$\aleph_0$-categoricity in first-order predicate calculus[1]

by

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I. *Basic concepts*

1. *Deductive systems.* By a *deductive system,* or simply a *system,* we will mean a pair (L, Σ) of a language L and a class Σ of sentences in L, closed under the consequence relation. The language L will always be assumed to be a language in first-order predicate calculus with identity, without predicate variables and without individual constants, and with a finite or denumerable number of predicate constants. The system (L, Σ) will in most cases be assumed to be complete and consistent, i.e. for any sentence P of L either P or $\sim P$ belongs to Σ but not both.

2. *Structures, models.* A *structure* associated with a language L may be defined in various different ways. It is enough here to state those properties of a structure that we will use.

Characterization. An L-structure M determines a class of objects U_M called the *universe of* M, and assigns to every predicate constant R in L of degree n a class of n-sequences from U_M, called the *extension* of R in M.

Def. When L is a language containing an identity relation I, we call M a *reduced structure,* if no pair (a, b), where a and b are different objects of U_M, belongs to the extension of I in M.

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[1] After this paper was completed, I was informed that an essential part of my results was already known. As appears from Mostowski's paper [3], T 10 has been proved by Ryll-Nardzewski, although his proof has not yet been published. From Mostowski's paper it is apparent that Ryll-Nardzewski's proof of the implication from (b) to (a) is the same as in this paper. Concerning the more difficult proof of the converse implication, Mostowski gives no information.

If M is any L-structure, there are obvious meanings of the phrase "P is true in M" for any sentence P in L, and of the phrase "$(a_1, \ldots, a_n)$ satisfies $A(x_{i1}, \ldots, x_{in})$", where $(a_1, \ldots, a_n)$ is a sequence of elements of U_M and $A(x_{i1}, \ldots, x_{in})$ is some formula of L containing no free variables other than $x_{i1}, \ldots, x_{in}$, and where a certain correlation between the variables and the objects is presupposed. A *model* of a system (L, Σ) is an L-structure M for which all the sentences of Σ are true. If M is any L-structure, there is a uniquely determined system (L, Σ_M) which is complete and such that M is a model of (L, Σ_M).

3. *Isomorphism, $\aleph_0$-categoricity.*

Def. Let M_1 and M_2 be two L-structures. By a *partial isomorphism relation* between M_1 and M_2 we understand a two-place relation Φ with its domain D_1 included in U_{M1}, and the converse domain D_2 included in U_{M2}, and such that whenever A is a formula in L and $\Phi a_i b_i$ for $i=1, \ldots, n$, then $(a_1, \ldots, a_n)$ satisfies A in M_1 if and only if $(b_1, \ldots, b_n)$ satisfies A in M_2. If a partial isomorphism relation Φ between M_1 and M_2 has U_{M1} for domain and U_{M2} for converse domain, Φ is called a *total* isomorphism and M_1 and M_2 are said to be *isomorphic*. If a partial isomorphism Φ between M_1 and M_2 has U_{M1} for its domain, M_2 is said to be an *arithmetical extension* of M_1 [5]. A system (L, Σ) is called *categorical* if all its models are isomorphic, and *$\aleph_0$-categorical*, if all its denumerable models are isomorphic.

Note. If the structures M_1 and M_2 are reduced structures, any partial isomorphism relation must be one-one. Otherwise it need not be. It is easy to see that to any given structure M there corresponds some structure M' which is reduced and which is isomorphic to M.

4. *Condition-sets.* A set Γ of formulas in L is called a *condition-set in* $(x_{i1}, x_{i2}, \ldots, x_{in})$, and a *condition-set of degree n*, if it contains no variables free other than $x_{i1}, \ldots, x_{in}$. (When we speak of *condition-sets* simply, we always mean condition-sets of some finite degree.) When speaking about formulas containing free variables we need to introduce a special consequence relation.

