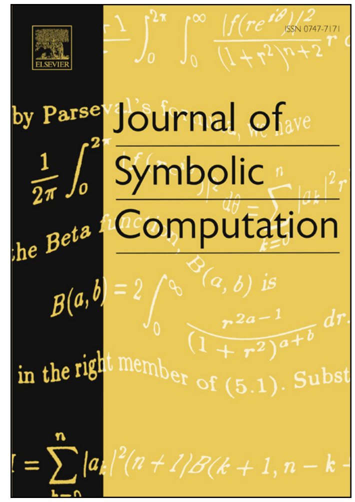

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PII: S0747-7171(17)30071-8
DOI: http://dx.doi.org/10.1016/j.jsc.2017.07.007
Reference: YJSCO 1803

To appear in: *Journal of Symbolic Computation*

Received date: 20 November 2016
Accepted date: 1 April 2017

Please cite this article in press as: Huang, C.-C., et al. Positive root isolation for poly-powers by exclusion and differentiation. *J. Symb. Comput.* (2017), http://dx.doi.org/10.1016/j.jsc.2017.07.007

Positive Root Isolation for Poly-Powers by Exclusion and Differentiation

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Abstract

We consider a class of univariate real functions—poly-powers—that extend integer exponents to real algebraic exponents for polynomials. Our purpose is to isolate positive roots of such a function into disjoint intervals, each contains exactly one positive root and together contain all, which can be easily refined to any desired precision. To this end, we first classify poly-powers into simple and non-simple ones, depending on the number of linearly independent exponents. For the former, based on Gelfond–Schneider theorem, we present two complete isolation algorithms—exclusion and differentiation. For the latter, their completeness depends on Schanuel's conjecture. We implement the two methods and compare them in efficiency via a few examples. Finally the proposed methods are applied to the field of systems biology to show the practical usefulness.

Keywords: real root isolation; generalized polynomial; transcendental number; interval arithmetic; systems biology

1. Introduction

Solving nonlinear equations captures a central position in Computer Algebra with extremely wide applications to many fields, such as physics, economics, control theory and systems biology. The foundation of equation-solving is locating real roots of univariate functions. Since those roots are not rational (even not algebraic) in general, what one can really hope for is to isolate them into disjoint intervals, each contains exactly one real root and together contain all, which can be easily refined to any desired precision for further use.

Polynomials are the principal objects for real root isolation. The early work can date back to the 18th century. Isaac Newton proposed a numerical iteration method for solving $\mathbb{R}$-polynomial equations (Newton, 1711). The iterative values could approach real roots at quadratic convergence rate. But it depended on the choosing of starting points. Later a large number of modifications were proposed and applied to solve nonlinear equations and systems. With the hypothesis of differentiability, Smale (1986) improved Newton's method by deducing consequences from data at a single point. This point of view had valuable features for computation. Newton's

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method could further solve the general polynomial system, provided the computational precision satisfied a bound polynomial in some parameters (Cucker and Smale, 1999).

However, due to the hardness of sign-determination of constants (Richardson, 1997), one cannot always distinguish between a root of multiplicity k and a cluster of k roots for a $\mathbb{R}$-polynomial (or $\mathbb{C}$-polynomial). So it yields the root clustering problem, a variant of the root isolation problem, that clusters all roots into disjoint small disks, each contained a multiset of roots, not an individual one. Dedieu and Yakoubsohn (1993) presented the exclusion algorithm for clustering real roots of $\mathbb{R}$-polynomials. Yap et al. (2013) exploited soft sign-determination that asked whether $\alpha > 0$ or $\alpha < 0$, but not $\alpha = 0$, and then established the first exact algorithm for analyzing root clusters of complex analytic functions. The bit-complexity of their algorithm was given for $\mathbb{C}$-polynomials in the recent paper (Becker et al., 2016), which was nearly the best one could hope for. For square-free $\mathbb{R}$-polynomials (which were obtained by the almost optimal method (Pan, 2002) for approximate polynomial factorization), the root clustering problem amounts to the root isolation problem, as they had no multiple real root. So real roots of square-free $\mathbb{R}$-polynomials could be efficiently isolated (Sagraloff and Mehlhorn, 2016; Pan and Tsigaridas, 2016).

To completely tackling multiple real roots, algebraic methods were designed to shoot this shortage. Using Sturm's theorem (Thomas, 1941), one could count the number of distinct real roots of a $\mathbb{R}$-polynomial, and thereby isolate them into disjoint intervals. Collins and Loos (1976) presented the differentiation method for $\mathbb{Q}$-polynomials that isolates real roots of the original function after those of the simpler derivative have been isolated. Collins and Akritas (1976) gave another method based on the sign rule of Descartes. These algebraic methods became rather popular, and had been incorporated into quantifier elimination over real closed fields $\mathfrak{T}(\mathbb{R}; <, =; +, \cdot; 0, 1)$ (Collins, 1975; Collins and Hong, 1991), polynomial system-solving (Rouillier, 1999; Cheng et al., 2007) and semi-algebraic system-solving (Xia and Yang, 2002).

Motivated by the profound Tarski's problem (Tarski, 1951)—deciding the first-order theory $\mathfrak{T}(\mathbb{R}; <, =; +, \cdot, \exp; 0, 1)$, many efforts had been put into the study of exponential functions. Richardson (1991) investigated the structure of real roots of exponential polynomials, elements of $\mathbb{Q}[x, e^x]$. Later Achatz et al. (2008) proposed the first complete real root isolation algorithm for such functions. Its rationale was the differentiation method, and its completeness was ensured by Lindemann's theorem (Siegel, 1949). Meantime, Strzeboński (2008, 2009) respectively studied the real root isolation of exp-log functions and tame elementary functions. But the completeness of the procedure depended on Schanuel's conjecture (Ax, 1971). Recently, these methods had been developed as decision procedure (McCallum and Weispfenning, 2012), cylindrical decomposition (Strzeboński, 2011), and quantifier elimination (Xu et al., 2015) for some fragments of multivariate formulas.

In this paper, we study a class of generalized polynomials, named poly-powers, closely related to polynomials and exponential polynomials (to be specified in the next section). It has the normal form

$$f(x) = \beta_0 + \beta_1 x^{\alpha_1} + \cdots + \beta_n x^{\alpha_n} \;, \tag{1}$$

where α_i $(1 \le i \le n)$ and β_i $(0 \le i \le n)$ are $\mathbb{R}_{\text{alg}}$-numbers with $0 < \alpha_1 < \cdots < \alpha_n$ and $n > 0$. For convenience, we assume w. l. o. g. that the input poly-power we study has a nonzero constant term, viz. $\beta_0 \neq 0$ in $f(x)$.

Our purpose is to isolate positive roots of poly-powers[1]. First of all, we perform factorization to simplify the input poly-power. To this end, we have to find a linearly independent base

[1] We do not concern complex roots of poly-powers. The reason is illustrated via an example. Even for the simplest

of its coefficients and exponents. Based on that, we get a multivariate polynomial representation of the input poly-power and thus perform the factorization. The union of positive roots of irreducible factors consists of those of the input poly-power. More importantly, factorization can help us classify poly-powers into simple and non-simple ones, depending on the number of linearly independent exponents in an irreducible poly-power. Then we propose two positive root isolation algorithms—exclusion and differentiation—for irreducible poly-powers. The former bisects intervals and excludes those subintervals that contain no positive root. The latter repeatedly differentiates poly-powers until the resulting poly-power has at most two terms, and lifts isolation intervals of derivatives to those of the original poly-power. The whole procedures are both established on algebraic manipulation, and hence are absolutely exact. The completeness is ensured by Gelfond–Schneider theorem for simple poly-powers, and by Schanuel's conjecture for non-simple ones. We implement the proposed methods with the `Maple` platform to get some heuristics on efficiency. Finally a case study in the field of Systems Biology is given to show the practical usefulness.

The contributions of this paper are three-fold:

1. **effectivity**—presenting complete isolation algorithms for simple poly-powers based on Gelfond–Schneider theorem;
2. **efficiency**—revising the definition of pseudo-derivatives for the differentiation method, so that the linearly independent base for the input poly-power can be reused;
3. **usefulness**—applying the proposed methods to a case study in systems biology.

The latter two are extending to our preliminary paper (Li et al., 2016).

Organization. First we recall some works on the functions similar to poly-powers in Section 2. In Section 3 we decompose the poly-power into simpler ones. Then we present two positive root isolation methods using exclusion and differentiation in Sections 4 & 5, and implement them in Section 6. A case study is given in Section 7. Finally Section 8 is the conclusion.

2. Related Work

The functions very similar to poly-powers are the so-called *generalized polynomials* in Bak et al. (1973) and *multiple power sums* in Xu et al. (2010b) that both extend integer exponents to real exponents for polynomials. Specifically, it has the form

$$b_0 + b_1 x^{a_1} + \cdots + b_n x^{a_n} \quad , \tag{2}$$

where a_i ($1 \leq i \leq n$) and b_i ($0 \leq i \leq n$) are real numbers. It is slightly more expressive than poly-powers. Bak et al. (1973) used generalized polynomials to estimate the convergence rate of functions in the spaces $\mathcal{L}_p[0,1]$ and $\mathcal{C}[0,1]$. Coste et al. (2005) proved the generalized Budan–Fourier theorem to count the number of complex roots of multiple power sums. Xu et al. (2010b) reproved the generalized Budan–Fourier theorem to count the number of positive roots, and gave

poly-power $x^\alpha - 1$ with $\alpha \in \mathbb{R}_{\mathrm{alg}} \setminus \mathbb{Q}$, it is impossible to obtain finitely many disjoint intervals such that each contains exactly one complex root and together contain all, since this poly-power has countably (infinitely) many distinct complex roots $\{\cos \frac{2k\pi}{\alpha} + \imath \sin \frac{2k\pi}{\alpha} \mid k \in \mathbb{Z}\}$. That is, the procedure is doomed to be nonterminating as its output is an infinite sequence of isolation intervals. Hence x is restricted to be positive in this paper, and thus the power function $x^\alpha = \exp(\alpha \cdot \ln x)$ can be uniquely determined as the default real-valued one.

the upper and the lower bounds of all positive roots. Note that the set of computable (real) numbers $\mathcal{R}$, including real algebraic numbers $\mathbb{R}_{\mathrm{alg}}$, is only a countable subset of real numbers $\mathbb{R}$. So we restrict those exponents into $\mathbb{R}_{\mathrm{alg}}$-numbers to rigorously ensure the computability.

From another viewpoint, the poly-power is subsumed by the exponential polynomial

$$\beta_0(t) + \beta_1(t)\mathrm{e}^{\alpha_1 t} + \cdots + \beta_n(t)\mathrm{e}^{\alpha_n t} \ , \tag{3}$$

where α_i $(1 \le i \le n)$ are $\mathbb{R}_{\mathrm{alg}}$-numbers and $\beta_i(t)$ $(0 \le i \le n)$ are $\mathbb{R}_{\mathrm{alg}}$-polynomials, since it be immediately obtained by the variable change $x = \mathrm{e}^t$ and restricting all $\mathbb{R}_{\mathrm{alg}}$-polynomials $\beta_i(t)$ to $\mathbb{R}_{\mathrm{alg}}$-numbers. Under Schanuel's conjecture, Chonev *et al.* gave a procedure to decide whether the exponential polynomial (3) has a real root in an appointed interval (see Subsection 2.1 of Chonev et al. (2016)). However, they only concerned deciding the existence of real roots, rather than the real root isolation.

3. Decomposition

In this section, we aim to decompose the input poly-power into simpler ones that preserve positive roots. Naturally one may relate it to polynomial factorization. However, since exponents in a poly-power are generally $\mathbb{R}_{\mathrm{alg}}$-numbers (unlike integers in a polynomial), there is no direct way to do the factorization. To resolve it, we will first find a $\mathbb{Z}$-linearly independent base that spans all exponents. Based on that, the poly-power can be transformed to an equivalent multivariate polynomial relation among algebraically independent powers, which can be easily factorized. These irreducible factors would be simpler poly-powers. Thus we complete the decomposition of the input poly-power.

