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Dividing a Cake Fairly

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If each of n people defines a (suitable) measure on a compact convex cake I, then there exists a division of I into n connected parts, and an assignment of these n parts to the n people, in such a way that the piece of cake assigned to each person is at least as large (in his own measure) as that assigned to anyone else. (The proof uses Brouwer's fixed-point theorem and Hall's theorem on systems of distinct representatives.) Slight progress is made towards finding algorithms for this problem.

1. INTRODUCTION

By a *cake* is meant a compact convex set in some Euclidean space. I shall take the space to be $\mathbb{R}$, so that the cake is simply a compact interval I, which without loss of generality I shall take to be $[0, 1]$. If you find this thought unappetizing, by all means think of a three-dimensional cake. Each point P of division of my cake will then define a plane of division of your cake: namely, the plane through P orthogonal to I.

A measure μ on I will be said to be *suitable*, or to satisfy *Condition C*, if

(C1) the μ-measurable sets are the Lebesgue-measurable subsets of I,

(C2) μ is a probability measure (so that $\mu(I) = 1$), and

(C3) μ is absolutely continuous with respect to Lebesgue measure.

Condition (C3) states that there is no set with positive μ-measure and Lebesgue measure zero, and it ensures that any set with positive μ-measure can be subdivided into sets with arbitrarily small μ-measure.

Suppose now that each of n people defines a suitable measure (in the above sense) on I. The classical cake-division problem (for which see, for example, [1]) is to divide the cake among the n people in such a way that each person receives at least $1/n$ of the cake (in his own measure). There are several known solutions to this problem; for the sake of completeness, three are given in Appendix 1. A deeper result [8], with an application to the division of inheritances, is that it is possible for each person to receive *more*

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than $1/n$ of the cake (in his own measure), except when the n measures are all identical.

The problem we shall consider here is to divide the cake among the n people in such a way that each person receives (in his own measure) *at least as much as anyone else.* If $n = 2$, this is clearly the same problem as the classical one, but if $n \geqslant 3$ it is a quite different, and much harder, problem. The problem for $n = 3$ was posed by Gamow and Stern [3]; the general problem has been disseminated by J. L. Selfridge, who also devised the algorithm described in Section 5.

In Theorem 1 (in the next section) I prove that there always exists a division of the cake into n (probably disconnected) portions all of which have measure exactly $1/n$ in each of the n measures. Theorem 2 (in Section 3) is a combinatorial result needed for the proof of the main theorem. Theorem 3 (in Section 4) is the main theorem (the result stated in the abstract), that there is a division into n connected portions satisfying the conditions of Gamow and Stern's problem. In Section 5 I consider algorithms for constructing such a division (without the restriction that the portions must be connected).

There is a problem on the division of wine among n people that is closely related to the cake-division problem. In this problem the n people have a jug of wine and n glasses (of assorted shapes and sizes), and must divide the wine among the n glasses, and then assign a glass to each person, in such a way that the conditions of Gamow and Stern's problem are satisfied. The snag is that each person forms a different estimate of the amount of wine in each glass at each stage; effectively we are given n^2 increasing functions, $f_{ij}(x)$ denoting the ith person's estimate of the volume of wine in glass j when the actual volume of wine in glass j is x. This problem is similar to the cake-division problem, but not the same; the inequality here is associated with the glasses, not with the wine, which unlike the cake is completely homogeneous. Also, a piece of cake has the same measure (in one of the n measures) wherever you put it, whereas the same portion of wine may appear to change size as it is poured from glass to glass; not even the total volume of wine appears to remain constant. We shall not mention this wine-division problem again, but remark here that Theorem 4, which is proved in Section 4 in order to prove Theorem 3, shows that there does exist a division of the wine among the glasses that satisfies the terms of this problem (always assuming, of course, that the glasses are large enough).

2. Divisions into Equal Parts

THEOREM 1. *Let $\mu_1,\dots,\mu_n$ be n suitable measures defined on the interval I. (That is, they satisfy Condition C.) Then there exists a partition of I into n*

measurable sets $X_1,..., X_n$ *such that, for each* i *and* j $(i, j = 1,..., n)$ $\mu_i(X_j) = 1/n$.

Proof. Dubins and Spanier [1] point out that the more general result, in which we are given $\alpha_1,..., \alpha_n$ such that $\sum \alpha_j = 1$, and require that $\mu_i(X_j) = \alpha_j$ for each i and j, is a special case of a theorem of Lyapunov ([6]; see also [5]). But Lyapunov's theorem provides no information about the number of cuts needed to effect the division. The proof of Theorem 1 given here uses the generalized ham-sandwich theorem of Stone and Tukey [10], and can also clearly be modified to prove the more general result.

