Available online at www.sciencedirect.com
ScienceDirect
Information and Computation 205 (2007) 1575–1607
Information and Computation
www.elsevier.com/locate/ic

Taylor approximation for hybrid systems☆

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Received 29 August 2005; revised 7 February 2007
Available online 10 August 2007

Abstract

We propose a new approximation technique for Hybrid Automata. Given any Hybrid Automaton H, we call $Approx(H,k)$ the Polynomial Hybrid Automaton obtained by approximating each formula ϕ in H with the formulae ϕ_k obtained by replacing the functions in ϕ with their Taylor polynomial of degree k. We prove that $Approx(H,k)$ is an over-approximation of H. We study the conditions ensuring that, given any $\epsilon > 0$, some k_0 exists such that, for all $k > k_0$, the "distance" between any vector satisfying ϕ_k and at least one vector satisfying ϕ is less than ϵ. We study also conditions ensuring that, given any $\epsilon > 0$, some k_0 exists such that, for all $k > k_0$, the "distance" between any configuration reached by $Approx(H,k)$ in n steps and at least one configuration reached by H in n steps is less than ϵ.

1. Introduction

Hybrid Automata [1,4] are a widely studied model for *hybrid systems* [25], i.e., dynamical systems combining discrete and continuous state changes. Hybrid Automata extend classic finite state machines with continuously evolving *variables*, and exhibit two kinds of state changes: discrete jump transitions, occurring instantaneously, and continuous flow transitions, occurring while time elapses. These two kinds of transitions are guarded by *jump conditions* and *activity functions*, respectively, which are formulae expressing constraints on the source and target value of the variables.

1.1. Reachability

Most of hybrid system applications are safety critical and require guarantees of safe operation. To analyze *safety properties* (i.e., properties requiring that a given set of *bad configurations* cannot be reached), the decidability of *reachability* problem (i.e., whether or not a given configuration can be reached) is determinant. Unfortunately, for most classes of hybrid systems, reachability is undecidable [14]. However, for some of these

☆ A preliminary version of this paper was presented at HSCC '05 [20]. The authors wish to thank Annalisa Panati for fruitful discussions.
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doi:10.1016/j.ic.2007.05.004

classes, computing the successors (or predecessors) of configurations sets is reasonably efficient, and, therefore, reachability *in a limited number of steps* is decidable.

There are also classes of hybrid systems for which the successors of configuration sets are not computable. A new methodology has been proposed in [13] to fill this gap. First of all, according to [13], an Hybrid Automaton H' is an *approximation* of another Hybrid Automaton H iff H' is obtained from H by weakening activity functions and jump conditions. In such a way, the set of all the possible computations of H' is a superset of the set of all the possible computations of H, and, hence, if we prove that a *bad* configuration cannot be reached by H' in n steps, then we can infer that it cannot be reached by H in n steps. In order to be sure that such a proof is possible, in [13] it is required that the approximation H' is in the class of the *Linear Hybrid Automata*, for which the successors of configuration sets are computable.

The notion of approximation is then strengthened in [13] with the notion of ϵ-*approximation*: Given any $\epsilon > 0$, H' is an ϵ-*approximation* of H iff, given any vector v' satisfying an activity function (resp. jump condition) in H', there is a vector v satisfying the corresponding activity function (resp. jump condition) in H such that the *distance* between v' and v is below ϵ. This notion of ϵ-approximation is motivated by the need to limit the *error* introduced by the approximation. Finally, any *approximation operator* γ mapping Hybrid Automata into their approximations is *asymptotically complete* iff, for any $\epsilon > 0$ and for any hybrid automaton H, an ϵ-approximation of H can be given by γ. In [13] an asymptotically complete approximation operator, called *rationally rectangular phase-portrait approximation operator*, is given which approximates any jump condition or activity function by a predicate satisfied by all points lying in a space consisting of a products of intervals with rational endpoints.

1.2. Our contribution

In the present paper, we propose a new approximation technique. Our idea is to weaken jump conditions and activity functions by replacing functions over variables with their polynomial of Taylor. More precisely, given any Hybrid Automaton H and natural k, $\mathbf{A}(H,k)$ is the set of the Hybrid Automata that are obtained by replacing in jump conditions and activity functions of H each function $f(\overrightarrow{x})$ over the variables $\overrightarrow{x}$ with the *polynomial of Taylor for* f *of degree* k *with respect to vector* $\overrightarrow{v}$, denoted $P^k(f, \overrightarrow{x}, \overrightarrow{v})$, where $\overrightarrow{v}$ is a vector in the domain of f. Of course, to define $\mathbf{A}(H,k)$ we require that all functions $f(\overrightarrow{x})$ are derivable k times. Notice that $\mathbf{A}(H,k)$ is in the class of *Polynomial Hybrid Automata*, for which computing the successors of configuration sets is decidable [29].

We shall prove that each Polynomial Hybrid Automaton H_k in $\mathbf{A}(H,k)$ is an approximation for H according to [13], i.e., that all jump conditions and activity functions of H_k are less demanding than those of H. We shall study the conditions ensuring that our approximation is asymptotically complete, in the sense that, for each $\epsilon > 0$ there exists some k_0 such that, for all $k > k_0$, $\mathbf{A}(H,k)$ contains only ϵ-approximations for H. We note that looking for more accurate approximations for H is in some sense mechanizable, since it simply requires taking increasing values for k.

Now, looking for ϵ-approximations for small values of ϵ is a strategy suggested in [13] to limit the error of the approximation. We observe that this analysis of the error is *syntactic*, in the sense that it does not consider the behavior of H and its approximation. In general, one expects explosion of the error. In fact, by approximating one activity function or one jump condition, an error is generated which implies the reachability of some configurations that were originally unreachable. If in these configurations the behaviors are once more affected by new errors caused by the approximation of other activity functions and jump conditions, error could explode dramatically. In this paper we take a step toward a *semantic* analysis of the error. We study conditions ensuring that, when k tends to the infinity, the behavior of any $H_k \in \mathbf{A}(H,k)$ gets close to the behavior of H. More precisely, these conditions ensure that, for each $\epsilon > 0$, there is some k_0 such that, for all $k > k_0$, if any $H_k \in \mathbf{A}(H,k)$ reaches a configuration c in n steps, then H reaches a configuration c' in n steps such that the *distance* between c and c' is below ϵ.

1.3. Related works

Approximation is a strategy widely used for the analysis of hybrid systems. However, the literature presents different notions of approximation, that we briefly comment on in this section. Several papers (see, e.g., [3,1,5,

14,15,17,18,19]) show that for some classes of Hybrid Automata it is possible to map a given automaton H into an approximation H' and a property P (like, for instance, reachability) over H into a property P' over H' such that: (1) H satisfies P iff H' satisfies P'; (2) the problem "H' satisfies P'" is decidable.

Unfortunately, there exist classes of Hybrid Automata for which the strategy previously described does not work, and different model checking strategies should be provided.

Several papers (see, e.g., [6,7,9,16,21,26,31,32]) study how one can compute under-approximations and/or over-approximations of the set of the reachable configurations.

Other papers extract finite-state *abstractions* from Hybrid Automata that are more accessible to analysis tools. Precisely, [2] exploits *predicate abstraction* for deriving finite-state models, whereas [30] exploits predicate abstraction plus *qualitative reasoning*. The idea is that, if model checking on the finite-state abstraction gives an answer, then this answer is always correct for the original system. To manage the case where model checking is inconclusive, [8] applies *counterexample-guided abstraction refinement* strategy. Namely, counterexamples that are execution paths of the abstraction and that are not execution paths of the original Hybrid Automaton are exploited for refining the abstraction. Another strategy for abstraction refinement has been proposed in [27] and based on constraint propagation.

Also [13] considers a class of Hybrid Automata for which the strategy of [3,1,5,14,15,17,18,19] does not work, but, instead of computing an approximation of the reachable set of configurations of the original Hybrid Automaton, or deriving a finite-state abstraction, the idea is to approximate syntactically the automaton, by weakening activity functions and jump conditions, so that the obtained automaton falls in the class of Linear Hybrid Automata, for which reachability in n steps is decidable.

As in [13], in this paper we approximate syntactically an automaton H with other automata, falling in a set denoted $\mathbf{A}(H,k)$. Moreover, we study also how close the behaviors of H and the automata in $\mathbf{A}(H,k)$ are.

Notice that syntactical approximations exploiting Taylor approximations have been also exploited in "algorithmic algebraic model checking" papers [22,23,24] for obtaining models suitable for studying bounded reachability [24], TCTL model checking [23], and model checking approximation conducted through bisimulation partitioning, polyhedra, grids and time discretization [22]. These papers refer to the preliminary version of our paper [20] for the analysis of the error introduced by Taylor approximation and the results on the over-approximation property.

1.4. Organization of the paper

The paper is organized as follows. In Sections 2 and 3 we recall the notions on the theory of Hybrid Automata and Taylor approximation that will be employed in the paper. In Section 4 we introduce our definition of approximation of an Hybrid Automaton, in Section 5 and Section 6 we do the syntactical analysis of the error, and in Section 7 we do the semantical analysis of the error. To study the practical impact of our proposal, in Section 8 we apply our approximation technique to the "navigation benchmark" of [10]. Finally, in Section 9 we outline some future developments of our paper.

2. Hybrid Automata

In this section we recall the formalism of Hybrid Automata (see, e.g., [25]).

2.1. Formulae

A *vector* of dimension n over a given set U is a tuple $\overrightarrow{u} = (u_1, \ldots, u_n)$ in U^n. We will write $\overrightarrow{u} \oplus (u)$ to denote the vector $(u_1, \ldots, u_n, u)$ of dimension $n+1$. Moreover, for vectors $\overrightarrow{u} = (u_1, \ldots, u_n)$ in U^n and $\overrightarrow{v} = (v_1, \ldots, v_m)$ in U^m, we will write $\overrightarrow{u} \oplus \overrightarrow{v}$ to denote the vector $(u_1, \ldots, u_n, v_1, \ldots, v_m)$ of dimension $n+m$. A *space* over U^n is a set of vectors of dimension n.

Let $X = \{x_1, \ldots, x_{|X|}\}$ be a finite set of *real variables*. We will write $\overrightarrow{X}$ to denote the vector $(x_1, \ldots, x_{|X|})$ over $X^{|X|}$.

Let F be a set of *function symbols* and $ar: F \to \mathbb{N}$ an *arity* mapping that assigns a natural $ar(f)$ to each symbol $f \in F$. A *term* over X and F has the form $f(x_{i_1}, \ldots, x_{i_{ar(f)}})$, where $f \in F$ and $x_{i_1}, \ldots, x_{i_{ar(f)}} \in X$. Term $f(x_{i_1}, \ldots, x_{i_{ar(f)}})$ will be also denoted with $f(\overrightarrow{x})$. Given a vector $\overrightarrow{u} = (u_1, \ldots, u_{ar(f)})$ over $\mathbb{R}^{ar(f)}$, we write $f(\overrightarrow{u})$ to denote $f(u_1, \ldots, u_{ar(f)})$.

Let us assume a unique *interpretation* I associating to each function symbol $f \in F$ a continuous function $I(f): \mathbb{R}^{ar(f)} \to \mathbb{R}$, such that $I(+)$ and $I(\cdot)$ are the sum and the multiplication over reals, and $I(-)$ is the negation. We explicitly require for these three functions since they are needed for building polynomials. Since I is unique, with abuse of notation sometimes we will use f instead of $I(f)$.

Definition 1. The set $\Phi(X, F)$ of the *formulae* over X and F is inductively defined as follows:

- $f(\overrightarrow{x}) \sim b \cdot x_i + c$ is in $\Phi(X, F)$ whenever $f(\overrightarrow{x})$ is a term over X and F, $\sim \in \{<, \leqslant, =, \geqslant, >\}$, $b, c \in \mathbb{R}$, and $x_i \in X$;
- $\neg\phi$ is in $\Phi(X, F)$ if ϕ is in $\Phi(X, F)$;
- $\phi_1 \vee \phi_2$ and $\phi_1 \wedge \phi_2$ are in $\Phi(X, F)$ if both ϕ_1 and ϕ_2 are in $\Phi(X, F)$;
- $\forall y \in I_y. \phi$ and $\exists y \in I_y. \phi$ are in $\Phi(X, F)$ if $y \in X$, I_y is a non-empty interval such that $\emptyset \subset I_y \subseteq \mathbb{R}$, and ϕ is in $\Phi(X, F)$.

A formula $\phi \in \Phi(X, F)$ is *polynomial* iff, for any subformula $f(\overrightarrow{x}) \sim b \cdot x_i + c$ of ϕ, $f(\overrightarrow{x})$ is a polynomial on variables $\overrightarrow{x}$.

Of course, we could simply use formulae $f(\overrightarrow{x}) \sim 0$ instead of $f(\overrightarrow{x}) \sim b \cdot x_i + c$. Our solution introduces flexibility, since, when f is not polynomial, we can either use formula $f(\overrightarrow{x}) \sim b \cdot x_i + c$ and approximate function f, or use formula $f(\overrightarrow{x}) - b \cdot x_i - c \sim 0$, with the term on the left side being the function over $\overrightarrow{x}$ and x_i to be approximated. As we shall show in Example 5, we let the user as more freedom as possible in choosing the functions to be approximated. For instance, given functions f and g, we can rewrite a formula $f(g(\overrightarrow{x})) \sim b \cdot x_i + c$ as $\exists y \in (-\infty, \infty). g(\overrightarrow{x}) = y \wedge f(y) \sim b \cdot x_i + c$. In the first case we approximate the composition of f and g, in the second case we approximate f and g separately.

Given formulae ϕ_1 and ϕ_2, let $\phi_1 \equiv \phi_2$ denote their syntactic identity.

For a formula $\phi \in \Phi(X, F)$, let $\mathbf{V}(\phi)$ denote the set of the variables appearing in ϕ. Formally:

$$
\begin{aligned}
&\mathbf{V}(f(x_{i_1}, \ldots, x_{i_{ar(f)}}) \sim b \cdot x_i + c) = \{x_{i_1}, \ldots, x_{i_{ar(f)}}, x_i\} \\
&\mathbf{V}(\neg\phi) = \mathbf{V}(\phi) \\
&\mathbf{V}(\phi_1 \vee \phi_2) = \mathbf{V}(\phi_1) \cup \mathbf{V}(\phi_2) \\
&\mathbf{V}(\phi_1 \wedge \phi_2) = \mathbf{V}(\phi_1) \cup \mathbf{V}(\phi_2) \\
&\mathbf{V}(\forall y \in I_y. \phi') = \mathbf{V}(\phi') \cup \{y\} \\
&\mathbf{V}(\exists y \in I_y. \phi') = \mathbf{V}(\phi') \cup \{y\}
\end{aligned}
$$

For a formula $\phi \in \Phi(X, F)$, let $\mathbf{FV}(\phi)$ denote the set of the *free* variables appearing in ϕ. Formally:

$$
\begin{aligned}
&\mathbf{FV}(f(x_{i_1}, \ldots, x_{i_{ar(f)}}) \sim b \cdot x_i + c) = \{x_{i_1}, \ldots, x_{i_{ar(f)}}, x_i\} \\
&\mathbf{FV}(\neg\phi) = \mathbf{FV}(\phi) \\
&\mathbf{FV}(\phi_1 \vee \phi_2) = \mathbf{FV}(\phi_1) \cup \mathbf{FV}(\phi_2) \\
&\mathbf{FV}(\phi_1 \wedge \phi_2) = \mathbf{FV}(\phi_1) \cup \mathbf{FV}(\phi_2) \\
&\mathbf{FV}(\forall y \in I_y. \phi') = \mathbf{FV}(\phi') \setminus \{y\} \\
&\mathbf{FV}(\exists y \in I_y. \phi') = \mathbf{FV}(\phi') \setminus \{y\}
\end{aligned}
$$

A function $v: X \to \mathbb{R}$ is called an *evaluation* over X. Given a vector $\overrightarrow{x} = (x_{i_1}, \ldots, x_{i_n})$ of variables over X^n, we write $v(\overrightarrow{x})$ for $(v(x_{i_1}), \ldots, v(x_{i_n}))$.

For an evaluation v over X, a variable $y \in X$, and a real c, the evaluation $v[y := c]$ is the evaluation such that $v[y := c](x) = v(x)$, for all $x \in X \setminus \{y\}$, and $v[y := c](y) = c$. Given vectors $\overrightarrow{x} = (x_{i_1}, \ldots, x_{i_n})$ over X^n and

$\overrightarrow{u} = (u_1, \ldots, u_n)$ over $\mathbb{R}^n$, notation $v[\overrightarrow{x} := \overrightarrow{u}]$ stays for $v[x_{i_1} := u_1] \ldots [x_{i_n} := u_n]$. Finally, we write $[\overrightarrow{X} := \overrightarrow{u}]$ to denote the evaluation v such that $v(\overrightarrow{X}) = \overrightarrow{u}$.

Let $\phi \in \Phi(X, F)$ and v be an evaluation over X. We write $v \models \phi$ to denote that *the evaluation v satisfies the formula ϕ*. Relation $\models$ is defined inductively with respect to ϕ as follows:

$$
\begin{array}{ll}
v \models f(\overrightarrow{x}) \sim b \cdot x_i + c & \text{iff } I(f)(v(\overrightarrow{x})) \sim b \cdot v(x_i) + c \\
v \models \neg\phi_1 & \text{iff } v \models \phi_1 \text{ does not hold} \\
v \models \phi_1 \vee \phi_2 & \text{iff either } v \models \phi_1 \text{ or } v \models \phi_2 \\
v \models \phi_1 \wedge \phi_2 & \text{iff both } v \models \phi_1 \text{ and } v \models \phi_2 \\
v \models \forall y \in I_y.\, \phi' & \text{iff } v[y := c] \models \phi' \text{ for all } c \in I_y \\
v \models \exists y \in I_y.\, \phi' & \text{iff } v[y := c] \models \phi' \text{ for some } c \in I_y
\end{array}
$$

For a formula ϕ in $\Phi(X, F)$, let $[\![\phi]\!]$ denote the set $\{v : X \to \mathbb{R} \mid v \models \phi\}$ of the evaluations over X satisfying ϕ.

Two formulae $\phi_1, \phi_2 \in \Phi(X, F)$ are *equivalent* iff $[\![\phi_1]\!] = [\![\phi_2]\!]$.

We will write $\exists y.\, \phi$ and $\forall y.\, \phi$ for $\exists y \in (-\infty, \infty).\, \phi$ and $\forall y \in (-\infty, \infty).\, \phi$, respectively.

Given reals m and M, we will write $x \in [m, M]$ for $x \geqslant m \wedge x \leqslant M$.

Notice that with $\Phi(X, F)$ we can express composition of functions and polynomials by introducing existential quantified variables, and we can express relations among functions. For instance, relation

$$f_1(g(x_1), h(x_2)) < f_2(k(x_2 \cdot (x_1)^2 + 2 \cdot x_1))$$

can be expressed by: $\exists y_1.\exists y_2.\exists y_3.\exists y_4.\exists y_5.\exists y_6.\exists y_7.\exists y_8.\exists y_{1,2}.\ y_1 = g(x_1) \wedge y_2 = h(x_2) \wedge y_3 = x_1 \cdot x_1 \wedge y_4 = x_2 \cdot y_3 \wedge y_5 = 2 \cdot x_1 \wedge y_6 = y_4 + y_5 \wedge y_7 = k(y_6) \wedge y_8 = f_2(y_7) \wedge y_{1,2} = f_1(y_1, y_2) \wedge y_{1,2} < y_8$.

Let us introduce a notion of normal form for formulae.

Definition 2. The subset of the *normal forms* in $\Phi(X, F)$ contains the formulae of the form $Q_1 y_1 \in I_{y_1}. \ldots Q_m y_m \in I_{y_m}.\, \phi$, where

- $Q_i \in \{\forall, \exists\}$, for $i = 1, \ldots, m$;
- ϕ neither contains quantifiers nor negations;
- ϕ contains only relations in $\{<, \leqslant\}$.

The following result is folklore.

