Infinitary Normalization

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Dedicated to Dov Gabbay, in celebration of his 60th anniversary.

ABSTRACT. In infinitary orthogonal first-order term rewriting the properties confluence (CR), Uniqueness of Normal forms (UN), Parallel Moves Lemma (PML) have been generalized to their infinitary versions CR^∞, UN^∞, PML^∞, and so on. Several relations between these properties have been established in the literature.

Generalization of the termination properties, Strong Normalization (SN) and Weak Normalization (WN) to SN^∞ and WN^∞ is less straightforward. We present and explain the definitions of these infinitary normalization notions, and establish that as a global property of orthogonal TRSs they coincide, so at that level there is just one notion of infinitary normalization. Locally, at the level of individual terms, the notions are still different. In the setting of orthogonal term rewriting we also provide an elementary proof of UN^∞, the infinitary Unique Normal form property.[1][2]

Keywords and phrases: term rewriting systems, infinitary term rewriting, normalization, strong normalization, infinite normal form, unique normal forms, ordinal numbers

1 Outline

We work in the framework of infinitary first-order term rewriting, dealing with transfinite rewrite sequences that may converge to a limit and the fundamental notion of an infinite normal form. Infinite normal forms can e.g. be seen to arise quite naturally as the limits of infinite processes generating "streams" of natural numbers, for example the primes or the fibonacci numbers.

[1]This paper appeared in the two-volume book of essays in honour of the 60th birthday of Dov Gabbay. Please use [Klop and de Vrijer, 2005] for reference purposes.

[2]This research is in the framework of the project INFINITY: Infinite objects, computation, modeling and reasoning, supported by NWO-BRICKS/FOCUS.

After recapitulating some of the basic definitions and facts, we consider the question of how to generalize the notions of weak normalization (WN) and strong normalization (SN) to the infinite setting, obtaining WN^{∞} and SN^{∞}, respectively. It turns out, somewhat surprisingly, that when applied to orthogonal term rewriting systems (OTRSs), these notions coincide. Moreover, although CR^{∞} is no longer a general property of infinitary orthogonal rewriting, we still have that—finite or infinite—normal forms are unique (UN^{∞}). The proofs of these facts use classical techniques of orthogonal term rewriting, such as the parallel moves lemma, generalized to the infinitary setting and an analysis of infinitary head normalization.

The notion of SN^{∞} was first introduced in [Kennaway, 1992]. A slightly different definition than ours is given there, in topological terms, in the context of abstract rewriting. Given our concrete approach to the notion of infinitary reduction the two definitions appear to amount to the same. Also UN^{∞} for orthogonal TRSs is not new here; in [Terese, 2003] it is shown as a consequence of an analysis of meaningless terms and Böhm trees in a broader setting, including infinitary lambda calculus. For first-order term rewriting systems a simpler proof of UN^{∞} can be given, making use of the infinitary parallel moves lemma. Note that PML^{∞} fails for the lambda calculus. The main technical contribution of the present paper is that WN^{∞} and SN^{∞} coincide as properties of infinitary orthogonal TRSs.

2 Introduction to infinitary term rewriting

We are concerned with the framework of first-order term rewriting and we assume familiarity with that area. For general background reading on term rewriting the reader may consult any standard text, for example [Baader and Nipkow, 1998], [Terese, 2003], [Klop, 1992], [Dershowitz and Jouannaud, 1990] and for more specific information on infinitary rewriting [Kennaway *et al.*, 1995], [Klop and de Vrijer, 1991] or the chapter Infinitary Rewriting by Kennaway and de Vries in [Terese, 2003]. Some of the basic notions will be recapitulated when and where needed, and the same for notation. We will also assume some familiarity with ordinal numbers.

2.1 Finitary and infinitary perspectives on term rewriting

One aspect of term rewriting is that it can be used to model computations with normal forms as the intended outcomes.

As a simple example consider the TRS $\mathcal{N}$ specifying the natural numbers with zero, successor and addition, with the reduction rules of Figure 1, which go back to [Dedekind, 1888].

Closed terms in this TRS represent arithmetical expressions involving the addition operator and the outcomes are the closed normal forms 0, $S(0)$,

$$\begin{array}{|lcl|}\hline A(x,0) & \to & x \\ A(x,S(y)) & \to & S(A(x,y)) \\ \hline \end{array}$$

Figure 1. Dedekind's rules

$S(S(0))$, etcetera, the *numerals*. Thus we have e.g. the reduction

$$\begin{array}{lcl} A(A(S(S(0)),0),S(0)) & \to & S(A(A(S(S(0)),0),0)) \\ & \to & S(A(S(S(0)),0)) \\ & \to & S(S(S(0))) \end{array}$$

modeling a computation of $(2 + 0) + 1$ with outcome 3. This example illustrates the interest in terminating reductions with finite terms (normal forms) as outcomes.

There is also an infinitary aspect to term rewriting. Again consider the TRS $\mathcal{N}$, its signature now expanded with a binary symbol P and a unary E and with the additional reduction rule $E(x) \to P(x, E(S(S(x))))$. Think of P as pairing, or a list-forming `cons` operator, which we will also denote by the infix symbol : for better readability. Beginning with the term $E(0)$ we now have the following infinite reduction

$$E(0) \;\to\; 0 : E(S(S(0))) \;\to\; 0 : S(S(0)) : E(S(S(S(S(0))))) \;\to\; \cdots$$

The consecutive terms in this reduction appear to converge in the limit to an infinite "term" representing the *stream* of even natural numbers, namely

$$0 : S(S(0)) : S(S(S(S(0)))) : S(S(S(S(S(S(0)))))) : \;\ldots$$

Likewise the stream of all naturals is generated from the term $N(0)$ using a unary symbol N with the reduction rule $N(x) \;\to\; x : N(S(x))$ and for example the constant stream of zeros $0 : 0 : 0 : 0 : \;\ldots$ can be obtained as the limit of an infinite reduction

$$Z \to 0 : Z \to 0 : 0 : Z \to 0 : 0 : 0 : Z \to \;\cdots$$

starting from a constant Z and using the single reduction rule $Z \to 0 : Z$

The finitary and infinitary perspectives on term rewriting give rise to a difference in emphasis on properties of term rewriting systems. If the finite normal forms are considered as the outcomes of computations, strong normalization is an attractive property: regardless of your reduction strategy, an outcome will always be found. The objective of generating streams as outcomes, on the other hand, is incompatible with strong normalization.

