The validity of weighted automata

(Fragments)

Sylvain Lombardy

LABRI, Université de Bordeaux

Jacques Sakarovitch

LTCI, CNRS/Telecom ParisTech

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Outline (of the fragments)

This work addresses, and proposes a solution to,
the problem of ε-transition removal in weighted automata.

The problem lies in effectivity.

The solution is based on a new, and more constrained,
definition of the validity of weighted automata.

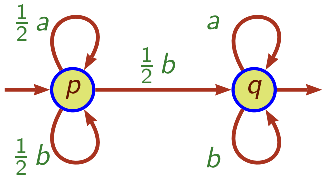
The definition insures that
algorithms are successful on valid automata.

In some (interesting) cases, we are able to establish that
success of algorithms implies validity of automata.

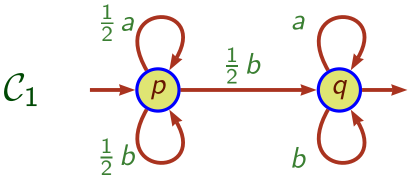
This solution provides a sound theoretical framework for
the algorithms implemented in VAUCANSON.

The weighted automaton model

$\mathcal{C}_1$



The weighted automaton model



- Weight of a path c: *product* of the weights of transitions in c
- Weight of a word w: *sum* of the weights of paths with label w

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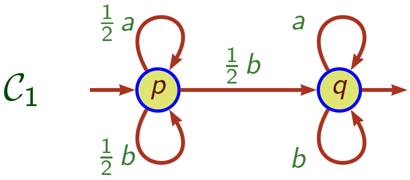

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$$\xrightarrow{1} p \xrightarrow{\frac{1}{2}\,b} p \xrightarrow{\frac{1}{2}\,a} p \xrightarrow{\frac{1}{2}\,b} q \xrightarrow{1}$$

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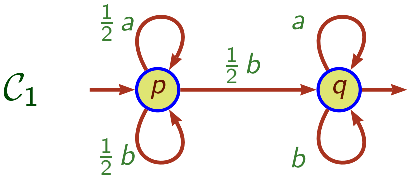

$|\mathcal{C}_1| \in \mathbb{Q}\langle\langle A^* \rangle\rangle$

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$$b\,a\,b \quad \longmapsto \quad \frac{1}{2} + \frac{1}{8} = \frac{5}{8} \;= \langle 0.101 \rangle_2 \qquad\qquad |\mathcal{C}_1|\colon A^* \longrightarrow \mathbb{Q}$$

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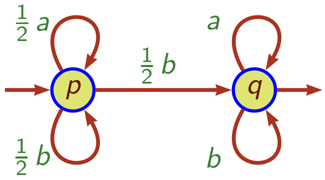

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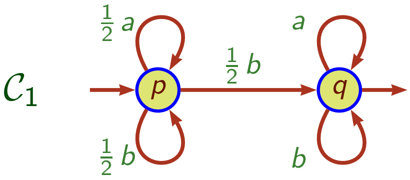
$$|\mathcal{C}_1| = \frac{1}{2}\,b + \frac{1}{4}\,a\,b + \frac{1}{2}\,b\,a + \frac{3}{4}\,b\,b + \frac{1}{8}\,a\,a\,b + \frac{1}{4}\,a\,b\,a + \frac{3}{8}\,a\,b\,b + \frac{1}{2}\,b\,a\,a + \ldots$$

The weighted automaton model



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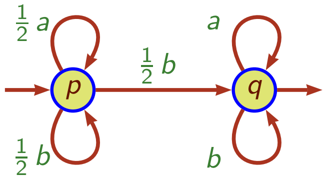

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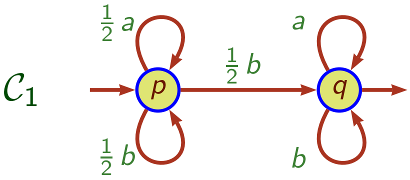
Every $\mathbb{K}$-automaton defines a series in $\mathbb{K}\langle\langle A^* \rangle\rangle$

whose coefficients are effectively computable

The weighted automaton model



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Every $\mathbb{K}$-automaton defines a series in $\mathbb{K}\langle\langle A^* \rangle\rangle$

whose coefficients are effectively computable

Where is the problem ?

