Necessary and Sufficient Ergodicity Condition for Open Synchronized Queueing Networks

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Abstract—**In this paper, a necessary and sufficient ergodicity condition for complex open queueing systems is given. The considered queueing networks belong to a particular class of unbounded Markov Stochastic Petri Nets. These systems may include synchronization features like fork and join arrivals and departures, feedback between behavior of different queues. Grouped and correlated arrivals and departures are also allowed. The paper includes an example and a proof of the ergodicity results.**

Index Terms—**Ergodicity conditions, Markov chains, open synchronized queueing networks, Petri nets, stochastic Petri nets.**

I. Introduction

THE study of open queueing systems with synchronization features is one of the main problems in the field of performance evaluation [2], [6].

An open synchronized queueing network (OSQN) is a particular Markov stochastic Petri net [16], [17] that can be viewed as an open Jackson queueing network with synchronization between servers:

- The synchronization rules of the underlying Petri net allow the modeling of dependencies between queues and more generally synchronization between processes.
- The net is open, for the underlying Petri net is unbounded (in an OSQN there can be several unbounded places).
- The transition firing rates are constant and the stochastic process associated with the net is a Markov process.

The following set of hypothesis introduces some limitations with regard to the notion of Markov stochastic Petri nets:

- The underlying Petri net does not include inhibitor arcs between the unbounded places and the input or output transitions of the unbounded places.
- The reachability graph of the underlying Petri net is strongly connected.
- The transition firing rates do not depend on unbounded place marks.

The complete study of such nets includes three steps. In the first step the qualitative properties of the state space and the trajectory space must be obtained from the analysis of the underlying Petri net. Then the ergodic properties of the net can be obtained using the properties of the measure of the trajectory space. This paper is devoted to this problem. Finally, an algorithm to compute the steady state probability distribution has to be found. This problem is still open in the case of OSQN with several unbounded places.

The study of OSQN with only one unbounded place has been done (ergodicity criteria and steady states probability distribution computation [7], [8]).

The OSQN qualitative properties, a necessary condition and a sufficient condition for the marking process ergodicity are given in [9].

This paper presents a necessary and sufficient condition for the marking process ergodicity of an OSQN with several unbounded places. The notation and some important results are recalled in the first section. Then we present the qualitative properties of OSQN. In the third section we state the ergodic results. We apply the ergodicity criteria to the example of a mutual exclusion server. Finally we give the proofs of the main results.

II. Stochastic Petri Nets

The concepts and the notations used in this paper are detailed in [8].

A. Petri Nets

In this paper the underlying Petri nets are the classical "valued" Petri nets (formerly called generalized) [1]. The underlying Petri net of an OSQN is denoted $R(P, T, V)$ where:

P is the set of places (with cardinality $|P|$).
T is the set of transitions (with cardinality $|T|$).
V is the set of arcs between places and transitions.

The set V is completely defined by the incidence matrices. These matrices define the valuation of each arc. The forward incidence matrix (valuation of the arcs between transitions and places) is denoted C^+. The backward incidence matrix (valuation of the arcs between places and transitions) is denoted C^-. The incidence matrix $C^+ - C^-$ is denoted C. We denote the jth incidence matrix column (for instance C^+) $C^+(\cdot, j)$ and the ith row $C^+(i, \cdot)$.

A marking is a $(|P|, 1)$ positive integer column vector. The marking M_k ith component is denoted $M_k(p_i)$ and called the mark of the place p_i in M_k.

Manuscript received August 31, 1987.
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IEEE Log Number 8826221.

0098-5589/89/0400-0367$01.00 © 1989 IEEE

Let M_0 be the initial marking of the Petri net. The behavior of the net is completely defined by the matrices C^- and C.

- *Condition:* A transition t_j is firable from a marking M_i if $M_i \geq C^-(\cdot, j)$.
- *Action:* The firing of t_j from M_i leads to a new marking M_k such that

$$M_k = M_i + C(\cdot, j).$$

Let $s = (t_{j_1}, t_{j_2}, \cdots, t_{j_n})$ be a sequence of transitions fired from a marking X_0 and $X_1, X_2, \cdots, X_n$ be the sequence of successive markings reached in s. The integer vector $N_{0n} = (N_{0n}(t_j))$ of the number of the occurrences of a transition t_j in the sequence s is called the characteristic vector of s.

According to this definition we can write for any reached marking X_n from X_0

$$X_n = X_0 + C \cdot N_{0n}.$$

This equation is called the net firing equation (or fundamental equation).

If $X_n = X_0$ then $C \cdot N_{0n} = C \cdot N_{00} = 0$ and s is a repetitive sequence of the net. So the nonnegative integer vectors g, solutions of the linear system $C \cdot g = 0$ associated with the net repetitive sequences are called t-invariants.

Nonnegative integer vectors f such that their transposed vector f^t satisfy the linear system of equations $f^t \cdot C = 0$, are such that any reachable marking X_n from X_0 satisfy the equation:

$$f^t \cdot X_n = f^t \cdot X_0.$$

f is called a p-invariant of the net. It defines an invariant linear relation verified by all markings.

B. Timed Petri Nets

A stochastic Petri net is first and foremost a timed Petri net. In a timed Petri net a firing time is associated with each transition firing. There are several ways to define a Petri net timed behavior. The one used for stochastic Petri nets is based upon the race model. In this case, a duration is associated with each transition. If several transitions can be fired from a marking, the fired transition is the one with the minimum duration.

According to a set of hypothesis for the timed behavior it is possible to build the timed net trajectory space. The set Ω of trajectories ω is defined by the infinite sequence of pairs $(X_n(\omega), \tau_n(\omega))$. For a given trajectory ω, $X_n(\omega)$ is the reached marking number n from $X_0(\omega) = M_0$ and $\tau_n(\omega)$ is the arrival time in this marking from $\tau_0(\omega) = 0$.

$$\Omega = \left[\omega\colon \omega = (X_0(\omega), \tau_0(\omega)), (X_1(\omega), \tau_1(\omega)), \cdots, (X_n(\omega), \tau_n(\omega)), \cdots,\right]$$

C. General Definition of Stochastic Petri Nets

Informally, a stochastic Petri net (SPN) is a timed Petri net with a random timed behavior. A formal definition of SPN must be such that all assumptions at a time t on the marking and the firing sequences are measurable with regard to a random space. The classical way is to define a probability measure of the trajectory space Ω. A study of the general definition of SPN can be found in [8].

D. Markov Stochastic Petri Nets

In this paper we consider the class of SPN such that the marking at time t is a continuous time homogeneous Markov process. This is the case if the probability of firing a transition t_j between t and $t + dt$ in any marking M_i is equal to $\lambda_{j,Mi} \cdot dt$. The real positive number $\lambda_{j,Mi}$ is called the transition firing rate of t_j in the marking M_i. We denote A as the generator of this Markov process.

It can be shown that the sojourn time in a marking M_i is exponentially distributed with mean $U_{M_i} = 1/\Sigma\, \lambda_{j,Mi}$ (the sum is taken over the set of the transition firable from M_i). The probability of firing t_j from M_i is $\lambda_{j,Mi} \cdot U_{M_i}$.

E. Marking and Firing Processes

The marking at time t for a given trajectory ω is defined by:

$$M(t, \omega) = X_n(\omega) \qquad \forall t \in [\tau_n(\omega), \tau_{n+1}(\omega)[.$$

In the same way the characteristic vector at time t is defined by:

$$N(t, \omega) = N_{0n}(\omega) \qquad \forall t \in [\tau_n(\omega), \tau_{n+1}(\omega)[.$$

As the trajectory space is measurable, the $(|P|, 1)$ column vector $M(t)$ is a random process and it is called a marking process.

The $(|T|, 1)$ column vector $N(t)$ is a random process and it is called the firing process.

The following properties can be derived from the firing equation of the underlying Petri net [8].

- *Trajectory Firing Equation:*

$$M(t, \omega) = M(0) + C \cdot N(t, \omega).$$

- *Timed Firing Equations:*

$$M(t) = M(0) + C \cdot N(t).$$

- *Expected Firing Equation:*

$$E(M(t)) = M(0) + C \cdot E(N(t))$$

where E is the expectation operator for the considered random space.

