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The Undecidability of Iterated Modal Relativization

Abstract. In dynamic epistemic logic and other fields, it is natural to consider relativization as an operator taking sentences to sentences. When using the ideas and methods of dynamic logic, one would like to iterate operators. This leads to *iterated relativization*. We are also concerned with the transitive closure operation, due to its connection to common knowledge. We show that for three fragments of the logic of iterated relativization and transitive closure, the satisfiability problems are Σ_1^1-complete. Two of these fragments do not include transitive closure. We also show that the question of whether a sentence in these fragments has a finite (tree) model is Σ_1^0-complete. These results go via reduction to problems concerning domino systems.

Keywords: Dynamic epistemic logic, iterated relativization, modal logic, undecidability

1. Introduction

This paper is concerned with the operation of *iterated relativization* as it appears in dynamic epistemic logic. This operation is motivated by the notion of an *epistemic program*: an algorithm whose steps use and change the epistemic states of agents. In many settings, we wish to consider programs defined by some sort of iteration operation. So analogous to the use of dynamic logic in other settings, we might wish to study the logical properties of the iteration operation in the setting of epistemic programs. This paper is concerned with just this issue.

In order to motivate the topic with an example, we begin with a discussion of iterated announcement in the well-known muddy children scenario. This discussion should be accessible to readers who have seen the Kripke semantics of modal logic. The actual work of the paper begins in Section 2. In that section we re-introduce the topic in more generality; readers who already know the motivation for the logical study of iterated relativization might wish to skip or skim the rest of this section.

We begin with a finite set $\mathcal{C}$ of children playing in the mud. As a result, some of them get mud on their foreheads. We assume that each child can see the others and hence knows the clean/dirty status of the others. We

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also assume that none of the children can see their own foreheads and that all children are perfect logical reasoners. Finally, all of these assumptions are taken to be common knowledge. An adult comes by and announces that at least one child is muddy. The adult then asks the children if they know whether they are muddy or not. The children simultaneously publicly announce whether or not they know. They repeat this, each time taking into account the previous announcement. (So in the terminology above, we have a simple example of an epistemic program.) It can be shown that if we begin with $n \geq 1$ dirty children, then there will be $n-1$ announcements of everyone's ignorance. Following this, there will be an announcement where all of the dirty children say that they know and the clean ones do not. We then have an announcement where the clean ones know too.

There are many discussions of this scenario in the literature. The topics of these discussions include the nature of knowledge and common knowledge, the assumptions needed to draw the conclusion we mentioned just above, and the appropriate mathematical models for this kind of work. This paper is not primarily about these topics. We are concerned with one particular aspect of the whole scenario: the *repetition* of the public announcement by the children of their own knowledge states. In order to bring out this issue, we must discuss the models and logical languages of interest.

For each set $\mathcal{C}$ of children, we have a language $\mathcal{L}_{\mathcal{C}}$. This language is the multi-agent modal logic with agent knowledge modalities for each child and with atomic sentences for the clean/dirty status of each. $\mathcal{L}_{\mathcal{C}}$ is interpreted on $\mathcal{L}_{\mathcal{C}}$*-models*

$$W \quad = \quad (W, (\xrightarrow{A})_{A\in\mathcal{C}}, (D_A)_{A\in\mathcal{C}}).$$

W is a set (of *worlds*), each $\xrightarrow{A}$ is a binary relation on W, and each D_A is a subset of W.

Here are the details on the syntax and semantics of $\mathcal{L}_{\mathcal{C}}$. We begin with atomic sentences d_A for each child, and the intended interpretation is that in a world w of a model W, $w \models \mathsf{d}_A$ iff A is dirty in w. Formally, $(W, w) \models \mathsf{d}_A$ iff $w \in D_A$. (Note that the semantics employs model-world pairs (W, w).) We have the usual boolean operations. So to say that child B is clean we would write $\neg\mathsf{d}_B$. Further, we have knowledge operations $\Box_A$ for $A \in \mathcal{C}$. Informally, $(W, w) \models \Box_A\varphi$ means that in w, A knows φ. Formally, the semantics is that $(W, w) \models \Box_A\varphi$ iff for all v such that $w \xrightarrow{A} v$, $(W, v) \models \varphi$. There is also a *common knowledge operator* $\Box^*$. Informally, $(W, w) \models \Box^*\varphi$ means that in w, φ is common knowledge. This is best modeled by the circular assertion:

$$\varphi \text{ is true, and every child knows (i).} \qquad \text{(i)}$$

Formally,

$$(W,w) \models \Box^* \varphi \quad \text{iff} \quad \text{for all } v \text{ such that } w \xrightarrow{*} v,\ (W,v) \models \varphi,$$

where $\xrightarrow{*}$ is the reflexive-transitive closure of $\bigcup_A (\xrightarrow{A})$.

To represent an announcement, we add an operation $\langle\varphi\rangle$ taking a sentence ψ to another sentence $\langle\varphi\rangle\psi$. The informal semantics is that φ is true, and after announcing it publicly to all agents, ψ is true. We model this by relativization; i.e., passing to a submodel. Formally, $(W,w) \models \langle\varphi\rangle\psi$ iff both $(W,w) \models \varphi$ and $(W^\varphi, w) \models \psi$, where W^φ is the submodel of W determined by $\{v \in W : (W,v) \models \varphi\}$.

This reification of announcements or other epistemic operations inside of modal languages is now an active field of modal logic. The papers [1, 2, 10, 11, 12, 19, 20] are the closest to our study, and these mention other works as well.

Finally, we have an iterated announcement operator $\langle\varphi^*\rangle$ defined by

$$(A,a) \models \langle\varphi^*\rangle\psi \quad \text{iff} \quad (A,a) \models \langle\varphi\rangle^n\psi \text{ for some } n.$$

This iteration operation is needed to formulate and study epistemic programs, just as some sort of iteration or recursion operation is needed in other computational settings. The central topic of this paper is the complexity of the satisfiability problem in languages which contain the iterated relativization operator.

At this point we have described the syntax and semantics of the languages $\mathcal{L}_\mathcal{C}$. In the rest of this section, we shall be concerned with the particular sentences in $\mathcal{L}_\mathcal{C}$ which are listed in Figure 1. These are based on sentences in Gerbrandy and Groeneveld [12]. Informally, the sentence vision says of a world w that every child A can see and therefore knows the status of all other children. Note that vision is finite since $\mathcal{C}$ is, and that it depends on the set $\mathcal{C}$ of children. The sentence $\mathsf{at\ least\ one}$ is straightforward and holds in a world w iff at least one child is dirty. The background statement $\mathsf{background}$ says of w that it is common knowledge that vision and $\mathsf{at\ least\ one}$ hold. Semantically, these two statements hold at all worlds reachable in zero or more steps from w. The intuition is that this is the background that the children have after the adult's announcement that at least one of them is dirty.

We must note that $\mathsf{background}$ is much weaker than what one would usually take to be the formalization of the overall background assumptions in the muddy children scenario. For example, one usually assumes the vision statement with the implications replaced by bi-implications. Also, at the

vision	$\bigwedge_{A\in\mathcal{C}}\bigwedge_{B\neq A}((\mathsf{d}_B \to \Box_A\mathsf{d}_B)\ \wedge\ (\neg\mathsf{d}_B \to \Box_A\neg\mathsf{d}_B))$
at least one	$\bigvee_{A\in\mathcal{C}}\mathsf{d}_A$
background	$\Box^*(\mathsf{vision} \wedge \mathsf{at\ least\ one})$
nobody knows	$\bigwedge_{A\in\mathcal{C}}(\neg\Box_A\mathsf{d}_A \wedge \neg\Box_A\neg\mathsf{d}_A)$
somebody knows	$\neg$nobody knows

Figure 1. Abbreviations in the discussion of the muddy children scenario, following [12].

outset one usually assumes the sentence nobody knows from Figure 1. We work with background as formulated in Figure 1 because it is already strong enough to make the following conditional valid. Let

$$\varphi_{\mathcal{C}} \quad\equiv\quad \mathsf{background} \to \langle \mathsf{nobody\ knows}^* \rangle \mathsf{somebody\ knows}. \tag{ii}$$

This $\varphi_{\mathcal{C}}$ says that some finite number of public announcements of everyone's ignorance will eventually result in the opposite: someone knowing their status. For this to hold, there must be some sort of background assumption. Again, ours is fairly weak. And the conclusion, too, is weaker than the usual assertion that *everyone* learns their state. The action involved is also weaker: instead of dealing with truthful announcements by everyone of their state, we only announce universal ignorance.

PROPOSITION 1.1. *For each finite set $\mathcal{C}$ of children, $\models \varphi_{\mathcal{C}}$. That is, $\varphi_{\mathcal{C}}$ holds in all worlds of all $\mathcal{L}_{\mathcal{C}}$-models.*

PROOF. Observe first that if $(W, w) \models \mathsf{background}$ and V is a submodel of W containing w, then $(V, w) \models \mathsf{background}$. This may be verified directly, and it also follows from the fact all of the $\Box$'s and $\Box^*$'s in the sentence background are within the scope of an even number of negations.

For all model-world pairs (W, w), let $n(W, w)$ be the largest number of clean children in any world reachable from w. Since the number of children is finite, $n(W, w)$ is a well-defined natural number.

Here is our second observation: assume that (W, w) is acceptable, and let v be a world reachable from w with exactly $n(W, w)$ clean children. Then

$$(W, v) \models \bigwedge_{A\in\mathcal{C}} (\mathsf{d}_A \to \Box_A\mathsf{d}_A).$$

To see this, let A be dirty in v. If $v \xrightarrow{A} u$, then by $\Box^*\mathsf{vision}$, for all $B \neq A$, B is clean in v iff B is clean in u. So by the maximality of $n(W, v)$, A must be dirty in u. This shows that $(W, v) \models \Box_A\mathsf{d}_A$.

Further, let $V = W^{\mathsf{nobody\ knows}}$. Our third observation is that for $w \in V$, $n(V, w) < n(W, w)$. To see this, recall that we construct V by passing to the submodel of W determined by the worlds in which nobody knows their status. As we have just seen, in worlds v with $n(W,w)$ clean children, the dirty children known their status. Moreover, in all such v, there is at least one dirty child, since $(W,w) \models \Box^*\mathsf{at\ least\ one}$. So all such v are *not* worlds of V. Turning things around and using the definition of $n(W,w)$, the worlds of V which are reachable from w in W have fewer than $n(W,w)$ clean children. *A fortiori*, the worlds of V which are reachable from w *in the submodel* V have fewer than $n(W,w)$ clean children.

Finally, we prove our proposition by induction on $n(W,w)$. So assume that $(W,w) \models \mathsf{background}$. If $(W,w) \models \mathsf{somebody\ knows}$, then we are done. Otherwise, let $V = W^{\mathsf{nobody\ knows}}$. Then $w \in V$, and we consider (V,w). Since (V,w) is a submodel of (W,w), $(V,w) \models \mathsf{background}$. From the third observation, $n(V,w) < n(W,w)$. So by our induction hypothesis, there is some j such that

$$(V,w) \models \langle \mathsf{nobody\ knows}\rangle^j \mathsf{somebody\ knows}.$$

And then

$$(W,w) \models \langle \mathsf{nobody\ knows}\rangle\langle \mathsf{nobody\ knows}\rangle^j \mathsf{somebody\ knows}.$$

That is, $(W,w) \models \langle \mathsf{nobody\ knows}\rangle^{j+1} \mathsf{somebody\ knows}$. ■

Remark. Statements like $\varphi_{\mathcal{C}}$ in (ii) have been considered in the literature before. Our treatment is based on the discussion in Gerbrandy and Groeneveld [12], but rather than formulate Proposition 1.1, they show something different. Here is how we would write it, making some changes from the original: Let $\mathcal{B}$ be a set of $n \geq 1$ children. Then

$$\models \left(\bigwedge_{A \in \mathcal{B}} \mathsf{d}_A \wedge \bigwedge_{A \notin \mathcal{B}} \neg \mathsf{d}_A \wedge \mathsf{background} \right) \rightarrow \neg \langle \mathsf{nobody\ knows}\rangle^{n-1} \neg \bigwedge_{A \in \mathcal{B}} \Box_A \mathsf{d}_A.$$

Note that our statement uses the iteration construct $\langle \psi^* \rangle$ explicitly. In addition to this difference, the validity proofs are also different.

A final comment: if $\mathcal{C}$ were infinite, then many of the points concerning $\varphi_{\mathcal{C}}$ in this section would be false. We would need infinitary conjunction to formulate the sentence $\varphi_{\mathcal{C}}$ in the first place. Even then, the analog of Proposition 1.1 would be false for infinite $\mathcal{C}$.

The point of Proposition 1.1 is that the statements $\varphi_{\mathcal{C}}$ in (ii) are natural *logical validities*. It makes sense to study the validities in $\mathcal{L}_{\mathcal{C}}$. The basic logic of announcements and common knowledge is known to be decidable, and indeed we have axiomatizations of it [2]. The main result of this paper is that adding the *iterated announcement* construct that gives us the $\langle \mathsf{nobody\ knows}^* \rangle$ operation results in logical systems whose set of satisfiable sentences are Σ^1_1-complete. In particular, such logical systems cannot be recursively axiomatized.

2. Preliminaries

At this point, we have seen some of the motivation for considering iterated announcement in languages formalizing epistemic logic. We now begin the actual work of the paper. In the sequel, we shall not be concerned at all with the muddy children scenario or the particular sentence in Proposition 1.1. Indeed, we are going to study more general matters. So we change some of the terminology, speaking of relativization rather than announcement, dropping the mention of "children" and indeed of any agents whatsoever, etc. We also wish to present our subject from a slightly different angle. In order to avoid confusion of notation and ideas, we therefore begin anew.