Choose a twisted curve in $\mathbb{R}^n$, such as the set of points

$$S = \{(t, t^2,..., t^n): t \in \mathbb{R}\},$$

with a continuous injection $f: I \to S$ such as

$$t \to (t, t^2,..., t^n).$$

If H is an oriented hyperplane in $\mathbb{R}^n$, and H^+ denotes the half-space on the positive side of H, then each of the quantities $\mu_i(f^{-1}(S \cap H^+))$ varies continuously with H as H moves around in $\mathbb{R}^n$ (since H always intersects S in at most a finite number of points). Thus the conditions of the generalized ham-sandwich theorem are satisfied, and that theorem tells us that there is a hyperplane H such that, for each i, $\mu_i(f^{-1}(S \cap H^+)) = \frac{1}{2}$. The points of $f^{-1}(S \cap H)$ now divide I into two portions (each the union of a number of subintervals) each of which has measure exactly $\frac{1}{2}$ in each of the n measures. If n is a power of 2, repeated bisection in this way achieves the required division in a finite number of steps; otherwise, the same result is achieved as the limit of an infinite process. (For example, if $n = 13$, then I is divided into 16 equal pieces, 13 of which are left intact; each of the remaining 3 pieces is further subdivided into 16 equal pieces; and so on.) ∎

If n is a power of 2, this method will effect the required division in $n - 1$ bisections (that is, in at most $n(n-1)$ cuts of I); but if n is not a power of 2, infinitely many cuts will be used. Fremlin [2] has proved that a finite number of cuts will always suffice. However, he has not been able to show that this number is bounded above by a function of n, since in his proof it depends on the measures $\mu_1,..., \mu_n$. The following conjecture would imply that the number of cuts needed for a given n is bounded above.

CONJECTURE. *For each positive integer* n *there is a positive integer* $f(n)$ *with the following property. If one is given* n *suitable measures on the interval* I *(that is, satisfying Condition C), and a real number* α *in* $[0, 1]$, *then there exists a set of at most* $f(n)$ *disjoint subintervals of* I *whose union has measure exactly* α *in each of the* n *measures.*

It is not difficult to see that $f(1) = 1$ and $f(2) = 2$; it is not known whether $f(3)$ exists.

3. A Combinatorial Theorem

Let $I := \{1,..., n\}$. (This terminology is usual in transversal theory, and there will surely be no confusion with the interval I of neighbouring sections.) Let $\mathbf{A}(I) := (A_1,..., A_n)$ be a family of n finite sets. If $K \subseteq I$, let $\mathbf{A}(K) := (A_i: i \in K)$ and let $A(K) := \bigcup (A_i: i \in K)$. (An obviously analogous terminology will be used for other families of sets. For example, if $\mathbf{B}(J)$ is a family of sets and $L \subseteq J$, then the expressions $\mathbf{B}(L)$ and $B(L)$ will be used without further explanation.) The family $\mathbf{A}(I)$ is said to have a *transversal* or *system of distinct representatives* if there exist n distinct elements $x_1,..., x_n$ such that $x_i \in A_i$ for each i ($i = 1,..., n$). A well-known theorem of Hall ([4]; see also [7]) states that $\mathbf{A}(I)$ has a transversal if and only if *Hall's condition* holds:

$$|A(K)| \geqslant |K| \qquad \text{for each subset } K \text{ of } I$$

(where $|\,|$ denotes cardinality, as usual).

Lemma 2.1. *Let $\mathbf{A}(I)$ be a finite family of finite sets, and let $c := \max\{|K| - |A(K)|: K \subseteq I\}$. (Evidently $c \geqslant 0$, since $|A(\varnothing)| = |\varnothing| = 0$.) Then the collection of subsets K of I for which $|A(K)| = |K| - c$ is closed under unions and intersections. (In particular, there is a unique maximal subset K of I for which $|A(K)| = |K| - c$.)*

Proof. Suppose that $|A(K_1)| = |K_1| - c$ and $|A(K_2)| = |K_2| - c$, and note that $A(K_1 \cup K_2) = A(K_1) \cup A(K_2)$ and $A(K_1 \cap K_2) \subseteq A(K_1) \cap A(K_2)$. So

$$\begin{aligned} |A(K_1 \cup K_2)| + |A(K_1 \cap K_2)| &\leqslant |A(K_1) \cup A(K_2)| + |A(K_1) \cap A(K_2)| \\ &= |A(K_1)| + |A(K_2)| \\ &= |K_1| + |K_2| - 2c \\ &= |K_1 \cup K_2| + |K_1 \cap K_2| - 2c. \end{aligned}$$

From the definition of c, the only possibility is that

$$|A(K_1 \cup K_2)| = |K_1 \cup K_2| - c \quad \text{and} \quad |A(K_1 \cap K_2)| = |K_1 \cap K_2| - c. \qquad \blacksquare$$

The following theorem will be needed in the next two sections.

