European Journal of Combinatorics 82 (2019) 102992

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Asymptotics of pattern avoidance in the Klazar set partition and permutation-tuple settings

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ARTICLE INFO

Article history:
Received 25 June 2018
Accepted 9 July 2019
Available online xxxx

ABSTRACT

We consider asymptotics of set partition pattern avoidance in the sense of Klazar. Our main result derives the asymptotics of the number of set partitions avoiding a given set partition within an exponential factor, which leads to a classification of possible growth rates of set partition pattern classes. We further define a notion of permutation-tuple avoidance, which generalizes notions of Aldred et al. and the usual permutation pattern setting, and similarly determine the number of permutation-tuples avoiding a given tuple to within an exponential factor. We note a seeming connection to previous results on hereditary properties of labelled graphs, prompting the question of if there is a generalization to ordered graphs.

1. Introduction

A fundamental question of pattern avoidance is that of asymptotics. That is, for some pattern p, how does the avoidance function $A_n(p)$, equal to the number of patterns of size n avoiding p, grow? More generally, what are the possible growth speeds of pattern classes? This has been especially well-studied in the most classical pattern avoidance area, that of permutations. The most famous result of this kind is the Marcus–Tardos Theorem, known earlier as the Stanley–Wilf Conjecture [16], and generalized repeatedly in later years.

Though initially only studied in the case of permutations, the concepts of pattern avoidance and the corresponding asymptotics have been found to apply to other structures as well. Klazar [12]

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https://doi.org/10.1016/j.ejc.2019.07.003

proposed a notion of set partition pattern avoidance[1] and proved several results about special cases involving the generating function of the avoidance sequence. Later, his conjecture about the case when the partitions have what this paper will refer to as permutability 1 (Klazar refers to these as *sufficiently restrictive partitions*, or srps) was proven independently by Klazar and Marcus [14] and Balogh, Bollobás, and Morris [3]. The most important result of this paper will be a generalization of those results, completely classifying the possible speeds of set partition pattern classes to an exponential factor.

We will see that the study of set partition pattern avoidance motivates a new notion of k-tuple permutation pattern avoidance, which generalizes classical permutation pattern avoidance (corresponding to $k = 1$) and the setting of pairs of permutations in Aldred et al. [1], which corresponds to $k = 2$. We will see that our set partition result implies a generalization of the Marcus–Tardos theorem to this setting.

As alluded to, the primary result of this paper is the following theorem:

Theorem 1.1. *All pattern classes of set partitions must grow as the Bell number B_n, eventually becomes* 0*, or grows within an exponential factor of $n^{n\left(1-\frac{1}{d}\right)}$ for some positive integer d.*

Our results will provide a mechanism for determining when a pattern class falls into a certain growth rate (parameterized by d). There will be a series of 'jumps' in growth rate, in the sense that there is a unique minimum pattern class T^d that grows within exponentially of $n^{n\left(1-\frac{1}{d}\right)}$ for each d, so that T^d grows at rate $c^n n^{n\left(1-\frac{1}{d}\right)}$ but if $T^d \not\subset \mathcal{C}$ then $\mathcal{C}$ grows at rate at most $c^n n^{n\left(1-\frac{1}{d-1}\right)}$.

T^d will turn out to be the pattern class consisting of set partitions of permutability at most $d-1$, where permutability (denoted by pm) is a certain statistic of permutations to be defined in Section 2.

Due to these observations, to prove Theorem 1.1 it will turn out to be sufficient to prove results in the case of the pattern class $Av(\pi)$ avoiding a single pattern. The main result here that we will prove is the following.

Theorem 1.2. *Let π be a set partition with $pm(\pi) \geq 1$. Then there exists a constant $c_2(\pi) > 1$ such that*

$$B_n(\pi) \leq c_2(\pi)^n n^{n\left(1-\frac{1}{pm(\pi)}\right)}$$

for all n. If $pm(\pi) = 0$, then there exists a constant $c_2(\pi)$ so that

$$B_n(\pi) \leq c_2(\pi)^n.$$

Similarly to the proof of the Stanley–Wilf conjecture, we will deduce this statement from an extremal one about forbidden ordered structures, which is a common generalization of two different generalizations of the Marcus–Tardos theorem about forbidden matrix patterns; one about hypermatrices, due to Klazar and Marcus [14], and the other about incidence in hypergraphs avoiding a 2-permutation graph, due to Balogh, Bollobás, and Morris [3], Klazar–Marcus [14]. We denote by $i(G)$ number of vertex–edge incidences in a hypergraph G; that is, the sum of the sizes of all hyperedges in G, $\sum_{E \text{ a hyperedge of } G} |E|$. We also use the notation $[n]$ for the set $\{1, \ldots, n\}$.

Theorem 1.3. *Let H be a fixed d-permutation hypergraph. Then for any $n \in \mathbb{Z}^+$ and hypergraph G on $[n]$ avoiding H, $i(G) = O(n^{d-1})$.*

Using a standard argument, we obtain the following as a corollary.

Lemma 1.4. *Let H be a fixed d-permutation hypergraph. There exists $c > 0$ such that for all $n \in \mathbb{Z}^+$, there are at most $c^{n^{d-1}}$ ordered hypergraphs on $[n]$ that avoid H.*

1 There are other possible definitions of set partition avoidance; for example, RGF-type pattern avoidance, studied in great detail by for example Mansour [15], which we will not discuss in this paper.

Section 2 will provide the necessary background for our results. Section 3 will discuss the new results in more detail, and describe the bounds we will prove for the single-pattern avoidance case. Section 4 will prove our lower bound. Sections 5, 6, and 7 form the heart of our argument, and respectively state our main lemma, prove the lemma, and then deduce the single-pattern upper bound given in Theorem 1.2 from the lemma. Section 8 will apply our results to parallel permutation pattern avoidance. Section 9 will give the simple deduction and generalization of Theorem 1.1 from the single-pattern case of Theorem 1.2. Finally, Section 10 will discuss possible (open) connections with hereditary graph properties, and Section 11 will describe additional possible directions in which to take our results.

2. Definitions and preliminary results

Definition 2.1. A *set partition* of n is a partition of $[n]$ into sets, where we ignore ordering of sets and ordering within the sets. We will write set partitions with slashes between the sets, as in $T_1/T_2/\cdots/T_m$ for some m. The *standard form* of a set partition is what is obtained from writing each T_i in increasing order, and then rearranging the sets so that $\min T_1 < \min T_2 < \cdots < \min T_m$. The T_i are called the *blocks* of the partition.

For example, 1635/24 and 24/1356 represent the same set partition but neither is in standard form; the standard form for this partition is 1356/24.

Definition 2.2. The *Bell number* B_n is the number of set partitions of $[n]$.

Definition 2.3. A set partition π of n *contains* a set partition π' of k in the Klazar sense (which we will use for the remainder of this paper) if there is a subset S of $[n]$ of cardinality k such that when π is restricted to the elements of S, the result is order-isomorphic to π'. We also say π' *occurs* in π. Otherwise, we say π *avoids* π'.

For example, 136/5/27 contains 14/23 because when we restrict 136/5/27 to the set $\{2, 3, 6, 7\}$, the result is 36/27, which is order-isomorphic to 23/14, standardizing to 14/23. However, it avoids 1/234.

We can think of containment in the following way: if we have some $f : [m] \to [n]$ and a set partition of $[n]$, we can take the pullback under f to get a partition of $[m]$, where a and b are in the same partition if and only if $f(a)$ and $f(b)$ are. Then π contains π' if and only if π' is the pullback of π under some order-preserving injection.

Note that this Klazar notion of avoidance differs from the RGF notion of pattern avoidance in set partitions, studied in detail by Mansour [15], where switching the order of the sets during standardization is not allowed.

We will be concerned with the enumeration of the number of partitions of a given length that avoid a particular pattern.

Definition 2.4. If π is some set partition of k, let $B_n(\pi)$ be the number of set partitions of n that avoid π. (Note that the notation is analogous to that for Bell numbers.)

Much of this paper is devoted to progress towards general asymptotic bounds for $B_n(\pi)$.

Definition 2.5. A *layered partition* is a partition $T_1/\cdots/T_m$ such that $\max T_i < \min T_{i+1}$ for all $i \in [m-1]$. Equivalently, each set T_i consists of an interval of consecutive integers.

For example, 12/3456/789 is layered while 13/2456/789 is not.

Alweiss [2] found the correct log-asymptotics for $B_n(\pi)$ in the case where π is layered. Earlier, Klazar and Marcus in [14] classified the cases where $B_n(\pi)$ grows at most exponentially, as a corollary of their Corollary 2.2.

An important notion in this paper will be relating set-partition pattern avoidance to tuple permutation pattern avoidance. To this end, we define the following notion.

Definition 2.6. Let $\sigma_1, \ldots, \sigma_d$ be permutations, $\sigma_i : [n] \to [n]$. We define the *set partition correspondent to* $(\sigma_1, \ldots, \sigma_d)$ to be the partition $T_1/\cdots/T_n$ of $(d+1)n$ such that $T_i = \{i, n + \sigma_1(i), 2n + \sigma_2(i), \ldots, dn + \sigma_d(i)\}$. It is easy to see that this is indeed a set partition, and we will write it $[\sigma_1, \ldots, \sigma_d]$.

Notice that a set partition of $(d+1)n$ is correspondent to some $(\sigma_1, \ldots, \sigma_d)$ if and only if every set in the partition contains exactly one element from each of $\{1, \ldots, n\}$, $\{n+1, \ldots, 2n\}$,..., $\{dn+1, \ldots, (d+1)n\}$. Klazar [12] referred to partitions of the form $[\sigma]$ (so $d = 1$) or partitions contained in any partition of this form as *srp*'s, and as previously alluded to, Klazar and Marcus proved in [14] that for π an srp, there exists $c > 0$ with $B_n(\pi) \le c^n$.

Now, we define what we will call *parallel pattern avoidance* for d-tuples of permutations $(\sigma_1, \ldots, \sigma_d)$. We denote by S_n the family of all permutations over $[n]$.

Definition 2.7. $\sigma_1, \ldots, \sigma_d \in S_n$ and $\sigma'_1, \ldots, \sigma'_d \in S_m$, $(\sigma_1, \ldots, \sigma_d)$ contains (respectively avoids) $(\sigma'_1, \ldots, \sigma'_d)$ if there exist (respectively do not exist) indices $c_1 < \cdots < c_m$ in $[n]$ such that for all i, $\sigma_i(c_1)\sigma_i(c_2)\cdots\sigma_i(c_m)$ is order-isomorphic to σ'_i.

