ON ORDERED GROUPS.*

By B. H. Neumann.

Certain axiomatic questions in geometry lead to the study of ordered division rings (cf. Hilbert [8]), and these in turn to the study of ordered groups: I only mention (without proof) that every ordered group can be embedded in (the multiplicative group of) an ordered division ring.

F. W. Levi [9], [10] has given necessary conditions (not sufficient), and also sufficient conditions (not necessary), for a group to be capable of being ordered. In the first section of this paper the necessary conditions are generalised, a typical result being: If two elements of an ordered group are not permutable with each other, then none of their powers are. A similar result is also proved for certain higher commutators. Just how far one can proceed in this direction remains an open question: Some light is thrown on it by an example.

The sufficient conditions of Levi (*loc. cit.*) can also be generalised. In the second section we derive a very general sufficient criterion (which is also necessary, but only trivially so). An order is actually constructed in a group when the criterion applies; if the group possesses an ordered factor group, then its order can be utilised in the construction.

The general criterion is an unwieldy weapon. It can be specialised in various ways (**3**). From one of these specialisations one sees that all free groups can be ordered. (This result has also been obtained by G. Birkhoff and independently A. Tarski; cf. G. Birkhoff [3].) More generally we show that the order of any ordered group can be refined to an order of a free group of which the given group is a homomorphic image. Some special constructions of new ordered groups from given ordered groups are also given.

In the fourth and final section these constructive methods are used to construct an ordered group which coincides with its commutator group. Such an example throws some light on the limitations of the various criteria and other results; it may also be of interest in itself, and is given in some detail.

1. Necessary conditions for ordered groups. We call a group G an 0-group if it can be fully ordered, i.e., if a transitive binary relation $a < b$ can be defined in G, such that of the three alternatives $a < b$, $a = b$, $b < a$

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one and only one takes place, and $a < b$ implies $at < bt$ and $ta < tb$ for all a, b, t in G. If G is an 0-group, and an order relation has been chosen for G, we call G simply an ordered group.[1] The group consisting of the unit element only is an "improper" ordered group (with void order).

We write G multiplicatively, denote the unit element by 1, and the order relation by $<$, even when dealing with several groups simultaneously: different order relations will be distinguished by the context. We also use the commutator notation of P. Hall [7]; thus

$$[x, y] = x^{-1}y^{-1}xy,$$
$$[x, y, z] = [[x, y], z],$$

and so on, with the corresponding notation for subgroups.

We call a group "locally infinite" if every element $\neq 1$ in it is of infinite order.[2] The group which consists of the unit element only is "improperly" locally infinite.

Levi [9] shows that an 0-group is locally infinite, and more generally that the equation $x^m = a$ for $a \varepsilon G$, m a natural number, has at most one solution x. One can show more generally:

1.1 LEMMA. *If a, b are elements of an 0-group G, and $[a^m, b] = 1$ for any integer $m \neq 0$, then $[a, b] = 1$.*

Proof. Assume that $[a, b] \neq 1$, and that $[a, b] > 1$ in some order of G. Now

$$[a^m, b] \equiv \prod_{\mu=m-1}^{\mu=0} (a^{-\mu}[a, b]a^{\mu})$$

and

$$[a^{-m}, b] \equiv \prod_{\mu=-m}^{\mu=-1} (a^{-\mu}[a, b]^{-1}a^{\mu})$$

for all $m > 0$. Hence $[a^m, b]$ is a product of conjugates of $[a, b]$; each of these is > 1, and therefore $[a^m, b] > 1$. Also $[a^{-m}, b]$ is a product of conjugates of $[a, b]^{-1}$; each of these is < 1, and therefore $[a^{-m}, b] < 1$. Hence if $[a, b] \neq 1$, then $[a^m, b] \neq 1$, and the lemma follows.

1.2 COROLLARY. *If two elements of an 0-group are not permutable with each other, then none of their powers ($\neq 1$) are permutable with each other.*

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[1] This is, of course, a special case of the o-groups of Everett and Ulam [4] and l-groups of G. Birkhoff [2].

[2] A group has a property locally if all its subgroups of finite rank (generated by a finite number of elements) have the property.

We can further extend 1.1 by establishing the following necessary condition for 0-groups.

1.3 LEMMA. *If a, b are elements of an 0-group G, and $[a^m, b, a] = 1$ for any integer $m \neq 0$, then $[a, b, a] = 1$.*

Proof. Let m again be a *positive* integer. Then we expand

$$[a^m, b, a] \equiv [\prod_{\mu=m-1}^{\mu=0} (a^{-\mu}[a, b]a^{\mu}), a]$$

$$\equiv \prod_{\mu=m-1}^{\mu=0} (t_\mu^{-1}[a^{-\mu}[a, b]a^{\mu}, a]t_\mu),$$

where t_μ are certain part products of $\prod_{\mu=m-1}^{\mu=0} (a^{-\mu}[a, b]a^{\mu})$. Then

$$[a^m, b, a] \equiv \prod_{\mu=m-1}^{\mu=0} ((a^{\mu}t_\mu)^{-1}[a, b, a](a^{\mu}t_\mu));$$

hence again $[a^m, b, a]$ is a product of conjugates of $[a, b, a]$, and therefore is $\gtrless 1$ if $[a, b, a] \gtrless 1$. Similarly

$$[a^{-m}, b, a] \equiv [\prod_{\mu=-m}^{\mu=-1} (a^{-\mu}[a, b]^{-1}a^{\mu}), a]$$

$$\equiv \prod_{\mu=-m}^{\mu=-1} (t'_\mu{}^{-1}[a^{-\mu}[a, b]^{-1}a^{\mu}, a]t'_\mu)$$

$$\equiv \prod_{\mu=-m}^{\mu=-1} ((a^{\mu}t'_\mu)^{-1}[[a, b]^{-1}, a](a^{\mu}t'_\mu))$$

$$\equiv \prod_{\mu=-m}^{\mu=-1} (([a, b]^{-1}a^{\mu}t'_\mu)^{-1}[a, b, a]^{-1}([a, b]^{-1}a^{\mu}t'_\mu)).$$

Thus $[a^{-m}, b, a]$ is a product of conjugates of $[a, b, a]^{-1}$ and therefore is $\lessgtr 1$ if $[a, b, a] \gtrless 1$. The lemma follows.