2.2 Convergence and limits defined

The structure of the *infinite terms* that arise as limits is most clearly displayed by representing them as infinite term trees. In Figure 2 infinite term trees are drawn for the streams of zeros and naturals, respectively generated by the terms Z and $N(0)$. Note that the function symbols Z and N themselves do not occur anymore in the limit terms, although they did occur in all finite approximations.

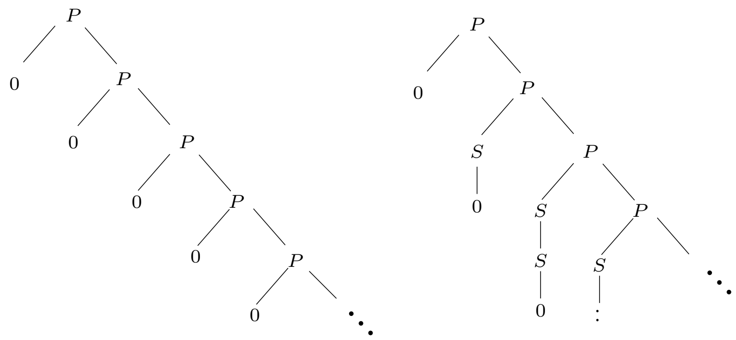

Figure 2. Infinite term trees

The question arises what the formal status of infinite terms is. We just offer a quick and informal answer. For a more extensive treatment we refer to [Terese, 2003]. First note that each finite term corresponds one-one to its *term tree*: a finite set of labeled positions satisfying the following three requirements:

1. the set of positions is closed under prefixes
2. each position is labeled by a function symbol or a variable
3. the arity of the function symbol at a position equals the number of outgoing edges at that position

So the constants (function symbols of arity 0) and the variables are at the endpoints of the term tree. Now we take an *infinite term tree* to be just a possibly infinite set of labeled positions, satisfying the same requirements 1-3. So in particular in an infinite term each position has a finite distance to the root.

It is sometimes convenient to use recursion equations to characterize infinite term trees. The example of the stream of zeros is then given by the equation $t = P(0,t)$, or shorter $t = 0 : t$, with the obvious semantics.[3]

Having extended our domain of rewriting to the infinite terms, we keep the notion of rewriting itself as it was: rewrite rules are pairs of finite terms.[4] However, now also infinite terms can be rewritten. A redex $C[l^\sigma]$ is still identified as a pattern occurring at a finite position with prefix C, but now the substitution σ may involve also infinite terms. So for example with the reduction rule $I(x) \to x$, the infinite term $I(I(I(I\ldots)))$ characterized by the equation $t = I(t)$ has a redex at each of its positions. Note that in this particular case all these redexes and their reducts happen to be the same term, all identical to the original term itself, which we will henceforth denote by I^ω.

In order to explain the notion of convergence we use, we consider again the example of the stream of the naturals. We have the infinite reduction

$$N(0) \to 0 : N(S(0)) \to 0 : S(0) : N(S(S(0))) \to \cdots$$

In Figure 3 we get a clear picture by drawing the term trees again.

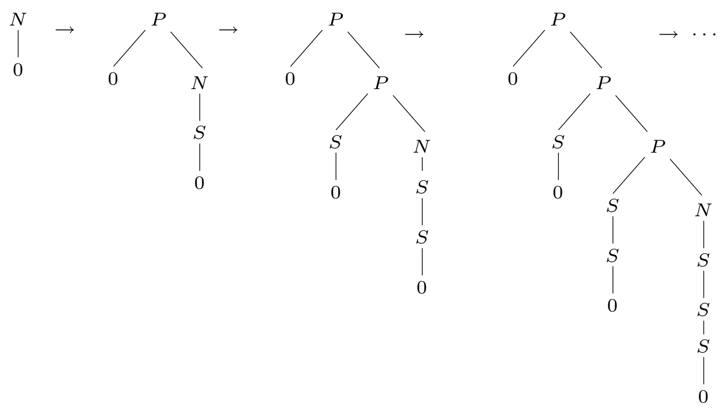

Figure 3. Converging infinite reduction

Looking at the terms as they evolve during this reduction, two things stand out. First, after each step a larger and larger prefix remains fixed throughout

[3]Of course one thinks here of a least fixed-point semantics. We note that it could in fact be implemented by the very techniques of infinitary term rewriting that this paper is about.

[4]One may also consider rewrite rules involving infinite terms. Most of the existing theory of infinitary rewriting extends to rewrite rules with infinite righthand side. For rules with infinite lefthand sides the situation is less clear.

the rest of the reduction and, secondly, the depth (i.e. the distance to the root) of the contracted redexes increases. In accordance with these observations we formulate the following two conditions:

1. For any depth n the prefix of positions up to depth n eventually gets fixed throughout the rest of the reduction.
2. The depth (i.e. the distance to the root) of the contracted redexes eventually grows beyond any finite value.

Note that condition 1 alone already would be sufficient to guarantee the existence of a well-defined limit, any position has a finite depth and hence will eventually be fixated. In the literature this is called *weak convergence* and it was used in the ground-breaking paper on infinitary term rewriting [Dershowitz *et al.*, 1991]. However, later developments have made it clear that it is better to require the stronger property 2, which is easily seen to imply 1. The reason is that with the stronger notion the resulting theory of infinitary term rewriting is much better behaved.[5] For details of this consult the background literature, e.g. [Terese, 2003]. Convergence according to the stronger notion 2 is usually called *strong convergence*, but since we will not be concerned anymore with alternative notions we will also call it *convergence* without more.

An example af a weakly but not strongly convergent reduction sequence can be given for example with the reduction rule $A(x) \to A(B(x)$:

$$A(x) \to A(B(x)) \to A(B(B(x))) \to A(B(B(B(x)))) \to \cdots$$

The infinite term A(B(B(B(. . .)))) could in principle be considered as the limit, but we will not do so because, as all redex contractions occur at the root, the reduction sequence is not strongly convergent.

It is illustrative to contrast this example of non-convergence with a "mirror" example, the TRS with the rule $A \to B(A)$. The reduction

$$A \to B(A) \to B(B(A)) \to B(B(B(A))) \to \cdots$$

is convergent, with as limit the infinite normal form B^ω. To indicate a convergent reduction of length ω we write $A \to^\omega B^\omega$.

2.3 Transfinite reductions

Reduction can proceed beyond the ordinal ω. An easy way to see this is by adding a passive pairing operator to the signature of the last example and considering the convergent reduction

$$P(A, A) \to P(B(A), A) \to P(B(B(A)), A) \to P(B(B(B(A))), A) \to \cdots$$

[5] More specifically, adopting 2 enables us to extend the notion of descendant of a subterm after a reduction to infinite reductions, where the critical point in the definition is of course the limit case.

