This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2020.2995173, IEEE Transactions on Automatic Control

Verification of Detectability for Unambiguous Weighted Automata

Aiwen Lai, Sébastien Lahaye, and Alessandro Giua, *Fellow, IEEE*

***Abstract*—In this note, we deal with the detectability problem for unambiguous weighted automata (UWAs). The problem is to determine if, after a finite number of observations, the set of possible states is reduced to a singleton. Four types of detectabilities, namely, strong detectability, detectability, strong periodic detectability, and periodic detectability are defined in terms of different requirements for current state estimation. We first construct a deterministic finite state automaton (called observer) over a weighted alphabet and prove that it can be used as the current-state estimator of the studied UWA. Finally, necessary and sufficient conditions based on the observer are proposed to verify detectabilities of a UWA.**

***Index Terms*—Discrete event systems, state estimation, weighted automata, detectability.**

I. INTRODUCTION

State estimation is a fundamental problem in discrete event systems (DESs), and it has been extensively studied in the framework of automata, e.g., [1], [2] and Petri nets (PNs), e.g., [3], [4], [5]. State estimation aims at accurately characterizing the possible states by observing system's behaviour, i.e., the output information obtained during the evolution of the system. State estimation plays an important role in some applications of DES, for instance, the verification of opacity, e.g., [6], [7], and fault diagnosis, e.g., [8], [9].

Recently, the problem of state estimation has been investigated in terms of detectability for non-probabilistic DES, e.g., [10], [11], [12], based on the construction of the observer automaton, and for probabilistic DES, e.g., [13], [14], [15], where the asymptotic behaviour is studied. Shu et al. [11] formulated the current-state estimation problem as detectability for non-deterministic finite state automata (NFAs). Four types of detectabilities, i.e., strong detectability, detectability, strong periodic detectability, periodic detectability, are defined to characterize different properties of current state estimation. An observer is constructed as to derive necessary and sufficient conditions for checking the detectabilities.

Note that classical automata and place/transition nets, where no quantitative information is associated with the occurrence of events, can only describe the logical behavior DESs. However, in various real-world systems, some quantitative information may be associated to state transitions to better characterize the system's evolution. Weighted automata (WAs) represent a well studied class of DES models [16], where transitions carry weights belonging to a semiring. The weight associated with a transition can model, e.g., the cost, the energy, the time needed or the probability associated with the execution of the transition. In [17], [18], it is shown that safe timed PNs under the preselection policy can be modeled by max-plus automata (WAs over the max-plus semiring). In [19], it is shown that bounded timed PNs under race policy can be represented by deterministic max-plus automata.

In this work, we present a formal approach to verify the detectability property for a restricted but important class of WAs, i.e., unambiguous weighted automata (UWAs), where no two or more paths leading from an initial state to the same state are labeled by the same string. Due to the influence of transition weights, the detectability properties of a UWA can be different from those of its underlying logical automaton, as it will be shown in Example 7. Note that many other fundamental problems that have been well solved for NFAs are still open or known to be undecidable in the framework of general WAs [20]. For example, the problem of deciding whether a given nondeterministic WA can be determinized is open. Besides, inequalities between behaviors (corresponding to the language containment problem in logical automata) for some classes of WAs, such as max-plus automata and min-plus automata, are undecidable. In the literature, such as [21], [22], the unambiguity is usually assumed to make the studied problem decidable. Similarly, in this paper, the unambiguity property is assumed to solve the detectability verification problem.

The main contributions of this paper are as follows. 1) We first define the concept of consistent states for an infinite sequence representing a trajectory of the system. Then, inspired by [11], we define four types of detectabilities for WAs. 2) A novel algorithm is introduced to construct a finite state automaton (called observer) for a given UWA, and we prove that this observer can be used for the current-state estimation of the studied system. Note that this algorithm is first presented in this paper, and is adapted to deal with the problem of initial-state detectability and initial-state opacity for WAs in [23]. 3) Necessary and sufficient conditions are derived from the constructed observer for verifying the four detectabilities properties of UWAs.

This paper is organized as follows. In Section II, some basics of WAs are recalled. Section III gives the definitions of four types of detectabilities of WAs. In Section IV, theorems (necessary and sufficient conditions) based on the construction of an observer are introduced for checking the detectabilities of a UWA. Finally, conclusions and future work are drawn in Section V.

A. Lai and S. Lahaye are with LARIS, Université d'angers, France *aiwen.lai@etud.univ-angers.fr* (A. Lai), *sebastien.lahaye@univ-angers.fr* (S. Lahaye)

A. Giua is with DIEE, University of Cagliari, Italy *giua@diee.unica.it*

II. BACKGROUND ON WEIGHTED AUTOMATA

In this section we recall some basics of WAs [24].

Definition 1. A semiring is a quintuple $\mathbb{S} = (\mathcal{D}, \oplus, \otimes, \varepsilon, e)$ composed by a set $\mathcal{D}$, two binary operations $\oplus$ and $\otimes$ on $\mathcal{D}$, and two constant values $\varepsilon, e \in \mathcal{D}$ satisfying the following four axioms:

- $(\mathcal{D}, \oplus)$ is a commutative monoid with zero element ε, i.e., $a \oplus b = b \oplus a$, $(a \oplus b) \oplus c = a \oplus (b \oplus c)$, and $\varepsilon \oplus a = a \oplus \varepsilon = a$ hold for any $a, b, c \in \mathcal{D}$;
- $(\mathcal{D}, \otimes)$ is a monoid with identity element e, i.e., $(a \otimes b) \otimes c = a \otimes (b \otimes c)$, $\varepsilon \otimes a = a \otimes \varepsilon = \varepsilon$, and $e \otimes a = a \otimes e = a$ hold for any $a, b, c \in \mathcal{D}$;
- the distributive laws $(a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c)$ and $c \otimes (a \oplus b) = (c \otimes a) \oplus (c \otimes b)$ hold for any $a, b, c \in \mathcal{D}$.

Example 1. A typical semiring is the set of nonnegative integers equipped with the usual addition and multiplication, i.e., $\mathbb{S} = (\mathbb{N}, +, \times, 0, 1)$. Other important semirings include [24]: Tropical semiring $(\mathbb{R} \cup \{+\infty\}, \min, +, +\infty, 0)$, max-plus semiring $(\mathbb{R} \cup \{-\infty\}, \max, +, -\infty, 0)$ and probability semiring $([0, 1], +, \times, 0, 1)$.

The set of matrices with m rows and n columns over $\mathbb{S} = (\mathcal{D}, \oplus, \otimes, \varepsilon, e)$ is denoted by $\mathbb{S}^{m\times n}$. For matrices $A, B \in \mathbb{S}^{m\times n}$ and $C \in \mathbb{S}^{n\times m}$, the matrix sum and product are defined in the following way:

$$[A \oplus B]_{ij} \triangleq A_{ij} \oplus B_{ij};$$
$$[A \otimes C]_{ij} \triangleq \bigoplus_{k=1}^{n} (A_{ik} \otimes C_{kj}).$$

Let E be an alphabet, i.e., a non-empty set of labels. A string defined on E is a finite sequence of labels in E. E^* represents the set of all the finite strings over alphabet E including the empty string, denoted by λ in this paper.

Definition 2. A WA over a semiring $\mathbb{S} = (\mathcal{D}, \oplus, \otimes, \varepsilon, e)$ is a tuple $G = (Q, E, \alpha, \mu)$ where

- Q and E are respectively a non-empty finite set of states and an alphabet;
- $\alpha \in \mathbb{S}^{1\times|Q|}$ specifies the initial weights. A state $q \in Q$ is said to be an initial state (belongs to $Q_i \subseteq Q$) iff $\alpha_q \neq \varepsilon$, and α_q is the corresponding initial weight;
- μ: $E \to \mathbb{S}^{|Q|\times|Q|}$ is a morphism representing the state transitions given by the family of matrices $\mu(a) \in \mathbb{S}^{|Q|\times|Q|}$, $a \in E$. For any string $\omega = e_1 \cdots e_k \in E^*$, we have $\mu(\omega) = \mu(e_1) \otimes \mu(e_2) \cdots \otimes \mu(e_k)$.

