Complexity collapse for unambiguous languages

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Paris

Existential quantifier $\rightsquigarrow$ projection

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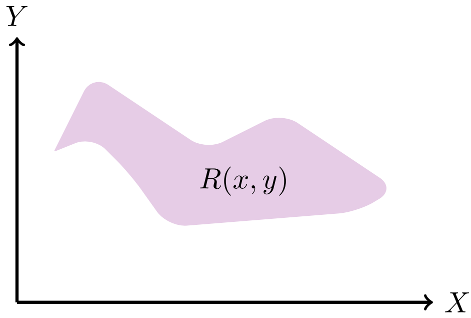
$$S = \Big\{ x \;:\; \exists_y \; R(x, y) \Big\}$$

Existential quantifier ⇝ projection



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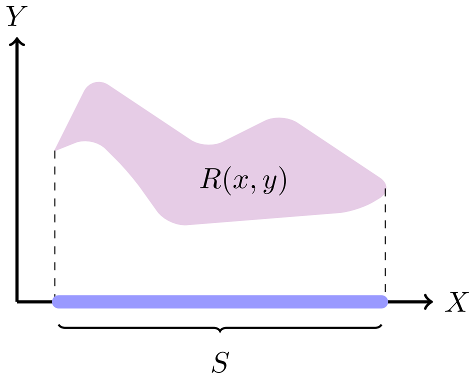

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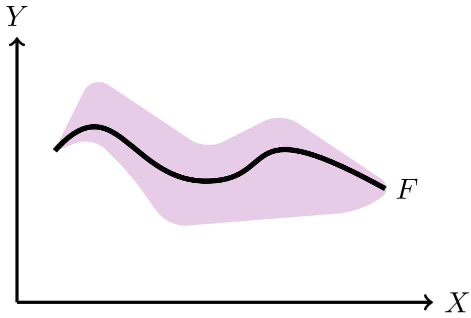

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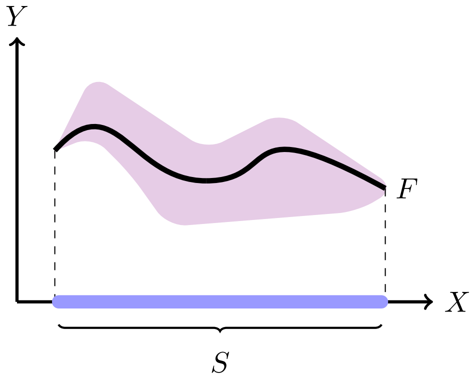

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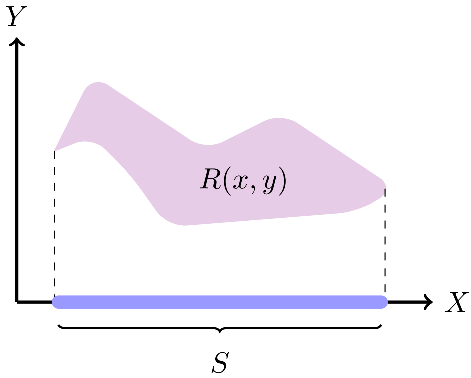

Theorem (~~Lebesgue~~, Souslin)

Projection of a Borel set may **not** *be Borel.*

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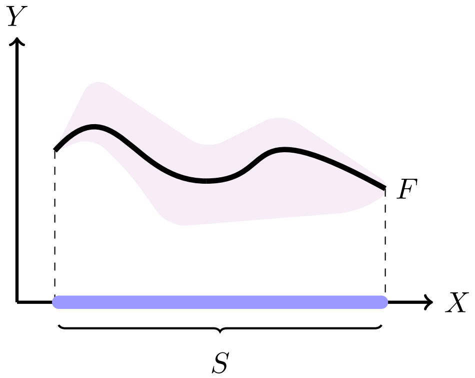

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Theorem (Lusin, Souslin)

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weak MSO	$\equiv$	Büchi $\cap$ (Büchi)^c (= weak)

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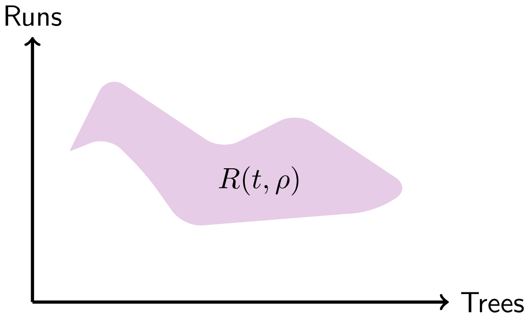

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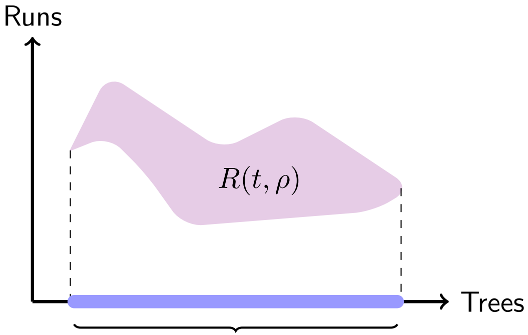

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Projection $\rightsquigarrow$ nondeterminism

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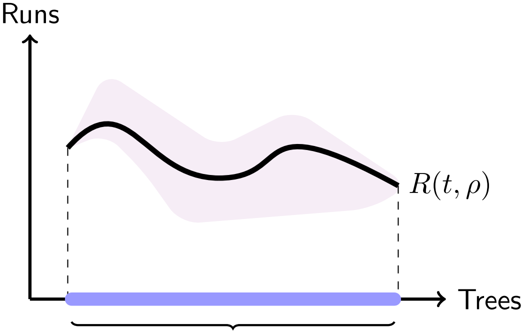
$\mathcal{A}$ is unambiguous if $\forall_t\ \exists_\rho^{\leq 1}\ \rho$ is accepting



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(Partial answer by Hummel [2012], [2013]:

There are unambiguous languages above $\mathbf{\Pi}^1_1$.)

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If $\mathcal{A}$ is unambiguous **and** *Büchi then $\mathrm{L}(\mathcal{A})$ is Borel.*

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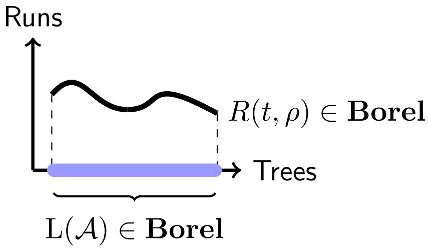

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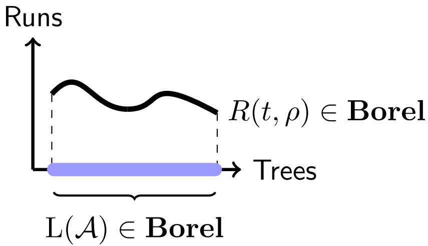

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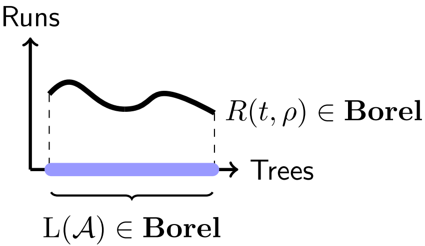

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But what if:

Conjecture (Skurczyński [1993])

If a $\mathrm{L}(\mathcal{A})$ is Borel then $\mathrm{L}(\mathcal{A})$ is weak MSO-definable.

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Hopefully a step towards upper bounds for unambiguous languages.