Let $\mathcal{A}$ denote the field extension (over $\mathbb{Q}$) generated by all $\mathbb{R}_{\mathrm{alg}}$-numbers $\alpha_1, \ldots, \alpha_n; \beta_0, \ldots, \beta_n$ appearing in the poly-power. (In fact, it suffices to require that $\mathcal{A}$ is generated by $\beta_0, \ldots, \beta_n$ for the purpose of decomposition. But it will bring much convenience in later if $\mathcal{A}$ is generated by $\alpha_1, \ldots, \alpha_n; \beta_0, \ldots, \beta_n$.) We expect to obtain a $\mathbb{Z}$-linearly independent base $\{\upsilon_1, \ldots, \upsilon_\ell\}$ of them, which fortunately can be completed by the following algorithm.

Algorithm 1 Computing a Simple Extension over $\mathbb{Q}$ (see Algorithm 2 of Loos (1983))

$$(\mu; p_1, \ldots, p_n) \Leftarrow \mathsf{Simple}(\mu_1, \ldots, \mu_n)$$

Input: Each μ_i is a $\mathbb{R}^+_{\mathrm{alg}}$-number, equipped with minimal polynomial ρ_i and isolation interval $\mathcal{I}_i$.
Output: μ is a $\mathbb{R}^+_{\mathrm{alg}}$-number, equipped with minimal polynomial ρ and isolation interval $\mathcal{I}$, such that each $\mu_i = p_i(\mu) \in \mathbb{Z}[\mu]$ with $\deg(p_i) < \deg(\rho)$ is a computable polynomial relation.

By invoking Algorithm 1 with $\mathsf{Simple}(\alpha_1, \ldots, \alpha_n, \beta_0, \ldots, \beta_n)$, we can obtain a simple extension representation $\mathbb{Q}(\mu) : \mathbb{Q}$ for $\mathcal{A}$, viz. $\mathbb{Q}(\mu) : \mathbb{Q} = \mathbb{Q}(\alpha_1, \ldots, \alpha_n, \beta_0, \ldots, \beta_n) : \mathbb{Q}$. Then the $\mathbb{R}_{\mathrm{alg}}$-numbers $1, \mu, \ldots, \mu^{l-1}$, where $l = \deg(\rho)$, must be $\mathbb{Z}$-linearly independent. Otherwise μ could be defined by a polynomial with degree less than l, which contradicts that ρ is the minimal polynomial of μ. So we have the representation

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} p_{1,0} & p_{1,1} & \cdots & p_{1,l-1} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n,0} & p_{n,1} & \cdots & p_{n,l-1} \end{pmatrix} \begin{pmatrix} 1 \\ \mu \\ \vdots \\ \mu^{l-1} \end{pmatrix} \ , \tag{4}$$

where all $p_{i,0}, p_{i,1}, \ldots, p_{i,l-1}$ are taken from coefficients of $\mathbb{Z}$-polynomials $\alpha_i = p_i(\mu) = p_{i,0} + p_{i,1}\mu + \cdots + p_{i,l-1}\mu^{l-1}$ $(0 \le i \le n)$. Note that there may still exist linear dependence between column vectors $(p_{1,j}, \ldots, p_{n,j})^{\mathrm{T}}$ $(0 \le j \le l-1)$. It implies that the size of the base $\{1, \mu, \ldots, \mu^{l-1}\}$ could be reduced by linearly combining base elements, provided that the coefficient matrix has rank less than l. After reduction, we obtain a $\mathbb{Z}$-linearly independent base $\{\nu_1, \ldots, \nu_\ell\}$, satisfying:

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} q_{1,1} & \cdots & q_{1,\ell} \\ \vdots & \ddots & \vdots \\ q_{n,1} & \cdots & q_{n,\ell} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_\ell \end{pmatrix}, \tag{5}$$

where $\nu_j \in \mathbb{R}^+_{\mathrm{alg}}$, $q_{i,j} \in \mathbb{Z}$, $\gcd(q_{1,j}, \ldots, q_{n,j}) = 1$ and the coefficient matrix has the full rank ℓ. Furthermore, by difference substitution[2], we can fix the $\mathbb{Z}$-linearly independent base $\{\upsilon_1, \ldots, \upsilon_\ell\}$, satisfying:

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} r_{1,1} & \cdots & r_{1,\ell} \\ \vdots & \ddots & \vdots \\ r_{n,1} & \cdots & r_{n,\ell} \end{pmatrix} \begin{pmatrix} \upsilon_1 \\ \vdots \\ \upsilon_\ell \end{pmatrix}, \tag{6}$$

where $\upsilon_j \in \mathbb{R}^+_{\mathrm{alg}}$, $r_{i,j} \in \mathbb{Z}^+$, $\gcd(r_{1,j}, \ldots, r_{n,j}) = 1$, and the coefficient matrix has the full rank ℓ, since $\alpha_1, \ldots, \alpha_n$ are positive.

Now the poly-power $f(x)$ can be represented as an element of the ring $\mathcal{A}[x^{\upsilon_1}, \ldots, x^{\upsilon_\ell}]$ with a nonzero constant term. We call by *associated* polynomial $g(x^{\upsilon_1}, \ldots, x^{\upsilon_\ell})$ the aforementioned polynomial representation of $f(x)$. Note that $\mathcal{A}[x^{\upsilon_1}, \ldots, x^{\upsilon_\ell}]$ is a unique factorization domain. The decomposition of $f(x)$ corresponds to the factorization of $g(x^{\upsilon_1}, \ldots, x^{\upsilon_\ell})$, which has been well implemented in existing materials, e. g. Loos (1983). A poly-power is said to be *irreducible* if its associated polynomial is irreducible.

Example 1. *Consider the poly-power*

$$\begin{aligned} f(x) = 20 &- \tfrac{20}{3}x + 15x^{\sqrt{3}} + \tfrac{9}{4}x^2 - (\tfrac{5}{2} + 4\sqrt{5})x^{\sqrt{3}+1} - 4x^{3\sqrt{2}-1} + \tfrac{45}{16}x^{2\sqrt{3}} + \tfrac{4\sqrt{5}}{3}x^{\sqrt{3}+2} \\ &+ (\tfrac{2}{3} + \tfrac{4\sqrt{3}}{5})x^{3\sqrt{2}} - 3\sqrt{5}x^{2\sqrt{3}+1} - \tfrac{\sqrt{5}}{9}x^{\sqrt{3}+3} - \tfrac{3}{2}x^{\sqrt{3}+3\sqrt{2}-1} - \tfrac{4\sqrt{3}}{15}x^{3\sqrt{2}+1} + \tfrac{\sqrt{5}}{2}x^{2\sqrt{3}+2} \\ &+ (\tfrac{3\sqrt{3}}{5} + \tfrac{4\sqrt{5}}{5})x^{\sqrt{3}+3\sqrt{2}} - \tfrac{9\sqrt{5}}{16}x^{3\sqrt{3}+1} + \tfrac{\sqrt{3}}{45}x^{3\sqrt{2}+2} + \tfrac{1}{5}x^{6\sqrt{2}-2} - (\tfrac{\sqrt{3}}{10} + \tfrac{2\sqrt{5}}{15})x^{\sqrt{3}+3\sqrt{2}+1} \\ &- \tfrac{4\sqrt{3}}{25}x^{6\sqrt{2}-1} + (\tfrac{9\sqrt{3}}{80} + \tfrac{3\sqrt{5}}{10})x^{2\sqrt{3}+3\sqrt{2}} + \tfrac{2\sqrt{3}}{75}x^{6\sqrt{2}} - (\tfrac{3\sqrt{3}}{5} + \tfrac{\sqrt{5}}{25})x^{6\sqrt{2}+\sqrt{3}-1} + \tfrac{\sqrt{3}}{125}x^{9\sqrt{2}-2} \ . \end{aligned}$$

Applying the above method to the exponents in $f(x)$, we get a $\mathbb{Z}$-linearly independent base

$$\{\nu_1, \nu_2, \nu_3\} = \{1, \sqrt{2}, \sqrt{3}\} \ ,$$

by which all exponents can be $\mathbb{Z}$-linearly expressed. Namely, $f(x)$ is equivalent to an element of the localization of the polynomial ring $\mathcal{A}[x^{\nu_1}, x^{-\nu_1}, x^{\nu_2}, x^{-\nu_2}, x^{\nu_3}, x^{-\nu_3}]$, where $\mathcal{A}$ is the field extension generated by all exponents and coefficients in $f(x)$.

[2] The approach can be exemplified as follows. Assume w. l. o. g. that a number $\alpha = q_1\nu_1 + q_2\nu_2 + q_3\nu_3$ with $0 < \nu_1 < \nu_2 < \nu_3$ is not linearly expressed with all nonnegative coefficients q_1, q_2, q_3. We can rewrite it as $\alpha = (q_1 + q_2 + q_3)\nu'_1 + (q_2 + q_3)\nu'_2 + q_3\nu'_3$ with variable changes $\nu'_1 = \nu$, $\nu'_2 = \nu_2 - \nu_1$ and $\nu'_3 = \nu_3 - \nu_2$. Repeating this process, we would eventually get a linear expression for α with all nonnegative coefficients, provided that α is positive. The interested reader can refer to Yang (2005) for more details. Certainly, there are many different approaches to reach this goal.

Note that there are some exponents linearly expressed by $\{\nu_1, \nu_2, \nu_3\}$ *with negative coefficients, like the exponent* $9\sqrt{2} - 2 = 9\nu_2 - 2\nu_1$ *in the last term. It implies that* $f(x)$ *cannot have a polynomial representation using the base* $\{\nu_1, \nu_2, \nu_3\}$*. Fortunately, we can always find another* $\mathbb{Z}$*-linearly independent base, saying*

$$\{\upsilon_1, \upsilon_2, \upsilon_3\} = \{1,\ \sqrt{2} - 1,\ \sqrt{3} - \sqrt{2}\}\ ,$$

by which all exponents are $\mathbb{Z}^+$*-linearly expressed. Now the* poly-power $f(x)$ *is equivalent to an element* $g(x^{\upsilon_1}, x^{\upsilon_2}, x^{\upsilon_3})$ *of the ring* $\mathcal{A}[x^{\upsilon_1}, x^{\upsilon_2}, x^{\upsilon_3}]$*. The latter is the associate polynomial of* $f(x)$*.*

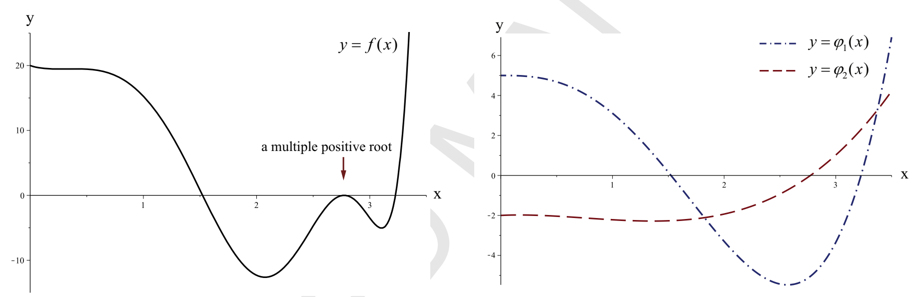
Finally, we have the factorization

$$g(x^{\upsilon_1}, x^{\upsilon_2}, x^{\upsilon_3}) = \varphi_1 \cdot \varphi_2^2\ ,$$

where $\varphi_1 = 5 - \sqrt{5}x^{\upsilon_3}x^{\upsilon_2}(x^{\upsilon_1})^2 + \frac{\sqrt{3}}{5}(x^{\upsilon_2})^3(x^{\upsilon_1})^3$ *and* $\varphi_2 = -2 + \frac{1}{3}x^{\upsilon_1} - \frac{3}{4}x^{\upsilon_3}x^{\upsilon_2}x^{\upsilon_1} + \frac{1}{5}(x^{\upsilon_2})^3(x^{\upsilon_1})^2$ *are irreducible* poly-powers *over* $\mathcal{A}[x^{\upsilon_1}, x^{\upsilon_2}, x^{\upsilon_3}]$*. In other words, we obtain two irreducible* poly-powers

$$\begin{aligned} \varphi_1 &= 5 - \sqrt{5}x^{\sqrt{3}+1} + \tfrac{\sqrt{3}}{5}x^{3\sqrt{2}}\ , \\ \varphi_2 &= -2 + \tfrac{1}{3}x - \tfrac{3}{4}x^{\sqrt{3}} + \tfrac{1}{5}x^{3\sqrt{2}-1}\ . \end{aligned}$$

whose positive roots are exactly the original poly-power $f(x)$*'s (see Figure 1).*



(a) The original poly-power

(b) Two irreducible poly-powers

Figure 1: Decomposition

4. Positive Root Isolation

In this section we aim to:

1. isolate positive roots for the irreducible poly-power whose associated polynomial has at most two variables;
2. isolate positive roots for the irreducible poly-power whose associated polynomial has at least three variables;
3. isolate positive roots for the reducible poly-power.

