On simple representations of stopping times and stopping time σ-algebras

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Abstract

There exists a simple, didactically useful one-to-one relationship between stopping times and adapted càdlàg (RCLL) processes that are non-increasing and take the values 0 and 1 only. As a consequence, stopping times are always hitting times. Furthermore, we show how minimal elements of a stopping time σ-algebra can be expressed in terms of the minimal elements of the σ-algebra of the underlying filtration. This facilitates an intuitive interpretation of stopping time σ-algebras. A tree example finally illustrates how these for students notoriously difficult to understand concepts, stopping times and stopping time σ-algebras, are easier to grasp by means of our results.

Key words: Adapted processes, *Début* Theorem, hitting time, stopped σ-algebra, stopping time, stopping time σ-algebra.

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1 Stopping times are hitting times: a natural representation

The possibly most popular example of a stopping time is the 'hitting time' of a set by a stochastic process, defined as the first time at which a certain pre-specified set is hit by the considered process. Often this example is considered for a Borel-measurable set and a one-dimensional real-valued adapted process on a discrete time axis. Similar results in more general set-ups are usually collectively called the *Début* Theorem; see Bass (2010, 2011) for a proof.

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Astonishingly, it seems to be less widely taught (and maybe known) that the inverse is true as well: for any stopping time there exists an adapted stochastic process and a Borel measurable set such that the corresponding hitting time will be exactly this stopping time. Furthermore, the stochastic process can be chosen very intuitively: it will be 1 until just before the stopping time is reached, from which on it will be 0. The 1-0-process therefore first hits the Borel set $\{0\}$ at the stopping time. As such, a one-to-one relationship is established between stopping times and adapted càdlàg processes that are non-increasing and take the values 0 and 1 only.

A 1-0-process as described above is obviously akin to a random traffic light which can go through three possible scenarios over time (here, 'green' stands for '1' and 'red' stands for '0'): (i) it stays red forever ('stopped immediately'); (ii) it is green at the beginning, then turns red, and stays red for the rest of the time ('stopped at some stage'); (iii) it stays green forever ('never stopped'). So, the traffic light can never change back to green once it has turned red (stopped once means stopped forever), and, for the adaptedness of the 1-0-process, it can only change based on information up to the corresponding point in time. This very intuitive interpretation of a stopping time as the time when such a 'traffic light' changes is considerably easier to understand than the concept of a random time which is 'known once it has been reached' – one of the verbal interpretations of the usual standard definition of a stopping time (see Def. 1 below).

While this representation and alternative definition of stopping times seems natural and didactically useful, it does not seem to be widely taught, or otherwise one would expect to find it in textbooks. However, there is no mentioning of it in standard textbooks on probability such as Billingsley (1995) or Bauer (2001) (Bauer as an example of a popular German stochastics textbook) or in standard textbooks on stochastic processes/calculus and stochastic mathematical finance such as Karatzas and Shreve (1991) or Bingham and Kiesel (2004).

We denote the time axis by $\mathbb{T}$, where $\mathbb{T} \subset \mathbb{R}$.

DEFINITION 1. *A stopping time w.r.t. to a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{T}}$ on a probability space $(\Omega, \mathcal{F}_\infty, \mathbb{P})$ is a random variable with values in $\mathbb{T} \cup \{+\infty\}$ such that*

$$\{\tau \leq t\} \in \mathcal{F}_t \quad (t \in \mathbb{T}). \tag{1}$$

(1) means that at any time $t \in \mathbb{T}$ one knows – based on information up to time t – if one has been stopped already, or not. Note that $\mathcal{F}_t \subset \mathcal{F}_\infty$ is assumed for $t \in \mathbb{T}$.

In the following, we call a real-valued stochastic process càdlàg (French: *continue à droite, limitée à gauche*) if for *all* $\omega \in \Omega$ the paths $X_\cdot(\omega) : \mathbb{T} \to \mathbb{R}$ have the property of being right-continuous with left-handed limits (RCLL).

