ON THE VALUE GROUP OF A DIFFERENTIAL VALUATION

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1. Introduction. If k is an ordinary differential field of characteristic zero whose subfield of constants is C, by a *differential valuation* of k is meant a valuation v of k that is trivial on C, for which C is a system of representatives of the residue class field, that satisfies the further condition that if a, b are nonzero elements of k and $v(a), v(b) > 0$, then also $v(a'b/b') > 0$ [1]. The significance of this notion is that with it various versions of L'Hospital's rule hold and one can do a considerable amount of asymptotic analysis.

It is known [1, Theorem 4] that if k is a differential field and v a differential valuation of k with value group Γ, then there is a map ψ from $\Gamma^* = \Gamma - \{0\}$ into Γ such that for all $a \in k^*$ with $v(a) \neq 0$ we have $\psi(v(a)) = v(a'/a)$ and

(a) if $\alpha \in \Gamma^*$ and $n \in \mathbf{Z}$, $n \neq 0$, then $\psi(n\alpha) = \psi(\alpha)$

(b) for any $\gamma \in \Gamma$, the set $\{\alpha \in \Gamma : \alpha = 0 \text{ or } \psi(\alpha) \geq \gamma\}$ is a subgroup of Γ

(c) for any $\alpha, \beta \in \Gamma^*$, $\psi(\beta) < \psi(\alpha) + |\alpha|$.

Theorem 1 of the present paper shows that if Γ is an arbitrary ordered abelian group and $\psi : \Gamma^* \to \Gamma$ a function satisfying conditions (a), (b), (c), then the pair (Γ, ψ) comes from a differential valuation of a differential field, as above, at least if suitable extra hypotheses are imposed. Our other main result, Theorem 2, uses the pair (Γ, ψ) associated with a differential valuation v of a differential field k to handle the problem of asymptotic integration: if $a \in k$, does there exist some $b \in k$ that approximates $\int a$, or, which should mean roughly the same thing, does there exist $b \in k$ such that $a - b'$ is small compared to a, that is such that $v(a - b') > v(a)$? For example, consider the differential field $k = \mathbf{R}(x, e^{x^2})$ whose derivation is the usual differentiation with respect to x, which can be identified with a Hardy field of germs of dif-

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ferentiable real-valued functions on neighborhoods of $+\infty$ in $\mathbf{R}$, which has a natural differential valuation corresponding to the place that sends each function into its limit as $x \to +\infty$ [1, Example 1]. The element $e^{x^2} \in k$ has a nonelementary integral $\int_0^x e^{t^2}\, dt$, but this can be approximated by $e^{x^2}/2x \in k$ in the above sense and also in the sense that the ratio of the two functions approaches 1 as $x \to +\infty$. (In fact, these two senses are equivalent, since $k(\int_0^x e^{t^2}\, dt)$ is also a Hardy field.) Our Section 3 below handles some general aspects of this theory.

2. Differential Fields with Given Value Group. For any torsion free abelian group Γ, the map $\gamma \to \gamma \otimes 1$ is an embedding of Γ in the $\mathbf{Q}$-vector space $\mathbf{Q}\Gamma = \Gamma \otimes_{\mathbf{Z}} \mathbf{Q}$, and if Γ is an ordered abelian group then there is a unique ordering of $\mathbf{Q}\Gamma$ that preserves the ordering of Γ. Suppose now that Γ is an ordered abelian group and that $\psi:\Gamma^* \to \Gamma$ is a function satisfying properties (a), (b), (c). We claim that there is a unique function from $(\mathbf{Q}\Gamma)^*$ into $\mathbf{Q}\Gamma$, which we also denote ψ, that extends the given ψ on Γ^* and that also satisfies (a), (b), (c): Property (a) clearly gives a well-defined definition of ψ on $(\mathbf{Q}\Gamma)^*$, property (b) for $\mathbf{Q}\Gamma$ is an immediate consequence of (b) for Γ, while property (c) for $\mathbf{Q}\Gamma$ is an immediate consequence of [1, Theorem 5]. Note that $\psi(\Gamma^*) = \psi((\mathbf{Q}\Gamma)^*)$ and that the cardinality of the latter set is at most the vector space dimension $\dim_{\mathbf{Q}} \mathbf{Q}\Gamma$. If (Γ, ψ) comes from a differential valuation v of a differential field k with subfield of constants C, then $\dim_{\mathbf{Q}} \mathbf{Q}\Gamma$ is just the rational rank of v, which is known to be at most deg tr k/C; thus $\psi(\Gamma^*)$ is finite if k is a finite extension field of C, in particular if k is generated as a differential extension field of C by a finite number of elements that are differentially algebraic over C.

