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Arithmetic properties of power series solutions of algebraic differential equations

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0. Introduction

In our previous work [13], we proved that a formal power-series solution of a certain p-adic non-linear differential equation converges p-adically in some non-trivial disk. We considered such a problem at a regular singular point and also at an irregular singular point. The problem at a regular singular point was more difficult. To deal with this difficulty, we utilized a Newton iteration procedure. This procedure was originally used by J. Moser and V. I. Arnol'd to deal with small divisors in celestial mechanics (cf. S. Sternberg [14]).

In the present work, we improve our previous results so that they imply the following Main Theorem (cf. Chapter 4):

Let $\bar{\mathbf{Q}}$ be the algebraic closure of the rational numbers. Let $y = \sum_{m=0}^{\infty} a_m x^m$ ($a_m \in \bar{\mathbf{Q}}$) be a formal solution of an algebraic differential equation. Then y has a positive ν-adic radius of convergence for every non-archimedean valuation $|\ |_\nu$ of $\bar{\mathbf{Q}}$.

Clearly this result provides a sequence of necessary conditions (one for each ν) for a power series y to satisfy an algebraic differential equation, i.e., $F(x, y, \cdots, y^{(n)}) = 0$ for some polynomial in $n + 2$ variables. An algebraic equation is viewed as a zero-th order algebraic differential equation. Thus, analytic functions which do not satisfy any algebraic differential equations are called transcendentally transcendental functions (or hypertranscendental functions).

O. Hölder [4] showed that the gamma function was transcendentally transcendental. A. Hurwitz [5], by an argument similar to H. Heine's proof [3] of G. Eisenstein's theorem on power-series expansions of algebraic functions, proved that if $y = \sum_{m=0}^{\infty} a_m x^m \in \mathbf{Q}[[x]]$ satisfies an algebraic differential

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equation then from some index N onward the primes in the denominator of a_m $(m \geqq N)$ must also occur as factors of $f(N)f(N+1)\cdots f(m)$ for a fixed polynomial f with rational integer coefficients. It follows from this result that if p_m denotes the largest prime in the denominator of a_m, then $\limsup_{m\to\infty} \log p_m/\log m$ is finite. G. Pólya [10] refined Hurwitz' argument and showed that if a transcendental entire function $y = \sum_{m=0}^{\infty} a_m x^m$ $(a_m \in \mathbf{Q})$ satisfies an algebraic differential equation then

$$\limsup_{m\to\infty} \frac{\log|a_m|}{m(\log m)^2} \quad \text{and} \quad \limsup_{m\to\infty} \frac{\log(\operatorname{den}(a_m))}{m(\log m)^2}$$

are finite, where $\operatorname{den}(a_m)$ denotes the denominator of a_m $(\operatorname{den}(0) = 1)$. In fact Pólya proved that if $y = \sum_{m=0}^{\infty} a_m x^m \in Q[[x]]$ satisfies an algebraic differential equation then

$$\text{(I)} \qquad 1 \leqq \operatorname{den}(a_m) \leqq e^{cm(\log m)^2} \qquad \text{for all} \quad m > 0$$

where c is a constant. Since $|a_m \operatorname{den}(a_m)| \geqq 1$ if $a_m \neq 0$, it follows that

$$\text{(II)} \qquad |a_m| \geqq e^{-cm(\log m)^2} \qquad \text{if} \quad a_m \neq 0\,.$$

If $a_m \in \bar{\mathbf{Q}}$ but not necessarily in $\mathbf{Q}$, then we need an upper estimate for $|a_m|$ in order to derive (II) from (I). In fact, if $y = \sum_{m=0}^{\infty} a_m x^m \in \bar{\mathbf{Q}}[[x]]$ satisfies an algebraic differential equation, then

$$\text{(III)} \qquad |a_m| \leqq e^{c'm(\log m)} \qquad \text{for all} \quad m > 0$$

where c' is a constant. Indeed, J. Popken [11] and K. Mahler [8] refined the Hurwitz-Pólya argument to establish (I), (II) and (III) for a power series with coefficients in $\underline{\mathbf{Q}}$. This was employed by them [12] to establish transcendence results for certain values of elliptic functions. Mahler [9] also proved a p-adic version of (II) where $|a_m|$ is replaced by $|a_m|_\nu$. A p-adic version of (III) is implicit in Mahler [8]. (We were directed to this result through an unpublished note of B. Dwork.) Our main result implies that if $y = \sum_{m=0}^{\infty} a_m x^m \in \bar{\mathbf{Q}}[[x]]$ satisfies an algebraic differential equation then, for every non-archimedean valuation $|\ \ |_\nu$ of $\bar{\mathbf{Q}}$, we have

$$\text{(III}'\text{)} \qquad |a_m|_\nu \leqq e^{c''m} \qquad \text{for all} \quad m > 0$$

where c'' is a constant which may depend on ν. The estimate (III′) was conjectured by Dwork.

In terms of proving functions transcendentally transcendental, the following chart summarizes the strength of the various results:

	Example of functions proved transcendentally transcendental
Hurwitz	$\sum_{m=0}^{\infty} \frac{x^m}{(m^m)!}$
Pólya	$\sum_{m=0}^{\infty} \frac{u_m x^m}{2^{[m(\log m)^{2+\varepsilon}]}}$
Popken-Mahler-Dwork	$\sum_{m=0}^{\infty} \frac{u_m x^m}{2^{[m(\log m)^{1+\varepsilon}]}}$
Sibuya-Sperber	$\sum_{m=0}^{\infty} \frac{u_m x^m}{2^{[m(\log m)^{\varepsilon}]}}$

where ε is an arbitrary positive number, u_m is an arbitrary odd integer, and $[\xi]$ denotes the largest integer k such that $k \leqq \xi$.

We note that although we do not discuss in the present work the effectiveness of our estimates (with varying primes ν), our results may be used to refine the archimedean estimates of Popken-Mahler. We intend to return to this question in another paper.

Our result may also be viewed as a sufficient condition for a power-series solution of a differential equation to converge p-adically. The first result in this direction was proved by E. Lutz [7] following a suggestion of A. Weil [16]. She proved that a formal power-series solution of a first order system without singularity converges p-adically in a non-trivial disk. D. N. Clark [2] proved that if the indicial polynomial of a p-adic linear differential equation has no roots which are so-called p-adic Liouville numbers then every formal power-series solution converges in a non-trivial disk. (We can construct p-adic numbers α such that $\sum_{m=0}^{\infty} x^m/(m-\alpha)$ diverges. Such an α is too well approximated by integers and is called a p-adic Liouville number (cf. Remark 2 following Theorem 1.2.1). Note that this divergent series satisfies the differential equation $xy' - \alpha y = (1-x)^{-1}$.) It is known that if $\alpha \in \bar{\mathbf{Q}}$ then α is not a p-adic Liouville number. Clark's result is valid even if the differential equation has singularity. We extend Clark's result to the non-linear case.

Our method is based on

(i) An estimate concerning linear systems (cf. Theorem 1.1.1) which was provided by B. Dwork;

(ii) A generalization (due to A. H. M. Levelt [6] and F. Baldassarri [1]) of the theorem of M. Hukuhara-H. Turrittin (cf. W. Wasow [15]);

(iii) A Newton iteration procedure to handle small divisors (cf. Sternberg [14]).

The results of Hurwitz, Pólya, Popken, Mahler, Dwork, Lutz, and Clark as well as the estimate due to Dwork (cf. (i)) were derived from detailed investigations of explicit recurrence formulas for coefficients of formal power-series solutions. We utilize deeper knowledge concerning the structure of differential equations.

In Chapter 1, we prove the estimate due to Dwork (cf. (i)) and three existence theorems. (These results may be regarded as the main results of the present work.) In Chapter 2, we prove three sufficient conditions for a power-series solution of a system of differential equations to converge p-adically. These three conditions correspond respectively to the three existence theorems of Chapter 1. In Chapter 3, a general non-linear system is reduced to a standard form so that the results of Chapter 2 may be utilized. In Chapter 4, an algebraic differential equation $F(x, y, \cdots, y^{(n)}) = 0$ is reduced to a system so that the results of Chapter 3 may be utilized. This completes the proof of the main theorem. Finally, in Chapter 5, we treat Pfaffian Systems.

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Notation

Ω denotes a complete field of characteristic zero with a rank one non-archimedean valuation extending the usual p-adic valuation on $\mathbf{Q}$ which will generally be written additively as ord normalized so that $\operatorname{ord} p = 1$. For convenience, we shall assume that its value group is dense in the real numbers. The associated absolute value will be normalized $|x| = p^{-\operatorname{ord} x}$.

$M_{n,m}(R) = n \times m$ matrices with coefficients in the ring R;

$M_n(R) = M_{n,n}(R)$;

$\mathfrak{D}^+(r) = \{x \in \Omega \mid \operatorname{ord} x \geqq r\}$;

$\mathfrak{D}(r) = \{x \in \Omega \mid \operatorname{ord} x > r\}$;

$A(r, \rho) =$ power-series in $\Omega[[x]]$ convergent on $\mathfrak{D}(r)$ and uniformly bounded by $p^{-\rho}$ on $\mathfrak{D}(r)$.

$= \{f \in \Omega[[x]] \mid f$ converges on $\mathfrak{D}(r)$ and satisfies $\operatorname{ord} f(x) \geqq \rho$ for $x \in \mathfrak{D}(r)\}$.

Note that if $\rho \geqq 0$, then $A(r, \rho)$ is a ring. Also, if $f = \sum_{n=0}^{\infty} f_n x^n$, observe that $f \in A(r, \rho)$ if and only if $\operatorname{ord} f_n \geqq -nr + \rho$ for all $n \geqq 0$.

$A(r^+, \rho) = \{f \in \Omega[[x]] \mid f$ converges on $\mathfrak{D}(r^+)$ and satisfies $\operatorname{ord} f(x) \geqq \rho$ for $x \in \mathfrak{D}(r^+)\}$;

$A_{n,m}(r, \rho) = M_{n,m}(A(r, \rho))$, for $\rho \geqq 0$;

$A_n(r, \rho) = M_n(A(r, \rho))$, for $\rho \geqq 0$;

$\sigma_{j,i}(a, b) = \sum_{l=j}^{j+i} (al + b)/2^l$;

$\sigma_j(a, b) = \sum_{l=j}^{\infty} (al + b)/2^l = (a(j + 1) + b)/2^{j-1}$,

in which we have used the identity

$$\sum_{l=j}^{\infty} lx^l = x \frac{d}{dx}\left(\frac{x^j}{1 - x}\right)$$

to evaluate the infinite series.

We will say that $\alpha \in \Omega$ satisfies condition A(k) with constants $(\tilde{c}_1, \tilde{c}_2)$ provided

A(k): $$\operatorname{ord}(m + \alpha) \leqq \tilde{c}_1 \log m + \tilde{c}_2\,, \qquad \text{for} \quad m \geqq 2^k\,.$$

1. Fundamental theorems

1.1. *A Theorem of B. Dwork.* Let $B(x)$ be an $n \times n$ matrix:

$$B(x) = \sum_{m=2^k}^{\infty} B_m x^m \quad (B_m \in M_n(\Omega))\,,$$

where k is a non-negative integer. Then if I denotes the $n \times n$ identity matrix, there is a unique matrix

$$Y(x) = I + \sum_{m=1}^{\infty} Y_m x^m \quad (Y_m \in M_n(\Omega))$$

such that as formal power series $xY'(x) = B(x)Y(x)$; in fact $Y(x) - I = 0(x^{2^k})$ as $x \to 0$.

THEOREM 1.1.1 (B. Dwork). *Assume that $B(x) \in A_n(r, R)$ where r and R are given real numbers, $R \geqq 0$. Then $Y(x)$ is convergent for $x \in \mathfrak{D}(r + k2^{-k} + c2^{-k})$ and $Y(x) - I \in A_n(r + k2^{-k} + c2^{-k}, R)$ for a constant c which is independent of p and k and may be chosen $c = 1.2$.*

To prove this theorem, we use the following lemma.

LEMMA 1.1.2 (B. Dwork). *Assume that r and υ are integers such that $r \geqq 0$ and $\upsilon \geqq 1$. Let $0 < m_1 < m_2 < \cdots < m_\upsilon = m$ be a strictly increasing sequence of positive integers. Then*

$$\frac{m}{p^r(p - 1)} + r\upsilon \geqq \operatorname{ord}(m_1 m_2 \cdots m_\upsilon) + \frac{S(m)}{p - 1}\,, \tag{1.1.1}$$

where if we write $m = a_0 + a_1 p + \cdots + a_s p^s$, then

$$S(m) = a_0 + a_1 + \cdots + a_s\,.$$

Proof. We proceed by induction on r and υ. More precisely we will show the assertion to be true if $r = 0$ or if $\upsilon = 1$. In addition if $r > 0$, $\upsilon > 1$, the lemma will hold provided it holds for all smaller values of r or υ.

It is convenient to prove the reduction step first. Fix $r > 0$, $\upsilon > 1$.

Case (i): If $p \nmid m_i$ for some $i<\upsilon$, then by use of the induction hypothesis on υ:

$$\operatorname{ord}(m_1 m_2 \cdots m_\upsilon) + \frac{S(m)}{p-1} = \operatorname{ord}(m_1 \cdots \hat{m}_i \cdots m_\upsilon) + \frac{S(m)}{p-1}$$
$$\leqq \frac{m}{p^r(p-1)} + r(\upsilon - 1) \leqq \frac{m}{p^r(p-1)} + r\upsilon ,$$

where the symbol ^ over a factor in a product denotes its deletion from the product.

Case (ii): If $p \nmid m_\upsilon$, we observe that

$$\operatorname{ord}(m_1 \cdots m_\upsilon) + \frac{S(m)}{p-1} = \operatorname{ord}(m_1 \cdots m_{\upsilon-1}) + \frac{S(m_{\upsilon-1})}{p-1} + \frac{S(m)}{p-1} - \frac{S(m_{\upsilon-1})}{p-1}$$
$$\leqq \frac{m_{\upsilon-1}}{p^r(p-1)} + r(\upsilon-1) + \frac{S(m)}{p-1} - \frac{S(m_{\upsilon-1})}{p-1}$$

by induction. Hence, to complete Case (ii), it remains to show that

$$\frac{S(m) - S(m')}{p-1} \leqq \frac{m - m'}{p^r(p-1)} + r$$

whenever $m > m' > 0$.

Set $m = m' + b$, $b > 0$. In terms of b, we want

$$\frac{b}{p^r(p-1)} + r \geqq \frac{S(m'+b) - S(m')}{p-1} .$$

Since $\binom{m'+b}{b} = (m'+b)!/m'!b! \in \mathbf{Z}$, we get

$$\operatorname{ord}\big((m'+b)!\big) \geqq \operatorname{ord}(m'!) + \operatorname{ord}(b!) ,$$

or

$$\frac{(m'+b) - S(m'+b)}{p-1} \geqq \frac{m' - S(m')}{p-1} + \frac{b - S(b)}{p-1} .$$

This yields $S(m') + S(b) \geqq S(m'+b)$, or

$$S(b) \geqq S(m'+b) - S(m') .$$

Hence, it suffices to show

$$\frac{b}{p^r(p-1)} + r \geqq \frac{S(b)}{p-1} . \tag{1.1.2}$$

Set $b = a_0 + a_1 p + \cdots + a_l p^l$. We can write $b = B + p^r N$, where

$$B = a_0 + a_1 p + \cdots + a_{r-1}p^{r-1}\,, \quad 0 \leqq a_i \leqq p-1\,, \qquad \text{and}$$
$$N = a_r + a_{r+1}p + \cdots + a_l p^{l-r}\,.$$

(Of course, if $r > l$, $N = 0$.) Then inequality (1.1.2) may now be phrased:

$$\frac{B + p^r N}{p^r(p-1)} + r \geqq \frac{S(B) + S(N)}{p-1}\,; \tag{1.1.3}$$

note that $S(b) = S(B) + S(N)$. Since $S(B) \leqq (p-1)r$, we have

$$r \geqq \frac{S(B)}{p-1}\,. \tag{1}$$

We see also $N \geqq S(N)$, and hence

$$\frac{p^r N}{p^r(p-1)} \geqq \frac{S(N)}{p-1}\,. \tag{2}$$

Now (1) + (2) implies (1.1.3).

Case (iii): If $p \mid m_i$ ($i = 1, 2, \cdots, \upsilon$), set $m_i = pn_i$. Then

$$\begin{aligned} \operatorname{ord}(m_1 \cdots m_\upsilon) + \frac{S(m_\cdot)}{p-1} &= \operatorname{ord}(n_1 \cdots n_\upsilon) + \frac{S(n_\cdot)}{p-1} + \upsilon \\ &\leqq \frac{n_\cdot}{p^{r-1}(p-1)} + (r-1)\upsilon + \upsilon = \frac{m_\cdot}{p^r(p-1)} + r\upsilon \end{aligned}$$

where the inequality arises by induction on r; note that $S(m_\cdot) = S(n_\cdot)$.

Case (iv): If $r = 0$, we get

$$\frac{m - S(m)}{p-1} = \operatorname{ord}(m!) \geqq \operatorname{ord}(m_1 \cdots m_\upsilon)\,.$$

Therefore, $m/(p-1) \geqq \operatorname{ord}(m_1 \cdots m_\upsilon) + S(m)/(p-1)$, as required.

Case (v): If $\upsilon = 1$, we need to show

$$\frac{m}{p^r(p-1)} + r \geqq \operatorname{ord} m + \frac{S(m)}{p-1} \qquad \text{for all} \quad r \geqq 0\,.$$

This is clear for $r = 0$ as above. Assume $r > 0$. If $p \nmid m$, then this is nothing more nor less than inequality (1.1.2), since $\operatorname{ord} m = 0$. If $p \mid m$, write $m = pn$. Then we need to prove

$$\frac{n}{p^{r-1}(p-1)} + r \geqq \operatorname{ord} n + 1 + \frac{S(n)}{p-1}\,.$$

Hence we can reduce to case $r - 1$, and by induction on r the case $\upsilon = 1$ is handled.

Q.E.D.

