Measure Properties of Game Tree Languages

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Abstract. We introduce a general method for proving measurability of topologically complex sets by establishing a correspondence between the notion of *game tree languages* from automata theory and the σ-algebra of $\mathcal{R}$*-sets*, introduced by A. Kolmogorov as a foundation for measure theory. We apply the method to answer positively to an open problem regarding the game interpretation of the probabilistic μ-calculus.

1 Introduction

Among logics for expressing properties of nondeterministic (including concurrent) processes, represented as transition systems, Kozen's modal μ-calculus [14] plays a fundamental rôle. This logic enjoys an intimate connection with parity games, which offers an intuitive reading of fixed-points, and underpins the existing technology for model-checking μ-calculus properties. An abstract setting for investigating parity games, using the tools of descriptive set theory, is given by so-called *game tree languages* (see, e.g. [2]). The language $\mathcal{W}_{i,k}$ is the set of parity games with priorities in $\{i \dots k\}$, played on an infinite binary tree structure, which are winning for Player $\exists$. The (i,k)-indexed sets $\mathcal{W}_{i,k}$ form a strict hierarchy of increasing topological complexity called the *index hierarchy* of game tree languages (see [5,1,2]). Precise definitions are presented in Section 2.

For many purposes in computer science, it is useful to add probability to the computational model, leading to the notion of probabilistic nondeterministic transition systems (PNTS's). In an attempt to identify a satisfactory analogue of Kozen's μ-calculus for expressing properties of PNTS's, the third author has recently introduced in [18,19] a quantitative fixed-point logic called *probabilistic μ-calculus with independent product* (pLμ). A central contribution of [19] is the definition of a game interpretation of pLμ, given in terms of a novel class of games

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generalizing ordinary two-player *stochastic* parity games. While in ordinary two-player (stochastic) parity games the outcomes are infinite sequences of game-states, in pLμ-games the outcomes are infinite trees, called *branching plays*, whose vertices are labelled with game-states. This is because in pLμ some of the game-states, called *branching states*, are interpreted as generating distinct game-threads, one for each successor state of the branching state, which continue their execution *concurrently* and *independently*. The winning set of a pLμ-game is therefore a collection of branching plays specified by a combinatorial condition associated with the structure of the game arena.

Unlike winning sets of ordinary two-player (stochastic) parity games, which are well-known to be Borel sets[4], the winning sets of pLμ-games generally belong to the $\mathbf{\Delta}^1_2$-class of sets in the projective hierarchy of Polish spaces [19, Theorem 4.20]. This high topological complexity is a serious concern because pLμ-games are *stochastic*, i.e. the final outcome (the branching play) is determined not only by the choices of the two players but also by the randomized choices made by a probabilistic agent. A pair of strategies for $\exists$ and $\forall$, representing a play up-to the choice of the probabilistic agent, only defines a probability measure on the space of outcomes. For this reason, one is interested in the *probability* of a play to satisfy the winning condition. Under the standard Kolmogorov's measure-theoretic approach to probability theory, a set has a well-defined probability only if it is a *measurable set*[5]. Due to a result of Kurt Gödel (see [10, § 25]), it is consistent with Zermelo-Fraenkel Set Theory with the Axiom of Choice (ZFC) that there exists a $\mathbf{\Delta}^1_2$ set which is not measurable. This means that it is not possible to prove (in ZFC) that all $\mathbf{\Delta}^1_2$-sets are measurable. However it may be possible to prove that a *particular* set (or family of sets) in the $\mathbf{\Delta}^1_2$-class is measurable. In [18] the author asks the following question[6]:

Question: are the winning sets of pLμ-games provably measurable?

This problem provided the original motivation of our work. We will answer positively to the question by developing a general method for proving measurability of topologically complicated sets.

This type of questions has been investigated since the first developments of measure theory, in late 19th century, as the existence of non-measurable sets (e.g. Vitali sets [10]) was already known. The measure-theoretic foundations of probability theory are based around the concept of a σ-algebra of measurable events on a space of potential outcomes. Typically, the σ-algebra is assumed to contain all open sets. Hence the minimal σ-algebra under consideration consists of all Borel sets whereas the maximal consists, by definition, of the collection of all measurable sets. The Borel σ-algebra, while simple to work with, lacks important classes of measurable sets (e.g. $\mathbf{\Pi}^1_1$-complete sets). On the other hand, the full σ-algebra of measurable sets may be difficult to work with since there

[4] See, e.g., Remark 10.57 in [3] for a discussion about measurability in this context.

[5] More precisely, *universally measurable*, see Section 2.

[6] Statement "is mG-UM($\mathit{\Gamma}_p$) true?", see Definition 5.1.18 and discussion at the end of Section 4.5 in [18]. See also Section 8.1 in [19].

is no constructive methodology for establishing its membership relation, i.e. for proving that a given set belongs to this σ-algebra.

This picture led to a number of attempts to find the largest σ-algebra, extending the Borel σ-algebra and including as many measurable sets as possible and, at the same time, providing practical techniques for establishing the membership relation. A general methodology for constructing such σ-algebras is to identify a family $\mathcal{F}$ of *safe* operations on sets which, when applied to measurable sets are guaranteed to produce measurable sets. When the operations considered have countable arity (e.g. countable union), the σ-algebra generated by the open sets closing under the operations in $\mathcal{F}$ admits a transfinite decomposition into ω_1 levels, and this allows the membership relation to be established inductively. The simplest case is given by the σ-algebra of Borel sets, with $\mathcal{F}$ consisting of the operations of complementation and countable union. Other less familiar examples include $\mathcal{C}$-sets studied by E. Selivanovski [20], Borel programmable sets proposed by D. Blackwell [4] and $\mathcal{R}$-sets proposed by A. Kolmogorov [13].

