\mathrm{tw}_d(c) = cd^{-1}$. Similarly, $\mathrm{tw}_a(b) = a^{-1}b$ and fixes the other generators, $\mathrm{tw}_{ac}(d) = dac$ and fixes the other generators, and and $\mathrm{tw}_{bd}(a) = bda$, fixing the other generators. The composition ψ has the effect

$$\psi = \mathrm{tw}_d\mathrm{tw}_a\mathrm{tw}_{ac}\mathrm{tw}_{bd}\colon \{a, b, c, d\} \to \{a^{-1}bdacd^{-1}a, a^{-1}b, cd^{-1}, dacd^{-1}\}.$$

The abelianization of H is $\mathbb{Z}^4$, and the linear map ψ^* induced on $\mathbb{Z}^4$ is

$$\psi^* = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

which is irreducible (characteristic polynomial $z^4 - 3z^3 + 5z^2 - 3z + 1$). The largest eigenvalues in modulus are $\lambda = 1.12174 \pm 1.30662i$. The paths α_i generated by the above construction are shown in Figure 2; the minimality in this case is clear from the figure (since the paths are self-similar with similarity constant λ, if there were any back-tracks they would be macroscopic). Figure 3 shows the path $\alpha_1 * \alpha_2 * \alpha_3 * \alpha_4$, Figure 4 shows $\alpha_2 * \alpha_4 * \alpha_1 * \alpha_3$, and Figure 5 shows the path $\alpha_1\alpha_2\alpha_3\alpha_4 = \alpha_2\alpha_4\alpha_1\alpha_3$. In Figure 6 another example is shown, in which the paths are increasing in the x-direction.

4.5. **From surface subgroups to topological 1-forms.** Let H be a subgroup of G isomorphic to the fundamental group of a closed surface Σ of Euler characteristic ≤ -2. We show how to associate to H a singular foliation of Σ with two topological 1-forms transverse to it.

By changing generators if necessary, we can suppose that H is generated by some number k of paths $\{a_1, \dots, a_k\}$, with a single relation $R = a_{i_1} \dots a_{i_{2k}}$. Let D be a $2k$-gon; consider its boundary ∂D to be the domain of the path $a_{i_1} * a_{i_2} * \dots * a_{i_{2k}}$.

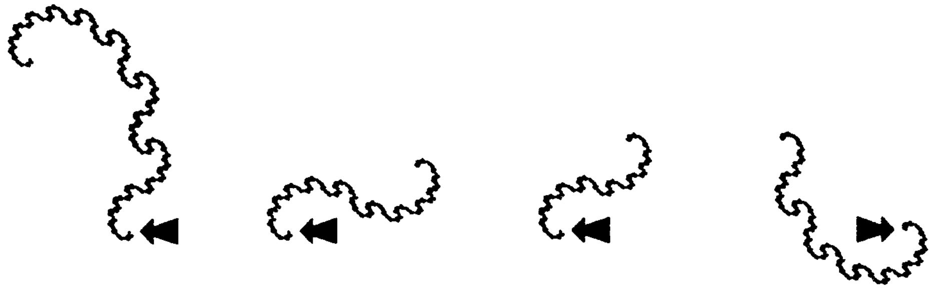

FIGURE 2. The paths $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, the arrows showing the origin of each path

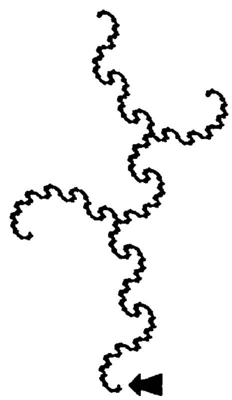

FIGURE 3. The concatenation $\alpha_1 * \alpha_2 * \alpha_3 * \alpha_4$

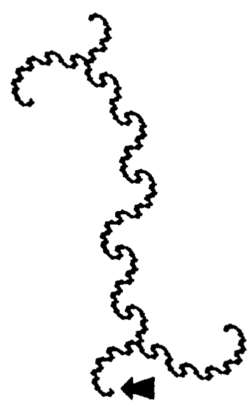

FIGURE 4. The concatenation $\alpha_2 * \alpha_4 * \alpha_1 * \alpha_3$

FIGURE 5. The concatenation $\alpha_1\alpha_2\alpha_3\alpha_4 = \alpha_2\alpha_4\alpha_1\alpha_3$

