J. Symbolic Computation (1992) 14, 265-282

An Approach to the Zero Recognition Problem by Buchberger Algorithm

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The class of "holonomic function" is considered. We present a quasi-algorithm that recognizes whether a holonomic function is zero or not. The algorithm consists of procedures which obtain differential operators that annihilate sums, products, and definite integrals with respect to parameters of holonomic functions. A Weyl algebra-analog of Buchberger's algorithm is used. A "holonomic" approach to the zero recognition algorithm was initiated by D.Zeilberger (1990) who realized it by Sylvester's dyalitic elimination. Our algorithm uses Buchberger's algorithm to improve Zeilberger's algorithm.

1. Introduction

We have two purposes. The first is to present an algorithm of obtaining differential equations for a definite integral with parameters. The algorithm is based on Buchberger algorithm (Buchberger 1982) . Computing differential equations for a definite integral has a wide range of applications to theoretical physics and " experimental " mathematics. If differential equations are obtained explicitly, we can compute the locus of singularities , local expansions of solutions, local monodromy, Hilbert function, free resolution ...etc.

The second purpose is to improve Zeilberger's (1990) zero recognition method for holonomic function by our first algorithm. Suppose that a function is annihilated by an ideal of differential operators and the solution space of the equations is finite dimensional. If the function and its derivatives are zero at a point, then the function is identically zero. Zeilberger's method is based on the above principle. Canonical forms do not exist for sufficiently rich classes of mathematical expressions (Caviness, 1970). We refer to Buchberger and Loos (1982) for a historical survey of canonical forms and zero recognition problem. The zero recognition of mathematical expression is , in general, impossible, but Zeilberger's " holonomic " point of view to the zero recognition problem is a breakthrough. The notion of "holonomic system" was initiated and developed by I.N. Bernstein and M. Kashiwara during the 1970's in pure mathematics. We refer to chapter 1 of Björk (1979) for the notion of holonomic systems. A pioneering work on holonomic systems and computer algebra is Galligo (1985) . The Author (1989) studied special function identities by holonomic system and Gröbner basis.

We will present conjectures and open problems in this paper. They are naturally obtained by considering " correctness " of the algorithms of this paper. We say, in this paper, that an algorithm is " correct " if holonomicity of the input implies holonomicity

0747-7171/92/080265+18 $08.00/0

of the output (see Theorem 2.1, 2.2, 2.3, 3.1 and 5.1). We have only a partial answer to the " correctness ".

The paper has four sections. Section two and three are devoted to the algorithm for computing differential equations for a definite integral. Section four describes Zeilberger's zero recognition algorithm. We describe an efficient method for obtaining differential equations for definite integral in Section five. The method is an improvement of Algorithm 2.1. The reader may find it profitable to read section two and three in parallel with Example 4.1 and 4.2 of the section four.

"Correctness" of Algorithm 2.1 and 5.1 is an open problem. Recently, a new and "correct" algorithm to obtain differential equations for a definite integral was found (Takayama, 1990). The algorithm is based on an infinite dimensional analog of Gröbner basis.

2. Integrals of holonomic functions

Let x be $(x_1, \cdots, x_n)$ and t be $(t_1, \cdots, t_m)$. The Weyl algebra

$$K\langle x, t, \partial_x, \partial_t\rangle$$

is denoted by A_{n+m} and

$$K\langle x, \partial_x\rangle$$

is denoted by A_n where K is a field of characteristic zero. $\mathcal{A}_{m+n}$ is the ring of differential operator with rational function coefficients :

$$K(x,t)\langle\partial_x, \partial_t\rangle$$

and $\mathcal{A}_n$ is

$$K(x)\langle\partial_x\rangle.$$

Assume that K is the field of complex numbers. Let f be a function of x that is a multivalued holomorphic function on $C^n \setminus \{\text{algebraic varieties}\}$. f is said a *holonomic function* if there exists an ideal $\wp$ of A_n that satisfies an equation $\wp f = 0$ and $A_n/\wp$ is a holonomic A_n module. If $A_n/\wp$ is holonomic A_n module, then $\mathcal{A}_n\wp$ is a zero dimensional ideal of $\mathcal{A}_n$,i.e.,

$$\dim_{K(x)}\mathcal{A}_n/(\mathcal{A}_n\wp) < +\infty$$

where $\mathcal{A}_n\wp$ is the ideal of the ring $\mathcal{A}_n$ generated by $\wp$. The dimension is denoted by $k(\wp)$.

Let f be a holonomic function of x and t , D be a domain in the t space and ∂D be the boundary of the domain D. In this chapter, we present an algorithm for obtaining an ideal of A_n which annihilates the function :

$$I(x) = \int_D f(x,t)dt.$$

An element p of $\wp \cap K\langle x, \partial_x, \partial_t\rangle$ can be written as

$$p_0(x,\partial_x) + \sum_{\alpha\neq 0, \alpha\in E(p)} p_\alpha(x,\partial_x)\partial_t{}^\alpha, \quad p_0, p_\alpha \in A_n \tag{2.1}$$

where $E(p)$ denotes the set of exponents of ∂_t in p. We put

$$\omega_\alpha = -\sum_{k=1,\alpha_k\neq 0}^{m} (-1)^k \partial_{t_1}{}^{\alpha_1}\cdots\partial_{t_k}{}^{\varepsilon(\alpha_k)}\cdots\partial_{t_m}{}^{\alpha_m} f dt_1\wedge\cdots\wedge dt_{k-1}\wedge dt_{k+1}\cdots\wedge dt_m \tag{2.2}$$

where $\alpha = (\alpha_1, \cdots, \alpha_m), \alpha \in E(p)$ and $\varepsilon(\alpha_k) = \alpha_k - 1$.

PROPOSITION 2.1. (c.f. Zeilberger, 1990) *In the above situation, if*

$$\int_{\partial D} \omega_\alpha = 0$$

for all $\alpha \in E(p) \setminus \{0\}$, *then*

$$p_{|\partial_t=0} I(x) = p_0 I(x) = 0.$$

PROOF. Since $p \in \wp$, then $pf = 0$. Integrating pf on D, we have

$$p_0 I + \sum_\alpha p_\alpha \int_D (\partial_t)^\alpha f dt = 0.$$

As $d\omega_\alpha = (\partial_t)^\alpha f dt$, Stoke's theorem implies that

$$\int_D (\partial_t)^\alpha f dt = \int_{\partial D} \omega_\alpha = 0$$

□

The following example is simple but contains essential ideas.

EXAMPLE 2.1. We compute a differential operator that annihilates the function :

$$I(x) = \int_{-\infty}^{\infty} f(x,t)dt, \quad f(x,t) = e^{-xt^2}.$$

We have $\ell_1 f = 0$ and $\ell_2 f = 0$ where

$$\ell_1 = \partial_x + t^2, \quad \ell_2 = \partial_t + 2xt.$$

We eliminate t from ℓ_1 and ℓ_2. An elimination can be done by

$$\ell_3 := 2x\ell_1 - t\ell_2, \quad \partial_t \ell_2 + 2x\ell_3 = 4x^2\partial_x + 2x + {\partial_t}^2.$$

Assume $x > 0$, then we have

$$(4x^2\partial_x + 2x)I(x) = 0.$$

One can see from Proposition 2.1 that if we provide an algorithm that obtains $\wp \cap K\langle x, \partial_x, \partial_t \rangle$, then we can compute an ideal which annihilates the function $I(x)$. The algorithm is nothing but Buchberger algorithm with lexico-total degree order which we will now explain.