If φ is a sentence of some language $\mathcal{L}$ and A is an $\mathcal{L}$-structure, then we write A^φ for the submodel of A determined by $\{a \in A : (A, a) \models \varphi\}$, the set of points of A satisfying φ. This definition applies for a wide variety of languages $\mathcal{L}$; we shall be interested in classical modal logic and some related languages. Specifically, we consider $\mathcal{L}(\mathsf{rel})$, the *extension of modal logic by relativization*: this language has sentences $[\varphi]\psi$ with the semantics

$$(A, a) \models [\varphi]\psi \quad \text{iff} \quad (a \in A^\varphi \text{ implies } (A^\varphi, a) \models \psi).$$

We define $\langle\varphi\rangle$ to be dual of $[\varphi]$. So $\langle\varphi\rangle\psi$ is $\neg[\varphi]\neg\psi$. That is,

$$(A, a) \models \langle\varphi\rangle\psi \quad \text{iff} \quad (a \in A^\varphi \text{ and } (A^\varphi, a) \models \psi),$$

and we also see that

$$(A, a) \models \langle\varphi\rangle\psi \quad \text{iff} \quad ((A, a) \models \varphi) \text{ and } (A, a) \models [\varphi]\psi).$$

The language $\mathcal{L}(\mathsf{rel})$ was proposed (with different names) by Plaza [19] and independently later by Gerbrandy [10, 11]. As one can see from the discussion in Section 1, $\mathcal{L}(\mathsf{rel})$ is important in connection with the modeling of *public announcements* in the multi-agent setting. But this paper settles technical questions and is therefore less interested in conceptual matters, so we shall not give further motivation for this or other logical systems. Getting back to $\mathcal{L}(\mathsf{rel})$, its originators noted that $\mathcal{L}(\mathsf{rel})$ is equivalent in expressive power to ordinary modal logic. One way to see this is to define a translation $t : \mathcal{L}(\mathsf{rel}) \to \mathcal{L}$. In stating this translation, we introduce some notation. Let ψ and φ be modal sentences, and suppose that φ is in negation normal form (i.e., all negations apply only to atomic sentences). Then we define φ^ψ by the following recursion: $p^\psi = p$, $(\neg p)^\psi = \neg p$, $(\varphi_1 \wedge \varphi_2)^\psi = \varphi_1^\psi \wedge \varphi_2^\psi$,

$(\varphi_1 \vee \varphi_2)^\psi = \varphi_1^\psi \vee \varphi_2^\psi$, $(\Box\varphi)^\psi = \Box(\psi \to \varphi^\psi)$, and $(\Diamond\varphi)^\psi = \Diamond(\psi \wedge \varphi^\psi)$. (For φ not in negation normal form, we set $\varphi^\psi = (\mathsf{nnf}\,\varphi)^\psi$, where $\mathsf{nnf}\,\varphi$ is the negation normal form of φ.)

PROPOSITION 2.1. *Let ψ and φ be modal sentences. Let A be a model, and let $a \in A^\psi$. Then $(A^\psi, a) \models \varphi$ iff $(A, a) \models \varphi^\psi$.*

PROOF. By induction on φ in negation normal form. Here, for example, is the induction step for $\Box\varphi$, assuming the result for φ. Let $a \in A^\psi$. Assume first that $(A^\psi, a) \models \Box\varphi$. Then for all $b \in A^\psi$ such that $a \to b$, we have by our induction hypothesis that $(A, a) \models \varphi^\psi$. In other words, $(A, a) \models \Box(\psi \to \varphi^\psi)$. That is $(A, a) \models (\Box\varphi)^\psi$. The converse is similar. ∎

Now we define the translation t of $\mathcal{L}(\mathsf{rel})$ to $\mathcal{L}$. The main clause is $([\varphi]\psi)_t = \varphi_t \to (\psi_t)^{\varphi_t}$. An induction using Proposition 2.1 shows that this map t preserves the semantics. And another induction shows that each φ_t is a purely modal sentence. Our conclusion at this point is that adding relativization to modal logic alone does not increase expressive power.

Things get more interesting when one adds further constructs. The first is the *common knowledge* (or *reflexive-transitive closure*) operator $\Box^*$, with the semantics

$$(A, a) \models \Box^*\varphi \quad \text{iff} \quad \text{for all } b \text{ such that } a \xrightarrow{*} b,\ (A, b) \models \varphi.$$

Here $\xrightarrow{*}$ is the reflexive-transitive closure of the accessibility relation of A. Call the resulting language $\mathcal{L}(\mathsf{rel}, \Box^*)$. In this case, the relevant results in this direction may be found in Baltag, Moss, and Solecki [2]. $\mathcal{L}(\mathsf{rel}, \Box^*)$ is more expressive than modal logic, and indeed more expressive than $\mathcal{L}(\Box^*)$; i.e., modal logic with the transitive closure operator $\Box^*$ added. In particular, one cannot express $[q]\Box^* p$ in $\mathcal{L}(\Box^*)$. But every sentence of $\mathcal{L}(\mathsf{rel}, \Box^*)$ is effectively equivalent to a sentence of propositional dynamic logic (PDL). This immediately implies the finite model property and indeed the decidability of $\mathcal{L}(\mathsf{rel}, \Box^*)$. (For these results on PDL, see, e.g., Harel, Kozen and Tiuryn [15].) Furthermore, there are sound and complete logical systems for this notion. Even more, it is possible to extend this positive result by generalizing the notion of relativization to many other types of "epistemic actions" on models. It would take us too far afield to get into this matter here, but one should see [3].

In this paper, we go one step further. We consider the *iterated relativization operator* $[\varphi^*]$. The semantics is given by

$$(A, a) \models [\varphi^*]\psi \quad \text{iff} \quad (A, a) \models [\varphi]^n\psi \text{ for all } n.$$

So we also have a dual operation $\langle\varphi^*\rangle$, and then

$$(A,a) \models \langle\varphi^*\rangle\psi \quad \text{iff} \quad (A,a) \models \langle\varphi\rangle^n\psi \text{ for some } n.$$

This is the operation that we used in Section 1. It is also convenient to note that

$$(A,a) \models [\varphi^*](\varphi \wedge \psi) \quad \text{iff} \quad (A,a) \models \langle\varphi\rangle^n\psi \text{ for all } n.$$

We define the logics $\mathcal{L}(\mathsf{rel}, \mathsf{rel}^*)$ and $\mathcal{L}(\mathsf{rel}, \mathsf{rel}^*, \Box^*)$ in the obvious ways.

Examples. Throughout this paper, we write D for $\Diamond\mathsf{True}$, so $[D^*]$ means $[(\Diamond\mathsf{True})^*]$. Semantically, the operation of relativizing by $\Diamond\mathsf{True}$ removes those points of a model which have no children. We use the letter D because this operation reminds us of the Cantor-Bendixson derivative of a set of real numbers, wherein one removes the isolated points. Indeed, we write A' for A^D and define $A^{(n)}$ by $A^{(0)} = A$ and $A^{(n+1)} = \left(A^{(n)}\right)'$.

We are interested in the iteration of the derivative operation, as in $\langle D^*\rangle\Box\mathsf{False}$. By induction on n, we see that $(A,a) \models \langle D\rangle^n\Box\mathsf{False}$ iff the longest path in A beginning at a has length exactly n. It follows that $(A,a) \models \langle D^*\rangle\Box\mathsf{False}$ iff there is some n such that all paths in A starting from a are of length at most n.

$[D^*]\Diamond\mathsf{True}$ is then the dual of $\langle D^*\rangle\Box\mathsf{False}$. It holds at a point a if there is no bound on the lengths of paths from a.

It was observed in [2] that $\mathcal{L}(\mathsf{rel}, \mathsf{rel}^*)$ does not have the finite model property because $[D^*]\Diamond\mathsf{True}$ is satisfiable but only by an infinite model. Nevertheless, the second author conjectured that the satisfiability problem for this logic was still decidable. This conjecture was refuted by the first author. The results of this paper show this in several ways. Specifically, we show that the satisfiability problem for the following fragments of $\mathcal{L}(\mathsf{rel}, \mathsf{rel}^*)$ and $\mathcal{L}(\mathsf{rel}, \mathsf{rel}^*, \Box^*)$ are Σ^1_1-complete:

1. The fragment generated by $[D^*]$, $\Box^*$, $\Box$, $\wedge$, and $\neg$.
2. The fragment generated by two iterated relativization operators $[D^*_x]$ and $[D^*_y]$, in addition to $\Box$, $\wedge$, $\neg$, and atomic sentences. Here D_x and D_y are two fixed modal sentences.
3. The fragment generated by $[D^*]$, arbitrary modal relativizations, $\Box$, $\wedge$, $\neg$, and atomic sentences.

One difference between these results is that in the first and third, we iterate only one very simple relativization D but we also add either the transitive closure operator $\Box^*$ or else complex modal relativizations. The second instead calls on the iteration of two particular (purely modal) sentences. Also, the first fragment does not use atomic sentences.

We also prove that the problems of deciding whether a sentence in $\mathcal{L}(\mathsf{rel}, \mathsf{rel}^*, \Box^*)$ has a finite model, or a finite tree model, are Σ^0_1-complete.

2.1. Domino systems

All of our Σ^1_1-hardness results go via reduction to the tiling problem for recurring domino systems. So we recall the basic definitions. The original paper on this is Harel [14], and the book by Blackburn, de Rijke, and Venema [5] has applications to modal logic.

Definition. A *domino system* is a tuple $\mathcal{D} = (Dominoes, H, V)$, where *Dominoes* is a finite set, and $H, V \subseteq Dominoes \times Dominoes$.

The *first quadrant* is the set $Q = N \times N$. A *tiling of* Q *by* $\mathcal{D}$ is a function $t : Q \to Dominoes$. The tiling t is *proper* if for all $n, m \in N$,

1. $H(t(n, m), t(n + 1, m))$.
2. $V(t(n, m), t(n, m + 1))$.

A *recurring domino system* is a pair $(\mathcal{D}, d_0)$ with $d_0 \in Dominoes$; a *proper tiling* of Q by $(\mathcal{D}, d_0)$ is a proper tiling of Q by $\mathcal{D}$ in which $t(n, 0) = d_0$ for infinitely many n.

We shall use the result of Harel [14] on the problem of deciding whether a recurring domino system $(\mathcal{D}, d_0)$ has a proper tiling: this problem is Σ^1_1-complete.

In Section 8, we prove a result by reduction to the periodicity problem for domino systems. We recall the relevant definitions later.

2.2. Translating $\mathcal{L}(\mathsf{rel}, \mathsf{rel}^*, \Box^*)$ into the Modal Iteration Calculus

In [7], Dawar, Grädel, and Kreutzer introduced a *Modal Iteration Calculus* MIC. Among other things, they showed that the satisfiability problem for MIC is undecidable. It was suggested by van Benthem [20] that adding iterated relativization to modal logic gives a fragment of MIC (see also Grädel, and Kreutzer [13]). We want to discuss this result, since MIC is the smallest previously-studied logical system containing $\mathcal{L}(\mathsf{rel}, \mathsf{rel}^*, \Box^*)$ that we are aware of.[1]

MIC is defined by adding two things to the syntax of modal logic: set variables $X_1, X_2, \ldots$, and a formula constructing operator

$$\mathbf{ifp}(X_j : X_1 \leftarrow \varphi_1, \ldots, X_k \leftarrow \varphi_k). \tag{iii}$$

[1]The results of this section are not needed in later sections of the paper.

Here the φ's are again formulas and $1 \leq j \leq k$. We understand **ifp** to be a variable binding operator, and $X_1, \ldots, X_k$ are bound in (iii). We define the semantics of all formulas of MIC. Assume that the free set variables of φ are among $Y_1, \ldots, Y_n$ and that $B_1, \ldots, B_n \subseteq A$ and $a \in A$. We define $(A, B_1, \ldots, B_n, a) \models \varphi$ by recursion on φ. For example, $(A, B_1, \ldots, B_n, a) \models Y_j$ iff $a \in B_j$. The main clause is for **ifp**-formulas as in (iii). Let the free set variables of each φ_i be included in the list $Y_1, \ldots, Y_n, X_1, \ldots, X_k$, and let $B_1, \ldots, B_n \subseteq A$. Define *iterates* $S_i^\alpha \subseteq A$ for α an ordinal and $1 \leq i \leq k$ by recursion on α: let $S_i^0 = \emptyset$,

$$S_i^{\alpha+1} \quad = \quad S_i^\alpha \ \cup \ \{a \in A : (A, B_1, \ldots, B_n, S_1^\alpha, \ldots, S_k^\alpha, a) \models \varphi_i\},$$

and for λ a limit ordinal, $S_i^\lambda = \bigcup_{\beta<\lambda} S_i^\beta$. For each i, the sequence S_i^α is an increasing sequence of subsets of A, and we set S_i^* to be its eventual value. Then

$$(A, B_1, \ldots, B_n, a) \models \mathbf{ifp}(X_j : \overline{X_i \leftarrow \varphi_i}) \quad \text{iff} \quad a \in S_j^*.$$

We now discuss the relation of the language $\mathcal{L}(\mathsf{rel}, \mathsf{rel}^*, \Box^*)$ to MIC. As it happens, van Benthem in [20] modified the semantics of sentences of the form $[\varphi^*]\psi$ to use iteration over *all ordinals* rather than only over the natural numbers. To get an exact match, we need to do a bit more.