We have that $P(A, A) \to^{\omega} P(B^{\omega}, A)$, but now the limit $P(B^{\omega}, A)$ is not a normal form, as it contains the redex A. So after the first ω steps another one is possible and so on, and by taking another limit we reach the normal form $P(B^{\omega}, B^{\omega})$, in $\omega + \omega$ steps. Notation: $A \to^{\omega+\omega} P(B^{\omega}, B^{\omega})$.

Now it is not difficult to construct longer reductions, for example the infinite term characterized by $t = P(A, t)$ has a reduction of length ω^2 to $t = P(B^{\omega}, t)$. As we will see in Example 3 below, convergent transfinite reductions can be constructed of any countable ordinal length.

DEFINITION 1. We sum up the notion of a transfinite reduction ρ of length β. It consists of rewrite steps $t_\alpha \to_{s_\alpha} t_{\alpha+1}$:

$$\rho : \quad t_0 \to_{s_0} t_1 \to_{s_1} \cdots\ t_\omega \to_{s_\omega} t_{\omega+1} \to_{s_{\omega+1}} \cdots$$

or in a more compact notation $\rho = (s_\alpha)_{\alpha<\beta}$. This only makes sense if for each limit ordinal $\lambda < \beta$ the *prefix* $\rho_\lambda = (s_\alpha)_{\alpha<\lambda}$ of ρ is convergent, with limit t_λ. Let d_α be the depth—in the term t_α—of the redex that is contracted in step s_α. Then for each limit ordinal $\lambda < \beta$ we must have that $(d_\alpha)_{\alpha<\lambda}$ tends to infinity.

Figure 4 depicts the course of the redex depths in a convergent reduction of length ω^2.

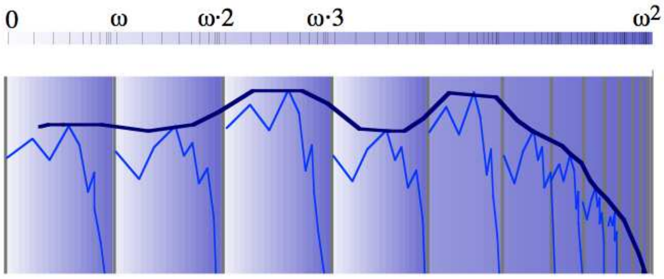

Figure 4. Convergence at each limit ordinal $\leq \omega^2$

Note that in Definition 1 we do not require convergence at the ordinal β, that is, convergence of the whole reduction. We get back to the topic of divergent transfinite reductions in Section 2.5.

2.4 Compression

Continuing the example of the last section, there is also a convergent reduction $P(A, A) \to^{\omega} P(B^{\omega}, B^{\omega})$, by alternating between performing steps in the left and right argument positions of P:

$$P(A, A) \to P(B(A), A) \to P(B(A), B(A)) \to P(B(B(A)), B(A)) \to \cdots$$

The transition from the reduction of length $\omega \cdot 2$ to that of length ω is called *compression*. A key result in the theory of infinitary rewriting is that for OTRSs compression of transfinite reductions to length ω is always possible. As a matter of fact, left-linearity of the TRS already suffices for having compression.

2.5 Convergence and divergence

For a transfinite reduction ρ of length λ with λ a limit ordinal there are two possibilities: either ρ is convergent or it is not. In the first case there is a limit term t_λ and we have that $\rho : t_0 \to^\lambda t_\lambda$. In the second case we say that ρ is *divergent*. Note again that for any reduction of length λ, convergent or divergent, all prefixes must be well-defined, meaning convergence at each limit ordinal $< \lambda$.

Now we look into the question of what constitutes a divergent reduction ρ of length λ. Let ρ again consist of steps s_α. Divergence of ρ means that there exists a finite number n such that for every $\alpha < \lambda$ there exists a $\beta > \alpha$ such that the step s_β has depth $\leq n$. If we then take N to be the smallest such n, we have that infinitely many steps s_β have depth N.

Conversely, if for some N there are infinitely many steps of depth N, then we have divergence. For let $X = \{\alpha \mid d_\alpha = N\}$ be an infinite set. Then we construct an infinite sequence of ordinals $\alpha_1, \alpha_2, \ldots$ such that for all i, $d_{\alpha_i} = N$ by taking α_1 the smallest element of X, α_2 the next smallest and so on. At the limit β of this increasing sequence we do not have convergence of ρ_β. We must then have $\beta = \lambda$, as otherwise a prefix of ρ would be ill-defined, and hence the reduction ρ is divergent.

We proved:

THEOREM 2. *A transfinite reduction is divergent if and only if for some N there are infinitely many steps at depth N.*

An immediate consequence of this theorem is that all convergent transfinite reductions have countable length. Namely for each N there can only be finitely many steps of depth N and a countable union of finite sets is countable.

But we can also directly prove a stronger result, namely that all reductions, no matter whether they are convergent or divergent, must be countable. For assume not. Then there is certainly a reduction $\rho = (s_\alpha)_{\alpha < \omega_1}$, where ω_1 is the first uncountable ordinal. Infinitary pigeon holing yields an $N \in \omega$ such that ρ has infinitely (even uncountably) many steps at depth N. As before we consider the infinite set $X = \{\alpha \mid d_\alpha = N\}$ and construct an infinite increasing sequence $\alpha_1, \alpha_2, \ldots$ of ordinals inside X. At the limit β of this increasing sequence we do not have convergence of the prefix ρ_β, but since β is a countable limit ordinal, $\beta < \omega_1$ and hence ρ is not well-defined.

EXAMPLE 3. It is instructive and also entertaining to play some more with the infinite term I^ω, which, as we already remarked, reduces only to itself and has at each position an identical redex. So, as a rewrite step is determined by its depth, a transfinite reduction $\rho = (s_\alpha)_{\alpha<\lambda}$ of I^ω can be identified with the sequence $(d_\alpha)_{\alpha<\lambda}$, where d_α is the depth of s_α.

The constant sequence $(0, 0, 0, \ldots)$, for example, codes the divergent reduction of length ω consisting of only root steps, whereas in the reduction $(0, 1, 2, \ldots)$ the depth of the steps tends to infinity, hence it converges. An example of a divergent reduction of length $\omega \cdot 2$ is $(0, 1, 2, \ldots\ 0, 0, 0, \ldots)$. We find converging and diverging reductions $(d_\alpha)_{\alpha<\omega^2}$ and $(d'_\alpha)_{\alpha<\omega^2}$ of length ω^2 by taking $d_{\omega\cdot n+m} = n + m$ and $d'_{\omega\cdot n+m} = m$, respectively.

