On the Union of Well-Founded Relations

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Abstract

We give a criterion for the union of well-founded (i.e., noetherian) relations to be well-founded, generalizing results of Geser and of Bachmair-Dershowitz. The proof is written in a calculational style and is conducted entirely in regular algebra.

Keywords: Well-founded relations, regular algebra, program termination.

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1 Introduction

A relation r is *well-founded* if there is no infinite sequence $n_0, n_1, \ldots$ such that $(n_i, n_{i+1}) \in r$ for all $i \geq 0$. Well-founded relations are the essence of induction. In particular, they are crucial for establishing the absence of infinite loops in a program. Well-founded relations are also used for proving termination of rewriting systems [4].

When it is difficult to establish well-foundedness directly, we may try a divide-and-conquer approach: First we decompose the given relation, then we establish that each part is well-founded, and finally, we attempt to deduce well-foundedness of the whole. For example, we may wish to prove that the union of two terminating sets of rewrite rules is itself terminating. Or, given that neither

$$\mathbf{do}\ b_1 \rightarrow p_1\ \mathbf{od} \quad \text{nor} \quad \mathbf{do}\ b_2 \rightarrow p_2\ \mathbf{od}$$

can loop indefinitely, we would like to prove the same for

$$\mathbf{do}\ b_1 \rightarrow p_1 \ [\!]\ b_2 \rightarrow p_2\ \mathbf{od}\,.$$

Both of these problems boil down to the question if the union of two well-founded relations is itself well-founded. So assume we are given two well-founded relations[1] b and r. The question is, then, what condition we can impose on b and r to ensure that $b \cup r$ is well-founded. Geser [3] discovered that it is sufficient to require that $b \cup r$ be *transitive*:

$$(b \cup r); (b \cup r) \ \subseteq \ b \cup r\,. \tag{1.1}$$

[1] Our use of the letters b and r follows the intuition that the two relations are given as sets of *blue* and *red* edges, respectively

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Bachmair and Dershowitz [1] found a different sufficient condition, which they name *quasicommutativity*:

$$r; b \;\subseteq\; b; r^* \, . \tag{1.2}$$

Yet another sufficient condition is that one of the two relations *absorbs* the other (we could not find out who discovered this; it seems to be folklore):

$$r; b \;\subseteq\; r \, . \tag{1.3}$$

The contribution of this paper is a sufficient condition which is much weaker than any of the above, namely

$$r; b \;\subseteq\; b; (b \cup r)^* \,\cup\, r \, .$$

Our result also generalizes previous ones in a different direction. The results in [1] and [3] were proved for concrete relations (sets of pairs), whereas we state and prove our result in an axiomatic framework for regular expressions of which relations are but one model.

2 Preliminaries

There are many different ways to choose axioms for the algebra of regular expressions. We follow [2] and define a regular algebra by the following axioms

(i) $(\mathcal{A}, \cup, \cap, \neg)$ is a complete Boolean algebra. The bottom element of $\mathcal{A}$ will be denoted by $\emptyset$.

(ii) $\mathcal{A}$ is endowed with an associative composition operator ";" that has an identity element id.

(iii) The composition operator ";" distributes (in each argument) over all disjunctions. In particular, $\emptyset; r = \emptyset = r; \emptyset$ for every $r \in \mathcal{A}$.

By the Knaster-Tarski theorem, every monotonic function $f : \mathcal{A} \to \mathcal{A}$ has a least fixed point μf and a greatest fixed point νf, and we have the following *induction rules*

$$\mu f \subseteq x \quad \Leftarrow \quad f(x) \subseteq x \qquad \text{and} \qquad x \subseteq \nu f \quad \Leftarrow \quad x \subseteq f(x) \, . \tag{2.1}$$

To save ourselves from having to invent a name for every function occurring in a fixed point expression we use $(x \mapsto E)$ to denote the function that maps every x to E (where E is an expression in x). With the aid of recursion we define two iteration operators. The first is known as the Kleene star

$$b^* \;=\; \mu(x \;\mapsto\; id \,\cup\, b; x) \, . \tag{2.2}$$

When b is a relation then b^* is its reflexive and transitive closure. An important property, which is used in some axiomatizations of the Kleene star, is the following *tail recursion lemma*:

$$r; b^* \;=\; \mu(x \;\mapsto\; r \,\cup\, x; b) \, . \tag{2.3}$$

The other iteration operator is defined by

$$b^\infty \;=\; \nu(x \mapsto b; x) \, . \tag{2.4}$$

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When we think of b as a relation describing a directed graph then b^∞ represents the set of all nodes that lie on an infinite path, in the sense that $(m,n) \in b^\infty$ if and only if there is an infinite sequence $m = m_0, m_1, m_2, \ldots$ such that $(m_i, m_{i+1}) \in b$ for all $i \geq 0$.

