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Continuity properties of preference relations

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Various types of continuity for preference relations on a metric space are examined constructively. In particular, necessary and sufficient conditions are given for an order-dense, strongly extensional preference relation on a complete metric space to be continuous. It is also shown, in the spirit of constructive reverse mathematics, that the continuity of sequentially continuous, order-dense preference relations on complete, separable metric spaces is connected to Ishihara's principle $\mathbf{BD}$-$\mathbb{N}$, and therefore is not provable within Bishop-style constructive mathematics alone.

1 Preferences and continuity

The notions of preference and utility play a fundamental role in traditional microeconomic theory. Each consumer is assumed to have a *consumption set* X – typically, but not essentially, a compact, convex subset of $\mathbb{R}^n$ – whose elements are the *consumption bundles*. Each component of the consumption bundle $\mathbf{x}$ is the amount of a corresponding good or service that the consumer may wish to purchase. It is assumed that there is a binary relation $\succ$ of *preference* on X, where "$\mathbf{x} \succ \mathbf{y}$" means that the consumer strictly prefers bundle $\mathbf{x}$ to bundle $\mathbf{y}$. The early developments of the theory went so far as to assume that the preferences were represented by a *utility function* $u : X \longrightarrow \mathbb{R}$, whereby $\mathbf{x} \succ \mathbf{y}$ if and only if $u(\mathbf{x}) > u(\mathbf{y})$. It was later realised that the existence of a utility function representing a given preference relation required justification; furthermore, one could not automatically assume that the utility function, if it existed, was continuous. This led to the study of necessary and sufficient conditions for the existence of continuous utility functions, a topic which has been explored in depth since the appearance of the pioneering work of Debreu in the 1950s [13, 14].

Early in the constructive analysis of preference relations it became clear that straightforward constructivisation of the classical proofs of the existence and continuity of utility functions was not possible. In fact, Debreu's famous theorem on this subject is false in recursive constructive mathematics [8]. Since the smoothest constructive path to an existence theorem for utility functions uses a very strong continuity condition on the preference relation [4], it would be interesting (maybe useful?) to have an existence theorem under weaker continuity conditions on preferences. To that end, it makes sense to examine the constructive connections between various types of continuity of preferences, analogous to ones for continuity of functions. We begin such an examination in this paper.

Let X be a set that is *inhabited* in the sense that we can construct an element of it. A binary relation $\succ$ on X is called a *preference relation* if it satisfies these two axioms:

$\mathbf{P}_1$: $\forall_{x,y\in X}(x \succ y \Rightarrow \neg(y \succ x))$;

$\mathbf{P}_2$: $\forall_{x,y\in X}(x \succ y \Rightarrow \forall_{z\in X}(x \succ z \vee z \succ y)$.

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The corresponding *preference-indifference relation* $\succcurlyeq$ is then defined by

$$\forall_{x,y\in X}(x \succcurlyeq y \Leftrightarrow \neg(y \succ x)).$$

(Of course, we write $y \prec x$ and $y \preccurlyeq x$ as equivalents of $x \succ y$ and $x \succcurlyeq y$, respectively.) The corresponding *reverse preference relation* $\succ_{\mathrm{rev}}$ is defined on X by

$$\forall_{x,y\in X}(x \succ_{\mathrm{rev}} y \Leftrightarrow y \succ x).$$

With each preference relation and each point a of X we associate

1. *the upper contour set*

$$[a, \rightarrow) = \{x \in X \,:\, x \succcurlyeq a\},$$

2. *the strict upper contour set*

$$(a, \rightarrow) = \{x \in X \,:\, x \succ a\},$$

3. *the lower contour set*

$$(\leftarrow, a] = \{x \in X \,:\, a \succcurlyeq x\},$$

4. and *the strict lower contour set*

$$(\leftarrow, a) = \{x \in X \,:\, a \succ x\}.$$

In this paper we are particularly interested in the case where (X, ϱ) is a metric space. The standard inequality on X is then defined by

$$\forall_{x,y\in X}(x \neq y \Leftrightarrow \varrho(x,y) > 0).$$

Corresponding to various types of continuity of functions between metric spaces there are types of continuity for a preference relation $\succ$ on X. We say that $\succ$ is

