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Quantaloids, Enriched Categories and Automata Theory

KIMMO I. ROSENTHAL
Department of Mathematics, Union College, Schenectady, N.Y. 12308, U.S.A.
email: rosenthk@gar.union.edu

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Abstract. This article is intended to be an survey article outlining how the theory of quantaloids and categories enriched in them provides an effective means of analyzing both automata and tree automata. The emphasis is on the unification of concepts and how categorical methods provide insight into various calculations and theorems, both illuminating the original presentation as well as yielding conceptually simpler proofs. Proofs will be omitted and the emphasis is on providing the reader (even a relatively inexperienced one) with an understanding of the basic constructions and results.

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1. Introduction

The possibility of analyzing automata using enriched categories goes back to Betti [2], and his subsequent work with Kasangian [5] extended these ideas to tree automata. Further developments along these lines appeared in [14–16].

A quantaloid is a locally small category whose hom-sets are complete lattices and whose composition preserves sups in both variables. They are a categorical generalization of quantales in that unital quantales are precisely the quantaloids with one object. Quantales have recently been objects of much interest in theoretical computer science. (For an introduction to quantales, as well as discussion of their role in modelling linear logic, see Rosenthal [23].) Quantaloids can also be viewed as the categories enriched in the symmetric, monoidal closed category Sl of sup-lattices. Quantaloids are thus locally partially ordered bicategories, allowing one to study them using the machinery of enriched category theory over a base bicategory, as developed by Street, Walters, Carboni, Betti, Kasangian and others (for some references, consult [3, 4, 8, 30–32]). The free quantaloid $P(A)$ on a locally small category A was investigated by Rosenthal [24], where the equivalence of $P(A)$-categories with relational presheaves on A was established. A relational presheaf on A is a lax functor $A^{op} \to Rel$, where Rel is the category (quantaloid) of sets and relations.

After presenting the basic ideas of quantaloid theory and the theory of categories enriched in quantaloids, we indicate how automata and tree automata can be effectively described as enriched categories. M-automata, where M is a monoid, are categories enriched in the quantale $P(M)$ of languages (subsets) of M. The work of Betti and Kasangian (mentioned above) shows how to interpret tree automata as certain $P(\mathbf{A})$-enriched categories, where $\mathbf{A}$ is an algebraic theory in the sense of Lawvere [18]. In both cases, the notion of the behavior of the automaton and the resulting recognizable language or forest, respectively, are obtained by utilizing the enriched category theoretical notion of bimodule.

Our exposition is geared towards two applications of categorical methods to automata theory. The first uses the theory of nuclei on a quantaloid to analyze the so-called syntactic congruence. This approach makes it clear that the passage from automata to tree automata can be analyzed as the passage from a one object category to a more general category. Thus, various attempts to to find an appropriate notion of 'syntactic monoid' for tree automata, as discussed in [10], are bound to be unsatisfactory. The appropriate notions are that of syntactic nucleus on a quantaloid and the associated syntactic quantaloid. These provide a unification of the two theories.

Our second application concentrates on the well-known relationship in the theory of computation between context-free languages and tree automata, namely that the languages arising as the yields of the recognizable forests (behaviors) of finite-state tree automata are precisely the context-free languages [10]. We shall make this correspondence clear and provide simple proofs by utilizing the enriched category approach to tree automata together with the work of Walters [33–35] on describing context-free languages. Walters constructed context-free languages utilizing multigraphs and the free category construction on a multigraph, as well as the algebraic theory Mon(A) of monoids augmented by constants A. Having been given a context-free grammar G and an object X of G, we use the notion of multirelation with constants A as a link to show how to construct a tree automaton, described as a relational presheaf $Mon(\mathrm{A}) \to Rel$, whose behavior results in the context-free language associated to X. Conversely, one can obtain a context-free language from any generalized $\mathbf{A}$-tree automaton, with $\mathbf{A}$ an algebraic theory (under some mild finiteness assumptions). The enriched category (i.e. relational presheaf) approach yields these results in an elegant way and indicates the central role played by the algebraic theory Mon(A), justifying the thesis that there are new insights that can be gained from the categorical perspective presented in this article.

We would be remiss if we did not make it clear that this article is by no means exhaustive on this subject and the reader is referred to [14–16] for further results. As the article is intended to be expository, almost all proofs will be omitted or only sketched. Many of the details can be found in [24] through [29].

2. Quantaloids

A quantaloid is a naturally arising example of a locally partially ordered bicategory. We require that the bicategory be locally complete as an ordered set and that composition distribute over suprema in both variables. Quantaloids can also be viewed as a generalization of unital quantales, in that a unital quantale is precisely a quantaloid with one object (and hence the terminology). For an overview of quantale theory, see Rosenthal [23]. Quantaloids have been previously considered in the work on process semantics by Abramsky and Vickers [1], by Pitts [22] in his analysis of geometric morphisms of topoi, and of course they arise, as we shall discover, in the enriched category approach to automata due to Betti and Kasangian [5]. There has been a series of papers on quantaloids [24–28] to which we shall be making frequent reference.

Let us now begin by looking at the formal definition of quantaloid and some examples.

DEFINITION 2.1. A *quantaloid* is a locally small category Q such that
(1) for $a, b \in Q$, the hom-set $Q(a, b)$ is a complete lattice,
(2) composition of morphisms in Q preserves sups in both variables.

(Pending our discussion of enriched categories in the next section, this says precisely that Q is enriched in the symmetric, monoidal, closed category **Sl** of sup-lattices.) The hom-sets $Q(a, a)$ are unital quantales and we shall use i_a to denote the identity morphism $a \to a$.

From (2) above, it follows that we have left and right residuation operations on morphisms described as follows.

If $f\colon a \to b$ and $g\colon a \to c$ are morphisms in Q, then there exists a morphism $f \to_l g\colon b \to c$ such that $h \circ f \leqslant g$ iff $h \leqslant f \to_l g$ for all $h\colon b \to c$.

Similarly, given $g\colon a \to c$ and $h\colon b \to c$, there is $h \to_r g\colon a \to b$ such that $h \circ f \leqslant g$ iff $f \leqslant h \to_r g$.

The following lemma summarizes some of the basic calculations with the residuation operations in a quantaloid. These calculations are central to the proofs of the results in Section 5.

LEMMA 2.1. *Let Q be a quantaloid and suppose we have morphisms $f\colon a \to b$, $g\colon a \to c$, $h\colon b \to c$, $k\colon d \to b$, $j\colon b \to e$ in Q. Then,*
(1) $h \circ (h \to_r g) \leqslant g$,
(2) $(f \to_l g) \circ f \leqslant g$,
(3) $k \to_r (h \to_r g) = h \circ k \to_r g$,
(4) $j \to_l (f \to_l g) = j \circ f \to_l g$.
If $i_a\colon a \to a$ and $i_b\colon b \to b$ denote the identity morphisms on a and b, then $i_b \to_r f = f = i_a \to_l f$.

There are two reasonable notions of quantaloid morphism depending on the context. We shall not be utilizing mappings between quantaloids, therefore we discuss them only briefly here.

DEFINITION 2.2. If Q and S are quantaloids, a *quantaloid homomorphism* is a functor F: $Q \to S$ such that on hom-sets it induces a sup-lattice morphism $Q(a, b) \to S(F(a), F(b))$.

F is a *lax morphism* if we only require that for composable morphisms f and g of Q, $F(f) \circ F(g) \leqslant F(f \circ g)$ and for the identity morphisms, $i_{F(a)} \leqslant F(i_a)$ for all $a \in Q$.

Let us now list several important examples of quantaloids.

EXAMPLES

(1) A quantaloid with one object is just a unital quantale, as remarked above.
(2) Rel, the category of sets and relations, is a quantaloid. The residuation operations in Rel are described as follows. Given relations R: $X \to Y$, S: $X \to Z$ and T: $Y \to Z$, we have that $R \to_l S = \{(y, z) \mid (x, y) \in R \Rightarrow (x, z) \in S$ for all $x \in X\}$ and that $T \to_r S = \{(x, y) \mid (y, z) \in T \Rightarrow (x, z) \in S$ for all $z \in Z\}$.
(3) **Sl**, the category of sup-lattices, is clearly a quantaloid.

(These latter two examples are 'large' quantaloids, in that they do not have a set of objects.)

We now present what for our purposes will be the central example of a quantaloid in this paper.

(4) Let A be a locally small category. Define a quantaloid $P(A)$ as follows. The objects of $P(A)$ are precisely those of A. Given $a, b \in A$, then $P(A)(a, b) = P(A(a, b))$, the power set of the hom-set $A(a, b)$. If S: $a \to b$ and T: $b \to c$ are sets of morphisms of A, let $TS = \{g \circ f \mid g \in T, f \in S\}$. This operation preserves unions in each variable and thus we have a quantaloid.

