Removing Algebraic Data Types from Constrained Horn Clauses Using Difference Predicates[*]

Emanuele De Angelis[1,3][0000−0002−7319−8439],
Fabio Fioravanti[1][0000−0002−1268−7829],
Alberto Pettorossi[2,3][0000−0001−7858−4032], and
Maurizio Proietti[3][0000−0003−3835−4931]

[1] DEC, University 'G. d'Annunzio', Chieti-Pescara, Italy
`fabio.fioravanti@unich.it`
[2] DICII, University of Rome 'Tor Vergata', Italy
`pettorossi@info.uniroma2.it`
[3] IASI-CNR, Rome, Italy
`{emanuele.deangelis,maurizio.proietti}@iasi.cnr.it`

Abstract. We address the problem of proving the satisfiability of Constrained Horn Clauses (CHCs) with Algebraic Data Types (ADTs), such as lists and trees. We propose a new technique for transforming CHCs with ADTs into CHCs where predicates are defined over basic types, such as integers and booleans, only. Thus, our technique avoids the explicit use of inductive proof rules during satisfiability proofs. The main extension over previous techniques for ADT removal is a new transformation rule, called *differential replacement*, which allows us to introduce auxiliary predicates corresponding to the lemmas used when making inductive proofs. We present an algorithm that applies the new rule, together with the traditional folding/unfolding rules, for the automatic removal of ADTs. We prove that if the set of the transformed clauses is satisfiable, then so is the set of the original clauses. By an experimental evaluation, we show that the use of the new rule significantly improves the effectiveness of ADT removal, and that our approach is competitive with respect to a state-of-the-art tool that extends the CVC4 solver with induction.

1 Introduction

Constrained Horn Clauses (CHCs) constitute a fragment of the first order predicate calculus, where the Horn clause format is extended by allowing *constraints*

on specific domains to occur in clause premises. CHCs have gained popularity as a suitable logical formalism for automatic program verification [3]. Indeed, many verification problems can be reduced to the satisfiability problem for CHCs.

Satisfiability of CHCs is a particular case of *Satisfiability Modulo Theories* (SMT), understood here as the general problem of determining the satisfiability of (possibly quantified) first order formulas where the interpretation of some function and predicate symbols is defined in a given constraint (or *background*) theory [2]. Recent advances in the field have led to the development of a number of very powerful SMT (and, in particular, CHC) *solvers*, which aim at solving satisfiability problems with respect to a large variety of constraint theories. Among SMT solvers, we would like to mention CVC4 [1], MathSAT [5], and Z3 [13], and among solvers with specialized engines for CHCs, we recall Eldarica [21], HSF [19], RAHFT [25], and Spacer [28].

Even if SMT algorithms for unrestricted first order formulas suffer from incompleteness limitations due to general undecidability results, most of the above mentioned tools work well in practice when acting on constraint theories, such as Booleans, Uninterpreted Function Symbols, Linear Integer or Real Arithmetic, Bit Vectors, and Arrays. However, when formulas contain universally quantified variables ranging over inductively defined *algebraic data types* (ADTs), such as lists and trees, then the SMT/CHC solvers often show a very poor performance, as they do not incorporate induction principles relative to the ADT in use.

To mitigate this difficulty, some SMT/CHC solvers have been enhanced by incorporating appropriate induction principles [38, 43, 44], similarly to what has been done in automated theorem provers [4]. The most creative step which is needed when extending SMT solving with induction, is the generation of the auxiliary lemmas that are required for proving the main conjecture.

An alternative approach, proposed in the context of CHCs [10], consists in transforming a given set of clauses into a new set: (i) where all ADT terms are removed (without introducing new function symbols), and (ii) whose satisfiability implies the satisfiability of the original set of clauses. This approach has the advantage of separating the concern of dealing with ADTs (at transformation time) from the concern of dealing with simpler, non-inductive constraint theories (at solving time), thus avoiding the complex interaction between inductive reasoning and constraint solving. It has been shown [10] that the transformational approach compares well with induction-based tools in the case where lemmas are not needed in the proofs. However, in some satisfiability problems, if suitable lemmas are not provided, the transformation fails to remove the ADT terms.

The main contributions of this paper are as follows.

(1) We extend the transformational approach by proposing a new rule, called *differential replacement,* based on the introduction of suitable *difference predicates*, which play a role similar to that of lemmas in inductive proofs. We prove that the combined use of the fold/unfold transformation rules [16] and the differential replacement rule is *sound*, that is, if the transformed set of clauses is satisfiable, then the original set of clauses is satisfiable.

(2) We develop a transformation algorithm that removes ADTs from CHCs by applying the fold/unfold and the differential replacement rules in a fully automated way.

(3) Due to the <mark>undecidability of the satisfiability problem for CHCs,</mark> our technique for ADT removal is incomplete. Thus, we evaluate its effectiveness from an experimental point of view, and in particular we discuss the results obtained by the implementation of our technique in a tool, called ADTREM. We consider a set of CHC satisfiability problems on ADTs taken from various benchmarks which are used for evaluating inductive theorem provers. The experiments show that ADTREM is competitive with respect to Reynolds and Kuncak's tool that augments the CVC4 solver with inductive reasoning [38].

The paper is structured as follows. In Section 2 we present an introductory, motivating example. In Section 3 we recall some basic notions about CHCs. In Section 4 we introduce the rules used in our transformation technique and, in particular, the novel differential replacement rule, and we show their soundness. In Section 5 we present a transformation algorithm that uses the transformation rules for removing ADTs from sets of CHCs. In Section 6 we illustrate the ADTREM tool and we present the experimental results we have obtained. Finally, in Section 7 we discuss the related work and make some concluding remarks.

2 A Motivating Example

Let us consider the following functional program *Reverse*, which we write using the OCaml syntax [30]:

```
type list = Nil | Cons of int * list;;
let rec append l ys = match l with
  | Nil -> ys     | Cons(x,xs) -> Cons(x,(append xs ys));;
let rec rev l = match l with
  | Nil -> Nil    | Cons(x,xs) -> append (rev xs) (Cons(x,Nil));;
let rec len l = match l with
  | Nil -> 0      | Cons(x,xs) -> 1 + len xs;;
```

The functions `append`, `rev`, and `len` compute list concatenation, list reversal, and list length, respectively. Suppose we want to prove the following property:

$$\forall \texttt{xs,ys. len (rev (append xs ys)) = (len xs) + (len ys)} \qquad (1)$$

Inductive theorem provers construct a proof of this property by induction on the structure of the list `l`, by assuming the knowledge of the following lemma:

$$\forall \texttt{x,l. len (append l (Cons(x,Nil))) = (len l) + 1} \qquad (2)$$

The approach we follow in this paper avoids the explicit use of induction principles and also the knowledge of *ad hoc* lemmas. First, we consider the translation of Property (1) into a set of constrained Horn clauses [10, 43], as follows[4]:

---

[4] In the examples, we use Prolog syntax for writing clauses, instead of the more verbose SMT-LIB syntax. The predicates `\=` (different from), `=` (equal to), `<` (less-than), `>=` (greater-than-or-equal-to) denote constraints between integers. The last argument of a Prolog predicate stores the value of the corresponding function.

```
1. false :- N2\=N0+N1, append(Xs,Ys,Zs), rev(Zs,Rs),
              len(Xs,N0), len(Ys,N1), len(Rs,N2).

2. append([],Ys,Ys).   3. append([X|Xs],Ys,[X|Zs]) :- append(Xs,Ys,Zs).
4. rev([],[]).          5. rev([X|Xs],Rs) :- rev(Xs,Ts), append(Ts,[X],Rs).
6. len([],N) :- N=0.    7. len([X|Xs],N1) :- N1=N0+1, len(Xs,N0).
```

The set of clauses 1–7 is satisfiable if and only if Property (1) holds. However, state-of-the-art CHC solvers, such as Z3 or Eldarica, fail to prove the satisfiability of clauses 1–7, because those solvers do not incorporate any induction principle on lists. Moreover, some tools that extend SMT solvers with induction [38, 43] fail on this particular example because they are not able to introduce Lemma (2).

To overcome this difficulty, we apply the transformational approach based on the fold/unfold rules [10], whose objective is to transform a given set of clauses into a new set without occurrences of list variables, whose satisfiability can be checked by using CHC solvers based on the theory of Linear Integer Arithmetic only. The soundness of the transformation rules ensures that the satisfiability of the transformed clauses implies the satisfiability of the original ones. We apply the *Elimination Algorithm* [10] as follows. First, we introduce a new clause:

```
8. new1(N0,N1,N2) :- append(Xs,Ys,Zs), rev(Zs,Rs),
                     len(Xs,N0), len(Ys,N1), len(Rs,N2).
```

whose body is made out of the atoms of clause 1 which have at least one list variable, and whose head arguments are the integer variables of the body. By folding, from clause 1 we derive a new clause without occurrences of lists:

```
9. false :- N2\=N0+N1, new1(N0,N1,N2).
```

We proceed by eliminating lists from clause 8. By unfolding clause 8, we replace some predicate calls by their definitions and we derive the two new clauses:

```
10. new1(N0,N1,N2) :- N0=0, rev(Zs,Rs), len(Zs,N1), len(Rs,N2).
11. new1(N01,N1,N21) :- N01=N0+1, append(Xs,Ys,Zs), rev(Zs,Rs),
                   len(Xs,N0), len(Ys,N1), append(Rs,[X],R1s), len(R1s,N21).
```

We would like to fold clause 11 using clause 8, so as to derive a recursive definition of new1 without lists. Unfortunately, this folding step cannot be performed because the body of clause 11 does not contain a variant of the body of clause 8, and hence the Elimination Algorithm fails to eliminate lists in this example.

