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Process Semantics for Place/Transition Nets with Inhibitor and Read Arcs

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Abstract. In this paper we introduce a truly concurrent semantics for P/T nets with inhibitor and read arcs, called henceforth Contextual P/T nets. The semantics is based on a proper extension of the notion of process to cope with read and inhibitor arcs: we show that most of the properties enjoined by the classical process semantics for P/T nets continue to hold and we substantiate the adequateness of our notion by comparing it with the step semantics.

Keywords: Contextual Nets, Step Semantics, Process Semantics

1. Introduction

Petri nets [17] are a very popular model to represent the behaviour of concurrent systems, widely used both by practitioners and theoreticians. The main attraction of the model lies in its

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intrinsically concurrent nature, providing an explicit way to represent the causal dependencies among the activities of a system.

However, in some cases the classical model has revealed to be not adequate to faithfully model the behaviour of some kinds of real systems, dealing e.g. with shared memory or priorities. This is mainly due to the fact that classical Petri nets allow to model only activities whose interaction with resources consists in consuming or producing them and this forbids Petri nets from being an adequate model when situations like the ones mentioned before appear.

Contextual nets extend classical nets with the ability to handle contexts: in a contextual net transitions do not only consume and produce tokens, but have also context conditions, specifying something that is necessary for the transition to fire, but is not affected by the firing. The ability for a transition to check for context conditions is obtained by enriching classical Petri nets with two new kinds of arcs: read arcs (also called positive contextual arcs or activator arcs), checking for presence of one token in the place for the transition to fire, and inhibitor arcs (sometimes called negative contextual arcs), testing for emptiness of the place.

Read arcs have been introduced in [12] and model the situations where a resource is read but not consumed by the transition. Hence, the same token (resource) can be used as context by many transitions at the same time, as well as by many occurrences of the same transition. An explicit representation of the testing for presence permits to directly specify a degree of concurrency greater than in classical nets. In fact, two transitions with disjoint presets but with a read arc on the same place may occur in any order, but also simultaneously. This feature is not present in classical nets, where we have to model the read operation as a consumption of one token followed by the production of a new one (a self-loop): while preserving the interleaving behaviour, this representation does not allow the simultaneous execution of the two transitions testing for presence of the same resource. Read arcs have been used to model concurrent access to shared data (e.g. the read operation in a database) [19, 7], to compare temporal efficiency in asynchronous systems [20] and to give a true concurrent semantics to Concurrent Constraint Programs [13, 4].

Inhibitor arcs, asking for absence of some elements for the transition to take place, have been introduced in [8] to solve a synchronization problem not expressible in classical Petri nets: the problem is concerned with two pairs of processes communicating through a shared buffer, with an allocation policy based on priorities. The expressiveness of inhibitor arcs has been deeply studied in [10]; in [16] it is shown that other extensions of classical nets presented in the literature (such as constraints [15], exclusive-or transitions [14] and switches [2]) are all equivalent to inhibitor arcs. Another interesting extension, proposed in [10] and shown there to be equivalent to inhibitor arcs, consists in associating a priority with each transition: if two transitions are both enabled at a given marking, then the transition with the highest priority will fire. In [3] it is shown how to simulate priorities with Petri nets (though not in the general case). Inhibitor arcs have also been employed for performance evaluation of distributed systems [1] and to provide π-calculus with a net semantics [5].

Despite of the increased expressive power added by inhibitor arcs, Contextual nets have been

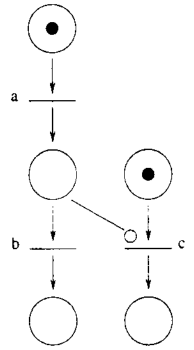

Figure 1 A contextual P/T system where the causal dependencies among occurrences of transitions depend on their temporal ordering.

ignored for a long time by theoreticians; probably one of the main reasons of this fact is that the causal dependencies among events are much more involved than in the classical case, and cannot be properly defined in terms of a unique partial order.

In this paper we tackle the problem of defining a truly concurrent semantics for Contextual nets, which faithfully models the different kinds of causal dependencies arising among occurrences of transitions, due to the presence of different kinds of arcs, and retains much of the pleasant properties fulfilled in the classical case.

Classical P/T systems are equipped with a widely accepted truly concurrent semantics based on processes [9]: each transition firing of the system is represented by an event of the process and each token produced in the system during its execution is represented by a condition in the process. Moreover, a process induces a partial order on its events.

As already pointed out by other authors [11, 21], we claim that a partial order is not sufficient to faithfully model the relations among the events corresponding to occurrences of transitions of a contextual P/T system. Consider the P/T system with inhibitor arcs in Figure 1: transition c is enabled at the initial marking; hence, if c is the first transition to fire, it is caused by no other event; however, if transition a fires first, then it produces a token in an inhibiting place of c, thus preventing c from firing; hence there is a sort of asymmetric conflict between a and c: while transition a can fire disregarding the behaviour of c, transition c can fire with no causes only if it precedes a. In case transition a fires first, c has to wait for b to fire; the firing of b empties the inhibiting place of c, thus enabling c again; as b recreates the conditions for c to fire, we say that in this case the firing of c causally depends on the firing of b. The example above shows us that we can correctly define the causal dependencies among events only if we know the temporal order in which (some of) them occur.

We are interested in a notion of process for contextual P/T systems that maintains the nice properties enjoyed by processes for standard P/T systems, namely the correspondences between slices of processes and reachable markings and between sets of independent events and steps, and the fact that a process univocally determines the causal dependencies among its events. To this aim, we introduce a notion of occurrence net, called enriched occurrence net, and one of process, and we prove that they satisfy the properties enunciated above. The key idea to make the last property to hold consists in partitioning the set of inhibitor arcs of an occurrence net into two parts: the before inhibitor arcs, requiring that the condition has not held yet for the event to happen, and the after inhibitor arcs, requiring that the condition no longer holds for the event to happen.

The paper is organized as follows. In the next section we recall some basic definitions that will be used thorough the paper, then, in section 3 we recall the notions of P/T nets and of process semantics. In section 4 we introduce Contextual P/T nets and their step semantics. In section 5 we develop the notion of process for Contextual nets and, in section 6, we compare it with firing sequences and step firing sequences. A discussion of our approach with respect to other presented in literature is the argument of the last section, where we also draw some conclusions.

Some preliminary ideas leading to the definition of processes for Contextual nets can be found in [6].

2. Basic definitions

Let ω be the set of natural numbers and $\omega^+ = \omega \setminus \{0\}$.

Relations

A relation over a set X is a subset of $X \times X$. The composition of two relations is defined as $R \circ R' = \{(x, z) \mid \exists y(xRy \wedge yR'z)\}$. We denote by R^+ (R^*) the transitive (reflexive and transitive) closure of R.

Partial orders

A (finite) *partial order* $(X, \leq)$ consists of a (finite) set X and a transitive, reflexive and antisymmetric relation $\leq$. As usual, we write $x < y$ for $x \leq y$ and $x \neq y$. Often $(X, <)$ is called strict partial order. The sets of *maximal* and *minimal* elements of X are defined as $Max(X) = \{x \in X \mid \forall y \in X(x \not< y)\}$ and $Min(X) = \{x \in X \mid \forall y \in X(y \not< x)\}$ respectively.

Let $c \subseteq X$; the *down-set* of c is $\downarrow c = \{x \in X \mid \exists y \in c(x \leq y)\}$; the *up-set* of c is $\uparrow c = \{x \in X \mid \exists y \in c(y \leq x)\}$.

The set $c \subseteq X$ is a co-set (or antichain) if $\forall x, y \in c(x \not< y \wedge y \not< x)$; c is a cut if it is a co-set and $\forall y \in X \setminus c \; \exists x \in c(x < y \vee y < x)$ (maximality condition).

The set $l \subseteq X$ is a li-set (or chain) if $\forall x, y \in l(x \leq y \vee y \leq x)$; l is a line if it is a li-set and $\forall y \in X \setminus l \; \exists x \in l(\neg x \leq y \wedge \neg y \leq x)$.

A *linearization* of $(X, \leq)$ is a complete sequence $x_1 \ldots x_n$ compatible with respect to $\leq$, i.e. $\{x_1, \ldots, x_n\} = X$, $x_i \leq x_j \Rightarrow i \leq j$ and $x_i = x_j \Rightarrow i = j$.

Multisets

Given a set S, a *finite multiset* over S is a function $m : S \rightarrow \omega$ such that the set $dom(m) = \{s \in S \mid m(s) \neq 0\}$ is finite. The *multiplicity* of an element s in m is given by the natural number $m(s)$. The set of all finite multisets over S, denoted by $\mathcal{M}_{fin}(S)$, is ranged over by m. A multiset

m such that $dom(m) = \emptyset$ is called *empty*. The set of all finite sets over S is denoted by $\wp_{fin}(S)$. The cardinality of a multiset is defined as $|m| = \sum_{s \in S} m(s)$.

We write $m \subseteq m'$ if $m(s) \leq m'(s)$ for all $s \in S$. The operator $\oplus$ denotes *multiset union*: $m \oplus m'(s) = m(s) + m'(s)$. The operator $\setminus$ denotes *multiset difference*: $m \setminus m'(s) =$ if $m(s) > m'(s)$ then $m(s) - m'(s)$ else 0. The *scalar product* of a number j with a multiset m is $(j \cdot m)(s) = j \cdot (m(s))$.

To lighten the notation, we sometimes use the following abbreviations. If m is a multiset containing only one occurrence of an element s (i.e. $dom(m) = \{s\}$ and $m(s) = 1$) then we denote m with s only. If m contains no duplicated occurrences of elements (i.e. $m(s) \leq 1$ for each $s \in S$) we sometimes use $dom(m)$ in place of m and vice versa. Let m be a multiset over S and m' a multiset over $S \cup S'$ such that $m'(s') = 0$ for each $s' \in S'$ and $m'(s) = m(s)$ for each $s \in S$; with abuse of notation, we sometimes use m in place of m' and vice versa.

Let $f : A \to B$ be a function. It is extended to a function from $\mathcal{M}_{fin}(A)$ to $\mathcal{M}_{fin}(B)$ in the following way: if $m \in \mathcal{M}_{fin}(A)$ then, for all $b \in B$, $(f(m))(b) = \sum_{f(a)=b} m(a)$.

3. P/T nets

We recall the definition of Place/Transition nets.

Definition 3.1. A P/T net is a tuple $N = (S, T, F)$, where

- S and T are the sets of *places* and *transitions*, such that $S \cap T = \emptyset$;
- $F : (S \times T) \cup (T \times S) \to \omega$ is the *flow function*. □

A P/T net is *finite* if both S and T are finite. A P/T net is of *finite synchronization* if, for all $t \in T$, the sets $\{s \in S \mid F(s,t) > 0\}$ and $\{s \in S \mid F(t,s) > 0\}$ are finite.

If $F(x,y) > 0$, we say that there is an arc from x to y with weight $F(x,y)$. If $F(s,t) = F(t,s) > 0$, we say that there is a *self-loop* on s and t.

A finite multiset over the set S of places is called a *marking*. Given a marking m and a place s, we say that the place s contains $m(s)$ *tokens*.

The *preset* of a transition t is the multiset ${}^\bullet t(s) = F(s,t)$, and represents the tokens to be "consumed"; the *postset* of t is the multiset $t^\bullet(s) = F(t,s)$, and represents the tokens to be "produced".

A transition t is *enabled* at m if ${}^\bullet t \subseteq m$; it is denoted by $m[t\rangle$. The execution of a transition t enabled at m produces the marking $m' = (m \setminus {}^\bullet t) \oplus t^\bullet$. This is usually written as $m[t\rangle m'$.

A P/T net is *T-restricted* if ${}^\bullet t \neq \emptyset$ and $t^\bullet \neq \emptyset$ for all transitions $t \in T$.

Definition 3.2. A P/T system is a tuple $N(m_0) = (S, T, F, m_0)$, where (S, T, F) is a P/T net and m_0 is a multiset over S, called the *initial marking*.

A *labelled* P/T net (system) over a set *Act* of labels is a tuple (S, T, F, l) $((S, T, F, m_0, l))$, where (S, T, F) $((S, T, F, m_0))$ is a P/T net (system) and $l : T \to Act$ is the labelling function. □

Any P/T system can be regarded as a labelled P/T system over the set T of transitions, where the labelling function is the identity function over T.

Remark 3.1. We will consider only nets with *finite synchronization* and with *finite initial markings*, because is technically easier to define the nonsequential semantics of such nets; from a philosophical point of view, a finite marking corresponds to consider a finite amount of resources, and finite synchronization corresponds to consider events that need a finite amount of resources to happen and produce a finite amount of resources. Moreover we will consider only T-restricted nets.

A *firing sequence starting at marking m* is defined inductively as follows:

- m is a firing sequence;
- if $m[t_1\rangle m_1 \dots [t_{n-1}\rangle m_{n-1}$ is a firing sequence and $m_{n-1}[t_n\rangle m_n$ then $m[t_1\rangle m_1 \dots [t_{n-1}\rangle m_{n-1}[t_n\rangle m_n$ is a firing sequence.

We simply call *firing sequence* a firing sequence starting at the initial marking m_0. Given a firing sequence $m[t_1\rangle \dots [t_n\rangle m_n$, we call $t_1 \dots t_n$ a *transition sequence starting at m*. We often write $m[t_1 \dots t_n\rangle m'$ to mean that there exist $m_1, \dots, m_{n-1}$ such that $m[t_1\rangle m_1 \dots m_{n-1}[t_n\rangle m'$.

