

REAL ZEROS OF MIXED RANDOM FEWNOMIAL SYSTEMS

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ABSTRACT. Consider a system $f_1(x) = 0, \dots, f_n(x) = 0$ of n random real polynomials in n variables, where each f_i has a prescribed set of exponent vectors in a set $A_i \subseteq \mathbb{Z}^n$ of cardinality t_i , whose convex hull is denoted P_i . Assuming that the coefficients of the f_i are independent standard Gaussian, we prove that the **expected number of zeros** of the random system in the positive orthant is at most $(2\pi)^{-\frac{n}{2}} V_0(t_1 - 1) \dots (t_n - 1)$. Here V_0 denotes the number of vertices of the Minkowski sum $P_1 + \dots + P_n$. We also derive a better bound in the unmixed case where all supports A_i are equal, improving upon Bürgisser et al. (SIAM J. Appl. Algebra Geom. 3(4), 2019). All arguments equally work for real exponent vectors.

1. INTRODUCTION

In many applications, we want to understand or find the positive real solutions of a system of multivariate polynomial equations, e.g., see [11, 17, 36]. Bezout's theorem, which bounds the number complex zeros in terms of degrees, usually highly overestimates the number of real zeros. This can be already seen from Descartes' rule of signs [10, p. 42], which implies that a real univariate polynomial with t terms has at most $t - 1$ positive zeros. In 1980, Khovanskii [20] obtained a far reaching generalization of Descartes' rule. He showed that the number of nondegenerate¹ positive solutions of a system $f_1(x) = 0, \dots, f_n(x) = 0$ of n real polynomial equations in n variables is bounded only in terms of n and the number t of distinct exponent vectors occurring in the system. This result in fact allows for any real exponents. Following Kushnirenko, one speaks of *fewnomial systems*, with the idea that the number t of terms is small, see [21].

Understanding the complex zeros of fewnomial systems is much simpler: the famous BKK-Theorem [2, 25] states that for given finite supports $A_1, \dots, A_n \subseteq \mathbb{Z}^n$ and Laurent polynomials $f_i(x) = \sum_{a \in A_i} c_i(a) x_1^{a_1} \dots x_n^{a_n}$ with generic complex coefficients $c_i(a)$, the number of complex solutions in $(\mathbb{C}^\times)^n$ of a corresponding system $f_1(x) = 0, \dots, f_n(x) = 0$ is given by $n!$ times the mixed volume of the Newton polytopes P_1, \dots, P_n , where P_i is defined as the convex hull of A_i .

Note that the number of real zeros has little to do with the metric properties of P_i : indeed, replacing A_i by a nonzero multiple $m_i A_i$ amounts to substituting x_i by $x_i^{m_i}$. Clearly, this does not change the number of positive real zeros of a fewnomial system, however P_i has been replaced by $m_i P_i$.

The bound on the number of real zeros obtained by Khovanskii is exponential in the number t . It is widely conjectured that this bound is far from optimal: in fact it is conjectured [30] that for fixed n , the number of nondegenerate positive solutions of a fewnomial system with t exponent vectors is bounded by a polynomial in t . Quite surprisingly, this question is open even for $n = 2$!

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¹i.e., the Jacobian of the system does not vanish at the zero.

For results in special cases, we refer to [4, 1, 36, 23, 24, 3]. Moreover, there is a very interesting connection to complexity theory [22, 5].

Given this state of affairs of real fewnomial theory, a possible way to advance is to ask what happens in generic situations. This can be made formal by considering *random* real fewnomial systems, see [35, 12, 31, 27, 8, 26]. Fix supports $A_1, \dots, A_n \subseteq \mathbb{Z}^n$ of cardinality t_1, \dots, t_n , respectively, and consider a system of n random polynomials $f_i(x)$ as above, but now the coefficients $c_i(a)$ are assumed to be independent standard Gaussian. Let us denote by $\mathbb{E}(A_1, \dots, A_n)$ the expectation of the number of nondegenerate positive real zeros of such system. Actually, we work in more generality, allowing any subsets A_i of \mathbb{R}^n ; see Section 4.

In [8] it was proven that $\mathbb{E}(A, \dots, A) \leq 2^{1-n} \binom{t}{n}$. The main result of the present paper is an extension of this to the mixed case, where the fewnomials may have different supports A_i . Our bound depends on the combinatorial structure of the Minkowski sum $P_1 + \dots + P_n$ through the number of its vertices. We remark that our proof is quite different from the one in [8], which is rather indirect.

Theorem 1.1. *If the $A_i \subseteq \mathbb{R}^n$ are finite nonempty sets of cardinality t_i and with convex hull P_i , for $i = 1, \dots, n$, then*

$$\mathbb{E}(A_1, \dots, A_n) \leq (2\pi)^{-\frac{n}{2}} V_0(t_1 - 1) \dots (t_n - 1).$$

Here V_0 denotes the number of vertices of the Minkowski sum $P := P_1 + \dots + P_n$.

The bound in this theorem looks similar to the one in a conjecture attributed to Kushnirenko, which states that the number of positive nondegenerate zeros is always bounded by $(t_1 - 1) \dots (t_n - 1)$. However, this was disproved in [16].²

In the unmixed situation, where all supports equal A , it is well known [12] that the expected number of positive zeros can be expressed by the volume of the image of the Veronese like map $\mathbb{R}_{>0}^n \rightarrow \mathbb{P}(\mathbb{R}^A)$ sending x to $[x^a]_{a \in A}$. This is a consequence of the kinematic formula for real projective spaces. In the mixed situation, there is no such simple characterization: we work with the more complicated kinematic formula for products of projective spaces (Theorem 3.2) that we derive from [18, 9]. After passing to exponential coordinates $w = \log x$, we bound the resulting integral over \mathbb{R}^n with a strategy inspired by the theory of toric varieties. The normal fan of the polytope P affords a decomposition of \mathbb{R}^n into the normal cones C at the vertices of P . The resulting integral over C can be bounded in terms of the characteristic function of the dual cone of C . Finally, an explicit a priori bound on the characteristic function (Proposition 2.4) completes the argument.

1.1. The univariate case and a conjecture. The univariate case ($n = 1$) was settled by Jindal et al. [19]. They showed that for any subset $S \subseteq \mathbb{R}$ of cardinality t , we have

$$(1.1) \quad \mathbb{E}(S) \leq \frac{2}{\pi} \sqrt{t-1}.$$

Moreover, they constructed a sequence $S_t \subseteq \mathbb{Z}$ of supports of cardinality t with $\mathbb{E}(S_t) \geq c\sqrt{t}$ for some constant $c > 0$. Consider for $t_1, \dots, t_n \geq 1$ the supports $A_1 := S_{t_1} \times 0 \dots \times 0, \dots, A_n := 0 \times \dots \times 0 \times S_{t_n}$. These supports describe a system of n equations, where the i th equation depends on x_i only. Therefore, $\mathbb{E}(A_1, \dots, A_n) = \mathbb{E}(S_{t_1}) \dots \mathbb{E}(S_{t_n})$, which with the above leads to the lower bound

$$(1.2) \quad \mathbb{E}(A_1, \dots, A_n) \geq c^n \sqrt{t_1 \dots t_n}.$$

We complement this by showing that for any $A = S_1 \times \dots \times S_n$ in product form, the expectation $\mathbb{E}(A, \dots, A)$ can be expressed in terms of the $\mathbb{E}(S_i)$ as follows.

²This conjecture was never published by Kushnirenko and apparently, he did not believe in it.

Proposition 1.2. *If $A = S_1 \times \dots \times S_n$ for finite $S_i \subseteq \mathbb{R}$, then*

$$\mathbb{E}(A, \dots, A) = \pi^n (\text{vol}(\mathbb{P}^n))^{-1} \mathbb{E}(S_1) \cdots \mathbb{E}(S_n).$$

We conjecture that the lower bound (1.2) is optimal in the following sense.

Conjecture 1. *Let $A_i \subseteq \mathbb{R}^n$ be finite nonempty sets of cardinality t_i with convex hull P_i , for $i = 1, \dots, n$. We denote by V_0 the number of vertices of $P_1 + \dots + P_n$. Then*

$$\mathbb{E}(A_1, \dots, A_n) \leq \kappa(V_0) \sqrt{t_1 \cdots t_n}$$

for some function $\kappa : \mathbb{N} \rightarrow \mathbb{N}$. In particular, for $A \subseteq \mathbb{R}^n$ of cardinality t , we have $\mathbb{E}(A, \dots, A) \leq \kappa(V_0) t^{\frac{n}{2}}$.