Def. If Γ_1 and Γ_2 are classes of formulas in L, we say that Γ_2

is a *c-consequence* [2] of Γ_1 if for every formula A of Γ_2 there are formulas $B_1, \ldots, B_n$ in Γ_1 such that the formula

$$(B_1 \cdot B_2 \ldots B_n) \supset A$$

is a valid formula in predicate calculus.

In other words: A c-consequence of Γ_1 is a class of sentences which one can derive from Γ_1 treating the free variables of Γ_1 as constants. When Γ_1 does not contain any free variables, the c-consequences of Γ_1 are the same as the consequences in the usual meaning. In the following, when using the term *consequence,* I will always mean c-consequence.

A condition-set Γ is called *c-consistent in* (L, Σ), or briefly *consistent in* (L, Σ), if no contradiction is a c-consequence of $\Sigma \cup \Gamma$; Γ is called *maximal c-consistent* (or *maximal consistent*), if it is c-consistent, contains the free variables $x_1, \ldots, x_n$, and is included in no other consistent condition-set in $x_1 \ldots, x_n$. (Clearly, any maximal consistent condition-set is included in some consistent condition-set of higher degree.)

If $a_1, \ldots, a_n$ are individual constants not belonging to L we can speak of *condition-sets in the symbols* $a_1, \ldots, a_n$ in the same way as if these symbols were variables.

II. *Condition-sets with finite basis*

When (L, Σ) is a complete consistent system, there are close connections between the maximal consistent condition-sets in (L, Σ), and the models of the system. We state first some obvious results:

T 1. a) Let (L, Σ) be a complete consistent system, M a model of (L, Σ) and $(a_1, \ldots, a_n)$ a sequence of elements in M. Then the set of those conditions in the variables $x_1, \ldots, x_n$, which are satisfied by $(a_1, \ldots, a_n)$, is a maximal consistent condition-set in (L, Σ).

b) If (L, Σ) is a complete consistent system, and Γ is a maximal consistent condition-set in (L, Σ), then there is some

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[2] A similar relation is called I-consequence in Los [2].

denumerable model M of (L, Σ) in which Γ is satisfied by some sequence.

On the other hand, as we shall see (T 7 and T 8), a model of a complete consistent system (L, Σ) does not in general exemplify all the maximal consistent condition-sets of (L, Σ). Condition-sets of a special type, those "with a finite basis", must however always be satisfied. We will now define this concept.

Def. Let (L, Σ) be a system, and Γ a condition-set in (L, Σ). A subclass Γ' of Γ is called a *basis of* Γ *in* (L, Σ), if $\Sigma, \Gamma' \rightarrow \Gamma$. (This symbolism shall be interpreted to state that Γ is a c-consequence of $\Sigma \cup \Gamma'$.) If Γ' is a finite subclass, it is called a *finite basis*.

If Γ is maximal consistent and has a finite basis, then it evidently even has a one-element basis. About condition-sets with finite basis we have the following result already announced:

T 2. If (L, Σ) is a complete consistent system, and Γ is a consistent condition-set with a finite basis in (L, Σ), then Γ must be satisfied in every model of (L, Σ).

As we saw in *T 1*, every sequence $(a_1, \ldots, a_n)$ of elements in a structure M determines a maximal consistent condition-set. It follows that we may apply the concept of basis to such sequences.

Def. Let M be any structure, (L, Σ) the corresponding system, $(a_1, \ldots, a_n)$ some sequence of elements in M, and Γ the maximal consistent condition-set in $(x_1, \ldots, x_n)$ determined by $(a_1, \ldots, a_n)$. Any basis of the set Γ in (L, Σ) will also be called a *basis of the sequence* $(a_1, \ldots, a_n)$ *in* M. If M is a structure in which every finite element-sequence has a finite basis, we say that M has *finite character*.

T 3. Let (L, Σ) be a complete consistent system, and M_1 and M_2 two denumerable models of (L, Σ), both with finite character. Then M_1 and M_2 are isomorphic.