Thus, now we depart from the Elimination Algorithm and we continue our derivation by observing that the body of clause 11 contains the *subconjunction* 'append(Xs,Ys,Zs), rev(Zs,Rs), len(Xs,N0), len(Ys,N1)' of the body of clause 8. Then, in order to find a variant of the whole body of clause 8, we may replace in clause 11 the remaining subconjunction 'append(Rs,[X],R1s), len(R1s,N21)' by the new subconjunction 'len(Rs,N2), diff(N2,X,N21)', where diff is a predicate, called *difference predicate*, defined as follows:

```
12. diff(N2,X,N21) :- append(Rs,[X],R1s), len(R1s,N21), len(Rs,N2).
```

From clause 11, by performing that replacement, we derive the following clause:

```
13. new1(N01,N1,N21) :- N01=N0+1, append(Xs,Ys,Zs), rev(Zs,Rs),
                 len(Xs,N0), len(Ys,N1), len(Rs,N2), diff(N2,X,N21).
```

Now, we can fold clause 13 using clause 8 and we derive a new clause without list arguments:

```
14. new1(N01,N1,N21) :- N01=N0+1, new1(N0,N1,N2), diff(N2,X,N21).
```

At this point, we are left with the task of removing list arguments from clauses 10 and 12. As the reader may verify, this can be done by applying the Elimination Algorithm without the need of introducing additional difference predicates. By doing so, we get the following final set of clauses without list arguments:

```
false :- N2\=N0+N1, new1(N0,N1,N2).
new1(N0,N1,N2) :- N0=0, new2(N1,N2).
new1(N0,N1,N2) :- N0=N+1, new1(N,N1,M), diff(M,X,N2).
new2(M,N) :- M=0, N=0.
new2(M1,N1) :- M1=M+1, new2(M,N), diff(N,X,N1).
diff(N0,X,N1) :- N0=0, N1=1.
diff(N0,X,N1) :- N0=N+1, N1=M+1, diff(N,X,M).
```

The Eldarica CHC solver proves the satisfiability of this set of clauses by computing the following model (here we use a Prolog-like syntax):

```
new1(N0,N1,N2) :- N2=N0+N1, N0>=0, N1>=0, N2>=0.
new2(M,N) :- M=N, M>=0, N>=0.
diff(N,X,M) :- M=N+1, N>=0.
```

Finally, we note that if in clause 12 we substitute the atom `diff(N2,X,N21)` by its model computed by Eldarica, namely the constraint 'N21=N2+1, N2>=0', we get exactly the CHC translation of Lemma (2). Thus, in some cases, the introduction of the difference predicates can be viewed as a way of automatically introducing the lemmas needed when performing inductive proofs.

3   Constrained Horn Clauses

Let *LIA* be the theory of linear integer arithmetic and *Bool* be the theory of boolean values. A *constraint* is a quantifier-free formula of $LIA \cup Bool$. Let $\mathcal{C}$ denote the set of all constraints. Let $\mathcal{L}$ be a typed first order language with equality [15] which includes the language of $LIA \cup Bool$. Let *Pred* be a set of predicate symbols in $\mathcal{L}$ not occurring in the language of $LIA \cup Bool$.

The integer and boolean types are said to be the *basic types*. For reasons of simplicity we do not consider any other basic types, such as real number, arrays, and bit-vectors, which are usually supported by SMT solvers [1, 13, 21]. We assume that all non-basic types are specified by suitable data-type declarations

(such as the `declare-datatypes` declarations adopted by SMT solvers), and are collectively called *algebraic data types* (ADTs).

An *atom* is a formula of the form $p(t_1, \ldots, t_m)$, where p is a typed predicate symbol in *Pred*, and $t_1, \ldots, t_m$ are typed terms constructed out of individual variables, individual constants, and function symbols. A *constrained Horn clause* (or simply, a *clause*, or a CHC) is an implication of the form $A \leftarrow c, B$ (for clauses we use the logic programming notation, where comma denotes conjunction). The conclusion (or *head*) A is either an atom or *false*, the premise (or *body*) is the conjunction of a constraint $c \in \mathcal{C}$, and a (possibly empty) conjunction B of atoms. If A is an atom of the form $p(t_1, \ldots, t_n)$, the predicate p is said to be a *head predicate*. A clause whose head is an atom is called a *definite clause*, and a clause whose head is *false* is called a *goal*.

We assume that, for every atom A which is the head of a clause, (i) each term of basic type occurring in A is a variable, and (ii) no variable of basic type occurs in A more than once. For instance, atom `p(X,[Y|T])` may occur as a head, while by Condition (i), the atoms `p(3,[Y|T])` and `p(X,[Y+Z|T])` may not. Conditions (i) and (ii) on head atoms can always be enforced at the expense of introducing new variables subject to constraints in the body of the clause. These conditions ensure that, when applying the unfolding rule (see Section 4), the unification of terms of basic type can be delegated to constraint solving.

We assume that all variables in a clause are universally quantified in front, and thus we can freely rename those variables. Clause C is said to be a *variant* of clause D if C can be obtained from D by renaming variables and rearranging the order of the atoms in its body. Given a term t, by *vars*(t) we denote the set of all variables occurring in t. Similarly for the set of all free variables occurring in a formula. Given a formula φ in $\mathcal{L}$, we denote by $\forall(\varphi)$ its *universal closure*.

Let $\mathbb{D}$ be the usual interpretation for the symbols in $LIA \cup Bool$, and let a $\mathbb{D}$-*interpretation* be an interpretation of $\mathcal{L}$ that, for all symbols occurring in $LIA \cup Bool$, agrees with $\mathbb{D}$.

A set P of CHCs is *satisfiable* if it has a $\mathbb{D}$-model and it is *unsatisfiable*, otherwise. Given two $\mathbb{D}$-interpretations $\mathbb{I}$ and $\mathbb{J}$, we say that $\mathbb{I}$ is *included* in $\mathbb{J}$ if for all ground atoms A, $\mathbb{I} \models A$ implies $\mathbb{J} \models A$. Every set P of definite clauses is satisfiable and has a *least* (with respect to inclusion) $\mathbb{D}$-model, denoted $M(P)$ [23]. If P is any set of constrained Horn clauses and Q is the set of the goals in P, then we define $Definite(P) = P \setminus Q$. We have that P is satisfiable if and only if $M(Definite(P)) \models Q$.

We will often use tuples of variables as arguments of predicates and write $p(X, Y)$, instead of $p(X_1, \ldots, X_m, Y_1, \ldots, Y_n)$, whenever the values of $m \ (\geq 0)$ and $n \ (\geq 0)$ are not relevant. Whenever the order of the variables is not relevant, we will feel free to identify tuples of distinct variables with finite sets, and we will extend to finite tuples the usual operations and relations on finite sets. Given two tuples X and Y of distinct elements, (i) their union $X \cup Y$ is obtained by concatenating them and removing all duplicated occurrences of elements, (ii) their intersection $X \cap Y$ is obtained by removing from X the elements which do not occur in Y, (iii) their difference $X \backslash Y$ is obtained by removing from X the

elements which occur in Y, and (iv) $X \subseteq Y$ holds if every element of X occurs in Y. For all $m \geq 0$, equality of m-tuples is defined as follows: $(u_1, \ldots, u_m) = (v_1, \ldots, v_m)$ iff $\bigwedge_{i=1}^{m} u_i = v_i$. The empty tuple $()$ is identified with the empty set $\emptyset$.

By $A(X, Y)$, where X and Y are disjoint tuples of distinct variables, we denote an atom A such that $vars(A) = X \cup Y$. Let P be a set of definite clauses. We say that the atom $A(X, Y)$ is *functional from X to Y with respect to P* if

(F1) $M(P) \models \forall X, Y, Z. \ A(X, Y) \wedge A(X, Z) \ \rightarrow \ Y = Z$

The reference to the set P of definite clauses is omitted, when understood from the context. Given a functional atom $A(X, Y)$, we say that X and Y are its *input* and *output* (tuples of) variables, respectively. The atom $A(X, Y)$ is said to be *total from X to Y with respect to P* if

(F2) $M(P) \models \forall X \exists Y. \ A(X, Y)$

If $A(X, Y)$ is a total, functional atom from X to Y, we may write $A(X; Y)$, instead of $A(X, Y)$. For instance, `append(Xs,Ys,Zs)` is a total, functional atom from `(Xs,Ys)` to `Zs` with respect to the set of clauses 1–7 of Section 2.

Now we extend the above notions from atoms to conjunctions of atoms. Let F be a conjunction $A_1(X_1; Y_1), \ldots, A_n(X_n; Y_n)$ such that: (i) $X = (\bigcup_{i=1}^{n} X_i) \setminus (\bigcup_{i=1}^{n} Y_i)$, (ii) $Y = (\bigcup_{i=1}^{n} Y_i)$, and (iii) for $i = 1, \ldots, n$, Y_i is disjoint from $(\bigcup_{j=1}^{i} X_j) \cup (\bigcup_{j=1}^{i-1} Y_j)$. Then, the conjunction F is said to be a *total, functional conjunction from X to Y* and it is also written as $F(X; Y)$. For $F(X; Y)$, the above properties (F1) and (F2) hold if we replace A by F. For instance, `append(Xs,Ys,Zs), rev(Zs,Rs)` is a total, functional conjunction from `(Xs,Ys)` to `(Zs,Rs)` with respect to the set of clauses 1–7 of Section 2.

4 Transformation Rules for Constrained Horn Clauses

In this section we present the rules that we propose for transforming CHCs, and in particular, for introducing difference predicates, and we prove their soundness. We refer to Section 2 for examples of how the rules are applied.

