# CONTRIBUTION TO THE MODIFIED Z-TRANSFORM THEORY *

BY

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SUMMARY

In this discussion, a useful theorem applied to the modified $z$-transform (**1, 2**)[2] is introduced and verified. This theorem is fairly general and covers a few special cases that found many applications in the study of pulsed-systems.

These mentioned cases are discussed extensively and documented with examples and a short table in the Appendix. Various other cases of the theorem and its extension are pointed out in this paper.

## CONVOLUTION THEOREMS

*Theorem 1. Complex Convolution Theorem for the Modified Z-Transform*[3]

$$\mathfrak{z}_m[f(t) \times h(t)] = \frac{1}{2\pi j}\int_\Gamma p^{-1}F^*(p, m)H^*\left(\frac{z}{p}, m\right) dp, \qquad 0 \leq m \leq 1, \quad z = e^{Ts}$$

$$= \frac{1}{2\pi j}\int_\Gamma p^{-1}H^*(p, m)F^*\left(\frac{z}{p}, m\right) dp \tag{1}$$

$$f(t) = h(t) = 0 \quad \text{for} \quad t < 0$$

where $\Gamma$ is a closed contour in a positive sense in the $p$-plane which encloses all the poles of $p^{-1}F^*(p, m)$, and $F^*(p, m) = \mathfrak{z}_m[f(t)]_{z=p}$, $H^*(z, m) = \mathfrak{z}_m[h(t)]$.

To prove the above theorem, we apply the complex convolution theorem to obtain the Laplace transform of multiplication of two time functions as follows (**1–4**):

$$\mathfrak{L}[f^*(t, m) \times h^*(t, m)] = \frac{T}{2\pi j}\int_{c-j(\pi/T)}^{c+j(\pi/T)} F^*(q, m)H^*(s - q, m)dq \tag{2}$$

where the line integral $c - j\frac{\pi}{T}$, to $c + j\frac{\pi}{T}$, lies on an analytic strip in

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[2] The boldface numbers in parentheses refer to the references appended to this paper.

[3] The complex convolution theorem for the $z$-transform case has been developed by Cypkin (**2**).

the complex $q$-plane that does not pass through or enclose the poles of $F^*(q, m)$ or of $H^*(s - q, m)$ and

$$F^*(q, m) = \mathcal{L}[f^*(t, m)]|_{s=q} \tag{3}$$

$$H^*(s, m) = \mathcal{L}[h^*(t, m)]. \tag{4}$$

The typical function $f^*(t, m)$ is defined as

$$\begin{aligned} f^*(t, m) &= f[t - (1 - m)T] \times \delta_T(t) \\ &= \sum_{n=0}^{\infty} f(nT - T + mT)\delta(t - nT) \qquad 0 \leq m \leq 1. \end{aligned} \tag{5}$$

The Laplace transform of $f^*(t, m)$ and $h^*(t, m)$ is our definition of the modified $z$-transform of $f(t)$ and $h(t)$, respectively, thus

$$\begin{aligned} F^*(s, m) = \mathcal{L}[f^*(t, m)] &= \mathfrak{z}_m[f(t)], \\ \mathcal{L}[h^*(t, m)] &= \mathfrak{z}_m[h(t)] = H^*(s, m). \end{aligned} \tag{6}$$

Because of the periodic nature of $\delta_T(t)$, the Laplace transform $F^*(s, m)$ is a rational function of $e^{Ts}$; thus, writing $F^*(s, m)$ in terms of $e^{Ts}$, replacing $e^{Ts}$ by $z$ we denote this function as $F^*(z, m)|_{z=e^{Ts}}$.

Changing the path of integration in Eq. 2 from the $q$-plane into the $p$-plane (1), (where $p = e^{Tq}$), and writing for $dq = \dfrac{de^{Tq}}{Te^{Tq}}$, we obtain:

$$\mathcal{L}[f^*(t, m)h^*(t, m)] = \frac{1}{2\pi j}\int_\Gamma p^{-1}F^*(p, m)H^*\left(\frac{z}{p}, m\right) dp. \tag{7}$$

It should be noted that the line-integral in Eq. 2 is changed into a closed contour $\Gamma$ in the $p$-plane as required by the change of the variable of integration. From the definition of the modified $z$-transform, the left side of Eq. 7 is equal to the $\mathfrak{z}_m[f(t)h(t)]$. Therefore:

$$\mathfrak{z}_m[f(t) \times h(t)] = \frac{1}{2\pi j}\int_\Gamma p^{-1}F^*(p, m)H^*\left(\frac{z}{p}, m\right) dp \tag{8}$$ [4]

which proves the theorem.

For the case $h(t) = f(t)$,

$$\mathfrak{z}_m[f(t)^2] = \frac{1}{2\pi j}\int_\Gamma p^{-1}F^*(p, m)F^*\left(\frac{z}{p}, m\right) dp. \tag{9}$$

[4] When $h(t) = \mu(t)$, the left side of this equation reduces to $F^*(z, m)$.