1. *pointwise continuous at* a if both the sets $(\leftarrow, a)$ and $(a, \rightarrow)$ are open in X;
2. *sequentially continuous at* a if for each $x \in X$ and each sequence $(x_n)_{n\geqslant 1}$ of points of X converging to x,

$$x \succ a \Rightarrow \exists_N \forall_{n\geqslant N}(x_n \succ a) \quad \text{and} \quad a \succ x \Rightarrow \exists_N \forall_{n\geqslant N}(a \succ x_n);$$

3. *nearly continuous at* a if for each $S \subset X$ and each x in the closure $\overline{S}$ of S,

$$x \succ a \Rightarrow \exists_{s\in S}(s \succ a) \quad \text{and} \quad a \succ x \Rightarrow \exists_{s\in S}(a \succ s);$$

4. *nondiscontinuous at* a if both the sets $(\leftarrow, a]$ and $[a, \rightarrow)$ are closed in X.

We say that $\succ$ is, for example, *sequentially continuous on* X if it is sequentially continuous at each point of X. It is straightforward to show that if $\succ$ is represented by a pointwise, sequential, or nearly continuous[1)] utility function, then $\succ$ itself has the corresponding continuity property on X.

Our aim is to investigate, within the framework of Bishop-style constructive mathematics BISH[2)], connections between these notions of continuity.

Proposition 1.1 *For preference relations on a metric space, pointwise continuity at a point implies sequential continuity, which implies near continuity, which implies nondiscontinuity.*

1) More on near continuity for functions is found in [9].

2) Mathematics that uses only intuitionistic logic and an appropriate set-theory such as that presented in [1]. For more on constructive analysis see [3, 10]. For background in the constructive theory of preference and utility, see [2, 4, 5].

P r o o f. Let $\succ$ be a preference relation on the metric space X, and let $a \in X$. Suppose that $\succ$ is pointwise continuous at a. If $x \succ a$, then there exists $r > 0$ such that $y \succ a$ whenever $\varrho(x, y) < r$. It follows that for every sequence $(x_n)_{n\geqslant1}$ of elements of X converging to x, we have $x_n \succ a$ for all sufficiently large n. The case $a \succ x$ is similarly handled. Hence $\succ$ is sequentially continuous at a.

Next suppose that $\succ$ is sequentially continuous at a. Let S be a subset of X, and let x be a point of $\overline{S}$ such that $x \succ a$. Then there exists a sequence $(x_n)_{n\geqslant1}$ of elements of S converging to x. Since $x \succ a$, from sequential continuity it follows that $x_n \succ a$ for all sufficiently large n. The case $a \succ x$ is similarly handled. Thus $\succ$ is nearly continuous at a.

To prove that near continuity implies nondiscontinuity, suppose that $\succ$ is nearly continuous at a, and consider a sequence $(x_n)_{n\geqslant1}$ in X that converges to x and satisfies $x_n \succcurlyeq a$ for all n. Assume that $a \succ x$. It is clear that x belongs to the closure of $[a, \rightarrow)$. Since $\succ$ is nearly continuous at a, it follows that $a \succ s$ for some $s \in [a, \rightarrow)$, a contradiction. Consequently, $\neg(a \succ x)$ and therefore $x \succcurlyeq a$ – that is, $x \in [a, \rightarrow)$. We prove similarly that the lower contour set at a is closed; so the preference relation $\succ$ is nondiscontinuous at a. □

An irreflexive binary relation R on a set X with an inequality $\neq$ is said to be *strongly extensional* if

$$\forall_{x,y\in X}(xRy \Rightarrow x \neq y).$$

Proposition 1.2 *A nearly continuous preference relation on a metric space is strongly extensional.*

P r o o f. Let $\succ$ be a nearly continuous preference relation on the metric space X, and let $a \succ b$. To prove that $a \neq b$, construct an increasing binary sequence $(\lambda_n)_{n\geqslant1}$ such that

$$\lambda_n = 0 \Rightarrow \varrho(a, b) < 1/n, \quad \lambda_n = 1 \Rightarrow \varrho(a, b) > 0.$$