1.4 COROLLARY. *If a, b are elements of an 0-group G and their commutator $[a, b]$ is not permutable with a, then no commutator $[a^m, b]$ of a power $(\neq 1)$ of a and b is permutable with any power $(\neq 1)$ of a.* Or: *if $[a, b, a] \neq 1$, then $[a^m, b, a^n] \neq 1$ for any $m \neq 0$, $n \neq 0$.*

This follows by applying Lemmas 1.1 and 1.3.

One will naturally look for further generalisations of these necessary conditions for 0-groups. Two directions suggest themselves: One, to decide whether $[a^m, b, a, a] = 1$ $(m \neq 0)$ entails $[a, b, a, a] = 1$; the other, to decide whether $[a^m, b, c] = 1$ $(m \neq 0)$ entails $[a, b, c] = 1$. The first of these questions I can not answer; the second is answered, negatively, by the following example.

1.5 *Example.* Let the group H be generated by elements

$$\cdots, b_{-1}, b_0, b_1, b_2, \cdots$$

$$\cdots, c_{-1}, c_0, c_1, c_2, \cdots$$

with the defining relations

1.51 $$[b_{\mu+1}, b_\mu] = c_\mu, \qquad \mu = 0, \pm 1, \pm 2, \cdots$$

1.52 $$[b_{\mu+\nu}, b_\mu] = 1, \qquad \mu = 0, \pm 1, \pm 2, \cdots \quad \nu = 2, 3, \cdots$$

1.53 $$[b_\mu, c_\nu] = 1, \qquad \mu, \nu = 0, \pm 1, \pm 2, \cdots$$

1.54 $$[c_\mu, c_\nu] = 1, \qquad \mu, \nu = 0, \pm 1, \pm 2, \cdots.$$

We define an order in H such that

1.55 $$1 < \cdots \ll c_{-1} \ll c_0 \ll c_1 \ll \cdots \ll b_{-1} \ll b_0 \ll b_1 \ll \cdots$$

where $x \ll y$ means that all powers of x lie between y^{-1} and y. Thus any element of H is > 1 if the highest-suffix b_μ in it appears with positive exponent, or if there is no b_μ in it, and the highest-suffix c_μ in it appears with positive exponent. One can satisfy oneself without difficulty that in this way H does become an ordered group. The mapping

$$b_\mu \to b_{\mu+1}, \qquad c_\mu \to c_{\mu+1}$$

clearly defines an automorphism of H *qua* group, and this automorphism leaves the order of H invariant. We now define G by adjoining this automorphism to H, i.e., we form $G = \{H, a\}$ with the relations

1.56 $$a^{-1}b_\mu a = b_{\mu+1}; \qquad a^{-1}c_\mu a = c_{\mu+1}; \qquad \mu = 0, \pm 1, \pm 2, \cdots$$

and we order G by making $a > 1$ and $a \gg h$ for all $h \in H$. It is again easy to see that G thus becomes an ordered group.

Now in G

$$[a, b_0, b_0] = [a^{-1}b_0^{-1}ab_0, b_0] = [b_1^{-1}b_0, b_0] = [b_1^{-1}, b_0] = c_0^{-1} < 1,$$

but when $m > 1$

$$[a^m, b_0, b_0] = [a^{-m}b_0^{-1}a^m b_0, b_0] = [b_m^{-1}b_0, b_0] = 1.$$

This example, therefore, proves

1.6 LEMMA. *In an 0-group,* $[a, b, b] \neq 1$ *is compatible with* $[a^m, b, b] = 1$ *for* $m > 1$.

and thus *a fortiori*

1.7 Corollary. *In an* 0-*group,* $[a, b, c] \neq 1$ *is compatible with* $[a^m, b, c] = 1$ *for* $m > 1$.

2. Sufficient conditions for ordered groups.

We use the following criterion for 0-groups, adapted from one given by Levi [9].

2.1 Lemma. *The group* G *is an* 0-*group if (and only if) it contains two subsets* s^+ *and* s^- *such that*

2.11 $$s^+ \cup s^- = G - \{1\},$$

i.e., every element of G*, except the unit element, lies in* s^+ *or in* s^-;

2.12 $$s^+ \cdot s^+ \subset s^+ \text{ and } s^- \cdot s^- \subset s^-,$$

i.e., s^+ *and* s^- *are semi-groups;*

2.13 $$t^{-1}s^+t \subset s^+ \text{ for all } t \in G,$$

i.e., s^+ *(and therefore also* s^-*) is self-conjugate in* G.

