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A NOTE ON CONTEXT-FREE LANGUAGES

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This paper describes regular and context-free grammars as certain morphisms of graphs, and the associated languages in terms of appropriate free category constructions applied to these graphs.

Introduction

This note contains a description of regular languages in terms of the notion of free category on a reflexive graph, and of context-free languages in terms of the notion of free category with products on a multigraph. More precisely, in each case a grammar is a morphism $\phi : \mathbf{G} \to \mathbf{H}$ of the appropriate kind of graph. Then arrows in the appropriate free category $\mathscr{F}\mathbf{G}$ are prescriptions for the construction of strings, while arrows on the appropriate free category $\mathscr{F}\mathbf{H}$ are simply strings in the language. The language defined by the grammar is the set of strings in the image of $\mathscr{F}\phi$. The problem of parsing is the problem of finding the inverse image of a string under $\mathscr{F}\phi$.

An early reference for context-free languages is [1]; a reference for category theory, which gives an exposition of Lawvere's work on the relation between calculi of terms and categories with products, is [3]; an analysis of paragraphs in terms of morphisms of reflexive graphs, analogous to my description below of regular grammars, appears in [2].

1. Regular languages

A *reflexive graph* (or 1-dimensional simplicial set) is a pair of sets $\mathbf{G}_0, \mathbf{G}_1$ and three functions

$$d_0, d_1 : \mathbf{G}_1 \to \mathbf{G}_0, \qquad s : \mathbf{G}_0 \to \mathbf{G}_1$$

such that $d_0 s = 1_{\mathbf{G}_0}$, $d_1 s = 1_{\mathbf{G}_0}$. The elements of $\mathbf{G}_0$ are called vertices or *objects*; the

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elements of $\mathbf{G}_1$ are called directed edges or *arrows*. If $a \in \mathbf{G}_1$ and $d_0\alpha = X$, $d_1\alpha = Y$ then we write $\alpha : X \to Y$ and call X the *domain* of α, Y the *codomain* of α; we denote sX by 1_X (the *identity* arrow of X). A *morphism* $\phi : \mathbf{G} \to \mathbf{H}$ *of reflexive graphs* is a pair of functions $\phi_0 : \mathbf{G}_0 \to \mathbf{H}_0$, $\phi_1 : \mathbf{G}_1 \to \mathbf{H}_1$ which preserves the domain, codomain and identity.

There is an obvious forgetful functor $\mathscr{U}$ from the category $\mathscr{CAT}$ of categories to the category $\mathscr{GRPH}_{\mathrm{refl}}$ of reflexive graphs, with a left adjoint $\mathscr{F}$. The objects of $\mathscr{F}\mathbf{G}$ are the same as those of $\mathbf{G}$; the arrows from X to Y in $\mathscr{F}\mathbf{G}$ are either identity arrows (if $X = Y$) or directed non-empty paths of non-identity arrows of $\mathbf{G}$, beginning at X and ending at Y.

Given an alphabet $\mathbf{A}$ there is an associated reflexive graph $\tilde{\mathbf{A}}$ with only one object I, and with arrows from I to I being the elements of $\mathbf{A}$ together with the identity arrow ε. Notice that a morphism $\mathbf{G} \to \tilde{\mathbf{A}}$ just means a labelling of each arrow of $\mathbf{G}$ by an element of $\mathbf{A}$ *or* by the identity arrow ε. Further, $\mathscr{F}\tilde{\mathbf{A}}$ has one object I, and $\mathrm{Hom}_{\mathscr{F}\tilde{\mathbf{A}}}(I, I)$ is the free monoid $\mathbf{A}^*$ on $\mathbf{A}$.

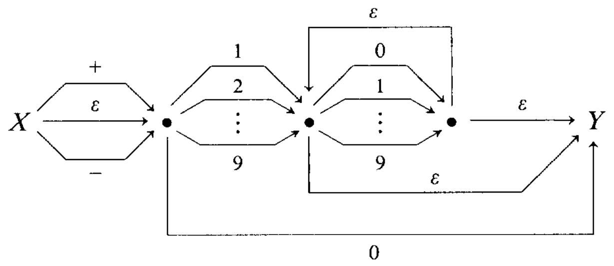
Definition. A *regular grammar* on a finite alphabet $\mathbf{A}$ is a morphism of finite reflexive graphs

$$\phi : \mathbf{G} \to \tilde{\mathbf{A}}.$$

Given two object X, Y in $\mathbf{G}$ there is an associated subset of $\mathbf{A}^*$, namely $\mathscr{F}\phi(\mathrm{Hom}_{\mathscr{F}\mathbf{G}}(X, Y))$. Subsets obtained in this way from regular grammars are called *regular languages*.

The meaning of this definition, and its relation with standard notions, is best clarified by the examination of an illustrative example.

