TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 327, Number 1, September 1991

CONTROL OF DEGENERATE DIFFUSIONS IN $\mathbf{R}^d$

OMAR HIJAB

ABSTRACT. An optimal regularity result is established for the viscosity solution of the degenerate elliptic equation

$$-Av + F(x, v, Dv) = 0,$$

$A = \frac{1}{2}\sum a_{ij}(x)\partial^2/\partial x_i\partial x_j$, $x \in \mathbf{R}^d$. We assume the equation is of *Bellman type*, i.e. $F(x, v, p) = \sup_{u\in U}[b(x, u)\cdot p + c(x, u)v - f(x, u)]$, $U \subset \mathbf{R}^d$. If we set $\lambda \equiv \inf_{x,u} c(x, u)$, then there exists $\lambda_0 \geq 0$ such that $0 < \lambda < \lambda_0$ implies v is Hölder, while $\lambda > \lambda_0$ implies v is Lipschitz. The following is established: Suppose the equation is also of *Lipschitz type*, i.e. suppose there is a Lipschitz function $u(x, v, p)$ such that the supremum in $F(x, v, p)$ is uniquely attained at $u = u(x, v, p)$; then there exists $\lambda_1 > \lambda_0$ such that $\lambda > \lambda_1$ implies v is $C^{1,1}$, i.e. Dv exists and is Lipschitz.

0. INTRODUCTION

Recall that a degenerate quasilinear equation

$$-Av + F(x, v, Dv) = 0, \qquad x \in \mathbf{R}^d, \tag{0.1}$$

$$A = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}, \tag{0.2}$$

is of *Bellman type* if

$$F(x, v, p) = \sup_{u\in U}(b(x, u)\cdot p + c(x, u)v - f(x, u))$$

for some parameter set $U \subset \mathbf{R}^d$ and smooth functions a, b, c, and f.

In this paper we are interested in the regularity of the bounded viscosity solution v of (0.1). Set $\lambda \equiv \inf_{x,u} c(x, u)$.

The following is known for appropriate $\lambda_0 \geq 0$ and $\lambda > \lambda_0$: The viscosity solution v of (0.1) exists and is unique in $C_b(\mathbf{R}^d)$ [8]. Moreover v is Lipschitz and semiconcave (see §1) and $Av \in L^\infty(\mathbf{R}^d)$ in the sense of Schwartz

Received by the editors May 1, 1989 and, in revised form, February 10, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 60G35, 49A10, 93E11, 93E20.
Key words and phrases. Nonlinear partial differential equation, diffusion, control, dynamic programming.
Supported by the National Science Foundation.

distributions [7]. In particular if a is nondegenerate, this together with the semiconcavity implies [7] the classical regularity result $v \in W^{2,\infty}(\mathbf{R}^d)$ [3, 5]. For a degenerate, there are additional results concerning the regularity of v in directions of nondegeneracy of a [5, 6].

The key to establishing these results is the variational representation of v as the value function

$$v(x) = \inf\{v^u(x) : \text{ all controls } u\}$$

of the control problem with dynamics

$$dx = -b(x, u)dt + \sigma(x)dw, \qquad x(0) = x_0 \in \mathbf{R}^d, \tag{0.3}$$

and cost criterion

$$v^u(x_0) = E\left(\int_0^\infty e^{-\int_0^t c(x(s), u(s))ds} f(x(t), u(t))\, dt\right). \tag{0.4}$$

Here w is a Brownian motion and σ satisfies $\sigma\sigma^* = a$.

Recall that the Bellman equation (0.1) is of *Lipschitz type* if the supremum in the nonlinearity F is uniquely attained at $u = \mathbf{u}(x, v, p)$, where $\mathbf{u}$ is Lipschitz in all variables.

When (0.1) is of Lipschitz type, it is known that, *as a consequence of* $v \in W^{2,\infty}(\mathbf{R}^d)$, the following holds [3] (see also §3). For each starting state x there exists a unique optimal control u ($v^u(x) = v(x)$), and optimal controls are characterized by the feedback (fixed point formula)

$$u(t) = \mathbf{u}(x(t), v(x(t)), Dv(x(t))), \qquad t \geq 0. \tag{0.5}$$

In the totally degenerate case, however, the existence of optimal controls, as functionals of the driving Brownian motion, let alone their characterization, is not as yet established, as v is not necessarily in $W^{2,\infty}(\mathbf{R}^d)$ in general.

The cut-off "discount-factor" λ_0 is necessary to assert the existence of $Dv^u(x)$ and $D^2v^u(x)$ for each fixed control u. In the particular case that b, σ do not depend on x, $\lambda_0 = 0$ [7].

In this paper we show that when (0.1) is of Lipschitz type and λ is sufficiently large, $\lambda > \lambda_1 > \lambda_0$, the above issues can be resolved, even when a is totally degenerate. In the simplest cases where b, σ do not depend on x and $\lambda_0 = 0$, λ_1 is related to the size of the Lipschitz constant of $\mathbf{u}$.

Assume (0.1) is of Lipschitz type. We establish (precise assumptions in §1), for λ sufficiently large,

(1) $v \in W^{2,\infty}(\mathbf{R}^d)$, i.e. $v \in C_b^1(\mathbf{R}^d)$ and Dv is Lipschitz;
(2) for each x there exists a control optimal at x;
(3) for each x there exists exactly one such control;
(4) for each x a control u is optimal at x iff the feedback (0.5) holds.

In particular, by choosing $a \equiv 0$, we see that the regularity result (1) holds for first-order Hamilton-Jacobi equations with nonlinearities of Bellman-Lipschitz type.

The above result also applies to the equation

$$-Av + H(x, Dv) + \lambda v = 0, \qquad x \in \mathbf{R}^d, \tag{0.6}$$

where $H(x, p) = \sup_{u\in U}(u \cdot p - L(x, u))$, L is strictly convex in u, $L_{uu} > 0$, and $U \subset \mathbf{R}^d$ is compact convex. Here the supremum is attained at $\mathbf{u}(x, v, p) = H_p(x, p)$; in this case the fact that $\mathbf{u}$ is Lipschitz is well known convexity theory and is recalled in the Appendix.

The regularity result (1) is best possible as a simple one-dimensional example shows (§5).

By introducing generalized (weak sense) controls, it is known [2, 4], that for each starting x there exists a generalized control u_x optimal at x, which can moreover be chosen in such a way that the family $x \mapsto u_x$ is Markovian. However the fact that for each x the optimal control u_x is Markovian in the sense (0.5) has not been established. As a consequence of the techniques in this paper, it follows that, for large λ, (0.5) characterizes optimal generalized controls as well. We do not formulate this result as we have no need here to go outside the category of (strong sense) controls.

In the control of diffusions under partial observations, an analogous infinite-dimensional control problem arises; there the state space is the space of probability measures $M(\mathbf{R}^d)$. As the techniques of this paper are purely probabilistic, and thus are not a priori restricted to finite dimensions, results analogous to (1)–(4) above are expected to hold. Of course (1) implies that D^2v exists almost everywhere on $\mathbf{R}^d$ and (0.1) holds almost everywhere. This result is not listed above as it is not probabilistic and hence cannot be immediately formulated in infinite dimensions. This work on partially observed control will appear elsewhere.

In §5 we give two proofs of the main result (1), keeping in mind that in infinite dimensions one technique may be more tractable than the other.

1. Dynamic programming

Throughout $|\cdot|$ denotes the euclidean norm of vectors in $\mathbf{R}^d$, $\|\cdot\|$ denotes the euclidean norm of matrices, $\|M\|^2 = \operatorname{trace}(M^*M)$.

Throughout $W^{2,\infty}(\mathbf{R}^d)$ denotes the Sobolev space of functions $v \in L^\infty(\mathbf{R}^d)$ whose first and second distributional derivatives Dv, D^2v are in $L^\infty(\mathbf{R}^d)$. Set

$$\|v\|_{W^{2,\infty}} = \operatorname*{ess\,sup}_{x\in\mathbf{R}^d}(|v(x)| + |Dv(x)| + \|D^2v(x)\|).$$

Let $C_b^{1,1}(\mathbf{R}^d)$ denote the space of differentiable functions v such that v, Dv are bounded on $\mathbf{R}^d$ and Dv is Lipschitz on $\mathbf{R}^d$. Set

$$\|v\|_{C^{1,1}} = \sup_{\substack{x, x'\in\mathbf{R}^d \\ x\neq x'}} \left(|v(x)| + |Dv(x)| + \frac{|Dv(x) - Dv(x')|}{|x - x'|}\right).$$

Since a function $f \in L^\infty(\mathbf{R}^d)$ has a distributional derivative Df in $L^\infty(\mathbf{R}^d)$ iff f is Lipschitz, $W^{2,\infty}(\mathbf{R}^d)$ and $C_b^{1,1}(\mathbf{R}^d)$ are isometric under the above norms.

Let $\Omega = \Omega^m = C_0([0, \infty); \mathbf{R}^m)$ be the set of continuous paths in $\mathbf{R}^m$ *starting from the origin*. Let $w : [0, \infty) \times \Omega \to \mathbf{R}^m$ denote the canonical map and let $\mathscr{F}_t = \sigma[w(s), 0 \le s \le t]$, $\mathscr{F} = \sigma[w(t), t \ge 0]$. Let W denote Wiener measure on $(\Omega, \mathscr{F})$. Then w is a $(\Omega, \mathscr{F}_t, W)$ Brownian motion.

