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Multiplicative relations in number fields

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In this paper, we obtain an explicit form of the currently best known inequality for linear forms in the logarithms of algebraic numbers. The results complete our previous investigations (*Bull. Austral. Math. Soc.* 15 (1976), 33-57) which were conditional on a certain independence condition on the algebraic numbers. The extra work needed to obtain unconditional results centres on the properties of multiplicative relations in number fields. In particular, we show that a set of multiplicatively dependent algebraic numbers always satisfies a relation with relatively small exponents.

1. Introduction

In a previous paper [8], we established certain inequalities satisfied by linear forms in the logarithms of algebraic numbers. These results, however, were conditional on the algebraic numbers satisfying a certain independence condition. In the present paper, we describe a method of eliminating that condition and of establishing unconditional results on the same lines as the theorems of [8]. Central to our proof is a new result on multiplicative relations in algebraic number fields. To state this, we introduce the following notation. If α is an algebraic number and its minimal defining polynomial is

$$(1) \qquad a_0x^d + a_1x^{d-1} + \ldots + a_d = a_0\left(x-\alpha^{(1)}\right) \ldots \left(x-\alpha^{(d)}\right) ,$$

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the a_j being relatively prime rational integers, we measure the "size" of α by the quantity

$$H(\alpha) = |a_0| \prod_{j=1}^{d} \max\{1, |\alpha^{(j)}|\} .$$

THEOREM 1. *Let* $\alpha_1, \ldots, \alpha_m$ *be multiplicatively dependent algebraic numbers in an algebraic number field* K *of degree* D *over* $\mathbb{Q}$ *and suppose that* $1 < H(\alpha_1) \le \ldots \le H(\alpha_m)$ *. Then there are a positive constant* $C_1 = C_1(m, D)$ *and integers* $t_1, \ldots, t_m$ *, not all zero, such that*

$$\alpha_1^{t_1} \ldots \alpha_m^{t_m} = 1 \quad \text{and} \quad \max_{1 \le j \le m} |t_j| \le C_1 \prod_{j=2}^{m} \log H(\alpha_j) .$$

We can take

$$C_1 = \left(\frac{3}{2} mD\right)^{m-1} (21D \log 6D)^{\min\{m, D+1\}} .$$

This is to say that, if $\alpha_1, \ldots, \alpha_m$ are known to be multiplicatively dependent, then they already satisfy a relation with relatively small exponents. Similar, though less sharp, results are given as consequences of the lengthy and deep principal arguments of Baker [*1*] and Stark [*11*]. In contrast, our result is proved by relatively elementary means. Our proof, in fact, generalises an argument of Stark [*11*, Lemma 7].

We now turn to the results on linear forms in logarithms. Throughout, $\alpha_1, \ldots, \alpha_n$ will denote n (≥ 2) non-zero algebraic numbers belonging to an algebraic number field K of degree D over $\mathbb{Q}$ and with heights respectively not exceeding $A_1, \ldots, A_n$ (with $\log \log A_j \ge 1$). We further suppose that $A_1 \le A_2 \le \ldots \le A_{n-1} = A' \le A_n = A$ and we set

$$\Omega' = (\log A_1) \ldots (\log A_{n-1}) \quad \text{and} \quad \Omega = \Omega'(\log A) .$$

In [*8*] we proved *inter alia* the following result.

Suppose there is a prime q *satisfying* $13 \le q \le 32(n+1)D$ *such that* $\left[K\left(\alpha_1^{1/q}, \ldots, \alpha_n^{1/q}\right) : K\right] = q^n$ *. Let* $\delta > 0$ *and write*

$$C = \left(32(n+1)D\right)^{9(n+1)}\ , \quad T = C\Omega' \log \Omega'\ , \textit{ and } \quad h = \left[\log\left(B'\delta^{-1}T\right)\right]\ .$$

Then, for any δ *with* $0 < \delta < hT$ *, the inequalities*

$$0 < |b_1 \log \alpha_1 + \ldots + b_n \log \alpha_n| < \min\{\exp(-hT \log A), \exp(-\delta B/B')\}$$

have no solutions in rational integers $b_1, \ldots, b_{n-1}$ *and* $b_n \neq 0$ *with absolute values at most* B *and* B' *respectively.*

In the present paper, we prove this same result without the initial independence condition on $\alpha_1, \ldots, \alpha_n$, but with a slightly larger value for the constant C , namely $C' = \left(25(n+1)D\right)^{10(n+1)}$. As immediate corollaries, we then have the following results.