We assume the monomials of A_{m+n} to be lexico-total degree ordered as follows.

$$\begin{array}{c} t^a x^b {\partial_t}^c {\partial_x}^d \succ t^\alpha x^\beta {\partial_t}^\gamma {\partial_x}^\delta \\ \Longleftrightarrow \\ t^a \succ_T t^\alpha \text{ or } (a = \alpha \text{ and } x^b {\partial_t}^c {\partial_x}^d \succ_t x^\beta {\partial_t}^\gamma {\partial_x}^\delta) \end{array} \tag{2.3}$$

where $a, b, c, d, \alpha, \beta, \gamma, \delta$ are multi-index, $\succ_T$ is the total degree order for t^a and $\succ_t$ is the total degree order for monomials $x^b {\partial_t}^c {\partial_x}^d$.

ALGORITHM 2.1. (Find differential equations of a definite integral with parameters.)

INPUT: $\wp$,an ideal of A_{m+n} such that $\wp f = 0$.
(1) Compute Gröbner basis of $\wp$ by the lexico-total degree order (2.3) .
Let the basis be G_0 and G be empty set.
(2) $G_1 := G_0 \cap K\langle x, \partial_x, \partial_t \rangle$.
(3) for all $p \in G_1$ do
(4) if $\int_{\partial D} \omega_\alpha = 0$ for all $\alpha \in E(p) \setminus \{0\}$ then
(5) $G := G \cup \{p_{|\partial_t = 0}\}$
else
(6) $G := G \cup \{$ return value of Algorithm 2.3 of the input: p $\}$.
endfor
OUTPUT: G, operators which annihilate the function $I(x)$.

Algorithm 2.1 is an arrangement of the elimination part of Zeilberger's method (1990) by Gröbner basis, however, it is well known that the lexicographic and lexico-total degree orders are not practical. They spend a lot of time and memory. The acceleration of the step (1) and (2) will be considered in Section five.

REMARK 2.1. Step (4) of Algorithm 2.1 needs a call to the zero recognition Algorithm 4.1 or to a formula database of the special functions or to an integration system. If Step (4) fails, then the zero recognition algorithm fails.

The theory of holonomic systems proves that the integral of a holonomic A_n module is holonomic (Bernstein, 1972 or see Björk, 1979) . Then, roughly speaking, the function $I(x)$ is a holonomic function, because the integral of the module $A_{n+m}/\wp$ is holonomic. Let M be a holonomic A_{n+1} module. An A_n module $M/\partial_{t_1} M$ is called the integral of M in the theory of holonomic system. However, Algorithm 2.1 does not compute it in general. Let $M = A_{n+1}/\wp$ be a holonomic A_{n+1} module. Put

$$\wp_1 = (\wp \cap K\langle \partial_{t_1}, x, \partial_x \rangle)_{|\partial_{t_1} = 0}.$$

The ideal $\wp_1$ annihilates $M/\partial_{t_1} M$, but the module $M/\partial_{t_1} M$ is not equal to $A_n/\wp_1$ in general. For example, let $\wp$ be $(\partial_{t_1}, \partial_x)$. Then $\wp_0 = (\partial_x)$, but $M/\partial_{t_1} M = 0$.

The computation of the integral $M/\partial_{t_1} M$ was an open problem. Recently, a new method of obtaining the integral of a module (Takayama, 1990) that is based on an infinite dimensional analog of Gröbner basis is found.

Case I: Condition (4) of Algorithm 2.1 is true for all α.

We consider "correctness" of Algorithm 2.1 under assuming Case I in the sequel. Zeilberger (1990) considered "correctness" in the case $n = m = 1$. We prove "correctness" in the case $m = 1$ and n is arbitrary (Theorem 2.2), however, it is a partial answer. The author conjectures "correctness" in the case that m and n are arbitrary.

We put $B_i = K\langle \partial_{t_i}, \cdots, \partial_{t_m}, x, \partial_x \rangle$ and $B_{m+1} = A_n$. Let $\wp$ be an ideal of A_{n+m}, $\wp_i, i = 1, \cdots, m$ be the ideal of B_{i+1} :

$$(B_1 \cap \wp)_{|\partial_{t_1} = \cdots = \partial_{t_i} = 0}$$

and $\wp_0 = B_1 \cap \wp$.

THEOREM 2.1. (1) *If $A_{n+m}/\wp$ is holonomic A_{n+m} module and*

$$\partial_{t_i} : B_i/\wp_{i-1} \longrightarrow B_i/\wp_{i-1}, \quad (i = 1, \cdots, m)$$

are injective, then $A_n/\wp_m$ is a holonomic A_n module.

(2) *Let $\wp$ be the input and G_1 be the set defined in (2) of Algorithm 2.1. The Gröbner basis of $B_1 \cap \wp$ by the total degree order $\succ_t$ is G_1 and the set $G = \{p_{|\partial_t=0} \mid p \in G_1\}$ generates the ideal $\wp_m$ in B_1.*

PROOF. (1). Put

$$\mathcal{F}_k = \{ \sum_{|\alpha|+|\beta|+|\gamma|+|\delta|\le k} a_{\alpha\beta\gamma\delta} t^\alpha x^\beta \partial_t{}^\gamma \partial_x{}^\delta, \quad a_{\alpha\beta\gamma\delta} \in K\}.$$

Let r and s be elements of B_1. If $r - s \notin B_1 \cap \wp \cap \mathcal{F}_k$, then $r - s \notin \wp \cap \mathcal{F}_k$. Since

$$\dim_K \mathcal{F}_k/\wp \cap \mathcal{F}_k = ck^{n+m} + O(k^{n+m-1}), \quad c \in \mathbf{R}, \quad k \to \infty,$$

then there exists a non-negative integer constant α such that

$$\dim_K B_1 \cap \mathcal{F}_k/B_1 \cap \wp \cap \mathcal{F}_k = dk^{n+m-\alpha} + O(k^{n+m-1-\alpha}), \quad d \in \mathbf{R}, \quad k \to \infty.$$

The right-hand side of the above is the Hilbert function of B_1 module $B_1/\wp_0$. Put $M_1 = B_1/\wp_0, S_k = \mathcal{F}_k \cap B_1/\wp_0 \cap \mathcal{F}_k$ and $T_k = S_k/\partial_{t_1} M_1 \cap S_k$. T_k is a filtration of B_2 module $M_1/\partial_{t_1} M_1$. It follows from

$$\partial_{t_1} S_{k-1} \subseteq \partial_{t_1} M_1 \cap S_k$$

that

$$\dim_K \partial_{t_1} M_1 \cap S_k \ge d(k-1)^{n+m-\alpha} + O((k-1)^{n+m-1-\alpha}).$$

Then we have

$$\begin{aligned} \dim_K T_k &= \dim_K S_k - \dim_K \partial_{t_1} M_1 \cap S_k \\ &\le ek^{n+m-1-\alpha} + O(k^{n+m-2-\alpha}), \quad e \in \mathbf{R}. \end{aligned}$$

Since ∂_{t_1} commutes with any element of B_1, then we have

$$M_1/\partial_{t_1} M_1 \simeq B_1/(\wp_0 + B_1\partial_{t_1}) \simeq B_2/\wp_1.$$

$B_2 \cap \mathcal{F}_k/\wp_1 \cap \mathcal{F}_k$ is a filtration of B_2 module $B_2/\wp_1$. Since

$$B_2 \cap \mathcal{F}_k/\wp_1 \cap \mathcal{F}_k \simeq T_k \quad (k >> 0),$$

we conclude that the Hilbert function for B_2 module $B_2/\wp_1$ is dominated by $O(k^{n+m-1-\alpha})$.