Proposition 3. *Given any formula $\phi \in \Phi(X, F)$, there exists a normal form equivalent to ϕ that can be constructed from ϕ.*

2.2. The formalism

Definition 4. An *Hybrid Automaton* over X and F is a tuple of the form $\langle \phi_{init}, Q, q_0, T, Act \rangle$, where

- $\phi_{\text{init}} \in \Phi(X, F)$ is the *initial condition.*
- Q is a finite *set of states.*
- $q_0 \in Q$ is the *initial state.*
- $T \subseteq Q \times \Phi(\{x_1, \ldots, x_{|X|}, x'_1, \ldots, x'_{|X|}\}, F) \times Q$ is a finite *set of transitions.* Variables $x'_1, \ldots, x'_{|X|}$ represent the new values taken by the variables $x_1, \ldots, x_{|X|}$ after the firing of the transition.
- $Act : Q \to \Phi(\{x_1, \ldots, x_{|X|}, t, x'_1, \ldots, x'_{|X|}\}, F)$ is the *activity function* assigning to each state q a formula $Act(q)$. The variable t represents time elapsing.[1]

[1] Note that invariants can be expressed by means of universal quantifiers.

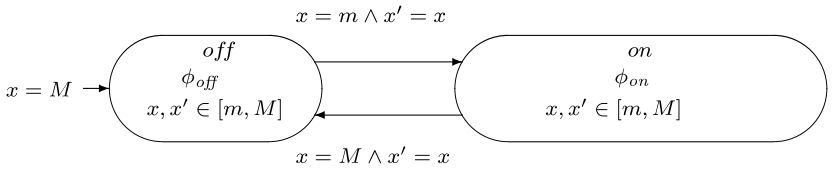

Fig. 1. The thermostat of [1].

Example 5 (*Alur et al. [1]*)**.** The Hybrid Automaton represented in Fig. 1 models a thermostat that controls the temperature of a room. The thermostat continuously senses the temperature and turns the heater on and off, aiming to keep the temperature between m and M degrees, where $M > m > 0$.

When the heater is off, the temperature, which is represented by variable x, decreases according to the exponential function xe^{-Kt}, where $K > 0$ is a constant determined by the room and t is the variable representing time. This is modeled by the activity function ϕ_{off}, which could be written in several ways. Let f_1 be the unary function such that $I(f_1)(x_1) = e^{x_1}$. Let f_2 be the binary function such that $I(f_2)(x_1, x_2) = x_1 \cdot e^{x_2}$. Let f_3 be the three arguments function such that $I(f_3)(x_1, x_2, x_3) = x_1 - x_2 \cdot e^{x_3}$. Let ψ denote the formula $x \in [m, M] \wedge x' \in [m, M]$. Among the ways in which we could write ϕ_{off}, we mention the following:

$$\phi^1_{\text{off}} \equiv \quad \exists y_1.\, \exists y_2.\ K \cdot t = -y_1 \wedge f_1(y_1) = y_2 \wedge x \cdot y_2 = x' \wedge \psi$$

$$\phi^2_{\text{off}} \equiv \quad \exists y.\ K \cdot t = -y \wedge f_2(x, y) = x' \wedge \psi$$

$$\phi^3_{\text{off}} \equiv \quad \exists y.\ K \cdot t = -y \wedge f_3(x', x, y) = 0 \wedge \psi$$

Of course the three ways are equivalent under the semantic point of view. The difference will emerge when non-polynomial functions are approximated with their Taylor polynomial, since we can approximate either f_1, or f_2, or f_3, as we shall discuss in Example 21. Notice that formulae above are not normal forms. They can be mapped to normal forms by replacing each subformula of the form $f(\overrightarrow{x}) = z$ with $f(\overrightarrow{x}) \leqslant z \wedge f^-(\overrightarrow{x}) \leqslant -z$.

When the heater is on, the temperature follows the function $xe^{-Kt} + h(1 - e^{-Kt})$, where $h > 0$ is a constant that depends on the power of the heater. This is modeled by the activity function ϕ_{on}, which could be written in several ways. Among the ways in which we could write ϕ_{on}, we mention the following:

$$\begin{aligned}\phi^1_{\text{on}} \equiv \quad & \exists y_1.\, \exists y_2.\, \exists y_3.\, \exists y_4.\, \exists y_5.\ K \cdot t = -y_1 \wedge f_1(y_1) = y_2 \wedge x \cdot y_2 = y_3 \wedge \\ & 1 - y_2 = y_4 \wedge h \cdot y_4 = y_5 \wedge y_3 + y_5 = x' \wedge \psi\end{aligned}$$

$$\phi^2_{\text{on}} \equiv \quad \exists y.\ K \cdot t = -y \wedge f_4(x', x, y) = 0 \wedge \psi$$

where f_4 is the three arguments function such that $f_4(x_1, x_2, x_3) = x_1 - (x_2 \cdot e^{x_3} + h \cdot (1 - e^{x_3}))$.

Definition 6 (*Fränzle [11,12]*)**.** An Hybrid Automaton is a *Polynomial Hybrid Automaton* iff ϕ_{init} is a polynomial formula, for each state q, $Act(q)$ is a polynomial formula, and, for each transition (q, ϕ, q'), ϕ is a polynomial formula.

Let us explain the behavior of an Hybrid Automaton H.

A *configuration* is a pair $(q, \overrightarrow{u})$, where q is a state in Q and $\overrightarrow{u}$ is a vector in $\mathbb{R}^{|X|}$ representing the value of the variables X. More precisely, $\overrightarrow{u}$ is a vector $(u_1, \ldots, u_{|X|})$ representing that each variable x_i assumes value u_i, for all $1 \leqslant i \leqslant |X|$. H can evolve from $(q, \overrightarrow{u})$ to another configuration $(q', \overrightarrow{u}')$, written $(q, \overrightarrow{u}) \to (q', \overrightarrow{u}')$, by performing either an activity step or a transition step, where

- an *activity step* describes the evolution from configuration $(q, \overrightarrow{u})$ due to remaining in state q and passing of time. In u units of time, the activity $Act(q)$ takes H to a new evaluation of the variables, more precisely:

  if $u \geqslant 0$ and $[\overrightarrow{X} := \overrightarrow{u}, t := u, \overrightarrow{X}' := \overrightarrow{u}'] \models Act(q)$, then $(q, \overrightarrow{u}) \to (q, \overrightarrow{u}')$

- a *transition step* describes the evolution from configuration $(q, \overrightarrow{u})$ due to the firing of a transition from state q. More precisely:

  if $(q, \phi, q') \in T$ and $[\overrightarrow{X} := \overrightarrow{u}, \overrightarrow{X}' := \overrightarrow{u}'] \models \phi$, then $(q, \overrightarrow{u}) \to (q', \overrightarrow{u}')$

A *run* r of H is a sequence of (activity and transition) steps $(q_0, \overrightarrow{u_0}) \to (q_1, \overrightarrow{u_1}) \to \cdots \to (q_i, \overrightarrow{u_i}) \cdots$, where q_0 is the initial state and $[\overrightarrow{X} := \overrightarrow{u_0}] \in [\![\phi_{\text{init}}]\!]$.

A configuration $(q, \overrightarrow{u})$ is *reachable in n steps* iff there is a run $(q_0, \overrightarrow{u_0}) \to (q_1, \overrightarrow{u_1}) \to \ldots \to (q_n, \overrightarrow{u_n}) \ldots$ such that $q_n = q$ and $\overrightarrow{u_n} = \overrightarrow{u}$. A configuration is *reachable* iff it is reachable in n steps for some $n \geqslant 0$.

2.3. Regions

A *region* R of an Hybrid Automaton H is a set of configurations of H. The set of the regions of H is denoted $\mathcal{R}(H)$.

The set of the configurations reachable by H from the configurations in a region R is denoted $Post(R, H)$. Formally:

$$Post(R, H) = \{(q', \overrightarrow{u}') \mid \exists (q, \overrightarrow{u}) \in R \text{ such that } (q, \overrightarrow{u}) \to (q', \overrightarrow{u}')\}$$

Let $Post^n(H)$ denote either the region $\{(q_0, \overrightarrow{u_0}) \mid [\overrightarrow{X} := \overrightarrow{u_0}] \in [\![\phi_{\text{init}}]\!]\}$, if $n = 0$, or the region $Post(Post^{n-1}(H), H)$, if $n > 0$. Moreover, let $Post(H)$ denote the region $\bigcup_{n \in \mathbb{N}} Post^n(H)$. The following result is folklore.

Theorem 7. *For each $n \in \mathbb{N}$, a configuration $(q, \overrightarrow{u})$ is reachable in n steps iff $(q, \overrightarrow{u}) \in Post^n(H)$. Hence $(q, \overrightarrow{u})$ is reachable iff $(q, \overrightarrow{u}) \in Post(H)$.*

The following result follows from Tarski's results [29].

Theorem 8. *If H is polynomial, then, for each $n \in \mathbb{N}$, it is decidable whether $(q, \overrightarrow{u}) \in Post^n(H)$.*

3. Taylor approximation

Let C_n denote the set of the possibly partial functions $f\colon \mathbb{R}^n \to \mathbb{R}$. Given a function $f \in C_n$, let $Dom(f) \subseteq \mathbb{R}^n$ denote the domain of f.

Let $D^i_j f$ denote the ith derivate of f with respect to the coordinate jth.

Let C^k_n denote the subset of the functions in C_n that are derivable k times, i.e., $f \in C^k_n$ iff, for any $j_1, \ldots, j_n$ with $j_1 + \cdots + j_n = k$, $(D^{j_1}_1 \ldots D^{j_n}_n f)$ exists.

Definition 9. Given a function $f \in C^k_n$ and a vector $\overrightarrow{v} \in Dom(f)$, the *polynomial of Taylor of degree k for f with respect to vector $\overrightarrow{v}$* is defined as follows:

$$P^k(f, \overrightarrow{x}, \overrightarrow{v}) = \sum_{j_1 + \cdots + j_n \leqslant k} \frac{\left((D^{j_1}_1 \ldots D^{j_n}_n f)(\overrightarrow{v})\right) \cdot \left((x_{i_1} - v_1)^{j_1} \ldots (x_{i_n} - v_n)^{j_n}\right)}{j_1! \cdot \cdots \cdot j_n!}$$

Given a vector $\overrightarrow{u} \in Dom(f)$, let $r^k(f, \overrightarrow{u}, \overrightarrow{v})$ denote the *remainder* (or *error*) $f(\overrightarrow{u}) - P^k(f, \overrightarrow{u}, \overrightarrow{v})$.

The intuition is that $P^k(f, \overrightarrow{x}, \overrightarrow{v})$ is a polynomial that approximates $f(\overrightarrow{x})$, and, for all $\overrightarrow{u} \in Dom(f)$, $r^k(f, \overrightarrow{u}, \overrightarrow{v})$ is the error of the approximation in $\overrightarrow{u}$. The following result, known as Lagrange Remainder Theorem, quantifies $r^k(f, \overrightarrow{u}, \overrightarrow{v})$.

Theorem 10 (*Lagrange*)**.** *Given a function $f \in C^{k+1}_n$ and two vectors $\overrightarrow{u}, \overrightarrow{v}$ in $Dom(f)$, there exists a vector $\overrightarrow{z}$ in $Dom(f)$ on the segment linking $\overrightarrow{u}$ and $\overrightarrow{v}$ such that*:

$$r^k(f, \overrightarrow{u}, \overrightarrow{v}) = \sum_{j_1 + \cdots + j_n = k+1} \frac{\left((D^{j_1}_1 \ldots D^{j_n}_n f)(\overrightarrow{z})\right) \cdot \left((u_1 - v_1)^{j_1} \ldots (u_n - v_n)^{j_n}\right)}{j_1! \cdot \cdots \cdot j_n!}$$

Our aim is to give an upper bound to $|r^k(f, \overrightarrow{u}, \overrightarrow{v})|$, under suitable hypothesis.

Definition 11. A function $f \in C_n^{k+1}$ is analytic in a set $S \subseteq Dom(f)$ with respect to a vector $\overrightarrow{v} \in Dom(f)$ if there exist two constants C and L such that, for all indexes $j_1, \ldots, j_n$ such that $j_1 + \cdots + j_n \leqslant k+1$, and for all vectors $\overrightarrow{u} \in S$, it holds that

$$\left|(D_1^{j_1} \ldots D_n^{j_n} f)(\overrightarrow{z})\right| \leqslant L \cdot C^{j_1+\cdots+j_n}$$

for all vectors $\overrightarrow{z} \in Dom(f)$ in the segment linking $\overrightarrow{u}$ and $\overrightarrow{v}$.

Intuitively, if f is analytic in S with respect to $\overrightarrow{v}$, C and L permits to having an upper bound to $\left|(D_1^{j_1} \ldots D_n^{j_n} f)(\overrightarrow{z})\right|$, for all $\overrightarrow{z}$ lying in the segment linking any $\overrightarrow{u} \in S$ to $\overrightarrow{v}$.

Definition 12. Assume a normal form $\psi \in \Phi(X, F)$ without any quantifier, and a function $f \in C_n^{k+1}$ appearing in ψ. We say that f is analytic in $[\![\psi]\!]$ with respect to a vector $\overrightarrow{v} \in Dom(f)$ iff f is analytic in the set $Dom(f) \cap \{v(x_{i_1}, \ldots, x_{i_n}) \mid v \in [\![\psi]\!]\}$ with respect to $\overrightarrow{v}$.

If f is analytic in $[\![\psi]\!]$ with respect to $\overrightarrow{v}$, the set S of Definition 11 where the derivatives of f have an upper bound contains the projections on the variables of f of the vectors satisfying ψ.

Definition 13. Assume a normal form $\phi \equiv Q_1 y_1 \in I_{y_1}. \ldots Q_m y_m \in I_{y_m}. \psi$, with ψ containing no quantifier, and a function $f \in C_n^{k+1}$ appearing in ψ. We say that f is analytic in $[\![\phi]\!]$ with respect to a vector $\overrightarrow{v} \in Dom(f)$ iff f is analytic in $[\![\psi]\!]$ with respect to $\overrightarrow{v}$.

If f is analytic in $[\![\phi]\!]$ with respect to $\overrightarrow{v}$, the set S of Definition 11 where the derivatives of f have an upper bound contains the projections on the variables of f of the vectors satisfying ψ, which is a possibly strict superset of the vectors that satisfy ϕ.

Example 14. Trigonometric functions are analytic in any $[\![\phi]\!]$ with respect to any v. As an example, for the function $\sin(x)$ it is sufficient to take the constants $L = C = 1$. Exponential and logarithmic functions are analytic in $[\![\phi]\!]$ with respect to any v, provided that ϕ constrains variables within finite intervals. As an example, for function e^{2x} and $[\![\phi]\!] \subseteq [0, 10]$, it is sufficient to take the constants $C = 2$ and $L = \max\{e^{20}, e^{2v}\}$.

Let us assume a function $f \in C_n^{k+1}$ analytic in $[\![\phi]\!]$ with respect to $\overrightarrow{v}$. Then, given $\hat{C}$ and $\hat{L}$ the minimal values satisfying the condition of Definition 11, for any k we shall denote with $C(f, \phi, \overrightarrow{v}, k)$ the value $\hat{L} \cdot \hat{C}^{k+1}$. Moreover, let $R^k(f, \overrightarrow{x}, \overrightarrow{v}, \phi)$ denote the polynomial

$$R^k(f, \overrightarrow{x}, \overrightarrow{v}, \phi) = \frac{C(f, \phi, \overrightarrow{v}, k) \cdot n^{k+1} \cdot \prod_{j=1}^{n} \left((x_{i_j} - v_j)^{2 \cdot \left\lceil \frac{k+1}{2} \right\rceil} + 1 \right)}{\left\lfloor \frac{k+1}{n} \right\rfloor !}$$

Otherwise, if f is not analytic in $[\![\phi]\!]$, let $R^k(f, \overrightarrow{x}, \overrightarrow{v}, \phi)$ be ∞.

Intuitively, for all vectors $\overrightarrow{u} \in Dom(f)$ such that $\overrightarrow{u} = v(\overrightarrow{x})$ for some $v \in [\![\psi]\!]$, it holds that $R^k(f, \overrightarrow{u}, \overrightarrow{v}, \phi)$ is an upper bound to $|r^k(f, \overrightarrow{u}, \overrightarrow{v})|$. Moreover, $R^k(f, \overrightarrow{u}, \overrightarrow{v}, \phi)$ gets close to 0 when k tends to the infinity. Let us formalize these two intuitions.

Proposition 15. *Let $\phi \equiv Q_1 y_1 \in I_{y_1}. \ldots Q_m y_m \in I_{y_m}. \psi \in \Phi(X, F)$ be a normal form, and $f \in F$ be analytic in $[\![\phi]\!]$ with respect to $\overrightarrow{v}$. It holds that, for all vectors $\overrightarrow{u}$ such that there exists some evaluation v with $v \in [\![\psi]\!]$ and $v(\overrightarrow{x}) = \overrightarrow{u}$:*

- $|r^k(f, \overrightarrow{u}, \overrightarrow{v})| \leqslant R^k(f, \overrightarrow{u}, \overrightarrow{v}, \phi)$,
- $\lim_{k \to \infty} R^k(f, \overrightarrow{u}, \overrightarrow{v}, \phi) = 0$.

Proof. Let us begin with proving the first property.

We know that, for all indexes $j_1, \ldots, j_n$ such that $j_1 + \cdots + j_n = k+1$, the following relation holds, for all $\overrightarrow{z}$ lying in the interval linking $\overrightarrow{u}$ and $\overrightarrow{v}$: $\left|(D_1^{j_1} \ldots D_n^{j_n} f)(\overrightarrow{z})\right| \leqslant C(f, \phi, \overrightarrow{v}, k)$.

For all $l \leqslant k+1$ and $u \in \mathbb{R}$, it holds that $u^l \leqslant u^{2\cdot\left\lceil \frac{k+1}{2} \right\rceil} + 1$. In fact, if $u \geqslant 1$, then $u^l \leqslant u^{2\cdot\left\lceil \frac{k+1}{2} \right\rceil}$. Otherwise, if $u < 1$, then $u^l < 1$. Hence we have that:

$$|r^k(f, \overrightarrow{u}, \overrightarrow{v})| \leqslant \sum_{j_1+\cdots+j_n=k+1} \frac{\left|(D_1^{j_1} \ldots D_n^{j_n} f)(\overrightarrow{z})\right| \cdot \left|(u_1 - v_1)^{j_1} \ldots (u_n - v_n)^{j_n}\right|}{j_1! \cdot \ldots \cdot j_n!}$$

$$\leqslant C(f, \phi, \overrightarrow{v}, k) \cdot \prod_{j=1}^{n} \left((u_j - v_j)^{2\cdot\left\lceil \frac{k+1}{2} \right\rceil} + 1 \right) \cdot \sum_{j_1+\ldots+j_n=k+1} \frac{1}{j_1! \cdot \ldots \cdot j_n!}$$

Now, for each k, it holds that $j_1! \cdot \ldots \cdot j_n! \geqslant \left(\left\lfloor \frac{k+1}{n} \right\rfloor\right)!$. In fact, $j_1 + \cdots + j_n = k+1$ implies that there exists some index $1 \leqslant h \leqslant n$ such that $j_h \geqslant \frac{k+1}{n}$. Therefore,

$$|r^k(f, \overrightarrow{u}, \overrightarrow{v})| \leqslant \frac{C(f, \phi, \overrightarrow{v}, k) \cdot \prod_{j=1}^{n} \left((u_j - v_j)^{2\cdot\left\lceil \frac{k+1}{2} \right\rceil} + 1 \right)}{\left\lfloor \frac{k+1}{n} \right\rfloor!} \cdot \sum_{j_1+\cdots+j_n=k+1} 1$$

$$= \frac{C(f, \phi, \overrightarrow{v}, k) \cdot \prod_{j=1}^{n} \left((u_j - v_j)^{2\cdot\left\lceil \frac{k+1}{2} \right\rceil} + 1 \right) \cdot n^{k+1}}{\left\lfloor \frac{k+1}{n} \right\rfloor!} = R^k(f, \overrightarrow{u}, \overrightarrow{v}, \phi)$$

Let us prove the second property. We have to prove that

$$\lim_{k\to\infty} R^k(f, \overrightarrow{u}, \overrightarrow{v}, \phi) = \lim_{k\to\infty} \frac{C(f, \phi, \overrightarrow{v}, k) \cdot n^{k+1} \cdot \prod_{j=1}^{n} \left((u_j - v_j)^{2\cdot\left\lceil \frac{k+1}{2} \right\rceil} + 1 \right)}{\left\lfloor \frac{k+1}{n} \right\rfloor!}$$

that is equal to 0.

