# Deciding Monotonic Games

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**Abstract.** In an earlier work [AČJYK00] we presented a general framework for verification of infinite-state transition systems, where the transition relation is monotonic with respect to a *well quasi-ordering* on the set of states. In this paper, we investigate extending the framework from the context of transition systems to that of *games*. We show that monotonic games are in general undecidable. We identify a subclass of monotonic games, called *downward closed* games. We provide algorithms for analyzing downward closed games subject to winning conditions which are formulated as safety properties.

## 1 Introduction

One of the main challenges undertaken by the model checking community has been to develop algorithms which can deal with infinite state spaces. In a previous work [AČJYK00] we presented a general framework for verification of infinite-state *transition systems*. The framework is based on the assumption that the transition relation is monotonic with respect to a *well quasi-ordering* on the set of states (configurations). The framework has been used both to give uniform explanations of existing results for infinite-state systems such as Petri nets, Timed automata [AD90], lossy channel systems [AJ96b], and relational automata [BBK77,Čer94]; and to derive novel algorithms for model checking of Broadcast protocols [EFM99,DEP99], timed Petri nets [AN01], and cache coherence protocols [Del00], etc.

A related approach to model checking is that of control [AHK97]. Behaviours of reactive systems can naturally be described as *games* [dAHM01,Tho02], where control problems can be reduced to the problem of providing winning strategies. Since the state spaces of reactive systems are usually infinite, it is relevant to try to design algorithms for solving games over infinite state spaces.

In this paper, we consider extending the framework of [AČJYK00] from the context of transition systems to that of games. This turns out to be non-trivial. In fact, for one of the simplest classes of monotonic transition systems, namely *Petri nets*, we show that the game problem is undecidable. The negative result holds for games with the simplest possible winning condition, namely that of *safety*. Such a game is played between two players $A$ and $B$, where player $A$ tries to avoid a given set of *bad* configurations, while player $B$ tries to force the play into such a configuration. On the other hand, we show decidability of the safety game problem for a subclass of monotonic games, namely *downward*

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*closed games*: if a player can make a move from a configuration $c_1$ to another configuration $c_2$, then all configurations which are larger than $c_1$ (with respect to the ordering on the state space) can also make a move to $c_2$. Typical examples of downward closed systems are those with *lossy behaviours* such as lossy channel systems [AJ96b] and lossy VASS [BM99].

We summarize our (un)decidability results as follows:

– Decidability of the safety problem for games where player $B$ has a downward closed behaviour (a $B$*-downward closed game*). Considering the case where only one player is downward closed is relevant, since it allows, for instance, modelling behaviours of systems where one player (representing the environment) may lose messages in a lossy channel system (a so called $B$-LCS game). In case player $A$ has a deterministic behaviour (has no choices), our algorithm for $B$-downward closed games degenerates to the symbolic backward algorithm presented in [AJ96b,AČJYK00] for checking safety properties. In fact, we give a characterization of the set of winning (and losing) configurations in such a game. Observe that this result implies decidability of the case when both players have downward closed behaviours.
– Decidability of the safety problem for $A$-downward closed games. In case player $B$ has a deterministic behaviour, our algorithm for $A$-downward closed games degenerates to the forward algorithms described in [AJ96b,AČJYK00] and [Fin94,FS98] for checking eventuality properties (of the form $\forall \diamond p$). However, in contrast to $B$-downward closed games, we show it is not possible to give a characterization of the set of winning (or losing) configurations.
– Decidability results for downward closed games do not extend to monotonic games. In particular we show that deciding safety properties for games based on VASS (Vector Addition Systems with States). is undecidable. VASS is a variant of Petri nets. The undecidability result holds even if both players are assumed to have monotonic behaviours.
– Undecidability of *parity games* for both $A$- and $B$-downward closed games. In a parity game, each configuration is equipped with a *rank* chosen from a finite set of natural numbers. The winning condition is defined by the parity of the lowest rank of a configuration appearing in the play. In particular, we show undecidability of parity games for both A-LCS and B-LCS games. On the other hand, if both players can lose messages, then the problem is decidable.

**Outline.** In the next Section, we recall some basic definitions for games. In Section 3, we introduce monotonic and downward closed games. We present a symbolic algorithm for solving the safety problem for *B-downward closed* games in Section 4; and apply the algorithm to $B$-LCS in Section 5. In Section 6, we consider *A-downward closed* games. In Section 7, we show that the safety problem is undecidable for monotonic games. In Section 8, we study decidability of parity games for the above models.

## 2 Preliminaries

In this section, we recall some standard definitions for games.

A game $G$ is a tuple $(C, C_A, C_B, \longrightarrow, C_F)$, where $C$ is a (possibly infinite) set of *configurations*, $C_A, C_B$ is a partitioning of $C$, $\longrightarrow \subseteq (C_A \times C_B) \cup (C_B \times C_A)$ is a set of *transitions*, and $C_F \subseteq C_A$ is a finite set of *final configurations*. We write $c_1 \longrightarrow c_2$ to denote that $(c_1, c_2) \in \longrightarrow$. For a configuration $c$, we define $Pre(c) = \{c' |\, c' \longrightarrow c\}$, and define $Post(c) = \{c' |\, c \longrightarrow c'\}$. We extend $Pre$ to sets of configurations such that $Pre(D) = \cup_{c \in D} Pre(c)$. The function $Post$ can be extended in a similar manner. Without loss of generality, we assume that there is no deadlock, i.e., $Post(c) \neq \emptyset$ for each configuration $c$. For a set $D \subseteq C_A$ of configurations, we define $\overset{A}{\sim} D$ to be the set $C_A \setminus D$. The operator $\overset{B}{\sim}$ is defined in a similar manner. For a set $D \subseteq C_A$, we use $\overline{Pre}(D)$ to denote $\overset{B}{\sim}\left(Pre\left(\overset{A}{\sim} D\right)\right)$. For $E \subseteq C_B$, we define $\overline{Pre}(E)$ in a similar manner.