COROLLARY 1.1.3. *Under the hypotheses of Lemma* 1.1.2, *we have*

$$\frac{m}{p^r(p-1)} + r\upsilon \geqq \operatorname{ord}(m_1 \cdots m_\upsilon)\,.$$

Proof of Theorem 1.1.1. By the Peano-Baker iteration scheme, the matrix $Y(x)$ of Theorem 1.1.1 is given by

$$Y = I + D^{-1}x^{-1}B + D^{-1}x^{-1}BD^{-1}x^{-1}B + \cdots ,$$

where D^{-1} denotes integration. Hence for $m \geqq 1$,

$$Y_m = \sum_{m_1+\cdots+m_l=m} \frac{B_{m_1}B_{m_2}\cdots B_{m_l}}{m_1(m_1+m_2)\cdots(m_1+\cdots+m_l)} .$$

Since $B(x) \in A_n(r, R)$, $\operatorname{ord} B_m \geqq -mr + R$. Therefore,

$$\operatorname{ord} Y_m \geqq -mr + R - \sup_{\upsilon\geq 1} \sup\{\operatorname{ord}\left(m_1(m_1+m_2)\cdots(m_1+\cdots+m_\upsilon)\right)\}$$

where the inner supremum runs over all υ-tuples of integers $(m_1, m_2, \cdots, m_\upsilon)$ with $m_i \geqq 2^k$ and $m = m_1 + m_2 + \cdots + m_\upsilon$. As a consequence $m \geqq 2^k \cdot \upsilon$. By Corollary 1.1.3, if $f(x) = \left(m/p^x(p-1)\right) + x\upsilon$, then

$$\operatorname{ord}\left(m_1(m_1+m_2)\cdots(m_1+m_2+\cdots+m_\upsilon)\right) \leqq f(t)$$

for all non-negative integers t. Define $\tau = \log(m/\upsilon)/\log p$, a positive real number, and set $t_0 = [\tau]$, its integral part. Then t_0 is a non-negative integer since $m \geqq 2^k\upsilon$. Let $\langle\tau\rangle$ denote the fractional part of the real number τ, so that $\tau = t_0 + \langle\tau\rangle$ and let $h(x) = \left(p^x/(p-1)\right) - x$. Then

$$\begin{aligned} f(t_0) &= \upsilon\left(t_0 + \frac{m}{\upsilon p^{t_0}(p-1)}\right) \\ &= \upsilon\left(\tau + h(\langle\tau\rangle)\right) , \end{aligned}$$

since $p^{\langle\tau\rangle} = (m/\upsilon)(1/p^{t_0})$. Therefore,

$$\operatorname{ord} Y_m \geqq -mr + R - \sup_{m/\upsilon\geq 2^k}\{\upsilon\left(\tau + h(\langle\tau\rangle)\right)\} .$$

Observe that $h(x) \leqq 1/(p-1)$ for $x \in [0, 1]$ since $h(0) = h(1) = 1/(p-1)$ and $h(x)$ is real analytic with a single critical value x_0, $p^{x_0} = (p-1)/\log p$, at which h is a minimum. Thus, if we set

$$g(\lambda) = \frac{1}{\lambda}\left(\frac{\log\lambda}{\log p} + \frac{1}{p-1}\right) ,$$

then

$$\operatorname{ord} Y_m \geqq -mr + R - m\sup_{\lambda\geq 2^k} g(\lambda) .$$

Observe that $g'(\lambda) < 0$ for $\lambda > e$, so that if $k \geqq 2$,

$$\sup_{\lambda\geq 2^k} g(\lambda) = k2^{-k}\frac{\log 2}{\log p} + \frac{2^{-k}}{p-1} \leqq (k+1)2^{-k} .$$

Furthermore $g(\lambda)$ is real analytic for $\lambda > 0$ with a simple critical value, a maximum, at

$$\lambda_0 = \exp\left(1 - \frac{\log p}{p-1}\right) > 1 .$$

Thus

$$\sup_{\lambda\geq 1}\{g(\lambda)\} = g(\lambda_0) = \frac{1}{e}\cdot\frac{1}{\log p}\cdot p^{1/(p-1)}$$
$$\leqq \frac{2}{e\log 2} < 1.1\ .$$

Therefore, for arbitrary $k \geqq 0$,

$$\sup_{\lambda\geq 2^k} g(\lambda) \leqq (k + 1.2)2^{-k}\ .$$

This completes the proof of the theorem.

1.2. *A special case.* In this section we consider a system of differential equations

(1.2.1) $$xy' + \alpha y = b(x) + B(x)y + F(x, y)\ ,$$

where $\alpha \in \Omega$, $y \in \Omega^n$ is an n-vector whose entries are $y_1, \cdots, y_n$ and

(i) $b(x) = \sum_{m=2^k}^{\infty} b_m x^m$ with $b_m \in \Omega^n$;

(ii) $B(x) = \sum_{m=2^k}^{\infty} B_m x^m$ with $B_m \in M_n(\Omega)$;

(iii) $F(x, y) = \sum_{|l|=2}^{\infty} F_l(x) y^l$ where $l = (l_1 \cdots l_n)$ with $l_j \in \mathbf{Z}_+$, $y^l = y_1^{l_1} y_2^{l_2} \cdots y_n^{l_n}$, $|l| = l_1 + l_2 + \cdots + l_n$, and $F_l(x) = \sum_{m=0}^{\infty} F_{l,m} x^m$ with $F_{l,m} \in \Omega^n$.

We will say that α satisfies condition A(k) with constants $(\tilde{c}_1, \tilde{c}_2)$ provided

A(k): $$\operatorname{ord}(m + \alpha) \leqq \tilde{c}_1 \log m + \tilde{c}_2\ , \qquad \text{for} \quad m \geqq 2^k\ .$$

The purpose of this section is to prove the following result

THEOREM 1.2.1. *Assume that α satisfies condition* A(k) *with constants* $(\tilde{c}_1, \tilde{c}_2)$. *Let* $c_1 = (\log 9/3)\tilde{c}_1$, $c_2 = \tilde{c}_2$. *Assume that* $b(x) \in A(r, \rho_0)^n$, $B(x) \in A_n(r, \rho_1)$, *and* $F_l(x) \in A(r, 0)^n$ *for all l where r is a given real number and ρ_0 and ρ_1 are given non-negative numbers. Then system* (1.2.1) *has a unique formal solution of the form*

(1.2.2) $$y = \varphi(x) = \sum_{m=2^k}^{\infty} \varphi_m x^m \qquad \textit{with} \quad \varphi_m \in \Omega^n$$

and $\varphi(x)$ has a non-trivial p-adic radius of convergence; in fact, if c is the constant of Theorem 1.1.1 *then*

$$\varphi(x) \in A\big(r + \sigma_k(1 + c_1, c + c_2), \rho_0\big)^n\ .$$

Remarks. (1) The assumption on $F(x, y)$ implies that it is convergent for $\operatorname{ord} x > r$ and $\operatorname{ord} y_j > 0$, $(j = 1, 2, \cdots, n)$.

(2) The assumption on α is crucial. Without it, counterexamples can be constructed even in the linear case (cf. Introduction). Clark [2] defines a p-adic non-Liouville number $\alpha \in \Omega$ by

$$\operatorname{ord}(m + \alpha) = 0(\log m)$$

as $m \to \infty$. It follows from this definition that elements of Ω which are algebraic over $\mathbf{Q}$ are p-adically non-Liouville. Clearly any p-adic non-Liouville number satisfies condition A(k) for an appropriate choice of constants.

Since αI is in the center of $M_n(\Omega)$, our proof of Theorem 1.2.1 is essentially identical to the scalar case (Theorem 2, [13]). We remark that in our previous work, we imposed a less restrictive condition (equation (1.2), [13]) on α than A(k). There is no obstacle to working with the less restrictive condition here as well. We impose condition A(k) only in the expectation that the results are then more conducive to obtaining effective estimates which will be useful in extracting archimedean information. Our idea is to establish an iteration procedure by which a solution of the non-linear equation is constructed as a series whose terms are themselves power-series solutions of a sequence of linear differential systems. The series will converge uniformly on a sufficiently small disk as well as in the x-adic topology. The terms of the series themselves converge on smaller and smaller domains of convergence. The rate at which these domains shrink in size decreases at a sufficiently rapid pace to ensure a non-trivial radius of convergence in the limit. This solution will be shown to be the unique power-series solution of type (1.2.2). The theorem then follows.

In somewhat more precise terms, we begin with a non-linear system (1.2.1) with $b(x) = 0(x^{2^k+1})$ as $x \to 0$ and take the linear part

$$x\psi' + \alpha\psi = b(x) + B(x)\psi \tag{1.2.3}$$

as an approximate equation. By Theorem 1.1.1, the solution of the equation

$$x\Phi' = B\Phi \tag{1.2.4}$$

involves a "controlled" loss (depending on k) in the radius of convergence. We use this solution Φ to change variables via $\psi = \Phi\gamma$ and reduce our linear problem (1.2.3) to solving

$$x\gamma' + \alpha\gamma = \tilde{b} \tag{1.2.5}$$

where $\tilde{b} = \Phi^{-1}b$. That equations of type (1.2.5) can be solved with only a small loss in the radius of convergence (depending on k) is proved in Lemma 1.2.2 below. Having solved the linear part (1.2.3) for $\psi(x) = 0(x^{2^k+1})$ as $x \to 0$, we obtain a new non-linear equation by substituting $y = \psi + z$ into the original equation and deriving the equation for z,

$$xz' + \alpha z = Bz + F(x, \psi + z) ,$$

and repeating the entire process. Using the Taylor expansion on the dependent variables in $F(x, y)$, this equation is precisely

$$xz' + \alpha z = F(x, \psi) + \{B(x) + F_y(x, \psi)\}z$$
$$+ \{F(x, \psi + z) - F(x, \psi) - F_y(x, \psi)z\} .$$

The new linear part can be treated by setting $z = \Phi w$ where Φ is the solution of (1.2.4) and solving

$$xw' + \alpha w = \Phi^{-1}F(x, \psi) + \Phi^{-1}F_y(x, \psi)\Phi w ,$$

a system having coefficients with a smaller radius of convergence but with k replaced by $k + 1$. This improvement is sufficient for the success of the iterative procedure in the proof of Theorem 1.2.1 below. We now proceed with the details.

LEMMA 1.2.2. *Suppose that a power series* $f(x) = \sum_{m=2^k}^{\infty} f_m x^m$ *with* $f_m \in \Omega^n$ *is an element of* $A(r, \rho)^n$ *where* r *and* ρ *are real numbers. Suppose that* α *satisfies condition* A(k), *with constants* $(\tilde{c}_1, \tilde{c}_2)$. *Let*

$$F(x) = \sum_{m=2^k}^{\infty} F_m x^m \qquad \text{with} \quad F_m \in \Omega^n \tag{1.2.6}$$

be the unique power-series solution of $xy' + \alpha y = f(x)$ *of the form* (1.2.6) *so that* $F_m = (m + \alpha)^{-1} f_m$. *Then* $F(x) \in A(r + c_1 k 2^{-k} + c_2 2^{-k}, \rho)^n$ *where* $c_1 =$ $(\log 9/3)\tilde{c}_1$, $c_2 = \tilde{c}_2$.

Proof. Our assumptions imply that $\operatorname{ord} f_m \geqq -mr + \rho$. Hence, using condition A(k), we have

$$\begin{aligned}\operatorname{ord}\left(\frac{1}{m + \alpha} f_m\right) &\geqq -mr + \rho - \operatorname{ord}(m + \alpha)\\ &\geqq -m(r + c_1 k 2^{-k} + c_2 2^{-k}) + \rho\\ &\quad + mc_1 k 2^{-k}\left(1 - \frac{\tilde{c}_1 \log m}{mc_1 k 2^{-k}}\right) + mc_2 2^{-k}\left(1 - \frac{1}{m2^{-k}}\right) .\end{aligned}$$

For $m \geqq 2^k$, we note that $1 - 2^k/m \geqq 0$. Observe that $\log x/x$ decreases for $x > e$. Therefore, if $k \geqq 2$ and $m \geqq 2^k$, then

$$1 - \frac{\tilde{c}_1 \log m}{mc_1 k 2^{-k}} = 1 - \frac{\tilde{c}_1 \log 2}{c_1} \frac{(\log m/m)}{(\log 2^k/2^k)} \geqq 0$$

provided $c_1 \geqq \tilde{c}_1 \log 2$. Now assume $k = 1$. We know $\log m/m \leqq \log 3/3$ for $m \geqq 3$. Hence, by checking the case $m = 2$, we establish

$$\sup \frac{\log m}{m} = \frac{\log 3}{3} ,$$

where the supremum runs over integers $m \geqq 2$. Therefore, for $m \geqq 2$,

$$\left(\frac{\log m/m}{\log 2/2}\right) \leqq \frac{2 \log 3}{3 \log 2} .$$

As a consequence, for $k = 1$, and $m \geqq 2$,

$$1 - \frac{\widetilde{c}_1 \log m}{mc_1 k 2^{-k}} \geqq 0$$

provided $c \geqq (\log 9/3)\widetilde{c}_1$. Finally once and for all set $c_1 = (\log 9/3)\widetilde{c}_1$; then

$$\operatorname{ord}\left(\frac{1}{m+\alpha} f_m\right) \geqq -m(r + c_1 k 2^{-k} + c_2 2^{-k}) + \rho$$

in all cases $k \geqq 1$, $m \geqq 2^k$. A simple check verifies this formula for any $k \geqq 0$, $m \geqq 2^k$. Q.E.D.

Definition 1.2.3. The system (1.2.1) is of type $(k+1, r, \alpha, \rho_0, \rho_1)$ provided the following assumptions, in addition to (1.2.1) (i–iii) hold:

(iv) $b(x) = \sum_{m=2^{k+1}}^{\infty} b_m x^m$ with $b_m \in \Omega^n$,

(v) $b(x) \in A(r, \rho_0)^n$, $B(x) \in A_n(r, \rho_1)$

and $F_l(x) \in A(r, 0)^n$ for all l where r is a real number and ρ_0 and ρ_1 are non-negative real numbers.

Under the assumption that k is so large that

$$\text{(B)} \qquad m + \alpha \neq 0\,, \qquad \text{for any} \quad m \geqq 2^k\,,$$

then (1.2.1) has a unique formal power-series solution of the form (1.2.2). Note that if α satisfies condition $\mathrm{A}(k_0)$ then condition (B) holds for α with any k, $k \geqq k_0$.

Theorem 1.2.1 is essentially a corollary of the following result.

THEOREM 1.2.4. *Assume that α satisfies conditions* $\mathrm{A}(k)$. *Assume further that system* (1.2.1) *is of type* $(j, r_j, \alpha, \rho_0, \rho_1)$ *with* $j \geqq k+1$. *Denote the system by* $S^{(j)}$ *and denote its unique solution of the form* (1.2.2) *by*

$$y_j = \varphi_j(x) = \sum_{m=2^j}^{\infty} \varphi_{j,m} x^m\,, \qquad \textit{with} \quad \varphi_{j,m} \in \Omega^n\,.$$

Then there exist functions

$$\Phi_j(x) = I + \sum_{m=2^{j-1}}^{\infty} \Phi_{j,m} x^m \qquad \textit{with} \quad \Phi_{j,m} \in M_n(\Omega)\,,$$
$$\gamma_j(x) = \sum_{m=2^j}^{\infty} \gamma_{j,m} x^m \qquad \textit{with} \quad \gamma_{j,m} \in \Omega^n\,,$$

such that

$$\Phi_j(x) - I \in A_n\big(r_j + (j-1)2^{-(j-1)} + c2^{-(j-1)}, \rho_1\big)\,,$$
$$\gamma_j(x) \in A\big(r_j + (j-1+c)2^{-(j-1)} + (c_1 j + c_2)2^{-j}, \rho_0\big)^n$$

and a system of differential equations $S^{(j+1)}$ *of type* $(j+1, r_{j+1}, \alpha, 2\rho_0, \rho_0)$ *such that*

$$r_{j+1} = r_j + (j-1+c)2^{-(j-1)} + (c_1 j + c_2)2^{-j}$$

and

$$\text{(1.2.7)} \qquad y_j = \Phi_j(y_{j+1} + \gamma_j)\,.$$

Proof. $\Phi_j(x)$ is constructed by Theorem 1.1.1 as a fundamental solution matrix of

$$xy' = By$$

satisfying $\Phi_j(x) - I \in A_n(r_j + (j-1+c)2^{-(j-1)}, \rho_1)$. As a consequence, if we transform equation (1.2.1) by the substitution $y = \Phi_j z$ then the equation for z is

$$xz' + \alpha z = \Phi_j(x)^{-1}\cdot b(x) + \textstyle\sum_{|l|\geqq 2}\Phi_j(x)^{-1}F_l(x)(\Phi_j(x)\cdot z)^l \ . \tag{1.2.8}$$

Let $\gamma_j(x)$ be the solution of

$$xz' + \alpha z = \Phi_j(x)^{-1}\cdot b(x) \ ,$$

given by Lemma 1.2.2. Hence, $\gamma_j(x) \in A(r_{j+1}, \rho_0)^n$. Define the system $S^{(j+1)}$ by substituting $z = \gamma_j + w$ into (1.2.8) and obtaining the equation for w. This completes the proof of Theorem 1.2.4. Q.E.D.

COROLLARY 1.2.5. *Under the hypotheses of Theorem* 1.2.4, *the formal solution* y_j *of the form* (1.2.2) *converges p-adically*; *in fact,*

$$y_j \in A(r_j + \sigma_{j-1}(1, c) + \sigma_j(c_1, c_2), \rho_0)^n \ .$$

Proof. Define in the notation of the statement of Theorem 1.2.4,

$$\psi_l(x) = \Phi_j(x)\Phi_{j+1}(x)\cdots\Phi_l(x)\alpha_l(x) \ .$$

The relation (1.2.7) implies that

$$y_j \equiv \textstyle\sum_{l=j}^{r}\psi_l(x) \pmod{x^{2^{r+1}}} \ . \tag{1.2.9}$$

Since

$$\Phi_l(x) - I \in A_n(r_l + (l-1+c)2^{-(l-1)}, \min(\rho_0, \rho_1))$$

and

$$\gamma_l(x) \in A(r_l + (l-1+c)2^{-(l-1)} + (c_1 l + c_2)2^{-l}, 2^{l-j}\rho_0)^n \ ,$$

where

$$r_l = r_j + \sigma_{j-1,l-j}(1, c) + \sigma_{j,l-j}(c_1, c_2) \ ,$$

we have $\psi_l(x) \in A(r_{l+1}, 2^{l-j}\rho_0)^n$. Hence, (1.2.9) implies that

$$\operatorname{ord}\varphi_{j,m} \geqq -mr_{l+1} + \rho_0 \ , \qquad \text{for} \quad 2^l \leqq m < 2^{l+1} \ ;$$

as a consequence,

$$y_j \in A(r_j + \sigma_{j-1}(1, c) + \sigma_j(c_1, c_2), \rho_0)^n \ . \qquad \text{Q.E.D.}$$

We observe that $y_j = \sum_{l=j}^{\infty}\psi_l(x)$ converges x-adically and the convergence is uniform on the closed disks $\mathcal{D}^+(r_j + \sigma_{j-1}(1, c) + \sigma_j(c_1, c_2) + \varepsilon)$ for $\varepsilon > 0$.