Suppose that $\varphi(X)$ is a formula with just X free. Let U, V, and W be new variables. Define $\mathbf{ifp}_\omega(X : X \leftarrow \varphi)$ to be

$$\mathbf{ifp}(V : U \leftarrow V, V \leftarrow (\neg W \wedge \varphi(\emptyset)) \vee (W \wedge \varphi(V) \wedge \neg\varphi(U)), W \leftarrow \mathsf{True}). \quad \text{(iv)}$$

Fix a model A, and consider the interpretation of (iv) in A. We write S^α for the iterates of the original system and U^α, V^α and W^α for the iterates of the new system. Clearly $U^1 = \emptyset$, $V^1 = \varphi(\emptyset) = S^1$, $W^0 = \emptyset$ and for $n \geq 1$, $W^n = A$. An induction on the natural number n shows that $U^{n+1} = V^n = S^n$ and $V^{n+1} = S^{n+1}$. We have already checked this for $n = 0$. Assume for n that $U^{n+1} = V^n = S^n$ and $V^{n+1} = S^{n+1}$. Then we see that $U^{n+2} = V^{n+1} = S^{n+1}$ (easily), and also that

$$\begin{array}{rcl} V^{n+2} & = & V^{n+1} \cup (\varphi(S^{n+1}) \setminus \varphi(S^n)) \\ & = & S^{n+1} \cup (\varphi(S^{n+1}) \setminus \varphi(S^n)) \\ & = & S^{n+2}. \end{array}$$

It follows that $U^\omega = V^\omega = S^\omega$. And then for $\alpha \geq \omega$ we see that $U^\alpha = V^\alpha = S^\omega$. The inductive step is that $V^{\alpha+1} = V^\alpha \cup (\varphi(S^\omega) \setminus \varphi(S^\omega)) = S^\omega \cup \emptyset = S^\omega$.

Our conclusion here is that for all formulas $\varphi(X)$ in the language, there is another formula $\mathbf{ifp}_\omega(X : X \leftarrow \varphi)$ (as in (iv)) whose interpretation in any model A is the ω-th inflationary iterate S^ω in A.

Furthermore, we note that MIC is closed under relativization in the following sense. If φ is a formula of MIC and ψ is a sentence of it, we define φ^ψ by the same recursion as earlier, except that we also add $X^\psi = X \wedge \psi$, and also

$$\mathbf{ifp}(X_j : \overline{X_i \leftarrow \varphi_i})^\psi \quad = \quad \mathbf{ifp}(X_j : \overline{X_i \leftarrow \varphi_i^\psi}).$$

From this, we define a translation t of $\mathcal{L}(\mathsf{rel}, \mathsf{rel}^*, \Box^*)$ into the sentences of MIC. The main clauses are

$$\begin{array}{lcl}
(\Diamond^*\varphi)_t & = & \mathbf{ifp}(X : X \leftarrow \varphi_t \vee \Diamond X) \\
([\varphi]\psi)_t & = & \mathbf{ifp}(Y : X \leftarrow \varphi_t, Y \leftarrow Z \wedge \psi_t^X, Z \leftarrow \mathsf{True}) \\
(\langle\psi^*\rangle\varphi)_t & = & \mathbf{ifp}_\omega(X : X \leftarrow \varphi_t^{\neg Y}, Y \leftarrow \neg(\psi_t^{\neg Y}))
\end{array}$$

We check that this works for the sentences $\langle\psi^*\rangle\varphi$, assuming that it works for ψ and φ. We write X^n for the n-th iteration of X, and similarly for Y. Fix a model A, and define subsets and submodels A_n by $A_0 = A$ and $A_{n+1} = (A_n)^\psi$. We first check by induction that $Y^n = -A_n$. This is clear for $n = 0$. Assuming that $Y^n = -A_n$, we have

$$\begin{array}{lcl}
Y^{n+1} & = & -(\psi^{A_n}) \cup Y^n \\
 & = & -A_{n+1} \cup (-A_n) \\
 & = & -A_{n+1}.
\end{array}$$

And then we also see that $X^n = \varphi^A \cup \varphi^{A_1} \cup \cdots \cup \varphi^{A_n}$. So we are after $X^\omega = \bigcup_{n<\omega} X^n$. Since we use $\mathbf{ifp}_\omega$, this is given by our formula above.

Implications. As a result of this translation and the Σ^1_1-hardness results to come, we have an improvement of Theorem 3.5 of [7]. That result exhibits an encoding of first-order arithmetic into the satisfiability problem for MIC. We also get the Σ^0_1-completeness of the finite satisfiability problem, and this appears to be new. Of perhaps more importance is that we have shown that some small fragments of MIC are undecidable. So the search for decidable expressive fragments of MIC that go beyond the modal μ-calculus will have to involve logics formed from different principles than the ones we study here.

3. The satisfiability problem for $\mathcal{L}(\mathsf{rel}, \mathsf{rel}^*, \Box^*)$ is in Σ^1_1

We sketch a proof that the set $\{\chi \in \mathcal{L}(\mathsf{rel}, \mathsf{rel}^*, \Box^*) : \chi \text{ is satisfiable}\}$ is Σ^1_1. This is in contrast for MIC, where the estimate of [7] gives Σ^1_2. What accounts for the difference is precisely that our semantics of the $[\varphi^*]\psi$ construct involves iteration over *numbers* rather than arbitrary ordinals. Were one to modify the semantics (as van Benthem does in [20]), then the Σ^1_1 upper bound is presumably false.

It will be useful to take True to be a primitive symbol of $\mathcal{L}(\mathsf{rel}, \mathsf{rel}^*, \Box^*)$. It is also worth remarking that the syntax of $\mathcal{L}(\mathsf{rel}, \mathsf{rel}^*, \Box^*)$ allows sentences of the form $[\varphi^*]\psi$. We often use abbreviations $[\varphi]^m\psi$, but these are exactly that: abbreviations. For example, $[p]^3q$ is an abbreviation for $[p][p][p]q$.

We define a map pre from $\mathcal{L}(\mathsf{rel}, \mathsf{rel}^*, \Box^*)^*$, the set of sequences from $\mathcal{L}(\mathsf{rel}, \mathsf{rel}^*, \Box^*)$, to $\mathcal{L}(\mathsf{rel}, \mathsf{rel}^*, \Box^*)$ as follows. Let $\mathrm{pre}(\lambda) = \mathsf{True}$, where λ is the empty sequence, $\mathrm{pre}(\varphi) = \varphi$, and for $n \geq 2$, $\mathrm{pre}(\varphi_1, \ldots, \varphi_n) = \varphi_1 \wedge [\varphi_1]\mathrm{pre}(\varphi_2, \ldots, \varphi_n)$. (pre stands for "precondition". The name comes from [3], where a generalization of this function plays an important role.)

Lemma 3.1. *Let A be any model, and define relations $R \subseteq A \times A$, $P \subseteq \omega \times A$, and $X \subseteq \mathcal{L}(\mathsf{rel}, \mathsf{rel}^*, \Box^*) \times A$ as follows: R is the accessibility relation of A, $P(k, x)$ iff $(A, x) \models p_k$, and $X(\varphi, x)$ iff $(A, x) \models \varphi$. Then $X(\mathsf{True}, x)$ holds for all x. And each instance of the following biconditionals also holds:*

$$\begin{array}{rcl}
X([\varphi_1]\cdots[\varphi_n]p_k, x) & \leftrightarrow & X(\mathrm{pre}(\varphi_1, \ldots, \varphi_n), x) \to P(k, x) \\
X([\varphi_1]\cdots[\varphi_n]\neg\psi, x) & \leftrightarrow & \\
 & & X(\mathrm{pre}(\varphi_1, \ldots, \varphi_n), x) \to \neg X([\varphi_1]\cdots[\varphi_n]\psi, x) \\
X([\varphi_1]\cdots[\varphi_n](\psi_1 \wedge \psi_2), x) & \leftrightarrow & \\
 & & X([\varphi_1]\cdots[\varphi_n]\psi_1, x) \wedge X([\varphi_1]\cdots[\varphi_n]\psi_2, x) \\
X([\varphi_1]\cdots[\varphi_n]\Box\psi, x) & \leftrightarrow & X(\mathrm{pre}(\varphi_1, \ldots, \varphi_n), x) \to \\
 & & (\forall y)(R(x, y) \to X([\varphi_1]\cdots[\varphi_n]\psi, y)) \\
X([\varphi_1]\cdots[\varphi_n]\Box^*\psi, x) & \leftrightarrow & (\forall m)X([\varphi_1]\cdots[\varphi_n]\Box^m\psi, x) \\
X([\varphi_1]\cdots[\varphi_n][\psi^*]\chi, x) & \leftrightarrow & (\forall m)X([\varphi_1]\cdots[\varphi_n][\psi]^m\chi, x)
\end{array} \qquad \text{(v)}$$

Moreover, each sentence of $\mathcal{L}(\mathsf{rel}, \mathsf{rel}^*, \Box^*)$ other than True is an instance of some (unique) sentence occurring on the left-hand side of one of these biconditionals.

Proof. All of the equivalences are special cases of results from [3].

The "moreover" assertion is checked by induction on φ. If φ is of the form p_k, $\neg\psi$, $\psi_1 \wedge \psi_2$, $\Box\psi$, $\Box^*\psi$, or $[\psi^*]\chi$, then we may take $n = 0$. If φ is of the form $[\psi]\chi$, then by induction hypothesis, χ is an instance of the left side of one of the biconditionals in (v). And then so is $[\psi]\chi$. ∎

Lemma 3.2. *There is a wellfounded relation $<$ on $\mathcal{L}(\mathsf{rel}, \mathsf{rel}^*, \Box^*)$ such that if φ_L occurs on the left-hand side of one of the biconditionals in (v), and φ_R occurs on the right-hand side of the same biconditional, then $\varphi_R < \varphi_L$.*

Remark. Here is an example of what we mean in this lemma, based on the fourth biconditional above. For all $\varphi_1, \ldots, \varphi_n, \psi$, we have $\mathrm{pre}(\varphi_1, \ldots, \varphi_n) < [\varphi_1]\cdots[\varphi_n]\Box\psi$, and we also have $[\varphi_1]\cdots[\varphi_n]\psi < [\varphi_1]\cdots[\varphi_n]\Box\psi$.

Concerning the last biconditional, we mean that for all m,

$$[\varphi_1]\cdots[\varphi_n][\psi]^m\chi \;<\; [\varphi_1]\cdots[\varphi_n][\psi^*]\chi.$$

PROOF. We obtain the relation $<$ as a lexicographic partial order (LPO) of $\mathcal{L}(\mathsf{rel}, \mathsf{rel}^*, \Box^*)$. For background on LPO, see the surveys by Dershowitz [8] and Plaisted [18]. We regard $\mathcal{L}(\mathsf{rel}, \mathsf{rel}^*, \Box^*)$ as an algebra of terms, using the constructors True, p_1, …, p_k, …, $\neg$, $\wedge$, $\Box$, $\Box^*$, rel, and rel^*. The latter are two-place function symbols: $\mathsf{rel}(\varphi,\psi)$ is an alternative for $[\varphi]\psi$, and $\mathsf{rel}^*(\varphi,\psi)$ is an alternative for $[\varphi^*]\psi$. In this proof, we shall let f and g range over these symbols. We define the ordering $<$ on the function symbols of $\mathcal{L}(\mathsf{rel}, \mathsf{rel}^*, \Box^*)$ to be the smallest transitive relation containing

$$\mathsf{True} \;< p_k, \wedge, \neg, \Box \;<\; \Box^* \;<\; \mathsf{rel} \;<\; \mathsf{rel}^*.$$

This wellfounded relation generates an LPO on $\mathcal{L}(\mathsf{rel}, \mathsf{rel}^*, \Box^*)$; as usual we denote this ordering by $<$ as well. Concretely, this is the smallest relation such that

(LPO1) If $(t_1,\ldots,t_n) < (s_1,\ldots,s_n)$ in the lexicographic ordering on n-tuples, and if $t_j < f(s_1,\ldots,s_n)$ for $1 \le j \le n$, then $f(t_1,\ldots,t_n) < f(s_1,\ldots,s_n)$.

(LPO2) If $t \le s_i$ for some i, then $t < f(s_1,\ldots,s_n)$.

(LPO3) If $g < f$ and $t_i < f(s_1,\ldots,s_n)$ for all $i \le m$, then $g(t_1,\ldots,t_m) < f(s_1,\ldots,s_n)$.

It is a general result on LPO that $<$ is wellfounded. It also has the subterm property: if φ is a strict subsentence of ψ, then $\varphi < \psi$.

We check by induction on $n \ge 1$ that

$$\mathrm{pre}(\varphi_1,\ldots,\varphi_n) \quad < \quad [\varphi_1]\cdots[\varphi_n]\psi.$$

This is where we use the assumption that $\wedge < \mathsf{rel}$. Further inductions show that $[\psi]^m\chi < [\psi^*]\chi$ for all m, and that $\Box^m\psi < \Box^*\psi$ for all m. (LPO3) is used in these, as are the assumptions that $\Box < \Box^*$ and $\mathsf{rel} < \mathsf{rel}^*$. These preliminary remarks establish the base case ($n = 0$) of an induction on n that if φ_L is the left-hand side of one of the biconditionals in (v), and φ_L is on the right side of the same biconditional, then $\varphi_R < \varphi_L$. The induction step follows easily from (LPO1), (LPO2) and the subterm property. ■

In the statement and proof of our result below, we assume a "nice" coding of $\mathcal{L}(\mathsf{rel}, \mathsf{rel}^*, \Box^*)$ by a subset of ω. We need to know that several functions are recursive in the codes. These include $\varphi_1,\ldots,\varphi_n \mapsto \mathrm{pre}(\varphi_1,\ldots,\varphi_n)$; $n,\varphi,\psi \mapsto [\varphi]^n\psi$; and $n,\varphi \mapsto \Box^n\varphi$.