We note that the σ-fields $\mathscr{F}_t$, $\mathscr{F}$ are not completed nor is the corresponding filtration made "right-continuous". Background on the relevant diffusion theory is [11, Chapter 4].

Fix a closed convex set $U \subset \mathbf{R}^d$. A *control* is a progressively measurable map $u : [0, \infty) \times \Omega \to U$. Let $\mathscr{C}$ denote the set of controls. We say $u_n \to u$ in $\mathscr{C}$ if $u_n \to u$ in $(dt \times W)$-probability on $[0, T] \times \Omega$ for all $T > 0$. With this topology, $\mathscr{C}$ is separable (Lemma A.1 in the Appendix).

Our assumptions in this paper are the following.

- (0.1) is of Lipschitz type;
- $U \subset \mathbf{R}^d$ is convex closed;
- $\sigma \in C_b^{1,1}(\mathbf{R}^d)$ is $(d \times m)$-matrix-valued;
- $b(\cdot, u)$, $c(\cdot, u)$, $f(\cdot, u)$, $u \in U$, lie in a bounded subset of $C_b^{1,1}(\mathbf{R}^d)$;
- b, c, f, b_x, c_x, and f_x are Lipschitz on $\mathbf{R}^d \times U$.

Throughout $\lambda = \inf_{x,u} c(x, u)$. For the special case of (0.6), these assumptions are implied by

- $U \subset \mathbf{R}^d$ is convex compact;
- $\sigma \in C_b^{1,1}(\mathbf{R}^d)$ is $(d \times m)$-matrix-valued;
- $L(\cdot, u)$, $u \in U$, lies in a bounded subset of $C_b^{1,1}(\mathbf{R}^d)$;
- L, L_x are Lipschitz on $\mathbf{R}^d \times U$;
- $L \in C^1(\mathbf{R}^d \times \mathbf{R}^d)$;
- For some $\varepsilon > 0, C > 0$,

$$\varepsilon |u' - u|^2 \le (L_u(x', u') - L_u(x, u)) \cdot (u' - u) + C|u' - u|\,|x' - x|, \qquad x, x' \in \mathbf{R}^d, \; u, u' \in U.$$

When $L \in C^2$, the above hypotheses on L are equivalent to the boundedness of L, L_x, L_u, L_{xx}, $L_{xu} = L_{ux}^t$, and the positivity $L_{uu} \ge \varepsilon I$ on $\mathbf{R}^d \times U$. The fact that the strict convexity of L implies (0.6) is of Lipschitz type is recalled in the Appendix (Lemmas A.2 and A.3).

To each starting state $x \in \mathbf{R}^d$ and control u corresponds the unique solution $x = x^u(t, \omega; x)$ of

$$x(t) = x - \int_0^t b(x(s), u(s))\, ds + \int_0^t \sigma(x(s))\, dw(s), \qquad t \ge 0. \tag{1.1}$$

Then x^u is an Ito process with coefficients $(a(x(t)), b(x(t), u(t)))$, where $a = \sigma\sigma^*$, and the map $(t, \omega, x, u) \mapsto x^u(t, \omega; x)$, $[0, \infty) \times \Omega \times \mathbf{R}^d \times \mathscr{C} \to \mathbf{R}^d$ can be chosen jointly measurable.

It is well known that the state process $x^u = x^u(t, \omega; x)$ can be chosen differentiable in $x \in \mathbf{R}^d$, $x(\cdot, \omega; \cdot) \in C^{0,1}([0, \infty) \times \mathbf{R}^d)$, for a.a.-$\omega$. In particular if $X(t) = X^u(t, \omega; x)$ denotes the $(d \times d)$-matrix-valued process obtained by differentiating $x^u(t, \omega; x)$ with respect to the initial state x then X is uniquely determined by

$$X(t) = I - \int_0^t b_x(x(s), u(s))X(s)\, ds + \int_0^t \sigma_x(x(s))X(s)\, dw(s), \qquad t \geq 0. \tag{1.2}$$

Here $\sigma_x X dw = \sigma_x^1 X dw_1 + \cdots + \sigma_x^m X dw_m$, where σ_x^k is the $d \times d$ matrix with (i, j)th entry $\partial\sigma_{ik}/\partial x_j$.

The cost $v^u(x)$ and the value function $v(x)$ are as in §0.

Lemma 1.1. *There is a constant $\lambda_0 \geq 0$ such that v^u is differentiable on $\mathbf{R}^d$ for $\lambda > \lambda_0$ and all controls u $(c = c(x, u))$,*

$$Dv^u(x) = E\left(\int_0^\infty e^{-\int_0^t c\,ds}\left[f_x(x, u)X - f(x, u)\int_0^t c_x(x, u)X\,ds\right] dt\right), \tag{1.3}$$

Dv^u is Lipschitz, and $\|v^u\|_{C^{1,1}} \leq C'$. Moreover the maps $(x, u) \mapsto v^u(x)$, $(x, u) \mapsto Dv^u(x)$ are continuous on $\mathbf{R}^d \times \mathscr{C}$.

Proof. By Lemma A.4, $E(\|X(t)\|^2) \leq Ce^{Ct}$, $t \geq 0$. Since f_x, c_x are bounded, differentiation under the integral sign yields (1.3). Thus v^u and Dv^u are bounded. We show Dv^u is Lipschitz.

Given starting states x, x' and a control u let x, x', X, X' denote the corresponding processes. Then $E(|x(t) - x'(t)|^2) \leq e^{Ct}|x - x'|^2$. Set $b = b(x, u)$, $b' = b(x', u)$, $b_x = b_x(x, u)$, $b'_x = b_x(x', u)$, $c = c(x, u)$, $c' = c(x', u)$, $c_x = c_x(x, u)$, $c'_x = c_x(x', u)$, $\sigma = \sigma(x)$, $\sigma' = \sigma(x')$, $\sigma_x = \sigma_x(x)$, $\sigma'_x = \sigma_x(x')$, $f = f(x, u)$, $f' = f(x', u)$, $f_x = f_x(x, u)$, $f'_x = f_x(x', u)$. Now

$$\begin{aligned}
&\frac{1}{6}|Dv^u(x) - Dv^u(x')|^2 \\
&\quad \leq \frac{1}{2}\left\{E\left(\int_0^\infty |e^{-\int_0^t c\,ds} f_x X - e^{-\int_0^t c'\,ds} f'_x X'|\,dt\right)\right\}^2 \\
&\qquad + \frac{1}{3}\left\{E\left(\int_0^\infty \left|e^{-\int_0^t c\,ds} f\int_0^t c_x X\,ds - e^{-\int_0^t c'\,ds} f'\int_0^t c'_x X'\,ds\right|dt\right)\right\}^2 \\
&\quad \leq \left\{E\left(\int_0^\infty |e^{-\int_0^t c\,ds} - e^{-\int_0^t c'\,ds}||f_x X|\,dt\right)\right\}^2 \\
&\qquad + \left\{E\left(\int_0^\infty e^{-\int_0^t c'\,ds}|f_x X - f'_x X'|\,dt\right)\right\}^2
\end{aligned}$$

(continues)

(continued)

$$
\begin{aligned}
&+\left\{E\left(\int_0^\infty |e^{-\int_0^t c\,ds}-e^{-\int_0^t c'\,ds}||f|\left(\int_0^t |c_x X|ds\right)dt\right)\right\}^2\\
&+\left\{E\left(\int_0^\infty e^{-\int_0^t c'\,ds}|f-f'|\left(\int_0^t |c_x X|\,ds\right)dt\right)\right\}^2\\
&+\left\{E\left(\int_0^\infty e^{-\int_0^t c'\,ds}|f'|\left(\int_0^t |c_x X-c_x' X'|\,ds\right)dt\right)\right\}^2\\
&=\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}.
\end{aligned}
$$

Since $c(x\,,\,u)\geq\lambda$ it follows that

$$
|e^{-\int_0^t c\,ds}-e^{-\int_0^t c'\,ds}|\leq e^{-\lambda t}\int_0^t |c-c'|\,ds.
$$

Thus by Cauchy-Schwarz, for λ large,

$$
\begin{aligned}
\mathrm{I}&\leq CE\left(\int_0^\infty e^{-\lambda t}\left(\int_0^t |c-c'|\,ds\right)^2 dt\right)E\left(\int_0^\infty e^{-\lambda t}\|X\|^2\,dt\right)\\
&\leq C\int_0^\infty e^{-\lambda t}t\int_0^t E(|x(s)-x'(s)|^2)ds\,dt\leq C|x-x'|^2.
\end{aligned}
$$

Now

$$
\begin{aligned}
\frac{1}{2}\mathrm{II}\leq &\left\{E\left(\int_0^\infty e^{-\lambda t}|f_x-f_x'|\,\|X\|\,dt\right)\right\}^2\\
&+\left\{E\left(\int_0^\infty e^{-\lambda t}|f_x'|\,\|X-X'\|\,dt\right)\right\}^2\\
=&\ \mathrm{II}'+\mathrm{II}''.
\end{aligned}
$$

