A NOTE ON THE IRRATIONALITY OF $\zeta(2)$ AND $\zeta(3)$

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1. *Introduction*

At the "Journées Arithmétiques" held at Marseille-Luminy in June 1978, R. Apéry confronted his audience with a miraculous proof for the irrationality of $\zeta(3) = 1^{-3}+2^{-3}+3^{-3}+\dots$. The proof was elementary but the complexity and the unexpected nature of Apéry's formulas divided the audience into believers and disbelievers. Everything turned out to be correct however. Two months later a complete exposition of the proof was presented at the International Congress of Mathematicians in Helsinki in August 1978 by H. Cohen. This proof was based on the lecture of Apéry, but contained ideas of Cohen and Don Zagier. For a more extensive record of this little history I refer to A. J. van der Poorten [1]. Apéry's proof will be published in *Acta Arithmetica.*

In this note we give another proof for the irrationality of $\zeta(3)$ which is shorter and, I think, more elegant. This proof is achieved by means of double and triple integrals, the shape of which is motivated by Apéry's formulas. Like Apéry's proof it also works for $\zeta(2)$, which is of course already known to be transcendental since it equals $\pi^2/6$. Most of the integrals that appear in the proof are improper. The manipulations with these integrals can be justified if one replaces $\int_0^1$ by $\int_\varepsilon^{1-\varepsilon}$ and by letting ε tend to zero.

2. Throughout this paper we denote the lowest common multiple of $1, 2, \dots, n$ by d_n. The value of d_n can be estimated by

$$d_n = \prod_{\substack{\text{Prime}\\ p\leqslant n}} p^{[\log n/\log p]} < \prod_{\substack{\text{Prime}\\ p\leqslant n}} p^{\log n/\log p} = n^{\pi(n)},$$

and the latter number is smaller than 3^n for sufficiently large n.

LEMMA 1. *Let r and s be non-negative integers. If $r > s$ then,*

(a) $$\int_0^1\int_0^1 \frac{x^r y^s}{1-xy}\,dxdy$$

is a rational number whose denominator is a divisor of $d_r^{\,2}$.

(b) $$\int_0^1\int_0^1 -\frac{\log xy}{1-xy}\,x^r y^s\,dxdy$$

is a rational number whose denominator is a divisor of $d_r^{\,3}$.

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If $r = s$, then

(c) $$\int_0^1\int_0^1 \frac{x^r y^r}{1-xy}\,dxdy = \zeta(2) - \frac{1}{1^2} - \ldots - \frac{1}{r^2},$$

(d) $$\int_0^1\int_0^1 -\frac{\log xy}{1-xy}\,x^r y^r\,dxdy = 2\left\{\zeta(3) - \frac{1}{1^3} - \ldots - \frac{1}{r^3}\right\}.$$

Remark. In case $r = 0$, we let the sums $1^{-2}+\ldots+r^{-2}$ and $1^{-3}+\ldots+r^{-3}$ vanish.

Proof. Let σ be any non-negative number. Consider the integral

$$\int_0^1\int_0^1 \frac{x^{r+\sigma} y^{s+\sigma}}{1-xy}\,dxdy. \tag{1}$$

Develop $(1-xy)^{-1}$ into a geometrical series and perform the double integration. Then we obtain

$$\sum_{k=0}^{\infty} \frac{1}{(k+r+\sigma+1)(k+s+\sigma+1)}. \tag{2}$$

Assume that $r > s$. Then we can write this sum as

$$\sum_{k=0}^{\infty} \frac{1}{r-s}\left\{\frac{1}{k+s+\sigma+1} - \frac{1}{k+r+\sigma+1}\right\} = \frac{1}{r-s}\left\{\frac{1}{s+1+\sigma} + \ldots + \frac{1}{r+\sigma}\right\}. \tag{3}$$

If we put $\sigma = 0$ then assertion (a) follows immediately. If we differentiate with respect to σ and put $\sigma = 0$, then integral (1) changes into

$$\int_0^1\int_0^1 \frac{\log xy}{1-xy}\,x^r y^s\,dxdy,$$

and summation (3) becomes

$$\frac{-1}{r-s}\left\{\frac{1}{(s+1)^2} + \ldots + \frac{1}{r^2}\right\}.$$

Assertion (b) now follows straight away.
Assume $r = s$, then by (1) and (2),

$$\int_0^1\int_0^1 \frac{x^{r+\sigma} y^{r+\sigma}}{1-xy}\,dxdy = \sum_{k=0}^{\infty} \frac{1}{(k+r+\sigma+1)^2}.$$

By putting $\sigma = 0$ assertion (c) becomes obvious. Differentiate with respect to σ and put $\sigma = 0$. Then we obtain

$$\int_0^1\int_0^1 \frac{\log xy}{1-xy}\,x^r y^r\,dxdy = \sum_{k=0}^{\infty} \frac{-2}{(k+r+1)^3},$$

which proves assertion (d).