THEOREM 2. *The inequalities*

$$0 < |b_1 \log \alpha_1 + \ldots + b_n \log \alpha_n| < \exp\left(-\left(25(n+1)D\right)^{10(n+1)}\Omega' \log \Omega' \log A \log B\right)$$

have no solutions in rational integers $b_1, \ldots, b_n$ $\left(b_n \neq 0\right)$ *with absolute values at most* B .

THEOREM 3. *Write* $C' = \left(25(n+1)D\right)^{10(n+1)}$ *and* $T = C'\Omega' \log \Omega'$. *If, for some* $\delta > 0$ *, there exist rational integers* $b_1, \ldots, b_{n-1}$ *with absolute values at most* B *such that*

$$0 < |b_1 \log \alpha_1 + \ldots + b_{n-1} \log \alpha_{n-1} - \log \alpha_n| < e^{-\delta B}\ ,$$

then $B < \delta^{-1}T \log\left(\delta^{-1}T\right) \log A$ *or* $B < C'^{-\frac{1}{2}}T \log\left(C'^{-\frac{1}{2}}T\right) \log A$ *according as* $\delta \leq C'^{-\frac{1}{2}}T$ *or* $\delta > C'^{-\frac{1}{2}}T$.

As detailed in [8], these results are best possible separately in A and B and best known in the remaining variables.

2. Multiplicative relations

We first prove Theorem 1. For this, we require a number of preliminary observations which we state as a series of lemmas.

In establishing an optimal form of Theorem 1, we will find that the quantity $H(\alpha)$ defined above is a more appropriate "size" of α than is the usual height of α, given by $\max|a_j|$, where the a_j are the coefficients of the minimal defining polynomial of α, as in (1). The two measurements are related as follows.

LEMMA 1. *If* α *is an algebraic number with minimal defining polynomial* (1), *then*

$$H(\alpha) \le \left\{\sum_{j=0}^{d} a_j^2\right\}^{\frac{1}{2}} \le d^{\frac{1}{2}} \max_{0\le j\le d} |a_j| .$$

This inequality seems to have been found by Landau [6], but it has had several rediscoverers. For some documentation and sharpenings, we refer to Ostrowski [7].

The next lemma is a classical result of Kronecker [5].

LEMMA 2. *If* α *is an algebraic integer and* $|\alpha^{(j)}| \le 1$ *for all the conjugates* $\alpha^{(j)}$ *of* α, *then* $\alpha = 0$ *or* α *is a root of unity.*

We also require the following refinement of Kronecker's Theorem.

LEMMA 3. *Let* α *be a non-zero algebraic integer of degree* d. *There is a positive constant* $C_2 = C_2(d)$ *such that, if* $\log|\alpha^{(j)}| \le C_2$ *for all the conjugates* $\alpha^{(j)}$ *of* α, *then* α *is a root of unity. We have the estimates*

$$\left(30d^2 \log 6d\right)^{-1} \le C_2 \le (\log 2)d^{-1} .$$

The existence of such a constant C_2, depending only on d, follows at once by a compactness argument. The example $\alpha = 2^{1/d}$ yields the upper bound and the much deeper lower estimate for C_2 follows from the work of Blanksby and Montgomery [3]. Schinzel and Zassenhaus [10] have conjectured that $C_2 = c/d$, for some absolute positive constant c.

The next two lemmas are well-known in the geometry of numbers.

LEMMA 4 (Minkowski's convex body theorem). *Let* S *be a convex region of* R^m *which is symmetrical about the origin and has volume greater than* 2^m. *Then* S *contains a point with integer coordinates other than the origin.*

(See, for example, Cassels [4], page 71.)