THEOREM 1. *Let Γ be an ordered abelian group and $\psi:\Gamma^* \to \Gamma$ a function with properties* (a), (b), (c). *Suppose that the ordered subset $\psi(\Gamma^*)$ of Γ, in the opposite ordering, is well-ordered. Let C be a field of characteristic zero such that the vector space dimension* $\dim_{\mathbf{Q}} C \geq \dim_{\mathbf{Q}} \mathbf{Q}\Gamma$. *Then there exists a differential field K whose subfield of constants is C and a differential valuation v of K whose value group is Γ such that for each $a \in K^*$ with $v(a) \neq 0$ we have $v(a'/a) = \psi(v(a))$.*

For each $\gamma \in \psi((\mathbf{Q}\Gamma)^*) = \psi(\Gamma^*)$ choose a set $\Sigma_\gamma \subset \Gamma$ whose canonical image in the $\mathbf{Q}$-vector space

$$\{\alpha \in \mathbf{Q}\Gamma : \alpha = 0 \quad \text{or} \quad \psi(\alpha) \geq \gamma\}/\{\alpha \in \mathbf{Q}\Gamma : \alpha = 0 \quad \text{or} \quad \psi(\alpha) > \gamma\}$$

is a $\mathbf{Q}$-basis of the latter vector space. We shall prove the theorem for any field C of characteristic zero such that $\dim_{\mathbf{Q}} C \geq \operatorname{card} \Sigma_\gamma$ for each $\gamma \in \psi(\Gamma^*)$, a slight weakening of the stated hypothesis. We have $\psi(\Sigma_\gamma) = \{\gamma\}$. Let $\Sigma = \cup(\Sigma_\gamma : \gamma \in \psi(\Gamma^*))$. We claim that Σ is a $\mathbf{Q}$-basis of $\mathbf{Q}\Gamma$. First, if Σ were linearly dependent we could find an integer $n \geq 1$ and elements $\alpha_1, \ldots, \alpha_n \in \Sigma$ and $c_1, \ldots, c_n \in \mathbf{Q}^*$ such that $c_1\alpha_1 + \cdots + c_n\alpha_n = 0$. Taking the canonical linear map of $\mathbf{Q}\Gamma$ with kernel

$$\{\alpha \in \mathbf{Q}\Gamma : \alpha = 0 \quad \text{or} \quad \psi(\alpha) > \min\{\psi(\alpha_1), \ldots, \psi(\alpha_n)\}\}$$