We can prove the following fact in the same way. If the Hilbert function of the filtered B_i module $B_i/\wp_{i-1}$ has order $O(k^p)$, then the Hilbert function of the filtered B_{i+1} module $B_{i+1}/\wp_i$ is dominated by $O(k^{p-1})$ where the filtrations are $\mathcal{F}_k \cap B_i/\mathcal{F}_k \cap \wp_{i-1}$.

It follows from the above fact that the Hilbert function of the filtered B_{m+1} module $B_{m+1}/\wp_m$ is dominated by $O(k^{n-\beta})$ where β is a non-negative constant. Since $\mathcal{F}_k \cap A_n/\mathcal{F}_k \cap \wp_m$ is a good filtration and $A_n = B_{m+1}$, we have $A_n/\wp_m$ is holonomic A_n module.

(2). Let G_0 be the Gröbner basis of $\wp$ by the order (2.3). An element $f \in B_1 \cap \wp$ can be reduced to zero by G_0. Since every head term of $G_0 \setminus G_0 \cap B_1$ contains the variable t, then f can be reduced to zero by $G_0 \cap B_1 = G_1$. Therefore G_1 is the Gröbner basis of $B_1 \cap \wp$. It is easy to prove $B_1 G = \wp_m$. □

If the condition (4) of Algorithm 2.1 is always true, then the ideal $\wp_m$ is generated by the output G of Algorithm 2.1 (Theorem 2.1 (2)). However Theorem 2.1 (1) can be applied if we have injectivity. Theorem 2.2 and 2.3 will give sufficient conditions for injectivity.

THEOREM 2.2. *If $m = 1$, the integral $I(x)$ exists, $[\partial_{t_1}{}^{\alpha} u]_{\partial D} = 0$ $(\forall \alpha \geq 0)$, $|D| = \infty$ and*

$$\wp = \mathrm{Ann}(f) = \{\ell \in A_{n+1} \mid \ell f = 0\},$$

then the map

$$\partial_{t_1} : B_1/\wp_0 \longrightarrow B_1/\wp_0$$

is injective.

PROOF. We assume ∂_{t_1} is not injective. Then we have an element p of B_1 such that $p \notin \wp_0$ and $\partial_{t_1} p \in \wp_0$. Since $\wp_0 = \wp \cap B_1$, then $p \notin \wp$. We have $pf \not\equiv 0$ and $\partial_{t_1} pf = 0$. Therefore pf is a function of x. We put $a(x) = pf$. If we write p as (2.1) , then we have

$$p_0 I(x) = \int_D a(x)dt = a(x)|D| = \infty \quad \text{or} \quad a(x) \equiv 0.$$

This is a contradiction. □

REMARK 2.2. Obtaining $\mathrm{Ann}(g)$ does not seems to be easy. We will give an example. Let λ be an indeterminate. Put $A_n(\lambda) = K(\lambda)\langle x, \partial_x \rangle$. Let g be f^λ where f is a polynomial in $x_1, \cdots, x_n$. It follows from the existence theorem of b functions (Bernstein, 1971 and Björk, 1979) that $\exists L \in A_n(\lambda)$, $\exists b \in K(\lambda)$ such that $Lf^{\lambda+1} = bf^\lambda$. Then we have

$$\frac{1}{b} Lf - 1 \in \mathrm{Ann}(f^\lambda).$$

Therefore the ideal generated by $\mathrm{Ann}(f^\lambda)$ and f is equal to $A_n(\lambda)$. It is well known that finding b functions is not easy in general ([Gal]), however, if we have generators of $\mathrm{Ann}(f^\lambda)$, then we can obtain a b function by applying Buchberger algorithm to $\mathrm{Ann}(f^\lambda)$ and f. Therefore obtaining $\mathrm{Ann}(g)$ does not seem to be easy.

For example, let f be

$$-x^2z^2 + x^4 + y^4 \quad \text{(Tacnode)} \ (x_1 = x, x_2 = y, x_3 = z, n = 3).$$

Trivial annihilators of f^λ are

$$\ell_0 = \sum_{i=1}^{n} x_i \partial_i - 4\lambda, \ \ell_i = f\partial_i - \lambda f_{x_i}, \ (i = 1, \cdots, n).$$

We can prove, by computing Gröbner basis, that

$$A_n(\lambda)(\ell_0, \cdots, \ell_n, f) = A_n(\lambda)(x, y^3, \ell_0) \neq A_n(\lambda).$$

Therefore $\ell_0, \cdots, \ell_n$ do not generate $\mathrm{Ann}(f^\lambda)$.

THEOREM 2.3. *If all head terms of the Gröbner basis of $\wp_{i-1}$ do not vanish as $\partial_{t_i} \to 0$, then the map ∂_{t_i} of Theorem 2.1 is injective.*

PROOF. If ∂_{t_i} is not injective, then there exists an element p of B_i such that $p \notin \wp_{i-1}$

and $\partial_{t_i} p \in \wp_{i-1}$. We can assume that the element p is irreducible. Since $\partial_{t_i} p \in \wp_{i-1}$, then $\partial_{t_i} p$ is reducible by the Gröbner basis of $\wp_{i-1}$. This means that a head term of an element of the Gröbner basis of $\wp_{i-1}$ can be divided by ∂_{t_i}. This is a contradiction. □

Case II: Condition (4) of Algorithm 2.1 is false for an α.

We will consider how to obtain an ideal that annihilates the function $I(x)$ and the "correctness" of Algorithm 2.1 under Case II in the sequel.

We solve the problem by assuming that there exists a given ideal $\wp_1$ of A_n such that

$$\wp_1 \int_{\partial D} \omega_\alpha = 0.$$

If ∂D has a simple shape (,for example, D is hyperpolyhedron), then $\int_{\partial D} \omega_\alpha$ is a holonomic function and we can compute $\wp_1$ by applying Algorithm 2.1 recursively. Therefore the above assumption is not absurd.

ALGORITHM 2.2. INPUT: $\wp_1$, an ideal of A_n that annihilates a function g.
$p \in K\langle x, \partial_x \rangle$.
(1) H := empty set.
(2) B := the Gröbner basis of the ideal $\mathcal{A}_n \wp_1$ in $\mathcal{A}_n$.
(3) $c := \dim_{K(x)} \mathcal{A}_n / \wp_1$
(4) if $c = \infty$ then Algorithm 2.2 fails.
(5) for $k = 1$ to n do
(6) for $i = 0$ to c do
(7) q_i :=Normal form of $(\partial_{x_k})^i p$ by B.
(8) Find a non-trivial set $\{d_i | d_i \in K[x], i = 0, \cdots, c\}$ such that $\sum_{i=0}^{c} d_i q_i = 0$.
(9) $H := H \cup \{\sum_{i=0}^{c} d_i (\partial_{x_k})^i\}$
(10) endfor;
(11) H :=Gröbner basis of the ideal generated by H in A_n
OUTPUT: The Gröbner basis H such that $Hpg = 0$.