The set of markings *reachable* from m, denoted by $[m\rangle$, is defined as the least set of markings such that

- $m \in [m\rangle$
- if $m_1 \in [m\rangle$ and, for some transition $t \in T$, $m_1[t\rangle m_2$ then $m_2 \in [m\rangle$.

We say that a marking m is reachable if m is reachable from the initial marking m_0.

A finite, non empty multiset over the set T is called a *step*. A step G is enabled at m if $m_1 \subseteq m$, where $m_1 = \bigoplus_t G(t) \cdot {}^\bullet t$. The execution of a step G enabled at m produces the marking $m' = (m \setminus m_1) \oplus m_2$, where $m_2 = \bigoplus_t G(t) \cdot t^\bullet$. This is written as $m[G\rangle m'$.

A *step firing sequence* is defined inductively as follows:

- m_0 is a step firing sequence;
- if $m_0[G_1\rangle m_1 \dots [G_{n-1}\rangle m_{n-1}$ is a step firing sequence and $m_{n-1}[G_n\rangle m_n$ then $m_0[G_1\rangle m_1 \dots [G_{n-1}\rangle m_{n-1}\ [G_n\rangle m_n$ is a step firing sequence.

Given a step firing sequence $m_0[G_1\rangle \dots [G_n\rangle m_n$, we call $G_1 \dots G_n$ a *step transition sequence.*

3.1. Process semantics

We recall the process semantics for P/T nets ([9]). In the following definition we introduce a class of nets where the flow relation takes values on $\{0, 1\}$, i.e. the weight on arcs are always equal to 1. With abuse of notation, we will use F to denote also the relation of which it is the characteristic function, i.e. the relation $\{(x, y) \mid x, y \in S \cup T \wedge F(x, y) = 1\}$.

As already done for transitions, we define the preset and postset of a place s respectively as ${}^\bullet s(t) = F(t, s)$ and $s^\bullet(t) = F(s, t)$.

Definition 3.3. An *occurrence net* is a net (B, E, F), whose places and transitions are called *conditions* and *events* respectively, such that:

1. B and E are finite;
2. the conditions are not branched, i.e. for all $b \in B$, $|{}^\bullet b| \leq 1$ and $|b^\bullet| \leq 1$;
3. the net is acyclic, i.e. for all $x, y \in B \cup E$ $(x, y) \in F^+$ implies $(y, x) \notin F^+$ (where F^+ is the transitive closure of F). □

Note that condition 2 above implies that the arc weights are always equal to 1, hence we can consider F as a relation.

Definition 3.4. Let $N = (S, T, F, m_0)$ be a P/T system. A *process* of N is a tuple $\pi = (B_\pi, E_\pi, F_\pi, \phi)$ where (B_π, E_π, F_π) is an occurrence net and $\phi : B_\pi \cup E_\pi \to S \cup T$ is a labelling function such that:

- conditions correspond to places and events to transitions, i.e. $\phi(B_\pi) \subseteq S$ and $\phi(E_\pi) \subseteq T$;
- minimal conditions corresponds to the initial marking, i.e. for all $s \in S$ we have $m_0(s) = |\{b \in B_\pi \mid {}^\bullet b = \emptyset \wedge \phi(b) = s\}|$;
- the neighborhood of transitions is respected, i.e. for all $s \in S$ and $e \in E_\pi$ $F(s, \phi(e)) = |\phi^{-1}(s) \cap {}^\bullet e|$ and $F(\phi(e), s) = |\phi^{-1}(s) \cap e^\bullet|$. □

4. Contextual P/T nets

Definition 4.1. A *contextual* P/T net is a tuple $N = (S, T, F, K, I)$ where

- (S, T, F) is a P/T net;
- $K \subseteq S \times T$ is the *read relation*;
- $I \subseteq S \times T$ is the *inhibiting relation*;
- $dom(F) \cap K = \emptyset$. □

The condition $dom(F) \cap K = \emptyset$ rules out nets that both consume and test for presence of a token in the same place. This restriction is not entirely technical, as we believe that it is reasonable to keep apart conditions to be tested for presence from conditions to be consumed.

If the set K of read arcs is empty, we call N *P/T net with inhibitor arcs* (or PTI net for short) and we omit the set K from the definition; if the set I of inhibitor arcs is empty, we call N *P/T net with read arcs* (or PTR net for short) and we omit set I from the definition. We sometimes abbreviate Contextual P/T net with CPT net.

The *read set* of a transition t is the set $\widehat{t} = \{s \in S \mid (s, t) \in K\}$, and represents the places to be "tested for presence of tokens". The *inhibitor set* of a transition t is the set ${}^\circ t = \{s \in S \mid (s, t) \in I\}$, and represents the places to be "tested for absence" of tokens.

This changes the definition of enabling: a transition t is enabled at m if ${}^\bullet t \subseteq m$, $\widehat{t} \subseteq dom(m)$ and $dom(m) \cap {}^\circ t = \emptyset$. The execution of a transition t enabled at m producing the marking m', written $m[t\rangle m'$, is defined as for P/T nets, hence the notion of firing sequence does not change.

Definition 4.2. A contextual P/T system is a tuple $N(m_0) = (S, T, F, K, I, m_0)$, where (S, T, F, K, I) is a contextual P/T net and m_0 is a multiset over S, called the *initial marking.* □

We adopt the usual notation to draw P/T e CPT nets: places are represented as circles, transitions as segments, flow arcs as directed segments (i.e. with an arrow at one end) and tokens as black dots inside the place. We represent a read arc as a line connecting the involved place and transition, and an inhibitor arc as a line terminating with a small circle on the transition side.

4.1. Step semantics

The definition of step semantics we present here is an adaptation to contextual P/T nets of the definition given for Contextual C/E nets in [12]. According to our definition, two transitions can happen in the same step iff they can happen in either order. We have to check that not all tokens in a place tested for presence by (an occurrence of) a transition are consumed by the others and that (an occurrence of) a transition does not produce tokens in a place tested for absence by another.

A step G is enabled at m iff

- $m_1 \oplus m_3 \subseteq m$, where $m_1 = \bigoplus_t G(t) \cdot {}^\bullet t$ and $m_3 = \bigcup_{t \in dom(G)} \widehat{t}$
- for all $t \in dom(G)$ ${}^\circ t \cap dom(m) = \emptyset$
- for all $t_1, t_2 \in dom(G)$, such that $t_1 = t_2 \Rightarrow G(t_1) \geq 2$, we have that $dom(t_1^\bullet) \cap {}^\circ t_2 = \emptyset$

The third condition ensures that, for each pair of occurrences of transitions in the step, it never happens that one occurrence puts a token in a place inhibiting the other one. The execution of a step G enabled at m producing the marking m', written $m[G\rangle m'$, is defined as for P/T nets.

The following Proposition enunciates an important property of our definition of step enabling: if a step G can fire then, for any way of dividing it in two sub-steps, these sub-steps can fire in sequence.

Proposition 4.1. *Let G_1, G_2 be steps and $G = G_1 \oplus G_2$. Then $m[G\rangle m'$ iff there exists m_1 such that $m[G_1\rangle m_1[G_2\rangle m'$.* □

Hence, given a firable step, any firing sequence obtained by sequentializing that step is firable:

Corollary 4.1. *Let G be a step of cardinality n. If $m[G\rangle m'$ then, for any sequence of transitions $t_1, \ldots, t_n$, such that $G(t) = |\{i \mid 1 \leq i \leq n \wedge t_i = t\}|$, we have $m[t_1\rangle \ldots [t_n\rangle m'$.* □

Finally, any marking reached by a step firing sequence can by reached by a firing sequence:

Corollary 4.2. *If $m[G_1\rangle m_1 \ldots [G_n\rangle m_n$ then m_n is reachable from m.* □

Now we will compare this notion of step with other notions appeared in the literature and illustrate the motivation of our choice.

There are two other alternative definitions of step in the literature: [21] proposes a notion of step for safe P/T nets with read arcs, whereas [11] proposes a different notion of step for elementary net systems with inhibitor arcs. We illustrate the differences of the three approaches through one example.

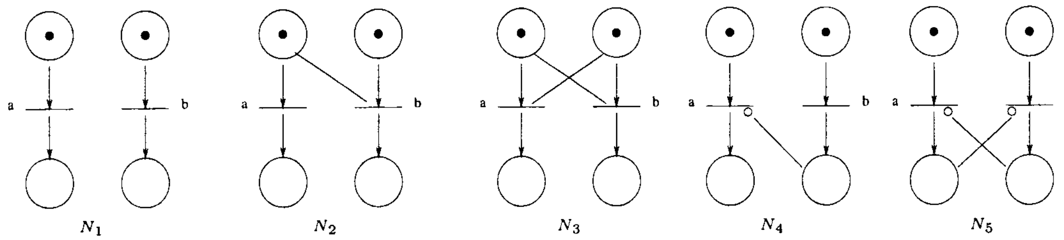

Figure 2 Contextual P/T systems behaving in different ways according to the three different versions of step semantics proposed in the literature.

Consider the nets in Figure 2: according to our definition of step, $\{a, b\}$ is a legal step for the system N_1 only; according to [21], it is a legal step also for system N_2 (and for system N_4, if we consider the natural extension of that definition to deal with inhibitor arcs); according to [11], it is a legal step for systems N_1, N_4 and N_5 (and also for N_2 and N_3, considering the natural extension of that definition to deal with read arcs).

As already pointed out in [12, 11], we think that the definitions of step presented in [11, 21] are more suitable when dealing with timed systems, whereas our approach is better when considering transition firing as instantaneous.

A comparison of the three approaches can be based on the preservation of properties of the step notion for standard P/T nets. We have seen above that our definition enjoys both the following interesting properties:

1. each marking reachable by a step firing sequence is reachable by a firing sequence;

   this means that the behaviour of the net w.r.t. reachability and liveness properties is completely captured by the interleaving semantic;
2. if a step is firable then any firing sequence obtained by sequentializing the step is firable;

   this means that two transitions that can be fired together can also be fired in any order.

The approach followed in [21] preserves the first property, whereas the second one is lost; in fact, if we consider the system N_2 of Figure 2, we have that step $\{a, b\}$ is enabled, but the firing sequence a followed by b is not enabled. Finally, according to the approach in [11] also the first property is lost; consider the system N_5 of Figure 2: the firing of step $\{a, b\}$ leads to a marking formed by putting one token in both the places in the bottom; the only enabled firing sequences are a and b, that cannot lead to the marking described above.

5. Process Semantics for Contextual P/T systems

In this section we show how to provide a truly concurrent semantics based on processes for contextual P/T systems.

We look for a definition of process maintaining the following pleasant properties enjoined by processes for P/T systems: slices of processes correspond to reachable markings, co-sets

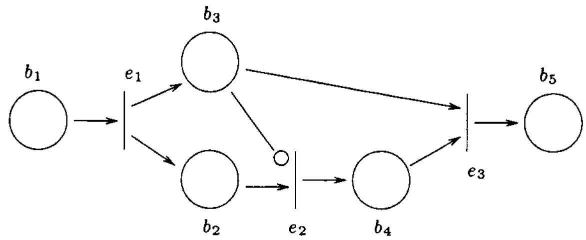

Figure 3 An occurrence net with contextual arcs with a slice not corresponding to a reachable marking.

of events correspond to steps (i.e. multisets of concurrently firable transitions) and, finally, a process univocally determines the causal dependencies among its events.

A first attempt to solve this problem is presented in [6]: it consists in enriching occurrence nets with inhibitor and read arcs; a process is formed by an occurrence net obeying the usual conditions for flow arcs, with a read arc for each read arc in the system and a set of inhibitor arcs for each inhibitor arc in the system. While for read arcs the things work correctly, we cannot say the same for inhibitor arcs, essentially because it may happen that the same occurrence net represents two executions that differ also for the causal relation among its events. Consider the "occurrence net" obtained by removing the tokens from the system in Figure 1, and regard it as a process of that system: a possible behaviour consists of firing event c, with no causes, followed by the firing of event a (with no causes) and event b (caused by a); if the first event to fire is a, at this point c is no longer enabled, and must wait for b to fire before becoming enabled again; thus, we can say that in this case c is causally dependent of b.

Hence we look for a different notion of occurrence net, in such a way that the two behaviours above illustrated correspond to two different occurrence nets. The problem is due to the fact that the reason of the absence of a condition from the current state (because the event producing the condition has not been fired yet or because the event removing the condition has been already fired) may influence the causal relation of events, and this information cannot be deduced from the occurrence net.

Consider the example above: in the first behaviour, c may happen because the token corresponding to the condition has not been produced yet, and it has no causes, whereas in the second behaviour c can happen because the token has been removed by event b.

To solve the problem, we partition the inhibitor arcs of the occurrence net in two sets: *before inhibitor arcs* and *after inhibitor arcs*. A before inhibitor arc, connecting a condition and an event, requires the the condition has not held yet for the event to happen; an after inhibitor arc requires that the condition no longer holds for the event to happen. As we said above, we look for a definition of process where each slice of a process corresponds to a reachable marking of the system.

This property does not hold if we adopt the notion of occurrence net in [6]; consider the occurrence net with contextual arcs in Figure 3: we have that $\{b_5\}$ may reasonably be considered a slice; however, it does not correspond to any reachable marking of the system obtained by putting one token in place b_1. For this property to hold, besides partitioning the set of inhibitor arcs, we need to add conditions to the definition, which rule out nets of the kind presented in Figure 3.