The weighted automaton model

We want to deal with automata whose transitions may be labelled by the empty word ε

A basic result in (classical) automata theory

Theorem

Every ε-NFA is equivalent to an NFA

A basic result in (classical) automata theory

Theorem

Every ε-NFA is equivalent to an NFA

Usefulness of ε-transitions:

Preliminary step for many constructions on NFA's:

- Product and star of position automata
- Thompson construction
- Construction of the universal automaton
- Computation of the image of a transducer
- ...

May correspond to the *structure* of the computations

Removal of ε-transitions is implemented in all automata software

A basic question in weighted automata theory

Question

Is every ε-WFA is equivalent to a WFA?

A basic question in weighted automata theory

Question

Is every ε-WFA is equivalent to a WFA?

certainly not !

A basic question in weighted automata theory

Question

Is every ε-WFA is equivalent to a WFA?

certainly not !

New questions

Which ε-WFAs have a *well-defined* behaviour?

How to compute the behaviour of an ε-WFA (when it is *well-defined*)?

How to decide if the behaviour of an ε-WFA is *well-defined*?

Behaviour of weighted automata

Infinite sums are given a meaning via a topology on $\mathbb{K}$

Topology on $\mathbb{K}$ defines a topology on $\mathbb{K}\langle\langle A^* \rangle\rangle$

Topology allows to define summable families in $\mathbb{K}\langle\langle A^* \rangle\rangle$

Behaviour of weighted automata

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$\mathcal{A} = \langle\, \mathbb{K}, Q, A, E, I, T \,\rangle$ possibly with ε-transitions

$\mathsf{P}_{\mathcal{A}}$ set of all paths in $\mathcal{A}$

$|\mathcal{A}|$ well-defined $\iff$ $\mathbf{WL}(\mathsf{P}_{\mathcal{A}})$ summable

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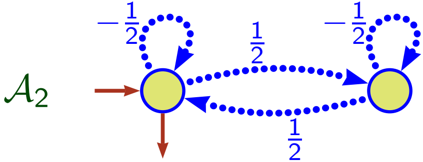
Definition taken in previous works (Lombardy, S. 03 –)

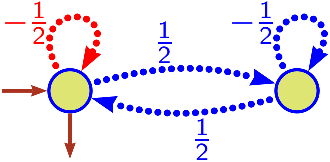
- Yields a consistent theory
- Two pitfalls for effectivity
  - *effective computation* of a summable family may not be possible
  - *effective computation* may give values to non summable families

Problems in computing the behaviour



Problems in computing the behaviour

$\mathcal{A}_2$



$$\left(-\frac{1}{2}\right)^* = \frac{2}{3}$$

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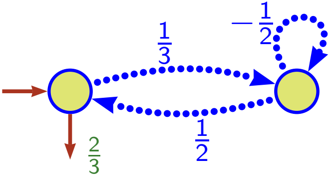

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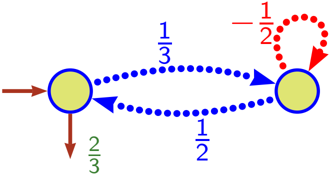

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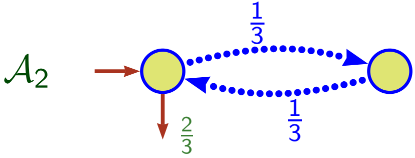

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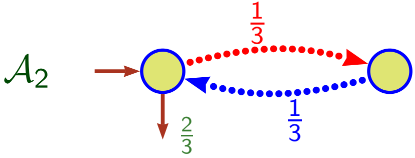

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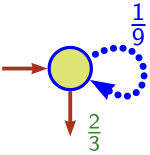

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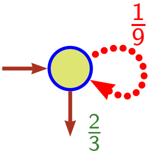

$$\left(\frac{1}{9}\right)^* = \frac{9}{8}$$

Problems in computing the behaviour

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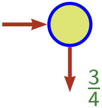

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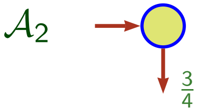

