Ehrenfeucht, Vaught, and the Decidability of the Weak Monadic Theory of Successor

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The decidability of the weak monadic theory of successor is usually considered as a consequence of the connection between monadic second-order logic and finite automata, as established around 1960 in papers of Büchi, Elgot, and Trakhtenbrot. However, there are several remarks and footnotes in papers of that time indicating that the result is also derivable from a theorem of A. Ehrenfeucht using an unpublished remark of R. L. Vaught. In the present note we review these hints and provide a proof along these lines. This simple argument is of methodological interest since it relies solely on first-order model theory and does not use finite automata.

1. HISTORICAL CONTEXT – AND SOME FOOTNOTES

Examples of decidable theories of arithmetic were known since the 1930's: Presburger and Skolem showed that the first-order theory of addition $\mathrm{FOTh}(\mathbb{N},+)$, respectively the first-order theory $\mathrm{FOTh}(\mathbb{N},\cdot)$ of multiplication, over the set of natural numbers is decidable. In the 1950's, Tarski asked for such a result in the framework of second-order logic. It is clear that even over the successor structure $(\mathbb{N},+1)$ the use of quantifiers over binary relations allows to define addition and multiplication and thus gives an undecidable theory (one observes that $x+y=z$ iff each binary relation containing $(0,x)$ and closed under simultaneously taking successor in both components contains (y,z); similarly multiplication is defined in terms of addition). Tarski pointed to monadic second-order logic MSO in which second-order variables and quantifiers are only admitted for unary ("monadic") relations, i.e. sets of natural numbers; he asked whether the corresponding theory $\mathrm{MSOTh}(\mathbb{N},+1)$ is decidable. A more restricted theory is obtained by considering weak monadic second-order logic WMSO in which the set quantifiers only range over finite sets, leading to the question whether $\mathrm{WMSOTh}(\mathbb{N},+1)$ is decidable.

Some progress was made by R. M. Robinson in [Rob58] who showed that an undecidable theory is obtained when besides successor also addition or just the double function d (with $d(n)=2n$) is admitted; so $\mathrm{MSOTh}(\mathbb{N},+1,d)$ and also $\mathrm{WMSOTh}(\mathbb{N},+1,d)$ are undecidable. Then Büchi and Elgot announced in their joint abstract [BE58] that $\mathrm{WMSOTh}(\mathbb{N},+1)$ is decidable. They wrote full papers separately, which appeared as [Büc60] and [Elg61]. The key in the proofs is an equivalence between weak monadic second-order logic and finite automata. Independently, Trakhtenbrot [Tra58] introduced monadic second-order logic as a "metalanguage" describing the behavior of finite automata (for a detailed account see [TB73, p.196 ff.]). The connection between monadic logic and finite automata proved to be most fruitful and led to strong decidability results during the subsequent decades.

However, just regarding the decidability of $\mathrm{WMSOTh}(\mathbb{N},+1)$, there are hints in the papers [BE58; Rob58; Büc60; Elg61] that an alternative proof is possible, based on work of Ehrenfeucht. Let us review these remarks.

Already in the abstract [BE58] we find such a hint following the statement on decidability of WMSOTh$(\mathbb{N}, +1)$ that appears as a corollary of the equivalence between WMSO over $(\mathbb{N}, +1)$ and finite automata:

> The corollary has been obtained by A. Ehrenfeucht and R. L. Vaught via Ehrenfeucht's theorem (unpublished) stating that the elementary theory of addition of ordinals is decidable.

In [Büc60], the same hint appears in a footnote accompanying the theorem on decidability of WMSOTh$(\mathbb{N}, +1)$:

> As R. L. Vaught remarks, this result can be obtained from a theorem of A. Ehrenfeucht; see Robinson [14].

The paper "Robinson [14]" is our reference [Rob58]. Robinson takes up the suggestions of Tarski and considers as "Problem 1" the question whether addition is definable in MSOTh$(\mathbb{N}, +1)$, and as "Problem 2" the question whether this theory is decidable. The "modified" Problems 1 and 2 are the corresponding questions on WMSOTh$(\mathbb{N}, +1)$. We find the following footnote:

> (Added in proof.) I have learned that A. Ehrenfeucht has proved that the arithmetical theory of the addition of ordinals is decidable, and that R. L. Vaught has shown how this result may be used to give a positive solution to the modified Problem 2 and hence a negative solution to the modified Problem 1. [. . .]

In [Elg61] Elgot there is an even stronger reference to Ehrenfeucht. Elgot derives the decidability of WMSOTh$(\mathbb{N}, +1)$ (for which he uses the term "set of true sentences of L^1_1") from his Theorem 5.3 (a) that establishes the bridge to automata. As a corollary he states:

> Corollary to 5.3(a) [3] Ehrenfeucht's Theorem (unpublished). The set of true sentences of L^1_1 is effective.

The footnote [3] says: "This result has been obtained independently by J. R. Büchi."