Proof. To simplify the proof, we will assume that M_1 and M_2 are reduced structures. Since every structure M is isomorphic to some reduced structure M', and the reduced structure M' has finite character if and only if M has, this assumption does not make the proof less general.

a) We first observe that we can find some partial isomorphism Φ between M_1 and M_2 with any given finite subclass F of U_{M1} as its domain. Let $(a_1, \ldots, a_n)$ be an enumeration of the elements of F, and let Γ be the corresponding maximal consistent condition-set. Since M_1 has finite character, Γ has a finite basis and is therefore satisfied by some sequence $(b_1, \ldots, b_n)$ in M_2 (T 2). The correlation Φ: $a_i \longleftrightarrow b_i$ is evidently a partial isomorphism.

b) We can also show that, under the stated conditions, any finite partial isomorphism between M_1 and M_2 can be extended to include any additional element of U_{M1} or U_{M2}. Let $a_1, \ldots, a_n$ be the elements in the domain of Φ, and $b_1, \ldots, b_n$ the corresponding elements in the converse domain. Let a be some additional element of U_{M1}. Since M_1 has finite character, there is some $(x_1, \ldots, x_{n+1})$-condition A in L which is a basis of the sequence $(a_1, \ldots, a_n\ a)$ in M_1. Then the condition $(Ey)A(x_1, \ldots, x_n\ y)$ is satisfied by $(a_1, \ldots, a_n)$ in M_1, and then also by $(b_1, \ldots, b_n)$ in M_2, since Φ is partial isomorphism. This means that there is an element b in U_{M2}, such that $(b_1, \ldots, b_n, b)$ satisfies the condition A. Since the set deducible from A is maximal consistent, the sequences $(a_1, \ldots, a_n, a)$ and $(b_1, \ldots, b_n, b)$ satisfy exactly the same conditions, and we see that the correlation extended with the correlation $\mathrm{a} \longleftrightarrow \mathrm{b}$ is again a partial isomorphism.

c) We can now prove the assertion of the theorem. By a) and b) we can find a sequence $\Phi_1, \ldots, \Phi_m \ldots$ of finite partial isomorphisms, where every Φ_{m+1} is an extension of the preceding Φ_m. Since M_1 and M_2 were assumed to be denumerable, we can evidently construct the sequence so that every element of U_{M1} or U_{M2} is included in Φ_n for some n. The sum of all these correlations must now be an isomorphism between M_1 and M_2. The domain is U_{M1} and the converse domain is U_{M2}; and since any finite sequences s_1 from U_{M1} and s_2 from U_{M2}, which correspond to each other by Φ, are already correlated by one of the partial isomorphism Φ_m, they must satisfy exactly the same conditions in L.

By the same method of proof we can also prove the following: *T 4*. If (L, Σ) is a complete consistent system, and M_0 is a denumerable model of (L, Σ) with finite character, then every other model M of (L, Σ) is an arithmetical extension of M_0.

It is clear, that if M is not itself denumerable and with finite character, this relation cannot hold the other way. So we get the following consequence of T 3–4:

T 5. If (L, Σ) is a complete consistent system, and has a model M_0 of finite character, then this model M_0 is within isomorphism uniquely characterized as the model of which every other model is an arithmetical extension.

The following theorem states a sufficient condition for $\aleph_0$-categoricity. We show later (T 9) that this condition is also necessary.

T 6. Let (L, Σ) be a complete consistent system. If every maximal consistent condition-set in (L, Σ) has a finite basis, then (L, Σ) is $\aleph_0$-categorical.

The theorem is an immediate consequence of T 4 by the observation (T 1b) that every model of (L, Σ) has finite character.

III. *Condition-sets without finite basis*

Our main goal is now to prove the converse of T 6. In other words, we want to show that if a complete consistent system (L, Σ) contains some maximal consistent condition-set without finite basis, then it is not $\aleph_0$-categorical. We use different methods of proof, according as the set of different maximal consistent condition-sets is denumerable or not.

T 7. Let (L, Σ) be a complete consistent system, such that the number of different maximal consistent condition-sets is nondenumerable. Then (L, Σ) is not $\aleph_0$-categorical.