We shall call $P(A)$ the free quantaloid on A. The reason for this terminology is that P defines a monad $LocSm \to LocSm$, where $LocSm$ is the category of locally small categories and functors. The algebras for this monad are precisely the quantaloids. See [24] for further details.

Note when A is a monoid M, i.e. a one-object category , we obtain $P(M)$, the power set of M, and the operation is the concatenation of subsets. For $A, B \subseteq M$, we have $AB = \{ab \mid a \in A, b \in B\}$.

One other important example from the literature worth mentioning occurs in the work of Walters (see [32] and [3]). If A be a locally small category, one can construct (using spans and cribles) a category of relations on A, which is a quantaloid. This construction plays a central role in analyzing sheaf theory from an enriched category point of view.

We now turn our attention to generalizing the concept of nucleus from the theory of frames and quantales to the level of quantaloids (see [13] for the frame case, and a discussion of quantic nuclei appears in [20] and [23]). This will be the key to our discussion of how to generalize the syntactic congruence from automata theory to tree automata.

DEFINITION 2.3. Let Q be a quantaloid. A *quantaloidal nucleus* is a lax morphism j: $Q \to Q$, which is the identity on the objects of Q and such that the maps $j_{a,b}$: $Q(a,b) \to Q(a,b)$ satisfy:
(1) $f \leqslant j_{a,b}(f)$ for all $f \in Q(a,b)$,
(2) $j_{a,b}(j_{a,b}(f)) = j_{a,b}(f)$ for all $f \in Q(a,b)$.

Note that from the definition of laxity (Definition 2.2), we obtain readily that $j_{b,c}(g) \circ j_{a,b}(f) \leqslant j_{a,c}(g \circ f)$ for all $g \in Q(b,c)$, $f \in Q(a,b)$.

Using the above formulas one can also show that

$$j_{a,c}(g \circ f) = j_{a,c}(g \circ j_{a,b}(f)) = j_{a,c}(j_{b,c}(g) \circ f).$$

Given a quantaloidal nucleus j on a quantaloid Q, let Q_j be the bicategory with the same objects as Q and with morphisms $f \in Q_j(a,b)$ iff $j_{a,b}(f) = f$. Then we obtain a quantaloid with composition in Q_j defined by:
if $f \in Q_j(a,b)$ and $g \in Q_j(b,c)$, then $g \circ_j f = j_{a,c}(g \circ f)$.
We record the following proposition.

PROPOSITION 2.1. *If j is a quantaloidal nucleus on a quantaloid Q, then Q_j is a quantaloid and j: $Q \to Q_j$ is a quantaloid homomorphism.*

In [4], Betti and Carboni indicate how Grothendieck topologies on a locally small category A arise as quantaloidal nuclei on the quantaloid $Rel(A)$ (mentioned earlier), which locally preserve meets.

One can show that every quantaloid arises from a free one via a quantaloidal nucleus. Given a quantaloid Q, we can define j: $P(Q) \to P(Q)$ by $j_{a,b}(B) = (\mathrm{sup}B) \downarrow = \{f \in Q(a,b) \mid f \leqslant \mathrm{sup}B\}$. It is not hard to verify that j is a quantaloidal nucleus and also that $j_{a,b}(B) = B$ iff $B = f\downarrow$ for some $f \in Q(a,b)$.

THEOREM 2.1. *Let Q be a quantaloid. Then, there exists a locally small category A and a quantaloidal nucleus j on $P(A)$ such that $Q \cong P(A)_j$.*

Proof. Verify that $Q \cong P(A)_j$ for the quantaloidal nucleus defined above. □

The theory of quantaloidal nuclei on free quantaloids $P(A)$ provides a generalization of the well-known notion of congruence on the category A. There is an adjunction between such congruences on A and nuclei on $P(A)$, which allows us to view nuclei as natural extensions of congruences. The following observations are useful in our analysis of the 'syntactic congruence' from automata theory in Section 5.

The congruences we consider will only be on morphisms with the same domain and codomain, so that objects never get identified.

DEFINITION 2.4. Let A be a locally small category. A congruence θ on A consists of a family $\{\theta_{a,b}\}$ where

(1) $\theta_{a,b}$ is a congruence on the hom-set $Q(a,b)$,
(2) if $(f,g) \in \theta_{a,b}$ and $h \in Q(b,c)$, then $(h \circ f, h \circ g) \in \theta_{a,c}$,
(3) if $(f,g) \in \theta_{a,b}$ and $k \in Q(d,a)$, then $(f \circ k, g \circ k) \in \theta_{d,b}$.

Let θ be a congruence on A. Define j_θ on $P(A)$ as follows. Suppose that $A \subseteq A(a,b)$. Let $j_{\theta,a,b}(A) = \{f \in A(a,b) \mid (g,f) \in \theta_{a,b}$ for some $g \in A\}$.

Conversely, suppose j is a quantaloidal nucleus on $P(A)$. Define θ_j on A by $(f,g) \in (\theta_j)_{a,b}$ iff $j_{a,b}(\{f\}) = j_{a,b}(\{g\})$. (We shall hereon in write this as $j_{a,b}(f) = j_{a,b}(g)$.)

Let $Con(A)$ denote the lattice of congruences on A and $N(P(A)$ denote the lattice of quantaloidal nuclei on $P(A)$.

The following proposition summarizes the key results.

PROPOSITION 2.2. (1) *If θ is a congruence on A, then j_θ: $P(A) \to P(A)$ is a quantaloidal nucleus.*
(2) *If j is a quantaloidal nucleus on $P(A)$, then θ_j is a congruence on A.*
(3) *The order-preserving map F: $Con(A) \to N(P(A))$ given by $F(\theta) = j_\theta$ is the left adjoint of the order-preserving map G: $N(P(A)) \to Con(A)$ given by $G(j) = \theta_j$.*

While $G(F(\theta)) = \theta$ holds for all congruences θ on A, $F(G(j))$ may fail to equal j in dramatic fashion. For example, let Q be a quantale and consider the quantic nucleus j on $P(Q)$ defined by $j(A) = (\mathrm{sup}A)\downarrow = \{b \in Q \mid b \leqslant \mathrm{sup}A\}$. Then, $(x,y) \in \theta_j$ iff $j(x) = j(y)$ iff $x = y$, i.e. θ_j is the diagonal relation on Q and hence $F(\theta_j)$ is the identity nucleus on $P(Q)$.

This is not surprising, as one would expect the notion of quantaloidal nucleus to be more general than that of congruence.

3. Enriched Categories

The theory of enriched categories over a base bicategory is an important tool for analyzing a wide variety of mathematical structures. This theory has been developed by many authors, including Betti, Carboni, Street, and Walters. We refer the reader to [3, 4, 6, 8, 30, 31] and [32]. These all discuss enriched categories, enriched functors and bimodules. For an excellent introduction to the case of the one-object base bicategory, see the seminal paper of Lawvere [19], where the important notion of Cauchy completeness is first introduced. Cauchy completeness is also discussed in [31], and the enriched category approach to sheaf theory using it is in [3, 4], and [32]. (The reader should also see Kelly [17].)

Since our interest lies in the case where the base bicategory is a quantaloid Q, we present the theory from that perspective, rather than in its most general form.

DEFINITION 3.1. Let Q be a quantaloid. A set X is a $Q-category$ iff it comes equipped with the following data:
(1) a function ρ: $X \to Obj(Q)$ assigning to $x \in X$ an object $\rho(x) \in Q$,
(2) an enrichment, which assigns to every pair of elements $x, y \in X$ a morphism $X(x, y)$: $\rho(x) \to \rho(y)$ in Q such that
 (a) $i_{\rho(x)} \leqslant X(x, x)$ for all $x \in X$,
 (b) $X(y, z) \circ X(x, y) \leqslant X(x, z)$ for all $x, y, z \in X$.

DEFINITION 3.2. Let X and Y be Q-categories. A $Q - functor$ is a function f: $X \to Y$ such that
(1) $\rho_X(x) = \rho_Y(f(x))$,
(2) $X(x, a) \leqslant Y(f(x), f(a))$ for all $x, a \in X$.

Let Q-Cat denote the category of Q-categories and Q-functors. We shall shortly consider the special case where Q is of the form $P(A)$ and obtain an alternate description using the notion of relational presheaf.

A more general notion of morphism than that of Q-functor is provided by the notion of a Q-bimodule.