We will occasionally say 'contains/avoids in parallel' to refer to this notion in particular.

For $d = 1$, parallel pattern avoidance reduces to the classical permutation pattern containment/avoidance. The idea of parallel avoidance for d-tuples of permutations also reduces to several other interesting concepts in special cases; for example, (σ_1, σ_2) avoids $(12, 21)$ if and only if $\sigma_1^{-1} \le \sigma_2^{-1}$ in the Weak Bruhat Order, which has been previously studied; for example, see [10] and $A007767$ in [18].

We now relate this to our topic of partition pattern avoidance.

Proposition 2.8. *Let $\sigma_1, \ldots, \sigma_d$ be permutations in S_n and $\sigma'_1, \ldots, \sigma'_d$ be permutations in S_m. The following two statements are equivalent:*

- *The d-tuple of permutations $(\sigma_1, \ldots, \sigma_d)$ contains the d-tuple of permutations $\sigma'_1, \ldots, \sigma'_d$.*
- *The set partition $[\sigma_1, \ldots, \sigma_d]$ contains the set partition $[\sigma'_1, \ldots, \sigma'_d]$.*

Proof. If $(\sigma_1, \ldots, \sigma_d)$ contains $(\sigma'_1, \ldots, \sigma'_d)$, we have indices $c_1 < \cdots < c_m$ in $[n]$ such that for all j, $\sigma_j(c_1)\cdots\sigma_j(c_m)$ is order-isomorphic to σ'_j. $[\sigma_1, \ldots, \sigma_d]$ has blocks $T_1, \ldots, T_n$ given by $T_i = \{i, n + \sigma_1(i), 2n + \sigma_2(i), \ldots, dn + \sigma_d(i)\}$. Restricting it to simply the elements in $T_{c_1}, \ldots, T_{c_m}$, we have blocks given by $\{c_i, n + \sigma_1(c_i), \ldots, dn + \sigma_d(c_i)\}$. We show that the partition restriction is order-isomorphic to $[\sigma'_1, \ldots, \sigma'_d]$. Since $c_i = \min T_{c_i}$, and the c_i are increasing, the block T_{c_i} must correspond to the ith block of $[\sigma'_1, \ldots, \sigma'_d]$, which is $\{i, m+\sigma'_1(i), \ldots, dm+\sigma'_d(i)\}$. Thus, we must show that $j_1 n + \sigma_{j_1}(c_{i_1}) < j_2 n + \sigma_{j_2}(c_{i_2})$ if and only if $j_1 m + \sigma'_{j_1}(i_1) < j_2 m + \sigma'_{j_2}(i_2)$. But since $1 \le \sigma_a(b) \le n$ and $1 \le \sigma'_a(b) \le m$ for all a, b, the first statement is equivalent to $j_1 < j_2$ or $j_1 = j_2 = j$ and $\sigma_j(c_{i_1}) < \sigma_j(c_{i_2})$, and the second is equivalent to $j_1 < j_2$ or $j_1 = j_2 = j$ and $\sigma'_j(i_1) < \sigma'_j(i_2)$. These are equivalent by the definition of pattern containment for k-tuples of permutations.

Now suppose $[\sigma_1, \ldots, \sigma_d]$ contains $[\sigma'_1, \ldots, \sigma'_d]$. Since all blocks of both partitions have size $d+1$, the blocks of the latter partition must correspond exactly to m blocks of the former, say blocks $T_{c_1}, \ldots, T_{c_m}$ with $c_1 < \cdots < c_m$. Now following the same argument as in the previous paragraph in reverse, we see that $(\sigma_1, \ldots, \sigma_d)$ contains $(\sigma'_1, \ldots, \sigma'_d)$ (at indices $c_1, \ldots, c_m$), as we showed the ordering information is exactly equivalent in both cases. $\square$

The concept of permutation-correspondent partitions gives us a useful statistic.

Definition 2.9. The *permutability* of a set partition π, which we will call $\mathrm{pm}(\pi)$, is the minimum d such that there exists a d-tuple of permutations $(\sigma_1, \ldots, \sigma_d) \in S_n^d$ such that the correspondent partition $[\sigma_1, \ldots, \sigma_d]$ contains π.

Notice that this definition implies that permutability is an increasing function on the poset of set partitions under containment. This is because if $\mathrm{pm}(\pi) = d$, then $\pi \in [\sigma_1, \ldots, \sigma_d]$ for some $\sigma_1, \ldots, \sigma_d$, and thus any π' contained in π is also contained in $[\sigma_1, \ldots, \sigma_d]$ and thus must have permutability at most d. Note that as one would expect, $[\sigma_1, \ldots, \sigma_d]$ has permutability d, as it has a block of size $d+1$, which is not contained in $[\sigma'_1, \ldots, \sigma'_{d-1}]$ for any choice of the σ'_i. If π is the set partition of $[n]$ with all blocks of size 1, then we set $\mathrm{pm}(\pi) = 0$. Also, one can suppose that the σ_i above are permutations of the numbers up to the number of blocks of π, as permutations on at least these many elements are needed, and surplus elements could be deleted without changing the order of the image of the smaller numbers.

Proposition 2.10. *A set partition of* $[n]$ *has permutability at most* $n-1$. *Equality is only attained by the single-block partition of* $[n]$.

Proof. Note that $\{1, n+1, \ldots, n^2-n+1\}, \{2, n+2, \ldots, n^2-n+2\}, \ldots, \{n, 2n, \ldots, n^2\}$ is a permutation-correspondent partition of permutability $n-1$. We show that it contains every set partition of $[n]$. Let $\pi = B_1/\cdots/B_m$ be a set partition of $[n]$ in standard form. Then letting $f : [n] \to [m]$ be the function that takes j to the index of the block of π containing j, we see that $\{(j-1)n + f(j)\}_{j\in[n]}$ constitutes an occurrence of π within our partition.

Further, if π does not consist of just a single block, let j_0 be the smallest element of $[n]$ not in B_1. We know that $1 \in B_1$. We then see that $\{(j-1)n+1\}_{j<j_0} \cup \{(j-2)n + f(j)\}_{j\geq j_0}$ gives an occurrence of π in our partition that does not use the largest element of each block. Removing those largest elements, we obtain a permutation-correspondent partition of permutability $n-2$, so all partitions π of $[n]$ that are not the single-block partition have permutability at most $n-2$. Since the single-block partition clearly has permutability at least $n-1$, we have proven the remark. □

3. Old and new results

One of the main goals of this paper is to determine as closely as possible the asymptotics of $B_n(\pi)$. It is not difficult to get a lower bound for $B_n(\pi)$, and we show the following.

Theorem 3.1. *Let* π *be a set partition with* $pm(\pi) \geq 1$. *Then there exists a constant* $c_1(\pi) > 0$ *such that*

$$B_n(\pi) \geq c_1(\pi)^n n^{n\left(1-\frac{1}{pm(\pi)}\right)}$$

for all n.

We will also prove the following upper bound, which will determine the growth rate of $B_n(\pi)$ to within an exponential factor.

Theorem 1.2. *Let* π *be a set partition with* $pm(\pi) \geq 1$. *Then there exists a constant* $c_2(\pi) > 1$ *such that*

$$B_n(\pi) \leq c_2(\pi)^n n^{n\left(1-\frac{1}{pm(\pi)}\right)}$$

for all n. *If* $pm(\pi) = 0$, *then there exists a constant* $c_2(\pi)$ *so that*

$$B_n(\pi) \leq c_2(\pi)^n.$$

Note that Klazar and Marcus proved Theorem 1.2 in the case where $\mathrm{pm}(\pi) = 1$ in [14].

The most general result of this paper deals with asymptotics of parallel avoidance. We first give the following definition.

Definition 3.2. If $\sigma_1, \ldots, \sigma_d$ are permutations of some $[m]$, we say that $S_n^d(\sigma_1, \ldots, \sigma_d)$ is the number of d-tuples of permutations $(\sigma'_1, \ldots, \sigma'_d)$ with $\sigma'_i \in S_n$ such that $(\sigma'_1, \ldots, \sigma'_d)$ avoids $(\sigma_1, \ldots, \sigma_d)$ in parallel.

The famous Marcus–Tardos Theorem [16], building on the work of Klazar [11], states the following (corresponding to the case $d = 1$).

Theorem 3.3 (*Marcus–Tardos [16]*). *Let $m \in \mathbb{N}$. For any permutation $\sigma \in S_m$, let $S_n(\sigma) = S_n^1(\sigma)$ be the number of permutations in S_n avoiding σ. Then for all σ there exists a constant c such that*

$$S_n(\sigma) \leq c^n.$$

Let $\sigma_1, \dots, \sigma_d$ be permutations, say in S_m. Then for every $(\sigma'_1, \dots, \sigma'_d)$ that avoids $(\sigma_1, \dots, \sigma_d)$, we have a corresponding set partition $[\sigma'_1, \dots, \sigma'_d]$ avoiding $[\sigma_1, \dots, \sigma_d]$ by Proposition 2.8. Thus, Theorem 1.2 should imply a corresponding bound on parallel permutation pattern avoidance. This turns out to suggest a natural generalization of Theorem 3.3 to d-tuples, in the form of the following.

Theorem 3.4. *Let $m > 1$ and let $\sigma_1, \dots, \sigma_d \in S_m$ be permutations. Then there exist constants $c_2 > c_1 > 0$ (depending on the σ_i) such that $c_1^n n^{n\frac{d^2-1}{d}} \leq S_n^d(\sigma_1, \dots, \sigma_d) \leq c_2^n n^{n\frac{d^2-1}{d}}$ for all n.*

4. Proof of Theorem 3.1

We will now prove Theorem 3.1. This will turn out to be a rather simple construction. The main idea is that if π has permutability d, then any permutation of permutability at most $d-1$ avoids π. First, we reduce to the case where $d|n$ by showing that $B_n(\pi)$ grows quite quickly.