The σ-algebra of $\mathcal{R}$-sets is, to our knowledge, the largest ever considered. Most measurable sets arising in ordinary mathematics are $\mathcal{R}$-sets belonging to the finite levels of the transfinite hierarchy of $\mathcal{R}$-sets. For example, all Borel sets, analytic sets, co-analytic sets and Selivanovski's $\mathcal{C}$-sets lie in the first two levels [8]. Thus, for most practical purposes, the following principle is valid:

Principle: " all practically useful measurable sets belong to the finite levels of the transfinite hierarchy of Kolomogorov's $\mathcal{R}$-sets."

Contributions. The definition of $\mathcal{R}$-sets in [13], formulated in terms of operations on sets and transformations on operations (Section 3), is purely set-theoretical. As a main technical contribution of this work, we provide an alternative game-theoretical characterization of the finite levels of the hierarchy of $\mathcal{R}$-sets in terms of game tree languages $\mathcal{W}_{i,k}$.

Theorem 1. *$\mathcal{W}_{k-1,2k-1}$ is complete for the k-th level of the hierarchy of $\mathcal{R}$-sets.*

As a consequence one can establish the measurability of a given set $A \subseteq X$ by constructing a continuous reduction to $\mathcal{W}_{i,k}$. This can be thought as a *coding* f of elements in X in terms of parity games with priorities in $\{i, \ldots, k\}$ such that $x \in A$ if and only if $f(x)$ is winning for Player $\exists$. Parity games are well-known and relatively simple to work with. Thus the proof method allows for easier applications. Since $\mathcal{R}$-sets exhaust the realm of reasonable measurable sets, and the sets $\mathcal{W}_{i,k}$ are complete among $\mathcal{R}$-sets, the method should cover most cases.

Additionally, in Section 6, we investigate the special $\aleph_1$-continuity property of measures on $\mathcal{W}_{i,k}$ with respect to the approximations $\mathcal{W}^{\alpha}_{i,k}$, crucially required in the proof of determinacy of pLμ-games of [19,18]. As observed in [18], the property follows from the set-theoretic Martin Axiom at $\aleph_1$ ($\mathrm{MA}_{\aleph_1}$). The problem of whether the property (and, as a consequence, the validity of the determinacy proof) holds in ZFC alone is left open in [18]. We provide a partial positive answer to this question proving the continuity property for $\mathcal{W}_{0,1}$ in ZFC alone.

Furthermore, we show that for higher ranks the property follows from a set-theoretic assumption weaker than $\mathrm{MA}_{\aleph_1}$ which, unlike $\mathrm{MA}_{\aleph_1}$, does not depend on cardinality assumptions such as the negation of the Continuum Hypothesis.

Applications. As already observed in [18, §5.4], the winning sets of pLμ-games reduce to game tree languages. Thus Theorem 1 settles the question posed in [18] about the measurability of pLμ winning sets. More generally, our result can find applications in solving similar problems. For example, in models of probabilistic concurrent computation (e.g. probabilistic Petri nets [15], probabilistic event structures [9], stochastic distributed games [21]), executions are naturally modelled by configurations of event structures (i.e. special kinds of acyclic graphs) and not by sequences. Many natural predicates on executions (e.g. the collection of *well-founded* graphs) are of high topological complexity.

Related Work. Beside the original work of Kolmogorov [13], the measure theoretic properties of $\mathcal{R}$-sets are investigated with set-theoretic methods by Lyapunov in [16]. A game-theoretic approach to $\mathcal{R}$-sets, closely related to this work, is developed by Burgess in [8] where the following characterization is stated as a remark without a formal proof: (1) every set $A\subseteq X$ belongs to a finite level of the hierarchy of $\mathcal{R}$-sets if and only if it is of the form $A=\Game(K)$, for some set $K \subseteq \omega^\omega$ which is a Boolean combination of F_σ sets, and (2) the levels of the hierarchy of $\mathcal{R}$-sets are in correspondence with the levels of the *difference hierarchy* (see [12, §22.E]) of F_σ sets. The operation $\Game$ is the so-called *game quantifier* (see [12, §20.D] and [6,7,11,17]). Admittedly, our characterization of $\mathcal{R}$-sets in terms of game tree languages $\mathcal{W}_{i,k}$, can be considered as a modern variant of the result of Burgess.[7] Having concrete examples of complete sets, however, sheds light on the concept of $\mathcal{R}$-sets and, in analogy with the study of complexity classes in computational complexity theory, may simplify further investigations. Lastly, it is suggestive to think that the origins of the concept of parity games, developed since the 80's in Computer Science to investigate ω-regular properties of transition systems, could be backdated to the original work of A. Kolmogorov.

2 Basic Notions from Descriptive Set Theory

We assume the reader is familiar with the basic notions of descriptive set theory and measure theory. We refer to [12] as a standard reference on these subjects.