We assume that the states of $G = (Q, E, \alpha, \mu)$ are well ordered by positive integers, i.e., $Q = \{1, 2, \cdots, |Q|\}$, and with a slight abuse of notation, we denote α_q the q^{th} element of α, and $\mu(a)_{qq'}$ the element in the q^{th} row and q'^{th} column of matrix $\mu(a)$.

Definition 3. A WA $G = (Q, E, \alpha, \mu)$ over a semiring $\mathbb{S} = (\mathcal{D}, \oplus, \otimes, \varepsilon, e)$ can be equivalently defined by $G = (Q, E, t, Q_i, \varrho)$, where $t : Q \times E \times Q \to \mathcal{D}$ is the transition function with $t(q, a, q') \triangleq \mu(a)_{qq'}$ for $q, q' \in Q$, $Q_i \triangleq \{q \in Q \mid \alpha_q \neq \varepsilon\}$ is the set of initial states, and $\varrho : Q_i \to \mathcal{D}$ is the function of initial weights $\varrho(q) \triangleq \alpha_q$ for $q \in Q_i$.

Definition 4. Given a WA G, a path of length k is defined as a sequence of transitions $\pi = (q_0, e_1, q_1)(q_1, e_2, q_2) \cdots (q_{k-1}, e_k, q_k)$, where $q_i \in Q$, $i = 0, \ldots, k$, $e_i \in E$ and $\mu(e_i)_{q_{i-1}q_i} \neq \varepsilon$ for $i = 1, \ldots, k$.

Path π is said to be labeled by $e_1e_2 \cdots e_k$, and π is a circuit if q_0 coincides with q_k. Let $p \overset{\omega}{\rightsquigarrow} q$ represents the set of paths labeled by string $\omega \in E^*$ from state p leading to q. For ω equals the empty string λ, $p \overset{\omega}{\rightsquigarrow} q$ is the empty set. For $P, R \subseteq Q$, we denote $P \overset{\omega}{\rightsquigarrow} R$ the union of $p \overset{\omega}{\rightsquigarrow} q$ for all $p \in P$ and $q \in R$.

Definition 5. Given an arbitrary path $\pi = (q_0, e_1, q_1)(q_1, e_2, q_2) \cdots (q_{k-1}, e_k, q_k)$ of a WA G, we define its weight, denoted by $\mathbb{W}(\pi)$, as

$$\mathbb{W}(\pi) = \begin{cases} \alpha_{q_0} \otimes \bigotimes\limits_{i=1,\ldots,k} \mu(e_i)_{q_{i-1}q_i}, & \text{if } q_0 \in Q_i; \\ \bigotimes\limits_{i=1,\ldots,k} \mu(e_i)_{q_{i-1}q_i}, & \text{otherwise.} \end{cases}$$

Definition 6. Given a WA G, we define the state vector $x(\omega) \in \mathbb{S}^{1\times|Q|}$, $\omega \in E^*$, to describe its dynamic evolution as

$$x(\omega) = \begin{cases} \alpha, & \text{if } \omega = \lambda; \\ \alpha \otimes \mu(\omega), & \text{otherwise.} \end{cases}$$

It can be verified that $x(\omega)_q$ corresponds to the sum $\oplus$ of weights of paths labeled by ω from an initial state to q:

$$x(\omega)_q = \bigoplus_{\pi \in Q_i \overset{\omega}{\rightsquigarrow} q} \mathbb{W}(\pi) = \bigoplus_{p \in Q_i} \alpha_p \otimes \mu(w)_{pq}.$$

We interpret $x(\omega)_q$ as the weight to reach state q according to ω. Depending on the interpretation, this weight can represent an amount of time, resources, or a cost.

Definition 7. Given an arbitrary path $\pi = (q_0, e_1, q_1)(q_1, e_2, q_2) \cdots (q_{k-1}, e_k, q_k)$ of a WA G with $q_0 \in Q_i$, we define the weighted sequence $\sigma(\pi) \in (E \times \mathcal{D})^*$ generated by π as: $\sigma(\pi) = (e_1, \tau_1)(e_2, \tau_2) \cdots (e_k, \tau_k)$, where τ_j is defined by $\tau_j = x(e_1 \cdots e_j)_{q_j}$, $j = 1, \ldots, k$.

Weighted sequence $\sigma(\pi)$ specifies a sequence of labels and their occurrence weights. We use $q_0 \overset{\sigma(\pi)}{\rightsquigarrow} q_k$ to represent that weighted sequence $\sigma(\pi)$ leads from q_0 to q_k.

Definition 8. A WA G is said to be unambiguous if $\forall q \in Q$, $\forall \omega \in E^*$, $|Q_i \overset{\omega}{\rightsquigarrow} \{q\}| \leq 1$.

In simple words, the unambiguity requires that for any state q of G and any string ω in E^*, there is at most one path labeled by ω leading to q from an initial state.

Remark 1. For a WA $G = (Q, E, \alpha, \mu)$ where all states are initial states, i.e., $Q_i = Q$, G is unambiguous iff every state in G has no two or more input transitions labeled by the same symbol. On the one hand, suppose (q', a, q) and (q'', a, q) with $q', q'' \in Q$, $q' \neq q''$, and $a \in E$, are two input transitions of state q in G. Since q' and q'' are initial states, then there exists a string $\omega = a$ such that $|Q_i \overset{\omega}{\rightsquigarrow} \{q\}| > 1$, i.e., G is not unambiguous. On the other hand, assume G is not unambiguous, i.e., $|Q_i \overset{\omega}{\rightsquigarrow} \{q\}| > 1$. Let a be the last label of ω. Then we know that there is more than one input transition of q that is labeled by symbol a.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2020.2995173, IEEE Transactions on Automatic Control

Example 2. The automaton in Fig. 1 is a WA G with $Q = \{1,2,3,4\}$, alphabet $E = \{u,a,b,c\}$, transitions $\mu(u)_{1,2} = 0.2$, $\mu(b)_{2,2} = 0.3$, $\mu(b)_{2,1} = 0.2$, $\mu(c)_{2,3} = 0.4$, $\mu(b)_{3,3} = 0.3$, $\mu(a)_{3,4} = 0.3$, $\mu(a)_{4,1} = 0.1$, $\mu(u)_{4,3} = 0.2$, $\alpha = (e,e,e,e)$. Note that all the non-listed coefficients in $\mu(u)$, $\mu(a)$, $\mu(b)$, and $\mu(c)$ are equal to ε, meaning that they do not model possible transitions in the automaton.

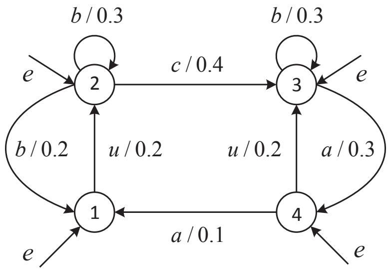

Fig. 1: Unambiguous weighted automaton G.

Consider an arbitrary word $\omega = e_1 \cdots e_k \in E^*$ and a state $q \in Q$ in UWA G. There exists at most one path labeled by ω from an initial state $p \in Q_i$ to q, and

$$x(\omega)_q = \begin{cases} \alpha_p \otimes \mu(\omega)_{pq}, & \text{if } q \text{ is reachable from p according to } \omega\ (i.e.|p \overset{\omega}{\rightsquigarrow} q| = 1);\\ \varepsilon, & \text{otherwise.} \end{cases}$$

As a result, the weighted sequence generated by path $\pi = (q_0, e_1, q_1)(q_1, e_2, q_2) \cdots (q_{k-1}, e_k, q_k)$ of a UWA (see Def. 7) can be simplified into $\sigma(\pi) = (e_1, \tau_1)(e_2, \tau_2) \cdots (e_k, \tau_k)$ where $\tau_1 = \alpha_{q_0} \otimes \mu(e_1)_{q_0 q_1}$, $\tau_i = \tau_{i-1} \otimes \mu(e_i)_{q_{i-1} q_i}$ for $i = 2, \ldots, k$.