LEMMA 5 (Minkowski's linear forms theorem). *Let* a_{ij} $(1 \le i,\ j \le m)$ *be real numbers and let* c_i $(1 \le i \le m)$ *be positive real numbers with*

$$c_1 c_2 \ldots c_m \ge |\det(a_{ij})| \ . \tag{2}$$

Then there are integers x_j $(1 \le j \le m)$, *not all zero, such that*

$$\left|\sum_{j=1}^{m} a_{1j}x_j\right| \le c_1 \ , \quad \left|\sum_{j=1}^{m} a_{ij}x_j\right| < c_i \quad (2 \le i \le m) \ . \tag{3}$$

We briefly recall the principle of the proof, since we shall need it again later on. If there is strict inequality in (2), the lemma follows at once from Lemma 4. If, on the other hand, (2) holds with equality, then the region defined by (3) is bounded. By Lemma 4, for each ε with $0 < \varepsilon < 1$, there are integers $x_{1\varepsilon}, \ldots, x_{n\varepsilon}$ not all zero, such that

$$\left|\sum_{j=1}^{m} a_{1j}x_{j\varepsilon}\right| < c_1 + \varepsilon < c_1 + 1 \ , \quad \left|\sum_{j=1}^{m} a_{ij}x_{j\varepsilon}\right| < c_i \quad (2 \le i \le m) \ .$$

There are only a finite number of possibilities for the $x_{j\varepsilon}$, and since ε can be chosen arbitrarily small, one of these possibilities must satisfy (3) as required. For further details, see, for example, Cassels [4], page 73.

Finally, we need the following technical inequality.

LEMMA 6. *Let* $\phi(m)$ *denote Euler's function. For any positive integer* m , *we have* $m < 4\phi(m) \log\log 6\phi(m)$.

The inequality can be checked directly for $1 \le m \le 100$. For $m > 100$, it follows by some easy manipulation from the inequality of Rosser and Schoenfeld [9], Theorem 15.

Proof of Theorem 1. We may suppose, without loss of generality, that any $m - 1$ of $\alpha_1, \ldots, \alpha_m$ are multiplicatively independent. Thus there is a unique set of relatively prime integers $t_1, \ldots, t_m$ such that

$$(4) \qquad \alpha_1^{t_1} \cdots \alpha_m^{t_m} = \zeta ,$$

where ζ is a root of unity, and t_k (say) $= \max|t_j| > 0$. We set

$$N_{K/Q}\alpha_j = \prod_p p^{e_{pj}} \quad (1 \le j \le m) ,$$

where the product is taken over all the rational primes. By Lemma 2, the relation (4) is equivalent to the system of linear equations

$$(5) \qquad \sum_{j=1}^{m} e_{pj}t_j = 0 \quad (p \text{ prime})$$

and

$$(6) \qquad \sum_{j=1}^{m} \log\left|\alpha_j^{(i)}\right| t_j = 0 \quad (1 \le i \le D) ,$$

where in (5) the index p runs through all rational primes and in (6) the index i runs through the D distinct embeddings of K into $\mathbb{C}$. The argument now divides into two cases according to the relative sizes of m and D .

First Case. Suppose that $m < D$. By Lemma 5, we can find integers $s_1, \ldots, s_m$ not all zero, such that

$$(7) \qquad \left|s_j - \frac{t_j}{t_k} s_k\right| < \frac{C_2}{m\log H(\alpha_j)} \quad (1 \le j \le m)$$

and

$$(8) \qquad |s_k| \le \left(\frac{m}{C_2}\right)^{m-1} \prod_{\substack{j=1 \\ j \ne k}}^{m} \log H(\alpha_j) ,$$