We start by introducing enriched occurrence net, that are basically occurrence nets with contextual arcs, to which we add a partition of inhibitor arcs.

Definition 5.1. An *enriched occurrence net* is a tuple $(B, E, F, K, I^{be}, I^{af})$, where

- $I^{be} \cap I^{af} = \emptyset$
- $(B, E, F, K, I^{be} \cup I^{af})$ is a contextual net
- B and E are finite
- the conditions are not branched, i.e. for all $b \in B$, $|^\bullet b| \leq 1$ and $|b^\bullet| \leq 1$
- the before and after requirements can be fulfilled, i.e.

  if $bI^{be}e$ then there exists $e' \in E$ such that $e'Fb$

  if $bI^{af}e$ then there exists $e' \in E$ such that bFe'
- the relation $F \cup \prec_k \cup \prec_i \cup \prec_{time}$ is acyclic, where $\prec_k = F \circ K$, $\prec_i = F^{-1} \circ I^{af}$ and $\prec_{time} = (K^{-1} \circ F) \cup ((I^{be})^{-1} \circ F^{-1})$. □

The before requirement ensures that, for any before inhibitor arc (b, e), there exists a state in which condition b has not been fulfilled yet, i.e. b does not hold at the beginning of the execution, but there is an event e' making b true. Analogously, the after requirement ensures that, for any after inhibitor arc (b, e), there exists a state in which b ceases to hold, i.e. there exists an event e' making b false.

It is easy to see that $\prec_k \subseteq E \times E$, $\prec_i \subseteq E \times E$ and $\prec_{time} \subseteq E \times E$.

The relations $\prec_k$, $\prec_i$, $\prec_{time}$ can be written in a more explicit form as follows:

- $e \prec_k e'$ iff there exists $b \in B$ such that eFb and bKe'
- $e \prec_i e'$ iff there exists $b \in B$ such that bFe and $bI^{af}e'$
- $e \prec_{time} e'$ iff there exists $b \in B$ such that (bKe and bFe') or ($bI^{be}e$ and $e'Fb$).

Besides the standard causal relation F^+, due to flow arcs, we introduce two new types of causal relation:

- $\prec_k$ is a causal relation due to the presence of read arcs: if $e \prec_k e'$, for e' to happen it is necessary that e produces the token corresponding to condition b;
- $\prec_i$ is a causal relation due to the presence of inhibitor arcs: if $e \prec_i e'$, for e' to happen it is necessary that e has removed the token corresponding to condition b.

Finally, we have seen above that the causal relation due to inhibitor arcs can change due to the temporal ordering of the occurrence of some events. It may also happen that some temporal ordering prevents an event in the occurrence sequence to happen; consider for example the net obtained by removing event b and the condition it produces from Figure 1: if the first event to occur is a, it prevents c to happen. We want that an enriched occurrence net represents a

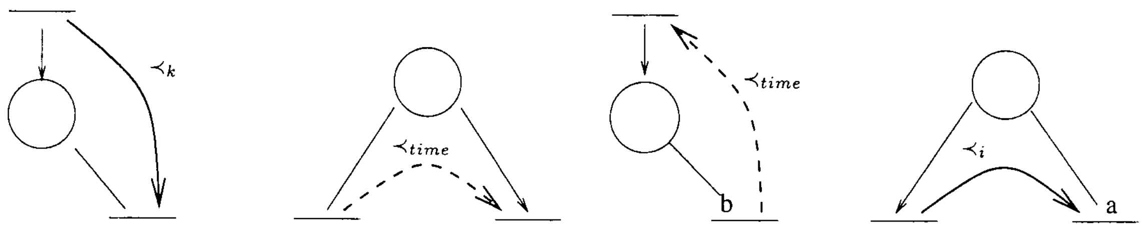

Figure 4. The relations $\prec_k$, $\prec_{time}$ and $\prec_i$

collection of runs where all events of the net happen (and their causal relations coincide); hence we introduce also a temporal relation between events, when this is necessary to permit all events to happen and to univocally determine the causal relations.

We require $e \prec_{time} e'$ if e tests for presence the condition corresponding to the token that e' removes, or if e' produces the token corresponding to the condition that e requires not to be produced to happen.

Consider again the net in Figure 3: we will show that, for any interpretation of the inhibitor arc (b_3, e_2), it is not an enriched occurrence net: if we regard (b_3, e_2) as a before-inhibitor arc, then we have the cycle $e_2 \prec_{time} e_1 F b_2 F e_2$; if we consider (b_3, e_2) as an after-inhibitor arc, then we get the cycle $e_3 \prec_i e_2 F b_4 F e_3$.

In Figure 4 the relations $\prec_k$, $\prec_i$ and $\prec_{time}$ are depicted. We represent before-inhibitor arcs as segments ending with a b on the transition side, and after-inhibitor arcs as segments ending with an a on the transition side.

Definition 5.2. Given an enriched occurrence net N, we define the precedence relation associated to N as $\prec = F \cup \prec_k \cup \prec_i \cup \prec_{time}$ and the order relation associated to N as $< = \prec^+$. □

Proposition 5.1. *$\leq$ ($<$) is a (strict) partial order over $B \cup E$.* □

Proposition 5.2. *$Max_{\leq}(N) \subseteq B$ and $Min_{\leq}(N) \subseteq B$.* □

The following proposition will be useful in the following:

Proposition 5.3. *Let $x \in B \cup E$ and $b \in B$.*

- *If $b < x$ then there exists $e \in E$ such that bFe and $e \leq x$.*
- *If $x < b$ then there exists $e \in E$ such that $x \leq e$ and eFb.* □

A useful corollary of the above proposition is that the maximal and minimal conditions w.r.t. $\leq$ and F^* (which is a partial order) coincide, thus in the following we will use Max and Min without indexes.

Proposition 5.4. *$Max_{\leq}(N) = Max_{F^*}(N)$ and $Min_{\leq}(N) = Min_{F^*}(N)$.* □

Definition 5.3. A *$\leq$-slice* is a cut of $(B \cup E, \leq)$ containing only elements of B. □

We can now introduce the notion of process for contextual P/T systems.

Definition 5.4. Let $N = (S, T, F, K, I, m_0)$ be a contextual P/T system. A *process* of N is a tuple $\pi = (B_\pi, E_\pi, F_\pi, K_\pi, I_\pi^{be}, I_\pi^{af}, \phi)$, where $(B_\pi, E_\pi, F_\pi, K_\pi, I_\pi^{be}, I_\pi^{af})$ is an enriched occurrence net and $\phi : B_\pi \cup E_\pi \to S \cup T$ is a labelling function such that

- conditions correspond to places and events to transitions, i.e. $\phi(B_\pi) \subseteq S$ and $\phi(E_\pi) \subseteq T$
- minimal conditions correspond to the initial marking, i.e. for all $s \in S$ we have $m_0(s) = |\{b \in B_\pi \mid {}^\bullet b = \emptyset \wedge \phi(b) = s\}|$
- for all $s \in S$ and $e \in E_\pi$, $F(s, \phi(e)) = |\phi^{-1}(s) \cap {}^\bullet e|$ and $F(\phi(e), s) = |\phi^{-1}(s) \cap e^\bullet|$
- for all $s \in S$ and $e \in E_\pi$, if $(s, \phi(e)) \in K$ then there exists $b \in \phi^{-1}(s)$ such that $(b, e) \in K_\pi$

  for all $b \in B_\pi$ and $e \in E_\pi$, if $(b, e) \in K_\pi$ then $(\phi(b), \phi(e)) \in K$

  for all $b, b' \in B_\pi$ and $e \in E_\pi$, if $b \neq b'$, $(b, e) \in K_\pi$ and $(b', e) \in K_\pi$ then $\phi(b) \neq \phi(b')$
- for all $s \in S$ and $e \in E_\pi$, if $(s, \phi(e)) \in I$ then, for all $b \in \phi^{-1}(s)$, $(b, e) \in I_\pi^{be} \cup I_\pi^{af} \cup F_\pi^{-1}$

  for all $b \in B_\pi$ and $e \in E_\pi$, if $(b, e) \in I_\pi^{be} \cup I_\pi^{af}$ then $(\phi(b), \phi(e)) \in I$. □

The first three items are the same as for processes for P/T nets. The condition on read arcs ensures that there exists exactly one contextual arc, in the enriched occurrence net, corresponding to a contextual arc of the system. Regarding inhibitor arc, each (before or after) inhibitor arc of the enriched occurrence net has a corresponding arc in the system; for each inhibitor arc (s, t) in the system, we connect (almost) all conditions corresponding to s to all events corresponding to t with a (before or after) inhibitor arc. The only case where we do not connect a condition b (corresponding to s) to an event e (corresponding to t) is when b is in the postset of e: as b starts to hold after e occurs, the only possibility is to put a before arc, but this arc makes $\prec_{time}$ reflexive, invalidating the acyclicity condition in the enriched occurrence net; however, as the condition b is made true by the occurrence of e, we are sure that e happens before condition b is fulfilled, hence thus making useless the presence of the before inhibitor arc. For this reason, with the requirement $(b, e) \in I_\pi^{be} \cup I_\pi^{af} \cup F_\pi^{-1}$, we ask for presence of an inhibitor arc only if there exists no flow arc from e to b.

Note that, for representing a read arc (s, t) of the system in the enriched occurrence net, we choose a suitable condition and the kind of arc is unique, whereas for inhibitor arcs we have no choice on conditions (we connect the event to all conditions corresponding to place s), but we can choose the kind of inhibitor arc to use.

It is easy to see that, if $K = \emptyset = I$, then our definition of process coincides with the classical one for P/T systems.

We illustrate how our definition of process works through some examples. Figure 6 represents the two maximal processes of the P/T system with read arcs in Figure 5; in the lower part of the Figure, the relations induced by each process on its event are depicted. Figure 8 reports the maximal processes (and the induced relations on events) of the P/T system with inhibitor arcs in Figure 7.

Now we define the prefix of a process: given a slice c, it is possible to define the portion above c of the process.

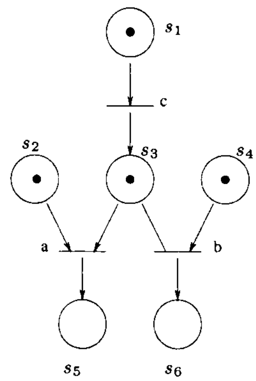

Figure 5. A P/T system with read arcs.

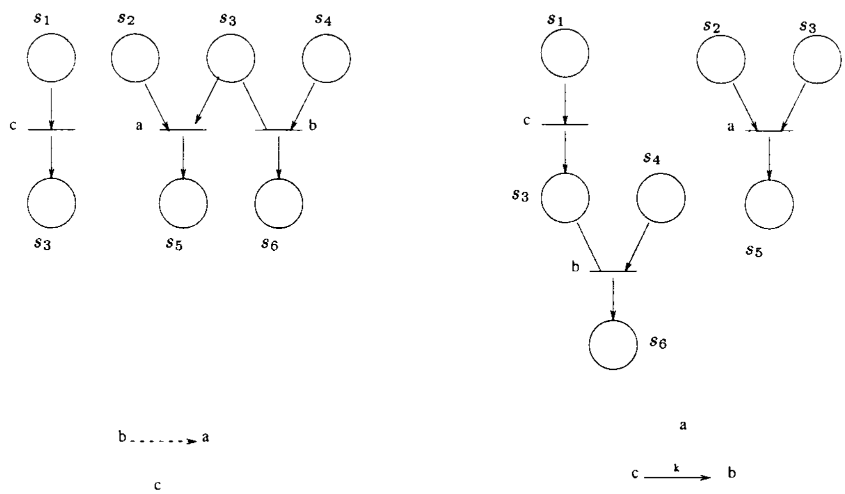

Figure 6 The maximal processes (and the induced relations on events) of the PTR system in Figure 5.

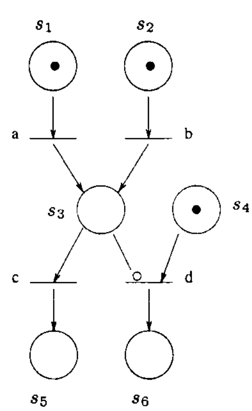

Figure 7. A P/T system with inhibitor arcs.

