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THE HALTING PROBLEM OF ONE STATE TURING MACHINES WITH n-DIMENSIONAL TAPE

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1. Introduction

SHANNON [1; pp. 161–163] showed that it is impossible to construct a universal TURING machine using only one internal state. FISCHER [2; p. 573] pointed out that if we use a weaker definition of universality, e.g. that of DAVIS [3; pp. 167–169] SHANNON's proof does not apply. However, since all definitions of universality agree that a universal TURING machine must have an unsolvable halting problem, in order to show the impossibility of a one state universal TURING machine it is sufficient to show that all one state TURING machines have a solvable halting problem. According to FISCHER [2] this has been done by R. ABBOTT and R. BOYD and also, independently, by HERMAN [5]. Here we shall do something more and show that all one state TURING machines with n-dimensional tape have a solvable halting problem.

Having solved the halting problem for one state TURING machines one wonders in what ways such machines can be generalised so that we retain the solvability of the halting problem.

We certainly cannot increase the number of states for as it was shown by SHANNON [1; pp. 158–160] there are two state universal TURING machines.

We cannot add an extra tape for, as was pointed out by SHEPHERDSON [4], any TURING machine can be simulated step by step by a one state TURING machine with two tapes. One of the tapes can be used (without moving it at all) to hold the state symbol.

Similarly, we cannot have one tape with two independently moving reading heads on it. This is because there are TURING machines with unsolvable halting problem which only use half of the computing tape. Then we can simulate step by step the working of such a machine by a one tape one state TURING machine with two reading heads, simply by using the originally unused position of tape to hold the state symbol.

So if a generalisation of a one state TURING machine is to have a solvable halting problem it can only have one state, one tape and one reading head. (We ignore questions like what happens if a one state, one tape TURING machine has two reading heads, one of which can only write, the other can only read.)

Two such generalisations suggest themselves. One is to allow the one state TURING machine to have an n-dimensional tape. The other is to allow the reading head to move to squares not immediately next to the one which was last scanned.

If a one state TURING machine is of the former kind it has a solvable halting problem as we shall show. Whether all one state TURING machines of the latter kind have a solvable halting problem is still unknown. In HERMAN [6] it is pointed out where the proof of the former goes wrong when applied to the latter.

There are some other generalisations which we must call improper since when applied the essential finite nature of TURING machines is lost. Such is for instance to allow an infinite alphabet for TURING machines. Such improper generalisations are also briefly discussed in HERMAN [6].

2. Basic definitions

In this paper we shall give a rather intuitive description of the formal proof given in HERMAN [6]. Readers willing but unable to reconstruct the formal proof will find a very detailed development there.

Here we shall restrict ourselves for the 2-dimensional case. The proof described works mutatis mutandis in the n-dimensional case as well. The general case is proved in HERMAN [6].

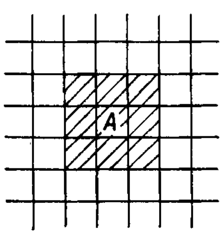

Fig. 1. The tape of a two dimensional TURING machine. The shaded area marks the nine squares neighbouring the square with A in it.

Fig. 2. The tape of a one dimensional TURING machine. The shaded area marks the three squares neighbouring the square with A in it.

A two dimensional one state TURING machine can be thought of as an infinite sheet of paper divided into squares. On these squares symbols are printed all belonging to a finite alphabet. Blank is also taken as a symbol, and at any given time all but a finite number of squares are blank.

Furthermore, we have a reading head, which scans at any given time exactly one square of the tape. It is sensitive to the symbol written there. Upon seeing the symbol the machine may either halt (switch itself off) or replace the symbol in the scanned square by another (possibly the same) symbol of the alphabet and then move to a neighbouring (possibly the same) square. (See Figs. 1 and 2.)

Whether the machine will halt and, if it will not, by what symbol it will replace the scanned symbol and to which neighbouring square it will move is entirely determined by the scanned symbol. If the machine does not halt the same operations are carried out depending on the new scanned symbol.

At any time the future history of the machine is determined by what is printed on the tape at the time together with the scanned square. This information is called a *description* of the machine.

A description is called *mortal* if when the machine starts in that description it will eventually halt. Otherwise it is called *immortal*.

The *halting problem* for 2-dimensional one state TURING machines is to give a decision procedure which will decide for an arbitrary 2-dimensional one state TURING machine M and an arbitrary description D of M whether or not D is mortal.

3. The solution of the halting problem for 2-dimensional one state Turing machines

Let M be the given machine and D the given description.

We shall distinguish cases depending on the nature of M. In each case the decision procedure will consist in working out the descriptions $D = D_0, D_1, D_2, \ldots$ which occur after D in succession up to a point q at which either the machine halts or it becomes clear that the machine will never halt.

Let X denote a *finite* square region of the tape which includes all non-blank squares and the scanned square of D.