Remark: We associate with the stochastic Petri net markings another stochastic process (X_n, τ_n), where X_n is the entered marking number n (random variable) and τ_n is the random arrival time in this marking. If the marking process is a continuous time homogeneous Markov process this process is a renewal Markov process.

F. Ergodicity of the Marking and the Firing Process

1) Definitions: The marking process is ergodic if there is a finite positive real vector M^* such that:

$$M^* = \lim_{t \to +\infty} E(M(t)) \underset{AS}{=} \lim_{t \to +\infty} \frac{\int_0^t M(s, \omega)\, ds}{t} < +\infty$$

where the notation $\underset{AS}{=}$ means equal for almost all $\omega \in \Omega$ and M^* is the steady state mean marking.

The firing process is ergodic if there is a finite positive real vector N^* such that:

$$N^* = \lim_{t \to +\infty} \frac{E(N(t))}{t} \underset{AS}{=} \lim_{t \to +\infty} \frac{N(t, \omega)}{t} < +\infty$$

where N^* is called the expected firing flow vector.

2) Some Properties of the Steady State Mean Marking and Expected Firing Flow:

a) P-Invariant Property: If the marking process is ergodic then for any p-invariant f:

$$f^t \cdot M^* = f^t \cdot M_0.$$

b) Balance Property: If the marking and the firing process are ergodic, for each place the expected input flow of tokens is equal to the expected output flow:

$$C \cdot N^* = 0.$$

c) Little's Formula: If the marking and the firing processes are ergodic, the mean sojourn time $T^*(p_i)$ of a token in a place p_i is finite and is given by:

$$T^*(p_i) = \frac{M^*(p_i)}{C^+(i, \cdot) \cdot N^*}.$$

d) Ergodic Theorem for Semi-Markov Stochastic Petri Nets: If the process (X_n, τ_n) is a positive recurrent, irreducible, aperiodic, Markov renewal process then the marking and the firing processes are ergodic.

III. Qualitative Properties of Open Synchronized Queueing Networks

A. Introduction

In this section we define the Synchronized Queueing Networks (SQN) and the Open Synchronized Queueing Networks (OSQN) and we state, without proof, the main qualitative properties of these nets. These properties are used in the ergodic study presented in the following sections. A more detailed presentation of these properties can be found in [9]. The complete presentation and the proofs of the qualitative properties of OSQN can be found in [10].

B. Definition

A Synchronized Queueing Network (SQN) is a Markov stochastic Petri net such that:

- Its reachability graph is strongly connected (for the initial marking of the underlying Petri net).
- The transition firing rates are independent of the marking.
- A synchronized queueing network is said to be open if at least one of its places is unbounded for the initial marking.

Remark: A Markov stochastic Petri net such that its reachability graph is strongly connected and such that its transitions firing rates depend only on the mark of bounded places, can be transformed into a SQN which has the same Markov generator. So all the results presented in this paper can be extended to this kind of nets.

C. Marking Space Dimension

When a Petri net has several unbounded places, in many cases the marks of these unbounded places can be linearly related. These linear relations are defined by the null space of the left operator associated with the incidence matrix C. These vectors are called real p-invariants since the p-invariants defined in Section II-A are restricted to nonnegative integer vectors.

We call marking space dimension (MSD) the number of unbounded places, the mark of which can vary independently.

When the number N of the unbounded places of an OSQN is greater than MSD, then the mark of $N - MSD$ unbounded places can be computed from the mark of the MSD others. The ergodicity of the marking process depends only on the study of the MSD independent unbounded places. Hence in the following sections we consider only the OSQN such that $MSD = N$. The unbounded places of an OSQN will be denoted $p_1, \cdots, p_{MSD}$.

D. Structure of the Reachability Graph

The reachability graph of an OSQN has a regular structure which can be described in several ways. This structure induces a regular structure on the Markov process generator associated with the OSQN. The study is mainly based on properties of this process, so we present the following property of the Petri net as a property of the generator.

Theorem: There is a numbering of the reachable markings of an OSQN such that the generator of the Markov process associated with this OSQN has a block tridiagonal structure (Fig. 1).

A suitable numbering can be deduced from the "lexical" ordering of vectors defined by:

$$M_i < M_k \Leftrightarrow \exists s \quad \forall r < s$$
$$M_i(p_r) = M_k(p_r) \quad \text{and} \quad M_i(p_s) < M_k(p_s).$$

Remarks: The block tridiagonal generator of an OSQN is characterized by 4 submatrices:

- A_0 for the irregular beginning boundary marking class.
- A, B, C for the repetitive regular classes.

If $MSD \geq 2$ then the order of the generator submatrices A, B, C is infinite.

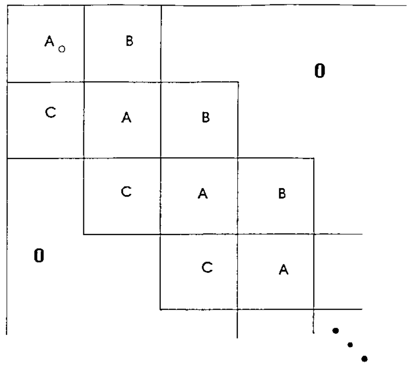

Fig. 1. The generator of an OSQN.

If $MSD = 1$ this order is finite and the Markov generator associated with an OSQN is called a quasi birth and death generator [14].

E. Saturated Nets

Let R be an OSQN and p_i an unbounded place of R. We denote by $R^{(i)}$ the stochastic Petri net obtained from R by deleting the place p_i and all its adjacent arcs. $R^{(i)}$ is called the saturated net according to p_i: $R^{(i)}$ has the same behavior as R when there is an infinite number of tokens in p_i.

Theorem: If p_1 is an unbounded place of an OSQN R then $R^{(1)}$ is a synchronized queueing network. The generator of $R^{(1)}$ is equal to $A + B + C$. If $MSD \geq 2$ then $R^{(1)}$ is an OSQN.

Remarks: The main point of this result is that the reachability graph of $R^{(1)}$ is strongly connected.

We have chosen the place p_1 to state the theorem. This choice is arbitrary. If we consider an OSQN R such that $MSD \geq 2$ and a saturated net $R^{(i)}$ of $R (i > 1)$, we can exchange the places p_1 and p_i. The new marking lexical ordering, leads to a new tridiagonal decomposition of the generator with submatrices $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ and the generator of $R^{(i)}$ is $A^{(i)} = \tilde{A} + \tilde{B} + \tilde{C}$.

Notation: All the objects related to a saturated net $R^{(i)}$ are superscripted with the index (i).

IV. Ergodicity Criteria for Open Synchronized Queueing Networks

A. Introduction

In this section we present a necessary and sufficient condition for the ergodicity of OSQN. This result generalizes the result given in [7] for OSQN with only one unbounded place. We then show how this result can be applied to a simple example. The last section is devoted to the proof of the results.

B. Ergodicity of the Firing Process

We denote $N^{(i)*}$ the saturated OSQN $R^{(i)}$ expected firing flow vector and $P^*(M_k)$ the steady state probability of the marking M_k.

Theorem: The firing process of all OSQN is ergodic, i.e.,

$$N^* = \lim_{t \to +\infty} \frac{E(N(t))}{t} \underset{AS}{=} \lim_{t \to +\infty} \frac{N(t, \omega)}{t} < +\infty.$$

- If the (X_n, τ_n) process is positive recurrent:

$$N^*(t_j) = \sum_{M_k} P^*(M_k)\lambda_{j,M_k}.$$

- If the marking process of the net is transient or null recurrent and if the unbounded place p_i is saturated:

$$N^*(t_j) = N^{(i)*}(t_j).$$

C. Ergodic Theorem for the Marking Process

Theorem:

If $\forall p_i i \in [1, MSD]\ C(i, \cdot) \cdot N^{(i)*} < 0$ then the process (X_n, τ_n) is positive recurrent.

If $\exists p_i i \in [1, MSD]\ C(i, \cdot) \cdot N^{(i)*} > 0$ then the process (X_n, τ_n) is transient.

