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The decidability of the intensional fragment of classical linear logic

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ABSTRACT

The intensional fragment of classical propositional linear logic combines modalities with contraction-free relevance logic — adding modalized versions of the thinning and contraction rules. This paper provides a *proof of the decidability* of this logic based on a sequent calculus formulation. Some related logics and some other fragments of linear logic are also shown decidable.

1. Introduction

Linear logic was introduced by Girard [15], and it has been applied, reformulated, extended and thoroughly investigated.[1] Theorem 3.7 in Lincoln et al. [24] claims that the propositional fragment is undecidable. However, the undecidability of linear logic would not imply the undecidability of its fragments. Indeed, it seems that the decidability of the intensional fragment (sometimes termed as *MELL*) has not been solved.[2]

Kripke [20] has shown that certain *relevance logics* are decidable. Detailed presentations of those results may be found in Belnap and Wallace [6], Anderson and Belnap [2, §13] and Dunn [13, §§3.6–3.9]. Meyer [27] proved that the non-distributive logic of relevant implication is decidable. The result has been further elaborated on in Thistlewaite et al. [35]. Bimbó and Dunn [10] extended the scope of decidable fragments to include the implicational logic of ticket entailment, and Bimbó [8, Ch. 9] provides proofs of decidability for further logics. This paper shows — using a version of the method exemplified by these results — that the *intensional fragment* of classical propositional linear logic is decidable. The modalization of the structural rules is peculiar to linear logic, which adds an extra layer to the decidability proof.

In Section 2, we carefully formulate sequent calculi for CLL_{int}, the intensional fragment of classical linear logic and RLL_{int}, intensional interlinear logic. The former is at the center of this paper, however, the latter is used in an essential way in

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[1] See also Belnap [5], Girard [16] and [17], Gunter and Gehlot [18], Kopylov [19], Lafont [22], Lincoln and Winkler [25], Martini and Masini [26], Meyer et al. [31], Nigam and Miller [32], Urquhart [37] and [38].

[2] *MELL* is a label used, for example, in Lincoln [23] and Di Cosmo and Miller [12]. The decidability of *MELL* is listed as an open problem by Y. Lafont on his web pages at the URL iml.univ-mrs.fr/~lafont/linear/decision/bienvenue.html (accessed on March 15th, 2015).

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the decidability proof of CLL_{int}. Section 3 provides proofs — in some detail — of the cut theorem for these calculi. These theorems ensure that the sequent calculi are properly formulated, and that cut-free proofs are sufficient. Section 4 contains the proof of the decidability of CLL_{int}, with an auxiliary proof of the decidability of RLL_{int}. In Section 5, we prove similar results for logics closely related to CLL_{int} and RLL_{int}. In the last section, we provide some concluding remarks about the importance of our results.

2. Sequent calculi

We introduce four sequent calculi that formalize two logics, the intensional fragment of linear logic and intensional interlinear logic.

2.1. Intensional fragment of classical linear logic: CLL_{int}

The intensional fragment of classical propositional linear logic is denoted by CLL_{int}.[3] The language of this logic contains a unary connective $^{\perp}$ (*negation*), three binary connectives $\multimap$ (*implication*), $\otimes$ (*fusion*) and $⅋$ (*fission*), and two unary modalities ! (*possibility*) and ? (*necessity*).[4] The atomic formulas comprise a denumerable set of *propositional variables*; *formulas* are defined by the following CFG, with the proviso that $\mathbb{P}$ is a non-terminal symbol that rewrites to a propositional variable.

$$\mathcal{A} ::= \mathbb{P} \mid (\mathcal{A}^{\perp}) \mid (\mathcal{A} \multimap \mathcal{A}) \mid (\mathcal{A} \otimes \mathcal{A}) \mid (\mathcal{A} ⅋ \mathcal{A}) \mid !\mathcal{A} \mid ?\mathcal{A}$$

We use $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$ as *metavariables* for formulas. Capital Greek letters stand for *multisets of formulas* (including the empty multiset of formulas).[5] The multiset $[\mathcal{A}_1, \ldots, \mathcal{A}_n]$ is denoted by $\mathcal{A}_1; \ldots; \mathcal{A}_n$; that is, we omit the brackets and separate the elements of the multiset by semicolons. This notation is in harmony with standard notation in sequent calculi for non-classical logics. $\Gamma; \Delta$ is a shorthand for the union of Γ and Δ, and $\mathcal{A}; \Gamma$ (or $\Gamma; \mathcal{A}$) is the union of Γ and $[\mathcal{A}]$ (the singleton multiset containing one copy of $\mathcal{A}$). The superscript modalities on multisets in the rules below indicate that in order for a rule to be applicable the formulas in the multiset must be appropriately modalized. For instance, $\Gamma^!$ is a multiset of formulas, in which each element $\mathcal{C}$ is of the form $!\mathcal{A}$, for some $\mathcal{A}$.

A *sequent* is a pair of multisets of formulas separated by $\vdash$. The *sequent calculus* CLL_{int} comprises the following *axiom* and *rules*. The last four rules are modalized structural rules. The rest of the rules are connective rules.

$$\mathcal{A} \vdash \mathcal{A} \quad \text{id}$$

$$\frac{\Gamma \vdash \Delta; \mathcal{A}}{\mathcal{A}^{\perp}; \Gamma \vdash \Delta}\ {}^{\perp}\vdash \qquad \frac{\mathcal{A}; \Gamma \vdash \Delta}{\Gamma \vdash \Delta; \mathcal{A}^{\perp}}\ \vdash^{\perp}$$

$$\frac{\mathcal{A}; \mathcal{B}; \Gamma \vdash \Delta}{\mathcal{A} \otimes \mathcal{B}; \Gamma \vdash \Delta}\ \otimes\vdash \qquad \frac{\Gamma \vdash \Delta; \mathcal{A} \qquad \Theta \vdash \Lambda; \mathcal{B}}{\Gamma; \Theta \vdash \Delta; \Lambda; \mathcal{A} \otimes \mathcal{B}}\ \vdash\otimes$$

$$\frac{\Gamma \vdash \Delta; \mathcal{A} \qquad \mathcal{B}; \Theta \vdash \Lambda}{\mathcal{A} \multimap \mathcal{B}; \Gamma; \Theta \vdash \Delta; \Lambda}\ \multimap\vdash \qquad \frac{\mathcal{A}; \Gamma \vdash \Delta; \mathcal{B}}{\Gamma \vdash \Delta; \mathcal{A} \multimap \mathcal{B}}\ \vdash\multimap$$

$$\frac{\mathcal{A}; \Gamma \vdash \Delta \qquad \mathcal{B}; \Theta \vdash \Lambda}{\mathcal{A} ⅋ \mathcal{B}; \Gamma; \Theta \vdash \Delta; \Lambda}\ ⅋\vdash \qquad \frac{\Gamma \vdash \Delta; \mathcal{A}; \mathcal{B}}{\Gamma \vdash \Delta; \mathcal{A} ⅋ \mathcal{B}}\ \vdash⅋$$

$$\frac{\mathcal{A}; \Gamma \vdash \Delta}{!\mathcal{A}; \Gamma \vdash \Delta}\ !\vdash \qquad \frac{\Gamma^! \vdash \Delta^?; \mathcal{A}}{\Gamma^! \vdash \Delta^?; !\mathcal{A}}\ \vdash!$$

$$\frac{\mathcal{A}; \Gamma^! \vdash \Delta^?}{?\mathcal{A}; \Gamma^! \vdash \Delta^?}\ ?\vdash \qquad \frac{\Gamma \vdash \Delta; \mathcal{A}}{\Gamma \vdash \Delta; ?\mathcal{A}}\ \vdash?$$

$$\frac{\Gamma \vdash \Delta}{!\mathcal{A}; \Gamma \vdash \Delta}\ !K\vdash \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta; ?\mathcal{A}}\ \vdash?K$$

$$\frac{!\mathcal{A}; !\mathcal{A}; \Gamma \vdash \Delta}{!\mathcal{A}; \Gamma \vdash \Delta}\ !W\vdash \qquad \frac{\Gamma \vdash \Delta; ?\mathcal{A}; ?\mathcal{A}}{\Gamma \vdash \Delta; ?\mathcal{A}}\ \vdash?W$$

The notion of a *proof* is usual; that is, a proof is a tree of occurrences of sequents where the leaves are instances of the axiom, and parent nodes are obtained by an application of a rule to their children. The term "tree" in this paper means a

3 "Intensional" is used in accordance with the use of this term in Bimbó and Dunn [9]. CLL_{int} is or is very closely related to *MELL* — depending on what exactly is meant by that acronym.

4 As a compromise between the unconventional notation of Girard [15] and the standard terminology in non-classical logics, I use Girard's symbols with the customary names of the connectives. A translation may be found in Avron [3], though the modalities here are switched for semantic reasons as explained in Bimbó and Dunn [9, Ch. 3].

5 By "multiset" we always mean a *finite* multiset; hence, we drop "finite." Many logics can be formulated as sequent calculi based on multisets — see e.g., Meyer and McRobbie [29] and [30] for related logics.

directed acyclic graph with a root, which is the usual sense of the term in logic.[6] A formula $\mathcal{A}$ is a *theorem* of CLL_{int} iff there is a proof in which the root is the sequent $\vdash \mathcal{A}$. We assume an *analysis* along the usual lines. We use the term "*principal formula*" and call the displayed subformulas of the principal formula *subalterns*. Other formulas are *parametric*, and a multiset of parametric formulas is a *parametric multiset*, for short. (See Definition 4 for more details.)

The use of multisets in sequents — together with the $(\multimap\vdash)$, $(\vdash\multimap)$ rules — means that $\multimap$ is the implication of BCI logic (when ! and ? are disregarded). The operation $\otimes$ is *abstractly residuated* with $\multimap$, and the two operations constitute a complete family. (See Bimbó and Dunn [9], Chaps. 1–5 for the definitions and uses of these notions.)

It is easy to see that the rules, save those for $\multimap$, are left–right *symmetric* with dual connectives standing for each other. This is a feature that Gentzen deemed to be delightful in *LK*, his sequent calculus for classical logic. Permitting multiple formulas on the right-hand side of the $\vdash$ in *all* rules is typical of sequent calculi that are variations on *LK*. Intuitionistic sequent calculi (or sequent calculi for *J*, intuitionistic logic) tend to restrict the multiplicity on the right-hand side of the turnstile. Gentzen's *LJ* is an example of such a calculus. We follow Troelstra [36] in calling the version of linear logic in which sequents do not adhere to an intuitionistic restriction *classical*.

Modal logic has a long history with certain normal modal logics such as K, T, B, $S4$ and $S5$ holding an enshrined central position. Assuming classical logic as a background, it is natural to view $(\vdash ?)$ as a rule for $\Diamond$ introduction on the right, because it can be informally read as "if $\mathcal{A}$ then $\Diamond\mathcal{A}$." The usual philosophical interpretation of this sentence is also plausible, because "what is actual, is possible" seems to make sense under the alethic meaning of $\Diamond$. However, outside of the context of classical logic (or of a Boolean algebra), the two monotone operators $\Diamond$ and $\Box$ become more similar.[7] Once the extensional connectives $\&$ (conjunction) and $\oplus$ (disjunction) are omitted too, the differences further diminish. From a semantical point of view, it is more appropriate to think of ! as $\Diamond$, and ? as $\Box$.[8] $!\mathcal{A} \multimap \mathcal{A}$, a theorem of CLL_{int}, can be given a corresponding informal reading, which is quite plausible in computer science and mathematics. For instance, if a value of a function could be computed by an algorithm, then the program better compute it. Similarly, if a mathematical structure *could have* some property, then it is often so defined that it *does have* the property in question. In other words, all possibilities are realized. (I promulgated "the ! is $\Diamond$, the ? is $\Box$" view in Bimbó [7] too.)

The interdefinability of $\Diamond$ and $\Box$ using negation is a common place in normal modal logics. The negation in linear logic is easily seen to be of period two, and allowing all forms of contraposition to be proved in the form $(\mathcal{X} \multimap \mathcal{Y}) \multimap \mathcal{Y}' \multimap \mathcal{X}'$ (where either the primed formula or its unprimed pair has an $^\perp$). Kripke [21] introduced rules into a sequent calculus for modal logics — where both $\Diamond$ and $\Box$ are primitive — that allow the interdefinability of the modalities to be proved. The $(\vdash !)$ and $(?\vdash)$ rules are analogs of Kripke's rules. Indeed, ! and ? are interdefinable in CLL_{int} by the usual "modal de Morgan laws."