THEOREM 3.3. *$\{\chi \in \mathcal{L}(\mathsf{rel}, \mathsf{rel}^*, \Box^*) : \chi$ is satisfiable$\}$ is Σ^1_1.*

PROOF. We claim that a sentence χ is satisfiable if there are sets $R \subseteq \omega \times \omega$, $P \subseteq \omega \times \omega$, and $X \subseteq \mathcal{L}(\mathsf{rel}, \mathsf{rel}^*, \Box^*) \times \omega$ such that $X(\mathsf{True}, y)$ holds for all y, and all instances of the biconditionals of (v) hold, and that there is some x such that $X(\chi, x)$.

In one direction, we appeal to a result stated in [7] that says that if χ is satisfiable, then it has a countable model A. (In [7] this was stated for MIC, based on an extension of a result originally shown by Flum [9]: the logic LFP has the Löwenheim-Skolem property. Since we know that $\mathcal{L}(\mathsf{rel}, \mathsf{rel}^*, \Box^*)$ is a sublogic of MIC, we now have this property for it.) So we may assume that the universe of A is ω, and now the rest follows from Lemma 3.1.

In the other direction, fix a sentence χ. Assume that we have R, P, and X. Let A be the model with universe ω whose structure is given by R and P in the obvious way. Let $<$ be a wellfounded relation as in Lemma 3.2. We argue by induction on $<$ that for all φ in the field of $<$, $X(\varphi, x)$ iff $(A, x) \models \varphi$. The induction is an easy consequence of Lemma 3.1. We also use the last assertion in Lemma 3.1 to know that all sentences belong to the field of $<$. In particular, the sentence χ with which we began belongs. And from this, the claim easily follows.

At this point, we have shown our claim. We conclude by noting that our condition on R, P and X in the first paragraph of this proof is arithmetic: it involves universal quantification over sequences from $\mathcal{L}(\mathsf{rel}, \mathsf{rel}^*, \Box^*)$ as well as application of some functions which we assume to be recursive. It follows that our equivalent formulation of satisfiability is Σ^1_1. ∎

4. $[D^*]$, $\Box^*$, $\Box$, and atomic sentences

In this section, we prove that satisfiability is Σ^1_1-complete for the language with $[D^*]$, $\Box^*$, $\Box$ and *atomic sentences*. We strengthen this in Section 5 to eliminate the atomic sentences. That is, we shall prove the following result:

THEOREM 4.1. *To every recurring domino system $(\mathcal{D}, d_0)$, we can effectively associate a sentence $\varphi_{\mathcal{D},d_0}$ of the language of $[D^*], \Box^*$, $\Box$, True, and the boolean connectives such that the following are equivalent:*

1. *There is a proper tiling of Q by $(\mathcal{D}, d_0)$.*
2. *$\varphi_{\mathcal{D},d_0}$ is satisfiable.*

COROLLARY 4.2. *The satisfiability problem for the fragment of Theorem 4.1 is Σ^1_1-complete.*

Fix a recurring domino system $(\mathcal{D}, d_0)$. We take a language with atomic sentences corresponding to the (finitely many) dominoes. Concretely, let d correspond to d.

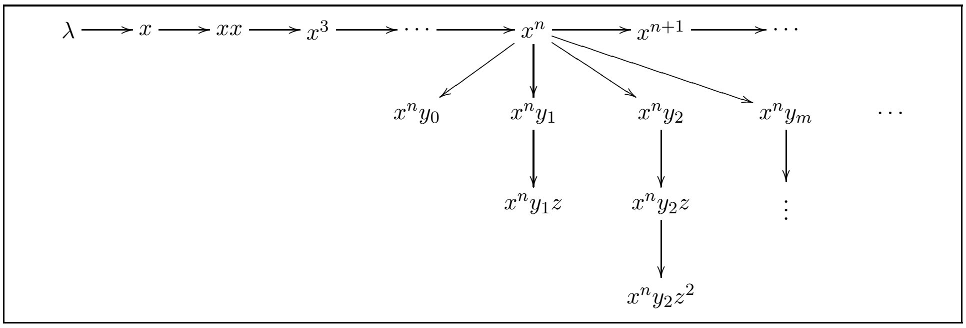

Figure 2. The frame F used in work on the fragment for $[D^*]$, $\Box^*$, $\Box$, and atomic sentences.

The intended frame for the first quadrant. Let x, $y_0, y_1, \ldots, y_m, \ldots$, and z be different symbols. We construct a frame F from a subset of $\{x, z, y_0, \ldots, y_m, \ldots\}^*$, the set of words on our symbols. The set of worlds of F is

$$\{x^n : 0 \leq n\} \ \cup \ \{x^n y_m z^p : 0 \leq n \text{ and } p \leq m\}.$$

We use standard notation here; for example, $x^5 y_3 z^0$ here really is the word $xxxxxy_3$. Note that F contains the empty word λ. The accessibility relation is given by $x^n \to x^{n+1}$, $x^n \to x^n y_m$, and $x^n y_m z^p \to x^n y_m z^{p+1}$. A picture of F may be found in Figure 2.

The points of the form x^n are called the *stalk* of F, since if one rotated our picture 90° it would be a single stalk with one branch of each finite length coming off of the each point of the stalk. Note that $(F, x^n y_m z^p) \models \Box\mathsf{False}$ iff $p = m$. And the derivative F' of F is isomorphic as a frame to F, via the map $x^n \mapsto x^n$ and $x^n y_{m+1} z^{p+1} \mapsto x^n y_m z^p$.

Tilings give models. Our intention is that $x^n y_m$ is a surrogate for the point (n, m) of the first quadrant. To get a model from a frame we need only specify the semantics of our atomic sentences. Let $t : Q \to \mathcal{D}$ be a tiling of the first quadrant. We construct a model F_t from t (and the underlying frame F described above) by declaring the atomic sentence d to be true at $x^n y_m$ iff $t(n, m) = d$. No other atomic sentences are true anywhere else.

The sentence $\varphi_{\mathcal{D}}$. In Figure 3 we list several sentences used in our this section. We begin with stalk, shorthand for $[D^*]\Diamond\mathsf{True}$. Note that $\models \mathsf{stalk} \to [D^*]\mathsf{stalk}$.

stalk	$[D^*]\Diamond\mathsf{True}$
χ_d	$\mathsf{d} \wedge \Box\mathsf{False}$
$\varphi_{\mathcal{D}}$	stalk
	$\wedge\Box^*(\mathsf{stalk} \to \Diamond\mathsf{stalk})$
	$\wedge\Box^*[D^*](\mathsf{stalk} \to \Diamond \bigvee_d \chi_d))$
	$\wedge\Box^*[D^*](\mathsf{stalk} \to \neg \bigvee_{\neg H(d,d')}(\Diamond\chi_d \wedge \Diamond\Diamond\chi_{d'}))$
	$\wedge\Box^*[D^*](\mathsf{stalk} \to \neg \bigvee_{\neg V(d,d')}(\Diamond\chi_d \wedge \Diamond\langle D\rangle\chi_{d'}))$
$\mathsf{recurring}(d_0)$	$\Box^*(\mathsf{stalk} \to \Diamond^*(\mathsf{stalk} \wedge \Diamond\chi_{d_0}))$
$\varphi_{\mathcal{D},d_0}$	$\varphi_{\mathcal{D}} \wedge \mathsf{recurring}(d_0)$

Figure 3. Abbreviations in the fragment for $[D^*]$, $\Box^*$, $\Box$, and atomic sentences.

The sentence χ_d will be our sentence saying of a node that it codes the domino d. But we must use derivatives to associate squares in the quadrant to the points of F.

We now consider the sentence $\varphi_{\mathcal{D}}$. In the third clause, we mean to take disjunctions over all pairs (d, d') such that $\neg H(d, d')$. Similarly for the last clause. That last clause may also be written without $\langle D\rangle$ as

$$\Box^*[D^*]\Bigg(\mathsf{stalk} \to \neg \bigvee_{\neg V(d,d')} (\Diamond\chi_d \wedge \Diamond(\mathsf{d}' \wedge \Diamond\mathsf{True} \wedge \Box\Box\mathsf{False}))\Bigg).$$

The intended models work. We next check that $(F_t, \lambda) \models \varphi_{\mathcal{D}}$, where λ again is the empty word. Recall that the stalk in F_t and in all its derivatives $F_t^{(m)}$ is the set of points of the form x^n for some n. This implies the first condition at the empty word λ. For the second, an induction on m shows that $x^n y_k z^p \in F_t^{(m)}$ iff $k - p \geq m$, and also that for $k \geq m$,

$$(F_t^{(m)}, x^n) \models \Diamond^{k-m}\mathsf{True} \wedge \neg\Diamond^{k-m+1}\mathsf{True}.$$

(when $k = m$, we intend that $\Diamond^0\mathsf{True} = \mathsf{True}$). As a result,

$$(F_t^{(m)}, x^n) \models \Diamond\chi_d \quad \text{iff} \quad d = t(n, m). \tag{vi}$$

This implies the second clause of $\varphi_{\mathcal{D}}$. And more crucially, the properness of the tiling t implies the last two clauses in $\varphi_{\mathcal{D}}$. (Indeed, the last two clauses could even be strengthened by dropping the mention of stalk. The point is that the only points with children satisfying any χ_d sentence are the points on the stalk. But the formulation as in Figure 3 will be needed in Section 5.)

Any model of $\varphi_{\mathcal{D}}$ gives a proper tiling. The significant direction is to show that models of $\varphi_{\mathcal{D}}$ gives proper tilings. There is a slightly stronger result that we use in the next section.

LEMMA 4.3. *Let* stalk *be* any *sentence so that* $\models \mathsf{stalk} \to [D^*]\mathsf{stalk}$. *Let* χ_d *be* any *sentences for* $d \in D$. *Let* $\varphi_{\mathcal{D}}$ *be as in Figure 3, using* stalk *and* χ_d. *Let* (A, a_0) *be an arbitrary model of* $\varphi_{\mathcal{D}}$. *There is a sequence in* A

$$a_0 \to a_1 \to \cdots \to a_n \to \cdots$$

with $a_n \models \mathsf{stalk}$ *for all* n. *Moreover,*

1. *There is a function* $t : Q \to \mathcal{D}$ *with the property that for* $(n, m) \in Q$, $(A^{(m)}, a_n) \models \Diamond\chi_{t(n,m)}$.
2. *Each such function* t *is a* proper *tiling of* Q *by* $\mathcal{D}$.

REMARK. This lemma says that every model of $\varphi_{\mathcal{D}}$ gives a proper tiling in all possible ways. It does not say that all models of $\varphi_{\mathcal{D}}$ are related in any way to the intended models, or that all models of $\varphi_{\mathcal{D}}$ give tilings in a unique or canonical way. For satisfiability, we need only know that proper tilings exist. This is the content of the present lemma.

PROOF. The sequence $a_0 \to a_1 \to \cdots$ exists by the first two clauses stalk and $\Box^*(\mathsf{stalk} \to \Diamond\mathsf{stalk})$ of $\varphi_{\mathcal{D}}$. The condition on the sentence stalk shows that $a_n \models [D^*][D^*]\Diamond\mathsf{True}$ as well; i.e., $a_n \models [D^*]\mathsf{stalk}$. Then the second clause of $\varphi_{\mathcal{D}}$ insures that for each n and m that there will be some $d \in D$ so that $(A^{(m)}, a_n) \models \Diamond\chi_d$. That is, some tiling t exists which satisfies the condition $(A^{(m)}, a_n) \models \Diamond\chi_{t(n,m)}$.

We turn to the second part. First, consider $t(n, m)$ and $t(n+1, m)$. By definition of t, $(A^{(m)}, a_n) \models \Diamond\chi_{t(n,m)}$ and $(A^{(m)}, a_{n+1}) \models \Diamond\chi_{t(n+1,m)}$. Since $a_n \to a_{n+1}$, we have

$$(A^{(m)}, a_n) \models \Diamond\chi_{t(n,m)} \wedge \Diamond\Diamond\chi_{t(n+1,m)}.$$

And since $(A, a_0) \models \varphi_{\mathcal{D}}$, we must have $H(t(n, m), t(n+1, m))$.

Second, consider $t(n, m)$ and $t(n, m+1)$. We have $(A^{(m)}, a_n) \models \Diamond\chi_{t(n,m)}$ and $(A^{(m+1)}, a_n) \models \Diamond\chi_{t(n,m+1)}$. Let $a_n \to b$ be such that $(A^{(m+1)}, b) \models \chi_{t(n,m+1)}$. Then $(A^{(m)}, b) \models \langle D\rangle\chi_{t(n,m+1)}$, so

$$(A^{(m)}, a_n) \models \Diamond\chi_{t(n,m)} \wedge \Diamond\langle D\rangle\chi_{t(n,m+1)}.$$

As above, we have $V(t(n, m), t(n, m+1))$. ∎

Recurring domino systems. If the original tiling t has d_0 infinitely often on the x-axis, then the intended model (F_t, λ) satisfies $\mathsf{recurring}(d_0)$ from Figure 3. Conversely, let $\varphi_{\mathcal{D},d_0} = \varphi_{\mathcal{D}} \wedge \mathsf{recurring}(d_0)$. If $(A, a_0) \models \varphi_{\mathcal{D},d_0}$, we may choose the path $a_0 \to a_1 \to \cdots$ so that for infinitely many i, $a_i \models \Diamond\chi_{d_0}$. Then we may arrange that the tiling that we get from this path has d_0 infinitely often on the x-axis.

Summary. Beginning with a tiling t, we constructed a sentence $\varphi_{\mathcal{D},d_0}$ with the property that models of $\varphi_{\mathcal{D},d_0}$ give proper tilings of $(\mathcal{D}, d_0)$. Conversely, every proper tiling of $(\mathcal{D}, d_0)$ gives a model of $\varphi_{\mathcal{D},d_0}$. This would complete the proof of Theorem 4.1 stated at the beginning of this section, except that we would like to strengthen the result to avoid the atomic sentences d corresponding to the dominoes.