Comments:

- The complete classification in the null recurrent case is not established. This classification from a theoretical point of view is very difficult, and practically uninteresting. We conjecture that if

$$\exists p_i i \in (1 \cdots MSD)\ C(i, \cdot) \cdot N^{(i)*} = 0$$

and

$$\forall p_j j \in (1 \cdots MSD)\ j \neq i\ C(i, \cdot) \cdot N^{(i)*} \leq 0$$

the process (X_n, τ_n) is null recurrent or transient.

- When the (X_n, τ_n) process associated with R is positive recurrent, the $(X_n, \tau_n)^{(i)}$ process associated with $R^{(i)}$ can be transient or null recurrent or positive recurrent.
- The computation of the ergodicity criterion implies the computation of the stationary distribution of all the open synchronized queueing networks $R^{(i)}$. Until now this computation is possible in the only case of OSQN having one unbounded place. Hence the ergodicity criterion can be numerically applied to the case of OSQN having at most two unbounded places.

V. Example

A. Description of the Example

This section is devoted to the computation of the saturation conditions of a simple system. It shows how the preceding ergodicity criterion can be completely computed.

We consider a queue with two classes of customers which share a single server in mutual exclusion. Arrivals of customers of the first class (resp. second class) are modeled in the OSQN R of Fig. 2 by the firing of the transition t_1 (resp. t_2). Tokens in place p_1 (resp. p_2) represent the first (resp. second) class of customers waiting (plus the customer in service).

The transition t_3 (resp. t_4) corresponds to the allocation of the server to the first (resp. second) class of customers.

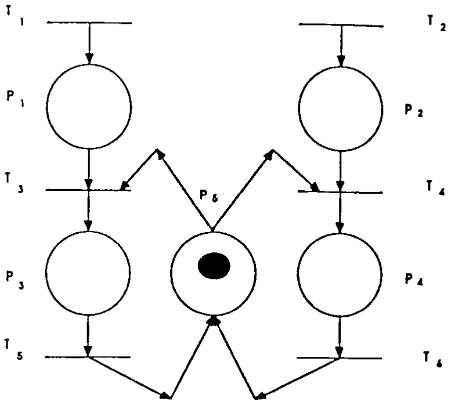

Fig. 2. The mutual exclusion OSQN.

The place p_5 represents a mutual exclusion semaphore. It allows only one customer to be served at a time. When the server is free (place p_5 marked) and they are one customer of each class claiming service (places p_1 and p_2 marked) the allocation is done according to the race policy of stochastic Petri nets [9], [13]. The end of a service is modeled by the firing of the transition t_5 or t_6, according to the class of the served customer.

We assume that the duration of the operation modeled by each transition t_j is exponentially distributed with rate λ_j.

B. Fully Saturated Net

The study of the stability conditions for the net R implies to compute the steady state behavior of the OSQN $R^{(1)}$ and $R^{(2)}$ obtained by deleting respectively the place p_1 or p_2. Fig. 3 represents the net $R^{(2)}$.

We must compute these two subnets expected firing flow vectors according to the cases where $R^{(1)}$ or $R^{(2)}$ are either transient or recurrent. Hence we have to study the stability criterion of $R^{(1)}$ and $R^{(2)}$ using the fully saturated net $R^{(1,2)}$. This OSQN is obtained by deleting the places p_1 and p_2 from the net R (Fig. 4).

As $R^{(1,2)}$ is a bounded stochastic Petri net, it has a finite number of reachable markings. Its steady state probability distribution is easily derived from the Chapman-Kolmogorov linear system. This leads to the expected firing flow vector $N^{(1,2)*}$:

$$N^{(1,2)*} = \begin{cases} \lambda_1 \\ \lambda_2 \\ \lambda_2 \Big/ \left(1 + \dfrac{\lambda_3}{\lambda_5} + \dfrac{\lambda_4}{\lambda_6}\right) \\ \lambda_4 \Big/ \left(1 + \dfrac{\lambda_3}{\lambda_5} + \dfrac{\lambda_4}{\lambda_6}\right) \\ \lambda_3 \Big/ \left(1 + \dfrac{\lambda_3}{\lambda_5} + \dfrac{\lambda_4}{\lambda_6}\right) \\ \lambda_4 \Big/ \left(1 + \dfrac{\lambda_3}{\lambda_5} + \dfrac{\lambda_4}{\lambda_6}\right) \end{cases} .$$

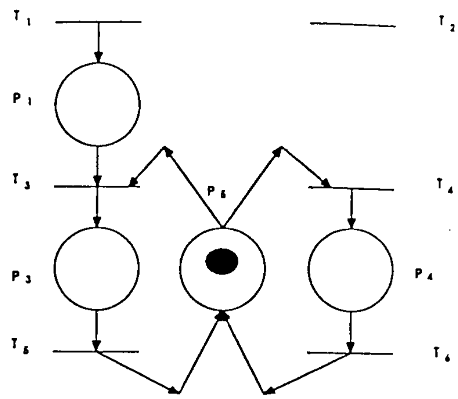

Fig. 3. The OSQN $R^{(2)}$ of the mutual exclusion net.

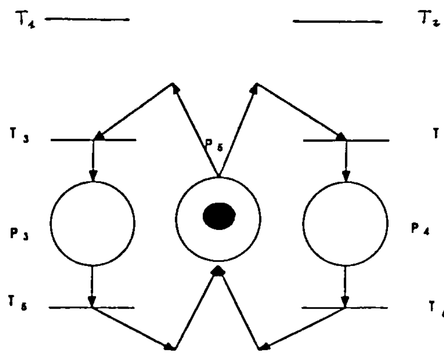

Fig. 4. The OSQN $R^{(1,2)}$ of the mutual exclusion net.

The stability conditions of $R^{(1)}$ and $R^{(2)}$ are:

The process $(X_n, t_n)^{(2)}$ associated with $R^{(2)}$ is positive recurrent if:

Condition A: $C(2, \cdot) \cdot N^{(1,2)*} < 0$

$$\Leftrightarrow \lambda_2 < \lambda_4 \Big/ \left(1 + \frac{\lambda_3}{\lambda_5} + \frac{\lambda_4}{\lambda_6}\right).$$

The process $(X_n, t_n)^{(1)}$ associated with $R^{(1)}$ is positive recurrent if:

Condition B: $C(1, \cdot) \cdot N^{(1,2)*} < 0$

$$\Leftrightarrow \lambda_1 < \lambda_3 \Big/ \left(1 + \frac{\lambda_3}{\lambda_5} + \frac{\lambda_4}{\lambda_6}\right).$$

C. Study of $R^{(1)}$ and $R^{(2)}$

Let $N^{(i)*}$ be the expected firing flow vector associated with $R^{(i)}$.

- If the corresponding process $(X_n, \tau_n)^{(i)}$ is transient or null recurrent then $N^{(i)*} = N^{(1,2)*}$.
- If $(X_n, \tau_n)^{(i)}$ is positive recurrent, the computation of $N^{(i)*}$ implies generally the computation of the steady state distribution of $R^{(i)}$. In the example presented here, we show that $N^{(i)*}$ can be derived from the invariant prop-

erties of the OSQN $R^{(i)}$

$$f^t \cdot M^{(i)*} = f^t \cdot M_0$$

$$C^{(i)} \cdot N^{(i)*} = 0$$

where f is a p-invariant and $C(i)$ is the incidence matrix of $R^{(i)}$.