In the special case $A = S_1 \times \dots \times S_n$, by combining (1.1) with Proposition 1.2, we obtain $\mathbb{E}(A, \dots, A) \text{vol}(\mathbb{P}^n) \leq 2^n \sqrt{t}$, which is much smaller than what Conjecture 1 predicts.

1.2. Improvement in unmixed case. We can improve on the dependence on n in the bound of Theorem 1.1 in the case where all supports are equal.

Theorem 1.3. *For $A \subseteq \mathbb{R}^n$ of cardinality $t \geq 1$ with convex hull P and V_0 vertices, we have*

$$\mathbb{E}(A, \dots, A) \leq (\text{vol}(\mathbb{P}^n))^{-1} V_0 \binom{t-1}{n}.$$

To compare this with the bound $\mathbb{E}(A, \dots, A) \leq 2^{1-n} \binom{t}{n}$ from [8], note that $\text{vol}(\mathbb{P}^n)^{-1} = \Gamma(\frac{n+1}{2}) \pi^{-\frac{n+1}{2}}$. Hence, for $n \rightarrow \infty$, the new bound goes asymptotically faster to zero than the old one.

Remark 1.4. The bound in [8] also holds for nonstandard centered Gaussian coefficients $c(a) \sim N(0, \sigma(a)^2)$. In this situation, our proof of Theorem 1.3 leads to an upper bound with the additional factor $(\max_a \sigma(a) / \min_a \sigma(a))^n$ (similarly for Theorem 1.1). We leave it as a challenge to remove this dependency.

1.3. Location of zeros. We finish with a result on the typical location of the zeros. It is well known that for random real polynomials, the positive real zeros x tend to accumulate around 1: see [12] for the dense and [19] for the sparse case. This means that $w = \log x$ accumulates around 0. We generalize this to multivariate systems as follows.

Theorem 1.5. *Fix a finite supports $A_1, \dots, A_n \subseteq \mathbb{R}^n$ and consider a random system (4.3) with independent standard Gaussian coefficients $c_i(a)$ for the stretched supports mA_i , where $m \in \mathbb{Z}_{>0}$. Fix $\varepsilon > 0$. Then the probability that the system has a zero $w \in \mathbb{R}^n$ with $\|w\| > \varepsilon$ goes to zero, as $m \rightarrow \infty$.*

There are sophisticated results on the distributions of *complex zeros* of random fewnomials systems [33, 34].

2. PRELIMINARIES

2.1. Metric property of charts of projective space. Consider the real projective space \mathbb{P}^m . We shall identify the tangent space $T_{[y]} \mathbb{P}^m$ at a point $[y] := [y_0 : \dots : y_m]$ with $\mathbb{R}y^\perp$. The standard Riemannian metric on \mathbb{P}^m is defined by $\langle v, w \rangle_{[y]} := \|y\|^{-2} \langle v, w \rangle$ for $v, w \in \mathbb{R}y^\perp$. We denote by P_y the orthogonal projection onto $\mathbb{R}y^\perp$.

Consider the affine chart $(\mathbb{P}^m)_{y_0 \neq 0} \rightarrow \mathbb{R}^m$, which maps $[y_0 : \dots : y_m]$ to $y_0^{-1}(y_1, \dots, y_m)$. Its inverse is given by

$$\pi : \mathbb{R}^m \rightarrow (\mathbb{P}^m)_{y_0 \neq 0}, (y_1, \dots, y_m) \mapsto [1 : y_1 : \dots : y_m].$$

By [6, Lemma 14.8], the derivative of π at $y' := (y_1, \dots, y_n)$ satisfies $D_{y'}\pi = \|\pi(y')\|^{-1}P_{y'}$, and therefore,

$$(2.1) \quad \|D_{y'}\pi\| \leq \|\pi(y')\|^{-1} \leq 1.$$

2.2. On the quantity σ . The relative position of two subspaces of a Euclidean vector space E can be quantified by a volume like quantity, which is crucial in the study of integral geometry in homogeneous spaces; see [18] and [9, §3.3]. To define this quantity, note first that there is an induced inner product on the exterior algebra $\Lambda(E)$ given by [9, (2.1)]

$$\langle v_1 \wedge \dots \wedge v_k, w_1 \wedge \dots \wedge w_k \rangle = \det(\langle v_i, w_j \rangle)_{1 \leq i, j \leq k}.$$

More concretely, $\|v_1 \wedge \dots \wedge v_n\| = |\det[v_1, \dots, v_n]|$, where $[v_1, \dots, v_n]$ denotes the matrix with columns $v_i \in E = \mathbb{R}^n$

Let V, W be linear subspaces of E of complementary dimensions. We define [9, (3.3)]

$$(2.2) \quad \sigma(V, W) := \|v_1 \wedge \dots \wedge v_k \wedge w_1 \wedge \dots \wedge w_m\| \in [0, 1],$$

where v_1, \dots, v_k and w_1, \dots, w_m are orthonormal bases of V and W , respectively. Clearly, $\sigma(V, W) = \sigma(W, V)$. Here are the extreme cases: $\sigma(V, W) = 0$ iff $V \cap W \neq 0$ and $\sigma(V, W) = 1$ iff V and W are orthogonal.

Proposition 2.1. *We have $\sigma(V^\perp, W^\perp) = |\det p|$, if the map $p: V^\perp \rightarrow W$ denotes the restriction of the orthogonal projection $E \rightarrow W$ to V^\perp . Moreover, $\sigma(V, W) = \sigma(V^\perp, W^\perp)$.*

Proof. Let ν_1, \dots, ν_m be an orthonormal basis of V^\perp . We decompose $\nu_i = \nu'_i + \nu''_i$ according to $E = W \oplus W^\perp$. Then $p(\nu_i) = \nu'_i$ and $|\det p| = \|\nu'_1 \wedge \dots \wedge \nu'_m\|$. If $\omega_1, \dots, \omega_k$ denotes an orthonormal basis of W^\perp , we have

$$\sigma(V^\perp, W^\perp) = \|\nu_1 \wedge \dots \wedge \nu_m \wedge \omega_1 \wedge \dots \wedge \omega_k\| = \|\nu'_1 \wedge \dots \wedge \nu'_m \wedge \omega_1 \wedge \dots \wedge \omega_k\| = \|\nu'_1 \wedge \dots \wedge \nu'_m\|,$$

This proves $\sigma(V^\perp, W^\perp) = |\det p|$.

For the second assertion, we use that $|\det p| = |\det q|$, where $q: W^\perp \rightarrow V$ denotes the restriction of the orthogonal projection $E \rightarrow V$ to W^\perp , see [7, Lemma 5.4]. \square

Clearly, the definition (2.2) can be extended to more than two subspaces; see [9, (3.5)]. But if $W = W_1 \oplus \dots \oplus W_n$ is an orthogonal decomposition, we can reduce to the case of two subspaces [9, Lemma A.6].

$$(2.3) \quad \sigma(V, W_1, \dots, W_n) = \sigma(V, W_1 + \dots + W_n).$$

2.3. Characteristic function of convex cones. We prove here an a priori upper bound on the characteristic function of a convex cone, which is a key ingredient in the proof of Theorem 1.1.

A convex cone $C \subseteq \mathbb{R}^n$ is called *proper* if it is n -dimensional and pointed, i.e., contained in a halfspace. It is well known that a convex $C \subseteq \mathbb{R}^n$ is proper iff its *dual cone*

$$C^* := \{x \in \mathbb{R}^n \mid \forall y \in C \langle x, y \rangle \geq 0\}$$

is proper. Let $g \in \text{GL}(n, \mathbb{R})$. Then $K := g(C)$ is a proper cone and $g^T(K^*) = C^*$. We denote by $\text{int}(C)$ the interior of C .

We assign to a proper cone $C \subseteq \mathbb{R}^n$ the function

$$(2.4) \quad v_C: \text{int}(C^*) \rightarrow \mathbb{R}_{>0}, \quad v_C(x) := \int_C e^{-\langle x, y \rangle} dy.$$

One calls v_C the *characteristic function* (or Koszul-Vinberg characteristic) of C^* . It is a useful analytic tool for investigating convex cones, e.g., see [13, I.3] and [15]. E.g., $\mathbb{R}_{>0}^n$ is self dual and $v_{\mathbb{R}_{>0}^n}(x) = (x_1 \cdot \dots \cdot x_n)^{-1}$ for $x \in \mathbb{R}_{>0}^n$.