Theorem 2. (a) *Let $I := \{1,..., n\}$, and let $\mathbf{A}(I)$ be a family of finite sets. Let J be a set of cardinality $m \geqslant n$ such that $A(I) \subseteq J$. Suppose that a* *sequence of families* $\mathbf{A}^{(0)}(I) = \mathbf{A}(I)$, $\mathbf{A}^{(1)}(I),\ldots, \mathbf{A}^{(t)}(I)$ *is constructed iteratively as follows. For each* $s \geqslant 0$, *if the family* $\mathbf{A}^{(s)}(I)$ *has a transversal, let* $t := s$; *the construction is finished. Otherwise, let* $K^{(s)}$ *be the unique largest subset of* I *for which* $|K^{(s)}| - |A^{(s)}(K^{(s)})|$ *is largest, and form the family* $\mathbf{A}^{(s+1)}(I)$ *from* $\mathbf{A}^{(s)}(I)$ *by adding one or more extra elements of* J *to one or more of the sets* $A_i^{(s)}$ *with* i *in* $K^{(s)}$. *(It is clear that this construction will terminate in a finite number of steps.) Then*

$$A^{(0)}(K^{(0)}) \cup \cdots \cup A^{(t-1)}(K^{(t-1)}) \neq J.$$

(b) *Suppose now that* $m \geqslant n+1$ *and that the above construction is continued until* $|A^{(t)}(K)| \geqslant |K| + 1$ *for every non-empty subset* K *of* I *(at which point it must terminate, since necessarily* $|A^{(t)}(\varnothing)| = |\varnothing|$*). Then the same conclusion holds.*

Proof. Define a new family $\mathbf{B}(J)$ dual to $\mathbf{A}(I)$ by

$$B_j := \{i : j \in A_i\} \qquad \text{for each } j \text{ in } J.$$

(If a bipartite graph is formed with vertex-set $I \cup J$ by drawing an edge between i (in I) and j (in J) whenever $j \in A_i$, then A_i is the set of neighbours of i and B_j the set of neighbours of j. This graph may help the reader to visualize some steps in the following proof.) Let $c := \max\{|K| - |A(K)| : K \subseteq I\}$ and let $d := c + m - n$. Note that, if K is a maximal subset of I for which $|A(K)|$ has a given value, and $L := J \backslash A(K)$, then $B(L) = I \backslash K$. (For, no edge joins a vertex of K to one in L, so that $B(L) \subseteq I \backslash K$. And if $i \in (I \backslash K) \backslash B(L)$, then we could add i to K without increasing $A(K)$, thereby violating the maximality of K.) Similarly, if L is a maximal subset of J for which $|B(L)|$ has a given value, and $K := I \backslash B(L)$, then $A(K) = J \backslash L$. In either case,

$$|K| - |A(K)| - c = |I \backslash B(L)| - |J \backslash L| - c = |L| - |B(L)| - d.$$

Thus $d = \max\{|L| - |B(L)| : L \subseteq J\}$, and $|K| - |A(K)| = c$ if and only if $|L| - |B(L)| = d$, where $L = J \backslash A(K)$ and $K = I \backslash B(L)$. Thus the unique maximal set K such that $|K| - |A(K)| = c$ corresponds to the unique minimal set L such that $|L| - |B(L)| = d$. Let the sets corresponding in this way to $K^{(0)},\ldots, K^{(t-1)}$ be $L^{(0)},\ldots, L^{(t-1)}$, where, for each s, $L^{(s)} = J \backslash A^{(s)}(K^{(s)})$ and $K^{(s)} = I \backslash B^{(s)}(L^{(s)})$, $\mathbf{B}^{(s)}(J)$ being the family of sets dual to $\mathbf{A}^{(s)}(I)$. For future reference note that, as long as $s \leqslant t-1$, then (writing c and d for what we should properly call $c^{(s)}$ and $d^{(s)}$) $c \geqslant 1$ and $d = c + m - n \geqslant 1$ in case (a), and $c \geqslant 0$ and $d = c + m - n \geqslant 1$ in case (b), so that in either case there is a subset L of J such that

$$|B^{(s)}(L)| = |L| - d < |L|. \tag{1}$$

The conclusion of the theorem is that $L^{(0)} \cap \cdots \cap L^{(t-1)} \neq \varnothing$.

Let $M^{(s)} := L^{(0)} \cap \cdots \cap L^{(s)}$ $(s = 0,\ldots, t-1)$. We first note that

$$\text{If } L \subseteq J \backslash L^{(s)}, \text{ then } |B^{(s)}(L) \backslash B^{(s)}(L^{(s)})| \geqslant |L| \qquad (s = 0,\ldots, t-1). \tag{2}$$

For otherwise

$$|L^{(s)} \cup L| - |B^{(s)}(L^{(s)} \cup L)| > |L^{(s)}| - |B^{(s)}(L^{(s)})|,$$

contrary to the definition of $L^{(s)}$ as a set for which $|L^{(s)}| - |B^{(s)}(L^{(s)})|$ is maximal. We now prove by induction on s that

$$\text{If } L \subseteq J \backslash M^{(s)}, \text{ then } |B^{(s)}(L)| \geqslant |L| \qquad (s = 0,\ldots, t-1). \tag{3}$$