DEFINITION 3.3. Let Q be a quantaloid. Let X and Y be Q-categories. A $Q - bimodule$ ϕ: $X \Rightarrow Y$ consists of an assignment to every $(x, y) \in X \times Y$ of a morphism $\phi(y, x)$: $\rho_Y(y) \to \rho_X(x)$ such that
(1) $\phi(y, x) \circ Y(y', y) \leqslant \phi(y', x)$ for all $x \in X$, $y, y' \in Y$,
(2) $X(x, x') \circ \phi(y, x) \leqslant \phi(y, x')$ for all $x, x' \in X$, $y \in Y$.

If ϕ: $X \Rightarrow Y$ and π: $Y \Rightarrow Z$ are bimodules, they can be composed to produce a bimodule $\pi \circ \phi$: $X \Rightarrow Z$ defined by $(\pi \circ \phi)(z, x) = \sup_{y \in Y}(\phi(y, x) \circ \pi(z, y))$. This composition is associative and the identity bimodules ι: $X \Rightarrow X$ are given by $\iota(x, x') = X(x, x')$.

We should also point out that every Q-functor f: $X \to Y$ gives rise to two bimodules, f^*: $X \to Y$ and f_*: $Y \to X$ defined by $f^*(y, x) = Y(y, f(x))$ and $f_*(x, y) = Y(f(x), y)$, respectively.

We thus obtain a category $Bim(Q)$ of Q-categories and Q-bimodules. We record the following proposition, where sups of bimodules are computed pointwise.

PROPOSITION 3.1. *If Q is a quantaloid, then $Bim(Q)$ is a quantaloid under bimodule composition.*

The residuations of bimodules are described as follows:

If ϕ: $X \Rightarrow Y$ and τ: $X \Rightarrow Z$ are bimodules, then $\phi \to_l \tau$: $Y \Rightarrow Z$ is defined by $(\phi \to_l \tau)(z, y) = \inf_{x \in X}(\phi(y, x) \to_r \tau(z, x))$.

If ω: $Y \Rightarrow Z$ is another bimodule, then $\omega \to_r \tau$: $X \Rightarrow Y$ is defined by $(\omega \to_r \tau)(y, x) = \inf_{z \in Z}(\omega(z, y) \to_l \tau(z, x))$.

Note that the reversal of order in the definition of composition, and the switching of $\to_r$ and $\to_l$ above, are due to how we have chosen to write composition in a quantaloid.

Sl-bimodules for a quantaloid Q provide non-symmetric models of $*$-autonomous categories and are thus of relevance to non-commutative linear logic (see Rosenthal [29]).

As a simple example of these concepts for the reader to consider, let $Q = 2$, the two-element Boolean algebra. Then, a Q-category is just a preordered set X, a Q-functor $X \to Y$ is an order-preserving function and a Q-bimodule $X \to Y$ is an order ideal, that is a relation from Y to X, which is downward closed with respect to the order on Y and upward closed with respect to the order on X. (See [7] for more on these ideals.)

We now wish to describe the category $P(A)$-Cat, of categories enriched in the quantaloid $P(A)$ with $P(A)$-functors as morphisms, in terms of lax functors $A^{op} \to Rel$, where recall Rel is the category of sets and relations. Following Ghilardi and Meloni ([11, 12]) we shall call these 'relational presheaves'. It should be pointed out that relational presheaves, as well as the appropriate notion of morphism (discussed below), were independently arrived at in [23], where they were called 'non-deterministic functors'.

We begin with a definition.

DEFINITION 3.4. Let A be a locally small category. A *relational presheaf* on A is a lax functor F: $A^{op} \to Rel$, where laxity means that
(1) $F(f) \circ F(g) \subseteq F(g \circ f)$ for all composable morphisms f, g of A,
(2) $\triangle_{F(a)} \subseteq F(1_a)$, where $\triangle$ is the diagonal relation and $a \in A$.

To every relational presheaf on A, we can associate a $P(A)$-category.

Let F: $A^{op} \to Rel$ be a relational presheaf. Define a $P(A)$-category X_F by setting $X_F[a] = F(a)$ and for $x \in F(a)$, $y \in F(b)$, define the enrichment $X_F(x, y) = \{f\colon a \to b \mid (y, x) \in F(f)\}$. Since F is a lax functor, (2) of Definition 3.1 is satisfied.

Conversely, if X is a $P(A)$-category, define a relational presheaf F_X by $F_X(a) = X[a]$ and if f: $a \to b$ is a morphism in A, then the relation $F(f)$ is defined by $(y, x) \in F(f)$ iff $f \in X(x, y)$. Laxity of F_X follows from the definition of $P(A)$-category and it is easy to see that $F_{X_F} = F$ and $X_{F_X} = X$ for all relational presheaves F and all $P(A)$-categories X.

We now wish to define the notion of morphism of relational presheaves, which will correspond to that of $P(A)$-functor. It should obviously be some kind of lax natural transformation and the appropriate definition requires the transition maps to actually be functions.

DEFINITION 3.5. Let F: $A^{op} \to Rel$ and G: $A^{op} \to Rel$ be relational presheaves on A. A *relational presheaf morphism* (or *rp-morphism*) R: $F \to G$ is a lax natural transformation such that R_a is a function from $F(a)$ to $G(a)$ for all $a \in A$.

Thus, for each $a \in A$, we have a function R_a: $F(a) \nrightarrow G(a)$ such that if f: $b \to a$ is a morphism in A, then $R_b \circ F(f) \subseteq G(f) \circ R_a$.

Let $P(A)$-Cat denote the category of $P(A)$-categories and $P(A)$-functors and let $R(A)$ denote the category of relational presheaves on A and rp-morphisms, (where composition of rp-morphisms R: $F \nrightarrow G$ and S: $G \nrightarrow H$ is defined by $(S \circ R)_a = S_a \circ R_a$).

If R: $F \nrightarrow G$ is an rp-morphism of relational presheaves on A, we define δ_R: $X_F \nrightarrow X_G$ as follows. If $x \in X_F[a] = F(a)$, let $\delta_R(x) = R_a(x)$.

Conversely, if $F, G \in R(A)$ and δ: $X_F \nrightarrow X_G$ is a $P(A)$-functor, we can define R_δ: $F \nrightarrow G$ by $R_{\delta,a}(x) = \delta(x)$ for $x \in F(a)$.

This leads to defining Φ: $R(A) \nrightarrow P(A)$-Cat by $\Phi(F) = X_F$ and if R: $F \nrightarrow G$, then $\Phi(R) = \delta_R$.

We have the following fundamental result relating categories enriched in $P(A)$ with relational presheaves on A. Details can be found in [24].

THEOREM 3.1. Φ: $R(A) \nrightarrow P(A)$-*Cat is an equivalence of categories.*

We wish to point out that this analysis could have been done utilizing covariant relational presheaves instead. Define F_X as before on objects and if f: $a \to b$ is a morphism of A, $(x, y) \in F(f)$ iff $f \in X(x, y)$. This is the dual of the relation in the contravariant case. Since the dual of a relation defines an involution on Rel, we obtain an equivalence of $P(A)$-Cat with covariant relational presheaves on A.

One application of the above equivalence is that the work of Ghilardi and Meloni [11] on modelling the operators of future and past possibility and necessity from modal and temporal logic using relational presheaves can be generalized to the setting of Q-enriched categories, where Q is an arbitrary quantaloid (see [27]).

We now have the necessary background to turn our attention to the applications of these concepts and ideas to automata and tree automata theory.

4. Automata and Tree Automata as Enriched Categories

The possibility of describing automata using enriched category theory goes back to Betti [2]. These ideas were further developed by Betti and Kasangian in [5]. (Also see [14, 15] and [16].) The presentation we shall adopt in this section owes a debt to the various descriptions contained in the above references.

Let M be a monoid. In classical automata theory, M is often taken to be a free monoid Σ^* on an alphabet Σ, however most of the theory readily extends to arbitrary monoids (see [9, 21]). We begin by defining the notion of a non-deterministic M-dynamics.

DEFINITION 4.1. Let M be a monoid. A *non-deterministic M-dynamics* is given by a set X together with a relation α with α: $X \times M \nrightarrow X$ (denoted $(x, a) \approx_\alpha y$) satisfying that

(1) $(x, e) \approx_\alpha x$ for all $x \in X$, where e is the identity of M,
(2) $(x, a) \approx_\alpha y$ and $(y, b) \approx_\alpha z$ implies $(x, ab) \approx_\alpha z$ for all $a, b \in M$ and all $x, y, z \in X$.

We shall omit explicit mention of α (and drop it as a subscript as well) unless it is needed in the context. We shall simply refer to X, where α is understood.

We think of X as the set of states of an automaton. The relationship $(x, a) \approx_\alpha y$ is to be interpreted as saying that y is a possible outcome of the the element a acting on the state x.

X is called *deterministic* iff α is actually a function, i.e α makes X into a right M-set.