The homogeneity property $v_C(tx) = t^{-n}v_C(x)$ for $t > 0$, $x \in \text{int}(C^*)$ is immediate to check. Moreover, the transformation formula implies the following invariance property: if $g \in \text{GL}(n, \mathbb{R})$ and $K := g(C)$, then $g^T(K^*) = C^*$ and

$$(2.5) \quad v_K(z) = |\det g| v_C(g^T z) \quad \text{for } z \in \text{int}(K^*).$$

Remark 2.2. The function $\log v_C$ is strictly convex and essentially equals Nesterov and Nemirovski's universal self-concordant barrier function [29, §2.5], see [15] for the proof.

The following is well known, e.g., see [15, Thm. 4.1]. Appendix A contains the proof for the sake of completeness.

Lemma 2.3. *We have $v_C(x) = n! \text{vol}\{y \in C \mid \langle x, y \rangle \leq 1\}$ for $x \in \text{int}(C^*)$.*

The following is essential for the proof of Theorem 1.1.

Proposition 2.4. *Let $C \subseteq \mathbb{R}^n$ be a proper cone. Then we have for $b_1, \dots, b_n \in C^*$.*

$$\|b_1 \wedge \dots \wedge b_n\| \cdot v_C(b_1 + \dots + b_n) \leq 1.$$

This bound is optimal.

Proof. We denote by $\text{cone}(b_1, \dots, b_n) \subseteq C^*$ the convex cone generated by b_1, \dots, b_n . Without loss of generality, we may assume that $b_1, \dots, b_n \in C^*$ is a basis of \mathbb{R}^n . Let b_1^*, \dots, b_n^* denote its dual basis, that is $\langle b_i^*, b_j \rangle = \delta_{ij}$. In matrix terminology, $[b_1^*, \dots, b_n^*]^T [b_1, \dots, b_n] = I_n$, hence

$$(2.6) \quad \det[b_1^*, \dots, b_n^*] \det[b_1, \dots, b_n] = \pm 1.$$

The definition of the dual basis implies that $\text{cone}(b_1^*, \dots, b_n^*)$ is the dual cone of $\text{cone}(b_1, \dots, b_n)$. Therefore, by duality, we get

$$C \subseteq \text{cone}(b_1, \dots, b_n)^* = \text{cone}(b_1^*, \dots, b_n^*).$$

Put $d := b_1 + \dots + b_n$ and let $y \in C$ such that $\langle d, y \rangle \leq 1$. Since $C \subseteq \text{cone}(b_1^*, \dots, b_n^*)$, we can write $y = \sum_i t_i b_i^*$ with $t_i \geq 0$. Moreover $\sum_i t_i = \langle d, y \rangle \leq 1$. Thus we have shown the inclusion

$$K := \{y \in C \mid \langle d, y \rangle \leq 1\} \subseteq \text{conv}\{0, b_1^*, \dots, b_n^*\}.$$

This implies the inequality of volumes

$$\text{vol}_n K \leq \text{vol}_n \text{conv}\{0, b_1^*, \dots, b_n^*\} = \frac{1}{n!} |\det[b_1^*, \dots, b_n^*]|.$$

Multiplying with $n! |\det[b_1, \dots, b_n]|$, using (2.6) and taking into account Lemma 2.3, the assertion follows.

The optimality is attained for $C = \mathbb{R}_{\geq 0}^n$ and $b_i = d_i e_i$ with $d_i > 0$. Indeed, we have

$$|\det[b_1, \dots, b_n]| v_C(d) = d_1 \cdot \dots \cdot d_n (d_1 \cdot \dots \cdot d_n)^{-1} = 1. \quad \square$$

2.4. Vertices and normal fan of sums of polytopes. We recall here some basic facts about polytopes and their normal fans; see [37, §7.1] for more details.

Let $P \subseteq \mathbb{R}^n$ be a full-dimensional polytope and v be a vertex of P . The *cone* P_v of P at v is defined as the convex cone generated by $P - v$. It is a proper cone. The dual cone of P_v , also called the *inner normal cone* of P at v , is defined as

$$P_v^* := \{y \in \mathbb{R}^n \mid \forall x \in P \langle x, y \rangle \geq 0\}.$$

The cone P_v^* is also proper. The union over all P_v^* equals \mathbb{R}^n . Moreover, for $v_1 \neq v_2$, we have $\dim(P_{v_1}^* \cap P_{v_2}^*) < n$. In fact, the P_v^* are the n -dimensional cones of the normal fan of P .

We will need the following result.

Lemma 2.5. *Let P_1, \dots, P_n be polytopes in \mathbb{R}^n . There is an injective map*

$$\text{Vert}(P_1 + \dots + P_n) \rightarrow \text{Vert}(P_1) \times \dots \times \text{Vert}(P_n), v \mapsto (v_1, \dots, v_n)$$

satisfying $v = v_1 + \dots + v_n$. Moreover, if we denote by Π_i the cone of P_i at the vertex v_i , then $\Pi := \Pi_1 + \dots + \Pi_n$ is the cone of $P_1 + \dots + P_n$ at the vertex $v_1 + \dots + v_n$. In particular, $\Pi^ = \Pi_1^* \cap \dots \cap \Pi_n^*$.*

Proof. To a nonzero weight $\omega \in \mathbb{R}^n$ we assign the face of P_i , given by

$$F(P_i, \omega) := \{w \in \mathbb{R}^n \mid \langle w, \omega \rangle = \min_{w' \in P_i} \langle w', \omega \rangle\}.$$

We have by [32, Thm. 1.7.5]

$$F(P_1 + \dots + P_n, \omega) = F(P_1, \omega) + \dots + F(P_n, \omega).$$

Suppose that $F(P_1 + \dots + P_n, \omega) = \{v\}$ is a vertex. Then all $F(P_i, \omega) = \{v_i\}$ are vertices and $v = v_1 + \dots + v_n$. The v_i are uniquely determined by v , see [14, Prop. 2.1]. Then the map $v \mapsto (v_1, \dots, v_n)$ is as required. The remaining assertions are clear. \square

3. RANDOM INTERSECTIONS IN PRODUCTS OF PROJECTIVE SPACES

3.1. The kinematic formula. We specialize here the general kinematic formula for homogeneous spaces from [9, Thm. A.2] to the case of products of real projective spaces (Theorem 3.2). For this purpose, we define the average scaling factor and we explain how to bound it in Lemma 3.5.

Consider the product $\Omega := \mathbb{P}^{m_1} \times \dots \times \mathbb{P}^{m_n}$ of real projective spaces. The product $G := O(m_1 + 1) \times \dots \times O(m_n + 1)$ of orthogonal groups acts transitively on Ω . So Ω is a homogeneous space and we have an induced transitive action of G on the tangent bundle of Ω . We focus on the special hypersurfaces H_1, \dots, H_n of Ω of the following shape

$$(3.1) \quad H_1 := \mathbb{P}^{m_1-1} \times \mathbb{P}^{m_2} \times \dots \times \mathbb{P}^{m_n}, \dots, H_n := \mathbb{P}^{m_1} \times \mathbb{P}^{m_2} \times \dots \times \mathbb{P}^{m_n-1}.$$

They are determined upon selecting hyperplanes \mathbb{P}^{m_i-1} in \mathbb{P}^{m_i} . Our goal is to investigate the average cardinality of the intersection $Z \cap H_1 \cap \dots \cap H_n$ of an n -dimensional smooth submanifold $Z \subseteq \Omega$ with random H_i corresponding to independently chosen uniform random hyperplanes in \mathbb{P}^{m_i} .

Fix a distinguished point $\omega \in \Omega$ and denote by K the stabilizer group of ω . E.g., take $\omega_i = [1 : 0 : \dots : 0]$ for all i . Notice that we have an induced action of K on the tangent space $T := T_\omega \Omega$, which we can identify with the standard action of $K = O(m_1) \times \dots \times O(m_n)$ on $T = \mathbb{R}^{m_1} \times \dots \times \mathbb{R}^{m_n}$. This induces an action of K on the Grassmann manifold $\text{Gr}(d, T)$ of linear subspaces of T with codimension d . Note that this action is transitive if $n = 1$, but not for $n \geq 2$.

We assign to an n -dimensional smooth submanifold $Z \subseteq \Omega$ a map

$$(3.2) \quad Z \rightarrow \text{Gr}(n, T)/K, \quad z \mapsto KgN_p Z$$

as follows. For given $p \in Z$ choose any $g \in G$ such that $gp = \omega$. The induced action of g maps the tangent space $T_p \Omega$ to $T_\omega \Omega = T$. This transports the normal subspace $N_p Z \subseteq T_p \Omega$ of Z at p to $gN_p Z \subseteq T$. Note that the K -orbit of the subspace $gN_p Z$ does not depend on the choice of g , which shows that the map (3.2) is well defined.