Example 3. Consider path $\pi = (1,u,2)(2,b,2)$ in UWA G depicted in Fig. 1. We have $\sigma(\pi) = (u, \alpha_1 \otimes \mu(u)_{1,2})(b, \alpha_1 \otimes \mu(u)_{1,2} \otimes \mu(b)_{2,2}) = (u, e \otimes 0.2)(b, e \otimes 0.2 \otimes 0.3)$.

Definition 9. Given a WA G, we define the generated weighted language $L(G)$ of G as:

$$L(G) = \{\sigma \in (E \times \mathcal{D})^* \mid \exists q \in Q, \\ \exists s \in E^*,\ \exists \pi \in Q_i \overset{s}{\rightsquigarrow} \{q\} : \sigma(\pi) = \sigma\}.$$

Definition 10. Given an alphabet $E = E_o \cup E_{uo}$, the projection operator on E_o is denoted by $P : E^* \to E_o^*$ and is defined as: $P(\lambda) = \lambda$; for each $a \in E$, $s \in E^*$ $P(sa) = P(s)a$, if $a \in E_o$, otherwise $P(sa) = P(s)$.

Now the projection operator $P : E^* \to E_o^*$ is extended to weighted sequences $P : (E \times \mathcal{D})^* \to (E_o \times \mathcal{D})^*$. For any $\sigma \in (E \times \mathcal{D})^*$, $P(\sigma)$ is the weighted sequence obtained from σ by removing all pairs corresponding to unobservable events.

In the rest, $P(L(G))$ represents the set of observable weighted sequences for WA G, i.e., the set of all possible observations that can be observed by an external agent. We emphasize that the language generated by a WA and its projection are originally defined in this paper. Let σ_1 and σ_2 be two weighted sequences, we denote by $\mathbb{E}_f(\sigma_1)$ the last event in σ_1, and by $\sigma_1 \cdot \sigma_2$ (or simply $\sigma_1\sigma_2$) the concatenation of σ_1 and σ_2.

III. PROBLEM STATEMENT

In this paper we restrict our attention to a WA G with identity initial weights[1]. The label set E is partitioned into two disjoint parts: the observable part E_o and the unobservable part E_{uo}. Considering that the firing of unobservable labels cannot be detected by an external agent, we assume that all unobservable labels are represented by symbol u, i.e., $E = E_o \cup \{u\}$. We also make the following assumptions on the studied weighted automaton G:

(A1). G is unambiguous (see Def. 8).

(A2). G is deadlock free, that is, for any state, there exists at least one output transition: $(\forall q \in Q)(\exists a \in E, q' \in Q)\mu(a)_{qq'} \neq \varepsilon$.

(A3). There is no circuit labeled only by unobservable labels in G.

Assumption (A1) requires that for any state q of G and any string $\omega \in E^*$, there is at most one path labeled by ω leading to q from the initial states. However, Assumption (A1) does not imply that an observed weighted sequence can be generated by a single path. For example, assuming that $\otimes$ is the usual addition $+$, the WA G in Fig. 1 is such that $\sigma_o = (b, 0.5)(b, 0.8)$ can be generated by two paths, i.e., $\pi_1 = (1,u,2)(2,b,2)(2,b,2)$ and $\pi_2 = (4,u,3)(3,b,3)(3,b,3)$. In addition, Assumption (A2) implies that the length of the generated weighted sequence becomes infinite as the system evolves indefinitely. By Assumption (A3), for an infinite sequence, the length of its projection is also infinite.

A. Consistent State

In a logical DES, a *trajectory* of the system consists in an infinite sequence of labels that the system may generate. The set of all possible trajectories of the system is called ω-language [25]. Similarly, given a WA G, a trajectory consists in an infinite sequence of (label, weight) pairs that G may generate. The set of all possible trajectories of G defines the ω-language $L^\omega(G)$, i.e., the set of infinite weighted sequences generated by G.

Definition 11. Given an observed weighted sequence $\sigma_o \in P(L(G))$, the set of all σ_o*−consistent states* is defined as

$$C(\sigma_o) = \{q \in Q \mid \exists \sigma \in L(G), \exists q_0 \in Q_i : \\ q_0 \overset{\sigma}{\rightsquigarrow} q, P(\sigma) = \sigma_o\}. \tag{1}$$

In simple words, a state q is consistent with observation σ_o, if there exists a generated weighted sequence σ leading to state q such that the projection of σ coincides with σ_o.

Example 4. Consider again the WA G in Fig. 1 with $\otimes = +$, $E_o = \{a,b,c\}$ and $E_{uo} = \{u\}$. Given $\sigma_o = (b, 0.5)(b, 0.8)$, it can be verified that the set of consistent states is $C(\sigma_o) = \{2,3\}$. In fact, two different paths from an initial state have weighted sequences that are consistent with σ_o, namely, $\pi_1 = (1,u,2)(2,b,2)(2,b,2)$ and $\pi_2 = (4,u,3)(3,b,3)(3,b,3)$. Considering π_1, we have

[1]The coefficients in α different from ε are all equal to e. This assumption is without loss of generality since an automaton with non-identity initial weights can always be transformed into an equivalent automaton with identity initial weights by adding new states and by considering these weights as state transitions durations associated to new fictive initial labels.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2020.2995173, IEEE Transactions on Automatic Control

$\sigma_1 = \sigma(\pi_1) = (u, 0.2)(b, 0.5)(b, 0.8)$ and $1 \overset{\sigma_1}{\rightsquigarrow} 2$. Therefore, state 2 is consistent with σ_o. Similarly, state 3 is a consistent state by considering π_2.

B. Detectability of Weighted Automata

In this subsection, we extend the detectability problem defined for logical DES modeled by classical automata in [11] to WAs framework. For an arbitrary finite or infinite sequence σ, we denote $Pref(\sigma)$ the set of all its prefixes.

Definition 12 (Strong Detectability)**.** A WA G is strongly detectable with respect to a projection P if one can determine, after a finite number of pair observations, the current state and subsequent states of the automaton for all trajectories of the automaton. That is,

$$(\exists n \in \mathbb{N})(\forall \sigma \in L^{\omega}(G))(\forall \sigma' \in Pref(\sigma))$$
$$|P(\sigma')| > n \Rightarrow |C(P(\sigma'))| = 1.$$

Definition 13 (Detectability)**.** A WA G is detectable with respect to a projection P if one can determine, after a finite number of pair observations, the current state and subsequent states of the automaton for some trajectories of the automaton. That is,

$$(\exists n \in \mathbb{N})(\exists \sigma \in L^{\omega}(G))(\forall \sigma' \in Pref(\sigma))$$
$$|P(\sigma')| > n \Rightarrow |C(P(\sigma'))| = 1.$$

Definition 14 (Strong Periodic Detectability)**.** A WA G is strongly periodically detectable with respect to a projection P if one can periodically determine the current state of the system for all trajectories of the automaton. That is,

$$(\exists n \in \mathbb{N})(\forall \sigma \in L^{\omega}(G))(\forall \sigma' \in Pref(\sigma))$$
$$(\exists \sigma'' \in (E \times \mathcal{D})^*)\sigma'\sigma'' \in Pref(\sigma)$$
$$\wedge |P(\sigma'')| < n \wedge |C(P(\sigma'\sigma''))| = 1.$$

Definition 15 (Periodic Detectability)**.** A WA G is periodically detectable with respect to a projection P if one can periodically determine the current state of the system for some trajectories of the automaton. That is,

$$(\exists n \in \mathbb{N})(\exists \sigma \in L^{\omega}(G))(\forall \sigma' \in Pref(\sigma))$$
$$(\exists \sigma'' \in (E \times \mathcal{D})^*)\sigma'\sigma'' \in Pref(\sigma)$$
$$\wedge |P(\sigma'')| < n \wedge |C(P(\sigma'\sigma''))| = 1.$$

IV. DETECTABILITY FOR UNAMBIGUOUS WEIGHTED AUTOMATA

In this section, inspired by the work in [11], an observer is constructed to derive necessary and sufficient conditions for verifying four detectabilities of a UWA.