A *play* $P$ (of $G$) from a configuration $c$ is an infinite sequence $c_0, c_1, c_2, \ldots$ of configurations such that $c_0 = c$, and $c_i \longrightarrow c_{i+1}$, for each $i \geq 0$. A play $c_0, c_1, c_2, \ldots$ is *winning* (for $A$) if there is no $j \geq 0$ with $c_j \in C_F$.

A *strategy* for player $A$ (or simply an $A$-*strategy*) is a partial function $\sigma_A : C_A \mapsto C_B$ such that $c \longrightarrow \sigma_A(c)$. A $B$-*strategy* is a partial function $\sigma_B : C_B \mapsto C_A$ and is defined in a similar manner to $\sigma_A$. A configuration $c \in C_A$ together with strategies $\sigma_A$ and $\sigma_B$ (for players $A$ and $B$ respectively) define a play $P(c, \sigma_A, \sigma_B) = c_0, c_1, c_2, \ldots$ from $c$ where $c_{2i+1} = \sigma_A(c_{2i})$, and $c_{2i+2} = \sigma_B(c_{2i+1})$, for $i \geq 0$. A similar definition is used in case $c \in C_B$ (interchanging the order of applications of $\sigma_A$ and $\sigma_B$ to the configurations in the sequence).

An $A$-strategy $\sigma_A$ is said to be *winning* from a configuration $c$, if for all $B$-strategies $\sigma_B$, it is the case that $P(c, \sigma_A, \sigma_B)$ is winning. A configuration $c$ is said to be *winning* if there is a winning $A$-strategy from $c$.

We shall consider the *safety problem* for games:

**The safety problem**
**Instance.** A game $G$ and a configuration $c$.
**Question.** Is $c$ winning?

## 3 Ordered Games

In this section, we introduce *monotonic* and *downward closed* games.

**Orderings.** Let $A$ be a set and let $\preceq$ be a quasi-order (i.e. a reflexive and transitive binary relation) on $A$. We say that $\preceq$ is a *well quasi-ordering (wqo)* on $A$ if there is no infinite sequence $a_0, a_1, a_2, \ldots$ with $a_i \not\preceq a_j$ for $i < j$. For $B \subseteq A$, we say that $B$ is *canonical* if there are no $a, b \in B$ with $a \neq b$ and $a \preceq b$. We use *min* to denote a function where, for $B \subseteq A$, the value of $min(B)$ is a canonical subset of $B$ such that for each $b \in B$ there is $a \in min(B)$ with $a \preceq b$. We say that $\preceq$ is *decidable* if, given $a, b \in A$ we can decide whether $a \preceq b$. A set $B \subseteq A$

is said to be *upward closed* if $a \in B$ and $a \preceq b$ imply $b \in B$. A *downward closed* set is defined in a similar manner.

**Monotonic Games.** An *ordered game* $G$ is a tuple $(C, C_A, C_B, \longrightarrow, C_F, \preceq)$, where $(C, C_A, C_B, \longrightarrow, C_F)$ is a game and $\preceq \subseteq (C_A \times C_A) \cup (C_B \times C_B)$ is a decidable wqo on the sets $C_A$ and $C_B$. The ordered game $G$ is said to be *monotonic* with respect to player $A$ (or simply $A$-*monotonic*) if, for each $c_1, c_2 \in C_A$ and $c_3 \in C_B$, whenever $c_1 \preceq c_2$ and $c_1 \longrightarrow c_3$, there is a $c_4$ with $c_3 \preceq c_4$ and $c_2 \longrightarrow c_4$. A $B$-*monotonic game* is defined in a similar manner. A *monotonic game* is both $A$-monotonic and $B$-monotonic.

**Downward Closed Games.** An ordered game $G = (C, C_A, C_B, \longrightarrow, C_F, \preceq)$ is said to be $A$-*downward closed* if, for each $c_1, c_2 \in C_A$ and $c_3 \in C_B$, whenever $c_1 \longrightarrow c_3$ and $c_1 \preceq c_2$, then $c_2 \longrightarrow c_3$. A $B$-*downward closed game* is defined in a similar manner. A game is *downward closed* if it is both $A$- and $B$-downward closed. Notice that each class of downward closed games is included in the corresponding class of monotonic games. For instance, each $A$-downward closed game is $A$-monotonic. From the definitions we get the following property.

**Lemma 1.** *For an $A$-downward closed game $G$ and any set $E \subseteq C_B$, the set $Pre(E)$ is upward closed. A similar result holds for $B$-downward closed games.*

## 4 B-Downward Closed Games

We present a symbolic algorithm for solving the safety problem for $B$-downward closed games. In the rest of this Section, we assume an $B$-downward closed game $G = (C, C_A, C_B, \longrightarrow, C_F, \preceq)$.