We can use a method that is similar to Algorithm 5.1 (2)(3) instead of varying the values of k in the above algorithm 2.2.

THEOREM 2.4. (A corollary of M. Kashiwara, 1978) *Let $\wp$ be an ideal of $\mathcal{A}_n$. If*

$$\dim_{K(x)} \mathcal{A}_n / \wp < +\infty,$$

then $A_n/(\wp \cap A_n)$ is holonomic A_n module.

PROOF. Kashiwara (1978) proved if a $\mathcal{D}$ module is holonomic outside of a codimension one variety, then the $\mathcal{D}$ module is holonomic inside the variety. It follows from this theorem that there exists an element p of $K[x]$ such that the A_n module :

$$K[x, p^{-1}] \otimes_{K[x]} (A_n / \wp \cap A_n)$$

is holonomic. Since

$$(A_n / \wp \cap A_n) \longrightarrow K[x, p^{-1}] \otimes_{K[x]} (A_n / \wp \cap A_n)$$

is injective, we conclude that $(A_n / \wp \cap A_n)$ is holonomic. □

We remark that if $(A_n/\wp \cap A_n)$ is holonomic, then $\wp$ is a zero dimensional ideal of $\mathcal{A}_n$. If $A_n/(\wp \cap A_n)$ is holonomic, then for each number $i \in [1, n]$ there exists an element of the ideal $\wp \cap A_n$ of the form :

$$\sum_{k=0}^{n_i} p^i_k(x)(\partial_{x_i})^k.$$

Therefore

$$\dim_{K(x)} \mathcal{A}_n/\wp \leq n_1 n_2 \cdots n_n < +\infty.$$

We remark that if $\wp_1$ is the zero dimensional ideal of $\mathcal{A}_n$, then the output H of Algorithm 2.2 generates a zero dimensional ideal of $\mathcal{A}_n$. It follows from Theorem 2.4 that if $A_n/\wp_1$ is holonomic , then $A_n/A_n \cap (\mathcal{A}_n H)$ is holonomic. Hence Algorithm 2.2 is " correct " ,i.e., we have holonomic output from holonomic input.

Step (8) of Algorithm 2.2 can be computed by Gaussian elimination. Suppose that $\{r_j | r_j \in \mathcal{A}_n, j = 1, \cdots, c\}$ is a basis of $\mathcal{A}_n/\wp_1$ as a linear space over the field $K(x)$ and r_j is m-irreducible by the Gröbner basis B. q_i of step (7) of Algorithm 2.2 can be written as

$$q_i = \sum_{j=1}^{c} m_{ij} r_j, \quad i = 0, \cdots, c, m_{ij} \in K(x).$$

(m_{ij}) is a $(c+1)$ by c matrix. By upper triangularization of the matrix (m_{ij}), we can compute d_i of Step (8).

Applications of Algorithm 2.2 yield an ideal which annihilates the function $I(x)$. (2.1) yields

$$p_0 I + \sum_{\alpha \in E(p), \alpha \neq 0} p_\alpha \int_{\partial D} \omega_\alpha = 0.$$

Algorithm 2.2 computes an ideal $\Re_\alpha$ which annihilates the function :

$$p_\alpha \int_{\partial D} \omega_\alpha$$

for each $\alpha \in E(p) \setminus \{0\}$. We apply Procedure 3.1 of Section 3 to the ideals $\Re_\alpha, \alpha \in E(p)$. Then we obtain an ideal $\Re_p$ which annihilates the function :

$$\sum_{\alpha \neq 0, \alpha \in E} p_\alpha \int_{\partial D} \omega_\alpha.$$

We have

$$\Re_p p_0 I = 0.$$

We apply the above procedure to each elements p of the output G_1 of Algorithm 2.1. An ideal :

$$\cup_{p \in G_1} \Re_p p_0$$

annihilates the function I.

We sum up the above procedure.

ALGORITHM 2.3. INPUT: p, of the form (2.1) .

(1) for $\alpha \in E(p) \setminus \{0\}$ do

(2) Call Algorithm 2.1 to obtain an ideal $\wp_\alpha$ that satisfies $\wp_\alpha \int_{\partial D} \omega_\alpha = 0$.
(3) Call Algorithm 2.2. The inputs are $\wp_\alpha$ and p_α. Let the output be H_α.
endfor
(4) Call Procedure 3.1. The inputs are $\{H_\alpha | \alpha \in E(p) \setminus \{0\}\}$. Let the output be H_p.
OUTPUT: $H_p p_0$, an ideal that annihilates the function I.

THEOREM 2.5. *In Algorithm 2.3, if $A_n/\wp_\alpha$ is holonomic for all $\alpha \in E(p) \setminus \{0\}$, then $A_n/(\mathcal{A}_n H_p \cap A_n)$ is holonomic.*

Finally, we mention two unsolved problems on "correctness".

1 What is the shape of ∂D, if we can compute $\wp_\alpha$ by Algorithm 2.1? Is the module $A_n/\wp_\alpha$ holonomic in this case?
2 We do not know what ideal the set $\cup_{p \in G_1} H_p p_0$ generates. Is Algorithm 2.1 "correct" in Case II?

3. Sum and product of holonomic functions

Let f and g be holonomic functions . In this chapter, we present procedures of obtaining ideals that annihilate $f+g$ and fg. These procedures are algorithmic statements of the well known fact that there exists a number N (resp. N') such that

$$(\partial_{x_i})^k (f+g) \quad k = 0, 1, \cdots, N$$

$$(\text{resp. } \ (\partial_{x_i})^k (fg) \quad k = 0, 1, \cdots, N')$$

are linearly dependent over the field $K(x)$. Therefore we only state the procedures without explanations.

We suppose that

$$\wp f = \Im g = 0$$

and $\wp$ and $\Im$ are zero dimensional ideals of $\mathcal{A}_n$. Let G and H be Gröbner bases of $\wp$ and $\Im$ by the total degree order.

PROCEDURE 3.1. INPUT: G and H.
$I :=$ empty set.
$N := k(\wp) + k(\Im) - 1$.
for $k = 1$ to n do
p_i :=normal form of $(\partial_{x_k})^i$ by G, $(i = 0, \cdots, N)$.
q_i :=normal form of $(\partial_{x_k})^i$ by H, $(i = 0, \cdots, N)$.
$q_i := q_i|_{\partial_x \to \partial_y}$.
Find a non-trivial set $\{r_i | r_i \in K[x], i = 0, \cdots, N\}$ such that
$\sum_{i=0}^N r_i(p_i + q_i) = 0$.
$I := I \cup \{\sum_{i=0}^N r_i (\partial_{x_k})^i)\}$.
endfor.
OUTPUT: I, generators of an ideal that annihilates the function $f+g$.

PROCEDURE 3.2. INPUT: G and H.
$I :=$ empty set.

