A Medvedev Characterization of Sets Recognized by Generalized Finite Automata

by

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1. In [2], I. T. Medvedev characterizes the regular events of S. C. Kleene [1] as the smallest class of sets containing a particular class of four primitive events and closed under a class of five primitive operations. In this paper the analogues of Medvedev's primitives are defined for terms (trees) over a ranked alphabet. It is then proved that the class of sets of terms generated by these primitives is identical with the regular sets of terms of J. W. Thatcher and J. B. Wright [3] which arise in generalized finite automata theory. In the process of extending Medvedev's results to trees it was discovered that, whereas his method required a minimum of four primitive sets and five primitive operations, the number of primitive events and primitive operations can each be diminished by one. In addition, involving the semigroup of an automaton has been avoided as it is unnecessary. Since the article of Medvedev is inaccessible to many readers for a variety of reasons, we include, in Section 3, a shorter direct proof of his result using a smaller class of primitives. The reader who is interested only in this result may omit Section 2.

2. The inductive definition of trees which we use is the usual description of terms from universal algebra Its use in automata theory originated with Thatcher and Wright [3]. A *ranked alphabet* is a pair $\langle \Sigma, \rho \rangle$, where Σ is a finite set and $\rho \colon \Sigma \to N$, the natural numbers. We denote the collection of n-ary symbols $\rho^{-1}(n)$, by Σ_n. Given a ranked alphabet $\langle \Sigma, \rho \rangle$, we construct the set $\mathscr{T}_\Sigma$ of Σ-*terms* (trees) inductively as follows:

(i) $\Sigma_0 \subseteq \mathscr{T}_\Sigma$,
(ii) if $t_1, t_2, \cdots, t_n \in \mathscr{T}_\Sigma$ and $f \in \Sigma_n$, then $ft_1t_2\cdots t_n \in \mathscr{T}_\Sigma$.

One can identify $\mathscr{T}_\Sigma$ with the set of rooted trees with node labels from Σ where a node is labeled with an element of Σ_n if the node has n successors. These terms serve as inputs to a class of finite automata which are defined via universal algebras.

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A Σ-*algebra*, where $\langle \Sigma, \rho \rangle$ is a ranked alphabet, is a pair $\langle A, \alpha \rangle$ consisting of a set A and a family α of operations on A indexed by Σ such that α_f is a $\rho(f)$-ary operation, that is, $\alpha_f \colon A^{\rho(f)} \to A$. A Σ-*automaton* is a triple $\mathfrak{A} = \langle A, \alpha, A_F \rangle$, where $\langle A, \alpha \rangle$ is a finite Σ-algebra and $A_F \subseteq A$. The elements of the set A are the states of $\mathfrak{A}$ and the elements of A_F are the accepting states of A. Given a Σ-automaton $\mathfrak{A} = \langle A, \alpha, A_F \rangle$, the evaluation of a Σ-term by $\mathfrak{A}$ is determined by the *output function* $h_{\mathfrak{A}} \colon \mathscr{T}_\Sigma \to A$ which is defined inductively as follows:

(i) If $\lambda \in \Sigma_0$, then $h_{\mathfrak{A}}(\lambda) = \alpha_\lambda(*)$, where $*$ is the only element of the singleton set A^0.

(ii) If $f \in \Sigma_n$ and $t_1, t_2, \cdots, t_n \in \mathscr{T}_\Sigma$, then $h_{\mathfrak{A}}(ft_1 \cdots t_n) = \alpha_f(h_{\mathfrak{A}}(t_1), \cdots, h_{\mathfrak{A}}(t_n))$.

The *behavior* of $\mathfrak{A}$, or the set of terms *recognized* by $\mathfrak{A}$, is the subset of $\mathscr{T}_\Sigma$ consisting of those terms whose output is an accepting state. Formally, the behavior of $\mathfrak{A}$, $bh(\mathfrak{A}) = \{t \in \mathscr{T}_\Sigma | h_{\mathfrak{A}}(t) \in A_F\}$.

In [3], Thatcher and Wright developed an algebraic characterization of recognizable sets. The following concepts are due to them.