We may assume that $\lambda_1 = 0$. If $\lambda_n = 0$, set $S_n = \{a\}$; if $\lambda_n = 1$, set $S_n = \{b\}$. Define

$$S = \bigcup_{n=1}^{\infty} S_n.$$

Then $b \in \overline{S}$. Since $\succ$ is nearly continuous, it follows that there exists $s \in S$ with $a \succ s$. Pick N such that $s \in S_N$. If $\lambda_N = 0$, then $s = a$, a contradiction. Hence $\lambda_N = 1$ and therefore $a \neq b$. □

Our next task is to produce necessary and sufficient conditions for the sequential continuity of a certain type of preference relation on a complete metric space (Proposition 1.5 below). We require the following two lemmas, which are reminiscent of Ishihara's tricks [15, 11].

Lemma 1.3 *Let $\succ$ be a strongly extensional preference relation on a complete metric space X. Let $a, b, x \in X$ satisfy $x \succ a \succ b$, and let $(x_n)_{n\geqslant1}$ be a sequence in X that converges to x. Then either $x_n \succ b$ for all n or else there exists n such that $a \succ x_n$.*

P r o o f. Construct an increasing binary sequence $(\lambda_n)_{n\geqslant1}$ such that

$$\lambda_n = 0 \Rightarrow \forall_{k\leqslant n}(x_k \succ b), \quad \lambda_n = 1 - \lambda_{n-1} \Rightarrow a \succ x_n.$$

We may assume that $\lambda_1 = 0$. If $\lambda_n = 0$, set $y_n = x$; if $\lambda_n = 1 - \lambda_{n-1}$, set $y_k = x_n$ for all $k \geqslant n$. Then $(y_n)_{n\geqslant1}$ is a Cauchy sequence in X and so converges to a limit $y \in X$. Either $y \succ a$ or $x \succ y$. In the first case, if there exists n such that $\lambda_n = 1 - \lambda_{n-1}$, then $y = x_n \prec a$, a contradiction; hence $\lambda_n = 0$, and therefore $x_n \succ b$, for all n. In the case $x \succ y$, the strong extensionality of $\succ$ yields $x \neq y$; so there exists N such that $x \neq y_N$. Then $\lambda_n = 1 - \lambda_{n-1}$, and therefore $x \succ a \succ x_n$, for some $n \leqslant N$. □

Lemma 1.4 *Under the hypotheses of Lemma* 1.3*, either $x_n \succ b$ for all sufficiently large n, or else $a \succ x_n$ for infinitely many n.*

P r o o f. In view of Lemma 1.3, we may assume that there exists $n_1 \geqslant 1$ such that $a \succ x_{n_1}$. Setting $\lambda_1 = 0$ and applying Lemma 1.3 recursively, construct an increasing binary sequence $(\lambda_k)_{k\geqslant1}$ and an increasing sequence $(n_k)_{k\geqslant1}$ of positive integers with the following properties:

1. If $\lambda_{k+1} = 0$, then there exists $n_{k+1} > n_k$ and $a \succ x_{n_{k+1}}$.
2. If $\lambda_{k+1} = 1 - \lambda_k$, then $x_n \succ b$ for all $n > n_k$, and $n_j = n_k$ for all $j \geqslant k + 1$.

If $\lambda_k = 0$, set $y_k = x$; if $\lambda_{k+1} = 1 - \lambda_k$, set $y_j = x_{n_k}$ for all $j \geqslant k+1$. Then $(y_k)_{k\geqslant 1}$ is a Cauchy sequence in X and so converges to a limit $y \in X$. Either $x \succ y$ or $y \succ a$. In the first case, by strong extensionality, $x \neq y$ and so there exists k such that $x \neq y_k$ and therefore $\lambda_k \neq 0$; whence $x_n \succ b$ for all $n \geqslant n_k$. In the remaining case, $y \succ a$, if, for some k, we have $\lambda_{k+1} = 1 - \lambda_k$, then $y = x_{n_k} \prec a$, a contradiction; hence $\lambda_k = 0$ for all k, the sequence $(n_k)_{k\geqslant 1}$ is strictly increasing, and $a \succ x_{n_k}$ for all k. □

A preference relation $\succ$ on a set X is said to be *order-dense* if for all $x, z \in X$ with $x \succ z$, there exists $y \in X$ such that $x \succ y \succ z$.

