Coordination Games on Weighted Directed Graphs

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Abstract

We study strategic games on weighted directed graphs, in which the payoff of a player is defined as the sum of the weights on the edges from players who chose the same strategy, augmented by a fixed non-negative integer bonus for picking a given strategy. These games capture the idea of coordination in the absence of globally common strategies.

We identify natural classes of graphs for which finite improvement or coalition-improvement paths of polynomial length always exist, and, as a consequence, a (pure) Nash equilibrium or a strong equilibrium can be found in polynomial time.

The considered classes of graphs are typical in network topologies: simple cycles correspond to the token ring local area networks, while open chains of simple cycles correspond to multiple independent rings topology from the recommendation G.8032v2 on the Ethernet ring protection switching. For simple cycles these results are optimal in the sense that without the imposed conditions on the weights and bonuses a Nash equilibrium may not even exist.

Finally, we prove that the problem of determining the existence of a Nash equilibrium or of a strong equilibrium in these games is NP-complete already for unweighted graphs and with no bonuses assumed. This implies that the same problems for polymatrix games are strongly NP-hard.

1 Introduction

1.1 Background

Nash equilibrium is a natural solution concept in game theory which has been widely used to reason about strategic interaction between rational agents. Although Nash's theorem guarantees existence of a mixed strategy Nash equilibrium for all finite games, pure strategy Nash equilibria need not always exist. In various games, for instance Cournot competition games or congestion games, pure Nash equilibria (from now, just Nash equilibria) do exist and correspond to natural outcomes.

In many scenarios of strategic interaction, apart from the question of the existence of Nash equilibria, an important concern is whether an equilibrium can be efficiently computed. In this context the concept of an *improvement path* is relevant. These paths are maximal paths constructed by starting at an arbitrary joint strategy and allowing at each stage

a single player who does not hold a best response to switch to a better strategy. By definition, every finite improvement path terminates in a Nash equilibrium.

In a seminal paper [38] identified the class of finite games in which every improvement path is guaranteed to be finite, which was coined as the *finite improvement property* (FIP). These are the games with which one can associate a *generalised ordinal potential*, a function on the set of joint strategies that properly tracks the qualitative change in players' payoffs resulting from a strategy change. Thus the FIP not only guarantees the existence of Nash equilibria but also ensures that it is possible to reach it from any initial joint strategy by a simple update dynamics amounting to a *local search*. This makes the FIP a desirable property. An important class of games that have the FIP are the *congestion games* that, as already noted in [43], actually have an *exact potential*, a function that exactly tracks the quantitative difference in players' payoffs.

However, the requirement that *every* improvement path is finite is very strong and only few classes of games have this property. [50] proposed a weakening of the FIP that stipulates that from any initial joint strategy only *some* improvement path is finite. Games for which this property holds are called *weakly acyclic games*. So in weakly acyclic games Nash equilibria can be reached through an appropriately chosen sequence of unilateral deviations of players, irrespective of the starting joint strategy.

Although the existence of a finite improvement path guarantees the existence of a Nash equilibrium, it does not necessarily result in an efficient algorithm to compute it. In fact, in various games improvement paths can be exponentially long. [23] showed that computing a Nash equilibrium in congestion games is PLS-complete. Even in the class of symmetric network congestion games, for which it is known that a Nash equilibrium can be efficiently computed [23], there are games in which some best response improvement paths are exponentially long [1]. Thus identifying natural classes of games in which starting from any joint strategy a Nash equilibrium can be reached by an efficiently generated improvement path of a polynomial length is of obvious interest.

1.2 Motivation

In game theory, coordination games are often used to model situations of cooperation, where players can increase their payoffs by coordinating on certain strategies. For two player games, this implies that coordinating strategies constitute Nash equilibria. The main characteristic of coordination is that players find it advantageous that other players follow their choice. In this paper, we study a simple class of multi-player coordination games, in which each player can choose to coordinate his actions within a certain neighbourhood. The neighbourhood structure is specified by a weighted directed graph the nodes of which are identified with the players. Each player has to his disposal a private set of colours that are his strategies.

The payoff of a player is defined as the sum of the weights of the edges from players who choose the same colour plus a fixed bonus for picking this particular colour. These games constitute a natural class of strategic games, which capture the following four key characteristics (see [2]):

- *Join the crowd property:* the payoff of each player weakly increases when more players choose his strategy (this is because the weights are assumed to be positive).

- *Local dependency:* the payoff of each player depends only on the choices made by a certain group of players (namely the neighbours in the given weighted directed graph).

- *Asymmetric strategy sets:* players may have different strategy sets.

- *Individual preferences:* the (positive) bonuses express players' private preferences.

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We call these games *weighted coordination games on graphs*, in short, just **coordination games**. In a companion paper, [2], to which we shall return briefly, we studied the same model of coordination but on undirected and unweighted graphs, and without bonuses. The use of directed graphs instead of undirected ones leads to completely different results. For instance, Nash equilibria do not need to exist then.

However, the motivation remains the same: to investigate ways of coordinating agents' actions when they can benefit from aligning their choices with other agents. Such circumstances arise in many natural situations, for instance when clients have to choose between multiple competing (for instance mobile phone) providers offering the same service and when it is beneficial to choose the same provider as friends or relatives did. Note that in this example the join the crowd and local dependency properties clearly hold. The asymmetry naturally arises when in different regions different services or products are offered.

The main focus of our study is the analysis for which combinations of (weighted) graphs and bonuses Nash equilibria exist and in case they do whether they can be efficiently computed. In coordination games on graphs each player tries to coordinate his choice with the group of players who form his neighbourhood. Consequently, it is natural to consider, in addition to Nash equilibria, an equilibrium concept that takes into account deviations of groups of players. This motivates our study of the existence of strong equilibria. Recall that in a strong equilibrium no coalition of players can profitably deviate, in the sense that every player of the coalition strictly improves his payoff. In analogy to the case of Nash equilibria we also study whether strong equilibria can be efficiently computed by means of improvement paths in which at each stage all players in a group can profitably deviate. We call such paths *coalitional improvement paths*, in short *c-improvement paths*.

1.3   Related work

The class of games that have the FIP, introduced in [38], was a subject extensive research. Prominent examples of such games are congestion games. Weakly acyclic games have received less attention, but the interest in them is growing. [36] showed that although congestion games with player specific payoff functions do not have the FIP, they are weakly acyclic. [15] improved upon this result by showing that a specific scheduling of players is sufficient to construct a finite improvement path beginning at an arbitrary starting point. According to this scheduling the players are free to choose their best response when updating their strategies.

Weak acyclicity of a game also ensures that certain modifications of the traditional no-regret algorithm yield an almost sure convergence to a Nash equilibrium [35]. [18] and more fully [19] showed that specific Internet routing games are weakly acyclic. In turn, [32] established that certain classes of network creation games are weakly acyclic and moreover that a specific scheduling of players can ensure that the resulting improvement path converges to a Nash equilibrium in $\mathcal{O}(n \log n)$ steps.

Some structural results also exist. [21] proved that the existence of a unique Nash equilibrium in every subgame implies that the game is weakly acyclic. A comprehensive classification of weakly acyclic games in terms of schedulers is provided in [6] and more extensively in [7], where it was also shown that games solvable by means of iterated elimination of never best responses to pure strategies are weakly acyclic. Finally, [37] provided a characterization of weakly acyclic games in terms of a weak potential and showed that every finite extensive form game with perfect information is weakly acyclic.

Strategic games we study here are related to various well-studied classes of games. In particular, coordination games on graphs form a natural subclass of *polymatrix games* [49]. These are multi-player games where the players' utilities are pairwise separable. Polymatrix games are well studied and they include classes of strategic form games with good computational properties like the two-player zero-sum games. [30] studied clustering games that are also polymatrix games based on undirected graphs. In this setup each player has the same set of strategies and as a result

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these games have, in contrast to ours, the FIP. A special class of polymatrix games was considered in [16], which coincide with the coordination games on undirected weighted graphs without bonuses. The authors showed that these games have an exact potential and that finding a pure Nash equilibrium is PLS-complete. However, the proof of the latter result crucially exploits the fact that the edge weights can be negative (which captures anti-coordination behaviour). In [5] it was shown how coordination and anti-coordination on simple cycles can be used to model and reason about the concept of self-stabilization introduced in [17] one of the main approaches to fault-tolerant computing.

When the graph is undirected and complete, coordination games on graphs are special cases of the monotone increasing congestion games that were studied in [44]. Further, when we omit bonuses, our coordination games become special cases of the *social network games* introduced and analysed in [45] provided one allows in them thresholds equal to 0. (Thresholds represent there the prices of the products and can be viewed as negative bonuses.) These games were associated with a threshold model of a social network introduced in [3] and based on weighted graphs with thresholds.

This paper can be seen as the companion paper of [4], and its full version [2], in which we studied the same model of coordination games but on undirected graphs. However, the use of undirected graphs instead of directed ones leads to completely different results. For example, the resulting games have the FIP and hence Nash equlibria always exist, in contrast to the directed case considered here.

A follow up on [4] was [42] in which polymatrix coordination games with bonuses (called *individual preferences*) were studied, focussing on α-approximate k-equilibria, which are the outcomes in which no group of at most k players can deviate in such a way that each member increases his payoff by at least a factor α. The players are nodes in undirected graphs. In these games, as in [4], Nash equilibria always exist.

Another generalisation is to analyse distributed coalition formation [28] where players have preferences over members of the same coalition. Such a generalisation of polymatrix game over subsets of players, called hypergraphical games, was introduced in [40]. Analysis of coalition formation games in the presence of constraints on the number of coalitions that can be formed was investigated in [48]. [47] studied a subclass of hypergraphical games where the underlying group interactions are restricted to coordination and anti-coordination. In this model, players' utilities depend not just on the groups that are formed by the strategic interaction, but also on the choice of action that the members of the group decide to coordinate on. It is shown that such games have a Nash equilibrium, which can be computed in pseudo-polynomial time. Moreover, in the pure coordination setting, when the game possesses a certain acyclic structure, strong equilibria exist and can be computed in polynomial time.

Coordination games on graphs are also related to *additively separable hedonic games (ASHG)* [13, 14], which were originally proposed in a cooperative game theory setting. In these games players are the nodes of a weighted graph and can form coalitions. The payoff of a node is defined as the total weight of all edges to neighbors that are in the same coalition. The work on these games mostly focused on computational issues, see, e.g., [11, 12, 10, 24]. The PLS-hardness result established in [24] does not carry over to our coordination games because it makes use of the negative edge weights.

In [2] we also mentioned related work on strategic games that involve colouring of the vertices of an undirected graph, in relation to the vertex colouring problem. In these games the players are nodes in a graph that choose colours. However, the payoff function differs from the one we consider here: it is 0 if a neighbour chooses the same colour and the number of nodes that chose the same colour otherwise. The reason is that these games are motivated by the question of finding the chromatic number of a graph. Representative references are [39], where it is shown that an efficient local search algorithm can be used to compute a good vertex colouring and [20], where this work is extended by analysing socially optimal outcomes and strong equilibria. Further, strong and k-equilibria in strategic games on graphs were also studied in Gourvès and Monnot [25, 26]. These games are related to, respectively, the MAX−CUT and MAX−k−CUT problems. These classes of games do not satisfy the join the crowd property, so these results are not

comparable with ours.

1.4 Our contributions

In this paper we identify various natural classes of weighted directed graphs for which the resulting games, possibly with bonuses, are weakly acyclic. Moreover, we prove that in these games, starting from any arbitrary joint strategy, improvement paths of polynomial length can be effectively constructed. So not only do these games have Nash equilibria, but they can also be efficiently computed by a simple form of local search. Since coordination games on graphs are polymatrix games, our results identify natural classes of polymatrix games in which Nash equilibria are guaranteed to exist and can be computed efficiently.

We start by analysing coordination games on simple cycles. Even in this limited setting, improvement paths of infinite length may exist. However, we show that finite improvement paths always exist when at most two nodes have bonuses or at most two edges have weights. We also show that without these restrictions Nash equilibria may not exist, so these results are optimal. We then extend this setting to *open chains* of simple cycles, i.e., simple cycles that form a chain and show the existence of finite improvement paths.

Most of our constructions involve a common, though increasingly more complex, proof technique. In each case we identify an easy to compute scheduling of players that, combined with an appropriate updating of strategies, guarantees that starting from an arbitrary initial joint strategy eventually a Nash equilibrium is reached in a polynomial number of steps.

We also study strong equilibria. In the restricted case of a weighted directed acyclic graphs (DAGs) we show that strong equilibria can be found along every coalitional improvement path. We also show that when only two colours are used, the coordination games do not necessarily have the FIP, but both Nash and strong equilibria can always be reached starting from an arbitrary initial joint strategy by, respectively, an improvement or a c-improvement path.

To deal with simple cycles we show that any finite improvement path can be extended by just one profitable coalitional deviation to reach a strong equilibrium. This allows us to strengthen the results on the existence of Nash equilibria to the case of strong equilibria. We also prove the existence of strong equilibria when the graphs are are open chains of cycles. Finally, we show that in some coordination games strong equilibria exist but cannot be reached from some initial joint strategies by any c-improvement path.

Building upon these results we study the complexity of finding and of determining the existence of Nash equilibria and strong equilibria. In particular we show that strong equilibrium in a coordination game on a simple cycle can be computed in linear time. However, the Nash existence problem, even for games on unweighted graphs and without bonuses turns out to be NP-complete.

Table 1 summarises our main results concerning the complexity of finding Nash and strong equilibria. We list here respectively: the length of the shortest improvement paths from an arbitrary initial joint strategy, the complexity of finding a Nash equilibrium (abbreviated to NE), the length of the shortest c-improvement paths starting from an arbitrary initial joint strategy, and the complexity of finding a strong equilibrium (abbreviated to SE). Here n is the number of nodes, $|E|$ the number of edges, and l the number of colours. In the case of open chain of cycles, m denotes the number of simple cycles in the chain and v the number of nodes in a simple cycle.

Most, though not all, results of this paper were reported earlier in shortened versions, as two conference papers, [8] and [46]. Some of these results, notably on bounds on the length of (c-)improvement paths, were improved.

graph/bonus/colouring	improvement path	NE	c-impr. path	SE
weighted simple cycles with ≤ 1 node with bonuses	$2n - 1$ [Thm. 3]	$\mathcal{O}(nl)$ [Thm. 28]	$2n$ [Cor. 19(i)]	$\mathcal{O}(nl)$ [Thm. 28]
simple cycles with bonuses with ≤ 1 non-trivial weight	$3n - 1$ [Thm.5]	$\mathcal{O}(nl)$ [Thm. 28]	$3n$ [Cor. 19(ii)]	$\mathcal{O}(nl)$ [Thm. 28]
weighted simple cycles with > 2 nodes with bonuses	Nash equilibrium may not exist [Example 6]			
weighted simple cycles with 2 nodes with bonuses	$3n$ [Thm. 7]	$\mathcal{O}(nl)$ [Thm. 28]	$3n$ [Cor. 19(iii)]	$\mathcal{O}(nl)$ [Thm. 28]
simple cycles with bonuses and 2 non-trivial weights	$4n - 1$ [Thm. 9]	$\mathcal{O}(nl)$ [Thm. 28]	$4n$ [Cor. 19(iiii)]	$\mathcal{O}(nl)$ [Thm. 28]
open chains of cycles	$3vm^3$ [Thm. 14]	$\mathcal{O}(vm^3l)$ [Thm. 29]	$4vm^4$ [Thm. 21]	$\mathcal{O}(v^2m^5l)$ [Thm. 31]
weighted DAGs with bonuses	$n - 1$ [Thm. 17]	$\mathcal{O}(nl + |E|)$ [Thm. 32]	$n - 1$ [Thm. 17]	$\mathcal{O}(nl + |E|)$ [Thm. 32]
two colours	$2n$ [Thm. 22]	$\mathcal{O}(n + |E|)$ [Thm. 33]	$2n$ [Thm. 24]	$\mathcal{O}(n^2 + n|E|)$ [Thm. 33]

Table 1: Bounds on the length of the shortest improvement and c-improvement paths for a given class of graphs or colouring and on the complexity of finding NE and SE. All edges are unweighted and there are no bonuses unless stated otherwise.

