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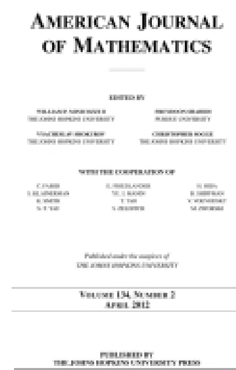
Bélair, Luc.
Macintyre, A. (Angus), 1941-
Scanlon, Thomas.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MODEL THEORY OF THE FROBENIUS ON THE WITT VECTORS

By Luc Bélair, Angus Macintyre, and Thomas Scanlon

Abstract. We give axiomatizations and prove quantifier elimination theorems for first-order theories of unramified valued fields with an automorphism having a close interaction with the valuation. We achieve an analogue of the classical Ostrowski theory of pseudoconvergence. In the outstanding case of Witt vectors with their Frobenius map, we use the ∂-ring formalism from Joyal.

0. Introduction. Our main objective is to understand the model theory of the rings of Witt vectors carrying the (relative) Frobenius automorphism. As generally happens in model theory, the objective is achieved by studying a much wider class of models, most of which have no particular mathematical interest.

A model for our enterprise is the work of Ax-Kochen and Ershov, henceforth AEK. In a fundamental series of papers [2], [13] they studied the model theory of henselian valued fields (K, v, k, Γ), where v: $K^* \to \Gamma$ is a henselian valuation with residue field k, subject only to the restrictions:

(a) K has characteristic 0

(b) if k has finite characteristic p then $v(p)$ is the least positive element in the value group.

We call valued fields satisfying conditions (a) and (b) *unramified*.

In this case, AEK showed that the theory of K is determined by those of Γ and k.

The most important case is when $K = \mathbb{Q}_p$ or an unramified algebraic extension of $\mathbb{Q}_p$, but the general setting also reveals information about variation in p, codified by taking ultraproducts of p-adic fields. In this way mixed characteristic theories converge to theories with characteristic zero residue fields (the pseudofinite fields of Ax [1]). One gets the famous AEK analogy between generic $\mathbb{Q}_p$ and generic $\mathbb{F}_p((t))$ (though, alas, the theory of fixed $\mathbb{F}_p((t))$ remains unknown).

Many subsequent authors refined the analysis, linking the type structure of K with those of Γ and k. Denef's angular component maps (see [9]) come in at the level of quantifier-elimination, and have remained prominent in recent very sophisticated work of Denef and Loeser on motivic integration (see [10]).

We will achieve something similar. The (K, v, k, Γ) of most importance to us are the completions of the maximal unramified extensions $\mathbb{Q}_p^{nr}$ of $\mathbb{Q}_p$, where

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$k = \mathbb{F}_p^{alg}$, the algebraic closure of $\mathbb{F}_p$. These are the fraction fields $W(\mathbb{F}_p^{alg})$ of the rings $W[\mathbb{F}_p^{alg}]$ of Witt vectors over $\mathbb{F}_p^{alg}$, and carry σ_p, the Witt Frobenius (see Example 1.2), satisfying $\sigma_p(x) \equiv x^p \pmod{p}$. A field with an automorphism is called a difference field. We will usually denote the distinguished automorphism by σ. To study the theory of those $(K, v, k, \Gamma, \sigma)$ we make a more general study of difference fields carrying a valuation. Though the model theory of difference fields has seen spectacular development (and applications) (see [7]), this does not help us much, except at the end of this paper when we study the variation in p of the completions of $\mathbb{Q}_p^{nr}$, and get information on types via the work of Hrushovski and Macintyre [15], [21] on variation of the Frobenius $x \mapsto x^p$.

Our main achievement is an analogue, in the setting of difference fields carrying a valuation, of the classical Ostrowski theory of pseudoconvergence (used by AEK). This is quite delicate, and requires some restrictions on how σ interacts with k and Γ. For example, we shall require that σ induces the identity map on the value group and that it reduces to some given automorphism $\bar{\sigma}$ on the residue field. Under these restrictions maximal immediate σ-extensions behave well, and from there we can proceed to analogues of the AEK results, for example showing that in the case k has characteristic 0 the theory is determined by of those of $(k, \bar{\sigma})$ and Γ. In the mixed case, with k of characteristic p , $v(p) = 1$, and $\bar{\sigma} =$ Frobenius, the theory is determined by those of k and Γ. In all cases we describe the types of K in terms of those of $(k, \bar{\sigma})$ and Γ, using angular components.

In the analysis of the Witt-Frobenius case, we make use of the ∂-ring formalism from Joyal's [16]. However we do not pursue the issue of an axiomatization in those terms.

The paper is organized as follows. In Section 1, we establish basic notation and assumptions and recall some key facts from valuation theory and the notion of angular component map. Section 2 contains a precise statement of the key result, a general Embedding Theorem. The main model theoretic results of this paper are applications of this Embedding Theorem: for valued difference fields for which it applies we obtain completeness and model-completeness theorems (Sections 9, 10), quantifier elimination (Section 11), completeness and decidability theorems when we vary p (Section 12). In Section 3, we axiomatize the basic properties involved in our work and single out the key base fields for which we are successful. In Section 4, we introduce the formalism of ∂-rings. In Section 5, we develop the theory of pseudoconvergence in the σ-setting. The main new feature here is the failure of continuity of σ-polynomials with respect to pseudoconvergence, in contrast to the purely algebraic setting where polynomials have this kind of continuity (Lemma 5.1). The crucial observation is that enough continuity can be preserved modulo an equivalence relation on pseudoconvergent sequences (Definition 5.3). This section is devoted to establishing the appropriate version of the new continuity (Theorem 5.9), and valued difference fields for which we have it will be called *pliable*. In Section 6, we establish some basic facts about the *σ-Hensel scheme*, which plays the role of Hensel's lemma in the classical setting, particularly with respect to pseudoconvergence. In Section 7, we establish the existence and uniqueness of maximal immediate σ-extensions for pliable valued difference fields, in analogy with the classical setting. In Section 8, we prove the Embedding Theorem, again in analogy with the classical situation (see e.g. [18]).

For basic model theory we refer to [23]. For the basic theory of pseudoconvergence and valuations we refer to [26], [18], [30], and for Witt vectors to [32]. For the basic theory of difference fields we refer to [8]. Our difference fields are the inversive difference fields of [8].

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1. Preliminaries. We will use boldface notation for multivariables and uples, e.g. $\boldsymbol{x} = (x_0, \dots, x_n)$. For a ring A, A^* will denote its multiplicative group of units.

We will be working with fields K of characteristic 0 with valuation $v\colon K^* \to \Gamma$ and residue field k, V will be the valuation ring, and $\bar{}$ will be used (in a variety of contexts) for reduction to k from V. In particular, if $F \in V[x_0, \dots, x_n]$, $\bar{F}$ is the reduced element in $k[x_0, \dots, x_n]$. The field K will carry in addition an automorphism σ, and we generally denote this structure as $(K, v, k, \Gamma, \sigma)$. We will denote by $\mathrm{Fix}(\sigma)$ the fixed field of σ.

From the outset we require σ be an isometry. (This terminology is taken from the literature [12].):

Axiom 1. $\forall x \;\; v(\sigma(x)) = v(x)$.

Definition 1.1. We say that $(K, v, k, \Gamma, \sigma)$ is a *valued field with isometry* if it satisfies Axiom 1.

On model theoretic grounds requiring that σ induce the identity on Γ is natural if we aim for the existence of model companions of theories of $(K, v, k, \Gamma, \sigma)$, by a result of Kikyo [17]. The axiom obviously implies that σ is continuous for the valuation topology, and σ reduces to an automorphism $\bar{\sigma}$ of k.

Example 1.2. Take k perfect, characteristic p, $K = W(k)$ = the field of fractions of the ring of Witt vectors $W[k]$. Let $\tau\colon k \to W[k]$ be the Teichmüller map, i.e. the (unique) multiplicative section of the reduction map. Every $x \in W(k)$ has a unique representation $x = \sum_{n \geq n_0} \tau(x_n) p^n, n_0 \in \mathbb{Z}$, and one has the automorphism $\sigma_p(x) = \sum_{n \geq n_0} \tau(x_n)^p p^n$, the *Witt Frobenius.*

Example 1.3. Let k be as in the previous example and f any automorphism of k. Then, as above, one has the automorphism of $W(k)$ given by $\sigma_f(x) = \sum_{n\geq n_0} \tau(f(x_n))p^n$. In fact, by the universal property of Witt vectors ([32], II.5, Prop. 10), any isometry of $W(k)$ is of this form.

Example 1.4. Take k a field of characteristic 0, $K = k((t))$, and f an automorphism of k. Then we get the automorphism of K defined by $\sigma_f(\sum x_n t^n) = \sum f(x_n)t^n$.

These examples satisfy another property, namely:

Axiom 2. $\forall x \exists y(\sigma(y) = y \wedge v(x) = v(y))$.

Definition 1.5. We say that $(K, v, k, \Gamma, \sigma)$ *has enough constants* if it satisfies Axiom 2.

One should note a minor logical difference between the two axioms. In our category of structures $(K, v, k, \Gamma, \sigma)$ substructures are fields closed under σ. Being a valued field with an isometry passes to substructures, while having enough constants does not.

We say an extension of valued fields carrying an automorphism is *immediate* if it is immediate as an extension of valued fields. We assume known the theory of pseudoconvergence, which elucidates these extensions. We use the variant where only eventual behavior is required: e.g. an ordinal-indexed (without a maximum) sequence $\{a_\rho\}$ is a pseudoconvergent series (henceforth p.c.) if there is an index ρ_0 such that for all $\rho_3 > \rho_2 > \rho_1 \geq \rho_0$ we have $v(a_{\rho_3} - a_{\rho_2}) > v(a_{\rho_2} - a_{\rho_1})$ ([26], or see [18]). We will use the notation $\{a_\rho\} \rightsquigarrow a$ for the statement that $\{a_\rho\}$ pseudoconverges to a (or, a is a pseudolimit of $\{a_\rho\}$). It will be very useful to use the notation γ_ρ for the eventual (δ-independent) value $v(a_\delta - a_\rho)$ for $\delta > \rho$. The *width* of $\{a_\rho\}$ is $\{\gamma \in \Gamma \cup \{\infty\}\colon\ \gamma > \gamma_\rho \text{ all } \rho\}$, and is important precisely because if $\{a_\rho\} \rightsquigarrow a$ then $\{a_\rho\} \rightsquigarrow b$ if and only if $v(a - b)$ is in the width of $\{a_\rho\}$.

A useful observation is that if $\{a_\rho\}$ is p.c. in a valued field, possibly with extra structure, then $\{a_\rho\}$ has a pseudolimit in an elementary extension.

One should observe that under immediate extensions having enough constants is preserved. The isometry axiom is also preserved, provided the automorphism is a valued field automorphism (as it will be if we work in an ambient valued field with an isometry).

We will extend the classical theory of henselisation, and we will make heavy use of the classical theory. We review the crucial fact. The property *henselian* (for fields) is first-order, and every valued field K has a henselisation $K \longrightarrow K^h$, immediate algebraic over K. Any isomorphism of valued fields extends *uniquely* to an isomorphism of their henselisations. A reference for all this material is [27].

This leads to a very useful lemma.

LEMMA 1.6. *If $(K, v, k, \Gamma, \sigma)$ is a valued field with an automorphism σ of valued fields, then σ extends* uniquely *to a valued field automorphism σ^h of K^h. If (K, σ) is a valued field with isometry having enough constants, then so is (K^h, σ^h).*

We will make some use of the so-called *coarse* valuation, a standard tool in the classical setting. The following lemma gathers the basic facts needed.

LEMMA 1.7. (The coarse valuation) *Suppose $(K, v, k, \Gamma, \sigma)$ is unramified and the characteristic of k is $p > 0$. Let Γ_0 be the convex subgroup of Γ generated by $v(p)$ and $\dot{v}$: $K^* \to \Gamma/\Gamma_0$ be the composition of v and the canonical quotient map $\Gamma \to \Gamma/\Gamma_0$.*

(i) *The map $\dot{v}$ is a valuation.*

(ii) *If $(K, v, k, \Gamma, \sigma)$ is a valued field with isometry (resp. having enough constants), then $(K, \dot{v}, \dot{k}, \Gamma/\Gamma_0, \sigma)$ is also a valued field with isometry (resp. also has enough constants), where $\dot{k}$ is the residue field for $\dot{v}$.*

(iii) *The residue field $\dot{k}$ of $\dot{v}$ has characteristic 0 and is isomorphic to a subfield of $W(k)$. If k is perfect and σ_f is the automorphism of $W(k)$ induced by $f = \bar{\sigma}$ as in Example 1.3, then $\dot{k}$ is isomorphic to a difference subfield of $(W(k), \sigma_f)$.*

Proof. (i) This is routine and well known (e.g. see [30], chap. 1).

(ii)–(iii). Let $\dot{V}$ be the valuation ring of $\dot{v}$, and $\dot{\mu}$ the maximal ideal. Then

$$\begin{aligned} \dot{V} &= \{x\colon\ v(x) \geq \gamma,\ \text{some } \gamma \in \Gamma_0\} \\ \dot{\mu} &= \{x\colon\ v(x) > \Gamma_0\}. \end{aligned}$$

No ramification gives $p \notin \dot{\mu}$, so $\dot{k} = \dot{V}/\dot{\mu}$ has characteristic 0. The field $\dot{V}/\dot{\mu}$ carries a valuation given by $v_0(x + \dot{\mu}) = v(x)$. The map σ induces $\dot{\sigma}$ on $\dot{k}$ by $\dot{\sigma}(x + \dot{\mu}) = \sigma(x) + \dot{\mu}$. It bears noting that the reduction map from V to k factors as the composition of the reduction map from the valuation ring of v_0 to k with the reduction map on $\dot{V}$ restricted to V so that $\dot{\sigma}$ induces $\bar{\sigma}$ on k. Obviously, $(\dot{k}, v_0, k, \Gamma_0, \dot{\sigma})$ is a valued field with isometry having enough constants, and has value group $\Gamma_0 = \mathbb{Z}v(p)$, so is isomorphic to a subfield of $W(k)$, and if k is perfect to a difference subfield of $(W(k), \sigma_f)$. □

Angular component maps are natural in the context of quantifier-elimination results in valued fields (mainly in the work of Denef's school (see [24])), and exist for $\aleph_1$-saturated K (see below).

Definition 1.8. Let (K, v, k, Γ) be a valued field. An *angular component map* (or *coefficient map*) is a map ac: $K^* \to k^*$ so that:

(i) ac is a multiplicative homomorphism;

(ii) the restriction of ac to $\{y\colon\ v(y) = 0\}$ is $y \mapsto \bar{y}$.

An angular component map corresponds to a splitting of the exact sequence (Krasner's *"corpoïde"*, see [4])

$$1 \longrightarrow k^* \longrightarrow K^*/1 + \max\ V \longrightarrow \Gamma \longrightarrow 0$$

where max V is the maximal ideal of V (see e.g. [4]). Note that ac exists whenever k^* is pure-injective, in particular when (K, v, k, Γ) is $\aleph_1$-saturated. Also, if $\pi\colon \Gamma \to K^*$ is a cross-section then $x \mapsto \overline{x/\pi(v(x))}$ is an angular component map.

In fact we will need only an angular component map ac on $\mathrm{Fix}(\sigma)$. Indeed, under the assumption that the valued field with isometry has enough constants, if ac is an angular component map on $\mathrm{Fix}(\sigma)$ it can be extended uniquely to the whole of K: for $x \in K$, let $y \in \mathrm{Fix}(\sigma)$ such that $v(x) = v(y)$, then $ac(x) = ac(y) \cdot \overline{y^{-1}x}$.

Definition 1.9. Let (K, v, k, Γ) now be an unramified valued field of characteristic 0 with k of characteristic $p > 0$ and consider the natural maps

$$\mathrm{res}_n\colon\ V \to V/(p^n), \qquad n = 1, 2 \ldots$$

A *system of angular component maps* is a system $ac_n\colon\ K^* \to (V/(p^n))^*$ such that:

(i) each ac_n is a multiplicative homomorphism;

(ii) the restriction of ac_n to $\{y\colon\ v(y) = 0\}$ is res_n;

(iii) ac_n is the composition $K^* \xrightarrow{ac_{n+1}} (V/(p^{n+1}))^* \to (V/(p^n))^*$ where $\to$ is the natural map.

Example 1.10. In $W(k)$, the maps $x \mapsto res_n(xp^{-v(x)})$ yield a system of angular component maps.

Again, suitable ac_n exist if k^* is pure-injective or if one has a normalized cross-section, thus under $\aleph_1$-saturation. As above, if $(K, v, k, \Gamma, \sigma)$ has enough constants, it suffices to have angular component maps ac_n for $\mathrm{Fix}(\sigma)$.

When using angular component maps for $(K, v, k, \Gamma, \sigma)$ we require in addition that they commute with the action of the distinguished automorphism. This is equivalent to asking that the angular component functions on K restrict to such functions on $\mathrm{Fix}(\sigma)$.

Remark 1.11. In particular, in the case of $W(\mathbb{F}_p^{alg})$ the fixed field is $\mathbb{Q}_p$, where the above angular component maps are definable (see [9]) and hence we have definability, more precisely $\exists$-definability, of the angular component maps in $(W(\mathbb{F}_p^{alg}), v, \mathbb{F}_p^{alg}, \mathbb{Z}, \sigma_p)$.

The following variant of an unpublished lemma of van den Dries (cf. [4], Lemma 3.6) suggests the flexibility of angular component maps.

LEMMA 1.12. *Let L_1' and L_2' be unramified valued fields equipped with one of the angular component maps above, say ac_*, and let $L_i \subset L_i'$, $i = 1$ and 2 be subvalued fields closed under ac_*. Let $\psi\colon\ L_1 \to L_2$ be a valued field isomorphism which respects ac_*, i.e. $\psi_{r,*}(ac_*(x)) = ac_*(\psi(x))$ where $\psi_{r,*}$ denotes the induced isomorphism between residue rings corresponding to ac_*, and let $\psi'\colon\ L_1' \to L_2'$ be a valued field isomorphism extending ψ. Suppose there exist a subgroup H of ${L_1'}^*$ such that $vL_1' = vL_1 + vH$ and a set of generators H_0 of H such that for all $h \in H_0$, $\psi'_{r,*}(ac_*(h)) = ac_*(\psi'(h))$. Then ψ' also respects ac_*.*

Proof. Note then that $\psi'_{r,*}(ac_*(z)) = ac_*(\psi'(z))$ for all $z \in H$. Now any $y \in L'_1$ can be written as $y = xzu$, for some $x \in L_1, z \in H, u \in L'_1$ such that $v(u) = 0$, and so we are done. □

Let $\psi\colon\ k \to k'$ be any map between fields k and k' and f a polynomial over k. Then f^{ψ} will denote the transform of f obtained by having ψ to act on the coefficients.

We will be dealing with simple extensions of (valued) difference fields. Suppose $(K_1, \sigma_1) \subseteq (K_2, \sigma_2)$ is an extension of difference fields. For $a \in K_2 \setminus K_1$, $K_1\langle a\rangle$ is the smallest difference subfield of (K_2, σ_2) containing K_1 and a. Clearly, its underlying field is $K_1(\{\sigma_2^j(a)\}_{j\in\mathbb{Z}})$. Here, as throughout this paper, σ^j stands for the j^{th} iterate of σ if $j \geq 0$, and the $(-j)^{\text{th}}$ iterate of σ^{-1} if $j \leq 0$.

It turns out that we need to consider only difference polynomials of one variable. Each polynomial $F(x_0, \ldots, x_n) \in K[x_0, \ldots, x_n]$ gives rise to a difference polynomial $G(x) = F(x, \sigma(x), \ldots, \sigma^n(x)))$ in the variable x over K, and we refer to $G(x)$ as a σ-polynomial. We put $\deg(G) := \deg(F) \in \mathbb{N} \cup \{\infty\}$, where $\deg(F)$ is the total degree of F. If G is not constant, that is, $G \notin K$, then let $F(x_0, \ldots, x_n)$ be as above with least possible n (which makes F unique), and put

$$\text{order}(G) := n, \quad \text{complexity}(G) := (n, \deg_{x_n}(F), \deg(F)) \in \mathbb{N}^3.$$

If $G \in K, G \neq 0$, then $\text{order}(G) = -\infty$ and $\text{complexity}(G) = (-\infty, 0, 0)$. Finally, $\text{order}(0) = -\infty$, and $\text{complexity}(0) = (-\infty, -\infty, -\infty)$. We order complexities lexicographically. For example, let $F_1, F_2, R \in K[x_0, \ldots, x_n]$ such that R is obtained by euclidean division of F_1 by F_2 with respect to x_n and clearing out denominators, then the σ-polynomial associated to R has lower complexity than the one associated to F_2. (Finer complexity measures would do, e.g. considering vector degrees of monomials.)

We say a is σ-transcendental over K if there is no nonzero G as above with $G(a) = 0$. Otherwise a is σ-algebraic over K. Note that nontrivial G may have infinitely many a with $G(a) = 0$ (e.g. $G(x) = \sigma(x) - x$).

For future use, we introduce some notation concerning σ-polynomials $G(x)$ as above.

Let $x_0, \ldots, x_n, y_0, \ldots, y_n$ be distinct indeterminates, and put $\boldsymbol{x} = (x_0, \ldots, x_n)$, $\boldsymbol{y} = (y_0, \ldots, y_n)$. For $\boldsymbol{l} \in \mathbb{N}^{n+1}$, let $|\boldsymbol{l}| = \sum_i l_i$. For a polynomial $F(\boldsymbol{x})$ over a field K we have a unique Taylor expansion in $K[\boldsymbol{x}, \boldsymbol{y}]$:

$$F(\boldsymbol{x} + \boldsymbol{y}) = \sum_{\boldsymbol{l}} F_{\boldsymbol{l}}(\boldsymbol{x}) \cdot \boldsymbol{y}^{\boldsymbol{l}},$$

where the sum is over all $\boldsymbol{l} = (l_0, \ldots, l_n) \in \mathbb{N}^{n+1}$, each $F_{\boldsymbol{l}} \in K[\boldsymbol{x}]$, with $F_{\boldsymbol{l}} = 0$ if $|\boldsymbol{l}| > \deg(F)$, and $\boldsymbol{y}^{\boldsymbol{l}} = y_0^{l_0} \cdots y_n^{l_n}$ (likewise, for $\boldsymbol{a}$ with components in any field we put $\boldsymbol{a}^{\boldsymbol{l}} = a_0^{l_0} \cdots a_n^{l_n}$). Thus, $\boldsymbol{l}! F_{\boldsymbol{l}} = \partial_{\boldsymbol{l}} F$, where $\partial_{\boldsymbol{l}}$ is the operator $\partial^{l_0}/\partial x_0 \ldots \partial^{l_n}/\partial x_n$ on $K[\boldsymbol{x}]$, and $\boldsymbol{l}! = l_0! \ldots l_n!$. We construe $\mathbb{N}^{n+1}$ as a monoid under pointwise addition, and let $\leq$ be the partial order on $\mathbb{N}^{n+1}$ induced by the natural order on $\mathbb{N}$. Define $\binom{\boldsymbol{l}}{\boldsymbol{j}}$ as $\binom{l_0}{j_0} \dots \binom{l_n}{j_n} \in \mathbb{N}$, when $\boldsymbol{j} \leq \boldsymbol{l}$ in $\mathbb{N}^{n+1}$. Then clearly:

LEMMA 1.13. *For $\boldsymbol{j}, \boldsymbol{l} \in \mathbb{N}^{n+1}$ we have $(F_{\boldsymbol{j}})_{\boldsymbol{l}} = \binom{\boldsymbol{j}+\boldsymbol{l}}{\boldsymbol{j}} F_{\boldsymbol{j}+\boldsymbol{l}}$.*

In particular, if $|\boldsymbol{l}| = 0, F_{\boldsymbol{l}} = F$, and if $|\boldsymbol{l}| = 1$, $F_{\boldsymbol{l}}$ is one of the $\frac{\partial F}{\partial x_i}$. Also, $\deg(F_{\boldsymbol{l}}) < \deg(F)$, if $|\boldsymbol{l}| \geq 1$ and $F \neq 0$.

Let now (K, σ) be a difference field, and x an indeterminate. When n is clear from context we set $\boldsymbol{\sigma}(x) = (x, \sigma(x), \dots, \sigma^n(x))$, and also $\boldsymbol{\sigma}(a) = (a, \sigma(a), \dots, \sigma^n(a))$ for $a \in K$. Then for F as above and $G(x) = F(\boldsymbol{\sigma}(x))$ we have the following identity in the ring of difference polynomials in the distinct indeterminates x and y over K:

$$\begin{aligned} G(x+y) &= F(\boldsymbol{\sigma}(x+y)) = F(\boldsymbol{\sigma}(x) + \boldsymbol{\sigma}(y)) \\ &= \sum_{\boldsymbol{l}} F_{\boldsymbol{l}}(\boldsymbol{\sigma}(x)) \cdot \boldsymbol{\sigma}(y)^{\boldsymbol{l}} \\ &= \sum_{\boldsymbol{l}} G_{\boldsymbol{l}}(x) \cdot \boldsymbol{\sigma}(y)^{\boldsymbol{l}}, \end{aligned}$$

where $G_{\boldsymbol{l}}(x) := F_{\boldsymbol{l}}(\boldsymbol{\sigma}(x))$. A key point will be that for $G \neq 0$ and $|\boldsymbol{l}| \geq 1$, $G_{\boldsymbol{l}}$ has lower complexity than G.

2. Statement of the main result: an embedding theorem.

The main result is a general embedding theorem, Theorem 2.2, which will give us a quantifier elimination result. We first isolate the relevant axioms for valued fields with an automorphism $(K, v, k, \Gamma, \sigma)$. We recall the two basic axioms presented above.

Axiom 1. (isometry) For all x, $v(\sigma(x)) = v(x)$.