We call the submanifold Z *cohomogeneous* if the map (3.2) is constant; see [9, A.5.1] and [28]. For instance, a product $Z = \mathcal{L}_1 \times \dots \times \mathcal{L}_n$ of lines \mathcal{L}_i in \mathbb{P}^{m_i} is cohomogeneous: indeed, the map (3.2) sends any point $p \in Z$ to the K -orbit of $\mathbb{R} \times \dots \times \mathbb{R}$.

Definition 3.1. *The average scaling factor function of the n -dimensional submanifold Z of $\mathbb{P}^{m_1} \times \dots \times \mathbb{P}^{m_n}$ is the function $\bar{\sigma}_Z: Z \rightarrow [0, 1]$ defined at $p \in Z$ by*

$$\bar{\sigma}_Z(p) := \mathbb{E}_{L_i} \sigma(gN_p Z, L_1 \times \dots \times L_n),$$

where $g \in G$ satisfies $gp = \omega$, and the expectation is taken over uniformly random lines L_i in $T = \mathbb{R}^{m_1} \times \dots \times \mathbb{R}^{m_n}$; see (2.2) for the definition of σ .

Note that due to the averaging over the K -orbit, the choice of g is irrelevant. The above definition is consistent with the one in [9, Def. A.1], since

$$(3.3) \quad \sigma(gN_p Z, L_1 \times \dots \times L_n) = \sigma(gN_p Z, L_1 \times 0 \times \dots \times 0, \dots, 0 \times \dots \times 0 \times L_n)$$

by (2.3); indeed note that the n lines $L_1 \times 0 \times \dots \times 0, \dots$ are pairwise orthogonal.

We introduce the notation

$$\rho_n := \mathbb{E} \|x\| = \sqrt{2} \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})} \leq \sqrt{n}$$

for standard Gaussian $x \in \mathbb{R}^n$ and note that [6, Lemma 2.25],

$$(3.4) \quad \frac{\text{vol}(\mathbb{P}^{m_i-1})}{\text{vol}(\mathbb{P}^{m_i})} = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{m_i+1}{2})}{\Gamma(\frac{m_i}{2})} = \frac{1}{\sqrt{2\pi}} \rho_{m_i},$$

We can now explicitly state the kinematic formula for products of real projective spaces.

Theorem 3.2. *For any n -dimensional submanifold Z of $\mathbb{P}^{m_1} \times \dots \times \mathbb{P}^{m_n}$, we have*

$$\mathbb{E}_{g \in G} \#(Z \cap g_1 H_1 \cap \dots \cap g_n H_n) = (2\pi)^{-\frac{n}{2}} \rho_{m_1} \dots \rho_{m_n} \int_Z \bar{\sigma}_Z dZ,$$

where the hypersurfaces H_i are defined in (3.1).

Proof. If $\sigma_K: Z \times H_1 \times \dots \times H_n \rightarrow [0, 1]$ denotes the average scaling function from [9, Def. A.1], then [9, Thm. A.2] states that

$$\mathbb{E}_{g \in G} \#(Z \cap g_1 H_1 \cap \dots \cap g_n H_n) = \frac{1}{\text{vol}(\Omega)^n} \int_{Z \times H_1 \times \dots \times H_n} \sigma_K d(Z \times H_1 \times \dots \times H_n).$$

By K -invariance and (3.3), we have $\sigma_K(z, y_1, \dots, y_n) = \bar{\sigma}_Z(z)$ for all $z \in Z$ and $y_i \in H_i$. Therefore,

$$\mathbb{E}_{g \in G} \#(Z \cap g_1 H_1 \cap \dots \cap g_n H_n) = \frac{\text{vol}(H_1) \dots \text{vol}(H_n)}{\text{vol}(\Omega)^n} \int_Z \bar{\sigma}_Z dZ.$$

Finally, (3.4) gives

$$\frac{\text{vol}(H_1) \dots \text{vol}(H_n)}{\text{vol}(\Omega)^n} = \prod_{i=1}^n \frac{\text{vol}(\mathbb{P}^{m_i-1})}{\text{vol}(\mathbb{P}^{m_i})} = \frac{\rho_{m_1} \dots \rho_{m_n}}{(2\pi)^{\frac{n}{2}}},$$

which completes the proof. \square

Example 3.3. A product $Z = \mathcal{L}_1 \times \dots \times \mathcal{L}_n$ of lines \mathcal{L}_i is cohomogeneous and we have $\bar{\sigma}_Z = (2/\pi)^{n/2} (\rho_{m_1} \dots \rho_{m_n})^{-1}$ by Theorem 3.2.

We shall focus on submanifolds Z arising as the image of an injective map

$$(3.5) \quad \psi: U \rightarrow \mathbb{P}^{m_1} \times \dots \times \mathbb{P}^{m_n}, \quad \psi(x) := (\psi_1(x), \dots, \psi_n(x)),$$

where the $\psi_i: U \rightarrow \mathbb{P}^{m_i}$ are smooth maps defined on an open subset $U \subseteq \mathbb{R}^n$. Let us denote by

$$J\psi(x) := \sqrt{\det((D_x \psi)^T D_x \psi)}$$

the *absolute Jacobian* of ψ at x . The transformation formula implies that

$$(3.6) \quad \int_Z \bar{\sigma}_Z dZ = \int_U \bar{\sigma}_Z(\psi(x)) J\psi(x) dx.$$

We next analyze the integrand on the right-hand side more closely.

Lemma 3.4. *Let $x \in U$ and put $T_i := T_{\psi_i(x)} \mathbb{P}^{m_i}$. Let $\lambda_1, \dots, \lambda_n$ be independent standard Gaussian linear forms on T_i . This defines the random linear forms $\lambda_i \circ D_x \psi_i$ on \mathbb{R}^n . Then*

$$\rho_{m_1} \cdots \rho_{m_n} \bar{\sigma}_Z(\psi(x)) J\psi(x) = \mathbb{E}_{\lambda_1, \dots, \lambda_n} \|(\lambda_1 \circ D_x \psi_1) \wedge \dots \wedge (\lambda_n \circ D_x \psi_n)\|.$$

Proof. To simplify notation, we assume w.l.o.g. that $\omega = \psi(x)$ is the distinguished point. We also identify T_i with \mathbb{R}^{m_i} . For $u_i \in T_i$ with $\|u_i\| = 1$ consider the line $L_i = \mathbb{R}u_i$ and the orthogonal projection $p_i: T_i \rightarrow L_i$, which is given by $p_i(w) = \mu_i(w)u_i$ with the linear form on T_i defined by $\mu_i(w) := \langle w, u_i \rangle$. Thus the orthogonal projection $p_L: T_1 \times \dots \times T_n \rightarrow L_1 \times \dots \times L_n$ is described by μ_1, \dots, μ_n . This implies that

$$(3.7) \quad |\det(p_L \circ D_x \psi)| = \|(\mu_1 \circ D_x \psi_1) \wedge \dots \wedge (\mu_n \circ D_x \psi_n)\|.$$

On the other hand, according to Proposition 2.1, we have

$$\sigma(L_1 \times \dots \times L_n, N_p Z) = |\det p'_L|,$$

where $p'_L: T_p Z \rightarrow L_1 \times \dots \times L_n$ denotes the restriction of p_L to $T_p Z$. Applying the determinant to the composition of $D_x \psi$ with p'_L , we get

$$J\psi(x) |\det p'_L| = |\det(p_L \circ D_x \psi)|.$$

By averaging over random lines L_i , we deduce from the definition of $\bar{\sigma}_Z$ and the above that

$$J\psi(x) \bar{\sigma}_Z(p) = J\psi(x) \mathbb{E}_{L_i} \sigma(N_p Z, L_1 \times \dots \times L_n) = J\psi(x) \mathbb{E}_{L_i} |\det p'_L| = \mathbb{E}_{L_i} |\det(p_L \circ D_x \psi)|.$$