Linear logic is sometimes classified as a relevance logic (e.g., in Bimbó [7]), however, not always (e.g., in Avron [4]). If we disregard the last six rules of CLL_{int} above, and we restrict the $(\vdash !)$ rule to a single formula on the right-hand side, then we see the contraction- and $\Diamond$-free intensional fragment of $R^{\Box}$, which is relevant implication with a modality but without contraction. (Of course, thinning, which is a rule typically derided in relevance logics — and for good reason — is absent in sequent calculi for various relevance logics.) The *real novelty* in linear logic is that the classical contraction and thinning rules are reinstated, however, only for certain modalized formulas. On the left-hand side of the $\vdash$, formulas starting with ! can be thinned in and contracted, whereas on the right-hand side, the same applies to ?'d formulas. The modalization of contraction limits the amount of possible contractions (hence, shrinks the proof-search space), at the same time, it adds some complications to the proof of the decidability.

2.2. Contraction-free sequent calculus: $[CLL_{\mathrm{int}}]$

The standard method to prove a logic decidable using a sequent calculus is to eliminate all contraction-like rules — without altering the set of provable sequents. If thinning is a rule in a calculus, then the axiom can be generalized to include arbitrary formulas both on the left and on the right. At the same time, principal formulas may be included in the premises. This method works for classical and intuitionistic logics — as was shown by Curry and Kleene. Alternatively, contraction can be absorbed into the other rules; most importantly, into the connective rules — without a more general axiom or a repetition of principal formulas. This idea reflects a *deep* understanding of sequent calculi, and it was introduced by Kripke to prove the decidability of certain logics that have no weakening. The latter approach is what we adopt here.

Contraction is very limited in linear logic. Only !'d formulas can be contracted on the left-hand side of the $\vdash$, and only ?'d formulas can be contracted on the right. Accordingly, the bracket notation in the definition of the modified CLL_{int} calculus, that we denote by $[CLL_{\mathrm{int}}]$, is context sensitive — there is no ambiguity though. The brackets are context sensitive exactly as the structural connective is, whether it is a comma or a semicolon. (The constraints imposed by [ ] are explained immediately after the list of rules.)

Our *contraction-free* calculus $[CLL_{\mathrm{int}}]$ is defined by the following axiom and rules.

[6] See the classic text Smullyan [34] or Bimbó [8, App. A] for formal definitions of trees.

[7] See for example, Dunn [14] for some modal logics in a negation-free context.

[8] Relational semantics for linear logic may be found, for example, in Allwein and Dunn [1] and in Bimbó and Dunn [9, Ch. 3].

$$\mathcal{A} \vdash \mathcal{A} \ \ \text{id}$$

$$\frac{\Gamma \vdash \Delta; \mathcal{A}}{\mathcal{A}^{\perp}; \Gamma \vdash \Delta}\ {}^{\perp}\!\vdash \qquad \frac{\mathcal{A}; \Gamma \vdash \Delta}{\Gamma \vdash \Delta; \mathcal{A}^{\perp}}\ \vdash^{\perp}$$

$$\frac{\mathcal{A}; \mathcal{B}; \Gamma \vdash \Delta}{\mathcal{A} \otimes \mathcal{B}; \Gamma \vdash \Delta}\ \otimes\vdash \qquad \frac{\Gamma \vdash \Delta; \mathcal{A} \qquad \Theta \vdash \Lambda; \mathcal{B}}{[\Gamma; \Theta] \vdash [\Delta; \Lambda; \mathcal{A} \otimes \mathcal{B}]}\ [\vdash\otimes]$$

$$\frac{\Gamma \vdash \Delta; \mathcal{A} \qquad \mathcal{B}; \Theta \vdash \Lambda}{[\mathcal{A} \multimap \mathcal{B}; \Gamma; \Theta] \vdash [\Delta; \Lambda]}\ [\multimap\vdash] \qquad \frac{\mathcal{A}; \Gamma \vdash \Delta; \mathcal{B}}{\Gamma \vdash \Delta; \mathcal{A} \multimap \mathcal{B}}\ \vdash\multimap$$

$$\frac{\mathcal{A}; \Gamma \vdash \Delta \qquad \mathcal{B}; \Theta \vdash \Lambda}{[\mathcal{A} \parr \mathcal{B}; \Gamma; \Theta] \vdash [\Delta; \Lambda]}\ [\parr\vdash] \qquad \frac{\Gamma \vdash \Delta; \mathcal{A}; \mathcal{B}}{\Gamma \vdash \Delta; \mathcal{A} \parr \mathcal{B}}\ \vdash\parr$$

$$\frac{\mathcal{A}; \Gamma \vdash \Delta}{[!\mathcal{A}; \Gamma] \vdash \Delta}\ [!\vdash] \qquad \frac{\Gamma^{!} \vdash \Delta^{?}; \mathcal{A}}{\Gamma^{!} \vdash \Delta^{?}; !\mathcal{A}}\ \vdash!$$

$$\frac{\mathcal{A}; \Gamma^{!} \vdash \Delta^{?}}{?\mathcal{A}; \Gamma^{!} \vdash \Delta^{?}}\ ?\vdash \qquad \frac{\Gamma \vdash \Delta; \mathcal{A}}{\Gamma \vdash [\Delta; ?\mathcal{A}]}\ [\vdash?]$$

$$\frac{\Gamma \vdash \Delta}{!\mathcal{A}; \Gamma \vdash \Delta}\ !K\vdash \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta; ?\mathcal{A}}\ \vdash?K$$

It may be useful to emphasize the obvious fact that the connectives denoted by different symbols are pairwise distinct. This means that in most of the rules the principal formula cannot coincide with a contracted formula. Furthermore, in the ($\vdash!$) and ($?\vdash$) rules, the modal restrictions on the rest of the sequent would exclude the possibility of the principal formula being contracted, even if arbitrary formulas could be contracted on either side. To sum up, the bracketing pertains primarily to cases when multisets of parametric formulas are joined. We detail the restrictions for the bracketed rules. The permitted contractions may never result in the *complete loss of a formula* from the multiset; nor the rules *force a contraction*.

([$\vdash\otimes$]) $[\Gamma; \Theta]$ is the multiset $\Gamma; \Theta$ with the proviso that if a formula of the form $!\mathcal{C}$ occurs in Γ and Θ in the premises of the rule, then one of the occurrences of $!\mathcal{C}$ may be omitted in the conclusion of the rule.
$[\Delta; \Lambda; \mathcal{A} \otimes \mathcal{B}]$ is the multiset $\Delta; \Lambda; \mathcal{A} \otimes \mathcal{B}$ with the proviso that if a formula of the form $?\mathcal{C}$ occurs in Δ and Λ in the premises of the rule, then one of the occurrences of $?\mathcal{C}$ may be omitted in the conclusion of the rule.

([$\multimap\vdash$]) $[\mathcal{A} \multimap \mathcal{B}; \Gamma; \Theta]$ is the multiset $\mathcal{A} \multimap \mathcal{B}; \Gamma; \Theta$ except that if a !'d formula, $!\mathcal{C}$ occurs both in Γ and Θ, then one of the occurrences may be omitted in the lower sequent of the rule.
$[\Delta; \Lambda]$ is the multiset $\Delta; \Lambda$, except that if a ?'d formula, $?\mathcal{C}$ occurs both in Δ and Λ, then one of the occurrences may be omitted in the lower sequent of the rule.

([$\parr\vdash$]) $[\mathcal{A} \parr \mathcal{B}; \Gamma; \Theta]$ is the multiset $\mathcal{A} \parr \mathcal{B}; \Gamma; \Theta$ in which a formula $!\mathcal{C}$ occurring both in Γ and Θ may have one fewer occurrence.
$[\Delta; \Lambda]$ is the multiset $\Delta; \Lambda$ in which a formula $?\mathcal{C}$ occurring both in Δ and Λ, then one of the occurrences may be omitted.

([$!\vdash$]) $[!\mathcal{A}; \Gamma]$ is the multiset $!\mathcal{A}; \Gamma$, where $!\mathcal{A}$ may occur as many times as in Γ.

([$\vdash?$]) $[\Delta; ?\mathcal{A}]$ is the multiset $\Delta; ?\mathcal{A}$, where $?\mathcal{A}$ may occur as many times as in Δ.

The notions of a *proof* and of a *theorem* are unchanged. The description of the rules may leave the impression that there is something unsettled in the rules. The rules are "more schematic" than rules formulated with metavariables in the sense that in ([$\vdash\otimes$]), ([$\multimap\vdash$]) and ([$\parr\vdash$]) the number of contractions can be arbitrarily large, because a sequent may be of arbitrarily large (but finite) size. However, this indeterminacy is exactly like instantiating metavariables with formulas (that can contain as many connectives as one wishes) and applying structural contraction rules in certain substructural logics (where contraction applies to sequences of formulas or to more refined structures built from formulas). Given a tree comprising (occurrences of) sequents, it is decidable whether the tree amounts to a proof; moreover, it can be determined if an application of a bracketed rule included contractions (and if it did, then how many and of which formulas).

2.3. *Intensional interlinear logic: RLL_{int} and $(\!|RLL_{\text{int}}|\!)$*

The definition of bracketing in [CLL_{int}] is complicated, because contraction is selective. A slight modification of CLL_{int} yields a much better behaved logic that we call *intensional interlinear logic* and denote by RLL_{int}.[9]

The *sequent calculus* RLL_{int} comprises the same axiom and rules as CLL_{int} does — save the ($!W\vdash$) and ($\vdash?W$) rules, which are replaced by ($W\vdash$) and ($\vdash W$). The latter rules are as follows.

$$\frac{\mathcal{A}; \mathcal{A}; \Gamma \vdash \Delta}{\mathcal{A}; \Gamma \vdash \Delta}\ W\vdash \qquad \frac{\Gamma \vdash \Delta; \mathcal{A}; \mathcal{A}}{\Gamma \vdash \Delta; \mathcal{A}}\ \vdash W$$

[9] The "*R*" in RLL_{int} may be thought to stand for "relevant," but ($!K\vdash$) and ($\vdash?K$) distort this feature of the logic.

In order to extract bounds for the "backward application" of the ($[!\vdash]$) and ($[\vdash ?]$) rules in the proof-search tree in CLL_{int}, we introduce $(\!|RLL_{\text{int}}|\!)$, which consists of the following axiom and rules.

$$\mathcal{A} \vdash \mathcal{A} \quad \text{id}$$

$$\frac{\Gamma \vdash \Delta; \mathcal{A}}{(\!|\mathcal{A}^{\perp}; \Gamma|\!) \vdash \Delta}\ (\!|{}^{\perp}\vdash|\!) \qquad \frac{\mathcal{A}; \Gamma \vdash \Delta}{\Gamma \vdash (\!|\Delta; \mathcal{A}^{\perp}|\!)}\ (\!|\vdash{}^{\perp}|\!)$$

$$\frac{\mathcal{A}; \mathcal{B}; \Gamma \vdash \Delta}{(\!|\mathcal{A} \otimes \mathcal{B}; \Gamma|\!) \vdash \Delta}\ (\!|\otimes\vdash|\!) \qquad \frac{\Gamma \vdash \Delta; \mathcal{A} \qquad \Theta \vdash \Lambda; \mathcal{B}}{(\!|\Gamma; \Theta|\!) \vdash (\!|\Delta; \Lambda; \mathcal{A} \otimes \mathcal{B}|\!)}\ (\!|\vdash\otimes|\!)$$

$$\frac{\Gamma \vdash \Delta; \mathcal{A} \qquad \mathcal{B}; \Theta \vdash \Lambda}{(\!|\mathcal{A} \multimap \mathcal{B}; \Gamma; \Theta|\!) \vdash (\!|\Delta; \Lambda|\!)}\ (\!|\multimap\vdash|\!) \qquad \frac{\mathcal{A}; \Gamma \vdash \Delta; \mathcal{B}}{\Gamma \vdash (\!|\Delta; \mathcal{A} \multimap \mathcal{B}|\!)}\ (\!|\vdash\multimap|\!)$$

$$\frac{\mathcal{A}; \Gamma \vdash \Delta \qquad \mathcal{B}; \Theta \vdash \Lambda}{(\!|\mathcal{A} \parr \mathcal{B}; \Gamma; \Theta|\!) \vdash (\!|\Delta; \Lambda|\!)}\ (\!|\parr\vdash|\!) \qquad \frac{\Gamma \vdash \Delta; \mathcal{A}; \mathcal{B}}{\Gamma \vdash (\!|\Delta; \mathcal{A} \parr \mathcal{B}|\!)}\ (\!|\vdash\parr|\!)$$

$$\frac{\mathcal{A}; \Gamma \vdash \Delta}{(\!|!\mathcal{A}; \Gamma|\!) \vdash \Delta}\ (\!|!\vdash|\!) \qquad \frac{\Gamma^{!} \vdash \Delta^{?}; \mathcal{A}}{\Gamma^{!} \vdash \Delta^{?}; !\mathcal{A}}\ \vdash!$$

$$\frac{\mathcal{A}; \Gamma^{!} \vdash \Delta^{?}}{?\mathcal{A}; \Gamma^{!} \vdash \Delta^{?}}\ ?\vdash \qquad \frac{\Gamma \vdash \Delta; \mathcal{A}}{\Gamma \vdash (\!|\Delta; ?\mathcal{A}|\!)}\ (\!|\vdash?|\!)$$

$$\frac{\Gamma \vdash \Delta}{!\mathcal{A}; \Gamma \vdash \Delta}\ !K\vdash \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta; ?\mathcal{A}}\ \vdash?K$$

The $(\!|\ \ |\!)$'s permit contractions of *arbitrary formulas*. The potential contractions fall into three groups: (1) two parametric multisets are joined, (2) a parametric multiset and the principal formula are combined and (3) two parametric multisets and the principal formula are joined. Cases (1) and (2) are similar to the cases for $[\ \ ]$ without a consideration for the shape of the formulas. In case (3), the principal formula may be contracted once or twice if it occurs in one or both parametric multisets; for other formulas, this case is like (1). All the contractions are optional and they cannot cause the disappearance of a formula.