gives a relation of linear dependence among those α_i's for which $\psi(\alpha_i) = \min\{\psi(\alpha_1), \ldots, \psi(\alpha_n)\}$, a contradiction. Therefore Σ is linearly independent. Next, if $\alpha_0 \in (\mathbf{Q}\Gamma)^*$ then we can find an element β_0 in the $\mathbf{Q}$-space spanned by $\Sigma_{\psi(\alpha_0)}$ such that either $\alpha_0 - \beta_0 = 0$ or $\psi(\alpha_0) < \psi(\alpha_0 - \beta_0)$. If $\alpha_0 - \beta_0 \neq 0$, we can find an element β_1 in the space spanned by $\Sigma_{\psi(\alpha_0 - \beta_0)}$ such that either $\alpha_0 - \beta_0 - \beta_1 = 0$ or $\psi(\alpha_0 - \beta_0) < \psi(\alpha_0 - \beta_0 - \beta_1)$. We can try to continue this process, thus getting $\beta_2, \beta_3, \ldots$, but by virtue of our well-ordering hypothesis this process must come to an end, giving $\alpha_0 = \beta_0 + \beta_1 + \cdots + \beta_n$ for some n, with each β_i in the $\mathbf{Q}$-space spanned by Σ. Thus Σ is a $\mathbf{Q}$-basis of $\mathbf{Q}\Gamma$. Our desired field K will be an algebraic extension field of $C(X)$, where X is some fixed set of indeterminates having the same cardinality as Σ. Let $v : X \to \Sigma$ be a fixed bijection. Also fix a function $h : X \to C^*$ with the property that for each $\gamma \in \psi(\Gamma^*)$, h maps the set $\{x \in X : v(x) \in \Sigma_\gamma\}$ onto a set of elements of C that are linearly independent over $\mathbf{Q}$; that such an h exists follows from our cardinality assumption. Now the multiplicative group of an algebraically closed field is divisible, hence an injective $\mathbf{Z}$-module. Thus for each $x \in X$ we can find a homomorphism φ_x of the group $\mathbf{Q}$ into $\overline{C(X)}^*$, where $\overline{C(X)}$ is some fixed algebraic closure of $C(X)$, such that $\varphi_x(1) = x$. For $x \in X$ and $a \in \mathbf{Q}$, we define the symbol x^a to be $\varphi_x(a)$. Thus the set of all expressions $x_1^{a_1}x_2^{a_2} \cdots x_n^{a_n}$, where n is some positive integer, each $x_i \in X$ and each $a_i \in \mathbf{Q}$, forms a multiplicative subgroup U_1 of $\overline{C(X)}$ and $x_1^{a_1}x_2^{a_2} \cdots x_n^{a_n}$ takes on its usual meaning if each $a_i \in \mathbf{Z}$. For any $u = x_1^{a_1} \cdots x_n^{a_n}$ in U_1, we define $v(u) = \Sigma_{i=1}^n a_i v(x_i) \in \mathbf{Q}\Gamma$, which is consistent with the previous definition of v on X. Since the elements of X are algebraically independent over C and $v(X) = \Sigma$ is a $\mathbf{Q}$-basis of $\mathbf{Q}\Gamma$, the map $v : U_1 \to \mathbf{Q}\Gamma$ is a group isomorphism. Now look at the ring $C[U_1] \subset \overline{C(X)}$. Any nonzero element of $C[U_1]$ can be written in the form $c_1u_1 + \cdots + c_ru_r$, where $u_1, \ldots, u_r$ are distinct elements of U_1 and each $c_i \in C^*$, and this representation is unique except for the order of the terms, again by the algebraic independence over C of the elements of X. We can therefore extend the map $v: U_1 \to \mathbf{Q}\Gamma$ to a well-defined map of the nonzero elements of $C[U_1]$ into $\mathbf{Q}\Gamma$ by setting $v(c_1u_1 + \cdots + c_ru_r) = \min\{v(u_1), \ldots, v(u_r)\}$, if $u_1, \ldots, u_r$ are distinct elements of U_1 and each $c_i \in C^*$, and we check immediately that for nonzero elements $y_1, y_2 \in C[U_1]$ we have $v(y_1y_2) = v(y_1) + v(y_2)$, $v(y_1 + y_2) \geq \min\{v(y_1), v(y_2)\}$. Thus the map $v: (C[U_1])^* \to \mathbf{Q}\Gamma$ extends to a valuation $v: (C(U_1))^* \to \mathbf{Q}\Gamma$. Thus there is a unique valuation v on $C(U_1)$ that is trivial on C and agrees with our original v on X, and the value group and residue class field of v are respectively $\mathbf{Q}\Gamma$ and C. Now let $T_1 = \{u \in U_1: v(u) \in \Gamma\}$. Then T_1 is a subgroup of U_1 and v induces an isomorphism between