Given two sets X and Y, we denote with X^Y the set of functions from Y to X. We denote with 2 and ω the two element set and the set of all natural numbers, respectively. The powerset of X will be denoted by both 2^X and $\mathcal{P}(X)$, as more convenient to improve readability. A topological space is *Polish* if it is separable and the topology is induced by a complete metric. A set is *clopen* if it

[7] The fact that $\mathcal{W}_{i,k}$ are $\mathcal{R}$-sets follows from the above formulation of Burgess' theorem. Also, our Theorem 1 can be easily inferred for $k = 1$. The case of $k = 2$ follows from Burgess's theorem in conjunction with [17]. Our proof of Theorem 1 yields an independent and formal argument backing the statement of Burgess' theorem.

is both closed and open. A space is *zero-dimensional* if the clopen subsets form a basis of the topology. In this work we limit our attention to zero-dimensional Polish spaces. Let X, Y be two topological spaces and $A \subseteq X$, $B \subseteq Y$ be two sets. We say that A is *Wadge reducible* to B, written as $A \leq_W B$, if there exists a continuous function $f\colon X \to Y$ such that $A = f^{-1}(B)$. Two sets A and B are *Wadge equivalent* (denoted $A \sim_W B$) if $A \leq_W B$ and $B \leq_W A$ hold. Given a family $\mathcal{C}$ of subsets of X, we say a set $A \in \mathcal{C}$ is *Wadge complete* if $B \leq_W A$ holds for all $B \in \mathcal{C}$. Given a Polish space X, we denote with $\mathcal{M}_{=1}(X)$ the Polish space of all Borel probability measures μ on X (see e.g. [12, Theorem 17.22]). A set $N \subseteq X$ is *μ-null* if there exists a Borel set $B \supseteq N$ such that $\mu(B) = 0$. A set $A \subseteq X$ is μ-measurable if $A = B \cup N$, for a Borel set B and a μ-null set N. A set $A \subseteq X$ is *universally measurable* if it is μ-measurable for all $\mu \in \mathcal{M}_{=1}(X)$. In what follows we omit the "universally" adjective.

Given two natural numbers $i < k$, the set $\mathrm{Tr}_{i,k}$ of all complete (i.e. without leaves) binary trees whose vertices are labelled by elements of $\{\exists, \forall\} \times \{i, \ldots, k\}$ is endowed with the standard 0-dimensional Polish topology (see e.g. [2]). Each $t \in \mathrm{Tr}_{i,k}$ can be interpreted as a two-player parity game with priorities in $\{i, \ldots, k\}$, with players $\exists$ and $\forall$ controlling vertices labelled by $\exists$ and $\forall$, respectively.

Definition 1. *Given two natural numbers $i < k$, the* game tree language $\mathcal{W}_{i,k}$ *is the subset of $\mathrm{Tr}_{i,k}$ consisting of all parity games admitting a winning strategy for $\exists$. The pair (i, k) is called the (Rabin–Mostowski)* index *of $\mathcal{W}_{i,k}$.*

Clearly, there is a natural Wadge equivalence between the languages $\mathcal{W}_{i,k}$ and $\mathcal{W}_{i+2,k+2}$. Therefore, we identify indices (i, k) and $(i + 2j, k + 2j)$ for every $i \leq k$ and $j \in \omega$. Indexes can be partially ordered by defining $(i, k) \subseteq (i', k')$ if and only if $\{i, \ldots, k\} \subseteq \{i', \ldots, k'\}$.

3 Definition and Basic Properties of $\mathcal{R}$–sets

As outlined in the introduction, the σ-algebra of $\mathcal{R}$-sets is generated by a family $\mathcal{F}$ of *operations* on subsets having countable arity. Following Kolmogorov, we define $\mathcal{F}$ as the family generated by the operation $\bigcup \circ \bigcap$ and closing under a *transformation* co-$\mathcal{R}$. It will be convenient to assume that the countably many inputs of an operation Γ are indexed by a countable set (called the *arena*) denoted by $\mathbb{A}_\Gamma$. Thus an operation Γ has type $\Gamma\colon \mathcal{P}(X)^{\mathbb{A}_\Gamma} \to \mathcal{P}(X)$. The operations of countable union and intersections are denoted by $\bigcup$ and $\bigcap$, respectively, and their arena is defined as $\mathbb{A}_{\bigcup} = \mathbb{A}_{\bigcap} = \omega$.

Definition 2. *Given two operations Γ and Θ their composition $\Theta \circ \Gamma$ is the operation with arena $\mathbb{A}_\Gamma \times \mathbb{A}_\Theta$ defined as: $\Theta \circ \Gamma(\{A_{s,s'} \mid s \in \mathbb{A}_\Gamma, s' \in \mathbb{A}_\Theta\}) = \Theta(\{\ \Gamma(\{A_{s,s'} \mid s \in \mathbb{A}_\Gamma\}) \mid s' \in \mathbb{A}_\Theta\})$.*

Definition 3. *A* basis *for an operation Γ is a set $N_\Gamma \subseteq 2^{\mathbb{A}_\Gamma}$ such that*

$$\Gamma(\{A_s :\ s \in \mathbb{A}_\Gamma\}) = \bigcup_{S \in N_\Gamma} \bigcap_{s \in S} A_s \tag{1}$$

Not all operations have a basis but a family $N \subseteq 2^{\mathbb{A}}$ uniquely determines an operation Γ with arena $\mathbb{A}$ and basis N. In what follows we only consider operations Γ with a basis. One can check that $N_{\bigcup} = \{\{n\} \mid n \in \omega\}$ and $N_{\bigcap} = \{\omega\}$.