3. Cut theorems

The cut rule is of exceptional importance in sequent calculi. However, we want to be able to consider cut-free proofs when looking for proofs. Hence, we want to establish the admissibility of the cut rule for each calculus that we defined in Section 2.

Definition 1 *(Single cut rule).* The *single cut rule* takes the following form.

$$\frac{\Gamma \vdash \Delta; \mathcal{C} \qquad \mathcal{C}; \Theta \vdash \Lambda}{\Gamma; \Theta \vdash \Delta; \Lambda}\ \text{cut}$$

In some proofs of the cut theorems the following versions of the cut are also helpful.

$$\frac{\Gamma \vdash \Delta; \mathcal{C} \qquad \mathcal{C}; \Theta \vdash \Lambda}{[\Gamma; \Theta] \vdash [\Delta; \Lambda]}\ [\text{cut}] \qquad \frac{\Gamma \vdash \Delta; \mathcal{C} \qquad \mathcal{C}; \Theta \vdash \Lambda}{(\!|\Gamma; \Theta|\!) \vdash (\!|\Delta; \Lambda|\!)}\ (\!|\text{cut}|\!)$$

In the [cut] rule, $[\Gamma; \Theta]$ is the multiset $\Gamma; \Theta$ — except that if $!\mathcal{A}$ occurs in Γ and Θ, then one of those occurrences may be omitted. Similarly, $[\Delta; \Lambda]$ is either $\Delta; \Lambda$ or if a formula of the form $?\mathcal{B}$ occurs both in Δ and Λ, then one of the $?\mathcal{B}$'s may be omitted. Similarly, for $(\!|\Gamma; \Theta|\!)$ and $(\!|\Delta; \Lambda|\!)$, but without a restriction on the shape of the contracted formulas.

The cut rules are related. The [cut] and the (|cut|) rules do not necessitate any contractions, hence, cut is an *instance* of [cut] and of (|cut|). On the other hand, [cut] is *derivable* in CLL_{int} from cut by applications of the $(!W\vdash)$ and $(\vdash ?W)$ rules, and (|cut|) is derivable in RLL_{int} from cut by applications of the $(W\vdash)$ and $(\vdash W)$ rules. It is opportune to record the following lemma at this point.

Lemma 2. *If* $\vdash \mathcal{A}$ *is provable in* $[CLL_{\text{int}}]$, *then it is* provable *in* CLL_{int}. *If* $\vdash \mathcal{A}$ *is provable in* $(\!|RLL_{\text{int}}|\!)$, *then it is* provable *in* RLL_{int}.

Proof. The axiom is the same in the calculi. Each rule of $[CLL_{\text{int}}]$ is a rule in CLL_{int} or it is derived from one by allowing contractions of !'d formulas on the left or ?'d formulas on the right. That is, the bracketed sequences of formulas are derivable by finitely many (including zero many) applications of the $(!W\vdash)$ or $(\vdash ?W)$ rules from the lower sequent of the similarly labeled rule. The reasoning is alike for $(\!|RLL_{\text{int}}|\!)$ and RLL_{int}. □

Of course, we need the converses of the claims in the lemma, which follow from Theorems 10 and 13, and from Theorems 12 and 14.

Definition 3 *(Degree of a formula).* The *degree* of a formula $\mathcal{A}$, denoted by $\delta(\mathcal{A})$, is inductively defined by clauses (1)–(3).

(1) If $\mathcal{A}$ is a propositional variable, then $\delta(\mathcal{A}) = 0$.
(2) If $\mathcal{A}$ is $\mathcal{B}^{\perp}$, $!\mathcal{B}$ or $?\mathcal{B}$, then $\delta(\mathcal{A}) = \delta(\mathcal{B}) + 1$.
(3) If $\mathcal{A}$ is $(\mathcal{B} \otimes \mathcal{C})$, $(\mathcal{B} \multimap \mathcal{C})$ or $(\mathcal{B} \parr \mathcal{C})$, then $\delta(\mathcal{A}) = \delta(\mathcal{B}) + \delta(\mathcal{C}) + 1$.

The least possible degree of a formula is 0, and the degree of a compound formula is a positive integer that is strictly greater than the degree of any proper subformula of the formula.

Given a proof, we assume that in each multiset repeated copies of a formula are indexed by positive integers $1, 2, 3, \ldots$ and a formula occurring once is subscripted by 1 (for reference). The analysis is defined relative to a fixed indexing throughout the proof. In particular, we assume that the formulas that are explicitly mentioned in the rules have a particular index. The role of the indexing is most prominent in the definition of the contraction measure and the rank of the cut, but it is also useful in talking about parametric formulas. An alternative way to think about the indexing is to say that we revert to sequences of formulas as antecedents and succedents in a sequent and we assume that there are permutation rules on the left and on the right, but we always omit permutation steps from a proof, which would be a step in the direction of the $G3$ calculi of Kleene.

Definition 4 *(Analysis).* The formulas in the rules are divided into three mutually disjoint categories, namely, *parametric formulas*, *subalterns* and *principal formulas*.

(1) The $\mathcal{A}$'s and $\mathcal{B}$'s displayed in the connective rules are *subalterns*. The $!\mathcal{A}$'s and $?\mathcal{A}$'s in the upper sequent of the rules $(!W\vdash)$ and $(\vdash ?W)$, respectively, are subalterns too. So are the $\mathcal{A}$'s in the $(W\vdash)$ and $(\vdash W)$ rules.
(2) The compound formulas displayed in the lower sequent of the rules are the *principal formulas*. The principal formula of the $(W\vdash)$ and $(\vdash W)$ rules is $\mathcal{A}$ in the lower sequents.
(3) The formulas that are in Γ, Δ, Θ and Λ are *parametric formulas*.

The analysis is fairly standard, possibly except the treatment of the (modalized) structural rules, in which we consider the formulas that are affected by the rules as subalterns (in the premises) and principal formulas (in the conclusions).

Definition 5 *(Ancestor).* Let an indexing and the analysis for a proof be fixed. The *ancestor* relation is the transitive closure of the *immediate ancestor* relation defined by (1)–(3).

(1) If $\mathcal{A}$ occurs in the lower sequent of a rule and it is parametric, then the matching $\mathcal{A}$ in the upper sequent is an immediate ancestor of $\mathcal{A}$.
(2) If $!\mathcal{A}$ is the principal formula of $(!W\vdash)$, then the two subalterns of the same shape in the upper sequent are immediate ancestors of $!\mathcal{A}$.
(3) If $?\mathcal{A}$ is the principal formula of $(\vdash ?W)$, then the two subalterns of the same shape in the upper sequent are immediate ancestors of $?\mathcal{A}$.
(4) If $\mathcal{A}$ is the principal formula of $(W\vdash)$ or $(\vdash W)$, then the two subalterns of the same shape in the upper sequent are immediate ancestors of $\mathcal{A}$.

A formula may *not have* ancestors in a proof, and a formula may *have many* ancestors. The formula and all its ancestors are of the same shape, and they all occur on the same side of the turnstile. One may think about the ancestors of a formula as certain occurrences of the same formula that are selected (due to the indexing) to be linked to the formula itself.

Definition 6 *(Rank of cut).* The *rank* of an application of a cut rule, denoted by ϱ, is the sum of the *left rank* (ϱ_l) and of the *right rank* (ϱ_r) of the cut. The left rank of the cut is the maximal number of consecutive sequents above the left premise of the cut in which ancestors of the cut formula occur in the succedent of the sequent plus 1. The right rank of the cut is the maximal number of consecutive sequents above the right premise of the cut in which ancestors the cut formula occur in the antecedent of the sequent plus 1.

The least possible value of ϱ_l is 1, and symmetrically, ϱ_r cannot be less than 1. For any cut in a proof, $\varrho \geq 2$.

Definition 7 *(Contraction measure).* If the cut formula is not of the form $!\mathcal{A}$ or $?\mathcal{A}$, then the *contraction measure* of the cut, denoted by μ, is 0. Otherwise, the contraction measure of the cut is the number of applications of the $(!W\vdash)$ or $(\vdash ?W)$ rules to ancestors of the cut formula.

For example, if the right premise of a cut with the cut formula being $!\mathcal{A}$ is by $(!W\vdash)$ with the principal formula being the cut formula, then μ is at least 1, because the rule affects the ancestors of $!\mathcal{A}$. If the cut were performed on the ancestors separately, then the contraction measure of those cuts would be strictly less than before, because the last contraction is subtracted from μ (plus each of the two ancestors may have a contraction measure greater than zero). In RLL_{int}, where $(W\vdash)$ and $(\vdash W)$ are unrestricted, the contraction measure of a cut is the number of applications of the $(W\vdash)$ and $(\vdash W)$ rules to ancestors of the cut formula.

Theorem 8 *(Cut theorem for CLL_{int}). The* cut *rule is* admissible *in CLL_{int}.*

Proof. The proof is by *triple induction* on the rank of the cut, on the contraction measure of the cut and on the degree of the cut formula. We outline some of the details of the proof — also with an eye toward referring back to this proof in the proof of Theorem 10. (We use $\mathcal{C}$ as a meta-variable for the cut formula.)

1. If $\delta(\mathcal{C})=0$ and $\varrho=2$, then $\mu=0$. The only possibility is that both premises are instances of (id) with the same atomic formula instantiating $\mathcal{A}$. By retaining one of the premises, the cut is eliminated.

2. If $\delta(\mathcal{C})=0$ and $\varrho>2$, then $\mu=0$. Either $\varrho_l>1$ or $\varrho_r>1$. The general shape of the segment of the proof ending in the cut and its transformation is one of the following two.

$$\text{cut}\ \frac{\text{rule}\ \dfrac{\begin{array}{c}\vdots\\ \Gamma'\vdash\Delta';p\end{array}}{\Gamma\vdash\Delta;p} \qquad \begin{array}{c}\vdots\\ p;\Theta\vdash\Lambda\end{array}}{\Gamma;\Theta\vdash\Delta;\Lambda} \quad\rightsquigarrow\quad \frac{\dfrac{\begin{array}{c}\vdots\\ \Gamma'\vdash\Delta';p\end{array} \qquad \begin{array}{c}\vdots\\ p;\Theta\vdash\Lambda\end{array}}{\Gamma';\Theta\vdash\Delta';\Lambda}\ \text{cut}}{\Gamma;\Theta\vdash\Delta;\Lambda}\ \text{rule}$$

$$\text{cut}\ \frac{\begin{array}{c}\vdots\\ \Gamma\vdash\Delta;p\end{array} \qquad \dfrac{\begin{array}{c}\vdots\\ p;\Theta'\vdash\Lambda'\end{array}}{p;\Theta\vdash\Lambda}\ \text{rule}}{\Gamma;\Theta\vdash\Delta;\Lambda} \quad\rightsquigarrow\quad \frac{\dfrac{\begin{array}{c}\vdots\\ \Gamma\vdash\Delta;p\end{array} \qquad \begin{array}{c}\vdots\\ p;\Theta'\vdash\Lambda'\end{array}}{\Gamma;\Theta'\vdash\Delta;\Lambda'}\ \text{cut}}{\Gamma;\Theta\vdash\Delta;\Lambda}\ \text{rule}$$

The two restricted rules could not have been used in place of the "rule," hence, there is no problem with swapping the cut rule and the "rule." In both cases, ϱ is decreased by 1.

3. If $\delta(\mathcal{C})>0$ and $\varrho=2$, then $\mu=0$. We consider three groups of cases.

3.1. If a premise is an instance of the axiom (id), then the tree rooted in the other premise suffices as the proof of the same end sequent. The cut is fully eliminated.