Proof. Let G possess two such subsets. Then, as they are semi-groups but do not contain the unit element, neither of them contains an element simultaneously with its inverse. But between them they contain all elements of G except 1. Hence of a pair of inverse elements one always belongs to s^+ and the other to s^-. Now we define an order relation in G by

2.14 $$a < b \text{ if and only if } a^{-1}b \in s^+.$$

Then if $a < b$, $b^{-1}a \in s^-$, and therefore $b \not< a$; also $b \neq a$, as the unit element is not in s^+. Hence of the three alternatives $a < b$, $a = b$, $b < a$, not more than one takes place. But one of them does take place, as of the two inverse elements $a^{-1}b$, $b^{-1}a$ one lies in s^+, unless $a^{-1}b = 1$. Also if $a < b$ and $b < c$, then $a^{-1}b \in s^+$, $b^{-1}c \in s^+$; hence $a^{-1}c \in s^+$, by 2.12, and $a < c$; which shows transitivity of the order relation. Finally if $a^{-1}b \in s^+$ then also $(at)^{-1}bt \in s^+$ because of 2.13, and $(ta)^{-1}tb \in s^+$ trivially; i.e. if $a < b$ then $at < bt$ and $ta < tb$. Hence G is ordered.

The converse is also true; for if G is an 0-group, we choose an order of G and then denote by s^+ the set of all elements > 1, by s^- the set of all elements < 1. Then 2.11-13 are easily checked.

We shall also use the following sufficient criterion, due to Levi [9] (*q.v.* for a proof):

2.2 Theorem. *A locally infinite abelian group is an* 0-*group.*

The most general sufficient criterion that we derive is the following.

2.3 THEOREM. *Let the group G possess a set of subgroups linearly ordered by inclusion:*

2.30 $$(G \supset) \cdots \supset H_\alpha \supset H_{\alpha'} \supset \cdots \supset \{1\}$$

(not necessarily all different) with the following properties:

2.31 *Each H_α is self-conjugate in G, i.e., 2.30 is a generalised normal series of G.*

2.32 *Each H_α except $\{1\}$ has an immediate successor $H_{\alpha'}$ in the series 2.30.*[3]

2.33 *For all terms of the series,*

$$[G, H_\alpha] \subset H_{\alpha'},$$

i.e., 2.30 is a generalised central series of any one of its terms.

2.34 *If 2.30 has a first term H_1, then G/H_1 is an 0-group.*

2.35 *$H_\alpha/H_{\alpha'}$ is (properly or improperly) locally infinite.*

2.36 *To every element $g \varepsilon H_1$ if 2.30 has a first term H_1, or to every element $g \varepsilon G$ if 2.30 has no first term, there is a minimal H_α containing it, i.e., $g = 1$ or there is an H_α with $g \varepsilon H_\alpha - H_{\alpha'}$. Then G is an 0-group.*

Proof. By 2.33 and 2.35 each $H_\alpha/H_{\alpha'}$ is abelian and locally infinite, hence, by 2.2, an 0-group. In each $H_\alpha/H_{\alpha'}$ we choose [4] an order. We denote by $s^+{}_\alpha H_{\alpha'}$ ($s^-{}_\alpha H_{\alpha'}$) the set of all elements of H_α which are greater (smaller) than the unit element (mod $H_{\alpha'}$) in this order. If 2.30 possesses a first term H_1, we also choose an order of G/H_1, and define $s^+{}_0 H_1$ ($s^-{}_0 H_1$) accordingly. Finally we introduce the union s^+ (s^-) of all $s^+{}_\alpha H_{\alpha'}$ ($s^-{}_\alpha H_{\alpha'}$)

2.41 $$s^+ = \bigcup_\alpha s^+{}_\alpha H_{\alpha'}, \qquad s^- = \bigcup_\alpha s^-{}_\alpha H_{\alpha'}.$$

Now let $g \neq 1$ be an element of G; then (by 2.36) either there is an H_α such that $g \varepsilon H_\alpha - H_{\alpha'}$, or 2.30 has a first term H_1 and $g \varepsilon G - H_1$. Hence g is greater or smaller than the unit element (mod $H_{\alpha'}$ or mod H_1) in H_α or G: hence $g \varepsilon s^+{}_\alpha H_{\alpha'} \cup s^-{}_\alpha H_{\alpha'}$ (or $s^+{}_0 H_1 \cup s^-{}_0 H_1$). In any case $g \varepsilon s^+ \cup s^-$, and

2.42 $$s^+ \cup s^- = G - \{1\}.$$

[3] Note that the series 2.30 need not be well-ordered; its order type is finite or made up, additively, from order types ω and $^*\omega + \omega$, with a finite tail. (For the notation *cf.* Fraenkel [5]).

[4] The axiom of choice is used. 2.2 requires well-order for its proof.

Let now g and h be two elements in s^+. Let $g \varepsilon H_\alpha - H_{\alpha'}$, $h \varepsilon H_\beta - H_{\beta'}$, and $H_\beta \subset H_\alpha$, say. Now if $H_\beta \neq H_\alpha$, i.e., $H_\beta \subset H_{\alpha'}$, then gh is congruent to g (mod $H_{\alpha'}$); then $gh \varepsilon s^+{}_\alpha H_{\alpha'}$ and $gh \varepsilon s^+$. If $H_\beta = H_\alpha$, then g and h are both in the same $s^+{}_\alpha H_{\alpha'}$, hence their product is. Hence again $gh \varepsilon s^+$. Correspondingly if g or h lies outside H_1. In any case

2.43 $$s^+ \cdot s^+ \subset s^+.$$

Similarly one proves

$$s^- \cdot s^- \subset s^-.$$

Finally let $g \varepsilon s^+$, let us say $g \varepsilon s^+{}_\alpha H_{\alpha'}$; and $t \varepsilon G$ arbitrary. Then [5]

$$t^{-1}gt = g \cdot [g, t] \;\varepsilon\; g \cdot [H_\alpha, G] \subset g \cdot H_{\alpha'}.$$

Thus

$$t^{-1}gt \cdot H_{\alpha'} = g \cdot H_{\alpha'},$$

and so

$$t^{-1}gt \;\varepsilon\; s^+{}_\alpha H_{\alpha'}.$$

If, on the other hand, 2.30 has a first term and $g \varepsilon s^+{}_0 H_1$ then also

$$t^{-1}gt \;\varepsilon\; s^+{}_0 H_1;$$

for $s^+{}_0$ is self-conjugate in G/H_1, and H_1 is self-conjugate in G; hence $s^+{}_0 H_1$ is self-conjugate in G. Thus we see that

2.44 $$t^{-1}s^+t \subset s^+ \qquad \text{for all } t \varepsilon G.$$

Combining 2.42-2.44 and 2.1, we see that G is an 0-group and the proof of the theorem is complete.

