

The Weisfeiler–Leman Dimension of Planar Graphs Is at Most 3

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We prove that the Weisfeiler–Leman (WL) dimension of the class of all finite planar graphs is at most 3. In particular, every finite planar graph is definable in first-order logic with counting using at most 4 variables. The previously best-known upper bounds for the dimension and number of variables were 14 and 15, respectively.

First, we show that, for dimension 3 and higher, the WL-algorithm correctly tests isomorphism of graphs in a minor-closed class whenever it determines the orbits of the automorphism group of every arc-colored 3-connected graph belonging to this class.

Then, we prove that, apart from several exceptional graphs (which have WL-dimension at most 2), the individualization of two appropriately chosen vertices of a colored 3-connected planar graph followed by the one-dimensional WL-algorithm produces the discrete vertex partition. This implies that the three-dimensional WL-algorithm determines the orbits of arc-colored 3-connected planar graphs.

As a byproduct of the proof, we get a classification of the 3-connected planar graphs with fixing number 3.

CCS Concepts: • **Mathematics of computing** → **Combinatorial algorithms**; • **Theory of computation** → **Finite Model Theory**; *Graph algorithms analysis*;

Additional Key Words and Phrases: First-order logic with counting, Weisfeiler–Leman algorithm, isomorphism testing, planar graphs

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1 INTRODUCTION

The Weisfeiler–Leman algorithm (WL-algorithm) is a fundamental algorithm used as a subroutine in graph isomorphism testing. For every positive integer k , there is a k -dimensional version of the algorithm, which colors all k -tuples of vertices in two given undirected input graphs and iteratively refines the color classes based on information of previously obtained colors.

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The algorithm has surprisingly strong links to notions that seem unrelated at first sight. For example, there is a precise correspondence to Sherali-Adams relaxations of certain linear programs [2, 14], there are duplicator-spoiler games capturing the same information as the algorithm [5], it is related to separability of coherent configurations [7], and there is a close correspondence between the algorithm and first-order logic with counting (C^k). More precisely, for two graphs G and G' , if the integer k is the smallest number such that the k -dimensional WL-algorithm distinguishes two graphs, then $k + 1$ is the smallest number of variables of a sentence in first-order logic with counting distinguishing the two graphs [5].

Exploiting these correspondences, the seminal construction of Cai, Fürer, and Immerman [5] shows that there are examples of pairs of graphs on n vertices that require a dimension of $\Omega(n)$ for the WL-algorithm to distinguish them.

However, for various graph classes, a bounded dimension suffices to distinguish every two non-isomorphic graphs from each other. While typically not very practical due to large memory consumption, this yields a polynomial-time algorithm to test isomorphism of graphs from such a class. In a tour de force, Grohe showed that for all graph classes with excluded minors, a bounded dimension of the WL-algorithm suffices to decide graph isomorphism [10, 12].

An ingredient in Grohe's proof deals with the special case of the class of planar graphs, for which he had separately proved a bound on the necessary dimension of the WL-algorithm earlier [8]. Thus, there is a k such that the k -dimensional WL-algorithm distinguishes every two non-isomorphic planar graphs from each other. In his Master's thesis [30] (see Reference [12, Subsection 18.4.4]), Redies analyzes Grohe's proof showing that this k can be chosen to be 14. For 3-connected planar graphs, it was also shown earlier by Verbitsky [32] that one can additionally require the quantifier rank to be logarithmic in the graph size. Feeling that this is far from optimal, Grohe asked in his book [12, Subsection 18.4.4] and also at the 2015 Dagstuhl meeting on the graph isomorphism problem [3] for a tight k . In this article, we show that $k = 3$ is sufficient.

We say that a graph is *identified* by the k -dimensional WL-algorithm if the algorithm distinguishes the graph from all other graphs. Following Grohe [12, Definition 18.4.3], we say that a graph class C has *WL-dimension* k if k is the smallest integer such that all graphs in C are identified by the k -dimensional WL-algorithm. With this terminology our main theorem reads as follows.

THEOREM 1. *The Weisfeiler–Leman dimension of the class of planar graphs is at most 3. (Equivalently, every planar graph is definable by a sentence in first-order logic with counting that uses only four variables.)*

Our proof is separated into two parts. The first part (Sections 3–5) constitutes a reduction from general graphs to 3-connected graphs, while the second part (Section 6) handles 3-connected planar graphs.

In the first part, which does not only concern planar graphs, we start by showing that for a hereditary graph class \mathcal{G} , for $k \geq 2$, if the k -dimensional WL-algorithm distinguishes every two vertex-colored non-isomorphic 2-connected graphs from each other, then it distinguishes every two non-isomorphic graphs in \mathcal{G} (see Theorem 5).

While it is tempting to believe that when requiring $k \geq 3$, a similar statement can be made about 3-connected graphs, there we need the additional assumption that \mathcal{G} is minor-closed. In fact, our proof also needs the more technical requirement that the WL-algorithm determines the vertex orbits.

We can then argue that for $k \geq 3$, under the assumption that \mathcal{G} is additionally minor-closed, if the k -dimensional WL-algorithm determines orbits on all arc-colored 3-connected graphs in \mathcal{G} , then it distinguishes every two non-isomorphic graphs in \mathcal{G} from each other (Theorem 13).

Table 1. Properties of the Planar 3-connected Graphs with Fixing Number 3

Name	$ V $	$ E $	$ F $	V-type	F-type
n -bipyramid ($n \geq 3$)	$n + 2$	$3n$	$2n$	$2\{n\} + n\{4\}$	$2n\{3\}$
tetrahedron	4	6	4	$4\{3\}$	$4\{3\}$
cube	8	12	6	$8\{3\}$	$6\{4\}$
triakis tetrahedron	8	18	12	$4\{3\} + 4\{6\}$	$12\{3\}$
icosahedron	12	30	20	$12\{5\}$	$20\{3\}$
rhombic dodecahedron	14	24	12	$8\{3\} + 6\{4\}$	$12\{4\}$
triakis octahedron	14	36	24	$8\{3\} + 6\{8\}$	$24\{3\}$
tetrakis hexahedron	14	36	24	$6\{4\} + 8\{6\}$	$24\{3\}$

To prove these two reductions, we employ several structural observations on decomposition trees. These allow us to cut off the leaves of the decomposition trees of 2- and 3-connected components, respectively, in an isomorphism-invariant way. This can be done without having to explicitly construct the corresponding decomposition trees (see Section 3).

In the second part, we show that the three-dimensional WL-algorithm identifies all (arc-colored) 3-connected planar graphs. More precisely, we argue that vertex orbits are determined, as required by our reduction. In fact, we show a stronger statement in that we do not need the full power of the three-dimensional WL-algorithm. Using Tutte’s Spring Embedding Theorem [31], we argue that if in an arc-colored 3-connected planar graph there are three vertices each with a unique color (so-called singletons) that share a common face then applying the one-dimensional WL-algorithm (usually called color refinement) yields a coloring of the graph in which all vertices are singletons. Since this would only give us a bound of $k = 4$, we show then that in most 3-connected planar graphs it suffices to individualize 2 vertices to get the same result. Our proof actually characterizes the exceptions, the graphs in which we need to individualize 3 vertices. We can handle these graphs separately to finish our proof.

The fixing number of a graph G is the minimum size of a set of vertices S such that the only automorphism that fixes S pointwise is the identity. It follows from generally known facts that the fixing number of a 3-connected planar graph is at most 3. Our proof, however, shows that the only 3-connected planar graphs with fixing number 3 are those depicted in Figure 1 on the next page (see also Corollary 26). The properties of these graphs are summarized in Table 1.

On the algorithmic side, we obtain a very easy algorithm to check isomorphism of 3-connected planar graphs. In fact, the arguments show that with the right cell selection strategy, individualization refinement algorithms (such as nauty and Traces [26], or Bliss), which currently constitute the fastest isomorphism algorithms in practice, have polynomial running time on 3-connected planar graphs.

Concerning lower bounds on the WL-dimension of planar graphs, it is not difficult to see that there are planar graphs with WL-dimension 2 (for example, the 6-cycle). However, the question of whether the maximum dimension of the class of planar graphs is 2 or 3 remains open.

Related work. There is an extensive body of work on isomorphism testing of planar graphs. Most notably, Hopcroft and Tarjan first exploited the decomposition of a graph into its 3-connected components to obtain an algorithm with quasi-linear running time [17–19], which led to a linear-time algorithm by Hopcroft and Wong [20]. More recent results show that isomorphism of planar graphs can be decided in logarithmic space [6].

There are also various results on the descriptive complexity of planar graph isomorphism. Grohe showed that FPC, fixed-point logic with counting, captures polynomial time on planar graphs [8]

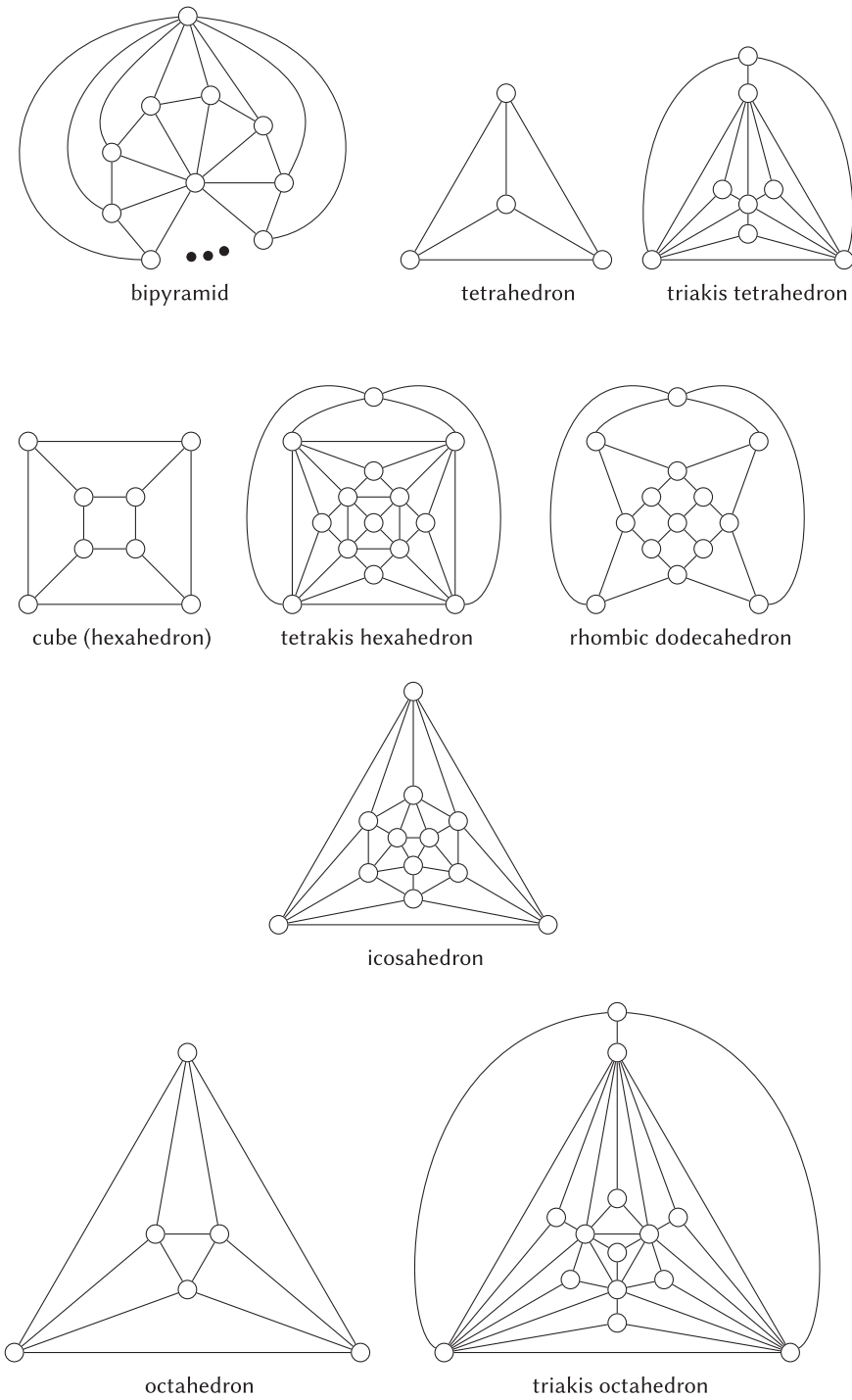


Fig. 1. The 3-connected planar graphs with fixing number 3.

and, more generally, on graphs of bounded genus [9]. For graphs parameterized by their treewidth, this had been shown in Reference [13].

Subsequent work shows that for 3-connected planar graphs and for graphs of bounded treewidth, it is possible to restrict the quantifier depth (or, equivalently, the number of iterations that the WL-algorithm performs until it terminates) to a polylogarithmic number, which translates to parallel isomorphism tests [15, 32]. For general graphs, recent results give new upper and lower bounds [4, 22]. Extending the results on planar graphs in the direction of dynamic complexity, Mehta shows that isomorphism of 3-connected planar graphs is in DynFO+ [27], where, in fact, no counting quantifiers are required.

While it is possible to describe precisely the graphs of WL-dimension 1 [1, 23] (i.e., graphs definable with a 2-variable sentence in first-order logic with counting), it appears difficult to make such statements for higher dimensions. However, for various graph classes for which the isomorphism problem is known to be polynomial-time solvable, one can give upper bounds on the dimension. For example, for cographs, interval graphs, and, more generally, for rooted directed-path graphs, it suffices to apply the two-dimensional WL-algorithm to decide isomorphism [7]. In general, isomorphism of graph classes with an excluded minor can be solved in polynomial time [29] and, in fact, a sufficiently high-dimensional WL-algorithm will decide isomorphism on such a class [10, 12]. More strongly, FPC captures polynomial time on graph classes with an excluded minor. In the proof of this result, structural graph theory and in particular decompositions play a central role. While our article uses very basic parts of these techniques and concepts, they are only implicit and we refer the reader to Reference [11] for a more systematic treatment.

2 PRELIMINARIES

All graphs in this article are finite simple graphs, that is, undirected graphs without loops. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. The *neighborhood* $N(X)$ of a subset of the vertices $X \subseteq V(G)$ is the set $\{u \in V(G) \setminus X \mid \exists v \in X \text{ s.t. } \{u, v\} \in E(G)\}$. For $X \subseteq V(G)$, we denote by $G[X]$ the *subgraph of G induced by X* , i.e., the graph with vertex set X and edge set $E(G) \cap \{\{u, v\} \mid u, v \in X\}$. The graph $G - X := G[V(G) \setminus X]$ is obtained from G by removing X . We write $G \cong H$ to indicate that G is *isomorphic* to H . A *minor* of G is a graph obtained by repeated vertex deletions, edge deletions, and edge contractions.

For a positive integer k , a graph G is *k -connected* if G has more than k vertices and for all $X \subseteq V(G)$ with $|X| < k$, the graph $G - X$ is connected. A *separator* $S \subseteq V(G)$ is a subset of the vertices such that $G - S$ is not connected. A vertex v is a *cut vertex* if $\{v\}$ is a separator, and a *2-separator* is a separator of size 2. A *2-connected component* of G is a subset S' of $V(G)$ such that the graph $G[S']$ is 2-connected and such that S' is maximal with respect to inclusion. We refer the reader to Reference [28] for more basic information on graphs, in particular on *planar graphs*, which are graphs that can be drawn in the plane without crossings.

A *vertex-colored graph* (G, λ) is a graph G with a function $\lambda : V \rightarrow C$, where C is an arbitrary non-empty set. We call λ a *vertex coloring* of G . Similarly, an *arc-colored graph* is a graph G with a function $\lambda : \{\{u, u\} \mid u \in V(G)\} \cup \{\{u, v\} \mid \{u, v\} \in E(G)\} \rightarrow C$. In this case, we call λ an *arc coloring*. We interpret $\lambda(u, u)$ as the vertex color of u and for $\{u, v\} \in E(G)$, we interpret $\lambda(u, v)$ as the color of the arc from u to v . In particular, it may be the case that $\lambda(u, v) \neq \lambda(v, u)$. However, while we allow such colorings, all graphs in this article are undirected. Furthermore, we treat every uncolored graph as a monochromatic colored graph.