3.2. If both premises are by a non-modal connective rule (with the same main connective in the principal formula), which is the cut formula, then there are four subcases. All these subcases are dealt with by replacing the cuts with cuts on formulas of lower degree. Here are the details when the connective is $\parr$. (We omit the details in the rest of the subcases.)

$$\text{cut}\ \frac{\vdash\parr\ \dfrac{\begin{array}{c}\vdots\\ \Gamma\vdash\Delta;\mathcal{A};\mathcal{B}\end{array}}{\Gamma\vdash\Delta;\mathcal{A}\parr\mathcal{B}} \qquad \dfrac{\begin{array}{c}\vdots\\ \mathcal{A};\Theta\vdash\Lambda\end{array} \qquad \begin{array}{c}\vdots\\ \mathcal{B};\Xi\vdash\Psi\end{array}}{\mathcal{A}\parr\mathcal{B};\Theta;\Xi\vdash\Lambda;\Psi}\ \parr\vdash}{\Gamma;\Theta;\Xi\vdash\Delta;\Lambda;\Psi} \quad\rightsquigarrow$$

$$\rightsquigarrow\quad \frac{\text{cut}\ \dfrac{\begin{array}{c}\vdots\\ \Gamma\vdash\Delta;\mathcal{A};\mathcal{B}\end{array} \qquad \begin{array}{c}\vdots\\ \mathcal{A};\Theta\vdash\Lambda\end{array}}{\Gamma;\Theta\vdash\Delta;\Lambda;\mathcal{B}} \qquad \begin{array}{c}\vdots\\ \mathcal{B};\Xi\vdash\Psi\end{array}}{\Gamma;\Theta;\Xi\vdash\Delta;\Lambda;\Psi}\ \text{cut}$$

3.3. If the cut formula is modalized, then there are four subcases to consider, because ! on the left and ? on the right may be introduced in two different ways. We give each segment of a proof and its transformation.

$$\text{cut}\ \frac{\vdash!\ \dfrac{\begin{array}{c}\vdots\\ \Gamma^!\vdash\Delta^?;\mathcal{A}\end{array}}{\Gamma^!\vdash\Delta^?;!\mathcal{A}} \qquad \dfrac{\begin{array}{c}\vdots\\ \mathcal{A};\Theta\vdash\Lambda\end{array}}{!\mathcal{A};\Theta\vdash\Lambda}\ !\vdash}{\Gamma^!;\Theta\vdash\Delta^?;\Lambda} \quad\rightsquigarrow\quad \frac{\begin{array}{c}\vdots\\ \Gamma^!\vdash\Delta^?;\mathcal{A}\end{array} \qquad \begin{array}{c}\vdots\\ \mathcal{A};\Theta\vdash\Lambda\end{array}}{\Gamma^!;\Theta\vdash\Delta^?;\Lambda}\ \text{cut}$$

The degree of the new cut is lower than that of the original cut.

$$\text{cut}\ \frac{\vdash!\ \dfrac{\begin{array}{c}\vdots\\ \Gamma^!\vdash\Delta^?;\mathcal{A}\end{array}}{\Gamma^!\vdash\Delta^?;!\mathcal{A}} \qquad \dfrac{\begin{array}{c}\vdots\\ \Theta\vdash\Lambda\end{array}}{!\mathcal{A};\Theta\vdash\Lambda}\ !K\vdash}{\Gamma^!;\Theta\vdash\Delta^?;\Lambda} \quad\rightsquigarrow\quad \frac{\dfrac{\begin{array}{c}\vdots\\ \Theta\vdash\Lambda\end{array}}{\Gamma^!;\Theta\vdash\Lambda}\ !K\vdash\text{'s}}{\Gamma^!;\Theta\vdash\Delta^?;\Lambda}\ \vdash ?K\text{'s}$$

The cut is fully eliminated. (The thicker lines indicate finitely many applications of one and the same rule.)

$$\dfrac{\dfrac{\overset{\vdots}{\Gamma \vdash \Delta; \mathcal{A}}}{\Gamma \vdash \Delta; ?\mathcal{A}}\,{\scriptstyle \vdash ?} \qquad \dfrac{\overset{\vdots}{\mathcal{A}; \Theta^! \vdash \Lambda^?}}{?\mathcal{A}; \Theta^! \vdash \Lambda^?}\,{\scriptstyle ?\vdash}}{\Gamma; \Theta^! \vdash \Delta; \Lambda^?}\,{\scriptstyle \mathrm{cut}} \quad \rightsquigarrow \quad \dfrac{\overset{\vdots}{\Gamma \vdash \Delta; \mathcal{A}} \qquad \overset{\vdots}{\mathcal{A}; \Theta^! \vdash \Lambda^?}}{\Gamma; \Theta^! \vdash \Delta; \Lambda^?}\,{\scriptstyle \mathrm{cut}}$$

$$\dfrac{\dfrac{\overset{\vdots}{\Gamma \vdash \Delta}}{\Gamma \vdash \Delta; ?\mathcal{A}}\,{\scriptstyle \vdash ?K} \qquad \dfrac{\overset{\vdots}{\mathcal{A}; \Theta^! \vdash \Lambda^?}}{?\mathcal{A}; \Theta^! \vdash \Lambda^?}\,{\scriptstyle ?\vdash}}{\Gamma; \Theta^! \vdash \Delta; \Lambda^?}\,{\scriptstyle \mathrm{cut}} \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\vdots}{\Gamma \vdash \Delta}}{\Gamma; \Theta^! \vdash \Delta}\,{\scriptstyle !K\vdash\text{'s}}}{\Gamma; \Theta^! \vdash \Delta; \Lambda^?}\,{\scriptstyle \vdash ?K\text{'s}}$$

The transformations and their justifications in the latter two cases are similar to the previous two.

4. If $\delta(\mathcal{C}) > 0$, $\varrho > 2$ and $\mu = 0$, then either $\varrho_l \geq 2$ or $\varrho_r \geq 2$. The left–right symmetry of CLL_{int} guarantees that it is sufficient to detail one or the other situation. Let us assume that $\varrho_l \geq 2$.

4.1. If the left premise is by one of the left rules save $(?\vdash)$, then $\mathcal{C}$ is parametric in the succedent of the left premise. Swapping the application of the left rule and the application of the cut rule ensures a decrease in the left rank by 1.

4.2. If the left premise is by $(?\vdash)$, then $\mathcal{C}$ is of the form $?\mathcal{D}$, because of the restriction on the succedent in the rule. The latter immediately excludes the possibility that the right premise is by $(\vdash !)$. ϱ_r may or may not be ≥ 2. If the right premise is an instance of the axiom, then the cut is eliminated as before.

If $\varrho_r \geq 2$, then $?\mathcal{D}$ is parametric if a right rule yields the right premise. $?\mathcal{D}$ is also parametric if one of the following left rules has been applied: $({}^{\perp}\vdash)$, $(\otimes\vdash)$, $(\multimap\vdash)$, $(\invamp\vdash)$, $(!\vdash)$, $(!K\vdash)$ and $(!W\vdash)$. Either way, moving the cut up on the right-hand side decreases ϱ_r, and preserves the applicability of $(?\vdash)$ on the left.

If both premises are by the $(?\vdash)$ rule, then $\varrho_r = 1$, that is, $?\mathcal{D}$ cannot be parametric in the antecedent of the right premise. The proof chunk ending in an application of the cut rule and its transformation are as follows.

$$\dfrac{\dfrac{\overset{\vdots}{\mathcal{A}; \Gamma^! \vdash \Delta^?; ?\mathcal{D}}}{?\mathcal{A}; \Gamma^! \vdash \Delta^?; ?\mathcal{D}}\,{\scriptstyle ?\vdash} \qquad \dfrac{\overset{\vdots}{\mathcal{D}; \Theta^! \vdash \Lambda^?}}{?\mathcal{D}; \Theta^! \vdash \Lambda^?}\,{\scriptstyle ?\vdash}}{?\mathcal{A}; \Gamma^!; \Theta^! \vdash \Delta^?; \Lambda^?}\,{\scriptstyle \mathrm{cut}} \quad \rightsquigarrow$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\overset{\vdots}{\mathcal{A}; \Gamma^! \vdash \Delta^?; ?\mathcal{D}} \qquad \dfrac{\overset{\vdots}{\mathcal{D}; \Theta^! \vdash \Lambda^?}}{?\mathcal{D}; \Theta^! \vdash \Lambda^?}\,{\scriptstyle ?\vdash}}{\mathcal{A}; \Gamma^!; \Theta^! \vdash \Delta^?; \Lambda^?}\,{\scriptstyle \mathrm{cut}}}{?\mathcal{A}; \Gamma^!; \Theta^! \vdash \Delta^?; \Lambda^?}\,{\scriptstyle ?\vdash}$$

The left rank of the new cut is one less than the left rank of the original cut.

4.3. If the left premise is by one of the right rules $(\vdash^{\perp})$, $(\vdash\otimes)$, $(\vdash\multimap)$ or $(\vdash\invamp)$, then $\mathcal{C}$ must be parametric in the succedent of the left premise. The applications of the cut rule and of the other rule may be permuted preserving the end sequent of the proof and decreasing ϱ_l.

If the left premise is by $(\vdash ?)$, $(\vdash ?K)$ or $(\vdash ?W)$, then $\mathcal{C}$ is parametric, because $\varrho_l > 1$ and $\mu = 0$. The cases are dealt with similarly to the previous ones.

If the left premise is by $(\vdash !)$, then $\mathcal{C}$ is parametric and of the form $?\mathcal{D}$. This rule is restricted, therefore, switching the application of the cut rule with the application of $(\vdash !)$ may not produce a proof. ϱ_r may or may not be > 1. If $\varrho_r = 1$, then the right premise is an instance of (id), thus the cut is eliminated.

If $\varrho_r > 1$, then it is by an application of one of fifteen rules, because $(\vdash !)$ cannot result in a sequent with $?\mathcal{D}$ in its antecedent. If the right premise is by a rule for ${}^{\perp}$, $\otimes$, $\multimap$, $\invamp$, or by a modalized structural rule, then the cut formula is parametric. So it is, if the rule applied is $(!\vdash)$ or $(\vdash ?)$. None of these rules is restricted, and the cut may be performed by swapping the cut rule with the rule yielding the right premise, which reduces ϱ_r.

The only remaining possibility is that the right premise is by the $(?\vdash)$ rule. The restriction on the rule implies that $\varrho_r = 1$. The original and transformed proof segments have the following shape.

$$\dfrac{\dfrac{\overset{\vdots}{\Gamma^! \vdash \Delta^?; \mathcal{A}; ?\mathcal{D}}}{\Gamma^! \vdash \Delta^?; !\mathcal{A}; ?\mathcal{D}}\,{\scriptstyle \vdash !} \qquad \dfrac{\overset{\vdots}{\mathcal{D}; \Theta^! \vdash \Lambda^?}}{?\mathcal{D}; \Theta^! \vdash \Lambda^?}\,{\scriptstyle ?\vdash}}{\Gamma^!; \Theta^! \vdash \Delta^?; \Lambda^?; !\mathcal{A}}\,{\scriptstyle \mathrm{cut}} \quad \rightsquigarrow$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\overset{\vdots}{\Gamma^! \vdash \Delta^?; \mathcal{A}; ?\mathcal{D}} \qquad \dfrac{\overset{\vdots}{\mathcal{D}; \Theta^! \vdash \Lambda^?}}{?\mathcal{D}; \Theta^! \vdash \Lambda^?}\,{\scriptstyle ?\vdash}}{\Gamma^!; \Theta^! \vdash \Delta^?; \Lambda^?; \mathcal{A}}\,{\scriptstyle \mathrm{cut}}}{\Gamma^!; \Theta^! \vdash \Delta^?; \Lambda^?; !\mathcal{A}}\,{\scriptstyle \vdash !}$$

5. If $\delta(\mathcal{C}) > 0$, $\varrho > 2$ and $\mu > 0$, we have to consider all the cases that we went through under **4.** (We omit repeating those cases.) Additionally, we have to consider when the left premise is by $(\vdash ?W)$ or the right premise is by $(!W\vdash)$ with $\mathcal{C}$ being the principal formula of the rule. These possibilities are symmetric. We detail the latter case.
5.1. If the left premise is the axiom and $\varrho_l = 1$, then the cut is eliminable. The left premise cannot be by the $(?\vdash)$ rule, because the rule's restriction conflicts with $!\mathcal{D}$ being in the succedent of the left premise. Except the other restricted rule, $(\vdash !)$, if the left premise is by any of the fourteen other rules, then $!\mathcal{D}$ is parametric in that rule. The general shape of the proof and the transformed proof are as follows.