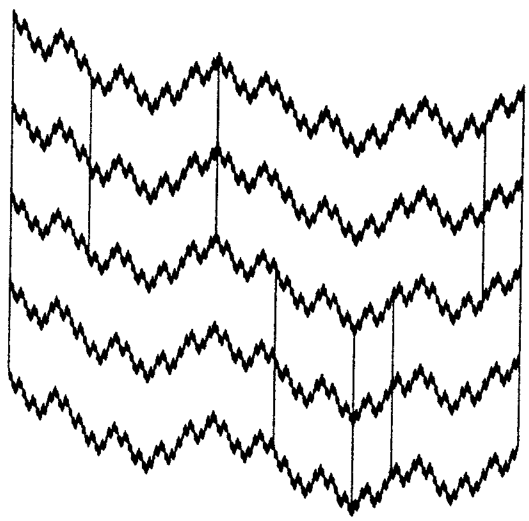

FIGURE 6. Part of the tiling contained in a strip

For each cancellation in going from $a_{i_1} * a_{i_2} * \ldots * a_{i_{2k}}$ to $a_{i_1}a_{i_2}\ldots a_{i_{2k}} = e$, put in a partial foliation of D with leaves joining corresponding points in the two halves of the cancellation. That is, if for example a suffix $a_{i_1}([s,t_1])$ cancels a prefix $a_{i_2}([0,t_1-s])$ of a_{i_2}, then put in chords in D for each $r \in [0, t_1-s]$ joining $s+r$ in the domain of a_{i_1} to t_1-s-r in the domain of a_{i_2}.

The union of these partial foliations gives a lamination L of D, that is, a closed collection of non-crossing arcs connecting boundary points of D. (There is an associated singular foliation obtained by collapsing the connected components of $D-L$ to singular leaves). There is a continuous function $\mathcal{T}\colon D \to \mathbb{R}^2$, which is constant on leaves of L and on connected components of $D-L$, defined on an arc or complementary region to be the common value at its boundary.

Define a surface Σ by gluing together the boundary of D in the standard way, gluing sides labeled a_i to each other isometrically and so the orientations agree.

This gluing results in a surface Σ whose fundamental group is H. The lamination L descends to a lamination on Σ which we also denote L.

The two coordinates of the map $\mathcal{T}$ from D to $\mathbb{R}^2$ define topological 1-forms ω_x, ω_y on Σ transverse of L. The x, y coordinates of the end-point maps of the paths a_i are the periods of ω_x, ω_y respectively. (They define the coordinates of ω_x, ω_y in cohomology.)

5. A non-polygonal tiling problem

A *tiling problem* is a finite set of tiles B (closed topological disks), such that one can tile the plane by using translates of copies of elements of B. Two tiling problems B, B' are said to be *Escher equivalent* [14] if there is a bijection $B \to B'$ which is obtained by altering the boundaries of the tiles B in a way which is compatible with the possible local arrangements of tiles in any tiling. (For a rigorous definition, see [14]. In particular, there is a choice of basepoints for the tiles such that for a tiling of the plane with translations V of tiles B, there is a tiling with the *same* set of translations V and tiles B'). The question which is answered incorrectly in [14] is, is every tiling problem Escher-equivalent to one in which the tiles are polygonal? We give here a counterexample.

Example. Figure 6 shows part of a tiling with four tiles a, b, c and d, which are approximate rectangles; each tile has congruent upper and lower boundaries, which are translations of one of the four generators of a genus-2 surface subgroup of G, with relation $abcd = bdac$. (This example comes from a pseudo-Anosov diffeomorphism of the surface Σ with oriented foliation; each generating path projects one-to-one to the x-axis.) The vertical edges of each tile are segments of length 1.

The tiling is meant to continue upward and downward in an infinite strip as indicated, which the upper tiles in the pattern *bdac* and the lower tiles in pattern *abcd*; the plane is then tiled by copies of this strip, adjacent strips being shifted vertically by small amounts with respect to each other.

A set of polygonal tiles B' equivalent to B must have the same set of identifications and so in particular the left and right boundaries of the tiles are straight line segments. (If two adjacent strips are displaced by arbitrary small amounts vertically with respect to each other then their common boundary must be a line). Furthermore, the paths along the upper boundaries of the tiles in B' must form the integrals of another topological 1-form for the associated surface with the same lamination. However such a 1-form can not give rise to a polygonal path, unless its integral is linear (since the associated lamination is minimal; the vertices of the polygonal path in this case would have to be dense), and if the path were linear, a straight boundary would allow other tilings for B' not occurring in B. Thus the tiles B are not polygonalizable.