We denote $N^{(i)*}(t_j)$ the transition t_j expected firing flow and $M^{(i)*}(p_k)$ the place p_k mean marking. The invariant relations of the OSQN $R^{(2)}$ are

$$M^{(2)*}(p_3) + M^{(2)*}(p_4) + M^{(2)*}(p_5) = 1$$

$$N^{(2)*}(t_1) = N^{(2)*}(t_3) = N^{(2)*}(t_5)$$

$$N^{(2)*}(t_4) = N^{(2)*}(t_6).$$

As the transitions t_1 and t_2 have no input place, then

$$N^{(2)*}(t_1) = \lambda_1$$

$$N^{(2)*}(t_2) = \lambda_2.$$

Finally, as the places p_3, p_4 and p_5 are safe (they contain at most one token) and as each of these places is respectively the only input place of transitions t_5, t_6, and t_4, we get

$$N^{(2)*}(t_5) = M^{(2)*}(p_3) \cdot \lambda_5$$

$$N^{(2)*}(t_4) = M^{(2)*}(p_5) \cdot \lambda_4$$

$$N^{(2)*}(t_6) = M^{(2)*}(p_4) \cdot \lambda_6.$$

Solving this system of equations with $M^{(2)*}$ and $N^{(2)*}$ as unknown, we obtain

$$N^{(2)*}(t_4) = \frac{\lambda_4\lambda_6(\lambda_5 - \lambda_1)}{\lambda_5(\lambda_4 + \lambda_6)}.$$

The same computation on the OSQN $R^{(1)}$ leads to

$$N^{(1)*}(t_3) = \frac{\lambda_3\lambda_5(\lambda_6 - \lambda_2)}{\lambda_6(\lambda_3 + \lambda_5)}.$$

D. Stability Conditions

When the process $(X_n, \tau_n)^{(2)}$ associated with $R^{(2)}$ is positive recurrent, then the process (X_n, τ_n) associated with R is positive recurrent if the following condition is fulfilled.

Condition C: $N^{(2)*}(t_2) < N^{(2)*}(t_4)$

$$\Leftrightarrow \lambda_2 < \frac{\lambda_4\lambda_6(\lambda_5 - \lambda_1)}{\lambda_5(\lambda_4 + \lambda_6)}.$$

When the process $(X_n, \tau_n)^{(1)}$ associated with $R^{(1)}$ is positive recurrent, then the process (X_n, τ_n) associated with R is positive recurrent if the following condition is fulfilled.

Condition D: $N^{(1)*}(t_1) < N^{(1)*}(t_3)$

$$\Leftrightarrow \lambda_1 < \frac{\lambda_3\lambda_5(\lambda_6 - \lambda_2)}{\lambda_6(\lambda_3 + \lambda_5)}.$$

So the process (X_n, τ_n) associated with R is positive recurrent if the two following properties are simultaneously true.

- Either $R^{(2)}$ is positive recurrent (condition A) and the condition C is fulfilled or $R^{(2)}$ is transient or null recurrent (not A) and B is fulfilled.
- Either $R^{(1)}$ is positive recurrent (condition B) and the condition D is fulfilled or $R^{(2)}$ is transient or null recurrent (not B) and A is fulfilled.

It can easily be shown that this set of logical properties is equivalent to C and D (Fig. 5).

Remark: The preceding stability condition is also true in the case of phase type distributions of arrivals in the places p_1 and p_2. As a matter of fact the transitions t_1 and t_2 can be replaced by two SQN such that the frequency arrival of tokens in p_1, p_2 are equal to λ_1, λ_2.

E. A Complex Mutual Exclusion Server

We present an application which illustrates the ability of the criterion to deal with complex synchronization or correlation features.

The example is derived from the mutual exclusion server by modifying the arrival processes (Fig. 6). The interarrivals durations in place p_1 are distributed according to the Erlang-K law. This is modeled by a subnet with two places p_0 and p_0' and two transitions t_0 and t_1. These transition firing rates are equal to $K\,\lambda_1$. The initial mark of p_0' is equal to $(K - 1)$ (Fig. 6).

A second class of customer arrival (in place p_2) is possible only when a customer of the first class is served (place p_3 is marked). These arrivals occur 2 by 2 according to an exponential distribution.

The computation of the ergodicity criterion is quite similar to the one presented for the mutual exclusion net. It leads to the four following conditions A, B, C, D. These conditions have the same interpretation than in the previous example.

Condition A: $C(2, \cdot) \cdot N^{(1,2)*} < 0 \Leftrightarrow \lambda_2 < \dfrac{\lambda_4\lambda_5}{\lambda_3}.$

Condition B: $C(1, \cdot) \cdot N^{(1,2)*} < 0$

$$\Leftrightarrow \lambda_1 < \lambda_3 \Big/ \left(1 + \frac{\lambda_3}{\lambda_5} + \frac{\lambda_4}{\lambda_6}\right).$$

Condition C: $C(2, \cdot) \cdot N^{(2)*} < 0$

$$\Leftrightarrow \lambda_2 < \frac{\lambda_4\lambda_6(\lambda_5 - \lambda_1)}{2\lambda_1(\lambda_4 + \lambda_6)}.$$

Condition D: $C(1, \cdot) \cdot N^{(1)*} < 0$

$$\Leftrightarrow \lambda_1 < \lambda_5 \Big/ \left(1 + \frac{2\lambda_2}{\lambda_6} + \frac{\lambda_5}{\lambda_3}\right).$$

VI. Proofs

A. Sketch of the Proofs

The study of the ergodicity is carried out in four steps.

a) In the first step we study the asymptotic behavior of

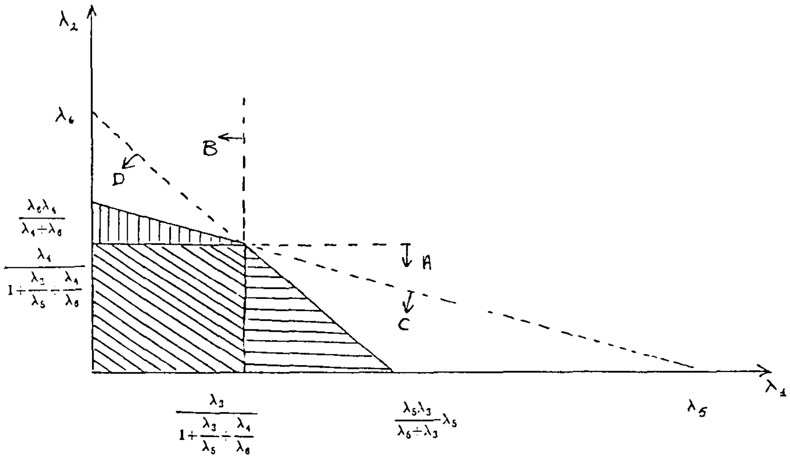

Fig. 5. Graphical representation of the ergodicity criterion.

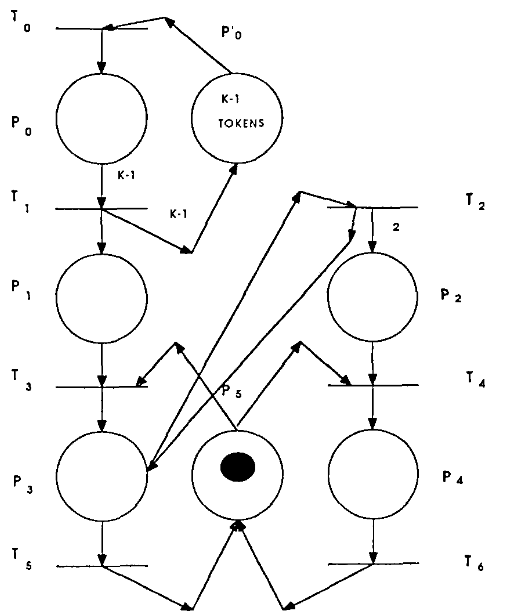

Fig. 6. An OSQN with feedback, grouped, and nonexponential arrivals.

a transient or null recurrent OSQN. In this case we show that there is always at least an unbounded place with an infinite mean mark. Such a place is called a saturated place. When the OSQN is transient we show that the convergence of the unbounded place mark to an infinite value is almost sure.

b) The state space of a transient OSQN can be completed by a boundary called the Martin exit boundary [4], [5], [11], such that the limiting behavior of the OSQN is determined by a probability distribution on the boundary. We show that the Martin boundary of an OSQN does not depend on the initial distribution. The measure on the boundary can be computed as the steady state distribution of the synchronized queueing network R^∞ obtained by deleting from R all the saturated places.

c) From this result we study the ergodicity of the firing process. If the (X_n, τ_n) process is positive recurrent the result is a consequence of the ergodic theorem for semi-Markov stochastic Petri nets. If the (X_n, τ_n) process is transient or null recurrent the expected firing flow vector of R is the expected firing flow vector of R^∞.

d) Finally the ergodicity criterion for the marking process is derived from the timed firing equation and from the study of the trajectories of the saturated nets.