$$\text{cut}\;\dfrac{\text{rule}\;\dfrac{\vdots \atop \Gamma' \vdash \Delta'; !\mathcal{D}}{\Gamma \vdash \Delta; !\mathcal{D}} \qquad \dfrac{\vdots \atop !\mathcal{D}; !\mathcal{D}; \Theta \vdash \Lambda}{!\mathcal{D}; \Theta \vdash \Lambda}\;!W\vdash}{\Gamma; \Theta \vdash \Delta; \Lambda} \quad \rightsquigarrow$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\vdots \atop \Gamma' \vdash \Delta'; !\mathcal{D} \qquad \dfrac{\vdots \atop !\mathcal{D}; !\mathcal{D}; \Theta \vdash \Lambda}{!\mathcal{D}; \Theta \vdash \Lambda}\;!W\vdash}{\Gamma'; \Theta \vdash \Delta'; \Lambda}\;\text{cut}}{\Gamma; \Theta \vdash \Delta; \Lambda}\;\text{rule}$$

The only remaining case is when the left premise is by the $(\vdash !)$ rule, hence, $\varrho_l = 1$. The proof we start with and the one we get look like the following.

$$\text{cut}\;\dfrac{\vdash !\;\dfrac{\vdots \atop \Gamma^! \vdash \Delta^?; \mathcal{D}}{\Gamma^! \vdash \Delta^?; !\mathcal{D}} \qquad \dfrac{\vdots \atop !\mathcal{D}; !\mathcal{D}; \Theta \vdash \Lambda}{!\mathcal{D}; \Theta \vdash \Lambda}\;!W\vdash}{\Gamma^!; \Theta \vdash \Delta^?; \Lambda} \quad \rightsquigarrow$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\vdots \atop \Gamma^! \vdash \Delta^?; !\mathcal{D} \qquad \dfrac{\vdots \atop \Gamma^! \vdash \Delta^?; !\mathcal{D} \qquad \vdots \atop !\mathcal{D}; !\mathcal{D}; \Theta \vdash \Lambda}{\Gamma^!; !\mathcal{D}; \Theta \vdash \Delta^?; \Lambda}\;\text{cut}}{\Gamma^!; \Gamma^!; \Theta \vdash \Delta^?; \Delta^?; \Lambda}\;\text{cut}}{\Gamma^!; \Theta \vdash \Delta^?; \Lambda}\;!W\vdash\text{'s}, \vdash ?W\text{'s}$$

The new applications of the cut rule have lower μ than the original one, but they maintain the same $\delta(\mathcal{C})$. □

A consequence of the cut theorem is that CLL_{int} has the *subformula property*. The logical components of the language are (unary and binary) connectives, which obviously, implies that every formula has *finitely many subformulas*, which is taken to be evident in the decision procedure.

The above proof of the cut theorem is new, though the theorem itself is not. The next theorem is completely new, and it states the admissibility of the [cut] rule in the new calculus $[CLL_{\text{int}}]$. The importance of our reformulation of CLL_{int} is that even restricted versions of the contraction rules $(W\vdash)$ and $(\vdash W)$ are problematic if we wish to base the decision procedure on the sequent calculus.

The notion of the contraction measure is omitted in connection to $[CLL_{\text{int}}]$, and the rank of the cut is defined anew.

Definition 9 *(Rank of cut).* The *rank* of an application of a cut rule, denoted by ϱ, is the sum of the *left rank* (ϱ_l) and of the *right rank* (ϱ_r) of the cut. The left rank of the cut is the maximal number of consecutive sequents above the left premise of the cut in which the cut formula occurs in the succedent of the sequent plus 1. The right rank of the cut is the maximal number of consecutive sequents above the right premise of the cut in which the cut formula occurs in the antecedent of the sequent plus 1.

Theorem 10 *(Cut theorem for* $[CLL_{\text{int}}]$*). The* [cut] *rule is* admissible *in* $[CLL_{\text{int}}]$.

Proof. The proof is by *double induction* on ϱ and δ. The previous proof serves as a blueprint for this proof.
1. If $\delta(\mathcal{C}) = 0$ and $\varrho = 2$, then the [cut] is eliminated.
2. If $\delta(\mathcal{C}) = 0$ and $\varrho > 2$, then either the left or the right premise is by a rule, in which $\mathcal{C}$ is parametric. $\delta(\mathcal{C}) = 0$ excludes the possibility that the rule is $(\vdash !)$ or $(?\vdash)$, which means that the [cut] rule may be permuted with the other rule. The same contractions (if any) that could be performed as part of the applications of the [cut] rule or the other rule can be performed in the modified proof.
3. If $\delta(\mathcal{C}) > 0$ and $\varrho = 2$, then one or both premises may be instances of (id). The [cut] is eliminable as usual.

If the premises are by a pair of rules $\langle(\vdash{}^\perp), ({}^\perp\vdash)\rangle$, $\langle([\vdash\otimes]), (\otimes\vdash)\rangle$, $\langle(\vdash \multimap), ([\multimap\vdash])\rangle$ or $\langle(\vdash \parr), ([\parr\vdash])\rangle$, then the [cut]s on the principal formulas are replaced by [cut]s on subalterns. It is easy to verify that all the earlier contractions (which could not involve the principal formulas) are available in the modified proofs.
3.1. If the cut formula is $!\mathcal{D}$ introduced by connective rules, then the given proof segment is the following.

$$\dfrac{{\scriptstyle \vdash!}\ \dfrac{\begin{array}{c}\vdots\\ \Gamma^! \vdash \Delta^?; \mathcal{D}\end{array}}{\Gamma^! \vdash \Delta^?; !\mathcal{D}} \qquad \dfrac{\begin{array}{c}\vdots\\ \mathcal{D}; \Theta \vdash \Lambda\end{array}}{[!\mathcal{D}; \Theta] \vdash \Lambda}\ {\scriptstyle [!\vdash]}}{[\Gamma^!; \Theta] \vdash [\Delta^?; \Lambda]}\ {\scriptstyle [\mathrm{cut}]}$$

It is not possible that the application of the ($[!\vdash]$) rule includes a contraction — as if Θ has absorbed $!\mathcal{D}$. For a contraction to happen, Θ must contain an occurrence of $!\mathcal{D}$, but then $\varrho_r > 1$. The [cut] is performed on the subalterns as before.

If the cut formula is $!\mathcal{D}$ and the right premise is by ($!K\vdash$), then we start with the following proof segment.

$$\dfrac{{\scriptstyle \vdash!}\ \dfrac{\begin{array}{c}\vdots\\ \Gamma^! \vdash \Delta^?; \mathcal{D}\end{array}}{\Gamma^! \vdash \Delta^?; !\mathcal{D}} \qquad \dfrac{\begin{array}{c}\vdots\\ \Theta \vdash \Lambda\end{array}}{!\mathcal{D}; \Theta \vdash \Lambda}\ {\scriptstyle !K\vdash}}{[\Gamma^!; \Theta] \vdash [\Delta^?; \Lambda]}\ {\scriptstyle [\mathrm{cut}]}$$

The contractions that result from the application of the [cut] rule cannot result in multisets $[\Gamma^!; \Theta]$ and $[\Delta^?; \Lambda]$ that do not contain Θ and Λ, respectively, as their submultisets. Therefore, the earlier transformation of the proof can be carried out — with just as many thinnings as needed.

The dual case with the cut formula being $?\mathcal{D}$ is completely symmetric, and we omit the details.

4. If $\delta(\mathcal{C}) > 0$ and $\varrho > 2$, then either $\varrho_l > 1$ or $\varrho_r > 1$. Let us assume the former. (The two possibilities are symmetric.)

4.1. The left premise may be by a left or by a right rule. If it is by an unrestricted left rule (i.e., by a left rule save ($?\vdash$)), then permuting the [cut] rule and the other rule yields the same end sequent. The permutation does not affect the contractions — if there were any in the given proof.

4.2. If the left premise is by ($?\vdash$), then the cut formula is parametric in the succedent, and it is of the form $?\mathcal{D}$. As in the proof of Theorem 8, we consider how the right premise came about. If the right premise is an instance of the axiom, then the [cut] is eliminated as usual. If $?\mathcal{D}$ is parametric in the antecedent of the right premise, then the [cut] is permuted with the rule. Lastly, if the right premise is also by the ($?\vdash$) rule, then all the contractions result from the application of the [cut] rule, which can be replicated when ϱ_l is reduced by moving the [cut] one step upward.

4.3. If the left premise is by a right rule, then we may simply swap the [cut] rule and the other rule unless the latter is ($\vdash!$) or ($[\vdash?]$). If the rule is ($\vdash!$), then we know that the cut formula is $?\mathcal{D}$, because the cut formula is parametric in the succedent of the left premise (otherwise, $\varrho_l = 1$ would be true). Then we scrutinize how the right premise is obtained.

If the right premise is (id), then the tree rooted in the left premise is the proof (without loss of potential contractions). If the right premise is by a structural rule or by a connective rule for $^\perp$, $\otimes$, $\multimap$ or $\parr$, then ϱ_r may be reduced as before. The right premise cannot be the result of an application of the ($\vdash!$) rule, because the cut formula occurs in the antecedent of the right premise. If the right premise is by ($?\vdash$), then the given proof segment and its transformation are as follows. All contractions are part of applications of the [cut] rule.

$${\scriptstyle [\mathrm{cut}]}\ \dfrac{{\scriptstyle \vdash!}\ \dfrac{\begin{array}{c}\vdots\\ \Gamma^! \vdash \Delta^?; \mathcal{A}; ?\mathcal{D}\end{array}}{\Gamma^! \vdash \Delta^?; !\mathcal{A}; ?\mathcal{D}} \qquad \dfrac{\begin{array}{c}\vdots\\ \mathcal{D}; \Theta^! \vdash \Lambda^?\end{array}}{?\mathcal{D}; \Theta^! \vdash \Lambda^?}\ {\scriptstyle ?\vdash}}{[\Gamma^!; \Theta^!] \vdash [\Delta^?; !\mathcal{A}; \Lambda^?]} \quad \rightsquigarrow$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma^! \vdash \Delta^?; \mathcal{A}; ?\mathcal{D}\end{array} \qquad \dfrac{\begin{array}{c}\vdots\\ \mathcal{D}; \Theta^! \vdash \Lambda^?\end{array}}{?\mathcal{D}; \Theta^! \vdash \Lambda^?}\ {\scriptstyle ?\vdash}}{[\Gamma^!; \Theta^!] \vdash [\Delta^?; \mathcal{A}; \Lambda^?]}\ {\scriptstyle [\mathrm{cut}]}}{[\Gamma^!; \Theta^!] \vdash [\Delta^?; !\mathcal{A}; \Lambda^?]}\ {\scriptstyle \vdash!}$$

If the rule is ($[\vdash?]$), then the principal formula may or may not be the cut formula. If the principal formula is not the cut formula, then we have the following proof segments.

$${\scriptstyle [\mathrm{cut}]}\ \dfrac{{\scriptstyle [\vdash?]}\ \dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash \Delta; \mathcal{A}; \mathcal{C}\end{array}}{\Gamma \vdash [\Delta; ?\mathcal{A}; \mathcal{C}]} \qquad \begin{array}{c}\vdots\\ \mathcal{C}; \Theta \vdash \Lambda\end{array}}{[\Gamma; \Theta] \vdash [\Delta; ?\mathcal{A}; \Lambda]} \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash \Delta; \mathcal{A}; \mathcal{C}\end{array} \quad \begin{array}{c}\vdots\\ \mathcal{C}; \Theta \vdash \Lambda\end{array}}{[\Gamma; \Theta] \vdash [\Delta; \mathcal{A}; \Lambda]}\ {\scriptstyle [\mathrm{cut}]}}{[\Gamma; \Theta] \vdash [\Delta; ?\mathcal{A}; \Lambda]}\ {\scriptstyle [\vdash?]}$$

The left rank is reduced by one; the contractions (if any) are retained. If the principal formula is the cut formula, then there is a parametric occurrence of a formula in the premise of the ($[\vdash?]$) rule that is of the shape of the cut formula. In general, we permute the [cut] and ($[\vdash?]$). That is, we have the next two proof segments.

$${\scriptstyle [\mathrm{cut}]}\ \dfrac{{\scriptstyle [\vdash?]}\ \dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash \Delta; ?\mathcal{C}; \mathcal{C}\end{array}}{\Gamma \vdash [\Delta; ?\mathcal{C}; ?\mathcal{C}]} \qquad \begin{array}{c}\vdots\\ ?\mathcal{C}; \Theta \vdash \Lambda\end{array}}{[\Gamma; \Theta] \vdash [\Delta; ?\mathcal{C}; \Lambda]} \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash \Delta; ?\mathcal{C}; \mathcal{C}\end{array} \quad \begin{array}{c}\vdots\\ ?\mathcal{C}; \Theta \vdash \Lambda\end{array}}{[\Gamma; \Theta] \vdash [\Delta; \mathcal{C}; \Lambda]}\ {\scriptstyle [\mathrm{cut}]}}{[\Gamma; \Theta] \vdash [\Delta; ?\mathcal{C}; \Lambda]}\ {\scriptstyle [\vdash?]}$$