Axiom 2. (enough constants) For all x, there is y so that $(\sigma(y) = y \wedge v(x) = v(y))$.

The $\boldsymbol{\sigma}$-Hensel Scheme. Let G be a σ–polynomial of order n. Let $\beta = \alpha + \varepsilon$, so $G(\beta) = G(\alpha) + \sum_{\boldsymbol{l} \geq 1} G_{\boldsymbol{l}}(\alpha) \cdot \boldsymbol{\sigma}(\varepsilon)^{\boldsymbol{l}}$. If $v(G(\alpha)) = \gamma + \min_{|\boldsymbol{l}|=1} v(G_{\boldsymbol{l}}(\alpha))$ and $v(G(\alpha)) < j \cdot \gamma + v(G_{\boldsymbol{l}}(\alpha))$, whenever $|\boldsymbol{l}| = j > 1$, then there is β in K with $v(\alpha - \beta) = \gamma$ and $G(\beta) = 0$.

Axiom R. For every $\lambda \in k^*$, the equation $\bar{\sigma}(x) = \lambda x$ has a non-zero solution in k.

Axiom RG. (Genericity Axiom) For each $n \in \mathbb{N}, n > 0, a_0, \dots, a_n, b \in k$ such that $a_0 a_n \neq 0$, and $F \in k[x_0, \dots, x_n], F \neq 0$, there is $x \in k$ such that $a_0 x + a_1 \bar{\sigma}(x) + \cdots + a_n \bar{\sigma}^n(x) = b$ and $F(\bar{\boldsymbol{\sigma}}(x)) \neq 0$.

Axioms R and RG allow us to proceed "generically" in the proof of the Embedding Theorem. This is further discussed in the next section.

Definition 2.1. We say that the valued field with isometry $(K, v, k, \Gamma, \sigma)$ is a *Witt-Frobenius case* if $\mathrm{char}(k) = p > 0$, $v(p)$ is the least positive element of Γ and $\bar{\sigma}(x) = x^p$.

Next we describe the embeddings. Let $(K_i, v_i, k_i, \Gamma_i, \sigma_i)$ for $i = 1$ and 2 be unramified valued fields with an automorphism and with appropriate angular component maps. Let L_i be difference subfields of the respective K_i. Namely, each L_i has residue field l_i, and value group G_i, for the induced valuation, and L_i is closed under the angular components in the obvious sense. We say a bijection $\psi\colon L_1 \to L_2$ is an *admissible isomorphism* if it has the following properties:

(A) ψ is an isomorphism of valued fields with isometry,

(B) the induced isomorphism $\psi_r\colon l_1 \longleftrightarrow l_2$ of difference fields is *elementary*, in the sense that for all formulas $\varphi(x_1, \dots x_n)$ of the language of difference fields,

$$k_1 \models \varphi(\alpha_1, \dots, \alpha_n) \iff k_2 \models \varphi(\psi_r(\alpha_1), \dots, \psi_r(\alpha_n)),$$

(C) the induced $\psi_v\colon G_1 \longleftrightarrow G_2$ is elementary, in the sense that for all formulas $\varphi_g(x_1, \dots x_n)$ of the language of ordered abelian groups,

$$\Gamma_1 \models \varphi_g(\gamma_1, \dots, \gamma_n) \iff \Gamma_2 \models \varphi_g(\psi_v(\gamma_1), \dots, \psi_v(\gamma_n)),$$

(D) ψ respects the angular component maps.

We can now state the Embedding Theorem.

THEOREM 2.2. (Embedding Theorem) *Let $(K_i, v_i, k_i, \Gamma_i, \sigma_i)$ for $i = 1$ and 2 be suitably saturated unramified valued difference fields with k_1 and k_2 perfect of the same characteristic p. Suppose either:*

(1) *Each K_i carries an angular component map $ac^{(i)}$ (resp. a system $ac_n^{(i)}$ of angular components) when $p = 0$ (resp. $p > 0$), and is a valued field with isometry which has enough constants and satisfies Axiom RG and the σ-Hensel scheme; or*

(2) *Each K_i is a Witt-Frobenius case, carries a system $ac_n^{(i)}$ of angular components, and is a valued field with isometry which has enough constants and satisfies Axiom R and the σ-Hensel scheme.*

Let L_i for $i = 1$ and 2 be small *difference subfields of the respective K_i. Namely, they have residue fields l_i and value groups G_i for the induced valuations, the L_i are closed under the angular components in the obvious sense, and K_i is* $(\max(card(l_i), card(G_i))^+$*-saturated.*

Assume we have an admissible isomorphism $\psi\colon L_1 \longleftrightarrow L_2$ and let $a \in K_1$. Then there exist $b \in K_2$ and an admissible isomorphism $\psi'\colon L_1\langle a\rangle \cong L_2\langle b\rangle$ extending ψ with $\psi'(a) = b$.

We formalize the hypotheses in the Embedding Theorem in terms of the *σ-AEK axioms*.

Definition 2.3. We say that the valued field with isometry $(K, v, k, \Gamma, \sigma)$ satisfies the σ-AEK axioms if it satisfies the hypotheses of the Embedding Theorem.

In the course of the proof of the Embedding Theorem, we work primarily with *pliable* valued fields with isometries. This is the context in which the theory of pseudoconvergent sequences works best for valued fields with isometries.

Definition 2.4. We say that a valued field with isometry is *pliable* if it is unramified, has enough constants, and is either a Witt-Frobenius case with an infinite residue field or the induced automorphism on the residue field satisfies no identities.

The proof of Theorem 2.2 will occupy us through Section 8.4.

Our arguments adapt easily to the case $\bar{\sigma}(x) = x^q$, q a finite power of $p > 0$.

Once one has proven Theorem 2.2, one deduces quantifier-elimination (see e.g. [23], Lemma 3.1.6 and Prop. 4.3.28), completeness and model-completeness (ibid., Lemma 2.4.11) and various model-theoretic consequences which are given beginning in Section 9. In particular, we produce an axiomatization for and deduce the decidability and model-completeness of the first-order theory of the Witt vectors $W(\mathbb{F}_p^{alg})$ with their Frobenius automorphism (Example 1.2).

3. Discussion of axioms and key base fields. During the work leading to the proof of the Embedding Theorem, further basic properties come into play. We also present them as axioms and indicate briefly their relevance and relationship to the main axioms appearing in the Embedding Theorem. We also single out some categories of fields we will be working with.

To discuss various closure properties of the residue field (viz. Axiom RG), it is convenient to phrase them in terms of difference operators. In the following (k, σ) is a difference field. We write $k[\sigma]$ for the *noncommutative* ring of linear difference operators over k. That is, $k[\sigma]$ is the associative ring generated by k and a symbol σ subject to the commutation rule $\sigma a = \sigma(a)\sigma$ for $a \in k$. This ring is right euclidean, therefore an Ore domain. Any nonzero $L \in k[\sigma]$ can be written as $L = \sum_{n=n_0}^{n_0+d} a_n \sigma^n$ for natural numbers n_0 and d and $a_n \in k$ with $a_{n_0} \cdot a_{n_0+d} \neq 0$. We call d the *essential degree* of L, ess.deg(L). Recall that the kernel of L is a vector space over Fix(σ). We consider a difference closed field (Ω, σ) (i.e. an existentially closed, or model of ACFA, see [7]) extending (k, σ) and write the fixed field of σ in Ω as $\mathrm{Fix}_\Omega(\sigma)$.

We rephrase Axiom RG in these terms and consider five additional properties.

Axiom RG. For all $L \in k[\sigma], \mathrm{ess.deg}(L) = d > 0$, and $F \in k[x_0, \dots, x_{d-1}]$, $F \neq 0$, and $b \in k$, there is some $x \in k$ for which $L(x) = b$ and $F(\boldsymbol{\sigma}(x)) \neq 0$.

Axiom R0. The fixed field Fix(σ) is infinite.

Axiom R1. For all $L \in k[\sigma], L \neq 0$ and $y \in k$, there is some $x \in k$ with $L(x) = y$.

Definition 3.1. We say that the difference field (L, σ) is *linearly difference closed* if it satisfies Axiom R1.

Axiom R2. For all $L \in k[\sigma], \text{ess.deg}(L) > 0$, there is some $x \in k^*$ with $L(x) = 0$.

The next axiom was pointed out by a referee as an alternative to Axiom RG.

Axiom R3. For all $L \in k[\sigma], L \neq 0$, we have $\dim_{\text{Fix}(\sigma)} \ker \text{L} = \text{ess.deg}(L)$.

Axiom R4. (No σ-identities) For all $n \in \mathbb{N}$ and $L \in k[X_0, \ldots, X_n] \setminus \{0\}$ there is some $x \in k$ for which $L(x, \sigma(x), \ldots, \sigma^n(x)) \neq 0$.

As we shall see, while these various axioms are not equivalent, there is a web of implications between them. We work with Axiom RG as it permits us to avoid accidental equalities and thereby develop a cleaner theory of immediate valued difference field extensions.

The basic properties which enable us to adapt to our context the classical tools are the conditions of having an isometry and enough constants, and either the residue field being infinite or satisfying Axiom R4, namely, those properties of pliability. Making the σ-Hensel scheme work requires Axiom R1.

Concerning Axiom R4, by [8] (page 201) if a difference field (k, σ) satisfies an identity, then it satisfies an identity of the form $\sigma^n(x) = x^{q^m}$, for some integers m and $n \neq 0$ with $q = \text{char}(k)$ if this is not zero and $q = 1$ if $\text{char}(k) = 0$. So in characteristic $p > 0$ we are essentially left with a power of the Frobenius map $x \mapsto x^p$, and in characteristic 0 with a σ of finite order . In particular, $(W(k), \sigma_p)$ satisfies no σ-identity provided k is infinite.

Axiom R and Axiom RG are used (only) in the proof of the Embedding Theorem in order to extend a basic admissible isomorphism $\psi\colon L_1 \to L_2$ to another one $\psi'\colon L_1' \to L_2'$ in a "generic way," where each L_i' is pliable with a linearly difference closed residue field.

Axioms R0, R2, R3 are discussed below to shed some light on Axioms R and RG. They appear again only at the end of the paper (Section 12).

LEMMA 3.2. (see [28], Prop. 5.3) *Suppose* $(K, v, k, \Gamma, \sigma)$ *is a valued field with isometry having enough constants for which the* σ*-Hensel scheme holds and the valuation is not trivial. Then* $(k, \bar{\sigma})$ *satisfies Axiom R1.*

LEMMA 3.3. *Suppose* (k, σ) *satisfies Axiom RG. Then* (k, σ) *satisfies Axiom R4.*

Proof. Consider a putative identity $F(x, \sigma(x), \ldots, \sigma^d(x)) \equiv 0$ for some nonzero polynomial F. By Axiom RG applied to $L = \sigma^{d+1} - 1$, $b = 1$, and F, there is some $c \in k$ with $\sigma^{d+1}(c) - c = 1$ and $F(\boldsymbol{\sigma}(c)) \neq 0$, showing that the identity $F(\boldsymbol{\sigma}(x)) \equiv 0$ fails. □

LEMMA 3.4. *Suppose $(K, v, k, \Gamma, \sigma)$ is a valued field with isometry satisfying Axiom R and the σ-Hensel scheme. Then $(K, v, k, \Gamma, \sigma)$ satisfies Axiom 2.*

Proof. Let $a \in K^*$. Then $a/\sigma(a)$ has valuation 0. Consider $\sigma(x) = x \cdot a/\sigma(a)$. By Axiom R, this can be solved in k, nontrivially, say by $\bar{x}_0$. Then by the σ-Hensel scheme there is a solution x in V with $\bar{x} = x_0$. So $v(x) = 0$, $\sigma(xa) = xa$ and $v(xa) = v(a)$. □

Axiom R is a particular case of Axiom R2, and has a rather different character from Axiom R1. In a Witt situation $W[k]$ it requires k to be closed under extracting $(p-1)^{\mathrm{st}}$ roots. The two axioms R, R1 are essentially independent (see Propositions 3.7, 3.8).

PROPOSITION 3.5. *Axiom R3 $\Longrightarrow$ Axiom R2.*

Proof. Let $\lambda \in k^*$. Set $P := \sigma - \lambda$. By Axiom R3, the dimension of the kernel of P on k is 1. In particular, there is a $a \in k^*$ with $P(a) = 0$, or $\sigma(a) = \lambda a$. □

PROPOSITION 3.6. *Axiom R1 + Axiom R2 $\Longrightarrow$ Axiom R3.*

Proof. Let $L \in k[\sigma]$ be a nonzero difference operator of essential degree $\mathrm{ess.deg}(L) = d$. The kernel of L on k is a vector space over $\mathrm{Fix}(\sigma)$ of dimension $e \leq d$. It is easy to find $M \in k[\sigma]$ with $\mathrm{ess.deg}(M) = e$ and $\ker M = \ker L$.

Indeed, any finite dimensional $\mathrm{Fix}(\sigma)$-vector subspace V of k is the kernel of some $M \in k[\sigma]$ with $\mathrm{ess.deg}(M) = \dim_{\mathrm{Fix}(\sigma)}(V)$. We check this by induction on $\dim V$ where in the case of $\dim V = 0$ we take M to be the identity operator. More generally, let $\alpha \in V$ be any nonzero element of V. The kernel of the operator $\Phi := (\sigma - \frac{\sigma(\alpha)}{\alpha})$ is exactly $\mathrm{Fix}(\sigma)\alpha$. The vector space $\Phi(V)$, thus, has dimension one less than that of V and by induction is the kernel of some $\Psi \in k[\sigma]$ with $\mathrm{ess.deg}(\Psi) = \dim \Phi(V)$. Set $M := \Psi \circ \Phi$.

Factoring, we can write $L = QM$ for some $Q \in k[\sigma]$. If $e \neq d$, then $\mathrm{ess.deg}(Q) > 0$ so by Axiom R2 there is some $y \in k^*$ with $Q(y) = 0$. By Axiom R1, there is some $x \in k$ with $M(x) = y$. As $y \neq 0, x \notin \ker M = \ker L$. However, $L(x) = QM(x) = Q(y) = 0$. With this contradiction, we see that $e = d$ as desired. □

PROPOSITION 3.7. *Axiom R2 + Axiom R0 $\not\Longleftrightarrow$ Axiom R1.*

Proof. Let p be any prime number. We produce an example of (k, σ) satisfying Axiom R2 for which σ is an automorphism and Axiom R1 fails and $\mathrm{Fix}(\sigma) = \mathbb{F}_p$. The example having an infinite fixed field is obtained by the compactness theorem of first-order logic (or taking an ultraproduct).

Let k be the direct limit of the finite fields $\mathbb{F}_{p^n}$ over all natural numbers n which are not divisible by p. Let $\sigma\colon\ k \to k$ be the Frobenius automorphism $x \mapsto x^p$.

Let $L \in k[\sigma]$ be a difference operator with ess.deg$(L) = d > 0$. Write $L(x) = \sum_{i=n_0}^{n_0+d} a_i\sigma^i$ (with $a_{n_0}a_{n_0+d} \neq 0$). As σ is an automorphism, L has a nontrivial zero if and only if $L' = \sum_{i=0}^{d} a_{n_0+i}\sigma^i$ has a nontrivial zero; and, in view of our choice of σ, this is so if and only if the polynomial $P(x) = \sum_{i=0}^{d} a_{n_0+i}x^{p^i-1}$ has a zero. As p does not divide $p^d - 1$, some irreducible factor, Q say, of P has degree not divisible by p. Hence, there is a root to Q in k. That is, there is a nonzero point in the kernel of L. So $(k, \bar{\sigma})$ satisfies Axiom R2. However, the difference polynomial $\sigma - \sigma^0$ is not surjective (e.g. 1 is not in its image). □

PROPOSITION 3.8. *Axiom R1 + Axiom R0* $\not\Longrightarrow$ *Axiom R.*

Proof. As before, we present examples with arbitrarily large fixed fields and conclude by compactness that such examples exist with infinite fixed fields.

Let p be a prime number greater than 2. Let $k \subset \mathbb{F}_p^{alg}$ be a subfield of the algebraic closure of $\mathbb{F}_p$ maximal with respect to the property that k has no solution to the equations $x^{p-1} = -1$. Note that if k_0 is some field and $\zeta \in k_0$ is a solution to $x^{p-1} = -1$ then all other solutions are in k_0 (having the form $\alpha\zeta$ for $\alpha \in \mathbb{F}_p^*$). Let $\zeta \in \mathbb{F}_p^{alg}$ be any solution to $x^{p-1} = -1$ and let $n = [k(\zeta) : k]$. It follows from the above observation that n divides $p - 1$. Indeed, if Q_1 and Q_2 are irreducible factors of $X^{p-1} + 1$ over k of degrees d_1 and d_2, respectively, then there are roots to $X^{p-1} + 1 = 0$ in $k[X]/Q_1$, a degree d_1 extension of k and in $\mathbb{F}_p[X]/Q_2$, a degree d_2 extension of k. By the above observation, all the roots must be in each of the fields. So $d_1 = d_2$.

Let $\sigma\colon\ k \to k$ be the Frobenius automorphism $x \mapsto x^p$. As 0 is the only solution to $\sigma(x) = -x$ in k, the difference field (k, σ) fails to satisfy Axiom R. However, it does satisfy Axiom R1. Indeed, let $a \in k$ and $L = \sigma^d + \sum_{i=0}^{d-1} a_i\sigma^i \in k[\sigma]$ with $d > 0$ and $a_i \in k$. The polynomial $P(X) := X^{p^d} + \sum a_i X^{p^i} - a$ has degree p^d. As $p \equiv 1 \bmod n$, $p^d \equiv 1 \bmod n$. Thus some irreducible factor Q of P over k has degree not divisible by n. Then $k[X]/(Q)$ contains a solution to $L(X) = a$ but no solution to $X^{p-1} = -1$. By maximality of k, the field $k[X]/(Q)$ is naturally isomorphic to k. □

Even though Axiom R1 is a consequence of the σ-Hensel scheme, this is not the case for Axiom R: for k as in the proof of the previous proposition, the Witt vectors $W(k)$ will satisfy the σ-Hensel scheme (see Cor. 6.3), but not Axiom R. Recall that (Ω, σ) is an existentially closed extension of (k, σ).

PROPOSITION 3.9. *Axiom R0 + Axiom R1 + Axiom R3* $\Longrightarrow$ *Axiom RG.*

Proof. Let $b \in k$ and $L \in k[\sigma]$ be a nonzero difference operator of essential degree ess.deg$(L) = d > 0$, and $F \in k[x_0, \dots, x_{d-1}], F \neq 0$. Let $N(k), N(\Omega)$ be respectively the kernel of L in k and Ω. Using Axiom R3, we can find $e_1, \dots, e_d \in N(k)$ linearly independent over Fix(σ), and using Axiom R1 we can find $a \in k$ with $L(a) = b$. The fields Fix(σ) and k are linearly independent over Fix(σ) in Ω. Thus $e_1, \dots, e_d$ remains a basis of $N(\Omega)$ over Fix$_\Omega(\sigma)$. Let $\psi\colon\ \mathbb{A}_k^d \to \mathbb{A}_k^1$ be the morphism $(x_1, \dots, x_d) \mapsto \sum_{i=1}^{d} x_i e_i$. Note that ψ restricted to $\mathbb{A}^d(\Omega)$ is an isomorphism between $(\Omega^d, +)$ and $(N(\Omega), +)$.

Let $X(\Omega) := \{x \in N(\Omega) \colon\ F(\boldsymbol{\sigma}(x)) = 0\}$. By construction, the σ-degree of $X(\Omega)$ is less than d. Hence, $\psi^{-1}(X(\Omega))$ is not generic and is therefore contained in a proper subvariety of $\mathbb{A}^d$. Let $Y \subset \mathbb{A}^d$ be the Zariski closure of $\psi^{-1}(X(\Omega))$.

As $\mathrm{Fix}(\sigma)$ is infinite, $\mathbb{A}^d(\mathrm{Fix}(\sigma)) = \psi^{-1}(N(k) + a))$ is Zariski dense in $\mathbb{A}^d$. Hence, there is some point $x \in N(k) + a$ with $\psi^{-1}(x) \notin Y(\Omega)$. That is, there is some point $x \in k$ with $L(x) = b$ and $F(\boldsymbol{\sigma}(x)) \neq 0$. □

We now show that Axiom RG implies all the other axioms on solutions to linear difference equations which we have considered.

PROPOSITION 3.10. *Axiom RG $\Longrightarrow$ Axioms R0, R1, R2, and R3.*

Proof. For Axiom R0, suppose that $\mathrm{Fix}(\sigma) = \mathbb{F}_q$. Applying Axiom RG to $L = \sigma - 1$, $b = 0$, and $F(x) = x^q - x$, we obtain $a \in k$ with $\sigma(a) = a$ and $a^q - a \neq 0$ contradicting the hypothesis that $\mathrm{Fix}(\sigma) = \mathbb{F}_q$.

To conclude Axiom R1, taking $F = 1$, we see that each instance of Axiom R1 becomes an instance of Axiom RG.

For Axiom R2, given L with $\mathrm{ess.deg}(L) > 0$, apply Axiom RG to L and $F(x_0) = x_0$.

Finally, for Axiom R3, let $L \in k[\sigma]$ be a nonzero difference operator of essential degree $\mathrm{ess.deg}(L) = d > 0$. Factoring by a sufficiently high power of σ on the right, we may asssume that $L = \sum_{i=0}^{d} a_i \sigma^i$, with $a_d a_0 \neq 0$.

As before, we can find linear difference operators Q and M in $k[\sigma]$ with $L = QM$, $\ker L = \ker M$, and $m := \mathrm{ess.deg}(M) = dim_{\mathrm{Fix}(\sigma)} \ker M$. If L witnesses the failure of Axiom R3, then $m < d$. Write $M = \sum_{i=0}^{m} b_i \sigma^i$. Let $F(x_0, \dots, x_{d-1}) = \sum_{i=0}^{m} b_i x_i$. Then by Axiom RG, there is some $a \in k$ with $L(a) = 0$ and $F(\boldsymbol{\sigma}(a)) \neq 0$. That is, $a \in \ker L \setminus \ker M$ contradicting our choice of M. □

4. The ∂-ring formalism. Our treatment of Witt vectors will depend on the formalism of ∂-rings of Joyal [16].

Let $(K, v, k, \Gamma, \sigma)$ be a Witt-Frobenius case. Our arguments adapt easily to the case $\bar{\sigma}(x) = x^q$, q a finite power of p, but for notational convenience we stick to the special case $q = p$.

Define $\partial_0(x) = x$ on V. Then define $\partial_1(x) = \frac{1}{p}(\sigma(x) - x^p)$. This is again a map $V \to V$, usually called ∂, and satisfying the axioms for a "p-derivation" ([6]) on V, namely:

$$\begin{aligned} \partial(1) &= 0 \\ \partial(x+y) &= \partial(x) + \partial(y) - \sum_{i=1}^{p-1} p^{-1} \binom{p}{i} \cdot x^i y^{p-i} \\ \partial(xy) &= x^p \partial(y) + y^p \partial(x) + p\partial(x)\partial(y). \end{aligned}$$

A ∂-ring is a commutative ring equipped with a unary operation ∂ satisfying the above identities. The Witt vectors functor $k \mapsto W[k]$ is right adjoint to the forgetful functor from ∂-rings to commutative rings. In a ∂-ring, the map $\sigma(x) = x^p + p\partial(x)$ is an endomorphism. In the theory of ∂-rings ([16]) there is a unique sequence of unary operations $\partial_0, \partial_1, \partial_2, \ldots$ (∂_0, ∂_1, as above) satisfying

$$\sigma^n(x) = \partial_0(x)^{p^n} + p\partial_1(x)^{p^{n-1}} + \cdots + p^n\partial_n(x)$$

In $W[k]$, the ∂_n yield the *components* (sometimes called *ghost coordinates*) of Witt vectors, namely, $x \in W[k]$ is identified with $(\overline{\partial_0(x)}, \overline{\partial_1(x)}, \ldots)$.