1.5 Potential applications

Coordination games constitute a natural and well studied model that represents various practical situations. The class of games we study in this paper models an extension of the coordination concept to a network setting, where the network is represented as a weighted directed graph, and where common strategies are not guaranteed to exist while the payoffs functions take care of individual preferences.

The classes of graphs that we consider are frequently used as network topologies. For example, the token ring local area networks are organised in directed simple cycles, while the open chains of simple cycles are supported by the recommendation G.8032v2 on the Ethernet ring protection switching.[1]

The basic technique that we use to show finite convergence to Nash equilibria is based on finite improvement paths of polynomial length. The concept of an improvement path is fundamental in the study of games but it also can be used to explain and analyse various real world applications. One such example is the Border Gateway Protocol (BGP) the purpose of which is to assign routes to the nodes of the Internet and to use them for routing packets.

Over the years, there has been extensive research in the network communications literature on how stable routing states are achieved and maintained in BGP in spite of strategic concerns. [22] and independently [27] observed that the operation of the BGP can be viewed as a best response dynamics in a natural class of routing games and finite improvement paths that terminate in Nash equilibria essentially translate to stable routing states. Following this observation the already mentioned [19] presented a game theoretic analysis of routing on the Internet in presence of 'misbehaving players' or backup edges.

Finally, coordination games on graphs are also relevant to cluster analysis. Its main objective is to organise a set of naturally related objects into groups according to some similarity measure. When adopting the game-theoretic perspective one can view possible cluster names as strategies and a satisfactory clustering of the considered graph

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[1] see http://www.beldensolutions.com/en/Company/Press/PR103EN0609/index.phtml

as an equilibrium in the coordination game associated with the considered graph. Clustering from a game theoretic perspective (using evolutionary games) was among others applied to car and pedestrian detection in images, and face recognition, see [41]. This approach was shown to perform very well against the state of the art.

1.6 Structure of the paper

In the next section we recall the relevant game-theoretic concepts and the notions of (c-)improvement paths, Nash and strong equilibria on which we focus. In Section 3 we introduce the class of games which forms the subject of this paper. The technical presentation starts in Section 4 in which we analyse the games the underlying graphs of which are (possibly weighted) simple cycles. In Section 5 we study open chains of simples cycles.

Then, in Section 6 we consider the problem of the existence of strong equilibria. Next, in Section 7, we study the complexity of finding and of determining the existence of Nash equilibria and strong equilibria. We conclude by summarising in Section 8 the results and stating a natural open problem.

2 Preliminaries

Throughout the paper $n > 1$ denotes the number of players. A **strategic game** $\mathcal{G} = (S_1, \ldots, S_n, p_1, \ldots, p_n)$ for n players, consists of a non-empty set S_i of **strategies** and a **payoff function** $p_i : S_1 \times \cdots \times S_n \to \mathbb{R}$, for each player i. We denote $S_1 \times \cdots \times S_n$ by S, call each element $s \in S$ a **joint strategy** and abbreviate the sequence $(s_j)_{j \neq i}$ to s_{-i}. Occasionally we write (s_i, s_{-i}) instead of s. We call a strategy s_i of player i a **best response** to a joint strategy s_{-i} of his opponents if for all $s_i' \in S_i$, $p_i(s_i, s_{-i}) \geq p_i(s_i', s_{-i})$. A joint strategy s is called a **Nash equilibrium** if each s_i is a best response to s_{-i}.

Fix a strategic game $\mathcal{G}$. We say that $\mathcal{G}$ satisfies the **positive population monotonicity (in short PPM)**, see [33], if for all joint strategies s and players i, j, $p_i(s) \leq p_i(s_i, s_{-j})$. (Note that (s_i, s_{-j}) refers to the joint strategy in which player j chooses s_i.) So if player j switches to player i's strategy and the remaining players do not change their strategies, then i's payoff weakly increases.

Next, by a **polymatrix game**, see [49], we mean a game $(S_1, \ldots, S_n, p_1, \ldots, p_n)$ in which for all pairs of players i and j there exists a *partial* payoff function a^{ij} such that for any joint strategy $s = (s_1, \ldots, s_n)$, the payoff of player i is given by $p_i(s) := \sum_{j \neq i} a^{ij}(s_i, s_j)$. So polymatrix games are strategic games in which the influence of a strategy selected by a player on the payoff of another player is always the same, regardless of what strategies other players select.

We call a non-empty subset $K := \{k_1, \ldots, k_m\}$ of the set of players $N := \{1, \ldots, n\}$ a **coalition**. Given a joint strategy s we abbreviate the sequence $(s_{k_1}, \ldots, s_{k_m})$ of strategies to s_K and $S_{k_1} \times \cdots \times S_{k_m}$ to S_K. We occasionally write (s_K, s_{-K}) instead of s.

Given two joint strategies s' and s and a coalition K, we say that s' is a **deviation of the players in** K from s if $K = \{i \in N \mid s_i \neq s_i'\}$. We denote this by $s \xrightarrow{K} s'$ and drop K if it is a singleton. If in addition $p_i(s') > p_i(s)$ holds for all $i \in K$, we say that the deviation s' from s is **profitable** and say that $s \xrightarrow{K} s'$ is a **c-improvement step**. Further, we say that a coalition K **can profitably deviate from** s if there exists a profitable deviation of the players in K from s. Next, we call a joint strategy s a **k-equilibrium**, where $k \in \{1, \ldots, n\}$, if no coalition of at most k players can profitably deviate from s. Using this definition, a **Nash equilibrium** is a 1-equilibrium and a **strong equilibrium**, see [9], is an n-equilibrium.

A **coalitional improvement path**, in short a **c-improvement path**, is a maximal sequence $\rho = (s^1, s^2, \ldots)$ of joint strategies such that for every $k \geq 1$ there is a coalition K such that $s^k \xrightarrow{K} s^{k+1}$ is a profitable deviation of the players in

K. If ρ is finite then by $last(\rho)$ we denote the last element of the sequence. Clearly, if a c-improvement path is finite, its last element is a strong equilibrium.

We say that $\mathcal{G}$ has the ***finite c-improvement property*** (***c-FIP***) if every c-improvement path is finite. Further, we say that the function $P : S \to A$, where A is a set, is a ***generalised ordinal c-potential***, also called ***generalised strong potential***, for $\mathcal{G}$ (see [29, 31]) if for some strict partial ordering $(P(S), \succ)$ the fact that s' is a profitable deviation of the players in some coalition from s implies that $P(s') \succ P(s)$. If a finite game admits a generalised ordinal c-potential then it has the c-FIP. The converse also holds, see, e.g., [2].

We say that $\mathcal{G}$ is ***c-weakly acyclic*** if for every joint strategy there exists a finite c-improvement path that starts at it. Thus games that are c-weakly acyclic have a strong equilibrium. We call a c-improvement path an ***improvement path*** if each deviating coalition consists of one player. The notion of a game having the ***FIP*** or being ***weakly acyclic*** is then defined by referring to the improvement paths instead of c-improvement paths.

In this paper we are interested in finding 'short' improvement and c-improvement paths. This motivates the following concept. We say that a game ***ensures improvement paths of length*** X (where X can also be expressed using the $\mathcal{O}(\cdot)$ function) if for each joint strategy there exists an improvement path that starts at it and is of length (at most) X. We use an analogous notion for the c-improvement paths.

Further, by a ***schedule*** we mean a finite or infinite sequence, each element of which is a player. Let ϵ denote the empty sequence and $seq : i$ the finite sequence seq extended by i. Given an initial joint strategy s a schedule generates an (not necessarily unique) initial fragment of an improvement path defined inductively as follows:

$$path(s, \epsilon) := s,$$

$$path(s, seq : i) := \begin{cases} path(s, seq) & \text{if } i \text{ holds a best response in the last element of} \\ & path(s, seq), \\ path(s, seq) \to s' & \text{otherwise,} \end{cases}$$

where s' is the result of updating the strategy of player i in the last element of $path(s, seq)$ to a best response.

Sometimes we additionally specify how players update their strategies to best responses, but even then the generated improvement paths do not need to be unique. The process of selecting a strategy is always linear in the number of strategies. To show that a game ensures short improvement paths we provide in each case an appropriate schedule. Note that an infinite schedule can generate a finite improvement path, which is the case when the last element of $path(s, seq)$ is a Nash equilibrium.

In the proofs we always mention the bounds on the improvement paths but actually these are bounds on the relevant prefixes of the defined schedules, which are always longer or of the same length.

3   Coordination games on directed graphs

We now define the class of games we are interested in. Fix a finite set M of l colours. A ***weighted directed graph*** (G, w) is a pair, where $G = (V, E)$ is a graph without self loops and parallel edges over the set of vertices $V = \{1, \ldots, n\}$ and w is a function that associates with each edge $e \in E$ a positive weight w_e. We say that a weight is ***non-trivial*** if it is different than 1.

Further, we say that a node j is an *in-neighbour* (from now on a ***neighbour***) of the node i if there is an edge $j \to i$ in E. We denote by N_i the set of all neighbours of node i in the graph G. A ***colour assignment*** is a function $C : V \to 2^M$ which assigns to each node of G a non-empty set of colours.

We also introduce the concept of a ***bonus***, which is a function β that assigns to each node i and colour $c \in M$ a non-negative integer $\beta(i, c)$. When stating our results, bonuses are assumed to be not present (or equivalently are

assumed to be all equal to 0), unless explicitly stated otherwise. We say that a bonus is ***non-trivial*** if it is different from the constant function 0.

Given a weighted graph (G, w), a colour assignment C and a bonus function β a strategic game $\mathcal{G}(G, w, C, \beta)$ is defined as follows:

- the players are the nodes,

- the set of strategies of player (node) i is the set of colours $C(i)$; we occasionally refer to the strategies as ***colours***,

- the payoff function for player i is

$$p_i(s) = \sum_{j \in N_i, \, s_i = s_j} w_{j \to i} + \beta(i, s_i).$$

So each node simultaneously chooses a colour and the payoff to the node is the sum of the weights of the edges from its neighbours that chose its colour augmented by the bonus to the node for choosing its colour. We call these games ***coordination games on weighted directed graphs***, from now on just ***coordination games***.

Note that because the weights are non-negative each coordination game satisfies the PPM. When the weights of all the edges are 1, we are dealing with a coordination game whose underlying graph is unweighted. In this case, we simply drop the function w from the description of the game and drop the qualification 'unweighted' when referring to the graph.

Similarly, when all the bonuses are 0, we obtain a coordination game without bonuses. Likewise, in the description of such a game we omit the function β. In a coordination game without bonuses when the underlying graph is unweighted, each payoff function is simply defined by

$$p_i(s) := |\{j \in N_i \mid s_i = s_j\}|.$$

Some results hold only for the games without bonuses but to prove them we still need bonuses, to take care of the induction step. The reason is that positive bonuses can model incoming edges from fixed colour source nodes, i.e., nodes with no incoming edges that have only one colour available to them.

Example 1. Consider the directed graph and the colour assignment depicted in Figure 1.

Take in the corresponding coordination game the joint strategy s that consists of the underlined colours. Then the payoffs are as follows:

- 0 for the nodes 1, 7, 8 and 9,

- 1 for the nodes 2, 4, 5, 6,

- 2 for the node 3.

Note that the above joint strategy is not a Nash equilibrium. In fact, this game has no Nash equilibrium. To see this observe that we only need to consider the strategies selected by the nodes 1, 2 and 3, since each of the nodes 4, 5 and 6 always plays a best response by selecting the strategy of its only predecessor and each of the nodes 7, 8, and 9 has just one strategy.

We now list all joint strategies for the nodes 1, 2 and 3 and in each of them underline a strategy that is not a best response to the choice of the other players: $(\underline{a}, a, b)$, $(a, a, \underline{c})$, $(a, c, \underline{b})$, $(a, \underline{c}, c)$, $(b, \underline{a}, b)$, $(\underline{b}, a, c)$, $(b, c, \underline{b})$ and $(\underline{b}, c, c)$.

In the above game edges were unweighted and no bonuses were used. In Example 6 we exhibit a much simpler graph with non-trivial weights bonuses admitted for which no Nash equilibrium exists. $\qquad\square$
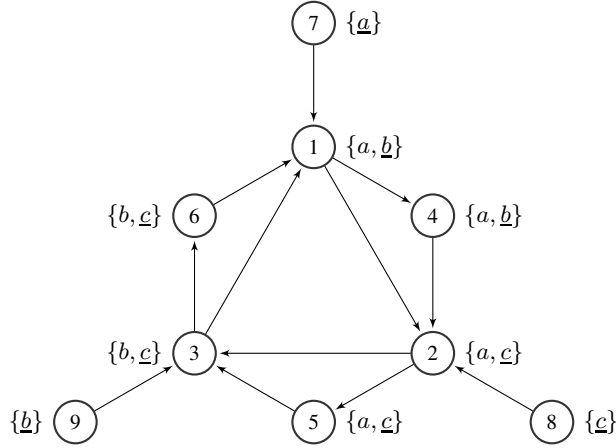
Figure 1: A coordination game with a selected joint strategy.

The above example of course rises several questions, for instance when a Nash equilibrium does exist, is the above example minimal in the number of colours, do there exist coordination games that have a Nash equilibrium but are not weakly acyclic, how difficult is to determine whether a Nash equilibrium exists, etc. We shall address these and other questions in the rest of the paper.

4   Simple cycles

Given that some of the coordination games have no Nash equilibria we consider special graph structures. In this section we focus on simple cycles. To fix the notation, suppose that the considered directed graph is $1 \to 2 \to \ldots \to n \to 1$. We begin with the following simple example showing that the coordinations games on a simple cycle do not have the FIP. Here and elsewhere, to increase readability, when presenting profitable deviations we underline the strategies that were modified.

Example 2. Suppose $n \geq 3$. Consider a coordination game on a simple cycle such that the nodes share at least two colours, say a and b. Take the joint strategy $(a, b, \ldots, b)$. Then both $(a, \underline{b}, b, \ldots, b) \to (a, a, b, \ldots, b)$ and $(\underline{a}, a, b, \ldots, b) \to (b, a, b, \ldots, b)$ are profitable deviations. After these two steps we obtain a joint strategy $(b, a, b, \ldots, b)$ that is a rotation of the initial one. Iterating we obtain an infinite improvement path. □

On the other hand a weaker result holds.

Theorem 3. *Every coordination game on a weighted simple cycle in which at most one node has bonuses ensures improvement paths of length $\leq 2n - 1$.*

Proof. First, assume that no node has bonuses. Fix an initial joint strategy. We construct the desired improvement path by scheduling the players clockwise, starting with player 1. We prove that after at most two rounds we reach a Nash equilibrium.

Phase 1. This phase lasts at most $n - 1$ steps. Each time we select a player who does not hold a best response and update his strategy to a best response. Such a modification affects only the payoff of the successor player, so after we considered player $n - 1$, in the current joint strategy s each of the players $1, 2, \ldots, n - 1$ holds a best response.

If at this moment the current strategy of player n is also a best response, then s is a Nash equilibrium and the improvement path terminates. Otherwise we move to the next phase.

Phase 2. We repeat the same process as in Phase 1, but starting with s and player n.

By the definition of the game the property that at least $n - 1$ players hold a best response continues to hold for all consecutive joint strategies and a Nash equilibrium is reached when the selected player holds a best response.

Suppose player n switches to a strategy c. Recall that $C(i)$ is the set of colours available to player i. Let

$$
n_0 := \begin{cases} n - 1 & \text{if } \forall i \in \{1, \ldots, n - 1\} : c \in C(i) \text{ and } s_i \neq c \\ \min\{i \in \{1, \ldots, n - 1\} \mid c \notin C(i) \text{ or } s_i = c\} - 1 & \text{otherwise.} \end{cases}
$$

The improvement path terminates after the players $1, \ldots, n_0$ successively switched to c as at this moment player $n_0 + 1$ holds a best response.