Note. There can be only a denumerable number of condition-sets with a finite basis. So if there are nondenumerably many maximal consistent condition-sets, there are also nondenumerably many maximal consistent condition-sets without finite basis.

Proof of T 7. In a denumerable model, there is only a denumerable number of finite sequences. Therefore for any denumerable model M_1 of (L, Σ) there is some maximal consistent condition-set Γ which is not satisfied in M_1. But according to T 1*b* there is some denumerable model M_2 of (L, Σ) in which Γ is satisfied. M_1 and M_2 are not isomorphic, and so (L, Σ) is not $\aleph_0$-categorical.

The proof in the denumerable case is more complicated, but it also gives us a stronger result.

T 8. Let (L, Σ) be a complete consistent system such that the set $\mathfrak{G}$ of the maximal consistent condition-sets without finite basis in (L, Σ) is (at most) denumerable. Then there is a denumerable model of (L, Σ) with finite character, i.e. a model in which none of the condition-sets $\Gamma \varepsilon \mathfrak{G}$ is satisfied.

For reference in the following proof we will first list two simple lemmas.

Lemma 1. If Φ, $A(a) \rightarrow \Psi$ where $A(a)$ is a sentence containing the individual constant a and Φ and Ψ are sets of sentences where a does not occur, then Φ, $(\mathrm{E}x)A(x) \rightarrow \Psi$.

Lemma 2. If in a structure M a sequence $(a_1, \ldots, a_n)$ is without finite basis, then there is some subsequence $(a_{i1}, \ldots, a_{ik})$ without repetitions, which is also without finite basis.

Example. Let a and b be different elements in the structure M. The sequence $(a\ a\ b\ b\ b)$ is a sequence in M with repetitions and $(a\ b)$ a subsequence without repetitions. We assume that $(a\ b)$ has a finite basis. This means that there is some condition $A(x_1\ x_2)$ which is satisfied by $(a\ b)$ and from which every other condition $B(x_1\ x_2)$ satisfied by $(a\ b)$ is a consequence. If we define the condition $C(x_1\ x_2\ x_3\ x_4\ x_5)$ as $A(x_1\ x_3) .\ x_1 = x_2 \cdot x_3 = = x_4 = x_5$, this is easily seen to be a basis for the sequence $(a\ a\ b\ b\ b)$.

Proof of T 8. In the construction of the model we will follow the procedure in Henkin's completeness proof [1], with some additional qualifications.

Henkin's construction of a model of a complete consistent system (L_0, Σ_0) can be described as follows: We suppose that we have an infinite number of infinite sequences $s_1\ s_2 \ldots$ of individual constants. We may write $s_1 = (a_{11}\ a_{12}\ a_{13} \ldots)$, $s_2 = (a_{21}\ a_{22}\ a_{23} \ldots)$, etc. The total set of individual constants a_{ij} we may call S. We form a series of successive extensions (L_1, Σ_1), (L_2, Σ_2) $\ldots$ of (L_0, Σ_0). The construction of each system (L_k, Σ_k) from the preceding system (L_{k-1}, Σ_{k-1}) can be described as the following three-step procedure:

k a) L_k is formed from L_{k-1} by adjunction of the symbols $a_{k1}, a_{k2}, \ldots$ of s_k to L_{k-1}.

k b) (L_k, Σ'_{k-1}) is a consistent system formed from (L_k, Σ_{k-1}) by adding to Σ_{k-1} a sentence $A(a_{ki})$ corresponding to every sentence $(Ex)A(x)$ in Σ_{k-1}.

kc) (L_k, Σ_k) is formed by extending (L, Σ'_{k-1}) to a complete consistent system.

The sum Σ_ω of all the sets Σ_n determines a model M with its elements in a one-one denotation correlation with the symbols of S and such that all the sentences of Σ_ω are true in M.

We observe: If $\alpha_1, \ldots, \alpha_n$ are individual constants adjoined to L_k (i.e. belonging to some of $s_1, \ldots, s_k$) then Σ_k contains a maximal consistent condition-set in the symbols $(\alpha_1, \ldots, \alpha_n)$, which is satisfied by the corresponding objects. We may denote this condition-set $\Sigma(\alpha_1, \ldots, \alpha_n)$. If M has finite character, then evidently every such set $\Sigma(\alpha_1, \ldots, \alpha_n)$ has a finite basis in (L, Σ).