We apply number-theoretic results to rule out multiple positive roots. Based on that and interval arithmetic, we will give two criteria respectively to check whether an interval contains exactly one positive root or none. If it has none, we would exclude it from our consideration. After checking sufficiently precise subintervals, we could eventually obtain the complete list of isolation intervals for all positive roots, each contains exactly one positive root.

4.1. Simple Case

Here we present a positive root isolation algorithm for an irreducible poly-power equivalent to a bivariate associated polynomial. That function is called the simple poly-power, an element of $\mathcal{A}[x^{\upsilon_1}, x^{\upsilon_2}]$. Specially, when the associated polynomial is univariate in $y = x^{\upsilon_1}$, the positive root isolation for the poly-power $f(x)$ would amount to that for the univariate polynomial $g(y)$, which can be plainly solved.

Let φ be an element of $\mathcal{A}[x^{\upsilon_1}, x^{\upsilon_2}]$. It can be rewritten as an element of $\mathcal{A}[y, y^{\upsilon}]$ with the substitution of $y = x^{\upsilon_1}$ and $\upsilon = \upsilon_2/\upsilon_1$, which is a simple poly-power too. Clearly, υ is irrational. Thus we can w. l. o. g. focus only on the positive root isolation for $\varphi \in \mathcal{A}[y, y^{\upsilon}]$ in this subsection.

We first recall a classic number-theoretic result that:

Theorem 2 (Gelfond–Schneider (1934), see Chapter 3 of Siegel (1949))**.** *The numbers β^{α} are transcendental for all algebraic numbers $\beta \in \mathbb{A} \setminus \{0, 1\}$ and all algebraic numbers $\alpha \in \mathbb{A} \setminus \mathbb{Q}$.*

Lemma 3. *Let φ be an irreducible element of $\mathcal{A}[y, y^{\upsilon}]$ with positive degree in y^{υ}. Then φ has no $\mathbb{R}^{+}_{\text{alg}}$-root with the only possible exception for $y = 1$.*

Proof. (The proof is similar to that of Theorem 3.5 of Achatz et al. (2008), but makes use of Theorem 2. To be self-contained, we give the proof below.)

Suppose that φ has a $\mathbb{R}^{+}_{\text{alg}}$-root y_0. The proof will be completed by showing $y_0 = 1$. Let f be the bivariate polynomial with algebraic coefficients, such that $f(y, y^{\upsilon}) = \varphi$. Clearly, $f(y_0, Y)$ is a univariate polynomial with algebraic coefficients, and it takes $Y = y_0^{\upsilon}$ as one of its roots. By Corollary 4 of Loos (1983) that all root of an $\mathbb{A}$-polynomial are algebraic, we have that y_0^{υ} is algebraic. Furthermore, by Theorem 2, we can infer that y_0 must be 1, since $y_0 \in \mathbb{A}$ is positive and $\upsilon \in \mathbb{A}$ is irrational then. $\square$

Lemma 4. *Let φ be an irreducible element of $\mathcal{A}[y, y^{\upsilon}]$ with positive degree in y^{υ}. Then φ has no multiple positive root with the only possible exception for $y = 1$.*

Proof. (The proof is similar to that of Theorem 2.7 of Achatz et al. (2008), but makes use of Theorem 2. To be self-contained, we give the proof below.)

Let the *strictly* 1*-shifted derivative* (*shifted derivative* for short, see Definition 1 of Gupta et al. (2013)) of φ be defined as

$$\psi = y \cdot \varphi' \quad . \tag{7}$$

Obviously, ψ shares the same positive roots with φ', and shares the same nonzero exponents with φ. Furthermore it can be easily verified that ψ is an element of $\mathcal{A}[y, y^{\upsilon}]$ since $\mathcal{A}$ is the field extension generated by $\alpha_1, \dots, \alpha_n; \beta_0, \dots, \beta_n$, and that φ, ψ are co-prime since φ is irreducible and ψ has less terms than φ. Assume that y_0 is a multiple positive root of φ, viz. $\varphi = 0$ and $\psi = 0$ simultaneously hold at $y = y_0$. After applying Sylvester's resultant to eliminate y polynomially in φ and ψ, the result would be a nonzero element of $\mathcal{A}[y^{\upsilon}]$ that is zero at $y = y_0$. It implies that y_0^{υ} is algebraic by Corollary 4 of Loos (1983), so is y_0. Finally we have $y_0 = 1$ by Theorem 2, since υ is irrational. $\square$

This lemma indicates that all positive roots (except 1) of φ are simple. It also implies that the shifted derivative ψ is nonzero at any positive root (except 1) of φ, and even in a sufficiently small neighbor of a positive root of φ. So we utilize the following criteria to determine whether an interval contains exactly one positive root or none.

Criterion 5. *A* poly-power φ *has no positive root in the interval* $\mathcal{I}$ *if the over-approximation (superset) of its range excludes zero, i. e.,* $0 \notin \mathrm{Rg}^+(\varphi, \mathcal{I})$.

Criterion 6. *A* poly-power φ *has exactly one positive root in the interval* $\mathcal{I}$ *excluding* 1 *if the over-approximation of the range of its shifted derivative* ψ *excludes zero and the under-approximation (subset) of its range includes zero, i. e.,* $0 \notin \mathrm{Rg}^+(\psi, \mathcal{I})$ *and* $0 \in \mathrm{Rg}^-(\varphi, \mathcal{I})$.

The correctness of the above two criteria is straightforward. Then we turn to algorithmically construct those approximations using the fact that all algebraic numbers are computable. The encoding of algebraic numbers is default—using minimal polynomials plus isolation intervals that can offer numerical approximations of any precision. In fact, Thom's lemma (Coste and Roy, 1988) gives another encoding of algebraic numbers, but it cannot directly offer numerical approximation. So we adopt the default one in the paper.

In what follows, we will construct the over-approximation $\mathrm{Rg}^+(f, \mathcal{I})$ of the range of a poly-power $f(x)$ on a rational interval $\mathcal{I} = (a, b) \subset (0, 1) \cup (1, +\infty)$. To get a safe approximation, we introduce the **interval arithmetic** rule:

- for $a > 1$, $a^{(\underline{\alpha}, \overline{\alpha})} = (a^{\underline{\alpha}}, a^{\overline{\alpha}})$ and $(a, b)^{(\underline{\alpha}, \overline{\alpha})} = (a^{\underline{\alpha}}, b^{\overline{\alpha}})$, where $(\underline{\alpha}, \overline{\alpha})$ is a rational interval approximation of α;
- for $a < 1$, $a^{(\underline{\alpha}, \overline{\alpha})} = (a^{\overline{\alpha}}, a^{\underline{\alpha}})$ and $(a, b)^{(\underline{\alpha}, \overline{\alpha})} = (a^{\overline{\alpha}}, b^{\underline{\alpha}})$;
- and others are standard.

For convenience, we denote by (α) the approximation $(\underline{\alpha}, \overline{\alpha})$. Assume w. l. o. g. that $f(x)$ is expressed in the form (1). Then, using interval arithmetic, we can obtain the over-approximation

$$\mathrm{Rg}^+(f, \mathcal{I}) = \beta_0 + \beta_1 \cdot \mathcal{I}^{(\alpha_1)} + \cdots + \beta_n \cdot \mathcal{I}^{(\alpha_n)} \quad . \tag{8}$$

On the other hand, we will construct an under-approximation $\mathrm{Rg}^-(f, \mathcal{I})$ under the premise that the derivative f' of f has no positive root in $\mathcal{I}$. We can see $f(a) \neq f(b)$ as f is monotonous in $\mathcal{I}$. Then, we can get two interval estimates V_{left} and V_{right} respectively for the values $f(x)$ at endpoints of $\mathcal{I}$:

$$\begin{aligned} V_{\mathrm{left}} &= \beta_0 + \beta_1 \cdot a^{(\alpha_1)} + \cdots + \beta_n \cdot a^{(\alpha_n)} \quad , \\ V_{\mathrm{right}} &= \beta_0 + \beta_1 \cdot b^{(\alpha_1)} + \cdots + \beta_n \cdot b^{(\alpha_n)} \quad . \end{aligned}$$

Moreover, V_{left} and V_{right} would be sign-invariant and disjoint, when (α_i)s $(1 \leq i \leq n)$ are sufficiently precise. Finally, we can get an under-approximation

$$\mathrm{Rg}^-(f, \mathcal{I}) = \begin{cases} (\sup V_{\mathrm{left}}, \inf V_{\mathrm{right}}) & \text{if } \sup V_{\mathrm{left}} < \inf V_{\mathrm{right}} \quad , \\ (\sup V_{\mathrm{right}}, \inf V_{\mathrm{left}}) & \text{if } \sup V_{\mathrm{right}} < \inf V_{\mathrm{left}} \quad . \end{cases} \tag{9}$$

Utilizing that premise that f' has no positive root in $\mathcal{I}$, the over-approximation can be refined as

$$\mathrm{Rg}^+(f, \mathcal{I}) = \begin{cases} (\inf V_{\mathrm{left}}, \sup V_{\mathrm{right}}) & \text{if } \inf V_{\mathrm{left}} < \sup V_{\mathrm{right}} \quad , \\ (\inf V_{\mathrm{right}}, \sup V_{\mathrm{left}}) & \text{if } \inf V_{\mathrm{right}} < \sup V_{\mathrm{left}} \quad . \end{cases} \tag{10}$$

If V_{left} and V_{right} have the opposite sign, we have $0 \in \mathrm{Rg}^-(f, \mathcal{I})$; if V_{left} and V_{right} have the same sign, we have $0 \notin \mathrm{Rg}^+(f, \mathcal{I})$. Anyway, either $0 \in \mathrm{Rg}^-(f, \mathcal{I})$ or $0 \notin \mathrm{Rg}^+(f, \mathcal{I})$ holds. It would yield a slight improvement on our preliminary paper (Li et al., 2016).

Thus we can obtain the desired over-approximations $\mathrm{Rg}^+(\varphi,\mathcal{I}),\mathrm{Rg}^+(\psi,\mathcal{I})$ and the under-approximation $\mathrm{Rg}^-(\varphi,\mathcal{I})$ of the range of the poly-powers φ,ψ on the interval $\mathcal{I}$. Those approximations are algebraic, and could be easily refined as rational intervals within any predefined error, provided that the length of $\mathcal{I}$ is sufficiently small and the irrational exponent υ of y^υ is sufficiently approached.

Example 7. *Consider the poly-power $\varphi_1 = 5 - \sqrt{5}x^{\sqrt{3}+1} + \frac{\sqrt{3}}{5}x^{3\sqrt{2}}$ in Example 1. Its shifted derivative is*

$$\psi_1 = -(\sqrt{15}+\sqrt{5})x^{\sqrt{3}+1} + \frac{3\sqrt{6}}{5}x^{3\sqrt{2}} \ .$$

Here we will estimate the ranges of φ_1 and ψ_1 on the intervals $(\frac{45}{32},\frac{103}{64}),(1,\frac{45}{32}),(\frac{103}{64},\frac{29}{16})$, and thereby infer the number of positive roots of φ_1 in them.