Proof of Theorem 1.2.1. Assume now that the hypotheses of Theorem 1.2.1 hold so that $b(x) = \sum_{m=2^k}^{\infty} b_m x^m$. Set $\lambda(x) = \sum_{m=2^k}^{\infty} (m+\alpha)^{-1} b_m x^m$. Then by Lemma 1.2.2, $\lambda(x) \in A\big(r + (c_1 k + c_2)2^{-k}, \rho_0\big)^n$ and $\lambda(x)$ satisfies

$$x\lambda' + \alpha\lambda = b(x) .$$

In (1.2.1), set $y = u + \lambda(x)$. Then

$$\begin{aligned} xu' + \alpha u = &\{B(x)\lambda(x) + F\big(x, \lambda(x)\big)\} + \{B(x) + F_y\big(x, \lambda(x)\big)\}u \\ &+ \{F\big(x, u + \lambda(x)\big) - F\big(x, \lambda(x)\big) - F_y\big(x, \lambda(x)\big)u\} . \end{aligned}$$

Therefore we are in the case of the corollary for $j = k+1$ (since the "constant" term $B(x)\lambda(x) + F\big(x, \lambda(x)\big) = 0(x^{2^{k+1}})$ as $x \to 0$). We must replace r_{k+1} by $r + \sigma_k(c_1, c_2)$ and ρ_1 by $\min(\rho_0, \rho_1)$. Then Theorem 1.2.1 follows from Corollary 1.2.5. Q.E.D.

1.3. *The first fundamental theorem.* In this section we consider a system of differential equations

$$xy' + Ay = b(x) + B(x)y + F(x, y) , \tag{1.3.1}$$

where (i) y is an n-vector whose entries are $y_1, \cdots, y_n$;

(ii) $$A = \begin{pmatrix} \alpha_1 I_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_s I_s \end{pmatrix}$$

where $\alpha_1, \cdots, \alpha_s \in \Omega$, $\alpha_i \neq \alpha_j$ $(i \neq j)$ and I_j is the $n_j \times n_j$ identity matrix $(n_1 + \cdots + n_s = n)$;

(iii) $$b(x) = \sum_{m=2^k}^{\infty} b_m x^m \quad \text{with} \quad b_m \in \Omega^n ;$$

(iv) $$B(x) = \begin{pmatrix} B_{11}(x) & \cdots & B_{1s}(x) \\ \cdots & \cdots & \cdots \\ B_{s1}(x) & \cdots & B_{ss}(x) \end{pmatrix},$$

where $B_{ij}(x)$ is an $n_i \times n_j$ matrix of the form:

$$B_{ij}(x) = \begin{cases} \sum_{m=0}^{\infty} B_{ijm} x^m & (i < j) , \\ \sum_{m=2^k}^{\infty} B_{ijm} x^m & (i \geqq j) , \end{cases}$$

with $B_{ijm} \in M_{n_i, n_j}(\Omega)$;

(v) $F(x, y) = \sum_{|l|=2}^{\infty} F_l(x) y^l$ where $F_l(x) = \sum_{m=0}^{\infty} F_{l,m} x^m$ with $F_{l,m} \in \Omega^n$.

THEOREM 1.3.1. *Assume that*

(i) $b(x) \in A(r, \rho_0)^n$, $B(x) \in A_n(r, \rho_1)$, $F_l(x) \in A(r, 0)^n$ *for all* l, $|l| \geqq 2$

where r is a given real number and ρ_0 and ρ_1 are given non-negative real numbers;

(ii) *each of the exponents* $\{\alpha_i\}_{i=1}^{s}$ *and each of the exponent-differences* $\{\alpha_i - \alpha_j\}_{i,j=1}^{s}$ *satisfy condition* A(k) *with constants* $(\tilde{c}_1, \tilde{c}_2)$.

Then, system (1.3.1) *has a unique formal solution of the form*

(1.3.2) $$y = \varphi(x) = \sum_{m=2^k}^{\infty} \varphi_m x^m \qquad (\varphi_m \in \Omega^n)$$

and $\varphi(x)$ *converges p-adically; in fact*

$$\varphi(x) \in A\big(r + \sigma_k(\mu_s, \tilde{\mu}_s), \rho_0\big)^n$$

where μ_s, *and* $\tilde{\mu}_s$ *are constants which depend only on* s *and the constants* $\tilde{c}_1$ *and* $\tilde{c}_2$ *and are given recursively in* (1.3.4) *below.*

Remark. If $s = 1$, Theorem 1.3.1 reduces to Theorem 1.2.1. Our proof proceeds by induction on s, and we assume the theorem for $s - 1$. To carry out the induction, it will be necessary to block-triangularize $B(x)$ by means of a change in variables. This is the purpose of the following lemma.

LEMMA 1.3.2. *Let* A *and* $B(x)$ *be as in Theorem* 1.3.1. *Assume that* $\alpha_1, \cdots, \alpha_s$ *satisfy condition* (ii) *of Theorem* 1.3.1 *for some* $k \in \mathbf{Z}_+$ *and some constants* $(\tilde{c}_1, \tilde{c}_2)$. *Then there exists an* $n \times n$ *matrix*

$$P(x) = I + \sum_{m=1}^{\infty} P_m x^m \qquad \big(P_m \in M_n(\Omega)\big)$$

such that

(i) $P(x)$ *converges on* $\mathfrak{D}\big(r + \sigma_k(\lambda_s, \tilde{\lambda}_s)\big)$ *and* $P(x) - I \in A_n\big(r + \sigma_k(\lambda_s, \tilde{\lambda}_s), \rho_1\big)$, *where* λ_s *and* $\tilde{\lambda}_s$ *are constants which depend only on* s *and the constants* $\tilde{c}_1$, $\tilde{c}_2$, *and are given recursively in* (1.3.4) *below;*

(ii) *the* $n \times n$ *matrix*

(1.3.3) $$C(x) = A - P^{-1}(x)[A - B(x)]P(x) - xP^{-1}(x)P'(x)$$

has the form

$$C(x) = \begin{bmatrix} C_{11}(x) & C_{12}(x) & \cdots & C_{1s}(x) \\ 0 & C_{22}(x) & \cdots & C_{2s}(x) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & C_{ss}(x) \end{bmatrix}$$

where $C_{ij}(x)$ *is an* $n_i \times n_j$ *matrix of the form*

$$C_{ij}(x) = \begin{cases} \sum_{m=0}^{\infty} C_{ijm} x^m & (i < j) \\ \sum_{m=2^k}^{\infty} C_{iim} x^m & (i = j) \end{cases}$$

with $C_{ijm} \in M_{n_i, n_j}(\Omega)$. *Furthermore,* $C(x) \in A_n\big(r + \sigma_k(\lambda_s, \tilde{\lambda}_s), \rho_1\big)$.

Remarks. 1. Let $c_1 = (\log 9/3)\tilde{c}_1$, $c_2 = \tilde{c}_2$ and recall that $c = 1.2$ is the constant of Theorem 1.1.1. Then the constants μ_s, $\tilde{\mu}_s$, λ_s and $\tilde{\lambda}_s$ which appear in the statements of Theorem 1.3.1 and Lemma 1.3.2 are defined by means of the following recursions:

$$\begin{aligned} \lambda_s &= \mu_{s-1} + \lambda_{s-1} , \\ \tilde{\lambda}_s &= \tilde{\mu}_{s-1} + \tilde{\lambda}_{s-1} , \\ \mu_s &= 2(1 + \lambda_s) + sc_1 , \\ \tilde{\mu}_s &= 2(1 + \lambda_s + \tilde{\lambda}_s + c) + sc_2 , \end{aligned} \tag{1.3.4}$$

with initial values $\lambda_1 = \tilde{\lambda}_1 = 0$, $\mu_1 = 1 + c_1$, $\tilde{\mu}_1 = c + c_2$.

2. It is useful to note that

$$\sum_{l \geqq k} \sigma_l(a, b) = \sigma_k(2a, 2(a + b)) .$$

3. Our purpose in constructing such a matrix P is that the transformation $y = Pz$ by property (ii) above takes the system (1.3.1) into the system

$$xz' + Az = \tilde{b}(x) + C(x)z + \tilde{F}(x, z) , \tag{1.3.5}$$

in which the non-constant part, $C(x)$, of the linear term is now block-upper-triangular.

Proof of Lemma 1.3.2. If $s = 1$ we may put $P(x) = I$ (note that $\lambda_1 = \tilde{\lambda}_1 = 0$). We proceed by induction on s. As a first step we shall show the existence of a transformation under which the block entries of the last row below the diagonal will vanish. The transformation we seek will be of the form $I + Q$ where Q is an $n \times n$ matrix of the form

$$Q = \left[\begin{array}{c|c} 0 & 0 \\ \hline Q_{s-1}\ Q_{s-2} \cdots Q_1 & 0 \end{array}\right]$$

with Q_{s-j} an $n_s \times n_j$ matrix. Since $Q^2 = 0$,

$$(I + Q)^{-1} = I - Q .$$

Set

$$\begin{aligned} U &= A - (I - Q)(A - B(x))(I + Q) - x(I - Q)Q' \\ &= B + QA - AQ + BQ - QB - QBQ - xQ' , \end{aligned}$$

where we have used the fact that $QAQ = QQ' = 0$. Set

$$U = \begin{pmatrix} U_{11} & \cdots & U_{1s} \\ \cdots & \cdots & \cdots \\ U_{s1} & \cdots & U_{ss} \end{pmatrix}$$

where U_{ij} is an $n_i \times n_j$ matrix. Then

$$\begin{aligned} U_{ij} &= B_{ij}(x) + B_{is}(x)Q_{s-j} , \qquad (i, j = 1, \cdots, s - 1) ; \\ U_{is} &= B_{is}(x) , \qquad (i = 1, \cdots, s - 1) ; \\ U_{sj} &= B_{sj}(x) + (\alpha_j - \alpha_s)Q_{s-j} + B_{ss}(x)Q_{s-j} - \textstyle\sum_{i=1}^{s-1} Q_{s-i}B_{ij}(x) \\ &\quad - \left(\textstyle\sum_{i=1}^{s-1} Q_{s-i}B_{is}(x)\right)Q_{s-j} - xQ'_{s-j} , \qquad (j = 1, 2, \cdots, s - 1) ; \\ U_{ss} &= B_{ss}(x) - \textstyle\sum_{i=1}^{s-1} Q_{s-i}B_{is}(x) . \end{aligned}$$

Setting $U_{sj} = 0$ $(j = 1, \cdots, s-1)$, we derive a system of differential equations

$$(1.3.6)\quad xQ_j' + (\alpha_s - \alpha_{s-j})Q_j = B_{s,s-j}(x) + B_{ss}(x)Q_j - \textstyle\sum_{i=1}^{s-1} Q_i B_{s-i,s-j}(x) - \left(\sum_{i=1}^{s-1} Q_i B_{s-i,s}(x)\right)Q_j \quad (j = 1, \cdots, s-1)\,.$$

We construct, by utilizing the induction hypothesis of Theorem 1.3.1,

$$Q_j(x) = \textstyle\sum_{m=2k}^{\infty} Q_{j,m}x^m \qquad (j = 1, \cdots, s-1)$$

with $Q_{j,m} \in M_{n_s,n_{s-j}}(\Omega)$ such that

(i) $Q_j(x) \in A_{n_s,n_{s-j}}(r + \sigma_k(\mu_{s-1}, \tilde{\mu}_{s-1}), \rho_1)$;

(ii) $Q_j(x)$ $(j = 1, \cdots, s-1)$ satisfy (1.3.6) on $\mathfrak{D}(r + \sigma_k(\mu_{s-1}, \tilde{\mu}_{s-1}))$.

Using the induction hypothesis of Lemma 1.3.2 with

$$\tilde{A} = \begin{bmatrix} \alpha_1 I_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_{s-1}I_{s-1} \end{bmatrix}$$

and

$$\tilde{B}(x) = \begin{bmatrix} U_{11}(x) & \cdots & U_{1,s-1}(x) \\ \cdots & \cdots & \cdots \\ U_{s-1,1}(x) & \cdots & U_{s-1,s-1}(x) \end{bmatrix},$$

we see that there exists an $(n - n_s) \times (n - n_s)$ matrix

$$\tilde{P}(x) = I + \textstyle\sum_{m=1}^{\infty} \tilde{P}_m x^m \quad \left(\tilde{P}_m \in M_{n-n_s}(\Omega)\right)$$

such that (using the recursive relations (1.3.4), namely $\lambda_s = \mu_{s-1} + \lambda_{s-1}$, $\tilde{\lambda}_s = \tilde{\mu}_{s-1} + \tilde{\lambda}_{s-1}$)

(i) $\tilde{P}(x)$ and $\tilde{P}^{-1}(x)$ converge on $\mathfrak{D}(r + \sigma_k(\lambda_s, \tilde{\lambda}_s))$;

(ii) $\operatorname{ord}(\tilde{P}(x) - I) \geqq \rho_1$ and $\operatorname{ord}(\tilde{P}^{-1}(x) - I) \geqq \rho_1$ for $\operatorname{ord} x \geqq r + \sigma_k(\lambda_s, \tilde{\lambda}_s)$;

(iii) the $(n - n_s) \times (n - n_s)$ matrix

$$\tilde{C}(x) = \tilde{A} - \tilde{P}^{-1}(x)[\tilde{A} - \tilde{B}(x)]\tilde{P}(x) - x\tilde{P}^{-1}(x)\tilde{P}'(x)$$

has the form

$$\tilde{C}(x) = \begin{bmatrix} C_{11}(x) & C_{12}(x) & \cdots & C_{1,s-1}(x) \\ 0 & C_{22}(x) & \cdots & C_{2,s-1}(x) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & C_{s-1,s-1}(x) \end{bmatrix},$$

where $C_{ij}(x)$ is an $n_i \times n_j$ matrix of the form

$$C_{ij}(x) = \begin{cases} \sum_{m=0}^{\infty} C_{ijm}x^m & (i < j) \\ \sum_{m=2k}^{\infty} C_{iim}x^m & (i = j) \end{cases}$$

with $C_{ijm} \in M_{n_i,n_j}(\Omega)$. Furthermore $\tilde{C}(x) \in A_{n-n_s}(r + \sigma_k(\lambda_s, \tilde{\lambda}_s), \rho_1)$.

Set

$$\hat{P}(x) = \begin{pmatrix} \tilde{P}(x) & 0 \\ 0 & I_s \end{pmatrix},$$

and

$$C(x) = A - \hat{P}^{-1}(x)\big(A - U(x)\big)\hat{P}(x) - x\hat{P}^{-1}(x)\hat{P}'(x) .$$

Then

$$C(x) = \begin{pmatrix} \tilde{C}(x) & V(x) \\ 0 & U_{ss}(x) \end{pmatrix},$$

where

$$V(x) = -\tilde{P}^{-1}(x)\begin{pmatrix} U_{1,s}(x) \\ \vdots \\ U_{s-1,s}(x) \end{pmatrix}.$$

Finally set

$$P(x) = \big(I + Q(x)\big)\hat{P}(x) .$$

Since $P'(x) = Q'(x)\hat{P}(x) + \big(I + Q(x)\big)\hat{P}'(x)$,

$$P^{-1}(x)P'(x) = \hat{P}(x)^{-1}\big(I - Q(x)\big)Q'(x)\hat{P}(x) + \hat{P}(x)^{-1}\hat{P}'(x) .$$

Therefore

$$C(x) = A - P^{-1}(x)\big(A - B(x)\big)P(x) - xP^{-1}(x)P'(x) .$$

This completes the proof of Lemma 1.3.2 for s assuming Theorem 1.3.1 for $s - 1$.

Let

$$\hat{C}(x) = \begin{pmatrix} C_{11}(x) & 0 \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & C_{ss}(x) \end{pmatrix}$$

be a block-diagonal matrix with entries $C_{ii}(x)$, $n_i \times n_i$ matrices, equal to the block-diagonal entries of the matrix $C(x)$ of Lemma 1.3.2. Then Theorem 1.1.1 implies the existence of a block-diagonal matrix $R(x)$ with the following properties

(i) $R(x)$ and $R(x)^{-1}$ converge on $\mathfrak{D}\big(r + \sigma_k(\lambda_s + 1, \tilde{\lambda}_s + c)\big)$;

(ii) $R(x) - I$ and $R(x)^{-1} - I$ belong to $A_n\big(r + \sigma_k(\lambda_s + 1, \tilde{\lambda}_s + c), \rho_1\big)$;

(iii) $xR'(x) = \hat{C}(x)R(x)$ on $\mathfrak{D}\big(r + \sigma_k(\lambda_s + 1, \tilde{\lambda}_s + c)\big)$.

