Minkowski Games

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We introduce and study Minkowski games. These are two-player games, where the players take turns to choose positions in $\mathbb{R}^d$ based on some rules. Variants include boundedness games, where one player wants to keep the positions bounded, and the other wants to escape to infinity; as well as safety games, where one player wants to stay within a prescribed set, while the other wants to leave it.

We provide some general characterizations of which player can win such games and explore the computational complexity of the associated decision problems. A natural representation of boundedness games yields coNP-completeness, whereas the safety games are undecidable.

CCS Concepts: • **Theory of computation** → *Computational geometry*; • **Software and its engineering** → Formal software verification;

Additional Key Words and Phrases: Control in $\mathbb{R}^d$, (stochastic) determinacy, polytopic/arbitrary, coNP-complete, undecidable

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1 INTRODUCTION

Minkowski games. In this article, we define and study *Minkowski games*. A *Minkowski play* is an infinite duration interaction between two players, called Player A and Player B, in the space $\mathbb{R}^d$. A *move* in a Minkowski play is a subset of $\mathbb{R}^d$. Player A has a set of moves $\mathcal{A}$ and Player B has a set of moves $\mathcal{B}$. The play starts in a position $v_0 \in \mathbb{R}^d$ and is played for an infinite number of rounds as follows. For a round starting in position v_{2n}, Player A chooses $A_n \in \mathcal{A}$ and Player B chooses a vector b_n in A_n. Next, Player B chooses $B_n \in \mathcal{B}$ and Player A chooses a vector a_n in B_n. Then a new round starts in the position $v_{2n+2} := v_{2n+1} + a_n$, where $v_{2n+1} := v_{2n} + b_n$. The *outcome* of a Minkowski play is thus an infinite sequence of vectors $v_0 v_1 \ldots v_n \ldots$ obtained during this infinite interaction. Each outcome is either winning for Player A or for Player B, and this is specified by a winning condition. A play whose outcome is winning is also said to be winning.

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We consider two types of winning conditions. First, we consider *boundedness*: an outcome $v_0 v_1 \dots v_n \dots$ is winning for Player A in the boundedness game if there exists a bounded subset Safe $\subseteq \mathbb{R}^d$ such that the outcomes stays in Safe, i.e., for all $i \geq 0$, $v_i \in$ Safe; otherwise, the outcome is winning for Player B. Second, we consider *safety*: given a subset Safe $\subseteq \mathbb{R}^d$, an outcome is winning for Player A if the outcome stays in Safe; otherwise, the outcome is winning for Player B.

$\mathcal{A}$ and $\mathcal{B}$ could have arbitrary cardinality, but unless stated otherwise, we will consider finite sets $\mathcal{A} = \{A_1, A_2, \dots, A_{n_A}\}$ and $\mathcal{B} = \{B_1, B_2, \dots, B_{n_B}\}$. Also, unless stated otherwise, both Safe in the safety Minkowski games and the moves in general will be bounded.

The Minkowski games are a natural mathematical abstraction to model the interaction between two agents taking actions, modeled by moves, with imprecision as the adversary resolves nondeterminism by picking a vector in the move chosen by the other player.[1] Perhaps more importantly, the appeal of Minkowski games comes also from their simple and natural definition. We provide in this article general results about these games and study several of their incarnations in which moves are given as (*i*) bounded rational polyhedral sets, (*ii*) sets defined using the first-order theory of the reals, or (*iii*) represented as compact or overt sets as defined in computable analysis [29]. Note that by Borel determinacy [20] all these games are determined, i.e., either of the players has a strategy that is winning for sure. Our results are as follows.

Results. We establish a necessary and sufficient condition for Player A to have a winning strategy in a *boundedness Minkowski game* with finitely many moves (Theorem 4.2) and give a simple proof (in comparison with Borel determinacy [20]) that these games are determined. We then turn our attention to computation complexity aspects of determining the winner of a game, i.e., who has a winning strategy. The necessary and sufficient condition that we have identified leads to a coNP solution when the moves are given as bounded rational polyhedral sets, and we provide matching lower bounds (Theorem 5.1). These results hold both for moves represented by sets of linear inequalities and moves represented as the convex hulls of a finite set of rational points. Additionally, we show that for every fixed dimension d, determining the winner can be done in polynomial time (Corollary 5.11). When the moves are defined using the first-order theory of the reals, determining the winner of a boundedness game is shown to be as hard as deciding truth of sentences in the first-order theory of the reals (Proposition 5.12). Finally, in the computable analysis setting, the problem is semi-decidable (Proposition 5.14), and this is the best that we can hope for.

We characterize the set of winning positions in a *safety Minkowski game*, even with infinite $\mathcal{A}$ and $\mathcal{B}$, as the greatest fixed point of an operator that removes the points where Player B can provably win (in finitely many rounds). We show that this greatest fixed point can be characterized by an approximation sequence of at most ω steps (Proposition 6.3), for finite $\mathcal{A}$ but even for infinite $\mathcal{B}$. This leads to semi-decidability in the general setting of computable analysis (Proposition 7.2). Then, we show that identifying the winner in a safety Minkowski game is undecidable even for finite sets of moves that are given as bounded rational polyhedral sets (Theorem 7.3). As a consequence, we consider a variant of the safety Minkowski games, called structural safety Minkowski games, where Player A must maintain safety from any starting vector within the set Safe. We show that deciding the winner in this variant is coNP-complete when the moves are defined as bounded rational polyhedral sets (Theorem 8.1).

When we consider (one-sided) Minkowski games, the first player chooses moves and the second player resolves nondeterminism on the moves by choosing a point in the set of points that

[1] See further discussions on the practical appeal of these games for modeling systems evolving in multi-dimensional spaces when we report on related works.

is defined by the move. This mathematical abstraction is well suited to model, for instance, the control of physical systems in which the control actions have some imprecision. There are two natural ways of resolving this imprecision. First, in the adversarial way, an adversary resolves the imprecision and the synthesis problem asks to find a strategy that is winning, for a safety or a boundedness objective, no matter how the imprecision is resolved by the adversary. Second, in the stochastic way, the imprecision is considered as a noise that is uniformly distributed in some neighborhood defined by action. In this second interpretation, the synthesis problem asks to find a strategy that is winning with probability one against the stochastic noise. Because of the nature of the objectives that we consider and the fact that our games have infinite duration, we show that the two interpretations in fact coincide, see Proposition 4.9 and Corollary 4.10.

Motivations and related works. Infinite duration games are commonly used as mathematical framework for modeling the controller synthesis problem for reactive systems [26]. For reactive systems embedded in some physical environment, games played on hybrid automata have been considered; see, e.g., Reference [14] and references therein. In such a model, one controller interacts with an environment whose physical properties are modeled by valuations of d real-valued variables (vectors in $\mathbb{R}^d$). Most of the problems related to the synthesis of controllers for hybrid automata are undecidable [14]. Restricted subclasses with decidable properties, such as timed automata and initialized rectangular automata have been considered [13, 19]. Most of the undecidability properties of those models rely on the coexistence of continuous and discrete evolutions of the configurations of hybrid automata. The one-sided version of the model of Minkowski games (where $\mathcal{B} = \{\{0\}\}$) can be seen as a restricted form of hybrid games in which each continuous evolution is of a unique fixed duration and space independent (such as in linear and rectangular hybrid automata). It is usually called discrete time control in this setting. While discrete-time control problems are known to be undecidable for linear hybrid automata, they are decidable for (bounded) rectangular automata [14]. We show in Remark 7.7 below how this positive result can be transferred to a subclass of Minkowski games.

To the best of our knowledge, the closest models to Minkowski games that have been considered in the literature so far are Robot games defined by Doyen et al. in Reference [8] and Bounded-Rate Multi-Mode Systems introduced by Alur et al. in References [1, 2]. Minkowski games generalize robot games: there the moves are always singletons given as integer vectors. While we show that the Safety problem is undecidable for bounded safety objectives, it is easy to show that this problem is actually decidable for robot games. However, in Reference [8] they investigate reachability of a specific position rather than safety conditions as we do here. Reachability was proven undecidable in Reference [22] even for two-dimensional robot games. Boundedness objectives have not been studied for Robot games.

Bounded-Rate Multi-Mode Systems (BRMMS) are a restricted form of hybrid systems that can switch freely among a finite number of modes. The dynamics in each mode is specified by a bounded set of possible rates. The possible rates are either given by convex polytopes or as finite set of vectors. There are several differences with Minkowski games. First BRMMS are asymmetric and are thus closer to the special case of one-sided Minkowski games. Second, the actions in BRMMS are given by a mode and a time delay $\delta \in \mathbb{R}$, while the time elapsing in our model can be seen as uniform and fixed. The ability to choose delays that are as small as needed makes the safety control problem for BRMMS with modes given as polytopes decidable, while we show that the safety Minkowski games with moves defined by polytopes are undecidable. The discrete time control of BRMMS, which is more similar to the safety Minkowski games, has been solved only for modes given as finite sets of vectors and left open for modes given as polytopes. Our undecidability results can be easily adapted to the discrete time control of BRMMS and thus solves the open question left in that paper. Boundedness objectives have not been studied for BRMMS.

In their article [15], Jurdzinski et al. consider the multi-dimensional extension of energy games first defined in Reference [4] and its complexity when the initial vector of energy levels is fixed and when the number of dimensions is fixed. Their problem can be seen as a special case of the safety problem for Minkowski games: their moves are vectors and so singleton sets, and their safety region is the set of all positive integer vectors. While the safety verification problem for multi-dimensional energy games with fixed initial state is 2ExpTime-C in general and PTime-C for fixed number of dimension, the safety problem for Minkowski games that we consider here is undecidable.

Structure of the article. Section 2 collects various basic mathematics, typically from linear algebra, that we are using in the article. It also defines the Minkowski games. Section 3 defines and studies auxiliary games, which will be used to decompose every more complex Minkowski game into a simpler Minkowski game and an (also simpler) auxiliary game. Section 4 characterizes the winner of a boundedness Minkowski game (also a probabilistic boundedness Minkowski game) in terms of simple convex geometry, and it describes the winning strategies. Section 5 studies the algorithmic complexity of finding the winner in the various settings. Section 6 collects a few properties of the winning region of the safety problem, depending on various restrictions on the game. Section 7 shows, among others things, that finding the winner of a safety Minkowski game is undecidable, even for a simple subclass. In Section 8, we consider structural safety games and prove coNP-completeness for the associated decision problem. Finally, Section 9 mentions a few open questions.

Conference version. A preliminary version of this work has been presented at STACS 2017 [18].

2 PRELIMINARIES

Linear constraints. Let $d \in \mathbb{N}_{>0}$, and $X = \{x_1, x_2, \ldots, x_d\}$ be a set of variables. A *linear term* on X is a term of the form $\alpha_1 x_1 + \alpha_2 x_2 + \ldots \alpha_d x_d$, where $x_i \in X$, $\alpha_i \in \mathbb{R}$ for all i, $1 \leq i \leq n$. A *linear constraint* is a formula $\alpha_1 x_1 + \alpha_2 x_2 + \ldots \alpha_d x_d \sim c$, and $\sim \in \{<, \leq, =, \geq, >\}$, that compares a linear term with a constant $c \in \mathbb{R}$. Given a valuation $v : X \to \mathbb{R}$, that can be seen equivalently as a vector in $\mathbb{R}^d$, we write $v \models \alpha_1 x_1 + \alpha_2 x_2 + \ldots \alpha_n x_n \sim c$ iff $\alpha_1 v(x_1) + \alpha_2 v(x_2) + \ldots \alpha_d v(x_d) \sim c$. Given a linear constraint $\phi \equiv \alpha_1 x_1 + \alpha_2 x_2 + \ldots \alpha_d x_d \sim c$, we write $[\![\phi]\!] = \{v \in \mathbb{R}^d \mid v \models \alpha_1 x_1 + \alpha_2 x_2 + \ldots \alpha_d x_d \sim c\}$. A linear constraint is rational, if all α_i and c are rational numbers.

Polyhedra, polytopes, convex hull. Given a finite set $\mathcal{H} = \{\phi_1, \phi_2, \ldots, \phi_n\}$ of linear constraints, we note $[\![\mathcal{H}]\!] = \{v \in \mathbb{R}^d \mid \forall \phi \in \mathcal{H} : v \models \phi\}$ the set of vectors that satisfies all the linear constraints in $\mathcal{H}$. Such a set is a convex set and is usually called a *polyhedron.* In the special case that is bounded, then it is called a *polytope.* We call a polytope rational, if all ϕ_i can be chosen rational. When a polytope is *closed*, then it is well-known that it can be represented not only by a finite set of linear inequalities that are all non-strict but also as the *convex hull* of a finite set of (extremal) vectors. The convex hull of a subset of a $\mathbb{R}$-vector space is noted and defined as follows:

$$\mathsf{CH}(\mathcal{V}) := \left\{ \sum_{i=0}^{n} \alpha_i x_i \mid n \in \mathbb{N} \wedge \sum_{i=0}^{n} \alpha_i = 1 \wedge \forall i (x_i \in \mathcal{V} \wedge \alpha_i \geq 0) \right\}.$$

Carathéodory's theorem says that for all $\mathcal{V} \subseteq \mathbb{R}^d$, every point in $\mathsf{CH}(\mathcal{V})$ is a convex combination of at most $d + 1$ points from $\mathcal{V}$. As a consequence, the n ranging over $\mathbb{N}$ in the definition of the convex hull can safely be replaced with fixed d.

Let P be a closed polytope. P has two families of representations: its H-representations are the finite sets of linear inequalities $\mathcal{H}$ such that $[\![\mathcal{H}]\!] = P$, and its V-representations are the finite sets of vectors $\mathcal{V}$ such that $\mathsf{CH}(\mathcal{V}) = P$. Some algorithmic operations are easier to perform on one representation or on the other. Unfortunately, in general, there cannot exist a polynomial time translation from one representation to the other unless $P = NP$. Nevertheless, such a polynomial time translation exists for fixed dimension:

Theorem 2.1 ([5]). *Let P be a rational closed polytope of fixed dimension $d \in \mathbb{N}$. There exists a polynomial time algorithm that given a H-representation of P, computes a V-presentation of P, and conversely, there exists a polynomial time algorithm that given a V-representation of P, computes a H-presentation of P.*

We denote by Ver(P) the extremal points, i.e., the vertices of a polytope P. It is the minimal set whose convex hull equals P. Note that a closed polytope is rational iff all its vertices are rational points.