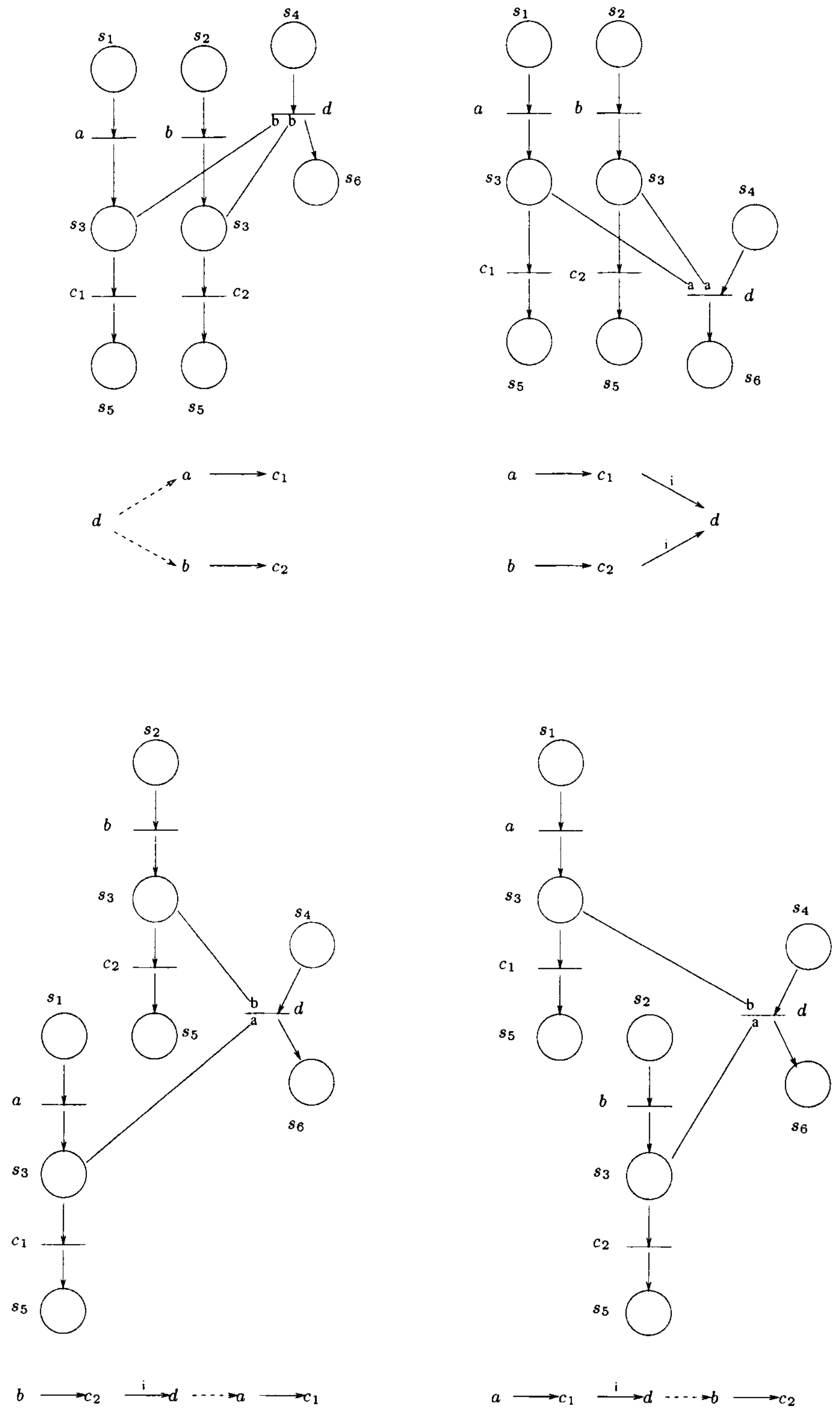

Figure 8 The maximal processes (and the induced relations on events) of the PTI system in Figure 7.

Definition 5.5. Let $\pi = (B, E, F, K, I^{be}, I^{af}, \phi)$ be a process of N and c be a $\leq$-slice of π. Then $\Downarrow (c, \pi) = (B \cap \downarrow c, E \cap \downarrow c, F \cap (\downarrow c \times \downarrow c), K \cap (\downarrow c \times \downarrow c), I^{be} \cap (\downarrow c \times \downarrow c), I^{af} \cap (\downarrow c \times \downarrow c), \phi|_{\downarrow c})$. □

We can check that the prefix of a process is still a process.

Proposition 5.5. *$\Downarrow (c, \pi)$ is a process of N.*

Proof:
The proof follows the structure of the definition of process:

- we show that $(B \cap \downarrow c, E \cap \downarrow c, F \cap (\downarrow c \times \downarrow c), K \cap (\downarrow c \times \downarrow c), I^{be} \cap (\downarrow c \times \downarrow c), I^{af} \cap (\downarrow c \times \downarrow c))$ is an enriched occurrence net; the only nontrivial condition is that the before and after requirements can be fulfilled:
  - suppose $(b, e) \in I^{be} \cap (\downarrow c \times \downarrow c)$; as $(b, e) \in I^{be}$, there exists $e' \in E$ such that $e'Fb$; hence $e' \leq b$, thus $e' \in E \cap \downarrow c$ and $(e', b) \in F \cap (\downarrow c \times \downarrow c)$.
  - suppose $(b, e) \in I^{af} \cap (\downarrow c \times \downarrow c)$; as $(b, e) \in I^{af}$, there exists $e' \in E$ such that bFe'; as bFe' and $bI^{af}e$, we have $e' \prec_i e$, hence $e' \leq e$; thus $e' \in E \cap \downarrow c$ and $(b, e') \in F \cap \downarrow (c \times \downarrow c)$.
- it is easy to see that minimal conditions correspond to initial markings.
- Let $e \in \downarrow c$; we show that ${}^\bullet e \subseteq \downarrow c$; let $b \in {}^\bullet e$; hence bFe; as $e \in \downarrow c$, there exists $b' \in c$ such that $e \leq b'$; hence $b \leq b'$, i.e. $b \in \downarrow c$.

  Let $e \in \downarrow c$; we show that $e^\bullet \subseteq \downarrow c$; let $b \in e^\bullet$; suppose $b \notin \downarrow c$; if b is unrelated to any element in c then c is not a cut, hence there exists $b' \in c$ such that $b' \leq b$; by Proposition 5.3 there exists e' such that $b' \leq e'$ and $e'Fb$; as conditions are unbranched, by eFb and $e'Fb$ we obtain $e = e'$, hence there exists $b' \in c$ such that $b' \leq e$; as $e \in \downarrow c$, there exists $b'' \in c$ such that $e \leq b''$; thus we have $b', b'' \in c$ such that $b' \leq b''$, contradicting the fact that c is a cut.
- Let $e \in \downarrow c$ and $(b, e) \in K$; we show that $b \in \downarrow c$; suppose $b \notin \downarrow c$; then there exists $b' \in c$ such that $b' \leq b$; by Proposition 5.3 there exists e' such that $b' \leq e'$ and $e'Fb$; as bKe and $e'Fb$, we have $e' \prec_k e$; as $e \in \downarrow c$, there exists $b'' \in c$ such that $e \leq b''$; hence there exists $b', b'' \in c$ such that $b' \leq e' \prec_k e \leq b''$, contradicting the fact that c is a cut.

□

6. Retrieving firing and step firing sequences

In this section we investigate the relationship of processes with respectively firing sequences and step firing sequences.

6.1. Retrieving firing sequences

Now we show how to associate a set of processes to a firing sequence. In the following, by linearization of π we mean a linearization of $(E, \leq)$.

Construction 6.1. Let $N = (S, T, F, K, I, m_0)$ be a contextual P/T system and $\sigma = m_0[t_1\rangle m_1 \dots [t_n\rangle m_n$ be a firing sequence of N. We construct a set $\Pi(\sigma)$ of objects, that will turn out to be processes of N. The construction proceeds by induction on the length of the firing sequence. The conditions will be of the form (s, i, k), where s is the place of N to which the condition corresponds, i denotes the step of the construction at which the condition is added and k is used to differentiate conditions, added at the same step, that correspond to the same place of N. The events will have the form (t, i), where t is the transition of N to which the event corresponds and i denotes the step of the construction at which the event is added.

If the firing sequence is m_0, then $\pi_0 = (B_0, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \phi_0)$, where

$$B_0 = \{(s, 0, i) \mid s \in S \wedge 1 \leq i \leq m_0(s)\} \text{ and } \phi_0 : (s, 0, i) \to s.$$

If the firing sequence is $m_0 \dots m_n[t_{n+1}\rangle m_{n+1}$, assume that $\pi_n = (B_n, E_n, F_n, K_n, I_n^{be}, I_n^{af}, \phi_n)$ is constructed from the firing sequence $m_0 \dots m_n$. Hence construct
$\pi_{n+1} = (B_{n+1}, E_{n+1}, F_{n+1}, K_{n+1}, I_{n+1}^{be}, I_{n+1}^{af}, \phi_{n+1})$ in the following way:

$$B_{n+1} = B_n \cup B_{n+1}^{new}$$

where $B_{n+1}^{new} = \{(s, n+1, i) \mid s \in S \wedge 1 \leq i \leq t_{n+1}^{\bullet}(s)\}$ and

$$E_{n+1} = E_n \cup \{(t_{n+1}, n+1)\}$$

For each $s \in S$, choose a set $Pre_{n+1}(s) \subseteq B_n$ such that
- for all $b \in Pre_{n+1}(s)$, $b^{\bullet_n} = \emptyset$ and $\phi_n(b) = s$
- $|Pre_{n+1}(s)| = {}^{\bullet}t_{n+1}(s)$

$$\begin{array}{rcl} F_{n+1} & = & F_n \\ & \cup & \{(b, (t_{n+1}, n+1)) \mid \exists s \in S : b \in Pre_{n+1}(s)\} \\ & \cup & \{((t_{n+1}, n+1), b) \mid b \in B_{n+1}^{new}\} \end{array}$$

For all $s \in \widehat{t}_{n+1}$, choose a condition $test_{n+1}(s) \in B_n$ such that $\phi_n(test_{n+1}(s)) = s$ and $test_{n+1}^{\bullet}(s) = \emptyset$. [1]

$$K_{n+1} = K_n \cup \{(test_{n+1}(s), (t_{n+1}, n+1)) \mid s \in \widehat{t}_{n+1}\}$$

$$I_{n+1}^{be} = I_n^{be} \cup \{(b, e) \mid e \in E_n \wedge b \in B_{n+1}^{new} \wedge \phi_{n+1}(b) \in {}^{\circ}\phi_{n+1}(e)\}$$

$$I_{n+1}^{af} = I_n^{af} \cup \{(b, (t_{n+1}, n+1)) \mid b \in B_n \wedge \phi_n(b) \in {}^{\circ}t_{n+1}\}.$$

$$\phi_{n+1} = \phi_n \cup \{((t_{n+1}, n+1), t_{n+1})\} \cup \{((s, n+1, i), s) \mid (s, n+1, i) \in B_{n+1}^{new}\}$$

□

[1] We need not to require $test_{n+1}(s) \notin Pre_{n+1}(s)$ because we consider contextual nets with $F \cap K = \emptyset$.

As happens for standard P/T systems, it may happen that more than one process corresponds to the same firing sequence; this is reflected in the above construction by the fact that, at each step, there may exist more than one set $Pre_{n+1}(s)$ (and more than one condition $test_{n+1}(s)$ as well) satisfying the required conditions, and different processes are constructed according to the choice of sets $Pre_{n+1}(s)$ (and conditions $test_{n+1}(s)$).

We have that each component of the tuple defining the process increases at each step, hence the following holds: $\prec_{k,j} \subseteq \prec_{k,j+1}$, $\prec_{i,j} \subseteq \prec_{i,j+1}$, $\prec_{time,j} \subseteq \prec_{time,j+1}$, $\prec_j \subseteq \prec_{j+1}$ and $<_j \subseteq <_{j+1}$.

To show that there exists at least one process corresponding to a firing sequence, we need to prove that it is always possible to find a set $Pre_{n+1}(s)$ and a condition $test_{n+1}(s)$ satisfying the required conditions; this can be easily derived by the fact that t_{n+1} is enabled at m_n and by the following proposition:

Proposition 6.1. $m_n(s) = |\{b \in B_n \mid b^{\bullet_n} = \emptyset \wedge \phi_n(b) = s\}|$ □

We show how to associate a natural number $\#(x)$ to each element $x \in B_n \cup E_n$, in such a way that $x < y$ implies $\#(x) < \#(y)$; this definition will be useful to prove that relation $F_n \cup \prec_{k,n} \cup \prec_{i,n} \cup \prec_{time,n}$ is acyclic.

Definition 6.1. Let $x \in B_n \cup E_n$. We define $\#(x) = \begin{cases} 2*i-1 & \text{if } x = (t,i) \\ 2*i & \text{if } x = (s,i,j) \end{cases}$. □

Proposition 6.2. *Let $x \in B_n \cup E_n$. Then $\#(x) \leq 2*n$.* □

Lemma 6.1. *Let $\pi_n = (N_n, \phi_n)$ be in $\Pi(\sigma)$. Then*

1. *N_n is an enriched occurrence net;*
2. *for all $x, y \in B_n \cup E_n$, $x <_n y$ implies $\#(x) < \#(y)$.*

Proof:
Let $N_n = (B_n, E_n, F_n, K_n, I_n^{be}, I_n^{af})$.

Acyclicity of relation $\prec_n = F_n \cup \prec_{k,n} \cup \prec_{i,n} \cup \prec_{time,n}$ follows from the fact that $x \prec_n y$ implies $\#(x) < \#(y)$, which derives from the following:

$$(1) \begin{cases} xF_ny & \Rightarrow & \#(x) < \#(y) \\ e \prec_{k,n} e' & \Rightarrow & \#(e) < \#(e') \\ e \prec_{i,n} e' & \Rightarrow & \#(e) < \#(e') \\ e \prec_{time,n} e' & \Rightarrow & \#(e) < \#(e') \end{cases}$$

Hence we will replace the proof of the last condition in the definition of occurrence net by condition (1). The proof is by induction on the length of σ, and follows the structure of the definition of enriched occurrence net.

If $\sigma = m_0$, then it is trivial to see that $(B_0, E_0, F_0, K_0, I_0^{be}, I_0^{af})$, as constructed in Definition 6.1 is an enriched occurrence net.