If $\langle \Sigma, \rho \rangle$ is a ranked alphabet, the following analogues of the complex product and Kleene closure operation are defined. If $\lambda \in \Sigma_0$, the λ-*product* of $U, V \subseteq \mathscr{T}_\Sigma$, denoted by $U \cdot_\lambda V$, is the set of all terms $t \in \mathscr{T}_\Sigma$ which can be obtained from a term $t' \in U$ by replacing each occurrence of λ in t' by an element of V. Again, for $\lambda \in \Sigma_0$, the λ-*closure* of $U \subseteq \mathscr{T}_\Sigma$, denoted by U^λ, is the set $\bigcup_{n=0}^{\infty} X_n$, where $X_0 = \{\lambda\}$ and $X_{n+1} = X_n \cup U \cdot_\lambda X_n$. These definitions lead to a notion of regularity for Σ-terms. The class of Σ-*regular* subsets of $\mathscr{T}_\Sigma$ is the smallest collection of subsets of $\mathscr{T}_\Sigma$ which contains all of the finite sets and is closed under the operations $\cup$, $\cdot_\lambda$ (λ-product), and ${}^\lambda$(λ-closure) for each $\lambda \in \Sigma_0$. In order that the notion of regularity for sets of Σ-terms correspond to analogous notions for ordinary finite automata, the alphabet $\langle \Sigma, \rho \rangle$ must be enlarged. A set $U \subseteq \mathscr{T}_\Sigma$ is *regular* if there is a ranked alphabet $\langle \Sigma', \rho' \rangle$ for which $\Sigma' - \Sigma \subseteq \Sigma_0'$, and U is Σ'-regular.

THEOREM 1. (Thatcher and Wright [3]) *A set $U \subseteq \mathscr{T}_\Sigma$ is recognizable by a Σ-automaton if and only if it is a regular set.*

We now present a Medvedev-type characterization of these recognizable sets. In order to describe the primitive sets of Σ-terms which correspond to those of Medvedev [2], we need the following definitions. The "root" or "top of the tree" function $\tau \colon \mathscr{T}_\Sigma \to \Sigma$ is given by: (i) if $\lambda \in \Sigma_0$, then $\tau(\lambda) = \lambda$; (ii) if $f \in \Sigma_n$ and $t_1, \cdots, t_n \in \mathscr{T}_\Sigma$, then $\tau(ft_1 \cdots t_n) = f$. The "next to the top of the tree" function $\eta \colon \mathscr{T}_\Sigma - \Sigma_0 \to \bigcup \{\Sigma^n | n = 1, 2, \cdots, \sup \rho(\Sigma)\}$ is given, for $f \in \Sigma_n$, $t_1, \cdots, t_n \in \mathscr{T}_\Sigma$, by $\eta(ft_1 \cdots t_n) = (\tau(t_1), \cdots, \tau(t_n))$. The following subsets of $\mathscr{T}_\Sigma$ are called *elementary events*: (i) Σ_0, (ii) for each $f \in \Sigma$, the set $\tau^{-1}(f)$, (iii) for each $(f_1, \cdots, f_k) \in \Sigma^k$, the set $\eta^{-1}(f_1, \cdots, f_k)$.

The notion of a *subtree* of a tree in $\mathscr{T}_\Sigma$ can be formalized as follows: (i) if $\lambda \in \Sigma_0$, the only subtree of λ is λ itself, (ii) if $f \in \Sigma_n$ and $t_1, \cdots, t_n \in \mathscr{T}_\Sigma$, then t' is a subtree of $t = ft_1 \cdots t_n$ provided that either $t' = t$ or t' is a subtree of some t_i.

The following operations on terms are called *elementary operations*: (i) $\cup$ (set union); (ii) $\cap$ (set intersection); (iii) projection; that is, the replacement of symbols in Σ_n by symbols from Σ'_n, where $\langle \Sigma', \rho' \rangle$ is a ranked alphabet; (iv) (t), where for $U \subseteq \mathscr{T}_\Sigma$, the set $(t)U$ consists of all trees all of whose subtrees are in U. Notice that the elementary events correspond to the events $E_{p=1}$, $E_{S_i}^{(p)}$, and $E_{S_i}^{(p-1)}$, respectively, of Medvedev [2] and that the elementary operations correspond to Medvedev's operations $\vee$, &, $[S_i \to S]$ and $(t)_{t \leq p}$, respectively. Moreover, our elementary events and elementary operations correspond to a proper subset of Medvedev's.