However, if $?\mathcal{C}$ does not occur in Δ or Λ above, then we may have to use a different transformation. If $\varrho_r > 1$, then the following works.

$$[\text{cut}]\ \dfrac{[\vdash ?]\ \dfrac{\begin{array}{c}\vdots\\ \Gamma\vdash\Delta;?\mathcal{C};\mathcal{C}\end{array}}{\Gamma\vdash[\Delta;?\mathcal{C};?\mathcal{C}]}\qquad \dfrac{\begin{array}{c}\vdots\\ ?\mathcal{C};\Theta'\vdash\Lambda'\end{array}}{?\mathcal{C};\Theta\vdash\Lambda}\ \text{rule}}{[\Gamma;\Theta]\vdash[\Delta;?\mathcal{C};\Lambda]}\qquad\rightsquigarrow$$

$$\rightsquigarrow\qquad \dfrac{\dfrac{[\vdash ?]\ \dfrac{\begin{array}{c}\vdots\\ \Gamma\vdash\Delta;?\mathcal{C};\mathcal{C}\end{array}}{\Gamma\vdash[\Delta;?\mathcal{C};?\mathcal{C}]}\qquad \begin{array}{c}\vdots\\ ?\mathcal{C};\Theta'\vdash\Lambda'\end{array}}{[\Gamma;\Theta']\vdash[\Delta;?\mathcal{C};\Lambda']}\ [\text{cut}]}{[\Gamma;\Theta]\vdash[\Delta;?\mathcal{C};\Lambda]}\ \text{rule}$$

If $\varrho_r = 1$ (because of the rule $(?\vdash)$), then the new proof has two [cut]s, if the application of the $([\vdash ?])$ rule contracted the principal formula. The first [cut] has lower ϱ_l, whereas the second [cut] has lower δ. In this [cut], each occurrence of a formula is matched by the same occurrence of the formula in the two $\Theta^!$'s and $\Lambda^?$'s; hence, $[\Gamma;\Theta^!]\vdash[\Delta;\Lambda^?]$ is a resulting sequence after the second [cut].

$$[\text{cut}]\ \dfrac{[\vdash ?]\ \dfrac{\begin{array}{c}\vdots\\ \Gamma\vdash\Delta;?\mathcal{C};\mathcal{C}\end{array}}{\Gamma\vdash\Delta;?\mathcal{C}}\qquad \dfrac{\begin{array}{c}\vdots\\ \mathcal{C};\Theta^!\vdash\Lambda^?\end{array}}{?\mathcal{C};\Theta^!\vdash\Lambda^?}\ ?\vdash}{[\Gamma;\Theta^!]\vdash[\Delta;\Lambda^?]}\qquad\rightsquigarrow$$

$$\rightsquigarrow\qquad \dfrac{[\text{cut}]\ \dfrac{\begin{array}{c}\vdots\\ \Gamma\vdash\Delta;?\mathcal{C};\mathcal{C}\end{array}\qquad \dfrac{\begin{array}{c}\vdots\\ \mathcal{C};\Theta^!\vdash\Lambda^?\end{array}}{?\mathcal{C};\Theta^!\vdash\Lambda^?}\ ?\vdash}{[\Gamma;\Theta^!]\vdash[\Delta;\mathcal{C};\Lambda^?]}\qquad \begin{array}{c}\vdots\\ \mathcal{C};\Theta^!\vdash\Lambda^?\end{array}}{[\Gamma;\Theta^!;\Theta^!]\vdash[\Delta;\Lambda^?;\Lambda^?]}\ [\text{cut}]\qquad\square$$

The proofs of the next two theorems proceed along the lines of the proofs of the previous two theorems. (We omit detailing them.)

Theorem 11 *(Cut theorem for RLL_{int}). The* cut *rule is* admissible *in RLL_{int}.*

Theorem 12 *(Cut theorem for $(\!|RLL_{\text{int}}|\!)$). The* $(\!|\text{cut}|\!)$ *rule is* admissible *in $(\!|RLL_{\text{int}}|\!)$.*

4. The decidability of CLL_{int}

The next theorem guarantees that the elimination of $(!W\vdash)$ and $(\vdash ?W)$ has been successful.

Theorem 13 *(Curry's lemma for $[CLL_{\text{int}}]$). If there is a proof of $\Gamma\vdash\Delta$ in $[CLL_{\text{int}}]$ where the height of the proof tree is χ, and $\Gamma'\vdash\Delta'$ could be obtained from $\Gamma\vdash\Delta$ by applications of $(!W\vdash)$ and $(\vdash ?W)$, then* there is a proof of $\Gamma'\vdash\Delta'$ *in $[CLL_{\text{int}}]$ with the height of the proof tree being χ' and $\chi'\le\chi$.*

Proof. The proof is by induction on the height of the proof tree, χ.
1. If $\Gamma\vdash\Delta$ is an instance of (id), then the claim is true, because no contraction is applicable to $\mathcal{A}\vdash\mathcal{A}$.
2. If $\Gamma\vdash\Delta$ is by a rule, then we may divide the rules into four groups. When no contractions of Γ and Δ are possible, the claim is automatically true — no matter which rule was applied.
2.1. If the rule is a connective rule that does not have contraction hidden in it, then the inductive hypothesis obviously yields the desired result. The rules that fall into this group are $(^\perp\vdash)$, $(\vdash{}^\perp)$, $(\vdash\multimap)$, $(\otimes\vdash)$, $(\vdash\parr)$, $(?\vdash)$ and $(\vdash !)$. (This case covers half of the rules.) We exhibit the details of two of the subcases; the others are similar.

The formulas that are contracted cannot be the principal formula of the rule. To illustrate what happens to parametric formulas, we make explicit a pair of $!\mathcal{C}$'s on the left-hand side of the $\vdash$, and a pair of $?\mathcal{D}$'s on the right. Either pair could be missing, and more pairs, or more formulas of the same shape could occur in the antecedent and in the consequent, respectively.

$$\otimes\vdash\ \dfrac{\begin{array}{c}\vdots\\ !\mathcal{C}; !\mathcal{C}; \mathcal{A}; \mathcal{B}; \Theta \vdash \Lambda; ?\mathcal{D}; ?\mathcal{D}\end{array}}{!\mathcal{C}; !\mathcal{C}; \mathcal{A}\otimes\mathcal{B}; \Theta \vdash \Lambda; ?\mathcal{D}; ?\mathcal{D}} \quad \overset{\text{i.h.}}{\rightsquigarrow} \quad \dfrac{\begin{array}{c}\vdots\\ !\mathcal{C}; \mathcal{A}; \mathcal{B}; \Theta \vdash \Lambda; ?\mathcal{D}\end{array}}{!\mathcal{C}; \mathcal{A}\otimes\mathcal{B}; \Theta \vdash \Lambda; ?\mathcal{D}}\ \otimes\vdash$$

$$\vdash!\ \dfrac{\begin{array}{c}\vdots\\ !\mathcal{C}; !\mathcal{C}; \Theta^{!} \vdash \Lambda^{?}; ?\mathcal{D}; ?\mathcal{D}; \mathcal{A}\end{array}}{!\mathcal{C}; !\mathcal{C}; \Theta^{!} \vdash \Lambda^{?}; ?\mathcal{D}; ?\mathcal{D}; !\mathcal{A}} \quad \overset{\text{i.h.}}{\rightsquigarrow} \quad \dfrac{\begin{array}{c}\vdots\\ !\mathcal{C}; \Theta^{!} \vdash \Lambda^{?}; ?\mathcal{D}; \mathcal{A}\end{array}}{!\mathcal{C}; \Theta^{!} \vdash \Lambda^{?}; ?\mathcal{D}; !\mathcal{A}}\ \vdash!$$

We note that although this rule is restricted, neither contraction can affect the applicability of the rule.

2.2. There are three non-modal connective rules, which have some potential contraction built in, namely, $([\multimap\vdash])$, $([\vdash\otimes])$ and $([\parr\vdash])$. The inductive hypothesis — together with an application of the same rule — guarantees the truth of the claim. We include here the details for the $([\parr\vdash])$ rule only.

$$[\parr\vdash]\ \dfrac{\begin{array}{c}\vdots\\ \mathcal{A}; !\mathcal{C}; !\mathcal{C}; \Theta \vdash \Lambda; ?\mathcal{D}; ?\mathcal{D}\end{array} \qquad \begin{array}{c}\vdots\\ \mathcal{B}; !\mathcal{C}'; !\mathcal{C}'; \Xi \vdash \Psi; ?\mathcal{D}'; ?\mathcal{D}'\end{array}}{[\mathcal{A}\parr\mathcal{B}; !\mathcal{C}; !\mathcal{C}; \Theta; !\mathcal{C}'; !\mathcal{C}'; \Xi] \vdash [\Lambda; ?\mathcal{D}; ?\mathcal{D}; \Psi; ?\mathcal{D}'; ?\mathcal{D}']} \quad \overset{\text{i.h.}}{\rightsquigarrow}$$

$$\overset{\text{i.h.}}{\rightsquigarrow} \quad \dfrac{\begin{array}{c}\vdots\\ \mathcal{A}; !\mathcal{C}; \Theta \vdash \Lambda; ?\mathcal{D}\end{array} \qquad \begin{array}{c}\vdots\\ \mathcal{B}; !\mathcal{C}'; \Xi \vdash \Psi; ?\mathcal{D}'\end{array}}{[\mathcal{A}\parr\mathcal{B}; !\mathcal{C}; \Theta; !\mathcal{C}'; \Xi] \vdash [\Lambda; ?\mathcal{D}; \Psi; ?\mathcal{D}']}\ [\parr\vdash]$$

2.3. There are two modal rules, which permit contractions; moreover, the principal formula may be one of the contracted formulas. However, we do not run into a problem in this case either, because the inductive hypothesis and the application of the same rule yields the desired sequent. We give the details of both cases.

$$[!\vdash]\ \dfrac{\begin{array}{c}\vdots\\ \mathcal{A}; !\mathcal{C}; !\mathcal{C}; \Theta \vdash \Lambda; ?\mathcal{D}; ?\mathcal{D}\end{array}}{[!\mathcal{A}; !\mathcal{C}; !\mathcal{C}; \Theta] \vdash \Lambda; ?\mathcal{D}; ?\mathcal{D}} \quad \overset{\text{i.h.}}{\rightsquigarrow} \quad \dfrac{\begin{array}{c}\vdots\\ \mathcal{A}; !\mathcal{C}; \Theta \vdash \Lambda; ?\mathcal{D}\end{array}}{[!\mathcal{A}; !\mathcal{C}; \Theta] \vdash \Lambda; ?\mathcal{D}}\ [!\vdash]$$

$$[\vdash?]\ \dfrac{\begin{array}{c}\vdots\\ !\mathcal{C}; !\mathcal{C}; \Theta \vdash \Lambda; ?\mathcal{D}; ?\mathcal{D}; \mathcal{A}\end{array}}{!\mathcal{C}; !\mathcal{C}; \Theta \vdash [\Lambda; ?\mathcal{D}; ?\mathcal{D}; ?\mathcal{A}]} \quad \overset{\text{i.h.}}{\rightsquigarrow} \quad \dfrac{\begin{array}{c}\vdots\\ !\mathcal{C}; \Theta \vdash \Lambda; ?\mathcal{D}; \mathcal{A}\end{array}}{!\mathcal{C}; \Theta \vdash [\Lambda; ?\mathcal{D}; ?\mathcal{A}]}\ [\vdash?]$$

2.4. Lastly, the rule may be either of the two structural rules. Neither of them permits contractions, because if the modal formula that is thinned into the sequent would be contracted, then the application of the thinning rule may be omitted. We illustrate the situation for $(\vdash?K)$.

$$\vdash?K\ \dfrac{\begin{array}{c}\vdots\\ !\mathcal{C}; !\mathcal{C}; \Theta \vdash \Lambda; ?\mathcal{D}; ?\mathcal{D}\end{array}}{!\mathcal{C}; !\mathcal{C}; \Theta \vdash \Lambda; ?\mathcal{D}; ?\mathcal{D}; ?\mathcal{A}} \quad \overset{\text{i.h.}}{\rightsquigarrow} \quad \dfrac{\begin{array}{c}\vdots\\ !\mathcal{C}; \Theta \vdash \Lambda; ?\mathcal{D}\end{array}}{!\mathcal{C}; \Theta \vdash \Lambda; ?\mathcal{D}; ?\mathcal{A}}\ \vdash?K$$

The above depicts what happens if $?\mathcal{D}$ and $?\mathcal{A}$ are distinct, furthermore, there is no occurrence of $?\mathcal{A}$ in Λ. If the occurrence $?\mathcal{A}$ could be contracted using $(\vdash?W)$, then we may simply omit the application of the $(\vdash?K)$ rule.

$$\vdash?K\ \dfrac{\begin{array}{c}\vdots\\ !\mathcal{C}; !\mathcal{C}; \Theta \vdash \Lambda; ?\mathcal{D}; ?\mathcal{D}\end{array}}{!\mathcal{C}; !\mathcal{C}; \Theta \vdash \Lambda; ?\mathcal{D}; ?\mathcal{D}; ?\mathcal{A}} \quad \overset{\text{i.h.}}{\rightsquigarrow} \quad \begin{array}{c}\vdots\\ !\mathcal{C}; \Theta \vdash \Lambda; ?\mathcal{D}\end{array} \qquad \square$$

Theorem 14 *(Curry's lemma for* $(\![RLL_{\text{int}}]\!)$*). If there is a proof of* $\Gamma\vdash\Delta$ *in* $(\![RLL_{\text{int}}]\!)$ *where the height of the proof tree is* χ*, and* $\Gamma'\vdash\Delta'$ *could be obtained from* $\Gamma\vdash\Delta$ *by applications of* $(W\vdash)$ *and* $(\vdash W)$*, then* there is a proof of $\Gamma'\vdash\Delta'$ *in* $(\![RLL_{\text{int}}]\!)$ *with the height of the proof tree being* $\chi'\leq\chi$.