6. Generalizations of G

6.1. **Non-minimal paths.** As we saw in our construction, the restriction of using only minimal paths in G can be a bit awkward. A possible alternative is to allow all continuous paths in G, but define two paths to be equivalent if they can be obtained from each other by removing or adding backtracks. There are two possibilities; take the equivalence generated by removing a single backtrack, or take the equivalence of removing all (even an infinite number of) backtracks.

In the first case, for a path such as that shown in Figure 7 there is no minimal path in its equivalence class. Any prefix of this path is equivalent to a segment, but the path itself is not equivalent to a segment. This causes problems in the classification proof (Theorem 1) since the resulting tree T is not Hausdorff.

In the second case, the backtracks might carry all the measure, as in the path in Figure 8. So it is not clear how to parametrize the resulting minimal path.

For these two groups the question of classifying the finitely-generated subgroups is open. For a third alternative definition, though, we can say something. Define a group G_2 by allowing all paths, with a weaker version of cancelling backtracks which is independent of parametrization, as follows.

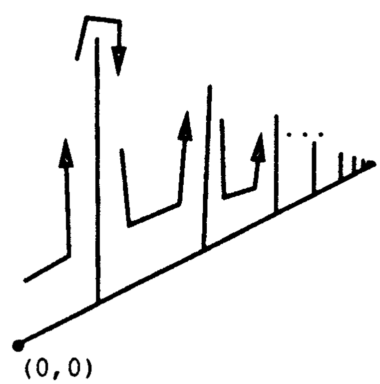

FIGURE 7. A badly non-minimal path

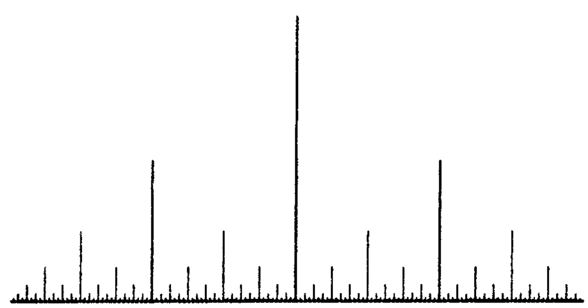

FIGURE 8. The backtracks may carry all the measure

Take all paths parametrized by $[0, 1]$ and starting at $(0, 0)$, with the following equivalence denoted $\sim_2$: $f \sim_2 g$ if there is a map $S\colon D \to \mathbb{R}^2$ satisfying

1. $S(e^{\pi it}) = f(t)$ for $0 \leq t \leq 1$
2. $S(e^{-\pi it}) = g(t)$ for $0 \leq t \leq 1$
3. D has a lamination L by chords such that S is constant on leaves and complementary regions of L.

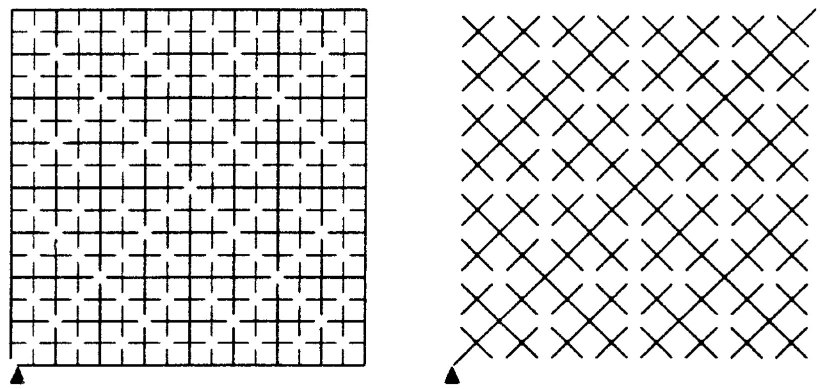

FIGURE 9. The trees inside and outside a space-filling curve

(This was in fact our original attempt at a definition of a group of paths G.)

We claim that the group G_2 of paths defined (using the usual inverse and concatenation product) with this equivalence is isomorphic to $\mathbb{R}^2$: any path is equivalent to the segment from $(0, 0)$ to its endpoint.

The idea of the proof is Figures 9a and 9b.

In the first path, walk through the maze from the starting point keeping the wall on your right. This makes a path equivalent to the path around the boundary of a square (indeed, the path is obtained from the boundary of the square by attaching backtracks).

For the second path, walk around the maze in the second figure keeping the wall on your left. This path is equivalent to the identity, since the maze is a tree.

Now do the same for the limiting mazes which become space-filling curves. In the limit, the two paths are the same path. (We leave it to the reader to construct the laminations of D giving an equivalence from the limit path to the boundary of the square, and the limit path to the identity.) So the boundary of the square is trivial.