**Scheme.** Given a configuration $c$ in $G$, we want to decide whether $c$ is winning or not. To do that, we introduce a scheme by considering a sequence of sets of configurations of the form:

$$s \; : \qquad D_0 \;,\; E_0 \;,\; D_1 \;,\; E_1 \;,\; D_2 \;,\; E_2 \;,\; \ldots$$

where $D_i \subseteq C_A$ and $E_i \subseteq C_B$. Intuitively, the sets $D_i$ and $E_i$ characterize the configurations (in $C_A$ and $C_B$ respectively) which are not winning. The elements of the sequence are defined by

$$D_0 = C_F \qquad\qquad E_0 = Pre(D_0)$$

$$D_{i+1} = D_i \cup \overline{Pre}(E_i) \qquad E_{i+1} = E_i \cup Pre(D_{i+1}) \qquad i = 0, 1, 2, \ldots$$

We say that $s$ *converges (at $\ell$)* if $D_{\ell+1} \subseteq D_\ell$ or $E_{\ell+1} \subseteq E_\ell$. In such a case, the set $D_\ell \cup E_\ell$ characterizes exactly the set of configurations which are not winning. The question of whether a given configuration $c$ is winning amounts therefore to whether $c \notin (D_\ell \cup E_\ell)$. To show that our characterization is correct, we show the following two Lemmas. The first Lemma shows that if $c$ appears in one of the generated sets then it is not a winning configuration. The second Lemma states that if the sequence converges, then the generated sets contain all non-winning configurations.

**Lemma 2.** *If $c \in D_i \cup E_i$, for some $i \geq 0$, then $c$ is not winning.*

**Lemma 3.** *If $s$ converges and $c \notin D_i \cup E_i$ for each $i \geq 0$, then $c$ is winning.*

Below, we present a symbolic algorithm based on the scheme above. We shall work with *constraints* which we use as symbolic representations of sets of configurations.

**Constraints.** An *A-constraint* denotes a (potentially infinite) set $[\![\phi]\!] \subseteq C_A$ of configurations. A *B-constraint* is defined in a similar manner. For constraints $\phi_1$ and $\phi_2$, we use $\phi_1 \sqsubseteq \phi_2$ to denote that $[\![\phi_2]\!] \subseteq [\![\phi_1]\!]$. For a set $\psi$ of constraints, we use $[\![\psi]\!]$ to denote $\bigcup_{\phi \in \psi} [\![\phi]\!]$. For sets of constrains $\psi_1$ and $\psi_2$, we use $\psi_1 \sqsubseteq \psi_2$ to denote that for each $\phi_2 \in \psi_2$ there is a $\phi_1 \in \psi_1$ with $\phi_1 \sqsubseteq \phi_2$. Notice that $\psi_1 \sqsubseteq \psi_2$ implies $[\![\psi_2]\!] \subseteq [\![\psi_1]\!]$. Sometimes, we identify constraints with their interpretations, so we write $c \in \phi$, $\phi_1 \subseteq \phi_2$, $\phi_1 \cap \phi_2$, $\neg\phi$, etc. We consider a particular class of $B$-constraints which we call *upward closed constraints*. An *upward closed constraint* is of the form $c\uparrow$, where $c \in C_B$, and has an interpretation $[\![c\uparrow]\!] = \{c' |\ c \preceq c'\}$.

A set $\Psi$ of $A$-constraints is said to be *effective* with respect to the game $G$ if

- The set $C_F$ is characterized by a finite set $\psi_F \subseteq \Psi$ (i.e. $[\![\psi_F]\!] = C_F$).
- For a configuration $c \in C_A$ and a constraint $\phi \in \Psi$, we can decide whether $c \in [\![\phi]\!]$.
- For each $\phi \in \Psi$, we can compute a finite set $\psi'$ of upward closed constraints such that $[\![\psi']\!] = Pre\,([\![\phi]\!])$. In such a case we use $Pre(\phi)$ to denote the set $\psi'$. Notice that $Pre\,([\![\phi]\!])$ is upward closed by Lemma 1. Also, observe that computability of $Pre(\phi)$ implies that, for a finite set $\psi \subseteq \Psi$, we can compute a finite set $\psi'$ of upward closed constraints such that $[\![\psi']\!] = Pre\,([\![\psi]\!])$.
- For each finite set $\psi$ of upward closed constraints, we can compute a finite set $\psi' \subseteq \Psi$ such that $[\![\psi']\!] = \overline{Pre}\,([\![\psi]\!])$. In such a case we use $\overline{Pre}(\psi)$ to denote the set $\psi'$.

The game $G$ is said to be *effective* if there is a set $\Psi$ of constraints which is effective with respect to $G$.