DEFINITION 4.2. Let (X, α) and (Y, β) be M-dynamics. A function $f: X \to Y$ is called an *M-morphism* iff $(x, a) \approx_\alpha x'$ implies that $(f(x), a) \approx_\beta f(x')$ for all $a \in M$, $x, x' \in X$.

Let M-Dyn denote the category of M-dynamics and M-morphisms. An M-dynamics gives rise to a relational presheaf F_X: $M^{op} \to$ *Rel*, where $F_X(M) = X$ and given $a \in M$, $F_X(a)$: $X \to X$ is the relation given by $(x, y) \in F_X(a)$ iff $(x, a) \approx_\alpha y$. From condition (2) in Definition 4.1, we readily obtain that $F_X(b) \circ F_X(a) \subseteq F_X(ab)$.

Conversely, a relational presheaf F: $M^{op} \to$ *Rel* leads to an M-dynamics X_F by $(x, a) \approx y$ iff $(x, y) \in F(a)$. The laxity of the functor F yields precisely the required conditions in Definition 4.1.

Recall that $P(M)$ is the quantale of subsets of M under the operation $A \circ B = AB = \{ab \mid a \in A, b \in B\}$. $P(M)$-Cat denotes the category of $P(M)$-enriched categories and $P(M)$-functors.

PROPOSITION 4.1. *If M is a monoid, there is an equivalence of categories $P(M)$-$Cat \cong M$-Dyn.*

Proof. In light of the above discussion, one can refer to Theorem 3.1, which showed the equivalence of $P(M)$-Cat with $R(M)$, the category of relational presheaves on M. □

With this equivalence established, we can now define the notion of an M-automaton and see how the additional structure required is expressible in the language of enriched category theory.

DEFINITION 4.3. If M is a monoid, an *M-automaton* is an M-dynamics X together with a set I of initial states and a set T of terminal states (where $I \subseteq X$ and $T \subseteq X$).

If (X, I, T) and (Y, J, S) are M-automata, then an *M-automata morphism* is an M-dynamics morphism $f: X \to Y$ such that $f(I) \subseteq J$ and $f(T) \subseteq S$.

Thus, we have a category M-Aut of M-automata and M-automata morphisms.

DEFINITION 4.4. Let (X, I, T) be an M-automaton. The *behavior* of (X, I, T) is given by $B(X, I, T) = \{a \in M \mid (a, i) \approx t$ for some $i \in I$, some $t \in T\}$.

We shall often write $B(X)$, where I and T are understood. Thus, the behavior is the set of all elements of M, which can act on an initial state to produce a terminal state.

We can associate $P(M)$-bimodules $\mathbf{I}$ and $\mathbf{T}$ to the initial and terminal states I and T as follows. ($\mathbf{1}$ here denotes the $P(M)$-category with one object $*$ and with $\mathbf{1}(*, *) = e$.)

$$\mathbf{I}\colon X \Rightarrow \mathbf{1} \text{ is given by } \mathbf{I}(*, x) = \{a \in M \mid a \in X(i, x) \text{ for some } i \in I\},$$

$$\mathbf{T}\colon \mathbf{1} \Rightarrow X \text{ is given by } \mathbf{T}(x, *) = \{a \in M \mid a \in X(x, t) \text{ for some } t \in T\}.$$

It is not hard to verify that these are bimodules using the fact that X is a $P(M)$-category.

LEMMA 4.1. *The composite bimodule* $\mathbf{I} \circ \mathbf{T}$*:* $\mathbf{1} \Rightarrow \mathbf{1}$ *evaluated at* $(*, *)$ *is precisely the behavior* $B(X, I, T)$ *of the automaton* (X, I, T).

Proof. $(\mathbf{I} \circ \mathbf{T})(*, *) = \bigcup_{x \in X} \mathbf{I}(*, x)\mathbf{T}(x, *) = \{ab \in M \mid a \in X(i, x)$ and $b \in X(x, t)$ for some $t \in T, i \in I\} \subseteq B(X, I, T)$, since $b \in X(x, t)$ and $a \in X(i, x)$ yields that $ab \in X(i, t)$. On the other hand, given an element $b \in B(X, I, T)$, let $x = i$ and $a = e$ to obtain the opposite containment. □

Following [14], one can define a category M-GenAut of generalized M-automata consisting of triples $(X, \mathbf{I}, \mathbf{T})$, where X is a $P(M)$-category and both $\mathbf{I}\colon X \Rightarrow \mathbf{1}$ and $\mathbf{T}\colon \mathbf{1} \Rightarrow X$ are bimodules. If $(X, \mathbf{I}, \mathbf{T})$ and $(Y, \mathbf{J}, \mathbf{S})$ are generalized automata, then using the notion of composition of bimodules (see Section 3) a morphism in M-GenAut is given by a $P(M)$-functor $f\colon X \to Y$ satisfying that $\mathbf{I} \subseteq \mathbf{J} \circ f_*$ and also $f_* \circ \mathbf{T} \subseteq \mathbf{S}$.

Note that $(\mathbf{J} \circ f_*)(*, x) = \{ab \mid \exists y$ with $a \in \mathbf{J}(*, y)$ and $(y, b) \approx f(x)\}$ and similarly $(f_* \circ \mathbf{T})(x, *) = \{cd \mid \exists y$ with $(y, c) \approx f(x)$ and $d \in \mathbf{T}(x, *)\}$.

PROPOSITION 4.2. *We have a functor* Φ*:* M*-Aut* $\to$ M*-GenAut defined on objects by* $\Phi(X, \mathrm{I}, \mathrm{T}) = (X, \mathbf{I}, \mathbf{T})$ *and on morphisms by* $\Phi(f) = f$*, where we view the automata morphism* f *as a* $P(M)$*-functor.*

Proof. All that needs to be verified is that an M-automata morphism f can, in fact, also be viewed as a morphism in the category M-GenAut. Suppose $a \in \mathbf{I}(*, x)$. Then, there exists $i \in I$ with $(i, a) \approx x$. We have $f(i) \in J$ and $(f(i), a) \approx f(x)$. Since $(f(x), e) \approx f(x)$, it follows that $ae = a \in (\mathbf{J} \circ f_*)(*, x)$ (recall $f_*(y, x) = Y(y, f(x))$).

Now consider a typical element $ab \in (f_* \circ \mathbf{T})(y, *)$. Then, there exists $x \in X$ with $a \in f_*(y, x)$ and $b \in \mathbf{T}(x, *)$. Thus, we have $(y, a) \approx f(x)$ and $(x, b) \approx t$ for some $t \in T$. Hence, $(f(x), b) \approx f(t)$ and so $(y, ab) \approx f(t)$, which is in $\mathbf{S}$. Thus, $ab \in \mathbf{S}(y, *)$, as desired. □

Kelly, Kasangian and Rossi [14] showed that this functor Φ has a right adjoint Ψ: M-GenAut $\to$ M-Aut. If $(X, \mathbf{I}, \mathbf{T})$ is a generalized M-automaton, we can define an M-automaton (X, I, T) by letting $I = \{x \in X \mid e \in \mathbf{I}(*, x)\}$ and similarly $T = \{x \in X \mid e \in \mathbf{T}(x, *)\}$. This will describe Ψ on objects. If f is a morphism in M-GenAut, $\Psi(f) = f$.

Before turning our attention to tree automata, we should also point out that the enriched category theoretical point of view is also useful in discussing recognizable sets and some of their algebraic properties. This is done by Kasangian and Rosebrugh in [16], where they utilize the notion of glueing along bimodules.

In order to generalize these ideas to the case of tree automata defined over an algebraic theory $\mathbf{A}$ (in the sense of Lawvere [18]), we must consider categories enriched in the free quantaloid $P(\mathbf{A})$. Classically, tree automata are described in universal algebraic terms (see [10]), but of course this can be done more efficiently with an invariant presentation using algebraic theories and the concomitant functorial description of algebras. Our discussion here shall basically follow the seminal paper of Betti and Kasangian [5] with a few changes in presentation.

Let $\mathbf{A}$ be an algebraic theory. The objects are represented by non-negative integers $[0], [1], \ldots, [n], \ldots$ where $[0]$ is the initial object, $[1]$ is the terminal object and $[n]$ is the n-fold coproduct of $[1]$. It follows that a map $[n] \to [m]$ can be thought of as an n-tuple of m-ary operations of the theory.

The maps $[1] \to [0]$ are the terms or trees of the theory.

Dually, algebraic theories can also be presented as categories where every object is the product of a fixed object. We shall find this alternative point of view convenient in Section 6, where the free category with products on a multigraph will be an intrinsic part of the development.

DEFINITION 4.5. A *non-deterministic* $\mathbf{A}$-*dynamics* is given by a relational presheaf $\mathbf{X}$: $A^{op} \to Rel$ satisfying that $\mathbf{X}[n] = (\mathbf{X}[1])^n$.