Proposition 1.5 *An order-dense preference relation $\succ$ on a complete metric space X is sequentially continuous if and only if it is both nondiscontinuous and strongly extensional.*

P r o o f. If $\succ$ is sequentially continuous, then by Propositions 1.1 and 1.2, it is both nondiscontinuous and strongly extensional.

Suppose, conversely, that $\succ$ has both these last two properties. Let the sequence $(x_n)_{n\geqslant 1}$ converge to x in X, and, to begin with, let $x \succ b$. Pick $a \in X$ such that $x \succ a \succ b$. By Lemma 1.4, either $x_n \succ b$ for all sufficiently large n, or else, as we suppose in order to obtain a contradiction, there is a strictly increasing sequence $(n_k)_{k\geqslant 1}$ of positive integers such that $a \succ x_{n_k}$ for each k. Since (this is our nondiscontinuity assumption) the set $(\leftarrow, a]$ is closed, it follows that $x \in (\leftarrow, a]$; that is, $a \nprec x$, which is a contradiction. Hence, in fact, $x_n \succ b$ for all sufficiently large n.

Finally suppose that $a \succ x$. Then the argument above shows that $x_n \succ_{\mathrm{rev}} a$ for the reverse preference relation $\succ_{\mathrm{rev}}$, and thus $a \succ x_n$, for all sufficiently large n. We have now proved that $\succ$ is sequentially continuous. □

In the case where the preference relation is represented by a utility function, Proposition 1.5 can be used to produce a weak version of [15, Theorem 1]; this is discussed in [6].

2 Sequential and pointwise continuity

With classical logic we can prove that, for a preference relation $\succ$ on the metric space X, sequential continuity at each point of X implies, and hence is equivalent to, pointwise continuity throughout X. To see this, suppose that $\succ$ is sequentially continuous at each point of X but is not pointwise continuous at $x \in X$. Then either $(\leftarrow, x)$ or $(x, \rightarrow)$ is not open in X. For example, if we assume that the latter set is not open, then we see that there exist a point $y \in X$ and a sequence $(y_n)_{n\geqslant 1}$ converging to y in X, such that $y \succ x \nsucc y_n$ for every n. Since $\succ$ is sequentially continuous at y, we must have $y_n \succ x$ for all sufficiently large n, a contradiction.

Clearly, this proof provides no indication of conditions that might ensure that sequential continuity constructively implies pointwise continuity. In seeking such conditions, and bearing in mind those preference relations that are represented by utility functions [4], we are guided by Ishihara's work [16] relating sequential and pointwise continuity of functions on a complete metric space.

A set S of positive integers is called *pseudobounded* if $\lim_{n\to\infty} n^{-1}s_n = 0$ for each sequence $(s_n)_{n\geqslant 1}$ in S. The following principle, introduced by Ishihara, holds classically and in both the intuitionistic and recursive models of constructive mathematics, is unprovable in a natural formalisation of BISH [19], and has proved extremely significant in constructive reverse mathematics:

$\mathbf{BD}$-$\mathbb{N}$ Every inhabited, countable, pseudobounded set of positive integers is bounded.

In particular, as Ishihara showed in [16, Theorem 4], $\mathbf{BD}$-$\mathbb{N}$ is equivalent to the proposition "Every sequentially continuous mapping of a complete, separable metric space into a metric space is pointwise continuous". For more on the role of $\mathbf{BD}$-$\mathbb{N}$ in constructive reverse mathematics, see [12, 17, 18].