Suppose now that a node has bonuses. Then we rename the nodes so that this is node n. Then the argument used in reasoning about Phase 2 remains correct. $\square$

As a side remark, note that the renaming of the players used at the end of the above proof is necessary as otherwise the used schedule can generate improvement paths that are longer than $2n - 1$.

Example 4. Suppose that $n \geq 5$ and that the simple cycle is unweighted. Assume that there are four colours a, b, c, d and consider the following colour assignment:

$$
C(1) = \ldots = C(n - 3) = C(n) = \{a, b, c, d\}, \ C(n - 2) = \{a, \overline{c}\}, C(n - 1) = \{c, d\},
$$

where the overline indicates the only positive bonus in the game.

Consider now the joint strategy $(b, \ldots, b, a, d, a)$. If we follow the clockwise schedule starting with player 1, there is only one improvement path, namely

$$
(\underline{b}, \ldots, b, a, d, a) \to^* (a, \ldots, a, a, d, \underline{a}) \to
$$
$$
(\underline{a}, \ldots, a, a, d, d) \to^* (d, \ldots, d, \underline{a}, d, d) \to (d, \ldots, d, c, \underline{d}, d) \to (d, \ldots, d, c, c, \underline{d}) \to
$$
$$
(\underline{d}, \ldots, d, c, c, c) \to^* (c, \ldots, c, c, c, c).
$$

In each joint strategy we underlined the strategy of the scheduled player from which he profitably deviates and each $\to^*$ refers to a sequence of $n - 3$ profitable deviations. So this improvement path is of length $3n - 5$ and thus longer than $2n - 1$ since $n \geq 5$. $\square$

Further, the following result holds.

Theorem 5. *Every coordination game with bonuses on a simple cycle in which at most one edge has a non-trivial weight ensures improvement paths of length $\leq 3n - 1$.*

Proof. We first assume that no edge has a non-trivial weight. As in the proof of Theorem 3 we schedule the players clockwise starting with player 1. However, we are now more specific about the strategies to which the players switch.

Let $MB(i)$ be the set of available colours to player i with the maximal bonus, i.e.,

$$MB(i) := \{c \in C(i) \mid \beta(i, c) = \max_{d \in C(i)} \beta(i, d)\}.$$

Below we stipulate that whenever the selected player i updates his strategy to a best response he always selects a strategy from $MB(i)$. Note that this is always possible, since the bonuses are non-negative integers. Indeed, suppose that the strategy of player's i predecessor is c. If $c \in MB(i)$, then player i selects c and otherwise he can select an arbitrary strategy from $MB(i)$.

Fix an initial joint strategy.

Phase 1. This phase is the same as in the proof of Theorem 3, except the above proviso. So when this phase ends the players $1, \ldots, n-1$ hold a best response. If at this moment the current joint strategy s is a Nash equilibrium, the improvement path terminates. Otherwise we move to the next phase.

Phase 2. We repeat the same process as in Phase 1, but starting with s and player n and proceeding at most n steps. From now on at each step at least $n-1$ players have a best response strategy. So if at a certain moment the scheduled player holds a best response, the improvement path terminates. Otherwise, the players $n, 1, \ldots, n-1$ successively update their strategies and after n steps we move to the final phase.

Phase 3. We repeat the same process as in Phase 2, again starting with player n. In the previous phase each player updated his strategy, so now in the initial joint strategy each player i holds a strategy from $MB(i)$. Hence each player can improve his payoff only if he switches to the strategy selected by his predecessor that also has the maximal bonus. Let c be the strategy to which player n switches and let

$$n_0 := \begin{cases} n-1 & \text{if } \forall i \in \{1, \ldots, n-1\} : c \in MB(i) \text{ and } s_i \neq c \\ \min\{i \in \{1, \ldots, n-1\} \mid c \notin MB(i) \text{ or } s_i = c\} - 1 & \text{otherwise.} \end{cases}$$

The improvement path terminates after the players $1, \ldots, n_0$ successively switched to c as at this moment player $n_0 + 1$ holds a best response.

If some edge has a non-trivial weight then we rename the players so that this edge is into the node n. Notice that we cannot now require that player n selects a best response from $MB(n)$, since the colour of his predecessor can yield a higher payoff due to the presence of the weight. So we drop this requirement for node n but maintain it for the other nodes.

Then at the beginning of Phase 3 we can only claim that each player $i \neq n$ holds a strategy from $MB(i)$, but this is sufficient for the remainder of the proof. $\square$

We would like to generalise the above two results to coordination games with bonuses on arbitrary weighted simple cycles. However, if we allow in a simple cycle non-trivial weights on three edges and associate bonuses with three nodes then some coordination games have no Nash equilibrium.

Example 6. Consider the weighted simple cycle and the colour assignment depicted in Figure 2, where the overlined colours have bonus 1.

The resulting coordination game does not have a Nash equilibrium. The list of joint strategies, each of them with an underlined strategy that is not a best response to the choice of other players, is the same as in Example 1: $(\underline{a}, a, b)$, $(a, a, \underline{c})$, $(a, c, \underline{b})$, $(a, \underline{c}, c)$, $(b, \underline{a}, b)$, $(\underline{b}, a, c)$, $(b, c, \underline{b})$ and $(\underline{b}, c, c)$. In fact, the game considered in that example simulates this game. $\square$
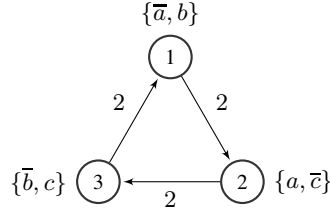
Figure 2: A coordination game without a Nash equilibrium

In what follows we show that this counterexample is minimal in the sense that if in a weighted simple cycle with bonuses at most two nodes have bonuses or at most two edges have non-trivial weights, then the coordination game has a Nash equilibrium. More precisely, we establish the following two results.

Theorem 7. *Every coordination game on a weighted simple cycle in which two nodes have bonuses ensures improvement paths of length $\leq 3n$.*

Proof. Relabel the nodes if necessary so that one of the nodes which has bonuses is node 1. Let k be the second node that has bonuses. Fix an initial joint strategy. We schedule, as before, the players clockwise, starting with player 1.

Phase 1. This phase lasts at most n steps. We repeatedly select the first player who does not hold a best response and update his strategy to a best response. A best response can be either the colour of the predecessor or, in the case of nodes 1 and k only, a colour with the maximal bonus. In case of equal payoffs of these two options we give a preference to the former. As in the previous proofs, a strategy update of a given node, affects only the payoff of the successor node. If at the end of Phase 1 the current strategy of player 1 is also a best response, then we reached a Nash equilibrium and the improvement path terminates. Otherwise we move on to the next phase.

Phase 2. In this phase we perform at most two rounds of clockwise updates of all the nodes, starting at player 1. We explicitly distinguish ten scenarios, which are defined as follows. (They also play important role in the proof of Theorem 14 in Section 5.) We focus on two types of strategy updates by the nodes with bonuses:

- an update to an inner colour (recorded as *i*), i.e., the colour of its predecessor, or

- an update to an outer colour (recorded as *o*), i.e., one of the colours with a maximal bonus.

If a colour is both inner and outer, then we record it as *i*. An *update scenario* is now a sequence of recordings of consecutive updates by the nodes with bonuses that is generated during the above two phases.

One possible update scenario is *(iooi)*, which takes place when player 1 first adopts the colour of its predecessor (*i*) and this colour then propagates until player k is reached. At this point player k adopts a different colour with the maximal bonus (*o*), and this colour propagates further until player 1 is reached again. Player 1 then adopts a different colour with the maximal bonus (*o*) which then propagates and is also adopted by player k (*i*). This propagation stops at a node j lying between the nodes k and 1. At this point a Nash equilibrium is reached because player j holds a best response and hence all players hold a best response.

In general, an update scenario has to stop after an *oi* or *ii* is recorded, because then the same colour is propagated throughout the whole cycle and no new colour is introduced. Moreover, an update string cannot contain *ooo* as a subsequence, because then the third update to an outer colour would yields the same payoff as the first one, so it

cannot be improving the payoff. It is now easy to enumerate all update scenarios satisfying these two constraints and these are as follows:

$$(o), (oi), (oo), (ooi), (i), (ii), (io), (ioi), (ioo), (iooi).$$

The only one of length 4 is the already considered update scenario *(iooi)*, which yields the longest sequence of profitable deviations in Phase 2, which is $2n$. □

Now consider coordination games on simple cycles with bonuses in which two edges have non-trivial weights. The following example shows that if we follow the clockwise schedule starting with player 1, then the bound $3n$ given by Theorem 7 does not need to hold.

Example 8. Suppose that $n \geq 5$, the weights of the edges $n - 3 \to n - 2$ and $n - 1 \to n$ are 2 and the weights of the other edges are 1. Let $C = \{a, b, c, d, e, f, g, h, i\}$. Define the colour and the bonus assignment as follows:

$$\begin{aligned}
C(1) &= C \setminus \{e\};\ \overline{f}, \overline{g}, \overline{i}, \\
C(2) &= C \setminus \{d\};\ \overline{e}, \overline{f}, \overline{g}, \overline{i}, \\
C(3) &= \ldots = C(n-3) = C, \\
C(n-2) &= C \setminus \{g, i\};\ \overline{h}, \\
C(n-1) &= C \setminus \{f\};\ \overline{g}, \overline{h}, \\
C(n) &= C \setminus \{h\};\ \overline{i},
\end{aligned}$$

where the overlined colours have bonus 1.

Consider now the joint strategy $(a, b, \ldots, b, c, c, d)$. If we follow the clockwise schedule starting at player 1, we can generate the following improvement path in which each player $i \neq n - 2, n$ always switches to a colour from $MB(i)$ (we cannot require it from players $n - 2$ and n because the weights equal 2):

$$\begin{aligned}
&(\underline{a}, b, \ldots, b, c, c, d) \to (d, \underline{b}, \ldots, b, c, c, d) \to (d, \overline{e}, \underline{b}, \ldots, b, c, c, d) \to (d, e, e, \underline{b}, \ldots, b, c, c, d) \to^* \\
&(\underline{d}, e, \ldots, e, e, e) \to (\overline{f}, \underline{e}, \ldots, e, e, e) \to^* (f, \ldots, f, \underline{e}, e) \to (f, \ldots, f, \overline{g}, \underline{e}) \to \\
&(\underline{f}, \ldots, f, g, g) \to^* (g, \ldots, g, \underline{f}, g, g) \to (g, \ldots, g, \overline{h}, \underline{g}, g) \to (g, \ldots, g, h, h, \underline{g}) \to \\
&(\underline{g}, \ldots, g, h, h, \overline{i}) \to^* (i, \ldots, i, h, h, i).
\end{aligned}$$

In each joint strategy we underlined the strategy of the scheduled player from which he profitably deviates and overlined the first occurrences of the newly introduced strategies. Each $\to^*$ refers to a sequence of $n - 3$ profitable deviations. So this improvement path is of length $4n - 3 > 3n - 1$. □

However, a slightly larger bound can be established.

Theorem 9. *Every coordination game on a simple cycle with bonuses in which two edges have non-trivial weights ensures improvement paths of length $\leq 4n - 1$.*

Proof. Rename the nodes so that the edges with a non-trivial weight are into the nodes k and n. We stipulate that each player $i \neq k, n$ always selects a best response from the set $MB(i)$ of available colours to player i with the maximal bonus. This is always possible for the reasons given in the proof of Theorem 5. As in the earlier proofs we construct the desired improvement path by scheduling the players clockwise, starting with player 1.

Phase 1. This phase lasts at most $2n - 1$ steps. If this way we do not reach a Nash equilibrium we move to the next phase.

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Phase 2. In this phase we continue the clockwise strategy updates for all the nodes starting with player n. We show that this can continue for at most two rounds.

In the second round of the previous phase each player $i \neq n$ updated his strategy, so at the beginning of this phase each player $i \neq k, n$ holds a strategy from $MB(i)$.

We focus on the strategy updates by the nodes k and n. To this end we reuse the reasoning used in the proof of Theorem 7 that involves the analysis of the update scenarios. So, as before, we distinguish between the updates of the nodes k and n to an inner colour (recorded as *i*) or to an outer colour (recorded as *o*) and consider the resulting update scenarios, so sequences of *i* and *o*.

For the same reasons as before an update scenario has to stop after an *oi* or *ii* is recorded, and it cannot contain contain *ooo* as a subsequence, as also here updates of a node to an outer colour yield the same payoff.

Therefore the same argument shows that the longest possible sequence of updates in this phase is $2n$. $\qquad\square$

5   Open chains of simple cycles

In this section we study directed graphs which consist of an open chain of $m \geq 2$ simple cycles. For simplicity, we assume that all cycles have the same number of nodes denoted by v. The results we show hold for arbitrary cycles as long as each cycle has at least 3 nodes. Formally, for $j \in \{1, 2, \ldots, m\}$, let $\mathcal{C}_j$ be the cycle $[j, 1] \to [j, 2] \to \ldots \to [j, v] \to [j, 1]$. An *open chain of cycles* $\mathcal{C}_1, \ldots, \mathcal{C}_m$ is a directed graph in which for all $j \in \{1, \ldots, m-1\}$ we have $[j, 1] = [j + 1, k]$ for some $k \in \{2, \ldots, v\}$. In other words, it consists of a sequence of m cycles such that any two consecutive cycles have exactly one node in common.

Any node that connects two cycles is called a *link node*. The node that connects $\mathcal{C}_j$ with $\mathcal{C}_{j+1}$, so $[j, 1]$, which is also $[j + 1, k]$, is called an *up-link node* in $\mathcal{C}_j$ and, at the same time, a *down-link node* in $\mathcal{C}_{j+1}$. The total number of nodes in such a graph is $n = vm - (m - 1)$. Figure 3 depicts an example of an open chain.
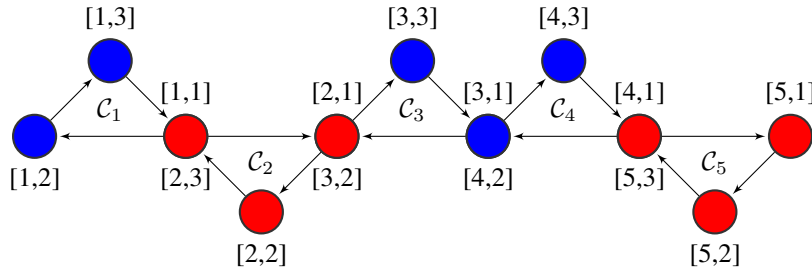

Figure 3: An open chain consisting of five cycles. Four nodes have double labels as they are link nodes. Each node can select either red or blue. The colouring of the nodes is an example of a joint strategy.

Throughout this section we assume a fixed coordination game on an open chain of cycles $\mathcal{C}_1, \ldots, \mathcal{C}_m$. We prove that such a game ensures improvement paths of polynomial length. The main idea of our construction is to build an improvement path by composing in an appropriate way the improvement paths for the simple cycles that form the open chain.

This is possible since, given a joint strategy, each cycle in the open chain can be viewed as a single cycle with at most two bonuses for which we know that an improvement path of length at most $3v$ exists due to Theorems 3 and 7. This is because the only nodes that have indegree two are the link nodes and given a joint strategy the edge to a

link node u from another cycle can be regarded as a bonus of 1 for the colour of the predecessor of u in another cycle. More formally, for a given joint strategy s and a cycle $\mathcal{C}_j$, we define the bonus function $\beta_j^s(u, c)$ as follows:

$$\beta_j^s(u, c) := \begin{cases} 1 & \text{if } u \text{ is a link node and } c = s(v), \\ & \text{where the node } v \text{ belongs to } \mathcal{C}_{j-1} \text{ or to } \mathcal{C}_{j+1} \text{ and } v \to u \text{ is an edge} \\ 0 & \text{otherwise.} \end{cases}$$

Further, to each improvement path χ in the coordination game on $\mathcal{C}_j$ with the bonus function β_j^s there corresponds a unique initial segment $\bar{\chi}$ of an improvement path in the coordination game on the open chain $\mathcal{C}_1, \ldots, \mathcal{C}_m$.