A. Construction of Observer

In this subsection, we first construct an observer, denoted by G_{obs}, for a UWA G. In detail, the observer G_{obs} is a finite state automaton (DFA) over a weighted alphabet $E_{obs} \subset E_o \times \mathcal{D}$, which is built so that it has the following structural properties: G_{obs} has only one initial state, and from a given state no two transitions of G_{obs} are labeled by the same weighted label $(a, t_a) \in E_{obs}$. That is, G_{obs} is a deterministic finite state automaton (DFA) over weighted alphabet E_{obs}. Let us stress, however, that from a state in G_{obs} there may exist several output transitions labeled by the same label $a \in E_o$ but with different weights t_a. In this case, the external agent can distinguish the transitions from the different associated weights. Then, we prove that G_{obs} can be used to estimate the consistent states for any infinite or finite weighted sequence observed from system G.

For any subset $y \subseteq Q$, its unobservable reach is denoted by $UR(y)$ and is defined as

$$UR(y) = \{q' \in Q \mid \exists q \in y,$$
$$\exists i \in \{0, 1, \ldots, |Q|-1\} : \mu(u^i)_{qq'} \neq \varepsilon\}.$$

Algorithm 1 details the construction of such an observer G_{obs} for a given UWA G. This algorithm is first presented here and it has been adapted to solve different problems in the recent contribution [23].

Algorithm 1 Construction of the observer of a UWA

Input: A UWA $G = (Q, E, t, Q_i, \varrho)$.
Output: Observer $G_{obs} = (Q_{obs}, E_{obs}, \delta_{obs}, q_{i,obs})$ of G.

1: Construct an NFA $G_o = (Q_o, E_{obs}, \delta_o, Q_i^o)$ from G as follows:
- $Q_i^o = Q_i$;
- $Q_o = Q_1 \cup Q_i$, where $Q_1 = \{q \in Q \mid \exists q' \in Q, \exists a \in E_o : \mu(a)_{q'q} \neq \varepsilon\}$ is the set of states in G that have at least one input transition marked by an observable label;
- E_{obs} consists of all weighted labels $(a, \tau) \in E_o \times \mathcal{D}$ for which $\exists q \in Q_o$, $\exists q' \in Q$, $\exists i \in \{0, 1, \ldots, |Q|-1\}$, s.t. $\mu(u^i a)_{qq'} = \tau$;
- $\delta_o \subseteq Q_o \times E_{obs} \times Q_o$ is the set of state transitions. $(q, (a, \tau), q') \in \delta_o$ iff $\exists i \in \{0, 1, \ldots, |Q|-1\}$, s.t. $\mu(u^i a)_{qq'} = \tau$

2: Calculate a DFA G'_{obs} through the determinization of NFA G_o as follows:
- $q'_{i,obs} = Q_i^o$;
- $\delta'_{obs} : 2^{Q_o} \times E_{obs} \to 2^{Q_o}$ is the state transition function. $\delta'_{obs}(q'_{obs}, (a, \tau))$ is defined as:
$$\delta'_{obs}(q'_{obs}, (a, \tau)) = \{q' \in Q_o \mid \exists q \in q'_{obs} :$$
$$q' \in \delta_o(q, (a, \tau))\}.$$
- Let $G'_{obs} = (Q'_{obs}, E_{obs}, \delta'_{obs}, q'_{i,obs}) = Ac(2^{Q_o}, E_{obs}, \delta'_{obs}, q'_{i,obs})$[2].

3: Observer G_{obs} is obtained by replacing each state q'_{obs} in G'_{obs} by pair $(q'_{obs}, UR(q'_{obs}))$ as follows.
- $q_{i,obs} = (q'_{i,obs}, UR(q'_{i,obs}))$;
- $Q_{obs} = \{(q'_{obs}, UR(q'_{obs})) \mid \exists q'_{obs} \in Q'_{obs}\}$;
- $\delta_{obs} : Q_{obs} \times E_{obs} \to Q_{obs}$ is the state transition function. $\delta_{obs}((q'_{obs}, UR(q'_{obs})), (a, \tau))$ is defined as: $\delta_{obs}((q'_{obs}, UR(q'_{obs})), (a, \tau)) = (\delta'_{obs}(q'_{obs}, (a, \tau)), UR(\delta'_{obs}(q'_{obs}, (a, \tau))))$;
- Let $G_{obs} = (Q_{obs}, E_{obs}, \delta_{obs}, q_{i,obs})$.

[2]We denote by the $Ac(G)$ the automaton obtained by removing all the states that are not accessible as well as transitions associated with such states in G.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2020.2995173, IEEE Transactions on Automatic Control

Step 1 of Algorithm 1 calculates a NFA G_o over weighted alphabet E_{obs}. The state set Q_o of automaton G_o is composed of the initial states of G and the states in G that have at least one input transition marked by an observable label. Besides, the duration of a sequence of unobservable labels along a path is added to the next observable label to capture the following phenomenon: *during the evolution of a system, we do not observe the occurrence of unobservable labels, but we keep track of the elapsed time corresponding to the occurrence of unobservable labels in the weighted alphabet* E_{obs}. Step 2 computes DFA G'_{obs} by applying the determinization process to NFA G_o. Step 3 constructs the observer of G from G'_{obs} by replacing each state q'_{obs} in G'_{obs} with pair $(q'_{obs}, UR(q'_{obs}))$.

Remark 2. By Algorithm 1, $|Q_{obs}|$ is bounded by $2^{|Q|}-1$, and $|E_{obs}|$ is bounded by $|Q|^3 \times |E_o|$. More precisely, because of Assumption (A3), the length of path labeled by unobservable events only is at most $|Q|-1$. Because of the unambiguity assumption, for each state $q \in Q$, each observable event $a \in E_o$, and each integer $i \in \{0, 1, 2, \ldots, |Q|-1\}$, there exist at most $|Q|$ paths labeled by $u^i a$ starting from state q. Since there are $|Q|$ states in G, $|Q|$ elements in $i = \{0, 1, 2, \ldots, |Q-1|\}$, and $|E_o|$ observable events, $|E_{obs}|$ is bounded by $|Q|^3 \times |E_o|$. Based on the above discussion, from Theorems 1–4 it follows that the verification of the four detectabilities of UWA G has exponential complexity in $|Q|$. $\square$

Example 5. Consider the UWA G in Fig. 1 and $\otimes = +$. After applying Algorithm 1, we obtain NFA G_o, DFA G'_{obs}, and observer G_{obs} shown in Figs. 2, 3 and 4, respectively.

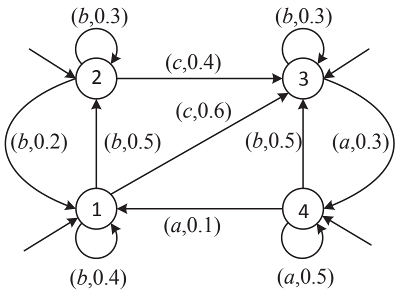

Fig. 2: Diagram of NFA G_o of G in Fig. 1.

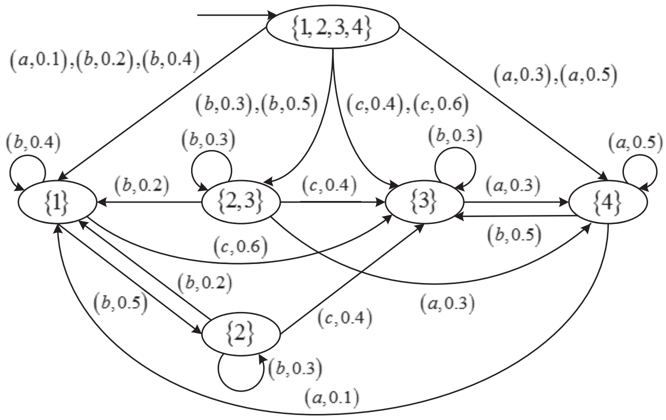

Fig. 3: DFA G'_{obs} of G in Fig. 1.