LEMMA 4.1. *Let k be a perfect field of characteristic p and σ an automorphism of $W[k]$. Then for all $x \in W[k]$, $\overline{\partial_n(\sigma(x))} = \bar{\sigma}(\overline{\partial_n(x)})$.*

Proof. Let $x \in W[k]$, then as in Example 1.3, $x = \sum_{n\geq 0} \tau\left(\overline{\partial_n(x)}^{p^{-n}}\right) \cdot p^n$ and

$$\sigma(x) = \sum_{n\geq 0} \tau\left(\bar{\sigma}\left(\overline{\partial_n(x)}^{p^{-n}}\right)\right) \cdot p^n = \sum_{n\geq 0} \tau\left(\left(\bar{\sigma}(\overline{\partial_n(x)})\right)^{p^{-n}}\right) \cdot p^n$$

so that $\overline{\partial_n(\sigma(x))} = \bar{\sigma}(\overline{\partial_n(x)})$. □

One shows:

LEMMA 4.2. *The map $V \to k^{n+1}$ given by $x \mapsto (\overline{\partial_0(x)}, \overline{\partial_1(x)}, \ldots, \overline{\partial_n(x)})$ is surjective.*

Let us write $(\boldsymbol{\partial})_n(x)$ for $(\partial_0(x), \ldots, \partial_n(x))$, and $(\bar{\boldsymbol{\partial}})_n(x)$ for $(\overline{\partial_0(x)}, \ldots, \overline{\partial_n(x)})$. This is supposed to suggest the $\boldsymbol{\sigma}$ and $\bar{\boldsymbol{\sigma}}$ notation. When n is understood we write $\boldsymbol{\partial}$ and $\bar{\boldsymbol{\partial}}$. We will use the ∂_i's in σ-polynomials:

$$F\left(x, \sigma(x), \ldots, \sigma^n(x)\right) = F\left(x, x^p + p\partial_1(x), \ldots, x^{p^n} + p\partial_1(x)^{p^{n-1}} + \cdots + p^n\partial_n(x)\right).$$

Definition 4.3. For fixed prime p, we will consider the polynomial functions $D_n, n \in \mathbb{N}$ (or D if no confusion arises) defined over $\mathbb{Z}$, from affine $(n+1)$-space to itself

$$D(y_0, \ldots, y_n) = (y_0, y_0^p + py_1, y_0^{p^2} + py_1^p + p^2y_2, \ldots, y_0^{p^n} + py_1^{p^{n-1}} + \cdots + p^n y_n).$$

Suppose for the moment that $F(\boldsymbol{y})$ is a homogeneous polynomial of degree m (the only case we will ever use). Now $F(D(\boldsymbol{y}))$ is also a polynomial in $\boldsymbol{y}$. Note that $F(D(\boldsymbol{y}))$ is not in general homogeneous, but it has no constant term and total degree at most mp^n. A moment's reflection on $D(\boldsymbol{y})$ shows:

LEMMA 4.4. (The universal linear maps) *For each prime p and integers m, n, for each $\boldsymbol{j}$ there is a linear function $\Lambda_{\boldsymbol{j}}(\{x_{\boldsymbol{l}}\}_{|\boldsymbol{l}|=m})$ with integer coefficients, so that* *for all homogeneous polynomials in* $\boldsymbol{y}$ *of degree* m *with generic coefficients* $\boldsymbol{c}$*,* $F(\boldsymbol{y},\boldsymbol{c}) = \sum_{|l|=m} c_l \cdot \boldsymbol{y}^l \in \mathbb{Z}[\boldsymbol{y},\boldsymbol{c}]$*, we have* $F(D(\boldsymbol{y})) = \sum_j d_j \cdot \boldsymbol{y}^j$*, where* $d_j = \Lambda_j(\{c_l\}_{|l|=m})$*.*

This lemma will be used in the following situation: $F(\boldsymbol{y})$ is given, so $F(\boldsymbol{\sigma}(x)) = F(D(\boldsymbol{\partial}(x)))$, and we want to find the coefficients of $\boldsymbol{\partial}(x)^{\boldsymbol{j}}$.

The following observation will be crucial in Theorem 6.10.

LEMMA 4.5. *Suppose* $(K, v, k, \Gamma, \sigma)$ *is a Witt-Frobenius case, and* $F(y_0, \dots, y_n)$ *is a polynomial over the valuation ring of* K *with at least one coefficient of valuation* 0*. Then* $F(D(\boldsymbol{y}))$*, still a polynomial over the valuation ring, has at least one coefficient of finite valuation. In particular there exists an integer* $N \geq 0$ *such that* $F(D(\boldsymbol{y}))$ *can be written as* $F(D(\boldsymbol{y})) = p^N f(\boldsymbol{y})$*, where* $f(\boldsymbol{y})$ *is a polynomial over the valuation ring with at least one coefficient of valuation* 0*.*

Proof. Consider the coarse valuation $\dot{v}$ on K (see Lemma 1.7). Then F has also its coefficients in the valuation ring of $\dot{v}$. Let $\dot{F}$ be its image under the residue map of $\dot{v}$. Recall that the residue field $\dot{k}$ of $\dot{v}$ is isomorphic to a valued subfield of $W(k)$, where the valuation on $\dot{k}$ can be identified with v. Whence $\dot{F}$ is a polynomial over the valuation ring of $\dot{k}$ with at least one coefficient of valuation 0. Now we get the required property for $\dot{F}(D(\boldsymbol{y}))$ directly in $W(k)$. But $\dot{F}(D(\boldsymbol{y}))$ is also the image of $F(D(\boldsymbol{y}))$ under the residue map of $\dot{v}$, so because of the identification of the valuation of $\dot{k}$ with v we get the desired property. □

5. Pseudoconvergence in the σ-setting.

5.1. Equivalent pseudoconvergent series. Let $(K, v, k, \Gamma, \sigma)$ be a valued field with isometry with enough constants. Let $\{a_\rho\}$ be a p.c. series in K and a a pseudolimit (maybe in an extension, by which we always mean an extension of valued fields with isometry and when working in the Witt-Frobenius case we mean that the extension is also a Witt-Frobenius case). A very important point in the pure valued field case is the following "pseudocontinuity" (see [18]).

LEMMA 5.1. *Let* $f \in K[x] \setminus K$*. Then* $\{f(a_\rho)\}$ *is p.c., with pseudolimit* $f(a)$*.*

Since the main idea of the proof will be needed later, it is worth recording the lemma on ordered abelian groups on which it depends.

LEMMA 5.2. *Let* $\{\gamma_\rho\}$ *be an increasing series of elements in an ordered abelian group* Γ*. Let* I *be a finite set, and for* $i \in I$ *let* $c_i + n_i x$*,* $c_i \in \Gamma, n_i \in \mathbb{Z}$*, be linear functions of* x *with* distinct n_i*. Then there is a* μ*, and an enumeration* $i_1, i_2, \dots$ *of* I *so that for* $\rho > \mu$*,* $c_{i_1} + n_{i_1}\gamma_\rho < c_{i_2} + n_{i_2}\gamma_\rho < \dots$*.*

Moreover, if γ is a positive element of Γ with only finitely many positive predecessors, there is a μ and an enumeration as above with $(c_{i_k}+n_{i_k}\gamma_\rho)-(c_{i_j}+n_{i_j}\gamma_\rho) > \gamma$ eventually, whenever $k < j$.

Unfortunately, Lemma 5.1 fails in the difference field situation. The reader may readily construct a counterexample using $G(x) = \sigma(x) - x$ instead of the polynomial f. Fortunately there is an alternative to Lemma 5.1, provided we make some extra assumptions on $(K, v, k, \Gamma, \sigma)$. We first need a natural notion of *equivalence* of p.c. series.

Definition 5.3. Two p.c. series $\{a_\rho\}$ and $\{\alpha_\delta\}$ over a valued field K are *equivalent* if for all extension fields L and $a \in L$ we have $\{a_\rho\} \rightsquigarrow a \Leftrightarrow \{\alpha_\delta\} \rightsquigarrow a$.

This is evidently an equivalence relation, and we have clearly:

LEMMA 5.4. *Two $\{a_\rho\}$ and $\{\alpha_\delta\}$ are equivalent if and only if they have a common limit in some extension and have the same width.*

A more explicit way to express the relation is given by:

LEMMA 5.5. *Two series $\{a_\rho\}$ and $\{\alpha_\delta\}$ are equivalent if and only if:*
(a) *for each ρ, eventually (in δ) $v(\alpha_\delta - a_{\rho+1}) > v(a_{\rho+1} - a_\rho)$ and*
(b) *for each δ, eventually (in ρ) $v(a_\rho - \alpha_{\delta+1}) > v(\alpha_{\delta+1} - \alpha_\delta)$.*

We now aim for a series of variations on the theme: if $(K, v, k, \Gamma, \sigma)$ satisfies some natural conditions, and $\{a_\rho\}$ from K is p.c. with limit a (perhaps in an extension valued field with isometry), then for each $G(x) = F(\boldsymbol{\sigma}(x))$ with F nonconstant, there is an equivalent p.c. $\{\alpha_\delta\}$ from K so that $\{G(\alpha_\delta)\} \rightsquigarrow G(a)$. We now develop the calculations needed.

5.2. The basic calculation. Let $\{a_\rho\}$ be given, with pseudolimit a. Let, as usual in these matters, $\gamma_\rho = v(a_\rho - a)$. The γ_ρ form an increasing series in Γ, $G(x)$ is given as $F(\boldsymbol{\sigma}(x))$, n = order of G, $G(x + y) = \sum_{l} G_l(x) \cdot \boldsymbol{\sigma}(y)^l$.

Now we try for an equivalent series $\{\alpha_\rho\} = \{a_\rho+\mu_\rho\theta_\rho\}$, from K, on which G behaves well. Here $\theta_\rho \in K$, $\mu_\rho \in K$, $v(\theta_\rho) = \gamma_\rho$, and (as K has enough constants) θ_ρ may be chosen in the fixed field. Let θ_ρ be so chosen with μ_ρ to be chosen later. We demand at least $v(\mu_\rho) = 0$.

Define d_ρ by $a_\rho - a = \theta_\rho d_\rho$, so d_ρ has value 0. So

$$\begin{aligned} \alpha_\rho - a &= \alpha_\rho - a_\rho + a_\rho - a \\ &= \theta_\rho(\mu_\rho + d_\rho), \end{aligned}$$

so if $v(\mu_\rho + d_\rho) = 0$ we will ensure $\{\alpha_\rho\} \rightsquigarrow a$. Since also $v(\mu_\rho + d_\rho) = 0$ implies that $\{a_\rho\}$ and $\{\alpha_\rho\}$ have the same width, it will imply they are equivalent. Note that d_ρ is forced on us, and it won't normally be in K.

Now

$$\begin{aligned}
G(\alpha_\rho) - G(a) &= \sum_{|\boldsymbol{l}|\geq 1} G_{\boldsymbol{l}}(a)\cdot \boldsymbol{\sigma}(\alpha_\rho - a)^{\boldsymbol{l}} \\
&= \sum_{m\geq 1}\sum_{|\boldsymbol{l}|=m} G_{\boldsymbol{l}}(a)\cdot \boldsymbol{\sigma}(\alpha_\rho - a)^{\boldsymbol{l}} \\
&= \sum_{m\geq 1}\sum_{|\boldsymbol{l}|=m} G_{\boldsymbol{l}}(a)\boldsymbol{\sigma}(\theta_\rho\cdot(\mu_\rho + d_\rho))^{\boldsymbol{l}} \\
&= \sum_{m\geq 1}\sum_{|\boldsymbol{l}|=m} G_{\boldsymbol{l}}(a)\boldsymbol{\sigma}(\theta_\rho)^{\boldsymbol{l}}\cdot \boldsymbol{\sigma}(\mu_\rho + d_\rho)^{\boldsymbol{l}} \\
&= \sum_{m\geq 1} H_m(\mu_\rho + d_\rho),
\end{aligned}$$

where H_m is a σ-polynomial over $K\langle a\rangle$ corresponding to the polynomial

$$\sum_{|\boldsymbol{l}|=m} G_{\boldsymbol{l}}(a)\cdot \boldsymbol{\sigma}(\theta_\rho)^{\boldsymbol{l}}\cdot \boldsymbol{x}^{\boldsymbol{l}} = F_m(\boldsymbol{x}).$$

Now note the value of the coefficients $G_{\boldsymbol{l}}(a)\cdot \boldsymbol{\sigma}(\theta_\rho)^{\boldsymbol{l}}$:

$$v(G_{\boldsymbol{l}}(a)\cdot \boldsymbol{\sigma}(\theta_\rho)^{\boldsymbol{l}}) = v(G_{\boldsymbol{l}}(a)) + m\gamma_\rho$$

for $|\boldsymbol{l}| = m$. Here we use the fact that σ is an isometry. We consider only m for which F_m is nonzero. For such m, pick $\boldsymbol{l}_m$ with $|\boldsymbol{l}_m| = m$ and $v(G_{\boldsymbol{l}_m}(a)\boldsymbol{\sigma}(\theta_\rho)^{\boldsymbol{l}_m})$ is minimal. Then write

$$F_m(\boldsymbol{x}) = G_{\boldsymbol{l}_m}(a)\boldsymbol{\sigma}(\theta_\rho)^{\boldsymbol{l}_m}\cdot f_m(\boldsymbol{x})$$

where f_m is a polynomial over the valuation ring of $K\langle a\rangle$, with one coefficient 1.

Now

$$\begin{aligned}
(1)\qquad v(H_m(\mu_\rho + d_\rho)) &= v(F_m(\boldsymbol{\sigma}(\mu_\rho + d_\rho))) \\
&= v(G_{\boldsymbol{l}_m}(a)\cdot \boldsymbol{\sigma}(\theta_\rho)^{\boldsymbol{l}_m}) + v(f_m(\boldsymbol{\sigma}(\mu_\rho + d_\rho))) \\
&= v(G_{\boldsymbol{l}_m}(a)) + m\gamma_\rho + v(f_m(\boldsymbol{\sigma}(\mu_\rho + d_\rho))).
\end{aligned}$$

Now suppose we can choose μ_ρ so that $v(\mu_\rho) = 0$, $v(\mu_\rho + d_\rho) = 0$ and $v(f_m(\boldsymbol{\sigma}(\mu_\rho + d_\rho))) = \beta_m$ is independent of ρ. Then we succeed in our project, since

$$v(G(\alpha_\rho) - G(a)) = v\left(\sum_{m\geq 1} H_m(u_\rho + d_\rho)\right)$$

$$v(H_m(\mu_\rho + d_\rho)) = v(G_{\boldsymbol{l}_m}(a)) + \beta_m + m\gamma_\rho.$$

So, applying Lemma 5.2, eventually for some fixed m,

$$v(G(\alpha_\rho) - G(a)) = v(G_{\boldsymbol{l}_m}(a)) + \beta_m + m\gamma_\rho,$$

so $G(\alpha_\rho) \rightsquigarrow G(a)$. (Note $v(G_{\boldsymbol{l}_m}(a))$ is independent of the choice of $\boldsymbol{l}_m$).

There are various ways to achieve this, as we will see.

5.3. Pseudocontinuity up to equivalence. Recall the notion of *pliable* $(K, v, k, \Gamma, \sigma)$ (Def. 2.4).

THEOREM 5.6. *Suppose $(K, v, k, \Gamma, \sigma)$ is a pliable and $\bar{\sigma}$ satisfies no identies on k. Suppose $\{a_\rho\}$ is p.c. in K and $\{a_\rho\} \rightsquigarrow a$, possibly in an extension. Let $\sum$ be a finite set of nonzero polynomials $F(x_0, \dots, x_n)$ over K. Then there is a p.c. $\{\alpha_\rho\}$ from K, equivalent to $\{a_\rho\}$, so that for each F in $\sum$, $\{G(\alpha_\rho)\} \rightsquigarrow G(a)$, where $G(x) = F(\boldsymbol{\sigma}(x))$. Furthermore, if one supposes only that $\{a_\rho\}$ is p.c., then there is an equivalent $\{a_\rho\}$ from K such that all $\{G(\alpha_\rho)\}$ are p.c.*

Proof. The last part of the theorem follows from the first by putting in an a, say in an elementary extension. To prove the theorem, let us first consider a single σ-polynomial $G(x)$, and as above let $\gamma_\rho = v(a_\rho - a)$, $\alpha_\rho = a_\rho + \mu_\rho\theta_\rho$, $v(\theta_\rho) = \gamma_\rho$, θ_ρ in the fixed field of K, μ_ρ to be chosen later, $a_\rho - a = \theta_\rho d_\rho$.

By the previous calculation, it suffices to find μ_ρ such that: $\bar{\mu}_\rho \neq 0$, $\bar{\mu}_\rho \neq -\bar{d}_\rho$, $v(f_m(\boldsymbol{\sigma}(\mu_\rho + d_\rho))) = 0$ and $\bar{\mu}_\rho$ in the residue field k of K. We want precisely an element μ of k so that $\bar{f}_m(\bar{\boldsymbol{\sigma}}(\mu + \bar{d}_\rho)) \cdot \mu \cdot (\mu + \bar{d}_\rho) \neq 0$; i.e. that $\bar{f}_m(\bar{\boldsymbol{\sigma}}(\mu) + \bar{\boldsymbol{\sigma}}(\bar{d}_\rho)) \cdot \mu \cdot (\mu + \bar{d}_\rho) \neq 0$. That is, the difference polynomial $\bar{f}_m(\bar{\boldsymbol{\sigma}}(x) + \bar{\boldsymbol{\sigma}}(\bar{d}_\rho)) \cdot x(x + \bar{d}_\rho)$ (which is over $K\langle a\rangle$) should not vanish on k. Note that the corresponding polynomial is $\bar{f}_m(\boldsymbol{x} + \bar{\boldsymbol{\sigma}}(\bar{d}_\rho)) \cdot x_0 \cdot (x_0 + \bar{d}_\rho)$ and this is *not* the zero polynomial since $\bar{f}_m \neq 0$ and the linear change of variables $\boldsymbol{y} := \boldsymbol{x} + \bar{\boldsymbol{\sigma}}(\bar{d}_\rho)$ is invertible.

Now it is not quite obvious that Axiom R4 allows us to select μ_ρ as required, since the polynomial is over $K\langle a\rangle$, and we need $\bar{\mu}_\rho \in k$. We conclude with the following lemma. □

LEMMA 5.7. *Let $(K_1, v_1, k_1, \Gamma_1, \sigma_1)$ be an extension, and $f(x_0, \dots, x_n)$ a nonzero polynomial over k_1. Then there is a y in k with $f(\bar{\boldsymbol{\sigma}}(y)) \neq 0$.*

Proof. Considering the monomials of degree at most the total degree of f as the basis of a finite dimensional k_1-vector space $V = \bigoplus k_1 x^{\boldsymbol{j}}$, we may construe $f(\boldsymbol{x})$ as $\boldsymbol{c} \cdot (\{\boldsymbol{x}^{\boldsymbol{j}}\})$, an inner product, where $\boldsymbol{c}$ is the vector of coefficients of f.

Consider the k-subspace S of V generated by all $(\{\bar{\boldsymbol{\sigma}}(y)^{\boldsymbol{j}}\})$ for y in k. To suppose that $f(\bar{\boldsymbol{\sigma}}(y))$ vanishes on k is to suppose that $\boldsymbol{c}$ is orthogonal to S which is not the zero subspace, as $(1, \dots, 1) \in S$. Choose a finite basis B for S. Then $\boldsymbol{c}$ is a nonzero solution of the system of linear equations $\boldsymbol{b} \cdot \boldsymbol{w} = 0$ for $\boldsymbol{b} \in B$. By elementary linear algebra there is a solution $\boldsymbol{c}'$ in k. Thus, $\boldsymbol{c}'$ is orthogonal to S, and the polynomial with $\boldsymbol{c}'$ as coefficients contradicts Axiom R4. □

For finitely many Gs the same proof works, since Axiom R4 clearly implies the analogous version in which several G occur.

5.4. Pseudocontinuity up to equivalence: the ∂-ring argument. In a Witt-Frobenius case, our treatment depends on the formalism of ∂-rings [16]. The argument adapts easily to the case $\bar{\sigma}(y) = y^q$ for q a finite power of p, but for notational convenience we stick to the special case $q = p$. Again, we need $(K, v, k, \Gamma, \sigma)$ to be pliable (Def. 2.4), and now the key point is that k be infinite.

THEOREM 5.8. *Suppose that $(K, v, k, \Gamma, \sigma)$ is a pliable Witt-Frobenius case. Suppose $\{a_\rho\}$ is p.c. in K and $\{a_\rho\} \rightsquigarrow a$, possibly in an extension. Let $\sum$ be a finite set of nonzero polynomials $F(x_0, \ldots, x_n)$ over K. Then there is a p.c. $\{\alpha_\rho\}$ from K, equivalent to $\{a_\rho\}$, so that for each F in $\sum$, $\{G(\alpha_\rho)\} \rightsquigarrow G(a)$, where $G(x) = F(\boldsymbol{\sigma}(x))$. Furthermore, suppose only that $\{a_\rho\}$ is p.c.: then there is an equivalent $\{a_\rho\}$ from K such that all $\{G(\alpha_\rho)\}$ are pseudoconvergent.*

Proof. Here, as in Theorem 5.6, the last part follows directly from the first, and we begin by considering only one $G(x)$. We revisit the basic calculation (5.2).

Recall that $\gamma_\rho = v(a_\rho - a)$, $\alpha_\rho = a_\rho + \mu_\rho\theta_\rho$, $v(\theta_\rho) = \gamma_\rho$, θ_ρ in the fixed field of K, μ_ρ to be chosen later, $a_\rho - a = \theta_\rho d_\rho$.

We go back to equation (1):

$$v(H_m(\mu_\rho + d_\rho)) = v(G_{l_m}(a)) + m\gamma_\rho + v(f_m(\boldsymbol{\sigma}(\mu_\rho + d_\rho))).$$

As before, it suffices to show that we can find μ_ρ such that $\bar{\mu}_\rho \neq 0, -\bar{d}_\rho$, and $v(f_m(\boldsymbol{\sigma}(\mu_\rho + d_\rho)))$ is independent of ρ.

We have $\boldsymbol{\sigma}(x) = D(\boldsymbol{\partial}(x))$, where D is our polynomial function from Section (4).

If the polynomial $f_m(D(y_0, \ldots, y_n))$ (over $K\langle a\rangle$) is zero, then $H_m(\boldsymbol{\sigma}(x + d_\rho))$ vanishes identically on the valuation ring of $K\langle a\rangle$, and we just ignore it, as will be seen to be harmless at end of our proof.

So let us assume $f_m(D(y_0, \ldots, y_n))$ is not zero. By dividing by a coefficient of lowest value, we can write $f_m(D(y_0, \ldots, y_n)) = \lambda_m \cdot g_m(y_0, \ldots, y_n)$, $\lambda_m \in K\langle a\rangle^*$, g_m over the valuation ring of $K\langle a\rangle$, and with one coefficient 1. By Lemma 4.5, $v(\lambda_m)$ is actually an integer. So

$$v(H_m(\mu_\rho + d_\rho)) = v(G_{l_m}(a)) + m\gamma_\rho + v(\lambda_m) + v(g_m(\boldsymbol{\partial}(\mu_\rho + d_\rho)))$$

and we will succeed if we can arrange $\bar{\mu}_\rho \neq 0$, $\bar{\mu}_\rho \neq -\bar{d}_\rho$ and $v(g_m(\boldsymbol{\partial}(\mu_\rho + d_\rho))) = 0$.

So consider the nonzero polynomial $y_0(y_0 + \bar{d}_\rho)\bar{g}_m(y_0, \ldots, y_n)$ over the residue field of $K\langle a\rangle$. Since k is infinite, there are $t_0, \ldots, t_n \in k$ on which this polynomial does not vanish. Now by Lemma 4.2 there is $t \in V$ with $\overline{\partial_i(t)} = t_i, i = 0, \ldots, n$, and then $\mu_\rho = t$ is our solution and we have proved the theorem.

This argument manifestly works as well with finitely many Gs. $\square$

5.5. Pseudoconvergence to 0. We will need a crucial refinement of the previous Theorems 5.6 and 5.8: roughly, in case $v(G(a_\rho)) \rightsquigarrow 0$ we need to be able to switch to an equivalent $\{\alpha_\rho\}$ such that $v(G(\alpha_\rho)) \rightsquigarrow 0$ still. We stay in pliable fields (Def. 2.4).

THEOREM 5.9. *Let $(K, v, k, \Gamma, \sigma)$ be pliable. Suppose $\{a_\rho\}$ is p.c. in K, and $\{a_\rho\} \rightsquigarrow a$, possibly in an extension, which is a Witt-Frobenius case if K is.*

Let $G(x)$ be a σ-polynomial over K, with $G(x) = F(\boldsymbol{\sigma}(x))$ as usual. Suppose that:

(i) $\{G(a_\rho)\} \rightsquigarrow 0$;

(ii) *for all $\boldsymbol{l}$ with $|\boldsymbol{l}| \geq 1$, such that $F_{\boldsymbol{l}}$ is not a constant, $\{G_{\boldsymbol{l}}(a_\rho)\}$ is p.c., but not to 0;*

(iii) *in a Witt-Frobenius case, we also assume that for each $m \leq$ total degree of F, and $\boldsymbol{j}$ of the appropriate length $\{\Lambda_{\boldsymbol{j}}(\{G_{\boldsymbol{l}}(a_\rho)\}_{|\boldsymbol{l}| \leq m})\}$ is p.c., but not to 0.*
Then there is $\{\alpha_\rho\}$ from K, equivalent to $\{a_\rho\}$ so that $\{G_{\boldsymbol{l}}(\alpha_\rho)\} \rightsquigarrow G_{\boldsymbol{l}}(a)$ if $|\boldsymbol{l}| \geq 0$ and $F_{\boldsymbol{l}}$ is not a constant, and $\{G(\alpha_\rho)\} \rightsquigarrow 0$.

Proof. Recall the notation of the basic calculation (5.2): $\gamma_\rho = v(a_\rho - a)$, $\alpha_\rho = a_\rho + \mu_\rho\theta_\rho$, $v(\theta_\rho) = \gamma_\rho$, θ_ρ in the fixed field of K, μ_ρ to be chosen later, $a_\rho - a = \theta_\rho d_\rho$.

We first prove the case in which $\bar{\sigma}$ satisfies no identities on k. By (ii), for each $|\boldsymbol{l}| \geq 1$, $v(G_{\boldsymbol{l}}(a_\rho))$ is eventually constant.