This follows from (2) if $s = 0$, since $M^{(0)} = L^{(0)}$. So suppose that $s > 0$, and let $L \subseteq J \backslash M^{(s)}$. Then we can write L as the disjoint union $L' \cup L''$, where

$$L' := L \backslash L^{(s)} \subseteq J \backslash L^{(s)}$$

and

$$L'' := L \cap L^{(s)} \subseteq J \backslash M^{(s-1)}.$$

Since, in graph-theoretic terms, we get from $s-1$ to s by adding one or more edges to the graph,

$$|B^{(s)}(L'')| \geqslant |B^{(s-1)}(L'')| \geqslant |L''|,$$

by the induction hypothesis of (3) applied to L''. Also, applying (2) to L',

$$|B^{(s)}(L') \backslash B^{(s)}(L'')| \geqslant |B^{(s)}(L') \backslash B^{(s)}(L^{(s)})| \geqslant |L'|.$$

Thus

$$|B^{(s)}(L)| = |B^{(s)}(L') \cup B^{(s)}(L'')| \geqslant |L'| + |L''| = |L|,$$

as required. It now follows that $M^{(s)} \neq \varnothing$ $(s = 0,\ldots, t-1)$. For, if $M^{(s)} = \varnothing$, then (3) says that $|B^{(s)}(L)| \geqslant |L|$ for each subset L of J, which violates (1). This completes the proof of Theorem 2. ∎

4. The Main Theorem

This section is devoted to a proof of the following theorem.

Theorem 3. *Let $\mu_1,\ldots,\mu_n$ be n suitable measures defined on the interval I. (That is, they satisfy Condition C.) Then there exists a partition of I into n subintervals $I_1,\ldots, I_n$ (in order along I) and a bijection π: $\{1,\ldots, n\} \to \{1,\ldots, n\}$ such that, for each i $(i = 1,\ldots, n)$, $\mu_i(I_{\pi(i)}) \geqslant \mu_i(I_j)$ for each j $(j = 1,\ldots, n)$.*

In terms of cake, $I_{\pi(i)}$ is of course the piece of cake that will be given to the person whose measure on the cake is μ_i. Now, the division of I in which the subintervals $I_1,..., I_n$ have lengths $\lambda_1,..., \lambda_n$ ($\sum \lambda_j = 1$) can be represented by the point $\boldsymbol{\lambda}$ with barycentric coordinates $\lambda_1,..., \lambda_n$ in a regular $(n-1)$-simplex S of unit altitude. If we define $f_{ij}(\boldsymbol{\lambda}) := \mu_i(I_j)$ for each i and j, then the functions $f_{ij}: S \to \mathbb{R}^+$ are continuous, and clearly $f_{ij}(\boldsymbol{\lambda}) = 0$ for each i if $\lambda_j = 0$. The result of Theorem 3 will therefore follow from Theorem 4. In fact, Theorem 4 proves a slightly stronger result, namely, that there is a division of the cake into n connected parts (depending only on the first $n-1$ measures) such that, whichever part the nth person finally chooses, the remaining $n-1$ parts can be given to the remaining $n-1$ people in such a way that each of them thinks he has at least as much cake as anyone else.

THEOREM 4. (a) *Let S be a regular $(n-1)$-simplex with $(n-2)$-faces $F_1,..., F_n$. Let $f_{ij}: S \to \mathbb{R}^+$ $(i, j = 1,..., n)$ be continuous functions such that, for each i and j, $f_{ij}(\boldsymbol{\lambda}) = 0$ if $\boldsymbol{\lambda} \in F_j$. Then there is a point $\boldsymbol{\lambda}$ in S (which we shall call a* satisfactory *point) and a bijection $\pi: \{1,..., n\} \to \{1,..., n\}$ such that, for each i and j,*

$$f_{i\pi(i)}(\boldsymbol{\lambda}) \geqslant f_{ij}(\boldsymbol{\lambda}). \tag{4}$$

(b) *In fact, we can even choose $\boldsymbol{\lambda}$ with the property that, for each l $(l = 1,..., n)$, there is a bijection $\pi_l: \{1,..., n\} \to \{1,..., n\}$ such that $\pi_l(n) = l$ and, for each i and j $(i \neq n)$ (4) holds.*

Proof. The proof is divided into three stages. In Stage 2 we shall define n functions $g_j: S \to \mathbb{R}^+$ $(j = 1,..., n)$. In Stage 3 we shall prove that these functions satisfy the following four conditions:

(G1) For each j, g_j is continuous.

(G2) For each j, $g_j(\boldsymbol{\lambda}) = 0$ if $\boldsymbol{\lambda} \in F_j$.

(G3) For each $\boldsymbol{\lambda}$ is S, there exists a j for which $g_j(\boldsymbol{\lambda}) = 0$.

(G4) A point $\boldsymbol{\lambda}$ in S is satisfactory if and only if $g_1(\boldsymbol{\lambda}) = \cdots = g_n(\boldsymbol{\lambda}) = 0$.

(In condition (G4), the term "satisfactory" has slightly different meanings according to whether we are trying to prove (a) or (b).)