Minkowski sum. For subsets $A, B \in \mathbb{R}^d$ their Minkowski sum $A + B$ is defined as $\{a + b \mid a \in A \wedge b \in B\}$. The Minkowski sum inherits commutativity and associativity from the usual sum of vectors. The set $\{0\}$ is the neutral element, but there are no inverse elements in general. If $A = \{a\}$, then $A + B$ (respectively, $B + A$) is written $a + B$ (respectively, $B + a$) in a slight abuse of notation. It is straightforward to prove that $\mathsf{CH}(A) + \mathsf{CH}(B) = \mathsf{CH}(A + B)$. Especially, if A and B are convex, so is $A + B$. While $A + A$ may be a strict superset of $2A := \{2a \mid a \in A\}$, in general, for convex A we find $A + A = 2A$. More generally, if A is convex, for all $n \in \mathbb{N}$ we find that the n-fold Minkowski sum of A with itself, i.e., $\{\sum_{i=1}^{n} a_i \mid (a_1, \ldots, a_n) \in A^n\}$, is equal to the product of the scalar n by A, i.e., $\{n \cdot a \mid a \in A\}$. Thus, writing $n \cdot A$ or nA is non-ambiguous.

Topological closure. The topological closure of a set S is denoted by $\overline{S}$.

Minkowski games and Strategies. We have described in the Introduction how the players interact in Minkowski games by choosing in each round a move and by resolving nondeterminism among the moves chosen by the other player. We now formally define the notion of strategy, which is based on the notion of history. Then, we define the notion of resulting outcome.

$$H_r := \mathbb{R}^d \times (\mathcal{A} \times \mathbb{R}^d \times \mathcal{B} \times \mathbb{R}^d)^*$$

are the *round histories.* They correspond to an initial position followed by a number of complete rounds, and

$$H : H_r \cup (H_r \times \mathcal{A}) \cup (H_r \times \mathcal{A} \times \mathbb{R}^d) \cup (H_r \times \mathcal{A} \times \mathbb{R}^d \times \mathcal{B})$$

are the *histories*, where the current round may or may not be complete. Note that when defining (round) histories, the order of display of the factors of the Cartesian product reflects the order of the choices in the Minkowski interaction.

When playing Minkowski games, players are applying *strategies*, based on histories. In a game with moves $\mathcal{A}$ and $\mathcal{B}$, strategies for the two players are defined as follows. A strategy for Player A is given by two functions λ_A^M and λ_A^P of the following types:

$$\lambda_A^M : H_r \to \mathcal{A}, \qquad \lambda_A^P : H_r \times \mathcal{A} \times \mathbb{R}^d \times \mathcal{B} \to \mathbb{R}^d.$$

The function λ_A^M prescribes a move in $\mathcal{A}$ depending on the round history, i.e., when a complete number of rounds have been played, starting from some initial position. The function λ_A^P prescribes a vector in $\mathbb{R}^d$ right after Player B has chosen a move $B \in \mathcal{B}$. The function should respect the following consistency constraint:

$$\lambda_A^P(v_0(A_0 b_0 B_0 a_0) \ldots (A_{n-1} b_{n-1} B_{n-1} a_{n-1}) A_n b_n B_n) \in B_n.$$

Almost symmetrically, a strategy for Player B is given by functions λ_B^M and λ_B^P of the following types:

$$\lambda_B^M : H_r \times \mathcal{A} \times \mathbb{R}^d \to \mathcal{B}, \qquad \lambda_B^P : H_r \times \mathcal{A} \to \mathbb{R}^d.$$

The function should respect the following consistency constraint:

$$\lambda_B^P(v_0(A_0b_0B_0a_0)\dots(A_{n-1}b_{n-1}B_{n-1}a_{n-1})A_n) \in A_n.$$

Given two strategies $(\lambda_A^M, \lambda_A^P)$ and $(\lambda_B^M, \lambda_B^P)$ and a starting position v_0, we inductively define

- $A_n := \lambda_A^M(v_0(A_0b_0B_0a_0)\dots(A_{n-1}b_{n-1}B_{n-1}a_{n-1}))$,
- $b_n := \lambda_B^P(v_0(A_0b_0B_0a_0)\dots(A_{n-1}b_{n-1}B_{n-1}a_{n-1})A_n)$,
- $B_n := \lambda_B^M(v_0(A_0b_0B_0a_0)\dots(A_{n-1}b_{n-1}B_{n-1}a_{n-1})A_nb_n)$,
- $a_n := \lambda_A^P(v_0(A_0b_0B_0a_0)\dots(A_{n-1}b_{n-1}B_{n-1}a_{n-1})A_nb_nB_n)$.

The resulting outcome is the sequence $(v_n)_{n\in\mathbb{N}} \in (\mathbb{R}^d)^{\mathbb{N}}$ inductively defined by $v_{2n+1} := v_{2n} + a_n$ and $v_{2n+2} := v_{2n+1} + b_n$. Given two strategies λ_A and λ_B, one for each player, and a position $v_0 \in \mathbb{R}^d$.

Winning conditions and variants of Minkowski games. By fixing the rule that determines who wins a Minkowski play, we obtain *Minkowski games*. Here, we consider three types of Minkowski games.

Definition 2.2. A *boundedness Minkowski game* is a pair $\langle\mathcal{A},\mathcal{B}\rangle$ of sets of moves in $\mathbb{R}^d$ for Player A and Player B. A play in a boundedness Minkowski game starts in some irrelevant $v_0 \in \mathbb{R}^d$, and the resulting outcome $v_0v_1\dots v_n\dots$ is winning for Player A if there exists a bounded subset Safe of $\mathbb{R}^d$ such that $v_i \in \mathsf{Safe}$ for all $i \in \mathbb{N}$; otherwise, Player B wins the game. The associated decision problem asks if Player A has a strategy λ_A, which is winning against all the strategies λ_B of Player B.

Definition 2.3. A *safety Minkowski game* is defined by $\langle\mathcal{A},\mathcal{B},\mathsf{Safe},v_0\rangle$, where $\mathcal{A}$ and $\mathcal{B}$ are sets of moves for Player A and Player B, $\mathsf{Safe} \subseteq \mathbb{R}^d$ (bounded unless stated otherwise), and $v_0 \in \mathsf{Safe}$ is the initial position. A play in a safety Minkowski game starts in v_0, and the resulting outcome $v_0v_1\dots v_n\dots$ is winning for Player A if $v_i \in \mathsf{Safe}$ for all $i \in \mathbb{N}$; otherwise, Player B wins the game. The associated decision problem asks if Player A has a strategy λ_A, which is winning against all the strategies λ_B of Player B.

Definition 2.4. A *structural safety Minkowski game* is defined by $\langle\mathcal{A},\mathcal{B},\mathsf{Safe}\rangle$, where $\mathcal{A}$, and $\mathcal{B}$ are sets of moves for Player A and Player B, and $\mathsf{Safe} \subseteq \mathbb{R}^d$. In such a game, the interaction between the two player starts from any position $v_0 \in \mathsf{Safe}$, and the resulting outcome $v_0v_1\dots v_n\dots$ is winning for Player A if $v_i \in \mathsf{Safe}$ for all $i \in \mathbb{N}$; otherwise, Player B wins the game. The associated decision problem asks if Player A has a strategy to win the safety game wherever it starts in Safe and against all the strategies of Player B.

A game is *single-sided* if $\mathcal{B} = \{\{0\}\}$, i.e., whenever Player B has only one trivial move. We use single-sided Minkowski games to show that several of our lower bounds hold for this subclass of games.

We sometimes say that a play is bounded or that a play stays within some set. What is meant is that the resulting outcome of the play is bounded or stays within some set.

3 AUXILIARY GAMES

We will make use of two kinds of auxiliary games in proving our results on Minkowski games. About these games, we mainly prove results that are sufficient for our purposes here but also some related results, since these games may be of independent interest. Our first auxiliary game captures the difference between a set and its convex hull for controlling some trajectory in $\mathbb{R}^d$. Player B plays points in some set $\mathsf{CH}(B) \subseteq \mathbb{R}^d$, which Player A has to approximate as well as possible while playing points in B. (Note that Player B plays first in this game.)

Definition 3.1. In the convex approximation game for $B \subseteq \mathbb{R}^d$ with error margin $E \subseteq \mathbb{R}^d$, in each turn j Player B first plays some $v_j \in \mathsf{CH}(B)$, then Player A follows with some $u_j \in B$. If for all $j \in \mathbb{N}$, $\sum_{i=0}^{j}(v_i - u_i) \in E$, then Player A wins, else Player B wins.

The precise nature of the error margins E allowing Player A to win will not be important for us in Section 4, important is that for bounded B there is some bounded E enabling Player A to win. However, we state results with explicit bounds, since these may be of independent interest. (Otherwise, we could have defined the convex approximation games as special cases of the boundedness Minkowski games.)

We will provide two results stating that if the error margin is large enough, Player A has a winning strategy in the convex approximation game. The first result (Proposition 3.3) makes no assumptions on the set B. Its proof relies on Lemma 3.2 below. The second result (Proposition 3.4) provides tighter bounds, but has some restrictions on B. Proposition 3.4 will not be used in the remainder of the article.

Lemma 3.2 ([28][2]). *For $B \subseteq \mathbb{R}^d$, we find that $B + d\mathsf{CH}(B) = (d+1)\mathsf{CH}(B)$.*

Proof. As $\mathsf{CH}(B)$ is convex, we recall that $(d+1)\mathsf{CH}(B)$ equivalently denotes the $d+1$-fold Minkowski sum of $\mathsf{CH}(B)$ with itself or the product of the scalar $d+1$ with $\mathsf{CH}(B)$, as mentioned in Section 2. Thus, the inclusion $B + d\mathsf{CH}(B) \subseteq (d+1)\mathsf{CH}(B)$ is trivial.

For the other direction, assume that $b \in (d+1)\mathsf{CH}(B)$. Then $(d+1)^{-1}b \in \mathsf{CH}(B)$. By Charathéodory's theorem, there are $d+1$ points $b_i \in B$ and scalars $\alpha_i \geq 0$ for $i \in \{0, \ldots, d\}$ with $\sum_{i=1}^{d} \alpha_i = 1$ and $\sum_{i=1}^{d} \alpha_i b_i = (d+1)^{-1}b$, i.e. $b = (d+1)\sum_{i=1}^{d} \alpha_i b_i$. W.l.o.g. assume that $\alpha_d \geq \alpha_i$ for all $i \leq d$. Then, in particular, $\alpha_d \geq (d+1)^{-1}$. Now, we can write

$$b = b_d + d\left[\frac{(d+1)\alpha_d - 1}{d} b_d + \sum_{i=0}^{d-1} \frac{d+1}{d} \alpha_i b_i\right].$$

The expression in square brackets is a convex combination of the b_i, thus we can conclude $b \in B + d\mathsf{CH}(B)$. ☐

The following proposition shows that for an arbitrary bounded set B, a player selecting displacements from B can mimic a trajectory created by a player selecting displacements from $\mathsf{CH}(B)$ up to some constant error bound. It uses Lemma 3.2, and is itself used in the proof of Theorem 4.2.

Proposition 3.3. *Pick $c \in d \cdot \mathsf{CH}(B)$.* Player A *has a winning strategy in the convex approximation game for $B \subseteq \mathbb{R}^d$ with error margin $d \cdot \mathsf{CH}(B) + \{-c\}$.*

Proof. We describe a strategy of Player A that ensures $\sum_{i=0}^{j}(v_i - u_i) \in d\mathsf{CH}(B) + \{-c\}$ inductively. The case $t = 0$ is satisfied, since $0 \in d\mathsf{CH}(B) + \{-c\}$ by choice of c. By induction hypothesis, we find that $v_j + \sum_{i=0}^{j-1}(v_i - u_i) + c \in \mathsf{CH}(B) + d\mathsf{CH}(B)$. By Lemma 3.2, there exists some $u_j \in B$ and $r \in d\mathsf{CH}(B)$, such that $v_j + \sum_{i=0}^{j-1}(v_i - u_i) + c = u_j + r$, i.e., $r - c = \sum_{i=0}^{j}(v_i - u_i) \in d\mathsf{CH}(B) + \{-c\}$ as desired. ☐

If we place some restrictions on the set B, then we can obtain better bounds. Essentially, our condition is that B contains the boundary of $\mathsf{CH}(B)$. Below $\mathrm{dist}(x, B) := \inf_{b \in B} \mathrm{dist}(x, b)$, where $\mathrm{dist}(x, b)$ is the Euclidian distance between x and b.

Proposition 3.4. *Let B be closed and satisfy* $\mathrm{dist}(x, B) = \mathrm{dist}(x, \mathsf{CH}(B))$ *for each* $x \notin \mathsf{CH}(B)$. *Let* $\rho := \max_{x \in \mathsf{CH}(B)} \mathrm{dist}(x, B)$ *and* $\overline{\mathrm{Ball}}(0, \rho) \subseteq E$, *where* $\overline{\mathrm{Ball}}(0, \rho)$ *is the close ball for the Euclidian*

[2]This result was provided by an anonymous contributor in the cited answer posted to math.stackexchange.com.

distance of centre 0 and radius ρ. Player A *has a winning strategy for the convex approximation game for B with error margin E.*

Proof. We describe a strategy of Player A that ensures $\sum_{i=0}^{j}(v_i - u_i) \in \overline{\mathrm{Ball}}(0, \rho)$ inductively. The case $j = 0$ is trivially satisfied. Assume that $\sum_{i=0}^{j-1}(v_i - u_i) \in \overline{\mathrm{Ball}}(0, \rho)$, and that Player B chose v_j in round j. Player A will play some $u_j \in B$ with $\mathrm{dist}(\sum_{i=0}^{j-1}(v_i - u_i) + v_j, u_j) = \mathrm{dist}(\sum_{i=0}^{j-1}(v_i - u_i) + v_j, B)$.

It remains to show that $\sum_{i=0}^{j}(v_i - u_i) \in \overline{\mathrm{Ball}}(0, \rho)$, i.e., that $\mathrm{dist}(\sum_{i=0}^{j-1}(v_i - u_i) + v_j, u_j) \leq \rho$. If $\sum_{i=0}^{j-1}(v_i - u_i) + v_j \in \mathrm{CH}(B)$, then this is true by definition of ρ. Else, by our various assumptions:

$$\begin{aligned}\mathrm{dist}\left(\sum_{i=0}^{j-1}(v_i - u_i) + v_j, u_j\right) &= \mathrm{dist}\left(\sum_{i=0}^{j-1}(v_i - u_i) + v_j, B\right)\\ &= \mathrm{dist}\left(\sum_{i=0}^{j-1}(v_i - u_i) + v_j, \mathrm{CH}(B)\right)\\ &\leq \mathrm{dist}\left(\sum_{i=0}^{j-1}(v_i - u_i) + v_j, v_j\right)\\ &= \mathrm{dist}\left(\sum_{i=0}^{j-1}(v_i - u_i), 0\right) \leq \rho. \qquad \square\end{aligned}$$

Our second class of auxiliary games is (up to some details such as the starting player) a special case of the convex approximation games:

Definition 3.5. In the d-dimensional $+1/-1$-game with threshold r, Player A plays positions $n_i \in \{1, \dots, d\}$ and Player B plays stochastic d-dimensional vectors $v_i \in \mathbb{R}^d$, i.e., its components have non-negative entries summing up to one. Player A wins if for all $j \in \mathbb{N}$ and all $k \in \{1, \dots, d\}$, we find that $|\{i \leq j \mid n_i = k\}| - \sum_{i=0}^{j}(v_i)_k \geq r$.