If we combine this with Lemma 1.3.2, then the transformation $z = R(x)w$ transforms (1.3.5) into

$$xw' + Aw = \tilde{\tilde{b}}(x) + \big(C(x) - \hat{C}(x)\big)w + \tilde{\tilde{F}}(x, w) \tag{1.3.7}$$

in which the non-constant part, $C(x) - \hat{C}(x)$, of the linear term is strictly block upper-triangular. We summarize these results:

LEMMA 1.3.3. *Under the same hypotheses as in Lemma* 1.3.2, *there exists an* $n \times n$ *matrix*

$$P(x) = I + \sum_{m=1}^{\infty} P_m x^m \qquad \big(P_m \in M_n(\Omega)\big)$$

such that

(i) $P(x)$ *and* $P(x)^{-1}$ *converge on* $\mathfrak{D}\big(r + \sigma_k(\lambda_s + 1, \tilde{\lambda}_s + c)\big)$;

(ii) $P(x) - I$ *and* $P(x)^{-1} - I$ *belong to* $A_n\big(r + \sigma_k(\lambda_s + 1, \tilde{\lambda}_s + c), \rho_1\big)$;

(iii) *the* $n \times n$ *matrix* $C(x)$ *defined by* (1.3.3) *has the form*

$$C(x) = \begin{pmatrix} 0 & C_{12}(x) & \cdots & C_{1s}(x) \\ \vdots & & & \vdots \\ \cdot & & & C_{s-1,s}(x) \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$$

where $C_{ij}(x) = \sum_{m=0}^{\infty} C_{ijm} x^m$ $\big(C_{ijm} \in M_{n_i, n_j}(\Omega)\big)$. *Furthermore*

$$C(x) \in A_n\big(r + \sigma_k(\lambda_s + 1, \tilde{\lambda}_s + c), \rho_1\big) .$$

We may now prove Theorem 1.3.1. The difficulty in this case (compared with Theorem 1.2.1) arises in solving the linear part

$$xy' + Ay = b + By$$

of a non-linear system (say (1.3.1)) with only a small loss in the radius of convergence. We first transform (1.3.1) via the change of variables $y = Pz$ which gives the new linear part

$$xz' + Az = \tilde{b} + C(x)z$$

with $C(x)$ block strictly upper-triangular as in Lemma 1.3.3. This equation may be solved by writing z in the form

$$z = \begin{pmatrix} z^{(1)} \\ \vdots \\ z^{(s)} \end{pmatrix}$$

where $z^{(i)}$ is an n_i-vector. We then may solve, in order, for $z^{(s)}, z^{(s-1)}, \cdots, z^{(1)}$ by utilizing Lemma 1.2.2. Having solved the linear part, the iterative procedure then follows the induction technique of Theorem 1.2.1. We proceed with the details of the proof.

Proof of Theorem 1.3.1. To complete the induction proof of Theorem 1.3.1 and Lemmas 1.3.2 and 1.3.3, we may assume the lemmas for s and the theorem for $s - 1$. On this basis, we must prove Theorem 1.3.1 for s. The existence and uniqueness of a formal solution of system 1.3.1 having the form (1.3.2) is a consequence of hypothesis (ii) of Theorem 1.3.1. Therefore in order to prove Theorem 1.3.1, it is sufficient to prove the existence of a convergent power-series solution of system (1.3.1) which satisfies all the required conditions.

Definition 1.3.4. The system (1.3.1) is of type $\left(k+1, r, \{\alpha_i\}_{i=1}^{s}, \rho_0, \rho_1\right)$ provided the following assumptions, in addition to (1.3.1) (i)-(v), hold

(vi) $b(x) = \sum_{m=2k+1}^{\infty} b_m x^m$, with $b_m \in \Omega^n$;

(vii) $b(x) \in A(r, \rho_0)^n$, $B(x) \in A_n(r, \rho_1)$,

and $F_l(x) \in A(r, 0)^n$ for all l where r is a real number and ρ_0 and ρ_1 are non-negative real numbers.

LEMMA 1.3.5. *Assume Lemmas* 1.3.2 *and* 1.3.3 *for s and Theorem* 1.3.1 *for $s-1$. Assume that each exponent $\{\alpha_i\}_{i=1}^{s}$ and each exponent-difference $\{\alpha_i - \alpha_j\}_{i,j=1}^{s}$ satisfy condition* A(k). *Assume further that system* (1.3.1) *is of type $\left(j, r_j, \{\alpha_i\}_{i=1}^{s}, \rho_0, \rho_1\right)$ with $j \geqq k+1$. Denote the system by $S^{(j)}$ and denote its unique solution of the form* (1.3.2) *by*

$$y_j = \varphi_j(x) = \sum_{m=2j}^{\infty} \varphi_{j,m} x^m \qquad with \quad \varphi_{j,m} \in \Omega^n .$$

Then there exist functions

$$\Phi_j(x) = I + \sum_{m=2j-1}^{\infty} \Phi_{j,m} x^m \qquad with \quad \Phi_{j,m} \in M_n(\Omega) ,$$
$$\gamma_j(x) = \sum_{m=2j}^{\infty} \gamma_{j,m} x^m \qquad with \quad \gamma_{j,m} \in \Omega^n ,$$

such that

$$\Phi_j(x) - I \in A_n\left(r_j + \sigma_{j-1}(1+\lambda_s, c+\tilde{\lambda}_s), \rho_1\right) ,$$
$$\gamma_j(x) \in A\left(r_j + \sigma_{j-1}(1+\lambda_s, c+\tilde{\lambda}_s) + s\sigma_{j,1}(c_1, c_2), \rho_0\right)^n$$

and a system of differential equations $S^{(j+1)}$ of type $\left(j+1, r_{j+1}, \{\alpha_i\}_{i=1}^{s}, 2\rho_0, \rho_0\right)$ such that

$$r_{j+1} = r_j + \sigma_{j-1}(1+\lambda_s, c+\tilde{\lambda}_s) + s\sigma_{j,1}(c_1, c_2)$$

and

$$y_j = \Phi_j \cdot (y_{j+1} + \gamma_j) . \tag{1.3.8}$$

Proof. Φ_j is constructed by Lemma 1.3.3 satisfying $\Phi_j - I \in A_n\big(r_j + \sigma_{j-1}(1+\lambda_s, c+\tilde{\lambda}_s), \rho_1\big)$, and so that the substitution of $y = \Phi_j z$ into (1.3.1) produces the following equation for z:

$$xz' + Az = \Phi_j^{-1} b(x) + \Gamma_j(x) z + \sum_{|l| \geqq 2} \Phi_j^{-1} F_l(x) \left(\Phi_j(x) \cdot z\right)^l \tag{1.3.9}$$

where $\Gamma_j(x)$ is block strictly upper-triangular. In particular, $\Gamma_j(x)$ has the form

$$\Gamma_j(x) = \begin{bmatrix} 0 & \Gamma_{j12}(x) & \cdots & \Gamma_{j1s}(x) \\ \vdots & & \ddots & \vdots \\ \cdot & & & \Gamma_{j,s-1,s}(x) \\ 0 & \cdots & \cdots & 0 \end{bmatrix}$$

where $\Gamma_{jkl}(x) = \sum_{m=0}^{\infty} \Gamma_{jklm} x^m$ with $\Gamma_{jklm} \in M_{n_k, n_l}(\Omega)$, and

$$\Gamma_j(x) \in A_n\big(r + \sigma_{j-1}(1 + \lambda_s, c + \tilde{\lambda}_s), \rho_1\big) .$$

One may then construct γ_j as a solution to

$$x\gamma' + A\gamma = \Phi_j^{-1} b(x) + \Gamma_j(x)\gamma$$

by utilizing (s times in succession) Lemma 1.2.2. Hence,

$$\gamma_j(x) = \sum_{m=2j}^{\infty} \gamma_{j,m} x^m , \qquad \text{with} \quad \gamma_{j,m} \in \Phi^n ,$$

and

$$\gamma_j(x) \in A(r_{j+1}, \rho_0)^n .$$

We define the system $S^{(j+1)}$ by substituting $z = \gamma_j + w$ into (1.3.9) and obtaining the equation for w. This completes the proof of Lemma 1.3.5.

COROLLARY 1.3.6. *Under the hypotheses of Lemma* 1.3.5, *the formal solution y_j of the system $S^{(j)}$ having the form* (1.3.2) *converges p-adically; in fact*

$$y_j \in A\big(r_j + \sigma_{j-1}(2(1 + \lambda_s), 2(1 + \lambda_s + \tilde{\lambda}_s + c)\big) + \sigma_j(sc_1\, sc_2), \rho_0\big)^n .$$

Proof. The proof of Corollary 1.3.6 proceeds precisely as does the proof of Corollary 1.2.5. We utilize Remarks 1 and 2 following the statement of Lemma 1.3.2 in computing the radius of convergence.

The last step, in proving Theorem 1.3.1, is, in the case $b(x) = \sum_{m=2k}^{\infty} b_m x^m$, to solve by application of Lemma 1.2.2 s times in succession,

$$x\lambda' + A\lambda = b(x) + B(x)\lambda$$

for $\lambda(x) \in A\big(r + s\sigma_{k,1}(c_1, c_2), \rho_0\big)^n$. In (1.3.1) we set $y = u + \lambda(x)$, to obtain a situation that satisfies the hypotheses of Corollary 1.3.6 for $j = k + 1$, with r_{k+1} replaced by $r + s\sigma_{k,1}(c_1, c_2)$, and ρ_1 by $\min(\rho_0, \rho_1)$. Then Theorem 1.3.1 follows from Corollary 1.3.6.

1.4. *The second fundamental theorem.* In this section we consider a system of differential equations

$$x\Gamma(x)y' + y = b(x) + B(x)y + F(x, y) , \tag{1.4.1}$$

where

(i) y is an n-vector whose entries are $y_1, \cdots, y_n$;

(ii) $$\Gamma(x) = \begin{bmatrix} x^{\sigma_1} A_1 & & 0 \\ & \ddots & \\ 0 & & x^{\sigma_\nu} A_\nu \end{bmatrix} ,$$

where $A_j \in M_{n_j}(\Omega)$ ($j = 1, \cdots, \nu$) and $\sigma_1, \cdots, \sigma_\nu$ are positive integers (note: $n_1 + \cdots + n_\nu = n$);

(iii) $b(x) = \sum_{m=N}^{\infty} b_m x^m$, $b_m \in \Omega^n$, $N > 0$ an integer;

(iv) $B(x) = \sum_{m=1}^{\infty} B_m x^m$, $B_m \in M_n(\Omega)$;

(v) $F(x, y) = \sum_{|l|=2}^{\infty} F_l(x) y^l$,

where

$$F_l(x) = \sum_{m=0}^{\infty} F_{l,m} x^m \ , \quad F_{l,m} \in \Omega^n \ .$$

We shall prove the following theorem:

THEOREM 1.4.1. *Assume that* $b(x) \in A(r, \rho_0)^n$, $B(x) \in A_n(r, 0)$, $F_l(x) \in A(r, 0)^n$, *and* $\Gamma(x) \in A_n(r, 0)$, *where* r *and* ρ_0 *are given real numbers*, $\rho_0 \geqq 0$. *Then system* (1.4.1) *has a unique formal solution of the form*

$$y = \varphi(x) = \sum_{m=N}^{\infty} \varphi_m x^m \qquad \text{with} \quad \varphi_m \in \Omega^n \tag{1.4.2}$$

and $\varphi(x) \in A(r, \rho_0)^n$.

Proof. The quantity φ_m is determined by

$$\varphi_m = b_m + \sum_{i=N}^{m} B_{m-i}\varphi_i + \sum_{|l|=2}^{[m/N]} \sum_{h=N|l|}^{m} F_{l,m-h}\gamma_{l,h} - \sum_{j=1}^{\nu} (m - \sigma_j) P_j \varphi_{m-\sigma_j} \ ,$$

where we have written

$$\varphi(x)^l = \sum_{m=N|l|}^{\infty} \gamma_{l,m} x^m \ , \quad (\gamma_{l,m} \in \Omega) \ ,$$

and

$$P_j = \begin{bmatrix} 0 & & & & & & 0 \\ & \cdot & & & & & \\ & & \cdot & & & & \\ & & & 0 & & & \\ & & & & A_j & & \\ & & & & & 0 & \\ & & & & & & \cdot \\ & & & & & & & \cdot \\ 0 & & & & & & & 0 \end{bmatrix}, \qquad j = 1, 2, \cdots, \nu \ .$$

(We use the convention $\varphi_m = 0$ for $m < N$.) Let

$$\varphi_m = \begin{bmatrix} \varphi_{1,m} \\ \vdots \\ \varphi_{n,m} \end{bmatrix}, \quad \varphi_{i,m} \in \Omega \ .$$

Then

$$\gamma_{l,h} = \sum_{h_1+\cdots+h_n=h} \Phi_{1,h_1}^{(l_1)} \Phi_{2,h_2}^{(l_2)} \cdots \Phi_{n,h_n}^{(l_n)}$$

where

$$\Phi_{i,m}^{(\mu)} = \sum_{m_1+\cdots+m_\mu=m} \varphi_{i,m_1} \varphi_{i,m_2} \cdots \varphi_{i,m_\mu} \ .$$

Observe that $\operatorname{ord} b_m \geqq -mr + \rho_0$, $\operatorname{ord} B_m \geqq -mr$, $\operatorname{ord} F_{l,m} \geqq -mr$ and

$\varphi_N = b_N$. Hence by the induction, $\operatorname{ord}\gamma_{l,h} \geqq -hr + \rho_0$, and one obtains $\operatorname{ord}\varphi_m \geqq -mr + \rho_0$. Q.E.D.

1.5. *The third fundamental theorem.* In this section we consider a system of differential equations

$$\begin{cases} xy' + Ay = b(x) + B(x)y + E(x)z + F(x, y, z) , \\ x\Gamma(x)z' + z = d(x) + H(x)y + D(x)z + G(x, y, z) , \end{cases} \tag{1.5.1}$$

where

(i) y is an n-vector whose entries are $y_1, \cdots, y_n$, and z is an $\tilde{n}$-vector whose entries are $z_1, \cdots, z_{\tilde{n}}$;

(ii)
$$A = \begin{pmatrix} \alpha_1 I_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_s I_s \end{pmatrix}$$

where $\alpha_1, \cdots, \alpha_s \in \Omega$, $\alpha_i \neq \alpha_j$ $(i \neq j)$, and I_j is the $n_j \times n_j$ identity matrix $(n_1 + \cdots + n_s = n)$;

(iii)
$$b(x) = \textstyle\sum_{m=2k}^{\infty} b_m x^m \text{ with } b_m \in \Omega^n;$$

(iv)
$$B(x) = \begin{pmatrix} B_{11}(x) & B_{12}(x) & \cdots & B_{1s}(x) \\ B_{21}(x) & B_{22}(x) & \cdots & B_{2s}(x) \\ \cdots & \cdots & \cdots & \cdots \\ B_{s1}(x) & B_{s2}(x) & \cdots & B_{ss}(x) \end{pmatrix}$$

where

$$B_{ij}(x) = \begin{cases} \sum_{m=0}^{\infty} B_{ijm} x^m & (i < j) , \\ \sum_{m=2k}^{\infty} B_{ijm} x^m & (i \geqq j) , \end{cases}$$

with $B_{ijm} \in M_{n_i, n_j}(\Omega)$;

(v)
$$\Gamma(x) = \begin{pmatrix} x^{\sigma_1} A_1 & & 0 \\ & \ddots & \\ 0 & & x^{\sigma_\nu} A_\nu \end{pmatrix}$$

where $A_j \in M_{\tilde{n}_j}(\Omega)$ $(j = 1, \cdots, \nu)$ and $\sigma_1, \cdots, \sigma_\nu$ are positive integers $(\tilde{n}_1 + \cdots + \tilde{n}_\nu = \tilde{n})$;

(vi) $d(x) = \sum_{m=2k}^{\infty} d_m x^m$ with $d_m \in \Omega^{\tilde{n}}$;

(vii)
$$D(x) = \textstyle\sum_{m=1}^{\infty} D_m x^m , \qquad E(x) = \sum_{m=0}^{\infty} E_m x^m , \qquad \text{and}$$
$$H(x) = \textstyle\sum_{m=2k}^{\infty} H_m x^m ,$$

with $D_m \in M_{\tilde{n}}(\Omega)$, $E_m \in M_{n,\tilde{n}}(\Omega)$ and $H_m \in M_{\tilde{n},n}(\Omega)$;

(viii)
$$F(x, y, z) = \textstyle\sum_{|l|+|\tilde{l}|=2}^{\infty} F_{l\tilde{l}}(x) y^l z^{\tilde{l}} ,$$
$$G(x, y, z) = \textstyle\sum_{|l|+|\tilde{l}|=2}^{\infty} G_{l\tilde{l}}(x) y^l z^{\tilde{l}} ,$$

where

$$F_{l\tilde{l}}(x) = \sum_{m=0}^{\infty} F_{l\tilde{l}m} x^m , \qquad G_{l\tilde{l}}(x) = \sum_{m=0}^{\infty} G_{l\tilde{l}m} x^m$$

with $F_{l\tilde{l}m} \in \Omega^n$ and $G_{l\tilde{l}m} \in \Omega^{\tilde{n}}$.

THEOREM 1.5.1. *Assume that*

(i) $b(x) \in A(r, \rho_0)^n$; $d(x) \in A(r, \rho_0)^{\tilde{n}}$; $B(x) \in A_n(r, \rho_1)$; $D(x) \in A_{\tilde{n}}(r, \rho_1)$; $E(x) \in A_{n,\tilde{n}}(r, \rho_1)$; $H(x) \in A_{\tilde{n},n}(r, \rho_1)$; $F_{l\tilde{l}}(x) \in A(r, 0)^n$; $G_{l\tilde{l}}(x) \in A(r, 0)^{\tilde{n}}$,

where ρ_0 and ρ_1 are given non-negative numbers and r is a given real number;

(ii) *on* $\mathfrak{P}(r)$, $\operatorname{ord}\Gamma(x) \geqq \max(0, -\operatorname{ord} A)$;

(iii) *each of the exponents* $\{\alpha_i\}_{i=1}^n$ *and each of the exponent differences* $\{\alpha_i - \alpha_j\}_{i,j=1}^n$ *satisfy condition* A(k) *with constants* $(\tilde{c}_1, \tilde{c}_2)$. *Then system* (1.5.1) *has a unique formal solution of the form*

$$\begin{cases} y = \varphi(x) = \sum_{m=2k}^{\infty} \varphi_m x^m , \\ z = \tilde{\varphi}(x) = \sum_{m=2k}^{\infty} \tilde{\varphi}_m x^m \end{cases} \tag{1.5.2}$$

with $\varphi_m \in \Omega^n$ and $\tilde{\varphi}_m \in \Omega^{\tilde{n}}$ such that $\varphi(x)$ (respectively $\tilde{\varphi}(x)$) is an n-vector (respectively $\tilde{n}$-vector) with entries in $A(r + \sigma_k(\mu_s, \tilde{\mu}_s), \rho_0)$ where μ_s and $\tilde{\mu}_s$ are given by (1.3.4).

To prove this theorem we need the following lemma.