We say a set $U \subseteq \mathscr{T}_\Sigma$ is *representable* if it can be obtained from the elementary events by a finite number of applications of the elementary operations.

THEOREM 2. *A set $U \subseteq \mathscr{T}_\Sigma$ is regular if and only if it is representable.*

The proof of this theorem is given in two parts. We first show that any representable set is regular. This is done by proving that any elementary event is regular and that the elementary operations preserve regularity.

LEMMA 1. *The elementary events are regular.*

Proof. The elementary event Σ_0 is trivially regular. To see that the elementary event $\tau^{-1}(f)$ is regular, $f \in \Sigma_n$, choose $x_1, x_2, \cdots, x_n$ not in Σ. Let Σ' be the ranked alphabet $\langle \Sigma \cup \{x_1, \cdots, x_n\}, \rho' \rangle$, where $\rho' | \Sigma = \rho$ and $\rho'(x_i) = 0$ for $i = 1, 2, \cdots, n$. Then

$$\tau^{-1}(f) = \{f x_1 x_2 \cdots x_n\} \cdot_{x_1} \mathscr{T}_\Sigma \cdot_{x_2} \mathscr{T}_\Sigma \cdot \cdots \cdot_{x_n} \mathscr{T}_\Sigma$$

so that $\tau^{-1}(f)$ is Σ'-regular, thus regular. Now let $(f_1, \cdots, f_n) \in \Sigma^n$ and let Σ' be defined as before. Then

$$\eta^{-1}(f_1, \cdots, f_n) = \bigcup_{g \in \Sigma_n} \{g x_1 \cdots x_n\} \cdot_{x_1} \tau^{-1}(f_1) \cdot \cdots \cdot_{x_n} \tau^{-1}(f_n),$$

thus $\eta^{-1}(f_1, \cdots, f_n)$ is regular.

LEMMA 2. *The elementary operations preserve regularity.*

Proof. That $\cup$, $\cap$ and projection preserve regularity is given in [3]. Let $U \subseteq \mathscr{T}_\Sigma$ be regular. We will prove that $(t)U$ is regular by constructing a Σ-automaton which recognizes $(t)U$. To this end, let $\mathfrak{A} = \langle A, \alpha, A_F \rangle$ be a Σ-automaton for which $bh(\mathfrak{A}) = U$. Construct a Σ-automaton $\mathscr{B} = \langle A_F \cup \{D\}, \beta, A_F \rangle$ where $D \notin A_F$ and β is given by $\beta_f(a_1, \cdots, a_n) = \alpha_f(a_1, \cdots, a_n)$ if for $i = 1, \cdots, n$, $a_i \in A_F$ and $\alpha_f(a_1, \cdots, a_n) \in A_F$, $\beta_f(a_1, \cdots, a_n) = D$ if $a_i = D$ for some i or if $\alpha_f(a_1, \cdots, a_n) \notin A_F$. The Σ-automaton $\mathscr{B}$ simulates $\mathfrak{A}$ on an input tree t as long as only accepting states of $\mathfrak{A}$ are encountered in evaluating t. If a non-accepting state of $\mathfrak{A}$ is encountered, $\mathscr{B}$ goes to the rejecting state D and remains there. The reader can easily supply the formal proof that $bh(\mathscr{B}) = (t)U$.

We now proceed to the proof of the second part of Theorem 2.

LEMMA 3. *Regular sets are representable.*

Proof. Let E be a regular set, $E \subseteq \mathscr{T}_\Sigma$, and let $\mathfrak{A} = \langle A, \alpha, A_F \rangle$ be a Σ-automaton for which $bh(\mathfrak{A}) = E$. Define a new ranked alphabet $\langle \Omega, \omega \rangle$, where

$$\Omega_n = \{(q_1, q_2, \cdots, q_n, f) | f \in \Sigma_n \text{ and } (q_1, \cdots, q_n) \in A^n\}$$

and $\omega(q_1, \cdots, q_n, f) = \rho(f)$. Notice that

$$\Omega_0 = \{(*, \lambda) | \lambda \in \Sigma_0 \text{ and } * \text{ is the only element in } A^0\}.$$