We will now show that (1) the term I^ω admits a convergent reduction of length any countable ordinal λ, and (2) if λ is a limit ordinal also a divergent reduction.

1. Take a countable λ. So there is a bijection $d : \lambda \to \omega$. Then the reduction $(d(\alpha))_{\alpha<\lambda}$ is a convergent reduction of length λ by Theorem 2.
2. We can use 1 to construct also a divergent reduction of length λ, if it is a limit ordinal. For consider a convergent reduction $\rho = (d_\alpha)_{\alpha<\lambda}$ and let $\alpha_1, \alpha_2, \ldots$ be an increasing sequence of ordinals with limit λ. Define $\rho' = (d'_\alpha)_{\alpha<\lambda}$, where $d'_{\alpha_i} = 0$ for all $i \in \omega$ and $d'_\alpha = d_\alpha$ for all other $\alpha < \lambda$. Then one easily sees that ρ'_β still converges at any limit ordinal $\beta < \lambda$, as it differs from ρ_β in at most finitely many places. Hence ρ' is well-defined and it obviously diverges at λ.

3 Normal forms and normalization properties

3.1 WN and SN in the finitary setting

Consider the TRS $\mathcal{T} = \{f(x) \to b,\ a \to f(a)\}$. In $\mathcal{T}$ we have an infinite reduction originating from the term a:

$$a \to f(a) \to f(f(a)) \to f(f(f(a))) \to \cdots$$

but on the other hand any term can be reduced to a normal form, in particular we have $a \twoheadrightarrow b$ via the reduction

$$a \to f(a) \to b$$

This is an example of a TRS that is *weakly normalizing* (WN), every term reduces to a normal form, but not *strongly normalizing* (SN), there are reductions that go on indefinitely, without ever reaching a normal form.[6]

[6]Term rewriting systems that are SN are also called *terminating*.

Of course there is the trivial implication SN $\implies$ WN and the above example shows that the converse does not hold. In this respect it is worth mentioning a classic result of [O'Donnell, 1977], specifying a situation where the converse *does* go through. It goes back to [Church, 1941], where it is proved that in the λI-calculus WN and SN are equivalent. A TRS is called *non-erasing* if in each rewrite rule all variables in the left-hand side occur also in the right-hand side.

THEOREM 4 (O'Donnell). *For non-erasing OTRSs we have WN $\iff$ SN.*

For the sake of completeness we add a recently discovered fact from [Ketema *et al.*, 2005]. Here AC is the property of *acyclicity*, there are no reduction cycles.

THEOREM 5. *For OTRSs we have WN $\implies$ AC.*

3.2 Infinite normal forms

A normal form, finite or infinite, is a term from which no rewrite step is possible, that is, a term without redex occurrences. Typical examples of infinite normal forms are the terms representing the streams of zeros and naturals, depicted in Section 2.2. Less standard examples in the TRS $\mathcal{N}$ of addition are the infinite term S^ω and the infinite binary tree labeled with A's, defined by the recursion equation $t = A(t,t)$.

A typical example of an infinite term that is not a normal form is the term I^ω from Section 2.2. As we already indicated this term does not reduce to a normal form either, as it can only reduce to itself. So I^ω is an example of a term that is not WN$^\infty$, where WN$^\infty$ is the property of reducing by a possibly transfinite reduction to an (infinitary) normal form.

3.3 The notion of infinitary strong normalization

Now we want to consider the question what SN$^\infty$ could mean. To keep the analogy with finitary SN it should be something like: no matter how you reduce, if you just keep going, in the end a normal form will always be reached. Naturally it might take any (countable) transfinite number of steps. This analysis leads us to the following provisional definition.[7]

> A term t has the property SN$^\infty$ if any maximal transfinite reduction from t reduces t to a normal form.

[7] In [Kennaway, 1992] a similar phrasing is rejected, apparently due to a subtle difference in perspective. Our transfinite reductions of length α presuppose (strong) convergence at all limit ordinals $\lambda < \alpha$ from the outset, whereas Kennaway also considers weak convergence, only to be eliminated later on. The resulting notions of SN$^\infty$ are the same.

The question that then has to be answered is of course: what are maximal reductions? But that is not difficult. There are just two types of transfinite reductions from t that cannot be prolonged:

1. The reductions that reduce t to a normal form.
2. The reductions that diverge.

Now the first possibility satisfies the provisional criterion, and only the second violates it. Hence it is clear what the definition of SN^∞ should be.

DEFINITION 6. A (finite or infinite) term is SN^∞ if it has no divergent reductions.

The notions of finitary SN and infinitary SN^∞ are independent, as we will point out in 1 and 2 below. Here especially the failure of $\mathrm{SN} \implies \mathrm{SN}^\infty$, although easy to understand, may come as a surprise.

1. $\mathrm{SN} \not\Rightarrow \mathrm{SN}^\infty$. Consider the two-rule TRS $\mathcal{N}$ for addition from Section 2.1. It is clearly SN. However SN^∞ fails as witnessed by the infinite term recursively defined by $t = A(t, 0)$. We have $t \to t$, yielding a divergent reduction, i.e. $\neg\mathrm{SN}^\infty$.

   For another, even simpler counterexample consider again the one-rule TRS $I(x) \to x$, trivially SN, and the infinite term I^ω that has no normal form.

2. $\mathrm{SN}^\infty \not\Rightarrow \mathrm{SN}$. There is a far from obvious couterexample here: take the fragment of Combinatory Logic (CL) consisting of the terms solely built by application from the combinator S, with the reduction rule $Sxyz \to xz(yz)$. This is an orthogonal TRS which is not SN, as e.g. the term $SSS(SSS)(SSS)$ has an infinite reduction. But it has the property SN^∞, according to [Waldmann, 2000].

3.4 Newman's Lemma

A typical application of SN in the finitary case is in Newman's Lemma: $\mathrm{WCR}\,\&\,\mathrm{SN} \implies \mathrm{CR}$. However, infinitary Newman Lemma fails, that is, we do not have the implication $\mathrm{WCR}\ \&\ \mathrm{SN}^\infty \implies \mathrm{CR}^\infty$. The following is an easy counterexample. An alternative counterexample can be found in [Kennaway, 1992].