Now

$$\begin{aligned} G(\alpha_\rho) &= G(a_\rho + \theta_\rho\mu_\rho) \\ &= G(a_\rho) + \sum_{m\geq 1} \sum_{|\boldsymbol{l}|=m} G_{\boldsymbol{l}}(a_\rho) \cdot \boldsymbol{\sigma}(\theta_\rho\mu_\rho)^{\boldsymbol{l}} \\ &= G(a_\rho) + \sum_{m\geq 1} \theta_\rho^m \cdot \sum_{|\boldsymbol{l}|=m} G_{\boldsymbol{l}}(a_\rho)\boldsymbol{\sigma}(\mu_\rho)^{\boldsymbol{l}} \\ &= G(a_\rho) + \sum_{m\geq 1} H_{m,\rho}(\mu_\rho), \end{aligned}$$

where $H_{m,\rho}$ is the σ-polynomial over K corresponding to

$$F_{m,\rho}(\boldsymbol{y}) = \sum_{|\boldsymbol{l}|=m} G_{\boldsymbol{l}}(a_\rho) \cdot \theta_\rho^m \cdot \boldsymbol{y}^{\boldsymbol{l}}.$$

The value of the coefficient $G_{\boldsymbol{l}}(a_\rho) \cdot \theta_\rho^m$ of $F_{m,\rho}$ is $v(G_{\boldsymbol{l}}(a_\rho)) + m\gamma_\rho$. Now write

$$F_{m,\rho}(\boldsymbol{y}) = c_{m,\rho} \cdot f_{m,\rho}(\boldsymbol{y}),$$

where $v(c_{m,\rho})$ is $m\gamma_\rho$+(the eventual minimum of $v(G_{\boldsymbol{l}}(a_\rho))$ for $|\boldsymbol{l}| = m$), and $f_{m,\rho}$ is a polynomial over the valuation ring V with at least one coefficient 1. Then, exactly as in the proof of Theorem 5.6, we could choose μ_ρ to work for G and all $G_{\boldsymbol{l}}$, and to satisfy $v(f_{m,\rho}(\boldsymbol{\sigma}(\mu_\rho))) = 0$ for all m such that $F_{m,\rho} \neq 0$. If such a choice is made,

$$v(G(\alpha_\rho)) = v(G(a_\rho) + \sum_{m \geq 1} \varepsilon_{m,\rho} \theta_\rho^m),$$

where $\varepsilon_{m,\rho} = \sum_{|\boldsymbol{l}|=m} G_{\boldsymbol{l}}(a_\rho) \cdot \boldsymbol{\sigma}(\mu_\rho)^{\boldsymbol{l}}$, and $v(\varepsilon_{m,\rho})$ is eventually constant and independent of the choice of μ_ρ. So we still have space to manoeuver. By Lemma 5.2 applied to the above situation, we would have eventually

$$\begin{aligned} v\left(\sum_{m \geq 1} \varepsilon_{m,\rho} \theta_\rho^m\right) &= \min_{m \geq 1} \left(v(\varepsilon_{m,\rho}) + m\gamma_\rho\right) \\ &= v(\varepsilon_{m_0,\rho}) + m_0 \gamma_\rho \end{aligned}$$

for a unique choice of $m_0 \geq 1$.

If $v(G(a_\rho)) \neq v(\varepsilon_{m_0,\rho}\theta_\rho^{m_0})$, we do nothing. If $v(G(a_\rho)) = v(\varepsilon_{m_0,\rho}\theta_\rho^{m_0})$, then replacing μ_ρ by a variable x consider

$$\begin{aligned} G(a_\rho) + \sum_{m \geq 1} \theta_\rho^m \cdot \sum_{|\boldsymbol{l}|=m} G_{\boldsymbol{l}}(a_\rho) \boldsymbol{\sigma}(x)^{\boldsymbol{l}} &= G(a_\rho) \left(1 + \sum_{|\boldsymbol{l}| \geq 1} G(a_\rho)^{-1} \theta_\rho^{|\boldsymbol{l}|} G_{\boldsymbol{l}}(a_\rho) \boldsymbol{\sigma}(x)^{\boldsymbol{l}}\right) \\ &= G(a_\rho)\ Q_\rho(\boldsymbol{\sigma}(x)), \end{aligned}$$

where $Q_\rho(y_0, \ldots, y_n)$ is a polynomial over V with one coefficient 1. So if we add the extra requirement that $\overline{Q}_\rho(\overline{\boldsymbol{\sigma}}(\overline{\mu}_\rho)) \neq 0$, easily fulfilled as before, we get that eventually

$$v(G(\alpha_\rho)) = \min\{v(G(a_\rho), v(\varepsilon_{m_0,\rho}) + m_0\gamma_\rho\}.$$

Now $v(\varepsilon_{m_0,\rho})$ is eventually constant, and both $v(G(a_\rho))$ and $v(\varepsilon_{m_0,\rho}) + m_0\gamma_\rho$ are eventually increasing. So $v(G(\alpha_\rho))$ is eventually increasing, i.e. $\{G(\alpha_\rho)\} \rightsquigarrow 0$, and we are done in this case.

We now prove the Witt-Frobenius case of the theorem.

We proceed as above to obtain $F_{m,\rho}(\boldsymbol{y}) = c_{m,\rho} \cdot f_{m\rho}(\boldsymbol{y})$, and now do the ∂-transformation on $f_{m,\rho}$. If $f_{m,\rho}(D(\boldsymbol{y}))$ is the zero polynomial then

$$\begin{aligned} H_{m,\rho}(x) &= F_{m,\rho}(\boldsymbol{\sigma}(x)) \\ &= c_{m,\rho} \cdot f_{m,\rho}(\boldsymbol{\sigma}(x)) \\ &= c_{m,\rho} \cdot f_{m,\rho}(D(\boldsymbol{\partial}(x))) \end{aligned}$$

vanishes on the valuation ring of $K\langle a\rangle$, and as in (5.4) we can ignore it.

So we consider m and ρ so that $f_{m,\rho}(D(\boldsymbol{y}))$, and so also $F_{m,\rho}(D(\boldsymbol{y}))$, is not the zero polynomial. Now we apply Lemma 4.4 to

$$F_{m,\rho}(\boldsymbol{y}) = \sum_{|\boldsymbol{l}|=m} G_{\boldsymbol{l}}(a_\rho) \cdot \theta_\rho^m \cdot \boldsymbol{y}^{\boldsymbol{l}}$$

to obtain

$$F_{m,\rho}(D(\boldsymbol{y})) = \theta_\rho^m \cdot \sum_{\boldsymbol{j}} \Lambda_{\boldsymbol{j}}(\{G_{\boldsymbol{l}}(a_\rho)\}_{|\boldsymbol{l}|=m}) \cdot \boldsymbol{y}^{\boldsymbol{j}}.$$

Now, by (iii), the $\Lambda_{\boldsymbol{j}}(\{G_{\boldsymbol{l}}(a_\rho)\}_{|\boldsymbol{l}|=m})$ are eventually constant in value. Let $\lambda_{m,\rho}$ be an eventual minimal value, and write

$$F_{m,\rho}(D(\boldsymbol{y})) = \theta_\rho^m \lambda_{m,\rho} \cdot g_{m,\rho}(\boldsymbol{y}),$$

where $g_{m,\rho}$ is a polynomial over V with at least one coefficient 1. Now, we can play the game of (5.4) to get μ_ρ satisfying all preceding constraints and $v(H_{m,\rho}(\mu_\rho)) = m\gamma_\rho + v(\lambda_{m,\rho})$. Note that $v(\lambda_{m,\rho})$ is eventually constant for each m and is independent of the choice of μ_ρ.

So we have again:

$$v(G(\alpha_\rho)) = v\left(G(a_\rho) + \sum_{m\geq 1} \varepsilon_{m,\rho} \cdot \theta_\rho^m\right),$$

where $\varepsilon_{m,\rho} = \sum_{|\boldsymbol{l}|=m} G_{\boldsymbol{l}}(a_\rho) \cdot \boldsymbol{\sigma}(\mu_\rho)^{\boldsymbol{l}}$, and $v(\varepsilon_{m,\rho}) = v(\lambda_{m,\rho})$ is eventually constant and independent of the choice of μ_ρ. By Lemma 5.2 we can eventually find an m_0 such that $v(\varepsilon_{m_0,\rho} \cdot \theta_\rho^{m_0}) < v(\varepsilon_{m,\rho} \cdot \theta_\rho^m)$ for all $m \neq m_0$, so that

$$\begin{aligned} v\left(\sum_{m\geq 1} \varepsilon_{m,\rho}\theta_\rho^m\right) &= \min_{m\geq 1}(v(\varepsilon_{m,\rho}) + m\gamma_\rho) \\ &= v(\varepsilon_{m_0,\rho}) + m_0\gamma_\rho. \end{aligned}$$

If $v(G(a_\rho)) \neq v(\varepsilon_{m_0,\rho}\theta_\rho^{m_0})$, we do nothing. If $v(G(a_\rho)) = v(\varepsilon_{m_0,\rho}\theta_\rho^{m_0})$, then consider

$$G(a_\rho) + \varepsilon_{m_0,\rho}\theta_\rho^{m_0} = G(a_\rho) + F_{m_0,\rho}(D(\boldsymbol{\partial}(\mu_\rho))).$$

Since $F_{m_0,\rho}(\boldsymbol{y})$ is homogeneous of degree $m_0 \geq 1$, $F_{m_0,\rho}(D(\boldsymbol{y}))$ has no constant term. So we can write

$$G(a_\rho) + F_{m_0,\rho}(D(\boldsymbol{y})) = G(a_\rho) \cdot Q_\rho(\boldsymbol{y}),$$

where $Q_\rho(y_0, \ldots, y_n)$ is a polynomial over V with constant term 1 and at least one other coefficient with valuation zero. The extra requirement that $v(Q_\rho(\boldsymbol{\partial}(\mu_\rho))) = 0$ can be fulfilled as before, and we get that eventually

$$v(G(\alpha_\rho)) = \min\{v(G(a_\rho), v(\varepsilon_{m_0,\rho}) + m_0\gamma_\rho\}$$

and we are done as in the previous case. □

We remark that we will also use a small variant of this theorem where we add finitely many more σ-polynomials to the $G_{\boldsymbol{l}}$ and $\Lambda_j\left(\{G_{\boldsymbol{l}}\}_{|\boldsymbol{l}|=m}\right)$ (see the proofs of Theorem 6.10 and Lemma 7.2).

6. Around the σ-Hensel Scheme.

6.1. Newton approximation. For the moment we consider the basic problem of how to start with $\alpha \in K$ and $G(\alpha) \neq 0$ and find $\beta \in K$ with $v(G(\beta)) > v(G(\alpha))$.

Definition 6.1. Suppose $G(x) = F(\boldsymbol{\sigma}(x))$ as usual, and $\alpha \in K$. We say α, G is in *σ-Hensel configuration* if

$$\begin{aligned} v(G(\alpha)) &= \gamma + \min_{|\boldsymbol{l}|=1} v(G_{\boldsymbol{l}}(\alpha)) \\ &< j \cdot \gamma + v(G_{\boldsymbol{l}}(\alpha)) \end{aligned}$$

whenever $|\boldsymbol{l}| = j > 1$.

Note that with G of order 0, we get one of the equivalent configurations of the standard Hensel scheme.

LEMMA 6.2. *Suppose $(K, v, k, \Gamma, \sigma)$ is a valued field with isometry with $(k, \bar{\sigma})$ linearly difference closed. Suppose α, G are in σ-Hensel configuration, with $v(G(\alpha)) = \gamma + \min_{|\boldsymbol{l}|=1} v(G_{\boldsymbol{l}}(\alpha))$. Then there is β with $v(\alpha - \beta) = \gamma, \beta, G$ in σ-Hensel configuration, and $v(G(\beta)) > v(G(\alpha))$.*

Proof. We try $\beta = \alpha + \varepsilon$, so

$$G(\beta) = G(\alpha) + \sum_{\boldsymbol{l} \geq 1} G_{\boldsymbol{l}}(\alpha) \cdot \boldsymbol{\sigma}(\varepsilon)^{\boldsymbol{l}}.$$

We first aim for $v(G(\beta)) > v(G(\alpha)) = \gamma + \min_{|\boldsymbol{l}|=1} v(G_{\boldsymbol{l}}(\alpha))$, suggesting that we aim for ε with

$$v\left(\sum_{|\boldsymbol{l}|=1} G_{\boldsymbol{l}}(\alpha) \cdot \boldsymbol{\sigma}(\varepsilon)^{\boldsymbol{l}} + G(\alpha)\right) > \gamma + \min_{|\boldsymbol{l}|=1} v(G_{\boldsymbol{l}}(\alpha)). \tag{2}$$

Pick λ so that $v(\lambda) = \gamma$ and let $\varepsilon = \lambda u$, for u a unit to be determined. Fix $\boldsymbol{l}_0$ with $|\boldsymbol{l}_0| = 1$, so $v(G_{\boldsymbol{l}_0}(\alpha)) = \min_{|\boldsymbol{l}|=1} v(G_{\boldsymbol{l}}(\alpha))$. Note that for $|\boldsymbol{l}| = 1$, $\boldsymbol{\sigma}(\varepsilon)^{\boldsymbol{l}} = \boldsymbol{\sigma}(\lambda)^{\boldsymbol{l}} \cdot \boldsymbol{\sigma}(u)^{\boldsymbol{l}}$ and $v(\boldsymbol{\sigma}(\lambda)^{\boldsymbol{l}}) = \gamma$.

So we want

$$v\left(\sum_{|\boldsymbol{l}|=1}\left(\frac{G_{\boldsymbol{l}}(\alpha)}{G_{\boldsymbol{l}_0}(\alpha)}\right)\cdot\frac{\boldsymbol{\sigma}(\lambda)^{\boldsymbol{l}}}{\lambda}\cdot\boldsymbol{\sigma}(u)^{\boldsymbol{l}}+\frac{G(\alpha)}{\lambda G_{\boldsymbol{l}_0}(\alpha)}\right) > 0.$$

Let

$$\begin{aligned} c_{\boldsymbol{l}} &= G_{\boldsymbol{l}}(\alpha)\cdot\boldsymbol{\sigma}(\lambda)^{\boldsymbol{l}}/\lambda\cdot G_{\boldsymbol{l}_0}(\alpha) \\ d &= G(\alpha)/\lambda\cdot G_{\boldsymbol{l}_0}(\alpha). \end{aligned}$$

$\sum_{|\boldsymbol{l}|=1} c_{\boldsymbol{l}}\cdot\boldsymbol{\sigma}(x)^{\boldsymbol{l}} + d$ is a nontrivial *linear* σ-polynomial over V, with $v(c_{\boldsymbol{l}_0}) = 0$, and we can find ε satisfying (2) if we can solve

$$\sum_{|\boldsymbol{l}|=1}\bar{c}_{\boldsymbol{l}}\,\bar{\boldsymbol{\sigma}}(y)^{\boldsymbol{l}}+\bar{d}=0$$

over k. Note that $v(d) = 0$, so $\bar{d} \neq 0$. Since $(k, \bar{\sigma})$ is linearly difference closed, let t be a solution of

$$\sum_{|\boldsymbol{l}|=1}\bar{c}_{\boldsymbol{l}}\cdot\bar{\boldsymbol{\sigma}}(y)^{\boldsymbol{l}}+\bar{d}=0$$

and choose u so that $\bar{u} = t$. Note that $v(\varepsilon) = v(\lambda) = \gamma$. Then (2) holds.
Now

$$\begin{aligned} v(G(\beta)) &= v(G(\alpha+\varepsilon)) \\ &= v\left(\sum_{|\boldsymbol{l}|=1}G_{\boldsymbol{l}}(\alpha)\cdot\boldsymbol{\sigma}(\varepsilon)^{\boldsymbol{l}}+G(\alpha)+\sum_{|\boldsymbol{l}|>1}G_{\boldsymbol{l}}(\alpha)\cdot\boldsymbol{\sigma}(\varepsilon)^{\boldsymbol{l}}\right). \end{aligned}$$

For $|\boldsymbol{l}| > 1$,

$$v(G_{\boldsymbol{l}}(\alpha)\cdot\boldsymbol{\sigma}(\varepsilon)^{\boldsymbol{l}}) = v(G_{\boldsymbol{l}}(\alpha)) + |\boldsymbol{l}|\cdot\gamma > v(G_{\boldsymbol{l}_0}(\alpha))+\gamma$$

by assumption. So

$$v\left(\sum_{|\boldsymbol{l}|>1}G_{\boldsymbol{l}}(\alpha)\cdot\boldsymbol{\sigma}(\varepsilon)^{\boldsymbol{l}}\right) > v(G_{\boldsymbol{l}_0}(\alpha))+\gamma.$$

So by (2),

$$v(G(\beta)) > v(G_{\boldsymbol{l}_0}(\alpha)) + \gamma = v(G(\alpha)).$$

Now we have to show that β, G is in σ-Hensel configuration.

For $|\boldsymbol{l}| \geq 1$,

$$G_{\boldsymbol{l}}(\beta) = G_{\boldsymbol{l}}(\alpha) + \sum_{|\boldsymbol{j}|\geq 1} (G_{\boldsymbol{l}})_{\boldsymbol{j}}(\alpha) \cdot \boldsymbol{\sigma}(\varepsilon)^{\boldsymbol{j}}.$$

Recall that

$$(G_{\boldsymbol{l}})_{\boldsymbol{j}}(\alpha) = \binom{\boldsymbol{l}+\boldsymbol{j}}{\boldsymbol{j}} G_{\boldsymbol{l}+\boldsymbol{j}}(\alpha),$$

so that if $|\boldsymbol{l}| \geq 1$ and $|\boldsymbol{j}| \geq 1$

$$\begin{aligned} v\left((G_{\boldsymbol{l}})_{\boldsymbol{j}}(\alpha) \cdot \boldsymbol{\sigma}(\varepsilon)^{\boldsymbol{j}}\right) &= v\left(\binom{\boldsymbol{l}+\boldsymbol{j}}{\boldsymbol{j}}\right) + v(G_{\boldsymbol{l}+\boldsymbol{j}}(\alpha)) + |\boldsymbol{j}| \cdot \gamma \\ &> v(G_{\boldsymbol{l}_0}(\alpha)) + (1 - (|\boldsymbol{l}| + |\boldsymbol{j}|)) \cdot \gamma + |\boldsymbol{j}| \cdot \gamma \\ &> v(G_{\boldsymbol{l}_0}(\alpha)) + (1 - |\boldsymbol{l}|) \cdot \gamma. \end{aligned}$$

In particular, if $|\boldsymbol{l}| = 1$, and $|\boldsymbol{j}| \geq 1$

$$v\left((G_{\boldsymbol{l}})_{\boldsymbol{j}}(\alpha) \cdot \boldsymbol{\sigma}(\varepsilon)^{\boldsymbol{j}}\right) > v(G_{\boldsymbol{l}_0}(\alpha)).$$

We conclude that

$$v(G_{\boldsymbol{l}_0}(\beta)) = v(G_{\boldsymbol{l}_0}(\alpha))$$

and in fact

$$v(G_{\boldsymbol{l}}(\beta)) = v(G_{\boldsymbol{l}}(\alpha))$$

if $|\boldsymbol{l}| = 1$ and $v(G_{\boldsymbol{l}}(\alpha)) = v(G_{\boldsymbol{l}_0}(\alpha))$. By the same argument, if $|\boldsymbol{l}| = 1$ and $v(G_{\boldsymbol{l}}(\alpha)) > v(G_{\boldsymbol{l}_0}(\alpha))$ then $v(G_{\boldsymbol{l}}(\beta)) > v(G_{\boldsymbol{l}_0}(\alpha)) = v(G_{\boldsymbol{l}_0}(\beta))$.
Let $v(G(\beta)) = \gamma_1 + v(G_{\boldsymbol{l}_0}(\beta))$, so $\gamma_1 > \gamma$. For $|\boldsymbol{l}| > 1$

$$G_{\boldsymbol{l}}(\beta) = G_{\boldsymbol{l}}(\alpha) + \sum_{|\boldsymbol{j}|\geq 1} (G_{\boldsymbol{l}})_{\boldsymbol{j}}(\alpha) \cdot \boldsymbol{\sigma}(\varepsilon)^{\boldsymbol{j}}$$

and as before for $|\boldsymbol{j}| \geq 1$

$$\begin{aligned} v((G_{\boldsymbol{l}})_{\boldsymbol{j}}(\alpha) \cdot \boldsymbol{\sigma}(\varepsilon)^{\boldsymbol{j}}) &> v(G_{\boldsymbol{l}_0}(\alpha)) + (1 - |\boldsymbol{l}| - |\boldsymbol{j}|) \cdot \gamma + |\boldsymbol{j}| \cdot \gamma \\ &= v(G_{\boldsymbol{l}_0}(\alpha)) + (1 - |\boldsymbol{l}|) \cdot \gamma. \end{aligned}$$

Also,

$$v(G_{\boldsymbol{l}}(\alpha)) > v(G_{\boldsymbol{l}_0}(\alpha)) + (1 - |\boldsymbol{l}|) \cdot \gamma$$

so

$$v(G_{\boldsymbol{l}}(\beta)) + |\boldsymbol{l}|\gamma > v(G_{\boldsymbol{l}_0}(\alpha)) + \gamma.$$

So, since $\gamma_1 > \gamma$,

$$v(G_{\boldsymbol{l}}(\beta)) + |\boldsymbol{l}| \cdot \gamma_1 > v(G_{\boldsymbol{l}_0}(\beta)) + \gamma_1.$$

Thus β, G is in σ-Hensel configuration. □

We note the following direct consequence.

COROLLARY 6.3. *Let $(K, v, k, \Gamma, \sigma)$ be a valued field with isometry such that $(k, \bar{\sigma})$ is linearly difference closed and (K, v) is a complete discrete valued field. Then whenever α, G is in σ-Hensel configuration, with*

$$v(G(\alpha)) = \gamma + \min_{|\boldsymbol{l}|=1} v(G_{\boldsymbol{l}}(\alpha))$$

there is a root β of G with $v(\alpha - \beta) = \gamma$.

Definition 6.4. (The σ-Hensel Scheme) We say $(K, v, k, \gamma, \sigma)$ satisfies the σ-Hensel scheme, or is σ-henselian, if whenever α, G is in σ-Hensel configuration with

$$v(G(\alpha)) = \gamma + \min_{|\boldsymbol{l}|=1} v(G_{\boldsymbol{l}}(\alpha))$$

then there is β in K with $v(\alpha - \beta) = \gamma$ and $G(\beta) = 0$.

Similar schemes were considered for difference operators by Duval [12].

It bears noting that just as the notion of "henselian" may be formulated in many different ways, so, too, may the condition of being "σ-henselian." The reader can consult [29] for another treatment.

Thus, $W(k)$ satisfies the σ-Hensel scheme if k satisfies: all equations

$$c_n x^{p^n} + c_{n-1} x^{p^{n-1}} + \cdots + c_0 = 0$$

with $c_i \in k$, $c_n \neq 0, c_0 \neq 0$, are solvable in k. This is equivalent to k not having any finite extension of degree divisible by p [31].

Note that if $(K, v, k, \gamma, \sigma)$ satisfies the σ-Hensel scheme then (K, v) is henselian. In analogy with a familiar, important result about henselian fields, we have:

THEOREM 6.5. *Suppose k has characteristic 0 and $(K, v, k, \Gamma, \sigma)$ satisfies the σ-Hensel scheme. Let K_0 be a difference subfield of K, maximal with respect to the property $x \in K_0^* \Rightarrow v(x) = 0$. Then (K_0, σ) is isomorphic to $(k, \bar{\sigma})$ via $x \mapsto \bar{x}$.*

Proof. Suppose that $x \mapsto \bar{x}$ is not surjective (it is clearly $1-1$).

Suppose $\alpha \in K, v(\alpha) = 0, v(\alpha - y) = 0$, all $y \in K_0$. If for all σ-polynomials $H(x)$ over the valuation ring of K_0 we have $v(H(\alpha)) = 0$, we are done. Otherwise, pick $G(x)$ over the valuation ring of K_0, of minimal complexity so that $v(G(\alpha)) > 0$. So for all $H(x)$ of lower complexity $v(H(x)) = 0$. It follows that α, G is in σ-Hensel configuration, with $\gamma = v(G(\alpha))$. So there is $\beta \in K$ with $G(\beta) = 0, v(\alpha - \beta) = \gamma$, so $\bar{\alpha} = \bar{\beta}$.

Now, by the standard minimality considerations,

$$y \in K_0\langle\beta\rangle^* \Rightarrow v(y) = 0$$

So $\beta \in K_0$, and $v(\alpha - \beta) > 0$, contradiction. $\square$

The previous theorem will be used in particular in the following situation with the coarse valuation (cf. Lemma 1.7).

LEMMA 6.6. *Suppose $(K, v, k, \Gamma, \sigma)$ is unramified with k of characteristic $p > 0$, and satisfies the σ-Hensel scheme. Let Γ_0 be the convex subgroup generated by $v(p)$ and $\dot{v}$ the coarse valuation. Then $(K, \dot{v}, \dot{k}, \Gamma/\Gamma_0, \sigma)$ also satisfies the σ-Hensel scheme.*

Proof. Suppose a, G is in σ-Hensel configuration for $\dot{v}$. So we have

$$\begin{aligned} \dot{v}(G(a)) &= \min_{|\bar{l}|=1} \dot{v}(G_{\bar{l}}(a)) + (\gamma + \Gamma_0) \\ &< \dot{v}(G_{\bar{j}}(a)) + |\bar{j}|(\gamma + \Gamma_0) \quad \text{if } |\bar{j}| > 1 \end{aligned}$$

for some $\gamma \in \Gamma$.

Since $\Gamma \to \Gamma/\Gamma_0$ is order-preserving, it is easily seen that for some γ_1 with $\gamma_1 - \gamma \in \Gamma_0$,

$$\begin{aligned} v(G(a)) &= \min_{|\bar{l}|=1} v(G_{\bar{l}}(a)) + \gamma_1 \\ &< v(G_{\bar{j}}(a)) + |\bar{j}| \cdot \gamma_1 \quad , \text{if } |\bar{j}| > 1 \end{aligned}$$

so there is β with $G(\beta) = 0, v(\beta - a) = \gamma$, so $v(\beta - a) = \gamma + \Gamma$. $\square$

6.2. Pseudoconvergence of σ-algebraic type. We consider a valued field with isometry $(K, v, k, \Gamma, \sigma)$. Recall the universal polynomials Λ_j from the ∂-ring formalism.

Definition 6.7. For a σ-polynomial G over K, $G(x) = F(\boldsymbol{\sigma}(x))$ as usual, we denote by $\mathcal{H}(G)$ the set of finitely many σ-polynomials obtained from G by closing under the following operations:

(i) taking all $G_{\boldsymbol{l}}$, for which the corresponding polynomials $F_{\boldsymbol{l}}$ are not constant and $|\boldsymbol{l}| \geq 1$;

(ii) in Witt-Frobenius case, taking also all nonconstant $\Lambda_j(\{G_{\boldsymbol{l}}\})$, for the corresponding polynomial $F_m(\boldsymbol{y}) = \sum_{|\boldsymbol{l}|=m} G_{\boldsymbol{l}}(\boldsymbol{x}) \cdot \boldsymbol{y}^{\boldsymbol{l}}$ with $m \leq$ total degree of F.
The set $\mathcal{H}(G)$ is finite since operations (i), (ii) yield σ-polynomials of lower complexity.