On the interval $\mathcal{I}' = (\frac{45}{32},\frac{103}{64})$, using the interval estimates of algebraic numbers $(\sqrt{2}) \subset (\frac{181}{128},\frac{91}{64})$, $(\sqrt{3}) \subset (\frac{113511}{65536},\frac{14189}{8192})$ and $(\sqrt{5}) \subset (\frac{285}{128},\frac{573}{256})$, we can calculate

$$\begin{aligned}\mathrm{Rg}^+(\varphi_1,\mathcal{I}') &= 5-(\sqrt{5})\cdot\left(\tfrac{45}{32},\tfrac{103}{64}\right)^{(\sqrt{3})+1} + \tfrac{(\sqrt{3})}{5}\cdot\left(\tfrac{45}{32},\tfrac{103}{64}\right)^{3\cdot(\sqrt{2})}\\ &\subset \left(5-\tfrac{573}{256}\cdot\left(\tfrac{103}{64}\right)^{\frac{22381}{8192}} + \tfrac{113511}{327680}\cdot\left(\tfrac{45}{32}\right)^{\frac{543}{128}}, 5-\tfrac{285}{128}\cdot\left(\tfrac{45}{32}\right)^{\frac{179047}{65536}} + \tfrac{14189}{40960}\cdot\left(\tfrac{103}{64}\right)^{\frac{273}{64}}\right)\\ &\subset (-1.75, 1.99) \ .\end{aligned}$$

Note that interval estimates of algebraic numbers can be easily offered to any desired precision. In fact, the above ones suffice in our examples. Similarly, we calculate

$$\mathrm{Rg}^+(\psi_1,\mathcal{I}') \subset (-16.20,-4.19) \ .$$

Furthermore, we can obtain two interval estimates V_{left} and V_{right} respectively for the values of φ_1 at endpoints of $\mathcal{I}'$:

$$\begin{aligned}V_{\text{left}} &= 5-(\sqrt{5})\cdot\left(\tfrac{45}{32}\right)^{(\sqrt{3})+1} + \tfrac{(\sqrt{3})}{5}\cdot\left(\tfrac{45}{32}\right)^{3\cdot(\sqrt{2})}\\ &\subset \left(5-\tfrac{573}{256}\cdot\left(\tfrac{45}{32}\right)^{\frac{22381}{8192}} + \tfrac{113511}{327680}\cdot\left(\tfrac{45}{32}\right)^{\frac{543}{128}}, 5-\tfrac{285}{128}\cdot\left(\tfrac{45}{32}\right)^{\frac{179047}{65536}} + \tfrac{14189}{40960}\cdot\left(\tfrac{45}{32}\right)^{\frac{273}{64}}\right)\\ &\subset (0.79, 0.84) \ ,\\ V_{\text{right}} &= 5-(\sqrt{5})\cdot\left(\tfrac{103}{64}\right)^{(\sqrt{3})+1} + \tfrac{(\sqrt{3})}{5}\cdot\left(\tfrac{103}{64}\right)^{3\cdot(\sqrt{2})}\\ &\subset \left(5-\tfrac{573}{256}\cdot\left(\tfrac{103}{64}\right)^{\frac{22381}{8192}} + \tfrac{113511}{327680}\cdot\left(\tfrac{103}{64}\right)^{\frac{543}{128}}, 5-\tfrac{285}{128}\cdot\left(\tfrac{103}{64}\right)^{\frac{179047}{65536}} + \tfrac{14189}{40960}\cdot\left(\tfrac{103}{64}\right)^{\frac{273}{64}}\right)\\ &\subset (-0.61,-0.53) \ .\end{aligned}$$

As $\sup V_{\text{right}} < \inf V_{\text{left}}$, *we have*

$$\mathrm{Rg}^-(\varphi_1,\mathcal{I}') = (\sup V_{\text{right}}, \inf V_{\text{left}}) \supset (-0.53, 0.79) \ .$$

It is clear that $0 \notin \mathrm{Rg}^+(\psi_1,\mathcal{I}')$ and $0 \in \mathrm{Rg}^-(\varphi_1,\mathcal{I}')$. Applying Criterion 6, we can know that φ_1 has exactly one positive root in the interval $(\frac{45}{32},\frac{103}{64})$.

For the interval $\mathcal{I}'' = (1,\frac{45}{32})$, we have

$$\mathrm{Rg}^+(\varphi_1,\mathcal{I}'') \subset (-0.34, 4.26) \quad \textit{and} \quad \mathrm{Rg}^+(\psi_1,\mathcal{I}'') \subset (-14.06, 0.25) \ .$$

Neither Criterion 5 nor Criterion 6 is applicable. Then we have to bisect $\mathcal{I}''$ to decide whether $\mathcal{I}''$ contains positive roots or none. On the subintervals $(1, \frac{77}{64})$ and $(\frac{77}{64}, \frac{45}{32})$, we have

$$\mathrm{Rg}^+(\varphi_1, (1, \tfrac{77}{64})) \subset (1.63, 3.54) \quad \textit{and} \quad \mathrm{Rg}^+(\varphi_1, (\tfrac{77}{64}, \tfrac{45}{32})) \subset (0.07, 2.80) \ .$$

Obviously, $0 \notin \mathrm{Rg}^+(\varphi_1, (1, \frac{77}{64})) \cup \mathrm{Rg}^+(\varphi_1, (\frac{77}{64}, \frac{45}{32}))$. *Applying Criterion 5 and Lemma 3, we can know that φ_1 has no positive root in the interval* $(1, \frac{45}{32})$.

For the interval $\mathcal{I}''' = (\frac{103}{64}, \frac{29}{16})$, we have

$$\mathrm{Rg}^+(\varphi_1, \mathcal{I}''') \subset (-3.76, 1.21) \quad \textit{and} \quad \mathrm{Rg}^+(\psi_1, \mathcal{I}''') \subset (-19.99, -3.64) \ .$$

Then we turn to calculate interval estimates for the values of φ_1 at endpoints of $\mathcal{I}'''$:

$$V_{\text{left}} \subset (-0.61, -0.53) \quad \textit{and} \quad V_{\text{right}} \subset (-2.05, -1.92) \ .$$

Since φ_1 is monotonous in $\mathcal{I}'''$, we get a refined $\mathrm{Rg}^+(\varphi_1, \mathcal{I}''') \subset (-2.05, -0.53)$. *Applying Criterion 5, we can know that φ_1 has no positive root in the interval* $(\frac{103}{64}, \frac{29}{16})$.

Now we are ready to state the exclusion method (see Algorithm 2) for isolating positive roots of a simple poly-power. Its rationale is that for a positive rational interval $\mathcal{I}$ excluding 1, we can bisect it and recursively tackle each subinterval, until one of the two criteria is applicable. Note that, to ensure termination, the approximation (υ) of υ should be dynamically refined during the execution process.

Algorithm 2 The positive root isolation for $\mathcal{A}[y, y^{\upsilon}]$ using Exclusion

$$\{\mathcal{I}_1, \ldots, \mathcal{I}_n\} \Leftarrow \mathsf{Isolate\_E}(\varphi, \mathcal{I})$$

Input: $\varphi \in \mathcal{A}[y, y^{\upsilon}]$ is a simple poly-power with positive degree in y^{υ}, which is defined on the rational interval $\mathcal{I} = (a, b) \subset (0, 1) \cup (1, +\infty)$.
Output: $\mathcal{I}_1, \ldots, \mathcal{I}_n$ are finitely many disjoint intervals, such that each contains exactly one positive root of φ in $\mathcal{I}$, and together contain all.

1: let $\psi = y \cdot \varphi'$ be the shifted derivative of φ;
2: **if** $0 \notin \mathrm{Rg}^+(\varphi, \mathcal{I})$ **then**
3:    **return** $\emptyset$; {Criterion 5 follows from the over-approximation (8).}
4: **else if** $0 \notin \mathrm{Rg}^+(\psi, \mathcal{I})$ **then**
5:    **if** $0 \in \mathrm{Rg}^-(\varphi, \mathcal{I})$ **then**
6:       **return** $\mathcal{I}$; {Criterion 6 follows from the under-approximation (9).}
7:    **else**
8:       **return** $\emptyset$; {Criterion 5 follows from the over-approximation (10).}
9:    **end if**
10: **else**
11:    $\mathsf{Isolate\_E}(\varphi, (a, \frac{a+b}{2}))$;
12:    $\mathsf{Isolate\_E}(\varphi, (\frac{a+b}{2}, b))$;
13: **end if**

Correctness. We first note that those midpoints $\frac{a+b}{2}$ cannot be any positive root of φ by Lemma 3, since they are rational ($\neq 1$). So splitting (a, b) into $(a, \frac{a+b}{2})$ and $(\frac{a+b}{2}, b)$ would not miss any positive root. Second, multiple roots of φ are ruled out by Lemma 4. Then we know that all positive roots of φ in (a, b) are simple. Since φ is analytic on $(0, +\infty)$ (i. e., it is infinitely differentiable and is converged by Taylor expansion), we can infer that φ has finitely many positive roots. Therefore, they would be isolated sooner or later by the under-approximation (9) after bisection, provided that the power function is sufficiently approached, which is guaranteed by the computability of algebraic numbers. The termination follows from the fact that the shifted derivative has no positive root in isolation intervals, when isolation intervals are sufficiently small; then those subintervals would be excluded by the over-approximation (10) if they are close to positive roots, and by the over-approximation (8) otherwise. $\square$

Example 8. *Consider the irreducible* poly-power $\varphi_1 = 5 - \sqrt{5}x^{\sqrt{3}+1} + \frac{\sqrt{3}}{5}x^{3\sqrt{2}}$ *in Example 1 again. Using the base* $\{1, \sqrt{2}-1, \sqrt{3}-\sqrt{2}\}$*, the associated polynomial of* φ_1 *is trivariate. However, we note that* φ_1 *has exactly two nonconstant terms. So it can be rewritten as a bivariate associated polynomial under the base* $\{\sqrt{2}, \sqrt{3}+1\}$*, which can be solved by Algorithm 2.*

By invoking Algorithm 2 with Isolate_E$(\varphi_1, (1, \frac{15}{2}))$*, we first get its shifted derivative* ψ_1 *as shown in Example 7. Then we compute the approximations* $\mathrm{Rg}^+(\varphi_1, \mathcal{I})$*,* $\mathrm{Rg}^+(\psi_1, \mathcal{I})$ *and* $\mathrm{Rg}^-(\varphi_1, \mathcal{I})$ *where* $\mathcal{I}$ *is initialized with* $(1, \frac{15}{2})$*, when they are necessary. Once Criterion 5 or Criterion 6 is applicable, we get an immediate return. Otherwise we bisect the interval and recursively invoke Algorithm 2 on subintervals. The final computational results are shown in Table 1, where*

- *'+' means* $\inf \mathrm{Rg}^+ > 0$*,*
- *'−' means* $\sup \mathrm{Rg}^+ < 0$*,*
- *'⋆' means the interval contains exactly one positive root,*
- *and the case indicates which approximation* (8)*,* (9) *or* (10) *is involved.*

The first four entries in Table 1 have been verified in Example 7.

Finally, we obtain that φ_1 *has two positive roots, which lie in* $(\frac{45}{32}, \frac{103}{64})$ *and* $(\frac{97}{32}, \frac{207}{64})$*.*

Interval	±/⋆	Case	Interval	±/⋆	Case	Interval	±/⋆	Case
$(1, \frac{77}{64})$	+	(8)	$(\frac{77}{64}, \frac{45}{32})$	+	(8)	$(\frac{45}{32}, \frac{103}{64})$	⋆	(9)
$(\frac{103}{64}, \frac{29}{16})$	−	(10)	$(\frac{29}{16}, \frac{129}{64})$	−	(10)	$(\frac{129}{64}, \frac{271}{128})$	−	(8)
$(\frac{271}{128}, \frac{71}{32})$	−	(8)	$(\frac{71}{32}, \frac{297}{128})$	−	(8)	$(\frac{297}{128}, \frac{155}{64})$	−	(8)
$(\frac{155}{64}, \frac{323}{128})$	−	(8)	$(\frac{323}{128}, \frac{21}{8})$	−	(8)	$(\frac{21}{8}, \frac{349}{128})$	−	(8)
$(\frac{349}{128}, \frac{181}{64})$	−	(8)	$(\frac{181}{64}, \frac{375}{128})$	−	(10)	$(\frac{375}{128}, \frac{97}{32})$	−	(10)
$(\frac{97}{32}, \frac{207}{64})$	⋆	(9)	$(\frac{207}{64}, \frac{55}{16})$	+	(10)	$(\frac{55}{16}, \frac{123}{32})$	+	(10)
$(\frac{123}{32}, \frac{17}{4})$	+	(10)	$(\frac{17}{4}, \frac{81}{16})$	+	(10)	$(\frac{81}{16}, \frac{47}{8})$	+	(8)
$(\frac{47}{8}, \frac{15}{2})$	+	(8)						

Table 1: Positive Root Isolation for φ_1

The remaining issue is to find the upper and the lower bounds for all positive roots of a poly-power. We refer to the following result:

Lemma 9 (Corollary 12 of Xu et al. (2010b))**.** *For a* poly-power $f(x)$ *in the form* (1)*, the number*

$$U = \max\left(1, \left(\sum_{i=0}^{n-1}\left|\frac{\beta_i}{\beta_n}\right|\right)^{\frac{1}{\tau}}\right) \quad \textit{with } \tau = \alpha_n - \alpha_{n-1}$$

is an upper bound for its positive roots.