*Evaluation of the Complex Convolution Integral* (*1, 2*)

To evaluate Eq. 8, we can make use of Cauchy's Integral Formula and evaluate the residues inside the circle which encloses only the poles of $p^{-1}F^*(p, m)$. An alternate procedure is to obtain the residues of $H^*\left(\frac{z}{p}, m\right)$ outside this circle, with path of integration taken in the negative sense, provided the integral is convergent. The choice between the two methods depends on the number of poles of each of the functions $F^*$ and $H^*$.

Making use of the first procedure and assuming $F^*(p, m)$ has simple poles, then

$$\mathfrak{z}_m[f(t) \times h(t)] = \sum_{n=1}^{N} \frac{A(p_n, m)}{p_n B'(p_n)} H^*\left(\frac{z}{p_n}, m\right) \tag{10}$$[5]

where

$$F^*(p, m) = \frac{A(p, m)}{B(p)} \tag{11}$$

$$\left.\begin{aligned} B(p) &= 0,\ p_1,\ p_2,\ \cdots\ p_n \\ B'(p_n) &= \left.\frac{dB}{dp}\right|_{p=p_n}. \end{aligned}\right. \tag{12}$$

In general for multiple poles cases, Eq. 8 becomes:

$$\mathfrak{z}_m[f(t) \times h(t)] = \sum_{\substack{\text{at poles of} \\ p^{-1}F^*(p,\, m)}} \text{Residue of } p^{-1}F^*(p, m)H^*\left(\frac{z}{p}, m\right) \tag{13}$$[5]

ILLUSTRATIVE EXAMPLES

*Example 1*

Assume

$$\left.\begin{aligned} f(t) &= e^{-\alpha t} \\ h(t) &= e^{-\beta t} \end{aligned}\right\} \tag{14}$$

The modified $z$-transform of each of the above functions can be obtained by using the following relation or through the use of tables (1, 2).

$$\begin{aligned} F^*(z, m) &= \mathfrak{L}[f[t - (1 - m)T] \times \delta_T(t)]|_{z=e^{Ts}} \\ &= \sum_{n=0}^{\infty} f(n - 1 + m)Tz^{-n}, \quad 0 \leq m \leq 1 \end{aligned} \tag{15}$$

Thus from tables (1) or from Eq. 15,

$$F^*(z, m) = \mathfrak{z}_m[e^{-\alpha t}] = \frac{e^{-\alpha m T}}{z - e^{-\alpha T}} \tag{16}$$

[5] For these cases it is assumed that the residue of the pole at the origin is zero. If such a residue exists, then it should be modified to include its value.

$$H^*(z, m) = \mathfrak{z}_m[e^{-\beta t}] = \frac{e^{-\beta m T}}{z - e^{-\beta T}}. \tag{17}$$

From Eq. 8

$$\mathfrak{z}_m[e^{-\alpha t}e^{-\beta t}] = \frac{1}{2\pi j}\int_\Gamma \frac{e^{-\alpha m T}}{p - e^{-\alpha T}} \times \frac{e^{-\beta m T}}{\dfrac{z}{p} - e^{-\beta T}} p^{-1}dp. \tag{18}$$

Since $F^*(p, m)$ has a simple pole at $p = e^{-\alpha T}$, Eq. 10 can be applied which yields:

$$\mathfrak{z}_m[e^{-\alpha t}e^{-\beta t}] = \frac{e^{-(\alpha+\beta)mT}}{e^{-\alpha T}\dfrac{z}{e^{-\alpha T}} - e^{-\beta T}} = \frac{e^{-(\alpha+\beta)mT}}{z - e^{-(\alpha+\beta)T}}, \quad 0 < m < 1. \tag{19}$$

*Example 2*

Find the modified $z$-transform of the following function (2, 5)

$$\mathfrak{z}_m[tf(t)]$$

In this case (1):

$$\mathfrak{z}_m[t] = \frac{mTz - mT + T}{(z - 1)^2} \tag{20}$$

$$\mathfrak{z}_m[f(t)] = F^*(z, m). \tag{21}$$

From Eq. 8

$$\mathfrak{z}_m[tf(t)] = \frac{1}{2\pi j}\int_\Gamma \frac{mTp - mT + T}{(p - 1)^2}\frac{1}{p} F^*\left(\frac{z}{p}, m\right) dp. \tag{22}$$

The function of the integrand in the region of integration has a multiple[6] at $p = 1$, thus the sum of the residues equals (1),

$$\begin{aligned}\mathfrak{z}_m[tf(t)] &= \frac{\partial}{\partial p}\frac{mTp - mT + m}{p} F^*\left(\frac{z}{p}, m\right)\Bigg|_{p=1} \\ &= F^*\left(\frac{z}{p}, m\right)\frac{\partial}{\partial p}[mT - mTp^{-1} + Tp^{-1}]\Bigg|_{p=1} \\ &\qquad + \frac{mTp - mT + T}{p}\frac{\partial}{\partial p} F^*\left(\frac{z}{p}, m\right)\Bigg|_{p=1} \\ &= F^*(z, m)[T(m - 1)] + m\frac{\partial}{\partial p} F^*\left(\frac{z}{p}, m\right)\Bigg|_{p=1}.\end{aligned} \tag{23}$$

[6] Since in most cases, $F^*\left(\frac{z}{p}, m\right)\Big|_{p=0} = 0$, it is assumed that the residue of the pole at the origin is zero.