Definition 6.8. Suppose $\{a_\rho\}$ from K is p.c. We say $\{a_\rho\}$ is of σ-algebraic type over K if there is a an equivalent p.c. series $\{a'_\rho\}$ from K and a σ-polynomial $G(x)$ over K so that $\{G(a'_\rho)\} \rightsquigarrow 0$, and for all $H \in \mathcal{H}(G)$, $\{H(a'_\rho)\}$ is pseudoconvergent.

The analogy of this definition with the classical case is not as direct as might be expected, but it will be sufficient for our purposes (see Theorem 7.5). A referee has pointed out that one can drop the extra assumptions on $\mathcal{H}(G)$, but at the cost of introducing stronger hypotheses in some of our intermediate results, i.e., in Theorem 5.9.

Definition 6.9. We say $\{a_\rho\}$ is of σ-transcendental type over K if it is not of σ-algebraic type over K.

6.3. From σ-algebraic type to σ-Hensel configuration.

THEOREM 6.10. *Let $(K, v, k, \Gamma, \sigma)$ be pliable. Suppose $\{a_\rho\}$ is p.c. in K, and $\{a_\rho\} \rightsquigarrow a$, possibly in an extension, which is a Witt-Frobenius case if K is.*

Suppose $\{a_\rho\}$ is of σ-algebraic type and $G(x)$ is a σ-polynomial of minimal complexity witnessing such a fact amongst all series equivalent to $\{a_\rho\}$.

Then there is $\{\alpha_\rho\}$ from K, equivalent to $\{a_\rho\}$, so that $\{G(\alpha_\rho)\} \rightsquigarrow 0$, and for all $H \in \mathcal{H}(G)$, $\{H(\alpha_\rho)\} \rightsquigarrow H(a)$, $\{H(\alpha_\rho)\} \not\rightsquigarrow 0$, and such that eventually α_ρ, G is in σ-Hensel configuration. Moreover, either $G(a) = 0$, or a, G is in σ-Hensel configuration.

Proof. Replacing $\{a_\rho\}$ by an equivalent series if necessary, we can assume G witnesses σ-algebraic type for $\{a_\rho\}$. Let $G(x) = F(\boldsymbol{\sigma}(x))$ as usual. By minimality, because of the inductive structure of $\mathcal{H}(G)$, we get that $\{G(a_\rho\} \rightsquigarrow 0$ and for all $H \in \mathcal{H}(G)$, $\{H(a_\rho)\}$ is p.c. but not to 0. We can then return to the calculations in the proof of Theorem 5.9. The proof produces an equivalent $\{\alpha_\rho\}$, so that for all $H \in \mathcal{H}(G)$, $\{H(\alpha_\rho)\} \rightsquigarrow H(a)$. By minimality of G, we also have $\{H(\alpha_\rho)\} \not\rightsquigarrow 0$, all H. In particular, $\{G_{\boldsymbol{l}}(\alpha_\rho)\} \not\rightsquigarrow 0$, all $|\boldsymbol{l}| \geq 1$, and for those $v(G_{\boldsymbol{l}}(\alpha_\rho)) = v(G_{\boldsymbol{l}}(a))$ eventually. Inspection of the proof of Theorem 5.9 yields considerably more. Let us suppose, without loss of generality, that $\{\alpha_\rho\}$ is just $\{a_\rho\}$. The proof actually gives the following:

For each $\boldsymbol{l}$ such that $F_{\boldsymbol{l}}$ is not a constant, there is a unique $m(\boldsymbol{l})) = m \geq 1$, and natural numbers $i_m, i_{|\boldsymbol{k}|,\boldsymbol{l}}$ such that

$$v(G_{\boldsymbol{l}}(a_\rho) - G_{\boldsymbol{l}}(a)) = \min_{|\boldsymbol{j}|=m} v((G_{\boldsymbol{l}})_{\boldsymbol{j}}(a)) + m\gamma_\rho + i_m < v((G_{\boldsymbol{l}})_{\boldsymbol{k}}(a)) + |\boldsymbol{k}| \cdot \gamma_\rho + i_{|\boldsymbol{k}|,\boldsymbol{l}}$$

whenever $|\boldsymbol{k}| \neq m$ and $|\boldsymbol{k}| \geq 1$ (the i's appear only in the Witt-Frobenius case).

Moreover, since $\{v(G(a_\rho))\}$ is increasing eventually we eventually get $v(G(a)) > v(G(a_\rho))$, and so eventually

$$\begin{aligned} v(G(a_\rho)) &= \min_{|\boldsymbol{j}|=m(\boldsymbol{0})} v(G_{\boldsymbol{j}}(a)) + m(\boldsymbol{0}) \cdot \gamma_\rho + i_m(\boldsymbol{0}) \\ v(G(a_\rho)) &= \min_{|\boldsymbol{j}|=m(\boldsymbol{0})} v(G_{\boldsymbol{j}}(a_\rho)) + m(\boldsymbol{0}) \cdot \gamma_\rho + i_m(\boldsymbol{0}). \end{aligned}$$

We claim that $m(\boldsymbol{0}) = 1$.

Suppose not, and choose $\boldsymbol{j}$ with $|\boldsymbol{j}| = m(\boldsymbol{0})$, and

$$v(G_{\boldsymbol{j}}(a)) = \min_{|\boldsymbol{k}|=m(\boldsymbol{0})} v(G_{\boldsymbol{k}}(a)).$$

Choose $\boldsymbol{l} \leq \boldsymbol{j}, |\boldsymbol{l}| = 1$. Now $v(G_{\boldsymbol{l}}(a_\rho)) = v(G_{\boldsymbol{l}}(a))$ eventually, so eventually

$$v(G_{\boldsymbol{l}}(a_\rho)) \leq v((G_{\boldsymbol{l}})_{\boldsymbol{k}}(a)) + |\boldsymbol{k}| \cdot \gamma_\rho + i_{|\boldsymbol{k}|,\boldsymbol{l}} - i_{m(\boldsymbol{0})},$$

so

$$v(G_{\boldsymbol{l}}(a_\rho)) \leq v\left(\binom{\boldsymbol{l}+\boldsymbol{k}}{\boldsymbol{k}}\right) + v(G_{\boldsymbol{l}+\boldsymbol{k}}(a)) + |\boldsymbol{k}| \cdot \gamma_\rho + i_{|\boldsymbol{k}|,\boldsymbol{l}} - i_{m(\boldsymbol{0})}$$

for all $\boldsymbol{k}$ with $|\boldsymbol{k}| \geq 1$.

Choose $\boldsymbol{k}$ so $\boldsymbol{l} + \boldsymbol{k} = \boldsymbol{j}$, and we have eventually

$$v(G_{\boldsymbol{l}}(a)) \leq v\left(\binom{\boldsymbol{j}}{\boldsymbol{k}}\right) + v(G_{\boldsymbol{j}}(a)) + (|\boldsymbol{j}| - |\boldsymbol{l}|) \cdot \gamma_\rho + i_{|\boldsymbol{j}-\boldsymbol{l}|,\boldsymbol{l}} - i_{m(\boldsymbol{0})}.$$

Since K is unramified we have eventually, by Lemma 5.2,

$$v(G_{\boldsymbol{l}}(a)) < v(G_{\boldsymbol{j}}(a)) + (|\boldsymbol{j} - \boldsymbol{l}|) \cdot \gamma_\rho + i_{|\boldsymbol{j}|,\boldsymbol{l}} - i_{m(\boldsymbol{0})}$$

$$v(G_{\boldsymbol{l}}(a)) + \gamma_\rho + i_{m(\boldsymbol{0})} < v(G_{\boldsymbol{j}}(a)) + |\boldsymbol{j}| \cdot \gamma_\rho + i_{|\boldsymbol{j}|,\boldsymbol{l}},$$

a contradiction. So $m(\boldsymbol{0}) = 1$. Hence we have eventually

$$v(G(a_\rho)) = \min_{|\boldsymbol{j}|=1} G_{\boldsymbol{j}}(a) + \gamma_\rho + i_{m(\boldsymbol{0})} < v(G_{\boldsymbol{k}}(a)) + |\boldsymbol{k}| \cdot \gamma_\rho + i_{|\boldsymbol{k}|,\boldsymbol{0}}$$

whenever $|\boldsymbol{k}| > 1$. Appealing again to Lemma 5.2 we get eventually

$$v(G(a_\rho)) = \min_{|\boldsymbol{j}|=1} v(G_{\boldsymbol{j}}(a)) + \gamma_\rho + i_{m(\mathbf{0})} < v(G_{\boldsymbol{k}}(a)) + |\boldsymbol{k}| \cdot \gamma_\rho + |\boldsymbol{k}| \cdot i_{m(\mathbf{0})}$$
$$< v(G_{\boldsymbol{k}}(a)) + |\boldsymbol{k}| \cdot (\gamma_\rho + i_{m(\mathbf{0})}),$$

so a_ρ, G are eventually in σ-Hensel configuration. Now suppose $G(a) \neq 0$. We already observed that $v(G(a)) > v(G(a_\rho))$ eventually. Then the same argument as in the last part of Lemma 6.2 shows that a, G are in σ-Hensel configuration (replacing β by a and α by a_ρ). $\square$

7. Maximal immediate $\boldsymbol{\sigma}$-extensions. We now develop a theory of maximal immediate σ-extensions.

Note that the classical theory tells us that the cardinality of (K, v, k, Γ) is bounded by a bound depending only on the cardinality of k and Γ (essentially the cardinality of the power series field $k((t^\Gamma)))$, so that by the usual maximality arguments, any field with isometry $(K, v, k, \Gamma, \sigma)$ has at least one maximal immediate σ-extension, and one maximal immediate σ-algebraic σ-extension.

Let $G(x)$ be a σ-polynomial over K and $\{a_\rho\}$ from K so that $\{a_\rho\} \rightsquigarrow a$ (a anywhere). We will use the notation

$$\{G(a_\rho)\} \rightsquigarrow_s G(a)$$

if $\{G(a_\rho)\} \rightsquigarrow G(a)$ and for all $H \in \mathcal{H}(G)$, $\{H(a_\rho)\} \rightsquigarrow H(a)$.

The next lemma is a routine extension of a result familiar in the polynomial setting.

LEMMA 7.1. *Suppose $(K, v, k, \Gamma, \sigma)$ is pliable. Let $\{a_\rho\}$ be p.c. of σ-transcendental type over K, with no pseudolimit in K. Then there is a proper immediate extension $(K\langle a\rangle, v, k, \Gamma, \sigma)$ with a σ-transcendental over K, such that $\{a_\rho\} \rightsquigarrow a$ and $(K\langle a\rangle, v, k, \Gamma, \sigma)$ is pliable. Conversely, suppose b is a σ-transcendental element over K in some pliable σ-extension of K and $\{a_\rho\} \rightsquigarrow b$. Then $(K\langle a\rangle, v, k, \Gamma, \sigma)$ and $(K\langle b\rangle, v, k, \Gamma, \sigma)$ are K-isomorphic by an isomorphism sending a onto b.*

Proof. For the first part, let $(K_2, v_2, k_2, \Gamma_2, \sigma_2)$ be an elementary extension of K containing a pseudolimit a of $\{a_\rho\}$. Let $K_1 = K\langle a\rangle$. If we show K_1 is immediate, the elementarity of K_2 clearly gives the rest. Let $G(x)$ be a σ-polynomial over K, not coming from a constant F. Use Theorems 5.6 and 5.8 to get an equivalent $\{a'_\rho\}$ so that $\{G(a'_\rho)\} \rightsquigarrow_s G(a)$. Now, $\{G(a'_\rho)\} \not\rightsquigarrow 0$, since $\{a_\rho\}$ is of σ-transcendental type. So $G(a) \neq 0$ and $v(G(a)) =$ eventual value $v(G(a'_\rho)) \in \Gamma$. Thus K_1 has the same value group as K. A similar, standard argument shows that it has the same residue field.

For the second part, suppose b is σ-transcendental over K such that $\{a_\rho\} \rightsquigarrow b$. We claim that the unique σ-isomorphism $K\langle a\rangle \cong K\langle b\rangle$ fixing K and sending a to b is an isomorphism of valued fields.

Let $H(x)$ be a non constant σ-polynomial over K. Use sections (5.2)-(5.4) to get $\{a'_\rho\} \sim \{a_\rho\}$ with $\{H(a'_\rho)\} \rightsquigarrow_s H(a)$ and *simultaneously* $\{H(a'_\rho)\} \rightsquigarrow_s H(b)$. This is straightforward in the Witt-Frobenius case, and not difficult to see in the case that $\bar{\sigma}$ satisfies no identities.

Now $\{H(a'_\rho)\} \not\rightsquigarrow 0$, since $\{a_\rho\}$ is not of σ-algebraic type. So $v(H(a'_\rho))$ is eventually constant and we have

$$\begin{aligned} v(H(b)) &= \text{eventual value } v(H(a'_\rho)) \\ &= v(H(a)). \end{aligned}$$

To conclude that $K\langle a\rangle \cong K\langle b\rangle$ is an isomorphism of valued fields, observe that an arbitrary element of $K\langle a\rangle$ is a quotient of elements of the form $\sigma^m(H(a))$ for some $m \in \mathbb{Z}$, H as above. □

As in the classical setting, the σ-algebraic case is trickier.

LEMMA 7.2. *Suppose $(K, v, k, \Gamma, \sigma)$ is pliable. Let $\{a_\rho\}$ be p.c. of σ-algebraic type over K, with no pseudolimit in K. Let G be a σ-polynomial of minimal complexity witnessing σ-algebraic type amongst all equivalent series. Then there is a proper immediate extension $(K\langle a\rangle, v, k, \Gamma, \sigma)$ of K, with $G(a) = 0$, such that $\{a_\rho\} \rightsquigarrow a$ and $(K\langle a\rangle, v, k, \Gamma, \sigma)$ is pliable. Conversely, suppose b is a solution to $G(x) = 0$ in some pliable σ-extension of K and $\{a_\rho\} \rightsquigarrow b$. Then $(K\langle a\rangle, v, k, \Gamma, \sigma)$ and $(K\langle b\rangle, v, k, \Gamma, \sigma)$ are K-isomorphic by an isomorphism sending a onto b.*

Proof. We may assume G witnesses σ-algebraic type for $\{a_\rho\}$.

We prove the first part. Let $G = F(\boldsymbol{\sigma}(x))$ have order n. Then $F(x_0, \ldots, x_n)$ is irreducible in $K[x_0, \ldots, x_n]$. For if $F = F_1 \cdot F_2$, we may use Theorem 5.9 to get an equivalent $\{a'_\rho\}$ such that $\{G(a'_\rho)\} \rightsquigarrow_s G(a)$, $\{G_1(a'_\rho)\} \rightsquigarrow_s G_1(a)$, $\{G_2(a'_\rho)\} \rightsquigarrow_s G_2(a)$ and $\{G(a'_\rho)\} \rightsquigarrow 0$, forcing one of $\{G_1(a'_\rho)\} \rightsquigarrow 0$ or $\{G_2(a'_\rho)\} \rightsquigarrow 0$, contrary to the minimality of G.

Let L be the field of fractions of the domain $K[x_0, \ldots, x_n]/(F)$.

We first give L the structure of an immediate extension of K.

Let $f(x_0, \ldots, x_n) \in K[x_0, \ldots, x_n]$. Clearly $hf - A = bF$ for some $h \in K[x_0, \ldots, x_{n-1}]$, $A, b \in K[x_0, \ldots, x_n]$, A of lower complexity than F.

For convenience, pick a pseudolimit a for $\{a_\rho\}$ in an elementary extension of $(K, v, k, \Gamma, \sigma)$.

"Define"

$$v(f) = v(A(\boldsymbol{\sigma}(a))) - v(h(\boldsymbol{\sigma}(a)))$$

Many things now have to be checked.

(i) *The map is well-defined.* Suppose $h_1 f - A_1 = b_1 F$, $h_1 \in K[x_0, \ldots, x_{n-1}]$, $A_1, b_1 \in K[x_0, \ldots, x_n]$, A_1 of lower complexity than F. Then

$$h_1 A - h A_1 \in (F)$$

and has lower x_n-degree than F, so = 0.
(This still leaves the "problem" that $A(\boldsymbol{\sigma}(a)), h(\boldsymbol{\sigma}(a))$ might both be 0.)

(ii) *$h(\boldsymbol{\sigma}(a)) \neq 0$, and $v(h(\boldsymbol{\sigma}(a))) \in \Gamma$.* If h is a constant, this is trivial. If not, go to equivalent $\{a'_\rho\}$ so that

$$\{h(\boldsymbol{\sigma}(a'_\rho))\} \rightsquigarrow_s h(\boldsymbol{\sigma}(a)).$$

But $h(\boldsymbol{\sigma}(x))$ has lower complexity than $G(x)$, so $\{h(\boldsymbol{\sigma}(a'_\rho))\} \not\rightsquigarrow 0$. So $h(\boldsymbol{\sigma}(a)) \neq 0$, and $v(h(\boldsymbol{\sigma}(a))) =$ eventual value $v(h(\boldsymbol{\sigma}(a'_\rho))) \in \Gamma$.

(iii) *$v(A(\boldsymbol{\sigma}(a))) \in \Gamma \cup \{\infty\}$, so $v(f) \in \Gamma \cup \{\infty\}$.* By an argument similar to (ii).

(iv) *Suppose for some equivalent $\{a'_\rho\}$ that $\{f(\boldsymbol{\sigma}(a'_\rho))\} \rightsquigarrow_s f(\boldsymbol{\sigma}(a))$, $\{f(\boldsymbol{\sigma}(a'_\rho))\} \not\rightsquigarrow 0$, $\{h(\boldsymbol{\sigma}(a'_\rho))\} \rightsquigarrow_s h(\boldsymbol{\sigma}(a))$, $\{A(\boldsymbol{\sigma}(a'_\rho)\} \rightsquigarrow_s A(\boldsymbol{\sigma}(a))$, and $\{b(\boldsymbol{\sigma}(a'_\rho))\} \rightsquigarrow_s b(\boldsymbol{\sigma}(a))$. Then $v(f) =$ the eventual value $v(f(\boldsymbol{\sigma}(a'_\rho))) = v(f(\boldsymbol{\sigma}(a)))$.* Consider the relation $h(\boldsymbol{\sigma}(a'_\rho))f(\boldsymbol{\sigma}(a'_\rho)) - A(\boldsymbol{\sigma}(a'_\rho)) = b(\boldsymbol{\sigma}(a'_\rho))F(\boldsymbol{\sigma}(a'_\rho))$. The valuation of the right-hand side is eventually increasing while the valuation of each of $h(\boldsymbol{\sigma}(a'_\rho))$, $f(\boldsymbol{\sigma}(a'_\rho))$, $A(\boldsymbol{\sigma}(a'_\rho))$ is eventually constant, so eventually we must have $v(h(\boldsymbol{\sigma}(a'_\rho))) + v(f(\boldsymbol{\sigma}(a'_\rho))) = v(A(\boldsymbol{\sigma}(a'_\rho)))$. But as we have seen previously, $v(f) =$ eventual value $v(A(\boldsymbol{\sigma}(a'_\rho)))-$ eventual value $v(h(\boldsymbol{\sigma}(a'_\rho)))$, whence the result.

(v) *v is a valuation extending that on K.*

(v.1) It clearly is an extension, and takes value ∞ on (F).

(v.2) That $v(f+g) \geq \min(v(f), v(g))$ is formal, given (i)–(iii).

(v.3) $v(fg) = v(f)+v(g)$: say $h_i \in K[x_0, \ldots, x_{n-1}], A_i, b_i \in K[x_0, \ldots, x_n], A_i$ of lower complexity than F such that $h_1 f - A_1 = b_1 F, h_2 g - A_2 = b_2 F, h_3 A_1 A_2 - A_3 = b_3 F$. Then $h_3 h_1 h_2 fg - A_3 = bF$, for some $b \in K[x_0, \ldots, x_n]$, and as in (iv), $v(A_1 A_2) = v(A_1(\boldsymbol{\sigma}(a))) + v(A_2(\boldsymbol{\sigma}(a)))$, so that by (i), and as in (ii) and (iii) we get:

$$\begin{aligned} v(fg) &= v(A_3(\boldsymbol{\sigma}(a))) - v(h_3(\boldsymbol{\sigma}(a))) - v(h_1(\boldsymbol{\sigma}(a))) - v(h_2(\boldsymbol{\sigma}(a))) \\ &= v(A_1 A_2) - v(h_1(\boldsymbol{\sigma}(a))) - v(h_2(\boldsymbol{\sigma}(a))) \\ &= v(A_1(\boldsymbol{\sigma}(a))) + v(A_2(\boldsymbol{\sigma}(a))) - v(h_1(\boldsymbol{\sigma}(a))) - v(h_2(\boldsymbol{\sigma}(a))) \\ &= v(f) + v(g). \end{aligned}$$

By the above arguments, we get a valuation on $K[x_0, \ldots, x_n]/(F)$, and extend it to L. Clearly the value group does not extend. Since $v(f) =$ eventual value $v\left(A(\boldsymbol{\sigma}(a'_\rho))/h(\boldsymbol{\sigma}(a'_\rho))\right)$, as in (ii), one shows likewise that the residue field does not extend.

Now let $v_i = x_i + (F)$ and consider the "map"

$$K(v_0, \ldots, v_{n-1}) \xrightarrow{\sigma} K(v_1, \ldots, v_n)$$

sending v_i to v_{i+1}, $(0 \leq i \leq n-1)$ and extending σ on K.

Again, there are things to check.

Each of $K(v_0, \ldots, v_{n-1})$ and $K(v_1, \ldots, v_n)$ has transcendence degree n over K. Suppose $f(v_0, \ldots, v_{n-1}) = 0$, $f \in K[x_0, \ldots, x_{n-1}]$. Then $f \in (F)$, impossible, unless $f = 0$.

Suppose $g(v_1, \ldots, v_n) = 0$, $g \in K[x_1, \ldots, x_n]$. So $g \in (F)$, say $g = g_1 \cdot F$. Then either g is constant, or by Theorem 5.9 there is some equivalent $\{a'_\rho\}$ so that $\{g(\boldsymbol{\sigma}(a'_\rho))\} \rightsquigarrow_s g(\boldsymbol{\sigma}(a))$, $\{g_1(\boldsymbol{\sigma}(a'_\rho))\} \rightsquigarrow g_1(\boldsymbol{\sigma}(a))$, and $\{G(a'_\rho) = F(\boldsymbol{\sigma}(a'_\rho))\} \rightsquigarrow 0$. This forces also $\{g(\boldsymbol{\sigma}(a'_\rho))\} \rightsquigarrow 0$.

But then

$$\{\sigma(g^{\sigma^{-1}}(\sigma^{-1}(\boldsymbol{\sigma}(a'_\rho))))\} \rightsquigarrow_s \sigma(g^{\sigma^{-1}}(\sigma^{-1}(\boldsymbol{\sigma}(a))))$$

$$\{\sigma(g^{\sigma^{-1}}(\sigma^{-1}(\boldsymbol{\sigma}(a'_\rho))))\} \rightsquigarrow 0$$

so

$$\{g^{\sigma^{-1}}\left(a'_\rho, \sigma(a'_\rho), \ldots, \sigma^{n-1}(a'_\rho)\right)\} \rightsquigarrow_s g^{\sigma^{-1}}\left(a, \sigma(a), \ldots, \sigma^{n-1}(a)\right)$$

$$\{g^{\sigma^{-1}}\left(a'_\rho, \sigma(a'_\rho), \ldots, \sigma^{n-1}(a'_\rho)\right)\} \rightsquigarrow 0$$

But $g^{\sigma^{-1}}\left(x, \sigma(x), \ldots, \sigma^{n-1}(x)\right)$ has lower complexity than G, a contradiction.

So

$$K(v_0, \ldots, v_{n-1}) \xrightarrow{\sigma} K(v_1, \ldots, v_n)$$

where $\sigma(v_i) = v_{i+1}$ and extending σ on K, is a well-defined field isomorphism between subfields of L.

Now, by earlier calculations, if $f \in K[x_0, \ldots, x_{n-1}]$

$$v(f(v_0, \ldots, v_{n-1})) = \text{ eventual value } v(f(\boldsymbol{\sigma}(a'_\rho)))$$

for suitable $\{a'_\rho\} \rightsquigarrow a$, and changing a'_ρ if necessary

$$\begin{aligned} v(\sigma(f(v_0,, \ldots, v_{n-1}))) &= v(f^\sigma(v_1, \ldots, v_n)) \\ &= \text{ eventual value } v(f^\sigma(\sigma(a'_\rho), \ldots, \sigma^n(a'_\rho))), \text{ by (iv)} \\ &= \text{ eventual value } v(f(a'_\rho, \ldots, \sigma^{n-1}(a'_\rho))). \end{aligned}$$

So $v(f) = v(\sigma(f))$, whence σ is value preserving.

Now consider the henselisation L^h of L, and the picture

$$\begin{tikzcd} & L^h & \\ & L \arrow[u] & \\ K(v_0,\dots,v_{n-1}) \arrow[ur] & & K(v_1,\dots,v_n). \arrow[ul] \\ & K \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Evidently L is algebraic over $K(v_0,\dots,v_{n-1})$. The proof that $K(v_1,\dots,v_n)$ has transcendence degree n, together with $F(v_0,\dots,v_n)=0$, gives v_0 algebraic over $K(v_1,\dots,v_n)$, so L is algebraic over $K(v_1,\dots,v_n)$. Thus

$$K(v_0,\dots,v_{n-1})^h = K(v_1,\dots,v_n)^h = L^h.$$

So σ extends uniquely to an automorphism σ of L^h, which is an immediate extension of K; σ is clearly value preserving. Note that $\{a_\rho\} \rightsquigarrow v_0$, since $v(v_0 - a_\rho) = v(a-a_\rho) = \gamma_\rho$ eventually.