Definition 4. *For a given operation Θ with arena $\mathbb{A}$ and basis N_Θ, we define a* dual *operation* co-Θ *with the same arena $\mathbb{A}$ and basis $N_{\text{co-}\Theta}$ defined as $N_{\text{co-}\Theta} \stackrel{\text{def}}{=} \left\{S \in 2^{\mathbb{A}} \mid \forall T \in N_\Theta \;\; T \cap S \neq \emptyset\right\}$. One can notice that equivalently we can define* co-$\Theta(\{A_s : \; s \in \mathbb{A}\}) = \bigcap_{S \in N_\Theta} \bigcup_{s \in S} A_s$.

As an illustration, the equalities co-$\bigcup = \bigcap$ and co-$\bigcap = \bigcup$ hold.

Definition 5. *The $\mathcal{R}$-transformation of an operation Θ with basis N_Θ is the operation $\mathcal{R}\Theta$, with arena $\mathbb{A}_{\mathcal{R}\Theta} = (\mathbb{A}_\Theta)^*$ (finite sequences of elements in $\mathbb{A}_\Theta$) uniquely determined by the basis:*

$$N_{\mathcal{R}\Theta} \stackrel{\text{def}}{=} \{S \subseteq (\mathbb{A}_\Theta)^* \mid \exists T \subseteq S.\;\; \epsilon \in T \wedge \forall t \in T \;\{v \in \mathbb{A}_\Theta : \;\; tv \in T\} \in N_\Theta\} \quad (2)$$

where ϵ denotes the empty sequence and tv the concatenation of $t \in (\mathbb{A}_\Theta)^*$ with $v \in \mathbb{A}_\Theta$. We denote with co-$\mathcal{R}$ the composition co-$(\mathcal{R}(\Theta))$ *and define the iteration*

$$\Theta_k \stackrel{\text{def}}{=} (\text{co-}\mathcal{R})^k \left(\bigcup \circ \bigcap\right).$$

Definition 6. *For a positive number $k \geq 1$, we say that a set $A \subseteq X$ is an $\mathcal{R}$-set of k-th level if and only if $A = \Theta_k(\{U_s : s \in \mathbb{A}_{\Theta_k}\})$ for some clopen sets $U_s \subseteq X$.*

In what follows by $\mathcal{R}$-sets we mean $\mathcal{R}$-sets of finite levels.

Lemma 1 ([8]). *The k–th level of $\mathcal{R}$-sets is closed under pre-images of continuous functions.*

We say that an operation Γ *preserves measurability* if for any family $\mathcal{E} = \{A_s\}_{s \in \mathbb{A}_\Gamma}$ of measurable sets, the set $\Gamma(\mathcal{E})$ is measurable. The following property motivates the notion of $\mathcal{R}$-sets:

Theorem 2 ([16, Theorem 4]). *If Γ and Θ preserve measurability then $\Gamma \circ \Theta$, $\mathcal{R}\Gamma$, and* co-Γ *preserve measurability.*

Corollary 1. *All $\mathcal{R}$-sets are measurable.*

4 Matryoshka games

In this section we define *Matryoshka games*, a variant of parity games which make it easier to establish a connection with the operations Θ_k defined in Section 3.

A Matryoshka game $\mathcal{G}$ is the familiar structure of a two-player parity game played on an infinite countably branching graph, extended with a *labelling function* assigning to each finite play (i.e. every sequence of game-states ending in a terminal state) a *play label*. Formally, a Matryoshka game $\mathcal{G}$ is a structure:

$$\mathcal{G} = \{V^{\mathcal{G}} = V^{\mathcal{G}}_{\exists} \sqcup V^{\mathcal{G}}_{\forall}, F^{\mathcal{G}}, E^{\mathcal{G}}, v^{\mathcal{G}}_{I}, \Omega^{\mathcal{G}}, \mathbb{A}^{\mathcal{G}}, \text{label}^{\mathcal{G}}\},$$

such that $\{V^{\mathcal{G}} = V^{\mathcal{G}}_{\exists} \sqcup V^{\mathcal{G}}_{\forall}, F^{\mathcal{G}}, E^{\mathcal{G}}, v^{\mathcal{G}}_I, \Omega^{\mathcal{G}}\}$ is a standard parity game with initial state $v^{\mathcal{G}}_I$, terminal positions $F^{\mathcal{G}} \subseteq V^{\mathcal{G}}$, and priority assignment $\Omega^{\mathcal{G}}$. Additionally, $\mathbb{A}^{\mathcal{G}}$ is a set of *play labels*, and $\mathrm{label}^{\mathcal{G}} \colon (V^{\mathcal{G}})^* F^{\mathcal{G}} \to \mathbb{A}^{\mathcal{G}}$ is a function assigning to finite plays their *play labels*. We assume that for every $v \in V^{\mathcal{G}}$ there is at least one $v' \in V^{\mathcal{G}} \cup F^{\mathcal{G}}$ such that $(v, v') \in E^{\mathcal{G}}$, so that the only terminal game-states are in $F^{\mathcal{G}}$. As for standard parity games, the pair (i, k) containing the minimal and maximal values of Ω is called the *index* of the game. By $P \in \{\exists, \forall\}$ we denote the *players* of the game. The opponent of P is denoted by $\bar{P}$.