First, we introduce the following notion of a *stratification* for a set of clauses. Let $\mathbb{N}$ denote the set of the natural numbers. A *level mapping* is a function $\ell \colon Pred \rightarrow \mathbb{N}$. For every predicate p, the natural number $\ell(p)$ is said to be the *level* of p. Level mappings are extended to atoms by stating that the level $\ell(A)$ of an atom A is the level of its predicate symbol. A clause $H \leftarrow c, A_1, \ldots, A_n$ is *stratified with respect to ℓ* if, for $i = 1, \ldots, n$, $\ell(H) \geq \ell(A_i)$. A set P of CHCs is *stratified with respect to ℓ* if all clauses in P are stratified with respect to ℓ. Clearly, for every set P of CHCs, there exists a level mapping ℓ such that P is stratified with respect to ℓ [32].

A *transformation sequence from P_0 to P_n* is a sequence $P_0 \Rightarrow P_1 \Rightarrow \ldots \Rightarrow P_n$ of sets of CHCs such that, for $i = 0, \ldots, n-1$, P_{i+1} is derived from P_i, denoted $P_i \Rightarrow P_{i+1}$, by applying one of the following Rules R1–R7. We assume that P_0 is stratified with respect to a given level mapping ℓ.

(R1) *Definition Rule.* Let D be the clause $newp(X_1, \ldots, X_k) \leftarrow c, A_1, \ldots, A_n$, where: (i) $newp$ is a predicate symbol in $Pred$ not occurring in the sequence $P_0 \Rightarrow P_1 \Rightarrow \ldots \Rightarrow P_i$, (ii) c is a constraint, (iii) the predicate symbols of $A_1, \ldots, A_n$

occur in P_0, and (iv) $(X_1, \ldots, X_k) \subseteq vars(c, A_1, \ldots, A_n)$. Then, $P_{i+1} = P_i \cup \{D\}$. We set $\ell(newp) = max\{\ell(A_i) \mid i = 1, \ldots, n\}$.

For $i = 0, \ldots, n$, by $Defs_i$ we denote the set of clauses, called *definitions*, introduced by Rule R1 during the construction of $P_0 \Rightarrow P_1 \Rightarrow \ldots \Rightarrow P_i$. Thus, $\emptyset = Defs_0 \subseteq Defs_1 \subseteq \ldots$. However, by using Rules R2–R7 we can replace a definition in P_i, for $i > 0$, and hence it may happen that $Defs_{i+1} \nsubseteq P_{i+1}$.

(R2) *Unfolding Rule.* Let C: $H \leftarrow c, G_L, A, G_R$ be a clause in P_i, where A is an atom. Without loss of generality, we assume that $vars(C) \cap vars(P_0) = \emptyset$. Let Cls: $\{K_1 \leftarrow c_1, B_1, \ldots, K_m \leftarrow c_m, B_m\}$, $m \geq 0$, be the set of clauses in P_0, such that: for $j = 1, \ldots, m$, (1) there exists a most general unifier ϑ_j of A and K_j, and (2) the conjunction of constraints $(c, c_j)\vartheta_j$ is satisfiable. Let $Unf(C, A, P_0) = \{(H \leftarrow c, c_j, G_L, B_j, G_R)\vartheta_j \mid j = 1, \ldots, m\}$. Then, by *unfolding C with respect to A*, we derive the set $Unf(C, A, P_0)$ of clauses and we get $P_{i+1} = (P_i \setminus \{C\}) \cup Unf(C, A, P_0)$.

When we apply Rule R2, we say that, for $j = 1, \ldots, m$, the atoms in the conjunction $B_j\vartheta_j$ are *derived* from A, and the atoms in the conjunction $(G_L, G_R)\vartheta_j$ are *inherited* from the corresponding atoms in the body of C.

(R3) *Folding Rule.* Let C: $H \leftarrow c, G_L, Q, G_R$ be a clause in P_i, and let D: $K \leftarrow d, B$ be a clause in $Defs_i$. Suppose that: (i) either H is *false* or $\ell(H) \geq \ell(K)$, and (ii) there exists a substitution ϑ such that $Q = B\vartheta$ and $\mathbb{D} \models \forall(c \rightarrow d\vartheta)$. By *folding C using definition D*, we derive clause E: $H \leftarrow c, G_L, K\vartheta, G_R$, and we get $P_{i+1} = (P_i \setminus \{C\}) \cup \{E\}$.

(R4) *Clause Deletion Rule.* Let C: $H \leftarrow c, G$ be a clause in P_i such that the constraint c is unsatisfiable. Then, we get $P_{i+1} = P_i \setminus \{C\}$.

(R5) *Functionality Rule.* Let C: $H \leftarrow c, G_L, F(X; Y), F(X; Z), G_R$ be a clause in P_i, where $F(X; Y)$ is a total, functional conjunction in $Definite(P_0) \cup Defs_i$. By *functionality*, from C we derive clause D: $H \leftarrow c, Y = Z, G_L, F(X; Y), G_R$, and we get $P_{i+1} = (P_i \setminus \{C\}) \cup \{D\}$.

(R6) *Totality Rule.* Let C: $H \leftarrow c, G_L, F(X; Y), G_R$ be a clause in P_i such that $Y \cap vars(H \leftarrow c, G_L, G_R) = \emptyset$ and $F(X; Y)$ is a total, functional conjunction in $Definite(P_0) \cup Defs_i$. By *totality*, from C we derive clause D: $H \leftarrow c, G_L, G_R$ and we get $P_{i+1} = (P_i \setminus \{C\}) \cup \{D\}$.

Since the initial set of clauses is obtained by translating a terminating functional program, the functionality and totality properties hold by construction and we do not need to prove them when we apply Rules R5 and R6.

(R7) *Differential Replacement Rule.* Let C: $H \leftarrow c, G_L, F(X; Y), G_R$ be a clause in P_i, and let D: $diff(Z) \leftarrow d, F(X; Y), R(V; W)$ be a definition clause in $Defs_i$, where: (i) $F(X; Y)$ and $R(V; W)$ are total, functional conjunctions with respect to $Definite(P_0) \cup Defs_i$, (ii) $W \cap vars(C) = \emptyset$, (iii) $\mathbb{D} \models \forall(c \rightarrow d)$, and (iv) $\ell(H) > \ell(diff)$. By *differential replacement*, we derive E: $H \leftarrow c, G_L, R(V; W), diff(Z), G_R$ and we get $P_{i+1} = (P_i \setminus \{C\}) \cup \{E\}$.

Note that no assumption is made on the set Z of variables, apart from the one deriving from the fact that D is a definition, that is, $Z \subseteq vars(d) \cup X \cup Y \cup V \cup W$.

Rule R7 has a very general formulation that eases the proof of the Soundness Theorem (see Theorem 1), which extends to Rules R1–R7 correctness results for transformations of (constraint) logic programs [16, 17, 39, 42]. In the transformation algorithm of Section 5, we will use a specific instance of Rule R7 which is sufficient for ADT removal (see, in particular, the Diff-Introduce step).

Theorem 1 (Soundness). *Let $P_0 \Rightarrow P_1 \Rightarrow \ldots \Rightarrow P_n$ be a transformation sequence using Rules* R1–R7*. Suppose that the following condition holds*:

(U) *for $i=1,\ldots,n-1$, if $P_i \Rightarrow P_{i+1}$ by folding a clause in P_i using a definition $D: H \leftarrow c, B$ in $Defs_i$, then, for some $j \in \{1,\ldots,i-1,i+1,\ldots,n-1\}$, $P_j \Rightarrow P_{j+1}$ by unfolding D with respect to an atom A such that $\ell(H) = \ell(A)$.*

If P_n is satisfiable, then P_0 is satisfiable.

Proof. See Appendix.

Thus, to prove the satisfiability of a set P_0 of clauses, it suffices: (i) to construct a transformation sequence $P_0 \Rightarrow \ldots \Rightarrow P_n$, and then (ii) to prove that P_n is satisfiable. Note, however, that if Rule R7 is used, it may happen that P_0 is satisfiable and P_n is unsatisfiable, that is, some false counterexamples to satisfiability, so-called *false positives*, may be generated, as we now show.

Example 1. Let us consider the following set P_1 of clauses derived by adding the definition clause D to the initial set $P_0 = \{$C,1,2,3$\}$ of clauses:

```
C. false :- X=0, Y>0, a(X,Y).
1. a(X,Y) :- X=<0, Y=0.        2. a(X,Y) :- X>0, Y=1.    3. r(X,W) :- W=1.
D. diff(Y,W) :- a(X,Y), r(X,W).
```

where: (i) a(X,Y) is a total, functional atom from X to Y, (ii) r(X,W) is a total, functional atom from X to W, and (iii) D is a definition in $Defs_1$. By applying Rule R7, from P_1 we derive the set $P_2 = \{$E,1,2,3,D$\}$ of clauses where:

```
E. false :- X=0, Y>0, r(X,W), diff(Y,W).
```

Now we have that P_0 is satisfiable, while P_2 is unsatisfiable.

5   An Algorithm for ADT Removal

Now we present Algorithm $\mathcal{R}$ for eliminating ADT terms from CHCs by using the transformation rules presented in Section 4 and automatically introducing suitable difference predicates. If $\mathcal{R}$ terminates, it transforms a set *Cls* of clauses into a new set *TransfCls* where the arguments of all predicates have basic type. Theorem 1 guarantees that if *TransfCls* is satisfiable, then also *Cls* is satisfiable.

Algorithm $\mathcal{R}$ (see Figure 1) removes ADT terms starting from the set *Gs* of goals in *Cls*. $\mathcal{R}$ collects these goals in *InCls* and stores in *Defs* the definitions of new predicates introduced by Rule R1.

Before describing the procedures used by Algorithm $\mathcal{R}$, let us first introduce the following notions.

Given a conjunction G of atoms, *bvars*(G) (or *adt-vars*(G)) denotes the set of variables in G that have a basic type (or an ADT type, respectively). We say that an atom (or clause) *has basic types* if *all* its arguments (or atoms, respectively)

Algorithm $\mathcal{R}$

Input: A set *Cls* of clauses;
Output: A set *TransfCls* of clauses that have basic types.