Let π be a set partition with $\mathrm{pm}(\pi) = d$. Assume $d > 1$, as the case $d = 1$ is trivial. First note that removing blocks containing one element from π does not change its permutability. This is because if a permutation-correspondent partition contains π minus a one-element block $\{s\}$, we may add one block to that partition to get a new permutation-correspondent partition of the same permutability containing π. For example, 12/34 is contained in 135/246 $=$ [12, 12], so 12/3/45 is contained in 147/268/359 $=$ [132, 123]; notice that the 147/268 is simply 135/246 while the 359 block is designed to add the 5, which corresponds to the 3 in 12/3/45. We may add any element we wish like this, as we may choose to insert any element into the ranges $\{1, \dots, k\}, \{k+1, \dots, 2k\}, \dots, \{(d-1)k+1, \dots, dk\}$ (where k is the permutation length) for the d elements of the added block.

If π' is π with all one-element blocks removed, any partition avoiding π' must avoid π since π contains π', so $B_n(\pi) \geq B_n(\pi')$ and $\mathrm{pm}(\pi') = d$ by the discussion in the previous paragraph. So it suffices to show the problem for π'; that is, we can assume without loss of generality that π has no blocks of size 1. This means that we can add any blocks of size 1 to a partition of $[n-i]$ avoiding π to get a partition of $[n]$ avoiding π. If we only range over partitions of $[n-i]$ with no blocks of size 1, the resulting partitions will all be distinct. Let $B'_n([\pi])$ be the number of partitions of $[n]$ avoiding $[\pi]$ with no blocks of size 1. Then since we can perform the process of adding single blocks in $\binom{n}{i}$ ways, we have $B_n(\pi) \geq \binom{n}{i} B'_{n-i}(\pi)$. This will allow us to reduce to the $d|n$ case, where the construction is quite simple.

Suppose n is a multiple of d, $n = dm$. Then if $\sigma_1, \dots, \sigma_{d-1} \in S_m$ are permutations, then $[\sigma_1, \dots, \sigma_{d-1}]$ will be a partition of $[m(d-1+1)] = [n]$, and by the definition of permutability, has permutability at most $d-1$. Therefore, it must avoid π, by the remarks after Definition 2.9. Since these all correspond to different partitions, and all blocks have size $d > 1$, we can count them to see that

$$B'_n(\pi) \geq m!^{d-1} = \left(\frac{n}{d}\right)!^{d-1}.$$

By Stirling's Approximation, there is a $c > 0$ such that $\left(\frac{n}{d}\right)! > c^{\frac{n}{d}} \left(\frac{n}{d}\right)^{\frac{n}{d}} = \left(\frac{c}{d}\right)^{\frac{n}{d}} n^{\frac{n}{d}}$. Substituting this in,

$$B'_n(\pi) \geq \left(\frac{c}{d}\right)^{\frac{(d-1)n}{d}} n^{\frac{(d-1)n}{d}} = c_0^n n^{n\left(1-\frac{1}{d}\right)},$$

where $c_0 = \left(\frac{c}{d}\right)^{\frac{d-1}{d}}$.

Now we use this to solve the case where $d \nmid n$. This is now just simple bounding using our increasingness property $B_n(\pi) \geq \binom{n}{i} B'_{n-i}(\pi)$. Let $n = dm + i$, $0 \leq i \leq d-1$. Since we are dealing with asymptotics we may assume that $n > d$. We have that since $n-i$ is a multiple of d, assuming $c_0 < 1$ without loss of generality for ease of manipulation,

$$\begin{aligned}
B_n(\pi) &\geq \binom{n}{i} B'_{n-i}(\pi) \\
&\geq \frac{(n-i)^i}{i!} c_0^{n-i} (n-i)^{(n-i)\left(1-\frac{1}{d}\right)} \\
&\geq \frac{c_0^n}{i!} (n-i)^{n\left(1-\frac{1}{d}\right)} \\
&= \frac{c_0^n}{i!} \left(1-\frac{i}{n}\right)^{n\left(1-\frac{1}{d}\right)} n^{n\left(1-\frac{1}{d}\right)} \\
&> \frac{c_0^n}{d!} \left(1-\frac{d}{n}\right)^{n\left(1-\frac{1}{d}\right)} n^{n\left(1-\frac{1}{d}\right)} \\
&= \frac{1}{d!} \left(c_0 \left(1-\frac{d}{n}\right)^{1-\frac{1}{d}}\right)^n n^{n\left(1-\frac{1}{d}\right)}.
\end{aligned}$$

Now, $c_0\left(1-\frac{d}{n}\right)^{1-\frac{1}{d}}$ is clearly bounded below by a constant for $n > d$, which concludes the proof of the theorem.

5. Ordered hypergraph pattern avoidance

We start this section by defining ordered hypergraph pattern avoidance.

Definition 5.1. Let G and H be hypergraphs whose vertex sets are totally ordered. Then G *contains* H if there exist both an order-preserving injection $V(H) \to V(G)$ and an injection $E(H) \to E(G)$ such that the two are compatible — that is, if $E \in E(H)$ is sent to $E' \in E(G)$, then every vertex of E is sent to a vertex of E' under the map of vertices (note that this map $V(E) \to V(E')$ need not be surjective). If G does not contain H, we as usual say that G *avoids* H.

Definition 5.2. Recall that the number of vertex–edge incidences in a hypergraph G is denoted by $i(G)$; that equals the sum of the sizes of all hyperedges in G, $\sum_{E \text{ a hyperedge of } G} |E|$. We will denote by $e(G)$ the number of hyperedges in G.

We also define a d-permutation hypergraph.

Definition 5.3. A *d-permutation hypergraph* is a hypergraph G on the vertex set $[kd]$ for some $k \in \mathbb{Z}^+$, such that the following properties are satisfied.

- G has k hyperedges, each of size d, such that each vertex is in exactly one hyperedge.
- Each hyperedge has exactly one vertex from each of $\{1, \ldots, k\}$, $\{k+1, \ldots, 2k\}$, …and $\{(d-1)k+1, \ldots, dk\}$.

In Section 2 of [14] and independently as Lemma 14 of [3], the following generalization of the Füredi–Hajnal conjecture [9] (which occurs when G is bipartite and was proved by Marcus and Tardos [16]) was proven.

Theorem 5.4 (*Balogh–Bollobás–Morris [3], Klazar–Marcus [14]*)**.** *Let H be a fixed 2-permutation hypergraph. Then for any $n \in \mathbb{Z}^+$ and hypergraph G on $[n]$ avoiding H, $i(G) = O(n)$.*

This was a key lemma in the proof of the $\mathrm{pm}(\pi) = 1$ case of Theorem 1.2. We prove the following generalization of this result to deduce Theorem 1.2 from it.

Theorem 1.3. *Let H be a fixed d-permutation hypergraph. Then for any $n \in \mathbb{Z}^+$ and hypergraph G on $[n]$ avoiding H, $i(G) = O(n^{d-1})$.*

Our proof most resembles the respective proof in [17] but very likely the methods of [3] and [14] could be modified in a similar fashion. Our arguments are completely self-contained. Before we prove Theorem 1.3, we show how it implies the following.

Lemma 1.4. *Let H be a fixed d-permutation hypergraph. There exists $c > 0$ such that for all $n \in \mathbb{Z}^+$, there are at most $c^{n^{d-1}}$ ordered hypergraphs on $[n]$ that avoid H.*

Proof of Lemma 1.4. This is a standard argument, as (for example) in the proof of Theorem 2.5 from [13], but we will include it for completeness.

Let G be a hypergraph on $[n]$ avoiding H. Create a new multihypergraph G' in the following manner. The vertices of G' correspond to adjacent pairs of vertices $I_1 = \{1, 2\}, I_2 = \{3, 4\}, \ldots$ of G (with a possible singleton at the end if n is odd), for a total of $\lceil \frac{n}{2} \rceil$ vertices.

For each edge E of G, we create an edge E' of G', which is the subset of $V(G')$ such that $a \in E'$ if and only if $I_a \cap E \neq \emptyset$.

Note that G' must avoid H, because any occurrence of H in G' can be used to create an occurrence of H in G (for each edge $E' \in G'$ that appears, consider the corresponding edge in E, and for each vertex a of E' that appears, take some element of $I_a \cap E$).

Let G'' be the hypergraph that results from deleting multiple edges from G'. Now, if G' contains an edge of size kd with multiplicity at least k, it contains H, as we can directly embed the kd vertices of H into the first kd vertices of this edge, and map the k edges of H to the k copies of this edge. This would contradict our assumption that G (and thus G' and G'') avoids H. So every edge of size at least kd occurs fewer than k times in G'.

Furthermore, if E' is an edge of G' with fewer than kd vertices, there are fewer than 3^{kd} edges of G that it can have arisen from. This is because for some $E \subset V(G)$ that gives rise to E', there are 3 choices for $I_a \cap E$ for each $a \in E'$ (the 3 nonzero subsets of I_a), and these choices completely determine E.

Therefore, every edge E' of G' occurs at most $\min(k, 3^{kd}) = 3^{kd} = O(1)$ times. By Theorem 1.3, $i(G'') = O(n^{d-1})$. Since G' is simply G'' but with duplicate edges, each of which occurs at most $O(1)$ times, we also have that $i(G') = O(n^{d-1})$.

Given a particular G'', how many possibilities are there for G? Firstly, we choose G'. For each edge of G'', we must choose how many times it occurs in G'. There are less than 3^{kd} choices for each edge, so we may simply bound this by $\left(3^{kd}\right)^{e(G'')} \leq 3^{kd \cdot i(G'')} = 3^{O(n^{d-1})}$.

Given G', we now need to choose G. Each edge of G' corresponds to an edge of G, and for each vertex–edge incidence $a \in E'$ in G', we have at most three choices in G, as we must choose some nonempty subset of I_a to be $E \cap I_a$, where E is the edge in G corresponding to E'. These choices completely determine G, so there are at most $3^{i(G')} = 3^{O(n^{d-1})}$ choices for G given G'.

Therefore, there are at most $3^{O(n^{d-1})}$ choices for G given G''. Now, G'' is some hypergraph on $\lceil \frac{n}{2} \rceil$ vertices avoiding H. So if $f(n)$ is the number of hypergraphs on n vertices that avoid H, we have shown that

$$f(n) \leq 3^{O(n^{d-1})} f\left(\left\lceil \frac{n}{2} \right\rceil\right).$$

This recurrence easily yields $f(n) = 3^{O(n^{d-1})}$, proving the Lemma. $\square$

6. Proof of Theorem 1.3

We will first show Theorem 1.3 in the case where G is t-uniform for a fixed t. In fact, we will prove something stronger by induction. First, we need to define the *projection* of a t-uniform hypergraph.