LEMMA 1.5.2. *Let A, $B(x)$, $\Gamma(x)$, $D(x)$, $E(x)$ and $H(x)$ be the same as in Theorem* 1.5.1. *Assume also that condition* (iii) *of Theorem* 1.5.1 *is satisfied. Then there exist an $n \times n$ matrix $\Phi(x)$ and an $\tilde{n} \times n$ matrix $\tilde{\Phi}(x)$ such that*

(i) $\Phi(x) = I + \sum_{m=2k}^{\infty} \Phi_m x^m$ *with* $\Phi_m \in M_n(\Omega)$, *and* $\tilde{\Phi}(x) = \sum_{m=2k}^{\infty} \tilde{\Phi}_m x^m$ *with* $\tilde{\Phi}_m \in M_{\tilde{n},n}(\Omega)$,

where I is the $n \times n$ identity matrix;

(ii) $\Phi(x)$, $\Phi^{-1}(x)$ *and* $\tilde{\Phi}(x)$ *converge on* $\mathfrak{P}(r + \sigma_k(1 + \lambda_s, c + \tilde{\lambda}_s))$ *where c is the constant of Theorem* 1.1.1 *and λ_s and $\tilde{\lambda}_s$ are given by* (1.3.4).

(iii) $\Phi(x) - I$ *and* $\Phi(x)^{-1} - I$ *belong to* $A_n(r + \sigma_k(1 + \lambda_s, c + \tilde{\lambda}_s), \rho_1)$, *and* $\tilde{\Phi}(x)$ *to* $A_{\tilde{n},n}(r + \sigma_k(1 + \lambda_s, c + \tilde{\lambda}_s), \rho_1)$;

(iv) *the system of differential equations*

$$\begin{cases} xy' + Ay = B(x)y + E(x)z , \\ x\Gamma(x)z' + z = H(x)y + D(x)z , \end{cases} \tag{1.5.3}$$

is reduced to

$$\begin{cases} x\eta' + A\eta = C(x)\eta + \hat{E}(x)\zeta , \\ x\Gamma(x)\zeta' + \zeta = \qquad F(x)\zeta , \end{cases} \tag{1.5.4}$$

by the transformation

(1.5.5) $$y = \Phi(x)\eta\,, \qquad z = \zeta + \tilde{\Phi}(x)\eta\,,$$

where

(i) $$\hat{E}(x) = \Phi^{-1}(x)E(x)$$

and

$$F(x) = D(x) - \Gamma(x)\tilde{\Phi}(x)\Phi^{-1}(x)E(x)\,;$$

(ii) $$C(x) = \begin{bmatrix} 0 & C_{12}(x) & C_{13}(x) & \cdots & C_{1s}(x) \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & C_{s-1,s}(x) \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}$$

where

(a) $C_{ij}(x) = \sum_{m=0}^{\infty} C_{ijm}x^m$ *with* $C_{ijm} \in M_{n_i,n_j}(\Omega)$;

(b) $C(x) \in A_n\big(r + \sigma_k(1 + \lambda_s, c + \tilde{\lambda}_s), \rho_1\big)$.

Proof. Consider the system

(1.5.6) $$x\Gamma(x)Q' + Q = H(x) + D(x)Q - \Gamma(x)Q\{B(x) - A\} - \Gamma(x)QE(x)Q$$

where Q is an $\tilde{n} \times n$ matrix to be determined. By Theorem 1.4.1 (note that assumption (ii) of Theorem 1.5.1 concerning $\operatorname{ord}\Gamma(x)$ is needed here), system (1.5.6) has a unique formal solution of the form

$$Q(x) = \sum_{m=2k}^{\infty} Q_m x^m \quad \big(Q_m \in M_{\tilde{n},n}(\Omega)\big)$$

such that $Q(x) \in A_{\tilde{n},n}(r, \rho_1)$. Set $z = \zeta + Q(x)y$. Then (1.5.3) becomes

(1.5.7) $$\begin{cases} xy' + Ay = \big(B(x) + E(x)Q(x)\big)y + E(x)\zeta, \\ x\Gamma(x)\zeta' + \zeta = \qquad\qquad\qquad\qquad F(x)\,\zeta, \end{cases}$$

where $F(x) = D(x) - \Gamma(x)Q(x)E(x)$.

Now applying Lemma 1.3.3 to A and $B(x) + E(x)Q(x)$, we can construct an $n \times n$ matrix

$$\Phi(x) = I + \sum_{m=2k}^{\infty} \Phi_m x^m \qquad \text{with} \quad \Phi_m \in M_n(\Omega)$$

such that

(i) $\Phi(x)$ and $\Phi^{-1}(x)$ converge for $\operatorname{ord} x > r + \sigma_k(1 + \lambda_s, c + \tilde{\lambda}_s)$;

(ii) $\Phi(x) - I$ and $\Phi^{-1}(x) - I$ belong to $A_n\big(r + \sigma_k(1 + \lambda_s, c + \tilde{\lambda}_s), \rho_1\big)$;

(iii) the $n \times n$ matrix

$$C(x) = A - \Phi^{-1}(x)[A - B(x) - E(x)Q(x)]\Phi(x) - x\Phi^{-1}(x)\Phi'(x)$$

satisfies all the requirements listed in Lemma 1.5.2.

Set $y = \Phi(x)\eta$. Then (1.5.7) becomes

$$\begin{cases} x\eta' + A\eta = C(x)\eta + \Phi^{-1}(x)E(x)\zeta \ , \\ x\Gamma(x)\zeta' + \zeta = \qquad\qquad F(x)\zeta \ . \end{cases}$$

To complete the proof of Lemma 1.5.2, it suffices to set $\tilde{\Phi}(x) = Q(x)\Phi(x)$.

Q.E.D.

Proof of Theorem 1.5.1. The existence and uniqueness of a formal solution of system (1.5.1) which has the form (1.5.2) is a consequence of assumption (iii) of Theorem 1.5.1. Therefore, in order to prove Theorem 1.5.1, it is sufficient to prove the existence of a convergent power-series solution of system (1.5.1) which satisfies all required conditions.

Definition 1.5.3. The system (1.5.1) is of type $(k+1, r, \{\alpha_i\}_{i=1}^s, \Gamma(x), \rho_0, \rho_1)$ provided the following assumptions, in addition to (1.5.1) (i)-(viii), hold;

(ix) $b(x) = \sum_{m=2k+1}^{\infty} b_m x^m$, with $b_m \in \Omega^n$,
$d(x) = \sum_{m=2k+1}^{\infty} d_m x^m$, with $d_m \in \Omega^{\tilde{n}}$;

(x) $b(x) \in A(r, \rho_0)^n$, $d(x) \in A(r, \rho_0)^{\tilde{n}}$, $B(x) \in A_n(r, \rho_1)$, $D(x) \in A_{\tilde{n}}(r, \rho_1)$, $E(x) \in A_{n,\tilde{n}}(r, \rho_1)$, $H(x) \in A_{\tilde{n},n}(r, \rho_1)$, $F_{l\tilde{l}}(x) \in A(r, 0)^{\tilde{n}}$, $G_{l\tilde{l}}(x) \in A(r, 0)^n$ for all l and $\tilde{l}$, where r is a real number and ρ_0 and ρ_1 are non-negative real numbers.

LEMMA 1.5.4. *Assume that each exponent* $\{\alpha_i\}_{i=1}^s$ *and each exponent-difference* $\{\alpha_i - \alpha_j\}_{i,j=1}^s$ *satisfy condition* A(k). *Assume further that system* (1.5.1) *is of type* $(j, r_j, \{\alpha_i\}_{i=1}^s, \Gamma(x), \rho_0, \rho_1)$ *with* $j \geqq k+1$. *Denote the system by* $S^{(j)}$ *and denote its unique solution of the form* (1.5.2) *by*

$$\begin{aligned} y_j &= \varphi_j(x) = \textstyle\sum_{m=2j}^{\infty} \varphi_{j,m} x^m \qquad \textit{with} \quad \varphi_{j,m} \in \Omega^n \ , \\ z_j &= \tilde{\varphi}_j(x) = \textstyle\sum_{m=2j}^{\infty} \tilde{\varphi}_{j,m} x^m \qquad \textit{with} \quad \tilde{\varphi}_{j,m} \in \Omega^{\tilde{n}} \ . \end{aligned}$$

Then there exist functions

$$\begin{aligned} \Phi_j(x) &= I + \textstyle\sum_{m=2j-1}^{\infty} \Phi_{j,m} x^m \ , && \textit{with} \quad \Phi_{j,m} \in M_n(\Omega) \ , \\ \tilde{\Phi}_j(x) &= \textstyle\sum_{m=2j-1}^{\infty} \tilde{\Phi}_{j,m} x^m \ , && \textit{with} \quad \tilde{\Phi}_{j,m} \in M_{\tilde{n},n}(\Omega) \ , \\ \gamma_j(x) &= \textstyle\sum_{m=2j} \gamma_{j,m} x^m \ , && \textit{with} \quad \gamma_{j,m} \in \Omega^n \ , \\ \tilde{\gamma}_j(x) &= \textstyle\sum_{m=2j} \tilde{\gamma}_{j,m} x^m \ , && \textit{with} \quad \tilde{\gamma}_{j,m} \in \Omega^{\tilde{n}} \ , \end{aligned}$$

such that

$$\begin{aligned} &\Phi_j(x) - I \in A_n\big(r_j + \sigma_j(1 + \lambda_s, c + \tilde{\lambda}_s), \rho_1\big) \ , \\ &\tilde{\Phi}_j(x) \in A_{\tilde{n},n}\big(r_j + \sigma_j(1 + \lambda_s, c + \tilde{\lambda}_s), \rho_1\big) \ , \\ &\gamma_j(x) \in A\big(r_j + \sigma_{j-1}(1 + \lambda_s, c + \tilde{\lambda}_s) + s\sigma_{j,1}(c_1, c_2), \rho_0\big)^n \ , \\ &\tilde{\gamma}_j(x) \in A\big(r_j + \sigma_{j-1}(1 + \lambda_s, c + \tilde{\lambda}_s), \rho_0\big)^{\tilde{n}} \ , \end{aligned}$$

and a system of differential equations $S^{(j+1)}$ *of type* $(j+1, r_{j+1}, \{\alpha_i\}_{i=1}^s, \Gamma(x), 2\rho_0, \rho_0)$ *such that*

$$r_{j+1} = r_j + \sigma_{j-1}(1 + \lambda_s, c + \tilde{\lambda}_s) + s\sigma_{j,1}(c_1, c_2)$$

and

(1.5.8)
$$\begin{cases} y_j = \Phi_j(y_{j+1} + \gamma_j) \\ z_j = z_{j+1} + \tilde{\gamma}_j + \tilde{\Phi}_j(y_{j+1} + \gamma_j) . \end{cases}$$

Proof. Φ_j and $\tilde{\Phi}_j$ are constructed by Lemma 1.5.2 satisfying

$$\Phi_j(x) - I \in A_n\big(r + \sigma_j(1 + \lambda_s, c + \tilde{\lambda}_s), \rho_1\big)$$

and

$$\tilde{\Phi}_j(x) \in A_{\tilde{n},n}\big(r + \sigma_j(1 + \lambda_s, c + \tilde{\lambda}_s), \rho_1\big)$$

and so that the substitutions $y = \Phi_j u$, $z = v + \tilde{\Phi}_j u$ produce the following system for u and v:

$$xu' + Au = \Phi_j^{-1}b + C_j(x)u + \Phi_j^{-1}Ev + \textstyle\sum_{|l|+|\tilde{l}|\geq 2}\Phi_j^{-1}F_{l\tilde{l}}(\Phi_j u)^l(v + \tilde{\Phi}_j u)^{\tilde{l}},$$
$$x\Gamma(x)v' + v = (d - \Gamma\tilde{\Phi}_j\Phi_j^{-1}b) + (D - \Gamma\tilde{\Phi}_j\Phi_j^{-1}E)v$$
$$+ \textstyle\sum_{|l|+|\tilde{l}|\geq 2}(G_{l\tilde{l}} - \Gamma\tilde{\Phi}_j\Phi_j^{-1}F_{l\tilde{l}})(\Phi_j u)^l(v + \tilde{\Phi}_j u)^{\tilde{l}}$$

where $C_j(x)$ is block strictly upper-triangular.

In particular, $C_j(x)$ has the form

$$C_j(x) = \begin{bmatrix} 0 & C_{j12}(x) & \cdots & C_{j1s}(x) \\ 0 & 0 & \ddots & \vdots \\ \vdots & & \ddots & \vdots \\ \vdots & & & C_{j,s-1,s}(x) \\ 0 & \cdots & \cdots & 0 \end{bmatrix}$$

where $C_{jkl}(x) = \sum_{m=0}^{\infty} C_{jklm}x^m$ with $C_{jklm} \in M_{n_k,n_l}(\Omega)$, and $C_j(x) \in A_n\big(r + \sigma_{j-1}(1 + \lambda_s, c + \tilde{\lambda}_s), \rho_1\big)$. One may then construct $\tilde{\gamma}_j$ by Theorem 1.4.1 satisfying

$$x\Gamma(x)\tilde{\gamma}_j' + \tilde{\gamma}_j = (d - \Gamma\tilde{\Phi}_j\Phi_j^{-1}b) + (D - \Gamma\tilde{\Phi}_j\Phi_j^{-1}E)\tilde{\gamma}_j .$$

Hence,

$$\tilde{\gamma}_j(x) = \textstyle\sum_{m=2^j}^{\infty}\tilde{\gamma}_{j,m}x^m , \qquad \text{with} \quad \tilde{\gamma}_{j,m} \in \Omega^{\tilde{n}}$$

and

$$\tilde{\gamma}_j(x) \in A\big(r_j + \sigma_{j-1}(1 + \lambda_s, c + \tilde{\lambda}_s), \rho_0\big)^{\tilde{n}} .$$

Finally, we construct γ_j by Lemma 1.2.2 satisfying

$$x\gamma_j' + A\gamma_j = \Phi_j^{-1}(b + E\tilde{\gamma}_j) + C_j\gamma_j .$$

Hence,

$$\gamma_j(x) = \textstyle\sum_{m=2^j}^{\infty}\gamma_{j,m}x^m , \qquad \text{with} \quad \gamma_{j,m} \in \Omega^n$$

and

$$\gamma_j(x) \in A\big(r_j + \sigma_{j-1}(1 + \lambda_s, c + \tilde{\lambda}_s) + s\sigma_{j,1}(c_1, c_2), \rho_0\big) .$$

We now define the system $S^{(j+1)}$ by substituting $u = \gamma_j + w$ and $v = \tilde{\gamma}_j + t$ and deriving the system for w and t. This completes the proof of Lemma 1.5.4. Q.E.D.

COROLLARY 1.5.5. *Under the hypotheses of Lemma* 1.5.4, *the formal solution* y_j, z_j *of the system* $S^{(j)}$ *having the form* (1.5.2) *converges p-adically; in fact* y_j *(respectively* z_j*) is an n-vector (respectively,* $\tilde{n}$*-vector) with entries in*

$$A\bigl(r_j + \sigma_{j-1}(2(1 + \lambda_s), 2(1 + \lambda_s + \tilde{\lambda}_s + c) + \sigma_j(sc_1, sc_2), \rho_0)\bigr) .$$

Proof. The proof is the same as the proof of Corollary 1.3.6.

The proof of Theorem 1.5.1 is now completed as is the proof of Theorem 1.3.1.

2. Convergence of power-series solutions

2.1. *Case* 1. In this section we consider a system of differential equations

$$xy' + Ay = b(x) + B(x)y + F(x, y) , \tag{2.1.1}$$

where

(i) y is an n-vector whose entries are $y_1, \cdots, y_n$;

(ii) $$A = \begin{bmatrix} \alpha_1 I_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_s I_s \end{bmatrix}$$

where $\alpha_1, \cdots, \alpha_s \in \Omega$, and I_j is the $n_j \times n_j$ identity matrix ($n_1 + \cdots + n_s = n$);

(iii) $b(x) = \sum_{m=1}^{\infty} b_m x^m$ with $b_m \in \Omega^n$;

(iv) $B(x) = \sum_{m=1}^{\infty} B_m x^m$ with $B_m \in M_n(\Omega)$;

(v) $F(x, y) = \sum_{|l|=2}^{\infty} F_l(x) y^l$, where $F_l(x) = \sum_{m=0}^{\infty} F_{l,m} x^m$ with $F_{l,m} \in \Omega^n$.

THEOREM 2.1.1. *Assume that*

(i) $b(x)$, $B(x)$ *and* $F_l(x)$ *are convergent for* $\operatorname{ord} x > r_0$, *where* r_0 *is a given real number;*

(ii) $\lim_{|l|\to+\infty}\{\operatorname{ord} F_l(x) + |l|r\} = +\infty$ *uniformly for* $\operatorname{ord} x > r_0$, *and* $r < r_1$ *where* r_1 *is a given real number;*

(iii) *system* (2.1.1) *admits a formal power-series solution*

$$\varphi(x) = \sum_{m=1}^{\infty} \varphi_m x^m \qquad with \quad \varphi_m \in \Omega^n ; \tag{2.1.2}$$

(iv) $\alpha_i - \alpha_j$ *is not a non-negative integer if* $i < j$.

(v) *each exponent* $\{\alpha_i\}_{i=1}^{s}$ *and each exponent-difference* $\{\alpha_i - \alpha_j\}_{i,j=1}^{s}$ *satisfy condition* A(k) *with constants* $(\tilde{c}_1, \tilde{c}_2)$ *for some choice of* k, $\tilde{c}_1$, $\tilde{c}_2$.

Then, the formal solution $\varphi(x)$ is convergent.

Proof. We shall reduce system (2.1.1) to a system which satisfies all the requirements listed in Theorem 1.3.1.

Step 1. Set $\tilde{\varphi}(x) = \sum_{m=1}^{2k-1} \varphi_m x^m$ and $y = \tilde{\varphi}(x) + u$. Then system (2.1.1) becomes

(2.1.3) $$xu' + Au = \tilde{b}(x) + \tilde{B}(x)u + \tilde{F}(x, u) ,$$

where

$$\begin{aligned} \tilde{b}(x) &= b(x) + B(x)\tilde{\varphi}(x) + F\big(x, \tilde{\varphi}(x)\big) - x\tilde{\varphi}'(x) - A\tilde{\varphi}(x) ; \\ \tilde{B}(x) &= B(x) + F_y\big(x, \tilde{\varphi}(x)\big) ; \\ \tilde{F}(x, u) &= F\big(x, \tilde{\varphi}(x) + u\big) - F\big(x, \tilde{\varphi}(x)\big) - F_y\big(x, \tilde{\varphi}(x)\big)u . \end{aligned}$$

Note that

(i) $\tilde{b}(x) = \sum_{m=2k}^{\infty} \tilde{b}_m x^m$ with $\tilde{b} \in \Omega^n$;

(ii) $\varphi(x) - \tilde{\varphi}(x)$ is a formal solution of system (2.1.3).