Let $Cls = Ds \cup Gs$, where Ds is a set of definite clauses and Gs is a set of goals;
$InCls := Gs$; $Defs := \emptyset$; $TransfCls := \emptyset$;
while $InCls \neq \emptyset$ **do**

- ■ *Diff-Define-Fold*(*InCls*, *Defs*, *NewDefs*, *FldCls*);
- ■ *Unfold*(*NewDefs*, *Ds*, *UnfCls*);
- ■ *Replace*(*UnfCls*, *Ds*, *RCls*);
   $InCls := RCls$; $Defs := Defs \cup NewDefs$; $TransfCls := TransfCls \cup FldCls$;

Fig. 1. The ADT Removal Algorithm $\mathcal{R}$.

have a basic type. An atom (or clause) *has ADTs* if *at least one* of its arguments (or atoms, respectively) has an ADT type.

Given a set (or a conjunction) S of atoms, *SharingBlocks*(S) denotes the partition of S with respect to the reflexive, transitive closure $\Downarrow_S$ of the relation $\downarrow_S$ defined as follows. Given two atoms A_1 and A_2 in S, $A_1 \downarrow_S A_2$ holds iff *adt-vars*$(A_1) \cap$ *adt-vars*$(A_2) \neq \emptyset$. The elements of the partition are called the *sharing blocks* of S.

A *generalization* of a pair (c_1, c_2) of constraints is a constraint $\alpha(c_1, c_2)$ such that $\mathbb{D} \models \forall(c_1 \rightarrow \alpha(c_1, c_2))$ and $\mathbb{D} \models \forall(c_2 \rightarrow \alpha(c_1, c_2))$ [18]. In particular, we consider the following generalization operator based on *widening* [7]. Suppose that c_1 is the conjunction $(a_1, \ldots, a_m)$ of atomic constraints, then $\alpha(c_1, c_2)$ is defined as the conjunction of all a_i's in $(a_1, \ldots, a_m)$ such that $\mathbb{D} \models \forall(c_2 \rightarrow a_i)$. For any constraint c and tuple V of variables, the *projection* of c onto V is a constraint $\pi(c, V)$ such that: (i) *vars*$(\pi(c, V)) \subseteq V$, and (ii) $\mathbb{D} \models \forall(c \rightarrow \pi(c, V))$. In our implementation, $\pi(c, V)$ is computed from $\exists Y.c$, where $Y = vars(c) \setminus V$, by a quantifier elimination algorithm in the theory of booleans and *rational* numbers. This implementation is safe in our context, and avoids relying on modular arithmetic, as is often done when eliminating quantifiers in LIA [37].

For two conjunctions G_1 and G_2 of atoms, $G_1 \lesssim G_2$ holds if $G_1 = (A_1, \ldots, A_n)$ and there exists a subconjunction $(B_1, \ldots, B_n)$ of G_2 (modulo reordering) such that, for $i = 1, \ldots, n$, B_i is an instance of A_i. A conjunction G of atoms is *connected* if it consists of a single sharing block.

■ *Procedure Diff-Define-Fold* (see Figure 2). At each iteration of the body of the **for** loop, the *Diff-Define-Fold* procedure removes the ADT terms occurring in a sharing block B of the body of a clause $C: H \leftarrow c, B, G'$ of *InCls*. This is done by possibly introducing some new definitions (using Rule R1) and applying the Folding Rule R3. To allow folding, some applications of the Differential Replacement Rule R7 may be needed. We have the following four cases.

• (**Fold**). We remove the ADT arguments occurring in B by folding C using a definition D introduced at a previous step. Indeed, the head of each definition introduced by Algorithm $\mathcal{R}$ is by construction a tuple of variables of basic type.

• (**Generalize**). We introduce a new definition $GenD: genp(V) \leftarrow \alpha(d, c), B$ whose constraint is obtained by generalizing (d, c), where d is the constraint

---

Procedure *Diff-Define-Fold*(*InCls*, *Defs*, *NewDefs*, *FldCls*)

Input: A set *InCls* of clauses and a set *Defs* of definitions;

Output: A set *NewDefs* of definitions and a set *FldCls* of clauses with basic types.

$NewDefs := \emptyset$; $FldCls := \emptyset$;

for each clause C: $H \leftarrow c, G$ in *InCls* **do**

  if C has basic types **then** $InCls := InCls \setminus \{C\}$; $FldCls := FldCls \cup \{C\}$

  else

    let C be $H \leftarrow c, B, G'$ where B is a sharing block in G that contains at least one atom that has ADTs;

    • (**Fold**) **if** in $Defs \cup NewDefs$ there is a (variant of) clause D: $newp(V) \leftarrow d, B$ such that $\mathbb{D} \models \forall (c \rightarrow d)$ **then** <u>fold</u> C using D and derive E: $H \leftarrow c, newp(V), G'$;

    • (**Generalize**) **else if** in $Defs \cup NewDefs$ there is a (variant of a) clause $newp(V) \leftarrow d, B$ and $\mathbb{D} \not\models \forall (c \rightarrow d)$ **then**

      <u>introduce definition</u> $GenD$: $genp(V) \leftarrow \alpha(d, c), B$;

      <u>fold</u> C using $GenD$ and derive E: $H \leftarrow c, genp(V), G'$;

      $NewDefs := NewDefs \cup \{GenD\}$;

    • (**Diff-Introduce**) **else if** in $Defs \cup NewDefs$ there is a (variant of a) clause D: $newp(U) \leftarrow d, B'$ such that: (i) $vars(C) \cap vars(D) = \emptyset$, and (ii) $B' \trianglelefteq B$ **then**

      take a maximal subconjunction M of B, if any, such that:

        (i) $B = (M, F(X; Y))$, for some connected conjunction M and non-empty conjunction $F(X; Y)$, (ii) $B'\vartheta = (M, R(V; W))$, for some substitution ϑ such that $W \cap vars(C) = \emptyset$, and (iii) for every atom A in $\{F(X; Y), R(V; W)\}$, $\ell(H) > \ell(A)$;

      <u>introduce definition</u> $\widehat{D}$: $diff(Z) \leftarrow \pi(c, X), F(X; Y), R(V; W)$

        where $Z = bvars(F(X; Y), R(V; W))$;

      $NewDefs := NewDefs \cup \{\widehat{D}\}$;

      <u>replace</u> $F(X; Y)$ by $(R(V; W), diff(Z))$ in C, and derive clause

        C': $H \leftarrow c, M, R(V; W), diff(Z), G'$;

      if $\mathbb{D} \models \forall (c \rightarrow d\vartheta)$

        then <u>fold</u> C' using D and derive E: $H \leftarrow c, newp(U\vartheta), diff(Z), G'$;

        else   <u>introduce definition</u> $GenD$: $genp(U') \leftarrow \alpha(d\vartheta, c), B'\vartheta$

            where $U' = bvars(B'\vartheta)$;

          <u>fold</u> C' using $GenD$ and derive E: $H \leftarrow c, genp(U'), diff(Z), G'$;

          $NewDefs := NewDefs \cup \{GenD\}$;

    • (**Project**) **else**

      <u>introduce definition</u> $ProjC$: $newp(V) \leftarrow \pi(c, V), B$ where $V = bvars(B)$;

      <u>fold</u> C using $ProjC$ and derive clause E: $H \leftarrow c, newp(V), G'$;

      $NewDefs := NewDefs \cup \{ProjC\}$;

  $InCls := (InCls \setminus \{C\}) \cup \{E\}$;

Fig. 2. The *Diff-Define-Fold* Procedure.

occurring in an already available definition whose body is B. Then, we remove the ADT arguments occurring in B by folding C using $GenD$.

• (**Diff-Introduce**). Suppose that B *partially matches* the body of an available definition D: $newp(U) \leftarrow d, B'$, that is, for some substitution ϑ, $B = (M, F(X; Y))$, and $B'\vartheta = (M, R(V; W))$. Then, we introduce a difference predicate through the new definition $\widehat{D}$: $diff(Z) \leftarrow \pi(c, X), F(X; Y), R(V; W)$, where

$Z = bvars(F(X;Y), R(V;W))$ and, by Rule R7, we replace the conjunction $F(X;Y)$ by $(R(V;W), diff(Z))$ in the body of C, thereby deriving C'. Finally, we remove the ADT arguments in B by folding C' using either D or a clause *GenD* whose constraint is a generalization of the pair $(d\vartheta, c)$ of constraints.

The example of Section 2 allows us to illustrate this (Diff-Introduce) case. With reference to that example, clause C: $H \leftarrow c, G$ that we want to fold is clause 11, whose body has the single sharing block B: '`append(Xs,Ys,Zs)`, `rev(Zs,Rs), len(Xs,N0), len(Ys,N1), append(Rs,[X],R1s), len(R1s,N21)`'. Block B partially matches the body '`append(Xs,Ys,Zs), rev(Zs,Rs), len(Xs,N0), len(Ys,N1)`, `len(Rs,N2)`' of clause 8 of Section 2 which plays the role of definition D: $newp(U) \leftarrow d, B'$ in this example. Indeed, we have that:

$M =$ (`append(Xs,Ys,Zs), rev(Zs,Rs), len(Xs,N0), len(Ys,N1)`),
$F(X;Y) =$ (`append(Rs,[X],R1s), len(R1s,N21)`), where $X =$ (`Rs,X`), $Y =$ (`R1s,N21`),
$R(V;W) =$ `len(Rs,N2)`, where $V =$ (`Rs`), $Y =$ (`N2`).