The lower bound L for all positive roots of the aforementioned $f(x)$ can be similarly obtained, since it is exactly the reciprocal of the upper bound for all positive roots of $x^{\alpha_n} \cdot f(\frac{1}{x})$, another poly-power.

Example 10. *Consider the two irreducible* poly-power*s* $\varphi_1 = 5 - \sqrt{5}x^{\sqrt{3}+1} + \frac{\sqrt{3}}{5}x^{3\sqrt{2}}$ *and* $\varphi_2 = -2 + \frac{1}{3}x - \frac{3}{4}x^{\sqrt{3}} + \frac{1}{5}x^{3\sqrt{2}-1}$ *in Example 1. For* φ_1*, we calculate an upper bound* $U = \frac{15}{2}$ *for its positive roots, since*

$$\max\left(1, \tfrac{5}{\sqrt{3}}\left(\sqrt{5}+5\right)^{\frac{1}{3\sqrt{2}-\sqrt{3}-1}}\right) < \max\left(1, \tfrac{5\cdot 65536}{113511}\left(\tfrac{573}{256}+5\right)^{\frac{1}{\frac{3\cdot 181}{128}-\frac{14189}{8192}-1}}\right) < \frac{15}{2} \ .$$

Correspondingly, we calculate a lower bound $L = 1$ *as the reciprocal of*

$$\max\left(1, \tfrac{1}{5}\left(\sqrt{5}+\tfrac{\sqrt{3}}{5}\right)^{\frac{1}{\sqrt{3}+1}}\right) = 1 \ ,$$

which is an upper bound for all positive roots of

$$x^{3\sqrt{2}} \cdot \varphi_1(\tfrac{1}{x}) = \tfrac{\sqrt{3}}{5} - \sqrt{5}x^{3\sqrt{2}-\sqrt{3}-1} + 5x^{3\sqrt{2}} \ .$$

We thereby know that all positive roots of φ_1 *have been isolated in Example 8. Similarly, we obtain that all positive roots of* φ_2 *are bounded by* $(1, \frac{13}{2})$*.*

Remark 11. *In Algorithm 2, the values of* φ *are expected to be nonzero at endpoints of* $\mathcal{I}$*, and then we can get sign-invariant approximations of the ranges of* $\varphi(\mathcal{I})$*. Obviously,* $\varphi(L)$ *and* $\varphi(U)$ *are nonzero owe to the definition of the upper and the lower bounds of positive roots. If* $\varphi(1) \neq 0$*, Algorithm 2 still works for* $(L, 1)$ *and* $(1, U)$*, because all positive roots of* φ *are simple then. Otherwise we have to find a rational number* $\epsilon > 0$ *such that* φ *has no positive root in the neighbor* $\delta(1; \epsilon) = (1-\epsilon, 1) \cup (1, 1+\epsilon)$*. Specifically, the constant* ϵ *can be chosen as any positive number less than*

$$\min\left(\frac{\varphi^{(K)}(1)}{\sup\limits_{y\in[1-\theta,1+\theta]}|\varphi^{(K+1)}(y)|}, \theta\right) \ ,$$

where K *is the minimal positive integer, satisfying the* K*th derivative* $\varphi^{(K)}(1) \neq 0$*, for any* $\theta > 0$*.*

Thereby, we can isolate all positive roots of a simple poly-power φ into disjoint intervals by invoking Algorithm 2 with Isolate_E$(\varphi, (L, 1))$ and Isolate_E$(\varphi, (1, U))$ if $\varphi(1) \neq 0$ (and with Isolate_E$(\varphi, (L, 1-\epsilon))$ and Isolate_E$(\varphi, (1+\epsilon, U))$ otherwise). It completes the positive root isolation for simple poly-powers.

4.2. Non-simple Case

Here we will extend the exclusion method (Algorithm 2) to the general poly-power φ, whose associated polynomial is allowed to be a product of arbitrarily many irreducible factors with arbitrarily many variables. Specifically, we have to isolate positive roots of an irreducible poly-power whose associated polynomial has more than two variables, and isolate those of a reducible poly-power. The only gap to be filled is showing that φ would have no multiple positive root. We solve it by another number-theoretic result:

Conjecture 12 (Schanuel (1960s), see Ax (1971))**.** *Let $\lambda_1, \ldots, \lambda_m$ be $\mathbb{Z}$-linearly independent complex numbers. Then the field extension*

$$\mathbb{Q}(\lambda_1, \mathrm{e}^{\lambda_1}, \ldots, \lambda_m, \mathrm{e}^{\lambda_m}) : \mathbb{Q}$$

has transcendence degree at least m.

By Conjecture 12, Chonev *et al.* recently proved a significant result on common roots:

Lemma 13 (Proposition 3 of Chonev et al. (2016))**.** *Let $\{\upsilon_1, \ldots, \upsilon_m\}$ and $\{\omega_1, \ldots, \omega_n\}$ be two $\mathbb{Z}$-linearly independent sets of $\mathbb{R}_{\mathrm{alg}}$-numbers, and ϕ_1, ϕ_2 two co-prime elements of $\mathbb{A}[t, \mathrm{e}^{t\upsilon_1}, \ldots, \mathrm{e}^{t\upsilon_m}, \mathrm{e}^{it\omega_1}, \ldots, \mathrm{e}^{it\omega_n}]$. Then ϕ_1 and ϕ_2 have no nonzero common real root.*

Corollary 14. *Let $\upsilon_1, \ldots, \upsilon_m$ be $\mathbb{Z}$-linearly independent $\mathbb{R}_{\mathrm{alg}}$-numbers, and φ_1, φ_2 two co-prime elements of $\mathcal{A}[x^{\upsilon_1}, \ldots, x^{\upsilon_m}]$. Then φ_1 and φ_2 have no common positive root with the only possible exception for $x = 1$.*

Proof. Suppose that φ_1 and φ_2 have a common positive root x_0. With the variable change $x = \mathrm{e}^t$, the resulting φ_1, φ_2 would be two co-prime elements ϕ_1, ϕ_2 of $\mathbb{A}[\mathrm{e}^{t\upsilon_1}, \ldots, \mathrm{e}^{t\upsilon_m}]$. By Lemma 13, ϕ_1 and ϕ_2 have no nonzero common real root. Hence the only possible common real root of ϕ_1 and ϕ_2 is zero, which implies $x_0 = 1$. $\square$

Corollary 15. *Let $\upsilon_1, \ldots, \upsilon_m$ be $\mathbb{Z}$-linearly independent $\mathbb{R}_{\mathrm{alg}}$-numbers, and φ an irreducible element of $\mathcal{A}[x^{\upsilon_1}, \ldots, x^{\upsilon_m}]$. Then φ has no multiple positive root with the only possible exception for $x = 1$.*

Proof. Suppose that φ has a multiple positive root x_0. Let ψ be the shifted derivative $\varphi' \cdot x$ of φ, another element of $\mathcal{A}[x^{\upsilon_1}, \ldots, x^{\upsilon_m}]$, so that $\psi(x_0) = \varphi'(x_0) = 0$. We know that φ and ψ are co-prime since φ is irreducible, and thereby infer $x_0 = 1$ by Corollary 14. $\square$

This corollary entails that an irreducible poly-power has no multiple positive root (except 1). Hence, under Conjecture 12, Algorithm 2 still works for the irreducible poly-power whose associated polynomial has more than two variables.

Example 16. *Consider the irreducible poly-power $\varphi_2 = -2 + \frac{1}{3}x - \frac{3}{4}x^{\sqrt{3}} + \frac{1}{5}x^{3\sqrt{2}-1}$ in Example 1. Its associated polynomial is trivariate. Under Conjecture 12, however, we can isolate its positive roots by Algorithm 2.*

We invoke Algorithm 2 with Isolate_E$(\varphi_2, (1, \frac{13}{2}))$*, as $(1, \frac{13}{2})$ is the bound of positive roots of φ_2 shown in Example 10. The computational procedure is similar to Example 8, and the final computational result is shown in Table 2. Finally, we obtain that φ_2 has exactly one positive root, which lies in $(\frac{19}{8}, \frac{49}{16})$.*

Interval	$\pm/\star$	Case	Interval	$\pm/\star$	Case	Interval	$\pm/\star$	Case
$(1, \frac{27}{16})$	$-$	(8)	$(\frac{27}{16}, \frac{65}{32})$	$-$	(8)	$(\frac{65}{32}, \frac{19}{8})$	$-$	(8)
$(\frac{19}{8}, \frac{49}{16})$	$\star$	(9)	$(\frac{49}{16}, \frac{15}{4})$	$+$	(10)	$(\frac{15}{4}, \frac{13}{2})$	$+$	(10)

Table 2: Positive Root Isolation for φ_2

On the other hand, Corollary 14 entails that two distinct irreducible poly-powers have no common positive root (except 1). Hence a square-free poly-power must have no multiple positive root (except 1). We can repeatedly bisect isolation intervals for positive roots of distinct irreducible poly-powers, until all of them become disjoint. It completes our positive isolation procedure, in which the termination depends on Conjecture 12.

Example 17. *Combining Example 8 with Example 16, we have that φ_1 has two positive roots in $\mathcal{I}_1 = (\frac{45}{32}, \frac{103}{64})$ and $\mathcal{I}_2 = (\frac{97}{32}, \frac{207}{64})$ respectively, and φ_2 has one in $\mathcal{I}_3 = (\frac{19}{8}, \frac{49}{16})$.*

Note that two distinct irreducible poly-powers have no common positive root (except 1) under Conjecture 12. So we know that $f(x) = \varphi_1 \cdot \varphi_2^2$ has three distinct positive roots in those intervals. However, the isolation intervals $\mathcal{I}_2$ and $\mathcal{I}_3$ are not disjoint, viz. $\mathcal{I}_2 \cap \mathcal{I}_3 \neq \emptyset$. In other words, $\mathcal{I}_2$ and $\mathcal{I}_3$ are not precise enough to distinguish two distinct roots of $f(x)$. Thus, we proceed to bisect $\mathcal{I}_2$ and $\mathcal{I}_3$ repeatedly, until the resulting isolation intervals $\hat{\mathcal{I}}_2 = (\frac{401}{128}, \frac{207}{64})$ and $\hat{\mathcal{I}}_3 = (\frac{87}{32}, \frac{49}{16})$ become disjoint.

Finally, we have that $f(x)$ has totally three distinct positive roots respectively in the interval $\mathcal{I}_1, \hat{\mathcal{I}}_3, \hat{\mathcal{I}}_2$, among which the one in $\hat{\mathcal{I}}_3$ has multiplicity 2 (see Figure 2).

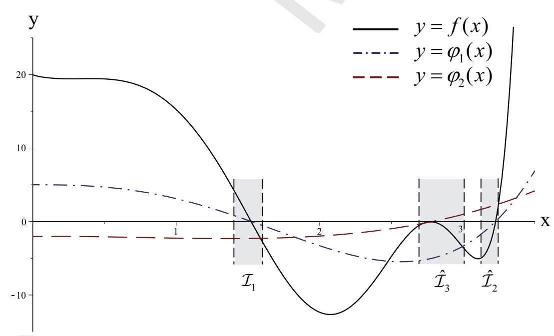


Figure 2: The Isolation Intervals for $f(x)$

5. An Alternative Positive Root Isolation

In this section, inspired by Collins and Loos (1976); Achatz et al. (2008), we apply another root isolation method—differentiation—to poly-power. Its rationale is that we isolate real roots of the original function after those of the simpler derivative have been isolated. Finally we discuss why other popular isolation methods are not directly suitable for poly-powers.

Before stating Algorithm 3, we would introduce some notations:

- shift(φ) is the shifted poly-power, by dividing φ by the power x^α of its least exponent, so that the result has a nonzero constant term; $\text{shift}^{-1}(\varphi)$ is the inverse that multiplies φ by that power x^α (this operator depends on the latest shift(φ), we abuse the notation here).
- gsfd(φ) is the greatest square-free divisor of φ, which can be obtained by factorization based on the associated polynomial of φ.