DEFINITION 2. *We call an adapted càdlàg process $X = (X_t)_{t\in\mathbb{T}}$ on $(\Omega, \mathcal{F}_\infty, \mathbb{F}, \mathbb{P})$ a 'stopping process' if*

$$X_t(\omega) \in \{0, 1\} \quad (\omega \in \Omega,\, t \in \mathbb{T}) \tag{2}$$

and

$$X_s \geq X_t \quad (s \leq t;\, s, t \in \mathbb{T}). \tag{3}$$

Obviously, $\{X_t = 1\} \in \mathcal{F}_t$ for $t \in \mathbb{T}$. For a finite or infinite discrete time axis given by $\mathbb{T} = \{t_k : \ k \in \mathbb{N}, t_k \geq t_j \text{ if } k \geq j\}$, an adapted process fulfilling (2) and (3) is automatically càdlàg. This follows from the observation that in this case $\mathbb{T}$ is bounded from below and either (1) finite as a set, or (2) is infinite and has one accumulation point, which lies not in $\mathbb{T}$, and therefore is bounded, or (3) has no accumulation point and is unbounded from above.

DEFINITION 3. *For a stopping process X on $(\Omega, \mathcal{F}_\infty, \mathbb{F}, \mathbb{P})$, define*

$$\tau^X(\omega) = \begin{cases} +\infty & \textit{if } X_t(\omega) = 1 \ \textit{ for all } t \in \mathbb{T}, \\ \min\{t \in \mathbb{T} : X_t(\omega) = 0\} & \textit{otherwise.} \end{cases} \tag{4}$$

The minimum in the lower case exists because of the càdlàg property for each path. By definition, it is clear that

$$X_t = \mathbf{1}_{\{\tau^X > t\}} \quad (t \in \mathbb{T}), \tag{5}$$

which by adaptedness of X and $\{\tau^X > t\} = \{\tau^X \leq t\}^C$ implies

$$\{\tau^X \leq t\} \in \mathcal{F}_t \quad (t \in \mathbb{T}). \tag{6}$$

Therefore, τ^X is a stopping time for any stopping process X. Clearly, τ^X is the first time of X hitting the Lebesgue-measurable set $\{0\}$.

DEFINITION 4. *For a stopping time τ, define a stochastic process $X^\tau = (X_t^\tau)_{t \in \mathbb{T}}$ by*

$$X_t^\tau = \mathbf{1}_{\{\tau > t\}} \quad (t \in \mathbb{T}). \tag{7}$$

By $\{\tau > t\} = \{\tau \leq t\}^C$ and (1), X^τ is an adapted process. One has $X_s^\tau = \mathbf{1}_{\{\tau > s\}} \geq \mathbf{1}_{\{\tau > t\}} = X_t^\tau$ for $s \leq t$. Because of $\lim_{t \downarrow \tau(\omega)} X_t^\tau(\omega) = 0 = X_{\tau(\omega)}^\tau$, X^τ is càdlàg and hence a stopping process.

THEOREM 1. *The mapping*

$$f : X \quad \longmapsto \quad \tau^X \tag{8}$$

is a bijection between the stopping processes and the stopping times on $(\Omega, \mathcal{F}_\infty, \mathbb{F}, \mathbb{P})$ such that

$$f^{-1} : \tau \quad \longmapsto \quad X^\tau, \tag{9}$$

and hence

$$\tau^{X^\tau} = \tau \quad \textit{and} \quad X^{\tau^X} = X. \tag{10}$$

Proof. That f maps stopping processes X to stopping times τ^X was seen in (6). That different stopping processes lead to different stopping times under f is obvious from (4); f is therefore an injection. For any stopping time τ, one has by (4) and (7) that

$$\tau^{X^\tau}(\omega) = \begin{cases} +\infty & \text{if } \mathbf{1}_{\{\tau > t\}}(\omega) = 1 \ \text{ for all } t \in \mathbb{T}, \\ \min\{t \in \mathbb{T} : \mathbf{1}_{\{\tau > t\}}(\omega) = 0\} & \text{otherwise.} \end{cases} \tag{11}$$

Since $\mathbf{1}_{\{\tau > t\}}(\omega) = 0$ if and only if $\tau(\omega) \leq t$, the right hand side of (11) is $\tau(\omega)$. Therefore, $\tau(\omega) = \tau^{X^\tau}(\omega)$, and f is a surjection and therefore is a bijection. From (5) and (7), we obtain $X_t^{\tau^X} = \mathbf{1}_{\{\tau^X > t\}} = X_t$ for $t \in \mathbb{T}$, which proves (9). $\square$

Note that there was no identification of almost surely identical stopping times or stopping processes in Theorem 1 or any of the definitions, however, one can obviously transfer all results to equivalence classes of almost surely identical objects.