Stage 1. We prove first that, *if* we can define functions $g_1,..., g_n$ satisfying the above four conditions, *then* the result of the theorem will follow. So suppose that the functions $g_1,..., g_n$ are given, and let $\mathbf{e}_1,..., \mathbf{e}_n$ be the unit vectors in the directions of the outward-facing normals to the faces

$F_1,\ldots, F_n$ of S. Define a function $G: S \to \mathbb{R}^{n-1}$ (which is continuous, by (G1)) by

$$G(\boldsymbol{\lambda}) := \boldsymbol{\lambda} + \sum_{j=1}^{n} g_j(\boldsymbol{\lambda})\mathbf{e}_j.$$

Let $H: \mathbb{R}^{n-1} \to S$ be the mapping that leaves points of S fixed and maps points outside S radially in (towards the barycentre) onto the boundary of S. Then HG is a continuous mapping from S to itself, and so by Brouwer's fixed-point theorem it has a fixed point $\boldsymbol{\lambda}$. Since each $\mathbf{e}_j$ has negative inner product with each of the other $\mathbf{e}_k$'s, it follows from (G2) that no point on the boundary of S is mapped radially outwards (from the barycentre) by G; so the fixed point $\boldsymbol{\lambda}$ of HG is necessarily a fixed point of G. Since $\sum \mathbf{e}_j = \mathbf{0}$ (and that is the *only* linear relation between the $\mathbf{e}_j$'s), it follows that $g_1(\boldsymbol{\lambda}) = \cdots = g_n(\boldsymbol{\lambda})$. It now follows from (G3) and (G4) that $\boldsymbol{\lambda}$ is a satisfactory point. So if we can prove the existence of the functions $g_1,\ldots, g_n$, the proof of the theorem will be complete.

Stage 2. We now describe the construction of the functions $g_1,\ldots, g_n$. We shall carry out the construction for a specific point $\boldsymbol{\lambda}$ in S, which will remain fixed throughout the discussion, so we shall actually define n numbers rather than n functions.

We shall define a sequence of tables of numbers, $f_{ij}^{(0)},\ldots, f_{ij}^{(t)}$ $(i, j = 1,\ldots, n)$, iteratively as follows. Define $f_{ij}^{(0)} := f_{ij}(\boldsymbol{\lambda})$ $(i, j = 1,\ldots, n)$. Suppose that the numbers $f_{ij}^{(s)}$ have already been defined for some $s \geqslant 0$. For each i $(i = 1,\ldots, n)$ let

$$f_{i\max}^{(s)} := \max\{f_{ij}^{(s)}: j = 1,\ldots, n\},$$
$$A_i^{(s)} := \{j: f_{ij}^{(s)} = f_{i\max}^{(s)}\},$$

and

$$\delta_i^{(s)} := \min\{f_{i\max}^{(s)} - f_{ij}^{(s)}: f_{ij}^{(s)} \neq f_{i\max}^{(s)}\}$$

if this exists; otherwise (that is, if all the $f_{ij}^{(s)}$'s are equal) it doesn't matter what $\delta_i^{(s)}$ is defined to be, since it will never be used: let it be 0.

Note that $A_i^{(0)}$ is the set of possible values for $\pi(i)$, so that the point $\boldsymbol{\lambda}$ that we are looking at is a satisfactory point (in the sense of part (a) of the theorem) if and only if the family of sets $(A_1^{(0)},\ldots, A_n^{(0)})$ has a transversal. And $\boldsymbol{\lambda}$ is a satisfactory point in the sense of part (b) of the theorem if and only if $|A^{(0)}(K)| \geqslant |K| + 1$ for every non-empty subset K of $\{1,\ldots, n-1\}$ (where of course $A^{(0)}(K) := \bigcup (A_i^{(0)}: i \in K)$), since (it is easy to see, using Hall's theorem) this is a necessary and sufficient condition for the family of sets $(A_1^{(0)},\ldots, A_{n-1}^{(0)}, \{l\})$ to have a transversal for every possible choice of l in $\{1,\ldots, n\}$. So if (for part (a)) the family of sets $(A_1^{(s)},\ldots, A_n^{(s)})$ has a transversal, or (for part (b)) $|A^{(s)}(K)| \geqslant |K|+1$ for every non-empty subset K of $\{1,..., n-1\}$, let $t := s$; the construction terminates. Otherwise, the construction continues as follows.

By Lemma 2.1 there is a unique largest subset $K^{(s)}$ of $\{1,..., n\}$ (for part (a)) or of $\{1,..., n-1\}$ (for part (b)) for which $|K^{(s)}| - |A^{(s)}(K^{(s)})|$ is largest. Let

$$\delta^{(s)} := \min\{\delta_i^{(s)}: i \in K^{(s)}\}.$$

Note that $\delta^{(s)} \neq 0$; for if $\delta^{(s)} = 0$, then $\delta_i^{(s)} = 0$ for some i in $K^{(s)}$, which means that $A^{(s)}(K^{(s)}) = A_i^{(s)} = \{1,..., n\}$ and the construction should already have terminated. Now define

$$f_{ij}^{(s+1)} := \begin{cases} f_{ij}^{(s)} - \delta^{(s)} & \text{if } i \in K^{(s)} \text{ and } j \in A_i^{(s)}, \\ f_{ij}^{(s)} & \text{otherwise.} \end{cases}$$

Note that $A_i^{(s+1)} \supseteq A_i^{(s)}$, with strict inequality if and only if $i \in K^{(s)}$ and $\delta^{(s)} = \delta_i^{(s)}$. It follows that at least one of the sets $A_i^{(s)}$ increases in size at each iteration, so that the construction terminates in a finite number of steps (at most n^2, in fact). Note also that column j of the table remains unchanged (that is, $f_{ij}^{(s+1)} = f_{ij}^{(s)}$ for all i) if and only if $j \notin A^{(s)}(K^{(s)})$.