T_1 and Γ. Setting $K = C(T_1)$, v induces a valuation of K that is trivial on C and has value group Γ and residue class field C. We now go about defining a derivation on K. There is a well-defined function $\xi: X \to U_1$ such that for each $x \in X$ we have $v(\xi(x)) = \psi(v(x))$. Since ψ maps Γ^* into Γ, we have $\xi(X) \subset T_1$. Since X is a transcendence base for K over C, there is a unique derivation of K that is trivial on C and such that for each $x \in X$ we have $x' = h(x)\xi(x)x$. This derivation makes K into a differential field such that for each $x \in X$ we have $v(x'/x) = v(\xi(x)) = \psi(v(x))$. Now let $T = C^*T_1$, a subgroup of the multiplicative group of K. We have $C^* \subset T$, $K = C(T)$, and the kernel of the homomorphism $v: T \to \Gamma$ is C^*. We now wish to show that any constant in T is in C. It suffices for this to show that the only constant in T_1 is 1. So suppose that $x_1, \ldots, x_n$ are distinct elements of X, that $a_1, \ldots, a_n \in \mathbf{Q}^*$, that $x_1^{a_1} \cdots x_n^{a_n} \in T_1$ and that $(x_1^{a_1} \cdots x_n^{a_n})' = 0$. Then $(x_1^{a_1} \cdots x_n^{a_n})'/(x_1^{a_1} \cdots x_n^{a_n}) = a_1x_1'/x_1 + \cdots + a_nx_n'/x_n = a_1h(x_1)\xi(x_1) + \cdots + a_nh(x_n)\xi(x_n) = 0$. Now each $a_ih(x_i) \in C^*$ and each $\xi(x_i) \in U_1$, and there is a partition $\mathcal{P}$ of the set $\{1, \ldots, n\}$ such that for $i, j \in \{1, \ldots, n\}$ we have $\xi(x_i) = \xi(x_j)$ if and only if i and j are in the same set in $\mathcal{P}$. Therefore for any set $I \in \mathcal{P}$ we have $\sum_{i \in I} a_ih(x_i) = 0$. For $i \in I \in \mathcal{P}$ we have $\psi(v(x_i)) = v(\xi(x_i))$, so that $v(x_i) \in \Sigma_{v(\xi(x_i))}$, which depends only on I. But the function $h: X \to C^*$ was constructed so that the elements $\{h(x_i)\}_{i \in I}$ are linearly independent over $\mathbf{Q}$, which implies that $a_i = 0$ for all $i \in I$, a contradiction. Therefore each constant in T is in C. We now verify that for each $u \in T$ such that $v(u) \neq 0$, we have $v(u'/u) = \psi(v(u))$. Since $T = C^*T_1 \subset C^*U_1$, we need only prove this for $u \in U_1$. Write $u = x_1^{a_1} \cdots x_n^{a_n}$, with $x_1, \ldots, x_n$ distinct elements of X and $a_1, \ldots, a_n \in \mathbf{Q}^*$. Then $v(u'/u) = v(\Sigma_{i=1}^n a_i x_i'/x_i) = v(\Sigma_{i=1}^n a_i h(x_i)\xi(x_i))$. We know that for $i = 1, \ldots, n$, $\xi(x_i)$ depends only on $\psi(v(x_i))$, so by the way the function h was defined we get $\Sigma_{i\in I} a_i h(x_i) \neq 0$ if I is any maximal subset of $\{1, \ldots, n\}$ for which $\xi(x_i)$ assumes a constant value. We therefore obtain $v(u'/u) = \min_{i=1,\ldots,n} v(\xi(x_i)) = \min_{i=1,\ldots,n} \psi(v(x_i))$. By our construction of the set Σ and property (b), the latter value is just $\psi(\Sigma_{i=1}^n a_i v(x_i)) = \psi(v(u))$, as was to be shown. We can now verify that if $a, b \in T$ and $v(a), v(b) > 0$, then $v(a'b/b') > 0$, by noting that $v(a'b/b') = v(a'/a) + v(a) - v(b'/b) = \psi(v(a)) + v(a) - \psi(v(b))$, which is positive by property (c). Thus Theorem 2 of [1] can be applied to our current situation, using the present v, C, T, and K, with the k and $\mathcal{C}$ of the quoted theorem both taken to be C. We conclude that v is a differential valuation of K and C is its subfield of constants. We already know that the value group of v is Γ, and all that remains to be verified is the statement that if $a \in K^*$ and $v(a) \neq 0$, then $v(a'/a) = \psi(v(a))$. This statement is already known for $a \in T$ and it follows for any $a \in K^*$ such that $v(a) \neq 0$ by choosing, as we can, some $a_1 \in T$ such that $v(a) = v(a_1)$ and noting that since v is a differential valuation of K we have $v(a') = v(a_1')$. This completes the proof.