Obviously, after performed the above operators, the resulting poly-power has the same positive roots as the original one.

Algorithm 3 The positive root isolation for poly-powers using differentiation

$$\{\mathcal{I}_1, \ldots, \mathcal{I}_n\} \Leftarrow \text{Isolate\_D}(\varphi, \mathcal{I})$$

Input: $\varphi \in \mathcal{A}[x^{\upsilon_1}, \ldots, x^{\upsilon_m}]$ is a poly-power on the rational interval $\mathcal{I} = (a, b) \subset (0, 1) \cup (1, +\infty)$.
Output: $\mathcal{I}_1, \ldots, \mathcal{I}_n$ are finitely many disjoint intervals, such that each contains exactly one positive root of φ in $\mathcal{I}$, and together contain all.

let $\psi_0 = \text{gsfd}(\varphi)$;
set $k = 0$;
while ψ_k has more than two terms **do**
  let $\psi_{k+1} = \text{gsfd}(x \cdot \text{shift}^{-1}((\text{shift}(\psi_k))'))$ be the pseudo-derivative;
  update k to $k + 1$;
end while
if $\psi_k(a) \cdot \psi_k(b) < 0$ **then**
  let the isolation interval $(a_1^{(k)}, b_1^{(k)})$ of ψ_k be (a, b);
  set $n^{(k)} = 1$;
else
  set $n^{(k)} = 0$;
end if
while $k > 0$ **do**
  for $i = 1$ **to** $n^{(k)}$ **do**
    repeat
      refine $(a_i^{(k)}, b_i^{(k)})$;
    until $0 \notin \text{Rg}^+(\psi_{k-1}, [a_i^{(k)}, b_i^{(k)}])$
  end for
  let $b_0^{(k)} = a$ and $a_{n^{(k)}+1}^{(k)} = b$;
  set $j = 0$;
  for $i = 0$ **to** $n^{(k)}$ **do**
    if $\psi_{k-1}(b_i^{(k)}) \cdot \psi_{k-1}(a_{i+1}^{(k)}) < 0$ **then**
      let $(a_{j+1}^{(k-1)}, b_{j+1}^{(k-1)}) = (b_i^{(k)}, a_{i+1}^{(k)})$;
      update j to $j + 1$;
    end if
  end for {Here all positive roots of ψ_{k-1} are isolated in $(a_1^{(k-1)}, b_1^{(k-1)}), \ldots, (a_j^{(k-1)}, b_j^{(k-1)})$.}
  set $n^{(k-1)} = j$;
  update k to $k - 1$;
end while
return $(a_1^{(0)}, b_1^{(0)}), \ldots, (a_{n^{(0)}}^{(0)}, b_{n^{(0)}}^{(0)})$;

Remark 18. *In fact, we can immediately interrupt the loop in Lines 3–6, if the current pseudo-derivative ψ_k has no positive root in $\mathcal{I}$. The latter can be partially determined by Theorem 7 of Xu et al. (2010b). Thereby it potentially improves the efficiency of Algorithm 3.*

Correctness. We first notice that the sequence of pseudo-derivatives (constructed by the loop in Lines 3–6) is finite, since ψ_{k+1} has at least one less term than ψ_k. We then prove the correctness by induction.

- Basically, we can easily isolate the unique positive root of the last pseudo-derivative (if it exists), since the last pseudo-derivative has two terms.

- Inductively, once isolation intervals $(a_1^{(k)}, b_1^{(k)}), \ldots, (a_{n^{(k)}}^{(k)}, b_{n^{(k)}}^{(k)})$ are known for all positive roots of ψ_k, we refine them by the loop in Lines 15–17 until their closure contain no positive root of ψ_{k-1}. Since the pseudo-derivative ψ_k shares the same positive roots of the derivative of ψ_{k-1} and ψ_{k-1} is square-free, ψ_{k-1} and ψ_k have no common positive root in $\mathcal{I}$ by Lemma 4 if φ is simple, and by Corollary 15 otherwise. So positive roots of ψ_{k-1} are separable with those of ψ_k in $\mathcal{I}$. It entails the termination of the refinement process. Then we check all intervals $(b_0^{(k)}, a_1^{(k)}), \ldots, (b_{n^{(k)}}^{(k)}, a_{n^{(k)}+1}^{(k)})$ complement to $[a_1^{(k)}, b_1^{(k)}], \ldots, [a_{n^{(k)}}^{(k)}, b_{n^{(k)}}^{(k)}]$ in $\mathcal{I}$, in each of which ψ_{k-1} is monotonous, and thus isolate all positive roots of ψ_{k-1}.

Repeating this process, we would eventually isolate all positive roots of ψ_0, which are exactly those of the input poly-power φ. □

Now we can see that the differentiation method is terminating for simple poly-powers by Gelfond–Schneider theorem, and is terminating for non-simple ones under Schanuel's conjecture. Decomposition in Section 3 is a profitable way to get potentially simpler factors of the input poly-power, thus rigorously ensures the termination in refinement. This situation is entirely same as the exclusion method in the previous section.

Remark 19. *There are many approaches to define those pseudo-derivatives. For instant, Achatz et al. (2008) defined pseudo-derivative as $\psi_{k+1} = \mathrm{gsfd}(\mathrm{shift}(\psi_k'))$. But it may be not an element of $\mathcal{A}[x^{\upsilon_1}, \ldots, x^{\upsilon_m}]$. Under our definition $\psi_{k+1} = \mathrm{gsfd}(x \cdot \mathrm{shift}^{-1}((\mathrm{shift}(\psi_k))'))$, we can reuse the $\mathbb{Z}$-linearly independent base for exponents of the input poly-power φ for decomposition, as $x \cdot \mathrm{shift}^{-1}((\mathrm{shift}(\psi_k))')$ keeps the same exponents as ψ_k and the coefficients in $x \cdot \mathrm{shift}^{-1}((\mathrm{shift}(\psi_k))')$ are elements in $\mathcal{A}$. The resulting ψ_{k+1} is still an element of $\mathcal{A}[x^{\upsilon_1}, \ldots, x^{\upsilon_m}]$. We do not need to compute any new $\mathbb{Z}$-linearly independent base for exponents of pseudo-derivatives. Additionally, the rational approximations for the base elements $\upsilon_1, \ldots, \upsilon_m$ can be also reused in the refinement process. Hence it improves the efficiency of Algorithm 3.*

Example 20. *Consider the poly-power $f(x)$ in Example 1 again. We invoke Algorithm 3 with* Isolate_D$(f(x), (1, \frac{15}{2}))$. *First we get the greatest square-free divisor of $f(x)$ as*

$$\begin{aligned}
\psi_0 &= \mathrm{gsfd}(f) = \varphi_1 \cdot \varphi_2 \\
&= -10 + \tfrac{5}{3}x - \tfrac{15}{4}x^{\sqrt{3}} + 2\sqrt{5}x^{\sqrt{3}+1} + x^{3\sqrt{2}-1} - \tfrac{\sqrt{5}}{3}x^{\sqrt{3}+2} - \tfrac{2\sqrt{3}}{5}x^{3\sqrt{2}} \\
&\quad + \tfrac{3\sqrt{5}}{4}x^{2\sqrt{3}+1} + \tfrac{\sqrt{3}}{15}x^{3\sqrt{2}+1} + (-\tfrac{3\sqrt{3}}{20} - \tfrac{\sqrt{5}}{5})x^{\sqrt{3}+3\sqrt{2}} + \tfrac{\sqrt{3}}{25}x^{6\sqrt{2}-1} .
\end{aligned}$$

We then construct the sequence of pseudo-derivatives $\psi_1, \ldots, \psi_9$ as follows, until the last one has two terms. Each pseudo-derivative is square-free, and has exponents that can be $\mathbb{Z}^+$-linearly *expressed by the base* $\{1,\ \sqrt{2}-1,\ \sqrt{3}-\sqrt{2}\}$.