where $C_2 = C_2(D)$ is the constant defined in Lemma 3. Let

$$(9) \qquad \alpha = \alpha_1^{s_1} \cdots \alpha_m^{s_m} .$$

By (5) and (7), for each index p , we have

$$\left|\sum_{j=1}^{m} e_{pj}s_j\right| \le \sum_{j=1}^{m} |e_{pj}|\cdot\left|s_j - \frac{t_j}{t_k}s_k\right| < \sum_{j=1}^{m} \frac{D\log H(\alpha_j)}{\log 2}\,\frac{C_2}{m\log H(\alpha_j)} \le 1\ ,$$

so the quantity on the left, being a rational integer, must be zero. Thus α is a unit. Similarly, by (6) and (7), for each index i ,

$$\left|\sum_{j=1}^{m} \log\left|\alpha_j^{(i)}\right|s_j\right| \le \sum_{j=1}^{m} \left|\log\left|\alpha_j^{(i)}\right|\right|\cdot\left|s_j - \frac{t_j}{t_k}s_k\right| < C_2\ ,$$

so by Lemma 3, α is a root of unity. Thus the relations (4) and (9) must be the same and, in particular, this shows that t_k is bounded by the quantity on the right in (8).

Second Case. Suppose that $m \ge D$. Consider the inequalities

$$\left|s_j - \frac{t_j}{t_k}s_k\right| < \frac{\log 2}{mD\log H(\alpha_j)} \quad (1 \le j \le m)\ , \tag{10}$$

$$\left|\sum_{j=1}^{m} \log\left|\alpha_j^{(i)}\right|\left(s_j - \frac{t_j}{t_k}s_k\right)\right| < C_2 \quad (1 \le i \le D)\ , \tag{11}$$

$$|s_k| \le \left(\frac{mD}{\log 2}\right)^{m-1}\left(\frac{\log 2}{C_2D}\right)^{D}\prod_{\substack{j=1\\ j\ne k}}^{m} \log H(\alpha_j)\ . \tag{12}$$

The region defined by (10) and (11) contains points $(s_1, \ldots, s_m)$ of $\mathbb{R}^m$ with arbitrarily large values of $|s_k|$. Further, (10) implies as in the first case that

$$\left|\sum_{j=1}^{m} \log\left|\alpha_j^{(i)}\right|\left(s_j - \frac{t_j}{t_k}s_k\right)\right| < \frac{\log 2}{D} \quad (1 \le i \le D)\ ,$$

and so we see that the region defined by (10), (11), and (12) in $\mathbb{R}^m$ has volume at least 2^m . By the sharpening of Minkowski's Theorem (Lemma 4) used in Lemma 5, it follows that we can find integers $s_1, \ldots, s_m$ not all zero, satisfying (10), (11), and (12). Proceeding as in the first case, we conclude that t_k is also bounded by the quantity on the right in (12).

Finally, we observe that the root of unity ζ in (4) has degree at most D , so by Lemma 6 it has order, N say, at most $4D \log\log 6D$. We now have the relation of multiplicative dependence

$$\alpha_1^{t_1 N} \ldots \alpha_m^{t_m N} = 1 ,$$

and collecting the various estimates obtained above gives the assertion of the theorem.

3. Linear forms in logarithms

As already indicated in the introduction, we shall deduce Theorems 2 and 3 from the following result.

THEOREM 4. *Let* $\alpha_1, \ldots, \alpha_n$ *be non-zero algebraic numbers belonging to an algebraic number field* K *of degree* D *over* $\mathbb{Q}$ *and with heights respectively not exceeding* $A_1, \ldots, A_n$ (*with* $\log\log A_j \geq 1$). *Suppose that* $A_1 \leq A_2 \leq \ldots \leq A_{n-1} \leq A_n$ *and write*

$$\Omega' = (\log A_1) \ldots (\log A_{n-1}) , \quad \Omega = \Omega'(\log A_n)$$

and

$$C_n = (25(n+1)D)^{10(n+1)} , \quad T = C_n\Omega' \log \Omega' , \quad h = [\log(B'\delta^{-1}T)] .$$