The procedure of the above proof is often easy to carry out in practice. Suppose for example that Γ is the subgroup of the lexicographically ordered group $\mathbf{R}^2$ given by

$$\Gamma = \{(m + n\pi, p) : m, n, p \in \mathbf{Z}\},$$

with ψ given by

$$\psi(m + n\pi, p) = \begin{cases} (-1, 0) & \text{if} \quad m \neq 0 \\ (-1, 1) & \text{if} \quad m = 0, (n, p) \neq (0, 0). \end{cases}$$

Properties (a), (b), (c) are easily checked. Following the procedure of the proof, we can choose, for example, $\Sigma_{(-1,0)} = \{(1, 0)\}$, $\Sigma_{(-1,1)} = \{(\pi, 0), (0, 1)\}$. Let $C = \mathbf{C}$ and let x_1, x_2, x_3 be indeterminates over $\mathbf{C}$ with $v(x_1) = (1, 0)$, $v(x_2) = (\pi, 0)$, $v(x_3) = (0, 1)$. The equation $v(\xi(x)) = \psi(v(x))$ gives $\xi(x_1) = x_1^{-1}$, $\xi(x_2) = \xi(x_3) = x_1^{-1}x_3$. Trying $h(x_1) = h(x_3) = 1$, $h(x_2) = a$, where a is an arbitrary element of

$\mathbf{C} - \mathbf{Q}$, and using the equation $x' = h(x)\xi(x)x$, we get $K = \mathbf{C}(x_1, x_2, x_3)$, with $x_1' = 1$, $x_2' = ax_1^{-1}x_2x_3$, $x_3' = x_1^{-1}x_3^2$. A specific complex variable solution of this system of equations is $x_1 = z$, $x_2 = (-1/\log z)^a$, $x_3 = -1/\log z$. Thus the given (Γ, ψ) is associated with a differential valuation of the field $\mathbf{C}\,(z, \log z, (-1/\log z)^a)$.

The well-ordering assumption on $\psi(\Gamma^*)$ made in the statement of Theorem 1 can be dropped if it turns out that the set Σ constructed in the proof spans $\mathbf{Q}\Gamma$. As an example where this happens, let $\Gamma = \bigoplus_{i\in\mathbf{Z}} \mathbf{Z}\,\epsilon_i$, where each $\epsilon_i > 0$ and each $\epsilon_i > n\epsilon_{i+1}$ for any $n \in \mathbf{Z}$ (so that Γ is isomorphic to the additive group of doubly infinite sequences of integers almost all of which are zero, lexicographically ordered) and let ψ be defined by $\psi(\epsilon_i) = \epsilon_0 + \epsilon_1 + \cdots + \epsilon_i$ if $i \geq 0$, $\psi(\epsilon_{-1}) = 0$, $\psi(\epsilon_i) = -\epsilon_{i+1} - \cdots - \epsilon_{-1}$ if $i < -1$. Choose $\Sigma = \{\epsilon_i\}_{i\in\mathbf{Z}}$, $C = \mathbf{R}$, indeterminates $\{x_i\}_{i\in\mathbf{Z}}$ over $\mathbf{R}$ such that each $v(x_i) = \epsilon_i$, and each $h(x_i) = -1$. We get $K = \mathbf{R}(\{x_i\}_{i\in\mathbf{Z}})$ with $x_i'/x_i = -x_0x_1 \cdots x_i$ if $i \geq 0$, $x_{-1}'/x_{-1} = -1$, $x_i'/x_i = -x_{i+1}^{-1} \cdots x_{-1}^{-1}$ if $i < -1$. A real variable solution of this system is $x_0 = 1/x$, $x_1 = 1/\log x$, $x_2 = 1/\log\log x, \ldots, x_{-1} = 1/e^x$, $x_{-2} = 1/\exp(e^x), \ldots$, so that (Γ, ψ) is associated with the differential valuation of the Hardy field $\mathbf{R}(\ldots, e^x, x, \log x, \log\log x, \ldots)$ as $x \to +\infty$.