B. Saturated Places

1) Infinite Mean Marking:

Theorem 1: If the (X_n, τ_n) process of an open synchronized queueing network R is transient or null recurrent then there is at least one unbounded place p_i such that:

$$\lim_{t \to +\infty} E(M(p_i, t)) = +\infty.$$

Proof: For the transient Markov process associated with R (R has an infinite number of states), we consider the infinite sequence of imbedded subsets of markings $S^{(m)}$ defined by

$$\forall m \in \mathbb{N}, S^{(m)} = [M_i : \forall j \in (1, \cdots, MSD)\, M_i(p_j) \leq m].$$

We denote by $\overline{S^{(m)}}$ the complementary subset of $S^{(m)}$.

$$\overline{S^{(m)}} = [M_i : \exists j \in (1, \cdots, MSD)\, M_i(p_j) > m].$$

We have:

$$\lim_{m \to +\infty} \overline{S^{(m)}} = \bigcap_{m \in N} \overline{S^{(m)}} = \varnothing.$$

We denote $P_{\overline{S^{(m)}}}(t)$ as the probability at time t of the $\overline{S^{(m)}}$ subset of markings.

The (X_n, τ_n) process associated with the considered OSQN is regular for there is only in each marking a finite number of firable transitions with a constant rate. For a regular transient or null recurrent Markov process all finite set of states has a null steady state probability distribution [3, pp. 251, 264].

For the $\overline{S^{(m)}}$ subset this property can be expressed by

$$\lim_{m \to +\infty} P_{\overline{S^{(m)}}}(t) = 1.$$

Hence for the measure of the trajectory space we have

$$\forall \epsilon \; \forall S^{(m)} \; \exists \; T(\epsilon, S^{(m)}): \forall t > T(\epsilon, S^{(m)})$$

$$P[\omega \in \Omega: M(p_1, t, \omega) \leq m, M(p_2, t, \omega) \leq m, \cdots, M(p_{MSD}, t, \omega) \leq m] < \epsilon.$$

This implies

$$\lim_{t \to +\infty} P[\omega \in \Omega: M(p_1, t, \omega) < +\infty, \cdots, M(p_{MSD}, t, \omega) < +\infty] = 0.$$

Hence there is at least one place p_i with an infinite steady state mean marking

$$\exists p_i \lim_{t \to +\infty} E(M(p_i, t)) = +\infty.$$

Remark: Theorem 1 does not show the almost sure convergence of the marking process to an infinite value for the place p_i. This is done in the following theorem.

2) Almost Sure Convergence: For an OSQN we denote by $\Omega I^{(1)}$ the subset of trajectories ω such that the mark of the unbounded place p_1 is "increasing."

$$\Omega I^{(1)} = [\omega: \forall m \in N \; \exists u \; \forall t > u, M(p_1, t, \omega) > m]$$

In the same way we define the "nonincreasing" trajectories for p_1:

$$\Omega NI^{(1)} = [\omega: \forall m \in N \; \forall u \; \exists t > u, M(p_1, t, \omega) \leq m].$$

Theorem 2: If the process (X_n, τ_n) associated with an OSQN is transient, and if p_1 is an unbounded saturated place, then

$$P(\Omega I^{(1)}) = 1.$$

Sketch of Proof: For a transient OSQN we consider a subset $E^{(k)}$ of the set of reachable markings. The mark of a standard place p_1 is bounded on $E^{(k)}$.

The probability vector $s^{(k)}$ in which the ith component is the probability to be infinitely often in $E^{(k)}$ starting from M_i is a harmonic function [11] of the OSQN imbedded Markov chain.

We show that this function can only be equal to 0 or 1 everywhere. If this vector is 1 then p_1 cannot be saturated. So this vector is equal to 0 and the increasing trajectory's probability is equal to 1.

Proof: The generator of the Markov process associated with the OSQN has a block tridiagonal structure.

We denote $V_0 V_1 \cdots V_n \cdots$ as the subsets of markings associated with the different blocks. An example of a V_n class is given in Fig. 7.

We denote $E^{(k)}$ as a subset of markings such that $E^{(k)} = \cup_{j=0,k} V_j$. $\overline{E^{(k)}}$ is the complementary subset of $E^{(k)}$ ($\overline{E^{(k)}} = \cup_{j=k+1,\infty} V_j$) and $s^{(k)}$ is the probability vector

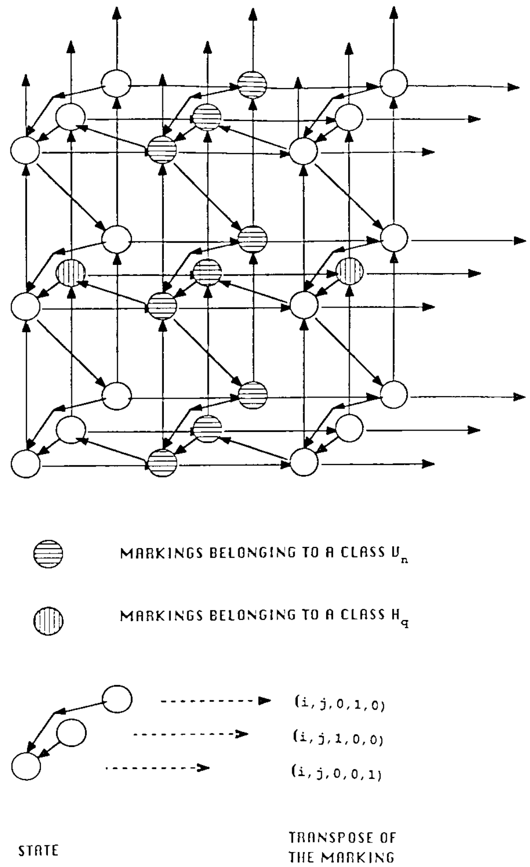

Fig. 7. The reachability graph of the mutual exclusion server.

defined by

$$s^{(k)}(i) = P(\omega \mid \forall u \; \exists t > u, M(p_1, t, \omega) \in E^{(k)} / M(0, \omega) = M_i).$$

If $\forall k$, $\forall i$, $s^{(k)}(i) = 0$ then $P(\Omega NI^{(1)}) = O$ and $P(\Omega I^{(1)}) = 1$ and if $\forall k$, $\forall i$, $s^{(k)}(i) = 1$ then $P(\Omega NI^{(1)}) = 1$ and $P(\Omega I^{(1)}) = 0$.

If the place p_1 is saturated, then the second case is impossible. If the first case is always true the property is proved. Hence we must show that $\exists k$, $\exists i$, $0 < s^{(k)}(i) < 1$ is impossible.

The proof relies on the property of the OSQN imbedded Markov chain. The marking process generator $\boldsymbol{A}$ is such that every row and column of $\boldsymbol{A}$ has a finite number of non-null entries (a finite number of transitions are firable in each marking and the number of marking predecessors is finite) and all transition rates are bounded. So all ergodicity properties of the chain, except those related to periodicity, are ergodicity properties of the process.

We denote $\boldsymbol{A}'$ the imbedded chain matrix which is stochastic.

$$\forall i, \forall j, \quad i \neq j$$

$$\boldsymbol{A}'(i, j) = -\boldsymbol{A}(i, j)/\boldsymbol{A}(i, i) \quad \text{and} \quad \boldsymbol{A}'(i, i) = 0$$

Clearly A' has the block tridiagonal structure of A. The non-null A' submatrices are superscript by $'$.