The preceding observation makes two important points. Firstly, a set M may be encoded as a relation $\{(m,n) \mid m \in M\}$. And secondly, a relation b is well-founded if and only if

$$b^\infty \ = \ \emptyset\,. \tag{2.5}$$

Compared to the definition in terms of infinite paths, this equation has the advantage of making sense not just for relations. We therefore take (2.5) as the definition of well-foundedness in regular algebra. By the induction rule for greatest fixed points, b is well-founded if and only if

$$x \subseteq b;x \qquad \Rightarrow \qquad x = \emptyset \tag{2.6}$$

holds for all $x \in \mathcal{A}$.

To highlight the intuition behind our proof we shall be using symbols and terminology from temporal logic. For any $r \in \mathcal{A}$ define an operator $\bigcirc_r$ by

$$\bigcirc_r x \ = \ r;x \qquad \text{for all } x \in \mathcal{A}\,. \tag{2.7}$$

The $\bigcirc_r$ operator will exclusively be applied to expressions that represent sets of nodes (in the sense explained above) and in that case $\bigcirc_r x$ represents the set of all nodes that have an outgoing red edge to a node in x. In contrast, the dual expression $\widetilde{\bigcirc}_r x$, which is defined by

$$\widetilde{\bigcirc}_r x \ = \ \neg \bigcirc_r \neg x \qquad \text{for all } x \in \mathcal{A}\,, \tag{2.8}$$

describes the set of all nodes with the property that *every* outgoing red edge leads to a node in x. There is a very nice law relating the two "next" operators. Namely, if we know that all red edges lead to nodes in x, and at least one red edge leads to a node in y, then there must be a red edge to a node in $x \cap y$:

$$\widetilde{\bigcirc}_r x \,\cap\, \bigcirc_r y \ \subseteq \ \bigcirc_r (x \cap y)\,. \tag{2.9}$$

The "next" operators have iterated forms (called *eventually* and *always*)

$$\Diamond_r \ = \bigcirc_{r^*} \qquad \text{and} \qquad \Box_r \ = \widetilde{\bigcirc}_{r^*}\,. \tag{2.10}$$

So $\Diamond_r x$ corresponds to the set of all nodes that have an outgoing (but possibly empty) red path to a node in x, whereas a node is in $\Box_r x$ if *all* red paths that depart from it lead to nodes in x. These operators are each other's duals

$$\neg \Diamond_r x \ = \ \Box_r \neg x\,. \tag{2.11}$$

A set of nodes that is invariant under relation r is also invariant under its reflexive-transitive closure

$$\bigcirc_r x = x \qquad \Rightarrow \qquad \Diamond_r x = x\,. \tag{2.12}$$

Since $\bigcirc_r r^\infty = r;r^\infty = r^\infty$, we have as a particular case

$$\Diamond_r r^\infty \ = \ r^\infty\,. \tag{2.13}$$

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Finally, the always operator satisfies the following *expansion law*:

$$\Box_r x \;=\; x \cap \widetilde{\bigcirc}_r \Box_r x\,. \tag{2.14}$$

All laws mentioned in this section are well-known properties of regular expressions and can be proved from axioms (i)–(iii).

3 The Theorem

Let b and r be well-founded elements of a regular algebra and assume

$$r; b \;\subseteq\; b; (b \cup r)^* \;\cup\; r\,. \tag{3.1}$$

Then $b \cup r$ is well-founded.

4 Proof

Throughout the proof we shall pretend that b and r are sets of blue and red edges forming a graph, because that will provide valuable intuition. The calculations will, of course, be rigorous. Let a node be called *infinite* if it gives birth to an infinite path (of blue and red nodes). As we have argued before, the set of all infinite nodes is then represented by the expression

$$inf \;=\; (b \cup r)^{\infty}\,. \tag{4.1}$$

We are required to show that $inf = \emptyset$.