We will now show that by performing the completion processes 1c, 2c, etc. in a special way we will obtain a model M with finite character.

If Γ is a condition-set $\{A_i(x_1, \ldots, x_n)\}$, and $(\alpha_1, \ldots, \alpha_n)$ is a sequence of symbols in S, then the corresponding elements in M will satisfy Γ if and only if all $A_i(\alpha_1, \ldots, \alpha_n)$ belong to Σ_ω. A sentence $\sim A_i(\alpha_1, \ldots, \alpha_n)$ will be called a *negative assignment to the pair* $\{\Gamma, (\alpha_1, \ldots, \alpha_n)\}$. In order that the model M has finite character, it is necessary and sufficient that no element-sequence in M shall satisfy any condition-set $\Gamma \varepsilon \mathfrak{G}$. But in view of Lemma 2 we may restrict ourselves to repetition-free sequences. We get then: M has finite character, if and only if Σ_ω contains some negative assignment to every pair (Γ, t), where $\Gamma \varepsilon \mathfrak{G}$ and t is a repetition-free symbol-sequence from S of appropriate degree. (The restriction to repetition-free symbol-sequences is made to avoid some trivial complications in the proof.)

If by S_k we denote the subset of S that is adjoined to L_k, namely the set of the elements from $s_1, \ldots, s_k$, we can give the following equivalent formulation: The model M has finite character if and only if for every k the set Σ_k contains some negative assignment for every pair (Γ, t) where $\Gamma \varepsilon \mathfrak{G}$, and t is a repetition-free sequence of symbols from S_k of appropriate degree. This condition on the system (L_k, Σ_k) we will call *the condition F*. We may also give another formulation: (L_k, Σ_k) fulfills the condition F, if every set $\Sigma(a_1, \ldots, a_n)$ contained in Σ_k (where the symbols $a_1, \ldots, a_n \varepsilon S$) has a finite basis in (L, Σ). Our task is now to show that we can construct the successive extensions (L_k, Σ_k) so that each of them satisfies the condition F. To show this is enough to prove the theorem.

We will make the proof by induction. We first observe that the condition is fulfilled trivially by the system $(L_0, \Sigma_0) = (L, \Sigma)$. We assume then that (L_{k-1}, Σ_{k-1}) has been constructed so as to satisfy the condition F. We will show that it follows that (L_k, Σ_k) can be so constructed. (To show this for arbitrary k is now sufficient to prove the theorem.)

The system (L_k, Σ'_{k-1}) is formed as usual. We now assume an enumeration $p_1, p_2, \ldots$ of all pairs (Γ, t) where $\Gamma \varepsilon \mathfrak{G}$ and t is a sequence of appropriate degree from S_k. We want to form a series of corresponding negative assignments in the following way: N_1 is a negative assignment to p_1, which is consistent with Σ'_{k-1}, and in general N_m is a negative assignment to p_m, which is consistent with $\Sigma'_{k-1}, N_1, \ldots, N_{m-1}$. (If we have an enumeration of all the sentences in L_ω, we can take N_m as the "first" – – if any – – sentence which satisfies these conditions.) If we have such a series, we can form (L_k, S_k) in two steps: We first form a system (L_k, Σ''_{k-1}) by adding the set $\{N_m\}$ to Σ'_{k-1}. (This system must be consistent, since every finite subset of $\{N_m\}$ is consistent with Σ'_{k-1}.) We then construct (L_k, Σ_k) as a complete consistent extension of (L_k, Σ''_{k-1}). This system clearly satisfies the condition F.