The following observation will be useful a number of times.

Note 10. *Consider a coordination game on an open chain and a joint strategy. Each node with payoff ≥ 1 plays a best response. The same claim holds for the just introduced coordination games on a simple cycle with at most two nodes with bonuses.*

Proof. The claim needs to be checked only for the link nodes and payoff 1. Suppose that in a coordination game on an open chain a payoff for some link node v is 1. This means that the colours selected by its two predecessors are different.

In the case of a game on a simple cycle $\mathcal{C}_j$, by definition for each s and j precisely one colour per link node yields a bonus 1, while the other bonuses are 0. So if a payoff for a link node v is 1, then the colour of its predecessor in $\mathcal{C}_j$ and the unique colour with the bonus are different.

So in both cases if v switches to another colour its payoff cannot increase. $\qquad\square$

We claim that Algorithm 1 below finds an improvement path of polynomial length. It repeatedly tries to correct the cycle with the least index in which some node does not play a best response.

To express this procedure we use the constructions explained in the proofs of Theorems 3 and 7. Further, for a joint strategy s that is not a Nash equilibrium we denote by $NBR(s)$ the least $j \in \{1, \ldots, m\}$ such that some node in $\mathcal{C}_j$ does not play a best response in s. In the example given in Figure 3 we have $NBR(s) = 1$.

Algorithm 1:

Input: A coordination game on an open chain of cycles $\mathcal{C}_1, \ldots, \mathcal{C}_m$ and an initial joint strategy s_0.
Output: A finite improvement path starting at s_0.

1   $\rho := s_0$;
2   $s := last(\rho)$;
3   **while** *s is not a Nash equilibrium* **do**
4      $j := NBR(s)$;
5      $\hat{s} :=$ the restriction of s to the nodes of $\mathcal{C}_j$;
6      $\chi :=$ the improvement path constructed in the proof of Theorem 3 or 7 for the coordination game on $\mathcal{C}_j$ with the bonus function β_j^s, starting at $\hat{s}$;
7      $\rho := \rho\bar{\chi}$;
8      $s := last(\rho)$
9   **return** ρ.

The execution of this algorithm, when dealing with a cycle $\mathcal{C}_j$, may 'destabilise' some lower cycles, and hence may require going back and forth along the sequence of cycles. In other words, the value of j may fluctuate. However, we can identify the minimum value below which j cannot drop.

To see this we introduce the following notion. Given a joint strategy s we assign to every cycle $\mathcal{C}_j$ one out of five possible *grades*, U+, +, U−, −, and ?, as follows:

$$grade^s(\mathcal{C}_j) := \begin{cases} \text{U+} & \text{if all its nodes play their best response in } s \text{ and } s([j,v]) = s([j,1]) \\ \text{+} & \text{if all its nodes play their best response in } s \text{ and } s([j,v]) \neq s([j,1]) \\ \text{U−} & \text{if } [j,2] \text{ is the only node that does not play a best response in } s \\ & \text{and } s([j,v]) = s([j,1]) \\ \text{−} & \text{if } [j,2] \text{ is the only node that does not play a best response in } s \\ & \text{and } s([j,v]) \neq s([j,1]) \\ \text{?} & \text{otherwise.} \end{cases}$$

Thus the grade ? means that for some $k \neq 2$ the node $[j,k]$ does not play a best response in s.

The following observation clarifies the relevance of the grade U+ and is useful for the subsequent considerations.

Note 11. *Suppose that after line 4 of Algorithm 1 the grade of a cycle $\mathcal{C}_i$ given s is* U+ *and $j > i$. Then from that moment on $j > i$ remains true and the grade of $\mathcal{C}_i$ remains* U+.

Proof. During each **while** loop iteration j can drop at most by 1, so the grade of $\mathcal{C}_i$ could be modified only if eventually after line 4 $j = i+1$ holds. The initial grade U+ of $\mathcal{C}_i$ implies that initially the colours of the nodes $[i,1]$ and $[i,v]$ are the same, and consequently the payoff for the node $[i,1]$ is ≥ 1 and it remains so whenever its other predecessor, belonging to $\mathcal{C}_j$, switches to another colour.

But $[i,1]$ is also the down-link node $[j,k]$ of $\mathcal{C}_j$. Hence by Note 10 the improvement path constructed in line 6 does not modify the colour of $[j,k]$, i.e., of the node $[i,1]$. So the grade of $\mathcal{C}_i$ remains U+ and hence if the **while** loop does not terminate right away, j increases after line 4. $\square$

In other words, $i < NBR(s) \rightarrow grade^s(\mathcal{C}_i) = \text{U+}$ is an invariant of the **while** loop of the algorithm.

Further, let $grade(s)$ be the sequence of grades given s assigned to each cycle, i.e.,

$$grade(s) := (grade^s(\mathcal{C}_1), \ldots, grade^s(\mathcal{C}_m)).$$

For instance, $grade(s) = (−, \text{U+}, ?, +, \text{U+})$ for the game and joint strategy s presented in Figure 3.

Suppose that Algorithm 1 selects j in line 4. It then constructs in line 6 the improvement path that starts in $\hat{s}$ defined in line 5, for the coordination game with bonuses on the cycle $\mathcal{C}_j$, as described in the proofs of Theorems 3 or 7. We now explain how this can change $grade(s)$. Note that only the grades of $\mathcal{C}_j$ and its adjacent cycles $\mathcal{C}_{j-1}$ and $\mathcal{C}_{j+1}$ (if they exist) can be affected.

Lemma 12. *The improvement path constructed in line 6 of Algorithm 1 modifies the grades of $\mathcal{C}_j$ and its adjacent cycles $\mathcal{C}_{j-1}$ and $\mathcal{C}_{j+1}$, if they exist, as explained in Figures 4, 5, 6, 7, and 8 below.*

Proof. We begin with some remarks and explanations. $NBR(s)$ returns the least index j of a cycle with a node that does not play a best response. So the initial grade of the cycle $\mathcal{C}_{j-1}$, if it exists, is + or U+ and the initial grade of the cycle $\mathcal{C}_j$ is U−, −, or ?. Moreover, the grade of $\mathcal{C}_j$ can only change to + or U+, because after line 6 all nodes in $\mathcal{C}_j$ play a best response. These observations allow us to limit the number of considered cases.

In the presented tables we list above the horizontal bar the initial situation for the discussed cycles and under the bar one or more outcomes that can arise. Further, the initial grade of C_{j+1} is a parameter x. If there are several options for the new grade of a given cycle, these are separated by /. Finally 'any' is an abbreviation for U+/ +/ U−/ −/ ?.

Figure 4 corresponds to the case when $j = 1$. In turn, Figures 5, 6, and 7 correspond to the cases when $1 < j < m$ and initially the grade of C_j is U−, −, or ?, respectively. Finally, Figure 8 corresponds to the case when $j = m$.

The cases considered in Figures 5 and 6 refer to the update scenarios defined in *Phase 2* in the proof of Theorem 7. They are concerned with the relation of the colour of the up-link node in the cycle C_j to the colour of its predecessor in this cycle.

−	x	U−	x	?	x
+/U+	x	U+	x	+/U+	any

Figure 4: Possible changes of the grades of C_j and C_{j+1} when $j = 1$.

case	+	U−	x	U+	U−	x
(i)	+	U+	x	U+	U+	x
(ii)	+/−/U+/U−	U+	x	impossible		
(io)	U−/U+	+/U+	x	impossible		
(ioi)	U−/U+	U+	any	impossible		
(ioo)	U−/U+	+	any	impossible		
(iooi)	impossible			impossible		

Figure 5: Possible changes of the grades of C_{j-1}, C_j, and C_{j+1} when $1 < j < m$ and the grade of C_j is U−.

case	+	−	x	U+	−	x
(o)	+	+	x	U+	+	x
(oi)	+/−/U+/U−	+/U+	x	impossible		
(oo)	U−/U+	+	x	impossible		
(ooi)	U−/U+	U+	any	impossible		

Figure 6: Possible changes of the grades of C_{j-1}, C_j, and C_{j+1} when $1 < j < m$ and the grade of C_j is −.

The justifications of these changes of the grades are lengthy and are provided in the appendix. □

We noted already in Note 11 that if a grade of a given cycle C_i is U+, then it will not be modified during the execution of the algorithm once $j > i$. This motivates the introduction of the following notion. Let $guard(s)$ be the largest $j \in \{1, \ldots, m\}$ such that given s the grade of C_j is U+ and the grade of each cycle $C_1, \ldots, C_{j-1}$ is either + or U+. If no such j exists, as it is the case in the example given in Figure 3, then we let $guard(s) = 0$.

Further, let *prefix(s)* be the longest prefix of *grade(s)* such that at most one of the grades it contains is −, U−, or ?. Moreover, this prefix stops after a cycle with grade ?. For the example given in Figure 3 we have *prefix(s)* = (−, U+).

Here is another example illustrating the introduced notions to which we shall return shortly.

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+	?	x		U+	?	x
+/–/U+/U–	+/U+	any		U+	+/U+	any

Figure 7: Possible changes of the grades of $\mathcal{C}_{j-1}$, $\mathcal{C}_j$, and $\mathcal{C}_{j+1}$ when $1 < j < m$ and the grade of $\mathcal{C}_j$ is ?.

case	+	U–		U+	U–
(i)	+	U+		U+	U+
(ii)	+/–/U+/U–	U+		impossible	
(io)	U–/U+	+/U+		impossible	
(ioi)	U–/U+	U+		impossible	
case	+	–		U+	–
(o)	+	+		U+	+
(oi)	+/–/U+/U–	+/U+		impossible	
(oo)	U–/U+	+		impossible	
(ooi)	U–/U+	U+		impossible	

+	?		U+	?
+/–/U+/U–	+/U+		U+	+/U+

Figure 8: Possible changes of the grades of $\mathcal{C}_{j-1}$ and $\mathcal{C}_j$ when $j = m$.

Example 13.

(i) Suppose that

$$grade(s) := (+, \text{U}+, \text{U}+, +, -, \text{U}-, ?).$$

Then $NBR(s) = 5$, $guard(s) = 3$, and prefix$(s) = (+, \text{U}+, \text{U}+, +, -)$.

(ii) Suppose that

$$grade(s) := (+, \text{U}+, -, +, \text{U}+, \text{U}-, \text{U}-, ?).$$

Then $NBR(s) = 3$, $guard(s) = 2$, and prefix$(s) = (+, \text{U}+, -, +, \text{U}+)$. $\square$

We are now in position to prove the appropriate result concerning open chains of cycles.

Theorem 14. *Every coordination game on an open chain of m cycles, each with v nodes, ensures improvement paths of length $\leq 3vm^3$.*

Proof. Let s_0 be an arbitrary initial joint strategy in this coordination game. We argue that starting at s_0, Algorithm 1 computes a finite improvement path ρ of length at most $3vm^3$. To prove it we introduce a progress measure μ defined on the current joint strategy that increases according to the lexicographic order each time the improvement path ρ is extended in line 7. μ is a quadruple the definition of which uses the *guard*, *NBR* and *prefix* functions defined above.

First, we set $\mu(s) = (m + 1, 0, 0, 0)$ if s is a Nash equilibrium. Otherwise

$$\mu(s) := \begin{cases} (guard(s), 1, 0, -NBR(s)) & \text{if } prefix(s) \text{ contains U– or if it contains U+} \\ & \text{somewhere after –} \\ (guard(s), 0, |prefix(s)|, -NBR(s)) & \text{otherwise.} \end{cases}$$

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For example, for s used in Example 13 we have respectively $\mu(s) = (3, 0, 5, -5)$ and $\mu(s) = (2, 1, 0, -3)$. An example run of Algorithm 1 and the value of the progress measure at each step are presented in Figure 9. It illustrates the fact that during the execution of the algorithm the index of the first cycle with no Nash equilibrium, i.e., the value of $NBR(s)$, can arbitrarily decrease.

$grade(s)$								$\mu(s)$
+	+	+	+	?	U+	U+	–	$(0, 0, 5, -5)$
+	+	+	–	+	?	U+	–	$(0, 0, 5, -4)$
+	+	–	+	+	?	U+	–	$(0, 0, 5, -3)$
+	U–	U+	?	+	?	U+	–	$(0, 1, 0, -2)$
U–	+	?	?	+	?	U+	–	$(0, 1, 0, -1)$
U+	+	?	?	+	?	U+	–	$(1, 0, 3, -3)$
U+	+	U+	?	+	?	U+	–	$(3, 0, 4, -4)$
U+	+	U+	+	+	?	U+	–	$(3, 0, 6, -6)$
U+	+	U+	+	+	U+	U+	–	$(7, 0, 8, -8)$
U+	+	U+	+	+	U+	U+	+	$(9, 0, 0, 0)$

Figure 9: The evolution of $grade(s)$ and $\mu(s)$ during an example run of Algorithm 1.

We now check using Lemma 12 that $\mu(s)$ increases w.r.t. the lexicographic ordering $<_{lex}$ each time one of the updates presented in Figures 4, 5, 6, 7, and 8 takes place. So throughout the analysis we assume that $j = NBR(s)$. Let s' denote the new joint strategy computed in line 8 of the algorithm. Note 11 implies that $guard(s) \leq guard(s')$. Further, thanks to the definition of $\mu(s')$ we can assume that s' is not a Nash equilibrium. Occasionally we use below the following simple fact.

Observation 1. Suppose that $\mu(s) = (guard(s), 0, |prefix(s)|, -NBR(s))$, $guard(s) \leq guard(s')$ and $|prefix(s)| < |prefix(s')|$. Then $\mu(s) <_{lex} \mu(s')$.

Re: Figure 4.
Then $guard(s) = 0$.
Case 1. The new grade of $\mathcal{C}_j$ is U+.
Then $guard(s) < guard(s')$ and hence $\mu(s) <_{lex} \mu(s')$.

Case 2. The new grade of $\mathcal{C}_j$ is +.

Subcase 1. $\mu(s) = (guard(s), 1, 0, -NBR(s))$.
Then the initial grade of $\mathcal{C}_j$ is – and $prefix(s)$ contains U+, say at position h. Hence $prefix(s')$ also contains U+ at position h and consequently $h \leq guard(s')$. But $guard(s) = 0$, so $\mu(s) <_{lex} \mu(s')$.

Subcase 2. $\mu(s) = (guard(s), 0, |prefix(s)|, -NBR(s))$.
If the initial grade of $\mathcal{C}_j$ is –, then $|prefix(s)| < |prefix(s')|$ since by assumption s' is not a Nash equilibrium. Otherwise the initial grade of $\mathcal{C}_j$ is ? and then $|prefix(s)| = 1$ by the definition of $prefix(s)$, while $1 < |prefix(s')|$. So in both cases by Observation 1 $\mu(s) <_{lex} \mu(s')$.

Re: Figure 5.
By definition $\mu(s) = (guard(s), 1, 0, -NBR(s))$.
Case 1. The new grade of $\mathcal{C}_{j-1}$ is +.

Then the case *(i)* or *(ii)* applies and hence the new grade of $\mathcal{C}_j$ is U+. So $guard(s) < guard(s')$ and hence $\mu(s) <_{lex} \mu(s')$.

Case 2. The new grade of $\mathcal{C}_{j-1}$ is U+.

Then $guard(s) < guard(s)$ and hence $\mu(s) <_{lex} \mu(s')$.

Case 3. The new grade of $\mathcal{C}_{j-1}$ is −.

Then the case *(ii)* applies and hence the new grade of $\mathcal{C}_j$ is U+. So $\mu(s') = (guard(s'), 1, 0, -NBR(s'))$. But $guard(s) \leq guard(s')$ and $-NBR(s) < -NBR(s')$, so $\mu(s) <_{lex} \mu(s')$.

Case 4. The new grade of $\mathcal{C}_{j-1}$ is U−.

Then $\mu(s') = (guard(s'), 1, 0, -NBR(s'))$ and $\mu(s) <_{lex} \mu(s')$ for the same reasons as in the previous case.

Re: Figure 6.