Take $\sigma = m_0 \dots m_n[t_{n+1}\rangle m_{n+1}$; by inductive hypothesis we know that for each $(N_n, \phi_n) \in \Pi(m_0 \dots m_n)$, N_n satisfies the required properties; we show that $(B_{n+1}, E_{n+1}, F_{n+1}, K_{n+1}, I_{n+1}^{be}, I_{n+1}^{af})$ satisfies the definition of process as well:

- as the new elements added to I^{be}_{n+1} and I^{af}_{n+1} contain the new conditions and the new event respectively, it is easy to see that $I^{be}_{n+1} \cap I^{af}_{n+1} = \emptyset$.
- if we add the contextual arc $(test_{n+1}(s), (t_{n+1}, n+1))$, then $s \in \widehat{t}$, hence $s \notin {}^{\bullet}t$. Thus, that arc cannot be added as a flow arc, hence $(B_{n+1}, E_{n+1}, F_{n+1}, K_{n+1}, I^{be}_{n+1} \cup I^{af}_{n+1})$ is a contextual net.
- we add a finite amount of elements to B_n and E_n to obtain B_{n+1} and E_{n+1}, hence they are finite.
- If we add a flow arc $(b, (t_{n+1}, n+1))$ then exists $s \in S$ such that $b \in Pre_{n+1}(s)$, hence $b^{\bullet_n} = \emptyset$; if we add a flow arc $((t_{n+1}, n+1), b)$ then $b \in B^{new}_{n+1}$, hence b has no other ingoing flow arcs; hence the conditions remain unbranched.
- if we add a before-inhibitor arc (b, e), then $b \in B^{new}_{n+1}$, and we have the flow arc $((t_{n+1}, n+1), b)$.

  if we add a after-inhibitor arc $(b, (t_{n+1}, n+1))$, then $b \in B_n$ and $\phi_{n+1}(b) \in {}^{\circ}t_{n+1}$; as t_{n+1} is enabled at m_n, we have $m_n(\phi_{n+1}(b)) = 0$; by Proposition 6.1 we have that $\{b' \in B_n \mid b'^{\bullet_n} = \emptyset \wedge \phi_{n+1}(b') = \phi_{n+1}(b)\} = \emptyset$, thus $b^{\bullet_n} \neq \emptyset$, i.e. there exists $e \in E_n$ such that bF_ne.

We show that property (1) is still true after adding the new arcs. Suppose $xF_{n+1}y$; we show that $\#(x) < \#(y)$; two cases can happen:

- if xF_ny then $\#(x) < \#(y)$ by inductive hypothesis;
- if the flow arc (x, y) has been added in the last step, i.e. $(x, y) \in F_{n+1} \setminus F_n$, two cases can happen:

  - the arc is $(b, (t_{n+1}, n+1))$, with $b \in B_n$; by proposition 6.2 we have $\#(b) \leq 2*n$ and $\#(t_{n+1}, n+1) = 2*(n+1) - 1 = 2*n+1 > \#(b)$.
  - the arc is $((t_{n+1}, n+1), b)$, with $b \in B^{new}_{n+1}$, hence $\#(b) = 2*(n+1) > 2*(n+1) - 1 = \#(t_{n+1}, n+1)$.

Let $e \prec_{k,n+1} e'$; then there exists b such that $eF_{n+1}b$ and $bK_{n+1}e'$; the following cases can happen:

- eF_nb and bK_ne'; hence $e \prec_{k,n} e'$ and $\#(e) < \#(e')$ by inductive hypothesis.
- eF_nb and the contextual arc (b, e') has been added in the last step; hence there exists $s \in S$ such that $b = test_{n+1}(s) \in B_n$ and $e' = (t_{n+1}, n+1)$; we have $\#(e) < \#(b)$ by inductive hypothesis; $\#(b) \leq 2*n$ and $\#(e') = 2*(n+1) - 1 = 2*n+1$, hence $\#(e) < \#(b) < \#(e')$.
- the flow arc (e, b) has been added in the last step; hence $b \in B^{new}_{n+1}$, i.e. $b \notin B_n$, hence the contextual arc (b, e') can neither be already present nor be added at the last step.

Let $e \prec_{i,n+1} e'$; then there exists b such that $bF_{n+1}e$ and $bI^{af}_{n+1}e'$. the following cases can happen:

- bF_ne and bI^{af}_ne'; hence $e \prec_{i,n} e'$ and $\#(e) < \#(e')$ by inductive hypothesis.
- eF_nb and the after-inhibitor arc (b, e') has been added in the last step; hence $\#(e) \leq 2*n$; as $e' = (t_{n+1}, n+1)$, $\#(e') = 2*(n+1) - 1 = 2*n+1 > \#(e)$.

- the flow arc (b, e) has been added in the last step; hence we have that $b \in Pre_{n+1}(s)$ for some s, hence b has no outgoing flow arcs in N_n.

  The after-inhibitor arc (b, e') cannot be already present in N_n, otherwise the after requirement was not fulfilled.

  Hence both arcs have been added in the last step, thus $e = e' = (t_{n+1}, n+1)$; as $(b, e') \in I^{af}_{n+1}$, we have $\phi_n(b) \in {}^{\circ}t_{n+1}$; as $(b, e) \in F_{n+1}$, we have that $b \in Pre_{n+1}(\phi_n(b))$, hence ${}^{\bullet}t_{n+1}(\phi_n(b)) > 0$; Hence t_{n+1} cannot fire, contradicting the fact that it is enabled at m_n.

Let $e \prec_{time,n+1} e'$; two cases can happen:

- there exists b such that $bK_{n+1}e$ and $bF_{n+1}e$; the following cases can happen:
  - bK_ne and bF_ne'; hence $e \prec_{time,n} e'$ and $\#(e) < \#(e')$ by inductive hypothesis.
  - bK_ne and the flow arc (b, e') has been added in the last step; hence $\#(e) \leq 2 * n$ and $e' = (t_{n+1}, n+1)$, hence $\#(e') = 2 * (n+1) - 1 = 2 * n + 1 > \#(e)$.
  - suppose the test arc (b, e) has been added in the last step; hence $e = (t_{n+1}, n+1)$, $\phi_n(b) \in \widehat{t_{n+1}}$ and b has no outgoing flow arcs; hence $(b, e') \notin F_n$; thus the flow arc (b, e') has been added in the last step, hence $b \in Pre_{n+1}(\phi_n(b))$, thus ${}^{\bullet}t_{n+1}(\phi_n(b)) > 0$; thus $(\phi_n(b), t_{n+1}) \in K$ and $(\phi_n(b), t_{n+1}) \in F$, contradicting the third item of the definition of contextual net.
- there exists b such that $bI^{be}_{n+1}e$ and $e'F_{n+1}b$; the following cases can happen:
  - bI^{be}_ne and $e'F_nb$; hence $e \prec_{time,n} e'$ and $\#(e) < \#(e')$ by inductive hypothesis.
  - both the before-inhibitor arc (b, e) and the flow arc (e', b) have been added in the last step; hence $e \in E_n$, thus $\#(e) \leq 2*n$; $e' = (t_{n+1}, n+1)$, hence $\#(e') = 2*n+1 > \#(e)$.
  - we show that it is not possible that only one of the two arcs has been added in the last step;

    if the before-inhibitor arc (b, e) has been added in the last step, then $b \in B^{new}_{n+1}$, hence $b \notin B_n$ thus we cannot have $(e', b) \in F_n$;

    if the flow arc (e', b) has been added in the last step, then $b \in B^{new}_{n+1}$ hence $b \notin B_n$ thus we cannot have $(b, e) \in I^{be}_n$.

□

Theorem 6.1. *Each $\pi \in \Pi(\sigma)$ is a process.*

Proof:
By induction on the length of σ;

If $\sigma = m_0$ then it is trivial to see that π_0, the only element in $\Pi(m_0)$, is a process.

Take $\sigma = m_0 \ldots m_n[t_{n+1}\rangle m_{n+1}$; by inductive hypothesis we know that each π_n is a process; we show that also each π_{n+1} is a process, following the structure of the definition of process:

- $(B_{n+1}, E_{n+1}, F_{n+1}, K_{n+1}, I^{be}_{n+1}, I^{af}_{n+1})$ is an enriched occurrence net by Lemma 6.1.

- it is easy to see that ϕ_{n+1} maps conditions on places and events to transitions;
- if condition b is added in step $n+1$, then $b \in B^{new}_{n+1}$, hence it has an ingoing flow arc; no ingoing arcs are added to already present conditions; hence the number of conditions, mapped on a state s, with no ingoing flow arcs does not change, thus, by inductive hypothesis, minimal conditions correspond to the initial marking;
- if event e was already present, no flow arcs incident on e are added in this step, hence the neighborhood condition continues to hold;

  if $e = (t_{n+1}, n+1)$, then $\phi_{n+1}(e) = t_{n+1}$; we have $\phi^{-1}_{n+1}(s) \cap {}^{\bullet_{n+1}}e = Pre_{n+1}(s)$, thus $|\phi^{-1}_{n+1}(s) \cap {}^{\bullet_{n+1}}e| = |Pre_{n+1}(s)| = F(s, t_{n+1}) = F(s, \phi_{n+1}(e))$; and we have $\phi^{-1}_{n+1}(s) \cap e^{\bullet_{n+1}} = \{(s, n+1, i) \mid (s, n+1, i) \in B^{new}_{n+1}\}$, hence $|\phi^{-1}_{n+1}(s) \cap e^{\bullet_{n+1}}| = |\{(s, n+1, i) \mid (s, n+1, i) \in B^{new}_{n+1}\}| = F(t_{n+1}, s) = F(\phi_{n+1}(e), s)$;
- suppose $(s, \phi_{n+1}(e)) \in K$;

  if $e \in E_n$ then by inductive hypothesis there exists $b \in \phi^{-1}_n(s) \subseteq \phi^{-1}_{n+1}(s)$ such that $(b, e) \in K_n \subseteq K_{n+1}$;

  if $e = (t_{n+1}, n+1)$, we have $s \in \widehat{t}_{n+1}$, hence there exists $test_{n+1}(s) \in B_n$, such that $\phi_n(test_{n+1}(s)) = s$ and $(test_{n+1}(s), e) \in K_{n+1}$; as $\phi_n(test_{n+1}(s)) = s$ and $\phi_{n+1}(test_{n+1}(s)) = \phi_n(test_{n+1}(s))$, we have $test_{n+1}(s) \in \phi^{-1}_{n+1}(s)$;

  take $(b, e) \in K_{n+1}$; if $(b, e) \in K_n$, then by inductive hypothesis $(\phi_{n+1}(b), \phi_{n+1}(e)) = (\phi_n(b), \phi_n(e)) \in K$; otherwise, $b = test_{n+1}(s)$, $e = (t_{n+1}, n+1)$ and $s \in \widehat{t}_{n+1}$ (i.e. $(s, t_{n+1}) \in K$); we have $\phi_{n+1}(b) = s$ and $\phi_{n+1}(e) = t_{n+1}$, hence $(\phi_{n+1}(b), \phi_{n+1}(e)) \in K$;

  take $(b, e), (b', e) \in K_{n+1}$; if $(b, e), (b', e) \in K_n$ then $\phi_n(b) \neq \phi_n(b')$, hence $\phi_{n+1}(b) \neq \phi_{n+1}(b')$.

  If the test arc (b, e) has been added in the last step, then $e = (t_{n+1}, n+1)$, hence also (b', e) has been added in the last step; in this step only one test arc $(test_{n+1}(s), (t_{n+1}, n+1))$ is added with $\phi_{n+1}(test_{n+1}(s)) = s$.
- suppose $(s, \phi_{n+1}(e)) \in I$; take $b \in \phi^{-1}_{n+1}(s)$; we have $\phi_{n+1}(b) = s \in {}^{\circ}\phi_{n+1}(e)$. Four cases can happen:

  - if $b \in B_n$ and $e \in E_n$, then by inductive hypothesis $(b, e) \in I^{be}_n \cup I^{af}_n \cup F^{-1}_n \subseteq I^{be}_{n+1} \cup I^{af}_{n+1} \cup F^{-1}_{n+1}$.
  - if $b \in B_n$and $e = (t_{n+1}, n+1)$, as $\phi_{n+1}(b) \in {}^{\circ}\phi_{n+1}(e)$, we have $(b, e) \in I^{af}_{n+1}$;
  - if $b \in B^{new}_{n+1}$ and $e \in E_n$, as $\phi_{n+1}(b) \in {}^{\circ}\phi_{n+1}(e)$, we have $(b, e) \in I^{be}_{n+1}$;
  - if $b \in B^{new}_{n+1}$ and $e = (t_{n+1}, n+1)$, then $(e, b) \in F_{n+1}$, hence $(b, e) \in F^{-1}_{n+1}$.

  Suppose now $(b, e) \in I^{be}_{n+1} \cup I^{af}_{n+1}$;

  if $(b, e) \in I^{be}_n \cup I^{af}_n$ then by inductive hypothesis $(\phi_{n+1}(b), \phi_{n+1}(e)) \in I$;

  if (b, e) has been added in the last step, we have that $\phi_{n+1}(b) \in {}^{\circ}\phi_{n+1}(e)$.

$\square$

The following theorem relates occurrences of transitions in the firing sequence to events of the corresponding process, and occurrences of markings to slices.