If G is an ordered group, H a self-conjugate subgroup of G, and $K = G/H$ is ordered so that $aH \leqq bH$ in the order of K whenever $a < b$ in the order of G, then we call the order of G a "refinement" of the order of K. Then the proof of Theorems 2.3 also shows:

2.5 COROLLARY. *If under the conditions of the criterion 2.3 the series 2.30 has a first term H_1, then any order of G/H_1 can be refined to an order of G.*

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[5] This is the only step in the proof which fully uses 2.33. One easily sees that 2.33 and 2.35 can be replaced by the following condition (which is not, however, equivalent to 2.33, 2.35)

2.33′ Each $H_\alpha/H_{\alpha'}$ is an 0-group, and in particular possesses an order which admits all inner automorphisms of G.

Then for the purposes of the proof such an order has to be chosen in each $H_\alpha/H_{\alpha'}$. This modification of the theorem generalizes another criterion due to Levi [9].

3. Special methods for constructing ordered groups. Theorem 2. 3 is somewhat unwieldy. Some special cases may, however, be of interest.

Let G be a group and denote by ${}^nG$ the terms of its lower central series:

$$ {}^0G = G, \qquad {}^{n+1}G = [G, {}^nG]. $$

Further denote by $Z_n(G)$ the terms of its upper central series:

$$ Z_0(G) = \{1\}, \qquad Z_{n+1}(G)/Z_n(G) \text{ the centre of } G/Z_n(G). $$

3. 1 THEOREM. *If G is such that*[6]

3. 11 $$ \bigcap_n {}^nG = \{1\}; $$

3. 12 ${}^nG/{}^{n+1}G$ *is locally infinite for* $n = 0, 1, 2, \cdots$; *then G is an* 0-*group.*

3. 2 THEOREM. *If G is such that*[7]

3. 21 $$ \bigcup_n Z_n(G) = G; $$

3. 22 $Z_{n+1}(G)/Z_n(G)$ *is locally infinite for* $n = 0, 1, 2, \cdots$; *then G is an* 0-*group.*

Both these theorems are easy consequences of 2. 3. The set of groups H_α, consisting of the terms of the lower or upper central series, is here finite or of order type ω (3. 1) or $^*\omega$ (3. 2), so that 2. 32 is satisfied. The definition of the central series assures 2. 31, 2. 33. Assumptions 3. 11 and 3. 21 entail 2. 36, 3. 12 and 3. 22 are simply 2. 35. 2. 34 is also satisfied, as a consequence of 2. 2. Hence 2. 3 applies.

3. 3 COROLLARY. (*Cf.* Birkhoff [3]) *All free groups (of finite or infinite rank) are* 0-*groups.*

3. 4 THEOREM. *If the ordered group G is represented as a factor group of a free group F with respect to a relation group R,*

3. 41 $$ G \cong F/R, $$

then the order of G can be refined to an order of F.

Proof. We use the intersection of R with the terms ${}^nF$ of the lower central series of F, putting

3. 42 $$ R_0 = (R \cap {}^0F =)\, R, \qquad R_n = R \cap {}^nF, \qquad n = 0, 1, 2, \cdots. $$

Now as R is self-conjugate in F, R_n is also self-conjugate in F, and

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[6] 3. 11 means that G is an N-group in the terminology of Baer [1].

[7] 3. 21 means that G is a Z-group in the terminology of Baer [1].

3.43 $$[F, R_n] \subset R_n \subset R.$$

Also

$$[F, R_n] \subset [F, {}^nF] = {}^{n+1}F.$$

Hence

3.44 $$[F, R_n] \subset R_{n+1}.$$

Let $a \varepsilon F$ have a (proper) power in R_{n+1}

$$a^k \varepsilon R_{n+1}, \qquad k \neq 0.$$

Then $a^k \varepsilon {}^{n+1}F$; but as $F/{}^{n+1}F$ is locally infinite, $a \varepsilon {}^{n+1}F$. Also $a^k \varepsilon R$; but as F/R is an 0-group and therefore locally infinite, also $a \varepsilon R$. Hence

$$a \varepsilon {}^{n+1}F \cap R = R_{n+1}.$$

This means that F/R_{n+1} is locally infinite, and therefore *a fortiori* R_n/R_{n+1} is locally infinite.

Finally

3.45 $$\bigcap_n R_n \subset \bigcap_n {}^nF = \{1\}.$$

Hence every element $r \varepsilon R$, $r \neq 1$ is in a smallest R_n,

$$r \varepsilon R_n - R_{n+1}.$$

The theorem follows now simply from Theorem 2.3 and Corollary 2.5.

We now give some fairly obvious results which can be used for constructing new ordered groups from given ordered groups. Detailed proofs are omitted.