Now take $K_1 = K\langle v_0\rangle$, finishing the proof of this first part.

For the second part, suppose b satisfies $G(b)=0$ and $\{a_\rho\} \rightsquigarrow b$. We first observe that $b,\sigma(b),\dots,\sigma^{n-1}(b)$ are algebraically independent over K. For if $H(b,\sigma(b),\dots,\sigma^{n-1}(b)) = 0$ for some non-zero $H \in K[x_0,\dots,x_{n-1}]$, we can go to an equivalent $\{a'_\rho\}$ such that $H(a'_\rho,\sigma(a'_\rho),\dots,\sigma^{n-1}(a'_\rho)) \rightsquigarrow_s H(b,\sigma(b),\dots,$ $\sigma^{n-1}(b))$, but that would contradict the minimality of G. Similarly, G is of minimal complexity such that $G(b)=0$. We then get a K-isomorphism of difference fields $K\langle a\rangle \simeq K\langle b\rangle$ sending a to b. Now the elements of these fields are described by σ-polynomials H, in a or b, of lower complexity than G. In particular, for any equivalent $\{a'_\rho\}$ necessarily $\{H(a'_\rho)\} \not\rightsquigarrow 0$. We can then argue as in the second part of Lemma 7.1 to conclude that this is also an isomorphism of valued fields. ◻

We note the following consequences. Recall that according to the classical theory, (K,v) has no proper immediate extension if and only if every p.c. series from K has a pseudolimit in K.

THEOREM 7.3. *Suppose (K,v,k,Γ,σ) is pliable. Then:*

(a) (K,v) *has a proper immediate extension if and only if* (K,v,k,Γ,σ) *has a proper immediate σ-extension which is pliable.*

(b) (K,v,k,Γ,σ) *has a proper immediate σ-extension which is σ-algebraic if and only if there is a p.c. $\{a_\rho\}$ of σ-algebraic type over K with no pseudolimit in K.*

(c) *Let $a \notin K$ be in some immediate pliable σ-extension of K, and let K_2 be a pliable σ-henselian extension of K such that every p.c. series from K_2 of length at most* card(Γ) *has a pseudolimit in K_2. Then $K\langle a\rangle$ embeds in K_2 over K.*

(d) *Let $a \notin K$ be in some immediate pliable σ-henselian σ-algebraic extension of K, and let K_2 be a pliable σ-henselian extension of K such that every p.c. series of σ-algebraic type over K_2 and of length at most* $\operatorname{card}(\Gamma)$ *has a pseudolimit in K_2. Then $K\langle a\rangle$ embeds in K_2 over K.*

Proof. (a) ($\Leftarrow$): Immediate. ($\Rightarrow$): If (K, v) has a proper immediate extension then there is some p.c. series $\{a_\rho\}$ from K having no pseudolimit in K. By either Lemma 7.1 or Lemma 7.2 depending on whether $\{a_\rho\}$ is of σ-transcendental or σ-algebraic type, there is an immediate pliable extension of $(K, v, k, \Gamma, \sigma)$ in which $\{a_\rho\}$ has a pseudolimit.

(b) ($\Leftarrow$): This is Lemma 7.2. ($\Rightarrow$): let a belong to a proper immediate σ-extension which is σ-algebraic. Say $G(a) = 0$, for some nonconstant σ-polynomial G. By the classical theory, there is a p.c. $\{a_\rho\}$ from K which pseudoconverges to a but has no pseudolimit in K. By Theorems 5.6 and 5.8 there is a p.c. $\{a'_\rho\} \sim \{a_\rho\}$ s.t. $\{G(a'_\rho)\} \rightsquigarrow_s G(a) = 0$. So $\{a'_\rho\}$ is of σ-algebraic type over K, and with no pseudolimit in K.

(c) Let K_1 be some immediate pliable σ-henselian extension of K containing a. By the classical theory, there is a p.c. $\{a_\rho\}$ from K such that $\{a_\rho\} \rightsquigarrow a$ and $\{a_\rho\}$ has no pseudolimit in K. By the assumption, there is $b \in K_2$ such that $\{a_\rho\} \rightsquigarrow b$.

If $\{a_\rho\}$ is of σ-transcendental type, then reasoning as in (b), a and b must both be σ-transcendental over K and we can apply Lemma 7.1.

If $\{a_\rho\}$ is of σ-algebraic type, let G be as in Lemma 7.2, namely, a minimal witness to σ-algebraicity. By Theorem 6.10, we get an equivalent $\{a'_\rho\}$ from K so that $\{G(a'_\rho)\} \rightsquigarrow 0$, $\{G(a'_\rho)\} \rightsquigarrow G(a)$, for all $H \in \mathcal{H}(G)$, $\{H(a'_\rho)\} \rightsquigarrow H(a)$, but not to 0, eventually a'_ρ, G are in σ-Hensel configuration, and either $G(a) = 0$ or a, G are in σ-Hensel configuration. Note that $\{a'_\rho\} \rightsquigarrow a$ and $v(G(a)) > v(G(a'_\rho))$ eventually. If $G(a) = 0$, we do nothing. Otherwise, by the σ-Hensel scheme, we get a' in K_1, such that $G(a') = 0$ and $v(a' - a) = v(G(a)) - \min_{|\boldsymbol{l}|=1} v(G_{\boldsymbol{l}}(a))$. But eventually, $v(G_{\boldsymbol{l}}(a'_\rho)) = v(G_{\boldsymbol{l}}(a)), |\boldsymbol{l}| \geq 1$, so eventually $v(a' - a) > \gamma_\rho$. Whence $\{a'_\rho\} \rightsquigarrow a'$. So in every case, we get a' in K_1, such that $\{a'_\rho\} \rightsquigarrow a'$ and $G(a') = 0$. Similarly, we get b' in K_2 with $\{a'_\rho\} \rightsquigarrow b'$ and $G(b') = 0$.

By Lemma 7.2, $K\langle a'\rangle$ is isomorphic to $K\langle b'\rangle$ as valued fields with isometry over K, with a' mapped to b'.

Now, a is immediate over $K\langle a'\rangle$. If it is not in $K\langle a'\rangle$, then we may repeat the argument and conclude by a standard maximality argument.

(d) By the proof of (b), there is a p.c. $\{a_\rho\}$ from K of σ-algebraic type pseudoconverging to a but with no pseudolimit in K. The calculations in (c) work now noting that every extension or p.c. series considered will be of σ-algebraic type. $\square$

We now tackle the issue of uniqueness of maximal immediate σ-extensions. At the same time, we consider the analogue for σ-algebraic immediate extensions.

LEMMA 7.4. *Let $(K, v, k, \Gamma, \sigma)$ be a valued field with isometry s.t. $(k, \bar{\sigma})$ is linearly difference closed. If $(K, v, k, \Gamma, \sigma)$ has no proper immediate σ-algebraic extensions then it satisfies the σ-Hensel scheme.*

Proof. Suppose first that $(K, v, k, \gamma, \sigma)$ has no proper immediate σ-extension. Suppose it is not σ-henselian, and start with a counterexample a, G in σ-Hensel configuration.

Let $a_0 = a$ and use the Newton approximation given earlier to start a p.c. $\{a_n\}_{n<\omega}$ in σ-Hensel configuration (cf. Def. 6.4 for notation). Note that for all $\boldsymbol{l}$ with $|\boldsymbol{l}| = 1$ such that

$$v(G_{\boldsymbol{l}}(a_0)) = \min_{|\boldsymbol{j}|=1} v(G_{\boldsymbol{j}}(a_0))$$

we have for each $n < \omega$

$$\begin{aligned} v(G_{\boldsymbol{l}}(a_n)) &= v(G_{\boldsymbol{l}}(a_0)) \\ &= \min_{|\boldsymbol{j}|=1} v(G_{\boldsymbol{j}}(a_n)). \end{aligned}$$

Fix, as in Section 6.1, $\boldsymbol{l}_0$ with $|\boldsymbol{l}_0| = 1$ so

$$v(G_{\boldsymbol{l}_0}(a_n)) = \min_{|\boldsymbol{j}|=1} v(G_{\boldsymbol{j}}(a_0)).$$

Now, $\{a_n\}$ must have a pseudolimit a_ω in K. As in the proof of Theorem 6.10, we get that a_ω, G are in σ-Hensel configuration. By continuing we eventually reach a contradiction.

Now suppose $(K, v, k, \Gamma, \sigma)$ has no proper immediate σ-algebraic extension. Since K is unramified, let (K_1, v_1) be the unique maximal immediate extension of (K, v). By uniqueness, σ extends to (K_1, v_1) and we get $(K_1, v_1, k_1, \Gamma_1, \sigma_1)$ which has no proper immediate σ-extension. By the first case, K_1 is σ-henselian. If a, G from K are in σ-Hensel configuration, there is an appropriate root of G in K_1, which is now forced to be in K. □

THEOREM 7.5. *Suppose $(K, v, k, \Gamma, \sigma)$ is pliable with $(k, \bar{\sigma})$ linearly difference closed. Then*

(i) $(K, v, k, \Gamma, \sigma)$ *has a maximal immediate σ-extension, which is pliable, and is unique up to isomorphism over* $(K, v, k, \Gamma, \sigma)$.

(ii) $(K, v, k, \Gamma, \sigma)$ *has a maximal σ-algebraic immediate extension, which is pliable, and is unique up to isomorphism over* $(K, v, k, \Gamma, \sigma)$.

Proof. We have already noticed the existence of both kinds of maximal immediate σ-extensions, and they will necessarily be pliable since they are immediate extensions of K. By the previous lemma they are also σ-henselian. The desired uniqueness then follows by a standard maximality argument using Theorem 7.3(c) and (d). □

We note the following:

LEMMA 7.6. *Let $(K, v, k, \Gamma, \sigma)$ be pliable with $(k, \bar{\sigma})$ linearly difference closed. Then the maximal immediate σ-extension of K embeds in any pliable σ-henselian extension of K which is* $\max(card(k), card(\Gamma))^+$*-saturated.*

Proof. Let K' be the maximal immediate σ-extension of K. The saturation assumption ensures that any p.c. $\{a_\rho\}$ from K or K' will have a pseudolimit in the saturated extension. The result then follows again from Theorem 7.3(c) by a standard maximality argument. □

8. Proof of the embedding theorem. In this section we complete the proof of the Embedding Theorem 2.2 which the reader should consult for notation.

We wish to exploit our work on pseudoconvergence and maximal immediate σ-extensions, but these do not apply directly to general L_i as above.

The main remaining work involves making the L_i pliable with a linearly difference closed residue field. We then exploit uniqueness of maximal immediate σ-extensions and fall into the trichotomy of the classical setting: extensions where only the residue field extends (so-called unramified), extensions where only the value group extends (so-called totally ramified), and immediate extensions.

We shall show a series of intermediate lemmas where we extend ψ to a σ-extension L_i' of L_i, still small, with a desired property. Concerning the issue of smallness, typically the basic step is to go to some $L_i' = L_i\langle a\rangle$, which is obviously small if L_i is, and then iterate the procedure $card(l_i)$ or $card(G_i)$ many times, which also preserves smallness. Eventually we iterate this process countably many times and take union of an increasing sequence of countably many small fields, which also preserves smallness. We will make no further reference to smallness in the proof.

In all cases, the extension ψ' of ψ should be admissible, i.e. satisfy conditions:

(A) ψ' is an isomorphism of valued fields with isometry.

(B) The induced isomorphism ψ_r': $l_1' \longleftrightarrow l_2'$ of difference fields is elementary.

(C) The induced ψ_v': $G_1' \longleftrightarrow G_2'$ is elementary.

(D) ψ' respects the angular component maps.

We note that by Lemma 1.6 and Lemma 1.12 we can always assume that L_i is henselian. Recall the notation f^ψ (Section 1).

8.1. Making a valued field with isometry pliable. We want to make the L_i of the Embedding Theorem pliable with linearly difference closed residue fields. This involves using Axioms R and RG of the ambient models and some dovetailing.

LEMMA 8.1. *Let ψ: $L_1 \to L_2$ be an admissible isomorphism, with L_i small, Witt-Frobenius case and henselian.*

(1) *Suppose* $\alpha \in k_1$ *is algebraic over* l_1. *Then there exist* $a \in K_1, b \in K_2$ *such that* $\bar{a} = \alpha$ *and* ψ *can be extended to an admissible isomorphism* $\psi' \colon L_1\langle a\rangle \to L_2\langle b\rangle$ *such that* $\psi'(a) = b$.

(2) *We can extend* ψ *to an admissible isomorphism* ψ' *between small valued subfields with isometry* L_i' *whose residue fields* l_i' *are relatively algebraically closed in* k_i.

Proof. Item (2) follows directly from item (1). To prove (1), the basic task is to select α algebraic over l_1, select a suitable $a \in K_1$ with $\bar{a} = \alpha$, and extend ψ to $L\langle a\rangle$.

Note that since L_1 is closed under σ and σ^{-1}, l_1 is perfect. For we are in a Witt situation and $\bar{\sigma}(x) = x^p$.

Now, select irreducible monic $f \in l_1[x]$ of degree n with $f(\alpha) = 0$. Since l_1 is perfect, $f'(\alpha) \neq 0$. Lift f to some monic $g \in L_1[x]$ of degree $n, \bar{g} = f$. Use Hensel's lemma to get $a \in K_1, g(a) = 0, \bar{a} = \alpha$, a unique with these properties. Also, g is clearly irreducible.

Then $L_1(a) = L_1[a] = \{h(a)\colon\ h \in L_1[x], \deg(h) < n\}$. Pick any such $h(x), h \neq 0$, and write $h = c \cdot h_1, c \in L_1, h_1$ over the valuation ring, and with at least one coefficient 1. Then $\bar{h}_1(\alpha) \neq 0$, so $v(h_1(a)) = 0$, and $v(h(a)) = v(c)$. So $L_1(a)$ is unramified over L_1, with residue field $l_1(\alpha)$.

Now, $\sigma(a)$ need not be in $L_1(a)$. However, we do know that $\sigma(a)$ is the unique root λ of g^σ such that $\bar{\lambda} = \bar{\sigma}(\alpha)$, a root of $f^{\bar{\sigma}}$. Now $f^{\bar{\sigma}}$ is irreducible over l_1, though perhaps not over $l_1(\alpha)$. So factor $f^{\bar{\sigma}}$ into coprime irreducibles over $l_1(\alpha)$, say $f^{\bar{\sigma}} = \Pi f_i$, and lift this to a factorization of g^σ over the henselian field $L_1(a)$, say $g^\sigma = \Pi g_i$, with $\bar{g}_i = f_i$. Now, $\bar{\sigma}(\alpha)$ is a root of f_1 say, and of no other f_i, so $\sigma(a)$ is a root of g_1, say, and of no other g_i. The g_i are of course irreducible.

So now we repeat our earlier procedure, working over $L_1(a)$ with f_1 and $\sigma(a)$, to get $L_1(a, \sigma(a))$, an unramified extension, with uniquely determined valuation structure, and residue field $l_1(\alpha, \bar{\sigma}(\alpha))$.

Now obtain $\sigma^{-1}(a)$ and then $\sigma^2(a), \sigma^{-2}(a)$, and so on, until we have an unramified extension $L_1\langle a\rangle$ with residue field $l_1\langle\alpha\rangle$.

Our task now is to find ψ', i.e. to find a suitable $b \in K_2$. We make essential use of ψ_r. The saturation allows us to extend ψ_r to an elementary map

$$\psi_r'\colon\ l_1\langle\alpha\rangle \longleftrightarrow l_2\langle\beta\rangle$$

with $\psi'(\alpha) = \beta$. Now we use ψ_r' to mimic what we just explained about $L_1\langle a\rangle$. Use Hensel's lemma to get a unique b so that $g^\psi(b) = 0$ and $\bar{b} = \psi_r'(\alpha) = \beta$. Then by our discussion of the valued field structure of $L_1(a)$ it is clear that ψ extends uniquely to a valued field isomorphism $\psi_1\colon\ L_1(a) \cong L_2(b)$ sending a to b. Also,$\psi_{1,r} = \psi_r' \,|_{l_1(\alpha)}$.

Now repeat to extend to $\psi_2\colon\ L_1(a, \sigma(a)) \cong L_2(b, \sigma(b))$, and so on till we get $\psi'\colon\ L_1\langle a\rangle \cong L_2\langle b\rangle$, as a map of valued fields, and $\psi'{}_r = \psi_r'$.

Finally, ψ' does respect σ, just by construction.

Since the extension is unramified, (C) and (D) are automatic, and ${\psi'}_r = \psi'_r$ gives (B). □

LEMMA 8.2. *Let ψ: $L_1 \to L_2$ be an admissible isomorphism, with L_i small.*

(1) *Given an inhomogeneous non trivial linear $\bar{\sigma}$-equation over l_1, there exist $a \in K_1, b \in K_2$ such that $\bar{a}$ is a solution to the given equation and ψ can be extended to an admissible isomorphism ψ': $L_1\langle a\rangle \to L_2\langle b\rangle$ such that $\psi'(a) = b$.*

(2) *We can extend ψ to an admissible isomorphism ψ' between small valued subfields with isometry L'_i whose residue fields are linearly difference closed.*

Proof. In the Witt-Frobenius case, this is subsumed by Lemma 8.1, since $\bar{\sigma}(x) = x^p$.

As (2) is a routine consequence of (1), it suffices to prove (1) in case we have Axiom RG. So, let

$$\overline{c}_n \cdot \bar{\sigma}^n(x) + \cdots + \overline{c}_0 \cdot x + \bar{d} = 0$$

be a linear $\bar{\sigma}$-equation over l_1, with $n \neq 0, \overline{c}_n \neq 0, \bar{d} \neq 0$, all c_i in L_1.

Use Axiom RG and saturation to get a solution α in k_1, so that α is not a root of any $F(\boldsymbol{\sigma}(x)) = 0$, $F \in l_1[x_0, \ldots, x_{n-1}], F \neq 0$. Now use the σ-Hensel scheme to lift α to a solution a (in K_1) of $c_n\sigma^n(x) + \cdots + d = 0$.

Consider $L_1(a, \sigma(a), \ldots, \sigma^{n-1}(a))$. This is closed under σ. But, noting for example that $\sigma^{-1}(c_n\sigma^n(a)+\cdots+d) = 0$ we get $\sigma^{-1}(a) \in L_1(a, \sigma(a), \ldots, \sigma^{n-1}(a))$, and see easily that $L_1(a, \sigma(a), \ldots, \sigma^{n-1}(a))$ is closed under σ^{-1}.

Now we check the valuation structure of $L_1(a, \sigma(a), \ldots, \sigma^{n-1}(a))$. Consider $f(a, \sigma(a), \ldots, \sigma^{n-1}(a))$, $f \in L_1[x_0, \ldots, x_{n-1}] \setminus \{0\}$. Write, as usual, $f = c \cdot f_1$, $c \in L_1, f_1$ over the valuation ring, one coefficient 1.

$$\begin{aligned} \overline{f}_1(\alpha, \bar{\sigma}(\alpha), \ldots, \bar{\sigma}^{n-1}(\alpha)) &\neq 0, \text{ so} \\ v(f_1(a, \sigma(a), \ldots, \sigma^{n-1}(a))) &= 0, \text{ so} \\ v(f(a, \sigma(a), \ldots, \sigma^{n-1}(a))) &= v(c). \end{aligned}$$

So v is uniquely determined, the extension is unramified over L_1, and has residue field $l_1(\alpha)$.

Now use saturation and the elementarity of ψ_r to find a match β in k_2 for α. Let b be an appropriate lifting. The existence of ψ' is routine. □

LEMMA 8.3. *Let ψ: $L_1 \to L_2$ be an admissible isomorphism, with L_i small. Suppose the L_i henselian and the l_i linearly difference closed.*

(1) *Let $c \in L_1$, then there exist $a \in K_1, b \in K_2$ such that $v(c) \in v(\mathrm{Fix}(L_1\langle a\rangle))$ and ψ can be extended to an admissible isomorphism ψ': $L_1\langle a\rangle \to L_2\langle b\rangle$ such that $\psi'(a) = b$.*

(2) *We can extend ψ to an admissible isomorphism ψ' between small valued subfields with isometry L_i' such that for all $c \in L_i$ there is $a \in L_i'$ such that $\sigma(a) = a$ and $v(c) = v(a)$.*

Proof. Again, (2) follows from (1) using Lemmas 8.2 and 1.6. To prove (1), this time our task is to take an element $c \in L_1^*$ and then obtain an extension containing an element in the fixed field with the same valuation as c.

Pick a nonzero solution α of the equation $\bar{\sigma}(y) = \overline{(c/\sigma(c))} \cdot y$ over k_1 using Axiom R. Use the σ-Hensel scheme to lift α to $a \in K_1$ with $\sigma(a) = (c/\sigma(c)) \cdot a$.

Then ca is in the fixed field, and $v(ca) = v(c)$, since $\alpha \neq 0$.

So we consider $L_1(a)$, which contains an element solving our problem. $L_1(a)$ is obviously closed under σ and σ^{-1}.

If α is transcendental over l_1, a by now routine argument shows that the valuation on $L_1(a)$ is uniquely determined, and that $L_1(a)$ is unramified over L_1, with residue field $l_1(\alpha)$.

Can α be chosen transcendental? We can vary α by multiplying by an element of $\mathrm{Fix}(\bar{\sigma})$, so by saturation we can choose α transcendental, *provided* $\mathrm{Fix}(\bar{\sigma})$ is infinite.

If $\mathrm{Fix}(\bar{\sigma})$ is infinite, the extension to ψ' is by now routine.

Suppose however $\mathrm{Fix}(\bar{\sigma})$ is finite, of characteristic p.

Go back to α. By Lemma 8.1, we may assume that l_1 is relatively algebraically closed in k_1 so the $\alpha \in l_1$. Let $a_0 \in L_1$ be any lifting of α. If $\sigma(a_0) - c/\sigma(c) \cdot a_0 = 0$, we are done (we need not extend). Otherwise, since all linear $\bar{\sigma}$-equations are solvable in l_1, we may do Newton approximation in L_1, to get a_1 with $v(a_1 - a_0) = v(\sigma(a_0) - c/\sigma(c) \cdot a_0)$ and $v(\sigma(a_1) - c/\sigma(c) \cdot a_1) > v(\sigma(a_0) - c/\sigma(c) \cdot a_0)$. If a_1 is a root of $\sigma(x) - c/\sigma(c) \cdot x$, we are done. Otherwise, we generate a p.c. $\{a_n\}_{n<\omega}$ in L_1 with

$$\begin{aligned} \gamma_n &= v(a_m - a_n) \text{ (for any } m > n) \\ &= v(\sigma(a_n) - c/\sigma(c) \cdot a_n) \end{aligned}$$

and $v(\sigma(x) - c/\sigma(c) \cdot x)$ increasing on $\{a_n\}_{n<\omega}$.

Note that for any pseudolimit a_ω (anywhere) of $\{a_n\}_{n<\omega}$

$$v(\sigma(a_\omega) - c/\sigma(c) \cdot a_\omega) > v(\sigma(a_n) - c/\sigma(c) \cdot a_n)$$

for each n.

Case 1. The series $\{a_n\}$ has no pseudolimit in L_1. Since L_1 is henselian, $\{a_n\}$ is of transcendental type over L_1, and for any pseudolimit a_ω of $\{a_n\}$ the isomorphism type of the valued field $L_1(a_\omega)$ is uniquely determined.

If some a_ω is a solution of $\sigma(x) = c/\sigma(c) \cdot x$, the field $L_1(a_\omega)$ is closed under σ, σ^{-1} and its isomorphism type as valued field with isometry is uniquely determined. Note that $\overline{a}_\omega = \alpha$. In this case the existence of the required ψ' is routine.

If some a_ω is not a solution, the σ-Hensel scheme gives us a solution a'_ω with

$$\begin{aligned} v(a'_\omega - a_\omega) &= v(\sigma(a_\omega) - c/\sigma(c) \cdot a_\omega) \\ &> v(\sigma(a_n) - c/\sigma(c) \cdot a_n) \end{aligned}$$

each n, so a'_ω is also a pseudolimit, and we go back to preceding paragraph.

Case 2. The series $\{a_n\}_{n<\omega}$ has a pseudolimit a_ω in L_1.

Newton approximate against a_ω. We continue our seach until we are driven into Case 1. □

LEMMA 8.4. *Let* ψ: $L_1 \to L_2$ *be an admissible isomorphism in the* non-Witt-Frobenius *case. Then we can extend* ψ *to an admissible isomorphism* ψ' *between small valued subfields with isometry* L'_i *whose residue fields satisfy Axiom R4.*

Proof. By saturation and Axiom R4 in K_1 we get $\alpha \in k_1$, $\bar{\sigma}$-transcendental over l_1. Lift α to a. Then by now familiar arguments show that the valuation on $L_1\langle a\rangle$ is uniquely determined, and $L_1\langle a\rangle$ is an unramified extension of L_1, with residue field $l_1\langle\alpha\rangle$; a is σ-transcendental over L_1. Obviously $l_1\langle\alpha\rangle$ satisfies Axiom R4. To lift ψ to $L_1\langle a\rangle$, we (as usual) use the fact that ψ_r is elementary to extend ψ_r to an elementary map ψ'_r: $l_1\langle\alpha\rangle \longleftrightarrow l_2\langle\beta\rangle$ for suitable $\beta \in k_2, \psi'(\alpha) = \beta$. Then lift β to b, and get ψ': $L_1\langle a\rangle \longleftrightarrow L_2\langle b\rangle$ with the right behavior for ${\psi'}_r$, etc. □

Finally, we obtain:

LEMMA 8.5. *Let* ψ: $L_1 \to L_2$ *be an admissible isomorphism, with* L_i *small. Then we can extend* ψ *to an admissible isomorphism* ψ' *between small valued subfields with isometry* L'_i *which are pliable with a linearly difference closed residue field. Furthermore, we can also make the* L'_i *with no immediate* σ*-extensions.*

Proof. Witt-Frobenius case: by Lemmas 8.2 and 8.3, we can construct a sequence of valued difference field extensions $L_i = L_{i,0} \subseteq L_{i,1} \subseteq L_{i,2} \subseteq \ldots$ and isomorphisms $\psi = \psi_0, \psi_1, \psi_2, \ldots$ such that

(1) ψ_j is an admissible isomorphism of $L_{1,j}$ onto $L_{2,j}$, and ψ_{j+1} extends ψ_j;

(2) $l_{i,j}$ is linearly difference closed if j is odd; and

(3) for all $c \in L_{i,j}$ there is $a \in L_{i,j+1}$ such that $\sigma(a) = a$ and $v(c) = v(a)$ if $j > 0$ is even.