This means that in each round Player A is putting a unit token on one out of d positions, while Player B is removing fractions of token summing up to one unit from the positions. Player B attempts to get some position below r, Player A wants to prevent this.

Proposition 3.6. Player A *has a winning strategy in the d-dimensional $+1/-1$-game with threshold $-d$.*

Proof. Consider the convex approximation game for

$$B := \{(1, 0, \dots, 0), (0, 1, \dots, 0), \dots, (0, 0, \dots, 1)\}.$$

Pick $c = (1, 1, \dots, 1)$. Then, by Proposition 3.3 Player A can force all components of the vectors $\sum_{i=0}^{j}(v_i - u_i)$ to not exceed $d - 1$. Since in a $+1/-1$-game, Player A moves first, whereas in a convex approximation game Player B moves first, we need to adjust this bounds by adding the maximum deviation possible through a single round: 1. $\square$

Proposition 3.7. Player B *has a winning strategy in the d-dimensional $+1/-1$-game with threshold $-H(d-1) + 1$ (where $H(n) := \sum_{i=1}^{n} i^{-1}$ is the nth harmonic number).*

Proof. For $k < d$, in the kth round there are at least $(d - k)$ positions never played by Player A so far. Player B plays each of these with weight $\frac{1}{d-k}$ each. In round $(d - 1)$, this gives total weight $-\sum_{i=1}^{d-1} i^{-1} = -H(d-1)$ to the position never played by Player A. $\square$

We leave the question open to precisely determine the following values:

$$\tau_d = -\sup\{r \mid \text{Player A wins the } +1/{-1}\text{-game with threshold } r\}.$$

What we will use is only that there are thresholds allowing Player A to win.

4 GENERAL RESULTS ON THE BOUNDEDNESS PROBLEM

To start this section, we consider the special case of one-sided boundedness Minkowski games and provide a sufficient (and necessary) condition for Player A to win. The proof showcases some ideas we will then use to fully characterize the general case. The characterization in the general case in particular implies that the condition for the one-sided case is necessary.

Proposition 4.1. *We consider a one-sided boundedness Minkowski game* $\langle \mathcal{A}, \{0\}\rangle$, *where* $\mathcal{A} = \{A_1, \dots, A_n\}$ *and such that* $0 \in \mathsf{CH}((x_i)_{1\le i\le n})$ *for all tuples* $(x_i)_{1\le i\le n}$ *in* $A_1 \times \cdots \times A_n$. *Then,* Player A *wins the boundedness game.*

Proof. We describe the current state by some list of pairs $(x_i, \alpha_i)_{i\le n}$ such that $x_i \in A_i$ and $\alpha_i \in [0,1]$. We keep two invariants satisfied throughout the play: First, it will always be the case that the current position is equal to $\sum_{i\le n} \alpha_i x_i$, which by boundedness of each A_i implies that Player A wins. Second, we maintain the invariant that there is some $k \le n$ with $\alpha_k = 0$. Initially, the choice of the x_i is arbitrary, and all α_i are 0. This ensures that the strategy we describe for Player A is well-defined.

On her turn, Player A plays some A_k for k with $\alpha_k = 0$. Player B reacts with some $x'_k \in A_k$, and we set $x_k := x'_k$ and $\alpha_k := 1$.

If immediately after the move, no α_i is currently 0, then we write a convex combination $0 = \sum_{i\le n} \beta_i x_i$, which is possible by assumption. Let $r := \max_{i\le n} \frac{\beta_i}{\alpha_i}$, so that $0 \le \alpha_i - r^{-1}\beta_i$ for all i, and $\alpha_i - r^{-1}\beta_i \le \alpha_i$, since the α_i and β_i are non-negative. Moreover, $\sum_{i\le n}(\alpha_i - r^{-1}\beta_i)x_i = \sum_{i\le n} \alpha_i x_i$ by the choice of the β_i. Therefore, performing the update $\alpha_i := \alpha_i - r^{-1}\beta_i$ leaves $\sum_{i\le n} \alpha_i x_i$ unchanged and ensures that $\alpha_i \in [0,1]$ remains true. Moreover, after the updating process, there is some $k \le n$ with $\alpha_k = 0$. Thus, the invariant is true again after the updating process. □

We introduce some notation to formulate the main lemma of this section, which is then summarized by Theorem 4.2. For some set of moves $\mathcal{B}$ let $\mathsf{CH}(\mathcal{B}) := \{\mathsf{CH}(B) \mid B \in \mathcal{B}\}$ and $\overline{\mathcal{B}} := \{\overline{B} \mid B \in \mathcal{B}\}$. We say that a strategy for Player B in a Minkowski game is *simple*, if it prescribes choosing always the same $B \in \mathcal{B}$, and if the choice $a_i \in A_i$ depends only on the choice of $A_i \in \mathcal{A}$ by Player A.

Theorem 4.2.

- *Boundedness Minkowski games are determined;*
- *the winner is the same for* $\langle \mathcal{A}, \mathcal{B}\rangle$ *and* $\langle \mathsf{CH}(\overline{\mathcal{A}}), \mathsf{CH}(\overline{\mathcal{B}})\rangle$;
- *if* Player B *has a winning strategy, he has a simple one;*
- Player A *wins iff* $0 \in (\mathsf{CH}\{a_i \mid i \le n\}) + \mathsf{CH}(\overline{B})$ *for all* $(a_i)_{i\le n}$ *with* $a_i \in A_i$ *and* $B \in \mathcal{B}$.

Proof. The claims follow from Lemma 4.3 below. □

Note that the determinacy of the boundedness Minkowski games also follows from Borel determinacy [20] (and also from techniques in Reference [30]), since the set of the winning plays for Player A is a Σ^0_2 set (for the usual product topology over discrete topology).

Lemma 4.3. *The following are equivalent for a boundedness Minkowski game* $\langle \mathcal{A} = \{A_1, \dots, A_n\}, \mathcal{B}\rangle$:

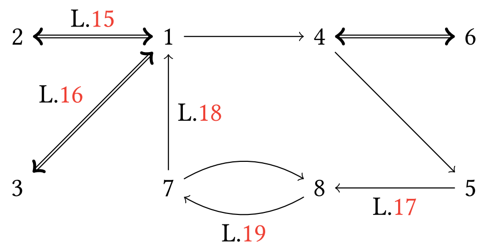

Fig. 1. Implications proved in Lemma 4.3.

(1) Player A *has a winning strategy in* $\langle \mathcal{A}, \mathcal{B} \rangle$.
(2) Player A *has a winning strategy in* $\langle \mathcal{A}, \mathsf{CH}(\mathcal{B}) \rangle$.
(3) Player A *has a winning strategy in* $\langle \mathcal{A}, \overline{\mathcal{B}} \rangle$.
(4) Player B *has no winning strategy in* $\langle \mathcal{A}, \mathcal{B} \rangle$.
(5) Player B *has no simple winning strategy in* $\langle \mathcal{A}, \mathcal{B} \rangle$.
(6) Player B *has no winning strategy in* $\langle \mathsf{CH}(\overline{\mathcal{A}}), \mathcal{B} \rangle$.
(7) *For all* $(a_i)_{i \leq n}$ *with* $a_i \in \mathsf{CH}(\overline{A}_i)$ *and* $B \in \mathcal{B}$, *we find that:*

$$0 \in (\mathsf{CH}\{a_i \mid i \leq n\}) + \mathsf{CH}(\overline{B})$$

(8) *For all* $(a_i)_{i \leq n}$ *with* $a_i \in A_i$ *and* $B \in \mathcal{B}$, *we find that:*

$$0 \in (\mathsf{CH}\{a_i \mid i \leq n\}) + \mathsf{CH}(\overline{B})$$

Proof. That (1) and (2) are equivalent is shown in Lemma 4.4 below. That (1) and (3) are equivalent is shown in Lemma 4.5 below. The implication from (1) to (4) is trivial, so is the implication from (4) to (5). That ¬(8) implies ¬(5) is Lemma 4.6. Using the equivalences of (1), (2), and (3), we see that it suffices to show that (7) implies (1) in the special case where all $B \in \mathcal{B}$ are closed and convex. This is the statement of Lemma 4.7. That (7) implies (8) is trivial, that (8) implies (7) is shown in Lemma 4.8. That (4) and (6) are equivalent then follows from (7), (8), and (4). being equivalent.

Figure 1 shows which implication is shown by which lemma. The edges without labels correspond to the implications discussed within the proof of Lemma 4.3. □

Lemma 4.4. Player A *has a winning strategy in* $\langle \mathcal{A}, \mathcal{B} \rangle$ *iff she has a winning strategy in* $\langle \mathcal{A}, \mathsf{CH}(\mathcal{B}) \rangle$.

Proof. Every strategy for Player A in $\langle \mathcal{A}, \mathcal{B} \rangle$ is also a valid strategy in $\langle \mathcal{A}, \mathsf{CH}(\mathcal{B}) \rangle$ (up to typecasting from B to $\mathsf{CH}(B)$), and if the strategy is winning in the former game, it is winning in the latter. Thus, we only need to show how to transform a winning strategy for Player A in $\langle \mathcal{A}, \mathsf{CH}(\mathcal{B}) \rangle$ to a winning strategy for Player A in $\langle \mathcal{A}, \mathcal{B} \rangle$.

Informally, we do this by using an auxiliary convex approximation game for each $B \in \mathcal{B}$. In the convex approximation game for B, Player A will consider the choices $a' \in \mathsf{CH}(B)$ prescribed to her by the strategy s' in $\langle \mathcal{A}, \mathsf{CH}(\mathcal{B}) \rangle$ as the moves of her opponent, and she will determine moves $a \in B$ according to some winning strategy for some suitable bounded set E_B (which she has by Proposition 3.3). The strategy s now chooses a for $\langle \mathcal{A}, \mathcal{B} \rangle$. If s' enforces that the play remains within some set E, then by linearity, s enforces that the outcome remains within $E - \bigoplus_{B \in \mathcal{B}} E_B$, which by finiteness of $\mathcal{B}$ is again a bounded set.

More specifically, let $s' = (\lambda^M_{A'}, \lambda^P_{A'})$ be a winning strategy for Player A in $\langle \mathcal{A}, \mathsf{CH}(\mathcal{B}) \rangle$, which ensures that the resulting outcome is bounded. For each $B \in \mathsf{CH}(\mathcal{B})$ let g_B be the convex approximation game for B. By Proposition 3.3 let s_B be a winning strategy for Player A in g_B with some error margin E_B. A strategy $s = (\lambda_A^M, \lambda_A^P)$ for Player A in $\langle \mathcal{A}, \mathcal{B} \rangle$ is defined as follows:

- $\lambda_A^M := \lambda_{A'}^M$,
- To define $\lambda_A^P(v_0(A_0b_0B_0a_0)\dots(A_{n-1}b_{n-1}B_{n-1}a_{n-1})A_nb_nB_n)$, let k be the number of times that B_n occurs in $B_0\dots B_n$, and let $\alpha_1\dots\alpha_k \in B_n^k$ be the sequence of prescriptions by $\lambda_{A'}^P$ at histories $v_0(A_0b_0B_0a_0)\dots(A_{i-1}b_{i-1}B_{i-1}a_{n-1})A_ib_iB_i$ with $B_i = B_n$. Let $\lambda_A^P(v_0, (b_0, B_0)\dots(b_{n-1}, b_{n-1}), B_n) := s_B(\alpha_1, \dots, \alpha_k)$.

Note that if Player B can make a sequence of choices when Player A plays according to s, he can make the same sequence of choices when she plays according to s'. Let $b_0a_0b_1a_1\dots$ be the vector sequence resulting from s and some sequence of choices by Player B, let $b_0^ca_0^cb_1^ca_1^c\dots$ be the vector sequence resulting from s' and the same sequence of choices by Player B, and let $v_0v_1\dots$ and $v_0^cv_1^c\dots$ be the respective resulting outcomes. For all n, we have $b_n = b_n^c$, since $\lambda_A^M = \lambda_{A'}^M$ and Player B makes the same choices in both games. However, for all $B \in \mathcal{B}$ and all $n \in \mathbb{N}$, we have $\sum_{i\le n, B_i=B} a_i^c - a_i \in E_B$ by definition of λ_A^P via s_B. So $v_n^c - v_n \in \bigoplus_{B\in\mathcal{B}} E_B$ for all n. Moreover $v_0^cv_1^c\dots$ is bounded, since s' is a winning strategy, so $v_0v_1\dots$ is also bounded. Since this holds for arbitrary sequences of choices by Player B, s is also a winning strategy for Player A. □

Lemma 4.5. Player A *has a winning strategy in* $\langle \mathcal{A}, \mathcal{B} \rangle$ *iff she has a winning strategy in* $\langle \mathcal{A}, \overline{\mathcal{B}} \rangle$.

Proof. Every strategy for Player A in $\langle \mathcal{A}, \mathcal{B} \rangle$ is also a valid strategy in $\langle \mathcal{A}, \overline{\mathcal{B}} \rangle$, and if the strategy is winning in the former game, it is winning in the latter. Thus, we only need to show how to transform a winning strategy s' for Player A in $\langle \mathcal{A}, \overline{\mathcal{B}} \rangle$ to a winning strategy s for Player A in $\langle \mathcal{A}, \mathcal{B} \rangle$.