$N := k(\wp) \cdot k(\Im)$.
for $k = 1$ to n do
  p_i :=normal form of $(\partial_{x_k})^i$ by G, $(i = 0, \cdots, N)$.
  q_i :=normal form of $(\partial_{x_k})^i$ by H, $(i = 0, \cdots, N)$.
  $q_i := q_i|_{x \to y, \partial_x \to \partial_y}$ $(i = 0, \cdots, N)$.
  Find a non-trivial set $\{r_i | r_i \in K[x], i = 0, \cdots, N\}$ such that
    $\sum_{i=0}^{N} r_i [(\sum_{j=0}^{i} \binom{i}{j} p_{i-j} q_j)|_{y \to x}] = 0$.
  $I := I \cup \{\sum_{i=0}^{N} r_i (\partial_{x_k})^i)\}$.
endfor.
OUTPUT: I, generators of an ideal that annihilates the function fg.

We can use a method that is similar to Algorithm 5.1 (2)(3) instead of varying the values of k in the above algorithms 3.1 and 3.2. We remark that

$$q_i|_{x \to y}$$

denotes the result of the substitution of x by y in q_i. It follows from Theorem 2.4 that we have a "correctness" theorem.

THEOREM 3.1. *If A_n/A_nG and A_n/A_nH are holonomic , then $A_n/(A_n \cap A_nI)$ is holonomic where I is the output of Procedure 3.1 or 3.2 and G and H are inputs.*

4. The zero recognition problem

In this section, we describe the final part of Zeilberger's (1990) zero recognition algorithm, and show an example by using our algorithms of section two and three.

Let $f_1, \cdots, f_\ell$ be holonomic functions and $\wp_1, \cdots, \wp_\ell$ be ideals that annihilate these functions. $\mathcal{P}_i$ denotes a functional with one or two arguments. The functional is an operation of sum or product or integration of functions. Let f be the holonomic function which is constructed by applications of $\mathcal{P}_1, \mathcal{P}_2, \cdots$ to $f_1, \cdots, f_\ell$ in the sequel. We will present a quasi-algorithm to decide whether f is identically zero or not.

EXAMPLE 4.1. Let $J_0(z)$ be the Bessel function of order 0. Put

$$f_1 = e^{-ax}, f_2 = J_0(bx), f_3 = (a^2 + b^2)^{-1/2}.$$

We define

$$\mathcal{P}_1(g, h) = gh, \mathcal{P}_2(g) = \int_0^\infty g dx, \mathcal{P}_3(g, h) = g - h.$$

Then we have

$$\int_0^\infty e^{-ax} J_0(bx) dx - (a^2 + b^2)^{-1/2} = \mathcal{P}_3(\mathcal{P}_2(\mathcal{P}_1(f_1, f_2)), f_3).$$

In the sequel, we assume that K is the field of complex numbers. It is well known that a holonomic function in A_n is a multivalued holomorphic function on $\mathbb{C}^n \setminus S$. S is an algebraic variety. If the holonomic function is annihilated by an ideal $\wp$, then S can be computed from $\wp$. The singularity S is denoted by $Sing(\wp)$. The method of the computation of $Sing(\wp)$ is well known, but we summarize it.

PROCEDURE 4.1. INPUT: Gröbner basis G of the ideal $\wp$ in $\mathcal{A}_n$.

(1) Let $\{(\partial_x)^\alpha | \alpha \in Q\}$ be representatives of $\mathcal{A}_n/(\mathcal{A}_n\wp)$.

(2) $\sum_{\beta \in Q} a_{i\alpha\beta}(\partial_x)^\beta$ be the normal form of $\partial_{x_i}(\partial_x)^\alpha$ by the basis G, where $\alpha \in Q$ and $a_{i\alpha\beta} \in K(x)$.

(3) $Sing(\wp)$:= the zeros of the polynomial which is the least common multiple of the denominators of $a_{i\alpha\beta}, 1 \le i \le n,\ \alpha, \beta \in Q$.

OUTPUT: $Sing(\wp)$.

If f is annihilated by an ideal $\wp$ and $x_0 \notin Sing(\wp)$, then it follows from the Cauchy's existence theorem of a solution of ordinary differential equation with parameters that the solution g of the differential equation $\wp g = 0$ is uniquely determined by $k(\wp)$ initial conditions at x_0. Therefore we have the following zero recognition algorithm.

ALGORITHM 4.1. (Quasi-algorithm for zero recognition problem, Zeilberger (1990)).
INPUT: $\wp_1, \cdots, \wp_\ell, f_1, \cdots, f_\ell, \mathcal{P}_1, \cdots, \mathcal{P}_q, f$

(1) Compute an ideal $\wp$ that annihilates the function f by calling Algorithm 2.1, Procedure 3.1 and 3.2.

(2) Compute $Sing(\wp)$ by calling Procedure 4.1.

(3) Select a point $x_0 \notin Sing(\wp)$.

(4) Let $\partial_x^\alpha, \alpha \in Q$ be representatives of $\mathcal{A}_n/(\mathcal{A}_n\wp)$.

(5) If $(\partial_x^\alpha f)(x_0) = 0$ for all $\alpha \in Q$ then $f \equiv 0$
else if $(\partial_x^\alpha f)(x_0) \neq 0$ for an $\alpha \in Q$ then $f \not\equiv 0$
else a failure (if we cannot prove either case).

EXAMPLE 4.2. This is the continuation of Example 4.1. Put

$$f = \mathcal{P}_3(\mathcal{P}_2(\mathcal{P}_1(f_1, f_2)), f_3).$$

We will prove

$$f \equiv 0$$

by Algorithm 4.1. A prototype system on REDUCE 3.3 computes this example.

First, we sum up the inputs to Algorithm 4.1. We put

$$\begin{aligned}
\wp_1 &= (\partial_x + a, \partial_a + x, \partial_b), \\
\wp_2 &= (b\partial_b^2 + \partial_b + x^2 b, x\partial_x^2 + \partial_x + xb^2, \partial_a), \\
\wp_3 &= ((a^2+b^2)\partial_a + a, (a^2+b^2)\partial_b + b).
\end{aligned}$$

$\wp_i$ annihilates f_i. The generators of the ideals are immediate consequence of the definitions of the exponential , Bessel and $\sqrt{\ }$ functions. $\wp_1$ and $\wp_2$ are ideals of $K(a,b,x)\langle\partial_a, \partial_b, \partial_x\rangle$ and $\wp_3$ is the ideal of $K(a,b)\langle\partial_a, \partial_b\rangle$.

