We sketch the proof that $p_a(\gamma) = 0$, where $\gamma = \gamma_\tau \otimes I$. It can be shown that there exists a partial isometry A in $C^U_{M_1}$ such that $\sigma^U_t(A) = \lambda^{it}A$ $(t \in \mathbb{R})$ and $\theta^U_1(A) = A$ (cf. the proof of [5, Theorem 4.4.2]). Then $A \in C^U_M$, $\sigma^U_t(A) = \lambda^{it}A$ $(t \in \mathbb{R})$, and $\gamma^U_\tau(A) = \lambda^{it}A$. On the other hand, $C^U_{R_\infty}$ contains a partial isometry B such that $\sigma^U_t(B) = \lambda^{it}B$ (cf. [5]). But then, $AB^* \in (C^U_{M \otimes R_\infty})_{\rho_U}$ and $\gamma^U_\tau(AB^*) = \lambda^{i\tau}AB^*$, and therefore $p_a(\gamma) = 0$ (cf. [5, Sec. 4.2]).

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VALUE OF THE STEINITZ CONSTANT

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A well-known lemma of Steinitz [1] states that for every normal m-dimensional space there exists a constant B such that for every collection of vectors $x_1, x_2, \ldots, x_n$ satisfying $x_1 + x_2 + \ldots + x_n = 0$, $\|x_i\| \leq 1$ $(i = 1, 2, \ldots, n)$ there exists a permutation π such that for all positive integers $k \leq n$ we have $\left\|\sum_{i=1}^{k} x_{\pi(i)}\right\| \leqslant B$. The smallest possible value of B is denoted by C and will be called the Steinitz constant. C depends only on m and the given norm.

It is shown in [2] that for a Euclidean space,

$$C \leqslant \sqrt{(4^m - 1)/3}. \tag{1}$$

A discussion of the Steinitz lemma and its applications is given in [3-5] and [6], where estimate (1) was rediscovered.

Up to now, only upper estimates exponential with respect to m have been known for C. We show that $C \leq m$ for every norm (not necessarily even symmetric).

<u>LEMMA.</u> Let K be a polyhedron in $\mathbf{R}^n$ defined by a system

$$\begin{cases} f_i(x) = a_i, & i = 1, 2, \ldots, p, \\ g_j(x) \leqslant b_j, & j = 1, 2, \ldots, q, \end{cases}$$

where the f_i and g_j are linear functions. Let x_0 be a vertex of K and $A = \{j : g_j(x_0) = b_j\}$.

Then $|A| \geq n - p$, where $|A|$ is the cardinality of the set A.

<u>Proof.</u> Assume the contrary. Then the system

$$\begin{cases} f_i(x) = 0, & i = 1, 2, \ldots, p, \\ g_j(x) = 0, & j \in A, \end{cases}$$

has a nontrivial solution x_1. The vectors $x_0 - \varepsilon x_1$ and $x_0 + \varepsilon x_1$ belong to K for sufficiently small $\varepsilon > 0$, which contradicts the fact that x_0 is a vertex.

<u>THEOREM 1.</u> Let an arbitrary norm be given in $\mathbf{R}^m$ and assume that $\|x_i\| \leq 1$ $(i = 1, 2, \ldots, n)$ and $x_1 + \ldots + x_n = x$.

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Then there exists a permutation π such that for all positive integers $k \leq n$

$$\left\| \sum_{i=1}^{k} x_{\pi(i)} - \frac{k-m}{n} x \right\| \leqslant m. \tag{2}$$