A morphism of $\mathbf{A}$-dynamics is just a relational presheaf morphism (see Definition 3.5). Thus, we have the category $\mathbf{A}$-Dyn of $\mathbf{A}$-dynamics and their morphisms. By Theorem 3.1, the category $R(\mathbf{A})$ of relational presheaves on $\mathbf{A}$ is equivalent to the category $P(\mathbf{A})$-Cat of categories enriched in the quantaloid $P(\mathbf{A})$ together with $P(\mathbf{A})$-functors.

We need to account for the additional requirement we have placed on our relational presheaves.

DEFINITION 4.6. Let $\mathbf{A}$ be an algebraic theory. A $P(\mathbf{A})$-category $\mathbf{X}$ is called *reachable* iff $\mathbf{X}[n] = \mathbf{X}[1]^n$ for all n. It is not hard to establish the following result, where $P(\mathbf{A})_{rch}$-Cat denotes the category of reachable $P(\mathbf{A})$-categories.

PROPOSITION 4.3. *There is an equivalence of categories* $P(\mathbf{A})_{rch}$-$Cat \cong \mathbf{A}$-*Dyn.*

We need to analyze the behavior of a tree automaton in terms of bimodules in a fashion analogous to our presentation for monoids. Let $[\tilde{n}]$ denote the trivial enriched category with one object over $[n]$.

DEFINITION 4.7. A *non-deterministic generalized* $\mathbf{A}$-*automaton* is a triple $(\mathbf{X}, \mathbf{I}, \mathbf{T})$, where $\mathbf{X}$ is a reachable $P(\mathbf{A})$-category and $\mathbf{I}\colon \mathbf{X} \to [\tilde{0}]$ and $\mathbf{T}\colon [\tilde{1}] \to \mathbf{X}$ are bimodules called the initial and terminal bimodules.

The bimodule $\mathbf{I}$ gives n-tuples of trees (terms) and the bimodule $\mathbf{T}$ provides operations to be performed on these trees. We can once again define the behaviour of $(\mathbf{X}, \mathbf{I}, \mathbf{T})$ in terms of bimodule composition.

DEFINITION 4.8. Let $(\mathbf{X}, \mathbf{I}, \mathbf{T})$ be an $\mathbf{A}$-automaton. The *behavior* $B(\mathbf{X}, \mathbf{I}, \mathbf{T})$ is the bimodule $\mathbf{I} \circ \mathbf{T}\colon [\tilde{1}] \to [\tilde{0}]$.

The behavior picks out a set of trees, that is an element of $P(A)([1], [0])$, which is called a forest in Gecseg and Steinby [10]. This forest is the result of applying the operations of $\mathbf{T}$ to the initial n-tuples of terms provided by the bimodule $\mathbf{I}$.

To obtain a category of tree automata, we can generalize the definition from the monoid case.

DEFINITION 4.9. If $(\mathbf{X}, \mathbf{I}, \mathbf{T})$ and $(\mathbf{Y}, \mathbf{J}, \mathbf{S})$ are non-deterministic $\mathbf{A}$-automata, a *morphism* of generalized $\mathbf{A}$-automata $(\mathbf{X}, \mathbf{I}, \mathbf{T}) \to (\mathbf{Y}, \mathbf{J}, \mathbf{S})$ is given by a $P(\mathbf{A})$-functor $f\colon \mathbf{X} \to \mathbf{Y}$ satisfying that $\mathbf{I} \subseteq \mathbf{J} \circ f_*$ and $f_* \circ \mathbf{T} \subseteq \mathbf{S}$.

Thus, we have the category $\mathbf{A}$-Aut of (generalized) $\mathbf{A}$-tree automata. There is the further related work of Betti and Kasangian [5] and and Kasangian and Rosebrugh [14] for the interested reader.

5. Syntactic Nuclei and the Syntactic Congruence

If M is a monoid and $A \subseteq M$, then there is a so-called syntactic congruence θ_A on M corresponding to A. Algebraically, it gives rise to the smallest quotient of M such that A is saturated under θ_A. In the theory of automata, A is recognizable by a finite automaton iff the quotient monoid M/θ_A is finite and the syntactic congruence and resulting monoid are related to the minimal automaton of A in that the transition monoid of the minimal automaton is isomorphic to M/θ_A. For more on the specifics of these constructions see [9] or [21].

In the theory of tree automata, the notion of recognizable set is replaced by that of a recognizable forest [10] and one can define analogues of the syntactic congruence and the minimal recognizer. In [10] there is some discussion of attempts to generalize the notion of syntactic monoid to the setting of tree automata, but any such attempt retaining the notion of monoid is bound to fail to capture the entire picture, as in moving from monoids to algebraic theories we are passing from one-object base categories to more general base categories; more precisely, from categories enriched in the quantale $P(M)$ to those enriched in the quantaloid $P(\mathbf{A})$, as indicated in the previous section. We shall argue, using the theory of nuclei on quantaloids, that in fact the appropriate notion is that of syntactic nucleus or the resulting syntactic quantaloid.

Earlier (Proposition 2.2), we described an adjunction between congruences on a locally small category A and quantaloidal nuclei on $P(A)$. This allows us to view nuclei as generalized congruences. With this in mind, we shall develop a general construction of the syntactic nucleus associated to a morphism (or family of morphisms) in a quantaloid Q. Utilizing this we will obtain a simultaneous generalization of both the monoid (automata) and algebraic theory (tree automata) cases.

We can now begin with the discussion of the syntactic nucleus. For detailed proofs of the results presented, see [25].

DEFINITION 5.1. Let Q be a quantaloid and let $f \in Q(c,d)$ and let $h \in Q(a,b)$. A morphism $g \in Q(a,b)$ is called *f-compatible with h* iff

$$h \to_r (x \to_r f) = g \to_r (x \to_r f) \quad \text{for all } x \in Q(b,d).$$

Now, define $j(f)\colon Q \to Q$, as follows: It is the identity on objects and for $h \in Q(a,b)$, let $j(f)_{a,b}(h) = \sup\{g \in Q(a,b) \mid g \text{ is } f\text{-compatible with } h\}$.

We have the following theorem which is the main result of this section. The proof in [25] utilizes various parts of Lemma 2.1.

THEOREM 5.1. *Let Q be a quantaloid and let $f \in Q(c,d)$. Then,*
(1) *$j(f)$ is a quantaloidal nucleus on Q.*
(2) *$j(f)_{c,d}(f) = f$.*
(3) *If j is any quantaloidal nucleus on Q satisfying $j_{c,d}(f) = f$, then $j \leqslant j(f)$.*
(4) *$Q_{j(f)}$ is the smallest quantaloidal quotient of Q containing f.*

This leads to the following definition.

DEFINITION 5.2. If Q is a quantaloid and $f \in Q(c,d)$ is a morphism of Q, then $j(f)$ is called the *syntactic nucleus* associated to f.

The choice for using right residuation was arbitrary. Thus, a natural question to ask is what happens if one considers the following. As before, let $f \in Q(c,d)$ and let $h \in Q(a,b)$. Now, define $J(f)_{a,b}(h) = \sup\{g \in Q(a,b) \mid h \rightarrow_l (y \rightarrow_l f) = g \rightarrow_l (y \rightarrow_l f)$ for all $y \in Q(c,a)\}$.

It can be shown that $J(f)$ is also a quantaloidal nucleus and in fact, $J(f) = j(f)$.

We should also point out that if F is a family of morphisms of Q, we can form the nucleus $j(F) = \bigcap\{j(f) \mid f \in F\}$. The resulting quotient $Q_{j(F)}$ is the smallest one containing all the morphisms in F.

We now need to justify the terminology syntactic nucleus and show that it indeed captures the syntactic congruence of both automata and tree automata theory.

Let us begin by considering the case of a monoid M and the syntactic congruence construction for an element A of the quantale $P(M)$.

DEFINITION 5.3. Let M be a monoid and let $A \subseteq M$. The *syntactic congruence* of A, which shall be denoted θ_A, is defined by $(s,t) \in \theta_A$ when $usv \in A$ iff $utv \in A$ for all $u, v \in M$.

As mentioned above, this is the largest congruence on M saturating A and one can define A to be recognizable iff θ_A is of finite index, that is M/θ_A is a finite quotient of M.

The syntactic nucleus on $P(M)$ associated with A, $j(A)$, is defined by $j(A)(C) = \bigcup\{B \in P(M) \mid B \rightarrow_r (X \rightarrow_r A) = C \rightarrow_r (X \rightarrow_r A)$ for all $X \subseteq M\}$.

Recall from Proposition 2.2 that we have the functor $G: N(P(M)) \to Con(M)$ defined by $(s,t) \in G(j)$ iff $j(s) = j(t)$ associating a congruence $G(j)$ on M to a quantic nucleus j on $P(M)$.