Now for any closed path β, a continuous map from the square to $\mathbb{R}^2$ sending the boundary of the square to β gives an equivalence from β to the identity. Thus $G_2 \cong \mathbb{R}^2$.

6.2. **Isometries.** One useful generalization of G is to include a base frame for each path. Formally define G_I to be the semi-direct product of G with S^1, with $e^{i\theta} \in S^1$ acting on a path f by sending it to the path $t \mapsto e^{i\theta} f(t)$.

Explicitly in G_I the product is defined by

$$(f, e^{i\theta})(g, e^{i\psi}) = (f \cdot (e^{i\theta} g), e^{i(\theta+\psi)})$$

where the $\cdot$ is the product in G.

This definition allows us to define "tiling groups" for tiling problems in which we are allowed to rotate the tiles.

In addition, one can adapt the proof of Theorem 1 to G_I, this allows us to construct an $\mathbb{R}$-tree on which a finitely-generated subgroup acts by isometries. However such a subgroup may now have fixed points of its action on the $\mathbb{R}$-tree (the stabilizer of a point is a subgroup of S^1). So we close with the

Question. What are the finitely generated subgroups of G_I?

7. Appendix

One annoying feature of pseudo-Anosov diffeomorphisms is that they are complicated to construct explicitly. Thus for example although it is generally accepted that diffeomorphisms of the type we need exist, and indeed "almost all" pseudo-Anosovs with transverse orientations should be irreducible on homology and have many roots outside the unit circle, it seems that no one has actually proved anything to that effect.

Here we give a construction (based on Rauzy induction), of a family of pseudo-Anosovs for orientable surfaces satisfying the required properties. The non-orientable case (which we omit) is similar.

Lemma 12. *For each orientable surface of genus $g \geq 2$ there is a pseudo-Anosov diffeomorphism with transversely oriented foliations whose action on homology is irreducible with at least two eigenvalues of modulus larger than* 1.

Proof. First, if a diffeomorphism ψ is irreducible on homology (the characteristic polynomial is irreducible) and has an eigenvalue not on the unit circle, then ψ is isotopic to a pseudo-Anosov diffeomorphism [8].

For the genus-2 surface, the example of Figure 6 is the automorphism of the group $\langle a, b, c, d | abcd = bdac \rangle$ given by:

$$a \to abdacda, \; b \to ab, \; c \to cd, \; d \to dacd.$$

Since it is positive (the image of a word with positive exponents has positive exponents), the corresponding diffeomorphism has transversely oriented unstable foliation. The action on homology has characteristic polynomial $z^4 - 7z^3 + 13z^2 - 7z + 1$, which is irreducible with two real roots > 1. So the genus-2 case is completed.

We now construct explicit examples of diffeomorphisms for genus $g \geq 3$. We use Rauzy induction (see e.g. [1]) on an interval exchange of $k = 2g$ intervals.

In particular, there are k intervals $a_1, \ldots, a_k$ such that the interval exchange $a_1 a_2 \ldots a_k \to a_k a_{k-1} \ldots a_1$ is conjugate by a similarity to the exchange obtained from it under the sequence of elementary Rauzy inductions $\beta^{n(k-1)} \alpha \beta^{k-2} \alpha \beta^{k-3} \alpha \ldots \alpha \beta \alpha$.

Here an "α" transformation on an interval exchange $b_1 \ldots b_k \to b_{i_1} \ldots b_{i_k}$ in which $|b_1| < |b_{i_1}|$ is the induced interval exchange (again having k intervals) on the subinterval of the domain $b_2 \ldots b_k$. Similarly a type "β" transformation (when $|b_1| > |b_{i_1}|$) is the induced transformation (again with k intervals) on the subinterval $b_{i_2} \ldots b_{i_k}$ of the range.

For example from the exchange $a_1 \ldots a_k \to a_k \ldots a_1$, if $|a_1| < |a_k|$ then replace the interval a_k with two intervals $a_1 a_k'$: this leads to the exchange $a_2 a_3 \ldots a_{k-1} a_1 a_k' \to a_k' a_{k-1} \ldots a_1$ which is the α-transformation applied to the original exchange.