We use π to denote the projection of $\mathscr{T}_\Omega \to \mathscr{T}_\Sigma$ induced by $(q_1, q_2, \cdots, q_n, f) \to f$. Let us construct two auxiliary representable sets over the alphabet $\langle \Omega, \omega \rangle$:

$$F_1 = \bigcup_{\alpha_f(\mathbf{q}) \in A_F} \tau^{-1}(\mathbf{q}, f),$$

$$F_2 = \bigcup_{\alpha_{f_i}(\mathbf{q}_i) = q_i} [\eta^{-1}((\mathbf{q}_1, f_1), \cdots, (\mathbf{q}_m, f_m)) \cap \tau^{-1}(q_1, \cdots, q_m, f)].$$

In the above, the notation $(\mathbf{q}, f)$ means that $\mathbf{q} \in A^k$, $f \in \Sigma_k$ so that $(\mathbf{q}, f) \in \Omega_k$. Let $F = F_1 \cap (t)\,[F_2 \cup \Omega_0]$. We will prove that $\pi(F) = E$.

(i) Let $t \in F$. If $t \in \Omega_0$, then $t = (*, \lambda)$, where $\lambda \in \Sigma_0$. Moreover, $\alpha_\lambda(*) \in A_F$ since $t \in F_1$, thus $\pi(t) = \lambda \in E$. Take as induction hypothesis that if $t \in (t)\,[F_2 \cup \Omega_0]$, then $h_{\mathfrak{A}}(\pi(t)) = \alpha_f(\mathbf{q})$, where $(\mathbf{q}, f) = \tau(t)$. Suppose that $t = (q_1, \cdots, q_m, f)t_1 \cdots t_m \in (t)\,[F_2 \cup \Omega_0]$. Then for $i = 1, 2, \cdots, m$, $h_{\mathfrak{A}}(\pi(t_i)) = \alpha_{f_i}(\mathbf{q}_i)$, where $\tau(t_i) = (\mathbf{q}_i, f_i)$. Then

$$\begin{aligned} h_{\mathfrak{A}}(\pi(t)) &= \alpha_f(h_{\mathfrak{A}}(\pi(t_1)), \cdots, h_{\mathfrak{A}}(\pi(t_m))) \\ &= \alpha_f(\alpha_{f_i}(\mathbf{q}_i), \cdots, \alpha_{f_m}(\mathbf{q}_m)) \\ &= \alpha_f(q_1, \cdots, q_m). \end{aligned}$$

Hence if $t \in F_1 \cap (t)\,[F_2 \cup \Omega_0]$, we have $h_{\mathfrak{A}}(\pi(t)) = \alpha_f(\mathbf{q})$, where $\tau(t) = (\mathbf{q}, f)$ and also $\alpha_f(\mathbf{q}) \in A_F$. Therefore $\pi(t) \in E$ and so $\pi(F) \subseteq E$.

(ii) If $t \in E$, we must construct an Ω-term $t^1 \in F$ for which $\pi(t') = t$. To this end we prove, by induction, that π maps $(t)\,[F_2 \cup \Omega_0]$ onto $\mathscr{T}_\Sigma$. If $t = \lambda \in \Sigma_0$ then $(*, \lambda) \in (t)\,[F_2 \cup \Omega_0]$ and $\pi(*, \lambda) = \lambda$. If $t = ft_1 \cdots t_m \in \mathscr{T}_\Sigma$, where $f \in \Sigma_m$ and $t_1, t_2, \cdots, t_m \in \mathscr{T}_\Sigma$, choose $t_1', t_2', \cdots, t_m' \in (t)\,[F_2 \cup \Omega_0]$ for which $\pi(t_i') = t_i$. Now let $t' = (q_1, q_2, \cdots, q_m, f)t_1't_2' \cdots t_m'$, where $q_i = h_{\mathfrak{A}}(t_i)$ for $i = 1, 2, \cdots, m$. Clearly $t' \in (t)\,[F_2 \cup \Omega_0]$ and $\pi(t') = t$. If $t \in E$, construct t' as above. Then $h_{\mathfrak{A}}(t) = h_{\mathfrak{A}}(\pi(t')) = \alpha_f(q_1, \cdots, q_m) \in A_F$ so that $t' \in F$. Thus $E \subset \pi(F)$.