The Weisfeiler–Leman algorithm (see Reference [5]). For $k \in \mathbb{N}$, a graph G and a coloring λ of the tuples in $(V(G))^k$, let (v_1, \dots, v_k) be a vertex k -tuple of G . We define ${}_0\chi_G^k(v_1, \dots, v_k)$ to be a tuple consisting of an encoding of $\lambda(v_1, \dots, v_k)$ and an encoding of the isomorphism class of

the colored graph obtained from $G[\{v_1, \dots, v_k\}]$ by coloring for $i \in \{1, \dots, k\}$ the vertex v_i with color i . That is, for a second graph G' , which is possibly equal to G , with coloring λ' and for a vertex k -tuple (v'_1, \dots, v'_k) of G' we have

$${}_0\chi_G^k(v_1, \dots, v_k) = {}_0\chi_{G'}^k(v'_1, \dots, v'_k)$$

if and only if $\lambda(v_1, \dots, v_k) = \lambda'(v'_1, \dots, v'_k)$ and there is an isomorphism from $G[v_1, \dots, v_k]$ to $G'[v'_1, \dots, v'_k]$ mapping v_j to v'_j for all $j \in \{1, \dots, k\}$.

We recursively define the color ${}_{i+1}\chi_G^k(v_1, \dots, v_k)$ by setting

$${}_{i+1}\chi_G^k(v_1, \dots, v_k) := ({}_i\chi_G^k(v_1, \dots, v_k); \mathcal{M}),$$

where \mathcal{M} is the multiset defined as

$$\mathcal{M} := \left\{ \left({}_i\chi_G^k(w, v_2, \dots, v_k), \dots, {}_i\chi_G^k(v_1, \dots, v_{k-1}, w) \right) \mid w \in V \right\}$$

if $k \geq 2$, and as $\mathcal{M} := \{({}_i\chi_G^1(w) \mid w \in N(v_1))\}$ if $k = 1$. That is, if $k = 1$, then the iteration is only over neighbors of v_1 .

There is a slight technical issue about the initial coloring. Suppose for a fixed k , we are given a graph G with a coloring λ' of $(V(G))^\ell$ for $\ell < k$. To turn it into a correct input for the k -dimensional WL-algorithm, we replace λ' with an appropriate coloring λ . For a vertex tuple (u_1, \dots, u_k) , we define $\lambda(u_1, \dots, u_k) := \lambda'(u_1, \dots, u_\ell)$. Note that λ preserves all information from λ' . If λ' is an arc coloring, then we define $\lambda(u_1, \dots, u_k)$ to be $(\lambda'(u_1, u_2), 0)$ if (u_1, u_2) is in the domain of λ' and to be $(1, 1)$ otherwise.

By definition, the coloring ${}_{i+1}\chi_G^k$ induces a refinement of the partition of the k -tuples of the vertices of the graph G with coloring ${}_i\chi_G^k$. Thus, there is some minimal i such that the partition induced by the coloring ${}_{i+1}\chi_G^k$ is not strictly finer than the one induced by the coloring ${}_i\chi_G^k$ on G . For this minimal i , we call ${}_i\chi_G^k$ the *stable* coloring of G and denote it by χ_G^k .

For $k \in \mathbb{N}$, the k -dimensional WL-algorithm takes as input a vertex coloring or an arc coloring λ of a graph G and returns the coloring χ_G^k . For two graphs G and G' , we say that the k -dimensional WL-algorithm *distinguishes* G and G' if its application to each of them results in colorings with differing color class sizes. More precisely, the graphs G and G' are distinguished if there is a color C in the range of χ_G^k such that the sets $\{\bar{v} \mid \bar{v} \in (V(G))^k, \chi_G^k(\bar{v}) = C\}$ and $\{\bar{w} \mid \bar{w} \in (V(G'))^k, \chi_{G'}^k(\bar{w}) = C\}$ have different cardinalities.

If two graphs are distinguished by the k -dimensional WL-algorithm for some k , then they are not isomorphic. However, if k is fixed, then the converse is not always true. There is a close connection between the WL-algorithm and first-order logic with counting (as well as fixed-point logic with counting). We refer the reader to existing literature (for example, References [5, 12]) for more information.

For improved readability, we will use the letter λ to denote arbitrary colorings that do not necessarily result from applications of the WL-algorithm.

3 DECOMPOSITIONS

For a graph G , define the set $P(G)$ to consist of the pairs (S, K) , where S is a separator of G of minimum cardinality and $K \subseteq V(G) \setminus S$ is the vertex set of a connected component of $G - S$.

We observe that if G is a connected graph that is not 2-connected, then $P(G)$ is the set of pairs $(\{s\}, K)$, where s is a cut vertex and $G[K]$ a connected component of $G - \{s\}$. In this case, we also write (s, K) instead of $(\{s\}, K)$. If G is 2-connected but not 3-connected, then all separators in $P(G)$ have size 2.

There is a natural partial order on $P(G)$ with respect to inclusion in the second component, i.e., we can define:

$$(S, K) \leq (S', K') \iff K \subseteq K'.$$

We define $P_0(G)$ to be the set of minimal elements of $P(G)$ with respect to \leq .

Remark 2. It immediately follows from the definitions that the sets $P(G)$ and $P_0(G)$ (and the corresponding partial orders) are isomorphism-invariant (i.e., preserved under isomorphisms).

Note that $P_0(G)$ is non-empty whenever G is not a complete graph. Also note that if G is not 2-connected, then for two distinct minimal elements (S, K) and (S', K') in $P_0(G)$, we have $K \cap K' = \emptyset$. Furthermore, in the case that G is connected but not 2-connected, the set $P_0(G)$ contains exactly the pairs (s, K) for which s is a cut vertex and $G[K]$ a connected component of $G - \{s\}$ that does not contain a cut vertex of G . These two observations can be generalized to graphs with a higher connectivity, but for this we need an additional requirement on the minimum degree as follows.

LEMMA 3. *Let G be a graph that is not $(k + 1)$ -connected and has minimum degree at least $\frac{3k-1}{2}$.*

- (1) *If $(S, K) \in P_0(G)$ and $(S', K') \in P(G)$ are distinct, then $K \subseteq K'$ or $(K \cup S) \cap (K' \cup S') = S \cap S'$.*
- (2) *If $(S, K), (S', K') \in P_0(G)$ are distinct, then $(K \cup S) \cap (K' \cup S') = S \cap S'$.*
- (3) *A pair $(S, K) \in P(G)$ is contained in $P_0(G)$ if and only if there is no separator S' of G of minimum cardinality with $S' \cap K \neq \emptyset$.*

PROOF. (Item 1) Assume that $(S, K) \in P_0(G)$ and $(S', K') \in P(G)$ are distinct. Note that $S = N(K)$ and $S' = N(K')$, since S and S' are separators of minimal size.

Suppose $K \not\subseteq K'$ and let $u_0 \in K \setminus K'$. First assume there exists a $v_0 \in (K \cup S) \cap K'$. Since K is connected and $S = N(K)$, there exists a path P from v_0 to u_0 that does not intersect $\bar{K} := V(G) \setminus (K \cup S)$ and whose inner vertices all lie in K . Since $u_0 \notin K'$, but $v_0 \in K'$, the path must contain vertices that lie in $S' \cap K$. Let v be the last vertex on P that lies in K' . Then its successor on P is some vertex $u \in K \cap S'$. Therefore, the graph $G - u$ is at most $(k - 1)$ -connected. However, u lies in K . By Corollary 1 in Reference [21], the graph $G - u$ is k -connected, yielding a contradiction.

Thus, we now know that $(K \cup S) \cap K' = \emptyset$. It suffices to show that $K \cap S' = \emptyset$. Supposing otherwise, choose some element $s \in K \cap S'$. This element has a neighbor $w \in K'$, since $N(K') = S'$. We cannot have $w \in \bar{K}$, since K and \bar{K} are separated by S . Thus, $w \in (K \cup S) \cap K'$, contradicting $(K \cup S) \cap K' = \emptyset$.

(Item 2) This follows by applying Item 1 twice.

(Item 3) If $(S, K) \in P(G)$ is not minimal, then there is $(S', K') \in P(G)$ with $K' \subsetneq K$. Then $S' \subseteq K \cup S$, since $S' = N(K')$ and $S = N(K)$. But $S \neq S'$, which shows that $S' \cap K \neq \emptyset$. Conversely, suppose $(S, K) \in P_0(G)$ and that there is a minimum-size separator S' of G with $S' \cap K \neq \emptyset$. Let $K' \subseteq V(G - S')$ be a vertex set such that $G[K']$ is a connected component of $G - S'$. Then (S, K) and (S', K') violate Item 1 of the lemma. \square

We remark that for a connected graph G that is not 3-connected, the elements of $P_0(G)$ correspond to the leaves in a suitable decomposition tree (i.e., the decomposition into 2- or 3-connected components) in the sense of Tutte. However, we will not require this fact.

In the following, we present a method to remove the vertices appearing in the second component of pairs in P_0 from graphs in such a way that the property whether two graphs are isomorphic is preserved. This will allow us to devise an inductive isomorphism test. In the next sections, we will then show that a sufficiently high-dimensional WL-algorithm in some sense implicitly performs this induction.

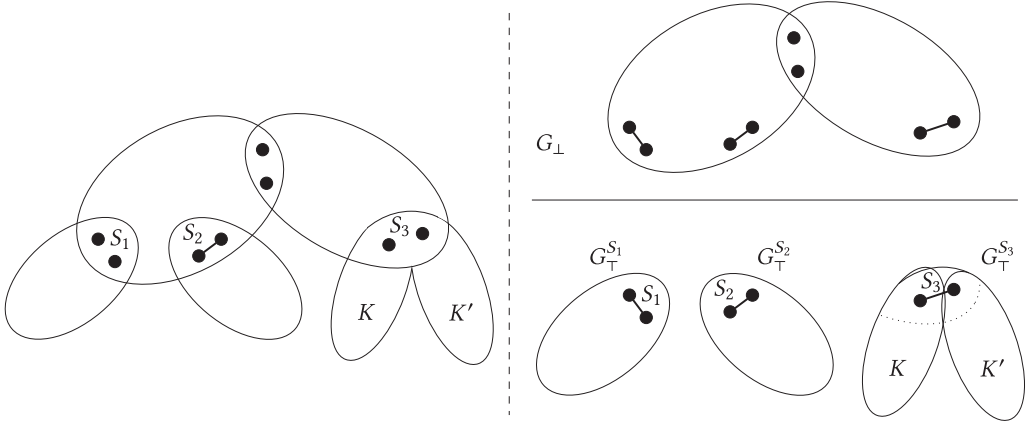


Fig. 2. The figure illustrates the notions of the graphs G_{\perp} and G_{\perp}^S . In the 2-connected graph on the left, the 2-separators are indicated and the respective decomposed graphs are shown on the right.

For a graph G and a set $S \subseteq V(G)$, we define the graph G_{\perp}^S as consisting of the vertices of S and those appearing together with S in $P_0(G)$. More precisely, G_{\perp}^S is the graph on the vertex set

$$V' = S \cup \bigcup_{(S,K) \in P_0(G)} K$$

with edge set $E' = E(G[V']) \cup \{\{s, s'\} \mid s, s' \in S \text{ and } s \neq s'\}$, see Figure 2, left and bottom right.¹ Note that if S is not a separator or does not appear as a separator in $P_0(G)$, then $V' = S$.

For an arc coloring λ of G , we define an arc coloring λ_{\perp}^S for the graph G_{\perp}^S as follows:

$$\lambda_{\perp}^S(v_1, v_2) := \begin{cases} (0, 0) & \text{if } \{v_1, v_2\} \subseteq S \text{ and } \{v_1, v_2\} \notin E(G) \\ (\lambda(v_1, v_2), 1) & \text{if } \{v_1, v_2\} \subseteq S \text{ and } \{v_1, v_2\} \in E(G) \\ (\lambda(v_1, v_2), 2) & \text{otherwise.} \end{cases}$$

If G is a vertex-colored graph with vertex coloring λ' , to obtain a coloring for G_{\perp}^S , then we define an arc coloring λ as $\lambda(v_1, v_2) := \lambda'(v_1)$ and let λ_{\perp}^S be as above.

For $(S, K) \in P_0(G)$, we also define $G_{\perp}^{(S,K)} := G_{\perp}^S[S \cup K]$, which differs from G_{\perp}^S in that only vertices from S and K are retained. Again we define a coloring $\lambda_{\perp}^{(S,K)}$ that is simply the restriction of λ_{\perp}^S to pairs (v_1, v_2) for which $v_1, v_2 \in S \cup K$.

Given a graph G , we define G_{\perp} (see Figure 2, left and top right) to be the graph with vertex set

$$V_{\perp} := V(G) \setminus \left(\bigcup_{(S,K) \in P_0(G)} K \right)$$

and edge set

$$E_{\perp} := E(G[V_{\perp}]) \cup \{\{s_1, s_2\} \mid \exists (S, K) \in P_0(G) \text{ s.t. } s_1, s_2 \in S, s_1 \neq s_2\}.$$

We observe that if G is not 2-connected, then G_{\perp} is equal to $G[V_{\perp}]$. In general, if for some k the graph G is not $(k+1)$ -connected but has minimum degree at least $\frac{3k-1}{2}$, then Lemma 3 applies. In particular, the various components whose vertex sets appear in $P_0(G)$ are disjoint. If G is not 3-connected, then this implies that G_{\perp} is a minor of G .

¹For the reader familiar with tree decompositions, we remark that this graph corresponds to the torso of the bag $S \cup K$ in a suitable tree decomposition. However, we will not require this point of view in the article.

In the following, we restrict our discussions to graphs that are not 3-connected. (We explicitly include graphs that are not 2-connected.) Given an arc coloring λ of G , we define an arc coloring λ_\perp of G_\perp as follows. Assume that $v_1, v_2 \in V(G_\perp)$. Let $S := \{v_1, v_2\}$, possibly with $v_1 = v_2$.

If S is a 2-separator of G but $S \not\subseteq E(G)$, then we set

$$\lambda_\perp(v_1, v_2) := \left(0, \text{ISOTYPE} \left(\left(G_\top^S, \lambda_\top^S \right)_{(v_1, v_2)} \right) \right).$$

Furthermore, if $v_1 = v_2$ or if $\{v_1, v_2\} \in E(G)$, then we set

$$\lambda_\perp(v_1, v_2) := \left(\lambda(v_1, v_2), \text{ISOTYPE} \left(\left(G_\top^S, \lambda_\top^S \right)_{(v_1, v_2)} \right) \right),$$

where $\text{ISOTYPE}((G_\top^S, \lambda_\top^S)_{(v_1, v_2)})$ denotes the isomorphism class of the colored graph $(G_\top^S, \lambda_\top^S)_{(v_1, v_2)}$ obtained from the arc-colored graph $(G_\top^S, \lambda_\top^S)$ by individualizing the vertices v_1 and v_2 . Thus, the graphs $(G_\top^{\{v_1, v_2\}}, \lambda_\top^{\{v_1, v_2\}})_{(v_1, v_2)}$ and $(G_\top^{\{v'_1, v'_2\}}, \lambda_\top^{\{v'_1, v'_2\}})_{(v'_1, v'_2)}$ have the same isomorphism type if and only if there is an isomorphism from the first graph to the second mapping v_1 to v'_1 and v_2 to v'_2 . Note that, by definition, the λ_\perp -colors of 2-separators of G are distinct from those of other pairs of vertices.

If not stated otherwise, then we implicitly assume that for a graph G with initial coloring λ , the corresponding graph G_\perp is a colored graph with initial coloring λ_\perp .

LEMMA 4. *For $k \in \{1, 2\}$, if G and G' are k -connected graphs that are not $(k + 1)$ -connected and that are of minimum degree at least $\frac{3k-1}{2}$ with arc colorings λ and λ' , respectively, then*

$$(G, \lambda) \cong (G', \lambda') \iff (G_\perp, \lambda_\perp) \cong (G'_\perp, \lambda'_\perp).$$

PROOF. (\implies) Suppose that φ is an isomorphism from (G, λ) to (G', λ') . Since $P_0(G)$ is isomorphism-invariant (Remark 1), we know that $\varphi(V(G_\perp)) = V(G'_\perp)$. We claim that φ induces an isomorphism from (G_\perp, λ_\perp) to $(G'_\perp, \lambda'_\perp)$. For this it suffices to observe that the definitions of G_\perp from G and λ_\perp from λ are isomorphism-invariant.