It seems that there is no written account (by Vaught or another author) on how the decidability of WMSOTh$(\mathbb{N}, +1)$ was derived from the decidability of the first-order theory of ordinal addition. (For example, in his lucid survey [Vau98] on Ehrenfeucht's work in model theory, Vaught does not mention this point.) Below we present a simple proof that serves this purpose.

2. DECIDABILITY OF WMSOTh$(\mathbb{N}, +1)$

In his celebrated paper [Ehr61] (following an announcement of the results in [Ehr57]), Ehrenfeucht introduces the model-theoretic games which today carry his name, both for first-order and weak monadic second-order logic. As an application he obtains results on the definability of ordinals, showing that precisely the ordinals $< \omega^\omega$ are FO-definable in the signature $\{<\}$, where $<$ denotes the ordinal ordering, and that precisely the ordinals $< \omega^{\omega^\omega}$ are FO-definable in the signature $\{<, +\}$, where $+$ denotes ordinal addition. (An ordinal β is FO-definable in the theory of ordinals if there is an FO-formula $\varphi_\beta(x)$ such that for any ordinal $\alpha > \beta$ we have $(\alpha, <) \models \varphi_\beta[\gamma]$ iff $\gamma = \beta$.) Ehrenfeucht showed that the same statement also holds for WMSO in place of FO.

Ehrenfeucht did not mention explicitly that from the results of his paper one obtains the decidability of the FO-theory of ordinals in either of these signatures. For the reader's convenience we sketch the argument for the FO-theory of ordinals in the signature $\{<\}$. (The reader will find a detailed account in Rosenstein's monograph

[Ros82], in particular in Chapters 6, 13, 15.) First, applying the Ehrenfeucht game, one verifies that no FO-sentence with signature $\{<\}$ can distinguish $(\omega^\omega, <)$ from the "structure" consisting of the class of all ordinals, equipped with the order between ordinals; so $\mathrm{FOTh}(\omega^\omega, <)$ coincides with the FO-theory of ordinals (see [Ros82, p. 269]). Now one shows that for each $n \in \mathbb{N}$ the ordinal ω^n is axiomatizable among all structures with signature $\{<\}$ by an FO-sentence φ_{ω^n} (see [Ros82, p. 262]). This means that for any sentence φ in the signature $\{<\}$ we have $(\omega^n, <) \models \varphi$ iff $\varphi_{\omega^n} \to \varphi$ is valid (validity meaning that every structure $(A, <)$ satisfies $\varphi_{\omega^n} \to \varphi$). Whether such a sentence $\varphi_{\omega^n} \to \varphi$ is valid can be checked in view of Gödel's completeness theorem ([Göd30]): The valid sentences coincide with those derivable in a proof calculus of first-order logic and are thus recursively enumerable. So, given such an enumeration, for any $\{<\}$-sentence φ, either $\varphi_{\omega^n} \to \varphi$ or $\varphi_{\omega^n} \to \neg\varphi$ will appear in the enumeration, giving the decision whether $(\omega^n, <) \models \varphi$ or not. It follows that also $\mathrm{FOTh}(\omega^\omega, <)$ is decidable: In order to check whether a sentence ψ, say of quantifier-depth n, is true in $(\omega^\omega, <)$ it suffices to note (again via the Ehrenfeucht game) that $(\omega^n, <)$ and $(\omega^\omega, <)$ satisfy the same sentences of quantifier-depth n. – In an analogous way one shows that $\mathrm{FOTh}(\omega^{\omega^\omega}, <, +)$ coincides with the first-order theory of addition of ordinals and is decidable.

In the sequel we do not work with ordinal addition in ω^{ω^ω} but prefer to work with the theory $\mathrm{FOTh}(\omega^\omega, <, +)$; this suffices for our purpose. Three preliminary remarks are useful.

First we recall *Cantor's normal form*: Each non-zero ordinal α has a unique representation as a sum of ω-powers, namely in the form

$$\alpha = \omega^{\beta_k} n_k + \ \ldots + \omega^{\beta_0} n_0$$

where $\beta_k > \ldots > \beta_0$ are ordinals and $n_0, \ldots, n_k$ are positive natural numbers, γn indicating the n-fold sum of γ. Note that the order of the terms is relevant; if the sum of ω-powers was written in reverse order it would be equal to $\omega^{\beta_k} n_k$: For $\beta < \gamma$, we have $\omega^\beta + \omega^\gamma = \omega^\gamma$, so ω^β is "absorbed" by ω^γ. For ordinals α_1, α_2 in general (including the case where the leading ω-powers are the same), we have $\alpha_1 \leq \alpha_2$ iff there exists α with $\alpha_1 + \alpha = \alpha_2$, which gives a definition of the order $<$ in terms of addition. (Indeed, Ehrenfeucht [Ehr61] just works with the signature $\{+\}$.)