$$
\begin{aligned}
\psi_1 = {} & \tfrac{5}{3}x - \tfrac{15\sqrt{3}}{4}x^{\sqrt{3}} + (2\sqrt{15}+2\sqrt{5})x^{\sqrt{3}+1} + (3\sqrt{2}-1)x^{3\sqrt{2}-1} + (-\tfrac{\sqrt{15}}{3} - \tfrac{2\sqrt{5}}{3})x^{\sqrt{3}+2} \\
& - \tfrac{6\sqrt{6}}{5}x^{3\sqrt{2}} + (\tfrac{3\sqrt{15}}{2} + \tfrac{3\sqrt{5}}{4})x^{2\sqrt{3}+1} + (\tfrac{\sqrt{6}}{5} + \tfrac{\sqrt{3}}{15})x^{3\sqrt{2}+1} \\
& + (-\tfrac{\sqrt{15}}{5} - \tfrac{3\sqrt{10}}{5} - \tfrac{9\sqrt{6}}{20} - \tfrac{9}{20})x^{\sqrt{3}+3\sqrt{2}} + (\tfrac{6\sqrt{6}}{25} - \tfrac{\sqrt{3}}{25})x^{6\sqrt{2}-1} \quad , \\
\psi_2 = {} & (\tfrac{15\sqrt{3}}{4} - \tfrac{45}{4})x^{\sqrt{3}} + (2\sqrt{15}+6\sqrt{5})x^{\sqrt{3}+1} + (-9\sqrt{2}+20)x^{3\sqrt{2}-1} + (-\sqrt{15} - \tfrac{5\sqrt{5}}{3})x^{\sqrt{3}+2} \\
& + (\tfrac{6\sqrt{6}}{5} - \tfrac{36\sqrt{3}}{5})x^{3\sqrt{2}} + (\tfrac{3\sqrt{15}}{2} + 9\sqrt{5})x^{2\sqrt{3}+1} + (\tfrac{\sqrt{6}}{5} + \tfrac{6\sqrt{3}}{5})x^{3\sqrt{2}+1} \\
& + (-\tfrac{6\sqrt{30}}{5} + \tfrac{\sqrt{15}}{5} + \tfrac{3\sqrt{10}}{5} + \tfrac{9\sqrt{6}}{20} - \tfrac{21\sqrt{5}}{5} - \tfrac{63\sqrt{3}}{20} - \tfrac{27\sqrt{2}}{10} + \tfrac{9}{20})x^{\sqrt{3}+3\sqrt{2}} \\
& + (-\tfrac{18\sqrt{6}}{25} + \tfrac{74\sqrt{3}}{25})x^{6\sqrt{2}-1} \quad , \\
\psi_3 = {} & (2\sqrt{15}+6\sqrt{5})x^{\sqrt{3}+1} + (9\sqrt{6} - 20\sqrt{3} + 69\sqrt{2} - 74)x^{3\sqrt{2}-1} + (-2\sqrt{15} - \tfrac{10\sqrt{5}}{3})x^{\sqrt{3}+2} \\
& + (-\tfrac{108\sqrt{6}}{5} + \tfrac{36\sqrt{3}}{5} - \tfrac{18\sqrt{2}}{5} + \tfrac{108}{5})x^{3\sqrt{2}} + (\tfrac{21\sqrt{15}}{2} + \tfrac{27\sqrt{5}}{2})x^{2\sqrt{3}+1} \\
& + (\tfrac{19\sqrt{6}}{5} + \tfrac{12\sqrt{3}}{5} - \tfrac{3\sqrt{2}}{5} - \tfrac{18}{5})x^{3\sqrt{2}+1} \\
& + (\tfrac{3\sqrt{30}}{5} - \tfrac{36\sqrt{15}}{5} - \tfrac{63\sqrt{10}}{5} - \tfrac{189\sqrt{6}}{20} + \tfrac{18\sqrt{5}}{5} + \tfrac{27\sqrt{3}}{10} + \tfrac{27\sqrt{2}}{20} - \tfrac{81}{5})x^{\sqrt{3}+3\sqrt{2}} \\
& + (\tfrac{462\sqrt{6}}{25} - \tfrac{58\sqrt{3}}{5} + \tfrac{54\sqrt{2}}{25} - \tfrac{222}{25})x^{6\sqrt{2}-1} \quad , \\
\psi_4 = {} & (-147\sqrt{6} + 168\sqrt{3} - 387\sqrt{2} + 622)x^{3\sqrt{2}-1} + (-2\sqrt{15} - \tfrac{10\sqrt{5}}{3})x^{\sqrt{3}+2} \\
& + (\tfrac{234\sqrt{6}}{5} - \tfrac{792\sqrt{3}}{5} + \tfrac{666\sqrt{2}}{5} - \tfrac{324}{5})x^{3\sqrt{2}} + (\tfrac{27\sqrt{15}}{2} + \tfrac{63\sqrt{5}}{2})x^{2\sqrt{3}+1} \\
& + (\tfrac{39\sqrt{6}}{5} + \tfrac{132\sqrt{3}}{5} - \tfrac{111\sqrt{2}}{5} - \tfrac{54}{5})x^{3\sqrt{2}+1} \\
& + (-\tfrac{111\sqrt{30}}{5} + \tfrac{54\sqrt{15}}{5} + \tfrac{117\sqrt{10}}{5} + \tfrac{351\sqrt{6}}{20} - \tfrac{396\sqrt{5}}{5} - \tfrac{297\sqrt{3}}{5} - \tfrac{999\sqrt{2}}{20} + \tfrac{243}{10})x^{\sqrt{3}+3\sqrt{2}} \\
& + (-\tfrac{2718\sqrt{6}}{25} + \tfrac{6346\sqrt{3}}{25} - \tfrac{2826\sqrt{2}}{25} + \tfrac{1962}{25})x^{6\sqrt{2}-1} \quad , \\
\psi_5 = {} & (6\sqrt{30} - \tfrac{28}{3}\sqrt{15} + 10\sqrt{10} - 16\sqrt{5})x^{\sqrt{3}+2} + (\tfrac{234\sqrt{6}}{5} - \tfrac{792\sqrt{3}}{5} + \tfrac{666\sqrt{2}}{5} - \tfrac{324}{5})x^{3\sqrt{2}} \\
& + (-\tfrac{81\sqrt{30}}{2} + 90\sqrt{15} - \tfrac{189\sqrt{10}}{2} + 144\sqrt{5})x^{2\sqrt{3}+1} + (\tfrac{78\sqrt{6}}{5} + \tfrac{264\sqrt{3}}{5} - \tfrac{222\sqrt{2}}{5} - \tfrac{108}{5})x^{3\sqrt{2}+1} \\
& + (\tfrac{6\sqrt{30}}{5} - \tfrac{342\sqrt{15}}{5} - \tfrac{216\sqrt{10}}{5} - \tfrac{162\sqrt{6}}{5} - \tfrac{234\sqrt{5}}{5} - \tfrac{351\sqrt{3}}{10} + \tfrac{27\sqrt{2}}{10} - \tfrac{1539}{10})x^{\sqrt{3}+3\sqrt{2}} \\
& + (\tfrac{19038\sqrt{6}}{25} - \tfrac{16308\sqrt{3}}{25} + \tfrac{5886\sqrt{2}}{25} - \tfrac{16956}{25})x^{6\sqrt{2}-1} \quad , \\
\psi_6 = {} & (-702\sqrt{6} + \tfrac{3312\sqrt{3}}{5} - \tfrac{3006\sqrt{2}}{5} + 1404)x^{3\sqrt{2}} + (-54\sqrt{30} + 54\sqrt{15} - 27\sqrt{10} + 126\sqrt{5})x^{2\sqrt{3}+1} \\
& + (\tfrac{936\sqrt{6}}{5} + \tfrac{312\sqrt{3}}{5} - \tfrac{336\sqrt{2}}{5} - \tfrac{2016}{5})x^{3\sqrt{2}+1} \\
& + (-\tfrac{1038\sqrt{30}}{5} + 144\sqrt{15} - 54\sqrt{10} - \tfrac{81\sqrt{6}}{2} - \tfrac{828\sqrt{5}}{5} - \tfrac{621\sqrt{3}}{5} - \tfrac{4671\sqrt{2}}{10} + 324)x^{\sqrt{3}+3\sqrt{2}} \\
& + (-\tfrac{160848\sqrt{6}}{25} + \tfrac{294336\sqrt{3}}{25} - \tfrac{176508\sqrt{2}}{25} + \tfrac{170424}{25})x^{6\sqrt{2}-1} \quad , \\
\psi_7 = {} & (-270\sqrt{30} + 630\sqrt{15} - 729\sqrt{10} + 612\sqrt{5})x^{2\sqrt{3}+1} + (\tfrac{936\sqrt{6}}{5} + \tfrac{312\sqrt{3}}{5} - \tfrac{336\sqrt{2}}{5} - \tfrac{2016}{5})x^{3\sqrt{2}+1} \\
& + (-54\sqrt{30} - \tfrac{828\sqrt{15}}{5} - \tfrac{3114\sqrt{10}}{5} - \tfrac{4671\sqrt{6}}{10} + 432\sqrt{5} + 324\sqrt{3} - \tfrac{243\sqrt{2}}{2} - \tfrac{1863}{5})x^{\sqrt{3}+3\sqrt{2}} \\
& + (\tfrac{1043856\sqrt{6}}{25} - \tfrac{1259424\sqrt{3}}{25} + \tfrac{137556\sqrt{2}}{5} - \tfrac{1229472}{25})x^{6\sqrt{2}-1} \quad ,
\end{aligned}
$$

$$
\begin{aligned}
\psi_8 = {} & \left(\tfrac{1608\sqrt{6}}{5} + \tfrac{9648\sqrt{3}}{5} - \tfrac{11664\sqrt{2}}{5} - \tfrac{3888}{5}\right)x^{3\sqrt{2}+1} \\
& + \left(180\sqrt{30} - \tfrac{2952\sqrt{15}}{5} + \tfrac{10404\sqrt{10}}{5} + \tfrac{7803\sqrt{6}}{5} - 3672\sqrt{5} - 2754\sqrt{3} + 405\sqrt{2} - \tfrac{6642}{5}\right)x^{\sqrt{3}+3\sqrt{2}} \\
& + \left(-\tfrac{11019816\sqrt{6}}{25} + \tfrac{17504064\sqrt{3}}{25} - \tfrac{15015528\sqrt{2}}{25} + \tfrac{18268848}{25}\right)x^{6\sqrt{2}-1}\ , \\
\psi_9 = {} & \left(\tfrac{9504\sqrt{30}}{5} - \tfrac{15408\sqrt{15}}{5} - \tfrac{7704\sqrt{10}}{5} - \tfrac{5778\sqrt{6}}{5} + \tfrac{9504\sqrt{5}}{5} + \tfrac{7128\sqrt{3}}{5} + \tfrac{21384\sqrt{2}}{5} - \tfrac{34668}{5}\right)x^{\sqrt{3}+3\sqrt{2}} \\
& + \left(\tfrac{74551824\sqrt{6}}{25} - \tfrac{101127024\sqrt{3}}{25} + 3393504\sqrt{2} - \tfrac{126630864}{25}\right)x^{6\sqrt{2}-1}\ .
\end{aligned}
$$

For the last pseudo-derivative ψ_9, it has no positive root in $(1, \frac{15}{2})$. We proceed to isolate positive roots of ψ_8, ψ_7, ψ_6: ψ_8, ψ_7 have no positive root in $(1, \frac{15}{2})$ while ψ_6 has unique one. The whole lifting process (i. e. Lines 13–29 of Algorithm 3) is shown in Table 3, where

- *'$+$' means* $\inf \mathrm{Rg}^+ > 0$,
- *'$-$' means* $\sup \mathrm{Rg}^+ < 0$,
- *'$\star$' means the interval contains exactly one positive root,*
- *and '$\rightsquigarrow$' denotes the refinement on the isolation interval.*

For other isolation methods, we have considered the one based on the sign rule of Descartes (Collins and Akritas, 1976), the one based on Budan–Fourier theorem (Coste et al., 2005), and the one based on Sturm's theorem (Thomas, 1941). All encounter technical bottlenecks.

1. For the first, it requires to regularly compute the shifted function $\varphi(x + c)$ for a poly-power $\varphi(x)$ and a rational number c, which would result in infinitely many terms and transcendental coefficients in $\varphi(x + c)$. Thus $\varphi(x + c)$ may be no longer in the scope of poly-powers, and bring new challenges in tackling multiple positive roots.
2. For the second, it requires to know the minimum distance (see Theorem 5 of Mignotte (1983)) between distinct positive roots of poly-powers, without which the termination of the bisection process is not ensured. To compute minimum distance, it seems that one has to develop the classical theory of *resultant* (see Section 5.8 of van der Waerden (1991)).
3. For the last, it requires to compute the remainder of two poly-powers, which is undefined, since exponents in poly-powers are not well-founded nonnegative integers then.

6. Experiments

In this section we implement the two proposed methods for isolating positive roots of poly-powers with the `Maple` platform. The prototype is demonstrated in Figure 3.

Note that the rational approximations for $\mathbb{R}_{\mathrm{alg}}$-elements in the base (6) will be widely used for constructing the approximations (8), (9) and (10). Therefore it is profitable to store those approximations of many different precision once in a table as a pretreatment (see the bolded label in Figure 3). When they are not good enough in the execution process, we would sent a request for better approximations.

The experiments have been conducted on a 64-bit Windows computer with a 2.20 GHz Intel Core i7 processor and 16 GB of RAMs. For an input poly-power we first factorize it by Algorithm 1. The factorization has been well implemented by existing materials, e. g. Loos (1983).

Derivative	Interval	$\pm/\star$
ψ_9	$(1, \frac{15}{2})$	$+$
ψ_8	$(1, \frac{15}{2})$	$+$
ψ_7	$(1, \frac{15}{2})$	$+$
ψ_6	$(a_1^{(6)}, b_1^{(6)}) = (1, \frac{15}{2}) \rightsquigarrow (\frac{269}{256}, \frac{141}{128})$	$\star$
ψ_5	$(1, \frac{269}{256})$	$-$
	$(a_1^{(5)}, b_1^{(5)}) = (\frac{141}{128}, \frac{15}{2}) \rightsquigarrow (\frac{11481}{8192}, \frac{3075}{2048})$	$\star$
ψ_4	$(1, \frac{11481}{8192})$	$-$
	$(a_1^{(4)}, b_1^{(4)}) = (\frac{3075}{2048}, \frac{15}{2}) \rightsquigarrow (\frac{61485}{32768}, \frac{504165}{262144})$	$\star$
ψ_3	$(a_1^{(3)}, b_1^{(3)}) = (1, \frac{61485}{32768}) \rightsquigarrow (\frac{1077293}{1048576}, \frac{553005}{524288})$	$\star$
	$(a_2^{(3)}, b_2^{(3)}) = (\frac{504165}{262144}, \frac{15}{2}) \rightsquigarrow (\frac{156842625}{67108864}, \frac{39576135}{16777216})$	$\star$
ψ_2	$(1, \frac{1077293}{1048576})$	$+$
	$(a_1^{(2)}, b_1^{(2)}) = (\frac{553005}{524288}, \frac{156842625}{67108864}) \rightsquigarrow (\frac{824451075}{536870912}, \frac{1734960135}{1073741824})$	$\star$
	$(a_2^{(2)}, b_2^{(2)}) = (\frac{39576135}{16777216}, \frac{15}{2}) \rightsquigarrow (\frac{23454341565}{8589934592}, \frac{11770297275}{4294967296})$	$\star$
ψ_1	$(1, \frac{824451075}{536870912})$	$+$
	$(a_1^{(1)}, b_1^{(1)}) = (\frac{1734960135}{1073741824}, \frac{23454341565}{8589934592}) \rightsquigarrow (\frac{37334022645}{17179869184}, \frac{308246841645}{137438953472})$	$\star$
	$(a_2^{(1)}, b_2^{(1)}) = (\frac{11770297275}{4294967296}, \frac{15}{2}) \rightsquigarrow (\frac{208766713845}{68719476736}, \frac{6700976800485}{2199023255552})$	$\star$
ψ_0	$(a_1^{(0)}, b_1^{(0)}) = (1, \frac{37334022645}{17179869184})$	$\star$
	$(a_2^{(0)}, b_2^{(0)}) = (\frac{308246841645}{137438953472}, \frac{208766713845}{68719476736})$	$\star$
	$(a_3^{(0)}, b_3^{(0)}) = (\frac{6700976800485}{2199023255552}, \frac{15}{2})$	$\star$

Table 3: The Lifting Process

So our experiments only focus on the irreducible poly-powers. We experiment with irreducible poly-powers of i) different numbers of terms and ii) different sizes of base (listed below).