For this, let s agree with s' on which moves $A \in \mathcal{A}$ Player A is choosing, and be such that if s' picks some $y' \in \overline{B} \in \overline{\mathcal{B}}$ in round n, then s picks some $y \in B$ with $\mathrm{dist}(y, y') < 2^{-n}$. As the notion of a play is linear, if s' enforces that the play stays within some set E, then s enforces that the play stays within $E + \overline{\mathrm{Ball}}(0, 2)$. □

Lemma 4.6. *Consider a boundedness Minkowski game* $\langle \mathcal{A} = \{a_1, \dots, a_n\}, \mathcal{B} \rangle$ *such that there are* $a_i \in A_i$, $B \in \mathcal{B}$ *with*

$$0 \notin (\mathrm{CH}\{a_i \mid i \le n\}) + \mathrm{CH}(\overline{B}).$$

Then, Player B *has a simple winning strategy, given by choosing* a_i *as response to* Player A *playing* A_i *and playing* B *whenever it is the turn of* Player B *to pick the move.*

Proof. Let u be the convex projection of 0 onto $(\mathrm{CH}\{a_i \mid i \le n\}) + \mathrm{CH}(\overline{B})$. After each round, the position will move by $|u|$ in direction u, hence the play will diverge. □

Lemma 4.7. *Consider a boundedness Minkowski game* $\langle \mathcal{A} = \{A_1, \dots, A_n\}, \mathcal{B} \rangle$ *such that every* $B \in \mathcal{B}$ *is closed and convex, and for all* $a_i \in \mathrm{CH}(A_i)$, $B \in \mathcal{B}$*, we find that*

$$0 \in (\mathrm{CH}\{a_i \mid i \le n\}) + B.$$

Then, Player A *has a winning strategy.*

Proof. We will reduce the boundedness Minkowski game satisfying these conditions to a $+1/-1$-game. Central to the reduction is that we can describe the current position in the boundedness Minkowski game in the form $x_1a_1 + \cdots + x_na_n + p$ with $x_i \ge 0$, $a_i \in \mathrm{CH}(A_i)$ and p being some fixed vector. Initially, we choose $x_i = n$, the $a_i \in \mathrm{CH}(A_i)$ arbitrarily, and p such that the resulting expression equals v_0. The values x_i will be considered as the positions in the $+1/-1$-game.

If Player B picks some $a_i' \in A_i$, then $x_i a_i + a_i' = (x_i + 1)[\frac{x_i}{x_i+1} a_i + \frac{1}{x_i+1} a_i']$, with the expression within [ ] being an element of $\mathsf{CH}(A_i)$. Thus, the choice of A_i by Player A can be considered as choosing the ith position in the $+1/-1$-game.

Given some move $B \in \mathcal{B}$ and the current value of the a_i, we know that there is some $b \in B$ with $b = -\sum_{i=1}^{n} \alpha_i a_i$, where $\alpha_i \geq 0$, $\sum_{i=1}^{n} \alpha_i = 1$. Player A will choose such a b, which corresponds to updating the x_i to $x_i - \alpha_i$. Thus, the choice by Player B can be seen as Player B making a move in the $+1/-1$-game.[3]

By Proposition 3.6, Player A has a strategy in the $+1/-1$-game that ensures that $x_i \geq 0$ remains true. As $\sum_{i=1}^{n} x_i = n^2$ remains constant throughout the $+1/-1$-game, we also have that $x_i \leq n^2$ is ensured. This in turn implies that the play in the Minkowski game remains bounded, i.e., Player A wins. □

Lemma 4.8. *Let B be compact and convex. If there are $a_i \in \mathsf{CH}(\overline{A_i})$, $i \leq n$ such that $0 \notin \mathsf{CH}\{a_1, \dots, a_n\} + B$, then there are $a_i' \in A_i$ with $0 \notin (\mathsf{CH}\{a_1', \dots, a_n'\}) + B$.*

Proof. First, note that $0 \notin (\mathsf{CH}\{a_1, \dots, a_n\}) + B$ is an open property in the a_i, hence it remains true under small perturbations of the a_i. Thus, replacing A_i with $\overline{A_i}$ does not change anything.

Second, $0 \notin (\mathsf{CH}\{a_1, \dots, a_n\}) + B$ is equivalent to $\mathsf{CH}\{a_1, \dots, a_n\} \cap -B = \emptyset$. It is known that in $\mathbb{R}^d$, disjoint compact convex sets are separated by hyperplanes. Let P be a hyperplane separating $\mathbb{R}^d$ into $L \supseteq \mathsf{CH}\{a_1, \dots, a_n\}$ and $U \supseteq -B$. Now, since $a_i \in L \cap \mathsf{CH}(A_i)$, we can conclude that $L \cap A_i \neq \emptyset$. Pick $a_i' \in L \cap A_i$. Then, $L \supseteq \mathsf{CH}\{a_1', \dots, a_n'\}$, hence $0 \notin (\mathsf{CH}\{a_1', \dots, a_n'\}) + B$. □

Probabilistic boundedness Minkowski games. Let us consider a variant of the boundedness Minkowski games where the decisions formerly made by Player B are now made by probability measures: points in a move of Player A are no longer chosen by Player B but by a probability measure, and moves of Player B are now drawn from another probability measure. So, in this new game, Player A is playing against nature.

More formally, a *probabilistic boundedness Minkowski game* is a triple $\langle \mathcal{M}, \mathcal{B}, \lambda \rangle$ where $\mathcal{B}$ is, as before, a finite set of moves in $\mathbb{R}^d$, and λ is a full-support probability measure over $\mathcal{B}$, and $\mathcal{M}$ is a finite set of probability measures over $\mathbb{R}^d$ with bounded support. (The support of a probability measure μ is the set $\operatorname{supp} \mu$ of points x such that $0 < \mu(U)$ for every neighborhood U of x. Full-support refers to the support being the whole domain.)

A play starts in some irrelevant $v_0 \in \mathbb{R}^d$, then Player A chooses some $\mu \in \mathcal{M}$ and some b_0 is drawn randomly according to μ, then some $B \in \mathcal{B}$ is drawn according to λ, and Player A chooses some a_1 in B, then Player A chooses some $\mu \in \mathcal{M}$, and so on. As before, the resulting outcome $v_0 v_1 \dots v_n \dots$ is winning for Player A if it is bounded.

Proposition 4.9. *Let $\langle \mathcal{M}, \mathcal{B}, \lambda \rangle$ be a probabilistic boundedness Minkowski game. Let $\mathcal{A}$ be the set of the supports of the probability measures in $\mathcal{M}$. Then* Player A *has a winning strategy in $\langle \mathcal{M}, \mathcal{B}, \lambda \rangle$ iff she has one in $\langle \mathcal{A}, \mathcal{B} \rangle$. Otherwise* Player A *looses with probability one.*

Proof. Let $\mathcal{M} = \{\mu_1, \dots, \mu_n\}$ and for all i let A_i be the support of μ_i.

If $0 \in (\mathsf{CH}\{a_i \mid i \leq n\}) + \mathsf{CH}(\overline{B})$ for all $(a_i)_{i \leq n}$ with $a_i \in A_i$ and $B \in \mathcal{B}$, then Player A has a winning strategy in $\langle \mathcal{A}, \mathcal{B} \rangle$ by Theorem 4.2, which is also winning in $\langle \mathcal{M}, \mathcal{B}, \lambda \rangle$.

Otherwise, let $(a_i)_{i \leq n}$ with $a_i \in A_i$ and $B \in \mathcal{B}$ be such that $0 \notin (\mathsf{CH}\{a_i \mid i \leq n\}) + \mathsf{CH}(\overline{B})$, and let l be the (positive) distance from 0 to the set $(\mathsf{CH}\{a_i \mid i \leq n\}) + \mathsf{CH}(\overline{B})$. Let $p > 0$ be less than $\lambda(B)$ for all $B \in \mathcal{B}$ and then $\mu_i(B(a_i, \frac{l}{2}))$ for all i, where $B(a_i, \frac{l}{2})$ is the open ball of center a_i and radius $\frac{l}{2}$. Note that $0 \notin (\mathsf{CH}(\cup_{i \leq n} B(a_i, \frac{l}{2}))) + \mathsf{CH}(\overline{B})$, and let c be the projection of 0 onto the convex set $(\mathsf{CH}(\cup_{i \leq n} B(a_i, \frac{l}{2}))) + \mathsf{CH}(\overline{B})$.

[3] Player B can not induce all moves available to him in the $+1/-1$-game by picking some $B \in \mathcal{B}$, but this is irrelevant for our purpose, as we are concerned with winning strategies of Player A.

Let $k \in \mathbb{N}$. The probability that the following happens k times in a row is at least p^{2k}: first, whichever μ_i is chosen by Player A, the points with positive probability measure are all in $B(a_i, \frac{l}{2})$; second, B is drawn from λ. In that case, the position moves by at least distance $k\|c\|$ in the direction of c. Thus, the probability that it never happens during an infinite play is zero. Moreover, if it happens it implies that the infinite play leaves $B(0, \frac{k\|c\|}{2})$ at some point. So, the play leaves $B(0, \frac{k\|c\|}{2})$ with probability one. Since a countable union of null sets is again a null set, the probability that the play leaves $B(0, \frac{k\|c\|}{2})$ for all $k \in \mathbb{N}$ is one. □

Corollary 4.10. *The following are equivalent for a probabilistic boundedness Minkowski game:*

(1) Player A *has a winning strategy.*
(2) Player A *has a strategy that wins with probability 1.*
(3) Player A *has a strategy that wins with positive probability.*

Slightly more generally, we could replace only some of the decision stages of Player B with a probability measure. Thanks to similar proof techniques, we could then state a modification of Proposition 4.9, where "Otherwise Player A looses with probability one" is replaced with "Otherwise Player B has a strategy winning with probability one."

5 COMPUTATIONAL COMPLEXITY OF THE BOUNDEDNESS PROBLEM

In the previous section, we have provided general results on boundedness Minkowski games. Here, we study the computational complexity of the associated decision problem.[4] To formulate complexity results, we need to consider classes of games that are defined in some effective way. We consider here three ways for the representation of sets of moves: by finite sets of linear constraints (or the convex hull of a finite set of vectors), by formulas in the first-order theory of the reals (that strictly extend the expressive power of linear constraints), and as compact sets or overt sets (closed sets with positive information) in the sense of computable analysis.

5.1 Moves Defined by Linear Constraints or as Convex Hulls

We prove the following main result in this section:

Theorem 5.1. *Given a boundedness Minkowski game $\langle \mathcal{A}, \mathcal{B} \rangle$ where moves in the sets of moves $\mathcal{A}$ and $\mathcal{B}$ are defined by finite sets of rational linear constraints or as convex hulls of a finite sets of rational vectors, deciding the winner is* coNP-complete. *The hardness already holds for one-sided boundedness games.*

We establish this result by showing how to reduce the 3-SAT problem to the complement of the boundedness Minkowski game problem. For that we need some intermediate results. A simple strategy λ_B for Player B is called a *vertex strategy*, if the $a_i \in A_i$ chosen by λ_B are always some vertex of A_i, where the set of the vertices of A_i is denoted $\mathrm{Ver}(A_i)$.

Corollary 5.2. *If* Player B *has a winning strategy in a boundedness Minkowski game $\langle \mathcal{A}, \mathcal{B} \rangle$ with closed moves in $\mathcal{A}$, then he has a winning vertex strategy.*

Proof. By Lemma 4.3 (4. ⇔ 6.), Player B wins $\langle \mathcal{A}, \mathcal{B} \rangle$ iff he wins $\langle \{\mathrm{Ver}(A) \mid A \in \mathcal{A}\}, \mathcal{B}, a_0 \rangle$. By Lemma 4.3 (¬4. ⇒ ¬5.) applied to the latter game, he then even has a simple winning strategy in $\langle \{\mathrm{Ver}(A) \mid A \in \mathcal{A}\}, \mathcal{B}, a_0 \rangle$. But this is just the definition of a vertex strategy. □

[4]For all our complexity results, all the encoding of numbers and vectors that we use are the natural ones; i.e., integer or rational numbers are encoded succinctly in binary.

As a consequence of the previous corollary and of the determinacy of boundedness Minkowski games (Corollary 4.2), to show that Player A has a winning strategy, it is sufficient to show that she can spoil all the vertex strategies of Player B. This is an important ingredient of the reduction below.

Lemma 5.3. *There is a polynomial time reduction from the 3SAT problem to the complement of the boundedness problem for one-sided Minkowski games with moves defined by closed polytopes.*

Proof. First, let us point out that the proof that we provide below works for both the H-representation and the V-representation. This is because the moves that we need to construct are all the convex hull of exactly three vectors. So, the H-representation of such a convex hull can be obtained in polynomial time.

Let $\Psi = \{C_1, C_2, \ldots, C_n\}$ be a set of clauses with three literals defined on the set of Boolean variables $X = \{x_1, x_2, \ldots, x_m\}$. Each C_i is of the form $\ell_{i1} \vee \ell_{i2} \vee \ell_{i3}$ where each ℓ_{ij} is either x or $\neg x$ with $x \in X$.

To define the set of moves $\mathcal{A}$ for Player A, we associate a move A_i with each clause C_i. The move is a subset of $\mathbb{R}^d$, where $d = 2 \cdot |X| = 2 \cdot m$, defined from C_i as follows. We associate with each variable $x_k \in X$ two dimensions of $\mathbb{R}^d$: $2k - 1$ and $2k$, and to each literal ℓ_{ij} a vector denoted by $\mathsf{Vect}(\ell_{ij})$ defined as follows. If the literal $\ell_{ij} = x_k$, then the vector $\mathsf{Vect}(\ell_{ij})$ has zeros everywhere but in dimension $2k - 1$ and $2k$ where it is, respectively, equal to 1 and -1. If the literal $\ell_{ij} = \neg x_k$, then the vector $\mathsf{Vect}(\ell_{ij})$ has zeros everywhere but in dimension $2k - 1$ and $2k$ where it is, respectively, equal to -1 and 1. So, for all literals ℓ_1 and ℓ_2, $\mathsf{Vect}(\ell_1) + \mathsf{Vect}(\ell_2) = \mathbf{0}$ if and only if $\ell_1 \equiv \neg\ell_2$ or $\ell_2 \equiv \neg\ell_1$. Finally, the move associated with the clause $C_i = \ell_{i1} \vee \ell_{i2} \vee \ell_{i3}$ is

$$A_i = \mathsf{CH}(\mathsf{Vect}(\ell_{i1}), \mathsf{Vect}(\ell_{i2}), \mathsf{Vect}(\ell_{i3})).$$

It remains to prove the correctness of our reduction. By Corollary 5.2, Player B has a winning strategy iff he has a winning vertex strategy, so we only need to consider the latter. We call a vertex strategy λ_B^v of Player B *valid*, iff there are no i_1, i_2 such that $\lambda_B^v(A_{i_1}) = -\lambda_B^v(A_{i_2})$. We will argue first that Player B has a valid vertex strategy iff there is a satisfying assignment for Ψ. Then, we argue that a vertex strategy for Player B is winning iff it is valid. The two parts together with Corollary 5.2 yield the desired claim that Player B has a winning strategy iff Ψ is satisfiable.

Claim: There is a valid vertex strategy iff there Ψ is satisfiable.

Given a satisfying truth assignment, we can pick some vertex strategy such that the vertex chosen always corresponds to some true literal. In particular, we never chose vertices corresponding to both x and $\neg x$ – but by construction of the moves, this ensures that the vertex strategy is valid. Conversely, a valid vertex strategy is never choosing vertices corresponding to both x and $\neg x$. Thus, we can obtain a truth assignment by making all literals corresponding to vertices chosen by the strategy true, and choosing arbitrarily for the remaining literals. This truth assignment satisfies Ψ by construction.