(\impliedby) Conversely, suppose that $\tilde{\varphi}$ is an isomorphism from (G_\perp, λ_\perp) to $(G'_\perp, \lambda'_\perp)$. Let $\{S_1, \dots, S_t\} := \{S \mid \exists K \text{ s.t. } (S, K) \in P_0(G)\}$ be the set of separators that appear in $P_0(G)$. Since $\tilde{\varphi}$ respects the colorings λ_\perp and λ'_\perp , we can conclude that

$$\{\tilde{\varphi}(S_1), \dots, \tilde{\varphi}(S_t)\} = \{S \mid \exists K \text{ s.t. } (S, K) \in P_0(G')\}.$$

For each $j \in \{1, \dots, t\}$, we choose an isomorphism φ_j from $G_\top^{S_j}$ to $G'^\top_{\tilde{\varphi}(S_j)}$ that maps each $s \in S_j$ to $\tilde{\varphi}(s) \in \tilde{\varphi}(S_j)$. We know that such an isomorphism exists, because $\tilde{\varphi}$ respects the colorings λ_\perp and λ'_\perp . We define a map φ from (G, λ) to (G', λ') by setting

$$\varphi(v) := \begin{cases} \tilde{\varphi}(v) & \text{if } v \in V(G_\perp) \\ \varphi_j(v) & \text{if there is a set } K \subseteq V(G) \text{ with } v \in K \text{ and } (S_j, K) \in P_0(G). \end{cases}$$

This map is well defined, since by Items 2 and 3 of Lemma 3, the elements in the second components of pairs in $P_0(G)$ are disjoint and not contained in $V(G_\perp)$. Moreover, the map is an isomorphism, since it respects all edges. Finally, by construction, it also respects the colors of vertices and arcs. \square

4 REDUCTION TO VERTEX-COLORED 2-CONNECTED GRAPHS

A *hereditary* graph class is a class of graphs that is closed under taking induced subgraphs. It is easy to see that for a hereditary graph class \mathcal{G} and $k \geq 2$, the k -dimensional WL-algorithm distinguishes all (vertex-colored) graphs in \mathcal{G} if it distinguishes all (vertex-colored) connected graphs

in \mathcal{G} . (By a vertex-colored graph in \mathcal{G} , we mean more precisely a colored graph whose underlying uncolored graph lies in \mathcal{G} .) For this, one simply has to observe that for two non-isomorphic connected components, the sets of colors that the WL-algorithm computes for their vertices are disjoint.

In this section, we show a stronger statement, replacing the assumption on connected graphs with an assumption on 2-connected graphs as follows.

THEOREM 5. *Let \mathcal{G} be a hereditary graph class. If, for $k \geq 2$, the k -dimensional Weisfeiler–Leman algorithm distinguishes every two non-isomorphic 2-connected vertex-colored graphs (G, λ) and (G', λ') with $G, G' \in \mathcal{G}$ from each other, then the k -dimensional Weisfeiler–Leman algorithm distinguishes all non-isomorphic vertex-colored graphs in \mathcal{G} .*

For the rest of this section, let \mathcal{G} be a hereditary graph class. Recall that for a graph $G \in \mathcal{G}$ with an initial vertex coloring or arc coloring λ , the coloring χ_G^k is the stable k -tuple coloring produced by the k -dimensional WL-algorithm on (G, λ) .

For ℓ vertices u_1, \dots, u_ℓ with $\ell < k$, we define

$$\chi_G^k(u_1, \dots, u_\ell) := \chi_G^k(u_1, \dots, u_\ell, \underbrace{u_\ell, \dots, u_\ell}_{k-\ell \text{ times}}),$$

that is, $\chi_G^k(u_1, \dots, u_\ell)$ is the color of the k -tuple resulting from extending the ℓ -tuple by repeating $k - \ell$ times its last entry.

To prove Theorem 5, we first show that the two-dimensional WL-algorithm distinguishes pairs of vertices that lie in a common 2-connected component from pairs that do not.

THEOREM 6. *Assume $k \geq 2$ and let G and H be two graphs. Let u and v be vertices from the same 2-connected component of G and let u' and v' be vertices that are not contained in a common 2-connected component of H . Then $\chi_G^k(u, v) \neq \chi_H^k(u', v')$.*

PROOF. To improve readability, in this proof we omit the superscripts k , i.e., we write χ_G and χ_H instead of χ_G^k and χ_H^k , respectively.

For an integer i and vertices x, y , denote by $W_i(x, y)$ the number of walks of length exactly i from x to y . (It will be clear from the context in which graph we count the number of walks.) By induction on i , it is easy to see that for $k \geq 2$ it holds that $W_i(x, y) \neq W_i(x', y')$ implies $\chi_G(x, y) \neq \chi_H(x', y')$ [33, p. 18]. Thus, it suffices to show that for some i , we have $W_i(u, v) \neq W_i(u', v')$. Since u' and v' are not contained in the same 2-connected component, there is a cut vertex w' such that every walk from u' to v' passes w' . Suppose that there does not exist a vertex w such that for all i the following hold:

- (1) $W_i(u, w) = W_i(u', w')$
- (2) $W_i(w, w) = W_i(w', w')$
- (3) $W_i(w, v) = W_i(w', v')$.

Then for every w it holds that $\chi_G(u, w) \neq \chi_H(u', w')$ or $\chi_G(w, w) \neq \chi_H(w', w')$ or $\chi_G(w, v) \neq \chi_H(w', v')$. If $\chi_G(w) \neq \chi_H(w')$, then $\chi_G(u, w) \neq \chi_H(u', w')$ and thus, for every vertex w , it holds that $\chi_G(u, w) \neq \chi_H(u', w')$ or $\chi_G(w, v) \neq \chi_H(w', v')$. In other words, there is no vertex w such that $(\chi_G(w, v), \chi_G(u, w)) = (\chi_H(w', v'), \chi_H(u', w'))$. By the definition of the WL-algorithm, this implies that $\chi_G(u, v) \neq \chi_H(u', v')$.

Now suppose that there is a vertex w such that for all i , Conditions 1, 2, and 3 hold. Then for every i , the number of walks of length i from u to v that pass w equals the number of walks from u' to v' that pass w' . However, there must be a walk from u to v that avoids w . Let d be its length. We have $W_d(u, v) > W_d(u', v')$ and thus $\chi_G(u, v) \neq \chi_H(u', v')$. \square

Next we argue that for $k \geq 2$, the k -dimensional WL-algorithm distinguishes cut vertices from other vertices.

COROLLARY 7. *Let $k \geq 2$ and assume G and H are connected graphs. Let $w \in V(G)$ and $w' \in V(H)$ be vertices such that $G - \{w\}$ is connected and $H - \{w'\}$ is disconnected. Then $\chi_G^k(w) \neq \chi_H^k(w')$.*

PROOF. Let u' and v' be two neighbors of w' not sharing a common 2-connected component in H . Note that such vertices do not exist for w in G .

It suffices to show that for all $u \in V(G)$, it holds that $\chi_G^k(u, w) \neq \chi_H^k(u', w')$. By Theorem 6, the color $\chi_H^k(u', w')$ encodes the existence of v' , which is a neighbor of w' and not contained in the same 2-connected component as u' . However, such a vertex does not exist in G for any choice of u and the fixed vertex w . \square

We prove Theorem 5 by induction over the sizes of the input graphs. The strategy is to show that on input (G, λ) , the WL-algorithm implicitly computes the graph (G_\perp, λ_\perp) and to then apply Lemma 4.

LEMMA 8. *Let $k \geq 2$ and assume G and H are connected graphs that are not 2-connected. For vertices $v \in V(G_\perp)$ and $w \in V(H) \setminus V(H_\perp)$, it holds that $\chi_G^k(v) \neq \chi_H^k(w)$.*

PROOF. Note that for a connected but not 2-connected graph G , a vertex $v \in V(G)$ is in $V(G_\perp)$ if and only if it is a cut vertex or there are at least two cut vertices that lie in the same 2-connected component as v . The equivalent statement holds for H . If v is a cut vertex of G , then the lemma follows immediately from Corollary 7.

If v is not a cut vertex, then there are at least two cut vertices u and u' lying in the same 2-connected component as v . Note that there are no such two vertices for w . By Corollary 7, the vertices u and u' obtain colors distinct from colors of non-cut vertices. Thus, also the colors $\chi_G^k(v, u)$ and $\chi_G^k(v, u')$ are distinct from all colors of edges from v to non-cut vertices. Moreover, Theorem 6 yields that the colors $\chi_G^k(v, u)$ and $\chi_G^k(v, u')$ encode that v, u, u' all share a common 2-connected component. This information about the existence of such u and u' is contained in the color $\chi_G^k(v)$ and thus, $\chi_G^k(v) \neq \chi_H^k(w)$. \square

LEMMA 9. *For graphs $G, G' \in \mathcal{G}$ with vertex colorings λ and λ' , respectively, assume $(s, K) \in P_0(G)$ and $(s', K') \in P_0(G')$. For $k \geq 2$, suppose the k -dimensional Weisfeiler–Leman algorithm distinguishes all non-isomorphic vertex-colored 2-connected graphs in \mathcal{G} . Assume there is no isomorphism from $(G_\perp^{(s, K)}, \lambda_\perp^{(s, K)})$ to $(G_\perp^{(s', K')}, \lambda_\perp^{(s', K')})$ that maps s to s' . Then*

$$\{\chi_G^k(s, v) \mid v \in K\} \cap \{\chi_{G'}^k(s', v') \mid v' \in K'\} = \emptyset.$$

PROOF. If $\chi_G^k(s) \neq \chi_{G'}^k(s')$, then the conclusion of the lemma is obvious. Thus, we can assume otherwise.

We have already seen with Corollary 7 that cut vertices obtain other colors than non-cut vertices. Thus, we can assume that G and G' are already colored in a way such that s and s' have a color different from the colors of vertices in $K \cup K'$. Also, by Theorem 6, without loss of generality, we may assume that vertex pairs in a common 2-connected component have different colors than vertex pairs that are not contained in a common 2-connected component.

For readability, we drop the superscripts (s, K) and (s', K') . We show by induction that for all $u, v \in K \cup \{s\}$ and all $u', v' \in K' \cup \{s'\}$ with $\{u, v\} \not\subseteq \{s\}$ and $\{u', v'\} \not\subseteq \{s'\}$, the following implication is true:

$$i\chi_{G_\perp}^k(u, v) \neq i\chi_{G_\perp}^k(u', v') \Rightarrow i\chi_G^k(u, v) \neq i\chi_{G'}^k(u', v'). \quad (1)$$

For $i = 0$, the claim follows from the definition of the colorings λ_\top and λ'_\top . For the induction step, assume that there exist vertices $u, v \in K \cup \{s\}$ and $u', v' \in K' \cup \{s'\}$ such that $\{u, v\} \not\subseteq \{s\}$, $\{u', v'\} \not\subseteq \{s'\}$ with ${}_i\chi_{G_\top}^k(u, v) = {}_i\chi_{G'_\top}^k(u', v')$ and ${}_{i+1}\chi_{G_\top}^k(u, v) \neq {}_{i+1}\chi_{G'_\top}^k(u', v')$ (otherwise, the statement holds trivially). Thus, there must be a color tuple (c_1, c_2) such that the sets

$$M := \{x \mid x \in V(G_\top) \setminus \{u, v\}, ({}_i\chi_{G_\top}^k(x, v), {}_i\chi_{G_\top}^k(u, x)) = (c_1, c_2)\}$$

and

$$M' := \{x' \mid x' \in V(G'_\top) \setminus \{u', v'\}, ({}_i\chi_{G'_\top}^k(x', v'), {}_i\chi_{G'_\top}^k(u', x')) = (c_1, c_2)\}$$

do not have the same cardinality. Let

$$D := \{({}_i\chi_G^k(x, v), {}_i\chi_G^k(u, x)) \mid x \in M\} \cup \{({}_i\chi_{G'}^k(x', v'), {}_i\chi_{G'}^k(u', x')) \mid x' \in M'\}.$$

By induction and by Theorem 6, we have that

$$\{x \mid x \in V(G) \setminus \{u, v\}, ({}_i\chi_G^k(x, v), {}_i\chi_G^k(u, x)) \in D\} = M$$

and

$$\{x' \mid x' \in V(G') \setminus \{u', v'\}, ({}_i\chi_{G'}^k(x', v'), {}_i\chi_{G'}^k(u', x')) \in D\} = M'$$

and, hence, these sets do not have the same cardinality. Thus, ${}_{i+1}\chi_G^k(u, v) \neq {}_{i+1}\chi_{G'}^k(u', v')$.

Having shown Implication (1), it suffices to show that

$$\{\chi_{G_\top}^k(s, v) \mid v \in K\} \cap \{\chi_{G'_\top}^k(s', v') \mid v' \in K'\} = \emptyset.$$

However, this follows directly from the assumption that the k -dimensional WL-algorithm distinguishes every pair of non-isomorphic vertex-colored 2-connected graphs in \mathcal{G} and that the graphs $(G_\top^{(s, K)}, \lambda_\top^{(s, K)})$ and $(G'_\top^{(s', K')}, \lambda'_\top^{(s', K')})$ are 2-connected. \square

With this, we can prove the following.

LEMMA 10. *Assume $k \geq 2$ and suppose the k -dimensional Weisfeiler–Leman algorithm distinguishes all non-isomorphic vertex-colored 2-connected graphs in \mathcal{G} . For two graphs $G, G' \in \mathcal{G}$ with vertex colorings λ, λ' , respectively, suppose $s \in V(G), s' \in V(G')$. Assume there is no isomorphism from $(G_\top^s, \lambda_\top^s)$ to $(G'_\top^{s'}, \lambda'^{s'})$ that maps s to s' . Then $\chi_G^k(s) \neq \chi_{G'}^k(s')$.*

PROOF. Assume otherwise that $\chi_G^k(s) = \chi_{G'}^k(s')$. Further suppose that

$$\{K_1, \dots, K_t\} = \{K \mid (s, K) \in P_0(G)\}$$

and that

$$\{K'_1, \dots, K'_{t'}\} = \{K' \mid (s', K') \in P_0(G')\}.$$

From $(G_\top^s, \lambda_\top^s)_{(s)} \not\cong (G'_\top^{s'}, \lambda'^{s'})_{(s')}$, we conclude that there is a vertex-colored graph (H, λ_H) such that the sets

$$I := \{i \mid (G_\top^{(s, K_i)}, \lambda_\top^{(s, K_i)}) \cong (H, \lambda_H)\}$$

and

$$I' := \{j \mid (G'_\top^{(s', K'_j)}, \lambda'^{(s', K'_j)}) \cong (H, \lambda_H)\}$$

have different cardinalities. Note that all K_i with $i \in I$ and all K'_j with $j \in I'$ have the same cardinality. By Lemma 9, for $v \in K_j$ with $j \in I$ and for $v' \in K'_{j'}$ with $j' \notin I'$, we have $\chi_G^k(s, v) \neq \chi_{G'}^k(s', v')$.

Letting $C := \{\chi_G^k(s, v) \mid \exists i \in I \text{ s.t. } v \in K_i\}$, the vertices s and s' do not have the same number of neighbors connected via an arc of color C . We conclude that $\chi_G^k(s) \neq \chi_{G'}^k(s')$. \square

COROLLARY 11. *Assume $k \geq 2$ and suppose the k -dimensional Weisfeiler–Leman algorithm distinguishes all non-isomorphic vertex-colored 2-connected graphs in \mathcal{G} . Let $G, G' \in \mathcal{G}$ be connected graphs that are not 2-connected with vertex colorings λ, λ' , respectively. If, for vertices $v_1, v_2 \in V(G_\perp)$ and $v'_1, v'_2 \in V(G'_\perp)$, it holds that $\chi_{G_\perp}^k(v_1, v_2) \neq \chi_{G'_\perp}^k(v'_1, v'_2)$, then $\chi_G^k(v_1, v_2) \neq \chi_{G'}^k(v'_1, v'_2)$.*

PROOF. By Lemma 8, with respect to the colorings χ_G^k and $\chi_{G'}^k$, the vertices in $V(G_\perp)$ and $V(G'_\perp)$ have other colors than the vertices in $V(G) \setminus V(G_\perp)$ and $V(G') \setminus V(G'_\perp)$. Thus, it suffices to show that the colorings χ_G^k and $\chi_{G'}^k$ refine the colorings λ_\perp and λ'_\perp , respectively, on the domains of those. For this, we first observe that by Corollary 14, we can assume that either both $\{v_1, v_2\}$ and $\{v'_1, v'_2\}$ are 2-separators or neither of them is. Thus, by the definition of λ_\perp and λ'_\perp , it suffices to show the following two statements.