By Cauchy-Schwarz, for λ large,

$$
\begin{aligned}
\mathrm{II}'&\leq E\left(\int_0^\infty e^{-\lambda t}|f_x-f_x'|^2\,dt\right)E\left(\int_0^\infty e^{-\lambda t}\|X\|^2\,dt\right)\\
&\leq CE\left(\int_0^\infty e^{-\lambda t}|x(t)-x'(t)|^2\,dt\right)\\
&\leq C|x-x'|^2.
\end{aligned}
$$

By Lemma A.4, for λ large, (here $\|\sigma_x-\sigma_x'\|^2=\|\sigma_x^1-\sigma_x^{1'}\|^2+\cdots+\|\sigma_x^m-\sigma_x^{m'}\|^2$)

$$
\begin{aligned}
\mathrm{II}''&\leq C\int_0^\infty e^{-\lambda t}\{E(\|X-X'\|)\}^2dt\\
&\leq C\int_0^\infty e^{-\lambda t}e^{Ct}E\left(\int_0^t (\|b_x-b_x'\|^2+\|\sigma_x-\sigma_x'\|^2)ds\right)dt\\
&\leq C\int_0^\infty e^{-(\lambda-C)t}E(|x(t)-x'(t)|^2)\,dt\\
&\leq C|x-x'|^2.
\end{aligned}
$$

Thus $\mathrm{II} \leq C|x-x'|^2$. Similarly for III, IV, V and so Dv^u is Lipschitz in x. The proof of the last statement is similar and is left to the reader. $\square$

A control u is *optimal at* x when $v^u(x) = v(x)$. A control is ε-*optimal at* x when $v^u(x) < v(x) + \varepsilon$.

Fix $T > 0$. For $\omega \in \Omega$ set $S_T\omega = \omega(\cdot + T) - \omega(T)$. Then for each control u there is a *unique* map $\theta_T u : [0, \infty) \times \Omega \times \Omega \to U$, measurable over $\mathscr{B}([0, \infty)) \times \mathscr{F} \times \mathscr{F}_T$, such that $\theta_T u(\cdot, \cdot, \omega)$ is a control for each ω and satisfies

$$u(t+T, \omega) = \theta_T u(t, S_T\omega, \omega), \qquad t \geq 0, \omega \in \Omega.$$

We refer to $\theta_T u$ as the $\mathscr{F}_T$-measurable family of controls "cut" from u at time T.

Conversely a control u and a $\mathscr{F}_T$-measurable family of controls $u'(\cdot, \cdot, \cdot)$ can be "pasted" together at time T to produce the control $u \otimes_T u'$ uniquely determined by the requirements that it equal u on $[0, T) \times \Omega$ and that $\theta_T(u \otimes_T u') = u'$.

We mention in passing that "cutting" and "pasting" can be done on three "levels". The simplest level is the deterministic case, where "cutting" and "pasting" are immediate, the next level is when controls are progressively measurable maps (strong sense controls), which is discussed above, and the next level is when controls are measures (weak sense controls), which we do not use in this paper. Here "cutting" is conditioning on $\mathscr{F}_T$ while "pasting" is in [11, §6.1].

By uniqueness of solutions to (1.1), for each $T > 0$ one has

$$x^u(t+T, \omega; x) = x^{\theta_T u(\cdot, \cdot, \omega)}(t, S_T\omega; x^u(T, \omega)), \qquad t \geq 0, \text{ a.a.-}\omega. \tag{1.4}$$

By uniqueness of solutions to (1.1), (1.2), it follows that for each $T > 0$

$$\begin{aligned} &X^{\theta_T u(\cdot, \cdot, \omega)}(t, S_T\omega; x^u(T, \omega)) \\ &\quad = X^u(t+T, \omega; x) X^u(T, \omega; x)^{-1}, \qquad t \geq 0. \end{aligned} \tag{1.5}$$

Given (1.4), (1.5), the following is straightforward.

Lemma 1.2. *For $T > 0$ and $\lambda > \lambda_0$*

$$\begin{aligned} &v^{\theta_T u}(x(T)) = E\left(\int_T^\infty e^{-\int_T^t c\,ds} f\,dt \,\middle|\, \mathscr{F}_T\right), \\ &Dv^{\theta_T u}(x(T)) \\ &\quad = E\left(\int_T^\infty e^{-\int_T^t c\,ds}\left[f_x X(\cdot|T) - f\left(\int_T^t c_x X(\cdot|T) ds\right)\right] dt \,\middle|\, \mathscr{F}_T\right) \end{aligned} \tag{1.6}$$

almost surely, where $X(t|T) = X(t)X(T)^{-1}$, $t \geq T$. $\square$

For completeness we derive Bellman's *dynamic programming principle* in this context [3, 6, 7, 10].

Since $(x, u) \mapsto v^u(x)$ is continuous on $\mathbf{R}^d \times \mathscr{C}$ and $\mathscr{C}$ is separable, the infimum in the definition of $v(x)$ can be taken over a countable set of controls.

Proposition 1.3. *For* $T > 0$ *and* $\lambda > 0$

$$v(x) = \inf E\left(\int_0^T e^{-\int_0^t c\,ds} f(x,u)\,dt + e^{-\int_0^T c\,dt} v(x(T))\right),$$

where the infimum is over all controls u.

Proof. Let $\varepsilon > 0$, $u \in \mathscr{C}$ be arbitrary, and for each ω choose a control $u'(\cdot,\cdot,\omega)$ that is ε-optimal at $x^u(T,\omega)$ as follows: Let $u_1, u_2, \ldots$ be a countable set of controls dense in $\mathscr{C}$, and for each ω let $n(\omega)$ be the first n such that u_n is ε-optimal at $x^u(T,\omega)$. Then it can be easily verified that $u'(\cdot,\cdot,\omega) = u_{n(\omega)}$ is a $\mathscr{F}_T$-measurable family of controls. Set $u'' = u \otimes_T u'$. Then $\theta_T u'' = u'$ and so by Lemma 1.2

$$\begin{aligned}
v(x) \le v^{u''}(x) &= E\left(\int_0^T e^{-\int_0^t c\,ds} f(x,u)\,dt + e^{-\int_0^T c\,dt} v^{\theta_T u''}(x(T))\right)\\
&\le E\left(\int_0^T e^{-\int_0^t c\,ds} f(x,u)\,dt + e^{-\int_0^T c\,dt} v(x(T))\right) + e^{-\lambda T}\varepsilon\\
&\le E\left(\int_0^T e^{-\int_0^t c\,ds} f(x,u)\,dt + e^{-\int_0^T c\,dt} v^{\theta_T u}(x(T))\right) + e^{-\lambda T}\varepsilon\\
&= E\left(\int_0^\infty e^{-\int_0^t c\,ds} f(x,u)\,dt\right) + e^{-\lambda T}\varepsilon\\
&= v^u(x) + e^{-\lambda T}\varepsilon.
\end{aligned}$$

Taking the infimum over u and letting $\varepsilon \downarrow 0$ the result follows. □

Corollary 1.4. *If* u *is optimal at* x *and* $T > 0$ *then* $\theta_T u$ *is optimal at* $x(T)$ *almost surely.* □

Corollary 1.5. *For* $\lambda > \lambda_0$ *the value function* $v \in C_b(\mathbf{R}^d)$ *is Lipschitz on* $\mathbf{R}^d$ *and semiconcave on* $\mathbf{R}^d$: *There is a constant* $C > 0$ *such that* $C|x|^2 - v(x)$ *is convex on* $\mathbf{R}^d$.

Proof. Clearly v is bounded. By Lemma 1.1 $|Dv^u(x)| \le C$ for all $x \in \mathbf{R}^d$ and all controls u. This implies $v(x) \le v^u(x) \le v^u(x') + C|x - x'|$; taking the infimum over u and reversing the roles of x, x', yields $|v(x) - v(x')| \le C|x - x'|$.

We also have $D^2 v^u(x) \le CI$ in the sense of Schwartz distributions; this implies $\frac{1}{2}C|x|^2 - v^u(x)$ is convex. Taking the supremum of this expression, the result follows. □

2. Hamilton's equations

Fix $x \in \mathbf{R}^d$ and suppose u is optimal at x. For constant $a \in U$ and for $\varepsilon > 0$ let u^ε be the control that equals a on $[0,\varepsilon) \times \Omega$ and u elsewhere. Let

$v^\varepsilon(x)$ denote the cost corresponding to the control u^ε and starting state x. Then $v^\varepsilon(x) \geq v^0(x) \equiv v^u(x)$ for $\varepsilon > 0$ and so $\frac{d}{d\varepsilon}\Big|_{\varepsilon=0^+} v^\varepsilon(x) \geq 0$.

Below we need to address the possibility that $\frac{1}{\varepsilon}\int_0^\varepsilon u(t, \omega)\, dt$ may not converge to $u(0, \omega)$ as $\varepsilon \downarrow 0$. Note that since u is progressively measurable $u(0)$ is a constant identically on Ω.