Finally, a standard Gaussian linear form on T_i is obtained as $\lambda_i = r_i \mu_i$ with independent random variables r_i and u_i , where u_i is uniformly random in the unit sphere of T_i and r_i^2 is χ^2 -distributed with m_i degrees of freedom. Thus $\mathbb{E} r_i = \rho_{m_i}$. Altogether, we obtain, using (3.7),

$$\begin{aligned} \rho_{m_1} \cdots \rho_{m_n} J\psi(x) \bar{\sigma}_Z(p) &= \rho_{m_1} \cdots \rho_{m_n} \mathbb{E} |\det(p_L \circ D_x \psi)| \\ &= \rho_{m_1} \cdots \rho_{m_n} \mathbb{E} \|(\mu_1 \circ D_x \psi_1) \wedge \dots \wedge (\mu_n \circ D_x \psi_n)\| \\ &= \mathbb{E} \|(\lambda_1 \circ D_x \psi_1) \wedge \dots \wedge (\lambda_n \circ D_x \psi_n)\|, \end{aligned}$$

which completes the proof. \square

3.2. Bounding the average scaling factor. In order to bound the quantity in Lemma 3.4, we use affine charts for the product of projective spaces. Let y_{i0}, \dots, y_{im_i} be coordinates for \mathbb{P}^{m_i} . Fix $0 \leq r_i \leq m_i$ for $i = 1, \dots, n$, and consider the inverse of the affine chart $\pi_{ir_i}: \mathbb{R}^{m_i} \rightarrow (\mathbb{P}^{m_i})_{y_{ir_i} \neq 0}$, see Subsection 2.1. We describe the maps ψ_i from (3.5) in these charts by smooth functions defined on open subsets of \mathbb{R}^n ,

$$(3.8) \quad \varphi_{ir_i}: \mathbb{R}^n \supseteq U_{ir_i} \rightarrow \mathbb{R}^{m_i},$$

satisfying $\psi_i := \pi_{ir_i} \circ \varphi_{ir_i}$. In order to simplify notation, we assume w.l.o.g. $r_i = 0$ and write $\pi_i := \pi_{i0}$, $\varphi_i := \varphi_{i0}$. In these charts, the combined map ψ of (3.5) is represented by a map

$$\varphi: U \rightarrow \mathbb{R}^{m_1} \times \dots \times \mathbb{R}^{m_n}, \quad \varphi(x) = (\varphi_1(x), \dots, \varphi_n(x))$$

defined on some open subset $U \subseteq \mathbb{R}^n$. We view the derivative $M(x) := D_x \varphi$ as a matrix of format $(m_1 + \dots + m_n) \times n$ with blocks $M_i(x) := D_x \varphi_i \in \mathbb{R}^{m_i \times n}$. For $1 \leq j_i \leq m_i$, $i = 1, \dots, n$, we denote by $M(x)_{j_1, \dots, j_n}$ the $n \times n$ submatrix of $M(x)$ obtained by selecting in the i th block the j_i th row.

Lemma 3.5. *Let $x \in U$ be such that $[y_i] := \psi_i(x) \in (\mathbb{P}^{m_i})_{y_{i0} \neq 0}$ for all i . Then*

$$\rho_{m_1} \cdots \rho_{m_n} \bar{\sigma}_Z(\psi(x)) J\psi(x) \leq \sum_{j_1, \dots, j_n} |\det M(x)_{j_1, \dots, j_n}|,$$

where the sum is over n -tuples $(j_1, \dots, j_n) \in [m_1] \times \dots \times [m_n]$.

Proof. From $\psi_i = \pi_i \circ \varphi_i$ we get $D\psi_i = D\pi_i \circ D\varphi_i$, where we drop arguments for notational simplicity. Let $\lambda_i: T_i \rightarrow \mathbb{R}$ be a linear form on $T_i = T_{\psi_i(x)} \mathbb{P}^{m_i}$. Then, defining $w_i := \lambda_i \circ D\pi_i$,

$$\lambda_i \circ D\psi_i = \lambda_i \circ D\pi_i \circ D\varphi_i = w_i \circ D\varphi_i.$$

If we identify $\lambda_i \circ D_x \psi_i$ with a vector in \mathbb{R}^n and w_i with a vector in \mathbb{R}^{m_i} , then we have the matrix product of formats $n \times \sum_i m_i$ and $\sum_i m_i \times n$,

$$(3.9) \quad R(x) := \begin{bmatrix} (\lambda_1 \circ D_x \psi_1)^T \\ \vdots \\ (\lambda_n \circ D_x \psi_1)^T \end{bmatrix} = \begin{bmatrix} w_1^T & 0 & \dots & 0 \\ 0 & w_2^T & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & w_n^T \end{bmatrix} \cdot \begin{bmatrix} M_1(x) \\ \vdots \\ M_n(x) \end{bmatrix}.$$

Lemma 3.4 tells us that

$$\rho_{m_1} \cdots \rho_{m_n} \bar{\sigma}_Z(\psi(x)) J\psi(x) = \mathbb{E}_{\lambda_i} |\det R(x)|,$$

where the expectation is over independent standard Gaussian λ_i . Note that the resulting random vector $w_i := \lambda_i \circ D\pi_i$ is not standard Gaussian anymore. However $\|D\pi_i\| \leq 1$ by (2.1), and Lemma 3.6 below imply that $\mathbb{E} w_{ij}^2 \leq 1$ for the j th component w_{ij} of w_i .

From Binet-Cauchy, we obtain from (3.9)

$$(\det R(x))^2 = \sum_{j_1, \dots, j_n} w_{1j_1}^2 \cdots w_{nj_n}^2 (\det M(x)_{j_1, \dots, j_n})^2,$$

where the sum is over all $(j_1, \dots, j_n) \in [m_1] \times \dots \times [m_n]$. Taking expectations yields

$$\mathbb{E}_w (\det R(x))^2 \leq \sum_{j_1, \dots, j_n} (\det M(x)_{j_1, \dots, j_n})^2.$$

We conclude that

$$\mathbb{E}_w |\det R(x)| \leq (\mathbb{E}_w (\det R(x))^2)^{\frac{1}{2}} \leq \sum_{j_1, \dots, j_n} |\det M(x)_{j_1, \dots, j_n}|,$$

which completes the proof. \square

Lemma 3.6. *Let $A \in \mathbb{R}^{p \times m}$ with $\|A\| \leq 1$. If $y \in \mathbb{R}^p$ is standard Gaussian, then the random variable $z := yA$ satisfies $\mathbb{E} |z_j|^2 \leq 1$ for all j .*

Proof. From $z_j = \sum_i y_i a_{ij}$ we get $z_j^2 = \sum_{i,k} y_i y_k a_{ij} a_{kj}$. Hence $\mathbb{E} z_j^2 = \sum_i a_{ij}^2$. Finally, $\sum_i a_{ij}^2 = \|A(e_j)\|^2 \leq \|A\|^2 \leq 1$. \square

4. MIXED RANDOM FEWNOMIAL SYSTEMS

We provide here the proofs of the assertions in the introduction. Let us first introduce some notation.

We assign to a real valued function $c: A \rightarrow \mathbb{R}$ on a finite nonempty subset $A \subseteq \mathbb{R}^n$ the real analytic function $F_{A,c}: \mathbb{R}^n \rightarrow \mathbb{R}$

$$(4.1) \quad F_{A,c}(w) := \sum_{a \in A} c(a) e^{\langle a, w \rangle}.$$

In the special case where A consists of integer vectors, $F_{A,c}$ arises from the Laurent polynomial $f_{A,c}(x) = \sum_{a \in A} c(a)x^a$ by a substitution: $F_{A,c}(w) = f_{A,c}(e^w)$. Generally, we have the following equivariance property: for $g \in \mathrm{GL}(n, \mathbb{R})$ and $b \in \mathbb{R}^n$,

$$(4.2) \quad F_{A+b, b.c}(w) = e^{\langle b, w \rangle} F_{A,c}(w), \quad F_{g(A), g.c}(w) = F_{A,c}(g^T w),$$

where $b.c(a) := c(a - b)$ and $(g.c)(a) := c(g^{-1}a)$.

Suppose now we have n such analytic functions encoded by $c_i: A_i \rightarrow \mathbb{R}$, for $i = 1, \dots, n$. Throughout, we denote by t_i the cardinality of A_i and by P_i its convex hull. We are interested in the number N of nondegenerate zeros $w \in \mathbb{R}^n$ of the system

$$(4.3) \quad F_{A_1, c_1}(w) = 0, \dots, F_{A_n, c_n}(w) = 0.$$

Our goal is to study the expected number of nondegenerate zeros for random coefficient functions. More specifically, we denote by $\mathbb{E}(A_1, \dots, A_n)$ the expectation of N , when all the coefficients $c_i(a)$, for $i \in [n]$ and $a_i \in A_i$, are independent standard Gaussians. Clearly, $\mathbb{E}(A_1, \dots, A_n)$ is invariant under permutations of the A_i . Also, $\mathbb{E}(A_1, \dots, A_n) = 0$ if $t_i = 1$ for some i . Moreover, we have $\mathbb{E}(A_1, \dots, A_n) = 0$ if $\dim(P_1 + \dots + P_n) < n$, see Lemma 4.2.