Claim: A vertex strategy is valid iff it is winning.

Assume that a vertex strategy λ_B^v is not valid, and that moves A_{i_1}, A_{i_2} witness this. Then, if Player A alternates between playing A_{i_1} and A_{i_2}, the resulting game remains bounded and is hence won by Player A. If Player B plays some valid vertex strategy λ_B^v, and Player A plays infinitely often some move A_i, then the position will diverge in the two dimensions associated with the literal $\lambda_B^v(A_i)$. By the pigeon hole principle, Player A has to play some move infinitely often. It follows that a valid vertex strategy is a winning strategy. □

We have now established the hardness part of Theorem 5.1. The coNP-membership part is covered by the following lemma.

Lemma 5.4. *Negative instances of the boundedness Minkowski games expressed with moves defined as sets of rational linear inequalities or convex hull of finite sets of rational vectors can be recognized by a nondeterministic polynomial time Turing machine.*

Proof. To show that Player A has no winning strategy, by Lemma 4.3 (1. ⇔ 8. ⇔ 9.), it suffices to exhibit $a_1 \in \mathrm{Ver}(A_1)$, $a_2 \in \mathrm{Ver}(A_2), \ldots, a_{n_A} \in \mathrm{Ver}(A_{n_A})$, and one $B \in \mathcal{B}$, such that

$$\mathbf{0} \notin \mathrm{CH}(a_1, a_2, \ldots, a_{n_A}) + \overline{B}.$$

If each A_i is given by a set of linear constraints, then each vertex in $\mathrm{Ver}(A_i)$ has a binary representation that is polynomial in the description of A_i, and so those points can be guessed in polynomial time.[5] If the A_i are given as convex hulls of a finite set of points, then we can obviously guess a vertex in polynomial time, too.

Finally, let us show that we can check in deterministic polynomial time that

$$\mathbf{0} \notin \mathrm{CH}(a_1, a_2, \ldots, a_{n_A}) + \overline{B}.$$

If B is given via rational linear inequalities, then this is equivalent to decide if the following set of linear constraints is unsatisfiable:

$$\begin{aligned}
&\textstyle\bigwedge_{i=1}^{i=n_A} 0 \le \alpha_i \le 1,\\
&\wedge \textstyle\sum_{i=1}^{i=n_A} \alpha_i = 1,\\
&\wedge x = \textstyle\sum_{i=1}^{i=n_A} \alpha_i a_i,\\
&\wedge y \in \overline{B},\\
&\wedge 0 = x + y.
\end{aligned}$$

If B is given as $\mathrm{CH}(b_1, \ldots, b_k)$ with rational b_j, then we need to decide whether

$$\mathbf{0} \notin \mathrm{CH}(a_1 + b_1, a_2 + b_1, \ldots, a_{n_A} + b_k).$$

Thus, the problem reduces to deciding feasibility of rational systems of inequalities, which is known to be decidable in polynomial time [6]. □

5.2 Fixed Dimension and Polytopic Moves

This section shows that given $d \in \mathbb{N}$ and a Minkowski game $\langle \mathcal{A}, \mathcal{B} \rangle$ with closed polytopic moves in $\mathbb{R}^d$, deciding which player has a winning strategy can be done in deterministic polynomial time. Note that for a fixed d, we can translate V-representations of (closed) polytopes into H-representations, and vice-versa (see Theorem 2.1), so w.l.o.g. we focus here on the V-representation of polytopes. In this setting, by Lemma 4.3 it suffices to consider games with finite moves, since the extremal points of a polytope are finitely many. The degree of the polynomial (upper-)bounding the algorithmic complexity will be $2d + 2$ in general, and $d + 1$ for single-sided games. By Lemma 4.3 again, Player B has a winning strategy iff there exist $a_1 \in A_1, \ldots, a_n \in A_n$ (the moves in $\mathcal{A}$) and $B \in \mathcal{B}$ such that $0 \notin \mathrm{CH}(a_1, \ldots, a_n) + \mathrm{CH}(B)$. Trying out all the tuples $(a_1, a_2, \ldots, a_n)$ cannot be done in polynomial time. Instead, let us rephrase the condition using a hyperplane separation result.

Observation 5.5. Player B *has a winning strategy iff there exist* $a_1 \in A_1, \ldots, a_n \in A_n$, $B \in \mathcal{B}$, *and a hyperplane separating* $\{a_1, \ldots, a_n\} + B$ *from* 0.

Trying out all the infinitely many hyperplanes is also unfeasible, so we will show how to restrict the search space to a small finite set of hyperplanes. Let us first give a very rough intuition. If a

[5] By definition, the elements of $\mathrm{Ver}(A_i)$ are solutions of systems of linear equations. As a consequence of the Gaussian elimination algorithm, those vertices are points with coordinates that are rational numbers with succinct binary encoding; i.e., they can be represented with a polynomial number of bits.

separating hyperplane exists, then we can "push" it away from 0 while retaining its separating property. Once a critical point (which cannot be passed without losing the property) is hit, the hyperplane can still be rotated around the point (or axis of two points, etc.). Ideally, the hyperplane would eventually settle while containing d affinely independent points from $\cup\mathcal{A} + B$, so it would suffice to check all the possible settling positions. There is a difficulty, though: rotating may go on and on without ever settling if $\cup\mathcal{A} + B$ does not contain d affinely independent points. This difficulty is overcome by adding finitely many "dummy" points to $\cup\mathcal{A} + B$. Adding the canonical basis will do just fine, as is done in Theorem 5.6 below.

Then, Algorithm 1 decides the winner of a Minkowski game with finite moves. It merely scans the search space related to the existential quantifier in Theorem 5.6, which then guarantees correctness.

Theorem 5.6. *Let $\langle\mathcal{A},\mathcal{B}\rangle$ be a Minkowski game with finite moves in $\mathbb{R}^d$, and let $C = \{e_1, \dots, e_d\}$ be the canonical basis of $\mathbb{R}^d$. The game is won by* Player B *iff there exist $B \in \mathcal{B}$ and affinely independent $x_1, \dots, x_d \in (\cup\mathcal{A} + B) \cup C$ s.t. for all $A \in \mathcal{A}$ there exists $a \in A$ s.t. the affine hull of $x_1, \dots, x_d$ separates $a + B$ from* 0.

Before proving Theorem 5.6, we recall some basics in linear/affine algebra and prove a few lemmas. For all $E \subseteq \mathbb{R}^d$ let $\mathsf{LS}(E) := \{\sum_{i=0}^{n} \alpha_i x_i \mid \forall i(x_i \in E \wedge \alpha_i \in \mathbb{R})\}$ be the linear span of E, and let $\mathsf{AH}(E) := \{\sum_{i=0}^{n} \alpha_i x_i \mid \forall i(x_i \in E \wedge \alpha_i \in \mathbb{R}) \wedge \sum_{i=0}^{n} \alpha_i = 1\}$ be its affine hull. Clearly $\mathsf{CH}(E) \subseteq \mathsf{AH}(E) \subseteq \mathsf{LS}(E)$ for all subsets $E \subseteq \mathbb{R}^d$. A set of points is called affinely independent if no point is in the affine hull of the others. Affine hulls are affine spaces, i.e., sums $x + L$ where $x \in \mathbb{R}^d$ and L is a linear subspace of $\mathbb{R}^d$. Also, $x + L = y + L$ for all $y \in x + L$. In particular, $\mathsf{AH}(E) = x + \mathsf{LS}(E - x)$ for all $x \in E$. The dimension of $x + L$ is defined as the dimension of L, so for all $x_1, \dots, x_n \in \mathbb{R}^d$,

$$\dim \mathsf{AH}(x_1, \dots, x_n) = \operatorname{rank}(x_2 - x_1, \dots, x_n - x_1) \tag{1}$$

and $\dim \mathsf{CH}(x_1, \dots, x_n) := \dim \mathsf{AH}(x_1, \dots, x_n)$. It is straightforward to show

$$\mathsf{LS}(x_1, \dots, x_n) = \mathsf{AH}(0, x_1, \dots, x_n). \tag{2}$$

Observation 5.7 states two inequalities about the rank of a finite set of vectors, and Lemma 5.8 characterizes when the second one is an equality.

Observation 5.7. $\operatorname{rank}(x_1, \dots, x_n) \leq 1 + \operatorname{rank}(x_2 - x_1, \dots, x_n - x_1) \leq 1 + \operatorname{rank}(x_1, \dots, x_n)$.

Proof. First inequality: if $x_1, \dots, x_k$ are linearly independent, so are $x_2 - x_1, \dots, x_k - x_1$. Second one: $\mathsf{AH}(x_1, \dots, x_n) \subseteq \mathsf{AH}(0, x_1, \dots, x_n) = \mathsf{LS}(x_1, \dots, x_n)$ by Equation (2). □

Lemma 5.8. $0 \in \mathsf{AH}(x_1, \dots, x_n)$ *iff* $\operatorname{rank}(x_1, \dots, x_n) = \operatorname{rank}(x_2 - x_1, \dots, x_n - x_1)$.

Proof. $0 \in \mathsf{AH}(x_1, \dots, x_n)$ iff $\mathsf{AH}(x_1, \dots, x_n) = \mathsf{AH}(0, x_1, \dots, x_n)$, i.e., iff $\mathsf{AH}(x_1, \dots, x_n) = \mathsf{LS}(x_1, \dots, x_n)$ by Equation (2), i.e., iff $\operatorname{rank}(x_2 - x_1, \dots, x_n - x_1) = \operatorname{rank}(x_1, \dots, x_n)$ by Equation (1). □

Lemma 5.9 below says that points from a larger set can be added to a smaller set while fulfilling two seemingly contradictory requirements: adding few enough points to preserve a small convex hull and adding enough points to generate a large linear span.

Lemma 5.9. *For all $S \subseteq E \subseteq \mathbb{R}^d$, if $0 \notin \mathsf{CH}(S)$, there exists $S' \subseteq E$ such that $0 \notin \mathsf{CH}(S \cup S')$ and $\mathsf{LS}(S \cup S') = \mathsf{LS}(E)$.*

Proof. Let $x_1, \dots, x_n \in E \setminus \mathsf{LS}(S)$ be as many linearly independent points as possible, so $\mathsf{LS}(S \cup S') = \mathsf{LS}(E)$, where $S' := \{x_1, \dots, x_n\}$. For all $y_1, \dots, y_k \in S$, if $0 = \sum_{i=1}^{n} \alpha_i x_i + \sum_{j=1}^{k} \beta_j y_j$ is a convex combination, so is $0 = \sum_{j=1}^{k} \beta_j y_j$ by linear independence. It follows that $0 \notin \mathsf{CH}(S \cup S')$. □

Lemma 5.9 above corresponds to, informally, a careful pushing and rotating the hyperplane. It is used in one of the cases in the proof of Lemma 5.10 below.

Lemma 5.10. *For all $S \subseteq E \subseteq \mathbb{R}^d$ such that $0 \notin \mathsf{CH}(S)$ and $\mathrm{rank}(E) = d$, there exist affinely independent $x_1, \ldots, x_d \in E$ such that $\mathsf{AH}(x_1, \ldots, x_d)$ separates S from 0.*

Proof. If $0 \notin \mathsf{AH}(E)$, then $\dim \mathsf{AH}(E) = d - 1$ by Lemma 5.8 and Observation 5.7, and any d affinely independent points in E witness the claim. Let us assume that $0 \in \mathsf{AH}(E)$, so $\dim \mathsf{CH}(E) = d$ by Lemma 5.8. By Lemma 5.9, we can assume w.l.o.g. that $\mathsf{LS}(S) = \mathbb{R}^d$, so $\dim \mathsf{AH}(S) \in \{d, d-1\}$ by Observation 5.7. If $\dim \mathsf{AH}(S) = d$, then among the vertices of a well-chosen facet of $\mathsf{CH}(S)$ there are d affinely independent points witnessing the claim. If $\dim \mathsf{AH}(S) = d - 1$, then $0 \notin \mathsf{AH}(S)$ by Lemma 5.8, so any d affinely independent points from S witness the claim. ◻

Proof of Theorem 5.6. The "if" implication is clear by Observation 5.5, so let us assume that the game is won by Player B. Let $A_1, \ldots, A_n$ be the elements of $\mathcal{A}$. By Lemma 4.3, there exist $a_1 \in A_1, \ldots, a_n \in A_n$ and $B \in \mathcal{B}$ such that $0 \notin \mathsf{CH}(a_1, \ldots, a_n) + \mathsf{CH}(B)$, which is equal to $\mathsf{CH}(\{a_1, \ldots, a_n\} + B)$. By Lemma 5.10 there exist affinely independent $x_1, \ldots, x_d \in (\cup_{i=1}^n A_i + B) \cup \{e_1, \ldots, e_d\}$ such that $\mathsf{AH}(x_1, \ldots, x_d)$ separates $\{a_1, \ldots, a_n\} + B$ from 0, i.e., each $a_i + B$ from 0. ◻

Corollary 5.11. *Consider Minkowski games with moves $A_1, \ldots, A_n$ for* Player A *and $B_1, \ldots, B_m$ for* Player B *that are finite sets of rational vectors. The algorithmic complexity of deciding the winner of the game is bounded from above by a multivariate polynomial of degree $2d + 2$ with leading term $\sum_{i,j} |A_i|^{d+1}|B_j|^{d+1}$.*

Proof. The time required for the rank computation on Line 5 of Algorithm 1 is a function of d, so we can ignore it. Given j, Line 9 is reached at most $(\sum_i |A_i||B_j| + d)^d(\sum_i |A_i|)$ times, and the time required to decide separation on Line 9 is of the form $f(d)|B_j|$, so the time required by the whole algorithm is of the form $\sum_j |B_j|(\sum_i |A_i||B_j| + d)^d(\sum_i |A_i|)$, which is equivalent to $\sum_{i,j} |A_i|^{d+1}|B_j|^{d+1}$. ◻

5.3 Moves Defined in the First-Order Theory of the Reals

In this subsection, we show that if the moves are definable in the first-order theory of the reals (over the signature $(\mathbb{R}, +, *, 0, 1, <)$), then so is Condition (8) in Lemma 4.3. Condition (8) was that for all $(a_i)_{i \le n}$ with $a_i \in A_i$ and $B \in \mathcal{B}$ it holds that

$$0 \in (\mathsf{CH}\{a_i \mid i \le n\}) + \mathsf{CH}(\overline{B}).$$

As the first-order theory is decidable, so is the question of who is winning a given boundedness Minkowski game with such moves.