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$$|\mathcal{A}_2| = I_2 \cdot \underline{E_2}^* \cdot T_2$$

$$\underline{E_2}^3 = \underline{E_2} \quad \Longrightarrow \quad \underline{E_2}^* \quad \text{undefined} \quad \Longrightarrow \quad |\mathcal{A}_2| \quad \text{undefined}$$

A chicken and egg problem

automaton	**algorithm**
$\mathcal{A}$	A
valid ?	**success ?**

A chicken and egg problem

automaton		**algorithm**
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valid	$\overset{?}{\Longleftarrow}$	**success**

Validity of weighted automata

$\mathcal{A} = \langle\, \mathbb{K}, Q, A, E, I, T \,\rangle$ possibly with ε-transitions

E^* *free monoid* generated by E

$\mathsf{P}_{\mathcal{A}}$ *set of paths* in $\mathcal{A}$ (local) rational subset of E^*

Definition
R rational family of paths of $\mathcal{A}$ $\quad R \in \mathrm{Rat}E^* \;\wedge\; R \subseteq \mathsf{P}_{\mathcal{A}}$

Definition
$\mathcal{A}$ is valid iff
$\forall R$ rational family of paths of $\mathcal{A}$, $\mathbf{WL}(R)$ is summable

Validity of weighted automata

Validity implies well-definition of behaviour

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Remark

If every (rational) subfamily of a summable family in $\mathbb{K}$ *is summable,*
then validity is equivalent to well-definition of behaviour

Eg. $\mathbb{R}$, $\mathbb{Q}$.

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Theorem

$\mathcal{A}$ *is valid iff the behaviour of every covering of* $\mathcal{A}$ *is well-defined*

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If $\mathcal{A}$ *is valid, then 'every' removal algorithm on* $\mathcal{A}$ *is successful*

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Reminder

We do not know yet how to decide whether
a $\mathbb{Q}$ - or an $\mathbb{R}$ -automaton is valid.

Deciding validity

Definition

$\mathbb{K}$ topological, ordered, positive, star-domain downward closed

$\mathbb{N}$, $\mathcal{N}$, $\mathbb{Q}_+$, $\mathbb{R}_+$, $\mathbb{Z}\mathsf{min}$, $\mathrm{Rat}\, A^*$,... are TOPS SDC

$\mathbb{N}_\infty$, (binary) positive decimals,... are not TOPS SDC

Theorem

$\mathbb{K}$ topological, ordered, positive, star-domain downward closed

A $\mathbb{K}$-automaton is valid if, and only if,

the ε-removal algorithm succeeds

Deciding validity

Definition

If $\mathcal{A}$ is a $\mathbb{Q}$- or $\mathbb{R}$-automaton,

then $\mathsf{abs}(\mathcal{A})$ is a $\mathbb{Q}_+$- or $\mathbb{R}_+$-automaton

Theorem

A $\mathbb{Q}$- or $\mathbb{R}$-automaton $\mathcal{A}$ is valid if and only if $\mathsf{abs}(\mathcal{A})$ is valid.

Conclusion

- Semiring structure is weak, topology does not help so much.
- This weakness imposes a restricted definition of validity,
  in order to guarantee success of validity algorithms.
- Axiomatic approach does not allow
  to deal wit most common numerical semirings: $\mathbb{Z}$min, $\mathbb{Q}$
- On 'usual' semirings,
  the new definition of validity coincides with the former one.

Conclusion (2)

- Apart the trivial cases, and the TOPS SCD case,
  decision of validity is never granted, and is to be established.
- On 'usual' semirings, validity is decidable.
- The new definition of validity
  fills the 'effectivity gap' left open by the former one.
- The algorithms implemented in VAUCANSON
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All's well, that ends well!

Hidden parts

- The removal algorithm itself:
  - Termination issues (weighted versus Boolean cases)
  - Complexity issues

- Automata and expressions validity

- 'Infinitary' axioms : *strong*, *star-strong* semirings

- Links with the 'axiomatic' approach (Bloom–Ésik–Kuich)

- References to previous work (on removal algorithm):
  - *locally closed* srgs (Ésik–Kuich), *k-closed* srgs (Mohri)
  - links with other algorithms:
    shortest-distance algorithm (Mohri),
    state-elimination method (Hanneforth–Higueira)