$s^{(k)}$ is a solution of the linear system of equations $s^{(k)} = A' \cdot s^{(k)}$. In Markov chain potential theory such a solution is called a harmonic (or regular) function of the chain [11]. As the OSQN reachability graph is strongly connected, either

$$\forall i, \forall k, s^{(k)}(i) = 0$$

or

$$\forall i, \forall k, s^{(k)}(i) \neq 0.$$

We consider the Markov chain matrix obtained from A' with $E^{(k)}$ made absorbing (Fig. 8). If we denote I as the identity matrix, the structure of the absorbing chain matrix is given by

$$\begin{array}{cc} & \begin{array}{cc} E^{(k)} & \overline{E^{(k)}} \end{array} \\ \begin{array}{c} E^{(k)} \\ \overline{E^{(k)}} \end{array} & \begin{bmatrix} I & 0 \\ Z'^{(k)} & Q' \end{bmatrix} \end{array}.$$

Let $F^{(k)}(i, j)$ the absorption probability by a marking $M_j \in E^{(k)}$ starting from a marking $M_i \in M$ and $F^{(k)}$ the associated matrix.

We have from [11, p. 107]:

$$F^{(k)} = \begin{array}{cc} & \begin{array}{cc} E^{(k)} & \overline{E^{(k)}} \end{array} \\ \begin{array}{c} E^{(k)} \\ \overline{E^{(k)}} \end{array} & \begin{bmatrix} I & 0 \\ (I - Q')^{-1} Z'^{(k)} & 0 \end{bmatrix} \end{array}.$$

Moreover the particular structure of $Z'^{(k)}$ induced by the structure of A implies

$$(I - Q')^{-1} Z'^{(k)} = \begin{array}{cc} & \begin{array}{ccccc} V_0 & V_1 & \cdots & V_{k-1} & V_k \end{array} \\ \begin{array}{c} V_{k+1} \\ V_{k+2} \\ \vdots \\ V_{k+n} \\ \cdot \end{array} & \begin{bmatrix} & & & & G_1 \\ & & & & G_2 \\ & & 0 & & \vdots \\ & & & & G_n \\ & & & & \cdot \end{bmatrix} \end{array}.$$

But, the theorem [11, p. 110] states that as $E^{(k)} \subset E^{(k+1)}$ then $F^{(k+1)} F^{(k)} = F^{(k)}$. This result applied to $F^{(k)}$ shows that G is a square matrix independent of k such that

$$G_n = G^n.$$

We denote $h^{(k)} = (h^{(k)}(i))$ the "hitting" vector, i.e., $h^{(k)}(i)$ is the probability to be absorbed by $E^{(k)}$ starting from M_i.

$$h^{(k)} = F^{(k)} \mathbf{1}$$

where $\mathbf{1}$ is a summing column vector, the components of which are equal to 1.

From [11, p. 110] we can write

$$s^{(k)} \neq \mathbf{1} \Rightarrow h^{(k)} \neq \mathbf{1}.$$

As $s^{(k)} \neq \mathbf{1}$ is an hypothesis we have $h^{(k)} \neq \mathbf{1}$ and $G\mathbf{1} \neq \mathbf{1}$. G is a strictly positive substochastic matrix.

But [11, p. 109],

$$s^{(k)} = F^{(k)} s^{(k)}.$$

We apply for example this equation to the subset $E^{(0)}$. $s^{(0)}$ can be decomposed into subvectors $s_n^{(0)}$ associated with markings of V_n classes.

$$s^{(0)} = \begin{bmatrix} s_0^{(0)} \\ s_1^{(0)} \\ \vdots \\ s_n^{(0)} \\ \cdot \end{bmatrix}.$$

From the preceding equation we get

$$s_n^{(0)} = G^n s_0^{(0)}.$$

As $s^{(0)}$ is a harmonic function,

$$s^{(0)} = A' s^{(0)}.$$

The first set of linear equations of this system according to the block tridiagonal structure of A' is $A'_0 s_0^{(0)} + B' s_1^{(0)} = s_0^{(0)}$ or $(A'_0 + B'G) s_0^{(0)} = s_0^{(0)}$. $A'_0 + B'$ is stochastic and $A'_0 + B' \cdot G$ is substochastic for $B' \cdot G \leq B'$ and $B' \cdot G \neq B'$. Hence, as we have $s_0^{(0)}(A'_0 + B' \cdot G) \leq s_0^{(0)}$ and $s_0^{(0)}(A'_0 + B' \cdot G) \neq s_0^{(0)}$, $\forall i, 0 < s^{(0)}(i) < 1$ is impossible. The same analysis can be applied to any $E^{(k)}$ subset of markings.

The probability measure of the increasing trajectories for a saturated place is equal to 1.

C. Asymptotic Behavior of a Transient OSQN

1) Saturated Net Markings: For any marking M_i, the vector

$$[M_i(p_2), \cdots, M_i(p_{MSD}), \cdots, M_i(p_{|P|})]$$

is a marking of the saturated net $R^{(1)}$. A marking of this saturated net is denoted $M_q^{(1)}$.

Let H_q be the subset of all the markings M_i of the OSQN defined for a given marking $M_q^{(1)}$ by:

$$H_q = [M_i: (M_i(p_2), \cdots, M_i(p_{|P|})) = M_q^{(1)}]$$

(an example of such a class H_q of markings is given in Fig. 7).

2) Theorem 3: If the (X_n, τ_n) process associated with an OSQN R is transient and the unbounded place p_1 is saturated we have:

$$\lim_{t \to +\infty} P_{H_q}(t) = \lim_{t \to +\infty} P^{(1)}_{M_q^{(1)}}(t).$$

In other words the limiting probability of the infinite subset of markings H_q in equal to the limiting probability of the marking $M_q^{(1)}$ in the saturated net $R^{(1)}$.

Remark: Theorem 3 can easily be interpreted. When the (X_n, τ_n) process of an OSQN is transient, its steady state behavior is characterized by the saturation of at least

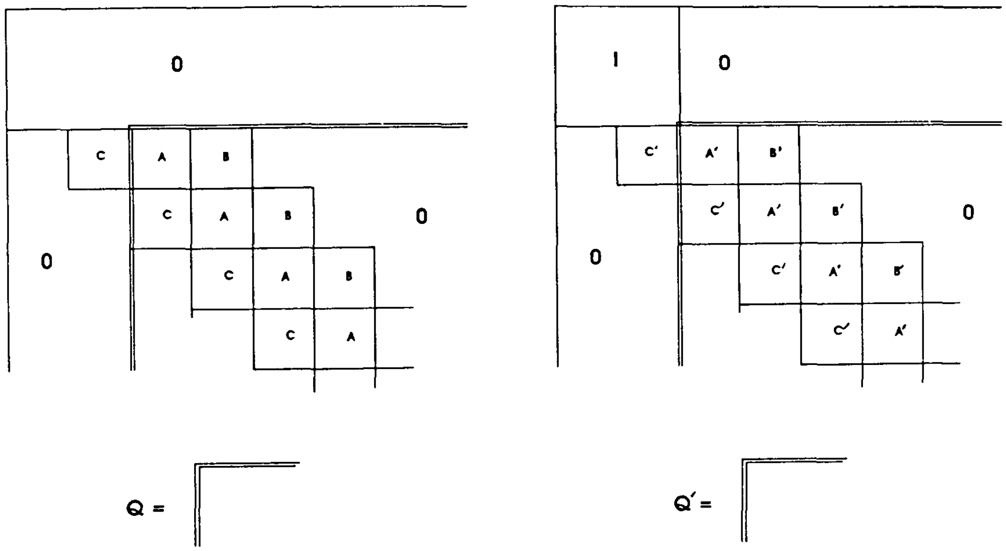

Fig. 8. The generator and imbedded Markov chain matrices of the absorbing process.

one of its unbounded places (say p_1 to simplify the notation). As a consequence this place p_1 is always marked in steady state and the firing conditions of the transitions related to p_1 are always satisfied with regard to p_1.

Hence the steady state behavior of the OSQN is exactly the steady state behavior of an OSQN obtained by deleting the place p_1 and all its related arcs. This net $R^{(1)}$ is the saturated net for the place p_1.

Proof:

a) Existence and Uniqueness of the Limiting Probabilities: The existence of a limiting probability for the H_q subsets comes from the theory of Martin exit boundary [5], [4], [11], [15]. For an irreducible transient Markov process with an infinite number of states, it is always possible to define a boundary of the state space such that with probability one the system tends to some point of the boundary.