Let u be a control. We say (t, ω) is a *Lebesgue point of* u provided $\frac{1}{\varepsilon}\int_t^{t+\varepsilon} |u(s, \omega) - u(t, \omega)|\, ds \to 0$ as $\varepsilon \downarrow 0$. Then for each ω, the point (t, ω) is a Lebesgue point of u for a.a. $t \geq 0$. Thus for a.a. $t \geq 0$, the set of ω's for which (t, ω) is a Lebesgue point has Wiener measure 1. We now work out the derivative of $v^\varepsilon(x)$.

Lemma 2.1. *Let* u *be optimal at* x *and suppose that* $(0, \omega)$ *is a Lebesgue point of* u *for* W*-a.a.* ω. *Suppose* $\lambda > \lambda_0$. *Then*

$$u(0) = \mathbf{u}(x, v(x), Dv^u(x)). \tag{2.1}$$

Proof. Let x^ε denote the state trajectory corresponding to u^ε. Then by definition of u^ε, x^ε satisfies

$$x^\varepsilon(t) = \xi^\varepsilon(t) - \int_0^t b(x^\varepsilon(s), u(s))\, ds + \int_0^t \sigma(x^\varepsilon(s))\, dw(s), \qquad t \geq 0, \tag{2.2}$$

where

$$\xi^\varepsilon(t) = x + \int_0^{t\wedge\varepsilon} [b(x^a(s), u(s)) - b(x^a(s), a)]\, ds, \qquad t \geq 0.$$

Since the initial data $\xi^\varepsilon(\cdot)$ is differentiable in the parameter ε at $\varepsilon = 0^+$, standard results on solutions of SDE's [6] imply the solution $x^\varepsilon(\cdot)$ is also differentiable in ε at $\varepsilon = 0^+$, in probability. Let $Z(t)$ denote the derivative at $\varepsilon = 0^+$. Differentiating (2.2) with respect to ε yields

$$Z(t) = \xi - \int_0^t b_x(x^u(s), u(s)) Z(s)\, ds + \int_0^t \sigma_x(x^u(s)) Z(s)\, dw(s), \qquad t \geq 0,$$

where $\xi = [b(x, u(0)) - b(x, a)]$. Hence by (1.2) it follows that

$$Z(t) = X(t)[b(x, u(0)) - b(x, a)], \qquad t \geq 0.$$

Now

$$\begin{aligned} v^\varepsilon(x) &= E\left(\int_0^\varepsilon e^{-\int_0^t c(x^a, a)\, ds} f(x^a(t), a)\, dt\right) \\ &\quad + E\left(\int_\varepsilon^\infty e^{-\int_0^t c(x^\varepsilon, u^\varepsilon) ds} f(x^\varepsilon(t), u(t))\, dt\right) \\ &= E\left(\int_0^\varepsilon e^{-\int_0^t c(x^a, a)\, ds} f(x^a(t), a)\, dt\right) \\ &\quad - E\left(e^{-\int_0^\varepsilon [c(x^a, \dot{a}) - c(x^a, u)]\, ds} \int_0^\varepsilon e^{-\int_0^t c(x^a, a)\, ds} f(x^a(t), u(t))\, dt\right) \\ &\quad + E\left(e^{-\int_0^\varepsilon [c(x^a, a) - c(x^a, u)]\, ds} \int_0^\infty e^{-\int_0^t [c(x^\varepsilon, u)\, ds} f(x^\varepsilon(t), u(t))\, dt\right). \end{aligned}$$

Differentiating v^ε with respect to ε, using (1.3) and the fact that $(0, \omega)$ is a Lebesgue point of u for a.a.-ω yields

$$\begin{aligned} &Dv^u(x)[b(x, a) - b(x, u(0))] \\ &\qquad + [c(x, a) - c(x, u(0))]v^u(x) - [f(x, a) - f(x, u(0))] \leq 0. \end{aligned}$$

Thus $u(0)$ attains the supremum in $F(x, v^u(x), Dv^u(x))$; since $v^u(x) = v(x)$, (2.1) follows. □

Corollary 2.2. *Let u be optimal at x and suppose $\lambda > \lambda_0$. Then*

$$u(t) = \mathbf{u}(x(t), v(x(t)), Dv^{\theta_t u}(x(t))), \qquad t \geq 0, \text{ a.s.-}(dt \times W). \tag{2.3}$$

Proof. From above we know that for a.a. $t \geq 0$, (t, ω) is a Lebesgue point of u for W-a.a. ω. Thus for each such (t, ω), $(0, \omega')$ is a Lebesgue point of $\theta_t u(\cdot, \cdot, \omega)$ for W-a.a. ω'. Also by Corollary 1.4, $\theta_t u(\cdot, \cdot, \omega)$ is optimal at $x(t, \omega)$ for W-a.a. ω. Since $\theta_t u(0) = u(t)$, applying Lemma 2.1 the result follows. □

For $t \geq 0$ set

$$p(t) = E\left(\int_t^\infty e^{-\int_t^s c\,dr}\left[f_x X(\cdot|t) - f\left(\int_t^s c_x X(\cdot|t)\,dr\right)\right] ds \,\middle|\, \mathscr{F}_t\right), \tag{2.4}$$

where as before $X(s|t) = X(s)X(t)^{-1}$. Then Lemma 1.2 states that for each $t \geq 0$,

$$Dv^{\theta_t u}(x(t)) = p(t), \quad \text{a.s.-}W.$$

Thus the feedback (2.3) reads

$$u(t) = \mathbf{u}(x(t), v(x(t)), p(t)), \qquad t \geq 0, \text{ a.s.-}(dt \times W). \tag{2.5}$$

Hamilton's equations consist of the closed system for $(x(t), p(t), X(t))$ given by (1.1), (1.2), (2.4), (2.5). In the simplest (but still interesting) case when b, σ do not depend on x, the equation for $X(t) \equiv I$ drops out.

3. The case $v \in W^{2,\infty}(\mathbf{R}^d)$

It is well known that the value function v is a generalized solution of the Bellman equation (0.1), in the sense of Schwartz distributions or in the sense of Crandall-Lions viscosity solutions [1, 8]. In particular, v is the unique viscosity solution of (0.1) in $C_b(\mathbf{R}^d)$. Set $F(x, v, p, u) = b(x, u) \cdot p + c(x, u)v - f(x, u)$. Then $F(x, v, p) = \sup_{u \in U} F(x, v, p, u)$.

We need the following lemmas. For completeness we state the well-known result of P.-L. Lions [7], specialized here to the quasi-linear case.

Lemma 3.1. *Suppose $\lambda > \lambda_0$. In the sense of Schwartz distributions, $Av \in L^\infty(\mathbf{R}^d)$ and (3.1) holds almost surely on $\mathbf{R}^d$. This together with the semi-concavity of v (Corollary 1.5) imply $v \in W^{2,\infty}(\mathbf{R}^d)$, when a is nondegenerate, $\langle a(x)\xi, \xi\rangle \geq \varepsilon|\xi|^2$, for all $x, \xi \in \mathbf{R}^d$.* □

Lemma 3.2. *Fix a starting x and control u and suppose $\lambda > \lambda_0$. Suppose $Dv(x^u(t))$ exists for all $t \geq 0$, a.s.-$(dt \times W)$. Then*

$$(3.1) \quad v^u(x) = v(x) + E\left(\int_0^\infty e^{-\int_0^t c(x,u)\,ds}[F(x, v(x), Dv(x)) - F(x, v(x), Dv(x), u)]\,dt\right).$$

Proof. Assume first that a is nondegenerate as in Lemma 3.1. Then by Ito's rule for $W^{2,\infty}$ functions [6] and (0.1)

$$e^{-\int_0^t c\,ds} v(x(t)) - \int_0^t e^{-\int_0^s c\,dr}[F - b(x, u)\cdot Dv(x) - c(x, u)v(x)]\,ds, \qquad t \geq 0,$$

is a martingale. Thus

$$(3.2) \quad 0 = v(x) - E(e^{-\int_0^T c\,dt} v(x(T))) + E\left(\int_0^T e^{-\int_0^t c\,ds}[F - b(x, u)\cdot Dv(x) - c(x, u)v(x)]\,dt\right),$$

where $F = F(x, v(x), Dv(x))$, for all $T > 0$.

Now we are going to change the underlying probability space. For emphasis below we write Ω^m, $\mathscr{F}_t^m$ to indicate the dimension of the driving noise w.