We consider first-order formulas with binary function symbols + and $\cdot$, constants 0 and 1 and binary relation symbol <. A move $A \subseteq \mathbb{R}^d$ is defined by some formula ϕ_A with d free variables $x_1, \ldots, x_k$ iff $A = \{(x_1, \ldots, x_k) \in \mathbb{R}^d \mid \phi(x_1, \ldots, x_n)\}$. If ϕ defines A, then

$$\phi_{\mathrm{conv}} = \exists a_1^1, \ldots, a_{d+1}^1, \ldots, a_{d+1}^d, \alpha_1, \ldots, \alpha_{d+1} \quad \bigwedge_{i=1}^{d+1} \phi(a_i^1, \ldots, a_i^d)$$
$$\wedge \sum_{i=1}^{d+1} \alpha_i = 1 \wedge \bigwedge_{i=1}^{d+1} 0 \le \alpha_i \wedge \bigwedge_{j=1}^{d} \left(x_j = \sum_{i=1}^{d+1} \alpha_i a_i^j \right)$$

ALGORITHM 1: FindWinner

```
Function FindWinner is
    input: d ∈ ℕ, polytopes A_1, ..., A_n, B_1 ..., B_m ⊆ ℚ^d
    output: the winner of the corresponding Minkowski game
    Let finite C be the canonical basis of ℝ^d;
    for 1 ≤ j ≤ m do
        for x_1, ..., x_d ∈ (∪_{i=1}^n A_i + B_j) ∪ C do
            if Rank(x_2 − x_1, ..., x_d − x_1) = d − 1 then
                w ← 0;
                for 1 ≤ i ≤ n do
                    for a_i ∈ A_i do
                        if Sep(x_1, ..., x_d, a_i + B_j) then w ← w + 1;
                    end for
                end for
                if w = n then return "Player B wins";
            end if
        end for
    end for
    return "Player A wins";
end

Function Rank is
    input: vectors x_1, ..., x_k ∈ ℚ^d for some d ∈ ℕ.
    output: the rank of {x_1, ..., x_n}
end

Function Sep is
    input: points x_1, ..., x_k ∈ ℚ^d, finite Y ⊆ ℚ^d
    output: whether the hyperplane including {x_1, ..., x_d} separates Y from 0
end
```

defines $\mathrm{CH}(A)$. Also, the formula

$$\phi_{\mathrm{cl}} = \forall \varepsilon \quad \varepsilon > 0 \Rightarrow \left(\exists a_1, \ldots, a_d \quad \phi(a_1, \ldots, a_d) \wedge \bigwedge_{i=1}^{d} a_i < x_i + \varepsilon \wedge x_i < a_i + \varepsilon \right)$$

defines $\overline{A}$. It then follows that Condition (8) in Lemma 4.3 is expressible as some formula ϕ_{win} obtained from the formulas ϕ_A, ϕ_B defining the moves in $\mathcal{A}$ and $\mathcal{B}$. Moreover, the length of the formula ϕ_{win} is polynomially bounded in the sum of the length of the ϕ_A, ϕ_B.

Proposition 5.12. *Deciding the winner of a boundedness Minkowski game with moves defined in the first-order theory of the reals is logspace-equivalent to deciding the truth of sentences in the first order theory of the reals.*

Proof. As explained above, deciding whether Player A wins reduces to deciding whether ϕ_{win} is true. Conversely, given some formula ϕ consider the one-dimensional one-sided Minkowski game $\langle 0, \{A\}, \{0\} \rangle$ where $A = \{x \mid (x = 0 \wedge \phi) \vee (x = 1 \wedge \neg\phi)\}$. Clearly, Player A wins $\langle 0, \{A\}, \{0\} \rangle$ iff ϕ is true. □

The exact computational complexity of deciding truth of sentences in the first order theory of the reals is open (cf. Reference [9]). In Reference [3], an EXPSPACE upper bound is provided, while Reference [10, Theorem 3] shows an exponential time lower bound.

5.4 The Computable Analysis Perspective

If we represent the sets involved in the boundedness Minkowski games via polyhedra or first-order formula, then we have only restricted expressivity available to us. Using notions from computable analysis [29], we can, however, consider computability for all boundedness Minkowski games with closed moves—and as Lemma 4.3 demonstrated, this is not a problematic restriction. The reader not interested in computable analysis can safely skip this subsection, as well as Section 7.1. The reader interested in computable analysis but not familiar with it might want to consult Reference [24] for a short introduction to the concepts used here.

We do have to decide on a representation for the sets. We have the spaces $\mathcal{A}(\mathbb{R}^d)$ of closed subsets, $\mathcal{K}(\mathbb{R}^d)$ of compact subsets and $\mathcal{V}(\mathbb{R}^d)$ of overt subsets available. In $\mathcal{A}(\mathbb{R}^d)$, a closed subset can be seen as being represented by an enumeration of rational balls exhausting its complement (i.e., by negative information). The space $\mathcal{K}(\mathbb{R}^d)$ adds to that some $K \in \mathbb{N}$ such that the set is contained in $[-K, K]^d$. In $\mathcal{V}(\mathbb{R}^d)$, a closed subset is instead represented by listing all rational balls intersecting it (i.e., by positive information). As the involved spaces are all connected, we cannot expect decidability, and instead turn our attention to semidecidability, i.e., truth values in the Sierpinski space $\mathbb{S}$.

A relevant property is that universal quantification over compact sets from $\mathcal{K}(\mathbb{R}^d)$ and existential quantification over overt sets from $\mathcal{V}(\mathbb{R}^d)$ preserve open predicates. We can use the former to find that:

Proposition 5.13. *The Minkowski sum $\alpha+ : \mathcal{A}(\mathbb{R}^d) + \mathcal{K}(\mathbb{R}^d) \to \mathcal{A}(\mathbb{R}^d)$ is computable.*

Proof. $\alpha\notin : \mathbb{R}^d \times \mathcal{A}(\mathbb{R}^d) \to \mathbb{S}$ is an open predicate by definition. Now note that $y \notin A + B$ iff $\forall z \in B\ y - z \notin A$. □

The Minkowski sum of two closed sets is not computable as a closed set: consider some $A \in \mathcal{A}(\mathbb{N})$. Then $0 \in A + (-A)$ iff $A \neq \emptyset$. If + were computable, then the former would be a Π^0_1-property, whereas the latter is Π^0_2-complete, and we find a contradiction.

It was already shown in Reference [17, Proposition 1.5] (also, Reference [31]) that convex hull is a computable operation on compact sets, but not on closed sets. Put together, we find that:

Proposition 5.14. *Consider boundedness Minkowski games, where moves $A \in \mathcal{A}$ are given as overt sets (i.e., in $\mathcal{V}(\mathbb{R}^d)$) and moves $B \in \mathcal{B}$ are given as compact sets (i.e., in $\mathcal{K}(\mathbb{R}^d)$). The set of games won by* Player B *constitutes a computable open subset.*

Proof. By Lemma 4.3, Player B can win iff for $\mathcal{A} = \{A_0, \ldots, A_n\}$, we find that there exists $a_i \in A_i$ and $B \in \mathcal{B}$:

$$0 \notin (\mathsf{CH}\{a_i \mid i \leq n\}) + \mathsf{CH}(\overline{B}).$$

As B is given as a compact set, we can compute $(\mathsf{CH}\{a_i \mid i \leq n\}) + \mathsf{CH}(\overline{B}) \in \mathcal{A}(\mathbb{R}^d)$. As before, $\alpha\notin : \mathbb{R}^d \times \mathcal{A}(\mathbb{R}^d) \to \mathbb{S}$ is an open predicate, and existential quantification over overt sets preserves open predicates. Thus, the entire requirements define a computably open subset of the space of Minkowski games. □

As there is a representation of the probability measures on a metric space [27], we can also consider the probabilistic boundedness Minkowski games in the computable analysis setting. As shown in Reference [25], given some overt set $A \in \mathcal{V}(\mathbb{R}^d)$, we can compute a probability measure μ on $\mathcal{V}(\mathbb{R}^d)$ such that $\operatorname{supp} \mu = A$, and conversely, given a measure μ, we can compute its support as an overt set. Thus, by Proposition 4.9, we see that Proposition 5.14 applies to probabilistic boundedness Minkowski games as well.

6 THE WINNING REGION OF THE SAFETY PROBLEM

We now turn our attention to safety Minkowski games. Given some move sets $\mathcal{A}$, $\mathcal{B}$ and the safe zone Safe, we want to understand for which initial positions $v_0 \in$ Safe Player A has a winning strategy in the safety Minkowski game $\langle \mathcal{A}, \mathcal{B}, \text{Safe} \rangle$. In a minor abuse of notation, we speak of **the** safety Minkowski game $\langle \mathcal{A}, \mathcal{B}, \text{Safe} \rangle$, and call the set of v_0 such that Player A has a winning strategy the *winning region* W.

Let us first note that these games are determined by Borel determinacy [20] (and also from techniques in Reference [11]), since the set of the winning plays for Player A is a closed set (for the usual product topology over discrete topology).

We give three kinds of general results: first (Lemma 6.2), the winning region is the greatest fixed point of an operator that removes the points where Player B can provably win (in finitely many rounds); second (Proposition 6.5), topological and finiteness assumptions about the game implies topological properties of the winning region and of its boundary; third (Lemma 6.6), the winning region is *stuck* inside Safe, i.e., in any direction there are arbitrarily small translations moving a point from W outside of Safe.

Let $\langle \mathcal{A}, \mathcal{B}, \text{Safe} \rangle$ be a safety game. Given E a target set, $f(E)$ is defined below as the positions from where Player A can ensure to fall in E after one round of the game.

Definition 6.1. For all $E \subseteq \mathbb{R}^d$ let

$$f(E) := \{x \in \mathbb{R}^d \mid \exists A \in \mathcal{A}, \forall a \in A, \forall B \in \mathcal{B}, \exists b \in B, x + a + b \in E\}$$

and let $g(E) := f(E) \cap \text{Safe}$.

Note that the fixed-point characterization of the winning region requires no assumption.

Lemma 6.2. *The winning region W of* Player A *is the greatest fixed point of g, even for infinite $\mathcal{A}$ and $\mathcal{B}$.*

Proof. First, note that every fixed point of g is included in W, since starting from there allows Player A to stay there for one round, and therefore forever. Therefore, it suffices to show that $g(W) = W$, which holds, since being in W is equivalent to being in Safe and able to reach W in one round. □

At the cost of a finiteness assumption on $\mathcal{A}$ below, we invoke the Kleene fixed point theorem and show that the winning region can be computed in ω many steps.

Proposition 6.3. *Let $S_0 := \mathbb{R}^d$, let $S_{n+1} := g(S_n)$ for all n, and let $S_\omega := \cap_{n \in \mathbb{N}} S_n$. S_ω is the greatest fixed point of g, even for infinite $\mathcal{B}$.*

Proof. First, note that f is monotone, and therefore so is g. To prove the lemma, it suffices to invoke (the dual of) the Kleene fixed point theorem, after proving (the dual of) the Scott continuity, namely, that if $(E_n)_{n \in \mathbb{N}}$ satisfy $E_{n+1} \subseteq E_n \subseteq \mathbb{R}^d$ for all n, then $g(\cap_n E_n) = \cap_n g(E_n)$. We prove it for f below, then it holds clearly for g, too.

$f(\cap_n E_n) \subseteq \cap_n f(E_n)$ by monotonicity. Conversely, let $x \in \cap_n f(E_n)$, so for all n there is $A_n \in \mathcal{A}$ such that $x + a + B \cap E_n \neq \emptyset$ for all $a \in A_n$ and $B \in \mathcal{B}$. By finiteness of $\mathcal{A}$ there is a constant subsequence (A) of (A_n), defined by some φ, so $x + a + B \cap E_{\varphi(n)} \neq \emptyset$ for all $a \in A$ and $B \in \mathcal{B}$. So $x \in f(\cap_n E_n)$, since $\cap_n E_{\varphi(n)} = \cap_n E_n$. □

The example below shows that the restriction to games with finitely many moves for $\mathcal{A}$ is necessary in Proposition 6.3. The game is essentially a single player game, i.e., Player B has no non-trivial choices to make.

Example 6.4. Consider the 5-dimensional safety Minkowski game $\langle \mathcal{A}_{\text{init}} \cup \mathcal{A}_{\text{up}} \cup \mathcal{A}_{\text{down}}, \{\{0\}\}, \text{Safe} \rangle$, where:

- $\mathsf{Safe} = [0,1]^2 \times \mathrm{CH}(\{(x,x,1) \in [0,1]^3\} \cup [0,1] \times \{(0,0)\})$.
- $\mathcal{A}_{\text{init}} = \{(0,+1,+2^{-k},0,0) \mid k \in \mathbb{N}\}$.
- $\mathcal{A}_{\text{up}} = \{(+2^{-k},0,0,+2^{-k},+1) \mid k \in \mathbb{N}\}$.
- $\mathcal{A}_{\text{down}} = \{(+2^{-k},0,0,-2^{-k},-1) \mid k \in \mathbb{N}\}$.

Then, $S_\omega = [0,1) \times \{(0,0,0,0\}$, but $W = \emptyset$.

Proof. Every finite non-losing play has to start with some move from $\mathcal{A}_{\text{init}}$, which sets the third component to some particular 2^{-k_0}. Afterwards, moves from $\mathcal{A}_{\text{up}}$ and $\mathcal{A}_{\text{down}}$ for the same value $k = k_0$ have to alternat; otherwise, the safety constraint is immediately violated. As any such move increased the first component by 2^{-k_0}, we find that after some finite number of moves the first component exceeds 1, and the game is lost. However, this number can be chosen arbitrarily large based on the choice of k_0. □

As a tangential remark, note that the example above could be adapted to a game with only finitely many moves, but cooperative players by joining all moves of the same type into one, and letting Player B choose k. Thus, considering non-zero sum games would also change the fixed point iteration.

Below, compactness of the winning region follows from topological and finiteness assumptions. Other assumptions (note the symmetry) make the interior of S_ω (denoted $\mathring{S}_\omega$) unreachable by Player A from its boundary. (The interior of a set is the largest open set included in the original set.) If $W = S_\omega$, then this gives its boundary the status of "almost-tie" region.

Proposition 6.5.

(1) *Let the elements of (possibly infinite) $\mathcal{B}$ be closed. If* Safe *is compact, then so are the S_n and S_ω.*
(2) *Let the elements of (possibly infinite) $\mathcal{A}$ be compact. In every game $\langle \mathcal{A}, \mathcal{B}, \mathsf{Safe}, v_0 \rangle$ with $v_0 \in S_\omega \setminus \mathring{S}_\omega$,* Player A *cannot force any end-of-the-round position inside $\mathring{S}_\omega$.*

Proof.