Theorem 6.2. *Let $\sigma = m_0 \ldots [t_n\rangle m_n$ be a firing sequence and $\pi_n \in \Pi(\sigma)$.*

1. *There is a bijective correspondence $\psi : \{1, \ldots, n\} \rightarrow E_n$ between occurrences of transitions in σ and events in π_n such that:*
   - $\phi_n(\psi(i)) = t_i$
   - $\psi(i) <_n \psi(j)$ *implies* $i < j$
2. *There exists a sequence of slices $c_0, \ldots, c_n$ such that*
   - $c_i \subseteq \downarrow c_{i+1}$
   - *for all* $s \in S$, $m_i(s) = |c_i \cap \phi_n^{-1}(s)|$
   - ${}^{\bullet_n}\psi(i+1) \subseteq c_i$
   - $bK_n\psi(i+1)$ *implies* $b \in c_i$
   - $bI_n^{be}\psi(i+1)$ *implies* $b \in (\uparrow c_i) \setminus c_i$
   - $bI_n^{af}\psi(i+1)$ *implies* $b \in (\downarrow c_i) \setminus c_i$
   - $c_{i+1} = c_i \setminus {}^{\bullet_n}\psi(i+1) \cup \psi(i+1)^{\bullet_n}$

Proof:
Take $\psi : i \rightarrow (t_i, i)$.

- $\phi_n(\psi(i)) = \phi_n(t_i, i) = t_i$
- if $\psi(i) <_n \psi(j)$, then $(t_i, i) <_n (t_j, j)$; by Lemma 6.1 we have $\#(t_i, i) < \#(t_j, j)$, i.e. $2 * i - 1 < 2 * j - 1$, hence $i < j$.

Take $c_i = Max(N_i)$; we show that c_i are slices of N_n. By proposition 5.2 $c_i \subseteq B_i \subseteq B_n$.

We show that c_i is a co-set in N_n: suppose $b, b' \in c_i$ and $b <_n b'$; As $b, b' \in B_i$, by Proposition 6.2 $\#(b) \leq 2 * i$ and $\#(b') \leq 2 * i$. As $b \not<_i b'$ and $b <_n b'$, there exist x, y and j such that $i < j \leq n$ and $b <_n x \prec_j y <_n b'$; take the least j satisfying the condition above; we have $x \prec_j y$ and $x \not\prec_{j-1} y$; we show that either $\#(x) > 2(j-1)$ or $\#(y) > 2(j-1)$; note that if an event e is added at step j then $\#(e) = 2j - 1 > 2(j-1)$, and if a condition b has been added at step j then $\#(b) = 2j > 2(j-1)$; the following cases can happen:

- xF_jy and this arc has been added at step j; by construction we have that y has been added at step j;
- $x \prec_{k,j} y$; hence $x, y \in E_j$ and there exists a condition b such that xF_jb and bK_jy; if the flow arc (x, b) has been added at step j, then event x has been added at step j; if the test arc (b, y) has been added at step j, then event y has been added at step j.
- $x \prec_{i,j} y$; hence there exists a condition b such that bF_jx and $bI_j^{af}y$; if the flow arc (b, x) has added at step j, then event x has been added at step j; if the after-inhibitor arc (b, y) has been added at step j, then event y has been added at step j.

- $x \prec_{time,j} y$; two cases can happen:
  - there exists a condition b such that bK_jx and bF_jy; if the test arc (b, x) has been added at step j then event x has been added at step j; if the flow arc (b, y) has been added at step j then event y has been added at step j.
  - there exists a condition b such that $bI_j^{be}x$ and yF_jb; if the before-inhibitor arc (b, x) has been added at step j then condition b has been added at step j; as yF_jb, we have that also event y has been added at step j; if the flow arc (y, b) has been added at step j then event y has been added at step j.

W.l.o.g. we suppose $\#(x) > 2(j-1)$; we have $x <_n b'$, with $\#(x) > 2(j-1) \geq 2i \geq \#(b')$, contradicting Lemma 6.1. Hence c_i is a co-set.

Now we show that c_i is a maximal co-set: suppose there exists $x \in B_n \cup E_n$ such that, for all $b \in c_i$, $x \not<_n b$ and $b \not<_n x$; two cases can happen:

- if $x \in B_i \cup E_i$, as c_i is a cut in N_i we have $x \in c_i$;
- otherwise, x has been added at a step $j > i$; hence there exists $b_{j-1} \in c_{j-1}$ such that $b_{j-1}F_j^+x$; if $b_{j-1} \notin c_{j-2}$ then there exists $b_{j-2} \in c_{j-2}$ such that $b_{j-2}F_j^2b_{j-1}$; we proceed in this way until we reach c_i; we have obtained a chain $b_iF_j^*b_{i+1}\dots b_{j-1}F_j^+x$, with $b_l \in c_l$ for $i \leq l \leq j-1$; hence $b_i <_n x$, with $b_i \in c_i$, contradiction.

Hence, if we take $c_i = Max(N_i)$ for $i = 1, \dots, n$, we have found a sequence $c_1, \dots, c_n$ of slices of N_n.

Now we show that these slices satisfy the conditions required by the Theorem.

- we have $c_i \subseteq B_i \cup E_i \subseteq B_{i+1} \cup E_{i+1}$; as $c_{i+1} = Max(N_{i+1})$, we have $c_i \subseteq\downarrow c_{i+1}$;
- by Proposition 6.1 we have $m_i(s) = |c_i \cap \phi_n^{-1}(s)|$;
- if $b \in {}^{\bullet_n}\psi(i+1)$ then exists s such that $b \in Pre_{i+1}(s)$, hence $b^{\bullet_i} = \emptyset$, thus $b \in Max(N_i) = c_i$;
- if $bK_n\psi(i+1)$ then exists s such that $b = test_{i+1}(s)$, hence $b^{\bullet_i} = \emptyset$, thus $b \in Max(N_i) = c_i$;
- if $bI_n^{be}\psi(i+1)$ then b has been added in a step $j > i+1$, hence $b \notin c_i$; moreover, there exists $b_j \in c_j$ such that $b_jF_n^2b$; if $b_j \notin c_{j-1}$, then there exists $b_{j-1} \in c_{j-1}$ such that $b_{j-1}F_n^2b_j$; proceeding in this way until we reach c_i, we obtain that there exists $b_i \in c_i$ such that $b_i <_n b$; hence $b \in (\uparrow c_i) \setminus c_i$.
- if $bI_n^{af}\psi(i+1)$, then $b \in B_i$ and $\phi_i(b) \in {}^{\circ}t_{i+1}$; from $b \in B_i$ we obtain $b \in\downarrow (c_i)$; if $b \in c_i$ then $b^{\bullet_i} = \emptyset$, hence by Proposition 6.1 we have $m_i(\phi_i(b)) > 0$, contradicting the enabledness of t_{i+1} at marking m_i;
- Take $b \in c_{i+1}$; we have $b^{\bullet_{i+1}} = \emptyset$; two cases can happen:
  - b has been added at step $i+1$; in this case we have $b \in \psi(i+1)^{\bullet_{i+1}}$.
  - $b \in B_i$; as $b^{\bullet_{i+1}} = \emptyset$, also $b^{\bullet_i} = \emptyset$, hence $b \in c_i$; moreover no outgoing flow arc has been added to b in this step, thus $b \notin {}^{\bullet_{i+1}}\psi(i+1)$.

in both cases, we have $b \in c_i \setminus {}^{\bullet_{i+1}}\psi(i+1) \cup \psi(i+1)^{\bullet_{i+1}}$.

Take $b \in c_i \setminus {}^{\bullet_{i+1}}\psi(i+1) \cup \psi(i+1)^{\bullet_{i+1}}$; two cases can happen:

- $b \in c_i$ and $b \notin {}^{\bullet_{i+1}}\psi(i+1)$; hence b has no outgoing flow arcs in N_i, and no outgoing arc has been added in step $i+1$, hence b has no outgoing flow arcs in N_{i+1} and $b \in c_{i+1}$;
- $b \in \psi(i+1)^{\bullet_{i+1}}$; in this case, b has been added in the last step, with no outgoing flow arc; hence $b \in c_{i+1}$.

in both cases, we have $b \in c_{i+1}$.

□

Theorem 6.3. *Let π be a process and $e_1, \ldots, e_n$ be a linearization of the set of events of π. Then there exists a firing sequence $\sigma = m_0[\phi(e_1)\rangle \ldots [\phi(e_n)\rangle m_n$ such that $\pi \in \Pi(\sigma)$.*

Proof:
By induction on $|E_\pi|$.

If $E_\pi = \emptyset$ it is easy to see that the firing sequence m_0 satisfies the requirements.

If $|E_\pi| = n > 0$, we first individuate a cut c corresponding to the state of the system before event e_n occurs; as $\Downarrow (c, \pi)$ is a process, we can construct the firing sequence $m_0 \ldots m_{n-1}$ by inductive hypothesis; hence, we show that the transition $\phi(e_n)$ is firable at m_{n-1}; finally, we show that π can be obtained from $\Downarrow (c, \pi)$ by applying the construction in Definition 6.1.

Let $c_1 = \{b \in B_\pi \mid b^\bullet = \emptyset \wedge \neg e_n F_\pi b\}$ and $c_2 = \{b \in B_\pi \mid b F_\pi e_n\}$. Let $c = c_1 \cup c_2$. We show that c is a slice of π.

First we show that c is a co-set: suppose $b, b' \in c$, with $b < b'$; by Proposition 5.3 there exists an event e such that $b F_\pi e < b'$; thus we have $b \notin c_1$, hence $b \in c_2$, and $b F_\pi e_n$. As conditions are unbranched, by $b F_\pi e$ and $b F_\pi e_n$ we get $e = e_n$, hence $e_n < b'$; we show that $b' \in c_1$: suppose $b' \in c_2$; hence $b' F_\pi e_n$, thus obtaining the cycle $e_n < b' F_\pi e_n$, contradicting the acyclicity of $<$; thus we have $b' \in c_1$. We know that $e_n < b'$; two cases can happen:

- $e_n F_\pi b'$; impossible because $b' \in c_1$ implies $\neg e_n F_\pi b'$;
- there exists e_i such that $e_n < e_i F_\pi b'$; as e_n is the last element of the linearization, $i < n$, contradicting the order-respecting property of linearizations.

Hence, if $b, b' \in c$ then $b \not< b'$.

We show that c is a cut: let $x \in B_\pi \cup E_\pi$; we know that $Max(\pi)$ is a slice, hence there exists $b \in Max(\pi)$ such that $x \leq b$. If $\neg e_n F_\pi b$, then $b \in c_1$ and we are done. Suppose $e_n F_\pi b$; the following cases can happen:

- $x = b$. By T-restrictedness there exists b' such that $b' F_\pi e_n$; hence $b' \in c_2$ and $b' < x$.
- $x = e_n$. By T-restrictedness there exists b' such that $b' F_\pi e_n$; hence $b' \in c_2$ and $b' < e_n$.
- $x < b$ and $x \neq e_n$. As $e_n F_\pi b$, and b has at most one ingoing flow arc, we have that $x < e_n F_\pi b$; The following cases can happen:

- There exists b' such that $x \leq b' F_\pi e_n$; we have $b' \in c_2$ and $x \leq b'$.
- There exists e_i such that $x \leq e_i \prec_k e_n$; hence there exists b' such that $e_i F_\pi b'$ and $b' K_\pi e_n$; two cases can happen:
  * $b'^\bullet = \emptyset$; as $e_i F_\pi b'$ and conditions have at most one ingoing arc, we have that $\neg e_n F_\pi b'$, hence $b' \in c_1$ and $x \leq b'$.
  * $b'^\bullet \neq \emptyset$; hence there exists e_j such that $b' F_\pi e_j$; as $b' K_\pi e_n$, we have $e_n \prec_{time} e_j$, but e_n is the last element of the linearization, hence $j < n$, contradicting the order-preserving property of linearizations.
- There exists e_i such that $x \leq e_i \prec_{time} e_n$; two cases can happen:
  * there exists b' such that $b' K_\pi e_i$ and $b' F_\pi e_n$; by T-restrictedness there exists b_1 such that $e_i F_\pi b_1$.
    If $b_1^\bullet = \emptyset$, by $e_i F_\pi b_1$ and the fact that conditions are unbranched we have $\neg e_n F_\pi b_1$, hence $b_1 \in c_1$ and we have finished.
    If $b_1 F_\pi e_n$ then $b_1 \in c_2$ and we have finished; otherwise, there exist an event $d_1 \neq e_n$ and a condition b_2 such that $b_1 F_\pi d_1 F_\pi b_2$; by acyclicity $b_1 \neq b_2$; if $b_2^\bullet = \emptyset$, we have $\neg e_n F_\pi b_2$ and $b_2 \in c_1$; if $b_2 F_\pi e_n$ then $b_2 \in c_2$; otherwise, proceeding in this way we obtain a chain $b_1 < b_2 < \ldots$ of distinct conditions; as the the set of conditions is finite, after at most $|B_\pi|$ steps we get a condition $b_m \in c_1 \cup c_2$, such that $x \leq e_i F_\pi b_1 < \ldots < b_m$.
  * There exists b' such that $e_n F_\pi b'$ and $b' I_\pi^{be} e_i$; by T-restrictedness there exists b_1 such that $e_i F_\pi b_1$; we proceed as in the item above.
- There exists e_i such that $x \leq e_i \prec_i e_n$; hence there exists b' such that $b' F_\pi e_i$ and $b' I_\pi^{af} e_n$; by T-restrictedness there exists b_1 such that $e_i F_\pi b_1$; now we proceed as in the first case of the item above.