If $0 < \delta < \min\{hT, C_n^{\frac{1}{2}}B'\}$, *then the inequalities*

$$0 < |b_1 \log \alpha_1 + \ldots + b_n \log \alpha_n| < \min\{\exp(-hT \log A_n), \exp(-\delta B/B')\}$$

have no solution in rational integers $b_1, \ldots, b_{n-1}$ *and* $b_n \neq 0$ *with absolute values at most* B *and* B' *respectively.*

Proof. We commence by supposing that, contrary to the assertion of the theorem, there are rational integers $b_1, \ldots, b_{n-1}$ and $b_n \neq 0$ with absolute values at most B and B' respectively such that

$$(13) \quad 0 < |b_1 \log \alpha_1 + \ldots + b_n \log \alpha_n| < \min\{\exp(-hT \log A_n), \exp(-\delta B/B')\} .$$

Suppose, in the first place, that $\alpha_1, \ldots, \alpha_{n-1}$ are multiplicatively dependent. By Theorem 1 and Lemma 1, it follows that we have a relation

(14) $$\alpha_1^{h_1} \cdots \alpha_{n-1}^{h_{n-1}} = 1$$

with integers $h_1, \ldots, h_{n-1}$ not all zero and having absolute values at most H', say, where

(15) $$H' = \left(60(n-1)D^3\right)^{n-1}\Omega' .$$

If $h_i \neq 0$, we can solve (14) for α_i and obtain

$$\left(\alpha_1^{b_1} \cdots \alpha_n^{b_n}\right)^{h_i} = \alpha_1^{b'_1} \cdots \alpha_{n-1}^{b'_{n-1}}\alpha_n^{b'_n} ,$$

with integers $b'_r = h_i b_r - h_r b_i$ $(1 \le r \le n-1)$ and $b'_n = h_i b_n$. In particular, $b'_i = 0$. After a little reorganisation, (13) yields a counterexample to the theorem with $n - 1$ logarithms instead of n. The theorem is trivial if $n = 1$, so we may suppose it to hold for $n - 1$ logarithms, whence if $\alpha_1, \ldots, \alpha_{n-1}$ are multiplicatively dependent, the theorem follows by induction on n.

Accordingly, we can suppose that $\alpha_1, \ldots, \alpha_{n-1}$ are multiplicatively independent. Let q be a prime with $16D \le q \le 32(n+1)D$. We claim that there are elements $\zeta_1, \ldots, \zeta_{n-1}$ in K with heights respectively not exceeding

$$A_1,\ (1+A_2)^{2D},\ \ldots,\ (1+A_{n-1})^{(n-1)D} ,$$

such that

$$\left[K\left(\zeta_1^{1/q}, \ldots, \zeta_{n-1}^{1/q}\right) : K\right] = q^{n-1}$$

and

(16) $$\alpha_1^{b_1} \cdots \alpha_{n-1}^{b_{n-1}} = \zeta_1^{b'_1} \cdots \zeta_{n-1}^{b'_{n-1}} ,$$

where $b'_1, \ldots, b'_{n-1}$ are rational integers of absolute value not exceeding BH'^2. This assertion will be justified in Section 4 below. Set $K' = K\left(\zeta_1^{1/q}, \ldots, \zeta_{n-1}^{1/q}\right)$. It follows from the work of [8] that our present theorem is proved if $\left[K'\left(\alpha_n^{1/q}\right) : K'\right] = q$. For we can apply Theorem 2 of [8], which we have stated in Section 1, to the algebraic numbers $\zeta_1, \ldots, \zeta_{n-1}$, α_n with δ replaced by δ/H'^2 , to obtain an inequality counter to the basic hypothesis (13). In order to compare these inequalities, we observe that the product of the logarithms of the heights of $\zeta_1, \ldots, \zeta_{n-1}$ does not exceed $(\frac{1}{2}nD)^{n-1}\Omega'$ and that the change in δ changes h to a quantity not exceeding $3h$. The new inequality clashes with (13) because $C_n > 3(\frac{1}{2}nD)^{n-1}C$, where C is the constant appearing in [8].