When the construction finally terminates with the table of numbers $f_{ij}^{(t)}$, let

$$g_j(\boldsymbol{\lambda}) := \sum_{i=1}^{n} (f_{ij}^{(0)} - f_{ij}^{(t)})$$

for each j ($j = 1,..., n$). This defines the functions $g_1,..., g_n: S \to \mathbb{R}^+$.

Stage 3. It remains to prove that these functions satisfy conditions (G1)–(G4). Condition (G4) is almost immediate from the construction, which terminates with $t = 0$ if and only if the point $\boldsymbol{\lambda}$ we started with was satisfactory. To prove condition (G3), note that $g_j(\boldsymbol{\lambda}) = 0$ if and only if $j \notin A^{(s)}(K^{(s)})$ for $s = 0,..., t-1$; the existence of such a j (for a fixed $\boldsymbol{\lambda}$) now follows from Theorem 2. Condition (G2) is also easy to check, since if $\boldsymbol{\lambda} \in F_j$ then $f_{ij}(\boldsymbol{\lambda}) = 0$ for each i, and so certainly $j \notin A^{(s)}(K^{(s)})$ for any $s < t$; thus $g_j(\boldsymbol{\lambda}) = 0$.

It remains to prove condition (G1), the continuity of the functions. To do this, suppose that $f'^{(0)}_{ij}$ and $f''^{(0)}_{ij}$ are two tables of numbers such that $|f'^{(0)}_{ij} - f''^{(0)}_{ij}| < \varepsilon$ for each i and j ($i, j = 1,..., n$). Suppose that we carry out the construction of Stage 2 for each of these tables of numbers so as to produce sets of numbers g'_j and g''_j ($j = 1,..., n$), and that the numbers of iterations required for these two constructions are t' and t'', respectively, where $0 \leqslant t' \leqslant n^2$ and $0 \leqslant t'' \leqslant n^2$.

LEMMA 4.1.

$$|g'_j - g''_j| \leqslant (3^{t'+t''} - 1)\, n\varepsilon < 3^{2n^2} n\varepsilon$$

for each j ($j = 1, \ldots, n$).

Proof. We shall append dashes to the symbols occurring in the construction of Stage 2 to denote the corresponding constructs when working with $f'^{(0)}_{ij}$ and $f''^{(0)}_{ij}$, and shall thus refer to (for example) $\delta'^{(0)}$, $A'^{(0)}(K'^{(0)})$ and $A''^{(0)}(K'^{(0)})$ without further explanation and in a manner that is hopefully self-explanatory.

We shall prove the result by induction on $t' + t''$. It is obvious if $t' + t'' = 0$, when $t' = t'' = 0$ and $g'_j = g''_j = 0$ for each j. So suppose that $t' + t'' > 0$; without loss of generality $t' > 0$. There are two cases.

Case 1. $\delta'^{(0)} \leqslant 2\varepsilon$. Then $|f'^{(1)}_{ij} - f''^{(0)}_{ij}| < 3\varepsilon$ for each i and j, and the table of numbers $f'^{(1)}_{ij}$ requires $t' - 1$ iterations for the construction of Stage 2. Since the contribution towards each g'_j of this first iteration is at most $2\varepsilon n$, it follows from the induction hypothesis that

$$|g'_j - g''_j| \leqslant 2\varepsilon n + (3^{t'+t''-1} - 1)n \cdot 3\varepsilon = (3^{t'+t''} - 1)\, n\varepsilon$$

as required.

Case 2. $\delta'^{(0)} > 2\varepsilon$. In this case, if $i \in K'^{(0)}$ and $f'^{(0)}_{ij} < f'^{(0)}_{i\max}$, then

$$f''^{(0)}_{ij} < f'^{(0)}_{ij} + \varepsilon \leqslant f'^{(0)}_{i\max} - \delta'^{(0)} + \varepsilon < f'^{(0)}_{i\max} - 2\varepsilon + \varepsilon < f''^{(0)}_{i\max}.$$

Thus $A''^{(0)}_i \subseteq A'^{(0)}_i$ if $i \in K'^{(0)}$, and so $A''^{(0)}(K'^{(0)}) \subseteq A'^{(0)}(K'^{(0)})$. Since $K''^{(0)}$ is the unique maximal set for which $|K''^{(0)}| - |A''^{(0)}(K''^{(0)})|$ takes its maximal value,

$$\begin{aligned} |K''^{(0)}| - |A''^{(0)}(K''^{(0)})| &\geqslant |K'^{(0)}| - |A''^{(0)}(K'^{(0)})| \\ &\geqslant |K'^{(0)}| - |A'^{(0)}(K'^{(0)})|. \end{aligned} \tag{5}$$