A *play* is defined as usual as a maximal path in the arena, i.e., either as a finite sequence in $\big(V^{\mathcal{G}}\big)^* F^{\mathcal{G}}$ or as an infinite sequence in $(V^{\mathcal{G}})^{\omega}$. Similarly, a strategy σ for Player P is a function $\sigma \colon \big(V^{\mathcal{G}}\big)^* V^{\mathcal{G}}_P \to V^{\mathcal{G}} \cup F^{\mathcal{G}}$ defined as expected.

The novelty in Matryoshka games is given by the set of play labels $\mathbb{A}^{\mathcal{G}}$ and the associated labelling function $\mathrm{label}^{\mathcal{G}}$. These are used to define *parametric* winning conditions in the Matryoshka game, as we now describe.

A set of play labels $X \subseteq \mathbb{A}^{\mathcal{G}}$ is called a *promise*. A finite play π is *winning for $\exists$ with promise X* if $\mathrm{label}(\pi) \in X$. An infinite play π is winning for $\exists$ if $\big(\limsup_{n\to\infty} \ \Omega^{\mathcal{G}}(\pi(n))\big)$ is even, as usual. If a play is not winning for $\exists$ then it is winning for $\forall$. A strategy σ for Player P is *winning in the Matryoshka game $\mathcal{G}$ with promise* X if, for every counter-strategy τ of $\overline{P}$, the resulting play $\pi(\sigma, \tau)$ is winning for P with promise X, in the sense just described. The following proposition directly follows from the well-known determinacy of parity games.

Proposition 1. *If $\mathcal{G}$ is a Matryoshka game with play labels $\mathbb{A}^{\mathcal{G}}$ and $X \subseteq \mathbb{A}^{\mathcal{G}}$ then exactly one of the players has a winning strategy in $\mathcal{G}$ with promise X.*

The point of having parametrized winning conditions in Matryoshka games is the possibility of defining set-theoretical operations with a direct game interpretation. Given a Polish space X, the *operation* on sets (see Section 2) associated with a Matryoshka game $\mathcal{G}$ has arena $\mathbb{A}^{\mathcal{G}}$ and is defined as follows:

$$\mathcal{G}(\mathcal{E}) \stackrel{\text{def}}{=} \big\{x \in X \ : \ \exists \text{ has a w. s. in } \mathcal{G} \text{ with promise } \{s \in \mathbb{A}^{\mathcal{G}} : x \in E_s\}\big\} \qquad (3)$$

where $\mathcal{E} = \{E_s : s \in \mathbb{A}^{\mathcal{G}}\}$ is a family of subsets of X.

We now sketch the definition of a Matryoshka game, called $\mathcal{G}_0$, whose associated operation is precisely the operation $(\bigcup \circ \bigcap)$ of Section 2. The structure of $\mathcal{G}_0$ is depicted in Figure 1. This is a simple two-steps game where $\exists$ chooses a number n and $\forall$ responds choosing a number m. Every play is finite and of the form $\langle \epsilon, n, n.m\rangle$. The set of play labels $\mathbb{A}^{\mathcal{G}_0}$ is defined as $\omega \times \omega$ and $\mathrm{label}^{\mathcal{G}}(\langle \epsilon, n, n.m\rangle) = (n, m)$.

We now introduce *transformations* on games which directly match the corresponding transformations on operations defined in Section 2. Due to space limitations we only sketch the definitions.

For a Matryoshka game $\mathcal{G}$ of index (i, k), we define co-$\mathcal{G}$ as the game obtained from $\mathcal{G}$ by replacing the sets $V_{\exists} \leftrightarrow V_{\forall}$ and increasing all priorities in Ω by 1. Note that the index of co-$\mathcal{G}$ is $(i+1, k+1)$, and that the sets of plays in the two games are equal. We define $\mathbb{A}^{\text{co-}\mathcal{G}} \stackrel{\text{def}}{=} \mathbb{A}^{\mathcal{G}}$ and $\mathrm{label}^{\text{co-}\mathcal{G}}(\pi) \stackrel{\text{def}}{=} \mathrm{label}^{\mathcal{G}}(\pi)$.

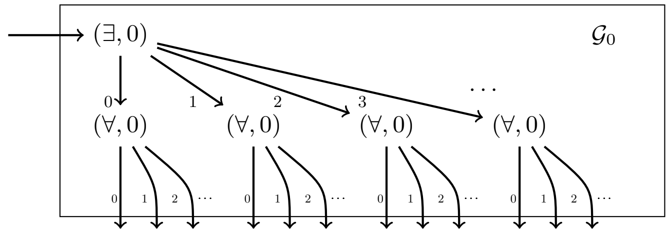

Fig. 1. The game $\mathcal{G}_0$ corresponding to the operation $\bigcup \circ \bigcap$.

We now define the $\mathcal{R}$ transformation on games. Let us take a Matryoshka game $\mathcal{G}$ of index (i, k). Let $2j$ be the minimal even number such that $k \leq 2j$. The game $\mathcal{RG}$ is depicted on Figure 2.