3.5 The (complete [8]) direct product of any set of 0-groups is an 0-group.

For we can well-order the direct factors, and chose an order in each. The direct product is then simply ordered by the convention that an element is $\gtrless 1$ according as the first component (in the well-order of the direct factors) $\neq 1$ [9] of the element is $\gtrless 1$ in the chosen order of the factor.

The following construction dispenses with well-order.

3.6 Given an ordered set of 0-groups, their *restricted* direct product [10] can be so ordered that the direct factors appear in the given set order.

For we can choose an order in each direct factor. The restricted direct

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[8] I. e. without restriction upon the number (or cardinal) of components $\neq 1$ of an element.

[9] 1 stands for the unit element of all the groups that occur; similarly $\lessgtr$ applies to the order chosen in the factors as well as to that under construction in the product.

[10] I. e., that in which every element has only a finite number of components $\neq 1$.

product is then simply ordered by the convention that an element is $\gtrless 1$ according as the last component $\neq 1$ (in the order of the set of factors) of the element is $\gtrless 1$ in the chosen order of the factor.

3.5 and 3.6 are in fact only special cases of known results. *Cf.* Hahn [6].

As an application of 3.6 we give, in some detail, the following construction, which will be used in the next section.

3.7 Starting from an ordered group B we form the restricted direct product of an ordered set of type $^*\omega + \omega$ of factors B_n, $n = 0, \pm 1, \pm 2 \cdots$ each isomorphic to B:

$$B^* = \cdots \times B_{-1} \times B_0 \times B_1 \times B_2 \times \cdots.$$

The elements are of the form

3.71 $$b^* = \cdots \times b_{-1,-1} \times b_{0,0} \times b_{1,1} \times b_{2,2} \times \cdots,$$

(where the first suffix distinguishes the direct factor in which the component lies, the second suffix the element of B which appears as this component); only a finite number of the components $b_{n,n}$ are different from the unit element, and the last one of these determines whether $b^* \gtrless 1$. Now B^* possesses an obvious automorphism (relating to its order as well as to the group operation), viz. that mapping each component B_n on its successor B_{n+1}. We denote this automorphism by a and extend B^* by means of it; i.e. we form all the products

3.72 $$g = a^\alpha b^*$$

with the transformation rule

3.73 $$a^{-1}b^*a = \cdots \times b_{-1,-2} \times b_{0,-1} \times b_{1,0} \times b_{2,1} \times \cdots$$

(where b^* is given by 3.71). The elements 3.72 form a group G, which we order first according to the power of a in it, then according to b^*. Thus

3.74 $$g \gtrless 1 \text{ if } \alpha \gtrless 0, \text{ or if } \alpha = 0 \text{ and } b^* \gtrless 1.$$

If B is given by a system of generators $b, b', \cdots$ and defining relations $r(b, b', \cdots) = r'(b, b', \cdots) = \cdots = 1$, then we can give G in the same manner

3.75 $$G = \{a, b, b', \cdots;\ r(b, b', \cdots) = r'(b, b', \cdots) = \cdots = 1,$$
$$[a^{-\lambda}ba^{\lambda}, a^{-\mu}ba^{\mu}] = [a^{-\lambda}ba^{\lambda}, a^{-\mu}b'a^{\mu}] = \cdots = 1,$$
$$(\lambda \neq \mu)\}.$$

The commutator relations are formed for all pairs of generators $b, b', \cdots$

of B, but may then be restricted to $\lambda = 0$, $\mu > 0$. If G is defined by 3.75, the order in G can be described without reference to B^*, solely in terms of a, B. To this end we represent an element of G in the form

3.76
$$g = a^{\alpha} \prod_{i=1}^{i=m} a^{-\mu_i} b_i a^{\mu_i}$$

where $\mu_1 > \mu_2 > \cdots > \mu_m$, all $b_i \neq 1$.[11] This is possible because transforms of elements of B by different powers of a are permutable with each other; α is the sum of exponents with which a appears in g.[12] Now the order is defined in G by

3.77
$$g \gtrless 1 \text{ if } \alpha \gtrless 0 \text{ or } \alpha = 0 \text{ and } b_1 \gtrless 1.$$

This construction can be extended by replacing the powers of a by the elements of an arbitrary ordered group A.

3.8 Let the ordered groups A and B be given by

3.81 $A = \{a, a', \cdots; \; q(a, a', \cdots) = q'(a, a', \cdots) = \cdots = 1\}$,

3.82 $B = \{b, b', \cdots; \; r(b, b', \cdots) = r'(b, b', \cdots) = \cdots = 1\}$.

Then we form the group

3.83
$$G = \{a, a', \cdots, b, b', \cdots; \quad 3.84 \text{—} .86\}$$

where the relations are

3.84
$$q(a, a', \cdots) = q'(a, a', \cdots) = \cdots = 1,$$

3.85
$$r(b, b', \cdots) = r'(b, b', \cdots) = \cdots = 1,$$

3.86
$$[b, a_1^{-1} b a_1] = [b, a_1^{-1} b' a_1] = \cdots = [b, a_2^{-1} b a_2] = \cdots = 1.$$

In the commutator relations 3.86 $a_1, a_2, \cdots$ range over all elements > 1 of A, and the elements of B involved range over all pairs of generators $b, b', \cdots$. An element $g \in G$ can be represented in the form

3.87
$$g = a_0 \prod_{\mu=1}^{\mu=m} (a_\mu^{-1} b_\mu a_\mu)$$

with $a_1 > a_2 > \cdots > a_m$ in A, and with all $b_\mu \neq 1$. This is possible because transforms of elements of B by different elements of A are permutable with each other. Then G is ordered by the convention

[11] If $m = 0$, the product is void.

[12] Easily seen to be an invariant as long as a and elements of B are chosen as the generators of G.