Secondly, as an inspection of Cantor's normal form shows, we see that the ω-powers α (of the form ω^β) are characterized by the condition that a sum of two ordinals $< \alpha$ is again $< \alpha$. We shall thus use the following first-order formula in the signature $\{<, +\}$:

$$\omega\text{-power}(x) \ := \ \forall y_1 \forall y_2((y_1 < x \wedge y_2 < x) \to y_1 + y_2 < x)$$

Third, we note that ω and ω^ω are FO-definable in the signature $\{<\}$, respectively $\{<, +\}$, by formulas $\varphi_\omega(x)$, $\varphi_{\omega^\omega}(x)$. The ordinal ω can be fixed to be the smallest ordinal $x \geq 1$ such that for each $y < x$ there is z with $y < z < x$. Then, using the formulas $\varphi_\omega(x)$ and ω-power(x), ω^ω is described as the smallest ordinal $x \geq \omega$ such that for each ω-power $y < x$ there is an ω-power z with $y < z < x$.

Hence from the decidability of $\mathrm{FOTh}(\omega^{\omega^\omega}, <, +)$ we conclude the decidability of $\mathrm{FOTh}(\omega^\omega, <, +)$ by relativization of quantifiers to the ordinals $\alpha < \omega^\omega$. Note that Cantor's normal form for ordinals $\alpha < \omega^\omega$ gives exponents β_i that are natural numbers.

We now derive the decidability of $\mathrm{WMSO}(\mathbb{N}, +1)$ from the decidability of $\mathrm{FOTh}(\omega^\omega, <, +)$. For this we translate each sentence φ of weak monadic second-order logic over $(\mathbb{N}, +1)$ into an FO-sentence φ' in signature $\{<, +\}$ such that φ is true in $(\mathbb{N}, +1)$ iff φ' is true in $(\omega^\omega, <, +)$. In other words, we interpret $\mathrm{WMSOTh}(\mathbb{N}, +1)$ in $\mathrm{FOTh}(\omega^\omega, <, +)$.

The idea is very simple: We code the natural numbers $0, 1, 2, \ldots$ by the ω-powers $\omega^0, \omega^1, \omega^2, \ldots$ and a non-empty finite set $\{m_1 < m_2 < \ldots < m_k\}$ of natural numbers by $\omega^{m_k} + \ldots + \omega^{m_1}$; the empty set is coded by the ordinal 0. So non-empty finite sets of natural numbers are coded by ordinals $< \omega^\omega$ where in the Cantor normal form coefficients > 1 are disallowed.

In order to specify the translation we note that quantifiers over natural numbers become quantifiers over ω-powers, and that the successor function becomes the function "next ω-power" (easily definable – we shall write $\mathrm{succ}(x)$ for the smallest ω-power $> x$). In order to handle set quantifiers and the membership relation between numbers and finite sets it remains to supply formulas defining the "ω-power-sums" (of the form $\omega^{m_k} + \ldots + \omega^{m_1}$ with $m_k > \ldots > m_1$) and the relation "x occurs as ω-power in the ω-power-sum y". This is done by the following formulas (where sometimes we write $\gamma > \beta$ instead of $\beta < \gamma$):

(1) ω-power-sum(y) :=
$$\neg\exists y_1, y_2, z(\omega\text{-power}(z) \wedge [(y = z + z + y_1 \wedge y_1 < \mathrm{succ}(z)) \vee (y = y_2 + z + z + y_1 \wedge y_2 + z > z > y_1)])$$
This excludes that in the Cantor normal form of y ω-power terms $\omega^m n$ occur with $n \geq 2$; observe that in this case one could either isolate from the leading term a sum of two identical ω-powers, or one could decompose the whole sum into two parts $y_2 + z$ and $z + y_1$ with an ω-power z, where y_2 has a greater leading ω-power than z and y_1 has a smaller leading ω-power than z or is 0.

(2) x occurs in ω-power-sum(y) :=
$$\omega\text{-power}(x) \wedge \omega\text{-power-sum}(y) \wedge \exists y_1, y_2[(x > y_1 \wedge y = x + y_1) \vee (y_2 > x > y_1 \wedge y = y_2 + x + y_1)]$$
Again, we distinguish the case where the ω-power x is the leading term in the Cantor normal form of y from the case where x is a later term.

Let us note that a related approach is pursued by Büchi in his later work [Büc65a; Büc65b] on the weak monadic theory of transfinite ordinals. There the first-order theory of ordinal addition is extended by a further binary relation. A finite subset $\{u_1 < \ldots < u_n\}$ of an ordinal α is represented by the ordinal $2^{u_n} + \ldots + 2^{u_1} < 2^\alpha$. Büchi uses the mutual interpretability of WMSOTh$(\alpha, <)$ and FOTh$(2^\alpha, +, E)$ where $E(x, y)$ means "x is a power of 2 and occurs in the representation of y as decreasing sum of powers of 2".

3. CONCLUSION

We have shown how to deduce the decidability of the weak monadic second-order theory of successor from a decidability result in first-order model theory, namely from Ehrenfeucht's result that FOTh$(\omega^\omega, <, +)$ is decidable. Perhaps the simplicity of the matter was a reason why this was not published so far. However, it is clearly of interest that the decidability of WMSOTh$(\mathbb{N}, +1)$ can be verified by a purely model-theoretic argument without any use of finite automata.

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