Hence we may assume that

(2.1.4) $$b(x) = \sum_{m=2k}^{\infty} b_m x^m ;$$

(2.1.5) $$\varphi(x) = \sum_{m=2k}^{\infty} \varphi_m x^m .$$

Step 2. We construct an $n \times n$ matrix,

$$P(x) = I + \sum_{m=1}^{\infty} P_m x^m \qquad \text{with} \quad P_m \in M_n(\Omega)$$

so that the $n \times n$ matrix

(2.1.6) $$C(x) = A - P^{-1}(x)[A - B(x)]P(x) - xP^{-1}(x)P'(x)$$

has the form

$$C(x) = \begin{pmatrix} C_{11}(x) & C_{12}(x) & \cdots & C_{1s}(x) \\ 0 & C_{22}(x) & \cdots & C_{2s}(x) \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & C_{ss}(x) \end{pmatrix},$$

where $C_{ij}(x) = \sum_{m=1}^{\infty} C_{ijm} x^m$ with $C_{ijm} \in M_{n_i, n_j}(\Omega)$. The construction of such a matrix $P(x)$ is essentially the same as in the proof of Lemma 1.3.2. The most important part of the proof is the construction of $Q_1, \cdots, Q_{s-1}$ by using system (1.3.6). Under assumption (iv), we have

$$m + (\alpha_s - \alpha_{s-j}) \neq 0$$

for $m = 0, 1, \cdots$. Hence, system (1.3.6) has a *formal* solution

$$Q_j(x) = \sum_{m=1}^{\infty} Q_{j,m} x^m \qquad \text{with} \quad Q_{j,m} \in M_{n_s, n_{s-j}}(\Omega) .$$

The rest of the construction is similar to the corresponding inductive step in the proof of Lemma 1.3.2. Note that we make no claims concerning the convergence of $P(x)$; we will only use a polynomial approximant of $P(x)$ (see (2.1.8) below).

Let $T_j(x)$ be an $n_j \times n_j$ matrix such that

$$\begin{cases} xT_j'(x) = C_{jj}(x)T_j(x) , \\ T_j(x) = I_j + \sum_{m=1}^{\infty} T_{j,m}x^m , \qquad T_{j,m} \in M_{n_j}(\Omega) . \end{cases}$$

Set

$$T(x) = \begin{bmatrix} T_1(x) & & 0 \\ & \ddots & \\ 0 & & T_s(x) \end{bmatrix}$$

and

$$\tilde{P}(x) = P(x)T(x) = I + \sum_{m=1}^{\infty} \tilde{P}_m x^m$$

where $\tilde{P}_m \in M_n(\Omega)$. Then, the $n \times n$ matrix

$$\tilde{C}(x) = A - \tilde{P}^{-1}(x)\big(A - B(x)\big)\tilde{P}(x) - x\tilde{P}^{-1}(x)\tilde{P}'(x)$$

has the form

$$\tilde{C}(x) = \begin{bmatrix} 0 & \tilde{C}_{12}(x) & \cdots & \tilde{C}_{1s}(x) \\ 0 & 0 & \cdots & \tilde{C}_{2s}(x) \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

where $\tilde{C}_{ij}(x) = \sum_{m=1}^{\infty} \tilde{C}_{ijm}x^m$, $\tilde{C}_{ijm} \in M_{n_i,n_j}(\Omega)$. Note that

$$\begin{aligned} \tilde{C}(x) &= A - T^{-1}(x)\big(A - C(x)\big)T(x) - xT^{-1}(x)T'(x) \\ &= T^{-1}(x)C(x)T(x) - xT^{-1}(x)T'(x) . \end{aligned}$$

Set

$$\Phi(x) = I + \sum_{m=1}^{2k-1} \tilde{P}_m x^m ,$$

and change system (2.1.1) to

(2.1.7) $$xu' + Au = \tilde{b}(x) + \tilde{B}(x)u + \tilde{F}(x, u)$$

by the transformation

(2.1.8) $$y = \Phi(x)u ,$$

where

$$\begin{aligned} \tilde{b}(x) &= \Phi^{-1}(x)b(x) ; \\ \tilde{B}(x) &= A - \Phi^{-1}(x)\big(A - B(x)\big)\Phi(x) - x\Phi^{-1}(x)\Phi'(x) ; \\ \tilde{F}(x, u) &= \Phi^{-1}(x)F\big(x, \Phi(x)u\big) . \end{aligned}$$

Note that $\Phi^{-1}(x)\varphi(x)$ is a formal solution of (2.1.7) and that

$$\tilde{B}(x) - \tilde{C}(x) = 0(x^{2k}) \qquad \text{as} \quad x \to 0 .$$

Hence we may assume that

(i) $b(x)$ has the form (2.1.4);

(ii) $\varphi(x)$ has the form (2.1.5);

(iii) $B(x)$ has the form

$$B(x) = \begin{pmatrix} B_{11}(x) & \cdots & B_{1s}(x) \\ \cdots & \cdots & \cdots \\ B_{s1}(x) & \cdots & B_{ss}(x) \end{pmatrix}$$

where

$$B_{ij}(x) = \begin{cases} \sum_{m=1}^{\infty} B_{ijm}x^m & (i < j) , \\ \sum_{m=2k}^{\infty} B_{ijm}x^m & (i \geqq j) , \end{cases}$$

with $B_{ijm} \in M_{n_i,n_j}(\Omega)$.

Step 3. A transformation

$$x = \alpha\xi , \qquad y = \beta u \quad (\alpha, \beta \in \Omega^*) \tag{2.1.9}$$

changes system (2.1.1) to

$$\xi \frac{du}{d\xi} + Au = \beta^{-1}b(\alpha\xi) + B(\alpha\xi)u + \beta^{-1}F(\alpha\xi, \beta u) . \tag{2.1.10}$$

We can choose α and β so that system (2.1.10) satisfies all the requirements listed in Theorem 1.3.1. Therefore the formal solution $\beta^{-1}\varphi(\alpha\xi)$ of (2.1.10) is convergent. Q.E.D.

2.2. *Case* 2. In this section we consider a system of differential equations

$$x\Gamma(x)y' + y = b(x) + B(x)y + F(x, y) , \tag{2.2.1}$$

where

(i) y is an n-vector whose entries are $y_1, \cdots, y_n$;

(ii)
$$\Gamma(x) = \begin{bmatrix} x^{\sigma_1}A_1 & & 0 \\ & \ddots & \\ 0 & & x^{\sigma_\nu}A_\nu \end{bmatrix}$$

where $A_j \in M_{n_j}(\Omega)$ ($j = 1, \cdots, \nu$) and $\sigma_1, \cdots, \sigma_\nu$ are positive integers (note: $n_1 + \cdots + n_\nu = n$);

(iii) $$b(x) = \sum_{m=1}^{\infty} b_m x^m , \quad b_m \in \Omega^n ;$$

(iv) $$B(x) = \sum_{m=1}^{\infty} B_m x^m , \quad B_m \in M_n(\Omega) ;$$

(v) $$F(x, y) = \sum_{|l|=2}^{\infty} F_l(x)y^l ,$$

where

$$F_l(x) = \sum_{m=0}^{\infty} F_{l,m}x^m , \quad F_{l,m} \in \Omega^n .$$

THEOREM 2.2.1. *Assume that*

(i) $b(x)$, $B(x)$ *and* $F_l(x)$ *are convergent for* $\operatorname{ord} x > r_0$, *where* r_0 *is a given real number*;

(ii) $\lim_{|l|\to+\infty}\{\operatorname{ord} F_l(x) + |l|r\} = +\infty$ *uniformly for* $\operatorname{ord} x > r_0$, *and* $r < r_1$ *where* r_1 *is a given real number*;

(iii) *a formal power series*

$$\varphi(x) = \sum_{m=1}^{\infty} \varphi_m x^m\ , \quad \varphi_m \in \Omega^n \tag{2.2.2}$$

is a formal solution of system (2.2.1). *Then the formal solution* $\varphi(x)$ *is convergent.*

Proof. A transformation

$$x = \alpha\xi\ , \qquad y = \beta u \quad (\alpha,\ \beta \in \Omega^*) \tag{2.2.3}$$

changes system (2.2.1) to

$$\xi\Gamma(\alpha\xi)\frac{du}{d\xi} + u = \beta^{-1}b(\alpha\xi) + B(\alpha\xi)u + \beta^{-1}F(\alpha\xi,\ \beta u)\ . \tag{2.2.4}$$

We can choose α and β so that system (2.2.4) satisfies all the requirements listed in Theorem 1.4.1. Therefore the formal solution $\beta^{-1}\varphi(\alpha\xi)$ of (2.2.4) is convergent. Q.E.D.

2.3. *Case* 3. In this section we consider a system of differential equations

$$\begin{cases} xy' + Ay = b(x) + B(x)y + E(x)z + F(x,\ y,\ z)\ , \\ x\Gamma(x)z' + z = d(x) + H(x)y + D(x)z + G(x,\ y,\ z)\ , \end{cases} \tag{2.3.1}$$

where

(i) y is an n-vector whose entries are $y_1, \cdots, y_n$, and z is an $\tilde{n}$-vector whose entries are $z_1, \cdots, z_{\tilde{n}}$;

(ii)
$$A = \begin{bmatrix} \alpha_1 I_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_s I_s \end{bmatrix}$$

where $\alpha_1, \cdots, \alpha_s \in \Omega$, and I_j is the $n_j \times n_j$ identity matrix $(n_1 + \cdots + n_s = n)$;

(iii)
$$\begin{aligned} b(x) &= \sum_{m=1}^{\infty} b_m x^m\ , & d(x) &= \sum_{m=1}^{\infty} d_m x^m\ , \\ B(x) &= \sum_{m=1}^{\infty} B_m x^m\ , & E(x) &= \sum_{m=0}^{\infty} E_m x^m\ , \\ H(x) &= \sum_{m=1}^{\infty} H_m x^m\ , & D(x) &= \sum_{m=1}^{\infty} D_m x^m\ , \end{aligned}$$

with $b_m \in \Omega^n$, $d_m \in \Omega^{\tilde{n}}$, $B_m \in M_n(\Omega)$, $E_m \in M_{n,\tilde{n}}(\Omega)$, $H_m \in M_{\tilde{n},n}(\Omega)$, $D_m \in M_{\tilde{n}}(\Omega)$;

(iv)
$$\Gamma(x) = \begin{bmatrix} x^{\sigma_1} A_1 & & 0 \\ & \ddots & \\ 0 & & x^{\sigma_\nu} A_\nu \end{bmatrix}$$

where $A_j \in M_{\tilde{n}_j}(\Omega)$ $(j = 1, \cdots, \nu)$ and $\sigma_1, \cdots, \sigma_\nu$ are positive integers $(\tilde{n}_1 + \cdots + \tilde{n}_\nu = \tilde{n})$;

(v)
$$F(x, y, z) = \sum_{|l|+|\tilde{l}|=2}^{\infty} F_{l,\tilde{l}}(x) y^l z^{\tilde{l}} ,$$
$$G(x, y, z) = \sum_{|l|+|\tilde{l}|=2}^{\infty} G_{l,\tilde{l}}(x) y^l z^{\tilde{l}} ,$$

where
$$F_{l,\tilde{l}}(x) = \sum_{m=0}^{\infty} F_{l,\tilde{l},m} x^m , \quad F_{l,\tilde{l},m} \in \Omega^n ,$$
$$G_{l,\tilde{l}}(x) = \sum_{m=0}^{\infty} G_{l,\tilde{l},m} x^m , \quad G_{l,\tilde{l},m} \in \Omega^n .$$

THEOREM 2.3.1. *Assume that*

(i) $b(x)$, $d(x)$, $B(x)$, $D(x)$, $E(x)$, $H(x)$, $F_{l\tilde{l}}(x)$ *and* $G_{l\tilde{l}}(x)$ *are convergent for* ord $x > r_0$, *where* r_0 *is a given real number*;

(ii)
$$\lim_{|l|+|\tilde{l}|\to+\infty} \{\operatorname{ord} F_{l,\tilde{l}}(x) + (|l| + |\tilde{l}|)r\} = +\infty$$

and

$$\lim_{|l|+|\tilde{l}|\to+\infty} \{\operatorname{ord} G_{l,\tilde{l}}(x) + (|l| + |\tilde{l}|)r\} = +\infty$$

uniformly for ord $x > r_0$, *and* $r < r_1$ *where* r_1 *is a given real number*;

(iii) *system* (2.3.1) *admits a formal solution*

$$\begin{cases} y = \varphi(x) = \sum_{m=1}^{\infty} \varphi_m x^m & \text{with} \quad \varphi_m \in \Omega^n , \\ z = \tilde{\varphi}(x) = \sum_{m=1}^{\infty} \tilde{\varphi}_m x^m & \text{with} \quad \tilde{\varphi}_m \in \Omega^{\tilde{n}} ; \end{cases} \tag{2.3.2}$$

(iv) $\alpha_i - \alpha_j$ *is not a non-negative integer if* $i < j$.

(v) *each of the exponents* $\{\alpha_i\}_{i=1}^{s}$ *and each of the exponent-differences* $\{\alpha_i - \alpha_j\}_{i,j=1}^{s}$ *satisfy condition* A(k_0) *with constants* $(\tilde{c}_1, \tilde{c}_2)$ *for some choice of* k_0, $\tilde{c}_1$, $\tilde{c}_2$. *Then the formal power series* $\varphi(x)$ *and* $\tilde{\varphi}(x)$ *are convergent.*

Proof. The main idea is to reduce system (2.3.1) to a system which satisfies all the requirements listed in Theorem 1.5.1. We can complete the proof of Theorem 2.3.1 by utilizing

(i) the idea of Step 1 of the proof of Theorem 2.1.1;

(ii) a construction of matrices Φ and $\tilde{\Phi}$ which are formal power series and transform the resulting system in the same way as do the corresponding (convergent) transformations of Lemma 1.5.2. (Cf. Step 2 of the proof of Theorem 2.1.1.)

(iii) the idea of Step 3 of the proof of Theorem 2.1.1.

3. Formal theory

3.1. *An application of a theorem due to Hukuhara-Turrittin.* Let Ω_0 be a subfield of Ω. Ω_0 need not be algebraically closed and need not be complete in the p-adic topology.

Let us consider a system of differential equations

$$x^{\sigma_0+1} u' = f(x, u) \tag{3.1.1}$$

where

(i) σ_0 is a positive integer;

(ii) u is an n_0-vector whose entries are $u_1, \cdots, u_{n_0}$;

(iii) $f(x, u) = \sum_{|l|=0}^{\infty} f_l(x) u^l$;

(iv) $f_l(x) = \sum_{m=0}^{\infty} f_{l,m} x^m$ with $f_{l,m} \in \Omega_0^{n_0}$;

(v) $f_l(x)$ converges for $\operatorname{ord} x > r_0$ and for all l;

(vi) $\lim_{|l|\to+\infty} \{\operatorname{ord} f_l(x) + |l| r\} = +\infty$ uniformly for $\operatorname{ord} x > r_0$ and $r > r_1$,

where r_0 and r_1 are given real numbers. We assume that system (3.1.1) admits a formal power-series solution

$$\varphi(x) = \sum_{m=1}^{\infty} \varphi_m x^m \qquad \text{with} \quad \varphi_m \in \Omega_0^{n_0} . \tag{3.1.2}$$

In this chapter, we shall reduce system (3.1.1) to a standard form to which the results of Chapter 2 apply (cf. § 3.2). For $N \in \mathbf{Z}_+$, set

$$\psi_N(x) = \sum_{m=1}^{N} \varphi_m x^m . \tag{3.1.2'}$$

In this section, we shall investigate algebraic properties of the system of linear differential equations

$$x^{\sigma_0+1} w' = f_u\big(x, \psi_N(x)\big) w \tag{3.1.3}$$

for a large N, where w is an n_0-vector (cf. Lemma 3.1.2). To do this, we shall first consider the system

$$x^{\sigma_0+1} w' = f_u\big(x, \varphi(x)\big) w . \tag{3.1.4}$$

Note that $f_u\big(x, \varphi(x)\big)$ is a formal power series in x whose coefficients are in $M_{n_0}(\Omega_0)$ (cf. Lemma 3.1.1).

(I) By virtue of a generalization, due to Levelt-Baldassarri, [6], [1] of a theorem of Hukuhara-Turrittin, there exists a finite extension $\widetilde{\Omega}_0$ of Ω_0, a positive integer h, and an $n_0 \times n_0$ matrix of formal power series

$$\widetilde{P}(\xi) = \sum_{m=0}^{\infty} \widetilde{P}_m \xi^m \qquad \text{with} \quad \widetilde{P}_m \in M_{n_0}(\widetilde{\Omega}_0) \tag{3.1.5}$$

such that (i) $\det \widetilde{P}(\xi) \neq 0$ as a formal power series in ξ, (ii) the transformation

$$w = \widetilde{P}(\xi)\widetilde{v} , \qquad x = \xi^h \tag{3.1.6}$$

changes system (3.1.4) to

$$\frac{d\widetilde{v}}{d\xi} = \widetilde{B}(\xi)\widetilde{v} \tag{3.1.7}$$

with

$$\widetilde{B}(\xi) = \begin{bmatrix} \widetilde{B}_1(\xi) & & & 0 \\ & \widetilde{B}_2(\xi) & & \\ & & \ddots & \\ 0 & & & \widetilde{B}_\tau(\xi) \end{bmatrix}$$

where $\tilde{B}_j(\xi)$ is an $m_j \times m_j$ matrix ($n_0 = m_1 + \cdots + m_\tau$) of the form:

$$\tilde{B}_j(\xi) = \lambda_j(\xi^{-1})I_j + \xi^{-1}J_j \tag{3.1.8}$$

such that

(i) $$\lambda_j(\xi^{-1}) = \sum_{m=1}^{\sigma_j+1} \lambda_{j,m}\xi^{-m}, \quad \lambda_{j,m} \in \tilde{\Omega}_0,$$

where $\sigma_1, \cdots, \sigma_\tau$ are nonnegative integers and

$$\lambda_{j,\sigma_j+1} \neq 0 \qquad \text{if} \quad \sigma_j > 0 ;$$

(ii) $\lambda_j(\xi^{-1}) - \lambda_i(\xi^{-1}) \notin \xi^{-1}\mathbf{Z}$ if $j \neq i$;

(iii) I_j is the $m_j \times m_j$ identity matrix;

(iv) J_j is a nilpotent matrix in $M_{m_j}(\tilde{\Omega}_0)$.