Let A be the real $A = \max\{2, |u_1 - v_1|, \ldots, |u_n - v_n|\}$. We note that $A^{k+2} \geqslant (u_j - v_j)^{2\cdot\left\lceil \frac{k+1}{2} \right\rceil} + 1$, for all $1 \leqslant j \leqslant n$. Let C and L be the constants such that $C(f, \phi, \overrightarrow{v}, k) = C \cdot L^{k+1}$. It holds that

$$0 \leqslant R^k(f, \overrightarrow{u}, \overrightarrow{v}, \phi) \leqslant \frac{C \cdot L^{k+1} \cdot n^{k+1} \cdot \prod_{j=1}^{n} \left(A^{k+2}\right)}{\left\lfloor \frac{k+1}{n} \right\rfloor!}$$

Thus we obtain

$$0 \leqslant R^k(f, \overrightarrow{u}, \overrightarrow{v}, \phi) \leqslant \frac{C \cdot A^n \cdot (A^n \cdot L \cdot n)^{k+1}}{\left(\left\lfloor \frac{k+1}{n} \right\rfloor\right)!}$$

Hence, it is sufficient to prove that for any M and n

$$\lim_{k\to\infty} \frac{M^k}{\lfloor \frac{k}{n} \rfloor!} = 0$$

Let $h = \frac{k}{n}$. It holds that

$$\lim_{k\to\infty} \frac{M^k}{\lfloor \frac{k}{n} \rfloor !} = 0 \quad \text{iff} \quad \lim_{h\to\infty} \frac{M^{n\cdot h}}{h!}$$

Now, since $M^{n\cdot h} = (M^n)^h$, it is sufficient to prove that, for any M,

$$\lim_{h\to\infty} \frac{M^h}{h!} = 0$$

which is well known. □

4. Appoximation of Hybrid Automata

Let us recall the notion of approximation of an Hybrid Automaton given in [13], which requires to replace formulae with less demanding ones.

Definition 16 (*Henzinger et al. [13]*)**.** An Hybrid Automaton H' is an *approximation* of an Hybrid Automaton H if H' is obtained from H by replacing each formula ϕ contained in the definition of H with a formula ϕ' such that $[\![\phi]\!] \subseteq [\![\phi']\!]$.

In this section we propose a notion of approximation that respects Definition 16. Let us begin with giving a notion of approximation for normal forms in $\Phi(X, F)$.

Definition 17. Given a normal form $\phi \in \Phi(X, F)$ and a natural k, if each function $f \in F \setminus \{+, \cdot, -\}$ that appears in ϕ is derivable $k+1$ times and is analytic in $[\![\phi]\!]$ with respect to a set of vectors V_f, then the *approximation of ϕ of degree k* is the set of formulae denoted $\mathbf{A}(\phi, k)$ that is defined inductively with respect to ϕ as follows:

(1) If $\phi \equiv f(\overrightarrow{x}) \sim b \cdot x_i + c$, then $\mathbf{A}(\phi, k)$ contains either ϕ, if f is a polynomial, or all formulae

$$\phi_{k,\overrightarrow{v}} \equiv P^k(f, \overrightarrow{x}, \overrightarrow{v}) - R^k(f, \overrightarrow{x}, \overrightarrow{v}, \phi) \sim b \cdot x_i + c$$

such that $\overrightarrow{v} \in V_f$, otherwise;
(2) If $\phi \equiv \phi^1 \wedge \phi^2$ then $\mathbf{A}(\phi, k) = \{\phi^1_k \wedge \phi^2_k \mid \phi^1_k \in \mathbf{A}(\phi^1, k) \text{ and } \phi^2_k \in \mathbf{A}(\phi^2, k)\}$;
(3) If $\phi \equiv \phi^1 \vee \phi^2$ then $\mathbf{A}(\phi, k) = \{\phi^1_k \vee \phi^2_k \mid \phi^1_k \in \mathbf{A}(\phi^1, k) \text{ and } \phi^2_k \in \mathbf{A}(\phi^2, k)\}$;
(4) If $\phi \equiv \exists y \in I_y.\, \phi'$ then $\mathbf{A}(\phi, k) = \{\exists y \in I_y.\, \phi'_k \mid \phi'_k \in \mathbf{A}(\phi', k)\}$;
(5) If $\phi \equiv \forall y \in I_y.\, \phi'$ then $\mathbf{A}(\phi, k) = \{\forall y \in I_y.\, \phi'_k \mid \phi'_k \in \mathbf{A}(\phi', k)\}$.

Let us prove that all formulae $\phi_k \in \mathbf{A}(\phi, k)$ are less demanding than ϕ.

Theorem 18. *Given a normal form ϕ and a natural k such that $\mathbf{A}(\phi, k)$ is defined, then it holds that $[\![\phi]\!] \subseteq [\![\phi_k]\!]$ for all $\phi_k \in \mathbf{A}(\phi, k)$.*

Proof. We reason by induction over ϕ. Let us begin with the base case $\phi \equiv f(\overrightarrow{x}) \sim b \cdot x_i + c$. We must prove that $[\![\phi]\!] \subseteq [\![\phi_{k,\overrightarrow{v}}]\!]$, for all $\phi_{k,\overrightarrow{v}}$ as in Definition 17.1. Let us assume that $v \in [\![\phi]\!]$. We know that, for all $\overrightarrow{u} \in Dom(f)$, $f(\overrightarrow{u}) = P^k(f, \overrightarrow{u}, \overrightarrow{v}) + r^k(f, \overrightarrow{u}, \overrightarrow{v})$. Hence, $v \in [\![\phi]\!]$ iff $P^k(f, v(\overrightarrow{x}), \overrightarrow{v}) + r^k(f, v(\overrightarrow{x}), \overrightarrow{v}) \sim b \cdot v(x_i) + c$.

By Proposition 15, $r^k(f, v(\overrightarrow{x}), \overrightarrow{v}) \in [-R^k(f, v(\overrightarrow{x}), \overrightarrow{v}, \phi), R^k(f, v(\overrightarrow{x}), \overrightarrow{v}, \phi)]$.

Since $\sim \in \{<, \leqslant\}$, this implies that $v \in [\![\phi_{k,\overrightarrow{v}}]\!]$.

Let us consider the case $\phi \equiv \phi^1 \vee \phi^2$. Each $\phi_k \in \mathbf{A}(\phi, k)$ has the form $\phi^1_k \vee \phi^2_k$, with $\phi^1_k \in \mathbf{A}(\phi^1, k)$ and $\phi^2_k \in \mathbf{A}(\phi^2, k)$. By the inductive hypothesis, $[\![\phi^1]\!] \subseteq [\![\phi^1_k]\!]$ and $[\![\phi^2]\!] \subseteq [\![\phi^2_k]\!]$. Hence, $[\![\phi]\!] = [\![\phi^1]\!] \cup [\![\phi^2]\!] \subseteq [\![\phi^1_k]\!] \cup [\![\phi^2_k]\!] = [\![\phi_k]\!]$.

The case $\phi \equiv \phi^1 \wedge \phi^2$ is analogous.

Let us consider the case $\phi \equiv \forall y \in I_y.\, \phi'$. Each $\phi_k \in \mathbf{A}(\phi, k)$ has the form $\forall y \in I_y.\, \phi'_k$, with $\phi'_k \in \mathbf{A}(\phi', k)$. Now, for all $v \in [\![\forall y \in I_y.\, \phi']\!]$, it holds that, for all $c \in I_y$, the following relation holds: $v[y := c] \in [\![\phi']\!]$. By the inductive hypothesis, this implies that $v[y := c] \in [\![\phi'_k]\!]$, thus implying that $v \in [\![\forall y \in I_y.\, \phi'_k]\!]$, and, as a consequence, $[\![\phi]\!] \subseteq [\![\phi_k]\!]$.

The case $\phi \equiv \exists y \in I_y.\, \phi'$ is analogous. □

Our notion of approximation for an Hybrid Automaton H exploits the notion of approximation for a normal form of Definition 17.

Definition 19. Let H be an Hybrid Automaton where, for each formula ϕ contained in the definition of H, the set $\mathbf{A}(\phi, k)$ is defined. The *approximation of degree k for H* is the set of the Polynomial Hybrid Automata denoted $\mathbf{A}(H, k)$ that are obtained from H by replacing each formula ϕ contained in the definition of H with some formula in $\mathbf{A}(\phi, k)$.

An immediate corollary of Theorem 18 states that Definition 19 respects Definition 16.

Corollary 20. *Given any Hybrid Automaton H and* $k \in \mathbb{N}$, *all Polynomial Hybrid Automata in* $\mathbf{A}(H, k)$ *are approximations of H according to Definition 16.*

Proof. Directly by Theorem 18. □

Example 21. Let us consider the thermostat of Example 5. Let us call H the automaton where ϕ_{off} and ϕ_{on} are the activity function ϕ^1_{off} and ϕ^1_{on}. The set $\mathbf{A}(H, 3)$ contains the automaton obtained from H by approximating all occurrences of f_1 in ϕ^1_{on} and ϕ^1_{off} by choosing the real 0 as vector $\overrightarrow{v}$. Since $D^k_{y_1} f_1(y_1) = e^{y_1}$, it holds that $P^3(f_1, y_1, 0) \equiv e^0 + e^0 \cdot y_1 + e^0 \cdot \frac{y_1^2}{2!} + e^0 \cdot \frac{y_1^3}{3!} = 1 + y_1 + \frac{y_1^2}{2} + \frac{y_1^3}{6}$. Moreover, $R^3(f_1, y_1, 0, \phi^1_{\text{off}}) = C(f_1, \phi^1_{\text{off}}, 0, 3) \cdot \frac{y_1^4+1}{24}$. Now, $C(f_1, \phi^1_{\text{off}}, 0, 3) = \max\{e^0, u\}$, where since in ϕ^1_{off} it holds that $x' = xe^{y_1}$ and $x, x' \in [m, M]$, we have that $u = \frac{M}{m}$.

As anticipated in Example 5, we could replace H by another automaton H' obtained by considering different functions in activity function formulae. For instance, as activity ϕ_{on} we could consider ϕ^2_{on}. In this case, there are automata in $\mathbf{A}(H', 3)$ where the function $f_4(x', x, y)$ is approximated by $P^3(f_4, (x', x, y), \overrightarrow{0})$, where $\overrightarrow{0}$ denotes $(0, 0, 0)$. We obtain that $P^3(f_4, (x', x, y), \overrightarrow{0}) = x' - x - x \cdot y - \frac{y^2 \cdot x}{2}$. Moreover, $R^3(f_4, (x', x, y), \overrightarrow{0}, \phi^2_{\text{on}}) = C(f_4, \phi^2_{\text{on}}, \overrightarrow{0}, 3) \cdot 3^4 \cdot (x^4 + 1) \cdot (x'^4 + 1) \cdot (y^4 + 1)$. Since for all k it holds that $\sum_{i_1+i_2+i_3=k+1} \frac{D^{i_1}_{x'} D^{i_2}_{x} D^{i_3}_{y} f_4(x'^{i_1} x^{i_2} y^{i_3})}{i_1! i_2! i_3!} = -e^y \cdot (x \cdot \frac{y^k}{k!}) - (x \cdot e^y + h \cdot e^y) \cdot \frac{y^{k+1}}{(k+1)!}$, since in ϕ^1_{on} it holds that $x \leqslant M$, and since $y = \ln\left(\frac{x'-h}{x-h}\right)$ implies that $|y| \leqslant |\ln\left(\frac{M-h}{m-h}\right)|$, we infer $C(f_4, \phi^2_{\text{on}}, \overrightarrow{0}, 3) \leqslant \max\left\{e^0, \frac{M-h}{m-h}\right\} \cdot \left(M \cdot \frac{\ln(\frac{M-h}{m-h})^3}{6}\right) + (M + h) \cdot \frac{\ln\left(\frac{M-h}{m-h}\right)^4}{24}$.

With the notion of syntactic approximation given in Definition 19, a notion of behavioral approximation can be associated. Intuitively, the behavior of an automaton H_k in $\mathbf{A}(H, k)$ approximates the behavior of H in the sense that all configurations that are reachable by H are reachable also by H_k, in the same number of steps. In Section 7 we will study conditions over H ensuring that, if k tends to the infinity, then the behavior of H_k gets "close" to the behavior of H.

Theorem 22. *Given any Hybrid Automaton H and* $k, n \in \mathbb{N}$, *if* $\mathbf{A}(H, k)$ *is defined, then, for all* $H_k \in \mathbf{A}(H, k)$:

$$Post^n(H) \subseteq Post^n(H_k)$$

Proof. The thesis follows from the following two facts:

- Operator *Post* is monotonic, meaning that $R \subseteq R'$ implies $Post(R, H) \subseteq Post(R', H)$.
- Each formula ϕ in H is replaced by a polynomial formula ϕ_k in $\mathbf{A}(H, k)$ such that $[\![\phi]\!] \subseteq [\![\phi_k]\!]$ (see Theorem 18). □

Notice that Theorem 22 implies that we have a sound method for proving that some *bad* configuration of H cannot be reached in n steps. In fact, it is computable whether some configuration can be reached in n steps

by a Polynomial Hybrid Automaton (see Theorem 8). Hence, if we prove that some bad configuration cannot be reached by some H_k in $\mathbf{A}(H,k)$ in n steps, then we infer that this configuration cannot be reached by H in n steps.

5. Syntactical analysis of the error

In [13], Definition 16 is strengthened by the notion of ϵ-approximation, which requires that any vector in $\mathbb{R}^n$ satisfying a formula ϕ' of the approximation H' must be "close" to at least one vector in $\mathbb{R}^n$ satisfying the corresponding formula ϕ in the original automaton H, where "close" means that the "distance" between the two vectors is bounded by ϵ. Intuitively, ϵ-approximations are motivated by the need to limit the error introduced by the approximation.

Here, we reformulate the notion of ϵ-approximation of [13] in terms of a notion of neighborhood of ray ϵ of a space in $\mathbb{R}^n$.

Given two vectors $\overrightarrow{u} = (u_1, \ldots, u_n)$ and $\overrightarrow{v} = (v_1, \ldots, v_n)$ in $\mathbb{R}^n$, let $d(\overrightarrow{u}, \overrightarrow{v})$ denote their *distance* $\sqrt{(u_1 - v_1)^2 + \cdots + (u_n - v_n)^2}$.

Given a vector $\overrightarrow{v}$ and a real $\epsilon > 0$, let $N(\overrightarrow{v}, \epsilon)$ denote the space of vectors $\{\overrightarrow{u} \mid d(\overrightarrow{v}, \overrightarrow{u}) \leqslant \epsilon\}$. This definition can be extended to spaces.

Definition 23. Given a space S in $\mathbb{R}^n$, with $n \in \mathbb{N}$, and a real $\epsilon \geqslant 0$, the *neighborhood of ray ϵ of space S* is the set of spaces

$$N(S,\epsilon) = \{S' \supseteq S \mid \forall \overrightarrow{v}' \in S' \ \ \exists \overrightarrow{v} \in S \text{ such that } d(\overrightarrow{v}, \overrightarrow{v}') \leqslant \epsilon\}$$

The following properties will not play any role in proving the results of the rest of the paper. We give them to demonstrate the solidity of Definition 23.

Proposition 24.
Given spaces S_1 and S_2, and some $\epsilon, \xi \geqslant 0$, it holds that:

(1) $S_1 \subseteq S_2$ *implies* $\forall S_1' \in N(S_1,\epsilon) \ \ \exists S_2' \in N(S_2,\epsilon)$ *such that* $S_1' \subseteq S_2'$;
(2) $\epsilon < \xi$ *implies* $N(S_1,\epsilon) \subset N(S_1,\xi)$;
(3) $\forall S' \in N(S_1 \cup S_2,\epsilon) \ \ \exists S_1' \in N(S_1,\epsilon), S_2' \in N(S_2,\epsilon)$ *such that* $S' = S_1' \cup S_2'$;
(4) $\forall S' \in N(S_1 \cap S_2,\epsilon) \ \ \exists S_1' \in N(S_1,\epsilon), S_2' \in N(S_2,\epsilon)$ *such that* $S' = S_1' \cap S_2'$;
(5) $N(S_1, 0) = \{S_1\}$.

Proof. Let us prove the five properties separately.

(1) Let $S_2' = S_1' \cup S_2$. First of all we have that $S_1' \cup S_2 \supseteq S_2$. It remains to prove that, for each $\overrightarrow{v}' \in (S_1' \cup S_2) \setminus S_2$, there is some $\overrightarrow{v} \in S_2$ such that $d(\overrightarrow{v}, \overrightarrow{v}') \leqslant \epsilon$. Hence, $\overrightarrow{v}' \in (S_1' \cup S_2) \setminus S_2$ implies $\overrightarrow{v}' \in S_1'$, and, since $S_1' \in N(S_1,\epsilon)$, there is some $\overrightarrow{\hat{v}} \in S_1$ such that $d(\overrightarrow{\hat{v}}, \overrightarrow{v}') \leqslant \epsilon$. Since $S_1 \subseteq S_2$, $\overrightarrow{\hat{v}} \in S_2$, and, therefore, $\overrightarrow{\hat{v}}$ is the vector $\overrightarrow{v}$ we were looking for.
(2) We have to prove that, given any $S_1' \in N(S_1,\epsilon)$, it holds that $S_1' \in N(S_1,\xi)$. First of all we note that $S_1' \in N(S_1,\epsilon)$ implies $S_1' \supseteq S_1$. It remains to prove that, for each $\overrightarrow{v}' \in S_1' \setminus S_1$, there is some $\overrightarrow{v} \in S_1$ such that $d(\overrightarrow{v}, \overrightarrow{v}') \leqslant \xi$. Since $S_1' \in N(S_1,\epsilon)$, we are sure that there is some $\overrightarrow{\hat{v}} \in S_1$ such that $d(\overrightarrow{\hat{v}}, \overrightarrow{v}') \leqslant \epsilon$. Since $\epsilon \leqslant \xi$, $d(\overrightarrow{\hat{v}}, \overrightarrow{v}') \leqslant \epsilon$ implies $d(\overrightarrow{\hat{v}}, \overrightarrow{v}') \leqslant \xi$, and, therefore, $\overrightarrow{\hat{v}}$ is the vector $\overrightarrow{v}$ we were looking for.
(3) Let $S_1' = S' \cap \{\overrightarrow{v}'' \mid \exists \overrightarrow{v} \in S_1 \mid d(\overrightarrow{v}, \overrightarrow{v}'') \leqslant \epsilon\}$, and $S_2' = S' \cap \{\overrightarrow{v}'' \mid \exists \overrightarrow{v} \in S_2 \mid d(\overrightarrow{v}, \overrightarrow{v}'') \leqslant \epsilon\}$. It is immediate that $S_1' \in N(S_1,\epsilon)$, $S_2' \in N(S_2,\epsilon)$ and $S' = S_1' \cup S_2'$.
(4) Let $S_1' = S' \cup S_1$, and $S_2' = S' \cup S_2$. It is immediate that $S_1' \in N(S_1,\epsilon)$, $S_2' \in N(S_2,\epsilon)$ and $S' = S_1' \cap S_2'$.
(5) Immediate. □

We can reformulate the notion of ϵ-approximation of [13] by exploiting Definition 23.

Definition 25. A formula $\phi' \in \Phi(X,F)$ is an *ϵ-approximation* of a formula $\phi \in \Phi(X,F)$ iff $\{v(\overrightarrow{X}) \mid v \in [\![\phi']\!]\} \in N(\{v(\overrightarrow{X}) \mid v \in [\![\phi]\!]\}, \epsilon)$.

Definition 26 (*Henzinger et al. [13], reformulated*)**.** An Hybrid Automaton H' is an *ϵ-approximation* of an Hybrid Automaton H if H' is obtained from H by replacing each formula ϕ contained in the definition of H with a formula ϕ' such that ϕ' is an ϵ-approximation of ϕ.