It follows that $t'' > 0$. Since the argument of Case 1 now disposes of the cases when $\delta''^{(0)} \leqslant 2\varepsilon$, we may suppose that $\delta''^{(0)} > 2\varepsilon$. By symmetry (reversing the rôles of $f'^{(0)}_{ij}$ and $f''^{(0)}_{ij}$ in the above argument), the first and last terms in (5) are equal. It follows that the first two terms are equal, which in turn shows that $K'^{(0)} \subseteq K''^{(0)}$ by the maximality of $K''^{(0)}$. By symmetry, $K'^{(0)} = K''^{(0)}$, and $A''^{(0)}_i = A'^{(0)}_i$ if $i \in K'^{(0)} = K''^{(0)}$ (interchanging $'$ and $''$ in the first two sentences of Case 2). It now follows from the definitions of $\delta'^{(0)}$ and $\delta''^{(0)}$ that $|\delta'^{(0)} - \delta''^{(0)}| < 2\varepsilon$, so that, for each j, the contributions to g'_j and g''_j caused by the first steps in the iterative constructions differ by less than $2\varepsilon n$, and, for each i and j, $|f'^{(1)}_{ij} - f''^{(1)}_{ij}| < 3\varepsilon$. Thus, by the induction hypothesis,

$$|g'_i - g''_j| < 2\varepsilon n + (3^{t'+t''-2} - 1)n \cdot 3\varepsilon < (3^{t'+t''} - 1)\, n\varepsilon,$$

as required. This completes the proof of Lemma 4.1. ∎

Since each of the functions f_{ij} is continuous, it follows immediately from Lemma 4.1 that each of the functions g_j is continuous, and this completes the proof of Theorem 4. ∎

5. Algorithms

In this section we consider algorithms for Gamow and Stern's cake-division problem, without the restriction (imposed in the last section) that the n portions of cake should be connected. The following algorithm of J. L. Selfridge works for three people, A, B and C, but it has not been extended to four or more people.

Algorithm

Person A divides the cake into three pieces that he thinks are equal. Person B cuts a bit off the largest piece (in his measure) in order to reduce it to the size of the second-largest piece. The bit cut off (known as the *decrement*—which may possibly be empty) is set on one side for the moment. The situation with the remaining three pieces is that A thinks that pieces 1 and 2 are equal largest, and B thinks that pieces 2 and 3 are equal largest; so whichever piece C wants, the other two pieces can clearly be given to A and B in such a way that everyone is happy. Let X be whichever of B and C gets piece 3, and let Y be the other of B and C (so that A, B and C are now known as A, X and Y).

We now come to dividing up the decrement. This is similar to the original problem except that we now have one further item of information: A does not care if X gets the whole of the decrement, since adding the whole of the decrement to piece 3 will just bring it up to the size of pieces 1 and 2 in A's measure. So Y divides the decrement into three pieces that he thinks are equal, X picks the largest of these in his measure, A picks the larger of the two remaining pieces, Y takes the one left over and everybody is happy. ∎

This algorithm gives rise to the concept of an *idol*. We say that B is an *idol* of A's if A doesn't care how much cake B gets. A division of the cake among n people will be called *satisfactory* if each of the n people is satisfied that he has at least as much cake (in his own measure) as each of the other $n - 1$ people who is not an idol of his. The following theorem is rather trivial; its proof is left to the reader.

THEOREM 5. *There is an algorithm for a satisfactory division if each of the n people has at least $n-2$ idols.* ∎

At the other end, we prove a theorem that is slightly less trivial, although it is still only a very small step towards an algorithm for the original problem. We shall require two lemmas.

LEMMA 6.1. *A division of the cake into n pieces has the property that, whichever piece P_n chooses, the remaining $n-1$ pieces can be divided among the remaining $n-1$ people in such a way that each of them thinks he has at least as much as anyone else (including P_n), if and only if it is possible to form a tree whose vertices are the n pieces of cake and whose edges are labelled $P_1,\dots,P_{n-1}$ in such a way that, for each i, the end-vertices of the edge labelled P_i are pieces of cake that P_i is happy to have (that is, they are equal largest in his measure).*

Proof. Let the set of pieces of cake that P_i is happy to have be $A_i \subseteq \{1,\dots,n\}$. Translating the problem into set-theory terminology as in Stage 2 of Theorem 4, what we have to prove is that $|A(K)| \geqslant |K|+1$, for every non-empty subset K of $\{1,\dots,n-1\}$, if and only if there is a tree whose vertices are $1,\dots,n$ and whose edges are labelled $A_1,\dots,A_{n-1}$ in such a way that, for each i, the end-vertices of the edge labelled A_i are elements of A_i. The proof of this is straightforward and is left to the reader. ∎

LEMMA 6.2. *Given any n pieces of cake, it is possible to cut bits off some* but not all *of them in such a way that the n pieces remaining satisfy the condition of Lemma* 6.1. *Moreover, there is an algorithm for constructing such a division.*