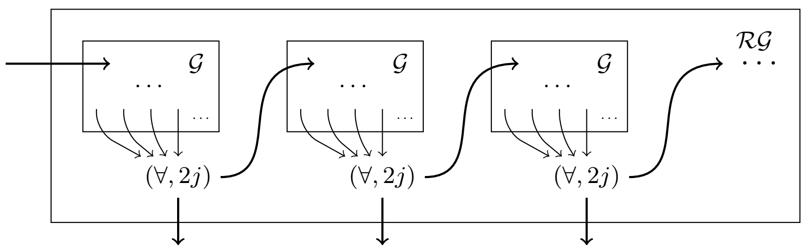

Fig. 2. The game $\mathcal{RG}$.

A play in the game $\mathcal{RG}$ starts from a first copy of $\mathcal{G}$. In this *inner* game, the play π can either be infinite (in which case π is a valid play in $\mathcal{RG}$ and is winning for Player P iff it is winning for P in $\mathcal{G}$) or terminate in a terminal state of $\mathcal{G}$. In this latter case, Player $\forall$ can either conclude the game $\mathcal{RG}$, or start another session of the inner game $\mathcal{G}$. Observe that if $\forall$ always chooses to start a new session, they lose because the even priority $2j$ is maximal in $\mathcal{RG}$.

The set of play labels $\mathbb{A}^{\mathcal{RG}}$ is defined as $\left(\mathbb{A}^{\mathcal{G}}\right)^*$, i.e., the set of finite sequences of play labels in $\mathcal{G}$. Let π be a play in $\mathcal{RG}$ that passes through n copies of $\mathcal{G}$ and then ends in a terminal position of $\mathcal{RG}$. In that case π can be decomposed into n plays $\pi_0, \ldots, \pi_{n-1}$ in $\mathcal{G}$. We then define the labeling function of $\mathcal{RG}$ as follows:

$$\mathrm{label}^{\mathcal{RG}}(\pi) \stackrel{\mathrm{def}}{=} \left(\mathrm{label}^{\mathcal{G}}(\pi_0),\ \mathrm{label}^{\mathcal{G}}(\pi_1), \ldots, \mathrm{label}^{\mathcal{G}}(\pi_{n-1})\right). \tag{4}$$

Given the basic Matryoshka game G_0 and the two transformations of games co- and $\mathcal{R}$, we can construct more and more complex "nested" games. This fact motivates the name of this class of games. We denote with $\mathcal{G}_k$ the game obtained from G_0 by iterating k-times the composed transformation co-$\mathcal{R}$.

By the definition, the game $\mathcal{G}_k$ for $k > 0$ consists of infinitely many copies of $\mathcal{G}_{k-1}$ and an additional set of new vertices as depicted on Figure 2. These new vertices are called the k*-layer* of the game. Therefore, by unfolding the definition, each vertex v of $\mathcal{G}_k$ is either a vertex of a copy of $\mathcal{G}_0$ or it belongs to a j-layer for some $1 \leq j \leq k$. Observe that if v is in a j-layer of $\mathcal{G}_k$ then

$$\Omega^{\mathcal{G}_k}(v) = k{+}j{-}1 \quad \text{and} \quad v \in V_{\forall}^{\mathcal{G}_k} \Leftrightarrow k{+}j{-}1 \equiv 0 \pmod 2. \tag{5}$$

We are now ready to state the expected correspondence between the operation Θ_k of Section 2 and the Matryoshka game $\mathcal{G}_k$.

Theorem 3. *For every $k \in \omega$ the basis N_{Θ_k} of the Θ_k operation equals the family* $\mathrm{promise}(\mathcal{G}_k) \stackrel{\mathrm{def}}{=} \{X \subseteq \mathbb{A}_k : \exists$ *has a winning strategy in $\mathcal{G}_k$ with promise X*$\}$.

Corollary 2. *For each k and $(E_s)_{s\in\mathbb{A}_k}$ we have $\Theta_k\big((E_s)_{s\in\mathbb{A}_k}\big) = \mathcal{G}_k\big((E_s)_{s\in\mathbb{A}_k}\big)$.*

5 Relation between $\mathcal{R}$–sets and the index hierarchy

In this section we prove the main result of this work, that is Theorem 1 stated in Section 1. As a preliminary step, it is convenient to define a variant of game tree languages defined on countable trees. This will simplify the connection with Matryoshka games which are played on countably branching structures. Let $\mathrm{Tr}^{\omega}_{i,k}$ be the space of labelled ω-trees $t \colon \omega^* \to \{\exists, \forall\} \times \{i, \ldots, k, \top, \bot\}$. Each $t \in \mathrm{Tr}^{\omega}_{i,k}$ is naturally interpreted as a parity game on the countable tree structure, with the possibility of terminating at leaves, labelled by $\top$ and $\bot$, which are winning for $\exists$ and $\forall$, respectively. We also require (1) that in the root there is a vertex (P, k) where $P = \exists$ if i is even and $P = \forall$ if i is odd and (2) that the tree is alternating, that is $\exists$ and $\forall$ make moves in turns.

Definition 7. $\mathcal{W}^{\omega}_{i,k} \subseteq \mathrm{Tr}^{\omega}_{i,k}$ *is the set of ω-trees such that $\exists$ has a w.s.*

An easy argument shows that dropping conditions (1) and (2) gives a Wadge equivalent language. The following routine lemma establishes the connection between ω-branching game tree languages $\mathcal{W}^{\omega}_{i,k}$ and binary (as in Section 2) game tree languages $\mathcal{W}_{i,k}$.