3.88 $$g \gtrless 1 \text{ if } a_0 \gtrless 1 \text{ or } a_0 = 1 \text{ and } b_1 \gtrless 1.$$

The proof that in this way G becomes an ordered group is omitted.

Finally we mention, without proof, a construction principle due to Steinitz [11].

3.9 Let a system Σ of ordered groups G_α be given with the property that to any two groups G_α, G_β in Σ there is a group G_γ in Σ which contains both G_α and G_β as subgroups [13] and continues the order of both. Then there is an ordered group G containing as subgroups all the groups in Σ, each with its order, and generated by them.

4. A perfect ordered group. Levi [10] shows that if an 0-group G is finitely generated then it is different from its commutator group G'. To show that the finiteness of the number of generators can not be dispensed with; to illustrate the limitations to our various sufficient criteria for 0-groups (2.3, 3.1, 3.2); and to demonstrate the application of our various constructive principles: we now construct an ordered group which is perfect,[14] i.e. coincides with its commutator group.

Starting from an infinite cycle

4.01 $$H_1 = \{b_1\}$$

we first define a series H_n of groups by repeated application of 3.7.

4.02 $$H_n = \{b_1, b_2, \cdots, b_n;\ [b_r, b_q^{-\lambda} b_s b_q^{\lambda}] = 1\ \ (q > r, s;\ \lambda \neq 0\ ^{15})\}.$$

Clearly H_n is obtained from H_{n-1} by adding the generator b_n and relations which entail that two elements of H_{n-1} transformed by different powers of b_n are permutable. The method of 3.7 can also be used to order H_n, when it can be seen that the order of H_n continues that of H_{n-1}; but we do not require the order at this stage.

We now consider the elements

4.03 $$c_\nu = [b_\nu, b_n] = b_\nu^{-1} \cdot b_n^{-1} b_\nu b_n, \qquad \nu = 1, 2, \cdots, n-1,$$

and denote by K_{n-1} the subgroup of H_n generated by these elements. We proceed to show that

4.04 $$K_{n-1} \cong H_{n-1};$$

more particularly, the mapping

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[13] A group G_γ may contain different subgroups isomorphic and similarly ordered to G_α; but G_α must be one of the subgroups of G_γ.

[14] We use "perfect" in its group-theoretical sense.

[15] Here as later λ may be restricted to positive values.

$$c_\nu \leftrightarrow b_\nu, \qquad \nu = 1, 2, \cdots, n-1,$$

defines an isomorphism between K_{n-1} and H_{n-1}. To see this we form any word $w(b_1, b_2, \cdots, b_{n-1})$ in the generators of H_{n-1}. Then

4.05 $\quad w(c_1, c_2, \cdots, c_{n-1}) = w(b_1^{-1}, b_2^{-1}, \cdots, b_{n-1}^{-1}) \cdot b_n^{-1} w(b_1, b_2, \cdots, b_{n-1}) b_n;$

for the expressions in c_ν^{-1} permute with those in $b_n^{-1} b_\nu b_n$. The two factors on the right-hand side of 4.05 lie in different components (viz. H_{n-1} and $b_n^{-1} H_{n-1} b_n$) of a direct product. Hence the left-hand side of 4.05 can equal the unit element only if both factors on the right-hand side do. Thus

$$w(c_1, c_2, \cdots, c_{n-1}) = 1$$

entails

$$w(b_1, b_2, \cdots, b_{n-1}) = 1,$$

and H_{n-1} is a homomorphic image of K_{n-1} under the mapping $c_\nu \to b_\nu$.

Conversely let $w(b_1, b_2, \cdots, b_{n-1}) = 1$. Then w is a product of conjugates (in H_{n-1}) of the left-hand sides of the defining relations for $b_1, b_2, \cdots, b_{n-1}$. Now these defining relations express the permutability of transforms by different powers of b_q, of any two elements expressible in terms of $b_1, b_2, \cdots, b_{q-1}$. From these relations then follows also the permutability of transforms by different powers of b_q^{-1}, of any two elements expressible in terms of $b_1^{-1}, b_2^{-1}, \cdots, b_{q-1}^{-1}$. Thus if

$$w(b_1, b_2, \cdots, b_{n-1}) = 1$$

is a relation connecting the generators, then

$$w(b_1^{-1}, b_2^{-1}, \cdots, b_{n-1}^{-1}) = 1$$

is a relation connecting their inverses. Hence in this case the whole right-hand side of 4.05 equals the unit element, and

$$w(c_1, c_2, \cdots, c_{n-1}) = 1$$

follows from

$$w(b_1, b_2, \cdots, b_{n-1}) = 1.$$

The mapping $c_\nu \leftrightarrow b_\nu$ generates, therefore, an isomorphism between K_{n-1} and H_{n-1}.[16]

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[16] Note that K_{n-1} does not, in general, contain all the elements $g_{n-1}^{-1} b_n^{-1} g_{n-1} b_n$. Thus it contains

$$c_1 c_2 = b_1^{-1} b_2^{-1} \cdot b_n^{-1} b_1 b_2 b_n,$$

but not

$$(b_1 b_2)^{-1} \cdot b_n^{-1} (b_1 b_2) b_n.$$

The intrinsic reason for the isomorphism of K_{n-1} and H_{n-1} is interesting, but beyond the scope of this paper.