Case 1. $\mu(s) = (guard(s), 1, 0, -NBR(s))$.

prefix(s) contains − at position j, so it contains U+ at some position $h > j$. Moreover, by the definition of *prefix*(s) all positions in it between j and h are + or U+.

So if the new grade of $\mathcal{C}_{j-1}$ is + or U+, then $j < guard(s')$ and hence $guard(s) < guard(s')$ since $guard(s) < NBR(s) = j$. So $\mu(s) <_{lex} \mu(s')$. Otherwise the new grade of $\mathcal{C}_{j-1}$ is − or U−. If it is −, then *prefix*(s') contains U+ at the position $h > j - 1$. So in both cases $\mu(s') = (guard(s'), 1, 0, -NBR(s'))$. But $-NBR(s) < -NBR(s')$, so $\mu(s) <_{lex} \mu(s')$.

Case 2. $\mu(s) = (guard(s), 0, |prefix(s)|, -NBR(s))$.

If the new grade of $\mathcal{C}_{j-1}$ is + or U+, then $|prefix(s)| < |prefix(s')|$ since we assumed that s' is not a Nash equilibrium. So by Observation 1 $\mu(s) <_{lex} \mu(s')$. If the new grade of $\mathcal{C}_{j-1}$ is − or U−, then $guard(s) = guard(s')$ and $|prefix(s)| = |prefix(s')|$ but $-NBR(s) < -NBR(s')$, so $\mu(s) <_{lex} \mu(s')$.

Re: Figure 7.

By the definition *prefix*(s) ends with ?, so $|prefix(s)| = j$ and $\mu(s) = (guard(s), 0, |prefix(s)|, -NBR(s))$.

If the new grade of $\mathcal{C}_{j-1}$ is + or U+, then $j < |prefix(s')|$, so by Observation 1 $\mu(s) <_{lex} \mu(s')$. If the new grade of $\mathcal{C}_{j-1}$ is − or U−, then $guard(s) = guard(s')$, $|prefix(s)| \leq |prefix(s')|$ and $-NBR(s) < -NBR(s')$, so $\mu(s) <_{lex} \mu(s')$.

Re: Figure 8.

The arguments for each case coincide with the arguments given for the corresponding cases concerning Figures 5, 6, and 7.

Let us now estimate the number of different values the progress measure μ has. If s is a Nash equilibrium, then $\mu(s) = (m + 1, 0, 0, 0)$, which accounts for one value.

Otherwise $guard(s) \in \{0, \ldots, m - 1\}$ and

$$guard(s) + 1 \leq NBR(s) \leq |prefix(s)| \leq m,$$

because by definition the index $NBR(s)$ cannot be smaller than $guard(s) + 1$ and the grade of the cycle with this index belongs to *prefix*(s). Therefore the number of values μ can take is

$$1 + \sum_{g=0}^{m-1} \sum_{p=g+1}^{m} (p - g) + \sum_{g=0}^{m-1} (m - g) = 1 + \sum_{g=0}^{m-1} \frac{(m - g)(1 + m - g)}{2} + \frac{m(1 + m)}{2} =$$

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$$1 + \sum_{x=1}^{m} \frac{x(1+x)}{2} + \frac{m(1+m)}{2} = 1 + \frac{m(m+1)(m+2)}{6} + \frac{m(1+m)}{2} =$$

$$1 + \frac{m(m+1)(m+5)}{6} \leq m^3 \ \text{ for } m \geq 2.$$

As a result, the length of the improvement path constructed by Algorithm 1 is at most $3vm^3$, because by Theorems 3 and Theorem 7 the improvement path in line 6 takes at most $3v$ improvement steps. $\qquad\square$

Finally, so far we assumed that we know the decomposition of the game graph into a chain of cycle in advance. In general the input may be an arbitrary graph and we would need to find this decomposition first. Fortunately this can be done in linear time as the following result shows.

Lemma 15. *Checking whether a given graph G is an open chain of cycles, and if so partitioning G into simple cycles $\mathcal{C}_1, \ldots, \mathcal{C}_m$ can be done in $\mathcal{O}(|G|)$ time.*

Proof. First note that if G is an open chain of cycles then there are no bidirectional edges and each of its nodes has either out- and in-degree values both equal to 1 or both equal to 2. These two conditions can be easily checked in linear time by simply going through all the nodes and their edges in G.

Assume now that G is an open chain of cycles. Then the above two conditions hold. We use them to partition G into cycles. First we construct the set A of all nodes in G with out- and in-degrees both equal to 2. This is the set of the link nodes of G. Next we build a new directed graph G' whose set of nodes is A and there is an edge from $u \in A$ to $v \in A$ iff v is reachable from u by traversing only nodes with out- and in-degree both equal to 1. Such a graph can be built using a single run of the depth first search algorithm starting from any node in A. Now note that the original graph G is an open chain of cycles if this graph G' is a line graph in which all edges are bidirectional. This condition can also be checked in linear time, by simply following all edges of G' in one direction.

To partition G into simple cycles we label one of the end nodes of G' as $[1, 1]$. Its only adjacent node we label as $[2, 1]$, the other adjacent node of $[2, 1]$ as $[3, 1]$, and so on until the node at the other end is of G' labeled as $[m - 1, 1]$. These are the labels of the link nodes. The labels of the remaining nodes in each cycle C_j for $j \in \{1, \ldots, m\}$ can then be simply inferred by following the edges in the original graph G. $\qquad\square$

6 Strong equilibria

In this section we study the existence of strong equilibria and the existence of finite c-improvements paths. To start with, we establish two results about the games that have the strongest possible property, the c-FIP.

First we establish a structural property of a coalitional deviation from a Nash equilibrium in our coordination games. It will be used to prove c-weak acyclicity for a class of games on the basis of their weak acyclicity. Note that such a result cannot hold for all classes of graphs because there exists a coordination game on an undirected graph which is weakly acyclic but has no strong equilibrium (see [2]).

Lemma 16. *Consider a coordination game. Any node involved in a profitable coalitional deviation from a Nash equilibrium belongs to a directed simple cycle that deviated to the same colour.*

Proof. Suppose that s' is profitable deviation of a coalition K from a Nash equilibrium s. It suffices to show that each node in K has a neighbour in K deviating to the same colour. Assume that for some player $i \in K$ it is not the case.

Then

$$p_i(s) < p_i(s'_K, s_{-K})$$
$$= \sum_{j \in N_j \cap K: s'_j = s'_i} w_{j \to i} + \sum_{j \in N_j \setminus K: s_j = s'_i} w_{j \to i} + \beta(i, s'_i)$$
$$\leq 0 + \sum_{j \in N_i: s_j = s'_i} w_{j \to i} + \beta(i, s'_i) = p_i(s'_i, s_{-i}),$$

which contradicts the fact that s is a Nash equilibrium. $\square$

Theorem 17. *Every coordination game with bonuses on a weighted directed acyclic graph (DAG) has the c-FIP and a fortiori a strong equilibrium. Further, every Nash equilibrium is a strong equilibrium. Finally, the game ensures both improvement paths and c-improvement paths of length $\leq n - 1$, where —recall— n is the number of nodes.*

Proof. Given a weighted DAG (V, E) on n nodes denote these nodes by $1, \ldots, n$ in such a way that for all $i, j \in \{1, \ldots, n\}$

$$\text{if } i < j \text{ then } (j \to i) \notin E. \tag{1}$$

So if $i < j$ then the payoff of the node i does not depend on the strategy selected by the node j.

Then given a coordination game whose underlying directed graph is the above weighted DAG and a joint strategy s we abbreviate the sequence $p_1(s), \ldots, p_n(s)$ to $p(s)$. We now claim that $p : S \to \mathbb{R}^n$ is a generalised ordinal c-potential when we take for the partial ordering $\succ$ on $p(S)$ the lexicographic ordering $>_{lex}$ on the sequences of reals.

So suppose that some coalition K profitably deviates from the joint strategy s to s'. Choose the smallest $j \in K$. Then $p_j(s') > p_j(s)$ and by (1) $p_i(s') = p_i(s)$ for $i < j$. By the definition of $>_{lex}$ this implies $p(s') >_{lex} p(s)$, as desired. Hence the game has the c-FIP.

The second claim is a direct consequence of Lemma 16 that implies that no coalition deviations are possible from a Nash equilibrium for DAGs.

Finally, to prove the last claim, given an initial joint strategy schedule the players in the order $1, \ldots, n$ and repeatedly update the strategy of each selected player to a best response. By (1) this yields an improvement path of length $\leq n - 1$. By the second claim this path is also a c-improvement path. $\square$

Example 2 shows that it is difficult to come up with other classes of directed graphs for which the coordination game has the FIP, let alone the c-FIP. However, the weaker property of c-weak acyclicity holds for the games on simple cycles considered in Section 4. Below we put $i \ominus 1 = i - 1$ if $i > 1$ and $1 \ominus 1 = n$.

Theorem 18. *Consider a coordination game with bonuses on a weighted simple cycle. Any finite improvement path is a finite c-improvement path or can be extended to it by a single profitable deviation of all players.*

Proof. Take a finite improvement path and denote by s the Nash equilibrium it reaches. If s is a strong equilibrium then we are done. Otherwise there exists a coalition K with a profitable deviation from s. By Lemma 16 K consists of all players and all of them switch to the same colour.

Let C be the set of common colours c such that a switching by all players to c is a profitable deviation from s. We just showed that C is non-empty. Select an arbitrary player i_0 and choose a colour from C for which player i_0 has a maximal bonus. Let s' be the resulting joint strategy.

We first claim that s' is a Nash equilibrium. Otherwise some player i can profitably deviate from s'_i to a colour c. Then $s'_{i \ominus 1} \neq c$, because all players hold the same colour in s'. So we have $p_i(s) < p_i(s') < p_i(c, s'_{-i}) = \beta(i, c) \leq p_i(c, s_{-i})$, which is a contradiction since s is a Nash equilibrium.

Next, we claim that s' is a strong equilibrium. Otherwise by the initial observation there is a profitable deviation of all players from s' to some joint strategy s'' in which all players switch to the same colour. So $p_{i_0}(s') < p_{i_0}(s'')$. Moreover, this profitable deviation is also a profitable deviation of all players from s, which contradicts the choice of i_0. $\square$

The above result directly leads to the following conclusions.

Corollary 19.
  (i) *Every coordination game on a weighted simple cycle in which at most one node has bonuses ensures c-improvement paths of length $\leq 2n$.*

 (ii) *Every coordination game with bonuses on a simple cycle in which at most one edge has a non-trivial weight ensures c-improvement paths of length $\leq 3n$.*

(iii) *Every coordination game on a weighted simple cycle in which two nodes have bonuses ensures c-improvement paths of length $\leq 3n + 1$.*

(iv) *Every coordination game on a simple cycle with bonuses in which two edges have non-trivial weights ensures c-improvement paths of length $\leq 4n$.*

Proof. By Theorems 3, 5, 7, 9, and 18. $\square$

We conclude this analysis of coordination games on simple cycles by the following observation that sheds light on Theorem 18 and is of independent interest.

Note 20. *Consider a coordination game with bonuses on a simple cycle with n nodes. Then every Nash equilibrium is a k-equilibrium for all $k \in \{1, \ldots, n-1\}$.*

Proof. Take a Nash equilibrium s. It suffices to prove that it is an $(n-1)$-equilibrium. Suppose otherwise. Then for some coalition K of size $\leq n-1$ and a joint strategy s', $s \overset{K}{\to} s'$ is a profitable deviation.

Take some $i \in K$ such that $i \ominus 1 \notin K$. We have $p_i(s') > p_i(s)$. Also $p_i(s'_i, s_{-i}) = p_i(s')$, since $s_{i\ominus 1} = s'_{i\ominus 1}$. So $p_i(s'_i, s_{-i}) > p_i(s)$, which contradicts the fact that s is a Nash equilibrium. $\square$

We now show that, as in the case of simple cycles, coordination games on open chains of cycles are c-weakly acyclic, so a fortiori have strong equilibria.

Theorem 21. *Every coordination game on a open chain of cycles of m simple cycles, each with v nodes, ensures c-improvement paths of length $4vm^4$.*

Proof. Assume the considered open chain of cycles $\mathcal{C}$ consists of the simple cycles $\mathcal{C}_j$, where $j \in \{1, 2, \ldots, m\}$. We begin with the following useful fact.

Observation 1. Suppose that in a joint strategy s for the coordination game on $\mathcal{C}$ a simple cycle $\mathcal{C}_i$ is unicoloured. Then in any profitable deviation from s the colours of the nodes in $\mathcal{C}_i$ do not change.

Proof The payoff of each node of the cycle $\mathcal{C}_i$ in s is ≥ 1. For the non-link nodes the payoff is then maximal, so none of these nodes can be a member of a coalition that profitably deviates. This implies that a link node cannot be a member of a coalition that profitably deviates either. Indeed, otherwise its payoff increases to 2 and hence in the new joint strategy its colour is the same as the colour of its predecessor j in the cycle $\mathcal{C}_i$, which is not the case, since we just explained that the colour of j does not change. $\square$

We now construct the desired c-improvement path ξ as an alternation of an improvement path guaranteed by Theorem 14 and a single profitable deviation by a coalition. Each time such a profitable coalitional deviation takes place, by Lemma 16 the deviating coalition includes a simple cycle $\mathcal{C}_i$ all nodes of which switch to the same colour.

By Observation 1 each time this is a different cycle, which is moreover disjoint from the previous cycles. This implies that the number of such profitable deviations in ξ is at most $\lceil m/2 \rceil$.

So ξ is finite and by Theorem 14 its length is at most $(\lceil m/2 \rceil + 1) \cdot 3vm^3 + \lceil m/2 \rceil$, where the first term counts the total length of at most $\lceil m/2 \rceil + 1$ improvement paths that separate at most $\lceil m/2 \rceil$ coalitional deviations, which is the second term of this expression. But $\lceil m/2 \rceil + 1 \leq m$ for $m \geq 2$, so $(\lceil m/2 \rceil + 1) \cdot 3vm^3 + \lceil m/2 \rceil \leq 3vm^4 + \lceil m/2 \rceil \leq 4vm^4$. $\qquad\square$

Example 2 shows that even when only two colours are used, the coordination game does not need to have the FIP. This is in contrast to the case of undirected graphs for which we proved in [2] that the coordination game does have the FIP. On the other hand, a weaker property does hold.

Theorem 22. *Every coordination game in which only two colours are used ensures improvement paths of length $\leq 2n$.*

Proof. We prove the result for a more general class of games, namely the ones that satisfy the PPM (the property defined in Section 2). Call the colours blue and red. When a node holds the blue colour we refer to it as a blue node, and the likewise for the red colour. Take a joint strategy s.

Phase 1. We consider a maximal sequence ξ of profitable deviations starting in s in which each node can only switch to blue. At each step the number of blue nodes increases, so ξ is of length at most n. Let s^1 be the last joint strategy in ξ. If s^1 is a Nash equilibrium, then ξ is the desired finite improvement path. Otherwise we move to the next phase.

Phase 2. We consider a maximal sequence χ of profitable deviations starting in s^1 in which each node can only switch to red. Also χ is of length at most n. Let s^2 be the last joint strategy in χ.

We claim that s^2 is a Nash equilibrium. Suppose otherwise. Then some node, say i, can profitably switch in s^2 to blue. Suppose that node i is red in s^1. In s^1 there are weakly more blue nodes than in s^2, so by the PPM also in s^1 node i can profitably switch to blue. This contradicts the choice of s^1.

Hence node i is blue in s^1, while it is red in s^2. So in some joint strategy s^3 from χ node i profitably switched to red. Then $s^3 = (i : b, s^3_{-i})$ and

$$p_i(i : b, s^3_{-i}) < p_i(i : r, s^3_{-i}) \leq p_i(i : r, s^2_{-i}) < p_i(i : b, s^2_{-i}),$$

where the weak inequality holds due to the PPM. But in s^3 there are weakly more blue nodes than in s^2, so by the PPM

$$p_i(i : b, s^2_{-i}) \leq p_i(i : b, s^3_{-i}).$$

This yields a contradiction. $\qquad\square$

In the coordination games in which only two colours are used Nash equilibria do not need to be strong equilibria.