It can be easily verified that for all forms of $F^*\left(\frac{z}{p}, m\right)\Big|_{p=1}$, the following is true

$$\frac{\partial}{\partial p} F^*\left(\frac{z}{p}, m\right)\Big|_{p=1} = -z\frac{\partial}{\partial z} F^*(z, m). \tag{24}$$

Therefore,

$$\mathfrak{z}_m[tf(t)] = T\left[(m-1)F^*(z, m) - z\frac{\partial}{\partial z} F^*(z, m)\right] = F_1^*(z, m). \tag{25}$$

Similarly for

$$\mathfrak{z}_m[t^2 f(t)] = T\left[(m-1)F_1^*(z, m) - z\frac{\partial}{\partial z} F_1^*(z, m)\right]. \tag{26}$$

By iterative extension of the above one can obtain, $\mathfrak{z}_m[t^k f(t)]$, $k$ = positive integer.

## APPLICATIONS OF THE THEOREM

Several special cases of wide interest can be obtained from the mentioned convolution theorem:

(*a*). *Application to obtain* $\mathfrak{z}[f(t)h(t)]$

The $z$-transform can be obtained from the modified $z$-transform by multiplying by $z$ and letting $m = 0$ in $F_1^*(z, m)$.

$$F_1^*(z) = zF_1^*(z, m)\big|_{m=0} = z\mathfrak{z}_m[f(t)h(t)]\big|_{m=0}. \tag{27}$$

Alternately, in integral form $F_1^*(z)$ can be represented (2):

$$F_1^*(z) = \mathfrak{z}[f(t)h(t)] = \frac{1}{2\pi j}\int_\Gamma \frac{F^*(p)}{p} \times H^*\left(\frac{z}{p}\right) dp \tag{28}$$

where

$$F^*(p) = \mathfrak{z}[f(t)]\big|_{z=p} \tag{29}$$

$$H^*(z) = \mathfrak{z}[h(t)]. \tag{30}$$

(*b*). *Application to obtain* $\sum_{n=0}^{\infty} [f(nT)]^2$

Letting $h(t) = f(t)$ in Eq. 28, we obtain

$$\mathfrak{z}[f^2(t)] = \sum_{n=0}^{\infty} [f(nT)]^2 z^{-n} = \frac{1}{2\pi j}\int_\Gamma \frac{F^*(p)}{p} F^*\left(\frac{z}{p}\right) dp. \tag{31}$$

Letting $z = 1$ in the above equation to obtain

$$\sum_{n=0}^{\infty} [f(nT)]^2 = \frac{1}{2\pi j} \int_\Gamma F^*(p) F^*(p^{-1}) p^{-1} dp, \tag{32}$$

where the contour-$\Gamma$ is along the unit circle in the $p$-plane, because all poles of $p^{-1}F^*(p)$ should be inside the unit circle in order for the summation to exist.

The above equation is used extensively in evaluating the sum of the square error at the sampling instants in sampled-data control systems (2) and also in obtaining the noise power gain (6). The evaluation of the above integral (without obtaining the poles of $F^*(p)$) can be obtained as ratio of two determinants which has been tabulated in some publications (2, 5).

(*c*). *Application to obtain* (5).

$$\int_0^\infty f(t)h(t)dt = \frac{T}{2\pi j} \int_0^1 \int_\Gamma F^*(p, m) H^*(p^{-1}, m) p^{-1} dp dm \tag{33}$$

or,

$$\int_0^\infty [f(t)]^2 dt = \frac{T}{2\pi j} \int_0^1 \int_\Gamma F^*(p, m) F^*(p^{-1}, m) p^{-1} dp dm \tag{34}$$

where the closed contour $\Gamma$ is the unit circle in the $p$-plane.

The above theorem is analogous to Parseval's theorem for Fourier transforms. It has been applied for the analysis of integral-square error in sampled-data control systems (5).

To show the above application from the general theorem we first obtain the following equivalent form of the integral in the left side of Eq. 33.

$$\begin{aligned} \int_0^\infty f(t)h(t)dt &= \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} f(t)h(t)dt \\ &= T \sum_{n=0}^{\infty} \int_0^1 f(nT - T + mT) h(nT - T + mT) dm, \end{aligned}$$

$$t = (n - 1 + m)T, \qquad n = \text{integer} \qquad 0 \leq m \leq 1 \tag{35}$$

On the other hand, the left side of Eq. 8 can be written as

$$\mathfrak{z}_m[f(t)h(t)] = \sum_{n=0}^{\infty} f(nT - T + mT) h(nT - T + mT) z^{-n}. \tag{36}$$