EXAMPLE 7. Consider the TRS R with the three rules:

$$\begin{array}{lcl} C & \to & A(C) \\ C & \to & B \\ A(B) & \to & B \end{array}$$

So R is not orthogonal. All reductions from C are depicted in Figure 5. There are two normal forms, A^ω and B. Hence UN^∞ does not hold and

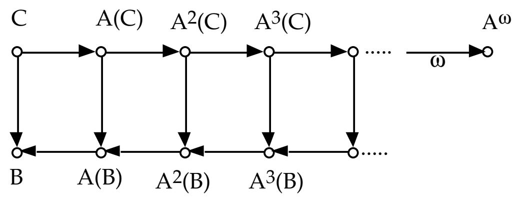

Figure 5. Reduction graph of C

neither does CR^∞. As all relevant terms of R are shown, it is clear that R is CR and also WCR. We also have SN^∞, as one easily sees.

4 Infinitary orthogonal rewriting

It is by now well-known that even for orthogonal TRSs infinitary confluence may fail. This is shown by the following example.

EXAMPLE 8. Consider the TRS with rules

$$\begin{array}{lcl} C & \to & A(B(C)) \\ A(x) & \to & x \\ B(x) & \to & x \end{array}$$

Then CR^∞ fails since C reduces in ω steps to A^ω and B^ω, infinite terms that both only reduce to themselves, so having no common reduct.

In this example it is essential that there are two collapsing rewrite rules and conditions can be given under which CR^∞ does go through, but we will not pursue this matter here, see [Kennaway *et al.*, 1995]. An important reason for the interest in confluence is that it implies uniqueness of normal forms. Below we will prove that despite the failure of CR^∞, in infinitary orthogonal rewriting we do have UN^∞, uniqueness of normal forms.

4.1 The Parallel Moves Lemma

A *parallel step* consists of the contractions of a possibly infinite set of disjoint redexes. This can be done in any order and will always result in a convergent reduction.

Fundamental is the infinitary parallel moves lemma (PML^∞) for OTRSs in this form:

THEOREM 9. *It is always possible to construct the finite or transfinite reduction diagram of a convergent reduction ρ against a parallel step p. The projection p/ρ is again a parallel step and ρ/p a convergent reduction.*

By way of example we give a more microscopic view of this construction for the case that ρ has length $\omega \cdot 2$. So we have

$$\rho: \quad t_0 \to_{s_0} t_1 \to_{s_1} \cdots \quad t_\omega \to_{s_\omega} t_{\omega+1} \to_{s_{\omega+1}} \cdots \quad t_{\omega\cdot 2}$$

Define $p_0 = p$ and $p_\alpha = p/\rho_\alpha$ (where ρ_α is the prefix of ρ of length α). Note that then also $p_{\alpha+1} = p_\alpha/s_\alpha$.) Finally let $S_\alpha = s_\alpha/p_\alpha$.

We then have locally at each t_α the diagram construction with initial term t_α of the single step $t_\alpha \to_{s_\alpha} t_{\alpha+1}$ against the parallel step p_α. The bottom and right residual steps are the S_α and $p_{\alpha+1}$. The complete diagram construction consists of collating the local diagrams. It is essential here that at the limits we have convergence. See Figure 6.

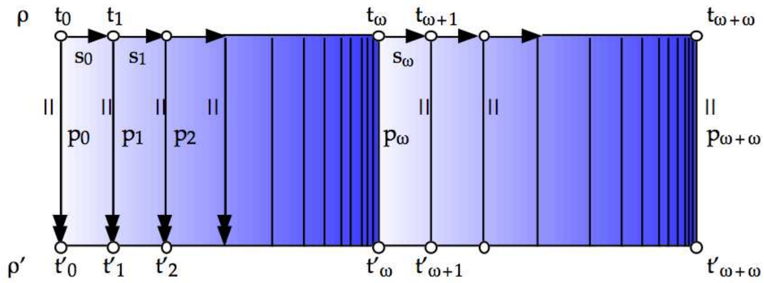

Figure 6. PML$^\infty$ for a transfinite reduction of length $\omega \cdot 2$

To appreciate that PML$^\infty$ is non-trivial we mention that for infinitary lambda calculus (λ^∞), this fundamental lemma fails.

One can make sense also of the projection of a divergent reduction ρ over a convergent reduction σ, given that the projections of ρ's prefixes exist. Let the ρ_α's ($\alpha < \lambda$) be the prefixes of ρ. Then, as the prefixes of a divergent reduction are convergent reductions, we can take ρ/σ as the union of the ρ_α/σ, $\alpha < \lambda$. A priori ρ/σ can be either convergent or divergent.

In fact, already the projections of convergent reductions σ and ρ over each other may be divergent. An example is the reduction diagram in Figure 7. It is obtained from the reductions $C \to^\omega A^\omega$ and $C \to^\omega B^\omega$ in the counterexample to CR$^\infty$ above. In Figure 7 the steps crossing a light layer are empty steps. The reduction $A^\omega \to A^\omega \to \cdots$ in the righthand side is a root reduction, hence divergent; likewise the reduction from B^ω at the bottom.

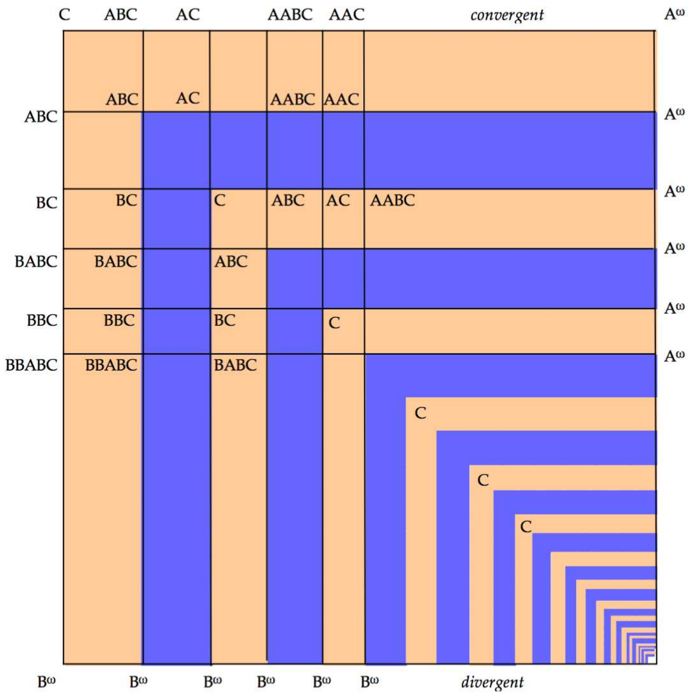

Figure 7. Non-confluent infinite reduction diagram

5 Unique normal forms: UN$^\infty$ for OTRSs

In this section we assume orthogonality for all TRSs and under this restriction we prove uniqueness of infinitary normal forms (UN$^\infty$).