Theorem 2.1 *If $\mathbf{BD}$-$\mathbb{N}$ holds, then every sequentially continuous, order-dense preference relation on a separable metric space is pointwise continuous.*

P r o o f. Fix a sequentially continuous, order-dense preference relation $\succ$ on a separable metric space X, and a sequence $(y_n)_{n\geqslant 1}$ dense in X. Given $a \in X$ and $x \in (a, \rightarrow)$, pick $z, z' \in X$ such that $x \succ z \succ z' \succ a$, and construct a mapping $\varphi : \mathbb{N} \times \mathbb{N} \longrightarrow \{0, 1\}$ such that

$$\varphi(m,n) = 0 \Rightarrow \varrho(y_m, x) < \frac{1}{n} \wedge z \succ y_m, \quad \varphi(m,n) = 1 \Rightarrow \varrho(y_m, x) > \frac{1}{n+1} \vee y_m \succ z'.$$

Then the set

$$S = \{0\} \cup \{n : \exists_m(\varphi(m,n) = 0)\}$$

is countable. We prove that it is pseudobounded. To that end, let $(s_n)_{n\geqslant 1}$ be a sequence in S. Let $\varepsilon > 0$. Construct a binary sequence $(\lambda_n)_{n\geq 1}$ such that

$$\lambda_n = 0 \Rightarrow n^{-1}s_n < \varepsilon, \quad \lambda_n = 1 \Rightarrow n^{-1}s_n > \varepsilon/2.$$

If $\lambda_n = 0$, set $\xi_n = x$. If $\lambda_n = 1$, choose m_n such that $\varphi(m_n, s_n) = 0$; setting $\xi_n = y_{m_n}$, we have $z \succ \xi_n$ and

$$\varrho(x, \xi_n) < \frac{1}{s_n} < \frac{2}{n\varepsilon}.$$

Clearly, the sequence $(\xi_n)_{n\geq 1}$ converges to x. Since $\succ$ is sequentially continuous and $x \succ z$, there exists N such that $\xi_n \succ z$ for all $n \geq N$. For such n, if $\lambda_n = 1$, then $z \succ y_{m_n} = \xi_n \succ z$, which is absurd. Hence $\lambda_n = 0$, and therefore $n^{-1}s_n < \varepsilon$, for all $n \geq N$. Since $(s_n)_{n\geq 1}$ in S and $\varepsilon > 0$ are arbitrary, we conclude that S is indeed pseudobounded.

Applying $\mathbf{BD}$-$\mathbb{N}$, compute N such that $s < N$ for all $s \in S$, and consider any m such that

$$\varrho(y_m, x) < 1/(N+1).$$

If $z' \succ y_m$, then $\varphi(m,N) \neq 1$, so $\varphi(m,N) = 0$ and therefore $N \in S$, contradicting our choice of N. Hence we have $y_m \succcurlyeq z'$. Since, by Proposition 1.1, $\succ$ is nondiscontinuous, we now see that $y \succcurlyeq z'$, and hence that $y \succ a$, whenever $\varrho(x,y) < 1/(N+1)$. This completes the proof that $(a, \rightarrow)$ is open. To prove that $(\leftarrow, a)$ is open also, we apply what we have just proved to the reverse preference relation $\succ_{\mathrm{rev}}$ associated with $\succ$. □

What about the converse of Theorem 2.1?

Theorem 2.2 *Suppose that every sequentially continuous, order-dense preference relation on a complete, separable metric space is pointwise continuous. Then* $\mathbf{BD}$-$\mathbb{N}$ *holds.*

Proof. Let S be an inhabited, countable, pseudobounded subset of $\mathbb{N}$. We may assume that $0 \in S$. Let

$$Z = \{0\} \cup \{2^{-m} : m \in S\} \quad \text{and} \quad X = \overline{Z} \cup [2,3].$$

The metric space X is complete. As in the proof of [7, Theorem 3.2], construct a sequentially continuous[3)] mapping $u : X \longrightarrow [0,1]$ with $u(0) = 0$, $u(2^{-n}) = 1$ for each $n \in S$, and $u(x) = x - 2$ for each $x \in [2,3]$. Let $\succ$ be the preference relation corresponding to u. Then $\succ$ is sequentially continuous and (in view of the intermediate-value theorem) order-dense. Moreover, $1 \succ 0$. Suppose that $\succ$ is pointwise continuous. Compute a positive integer N such that if $x \in X$ and $|x| < 2^{-N}$, then $1 \succ x$. If $n \in S$ and $n > N$, then $u(2^{-n}) = 1$ and $1 \succ 2^{-n}$, which is absurd. We conclude that $n \leqslant N$ for all $n \in S$. □

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3) Actually, the mapping u has a stronger property of *uniform sequential continuity*, which we do not need here.

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