3. Asymptotic Integration. Let k be a differential field with subfield of constants C and let v be a differential valuation of k. Given a nonzero element $a \in k$, we may want to find an integral of a in k, or, if this is not possible, an element $b \in k$ whose derivative is near a in an appropriate sense, which is here that $v(a - b') > v(a)$. If this can be done, we may be able to find a $c \in k$ such that $v((a - b') - c') = v(a - (b + c)') > v(a - b')$, and with luck we may be able to find a sequence $0 = b_0, b_1, b_2, \ldots$, in k such that the sequence $v(a - b_i')$ increases in $\Gamma = v(k^*)$, possibly without bound. Whether or not the sequence $v(a - b_i')$ is bounded, the various approximations b_i to $\int a$ may be useful for whatever task is at hand. Thus the principal question is whether, given $a \in k^*$, there is a $b \in k$ such that $v(a - b') > v(a)$. If we have such a b, then $v(a) = v(b')$. Conversely, if $b \in k$ is such that $v(a) = v(b')$, then $v(a/b') = 0$, so there exists some $c \in C$ such that $v((a/b') - c) > 0$, or $v(a - (cb)') > v(b') = v(a)$. Therefore the question is that of finding $b \in k$ such that $v(b') = v(a)$. If $v(b) = 0$, then for some $c_1 \in C$ we have $v(b - c_1) > 0$ while $(b - c_1)' = b'$, so the question is that of finding $b \in k$ such that $v(b) \neq 0$ and $v(b') = a$, or, since $v(b') = \psi(v(b)) + v(b)$, the question is of finding $\beta \in \Gamma^*$ such that $v(a) = \psi(\beta) + \beta$. Thus the nonzero elements of k which are "asymptotically integrable" are those elements whose values are of the form $\psi(\beta) + \beta$, for some $\beta \in \Gamma^*$.

THEOREM 2. *Let Γ be an ordered abelian group and $\psi: \Gamma^* \to \Gamma$ a function with properties* (a), (b), (c). *If $\psi(\Gamma^*)$ has a maximal element α, then there exists no $\beta \in \Gamma^*$ such that $\psi(\beta) + \beta = \alpha$. If the set $\psi(\Gamma^*)$ is well-ordered, then there is at most one element $\alpha \in \Gamma$ such that $\alpha \neq \psi(\beta) + \beta$ for any $\beta \in \Gamma^*$.*

If $\alpha_0 \in \Gamma^*$ is such that $\alpha = \psi(\alpha_0)$ is maximal in $\psi(\Gamma^*)$ and if $\beta \in \Gamma^*$ is such that $\psi(\beta) + \beta = \alpha = \psi(\alpha_0)$, then we must have $\psi(\beta) \leq \psi(\alpha_0)$, so that $\beta \geq 0$, implying $\beta > 0$ and contradicting $\psi(\alpha_0) < \psi(\beta) + |\beta|$. This proves the first contention. Now suppose that our well-ordering condition holds and that $\alpha \in \Gamma$ is not of the form $\psi(\beta) + \beta$ for any $\beta \in \Gamma^*$. If for some $\gamma \in \psi(\Gamma^*)$ we had $\alpha - \gamma \in \psi^{-1}(\gamma)$, then we would have $\psi(\alpha - \gamma) + (\alpha - \gamma) = \gamma + (\alpha - \gamma) = \alpha$, which is impossible. Hence $\alpha - \gamma \notin \psi^{-1}(\gamma)$ for each $\gamma \in \psi(\Gamma^*)$. We claim that if $\gamma_1, \gamma_2 \in \psi(\Gamma^*)$ and $\gamma_1 \leq \gamma_2$ then $\alpha - \gamma_2 \notin \psi^{-1}(\gamma_1)$. For otherwise we could find $\gamma_1, \gamma_2 \in \psi(\Gamma^*)$ with $\gamma_1 \leq \gamma_2$ and $\alpha - \gamma_2 \in \psi^{-1}(\gamma_1)$ such that γ_2 is minimal for all pairs (γ_1, γ_2) with these properties. Since $\alpha - \gamma_2 \notin \psi^{-1}(\gamma_2)$ we must have $\gamma_1 < \gamma_2$. By the minimality property of γ_2, for each $\gamma \in \psi(\Gamma^*)$ such that $\gamma \leq \gamma_1$ we have $\alpha - \gamma_1 \notin \psi^{-1}(\gamma)$. But either $\alpha - \gamma_1 = 0$ or $\alpha - \gamma_1 \in \psi^{-1}(\psi(\alpha - \gamma_1))$, so that

$$\alpha - \gamma_1 \in \cup\{\psi^{-1}(\gamma): \gamma \in \psi(\Gamma^*), \gamma > \gamma_1\} \cup \{0\}.$$