Now similarly H_{n-1} contains in its commutator group a subgroup isomorphic to H_{n-2}; hence K_{n-1} contains in its commutator group a subgroup[17] L_{n-2} isomorphic to H_{n-2}; and so it goes on. The idea of the construction is now to consider a sequence of groups $\cdots, L, K,$ rather than that of the groups H; in this way we ensure that each term of the sequence lies in the commutator group of its successor.[18] To do this we define groups G_n each isomorphic to H_n; but such that an isomorphism from G_n to H_n maps G_{n-1} on K_{n-1}, not on H_{n-1}. We define

4.06 $\quad G_n = \{a_{11}, a_{21}, a_{22}, a_{31}, a_{32}, a_{33}, \cdots, a_{n1}, \cdots, a_{nn};\ 4.07,\ .08\}$
with the relations

4.07 $\quad [a_{pr}, a_{pq}^{-\lambda} a_{ps} a_{pq}^{\lambda}] = 1$ for $n \geqq p \geqq q > r, s;\ \lambda \neq 0;$

4.08 $\quad [a_{pq}, a_{pp}] = a_{p-1,q}$ for $n \geqq p > q.$

It is seen from 4.08 that G_n can be generated by $a_{n1}, a_{n2}, \cdots, a_{nn}$; and those relations 4.07 for which $p = n$ are the same as the defining relations of H_n in 4.02. Therefore the mapping

$$b_1 \to a_{n1}, b_2 \to a_{n2}, \cdots, b_n \to a_{nn}$$

generates a homomorphism of H_n onto G_n.

To show that this homomorphism is an isomorphism we prove that all the relations 4.07 follow already from those for which $p = n$, together with 4.08. The set of relations 4.07 for which p has a certain fixed value, $p = m$, say, will be denoted by 4.07_m for short; similarly we denote by 4.08_m those relations 4.08 for which $p = m$. We show that 4.07_{m-1} follow from 4.07_m and 4.08_m; then 4.07_p for $p = 1, 2, \cdots, n-1$ follow from 4.07_n together with 4.08.

Consider any word w formed of $m-1$ generators, $w(x_1, x_2, \cdots, x_{m-1})$. Then, using 4.08,

$$\begin{aligned} w(a_{m-1,1}, a_{m-1,2}, \cdots, a_{m-1,m-1}) &= w([a_{m1}, a_{mm}], [a_{m2}, a_{mm}], \cdots [a_{mm-1}, a_{mm}]) \\ &= w(a_{m1}^{-1} \cdot a_{mm}^{-1} a_{m1} a_{mm}, a_{m2}^{-1} \cdot a_{mm}^{-1} a_{m2} a_{mm}, \cdots, a_{mm-1}^{-1} \cdot a_{mm}^{-1} a_{mm-1} a_{mm}). \end{aligned}$$

By 4.07_m each $a_{mm}^{-1} a_{ms} a_{mm}$ permutes with each a_{mr}, for $r, s = 1, 2, \cdots, m-1$. Hence

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[17] L_{n-2} is not a subgroup of H_{n-1}; hence its relation to H_{n-1} is not the same as that of K_{n-1} to H_n.

[18] If two sequences of groups are given

$$A_1 \subset A_2 \subset \cdots \text{ and } B_1 \subset B_2 \cdots$$

such that $A_1 \cong B_1$, $A_2 \cong B_2, \cdots$, and if A and B are the groups generated by these sequences by the Steinitz method (*cf.* 3.9), then A and B need not be isomorphic.

4.09 $$w(a_{m-1,1}, \cdots, a_{m-1,m-1}) = w(a_{m1}^{-1}, \cdots, a_{mm-1}^{-1}) \cdot a_{mm}^{-1} w(a_{m1}, \cdots, a_{mm-1}) a_{mm}.$$

We apply this in particular to the left-hand side of 4.07$_{m-1}$, and obtain for $m-1 \geq q > r, s$; $\lambda \neq 0$,

4.10 $$[a_{m-1,r}, a_{m-1,q}^{-\lambda} a_{m-1,s} a_{m-1,q}^{\lambda}]$$
$$= [a_{mr}^{-1}, a_{mq}^{\lambda} a_{ms}^{-1} a_{mq}^{-\lambda}] \cdot a_{mm}^{-1} [a_{mr}, a_{mq}^{-\lambda} a_{ms} a_{mq}^{\lambda}] a_{mm}.$$

Here both commutators on the right-hand side equal the unit element by 4.07$_m$, and 4.07$_{m-1}$ follows.

Hence all the relations in G_n follow from 4.07$_n$ and 4.08. The latter are only explicit definitions of the generators a_{pq}, $p < n$, one for each, and no relations between $a_{n1}, a_{n2}, \cdots, a_{nn}$ can follow from them. If we generate G_n by means of $a_{n1}, a_{n2}, \cdots, a_{nn}$ only, then 4.07$_n$ form a *complete* system of defining relations. Therefore the mapping generated by

4.11 $$a_{n1} \leftrightarrow b_1, a_{n2} \leftrightarrow b_2, \cdots, a_{nn} \leftrightarrow b_n$$

is an isomorphism between G_n and H_n. It is seen without difficulty that this isomorphism maps G_{n-1} on K_{n-1}; but the groups H_n and K_{n-1} are now no longer needed.