There is the well-known distinction between *root* (or *head*) reduction and *non-root* reduction, also called *internal* reduction. In a root (head) step the root of the term is part of the contracted redex, in an internal step the root is left untouched. We will generalize this notion of internal reduction relative to the root position to internal reduction relative to an arbitrary prefix C. This is called *C-stable reduction* or ι_C-reduction and it leaves all of C untouched.

DEFINITION 10.

1. Let C be a prefix of t. A rewrite step from t is *C-stable* (a ι_C-step) if the contracted redex lies below C. Idem for parallel steps.

2. *C-stable* reduction or *ι_C-reduction* is reduction consisting of only C-stable steps.

3. A term is called *C-stable* if it allows only C-stable reduction.

4. If C is the "full" prefix up to depth n then C-stable reduction, is also called *n-stable* reduction. Accordingly a term allowing only n-stable reduction is called *n-stable*.

So internal or root-stable reduction is the same as 1-stable reduction, or C-stable reduction where C is just the root. A root-stable term is also called a *head normal form*.

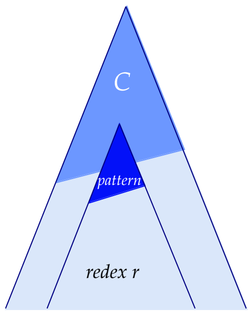

Figure 8. Redex r overlapping with prefix C

LEMMA 11.

1. *If redex r overlaps with C and $t \to s$ by a ι_C-step, then r has a unique residual in s, at the same position and still a redex.*

2. *The same for $\rho : t \twoheadrightarrow s$.*

3. *The same for $\rho : t \to^\alpha s$.*

Proof.

1. This is the crux of orthogonal rewriting. Not only C is stable, but also $C \cup \pi(r)$, the union of C with the pattern $\pi(r)$ of r.

2. Repetition of 1.

3. If the prefix $C \cup \pi(r)$ is stable in ρ, it will also be present in the limit (by the very definition of what a limit is).

■

PROPOSITION 12. *If t reduces to an infinite normal form by ι_C-reduction, then no redex in t overlaps with C.*

Proof. By Lemma 11(3). ■

It is easy to see that projection of (parallel) ι_C-steps over each other yields ι_C-steps as residuals.

LEMMA 13. *Projection of a ι_C-reduction over a (parallel) ι_C-step yields a ι_C-reduction again.*

PROPOSITION 14.

1. *If t reduces to an infinite normal form by ι_C-reduction ρ and $t \to s$, then C is a prefix of s and no redex in s overlaps with C.*
2. *If t reduces to an infinite normal form by ι_C-reduction, then t is C-stable.*
3. *If t reduces to an infinite normal form by n-stable reduction, then t is n-stable.*

Proof.

1. No redex in t overlaps with C because of Proposition 12. Hence $t \to_p s$ is a ι_C-step. Then by Lemma 4 the projection ρ/p is a ι_C-reduction again, by PML$^\infty$ to a normal form. Apply Proposition 12 again.
2. By Proposition 12 no redex in t itself overlaps with C and by repeating (1) we see that this property is preserved under reduction.
3. This is a special case of (2).

■

THEOREM 15. *For OTRSs we have UN$^\infty$.*

Proof. Consider two reductions ρ and σ from the same term t, both converging to infinitary normal forms, say nf_1 and nf_2, respectively.

$$\begin{array}{ll} \rho: & t_0 \to_{s_0} t_1 \to_{s_1} \cdots nf_1 \\ \sigma: & t'_0 \to_{s'_0} t'_1 \to_{s'_1} \cdots nf_2 \end{array}$$

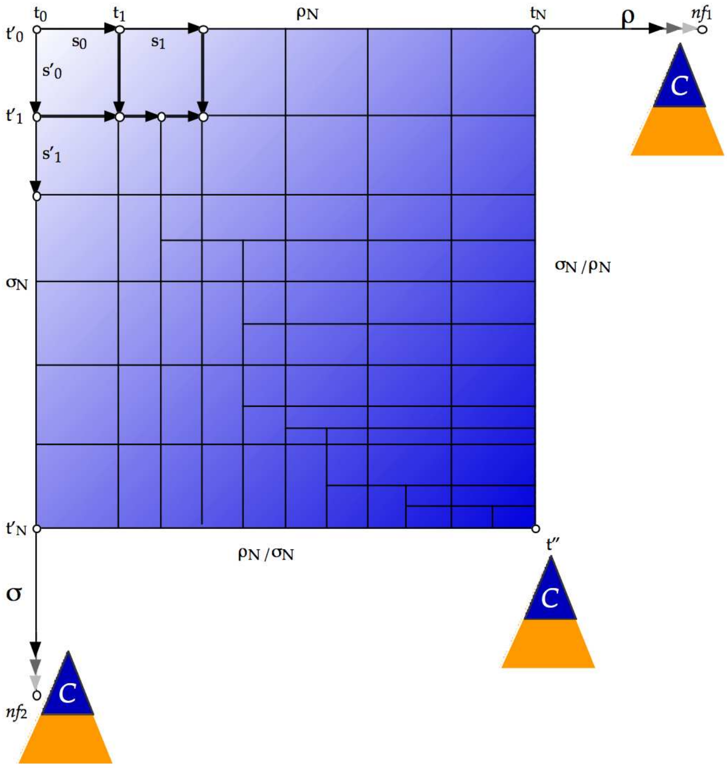

Figure 9. Proof of UN^{∞}

By compression we may assume that ρ and σ have length at most ω. We show that all finite prefixes of the normal forms nf_1 and nf_2 are identical, hence $nf_1 = nf_2$. The proof is depicted in Figure 9.

Let C be a finite prefix of nf_1, say of depth n. Both in ρ and in σ all redex activity will eventually be below depth n, say after N_1 and N_2 steps respectively, with $N = \max(N_1, N_2)$. Now note that by Proposition 14(3), the terms t_N and t'_N are n-stable, as both reduce to infinite normal form by n-stable reduction. And in particular t_N will be C-stable.