Then $L'_i = \bigcup_j L_{i,j}$ and $\psi' = \bigcup_j \psi_j$ yield the desired extensions. In the non-Witt-Frobenius case, do the same dovetailing using in addition Lemma 8.4.

Assume now that each L_i is pliable with a linearly difference closed residue field. By Lemma 7.6, let L'_i be a copy in K_i of the maximal immediate σ-extension of L_i. By Theorem 7.5 we can now extend ψ to the L'_i, which are also pliable with a linearly difference closed residue field (since σ-henselian). □

8.2. Unramified extensions.

LEMMA 8.6. *Let ψ: $L_1 \to L_2$ be an admissible isomorphism, with L_i small. Suppose L_i are pliable and σ-henselian. Let $\alpha \in k_1 \setminus l_1$. Then there exists $a \in K_1, b \in K_2$ such that $\bar{a} = \alpha$ and ψ can be extended to an admissible isomorphism ψ': $L_1\langle a\rangle \to L_2\langle b\rangle$ such that $\psi'(a) = b$ and $L_1\langle a\rangle/L_1$ is an unramified extension.*

Proof. We want to extend ψ to L_i' whose residue field extends that of L_i.

Case 1. The k_i are of characteristic 0. Use Theorem 6.5 to select difference subfields, $l_i^\sharp$ representing l_i, and then difference subfields $k_i^\sharp \supseteq l_i^\sharp$ of K_i, representing k_i.

Then ψ_r induces naturally $\psi_r^\sharp$: $l_1^\sharp \longleftrightarrow l_2^\sharp$ elementary in the setting of the valued difference fields $k_i^\sharp$ (under σ).

Let $\alpha \in k_1 \backslash l_1$, and $\alpha^\sharp$ the corresponding element of $k_1^\sharp$. Consider $L_1\langle\alpha^\sharp\rangle$. We have to detect its isomorphism type and match it in K_2.

Pick a basis B of $l_1^\sharp\langle\alpha^\sharp\rangle$ over $l_1^\sharp$. Then any element of $L_1\langle\alpha^\sharp\rangle$ can be written as a quotient of elements having the form $x = \sum \lambda_i \cdot b_i$, $\lambda_i \in L_1$, $b_i \in B$, with the b_i distinct. As usual, if $x \neq 0$, $x = \lambda_{i_0} \cdot \sum(\lambda_i/\lambda_{i_0}) \cdot b_i$ for some i_0, where $v(\lambda_i/\lambda_{i_0}) \geq 0$. Now $\lambda_i/\lambda_{i_0} = \mu_i + \varepsilon_i, \mu_i \in l_1^\sharp, v(\varepsilon_i) > 0$. Also, $\mu_{i_0} = 1$.

Thus $\sum(\lambda_i/\lambda_{i_0}) \cdot b_i$ has the same residue as $\sum \mu_i \cdot b_i$, and this is nonzero, by the basis property of B. So $v(x) = v(\lambda_{i_0})$, so the valuation is uniquely determined by B.

Also, $\sigma(x) = \sum \sigma(\lambda_i) \cdot \sigma(b_i)$, again uniquely determined by B and $l_1\langle\alpha\rangle$.

Thus to extend to ψ' we let $\alpha^\sharp$ be sent to $\psi_r^\sharp(\alpha^\sharp)$, and match B to $\psi_r^\sharp(B)$.

The earlier calculations show that $L_1\langle\alpha^\sharp\rangle$ is unramified over L_1, and has residue field (naturally isomorphic to) $l_1^\sharp\langle\alpha^\sharp\rangle$. So (A), (B), (C), (D) are taken care of.

Case 2. The k_i are of characteristic $p > 0$. To simplify notation we can assume w.l.o.g. that $k_1 = k_2$, by taking suitable ultrapowers (see [23], p. 69).

We work with the coarse valuations $\dot{v}$, which are σ-henselian on the L_i and K_i by Lemma 6.6; ψ of course respects $\dot{v}$, but we do not claim its reduction is elementary on the residue fields of L_1 and L_2 for $\dot{v}$. Now use Theorem 6.5 to get a copy of $\dot{l}_1, \dot{l}_1^\sharp$ say, as a difference subfield of L_1, and similarly a copy of $\dot{k}_1$, $\dot{k}_1^\sharp$, with $\dot{l}_1^\sharp \subseteq \dot{k}_1^\sharp$.

By saturation $\dot{k}_1^\sharp$ is a complete discrete valued field whose residue field naturally identifies with k_1, so $\dot{k}_1^\sharp$ is isomorphic to $W(k_1)$ by a unique isomorphism which is compatible with the residue map onto k_1, and via this isomorphism σ identifies with the automorphism of $W(k_1)$ induced by $\bar{\sigma}$ as in Example 1.3. Note that this isomorphism sends $(\dot{l}_1^\sharp, \sigma)$ onto a difference subfield.

Now let c be an element of K_1 with $v(c) \geq 0$ so that $\bar{c} \notin \underline{l_1}$, and let $c^\sharp$ be the element of $\dot{k}_1^\sharp$ corresponding to the $\dot{v}$ residue class of c. Note $\overline{c^\sharp} = \bar{c}$.

Then by the argument in Case 1, L_1 and $\dot{l}_1^\sharp\langle c^\sharp\rangle$ are linearly disjoint over $\dot{l}_1^\sharp$ and $\dot{v}$ on the compositum $L_1\langle c^\sharp\rangle$ is determined by $(\dot{l}_1^\sharp\langle c^\sharp\rangle, \dot{v})$. Also, σ is determined, and the $\dot{v}$ value group is not extended, so in fact the extension is unramified for $\dot{v}$.

Now we look more closely at the argument of Case 1. Note that in the present case we may pick a basis B (for $\dot{l}_1^\sharp\langle c^\sharp\rangle$ over $\dot{l}_1^\sharp$) with $v(b) \geq 0$ all $b \in B$ (for $\dot{v}(b) = 0$, so $v(b) \in \mathbb{Z}$, and if $v(b) = -m,\ m \geq 0$, replace b by $p^m \cdot b$). Now consider $x \neq 0$ in $L_1\langle c^\sharp\rangle$, of the form $x = \sum \lambda_i \cdot b_i,\ b_i \in B, \lambda_i \in L_1$. Pick i_0 with $v(\lambda_{i_0})$ minimal among the $v(\lambda_i)$. Then as before (since $v \geq 0 \to \dot{v} \geq 0$)

$$\dot{v}(x) = \dot{v}(\lambda_{i_0}),\ \text{and}$$

$$\dot{v}\left(\sum (\lambda_i/\lambda_{i_0}) \cdot b_i\right) = 0,\ \text{so}$$

$$v\left(\sum (\lambda_i/\lambda_0) \cdot b_i\right) \in \mathbb{Z},\ \text{and is} \geq 0.$$

Now we revert to the ∂-formalism, more precisely to the components of Witt vectors.

We have (cf. Lemma 4.2)

$$v\left(\sum (\lambda_i/\lambda_0) \cdot b_i\right) = n$$

$$\Longleftrightarrow$$

$$\overline{\partial_n\left(\sum (\lambda_i/\lambda_0) \cdot b_i\right)} \neq 0 \text{ and } \overline{\partial_j\left(\sum (\lambda_i/\lambda_0) \cdot b_i\right)} = 0 \text{ for } 0 \leq j < n.$$

(Moreover, if $v(y) \geq 0$ then $\dot{v}(y) > 0$ iff $\overline{\partial_j(y)} = 0\ j = 0, 1, 2, \ldots$.) Now, by [16], (for fixed p) $\partial_n(x+y)$ and $\partial_n(x \cdot y)$ are given by universal polynomials (over $\mathbb{Z}$) in the $\partial_j(x), \partial_j(y), 0 \leq j \leq n$. So once one knows all $\overline{\partial_j(b_i)}$, one has determined all $\overline{\partial_n\left(\sum (\lambda_i/\lambda_0) \cdot b_i\right)}$, and so knows all $v\left(\sum (\lambda_i/\lambda_0) \cdot b_i\right)$, and thus v on $L_1\langle c^\sharp\rangle$.

In fact, since by Lemma 4.1

$$\overline{\partial_n(\sigma(y))} = \bar{\sigma}(\overline{\partial_n(y)})$$

for all $y \in \dot{k}_1^\sharp$, it is clear that one knows v on $L_1\langle c^\sharp\rangle$ once one knows the sequence

$$\langle \overline{\partial_0(c^\sharp)}, \ldots \overline{\partial_n(c^\sharp)}, \ldots \rangle$$

or, more precisely, its type over l_1. Now we bring the elementarity of ψ_r into play (with saturation), to extend ψ_r to an elementary ψ_r' defined on $l_1(\overline{\partial_0(c^\sharp)}, \ldots \overline{\partial_n(c^\sharp)}, \ldots)$. Again by saturation there is $d^\sharp$ in K_1 with $v(d^\sharp) \geq 0$ and $\overline{\partial_n(d^\sharp)} = \psi_r'(\overline{\partial_n(c^\sharp)})$ for each n.

We now claim that ψ extends to ψ': $L_1\langle c^\sharp\rangle \cong L_2\langle d^\sharp\rangle$, and $\psi'{}_r = \psi_r'$.

First, via the components of Witt vectors $(\bar{\partial}_n)$ and the identifications above, we have an isomorphism of difference fields

$$\psi' \colon \ \dot{l}_1^\sharp \langle c^\sharp \rangle \cong \psi(\dot{l}_1^\sharp)\langle d^\sharp \rangle.$$

Consider an abstract σ-polynomial $G(x)$ over L_1, and write $G(x) = e \cdot G_1(x),\ e \in L_1,\ G_1$ over the valuation ring with one coefficient equal to 1. If G is over $\dot{l}_1^\sharp$ and $G(c^\sharp) \neq 0$ we know

$$\dot{v}(G(c^\sharp)) = 0, \text{ so}$$

$$v(G(c^\sharp)) \in \mathbb{Z} \text{ and } v(G_1(c^\sharp)) \geq 0.$$

By the preceding analysis in terms of the ∂_n, and the choice of $d^\sharp$, $G^\psi(d^\sharp) \neq 0$ and

$$v(G^\psi(d^\sharp)) = v(G(c^\sharp)) \in \mathbb{Z}.$$

So $\dot{v}$ is 0 on $\psi(\dot{l}_1^\sharp)\langle d^\sharp \rangle$, and clearly the latter is a σ-copy of the $\dot{v}$ residue field. Moreover we have at least extended ψ on $\dot{l}_1^\sharp$ to $\psi' \colon\ \dot{l}_1^\sharp \langle c^\sharp \rangle \cong \psi(\dot{l}_1^\sharp)\langle d^\sharp \rangle$ preserving σ and valuation, and with $\psi'(c^\sharp) = d^\sharp$. Now we use $\psi'(B)$ as a basis for $\psi(\dot{l}_1^\sharp)\langle d^\sharp \rangle$ over $\psi(\dot{l}_1^\sharp)$, establish the linear disjointness from L_2 over $\psi(\dot{l}_1^\sharp)$, and again using the data on the $\overline{\partial_n(c^\sharp)}$ and $\overline{\partial_n(d^\sharp)}$ we extend ψ to the required $\psi' \colon\ L_1\langle c^\sharp \rangle \cong L_2\langle d^\sharp \rangle$ preserving σ and v. It is clear that the respective residue fields are $l_1\langle \bar{c} \rangle$ and $l_2\langle \overline{d^\sharp} \rangle$, and ${\psi'}_r = \psi_r$. Also, the extensions are unramified, and we are done, since c was arbitrary with $v(c) \geq 0$. □

8.3. Totally ramified extensions.

LEMMA 8.7. *Let $\psi \colon\ L_1 \to L_2$ be an admissible isomorphism, with L_i small. Suppose L_i are pliable and σ-henselian. Let $\gamma \in vK_1 \setminus vL_1$. Then there exist $a \in K_1, b \in K_2$ such that $v(a) = \gamma$ and ψ can be extended to an admissible isomorphism $\psi' \colon\ L_1\langle a \rangle \to L_2\langle b \rangle$ with $\psi'(a) = b$ and* $\mathrm{res} L_1\langle a \rangle = l_1$.

Proof. We want to extend ψ to L_i' whose value group extends that of L_i.

The task here is to go from L_1 to some L_1', to introduce in L_1' an element a so $v(a) \in \Gamma_1$ is prescribed.

Case 1. The element $\gamma \in \Gamma_1$ is rationally independent of $G_1 = v(L_1^*)$. Choose $y \in K_1, \sigma(y) = y, v(y) = \gamma$ (using the fact that K_1 has enough constants). Then $L_1(y)$ is a difference subfield and because of the assumption on $\gamma, v(\sum c_j y^j) = \min v(c_j) + j\gamma$ for $c_j \in L_1$. Thus v is uniquely determined, and the new field has value group generated by G_1 and γ, and the residue field is not extended. This time, use elementarity of ψ_v to extend it to $(\psi_v)'$ on the group generated by G_1 and γ. It is routine to extend ψ to ψ' with $(\psi')_v = (\psi_v)'$.

Finally, we deal with angular components. Earlier, we just had to choose $\psi'(y)$ so that $\sigma(\psi'(y)) = \psi'(y)$ and $v(\psi'(y)) = (\psi_v)'(v(y))$. To get angular components to match up we need only ensure that $\psi'(y)$ satisfies the extra constraints (recall that the residue field is not extended, and therefore neither are the residue rings modulo p^n for $n \in \mathbb{N}$):

$$ac_n(\psi'(G(y))) = \overline{\psi'_n}(ac_n(G(y)))$$

for G a σ-polynomial over L_1. So by Lemma 1.12 we need just

$$\psi'_{r,n}(ac_n(y)) = ac_n(\psi'(y)).$$

Thus in addition to fixing $v(\psi'(y))$ we need to fix $ac_n(\psi'(y))$ in the above compatible way. By this compatibility, and saturation, it is enough to get, for any n, some y' in K_2 with $\sigma(y') = y'$, $v(y') = \psi'_v(\gamma)$ and $ac_n(y') = \psi_{r,n}(ac_n(y))$. But this is always true, as ac_n restricts to $\mathrm{Fix}(\sigma)$ and we can always scale by elements of value 0, and res_n is surjective.

Case 2. For some $n \in \mathbb{N}$, assumed minimal, $n \cdot \gamma \in G_1$. This time we use ideas already familiar from quantifier-elimination in the valued field case (for example, [3]). Let $\gamma_1 = n \cdot \gamma$.

We exploit the remark which concluded Case 1. There is, for any m, an element y_m in L_1 of value γ with $\sigma(y_m) = y_m$ and with $ac_m(y_m) = 1$. So by Hensel's lemma, for $m \geq 2v(n) + 1$ there is a unique w_m in K_1 with $w_m^n = y_m$ and $ac_m(w_m) = 1$.

Now, by uniqueness, $\sigma(w_m) = w_m$. Fix m as above, and let $w = w_m$. Consider the difference subfield $L_1(w) = L_1[w]$. By the minimality of n,

$$\begin{aligned} v\left(\sum_{r=0}^{n-1} c_r \cdot w^t\right) &= \min\left(v(c_r) + rv(w)\right) \\ &= v(c_{r_0}) + r_0 \cdot \gamma, \end{aligned}$$

say (for $c_r \in L_1$), and so is uniquely determined by $v(w) = \gamma$.

Also, the residue field is not extended. Now, Hensel's lemma provides a unique t with $t^n = \psi(y_m)$ and $ac_m(t) = 1$. There is a unique ψ' (extending ψ)

$$\psi'\colon\ L_1(w) \cong L_2(t)$$

$$\psi'(w) = t.$$

ψ' clearly satisfies (A),(B),(C). For (D), concerning angular components, we argue that by uniqueness

$$ac_k(t) = {\psi'}_{r,k}(ac_k(w))$$

for all $k \geq m$. Then, by Lemma 1.12 again, we are done.

8.4. Epilogue. We can now conclude the proof of the Embedding Theorem. We have an admissible isomorphism $\psi\colon L_1 \cong L_2$, the L_i small.

An $\alpha \in K_1$ is now given, and we have to find an extension ψ' of ψ, again satisfying the above properties, with $\alpha \in \mathrm{dom}(\psi')$.

By Lemma 8.5 we can always assume that L_i is pliable and σ-henselian. Then, by Lemma 8.6 and Lemma 8.7 we can define a sequence of small valued difference field extensions $L_i = L_{i,0} \subseteq L_{i,1} \subseteq \ldots$ and isomorphisms $\psi = \psi_0, \psi_1, \ldots$ such that

(1) ψ_j is an admissible isomorphism of $L_{1,j}$ onto $L_{2,j}$ and ψ_{j+1} extends ψ_j ;
(2) the residue field of $L_{1,j}\langle\alpha\rangle$ is contained in the residue field of $L_{1,j+1}$;
(3) the value group of $L_{1,j}\langle\alpha\rangle$ is contained in the value group of $L_{1,j+1}$;
(4) $L_{1,j}$ is pliable and σ-henselian.

Let $L_{i,\omega} = \bigcup_j L_{i,j}$ and $\psi_\omega = \bigcup_j \psi_j$. Then $L_{i,\omega}$ are pliable and σ-henselian (so their residue fields are linearly difference closed), ψ_ω is an admissible isomorphism of $L_{1,\omega}$ onto $L_{2,\omega}$ and now $L_{1,\omega}\langle\alpha\rangle$ is an immediate extension of $L_{1,\omega}$. As in the proof of Lemma 8.5, let L_1' be a maximal σ-extension of $L_{1,\omega}$ inside K_1 and containing α, and let L_2' be a maximal immediate σ-extension of $L_{2,\omega}$ inside K_2. By Theorem 7.5, we can now extend ψ_ω to an admissible isomorphism $\psi'\colon L_1' \to L_2'$, and we are done. □

9. Completeness and model completeness. Recall that $(K, v, k, \Gamma, \sigma)$ satisfies the σ-AEK axioms if it is an unramified valued difference field for which the Embedding Theorem applies (Definition 2.3).

THEOREM 9.1. *Suppose $(K, v, k, \Gamma, \sigma)$ satisfies the σ-AEK axioms. Then the elementary theory of $(K, v, k, \Gamma, \sigma)$ is determined by the elementary theory of $(k, \bar{\sigma})$ and the elementary theory of Γ.*

Proof. It is clear that on any suitably saturated model there is a system $\{ac_n\}_{n\in\mathbb{N}}$ such that if k has characteristic p then $ac_n(p^m) = 1$ all m. Consider $(K_i, v_i, k_i, \Gamma_i, \sigma_i)$ $i = 1, 2$ with $(k_1, \bar{\sigma}_1) \equiv (k_2, \bar{\sigma}_2)$ and $\Gamma_1 \equiv \Gamma_2$. Let $\psi\colon \mathbb{Q} \longleftrightarrow \mathbb{Q}$ be the identity, use the above ac_n, and apply the Embedding Theorem. □

COROLLARY 9.2. *Suppose $(K, v, k, \Gamma, \sigma)$ satisfies the σ-AEK axioms. Then $(K, v, k, \Gamma, \sigma)$ is decidable if and only if $(k, \bar{\sigma})$ and Γ are.*

Example 9.3. The elementary theory of $W(\mathbb{F}_p^{alg})$ with the Witt Frobenius is axiomatized by "Witt-Frobenius case," the σ-AEK axioms, residue field algebraically closed of characteristic p, value group a $\mathbb{Z}$-group with unit $v(p)$. Call this theory WF_p. One actually has an axiomatization as a difference field since the p-valuation is algebraically definable. So this theory of (valued) difference field is decidable.

THEOREM 9.4. *Let $\sum$ be a set of sentences for a class of $(K, v, k, \Gamma, \sigma)$ satisfying the σ-AEK axioms, saying that $(k, \bar{\sigma})$ satisfies $\sum_1$, and Γ satisfies $\sum_2$, where $\sum_1$ and $\sum_2$ are model-complete. Then $\sum$ is model-complete.*

Proof. This is only a minor variant of Theorem 9.1. This time one has to consider a small model L_1 and ψ: $L_1 \longleftrightarrow L_1$ the identity, and extend ψ to a bigger model L_1' (all inside a suitably saturated K_1). By blowing up we can put a system ac_n on L_1 without loss of generality (assuming $L_1 \not\prec K_1$), and extend the ac_n to (a new) K_1 suitably saturated. Then the Embedding Theorem gives the result. □

Example 9.5. The elementary theory of $W(\mathbb{F}_p^{alg})$ with the Witt Frobenius.

We now know that WF_p, the elementary theory of $W(\mathbb{F}_p^{alg})$ with the Witt Frobenius, is model-complete. The next proposition ensures that WF_p is the model companion of the theory of "Witt-Frobenius case" valued fields with isometry, i.e. unramified satisfying $\bar{\sigma}(x) = x^p$. Call this theory T_p.

PROPOSITION 9.6. (Model Companion) *Every model of T_p embeds in a model of WF_p.*

Proof. Let (K, v, σ) be a model of T_p. We have seen that we can go to henselisations, so we can assume (K, v) is henselian. Using the ramification theory of general valuations (see [30], chap. 3) we can pass to the maximal unramified extension (K', v) of (K, v) inside its algebraic closure. This is a Galois extension whose residue field is the algebraic closure of the residue field of K, its value group is the same as K, and σ extends to K' (see Lemma 8.1), $v(\sigma(x)) = v(x)$ is automatically fulfilled. We are now in position to use the same arguments as in Lemma 8.3 but now working inside the (unique) maximal immediate extension of (K', v): by uniqueness σ extends and we will have a valued field with isometry satisfying the σ-Hensel scheme by Lemma 7.4. We have now extended (K, v, σ) to a model of T_p which is henselian, has an algebraically closed residue field and has enough constants. To get a $\mathbb{Z}$-group, apply the argument as in p-adic fields but with minor adjustments for σ (see [25], §3, Thm. 3.1). Finally, to get the σ-Hensel scheme, go to the maximal immediate extension as before. □

10. Teichmüller lifts. In this section we discuss an application of our main results to the theory of Teichmüller lifts. This material is not used in the sequel. The reader may wish to consult [14] for background on formal groups.

The usual Teichmüller map is a multiplicative section of the residue map for the ring of Witt vectors of a perfect field. In terms of limits, if k is a perfect field of characteristic $p > 0$, then the Teichmüller map $\tau\colon k \to W[k]$ is defined by

$$\tau(x) := \lim_{\substack{n\to\infty \\ (\pi(\widetilde{x}))^{p^n}=x}} \widetilde{x}^{p^n}$$

where $\pi\colon W[k] \to k$ is the reduction map. The map τ plays a central rôle in the theory of the Witt vectors. For instance, every element of $W[k]$ may be developed (uniquely) as a power series in p with coefficients from the image of τ. That is, for any $x \in W[k]$ there is a uniquely associated sequence $(x_i)_{i\in\omega}$ of elements of k for which $x = \sum_{i=0}^{\infty} \tau(x_i)p^i$.

Work of van den Dries [11] shows that the theory of $(W(k), k, \pi, \tau, +, \times)$ is determined by the theory of k and is, in particular, decidable when k is decidable.

One can define τ, as well as other analogous Teichmüller maps, using the Witt-Frobenius. Indeed, if $\sigma\colon W[k] \to W[k]$ is an automorphism lifting the Frobenius, then τ may be defined by

$$\tau(x) = y \Longleftrightarrow \pi(y) = x \;\&\; \sigma(y) = y^p.$$

So, the structure $(W[k], k, \pi, \tau, +, \times)$ is interpretable in $(W[k], v, \sigma, +, \times)$ and van den Dries' relative completeness and decidability theorems follow from our main theorem (at least in the cases where our axioms on solutions of residual linear difference equations hold).

There are other Teichmüller maps interpretable in $(W[k], v, \sigma, +, \times)$. Suppose, for instance, that G is a semiabelian scheme over $W[k]$.

We define the Teichmüller map $\tau_G\colon G_k(k) \to G(W[k])$ by

$$\tau_G(x) := \lim_{\substack{n\to\infty \\ [p^n]\pi(\widetilde{x})=x}} [p^n](\widetilde{x}).$$

It is not true that for every such G the map τ_G is definable in the valued difference field $(W(k), \sigma, v, +, \times)$, but it is for sufficiently nice G. Suppose that there is an isogeny $\psi\colon G \to G^{(\sigma)}$ which restricts to the Frobenius morphism $F\colon G_k \to G_k^{(p)}$ on the special fibre. Such an isogeny exists when G is the multiplicative group, in which case ψ is just $x \mapsto x^p$, and more generally for canonical lifts. In this case τ_G may be defined by

$$\tau_G(x) = y \Longleftrightarrow \pi(y) = x \;\&\; \sigma(y) = \psi(y).$$

Let us check that that this formula correctly defines τ_G. Using the limit formula for τ_G and the continuity of σ and ψ, one sees that τ_G commutes with the σ and ψ in the sense that $\sigma \circ \tau_G = \tau_{G^{(\sigma)}} \circ F$ and that the Teichmüller maps are homomorphisms. It follows that for any $x \in G_k(k)$ that $(\sigma -_G \psi)(\tau_G(x)) = \tau_{G^{(\sigma)}}((F - F)(x)) = 0$. Conversely, we note that there is only one solution to $\sigma(y) = \psi(y)$ and $\pi(y) = 0$. Indeed, as ψ restricts to the Frobenius, all of the eigenvalues of its differential have positive p-adic valuation. Using a formal group law for G, we may express the kernel of reduction as $\widehat{G}(pW[k]) \cong ((pW[k])^g, \oplus_G)$ where $g = \dim G$ and $\oplus_G$ is the formal group law. By the above observation, relative to these coordinates, $\psi(x_1p^t, \ldots, x_gp^t) \equiv (0, \ldots, 0) \pmod{p^{t+1}}$ while $\sigma(x_1p^t, \ldots, x_gp^t) \equiv (x_1^pp^t, \ldots, x_g^pp^t) \pmod{p^{t+1}}$. Clearly, zero is the only solution to $\sigma(y) = \psi(y)$ in the kernel of reduction. Thus, for any $x \in G_k(k)$, $\tau_G(y)$ is the unique solution to $\pi(y) = x$ and $\sigma(y) = \psi(y)$.