It therefore remains to treat the possibility $\left[K'\left(\alpha_n^{1/q}\right) : K'\right] \neq q$. If $\log A_n > 3nD\Omega'$, we assert that we can find γ in K with height at most $A_n^{\frac{1}{2}}$ such that

$$\zeta_1^{b'_1} \cdots \zeta_{n-1}^{b'_{n-1}} \alpha_n^{b_n} = \zeta_1^{b''_1} \cdots \zeta_{n-1}^{b''_{n-1}} \gamma^{b'_n} \tag{17}$$

where the $b''_1, \ldots, b''_{n-1}$ and b'_n are rational integers of absolute value not exceeding $BH'^2 + qB'$ and qB' respectively. This, too, follows from the construction of Section 4, as shown below. If now $[K'(\gamma^{1/q}) : K'] = q$, then the proof is completed by applying Theorem 2 of [8], as before. If not, we can repeat the argument and, after at most $2 \log \log A_n$ repetitions, we can arrange that $\log A_n < 3nD\Omega'$. With this bound on A_n , we can apply our previous argument to the numbers $\alpha_1, \ldots, \alpha_n$. To do this, replace the quantity H' defined in (15) by $H = H'(\log A_n) < H'^2$. If $\alpha_1, \ldots, \alpha_n$ are multiplicatively dependent, the theorem is proved by induction. If, on the other hand, they are multiplicatively independent, then we can find $\zeta_1, \ldots, \zeta_n$ in K with heights respectively not exceeding

$$A_1,\ (1+A_2)^{2D},\ \ldots,\ (1+A_n)^{nD}$$

such that

$$\left[K\left(\zeta_1^{1/q}, \ldots, \zeta_n^{1/q}\right) : K\right] = q^n$$

and

$$\alpha_1^{b_1} \ldots \alpha_n^{b_n} = \zeta_1^{b_1'} \ldots \zeta_n^{b_n'} ,$$

where $b_1', \ldots, b_{n-1}'$ and b_n' are rational integers with absolute values not exceeding BH^2 and $B'H^2$ respectively. Since $\zeta_1, \ldots, \zeta_n$ satisfy the independence condition, we can apply Theorem 2 of [8], with δ replaced by δ/H^2 . As before, the resultant inequality is incompatible with (13); the final step requires $C_n > 5(\frac{1}{2}nD)^n C$.

4. Eliminating the independence condition

We now show how to construct the numbers in equations (16) and (17) and thereby fill in the gaps in the proof of Theorem 4. For this purpose, we require two lemmas of Baker and Stark [2].

LEMMA 7. *Let* $\alpha_1, \ldots, \alpha_m$ *be non-zero elements of an algebraic number field* K *and let* $\alpha_1^{1/q}, \ldots, \alpha_m^{1/q}$ *denote fixed* q*-th roots for some prime* q . *Write* $K' = K\left(\alpha_1^{1/q}, \ldots, \alpha_{m-1}^{1/q}\right)$. *Then either* $K'\left(\alpha_m^{1/q}\right)$ *is an extension of* K' *of degree* q *, or we have*

$$\alpha_m = \alpha_1^{j_1} \ldots \alpha_{m-1}^{j_{m-1}}\gamma^q$$

for some γ *in* K *and some integers* $j_1, \ldots, j_{m-1}$ *with* $0 \le j_r < q$.

(See [2], Lemma 3.)

LEMMA 8. *Suppose that* α *and* β *are elements of an algebraic number field of degree* D *and that* $\alpha = \beta^q$ *for some positive integer* q . *If* $a\alpha$ *is an algebraic integer for some positive rational integer* a *and if* b *is the leading coefficient in the minimal defining polynomial of* β *, then* $b \le a^{D/q}$.

(See [2], Lemma 4.)