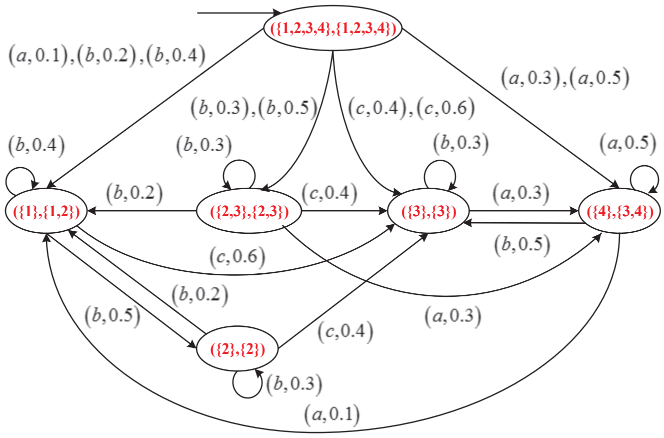

Fig. 4: Observer G_{obs} of G in Fig. 1.

Let E^*_{obs} be the set of strings over weighted alphabet E_{obs} including (λ, e) corresponding to empty string λ and the identity weight value. The language generated by the observer $G_{obs} = (Q_{obs}, E_{obs}, \delta_{obs}, q_{i,obs})$ is defined as:

$$L(G_{obs}) = \{\omega \in E^*_{obs} \mid \exists q_{obs} \in Q_{obs} : \delta_{obs}(q_{i,obs}, \omega) = q_{obs}\}.$$

It should be noticed that the language generated by G_{obs} is a subset of E^*_{obs}, i.e., $L(G_{obs}) \subseteq E^*_{obs}$. While the language generated by G is a subset of $(E \times \mathcal{D})^*$, i.e., $L(G) \subseteq (E \times \mathcal{D})^*$. For any observed weighted sequence $\sigma_o = (a_1, \tau_1)(a_2, \tau_2) \cdots (a_n, \tau_n) \in P(L(G))$, we define $\sigma_o^{elem} = (a_1, \tau_1)(a_2, \tau_2 \otimes^{-1} \tau_1) \cdots (a_n, \tau_n \otimes^{-1} \tau_{n-1})$ to denote its equivalent notation in E^*_{obs}, where $\tau_k \otimes^{-1} \tau_{k-1}$ represents the weight[3] for the elementary transition according to a_k, $k = 2, 3, \ldots, n$. We denote $P(L(G))^{elem}$ the equivalent notation of $P(L(G))$, that is,

$$P(L(G))^{elem} = \{\sigma \in (E_o \times \mathcal{D})^* \mid \exists \sigma_o \in P(L(G)) : \sigma_o^{elem} = \sigma\}.$$

The following lemma states the equivalence between the observed language generated by G and the language generated by observer G_{obs}. It follows immediately from the construction process of observer G_{obs} presented in Algorithm 1. Hence the proof here is omitted.

Lemma 1. The projection of language $L(G_{obs})$ generated by G_{obs} coincides with $P(L(G))^{elem}$, that is, $L(G_{obs}) = P(L(G))^{elem}$.

For any pair (y, y') of state sets, where $y, y' \subseteq Q$, we use $P_{roj1}((y, y'))$ to represent its first element, i.e., $P_{roj1}((y, y')) = y$. Similarly, $P_{roj2}((y, y')) = y'$ denotes the second element of this pair.

Proposition 1. For any $\sigma_o = (a_1, \tau_1)(a_2, \tau_2) \cdots (a_k, \tau_k) \in P(L(G))$, the set of consistent states is given by $C(\sigma_o) = P_{roj2}(\delta_{obs}(q_{i,obs}, {\sigma_o}^{elem})) = P_{roj2}(\delta_{obs}(q_{i,obs}, (a_1, \tau_1)(a_2, \tau_2 \otimes^{-1} \tau_1) \cdots (a_k, \tau_k \otimes^{-1}$

[3]Note that $\tau_k \otimes^{-1} \tau_{k-1}$ is defined as the value $x \in \mathcal{D}$ such that $x \otimes \tau_{k-1} = \tau_k$, with τ_k and $\tau_{k-1} \in \mathcal{D} \setminus \{\varepsilon\}$.

$\tau_{k-1})))$, where $q_{i,obs}$ and δ_{obs} are the initial state and transition function of observer G_{obs}.

Proof. For a non-empty sequence $\sigma_o \in P(L(G))$, we first define $C'(\sigma_o)$ as follows.

$$\begin{aligned} C'(\sigma_o) = \{q \in Q \mid \exists \sigma \in L(G), \exists q_0 \in Q_i : \\ q_0 \overset{\sigma}{\rightsquigarrow} q,\ P(\sigma) = \sigma_o,\ \mathbb{E}_f(\sigma) = \mathbb{E}_f(\sigma_o)\}. \end{aligned} \tag{2}$$

In simple words, $C'(\sigma_o)$ consists of all those states q such that there exists a generated weighted sequence σ ending with the last label in σ_o leading to q, and the projection of σ coincides with σ_o. Note that for empty sequence (λ, e), we define $C'((\lambda, e)) = Q_i$. Then we prove that for any $\sigma_o \in P(L(G))$, $C'(\sigma_o) = \delta'_{obs}(q'_{i,obs}, {\sigma_o}^{elem})$ holds, where $q'_{i,obs}$ and δ'_{obs} are the initial state and transition function of G'_{obs} defined in Step 2 of Algorithm 1. This can be done by induction on the length of the observed weighted sequence.

(Base step.) From Step 2 of Algorithm 1, we know that $q'_{i,obs} = Q_i$, which coincides with $C'(\lambda, e)$.

(Inductive step.) Let $\sigma_o = (a_1, \tau_1)(a_2, \tau_2) \cdots (a_n, \tau_n) \in P(L(G))$ be an observed weighted sequence. Assume that $C'(\sigma_o) = \delta'_{obs}(q'_{i,obs}, \sigma_o^{elem}) \in Q'_{obs}$. We need to prove that if a new pair (a, τ) is observed in continuation of σ_o, then $C'(\sigma_o \cdot (a, \tau)) = \delta'_{obs}(C'(\sigma_o), (a, \tau \otimes^{-1} \tau_n))$.

According to Algorithm 1, we have

$$\begin{aligned} \delta'_{obs}(C'(\sigma_o), (a, \tau \otimes^{-1} \tau_n)) = \{q \in Q \mid \exists q' \in C'(\sigma_o), \\ \exists i \in \{0, 1, \ldots, |Q|-1\} : \mu(u^i a)_{q'q} = \tau \otimes^{-1} \tau_n\}. \end{aligned}$$

That is, $\delta'_{obs}(C'(\sigma_o), (a, \tau \otimes^{-1} \tau_n))$ consists of all states that can be reached from a state in $C'(\sigma_o)$ via a path labeled by $u^i a$, $i \in \{0, 1, \ldots, |Q|-1\}$, and the weight associated with the path is equal to $\tau \otimes^{-1} \tau_n$. Besides, from Eq. (2), we know that $C'(\sigma_o)$ contains all states that can be reached by some generated weighted sequences ending with the last label in σ_o, and the projection of these generated weighted sequences is equal to σ_o. Therefore, $\delta'_{obs}(C'(\sigma_o), (a, \tau \otimes^{-1} \tau_n))$ consists of all states q that can be reached by a weighted sequence σ from an initial state such that $\mathbb{E}_f(\sigma) = \mathbb{E}_f(\sigma_o \cdot (a, \tau)) = a$ and $P(\sigma) = \sigma_o \cdot (a, \tau)$. This coincides with the definition of $C'(\sigma_o \cdot (a, \tau))$. That is, $\delta'_{obs}(C'(\sigma_o), (a, \tau \otimes^{-1} \tau_n)) = C'(\sigma_o \cdot (a, \tau))$. Therefore, for any $\sigma_o \in P(L(G))$, $C'(\sigma_o) = \delta'_{obs}(q'_{i,obs}, {\sigma_o}^{elem})$ holds.