To define order in G_n we proceed as in 3.7; but in order to show that this order continues the order correspondingly defined for a subgroup $G_m \subset G_n$ $(m < n)$, we define the order simultaneously in the subgroup. To order G_m we form the chain of subgroups

4.12 $$G_{m1} = \{a_{m1}\}, G_{m2} = \{a_{m1}, a_{m2}\}, \cdots, G_{mm} = \{a_{m1}, a_{m2}, \cdots, a_{mm}\}.$$

Then

4.13 $$G_{m1} \subset G_{m2} \subset \cdots \subset G_{mm} = G_m.$$

We proceed by induction. G_{m1} can be trivially ordered: $a_{m1}^{\lambda} \gtrless 1$ according as $\lambda \gtrless 0$. We assume that $G_{m,q-1}$ has been ordered already. Now let $g \neq 1$ be an element of $G_{m,q}$. Then g can be expressed in the form (cf. 3.7)

4.14 $$g = a_{mq}^{\lambda} \prod_i a_{mq}^{-\mu_i} g_i a_{mq}^{\mu_i},$$

where $\mu_1 > \mu_2 > \cdots$ and all $g_i \in G_{m,q-1}$.[19] Then we define

4.15 $$g \gtrless 1 \text{ if } \lambda \gtrless 0 \text{ or } \lambda = 0 \text{ and } g_1 \gtrless 1 \text{ (in } G_{m,q-1}).$$

It is easy to confirm the usual properties of this order relation, and we omit the proof.

[19] The product Π_i may consist of a single factor, or be absent.

To compare the order relations in G_m and G_{m-1} let $g \varepsilon G_{m-1}$ be expressed as a word

4.16 $$g = w(a_{m-1,1}, a_{m-1,2}, \cdots, a_{m-1,m-1}).$$

Then in G_m it can be expressed in the form 4.09, or preferably in the form

4.17 $$g = a_{mm}^{-1} w(a_{m1}, \cdots, a_{mm-1}) a_{mm} \cdot w(a_{m1}^{-1}, \cdots, a_{mm-1}^{-1}).$$

This is of the form 4.14 with $q = m$, $\lambda = 0$, two factors in the product, $\mu_1 = 1$, $\mu_2 = 0$. Hence $g \gtrless 1$ according as $w(a_{m1}, \cdots, a_{mm-1}) \gtrless 1$. But it is clear that

$$w(a_{m1}, \cdots, a_{mm-1}) \gtrless 1$$

according as

$$w(a_{m-1,1}, \cdots, a_{m-1,m-1}) \gtrless 1;$$

for the first suffix m of the generators does not enter the definition 4.15 at all. Hence $g \gtrless 1$ *qua* element of G_m according as $g \gtrless 1$ *qua* element of G_{m-1}. By induction one then sees that the order of G_n coincides in G_m $(m < n)$ with the order of G_m.

We now have all the material together to construct the example, by applying Steinitz' method 3.9 to the series

$$G_1, G_2, \cdots, G_n, \cdots.$$

4.2 *Example.* Let G_ω be the group generated by

$$a_{11}, a_{21}, a_{22}, a_{31}, a_{32}, a_{33}, \cdots, a_{n1}, \cdots, a_{nn}, \cdots$$

with the defining relations

4.21 $$[a_{pr}, a_{pq}^{-\lambda} a_{ps} a_{pq}^{\lambda}] = 1 \text{ for } p \geqq q > r, s;\ \lambda \neq 0;$$

4.22 $$[a_{pq}, a_{pp}] = a_{p-1,q} \text{ for } p > q.$$

Relations 4.22 ensure that $G'_\omega = G_\omega$. Let G_ω be ordered by the definition 4.15 when the element $g \varepsilon G_\omega$ is expressed in the form 4.14. Then G_ω is a perfect ordered group.

As G_ω coincides with its commutator group, its lower central series is stillborn. So is its upper central series, for the center of G_ω is easily seen to be $\{1\}$. One can show even more:

4.3 Lemma. *Every element > 1 in G_ω has arbitrarily large conjugates: if $1 < g < h$ in G_ω, then there is an element $t \varepsilon G_\omega$ such that $t^{-1} g t > h$.*

Proof. Let m be such that g and h both lie in G_{m-1}. We express them in the next higher group G_m, using the representation 4. 17, but abbreviating it to

4. 31 $$g = a_{mm}{}^{-1} g_1 a_{mm} \cdot \bar{g}_1,$$

4. 32 $$h = a_{mm}{}^{-1} h_1 a_{mm} \cdot \bar{h}_1,$$

where g_1, $\bar{g}_1$, h_1, $\bar{h}_1$ are words in the generators $a_{m1}, a_{m2}, \cdots, a_{mm-1}$, and we also know that $g_1 > 1$. We put $t = a_{mm}$. Then

4. 33 $$t^{-1} g t \cdot h^{-1} = a_{mm}{}^{-2} g_1 a_{mm}{}^{2} \cdot a_{mm}{}^{-1} \bar{g}_1 h_1{}^{-1} a_{mm} \cdot \bar{h}_1{}^{-1} > 1,$$

because $g_1 > 1$; and the result follows.

From this we see immediately:

4. 4 LEMMA. *If the self-conjugate subgroup $H \subset G_\omega$ contains with any element h also all the elements between h and its inverse,*[20] *then $H = \{1\}$ or $H = G_\omega$.*

This lemma allows us to show that G_ω (in the given order) is what one would call "ordinally simple":

4. 5 THEOREM. *If G_ω is mapped homomorphically on an ordered group G^* such that $g < h$ in G_ω implies $g^* \leqq h^*$ for the homomorphic images of g and h in G^*, then either the homomorphism is trivial, i. e., $G^* = \{1\}$, or the homomorphism is an isomorphism.*

Proof. Let the kernel of the homomorphism be the self-conjugate subgroup H of G_ω. If $H \neq \{1\}$, $1 < h \,\varepsilon\, H$, and $1 < g < h$, $g \,\varepsilon\, G_\omega$ arbitrary, then $1 \leq g^* \leq h^* = 1$; hence $g \,\varepsilon\, H$. Then $H = G_\omega$ by 4. 4, and the homomorphism is trivial. On the other hand, if $H = \{1\}$, then the homomorphism is an isomorphism; which proves the theorem.

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