5. Eliminating atomic sentences

We eliminate the atomic sentences by making the models more complicated and doing extra work in the coding. The overall strategy is to *redefine* χ_d to be a certain sentence built only from $[D^*]$, $\Box^*$, $\Box$, and the boolean connectives. We only need to show that proper tilings from a domino system $\mathcal{D}$ give models of $\varphi_{\mathcal{D}}$, or rather the version of $\varphi_{\mathcal{D}}$ obtained by the redefinition. We only need to find *some* model of $\varphi_{\mathcal{D}}$; this was the easy part in the previous section. Then Lemma 4.3 tells us that *any* model of $\varphi_{\mathcal{D}}$ gives a proper tiling in each of its paths through the stalk and in each sequence of choices along the path.

Again, we fix a domino system $\mathcal{D}$ for the remainder of this section. It will be convenient to take the dominoes to be a set of the form $\{2, 3, \ldots, K\}$. That is, $d \geq 2$ for $d \in$ *Dominoes*. The reason for this will become clear as we develop our coding.

Models from proper tilings. Let $t : Q \to \mathcal{D}$ be a proper tiling of Q by $\mathcal{D}$. We construct a frame G_t as follows. We again begin with infinitely many different symbols $x, y_0, y_1, \ldots, y_m, \ldots$ and z. The set of worlds of G_t is the following set of words:

$$\{x^n : 0 \leq n\} \ \cup \ \{x^n y_m^q z^p : 0 \leq n, 0 \leq m, 1 \leq q \leq t(n,m), 0 \leq p \leq m+1\}.$$

The accessibility relation is given by $u \to v$ iff both belong to G_t and if v is a one-letter extension of u.

Some examples of the coding. We take $n = 5$, $m = 2$. Suppose that $t(5, 2) = 4$. Then the model would contain the points in Figure 4 as an induced substructure.

There are other arrows from $\lambda, x, \ldots, x^5$, but for all of the other points shown, there are no other arrows besides what is in the figure. The derivative operation removes $x^5y_2z^3, \ldots, x^5y_2^4z^3$. The second derivative removes $x^5y_2z^2, \ldots, x^5y_2^4z^2$. Recalling that $n = 5$, $m = 2$, and $t(5, 2) = 4$. we have

$$(G_t^{(m)}, x^n y_m) \models \Diamond\Box\mathsf{False} \ \wedge \ \Diamond^{t(n,m)}\Box\mathsf{False} \ \wedge \ \neg\Diamond^{t(n,m)+1}\Box\mathsf{False}.$$

$$\begin{tikzcd}
\lambda \arrow[r] & x \arrow[r] & \cdots \arrow[r] & x^5 \arrow[r] & x^5y_2 \arrow[r] \arrow[d] & x^5y_2y_2 \arrow[r] \arrow[d] & x^5y_2y_2y_2 \arrow[r] \arrow[d] & x^5y_2^4 \arrow[d] \\
& & & & x^5y_2z \arrow[d] & x^5y_2y_2z \arrow[d] & x^5y_2y_2y_2z \arrow[d] & x^5y_2^4z \arrow[d] \\
& & & & x^5y_2z^2 \arrow[d] & x^5y_2y_2z^2 \arrow[d] & x^5y_2y_2y_2z^2 \arrow[d] & x^5y_2^4z^2 \arrow[d] \\
& & & & x^5y_2z^3 & x^5y_2y_2z^3 & x^5y_2y_2y_2z^3 & x^5y_2^4z^3
\end{tikzcd}$$

Figure 4. Part of G_t with $n = 5$, $m = 2$, and $t(5, 2) = 4$.

This is the key point for our coding. Taking a third derivative leaves only the top row. Then the next four derivatives remove in turn $x^5y_2^4, \ldots, x^5y_2$. One can check that again for $n = 5$, $m = 2$, and $t(5, 2) = 4$, if $r \neq m$,

$$(G_t^{(r)}, x^n y_m) \models \neg(\Diamond\Box\mathsf{False} \wedge \Diamond^{t(n,m)}\Box\mathsf{False}).$$

As we shall see, this holds for all n and m, using the assumption that $t(n, m) \geq 2$.

The sentences χ_d and $\varphi_{\mathcal{D}}$. Recall that $D = \{2, \ldots, K\}$. For $d \in D$, let

$$\chi_d \quad = \quad \Diamond\Box\mathsf{False} \ \wedge \ \Diamond^d\Box\mathsf{False} \ \wedge \ \neg\Diamond^{d+1}\Box\mathsf{False}.$$

We again take $\mathsf{stalk} = [D^*]\Diamond\mathsf{True}$. Observe that these sentences are defined independently of the intended models. Then we construct $\varphi_{\mathcal{D}}$ from these sentences exactly as in Figure 3. It remains to show that $(G_t, \lambda) \models \varphi_{\mathcal{D}}$.

Derivatives. Recall that G_t is

$$\{x^n : n \geq 0\} \ \cup \ \{x^n y_m^q z^p : 0 \leq n, 0 \leq m, 1 \leq q \leq t(n, m), 0 \leq p \leq m + 1\}.$$

By induction on $r \geq 0$ we see that $G_t^{(r)}$ is

$$\begin{aligned} \{x^n : n \geq 0\} \cup \{x^n y_m^q z^{p-r} : 0 \leq n, 0 \leq m, 1 \leq q \leq t(n, m), r \leq p \leq m+1\} \\ \cup \ \{x^n y_m^q : 0 \leq n, 0 \leq m, 1 \leq q \leq t(n, m) + m + 1 - r, m + 1 < r\}. \end{aligned}$$

It follows that $u \models \mathsf{stalk}$ iff u is of the form x^n for some n. This immediately gives the first clause of $\varphi_{\mathcal{D}}$. For all n and m, $G_t^{(m)}$ contains a submodel

$$\begin{tikzcd}
x^n \arrow[r] & x^n y_m \arrow[r] \arrow[d] & \cdots \arrow[r] & x^n y_m^{t(n,m)} \arrow[d] \\
& x^n y_m z & & x^n y_m^{t(n,m)} y
\end{tikzcd}$$

Except for x^n, none of the points above have any other neighbors besides the ones shown. As a result, $(G_t^{(m)}, x^n y_m) \models \chi_{t(n,m)}$. This for all n and m shows that (G_t, λ) satisfies the condition $\Box^*[D^*](\mathsf{stalk} \to \Diamond \bigvee_d \chi_d)$.

For the same n and m, the only d such that $(G_t^{(m)}, x^n y_m) \models \chi_d$ is $t(n,m)$. This follows easily from the definition of χ_d.

We claim in addition that if $k \neq m$, then for $(G_t^{(m)}, x^n y_k)$ satisfies no sentence χ_d. (Actually, $x^n y_k$ only belongs to $G_t^{(m)}$ when $m \leq t(n,k) + k$.) When $k > m$, $(G_t^{(m)}, x^n y_k) \models \neg\Diamond\Box\mathsf{False}$. And when $k < m \leq t(n,k) + k$, the relevant submodel of $G_t^{(m)}$ is

$$x^n \longrightarrow x^n y_k \longrightarrow \cdots \longrightarrow x^n y_k^{t(n,k)+k+1-m}$$

The only way to have $(G_t^{(m)}, x^n y_k) \models \Diamond\Box\mathsf{False}$ is if $t(n,k) + k + 1 - m = 1$. Then $(G_t^{(m)}, x^n y_k) \models \neg\Diamond^2\mathsf{True}$. Hence for all $d \geq 2$, $(G_t^{(m)}, x^n y_k) \models \neg\chi_d$.

Now we see that the same equation as (vi) holds:

$$(G_t^{(m)}, x^n) \models \Diamond\chi_d \quad \text{iff} \quad d = t(n,m). \tag{vii}$$

We check the last clause of $\varphi_{\mathcal{D}}$ holds; the third clause is similar. The only points satisfying stalk in any derivative are the x^n points. Suppose toward a contradiction that d and d' are such that $\neg V(d,d')$ and yet $(G_t^{(m)}, x^n) \models \Diamond\chi_d \wedge \langle D\rangle\Diamond\chi_{d'}$. Then $(G_t^{(m+1)}, x^n) \models \Diamond\chi_{d'}$. So by (vii), $d = t(n,m)$ and $d' = t(n,m+1)$. But this contradicts the properness of the tiling t.

6. Two iterated modal derivatives, modal logic, but no $\Box^*$

In this section, we prove the following result:

THEOREM 6.1. *There are two modal sentences D_x and D_y such that to every recurring domino system $(\mathcal{D}, d_0)$, we can effectively associate a sentence $\varphi_{\mathcal{D},d_0}$ built from $[D_x^*]$, $[D_y^*]$, $\Box$, True, atomic sentences and the boolean connectives, such that the following are equivalent:*

1. *There is a proper tiling of Q by $(\mathcal{D}, d_0)$.*
2. *$\varphi_{\mathcal{D},d_0}$ is satisfiable.*

We again take atomic sentences d corresponding to the (finitely many) dominoes. We also take new atomic sentences north and east. From all these we form the sentences listed in Figure 5.

We define sentences D_x and D_y to be $\mathsf{x} \to \Diamond\mathsf{True}$ and $\mathsf{y} \to \Diamond\mathsf{True}$, respectively. For any model A, let $D_x(A) = A^{D_x}$ and $D_y(A) = A^{D_y}$. Intuitively, $D_x(A)$ is A after deleting the set of x-points of A which are endpoints.

x	$\mathsf{east} \wedge \neg\mathsf{north}$
y	$\mathsf{north} \wedge \neg\mathsf{east}$
D_x	$\mathsf{x} \to \Diamond\mathsf{True}$
D_y	$\mathsf{y} \to \Diamond\mathsf{True}$
square_1	$\mathsf{east} \wedge \mathsf{north} \wedge \Box(\mathsf{x} \vee \mathsf{y})$
square_2	$\Box(\mathsf{x} \to \langle D_x^* \rangle \Box\mathsf{False}) \wedge \Box(\mathsf{y} \to \langle D_y^* \rangle \Box\mathsf{False})$
χ_d	$\mathsf{d} \wedge \Diamond(\mathsf{x} \wedge \Box\mathsf{False}) \wedge \Diamond(\mathsf{y} \wedge \Box\mathsf{False})$
tiling_1	$\neg\mathsf{east} \wedge \neg\mathsf{north} \wedge \Box(\mathsf{square}_1 \wedge \mathsf{square}_2)$
tiling_2	$[D_x^*][D_y^*] \bigvee_d \Diamond\chi_d$
proper_1	$[D_x^*][D_y^*]\neg \bigvee_{\neg H(d,d')}(\Diamond\chi_d \wedge \langle D_x \rangle \Diamond\chi_{d'})$
proper_2	$[D_x^*][D_y^*]\neg \bigvee_{\neg V(d,d')}(\Diamond\chi_d \wedge \langle D_y \rangle \Diamond\chi_{d'})$
$\mathsf{recurring}(d_0)$	$[D_x^*]\langle D_x^* \rangle \Diamond\chi_{d_0}$
$\varphi_{\mathcal{D},d_0}$	$\mathsf{tiling}_1 \wedge \mathsf{tiling}_2 \wedge \mathsf{proper}_1 \wedge \mathsf{proper}_2 \wedge \mathsf{recurring}(d_0)$

Figure 5. Sentences used in the fragment with $[D_x^*]$, $[D_y^*]$ and modal logic.

Intended models for the squares in Q. Our intended model for the square (i,j) is $W_{i,j}$ as shown below:

$$-(i+1) \longleftarrow \cdots \longleftarrow -1 \longleftarrow 0 \longrightarrow 1 \longrightarrow \cdots \longrightarrow j+1$$

with $0 \models \mathsf{north} \wedge \mathsf{east}$; $1, \ldots, j+1 \models \mathsf{y}$; and $-1, \ldots, -i, -(i+1) \models \mathsf{x}$.

Observation. $\langle W_{i,j}, 0 \rangle \models \mathsf{square}_1 \wedge \mathsf{square}_2$. This is trivial for square_1. Note that $D_x(W_{i+1,j}) = W_{i,j}$ and $D_y(W_{i,j+1}) = W_{i,j}$. These imply that

$$\langle W_{i,j}, 0 \rangle \models \Box(\mathsf{x} \to \langle D_x \rangle^i \Box\mathsf{False}) \wedge \Box(\mathsf{y} \to \langle D_y \rangle^j \Box\mathsf{False}).$$

This implies square_2.

Intended models for the tilings. Let $t : Q \to D$. We encode t as the model $T = T(t)$ whose set of worlds is

$$\{*\} + \sum_{N \times N} W_{i,j}.$$

That is, disjoint copies of all of the models $W_{i,j}$ from above together with a new point $*$. We write $0^{i,j}$ for the 0 of $W_{i,j}$ The accessibility relation has $* \to 0^{i,j}$ for all i,j. The copies $W_{i,j}$ are just as before. The atomic sentences are the same as before except now we must take care of the sentences corresponding to the dominoes. We specify that $0^{i,j} \models \mathsf{d}$ iff $t(i,j) = d$. The top point $*$ satisfies nothing.

The intended models satisfy $\varphi_{\mathcal{D},d_0}$. We check that for all d_0-recurring tilings t, $(T(t),*) \models \varphi_{\mathcal{D},d_0}$. The main point is that

$$D_x^n(D_y^m(T(t),*)) \quad = \quad (T(\lambda rs.t(r+n,s+m)),*).$$

This implies all of our properties.

Any model of $\varphi_{\mathcal{D},d_0}$ gives a proper recurrent tiling.