Let σ_n be the $d\times(m+d)$ matrix $(\sigma, \frac{1}{n}I)$. Let v_n denote the value function corresponding to b, c, f, and σ_n. Then (3.2) applies, *relative to* Ω^{m+d}, and we obtain

(3.2_n)

$$0 = v_n(x) - E(e^{-\int_0^T c\,dt} v_n(x_n(T))) + E\left(\int_0^T e^{-\int_0^t c\,ds}[F_n - b(x_n, u)\cdot Dv_n(x_n) - c(x_n, u)v_n(x)]\,dt\right),$$

where $F_n = F(x_n, v_n(x_n), Dv_n(x_n))$ and $c = c(x_n, u)$. It is important to realize that in (3.2_n) x_n is $\mathscr{F}_t^{m+d}$-progressively measurable, even though u is only $\mathscr{F}_t^m$-progressively measurable. However x is $\mathscr{F}_t^m$-progressively measurable. In fact (3.2_n) holds for u $\mathscr{F}_t^{m+d}$-progressively measurable, but we do not use this. Now as $n \to \infty$ $x_n(t) \to x(t)$ in probability since $\sigma_n \to \sigma$ uniformly. Since $v_n \to v$ locally uniformly, it follows that $v_n(x_n(t)) \to v(x(t))$ in probability. Since Dv exists at $x(t)$ a.s. and v_n are uniformly semiconcave (Corollary 1.5), it also follows that $Dv_n(x_n(t)) \to Dv(x(t))$ in probability. Passing to the limit, it follows that (3.2) holds for degenerate systems, on Ω^m. Letting $T \uparrow \infty$ we obtain

$$(3.3) \quad 0 = v(x) + E\left(\int_0^\infty e^{-\int_0^t c\,ds}[F - b(x, u)\cdot Dv(x) - c(x, u)v(x)]\,dt\right).$$

Combining (3.3) with (0.4), the result follows. □

Lemma 3.3. *Suppose $v \in W^{2,\infty}(\mathbf{R}^d)$ and $\lambda > \lambda_0$. Then for each x there exists exactly one control u optimal at x. Moreover a control u is optimal iff*

$$u(t) = \mathbf{u}(x(t), v(x(t)), Dv(x(t))), \qquad t \geq 0, \text{ a.s.-}(dt \times W). \tag{3.4}$$

Proof. Since $v \in W^{2,\infty} = C_b^{1,1}$, v and Dv are Lipschitz. Hence $\mathbf{u}(x) = \mathbf{u}(x, v(x), Dv(x))$ is Lipschitz. Thus there exists a unique solution x^* to the equation

$$x^*(t) = x - \int_0^t b(x^*(s), \mathbf{u}(x^*(s)))\, ds + \int_0^t \sigma(x^*(s))\, dw(s), \qquad t \geq 0 \tag{3.5}$$

(here we are in Ω^m). Setting $u^*(t) = \mathbf{u}(x^*(t))$ yields a control satisfying the feedback (3.4). Since $F(x, v(x), Dv(x)) = F(x, v(x), Dv(x), \mathbf{u}(x))$, u^* is optimal by Lemma 3.2. Conversely if u satisfies the feedback (3.4) then x^u solves (3.5) so $x^u = x^*$ so $u = u^*$. The result follows. □

4. An a priori estimate

In the next section we establish $v \in W^{2,\infty}$ for λ large. Since we do not know this yet, the results in §3 do not apply and hence we do not yet know that optimal controls exist. To this end we need a simple estimate.

In what follows $\mathbf{u}(x, p) \equiv \mathbf{u}(x, v(x), p)$. Since v is Lipschitz, it follows that $\mathbf{u}(x, p)$ is Lipschitz. Let x, x' be two starting states and suppose optimal controls u, u' (on Ω^m) exist at x, x' respectively; let x, p, X, x', p', X' be the corresponding processes. Then we have the optimal feedbacks

$$u = \mathbf{u}(x, p), \qquad u' = \mathbf{u}(x', p').$$

Set

$$\rho(t) = E(|x(t) - x'(t)|^2 + |p(t) - p'(t)|^2).$$

The technique in the following lemma is the key observation on which the results of this paper turn.

Lemma 4.1. *There exists $\lambda_1 > \lambda_0$ such that for $\lambda > \lambda_1$,*

$$\rho(t) \leq C|x - x'|^2 e^{Ct}.$$

Proof. To begin note $\rho(t) \leq Ce^{Ct}$. Set $b = b(x, u)$, $b' = b(x', u')$, $b_x = b_x(x, u)$, $b'_x = b_x(x', u')$, $c = c(x, u)$, $c' = c(x', u')$, $c_x = c_x(x, u)$, $c'_x = c_x(x', u')$, $\sigma = \sigma(x)$, $\sigma' = \sigma(x')$, $\sigma_x = \sigma_x(x)$, $\sigma'_x = \sigma_x(x')$, $f = f(x, u)$, $f' = f(x', u')$, $f_x = f_x(x, u)$, $f'_x = f_x(x', u')$. Then since $\mathbf{u}$ is Lipschitz, $E(|u(t) - u'(t)|^2) \leq C\rho(t)$; this implies

$$E(|\psi(t) - \psi'(t)|^2) \leq C\rho(t),$$

where ψ is any one of the components of b, b_x, c, c_x, σ, σ_x, f, f_x. Now by Ito's rule

$$d|x - x'|^2 = 2(b - b')(x - x')dt + 2(\sigma - \sigma')(x - x')dw + \|\sigma - \sigma'\|^2 dt;$$

it follows that

$$E(|x(t)-x'(t)|^2) \le |x-x'|^2 + \frac{C}{2}\int_0^t \rho(s)\,ds. \tag{4.1}$$

Moreover $X(s|t) = X(s)X(t)^{-1}$, $s \ge t$, satisfies

$$X(s|t) = I - \int_t^s b_x(r)X(r|t)\,dr + \int_t^s \sigma_x(r)X(r|t)\,dw(r), \qquad s \ge t;$$

this implies (Corollary A.5)

$$E(\|X(s|t)\|^2|\mathscr{F}_t) \le Ce^{C(s-t)}, \qquad s \ge t. \tag{4.2}$$

Let $X'(s|t)$ denote the corresponding primed quantity. Now

(4.3)

$$\begin{aligned}
&\frac{1}{6}E(|p(t)-p'(t)|^2) \\
&\quad\le \frac{1}{2}E\left\{E\left(\left(\int_t^\infty |e^{-\int_t^s c\,dr} f_x X(\cdot|t) - e^{-\int_t^s c'\,dr} f'_x X'(\cdot|t)|ds\right|\mathscr{F}_t\right)^2\right\} \\
&\qquad + \frac{1}{3}E\left\{E\left(\left(\int_t^\infty \left|e^{-\int_t^s c\,dr} f \int_t^s c_x X(\cdot|t)\,dr\right.\right.\right.\right. \\
&\qquad\qquad \left.\left.\left.\left. -e^{-\int_t^s c'\,dr} f' \int_t^s c'_x X'(\cdot|t)\,dr\right| ds\right|\mathscr{F}_t\right)^2\right\} \\
&\quad\le E\left\{E\left(\left(\int_t^\infty |e^{-\int_t^s c\,dr} - e^{-\int_t^s c'\,dr}||f_x X(\cdot|t)|\,ds\right|\mathscr{F}_t\right)^2\right\} \\
&\qquad + E\left\{E\left(\left(\int_t^\infty e^{-\int_t^s c'\,dr}|f_x X(\cdot|t) - f'_x X'(\cdot|t)|\,ds\right|\mathscr{F}_t\right)^2\right\} \\
&\qquad + E\left\{E\left(\left(\int_t^\infty |e^{-\int_t^s c\,dr} - e^{-\int_t^s c'\,dr}||f|\left(\int_t^s |c_x X(\cdot|t)|\,dr\right)ds\right|\mathscr{F}_t\right)^2\right\} \\
&\qquad + E\left\{E\left(\left(\int_t^\infty e^{-\int_t^s c'\,dr}|f-f'|\left(\int_t^s |c_x X(\cdot|t)|\,dr\right)ds\right|\mathscr{F}_t\right)^2\right\} \\
&\qquad + E\left\{E\left(\left(\int_t^\infty e^{-\int_t^s c'\,dr}|f'|\left(\int_t^s |c_x X(\cdot|t) - c'_x X'(\cdot|t)|dr\right)ds\right|\mathscr{F}_t\right)^2\right\} \\
&\quad = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V}.
\end{aligned}$$

Since $c(x,u) \ge \lambda$ it follows that

$$|e^{-\int_t^s c\,dr} - e^{-\int_t^s c'\,dr}| \le e^{-\lambda(s-t)}\int_t^s |c-c'|\,dr.$$

Thus by Cauchy-Schwarz, for λ large,

$$\begin{aligned}
\mathrm{I} \le CE\Bigg\{ & E\left(\int_t^\infty e^{-\lambda(s-t)}\left(\int_t^s |c-c'|\,dr\right)^2 ds\,\Bigg|\,\mathscr{F}_t\right) \\
& \times E\left(\int_t^\infty e^{-\lambda(s-t)}\|X(s|t)\|^2\,ds\,\Bigg|\,\mathscr{F}_t\right)\Bigg\} \\
\le{} & C\int_t^\infty e^{-\lambda(s-t)}(s-t)\int_t^s \rho(r)\,dr\,ds.
\end{aligned}$$

Similarly

$$\begin{aligned}
\frac{1}{2}\mathrm{II} \le{} & \left\{E\left(\int_t^\infty e^{-\lambda(s-t)}|f_x-f_x'|\,\|X(\cdot|t)\|\,ds\,\Bigg|\,\mathscr{F}_t\right)^2\right\} \\
& + C\left\{E\left(\int_t^\infty e^{-\lambda(s-t)}\|X(\cdot|t)-X'(\cdot|t)\|\,ds\,\Bigg|\,\mathscr{F}_t\right)^2\right\} \\
={} & \mathrm{II}' + \mathrm{II}''.
\end{aligned}$$