The following proposition justifies our use of the terminology 'syntactic nucleus' by showing that we can recover the congruence θ_A from the nucleus $j(A)$. The proof is a straightforward calculation using the definition of residuation in $P(M)$.

(Since the notion of nucleus is more general than that of congruence, one would not necessarily expect to obtain $j(A)$ from θ_A.) Recall from Section 2.

PROPOSITION 5.1. *Let M be a monoid and let $A \subseteq M$. Let θ_A denote the syntactic congruence of A on M and let $j(A)$ denote the syntactic nucleus on $P(M)$. Then, $G(j(A)) = \theta_A$.*

Turning our attention to the tree automata case, in Section 3 we discussed how tree automata can be viewed as $P(\mathbf{A})$-enriched categories, where $\mathbf{A}$ is an algebraic theory. We do not need all the technical details concerning the bimodules $\mathbf{I}$ and $\mathbf{T}$ in our discussion (although they will be central in Section 6), but can in fact just consider the notion of an $\mathbf{A}$-dynamics.

There is discussion of the syntactic congruence for tree automata in [10], where they are limited by their universal algebraic, as opposed to categorical, perspective. The use of quantaloidal nuclei clearly indicates the way in which the monoid (automata) case should be generalized by providing a general framework including both examples in a natural way. The syntactic nucleus construction is more general in that it applies not just to forests (morphisms $[1] \to [0]$ in the quantaloid $P(\mathbf{A})$), but in fact to any morphism of $P(\mathbf{A})$ and natural in its use of the residuation operations of a quantaloid.

We need to analyze the residuation operations in $P(\mathbf{A})$ relative to forests. This will lead to an understanding of the syntactic nucleus in this setting.

Let B: $[1] \to [0]$ be a forest and let X: $[n] \to [0]$ be a morphism of $P(\mathbf{A})$. Thus, X is a set of n-tuples $(x_1, x_2, \ldots, x_n)$ of terms x_i: $[1] \to [0]$.

Then, $X \to_r B$ is a map $[1] \to [n]$ and is defined by $X \to_r B = \{f \mid f$ is an n-ary operation and $f(x_1, x_2, \ldots, x_n) \in B$ for all $x_1, x_2, \ldots, x_n \in X\}$.

If Y: $[1] \to [n]$ is a set of n-ary operations and H: $[m] \to [n]$ is a set of m-tuples of n-ary operations, then $H \to_r Y$: $[1] \to [m]$ is defined by $H \to_r Y = \{g \mid g$ is an m-ary operation and $g(h_1, h_2, \ldots, h_m) \in Y$ for all $(h_1, h_2, \ldots, h_m) \in H\}$.

It follows that $g \in H \to_r (X \to_r B)$ iff g is an m-ary operation and given $(h_1, h_2, \ldots, h_m) \in H$ (where each h_i is n-ary), and given $(x_1, x_2, \ldots, x_n) \in X$ (where each x_i is a term), we have:

$$g(h_1(x_1, x_2, \ldots, x_n), h_2(x_1, x_2, \ldots, x_n), \ldots, h_m(x_1, x_2, \ldots, x_n)) \in B.$$

Thus, for n-ary operations $k_1, k_2, \ldots, k_m$, we have $(k_1, k_2, \ldots, k_m) \in j_B(H)$ iff:

$$\begin{aligned} &g(h_1(x_1, x_2, \ldots, x_n), h_2(x_1, x_2, \ldots, x_n), \ldots, h_m(x_1, x_2, \ldots, x_n)) \in B \quad \text{iff} \\ &g(k_1(x_1, x_2, \ldots, x_n), k_2(x_1, x_2, \ldots, x_n), \ldots, k_m(x_1, x_2, \ldots, x_n)) \in B, \\ &\text{for all } (h_1, h_2, \ldots, h_m) \in H. \end{aligned}$$

Now that we have an explicit calculation of $j_B(H)$, let us consider some special cases.

Suppose h, k: $[1] \to [n]$ are n-ary operations. Then, using j_B we obtain $j_B(h) = j_B(k)$ iff for all unary operations f, for all n-tuples of terms (trees) $(x_1, x_2, \ldots, x_n)$, we have

$$f(h(x_1, x_2, \ldots, x_n)) \in B \quad \text{iff } f(h(x_1, x_2, \ldots, x_n)) \in B.$$

If we simplify this to the case where $n = 0$, i.e. h and k are trees, then we obtain from j_B the congruence θ_B on $\mathbf{A}([1], [0])$ the free $\mathbf{A}$-algebra of all $\mathbf{A}$-trees, defined by $(h, k) \in \theta_B$ iff $j_B(h) = j_B(k)$ iff for all unary f, $f(h) \in B$ iff $f(k) \in B$.

This is precisely the congruence on $\mathbf{A}([1], [0])$ described in [10, pp. 89–90] which results in the minimal recognizer of the forest B. One can also realize, after a moment's reflection, that this is the congruence described on pp. 94–95 of [10]. B is recognizable precisely when this congruence is of finite index.

We have seen that the same residuation calculations, for the quantaloids $P(M)$ and $P(\mathbf{A})$, are behind the construction of the syntactic congruence in the theory of automata as well as its generalization to tree automata. The categorical approach clarifies exactly the way in which the two are related.

6. Context-Free Languages and Tree Automata

It is well-known in the theory of computation that there is a close relationship between context-free languages and tree automata. The languages which arise as the yields of the recognizable forests (behaviors) of finite-state tree automata are precisely the context-free languages [10]. We shall show that this correspondence can made clear via the enriched category (equivalently, relational presheaf) approach to tree automata described in Section 4. This approach will also allow us to provide simple proofs and furthermore it will indicate the central role played by the algebraic theory $Mon(\mathrm{A})$ of monoids augmented by a set of constants A. Central to this work is Walters' categorical approach to context-free languages [33, 35].

Walters showed that a context-free grammar on an alphabet A can be realized as a multigraph morphism $G \to \tilde{A}$, where $\tilde{A}$ is the multigraph associated to the set A. Given an object X of G, the context-free language associated with X is constructed by considering the free categories with products on G and $\tilde{A}$ and the algebraic theory $Mon(\mathrm{A})$ of monoids augmented by constants A, which arises as a quotient in this setting. We use the notion of a *multirelation with constants* A as a bridge between context-free grammars with alphabet A and relational presheaves $Mon(\mathrm{A}) \to Rel$. The algebraic theory $Mon(\mathrm{A})$ plays the key role and an analysis of the case of a general algebraic theory reduces to looking at it.

Since we shall be working with free categories with products on a multigraph, we consider algebraic theories as having as objects products of a single fixed object and consider covariant (as opposed to contravariant) relational presheaves.

We begin with the definitions of multigraph and multigraph morphism in order to summarize the aforementioned work of Walters.

DEFINITION 6.1. A *multigraph* G consists of a set of objects G_* and for every $X \in G_*$, and for every list $X_1, X_2, \ldots, X_n$ of elements of G_* with $n \geqslant 0$, a set of multiarrows $X_1 X_2 \cdots X_n \to X$.

If $n = 0$, we write multiarrows as $1 \to X$.

DEFINITION 6.2. If G and H are multigraphs, a multigraph morphism, denoted ϕ: $G \to H$, consists of a function ϕ_*: $G_* \to H_*$ together with an assignment for each multiarrow α: $X_1 X_2 \cdots X_n \to X$ in G, of a multiarrow in H

$$\phi(\alpha)\colon\ \phi_*(X_1)\phi_*(X_2)\cdots\phi_*(X_n) \to \phi_*(X).$$

Let A be a set. We can associate to it a multigraph $\tilde{A}$ with a single object M and a single multiarrow ι_n: $M^n \to M$ for each $n \geqslant 1$ and a multiarrow a: $1 \to M$ for each $a \in$ A. Walters [33] (also see [35]) analyzed the notion of a context-free grammar with alphabet A as follows.

DEFINITION 6.3. A context-free grammar on a set A is a multigraph morphism ϕ: $G \to \tilde{A}$, where G is a multigraph with a finite number of objects and multiarrows.

We denote by $F_\times(G)$ the free category with (strictly associative finite) products [34].

The context-free grammar ϕ: $G \to \tilde{A}$ induces a morphism of categories with products $F_\times(\phi)$: $F_\times(G) \to F_\times(\tilde{A})$.