Example. Let $\mathbf{A}$ be the set of digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ together with the signs $+, -$. Then the following labelled diagram represents a morphism from a reflexive graph $\mathbf{G}$ to $\tilde{\mathbf{A}}$:



The graph $\mathbf{G}$ is what remains when the labels are removed, and identity arrows are added (the identity arrows of $\mathbf{G}$ have been suppressed in the diagram). The reason that reflexive graphs are considered is that while only the non-identity arrows of $\mathbf{G}$ are displayed in the diagram, some of these are labelled with the identity arrow of $\tilde{\mathbf{A}}$.

Now some typical paths from X to Y in $\mathscr{F}\mathbf{G}$ (writing composition from left to right) are

$$+73\varepsilon 8\varepsilon 9\varepsilon 0\varepsilon \quad \text{and} \quad \varepsilon 73\varepsilon 8\varepsilon 9\varepsilon 0\varepsilon \quad \text{and} \quad -0$$

and their images under $\mathscr{F}\phi$ are

$$+73890 \quad \text{and} \quad 73890 \quad \text{and} \quad -0$$

respectively. It is clear that the subset of $\mathbf{A}^*$ defined by this grammar consists of integers (or arbitrary length) with leading zeros suppressed and with an optional plus or minus sign.

2. Context-free languages

A *multigraph* $\mathbf{G}$ is a sequence of sets $\mathbf{G}_*, \mathbf{G}_0, \mathbf{G}_1, \mathbf{G}_2, \dots$ and for $n = 0, 1, 2, \dots$ functions

$$d_1, d_2, \dots, d_n, c : \mathbf{G}_n \to \mathbf{G}_*.$$

The elements of $\mathbf{G}_*$ are called vertices or *objects*; the elements of $\mathbf{G}_n$ are called directed edges or *arrows*. If α is in $\mathbf{G}_n$ and $d_i\alpha = X_i$, $c\alpha = Y$, then we write $\alpha : X_1X_2\cdots X_n \to Y$ and call $X_1X_2\cdots X_n$ the *domain* of α, Y the *codomain* of α; when α is in $\mathbf{G}_0$ and $c\alpha = Y$ we write $\alpha : 1 \to Y$. A *morphism* $\phi : \mathbf{G} \to \mathbf{H}$ *of multigraphs* is a sequence of functions $\phi_* : \mathbf{G}_* \to \mathbf{H}_*$, $\phi_0 : \mathbf{G}_0 \to \mathbf{H}_0$, $\phi_1 : \mathbf{G}_1 \to \mathbf{H}_1, \dots$ which preserves the operations $d_1, d_2, \dots$ and c.

There is an forgetful functor $\mathscr{U}_\times$ from the category $\mathscr{CAT}_\times$ of categories with assigned strictly-associative finite products (and functors preserving the assigned products) to the category $\mathscr{MULT}$ of multigraphs. If $\mathbf{C}$ is such a category with products, then the objects of $\mathscr{U}_\times\mathbf{C}$ are the objects of $\mathbf{C}$, and the arrows of $\mathscr{U}_\times\mathbf{C}_n$ are the arrows in $\mathbf{C}$ from an assigned n-ary product $X_1 \times X_2 \times X_3 \times \cdots \times X_n$ of objects of $\mathbf{C}$ to a single object Y of $\mathbf{C}$. The functor $\mathscr{U}_\times$ has a left adjoint $\mathscr{F}_\times$. The objects of $\mathscr{F}_\times\mathbf{G}$ are strings of objects in $\mathbf{G}$; the arrows of $\mathscr{F}_\times\mathbf{G}$ from $X_1X_2X_3\cdots X_n$ to $Y_1Y_2\cdots Y_m$ are m-tuples of *terms* and composition is substitution of terms. To be more explicit, arrows of $\mathscr{F}_\times\mathbf{G}$ are defined inductively as follows. For each object X of $\mathbf{G}$ take an infinite sequence $x_1, x_2, x_3, \dots$ of *variables* of that type. Then

(i) x_i is an arrow in $\mathscr{F}_\times\mathbf{G}$ from any string containing at least i occurences of X to X,

(ii) given for each $j = 1, 2, 3, \dots, m$ an arrow $\alpha_j : S \to X_j$ in $\mathscr{F}_\times\mathbf{G}$, where S is a string and X_j is an object of $\mathbf{G}$, then $\alpha_1, \alpha_2, \dots, \alpha_m$ is an arrow in $\mathscr{F}_\times\mathbf{G}$ from S to $X_1X_2\cdots X_m$,

(iii) given $\alpha : S \to X_1X_2\cdots X_n$ an arrow in $\mathscr{F}_\times\mathbf{G}$ and $\beta : X_1X_2\cdots X_n \to Y$ an arrow in $\mathbf{G}$, then $\beta(\alpha)$ is an arrow in $\mathscr{F}_\times\mathbf{G}$ from S to Y.