**Symbolic Algorithm.** Given a constraint system $\Psi$ which is effective with respect to the game $G$, we can solve the safety game problem by deriving a symbolic algorithm from the scheme described above. Each $D_i$ will be characterized by a finite set of constraints $\psi_i \in \Psi$ , and each $E_i$ will be represented by a finite set of upward closed constraints $\psi'_i$. More precisely:

$$\psi_0 \;=\; \psi_F \qquad\qquad \psi'_0 \;=\; Pre(\psi_0)$$

$$\psi_{i+1} \;=\; \psi_i \;\cup\; \overline{Pre}(\psi'_i) \qquad \psi'_{i+1} \;=\; \psi'_i \;\cup\; Pre(\psi_{i+1}) \qquad i = 0, 1, 2, \ldots$$

The algorithm terminates in case $\psi'_j \sqsubseteq \psi'_{j+1}$. In such a case, a configuration $c$ is not winning if and only if $c \in [\![\psi_j]\!] \cup [\![\psi'_j]\!]$. This gives an effective procedure for deciding the safety game problem according to the following

- Each step can be performed due to effectiveness of $\Psi$ with respect to $G$.
- For a configuration $c \in C_A$ and a constraint $\phi \in \psi_i$, we can check $c \in [\![\phi]\!]$ due to effectiveness of $\Psi$ with respect to $G$. For a configuration $c \in C_B$ and a constraint $\phi \in \psi'_i$, we can check $c \in [\![\phi]\!]$ due to decidability of $\preceq$.
- For a configuration $c$ and an upward closed constraint $\phi = c' \uparrow$, we can check $c \in [\![\phi]\!]$, since $\preceq$ is decidable and since $c \in [\![\phi]\!]$ if and only if $c' \preceq c$.
- The termination condition can be checked due to decidability of $\preceq$ (which implies decidability of $\sqsubseteq$).
- Termination is guaranteed due to well quasi-ordering of $\preceq$ (which implies well quasi-ordering of $\sqsubseteq$).

From this we get the following

**Theorem 1.** *The safety problem is decidable for the class of effective $B$-downward closed games.*

## 5 B-LCS

In this section, we apply the symbolic algorithm presented in Section 4 to solve the safety game problem for *B-LCS* games: games between two players operating on a finite set of channels (unbounded FIFO buffers), where player $B$ is allowed to lose any number of messages after each move.

A *B-lossy channel system (B-LCS )* is a tuple $(S, S_A, S_B, L, M, T, S_F)$, where $S$ is a finite set of *(control) states*, $S_A, S_B$ is a partitioning of $S$, $L$ is a finite set of *channels*, $M$ is a finite *message alphabet*, $T$ is a finite set of *transitions*, and $S_F \subseteq S_A$ is the set of *final states*. Each transition in $T$ is a triple $(s_1, op, s_2)$, where

- either $s_1 \in S_A$ and $s_2 \in S_B$, or $s_1 \in S_B$ and $s_2 \in S_A$.
- *op* is of one of the forms: $\ell!m$ (sending message $m$ to channel $\ell$), or $\ell?m$ (receiving message $m$ from channel $\ell$), or *nop* (not affecting the contents of the channels).

A B-LCS $\mathcal{L} = (S, S_A, S_B, L, M, T, S_F)$ induces a $B$-downward closed game $G = (C, C_A, C_B, \longrightarrow, C_F, \preceq)$ as follows:

- *Configurations:* Each configuration $c \in C$ is a pair $(s, w)$, where $s \in S$, and $w$, called a *channel state*, is a mapping from $L$ to $M^*$. In other words, a configuration is defined by the control state and the contents of the channels. We partition the set $C$ into $C_A = \{(s, w) \,|\, s \in S_A\}$ and $C_B = \{(s, w) \mid s \in S_B\}$.
- *Final Configurations:* The set $C_F$ is defined to be $\{(s, w) \mid s \in S_F\}$.
- *Ordering:* For $x_1, x_2 \in M^*$, we use $x_1 \preceq x_2$ to denote that $x_1$ is a (not necessarily contiguous) substring of $x_2$. For channel states $w_1, w_2$, we use $w_1 \preceq w_2$ to denote that $w_1(\ell) \preceq w_2(\ell)$ for each $\ell \in L$. We use $(s_1, w_1) \preceq (s_2, w_2)$ to denote that both $s_1 = s_2$ and $w_1 \preceq w_2$. The ordering $\preceq$ is decidable and wqo (by Higamn's Lemma [Hig52]).

– *Non-loss transitions*: $(s_1, w_1) \longrightarrow (s_2, w_2)$ if one of the following conditions is satisfied
  - There is a transition in $T$ of the form $(s_1, \ell!m, s_2)$, and $w_2$ is the result of appending $m$ to the end of $w_1(\ell)$.
  - There is a transition in $T$ of the form $(s_1, \ell?m, s_2)$, and $w_1$ is the result of appending $m$ to the head of $w_2(\ell)$.
  - There is a transition in $T$ of the form $(s_1, nop, s_2)$, and $w_2 = w_1$.

– *Loss transitions:* If $s_1 \in S_B$ and $(s_1, w_1) \longrightarrow (s_2, w_2)$ according to one of the previous two rules then $(s_1, w_1) \longrightarrow (s_2', w_2')$ for each $(s_2', w_2') \preceq (s_2, w_2)$.

**Remark.** To satisfy the condition that there are no deadlock states in games induced by $B$-LCS, we can always add two "winning" states $s_1^* \in S_A$, $s_2^* \in S_B$, and two "losing" states $s_3^* \in S_A$, $s_4^* \in S_B$, where $s_3^* \in S_F$, and $s_1^* \notin S_F$. We add four transitions $(s_1^*, nop, s_2^*)$, $(s_2^*, nop, s_1^*)$, $(s_3^*, nop, s_4^*)$, and $(s_4^*, nop, s_3^*)$. Furthermore, we add transitions $(s, nop, s_4^*)$ for each $s \in S_A$, and $(s, nop, s_1^*)$ for each $s \in S_B$. Intuitively, if player $A$ enters a configuration, where he has no other options, then he is forced to move to $s_4^*$ losing the game. A similar reasoning holds for player $B$.