Equation (4.2) implies the following invariance properties

$$(4.4) \quad \begin{aligned} \mathbb{E}(A_1 + b_1, \dots, A_n + b_n) &= \mathbb{E}(A_1, \dots, A_n), \\ \mathbb{E}(g(A_1), \dots, g(A_n)) &= \mathbb{E}(A_1, \dots, A_n), \end{aligned}$$

where $b_1, \dots, b_n \in \mathbb{R}^n$ and $g \in \mathrm{GL}(n, \mathbb{R})$. In particular, \mathbb{E} is invariant under replacing A_i by $\lambda_i A_i$ for $\lambda_i \in \mathbb{R}^\times$.

Our main result is Theorem 1.1 stated in the introduction. Note that it gives the correct answer $\mathbb{E}(A_1, \dots, A_n) = 0$ if $t_i = 1$ for some i .

Example 4.1. In the case $t_1 = \dots = t_n = 2$, the P_i are segments. If they are linearly independent, $P_1 + \dots + P_n$ is a parallelepiped with 2^n vertices. Thus, Theorem 1.1 gives $\mathbb{E}(A_1, \dots, A_n) \leq (2/\pi)^{\frac{n}{2}}$. This can be easily verified directly as follows. Suppose $A_i = \{a_i, b_i\}$. We claim that $\mathbb{E}(A_1, \dots, A_n) = 2^{-n}$ if $b_1 - a_1, \dots, b_n - a_n$ are linearly independent. For showing this, by the invariance properties (4.4), it suffices to consider the case where $A_i = \{0, e_i\}$. Then (4.3) amounts to the system $c_i(0) + c_i(e_i)e^{w_i} = 0$, for $i = 1, \dots, n$, which has a solution iff $c_i(0)c_i(e_i) < 0$, for all i . This happens with probability 2^{-n} , hence indeed $\mathbb{E}(A_1, \dots, A_n) = 2^{-n}$.

4.1. Proof of Theorem 1.1. Let us look at a special instance of (3.5). To the given finite nonempty subsets $A_1, \dots, A_n \subseteq \mathbb{R}^n$, we assign the maps

$$\psi_i: \mathbb{R}_{>0}^n \rightarrow \mathbb{P}(\mathbb{R}^{A_i}) \simeq \mathbb{P}^{m_i}, \quad \psi_i(x) := [x^{a_i}]_{a_i \in A_i},$$

where $m_i := \#A_i - 1$. Recall that P_i denotes the convex hull of A_i and put $P := P_1 + \dots + P_n$. We consider the combined map

$$(4.5) \quad \psi: \mathbb{R}_{>0}^n \rightarrow \mathbb{P}^{m_1} \times \dots \times \mathbb{P}^{m_n}, \quad \psi(x) := (\psi_1(x), \dots, \psi_n(x)).$$

Lemma 4.2. *The map ψ is injective iff P is n -dimensional. Moreover, if P is not n -dimensional, then $\mathrm{rank} D_x \psi < n$ for all $x \in \mathbb{R}_{>0}^n$.*

Proof. Assume $\psi(\exp(w)) = \psi(\exp(w'))$ for $w \neq w' \in \mathbb{R}^n$. Then there are $c_i \in \mathbb{R}$ such that for all $a_i \in A_i$ we have $\langle a_i, w - w' \rangle = c_i$. Hence, $\langle x, w - w' \rangle = c_i$ for all $x_i \in P_i$. It follows that $\langle x, w - w' \rangle = c_1 + \dots + c_n$ for all $x \in P$. Hence $\dim P < n$.

Conversely, assume there is a nonzero $w \in \mathbb{R}^n$ and $c \in \mathbb{R}$ such that $\langle x, w \rangle = c$ for all $x \in P$. Then there are $c_i \in \mathbb{R}$ such that $\langle x_i, w \rangle = c_i$ for all $x_i \in P_i$. It follows that for any $x \in \mathbb{R}_{>0}^n$ and any $s \in \mathbb{R}$ we have

$$\psi_i(e^{sw} x) = [(e^{sw})^{a_i} x^{a_i}]_{a_i \in A_i} = [e^{s\langle a_i, w \rangle} x^{a_i}]_{a_i \in A_i} = [e^{sc_i} x^{a_i}]_{a_i \in A_i} = \psi_i(x).$$

Hence ψ is not injective. Moreover, w is in the kernel of the derivative of ψ_i at x . \square

We denote by Z the image of ψ . Then we can write

$$\mathbb{E}(A_1, \dots, A_n) = \mathbb{E}_{g \in G} \#(Z \cap g_1 H_1 \cap \dots \cap g_n H_n),$$

where the hypersurfaces H_i are defined in (3.1). By Theorem 3.2 and (3.6), this can be expressed as

$$(4.6) \quad \mathbb{E}(A_1, \dots, A_n) = (2\pi)^{-\frac{n}{2}} \rho_{m_1} \cdots \rho_{m_n} \int_{\mathbb{R}_{>}^n} (\bar{\sigma}_Z \circ \psi) J\psi \, dx.$$

We make the coordinate change $\mathbb{R}^n \rightarrow \mathbb{R}_{>0}^n, (w_1, \dots, w_n) \mapsto x = (e^{-w_1}, \dots, e^{-w_n})$, which has the absolute Jacobian $x_1 \cdots x_n$, and obtain (slightly abusing notation)

$$(4.7) \quad \int_{\mathbb{R}_{>}^n} (\bar{\sigma}_Z \circ \psi) J\psi \, dx = \int_{\mathbb{R}^n} x_1 \cdots x_n (\bar{\sigma}_Z \circ \psi) J\psi \, dw.$$

Recall from Subsection 2.4 that each vertex v of P defines the inner normal cone $C_v := P_v^*$. We can write

$$(4.8) \quad \mathbb{R}^n = \bigcup_v C_v$$

as the union over the vertices v of P . Moreover, we know that $\dim(C_v \cap C_{v'}) < n$ for different vertices v, v' . Therefore, we can rewrite (4.7) as the sum

$$\sum_v \int_{C_v} x_1 \cdots x_n (\bar{\sigma}_Z \circ \psi) J\psi \, dw.$$

over the V_0 many vertices v of P .

Fix now a vertex v of P . According to Lemma 2.5, there are vertices v_i of P_i , for $i = 1, \dots, n$, satisfying $v = v_1 + \dots + v_n$. Note that $a_i \in A_i$.

We define the map $\varphi_i: \mathbb{R}_{>0}^n \rightarrow \mathbb{R}^{A_i \setminus \{v_i\}}$ by

$$\varphi_i(x) = (x^{a_i - v_i})_{a_i \in A_i \setminus \{v_i\}} \in \mathbb{R}^{A_i \setminus \{v_i\}} \simeq \mathbb{R}^{m_i}.$$

Note that φ_i expresses ψ_i in the affine chart $\mathbb{P}(\mathbb{R}^{A_i})_{y_{iv_i} \neq 0} \rightarrow \mathbb{R}^{a_i \in A_i \setminus \{v_i\}}$, which maps $[y_{ia_i}]_{a_i \in A_i}$ to $y_{iv_i}^{-1} (y_{ia_i})_{a_i \in A_i \setminus \{v_i\}}$. So we are in the setting of Subsection 3.2 and φ_i is an instance of (3.8). The rows of the matrix $M(x) := D_x \varphi$ are labeled by the disjoint union $A_1 \sqcup \dots \sqcup A_n$ and $M(x)$ has n columns. For any n -tuple (a_1, \dots, a_n) with $a_i \in A_i \setminus \{v_i\}$, we denote by $M(x)_{a_1, \dots, a_n}$ the $n \times n$ submatrix of $M(x)$, obtained by selecting from $M(x)$ the rows numbered by a_1, \dots, a_n . We apply Lemma 3.5 to bound

$$\rho_{m_1} \cdots \rho_{m_n} \int_{C_v} x_1 \cdots x_n (\bar{\sigma}_Z \circ \psi) J\psi \, dw \leq \sum_{a_1, \dots, a_n} \int_{C_v} x_1 \cdots x_n |\det M(x)_{a_1, \dots, a_n}| \, dw,$$

where the sum runs over all tuples (a_1, \dots, a_n) with $a_i \in A_i \setminus \{v_i\}$. So there are $m_1 \cdots m_n$ many summands. To prove Theorem 1.1, it is sufficient to show that

$$(4.9) \quad \int_{C_v} x_1 \cdots x_n |\det M(x)_{a_1, \dots, a_n}| \, dw \leq 1$$

for each vertex v and each selection (a_1, \dots, a_n) .