- (1) If $v_1 = v_2$ and $v'_1 = v'_2$ and also $\text{ISOTYPE}((G_\perp^{\{v_1\}}, \lambda_\perp^{\{v_1\}})_{(v_1)}) \neq \text{ISOTYPE}((G'_\perp^{\{v'_1\}}, \lambda'_\perp^{\{v'_1\}})_{(v'_1)})$, then $\chi_G^k(v_1) \neq \chi_{G'}^k(v'_1)$.
- (2) If $\{v_1, v_2\} \in E(G)$ and $\{v'_1, v'_2\} \in E(G')$ and also $\lambda_\perp(v_1, v_2) \neq \lambda'_\perp(v'_1, v'_2)$, then $\chi_G^k(v_1, v_2) \neq \chi_{G'}^k(v'_1, v'_2)$.

For the first item, from $\text{ISOTYPE}((G_\perp^{\{v_1\}}, \lambda_\perp^{\{v_1\}})_{(v_1)}) \neq \text{ISOTYPE}((G'_\perp^{\{v'_1\}}, \lambda'_\perp^{\{v'_1\}})_{(v'_1)})$, we know that v_1 and v'_1 must be cut vertices. Thus, the statement is exactly Lemma 10. For the second item, from the definition of λ_\perp and λ'_\perp we obtain $\lambda(v_1, v_2) \neq \lambda'(v'_1, v'_2)$, which implies $\chi_G^k(v_1, v_2) \neq \chi_{G'}^k(v'_1, v'_2)$. \square

PROOF OF THEOREM 5. Let $G, G' \in \mathcal{G}$ be non-isomorphic graphs with vertex colorings λ and λ' , respectively. It suffices to show the statement in case G and G' are connected. We proceed by induction on $|V(G)| + |V(G')|$. The base case $|V(G)| + |V(G')| = 2$ is trivial. For the induction step, if both graphs are 2-connected, then the statement follows directly from the assumptions. If exactly one of the graphs is 2-connected, then exactly one of the graphs has a cut vertex and the statement follows from Lemma 7. Thus, suppose both graphs are not 2-connected, but connected. Since $(G, \lambda) \not\cong (G', \lambda')$, we know by Lemma 4 that $(G_\perp, \lambda_\perp) \not\cong (G'_\perp, \lambda'_\perp)$. By induction, the k -dimensional WL-algorithm distinguishes (G_\perp, λ_\perp) from $(G'_\perp, \lambda'_\perp)$. Furthermore, by Lemma 8, the vertices in $V(G_\perp)$ and $V(G'_\perp)$ have other colors than the vertices in $V(G) \setminus V(G_\perp)$ and $V(G') \setminus V(G'_\perp)$. Moreover, by Corollary 11, the partition of the vertices and arcs induced by the coloring χ_G^k restricted to vertex tuples with elements from $V(G_\perp)$ is finer than the partition induced by $\chi_{G_\perp}^k$ on the graph G_\perp with initial coloring λ_\perp . Similarly, the partition induced by $\chi_{G'}^k$ restricted to vertex tuples with elements from $V(G'_\perp)$ is finer than the partition induced by $\chi_{G'_\perp}^k$ on G'_\perp with initial coloring λ'_\perp . Thus, the k -dimensional WL-algorithm distinguishes (G, λ) from (G', λ') . \square

5 REDUCTION TO ARC-COLORED 3-CONNECTED GRAPHS

In this section, our aim is to weaken the assumption from Theorem 5, which requires that 2-connected graphs are distinguished, to an assumption of 3-connected graphs being distinguished.

The strategy to prove our reduction follows similar ideas as those used in Section 4. It relies on the assumption that the input consists of vertex-colored 2-connected graphs, which we can make without loss of generality by the reduction from the last section. Now we consider the decomposition of vertex-colored 2-connected graphs into their so-called “3-connected components.”

Most of the results stated in Section 4 have analogous formulations for the three- or higher-dimensional WL-algorithm on 2-connected graphs. But a 3-connected component of a 2-connected graph G is a minor of G and not necessarily a subgraph. Thus, we require that the graph class \mathcal{G} is minor-closed. Furthermore, to enable the inductive approach, we will now have to consider graphs G in which the 2-tuples (u, v) with $\{u, v\} \in E(G)$, i.e., the arcs, are also colored. However, it turns out that it is not sufficient to require that arc-colored graphs are distinguished. In fact, we need the following stronger property.

Definition 12. Let \mathcal{H} be a set of graphs. We say that the k -dimensional WL-algorithm *determines orbits* on all arc-colored graphs in \mathcal{H} if for every pair of arc-colored graphs $(G, \lambda), (G', \lambda')$ with $G, G' \in \mathcal{H}$ and all vertices $s \in V(G)$ and $s' \in V(G')$ the following holds: there exists an isomorphism from (G, λ) to (G', λ') mapping s to s' if and only if $\chi_G^k(s) = \chi_{G'}^k(s')$.

Note that for $G' = G$, the vertex color classes obtained by an application of the k -dimensional WL-algorithm to an arc-colored graph (G, λ) are the orbits of the automorphism group of G with respect to λ .

The main result in this section is the following reduction theorem.

THEOREM 13. *Let \mathcal{G} be a minor-closed graph class and assume $k \geq 3$. Suppose the k -dimensional Weisfeiler–Leman algorithm determines orbits on all arc-colored 3-connected graphs in \mathcal{G} . Then the k -dimensional Weisfeiler–Leman algorithm distinguishes all non-isomorphic arc-colored graphs in \mathcal{G} .*

The next corollary states that the three-dimensional WL-algorithm distinguishes 2-separators from other pairs of vertices.

COROLLARY 14. *Assume $k \geq 3$ and let G and H be 2-connected graphs. Let u, v, u', v' be vertices such that $G - \{u, v\}$ is disconnected and $H - \{u', v'\}$ is connected. Then $\chi_G^k(u, v) \neq \chi_H^k(u', v')$.*

PROOF. Consider the connected graphs $G - \{u\}$ and $H - \{u'\}$. In the first graph, v is a cut vertex but in the second graph v' is not a cut vertex. Thus, by Corollary 7, we have that $\chi_{G-\{u\}}^{k-1}(v) \neq \chi_{H-\{u'\}}^{k-1}(v')$ and thus $\chi_G^k(u, v) \neq \chi_H^k(u', v')$. \square

Just as we did in the previous section, we want to apply a recursive strategy that relies on Lemma 4. However, to apply that lemma we require a minimum degree of 3. The following lemma states that vertices of degree 2 can be removed.

LEMMA 15. *Let \mathcal{G} be a minor-closed graph class and assume $k \geq 2$. Suppose the k -dimensional Weisfeiler–Leman algorithm distinguishes all arc-colored graphs in \mathcal{G} of minimum degree at least 3. Then the k -dimensional Weisfeiler–Leman algorithm distinguishes all non-isomorphic arc-colored graphs in \mathcal{G} .*

SKETCH OF THE PROOF. The proof is a basic exercise regarding the WL-algorithm. Let (G, λ) be an arc-colored graphs of minimum degree less than 3 with $G \in \mathcal{G}$ and let (G', λ') be a second arc-colored graph with $G \in \mathcal{G}$ and that is not isomorphic to G .

As the *base case*, we first observe that the two-dimensional WL-algorithm decides isomorphism for graphs whose degree sequences differ, as well as for graphs of maximum degree at most 2 (i.e., disjoint unions of cycles, isolated edges and isolated vertices). In particular, we can assume without loss of generality that G' has the same degree sequence as G .

Thus, assume both G and G' contain vertices of degree at least 3. For two vertices u and v , call a path from u to v a *chain* if all its inner vertices have degree 2 in the surrounding graph. Note that, since vertices of different degrees obtain different colors, the color $\chi^k(u, v)$ implicitly encodes, for each i , the number of chains of length exactly i from u to v .

We describe an algorithmic reduction to the base case.

Step 1: Repeat the following procedure recursively on its own output while the resulting graphs have the same degree sequences and contain vertices of degree 1 and vertices of degree at least 3. Assign to each vertex $v \in V(G)$ a color encoding the tuple containing for each pair C, C' of a vertex color C and an arc color C' the number

$$\left| \left\{ w \mid w \in N(\{v\}), w \text{ is a vertex of degree 1 and has color } C \text{ and the arc } (v, w) \text{ has color } C' \right\} \right|$$

and remove all vertices of degree 1 from G . Construct a colored subgraph of G' in an analogous fashion.

Step 2: Let (H, λ_H) and $(H', \lambda_{H'})$ be the output of Step 1. If H and H' have different degree sequences, then output NO. If they have maximum degree at most 2, then apply the base case to determine the output. If both graphs have minimum degree 2 and they contain vertices of degree at least 3, then define (J, λ_J) and $(J', \lambda_{J'})$ as the minors of H and H' obtained by only retaining vertices of degree at least 3, connecting two such vertices with an edge if there is a chain between them, and assigning the edge a color that encodes the multiset of lengths of (colored) chains between the vertices. Call Step 1 on input (J, λ_J) and $(J', \lambda_{J'})$.

Clearly, the algorithm terminates and outputs NO if and only if the final computed graphs are non-isomorphic. Now an induction over the number of recursive calls shows that the k -dimensional WL-algorithm distinguishes (G, λ) and (G', λ') if and only if the above algorithm outputs NO. \square

The lemma allows us to focus on graphs with minimum degree 3. Doing so, in analogy to Lemma 8, the following proposition gives a characterization of the vertices in $V(G_\perp)$.

PROPOSITION 16. *Let G be a 2-connected graph of minimum degree at least 3 that is not 3-connected. Then $x \notin V(G_\perp)$ if and only if there exists a vertex u contained in a separator of G of minimal size such that $x \notin V((G - \{u\})_\perp)$ and such that the (unique) 2-connected component containing x in $G - \{u\}$ has exactly one vertex belonging to a separator of G of minimal size.*

PROOF. If $x \notin V(G_\perp)$, then $x \in K$ for some $(\{u, s\}, K) \in P_0(G)$. Then $(s, K) \in P_0(G - \{u\})$ and thus, $x \notin V((G - \{u\})_\perp)$. Moreover, by Item 3 of Lemma 3, no vertex of K is contained in a separator of G of minimal size, and no vertex of K is a cut vertex of $G - \{u\}$. This implies that s is the only vertex in the 2-connected component of x in $G - \{u\}$ that belongs to a separator of G of minimal size.

Conversely, suppose u is a vertex in a separator of G of minimal size such that $x \notin V((G - \{u\})_\perp)$ and the 2-connected component of x in $G - \{u\}$ has exactly one vertex belonging to a separator of G of minimal size. Since $x \notin V((G - \{u\})_\perp)$, there is $(s, K) \in P_0(G - \{u\})$ with $x \in K$. Then, by Item 3 of Lemma 3, no vertex of K is a cut vertex of $G - \{u\}$. Thus, $K \cup \{s\}$ is a 2-connected component of $G - \{u\}$. Since, by assumption, s is the only vertex in this 2-connected component that is contained in a separator of G of minimal cardinality, by Item 3 of Lemma 3, we have $(\{u, s\}, K) \in P_0(G)$. \square

LEMMA 17. *Assume $k \geq 3$ and let G, G' be 2-connected graphs of minimum degree at least 3 that are not 3-connected. Then for all $v \in V(G_\perp)$ and $v' \in V(G') \setminus V(G'_\perp)$, it holds that $\chi_G^k(v) \neq \chi_{G'}^k(v')$.*

PROOF. Suppose that $v \in V(G_\perp)$ and $v' \in V(G') \setminus V(G'_\perp)$. Let u be a vertex contained in some minimal separator of G' such that $v' \notin V((G' - \{u\})_\perp)$ and such that the 2-connected component of v' in $G' - \{u\}$ has exactly one vertex that is contained in a separator of G' of minimal size. Such a vertex exists by Proposition 16. We argue that $\chi_G^k(v, t) \neq \chi_{G'}^k(v', u)$ for all $t \in V(G)$.

If t is not contained in any minimal separator, then this follows from Corollary 14. Otherwise, we know by Proposition 16 that $v \in V((G - \{t\})_\perp)$ or the 2-connected component of v in $G - \{t\}$

does not have exactly one vertex that is contained in a separator of G of minimal size. In the first case, we use Lemma 8, and in the second case, we use Theorem 6 and Corollary 14 to conclude that $\chi_{G-\{t\}}^{k-1}(v) \neq \chi_{G'-\{u\}}^{k-1}(v')$ and thus $\chi_G^k(v, t) \neq \chi_{G'}^k(v', u)$. \square

LEMMA 18. *For $k \geq 3$, suppose the k -dimensional Weisfeiler–Leman algorithm determines orbits on all arc-colored 3-connected graphs in \mathcal{G} . Suppose $G, G' \in \mathcal{G}$ are arc-colored 2-connected graphs of minimum degree at least 3. Assume that $(\{s_1, s_2\}, K) \in P_0(G)$ and $(\{s'_1, s'_2\}, K') \in P_0(G')$.*

Suppose that no isomorphism from $(G_{\top}^{(\{s_1, s_2\}, K)}, \lambda_{\top}^{(\{s_1, s_2\}, K)})$ to $(G'_{\top}^{(\{s'_1, s'_2\}, K')}, \lambda'_{\top}^{(\{s'_1, s'_2\}, K')})$ maps s_1 to s'_1 and s_2 to s'_2 . Then

$$\{\chi_G^k(s_1, s_2, v) \mid v \in K\} \cap \{\chi_{G'}^k(s'_1, s'_2, v) \mid v \in K'\} = \emptyset.$$

PROOF. This is an adaption of the proof of Lemma 9.

If $\chi_G^k(s_1, s_2) \neq \chi_{G'}^k(s'_1, s'_2)$, then the conclusion of the lemma is obvious. Thus, we can assume otherwise. We have already seen with Corollary 14 that 2-separators obtain other colors than other pairs of vertices. Thus, with Item 3 of Lemma 3, we can assume that G and G' are already colored in a way such that $(s_1, s_2), (s_2, s_1), (s'_1, s'_2), (s'_2, s'_1)$ have colors different from the colors of pairs of vertices (t_1, t_2) with $\{t_1, t_2\} \cap (K \cup K') \neq \emptyset$. Also, we may presuppose without loss of generality that s_1, s_2, s'_1, s'_2 have colors different from colors of vertices that are not contained in any 2-separator of any of the graphs. Moreover, by Theorem 6, without loss of generality, we may assume that vertex pairs in different 2-connected components of $G - \{s_1\}$ have other colors than vertex pairs $\{u, v\}$ that, for every vertex s contained in a 2-separator, lie in a common 2-connected component of $G - \{s\}$. Analogous statements hold for $G - \{s_2\}$, $G' - \{s'_1\}$, and $G' - \{s'_2\}$, respectively. In particular, without loss of generality, pairs of vertices that are both contained in K have other colors than pairs of vertices from G_{\top} that are not contained in a common 3-connected component of G (and similarly for K' in G').

For readability, we drop the superscripts $(\{s_1, s_2\}, K)$ and $(\{s'_1, s'_2\}, K')$.

We show by induction that for all $i \in \mathbb{N}$, all $u, v, w \in K \cup \{s_1, s_2\}$ and all $u', v', w' \in K' \cup \{s'_1, s'_2\}$ with $\{u, v, w\} \not\subseteq \{s_1, s_2\}$ and $\{u', v', w'\} \not\subseteq \{s'_1, s'_2\}$, the following implication holds:

$$i\chi_{G_{\top}}^k(u, v, w) \neq i\chi_{G'_{\top}}^k(u', v', w') \Rightarrow i\chi_G^k(u, v, w) \neq i\chi_{G'}^k(u', v', w'). \quad (2)$$

For the induction base with $i = 0$, suppose $0\chi_{G_{\top}}^k(u, v, w) \neq 0\chi_{G'_{\top}}^k(u', v', w')$. Then either there is no isomorphism from $G_{\top}[\{u, v\}]$ to $G'_{\top}[\{u', v'\}]$ mapping u to u' and v to v' , or $\lambda_{\top}(u, v, w) \neq \lambda'_{\top}(u', v', w')$.