We call Procedure 3.2 to compute an ideal which annihilates $\mathcal{P}_1(f_1, f_2)$. The ideal is generated by

$$\left\{ \begin{array}{l}
x\partial_x^2 + (2ax+1)\partial_x + (a^2+b^2)x + a \\
\partial_a + x \\
b\partial_b^2 + \partial_b + x^2 b
\end{array} \right. \tag{4.1}$$

We call Algorithm 2.1 . (4.1) is the input. We consider the following ring to avoid the tiresome notation:

$$K\langle a, b, x, \partial_a, \partial_b, \partial_x\rangle.$$

We apply Algorithm 2.1 by putting $n = 2, m = 1, x_1 = a, x_2 = b$ and $t_1 = x$. A Gröbner basis of (4.1) by the order (2.3) is

$$-a\partial_a\partial_x + b\partial_b\partial_x - (a^2+b^2)\partial_a - a, \tag{4.2}$$

$$\partial_a + x, \tag{4.3}$$

$$b(\partial_a^2 + \partial_b^2) + \partial_b, \tag{4.4}$$

$$a\partial_a + b\partial_b + \partial_a\partial_x + 1. \tag{4.5}$$

Then G_1 of Algorithm 2.1 is

$$\{(4.2),(4.4),(4.5)\}. \tag{4.6}$$

$$\int_0^\infty \partial_x(e^{-ax}J_0(bx))dx = \int_{\partial[0,\infty]} e^{-ax}J_0(bx)dx = -1, a > 0,$$

then the condition of the step (4) of Algorithm 2.1 is not satisfied, so we call Algorithm 2.2 in twice. The first input of Algorithm 2.2 is $\wp_1 = (\partial_a, \partial_b)$ and $p = \partial_a$. $H = \{1\}$ is the output. The second input is $\wp_1 = (\partial_a, \partial_b)$ and $p = \partial_b$. $H = \{1\}$ is the output. Therefore

$$\left\{\begin{array}{lcl} (4.2)_{|\partial_x \to 0} & = & (a^2+b^2)\partial_a + a, \\ (4.4)_{|\partial_x \to 0} & = & b(\partial_a^2 + \partial_b^2) + \partial_b, \\ (4.5)_{|\partial_x \to 0} & = & a\partial_a + b\partial_b + 1 \end{array}\right. \tag{4.7}$$

is an ideal that annihilates the function $\mathcal{P}_2(\mathcal{P}_1(f_1, f_2))$. (4.7) is the output of Algorithm 2.1. A Gröbner basis of (4.7) in $K(a,b)\langle\partial_a, \partial_b\rangle$ by the total degree order is

$$\left\{\begin{array}{l} (a^2+b^2)\partial_a + a, \\ a\partial_a + b\partial_b + 1. \end{array}\right. \tag{4.8}$$

We call Procedure 3.1 to obtain an ideal which annihilates $\mathcal{P}_3(\mathcal{P}_2(\mathcal{P}_1(f_1, f_2))$. The ideal is also generated by (4.8) . We have completed step (1) of Algorithm 4.1.

We proceed to step (2). We call Procedure 4.1 and we have

$$\left\{\begin{array}{ll} \partial_a & \longrightarrow_{(4.8)} \frac{-a}{(a^2+b^2)}, \\ \partial_b & \longrightarrow_{(4.8)} \frac{-b}{(a^2+b^2)}. \end{array}\right.$$

Therefore

$$Sing((4.8)) = \{(a,b) | a^2 + b^2 = 0\}.$$

We select $x_0 = (1,0) \notin Sing((4.8))$. We have

$$\begin{array}{ll} & \mathcal{P}_3(\mathcal{P}_2(\mathcal{P}_1(f_1, f_2))(1,0) - 1/\sqrt{1^2+0^2} \\ = & \int_0^\infty e^{-x}J_0(0)dx - 1/\sqrt{1} \\ = & \int_0^\infty e^{-x}dx - 1 = 0. \end{array} \tag{4.9}$$

Therefore we conclude that $f \equiv 0$.

We remark that the step (4) of Algorithm 4.1 ((4.9) in Example 4.2) needs , for example , a formula database for special functions and an integration system.

5. A fast elimination algorithm of t

We describe an acceleration of step (1) of Algorithm 2.1 and a method to save memory. The lexico-total degree order uses much time and much memory. We can obtain a Gröbner basis with lexicographic order from the Gröbner basis with total degree order by solving a linear indefinite equation in the case of a zero dimensional ideal of the polynomial ring (Buchberger, 1985). We use a modification of the method in this section, however it follows from Bernstein's inequality that any non-trivial ideal of the Weyl algebra is not zero dimensional. We overcome this difficulty by a localization with respect to variables x and ∂_t.

Consider a ring:

$$\hat{A}_{m+n} = K(x, \partial_t)\langle \partial_x, t\rangle.$$

We have a relation:

$$t_i f(\partial_{t_i}) = f(\partial_{t_i})t_i - \dot{f}(\partial_{t_i}),\ f \in K(\partial_t).$$

We reduce the problem of obtaining $\wp \cap K\langle x, \partial_x, \partial_t\rangle$ to obtaining

$$\wp \cap K(x, \partial_t)\langle \partial_x\rangle. \tag{5.1}$$

We can use a modification of the method 6.11 of Buchberger (1985) in the ring $\hat{A}_{m+n}$.

We assume that monomials of $\hat{A}_{m+n}$ are ordered as

$$\partial_x^\alpha t^\beta \succ \partial_x^\gamma t^\delta \Longleftrightarrow (\alpha, \beta) \succ (\gamma, \delta) \ \ \text{(total degree order in } \mathrm{N}_0^{n+m}). \tag{5.2}$$

We remark that $K(x, \partial_t)$ is the coefficient field. We can define the notion of Gröbner basis in the ring $\hat{A}_{m+n}$ (See Takayama (1989)).

REMARK 5.1. The Fourier transforms F and F^{-1} are defined as follows.

$$F : \begin{array}{ll} t & \longrightarrow -i\partial_t \\ \partial_t & \longrightarrow -it \end{array}, \quad F^{-1} : \begin{array}{ll} t & \longrightarrow i\partial_t \\ \partial_t & \longrightarrow it. \end{array}$$

(5.1) is reduced to obtaining

$$F(\wp) \cap K(x, t)\langle \partial_x\rangle.$$

This is called the ODE section problem (See Takayama (1989,2)). A fast algorithm for obtaining $F(\wp) \cap K(t, x)\langle \partial_x\rangle$ was given in Takayama (1989,2) by assuming that $F(f)$ is hypergeometric function (c.f. Zeilberger (1991)). The algorithm of this paper can be applied to the ODE section problem.

Suppose that $A_{n+m}/\wp$ is holonomic, then $\hat{A}_{m+n}\wp$ is the zero dimensional ideal of $\hat{A}_{m+n}$, i.e., $k(\hat{A}_{m+n}\wp) < +\infty$. We assume that $A_{n+m}/\wp$ is holonomic and a function f_0 is annihilated by $\wp$ in the sequel.

Let f and g be elements of $\hat{A}_{m+n} \cap A_{n+m}$. We put

$$\text{head}(f) = \text{head term of } f \text{ by the order (5.2)} = f_{\alpha\beta}(x, \partial_t)\partial_x^\alpha t^\beta,\ f_{\alpha\beta} \in K[x, \partial_t],$$

$$\text{head}(g) = \text{head term of } g \text{ by the order (5.2)} = g_{\gamma\delta}(x, \partial_t)\partial_x^\gamma t^\delta,\ g_{\gamma\delta} \in K[x, \partial_t],$$

$$\xi = \text{lcm}(\alpha, \gamma) \ \text{ and } \ \eta = \text{lcm}(\beta, \delta)$$

where

$$\mathrm{lcm}(\alpha,\gamma) = (\max(\alpha_1,\gamma_1),\cdots,\max(\alpha_n,\gamma_n)).$$

DEFINITION 5.1.

$$\mathrm{sp}(f,g) = \frac{\mathrm{lcm}(f_{\alpha\beta},g_{\gamma\delta})}{f_{\alpha\beta}}\partial_x^{\xi-\alpha}t^{\eta-\beta}f - \frac{\mathrm{lcm}(f_{\alpha\beta},g_{\gamma\delta})}{g_{\gamma\delta}}\partial_x^{\xi-\gamma}t^{\eta-\delta}g.$$

DEFINITION 5.2.

$$f \longrightarrow h \text{ by } g \quad (f \text{ can be reduced to } h \text{ by } g)$$

$\Longleftrightarrow \mathrm{lcm}(\alpha,\gamma) = \alpha,\ \mathrm{lcm}(\beta,\delta) = \beta$ and $h = \mathrm{sp}(f,g)$.