Our aim is to study the conditions over the formulae in H ensuring that, given any $\epsilon > 0$, there exists some $k_0 \in \mathbb{N}$ such that, for all $k > k_0$, it holds that the set $\mathbf{A}(H,k)$ contains only ϵ-approximations for H.

In the next two subsections we analyze two examples showing that the existence of such a k_0 is, in general, not guaranteed.

5.1. Bounded formulae

The first example suggests that we can manage only formulae ϕ constraining variables within bounded intervals, thus avoiding variables that can tend to the infinity. (This does not represent a critical issue, since the definition of convergence for Taylor series requires that variables belong to bounded intervals.)

Example 27. Let us consider the following formula:

$$\phi \equiv -\sin(x') \leqslant 0$$

Note that $-\sin(x') \leqslant 0$ is a normal form, since it respects the schema $f(x') \leqslant b \cdot x_i + c$, with $I(f)(u) = -\sin(u)$ and $b = c = 0$.

Let us take any $0 < \epsilon < \frac{\pi}{2}$. We can show that there is no k such that $\mathbf{A}(\phi,k)$ contains ϵ-approximations for ϕ. We have that:

$$[\![\phi]\!] = \{v \mid v(x') \in \{[2i \cdot \pi, (2i+1) \cdot \pi] \mid i \in \mathbb{Z}\}\}$$

Given any $k \in \mathbb{N}$, and any $v \in Dom(\sin(x'))$, the set $\mathbf{A}(\phi,k)$ contains all the formulae $\phi_{k,v}$ of the following form:

$$P^k(-\sin(x'),x',v) - R^k(-\sin(x'),x',v,\phi) \leqslant 0$$

Since $P^k(-\sin(x'),x',v) - R^k(-\sin(x'),x',v,\phi)$ is a polynomial, it holds that

$$\lim_{x' \to \infty} |P^k(-\sin(x'),x',v) - R^k(-\sin(x'),x',v,\phi)| = \infty$$

Since $[\![\phi]\!] \subseteq [\![\phi_{k,v}]\!]$, this last property and the form of $[\![\phi]\!]$ imply either relation $\lim_{x' \to \infty} P^k(-\sin(x'),x',v) - R^k(-\sin(x'),x',v,\phi) = -\infty$, or relation $\lim_{x' \to -\infty} P^k(-\sin(x'),x',v) - R^k(-\sin(x'),x',v,\phi) = -\infty$.
Therefore, there exists some x_0 such that either all evaluations v with $v(x') \in (-\infty, -x_0]$ are in $[\![\phi_{k,v}]\!]$, or all evaluations v with or $v(x') \in (x_0, \infty]$ are in $[\![\phi_{k,v}]\!]$. Given any $u = (2 \cdot i + \frac{3}{2}) \cdot \pi$, for some $i \in \mathbb{Z}$ such that $u \in (-\infty, -x_0]$ or $u \in [x_0, \infty)$, and the evaluation v such that $v(x') = u$, it holds that $v \in [\![\phi_{k,v}]\!]$, but there is no $v' \in [\![\phi]\!]$ with $d(u, v'(x')) \leqslant \epsilon$, thus implying $[\![\phi_{k,v}]\!] \notin N([\![\phi]\!], \epsilon)$.

As suggested by Example 27, let us introduce the notion of bounded formula.

Definition 28. Given a partition $\{y_1, \ldots, y_m\}, \{x_{h_1}, \ldots, x_{h_n}\}$ of X, a normal form $\phi \in \Phi(X,F)$ is *bounded* iff it is of the form

$$Q_1 y_1 \in [l_{y_1}, r_{y_1}]. \ \cdots Q_m y_m \in [l_{y_m}, r_{y_m}]. \ \left(\phi' \wedge \bigwedge_{i \in [1,n]} x_{h_i} \in [l_i^\phi, u_i^\phi] \right)$$

where $Q_1, \ldots, Q_m \in \{\exists, \forall\}$, $l_{y_1} \leqslant r_{y_1}, \ldots, l_{y_m} \leqslant r_{y_m}$, and $l_1^\phi \leqslant u_1^\phi, \ldots, l_n^\phi \leqslant u_n^\phi$.

5.2. Avoiding operators <and>

The second example suggests to take care with formulae of the form $f(\overrightarrow{x}) \sim c$, where $\sim \in \{<, >\}$, since these kind of formulae describe open sets.

Example 29. Let us consider the following normal form:

$$\phi \equiv -\sin(x') < -1$$

Let us take any $\epsilon > 0$. We can show that, for all k of the form $k = 4h + 2$, with $h \in \mathbb{N}$, $\mathbf{A}(\phi, k)$ contains some formulae that are not ϵ-approximations for ϕ.
We have that $[\![\phi]\!] = \emptyset$.
Given any $k \in \mathbb{N}$, and any $v \in Dom(\sin(x'))$, the set $\mathbf{A}(\phi, k)$ contains all the formulae $\phi_{k,v}$ of the following form:

$$P^k(-\sin(x'), x', v) - R^k(-\sin(x'), x', v, \phi) < -1$$

For each k of the form $k = 4h + 2$, it holds that $D^{k+1}(-\sin(x')) = \cos(x')$. Lagrange Remainder Theorem ensures that, for all $u \in \mathbb{R}$:

$$-\sin(u) - P^k(-\sin(x'), u, v) = \cos(\xi)\frac{(u-v)^{k+1}}{(k+1)!}, \text{ for some } u \leqslant \xi \leqslant v$$

Let $u = (2i + \frac{1}{2}) \cdot \pi$ and $2i \cdot \pi < v < u$. Since $\sin(u) = 1$ and $\cos(\xi)\frac{(u-v)^{k+1}}{(k+1)!} \geqslant 0$ for all $v \leqslant \xi \leqslant u$, we have that $P^k(-\sin(x'), u, v) \leqslant -1$.

Now, $C(-\sin(x'), \phi, v, k) = 1$, which implies that $R^k(-\sin(x'), u, v, \phi)$ has value $\frac{(u-v)^{2\left\lceil \frac{k+1}{2} \right\rceil}+1}{(k+1)!}$, which is strictly > 0.
Hence, $P^k(-\sin(x'), x', v) - R^k(-\sin(x'), x', v, \phi) < -1$ is satisfied in a neighborhood of u, thus implying that $[\![\phi_{k,v}]\!] \neq \emptyset$ and $[\![\phi_{k,v}]\!] \notin N([\![\phi]\!], \epsilon)$.

Notice that if we consider the formula $-\sin(x') \leqslant -1$ instead of $-\sin(x') < -1$, the argument of Example 29 falls. In fact, in such a case the evaluation v with $v(x') = (2i + \frac{1}{2}) \cdot \pi$ satisfies $v \in [\![\phi]\!]$, for all $i \in \mathbb{N}$, and, if k tends to the infinity, $R^k(-\sin(x'), x', v, \phi)$ tends to 0 and $P^k(-\sin(x'), x', v) - R^k(-\sin(x'), x', v, \phi) \leqslant -1$ is satisfied only for values u' that tend to $(2i + \frac{1}{2}) \cdot \pi$, for some $i \in \mathbb{N}$. If we fix ϵ, we can choose some k_0 such that, if $k > k_0$, $d(u, u') \leqslant \epsilon$.

6. Results on syntactical analysis of the error

First of all let us prove that for all formulae ϕ satisfying the restrictions suggested by Example 27 and Example 29, and for all $\epsilon > 0$, there exists some k_0 such that, for all $k > k_0$, $\mathbf{A}(\phi, k)$ contains only ϵ-approximations of ϕ.

Before giving the formal proof of this result, let us explain the main idea for the base case $f(x) \sim c$, which is an instance of $f(\overrightarrow{x}) \sim b \cdot x_i + c$ with f unary and $b = 0$.

Let ϕ be the bounded normal form $f(x) \leqslant c \wedge x \in [l_x, u_x]$, with f a non-polynomial function. Let us assume that the graphic of f, restricted in the interval $[l_x, u_x]$, is that depicted in Fig. 2 (black line). Intervals where $f(x) \leqslant c$ is satisfied are graphically represented by means of bold lines on the abscissa axis.

The idea is to split $Dom(f) \cap [l_x, u_x]$ in $m \geqslant 0$ closed intervals $S_1, \ldots, S_m$ of size less than ϵ, and to show that there is a k_0 such that, for all $k > k_0$, for each value z and evaluation v such that $v(x) = z$ and $v \in [\![\phi_k]\!]$ for some $\phi_k \in \mathbf{A}(\phi, k)$, there are a value z' and an evaluation v' such that $v'(x) = z'$, $v' \in [\![\phi]\!]$ and z' and z lye in the same interval S_i. In fact, in this case, z and z' have distance less than ϵ and, by the arbitrarity of z, we infer that ϕ_k is an ϵ-approximation of ϕ. For instance, in Fig. 2 these intervals are S_1, S_2, S_3, S_4. In Fig. 2 we represent also the functions $g(x)$ (bright grey line) and $h(x)$ (dark grey line). Both $f(x) \leqslant c \wedge x \in [l_x, u_x]$ and $g(x) \leqslant c \wedge x \in [l_x, u_x]$ are satisfied only by values in S_1 and S_4, thus implying that $g(x) \leqslant c \wedge x \in [l_x, u_x]$ is an ϵ-approximation for ϕ. This does not hold for $h(x) \leqslant c \wedge x \in [l_x, u_x]$, since $h(x) \leqslant c$ is satisfied also by values in S_2 and S_3.

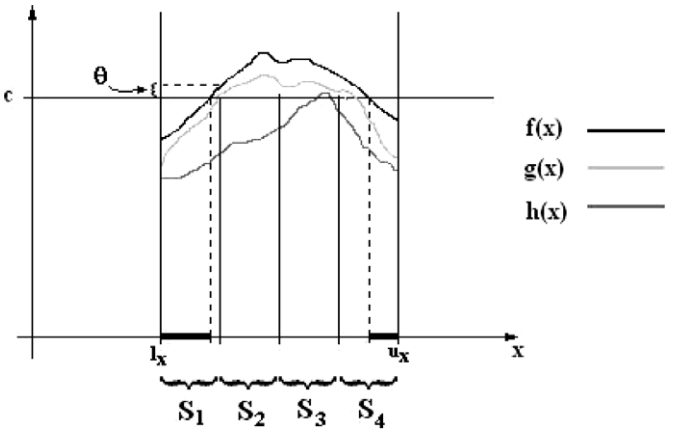

Fig. 2. The function f.

Let us consider the intervals $S_{i_1}, \ldots, S_{i_l}$ in $\{S_1 \ldots, S_m\}$ such that no evaluation in $[\![\phi]\!]$ maps x to $S_{i_1} \cup \cdots \cup S_{i_l}$. For instance, in Fig. 2 these intervals are S_2 and S_3. It suffices to prove that there exists a k_0 such that, for all $k > k_0$ and $\phi_k \in \mathbf{A}(\phi, k)$, all evaluations $v \in [\![\phi_k]\!]$ are such that $v(x) \notin S_{i_1} \cup \cdots \cup S_{i_l}$. Let $\vartheta = \min\{f(u) - c \mid u \in S_{i_1} \cup \cdots \cup S_{i_l}\}$ (in Fig. 2 we have depicted ϑ for S_1). The continuity of f and the fact that all these intervals are closed ensure that ϑ exists, and that $\vartheta > 0$. For each k, all formula in $\mathbf{A}(\phi, k)$ have the form $P^k(f, x, v) - R^k(f, x, v, \phi) \leqslant c$, for some $v \in Dom(f)$. It suffices to prove that $P^k(f, u, v) - R^k(f, u, v, \phi) \geqslant c$ for all $u \in S_{i_1} \cup \cdots \cup S_{i_l}$. It holds that $\max\{R^k(f, u, v, \phi) \mid u \in S_{i_1} \cup \cdots \cup S_{i_l}\} = R^k(f, u_0, v, \phi)$, with u_0 the upper bound of $S_{i_1} \cup \cdots \cup S_{i_l}$. Let e_k denote such a value $R^k(f, u_0, v, \phi)$. Actually, e_k is an upper bound in $S_{i_1} \cup \cdots \cup S_{i_l}$ of the error introduced by the approximation. Moreover, by Proposition 15 we can find a k_0 such that, for all $k > k_0$, $e_k < \frac{\vartheta}{2}$. This means that we have a bound to the error for $k > k_0$ which implies that $P^k(f, u, v) - R^k(f, u, v, \phi) \geqslant c$ for all $u \in S_{i_1} \cup \cdots \cup S_{i_l}$. In fact, for all $u \in S_{i_1} \cup \cdots \cup S_{i_l}$ it holds that $f(u) = P^k(f, u, v) + r^k(f, u, v) \geqslant \vartheta + c$. Since $|r^k(f, u, v)| \leqslant e_k < \frac{\vartheta}{2}$, this implies that it cannot happen that $P^k(f, u, v) - R^k(f, u, v, \phi) \leqslant c$. Summarizing, each $\phi_k \in \mathbf{A}(\phi, k)$ is such that all $v \in [\![\phi_k]\!]$ cannot map x to $S_{i_1} \cup \cdots \cup S_{i_l}$, as required.

The more general case $f(\vec{x}) \leqslant b \cdot x_i + c$ is an immediate extension of the case $f(x) \leqslant c$. Conjunction, disjunction and existential quantifications can be proved by induction. The case $\forall y \in [l_y, u_y].\, \phi$ is more complex, since the inductive hypothesis cannot exploited in any trivial way. Actually, by the inductive hypothesis we can assume that there exists a k_0 such that, for all $k > k_0$, $[\![\phi_k]\!]$ is close enough to $[\![\phi]\!]$. Unfortunately, this does not ensure that $[\![\forall y \in [l_y, u_y].\, \phi_k]\!]$ is close enough to $[\![\forall y \in [l_y, u_y].\, \phi]\!]$, since the universal quantifier does not preserve the approximation property. To have an idea of the motivation, let us consider formulae $\phi \equiv x \in [0,1] \wedge (y \in [2,3] \vee y \in [4,5])$ and $\psi \equiv x \in [0,1] \wedge y \in [2,5]$, with $[\![\phi]\!]$ containing all the evaluations mapping (x, y) to any value in $([0,1] \times [2,5]) \setminus B$, where B is the rectangle $[0,1] \times (3,4)$, and $[\![\psi]\!]$ containing all the evaluations mapping (x, y) to any value in $[0,1] \times [2,5]$. If we choose a value $\epsilon > 1$ (namely, if ϵ is greater than the height of the rectangle B), then ψ is an ϵ-approximation of ϕ. Let us consider formulae $\forall y \in [2,5].\, \phi$ and $\forall y \in [2,5].\, \psi$. It holds that $[\![\forall y \in [2,5].\, \phi]\!] = \emptyset$, and $[\![\forall y \in [2,5].\, \psi]\!]$ contains all the evaluations mapping x to any value in $[0,1]$, thus implying that $\forall y \in [2,5].\, \psi$ is not an ϵ-approximation of $\forall y \in [2,5].\, \phi$.

A similar problem arises if we have a formula $\phi \in \Phi(X, F)$, an ϵ-approximation ψ, and two spaces D_1 and D_2 containing all the possible values that the evaluations in $[\![\psi]\!] \setminus [\![\phi]\!]$ can be assign to X, such that D_1 and D_2 have empty intersection but have non-empty intersection when projecting on variables in $\mathbf{FV}(\phi) \setminus \{y\}$. In this case, the set D obtained as the intersection of D_1 and D_2 projected on the variables $X \setminus \{y\}$ may be such that the values in D can be assigned to the variables $X \setminus \{y\}$ by all evaluations in $[\![\forall y \in [l_y, r_y].\, \psi]\!]$ and by no evaluations in $[\![\forall y \in [l_y, r_y].\, \phi]\!]$, thus implying that $\forall y \in [l_y, r_y].\, \psi$ is not an approximation of $\forall y \in [l_y, r_y].\, \phi$ for any ϵ bounded by the maximal distance between any two points in D. As an example of this case, let us take the formulae

$$\phi \equiv (x \geqslant 3 \vee y \leqslant 3) \wedge (x \leqslant 2 \vee y \geqslant 1) \wedge x \in [0,5] \wedge y \in [0,4]$$

$$\psi \equiv (x \geqslant 2.1 \vee y \leqslant 3) \wedge (x \leqslant 2.9 \vee y \geqslant 1) \wedge x \in [0,5] \wedge y \in [0,4]$$

and the reals $\epsilon = 1$, $l_y = 0$ and $r_y = 4$. In this case D_1 and D_2 are the rectangles $(2.1, 3) \times (3, 4)$ and $(2, 2.9) \times (0, 1)$, and D is the interval $(2.1, 2.9)$. It clearly holds that $\{v((x, y)) \mid v \in [\![\psi]\!]\} \in N(\{v((x, y)) \mid v \in [\![\phi]\!]\}, \epsilon)$, but $\{v(x) \mid v \in [\![\forall y \in [l_y, r_y].\psi]\!]\} = (2.2, 2.9) \notin N(\{v(x) \mid v \in [\![\forall y \in [l_y, r_y].\phi]\!]\}, \epsilon) = \emptyset$.

The problems emphasized before can be solved by improving the approximation only if the rectangle B in the first example, and the interval D in the second are finite open sets. Fortunately, this condition holds, as we prove in a ad-hoc Lemma (Lemma 31 below).

Theorem 30. *Given any normal form $\phi \in \Phi(X, F)$ such that*:

(1) *ϕ is bounded*;
(2) *each subformula $f(\overrightarrow{x}) \sim b \cdot x_i + c$ in ϕ is such that $\sim$ is $\leqslant$,*

then, for each $\epsilon > 0$, there exists some k_0 such that, for each $k > k_0$, the set $\mathbf{A}(\phi, k)$ contains only ϵ-approximations for ϕ.

Proof. Let us begin with proving the following Lemma.

Lemma 31. *If $\phi \in \Phi(X, F)$ is bounded, and, for each formula $f(\overrightarrow{x}) \sim b \cdot x_i + c$ in ϕ it holds that $\sim$ is $\leqslant$ (resp. $\sim$ is $<$), then $\{v(\boldsymbol{FV}(\phi)) \mid v \in [\![\phi]\!]\}$ is a closed set (resp. open set).*

Proof. Let us consider first the case where for each formula $f(\overrightarrow{x}) \sim b \cdot x_i + c$ in ϕ, it holds that $\sim$ is $\leqslant$.

Let $\phi \equiv Q_1 y_1 \in [l_{y_1}, r_{y_1}]. \cdots Q_m y_m \in [l_{y_m}, r_{y_m}]. (\phi' \wedge \bigwedge_{i \in [1,n]} x_{h_i} \in [l_i^\phi, u_i^\phi])$.

Without loss of generality, we can assume that x_{h_i} is x_i for all $1 \leqslant i \leqslant n$.

Let $\overrightarrow{x}$ denote $(x_1, \ldots, x_n)$ and and $\overrightarrow{y}$ denote $(y_1, \ldots, y_m)$.

Without loss of generality, we can assume that, if ϕ contains a formula $f(\overrightarrow{z}) \leqslant b \cdot x_i + c$, then $\overrightarrow{z} = \overrightarrow{x} \oplus \overrightarrow{y}$. In fact, in general, we have that $f(\overrightarrow{z}) \leqslant b \cdot x_i + c$ iff $f'(\overrightarrow{z} \oplus (z)) \leqslant b \cdot x_i + c$, where f' is the function such that $f'(\overrightarrow{v} \oplus (c)) = f(\overrightarrow{v})$, for all c. Obviously, f' is continuous since f is continuous.

Without loss of generality, we can assume that, for each formula $f(\overrightarrow{x} \oplus \overrightarrow{y}) \leqslant b \cdot x_i + c$ appearing in ϕ, it holds that $b = c = 0$. In fact, we can rewrite all $f(\overrightarrow{x} \oplus \overrightarrow{y}) \leqslant b \cdot x_i + c$ in ϕ into $f'(\overrightarrow{x} \oplus \overrightarrow{y}) \leqslant 0$, where f' is the function such that $f'(\overrightarrow{v}) = f(\overrightarrow{v}) - b \cdot v_i - c$. Notice that, since f is continuous, then also f' is continuous.