(1) Let E be a closed set and let $x \notin f(E)$. For all $A \in \mathcal{A}$ let a_A and B_A be such that $x + a_A + B_A \cap E = \emptyset$. Since B_A and E are closed, there is $r_A > 0$ such that $B(x, r_A) + a_A + B_A \cap E = \emptyset$. Let $r := \min_A r_A$. For all $y \in B(x,r)$ for all $A \in \mathcal{A}$, we find $y + a_A + B_A \cap E = \emptyset$. It shows that $\mathbb{R}^d \setminus f(E)$ is open, so f preserves closeness, and so does g. Therefore S_n is compact (closed and bounded) for all n, and so is S_ω by intersection.
(2) It suffices to show that $\forall A \in \mathcal{A} \exists a \in A \exists B \in \mathcal{B} \forall b \in B, x + a + b \notin \mathring{S}_\omega$, so let $A \in \mathcal{A}$. Toward a contradiction, let us assume that $\forall a \in A, \forall B \in \mathcal{B} \exists b \in B, x + a + b \in \mathring{S}_\omega$. Let us fix $B \in \mathcal{B}$ for now, so $\forall a \in A, \exists b_a \in B, x + a + b_a \in \mathring{S}_\omega$. Since $\mathring{S}_\omega$ is open, $\forall a \in A, \exists b_a \in B \exists r_a > 0, x + a + b_a + B(0, 2r_a) \subseteq \mathring{S}_\omega$. The open balls $\{B(a, r_a)\}_{a \in A}$ form a cover of A, so by compactness let finitely many $a_1, \ldots, a_k$ be such that the $B(a_i, r_{a_i})$ still cover A. For all $a \in A$, we can thus define $i(a) := \min\{i \mid a \in B(a_i, r_{a_i})\}$ and $b'_a := b_{a_{i(a)}}$. Let $r := \min_i\{r_{a_i}\}$. For all $a \in A$ and $\delta \in \mathbb{R}^d$ such that $\|\delta\| < r$, we have $x + \delta + a + b'_a = x + \delta + a_{i(a)} + (a - a_{i(a)}) + b_{a_{i(a)}}$. Since $\|a - a_{i(a)}\| < r_{a_{i(a)}}$ by definition of the cover and $\|\delta\| < r \leq r_{a_{i(a)}}$, we find $x + \delta + a + b'_a \in \mathring{S}_\omega$. So $\forall y \in B(x,r) \forall a \in A \exists b \in B, y + a + b \in \mathring{S}_\omega$. Just before letting B range over $\mathcal{B}$ again, let $r_B := r$. Since $\mathcal{B}$ is finite, let $r' := \min_{B \in \mathcal{B}}\{r_B\}$. Therefore $\forall y \in B(x, r') \forall a \in A \forall B \in \mathcal{B} \exists b \in B, y + a + b \in \mathring{S}_\omega$, so $x \in \mathring{S}_\omega$, contradiction. □

Finally, we give geometrical properties of the winning region with respect to Safe. Lemma 6.6, requiring no assumption, says the following: seen as a physical object, W cannot move by (continuous) translation to another position while always remaining entirely in Safe.

Lemma 6.6. *For all $t \in \mathbb{R}^d \setminus \{0\}$, either $\mathbb{R}^+ \cdot t + W \subseteq W$, or for all $\epsilon > 0$ there is $0 < \epsilon' \leq \epsilon$ such that $(\epsilon' \cdot t + W) \cap \mathrm{Safe}^C \neq \emptyset$, where $\mathrm{Safe}^C := \mathbb{R}^d \setminus \mathrm{Safe}$.*

Proof. The proof has two parts. First, let us prove the following claim about general sets: if $S \subseteq T \subseteq \mathbb{R}^d$ are such that $\forall t \in \mathbb{R}^d (t + S \subseteq T \Rightarrow t + S \subseteq S)$, then for all $t \in \mathbb{R}^d \setminus \{0\}$, either $\mathbb{R}^+ \cdot t + S \subseteq S$, or for all $\epsilon > 0$ there is $0 < \epsilon' \leq \epsilon$ such that $(\epsilon' \cdot t + S) \cap T^C \neq \emptyset$. Let $t \in \mathbb{R}^d \setminus \{0\}$ and let $x + \epsilon_0 \cdot t \notin S$ for some $\epsilon_0 > 0$ and $x \in S$. Toward a contradiction, let $\epsilon_1 > 0$ be such that $(S + \epsilon \cdot t) \subseteq T$ for all $0 < \epsilon \leq \epsilon_1$. Let $n \in \mathbb{N}$ be such that $\frac{\epsilon_0}{n} \leq \epsilon_1$. On the one hand $\frac{\epsilon_0}{n} \cdot t + S \subseteq S$, so $\frac{k\epsilon_0}{n} \cdot t + S \subseteq S$ for all $k \in \mathbb{N}$, by induction on k. On the other hand, there exists a natural $0 < k \leq n$ such that $\neg(\frac{k\epsilon_0}{n} \cdot t + S \subseteq S)$, contradiction, and the claim is proved.

Let us now prove that for all $t \in \mathbb{R}^d$, if $(t + W) \subseteq \mathrm{Safe}$ then $t + W \subseteq W$. If $(t + W) \subseteq \mathrm{Safe}$, then Player A can stay in Safe when starting in $t + W$, simply by using a winning strategy for W up to translation by t. So $t + W \subseteq W$. Invoking the above claim shows the lemma. □

The following is a corollary of Lemma 6.6: it says that if Safe is bounded and convex, the image of W by any non-zero translation is no longer included in Safe.

Corollary 6.7. *If Safe is bounded and convex, then $(t + W) \cap \mathrm{Safe}^C \neq \emptyset$ for all $t \in \mathbb{R}^d \setminus \{0\}$.*

Proof. Toward a contradiction, let $t \in \mathbb{R}^d \setminus \{0\}$ be such that $(t + W) \cap \mathrm{Safe}^C = \emptyset$. By Lemma 6.6 let $0 < \epsilon < 1$ be such that $(\epsilon \cdot t + W) \cap \mathrm{Safe}^C \neq \emptyset$, which is witnessed by some $x \in W$ such that $\epsilon \cdot t + x \notin \mathrm{Safe}$. By convexity $t + x \notin \mathrm{Safe}$, contradiction. □

7 COMPUTATIONAL COMPLEXITY OF THE SAFETY PROBLEMS

In this section, we prove two results regarding the decidability of the winner in safety Minkowski games. Proposition 7.2 in Section 7.1 shows that under general conditions phrased in the language of computable analysis, it is semidecidable whether Player B has a winning strategy. The reader not interested in computable analysis can safely skip this subsection, and proceed to Section 7.2, where we show that even for safety Minkowski games with moves and safe zone defined as a set of rational linear inequalities, it is undecidable which player has a winning strategy (Theorem 7.3).

7.1 Semidecidability of the Safety Problem

Similar to our observations in Section 5.4 on the boundedness games, we can derive semidecidability of the winner in safety Minkowski games from the general mathematical considerations, provided that we represent the sets involved in the appropriate way. We make use of the characterizations of $\mathcal{V}(\mathbb{R}^d)$ and $\mathcal{K}(\mathbb{R}^d)$ via the preservation of open predicates under quantification developed in Reference [24, Section 10].

Observation 7.1. *Consider finite $\mathcal{A}$ of moves from $\mathcal{V}(\mathbb{R}^d)$ and finite $\mathcal{B}$ of moves from $\mathcal{K}(\mathbb{R}^d)$. Let $\mathrm{Safe} \in \mathcal{A}(\mathbb{R}^d)$. Then the function g from Definition 6.1 is well-defined and computable from the parameters as a function $g : \mathcal{A}(\mathbb{R}^d) \to \mathcal{A}(\mathbb{R}^d)$.*

Proposition 7.2. *Given a safety Minkowski game $\langle \mathcal{A}, \mathcal{B}, \mathrm{Safe}, v_0 \rangle$ with $\mathcal{A}$ being a finite set of overt sets (i.e., from $\mathcal{V}(\mathbb{R}^d)$, aka, represented by positive information), $\mathcal{B}$ being a finite set of compact sets (i.e., from $\mathcal{K}(\mathbb{R}^d)$, aka, represented by negative information and a bound) and Safe being given as an element of $\mathcal{A}(\mathbb{R}^d)$ (aka, given by negative information), we can semidecide (recognize) if Player B has a winning strategy.*

Proof. By Observation 7.1, we can compute the function $g : \mathcal{A}(\mathbb{R}^d) \to \mathcal{A}(\mathbb{R}^d)$ defined in Definition 6.1. As $\mathcal{A}(\mathbb{R}^d)$ is effectively closed under countable intersection, we then can compute $S_\omega \in \mathcal{A}(\mathbb{R}^d)$. By Proposition 6.3, this is the greatest fixed point of g, and by Lemma 6.2, the greatest fixed point is the winning region of Player A. Thus, Player B wins iff $v_0 \notin S_\omega$, and by definition of $\mathcal{A}(\mathbb{R}^d)$, this is semidecidable. □

7.2 Undecidability of the Safety Problem

In the remainder of this section, we prove that safety Minkowski games are undecidable even if moves and safe zone are defined as a set of rational linear inequalities.

Theorem 7.3. *There is $d \in \mathbb{N}$, a rational convex polytope* Safe *and a finite family $\mathcal{A}$ of rational closed convex polytopes all in $\mathbb{R}^d$ such that it is undecidable whether* Player A *has a winning strategy in the one-sided safety Minkowski game $\langle \mathcal{A}, \text{Safe}, v_0 \rangle$, given v_0 as a rational vector.*

To establish this theorem, we provide a reduction from the control state reachability problem for two counter machines to the problem of deciding if Player B has a winning strategy in a safety Minkowski game. As the first step, we introduce a slightly more general version of one-sided Minkowski games, and demonstrate a reduction to one-sided safety Minkowski games:

Definition 7.4. A safety-reachability one-sided Minkowski game is given by a tuple $\langle \mathcal{A}, \text{Safe}, \text{Goal}, v_0 \rangle$, where $\langle \mathcal{A}, \text{Safe}, v_0 \rangle$ is some d-dimensional safety one-sided Minkowski game, and Goal $\subseteq$ Safe. It is played like the safety Minkowski game, and if Player A wins $\langle \mathcal{A}, \text{Safe}, v_0 \rangle$, then she wins $\langle \mathcal{A}, \text{Safe}, \text{Goal}, v_0 \rangle$. If the play enters Goal prior to leaving Safe for the first time, then also Player A wins. Else Player B wins.

Proposition 7.5. *Given a d-dimensional safety-reachability one-sided Minkowski game $\langle \mathcal{A}, \text{Safe}, \text{Goal}, v_0 \rangle$, we define the associated $d + 1$-dimensional safety one-sided Minkowski game $\langle \mathcal{A}', \text{Safe}', v_0' \rangle$ as follows:*

(1) $v_0' := \langle v_0, 0 \rangle$,
(2) $\text{Safe}' := \text{CH}\left((\text{Safe} \times \{0\}) \cup (\text{Goal} \times \{1\})\right)$,
(3) $\mathcal{A}' := \{A \times \{0\} \mid A \in \mathcal{A}\} \cup \{\{(0, \ldots, 0, 1)\}, \{(0, \ldots, 0, -1)\}\}$.

Now Player A *(respectively,* Player B*) has a winning strategy in the original game iff she (respectively, he) has one in the associated game.*

Proof. Every play in $\langle \mathcal{A}', \text{Safe}', v_0' \rangle$ where Player A never chooses one of the moves in $\{\{(0, \ldots, 0, 1)\}, \{(0, \ldots, 0, -1)\}\}$ is also a valid play in $\langle \mathcal{A}, \text{Safe}, \text{Goal}, v_0 \rangle$ after projection, and Player B has no additional options to deviate in the latter. Moreover, if the play is won for Player A in $\langle \mathcal{A}', \text{Safe}', v_0' \rangle$, then it is also winning for her in $\langle \mathcal{A}, \text{Safe}, \text{Goal}, v_0 \rangle$.

By construction of Safe$'$, the first time Player A uses a move from $\{\{(0, \ldots, 0, 1)\}, \{(0, \ldots, 0, -1)\}\}$, it has to be $\{(0, \ldots, 0, 1)\}$, and the first d components of the position need to fall into Goal. Thus, in the corresponding play in $\langle \mathcal{A}, \text{Safe}, \text{Goal}, v_0 \rangle$, Player A has already won. Conversely, if the play reaches Goal in $\langle \mathcal{A}, \text{Safe}, \text{Goal}, v_0 \rangle$, then Player A can continue the play in $\langle \mathcal{A}', \text{Safe}', v_0' \rangle$ by alternating the moves $\{(0, \ldots, 0, 1)\}$ and $\{(0, \ldots, 0, -1)\}$ and win. □

We recall some preliminaries on two-counter machines:

2CM and the control state reachability problem. A two-counter machine, 2CM for short, is defined by a finite directed graph (Q, E) with labeled edges. Vertices have out-degree 0, 1, or 2. If the out-degree is 1, then the corresponding edge is labeled with one of INC^i, DEC^i for $i \in \{0, 1\}$. If the out-degree is 2, then one outgoing edge is labeled with isZero?^i and the other with isNotZero?^i for some $i \in \{0, 1\}$. There is a designated starting vertex $q_0 \in Q$.

A finite or infinite path through the graph is a *valid computation starting from n_0 and n_1* if the following is true: the path starts at q_0. If one starts with $c_0 := n_0$ and $c_1 := n_1$ and increments (decrements) c_i by 1 whenever encountering a label INC^i (DEC^i), then at the moment an edge labeled with isZero?^i (isNotZero?^i) is passed, we find that $c_i = 0$ ($c_i \neq 0$). Moreover, we demand that a decrement command is never encountered for a counter with value 0.

Theorem 7.6 ([21, Theorem Ia]). *There is a* 2CM *such that it is undecidable whether there exists an infinite valid computation starting from n_0 and n_1 (where n_0 and n_1 are the input).*

We will slightly modify the *2CM* to simplify the construction. We subdivide every edge by adding another vertex on it. If the original edge was labeled INC^i (DEC^i), then the two new edges will be labeled INCa^i and INCb^i (DECa^i and DECb^i). If the original edge was labeled isZero?^{i} or isNotZero?^{i}, then we move the label to the newly-added vertex.

Now, we are ready to reduce the non-halting problem of modified 2CM's to the existence of a winning strategy for Player A in a safety-reachability one-sided Minkowski game, which constitutes the proof of Theorem 7.3. The general idea of the reduction is as follows. First, Player A is forced to simulate the computation of the 2CM to avoid violating the safety condition of the safety Minkowski game. The value of each counter c_i, $i \in \{1, 2\}$, is coded in some dimension y_i such that when the counter c_i is equal to $k \in \mathbb{N}$ then the value of $y_i = \frac{1}{2^k}$. The role of Player B is restricted to assist Player A to multiply or divide the x_i by 2. Her failure to operate as intended will let the play reach Goal. Additionally, each vertex Q is associated with one dimension that will be non-zero iff the computation is currently in that vertex.