In the first case, by definition of the graphs G_{\top} and G'_{\top} and their colorings, we immediately get $0\chi_G^k(u, v, w) \neq 0\chi_{G'}^k(u', v', w')$, since an isomorphism from $G[\{u, v\}]$ to $G'[\{u', v'\}]$ that maps u to u' and v to v' would induce an isomorphism from $G_{\top}[\{u, v\}]$ to $G'_{\top}[\{u', v'\}]$ with the same mappings. In the second case, since G and G' are arc-colored graphs, we have $\lambda_{\top}(u, v, w) = \lambda_{\top}(u, v) \neq \lambda'_{\top}(u', v') = \lambda'_{\top}(u', v', w')$. If $\lambda(u, v) \neq \lambda'(u', v')$, then $0\chi_G^k(u, v, w) \neq 0\chi_{G'}^k(u', v', w')$. Otherwise, from the definitions of λ_{\top} and λ'_{\top} , we conclude that $(u, v) \in E(G) \not\leftrightarrow (u', v') \in E(G')$ or $\{u, v\} \subseteq \{s_1, s_2\} \not\leftrightarrow \{u', v'\} \subseteq \{s'_1, s'_2\}$. In the first subcase, it follows that $0\chi_G^k(u, v, w) \neq 0\chi_{G'}^k(u', v', w')$. In the second subcase, since $(\{s_1, s_2\}, K) \in P_0(G)$ and $(\{s'_1, s'_2\}, K') \in P_0(G')$, we know with Item 3 from Lemma 3 that either at least one of u and v or at least one of u' and v' is a vertex not contained in any 2-separator. Since we have assumed that s_1, s_2, s'_1, s'_2 have colors that are different from the colors of vertices not contained in any 2-separator, we also conclude that $0\chi_G^k(u, v, w) \neq 0\chi_{G'}^k(u', v', w')$.

For the induction step, assume there exist vertices $u, v, w \in K \cup \{s_1, s_2\}$ and $u', v', w' \in K' \cup \{s'_1, s'_2\}$ such that $\{u, v, w\} \not\subseteq \{s_1, s_2\}$ and $\{u', v', w'\} \not\subseteq \{s'_1, s'_2\}$ with ${}_i\chi_{G_\top}^k(u, v, w) = {}_i\chi_{G'_\top}^k(u', v', w')$ and ${}_{i+1}\chi_{G_\top}^k(u, v, w) \neq {}_{i+1}\chi_{G'_\top}^k(u', v', w')$.

Thus, there must be a color triple (c_1, c_2, c_3) such that the sets

$$M := \left\{ x \mid x \in V(G_\top) \setminus \{u, v, w\}, \right. \\ \left. ({}_i\chi_{G_\top}^k(x, v, w), {}_i\chi_{G_\top}^k(u, x, w), {}_i\chi_{G_\top}^k(u, v, x)) = (c_1, c_2, c_3) \right\}$$

and

$$M' := \left\{ x' \mid x' \in V(G'_\top) \setminus \{u', v', w'\}, \right. \\ \left. ({}_i\chi_{G'_\top}^k(x', v', w'), {}_i\chi_{G'_\top}^k(u', x', w'), {}_i\chi_{G'_\top}^k(u', v', x')) = (c_1, c_2, c_3) \right\}$$

do not have the same cardinality. Let

$$D := \left\{ ({}_i\chi_G^k(x, v, w), {}_i\chi_G^k(u, x, w), {}_i\chi_G^k(u, v, x)) \mid x \in M \right\} \cup \\ \left\{ ({}_i\chi_{G'}^k(x', v', w'), {}_i\chi_{G'}^k(u', x', w'), {}_i\chi_{G'}^k(u', v', x')) \mid x' \in M' \right\}.$$

By induction and by Theorem 6, we have that

$$\left\{ x \mid x \in V(G) \setminus \{u, v, w\}, ({}_i\chi_G^k(x, v, w), {}_i\chi_G^k(u, x, w), {}_i\chi_G^k(u, v, x)) \in D \right\} = M$$

and

$$\left\{ x' \mid x' \in V(G') \setminus \{u', v', w'\}, ({}_i\chi_{G'}^k(x', v', w'), {}_i\chi_{G'}^k(u', x', w'), {}_i\chi_{G'}^k(u', v', x')) \in D \right\} = M'.$$

Hence, these sets do not have the same cardinality. Thus, ${}_{i+1}\chi_G^k(u, v, w) \neq {}_{i+1}\chi_{G'}^k(u', v', w')$.

Having shown Implication (2), it suffices to show that

$$\left\{ \chi_{G_\top}^k(s_1, s_2, v) \mid v \in K \right\} \cap \left\{ \chi_{G'_\top}^k(s'_1, s'_2, v') \mid v' \in K' \right\} = \emptyset.$$

For this, it suffices to prove that $\chi_{G_\top}^k(s_1) \neq \chi_{G'_\top}^k(s'_1)$ holds.

The graphs $(G_\top^{(\{s_1, s_2\}, K)}, \lambda_\top^{(\{s_1, s_2\}, K)})$ and $(G'_\top^{(\{s'_1, s'_2\}, K')}, \lambda'_\top^{(\{s'_1, s'_2\}, K')})$ are 3-connected. Thus, if we had that $\chi_{G_\top}^k(s_1) = \chi_{G'_\top}^k(s'_1)$, there would have to be an isomorphism from $(G_\top^{(\{s_1, s_2\}, K)}, \lambda_\top^{(\{s_1, s_2\}, K)})$ to $(G'_\top^{(\{s'_1, s'_2\}, K')}, \lambda'_\top^{(\{s'_1, s'_2\}, K')})$ that maps s_1 to s'_1 , since we have assumed that the k -dimensional WL-algorithm determines orbits on all arc-colored 3-connected graphs in \mathcal{G} . However, by definition of $\lambda_\top^{(\{s_1, s_2\}, K)}$ and $\lambda'_\top^{(\{s'_1, s'_2\}, K')}$, this isomorphism would also map s_2 to s'_2 , contradicting the assumptions of the lemma. \square

Using the lemma, we can show the following.

LEMMA 19. *Assume $k \geq 3$ and suppose the k -dimensional Weisfeiler–Leman algorithm determines orbits on all arc-colored 3-connected graphs in \mathcal{G} . Assume $G, G' \in \mathcal{G}$ are arc-colored 2-connected graphs of minimum degree at least 3 and let $\{s_1, s_2\} \subseteq V(G)$ and $\{s'_1, s'_2\} \subseteq V(G')$ be 2-separators of G and G' , respectively.*

Suppose that no isomorphism from $(G_\top^{\{s_1, s_2\}}, \lambda_\top^{\{s_1, s_2\}})$ to $(G'_\top^{\{s'_1, s'_2\}}, \lambda'_\top^{\{s'_1, s'_2\}})$ maps s_1 to s'_1 and s_2 to s'_2 . Then $\chi_G^k(s_1, s_2) \neq \chi_{G'}^k(s'_1, s'_2)$.

PROOF. This proof is similar to the proof of Lemma 10. Suppose that

$$\{K_1, \dots, K_t\} = \left\{ K \mid (\{s_1, s_2\}, K) \in P_0(G) \right\}$$

and that

$$\{K'_1, \dots, K'_{l'}\} = \{K' \mid (\{s'_1, s'_2\}, K') \in P_0(G')\}.$$

Since there is no isomorphism from $(G_{\top}^{\{s_1, s_2\}}, \lambda_{\top}^{\{s_1, s_2\}})$ to $(G_{\top}^{\{s'_1, s'_2\}}, \lambda_{\top}^{\{s'_1, s'_2\}})$ that maps s_1 to s'_1 and s_2 to s'_2 , there is an arc-colored graph (H, λ_H) such that the sets

$$I := \left\{ i \mid \left(G_{\top}^{\{s_1, s_2\}, K_i}, \lambda_{\top}^{\{s_1, s_2\}, K_i} \right)_{(s_1, s_2)} \cong (H, \lambda_H) \right\}$$

and

$$I' := \left\{ j \mid \left(G_{\top}^{\{s'_1, s'_2\}, K'_j}, \lambda_{\top}^{\{s'_1, s'_2\}, K'_j} \right)_{(s'_1, s'_2)} \cong (H, \lambda_H) \right\}$$

have different cardinalities. Note that all K_i with $i \in I$ and all K'_j with $j \in I'$ have the same cardinality. We know by Lemma 18 that for $v \in K_i$ with $i \in I$ and $v' \in K'_j$ with $j \notin I'$, we have $\chi_G^k(s_1, s_2, v) \neq \chi_{G'}^k(s'_1, s'_2, v')$. Letting $C := \{\chi_G^k(s_1, s_2, v) \mid \exists i \in I \text{ s.t. } v \in K_i\}$, the sets $\{v \mid \chi_G^k(s_1, s_2, v) \in C\}$ and $\{v' \mid \chi_{G'}^k(s'_1, s'_2, v') \in C\}$ do not have the same cardinality. We conclude that $\chi_G^k(s_1, s_2) \neq \chi_{G'}^k(s'_1, s'_2)$. \square

COROLLARY 20. *Assume $k \geq 3$ and suppose the k -dimensional Weisfeiler–Leman algorithm determines orbits on all arc-colored 3-connected graphs in \mathcal{G} . Let $G, G' \in \mathcal{G}$ be 2-connected graphs of minimum degree at least 3 with arc colorings λ and λ' , respectively. If, for vertices $v_1, v_2 \in V(G_{\perp})$ and $v'_1, v'_2 \in V(G'_{\perp})$, it holds that $\chi_{G_{\perp}}^k(v_1, v_2) \neq \chi_{G'_{\perp}}^k(v'_1, v'_2)$, then $\chi_G^k(v_1, v_2) \neq \chi_{G'}^k(v'_1, v'_2)$.*

PROOF. We may assume without loss of generality that $\lambda(v_1, v_1) = \lambda'(v'_1, v'_1)$ and $\lambda(v_2, v_2) = \lambda'(v'_2, v'_2)$. Furthermore, by Lemma 17, with respect to the colorings χ_G^k and $\chi_{G'}^k$, the vertices in $V(G_{\perp})$ and $V(G'_{\perp})$ have other colors than the vertices in $V(G) \setminus V(G_{\perp})$ and $V(G') \setminus V(G'_{\perp})$. Thus, it suffices to show that the colorings χ_G^k and $\chi_{G'}^k$ refine the colorings λ_{\perp} and λ'_{\perp} , respectively, on the domains of those. For this, by the definition of λ_{\perp} and λ'_{\perp} , it suffices to show the following two statements.

- (1) If $\{v_1, v_2\}$ and $\{v'_1, v'_2\}$ are 2-separators and we have $\text{ISOTYPE}((G_{\top}^{\{v_1, v_2\}}, \lambda_{\top}^{\{v_1, v_2\}})_{(v_1, v_2)}) \neq \text{ISOTYPE}((G_{\top}^{\{v'_1, v'_2\}}, \lambda_{\top}^{\{v'_1, v'_2\}})_{(v'_1, v'_2)})$, then $\chi_G^k(v_1, v_2) \neq \chi_{G'}^k(v'_1, v'_2)$.
- (2) If $v_1 = v_2$ and $v'_1 = v'_2$, or $\{v_1, v_2\} \in E(G)$ and $\{v'_1, v'_2\} \in E(G')$ but $\{v_1, v_2\}$ and $\{v'_1, v'_2\}$ are not 2-separators, then if $\lambda_{\perp}(v_1, v_2) \neq \lambda'_{\perp}(v'_1, v'_2)$, then $\chi_G^k(v_1, v_2) \neq \chi_{G'}^k(v'_1, v'_2)$.

The first item is exactly Lemma 19. For the second item, from the definition of λ_{\perp} and λ'_{\perp} we obtain $\lambda(v_1, v_2) \neq \lambda'(v'_1, v'_2)$, which implies $\chi_G^k(v_1, v_2) \neq \chi_{G'}^k(v'_1, v'_2)$. \square

PROOF OF THEOREM 13. Let $G, G' \in \mathcal{G}$ be non-isomorphic graphs with arc colorings λ and λ' , respectively, such that $(G, \lambda) \not\cong (G', \lambda')$ and suppose the k -dimensional WL-algorithm determines orbits on all arc-colored 3-connected graphs in \mathcal{G} . We prove the statement by induction on $|V(G)| + |V(G')|$. The base case $|V(G)| + |V(G')| = 2$ is trivial. For the induction step, if both graphs are 3-connected, then the statement follows directly from the assumptions, since “determines orbits” is a stronger assumption than “distinguishes.” If exactly one of the graphs is 3-connected, then exactly one of the graphs has a 2-separator and the statement follows from Corollary 14.

Thus, suppose both graphs are not 3-connected and assume all pairs of arc-colored graphs (H, λ_H) and $(H', \lambda_{H'})$ with $|V(H)| + |V(H')| < |V(G)| + |V(G')|$ are distinguished. By Theorem 5, with \mathcal{G} being the class of graphs containing every graph isomorphic to G, G' or a minor of one of them, it suffices to show the statement for the case that G and G' are 2-connected. If G and G' do not have the same minimum degree, then they are distinguished by their degree sequences.

Suppose both G and G' have minimum degree at least 3. Then, since (G, λ) and (G', λ') are not isomorphic, we know by Lemma 4 that $(G_\perp, \lambda_\perp) \not\cong (G'_\perp, \lambda'_\perp)$. Therefore, (G_\perp, λ_\perp) and $(G'_\perp, \lambda'_\perp)$ are distinguished by the k -dimensional WL-algorithm by induction assumption. By Lemma 17, the vertices in $V(G_\perp)$ and $V(G'_\perp)$ have other colors than the vertices in $V(G) \setminus V(G_\perp)$ and $V(G') \setminus V(G'_\perp)$. Moreover, by Corollary 20, the partition of the vertices and arcs induced by the coloring χ_G^k restricted to $V(G_\perp)$ is finer than the partition induced by λ_\perp . Similarly, the partition induced by $\chi_{G'}^k$ on $V(G'_\perp)$ is finer than the partition induced by λ'_\perp . Thus, since the k -dimensional WL-algorithm distinguishes (G_\perp, λ_\perp) from $(G'_\perp, \lambda'_\perp)$, it also distinguishes (G, λ) from (G', λ') .

By Lemma 15, the case that G and G' do not have minimum degree at least 3 reduces to the case of minimum degree at least 3, letting \mathcal{G} be the class of graphs containing every graph isomorphic to G , G' or a minor of one of them. \square

In the last two sections, we have concerned ourselves with graphs being distinguished (referring to two input graphs from a class) rather than graphs being identified (referring to one input graph from a class and another input graph being arbitrary). However, the theorems we prove also have corresponding versions concerning the latter notion.

For a graph G , the *Weisfeiler–Leman dimension* (WL-dimension) of G is the least integer k such that the k -dimensional WL-algorithm distinguishes G from every non-isomorphic graph G' . Similarly, for a graph class \mathcal{G} , the *Weisfeiler–Leman dimension* of \mathcal{G} is the least integer k such that every graph in \mathcal{G} has WL-dimension at most k .

LEMMA 21. *Let \mathcal{G} be a minor-closed graph class. The Weisfeiler–Leman dimension of graphs in \mathcal{G} is at most $\max\{3, k\}$, where k is the minimal number ℓ such that the ℓ -dimensional Weisfeiler–Leman algorithm determines orbits on all arc-colored 3-connected graphs in \mathcal{G} and identifies such graphs.*

The proof follows almost verbatim the lines of the entire proof of Theorem 13 outlined in the last two sections, replacing “distinguishes” with “identifies.”

6 ARC-COLORED 3-CONNECTED PLANAR GRAPHS

Let G be a 3-connected planar graph. We show that, typically, we can individualize two vertices in G so that applying the one-dimensional WL-algorithm yields a discrete graph. There are some 3-connected planar graphs for which this is not the case. However, we can precisely determine the collection of such exceptions.

Definition 22. A graph G is an *exception* if G is a 3-connected planar graph in which there are no two vertices v, w in G such that $\chi_{G(v, w)}^1$ is the discrete coloring.

Here and in the following, we denote by $G_{(v_1, v_2, \dots, v_t)}$ the colored graph obtained from the (un-colored) graph G by individualizing the vertices v_1, v_2, \dots, v_t in that order. More specifically, we let $G_{(v_1, v_2, \dots, v_t)}$ be the colored graph (G, λ) with

$$\lambda(v) := \begin{cases} i & \text{if } v = v_i \\ 0 & \text{if } v \notin \{v_1, \dots, v_t\}. \end{cases}$$

As before, χ_G^1 denotes the stable coloring of the one-dimensional WL-algorithm applied to the graph G .