Definition 6.1. Let G be a t-uniform ordered hypergraph, and let J be a subset of $[t]$ of cardinality a. For a hyperedge $E \in G$, let $Proj_J E$ be the hyperedge of cardinality a given by deleting the ith vertex of E for all $i \notin J$. Let $Proj_J G$ be the a-uniform hypergraph given by the same vertex set as G and the hyperedges $Proj_J E$ for all hyperedges $E \in G$ (only counting multiple hyperedges once).

Observation. *If G is t-uniform, G contains $Proj_J G$ for any $J \subset [t]$.*

Our strengthening of Theorem 1.3 for t-uniform hypergraphs is the following.

Lemma 6.2. *Fix $t, d, k \in \mathbb{Z}^+$. Then there exists a constant $c_{t,d,k}$ such that for all n and all t-uniform hypergraphs G on $[n]$ with $e(G) > c_{t,d,k}n^{d-1}$, there exists $J \subset [t]$ with $|J| = d$ such that $Proj_J G$ contains every d-permutation hypergraph on kd vertices.*

If G avoids H, then by the above Observation $Proj_J G$ must also avoid H for all J, and since $i(G) = t \cdot e(G)$, Lemma 6.2 is indeed a strengthening of Theorem 1.3 for t-uniform hypergraphs.

The proof of Lemma 6.2 will span over several pages.

Proof of Lemma 6.2. The proof will be induction on t, d, n (while k is fixed).

The base cases of $t < d$ or $d = 1$ are simple. If $t < d$, then $e(G) \leq \binom{n}{t} \leq n^t \leq n^{d-1}$. If $d = 1$, then, by the definition of avoidance, if the conclusion does not hold, $Proj_J G$ must have less than k hyperedges for all J of size 1, where k is the number of hyperedges (which in this case just consist of a single vertex) of H. That is, for any $i \in [t]$, there are only $k - 1$ choices for the ith vertex of the hyperedges of G. Thus, $e(G) \leq (k-1)^t = O(1)$, as desired.

We now proceed to the inductive step. Suppose that G is a t-uniform hypergraph on vertex set $[n]$ that does not satisfy the conclusion of the lemma (that is, there is no J that satisfies the conditions of the lemma). We wish to show that $e(G) = O(n^{d-1})$.

Now, for some positive integer s (which will be potentially large, but fixed independently of n), divide the vertices of G (that is, the set $[n]$) up into *intervals* of size s, with the remainder in another interval (that is, our intervals are $\{1, \dots, s\}, \{s+1, \dots, 2s\}, \dots, \{\left(\left\lfloor \frac{n}{s} \right\rfloor - 1\right) s + 1, \dots, \left\lfloor \frac{n}{s} \right\rfloor s\}, \{\left\lfloor \frac{n}{s} \right\rfloor s + 1, \dots, n\}$). Call these intervals $I_1, \dots, I_{\lceil \frac{n}{s} \rceil}$.

Suppose E is a hyperedge which has at least two vertices in the same interval. Then let $f(E)$ be the smallest i such that the ith and $(i+1)$st vertices of E lie in the same interval. Let G_0 be the hypergraph on $V(G) = [n]$ containing exactly the hyperedges of G which have at least two vertices in the same interval.

Proposition 6.3. $e(G_0) \leq c_{t-1,d,k}(t-1)(s-1)n^{d-1} = O(n^{d-1})$.

Proof. Since $f(E) \in [t-1]$ for all hyperedges $E \in G_0$, by the Pigeonhole Principle, at least $\frac{e(G_0)}{t-1}$ hyperedges of G_0 must map to the same number under f. Let G_1 be a hypergraph on $[n]$ with at least $\frac{e(G_0)}{t-1}$ hyperedges all of which map to the same $i_0 \in [t-1]$, i.e., $f(E) = i_0$ for all $E \in G_1$. Then for all $E \in G_1$, by definition, the 1st, $\dots$, i_0th elements of E are in different intervals, and the i_0th and (i_0+1)st are in the same interval.

Consider the hypergraph $G_2 = Proj_{[t]\setminus\{i_0+1\}} G_1$. Given any hyperedge $E_2 \in G_2$, it may correspond to multiple hyperedges in G_1. But if $E_1 \in G_1$ corresponds to $E_2 \in G_2$, all of E_1's vertices are determined, except for the (i_0+1)st vertex, which must be in the same interval as the i_0th (which is determined). Thus, there are at most $s-1$ choices for E_1 given E_2. So at most $s-1$ hyperedges of G_1 can correspond to any given hyperedge of G_2, which implies that

$$e(G_2) \geq \frac{e(G_1)}{s-1} \geq \frac{e(G_0)}{(t-1)(s-1)}.$$

Since G_1 is obtained from G by deleting some hyperedges, $Proj_{[t]\setminus\{i_0+1\}} G$ contains $G_2 = Proj_{[t]\setminus\{i_0+1\}} G_1$. If there is a $J' \subset [t-1]$ with $|J'| = d$ such that $Proj_{J'} G_2$ contains every d-permutation hypergraph on kd vertices, then $Proj_{J'} Proj_{[t]\setminus\{i_0+1\}} G$ also contains every d-permutation hypergraph on kd vertices. But the composition of two projections is itself a projection, in this case, by some $J \subset [t]$ with $|J| = d$. Thus, the conclusion of the lemma holds for G, contradicting our assumption.

Therefore, there is no $J' \subset [t-1]$ with $|J'| = d$ such that $Proj_{J'}G_2$ contains every d-permutation hypergraph on kd vertices. By the inductive hypothesis (on t), there exists $c_{t-1,d,k}$ such that $e(G_2) \le c_{t-1,d,k}n^{d-1}$. Thus, $e(G_0) \le (t-1)(s-1)e(G_2) \le c_{t-1,d,k}(t-1)(s-1)n^{d-1}$. $\square$

Let G' be the hypergraph obtained from G by removing the hyperedges of G_0, thus, G' contains the hyperedges of G all of whose vertices are in distinct intervals. We divide the hyperedges of G' into *blocks* depending on which intervals the vertices of each hyperedge lie in; that is, E and E' are in the same block if and only if for all $i \in [t]$, the ith vertex of E and E' are in the same interval. Thus, there are $\lceil \frac{n}{s} \rceil^t$ possible blocks (some blocks may contain no hyperedges).

Let b be a block and G_b be the subgraph of G' with just the hyperedges of that block; thus, $E(G') = \dot\cup_b E(G_b)$. For $J \subset [t]$ with $|J| = d-1$, we say that b is *J-wide* if $Proj_J G_b$ contains H for every $(d-1)$-permutation hypergraph H on $k(d-1)$ vertices. If there is no such J, we say that b is *thin* and, by the inductive hypothesis, there exists $c_{t,d-1,k}$ (not dependent on s) such that $e(G_b) \le c_{t,d-1,k}s^{d-2}$.

Now we will bound the number of J-wide blocks. Fix a particular $J \subset [t]$ with $|J| = d-1$. We partition the blocks into *J-blockcolumns*; two blocks b and b' are in the same J-blockcolumn if the intervals corresponding to the ith vertices of the hyperedges in b and b' are the same for all $i \in J$. That is, a blockcolumn is obtained by fixing for every $i \in J$ which interval the ith vertex belongs to. Since $|J| = d-1$, there are at most $\binom{\lceil \frac{n}{s} \rceil}{d-1}$ J-blockcolumns.

Proposition 6.4. *Every J-blockcolumn can have at most $(k-1)^{t-d+1}\binom{s}{k}^{(d-1)(k!)^{d-2}}$ blocks that are J-wide.*

Proof. Suppose for the sake of contradiction that a particular J-blockcolumn has at least $(k-1)^{t-d+1}\binom{s}{k}^{(d-1)(k!)^{d-2}} + 1$ J-wide blocks. For every $(d-1)$-permutation hypergraph H on $k(d-1)$ vertices, we know that H is contained in $Proj_J G_b$ for any of these blocks b. A copy of H in $Proj_J G_b$ can occur in $\binom{s}{k}^{d-1}$ possible places (by a place we mean the injection of vertex sets given by containment), as for every $i \in [d-1]$ we must choose the k locations in the corresponding interval that the ith vertices of the hyperedges of H are mapped to (there are k such vertices). There are $(k!)^{d-2}$ possible $(d-1)$-permutation hypergraphs on $k(d-1)$ vertices, as for each set of k vertices, except the first, we can match them up with the first k by any permutation. Thus, there are $\binom{s}{k}^{(d-1)(k!)^{d-2}}$ ways that the copies of all the $(d-1)$-permutation hypergraphs on $k(d-1)$ vertices can occur in a block. Since two blocks in the same blockcolumn by the definition of blockcolumn have the same relevant intervals, by the Pigeonhole Principle, our blockcolumn must contain $(k-1)^{t-d+1}+1$ blocks where the copies of every H occur on the exact same vertices.

Thus, again by the Pigeonhole Principle, there is some $i_0 \in J$ such that among these $(k-1)^{t-d+1}+1$ blocks, there are k blocks such that the i_0th vertices of the hyperedges of the k blocks are in k different intervals. Call these blocks $b_1, \dots, b_k$, and assume they are sorted in increasing order of the interval the i_0th vertex is in. Let $J' = J \cup \{i_0\}$.

Claim 6.5. $\mathbf{G} = Proj_{J'}G$ *contains every d-permutation hypergraph H on kd vertices.*

Proof. We now translate our conditions on $b_1, \dots, b_k$ to conditions on blocks of $\mathbf{G}$. Suppose that i_0 is the j_0th smallest element of J', and let $J_0 = J' \setminus \{j_0\}$, so that $Proj_J$ and $Proj_{J_0} \circ Proj_{J'}$ are the same operator. The blocks $b_1, \dots, b_k$ will translate to blocks of $\mathbf{G}$, say $b'_1, \dots, b'_k$. These blocks will have the property that for any $(d-1)$-permutation hypergraph H' on $k(d-1)$ vertices, $Proj_{J_0}\mathbf{G}_{b'_i}$ contains a copy of H' for any i, moreover, these copies are located at exactly the same position for each i. Furthermore, the j_0th vertices of the blocks are all in different intervals.

In particular, $Proj_{J_0}\mathbf{G}_{b'_i}$ contains a copy of $Proj_{J_0}H$ in the same location for all i. We will use these copies to construct a copy of H in $\mathbf{G}$.