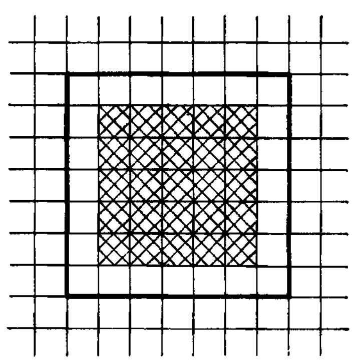

Fig. 3. Shaded area is X. Outside X all squares are blank. X' is the area inside the thick line.

Let X' denote X together with a belt of blank squares around X (see Fig. 3).

During the computation the only way the reading head can move from a square of X to a square outside X' is through a square of the belt $X' - X$.

Case α. Once the reading head scans an empty square it will never move to another square.

Since there are only finitely many letters in the alphabet we can quickly find out whether or not this is the case.

In Case α the reading head can never move outside X'. Since there are only finitely many different descriptions in which all squares outside X' are blank and unscanned, by following the computation $D_0, D_1, D_2, \ldots$ by M we shall sooner or later get to a description D_q for which either (i) the machine halts or (ii) D_q is the same as D_p, for some $p < q$. In the first case D is not immortal, while in the second case it is.

If Case α does not apply then, if the reading head is scanning a square with a blank in it, it will sooner or later move out of that square. Let us call the direction in which it first moves the *positive principal direction*, the direction opposite to it the *negative principal direction*, and these two directions, together with not moving at all, the *principal direction.*

We denote by Y the set of all squares on the tape which lie in the negative principal direction from X.

Case β'. Whenever the machine is scanning a square which at some time before has been blank, it will either halt or move in the principal direction.

If we are in Case β', while the machine is outside X' it will always be scanning squares which lie in the principal direction from the element of X' which has last been scanned.

Note that if we deal with (one-dimensional) one state TURING machines the equivalents of Cases α and β' are the only cases which can occur. So after we solved the halting problem in Case β' the halting problem for one state TURING machines is also solved.

Let us follow the computation $D_0, D_1, D_2, \ldots$ by M. If the machine halts or at some point q we recognise that $D_q = D_p$, for some $p < q$, the halting problem is solved. Otherwise the machine cannot stay within X' for more than a fixed finite number of steps in succession.

Nor can it stay outside X' for ever without us recognising this within a finite number of steps.

For if the machine is outside X' but not in Y, then by the definition of principal direction it will never halt. This happens in Fig. 4 b) and c).

Now suppose that the machine stays for ever outside X', but the square of X' which it last scanned is in Y. Such is the situation in Fig. 4a). In this case it will forever stay in the half line (heavily shaded in the Figure) determined by the principal direction. Suppose that we are at the time when the reading head last scans a square of X' and that this half line has the symbols $a\,b\,c \ldots k$ followed by blanks printed in its squares, starting from the square of X' [Fig. 5a)].

This word $a\,b\,c \ldots k$ is determined solely by the number of times the square which is now being scanned has been scanned.

From now on the reading head will always scan squares outside X'. Sooner or later the machine must arrive at a situation when the word $a\,b\,c \ldots k$ is shifted just one place in the negative principal direction [Fig. 5b)].