It suffices to prove that each formula $Q_i y_i \in [l_{y_i}, r_{y_i}]. \cdots Q_m y_m \in [l_{y_m}, r_{y_m}].(\phi' \wedge \bigwedge_{i \in [1,n]} x_i \in [l_i^\phi, u_i^\phi])$ is equivalent to a formula $g(\overrightarrow{x} \oplus (y_1, \ldots, y_{i-1})) \leqslant 0 \wedge \bigwedge_{i \in [1,n]} x_i \in [l_i^\phi, u_i^\phi]$, for g a continuous function. In fact, in this case, it follows that ϕ is equivalent to formula $g(\overrightarrow{x}) \leqslant 0 \wedge \bigwedge_{i \in [1,n]} x_i \in [l_i^\phi, u_i^\phi]$ over $\overrightarrow{x}$ and F, for a continuous g, which denotes a closed set. Let us reason by induction over $\alpha = |\{i, i+1, \ldots, m\}|$. Let us consider the base case $\alpha = 0$. We can reason by induction over the form of ϕ'.

- If $\phi' \equiv f(\overrightarrow{x} \oplus \overrightarrow{y})$, then the thesis is immediate, since f is the continuous function g we were looking for.
- If $\phi' \equiv f(\overrightarrow{x} \oplus \overrightarrow{y}) \leqslant 0 \vee f'(\overrightarrow{x} \oplus \overrightarrow{y}) \leqslant 0$, then ϕ' is equivalent to $f''(\overrightarrow{x} \oplus \overrightarrow{y}) \leqslant 0$, where $f''(\overrightarrow{v}) = \min\{f(\overrightarrow{v}), f'(\overrightarrow{v})\}$. Since both f and f' are continuous, then also f'' is continuous, and it is the function g we were looking for.
- The case $\phi' \equiv f(\overrightarrow{x} \oplus \overrightarrow{y}) \leqslant 0 \wedge f'(\overrightarrow{x} \oplus \overrightarrow{y}) \leqslant 0$ is analogous. We should use max instead of min.

Let us consider now the induction step. Namely, we prove that if the thesis holds for α then the thesis holds for $\alpha + 1$. Let $j = m - (\alpha + 1)$. Assume first that $Q_j = \exists$. In this case, we are dealing with formula $\psi \equiv \exists y_j \in [l_{y_j}, u_{y_j}]. \psi'$, where, by the inductive hypothesis, ψ' is equivalent to $f(\overrightarrow{x} \oplus (y_1, \ldots, y_j)) \leqslant 0 \wedge \bigwedge_{i \in [1,n]} x_i \in [l_i^\phi, u_i^\phi]$, for some continuous f. It holds that ψ is equivalent to $f'(\overrightarrow{x} \oplus (y_1, \ldots, y_{j-1})) \leqslant 0 \wedge \bigwedge_{i \in [1,n]} x_i \in [l_i^\phi, u_i^\phi]$, where for all $\overrightarrow{v}$ with $|\overrightarrow{v}| = |\overrightarrow{x}| + j - 1$, $f'(\overrightarrow{v}) = min\{f(\overrightarrow{v} \oplus (c)) \mid c \in [l_{y_j}, u_{y_j}]\}$. We have to prove that f' is continuous, since, in that case, it is the function g we were looking for.

By Heine-Cantor Theorem, since f is continuous and $[l_{y_j}, r_{y_j}]$ is a closed bounded set, then f is uniformly continuous in that set. Actually, we can prove that f' is uniformly continuous. Let us recall that a function h is uniformly continuous if, for any $\epsilon > 0$, there exists $\delta > 0$ such that, for any $\overrightarrow{v}, \overrightarrow{v}' \in Dom(h)$, it holds that $d(\overrightarrow{v}, \overrightarrow{v}') < \delta$ implies $|h(\overrightarrow{v}) - h(\overrightarrow{v}')| < \epsilon$. Let us take any $\epsilon > 0$. Let δ be the value such that $d(\overrightarrow{v} \oplus (c), \overrightarrow{v}' \oplus (c')) < \delta$ implies $|f(\overrightarrow{v} \oplus (c)) - f(\overrightarrow{v}' \oplus (c'))| < \epsilon$. We can prove that δ is such that $d(\overrightarrow{v}, \overrightarrow{v}') < \delta$ implies

$|f'(\overrightarrow{v}) - f'(\overrightarrow{v}')| < \epsilon$. In fact, given arbitrary $\overrightarrow{u}, \overrightarrow{u}'$ such that $d(\overrightarrow{u}, \overrightarrow{u}') < \delta$, it holds that $|f'(\overrightarrow{u}) - f'(\overrightarrow{u}')| = |f(\overrightarrow{u} \oplus (c)) - f(\overrightarrow{u}' \oplus (c'))|$, for suitable values of c and c'. We have two cases: Either $f(\overrightarrow{u} \oplus (c)) - f(\overrightarrow{u}' \oplus (c')) \geqslant 0$, or $f(\overrightarrow{u} \oplus (c)) - f(\overrightarrow{u}' \oplus (c')) < 0$. We consider only the former one, the latter being analogous. If $f(\overrightarrow{u} \oplus (c)) - f(\overrightarrow{u}' \oplus (c')) \geqslant 0$, then $|f'(\overrightarrow{u}) - f'(\overrightarrow{u}')| = f(\overrightarrow{u} \oplus (c)) - f(\overrightarrow{u}' \oplus (c'))$. By construction of f', we have that $f(\overrightarrow{u} \oplus (c)) \leqslant f(\overrightarrow{u} \oplus (c'))$, which implies, $f(\overrightarrow{u} \oplus (c)) - f(\overrightarrow{u}' \oplus (c')) \leqslant f(\overrightarrow{u} \oplus (c')) - f(\overrightarrow{u}' \oplus (c'))$. Moreover, $d(\overrightarrow{u} \oplus (c'), \overrightarrow{u}' \oplus (c')) = d(\overrightarrow{u}, \overrightarrow{u}') < \delta$, which implies $f(\overrightarrow{u} \oplus (c')) - f(\overrightarrow{u}' \oplus (c')) < \epsilon$, thus implying that f' is uniformly continuous, and, therefore, continuous.

The case $Q_j = \forall$ is analogous. We should use max instead of min.

Let us consider the case where for each formula $f(\overrightarrow{x}) \sim b \cdot x_i + c$ in ϕ it holds that $\sim$ is $<$. To prove that $\{v(\mathbf{FV}(\phi)) \mid v \in [\![\phi]\!]\}$ is open, it suffices to prove that $\mathbb{R}^{|\mathbf{FV}(\phi)|} \setminus \{v(\mathbf{FV}(\phi)) \mid v \in [\![\phi]\!]\} = \{v(\mathbf{FV}(\phi)) \mid v \in [\![\neg\phi]\!]\}$ is closed. By Proposition 3, $\neg\phi$ can be rewritten into a normal form ϕ' such that, for each formula $f(\overrightarrow{x}) < b \cdot x_i + c$ in ϕ, there is the formula $f^-(\overrightarrow{x}) \leqslant -b \cdot x_i - c$ in ϕ'. Since $\leqslant$ is the only relation symbol in ϕ', we have already proved that $\{v(\mathbf{FV}(\phi)) \mid v \in [\![\phi']\!]\}$ is closed, which completes the proof. □

We are ready to prove Theorem 30. Let us reason by induction over ϕ.

Proof.

Base case: ϕ has no quantifier.

Since ϕ has no quantifier, we can rewrite ϕ in disjunctive normal form as usual. Hence, we can assume that ϕ has the form:

$$\phi \equiv \bigvee_{1 \leqslant i \leqslant h} \left(\left(\bigwedge_{1 \leqslant j \leqslant n_i} f_{i,j}(\overrightarrow{x_{i,j}}) \leqslant b_{i,j} \cdot x_{i,j} + c_{i,j} \right) \wedge \psi_i \right) \tag{1}$$

where, for all $1 \leqslant i \leqslant h$ and $1 \leqslant j \leqslant n_i$, $\overrightarrow{x_{i,j}}$ is a vector over variables in X, $x_{i,j}$ is a variable in X, and $f_{i,j}$ is not a polynomial, and, for all $1 \leqslant i \leqslant h$, ψ_i contains the conjunction of formulae of the form $p(\overrightarrow{x}') \leqslant b \cdot x'_i + c$, with p a polynomial function. Of course, the disjunction above contains the constraints $x_i \in [l_i^\phi, u_i^\phi]$, for all $1 \leqslant i \leqslant |X|$, which appear in all ψ_i.

Let us reason by induction over h.

Let us consider first the base case where $h = 1$, namely ϕ has the form

$$\phi \equiv \left(\bigwedge_{1 \leqslant j \leqslant n_1} f_{1,j}(\overrightarrow{x_{1,j}}) \leqslant b_{1,j} \cdot x_{1,j} + c_{1,j} \right) \wedge \psi_1$$

Each $\phi_k \in \mathbf{A}(\phi, k)$ has the form

$$\phi_k \equiv \left(\bigwedge_{1 \leqslant j \leqslant n_1} P^k(f_{1,j}, \overrightarrow{x_{1,j}}, \overrightarrow{v_{1,j}}) - R^k(f_{1,j}, \overrightarrow{x_{1,j}}, \overrightarrow{v_{1,j}}, \phi) \leqslant b_{1,j} \cdot x_{1,j} + c_{1,j} \right) \wedge \psi_1$$

where, for all $1 \leqslant j \leqslant n_1$, $\overrightarrow{v_{1,j}}$ is a vector in $Dom(f_{1,j})$.

We are looking for a k_0 such that, if $k > k_0$, then, given any vector $\overrightarrow{z}$ and evaluation $v \in [\![\phi_k]\!] \setminus [\![\phi]\!]$ such that $v(\overrightarrow{X}) = \overrightarrow{z}$, there exist some vector $\overrightarrow{z}'$ and evaluation $v' \in [\![\phi]\!]$ such that $v'(\overrightarrow{X}) = \overrightarrow{z}'$ and $d(\overrightarrow{z}, \overrightarrow{z}') \leqslant \epsilon$.

Since all constraints $x_i \in [l_i^\phi, u_i^\phi]$, for all $1 \leqslant i \leqslant |X|$, are in ψ_1 and, therefore, in ϕ_k, it follows that $\overrightarrow{z}$ is in $[l_1^\phi, u_1^\phi] \times \cdots \times [l_{|X|}^\phi, u_{|X|}^\phi]$.

Let $S_1, \ldots, S_m$ be m spaces in $\mathbb{R}^n$ such that:

- $\left[l_1^\phi, u_1^\phi\right] \times \cdots \times \left[l_{|X|}^\phi, u_{|X|}^\phi\right] = S_1 \cup \cdots \cup S_m$
- for all $1 \leqslant i \leqslant m$, there exist $d_{i,1}, \ldots, d_{i,|X|}$ and $\delta \leqslant \frac{\epsilon}{\sqrt{|X|}}$ such that $S_i = \left[l_1^\phi + d_{i,1}, l_1^\phi + d_{i,1} + \delta\right] \times \cdots \times \left[l_{|X|}^\phi + d_{i,|X|}, l_{|X|}^\phi + d_{i,|X|} + \delta\right]$.

We note that $\delta \leqslant \frac{\epsilon}{\sqrt{|X|}}$ implies $\sqrt{|X| \cdot \delta^2} \leqslant \epsilon$, and, therefore, for all $1 \leqslant i \leqslant m$ and $\overrightarrow{u}, \overrightarrow{v} \in S_i$, $d(\overrightarrow{v}, \overrightarrow{u}) \leqslant \epsilon$.

Now, given a vector $\overrightarrow{u} = (u_1, \ldots, u_{|X|})$ over $\mathbb{R}^{|X|}$, and a term $f(x_{i_1}, \ldots, x_{i_{ar(f)}})$, we write $f(\overrightarrow{u})$ to denote $f(u_{i_1}, \ldots, u_{i_{ar(f)}})$.

Let $f_{\max}(\overrightarrow{X})$ be the function such that, for all $\overrightarrow{u} \in \mathbb{R}^{|X|}$:

$$f_{\max}(\overrightarrow{u}) = \max\{f_{1,j}(\overrightarrow{u}) - b_{1,j} \cdot u_{1,j} - c_{1,j} \mid 1 \leqslant j \leqslant n_1\}$$

Since all $f_{1,j}$ are continuous, also function $f_{\max}$ is continuous. Moreover, for all $\overrightarrow{u} \in \mathbb{R}^{|X|}$, it holds that $f_{\max}(\overrightarrow{u}) \leqslant 0$ iff $\bigwedge_{1 \leqslant j \leqslant n_1} f_{1,j}(\overrightarrow{u}) \leqslant b_{1,j} \cdot u_{1,j} + c_{1,j}$.

Let $S_{i_1}, \ldots, S_{i_l}$ be the spaces in $\{S_1, \ldots, S_m\}$ such that $f_{\max}(\overrightarrow{u}) > 0$ for all $\overrightarrow{u} \in S_{i_1} \cup \cdots \cup S_{i_l}$, namely, no v in $[\![\phi]\!]$ maps $\overrightarrow{X}$ to any value in $S_{i_1} \cup \cdots \cup S_{i_l}$. Let $\vartheta = \inf\{f_{\max}(\overrightarrow{u}) \mid \overrightarrow{u} \in S_{i_1} \cup \cdots \cup S_{i_l}\}$. It holds that $\vartheta \neq 0$. In fact, if ϑ would be 0, then there would exist some $\overrightarrow{v}$ such that $\lim_{\overrightarrow{u} \longrightarrow \overrightarrow{v}} f_{\max}(\overrightarrow{u}) = 0$. Moreover, since $S_{i_1}, \ldots, S_{i_l}$ are closed sets, $\overrightarrow{v} \in S_{i_1} \cup \cdots \cup S_{i_l}$, and, since $f_{\max}$ is continuous, $f_{\max}(\overrightarrow{v}) = 0$, which contradicts that $f_{\max}(\overrightarrow{u}) > 0$ for all $\overrightarrow{u} \in S_{i_1} \cup \cdots \cup S_{i_l}$.

Now, for all $1 \leqslant j \leqslant n_1$ and for all $\overrightarrow{u_{1,j}} \in Dom(f_{1,j})$, it holds that $f_{1,j}(\overrightarrow{u_{1,j}}) = P^k(f_{1,j}, \overrightarrow{u_{1,j}}, \overrightarrow{v_{1,j}}) + r^k(f_{1,j}, \overrightarrow{u_{1,j}}, \overrightarrow{v_{1,j}})$, where the remainder satisfies (by Theorem 10 and Proposition 15) $r^k(f_{1,j}, \overrightarrow{u_{1,j}}, \overrightarrow{v_{1,j}}) \in [-R^k(f_{1,j}, \overrightarrow{u_{1,j}}, \overrightarrow{v_{1,j}}, \phi),$

$R^k(f_{1,j}, \overrightarrow{u_{1,j}}, \overrightarrow{v_{1,j}}, \phi)]$. Let us take the vector $\overrightarrow{t}$ that satisfies relation $(u_i - v_i)^{2 \cdot \left\lceil \frac{k+1}{2} \right\rceil} + 1 \leqslant (t_i - v_i)^{2 \cdot \left\lceil \frac{k+1}{2} \right\rceil} + 1$ for all $1 \leqslant i \leqslant |X|$ and for all $\overrightarrow{u} \in [l^\phi_1, u^\phi_1] \times \cdots \times [l^\phi_{|X|}, u^\phi_{|X|}]$. Vector $\overrightarrow{t}$ exists since all $S_1, \ldots, S_m$ are closed sets. Let us take the real e_k such that e_k is equal to $e_k = \max\{R^k(f_{1,j}, \overrightarrow{t}, \overrightarrow{v_{1,j}}, \phi) \mid 1 \leqslant j \leqslant n_1\}$. By the definition of R^k, it holds that $e_k \geqslant R^k(f_{1,j}, \overrightarrow{u}, \overrightarrow{v_{1,j}}, \phi)$ for all $1 \leqslant j \leqslant n_1$ and $\overrightarrow{u} \in S_{i_1} \cup \cdots \cup S_{i_l}$. By Proposition 15 there exists some k_0 such that, for all $k > k_0$, $e_k < \frac{\vartheta}{2}$.

We are able to prove that the vector $\overrightarrow{z}$ chosen above satisfies $\overrightarrow{z} \notin S_{i_1} \cup \cdots \cup S_{i_l}$. In fact, if $\overrightarrow{z}$ would be in $S_{i_1} \cup \cdots \cup S_{i_l}$, then we could infer two informations. From $\overrightarrow{z} = v(\overrightarrow{X})$ and $v \in [\![\phi_k]\!]$ we can infer $P^k(f_{1,j}, \overrightarrow{z}, \overrightarrow{v_{1,j}}) - R^k(f_{1,j}, \overrightarrow{z}, \overrightarrow{v_{1,j}}, \phi) \leqslant b_{1,j} \cdot z_{1,j} + c_{1,j}$ for all $1 \leqslant j \leqslant n_1$. The second information is that, since $\vartheta = \inf\{f_{\max}(\overrightarrow{u}) \mid \overrightarrow{u} \in S_{i_1} \cup \cdots \cup S_{i_l}\}$, then $f_{1,j}(\overrightarrow{z}) = P^k(f_{1,j}, \overrightarrow{z}, \overrightarrow{v_{1,j}}) + r^k(f_{1,j}, \overrightarrow{z}, \overrightarrow{v_{1,j}}) \geqslant \vartheta + b_{1,j} \cdot z_{1,j} + c_{1,j}$ for some $1 \leqslant j \leqslant n_1$. Hence, for such j, $r^k(f_{1,j}, \overrightarrow{z}, \overrightarrow{v_{1,j}}) + R^k(f_{1,j}, \overrightarrow{z}, \overrightarrow{v_{1,j}}, \phi) \geqslant \vartheta$, which contradicts $|r^k(f_{1,j}, \overrightarrow{z}, \overrightarrow{v_{1,j}}) + R^k(f_{1,j}, \overrightarrow{z}, v_{1,j}, \phi)| \leqslant |r^k(f_{1,j}, \overrightarrow{z}, \overrightarrow{v_{1,j}})| + R^k(f_{1,j}, \overrightarrow{z}, v_{1,j}, \phi) < e_k + e_k < \frac{\vartheta}{2} + \frac{\vartheta}{2} = \vartheta$. Hence, $\overrightarrow{z}$ is in some space S_i such that $S_i \notin \{S_{i_1}, \ldots, S_{i_l}\}$. Since there exists some $v' \in [\![\phi]\!]$ with $v'(\overrightarrow{x}) = z' \in S_i$, and since $\overrightarrow{z}, \overrightarrow{z}'$ are in some $S_i \notin \{S_{i_1}, \ldots, S_{i_l}\}$ and satisfy $d(\overrightarrow{z}, \overrightarrow{z}') \leqslant \epsilon$, the thesis holds.

Let us consider the inductive step with $h > 1$, namely the formula ϕ has the form $\phi \equiv \bigvee_{1 \leqslant i \leqslant h} \phi^i$, where $\phi^i \equiv \left(\bigwedge_{1 \leqslant j \leqslant n_i} f_{i,j}(\overrightarrow{x_{i,j}}) \leqslant b_{i,j} \cdot x_{i,j} + c_{i,j}\right) \wedge \psi_i$. By the inductive hypothesis, for all $1 \leqslant i \leqslant h$, there exists some k^i_0 such that, for all $k > k^i_0$, all formulae $\phi^i_k \in \mathbf{A}(\phi^i, k)$ satisfy $\{v(\overrightarrow{X}) \mid v \in [\![\phi^i_k]\!]\} \in N(\{v(\overrightarrow{X}) \mid v \in [\![\phi^i]\!]\}, \epsilon)$. We can prove the thesis for $k_0 = \max\{k^1_0, \ldots, k^h_0\}$. Let us consider any $\phi_k \in \mathbf{A}(\phi, k)$. We know that $\phi_k \equiv \bigvee_{1 \leqslant i \leqslant h} \phi^i_k$, with $\phi^i_k \in \mathbf{A}(\phi^i, k)$. Let us take any vector $\overrightarrow{z}$ and evaluation v such that $v(\overrightarrow{X}) = \overrightarrow{z}$, $v \in [\![\phi_k]\!]$, and $k > k_0$. It holds that $v \in [\![\phi^i_k]\!]$ for some $1 \leqslant i \leqslant h$. Since $k > k_0$, then $k > k^i_0$, and, therefore, there exist some vector $\overrightarrow{z}'$ and evaluation v' with $v'(\overrightarrow{X}) = \overrightarrow{z}'$ and $v' \in [\![\phi^i]\!]$ such that $d(\overrightarrow{z}, \overrightarrow{z}') \leqslant \epsilon$. Since $[\![\phi^i]\!] \subseteq [\![\phi]\!]$, the thesis follows.