Index the k hyperedges of H, $E_1, \dots, E_k$, in increasing order of their j_0th vertex. Now, our copies of $Proj_{J_0}H$ inside each $Proj_{J_0}\mathbf{G}_{b'_i}$ give us compatible maps $E(Proj_{J_0}H) \to E(Proj_{J_0}\mathbf{G}_{b'_i})$ and $V(Proj_{J_0}H) \to V(Proj_{J_0}\mathbf{G}_{b'_i})$, where the second map is the same for all i by our construction. Thus, for each hyperedge $E_j \in H$, we can consider the hyperedge it maps to in $Proj_{J_0}\mathbf{G}_{b'_i}$, which in turn will be a projection of a hyperedge in $\mathbf{G}$, which we denote by $E_{i,j}$.

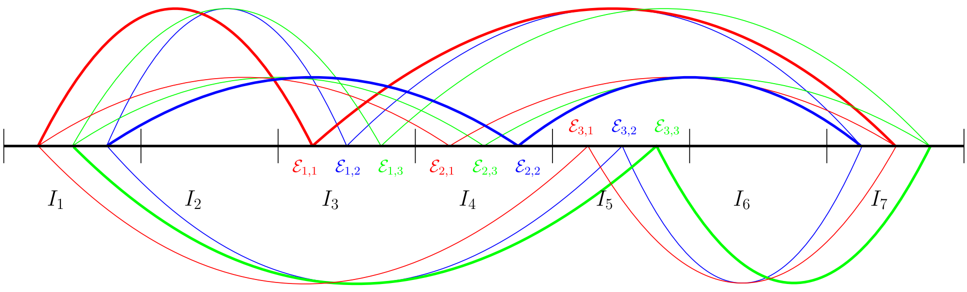

Fig. 1. Example of position of hyperedges $E_{i,j}$. Colour classes represent hyperedges whose $Proj_{J_0}$ image is the same (e.g., red is $E_{1,1}$, $E_{2,1}$, $E_{3,1}$). Hyperedges whose middle (j_0th) vertex is in the same interval, belong to the same block (e.g., the middle vertex of $E_{1,1}$, $E_{1,2}$, $E_{1,3}$ are all in I_3). The three bold hyperedges form a 3-permutation hypergraph H.

By our construction, we know that the following hold (see Fig. 1).

(1) $Proj_{J_0} E_{i,j}$ is independent of i, i.e., two hyperedges, $E_{i,j}$ and $E_{i',j}$, differ only in their j_0th vertex. (This holds as the copies of $Proj_{J_0} H$ occur in the same place in all blocks i.)
(2) For any i, the hyperedges $Proj_{J_0} E_{i,j}$ (over all j) give us a copy of $Proj_{J_0} H$, with $Proj_{J_0} E_{i,j}$ corresponding to hyperedge E_j.
(3) For any fixed i, the j_0th vertices of $E_{i,j}$, $v_{i1}, \ldots, v_{ik}$, are in the same interval. These intervals “increase” with i, i.e., $v_{i,j}$ is in an earlier interval than $v_{(i+1),j'}$ for all i, j, j'.

We now claim that $E_{i,i}$, $1 \le i \le k$, forms a copy of H inside G. We know that all, except possibly the j_0th vertices of the hyperedges, are in the correct place by (1) and (2). All j_0th vertices are greater than all $(j_0 - 1)$th vertices and less than all $(j_0 + 1)$th vertices by (3). Finally, the j_0th vertices are in the correct order because the $v_{i,i}$ will be sorted in increasing order of i by (3), and we chose the hyperedges in H to be sorted in increasing order as well. This proves the claim. □

Since $\mathbf{G} = Proj_{J'} G$, we have that $Proj_{J'} G$ contains H. Since H was an arbitrary d-permutation hypergraph on kd vertices (and J and i_0 were chosen independently of H), we have that $Proj_{J'} G$ contains all d-permutation hypergraphs on kd vertices, which is a contradiction. Thus, our assumption must be false and every J-blockcolumn must have at most $(k-1)^{t-d+1}\binom{s}{k}^{(d-1)(k!)^{d-2}}$ blocks that are J-wide, which finishes the proof of the proposition. □

Note that since (for a particular J) the J-blockcolumns are chosen by fixing $d-1$ distinct intervals in increasing order, and there are $\lceil \frac{n}{s} \rceil$ intervals, there are at most $\binom{\lceil \frac{n}{s} \rceil}{d-1}$ J-blockcolumns. Thus, the total number of J-wide blocks is at most $(k-1)^{t-d+1}\binom{s}{k}^{(d-1)(k!)^{d-2}}\binom{\lceil \frac{n}{s} \rceil}{d-1}$. Since there are $\binom{t}{d-1}$ choices for J, the total number of blocks that are J-wide for some choice of J is at most

$$\binom{t}{d-1}(k-1)^{t-d+1}\binom{s}{k}^{(d-1)(k!)^{d-2}}\binom{\lceil \frac{n}{s} \rceil}{d-1},$$

and thus, the number of hyperedges in G' in blocks that are not thin (i.e., J-wide for some J) is at most

$$s^t\binom{t}{d-1}(k-1)^{t-d+1}\binom{s}{k}^{(d-1)(k!)^{d-2}}\binom{\lceil \frac{n}{s} \rceil}{d-1}$$

(since each block may contain at most s^t hyperedges).

Now, we bound the number of nonempty thin blocks. Form a new ordered hypergraph G_s from G' in the following manner: G_s will have $\lceil \frac{n}{s} \rceil$ vertices corresponding to the intervals in $[n] = V(G')$. The hyperedges will correspond to nonempty thin blocks in the following manner: every nonempty block corresponds to a choice of t intervals, in which the corresponding vertices of each hyperedge of the block will reside. For each such nonempty thin block, we add a hyperedge to G_s whose t vertices will be the t intervals corresponding to that block. So G_s will also be t-uniform.

Proposition 6.6. $e(G_s) \le c_{t,d,k}\lceil \frac{n}{s}\rceil^{d-1}$.

Proof. Using the induction hypothesis (on n), it is enough to show that there is no $J \subset [t]$, $|J| = d$, such that $Proj_J G_s$ contains all d-permutation hypergraphs H on kd vertices. Suppose the contrary. For each such H, this gives a set of k hyperedges in $Proj_J G_s$ (and thus, k hyperedges in G_s) that exhibit the containment. These correspond to k hyperedges of G, and since orders in G_s are preserved in G, projecting these k hyperedges by J will also give a copy of H in $Proj_J G$. Thus, $Proj_J G$ contains all d-permutation hypergraphs H on kd vertices, so $Proj_J G$ does as well, again a contradiction. □

We now put these parts together. We have shown the following:

1. G has at most $c_{t-1,d,k}(t-1)(s-1)n^{d-1}$ hyperedges with vertices in the same interval.
2. We may divide the remaining hyperedges into blocks. There are at most
$$s^t\binom{t}{d-1}(k-1)^{t-d+1}\binom{s}{k}^{(d-1)(k!)^{d-2}}\binom{\lceil\frac{n}{s}\rceil}{d-1}$$
edges in non-thin blocks.
3. There are at most $c_{t,d,k}\lceil \frac{n}{s}\rceil^{d-1}$ nonempty thin blocks and each has at most $c_{t,d-1,k}s^{d-2}$ hyperedges.

Combining these, we obtain a bound.

$$\begin{aligned}|E(G)| &\le c_{t,d-1,k}s^{d-2}c_{t,d,k}\left\lceil\frac{n}{s}\right\rceil^{d-1} + c_{t-1,d,k}(t-1)(s-1)n^{d-1}\\ &+ s^t\binom{t}{d-1}(k-1)^{t-d+1}\binom{s}{k}^{(d-1)(k!)^{d-2}}\binom{\lceil\frac{n}{s}\rceil}{d-1}\\ &= c_{t,d-1,k}s^{d-2}c_{t,d,k}\left\lceil\frac{n}{s}\right\rceil^{d-1} + O\left(n^{d-1}\right),\end{aligned}$$

where the hidden constant in the O notation does not depend on $c_{t,d,k}$. Choosing the constant s to be greater than $c_{t,d-1,k}$, the right hand side will be less than $c_{t,d,k}n^{d-1}$ for any sufficiently large constant $c_{t,d,k}$, completing the proof of Lemma 6.2. □

We now use Lemma 6.2 to prove Theorem 1.3.

Take some d-permutation hypergraph H on kd vertices, and let G be a hypergraph on $[n]$ that avoids H. Note that the hyperedges of G of size at least kd can be repeated at most $k-1$ times, as k copies of the same hyperedge of size kd would contain a copy of H. In fact, any kd vertices can occur together in at most $k-1$ hyperedges. With a similar argument and *ordered downshifts*, we can prove that a minimal G cannot have hyperedges larger than $2kd+2k-3$. This will finish the proof of Theorem 1.3, as we can just use Lemma 6.2 for $t = 1, \dots, 2kd+2k-3$.

For any $x \in V(G)$ and $E \in E(G)$, let $E_{\le x} = \{v \in E \mid v \le x\}$ and $E_{>x} = \{v \in E \mid v > x\}$. Thus $E = E_{\le x} \cup E_{>x}$. Suppose that none of $E_{\le x}$ and $E_{>x}$ are in G. Define a new graph G_x on the same vertex set as G with edge set $E(G_x) = E(G) \setminus \{E\} \cup \{E_{\le x}, E_{>x}\}$. Note that we have $i(G_x) = i(G)$. Moreover, notice that if G avoided the d-permutation hypergraph H, then so will G_x, as only one of $E_{\le x}$ and $E_{>x}$ could be mapped to a hyperedge of H since the first vertex of any hyperedge of H always precedes the last vertex of any other hyperedge of H. But in this case we would also get a copy of H in G, as to whichever hyperedge of H $E_{\le x}$ or $E_{>x}$ was mapped onto, E can be also mapped onto that.

Say that G is *ordered downwards closed* if $E \in E(G)$ implies that $E_{\le x}$ or $E_{>x}$ is also in G. The above argument proved that the maximum of $i(G)$ among hypergraphs avoiding H is attained on an ordered downwards closed hypergraph, thus it is sufficient to prove the following.