By Cauchy-Schwarz and A.5, for λ large,

$$\begin{aligned}
\mathrm{II}' \le E\Bigg\{ & E\left(\int_t^\infty e^{-\lambda(s-t)}|f_x-f_x'|^2\,ds\,\Bigg|\,\mathscr{F}_t\right) \\
& \times E\left(\int_t^\infty e^{-\lambda(s-t)}\|X(\cdot|t)\|^2\,ds\,\Bigg|\,\mathscr{F}_t\right)\Bigg\} \\
\le{} & C\int_t^\infty e^{-\lambda(s-t)}\rho(s)\,ds.
\end{aligned}$$

Similarly using A.5, for λ large,

$$\mathrm{II}'' \le C\int_t^\infty e^{-\lambda(s-t)}e^{C(s-t)}\left(\int_t^s \rho(r)\,dr\right)ds.$$

Continuing in this manner yields

$$\mathrm{III} \le C\int_t^\infty e^{-\lambda(s-t)}(s-t)\left(\int_t^s \rho(r)\,dr\right)ds,$$

$$\mathrm{IV} \le C\int_t^\infty e^{-\lambda(s-t)}\rho(s)\,ds,$$

$$\mathrm{V} \le C\int_t^\infty e^{-\lambda(s-t)}\int_t^s\left(\rho(r)+e^{C(r-t)}\left(\int_t^r \rho(q)\,dq\right)\right)dr\,ds.$$

Combining (4.1), (4.3) yields

$$\rho(t) \le (K\rho)(t) + |x-x'|^2, \tag{4.4}$$

where K is the positive linear operator

$$\begin{aligned}
Kf(t) ={} & \frac{C}{2}\int_0^t f(s)\,ds \\
& + C\int_t^\infty e^{-(\lambda-C)(s-t)}\left(f(s)+C\int_t^s\left(f(r)+C\int_t^r f(q)\,dq\right)dr\right)ds.
\end{aligned}$$

Now if K were causal (4.4) would be a Volterra inequality which can be iterated in the usual way. However K is not causal. Nevertheless (4.4) is a Fredholm inequality which can be iterated and solved in the usual way when the norm of $K = K_\lambda$ is small enough; but this happens when λ is large enough.

Specifically when $\lambda > 8C$, $f(t) \le e^{Ct}$ implies $Kf(t) \le \delta e^{Ct}$, where $\delta = \frac{1}{2} + \frac{3C}{\lambda - 2C} < 1$. Iterating (4.4) yields

$$\begin{aligned}\rho(t) &\le (I + K + K^2 + \cdots + K^{n-1})|x - x'|^2 + K^n \rho(t) \\ &\le (1 + \delta + \delta^2 + \cdots + \delta^{n-1})|x - x'|^2 e^{Ct} + \delta^n C e^{Ct} \\ &\to \frac{1}{1-\delta}|x - x'|^2 e^{Ct} \quad \text{as } n \uparrow \infty. \quad \square\end{aligned}$$

Since $p(0) = Dv^u(x)$, setting $t = 0$ in Lemma 4.1 yields

Corollary 4.2. *Let $\lambda > \lambda_1$. $|Dv^u(x) - Dv^{u'}(x')| \le C|x - x'|$ for any two controls u, u' optimal at x, x' respectively. Moreover u_n, u optimal at x_n, x and $x_n \to x$ implies $u_n \to u$ in $\mathscr{C}$.* $\square$

5. The main results

Theorem 5.1. *$\lambda > \lambda_1$ implies $v \in W^{2,\infty}(\mathbf{R}^d) = C_b^{1,1}(\mathbf{R}^d)$: v is differentiable, v and Dv are bounded, Dv is Lipschitz on $\mathbf{R}^d$, and $\|v\|_{C^{1,1}} \le C$.*

First Proof. First we assume a is nondegenerate. Then we use Lemma 4.1 to derive an estimate on the $C^{1,1}$ norm of v independent of the ellipticity constant. Then we pass to the limit.

Assume a is nondegenerate. Then $v \in W^{2,\infty}$ and optimal controls exist. Fix $x_0 \in \mathbf{R}^d$. For $\xi \in \mathbf{R}^d$, let $x_t = x_0 + t\xi$. Let u_t, $t \ge 0$, denote the control optimal at x_t; then

$$\frac{v(x_t) - v(x_0)}{t} \le \frac{v^{u_0}(x_t) - v^{u_0}(x_0)}{t}$$

and so

$$\limsup_{t\downarrow 0} \frac{v(x_t) - v(x_0)}{t} \le Dv^{u_0}(x_0)\xi.$$

Also

$$\frac{v(x_t) - v(x_0)}{t} \ge \frac{v^{u_t}(x_t) - v^{u_t}(x_0)}{t} = Dv^{u_t}(x_t^*),$$

so

$$\liminf_{t\downarrow 0} \frac{v(x_t) - v(x_0)}{t} \ge \liminf_{t\downarrow 0} Dv^{u_t}(x_t^*)\xi = Dv^{u_0}(x_0)\xi,$$

by Corollary 4.2 and Lemma 1.1. Thus for each x, $Dv(x) = Dv^u(x)$, where u is the control optimal at x. Hence by Corollary 1.5 v and Dv are bounded on $\mathbf{R}^d$ and by Corollary 4.2 again $|Dv(x) - Dv(x')| = |Dv^u(x) - Dv^{u'}(x)| \le C|x - x'|$. This establishes $\|v\|_{C^{1,1}} \le C$ when a is nondegenerate, with C independent of the ellipticity constant.

For the general case we now set, as in Lemma 3.2, $\sigma_n = (\sigma, \frac{1}{n}I)$. Then $a_n = \sigma_n\sigma_n^*$ is nondegenerate. By the above, $\|v_n\|_{C^{1,1}} \le C$ with C independent of n. Since $v_n \to v$, the result follows by passing to a subsequence. □

Second Proof. This proof avoids totally the results in §3 and P.-L. Lions' theorem; the price we pay is the use of relaxed generalized (weak sense) controls as well as the fact that this proof works well only when $c(x, u) = c(x)$ does not depend on u. A *generalized control* is a control where the underlying probability space and Wiener process is allowed to depend on the control [2, 4]. Because the dependence of $b(x, u)$ ($f(x, u)$) on u is not necessarily affine (convex respectively), one needs to extend the definition to the class of *relaxed* generalized controls [2, 4] to force b, f to be affine and convex respectively, albeit at the cost of replacing U by $\tilde{U} = M(U)$. Writing $v^u(x) = E^P(\Phi(x, u))$, where P denotes the law of $(x(\cdot), u(\cdot))$, and imposing an appropriate weak topology on the space of deterministic controls $u(\cdot)$, the lower semicontinuity of Φ follows as well as the compactness of the laws $\{P\}$. Without giving the precise definitions, this implies the existence for each x of a relaxed generalized control optimal at x, even in the degenerate case. Given two such controls u, u' optimal at x, x' respectively, defined a priori on two different probability spaces, one constructs a single probability space supporting both controls, following Yamada-Watanabe (see [11]). They then can be compared pathwise and the results in §4 hold with no change whatsoever. This establishes Corollary 4.2 when u, u' are relaxed generalized controls optimal at x, x' respectively and so we see that there is a Lipschitz function F on $\mathbf{R}^d$ such that $F(x) = Dv^u(x)$ for all x. By the argument in the first proof we see that $Dv(x)$ exists and equals $Dv^u(x) = F(x)$ for all x. The result follows. □

Corollary 5.2. *Let $\lambda > \lambda_1$. Then for each x there is exactly one* (*strong*) *control optimal at x.*

Proof. Combine Theorem 5.1 and Lemma 3.3. □

The above regularity result is best possible (using these kinds of techniques). For example take the one-dimensional system $f(x, u) = \frac{1}{2}u^2 + \phi(x)$, $\phi \in C_0^\infty(\mathbf{R})$, equal to $\frac{1}{2}x^2$ near $x = 0$, $U = [0, 1] \subset \mathbf{R}$, $b(x, u) = u$, $c(x, u) \equiv \lambda$, $\sigma(x) \equiv 0$. The value function then satisfies

$$H(v') - \phi(x) + \lambda v = 0, \qquad \text{where } H(p) = \begin{cases} 0, & p \le 0, \\ \frac{1}{2}p^2, & 0 \le p \le 1, \\ p - \frac{1}{2}, & p \ge 1. \end{cases}$$

Then near and to the left of zero, the value function is $v(x) = \frac{1}{2\lambda}x^2$; near and to the right of zero, the value function is $v(x) = \frac{1}{2}kx^2$, where k solves the (Riccati) equation $k^2 + \lambda k - 1 = 0$. This shows that v' is Lipschitz near zero but v'' does not exist at zero, for all $\lambda > 0$.

We conclude with some philosophical remarks concerning regularity which may shed some light on Theorem 5.1. We have shown above that, for λ sufficiently large, $v \in C_b^{1,1} = W^{2,\infty}$. However the identification $C_b^{1,1} = W^{2,\infty}$ is measure-theoretic, i.e. makes use of a reference measure on $\mathbf{R}^d$. Since probabilistic methods do not a priori interact with base measures on the state space (they involve the measure on Ω), there is no way these methods imply directly that the value function v is in $W^{2,\infty}$.