If we let $z = 1$, multiply by $T$ and integrate $\omega.r.$ to $m$ we get:

$$T \int_0^1 \mathfrak{z}_m[f(t)h(t)]\big|_{z=1} dm = T \int_0^1 \sum_{n=0}^{\infty} f(nT - T + mT)h(nT - T + mT)dm. \quad (37)$$

Interchanging the order of summation and integration and using Eq. 8, we obtain

$$T \sum_{n=0}^{\infty} \int_0^1 f(nT - T + mT)h(nT - T + mT)dm = \frac{T}{2\pi j} \int_0^1 \int_\Gamma F^*(p, m)H^*(p^{-1}, m)p^{-1}dpdm. \quad (38)$$

From Eq. 35, the above equals

$$\int_0^\infty f(t)h(t)dt = \frac{T}{2\pi j} \int_0^1 \int_\Gamma F^*(p, m)H^*(p^{-1}, m)p^{-1}dpdm. \quad (39)^7$$

When $h(t) = f(t)$, the above yields Eq. 34. The evaluation of this integral is simplified if the order of integration in $z$ and $m$ is interchanged.[8] This is possible since the integrand is continuous along $\Gamma$ and in the interval $0 \leq m \leq 1$. The evaluation of the first integral (5) is carried similar to Eq. 32.

Noting Example 2, for $\mathfrak{z}_m[tf(t)]$ and letting $h(t)$ in Eq. 39 be $[tf(t)]$, we get

$$\int_0^\infty t[f(t)]^2dt = \frac{T^2}{2\pi j} \int_0^1 \int_\Gamma \left[(m - 1)F^*(p, m) - p \frac{\partial}{\partial p} F^*(p, m)\right] \times F^*(p^{-1}, m)p^{-1}dpdm. \quad (40)$$

Similar relations can be obtained for higher time moments.

(*d*). *Application to obtain*

$$\mathfrak{z}_m\left[\int_0^t f(t)h(t)dt\right] = \frac{T}{2\pi j(z - 1)} \int_0^1 \int_\Gamma p^{-1}F^*(p, m)H^*\left(\frac{z}{p}, m\right) dpdm + \frac{T}{2\pi j} \int_0^m \int_\Gamma p^{-1}F^*(p, m)H^*\left(\frac{z}{p}, m\right) dpdm. \quad (41)$$

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[7] As a special case when $h(t) = \mu(t) =$ unit step, we obtain

$$\int_0^\infty f(t)dt = T \int_0^1 F^*(p, m)\Big|_{p=1} dm, \qquad \text{where} \qquad F^*(p, m) = \mathfrak{z}_m[f(t)]\Big|_{z=p}.$$

[8] See Theorem 2 for evaluation of the integral $\omega.r.$ to $m$.

The above relationship can be obtained from writing the integral in a summation form as follows:

$$\int_0^t f(t)h(t)dt = T\left[\sum_{n=0}^{n}\int_0^1 f(n-1+m)Th(n-1+m)Tdm \right.$$
$$\left. + \int_0^m f(n-1+m)Th(n-1+m)Tdm\right]. \quad (42)$$

The modified $z$-transform of the above yields[9]

$$\mathfrak{z}_m\left[\int_0^t f(t)h(t)dt\right]$$
$$= \frac{T}{z-1}\int_0^1 \sum_{n=0}^{\infty} f(n-1+m)Th(n-1+m)Tz^{-n}dm$$
$$+ T\int_0^m \sum_{n=0}^{\infty} f(n-1+m)Th(n-1+m)Tz^{-n}dm \quad (43)$$

but the infinite summations are the modified $z$-transform of $f(t)\ h(t)$, thus:

$$\mathfrak{z}_m\left[\int_0^t f(t)h(t)dt\right] = \frac{T}{z-1}\int_0^1 F_1^*(z, m)dm + T\int_0^m F_1^*(z, m)dm \quad (44)$$

where

$$F_1^*(z, m) = \mathfrak{z}_m[f(t)h(t)] = \frac{1}{2\pi j}\int_\Gamma F^*(p, m)p^{-1}H^*\left(\frac{z}{p}, m\right)dp. \quad (45)$$

For the special case when $h(t) = \mu(t) =$ unit step, the above yields

$$\mathfrak{z}_m\left[\int_0^t f(t)dt\right] = \frac{T}{z-1}\int_0^1 F^*(z, m)dm + T\int_0^m F^*(z, m)dm \quad (46)$$

where

$$F^*(z, m) = \mathfrak{z}_m[f(t)]. \quad (47)$$

Equation 46 also yields the modified $z$-transform (2) of $\frac{F(s)}{s}$. Extension to obtain $\mathfrak{z}_m\left[\frac{F(s)}{s^k}\right]$, by iterative process is straightforward.