Proof. The structure of the proof of this theorem is like the structure of the proof of Theorem 13. We omit the details. $\square$

Definition 15 *(Modally cognate sequents).* Let $\Gamma\vdash\Delta$ and $\Theta\vdash\Lambda$ be sequents. They are *modally cognate sequents* iff (1)–(3) hold of them.

(1) Γ and Θ contain the same formulas, that is, they may differ only in the number of copies of a formula. Similarly, Δ and Λ are the same if viewed as sets.
(2) If $\mathcal{A}$ is not $!\mathcal{C}$, then $\mathcal{A}$ occurs as many times in Γ as in Θ.
(3) If $\mathcal{B}$ is not $?\mathcal{D}$, then $\mathcal{B}$ occurs as many times in Δ as in Λ.

The usual notion of *cognate sequents* results, if we omit (2) and (3) from this definition.

As an example, let us consider a few pairs of sequents. We assume that $\mathcal{A}_1, \mathcal{A}_2, \dots$ are distinct formulas and none of them is of the form $!\mathcal{C}$ or $?\mathcal{D}$. The sequents

(α) $\mathcal{A}_1; \mathcal{A}_1; \mathcal{A}_2 \vdash \mathcal{A}_3; ?\mathcal{A}_4; !\mathcal{A}_4$ (β) $\mathcal{A}_1; \mathcal{A}_3 \vdash ?\mathcal{A}_4; ?\mathcal{A}_4; ?\mathcal{A}_5$

are (obviously) not modally cognate. $\mathcal{A}_2$ is in the antecedent of α but not in the antecedent of β. These two sequents are not cognate in the usual sense either.[10] The next two sequents, on the other hand, are modally cognate (therefore, also cognate).

(γ) $\mathcal{A}_1; \mathcal{A}_1; !\mathcal{A}_2; ?\mathcal{A}_3 \vdash \mathcal{A}_2; ?\mathcal{A}_4; ?\mathcal{A}_4; ?\mathcal{A}_5$

(η) $\mathcal{A}_1; \mathcal{A}_1; !\mathcal{A}_2; !\mathcal{A}_2; ?\mathcal{A}_3 \vdash \mathcal{A}_2; ?\mathcal{A}_4; ?\mathcal{A}_5; ?\mathcal{A}_5$

All the formulas that are not prefixed with ! occur the same number of times in the antecedents of γ and η; all the formulas that are not prefixed with ? occur the same number of times in the succedents of the two sequents. It may be useful to emphasize that *neither* sequent can be obtained from the other one by applications of the $(!W\vdash)$ and $(\vdash ?W)$ rules (or by applications of the $(W\vdash)$ and $(\vdash W)$ rules).

Observation 16. *A pair of modally cognate sequents is* cognate.

If we approach the modally cognate sequents starting from cognate sequents, then we can say that they must be cognate if we restrict our attention to !'d formulas on the left and ?'d formulas on the right, but otherwise, they must be identical.

Definition 17 *(Modally irredundant sequences of sequents).* Let $\langle \Gamma_1 \vdash \Delta_1, \dots, \Gamma_n \vdash \Delta_n, \dots \rangle$ be a (finite or infinite) sequence of modally cognate sequents. The sequence is *modally irredundant* iff for no pair of sequents $\Gamma_i \vdash \Delta_i$ and $\Gamma_j \vdash \Delta_j$ where $i < j$, the sequents are identical or $\Gamma_i \vdash \Delta_i$ is obtainable from $\Gamma_j \vdash \Delta_j$ by applications of the $(!W\vdash)$ and $(\vdash ?W)$ rules.

A sequence of cognate sequents is *irredundant* under a similar condition, but with $(W\vdash)$ and $(\vdash W)$ replacing $(!W\vdash)$ and $(\vdash ?W)$, respectively.

Theorem 18 *(Kripke's lemma). If a sequence of modally cognate sequents is irredundant, then it is* finite.

Proof. The proof of this lemma is easy — given that Kripke's original lemma for cognate sequents has been proved. Detailed proofs of Kripke's lemma are included, for example, in Anderson and Belnap [2, §13] and Riche and Meyer [33].[11] Since modally cognate sequents are cognate sequents, the claim follows. □

Theorem 19 *(König's lemma). A finitely branching infinite tree has an* infinite branch.

This is an old and widely known lemma, therefore, we do not include its proof either.

Lemma 20. *If $\mathcal{A}$ is a theorem of CLL_{int}, then $\mathcal{A}$ is a* theorem *of RLL_{int}. Similarly, if $\mathcal{A}$ is a theorem of $[CLL_{\text{int}}]$, then $\mathcal{A}$ is a* theorem *of $(\!|RLL_{\text{int}}|\!)$.*

Proof. We assume a 1–1 match — the one suggested by our notation — between the symbols of the logics. $(!W\vdash)$ is a special instance of $(W\vdash)$, and $(\vdash ?W)$ is a special instance of $(\vdash W)$. A similar relationship holds between rules in $[CLL_{\text{int}}]$ and $(\!|RLL_{\text{int}}|\!)$, because none of the permitted contractions are forced; they are merely allowed. □

Theorem 21 *(Decidability of RLL_{int}). The logic RLL_{int} is* decidable.

Proof. The proof is via a finite proof-search tree, in which "backward applications" of the rules in $(\!|RLL_{\text{int}}|\!)$ produce new leaves. Branches are aborted if they would become redundant. There are finitely many rules (with finitely many potential contractions), finitely many subformulas and sequents are pairs of finite multisets. These features guarantee that each node in the proof-search tree has a finite outdegree. Kripke's lemma guarantees that all the branches are finite, thus, by König's lemma, the proof-search tree is finite. The completed proof-search tree either contains a proof of the sequent in its root, or it does not. (It is decidable if it does or does not.) The construction procedure of the proof-search tree guarantees that if a formula is a theorem of RLL_{int}, then the proof-search tree will contain a proof. □

[10] The original definition of cognate sequents goes back to the 1950s and is due to Kleene. Various — but equivalent modulo the underlying datatype — definitions of cognate sequents may be found, for example, in Anderson and Belnap [2], Dunn [13] and [8].

[11] Kripke's lemma is, perhaps, not sufficiently well known. However, it can be seen to be equivalent to other lemmas from discrete mathematics — as pointed out in Riche and Meyer [33]. Therefore, we do not include a proof of Kripke's lemma here.

We can utilize the decidability of RLL_{int} in at least two ways, the second of which is that we concentrate on. First, however, we mention that if $\mathcal{A}$ is not a theorem of RLL_{int}, then $\mathcal{A}$ is surely not a theorem of CLL_{int}. Second, the proof-search produces *all* the proofs in $(\!|RLL_{\mathrm{int}}|\!)$ in which the proof tree has no branch in which a (sub)sequence of sequents is redundant. We cannot simply transpose all proofs from $(\!|RLL_{\mathrm{int}}|\!)$ into $[CLL_{\mathrm{int}}]$, because some of the built-in contractions in the former are absent in the latter calculus. Indeed, the set of theorems of CLL_{int} is a *proper* subset of the set of theorems of RLL_{int}. However, given a proof in $(\!|RLL_{\mathrm{int}}|\!)$, all the contractions can be made explicit — in effect, turning the proof into a proof in RLL_{int}. Furthermore, it is possible to determine the number of contractions that have been applied to the ancestors of a formula such as the immediate proper subformulas of $!\mathcal{A}$ and $?\mathcal{A}$ (on the appropriate side).

Definition 22 *(Heap number).* The *heap number* of $!\mathcal{A}$ on the left of the turnstile (of $?\mathcal{A}$ on the right of the turnstile) that is a subformula of a formula in the sequent $\Gamma \vdash \Delta$ is the maximal number of contractions applied in any proof of $\Gamma \vdash \Delta$ in $(\!|RLL_{\mathrm{int}}|\!)$ to the subaltern $\mathcal{A}$ and its ancestors.[12]

The heap number tells us how many copies of $\mathcal{A}$ would be accumulated in a proof, if all the contractions would be part of the application of the $([\,!\vdash])$ (or respectively, of the $[\,\vdash ?])$) rule.

Now we have all the components that we need for the proof of our main theorem.

Theorem 23 *(Decidability of CLL_{int}). The logic CLL_{int} is* decidable.

Proof. We build a proof-search tree in $[CLL_{\mathrm{int}}]$. In addition to not continuing with a branch if it would become modally redundant, we limit the expansion of branches by the heap numbers of $!\mathcal{A}$ and $?\mathcal{A}$. Namely, at most heap-number-many "backward applications" of $([\,!\vdash])$ and $([\,\vdash ?])$ are permitted (with the principal formulas $!\mathcal{A}$ and $?\mathcal{A}$). On one hand, this guarantees that the procedure does not create an infinite sequence of modally irredundant (but redundant) sequents. On the other hand, we have all the copies of the subaltern $\mathcal{A}$ that could be used to produce a proof in $[CLL_{\mathrm{int}}]$. □

It is interesting to note that the proof of the decidability of intensional interlinear logic is smoother than the analogous proof for CLL_{int}. This shows that RLL_{int} is a *more coherent* logic than CLL_{int}, that is, from the point of view of its proof-theoretic features, this semi-relevant logic is *superior* to *MELL*.

5. Constants and intuitionistic fragments

The pair of zero-ary connectives (or constants) $\mathbf{1}$ and $\bot$ can be included in CLL_{int} and RLL_{int} by the following axioms and rules. (We denote the expanded calculi by $CLL^{\mathbf{1}\bot}_{\mathrm{int}}$ and $RLL^{\mathbf{1}\bot}_{\mathrm{int}}$, respectively.)

$$\vdash \mathbf{1} \;\; \vdash\!\mathbf{1} \qquad\qquad \mathcal{A} \vdash \mathcal{A} \;\; \mathrm{id} \qquad\qquad \bot \vdash \;\; \bot\!\vdash$$

$$\frac{\Gamma \vdash \Delta}{\mathbf{1}; \Gamma \vdash \Delta}\;\mathbf{1}\!\vdash \qquad\qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta; \bot}\;\vdash\!\bot$$

These constants are identity elements for certain connectives in these logics. In relevance logics, they are sometimes called *Ackermann constants* and denoted by $\boldsymbol{t}$ and $\boldsymbol{f}$. However, we continue to use the original notation from linear logic, that is, $\mathbf{1}$ and $\bot$.

Neither axiom and neither rule has to incorporate contractions. Therefore, $[CLL^{\mathbf{1}\bot}_{\mathrm{int}}]$ and $(\!|RLL^{\mathbf{1}\bot}_{\mathrm{int}}|\!)$ is obtained by the same additions (from $[CLL_{\mathrm{int}}]$ and $(\!|RLL_{\mathrm{int}}|\!)$, respectively).