We show decidability of the safety problem for $B$-LCS using Theorem 1. To do that we first describe upward closed constraints for $B$-LCS, and then introduce constraints which are effective with respect to $B$-LCS. We introduce upward closed constraints in several steps. First, we define upward closed constraints on words, and then generalize them to channel states and configurations. An *upward-closed constraint* over $M$ is of the form $X\uparrow$ where $X \subseteq M^*$, and has an interpretation $[\![X\uparrow]\!] = \{x \mid \exists x' \in X.\ x' \preceq x\}$. An upward closed constraint $\phi$ over channel states is a mapping from $L$ to upward closed constraints over $M$, with an interpretation $[\![\phi]\!] = \{w \mid \forall \ell \in L.\ w(\ell) \in [\![\phi(\ell)]\!]\}$. We use $w\uparrow$ to denote the upward closed constraint $\phi$ over channel states where $\phi(\ell) = w(\ell)\uparrow$ for each $\ell \in L$. An upward closed constraint $\phi$ (over configurations) is of the form $(s, \phi')$, where $s \in S$ and $\phi'$ is an upward closed constraint over channel states, with an interpretation $[\![\phi]\!] = \{(s, w) \mid w \in [\![\phi']\!]\}$. We use $(s, w)\uparrow$ to denote the upward closed constraint $(s, w\uparrow)$.

We introduce *extended upward-closed constraints* which we show to be effective with respect to $B$-LCS games. An *extended upward-closed constraint* over $M$ is of the form $x \bullet \phi$, where $x \in M^*$ and $\phi$ is an upward closed constraint over $M$, and has an interpretation $[\![x \bullet \phi]\!] = \{x \bullet x' \mid x' \in [\![\phi]\!]\}$. Extended upward closed constraints are generalized to channel states and configurations in a similar manner to above.

In the rest of this section we prove the following lemma

**Lemma 4.** *Extended upward closed constraints are effective for B-LCS games.*

From Theorem 1 and Lemma 4 we get the following

**Theorem 2.** *The safety problem is decidable for B-LCS games.*

We devote the rest of this section to the proof of Lemma 4. This is achieved as follows:

- The set $C_F$ is characterized by the (finite) set of constraints of the form $(s, \phi)$ where $s \in S_F$ and $\phi$ is an extended upward closed constraint over channels, where $\phi(\ell) = \epsilon \bullet \epsilon \uparrow$ for each $\ell \in L$ (notice that $[\![\epsilon \bullet \epsilon \uparrow]\!] = M^*$).
- For a configuration $c \in C_A$ and an extended constraint $\phi$ we can check whether $c \in [\![\phi]\!]$. (Lemma 5).
- For each extended upward closed constraint $\phi$, we can compute a finite set $\psi$ of upward closed constraints, such that $\psi = Pre(\phi)$ (Lemma 7).
- For each finite set $\psi_1$ of upward closed constraints, we can compute a finite set $\psi_2$ of extended upward closed constraints, such that $\psi_2 = \overline{Pre}(\psi_1)$. (Lemma 6).

For words $x_1, x_2$, we use $x_1 \sqcap x_2$ to denote the (finite) set of minimal (with respect to $\preceq$) words $x_3$ such that $x_1 \preceq x_3$ and $x_2 \preceq x_3$.

**Lemma 5.** *For a configuration $c \in C_A$ and an extended constraint $\phi$ we can check whether $c \in [\![\phi]\!]$.*

**Lemma 6.** *For each finite set $\psi_1$ of upward closed constraints, we can compute a finite set $\psi_2$ of extended upward closed constraints, such that $\psi_2 = \overline{Pre}(\psi_1)$.*

**Lemma 7.** *For each extended upward closed constraint $\phi$, we can compute a finite set $\psi$ of upward closed constraints, such that $\psi = Pre(\phi)$.*

**Remark.** Theorem 1 holds also in the case where both players can lose messages. In fact, we can show that negation constraints are effective with respect to such games.

## 6 *A*-Downward Closed Games

We present an algorithm for solving the safety problem for $A$-downward closed games. We use the algorithm to prove decidability of the safety problem for a variant of lossy channel games, namely *A-LCS*.

An $A$-downward closed game is said to be *effective* if for each configuration $c$ we can compute the set $Post(c)$. Observe that this implies that the game is finitely branching.

Suppose that we want to check whether a configuration $c_{init} \in C_A$ is winning. The algorithm builds an AND-OR tree, where each node of the tree is labelled with a configuration. OR-nodes are labelled with configurations in $C_A$, while AND-nodes are labelled with configurations in $C_B$.

We build the tree successively, starting from the root, which is labelled with $c_{init}$ (the root is therefore an OR-node). At each step we pick a leaf with label $c$ and perform one of the following operations:

- If $c \in C_F$ then we declare the node *unsuccessful* and close the node (we will not expand the tree further from the node).
- If $c \in C_A$, $c \notin C_F$, and there is a predecessor of the node in the tree with label $c'$ where $c' \preceq c$ then we declare the node *successful* and close the node.
- Otherwise, we add a set of successors, each labelled with an element in $Post(c)$. This step is possible by the assumption that the game is effective.