Proposition 6.7. *If G is ordered downwards closed avoiding the d-permutation hypergraph H, then all of its hyperedges have size at most $2kd + 2k - 3$.*

Proof. Suppose that G has a hyperedge E of size $\geq 2kd + 2k - 2$. Consider the kdth, $(kd + 1)$st, …, $(kd+2k-2)$th vertices of E as x and check whether $E_{\leq x}$ or $E_{>x}$ is in G. Without loss of generality, for k of the above vertices $E_{\leq x}$ is in G. But then the first kd vertices of E are in k different hyperedges, which means that we can find any H in them. □

7. Proof of Theorem 1.2

We now will use Theorem 1.3 to prove Theorem 1.2.

First note that the case $\mathrm{pm}(\pi) = 0$ is simple, as when $\pi = 1/2/\cdots/k$, π' avoids π if and only if π' has at most $k - 1$ blocks. Thus $B_n(\pi) \leq (k - 1)^n$, so we may simply let c_2 equal $k - 1$. For the remainder of the proof, we assume $\mathrm{pm}(\pi) \geq 1$.

Note that if π is contained in π', then $B_n(\pi') \geq B_n(\pi)$. Thus, since every permutability-d partition is contained in $[\sigma_1, \ldots, \sigma_d]$ for some permutations $\sigma_1, \ldots, \sigma_d$ (by definition of permutability), it suffices to show Theorem 1.2 in the case where $\pi = [\sigma_1, \ldots, \sigma_d]$.

Let $\sigma_1, \ldots, \sigma_d \in S_k$, $\pi = [\sigma_1, \ldots, \sigma_d]$ be the corresponding partition of $[(d + 1)k]$, and H be the $(d + 1)$-permutation hypergraph on $[(d + 1)k]$ vertices with edges $\{i, k + \sigma_1(i), \ldots, dk + \sigma_d(i)\}$ for $1 \leq i \leq k$.

We want to show that there exists $c_2 > 0$ such that $B_n([\sigma_1, \ldots, \sigma_d]) \leq c_2^n n^{n\left(1-\frac{1}{d}\right)}$ for all $n \in \mathbb{Z}^+$.

Note that H is in essence the hypergraph corresponding to the set partition $[\sigma_1, \ldots, \sigma_d]$; the edges correspond to blocks. We can formalize this in the following definition.

Definition 7.1. Let π be a set partition of $[n]$. Then the *hypergraph corresponding to* π is simply the 1-regular hypergraph whose edges are exactly given by the blocks of π.

Note that in the case of hypergraphs corresponding to set partitions, the notion of set partition avoidance is exactly the same as that of hypergraph avoidance. Take any set partition π' on $[n]$ avoiding π, and let G be the hypergraph on n vertices corresponding to π'. Then by this observation G must avoid H.

Given a positive integer s (possibly depending on n), we may construct a new hypergraph G' on $[s]$ as follows. First, we divide $[n]$ into s intervals $I_1, \ldots, I_s$ (in increasing order) so that each has size $\left\lfloor \frac{n}{s} \right\rfloor$ or $\left\lceil \frac{n}{s} \right\rceil$ (the number of each depends on the value of n modulo s). For each edge $E \in G$, we construct an edge E' on the vertex set $[s]$ by the rule that $j \in E'$ if and only if I_j contains at least one vertex of E. Finally, we remove duplicate edges to obtain G'.

For example, if G is the hypergraph $\{1, 4\}, \{2, 5, 6\}, \{3\}$ on $[6]$ and $s = 2$, then $I_1 = \{1, 2, 3\}$ and $I_2 = \{4, 5, 6\}$, and G' will be on the vertex set $[2]$ and have edges $\{1, 2\}$ and $\{1\}$.

Suppose that G' contained H. Then we can find k edges $E'_1, \ldots, E'_k$ in G', and for each edge E'_i, $d+1$ vertices, $v'_{i,1}, \ldots, v'_{i,d+1}$, that give the containment. But each edge E'_i must arise from at least one $E_i \in G$. Choose such an E_i for each E'_i. Then every vertex $v'_{i,j}$ must have at least one corresponding $v_{i,j} \in I_{v'_{i,j}} \cap E_i$, by the definition of G. Choose such a $v_{i,j}$ for every $v'_{i,j}$. Then the edges E_i and the vertices $v_{i,j}$ represent a copy of H in G, as the $v_{i,j}$ have relative ordering the same as that of $v'_{i,j}$ since $I_1, \ldots, I_s$ are arranged in increasing order. This contradicts our assumption that G does not contain H, so G' must in fact not contain H either.

Note that G' need not be 1-regular, as in the example above, so we need to bound the total number of hypergraphs on s vertices avoiding H. It follows from Lemma 1.4 that there are at most c^{s^d} possibilities for G', since H is a $d + 1$-permutation hypergraph.

We now bound the number of set partitions π, and corresponding hypergraphs G on $[n]$ that can correspond to a given G' on $[s]$. It is clear that $i(G) \geq i(G')$ for any G on $[n]$ corresponding to G' on $[s]$ (as G' is formed by contracting parts of edges and deleting duplicates), so since $i(G) = n$ (as G corresponds to a set partition), $i(G') \leq n$.

The number of blocks of each size of π corresponds to an integer partition of n, and it is well known that there are $e^{O(n)}$ integer partitions of n (in fact, $e^{O(\sqrt{n})}$). Now, fix an integer partition of n, and suppose i occurs c_i times. We want to bound the number of partition-correspondent hypergraphs G with c_i edges of size i that correspond to G'. By counting vertices we see that $\sum_{i=1}^{n} ic_i = n$.

Each edge E of size i of G corresponds to some edge E' of size at most i of G'. By a weak bounding argument, there are at most n edges of G', as $i(G') \le n$. Once one of these at most n edges is chosen to be E', of size at most i, this gives i size-$\frac{n}{s}$ intervals where the vertices of E can lie. Thus, there are at most $\binom{\frac{in}{s}}{i}$ choices for E given E', giving at most $n\binom{\frac{in}{s}}{i}$ total choices for E. Choosing all size-i edges simultaneously, and dividing by $c_i!$ to account for the fact that the edges are not distinguishable, we obtain that there are at most $\frac{\left(n\binom{\frac{in}{s}}{i}\right)^{c_i}}{c_i!}$ ways to choose all edges of size i simultaneously. (Some choices of edges contradict each other – for example, if they share a vertex of G – but this will only decrease the number of options.) Therefore, the total number of ways to choose the set partition π to correspond to G' is at most

$$e^{O(n)} \max_{c_1+2c_2+\cdots+nc_n=n} \prod_{i=1}^{n} \frac{\left(n\binom{\frac{in}{s}}{i}\right)^{c_i}}{c_i!}.$$

Since there are at most c^{s^d} ways to choose G', this implies that

$$B_n(\pi) \le c^{s^d} e^{O(n)} \max_{c_1+2c_2+\cdots+nc_n=n} \prod_{i=1}^{n} \frac{\left(n\binom{\frac{in}{s}}{i}\right)^{c_i}}{c_i!}.$$

By a (very) weak form of Stirling's Approximation, $i! > \frac{i^i}{e^i}$ for all i (for example, using $\left(1+\frac{1}{i}\right)^i < e$ and telescoping the left hand product from $i = 1$ to $i = n-1$). Therefore,

$$\binom{\frac{in}{s}}{i} \le \frac{\left(\frac{in}{s}\right)^i}{i!} = \frac{i^i}{i!}\left(\frac{n}{s}\right)^i < \left(\frac{en}{s}\right)^i.$$

Thus

$$\begin{aligned}
\max_{c_1+2c_2+\cdots+nc_n=n} \prod_{i=1}^{n} \frac{\left(n\binom{\frac{in}{s}}{i}\right)^{c_i}}{c_i!} &\le \max_{c_1+2c_2+\cdots+nc_n=n} \prod_{i=1}^{n} \frac{\left(n\left(\frac{en}{s}\right)^i\right)^{c_i}}{c_i!} \\
&\le \max_{c_1+2c_2+\cdots+nc_n=n} \prod_{i=1}^{n} \frac{n^{c_i}\left(\frac{en}{s}\right)^{ic_i}}{c_i!} \\
&= \left(\frac{en}{s}\right)^n \max_{c_1+2c_2+\cdots+nc_n=n} \prod_{i=1}^{n} \frac{n^{c_i}}{c_i!}.
\end{aligned}$$

Substituting this into our bound for $B_n(\pi)$, we obtain

$$\begin{aligned}
B_n(\pi) &\le c^{s^d} e^{O(n)} \left(\frac{en}{s}\right)^n \max_{c_1+2c_2+\cdots+nc_n=n} \prod_{i=1}^{n} \frac{n^{c_i}}{c_i!} \\
&= \frac{c^{s^d}}{s^n} e^{O(n)} n^n \max_{c_1+2c_2+\cdots+nc_n=n} \prod_{i=1}^{n} \frac{n^{c_i}}{c_i!}.
\end{aligned}$$

The fraction on the left is the only part of this expression that depends on s, so we may choose s to minimize it. The minimum occurs when s is within a constant factor of $n^{\frac{1}{d}}$, so since we do not know the value of c, we will simply choose $s = n^{\frac{1}{d}}$. (Not coincidentally, this minimization is analogous to Brightwell's in [7], with the same result of $s \approx n^{\frac{1}{d}}$.)

Substituting this value of s, we see that $c^{s^d} = e^{O(n)}$, so we obtain the bound

$$B_n(\pi) \leq e^{O(n)} n^{n\left(1-\frac{1}{d}\right)} \max_{c_1+2c_2+\cdots+nc_n=n} \prod_{i=1}^{n} \frac{n^{c_i}}{c_i!}.$$

Therefore, to finish the problem and show that the right hand side is within an exponential factor of $n^{n\left(1-\frac{1}{d}\right)}$, it simply suffices to show that $\max_{c_1+2c_2+\cdots+nc_n=n} \prod_{i=1}^{n} \frac{n^{c_i}}{c_i!} = e^{O(n)}$, or in other words, that $\max_{c_1+2c_2+\cdots+nc_n=n} \sum_{i=1}^{n} (c_i \ln(n) - \ln(c_i!)) = O(n)$.