The only thing, which now remains to be proved, is that under the assumptions made we can always find such a complete series $N_1, N_2, \ldots$ of negative assignments, which is consistent with Σ'_{k-1}. To prove this, we assume that there are such assignments $N_1, \ldots, N_{m-1}$, but that it is not possible to add a negative assignment of p_m without introducing a contradiction. If $p_m = (\Gamma, (\alpha_1, \ldots, \alpha_n))$, where $\Gamma = \{A_i(x_1, \ldots, x_n)\}$ this means that the set $\{A_i(\alpha_1, \ldots, \alpha_n)\}$ is deducible from $\Sigma'_{k-1}, N_1, \ldots, N_{m-1}$. If we let $b_1, b_2, \ldots$ be the symbols of s_k, this can be written:

$$(1) \qquad \Sigma_{k-1}, \{Q_r(b_r)\}, B \rightarrow \{A_i(\alpha_1, \ldots, \alpha_n)\}$$

where the statements $Q_r(b_r)$ are the statements of Σ'_{k-1}, which correspond to existential statements $(Ex)Q_r(x)$ in Σ_{k-1}, and B is the conjunction of the statements $N_1, \ldots, N_{m-1}$.

We will show that this implies that the set $\{A_i(x_1, \ldots, x_n)\}$ has a finite basis, and thus that our assumptions are impossible.

It is important to observe in relation (1), that the left-hand side is consistent, that Σ_{k-1} contains no symbols from s_k, each sentence $Q_r(b_r)$ contains no symbol from s_k except b_r, and the set $\{A_i(\alpha_1, \ldots, \alpha_n)\}$ contains no symbols from S except $\alpha_1, \ldots, \alpha_n$, some of which may belong to s_k and some to S_{k-1}. We see now that we may apply Lemma 1 to effect some simplifications in the relation (1). Every sentence $Q_r(b_r)$, such that b_r does not occur anywhere else, may be dropped, – – since it may be replaced by the corresponding existential statement, and this is already in Σ_{k-1}. This means that all except a finite number of the sentences $Q_r(b_r)$ may be dropped. We write the remaining ones in a conjunction together with B, and we have a relation:

$$(2) \qquad \Sigma_{k-1}, C \rightarrow \{A_i(\alpha_1, \ldots, \alpha_n)\}$$

where the left-hand side is still consistent.

Let $\beta_1, \ldots, \beta_q$ be those symbols of S_{k-1} which occur in C or in $\{A_i(\alpha_1, \ldots, \alpha_n)\}$. (There can be only a finite number of them.) By again applying Lemma 1 and observing that (L_{k-1}, Σ_{k-1}) is complete, we see that we can, in the relation (2), dispense with all sentences of Σ_{k-1} which contain symbols of S_{k-1} other than $\beta_1, \ldots, \beta_q$. But the remaining subset of Σ_{k-1} is $\Sigma(\beta_1, \ldots, \beta_q)$. We write this result:

$$(3) \qquad \Sigma(\beta_1, \ldots, \beta_q), C \rightarrow \{A_i(\alpha_1, \ldots, \alpha_n)\}$$

But we know that (L_{k-1}, Σ_{k-1}) satisfies the condition F, and this means that $\Sigma(\beta_1, \ldots, \beta_q)$ must have a finite basis in $(L,$

Σ). If we let D be a basis we have $\Sigma, D \rightarrow \Sigma\ (\beta_1, \ldots, \beta_q)$, and (3) can be replaced by:

$$\Sigma, C, D \rightarrow \{A_i(\alpha_1, \ldots, \alpha_n\} \tag{4}$$

If here we conjoin C and D and eliminate all constants except $\alpha_1, \ldots, \alpha_n$, we get:

$$\Sigma_0, E(\alpha_1, \ldots, \alpha_n) \rightarrow \{A_i(\alpha_1, \ldots, \alpha_n)\} \tag{5}$$

From this it follows that the set $\Gamma = \{A_i(x, \ldots, x_n)\}$ has a finite basis, which is contrary to our assumption. With this we have shown that the desired series $N_1, N_2, \ldots$ of negative assignments exists. This completes the proof of the theorem.