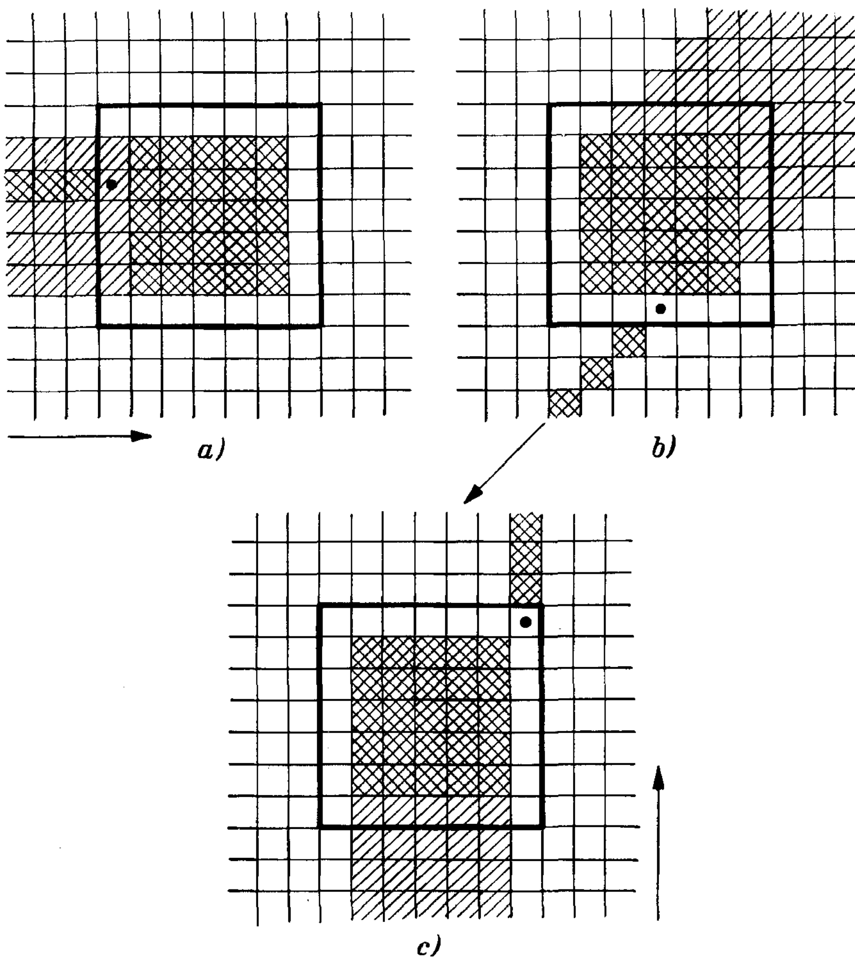

Fig. 4. The arrow indicates the positive principal direction. The heavy dot indicates the square of X' which was last scanned. The heavily shaded regions are X together with the half line in which the reading head must stay before returning to X'. The lightly shaded region is Y.

Conversely, if such a word shift occurs, without the machine ever returning within X', then clearly the word will be shifted over and over again and the reading head will forever stay outside X'. Also the machine will never halt.

So the reading head stays forever outside X' if and only if such word shift occurs, and this can be recognised within a finite number of steps.

The only situation which will not be recognised within a finite number of steps is when the reading head crosses the boundary between X' and Y infinitely often. Every time it leaves X' it will do so via a square of the belt $X' - X$ and sooner or later it will return via the same square. Furthermore, this square, which was initially blank, will at any time have been scanned at least as often as any other square in the half line in the negative principal direction from it. So if the machine will halt, it will do so within X'.

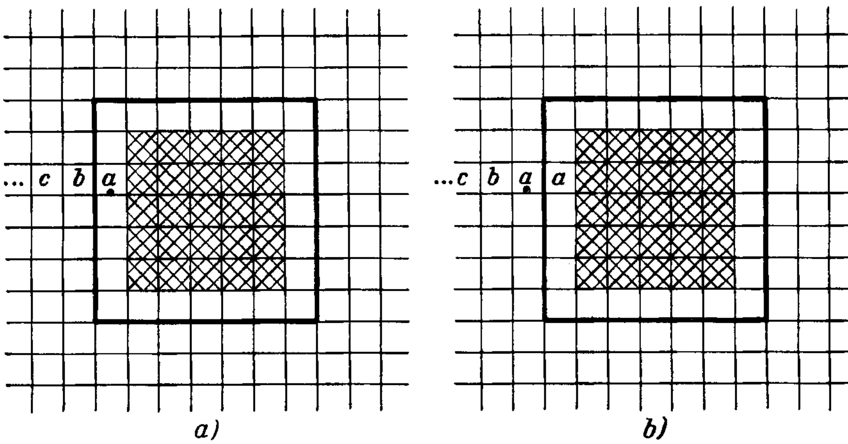

Fig. 5. Unscanned squares may or may not have non-blank symbols in them. Scanned square has a dot in it.

Since X' is finite, at any time we can observe all the squares in X' which may ever be scanned by the reading head and decide whether any of these squares can possibly contain, in the future, a symbol sensing which the machine will halt. If that cannot happen then the machine will never halt. However, such a possibility of halting does not imply certainty, since the reading head may get trapped outside X' before it happens. However, if it does not, then the machine will in a finite number of steps get to scanning the symbol seeing which it halts.

To summarise: In Case β' the machine will halt after a finite length of time or we can recognise in a finite length of time that it will never halt because a description is the same as an earlier description or the machine is trapped outside X' or there is no possibility of halting within X' in the future.

Case β''. In this case the reading head will move from a square which at some time had the blank symbol in it in a direction other than the principal direction, provided that square is scanned often enough. We call the first such direction the secondary direction.

In this case we follow the computation $D_0, D_1, D_2, \ldots$ by M.

If the machine halts or if $D_q = D_p$ for some $p < q$ or if the reading head scans a square outside X not in Y, there is no need to look further.

Otherwise, let $\bar{M}$ be the 2-dimensional one state TURING machine defined by the same rules as M, except that it halts on every symbol sensing which M would move in a direction other than the principal one. Then $\bar{M}$ is a machine to which Case β' applies and so its halting problem is solvable. For each description D_q of M, decide whether D_q is an immortal description of $\bar{M}$. If it is, then D_q is an immortal description of M as well, and so D is an immortal description of M.

One of the cases described above must occur after a finite number of steps. For if M never halts, and D_q is never equal to D_p for $p < q$ and the reading head always scans an element of Y or X, there must be a half line in $X \cup Y$ (in the critical direction) in which squares which are further than any fixed distance from X are eventually scanned. But on a little reflection we see that unless the reading head gets trapped in this half line the same can be said for the half line neighbouring the line mentioned above in the secondary direction. Now this argument can be repeated, and since $X \cup Y$ has only finitely many such half lines we see that sooner or later we get outside $X \cup Y$.

To summarise: In Case β'' the machine will halt in a finite length of time, or we can recognise in a finite length of time that it will never halt because a description is the same as an earlier description, or because the reading head scans a square outside $X \cup Y$ or the description at the time is immortal for the machine $\bar{M}$.

Since Cases α, β' and β'' exhaust all possibilities, the proof is complete.

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