As earlier, $c_i! \geq \left(\frac{c_i}{e}\right)^{c_i}$, and taking logarithms $\ln(c_i!) > c_i \ln c_i - c_i$. Substituting,

$$\max_{c_1+2c_2+\cdots+nc_n=n} \sum_{i=1}^{n} (c_i \ln(n) - \ln(c_i!)) \leq \max_{c_1+2c_2+\cdots+nc_n=n} \sum_{i=1}^{n} (c_i(\ln(n)+1) - c_i \ln(c_i)).$$

Now, the function $c_i(\ln n + 1) - c_i \ln c_i$ is concave in c_i, so we may use Lagrange multipliers to maximize our expression subject to the restriction $\sum_{i=1}^{n} ic_i = n$. (The extrema of the domain, where all but one c_i is 0, clearly satisfy the desired inequality.)

We have that

$$\frac{\partial}{\partial c_j}\left(\sum_{i=1}^{n} (c_i(\ln(n)+1) - c_i \ln(c_i))\right) = \ln n - \ln c_i$$

and

$$\frac{\partial}{\partial c_j}\left(\sum_{i=1}^{n} ic_i\right) = j$$

Thus using Lagrange multipliers, the optimum occurs where the vectors $(\ln n - \ln c_j)_{1\leq j\leq n}$ and $(j)_{1\leq j\leq n}$ are proportional; that is, $c_i = \frac{n}{a^i}$ for some $a = a(n) > 0$ depending on n. For this value of c_i,

$$\sum_{i=1}^{n} (c_i(\ln(n)+1) - c_i \ln(c_i)) = \sum_{i=1}^{n} (c_i + ic_i \ln(a)) \leq n \ln(a) + n,$$

as $\sum_{i=1}^{n} c_i \leq \sum_{i=1}^{n} ic_i = n$.

Since this set of c_i are the optimum, we have in fact shown that

$$\max_{c_1+2c_2+\cdots+nc_n=n} \sum_{i=1}^{n} (c_i(\ln(n)+1) - c_i \ln(c_i)) \leq n(\ln a + 1).$$

To show that this expression is $O(n)$, it suffices to show that a is bounded (recall that a is a function of n).

The value of a is determined by the fact that $c_i = \frac{n}{a^i}$ is a solution to the given condition $\sum_{i=1}^{n} ic_i = n$. Therefore, $n = \sum_{i=1}^{n} \frac{in}{a^i}$, so $\sum_{i=1}^{n} \frac{i}{a^i} = 1$. It is clear from this that $a > 1$. Thus

$$\begin{aligned} 1 &= \sum_{i=1}^{n} \frac{i}{a^i} \\ &< \sum_{i=1}^{\infty} \frac{i}{a^i} \\ &= \frac{\frac{1}{a}}{\left(1-\frac{1}{a}\right)^2} \\ &= \frac{a}{(a-1)^2}. \end{aligned}$$

Therefore, $(a-1)^2 < a$, so $a^2 - 3a + 1 < 0$, so $a < \frac{3+\sqrt{5}}{2}$. This shows that a is bounded, and putting everything together we see that we have shown that

$$\begin{aligned} \max_{c_1+2c_2+\cdots+nc_n=n} \sum_{i=1}^{n} (c_i \ln(n) - \ln(c_i!)) &\le \max_{c_1+2c_2+\cdots+nc_n=n} \sum_{i=1}^{n} (c_i(\ln(n)+1) - c_i \ln(c_i)) \\ &\le n(\ln a + 1) \\ &\le n\left(\ln\left(\frac{3+\sqrt{5}}{2}\right)+1\right) \\ &= O(n), \end{aligned}$$

so we have completed the proof.

8. Proof of Theorem 3.4

We now turn in the direction of parallel avoidance, by proving Theorem 3.4.

Note that the upper bound follows quite simply from Theorem 1.2, as given $\sigma_1, \ldots, \sigma_d \in S_m$, each $(\sigma'_1, \ldots, \sigma'_d) \in S_n^d$ that avoids $(\sigma_1, \ldots, \sigma_d)$ yields a different set partition $[\sigma'_1, \ldots, \sigma'_d]$ of $[(d+1)n]$ avoiding $[\sigma_1, \ldots, \sigma_d]$, which has permutability d. Thus

$$S_n^d(\sigma_1, \ldots, \sigma_d) \le B_{(d+1)n}([\sigma_1, \ldots, \sigma_d]) \le c_2^n((d+1)n)^{\left(1-\frac{1}{d}\right)(d+1)n}$$

for some $c_2 > 0$, which gives the desired upper bound.

Now, we show the lower bound, which will turn out to follow easily from previously known results on random orders. Let $\sigma_1, \ldots, \sigma_d \in S_m$ with $m > 1$. Then restricting σ_i to their first two elements will yield some permutation that is an element of S_2; that is, either 12 or 21. Thus

$$S_n^d(\sigma_1, \ldots, \sigma_d) \ge S_n^d(\sigma'_1, \ldots, \sigma'_d)$$

where σ'_i is either 12 or 21 for all i. Now, some permutation π contains 21 (that is, has an inversion) in exactly the indices where the complement of π (if $\pi \in S_n$, the complement of π is given by replacing each i by $n+1-i$) contains 12. Thus, replacing, say, all first permutations by their complement gives a bijection between $S_n^d(12, \sigma'_2, \ldots, \sigma'_d)$ and $S_n^d(21, \sigma'_2, \ldots, \sigma'_d)$. Doing this for all indices, we see that we can replace each 21 by a 12, so

$$S_n^d(\sigma_1, \ldots, \sigma_d) \ge S_n^d(12, \ldots, 12).$$

Thus, it suffices to prove the lower bound when all permutations are 12; that is, it suffices to show that there exists $c_1 > 0$ with

$$c_1^n n^{n\frac{d^2-1}{d}} \le S_n^d(12, \ldots, 12).$$

We first translate to the language of probabilities. Let $q_d(n)$ be the probability that randomly chosen $\sigma_1, \ldots, \sigma_d \in S_n$ will have $(\sigma_1, \ldots, \sigma_d)$ avoiding $(12, \ldots, 12)$. Note that since there are $n!^k$ ways to choose k permutations in S_n, $q_d(n) = \frac{S_n^d(12,\ldots,12)}{n!^d}$. We know that $n!$ is within an exponential factor of n^n by Stirling's approximation, so if we divide the desired statement by $n!^d$, we obtain that we want to show

$$c_1^n n^{-\frac{n}{d}} \le q_d(n)$$

for all $n \in \mathbb{N}$ for some constant $c_1 > 0$.

We now translate the problem into the language of random $(d+1)$-dimensional orderings as follows. Let $p_1, \ldots, p_n$ be random points (in the usual sense) in $[0, 1]^{d+1}$. We can sort them by their first coordinates. Once this is done, looking at the ordering of the ith coordinates of all n points for some fixed $2 \le i \le d+1$ will generate a permutation, so we get d permutations $\sigma_1, \ldots, \sigma_d$ given by these orderings. It is easy to see that these permutations are independently and uniformly randomly chosen.

Now, we consider the (random) poset, also known as the random $(d+1)$-dimensional order $P_{d+1}(n)$, on these points as follows. We say that $p_i < p_j$ if and only if all coordinates of p_i are less than those of p_j. Suppose p_i has the a_ith smallest first coordinate, and similarly p_j has the a_jth smallest. Then the condition that $p_i < p_j$ corresponds to (looking at the first coordinate) the condition that $a_i < a_j$, and (looking at the other d coordinates) the condition that $\sigma_\ell(a_i) < \sigma_\ell(a_j)$ for all $\ell \in [d]$. This idea of relating sets of d permutations to random $(d+1)$-dimensional orderings seems to go back to Winkler [20].

By definition, $(\sigma_1, \ldots, \sigma_d)$ avoids $(12, \ldots, 12)$ if and only if there is no $b_1 < b_2$ with $\sigma_i(b_1) < \sigma_i(b_2)$ for all i, and we can see by the previous paragraph that this is in turn equivalent to there being no pair of comparable elements in $P_{d+1}(n)$; that is, $P_{d+1}(n)$ is an antichain. Crane and Georgiou [8] derived from results of Brightwell [7] that this probability is at least $\left(\frac{1}{e} + o(1)\right)^n n^{-\frac{n}{d}}$, finishing the proof of the lower bound.

9. Set partition pattern classes

We may consider pattern classes of partitions; that is, collections of set partitions that are closed downward under containment, in the following sense.

Definition 9.1. A *pattern class* $\mathcal{C}$ of set partitions is a set of set partitions such that if $A \in \mathcal{C}$ and A contains B, $B \in \mathcal{C}$.

For example, if π is a set partition, the set partitions that avoid π form a pattern class. For a pattern class $\mathcal{C}$, we can let $\mathcal{C}_n \subset \mathcal{C}$ consist of the partitions of $[n]$ in $\mathcal{C}$. We then can consider the growth rate of $\mathcal{C}$ by looking at the sequence $|\mathcal{C}_n|$.

Theorem 1.2 allows us to prove the following result, classifying the growth rate of set partition pattern classes to within an exponential factor, and thus implying Theorem 1.1.

Corollary 9.2. *Let $\mathcal{C}$ be a nonempty pattern class of set partitions, not containing all set partitions, and let d be the smallest positive integer such that there exists a set partition of permutability d not in $\mathcal{C}$. Then there exists $c_2 > c_1 > 0$ such that for all $n \in \mathbb{Z}^+$,*

$$c_1^n n^{n\left(1-\frac{1}{d}\right)} \le |\mathcal{C}_n| \le c_2^n n^{n\left(1-\frac{1}{d}\right)}.$$

Proof. Let π have permutability d, and not be contained in $\mathcal{C}$. Then all elements of $\mathcal{C}$ must avoid π, by the definition of pattern class. Thus, $|\mathcal{C}_n| \le B_n(\pi)$, so Theorem 1.2 proves the upper bound.

For the lower bound, it suffices to notice that the argument of Theorem 3.1 gives in fact a lower bound on the number of partitions of $[n]$ of permutability at most $d-1$, and by our assumption all of these partitions are contained in $\mathcal{C}$. This proves the corollary. □

Proof of Theorem 1.1. Immediate from Corollary 9.2. □

Theorem 1.1 shows that there is an infinite sequence of 'jumps' in pattern class growth rates, from $n^{n\left(1-\frac{1}{d-1}\right)}$ to $n^{n\left(1-\frac{1}{d}\right)}$ (modulo an exponential factor) for all d.

Say a pattern class has *growth rate d* if it grows within an exponential factor of $n^{n\left(1-\frac{1}{d}\right)}$. Any pattern class $\mathcal{C}$ of growth rate d must contain all set partitions of permutability at most d, by Corollary 9.2. Since the set partitions of permutability at most d form a pattern class (which must have growth rate d by Corollary 9.2), we have the following.

