Theoretical Computer Science 23 (1983) 333–337
North-Holland Publishing Company

NOTE

TURING MACHINES WITH LINEAR ALTERNATION, THEORIES OF BOUNDED CONCATENATION AND THE DECISION PROBLEM OF FIRST ORDER THEORIES

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Abstract. L. Berman [1] has proved completeness results for the theories $\mathrm{Th}(R, +, 0)$ and $\mathrm{Th}(N, +, 0)$ in time classes with linear alternation. These results exemplify the following phenomenon: A lower bound result of $\mathrm{NTIME}(t(n))$ and an upper bound result of $\mathrm{DSPACE}(t(n))$ for the decision problem of a first order theory T are likely to yield a completeness result for the intermediate class with linear alternation $\mathrm{LATIME}(t(n))$ $(=\mathrm{STA}(-, t(n), n)$ in the notation of Berman [1]).

We shall prove that the theory $\mathrm{BCT}(2^*|t(n))$ of t-bounded concatenation is complete in the class $\mathrm{LATIME}(t(\mathrm{O}(n)))$, whenever t satisfies: $t(1)\geq 2$ and $t(m_1+m_2)\geq t(m_1)\cdot t(m_2)$ for $m_1, m_2>0$. This yields a sequence of theories which are complete in the classes of the intermediate hierarchy of elementary recursive sets based on time classes with linear alternation. In particular, we shall show that the theory $\mathrm{Th}(2^*, r_0, r_1)$ of the binary tree is complete in $\mathrm{LATIME}(2^{\mathrm{O}(n)})$. Two further examples will be provided.

1. Time classes with linear alternation and the intermediate hierarchy

Chandra and Stockmeyer [4] resp. Kozen [11] have extended the concept of nondeterministic Turing machines to alternating Turing machines. An alternating Turing machine (=ATM) M is a nondeterministic Turing machine with two disjoint sets of states, existential and universal states, and a distinguished accepting state. An input w is accepted by M (i.e. $w\in L(M)$), if there exists a finite accepting computation tree of M for w. For a definition of this notion the reader is referred to Berman [1] or Chandra, Kozen and Stockmeyer [5]. A language L belongs to the *time class with linear alternation* $\mathrm{LATIME}(t(n))$ $(=\mathrm{STA}(-, t(n), n)$ in the notation of Berman [1]), if L is accepted by an ATM M which on inputs of length n makes at most $t(n)$ steps and involves at most n alternations. For a precise definition the reader is again referred to [1] or [5].

It follows from results in Chandra, Kozen and Stockmeyer [5] that the class $\mathrm{LATIME}(t(n))$ sits between the corresponding time and space classes if $t(n)\geq n$ for all n:

$$\mathrm{NTIME}(t(n))\subseteq \mathrm{LATIME}(t(n))\subseteq \mathrm{DSPACE}(t(n)).$$

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It is well known that the class E of elementary recursive sets can be obtained as the union of the following hierarchies:

$$E = \bigcup_k \mathrm{NTIME}(e_k(\mathrm{O}(n))) = \bigcup_k \mathrm{DSPACE}(e_k(\mathrm{O}(n))),$$

where $e_1(n) = 2^n$ and $e_{k+1}(n) = 2^{e_k(n)}$. Thus we obtain the *intermediate hierarchy for* E as follows:

$$E = \bigcup_k \mathrm{LATIME}(e_k(\mathrm{O}(n))).$$

The time classes with linear alternation are very useful in classifying the complexity of the decision problem of first order theories for two reasons. Alternation is closely related to quantification and propositional operations, and a sentence of length n contains at most n quantifiers and propositional operations. For the first time this was observed by Berman [1] who proved that the theory $\mathrm{Th}(R, +, 0)$ of real addition and the Presburger Arithmetic $\mathrm{Th}(N, +, 0)$ are complete in $\mathrm{LATIME}(e_1(\mathrm{O}(n)))$ resp. $\mathrm{LATIME}(e_2(\mathrm{O}(n)))$ w.r.t. polynomial time reductions. Other examples can be found in Kozen [12]. These observations may be summarized as follows:

(*) A proof of a lower bound of $\mathrm{NTIME}(t(n))$ and a proof of an upper bound of $\mathrm{DSPACE}(t(n))$ for the decision problem of a first order theory T, when examined, are likely to yield a proof of a completeness result for $\mathrm{LATIME}(t(n))$.