Recalling that $\alpha - \gamma_2 \in \psi^{-1}(\gamma_1)$ and using property (b) we obtain $\gamma_2 - \gamma_1 = (\alpha - \gamma_1) - (\alpha - \gamma_2) \in \psi^{-1}(\gamma_1)$. Therefore property (c) implies that $\gamma_2 < \psi(\gamma_2 - \gamma_1) + |\gamma_2 - \gamma_1| = \gamma_1 + (\gamma_2 - \gamma_1) = \gamma_2$, which is false. This proves that for each $\gamma_1 \leq \gamma_2$ in $\psi(\Gamma^*)$ we have $\alpha - \gamma_2 \notin \psi^{-1}(\gamma_1)$, so that

$$\alpha - \gamma_2 \in \cup\{\psi^{-1}(\gamma): \gamma \in \psi(\Gamma^*), \gamma > \gamma_2\} \cup \{0\}.$$

Thus if $\alpha_1, \alpha_2 \in \Gamma$ are neither of the form $\psi(\beta) + \beta$ for any $\beta \in \Gamma^*$, the statement $\alpha_1 - \alpha_2 = (\alpha_1 - \gamma_2) - (\alpha_2 - \gamma_2)$ and property (b) imply

$$\alpha_1 - \alpha_2 \in \cup\{\psi^{-1}(\gamma): \gamma \in \psi(\Gamma^*), \gamma > \gamma_2\} \cup \{0\}$$

for all $\gamma_2 \in \psi(\Gamma^*)$. If $\alpha_1 \neq \alpha_2$ we get a contradiction by taking $\gamma_2 = \psi(\alpha_1 - \alpha_2)$, which completes the proof.

COROLLARY. *Let k be a differential field, C its subfield of constants, and let v be a differential valuation of k. Suppose that $k \neq C$ and that each element of k is differentially algebraic over C. Then for any $a \in k^*$ there exists a $b \in k$ such that $v(a - b') > v(a)$ except in the case where* $\max\{v(u'/u)$: $u \in k^*$, $v(u) \neq 0\}$ *exists and $v(a)$ is this maximum.*

For any $a, s \in k^*$ with $v(s) \neq 0$, the differential field $C\langle a, s\rangle$ generated by C, a and s is of finite transcendence degree over C, so that the value group $v((C\langle a, s\rangle)^*)$ is of finite rational rank. Thus the set $\{v(u'/u): u \in (C\langle a, s\rangle)^*, v(u) \neq 0\}$ is finite. The theorem tells us that for some nonzero $\beta \in v((C\langle a, s\rangle)^*)$ we have $v(a) = \psi(\beta) + \beta$, which implies the existence of some $b \in C\langle a, s\rangle$ such that $v(a - b') > v(a)$, except in the case where $v(a) = \max\,\{v(u'/u): u \in (C\langle a, s\rangle)^*, v(u) \neq 0\}$, which implies in particular that $v(a)$ is of the form $v(u'/u)$ for some $u \in k^*$ such that $v(u) \neq 0$ and that $v(a) \geq v(s'/s)$. Since s was an arbitrary element of k^* such that $v(s) \neq 0$, we are done.

For example, consider the sequence of Hardy fields

$$\mathbf{R}(x) \subset \mathbf{R}(x, \log x) \subset \mathbf{R}(x, \log x, \log\log x) \subset \cdots$$

of germs of differentiable real-valued functions on neighborhoods of $+\infty$ in $\mathbf{R}$, differentiation being the usual differentiation with respect to x. According to [1, (*)], in a Hardy field we get a maximal ψ from a minimal positive element of the value group. Elements of the above sequence of fields with minimal positive values are

$$1/x,\ 1/\log x,\ 1/\log\log x,\ \ldots$$

respectively, which have logarithmic derivatives

$$-1/x,\ -1/x\log x,\ -1/x\log x\log\log x,\ \ldots$$

respectively, whose values are precisely the values of those elements of the respective fields which cannot be asymptotically integrated in the same field, each element, however, having an integral in the next field.

In the union Hardy field $\mathbf{R}(x, \log x, \log\log x, \ldots)$, all elements can be asymptotically integrated.

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