The procedure terminates by Köning's Lemma and by well quasi-ordering of $\preceq$. The resulting tree is evaluated interpreting AND-nodes as conjunction, OR-nodes as disjunction, successful leaves as the constant true and unsuccessful leaves as the constant false. The algorithm answers "yes" if and only if the resulting tree evaluates positively.

**Theorem 3.** *The safety problem is decidable for effective A-downward closed games.*

**$A$-LCS** An *$A$-LCS* has the same syntax as a $B$-LCS. The game induced by an $A$-LCS has a similar behaviour to that induced by a $B$-LCS. The difference is that in the definition of the *loss transitions*:

– If $s_1 \in S_A$ and $(s_1, w_1) \longrightarrow (s_2, w_2)$ according to a non-loss transition then $(s_1, w_1) \longrightarrow (s_2', w_2')$ for each $(s_2', w_2') \preceq (s_2, w_2)$.

It is straightforward to check that a game induced by an $A$-LCS is $A$-downward closed and effective. This gives the following.

**Theorem 4.** *The safety problem is decidable for A-LCS games.*

Although the safety problem is decidable for $A$-LCS games, it is not possible to give a characterization of the set of winning configurations as we did for $B$-LCS. By a similar reasoning to Lemma 1, the set $Pre(\overset{B}{\sim} E_i)$ is upward closed and therefore can be characterized by a finite set of upward closed constraints for each $i \geq 0$. In turn, the set $\bigcup_{i \geq 0} D_i$ can be characterized by a finite set of negation constraints. We show that we cannot compute a finite set of negation constraints $\psi$ such that $[\![\psi]\!] = \bigcup_{i \geq 0} D_i$, as follows.

We reduce an uncomputability result reported in [BM99] for transition systems induced by lossy channel systems. The results in [BM99] imply that we cannot characterize the set of configurations $c$ satisfying the property $c \models \exists_\infty \Box \neg S_F$, i.e., we cannot characterize the set of configurations from which there is an infinite computation which never visits a given set $S_F$ of control states. Given a lossy channel system $\mathcal{L}$ (inducing a transition system) and a set $S_F$ of states, we derive an $A$-LCS $\mathcal{L}'$ (inducing an $A$-downward closed game). For each configuration $c$ in $\mathcal{L}$, it is the case that $c \models \exists_\infty \Box \neg S_F$ if and only if the configuration corresponding to $c$ is winning in the game induced by $\mathcal{L}'$. Intuitively, player $A$ simulates the transitions of $\mathcal{L}$, while player $B$ follows passively. More precisely, each state $s$ in $\mathcal{L}$ has a copy $s \in C_A$ in $\mathcal{L}'$. For each transition $t = (s_1, op, s_2)$ in $\mathcal{L}$, there is a corresponding "intermediate state" $s_t \in C_B$ and two corresponding transitions $(s_1, op, s_t)$ and $(s_t, nop, s_2)$ in $\mathcal{L}'$. Furthermore, we have two state $s_1^* \in C_A$ and $s_2^* \in C_B$ which are losing (defined in a similar manner to Section 5). Each configuration in $C_A$ can perform a transition labelled with $nop$ to $s_2^*$. It is straightforward to check that a configuration $c$ is winning in $\mathcal{L}'$ if and only if $c \models \exists_\infty \Box \neg F$.

From this, we get the following:

**Theorem 5.** *We cannot compute a finite set of negation constraints characterizing the set of non-winning configurations in an A-LCS (although such a set always exists).*

## 7 Undecidability of Monotonic Games

We show that the decidability of the safety problem does not extend from downward closed games to monotonic games. We show undecidability of the problem for a particular class of monotonic games, namely *VASS games*. In the definition of VASS games below, both players are assumed to have monotonic behaviours. Obviously, this implies undecidability for $A$- and $B$-monotonic games.

In fact, it is sufficient to consider VASS with two dimensions (two variables). Let $\mathcal{N}$ and $\mathcal{I}$ denote the set of natural numbers and integers respectively.

**VASS Games** A *(2-dimensional) VASS (Vector Addition System with States) game* $\mathcal{V}$ is a tuple $(S, S_A, S_B, T, S_F)$, where $S$ is a finite set of *(control) states*, $S_A, S_B$ is a partitioning of $S$, $T$ is a finite set of *transitions*, and $S_F \subseteq S$ is the set of *final states*. Each transition is a triple $(s_1, (a, b), s_2)$, where

- either $s_1 \in S_A$ and $s_2 \in S_B$, or $s_1 \in S_B$ and $s_2 \in S_A$.
- $a, b \in \mathcal{I}$. The pair $(a, b)$ represents the change made to values of the variables during the transition.

A VASS $\mathcal{V} = (S, S_A, S_B, T, S_F)$ induces a monotonic game $G = (C, C_A, C_B, \longrightarrow, C_F, \preceq)$ as follows:

- Each configuration $c \in C$ is a triple $(s, x, y)$, where $s \in S$ and $x, y \in \mathcal{N}$. In other words, a configuration is defined by the state and the values assigned to the variables.
- $C_A = \{(s, x, y) \mid s \in S_A\}$.
- $C_B = \{(s, x, y) \mid s \in S_B\}$.
- $(s_1, x_1, y_1) \longrightarrow (s_2, x_2, y_2)$ iff $(s_1, (a, b), s_2) \in T$, and $x_2 = x_1 + a$, and $y_2 = y_1 + b$. Observe that since $x_2, y_2 \in \mathcal{N}$, we implicitly require $x_2 \geq 0$ and $y_2 \geq 0$; otherwise the transition is blocked.
- $C_F = \{(s, x, y) \mid s \in S_F\}$.
- $(s_1, x_1, y_1) \preceq (s_2, x_2, y_2)$ iff $s_1 = s_2$, $x_1 \leq x_2$, and $y_1 \leq y_2$.