2. Theories of bounded concatenation

Computations of Turing machines may be represented as concatenation of words which represent configurations of the Turing machine. Therefore in 1975 A.R. Meyer introduced the theories of bounded concatenation as a uniform method for proving lower bounds for the complexity of first order theories (cf. Fleischmann, Mahr and Siefkes [9, 10]; Bruss and Meyer [3]).

Let Con resp. Con_k be the concatenation relation on Σ^* resp. $\Sigma^{\leq k}$ for a finite alphabet Σ. The full concatenation theory $\mathrm{CT}(\Sigma^*) = \mathrm{Th}(\Sigma^*, \mathrm{Con}, (a : a \in \Sigma))$ is undecidable (cf. [10]). However, the *t-bounded concatenation theory* $\mathrm{BCT}(\Sigma^* | t(n)) = \mathrm{Th}(\Sigma^*, (\mathrm{Con}_{t(n)} : n \in N), (a : a \in \Sigma))$ is decidable whenever $t : N \to N$ is computable. There is a uniform polynomial time reduction to $\mathrm{BCT}(\Sigma^* | t(n))$ for each theory $\mathrm{Th}(\Sigma^{\leq k}, \mathrm{Con}_{t(k)}, (a : a \in \Sigma))$. It should be noted that the predicate c_n associated with the relation $\mathrm{Con}_{t(n)}$ contains the parameter n in unary notation! As a variant of $\mathrm{BCT}(\Sigma^* | t(n))$ we consider the theory $\mathrm{BCT}_r(\Sigma^* | t(n))$, where the relation $\mathrm{Con}^r_{t(n)}$ replaces $\mathrm{Con}_{t(n)}$ and Con^r_k is defined as follows: $(u, v, w) \in \mathrm{Con}^r_k$ iff $(u, v, w) \in \mathrm{Con}$ and $|w| \leq k$. Adding the equal length relation El we obtain the theories $\mathrm{BCT}^{\mathrm{El}}(\Sigma^* | t(n))$ and $\mathrm{BCT}^{\mathrm{El}}_r(\Sigma^* | t(n))$. Using the obvious polynomial time reductions we get:

$$\begin{aligned} &\mathrm{BCT}(\Sigma^*|t(n)) \leq_p \mathrm{BCT}_r(\Sigma^*|t(n)) \leq_p \mathrm{BCT}_r^{\mathrm{El}}(\Sigma^*|t(n)),\\ &\mathrm{BCT}(\Sigma^*|t(n)) \leq_p \mathrm{BCT}^{\mathrm{El}}(\Sigma^*|t(n)) \leq_p \mathrm{BCT}_r^{\mathrm{El}}(\Sigma^*|t(n)). \end{aligned} \tag{1}$$

The theorem below shows that the theories of bounded concatenation can be used to measure the complexity of first order theories by means of syntactic reductions. In particular these theories constitute a set of examples for the observation (*).

Theorem 1. *The theories* $\mathrm{BCT}(2^*|t(n))$, $\mathrm{BCT}_r(2^*|t(n))$, $\mathrm{BCT}^{\mathrm{El}}(2^*|t(n))$ *and* $\mathrm{BCT}_r^{\mathrm{El}}(2^*|t(n))$ *are complete in* $\mathrm{LATIME}(t(\mathrm{O}(n)))$ *w.r.t. polynomial time reductions, whenever t satisfies*: $t(m_1+m_2) \geq t(m_1)\cdot t(m_2)$ *for* $m_1, m_2 > 0$ *and* $t(1) \geq 2$.

This improves the lower bound of $\mathrm{NTIME}(t(n))$ for $\mathrm{BCT}(2^*|t(n))$ in Fleischmann, Mahr and Siefkes [10]. As a corollary we obtain a sequence of theories which are complete in the classes of the intermediate hierarchy of E defined above.

Corollary 2. *The statement in Theorem 1 holds for the functions* e_k *with* $k \geq 1$.

In particular, we are able to improve the results in Ferrante and Rackoff [7] on the theory $\mathrm{Th}(2^*, r_0, r_1, \mathrm{El})$, where r_0, r_1 are the successor operations on 2^*.

Corollary 3. *The theories* $\mathrm{Th}(2^*, r_0, r_1)$ *and* $\mathrm{Th}(2^*, r_0, r_1, \mathrm{El})$ *are complete in* $\mathrm{LATIME}(2^{\mathrm{O}(n)})$.