LEMMA 6.2. *Suppose that $(W,w) \models \langle D_x^* \rangle \Box \mathsf{False}$. Then every point u of W reachable from w via a path of length ≥ 1 satisfies x.*

PROOF. Suppose not. Let u be of minimal distance from w such that for some v, $u \models \mathsf{x}$, $u \to v$, and $v \models \neg\mathsf{x}$. We have a sequence $w = w_0 \to w_1 \to \cdots \to w_{n+1} = u$. By minimality, $w_1, \ldots, w_{n-1}$ all satisfy x. But D_x maintains $w_1, \ldots, w_{n+1} = u$, and v. Indeed, for all n, D_x^n maintains all of these. *A fortiori*, $(W,w) \models [D_x^*]\Diamond\mathsf{True}$. This is a contradiction. ■

LEMMA 6.3. *Let $\varphi_{\mathcal{D},d_0}$ be as in Figure 5. Let $(A,a_0) \models \varphi_{\mathcal{D},d_0}$. Then*

1. *There is a function $t : Q \to \mathcal{D}$ with the property that for $(n,m) \in Q$, $(A,a_0) \models \langle D_x \rangle^n \langle D_y \rangle^m \Diamond \chi_{t(n,m)}$.*
2. *Each such function t is a* proper *tiling of Q by $\mathcal{D}$.*
3. *There is some t such that d_0 occurs infinitely often on the x-axis.*

PROOF. We may assume that each point of A is reachable from a_0. By tiling_1, a_0 does not satisfy x or y. Therefore $a_0 \in D_x^n(D_y^m(A))$, for all n and m. Now the existence of t is immediate from tiling_2. To check that t is proper, consider $t(n,m)$ and $t(n+1,m)$. As we know,

$$\begin{array}{l} (A,a_0) \models \langle D_x \rangle^n \langle D_y \rangle^m \Diamond \chi_{t(n,m)}, \text{ and} \\ (A,a_0) \models \langle D_x \rangle^{n+1} \langle D_y \rangle^m \Diamond \chi_{t(n+1,m)}. \end{array} \tag{viii}$$

The heart of the matter is that the two derivatives commute on A. To prove this, we explicitly determine $D_x^p D_y^q(A,a_0)$ and $D_y^q D_x^p(A,a_0)$. Note that A consists of a_0, (satisfying $\neg\mathsf{east} \wedge \neg\mathsf{north}$), its children (all satisfying $\mathsf{east} \wedge \mathsf{north}$), and the descendants of its children. All of them satisfy $\mathsf{x} \vee \mathsf{y}$. By Lemma 6.2 (or the version of it with y replacing x throughout), if some point $b \in A$ satisfies x (or y) then so do all the children of b. It follows from this that A is partitioned into three sets:

$$\{a_0\} \ \cup \ \{b \in A : a_0 \to b\} \ \cup \ \{b \in A : b \models \mathsf{x}\} \ \cup \ \{b \in A : b \models \mathsf{y}\}.$$

Again, the points in the last two groups have all their children in the same group. It follows that

$$\begin{aligned} D_x^p D_y^q(A, a_0) &= \{a_0\} \cup \{b \in A : a_0 \to b\} \\ &\cup \{b \in A : b \models \mathsf{x} \wedge \Diamond^p \mathsf{True}\} \\ &\cup \{b \in A : b \models \mathsf{y} \wedge \Diamond^q \mathsf{True}\}. \end{aligned}$$

And this set is also exactly $D_y^q D_x^p(A, a_0)$. We apply this to the second equation of (viii). First, we take $p = n+1$ and $q = m$. Then we take $p = n$ and $q = m$, and read the equation backwards. The upshot is that

$$(A, a_0) \models \langle D_x \rangle^n \langle D_y \rangle^m \langle D_x \rangle \Diamond \chi_{t(n+1,m)}.$$

Recall that $(A, a_0) \models \mathsf{proper}_1$. This implies $H(t(n, m), t(n + 1, m))$.

Similar work shows $V(t(n, m), t(n, m + 1))$ for all n and m. Indeed, this work is simpler because one does not have to know that the derivative operations commute.

Finally, the sentence $\mathsf{recurring}(d_0)$ implies that there is some tiling t such that for infinitely many n, $t(n, 0) = d_0$. ■

7. $[D^*]$, modal relativization, and $\Box$; but no $\Box^*$

We next get undecidability for the fragment with $[D^*]$, modal announcements, and the usual modal apparatus. Crucially, the fragment does not include $\Box^*$.

THEOREM 7.1. *There are fixed model sentences $\varphi_1, \ldots, \varphi_3$ such that we can effectively associate to every recurring domino system $(\mathcal{D}, d_0)$ a sentence $\varphi_{\mathcal{D},d_0}$ of the language of $[D^*]$, $[\varphi_1], \ldots, [\varphi_3]$, $\Box$, True, atomic sentences and the boolean connectives such that the following are equivalent:*

1. *There is a proper tiling of Q by $(\mathcal{D}, d_0)$.*
2. *$\varphi_{\mathcal{D},d_0}$ is satisfiable.*

7.1. Frames and models

Fix a recurring domino system $(\mathcal{D}, d_0)$. We take a language with atomic sentences corresponding to the (finitely many) dominoes. Concretely, let d correspond to d. We also take new symbols root, column, a, b, red, blue, and yellow. The role of red, blue, and yellow will be to fix the order of the columns, which would otherwise be lost when trying to interpret the encoding tiling. We will require that red columns are followed by blue, blue by yellow, and yellow by red.

The intended frame for the first quadrant. This time, we construct a frame F by taking the set of symbols $\{x_0, \ldots, x_i, \ldots, y, z_0, \ldots, z_m, \ldots\}$ and from these the set of worlds

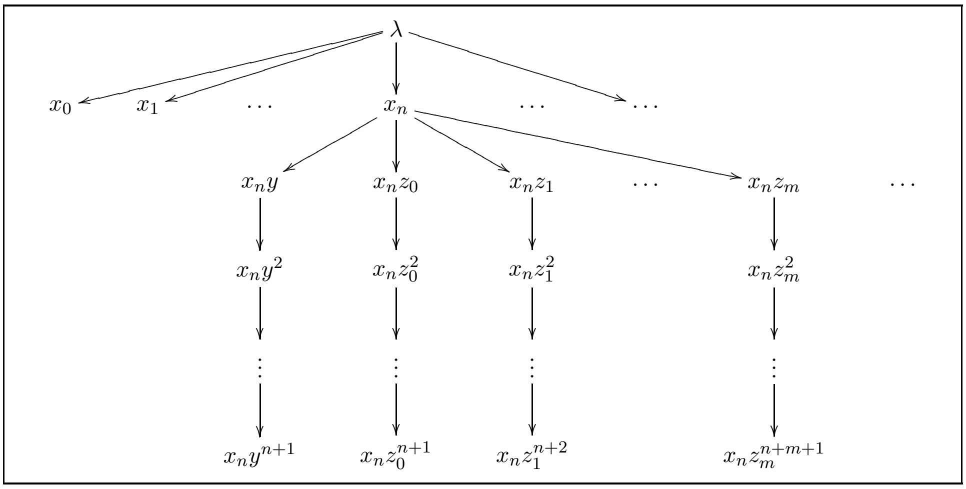

Figure 6. The intended model for the fragment with $[D^*]$ and modal relativizations.

$$\begin{aligned}\{\lambda\} \ \cup \ \{x_i : 0 \leq i\} \ &\cup \ \{x_iy^p : 0 \leq i, 1 \leq p \leq i+1\} \\ &\cup \ \{x_iz_m^p : 0 \leq i, 0 \leq m, 1 \leq p \leq i+m+1\}.\end{aligned}$$

The accessibility relation is given by $\lambda \to x_i$, $x_iy^p \to x_iy^{p+1}$, and $x_nz_m^p \to x_nz_m^{p+1}$. The picture is shown in Figure 6.

Now fix a proper recurring tiling t of Q by $\mathcal{D}$. We get a model F_t as follows:

1. $\lambda \models \mathsf{root}$.
2. $x_n \models \mathsf{column}$.
3. $x_n \models \mathsf{red}$ iff $n \equiv 0 \pmod 3$,
   $x_n \models \mathsf{blue}$ iff $n \equiv 1 \pmod 3$, and
   $x_n \models \mathsf{yellow}$ iff $n \equiv 2 \pmod 3$.
4. $x_ny^p \models \mathsf{a}$ for $1 \leq p \leq n+1$.
5. $x_nz_m^p \models \mathsf{b}$, for $1 \leq p \leq n+m+1$.
6. $x_nz_m \models t(n,m)$.

The sentence $\varphi_{\mathcal{D}}, d_0$. We consider the sentences in Figure 7.

The intended models work. First, note that $\mathsf{structure}$ is true at all worlds of F. Thus relativizing by it does no work. By induction on $n \geq 0$, $F_t^{(n)}$ is

$$\begin{aligned}\{\lambda\} \ \cup \ \{x_i : 0 \leq i\} \ &\cup \ \{x_iy^p : 0 \leq i, 1 \leq p \leq i+1-n\} \\ &\cup \ \{x_iz_j^p : 0 \leq i, 1 \leq p \leq i+j+1-n\}.\end{aligned}$$

structure	"exactly one of $\{\mathsf{root}, \mathsf{column}, \mathsf{a}, \mathsf{b}\}$ holds" $\wedge(\mathsf{root} \to \Box\mathsf{column}) \wedge (\mathsf{column} \to \Box(\mathsf{a} \vee \mathsf{b})) \wedge (\mathsf{a} \to \Box\mathsf{a}) \wedge (\mathsf{b} \to \Box\mathsf{b})$ $\wedge(\mathsf{column} \to$ "exactly one of $\{\mathsf{red}, \mathsf{blue}, \mathsf{yellow}\}$")
activecol	$\mathsf{column} \to (\Diamond\mathsf{a} \wedge \neg\Diamond\Diamond\mathsf{a})$
nextcol	$\mathsf{column} \to (\Diamond\Diamond\mathsf{a} \wedge \neg\Diamond\Diamond\Diamond\mathsf{a})$
twocols	$\mathsf{activecol} \vee \mathsf{nextcol}$
χ_d	$\mathsf{b} \wedge \mathsf{d} \wedge \Box\mathsf{False}$
α	$\langle\mathsf{structure}\rangle[D^*](D \wedge \langle\mathsf{activecol}\rangle[D^*] \bigvee_d \Diamond\Diamond\chi_d)$
β	$[\mathsf{structure}][D^*][\mathsf{activecol}][D^*]\neg \bigvee_{\neg V(d,d')}(\Diamond\Diamond\chi_d \wedge \langle D\rangle\Diamond\Diamond\chi_{d'})$
γ_1	$[\mathsf{structure}]\Box(\mathsf{activecol} \to \mathsf{red})$
γ_2	$[\mathsf{structure}][D^*]$ $(\Diamond(\mathsf{activecol} \wedge \mathsf{red}) \to \Box(\mathsf{nextcol} \to \mathsf{blue}))$ $\wedge(\Diamond(\mathsf{activecol} \wedge \mathsf{blue}) \to \Box(\mathsf{nextcol} \to \mathsf{yellow}))$ $\wedge(\Diamond(\mathsf{activecol} \wedge \mathsf{yellow}) \to \Box(\mathsf{nextcol} \to \mathsf{red})))$
δ	$[\mathsf{structure}][D^*][\mathsf{twocols}][D^*]\neg \bigvee_{\neg H(d,d')}$ $(\Diamond(\mathsf{red} \wedge \Diamond\chi_d) \wedge \langle D\rangle\Diamond(\mathsf{blue} \wedge \Diamond\chi_{d'}))$ $\vee(\Diamond(\mathsf{blue} \wedge \Diamond\chi_d) \wedge \langle D\rangle\Diamond(\mathsf{yellow} \wedge \Diamond\chi_{d'}))$ $\vee(\Diamond(\mathsf{yellow} \wedge \Diamond\chi_d) \wedge \langle D\rangle\Diamond(\mathsf{red} \wedge \Diamond\chi_{d'}))$
recurring(d_0)	$[\mathsf{structure}]\langle\mathsf{activecol}\rangle[D^*]\langle D^*\rangle\Diamond\Diamond\chi_{d_0}$
$\varphi_{\mathcal{D},d_0}$	$\mathsf{root} \wedge \alpha \wedge \beta \wedge \gamma_1 \wedge \gamma_2 \wedge \delta \wedge \mathsf{recurring}(d_0)$

Figure 7. Sentences used in the fragment with $[D^*]$ and modal relativizations.

In $F_t^{(n)}$, the only point satisfying $\mathsf{column} \wedge \Diamond\mathsf{a} \wedge \neg\Diamond\Diamond\mathsf{a}$ is x_n. It follows that the part of $F_t^{(n)(\mathsf{activecol})}$ reachable from λ is

$$\{\lambda, x_n, x_n y\} \ \cup \ \{x_n z_j^p : 0 \leq j, 1 \leq p \leq j+1\}. \tag{ix}$$

For all $n \geq 0$ and $m \geq 1$, the part of $F_t^{(n)(\mathsf{activecol})(m)}$ accessible from λ is

$$\{\lambda, x_n\} \ \cup \ \{x_n z_j^p : m \leq j, 1 \leq p \leq j+1-m\}.$$

The only point of $F_t^{(n)(\mathsf{activecol})(m)}$ which satisfies any χ_d sentence is $x_n z_m$. This easily implies that $(F_t^{(n)}, \lambda) \models \alpha$. Appealing to the properness of the tiling t, we see also that $(F_t^{(n)}, \lambda) \models \beta$.