$F_\times(\tilde{A})$ has as objects M^n and its morphisms correspond to bracketings of variables and elements of A. We let Mon(A) denote the algebraic theory of monoids augmented with constants A. Departing from the notation of Section 3, we shall suggestively (since we are considering monoids) denote its unique object by M as well, and denote its other objects by M^n, using 1 instead of [0], as we are now working with products rather than coproducts. There is a functor

$$\psi\colon\ F_\times(\tilde{A}) \to Mon(A)$$

which assigns M to M and M^n to M^n and identifies different bracketings equivalent under associativity. Thus, Mon(A) arises as a quotient category of $F_\times(\tilde{A})$ and the free monoid on A, $A^* = \mathrm{Hom}_{Mon(A)}(1, M)$.

Walters gave the following definition of context-free languages, which does coincide with the classical notion.

DEFINITION 6.4. Let $G \to \tilde{A}$ be a context-free grammar and let $X \in G_*$. The *context-free language associated to* X is the subset $\psi(F_\times(\phi)(\mathrm{Hom}(1, X)))$ of A* where the Hom is taken in $F_\times(G)$.

The category *Rel* of sets and relations is a multigraph with a multiarrow $X_1X_2 \cdots X_n \to X$ being an $(n+1)$-ary relation $R \subseteq (X_1 \times X_2 \times \cdots \times X_n \times X)$. By a *multirelation*, we mean a multigraph morphism $J \to$ *Rel*, where J is the multigraph with exactly one object M and one multiarrow ι_n: $M^n \to M$ for each $n \geqslant 0$. These were studied by Ghilardi and Meloni [12]. We slightly generalize this as follows.

DEFINITION 6.5. A *multirelation with constants* A is a multigraph morphism $\tilde{A} \to$ *Rel*.

If ϕ: $G \to \tilde{A}$ is a context-free grammar, we obtain **G**, a multirelation with constants A, defined by $\mathbf{G}(M) = G_*$ and the multiarrow $\mathbf{G}(\iota_n)$: $(G_*)^n \to G_*$

is defined for $n \geqslant 1$ by letting $(X_1, X_2, \ldots, X_n, X) \in \mathbf{G}(\iota_n)$ iff there exists a multiarrow $X_1 X_2 \cdots X_n \;\to X$ in G.

For each $a \in A$, define the subset $\mathbf{G}(a)$: $1 \to G_*$ by $X \in \mathbf{G}(a)$ iff there is a multiarrow α: $1 \to X$ in G such that $\phi(\alpha) = a$.

Conversely a multirelation K: $\tilde{A} \to Rel$, with appropriate finiteness conditions, gives rise to a context-free grammar $\overline{K}$ having objects $K(M)$ and for $n \geqslant 1$, the unique multiarrow $X_1 X_2 \cdots X_n \;\to X$ iff $(X_1, X_2, \ldots, X_n, X) \in K(\iota_n)$. When $n = 0$, then $\overline{K}$ has a multiarrow a: $1 \to X$ whenever $X \in K(a)$. There is an obvious multigraph morphism $\overline{K} \to \tilde{A}$.

Beginning with a context-free grammar ϕ: $G \to \tilde{A}$, the context-free languages determined by $\bar{\phi}$: $\overline{\mathbf{G}} \to \tilde{A}$ are the same as those of ϕ: $G \to \tilde{A}$.

Since the emphasis in this section revolves around free categories with products, we shall use covariant relational presheaves $\mathbf{A} \to Rel$; in particular we are interested in relational presheaves on the algebraic theory Mon(A) of monoids augmented with constants A.

First observe that the multirelation $\mathbf{G}$ described above, which is associated to a context-free grammar G extends to a relational presheaf $\overline{\mathbf{G}}$: $F_\times(\tilde{A}) \to Rel$, where $\overline{\mathbf{G}}(M) = G_*$ and given a morphism γ: $M^n \to M^k$ in $F_\times(\tilde{A})$,

$((X_1, X_2, \ldots, X_n), (Y_1, Y_2, \ldots, Y_k) \in \overline{\mathbf{G}}(\gamma)$ iff there exists

α: $((X_1 \times X_2 \times \cdots \times X_n) \to (Y_1 \times Y_2 \times \cdots \times Y_k)$ such that $F_\times(\phi)(\alpha) = \gamma$.

For notational convenience we shall let ξ denote the composite functor

$$(\psi \circ F_\times(\phi)) \colon F_\times(G) \to F_\times(\tilde{A}) \to Mon(A).$$

Given $X \in G_*$, the relational presheaf $\overline{\mathbf{G}}$ can be extended to a relational presheaf $\overline{\mathbf{G}}$: Mon(A) $\to Rel$, as follows.

Let $\overline{\mathbf{G}}(M) = \Gamma$ and $\overline{\mathbf{G}}(M^n) = \Gamma^n$, where $\Gamma = \{w \mid w = X$ or $w = W_1 W_2 \cdots W_k$ is a subword of the domain of a multiarrow in $G_*\}$.

We shall use $[w]$ to denote $W_1 \times W_2 \times \cdots \times W_k$ in $F_\times(G)$.

It suffices to define $\overline{\mathbf{G}}$ for the multiplication μ: $M^2 \to M$ and the constants σ: $1 \to M$ (i.e. $\sigma \in A^*$) in Mon(A).

Define $((w, u), v) \in \overline{\mathbf{G}}(\mu)$ iff there exists α: $[w] \times [u] \to [v]$ in $F_\times(G)$ and $w \in \overline{\mathbf{G}}(\sigma)$ iff there exists σ_i: $1 \to W_i$ in G_* (where $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$).

It is not hard to see that these definitions lead to a relational presheaf.

We shall now give explicit descriptions of the initial and terminal bimodules in this concrete setting.

DEFINITION 6.6. Let $\mathbf{X}$ be a Mon(A)-dynamics, i.e. a relational presheaf $\mathbf{X}$: Mon(A) $\to Rel$ satisfying that $\mathbf{X}(M^n) = \mathbf{X}(M)^n$.

(1) An *initial bimodule* $\mathbf{I}$ for X is given by the following:

For $(x_1, x_2, \ldots, x_n) \in X(M)^n$, an assignment of a subset $\mathbf{I}(x_1, x_2, \ldots, x_n)) \subseteq (A^*)^n$ such that given $(y_1, y_2, \ldots, y_k) \in X(M)^k$ with $((x_1, x_2, \ldots, x_n)),$

$(y_1, y_2, \ldots, y_k)) \in X(\gamma)$, where γ: $M^n \to M^k$ is a morphism in Mon(A), then

$$(\gamma \circ \mathbf{I})(x_1, x_2, \ldots, x_n) \subseteq \mathbf{I}(y_1, y_2, \ldots, y_k).$$

Note $(\gamma \circ \mathbf{I})(x_1, x_2, \ldots, x_n) = \{\gamma \circ \lambda \mid \lambda \in \mathbf{I}(x_1, x_2, \ldots, x_n)\}$.
A *terminal bimodule* $\mathbf{T}$ for X is given by the following:

For $(x_1, x_2, \ldots, x_n) \in X(M)^n$, an assignment of a subset $\mathbf{T}(x_1, x_2, \ldots, x_n))$ of the set of n-ary operations of Mon(A) such that given $(y_1, y_2, \ldots, y_k) \in X(M)^k$ with $((y_1, x_2, \ldots, y_n)), (x_1, x_2, \ldots, x_k)) \in X(\lambda))$, where λ: $M^k \to M^n$ is a morphism in Mon(A), then $\mathbf{T}(x_1, x_2, \ldots, x_n) \circ \lambda \subseteq \mathbf{T}(y_1, y_2, \ldots, y_k)$.
Note $\mathbf{T}(x_1, x_2, \ldots, x_n) \circ \lambda = \{\alpha \circ \lambda \mid \alpha \in \mathbf{T}(x_1, x_2, \ldots, x_n)$.

We reiterate here the definition of a tree automaton for the benefit of the reader.

DEFINITION 6.7. A *non-deterministic generalized* Mon(A)*-tree automaton* is a triple $(\mathbf{X}, \mathbf{I}, \mathbf{T})$ where $\mathbf{X}$ is a Mon(A)-dynamics, $\mathbf{I}$ is an initial bimodule and $\mathbf{T}$ is a terminal bimodule.

The *behavior* $B(\mathbf{X}, \mathbf{I}, \mathbf{T})$ of is the subset of A^* defined by $B = \{\alpha \circ \beta \mid$ there exists $(x_1, x_2, \ldots, x_n) \in \mathbf{X}(M)^n$ with $\alpha \in \mathbf{T}(x_1, x_2, \ldots, x_n)$ and $\beta \in \mathbf{I}(x_1, x_2, \ldots, x_n)\}$.

Note that by taking the covariant point of view, the behavior is now described by $\mathbf{T} \circ \mathbf{I}$ (as opposed to $\mathbf{I} \circ \mathbf{T}$).

Note that in the above, $\alpha \circ \beta$: $1 \to M$ in Mon(A) and hence is an element of A^*.