Theorem 24 *(Cut theorems for $CLL^{\mathbf{1}\bot}_{\mathrm{int}}$ and $[CLL^{\mathbf{1}\bot}_{\mathrm{int}}]$). The* cut *and the* [cut] *rule, respectively, is* admissible *in $CLL^{\mathbf{1}\bot}_{\mathrm{int}}$ and in $[CLL^{\mathbf{1}\bot}_{\mathrm{int}}]$.*

Proof. The proofs are straightforward extensions of the proofs of Theorems 8 and 10, or they may constructed as elsewhere in the literature. We do not repeat the details here. □

We do not include the details of the proof of the next theorem either, which can be constructed like those of Theorems 11 and 12.

Theorem 25 *(Cut theorem for $RLL^{\mathbf{1}\bot}_{\mathrm{int}}$ and $(\!|RLL^{\mathbf{1}\bot}_{\mathrm{int}}|\!)$). The* cut *rule is* admissible *in $RLL^{\mathbf{1}\bot}_{\mathrm{int}}$ and $(\!|RLL^{\mathbf{1}\bot}_{\mathrm{int}}|\!)$.*

Theorem 26 *(Curry's lemma for $[CLL^{\mathbf{1}\bot}_{\mathrm{int}}]$). If there is a proof of $\Gamma \vdash \Delta$ in $[CLL^{\mathbf{1}\bot}_{\mathrm{int}}]$ where the height of the proof tree is χ, and $\Gamma' \vdash \Delta'$ could be obtained from $\Gamma \vdash \Delta$ by applications of $(!W\vdash)$ and $(\vdash ?W)$, then* there is a proof of $\Gamma' \vdash \Delta'$ *in $[CLL^{\mathbf{1}\bot}_{\mathrm{int}}]$ with the height of the proof tree being χ' and $\chi' \leq \chi$.*

[12] It would be delightful to be able to connect our heap number to HEAP in Meyer [28]; unfortunately, not so fast. However, both of them are *finite*.

Proof. The proof mostly comprises steps from the proof of Theorem 13. We add in Case **1** that neither of the new axioms permits applications of the $(!W\vdash)$ or $(\vdash ?W)$ rules. For Case **2.1.**, we note that neither $\mathbf{1}$ nor $\perp$ may be contracted, and there is no contraction added to the $(\mathbf{1}\vdash)$ or $(\vdash\perp)$ rules in $[CLL^{\mathbf{1}\perp}_{\mathrm{int}}]$. For example, we have the immediate step below for $\mathbf{1}$. (We assume the same conventions for illustrative purposes as in the proof of Theorem 13.)

$$\vdash\mathbf{1}\ \frac{\begin{matrix}\vdots\\ !\mathcal{C};\, !\mathcal{C};\, \Gamma\vdash\Delta;\, ?\mathcal{D};\, ?\mathcal{D}\end{matrix}}{\mathbf{1};\, !\mathcal{C};\, !\mathcal{C};\, \Gamma\vdash\Delta;\, ?\mathcal{D};\, ?\mathcal{D}} \quad \overset{\text{i.h.}}{\rightsquigarrow} \quad \frac{\begin{matrix}\vdots\\ !\mathcal{C};\, \Gamma\vdash\Delta;\, ?\mathcal{D}\end{matrix}}{\mathbf{1};\, !\mathcal{C};\, \Gamma\vdash\Delta;\, ?\mathcal{D}}\ \vdash\mathbf{1}$$

The situation with the $(\vdash\perp)$ rule is alike (and we omit the details). □

The truth of the following theorem should be obvious at this point. (We do not include the details of its proof.)

Theorem 27 *(Curry's lemma for $(\!|RLL^{\mathbf{1}\perp}_{\mathrm{int}}|\!)$). If there is a proof of $\Gamma\vdash\Delta$ in $(\!|RLL^{\mathbf{1}\perp}_{\mathrm{int}}|\!)$ where the height of the proof tree is χ, and $\Gamma'\vdash\Delta'$ could be obtained from $\Gamma\vdash\Delta$ by applications of $(W\vdash)$ and $(\vdash W)$, then* there is a proof of *$\Gamma'\vdash\Delta'$ in $(\!|RLL^{\mathbf{1}\perp}_{\mathrm{int}}|\!)$ with the height of the proof tree being χ' and $\chi'\leq\chi$.*

Theorem 28 *(Decidability of $RLL^{\mathbf{1}\perp}_{\mathrm{int}}$). The logic $RLL^{\mathbf{1}\perp}_{\mathrm{int}}$ is* decidable*.*

Proof. The proof is a straightforward extension of the proof of Theorem 21. □

Theorem 29 *(Decidability of $CLL^{\mathbf{1}\perp}_{\mathrm{int}}$). The logic $CLL^{\mathbf{1}\perp}_{\mathrm{int}}$ is* decidable*.*

Proof. We note that the same kind of relationship that held between RLL_{int} and $(\!|RLL_{\mathrm{int}}|\!)$, on one hand, and CLL_{int} and $[CLL_{\mathrm{int}}]$, on the other, obtains between $RLL^{\mathbf{1}\perp}_{\mathrm{int}}$ and $(\!|RLL^{\mathbf{1}\perp}_{\mathrm{int}}|\!)$, on one hand, and $CLL^{\mathbf{1}\perp}_{\mathrm{int}}$ and $[CLL^{\mathbf{1}\perp}_{\mathrm{int}}]$, on the other. With this observation in hand, the proof of this theorem is exactly as the proof of the decidability of CLL_{int}. □

It has been noted by many people that $CLL^{\mathbf{1}\perp}_{\mathrm{int}}$ (or its extension with the extensional connectives, i.e., "full linear logic") is not akin to intuitionistic logic — despite the fact that there is a translation of J into full linear logic. In a sequent calculus formulation, the "intuitionistic character" of a logic often reveals itself when the right-hand side of a sequent is restricted to contain at most one formula.[13] (For now, we exclude $\invamp$ from the language.)

The *intensional fragment of intuitionistic linear logic with* $\mathbf{1}$ *and* $\perp$, denoted by $ILL^{\mathbf{1}\perp}_{\mathrm{int}}$, is defined by the axioms and rules of $CLL^{\mathbf{1}\perp}_{\mathrm{int}}$ (except the rules $(\invamp\vdash)$ and $(\vdash\invamp)$) with the proviso that no sequent can have a multiset with two or more formula occurrences on the right-hand side of the turnstile. (The condition excludes the upper sequent of the $(\vdash ?W)$ rule from the set of sequents, therefore, the rule itself may be thought to be excluded too.)

The contraction-free version of $ILL^{\mathbf{1}\perp}_{\mathrm{int}}$, denoted by $[ILL^{\mathbf{1}\perp}_{\mathrm{int}}]$ is defined by a similar omission and restriction from $[ILL^{\mathbf{1}\perp}_{\mathrm{int}}]$. We only repeat and clarify four rules here, the analogs of which have bracketed multisets in the succedent of their conclusion. (X is the empty multiset or a multiset with one copy of one formula.)

$$\frac{\Gamma\vdash\mathcal{A}\qquad \mathcal{B};\Theta\vdash X}{[\mathcal{A}\multimap\mathcal{B};\Gamma;\Theta]\vdash X}\ [\multimap\vdash] \qquad \frac{\Gamma\vdash\mathcal{A}\qquad \Theta\vdash\mathcal{B}}{[\Gamma;\Theta]\vdash\mathcal{A}\otimes\mathcal{B}}\ [\vdash\otimes]$$

$$\frac{\Gamma\vdash\mathcal{A}}{\Gamma\vdash ?\mathcal{A}}\ \vdash ? \qquad \frac{\Gamma\vdash\mathcal{C}\qquad \mathcal{C};\Theta\vdash X}{[\Gamma;\Theta]\vdash X}\ [\mathrm{cut}]_2$$

Theorem 30 *(Cut theorem for $ILL^{\mathbf{1}\perp}_{\mathrm{int}}$ and $[ILL^{\mathbf{1}\perp}_{\mathrm{int}}]$). The* cut *rule is* admissible *in $ILL^{\mathbf{1}\perp}_{\mathrm{int}}$, and the* $[\mathrm{cut}]_2$ *rule is* admissible *in $[ILL^{\mathbf{1}\perp}_{\mathrm{int}}]$.*

Proof. The proofs are similar to the proofs of Theorems 8 and 10, and we omit the details. □

The *intensional intuitionistic interlinear logic with the Ackermann constants* is denoted by $XLL^{\mathbf{1}\perp}_{\mathrm{int}}$, and it is obtained from $ILL^{\mathbf{1}\perp}_{\mathrm{int}}$ by replacing $(!W\vdash)$ with its unrestricted version $(W\vdash)$. $(\!|XLL^{\mathbf{1}\perp}_{\mathrm{int}}|\!)$ is related to $XLL^{\mathbf{1}\perp}_{\mathrm{int}}$ like $(\!|RLL^{\mathbf{1}\perp}_{\mathrm{int}}|\!)$ is to $RLL^{\mathbf{1}\perp}_{\mathrm{int}}$ — while adhering to the intuitionistic restriction on the right-hand side of the turnstile. We give five rules here to clarify some of the less obvious rules. (X is either empty or $\mathcal{A}^{\perp}$, and Y is either empty or a singleton multiset.)

$$\frac{\mathcal{A};\Gamma\vdash X}{\Gamma\vdash\mathcal{A}^{\perp}}\ \vdash^{\perp} \qquad \frac{\Gamma\vdash\mathcal{A}\qquad \Theta\vdash\mathcal{B}}{(\!|\Gamma;\Theta|\!)\vdash\mathcal{A}\otimes\mathcal{B}}\ (\!|\vdash\otimes|\!) \qquad \frac{\Gamma\vdash\mathcal{A}\qquad \mathcal{B};\Theta\vdash Y}{(\!|\mathcal{A}\multimap\mathcal{B};\Gamma;\Theta|\!)\vdash Y}\ (\!|\multimap\vdash|\!)$$

$$\frac{\Gamma\vdash\mathcal{A}}{\Gamma\vdash ?\mathcal{A}}\ \vdash ? \qquad \frac{\Gamma\vdash\mathcal{C}\qquad \mathcal{C};\Theta\vdash Y}{(\!|\Gamma;\Theta|\!)\vdash Y}\ (\!|\mathrm{cut}|\!)_2$$

[13] This was Gentzen's idea, but there are variations on this theme. See, for example, Curry [11] for a calculus with a variable number of formulas on the right-hand side of the turnstile.

Theorem 31 *(Cut theorem for $XLL^{\mathbf{1}\perp}_{\mathrm{int}}$ and $(\!|XLL^{\mathbf{1}\perp}_{\mathrm{int}}|\!)$). The* cut *rule is* admissible *in $XLL^{\mathbf{1}\perp}_{\mathrm{int}}$. The $(\!|\mathrm{cut}|\!)_2$ rule is* admissible *in $(\!|XLL^{\mathbf{1}\perp}_{\mathrm{int}}|\!)$.*

Proof. The proofs are similar to the proofs of Theorems 11 and 12. □

We state the next two theorems without detailing their proofs.

Theorem 32 *(Curry's lemma for $[ILL^{\mathbf{1}\perp}_{\mathrm{int}}]$ and $(\!|XLL^{\mathbf{1}\perp}_{\mathrm{int}}|\!)$). If there is a proof of $\Gamma \vdash X$ in $[ILL^{\mathbf{1}\perp}_{\mathrm{int}}]$ (in $(\!|XLL^{\mathbf{1}\perp}_{\mathrm{int}}|\!)$) where the height of the proof tree is χ, and $\Gamma' \vdash X$ could be obtained from $\Gamma \vdash X$ by applications of $(!W\vdash)$ (by applications of $(W\vdash)$), then* there is a proof of *$\Gamma' \vdash X$ in $[ILL^{\mathbf{1}\perp}_{\mathrm{int}}]$ (in $(\!|XLL^{\mathbf{1}\perp}_{\mathrm{int}}|\!)$) with the height of the proof tree being χ' and $\chi' \leq \chi$.*

Theorem 33 *(Decidability of $ILL^{\mathbf{1}\perp}_{\mathrm{int}}$ and $XLL^{\mathbf{1}\perp}_{\mathrm{int}}$). The logics $XLL^{\mathbf{1}\perp}_{\mathrm{int}}$ and $ILL^{\mathbf{1}\perp}_{\mathrm{int}}$ are* decidable*.*

6. Conclusions

We have proved that the intensional fragment of classical linear logic (which is sometimes abbreviated as *MELL*) is decidable. This closes the last open question concerning the decidability of a prominent fragment of linear logic. The result can be extended to closely related logics such as intensional interlinear logic, intensional intuitionistic linear logic — with or without the Ackermann constants. The core of our proof is an application of Kripke's and Curry's ideas. Namely, contraction is expunged. However, the elimination of contraction is carried out not by loosening the axiom and the concept of a sequent, but by cautiously strengthening some connective rules.

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