LEMMA 23. *Let G be a 3-connected planar graph and let v_1, v_2, v_3 be vertices of G . If v_1, v_2, v_3 lie on a common face, then $\chi_{G(v_1, v_2, v_3)}^1$ is a discrete coloring.*

PROOF. We intend to use the Tutte’s Spring Embedding Theorem [31] (see Reference [25, Section 12.3]). Let v_1, v_2, v_3 be vertices on a common face of G . Let $\mu_0 : V(G) \setminus \{v_1, v_2, v_3\} \rightarrow \mathbb{R}^2$ be an

arbitrary mapping that satisfies $\mu_0(v_1) = (0, 0)$, $\mu_0(v_2) = (1, 0)$, and $\mu_0(v_3) = (0, 1)$. For $i \in \mathbb{N}$, we define μ_{i+1} recursively by setting

$$\mu_{i+1}(v) = \begin{cases} \frac{1}{d(v)} \sum_{w \in N(v)} \mu_i(w) & \text{if } v \notin \{v_1, v_2, v_3\}, \\ \mu_i(v) & \text{otherwise.} \end{cases}$$

Then Tutte's result says that this recursion converges to a barycentric planar embedding of G , that is, an embedding in which every vertex not in $\{v_1, v_2, v_3\}$ is contained in the convex hull of its neighbors [25, 31]. This implies that after a finite number of steps, the barycentric embedding is injective, i.e., no two vertices are mapped to the same image.

From the theorem, we only require the fact that for some i , the map μ_i is injective. Choose μ_0 with the requirements above and so that all vertices in $V(G) \setminus \{v_1, v_2, v_3\}$ have the same image. For example, set $\mu_0(v) = (1, 1)$ for $v \in V(G) \setminus \{v_1, v_2, v_3\}$. We argue the following statement by induction on i . For every pair of vertices v and v' , it holds that

$$\mu_i(v) \neq \mu_i(v') \Rightarrow {}_i\chi_{G(v_1, v_2, v_3)}^1(v) \neq {}_i\chi_{G(v_1, v_2, v_3)}^1(v'),$$

where ${}_i\chi_{G(v_1, v_2, v_3)}^1(x)$ denotes the color of vertex x after the i th iteration of the one-dimensional WL-algorithm applied to $G_{(v_1, v_2, v_3)}$.

For $i = 0$, the statement holds by the definition of μ_0 and the fact that v_1 , v_2 , and v_3 are singletons in $G_{(v_1, v_2, v_3)}$. For $i > 0$, if $\sum_{w \in N(v)} \mu_i(w) \neq \sum_{w' \in N(v')} \mu_i(w')$, then $\{\mu_i(w) \mid w \in N(v)\}$ and $\{\mu_i(w') \mid w' \in N(v')\}$ are different and thus, by induction, the multisets $\{{}_i\chi_{G(v_1, v_2, v_3)}^1(w) \mid w \in N(v)\}$ and $\{{}_i\chi_{G(v_1, v_2, v_3)}^1(w') \mid w' \in N(v')\}$ are different.

By the Spring Embedding Theorem, for some i , the map μ_i is injective, implying that ${}_i\chi_{G(v_1, v_2, v_3)}^1$ and therefore also $\chi_{G(v_1, v_2, v_3)}^1$ are discrete colorings. \square

From the lemma, one can directly conclude that for $k \geq 4$, the k -dimensional WL-algorithm determines the vertex orbits of every 3-connected planar graph.

COROLLARY 24. *For $k \geq 4$, the k -dimensional Weisfeiler–Leman algorithm determines orbits on all arc-colored 3-connected planar graphs and identifies these graphs.*

PROOF. Let G be an arc-colored 3-connected planar graph and let $n := |G|$ be its size. Then by Lemma 23, there are vertices v_1, v_2, v_3 such that $\chi_{G(v_1, v_2, v_3)}^1$ is discrete, since the additional arc coloring can only refine the stable coloring of the uncolored graph. This implies that the multi-set $C := \{\chi_G^4(v_1, v_2, v_3, x) \mid x \in V(G)\}$ contains n different colors. Let H be a second arc-colored graph. If H contains vertices v'_1, v'_2, v'_3 such that $\{\chi_H^4(v'_1, v'_2, v'_3, x') \mid x' \in V(H)\} = C$, then G and H are isomorphic via an isomorphism that maps v_1 to v'_1 . Otherwise, the color $\chi_G^4(v_1, v_2, v_3, v_3)$ is for all $v'_1, v'_2, v'_3 \in V(H)$ different from the color $\chi_H^4(v'_1, v'_2, v'_3, v'_3)$, implying that the sets of vertex colors computed by the four-dimensional WL-algorithm in G and H are disjoint and thus, the two graphs are distinguished. Hence, G is identified. \square

The statement of the corollary also holds for $k = 3$. The proof amounts to proving the following theorem.

THEOREM 25. *If G is an exception (i.e., G is a 3-connected planar graph without a pair of vertices v, w such that $\chi_{G(v, w)}^1$ is discrete), then G is isomorphic to one of the graphs in Figure 1.*

Before we present the lengthy proof of the theorem, we state its implications.

The fixing number of a graph G is the minimum size of a set of vertices S such that the only automorphism that fixes S pointwise is the identity.

COROLLARY 26. *Let G be a 3-connected planar graph. The fixing number of G is at most 3 with equality attained if and only if G is isomorphic to an exception (i.e., one of the graphs depicted in Figure 1).*

PROOF. If in a given graph G , there is a set of ℓ vertices such that individualizing all vertices in the set and then applying the one-dimensional WL-algorithm yields a discrete coloring, then ℓ is an upper bound on the fixing number of G . By Definition 22, a graph that is not an exception has fixing number at most 2. To conclude the corollary, it thus suffices to check that all exceptions have fixing number 3, which can be done using Theorem 25. \square

COROLLARY 27. *For $k \geq 3$, the k -dimensional Weisfeiler–Leman algorithm determines orbits on all arc-colored 3-connected planar graphs and identifies these graphs.*

PROOF. Let G be an arc-colored 3-connected planar graph that is not an exception. Then there are vertices v and w such that $\chi_{G(v,w)}^1$ is discrete. Analogously to the proof of Corollary 24, we obtain that for every second arc-colored graph H , the three-dimensional WL-algorithm only assigns equal colors to a vertex of G and a vertex of H if there is an isomorphism mapping the one to the other. Thus, G is identified and its orbits are determined.

A thorough but tedious case analysis by hand for each exception (which could also be performed by a computer, e.g., using COCO2P [24]) shows that every arc-colored exception is distinguished from all non-isomorphic arc-colored graphs and that on each arc-colored exception, the stable coloring under the three-dimensional WL-algorithm induces the orbit partition on the vertices. \square

The task in the remainder of this section is to show Theorem 25. The proof of the theorem is a lot more involved than the proof of Corollary 24. Thus, at the expense of increasing k by 1 from 3 to 4 in Theorem 1, the reader may skip the following lengthy exposition.

To show Theorem 25, we characterize the exceptions in a case-by-case analysis with respect to the existence of vertices of certain degrees. The following two lemmas serve as general tools to deduce information about the structure of the input graphs.

For a subgraph G' of a graph G , we say that $v \in G'$ is *saturated in G' with respect to G* if $d_{G'}(v) = d_G(v)$. Thus, if a vertex is saturated, then its neighbors in G and in G' are the same.

LEMMA 28. *Let G be a 3-connected planar graph.*

- (1) *Let G' be a subgraph of G and suppose that the sequence v_1, \dots, v_t forms a face cycle in the planar embedding of G' induced by a planar embedding of G . If in $\{v_1, \dots, v_t\}$, there are at most two vertices that are not saturated in G' with respect to G , then it holds that $V(G) = \{v_1, \dots, v_t\}$ or v_1, \dots, v_t is a face cycle of G .*
- (2) *If v_1, \dots, v_t is a 3-cycle that contains a vertex of degree 3 in G or v_1, \dots, v_t is an induced 4-cycle of G that contains at least two vertices of degree 3 in G , then $V(G) = \{v_1, \dots, v_t\}$ or v_1, \dots, v_t is a face cycle of G .*

PROOF SKETCH. For Item 1, suppose v_1, \dots, v_t forms a face cycle of G' and there are at most two vertices v_i and v_j that are not saturated in G' with respect to G . Assume v_1, \dots, v_t is not a face cycle in G . Then the vertices v_i and v_j are the only ones among v_1, \dots, v_t that have neighbors in G inside the region of the plane corresponding to the face cycle of G' formed by v_1, \dots, v_t . Therefore, if $V(G) \neq \{v_1, \dots, v_t\}$, then $\{v_i, v_j\}$ is a separator of G , which contradicts the 3-connectivity.

For Item 2, consider an induced 4-cycle v_1, \dots, v_4 in which there exist two vertices v_i and v_j of degree 3 in G . If v_1, \dots, v_4 is not a face cycle of G , then $\{v_1, \dots, v_4\} \setminus \{v_i, v_j\}$ is a separator of size 2. The argument for a 3-cycle v_1, v_2, v_3 is similar. \square

LEMMA 29. *Let G be an exception and let v be a vertex of G . Let $u_1, \dots, u_{d(v)}$ be the cyclic ordering of the neighbors of v induced by a planar embedding of G . Then every pair of vertices u_i and u_{i+1} has a common neighbor of degree $d(v)$ other than v .*

PROOF. If u_i and u_{i+1} do not have a common neighbor of degree $d(v)$ other than v , then the coloring $\chi_{G(u_i, u_{i+1})}^1$ has the three singletons u_i, v, u_{i+1} , which lie on a common face. Thus, by Lemma 23, the coloring $\chi_{G(u_i, u_{i+1})}^1$ is discrete, contradicting the assumption that G is an exception. \square

For an exception G with a distinguished vertex v and $N(v) := \{u_1, \dots, u_t\}$, to reason about planarity, we will often consider the graph G^v obtained from G by inserting edges between every pair u_i and u_{i+1} . It is easy to see that since G is planar, the graph G^v is also planar.

Now we can start determining the structure of the exceptions. Since every exception is, by definition, 3-connected, no exception has a vertex of degree 1 or 2. Since every planar graph has a vertex of degree at most 5, every exception contains a vertex of degree 3, 4, or 5. We begin with the exceptions that contain a vertex of degree 5.

LEMMA 30. *If G is an exception that has a vertex of degree 5, then it is isomorphic to the icosahedron or the bipyramid on seven vertices.*

PROOF. To simplify notation, we let $\chi_G := \chi_G^1$ in the course of this proof. Let G be an exception with a vertex v of degree 5. Let $N := N(v)$ be the set of neighbors of v , and let (u_1, \dots, u_5) be their cyclic ordering induced by a planar embedding of G . For convenience, we will take indices modulo 5.

By Lemma 29, every pair of vertices u_i, u_{i+1} has a common neighbor $x_{i, i+1}$ of degree 5 other than v . (We remark that the vertices $x_{i, i+1}$ are not necessarily distinct or unique.)

CLAIM 1. *For all $i \in \{1, \dots, 5\}$, every pair of vertices u_i, u_{i+2} has a common neighbor $x_{i, i+2}$ of degree 5 other than v .*

PROOF. We show the claim for $i = 1$, the other cases follow analogously. Assume that u_1 and u_3 do not have a common neighbor of degree 5 in G other than v . Thus, in the coloring $\chi_{G(u_1, u_3)}$, the vertex v is a singleton. If there are two consecutive vertices in N that are singletons, then $\chi_{G(u_1, u_3)}$ is discrete by Lemma 23, which contradicts the assumption that G is an exception. It follows that N is the union of three color classes of the coloring $\chi_{G(u_1, u_3)}$, one of which is $\{u_2, u_4, u_5\}$.

Case A: $x_{1,2} \in N$ or $x_{2,3} \in N$. We only consider the case that $x_{1,2} \in N$, since the case $x_{2,3} \in N$ is analogous. We know that u_2, u_4 and u_5 have the same degree in $G[N]$. Thus, the vertices u_1 and u_3 must have the same degree in $G[N]$. Indeed, otherwise one of them would have a unique degree in $G[N]$ and then some u_i would be a singleton in $\chi_{G(v)}$, rendering $\chi_{G(v, u_{i+1})}$ discrete by Lemma 23.

Now suppose first $x_{1,2} = u_4$ or $x_{1,2} = u_5$. Either way, one and thus all of the vertices u_2, u_4, u_5 are adjacent to u_1 , because they form a color class of $\chi_{G(u_1, u_3)}$. It follows that the degree of u_1 and hence also of u_3 in $G[N]$ is at least 3. Thus, u_1 and u_3 have a common neighbor among $\{u_2, u_4, u_5\}$, which must have degree 5 in G , since $x_{1,2}$ has degree 5.

In the case $x_{1,2} = u_3$, we can deduce that all vertices of u_2, u_4, u_5 are adjacent to u_3 and thus treat the case analogously by simply swapping the roles of u_1 and u_3 .

Case B: $x_{1,2} \notin N$ and $x_{2,3} \notin N$. Since u_2 and u_4 have the same color and u_1 has a unique color in $\chi_{G(u_1, u_3)}$, the vertices u_1 and u_4 have a common neighbor $x \notin N$ of degree 5 other than v . Similarly, u_3 and u_5 have a common neighbor $x' \notin N$ of degree 5 other than v . To show that $x = x'$, consider the graph $G' := G^v$ obtained from G by inserting edges between every pair u_i and u_{i+1} .

If $x \neq x'$, then G' contains a $K_{3,3}$ minor (by contracting the paths of length 2 from u_1 to u_4 via x , and from u_3 to u_5 via x'), contradicting its planarity.

Thus, we have $x = x'$, and therefore, x is a vertex of degree 5 adjacent to u_1 and to u_3 . \lrcorner

Having proved the claim, we now finish the proof of Lemma 23. Again, we distinguish two cases.

Case 1: $G[N]$ is non-empty and some vertex in N has degree 5 in G . Due to planarity, there can be at most one vertex in N that has degree 4 within $G[N]$. Indeed, if there were two such vertices u_i and u_j , then each of v , u_i , u_j would be adjacent to all vertices in $N \setminus \{u_i, u_j\}$, yielding a $K_{3,3}$ minor. However, as seen in the analysis of Case A of Claim 1, no vertex in N can have a unique degree in $G[N]$ and thus, $G[N]$ has a maximum degree of at most 3.

Suppose that $G[N]$ contains an edge $\{u_i, u_{i+2}\}$ for some i , say $i = 1$. Due to Claim 1, the vertex u_2 must share a common neighbor with each of u_4 and u_5 apart from v . By the planarity of G' , these common neighbors are in $\{u_1, u_3\}$. However, the two common neighbors must be different, since otherwise, the respective vertex has degree $|\{u_1, u_2, u_4, u_5\}| = |\{u_3, u_2, u_4, u_5\}| = 4$ in $G[N]$. Thus, u_2 is adjacent to u_1 and u_3 , and both u_1 and u_3 have degree 3 in $G[N]$.

By the planarity of G' , the vertex u_2 has degree 2 in $G[N]$. Suppose u_4 has degree 3 in $G[N]$. Then it must be adjacent to u_1 , u_3 and u_5 . Since u_5 cannot have a unique degree in $G[N]$, it must be adjacent to u_3 , i.e., it must have degree 2 in $G[N]$. (By planarity, it cannot be adjacent to u_1 .) However, this would give u_3 a degree of 4 in $G[N]$, yielding a contradiction. The case that u_5 has degree 3 in $G[N]$ is symmetric. Thus, u_1 and u_3 are the only vertices of degree 3 in $G[N]$.

Then u_2 is the only vertex of N that is adjacent to two vertices of degree 3 in $G[N]$, making u_2 a singleton in $\chi_{G(v)}$ and yielding a contradiction.

We conclude that there is no edge of the form $\{u_i, u_{i+2}\}$. This implies that in $G[N]$, there is no vertex of degree 1. (Otherwise, we could individualize this vertex and v , and together with their unique common neighbor, they would yield a discrete coloring by Lemma 23.) Consequently, since $G[N]$ is non-empty, we conclude that u_1, u_2, u_3, u_4, u_5 is an induced cycle in $G[N]$. Since some vertex in N has degree 5 and within $G[N]$ the two neighbors of each vertex must have the same degree, we conclude that all vertices in N have degree 5.