The proof is structured as follows. We color certain nodes red (4.2) and use the non-existence of infinite blue paths to show that, from every infinite node, there is a finite blue path to a red node (4.3). Then we use assumption (3.1) to show that from every red node there is a red edge to a red node. Since no infinite red paths exist, it then follows that there are no red nodes at all. And since we already know that every infinite node is connected to a red node, we conclude that no infinite nodes exist.

Let us color all nodes x red that satisfy the following two conditions:

- x is infinite;
- there is no blue edge from x to any infinite node.

In a formula,

$$red \;=\; inf \;\cap\; \neg \bigcirc_b inf\,. \tag{4.2}$$

Now assume x is an infinite node. Then let us follow blue edges to infinite nodes as long as possible. Since there is no infinite blue path (we assume that b is well-founded), we will get stuck eventually. In other words, we will arrive, via a blue path, at a red node. So we expect

$$inf \;\subseteq\; \Diamond_b red\,. \tag{4.3}$$

Now let us prove it. The only calculational way of exploiting the well-foundedness of b is by appealing to (2.6). This rule allows us to prove something of the form $x = \emptyset$, so we have to start by shunting $\Diamond_b red$ to the left hand side:

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$$
\begin{array}{ll}
 & inf \;\subseteq\; \Diamond_b red \\
\Leftrightarrow & \quad \{ \text{ Boolean algebra } \} \\
 & inf \cap \neg\Diamond_b red \;=\; \emptyset \\
\Leftrightarrow & \quad \{ \text{ Duality of } \Diamond \text{ and } \Box \text{ (2.11) } \} \\
 & inf \cap \Box_b \neg red \;=\; \emptyset \\
\Leftarrow & \quad \{ \; b \text{ is well-founded (2.6) and (2.7) } \} \\
 & inf \cap \Box_b \neg red \;\subseteq\; \bigcirc_b(inf \cap \Box_b \neg red) \;\; .
\end{array}
$$

This inequation is proved by the following calculation:

$$
\begin{array}{ll}
 & inf \cap \Box_b \neg red \\
= & \quad \{ \text{ Expand } \Box_b \text{ (2.14) } \} \\
 & inf \cap \neg red \cap \widetilde{\bigcirc}_b \Box_b \neg red \\
= & \quad \{ \text{ Definition of } red \text{ (4.2), de Morgan } \} \\
 & inf \;\cap\; (\neg inf \cup \bigcirc_b inf) \;\cap\; \widetilde{\bigcirc}_b \Box_b \neg red \\
\subseteq & \quad \{ \text{ Complement rule of Boolean algebra } \} \\
 & \bigcirc_b inf \;\;\cap\;\; \widetilde{\bigcirc}_b \Box_b \neg red \\
\subseteq & \quad \{ \text{ Combine } \bigcirc \text{ and } \widetilde{\bigcirc} \text{ (2.9) } \} \\
 & \bigcirc_b(inf \cap \Box_b \neg red) \;\; .
\end{array}
$$

This completes the proof of (4.3). Our next goal is to show that every red node has an outgoing edge to a red node:

$$red \;\subseteq\; \bigcirc_r red\,. \tag{4.4}$$

So let n be a red node. Since red nodes are infinite, there is certainly an edge from n to some infinite node m_0, and since n is red, that edge must be red. By virtue of (4.3) we can find a blue path from m_0 to some red node m_k.

$$n \xrightarrow{r} m_0 \xrightarrow{b} \cdots \xrightarrow{b} m_k$$

Now if there were only one blue edge in this path (i.e. $k = 1$), then we could use assumption (3.1) and replace the path from n to m_k with one of the following

$$n \xrightarrow{b} \bullet \xrightarrow{(b \cup r)^*} m_k \qquad \text{or} \qquad n \xrightarrow{r} m_k$$

and the first alternative would be ruled out because n is a red node and therefore has no outgoing blue edge to any infinite node. Thus we would indeed obtain a red edge from n to m_k and m_k is again red. As it is, we have to do induction on the number of blue edges in the first picture. We start with the red edge from n to m_0, then we deduce the existence of a red edge from n to m_1 and so on. In this way, we can strengthen assumption (3.1) to

$$r; b^* \;\subseteq\; b; (b \cup r)^* \cup r\,. \tag{4.5}$$

We will now prove that (4.5) is, indeed, a consequence of (3.1).