Inductive step: $\phi \equiv \exists y \in [l_y, r_y].\phi'$.

By the inductive hypothesis, there exists some k'_0 such that, for all $k > k'_0$, all formulae $\psi'_k \in \mathbf{A}(y \in [l_y, r_y] \wedge \phi', k)$ satisfy $\{v(\overrightarrow{X}) \mid v \in [\![\psi'_k]\!]\} \in N(\{v(\overrightarrow{X}) \mid v \in [\![y \in [l_y, r_y] \wedge \phi']\!]\}, \epsilon)$. It holds that $\mathbf{A}(y \in [l_y, r_y] \wedge \phi', k) = \{y \in [l_y, r_y] \wedge \phi'_k \mid \phi'_k \in \mathbf{A}(\phi', k)\}$. We can prove the thesis for $k_0 = k'_0$. Each $\phi_k \in \mathbf{A}(\phi, k)$ has the form $\phi_k \equiv \exists y \in [l_y, r_y].\phi'_k$, for some $\phi'_k \in \mathbf{A}(\phi', k)$. Let us take any evaluation $v \in [\![\phi_k]\!]$ and $k > k_0$. There is some real $l_y \leqslant c \leqslant r_y$ such that $v[y := c] \in [\![y \in [l_y, r_y] \wedge \phi'_k]\!]$. Since $k_0 > k'_0$, there exists some evaluation v' with $v'(y) = v(y)$ and real $l_y \leqslant c' \leqslant r_y$ such that $v'[y := c'] \in [\![y \in [l_y, r_y] \wedge \phi']\!]$ and $d(v[y := c](\overrightarrow{X}), v'[y := c'](\overrightarrow{X})) < \epsilon$. Since $v(y) = v'(y)$, this implies that $d(v(\overrightarrow{X}), v'(\overrightarrow{X})) < \epsilon$. Since $l_y \leqslant c' \leqslant r_y$ and, as a consequence, $v' \in [\![\phi]\!]$, the proof is complete.

Inductive step: $\phi \equiv \forall y \in [l_y, r_y].\phi'$.

All $\phi_k \in \mathbf{A}(\phi, k)$ have the form $\forall y \in [l_y, r_y].\phi'_k$, where $\phi'_k \in \mathbf{A}(\phi', k)$.

Let m be the natural such that $m = |\mathbf{FV}(\phi)|$. W.l.o.g. we can assume that $\mathbf{FV}(\phi) = \overrightarrow{x} = \{x_1, \ldots, x_m\}$. Since ϕ is bounded, we are sure that $\{v(\overrightarrow{x}) \mid v \in [\![\phi]\!]\} \subseteq [l^\phi_1, u^\phi_1] \times \cdots \times [l^\phi_m, u^\phi_m]$. The set $(l^\phi_1 - 1, u^\phi_1 + 1) \times \cdots \times (l^\phi_m -$

$1, u_m^\phi + 1) \cap \{v(\overrightarrow{x}) \mid v \in [\![\neg\phi]\!]\}$, which is open since all the intervals $(l_i^\phi - 1, u_i^\phi + 1)$ are open, $\{v(\overrightarrow{x}) \mid v \in [\![\neg\phi]\!]\}$ is open (by Lemma 31), and the class of the open sets is closed under finite intersection, can be partitioned in the (possibly infinite) open, bounded, connected and pairwise disjoint sets $C_1, \ldots, C_l, \ldots$, such that $\{v(\overrightarrow{x}) \mid v \in [\![\phi]\!]\} = [l_1^\phi, u_1^\phi] \times \cdots \times [l_m^\phi, u_m^\phi] \setminus (C_1 \cup \cdots \cup C_l \cup \cdots)$.

As in the proof of the base case, let $S_1, \ldots, S_h$ be h spaces in $\mathbb{R}^m$ such that:

- $[l_1^\phi, u_1^\phi] \times \cdots \times [l_m^\phi, u_m^\phi] = S_1 \cup \cdots \cup S_h$
- for all $1 \leqslant i \leqslant h$, there exist $d_{i,1}, \ldots, d_{i,m}$ and $\delta \leqslant \frac{\epsilon}{\sqrt{m}}$ such that $S_i = [l_1^\phi + d_{i,1}, l_1^\phi + d_{i,1} + \delta] \times \cdots \times [l_m^\phi + d_{i,m}, l_m^\phi + d_{i,m} + \delta]$.

We have already argued that, for all $1 \leqslant i \leqslant h$ and $\overrightarrow{u}, \overrightarrow{v} \in S_i$, $d(\overrightarrow{v}, \overrightarrow{u}) \leqslant \epsilon$.

The target is to show that, given any vector $\overrightarrow{z}$ and evaluation v with $v(\overrightarrow{x}) = \overrightarrow{z}$ and $v \in [\![\phi_k]\!]$, there exists some $1 \leqslant i \leqslant h$ such that $\overrightarrow{z} \in S_i$ and $S_i \cap \{v(\overrightarrow{x}) \mid v \in [\![\phi]\!]\} \neq \emptyset$, which implies the thesis.

For each $1 \leqslant i \leqslant h$, if $S_i \cap \{v(\overrightarrow{x}) \mid v \in [\![\phi]\!]\} = \emptyset$, then, since $S_i \subseteq (l_1^\phi - 1, u_1^\phi + 1) \times \cdots \times (l_m^\phi - 1, u_m^\phi + 1) \cap \{v(\overrightarrow{x}) \mid v \in [\![\neg\phi]\!]\}$ and sets $C_1, \ldots, C_l, \ldots$ are pairwise disjoint, there exists some C_j such that $S_i \subseteq C_j$. Since S_i is a closed set and C_j is an open set, the inclusion is strict, i.e., $S_i \subset C_j$. Let A be the least subset of the indexes $\{1, \ldots, l, \ldots\}$ such that, for all $1 \leqslant i \leqslant h$, if $S_i \cap \{v(\overrightarrow{x}) \mid v \in [\![\phi]\!]\} = \emptyset$, then $S_i \subset C_j$, for some $j \in A$. It is immediate that $|A| \leqslant h$. It follows that, given any $k \in \mathbb{N}$, then, for all $\phi_k \in \mathbf{A}(\phi, k)$, $\{v(\overrightarrow{x}) \mid v \in [\![\phi_k]\!]\} \in N(\{v(\overrightarrow{x}) \mid v \in [\![\phi]\!]\}, \epsilon)$ iff $\{v(\overrightarrow{x}) \mid v \in [\![\phi_k]\!]\} \setminus \left(\bigcup_{i \notin A} C_i\right) \in N([l_1^\phi, u_1^\phi] \times \cdots \times [l_m^\phi, u_m^\phi] \setminus \left(\bigcup_{i \in A} C_i\right), \epsilon)$.

Now, also the set $(l_1^\phi - 1, u_1^\phi + 1) \times \cdots \times (l_m^\phi - 1, u_m^\phi + 1) \times (l_y - 1, r_y + 1) \cap \{v(\overrightarrow{x} \oplus (y)) \mid v \in [\![\neg\phi']\!]\}$ is open and can be partitioned into (possibly infinite) open, bounded, connected and pairwise disjoint sets $D_1, \ldots, D_l, \ldots$ such that $\{v(\overrightarrow{x} \oplus (y)) \mid v \in [\![\phi']\!]\} \cap \mathbb{R}^m \times [l_y, r_y] = [l_1^\phi, u_1^\phi] \times \cdots \times [l_m^\phi, u_m^\phi] \times [l_y, r_y] \setminus (D_1 \cup \cdots \cup D_l \cup \cdots)$.

Given any set C_i and any vector $\overrightarrow{v} \in C_i$, there exists at least one set D_j such that $\overrightarrow{v} \oplus c \in D_j$, for some $l_y - 1 < c < r_y + 1$. Hence, for such a C_i, there exists a (possibly infinite) set of indexes A^i such that $C_i = \{\overrightarrow{v} \mid \overrightarrow{v} \oplus (c) \in D_j \text{ and } j \in A^i\}$. If $C_i \supset S_k$ for some $1 \leqslant k \leqslant h$, then we can prove that there exists a finite set of indexes $B^i \subseteq A^i$ such that $S_k \subset \{\overrightarrow{v} \mid \overrightarrow{v} \oplus (c) \in D_j \text{ with } j \in B^i\}$.

In fact, by contradiction let us assume that such a set B^i does not exist. In this case, since $S_k \times [l_y, r_y]$ is bounded and closed, a well known result on bounded and closed sets ensures that we can construct an infinite succession of vectors $\{\overrightarrow{z_i}\}_{i=1,\ldots,l,\ldots}$, with $\overrightarrow{z_i} \in D_i \cap S_k \times [l_y, r_y]$ and $\overrightarrow{z_i} \neq \overrightarrow{z_j}$, for any $i \neq j$, that converges to a vector $\overrightarrow{v} \oplus (c) \notin D_1 \cup \cdots \cup D_l \cup \cdots$ in $S_k \times [l_y, r_y]$. Hence, by definition of convergence, given any $\delta > 0$, $N(\overrightarrow{v} \oplus (c), \delta)$ contains infinite vectors of $\{\overrightarrow{z_i}\}_{i=1,\ldots,l,\ldots}$. Hence, since $\overrightarrow{z_i} \in D_i \cap S_k \times [l_y, r_y]$, an infinite number of sets in $D_1, \ldots, D_l, \ldots$ lie in $N(\overrightarrow{v} \oplus (c), \delta)$. But, since $S_k \subset C_i$, then $\overrightarrow{v} \in C_i$, which contradicts the fact that $\overrightarrow{v} \oplus (c) \notin D_l$, for any l.

Let $1 \leqslant i \leqslant h$ and $l_1, l_2 \in B^i$, with $l_1 \neq l_2$. We write $l_1 \otimes l_2$ to denote the set of the vectors $\overrightarrow{v}$ such that there exist c, c' such that $\overrightarrow{v} \oplus (c) \in D_{l_1}$ and $\overrightarrow{v} \oplus (c') \in D_{l_2}$. If $l_1 \otimes l_2 \neq \emptyset$, then M_{l_1,l_2} denotes $\sup\{d(\overrightarrow{u}, \overrightarrow{v}) \mid \overrightarrow{u}, \overrightarrow{v} \in l_1 \otimes l_2\}$.

Let err_1 be the value

$$err_1 = \min\{\epsilon, \inf\{M_{l_1,l_2} \mid l_1, l_2 \in B^i, 1 \leqslant i \leqslant n, l_1 \otimes l_2 \neq \emptyset\}\}$$

Since all B^i are finite and all sets D_l are open and pairwise disjoint, $err_1 > 0$.

Let $1 \leqslant i \leqslant h$ and $l \in B^i$ be an index such that $D_l \cap \{\overrightarrow{v} \oplus (c) \mid \overrightarrow{v} \notin S_i\} \neq \emptyset$. Let G_l^i denote $\sup\{d(\overrightarrow{u}, \overrightarrow{v}) \mid \overrightarrow{u} \oplus (c), \overrightarrow{v} \oplus (c') \in D_l \setminus S_i \times [l_y, r_y]\}$.

Let err_2 be the value

$$err_2 = \min\{\epsilon, \inf\{G_l^i | 1 \leqslant i \leqslant h, l \in B^i, D_l \cap \{\overrightarrow{v} \oplus (c) \mid \overrightarrow{v} \notin S_i\} \neq \emptyset\}\}$$

Since all sets C_l are open and all sets S_i are closed, it holds that $err_2 > 0$.

Let $\overrightarrow{v}, \overrightarrow{u}$ be two vectors. With $\text{seg}(\overrightarrow{v}, \overrightarrow{u})$ we denote the set of the vectors in the segment linking $\overrightarrow{v}, \overrightarrow{u}$.

Let K_l denote the value

$$\sup\{c - c' \mid \mathrm{seg}(\overrightarrow{v} \oplus (c), \overrightarrow{v} \oplus (c')) \subseteq D_l \text{ and } N(\overrightarrow{v}, \epsilon) \subseteq \{\overrightarrow{v} \mid \overrightarrow{v} \oplus (c) \in D_l\}\}$$

Let err_3 be the value

$$err_3 = \min\left\{\epsilon, \inf\left\{K_l \mid l \in \bigcup_{i=1}^{h} B^i\right\}\right\}$$

Since all B^i are finite and all sets D_l are open and pairwise disjoint, $err_3 > 0$.

Let $\epsilon' \in \left(0, \min\{\frac{err_1}{2}, err_2, \frac{err_3}{2}, \epsilon\}\right]$. By the inductive hypothesis, we can find a k_0 such that, for any $k \geqslant k_0$, all $\phi'_k \in \mathbf{A}(\phi', k)$ satisfy $\{v(\overrightarrow{x} \oplus (y)) \mid v \in [\![\phi'_k \wedge y \in [l_y, r_y]]\!]\} \in N(\{v(\overrightarrow{x} \oplus (y)) \mid v \in [\![\phi' \wedge y \in [l_y, r_y]]\!]\}, \epsilon')$. Since $\epsilon' < \epsilon$, it is sufficient to prove that $\{v(\overrightarrow{x}) \mid v \in [\![\phi_k]\!]\} \in N(\{v(\overrightarrow{x}) \mid v \in [\![\phi]\!]\}, \epsilon')$, for all $\phi_k \in \mathbf{A}(\phi, k)$,

Since $\epsilon' < \frac{err_1}{2}$, $\epsilon' < \frac{err_3}{2}$, and all C_i with $i = 1, \ldots, l, \ldots$ are such that $C_i \cap \{v(\overrightarrow{x}) \mid v \in [\![\phi]\!]\} = \emptyset$ it holds that, for any $j \in A$, there exists a connected open set $C'_j \subseteq C_j$ such that $C'_j \cap \{v(\overrightarrow{x}) \mid v \in [\![\phi_k]\!]\} = \emptyset$. Since $\epsilon' < err_2$, then $C_j \supset S_i$ iff $C'_j \supset S_i$. Hence, if $\overrightarrow{z}$ is a vector such that $\overrightarrow{z} \in \{v(\overrightarrow{x}) \mid v \in [\![\phi_k]\!]\}$, then there exists some set S_i, with $1 \leqslant i \leqslant h$, such that $\overrightarrow{z} \in S_i$ and $S_i \cap \{v(\overrightarrow{x}) \mid v \in [\![\phi]\!]\} \neq \emptyset$. Since $d(\overrightarrow{u}, \overrightarrow{v}) < \epsilon$ for all $\overrightarrow{u}, \overrightarrow{v} \in S_i$, this implies the thesis. □

The result above can be immediately extended to automata.

Corollary 32. *Given any Hybrid Automaton H such that each formula contained in the definition of H satisfies the hypothesis of Theorem 30, then, for each $\epsilon > 0$, there exists some k_0 such that, for each $k > k_0$, the set $\mathbf{A}(H, k)$ contains only ϵ-approximations for H.*

Proof. By Theorem 30 and the fact that the number of the formulae contained in the definition of H is finite. □

Let us show that if the hypothesis of Theorem 30 are strengthened, then formulae with strict inequalities can be admitted. Let us consider a formula $\phi \equiv f(x) < c$. Problems arise when there is a real u such that $f(u) = c$ and $f(u') > c$ in a neighborhood N of u (i.e., c is a local minimum for f). For instance, this happens if we take the formula $-\sin(x) < -1$ of Example 29, where -1 is the minimum of the function $-\sin(x)$ in a neighborhood of $\frac{\pi}{2}$. The problem is that $f(u') < c$ does not hold for any u' in N, but it could happen that $P^k(f, u, v) - R^k(f, u, v, \phi) < c$, thus implying that $P^k(f, u, v) - R^k(f, u, v, \phi) < c$ is not an ϵ-approximation of $f(x) < c$ for any ϵ bounded by the ray of N.

To prevent this problem, we require that c is not a local minimum in a neighborhood of u for any function g. Hence we require that all functions appearing in a formula in a neighborhood of u are increasing (or decreasing).

Theorem 33. *Consider a normal form $\phi \in \Phi(X, F)$ for which $\mathbf{A}(\phi, k)$ exists for all $k > 0$. If the following conditions hold*:

(1) *ϕ is bounded and has no universal quantification*;
(2) *for any subformula $f(\overrightarrow{x}) < b \cdot x_i + c$ in ϕ, where f is not a polynomial, and vector $\overrightarrow{u} = (u_1, \ldots, u_{|X|})$, if $f(\overrightarrow{u}) = b \cdot u_i + c$, then there exists a variable $x_j \in \overrightarrow{x}$ such that*:
   (a) *the component u_j of $\overrightarrow{u}$ is such that $u_j \in (l_j^{\phi}, u_j^{\phi})$*;
   (b) *one of the following facts hold*:
      (i) *for all $g(\overrightarrow{x}') \sim b' \cdot x'_i + c'$ appearing in ϕ such that $g(\overrightarrow{u}) = b' \cdot u'_i + c'$, there exists a neighborhood $N(\overrightarrow{u}, \epsilon)$ with $\epsilon > 0$ where the function $g(\overrightarrow{x}) - b' \cdot x'_i - c'$ is strictly increasing on x_j*;
      (ii) *for all $g(\overrightarrow{x}') \sim b' \cdot x'_i + c'$ appearing in ϕ such that $g(\overrightarrow{u}) = b' \cdot u'_i + c'$, there exists a neighborhood $N(\overrightarrow{u}, \epsilon)$ with $\epsilon > 0$ where the function $g(\overrightarrow{x}) - b' \cdot x'_i - c'$ is strictly decreasing on x_j*,

then, for each $\epsilon > 0$, there exists some k_0 such that, for each $k > k_0$, the set $\mathbf{A}(\phi, k)$ contains only ϵ-approximations of ϕ.

Proof. Let us reason by induction over ϕ.

Base case: ϕ has no quantifier.

As in the proof of Theorem 30, since ϕ has no quantifier, we can rewrite ϕ in disjunctive normal form as usual, and we can assume that ϕ has the form:

$$\bigvee_{1\leqslant i\leqslant m}\left(\left(\bigwedge_{1\leqslant j\leqslant m_i} g^1_{i,j}(\overrightarrow{x_{i,j}}^1)\leqslant b^1_{i,j}\cdot x^1_{i,j}+c^1_{i,j}\wedge\bigwedge_{1\leqslant j\leqslant n_i} g^2_{i,j}(\overrightarrow{x_{i,j}}^2)< b^2_{i,j}\cdot x^2_{i,j}+c^2_{i,j}\right)\wedge\psi_i\right)$$

where polynomial formulae appear only in ψ_i, and all ψ_i contain the constraints $x_i\in[l^\phi_i,u^\phi_i]$ for all $1\leqslant i\leqslant|X|$.