Example 23. Consider a bidirectional cycle $1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4 \leftrightarrow 1$ in which each node has two colours, a and b. Then (a, a, b, b) is a Nash equilibrium, but it is not a strong equilibrium because of the profitable deviation to (a, a, a, a), which is a strong equilibrium. $\qquad\square$

On the other hand the following counterpart of the above result holds for the c-improvements paths.

Theorem 24. *Every coordination game in which only two colours are used ensures c-improvement paths of length $\leq 2n$.*

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Proof. As in the above proof we establish the result for the games that satisfy the PPM. We retain the terminology of blue and red colours, that we abbreviate to b and r. Take a joint strategy s.

Phase 1. We consider a maximal sequence ξ of profitable deviations of the coalitions starting in s in which the nodes can only switch to blue. At each step the number of blue nodes increases, so ξ is of length at most n. Let s^1 be the last joint strategy in ξ. If s^1 is a strong equilibrium, then ξ is the desired finite c-improvement path. Otherwise we move to the next phase.

Phase 2. We consider a maximal sequence χ of profitable deviations of the coalitions starting in s^1 in which the nodes can only switch to red. Also χ is of length at most n. Let s^2 be the last joint strategy in χ.

We claim that s^2 is a strong equilibrium. Suppose otherwise. Then for some joint strategy s', $s^2 \xrightarrow{K} s'$ is a profitable deviation of some coalition K. Let L be the set of nodes from K that switched in this deviation to blue. By the definition of s^2 the set L is non-empty.

Given a set of nodes M and a joint strategy s we denote by $(M : b, s_{-M})$ the joint strategy obtained from s by letting the nodes in M to select blue, and similarly for the red colour. Also it should be clear which joint strategy we denote by $(M : b, P \setminus M : r, s_{-P})$, where $M \subseteq P$.

We claim that $s^2 \xrightarrow{L} (L : b, s^2_{-L})$ is a profitable deviation of the players in L. Indeed, we have for all $i \in L$

$$p_i(s^2) < p_i(L : b, s^2_{-L}), \tag{2}$$

since by the assumption $p_i(s^2) < p_i(s')$ and by the PPM $p_i(s') \leq p_i(L : b, s^2_{-L})$.

Let M be the set of nodes from L that are red in s^1. Suppose that M is non-empty. We show that then for all $i \in M$

$$p_i(M : r, L \setminus M : b, s^1_{-L}) < p_i(M : b, L \setminus M : b, s^1_{-L}). \tag{3}$$

Indeed, we have for all $i \in M$

$$
\begin{aligned}
p_i(M : r, L \setminus M : b, s^1_{-L}) &\leq p_i(M : r, L \setminus M : b, s^2_{-L}) \\
\leq \quad p_i(M : r, L \setminus M : r, s^2_{-L}) &< p_i(M : b, L \setminus M : b, s^2_{-L}) \\
\leq \quad p_i(M : b, L \setminus M : b, s^1_{-L}),
\end{aligned}
$$

where the weak inequalities hold due to the PPM and the strict inequality holds by the definition of L.

But $s^1 = (M : r, L \setminus M : b, s^1_{-L})$, so (3) contradicts the definition of s^1. Thus M is empty, i.e., all nodes from L are blue in s^1.

Let i be a node from L that as first turns red in χ. So in some joint strategy s^3 from χ node i profitably switched to red in a profitable deviation to a joint strategy s^4. Then $s^3 = (L : b, s^3_{-L})$, $s^4 = (i : r, s^4_{-i})$ and

$$p_i(L : b, s^3_{-L}) < p_i(i : r, s^4_{-i}) \leq p_i(s^2) < p_i(L : b, s^2_{-L}),$$

where the weak inequality holds due to the PPM and the strict inequalities hold by the definition of i and (2). But in $(L : b, s^3_{-L})$ there are weakly more blue nodes than in $(L : b, s^2_{-L})$, so by the PPM

$$p_i(L : b, s^2_{-L}) \leq p_i(L : b, s^3_{-L}).$$

This yields a contradiction. (The final step in this proof in [8] contained a bug that is now corrected.) $\qquad\square$
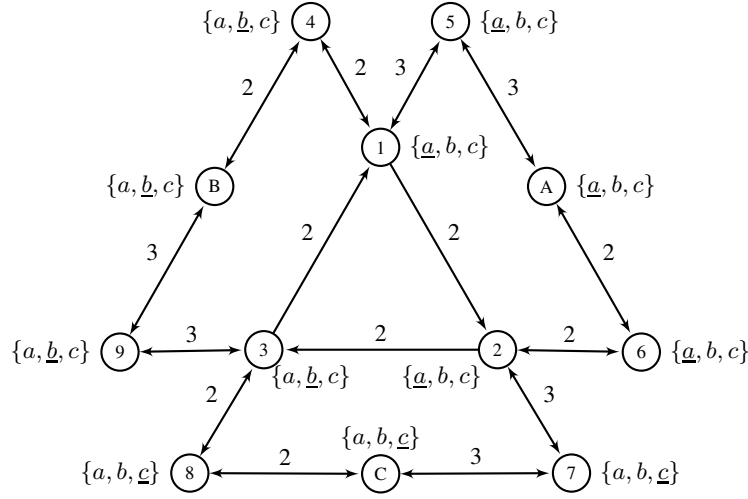
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Figure 10: A coordination game with strong equilibria unreachable from a given initial joint strategy.

When the underlying graph is symmetric and the set of strategies for every node is the same, the existence of strong equilibrium for coordination games with two colours follows from Proposition 2.2 in [34]. Theorem 24 shows a stronger result, namely that these games are c-weakly acyclic. Example 1 shows that when three colours are used, Nash equilibria, so a fortiori strong equilibria do not need to exist.

Finally, note that sometimes strong equilibria exist even though the coordination game is not c-weakly acyclic.

Example 25. Consider the coordination game depicted in Figure 10.

Note that the underlying graph is strongly connected and that all edges except $1 \rightarrow 2$, $2 \rightarrow 3$ and $3 \rightarrow 1$ are bidirectional. Although the graph is weighted, the weighted edges can be replaced by unweighted ones by adding auxiliary nodes without affecting the strong connectedness of the graph. The behaviour of the game on this new unweighted graph will be analogous to the one considered.

Let us analyse now the initial joint strategy s that consists of the underlined colours in Figure 10. We argue that the only nodes that can profitably switch colours (possibly in a coalition) are the nodes 1, 2 and 3 and that this is the case independently of their strategies.

First consider the nodes A, B, and C. They have the maximum possible payoff of 5, independently of the strategies of the nodes 1, 2 and 3, so none of them can be a member of a profitably deviating coalition.

Further, each node from the set $\{4, \ldots, 9\}$ has two neighbours, each with the same weight. One of them is from the set $\{A, B, C\}$ with whom it shares the same colour, which results in the payoff of 2. So for each node from $\{4, \ldots, 9\}$ a possible profitable coalitional deviation has to involve a neighbour from $\{A, B, C\}$.

Therefore, the only nodes that can profitably deviate are nodes 1, 2 and 3. Moreover, this will continue to be the case in any joint strategy resulting from a sequence of profitable coalitional deviations starting from s. (Another way to look at it by arguing that the restriction of s to the nodes $\{A, B, C, 4, \ldots, 9\}$ is a strong equilibrium in the game on these nodes in which we add to the nodes from $\{4, 6, 8\}$ bonuses 2 and to the nodes from $\{5, 7, 9\}$ bonuses 3.)

So it suffices to analyse the weighted simple cycle and the colour assignment depicted in Figure 11, with the non-trivial bonuses mentioned above the colours.

However, the resulting coordination game does not have a Nash equilibrium and a fortiori no strong equilibrium.
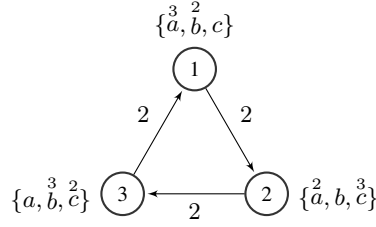
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Figure 11: A coordination game without a Nash equilibrium

To see it first notice that each of the nodes can secure a payoff at least 3, while selecting a colour with a trivial bonus it can secure a payoff of at most 2. So we do not need to analyse joint strategies in which a node selects a colour with a trivial bonus. This leaves use with the following list of joint strategies: $(\underline{a}, a, b)$, $(a, a, \underline{c})$, $(a, c, \underline{b})$, $(a, \underline{c}, c)$, $(b, \underline{a}, b)$, $(\underline{b}, a, c)$, $(b, c, \underline{b})$ and $(\underline{b}, c, c)$. In each of them, as in Examples 1 and 6, we underlined a strategy that is not a best response to the choice of other players. This means that no c-improvement path in this game terminates.

Consequently no c-improvement path in the original game that starts with s terminates. Therefore, the original game is neither weakly acyclic nor c-weakly acyclic. On the other hand, it has three trivial strong equilibria in which all players pick the same colour. □

Note that in the game considered in this example all players have the same sets of strategies. We can summarise this example informally as follows. There exists a graph with the same set of alternatives (called colours) for all nodes and an initial situation (modelled by a colour assignment) starting from which no stable outcome (modelled as a Nash equlibrium) can be achieved even if forming coalitions is allowed.

7   Complexity issues

Finally, we study the complexity of finding Nash equilibria and strong equilibria, and of determining their existence. The results obtained so far provide bounds on the length of short (c-)improvement paths. But in each proof we actually provide bounds on the length of the corresponding schedule, a notion defined in Section 2. This allows us to determine in each case the complexity of finding a Nash equilibrium or a strong equilibrium, by analysing the cost of finding a profitable deviation from a given joint strategy.

We assume that the colour assignment C is given as a $\{0, 1\}$-matrix of size $V \times M$, such that (i, c) entry is 1 iff colour c is available to node i. The bonus function β, if present, is represented by another matrix of size $V \times M$, where the (i, c) entry holds the value of $\beta(i, c)$. The game graph is represented using adjacency lists, where for each node we keep a list of all outgoing and incoming edges and, if the graph is weighted, their weights are represented in binary. As usual, we provide the time complexity in terms of the number of arithmetic operations performed. All our algorithms operate only on numbers that are linear in the size of the input, so the actual number of bit operations is at most polylogarithmically higher.

Below, as in Table 1 in Section 1, n is the number of nodes, $|E|$ the number of edges, l the number of colours, and in the case of the open chains of cycles m the number of simple cycles in a chain and v the number of nodes in each cycle.

Lemma 26. *Consider a coordination game. Given a joint strategy a best response for a player i can be computed in time $\mathcal{O}(l + e_i)$, where e_i is the number of incoming edges to node i.*

Proof. We first calculate for each colour the sum of the weights on all edges from neighbors of player i with that colour. This can be done by simply iterating over all e_i incoming edges. We then iterate over all of these l values to select any colour with the highest such a value. □

When we only care about the current payoff of player i, then there is no need to iterate over all l colours and we get the following.

Lemma 27. *Consider a coordination game. Given a joint strategy the payoff of player i can be computed in time $\mathcal{O}(1 + e_i)$, where e_i is the number of incoming edges to node i.*

Proof. It suffices to iterate over all e_i incoming edges the sum the weights of all edges from neighbors of player i with the same colour. The term 1 is needed to cover the case of nodes with no neighbours. □

We first deal with the complexity of finding a Nash equilibrium and a strong equilibrium for the most involved versions of simple cycles considered in Section 4.

Theorem 28. *Consider a coordination game on a simple cycle that is either weighted with at most two nodes with bonuses or with bonuses with at most two edges having non-trivial weights. Both a Nash equilibrium and a strong equilibrium can be computed in time $\mathcal{O}(nl)$.*

Proof. In both cases, due to Theorems 7 and 9, to compute a Nash equilibrium it suffices to follow a schedule of length $\mathcal{O}(n)$. At each step of this schedule it suffices to consider only the deviations to a colour with the maximal bonus. We can find such colours in time $\mathcal{O}(l)$ and then simply follow the $\mathcal{O}(l)$ procedure given in Lemma 26 for finding a best response within this narrowed down set. We conclude that computing a Nash equilibrium can be done in time $\mathcal{O}(nl)$.

Finally, to compute a strong equilibrium we first compute a Nash equilibrium and subsequently check whether there is a profitable deviation of all nodes to a single colour. By Theorem 18 one of these two joint strategies is a strong equilibrium.

The latter step involves iterating over all l colours and computing for each of them the payoff of all nodes when they all hold this single colour, assuming such a colour is shared by all nodes. Each iteration takes $\mathcal{O}(n)$ time, which results in total $\mathcal{O}(nl)$ time, as well. □

Theorem 29. *Consider a coordination game on an open chain of cycles. A Nash equilibrium can be computed in time $\mathcal{O}(vm^3l)$.*

Proof. From Theorem 14 it follows that for an open chain of cycles there exists an improvement path of length at most $3vm^3$. Due to Lemma 26 computing each best response can be done in time $\mathcal{O}(l)$. It follows that a Nash equilibrium can be computed in time $\mathcal{O}(vm^3l)$. □

In a further analysis we shall use the following algorithm.
The following lemma justifies the claim made in the description of the output of this algorithm.

Lemma 30. *Consider a strategic game that satisfies the PPM property, a joint strategy s and a strategy c. Algorithm 2 computes a maximal coalition that can profitably deviate from s to c, if there exists one, and otherwise returns the empty set.*

Proof. First note that due to line 4 the algorithm always terminates. Suppose that A^* is a maximal coalition that can profitably deviate to c. So $s \xrightarrow{A^*} s^*$, where $s_i^* = c$ for $i \in A^*$. Consider the execution of the above algorithm. Then $A^* \subseteq A$ after line 1. By the PPM property no player from A^* can be removed in line 4, because otherwise it could not

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Algorithm 2:

    Input: A strategic game $(S_1, \ldots, S_n, p_1, \ldots, p_n)$ that satisfies the PPM property, a joint strategy s, and a strategy c.

    Output: A maximal coalition that can profitably deviate to c, if there exists one, and otherwise the empty set.

1 $A := \{i \in \{1, \ldots, n\} \mid c \in S_i\};$ *(i.e., A is the set of players that can select c)*

2 **while** $A \neq \emptyset$ and $s \xrightarrow{A} s'$, where $s_i' = c$ for $i \in A$, is not a profitable deviation **do**

3   choose some $a \in A$ such that $p_a(s) \geq p_a(s')$;

4      $A := A \setminus \{a\};$

5 **return** A

---

profit from the deviation $s \xrightarrow{A^*} s^*$ either. So the coalition A the algorithm returns contains A^* and a fortiori is non-empty. Hence the **while** loop was exited because $s \xrightarrow{A} s'$, where $s_i' = c$ for $i \in A$, is a profitable deviation. By the maximality of A^* we get $A = A^*$.

If no coalition can profitably deviate to c, then the **while** loop is exited because $A = \emptyset$ and the algorithm returns the empty set. □

This lemma and Theorem 21 allow us to derive the following result.

Theorem 31. *Consider a coordination game on an open chain of cycles. A strong equilibrium can be computed in time $\mathcal{O}(vm^4 l)$.*

Proof. By Theorem 21 it follows that for an open chain of cycles there exists a c-improvement path of length at most $4vm^4$. Moreover, such a path consists of $\mathcal{O}(vm^4)$ single-player improvement steps and $\mathcal{O}(m)$ of c-improvement steps. By Lemma 26, executing the former steps can be done in time $\mathcal{O}(vm^4 l)$. It remains to estimate the latter.

All considered c-improvement steps are from a Nash equilibrium. So by Lemma 16, any node involved in a c-improvement step belongs to a directed simple cycle that deviated to the same colour. It follows that in any c-improvement step, nodes that deviate to two different colours cannot be adjacent to each other and so do not influence each other payoffs. Therefore, any multicolour c-improvement step can be split into a sequence of unicolour c-improvement steps (one for each deviating colour).