By comparing Eqs. (1) and (2), we know that $C(\sigma_o) = UR(C'(\sigma_o))$. Besides, according to Algorithm 1, we have $P_{roj2}(\delta_{obs}(q_{i,obs}, {\sigma_o}^{elem})) = UR(C'(\sigma_o))$. Hence $C(\sigma_o) = P_{roj2}(\delta_{obs}(q_{i,obs}, {\sigma_o}^{elem}))$. □

B. Verification of Detectabilities

This subsection derives necessary and sufficient conditions from observer G_{obs} to verify detectabilities of a UWA G. Special attention must be paid to states belonging to elementary circuits because the trajectories of observer $G_{obs} = (Q_{obs}, E_{obs}, \delta_{obs}, q_{i,obs})$ can visit these states indefinitely. We denote by S_{ci} the set of all elementary circuits of G_{obs}:

$$\begin{aligned} S_{ci} = \{(q_{obs}, s) \in Q_{obs} \times E^*_{obs} \mid \delta_{obs}(q_{obs}, s) = q_{obs} \wedge \\ |s| \geq 1 \wedge (\forall s' \in Pref(s) \text{ s.t. } s' \neq s : \delta_{obs}(q_{obs}, s') \neq q_{obs})\}. \end{aligned}$$

Definition 16. A state $q_{obs} \in Q_{obs}$ — which by definition is a pair of subsets of Q — is said to be *singleton* if $|P_{roj2}(q_{obs})| = 1$, where $P_{roj2}(q_{obs})$ is the second element of q_{obs}. We denote by Q^{single}_{obs} the set of all singleton states in G_{obs}, i.e., $Q^{single}_{obs} = \{q_{obs} \in Q_{obs} \mid |P_{roj2}(q_{obs})| = 1\}$.

Theorem 1 (Criterion for Checking Strong Detectability)**.** A UWA G is strongly detectable with respect to projection P iff any state reachable from any elementary circuit in observer G_{obs} belongs to Q^{single}_{obs}, that is: $(\forall (q_{obs}, s) \in S_{ci})(\forall q' \in Q_{obs}$ s.t. $\exists \omega \in E^*_{obs}, \delta_{obs}(q_{obs}, \omega) = q') q' \in Q^{single}_{obs}$.

Proof. (If) Assume that G is not strongly detectable with respect to projection P, namely,

$$\begin{aligned} (\forall n \in \mathbb{N})(\exists \sigma \in L^\omega(G))(\exists \sigma' \in Pref(\sigma)) \\ |P(\sigma')| > n \wedge |C(P(\sigma'))| \neq 1. \end{aligned}$$

Let $n = |Q_{obs}|$. Consider σ, σ' such that $\sigma' \in Pref(\sigma)$, $|P(\sigma')| > n$ and $|C(P(\sigma'))| \neq 1$. Since $|P(\sigma')| > n = |Q_{obs}|$, the sequence $P(\sigma')^{elem} \in E^*_{obs}$ must visit at least one elementary circuit $(q_{obs}, s) \in S_{ci}$ in G_{obs}. Let $v \in E^*_{obs}$ such that $\delta_{obs}(q_{i,obs}, v) = q_{obs}$ and $\omega \in E^*_{obs}$ such that $P(\sigma')^{elem} = v\omega$. By Proposition 1, we have $\delta_{obs}(q_{obs}, \omega) = C(P(\sigma'))$. By assumption $|C(P(\sigma'))| \neq 1$, we know that $\delta_{obs}(q_{obs}, \omega) \notin Q^{single}_{obs}$. Therefore, we can claim that $(\exists (q_{obs}, s) \in S_{ci})(\exists q' \in Q_{obs}$ s.t. $\exists \omega \in E^*_{obs}, \delta_{obs}(q_{obs}, \omega) = q') q' \notin Q^{single}_{obs}$.

(Only If) Assume $(\exists (q_{obs}, s) \in S_{ci})(\exists q' \in Q_{obs}$ s.t. $\exists \omega \in E^*_{obs}, \delta_{obs}(q_{obs}, \omega) = q') q' \notin Q^{single}_{obs}$. Let $v \in E^*_{obs}$ such that $\delta_{obs}(q_{i,obs}, v) = q_{obs}$. By assumption $\delta_{obs}(q_{obs}, \omega) \notin Q^{single}_{obs}$, we have $\delta_{obs}(q_{i,obs}, \nu s^j \omega) \notin Q^{single}_{obs}$ for $j \in \mathbb{N}$. Define σ' such that $P(\sigma')^{elem} = vs^j\omega$, by Proposition 1 and Lemma 1, we have $C(P(\sigma')) \neq 1$. Since j can be any arbitrary integer, we can conclude that

$$\begin{aligned} (\forall n \in \mathbb{N})(\exists \sigma \in L^\omega(G))(\exists \sigma' \in Pref(\sigma)) \\ |P(\sigma')| > n \wedge |C(P(\sigma'))| \neq 1. \end{aligned}$$

That is, UWA G is not strongly detectable. □

Theorem 2 (Criterion for Checking Detectability)**.** A UWA G is detectable with respect to projection P iff there is at least one elementary circuit composed of states which all belong to Q^{single}_{obs} in observer G_{obs}, that is: $(\exists (q_{obs}, s) \in S_{ci})(\forall \omega \in Pref(s)) \delta_{obs}(q_{obs}, \omega) \in Q^{single}_{obs}$.

Proof. (If) We assume that $(\exists (q_{obs}, s) \in S_{ci})(\forall \omega \in Pref(s)) \delta_{obs}(q_{obs}, \omega) \in Q^{single}_{obs}$. Let $v \in E^*_{obs}$ such that $\delta_{obs}(q_{i,obs}, v) = q_{obs}$. Define $\sigma \in P(L(G))$ such that $P(\sigma)^{elem} = vs^j$ where j is an infinite integer. By assumption $\delta_{obs}(q_{obs}, \omega) \in Q^{single}_{obs}$, we then have $\delta_{obs}(q_{i,obs}, vs^j\omega) \in Q^{single}_{obs}$. Let $n = |v| \in \mathbb{N}$, then by Proposition 1 and Lemma 1, we can claim that

$$\begin{aligned} (\exists n \in \mathbb{N})(\exists \sigma \in L^\omega(G))(\forall \sigma' \in Pref(\sigma)) \\ |P(\sigma')| > n \Rightarrow |C(P(\sigma'))| = 1. \end{aligned}$$

That is, UWA G is detectable.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2020.2995173, IEEE Transactions on Automatic Control

(Only If) Assume that G is detectable with respect to projection P, that is,

$$(\exists n \in \mathbb{N})(\exists \sigma \in L^{\omega}(G))(\forall \sigma' \in Pref(\sigma))$$
$$|P(\sigma')| > n \Rightarrow |C(P(\sigma'))| = 1.$$

Since σ is an infinite sequence, then $P(\sigma)^{elem}$ must visit at least one elementary circuit $(q_{obs}, s) \in S_{ci}$ in G_{obs} infinitely often. Let $v \in E_{obs}^*$ such that $\delta_{obs}(q_{i,obs}, v) = q_{obs}$. By assumption $|C(P(\sigma'))| = 1$ and Proposition 1, we have $(\forall \omega \in Pref(s))\delta_{obs}(q_{i,obs}, v\omega) \in Q_{obs}^{single}$, which is equivalent to $(\forall \omega \in Pref(s))\delta_{obs}(q_{obs}, \omega) \in Q_{obs}^{single}$. Thus, $(\exists (q_{obs}, s) \in S_{ci})(\forall \omega \in Pref(s))\delta_{obs}(q_{obs}, \omega) \in Q_{obs}^{single}$. □

Theorem 3 (Criterion for Checking Strong Periodic Detectability)**.** A UWA G is strongly periodically detectable with respect to projection P iff each elementary circuit of observer G_{obs} contains at least one state belonging to Q_{obs}^{single}, that is: $(\forall (q_{obs}, s) \in S_{ci})(\exists \omega \in Pref(s))\delta_{obs}(q_{obs}, \omega) \in Q_{obs}^{single}$.