THEOREM 1. *$\zeta(2)$ is irrational.*

Proof. For a positive integer n consider the integral

$$\int_0^1\int_0^1 \frac{(1-y)^n P_n(x)}{1-xy}\,dxdy, \tag{4}$$

where $P_n(x)$ is the Legendre-type polynomial given by $n!P_n(x) = \left\{\frac{d}{dx}\right\}^n x^n(1-x)^n$.

Note that $P_n(x) \in \mathbb{Z}[x]$. In this proof we shall denote the double integration by the single sign $\int$. It is clear from Lemma 1 that integral (4) equals $(A_n + B_n\zeta(2))d_n^{-2}$ for some $A_n \in \mathbb{Z}$ and $B_n \in \mathbb{Z}$. After an n-fold partial integration with respect to x integral (4) changes into

$$(-1)^n \int \frac{y^n(1-y)^n x^n(1-x)^n}{(1-xy)^{n+1}}\,dxdy. \tag{5}$$

It is a matter of straightforward computation to show that

$$\frac{y(1-y)x(1-x)}{1-xy} \leqslant \left\{\frac{\sqrt{5}-1}{2}\right\}^5 \quad \text{for all} \quad 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1.$$

Hence integral (4) is bounded in absolute value by

$$\left\{\frac{\sqrt{5}-1}{2}\right\}^{5n} \int \frac{1}{1-xy}\,dxdy = \left\{\frac{\sqrt{5}-1}{2}\right\}^{5n} \zeta(2).$$

Since integral (5) is non-zero we have

$$0 < |A_n + B_n\zeta(2)| d_n^{-2} < \left\{\frac{\sqrt{5}-1}{2}\right\}^{5n} \zeta(2),$$

and hence

$$0 < |A_n + B_n\zeta(2)| < d_n^2 \left\{\frac{\sqrt{5}-1}{2}\right\}^{5n} \zeta(2) < 9^n \left\{\frac{\sqrt{5}-1}{2}\right\}^{5n} \zeta(2) < \left\{\frac{5}{6}\right\}^n$$

for sufficiently large n. This implies the irrationality of $\zeta(2)$, for if $\zeta(2)$ was rational the expression in modulus signs would be bounded below independently of n.

THEOREM 2. *$\zeta(3)$ is irrational.*

Proof. Consider the integral

$$\int_0^1\int_0^1 \frac{-\log xy}{1-xy} P_n(x)P_n(y)\,dxdy, \tag{6}$$

where $n!P_n(x) = \left\{\frac{d}{dx}\right\}^n x^n(1-x)^n$. It is clear from Lemma 1 that integral (6) equals $(A_n+B_n\zeta(3))d_n^{-3}$ for some $A_n \in \mathbb{Z}$, $B_n \in \mathbb{Z}$. By noticing that

$$\frac{-\log xy}{1-xy} = \int_0^1 \frac{1}{1-(1-xy)z}\,dz,$$

integral (6) can be written as

$$\int \frac{P_n(x)P_n(y)}{1-(1-xy)z}\,dxdydz,$$

where $\int$ denotes the triple integration. After an n-fold partial integration with respect to x our integral changes into

$$\int \frac{(xyz)^n(1-x)^n\,P_n(y)}{(1-(1-xy)z)^{n+1}}\,dxdydz.$$

Substitute

$$w = \frac{1-z}{1-(1-xy)z}.$$

We obtain

$$\int (1-x)^n(1-w)^n \frac{P_n(y)}{1-(1-xy)w}\,dxdydw.$$

After an n-fold partial integration with respect to y we obtain

$$\int \frac{x^n(1-x)^n\,y^n(1-y)^n\,w^n(1-w)^n}{(1-(1-xy)w)^{n+1}}\,dxdydw. \tag{7}$$

It is straightforward to verify that the maximum of

$$x(1-x)y(1-y)w(1-w)(1-(1-xy)w)^{-1}$$

occurs for $x = y$ and then that

$$\frac{x(1-x)y(1-y)w(1-w)}{1-(1-xy)w} \leqslant (\sqrt{2}-1)^4 \quad \text{for all} \quad 0 \leqslant x, y, w \leqslant 1.$$

Hence integral (6) is bounded above by

$$(\sqrt{2}-1)^{4n}\int \frac{1}{1-(1-xy)w}\,dxdydw = (\sqrt{2}-1)^{4n}\int_0^1\int_0^1 \frac{-\log xy}{1-xy}\,dxdy = 2(\sqrt{2}-1)^{4n}\,\zeta(3).$$

Since integral (7) is not zero we have

$$0 < |A_n + B_n \zeta(3)| d_n^{-3} < 2\zeta(3)(\surd 2 - 1)^{4n},$$

and hence

$$0 < |A_n + B_n \zeta(3)| < 2\zeta(3) d_n^{3} (\surd 2 - 1)^{4n} < 2\zeta(3) 27^n (\surd 2 - 1)^{4n} < \left(\frac{4}{5}\right)^n$$

for sufficiently large n, which implies the irrationality of $\zeta(3)$.

Reference

1. A. J. van der Poorten, "A proof that Euler missed . . . Apéry's proof of the irrationality of $\zeta(3)$". *To appear.*

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