We are now ready for our first main result.

THEOREM 6.1. *Let* ϕ: $G \to \tilde{A}$ *be a context-free grammar and let* $X \in G_*$. *Then, there is a non-deterministic* $Mon(A)$*-tree automaton* $(\mathbf{X}, \mathbf{I}, \mathbf{T})$ *such that its behavior* B *is the context-free language associated to* X.

Proof. Consider the Mon(A)-dynamics $\overline{\mathbf{G}}$: Mon(A) $\to$ *Rel* described above. Define bimodules $\mathbf{I}$ and $\mathbf{T}$ as follows. Let

$$\mathbf{I}(w_1, w_2, \ldots, w_n) = \{(\sigma_1, \sigma_2, \ldots, \sigma_n) \mid w_i \in \overline{\mathbf{G}}(\sigma_i)\}.$$

$$\mathbf{T}(w_1, w_2, \ldots, w_n) = \{\beta\colon M^n \to M \mid ((w_1, w_2, \ldots, w_n, X) \in \overline{\mathbf{G}}(\beta)\}.$$

The fact that these satisfy the requirements for initial and terminal bimodules respectively, is immediate (by the functorality of ξ) and it is not hard to see that the behavior $B = \mathbf{T} \circ \mathbf{I}$ is precisely the context-free language associated to the object X. □

We now wish to show that for any algebraic theory $\mathbf{A}$, the behavior, under some mild finiteness restrictions, of any $\mathbf{A}$-tree automaton gives rise to a context-free language in a natural way. Since we are taking the perspective of products in this section, we recall for the reader that we now wish to consider an algebraic theory $\mathbf{A}$ with objects represented as products M^n of a single object M. The morphisms $M^n \to M$ are n-ary operations of the theory and morphisms $M^n \to M^k$ are k-tuples of n-ary operations. Composition is given by substitution of operations. Elements of $\mathbf{A}(1, M)$ are called the terms of $\mathbf{A}$, which we shall denote by $\tau(\mathbf{A})$.

We shall assume that $\mathbf{A}$ has a distinguished set of constants denoted A. The notions of initial and terminal bimodule from Definition 6.6 can be directly carried over by replacing Mon(A) by $\mathbf{A}$ and using $\tau(\mathbf{A})$ for A^* in the definition of $\mathbf{I}$.

This leads to the following definition.

DEFINITION 6.8. A *non-deterministic generalized* $(\mathbf{A}, A)$*-tree automaton* is a triple $(\mathbf{X}, \mathbf{I}, \mathbf{T})$ where $\mathbf{X}$ is a $\mathbf{A}$-dynamics, $\mathbf{I}$ is an initial bimodule and $\mathbf{T}$ is a terminal bimodule.

Classically, it is required that for the dynamics $\mathbf{X}$, $\mathbf{X}(M)$ is a finite set. The further data consists of an initial assignment of constants and a set of final states. Obviously, the notions of initial and terminal bimodule that we have been using are much more general.

In order to describe the connection between context-free languages and tree automata, [10] introduces the concept of a recursively defined yield function denoted yd: $\mathbf{A}(1, M) \to Mon(\mathrm{A})(1, M) = A^*$. It assigns to every forest a language in the alphabet A and it is presented in universal algebraic terms by: $yd(a) = a$ for every $a \in \mathrm{A}$ and for a constant $c \notin \mathrm{A}$, $yd(c) = e$, the empty word. Upon being given a more general term such as $\omega = \sigma(\tau_1, \tau_2, \dots, \tau_n)$, then $yd(\omega) = yd(\tau_1) \cdot yd(\tau_2) \cdot \cdots \cdot yd(\tau_n)$, where $\cdot$ denotes concatenation. Thus, in effect, the yield function reduces an m-ary operation of the given theory to concatenation in A^*.

Applying the yield function to the forest B recognized by a finite-state $(\mathbf{A}, A)$-tree automaton results in the language $yd(B)$ *recognized* by the automaton and this language will be context-free.

In our categorical approach, we shall associate to every finite-state $(\mathbf{A}, \mathrm{A})$-tree automaton $(\mathbf{X}, \mathbf{I}, \mathbf{T})$ a multirelation with constants A, and hence, as described earlier, a context-free grammar. Then, by again making use of the algebraic theory Mon(A) and by applying the constructions of Theorem 6.1, we shall directly obtain the yield of the behavior of $(\mathbf{X}, \mathbf{I}, \mathbf{T})$.

We first need to suitably restrict our generalized tree automata.

DEFINITION 6.9. Let $\mathbf{A}$ be an algebraic theory with a distinguished set of constants A. A non-deterministic generalized $(\mathbf{A}, A)$-tree automaton $(\mathbf{X}, \mathbf{I}, \mathbf{T})$ is called a *finite-state recognizer* iff

(1) $X = \mathbf{X}(M)$ is a finite set

(2) Given $a_1, a_2, \ldots, a_n \in A$ and $x_1, x_2, \ldots, x_n \in X$, then

$$(a_1, a_2, \ldots, a_n) \in \mathbf{I}((x_1, x_2, \ldots, x_n) \quad \text{iff } a_i \in \mathbf{I}(x_i) \text{ for all } i = 1, 2, \ldots, n.$$

An initial bimodule **I**, as defined in Definition 6.6, need not satisfy the second requirement.

If $x \in X$, let $\mathbf{T}_x$ be defined by $\alpha \in \mathbf{T}_x(x_1, x_2, \ldots, x_n)$ for $x_1, x_2, \ldots, x_n \in X$ iff $\alpha \in \mathbf{T}(x_1, x_2, \ldots, x_n)$ and $(x_1, x_2, \ldots, x_n, x) \in \mathbf{X}(\alpha)$.

One can check fairly easily that $\mathbf{T}_x$ qualifies as a terminal bimodule. We now specialize the defintion of behavior.

DEFINITION 6.10. The *x-behavior* of $(\mathbf{X}, \mathbf{I}, \mathbf{T})$ is given by B_x, the behavior of $(\mathbf{X}, \mathbf{I}, \mathbf{T}_x)$.

$$B_x = \{\alpha \circ \beta \mid \exists (x_1, x_2, \ldots, x_n) \in \mathbf{X}(M)^n \quad \text{with } \alpha \in \mathbf{T}_x(x_1, x_2, \ldots, x_n), \\ \beta \in \mathbf{I}(x_1, x_2, \ldots, x_n)\}.$$

Given a finite-state recognizer $(\mathbf{X}, \mathbf{I}, \mathbf{T})$, define $\overline{\mathbf{X}}$, a multirelation with constants A, as follows:
(1) $(x_1, x_2, \ldots, x_n, x) \in \overline{\mathbf{X}}$ iff there exists $\alpha \in \mathbf{T}_x(x_1, x_2, \ldots, x_n)$,
(2) if $a \in A$, define a subset $\overline{\mathbf{X}}(a)$ of X by $x \in \overline{\mathbf{X}}$ iff $a \in \mathbf{I}(x)$.

Let $\mathbf{G}_\mathbf{X}$ denote the associated context-free grammar and let $x \in X$. As described in Theorem 6.1, we can construct a Mon(A)-tree automaton, which we shall denote $(\overline{\mathbf{G}}_\mathbf{X}, \mathbf{I}_\mathbf{X}, \mathbf{T}_\mathbf{X})$, so that the behavior of $(\overline{\mathbf{G}}_\mathbf{X}, \mathbf{I}_\mathbf{X}, \mathbf{T}_\mathbf{X})$, denoted by $B_{\mathbf{X},x}$ is the context-free language associated to x for the grammar $\mathbf{G}_\mathbf{X}$.

Note that, in effect, passing to the algebraic theory Mon(A) is doing the work of the yield function described above. The following result is now straightforward to obtain.

THEOREM 6.2. *Let* $(\mathbf{X}, \mathbf{I}, \mathbf{T})$ *be a finite-state* $(\mathbf{A}, A)$*-recognizer with* $\mathbf{X}(M) = X$*, and for* $x \in X$*, let* B_x *denote the x-behavior of* $(\mathbf{X}, \mathbf{I}, \mathbf{T})$*. Then,* $yd(B_x) = B_{\mathbf{X},x}$*, the behavior of the Mon(A)-recognizer* $(\overline{\mathbf{G}}_\mathbf{X}, \mathbf{I}_\mathbf{X}, \mathbf{T}_\mathbf{X})$.

COROLLARY 6.1. *The yield of a finite-state* $(\mathbf{A}, A)$*-recognizer is a context-free language.*

Proof. The fact that the behavior B of $(\mathbf{X}, \mathbf{I}, \mathbf{T})$ is also context-free follows readily from the closure properties of context-free languages and the fact that each B_x is context-free for $x \in X$.

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