Proof. (If) Assume that G is not strongly periodically detectable with respect to projection P, namely,

$$(\forall n \in \mathbb{N})(\exists \sigma \in L^{\omega}(G))(\exists \sigma' \in Pref(\sigma))$$
$$(\forall \sigma'' \in (E \times \mathcal{D})^*)\sigma'\sigma'' \in Pref(\sigma)$$
$$\wedge |P(\sigma'')| < n \Rightarrow |C(P(\sigma'\sigma''))| \neq 1.$$

From the construction procedure of G_{obs}, we know that the length of a sequence involving no circuit in G_{obs} is bounded by $|Q_{obs}|-1$. Now let $n = |Q_{obs}|+1$ and $|P(\sigma'')| = |Q_{obs}|$. Then, sequence $P(\sigma)^{elem}$ must visit at least one elementary circuit $(q_{obs}, s) \in S_{ci}$ in G_{obs} such that it is the last elementary circuit passed by $P(\sigma'\sigma'')^{elem}$. Since $(\forall \sigma'' \in (E \times \mathcal{D})^*)\sigma'\sigma'' \in Pref(\sigma) \wedge |P(\sigma'')| < |Q_{obs}|+1 \Rightarrow |C(P(\sigma'\sigma''))| \neq 1$, by Proposition 1, we have $(\forall \omega \in Pref(s))\delta_{obs}(q_{obs}, \omega) \notin Q_{obs}^{single}$. Otherwise, there must exist $\sigma'' \in (E \times \mathcal{D})^*$ such that $\sigma'\sigma'' \in Pref(\sigma)$, $|P(\sigma'')| < |Q_{obs}|+1$, and $|C(P(\sigma'\sigma''))| = 1$, which contradicts the above assumption. Therefore, $(\exists (q_{obs}, s) \in S_{ci})(\forall \omega \in Pref(s))\delta_{obs}(q_{obs}, \omega) \notin Q_{obs}^{single}$.

(Only If) Assume that $(\exists (q_{obs}, s) \in S_{ci})(\forall \omega \in Pref(s))\delta_{obs}(q_{obs}, \omega) \notin Q_{obs}^{single}$. Let $v \in E_{obs}^*$ such that $\delta_{obs}(q_{i,obs}, v) = q_{obs}$. Define $\sigma \in L(G)$ such that $P(\sigma)^{elem} = vs^j$ where j is an infinite integer. Let $\sigma' \in Pref(\sigma)$ such that $P(\sigma')^{elem} = v$. Then, by Proposition 1 and Lemma 1, for any $n \in \mathbb{N}$ and any $\sigma'' \in (E \times \mathcal{D})^*$ such that $\sigma'\sigma'' \in Pref(\sigma)$ and $|P(\sigma'')| < n$, $C(P(\sigma'\sigma''))$ corresponds to a state $\delta_{obs}(q_{obs}, \omega)$, with $\omega \in Pref(s)$, in elementary circuit (q_{obs}, s). Since all states in (q_{obs}, s) are not singleton, then $|C(P(\sigma'\sigma''))| \neq 1$. This implies that

$$(\forall n \in \mathbb{N})(\exists \sigma \in L^{\omega}(G))(\exists \sigma' \in Pref(\sigma))$$
$$(\forall \sigma'' \in (E \times \mathcal{D})^*)\sigma'\sigma'' \in Pref(\sigma)$$
$$\wedge |P(\sigma'')| < n \Rightarrow |C(P(\sigma'\sigma''))| \neq 1.$$

That is, UWA G is not strongly periodically detectable. □

Theorem 4 (Criterion for Checking Periodic Detectability)**.** A UWA G is periodically detectable with respect to projection P iff there is at least one elementary circuit in observer G_{obs} which contain at least one state belonging to Q_{obs}^{single}, that is: $(\exists (q_{obs}, s) \in S_{ci})(\exists \omega \in Pref(s))\delta_{obs}(q_{obs}, \omega) \in Q_{obs}^{single}$.

Proof. (If) Assume $(\exists (q_{obs}, s) \in S_{ci})(\exists \omega \in Pref(s))\delta_{obs}(q_{obs}, \omega) \in Q_{obs}^{single}$. Let $v \in E_{obs}^*$ such that $\delta_{obs}(q_{i,obs}, v) = q_{obs}$. Define σ such that $P(\sigma)^{elem} = vs^j$ where j is an infinite integer. Let $n = |vs|$, according to Proposition 1 and Lemma 1, we know that for all $\sigma' \in Pref(\sigma)$, there must exist $\sigma'' \in (E \times \mathcal{D})^*$ such that $\sigma'\sigma'' \in Pref(\sigma)$, $|P(\sigma'')| < n$, and $P(\sigma'\sigma'')$ reaches the singleton state $\delta_{obs}(q_{obs}, \omega)$ in circuit (q_{obs}, s), i.e., $|C(P(\sigma'\sigma''))| = 1$. This implies that

$$(\exists n \in \mathbb{N})(\exists \sigma \in L^{\omega}(G))(\forall \sigma' \in Pref(\sigma))$$
$$(\exists \sigma'' \in (E \times \mathcal{D})^*)\sigma'\sigma'' \in Pref(\sigma)$$
$$\wedge |P(\sigma'')| < n \wedge |C(P(\sigma'\sigma''))| = 1.$$

That is, UWA G is periodically detectable.

(Only If) Assume that G is periodically detectable with respect to projection P, that is,

$$(\exists n \in \mathbb{N})(\exists \sigma \in L^{\omega}(G))(\forall \sigma' \in Pref(\sigma))$$
$$(\exists \sigma'' \in (E \times \mathcal{D})^*)\sigma'\sigma'' \in Pref(\sigma)$$
$$\wedge |P(\sigma'')| < n \wedge |C(P(\sigma'\sigma''))| = 1.$$

Given an infinite sequence σ satisfying the conditions in the above equation, then $P(\sigma)^{elem}$ must visit at least one elementary circuit $(q_{obs}, s) \in S_{ci}$ of G_{obs} in which $|C(P(\sigma'\sigma''))| = 1$ holds for any $\sigma' \in Pref(\sigma)$ and for some $\sigma'' \in (E \times \mathcal{D})^*$. Let $v \in E_{obs}^*$ such that $\delta_{obs}(q_{i,obs}, v) = q_{obs}$. By Proposition 1, we have $(\exists \omega \in Pref(s))\delta_{obs}(q_{i,obs}, v\omega) \in Q_{obs}^{single}$, which is equivalent to $(\exists \omega \in Pref(s))\delta_{obs}(q_{obs}, \omega) \in Q_{obs}^{single}$. Hence, $(\exists (q_{obs}, s) \in S_{ci})(\exists \omega \in Pref(s))\delta_{obs}(q_{obs}, \omega) \in Q_{obs}^{single}$. □

Remark 3. By Theorems 1–4, in order to check detectabilities of UWA $G = (Q, E, \alpha, \mu)$, we need to find out all states belonging to elementary circuits in $G_{obs} = (Q_{obs}, E_{obs}, \delta_{obs}, q_{i,obs})$. This can be done by finding all the strongly connected components (SCCs) in G_{obs}, with a complexity that is linear in the number of states and transitions of G_{obs}, i.e., $\mathcal{O}(|Q_{obs}|+|Q_{obs}|\times|E_{obs}|)$. A SCC is said to be valid if it contains multiple nodes or contains only one node and the node has a self-loop.

Strong detectability can be verified by checking if all states that are reachable from any state belonging to the elementary circuits are singleton, with complexity $\mathcal{O}(|Q_{obs}|\times[|Q_{obs}|+|Q_{obs}|\times|E_{obs}|]) = \mathcal{O}(|Q_{obs}|^2\times|E_{obs}|)$. Detectability can be verified as follows: *for any valid SCC, we first identify all its singleton states, delete all the other non-singleton states and the associated transitions to obtain a subgraph of this SCC. Then we check if the remaining subgraph is a SCC (i.e. contains a circuit).* The complexity of the above operations is linear in the number of states and the transitions in the remaining subgraph of the SCC, i.e., $\mathcal{O}(|Q_{obs}+|Q_{obs}|\times|E_{obs}|) = \mathcal{O}(|Q_{obs}|\times|E_{obs}|)$. Since there are at most $|Q_{obs}|$ SCCs in G_{obs}, the detectability can be verified with a complexity $\mathcal{O}(|Q_{obs}|^2\times|E_{obs}|)$. Similar to detectability verification, strong periodic detectability can be verified with a complexity $\mathcal{O}(|Q_{obs}|^2 \times |E_{obs}|)$. More precisely, we can verify strong periodic detectability as follows: *for any valid SCC, we first identify all its non-singleton states, delete all the other singleton states and the associated transitions to obtain a subgraph of this SCC. Then we check if the subgraph contains a (elementary) circuit consisting only of non-singleton states*. Periodic detectability can be verified by checking if there exists a valid SCC such that it contains a singleton state. The complexity of doing this is $\mathcal{O}(|Q_{obs}|^2)$ as there are at most $|Q_{obs}|$ valid SCCs and the number of states in each valid SCC is bounded by $|Q_{obs}|$.