(II) We shall change $\xi^{-1}J_j$ in (3.1.8) to J_j. To do this, we assume without loss of generality that

$$J_j = \begin{bmatrix} 0 & a_{2j} & & & 0 \\ & 0 & \ddots & a_{3j} & \\ & & \ddots & \ddots & \ddots \\ & & & \ddots & a_{m_j,j} \\ 0 & & & & 0 \end{bmatrix}$$

with $a_{ij} \in \tilde{\Omega}_0$. We may set (although this is not always the most efficient choice possible)

$$T(\xi) = \begin{bmatrix} T_1(\xi) & & 0 \\ & \ddots & \\ 0 & & T_\tau(\xi) \end{bmatrix}$$

where

$$T_j(\xi) = \begin{bmatrix} 1 & & & & 0 \\ & \xi & & & \\ & & \xi^2 & & \\ & & & \ddots & \\ 0 & & & & \xi^{m_j-1} \end{bmatrix}.$$

Then the transformation

$$\tilde{v} = T(\xi)v$$

changes system (3.1.7) to

$$\frac{dv}{d\xi} = B(\xi)v \tag{3.1.9}$$

with

$$B(\xi) = \begin{bmatrix} B_1(\xi) & & 0 \\ & \ddots & \\ 0 & & B_\tau(\xi) \end{bmatrix} \tag{3.1.10}$$

where

(3.1.8′)
$$B_j(\xi) = \lambda_j(\xi^{-1})I_j + \xi^{-1}D_j + J_j$$

with

$$D_j = -\begin{bmatrix} 0 & & & & 0 \\ & 1 & & & \\ & & 2 & & \\ & & & \ddots & \\ 0 & & & & m_j - 1 \end{bmatrix}.$$

In fact,

$$B_j(\xi) = T_j^{-1}(\xi)\tilde{B}_j(\xi)T_j(\xi) - T_j^{-1}(\xi)\frac{dT_j(\xi)}{d\xi}.$$

Thus, we have proved the following lemma:

LEMMA 3.1.1. *Set*

$$P(\xi) = \tilde{P}(\xi)T(\xi) = \sum_{m=0}^{\infty} P_m\xi^m.$$

Then

(i) $P_m \in M_{n_0}(\tilde{\Omega}_0)$;

(ii) $\det P(\xi) \neq 0$ *as a formal power series in* ξ;

(iii) *the transformation*

(3.1.11)
$$w = P(\xi)v, \qquad x = \xi^h$$

changes system (3.1.4) *to system* (3.1.9).

Observe that $P(\xi)$ satisfies the differential equation:

(3.1.12)
$$\frac{dP(\xi)}{d\xi} = \frac{h \cdot f_u(\xi^h, \varphi(\xi^h))}{\xi^{h\sigma_0+1}}P(\xi) - P(\xi)B(\xi),$$

and that system (3.1.4) becomes

$$\frac{dw}{d\xi} = \frac{h \cdot f_u(\xi^h, \varphi(\xi^h))}{\xi^{h\sigma_0+1}}w$$

by the transformation $x = \xi^h$.

Now we return to (3.1.3). We shall first choose N. To do this, set $\det P(\xi) = \sum_{m=m_0}^{\infty}\alpha_m\xi^m$, where $\alpha_m \in \tilde{\Omega}_0$ and $\alpha_{m_0} \neq 0$. Choose N so large that

(3.1.13)
$$N \geqq m_0 + 1 + \max\{h\sigma_0, \sigma_1, \cdots, \sigma_\tau\}.$$

Set

$$Q_N(\xi) = \sum_{m=0}^{N} P_m\xi^m.$$

Then since $Q_N = P + (Q_N - P) = P\{I + P^{-1}(Q_N - P)\}$, we get

$$Q_N^{-1}(\xi) = \{I + P^{-1}(\xi)(Q_N(\xi) - P(\xi))\}^{-1}P^{-1}(\xi)$$

where $\{I + P^{-1}(\xi)(Q_N(\xi) - P(\xi))\}^{-1}$ is a power series in ξ with coefficients in $M_{n_0}(\widetilde{\Omega}_0)$.

Set

$$\frac{dQ_N(\xi)}{d\xi} = \frac{h \cdot f_u(\xi^h, \psi_N(\xi^h))}{\xi^{h\sigma_0+1}} Q_N(\xi) - Q_N(\xi)A_N(\xi) . \tag{3.1.14}$$

Then from (3.1.12) we derive

$$A_N = (-Q_N^{-1})\Big[\frac{h \cdot f_u(\xi^h, \psi_N(\xi^h))}{\xi^{h\sigma_0+1}}(P - Q_N) + h\frac{f_u(\xi^h, \varphi(\xi^h)) - f_u(\xi^h, \psi_N(\xi^h))}{\xi^{h\sigma_0+1}} P$$
$$- PB - \Big(\frac{dP}{d\xi} - \frac{dQ_N}{d\xi}\Big)\Big] .$$

Note that solving (3.1.14) for $A_N(\xi)$, we obtain

$$A_N(\xi) = Q_N(\xi)^{-1}\frac{h \cdot f_u(\xi^h, \psi_N(\xi^h))}{\xi^{h\sigma_0+1}} Q_N(\xi) - Q_N(\xi)^{-1}\frac{dQ_N(\xi)}{d\xi} .$$

Hence $A_N(\xi)$ is analytic in a disk punctured at $\xi = 0$. We summarize these results.

LEMMA 3.1.2. *If we assume* (3.1.13), *the transformation*

$$w = Q_N(\xi)v , \qquad x = \xi^h \tag{3.1.15}$$

changes system (3.1.3) *to*

$$\frac{dv}{d\xi} = A_N(\xi)v \tag{3.1.16}$$

where

$$A_N(\xi) = B(\xi) + \xi E_N(\xi) \tag{3.1.17}$$

with

$$E_N(\xi) = \sum_{m=0}^{\infty} E_{N,m}\xi^m \quad \big(E_{N,m} \in M_{n_0}(\widetilde{\Omega}_0)\big) .$$

The matrix $B(\xi)$ *is given in Lemma* 3.1.1 *and the power series* $E_N(\xi)$ *is convergent for* $\operatorname{ord}\xi > r_2$ *for some real number* r_2.

3.2. *A standard form.* Let us fix an integer M satisfying (3.1.13) and set $N = 2M$. Change system (3.1.1) by the transformation

$$u = \psi_N(x) + \tilde{u} \tag{3.2.1}$$

to derive

$$x^{\sigma_0+1}\tilde{u}' = \{f(x, \psi_N(x)) - x^{\sigma_0+1}\psi_N'(x)\}$$
$$+ f_u(x, \psi_N(x))\tilde{u} + \sum_{|l|=2}^{\infty} \widetilde{F}_l(x)\tilde{u}^l , \tag{3.2.2}$$

where

$$\sum_{|l|=2}^{\infty} \widetilde{F}_l(x)\tilde{u}^l = f(x, \psi_N(x) + \tilde{u}) - f(x, \psi_N(x)) - f_u(x, \psi_N(x))\tilde{u} . \tag{3.2.3}$$

Set

(3.2.4) $$\tilde{u} = x^M w\ .$$

Then (3.2.2) becomes

(3.2.5) $$x^{\sigma_0+1}w' = \tilde{g}(x) + \{f_u(x, \psi_N(x)) - Mx^{\sigma_0}I\}w + \sum_{|l|=2}^{\infty} x^{M(|l|-1)}\tilde{F}_l(x)w^l\ ,$$

where

(3.2.6) $$\tilde{g}(x) = x^{-M}\{f(x, \psi_N(x)) - x^{\sigma_0+1}\psi_N'(x)\}\ .$$

Since $\varphi(x)$ is a formal power-series solution of (3.1.1) and $\psi_N(x)$ is given by (3.1.2′), we have

(3.2.7) $$\tilde{g}(x) = \sum_{m=M+1}^{\infty} \tilde{g}_m x^m \qquad \text{with} \quad \tilde{g}_m \in \tilde{\Omega}_0^{n_0}\ .$$

The transformation

(3.2.8) $$x = \xi^h$$

changes (3.2.5) to

(3.2.9) $$\frac{dw}{d\xi} = \frac{h\cdot\tilde{g}(\xi^h)}{\xi^{h\sigma_0+1}} + \left\{\frac{h\cdot f_u(\xi^h, \psi_N(\xi^h))}{\xi^{h\sigma_0+1}} - \xi^{-1}(hM)I\right\} w + \frac{h}{\xi^{h\sigma_0+1}}\sum_{|l|=2}^{\infty} \xi^{hM(|l|-1)}\tilde{F}_l(\xi^h)w^l\ .$$

Finally set

(3.2.10) $$w = Q_N(\xi)v$$

to derive

(3.2.11) $$\frac{dv}{d\xi} = g(\xi) + A(\xi)v + F(\xi, v)$$

where

(3.2.12) $$g(\xi) = \frac{h}{\xi^{h\sigma_0+1}}Q_N^{-1}(\xi)\tilde{g}(\xi^h) = \sum_{m=1}^{\infty} g_m\xi^m \qquad \text{with} \quad g_m \in \tilde{\Omega}_0^{n_0}\ ;$$

(3.2.13) $$A(\xi) = A_N(\xi) - \xi^{-1}(hM)I = B(\xi) - \xi^{-1}(hM)I + \xi E_N(\xi)\ ;$$

(3.2.14) $$F(\xi, v) = Q_N^{-1}(\xi)\left\{\frac{h}{\xi^{h\sigma_0+1}}\sum_{|l|=2}^{\infty} \xi^{hM(|l|-1)}\tilde{F}_l(\xi^h)(Q_N(\xi)v)^l\right\} = \sum_{|l|=2}^{\infty} F_l(\xi)v^l$$

with

$$F_l(\xi) = \sum_{m=0}^{\infty} F_{l,m}\xi^m\ , \quad F_{l,m} \in \tilde{\Omega}_0^{n_0}\ .$$

There exist two real numbers r_3 and r_4 such that

(i) $g(\xi)$, $E_N(\xi)$ and $F_l(\xi)$ converge for $\operatorname{ord}\xi > r_3$;

(ii) $\lim_{|l|\to+\infty}\{\operatorname{ord} F_l(\xi) + |l|r\} = +\infty$ uniformly for $\operatorname{ord}\xi > r_3$ and $r > r_4$.

The formal power series

(3.2.15) $$\gamma(\xi) = \sum_{m=1}^{\infty} \gamma_m \xi^m , \quad \gamma_m \in \widetilde{\Omega}_0^{n_0} ,$$

defined by

(3.2.16) $$\gamma(\xi) = \xi^{-hM} Q_N^{-1}(\xi)\{\varphi(\xi^h) - \psi_N(\xi^h)\}$$

is a formal solution of system (3.2.11). If $\gamma(\xi)$ is convergent, then $\varphi(x)$ is also convergent.

Thus we proved the following theorem.

THEOREM 3.2.1. *The transformation*

(3.2.17) $$\begin{cases} u = \psi_N(x) + x^M Q_N(\xi)v , \\ x = \xi^h \end{cases}$$

changes system (3.1.1) *to* (3.2.11). *The power series* $\gamma(\xi)$ *given by* (3.2.16) *is a formal solution of system* (3.2.12).

The matrix $B(\xi)$ in (3.2.13) is given by (3.1.10) and (3.1.8′). Cases 1, 2, and 3 of Chapter 2, respectively, correspond to the following three cases:

(1) $\sigma_j = 0$ for all j;

(2) $\sigma_j > 0$ for all j;

(3) $\sigma_j = 0$ for some j and $\sigma_j > 0$ for some other j.

In fact, if $\sigma_j = 0$ for all j (i.e., Case (1)), we have

$$\lambda_j(\xi^{-1}) = \xi^{-1}\lambda_{j,1} \qquad \text{for all} \quad j ,$$

and hence

$$B_j(\xi) = \xi^{-1}(\lambda_{j,1} I_j + D) + J_j .$$

Therefore, we can write (3.2.13) as

$$A(\xi) = \xi^{-1}\Lambda + J + \xi E_N(\xi) ,$$

where Λ is a diagonal matrix in $M_{n_0}(\widetilde{\Omega}_0)$ and

$$J = \begin{bmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_\tau \end{bmatrix} .$$

This implies that Case (1) corresponds to Case 1 of Chapter 2.

If $\sigma_j > 0$ for all j (i.e., Case (2)), we have

$$\lambda_j(\xi^{-1}) = \lambda_{j,\sigma_j+1}\xi^{-\sigma_j-1}\{1 + \xi\mu_j(\xi)\} \qquad \text{for all} \quad j ,$$

where $\mu_j(\xi) \in \widetilde{\Omega}_0[\xi]$, and hence

$$B_j(\xi) = \lambda_{j,\sigma_j+1}\xi^{-\sigma_j-1}\{I_j - \xi C_j(\xi)\}$$

where $C_j(\xi) \in M_{m_j}(\widetilde{\Omega}_0[\xi])$. Therefore, we can write

$$\xi\Gamma(\xi)A(\xi) = -I + \xi\{C(\xi) - (hM)\xi^{-1}\Gamma(\xi) + \xi\Gamma(\xi)E_N(\xi)\}$$

where

$$\Gamma(\xi) = -\begin{bmatrix} \xi^{\sigma_1}A_1 & & 0 \\ & \ddots & \\ 0 & & \xi^{\sigma_\tau}A_\tau \end{bmatrix}, \qquad A_j = \frac{1}{\lambda_{j,\sigma_j+1}} I_j ,$$

and

$$C(\xi) = \begin{bmatrix} C_1(\xi) & & 0 \\ & \ddots & \\ 0 & & C_\tau(\xi) \end{bmatrix}.$$

Hence, Case (2) corresponds to Case 2 of Chapter 2.

Similarly, Case (3) corresponds to Case 3 of Chapter 2.

The convergence of $\gamma(\xi)$ follows from the results of Chapter 2 if the quantities $-\lambda_{j,1}$ (for all j such that $\sigma_j = 0$) satisfy the conditions imposed on the quantities α_j of Chapter 2. Thus, we proved the following result.

THEOREM 3.2.2. *If the quantities* $-\lambda_{j,1}$ (*defined in* (3.1.8) *above*) *and their differences for those* j *such that* $\sigma_j = 0$ *satisfy condition* A(k) *with constants* $\tilde{c}_1$ *and* $\tilde{c}_2$, *then the formal solution* $\varphi(x)$ (3.1.2) *of the system* (3.1.1) *converges p-adically in a non-trivial disk of convergence.*

Definition 3.2.3. A field Ω_0 with the property that every element α of its algebraic closure satisfies condition A(k) with constants ($\tilde{c}_1$, $\tilde{c}_2$) (where k, $\tilde{c}_1$, $\tilde{c}_2$ may depend on α) will be called closed under the A(k) condition.

Since Liouville numbers are transcendental, all fields of algebraic numbers are closed under the A(k) property. In Chapter 5 we will exploit the fact that Liouville numbers belong to $\mathbf{Z}_p$ to provide another important example of a field closed under the A(k) condition.

COROLLARY 3.2.4. *Assume the field* Ω_0 *is closed under the* A(k)*-condition, and that the formal power-series solution* $\varphi(x)$ (3.1.2) *of system* (3.1.1) *has coefficients in* Ω_0. *Then* $\varphi(x)$ *converges p-adically in a non-trivial disk of convergence.*

4. Algebraic differential equations

4.1. *Main theorem.* Let $\varphi(x) = \sum_{m=0}^{\infty} \varphi_m x^m$ be a formal power series in x with coefficients in $\mathbf{Q}$, a fixed algebraic closure of $\mathbf{Q}$. Let Ω_0 be the field $\mathbf{Q}(\{\varphi_m\}_{m\geqq 0})$ generated over $\mathbf{Q}$ by the coefficients of $\varphi(x)$. We assume that φ is the solution of an algebraic differential equation. In particular, we assume the existence of a non-zero polynomial $F(x_0, x_1, \cdots, x_{n+1})$ in $n+2$ variables $x_0, \cdots, x_{n+1}$ with coefficients in a field K, $K \supseteqq \Omega_0$ such that

$$F(x, \varphi(x), \varphi'(x) \cdots \varphi^{(n)}(x)) = 0 . \tag{4.1.1}$$

We shall prove in this chapter the following theorem.

THEOREM 4.1.1. *Let $\varphi(x)$ be a formal power series having coefficients in $\bar{\mathbf{Q}}$ and satisfying an algebraic differential equation. Then $\varphi(x)$ converges ν-adically in a non-trivial disk of convergence for every non-archimedean valuation ν of Ω_0.*

Remark. The result is independent of the normalization chosen for the valuation ν. We shall assume that $|\ |_\nu$ is an extension of the usual p-adic valuation so that in additive notation $\mathrm{ord}_\nu(p) = 1$. Since the valuation will be fixed (though arbitrary) throughout this chapter we shall drop the subscript and write simply "ord".

4.2. *Proof of the main theorem. Part* I. It is known (for example [8], [9]) that under the hypotheses of Theorem 4.1.1, $\varphi(x)$ in fact satisfies a non-trivial algebraic differential equation defined over Ω_0. Thus we may assume, with no loss in generality, that $F \in \Omega_0[x_0, \cdots, x_{n+1}]$.

Let n be the smallest non-negative integer which satisfies the following condition:

CONDITION (C). *There exists an element F of $\Omega_0[x_0, \cdots, x_{n+1}]$ such that*
(i) $F \neq 0$ *as an element of* $\Omega_0[x_0, \cdots, x_{n+1}]$;
(ii) $F(x, \varphi(x), \cdots, \varphi^{(n)}(x)) = 0$ *as an element of* $\Omega_0[[x]]$.