For example in the study of Bellman equations on Hilbert space there is no natural reference measure. Hence optimal regularity must be formulated in terms of $C^{1,1}$ [9]. Another example is partially observed control where the state space is the space of probability measures $M(\mathbf{R}^d)$. Here again there are no natural Sobolev spaces and one expects optimal regularity to be formulated in terms of $C^{1,1}$. By contrast, on Wiener space Ω there is a natural reference measure and so the Malliavin Calculus can be formulated in terms of Sobolev spaces on Ω.

A. Appendix

Lemma A.1. *$\mathscr{C}$ is separable.*

Proof. Let $\mathscr{C}_b$ be the bounded controls. Since any control can be approximated in probability by bounded controls, it is enough to show $\mathscr{C}_b$ is separable. Next since $\mathscr{C}_b \subset L^2 = L^2([0, \infty) \times \Omega, e^{-t}dt \times W)$, it is enough to show that $\mathscr{C}_b$ is separable in L^2-norm. Now note that $[0, \infty) \times \Omega$ is a Polish space. Thus we can use Lusin's theorem to produce a map $f \in C_b([0, \infty)\times\Omega; U)$ that approximates any given $u \in \mathscr{C}_b$ in L^2-norm. Given $a \in \mathbf{R}$ let $a^n = (a\wedge n)\vee(-n)$; given $v = (v_1, \dots, v_m) \in \mathbf{R}^m$ let v^n be the vector whose ith component is $(v_i)^n$. Given $\omega \in \Omega$, let $\omega^n(t) = (\omega(t))^n$ and let ω_n denote the piecewise linear interpolation of ω over a partition of mesh $\frac{1}{n}$. Then for all ω the path $(\omega^n)_n$ is uniformly bounded by nm and is Lipschitz in t with Lipschitz constant $2mn^2$. Thus the set $\{(t \wedge n, (\omega^n)_n) : t \geq 0, \omega \in \Omega\} \subset [0, \infty) \times \Omega$ is compact and so $g_n(t, \omega) = f(t \wedge n, (\omega^n)_n)$ is bounded and uniformly continuous and approximates f in L^2-norm for large n. Since the space of bounded uniformly continuous maps on $[0, \infty)\times\Omega$ is separable in the sup norm, we conclude there is a sequence of maps $f_1, f_2, \dots$ in $C_b([0, \infty)\times\Omega; U)$ such that any bounded control can be approximated arbitrarily closely in L^2-norm by elements of this sequence. Let $u_1, u_2, \dots$ denote the L^2-projections of $f_1, f_2, \dots$ onto the Hilbert subspace of all progressively measurable maps. Since U is closed convex the projections are controls and the result follows. □

Let $U_r = U \cap \{u \mid |u| \leq r\}$. Lemmas A.2 and A.3 establish the fact that $F(x, v, p) = H(x, p) + \lambda v$ in (0.6) is of Lipschitz type when L is strictly convex.

Lemma A.2. *Let* $U \subset \mathbf{R}^d$ *be closed and convex. Let* $L \in C^1(\mathbf{R}^d \times \mathbf{R}^d)$ *satisfy*

(L.1) $$\inf_{x\in\mathbf{R}^d} \frac{L(x, u)}{|u|} \to +\infty \quad \text{as } |u| \to \infty, u \in U,$$

(L.2) $$\sup_{x\in\mathbf{R}^d} \sup_{u\in U_r} (|L(x, u)| + |L_x(x, u)|) < +\infty,$$

and

(L.3) $$(L_u(x, u') - L_u(x, u)) \cdot (u' - u) \geq \varepsilon_r |u' - u|^2, \qquad x \in \mathbf{R}^d, u, u' \in U_r,$$

for some $\varepsilon_r > 0$, *both for all* $r > 0$. *Set* $H(x, p) = \sup_{u\in U}(p \cdot u - L(x, u))$. *Then* $H \in C^1(\mathbf{R}^d \times \mathbf{R}^d)$,

(H.1) $$\sup_{x\in\mathbf{R}^d} \sup_{|p|\leq r} (|H(x, p)| + |H_x(x, p)| + |H_p(x, p)|) < +\infty,$$

for all $r > 0$,

(H.2) $$L(x, u) - p \cdot u + H(x, p) \geq 0, \qquad x, p \in \mathbf{R}^d, u \in U,$$

with equality iff

(H.3) $$u = H_p(x, p),$$

and $H_x(x, p) = -L_x(x, H_p(x, p))$. *In particular* H_x *and* H_p *are bounded on* $\mathbf{R}^d \times \mathbf{R}^d$ *and* H *has at most linear growth in* p *on* $\mathbf{R}^d \times \mathbf{R}^d$, *when* U *is compact, and* $H_p(x, \cdot) : \mathbf{R}^d \to \mathbf{R}^d$ *is a global homeomorphism with inverse* $L_u(x, \cdot)$ *for all* $x \in \mathbf{R}^d$, *when* $U = \mathbf{R}^d$.

Proof. To begin, (H.2) holds by definition of H.

Given x, p, let u_n be a sequence in U such that $p \cdot u_n - L(x, u_n)$ approaches the supremum in the definition of H. Then by (L.1), u_n must lie in a bounded subset of $\mathbf{R}^d$. Hence the supremum is attained at some point in U. Now suppose that the supremum is attained at two points u and u'. By (L.3) the function $t \to L(x, (1 - t)u + tu')$ is strictly convex on $0 \leq t \leq 1$. This implies that the supremum is not attained at either u or u'. Hence for given x, p, the supremum is uniquely attained at some point in U; call it $u^*(x, p)$.

Since H is a supremum of continuous functions, it follows that H is lower semicontinuous. To establish upper semicontinuity, let $x_n \to x$, $p_n \to p$. Then by (L.1), $u_n^* = u^*(x_n, p_n)$ lies in a bounded subset of $\mathbf{R}^d$. Hence by passing to a subsequence, $H(x_n, p_n) = p_n \cdot u_n^* - L(x_n, u_n^*) \to p \cdot u' - L(x, u') \leq H(x, p)$. This implies upper semicontinuity, and also implies the continuity of u^*.

Now let $\xi \in \mathbf{R}^d$ and set $p_t = p + t\xi$, $u_t^* = u^*(x, p_t)$. Then $H(x, p_t) \geq p_t \cdot u_0^* - L(x, u_0^*)$, $H(x, p_0) = p_0 \cdot u_0^* - L(x, u_0^*)$. Subtracting these two expressions, dividing by t, and letting $t \downarrow 0$ yields

$$\liminf_{t\downarrow 0} \frac{H(x, p + t\xi) - H(x, p)}{t} \geq u^*(x, p) \cdot \xi.$$

Reversing the roles of p_t, p_0 yields

$$\limsup_{t\downarrow 0} \frac{H(x, p+t\xi) - H(x, p)}{t} \le u^*(x, p)\cdot\xi,$$

where we have used the fact that $u_t^* \to u_0^*$ as $t \downarrow 0$. This shows $H_p(x, p) = u^*(x, p)$ and establishes (H.3). An almost identical argument shows that $H_x(x, p) = -L_x(x, u^*(x, p)) = -L_x(x, H_p(x, p))$.

By (L.2) it follows that H is bounded below on $\mathbf{R}^d \times \{p : |p| \le r\}$ for all $r > 0$. By (L.1) it therefore follows that H_p and hence H, H_x are bounded on $\mathbf{R}^d \times \{p : |p| \le r\}$ for all $r > 0$. This establishes (H.1).

Now if $U = \mathbf{R}^d$ then U is open; since the supremum defining H is attained at $u^* = u^*(x, p)$ we must have $p - L_u(x, u^*) = 0$. Thus $L_u(x, \cdot)$ is onto. If $L_u(x, u) = p$ for some other u then the function $f(t) = -(p\cdot u_t) - L(x, u_t)$, $u_t = (1-t)u + tu^*$, is strictly convex on $0 \le t \le 1$ and satisfies $f'(0) = f'(1) = 0$. Thus $u = u^*$, showing that L_u is one-to-one. Since $u^* = H_p(x, p)$ this establishes $L_u(x, \cdot)^{-1} = H_p(x, \cdot)$ for all $x \in \mathbf{R}^d$.

If U is compact then $H_p \in U$ is bounded and hence by (L.2) so is H_x. □

We note that assumption (L.3) above is equivalent to

$$L_{uu}(x, u) \ge \varepsilon_r I, \qquad x \in \mathbf{R}^d,\ u \in U_r,$$

when $L \in C^2$. Moreover when U is compact (L.1) is equivalent to L bounded below on $\mathbf{R}^d \times \mathbf{R}^d$ because one can always obtain (L.1) in this case by modifying L off U.