2 Minimal elements of stopping time σ-algebras

Another concept students of stochastic processes and probability are usually struggling with are the so-called stopping time σ-algebras.

DEFINITION 5. *Let τ be a stopping time on a filtered probability space $(\Omega, \mathcal{F}_\infty, (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbb{P})$. We define the stopping time σ-algebra w.r.t. τ as*

$$\mathcal{F}_\tau = \{F \in \mathcal{F}_\infty : F \cap \{\tau \le t\} \in \mathcal{F}_t \text{ for all } t \in \mathbb{T}\}. \tag{12}$$

It is well known and straightforward to show that $\mathcal{F}_\tau$ indeed is a σ-algebra. Note again that $\mathcal{F}_t \subset \mathcal{F}_\infty$ is assumed for $t \in \mathbb{T}$.

DEFINITION 6. *For a measurable space $(\Omega, \mathcal{F})$, we define the set of the minimal elements in the σ-algebra $\mathcal{F}$ by*

$$\mathcal{A}(\mathcal{F}) = \{A \in \mathcal{F} : A \ne \emptyset, \text{ and if } F \in \mathcal{F} \text{ and } F \subset A, \text{ then } F = A\}. \tag{13}$$

Eq. (13) means that $A \in \mathcal{A}(\mathcal{F})$ can not be 'split' in $\mathcal{F}$, which is why elements of $\mathcal{A}(\mathcal{F})$ are also referred to as 'atoms' of $\mathcal{F}$. Obviously, $\mathcal{A}(\mathcal{F}) \subset \mathcal{F}$. For $|\mathcal{F}| < +\infty$, it is therefore easy to see that $\mathcal{A}(\mathcal{F})$ is a partition of Ω and that

$$\mathcal{F} = \sigma(\mathcal{A}(\mathcal{F})) \tag{14}$$

since any non-empty $F \in \mathcal{F}$ can be written as a finite union of elements in $\mathcal{A}(\mathcal{F})$.

DEFINITION 7. *For $t \in \mathbb{T} \cup \{+\infty\}$, we denote the set of minimal elements in $\mathcal{F}_t$ by*

$$\mathcal{A}_t = \mathcal{A}(\mathcal{F}_t) = \{A \in \mathcal{F}_t : A \ne \emptyset, \text{ and if } F \in \mathcal{F}_t \text{ and } F \subset A, \text{ then } F = A\}. \tag{15}$$

Further, we define

$$\mathcal{A}_\tau^t = \{A \in \mathcal{A}_t : A \subset \{\tau = t\}\} \quad (t \in \mathbb{T} \cup \{+\infty\}), \tag{16}$$

$$\mathcal{A}_\tau = \bigcup_{t\in\mathbb{T}\cup\{+\infty\}} \mathcal{A}_\tau^t. \tag{17}$$

Note that the $\mathcal{A}_\tau^t$ are disjoint for $t \in \mathbb{T} \cup \{+\infty\}$.

THEOREM 2. *The elements of $\mathcal{A}_\tau$ are minimal elements of $\mathcal{F}_\tau$, i.e. $\mathcal{A}_\tau \subset \mathcal{A}(\mathcal{F}_\tau)$. If $|\mathcal{F}_\infty| < +\infty$, then $\mathcal{A}_\tau$ is the set of all minimal elements of $\mathcal{F}_\tau$, i.e. $\mathcal{A}_\tau = \mathcal{A}(\mathcal{F}_\tau)$, and $\mathcal{F}_\tau = \sigma(\mathcal{A}_\tau) = \sigma(\mathcal{A}(\mathcal{F}_\tau))$.*