We can avoid deadlock in VASS games in a similar manner to Section 5.

**Theorem 6.** *The safety problem is undecidable for VASS games.*

Undecidability is shown through a reduction from an undecidable problem for *2-counter machines*.

**2-Counter Machines.** A *2-counter machine* $M$ is a tuple $(S_M, T_M)$, where $S_M$ is a finite set of *states*, and $T_M$ is a finite set of *transitions*. Each transition is a triple of the form $(s_1, (a, b), s_2)$, or $(s_1, x = 0?, s_2)$, or $(s_1, y = 0?, s_2)$, where $s_1, s_2 \in S_M$.

A *configuration* of $M$ is a triple $(s, x, y)$ where $s \in S_M$ and $x, y \in \mathcal{N}$. We define a transition relation $\longrightarrow$ on configurations such that $(s_1, x_1, y_1) \longrightarrow (s_2, x_2, y_2)$ iff either

- $(s_1, (a, b), s_2) \in T_M$, and $x_2 = x_1 + a$, and $y_2 = y_1 + b$; or

- $(s_1, x = 0?, s_2) \in T_M$ and $x_2 = x_1 = 0$, and $y_2 = y_1$; or
- $(s_1, y = 0?, s_2) \in T_M$ and $x_2 = x_1$, and $y_2 = y_1 = 0$.

The *2-counter reachability problem* is defined as follows

**2-counter reachability problem**
**Instance.** A 2-counter machine $M = (S_M, T_M)$ and two states $s_{init}, s_f \in S_M$.
**Question.** Is there a sequence

$$(s_0, x_0, y_0) \longrightarrow (s_1, x_1, y_1) \longrightarrow (s_2, x_2, y_2) \longrightarrow \cdots \longrightarrow (s_n, x_n, y_n)$$

of transitions such that $s_0 = s_{init}$, $x_0 = 0$, $y_0 = 0$, and $s_n = s_f$?

It is well-known that the 2-counter reachability problem is undecidable. In the following, we show how to reduce the 2-counter reachability problem to the safety problem for VASS games. Given a 2-counter machine $M = (S_M, T_M)$ and two states $s_{init}, s_f \in S_M$, we construct a corresponding VASS game, such that the reachability problem has a positive answer if and only if the game problem has a negative answer. Intuitively, player $B$ emulates the moves of the 2-counter machine, while player $A$ is passive. Tests for equality with 0 cannot be emulated directly by a VASS system. This means that player $B$ could try to make moves not corresponding to an actual move of the 2-counter machine. However, if player $B$ tries to "cheat", i.e. make a forbidden move, then we allow player $A$ to go into a winning escape loop. This means that player $B$ always chooses to make legal moves. Furthermore, we add an escape loop accessible when the system has reached the final state. This loop is winning for player $B$. Thus, player $B$ wins whenever the final state is reachable. More formally, we define the VASS game $\mathcal{V} = (S, S_A, S_B, T, S_F)$ as follows:

- $S_A = \{s_t^A |\ t \in T_M\} \cup \{s_*^A, s_{reached}^A, s_{init}^A\}$. In other words, for each transition $t \in T_M$ there is a state $s_t^A \in S_A$. We also add three special states $s_*^A$,$s_{reached}^A$ and $s_{init}^A$ to $S_A$.
- $S_B = \{s^B |\ s \in S_M\} \cup \{s_*^B\}$. In other words, for each state in $s \in S_M$ there is a corresponding state $s^B \in S_B$. We also add a special state $s_*^B$ to $S_B$.
- For each transition $t$ of the form $(s_1, (a, b), s_2) \in T_M$, there are two transitions in $T$, namely $(s_1^B, (a, b), s_t^A)$ and $(s_t^A, (0, 0), s_2^B)$. Player $B$ chooses a move, and player $A$ follows passively.
- For each transition $t$ of the form $(s_1, x = 0?, s_2) \in T_M$, there are three transitions in $T$, namely $(s_1^B, (0, 0), s_t^A)$, $(s_t^A, (0, 0), s_2^B)$, and $(s_t^A, (-1, 0), s_*^B)$. Player $B$ may cheat here. However, if this is the case, player $A$ will be allowed to move to $s_*^B$, which is winning.
- Transitions of the form $(s_1, y = 0?, s_2) \in T_M$ are handled in a similar manner to the previous case.
- There are five additional transitions in $T$, namely an initializing transition $(s_{init}^A, (0, 0), s_{init}^B)$; an escape loop to detect that the final state has been reached $\left(s_f^B, (0, 0), s_{reached}^A\right)$ and $\left(s_{reached}^A, (0, 0), s_f^B\right)$; a loop to detect illegal moves $(s_*^B, (0, 0), s_*^A)$ and $(s_*^A, (0, 0), s_*^B)$.
- $S_F = \{s_{reached}^A\}$.