As an obvious consequence of $T\,7$ and $T\,8$ we now get the desired converse of $T\,6$:

T 9. Let (L, Σ) be a complete consistent system. If some maximal consistent condition-set in this system is without finite basis, then the system is not $\aleph_0$-categorical.

The Boolean algebra $B(\Sigma, x_1, x_2, \ldots, x_n)$

By $L(x_1, \ldots, x_n)$ we will denote the class of those formulas in L which contain no free variables except $x_1, \ldots, x_n$. Given some class Σ of sentences of L, the formulas of $L(x_1, \ldots, x_n)$ form a Boolean algebra, if we regard conjunction as intersection, disjunction as sum, negation as complement, and consider two formulas A and B identical, if $A \equiv B$ is a consequence of Σ [4]. This Boolean algebra we designate $B(\Sigma, x_1, \ldots, x_n)$. The maximal consistent condition-sets in (L, Σ) in the variables $x_1, \ldots, x_n$ are the maximal (dual) ideals in this Boolean algebra. The maximal consistent condition-sets with finite basis are principal ideals in $B(\Sigma, x_1, \ldots, x_n)$.

T 10. (Theorem of Ryll-Nardzewski). Let (L, Σ) be a complete consistent system. The following two statements about (L, Σ) are then equivalent:

(a) (L, Σ) is $\aleph_0$-categorical.

(b) For every n, the algebra $B(\Sigma, x_1, \ldots, x_n)$ is finite, *Proof.* Assume that (b) does not hold, i.e. for some n, $B(\Sigma, x_1, \ldots, x_n)$ is infinite. It is known that every infinite Boolean algebra has some non-principal maximal ideal. Translated to the terminology we have used, this means that there is some maximal consistent condition-set in (L, Σ) without finite basis. By the preceding theorem it follows that (L, Σ) is not $\aleph_0$-categorical. To prove the implication the other way we assume that (b) holds. It follows that all maximal ideals are principal ideals, i.e. all maximal consistent condition-sets in (L, Σ) have a finite basis. Statement (a) follows then by *T* 6.

We may restate essentially the same result in terms of structures.

Def. If $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are two sequences of elements in an L-structure M, we say that they *have the same type* in M if they determine the same maximal consistent condition-set.

T 11. Given an L-structure M, the corresponding complete system (L, Σ_M) is $\aleph_0$-categorical if and only if for every n the number of different types of n-sequences in M is finite.

Proof. Assume that there are m different types of n-sequences. If $A(x_1, \ldots, x_n)$ and $B(x_1, \ldots, x_n)$ are two conditions which are satisfied by the same types of sequences, then

$$A(x_1, \ldots, x_n) \equiv B(x_1, \ldots, x_n)$$

holds generally for M and is therefore a consequence of Σ_M. It follows that there can be at most 2^m non-equivalent conditions in $(x_1, \ldots, x_n)$, i.e. that $B(\Sigma_M, x_1, \ldots, x_n)$ is finite. Since this holds for every n, (L, Σ_M) is $\aleph_0$-categorical according to the preceding theorem. The converse is immediate.

This theorem gives an easy test to decide for any given M whether (L, Σ_M) is $\aleph_0$-categorical. Take, for instance, M to be the structure of natural numbers in terms of the successor relation. Obviously each element has a different type, i.e. there are infinitely many types of 1-sequences. It follows by the theorem that (L, Σ_M) is not $\aleph_0$-categorical.

IV. *Summary*

We have considered complete consistent systems in the first-oder predicate calculus with identity, and have studied the set of the models of such a system by means of the *maximal consistent condition-sets* associated with the system. The results may be summarized thus: (a) A complete consistent system is $\aleph_0$-categorical (=categorical in the denumerable domain) if and only if for every n, the number of different conditions in n variables is finite (T 10). (b) If a complete consistent system has a model M with finite character (i.e. a model M such that every maximal consistent condition-set satisfied in M has a finite basis), then this model M is uniquely characterized by the property that every other model is an arithmetic extension of M (T 5). (c) Every complete consistent system, which has only a denumerable number of different associated maximal consistent condition-sets, has a model with finite character (T 8).

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