All the moves and invariants that we use are definable by finite sets of linear constraints.

Defining the reduction. We are given a modified 2CM with vertex set Q (called control states) and edges E. The associated safety-reachability Minkowski game will be played in $\mathbb{R}^{4+|Q|}$. The first four dimensions are (x_0, y_0, x_1, y_1), where the y_i encode the counter values, and the x_i are auxiliary values. The remaining $|Q|$ dimensions shall be indexed with the states q.

Every instruction $e \in E$ corresponds to some move A_e for Player A. The move A_e will always decompose as $A_e = A_e^{xy} \times \{a_e^Q\}$. If e is an edge from q_i to q_f, then $a_e^Q \in \mathbb{R}^{|Q|}$ will have -1 at component q_i, $+1$ at component q_f and 0 elsewhere.

Label of e	Value of A_e^{xy}	Label of e	Value of A_e^{xy}
—	$\{(0,0,0,0)\}$		
INCa^0	$\mathsf{CH}\{(0,0),(1,-1)\} \times \{(0,0)\}$	DECa^0	$\mathsf{CH}\{(0,0),(1,0)\} \times \{(0,0)\}$
INCa^1	$\{(0,0)\} \times \mathsf{CH}\{(0,0),(1,-1)\}$	DECa^1	$\{(0,0)\} \times \mathsf{CH}\{(0,0),(1,0)\}$
INCb^0	$\mathsf{CH}\{(0,0),(-1,0)\} \times \{0,0\}$	DECb^0	$\mathsf{CH}\{(0,0),(-1,1)\} \times \{(0,0)\}$
INCb^1	$\{0,0\} \times \mathsf{CH}\{(0,0),(-1,0)\}$	DECb^1	$\{(0,0)\} \times \mathsf{CH}\{(0,0),(-1,1)\}$

It remains to define the sets Safe and Goal. For that, let Q_z^i be the set of states labeled with isZero?^{i}, and let Q_n^i be the set of states labeled with isNotZero?^{i}. Let Q_o be the set of unlabeled states with non-zero outdegree. Let e_q be the $|Q|$-dimensional vector having 1 in component q and 0 elsewhere.

$$\begin{aligned}\mathsf{Safe} := \mathsf{CH}\Bigg[&\left(\bigcup_{q\in Q_o}[0,1]^4\times\{e_q\}\right)\cup\left(\bigcup_{q\in Q_n^0}[0,1]\times[0,0.7]\times[0,1]^2\times\{e_q\}\right)\\ &\cup\left(\bigcup_{q\in Q_n^1}[0,1]^3\times[0,0.7]\times\{e_q\}\right)\cup\left(\bigcup_{q\in Q_z^0}[0,1]\times\{1\}\times[0,1]^2\times\{e_q\}\right)\\ &\cup\left(\bigcup_{q\in Q_z^1}[0,1]^3\times\{1\}\times\{e_q\}\right)\Bigg]\end{aligned}$$

$$\mathsf{Goal} := \mathsf{Safe} \cap (\{(x,y)\in\mathbb{R}^2 \mid y \neq x \neq 0\}\times\mathbb{R}^{2+|Q|} \cup \mathbb{R}^2\times\{(x,y)\in\mathbb{R}^2 \mid y\neq x\neq 0\}\times\mathbb{R}^{|Q|})$$

The starting position of the game is as follows: $(0, 2^{-n_0}, 0, 2^{-n_1}, 0, \ldots, 0, 1, 0, \ldots,)$, where n_0 and n_1 are the starting values for the counters, and the unique 1 in the latter part is found at the index corresponding to the starting state of the *2CM*.

Correctness of the reduction. We claim that Player A has a winning strategy in the constructed game, iff the (modified) *2CM* has a valid infinite computation path. As moves correspond to edges, every sequence of moves chosen by Player A in the game can be seen as a sequence of edges for the *2CM*.

First, we argue that every sequence of edges that is not a path induces a losing strategy in the game. As the values of the components associated with the control states must remain between 0 and 1, and every move has components -1, $+1$ somewhere and 0 elsewhere it follows that every non-losing sequences of moves ensure that exactly one state-component q_i of the position is 1, and the others are 0. Every move coming from an edge not having the initial state q_i will lose immediately.

Next, we shall explain how the moves for INCa^i and INCb^i together cause the desired effect. If the current relevant part of the position is $(0, 2^{-k})$, then after the move INCa^i Player B may pick any $(x, y) \in (0, 2^{-k}) + \mathrm{CH}\{(0, 0), (1, -1)\}$, in other words, Player B picks some $t \in [0, 1]$ and sets the position to $(t, 2^{-k} - t)$. If Player B picks $t = 0$, then Player A can repeat the same move. By the definition of Goal, the only other safe choice for Player B is to pick $t = 2^{-k-1}$, i.e., to set the position to $(2^{-k-1}, 2^{-k-1})$. The move associated with INCb^i follows, which means that Player B gets to pick some $(2^{-k-1} - t, 2^{-k-1})$. Again, choosing $t = 0$ lets Player A repeat her move, and the only other choice compatible with avoiding Goal is to move to $(0, 2^{-k-1})$.

The construction for DECa^i, DECb^i works similarly: starting at $(0, 2^{-k})$ for $k \neq 0$, Player B can only remain, enter Goal or move to $(2^{-k}, 2^{-k})$ if Player A plays a move corresponding to DECa^i. The subsequent DECb^i move allows Player B to remain, enter Goal or to move to $(0, 2^{-k+1})$. If a DECa^i, DECb^i-pair is encountered starting at $(0, 1)$, then Player B can force the play to leave Safe, corresponding to our convention that decrementing a counter at value 0 terminates the computation of the 2CM.

Finally, we need to discuss (conditional) halting: by the construction of Safe, if a vertex with out-degree 0 is reached, or a vertex labeled with an unsatisfied condition, then the play is losing for Player A. Thus, winning strategies of Player A correspond exactly to infinite non-halting computations of the 2CM.

Remark 7.7 (On the Existence of a Finite Bisimilarity Quotient). In line with the undecidability result above, it can be shown that safety Minkowski games with safety region and moves defined by linear inequalities have in general no finite bisimilarity quotient. In contrast, it is an easy exercise to establish, by application of definitions and results in Reference [14], that every safety Minkowski game with a safety region and moves defined as finite union of rational multi-rectangles has a finite bisimilarity quotient. This finite bisimilarity quotient can then be used to show that the fixed point defining the set of winning states for Player A is effectively computable. Rational multi-rectangular sets in $\mathbb{R}^d$ are defined as finite union of sets defined by constraints of the form $\bigwedge_{i=1}^{i=d} x_i \in [a_i, b_i]$ where $a_i, b_i \in \mathbb{Q}$ are the rational bounds of an closed non-empty interval in $\mathbb{R}$.

8 STRUCTURAL SAFETY GAMES

The undecidability result of the previous section for safety game with polytopic sets motivates the study of *structural safety Minkowski games*. In a (one-sided) structural safety game, there is no designated initial state and Player A is asked to be able to maintain the system safe starting from any point in the safe region. It is not difficult to see that this stronger requirement makes the game equivalent to a "one round" game. Indeed, if Player A can maintain safety from all positions within

Safe, then it means that after one round of the game, the game is again within Safe, from which Player A can win for one more round, and so on.

We establish in this section the exact complexity of the structural safety games when moves and the set Safe are polytopic.

Theorem 8.1. *Given a one-sided structural safety Minkowski game* $\langle \mathcal{A}, \mathcal{B}, \text{Safe}\rangle$ *where moves and the set* Safe *are rational polytopic, it is* coNP-Complete *to decide if* Player A *has a winning strategy from all positions in* Safe.

To prove this theorem, we first show that when Player B wins the structural safety Minkowski game then there exists a position v_0 and vertex strategy that is winning. This establishes membership of the decision problem to coNP.

Lemma 8.2. *Given a one-sided structural safety Minkowski game* $\langle \mathcal{A}, \text{Safe}\rangle$, *where moves in* $\mathcal{A}$ *and the set* Safe *are rational polytopic, if there is no winning strategy for* Player A, *then there exists a rational position* $v \in$ Safe *with polynomial size binary representation and a vertex strategy of* Player B *that is winning for* Player B.

Proof. As there is no winning strategy for Player A in the structural safety game then, by definition, there exists $v \in$ Safe such that for all $A \in \mathcal{A}$, $v + A \not\subseteq$ Safe. As Safe is convex, it must be the case that for each A, there exists $a \in \text{Ver}(A)$ such that $v + a \notin$ Safe. Let us note V this set of vertices. As all vertices $a \in V$ are such that $v + a$ is outside Safe, it must be the case that $v + a$ violates at least one of the linear constraints that define Safe. Let ϕ_a be one such constraint defining Safe and violated by $v + a$ for move A. So, we can deduce that the following system of inequalities is satisfiable: $x \in \text{Safe} \wedge \bigwedge_{a \in V} x + a \notin [\![\phi_a]\!]$. Because all the vertices a are definable by polynomial size binary representations and all constraints in the inequalities ϕ_a are defined with polynomial size binary presentable coefficients, then by classical results on solutions of systems of linear inequalities, see, e.g., Reference [23], there exists a value v for x with a polynomial size binary representation. □

The hardness is established by the following lemma.

Lemma 8.3. *There is a polynomial time reduction from the 3SAT problem to the complement of the structural safety problem for one-sided Minkowski games with moves defined by rational closed polytopes.*

Proof. Let $\Psi = \{C_1, C_2, \ldots, C_n\}$ be a 3SAT instance where each clause $C_i \equiv \ell_{i1} \vee \ell i2 \vee \ell i3$ are literals built from the set of variables $X = \{x_1, x_2, \ldots, x_m\}$. A literal ℓ_{ij} is *positive* if it is of the form x for some $x \in X$, and it is *negative* if it is of the form $\neg x$ for some $x \in X$. We associate with each clause C_i, a move $A_i \subseteq \mathbb{R}^{2m}$ and each propositional variable $x_j \in X$ is associated with two dimensions related to real-valued variables x_{j1}, x_{j2} in the sequel. The move A_i is defined as the convex hull of the the three vectors $v(\ell_{ij})$ defined as follows:

$$v(\ell_{ij})(k) = \begin{cases} 0 & \text{if } k \neq 2i-1 \wedge k \neq 2i \\ 1 & \text{if } k = 2i-1 \wedge \ell_{ij} \text{ is positive, or } k = 2i \wedge \ell_{ij} \text{ is negative} \\ -1 & \text{if } k = 2i-1 \wedge \ell_{ij} \text{ is a negative, or } k = 2i \wedge \ell_{ij} \text{ is positive} \\ \end{cases}$$

and Safe is defined by the following set of linear constraints:

$$\bigwedge_{x_j \in X} -1 \leq x_{j1} \leq 1 \wedge -1 \leq x_{j2} \leq 1 \wedge x_{j1} + x_{j2} = 0.$$

We prove the correctness of our reduction as follows. **First**, we establish that if Ψ is satisfiable then Player B wins the one-sided structural safety game that we have constructed.

Let $f : X \to \{0, 1\}$ be a valuation of the propositional variables in X such that $f \models \Psi$. We construct v_0 as follows: for all $x_j \in X$, $v_0(x_{j1}) = 1$ if $f(x_j) = 1$, and otherwise $v_0(x_{j1}) = -1$, and $v_0(x_{j2}) = 1$ if $f(x_j) = 0$, and otherwise $v_0(x_{j2}) = -1$.

Now, let us show that for all modes $A_i \in \mathcal{A}$, we have that

$$(v_0 + A_i) \cap \overline{\mathsf{Safe}} \neq \emptyset.$$

This is the case because A_i is associated with C_i. As $f \models \Psi$, we know that there is a literal ℓ_{ij} such that $f \models \ell_{ij}$. Assume that $\ell_{ij} = x_k$ (the case for $\neg x_k$ is symmetric). Because $f(x_k) = 1$, we have that $v_0(x_{k1}) = 1$. Now, A_i contains a vertex $a = (0, \ldots, 0, 1, -1, 0, \ldots, 0)$, i.e., $a(k1) = 1$, and $a(k2) = -1$. Clearly, $v_0(x_{k1}) + a(x_{k1}) = 2$, and so if Player B chooses $a \in A_i$, the next position is outside of Safe.

Second, assume that there is v and $\lambda_2^v : \mathcal{A} \to \mathbb{R}^{2m}$ a vertex strategy of Player B. This is w.l.o.g. by Lemma 8.2. Note that we can further assume that $v(x_{k1}) \neq 0$, and $v(x_{k2}) \neq 0$ for all k, $1 \leq k \leq m$. This is because if $v(x_{k1}) = 0$ then $v(x_{k_2} = 0$ and so by definition of Safe and the moves, it is the case that those two dimensions are not responsible for the violation of safety. So, we can assume that all dimension in v are nonzero.

Now, we define $f^v : X \to \{0, 1\}$, $f^v(x_k) = 1$ if and only if $v(x_{k1}) > 0$. Let us now prove that $f^v \models \Psi$. Let C_i be a clause $\ell_{i1} \vee \ell_{i2} \vee \ell_{i3}$. We know that $v + \lambda_2^v(A_i) \not\subseteq$ Safe. This means that there is a vertex a_{ij} of A_i such that $v + a_{ij} \notin$ Safe. This vertex corresponds to the literal ℓ_{ij} and $f^v \models \ell_{ij}$ by construction of the moves and Safe. ◻

9 OPEN QUESTIONS

By comparing the results from Sections 5.1 and 5.2, we see that while deciding the winner in a boundedness Minkowski game is coNP-complete, in general, it becomes polynomial-time if the dimension of the ambient space is fixed. Thus, it makes a good candidate for an investigation in the setting of parameterized complexity [7]. Is the problem fixed-parameter tractable? Is it hard for some W[n]-class? While parameterized complexity of geometric problems with the dimension as a parameter has not received much attention so far, some hardness results are found in References [12, 16].

In Section 7, we showed that from some dimension d onwards, it becomes undecidable to determine the winner in a safety Minkowski game defined via sets of linear constraints defining open and closed convex polytopes. This gives immediate rise to two questions: first, what happens for small dimensions? Given that our construction needs essentially two dimensions per instruction, and two per counter, an optimal value is presumably obtained by using universal machine having more than two counters. Second, what happens if we restrict our attention to games defined via sets of linear constraints that are all non strict (defining closed convex polytopes only)?

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