A more intuitive description of the set F may be useful to the reader. A term (tree) t' appears in F_2 exactly when each node is labeled with something of the form $(q_1, \cdots, q_n, f) \in \Omega_n$, where q_i is the output of the subtree of $\pi(t')$ attached to the ith successor node of f. If t' is also in F_1, then the top, or root, of t' is labeled with $(q_1, \cdots, q_n, f)$, where $\alpha_f(q_1, \cdots, q_n) \in A_F$. Thus $t' \in F$ exactly when $\pi(t')$ can be evaluated by the Σ-automaton $\mathfrak{A}$ and the resulting output is in A_F.

3. We now present a short, direct proof of a generalization of the theorem of Medvedev [2].

If Σ is a finite alphabet, the following subsets of Σ^* are *elementary events*: (i) Σ, (ii) for each $\sigma \in \Sigma$, the set $\Sigma^*\sigma$, (iii) for each $\sigma \in \Sigma$, the set $\Sigma^*\sigma\Sigma$, and the following operations on subsets of Σ^* are *elementary operations*: (i) $\cup$ (set union), (ii) $\cap$ (set intersection), (iii) (t), where if $U \subseteq \Sigma^*$, $(t)U = \{x \in \Sigma^* |$ every prefix of x is in $U\}$, (iv) projection, that is, the replacement of elements of Σ by elements of another finite alphabet.

These events and operations are a proper subset of those used by Medvedev. He uses the additional event Σ^2 and the additional operation $\sim$ (set complement in Σ^*).

A subset $U \subseteq \Sigma^*$ is *representable* if it can be obtained from the elementary events by finitely many applications of the elementary operations.

THEOREM 3. *A $U \subseteq \Sigma^*$ is representable if and only if it is regular (in the sense of Kleene* [1]).

Note. Kleene's regular sets do not contain the null word, thus "regular" in Theorem 3 could be replaced by "ε-free regular". A proof of Theorem 3 for regular sets which may contain the null word can be obtained by specializing the methods of Section 2 to initialized monadic algebras.

Proof. The elementary events are regular and the elementary operations preserve regularity so that the "only if" statement is clear from the usual literature on regular events. To prove the converse, let $\mathfrak{A} = \langle Q, \delta, q_0, F\rangle$ be a Σ-automaton which accepts U and let Ω be the finite alphabet $Q \times \Sigma$. Define the auxiliary representable sets

$$B_1 = \bigcup_{\delta(q,\sigma)\in F} \Omega^*(q, \sigma)$$

$$B_2 = \bigcup_{\delta(q,\sigma) = q'} [\Omega^*(q, \sigma)\Omega \cap \Omega^*(q', \sigma')]$$

$$B_3 = \Big[\bigcup_{\sigma\in\Sigma} \Omega^*(q_0\ \sigma)\Big] \cap \Omega.$$

Now let $B = B_1 \cap (t)\,[B_2 \cup B_3]$ and let $\pi\colon \Omega^* \to \Sigma^*$ be the projection determined by $(q\ \sigma) \to \sigma$. We assert that $\pi(B) = U$. Let $x = (q_1, \sigma_1)\cdots(q_m, \sigma_m) \in B$. Then $(q_1, \sigma_1) \in B_3$ so that $q_1 = q_0$. Now notice that for $i = 1,\cdots, m-1$, we have $\delta(q_i, \sigma_i) = q_{i+1}$ since $x \in (t)\,[B_2 \cup B_3]$. Moreover, $\delta(q_m, \sigma_m) \in F$ since $x \in B_1$. Hence $\pi(x) = \sigma_1\cdots\sigma_m \in U$. Conversely, if $y = \sigma_1\cdots\sigma_m \in U$, define $x = (q_1, \sigma_1)\cdots(q_m, \sigma_m) \in B$ as follows. Set $q_1 = q_0$ and inductively define $q_{i+1} = \delta(q_i, \sigma_i)$ for $i = 1, 2,\cdots, m-1$. Clearly $\pi(x) = y$ and it is easy to see that $x \in (t)\,[B_2 \cup B_3]$. Since $y \in U$, the state $\delta(q_0, y) = \delta(q_m, \sigma_m) \in F$, and so $x \in B$.

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