Example 6. Consider observer G_{obs} in Fig. 4 of UWA G in Fig. 1. By Theorems 1 and 3, G is not strongly detectable and not strongly periodically detectable since elementary circuit $(\{2,3\},\{2,3\}) \stackrel{(b,0.3)}{\longrightarrow} (\{2,3\},\{2,3\})$ has its state in $Q_{obs} \setminus Q_{obs}^{single}$. Because circuit $(\{3\},\{3\}) \stackrel{(b,0.3)}{\longrightarrow} (\{3\},\{3\})$ has all its states in Q_{obs}^{single}, by Theorems 2 and 4, G is detectable and periodically detectable. □

Example 7. Consider UWA G in Fig. 5 whose observer G_{obs} is presented in Fig. 6. By Theorems 1-4, we know that G is strongly detectable (hence, detectable, strongly periodically detectable and periodically detectable) since all states reachable from any elementary circuit of G_{obs} are entirely within Q_{obs}^{single}. Note that the underlying logical automaton is not strongly detectable. In fact, for observation b^k with $k \in \{1, 2, \cdots\}$, the logical automaton may always have two possible current states, i.e. 2 and 3. □

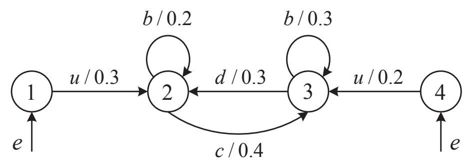

Fig. 5: Unambiguous weighted automaton G.

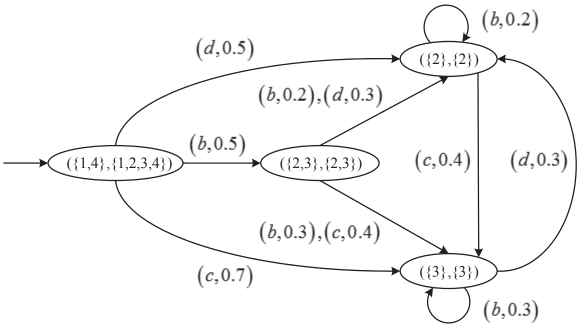

Fig. 6: Observer G_{obs} of automaton G in Fig. 5.

V. Conclusion

In this paper, we deal with the detectability problem for UWAs. An algorithm is first proposed to build an observer of a UWA, and then necessary and sufficient conditions are proposed for checking four types of detectabilities. As a future work, we plan to investigate the detectability problem for more general WAs.

References

[1] X. Yin and S. Lafortune, "Codiagnosability and coobservability under dynamic observations: Transformation and verification," *Automatica*, vol. 61, pp. 241–252, 2015.

[2] A. Lai, S. Lahaye, and A. Giua, "State estimation of max-plus automata with unobservable events," *Automatica*, vol. 105, pp. 36–42, 2019.

[3] D. Corona, A. Giua, and C. Seatzu, "Marking estimation of Petri nets with silent transitions," *IEEE Trans. Autom. Control*, vol. 52, pp. 1695–1699, 2007.

[4] F. Basile, M. Cabasino, and C. Seatzu, "State estimation and fault diagnosis of labeled time Petri net systems with unobservable transitions," *IEEE Trans. Autom. Control*, vol. 60, no. 4, pp. 997–1009, 2015.

[5] Y. Ru and C. N. Hadjicostis, "Sensor selection for structural observability in discrete event systems modeled by Petri nets," *IEEE Trans. Autom. Control*, vol. 55, no. 8, pp. 1751–1764, 2010.

[6] A. Saboori and C. N. Hadjicostis, "Verification of initial-state opacity in security applications of discrete event systems," *Inform. Sci.*, vol. 246, pp. 115–132, 2013.

[7] Y. Tong, Z. Li, C. Seatzu, and A. Giua, "Verification of state-based opacity using Petri nets," *IEEE Trans. Autom. Control*, vol. 62, no. 6, pp. 2823–2837, 2017.

[8] M. Sampath, R. Sengupta, S. Lafortune, K. Sinnamohideen, and D. Teneketzis, "Diagnosability of discrete-event systems," *IEEE Trans. Autom. Control*, vol. 40, no. 9, pp. 1555–1575, 1995.

[9] M. P. Cabasino, A. Giua, M. Pocci, and C. Seatzu, "Discrete event diagnosis using labeled Petri nets. an application to manufacturing systems," *Control Eng. Pract.*, vol. 19, no. 9, pp. 989–1001, 2011.

[10] S. Shu, F. Lin, and H. Ying, "Detectability of discrete event systems," *IEEE Trans. Autom. Control*, vol. 52, no. 12, pp. 2356–2359, 2007.

[11] S. Shu and F. Lin, "Generalized detectability for discrete event systems," *Syst. Control Lett.*, vol. 60, no. 5, pp. 310–317, 2011.

[12] ——, "I-detectability of discrete-event systems," *IEEE Trans. Autom. Sci. Eng.*, vol. 10, no. 1, pp. 187–196, 2013.

[13] C. Keroglou and C. N. Hadjicostis, "Verification of detectability in probabilistic finite automata," *Automatica*, vol. 86, pp. 192–198, 2017.

[14] X. Yin, "Initial-state detectability of stochastic discrete-event systems with probabilistic sensor failures," *Automatica*, vol. 80, pp. 127–134, 2017.

[15] P. Zhao, S. Shu, F. Lin, and B. Zhang, "Detectability measure for state estimation of discrete event systems," *IEEE Trans. Autom. Control*, vol. 64, no. 1, pp. 433–439, 2019.

[16] S. Gaubert, "Performance evaluation of (max,+) automata," *IEEE Trans. Autom. Control*, vol. 40, no. 12, pp. 2014–2025, 1995.

[17] S. Gaubert and J. Mairesse, "Modeling and analysis of timed petri nets using heaps of pieces," *IEEE Trans. Autom. Control*, vol. 44, no. 4, pp. 683–697, 1999.

[18] S. Lahaye, J. Komenda, and J.-L. Boimond, "Compositions of (max,+) automata," *Discrete Event Dyn. Syst.*, vol. 25, no. 1-2, pp. 323–344, 2015.

[19] J. Komenda, S. Lahaye, and J.-L. Boimond, "Determinization of timed petri nets behaviors," *Discrete Event Dyn. Syst.*, vol. 26, no. 3, pp. 413–437, 2016.

[20] S. Almagor, U. Boker, and O. Kupferman, "What's decidable about weighted automata?" in *ATVA'11*. Springer, 2011, pp. 482–491.

[21] M.-P. Béal, E. Czeizler, J. Kari, and D. Perrin, "Unambiguous automata," *Math. Comput. Sci.*, vol. 1, no. 4, pp. 625–638, 2008.

[22] I. Klimann, S. Lombardy, J. Mairesse, and C. Prieur, "Deciding unambiguity and sequentiality from a finitely ambiguous max-plus automaton," *Theo. Comput. Sci.*, vol. 327, no. 3, pp. 349–373, 2004.

[23] A. Lai, S. Lahaye, and Z. Li, "Initial-state detectability and initial-state opacity of unambiguous weighted automata," *Automatica (submitted)*, author version downloadable from http://tiny.cc/z6hn4y.

[24] M. Droste, W. Kuich, and H. Vogler, *Handbook of weighted automata*. Springer Science & Business Media, 2009.

[25] J. G. Thistle and W. M. Wonham, "Control of infinite behavior of finite automata," *SIAM J. Control Optim.*, vol. 32, no. 4, pp. 1075–1097, 1994.