Hence c is a cut. Moreover:

- for each event x, if $x \neq e_n$ then $x \in \downarrow c$;
- for each condition x, if $\neg e_n F_\pi x$ then $x \in \downarrow c$.

By Proposition 5.5 we have that $\Downarrow (c, \pi)$ is a process of N.

We have that $E_\pi \cap \downarrow c = \{e_1, \ldots, e_{n-1}\}$, hence the process $\Downarrow (c, \pi)$ contains $n-1$ events; by inductive hypothesis there exists a firing sequence $\sigma' = m_0 \ldots [t_{n-1}\rangle m_{n-1}$ such that $\Downarrow (c, \pi) \in \Pi(\sigma')$.

Let $t_n = \phi(e_n)$; we show that t_n is enabled at m_{n-1}.

By Proposition 6.1 we have that $m_{n-1}(s) = |A_s|$, where $A_s = \{b \in B \cap \downarrow c \mid \forall e \in E \cap \downarrow c (\neg b F_\pi e) \wedge \phi(b) = s\}$; it is easy to see that $A_s = \phi^{-1}(s) \cap c$.

By definition of process we have $F(s, t_n) = |\phi^{-1}(s) \cap {}^\bullet e_n| = |\phi^{-1}(s) \cap c_2|$; as $c_2 \subseteq c$, we have $F(s, t_n) = |\phi^{-1}(s) \cap c_2| \leq |\phi^{-1}(s) \cap c| = m_{n-1}(s)$. Hence ${}^\bullet t_n \subseteq m_{n-1}$.

Let $(s, t_n) \in K$; by definition of process there exists $b \in \phi^{-1}(s)$ such that $(b, e_n) \in K_\pi$; we show that $b \in c$: two cases can happen:

- $b^\bullet = \emptyset$; if $e_n F_\pi b$, as $b K_\pi e_n$ we obtain the cycle $e_n \prec_k e_n$; hence $\neg e_n F_\pi b$ thus $b \in c_1$.
- $b^\bullet \neq \emptyset$; hence there exists e_i such that bFe_i; from $bF_\pi e_i$ and $bK_\pi e_n$ we get $e_n \prec_{time} e_i$, but e_n is the last element of the linearization, hence $i < n$,contradicting the order-preserving property of linearizations.

Hence $b \in c$; thus $b \in A_s$, i.e. $m_{n-1}(s) > 0$, i.e. $s \in dom(m_{n-1})$; thus we have $\widehat{t_n} \subseteq dom(m_{n-1})$.

Let $(s, t_n) \in I$; by definition of process, for all $b \in \phi^{-1}(s)$ we have $(b, e_n) \in I_\pi^{be} \cup I_\pi^{af} \cup F_\pi^{-1}$. Suppose $m_{n-1}(s) > 0$; hence there exists $b \in \phi^{-1}(s) \cap c$; as $b \in \phi^{-1}(s)$, we have $(b, e_n) \in I_\pi^{be} \cup I_\pi^{af} \cup F_\pi^{-1}$; the following cases can happen:

- $(b, e_n) \in I_\pi^{be}$; by definition of enriched occurrence net there exists e_i such that $e_i F_\pi b$; we obtain $e_n \prec_{time} e_i$, but e_n is the last element of the linearization, hence $i < n$, contradicting the order-preserving property of linearizations.
- $(b, e_n) \in I_\pi^{af}$; by definition of enriched occurrence net there exists e_i such that $bF_\pi e_i$; and we know that $b \in c$; if $b \in c_1$ then $b^\bullet = \emptyset$, which leads to a contradiction; if $b \in c_2$ then $bF_\pi e_n$; as conditions are unbranched, from $bF_\pi e_i$ we obtain $e_n = e_i$, but from $bF_\pi e_i$ and $bI_\pi^{af} e_n$ we get $e_i \prec_i e_n = e_i$, obtaining a cycle, and we obtain again a contradiction.
- $(b, e_n) \in F_\pi^{-1}$; in this case $e_n F_\pi b$, hence $b \notin c_1$; as $b \in c$, we obtain $b \in c_2$, hence $bF_\pi e_n$, obtaining a cycle, which is again a contradiction.

Hence we have $m_{n-1}(s) = 0$; thus ${}^\circ t_n \cap dom(m_{n-1}) = \emptyset$. Hence there exists a marking m_n such that $m_0 \ldots m_{n-1}[t_n\rangle m_n$ is a firing sequence.

Now we show that π can be obtained from $\Downarrow (c, \pi)$, by applying the step of the inductive definition of processes corresponding to the firing of t_n.

We have seen above that $\downarrow c = \{e_1, \ldots, e_{n-1}\} \cup \{b \in B \mid \neg e_n F_\pi b\}$; the event we add is e_n; take $B_n^{new} = \{b \in B \mid e_n F_\pi b\} = e_n^\bullet$. By definition of process, we have that $t_n^\bullet(s) = F(t_n, s) = |\phi^{-1}(s) \cap e_n^\bullet|$; hence, for each s, B_n^{new} contains the right quantity of conditions mapped on s.

We take $Pre_n(s) = \phi^{-1}(s) \cap c_2$; by definition of $\Downarrow$ we have that $c_2 \subseteq Max(\Downarrow (c, \pi))$; moreover we have seen above that $F(s, t_n) = |\phi^{-1}(s) \cap c_2|$, hence ${}^\bullet t_n(s) = F(s, t_n) = |\phi^{-1}(s) \cap c_2| = |Pre_n(s)|$; thus $Pre_n(s)$ satisfies the required conditions. Moreover, $c_2 = {}^\bullet e_n$ and $B_n^{new} = e_n^\bullet$, hence the flow arcs are correct.

For each $s \in \widehat{t_n}$ we have seen above that there exists $b \in c \cap \phi^{-1}(s)$ such that $(b, e_n) \in K_\pi$; hence we take $test_n(s) = b$; by definition of process, this b is unique, hence the read arcs are correct.

We show that the set of inhibitor arcs of π that were not present in $\Downarrow (c, \pi)$ coincides with the set of inhibitor arcs added in the step of the inductive definition of processes.

Take $(b, e_i) \in I_\pi^{be}$. If this arc is not present in $\Downarrow (c, \pi)$, then two cases can happen:

- $b \not\in\downarrow c$; we have $b \in B_n^{new}$ and $e_n F_\pi b$. As $e_n F_\pi b$ and $bI_\pi^{be} e_i$, we have $e_i \prec_{time} e_n$, hence $e_n \neq e_i$, thus $e_i \in\downarrow c$. By definition of process, we have that $(\phi(b), \phi(e_i)) \in I$.

  Summarizing, we have $e_i \in E \cap \downarrow c$, $b \in B_n^{new}$ and $(\phi(b), \phi(e_i)) \in I$, hence this arc is added also in the step of the inductive definition of process.

- $e_i \not\in\downarrow c$, hence we have $e_i = e_n$; by the before condition, there exists e_j such that $e_j F_\pi b$; as $b I_\pi^{be} e_n$, we have $e_n \prec_{time} e_j$; but $n > j$, contradicting the order preserving property of linearizations.

Take $(b, e_i) \in I_\pi^{af}$. If this arc is not present in $\Downarrow (c, \pi)$, then two cases can happen:

- $b \not\in\downarrow c$, hence $b \in B_n^{new}$, and $e_n F_\pi b$; as $b I_\pi^{af} e_i$, by the after requirement there exists e_j such that $b F_\pi e_j$; hence we have $e_n F_\pi^2 e_j$, contradicting the order preserving property of linearizations.
- $e_i \not\in\downarrow c$, hence we have $e_i = e_n$; we show that $b \notin B_n^{new}$: if $b \in B_n^{new}$, then $e_n F_\pi b$; as $b I_\pi^{af} e_n$, by the after requirement there exists e_j such that $b F_\pi e_j$; hence we have $e_n F_\pi^2 e_j$, contradicting the order preserving property of linearizations. Hence $b \in B \cap \downarrow c$. By definition of process, we have that $(\phi(b), t_n) \in I$.

  Summarizing, we have $e_i = e_n$, $b \in B \cap \downarrow c$ and $(\phi(b), t_n) \in I$, hence this arc is added also in the step of the inductive definition of process.

Now we show that each inhibitor arc (b, e_i) that must be added by the inductive definition of process is actually present in π; two cases can happen:

- (b, e_i) is a before arc, $e_i \in E \cap \downarrow c$, $b \in B_n^{new}$ and $(\phi(b), \phi(e_i)) \in I$. As $(\phi(b), \phi(e_i)) \in I$, by definition of process we have $(b, e_i) \in I_\pi^{be} \cup I_\pi^{af} \cup F_\pi^{-1}$. If $(b, e_i) \in I_\pi^{af}$, by the after requirement in the definition of enriched occurrence net there exists e_j such that $b F_\pi e_j$; as $b \in B_n^{new}$, we have $e_n F_\pi b$; thus we obtain $e_n < e_j$, but $j < n$, contradicting the order preserving property of linearizations. If $(e_i, b) \in F_\pi$, as $b \in B_n^{new}$ we have $e_n F_\pi b$; as conditions are unbranched, we get $e_i = e_n$, contradicting the fact that $e_i \in E \cap \downarrow c$. Thus we obtain $(b, e_i) \in I_\pi^{be}$.
- (b, e_i) is an after arc, $e_i = e_n$, $b \in B_\pi \cap \downarrow c$ and $(\phi(b), \phi(e_i)) \in I$. As $(\phi(b), \phi(e_n)) \in I$, by definition of process we have $(b, e_n) \in I_\pi^{be} \cup I_\pi^{af} \cup F_\pi^{-1}$. If $(b, e_n) \in I_\pi^{be}$, by the before requirement in the definition of enriched occurrence net there exists e_j such that $e_j F_\pi b$; as $(b, e_n) \in I_\pi^{be}$, we get $e_n \prec_{time} e_i$, but $i < n$, contradicting the order preserving property of linearizations. If $(e_n, b) \in F_\pi$, then $b \in B_n^{new}$, contradicting the fact that $b \in B_\pi \cap \downarrow c$. Thus we obtain $(b, e_i) \in I_\pi^{af}$.

Hence the set of before (after) inhibitor arcs that are present in π and not in $\Downarrow (c, \pi)$ coincides with the set of before (after) inhibitor arcs added in the step of the inductive definition of process. □

Proposition 6.3. *Let π be a process and c a slice of π; then $m(s) = |c \cap \phi^{-1}(s)|$ is a reachable marking. Moreover, there exists a linearization $e_1, \ldots, e_n$ of $E \cap \downarrow c$ such that $m_0[\phi(e_1)\rangle \ldots m_{n-1}[\phi(e_n)\rangle m_n$ is a firing sequence.*

Proof:
By Proposition 5.5, $\Downarrow (c, \pi)$ is a process. Take a linearization $e_1, \ldots, e_n$ of $E \cap \downarrow c$; by Theorem 6.3 there exists a firing sequence $\sigma = m_0[\phi(e_1)\rangle \ldots m_{n-1}[\phi(e_n)\rangle m_n$ such that $\Downarrow (c, \pi) \in \Pi(\sigma)$. We have that $c = Max(\Downarrow (c, \pi))$; by Proposition 6.1 we have that $m_n(s) = |c \cap \phi^{-1}(s)|$. □

6.2. Retrieving step firing sequences

We show how to associate a set of processes to a step firing sequence. Some proofs are an easy generalization of those seen in the case of retrieving firing sequences and are then omitted.

Construction 6.2. Let $N = (S, T, F, K, I, m_0)$ be a contextual P/T system and $\sigma = m_0[G_1\rangle m_1 \ldots [G_n\rangle m_n$ be a step firing sequence of N. We construct a set $\Pi_{step}(\sigma)$ of objects, that will turn out to be processes of N. The construction proceeds by induction on the length of the step firing sequence. The conditions will be of the form (s, i, h, k), where s is the place of N to which the condition corresponds, i denotes the step of the construction at which the condition is added, h identifies the occurrence of transition in the step that generates the condition and k is used to differentiate conditions, produced at the same step by the same occurrence of transition, that correspond to the same place of N. The events will have the form (t, i, h), where t is the transition of N to which the event corresponds, i denotes the step of the construction at which the event is added and h identifies the occurrence of transition in the step to which the event corresponds.