First, we confirm the claim surrounding (16). As above, let $\alpha_1, \ldots, \alpha_{n-1}$ be multiplicatively independent and let q be a prime with $16D \le q \le 32(n+1)D$. Choose m minimal such that

$$\left[K\left(\alpha_1^{1/q}, \ldots, \alpha_m^{1/q}\right) : K\right] \neq q^m .$$

By Lemma 7,

$$\alpha_m = \alpha_1^{j_1} \ldots \alpha_{m-1}^{j_{m-1}}\gamma^q \tag{18}$$

for some γ in K and some integers $j_1, \ldots, j_{m-1}$ with $0 \le j_r < q$. Starting with (18), we now construct, as far as possible, a sequence $\gamma_1 = \gamma, \gamma_2, \gamma_3, \ldots$ of elements of K such that

$$\gamma_l = \alpha_1^{j_{l1}} \ldots \alpha_{m-1}^{j_{l,m-1}}\gamma_{l+1}^q \quad (l = 1, 2, \ldots) , \tag{19}$$

where the integers j_{lr} satisfy $0 \le j_{lr} < q$ $(1 \le r \le m-1)$. Then we have

$$\alpha_m = \alpha_1^{s_{l1}} \ldots \alpha_{m-1}^{s_{l,m-1}}\gamma_l^{q^l} ,$$

and indeed there is a ζ_m in K such that

$$\alpha_m = \alpha_1^{t_{m1}} \ldots \alpha_{m-1}^{t_{m,m-1}}\zeta_m^{q^l} , \tag{20}$$

where the integers t_{mr} satisfy $|t_{mr}| < \frac{1}{2}q^l$ $(1 \le r \le m-1)$. From (20), a denominator for $\zeta_m^{q^l}$ is bounded by

$$A_1^{|t_{m1}|} \ldots A_{m-1}^{|t_{m,m-1}|}A_m ,$$

whence by Lemma 8, the leading coefficient of the minimal defining polynomial of ζ_m is bounded by

$$A_1^{D/2} \ldots A_{m-1}^{D/2}A_m^{D/q^l} .$$

Moreover, each conjugate of ζ_m has absolute value at most

$$(1+A_1)^{\frac{1}{2}} \ldots (1+A_{m-1})^{\frac{1}{2}}(1+A_m)^{1/q^l} ,$$

so the height of ζ_m is bounded by

$$(1+A_1)^D \ldots (1+A_{m-1})^D(1+A_m)^{2D/q^l} < (1+A_m)^{mD} ; \tag{21}$$

if $m = 1$, then plainly A_1 already bounds the height of ζ_1 .

Now consider the sequence (19). Set

$$H_m = (60D^3m)^m(\log A_1) \ldots (\log A_m).mD \log(1+A_m) .$$

If the sequence (19) fails to terminate for some l with $q^l \le H_m$, then choose ζ_m corresponding to some l with $q^l > H_m$. By Theorem 1, the multiplicatively dependent numbers $\alpha_1, \ldots, \alpha_m$, ζ_m satisfy a further non-trivial relation

$$\alpha_1^{h_1} \ldots \alpha_m^{h_m}\zeta_m^{h_0} = 1 , \tag{22}$$

with integers $h_0, \ldots, h_m$ not all zero and having absolute values not exceeding H_m . On eliminating ζ_m from the relations (20) and (22), we obtain a non-trivial relation

$$\alpha_1^{s_1} \ldots \alpha_m^{s_m} = 1 ,$$

where $s_r = q^l h_r - t_{mr}h_0$ $(1 \le r \le m-1)$ and $s_m = q^l h_m + h_0$. If $m < n$, this is a contradiction since $\alpha_1, \ldots, \alpha_{n-1}$ are multiplicatively independent. Thus, if $m < n$, the sequence (19) terminates at γ_l for some l with $q^l = G_m < H_m$ and the corresponding ζ_m satisfies $\left[K\left(\alpha_1^{1/q}, \ldots, \alpha_{m-1}^{1/q}, \zeta_m^{1/q}\right) : K\right] = q^m$. We observe that

$$\alpha_1^{b_1} \cdots \alpha_n^{b_n} = \alpha_1^{b_1'} \cdots \alpha_{m-1}^{b_{m-1}'} \zeta_m^{b_m'} \alpha_{m+1}^{b_{m+1}} \cdots \alpha_n^{b_n} , \tag{23}$$

with integers b_r' given by $b_r' = b_r + b_m t_{mr}$ $(1 \le r \le m-1)$ and $b_m' = b_m G_m$.