The component (row) of the derivative $D_x \varphi_i$ corresponding to $a_i \in A_i \setminus \{v_i\}$ is given by

$$(D_x \varphi_i)_{a_i} = x^{a_i - v_i} (a_i - v_i) \text{diag}(x_1^{-1}, \dots, x_n^{-1}).$$

Hence the $n \times n$ -submatrix $M(x)_{a_1, \dots, a_n}$ of $M(x)$ is given by

$$M(x)_{a_1, \dots, a_n} = \text{diag}(x^{a_1-v_1}, \dots, x^{a_n-v_n}) \begin{bmatrix} a_1 - v_1 \\ \vdots \\ a_n - v_n \end{bmatrix} \text{diag}(x_1^{-1}, \dots, x_n^{-1}).$$

Therefore, setting $b_i := a_i - v_i$, we get

$$x_1 \cdots x_n \det(M(x)_{a_1, \dots, a_n}) = x^{b_1 + \dots + b_n} \det[b_1, \dots, b_n].$$

Let us write Π_i for the cone of P_i at the vertex v_i . By definition, $b_i \in \Pi_i^*$. By Lemma 2.5, $\Pi := \Pi_1 + \dots + \Pi_n$ equals the cone of the polytope $P = P_1 + \dots + P_n$ at the vertex $v = v_1 + \dots + v_n$. Hence $b_i \in \Pi_i^* \subseteq \Pi_1^* \cap \dots \cap \Pi_n^* = \Pi^* = C_v$.

We can therefore rewrite the left-hand side of (4.9) as

$$(4.10) \quad \int_{C_v} x_1 \cdots x_n |\det M(x)_{a_1, \dots, a_n}| dw = \int_{C_v} e^{-\langle b_1 + \dots + b_n, w \rangle} |\det[b_1, \dots, b_n]| dw.$$

By Proposition 2.4, this is at most 1. This shows claim (4.9) and finishes the proof of Theorem 1.1. \square

4.2. Proof of Proposition 1.2. For finite $S_i \subseteq \mathbb{R}$, put $A := S_1 \times \dots \times S_n$, and consider

$$(4.11) \quad \begin{aligned} \psi_i &: \mathbb{R}_{>0} \rightarrow \mathbb{P}(\mathbb{R}^{S_i}), x_i \mapsto [x_i^{a_i}]_{a_i \in S_i}, \\ \psi &: \mathbb{R}_{>0}^n \rightarrow \mathbb{P}(\mathbb{R}^A), x \mapsto [x^a]_{a \in A} \end{aligned}$$

with images Z_i and Z , respectively. The kinematic formula for real projective space [9, Cor. A.3] gives

$$\mathbb{E}(S_i) = \frac{\text{vol}(Z_i)}{\text{vol}(\mathbb{P}^1)}, \quad \mathbb{E}(A, \dots, A) = \frac{\text{vol}(Z)}{\text{vol}(\mathbb{P}^n)}.$$

The key insight is that Z is obtained as the image of $Z_1 \times \dots \times Z_n$ under the *Segre embedding*

$$\mathbb{P}(\mathbb{R}^{S_1}) \times \dots \times \mathbb{P}(\mathbb{R}^{S_n}) \rightarrow \mathbb{P}(\mathbb{R}^{S_1} \otimes \dots \otimes \mathbb{R}^{S_n}) \simeq \mathbb{P}(\mathbb{R}^A),$$

which is isometric (see Appendix B). Therefore, we have $\text{vol}(Z) = \text{vol}(Z_1) \cdots \text{vol}(Z_n)$, which completes the proof of Proposition 1.2. \square

4.3. Proof of Theorem 1.3. Given is a finite subset $A \subseteq \mathbb{R}^n$ with convex hull P . By Lemma 4.2 we can w.l.o.g. assume that $\dim P = n$. Consider the injective map

$$(4.12) \quad \psi: \mathbb{R}_{>0}^n \rightarrow \mathbb{P}(\mathbb{R}^A), \psi(x) := [x^a]_{a \in A}$$

with image $Z \subseteq \mathbb{P}(\mathbb{R}^A)$. The kinematic formula for real projective space [9, Cor. A.3] is considerably simpler than the one in Theorem 3.2, since $O(m)$ acts transitively on the Grassmann manifolds $\text{Gr}(k, \mathbb{R}^m)$: we have

$$(4.13) \quad \mathbb{E}(A, \dots, A) = \frac{\text{vol}(Z)}{\text{vol}(\mathbb{P}^n)} = \frac{1}{\text{vol}(\mathbb{P}^n)} \int_{\mathbb{R}_{>0}^n} J\psi(x) dx.$$

We now proceed as in the proof of Theorem 1.3. We make the coordinate change $x = e^{-w}$ and decompose the resulting integral according to the decomposition (4.8) of \mathbb{R}^n into the full dimensional cones C_v corresponding to vertices v . Thus

$$\int_{\mathbb{R}_{>0}^n} J\psi(x) dx = \int_{\mathbb{R}^n} x_1 \cdots x_n J\psi(x) dw = \sum_{C_v} \int_{C_v} x_1 \cdots x_n J\psi(x) dw$$

For a fixed vertex v of P , we consider the map $\varphi: \mathbb{R}_{>0}^n \rightarrow \mathbb{R}^{A \setminus \{v\}}$ defined by

$$(4.14) \quad \varphi(x) = (x^{a-v})_{a \in A \setminus \{v\}}.$$

Then we have $\psi(x) = \pi(\varphi(x))$, where π is the inverse of the chart $\mathbb{P}(\mathbb{R}^A)_{y_v \neq 0} \rightarrow \mathbb{R}^{A \setminus \{v\}}$. It is easy to verify that $J\psi(x) \leq J\varphi(x)$ using $\|D_{\varphi(x)}\pi\| \leq 1$, see (2.1).

Let us view $\mathcal{M}(x) := D_x\varphi$ as a matrix whose rows are labelled by elements of $A \setminus \{v\}$. and denote by $\mathcal{M}(x)_{a_1, \dots, a_n}$ the submatrix of $\mathcal{M}(x)$ obtained by selecting the rows labelled by the a_i . Binet-Cauchy implies that

$$J\varphi(x)^2 = \det(\mathcal{M}(x)^T \mathcal{M}(x)) = \sum_{a_1, \dots, a_n} (\det \mathcal{M}(x)_{a_1, \dots, a_n})^2,$$

with the sum running over all n -element subsets $\{a_1, \dots, a_n\}$ of $A \setminus \{v\}$, of which there are $\binom{t-1}{n}$ many. This implies $J\varphi(x) \leq \sum_{a_1, \dots, a_n} |\det \mathcal{M}(x)_{a_1, \dots, a_n}|$. We have arrived at

$$\int_{C_v} x_1 \cdots x_n J\psi(x) dw \leq \sum_{a_1, \dots, a_n} \int_{C_v} x_1 \cdots x_n |\det \mathcal{M}(x)_{a_1, \dots, a_n}| dw \leq \binom{t-1}{n},$$

where the right-hand inequality follows from Proposition 2.4 as in (4.10). \square

4.4. Proof of Theorem 1.5. The key observation is the following. Define for $\varepsilon > 0$

$$D_\varepsilon := \{x \in \mathbb{R}^n \mid \|x\| \geq \varepsilon\}.$$

Lemma 4.3. *Let $C \subseteq \mathbb{R}^n$ be a proper cone, $d \in \text{int}(C^*)$, and $\varepsilon > 0$. Then*

$$\lim_{m \rightarrow \infty} m^n \int_{C \cap B_\varepsilon} e^{-m\langle d, w \rangle} dw = 0$$

Proof. Since $\cap_{m \geq 1} D_{m\varepsilon} = \emptyset$, basic integration theory implies

$$\lim_{m \rightarrow \infty} \int_{C \cap D_{m\varepsilon}} e^{-\langle d, u \rangle} du = 0.$$

Making the change of variables $u = mw$ shows the claim. \square

We now observe the following. Let $U \subseteq \mathbb{R}_{>0}^n$ be open. Analogously as for (4.6), one shows that

$$(2\pi)^{-\frac{n}{2}} \rho_{m_1} \cdots \rho_{m_n} \int_U x_1 \cdots x_n (\bar{\sigma}_Z \circ \psi) J\psi dw.$$

equals the expected number of nondegenerate zeros in U of the random system (4.3).