Consider now a Nash equilibrium s that is not a strong equilibrium. Each coordination game satisfies the PPM property, so Lemma 30 implies that by executing Algorithm 2 for each colour c in turn we eventually find a maximal coalition that can profitably deviate from s to the same colour or determine that no such coalition exists.

Let us now estimate the time complexity of executing Algorithm 2. Executing the assignment in line 1 can be done in $\mathcal{O}(vm)$ time. Computing the payoffs of every node in s and s' in line 2 can be done in $\mathcal{O}(vm)$ time due to Lemma 27. The **while** loop can be reentered at most vm times, because there are at most vm nodes in A. Further, because we are dealing with an open chain of cycles each removal of a node from A affects the payoff of at most two other players. So updating the payoffs of all players in s' can be done in $\mathcal{O}(1)$ time. Therefore executing the **while** loop takes in total $\mathcal{O}(vm)$ time. This is also the time complexity of executing the algorithm, since line 5 takes only $\mathcal{O}(1)$ time.

To find a unicolour profitable deviation from a Nash equilibrium that is not a strong equilibrium, in the worst case Algorithm 2 has to be executed for each colour. So each such c-improvement step takes in total $\mathcal{O}(vml)$ time. As there are $\mathcal{O}(m)$ of these c-improvement steps, their execution takes in total $\mathcal{O}(vm^2 l)$ time. So the execution of these steps

is dominated by the executions of the already considered single-player improvement steps that take in total $\mathcal{O}(vm^4l)$ time, which is then also the time bound for computing a strong equilibrium. $\qquad\square$

Finally, we deal with the cases of weighted DAGs and games with two colours.

Theorem 32. *Consider a coordination game on a weighted DAG. Both a Nash equilibrium and a strong equilibrium can be computed in time* $\mathcal{O}(nl + |E|)$.

Proof. Consider a weighted DAG (V, E). The procedure given in Theorem 17 first relabels the nodes using $\{1, \ldots, n\}$ in such a way that for all $i, j \in \{1, \ldots, n\}$ if $i < j$, then $(j \to i) \notin E$. Such a relabelling can be done in time $\mathcal{O}(n + |E|)$ by means of a topological sort of nodes using a DFS algorithm. Next, the schedule that we will use is simply $1, \ldots, n$. Due to Lemma 26, given a joint strategy the best response for a player i can be computed in $\mathcal{O}(l + e_i)$ time, where e_i is the number of incoming edges to node i.

Thus a Nash equilibrium can be constructed in time $\mathcal{O}(\sum_{i \in V}(l + e_i)) = \mathcal{O}(nl + |E|)$. By Theorem 17 every Nash equilibrium is also a strong equilibrium. $\qquad\square$

Theorem 33. *Consider a coordination game on a graph* (V, E) *in which only two colours are used.*

(i) *A Nash equilibrium can be computed in time* $\mathcal{O}(n + |E|)$.

(ii) *A strong equilibrium can be computed in time* $\mathcal{O}(n^2 + n|E|)$.

Proof. Given node i we denote by e_i the number of incoming edges to i and by e_i' the number of outgoing edges from i.

(i) The proof of Theorem 22 provides a procedure that follows two phases to construct a Nash equilibrium. Each phase constructs a maximal sequence of profitable deviations to a specific colour.

Note that by Lemma 27, given a joint strategy, the payoff of player i can be computed in $\mathcal{O}(1+e_i)$ time. Therefore, a profitable deviation from any joint strategy (if it exists) can be found in time $\sum_{i \in V} \mathcal{O}(1 + e_i) = \mathcal{O}(n + |E|)$.

This would yield time complexity of $\mathcal{O}(n^2 + n|E|)$ for both the first and the second phase, because each phase consists of at most n profitable deviations. To reduce this to $\mathcal{O}(n + |E|)$ we proceed as follows.

For each player i, given a joint strategy of its opponents, let (r_i, b_i) be its payoffs for selecting, respectively, red and blue colours. By Lemma 27, given an initial joint strategy, these pairs of payoffs for all players can be calculated in time $\sum_{i \in V} \mathcal{O}(1 + e_i) = \mathcal{O}(n + |E|)$. In the first phase, where players switch colour from red to blue only, we simultaneously create a list L of all players i whose current colour is red and $r_i < b_i$ holds.

We then repeatedly remove a player i from L and switch its colour to blue. This change affects the payoffs of e_i' other players. More precisely, $e_i' = |\{j \in V \mid i \in N_j\}|$ and for any j such that $i \in N_j$, the pair (r_j, b_j) is updated to $(r_j - w_{i \to j}, b_j + w_{j \to i})$. If after this change $r_j < b_j$ holds and player j holds colour red then we add player j to the list L. Note that no player has be removed from L as a result of the deviation of player i due to the PPM property of our games. Therefore, after a deviation of player i, the time needed to update all values of (r_j, b_j) and the list L is $\mathcal{O}(1 + e_i')$.

The first phase ends when L becomes empty. Then we rebuild the list by switching the role of the colours and proceed in the analogous way. In particular, from that moment on we add a player i to the list if $b_i < r_i$.

In each phase each player can switch its colour at most once, so the complexity of each phase, as well as both of them, is $\sum_{i \in V} \mathcal{O}(1 + e_i') = \mathcal{O}(n + |E|)$.

(ii) The existence of c-improvement paths of length at most $2n$ is guaranteed by Theorem 24. We now estimate time complexity of computing the c-improvement steps in the sequences ξ and χ, as defined in the proof of Theorem 24.

In each such step a coalition is selected that deviates profitably to a single colour, blue or red, the joint strategy is modified, and the payoffs of the players are appropriately modified. Without loss of generality we can assume that each time a maximal coalition is selected. By Lemma 30 such a coalition can be computed using Algorithm 2. So it suffices to determine the complexity of Algorithm 2 and of the computation of the new joint strategy and the modified payoffs in case of coordination games with two colours.

The complexity of executing the assignment in line 1 is $\mathcal{O}(n)$. To evaluate the condition of the **while** loop in line 2, we first calculate $p_i(s)$ and $p_i(s')$ for every player i. By Lemma 27 all these values and the set of players $A' := \{i \in A \mid p_i(s) \geq p_i(s')\}$, for which the deviation to s' is not profitable, can be calculated in time $\sum_{i \in V} \mathcal{O}(1 + e_i) = \mathcal{O}(n + |E|)$. Note that the body of the **while** loop is executed as long as $A' \neq \emptyset$.

After each removal of a node $a \in A'$ from A in line 4 (and as a result from A'), the payoffs $p_i(s')$ of at most e_a other players are affected and by Lemma 27 updating them takes time $\mathcal{O}(1 + e_a)$. At the same time, if for any of these e_a players, the deviation to s' is not longer profitable, i.e., $p_i(s) \geq p_i(s')$ holds, then we add him to A'. Note that no player has to be removed from A' after a deviation of player a due to the PPM property of our games.

Now, each player is removed in line 4 at most once, so the total time needed to execute this **while** loop is $\sum_{i \in V} \mathcal{O}(1 + e'_i) = \mathcal{O}(n + |E|)$ time. Finally, line 5 takes $\mathcal{O}(1)$ time. So for both colours the execution of Algorithm 2 takes $\mathcal{O}(n + |E|)$ time. Once the algorithm returns the empty set we switch the colours and move to the second phase. This phase ends when the algorithm returns the empty set. By Theorem 24 it follows that a strong equilibrium can be computed in time $\mathcal{O}(n^2 + n|E|)$. □

Finally, we study the complexity of determining the existence of Nash equilibria and of strong equilibria. We already noticed in Example 1 that some coordination games have no Nash equilibria. In general, the following holds.
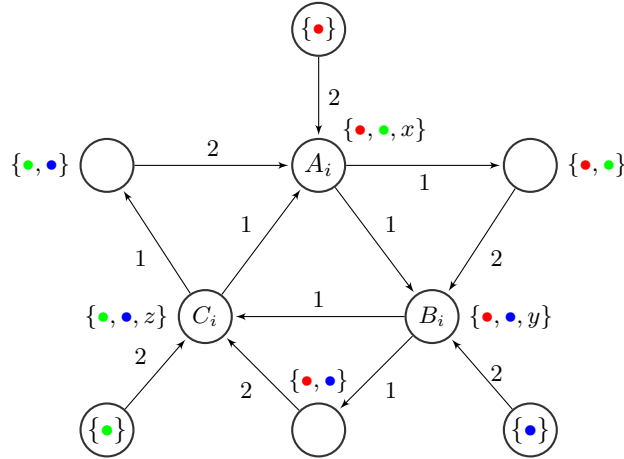

Figure 12: Gadget D_i with three parameters $x, y, z \in \{\top, \bot\}$ and three distinguished nodes A_i, B_i, C_i.

Theorem 34. *The Nash equilibrium existence problem in coordination games without bonuses (on unweighted graphs) is NP-complete.*

Proof. The problem is in NP, since we can simply guess a colour assignment and checking whether it is a Nash equilibrium can be done in polynomial time.
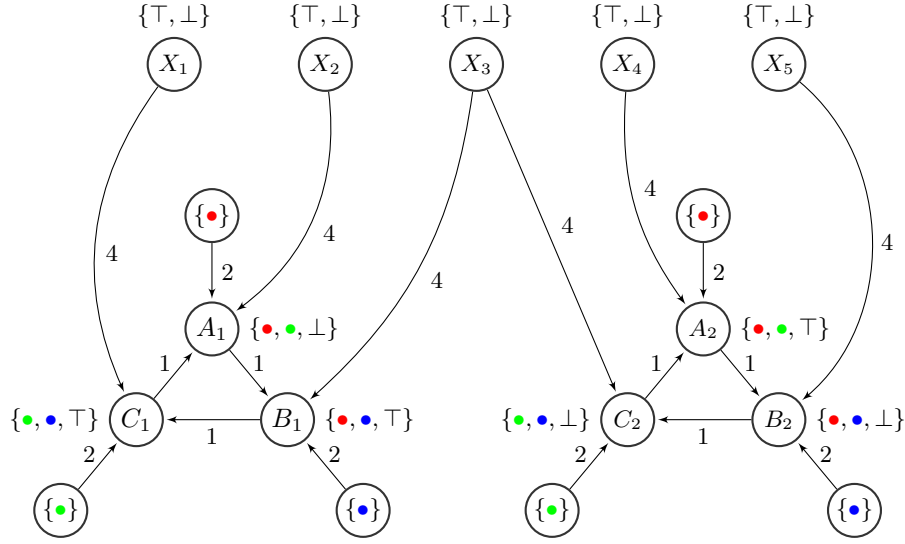
Figure 13: The game $\mathcal{G}_\phi$ corresponding to the formula $\phi = (\neg x_2 \vee x_3 \vee x_1) \wedge (x_4 \vee \neg x_5 \vee \neg x_3)$, where in each gadget the nodes of indegree 1 are omitted.

To prove NP-hardness we first provide a reduction from the 3-SAT problem, which is NP-complete, to coordination games on directed graphs with natural number weights. Assume we are given a 3-SAT formula

$$\phi = (a_1 \vee b_1 \vee c_1) \wedge (a_2 \vee b_2 \vee c_2) \wedge \ldots \wedge (a_k \vee b_k \vee c_k)$$

with k clauses and n propositional variables $x_1, \ldots, x_n$, where each a_i, b_i, c_i is a literal equal to x_j or $\neg x_j$ for some j. We will construct a coordination game $\mathcal{G}_\phi$ of size $\mathcal{O}(k)$ with natural number weights such that $\mathcal{G}_\phi$ has a Nash equilibrium iff ϕ is satisfiable.

First, for every propositional variable x_i we have a corresponding node X_i in $\mathcal{G}_\phi$ with two possible colours $\top$ and $\bot$. Intuitively, for a given truth assignment, if x_i is true then $\top$ should be chosen for X_i and otherwise $\bot$ should be chosen. In our construction we make use of a gadget, denoted by $D_i(x, y, z)$, with three parameters $x, y, z \in \{\top, \bot\}$ and i used just for labelling purposes, and presented in Figure 12. This gadget behaves similarly to the game without Nash equilibrium analysed in Example 1.

What is important is that for all possible parameters values, the gadget $D_i(x, y, z)$ does not have a Nash equilibrium. Indeed, each of the nodes A_i, B_i, or C_i can always secure a payoff 2, so selecting $\top$ or $\bot$ is never a best response and hence in no Nash equilibrium a node chooses $\top$ or $\bot$. The rest of the reasoning is as in Example 1.

For any literal l let

$$\mathrm{pos}(l) := \begin{cases} \top & \text{if } l \text{ is a positive literal} \\ \bot & \text{otherwise.} \end{cases}$$

For every clause $(a_i \vee b_i \vee c_i)$ in ϕ we add to the game graph $\mathcal{G}_\phi$ the $D_i(\mathrm{pos}(a_i), \mathrm{pos}(b_i), \mathrm{pos}(c_i))$ instance of the gadget. Finally, for every literal a_i, b_i, or c_i in ϕ, which is equal to x_j or $\neg x_j$ for some j, we add an edge from X_j to A_i, B_i, or C_i, respectively, with weight 4. We depict an example game $\mathcal{G}_\phi$ in Figure 13. (This Figure corrects the corresponding figure in [8]).

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We claim that $\mathcal{G}_\phi$ has a Nash equilibrium iff ϕ is satisfiable.

($\Rightarrow$) Assume there is a Nash equilibrium s in the game $\mathcal{G}_\phi$. We claim that the truth assignment $\nu : \{x_1, \ldots, x_n\} \to \{\top, \bot\}$ that assigns to each x_j the colour selected by the node X_j in s makes ϕ true. Fix $i \in \{1, \ldots, k\}$. We need to show that ν makes one of the literals a_i, b_i, c_i of the clause $(a_i \vee b_i \vee c_i)$ true.

From the above observation about the gadgets it follows that at least one of the nodes A_i, B_i, C_i selected in s the same colour as its neighbour X_j. Without loss of generality suppose it is A_i. The only colour these two nodes, A_i and X_j, have in common is $\mathrm{pos}(a_i)$. So X_j selected in s $\mathrm{pos}(a_i)$, which by the definition of ν equals $\nu(x_j)$. Moreover, by construction x_j is the variable of the literal a_i. But $\nu(x_j) = \mathrm{pos}(a_i)$ implies that ν makes a_i true.

($\Leftarrow$) Assume ϕ is satisfiable. Take a truth assignment $\nu : \{x_1, \ldots, x_n\} \to \{\top, \bot\}$ that makes ϕ true. For all j, we assign the colour $\nu(x_j)$ to the node X_j. We claim that this assignment can be extended to a Nash equilibrium in $\mathcal{G}_\phi$.

Fix $i \in \{1, \ldots, k\}$ and consider the $D_i(\mathrm{pos}(a_i), \mathrm{pos}(b_i), \mathrm{pos}(c_i))$ instance of the gadget. The truth assignment ν makes the clause $(a_i \vee b_i \vee c_i)$ true. Suppose without loss of generality that ν makes a_i true. We claim that then it is always a unique best response for the node A_i to select the colour $\mathrm{pos}(a_i)$.

Indeed, let j be such that $a_i = x_j$ or $a_i = \neg x_j$. Notice that the fact that ν makes a_i true implies that $\nu(x_j) = \mathrm{pos}(a_i)$. So when node A_i selects $\mathrm{pos}(a_i)$, the colour assigned to X_j, its payoff is 4.

This partial assignment of colours can be completed to a Nash equilibrium. Indeed, remove from the directed graph of $\mathcal{G}_\phi$ all X_j nodes and the nodes that secured the payoff 4, together with the edges that use any of these nodes. The resulting graph has no cycles, so by Theorem 17 the corresponding coordination game has a Nash equilibrium. Combining both assignments of colours we obtain a Nash equilibrium in $\mathcal{G}_\phi$.