Let $\mathcal{S}$ be the set of all elements of $\Omega_0[x_0, \cdots, x_{n+1}]$ which satisfy (i) and (ii) of Condition (C). Set

$$d(F) = \text{the degree of } F \text{ in } x_{n+1}$$

for $F \in \Omega_0[x_0, \cdots, x_{n+1}]$. Then $d(F) > 0$ for every $F \in \mathcal{S}$.

This means that if $F \in \mathcal{S}$,

$$\frac{\partial F}{\partial x_{n+1}} \neq 0 \quad \text{as an element of} \quad \Omega_0[x_0, \cdots, x_{n+1}] \,. \tag{4.2.1}$$

Let $F \in \mathcal{S}$ be such that

$$d(F) = \min_{\tilde{F} \in \mathcal{S}} d(\tilde{F}) \,.$$

Set

$$G(x_0, \cdots, x_{n+1}) = \frac{\partial F}{\partial x_{n+1}}(x_0, \cdots, x_{n+1}) \,. \tag{4.2.2}$$

Since $d(G) < d(F)$, we have $G \notin \mathcal{S}$. Then (4.2.1) implies

$$G(x, \varphi(x), \cdots, \varphi^{(n)}(x)) \neq 0 \tag{4.2.3}$$

as an element of $\Omega_0[[x]]$. Set

$$G(x, \varphi(x), \cdots, \varphi^{(n)}(x)) = \sum_{m=m_0}^{\infty} a_m x^m \,, \tag{4.2.4}$$

where $a_m \in \Omega_0$ and

$$a_{m_0} \neq 0 . \tag{4.2.5}$$

Set

$$\varphi(x) = \sum_{m=0}^{\infty} \gamma_m x^m , \quad \gamma_m \in \Omega_0$$

and

$$\varphi_N(x) = \sum_{m=0}^{N} \gamma_m x^m .$$

Then if

$$N \geqq m_0 + n , \tag{4.2.6}$$

we have

$$G\big(x, \varphi_N(x), \cdots, \varphi_N^{(n)}(x)\big) = a_{m_0} x^{m_0} \Phi_N(x) \tag{4.2.7}$$

where

$$\Phi_N \in \Omega_0[x] \tag{4.2.8}$$

and

$$\Phi_N(0) = 1 . \tag{4.2.9}$$

4.3. *Proof of the main theorem, Part* II. Since $F\big(x, \varphi(x), \cdots, \varphi^{(n)}(x)\big) = 0$, we have

$$\frac{\partial F}{\partial x_{n+1}}\big(x, \varphi(x), \cdots, \varphi^{(n)}(x)\big)\varphi^{(n+1)}(x) + \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} \varphi^{(i)} + \frac{\partial F}{\partial x_0} = 0 . \tag{4.3.1}$$

If we set

$$H(x_0, \cdots, x_{n+1}) = \frac{\partial F}{\partial x_0}(x_0, \cdots, x_{n+1}) + \sum_{i=i}^{n} x_{i+1} \frac{\partial F}{\partial x_i}(x_0, \cdots, x_{n+1}) ,$$

we have $H \in \Omega_0[x_0, \cdots, x_{n+1}]$, and we can write (4.3.1) as

$$G\big(x, \varphi(x), \cdots, \varphi^{(n)}(x)\big)\varphi^{(n+1)}(x) + H\big(x, \varphi(x), \cdots, \varphi^{(n)}(x)\big) = 0 \tag{4.3.2}$$

as an element of $\Omega_0[[x]]$. This means that the differential equation

$$G(x, u, u', \cdots, u^{(n)})u^{(n+1)} + H(x, u, u', \cdots, u^{(n)}) = 0 \tag{4.3.3}$$

admits a formal power-series solution

$$u = \varphi(x) . \tag{4.3.4}$$

We change the differential equation (4.3.3) by a transformation

$$u = \varphi_N(x) + x^M w \tag{4.3.5}$$

where

$$M \geqq m_0 + n \quad \text{and} \quad N \geqq M + n . \tag{4.3.6}$$

Note that

$$u^{(i)} = \varphi_N^{(i)}(x) + \sum_{l=0}^{i} \binom{i}{l} \frac{M!}{(M+l-i)!} x^{M+l-i} w^{(l)} , \qquad i = 0, 1, \cdots, n+1$$

and that

$$M + l - i \geqq m_0 \quad \text{if} \quad 0 \leqq l \leqq i \leqq n .$$

Therefore, if we put

$$G(x, u, u', \cdots, u^{(n)}) = \tilde{J}(x, w, w', \cdots, w^{(n)}) ,$$

then

$$\tilde{J}(x, w, w', \cdots, w^{(n)}) = a_{m_0} x^{m_0} J(x, w, w', \cdots, w^{(n)})$$

where

$$J(x_0, x_1, \cdots, x_{n+1}) \in \Omega_0[x_0, x_1, \cdots, x_{n+1}]$$

and

(4.3.7) $$J(0, 0, \cdots, 0) = 1 . \qquad \text{(Cf. (4.2.7) and (4.2.9).)}$$

The differential equation (4.3.3) becomes

(4.3.8) $$a_{m_0} x^{m_0+M} J(x, w, w', \cdots, w^{(n)}) w^{(n+1)} - K(x, w, w', \cdots, w^{(n)}) = 0$$

by the transformation (4.3.5), where

(4.3.9) $$K(x_0, x_1, \cdots, x_{n+1}) \in \Omega_0[x_0, \cdots, x_{n+1}] .$$

The differential equation (4.3.8) admits a formal solution

(4.3.10) $$w = \frac{\varphi(x) - \varphi_N(x)}{x^M} = \sum_{m=N+1}^{\infty} \gamma_m x^{m-M} .$$

Note that

(4.3.11) $$N + 1 - M \geqq n + 1 .$$

Finally, write the differential equation (4.3.8) as

(4.3.12) $$x^{m_0+M} w^{(n+1)} = \frac{K(x, w, \cdots, w^{(n)})}{a_{m_0} J(x, w, \cdots, w^{(n)})}$$

and then change (4.3.12) to a system by

(4.3.13) $$\begin{cases} y_1 = w \\ y_2 = w' \\ \quad \cdots \\ y_{n+1} = w^{(n)} . \end{cases}$$

Since $J(0, \cdots, 0) \neq 0$ (cf. (4.3.7)), the system that arises is of the type (3.1.1). Since Ω_0 is closed under the A(k) condition, Corollary 3.2.4 concludes the proof of Theorem 4.1.1. Q.E.D.

Remarks. 1) The above argument actually yields a slightly stronger result than Theorem 4.1.1. Suppose that $\varphi(x) \in \bar{\mathbf{Q}}[[x]]$ and that

$$F(\varphi, \varphi', \cdots, \varphi^{(n)}) = 0$$

where F is a non-trivial polynomial with coefficients which belong to $\mathbf{Q}[[x]]$ and converge ν-adically (for ν a non-archimedean valuation of $\bar{\mathbf{Q}}$). Then φ converges ν-adically on a non-trivial disk of convergence.

2) Suppose that $\varphi(x) = (\varphi_1(x), \cdots, \varphi_n(x))$ is an n-vector in $(\bar{\mathbf{Q}}[[x]])^n$ and that $\varphi(x)$ satisfies an m^{th} order $n \times n$ algebraic system

$$F_i(x, \varphi_1, \cdots, \varphi_1^{(m)}, \varphi_2, \cdots, \varphi_2^{(m)}, \cdots, \varphi_n, \cdots \varphi_n^{(m)}) = 0 \qquad (\text{for } i = 1, \cdots, n)$$

where each F_i is a polynomial with coefficients in $\bar{Q}$ in $nm + n + 1$ variables $F_i = F_i(Z_0, Z_1, \cdots, Z_{mn+n}) \in Q[Z_0, Z_1, \cdots, Z_{mn+n}]$ and $\varphi(x)$ satisfies the Jacobian condition (with respect to the respective m^{th} derivatives):

$$\det \frac{\partial F_i}{\partial Z_{j(m+1)}}(x, \varphi_1, \cdots, \varphi_1^{(m)}, \cdots, \varphi_n, \cdots, \varphi_n^{(m)})_{i,j=1,\cdots,n} \neq 0$$

as an element in $\bar{\mathbf{Q}}[[x]]$. Then the argument of Chapter 4 enables us to reduce to a system of the form of (3.1.1) so that $\varphi(x)$ converges ν-adically for every non-archimedean valuation ν of $\bar{\mathbf{Q}}$.

5. Certain partial differential systems

5.1. *Main results.* We consider in this chapter an extension of our results to certain partial differential systems, i.e., Pfaffian systems. As in Chapter 3, Ω_0 will denote a subfield of Ω. Ω_0 need not be algebraically closed nor complete. We denote by x the n-vector $(x_1, \cdots, x_n)$ and by y the m-vector $(y_1, \cdots, y_m)$. We shall denote by $\Omega\{x\}$ the ring of power series with coefficients in Ω which converge p-adically on some non-trivial polydisk of convergence $\mathfrak{D}(r_1) \times \mathfrak{D}(r_2) \times \cdots \times \mathfrak{D}(r_n) \subseteqq \Omega^n$.

THEOREM 5.1.1. *Let Ω_0 be closed under the* A(k) *condition. Let $\varphi(x)$ be an m-vector of formal power series in $x_1, \cdots, x_n$ having coefficients in Ω_0. We assume that $\varphi(x)$ satisfies the following Pfaffian system:*

$$\text{(L):} \qquad x^{\sigma_i}\frac{\partial y}{\partial x_i} = b_i(x) + B_i(x)y + F_i(x, y) , \qquad i = 1, \cdots, n ,$$

where

$$\sigma_i = (\sigma_{i1}, \cdots, \sigma_{in}) \in (\mathbf{Z}_+)^n , \qquad x^{\sigma_i} = x_1^{\sigma_{i1}} \cdots x_n^{\sigma_{in}} ,$$
$$b_i(x) \in \Omega\{x\}^m , \qquad B_i(x) \in M_m(\Omega\{x\}) ,$$
$$F_i(x, y) = \sum_{|l| \geqq 2} F_{il}(x) y^l \in \Omega\{x, y\}^m ,$$

where

$$l = (l_1, \cdots, l_m) \in (\mathbf{Z}_+)^m , \qquad |l| = l_1 + \cdots + l_m \quad \textit{and} \quad y^l = y_1^{l_1}, \cdots, y_m^{l_m} .$$

Then the formal power series $\varphi(x)$ converges p-adically in a non-trivial polydisk.

Remark. We do not assume that the system (L) is completely integrable. The assumption that φ satisfies this system is sufficient to prove the convergence of φ.

Before proceeding with the proof of the theorem we state an immediate "global" corollary.

COROLLARY 5.1.2. *Assume that $\varphi(x)$ is an m-vector of formal power series in $x_1, \cdots, x_n$ having all coefficients in $\bar{Q}$. Let Ω_0 be the field generated by adjoining to Q the coefficients of $\varphi(x)$. Let $K \supseteqq \Omega_0$. Assume that $\varphi(x)$ satisfies the following systems of partial differential systems*:

$$(\tilde{\mathrm{L}}): \qquad f_i(x)\frac{\partial y}{\partial x_i} = b_i(x) + B_i(x)y + F_i(x, y)\ , \qquad i = 1, \cdots, n\ ,$$

where $b_i(x) \in K[x]^m$, $B_i(x) \in M_m(K[x])$, $F_i(x, y) \in K[x, y]^m$ such that $F_i(x, y) = \sum_{|l|\geqq 2} F_{il}(x)y^l$ where we use multi-index notation as in the theorem, and $f_i(x) \in M_m(K[x])$. We furthermore assume that

$$\det f_i(x) = x^{\sigma_i}\hat{f}_i(x) \qquad \textit{with} \quad \hat{f}_i(0) \neq 0\ . \tag{5.1.1}$$

Under these hypotheses, the formal solution $\varphi(x)$ converges ν-adically in a non-trivial polydisk of convergence for every non-archimedean valuation ν of Ω_0.

Proof. By standard "domain of rationality" arguments, we may replace K by $\bar{Q}$. (For example, if $\{Z_\beta\}$ is a transcendence basis for K over Ω_0, we may specialize the Z_β to elements of Ω_0 so that (5.1.1) still holds. We remark that if the system $(\tilde{\mathrm{L}})$ is completely integrable, the above change of coefficients also will not destroy this property.) Then inverting the $f_i(x)$ puts us in the situation of a system (L) in which $b_i(x)$, $F_i(x, y)$ converge ν-adically for every non-archimedean valuation ν of $\bar{\mathbf{Q}}$. Q.E.D.

5.2. *Proof of Theorem* 5.1.1. Our proof of Theorem 5.1.1 will hinge in an essential way on the following lemma.

LEMMA 5.2.1. *Let $L \subseteqq \Omega$ be a field which is closed under the* A(k) *condition. Let $\{\alpha_1, \cdots, \alpha_n\} \subseteqq \Omega$ be a set of n units whose reductions are algebraically independent over E, the residue class field of L. Then $\Gamma = L(\alpha_1, \cdots, \alpha_n)$ is also closed under the* A(k) *condition.*

Proof. Suppose not. Then there exists an element γ algebraic over Γ, such that $\operatorname{ord}(\gamma + m) \neq 0(\log m)$ as $m \to +\infty$. This implies, in particular, that $\gamma \in \mathbf{Z}_p$. Let $f(z)$ be a non-trivial polynomial with coefficients in $L[\alpha_1, \cdots, \alpha_n]$ satisfied by γ:

$$f(\gamma) = \sum A_{\lambda j}\alpha^\lambda\gamma^j = 0$$

with $A_{\lambda j} \in L$ and where $\lambda = (\lambda_1, \cdots, \lambda_n)$ runs through a finite set Λ of elements of $(\mathbf{Z}_+)^n$ and $j = 0, 1, \cdots, J$. Thus $\sum_{\lambda \in \Lambda} c_\lambda \alpha^\lambda = 0$ (where $c_\lambda = \sum_{j=0}^J A_{\lambda j}\gamma^j$) is an algebraic relation among $\alpha_1, \cdots, \alpha_n$ which is defined over $L(\gamma)$. Since $\gamma \in \mathbf{Z}_p$, we observe that L and $L(\gamma)$ both have residue class field E. If $c_\lambda \neq 0$ for some $\lambda \in \Lambda$ then let $|c_{\tilde{\lambda}}| = \sup_{\lambda \in \Lambda} |c_\lambda|$ for some $\tilde{\lambda} \in \Lambda$. Reducing the equation $\sum_{\lambda \in \Lambda} (c_{\tilde{\lambda}}^{-1} c_\lambda)\alpha^\lambda = 0$ then violates our hypothesis on $\{\alpha_1, \cdots, \alpha_n\}$. As a consequence, we obtain at least one non-trivial relation

$$\sum_{j=0}^J A_{\lambda j}\gamma^j = 0 , \quad A_{\lambda j} \in L$$

which shows γ is algebraic over L and contradicts our assumption that L is closed under the A(k) condition. Q.E.D.

We now complete the proof of the theorem. We may assume without loss of generality that $b_i(x)$, $B_i(x)$ and $F_i(x, y)$ are defined over $\bar{\Omega}_0$, the algebraic closure of Ω_0. (Furthermore if the system (L) is completely integrable, then the system defined over $\bar{\Omega}_0$ may also be taken to be completely integrable.) We take L now, in the notation of Lemma 5.2.1, to be $\bar{\Omega}_0$ so that $\Gamma = \bar{\Omega}_0(\alpha_1, \cdots, \alpha_n)$. Consider the substitution $x = \alpha t$ (i.e., $x_i = \alpha_i t$ for $i = 1, 2, \cdots, n$), where t is a fixed transcendental over Ω. We express φ in terms of its homogeneous components by

$$\varphi(x) = \sum_{j=0}^\infty \varphi_j(x) \tag{5.2.1}$$

where $\varphi_j(x)$ is a homogeneous polynomial of degree j in $\Omega_0[x]^m$. Set

$$\psi(t) = \varphi(\alpha t) = \sum_{j=0}^\infty t^j \varphi_j(\alpha) . \tag{5.2.2}$$

We will use the following equation for ψ:

$$\frac{d\psi}{dt} = \sum_{i=1}^n \alpha_i \frac{\partial \varphi}{\partial x_i}(\alpha t) \tag{5.2.3}$$

and also the fact that φ satisfies the systems (L). By (L),

$$\alpha^{\sigma_i} t^{|\sigma_i|} \frac{\partial \varphi}{\partial x_i}(\alpha t) = b_i(\alpha t) + B_i(\alpha t)\varphi(\alpha t) + F_i(\alpha t, \varphi(\alpha t))$$

so that ψ is a formal power-series solution in t with coefficients in Γ of a system of type (3.1.1). Hence by Corollary 3.2.4, $\psi(t)$ converges p-adically in a non-trivial disk of convergence, so that

$$\operatorname{ord} \varphi_j(\alpha) \geqq -rj \qquad \text{for some real number } r . \tag{5.2.4}$$

If we write $\varphi_j(x) = \sum_{|\mu|=j} \varphi_{j,\mu} x_1^{\mu_1} \cdots x_n^{\mu_n}$ with $\varphi_{j,\mu} \in \Omega_0$ then

$$\operatorname{ord} \varphi_j(\alpha) = \operatorname{Inf}_{|\mu|=j} \{\operatorname{ord} \varphi_{j,\mu}\} . \tag{5.2.5}$$

For let $\operatorname{ord} \varphi_{j,\tilde{\mu}} = \operatorname{Inf}_{|\mu|=j} \{\operatorname{ord} \varphi_{j,\mu}\}$ for some $\tilde{\mu} \in (\mathbf{Z}_+)^n$, $|\tilde{\mu}| = j$. Then

$$\varphi_{j,\tilde{\mu}}^{-1} \varphi_j(x) = \sum_{|\mu|=j} \varphi_{j,\tilde{\mu}}^{-1} \varphi_{j,\mu} \alpha_1^{\mu_1} \cdots \alpha_n^{\mu_n}$$

where the coefficients on the right are integers at least one of which is a unit. Thus the hypothesis on $\{\alpha_1, \cdots, \alpha_n\}$ implies (5.2.5). As a consequence of (5.2.4) and (5.2.5)

$$\operatorname{ord} \varphi_{j\mu} \geqq -rj ,$$

so that $\varphi(x)$ converges for $\operatorname{ord} x_i > r$, $i = 1, \cdots, n$.

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