In this example, ϑ is the identity substitution. Moreover, the condition on the level mapping ℓ required in the *Diff-Define-Fold* Procedure of Figure 2 can be fulfilled by stipulating that $\ell(\texttt{new1}) > \ell(\texttt{append})$ and $\ell(\texttt{new1}) > \ell(\texttt{len})$. Thus, the definition $\widehat{D}$ to be introduced is:

```
12. diff(N2,X,N21) :- append(Rs,[X],R1s), len(R1s,N21), len(Rs,N2).
```

Indeed, we have that: (i) the projection $\pi(c, X)$ is $\pi(\texttt{N01=N0+1}, (\texttt{Rs,X}))$, that is, the empty conjunction, (ii) $F(X;Y), R(V;W)$ is the body of clause 12, and (iii) the head variables `N2`, `X`, and `N21` are the integer variables in that body. Then, by applying Rule R7, we replace in clause 11 the conjunction '`append(Rs,[X],R1s)`, `len(R1s,N21)`' by the new conjunction '`len(Rs,N2), diff(N2,X,N21)`', hence deriving clause C', which is clause 13 of Section 2. Finally, by folding clause 13 using clause 8, we derive clause 14 of Section 2, which has no list arguments.

• (**Project**). If none of the previous three cases apply, then we introduce a new definition *ProjC*: $newp(V) \leftarrow \pi(c, V), B$, where $V = bvars(B)$. Then, we remove the ADT arguments occurring in B by folding C using *ProjC*.

The *Diff-Define-Fold* procedure may introduce new definitions with ADTs in their bodies, which are added to *NewDefs* and processed by the *Unfold* procedure. In order to present this procedure, we need the following notions.

The application of Rule R2 is controlled by marking some atoms in the body of a clause as *unfoldable*. If we unfold with respect to atom A clause C: $H \leftarrow c, L, A, R$ the marking of the clauses in $Unf(C, A, Ds)$ is handled as follows: the atoms derived from A are not marked as unfoldable and each atom A'' inherited from an atom A' in the body of C is marked as unfoldable iff A' is marked as unfoldable.

An atom $A(X;Y)$ in a conjunction $F(V;Z)$ of atoms is said to be a *source atom* if $X \subseteq V$. Thus, a source atom corresponds to an innermost function call in a given functional expression. For instance, in clause 1 of Section 2, the source atoms are `append(Xs,Ys,Zs)`, `len(Xs,N0)`, and `len(Ys,N1)`. Indeed, the body of clause 1 corresponds to `len(rev(append xs ys))` $\neq$ `(len xs)+(len ys)`.

An atom $A(X;Y)$ in the body of clause C: $H \leftarrow c, L, A(X;Y), R$ is a *head-instance* with respect to a set Ds of clauses if, for every clause $K \leftarrow d, B$ in Ds such that: (1) there exists a most general unifier ϑ of $A(X;Y)$ and K, and (2) the constraint $(c,d)\vartheta$ is satisfiable, we have that $X\vartheta = X$. Thus, the input variables of $A(X;Y)$ are not instantiated by unification. For instance, the atom `append([X|Xs],Ys,Zs)` is a head-instance, while `append(Xs,Ys,Zs)` is not.

In a set *Cls* of clauses, predicate p *immediately depends on* predicate q, if in *Cls* there is a clause of the form $p(\ldots) \leftarrow \ldots, q(\ldots), \ldots$ The *depends on* relation is the transitive closure of the *immediately depends on* relation. Let $\prec$ be a well-founded ordering on tuples of terms such that, for all terms t, u, if $t \prec u$, then, for all substitutions ϑ, $t\vartheta \prec u\vartheta$. A predicate p is *descending* with respect to $\prec$ if, for all clauses, $p(t;u) \leftarrow c, p_1(t_1;u_1), \ldots, p_n(t_n;u_n)$, for $i = 1, \ldots, n$, if p_i depends on p then $t_i \prec t$. An atom is descending if its predicate is descending. The well-founded ordering $\prec$ we use in our implementation is based on the *subterm* relation and is defined as follows: $(x_1, \ldots, x_k) \prec (y_1, \ldots, y_m)$ if every x_i is a subterm of some y_j and there exists x_i which is a strict subterm of some y_j. For instance, the predicates `append`, `rev`, and `len` in the example of Section 2 are all descending.

■ *Procedure Unfold* (see Figure 3) repeatedly applies Rule R2 in two phases. In Phase 1, the procedure unfolds the clauses in *NewDefs* with respect to at least one source atom. Then, in Phase 2, clauses are unfolded with respect to head-instance atoms. Unfolding is repeated only w.r.t descending atoms. The termination of the *Unfold* procedure is ensured by the fact that the unfolding with respect to a non-descending atom is done at most once in each phase.

Procedure *Unfold*(*NewDefs*, *Ds*, *UnfCls*)
Input: A set *NewDefs* of definitions and a set *Ds* of definite clauses;
Output: A set *UnfCls* of clauses.

UnfCls := *NewDefs*; Mark as unfoldable a nonempty set of source atoms in the body of each clause of *UnfCls*;
- **while** there exists a clause C: $H \leftarrow c, L, A, R$ in *UnfCls*, for some conjunctions L and R, such that A is an unfoldable atom **do**
  UnfCls := (*UnfCls* − {C}) ∪ *Unf*(C, A, Ds);
- Mark as unfoldable all atoms in the body of each clause in *UnfCls*;
- **while** there exists a clause C: $H \leftarrow c, L, A, R$ in *UnfCls*, for some conjunctions L and R, such that A is a head-instance atom with respect to *Ds* and A is either unfoldable or descending **do**
  UnfCls := (*UnfCls* − {C}) ∪ *Unf*(C, A, Ds);

Fig. 3. The *Unfold* Procedure.

■ *Procedure Replace* simplifies some clauses by applying Rules R5 and R6 as long as possible. *Replace* terminates because each application of either rule decreases the number of atoms.

Thus, each execution of the *Diff-Define-Fold*, *Unfold*, and *Replace* procedures terminates. However, Algorithm $\mathcal{R}$ might not terminate because new predicates may be introduced by *Diff-Define-Fold* at each iteration of the **while-do** of $\mathcal{R}$. Soundness of $\mathcal{R}$ follows from soundness of the transformation rules (see Appendix).

Theorem 2 (Soundness of Algorithm $\mathcal{R}$). *Suppose that Algorithm $\mathcal{R}$ terminates for an input set Cls of clauses, and let TransfCls be the output set of clauses. Then, every clause in TransfCls has basic types, and if TransfCls is satisfiable, then Cls is satisfiable.*

Algorithm $\mathcal{R}$ is not complete, in the sense that, even if *Cls* is a satisfiable set of input clauses, then $\mathcal{R}$ may not terminate or, due to the use of Rule R7, it may terminate with an output set *TransfCls* of unsatisfiable clauses, thereby generating a false positive (see Example 1 in Section 4). However, due to well-known undecidability results for the satisfiability problem of CHCs, this limitation cannot be overcome, unless we restrict the class of clauses we consider. The study of such restricted classes of clauses is beyond the scope of the present paper and, instead, in the next section, we evaluate the effectiveness of Algorithm $\mathcal{R}$ from an experimental viewpoint.

6   Experimental Evaluation

In this section we present the results of some experiments we have performed for assessing the effectiveness of our transformation-based CHC solving technique. We compare our technique with the one proposed by Reynolds and Kuncak [38], which extends the SMT solver CHC4 with inductive reasoning.

Implementation. We have developed the ADTREM tool for ADT removal, which is based on an implementation of Algorithm $\mathcal{R}$ in the VeriMAP system [8].

Benchmark suite and experiments. Our benchmark suite consists of 169 verification problems over inductively defined data structures, such as lists, queues, heaps, and trees, which have been adapted from the benchmark suite considered by Reynolds and Kuncak [38]. These problems come from benchmarks used by various theorem provers: (i) 53 problems come from CLAM [22], (ii) 11 from Hip-Spec [6], (iii) 63 from IsaPlanner [14, 24], and (iv) 42 from Leon [41]. We have performed the following experiments, whose results are summarized in Table 1[5]. (1) We have considered Reynolds and Kuncak's **dtt** encoding of the verification problems, where natural numbers are represented using the built-in SMT type *Int*, and we have discarded: (i) problems that do not use ADTs, and (ii) problems that cannot be directly represented in Horn clause format. Since ADTREM does not support higher order functions, nor user-provided lemmas, in order to make a comparison between the two approaches on a level playing field, we have replaced higher order functions by suitable first order instances and we have removed all auxiliary lemmas from the input verification problems. We have also replaced

---

[5] The tool and the benchmarks are available at `https://fmlab.unich.it/adtrem/`.

the basic functions recursively defined over natural numbers, such as the *plus* and *less-or-equal* functions, by LIA constraints.

(2) Then, we have translated each verification problem into a set, call it P, of CHCs in the Prolog-like syntax supported by ADTREM by using a modified version of the SMT-LIB parser of the ProB system [31]. We have run Eldarica and Z3,[6] which use no induction-based mechanism for handling ADTs, to check the satisfiability of P. Rows 'Eldarica' and 'Z3' show the number of solved problems, that is, problems whose CHC encoding has been proved satisfiable.

(3) We have run algorithm $\mathcal{R}$ on P to produce a set T of CHCs without ADTs. Row '$\mathcal{R}$' reports the number of problems for which Algorithm $\mathcal{R}$ terminates.

(4) We have converted T into the SMT-LIB format, and then we have run Eldarica and Z3 for checking its satisfiability. Rows 'Eldarica$_{noADT}$' and 'Z3$_{noADT}$' report outside round parentheses the number of solved problems. There was only one false positive (that is, a satisfiable set P of clauses transformed into an unsatisfiable set T), which we have classified as an unsolved problem.

(5) In order to assess the improvements due to the use of the differential replacement rule we have applied to P a modified version, call it $\mathcal{R}^\circ$, of the ADT removal algorithm $\mathcal{R}$ that *does not* introduce difference predicates, that is, the *Diff-Introduce* case of the *Diff-Define-Fold* Procedure of Figure 2 is never executed. The number of problems for which $\mathcal{R}^\circ$ terminates and the number of solved problems using Eldarica and Z3 are shown within round parentheses in rows '$\mathcal{R}$', 'Eldarica$_{noADT}$', and 'Z3$_{noADT}$', respectively.