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$$
\begin{array}{ll}
 & r;b^* \;\subseteq\; b;(b\cup r)^* \,\cup\, r \\
\Leftrightarrow & \quad \{ \text{ Tail recursion lemma (2.3) } \} \\
 & \mu(x \;\mapsto\; r \,\cup\, x;b) \;\subseteq\; b;(b\cup r)^* \,\cup\, r \\
\Leftarrow & \quad \{ \text{ Induction rule (2.1) } \} \\
 & r \,\cup\, (b;(b\cup r)^* \,\cup\, r)\,;b \;\subseteq\; b;(b\cup r)^* \,\cup\, r \\
\Leftrightarrow & \quad \{ \text{ Simplify, using distributivity and } (b\cup r)^*;b \subseteq (b\cup r)^* \} \\
 & r;b \;\subseteq\; b;(b\cup r)^* \,\cup\, r \;\; .
\end{array}
$$

Since this is precisely assumption (3.1), we have established claim (4.5). In order to rewrite (4.5) in terms of the temporal operators we multiply both sides with x (from the right) and distribute. This yields

$$\bigcirc_r \Diamond_b x \;\subseteq\; \bigcirc_b \Diamond_{b\cup r} x \,\cup\, \bigcirc_r x\,. \tag{4.6}$$

We now return to our goal of proving (4.4), that every red node has an outgoing edge to a red node.

$$
\begin{array}{ll}
 & \mathit{red} \\
= & \quad \{ \text{ def. of } \mathit{red} \text{ (4.2) } \} \\
 & \mathit{inf} \,\cap\, \neg\bigcirc_b \mathit{inf} \\
= & \quad \{ \; \mathit{inf} = \bigcirc_{b\cup r} \mathit{inf} \text{ by (4.1) } \} \\
 & \bigcirc_{b\cup r} \mathit{inf} \,\cap\, \neg\bigcirc_b \mathit{inf} \\
= & \quad \{ \text{ Distributivity } \} \\
 & (\bigcirc_b \mathit{inf} \,\cup\, \bigcirc_r \mathit{inf}\,) \;\cap\; \neg\bigcirc_b \mathit{inf} \\
= & \quad \{ \text{ Complement rule of Boolean algebra } \} \\
 & \bigcirc_r \mathit{inf} \,\cap\, \neg\bigcirc_b \mathit{inf} \\
\subseteq & \quad \{ \; \mathit{inf} \subseteq \Diamond_b \mathit{red} \text{ by (4.3) } \} \\
 & \bigcirc_r \Diamond_b \mathit{red} \,\cap\, \neg\bigcirc_b \mathit{inf} \\
\subseteq & \quad \{ \text{ (4.6) } \} \\
 & (\bigcirc_r \mathit{red} \,\cup\, \bigcirc_b \Diamond_{b\cup r} \mathit{red}) \;\cap\; \neg\bigcirc_b \mathit{inf} \\
\subseteq & \quad \{ \; \mathit{red} \subseteq \mathit{inf} \text{ by (4.2) } \} \\
 & (\bigcirc_r \mathit{red} \,\cup\, \bigcirc_b \Diamond_{b\cup r} \mathit{inf}\,) \;\cap\; \neg\bigcirc_b \mathit{inf} \\
= & \quad \{ \; \Diamond_{b\cup r} \mathit{inf} = \mathit{inf} \text{ by (4.1) and (2.13) } \} \\
 & (\bigcirc_r \mathit{red} \,\cup\, \bigcirc_b \mathit{inf}\,) \;\cap\; \neg\bigcirc_b \mathit{inf} \\
\subseteq & \quad \{ \text{ Complement Rule of Boolean algebra } \} \\
 & \bigcirc_r \mathit{red} \;\; .
\end{array}
$$

Now the well-foundedness of r implies $\mathit{red} = \emptyset$ (we are using the characterization (2.6) here with r in place of b). Since $\Diamond_b \emptyset = b^*;\emptyset = \emptyset$, it follows from (4.3) that $\mathit{inf} = \Diamond_b \mathit{red} = \emptyset$. The proof is now complete.

5 Final Remarks

Our presentation may suggest that the theorem was discovered by looking at paths through graphs and that the calculational proof was added as an afterthought. However, the converse is true. We started by trying to re-prove a known theorem (that transitivity of the union is a sufficient condition) in the relational calculus. It took

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us a long time, but when we finally found it, it turned out to be a proof of our, stronger, result. Only after the theorem was discovered, we searched for a proof in terms of paths, which we then used to restructure and improve the presentation of the calculation. The morale of this little story is that the relational (or regular) calculus is not only useful for concisely presenting and proving known facts but also actively helps with discovering new theorems.

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