(*c*) *Application to obtain*

$$F(s) = T\int_0^1 z^{-m+1}F^*(z, m)dm,$$

[9] Noting that (1), $\frac{1}{z-1}F^*(z) = \mathfrak{z}[\sum_{k=0}^{k-1} f(kT)]$.

where

$$F^*(z, m) = \mathfrak{z}_m[f(t)],\dagger \qquad F(s) = \mathcal{L}[f(t)]. \tag{48}$$

To show the above, we multiply the left side of Eq. 33 by $e^{-st}$ and integrate to obtain

$$\int_0^\infty f(t)h(t)e^{-st}dt$$

$$= T\sum_{n=0}^{\infty} e^{-nTs}\int_0^1 f(nT - T + mT)h(nT - T + mT)e^{(1-m)sT},$$

$$t = (n - 1 + m)T, \qquad 0 \leq m \leq 1 \tag{49}$$

or

$$\int_0^\infty f(t)h(t)e^{-st}dt$$

$$= T\int_0^1 z^{-m+1}\sum_{n=0}^{\infty} f(nT - T + mT)h(nT - T + mT)z^{-n}dm \tag{50}$$

but

$$\sum_{n=0}^{\infty} f(nT - T + mT)h(nT - T + mT)z^{-n} = \mathfrak{z}_m[f(t)h(t)],$$

therefore from Eq. 8 we get

$$F_1(s) = \int_0^\infty f(t)h(t)e^{-st}dt$$

$$= T\int_0^1 z^{-m+1}\int_\Gamma F^*(p, m)H^*\left(\frac{z}{p}, m\right)p^{-1}dpdm \tag{51}$$

or

$$F_1(s) = T\int_0^1 z^{-m+1}F_1{}^*(z, m)dm, \qquad 0 \leq m \leq 1 \tag{52}$$

where

$$F_1{}^*(z, m) = \mathfrak{z}_m[f(t)h(t)] \qquad \text{and} \qquad F_1(s) = \mathcal{L}[f(t)h(t)]. \tag{53}$$

The above relation shows the equivalence between the Laplace transform and the modified $z$-transforms. The above has been shown derived differently by Cypkin (2). It is obtained here in a simple manner as a special case of the convolution theorem.

† $F^*(z, m)$ can be directly obtained from $F(s)$ by using the following convolution integral (1):

$$F^*(z, m) = \frac{z^{-1}}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(p)\,\frac{e^{mpT}}{1 - e^{-T(s-p)}}\,dp\Big|_{z=e^{Ts}}.$$

*Example*

Knowing $F^*(z, m) = \dfrac{e^{-amT}}{z - e^{-aT}}$, obtain $F(s)$. From Eq. 52, $F(s)$ equals

$$F(s) = T \int_0^1 z^{-m} \frac{ze^{-amT}}{z - e^{-aT}}\, dm$$

$$= \frac{Ts}{z - e^{-aT}} \int_0^1 e^{-mT(a+s)dm} \bigg|_{z=e^{Ts}} = \frac{1}{s+a}. \quad (54)$$

(*f*) *Application to obtain*

$$F(s) = \frac{T}{2\pi j} \int_{c-j(\pi/T)}^{c+j(\pi/T)} F^*(q) \frac{1}{s-q}\, dq, \quad \text{where} \quad F^*(q) = F^*(s)\,|_{s=q}. \quad (55)$$

Knowing $F^*(s) = \mathfrak{L}[f^*(t)]$ from the above equation, to obtain a continuous function $f(t)$, whose values coincide with $f^*(t)$ at the sampling instants only, the following condition should be satisfied:

$$f(t)u(t) = f^*(t)u(t), \qquad u(t) = \text{unit step} \quad (56)$$

or in Laplace transforms,

$$F(s) = \mathfrak{L}[f(t)u(t)] = \mathfrak{L}[f^*(t)u(t)]. \quad (57)$$

Using Eq. 2 to evaluate the right side of Eq. 57, we obtain (2)

$$F(s) = \frac{T}{2\pi j} \int_{c-j(\pi/T)}^{c+j(\pi/T)} F^*(q) \frac{1}{s-q}\, dq. \quad (58)$$

In evaluating the above equation only the poles in the linearized strip between $c - j\dfrac{\pi}{T}$ to $c + j\dfrac{\pi}{T}$ are considered.

*Example*

$$f^*(t) = e^{-\alpha nT}\delta_T(t), \qquad F^*(q) = F^*(s)\bigg|_{s=q} = \frac{e^{Tq}}{e^{Tq} - e^{-\alpha T}}. \quad (59)$$

Evaluating Eq. 58, for the poles of $F^*(q)$ we obtain

$$F(s) = T \frac{e^{Tq}}{\dfrac{d}{dq}[e^{Tq} - e^{-\alpha T}]} \times \frac{1}{s-q}\bigg|_{q=-\alpha} = \frac{1}{s+\alpha}. \quad (60)$$

Alternately evaluating for the pole $q = s$, we obtain

$$F(s) = T \times \frac{1}{T} \sum_{k=-\infty}^{k=\infty} \frac{1}{q + \alpha + jk\omega_r}\bigg|_{q=s,\ k=0} = \frac{1}{s+\alpha}, \qquad \omega_r = \frac{2\pi}{T}. \tag{61}$$