(6) Finally, we have run the **cvc4+ig** configuration of the CVC4 solver extended with inductive reasoning [38] on the 169 problems in SMT-LIB format obtained at Step (1). Row 'CVC4+Ind' reports the number of solved problems.

	CLAM	HipSpec	IsaPlanner	Leon	Total
number of problems	53	11	63	42	169
Eldarica	0	2	4	9	15
Z3	6	0	2	10	18
$\mathcal{R}$	(18) 36	(2) 4	(56) 59	(18) 30	(94) 129
Eldarica$_{noADT}$	(18) 32	(2) 4	(56) 57	(18) 29	(94) 122
Z3$_{noADT}$	(18) 29	(2) 3	(55) 56	(18) 26	(93) 114
CVC4+Ind	17	5	37	15	74

Table 1. *Experimental results.* For each problem we have set a timeout limit of 300 seconds. Experiments have been performed on an Intel Xeon CPU E5-2640 2.00GHz with 64GB RAM under CentOS.

Evaluation of Results. The results of our experiments show that ADT removal considerably increases the effectiveness of CHC solvers without inductive reasoning support. For instance, Eldarica is able to solve 15 problems out of 169, while

---

[6] More specifically, Eldarica v2.0.1 and Z3 v4.8.0 with the Spacer engine [27].

it solves 122 problems after the removal of ADTs. When using Z3, the improvement is slightly lower, but still very significant because the number of problems solved by Z3 rises from 18 to 114. Note also that, when the ADT removal terminates (129 problems out of 169), the solvers are very effective (about 95% successful verifications for Eldarica and about 88% for Z3). The improvements specifically due to the use of the difference replacement rule are demonstrated by the increase of the number of problems for which the ADT removal algorithm terminates (from 94 to 129), and of the number of problems solved (from 94 to 122, for Eldarica, and from 93 to 114, for Z3).

ADTREM compares favorably to CVC4 extended with induction (compare rows 'Eldarica$_{noADT}$' and 'Z3$_{noADT}$' to row 'CVC4+Ind'). Interestingly, the effectiveness of CVC4 may be increased if one extends the problem formalization with extra lemmas which may be used for proving the main conjecture. Indeed, CVC4 solves 100 problems when auxiliary lemmas are added, and 134 problems when, in addition, it runs on the **dti** encoding, where natural numbers are represented using both the built-in type *Int* and the ADT definition with the zero and successor constructors. Our results show that in most cases ADTREM needs neither those extra axioms nor that sophisticated encoding.

Finally, in Table 2 we report some problems solved by ADTREM with Eldarica that are not solved by CVC4 with induction (using any encoding and auxiliary lemmas), or vice versa. For details, see `https://fmlab.unich.it/adtrem/`.

Problem	Property proved by ADTREM *and not by* CVC4
CLAM goal6	$\forall x, y.\, len(rev(append(x, y))) = len(x) + len(y)$
CLAM goal49	$\forall x.\, mem(x, sort(y)) \Rightarrow mem(x, y)$
IsaPlanner goal52	$\forall n, l.\, count(n, l) = count(n, rev(l))$
IsaPlanner goal80	$\forall l.\, sorted(sort(l))$
Leon heap-goal13	$\forall x, l.\, len(qheapsorta(x, l)) = hsize(x) + len(l)$

Problem	Property proved by CVC4 *and not by* ADTREM
CLAM goal18	$\forall x, y.\, rev(append(rev(x), y)) = append(rev(y), x)$
HipSpec rev-equiv-goal4	$\forall x, y.\, qreva(qreva(x, y), nil) = qreva(y, x)$
HipSpec rev-equiv-goal6	$\forall x, y, z.\, append(qreva(x, y), z) = qreva(x, append(y, z))$

Table 2. A comparison between ADTREM with Eldarica and CVC4 with induction.

7   Related Work and Conclusions

Inductive reasoning is supported, with different degrees of human intervention, by many theorem provers, such as ACL2 [26], CLAM [22], Isabelle [33], HipSpec [6], Zeno [40], and PVS [34]. The combination of inductive reasoning and SMT solving techniques has been exploited by many tools for program verification [29, 36, 38, 41, 43, 44].

Leino [29] integrates inductive reasoning into the Dafny program verifier by implementing a simple strategy that rewrites user-defined properties that may benefit from induction into proof obligation to be discharged by Z3. The advantage of this technique is that it fully decouples inductive reasoning from SMT solving. Hence, no extensions to the SMT solver are required.

In order to extend CVC4 with induction, Reynolds and Kuncak [38] also consider the rewriting of formulas that may take advantage from inductive reasoning, but this is done dynamically, during the proof search. This approach allows CVC4 to perform the rewritings lazily, whenever new formulas are generated during the proof search, and to use the partially solved conjecture, to generate lemmas that may help in the proof of the initial conjecture.

The issue of generating suitable lemmas during inductive proofs has been also addressed by Yang et al. [44] and implemented in ADTIND. In order to conjecture new lemmas, their algorithm makes use of a syntax-guided synthesis strategy driven by a grammar, which is dynamically generated from user-provided templates and the function and predicate symbols encountered during the proof search. The derived lemma conjectures are then checked by the SMT solver Z3.

In order to take full advantage of the efficiency of SMT solvers in checking satisfiability of quantifier-free formulas over LIA, ADTs, and finite sets, the Leon verification system [41] implements an SMT-based solving algorithm to check the satisfiability of formulas involving recursively defined first-order functions. The algorithm interleaves the unrolling of recursive functions and the SMT solving of the formulas generated by the unrolling. Leon can be used to prove properties of Scala programs with ADTs and integrates with the Scala compiler and the SMT solver Z3. A refined version of that algorithm, restricted to *catamorphisms*, has been implemented into a solver-agnostic tool, called RADA [36].

In the context of CHCs, Unno et al. [43] have proposed a proof system that combines inductive theorem proving with SMT solving. This approach uses Z3-PDR [20] to discharge proof obligations generated by the proof system, and has been applied to prove relational properties of OCaml programs.

The distinctive feature of the technique presented in this paper is that it does not make use of any explicit inductive reasoning, but it follows a transformational approach. First, the problem of verifying the validity of a universally quantified formula over ADTs is reduced to the problem of checking the satisfiability of a set of CHCs. Then, this set of CHCs is transformed with the aim of deriving a set of CHCs over basic types (i.e., integers) only, whose satisfiability implies the satisfiability of the original set. In this way, the reasoning on ADTs is separated from the reasoning on satisfiability, which can be performed by specialized engines for CHCs on basic types (e.g. Eldarica [21] and Z3-Spacer [28]). Some of the ideas presented here have been explored in [11, 12], but there neither formal results nor an automated strategy were presented.

A key success factor of our technique is the introduction of difference predicates, which can be viewed as the transformational counterpart of lemma generation. Indeed, as shown in Section 6, the use of difference predicates greatly

increases the power of CHC solving with respect to previous techniques based on the transformational approach, which do not use difference predicates [10].

As future work, we plan to apply our transformation-based verification technique to more complex program properties, such as relational properties [9].

References

1. C. Barrett, C. L. Conway, M. Deters, L. Hadarean, D. Jovanovic, T. King, A. Reynolds, and C. Tinelli. CVC4. In G. Gopalakrishnan and S. Qadeer, editors, *Computer Aided Verification, Proceedings of the 23rd International Conference CAV '11, Snowbird, UT, USA, July 14–20, 2011*, Lecture Notes in Computer Science 6806, pages 171–177. Springer, 2011.

2. C. W. Barrett and C. Tinelli. Satisfiability modulo theories. In E. M. Clarke, T. A. Henzinger, H. Veith, and R. Bloem, editors, *Handbook of Model Checking*, pages 305–343. Springer, 2018.

3. N. Bjørner, A. Gurfinkel, K. L. McMillan, and A. Rybalchenko. Horn clause solvers for program verification. In L. D. Beklemishev, A. Blass, N. Dershowitz, B. Finkbeiner, and W. Schulte, editors, *Fields of Logic and Computation II - Essays Dedicated to Yuri Gurevich on the Occasion of His 75th Birthday*, Lecture Notes in Computer Science 9300, pages 24–51, Switzerland, 2015. Springer.

4. A. Bundy. The automation of proof by mathematical induction. In A. Robinson and A. Voronkov, editors, *Handbook of Automated Reasoning*, volume I, pages 845–911. North Holland, 2001.

5. A. Cimatti, A. Griggio, B. Schaafsma, and R. Sebastiani. The MathSAT5 SMT Solver. In N. Piterman and S. Smolka, editors, *Proceedings of the 19th International Conference on Tools and Algorithms for the Construction and Analysis of Systems, TACAS '13*, Lecture Notes in Computer Science 7795, pages 93–107. Springer, 2013.

6. K. Claessen, M. Johansson, D. Rosén, and N. Smallbone. Automating inductive proofs using theory exploration. In M. P. Bonacina, editor, *Automated Deduction – CADE-24, Proceedings of the 24th International Conference on Automated Deduction, Lake Placid, NY, USA, June 9–14, 2013*, Lecture Notes in Artificial Intelligence 7898, pages 392–406. Springer, 2013.

7. P. Cousot and N. Halbwachs. Automatic discovery of linear restraints among variables of a program. In *Proceedings of the Fifth ACM Symposium on Principles of Programming Languages, POPL '78*, pages 84–96. ACM, 1978.

8. E. De Angelis, F. Fioravanti, A. Pettorossi, and M. Proietti. VeriMAP: A Tool for Verifying Programs through Transformations. In *Proceedings of the 20th International Conference on Tools and Algorithms for the Construction and Analysis of Systems, TACAS '14*, Lecture Notes in Computer Science 8413, pages 568–574. Springer, 2014. Available at http://www.map.uniroma2.it/VeriMAP.

