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SYSTEMS OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH BOUNDED COEFFICIENTS MAY HAVE VERY OSCILLATING SOLUTIONS

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ABSTRACT. An elementary example shows that the number of zeroes of a component of a solution of a system of linear ordinary differential equations cannot be estimated through the norm of coefficients of the system.

Bounds for oscillations. In [1] it was shown that a linear ordinary differential equation of order n, with real analytic coefficients bounded in a neighborhood of the interval $[-1, 1]$, admits a uniform upper bound for the number of isolated zeros of a solution defined on this interval. The analyticity condition can be relaxed; only the boundedness of the coefficients matters. Probably, the simplest result in this spirit is the following theorem for the linear ordinary differential equation

$$y^{(n)}(t) + a_1(t)y^{(n-1)}(t) + \cdots + a_n(t)y(t) = 0 \tag{1}$$

with continuous coefficients on $[\alpha, \beta] \subset \mathbb{R}$.

Theorem 1 ([3, 4]). *If the coefficients of the differential equation* (1) *are uniformly bounded by the constant* $C \geq 1$ *(that is,* $\max\{|a_i(t)| : i = 1, \dots, n\} \leq C$*), then a solution defined on* $[\alpha, \beta]$ *cannot have more than* $n - 1 + \frac{n}{\ln 2} C|\beta - \alpha|$ *isolated zeros.*

An analog of this result for a *system* of ordinary differential equations, viewed as a vector field in space, would concern the number of isolated intersections between integral trajectories of the vector field and hyperplanes (or, more generally, hypersurfaces). For *polynomial* systems of degree d on $\mathbb{R}^n$ of the form

$$\dot{x}_i = v_i(t, x), \quad i = 1, \dots, n, \qquad v_i(t, x) = \sum_{k+|\alpha| \leq d} v_{ik\alpha} t^k x^\alpha, \tag{2}$$

and algebraic hypersurfaces given by $\{P = 0\}$ where $P = P(t, x)$ is a polynomial of degree d, the following theorem, proved in [3] (see also [2]), gives a bound for the number of isolated intersections in case the magnitude of the domain of the solution and the amplitude of the solution are controlled by the *height* of the polynomial system, that is, the number $\max\{|v_{ik\alpha}| : k + |\alpha| \leq d, i = 1, \dots, n\}$.

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Theorem 2. *Suppose that the height of system* (2) *is bounded by the positive constant* C. *If* γ *is an orbit of the system contained in the box* $B_C = \{(t, x) \in \mathbb{R}^{n+1} : |t| < C, |x_i| < C\}$, *then the number of isolated intersections of* γ *and* $\{P = 0\}$ *is at most* $(2 + R)^B$ *where* $B = B(n, d)$ *is an explicit elementary function of* d *and* n *whose growth rate is smaller than* $\exp\exp\exp\exp(4n \ln d + O(1))$ *as* $d, n \to \infty$.

As mentioned in [3], Theorem 2 is nontrivial even for linear systems

$$\dot{x} = A(t)x, \qquad x \in \mathbb{R}^n, \quad A(t) = \sum_{k=0}^{d} A_k t^k, \tag{3}$$

and linear hyperplanes $\{\sum_{i=1}^n p_i x_i = 0\}$. In this case, the box condition reduces to the requirement that $t \in [-C, C]$; the height condition reduces to the uniform boundedness of the norms of the matrix coefficients $A_k \in \mathrm{Mat}_{n\times n}(\mathbb{R})$.

Corollary 3. *If, for system* (3), $\max\{\|A_k\| : k = 0, \dots, d\} < C$, *then there is a uniform bound* (*expressible as an elementary function of* C) *for the number of isolated zeros of every component of every (vector) solution defined on the interval* $[-C, C]$.

A comparison of Theorems 1 and 2 suggests the following question: Can the height condition on the polynomial vector field in Theorem 2 be replaced, for instance, by a bound on the norm $\max_{i=1,\dots,n,\,(t,x)\in B_C} |v_i(t, x)|$; or, in Corollary 3, can it be replaced by a bound on the norm $\max_{t\in[-C,C]} \|A(t)\|$? We will show that *this is impossible.*

The example. For each integer d (no matter how large), there is a linear 2×2 system (3) of degree $2d$ with $\max_{t\in[-1,1]} \|A(t)\| \le 1$ such that a component of one of its solutions in the box B_1 has d isolated zeros in the interval $[-1, 1]$.

Let $t_1, \dots, t_d$ be distinct numbers in the interval $[-1, 1]$ and let

$$a(t) := \lambda(t - t_1) \cdots (t - t_d)$$

where λ is a number chosen so small that $|a(t)| + |\dot{a}(t) + a^2(t)| < 1$ whenever $t \in [-1, 1]$. While the solution $\phi_1(t) = \exp(\int_0^t a(s)\, ds)$ of the differential equation $\dot{x}_1 = a(t)x_1$ has no zeroes, its derivative $\phi_2 = \dot{\phi}_1 = a(t)\phi_1$ has d zeros and also satisfies the equation $\dot{\phi}_2 = (\dot{a} + a^2)\phi_1$. Hence, the supremum over $[-1, 1]$ of the coefficient matrix of the degree $2d$ polynomial linear system

$$\dot{x}_1 = a(t)x_1, \quad \dot{x}_2 = (\dot{a}(t) + a(t)^2)x_1 \tag{4}$$

is bounded by 1, and the second component of the solution $t \mapsto (\phi_1(t), \phi_2(t))$ has d isolated zeros in this interval. Moreover, because the system is linear, a constant multiple of this solution is in the box B_1.

Remark 1. The example shows that the bound stated in Theorem 1 cannot be extended to derivatives of solutions. Also, by choosing λ sufficiently small, the coefficients of system (4) can be made uniformly small in every preassigned *complex* neighborhood of the real segment $[-1, 1]$. Hence, the bounds for oscillation with respect to hyperplanes cannot be achieved in the spirit of [1] by imposing bounds for analytic coefficients in the complex domain.

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