Thus, if a vertex fulfills Case 1, then all its neighbors also fulfill Case 1. Therefore, being connected, the entire graph G must be a 5-regular triangulated planar graph, since we have restricted ourselves to connected graphs. Being 5-regular, the graph has $m = 5n/2$ edges. The Euler formula states that $2 = n - m + f$, where f is the number of faces of a plane embedding of the graph. Since G is triangulated, every face cycle contains exactly 3 edges and every edge is contained in exactly 2 faces. Thus, we have $f = 2m/3$, and hence, $m = 3n - 6$. We conclude that $n = 12$. There is only one 5-regular graph on 12 vertices, namely the icosahedron (see, for example, Reference [16]).

Case 2: $G[N]$ is empty or no vertex in N has degree 5 in G . From Claim 1, we already know that every pair of vertices u_i and u_{i+2} has a common neighbor of degree 5 other than v . In the current subcase, this vertex cannot be in N . Due to planarity, all these common neighbors for different i must be equal to a single vertex y adjacent to all vertices of N .

Observing that y has degree 5, consider the subgraph $H := G[\{v, y, u_1, \dots, u_5\}]$ of G . With the described cyclic ordering of the vertices in N , there is only one planar drawing of H up to equivalence. In this drawing, every face is a 4-cycle or a 3-cycle containing y and v . Since y and v have degree 5 in G , but they already have degree 5 in H , they are saturated. Thus, since both y and v belong to each face of H , by Item 1 of Lemma 28, no interior of a face of the drawing of H contains vertices of G . Therefore, $G = H = G[\{v, y, u_1, \dots, u_5\}]$. Since G is 3-connected, $G[N]$ cannot be empty. Similarly as in Case 1, we conclude that u_1, \dots, u_5 is a cycle rendering G the bipyramid on seven vertices. \square

Hence, we have characterized all exceptions that contain a vertex of degree 5. We continue with the exceptions that contain a vertex of degree 3.

LEMMA 31. *If G is an exception that has a vertex of degree 3, then it is isomorphic to a tetrahedron, a cube, a rhombic dodecahedron, a triangular bipyramid, a triakis tetrahedron, or a triakis octahedron.*

PROOF. Assume G is a 3-connected planar graph with a vertex of degree 3 and that G does not have two vertices whose individualization followed by an application of the one-dimensional WL-algorithm produces the discrete partition.

Let v be a vertex of degree 3 in G and let $N := N(v) = \{u_1, u_2, u_3\}$ be its neighbors. By Lemma 23, no vertex of N can have a unique degree in G . Thus, we know that the graph $G[N]$ is either a triangle or empty.

By Lemma 29, for $i \in \{1, 2, 3\}$ (indices always taken modulo 3), the pair u_i and u_{i+1} has a common neighbor $x_{i,i+1}$ of degree 3 other than v . (As in the previous proof, these $x_{i,i+1}$ are not necessarily distinct or unique.) If $x_{i,i+1} \in N$, then $x_{i,i+1} = u_{i+2}$ and $N(\{u_{i+2}, v\}) = \{u_i, u_{i+1}\}$. Thus, unless G only has 4 vertices (in which case it is the tetrahedron), the set $\{u_i, u_{i+1}\}$ forms a 2-separator, which contradicts G being 3-connected. We can therefore assume, for all $i \in \{1, 2, 3\}$, that $x_{i,i+1} \notin N$.

Either $v, u_i, x_{i,i+1}, u_{i+1}$ forms a face cycle of G , or the vertices u_i and u_{i+1} are adjacent and both $u_i, u_{i+1}, x_{i,i+1}$ and u_i, u_{i+1}, v are face cycles. Indeed, the vertex $x_{i,i+1}$ has degree 3, so the claim follows directly from Item 2 of Lemma 28.

Case 1: $G[N]$ is empty. In this case, every face incident to v is a 4-cycle consisting of two non-adjacent vertices v and $x_{i,i+1}$ of degree 3 in G and two other vertices u_i and u_{i+1} of degree $d \geq 3$. By analogous arguments as for v , for every $i \in \{1, 2, 3\}$, the graph $G[N(x_{i,i+1})]$ must be either empty or a triangle. It cannot be a triangle, because the edge $\{u_i, u_{i+1}\}$ is not present. So, by successively replacing v with its opposite vertices $x_{i,i+1}$ in the incident 4-cycles, we conclude that G is a $(3, d)$ -biregular quadrangulation.

Case 1.1: G is 3-regular. In this subcase, G must be a 3-regular quadrangulation. Such a graph has $m = 3n/2$ edges, since it is 3-regular, but, by the Euler formula, also $m = 2n - 4$ edges, since it is a quadrangulation. Thus, $n = 8$. It is easy to verify that the only 3-regular planar quadrangulation on 8 vertices is the cube.

Case 1.2: G is not 3-regular. Then G is bipartite and biregular with degrees 3 and d . Let n_3 and n_d be the number of vertices of degree 3 and d , respectively. Then $3n_3 = dn_d$ by double-counting and $dn_d = m = 2n - 4$, since G is a quadrangulation. It follows that $dn_d = 2(n_3 + n_d) - 4 = 2(dn_d/3 + n_d) - 4$, which gives that $4 = n_d(2 - d/3)$. Thus, $d \leq 5$ and by 3-connectedness, $d \in \{3, 4, 5\}$. The case $d = 3$ is Case 1.1 and $d = 5$ cannot occur due to Lemma 30, since neither the icosahedron nor the bipyramid on 7 vertices has vertices of degree 3.

We conclude that G is a $(3, 4)$ -biregular quadrangulation. We have $3n_3 = 4n_4$. Then, since $n = n_3 + n_4$, it holds that $m = 3n_3 = 3 \cdot 4n/7$, but, by the Euler formula, also $m = 2n - 4$. Thus, $n = 14$ and $n_3 = 8$.

Every vertex of degree 3 is incident to 3 faces and thus has 3 “opposite vertices” in the bipartite graph G . Thus, if we add an edge between every pair of vertices of degree 3 that lie on a common face and remove all vertices that originally had degree 4, we obtain a new 3-regular planar quadrangulation on $n_3 = 8$ vertices. The only such graph is the cube. Undoing the modification, we deduce that G is the rhombic dodecahedron.

This concludes the case that $G[N]$ is empty.

Case 2: $G[N]$ is a triangle. In this case, since we know that all $x_{i,i+1}$ have degree 3, by substituting $x_{i,i+1}$ for v , we obtain that every face of G is a 3-cycle consisting of a vertex of degree 3 and two other vertices of equal degree d .

Case 2.1: G is 3-regular, i.e., $d = 3$. Then G is a 3-regular triangulation. Thus, $m = 3n/2$ and, by the Euler formula, $m = 3n - 6$, hence $n = 4$. We conclude that G is the tetrahedron.

Case 2.2: G is not 3-regular. Consider the graph \tilde{G} obtained from G by removing all vertices of degree 3. The resulting graph is a planar $d/2$ -regular triangulation. Indeed, since v was chosen arbitrarily among all vertices of degree 3 and the vertices $x_{i,i+1}$ have degree 3 as well, it is easy to see that the resulting graph is a triangulation. Moreover, one can verify that in G , for every vertex of degree d , in the cyclic ordering of its neighbors, the degrees 3 and d alternate. Thus, a deletion of the vertices of degree 3 halves the degrees of each of the other vertices.

Since for $d' > 5$, no d' -regular graph is planar, it follows that $d/2 \in \{2, 3, 4, 5\}$. For $d/2 = 2$ we obtain a single 3-cycle. This implies that G is a triangular bipyramid.

For $d/2 = 3$, since \tilde{G} is a triangulation, all vertices in N have (in \tilde{G}) a single common neighbor w and every vertex in $G'[N \cup \{w\}]$ is saturated with respect to G' . Thus, we conclude that \tilde{G} is a tetrahedron. This implies that G is a triakis tetrahedron.

For $d/2 = 4$, we obtain that \tilde{G} must be an octahedron. This implies that G is the triakis octahedron. For $d/2 = 5$, we would obtain an icosahedron. This would imply that G is the triakis icosahedron. However, in this solid, there are two vertices u and u' (namely vertices of degree 3 of distance 2 that only have one common neighbor) such that $\chi_{G(u,u')}^1$ is discrete. Thus, this graph is not an exception. \square

It remains to examine the exceptions that contain no vertices of degree 3 or 5. Every such exception must contain a vertex of degree 4.

LEMMA 32. *If G is an exception that has a vertex of degree 4, then it is isomorphic to a bipyramid or a tetrakis hexahedron.*

PROOF. Assume G is an exception with a vertex of degree 4. First we make two observations that hold for every vertex u of degree 4 with neighbors v_1, v_2, v_3, v_4 in cyclic order.

OBSERVATION 1. *It holds that $d(v_1) = d(v_3)$ and $d(v_2) = d(v_4)$. Otherwise, we can individualize u and a neighbor v_i of u so that v_{i+1} and v_{i-1} refine to a singleton class in the coloring resulting from an application of the one-dimensional WL-algorithm, which by Lemma 23 yields a contradiction to the assumption of G being an exception.*

OBSERVATION 2. *By a similar argument, the induced graph $G[v_1, v_2, v_3, v_4]$ either is empty or contains a cycle in which v_i is adjacent to v_{i+1} for all $i \in \{1, 2, 3, 4\}$ (indices taken modulo 4).*

Due to Observation 2, if G is 4-regular, then either G is a triangulation or every face is of size at least 4. In the first case, G has $n = 6$ vertices, since $m = 4n/2$ and, by the Euler formula, $m = 3n - 6$. Thus, G is the octahedron, which is a bipyramid.

The second case cannot occur. Indeed, a planar graph without triangle faces has at most $2m/4 = m/2$ faces, since every edge is contained in exactly 2 faces and every face cycle contains at least 4 edges. Thus, by the Euler formula, the graph has at most $2n - 4$ edges, but a 4-regular graph has $4n/2 = 2n$ edges.

We can thus assume that G is not 4-regular. Hence, the graph has a vertex v of degree other than 4 that is adjacent to a vertex of degree 4. By Lemmas 30 and 31, we can assume that G neither has a vertex of degree 5 nor a vertex of degree 3. Thus, we can assume that v has degree at least 6. Let $N := N(v)$ be the set of neighbors of v , and let $N_4 \subseteq N$ be those neighbors of v that are of degree 4. Suppose (u_1, \dots, u_t) is the cyclic ordering of N_4 induced by the cyclic ordering of N obtained from a planar embedding of G .

Case 1: $E(G[N_4])$ is non-empty. Assume there are $u, u' \in N_4$ that are adjacent, i.e., $E(G[N_4])$ is non-empty. According to Observation 1, each vertex in N_4 must have two neighbors in G of degree $d(v) \neq 4$, i.e., outside of N_4 , and thus, $G[N_4]$ has maximum degree 2. We argue that $G[N_4]$ cannot have a vertex of degree 1. Indeed, if u_i were a vertex of degree 1 in $G[N_4]$, then u_i would have two neighbors of degree $d(v)$ and two neighbors of degree 4, one of which contained in N_4 , i.e., adjacent to v , and one of which non-adjacent to v . This means that in $\chi_{G(v, u_i)}^1$, all neighbors of u_i would be singletons, since the two neighbors of u_i of degree 4 would disagree on being adjacent to v or not. This is impossible by Lemma 23. We conclude that $G[N_4]$ has only vertices of degrees 2 and 0. Since $G[N_4]$ is non-empty, this implies that there is some cycle in $G[N_4]$. Assume this cycle has an edge $\{u_i, u_j\}$ connecting two vertices that are not consecutive in the cyclic order (u_1, \dots, u_t) .

Let u_i^+ and u_i^- be the vertices of $G[N]$ following and preceding, respectively, the vertex u_i in the cyclic ordering of N (so they may or may not have degree 4). By Lemma 29, there must be vertices x^+ and x^- of degree $d(v) \neq 4$ such that x^+ is adjacent to both u_i and u_i^+ , and x^- is adjacent to both u_i and u_i^- . However, $x^+ \neq x^-$, since the cycle v, u_i, u_j separates u_i^+ from u_i^- . We conclude that u_i has the following five neighbors: the vertex v , two neighbors in N_4 , as well as x^+ and x^- . However, u_i has degree 4, which gives a contradiction. We conclude that u_i is adjacent to u_{i+1} for all $i \in \{1, \dots, t\}$.

Finally, by Observation 1, every pair of vertices $\{u_i, u_{i+1}\}$ must have a common neighbor of degree $d(v)$ other than v , since otherwise, all neighbors of u_i would be singletons in $\chi_{G(u_i, u_{i+1})}^1$. Since u_i is adjacent to v, u_{i-1} , and u_{i+1} and has degree 4, it can only have one further neighbor. Thus, all these common neighbors for the pairs $\{u_i, u_{i+1}\}$ are indeed the same vertex x . Consider $\tilde{G} := G[\{u_1, \dots, u_t, v, x\}]$. The graph \tilde{G} is a bipyramid, in particular, it is 3-connected and every face is a 3-cycle with two vertices saturated in \tilde{G} with respect to G . Item 1 of Lemma 28 implies that $G = \tilde{G}$. We conclude that G is isomorphic to a bipyramid.

Case 2: $E(G[N_4])$ is empty. We now assume that the degree 4 neighbors of v form an independent set.

CLAIM 1. *For every $i \in \{1, \dots, t\}$, there is a vertex $x_{i,i+1}$ such that either the sequence $v, u_i, x_{i,i+1}, u_{i+1}$ forms a face cycle or both $v, u_i, x_{i,i+1}$ and $v, u_{i+1}, x_{i,i+1}$ form face cycles.*

PROOF. Without loss of generality, we show the claim for $i = 2$. We first argue that there are vertices $u' \in N_4$ and x' such that v, u_2, x', u' is a 4-cycle. For this, let x' be the first neighbor preceding v in the cyclic ordering among the neighbors of u_2 . Then, by Lemma 29, there must be a vertex u' other than u_2 of degree 4 that is adjacent to x' and v .

Let $u_j \in N_4$ be the first neighbor of v following u_2 in the cyclic ordering of vertices in N_4 that has a common neighbor with u_2 other than v . Note that u' is a candidate, thus, the existence of u_j is guaranteed.

We choose a common neighbor x of u_2 and u_j so that it is closest to v : more precisely, for a common neighbor $x \neq v$ of u_2 and u_j , consider the cycle u_2, v, u_j, x . It bounds two areas, one of which contains the vertices of N that follow u_2 but precede u_j , while the other one contains the vertices that follow u_j but precede u_2 in the cyclic ordering of N . (One of these sets may be empty.) We choose x so that the first of these areas is minimal with respect to inclusion and we call this area A .

We claim that the 4-cycle v, u_2, x, u_j is a face cycle of G or a face cycle after removing the diagonal $\{v, x\}$ (i.e., the 3-cycles v, u_2, x and v, u_j, x are faces). Note that the edge $\{u_2, u_j\}$ cannot be present, since $G[N_4]$ is empty. Now, toward a contradiction, suppose that u_2 has a neighbor that

lies within A . Choose as such a neighbor z the vertex that precedes v in the cyclic ordering of the neighbors of u_2 . Then by Lemma 29, for some vertex $\bar{u} \in N_4$, the sequence v, u_2, z, \bar{u} forms a 4-cycle. Now, by planarity, since z lies in A , the vertex \bar{u} cannot lie outside A . Thus, either \bar{u} precedes u_j in the cyclic ordering of N_4 starting from u_2 , or $\bar{u} = u_j$. In both cases, the cycle through v, u_2, z, u_j bounds an area that is a proper subset of A , which contradicts the minimal choices of u_j and x .

Finally, assume that u_j has a neighbor z that lies within A . Choose z to be the vertex that follows v in the cyclic ordering of neighbors of u_j . Since $\{v, x\}$ is not a separator and u_2 does not have a neighbor inside A , there must be a path from z to u_2 avoiding both v and x , thus a path via u_j that leaves A . Hence, the vertex u_j must have a neighbor outside of A . Therefore, since u_j has degree 4 and is also adjacent to v and x , the vertex z is the only neighbor of u_j inside A .