We construct a Gröbner basis for $\hat{A}_{m+n}\wp$ in $\hat{A}_{m+n}$ by using Buchberger algorithm and the total degree order (5.2) . If we use s-polynomial and reduction as defined by 5.1 and 5.2 in Buchberger algorithm, then it follows from $\mathrm{sp}(f,g) \in A_{n+m}$, $f,g \in A_{n+m}$ that the obtained Gröbner basis consists of elements of $\wp \subseteq A_{n+m}$. Suppose that $\{\partial_x^\alpha t^\beta | (\alpha,\beta) \in Q\}$ is a basis of $\hat{A}_{m+n}/\hat{A}_{m+n}\wp$ as a linear space over the field $K(x,\partial_t)$. Let the Gröbner basis be $G = \{g^1,\cdots,g^p\}$ where

$$g^i = \sum_{(\alpha,\beta)\in Q} a^i_{\alpha\beta}\partial_x^\alpha t^\beta, \quad a^i_{\alpha\beta} \in K[x,\partial_t]. \tag{5.3}$$

We have $g^i \in \wp \subset A_{n+m}$. Since $A_{n+m}/\wp$ is holonomic, $\#Q < +\infty$.

Let us explain a method to obtain (5.1) . We reduce ∂_x^k by the Gröbner basis G. Suppose that

$$\partial_x^k \longrightarrow h^1 \longrightarrow \cdots \longrightarrow h^{q+1} = c^k$$

where

$$c^k = \sum_{(\alpha,\beta)\in Q} c^k_{\alpha\beta}\partial_x^\alpha t^\beta, \quad c^k_{\alpha\beta} \in K[x,\partial_t].$$

We have

$$\begin{aligned} c^k &= \mathrm{sp}(h^q, g^{i_q}) \\ &= b^q h^q - \hat{b}^q g^{i_q}, \quad {}^\exists\hat{b}^q \in A_{n+m}, {}^\exists b^q \in K[x,\partial_t] \\ &= b^q \mathrm{sp}(h^{q-1}, g^{i_{q-1}}) - \hat{b}^q g^{i_q} \\ &\cdots \\ &= b_0^k\partial_x^k - \textstyle\sum_{i=1}^p b_i^k g^i, \quad b_0^k \in K[x,\partial_t], b_i^k \in A_{n+m}. \end{aligned}$$

Let $\hat{Q}$ be $k(\hat{A}_{m+n}\wp) + 1$ different exponents set. Since $\#\hat{Q} = \#Q + 1$, then

$$\{b_0^k\partial_x^k \equiv c^k \bmod \wp | k \in \hat{Q}\}$$

is linearly dependent over the field $K(x,\partial_t)$. Therefore the system of linear indefinite equations:

$$\sum_{k\in\hat{Q}} d_k c^k_{\alpha\beta} = 0, \quad (\alpha,\beta) \in Q \tag{5.4}$$

has a non-trivial solution $d_k \in K[x,\partial_t], k \in \hat{Q}$. Since $g^i \in \wp$, we have

$$e_{\hat{Q}} := \sum_{k\in\hat{Q}} d_k b_0^k \partial_x^k \in \wp \cap K\langle x,\partial_t,\partial_x\rangle. \tag{5.5}$$

If $e_{\hat{Q}} = \partial_t^{\epsilon} \hat{e}_{\hat{Q}}$ and $\hat{e}_{\hat{Q}} f_0 = 0$, we put

$$e_{\hat{Q}} = \hat{e}_{\hat{Q}}. \tag{5.6}$$

We remark that if $\hat{e}_{\hat{Q}} \longrightarrow^* 0$ by a Gröbner basis of $\wp$ in the Weyl algebra A_{n+m}, then $\hat{e}_{\hat{Q}} f_0 = 0$.

Let I be a monoideal of N_0^n. $\#(\mathrm{N}_0^n \setminus I)$ is denoted by $k(I)$. $G(I)$ is the set of minimum generators of the monoideal I. For any number N, the number of monoideals that satisfy the condition $k(I) = N$ is finite.

ALGORITHM 5.1. (An improvement of step (1) and (2) of Algorithm 2.1.)
(1) $G :=$ a Gröbner basis of $\hat{A}_{m+n}\wp$ in $\hat{A}_{m+n}$ by the order (5.2)
(2) Select a monoideal $I \subseteq \mathrm{N}_0^n$ such that

$$k(I) \leq k(\hat{A}_{m+n}\wp).$$

(The monoideal must not be tested yet.)
(3) for all $\gamma \in G(I)$ do
(4) $\hat{Q} := \{\gamma\} \cup (\mathrm{N}_0^n \setminus I)$
(5) Reduce ∂_x^k, $k \in \hat{Q}$ by the Gröbner basis G and obtain (5.4)
(6) Solve (5.4)
(7) if $d_\gamma = 0$ then goto (2)
(8) else obtain $e_{\hat{Q}}$ of (5.5) or (5.6)
(9) endfor
(10) $G_1 := \{e_{\hat{Q}} | \hat{Q} = \{\gamma\} \cup (\mathrm{N}_0^n \setminus I), \gamma \in G(I)\}$

We have the following Theorem for the output G_1 of Algorithm 5.1.

THEOREM 5.1. *If $A_{n+m}/\wp$ is holonomic, then $G_1 \subseteq \wp$ and G_1 generates a zero dimensional ideal of the ring $K(x, \partial_t)\langle\partial_x\rangle$,i.e., the module:*

$$K(\partial_t)\langle x, \partial_x\rangle / (K(\partial_t, x)\langle\partial_x\rangle G_1 \cap K(\partial_t)\langle x, \partial_x\rangle)$$

is holonomic $K(\partial_t)\langle x, \partial_x\rangle$ module.

PROOF. We can easily prove $G_1 \subseteq \wp$. Since $A_{n+m}/\wp$ is holonomic, $k(\hat{A}_{m+n}\wp)$ is finite. Therefore $\hat{A}_{m+n}\wp \cap K(x, \partial_t)\langle\partial_x\rangle$ is a zerodimensional ideal of $K(x, \partial_t)\langle\partial_x\rangle$ and

$$\dim_{K(x,\partial_t)} k(x, \partial_t)\langle\partial_x\rangle / \left(\hat{A}_{m+n}\wp \cap K(x, \partial_t)\langle\partial_x\rangle\right) \leq k(\hat{A}_{m+n}\wp).$$

Hence there exists a monoideal I such that $k(I) \leq k(\hat{A}_{m+n}\wp)$ and $d_\gamma \neq 0$, $^\forall\gamma \in G(I)$. We have

$$\dim_{K(x,\partial_t)} k(x, \partial_t)\langle\partial_x\rangle / \left(\hat{A}_{m+n}\wp \cap K(x, \partial_t)\langle\partial_x\rangle\right) = k(I)$$

in the case. □

REMARK 5.2. It follows from Theorem 5.1 that the output G_1 of Algorithm 5.1 is a very large system of differential operators. However the output G_1 of Algorithm 5.1 is not equal to $G_0 \cap K\langle x, \partial_x, \partial_t\rangle$ in general where G_0 is a Gröbner basis of $\wp$ in A_{n+m} by the lexico-total degree order. However we have $G_1 f_0 = 0$. Therefore we can obtain differential operators that annihilate the integral of f_0. The algorithm works well. We want to prove that $A_n/A_n(G_{1|_{\partial_t \to 0}})$ is holonomic, but we cannot prove and cannot find a counter example.