Proof. The existence of such a division follows from Theorem 4(b), since the division in which proportion v_i of the ith piece remains, for each i, and $\max\{v_i : i=1,\dots,n\}=1$, can be represented by the point $\boldsymbol{\lambda}$ with barycentric coordinates

$$(\lambda_1,\dots,\lambda_n)=\left(\frac{v_1}{\sum v_i},\dots,\frac{v_n}{\sum v_i}\right)$$

in an $(n-1)$-simplex. And given that such a division exists, Lemma 6.1 shows that it can be constructed in a finite number of steps. For, there are finitely many possible trees whose vertices are the n pieces of cake, there are finitely many ways of labelling the edges of each tree $P_1,\dots,P_{n-1}$, and there are finitely many possible choices for a piece of cake (which we shall call the *root* of the tree) that is to be left intact. And given any one of this finite number of situations, it is a finite process to test whether or not the corresponding division exists and satisfies our criteria, simply by working outwards from the root throughout the tree cutting bits off the pieces as we go. So by this highly inelegant but finite process, the required division can be constructed. ∎

THEOREM 6. *If there is an algorithm that will effect a satisfactory division of the cake in the case when each of the n people has at least one idol, then there is an algorithm that will work for the original problem without idols.*

Proof. P_1 divides the cake into n pieces that he thinks are equal. By the method of Lemma 6.2, bits are cut off some (but not all) of these in such a way that there is a satisfactory assignment of the n pieces remaining to the n people. (It does not matter which $n-1$ people carry out this division. The only point of the strong requirement in Lemma 6.1, that P_n should be allowed an arbitrary choice, is to ensure that the division can be carried out in a finite number of steps.) This process creates up to $n-1$ "decrements," each of which we now treat as a separate cake in its own right. However, since P_1 evidently received a piece of cake that had not been decremented (or, at least, whose decrement had value zero in P_1's measure), it follows that, for each of these decrements, P_1 has an idol. The procedure for dividing up each of the decrements is the same as for the cake itself, except that this time it is P_2 who carries out the initial division. After this division we are left with at most $(n-1)^2$ "decrements of decrements," for each of which P_1 has an idol and P_2 has an idol. Each of these is now divided as before, with P_3 this time making the initial division; and so on. At the end of the entire process we are left with at most $(n-1)^n$ "decrements of decrements of $\cdots$ of decrements," for each of which each of $P_1,\ldots, P_n$ has an idol. If there exists an algorithm for carrying out a satisfactory division of each of these, then the division of the original cake can be completed. ∎

Note that, if $n=3$, Theorems 5 and 6 meet, and provide an algorithm for dividing the cake, albeit a rather more complicated one than Selfridge's algorithm mentioned at the beginning of this section. But for $n \geqslant 4$, no such algorithm is known. I cannot see how to continue the algorithm of Theorem 6 so as to leave only decrements for which one person has at least two idols.

APPENDIX 1: THE CLASSICAL CAKE-DIVISION PROBLEM

This problem is to divide a cake among n people in such a way that each person receives (in his own measure) at least $1/n$ of the total. Undoubtedly the neatest solution (mentioned by Dubins and Spanier in [1]) is that of Banach and Knaster. In this solution, a knife is moved slowly across the top of the cake, while the n people watch and estimate how much cake the knife would cut off if it were to descend. As soon as anyone thinks that the knife would cut off $1/n$ of the cake, he shouts; the knife then descends, and the piece cut off is given to the person who shouted (or to one of them, if several people shouted simultaneously), who is happy because he thinks he has been given exactly $1/n$ of the cake. The process then continues with the remaining $n-1$ people, each of whom thinks that what is left constitutes at least $(n-1)/n$ of the cake (as otherwise he would have shouted earlier).

While this solution is neat, not everybody would agree that it constitutes a finite algorithm. The following solution, shown to me by a class-mate at school circa 1961, is essentially an algorithmized version of Banach and Knaster's solution. In this solution the n people sit round a circular table. One of them cuts himself a piece of cake which he thinks is $1/n$ of the total. This is then passed round the table and scrutinized by each of the other $n-1$ people in turn. Anyone who thinks that it is bigger than $1/n$ may cut a bit off it to reduce it to exactly $1/n$ (the bit cut off being miraculously restored to the main body of the cake), but the last person to cut a bit off it is obliged to accept it. As in Banach and Knaster's solution, the person who accepts this piece of cake thinks that it is $1/n$ of the total, while everyone else thinks that what is left is at least $(n-1)/n$ of the total, and the process then continues.

A third solution is given by Saaty in [9]. This solution is the reverse of the previous one, in that it starts with two people and works up to n, instead of the other way round. The first two people, A and B, divide the whole cake between them by the classic method of "I divide, you choose." That is, A divides the cake into two pieces that he thinks are equal, and B chooses the one that he thinks is larger. When C arrives, A and B each divide their portions into three pieces that they think are equal, and C chooses the largest (in his estimation) of A's three pieces and the largest of B's three pieces. The process continues in this way until all n people have arrived. It is easy to see that they are then all satisfied that they have at least $1/n$ of the cake. (For large n, this algorithm involves many more cuts than the preceding one, but it provides a better method of coping with the unexpected late-arriving guest.)

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