Finally we present two further examples. Part (2) improves an upper bound of Michel [13].

Proposition 4. (1) *The theory* $\mathrm{Th}(P_{\mathrm{fin}}(N), \subseteq, 0)$ *of finite subsets of N belongs to* $\mathrm{LATIME}(2^{\mathrm{O}(n^2)})$ *and is complete in* $\bigcup_k \mathrm{LATIME}(2^{\mathrm{O}(n^k)})$.

(2) *The theory* $\mathrm{Th}(N^+, |, 1)$ *of the divisibility relation on* $N^+ = N - \{0\}$ *belongs to* $\mathrm{LATIME}(2^{\mathrm{O}(n^2 \log(n))})$ *and is complete in* $\bigcup_k \mathrm{LATIME}(2^{\mathrm{O}(n^k)})$.

In this note we do not prove these results. But we shall give references and indicate how the proofs of the cited results can be modified to yield our results.

Applying the method of Berman [1] for describing ATM computations with linear alternation to the proof of the lower bound $\mathrm{NTIME}(t(n))$ in Fleischmann, Mahr and Siefkes [10], one is able to prove

$$\mathrm{LATIME}(t(n)) \subseteq_p \mathrm{BCT}(2^*|(t(n)+2)^2). \tag{2}$$

Replacing the bound 2^{3m+k} by $t(3m+k)$ in the proof of Ferrante and Rackoff [7] for the upper bound of $\mathrm{DSPACE}(2^{\mathrm{O}(n)})$ for the theory $\mathrm{Th}(2^*, r_0, r_1, \mathrm{El})$ one can prove under the assumptions on t:

$$\mathrm{BCT}_r^{\mathrm{El}}(2^*|t(n)) \equiv \mathrm{LATIME}(t(3n+2)). \tag{3}$$

The combination of (1), (2) and (3) yields the proof of Theorem 1. Corollary 2 holds since the functions e_k satisfy the conditions on t. To prove Corollary 3 one produces a reduction $\mathrm{BCT}(2^*|2^n) \leqslant_p \mathrm{Th}(2^*, r_0, r_1)$ and obtains a proof of $\mathrm{Th}(2^*, r_0, r_1, \mathrm{El}) \in \mathrm{LATIME}(2^{O(n)})$ from the proof in Ferrante and Rackoff [7] mentioned above.

The structure $(P_{\mathrm{fin}}(N), \subseteq, \emptyset)$ resp. $(N^+, |, 1)$ is isomorphic to the countable weak direct power $(\tilde{2}, \tilde{\leqslant}, \tilde{0})$ resp. $(\tilde{N}, \tilde{\leqslant}, \tilde{0})$ of $(2, \leqslant, 0)$ resp. $(N, \leqslant, 0)$. The method of Ferrante and Rackoff [7] for countable weak direct powers yields the upper bounds in Proposition 4. A result in Kozen [12] yields the lower bounds since one may verify:

$$\mathrm{Th}(\text{finite BA's}) \leqslant_p \mathrm{Th}(\tilde{2}, \tilde{\leqslant}, \tilde{0}) \leqslant_p \mathrm{Th}(\tilde{N}, \tilde{\leqslant}, \tilde{0}).$$

A lemma of Cook [2] yields the completeness results.

3. Remarks and questions

The theories of bounded concatenation are closely related to the theories of addition and bounded multiplication (cf. [8]), since the former can be reduced to the latter.

The conditions on t imply $t(n) \geqslant 2^n$ for all n. Therefore Theorem 1 cannot be applied to $\mathrm{LAPTIME} = \bigcup_k \mathrm{LATIME}(O(n^k))$, a subclass of APTIME = PSPACE by [5]. However, QBF belongs to $\mathrm{LATIME}(n)$ (cf. [14]) and is thus even LAPTIME-complete. Moreover, it can be shown that the theory $\mathrm{Th}(2^*, \leqslant_{\mathrm{lex}})$ of the lexicographical ordering – which might be called a weak concatenation theory – is LAPTIME-complete. The complexity of $\mathrm{Th}(2^*, r_0, r_1, \leqslant_{\mathrm{lex}})$, formulated as a question in [6], remains open. Is the theory complete in any class $\mathrm{LATIME}(t(O(n)))$? More generally, one might ask for a natural example of a structure A such that $\mathrm{Th}(A)$ is not complete in any class $\mathrm{LATIME}(t(O(n)))$.

An extended version of this note will appear in [16].

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