We follow the proof of Theorem 1.1. Note that stretching the support does not change the Newton polytopes P_i and $P = P_1 + \dots + P_n$. Fix a vertex v of P . According to Lemma 2.5, there are vertices v_i of P_i , for $i = 1, \dots, n$, satisfying $v = v_1 + \dots + v_n$. Tracing the proof of Theorem 1.1, one sees that it is sufficient to show that (compare (4.10)) for any selection $a_1 \in A_1 \setminus \{v_1\}, \dots, a_n \in A_n \setminus \{v_n\}$, the vectors $b_i = a_i - v_i$ satisfy

$$\lim_{m \rightarrow \infty} \int_{C_v} e^{-m\langle b_1 + \dots + b_n, w \rangle} |\det[m b_1, \dots, m b_n]| dw = 0.$$

However, this is a consequence of Lemma 4.3. \square

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APPENDIX A. PROOF OF LEMMA 2.3

We fix $x \in \text{int}(C^*)$. For $t \geq 0$ we define the $n - 1$ -dimensional slice

$$C_t := \{y \in C \mid \langle x, y \rangle = t\|x\|\}.$$

By Fubini, we get

$$v_C(x) = \int_C e^{-\langle x, y \rangle} dy = \int_0^\infty \text{vol}_{n-1}(C_t) e^{-t\|x\|} dt = \text{vol}_{n-1}(C_1) \int_0^\infty t^{n-1} e^{-t\|x\|} dt.$$

Note that

$$\int_0^\infty t^{n-1} e^{-t\|x\|} dt = \frac{1}{\|x\|^n} \int_0^\infty s^{n-1} e^{-s} ds = \frac{1}{\|x\|^n} \Gamma(n) = \frac{(n-1)!}{\|x\|^n}.$$

Moreover, setting $K := \{y \in C \mid \langle x, y \rangle \leq 1\}$, we have

$$\text{vol}_{n-1}(C_1) = n \text{vol}_n\{y \in C \mid \langle x, y \rangle \leq \|x\|\} = n \|x\|^n \text{vol}_n(K).$$

It follows that

$$v_C(x) = n \|x\|^n \text{vol}_n(K) \frac{(n-1)!}{\|x\|^n} = n! \text{vol}_n(K),$$

completing the proof. □

APPENDIX B. THE SEGRE EMBEDDING IS ISOMETRIC

Consider the Segre embedding

$$S: \mathbb{P}(\mathbb{R}^m) \times \mathbb{P}(\mathbb{R}^n) \rightarrow \mathbb{P}(\mathbb{R}^{m \times n}), ([x], [y]) \mapsto [x_i y_j].$$

It is well known that S is a smooth embedding. If we endow the real projective space with the standard Riemannian metric (see § 2.1), then S is isometric. This is also true for the Segre embedding with several factors. We provide the proof for lack of reference.

Proposition B.1. *The Segre embedding is isometric.*

Proof. For notational simplicity, we restrict ourselves to the case of two factors. We need to show that the derivatives of S preserve the inner products. By orthogonal invariance, it suffices to consider the derivative at $([e_0], [e_0])$, which is mapped to $[E_{00}]$. We can isometrically identify the tangent spaces at these points with $\mathbb{R}^{m-1} \times \mathbb{R}^{n-1}$ and \mathbb{R}^{mn-1} , respectively. Then derivative of S at $([e_0], [e_0])$ is given by

$$((v_1, \dots, v_{m-1}), (w_1, \dots, w_{n-1})) \mapsto \begin{bmatrix} 0 & v_1 & \dots & v_{m-1} \\ w_1 & 0 & \dots & 0 \\ \vdots & 0 & \ddots & 0 \\ w_{n-1} & 0 & \dots & 0 \end{bmatrix}.$$

Clearly, this map preserves the inner products. □

APPENDIX C. SUPPLEMENT

It is instructive to see how (4.13) directly follows from the more general kinematic formula in Theorem 3.2. Consider the injective map ψ from (4.12) with image $Z \subseteq \mathbb{P}(\mathbb{R}^A)$. We use ψ to define the map

$$(C.1) \quad \psi_d: \mathbb{R}_{>0}^n \rightarrow (\mathbb{P}(\mathbb{R}^A))^n, \quad x \mapsto (\psi(x), \dots, \psi(x)).$$

The image $Z_d = \{(y, \dots, y) \mid y \in Z\} \subseteq (\mathbb{P}^m)^n$ of ψ_d is the diagonal embedding of Z in the product of projective spaces. By Theorem 3.2 and (3.6) we have

$$\mathbb{E}(A, \dots, A) = (2\pi)^{-\frac{n}{2}} \rho_m^n \int_{\mathbb{R}_{>0}^n} (\bar{\sigma}_{Z_d} \circ \psi_d) J\psi_d dx.$$

Via Lemma C.1 below, we indeed conclude that

$$\mathbb{E}(A, \dots, A) = \frac{1}{\text{vol}(\mathbb{P}^n)} \int_{\mathbb{R}_{>0}^n} J\psi dx = \frac{\text{vol}(Z)}{\text{vol}(\mathbb{P}^n)},$$

which is (4.13).

Lemma C.1. *For $x \in \mathbb{R}_{>0}^n$ we have*

$$\rho_m^n \bar{\sigma}_{Z_d}(\psi_d(x)) J\psi_d(x) = \frac{(2\pi)^{\frac{n}{2}}}{\text{vol}(\mathbb{P}^n)} J\psi(x).$$

Proof. Lemma 3.4 applied to the map ψ_d from (C.1) gives

$$(C.2) \quad \rho_m^n \bar{\sigma}_{Z_d}(\psi_d(x)) J\psi_d(x) = \mathbb{E}_{\lambda_1, \dots, \lambda_n} \|(\lambda_1 \circ D_x \psi) \wedge \dots \wedge (\lambda_n \circ D_x \psi)\|,$$

where the λ_i are standard Gaussian linear forms on $T_{\psi(x)}\mathbb{P}^m$. Take an isometry $T_{\psi(x)}\mathbb{P}^m \simeq \mathbb{R}^m$, view $\lambda_i \in \mathbb{R}^m$ as a vector, and view $\Delta := D_x \psi$ as a matrix in $\mathbb{R}^{m \times n}$. We note that $J\psi(x) = \sqrt{\det(\Delta^T \Delta)}$. The right-hand side of (C.2) can be written as the expectation $\mathbb{E}_{\lambda_i} |\det R(x)|$, with the matrix

$$(C.3) \quad R(x) := \begin{bmatrix} \lambda_1^T \circ D_x \psi \\ \vdots \\ \lambda_n^T \circ D_x \psi \end{bmatrix} = \begin{bmatrix} \lambda_1^T \\ \vdots \\ \lambda_n^T \end{bmatrix} \cdot \Delta.$$

We thus need to prove that

$$(C.4) \quad \mathbb{E}_{\lambda_i} |\det R(x)| = \frac{(2\pi)^{\frac{n}{2}}}{\text{vol}(\mathbb{P}^n)} \sqrt{\det(\Delta^T \Delta)}.$$

In order to show this, by the singular value decomposition, we may assume that $\Delta = \begin{bmatrix} D \\ 0 \end{bmatrix}$, where $D = \text{diag}(\sigma_1, \dots, \sigma_n)$. Note that $\sqrt{\det(\Delta^T \Delta)} = \sigma_1 \cdots \sigma_n$. Then (C.3) can be written as $R(x) = \Lambda D$, where $\Lambda \in \mathbb{R}^{n \times n}$ is a standard Gaussian *square* matrix and we get $\mathbb{E}_{\Lambda} |\det(R(x))| = \sigma_1 \cdots \sigma_n \mathbb{E}_{\Lambda} |\det \Lambda|$. It is well known that $\mathbb{E}_{\Lambda} |\det \Lambda| = \rho_n \rho_{n-1} \cdots \rho_1$, e.g., see [6, Cor. 4.11]. On the other hand [6, Lemma 2.25]

$$\rho_m = \sqrt{2\pi} \frac{\text{vol}(\mathbb{P}^{m-1})}{\text{vol}(\mathbb{P}^m)},$$

hence $\rho_n \rho_{n-1} \cdots \rho_1 = \frac{(2\pi)^{\frac{n}{2}}}{\text{vol}(\mathbb{P}^n)}$. We have thus verified (C.4). \square