(*g*) *Application to obtain*

$$F^*(z, m) = \frac{1}{n} \sum_{k=0}^{n-1} F^*(z_n e^{j2\pi(k/n)}, m), \qquad z_n = e^{(T/n)s} = z^{(1/n)} \tag{62}$$

or

$$F^*(z) = \frac{1}{n} \sum_{k=0}^{n-1} F^*(z_n e^{j2\pi(k/n)}). \tag{63}$$

The above theorem signifies that, knowing the modified $z$-transform with respect to the period $(T/n)$ (when $n$ integer is larger than unity), one can obtain from it the modified $z$-transform of $F^*(z, m)$ with respect to the period $T$. This derivation is of importance in the analysis of multi-rate sampled-data feedback systems (1, 2).

The inverse of Eq. 62 indicates a time function which is formed by the product of the inverse of $F^*(z_n, m)$ and a sampled step-function with period $T$, that is, $f(k, m)T = f(k, m)T/n \times \mu(kT)$. Thus using the convolution integral we obtain:

$$F^*(z, m) = \frac{1}{2\pi j} \int_\Gamma F^*(p_n, m) p_n^{-1} \frac{1}{1 - (z_n/p_n)^{-n}}\, dp_n \tag{64}$$

where

$$F^*(p_n, m) = F^*(z_n, m)\big|_{z_n = p_n} = \mathcal{L}[f^*(t, m)] = \mathfrak{z}_m[f(t)]\big|_{z_n = e^{(T/n)s}} \tag{65}$$

and

$$\frac{1}{1 - z^{-1}}\bigg|_{z=e^{Ts}} = \frac{1}{1 - z_n^{-n}}\bigg|_{z_n = e^{(T/n)s}} = \mathcal{L}[\delta_T(t)] = \mathfrak{z}[u(t)]\bigg|_{z = z_n^n}. \tag{66}$$

Equation 64 can be written as:

$$F^*(z, m) = \frac{1}{2\pi j} \int_\Gamma F^*(p_n, m) p_n^{-1} \frac{-1}{(p_n/z_n)^n - 1}\, dp_n. \tag{67}$$

Integrating in a negative sense w.r. to the poles of $\dfrac{1}{(p_n/z_n)^n - 1}$, which are

$$(p_n/z_n)^n = e^{j2\pi k}, \qquad k = 0, 1, 2, \cdots, n-1,$$

we obtain the sum of the residues

$$= -\sum_{k=0}^{n-1} F^*(p_n, m)p_n^{-1} \frac{1}{\dfrac{d}{dp}(p_n/z_n)^n - 1}\Bigg|_{p_n = z_n e^{j2\pi(k/n)}}. \quad (68)$$

Thus,

$$F^*(z, m) = \frac{1}{n}\sum_{k=0}^{n-1} F^*(z_n e^{j2\pi(k/n)}, m). \quad (69)$$

For the case of $z$-transform, letting $m = 1$ in Eq. 69, we obtain

$$F^*(z) = \frac{1}{n}\sum_{k=0}^{n-1} F^*(z_n e^{j2\pi(k/n)}). \quad (70)$$

Further application of the theorem in the analysis of modulated discrete signals is of further use in the study of modulated sampled-data systems.

### ADDITIONAL THEOREMS

*Theorem 1a*

$$\mathfrak{z}_m\left[\frac{f(t)}{t}\right] = \frac{z^{m-1}}{T}\int_z^\infty z^{-m}F^*(z, m)dz, \qquad 0 < m < 1$$

$$\lim_{t\to 0} f(t) = 0, \qquad t = (n - 1 + m)T. \quad (71)$$

This theorem can also be derived from Eq. 8 as a special case when $h(t) = \frac{1}{t}$. However in using the convolution integral it is necessary to know $\mathfrak{z}_m\left[\frac{1}{t}\right]$. To avoid this, the above is derived differently as follows:

$$F^*(z, m) = \sum_{n=0}^{\infty} f(nT - T + mT)z^{-n} = \sum_{n=1}^{\infty} f(nT - T + mT)z^{-n}, \quad (72)$$

since it is assumed that

$$f(t)\big|_{t=0} = 0.$$

Multiply the above equation by $z^{-m}$ and integrate to obtain

$$\int z^{-m}F^*(z, m)dz = \int \sum_{n=0}^{\infty} f(nT - T + mT)z^{-(n+m)}$$

$$= \sum_{n=1}^{\infty} \frac{f(nT - T + mT)z^{-(n+m-1)}}{-n - m + 1}, \qquad 0 < m < 1 \quad (73)$$

or,

$$\frac{1}{T} z^{m-1} \int_z^\infty z^{-m} F^*(z, m) dz = \sum_{n=1}^{\infty} \frac{f(nT - T + mT) z^{-n}}{(nT - T + mT)}. \quad (74)$$