Given an alphabet $\mathbf{A}$ there is an associated multigraph $\tilde{\mathbf{A}}$ with only one object M, and with one arrow μ_n from M^n to M for each n, together with the elements of **A** as arrows from 1 to M. Notice that a morphism $\mathbf{G} \to \tilde{\mathbf{A}}$ just means a labelling of each arrow of **G** by an element of **A** (for arrows in $\mathbf{G}_0$) *or* by the arrow $\mu_n : M^n \to M$.

Further, $\mathscr{F}_\times \tilde{\mathbf{A}}$ has objects M^n ($n = 0, 1, 2, 3, \ldots$). To see what the arrows in $\mathscr{F}_\times \tilde{\mathbf{A}}$ are like, consider the following three arrows from M^3 to M:

$$\mu_2(\mu_2(x_1, x_2), x_3) \quad \text{and} \quad \mu_3(x_1, x_2, x_3) \quad \text{and} \quad \mu_2(x_1, \mu_2(x_2, x_3))$$

where x_1, x_2, x_3 are variables of type M. These arrows may be identified with the three different bracketings of x_1, x_2, x_3. More generally, arrows in $\mathscr{F}_\times \tilde{\mathbf{A}}$ may be identified with bracketings of variables and elements of the alphabet **A**.

Consider $\mathbf{Mon_A}$, the algebraic theory of monoids augmented by the set **A** of constants. $\mathbf{Mon_A}$ is a quotient category of $\mathscr{F}_\times \tilde{\mathbf{A}}$; the quotient (product preserving) functor ψ from $\mathscr{F}_\times \tilde{\mathbf{A}}$ to $\mathbf{Mon_A}$ is the identity on objects, and identifies different bracketings which are equivalent under associativity. It is clear that $\mathrm{Hom}_{\mathbf{Mon_A}}(1, M)$ is the free monoid $\mathbf{A}^*$ on **A**.

Definition. A *context-free grammar*, on a finite alphabet **A** is a morphism of multigraphs

$$\phi : \mathbf{G} \to \tilde{\mathbf{A}},$$

where **G** is a multigraph with only a finite number of objects and arrows. Given an object E in **G** there is an associated subset of $\mathbf{A}^*$, namely

$$\psi \mathscr{F}_\times \phi(\mathrm{Hom}_{\mathscr{F}_\times \mathbf{G}}(1, E)).$$

Subsets obtained in this way from context-free grammars are called *context-free languages*.

Again the meaning of the definition, and its relation with standard notions, is best clarified by the examination of a illustrative example.

Example. Let **A** be the set of characters $a, b, c, \ldots, x, y, z$ together with the symbols $+, [,]$. Let $\mathbf{G}_*$ be the set E, L, R, S, C. Then the following diagram represents a morphism from a multigraph **G** to $\tilde{\mathbf{A}}$:

$$\begin{array}{ll}
a, b, c, \ldots, z : 1 \to C, & \mu_3 : ESE \to E, \\
[: 1 \to L, & \mu_3 : LER \to E, \\
] : 1 \to R, & \mu_1 : C \to E. \\
+ : 1 \to S, &
\end{array}$$

The names given to the arrows in the diagram are the labels (in this example no ambiguity arises from naming the arrows by the labels).

Now a typical arrow from 1 to E in $\mathscr{F}_\times\mathbf{G}$ is

$$\mu_3(\mu_3([, \mu_3(\mu_1(a), +, \mu_1(b)),]), +, \mu_1(a)).$$

The image under $\psi\mathscr{F}\phi$ is

$$[a+b]+a.$$

It is clear that the subset of $\mathbf{A}^*$ defined by this grammar consists of arithmetic expressions (or arbitrary length) built from the alphabet $\mathbf{A}$ using square brackets and plus signs, and that the arrows in $\mathscr{F}_\times\mathbf{G}$ are parse trees for the arithmetic expressions.

3. Remarks

3.1. The relation between regular and context-free grammars, as defined above, is as follows. Given a regular grammar form a multigraph with an object $E_{X,Y}$ for each pair of objects X, Y of the reflexive graph and two types of arrows.

(i) For each triple of objects X, Y, Z of the reflexive graph, take an arrow $E_{X,Y}E_{Y,Z} \to E_{X,Z}$.

(ii) For each arrow of the reflexive graph from X to Y, take an arrow $1 \to E_{X,Y}$. Then label the arrows of the multigraph as follows: label arrows of type (i) with μ_2, and the arrows of type (ii) by their label in the reflexive graph, except that the arrows labelled by identities in the reflexive graph are now labelled by μ_0. The language defined by taking the object $E_{X,Y}$ in the resulting context-free grammar is the same as the language defined by the pair of objects X, Y of the regular grammar.

3.2. In the consideration of context-free languages it may sometimes be useful to consider an alphabet augmented by function symbols, rather than the alphabet of constants considered here.

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References

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