It is not hard to check that the above sequence of elementary inductions (if allowed) leads to an exchange on k intervals with the same permutation, $a_1' a_2' \ldots a_k' \to a_k' a_{k-1}' \ldots a_1'$. The vector of new lengths $(|a_1'|, \ldots, |a_k'|)$ is obtained by applying the linear mapping M^{-1} to the vector of original lengths $(|a_1|, \ldots, |a_k|)$, where M is the nonnegative integer matrix:

$$\begin{pmatrix} 1+n & 3n & 6n & 12n & 24n & \dots & 3\cdot 2^{k-3}n & 2^{k-2}n \\ 0 & 2 & 3 & 6 & 12 & \dots & 3\cdot 2^{k-4} & 2^{k-3} \\ 0 & 0 & 2 & 3 & 6 & \dots & 3\cdot 2^{k-5} & 2^{k-4} \\ \vdots & & & & & & & \vdots \\ 0 & 0 & 0 & 0 & & \dots & 2 & 1 \\ 1 & 1 & 1 & 1 & & \dots & 1 & 1 \end{pmatrix}.$$

Now M is the same as the action on homology of the associated pseudo-Anosov diffeomorphism. The characteristic polynomial of M can be obtained by row reduction and expanding along the last row: it is $q_n(z) = u(z) + nw(z)$, where

$$u(z) = (1-z)\left((1-z)(2-z)^{k-2} - \frac{(2-z)^{k-2} - (1-2z)^{k-2}}{1+z}\right),$$

$$w(z) = (1-z)(2-z)^{k-2} - \frac{(2-z)^{k-2} - (1-2z)^{k-2}}{1+z} - (1-2z)^{k-2}.$$

Lemma 13. *There exists infinitely many n so that $q_n(z)$ is irreducible.*

Proof. Note that $u(z) - (1-z)w(z) = (1-z)(1-2z)^{k-2}$. So for any integer z_0 we have $u(z_0) \wedge w(z_0)$ divides $(1-z_0)(1-2z_0)^{k-2}$ (here $\wedge$ denotes gcd). Now

$$-(1-2z_0)^{k-2} \equiv w(z_0) \bmod (1-z_0),$$

and since $1-2z_0$ is prime to $1-z_0$, we have that $(1-z_0)$ is prime to $w(z_0)$. Also if $2z_0-1$ is a prime number greater than 5 then $2-z_0 \not\equiv 0 \bmod (1-2z_0)$, and we have

$$w(z_0) \equiv \frac{-z_0^2(2-z_0)^{k-2}}{1+z_0} \bmod (1-2z_0),$$

so $1-2z_0$ is also prime to $w(z_0)$, and we conclude that, for $2z_0-1$ prime >5, $u(z_0) \wedge w(z_0) = 1$.

By a theorem of Dirichlet, an arithmetic progression $a+nb$, $n \in \mathbb{Z}$, where a, b are relatively prime, contains infinitely many primes. So there exist arbitrarily large n such that $q_n(z_0) = u(z_0) + nw(z_0)$ is prime.

Suppose $z_0 > 3k3^k$ such that $2z_0 - 1$ is prime and choose n much larger than z_0.

Each coefficient of u and w is at most 3^k, so the sum of the coefficients is at most $k3^k$. Also u and w are monic and of degree k and $k-1$ respectively. So if r is a root of $q_n(z) = u(z) + nw(z)$ for which $|r| > 2k3^k$, then $|r|$ is of the same order as n.

Now if q_n is reducible, $q_n(z) = q'(z)q''(z)$, then since $q_n(z_0)$ is prime, one of $|q'(z_0)|, |q''(z_0)|$ (say $|q'(z_0)|$) is equal to 1. But $|q'(z_0)| = \prod_{roots\ r_i} |z_0 - r_i|$, and if $|r_i| > 2k3^k$ then $|r_i| \sim n$ so $|z_0 - r_i| > 1$, and if $|r_i| \leq 2k3^k$ then again $|r_i - z_0| > 1$, so $|q'(z_0)|$ cannot be 1. •

It remains to show

Lemma 14. *For m large q_m has more than one root of modulus > 1.*

Proof. First we find the roots of $w(z)$. Except for $z = 0$ the roots of w satisfy

$$z = -\left(\frac{1-2z}{2-z}\right)^{k-2}.$$

Note that the map $f(z) = -\left(\frac{1-2z}{2-z}\right)^{k-2}$ satisfies

$$f((-\infty,-1]) = (-2^{k-2},-1]$$

and $f'(-1) = (k-2)/3$. Since $k \geq 6$, $f'(-1) > 1$ and so f has a fixed point on $(-\infty,-1)$, which is a root of w. So $nw(z)$ maps a neighborhood of this root around a large disk containing the origin. Thus $u(z) + nw(z)$ will also map this neighborhood (with the same degree) around the origin. So $q_n(z)$ for n sufficiently large has a root near this root of $w(z)$. Since we already know that q_n has a real positive root > 1 (the expansion of the corresponding pseudo-Anosov mapping), this completes the proof. •

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