Lemma 2. *For $i < k$ the language $\mathcal{W}_{i,k}$ is Wadge equivalent to $\mathcal{W}^{\omega}_{i+1,k}$. In particular $\mathcal{W}_{0,1} \sim_W \mathcal{W}^{\omega}_{1,1}$ and $\mathcal{W}_{1,3} \sim_W \mathcal{W}^{\omega}_{0,1}$.*

The fact that $\mathcal{W}_{i,k}$ corresponds to $\mathcal{W}^{\omega}_{i+1,k}$ reflects the cost of the translation of ω-branching games into binary games: an extra priority is required to mimic countably many choices by iterating binary choices. Thanks to this lemma, in Theorem 1 one can replace the languages $\mathcal{W}_{k-1,2k-1}$ with the languages $\mathcal{W}^{\omega}_{k,2k-1}$.

First, we show that every $\mathcal{W}^{\omega}_{k,2k-1}$ is indeed an $\mathcal{R}$-set. We will do so by explicitly constructing a family $\mathcal{E}_k = \{E_s \mid s \in \mathbb{A}_k\}$ of clopen sets in $\mathrm{Tr}^{\omega}_{k,2k-1}$ such that $\Theta_k(\mathcal{E}_k) = \mathcal{W}^{\omega}_{k,2k-1}$, where $\mathbb{A}_k$ is the arena of the operation Θ_k. The construction requires some effort. First we recall, from Section 3 that the arena of the operation $\bigcup \circ \bigcap$ is $\mathbb{A}_0 = \{\langle n, m\rangle : n, m \in \omega\}$ (pairs of natural numbers) and from the definition of the transformation $\mathcal{R}$ we have $\mathbb{A}_k = \big(\mathbb{A}_{k-1}\big)^*$. Thus, for all $k \in \omega$, $\mathbb{A}_k$

is a set of nested sequences of pairs of natural numbers. For a sequence $s \in \mathbb{A}_k$ we define the maps `flatten` and `prioritiesMap` such that $\texttt{flatten}(s) \in \mathbb{A}_0^\star$ and $\texttt{prioritiesMap}(s) \in \omega^*$. The map `flatten` takes a nested sequence in $\mathbb{A}_k$ and returns the "flattened" sequence, that is all the braces are removed, for example $\texttt{flatten}\big((((\langle 2, 15\rangle)), ((\langle 7, 5\rangle), (\langle 6, 4\rangle))))\big) = (\langle 2, 15\rangle, \langle 7, 5\rangle, \langle 6, 4\rangle)$. The function `prioritiesMap` computes the number of closing brackets after each pair of natural numbers: e.g., $\texttt{prioritiesMap}\Big((((\langle 2, 15\rangle)), ((\langle 7, 5\rangle), (\langle 6, 4\rangle))))\Big) = (2, 1, 3)$.

We also define $\texttt{treeMap}(t, s)$ where $t \in \mathrm{Tr}^\omega_{k,2k-1}$ and $s \in \mathbb{A}_k$. Since we limited our attention to alternating trees, each vertex in the ω-branching tree t can be identified with a sequence of pairs of natural numbers. Then, if $s \in \mathbb{A}_k$, the function $\texttt{treeMap}(t, s)$ computes first $\texttt{flatten}(s)$ and returns the sequence of priorities assigned to the vertices along the path of t indicated by $\texttt{flatten}(s)$. On Figure 3 we have an example of a tree t where

$$\texttt{treeMap}\big(t, (((\langle 2, 15\rangle)), ((\langle 7, 5\rangle), (\langle 6, 4\rangle))))\big) = (2, 1, 3).$$

Define $\mathcal{E}_k = \{E_s : s \in \mathbb{A}_k\}$ such that for $t \in \mathrm{Tr}^\omega_{k,2k-1}$ we have $t \in E_s$ iff for

- $v = \texttt{prioritiesMap}(s)$,
- $b = \texttt{treeMap}(t, s)$,
- $L = \min\{k \in \omega : v(k) \neq b(k)\}$

$v \neq b$ holds, and either $b(L) = \top$ or

$$\min(b(L), v(L)) \equiv 0 \pmod 2. \tag{6}$$

It is simple to verify that the sets E_s are indeed clopen in the space $\mathrm{Tr}^\omega_{k,2k-1}$ (for a definition of the topology see, e.g. [2]).

Theorem 4. $\forall_{k \geq 1}\ \Theta_k(\mathcal{E}_k) = \mathcal{W}^\omega_{k,2k-1}$.

Proof. The proof is based on Matryoshka games. Consider a tree $t \in \mathrm{Tr}^\omega_{k,2k-1}$ and assume that Player $P \in \{\exists, \forall\}$ has a winning strategy σ on the tree t. We claim that P has a winning strategy in the Matryoshka game $\mathcal{G}_k$ with promise $\mathcal{E}_k$. From this fact the theorem will follow by an application of Corollary 2 and Proposition 1. We consider the case $P = \exists$. The opposite case is analogous.

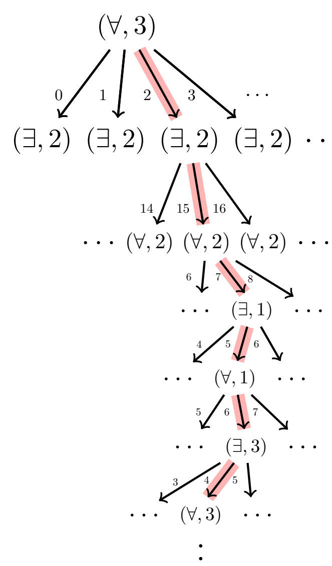

Fig. 3. An illustration of `treeMap`.