If $\frac{f(n-1+m)T}{(n-1+m)T} z^{-0}$ is added to both sides of Eq. 74, then this term has a contribution only at $m = 1$, which is the $\lim_{t \to 0} \frac{f(t)}{t}$, thus the right side of the above equation is $\mathfrak{z}_m \left[ \frac{f(t)}{t} \right]$,

$$\mathfrak{z}_m \left[ \frac{f(t)}{t} \right] = \frac{1}{T} z^{m-1} \int_z^\infty z^{-m} F^*(z, m) dz + \lim_{t \to 0} \frac{f(t)}{t}, \quad 0 < m \leq 1. \quad (75)$$ [10]

For the case of $z$-transform we obtain, by substituting $m = 1$ in Eq. 75

$$\mathfrak{z} \left[ \frac{f(t)}{t} \right] = \frac{1}{T} \int_z^\infty z^{-1} F^*(z) dz + \lim_{t \to 0} \frac{f(t)}{t},$$

where

$$F^*(z) = F^*(z, m) \mid_{m=1}. \quad (76)$$

Extension of Eq. 75 to obtain $\mathfrak{z}_m \left[ \frac{f(t)}{t^k} \right]$ is evident (2).

Special cases which stem from this theorem are indicated in the following.

(*a*)

$$\mathfrak{z} \left[ \frac{f(t)}{t + hT} \right] = \frac{1}{T} z^h \int_z^\infty z^{-(1+h)} F^*(z) dz \quad (77)$$

where $h$ is a positive constant larger than zero. The above is obtained from Eq. 75 by substituting for $m - 1 = h$, and using the $z$-transform case.

(*b*) When $z = 1$, relation (*a*) yields:

$$\sum_{n=0}^{\infty} \frac{f(nT)}{(n+h)T} = \frac{1}{T} \int_1^\infty z^{-(1+h)} F^*(z) dz, \qquad h > 0. \quad (78)$$

For the case $h = 0$, the above is modified to yield

$$\sum_{n=0}^{\infty} \frac{f(nT)}{nT} = \frac{1}{T} \int_1^\infty z^{-1} F^*(z) dz + \lim_{n \to 0} \frac{f(nT)}{(nT)}. \quad (79)$$

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[10] When $m = 0$, we obtain $\mathfrak{z}\left(\frac{f(t-T)}{t-T}\right) = z^{-1}\mathfrak{z}\left(\frac{f(t)}{t}\right)$.

*Example*

Obtain, using Eq. 75,

$$\mathfrak{Z}_m\left[\frac{1-e^{-\alpha t}}{t}\right],\quad f(t) = 1 - e^{-\alpha t},\quad F^*(z,m) = \frac{1}{z-1} - \frac{e^{-\alpha m T}}{z - e^{-\alpha T}} \tag{80}$$

$$\int z^{-m}F^*(z,m)dz = \int \left\{\frac{z^{-m}}{z-1} - \frac{z^{-m}e^{-\alpha m T}}{z-e^{-\alpha T}}\right\}dz$$

$$= \int [1 + z^{-1} + z^{-2} + \cdots]z^{-m-1}dz$$

$$- \int [1 + z^{-1}e^{-\alpha T} + \cdots]z^{-m-1}e^{-\alpha m T}dz \tag{81}$$

for $0 < m < 1$

$$\int_z^\infty z^{-m}F^*(z,m)dz$$

$$= -\left[\frac{z^{-m}}{m} + \frac{e^{-\alpha T}z^{-m-1}}{m+1} + \frac{e^{-2\alpha T}z^{-m-2}}{m+2} + \cdots\right]e^{-\alpha m T}$$

$$+ \frac{z^{-m}}{m} + \frac{z^{-m-1}}{m+1} + \frac{z^{-m-2}}{m+2} + \cdots. \tag{82}$$

Finally,

$$\mathfrak{Z}_m\left[\frac{1-e^{-\alpha T}}{t}\right] = \frac{z^{-1}}{mT}[1 - e^{-\alpha m T}] + \frac{z^{-2}}{(m+1)T}(1 - e^{-\alpha T}e^{-\alpha m T})$$

$$+ \frac{z^{-3}}{(m+2)T}(1 - e^{-2\alpha T}e^{-\alpha m T}) + \cdots +,\qquad 0 < m \leq 1. \tag{83}$$ [11]

For the $z$-transform let $m = 1$ in Eq. 83, to obtain after closing the series

$$\mathfrak{Z}\left[\frac{1-e^{-\alpha T}}{t}\right] = \alpha + \frac{1}{T}\ln\frac{z - e^{-\alpha T}}{z-1}. \tag{84}$$

A short table of $z$-transforms is given in the Appendix.

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[11] It is of interest to note that when $\alpha \to \infty$, we obtain

$$\mathfrak{Z}_m\left[\frac{1}{t}\right] = \frac{z^{-1}}{mT} + \frac{z^{-2}}{(m+1)T} + \frac{z^{-3}}{(m+2)T} + \cdots \frac{z^{-n}}{(m+n-1)T} + \delta_a(t) \quad \text{for} \quad 0 < m \leq 1.$$

For the case $m = 1$, we obtain

$$\mathfrak{Z}\left[\frac{1}{t}\right] = \delta_a(t) + \frac{1}{T}\ln\frac{z}{z-1},\qquad \text{where}\qquad \delta_a(t) = \lim_{\substack{\alpha\to\infty\\ t\to 0}}\left[\frac{1-e^{-\alpha t}}{t}\right].$$

In some publications (2) the $z$-transform of $\left[\frac{1}{t}\right]$ is defined only for $t > 0$, thus the infinite value of $\delta_a(t)$ is ignored.