Given a set $\Gamma = [\gamma_k]$ of boundary points, the probability $P_k^{(i)}$ that starting from an initial state M_i the system converge to a point of Γ can be computed. The $P_k^{(i)}$ are the absorption probabilities for Γ and are solutions of a linear system [5, p. 418]. If the Markov chain is irreducible (i.e., the associated graph is strongly connected) the Martin exit boundary is independent of the initial distribution [4].

b) Limiting Probabilities for the H_q Subsets: As the Markov process associated with an OSQN is regular, the exit boundary cannot be reached in a finite time. The backward and forward differential Chapman–Kolmogorov equations have a single solution which is associated with the minimal process [5], [4]. So we use these equations to compute the steady state distribution on the Martin boundary.

According to the block tridiagonal structure of A the Chapman–Kolmogorov equations are

$$P_0'(t) = P_0(t) \cdot A_0 + P_1(t) \cdot C$$

$$P_1'(t) = P_0(t) \cdot B + P_1(t) \cdot A + P_2(t) \cdot C$$

$$\cdots$$

$$P_n'(t) = P_{n-1}(t) \cdot B + P_n(t) \cdot A + P_{n+1}(t) \cdot C.$$

But the class of markings V_0, V_1 are such that $M_i(p_1) < k$ for some integer k. From Theorem 6.2.2 we have

$$\lim_{t \to +\infty} P_0(t) = \lim_{t \to +\infty} P_1(t) = 0.$$

The sum of the preceding equations leads to

$$\lim_{t \to +\infty} \sum_{n=0}^{+\infty} P_n'(t) = \lim_{t \to +\infty} \sum_{n=0}^{+\infty} P_n(t) \cdot [A + B + C].$$

From the properties of the saturated nets (Section III-E) the Chapman–Kolmogorov equations of the saturated net are

$$P^{(1)'}(t) = P^{(1)}(t) \cdot [A + B + C].$$

The two systems are identical and have an unique solution. By definition of the markings of the H_q subsets, each marking of a V_i subset belongs to a single H_q subset. Hence we have

$$\lim_{t \to +\infty} \sum_{n=0}^{+\infty} P_n(t) = \lim_{t \to +\infty} [P_{H0(t)}, \cdots, P_{H_q}(t), \cdots].$$

Hence the limiting probabilities for the H_q subsets are the limiting probabilities for the $M_q^{(1)}$ markings of the saturated net.

3) Computation of the Limiting Probabilities: The probability distribution on the Martin exit boundary can be computed from the steady state probability distribution of the OSQN R^∞ obtained by deleting all the OSQN R saturated places. As a matter of fact the preceding analysis of the asymptotic behavior of an OSQN shows that if two places p_1 and p_j are saturated in R, then p_j is saturated in $R^{(1)}$. The steady state mean marking of p_j has the same value in the two nets for it is defined by the same limiting expression.

This property implies that if we delete successively all the saturated places from R all the nets obtained at each step have the same boundary distribution. At the end of this iterative procedure we obtain a SQN R^∞ such that this net has no saturated place. Hence the process (X_n, τ_n) associated with R^∞ is positive recurrent. The steady state probability distribution of R^∞ defines the Martin exit boundary distribution and the limiting probabilities for the H_q subsets.

D. Ergodicity of the Firing Process

Theorem 6.4: The firing process of an OSQN is ergodic i.e.,

$$N^* = \lim_{t\to+\infty} \frac{E(N(t))}{t} \underset{AS}{=} \lim_{t\to+\infty} \frac{N(t, \omega)}{t} < +\infty.$$

- If the (X_n, τ_n) process is positive recurrent then

$$N^*(t_j) = \sum_{M_k} P^*(M_k)\,\lambda_{j,Mk}.$$

- If the marking process of the net is transient or null recurrent and if the unbounded place p_1 is saturated then

$$N^*(t_j) = N^{*(1)}(t_j).$$

Proof: The proof is decomposed according to the three possible ergodicity properties of the (X_n, τ_n) process: positive recurrent, transient, and null recurrent.

Positive Recurrent Case: The proof has been given for stochastic Petri net such that the (X_n, τ_n) process is positive recurrent [8].

Transient Case: For transient OSQN the proof is similar but relies on the existence of the Martin boundary distribution.

The set of markings such that a given transition t_j is firable can be partitioned into subsets C_q. In each marking of C_q the same transition sets are firable. We denote λ_{C_q} the sum of firing rates taken over the transitions firable in each marking belonging to C_q.

Let $N(t_j, t, \omega)$ (resp. $N_k(t_j, t, \omega)$) be the number of firings of a transition t_j between 0 and t on a given trajectory ω (resp. in the marking M_k).

Let $W_k(t, \omega)$ be the number of visits to the marking M_k on a trajectory ω.

Let $T_k(t, \omega)$ be the cumulative sojourn time in the marking M_k between 0 and t on the trajectory ω.

We can write

$$\frac{N(t_j, t, \omega)}{t} = \sum_{C_q} \sum_{M_k\in C_q} \frac{N_k(t_j, t, \omega)}{t}$$

$$= \sum_{C_q} \left[\frac{\sum_{M_k\in C_q} N_k(t_j, t, \omega)}{\sum_{M_k\in C_q} W_k(t, \omega)} \cdot \frac{\sum_{M_k\in C_q} W_k(t, \omega)}{\sum_{M_k\in C_q} T_k(t, \omega)} \cdot \frac{\sum_{M_k\in C_q} T_k(t, \omega)}{t} \right].$$

This decomposition is allowed for the Markov process associated with the OSQN is a transient regular Markov process. In this case the cumulated sojourn time and the number of visits in a marking for a given trajectory are almost surely finite. The limit of the three terms are:

a)

$$\lim_{t\to+\infty} \sum_{M_k\in C_q} \frac{T_k(t, \omega)}{t} \underset{AS}{=} \lim_{t\to+\infty} P_{C_q}(t)$$

where $\lim_{t\to+\infty} P_{C_q}(t)$ is the probability of the subset of the Martin boundary to which converges the markings of C_q. Two cases must be considered.

- C_q is visited only finitely many times and the limiting probability of C_q is zero.
- C_q contains at least a subset of markings which converges to a non-null probability distribution boundary point. Hence the limiting probability of C_q is not zero. We consider only this case in the study of the other terms.

b)

$$\lim_{t\to+\infty} \frac{\sum_{M_k\in C_q} W_k(t, \omega)}{\sum_{M_k\in C_q} T_k(t, \omega)} \underset{AS}{=} \lambda_{C_q}$$

As a matter of fact the class C_q, which contains an infinite number of markings, is visited almost surely infinitely often on a given trajectory ω. The sojourn time in each marking is a random variable exponentially distributed with mean $1/\lambda_{C_q}$.

c)

$$\lim_{t\to+\infty} \frac{\sum_{M_k\in C_q} N_k(t_j, t, \omega)}{\sum_{M_k\in C_q} W_k(t, \omega)} \underset{AS}{=} \frac{\lambda_j}{\lambda_{C_q}}.$$

In each marking of C_q, the probability to fire t_j is determined by a Bernoulli's trial with probability of success λ_j/λ_{C_q}.

Since in the long run the number of such independent experiences is almost surely infinite, the property is verified.

From the three preceding limits we get

$$\lim_{t \to +\infty} \frac{N(t_j, t, \omega)}{t} \underset{AS}{=} \lim_{t \to +\infty} \frac{E(N(t))}{t} = \lambda_j \lim_{t \to +\infty} \sum_{C_q} P_{C_q}(t).$$

As a consequence the limiting probabilities of the C_q classes are defined by the boundary distribution. If the place p_1 is saturated this distribution can be computed from $R^{(1)}$ and

$$N^*(t_j) = N^{*(1)}(t_j).$$

Null Recurrent Case: If the (X_n, τ_n) process is null recurrent, there is at least an unbounded place p_1 such that (Theorem 1) $\lim_{t \to +\infty} E(M(p_1, t)) = +\infty$. Or

$$P(\omega \mid \lim_{t \to +\infty} M(p_1, t, \omega) = +\infty) > 0.$$

Hence the limiting behavior of the OSQN is the behavior of $R^{(1)}$ with a non-null probability (this property can be stated more precisely using the Martin boundary of recurrent Markov processes)

$$P\left(\omega \mid \lim_{t \to +\infty} \frac{N(t, \omega)}{t} = N^{(1)*}\right) > 0.$$

But as $N(t)/t$ is the ratio of two functionals of the null recurrent Markov process associated with the OSQN, we have [3, pp. 264–269]

$$\lim_{t \to +\infty} \frac{N(t, \omega)}{t} \underset{AS}{=} N^{(1)*}.$$

E. Ergodic Theorem for the Marking Process

In this section we prove the following theorem.