We will simulate the game on t in the Matryoshka game $\mathcal{G}_k$. A play in $\mathcal{G}_k$ consists of playing pairs of numbers (corresponding to moves in t) in the copies of $\mathcal{G}_0$ and, additionally, deciding whether to *exit* an j-layer of the game or not. We say that a play in $\mathcal{G}_k$ is *fair* if whenever the players encounter a priority $k+j$ in t then they exit exactly j first layers of $\mathcal{G}_k$ (i.e. the layer $j+1$ is reached) and if they encounter a symbol $\bot$ or $\top$ then the players exit all the layers of $\mathcal{G}_k$.

Let $\exists$ use the original strategy σ in the copies of $\mathcal{G}_0$ and play "fairly" as long as $\forall$ does. If $\forall$ also plays "fairly" then the play is winning for $\exists$: either $\top$ is reached in t and $\exists$ wins since $t \in E_s$ or the play is infinite and $\exists$ wins by the parity condition — the priorities visited in $\mathcal{G}_k$ agree with those visited in t, see (5).

If $\forall$ does not play "fairly" (i.e. when a priority $k+j$ is reached in t they don't exit the l-layer of $\mathcal{G}_k$ with $l \leq j$ or they exit the $(j+1)$-layer of $\mathcal{G}_k$) then $\exists$ uses the following counter-strategy: whenever possible $\exists$ exits the current layer of $\mathcal{G}_k$. There are two possible developments of such a play. The first case is that $\forall$ allows to exit the whole game and then $\exists$ wins thanks to (6). Now assume that $\forall$ never allows the game to reach a terminal position. In that case, let j be maximal such that the j-layer of $\mathcal{G}_k$ is visited infinitely often. By (5) we know that the limit superior of the priorities visited in $\mathcal{G}_k$ is $k+j-1$ and, since $\forall$ is the owner of the vertices in the j-layer of $\mathcal{G}_k$, it holds that $k+j-1 \equiv 0 \pmod 2$. Therefore, $\exists$ wins the play by the parity condition. $\square$

Theorem 5. *Let $L = \Theta_k(E_s)$ be a set obtained using the Θ_k operation applied to a family of clopen subsets $(E_s)_{s \in \mathbb{A}_k}$ with $E_s \subseteq Y$ in a Polish space Y. Then, there exists a continuous reduction $f\colon Y \to \mathrm{Tr}^{\omega}_{k,2k-1}$ such that $f^{-1}\left(\mathcal{W}^{\omega}_{k,2k-1}\right) = L$.*

Sketch. The operation Θ_k is presented as the corresponding Matryoshka game using Theorem 3 and Corollary 2. This is a parametrized family of parity games and thus continuously reducible to $\mathcal{W}^{\omega}_{k,2k-1}$. $\square$

Theorems 4 and 5 imply that the language $\mathcal{W}^{\omega}_{k,2k-1}$ is complete for the k-th level of the hierarchy of $\mathcal{R}$-sets. Theorem 1 follows from Lemmas 1 and 2.

6 Continuity of measures on $\mathcal{W}_{i,k}$

For an odd $k \in \omega$ the language $\mathcal{W}_{i,k}$ admits a natural transfinite decomposition into simpler approximant sets $\mathcal{W}^{\alpha}_{i,k}$, for $\alpha < \omega_1$ (see [18, §6.2,3]). The proof of determinacy of pLμ games of [18] relies on the following special *continuity property*: $\sup_{\alpha<\omega_1} \mu\big(\mathcal{W}^{\alpha}_{i,k}\big) = \mu\big(\mathcal{W}_{i,k}\big)$. Since the increasing chain $\mathcal{W}^{\alpha}_{i,k}$ is uncountable, the property does not follow from the standard σ-continuity of measures. As observed in [18], the property follows from Martin Axiom at $\aleph_1$ ($\mathrm{MA}_{\aleph_1}$). The problem of whether the property holds in ZFC alone is left open (see Item 2 of Section 8.2 in [18]). The following theorem gives a partial answer to this problem.

Theorem 6. *The continuity property holds in* ZFC *for $\mathcal{W}_{0,1}$. Let k be an odd number, $i < k$. For $\mathcal{W}_{i,k}$ the continuity property holds assuming the determinacy of Harrington's games*[8] *with arbitrary analytic winning sets.*

6.1 Improvement

After submitting the paper the authors have realised that it is possible to prove the above theorem without the additional assumption of determinacy. This proof will be included in the journal version of the paper.

[8] See, e.g., [10, Section 33.5] for details about this type of games.

7 Conclusion

The notion of $\mathcal{R}$-sets is a robust concept and admits natural variations. One can equivalently work in arbitrary (not zero-dimensional) Polish spaces and start from a basis of, e.g. Borel sets rather than clopens. The family of operations $\Theta_k = (\text{co-}\mathcal{R})^k(\bigcup \circ \bigcap)$ can be replaced by, e.g. either $(\text{co-}\mathcal{R})^k(\bigcup)$ or $(\text{co-}\mathcal{R})^k(\bigcap)$. Similarly, one can consider binary rather than countably branching, Matryoshka games. The notion of $\mathcal{R}$-sets remains unchanged in these alternative setups.

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