Lemma A.3. *Continuing Lemma* A.2, *assume that* L_x *is Lipschitz on* $\mathbf{R}^d \times U_r$ *and*

(L.4) $\qquad \varepsilon_r|u' - u|^2 \le (L_u(x', u') - L_u(x, u))\cdot(u' - u) + C|x' - x|\,|u' - u|,$

$x \in \mathbf{R}^d$, $u, u' \in U_r$, *for some* $\varepsilon_r > 0$, *both for all* $r > 0$. *Then* H_x, H_p *are Lipschitz on* $\mathbf{R}^d \times \{p : |p| \le r\}$ *for all* $r > 0$. *In particular if* L_x *is globally Lipschitz on* $\mathbf{R}^d \times \mathbf{R}^d$ *and* ε *in* (L.4) *can be chosen independent of* r, *then* H_x, H_p *are globally Lipschitz on* $\mathbf{R}^d \times \mathbf{R}^d$.

Proof. Choose $c_k \in \mathbf{R}$, $\xi_k \in \mathbf{R}^d$, $k \ge 1$, such that U is the intersection of the half-spaces $\{u : \xi_k \cdot u \le c_k\}$, $k \ge 1$. For $n \ge 1$ set

$$L^n(x, u) = L(x, u) + n\sum_{k=1}^{n}(0 \vee (\xi_k\cdot u - c_k))^2$$

and for $n \ge 0$ set

$$H^n(x, p) = \sup_{u\in\mathbf{R}^d}(p\cdot u - L^n(x, u)),$$

where $L^0 = L$; let H be defined as in Lemma A.2. Then $L^n \geq L$ satisfies (L.1) uniformly in $n \geq 1$. Thus $u_n^* = H_p^n(x, p)$ lies in a bounded subset of $\mathbf{R}^d$. By passing to a subsequence, we have $H^n(x, p) = p \cdot u_n^* - L^n(x, u_n^*) \leq p \cdot u_n^* - L(x, u_n^*) \to p \cdot u' - L(x, u')$ for some $u' \in \mathbf{R}^d$. We claim $u' \in U$. If not then for some $k \geq 1$, $\varepsilon > 0$ one has $\xi_k \cdot u' > c_k + \varepsilon$; thus for n sufficiently large $-\infty < H(x, p) \leq H^n(x, p) \leq H^0(x, p) - n\varepsilon^2 \to -\infty$ as $n \uparrow \infty$, a contradiction. Thus $u' \in U$. This yields $\limsup_n H^n(x, p) \leq H(x, p)$. Since $H(x, p) \leq H^n(x, p)$, we conclude that $H^n(x, p) \to H(x, p)$. Moreover the above argument also shows $H_p^n(x, p) \to H_p(x, p)$. Since $L_x^n = L_x$, it also follows that $H_x^n(x, p) \to H_x(x, p)$.

Now (L.4) for L implies (L.4) for L^n with the same ε_r; thus

$$|u' - u| \leq C_r' |L_u^n(x', u') - L_u^n(x, u)| + C_r' |x' - x|,$$

for all x, x', $u, u' \in U_r$. Since $H_p^n(x, \cdot)$ is the inverse of $L_u^n(x, \cdot)$, inserting $u' = H_p^n(x', p')$, $u = H_p^n(x, p)$, yields

$$|H_p^n(x', p') - H_p^n(x, p)| \leq C_r'' |x' - x| + C_r'' |p' - p|$$

for all x, x', $|p| \leq r$, $|p'| \leq r$. Letting $n \to \infty$ we obtain H_p is Lipschitz on $\mathbf{R}^d \times \{p : |p| \leq r\}$ for all $r > 0$. Since $H_x^n(x, p) = -L_x^n(x, H_p^n(x, p)) = -L_x(x, H_p^n(x, p))$ we also have the same result for H_x. Finally if the constants do not depend on r, it is clear that the global result holds. □

We note that Lemma A.3 is optimal: When L is C^∞, H_p is at best Lipschitz in general; if however one also has $U = \mathbf{R}^d$ then H is also C^∞ and in this case $H_{pp} = L_{uu}^{-1}$. Also if L is C^2 then the Lipschitz conditions in Lemma A.3 are equivalent to boundedness conditions on L_{xu}, L_{xx}.

Lemma A.4. *Let $B_0, \dots, B_m$, $B_0', \dots, B_m'$, be bounded $(d \times d)$-matrix-valued processes and suppose X, X' satisfy*

$$dX = B_0 X\,dt + B_1 X\,dw_1 + \cdots + B_m X\,dw_m, \quad X(0) = I,$$
$$dX' = B_0' X'\,dt + B_1' X'\,dw_1 + \cdots + B_m' X'\,dw_m, \quad X'(0) = I.$$

Then $E(\|X(t)\|^2) \leq Ce^{Ct}$, $t \geq 0$, and

$$\{E(\|X(t) - X'(t)\|)\}^2 \leq Ce^{Ct} E\left(\int_0^t \left(\sum_{k=0}^m \|B_k(s) - B_k'(s)\|^2\right) ds\right).$$

Proof. The reader may find the proof more enlightening by first checking the case $d = 1$, $m = 1$. For simplicity we assume $m = 1$ and we set $B_0 = A$, $B_0' = A'$, $B_1 = B$, $B_1' = B'$. Set $\langle P, Q \rangle = \operatorname{trace}(P^* Q)$. Then $2\langle P, Q \rangle \leq \|P\|^2 + \|Q\|^2$, $\|PQ\| \leq \|P\| \, \|Q\|$. Set $Q = X' X^{-1}$. Then by Cauchy-Schwarz

(A.1) $$\{E(\|X(t) - X'(t)\|)\}^2 \leq E(\|Q(t) - I\|^2) E(\|X(t)\|^2).$$

We first estimate $E(\|Q_0(t)\|^2)$ where $Q_0 = Q - I$.

Now by Ito's rule

$$dQ_0 = (-Q(A - B^2) + A'Q - B'QB)dt + (B'Q - QB)dw.$$

Then

$$\begin{aligned} d\|Q_0\|^2 &= 2\langle Q_0, -Q_0(A - B^2) + A'Q_0 - B'Q_0B\rangle dt \\ &\quad + 2\langle Q_0, (A' - A) - (B' - B)B\rangle dt + \|B'Q - QB\|^2 dt + (\cdots)dw \\ &\le C\|Q_0\|^2 dt + C(\|A' - A\|^2 + \|B' - B\|^2)dt + (\cdots)dw \qquad \text{(A.2)} \\ &\le C\|Q_0\|^2 dt + C dt + (\cdots)dw; \qquad \text{(A.3)} \end{aligned}$$

here we used $\|B'Q - QB\|^2 \le 2\|B'Q_0 - Q_0B\|^2 + 2\|B' - B\|^2 \le C\|Q_0\|^2 + 2\|B' - B\|^2$. Taking expectations in (A.3) yields $E(\|Q_0(t)\|^2) \le Ce^{Ct}$. Choosing $A \equiv B \equiv 0$ yields $E(\|X'(t) - I\|^2) \le Ce^{Ct}$ which implies the first assertion. Taking expectations in (A.2) and using (A.1) yields the second assertion. □

Corollary A.5. *With notation as in Lemma* 4.1, $E(\|X(s|t)\|^2|\mathscr{F}_t) \le Ce^{C(s-t)}$, $s \ge t$, *and*

$$E\{E(\|X(s|t) - X'(s|t)\| \mid \mathscr{F}_t)^2\} \le Ce^{C(s-t)} \int_t^s \rho(r)\, dr, \qquad s \ge t.$$

Proof. By (1.2), (1.4), (1.5) it is enough to consider the case $t = 0$. In this case the result follows from Lemma A.4. □

References

1. M. G. Crandall and P.-L. Lions, *Viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc. **277** (1983), 1–42.
2. N. El-Karoui, D. Hùù Nguyen, and M. Jeanblanc-Piqué, *Compactification methods in the control of degenerate diffusions: Existence of an optimal Markovian control*, Stochastics **20** (1987), 169-219.
3. W. H. Fleming and R. W. Rishel, *Deterministic and stochastic optimal control*, Springer-Verlag, New York, 1975.
4. U. G. Haussmann, *Existence of optimal Markovian controls for degenerate diffusions*, Proc. Third Bad Honnef Conference on Stochastic Differential Equations, Lecture Notes in Control, vol. 78, Springer-Verlag, New York, 1985.
5. N. V. Krylov, *Control of a solution of a stochastic integral equation with degeneration*, Theory Probab. and Appl. **17** (1972), 114–131.
6. ——, *Controlled diffusion processes*, Springer-Verlag, New York, 1980.
7. P.-L. Lions, *Control of diffusion processes in* $\mathbf{R}^N$, Comm. Pure and Appl. Math. **34** (1981), 121–147.
8. ——, *Optimal control of diffusion processes.* II, Comm. Partial Differential Equations **8** (1983), 1229–1276.
9. ——, *Viscosity solutions of fully nonlinear second-order equations and optimal stochastic control in infinite dimensions, Part* I: *The case of bounded stochastic evolutions*, Preprint, 1988.

10. M. Nisio, *Some remarks on stochastic optimal controls*, Proc. 3rd Japan–USSR Symposium on Probability Theory, Springer-Verlag, New York, pp. 446–460.

11. D. W. Stroock and S. R. S. Varadhan, *Multidimensional diffusion processes*, Springer-Verlag, New York, 1979.

DEPARTMENT OF MATHEMATICS, TEMPLE UNIVERSITY, PHILADELPHIA, PENNSYLVANIA 19122