9. E. De Angelis, F. Fioravanti, A. Pettorossi, and M. Proietti. Relational verification through Horn clause transformation. In X. Rival, editor, *Proceedings of the 23rd International Symposium on Static Analysis, SAS '16, Edinburgh, UK, September 8–10, 2016*, Lecture Notes in Computer Science 9837, pages 147–169. Springer, 2016.

10. E. De Angelis, F. Fioravanti, A. Pettorossi, and M. Proietti. Solving Horn clauses on inductive data types without induction. *Theory and Practice of Logic Programming*, 18(3-4):452–469, 2018. Special Issue on *ICLP '18*.

11. E. De Angelis, F. Fioravanti, A. Pettorossi, and M. Proietti. Lemma generation for Horn clause satisfiability: A preliminary study. In A. Lisitsa and A. P. Nemytykh, editors, *Proceedings Seventh International Workshop on Verification and Program Transformation, VPT@Programming 2019, Genova, Italy, 2nd April 2019*, volume 299 of *EPTCS*, pages 4–18, 2019.

12. E. De Angelis, F. Fioravanti, A. Pettorossi, and M. Proietti. Proving properties of sorting programs: A case study in Horn clause verification. In E. De Angelis, G. Fedyukovich, N. Tzevelekos, and M. Ulbrich, editors, *Proceedings of the Sixth Workshop on Horn Clauses for Verification and Synthesis and Third Workshop on Program Equivalence and Relational Reasoning, HCVS/PERR@ETAPS 2019, Prague, Czech Republic, 6–7th April 2019*, volume 296 of *EPTCS*, pages 48–75, 2019.

13. L. M. de Moura and N. Bjørner. Z3: An efficient SMT solver. In *Proceedings of the 14th International Conference on Tools and Algorithms for the Construction and Analysis of Systems, TACAS '08*, Lecture Notes in Computer Science 4963, pages 337–340. Springer, 2008.

14. L. Dixon and J. D. Fleuriot. IsaPlanner: A prototype proof planner in Isabelle. In F. Baader, editor, *Automated Deduction – CADE-19, Proceedings of the 19th International Conference on Automated Deduction, Miami Beach, FL, USA, July 28 – August 2, 2003*, Lecture Notes in Computer Science 2741, pages 279–283. Springer, 2003.

15. H. Enderton. *A Mathematical Introduction to Logic.* Academic Press, 1972.

16. S. Etalle and M. Gabbrielli. Transformations of CLP modules. *Theoretical Computer Science*, 166:101–146, 1996.

17. F. Fioravanti, A. Pettorossi, and M. Proietti. Transformation rules for locally stratified constraint logic programs. In K.-K. Lau and M. Bruynooghe, editors, *Program Development in Computational Logic*, Lecture Notes in Computer Science 3049, pages 292–340. Springer-Verlag, 2004.

18. F. Fioravanti, A. Pettorossi, M. Proietti, and V. Senni. Generalization strategies for the verification of infinite state systems. *Theory and Practice of Logic Programming. Special Issue on the 25th Annual GULP Conference*, 13(2):175–199, 2013.

19. S. Grebenshchikov, N. P. Lopes, C. Popeea, and A. Rybalchenko. Synthesizing software verifiers from proof rules. In *Proceedings of the ACM SIGPLAN Conference on Programming Language Design and Implementation, PLDI '12*, pages 405–416, 2012.

20. K. Hoder and N. Bjørner. Generalized property directed reachability. In A. Cimatti and R. Sebastiani, editors, *Proceedings of the 15th International Conference on Theory and Applications of Satisfiability Testing, SAT '12*, Lecture Notes in Computer Science 7317, pages 157–171. Springer, 2012.

21. H. Hojjat and P. Rümmer. The ELDARICA Horn solver. In N. Bjørner and A. Gurfinkel, editors, *2018 Formal Methods in Computer Aided Design, FMCAD 2018, Austin, TX, USA, October 30 – November 2, 2018*, pages 1–7. IEEE, 2018.

22. A. Ireland and A. Bundy. Productive use of failure in inductive proof. *Journal of Automated Reasoning*, 16(1):79–111, Mar. 1996.

23. J. Jaffar and M. Maher. Constraint logic programming: A survey. *Journal of Logic Programming*, 19/20:503–581, 1994.

24. M. Johansson, L. Dixon, and A. Bundy. Case-analysis for rippling and inductive proof. In M. Kaufmann and L. C. Paulson, editors, *Interactive Theorem Proving*, Lecture Notes in Computer Science 6172, pages 291–306. Springer, 2010.

25. B. Kafle, J. P. Gallagher, and J. F. Morales. RAHFT: A tool for verifying Horn clauses using abstract interpretation and finite tree automata. In *Computer Aided Verification, Proceedings of the 28th International Conference CAV '16, Toronto, ON, Canada, July 17–23, 2016, Proceedings, Part I*, Lecture Notes in Computer Science 9779, pages 261–268. Springer, 2016.

26. M. Kaufmann, P. Manolios, and J. S. Moore. *Computer-Aided Reasoning: An Approach.* Kluwer Academic Publishers, 2000.

27. A. Komuravelli, A. Gurfinkel, and S. Chaki. SMT-based model checking for recursive programs. In *Computer Aided Verification, Proceedings of the 26th International Conference CAV '14, Vienna, Austria, July 18–22, 2014*, Lecture Notes in Computer Science 8559, pages 17–34. Springer, 2014.

28. A. Komuravelli, A. Gurfinkel, S. Chaki, and E. M. Clarke. Automatic abstraction in SMT-based unbounded software model checking. In N. Sharygina and H. Veith, editors, *Computer Aided Verification, Proceedings of the 25th International Conference CAV '13, Saint Petersburg, Russia, July 13–19, 2013*, Lecture Notes in Computer Science 8044, pages 846–862. Springer, 2013.

29. K. Leino. Automating induction with an SMT solver. In V. Kuncak and A. Rybalchenko, editors, *Verification, Model Checking, and Abstract Interpretation, Proceedings of the 13th International Conference VMCAI 2012, Philadelphia, PA, USA, January 22–24, 2012*, pages 315–331. Springer, 2012.

30. X. Leroy, D. Doligez, A. Frisch, J. Garrigue, D. Rémy, and J. Vouillon. The OCaml system, Release 4.06. Documentation and user's manual, Institut National de Recherche en Informatique et en Automatique, France, 2017.

31. M. Leuschel and M. Butler. ProB: A model checker for B. In *FME 2003: Formal Methods*, pages 855–874. Springer, 2003.

32. J. W. Lloyd. *Foundations of Logic Programming.* Springer-Verlag, Berlin, 1987. Second Edition.

33. T. Nipkow, M. Wenzel, and L. C. Paulson. *Isabelle/HOL: A Proof Assistant for Higher-Order Logic.* Springer, 2002.

34. S. Owre, J. M. Rushby, and N. Shankar. PVS: A prototype verification system. In D. Kapur, editor, *Automated Deduction – CADE-11, Proceedings of the 11th International Conference on Automated Deduction, Saratoga Springs, NY, USA, June 15–18, 1992*, pages 748–752. Springer, 1992.

35. A. Pettorossi and M. Proietti. Totally correct logic program transformations via well-founded annotations. *Higher-Order and Symbolic Computation*, 21:193–234, 2008.

36. T.-H. Pham, A. Gacek, and M. W. Whalen. Reasoning about algebraic data types with abstractions. *J. Autom. Reason.*, 57(4):281–318, Dec. 2016.

37. M. O. Rabin. Decidable theories. In J. Barwise, editor, *Handbook of Mathematical Logic*, pages 595–629. North-Holland, 1977.

38. A. Reynolds and V. Kuncak. Induction for SMT solvers. In D. D'Souza, A. Lal, and K. G. Larsen, editors, *Verification, Model Checking, and Abstract Interpretation, Proceedings of the 16th International Conference VMCAI 2015, Mumbai, India, January 12–14, 2015*, Lecture Notes in Computer Science 8931, pages 80–98. Springer, 2015.

39. H. Seki. On inductive and coinductive proofs via unfold/fold transformations. In D. De Schreye, editor, *Proceedings of the 19th International Symposium on Logic Based Program Synthesis and Transformation (LOPSTR '09), Coimbra, Portugal, September 9–11, 2009*, Lecture Notes in Computer Science 6037, pages 82–96. Springer, 2010.

40. W. Sonnex, S. Drossopoulou, and S. Eisenbach. Zeno: An automated prover for properties of recursive data structures. In C. Flanagan and B. König, editors, *Proceedings of the 18th International Conference on Tools and Algorithms for the Construction and Analysis of Systems, TACAS '12, Tallinn, Estonia, March 24 – April 1, 2012*, pages 407–421. Springer, 2012.

41. P. Suter, A. S. Köksal, and V. Kuncak. Satisfiability modulo recursive programs. In E. Yahav, editor, *Proceedings of the 18th International Symposium on Static Analysis, SAS '11*, Lecture Notes in Computer Science 6887, pages 298–315. Springer, 2011.

42. H. Tamaki and T. Sato. A generalized correctness proof of the unfold/fold logic program transformation. Technical Report 86-4, Ibaraki University, Japan, 1986.

43. H. Unno, S. Torii, and H. Sakamoto. Automating induction for solving Horn clauses. In R. Majumdar and V. Kuncak, editors, *Computer Aided Verification, Proceedings of the 29th International Conference CAV '17, Heidelberg, Germany, Part II*, Lecture Notes in Computer Science 10427, pages 571–591. Springer, 2017.