Projecting the finite reductions ρ_N and σ_N yields a common reduct t'' of t_N and t'_N. Because of the n-stability of t_N and $t_{N'}$, t'' will share its prefix up to depth n with both these terms. So in particular C will also be a prefix of t'_N, and hence of nf_2. ■

As a corollary we now have the following theorem from [Dershowitz *et al.*, 1991].[8] The proof is immediate, analogous to that of UN & SN $\Longrightarrow$ CR for the finitary case.

COROLLARY 16. *For OTRSs the implication* $\mathrm{SN}^\infty \Longrightarrow \mathrm{CR}^\infty$ *holds.*

6 The equivalence of $\mathbf{SN}^\infty$ and $\mathbf{WN}^\infty$

Also in this section we assume orthogonality. We show that as properties of an OTRS the notions SN^∞ and WN^∞ are equivalent.

PROPOSITION 17. *Consider a transfinite divergent reduction ρ of length λ, containing infinitely many head steps and a coinitial parallel step p. Then the projections $\sigma_\alpha = \rho_\alpha/p$ ($\alpha < \lambda$) are the prefixes of a divergent reduction σ of length λ, also containing infinitely many head steps.*

Proof. Let s_α be the step in ρ performed at ordinal α. So we have

$$\rho : \ t_0 \to_{s_0} t_1 \to_{s_1} \ \cdots \ t_\omega \to_{s_\omega} t_{\omega+1} \to_{s_{\omega+1}} \cdots$$

Define $p_0 = p$ and $p_\alpha = p/\rho_\alpha$ (Note that then also $p_{\alpha+1} = p_\alpha/s_\alpha$.) Finally let $S_\alpha = s_\alpha/p_\alpha$.

There are two possibilities:

1. For all ordinals α such that s_α is a head step, p_α is internal. Then each of the corresponding S_α's is a head step as well: infinitely many.

2. For some α such that s_α is a head step, p_α is a head step too. Then $p_{\alpha+1} = \varnothing$ and hence $p_\beta = \varnothing$ for all $\beta > \alpha$. Hence $S_\beta = s_\beta$ for all $\beta > \alpha$. Since the set $\{s_\beta \mid \beta > \alpha\}$ contains infinitely many head steps, so does $\{S_\beta \mid \beta > \alpha\}$.

In both cases σ contains infinitely many head steps, hence diverges. ■

COROLLARY 18. *Projection of a ρ containing infinitely many head steps over any finite reduction yields a σ containing infinitely many head steps.*

PROPOSITION 19. *Assume t has a divergent reduction ρ containing infinitely many head steps. Then t does not reduce to a head normal form.*

---

[8] As a matter of fact, in [Dershowitz *et al.*, 1991] instead of SN^∞ a notion is used that is called there *top termination* and which corresponds to our notion SHN^∞ (see Section 7). However, a result of the present paper is that as properties of OTRSs the notions SN^∞ and SHN^∞ are equivalent.

Proof. For a proof by contradiction suppose t has a transfinite reduction to head normal form h. Claim: t can be reduced to h in a finite reduction φ. For, suppose not. Then by compression we may assume that $t \to^{\omega} h$. For this reduction to be convergent the head must be in rest after finitely many steps. By Corollary 18 projection of ρ over φ yields a divergent reduction from h containing infinitely many head steps. This contradicts the assumption that h is a head normal form. ∎

COROLLARY 20. *A term t cannot both have a normal form and a reduction containing infinitely many head steps.*

PROPOSITION 21. *Consider an OTRS $\mathcal{T}$ and suppose there exists a divergent reduction ρ in $\mathcal{T}$. Then there is in $\mathcal{T}$ also a divergent reduction σ containing infinitely many head steps.*

Proof. Let $\rho = (s_\alpha)_{\alpha<\lambda}$, as in the display in the proof of Proposition 17 above. Divergence of ρ implies that λ is a limit ordinal and that for some n we have that for every $\alpha < \lambda$ there exists a $\beta > \alpha$ such that the step s_β has depth $\leq n$. Let N be the smallest such n.

If $N = 0$, then we are done, take $\sigma = \rho$.

Otherwise $N > 0$ and then, as ρ has only finitely many steps with depth $< N$, there exists an ordinal $\Gamma < \lambda$ such that for all $\beta > \Gamma$ the depth of step s_β is $\geq N$. Beyond Γ the prefix of the term t_Γ consisting of all positions up to depth $N-1$ will be fixed throughout the rest of the reduction ρ. So there is a fixed finite set of positions of depth N, uniform for all terms t_β, $\beta > \Gamma$. At these positions all infinitely many steps s_β, $\beta > \Gamma$ of depth N must take place and by pigeon holing at least one of these positions, say P, then will have infinitely many of these steps. Now consider the reduction from the term $t_\Gamma|_P$ consisting of all steps s_β, $\beta > \Gamma$ that take place at or below P. That will be a transfinite reduction σ containing infinitely many root steps, which is therefore also divergent. ∎

THEOREM 22. *For OTRSs we have* $\mathrm{SN}^\infty(\mathcal{T}) \iff \mathrm{WN}^\infty(\mathcal{T})$.

Proof. $(\Rightarrow)$ is trivial.

$(\Leftarrow)$ Assume that $\mathcal{T}$ is not SN^∞. That is, in $\mathcal{T}$ there exists a divergent reduction. Then by Proposition 21 there exists in $\mathcal{T}$ also a reduction with infinitely many head steps. By Corollary 20 it then follows that $\mathcal{T}$ is not WN^∞. ∎

On reflection, recalling O'Donnell's Theorem 4 (OD) which states that in the finitary case the equivalence of WN and SN follows from non-erasingness, it is remarkable that according to Theorem 22 in the infinitary setting the equivalence holds without more. Here it is crucial to keep in mind that Theorem 22 is about WN^∞ and SN^∞ as properties of an OTRS $\mathcal{T}$. At the level of terms the infinitary case deviates from finitary as well, but quite in the opposite direction: for terms O'Donnell's Theorem does not hold at all. Abbreviating the property of non-erasingness by NE, the infinitary version of O'Donnell's Theorem for terms would read:

$$\mathrm{OD}^\infty : \quad \mathrm{NE} \implies (\, \mathrm{WN}^\infty(t) \iff \mathrm{SN}^\infty(t) \,)$$

Failure of OD^∞ is demonstrated by the following example.

EXAMPLE 23. Consider the term $a(c)$ in the non-erasing orthogonal TRS $\mathcal{T} = \{c \to c,\ a(x) \to b(a(x))\}$; it violates OD^∞ by being WN^∞ but not SN^∞.

WN^∞: The infinite reduction $a(c) \to b(a(c)) \to b(b(a(c))) \to \cdots$ reduces $a(c)$ to its infinitary normal form b^ω.