Proof. (i) $\mathcal{A}_\tau \subset \mathcal{F}_\tau$: Let $A \in \mathcal{A}_\tau$, so, for some $s \in \mathbb{T} \cup \{+\infty\}$, $A \in \mathcal{A}_s$ and $A \subset \{\tau = s\}$ and therefore $A \cap \{\tau = s\} = A$. Assume now $t < s$ for some $t \in \mathbb{T}$. Then $A \cap \{\tau \leq t\} = A \cap \{\tau = s\} \cap \{\tau \leq t\} = \emptyset \in \mathcal{F}_t$. For $t \in \mathbb{T}$, assume now $t \geq s$. Then $A \cap \{\tau \leq t\} = A \cap \{\tau = s\} \cap \{\tau \leq t\} = A \cap \{\tau = s\} = A \in \mathcal{F}_s \subset \mathcal{F}_t$. Hence, $A \cap \{\tau \leq t\} \in \mathcal{F}_t$ for all $t \in \mathbb{T}$, and therefore $\mathcal{A}_\tau \subset \mathcal{F}_\tau$. (ii) $A \in \mathcal{A}_\tau$, $F \in \mathcal{F}_\tau$ and $F \subset A$ implies $F = A$: (a) Assume that $A \in \mathcal{A}_\tau^\infty$ and $F \in \mathcal{F}_\tau$ with $F \subset A$. Clearly, $A \in \mathcal{A}_\infty$, implying $F = A$ since $F \in \mathcal{F}_\infty$. (b) Assume $A \in \mathcal{A}_\tau^t$ for some $t \in \mathbb{T}$ and $F \in \mathcal{F}_\tau$ with $F \subset A$. Therefore, $A \in \mathcal{A}_t$ and $F \subset A \subset \{\tau = t\}$, and hence $F \cap \{\tau = t\} = F$. As $F \in \mathcal{F}_\tau$, one has $F \cap \{\tau \leq t\} \in \mathcal{F}_t$. Since $\{\tau = t\} \in \mathcal{F}_t$, $F \cap \{\tau \leq t\} \cap \{\tau = t\} = F \cap \{\tau = t\} = F \in \mathcal{F}_t$, but $A \in \mathcal{A}_t$, and therefore $F = A$. (i) and (ii) prove the first statement of the theorem. Assume now $|\mathcal{F}_\infty| < +\infty$. $\mathcal{A}_\tau$ is then a partition of Ω, because any two distinct sets in $\mathcal{A}_\tau$ are disjoint, and, since $\{\tau = t\} \in \mathcal{F}_t$ for $t \in \mathbb{T} \cup \{+\infty\}$, one has $\bigcup \mathcal{A}_\tau^t = \{\tau = t\}$, so $\bigcup \mathcal{A}_\tau = \Omega$. This proves the second statement. The third statement follows by Eq. (14). $\square$

The following result is well known.

PROPOSITION 1. *For $|\mathcal{F}_\infty| < +\infty$, $\sigma(\tau) \subset \mathcal{F}_\tau$ and, in general, $\sigma(\tau) \neq \mathcal{F}_\tau$.*

Proof. $\{\tau = t\}$ ($t \in \mathbb{T}$) and $\{\tau = +\infty\}$ are the minimal elements of $\sigma(\tau)$ if they are non-empty, but it is well known and straightforward to see that these sets are elements of $\mathcal{F}_\tau$, too. Therefore, $\sigma(\tau) \subset \mathcal{F}_\tau$. It is an easy exercise to find examples where $\sigma(\tau) \neq \mathcal{F}_\tau$ (see example below). $\square$

We can interpret the filtered probability space $(\Omega, \mathcal{F}_\infty, (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbb{P})$ as a random 'experiment' or 'experience' that develops over time.

The common interpretation of the filtration $(\mathcal{F}_t)_{t\in\mathbb{T}}$ is that if it is possible to repeat the experiment arbitrarily often until time $t \in \mathbb{T}$, the maximum information that can be obtained about the experiment is $\mathcal{F}_t$. Hence, $\mathcal{F}_t$ represents (potentially) available information up to time t.

Using Theorem 2 for $|\mathcal{F}_\infty| < +\infty$, we can interpret the stopping time σ-algebra $\mathcal{F}_\tau$ as the maximum information that can be obtained from repeatedly carrying out the experiment up to the random time τ. This is straightforward from the definition of the $\mathcal{A}_\tau^t$ in (16). While this interpretation is, of course, generally known, minimal sets are usually not used to derive it. However, the example below will illustrate that this is a very natural way of interpreting stopping time σ-algebras.