$$f_1(x) = 5 - \sqrt{5}x^{\sqrt{3}+1} + \frac{\sqrt{3}}{5}x^{3\sqrt{2}} \;,$$
$$f_2(x) = -2 + \frac{1}{3}x - \frac{3}{4}x^{\sqrt{3}} + \frac{1}{5}x^{3\sqrt{2}-1} \;,$$
$$f_3(x) = 1 + x^2 - 5x^{\sqrt{7}} + \frac{1}{2}x^{3\sqrt{7}+1} \;,$$
$$f_4(x) = 2 - 5x^{\sqrt{7}} - x^{\sqrt{3}+\sqrt{7}} + \frac{1}{2}x^{3\sqrt{2}+1} \;,$$
$$f_5(x) = -2 + 2\sqrt{7}x - 3x^{\sqrt{7}} - x^{\sqrt{3}+\sqrt{7}} + \frac{1}{2}x^{3\sqrt{2}+1} \;,$$
$$f_6(x) = -1 - 60x^{\sqrt{\sqrt{7}+1}} + 22\sqrt{7}x^{\sqrt{3}} + x^{2\sqrt[3]{7}} + x^{3\sqrt[5]{4}} \;,$$
$$f_7(x) = -1 - x^{\sqrt[7]{3}} + 40\sqrt{2}x^{\sqrt{3}} - 60x^{\sqrt{\sqrt{7}+1}} + x^{\sqrt{7}} + x^{2\sqrt[3]{7}} + \frac{7}{2}x^{3\sqrt[5]{4}} - x^5 \;,$$
$$f_8(x) = 10 - x^{\sqrt[7]{3}} + 40\sqrt{2}x^{\sqrt{3}} - 60x^{\sqrt{\sqrt{7}+1}} + x^{\sqrt{7}} + x^{2\sqrt[3]{7}} + \frac{7}{2}x^{3\sqrt[5]{4}} - 2x^5 \;,$$

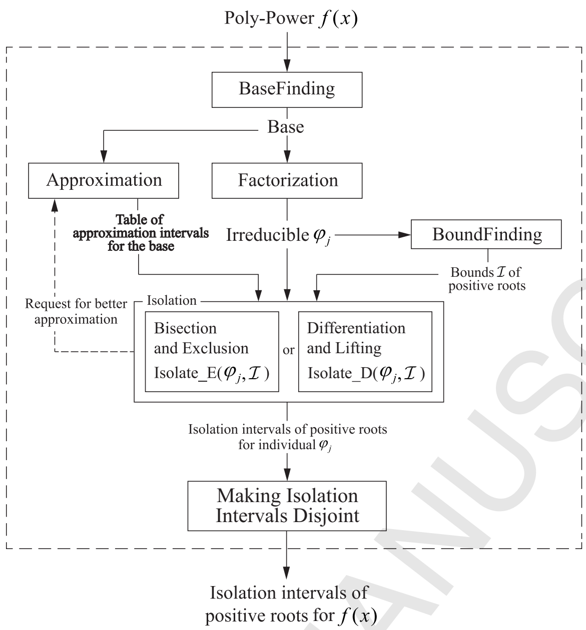


Figure 3: The Implementation Prototype

$$f_9(x) = \tfrac{1}{10} - 5\sqrt{2}x^{\sqrt{3}} + x^{\frac{6}{7}\sqrt{2}} - 2x^{\sqrt{2}+\frac{1}{2}} + 20x^{\sqrt{7}} - 2x^{2\sqrt{3}} - 7x^{2\sqrt{2}+1} - \tfrac{7}{2}x^{\sqrt{2}+\sqrt{3}+1} + 2x^5 - \tfrac{5}{7}x^{3\sqrt{3}} ,$$

$$f_{10}(x) = 20 - x^{\sqrt[7]{3}} + \sqrt{2}x^{\sqrt{3}} + 2x^{\sqrt{\sqrt{7}+1}} - 20x^{\sqrt{7}} + 2x^{2\sqrt[3]{6}} + x^{2\sqrt[3]{7}} + \tfrac{7}{2}x^{3\sqrt[5]{4}} - 3x^5 + 2x^{3\sqrt{3}} .$$

The experimental results are shown in Table 4, in which the columns give the features of the input poly-powers:

- the number of terms (Terms),
- the size of the base for exponents and coefficients (Base),
- the number of distinct positive roots (Roots),

and the analytical indicators for the two methods Isolate_E and Isolate_D:

- the running time (in seconds) for isolating positive roots (Time),
- the maximal bisection depth (Depth).

From the experimental results, we can seen that both two methods can solve these instances within one second. The differentiation method is relatively efficient. Most of their computational amounts are spent in bisecting intervals and computing approximations over subintervals.

Specifically, the exclusion method has to search the whole interval $\mathcal{I}$ within the upper and lower bounds for positive roots. The precision of the over-approximation (8) is sensitive to the length of $\mathcal{I}$. If the resulting interval of the approximation is not decisive, bisecting $\mathcal{I}$ would be

poly-power				Isolate_E		Isolate_D	
No.	Terms	Base	Roots	Time	Depth	Time	Depth
1	3	4	2	0.152	6	0.028	7
2	4	3	1	0.036	5	0.035	5
3	4	2	2	0.194	6	0.034	6
4	4	4	2	0.217	6	0.046	7
5	5	4	3	0.410	7	0.068	5
6	5	6	3	0.428	11	0.072	6
7	8	8	4	0.511	11	0.059	8
8	8	8	1	0.113	6	0.030	7
9	10	4	5	0.791	13	0.122	10
10	10	9	2	0.147	6	0.037	7

Table 4: Experimental Results

required. After each bisection, we need to put two subintervals into consideration. In the worst case, the number of all involved subintervals is exponential w.r.t. the maximal bisection depth.

However, the differentiation method only refines isolation intervals for pseudo-derivatives. A poly-power in the form (1) has at most n distinct positive roots, and all its pseudo-derivatives have at most $(n-1)+\cdots+1$ distinct positive roots. Totally there are at most $\frac{n(n-1)}{2}$ isolation intervals to be refined. After each bisection, only one subinterval (containing the positive root) is remained for further consideration. In the worst case, the number of all involved subintervals is linear w.r.t. the maximal bisection depth.

The maximal bisection depth has no obvious disparity between two methods. Hence we suggest that the differentiation method is suitable for sparse poly-powers, while the exclusion method is suitable for poly-powers within small root bounds.

7. Application

To show the practical usefulness, we apply the positive root isolation of poly-powers to a case study in Systems Biology, nutrient flow in an aquarium (see (Gustafson, 1998, pp. 589–590)). It is described as follows.

> *Consider a vessel of water containing a radioactive isotope, to be used as a tracer for the food chain, which consists of aquatic plankton varieties—phytoplankton and zooplankton. Let*
>
> - $\xi_1(t)$ *be isotope concentration in the water,*
> - $\xi_2(t)$ *be isotope concentration in phytoplankton,*
> - *and* $\xi_3(t)$ *be isotope concentration in zooplankton.*
>
> *Typically, the system of differential equations is*

$$\begin{pmatrix}\xi_1'(t)\\ \xi_2'(t)\\ \xi_3'(t)\end{pmatrix} = \begin{pmatrix}-3 & 6 & 5\\ 2 & -12 & 0\\ 1 & 6 & -5\end{pmatrix}\begin{pmatrix}\xi_1(t)\\ \xi_2(t)\\ \xi_3(t)\end{pmatrix}. \tag{11}$$

After solving it with the initial radioactive isotope concentrations $\xi_1(0) = 100, \xi_2(0) = 0, \xi_3(0) = 0$*, we get the solution*

$$\begin{cases} \xi_1(t) = \frac{3000}{47} + (\frac{850}{47} + \frac{725\sqrt{6}}{141})e^{(-10+\sqrt{6})t} + (\frac{850}{47} - \frac{725\sqrt{6}}{141})e^{(-10-\sqrt{6})t} \\ \xi_2(t) = \frac{500}{47} + (-\frac{250}{47} + \frac{1100\sqrt{6}}{141})e^{(-10+\sqrt{6})t} + (-\frac{250}{47} - \frac{1100\sqrt{6}}{141})e^{(-10-\sqrt{6})t} \\ \xi_3(t) = \frac{1200}{47} - (\frac{600}{47} + \frac{1825\sqrt{6}}{141})e^{(-10+\sqrt{6})t} + (-\frac{600}{47} + \frac{1825\sqrt{6}}{141})e^{(-10-\sqrt{6})t} \end{cases} .$$

The systems biological problem we studied is "when the isotope concentration in phytoplankton is exactly the square root of eight times that in zooplankton after the isotope concentration in water reaches its quarter-life", i. e.

$$(\xi_2(t))^2 = 8\xi_3(t) \wedge \xi_1(t) < 75 \ .$$

With the variable change $x = e^t$, we reduce the systems biological problem to the positive root isolation of simple poly-powers (after a proper normalization):

$$\begin{aligned} \phi_1 &= e^{(20+2\sqrt{6})t} \cdot [\xi_2^2 - 8\xi_3] \\ &= (\frac{550000\sqrt{6}}{6627} + \frac{2607500}{6627}) - \frac{95000}{141}x^{2\sqrt{6}} + (-\frac{550000\sqrt{6}}{6627} + \frac{2607500}{6627})x^{4\sqrt{6}} \\ &\quad + (-\frac{595400\sqrt{6}}{2209} - \frac{24400}{2209})x^{10+\sqrt{6}} + (\frac{595400\sqrt{6}}{2209} - \frac{24400}{2209})x^{10+3\sqrt{6}} - \frac{201200}{2209}x^{20+2\sqrt{6}} \ , \\ \phi_2 &= e^{(10+\sqrt{6})t} \cdot [\xi_1 - 75] \\ &= (\frac{850}{47} - \frac{725\sqrt{6}}{141}) + (\frac{850}{47} + \frac{725\sqrt{6}}{141})x^{2\sqrt{6}} - \frac{525}{47}x^{10+\sqrt{6}} \ . \end{aligned}$$

By invoking Algorithm 2, we get two isolation intervals $(\frac{2647}{2560}, \frac{829}{800})$ and $(\frac{7763}{6400}, \frac{487}{400})$ for ϕ_1's positive roots (greater than 1), and one isolation interval $(\frac{217}{200}, \frac{117}{100})$ for ϕ_2's. So the desired critical time t lies in $(0.193, 0.197) \supset (\ln\frac{7763}{6400}, \ln\frac{487}{400})$, which can be easily refined to any desired precision.

Through the example, we can see that for the linear dynamical system

$$\vec{\xi}'(t) = \mathbf{A} \cdot \vec{\xi}(t) \ , \tag{12}$$

if the rational matrix $\mathbf{A}$ has simple real eigenvalues λ_i only, its solution can be expressed as exponential polynomials $\beta_0 + \beta_1 e^{\alpha_1 t} + \cdots + \beta_n e^{\alpha_n t}$, where all α_i and all β_i are $\mathbb{R}_{\text{alg}}$-numbers (Kailath, 1980), and can be reformulated as poly-powers in $x = e^t$. So the proposed methods in this paper can be properly used to study the behavior of such linear dynamic systems. As far as we know, it is unreported before, saying in the existing materials, i. e. Anai and Weispfenning (2001); Lafferriere et al. (2001); Xu et al. (2010a, 2013).

8. Conclusion

In this paper, we presented two methods to isolate positive roots of poly-powers. They are complete for simple poly-powers, and are "conditionally" complete (refer to Schanuel's conjecture) for non-simple ones. Thereby we made full use of a number-theoretic roadmap from the solvability to its known frontier. An immediate application of our work was shown in the reachability problem for linear dynamical systems.

For further work, we are interested in the minimum distance estimation of distinct positive roots of poly-powers, which would yield another isolation method. Besides, implementing the proposed methods with binary arithmetic would bring a chance to analyze complexity.

Acknowledgments

The authors thank Zhenbing Zeng for their insightful suggestions on this work. The authors are also grateful to the reviewers whose careful comments helped us to improve the accurateness. This work is funded by NSFC (No. 11401218 & No. 11435005), CSC (No. 201606145008), the Open Project of Shanghai Key Laboratory of Trustworthy Computing (No. 07dz22304201403), and the Science and Technology Commission of Shanghai Municipality (No. 14DZ2260800).

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