Let us reason by induction over m. Let us consider first the base case where $m=1$, namely ϕ has the form:

$$\bigwedge_{1\leqslant j\leqslant m_1} g^1_{1,j}(\overrightarrow{x_{1,j}}^1)\leqslant b^1_{1,j}\cdot x^1_{1,j}+c^1_{1,j}\wedge\bigwedge_{1\leqslant j\leqslant n_1} g^2_{1,j}(\overrightarrow{x_{1,j}}^2)< b^2_{1,j}\cdot x^2_{1,j}+c^2_{1,j}\wedge\psi_1$$

Each $\phi_k\in\mathbf{A}(\phi,k)$ has the form $\phi_k\equiv\phi^1_k\wedge\phi^2_k\wedge\psi_1$, where

$$\phi^1_k\equiv\bigwedge_{1\leqslant j\leqslant m_1} P^k(g^1_{1,j},\overrightarrow{x_{1,j}}^1,\overrightarrow{v^1_{1,j}})-R^k_{1,j,1}\leqslant b^1_{1,j}\cdot x^1_{1,j}+c^1_{1,j}$$

$$\phi^2_k\equiv\bigwedge_{1\leqslant j\leqslant n_1} P^k(g^2_{1,j},\overrightarrow{x_{1,j}}^2,\overrightarrow{v^2_{1,j}})-R^k_{1,j,2}\leqslant b^2_{1,j}\cdot x^2_{1,j}+c^2_{1,j}$$

and, for all $1\leqslant j\leqslant m_1$, $\overrightarrow{v^1_{1,j}}$ is in $Dom(g^1_{1,j})$ and $R^k_{1,j,1}=R^k(g^1_{1,j},\overrightarrow{x_{1,j}}^1,\overrightarrow{v^1_{1,j}},\phi)$, and for all $1\leqslant j\leqslant n_1$, $\overrightarrow{v^2_{1,j}}$ is in $Dom(g^2_{1,j})$ and $R^k_{1,j,2}=R^k(g^2_{1,j},\overrightarrow{x_{1,j}}^2,\overrightarrow{v^2_{1,j}},\phi)$.

We are looking for a k_0 such that, if $k>k_0$, then, given any evaluation v and vector $\overrightarrow{z}$ such that $v(\overrightarrow{X})=\overrightarrow{z}$ and $v\in[\![\phi_k]\!]\setminus[\![\phi]\!]$, there exists some evaluation v' and vector $\overrightarrow{z}'$ such that $v'(\overrightarrow{X})=\overrightarrow{z}'$, $v'\in[\![\phi]\!]$ and $d(\overrightarrow{z},\overrightarrow{z}')\leqslant\epsilon$.

It holds that $\overrightarrow{z}$ is in $[l^\phi_1,u^\phi_1]\times\cdots\times[l^\phi_{|X|},u^\phi_{|X|}]$.

As in the proof of Theorem 30, let $S_1,\ldots,S_m$ be m spaces in $\mathbb{R}^{|X|}$ such that:

- $[l^\phi_1,u^\phi_1]\times\cdots\times[l^\phi_{|X|},u^\phi_{|X|}]=S_1\cup\cdots\cup S_m$
- for all $1\leqslant i\leqslant m$, there exist $d_{i,1},\ldots,d_{i,|X|}$ and $\delta\leqslant\frac{\epsilon}{\sqrt{2|X|}}$ such that $S_i=[l^\phi_1+d_{i,1},l^\phi_1+d_{i,1}+\delta]\times\cdots\times[l^\phi_{|X|}+d_{i,|X|},l^\phi_{|X|}+d_{i,|X|}+\delta]$.

We note that $\delta\leqslant\frac{\epsilon}{\sqrt{2|X|}}$ implies $\sqrt{|X|\cdot\delta^2}\leqslant\frac{\epsilon}{2}$, and, therefore, for all $1\leqslant i\leqslant m$ and $\overrightarrow{u},\overrightarrow{v}\in S_i$, $d(\overrightarrow{v},\overrightarrow{u})\leqslant\frac{\epsilon}{2}$.

Let $g^1_{\max}(\overrightarrow{X})$ and $g^2_{\max}(\overrightarrow{X})$ be the functions such that:

$$g^1_{\max}(\overrightarrow{X})=\max\{g^1_{1,j}(\overrightarrow{x^1_{1,j}})-b^1_{1,j}\cdot x^1_{1,j}-c^1_{1,j}\,|\,1\leqslant j\leqslant m_1\}$$

$$g^2_{\max}(\overrightarrow{X})=\max\{g^2_{1,j}(\overrightarrow{x^2_{1,j}})-b^2_{1,j}\cdot x^2_{1,j}-c^2_{1,j}\,|\,1\leqslant j\leqslant n_1\}$$

Since all $g^h_{i,j}$ are continuous, also the functions $g^1_{\max}$ and $g^2_{\max}$ are continuous. Moreover, since all $g^h_{i,j}$ satisfy conditions 2 of the hypothesis, it can be proved that also functions $g^1_{\max}$ and $g^2_{\max}$ satisfy it. Finally, for all $\overrightarrow{u}\in\mathbb{R}^{|X|}$, it holds that $g^1_{\max}(\overrightarrow{u})\leqslant 0$ iff $\bigwedge_{1\leqslant j\leqslant m_1}\left(g^1_{1,j}(\overrightarrow{u})\leqslant b^1_{1,j}\cdot u^1_{1,j}+c^1_{1,j}\right)$, and that $g^2_{\max}(\overrightarrow{u})<0$ iff $\bigwedge_{1\leqslant j\leqslant n_1}\left(g^2_{1,j}(\overrightarrow{u})< b^2_{1,j}\cdot u^2_{1,j}+c^2_{1,j}\right)$.

Let $h \in \{1,2\}$. Let $S^h_{i_1}, \ldots, S^h_{i_{l_h}}$ be the spaces in $\{S_1, \ldots, S_m\}$ such that $g^h_{\max}(\overrightarrow{u}) > 0$ for all $\overrightarrow{u} \in S^h_{i_1} \cup \cdots \cup S^h_{i_{l_h}}$. It holds that no evaluation $v \in [\![\phi]\!]$ maps $\overrightarrow{X}$ to $S^h_{i_1} \cup \cdots \cup S^h_{i_{l_h}}$. Let $\vartheta^h = \inf\{g^h_{\max}(\overrightarrow{u}) \mid \overrightarrow{u} \in S^h_{i_1} \cup \cdots \cup S^h_{i_{l_h}}\}$. As in the proof of Theorem 30, we can prove that $\vartheta^h \neq 0$.

For all $g^h_{1,j}$ and vectors $\overrightarrow{u^h_{1,j}} \in Dom(g^h_{1,j})$, it holds that $g^h_{1,j}(\overrightarrow{u^h_{1,j}})$ is equal to $P^k(g^h_{1,j}, \overrightarrow{u^h_{1,j}}, \overrightarrow{v^h_{1,j}}) + r^k(g^h_{1,j}, \overrightarrow{u^h_{1,j}}, \overrightarrow{v^h_{1,j}})$, where the error $r^k(g^h_{1,j}, \overrightarrow{u^h_{1,j}}, \overrightarrow{v^h_{1,j}})$ satisfies, by Theorem 10 and Proposition 15,

$$r^k\left(g^h_{1,j}, \overrightarrow{u^h_{1,j}}, \overrightarrow{v^h_{1,j}}\right) \in \left[-R^k\left(g^h_{1,j}, \overrightarrow{u^h_{1,j}}, \overrightarrow{v^h_{1,j}}, \phi\right), R^k\left(g^h_{1,j}, \overrightarrow{u^h_{1,j}}, \overrightarrow{v^h_{1,j}}, \phi\right)\right].$$

Let $\overrightarrow{t}$ be the vector such that, for all $\overrightarrow{u} \in S_1 \cup \cdots \cup S_m$, $(u_i - v_i)^{2\cdot\left\lceil \frac{k+1}{2} \right\rceil} + 1 \leqslant (t_i - v_i)^{2\cdot\left\lceil \frac{k+1}{2} \right\rceil} + 1$ for all $1 \leqslant i \leqslant |X|$. Vector $\overrightarrow{t}$ exists since $S_1, \ldots, S_m$ are closed sets. As in the proof of Theorem 30, let us define

$$e^1_k = \max\left\{R^k(g^1_{1,j}, \overrightarrow{t}, \overrightarrow{v^1_{1,j}}, \phi) \mid 1 \leqslant j \leqslant m_1\right\}$$

$$e^2_k = \max\left\{R^k(g^2_{1,j}, \overrightarrow{t}, \overrightarrow{v^2_{1,j}}, \phi) \mid 1 \leqslant j \leqslant n_1\right\}$$

By the definition of R^k it holds that $e^1_k \geqslant R^k(g^1_{1,j}, \overrightarrow{u}, \overrightarrow{v^1_{1,j}}, \phi)$ for all $1 \leqslant j \leqslant m_1$ and $\overrightarrow{u} \in S^1_{i_1} \cup \cdots \cup S^1_{i_{l_1}}$, and that $e^2_k \geqslant R^k(g^2_{1,j}, \overrightarrow{u}, \overrightarrow{v^2_{1,j}}, \phi)$ for all $1 \leqslant j \leqslant n_1$ and $\overrightarrow{u} \in S^2_{i_1} \cup \cdots \cup S^2_{i_{l_2}}$. By Proposition 15 there exists some k^h_0 such that, for all $k > k^h_0$, $e^h_k < \vartheta^h/2$. Let $k_0 = \max\{k^1_0, k^2_0\}$.

We can prove that that $\overrightarrow{z} \notin S^h_{i_1} \cup \cdots \cup S^h_{i_{l_h}}$ for any $h \in \{1,2\}$. In fact, if $\overrightarrow{z}$ would be in $S^h_{i_1} \cup \cdots \cup S^h_{i_{l_h}}$, then we can infer that $P^k(g^h_{1,j}, \overrightarrow{z}, \overrightarrow{v^h_{1,j}}) - R^k(g^h_{1,j}, \overrightarrow{z}, \overrightarrow{v^h_{1,j}}, \phi) \leqslant b^h_{1,j} \cdot z^h_{1,j} + c^h_{1,j}$ for all $g^h_{1,j}$ (which follows from $v(\overrightarrow{X}) = \overrightarrow{z}$ and $v \in [\![\phi_k]\!]$) and that $P^k(g^h_{1,j}, \overrightarrow{z}, \overrightarrow{v^h_{1,j}}) + r^k(g^h_{1,j}, \overrightarrow{z}, \overrightarrow{v^h_{1,j}}) \geqslant b^h_{1,j} \cdot z^h_{1,j} + c^h_{1,j} + \vartheta^h$ (which follows from $\overrightarrow{z} \in S^h_{i_1} \cup \cdots \cup S^h_{i_l,h}$), for some $g^h_{1,j}$. Hence, for these indexes j and h, $r^k(g^h_j, \overrightarrow{z}, \overrightarrow{v^h_{1,j}}) + R^k(g^h_j, \overrightarrow{z}, v^h_{1,j}, \phi) \geqslant \vartheta^h$, which contradicts $|r^k(g^h_j, \overrightarrow{z}, \overrightarrow{v^h_{1,j}}) + R^k(g^h_j, \overrightarrow{z}, \overrightarrow{v^h_{1,j}}, \phi)| \leqslant |r^k(g^h_j, \overrightarrow{z}, \overrightarrow{v^h_{1,j}})| + |R^k(g^h_j, \overrightarrow{z}, \overrightarrow{v^h_{1,j}}, \phi)| < e^h_k + e^h_k < \frac{\vartheta^h}{2} + \frac{\vartheta^h}{2} = \vartheta^h$. Hence, $\overrightarrow{z}$ is in some space S_i such that there exists some $\overrightarrow{z}' \in S_i$ such that either $g^1_{\max}(\overrightarrow{z}') \leqslant 0$ and $g^2_{\max}(\overrightarrow{z}') < 0$ or $g^1_{\max}(\overrightarrow{z}') \leqslant 0$ and $g^2_{\max}(\overrightarrow{z}') = 0$.

In the first case, there exists some evaluation $v' \in [\![\phi]\!]$ with $v'(\overrightarrow{X}) = \overrightarrow{z}'$, and, since $\overrightarrow{z}, \overrightarrow{z}' \in S_i$, $d(\overrightarrow{z}, \overrightarrow{z}') < \frac{\epsilon}{2} \leqslant \epsilon$, and v' and $\overrightarrow{z}'$ are the evaluation and the vector we were looking for.

In the second case, by condition 2b of the hypothesis, $g^2_{\max}$ is either strictly increasing or strictly decreasing on the jth component in some neighborhood $N(\overrightarrow{z}', \delta_2)$ of $\overrightarrow{z}'$, with $\delta_2 > 0$. Moreover, by condition 2a of the hypothesis, $\overrightarrow{z}'_j$ is in (l^ϕ_j, u^ϕ_j). Let us suppose that $g^2_{\max}$ is strictly increasing on the jth component in $N(\overrightarrow{z}', \delta_2)$, namely we are in the case 2b.i; the other case is similar.

If $g^1_{\max}(\overrightarrow{z}') = 0$, then, by condition 2b.i of the hypothesis, there exists a ray $\delta_1 > 0$ such that $g^1_{\max}$ is strictly increasing on the jth component in $N(\overrightarrow{z}', \delta_1)$. Otherwise, if $g^1_{\max}(\overrightarrow{z}') < 0$, then, since $g^1_{\max}$ is continuous, there exists a ray $\delta_1 > 0$ such that $g^1_{\max}$ is less or equal to 0 in $N(\overrightarrow{z}', \delta_1)$.

Since z'_j is in (l^ϕ_j, u^ϕ_j), i.e., since z'_j is not in the border of $[l^\phi_j, u^\phi_j]$, we are sure that:

$$\left(N(\overrightarrow{z}', \min(\delta_1, \delta_2)) \setminus \{\overrightarrow{z}'\}\right) \cap S_i \neq \emptyset.$$

Therefore, there exists $\delta \in (0, min(\delta_1, \delta_2)]$ such that $N(\overrightarrow{z}', \delta) \subseteq N(\overrightarrow{z}', \delta_1) \cap N(\overrightarrow{z}', \delta_2) \cap S_i$. Hence it holds that $g^2_{\max}$ is strictly increasing on the jth component in $N(\overrightarrow{z}', \delta)$. Moreover, if $g^1_{\max}(\overrightarrow{z}') = 0$, then also $g^1_{\max}$ is strictly increasing on the j^{th} component in $N(\overrightarrow{z}', \delta)$, and, otherwise, if $g^1_{\max}(\overrightarrow{z}') < 0$, then $g^1_{\max}$ is less or equal to 0 in $N(\overrightarrow{z}', \delta)$.

Since $g^2_{\max}$ is strictly increasing on j in $N(\overrightarrow{z}', \delta)$ and $g^2_{\max}(\overrightarrow{z}') = 0$, then for all $\overrightarrow{z}'' \in N(\overrightarrow{z}', \delta)$ such that $z''_j < z'_j$, we have that $g^2_{\max}(\overrightarrow{z}'') < 0$. If $g^1_{\max}(\overrightarrow{z}') = 0$ and $g^1_{\max}$ is strictly increasing on j in $N(\overrightarrow{z}', \delta)$, then $g^1_{\max}(\overrightarrow{z}'') < 0$. If $g^1_{\max}(\overrightarrow{z}') < 0$, and $g^1_{\max}$ is less or equal to 0 in $N(\overrightarrow{z}', \delta)$, then $g^1_{\max}(\overrightarrow{z}'') \leqslant 0$. Hence, in both cases there exists a vector $\overrightarrow{z}''$ and en evaluation v'' such that $v''(\overrightarrow{X}) = \overrightarrow{z}''$, $v'' \in [\![\phi]\!]$ and, since $\overrightarrow{z}', \overrightarrow{z}'' \in S_i$,

$d(\overrightarrow{z}', \overrightarrow{z}'') < \frac{\epsilon}{2}$. Now, $d(\overrightarrow{z}, \overrightarrow{z}'') \leqslant d(\overrightarrow{z}, \overrightarrow{z}') + d(\overrightarrow{z}', \overrightarrow{z}'') < \frac{\epsilon}{2} + \frac{\epsilon}{2} \leqslant \epsilon$, and $\overrightarrow{z}''$ and v'' are the vector and the evaluation we were looking for.

The proof of the inductive step where $m > 1$ is as in Theorem 30.

Inductive step: $\phi \equiv \exists y \in [l_y, r_y].\phi'$.
The proof is as in Theorem 30. □

The result above can be immediately extended to automata.

Corollary 34. *Given any Hybrid Automaton H such that each formula contained in the definition of H satisfies the hypothesis of Theorem 33, then, for each $\epsilon > 0$, there exists some k_0 such that, for each $k > k_0$, the set $\mathbf{A}(H, k)$ contains only ϵ-approximations for H.*

Proof. By Theorem 33 and the fact that the number of the formulae contained in the definition of H is finite. □

The following example shows that conditions 2b.i and 2b.ii of Theorem 33 cannot be relaxed by simply requiring that "for each $g(\overrightarrow{x}') \sim b' \cdot x_i' + c'$ appearing in ϕ such that $g(\overrightarrow{u}) = b' \cdot u_i' + c'$ there exists a neighborhood $N(\overrightarrow{u}, \epsilon)$ with $\epsilon > 0$ where the function $g(\overrightarrow{x}') - b' \cdot x_i' - c'$ is strictly monotonic on x_j".

Example 35. Let us consider the following normal form:

$$\phi \equiv 3^{x'} < 27 \wedge -x' \leqslant -3 \wedge \phi', \text{ where } \phi' \equiv x' \in [0, 100]$$

Since $-x'$ is decreasing and $3^{x'}$ is increasing, condition 2b of Theorem 33 is violated for $u = 3$.
Let us take any $\epsilon > 0$. We can show that there is no k such that $\mathbf{A}(\phi, k)$ contains ϵ-approximations of ϕ.
We have that $[\![\phi]\!] = \emptyset$.
Given any $k \in \mathbb{N}$, and any $v \in \mathbb{R}$, $\mathbf{A}(\phi, k)$ contains all the formulae $\phi_{k,v}$ of the following form:

$$P^k(3^{x'}, x', v) - R^k(3^{x'}, x', v, \phi) \leqslant 27 \wedge -x' \leqslant -3 \wedge \phi'$$

For each $k \in \mathbb{N}$, it holds that $D^k(3^{x'}) = \ln(3)^k \cdot 3^{x'}$. Lagrange Remainder Theorem ensures that, for all $u \in \mathbb{R}$, $3^u - P^k(3^{x'}, u, v) = \ln(3)^{k+1} \cdot 3^{\xi} \cdot \frac{(u-v)^{k+1}}{(k+1)!}$, for some ξ in the interval linking u and v. Hence, if $u > v$, it holds that $3^u - P^k(3^{x'}, u, v) > 0$, thus implying $3^u > P^k(3^{x'}, u, v)$. Now, it holds that $C(3^{x'}, \phi, v, k) = 3^{100} \cdot \ln(3)^{k+1}$ and $R^k(3^{x'}, x', v, \phi) = 3^{100} \cdot \ln(3)^{k+1} \cdot \frac{(u-v)^{2\left\lceil \frac{k+1}{2} \right\rceil}}{(k+1)!}$. Since $R^k(3^{x'}, u, v, \phi)$ is greater than 0 for all $u \neq v$, it holds that $P^k(3^{x'}, u, v) - R^k(3^{x'}, u, v, \phi) < 3^u$, thus implying that, for each k, there exists $e_k > 0$ such that $P^k(3^{x'}, u, v) - R^k(3^{x'}, u, v, \phi) < 27$ iff $u < 3 + e_k$. Now, $\{v(x) \mid v \in [\![\phi_k]\!]\} = (3, 3 + e_k)$ is not empty, thus implying $[\![\phi_k]\!] \notin N([\![\phi]\!], \epsilon)$.

The following example shows that Theorem 33 does not hold if we admit universal quantifiers.

Example 36. Let us consider the following normal form:

$$\phi \equiv \forall y \in [-1, 1].\, x' \in [-2, 1] \wedge -(x'^2 + y^2) < -1 \wedge -3^{x'+1} \leqslant -y$$

The formula ϕ satisfies condition 2 of Theorem 33, but it does not satisfy condition 1.
Let us take any $\epsilon > 0$.
We can show that there is no k such that $\mathbf{A}(\phi, k)$ contains ϵ-approximations of ϕ.
It holds that $[\![\phi]\!] = \emptyset$. In fact, fixed $x' \in [-1, 1]$, for all $y \in [-1, 1]$, $-(x'^2 + y^2) < -1$ does not hold, and, fixed $x' \in [-2, -1)$, for all $y \in [-1, 1]$, $-3^{x'+1} \leqslant -y$ does not hold.
Given any $k > 0$, and any $v \in \mathbb{R}$, $\mathbf{A}(\phi, k)$ contains all the formulae $\phi_{k,v}$ of the following form

$$\forall y \in [-1, 1].x' \in [-2, 1] \wedge -(x'^2 + y^2) < -1 \wedge \psi$$

where ψ has the form:

$$P^k(-3^{x'+1}, x', v) - R^k(-3^{x'+1}, x', v, \psi) \leqslant -y$$

For each $k \in \mathbb{N}$, it holds that $D^k(-3^{x'+1}) = -\ln(3)^k \cdot 3^{x'+1}$. Lagrange Remainder Theorem ensures that, for all $u \in \mathbb{R}$, $-3^{u+1} - P^k(-3^{x'+1}, u, v) = -\ln(3)^{k+1} \cdot 3^{\xi+1} \cdot \frac{(u-v)^{k+1}}{(k+1)!}$, for some ξ in the interval linking u and v. Hence, if $u < v$ and k even, it holds that $-3^{u+1} - P^k(-3^{x'+1}, u, v) > 0$, thus implying $-3^{u+1} > P^k(-3^{x'+1}, u, v)$. Now, it holds that $C(-3^{x'+1}, \phi, v, k) = 3^{101} \cdot \ln(3)^{k+1}$ and $R^k(-3^{x'+1}, x', v, \psi) = 3^{101} \cdot \ln(3)^{k+1} \cdot \frac{(u-v)^{2\left\lceil \frac{k+1}{2} \right\rceil}}{(k+1)!}$.