EXAMPLE 5.1. We consider a problem in A_{1+1}. Put $t = t_1$ and $x = x_1$. Suppose that a left ideal $\wp$ is generated by

$$f_1 = t^2 + x^2 \text{ and } f_2 = x\partial_x + t\partial_t.$$

The Gröbner basis of $\wp$ in A_{1+1} by the lexicographic order $t \succ \partial_t \succ x \succ \partial_x$ is

$$f_1, f_2, -xt\partial_t + x^2\partial_t + 2t, f_3 = -x^2\partial_t^2 - x^2\partial_x^2 + 2x\partial_x - 2.$$

Then $\wp \cap K\langle x, \partial_t, \partial_x \rangle$ is generated by f_3.

We use Algorithm 5.1. Since head$(f_1) = t^2$ and head$(f_2) = x\partial_x$ by the order (5.2), then sp$(f_1, f_2) = x\partial_x f_1 - t^2 f_2 \longrightarrow^* 0$. Therefore f_1 and f_2 is a Gröbner basis of $K(x, \partial_t)\langle \partial_x, t\rangle$ by the total degree order (5.2). We have $\dim_{K(x,\partial_t)} \hat{A}_2/\hat{A}_2\wp = 2$ and $\{1, t\}$ is a basis of $\hat{A}_2/\hat{A}_2\wp$ as a linear space over $K(x, \partial_t)$. We have

$$1 \equiv 1 \text{ and } x\partial_x \equiv -t\partial_t \bmod \hat{A}_2\wp$$

and

$$x\partial_x^2 \equiv -2t\partial_t - 2 - x^2\partial_t^2 \bmod \hat{A}_2\wp.$$

The linear indefinite equations (5.4) are

$$\begin{aligned} &d_0 - d_2(2 + x^2\partial_t^2) = 0 \\ &d_1 + 2d_2 = 0. \end{aligned}$$

A solution is $d_0 = 2 + x^2\partial_t^2, d_2 = 1$ and $d_1 = -2$. The output is $d_0 + d_1 x\partial_x + d_2 x\partial_x^2 = -f_3$.

6. Appendix: An elementary proof of Theorem 2.4

Theorem 2.4 can be proved by an elementary method. As we cannot find an elementary proof in the literature, we shall state it.

Let $s_1, \cdots, s_m$ be a basis of $\mathcal{A}_n/\wp$ as a linear space over the field $K(x)$. We can assume that s_i are monomials of ∂_j. Since $\{s_i\}$ is a basis, there exists $p \in K[x]$ and $q^y_{kj} \in K[x]$, $1 \le y \le n, 1 \le k, j \le m$ such that

$$p\partial_y s_k = \sum_{j=1}^{m} q^y_{kj} s_j.$$

Since $\partial_y \partial_z s_k = \partial_z \partial_y s_k$ in $\mathcal{A}_n/\wp$, we have

$$p(\partial_z q^y_{kj}) + \sum_{i=1}^{m} q^y_{ki} q^z_{ij} + (\partial_z p) q^y_{kj} = p(\partial_y q^z_{kj}) + \sum_{i=1}^{m} q^z_{ki} q^y_{ij} + (\partial_y p) q^z_{kj}. \quad (6.1)$$

$p\partial_y s_k - \sum_{j=1}^{m} q^y_{kj} s_j$ is an element of $\wp \cap A_n$. We have

$$\begin{aligned} \partial_y(g \otimes s_k) &= (\partial_y g) \otimes s_k + g \otimes (\partial_y s_k) \\ &= (\partial_y g) \otimes s_k + g p^{-1} \otimes p\partial_y s_k \\ &= (\partial_y g) \otimes s_k + g p^{-1} \otimes \textstyle\sum_{j=1}^{m} q^y_{kj} s_j \end{aligned}$$

in $K[x, p^{-1}] \otimes_{K[x]} (A_n/\wp \cap A_n)$. The above leads us to the following A_n module structure of

$$M = \sum_{k=1}^{m} K[x, p^{-1}] s_k.$$

An element f of M is denoted by $(f_1, \cdots, f_m)$ where

$$f = \sum_{k=1}^{m} f_k s_k, \ f_k \in K[x, p^{-1}].$$

Assume that

$$M \ni f = (f_1, \cdots, f_m), \ g = (g_1, \cdots, g_m).$$

We define $f + g = (f_1 + g_1, \cdots, f_m + g_m)$. For $x_i \in A_n$, we define $x_i f = (x_i f_1, \cdots, x_i f_m)$. Assume $f = (f_1, \cdots, f_m) p^{-v}$. We define

$$\text{k}-\text{th component of } \partial_y f = \partial_y (f_k p^{-v}) + p^{-v-1} \sum_{j=1}^{m} f_j q_{jk}^y.$$

PROPOSITION 6.1. *M is left A_n module.*

We can prove Proposition 6.1 by using (6.1) .
Put

$$T = \max_{1 \le j,k \le m, \ 1 \le y \le n} \{\deg p + 1, \deg q_{jk}^y\}$$

and

$$\Omega_v = \{p^{-v}(f_1, \cdots, f_m) | \deg f_i \le vT\}.$$

Since $\deg p + 1 \le T$, then $x_i \Omega_v \subseteq \Omega_{v+1}$. Let f be $p^{-v}(f_1, \cdots, f_m)$, $\deg f_i \le vT$. The k-th component of $\partial_y f$ is equal to

$$p^{-v-1}[(\partial_y f_k) p + v f (\partial_y p) + \sum_{j=1}^{m} f_j q_{jk}^y].$$

Since $\max\{vT - 1 + \deg p, vT + \deg p - 1, vT + T\} \le (v+1)T$, we conclude that $\partial_y \Omega_v \subseteq \Omega_{v+1}$. Therefore Ω_v is a filtration of M. It follows from $\dim_K \Omega_v = m \cdot \binom{vT+n}{n}$ that M is a holonomic A_n module.

Consider the map:

$$\varphi : M \ni (f_1, \cdots, f_m) \longmapsto \sum_{i=1}^{m} f_i \otimes s_i \in K[x, p^{-1}] \otimes_{K[x]} A_n / (A_n \cap \wp).$$

The map φ is an A_n homomorphism and surjective. Therefore $K[x, p^{-1}] \otimes_{K[x]} A_n / (A_n \cap \wp)$ is holonomic by the filtration $\varphi(\Omega_v)$.

Acknowledgement. The author thanks to Professor M. Kashiwara who spent his precious time answering to my questions and to the editor for correcting grammatical errors.

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