We sequentially replace $\alpha_1, \ldots, \alpha_{n-1}$ by $\zeta_1, \ldots, \zeta_{n-1}$. At each stage, we write the analogue of (20) as

$$\alpha_m = \zeta_1^{t_{m1}} \cdots \zeta_{m-1}^{t_{m,m-1}} \zeta_m^{G_m} = \alpha_1^{t_{m1}'/G_1} \cdots \alpha_{m-1}^{t_{m,m-1}'/G_{m-1}} \zeta_m^{G_m} ,$$

where the t_{mr} are integers and ζ_m is chosen to make $|t_{mr}'| < \frac{1}{2}G_m$ and $\left[K\left(\zeta_1^{1/q}, \ldots, \zeta_m^{1/q}\right) : K\right] = q^m$. This makes $|t_{mr}| < \frac{1}{4}(m+1)G_m$, so, by repeated use of (23), we finally obtain (16) as asserted with the exponents b_r' satisfying

$$|b_r'| \le nB\left(G_r + G_{r+1} + \ldots + G_{n-1}\right) \le BH'^2 \quad (1 \le r \le n-1) .$$

The same construction yields the assertion (17). Indeed, suppose that $\left[K'\left(\alpha_n^{1/q}\right) : K'\right] \ne q$ where, as before, $K' = K\left(\zeta_1^{1/q}, \ldots, \zeta_{n-1}^{1/q}\right)$, and that $\log A_n > 3nD\Omega'$. We apply the above argument with $m = n$. The equation (18) takes the form

$$\alpha_n = \zeta_1^{j_1} \cdots \zeta_{n-1}^{j_{n-1}} \gamma^q = \alpha_1^{t_1/G_1} \cdots \alpha_{n-1}^{t_{n-1}/G_{n-1}} \gamma^q ,$$

where the j_r are integers and γ is chosen so that $|t_r| < \frac{1}{2}q$. Proceeding as before, we see that the height of γ is bounded by the expression (21) and this is at most $A_n^{\frac{1}{2}}$, as required in (17). Moreover, the substitution (24) leads to the bound asserted for the exponents in (17).

This completes the proof of Theorem 4.

5. Conclusion

Only one further remark is needed to establish Theorems 2 and 3. This is the following trivial inequality; it is actually best possible in A, but trivial in B.

LEMMA 9. *Either* $\alpha_1^{b_1} \ldots \alpha_n^{b_n} = 1$, *or*

$$|b_1 \log \alpha_1 + \ldots + b_n \log \alpha_n| > \exp(-2nDB \log A) ,$$

where $b_1, \ldots, b_n$ *are rational integers with absolute values at most* B.

Proof. Let a_j be the leading coefficient, supposed positive, of the minimal defining polynomial of α_j or α_j^{-1}, according as b_j is positive or negative. Then

$$a_1^{|b_1|} \ldots a_n^{|b_n|}\left(\alpha_1^{b_1} \ldots \alpha_n^{b_n}-1\right)$$

is an integer of K and its conjugates are bounded above by $2A^{nB}(1+A)^{nB}$, whence

$$\left|\alpha_1^{b_1} \ldots \alpha_n^{b_n}-1\right| > \exp\left(-\tfrac{3}{2} nDB \log A\right) ,$$

and the assertion follows immediately.

Proofs of Theorems 2 and 3. Define the quantities C_n and T as in Theorem 4. If $T > C_n^{\frac{1}{2}}B$, Theorem 2 follows from Lemma 9, while if $T \le C_n^{\frac{1}{2}}B$, Theorem 2 follows from Theorem 4 with $B' = B$ and $\delta = T$. Again, if $\delta < C_n^{\frac{1}{2}}$, Theorem 3 follows at once from Theorem 4 and the assertion of Theorem 2 for $\delta \ge C_n^{\frac{1}{2}}$ is actually a weaker claim.

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* The authors have not had access to [5] and [6], which are quoted at second hand. Editor.

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