44. W. Yang, G. Fedyukovich, and A. Gupta. Lemma synthesis for automating induction over algebraic data types. In T. Schiex and S. de Givry, editors, *Principles and Practice of Constraint Programming, Proceedings of the 25th International Conference CP 2019, Stamford, CT, USA, September 30 – October 4, 2019*, Lecture Notes in Computer Science 11802, pages 600–617. Springer, 2019.

8 Appendix

In this appendix we show the proofs of the results presented in Sections 4 and 5. First, we recall some definitions and facts from the literature [35]. The least $\mathbb{D}$-model of a set P of clauses is the set, denoted $M(P)$, of all ground atoms which are true in P [23].

A *reverse-implication-based transformation sequence* is a sequence $P_0 \Rightarrow P_1 \Rightarrow \ldots \Rightarrow P_n$ of sets of clauses where, for $i = 0, \ldots, n{-}1$, P_{i+1} is derived from P_i by applying one of the following rules (see Section 4): Definition (Rule R1), Unfolding (Rule R2), Folding (Rule R3), and the following rule, called *Body Weakening* (Rule W).

(Rule W) *Body Weakening.* Let C: $H \leftarrow c, c_1, G_L, G_1, G_R$ be a clause in P_i, and suppose that the following condition holds for some constraint c_2 and conjunction G_2 of atoms:

$$M(Definite(P_0) \cup Defs_i) \models \forall (c_1 \wedge G_1 \rightarrow \exists Y. c_2 \wedge G_2)$$

where $Y = vars(\{c_2, G_2\}) \setminus vars(\{H, c, G_L, G_R\})$. Suppose also that $\ell(H) > \ell(A)$, for every atom A occurring in G_2 and not in G_1. By *body weakening*, from clause C we derive clause D: $H \leftarrow c, c_2, G_L, G_2, G_R$, and we get $P_{i+1} = (P_i \setminus \{C\}) \cup \{D\}$.

Theorem 3. *If $P_0 \Rightarrow \ldots \Rightarrow P_n$ is a reverse-implication-based transformation sequence for which Condition* (U) *of Theorem* 1 *holds. For $i = 0, \ldots, n$, let $\mathcal{D}_i = Definite(P_i)$. Then $M(\mathcal{D}_0 \cup Defs_n) \subseteq M(\mathcal{D}_n)$.*

The proof of this theorem [35] is based on the fact that, for $i = 0, \ldots, n{-}1$, $M(\mathcal{D}_0 \cup Defs_i) \models \mathcal{D}_i \leftarrow \mathcal{D}_{i+1}$, hence the term *reverse-implication-based* transformation sequence. Note that, in particular, if we apply the body weakening rule to a clause C of a given set P_i of clauses whereby we replace a conjunction G_1 in the body of C by a new a conjunction G_2 such that $M(\mathcal{D}_0 \cup Defs_i) \models \forall (G_1 \rightarrow G_2)$, then we derive a new clause D such that $M(\mathcal{D}_0 \cup Defs_i) \models C \leftarrow D$.

Theorem 4. *Suppose that $P_0 \Rightarrow \ldots \Rightarrow P_n$ is a reverse-implication-based transformation sequence and Condition* (U) *of Theorem* 1 *holds. If P_n is satisfiable, then P_0 is satisfiable.*

Proof. First, we observe that P_0 is satisfiable iff $P_0 \cup Defs_n$ is satisfiable. Indeed, we have that: (i) if $\mathcal{M}$ is a $\mathbb{D}$-model of P_0, then the $\mathbb{D}$-interpretation $\mathcal{M} \cup \{newp(a_1, \ldots, a_k) \mid newp$ is a head predicate in $Defs_n$ and $a_1, \ldots, a_k$ are ground terms$\}$ is a $\mathbb{D}$-model of $P_0 \cup Defs_n$, and (ii) if $\mathcal{M}$ is a $\mathbb{D}$-model of $P_0 \cup Defs_n$, then all clauses of P_0 are true in $\mathcal{M}$, and hence $\mathcal{M}$ is a $\mathbb{D}$-model of P_0.

Then, let us consider a new sequence $P'_0 \Rightarrow \ldots \Rightarrow P'_n$ obtained from the transformation sequence $P_0 \Rightarrow \ldots \Rightarrow P_n$ by replacing each occurrence of *false* in the head of a clause by a fresh, new predicate symbol, say f. $P'_0, \ldots, P'_n$ are sets of definite clauses, and thus, for $i = 0, \ldots, n$, $Definite(P'_i) = P'_i$. The sequence $P'_0 \Rightarrow \ldots \Rightarrow P'_n$ satisfies the hypotheses of Theorem 3, and hence $M(P'_0) \cup Defs_n \subseteq M(P'_n)$. We have that:

P_n is satisfiable
implies $P'_n \cup \{\neg f\}$ is satisfiable
implies $f \notin M(P'_n)$
implies, by Theorem 3, $f \notin M(P'_0) \cup Defs_n$
implies $P'_0 \cup Defs_n \cup \{\neg f\}$ is satisfiable
implies $P_0 \cup Defs_n$ is satisfiable
implies P_0 is satisfiable. $\hfill\square$

Now, (i) in order to prove Theorem 1 of Section 4, which states the soundness of Rules R1–R7, we show that Rules R4–R7 are all instances of Rule W.

An application of Rule R4 (Clause Deletion) whereby clause C: $H \leftarrow c, G$ is deleted whenever the constraint c is unsatisfiable, is equivalent to the replacement of the body of clause C by *false*. Indeed, if c is unsatisfiable, we have that:

$M(\mathit{Definite}(P_0) \cup Defs_i) \models \forall (c \wedge G \rightarrow \mathit{false})$

and the applicability condition of Rule W trivially holds. Also the condition:

$\ell(H) > \ell(A)$, for every atom A in *false*

trivially holds, because there are no atoms in *false*. Thus, the replacement of the body of clause $H \leftarrow c, G$ by *false* can be performed by applying Rule W.

Let us now consider Rule R5 (Functionality). Recall that by $F(X;Y)$ we denote a conjunction of atoms that defines a total, functional relation from X to Y. When Rule R5 is applied whereby a conjunction $F(X;Y), F(X;Z)$ is replaced by $Y = Z, F(X;Y)$, it is the case that

$M(\mathit{Definite}(P_0) \cup Defs_i) \models \forall (F(X;Y) \wedge F(X;Z) \rightarrow Y = Z)$

holds. When this replacement is performed, also the applicability condition of Rule W on the levels of atoms trivially holds, and thus Rule R5 is an instance of Rule W.

An application of Rule R6 (Totality) replaces a conjunction $F(X;Y)$ by *true* (that is, the empty conjunction), which is implied by any formula. Hence, Rule R6 is an instance of Rule W.

Rule R7 (Differential Replacement), is an instance of Rule W as a consequence of the following lemma.

Lemma 1. *Let us consider a transformation sequence $P_0 \Rightarrow \ldots \Rightarrow P_i$ and a clause C: $H \leftarrow c, G_L, F(X;Y), G_R$ in P_i. Let us assume that by applying Rule R7 on clause C using the definition clause*

D: $\mathit{diff}(Z) \leftarrow d, F(X;Y), R(V;W)$,

where: *(i)* $W \cap vars(C) = \emptyset$, *and (ii)* $\mathbb{D} \models \forall (c \rightarrow d)$, *we derive clause*

E: $H \leftarrow c, G_L, R(V;W), \mathit{diff}(Z), G_R$

and we get the new set $P_{i+1} = (P_i \setminus \{C\}) \cup \{E\}$ of clauses. Then,

$M(\mathit{Definite}(P_0) \cup Defs_i) \models \forall (c \wedge F(X;Y) \rightarrow \exists W. (R(V;W) \wedge \mathit{diff}(Z)))$.

Proof. Let $\mathcal{M}$ denote $M(\mathit{Definite}(P_0) \cup \mathit{Defs}_i)$. Since $R(V;W)$ is a total, functional conjunction from V to W with respect to $\mathit{Definite}(P_0) \cup \mathit{Defs}_i$, we have:

$$\mathcal{M} \models \forall \, (c \wedge F(X;Y) \to \exists W.\, R(V;W)) \tag{1}$$

Since, by Condition (i), none of the variables in W occurs in C, from definition D, we get:

$$\mathcal{M} \models \forall \, (d \wedge F(X;Y) \wedge R(V;W) \to \mathit{diff}(Z)) \tag{2}$$

From (1) and (2), by Condition (ii), we get the thesis. $\qquad\square$

From Lemma 1 it follows that Rule R7, which replaces in the body of clause C: $H \leftarrow c, G_L, F(X;Y), G_R$ the conjunction $F(X;Y)$ by the new conjunction $R(V;W), \mathit{diff}(Z)$, whenever Condition $\ell(H) > \ell(\mathit{diff})$ holds, is an instance of Rule W.

Having proved Lemma 1), we have that Rules R4–R7 are all instances of Rule W. Thus, we get the following lemma.

Lemma 2. *Suppose that $P_0 \Rightarrow \ldots \Rightarrow P_n$ is a transformation sequence using Rules* R1–R7, *and suppose also that Condition* (U) *of Theorem* 1 *holds. Then $P_0 \Rightarrow \ldots \Rightarrow P_n$ is a reverse-implication-based transformation sequence.*

Now Theorem 1 of Section 4 follows from Lemma 2 and Theorem 4. $\qquad\square$

Finally, we prove the soundness of Algorithm $\mathcal{R}$, that is, Theorem 2 of Section 5.

Each procedure used in Algorithm $\mathcal{R}$ consists of a sequence of applications of Rules R1–R7. Thus, the thesis follows from Theorem 1, if Condition (U) of the hypothesis of that theorem holds. This condition is satisfied if for each application of the *Unfold* procedure, the nonempty set of source atoms that are marked as unfoldable in the body of a clause, includes at least one atom with the same level as the head of the clause. This property can always be enforced by dynamically constructing the level mapping ℓ during the execution of Algorithm $\mathcal{R}$. $\qquad\square$