*Theorem 2*

$$T \int_0^1 F^*(z, m)H^*(z^{-1}, m)dm = \mathfrak{z}[F(s)H(-s)], \quad 0 \leq m \leq 1. \quad (85)^{12}$$

The above theorem is an extension of the following theorem which is discussed in the literature (1, 5, 7, 8).

$$T \int_0^1 F^*(z, m)F^*(z^{-1}, m)dm = \mathfrak{z}[F(s) \times F(-s)]. \tag{86}$$

Following similar steps for the derivation (5) of Eq. 86, one can easily obtain the theorem in Eq. 85.

The use of the above theorem is of advantage in the evaluation of the integral ω.r. to $m$ in Eq. 39. Thus from Eqs. 39 and 85 we obtain

$$\begin{aligned} \int_0^\infty f(t)h(t)dt &= \frac{T}{2\pi j} \int_\Gamma \int_0^1 F^*(p, m)H^*(p^{-1}, m)dmp^{-1}dp \\ &= \frac{1}{2\pi j} \int_\Gamma \mathfrak{z}[F(s) \times H(-s)]|_{z=p}p^{-1}dp. \end{aligned} \tag{87}$$

## CONCLUSION

The complex convolution integral for both the $z$-transform and modified $z$-transform has been discussed in detail. The applications of the theorems for various fields such as sampled data systems, digital control systems, circuit theory and to the operational solution of difference equations both linear and non-linear, are possible using the techniques of this paper. It is of particular importance that the extension of the convolution integral for solving certain types of non-linear difference equations is warranted.

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[12] $\mathfrak{z}[F(s)H(-s)]$ symbolizes the $z$-transform of a time function whose Laplace transform is $[F(s)H(-s)]$.

**APPENDIX: A SHORT TABLE OF Z-TRANSFORMS OF CERTAIN FUNCTIONS**

| $f(t)$ | $F^*(z,m)$ | $F^*(z)$ |
|---|---|---|
| $\frac{1-e^{-\alpha t}}{t}$ | $\frac{z^{-1}}{mT}[1-e^{-\alpha mT}] + \frac{z^{-2}}{(m+1)T}[1-e^{-\alpha T}e^{-\alpha mT}] + \frac{z^{-n}}{(m+n-1)T}[1-e^{-\alpha(n-1)T}e^{-\alpha mT}] + \cdots$ | $\alpha+\frac{1}{T}\ln\frac{z-e^{-\alpha T}}{z-1}, \quad \alpha>0$ |
| $\frac{1}{t}, \quad t>0$ | $\frac{z^{-1}}{mT}+\frac{z^{-2}}{(m+1)T}+\cdots\frac{z^{-n}}{(m+n-1)T} = z^{m-1}\int_0^z \frac{p^{-m}}{1-p}dp$ | $\frac{1}{T}\ln\frac{z}{z-1}$ |
| $\frac{\sin\alpha t}{t}$ | $\frac{1}{T}e^{-(1-m)Ts}\sum_{k=-\infty}^{k=\infty}\left[\tan^{-1}\frac{\alpha}{s+jk\omega_r}\right]e^{jmkT\omega_r}$ <br> $z=e^{Ts}, \quad \omega_r=\frac{2\pi}{T}, \quad k=\text{integer}$ | $\alpha+\frac{1}{T}\left[\frac{\pi}{2}-\tan^{-1}\frac{z-\cos\alpha T}{\sin\alpha T}\right], \quad \alpha>0$ |
| $\frac{f(t)}{t}$ | $\frac{1}{T}z^{m-1}\int_z^\infty z^{-m}F^*(z,m)dz+\lim_{t\to 0}\frac{f(t)}{t},$ <br> $0\le m\le 1$ | $\frac{1}{T}\int_z^\infty z^{-1}F^*(z)dz+\lim_{t\to 0}\frac{f(t)}{t}$ |
| $\int_0^t f(t)dt$ | $\frac{T}{z-1}\int_0^1 F^*(z,m)dm+T\int_0^m F^*(z,m)dm$ | $\frac{Tz}{z-1}F^*(z)$ |
| $t^k f(t)$ | $T\left[(m-1)F_1^*(z,m)-z\frac{\partial}{\partial z}F_1^*(z,m)\right]$ | $-Tz\frac{\partial}{\partial z}F_1^*(z)$ |
| $k>0$, and integer | $F_1^*(z,m)=\mathfrak{z}_m[t^{k-1}f(t)]$ | $F_1^*(z)=\mathfrak{z}[t^{k-1}f(t)]$ |