Moreover, the part of $F_t^{(n)(\mathsf{nextcol})}$ reachable from λ is

$$\{\lambda, x_{n+1}, x_{n+1}y, x_{n+1}y^2\} \ \cup \ \{x_{n+1} z_j^p : 0 \leq j, 1 \leq p \leq j+2\}. \tag{x}$$

From (ix) and (x) and the definition of F_t, we easily see that $(F_t^{(n)}, \lambda) \models \gamma_1 \wedge \gamma_2$. Continuing our discussion, we see that for all m, the part of $F_t^{(n)(\mathsf{twocols})(m)}$ reachable from λ is

$$\begin{aligned} \{\lambda, x_n, x_{n+1}\} \ &\cup \ \{x_n z_j^p : m \leq j, 1 \leq p \leq j+1-m\} \\ &\cup \ \{x_{n+1} z_j^p : m+1 \leq j, 1 \leq p \leq j+2-m\}. \end{aligned} \tag{xi}$$

(For $0 \leq m \leq 2$ we also have $x_n y$, $x_{n+1} y$ and $x_{n+1} y^2$. But these are not relevant to our discussion, and we shall ignore them.)

We now come to the most critical part of the verification, the part about δ. We may assume, without loss of generality, that $x_n \models \mathsf{red}$. Therefore $x_{n+1} \models \mathsf{blue}$. Also note that $(F_t^{(n)(\mathsf{twocols})(m+1)}, x_{n+1}) \models \Diamond\chi_{t(n+1,m)}$, so that $(F_t^{(n)(\mathsf{twocols})(m)}, \lambda) \models \langle D\rangle\Diamond\Diamond\chi_{t(n+1,m)}$. Moreover, the only sentence of one of the forms listed in the statement of δ which is satisfied by $(F_t^{(n)(\mathsf{twocols})(m)}, \lambda)$ is

$$\Diamond(\mathsf{red} \wedge \Diamond\chi_{t(n,m)}) \wedge \langle D\rangle\Diamond(\mathsf{blue} \wedge \Diamond\chi_{t(n+1,m)}).$$

And for this sentence, we do have $H(t(n,m), t(n+1,m))$. This concludes the verification of δ.

At this point, we can explain the need for red, blue and yellow. Suppose we drop the colors from the statement of δ. Then $(F_t^{(n)(\mathsf{twocols})(m)}, \lambda)$ might satisfy a sentence of the form

$$\Diamond\Diamond\chi_{t(n,m)} \wedge \langle D\rangle\Diamond\Diamond\chi_{t(n+1,m)}.$$

where $H(t(n,m), t(n+1,m))$ is false. The problem is that in (xi), we have no way to know which nodes code squares in the n-th column of the desired model, and which code squares in the $(n+1)$-st column. That is, once the derivatives have eliminated the a-points, we have no way to tell right from left in $F_t^{(n)(\mathsf{twocols})(m)}$.

Returning to the final point concerning the intended models, the fact that $t(n,0) = d_0$ for infinitely many n implies that $(F_t^{(n)}, \lambda) \models \mathsf{recurring}(d_0)$.

Any model of $\varphi_{\mathcal{D},d_0}$ gives a proper tiling. Let $(A, w) \models \varphi_{\mathcal{D},d_0}$. We may assume that every element of A is reachable from w in finitely many transitions along $\to$. Indeed, throughout this proof, in *all* models, we assume that every point is reachable from w.

The sentence $\mathsf{structure}$ is *universal*; that is, it may be written in terms of atomic sentences and their negations using $\wedge$, $\vee$, and $\Box$ (but not $\Diamond$). Any such universal sentence ρ has the property that $[\rho]\Box^*\rho$ is valid; that is, for all (X, x), if $(X, x) \models \rho$, then after relativizing with ρ, we see that ρ holds at all points reachable from x. Therefore, $(A, w)^{\mathsf{structure}} \models \Box^*\mathsf{structure}$. Of course, this point could be verified directly without using the more general fact concerning positivity.

At this point, we know something about the structure of $(A, w)^{\mathsf{structure}}$. First, $w \models \mathsf{root}$, and w is the only point with this property. The children of w satisfy column and they are the only points to do so. They also satisfy some color sentence. The rest of the model consists of a-points and b-points; the children of each of these types is again of the same type.

For the rest of this argument, we save on notation by replacing (A, w) by $(A, w)^{\mathsf{structure}}$; thus we assume that our remarks in the previous paragraph apply to the original (A, w).

As a consequence of α, for each n, $(A^{(n)}, w) \models \langle\mathsf{activecol}\rangle\Diamond\mathsf{True}$. So $(A^{(n)}, w) \models \Diamond\mathsf{activecol}$ also. Let

$$\begin{array}{lcl} C_n & = & \{x : w \to x, (A^{(n)}, x) \models \mathsf{activecol}\}, \\ N_n & = & \{x : w \to x, (A^{(n)}, x) \models \mathsf{nextcol}\}. \end{array}$$

Each C_n is nonempty. Note that the following are equivalent:

1. $(A^{(n)}, x) \models \mathsf{nextcol} = (\mathsf{column} \to (\Diamond\Diamond\mathsf{a} \wedge \neg\Diamond\Diamond\Diamond\mathsf{a}))$.
2. $(A^{(n+1)}, x) \models \mathsf{activecol} = (\mathsf{column} \to (\Diamond\mathsf{a} \wedge \neg\Diamond\Diamond\mathsf{a}))$.

In other words, $N_n = C_{n+1}$. By γ_1, for each $x \in C_0$ we have $x \models \mathsf{red}$. By an induction using γ_2, we see that each $x \in C_n$ satisfies the same color and that the colors cycle through red, blue, yellow, red, ..., as desired.

Now we define a tiling t from (A, w). We know from α that

$$(A^{(n)(\mathsf{activecol})(m)}, w) \models \Diamond\Diamond\chi_d,$$

for some domino d. We choose one such d and define $t(n, m) = d$. The main point of the construction is to make sure that t is a proper tiling. The fact that $V(t(n, m), t(n, m + 1))$, for all n and m, comes from β. The hard work comes in checking that for all n and m, $H(t(n, m), t(n + 1, m))$.

$A^{(n+1)(\mathsf{activecol})}$ consists of w, C_{n+1}, the a-children in $A^{(n+1)}$ of the elements of C_{n+1} (these last are end nodes of $A^{(n+1)}$, since $\Diamond\mathsf{a} \wedge \neg\Diamond\Diamond\mathsf{a}$ holds on C_{n+1}), and all of the b-descendants in $A^{(n+1)}$ of the elements of C_{n+1} (these are exactly the b-nodes in the original A which are descendants of some element of C_{n+1} and which satisfy $\Diamond^{n+1}\mathsf{True}$).

$A^{(n)(\mathsf{nextcol})}$ consists of w, the set N_n, the a-descendants in $A^{(n)}$ of the elements of N_n (but recall that elements of N_n all satisfy $\Diamond^2\mathsf{a} \wedge \neg\Diamond^3\mathsf{a}$ in $A^{(n)}$), and all of the b-descendants in $A^{(n)}$ of the elements of N_n (these are exactly the b-nodes in the original A which are descendants of some element of N_n and which satisfy $\Diamond^n\mathsf{True}$).

It follows from these observations and from the fact that $N_n = C_{n+1}$ that

$$A^{(n)(\mathsf{nextcol})(1)} \quad = \quad A^{(n+1)(\mathsf{activecol})}.$$

So by induction on m,

$$A^{(n+1)(\mathsf{activecol})(m)} \quad = \quad A^{(n)(\mathsf{nextcol})(m+1)} \quad \subseteq \quad A^{(n)(\mathsf{twocols})(m+1)}. \qquad \text{(xii)}$$

To conclude, we fix n and m and check that $H(t(n,m), t(n+1,m))$. Let $x \in C_n$, $y \in A^{(n)(\mathsf{activecol})(m)}$, $u \in C_{n+1}$, and $v \in A^{(n+1)(\mathsf{activecol})(m)}$ be such that

1. $w \to x \to y$.
2. $(A^{(n)(\mathsf{activecol})(m)}, y) \models \chi_{t(n,m)}$.
3. $w \to u \to v$.
4. $(A^{(n+1)(\mathsf{activecol})(m)}, v) \models \chi_{t(n+1,m)}$.

The last point here tells us that $(A^{(n)(\mathsf{nextcol})(m)}, v) \models \langle D \rangle \chi_{t(n+1,m)}$. Without loss of generality, take $x \models \mathsf{red}$. So $u \models \mathsf{blue}$. By (xii) and the points above,

$$(A^{(n)(\mathsf{twocols})(m)}, w) \models \Diamond(\mathsf{red} \wedge \Diamond\chi_{t(n,m)}) \wedge \langle D \rangle \Diamond(\mathsf{blue} \wedge \Diamond\chi_{t(n+1,m)}).$$

We see from δ that $H(t(n,m), t(n+1,m))$, as desired.

The recurrence condition is easy to check.

This concludes the proof of Theorem 7.1.

8. Undecidability of satisfiability on finite (tree) models

In this section, we show that the problem of determining whether a sentence ψ of our language $\mathcal{L}(\mathsf{rel}, \mathsf{rel}^*, \Box^*)$ has a *finite* model is Σ^0_1-complete. The same work shows that the problem of determining whether ψ has a finite *tree* model is also Σ^0_1-complete. Note that the relation $A \models \varphi$ is decidable for sentences $\varphi \in \mathcal{L}(\mathsf{rel}, \mathsf{rel}^*, \Box^*)$ and finite models A. Therefore, the set of φ which have a finite model is Σ^0_1. The proof that this problem is Σ^0_1-hard goes by reduction from the problem of deciding whether a domino system has a *periodic* tiling of the first quadrant.

Definition. A *rectangle* is a subset of the first quadrant of the form

$$R \quad = \quad \{0, \ldots, r\} \times \{0, \ldots s\}. \qquad \text{(xiii)}$$

Let $\mathcal{D} = (Dominoes, H, V)$, be a domino system. A *repeatable rectangle* (for $\mathcal{D}$) is a pair (R, t), where R is a rectangle, and $t : R \to Dominoes$ satisfies the following conditions:

1. $H(t(n,m), t(n+1,m))$ for $0 \le n < r$ and $0 \le m \le s$.

2. $V(t(n,m), t(n,m+1))$ for $0 \leq n \leq r$ and $0 \leq m < s$.
3. $H(t(r,m), t(0,m))$ for $0 \leq m \leq s$.
4. $V(t(n,s), t(n,0))$ for $0 \leq n \leq r$.

A repeatable rectangle is just a witness to the existence of a periodic tiling of the plane or first quadrant.

PROPOSITION 8.1. *The question of whether a domino system has a repeatable rectangle is Σ^0_1-complete.*

This result is originally due to Berger [4]. It appears as Theorem 3.1.7 of [6] with a proof in Cyril Allauzen and Bruno Durand's appendix of [6]. Another reference on this matter is Lin [16]. (Incidentally, Lin's paper is in English but seems not to be known to later workers on tiling.)

THEOREM 8.2. *For every domino system $\mathcal{D}$ we can effectively find a sentence $\varphi_{\mathcal{D}}$ of $\mathcal{L}(\mathsf{rel}, \mathsf{rel}^*, \Box^*)$ such that the following are equivalent:*

1. $\mathcal{D}$ has a repeatable rectangle.

2. $\varphi_{\mathcal{D}}$ is satisfied on a finite tree.

3. $\varphi_{\mathcal{D}}$ is satisfied on some (finite or infinite) model.

Moreover, such a $\varphi_{\mathcal{D}}$ can be found in the fragment of $\mathcal{L}(\mathsf{rel}, \mathsf{rel}^*, \Box^*)$ considered in each of Sections 4, 6 and 7.

We shall prove this result in this section for just one of our fragments, the one of Section 4. We shall not attempt to work without atomic sentences, and indeed this time around we need more atomic sentences than before. The intended models are basically the obvious finite versions of the stalk models which we have seen. But the sentences that encode the models are substantially more complicated.

Incidentally, when one changes to a different fragment, many details in the proof of Theorem 8.2 change. We believe that it would be possible to encode an undecidable problem into all of our fragments in such a way as to make it easier to go in a "fragment-independent" way from the Σ^1_1-completeness results on general satisfiability to the Σ^0_1-completeness results for finite satisfiability. However, to do this, one would need to encode a new tiling problem created just for this purpose. We opt for quoting a known tiling problem (in Proposition 8.1), and so we only give the details of the finite satisfiability result in one fragment.

The intended finite frames corresponding to rectangles. We use the notation from Section 4. Recall that associated to the first quadrant Q we have a frame F. For the rectangle R as in (xiii), we let F_R be the subframe of F determined by

$$\{x^n : 0 \leq n \leq r\} \ \cup \ \{x^n y_m z^p : 0 \leq n \leq r; 0 \leq m \leq s; \text{ and } 0 \leq p \leq m\}.$$

These are just the points of the original frame that figure into the coding of the points in R. We keep the accessibility relation $\rightarrow$ exactly as before.

Repeatable rectangles give models. Let (R, t) be a repeatable rectangle, so $t : R \rightarrow \mathit{Dominoes}$. As in our earlier work, we take atomic sentences d for $d \in \mathit{Dominoes}$. This time we take stalk to be an atomic sentence, not an abbreviation. We also need atomic sentences hmax and vmax that are true of points coding squares that are "rightmost" and "uppermost". The purposes of these are perhaps best gleaned from the intended models.

We construct a model $F_{(R,t)}$ from t (and the underlying frame F_R described above) by declaring

$$\begin{array}{rl} x^n & \models \mathsf{stalk}, \\ \text{if } t(n,m) = d, \text{ then } x^n y_m & \models \mathsf{d}, \\ x^r y_m & \models \mathsf{hmax}, \\ \text{and } x^n y_s & \models \mathsf{vmax}, \end{array}$$

for all $0 \leq n \leq r$ and $0 \leq m \leq s$. No other atomic sentences are true at any other points.

The sentence $\varphi_{\mathcal{D}}$. See Figure 8. We might note that there are natural sentences which are true in the intended models but which we do not take as conjuncts of $\varphi_{\mathcal{D}}$. Among these are $\mathsf{stalk} \leftrightarrow \neg \bigvee_d \mathsf{d}$ and $\mathsf{hmax} \vee \mathsf{vmax} \rightarrow \neg \mathsf{stalk}$. The reasons for not incorporating these into $\varphi_{\mathcal{D}}$ are: (a) the proof goes through without them; and (b), the argument would not be substantially shorter if we did add the extra clauses.