This example of an interpretable Teichmüller map may be generalized slightly to the case of quasi-canonical lifts for which there is an isogeny $\vartheta\colon G \to G^{(\sigma^n)}$ restricting to the p^n-Frobenius on the special fibre.

11. Quantifier elimination. In this section we state precisely the quantifier elimination theorems and point out some formal equivalences between them.

We have to look more closely at Denef's angular components.

While some angular component functions are already definable in the language of valued fields for some valued fields (e.g. $\mathbb{Q}_p$, cf. remark 1.11), there are henselian valued fields in which these functions are not definable (take for example $\mathbb{C}((t))$). This issue is avoided in the work of Basarab and Kuhlmann by considering "mixed structures" or "additive-multiplicative congruences" [3], [20]. If K is a valued field and $I \subseteq V_K$ is a proper ideal (in our case the maximal ideal $\mathfrak{m}$ of V, or some $p^n\mathfrak{m}$), then we set $K_I := K^*/(1+I)$ and $\pi_I\colon K^* \to K_I$ the quotient map. The structure K_I is more than just a group under multiplication. It continues to carry the valuation of K and addition on K leaves a trace on K_I in the form of a ternary relation $A_I(x, y, z) \Leftrightarrow (\exists \tilde{x}, \tilde{y} \in K^*)\pi_I(\tilde{x}) = x \;\&\; \pi_I(\tilde{y}) = y \;\&\; \pi_I(\tilde{x} + \tilde{y}) = z$. Quantifier elimination relative to the mixed structures (for an appropriate choice of ideals) holds in fairly general henselian fields. There is a price to be paid for working only with structures interpretable in the language of valued fields: the structure of the class of definable sets in the mixed structures may be obscure.

Angular components and mixed structures adapt to valued difference fields without any substantial changes. If (K, v, σ) is a valued difference field and $I \subseteq V_K$ is a proper ideal, then σ induces an automorphism of K_I which we continue to denote by σ. We say that the angular component map $ac_I\colon K^* \to (V_K/I)^*$ is compatible with σ if it commutes with σ. A simple calculation as in section 1 shows that, in our context, this is equivalent to having ac restrict to $\mathrm{Fix}(\sigma)$ in the natural way. After stating precisely the elimination theorems for those formalisms, we show how they follow from each other. Our embedding theorem yielding elimination of quantifiers used the angular component maps.

We need to use many-sorted first-order logic, where variables, constant symbols, function symbols and relation symbols have prescribed sorts. Terms and formulas are built in the usual manner, and the classical results hold (see [19]).

Let μ be a multisorted signature and $\mathcal{L}(\mu)$ the associated first-order language. If f is a function symbol of μ, let $\mathrm{dom}(f)$ designate the sequence of sorts for the domain of f, and $\mathrm{rng}(f)$ the sort for the range of f. If R is a relation symbol, let $\mathrm{fld}(R)$ designate the sequence of sorts for the domain of R. Let Σ be a set of μ-sort symbols. We define a new signature $\mu' := \mu_{qf-\Sigma}$ having the same constant, function, and relation symbols plus new relation symbols $R_\phi(\sigma)$ for each $\phi \in \mathcal{L}(\mu)$ whose free variables are all of sorts belonging to Σ. Fixing an ordering of the free variables of ϕ, $x_{S_0}^{(i_0)}, \ldots, x_{S_{m-1}}^{i_{m-1}}$, we define $\mathrm{fld}(R_\phi)$ to be $\langle S_0, \ldots, S_{m-1} \rangle$. The theory $T_{qf-\mu,\Sigma}$ is generated by the (universal closures of) the formulas $\phi \leftrightarrow R_\phi$, for $\phi \in \mathcal{L}(\mu)$ as above.

Definition 11.1. Let μ be a multisorted signature, Σ a set of μ-sort symbols, and T an $\mathcal{L}(\mu)$-theory. We say that T eliminates quantifiers relative to Σ if for any formula $\varphi \in \mathcal{L}(\mu)$ there is a quantifier-free formula $\psi \in \mathcal{L}(\mu_{qf-\Sigma})$ such that $T \cup T_{qf-\Sigma,\mu} \vdash \varphi \leftrightarrow \psi$.

We intend to prove not only relative quantifier elimination results in a fixed language but rather such results for *any* expansion of the language of valued difference fields by predicates on the mixed structures.

Definition 11.2. Let μ be a multisorted signature, Σ be a nonempty set of μ-sorts, and T an $\mathcal{L}(\mu)$-theory. We say that T *resplendently eliminates quantifiers relative to* Σ if for any model $M \models T$ and any signature $\tau \supseteq (\mu|_\Sigma)$ with exactly Σ as sorts and only new predicates, and any expansion M' of $M|_\Sigma$ to a τ-structure and any formula $\phi \in \mathcal{L}(\mu \cup \tau)$, there is some quantifier-free $\psi \in \mathcal{L}((\mu \cup \tau)_{qf-\Sigma})$ such that $T \cup T_{qf-\Sigma,\mu\cup\tau} \vdash \phi \leftrightarrow \psi$.

LEMMA 11.3. *Let $\mu \subseteq \mu'$ be multisorted signatures. Let Σ be a set of μ-sort symbols. We suppose that the only difference between μ and μ' is that there may be new function symbols in μ'. Let T be an $\mathcal{L}(\mu)$-theory and $T' \supseteq T$ a $\mathcal{L}(\mu')$ theory which eliminates quantifiers in $\mathcal{L}(\mu')$ and with $T = T'|_{\mathcal{L}(\mu)}$.*

We make the following assumptions.

- *If f is a new function symbol of μ', then* $\mathrm{dom}(f) \in {}^{<\omega}\Sigma$ *and* $\mathrm{rng}(f) \in \Sigma$.
- *If R is a μ-relation symbol, then either all sorts of its domain belong to Σ or all do not.*
- *If t is a μ-term with all sorts of its variables belonging to Σ, then* $\mathrm{rng}(t) \in \Sigma$.

If in every model $M \models T$ every $\mathcal{L}(\mu)_M$-definable subset of $(M|_\Sigma)^n$ is already definable in $\mathcal{L}(\mu)_{(M|_\Sigma)}$, then T eliminates quantifiers in $\mathcal{L}(\mu)$ relative to Σ.

In what follows we write $\mathrm{tp}(\boldsymbol{a})$ for the type of the tuple $\boldsymbol{a}$ and $\mathrm{qftp}(\boldsymbol{a})$ for its quantifier-free type.

Proof. Let $\phi \in \mathcal{L}(\mu)$ be a formula. By hypothesis, there is some quantifier-free $\psi \in \mathcal{L}(\mu')$ such that $T' \vdash \phi \leftrightarrow \psi$. Write ϕ as $\phi(\boldsymbol{x}, \boldsymbol{y})$ with $\boldsymbol{x}$ a tuple of variables ranging over sorts not in Σ and $\boldsymbol{y}$ a tuple of variables ranging over sorts in Σ. By our hypotheses on μ and μ', up to equivalence over T', we have $\psi(\boldsymbol{x}, \boldsymbol{y}) = \bigvee_{j=1}^{n} \theta_j(\boldsymbol{x}, \boldsymbol{y}) \ \wedge \ \vartheta_j(\boldsymbol{\alpha}(\boldsymbol{t}(\boldsymbol{x}), \boldsymbol{y}))$, where $\theta_j \in \mathcal{L}(\mu)$ and $\vartheta_j \in \mathcal{L}(\mu \mid \Sigma)$ are quantifier-free formulas, $\boldsymbol{\alpha}$ is a tuple of μ'-terms, and $\boldsymbol{t}(\boldsymbol{x})$ is a tuple of $\mathcal{L}(\mu)$-terms with range in Σ. By considering each formula $\phi \wedge \theta_j$ separately, it suffices to work in the case that $n = 1$ so that we drop the subscripts from θ and ϑ.

If $\theta(\boldsymbol{x}, \boldsymbol{y}) \wedge \vartheta(\boldsymbol{\alpha}(\boldsymbol{t}(\boldsymbol{x}), \boldsymbol{y}))$ is not equivalent (modulo $T' \cup T_{qf-\mu,\Sigma}$) to a quantifier-free $\mathcal{L}(\mu \cup \mu_{qf-\Sigma,\mu})$-formula, then it is consistent with T' that there be $(\boldsymbol{x}', \boldsymbol{y}')$ with $\mathrm{tp}_{\mathcal{L}(\mu|_\Sigma)}(\boldsymbol{t}(\boldsymbol{x}), \boldsymbol{y}) = \mathrm{tp}_{\mathcal{L}(\mu|_\Sigma)}(\boldsymbol{t}(\boldsymbol{x}'), \boldsymbol{y}')$ and $\mathrm{qftp}_{\mathcal{L}(\mu)}(\boldsymbol{x}, \boldsymbol{y}) = \mathrm{qftp}_{\mathcal{L}(\mu)}(\boldsymbol{x}', \boldsymbol{y}')$ and $\phi(\boldsymbol{x}, \boldsymbol{y}) \wedge \neg\phi(\boldsymbol{x}', \boldsymbol{y}')$.

As no new structure is induced on Σ (relative to T), $\mathrm{tp}_{\mathcal{L}(\mu|\Sigma)}(\boldsymbol{t}(\boldsymbol{x}), \boldsymbol{y}) \vdash \mathrm{tp}_{\mathcal{L}(\mu)}(\boldsymbol{t}(\boldsymbol{x}), \boldsymbol{y}))$. Thus, we can find such $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{x}', \boldsymbol{y}'$ in some model $M \models T$ (which we may presume to be a reduct to $\mathcal{L}(\mu)$ of a model of T') and a $\mathcal{L}(\mu)$-automorphism $\tau\colon M \to M$ with $\tau(\boldsymbol{t}(\boldsymbol{x}), \boldsymbol{y}) = (\boldsymbol{t}(\boldsymbol{x}'), \boldsymbol{y}')$. If we suppose $M \models \phi(\boldsymbol{x}, \boldsymbol{y})$, then as $(\boldsymbol{x}, \boldsymbol{y}), (\boldsymbol{x}', \boldsymbol{y}')$, and $(\tau(\boldsymbol{x}), \tau(\boldsymbol{y}))$ all have the same $\mathcal{L}$-quantifier free type, we have $M \models \theta(\boldsymbol{x}, \boldsymbol{y}) \wedge \theta(\boldsymbol{x}', \boldsymbol{y}') \wedge \theta(\tau(\boldsymbol{x}), \tau(\boldsymbol{y}))$. We have then

$$\begin{aligned} \phi(\boldsymbol{x}, \boldsymbol{y}) &\Rightarrow \phi(\tau(\boldsymbol{x}), \tau(\boldsymbol{y})) \\ &\Rightarrow \theta(\tau(\boldsymbol{x}), \tau(\boldsymbol{y})) \wedge \vartheta(\boldsymbol{\alpha}(\tau(\boldsymbol{t}(\boldsymbol{x})), \tau(\boldsymbol{y}))) \\ &\Rightarrow \theta(\boldsymbol{x}', \boldsymbol{y}') \wedge \vartheta(\boldsymbol{\alpha}(\boldsymbol{t}(\boldsymbol{x}'), \boldsymbol{y}')) \\ &\Rightarrow \phi(\boldsymbol{x}', \boldsymbol{y}') \end{aligned} \qquad \square$$

Let us return now to valued fields with isometry to explain how we shall consider them as multisorted structures for the current discussion. The multisorted signature of valued fields with isometry μ_{isom} is defined as follows. There are sort symbols K, Γ, k, "as before", viz. to be interpreted as the base valued field, the value group, the residue field, and now extra sorts K_n for each $n \in \omega$, to be interpreted, by abuse of notation, as the previous $K_{p^n \mathfrak{m}}$, where p is a constant symbol to designate the residue characteristic in the case of positive residue characteristic and 1 when the residue characteristic is zero. There is a function symbol σ for the distinguished automorphism, v for the valuation map. There are the various required symbols for K construed as a ring, Γ as an ordered abelian group plus a constant $\infty = v(0)$, k as a ring plus a constant $\bar{\infty}$ to be interpreted as the reduction of any element not in the valuation ring, and another constant $\bar{p}$ for the reduction of p. The sort K_n is construed as the truncated ring already described, with a full multiplication and a ternary predicate for addition whenever it is defined and a constant p_n for the reduction of p. We have extra function symbols ρ for the previous reduction map $\bar{\ }$ extended as just prescribed, v_n for the induced valuation on K_n, $\bar{\sigma}$ for the reduction of σ, σ_n for the reduction of σ in K_n, π_n for the quotient map $K \to K_n$, $\pi_{n,m}$ for the natural quotient map $K_n \to K_m$, ρ_n for the induced reduction map on K_n. So, as μ_{isom}-structures, our valued fields with isometry can be presented, by abuse of notation, as follows $(K, k, \Gamma, K_n, v, v_n, \sigma, \bar{\sigma}, \sigma_n, p, \bar{p}, p_n, \infty, \bar{\infty}, \rho, \rho_n, \pi_n, \pi_{n,m})$.

Let $(L, v, k, \Gamma, \sigma)$ be a valued field with isometry. In the case that p^L is a unit, the maps $\pi_{n,m}\colon K_n^L \to K_m^L$ are isomorphisms so that there is no need to go beyond the reduct to (K, Γ, K_0). In order to make a uniform statement, we do not explicitly specify the value of p. However, in mixed characteristic, we require that $v(p^L)$ is at least that of p, the residue characteristic.

For our discussion, we will construe angular component maps as splittings of the exact sequences

$$1 \longrightarrow (V/p^n\mathfrak{m})^* \longrightarrow K^*/(1+p^n\mathfrak{m}) \longrightarrow \Gamma \longrightarrow 0$$

as in Section 1. We thus set the language of valued difference fields with angular components by adjoining to $\mathcal{L}(\mu_{\mathrm{isom}})$ function symbols ac_n having domain sort and range sort K_n, as the natural range of an angular component ac_n is the group $(V/p^n\mathfrak{m})^*$ which is now definable as the kernel of the induced valuation v_n on K_n.

We can now state the elimination theorems.

THEOREM 11.4. *The theory of σ-henselian fields satisfying the σ-AEK axioms with angular component functions resplendently eliminates quantifiers relative to $\{\Gamma, k\}$ (and also relative to $\{K_n\colon n \in \omega\}$).*

We have a similar statement without angular component functions.

THEOREM 11.5. *The theory of σ-henselian fields satisfying the σ-AEK axioms resplendently eliminates quantifiers relative to $\{\Gamma, k\} \cup \{K_n\colon n \in \omega\}$.*

For residue characteristic zero, we get quantifier elimination relative to just K_0.

THEOREM 11.6. *The theory of σ-henselian fields satisfying the σ-AEK axioms and of residue characteristic zero with an angular component function, in the reduct to $\{K, K_0, k, \Gamma\}$, resplendently eliminates quantifiers relative to $\{\Gamma, k\}$ (and also relative to $\{K_0\}$).*

Likewise, we have a better statement for residue characteristic zero without angular components.

THEOREM 11.7. *The theory of σ-henselian fields satisfying the σ-AEK axioms and residue characteristic zero, in the reduct to $\{K, K_0, k, \Gamma\}$, resplendently eliminates quantifiers relative to K_0.*

Theorems 11.4, 11.6 follow directly from the Embedding Theorem. More general theorems are proved in [29] in the slightly different setting of valued D-fields.

As the forms of Theorems 11.4 to 11.7 are so similar, it should come as no surprise to the reader that there are formal implications between these statements.

Before proving the formal implications we need a lemma on expansions of valued fields with isometry to valued fields with isometry with angular components, which follows from the remarks in section 1 (or see [29]).

LEMMA 11.8. *If K is a valued field with isometry and enough constants and τ: $\bigcup_{n\in\omega} K_n \to \bigcup_{n\in\omega} K_n$ is an automorphism of the reduct to $\{K_n\colon n \in \omega\}$, then there is an elementary extension $(K^\star, \tau^\star)$ of (K, τ) which admits the structure of a valued field with isometry with angular components with respect to which τ^* is an automorphism.*

PROPOSITION 11.9. *Theorem 11.4 implies Theorem 11.5 and Theorem 11.6 implies Theorem 11.7.*

Proof. We begin with Theorem 11.4 $\Rightarrow$ Theorem 11.5. Let $\mathcal{L}' \supseteq \mathcal{L}(\mu_{\mathrm{isom}})$ be some expansion of the language of valued fields with isometry by relations on $\{K_n\colon n \in \omega\}$. Let K be a σ-henselian field satisfying the σ-AEK axioms, considered as an $\mathcal{L}'$-structure. By Lemma 11.8 applied to $\tau = \mathrm{id}$ there is an elementary extension $K^\star \succeq K$ which admits an expansion to $\mathcal{L}'(\{ac_n\colon n \in \omega\})$.

We check that the hypotheses of Lemma 11.3 apply. As K is a σ-henselian field satisfying the σ-AEK axioms, the main point is that K induces no new structure on $\{K_n\colon n \in \omega\}$. For this it suffices to show that every automorphism of $\{K_n\colon n \in \omega\}$ extends to an $\mathcal{L}'$-automorphism of some elementary extension of K.

Let K be considered as an $\mathcal{L}'$-structure and let τ be any automorphism of $K\!\restriction_{\{K_n\colon n\in\omega\}}$. By Lemma 11.8 there is a σ-henselian field $K^\star$ with angular components considered as an $\mathcal{L}'(\{ac_n\colon n \in \omega\})$-structure such that $K \preceq (K^\star\!\restriction_{\mathcal{L}})$ and there is some $\tau^\star \in \mathrm{Aut}_{\mathcal{L}'(\{ac_n\colon n\in\omega\})}(K^\star\!\restriction_{\{K_n\colon n\in\omega\}})$ with $\tau^\star \supseteq \tau$. By our hypotheses and Theorem 11.4, there is an elementary extension $L \succeq K^\star$ on which $\tau^\star$ extends to an automorphism of all of L. This automorphism is, of course, also an automorphism of $L\!\restriction_{\mathcal{L}}$. Thus, every $\mathcal{L}'$-automorphism of $K\!\restriction_{\{K_n\colon n\in\omega\}}$ extends to some $\mathcal{L}'$-automorphism of some elementary extension of K. Thus, the implication Theorem 11.4 $\Rightarrow$ Theorem 11.5 now follows from Lemma 11.3.

The argument in the case of Theorem 11.6 $\Rightarrow$ Theorem 11.7 is the same as that in the case of Theorem 11.4 $\Rightarrow$ Theorem 11.5 with the exception that we talk only about K_0 instead of all the mixed structures. $\square$

We also observe that quantifier elimination with angular components also follows from quantifier elimination with the mixed structures, in our unramified case.

PROPOSITION 11.10. *Theorem 11.5 implies Theorem 11.4 and Theorem 11.7 implies Theorem 11.6 .*

Proof. To see this we need to reinterpret the quantifier elimination with mixed structures in terms of embeddings. Then, arguments as in [4] (Lemma 4.3 and Theorem 5.3), apply mutatis mutandis to recover quantifier elimination with angular components. The point is the interpretation of the angular component map as a splitting of the corresponding exact sequence of the mixed structure. An embedding of the underlying groups will then readily induce an embedding of the corresponding exact sequence. To take care of the extra structure one uses finite ramification. In the end, the embedding of valued fields is compatible with angular components by construction. □

Finally, we note that since in $W(\mathbb{F}_p^{\mathrm{alg}})$ angular components are existentially definable relative to the fixed field $\mathbb{Q}_p$ (remark 1.11), if $\varphi(\boldsymbol{x})$ is a "relative" quantifier free formula given by Theorem 11.4, with base field free variables $\boldsymbol{x}$, then by replacing the ac_n's by their definition we get an equivalent formula $\varphi(\boldsymbol{x}, \boldsymbol{x}')$, where $\boldsymbol{x}'$ are new base field variables and the ac_n's do not occur, but the only occurences of base field quantifiers are of the form $\exists x'(\sigma(x') = x' \wedge \cdots)$, i.e., $\exists x' \in \mathrm{Fix}(\sigma)$.

12. From characteristic p to characteristic 0. Fix a Witt-Frobenius case $(K_p, v_p, k_p, \Gamma_p, \sigma_p)$ satisfying the σ-AEK axioms for each prime p, and let $(K, v, k, \Gamma, \sigma)$ be a nonprincipal ultraproduct of the $(K_p, v_p, k_p, \Gamma_p, \sigma_p)$. Note that $(K_p, v_p, k_p, \Gamma_p, \sigma_p)$ is elementarily equivalent to $W(k_p)$ with the Witt Frobenius.

LEMMA 12.1. *The difference field $(k, \bar{\sigma})$ satisfies no $\bar{\sigma}$-identities.*

Proof. It follows from our previous discussion on σ-identities (section 3), but in this case there is a simple direct argument: consider any putative $\bar{\sigma}$-identity

$$F(x, \bar{\sigma}(x), \ldots \bar{\sigma}^{(n)}(x)) \equiv 0, \quad x \in k$$

i.e.,

$$F(x, x^p, \ldots x^{p^n}) \equiv 0, \quad x \in k.$$

Now if $F(x_0, \ldots x_n) = \sum c_{\boldsymbol{l}} \cdot \boldsymbol{x}^{\boldsymbol{l}}$, and $p > \max |\boldsymbol{l}|$, $F(x, x^p, \ldots x^{p^n})$ is not the zero polynomial over k_p, so, since k_p is infinite, doesn't vanish identically. □

To get Axiom RG satisfied in the ultraproduct we (apparently) need to make some assumptions on the k_p. Since K_p is σ-henselian, k_p is linearly difference closed, and if we require Axiom R2 or Axiom R3 for each $(k_p, \bar{\sigma}_p)$, by Propositions 3.9 and 3.6 this will make Axiom RG true in the ultraproduct. In particular:

LEMMA 12.2. *If the k_p are algebraically closed, the ultraproduct satisfies Axiom RG.*

Now consider the characteristic p field $(k_p((t)), v_p, k_p, \mathbb{Z}, \sigma_{p,t})$ where $\sigma_{p,t}$ is genuine Frobenius on k_p, and $\sigma_{p,t}(t) = t$, as in Example 1.4. The following theorem subsumes the AEK theorem relating $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$ (by considering the $\mathrm{Fix}(\sigma)$): taking ultraproducts we get fields satisfying the σ-AEK axioms with identical residue fields and value groups, whence elementarily equivalent.

THEOREM 12.3. *If D is any nonprincipal ultraproduct on the primes, then*

$$\prod_D (W(k_p), \upsilon_p, k_p, \mathbb{Z}, \sigma_p) \equiv \prod_D (k_p((t)), \upsilon_p, k_p, \mathbb{Z}, \sigma_{p,t}).$$

COROLLARY 12.4. *Any sentence true in all* $(W(\mathbb{F}_p^{alg}), \sigma_p)$ *is true in all but finitely many* $(\mathbb{F}_p^{alg}((t)),\ \sigma_{p,t})$, *and vice versa.*

The model companion of the theory of difference fields of characteristic 0 is known as $ACFA_0$ ("algebraically closed fields with an automorphism", see [7]). Nonprincipal ultraproducts of the $(W(\mathbb{F}_p^{alg}), \upsilon_p, \mathbb{F}_p^{alg}, \mathbb{Z}, \sigma_p)$ lead to $(K, \upsilon, k, \Gamma, \sigma)$ where k is algebraically closed of characteristic 0, and $(k, \bar{\sigma}) \models ACFA_0$ (by unpublished work of Hrushovski and Macintyre [15], [22]). Since $ACFA_0$ is decidable we get:

THEOREM 12.5. *The theory of the class of all* $(W(\mathbb{F}_p^{alg}), \upsilon_p, \mathbb{F}_p^{alg}, \mathbb{Z}, \sigma_p)$ *is decidable.*

Note. Using the (quite intricate) quantifier-elimination for $ACFA_0$ given in [21], one can give one for the class of all $(W(\mathbb{F}_p^{alg}), \upsilon_p, \mathbb{F}_p^{alg}, \mathbb{Z}, \sigma_p)$, but in view of the complexity of this we omit the details.

UNIVERSITÉ DU QUÉBEC – UQAM, DÉP. DE MATHÉMATIQUES, CASE POSTALE 8888, SUCCURSALE CENTRE-VILLE, MONTRÉAL, QUÉBEC, H3C 3P8
E-mail: belair.luc@uqam.ca

UNIVERSITY OF EDINBURGH, DEPARTMENT OF MATHEMATICS AND STATISTICS, EDINBURGH, SCOTLAND
Current address: QUEEN MARY UNIVERSITY OF LONDON, LONDON, ENGLAND
E-mail: angus@dcs.qmul.ac.uk

UNIVERSITY OF CALIFORNIA AT BERKELEY, DEPARTMENT OF MATHEMATICS, EVANS HALL, BERKELEY, CA 94720-3840
E-mail: scanlon@math.berkeley.edu

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