Theorem 6.5:

a) If the process (X_n, τ_n) is recurrent then

$$\forall p_i \, i \in [1, MSD] \; C(i, \cdot) \cdot N^{(i)*} \leq 0.$$

b) If the process (X_n, τ_n) is transient then

$$\exists p_i \, C(i, \cdot) \cdot N^{(i)*} \geq 0.$$

c) If the process (X_n, τ_n) is null recurrent then

$$\exists p_i \, C(i, \cdot) \cdot N^{(i)*} = 0.$$

Remark: Taking the negation of the preceding propositions we can express the main result of the paper as an analytical criterion for the classification of OSQN:

If $\forall p_i \, i \in [1, MSD] \; C(i, \cdot) \cdot N^{(i)*} < 0$ then the process (X_n, τ_n) is positive recurrent.

If $\exists p_i \, i \in [1, MSD] \; C(i, \cdot) \cdot N^{(i)*} > 0$ then the process (X_n, τ_n) is transient (there is at least one place with an infinite mean marking).

Proof:

Preliminary Remark: If the process (X_n, τ_n) is positive recurrent, then for any initial distribution the probability of reaching in a finite time a marking in class V_1 is equal to one.

We consider a modification of the initial Markov process in which all states of the classes V_0 and V_1 are absorbing states. The generator of this process is the infinite matrix given Fig. 8. Then for any initial distribution $P(0)$ with support in $V_i \, i > 1$, the absorption in a finite time is almost sure. Q is the subgenerator relative to the transient states (Fig. 8). The mean absorption time vector T and the probability of absorption vector $P(\infty)$ are the unique convergent solution of the following set of equations

$$T \cdot Q + P(0) = 0$$

$$P(\infty) = T \cdot (C, 0, 0, \cdots).$$

a) For an OSQN such that its associated (X_n, τ_n) process is positive recurrent we define a random process on the set of positive, zero, or negative integers $M^{(1)}(p_1, t)$ obtained from the firing process $N^{(1)}(t)$ of the saturated net $R^{(1)}$ by

$$M^{(1)}(p_1, t) = M(p_1, 0) + C(1, \cdot) \cdot N^{(1)}(t).$$

In this process the Petri net firing rules are not applied to the place p_1. Hence the "mark" of this place can be negative.

The random vector $(M^{(1)}(p_1, t), M^{(1)}(p_2, t), \cdots, M^{(1)}(p_{|P|}, t))$ is a Markov process, the generator of which is given by Fig. 9. This process is called the symmetrized process according to the place p_1.

The preceding definitions imply the following two properties:

$$\lim_{t \to +\infty} \frac{E(C(1, \cdot) \cdot N^{(1)}(t))}{t} = C(1, \cdot) \cdot N^{(1)*}$$

and

$$\lim_{t \to +\infty} E(M^{(1)}(p_1, t)) < +\infty => C(1, \cdot) \cdot N^{(1)*} \leq 0.$$

If we modify the symmetrized process by setting all the states of any class V_k as absorbing states, then for any initial distribution with support in a class V_i, $i > k$, the probability of absorption in a finite time is equal to one. As a matter of fact the absorption probability vector in the symmetrized process in the solution of the same system of equation as $P(\infty)$ in the considered positive recurrent OSQN.

So for any initial distribution of the symmetrized process, the support of which is in $V_i \, i > k$ then:

$$\forall \epsilon \quad 0 \leq \epsilon < 1 \quad \exists t < +\infty$$

$$P(M^{(1)}(p_1, t, \omega) \in V_k) \leq 1 - \epsilon.$$

As the generator of the symmetrized process has the same repetitive structure from $-\infty$ to $+\infty$, we can choose $i > k$ arbitrarily.

Hence for a given probability distribution with support in the V_i classes ($i > k$), in the symmetrized process the return in a finite time in a marking of any class V_k, V_{k-1},

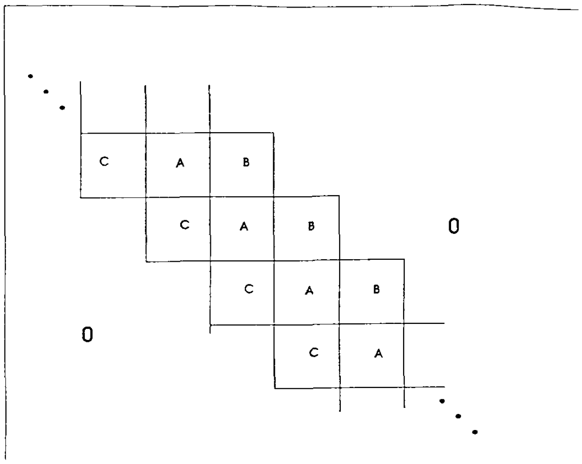

Fig. 9. The symmetrized process generator.

$\cdots$ is an almost sure event. So the limit of $E(M^{(1)}(p_1, t))$ cannot be $+\infty$ and

$$C(1, \cdot) \cdot N^{(1)*} \leq 0.$$

As this proof does not depend on the choice of the unbounded place p_1 we get if the process (X_n, τ_n) is positive recurrent then

$$\forall p_i,\ i \in [1, MSD]\ C(i, \cdot) \cdot N^{(1)*} \leq 0.$$

b) If the OSQN is transient or null recurrent then, for at least one place p_i, $\lim_{t \to +\infty} E(M(p_i, t)) = +\infty$. So $\lim_{t \to +\infty} E(M(p_i, t))/t \geq 0$. As a consequence of the firing equation of the OSQN:

$$E(M(t)) = M(0) + C \cdot E(N(t))$$

we have $\lim_{t \to +\infty} E(N(t))/t \geq 0$. The ergodic theorem for the firing process states that in this case $N^* = N^{(i)*}$. So, $C \cdot N^{(i)*} \geq 0$.

c) In the null recurrent case we have from b) for at least one unbounded place p_i

$$C(i, \cdot) \cdot N^{(1)*} \geq 0.$$

But, from a) for a recurrent OSQN

$$\forall p_i\ i \in [1, MSD]\ C(i, \cdot) \cdot N^{(1)*} \leq 0.$$

Hence, the only possible case for a null recurrent OSQN is

$$\exists p_i,\ i \in [1, MSD]\ C(i, \cdot) \cdot N^{(1)*} = 0.$$

VII. Conclusion

The criterion presented in this paper generalizes the stability condition for Jackson queueing networks and the Lavenberg criterion [12].

The considered models may include several open queues, the behavior of which are described by a complex synchronization scheme. This scheme allows grouped arrivals or departures and the arrival and departure processes can be correlated.

Nevertheless, some synchronizations schemes cannot be described by Petri nets. For example, in a Petri net the enabling condition of a transition cannot be the null mark of an unbounded place. In order to model such behavior the notion of inhibitor arcs must be introduced [1]. We conjecture that it is possible to prove a similar result for a particular class of OSQN with inhibitor arcs.

Our result is not valid for stochastic Petri nets, the firing rates of which are functions of the mark of the unbounded places. We conjecture that it is possible to extend the criterion to monotonic bounded firing rates.

The example presented in this paper shows that, in some particular cases, the use of SPN invariants properties authorize the ergodicity criterion computation. In the general case, the steady state probability distribution of OSQN having several unbounded places must be computed. If the marking space dimension is two, the computation of the criterion implies to obtain the probability distribution of two OSQN each with $MSD = 1$ (complex queues). Several numerical solutions of this problem are presented in [7].

A general method to compute the OSQN steady state probability distribution, which is the key to the practical use of these nets, must be found.

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