The intended models work. We verify some of the clauses of $\varphi_{\mathcal{D}}$. As in our earlier work, we first check that

$$(F^{(m)}_{R,t}, x^n) \models \Diamond \chi_d \quad \text{iff} \quad d = t(n,m). \tag{xiv}$$

These are the only points that satisfy $\Diamond \chi_d$. Moreover, the only points of $F^{(m)}_{R,t}$ satisfying active are those of the form $x^n y_m$.

We remind the reader that we write λ for x^0. So with $m = 0$, we have $(F_{R,t}, \lambda) \models \Box^*(\mathsf{stalk} \rightarrow \Diamond \mathsf{active})$ via the points $x^n y_0$.

stalk	this is now an atomic sentence
structure	$\bigwedge_{d\neq d'}(\mathsf{d}\to\neg\mathsf{d'})\wedge(\Diamond^*\mathsf{hmax}\to\mathsf{stalk}\ \mathsf{xor}\ \mathsf{hmax})$
χ_d	$\mathsf{d}\wedge\Box\mathsf{False}$
active	$\bigvee_d\chi_d$
$\varphi_{\mathcal{D}}$	$\mathsf{stalk}\wedge\Box^*(\mathsf{stalk}\to\Diamond\mathsf{active})\wedge\Box^*\mathsf{structure}$
	$\wedge\Diamond^*\mathsf{hmax}$
	$\wedge\langle D^*\rangle\Diamond(\mathsf{active}\wedge\mathsf{vmax})$
	$\wedge[D^*](\Diamond^*(\mathsf{stalk}\wedge\Diamond(\mathsf{active}\wedge\mathsf{vmax}))\to\Box^*(\mathsf{stalk}\to\Box(\mathsf{active}\to\mathsf{vmax})))$
	$\wedge[D^*](\Diamond^*(\mathsf{stalk}\wedge\Diamond(\mathsf{active}\wedge\neg\mathsf{vmax}))\to\Box^*(\mathsf{stalk}\to\langle D\rangle\Diamond\mathsf{active}))$
	$\wedge\Box^*(\Diamond\mathsf{hmax}\to\Box\mathsf{hmax})$
	$\wedge\Box^*[D^*]\neg\bigvee_{\neg H(d,d')}(\Diamond\chi_d\wedge\Diamond\Diamond\chi_{d'})$
	$\wedge\Box^*[D^*]\neg\bigvee_{\neg V(d,d')}(\Diamond\chi_d\wedge\Diamond\langle D\rangle\chi_{d'})$
	$\wedge[D^*]\bigwedge_d(\Diamond\chi_d\to\Box^*(\mathsf{active}\wedge\mathsf{hmax}\to\bigvee_{d':H(d',d)}\mathsf{d'}))$
	$\wedge\Box^*\bigwedge_d(\Diamond\chi_d\to[D^*](\Diamond\mathsf{vmax}\to\Box(\mathsf{active}\to\bigvee_{d':V(d',d)}\mathsf{d'})))$

Figure 8. Sentences in the finite model result for the fragment $[D^*]$, $\Box^*$, $\Box$, and atomic sentences.

Next, we check all of the clauses of $\varphi_{\mathcal{D}}$ mentioning hmax. The points where hmax holds are those of the form $x^r y_m$. And the only path from λ to a point of this form is $\lambda\to x\to\cdots\to x^r\to x^r y_m$. This implies that $(A,\lambda)\models\Box^*(\Diamond^*\mathsf{hmax}\to\mathsf{stalk}\ \mathsf{xor}\ \mathsf{hmax})$.

Taking $n=0$ in (xiv), we see that for each m, $(F^{(m)}_{R,t},\lambda)\models\Diamond\chi_{t(0,m)}$. And the only point of $F^{(m)}_{R,t}$ satisfying $\mathsf{active}\wedge\mathsf{hmax}$ is $x^r y_m$. Let $d'=t(r,m)$. Then $x^r y_m\models\mathsf{d'}$. And by the assumption that R is a repeatable rectangle, we have $H(t(r,m),t(0,m))$. This discussion shows that $(F_{R,t},\lambda)$ satisfies the last sentence involving hmax.

Finally, we check the clauses mentioning vmax. We have $(F_{R,t},\lambda)\models\langle D\rangle^*\Diamond(\mathsf{active}\wedge\mathsf{vmax})$ because $\lambda\to x^0y_s$ and $(F^{(s)}_{R,t},x^0y_s)\models\mathsf{active}\wedge\mathsf{vmax}$. The two long conditions on vmax are actually easy to check in the intended models. The first says informally that as we take derivatives, if any stalk point has a child which is active and satisfies vmax, then all stalk points have such a child. The second says that if any stage a stalk point has a child which is active but does not satisfy vmax (so the stage is below s), then at this stage all stalk points have an active child in the *next* derivative. We omit the argument for the last vmax condition.

Any model of $\varphi_{\mathcal{D}}$ gives a repeatable rectangle. We are checking (3)$\Longrightarrow$(1) in Theorem 8.2. Let $\varphi_{\mathcal{D}}$ be as in Figure 8. Let (A,a_0) be an arbitrary model of $\varphi_{\mathcal{D}}$. We note that for all k, $(A^{(k)},a)\models\mathsf{structure}$.

LEMMA 8.3. *Then there are numbers r and s and points a_n and $b_{n,m}$ for $0\le n\le r$ and $0\le m\le s$ such that*

1. *a_0 is the given point that satisfies $\varphi_{\mathcal{D}}$ in A.*
2. $a_0 \to \cdots \to a_n \to \cdots \to a_r$.
3. $a_r \models \Diamond\mathsf{hmax}$.
4. $a_n \models \mathsf{stalk}$.
5. $a_n \to b_{n,m}$.
6. $(A^{(m)}, b_{n,m}) \models \mathsf{active}$.
7. $b_{n,s} \models \mathsf{vmax}$.
8. $b_{r,m} \models \mathsf{hmax}$.

Proof. Let r be least such that $(A, a_0) \models \Diamond^{r+1}\mathsf{hmax}$. From a_0 and r, we get the a-points so that parts (1)–(3) hold. We need to check in (4) that each $a_n \models \mathsf{stalk}$. Certainly $a_n \models \Diamond^*\mathsf{hmax}$. So by $\mathsf{structure}$, $a_n \models \mathsf{stalk}\ \mathsf{xor}\ \mathsf{hmax}$. By minimality of r, no a_n can satisfy hmax. Let s be least so that $(A, a_0) \models \langle D\rangle^s\Diamond(\mathsf{active} \wedge \mathsf{vmax})$. It is possible that $r = 0$ or $s = 0$.

Claim. For $0 \leq n \leq r$ and $0 \leq m \leq s$, $(A, a_n) \models \langle D\rangle^m\Diamond\mathsf{active}$. For $m < s$, $(A, a_n) \models \langle D\rangle^m\Box(\mathsf{active} \to \neg\mathsf{vmax})$.

Proof. By induction on m. For $m = 0$, one of the clauses of $\varphi_{\mathcal{D}}$ is $\Box^*(\mathsf{stalk} \to \Diamond\mathsf{active})$. We already know that $a_n \models \mathsf{stalk}$, and so $(A, a_n) \models \Diamond\mathsf{active}$. Suppose in addition that $0 < s$. We claim that for all n, $(A, a_n) \models \Box(\mathsf{active} \to \neg\mathsf{vmax})$. For suppose not. Then $(A, a_n) \models \Diamond(\mathsf{active} \wedge \mathsf{vmax})$. One of the clauses in $\varphi_{\mathcal{D}}$ is

$$[D^*](\Diamond^*(\mathsf{stalk} \wedge \Diamond(\mathsf{active} \wedge \mathsf{vmax})) \to \Box^*(\mathsf{stalk} \to \Box(\mathsf{active} \to \mathsf{vmax}))).$$

Also, $(A, a_n) \models \mathsf{stalk} \wedge \Diamond(\mathsf{active} \wedge \mathsf{vmax})$. So $(A, a_0) \models (\mathsf{stalk} \to \Box(\mathsf{active} \to \mathsf{vmax})$. This in turn implies that $(A, a_0) \models \Diamond(\mathsf{active} \wedge \mathsf{vmax})$. Looking back to the definition of s, we see that $s = 0$. This is a contradiction.

Now assume our claim for m. By this induction hypothesis, $(A^{(m)}, a_n) \models \Diamond(\mathsf{active} \wedge \neg\mathsf{vmax})$. As we know, $(A^{(m)}, a_n) \models \mathsf{stalk}$. Another clause in $\varphi_{\mathcal{D}}$ is

$$[D^*](\Diamond^*(\mathsf{stalk} \wedge \Diamond(\mathsf{active} \wedge \neg\mathsf{vmax})) \to \Box^*(\mathsf{stalk} \to \langle D\rangle\Diamond\mathsf{active})).$$

So we see that $(A^{(m)}, a_n) \models \langle D\rangle\Diamond\mathsf{active}$. That is, $(A, a_n) \models \langle D\rangle^{m+1}\Diamond\mathsf{active}$. And exactly as above, if $m + 1 < s$, then $(A, a_n) \models \langle D\rangle^{m+1}\Box(\mathsf{active} \to \neg\mathsf{vmax})$. ■

For $0 \leq n \leq r$ and $0 \leq m \leq s$, let $b_{n,m}$ be such that $a_n \to b_{n,m}$, $(A^{(m)}, b_{n,m}) \models \mathsf{active}$, and in addition with $(A^{(s)}, b_{n,s}) \models \mathsf{vmax}$. Most of the parts of our lemma are immediate. We verify in the last part that

$b_{r,m} \models \mathsf{hmax}$. For this, recall that $a_r \models \Diamond\mathsf{hmax}$. One of the clauses in $\varphi_{\mathcal{D}}$ is that $\Box^*(\Diamond\mathsf{hmax} \to \Box\mathsf{hmax})$. So $a_r \models \Box\mathsf{hmax}$. Since $a_r \to b_{r,m}$, we are done. ■

We continue with the proof of Theorem 8.2. Fix n, m and any points a_n and $b_{n,m}$ as in Lemma 8.3. Let R be the rectangle $\{(n,m) : 0 \leq n \leq r \text{ and } 0 \leq m \leq s\}$. Define t on R by:

$$t(n,m) \quad = \quad \text{the unique } d \text{ such that } (A^{(m)}, b_{n,m}) \models \mathsf{d}. \tag{xv}$$

So $(A^{(m)}, b_{n,m}) \models \chi_{t(n,m)}$.

LEMMA 8.4. *(R,t) is a repeatable rectangle for $\mathcal{D}$.*

PROOF. There are four conditions. The first two have to do with t working correctly "inside" R. The arguments here are the same as in Lemma 4.3, so we omit them. Instead we check the periodicity conditions, which we repeat below:

3. $H(t(r,m), t(0,m))$ for $0 \leq m \leq s$.
4. $V(t(n,s), t(n,0))$ for $0 \leq n \leq r$.

Here is an argument for (3). Consider $b_{0,m}$ and $b_{r,m}$. As we know, $(A^{(m)}, b_{0,m}) \models \chi_{t(0,m)}$, and $(A^{(m)}, b_{r,m}) \models \chi_{t(r,m)}$. Using $b_{0,m}$, we see that

$$(A^{(m)}, a_0) \models \Box^*(\mathsf{active} \wedge \mathsf{hmax} \to \bigvee_{d' : H(d', t(0,m))} \mathsf{d}').$$

As we know from parts (6) and (8) of Lemma 8.3, $(A^{(m)}, b_{r,m}) \models \mathsf{active} \wedge \mathsf{hmax}$. So there is some d' such that $H(d', t(0,m))$ and $b_{r,m} \models \mathsf{d}'$. By (xv), $d' = t(r,m)$. This means that $H(t(r,m), t(0,m))$, as desired.

Finally, we check the periodicity condition (4). Consider $b_{n,0}$ and $b_{n,s}$. This time we have $(A, a_n) \models \Diamond(\mathsf{active} \wedge \chi_{t(n,0)})$. By our last periodicity clause in $\chi_{\mathcal{D}}$, we have

$$(A^{(s)}, a_n) \models \Diamond\mathsf{vmax} \to \Box(\mathsf{active} \to \bigvee_{d' : V(d', t(n,0))} d').$$

Now $(A^{(s)}, a_n) \models \Diamond(\mathsf{vmax} \wedge \mathsf{active})$ via $b_{n,s}$. So there is some d' such that $V(d', d)$ and $b_{n,s} \models \mathsf{d}'$. Again by (xv), $d' = t(n,s)$. This proves $V(t(n,s), t(n,0))$. ■

This completes the proof of Theorem 8.2. And from the theorem and Proposition 8.1, we infer the Σ^0_1-completeness of the question of whether a sentence of $\mathcal{L}(\mathsf{rel}, \mathsf{rel}^*, \Box^*)$ has a finite (tree) model. *A fortiori*, the same holds for MIC.

Open problems. We only removed the atomic sentences from the fragment of Section 3. So it is open to re-work the remaining results without atomic sentences.

We conclude with another problem raised by our work, a problem which we find more interesting. We do not know whether the satisfiability problem for the fragment determined by $[D^*]$, $\Box$, $\wedge$, $\neg$, and True is decidable. If it were Σ^1_1-complete, then the result would subsume the parallel results for the fragments in this paper. And if it were decidable (with atomic sentences), then it would be "maximal" in the sense that adding any of the following features would destroy decidability: the transitive closure operation $\Box^*$, another iterated derivative, or relativization by modal sentences.

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