3 Example

In Figure 1, we see the usual interpretation of a discrete time finite space filtration as a stochastic tree. In such a setting, the set of paths of maximal length represents Ω. In this case, $\Omega = \{\omega_1, \ldots, \omega_8\}$. A path up to some node at time t (here $\mathbb{T} = \{0, 1, 2, 3\}$ and $\mathcal{F}_3 = \mathcal{F}_\infty = \mathcal{P}(\Omega)$) represents the set of those paths of full length that have this path up to time t in common. The paths up to time t represent the minimal elements (atoms) $\mathcal{A}_t$ of $\mathcal{F}_t$. For instance, in the example of Fig. 1,

$$\mathcal{A}_1 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \{\omega_5, \ldots, \omega_8\}\}. \tag{18}$$

A stopping time τ (see Fig. 1) and the corresponding stopping process X^τ are given (X^τ is given by the values of the nodes of the tree). The intuitive interpretation of the stopping time as the random time at which X^τ jumps to ('hits') 0 becomes clear (see boxed zeros). Furthermore, the paths up to those boxed zeros represent the minimal elements $\mathcal{A}_\tau$ of the stopping time σ-algebra $\mathcal{F}_\tau$. This follows of course from the fact that for any $A \in \mathcal{A}_\tau^t$ one has $A \subset \{\tau = t\}$ by (16) and therefore $X_t^\tau(A) = 0$, but $X_s^\tau(A) = 1$ for $s < t$. So, in a tree example such as the given one, the 'frontier of first zeros' (or, preciser, the paths leading up to it) describes the stopping time σ-algebra (as well as the stopping time itself). Therefore, it becomes very obvious in what sense $\mathcal{F}_\tau$ contains the information in the system that can be explored up to time τ. In the case of the example,

$$\mathcal{A}_\tau = \{\{\omega_1, \omega_2\}, \{\omega_3\}, \{\omega_4\}, \{\omega_5, \omega_6\}, \{\omega_7\}, \{\omega_8\}\}. \tag{19}$$

It is also easy to see that in this case the minimal elements of $\sigma(\tau)$, denoted by $\mathcal{A}(\sigma(\tau))$, are given by

$$\mathcal{A}(\sigma(\tau)) = \{\{\omega_1, \omega_2\}, \{\omega_5, \omega_6\}, \{\omega_3, \omega_4, \omega_7, \omega_8\}\}. \tag{20}$$

Therefore, $\mathcal{A}_\tau \neq \mathcal{A}(\sigma(\tau))$ and, hence, $\sigma(\tau) \neq \mathcal{F}_\tau$, as stated in Prop. 1.

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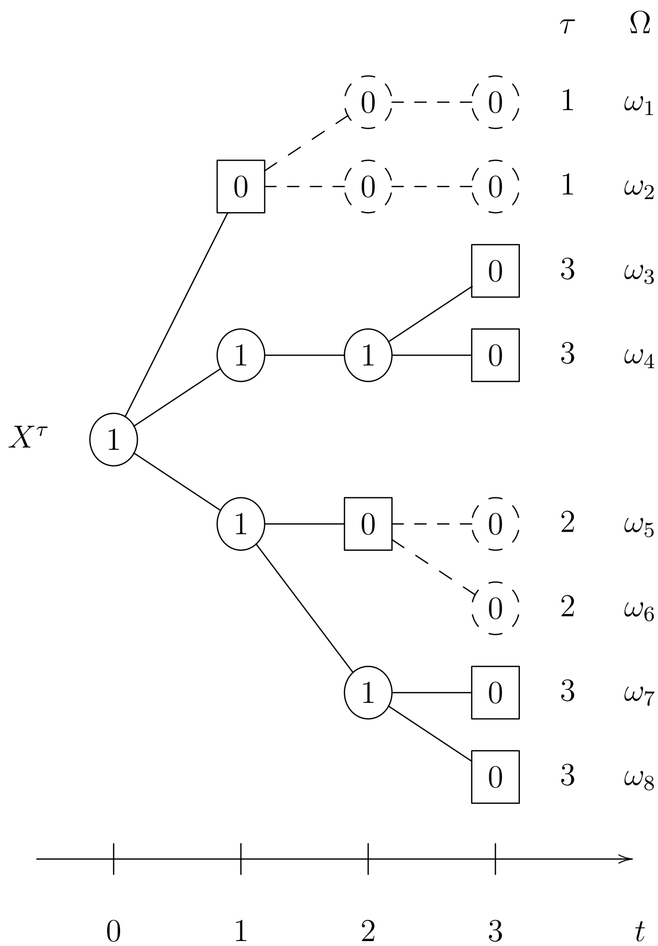

Figure 1: Example.