If the step firing sequence is m_0, then $\pi_0 = (B_0, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \phi_0)$, where

$$B_0 = \{(s, 0, 0, i) \mid s \in S \wedge 1 \leq i \leq m_0(s)\} \text{ and } \phi_0 : (s, 0, 0, i) \to s.$$

If the step firing sequence is $m_0 \ldots m_n[G_{n+1}\rangle m_{n+1}$, assume that $\pi_n = (B_n, E_n, F_n, K_n, I_n^{be}, I_n^{af}, \phi_n)$ is constructed from the step firing sequence $m_0 \ldots m_n$. Hence construct $\pi_{n+1} = (B_{n+1}, E_{n+1}, F_{n+1}, K_{n+1}, I_{n+1}^{be}, I_{n+1}^{af}, \phi_{n+1})$ in the following way:

Take a sequence of transitions $t_{n+1}^1 \ldots t_{n+1}^{|G_{n+1}|}$ such that, for all $t \in T$, $G_{n+1}(t) = |\{h \mid 1 \leq h \leq |G_{n+1}| \wedge t_{n+1}^h = t\}|$.

$$B_{n+1} = B_n \cup \bigcup_{h=1}^{|G_{n+1}|} B_{n+1}^{new,h}$$

where, for $h = 1, \ldots, |G_{n+1}|$, $B_{n+1}^{new,h} = \{(s, n+1, h, i) \mid s \in S \wedge 1 \leq i \leq (t_{n+1}^h)^\bullet(s)\}$

$$E_{n+1} = E_n \cup \{(t_{n+1}^h, n+1, h) \mid 1 \leq h \leq |G_{n+1}|\}$$

For each $s \in S$ and $h = 1, \ldots, |G_{n+1}|$, choose a set $Pre_{n+1}^h(s) \subseteq B_n$ such that

- for all $b \in Pre_{n+1}^h(s)$, $b^{\bullet_n} = \emptyset$ and $\phi_n(b) = s$
- $|Pre_{n+1}^h(s)| = {}^\bullet t_{n+1}^h(s)$
- $Pre_{n+1}^{h_1} \cap Pre_{n+1}^{h_2} = \emptyset$ for $h_1 \neq h_2$, $1 \leq h_1, h_2 \leq |G_{n+1}|$

$$\begin{array}{rcl} F_{n+1} & = & F_n \\ & \cup & \{(b, (t_{n+1}^h, n+1, h)) \mid \exists s \in S : b \in Pre_{n+1}^h(s)\} \\ & \cup & \{((t_{n+1}^h, n+1, h), b) \mid b \in B_{n+1}^{new,h}\} \end{array}$$

For $h = 1, \ldots, |G_{n+1}|$, for all $s \in \widehat{t}_{n+1}^h$, choose a condition $test_{n+1}^h(s) \in B_n$ such that

- $\phi_n(test^h_{n+1}(s)) = s$
- $(test^h_{n+1})^\bullet(s) = \emptyset$
- $test^{h_1}_{n+1} \notin Pre^{h_2}_{n+1} = \emptyset$ for $1 \leq h_1, h_2 \leq |G_{n+1}|$

$$K_{n+1} = K_n \cup \{(test^h_{n+1}(s), (t^h_{n+1}, n+1, h)) \mid s \in \widehat{t}^h_{n+1}\}$$

$$I^{be}_{n+1} = I^{be}_n \cup \{(b, e) \mid e \in E_n \wedge b \in \bigcup_{h=1}^{|G_{n+1}|} B^{new,h}_{n+1} \wedge \phi_{n+1}(b) \in {}^\circ\phi_{n+1}(e)\}$$

$$I^{af}_{n+1} = I^{af}_n \cup \{(b, (t^h_{n+1}, n+1, h)) \mid 1 \leq h \leq |G_{n+1}| \wedge b \in B_n \wedge \phi_n(b) \in {}^\circ t^h_{n+1}\}$$

$$\phi_{n+1} = \phi_n \cup \{((t^h_{n+1}, n+1, h), t^h_{n+1})\} \cup \{((s, n+1, h, i), s) \mid (s, n+1, h, i) \in B^{new,h}_{n+1}\}.$$

□

Note that Proposition 6.1 continues to hold, hence, as by definition of enabling of step G_{n+1} we have that $\bigoplus_t(G_{n+1}(t) \cdot {}^\bullet t) \oplus \bigcup_{t \in dom(G)} \widehat{t} \subseteq m_n$, it is always possible to find sets $Pre^h_{n+1}(s)$ and conditions $test^h_{n+1}(s)$, for $h = 1, \ldots, |G_{n+1}|$, satisfying the required conditions.

We show how to associate a natural number $\#(x)$ to each element $x \in B_n \cup E_n$, in such a way that $x < y$ implies $\#(x) < \#(y)$.

Definition 6.2. Let $x \in B_n \cup E_n$. We define $\#(x) = \begin{cases} 2 * i - 1 & \text{if } x = (t, i, h) \\ 2 * i & \text{if } x = (s, i, h, j) \end{cases}$ □

Proposition 6.4. *Let $x \in B_n \cup E_n$. Then $\#(x) \leq 2 * n$.* □

Lemma 6.2. *Let $\pi_n = (N_n, \phi_n)$ be in $\Pi_{step}(\sigma)$. Then*

1. *N_n is an enriched occurrence net;*
2. *for all $x, y \in B_n \cup E_n$, $x < y$ implies $\#(x) < \#(y)$.*

Proof:

The proof is similar to the proof of Lemma 6.1. Let $\sigma = m_0 \ldots m_n[G_{n+1}\rangle m_{n+1}$ and $N_{n+1} = (B_{n+1}, E_{n+1}, F_{n+1}, K_{n+1}, I^{be}_{n+1}, I^{af}_{n+1})$.

The main difference is in the proof of the following facts:

$$\begin{array}{lll} e \prec_{k,n+1} e' & \Rightarrow & \#(e) < \#(e') \\ e \prec_{i,n+1} e' & \Rightarrow & \#(e) < \#(e') \\ e \prec_{time,n+1} e' & \Rightarrow & \#(e) < \#(e') \end{array}$$

Besides the cases considered in the proof of Lemma 6.1, we have also to consider the case where e and e' have been added at the same step. Now we show how to deal with that case.

Let $e, e' \in E_{n+1}$, with $\#(e) = \#(e')$ (i.e. the two events have been added in the same step). We show that $(e, e') \notin \prec_{k,n+1} \cup \prec_{i,n+1} \cup \prec_{time,n+1}$. If $e, e' \in E_n$ then the result is true by inductive hypothesis. We consider the case where $e, e' \in E_{n+1} \setminus E_n$.

- Suppose $e \prec_{k,n+1} e'$; hence there exists $b \in B_{n+1}$ such that $eF_{n+1}b$ and $bK_{n+1}e'$; from $eF_{n+1}b$, by construction of the process we have $b \in B_{n+1}^{new,h}$ for some h, hence $\#(b) = 2*(n+1)$; from $bK_{n+1}e'$, by construction we have $b \in B_n$, hence $\#(b) \leq 2*n$, contradiction.
- Suppose $e \prec_{i,n+1} e'$; hence there exists $b \in B_{n+1}$ such that $bF_{n+1}e$ and $bI_{n+1}^{af}e'$. Let $e = (t_{n+1}^h, n+1, h)$ and $e' = (t_{n+1}^{h'}, n+1, h')$, with $1 \leq h, h' \leq |G_{n+1}|$. From $bF_{n+1}e$, by construction we get $b \in Pre_{n+1}^h(\phi_n(b))$, hence ${}^\bullet t_{n+1}^h(\phi_n(b)) > 0$; as step G_{n+1} is enabled at m_n, and $G_{n+1}(t_{n+1}^h) > 0$, we have $m_n(\phi_n(b)) > 0$. From $bI_{n+1}^{af}e'$ we get $\phi_n(b) \in {}^\circ t_{n+1}^{h'}$; thus we have $t_{n+1}^{h'} \in dom(G_{n+1})$ and ${}^\circ t_{n+1}^{h'} \cap dom(m_n) \neq 0$, hence step G_{n+1} is not enabled at m_n, contradiction.
- Suppose $e \prec_{time,n+1} e'$; two cases can happen:
  - there exists $b \in B_{n+1}$ such that $bK_{n+1}e$ and $bF_{n+1}e'$. Let $e = (t_{n+1}^h, n+1, h)$ and $e' = (t_{n+1}^{h'}, n+1, h')$, with $1 \leq h, h' \leq |G_{n+1}|$.
    From $bF_{n+1}e'$ we get $b \in Pre_{n+1}^h(\phi_n(b))$; from $bK_{n+1}e$ we get $b = test_{n+1}^{h'}(\phi_n(b))$; hence $test_{n+1}^{h'}(\phi_n(b)) \in Pre_{n+1}^h(\phi_n(b))$, contradicting the requirements in the construction of the process.
  - there exists $b \in B_{n+1}$ such that $e'K_{n+1}b$ and $bI_{n+1}^{be}e$. This case is impossible because, by construction, if $(b, e) \in I_{n+1}^{be}$ then $e \in E_n$.

□

As events corresponding to occurrences of transitions in a step G are added at the same moment in the construction in Definition 6.2 (i.e. they have the same number $\#$), Lemma 6.2 tells us that the set of these events is a co-set of the process.

Theorem 6.4. *Each $\pi \in \Pi_{step}(\sigma)$ is a process.* □

To construct a step firing sequence starting from a process, we need the following generalization of the notion of linearization to steps:

Definition 6.3. Let $(X, \leq)$ be a partial order. A *step linearization* of $(X, \leq)$ is a complete sequence of disjoint subsets $X_1 \dots X_k$ compatible with $\leq$, i.e. $X = \bigcup_{i+1}^k X_i$, $X_i \cap X_j = \emptyset$ for $i \neq j$ and, for all $x \in X_i$ and $y \in X_j$, $x < y \Rightarrow i < j$. □

Theorem 6.5. *Let π be a process and $E_1, \dots, E_n$ be a step linearization of the set of events of π. Then there exists a step firing sequence $\sigma = m_0[\phi(E_1)\rangle \dots [\phi(E_n)\rangle m_n$ such that $\pi \in \Pi_{step}(\sigma)$.* □

According to Theorems 6.4 and 6.5, we get a close correspondence between steps and co-sets of events: any co-set of events of a process corresponds to a step in the corresponding step firing sequence, and vice versa.

In Subsection 4.1 we recalled some different notions of step semantics, proposed in the literature; according to [21], a step roughly corresponds to a set of causally unrelated events, for which there exists at least one possible sequential execution. We claim that steps à la Vogler [21] can be retrieved from our processes, as set of events unrelated w.r.t. the ordering relation $(F \cup \prec_k \cup \prec_i)^+$.

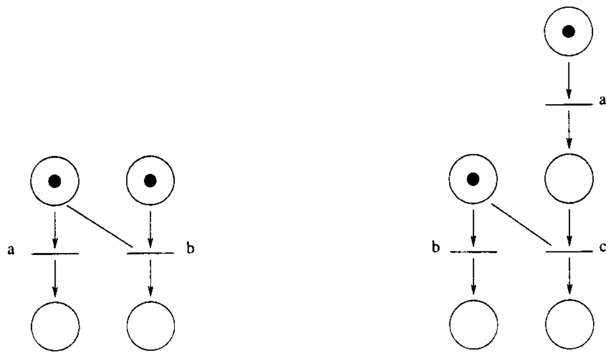

Figure 9 Two nets illustrating the differences of the causal relation induced by our approach and by the one adopted in [12] (on the left) and the one adopted in [21] (on the right).

7. Conclusions

In this paper we have defined a process semantics for Contextual nets. There are some other approaches in literature that we briefly compare with ours.

In [12] a process semantics for C/E nets with read arcs (called there positive contexts) is presented (C/E nets are a class of nets permitting the presence of at most one token for place in any reachable marking; for a formal definition, see e.g. [18]). The main difference w.r.t. our approach is that only one relation (called causal relation) is defined on events there; consider the net on the left in Figure 9: according to [12], event a is causally dependent on b, while we think that event a does not depend on the behaviour of event b, in fact it can occur even if b has not occurred yet; the only thing we can say is that if both of them occur in a computation then b happened before a, hence we relate them with a temporal precedence relation.

In [22] a notion of process for P/T nets with positive contexts is presented, which is based on a notion of enriched occurrence net rather different from ours, as there places and transitions concerning the context tokens are added with the aim of capturing more precisely how the tokens are used. However, as in [12], only a causal relation is considered.

In [19] a process semantics for both C/E nets with read arcs and P/T nets with read arcs and flow arcs with weight 1 is presented, following the lines of [12]. Here two different kinds of dependencies are defined: the first one, called functional dependency, coincides with our $(F \cup \prec_k)^*$ and is intended to "model the flow of data in the net"; the second one, called causal dependency, coincides with our $(F \cup \prec_k \cup \prec_{time})^*$ and "deals with the order in which the transitions may take place".

In [21] a process semantics for safe P/T nets with read arcs is proposed in a framework where activities have durations, and related to ST-traces, that are sequences of transition starts and ends. A relational structure composed of two relations is derived from processes: $\prec$ (called causality) and $\sqsubset$; the intended meaning of $e \prec f$ is that e ends before f starts, whereas the intended meaning of $e \sqsubset f$ is that e starts before f starts. At a first glance, it seems that $\prec$ is similar to the union of F^+ and $\prec_k$, and that $\sqsubset$ is similar to our $<$; however, there are some differences, mainly due to the fact that we are trying to capture the causal dependencies among events, whereas in [21] there is more interest for the temporal relations between startings and endings of transitions. Consider the net on the right in figure 9: according to [21], we have $a \prec c$ and $c \sqsubset b$; even in our approach we have a causal relation between a and c (aF^2b), and a temporal precedence of c w.r.t. b ($c \prec_{time} b$). However, according to [21], we also have that $a \prec b$, whereas in our approach there is no causal dependency between a and b.

In [11] a process semantics for elementary net systems with inhibitor arcs is defined; a peculiarity of that work is that, while dealing with the semantics of nets with inhibitor arcs, processes are based on occurrence nets enriched with read arcs (called there activator arcs). Two relations are derived from processes: the first one, called causality and denoted by $\prec$, coincides with the union of our flow and read causality $F^+ \cup \prec_k$; the second one, called weak causality and denoted by $\sqsubset$, coincides with our temporal precedence $\prec_{time}$.

A straightforward generalization of the notion of Contextual nets is to equip read and inhibitor arcs with weights. It is easy to adapt our notions to this case.

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