Let $G = (C, C_A, C_B, \longrightarrow, C_F, \preceq)$ be the monotonic game induced by $\mathcal{V}$. We show that there is a sequence

$(s_0, x_0, y_0) \longrightarrow (s_1, x_1, y_1) \longrightarrow (s_2, x_2, y_2) \longrightarrow \cdots \longrightarrow (s_n, x_n, y_n)$ of transitions in $M$ with $s_0 = s_{init}$, $x_0 = 0$, $y_0 = 0$, and $s_n = s_f$ iff the configuration $\left(s^A_{init}, 0, 0\right)$ is not winning in $G$.

## 8 Parity Games

A *parity game* $G$ *of degree* $n$ is a tuple $(C, C_A, C_B, \longrightarrow, r)$ where $C, C_A, C_B, \longrightarrow$ are defined as in games (Section 2), and $r$ is a mapping from $C$ to the set $\{0, \ldots, n\}$ of natural numbers. We use $C^k$ to denote $\{c|\ r(c) = k\}$. The sets $C^k_A$ and $C^k_B$ are defined in a similar manner. We call $r(c)$ the *rank* of $c$. Abusing notation, we define the *rank* $r(P)$ of a play $P = c_0, c_1, c_2, \ldots$ to be $\min\{r(c_0), r(c_1), r(c_2) \ldots\}$. We say that $P$ is *parity winning* if $r(P)$ is even. We say that $c$ is *parity winning* if there is an $A$-strategy $\sigma_A$ such that, for each $B$-strategy $\sigma_B$, it is the case that $P(c, \sigma_A, \sigma_B)$ is parity winning.

**The parity problem**
**Instance.** A parity game $G$ and a configuration $c$ in $G$.
**Question.** Is $c$ (parity) winning?

**Remark.** Notice that our definition of parity games considers parity of configurations which *appear* in the play, rather than the configurations which appear *infinitely often* (which is the standard definition). Our undecidability result can be extended for the latter case, too.

We show below that the parity problem is undecidable for $A$-downward closed games. In particular, we show undecidability of the problem for $A$-LCS games. The proof for $B$-downward closed games is similar.

**Theorem 7.** *The parity problem is undecidable for A-LCS games.*

In [AJ96a] we show undecidability of the *recurrent state problem*, for transition systems based on lossy channel systems.

**Recurrent State Problem**
**Instance.** A lossy channel systems $\mathcal{L}$ and a control states $s_{init}$.
**Question.** Is there a channel state $w$ such that there is an infinite computation starting from $(s_{init}, w)$?

We reduce the recurrent state problem for LCS to the parity problem for $A$-LCS. We construct a new $\mathcal{L}'$ to simulate $\mathcal{L}$. Intuitively, we let player $A$ choose the moves of the original system, while player $B$ follows passively. An additional loop at the beginning of $\mathcal{L}'$ allows us to guess the initial contents $w$ of the channels. If the system deadlocks, then player $B$ wins. So the only way for player $A$ to win is to make the system follow an infinite sequence of moves. More formally, $\mathcal{L}' = (S, S_A, S_B, L, M, T, S_F)$ is defined as follows. For each control state $s$ in $\mathcal{L}$, we create a control state $s^A \in S_A$. For each transition $t$ in $\mathcal{L}$, we create a control state $s^B_t \in S_B$. For each transition $t = (s_1, op, s_2)$ in $\mathcal{L}$ there are two transitions

$\left(s_1^A, op, s_t^B\right)$ and $\left(s_t^B, nop, s_2^A\right)$ in $\mathcal{L}'$. Furthermore, there are five additional states $s_1^*, s_4^* \in S_A$, $s_2^*, , s_3^*, , s_5^* \in S_B$, together with the following transitions:

- Two transitions $(s_1^*, \ell!m, s_2^*)$ and $(s_2^*, nop, s_1^*)$ for each $m \in M$ and $\ell \in L$. These two allow to build up the initial channel contents.
- Two transitions $(s_1^*, nop, s_3^*)$ and $\left(s_3^*, nop, s_{init}^A\right)$. This is to get to the initial state of $\mathcal{L}$ when the channel content is ready.
- A transition $\left(s^A, nop, s_5^*\right)$ for each control state $s$ in $\mathcal{L}$. This transition is only taken when $\mathcal{L}$ is deadlocked.
- Two transitions $(s_4^*, nop, s_5^*)$, and $(s_5^*, nop, s_4^*)$. This loop indicates a deadlock in $\mathcal{L}$.

The ranks of the configurations are defined as follows:

- $r\left((s_1^*, w)\right) = r\left((s_2^*, w)\right) = r\left((s_3^*, w)\right) = 3$, for each $w$.
- $r\left(\left(s^A, w\right)\right) = r\left(\left(s_t^B, w\right)\right) = 2$, for each $w$, each transition $t$ in $\mathcal{L}$, and each control state $s$ in $\mathcal{L}$.
- $r\left((s_4^*, w)\right) = r\left((s_5^*, w)\right) = 1$, for each $w$.

We show that $(s_1^*, \epsilon)$ is parity-winning if and only if there exists a $w$ and an infinite sequence starting from $(s_{init}, w)$.

**Remarks.**

- In case both players can lose messages, we can show that the parity problem is decidable. The reason is that the best strategy for each player is to empty the channels after the next move. The problem can therefore be reduced into an equivalent problem over finite-state graphs.
- Using results in [May00], we can strengthen Theorem 7, showing undecidability for $A$-VASS (and $B$-VASS) games. Such games are special cases of the ones reported here where the message alphabet is of size one (each channel behaves as a *lossy counter*).

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