To conclude the result for coordination games without weights notice that an edge with a natural number weight w can be simulated by adding w extra players to the game. More precisely, an edge $(i \to j)$ with the weight w can be simulated by the extra set of players $\{i_1, \ldots, i_w\}$ and the following $2 \cdot w$ unweighted edges: $\{(i \to i_1), (i \to i_2), \ldots, (i \to i_w), (i_1 \to j), (i_2 \to j), \ldots, (i_w \to j)\}$. Given a colour assignment in the original game with the weighted edges, we then assign to each of the new nodes $i_1, \ldots, i_w$ the colour set of the node i. Then the initial coordination game has a Nash equilibrium iff the new one, without weights, has one. Further, the new game can be constructed in linear time. $\square$

Corollary 35. *The strong equilibrium existence problem in coordination games without bonuses (on unweighted graphs) is NP-complete.*

Proof. It suffices to note that in the above proof the ($\Rightarrow$) implication holds for a strong equilibrium, as well, while in the proof of the ($\Leftarrow$) implication by virtue of Theorem 17 actually a strong equilibrium is constructed. $\square$

An interesting application of Theorem 34 is in the context of polymatrix games introduced in Section 2. It was shown in [45] that deciding whether a polymatrix game has a Nash equilibrium is NP-complete. We can strengthen this result by showing that the problem is strongly NP-hard, i.e., NP-hard even if all input numbers are bounded by a polynomial in the size of the input.

Theorem 36. *Deciding whether a given polymatrix game has a Nash equilibrium is strongly NP-complete.*

Proof. Any coordination game $\mathcal{G} = (G, C)$ on an unweighted graph $G = (V, E)$ can be viewed as a polymatrix game $\mathcal{P}$ whose values of all partial payoffs functions are equal either 0 or 1. Specifically, the set of players in $\mathcal{P}$ is the same as in $\mathcal{G}$, i.e., V. The strategy set S_i of player i is simply $C(i)$. We define

$$a^{ij}(s_i, s_j) := \begin{cases} 1 & \text{if } j \in N_i \text{ and } s_i = s_j \\ 0 & \text{otherwise} \end{cases}$$

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where, as before, N_i is the set of neighbours of node i in the assumed directed graph G.

Notice that the payoffs in both games are the same since for any joint strategy $s = (s_1, \ldots, s_n)$

$$p_i^{\mathcal{P}}(s) = \sum_{j \neq i} a^{ij}(s_i, s_j) = |\{j \in N_i \mid s_i = s_j\}| = p_i^{\mathcal{G}}(s).$$

NP-hardness follows, because this problem was shown to be NP-hard for coordination games on unweighted graphs in Theorem 34.

As all numerical inputs are assumed to be 0 or 1 they are obviously bounded by a polynomial in the size of the input. So strong NP-hardness follows. As shown in [45], deciding whether a given polymatrix game has a Nash equilibrium is in NP, which together implies strong NP-completeness of this problem. $\qquad\square$

8 Conclusions

In this paper we studied natural coordination games on weighted directed graphs, in presence of bonuses representing individual preferences. In our presentation we focussed on the existence of Nash and strong equilibria and on ways of computing them efficiently in case they exist. To this end we extensively used improvement and coalitional improvement (in short c-improvement) paths that can be seen as an instance of a local search.

We identified natural classes of graphs for which coordination games have improvement or c-improvement paths of polynomial length. For simple cycles these results are optimal in the sense that lifting any of the imposed restrictions may result in coordination game without a Nash equilibrium.

In proving our results we used increasingly more complex ways of constructing (c-)improvement paths of polynomial length. In particular, the construction in the proof of Theorem 14 relied on the constructions considered in the proofs of Theorems 3 and 7.

For the class of graphs we considered local search in the form of the (c-)improvement paths turns out to be an efficient way of computing a Nash equilibrium or a strong equilibrium. But this is not true in general. In fact, Example 25 shows that this form of local search does not guarantee that a Nash equilibrium or a strong equilibrium can be found, even when the underlying graph is strongly connected and all nodes have the same set of colours. We also showed that the existence problem both for Nash and strong equilibria is NP-complete even for the coordination games on unweighted graphs and without bonuses.

There are other directed graphs than the ones we considered here, for which the coordination games are weakly or c-weakly acyclic. For example, we proved in [2] that the coordination games on complete graphs have the c-FIP and the proof carries through to the complete directed graphs. In turn, in [8] we showed that that every coordination game on a directed graph in which all strongly connected components are simple cycles is c-weakly acyclic. Further, in [46] weighted open chains of cycles, closed chains of cycles, and simple cycles with appropriate cross-edges were considered.

For some of these classes of graphs some problems remain open, for instance the existence of finite c-improvement paths for weighted open chains of cycles. A rigorous presentation of the proofs of weak acyclicity and c-weak acyclicity for the corresponding coordination games is lengthy and quite involved. We plan to present them in a sequel paper.

Finally, we believe that the following generalisation of several of our results is true.

Conjecture Coordination games on graphs with all nodes of indegree ≤ 2 are c-weakly acyclic.

Extensive computer simulations seem to support this conjecture. However, our techniques do not seem to adapt easily to this bigger class of graphs.

A Remainder of the proof of Lemma 12.

For the convenience of the reader we recall here Figures 4, 5, 6, 7, and 8.

$-$	x		U$-$	x		?	x
+/U+	x		U+	x		+/U+	any

Figure 4: Possible changes of the grades of $\mathcal{C}_j$ and $\mathcal{C}_{j+1}$ when $j = 1$.

case	+	U$-$	x	U+	U$-$	x
(i)	+	U+	x	U+	U+	x
(ii)	+/$-$/U+/U$-$	U+	x	impossible		
(io)	U$-$/U+	+/U+	x	impossible		
(ioi)	U$-$/U+	U+	any	impossible		
(ioo)	U$-$/U+	+	any	impossible		
(iooi)	impossible			impossible		

Figure 5: Possible changes of the grades of $\mathcal{C}_{j-1}, \mathcal{C}_j$, and $\mathcal{C}_{j+1}$ when $1 < j < m$ and the grade of $\mathcal{C}_j$ is U$-$.

case	+	$-$	x	U+	$-$	x
(o)	+	+	x	U+	+	x
(oi)	+/$-$/U+/U$-$	+/U+	x	impossible		
(oo)	U$-$/U+	+	x	impossible		
(ooi)	U$-$/U+	U+	any	impossible		

Figure 6: Possible changes of the grades of $\mathcal{C}_{j-1}, \mathcal{C}_j$, and $\mathcal{C}_{j+1}$ when $1 < j < m$ and the grade of $\mathcal{C}_j$ is $-$.

We provide here justifications of the changes of the grades in these Figures.
Re: Figure 4.
Case 1. The initial grade of $\mathcal{C}_j$ is $-$.

This corresponds to the situation at the beginning of *Phase 2* in the proof of Theorem 3 when exactly one node has a bonus. This phase starts with the node $[j, 2]$ and ends after at most $n - 1$ steps. So the colour of $[j, 1]$ is not modified and consequently the payoff to the down-link node $[j + 1, k]$ of $\mathcal{C}_{j+1}$ is not modified. Further the new grade of $\mathcal{C}_j$ can be either + or U+ depending whether at the end of this phase the colours of $[j, v]$ and $[j, 1]$ differ.

Case 2. The initial grade of $\mathcal{C}_j$ is U$-$.

The reasoning is the same as in **Case 1**. However, the colour of $[j, v]$ is now not modified. The reason is that the only colour that is propagated is that of $[j, 1]$ and initially it is also the colour of $[j, v]$. So the new grade of $\mathcal{C}_j$ is now U+.

Case 3. The initial grade of $\mathcal{C}_j$ is ?.

This corresponds to the situation at the beginning of *Phase 1* in the proof of Theorem 3 when exactly one node has a bonus. The constructed improvement path ends after at most $2n - 1$ steps, so in the process the colour of $[j, 1]$ can

+	?	x	U+	?	x
+/–/U+/U–	+/U+	any	U+	+/U+	any

Figure 7: Possible changes of the grades of $\mathcal{C}_{j-1}$, $\mathcal{C}_j$, and $\mathcal{C}_{j+1}$ when $1 < j < m$ and the grade of $\mathcal{C}_j$ is ?.

case	+	U–		U+	U–
(i)	+	U+		U+	U+
(ii)	+/–/U+/U–	U+		impossible	
(io)	U–/U+	+/U+		impossible	
(ioi)	U–/U+	U+		impossible	
case	+	–		U+	–
(o)	+	+		U+	+
(oi)	+/–/U+/U–	+/U+		impossible	
(oo)	U–/U+	+		impossible	
(ooi)	U–/U+	U+		impossible	

+	?		U+	?
+/–/U+/U–	+/U+		U+	+/U+

Figure 8: Possible changes of the grades of $\mathcal{C}_{j-1}$ and $\mathcal{C}_j$ when $j = m$.

change. If it does, then the grade of the cycle $\mathcal{C}_{j+1}$ can change arbitrarily. In particular, it can become U+ or U– if the down-link node of $\mathcal{C}_{j+1}$ is $[j+1, v]$.

Further the new grade of $\mathcal{C}_j$ can be either + or U+, for the same reasons as in **Case** 1.

Re: Figure 5.

The assumption that the grade of $\mathcal{C}_j$ is initially U– means that initially the colours of $[j, 1]$ and its predecessor $[j, v]$ in this cycle are the same. Then the construction in line 6 of the improvement path for the considered coordination game for $\mathcal{C}_j$ with bonuses for the link nodes corresponds to any update scenario presented in *Phase 2* of the proof of Theorem 7 that starts with *i*. There are six such scenarios to consider.

Case *(i)*.

This means that the propagation of the colour of the up-link node of $\mathcal{C}_j$ stops before the down-link node of $\mathcal{C}_j$ is reached. So the improvement path constructed in line 6 does not change the colours of the link nodes of $\mathcal{C}_j$ and of the predecessor $[j, v]$ of the up-link node $[j, 1]$. Hence the grades of $\mathcal{C}_{j-1}$ and $\mathcal{C}_{j+1}$ remain unchanged and the grade of $\mathcal{C}_j$ becomes U+.

The remaining cases consider the situations in which the down-link node of $\mathcal{C}_j$ switches to another colour. We now claim that in these cases the grade of $\mathcal{C}_{j-1}$ is initially +. Indeed, if this grade is initially U+, then the payoff to the up-link node $[j-1, 1]$ of $\mathcal{C}_{j-1}$ is ≥ 1. But $[j-1, 1]$ is also the down-link node of $\mathcal{C}_j$, so the claim follows by Note 10.

Case *(ii)*.

This means that the propagation of the new colour of the up-link node of $\mathcal{C}_j$ stops between the down-link and up-link nodes of $\mathcal{C}_j$ and that the down-link node adopted the colour of the up-link node. So the improvement path constructed in line 6 does not change the colours of $[j, 1]$ and its predecessor $[j, v]$.

Hence the grade of C_j becomes U+ and the grade of C_{j+1} remains unchanged. On the other hand, the grade of C_{j-1} can remain unchanged or change from + to −, U+ of U− because of the new colour of the up-link node $[j-1,1]$ of C_{j-1}.

Case *(io)*.

This means that the propagation of the colours stops between the down-link and up-link nodes of C_j but now the down-link node (so $[j-1,1]$) adopted the colour of its predecessor $[j-1,v]$ in C_{j-1}. So as in the previous case the grade of C_{j+1} remains unchanged.

However, the grade of C_j can now also become + if this propagation of the colours changes the colour of the predecessor $[j,v]$ of the up-link node $[j,1]$. Further, the grade of C_{j-1} now changes from + to U− or U+ because the new colour of $[j-1,1]$ is now the colour of $[j-1,v]$ and as a result the node $[j-1,2]$ can now become the only node that does not play a best response.

Case *(ioi)*.

This means that the propagation of the colours now stops between the up-link and down-link nodes of C_j but now the down-link node (so $[j-1,1]$) adopted the colour of its predecessor $[j-1,v]$ in C_{j-1} and subsequently the up-link node $[j,1]$ of C_j adopted the colour of its predecessor $[j,v]$ in C_j. So the grade of C_j now becomes U+.

Further, the grade of C_{j-1} now changes from + to U− or U+ for the same reasons as in the previous case. Finally, the grade of C_{j+1} can now change arbitrarily for the same reasons as in Case 3 concerning Figure 4.

Case *(ioo)*.

This case is similar to the previous one, with the difference that in the second round of the propagation of the colours the up-link node $[j,1]$ of C_j adopted the colour of its predecessor in C_{j+1} instead of the colour of its predecessor $[j,v]$ in C_j. Consequently, the grade of C_j now becomes +. Further, the grade of C_{j-1} can now change from + to U− or U+, while the grade of C_{j+1} can now change arbitrarily, both for the same reason as in the previous case.

Case *(iooi)*.

This case cannot occur. Indeed, it would imply that the down-link node in C_j first switches to the colour of its predecessor in C_{j-1} and later switches to different colour. But the second switch is not possible due to Note 10.

Re: Figure 6.

The assumption that the grade of C_j is initially − means that initially the colours of $[j,1]$ and its predecessor $[j,v]$ in this cycle differ. Then the construction in line 6 of the improvement path for the considered coordination game for C_j with bonuses for the link nodes corresponds to any update scenario presented in *Phase 2* of the proof of Theorem 7 that starts with *o*. There are four such scenarios to consider.

Case *(o)*.

The reasoning is the same as in **Case** *(i)* above with the difference that the grade of C_j becomes now + as the colours of $[j,1]$ and $[j,v]$ do not change and hence remain different.

In the remaining cases the grade of C_{j-1} is initially + for the reasons given after **Case** *(i)* above.

Case *(oi)*.

This case is analogous to **Case** *(ii)* above. In particular, the improvement path constructed in line 6 does not change the colours of $[j,1]$ and its predecessor $[j,v]$. Hence the grade of C_j becomes + and the grade of C_{j+1} remains unchanged, while the grade of C_{j-1} can remain unchanged or change from + to −, U+ of U−.

Case *(oo)*.

This case is analogous to **Case** *(io)* above. So, as in that case, the grade of $\mathcal{C}_{j+1}$ remains unchanged and the grade of $\mathcal{C}_{j-1}$ now changes from + to U− or U+. However, the grade of $\mathcal{C}_j$ can now also become U+ if this propagation of the colours changes the colour of $[j, v]$ to the colour of its successor $[j, 1]$.

Case *(ooi)*.

This case is analogous to **Case** *(ioi)* above. So, as in that case the grade of $\mathcal{C}_j$ now becomes U+, the grade of $\mathcal{C}_{j-1}$ changes from + to U− or U+, and the grade of $\mathcal{C}_{j+1}$ can change arbitrarily.

Re: Figure 7.

This case corresponds to the situation at the beginning of *Phase 1* in the proof of Theorem 7. The constructed improvement path ends after at most $3n$ steps, so in the process the colour of $[j, 1]$ can change. Therefore, as in **Case 3** concerning Figure 4, the grade of the cycle $\mathcal{C}_{j+1}$ can change arbitrarily, while the grade of $\mathcal{C}_j$ can become either + or U+.

Finally, if initially the grade of $\mathcal{C}_{j-1}$ is +, then as in **Case** *(ii)*, its grade can remain unchanged or change to −, U+ of U−. Further, if initially this grade is U+, then by the argument used in the proof of Note 11 the grade does not change.

Re: Figure 8.

We reduce the analysis for this case to the previous three cases by extending the open chain with a new cycle $\mathcal{C}_{m+1}$ in which all new nodes have to their disposal colours that all differ from the colours available to the nodes of $\mathcal{C}_m$. Then in Algorithm 1 the bonus function for the up-link node of $\mathcal{C}_m$ is always 0 on the colours available to it, and consequently for $j = m$ the improvement path constructed in line 6 of Algorithm 1 is the same as for the original open chain. So for the case when $j = m$ we can use Figures 5, 6, and 7 with the last columns always omitted. This yields Figure 8.

A perceptive reader can inquire why the row corresponding to the case *(ioo)* is missing. The reason is that it deals with the situation when the up-link node of $\mathcal{C}_j$ switches to an outer colour, i.e, a colour of its predecessor in $\mathcal{C}_{j+1}$. But for $j = m$ this cannot happen by the choice of the colours for the new nodes.

Acknowledgements

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