By Lemma 29, there is a vertex $\bar{u} \in N_4$ such that the sequence v, \bar{u}, z, u_j forms a 4-cycle. By the minimality of x , we cannot have that $\bar{u} = u_2$. Consider the coloring $\chi_{G(u_2, \bar{u})}^1$. In this coloring, v is a singleton. Indeed, since z lies in A , the vertex \bar{u} must also lie in A . Thus, by the minimality of u_j , the vertices u_2 and \bar{u} cannot have a common neighbor apart from v . Furthermore, u_j is the only vertex in N_4 that has simultaneously a common neighbor with u_2 other than v and a common neighbor with \bar{u} other than v . Hence, u_j is a singleton with respect to $\chi_{G(u_2, \bar{u})}^1$. Moreover, since $N(u_j) \cap A = \{z\}$ and $\{u_2, x\} \cap N(\bar{u}) = \emptyset$, the vertices \bar{u} and u_j only have one common neighbor other than v , namely z , which is thus a singleton as well. The singletons v, u_j , and z lie on a common face by the choice of z , which, by Lemma 23, yields a contradiction to the assumption of G being an exception.

Thus, neither u_2 nor u_j have a neighbor inside the cycle. This implies that the sequence v, u_2, x, u_j forms a face cycle or it becomes a face cycle after removing the diagonal $\{v, x\}$, since otherwise, the set $\{v, x\}$ would be a separator. We conclude that $u_j = u_3$ and that the vertex $x_{2,3} := x$ justifies the claim. \square

Overall, the claim implies that in the cyclic ordering of N , between every pair $u_i, u_{i+1} \in N_4$, there is at most one vertex, namely $x_{i,i+1}$. Thus, at least every second neighbor of v is of degree 4 (in particular $|N| \leq 2|N_4|$) and, thus, being of degree at least 6, the vertex v has at least 3 neighbors of degree 4, i.e., $|N_4| \geq 3$.

Recall that u_1 and u_3 are the two vertices that are closest to u_2 in the cyclic ordering of N_4 . In the following, we call an edge of G a *diagonal* if neither of its endpoints has degree 4.

We distinguish several cases according to the size of N_4 .

Case 2.1: $|N_4| = 3$. Since v has degree at least 6 and $|N_4| \geq |N|/2$, we conclude that v also has exactly 3 neighbors of degree larger than 4. Thus, the degree of v is 6. Let $u_1, t_1, u_2, t_2, u_3, t_3$ be the neighbors of v in the cyclic ordering. Then, by Claim 2, the vertices v, u_i, t_i form a face cycle for every $i \in \{1, 2, 3\}$. Likewise, the vertices v, t_i, u_{i+1} form a face cycle. Thus, the graph induced by $N \cup \{v\}$ is a wheel with 7 vertices. By Observation 2, the neighborhood of u_i contains a cycle (because u_i has degree 4). Moreover, by Observation 1, the vertices t_1, t_2 , and t_3 all have the same degree d . Recall that our initial assumption implies that G contains no vertex of degree 3 or 5.

We argue that $d = 6$. Since all u_i have degree 4, if every pair of vertices u_i and u_{i+1} had a common neighbor other than v and t_i , this would have to be a single vertex x adjacent to u_1, u_2 , and u_3 (it cannot be adjacent to any t_i since $t_i \notin N_4$). However, such x does not exist. Indeed, otherwise $\tilde{G} := G[u_1, u_2, u_3, t_1, t_2, t_3, x, v]$ would be a 3-connected graph in which every face cycle is either a triangle with a saturated vertex or a 4-cycle with two saturated vertices. Then Item 2 of Lemma 28 would imply $G = \tilde{G}$, which cannot be, since \tilde{G} has a vertex of degree 3, namely x .

Thus, some pair u_i, u_{i+1} does not have a common neighbor different from v and t_i , which in turn implies $d(t_i) = d(v) = 6$, using Lemma 29. Therefore, all neighbors of v have degree 4 or 6. More strongly, we conclude that the vertex degrees appearing among the neighbors of t_i are the same

as the vertex degrees appearing among neighbors of v , including multiplicities. If this did not hold for every $i \in \{1, 2, 3\}$, then there would be a vertex $u_j \in N_4$ such that in $\chi_{G(u_j, v)}^1$, every u_i would obtain a unique color, resulting also in a unique color for some t_i . Thus, by Lemma 23, the graph G would not be an exception. Hence, each t_i also has 3 neighbors of degree 4.

By Observation 1, all vertices in N_4 only have neighbors of degree 6 (since we already know that they have 3 neighbors of degree 6). By a similar argument, these degree 6 vertices have themselves 3 neighbors of degree 4 and 3 neighbors of degree 6. We conclude that the entire graph is (4, 6)-biregular. Since every face incident with v is a triangle, and v is arbitrary among the degree 6 vertices, we deduce that G is a triangulation. Moreover, every vertex of degree 4 has exactly 4 neighbors of degree 6 and every vertex of degree 6 has exactly 3 neighbors of degree 4. By double-counting the edges incident to both a vertex of degree 4 and a vertex of degree 6, we conclude that $4n_4 = 3n_6$, where n_i is the number of vertices of G of degree i . Thus, $n_6 = 4n_4/3$, and since $n_4 + n_6 = n = |G|$, we obtain $n_4 = 3n/7$. By the structure of G , its number of edges is $4n_4 + 3n_6/2$, which equals $n + 3n_4 + n_6/2 = n + 11n_4/3$. Indeed, $4n_4$ is the number of edges incident to vertices of degree 4 and $3n_6/2$ is the number of edges incident only to vertices of degree 6. We conclude that G has $18n/7$ edges. Since G is a triangulation, by the Euler formula, it has $3n - 6$ edges. Thus, $18n/7 = 3n - 6$, and hence, G is a graph on 14 vertices. Furthermore, the graph \tilde{G} induced by the vertices of degree 6 is a 3-regular graph on $14 - 6 = 8$ vertices. By the structure of the triangulated graph G , all faces in the induced drawing of \tilde{G} are 4-cycles. We conclude that \tilde{G} is the cube. (There is only one triangle-free planar 3-regular graph on 8 vertices.) Each face of \tilde{G} contains, within G , a vertex of degree 4. We conclude that G is the tetrakis hexahedron.

Case 2.2: $|N_4| = 4$. In this case, v must be adjacent to some of the $x_{i, i+1}$, since otherwise, v has degree 4. Thus, for every $i \in \{1, 2, 3, 4\}$, the diagonal $\{v, x_{i, i+1}\}$ must be present, since by Observation 2, each neighborhood $N(u_i)$ contains a cycle in which either every or no vertex is adjacent to its predecessor and its successor in the cyclic ordering of $N(u_i)$. It follows that v must have degree 8 and that $|N_4| = |N|/2$. By a similar argument, using the presence of the diagonals $\{v, x_{i, i+1}\}$ and Observation 2, for every neighbor of v of degree other than 4, at most half of its neighbors have degree 4. We obtain that G is a triangulation that has at most as many vertices of degree 4 as of degree at least 8. (Every vertex of degree $d \geq 8$ is adjacent to at most $d/2$ vertices of degree 4 and every vertex of degree 4 is adjacent to 4 vertices of degree at least 8.) Let n_d be the number of vertices of degree d and let $n_{\geq 8}$ be the number of vertices of degree at least 8. Then, by the 3-connectivity of G and Lemmas 30 and 31, we can assume $n_1 = n_2 = n_3 = n_5 = 0$ and thus, $n = n_4 + n_{\geq 8} + n_6 + n_7$. We show that G has at least $3n$ edges. Indeed, by summing over the degrees, we have $6n = 6n_4 + 6n_{\geq 8} + 6n_6 + 6n_7 \leq 4n_4 + 8n_{\geq 8} + 6n_6 + 7n_7 \leq 2m$, where the first inequality follows from our observation that $n_4 \leq n_{\geq 8}$. Thus, $m \geq 3n$, which contradicts the bound $m \leq 3n - 6$ obtained from the Euler formula.

Case 2.3: $|N_4| \geq 5$. Recall that $t = |N_4|$. We first show the following claim.

CLAIM 2. *For each $i \in \{1, \dots, t\}$, the vertices u_i and u_{i+2} have a common neighbor other than v .*

PROOF. We show the statement for $i = 1$. Assuming otherwise implies that v is a singleton in $\chi_{G(u_1, u_3)}^1$. We first argue that in this coloring, the vertex u_2 is also a singleton. Again, assume otherwise. Then there must be a vertex $u \in N_4 \setminus \{u_2\}$ that has the same color as u_2 . By Claim 2, for $i \in \{1, 2\}$, the vertices u_i and u_{i+1} have a common neighbor $x_{i, i+1}$ so that the sequence $v, u_i, x_{i, i+1}, u_{i+1}$ forms a face cycle or a face cycle after removing a diagonal.

Thus, in order not to be distinguished from u_2 , the vertex u must have a neighbor $y_{1,2}$ other than v that is adjacent to u_1 and a neighbor $y_{2,3}$ other than v that is adjacent to u_3 . Moreover,

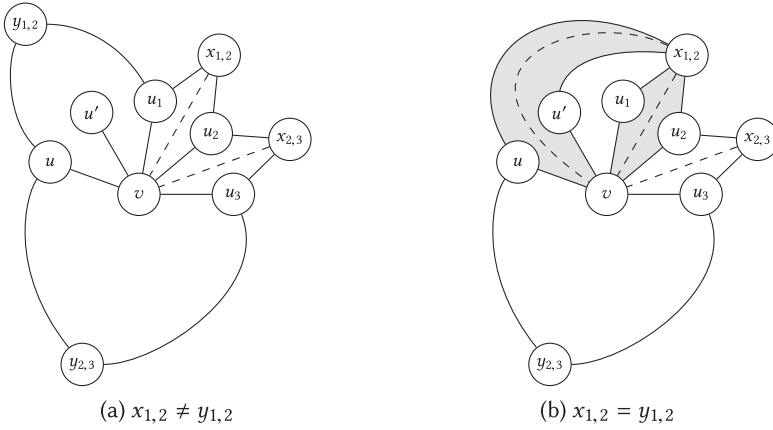


Fig. 3. An illustration of Case 2.3 in the proof of Lemma 3.2. Some possible diagonals are shown as dashed lines. The highlighted regions in the drawing on the right cannot contain other vertices implying that $\{x_{1,2}, v\}$ separates u_1 from u .

for $i \in \{1, 2\}$, the vertex $y_{i,i+1}$ should have the same color in $\chi_{G(u_1, u_3)}^1$ as $x_{i,i+1}$. See Figure 3(a). (Note that $y_{1,2} \neq y_{2,3}$, since we have assumed that u_1 and u_3 do not have a common neighbor other than v .)

Since it holds that $|N_4| \geq 5$, we know that $u_4 \neq u_t$. Therefore, $u \neq u_4$ or $u \neq u_t$. By symmetry, we can assume the latter. (To see the symmetry, recall that u_4 is the successor of u_3 in N_4 and u_t is the predecessor of u_1 in N_4 .) Note that the cycle $v, u, y_{1,2}, u_1$ separates u_t from u_2 . Consider the area A' bounded by the cycle $v, u, y_{1,2}, u_1$, which contains u_t . Inside A' lies the vertex $u' \in N_4$ that follows u in the cyclic ordering of N_4 .

Consider the set $M := N(\{u_3\}) \cup \{u_1, u_3\}$ and note that M is a union of color classes. By the assumption of Case 2 and since $N(\{u_1\}) \cap N(\{u_3\}) = \{v\}$, the cycle $v, u, y_{1,2}, u_1$ contains only the two vertices v and u_1 of M and there are no vertices in the interior of A' that are in M . Due to the 3-connectivity of G , there must be a path from u_3 to u' that avoids v and u_1 . Therefore, there must be a path from u or $y_{1,2}$ to u' that avoids M (and contains neither u nor $y_{1,2}$ as an inner vertex). Note that u' does not have the same color as u_2 , since it cannot share a neighbor other than v with u_3 . Indeed, the vertex would have to be $y_{1,2}$, but then u_1 and u_3 would have $y_{1,2}$ as a common neighbor apart from v .

Suppose $x_{1,2} \neq y_{1,2}$. Then every path from u_2 to a vertex in $N_4 \setminus \{u_1, u_2, u_3, u\}$ that does not have an inner vertex in M must contain u or $y_{1,2}$ as an inner vertex. The same holds for every path from $x_{1,2}$ to a vertex in $N_4 \setminus \{u_1, u_2, u_3, u\}$. However, this condition can be detected by the k -dimensional WL-algorithm and as described in the previous paragraph, at least one of u and $y_{1,2}$ does not satisfy it. This implies that u and u_2 do not have the same color, or that $x_{1,2}$ and $y_{1,2}$ do not have the same color, contradicting our construction. We conclude that $x_{1,2} = y_{1,2}$.

By Claim 1, the vertices u and u' have a common neighbor other than v . If this neighbor were not $x_{1,2}$, then u could not have the same color as u_2 , since there would be a path from u to $u' \in N_4 \setminus \{u_1, u_2, u_3, u\}$ none of whose inner vertices would be contained in $M \cup \{x_{1,2}\}$, whereas there would be no such path from u to any vertex in $N_4 \setminus \{u_1, u_2, u_3, u\}$. Figure 3(b) depicts this situation. We conclude that u' is a neighbor of $y_{1,2} = x_{1,2}$. Then, however, the pair $\{x_{1,2}, v\}$ separates u_1 from u , since by Claim 1, neither u' nor u_2 have a neighbor both inside and outside of the cycle $v, u', x_{1,2}, u_2$. Therefore, $\{x_{1,2}, v\}$ is a 2-separator, contradicting the 3-connectivity of G .

Up to this point, regarding our efforts to prove the claim, we have shown that u_2 is a singleton in $\chi_{G(u_1, u_3)}^1$. If $x_{1,2}$ were not adjacent to v , or $x_{2,3}$ were not adjacent to v , then $\chi_{G(u_1, u_3)}^1$ would have three singletons lying on the same face. So assume otherwise. We can also assume that neither $x_{1,2}$ nor $x_{2,3}$ is a singleton. But this cannot be, because the copies rendering $x_{1,2}$ and $x_{2,3}$ non-singletons (i.e., the other, necessarily existing vertices that have the same color as $x_{1,2}$ and $x_{2,3}$, respectively) should also be non-equal and adjacent to u_2 , which would force u_2 to have degree at least 5. This proves the claim. \dashv

Since $G[N_4]$ is empty, every common neighbor of u_i and u_{i+2} other than v must be equal to every common neighbor of u_{i+1} and u_{i+3} other than v . This means that there is a vertex v' other than v adjacent to all vertices of N_4 .

Consider now the area bounded by the cycle v, u_i, v', u_{i+1} , which does not contain u_{i+2} . If both u_i and u_{i+1} have two neighbors inside this area, then they do not have any neighbors outside the area, making $\{v, v'\}$ a separator. If u_i only has one neighbor inside the area, then this neighbor coincides with $x_{i,i+1}$ (since v' cannot be $x_{i,i+1}$ in this case, because v, u_i, v', u_{i+1} would not be a face cycle, even after removing the possible diagonal $\{v, v'\}$) and must therefore be adjacent to u_{i+1} . We conclude that $v, u_i, x_{i,i+1}, u_{i+1}$ forms a face cycle or becomes a face cycle after removing the diagonal $\{x_{i,i+1}, v\}$. A symmetric argument can be applied with respect to v' in place of v . It follows that inside the cycle v, u_i, v', u_{i+1} , there is at most one vertex, namely $x_{i,i+1}$. However, we already ruled out vertices of degree 3 at the beginning of the proof, so $x_{i,i+1}$ must be adjacent to all vertices of the cycle and thus has degree 4. This cannot be, since it would then lie in N_4 . We conclude that u_i does not have a neighbor inside the cycle v, u_i, v', u_{i+1} . A similar observation holds for the cycle v, u_{i-1}, v', u_i . Still, due to the 3-connectivity of G , the vertex u_i must have some neighbors within the area bounded by the cycle v, u_i, v', u_{i+1} or within the area bounded by the cycle v, u_{i-1}, v', u_i , yielding the final contradiction. \square

PROOF OF THEOREM 25. Recalling that every 3-connected planar graph has a vertex of degree 3, 4, or 5, the proof follows immediately by combining Lemma 23 with Lemmas 30, 31, and 32. \square

We can finally prove Theorem 1.

PROOF OF THEOREM 1. Let $k \geq 3$. By Corollary 27, the k -dimensional WL-algorithm determines orbits on all arc-colored 3-connected planar graphs and identifies them. Thus, by Lemma 21, since the class of all planar graphs is minor-closed, its WL-dimension is at most 3. \square

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