Corollary 9.3. *The set partitions of permutability at most d form the minimum pattern class of growth rate d, in the sense that they are contained in every other pattern class of growth rate d.*

10. Hereditary properties of graphs

Set partitions may be thought of as ordered graphs (graphs together with a total ordering on the vertices) in which every connected component is a clique (the blocks are just given by the sets of vertices in connected components). Note that in this setting, set partition containment becomes the relation of taking an induced subgraph. Since all induced subgraphs also have all connected components cliques, they also correspond to set partitions. Thus all pattern classes of set partitions correspond to hereditary properties (properties closed under taking an induced subgraph) of ordered graphs. Thus Theorem 1.2 motivates the question: What may be said about factorial growth rates of hereditary properties of ordered graphs? The first superexponential jump (from c^n to $c^n n^{\frac{n}{2}}$) is conjectured and proven in special cases in [3], but this problem still appears to be open, as well as that of higher jumps (such as those that exist in the set partition case, as given by Theorem 1.1).

For *labelled* graphs, the situation is better understood. In Theorem 28 of [4] and Theorem 2 of [5], Balogh, Bollobás, and Weinrich showed the following.

Theorem 10.1 (*Balogh–Bollobás–Weinrich [4,5]*)**.** *Let $\mathcal{C}$ be a hereditary property of graphs, graded by number of vertices. Then either $|\mathcal{C}_n| \geq B_n$ for all sufficiently large n, $\mathcal{C}_n$ is eventually empty, or $|\mathcal{C}_n| = n^{n\left(1-\frac{1}{d}+o(1)\right)n}$ for some $d \in \mathbb{Z}^+$.*

This result is extremely similar in nature to Theorem 1.1. Since hereditary properties of labelled graphs are also hereditary properties of ordered graphs (where we take the labelling to be an ordering), there may be a common generalization of these two results in the language of hereditary properties of ordered graphs.

However, the naive generalization is false. Consider the hereditary (in fact, monotone!) graph property $\mathcal{C}^S$ given by taking all ordered graphs G satisfying the following two properties:

(1) The connected components of G are stars (on at least 2 vertices) and isolated vertices.
(2) Let the stars of G be $S_1, \dots, S_r$. If a_i is the vertex in S_i with smallest label, then a_i is connected to every other vertex of S_i (since S_i is a star, the other vertices will all be leaves).
(3) With notation as in the last condition, $a_1, \dots, a_r$ are the r smallest labels that appear in $S_1 \cup \dots \cup S_r$. That is, there is no i, j and $b_j \in S_j$ such that $b_j < a_i$.

Call $\mathcal{C}^S$ the *minimal-vertex star* property.

Proposition 10.2. *$\mathcal{C}^S$ is monotone (thus hereditary) and satisfies $B_n > |\mathcal{C}_n^S|$ for $n \geq 4$ and $|\mathcal{C}_n^S| = n^{(1+o(1))n}$.*

Proof. Note that (1) is clearly preserved by taking subgraphs. Since a substar of a star has the same 'centre' (the same vertex connected to all other vertices), (2) and (3) are also preserved under taking subgraphs. So $\mathcal{C}^S$ is a monotone graph property.

To show $|\mathcal{C}_n^S| < B_n$, we show an injection from elements of $\mathcal{C}_n^S$ to set partitions of $[n]$. For a graph $G \in C_n^S$, we take it to the partition of $[n]$ given by looking at the connected components of G. This is an injection, as we may recreate G from the set partition by letting the parts of size more than 1 correspond to stars, and putting the lowest-labelled vertex in that part in the centre of the star. But by (3), it is not surjective, as $12/34/5/6/\cdots/n$ is not in the image of our injection, as the graph we would recreate from the set partition does not satisfy property (3).

Now we show that $|\mathcal{C}_n^S| = n^{(1+o(1))n}$. The upper bound follows from the fact that $B_n < n^n$ for $n > 1$. For the lower bound, fix some r. Let vertices $1, \dots, r$ have no edges between them. For $r+1 \leq i \leq n$, put an edge between vertex i and exactly one vertex in $\{1, \dots, r\}$. Then the graph is a disjoint union of stars centred at $1, \dots, r$, so (1), (2), and (3) are all satisfied. For each of $n-r$ vertices, we have r choices of where to connect it to, so $|\mathcal{C}_n^S| \geq r^{n-r}$ for all r. Letting $r = \left\lfloor \frac{n}{\ln n} \right\rfloor$, we see that

$$|\mathcal{C}_n^S| \geq \left(\frac{n}{\ln n} - 1\right)^{n-\frac{n}{\ln n}} \geq \left(\frac{n}{2\ln n}\right)^n n^{-\frac{n}{\ln n}} = (2e)^{-n}\left(\frac{n}{\ln n}\right)^n.$$

This shows that $\mathcal{C}_n^S = n^{(1+o(1))n}$, as desired. □

Note that by Theorems 1.1 and 10.1, no growth rate like that of $\mathcal{C}_n^S$ is possible in the cases of labelled graphs or of set partitions. However, B_n is within an exponential factor of $\left(\frac{n}{\ln n}\right)^n$ (see for example [19], Section 6.2), so since we have shown that $(2e)^{-n}\left(\frac{n}{\ln n}\right)^n \leq |\mathcal{C}_n^S| \leq B_n$, we may conclude that $|\mathcal{C}_n^S|$ is within an exponential factor of B_n. In other words, $|\mathcal{C}_n^S|$ does not dip 'too much' below the Bell numbers. This motivates the following question.

Question. *Does some modification of Theorems 1.1 and 10.1 hold for hereditary (or even monotone) properties of ordered graphs?*

A subcase of this question is answered by Balogh, Bollobás, and Morris in Theorems 4 and 5 of [3], where they show the following.

Theorem 10.3 (*Balogh–Bollobás–Morris [3]*)**.** *Let $\mathcal{C}$ be an ordered graph property such that either of the following holds*

1. *$\mathcal{C}$ is monotone, or*
2. *$\mathcal{C}$ is hereditary, and there is some t such that neither K_t nor (any ordering of) $K_{t,t}$ is contained in $\mathcal{C}$.*

Then, either there exists $c > 0$ such that $|\mathcal{C}_n| \leq c^n$ for all n, or there is some $a > 0$ such that $|\mathcal{C}_n| \geq a^n n^{\frac{n}{2}}$ for all n. In the second case, $\mathcal{C}$ must contain the property $\mathcal{P}$ of graphs that correspond to set partitions of permutability 0 *or* 1*.*

Note that we say that a graph G corresponds to a set partition π if i, j are connected in the graph if and only if i, j are in the same set in π.

Theorem 10.3 shows that there is a jump from exponential to within exponential of $n^{\frac{n}{2}}$ for monotone properties and certain hereditary properties. However, the question of whether the jumps from $c^n n^{n\left(1-\frac{1}{d}\right)}$ to $c^n n^{n\left(1-\frac{1}{d+1}\right)}$ occur for $d > 1$ appears open, as does the question of whether there exists a final jump analogous to the Bell number jump in the labelled graph case.

11. Further directions

Apart from what is discussed in the previous section, there are several possible directions to attempt to extend these results.

One such direction is the computation of the correct exponential factor for parallel avoidance in simple cases. That is, does $\lim_{n\to\infty} \sqrt[n]{\frac{S_n^d(\sigma_1,\ldots,\sigma_d)}{(n!)^{\frac{d^2-1}{d}}}}$ exist, and if so what is its value in simple cases? From results in [8] on $Q_3(n)$ we can see that in the $d = 2$ case with $\sigma_1 = \sigma_2 = 12$, we have a lower bound of 1 and an upper bound of $3\sqrt{3(3\log 3 - 4\log 2)} \approx 3.76$. Weaker results in [8] apply for more than two copies of 12, but no other cases appear to have been studied, and the problem appears quite difficult even in these cases. Similarly, we may try to compute the correct exponential factor for set partition avoidance in simple cases, though this seems similarly difficult. Another problem may be to prove the existence of the limit above, and similarly for set partitions, although the fact that this problem is open for the case of one permutation is not encouraging.

Another natural (perhaps more tractable) question is whether it is possible to classify the growth rates of pattern classes of d-tuples of permutations in a similar way to this paper's treatment of set partitions. When the pattern class has no basis elements, the answer is obviously $n!^d$, and with exactly one basis element, Theorem 3.4 shows the speed of the pattern class is within exponential of $(n!)^{\frac{d^2-1}{d}}$. However, not all (proper) pattern classes grow at this rate; the class given by avoiding

(21, 12) and (21, 21) simply grows as $n!$, as the first element of any pair in this pattern class must be the identity. Indeed, the product of a pattern class of d tuples and a pattern class of d'-tuples will be a pattern class of $d + d'$-tuples, and using this, for any d, we may form a pattern class of d-tuples that grows within an exponential factor of $n^{\alpha n}$, where

$$\alpha = d - \sum_{i=1}^{m} \frac{1}{d_i},$$

with $m \in \mathbb{Z}^+$ and $\sum d_i \leq d$. Are any other growth rates possible?

The remark at the end of Section 2, when taken with Theorems 3.1 and 1.2, shows an asymptotic version of a conjecture of Bloom and Sarancino [6] that the single-block partition is easiest to avoid among all partitions of the same $[k]$. Their stronger conjecture is that this holds for every n, not just as n tends to infinity.

A more tangential potential notion for further study is that of the permutability statistic and its distribution. To the authors' knowledge, this statistic has not explicitly appeared before in the literature, and given its strong connection to asymptotics, it may be worthwhile to study.

Acknowledgements

The first author would like to thank Ryan Alweiss for inspiring his interest in the problem of the asymptotics of set partition pattern avoidance, Adam Hammett for extremely helpful correspondence about the link to random k-dimensional orders and previous work done in that area, Vít Jelínek for remarking on the connection to ordered graphs, Mitchell Lee for conversations about parallel avoidance, and Adam Marcus and Mitchell Keller for their insight into their previous work.

The authors would also like to acknowledge that this research has started on MathOverflow.

The first author's research is supported by an NDSEG Graduate Fellowship, and by Harvard University under a graduate teaching fellowship.

The second author's research is supported by the Marie Skłodowska-Curie action of the EU, under grant IF 660400 and by the Lendület program of the Hungarian Academy of Sciences (MTA), under grant number LP2017-19/2017.

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