Proof. By induction, we construct a chain of sets

$$A_m \subset A_{m+1} \subset \dots \subset A_n = \{1, 2, \dots, n\}$$

and numbers λ_k^i ($k = m, m+1, \dots, n$; $i \in A_k$) with the properties:

$$|A_k| = k, \quad 0 \leqslant \lambda_k^i \leqslant 1, \quad \sum_{i \in A_k} \lambda_k^i = k - m, \quad \sum_{i \in A_k} \lambda_k^i x_i = \frac{k-m}{n} x.$$

$$k = n. \quad A_n = \{1, 2, \dots, n\}, \quad \lambda_n^i = \frac{n-m}{n}.$$

Inductive Step $k+1 \to k$. Consider the set K of collections $\{\mu_i, i \in A_{k+1}\}$ with the properties:

$$0 \leqslant \mu_i \leqslant 1, \quad \sum_{i \in A_{k+1}} \mu_i = k - m, \quad \sum_{i \in A_{k+1}} \mu_i x_i = \frac{k-m}{n} x. \tag{3}$$

K is convex and compact in R^{k+1} and nonempty (we can take $\mu_i = \frac{k-m}{k+1-m} \lambda_{k+1}^i$). Let $\{\bar{\mu}_i, i \in A_{k+1}\}$ be a vertex of K. By the lemma $|\{i: 0 < \bar{\mu}_i < 1\}| \leq m + 1$. Using (3), we get that $\{i: \bar{\mu}_i = 0\} \neq \phi$. Let $\bar{\mu}_j = 0$. We put $A_k = A_{k+1} \setminus \{j\}$, $\lambda_k^i = \bar{\mu}_i$ ($i \in A_k$), which completes the construction.

We put $\{\pi(i)\} = A_i \setminus A_{i-1}$ ($i = m+1, \dots, n$), with π otherwise arbitrary. For $k \leq m$, inequality (2) to be proved is obvious. For $k \geq m+1$,

$$\left\| \sum_{i=1}^{k} x_{\pi(i)} - \frac{k-m}{n} x \right\| = \left\| \sum_{i \in A_k} (1 - \lambda_k^i) x_i \right\| \leqslant \sum_{i \in A_k} (1 - \lambda_k^i) = m.$$

Remark. Symmetry of the norm has not been used anywhere.

For the case $x = 0$ and a symmetric norm (Steinitz' lemma), a more cumbersome proof is given in [7].

The following example shows that the estimate obtained is best possible. Take the unit ball to be a regular (unsymmetric) simplex with center at zero. Put $n = m + 1$, and let x_i be the i-th vertex of the simplex. Then $\Sigma x_i = 0$ and $\| x_{\pi(1)} + \dots + x_{\pi(n-1)} \| = m$ for every permutation π.

The maximal lower estimate for the Steinitz constant known to the author for the case of a symmetric norm ($(m+1)/2$ in the space l_1, $C \geq (m+3)^{1/2}/2$ for a Euclidean space) is attained on the collection of vectors $B_k = \{e_i$ ($i = 1, \dots, m-1$); a_i^k ($i = 1, \dots, k$); b_i^k ($i = 1, \dots, k$)$\}$, $k \geq (m-1)/2$ as $k \to \infty$, where the e_i are unit vectors; $a_i^k(j) = b_i^k(j) = -1/2k$ ($j = 1, \dots, m-1$, $i = 1, \dots, k$), $a_i^k(m) = -b_i^k(m) = 1 - (m-1)/2k$ ($i = 1, \dots, k$).

We state without proof another assertion relevant to the topic considered.

THEOREM 2. Let $x_i \in R^m$, $\|x_i\| \leqslant 1$ ($i = 1, 2, \dots, n$), $\Sigma x_i = x$. Then $\forall \rho$ ($0 \leqslant \rho \leqslant 1$) $\exists A \subseteq \{1, 2, \dots, n\}$, for which $\left\| \sum_{i \in A} x_i - \rho x \right\| < m$, and in the case of a symmetric norm $\left\| \sum_{i \in A} x_i - \rho x \right\| \leqslant m/2$.

Both the estimates are exact. For the proof put $n = m$ and let $\{x_i\}$ be the standard basis in R^m. In the symmetric case, we define an l_1-norm and set $\rho = 1/2$. For the unsymmetric case, let the unit ball be the convex hull of the vectors $\{x_1, \dots, x_n, -\varepsilon x_1, \dots, -\varepsilon x_n\}$ and $\rho = \varepsilon$, where ε is a small positive number.

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