$\neg\mathrm{SN}^\infty$: There is also the divergent reduction $a(c) \to a(c) \to a(c) \to \cdots$

For an intuitive explanation first note that $a(c)$ is a counterexample to OD^∞ in particular in the sense that it does not satisfy the implication

$$\mathrm{WN}^\infty(t) \,\&\, \neg\mathrm{SN}^\infty(t) \quad \implies \quad \neg\mathrm{NE}$$

In the finitary case, erasure ($\neg$NE) is "needed" for getting rid of the part of a $\neg$SN-term that generates an infinite reduction, in order to pass to a normal form. One wonders why the same would not be needed as well in the infinitary case. Well, look at the example again. The c is not erased literally, but in the infinite reduction it is pushed over the edge of infinity, so to say, with the same effect as erasure: the potentially divergent part c has disappeared in the infinite normal form b^ω.

7 Weak and Strong Head Normalization

We also consider the notions of *weak* and *strong head normalization* (WHN and SHN). A head normal form is a term which is root-stable, as defined in Section 5. Then we have for a term t:

WHN: There is a reduction of t to a head normal form.

SHN: In each infinite reduction of t after a finite number of steps a head normal form is reached.

The infinitary versions then follow naturally.

WHN$^\infty$: There is a possibly transfinite reduction of t to a head normal form.

SHN$^\infty$: In each maximal transfinite reduction of t, no matter whether converging or diverging, at some point a head normal form is reached.

We restrict attention to orthogonal systems again. Then, if a head normal form can be reached by an infinite reduction, by compression it can be reached already in a reduction of length $\leq \omega$. In this reduction the root becomes stable after finitely many steps. So for finite terms there is no difference between finitary and infinitary WHN. As to SHN$^\infty$, again one easily sees that for finite terms it is equivalent to finitary SHN. By contrast note that I^ω is an infinite term that is neither SHN$^\infty$ nor WHN$^\infty$ in the TRS with rule $I(x) \to x$, notwithstanding the TRS being both WHN and SHN.

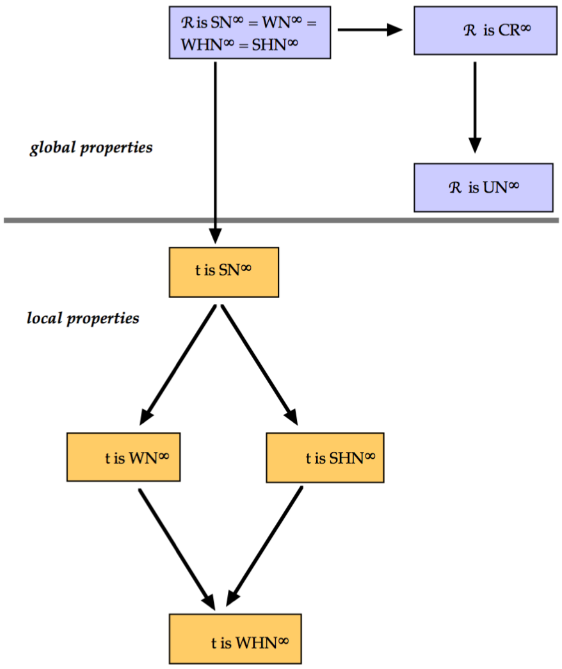

Figure 10. Survey of infinitary properties

The following two examples show that the four implications between the local properties SN$^\infty$, WN$^\infty$, SHN$^\infty$ and WHN$^\infty$ as in Figure 10 are strict.

EXAMPLES 24.

1. In the TRS with rules $\{c \to c, a(x) \to b\}$ the term $a(c)$ is WN^{∞}, $\neg\mathrm{SN}^{\infty}$, WHN^{∞} and $\neg\mathrm{SHN}^{\infty}$.

2. In the TRS with the single rule $c \to c$ and additional unary function symbol e the term $e(c)$ is SHN^{∞}, $\neg\mathrm{SN}^{\infty}$, WHN^{∞} and $\neg\mathrm{WN}^{\infty}$

This was all at the level of terms. At the global level of orthogonal TRSs the notions of WHN^{∞} and SHN^{∞} both coincide with WN^{∞} and SN^{∞}, which we showed to be the same.

We conclude this section on head normalization by noting that the implication WN $\implies$ AC (Theorem 5) can be strengthened to WHN $\implies$ AC. This was proved in [Ketema *et al.*, 2004].

8 Concluding remarks

There are two directions in which we would like to see the subject of this paper extended. The first is to relax the requirement of orthogonality to weak orthogonality. A TRS is weakly orthogonal if the reduction rules are left-linear and critical pairs $\langle t, s\rangle$ generated by overlapping rules are trivial, i.e. of the form $\langle t, t\rangle$. An example of a weakly orthogonal TRS is obtained by adding to the TRS for addition as in this paper a function symbol P for predecessor, with extra reduction rules $S(P(x)) \to x$ and $P(S(x)) \to x$.

The second direction is the extension to the higher-order case, where bound variables are present, as in λ-calculus. For $\lambda\beta$-calculus the property UN^{∞} does hold, see [Terese, 2003], but for general orthogonal higher-order systems UN^{∞} is a conjecture only. The infinitary λ-calculus itself is fully covered in [Terese, 2003]. Work on general infinitary higher-order rewriting has already begun in [Ketema and Simonsen, 2005].

A combination of these two directions is found in the $\lambda\beta\eta$-calculus, which is a higher-order weakly orthogonal TRS. Unfortunately, because the η-rule tests for the absence of a variable[9], something which may happen only in the limit, fundamental theorems such as compression do not hold for the infinitary $\lambda\beta\eta$-calculus. Recently, Severi and de Vries have considered both η-reduction and η-expansion with the aim of generating and studying models by means of transfinite rewriting techniques [Severi and de Vries, 2002; Severi and de Vries, 2005].

Figure 11 summarizes the situation (OCRS stands for Orthogonal Combinatory Reduction System).

[9] In higher-order terminology: the rule is not fully-extended.

	PML	CR	UN	PML^∞	CR^∞	UN^∞
OTRS	*yes*	*yes*	*yes*	*yes*	*no*	*yes*
w.o. TRS	*yes*	*yes*	*yes*	*?*	*no*	*?*
$\lambda\beta$	*yes*	*yes*	*yes*	*no*	*no*	*yes*
OCRS	*yes*	*yes*	*yes*	*no*	*no*	*?*

Figure 11. Some open questions in infinitary rewriting

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