Since $R^k(-3^{x'+1}, u, v, \psi) > 0$, for all $u \neq v$, it holds that $P^k(-3^{x'+1}, u, v) - R^k(-3^{x'+1}, u, v, \psi)$ is less or equal than $-3^{u+1} - R^k(-3^{x'+1}, u, v, \psi)$. Hence, if $u = -1$, $P^k(-3^{x'+1}, u, v) - R^k(-3^{x'+1}, u, v, \psi) \leqslant 1$. Therefore, there exists some $\delta > 0$ such that $\{v(x) \mid v \in [\![\psi]\!]\} = (-1-\delta, -1)$, Moreover, since all evaluations in $[\![\psi]\!]$ are in $[\![\phi_{k,v}]\!]$, it holds that $[\![\phi_{k,v}]\!]$ is not empty, and it follows that $[\![\phi_{k,v}]\!] \notin N([\![\phi]\!], \epsilon)$.

7. Semantical analysis of the error

The notion of ϵ-approximation permits us to do a syntactical analysis of the error. Our target is to do a semantical analysis of the error, i.e., we aim to measure how the behavior of the automata in $\mathbf{A}(H, k)$ is close to the behavior of H.

We define before the preliminary notion of neighborhood of a region.

Definition 37. Given a region R and a real $\epsilon \geqslant 0$, the *neighborhood of ray ϵ of region R* is the set of regions

$$N(R, \epsilon) = \{R' \supseteq R \mid \forall (q', \overrightarrow{v}') \in R' \; \exists \, (q, \overrightarrow{v}) \in R \,.\, q = q' \text{ and } d(\overrightarrow{v}, \overrightarrow{v}') \leqslant \epsilon\}$$

The following properties, analogous to those of Proposition 24, will not play any role in proving the results of the rest of the paper. We give them to demonstrate the solidity of Definition 37.

Proposition 38. *Given regions R_1 and R_2, and some $\epsilon, \xi \geqslant 0$, it holds that*:

(1) $R_1 \subseteq R_2$ *implies* $\forall R_1' \in N(R_1, \epsilon) \;\; \exists R_2' \in N(R_2, \epsilon)$ *such that* $R_1' \subseteq R_2'$;
(2) $\epsilon < \xi$ *implies* $N(R_1, \epsilon) \subset N(R_1, \xi)$;
(3) $\forall R' \in N(R_1 \cup R_2, \epsilon) \;\; \exists R_1' \in N(R_1, \epsilon), R_2' \in N(R_2, \epsilon)$ such that $R' = R_1' \cup R_2'$;
(4) $\forall R' \in N(R_1 \cap R_2, \epsilon) \;\; \exists R_1' \in N(R_1, \epsilon), R_2' \in N(R_2, \epsilon)$ such that $R' = R_1' \cap R_2'$;
(5) $N(R_1, 0) = \{R_1\}$.

Let us prove that, under a suitable hypothesis, for all $n \in \mathbb{N}$, if k tends to the infinity, then the behavior of length at most n of each automaton $H_k \in \mathbf{A}(H, k)$ gets close to the behavior of H, in the sense that $Post^n(H_k)$ is in a neighborhood of $Post^n(H)$ of ray arbitrarily small. This comes from the fact that the region reached after n steps can be expressed by means of a formula by using existential quantifications.

Theorem 39. *Consider an Hybrid Automaton H such that each formula contained in the definition of H satisfies the hypothesis of Theorem 30. For each $\epsilon > 0$ and $n \in \mathbb{N}$, there exists some k_0 such that, for all $k > k_0$, it holds that*:

$$\forall H_k \in \mathbf{A}(H, k)\text{: } Post^n(H_k) \in N(Post^n(H), \epsilon)$$

Proof. We have to prove that, for each $\epsilon > 0$ and $n \in \mathbb{N}$, there exists some k_0 such that, for all $k > k_0$, given any configuration $(q_n, \overrightarrow{v_n}) \in Post^n(H_k) \setminus Post^n(H)$, for any $H_k \in \mathbf{A}(H, k)$, there is some configuration $(s_n, \overrightarrow{u_n}) \in Post^n(H)$ with $s_n = q_n$ and $d(\overrightarrow{v_n}, \overrightarrow{u_n}) \leqslant \epsilon$.

With abuse of notation, given a vector $\overrightarrow{v}$ and a formula $\phi \in \Phi(X, F)$, we write $\overrightarrow{v} \in [\![\phi]\!]$ to denote that there exists some evaluation $v \in [\![\phi]\!]$ mapping variables X to $\overrightarrow{v}$.

Since $(q_n, \overrightarrow{v_n}) \in Post^n(H_k)$ for some $H_k \in \mathbf{A}(H, k)$, there exists a formula $\phi_k^0 \in \mathbf{A}(\phi_{\text{init}}, k)$, where ϕ_{init} is the initial condition of H, formulae $\phi_k^1 \in \mathbf{A}(\phi^1, k), \ldots, \phi_k^n \in \mathbf{A}(\phi^n, k)$, states $q_0, \ldots, q_{n-1}$ in H and vectors $\overrightarrow{v_0}, \ldots, \overrightarrow{v_{n-1}}$, such that:

- $\overrightarrow{v_0} \in [\![\phi_k^0]\!]$;
- for each $0 \leqslant i \leqslant n-1$:
  - either $q_i \overset{\phi^{i+1}}{\rightarrow} q_{i+1}$ is a transition in H (and, therefore, $q_i \overset{\phi_k^{i+1}}{\rightarrow} q_{i+1}$ is a transition in H_k) and $(\overrightarrow{v_i}, \overrightarrow{v_{i+1}}) \in [\![\phi_k^{i+1}]\!]$
  - or $q_{i+1} = q_i$, ϕ^{i+1} is the activity function of state q_i in H (and, therefore, ϕ_k^{i+1} is the activity function of state q_i in H_k), and $(\overrightarrow{v_i}, v, \overrightarrow{v_{i+1}}) \in [\![\phi_k^{i+1}]\!]$ for some $v \in \mathbb{R}$.

Let us consider the following formula

$$\psi_k^1 \equiv \left(\phi_k^0 \wedge \phi_k^1\right)[t := t_1]\left[x_1 := x_1^0, \ldots, x_m := x_m^0, x_1' := x_1^1, \ldots, x_m' := x_m^1\right]$$

the following formulae for all $2 \leqslant i \leqslant n$,

$$\psi_k^i \equiv \phi_k^i\,[t := t_i]\left[x_1 := x_1^{i-1}, \ldots, x_m := x_m^{i-1}, x_1' := x_1^i, \ldots, x_m' := x_m^i\right]$$

and, finally, the following formula

$$\Gamma_k^n \equiv \psi_k^1 \wedge \psi_k^2 \wedge \cdots \wedge \psi_k^n$$

Formula Γ_k^n is a normal form in $\phi(\hat{X}, F)$, where $\hat{X}$ is the set of variables $\hat{X} = \{t_i \mid 1 \leqslant i \leqslant n \text{ and } \phi_i \text{ is an activity function}\} \cup \overrightarrow{x_i^0} \cup \cdots \cup \overrightarrow{x_i^n}$

It holds that $\overrightarrow{t} \oplus \overrightarrow{v_0} \oplus \cdots \oplus \overrightarrow{v_n} \in [\![\Gamma_k^n]\!]$, for some $\overrightarrow{t}$.

Let us consider the following formula

$$\psi^1 \equiv \left(\phi_{\text{init}} \wedge \phi^1\right)[t := t_1]\left[x_1 := x_1^0, \ldots, x_m := x_m^0, x_1' := x_1^1, \ldots, x_m' := x_m^1\right]$$

the following formulae for all $2 \leqslant i \leqslant n$,

$$\psi^i \equiv \phi^i\,[t := t_i]\left[x_1 := x_1^{i-1}, \ldots, x_m := x_m^{i-1}, x_1' := x_1^i, \ldots, x_m' := x_m^i\right]$$

and, finally, the following formula

$$\Gamma^n \equiv \psi^1 \wedge \psi^2 \wedge \cdots \wedge \psi^n$$

Let us observe that $\Gamma_k^n \in \mathbf{A}(\Gamma^n, k)$.

Since we have assumed that $\phi_{\text{init}}, \phi^1, \ldots, \phi^n$ satisfy the hypothesis of Theorem 30, we can immediately infer that also Γ^n satisfies these hypothesis.

Hence, by applying Theorem 30, we are sure that, for each $\epsilon > 0$, there exists some k_0 such that, for all $k > k_0$ it holds that $\overrightarrow{t} \oplus \overrightarrow{v_0} \oplus \cdots \oplus \overrightarrow{v_n} \in [\![\Gamma_k^n]\!]$ implies that there is some $\overrightarrow{\tau} \oplus \overrightarrow{u_0} \oplus \cdots \oplus \overrightarrow{u_n} \in [\![\Gamma^n]\!]$ such that $d(\overrightarrow{t} \oplus \overrightarrow{v_0} \oplus \cdots \oplus \overrightarrow{v_n}, \overrightarrow{\tau} \oplus \overrightarrow{v_0} \oplus \cdots \oplus \overrightarrow{u_n}) \leqslant \epsilon$, which implies $d(\overrightarrow{v_n}, \overrightarrow{u_n}) \leqslant \epsilon$.

Since $\overrightarrow{\tau} \oplus \overrightarrow{u_0} \oplus \cdots \oplus \overrightarrow{u_n} \in [\![\Gamma^n]\!]$ implies $(q_n, \overrightarrow{u_n}) \in Post^n(H)$, $(q_n, \overrightarrow{u_n})$ is the configuration we were looking for. □

We show that the result of Theorem 39 does not hold if we take the hypothesis of Theorem 33.

Example 40. Let H be the Hybrid Automaton in Fig. 3 and $\epsilon > 0$. H satisfies the hypothesis of Theorem 33, but the formula $3^{x'} < 27$ in the activity function of state q_1 violates the second condition of Theorem 30.

We have already seen in Example 35 that, given any $H_k \in \mathbf{A}(H, k)$ obtained by replacing the activity formula $3^{x'} < 27$ with its Taylor approximation $P^k(3^{x'}, x', v) - R^k(3^{x'}, u, v, \phi) < 27$, for some $v \in \mathbb{R}$, then there exists a value $e_k > 0$ such that, for all $u < 3 + e_k$, it holds that $P^k(3^{x'}, u, v) - R^k(3^{x'}, u, v, \phi) < 27$. Hence, $Post^1(H_k)$ contains some configurations (q_1, v_1) with $v_1 \geqslant 3$, and, therefore, $Post^2(H_k)$ contains all (q_2, v) such that $v \leqslant 1$.

We note that no configuration (q_2, v) can be reached by H, for any v. Summarizing, $Post^2(H_k) \notin N(Post^2(H), \epsilon)$.

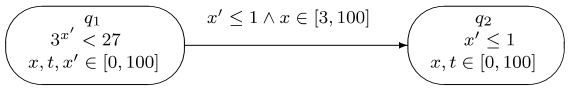

Fig. 3. An Hybrid Automaton.

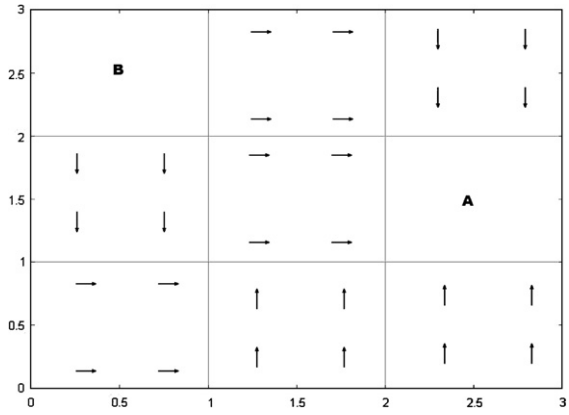

Fig. 4. The grid.

8. Experiments

In this section we apply our approximation technique to the "navigation benchmark" proposed in [10]. We employ `redlog` [28] tool to check whether Polynomial Hybrid Automata satisfy or not properties expressed by formulae.

An object moves in a grid contained in the $\mathbb{R}^2$ plane, starting from an initial position (x_0, y_0) and with an initial velocity (v_{0x}, v_{0y}). The desired velocity v_d is determined by the position of the object in the grid. More precisely, the grid is partitioned in a finite number of cells, and the desired velocity depends on the cell. In Fig. 4 we show a grid of size 3×3 partitioned into 9 cells. The arrow "$\rightarrow$" represents the desired velocity $(1, 0)$, the arrow "$\downarrow$" represents the desired velocity $(0, -1)$, the arrow "$\leftarrow$" represents the desired velocity $(-1, 0)$, and the arrow "$\uparrow$" represents the velocity $(0, 1)$. In each cell, the actual velocity v of the object is determined by the differential equation $\dot{v} = C(v - v_d)$, where $C \in \mathbb{R}^{2\times 2}$. As in [10], let us consider $C = \begin{pmatrix} -1.2 & 0.1 \\ 0.1 & -1.2 \end{pmatrix}$.

Hence, given the initial velocity (v_{0x}, v_{0y}) and a desired velocity $v_d = (v_{dx}, v_{dy})$, the solution $(v_x(t), v_y(t))$ of the differential equation expressing the actual velocity after t units of time is the following

$$\begin{cases} v_x(t) = \frac{v_{0x}+v_{0y}-v_{dx}-v_{dy}}{2} e^{-1.1t} + \frac{v_{0x}-v_{0y}-v_{dx}+v_{dy}}{2} e^{-1.3t} + v_{dx} \\ v_y(t) = \frac{v_{0x}+v_{0y}-v_{dx}-v_{dy}}{2} e^{-1.1t} - \frac{v_{0x}-v_{0y}-v_{dx}+v_{dy}}{2} e^{-1.3t} + v_{dy} \end{cases}$$

Given the initial position (x_0, y_0) of the object, the actual position $(x(t), y(t))$ of the object can be computed by integrating $v_x(t)$ and $v_y(t)$, until the object crosses the border of the cell. When the object enters another cell in a point (x_1, y_1) and with a velocity (v_{1x}, v_{1y}), this machinery can be repeated by taking (x_1, y_1) as initial position and the velocity (v_{1x}, v_{1y}) as initial velocity.

Overall, both the initial position (x_0, y_0) of the object and its initial velocity (v_{0x}, v_{0y}) determine its trajectory.

The problem is to check whether, along this trajectory, the object reaches the "target" cell **A** avoiding the "bad" cell **B** . Also the part of $\mathbb{R}^2$ outside the grid is to be avoided.

In Fig. 5 we depict the trajectory of the object in the case in which the initial position is $(x_0, y_0) = (1.6, 2.6)$ and the initial velocity is $(v_{0x}, v_{0y}) = (-0.2, -0.1)$.

The system can be immediately modeled with an Hybrid Automaton having a state for each cell, where the state modeling the bad cell **B** models also the space in $\mathbb{R}^2$ outside the grid. We have exploited our approximation technique to approximate exponential functions.

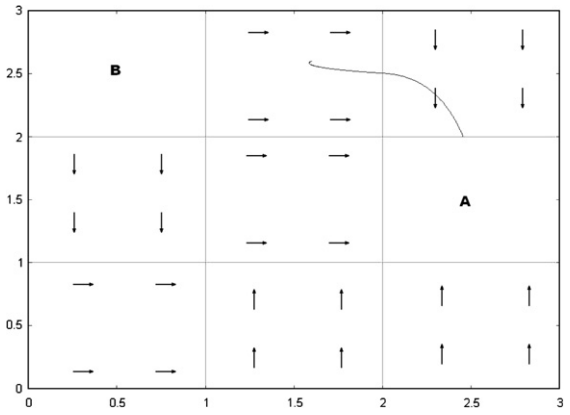

Fig. 5. The first trajectory.

As an example the approximation for $k = 6$ of the activity function of the starting location is provided in Fig. 6.

Variables `vdx` and `vdy` represents the desired velocity . Variables `vx1` and `vy1` represent the starting velocity and variables `vx2` and `vy2` the exiting velocity. Variables `x1` and `y1` represent the starting position and variables `x2` and `y2` the exiting position. The variable `t` represents the time elapsed in the starting cell. Variables `expa` and `erra` are, respectively, the polynomial of Taylor of degree 6 computed in 0 and the error of the function $e^{-1.1t}$. Similarly, variables `expb` and `errb` are, respectively, the polynomial of Taylor of degree 6 computed in 0 and the error of the function $e^{-1.3t}$. Variables `ca` and `cb` are auxiliary variables for computing `vx2` and `vy2`. Finally, variables `cx` and `cy` are auxiliary variables for computing `x2` and `y2`.

In this way, the trajectory of the original system is approximated by a space enclosed in $\mathbb{R}^2$. By exploiting `gnuplot` and `OpenOffice` we obtained a graphical representation of the spaces that approximate the trajectory for several values of k. In Fig. 7 we show these representations for $k = 3, 4, 5, 6$.

By applying `redlog` we checked whether our approximations behave like the original trajectory, namely they permit reaching the target cell **A** without touching the bad cell **B**, by crossing the same cells that are crossed by the original trajectory.

We have considered $k = 1, \ldots, 6$.

If $k = 1$, or $k = 2$, or $k = 3$, `redlog` computes that the object, starting from (x_0, y_0), exits from the starting cell and may enter in the bad cell **B**. Hence, these approximations should be rejected. If $k = 4$, or $k = 5$, or $k = 6$, `redlog` computes that the object, starting from (x_0, y_0), cannot leave the starting cell entering in the bad cell **B**. These results and the `redlog` computation times are summarized in Fig. 9. (We used a laptop equipped with an `Intel Pentium` l.4 GHz and 256 MB DDR-SDRAM.)

In Fig. 8 we write, as an example, the command used for computing that the bad cell **B** cannot be touched with $k = 6$. Command `rlqe ex` applied to a variable x requires to `redlog` to apply the quantifier elimination algorithm on x.

The formula `a` is the formula in Fig. 6 with bounds to variables as required by our results. The bounds have been computed manually by studying the functions $x(t)$ and $y(t)$. We require also that `x2 = 1` to check whether the cell **B** can be touched. As shown in Fig. 9, `redlog` answers *false* in 0.260s.

Then, for $k = 4, 5, 6$, `redlog` checks that the border of the starting cell is crossed by the object in a point (x, y) such that $x = 2$, $y \neq 2$, and $y \neq 3$, thus implying that the object enters the same cell in the top-right corner of the grid that is entered also in the original trajectory. We have used `redlog` also to compute bound values of y, v_x and v_y when the object enters this top-right cell (we have checked with up to 2 decimal places). Computation times and bound values are summarized in Fig. 10. (As an example, for $k = 4$ we have found that $2.47 < y < 2.52$, $0.54 < v_x < 0.80$ and $-0.07 < v_y < -0.03$.) Notice that the size of the intervals of the possible values of y, v_x, and v_y decreases with the growth of k.

```
vdx = 1 and vdy = 0
  and
vx1 = -0.2 and vy1 = -0.1
  and
x1 = 1.6  and  y1 = 2.6
  and
expa = 1-1.1*t+((-1.1)^2*t^2)/2+((-1.1)^3*t^3)/6+((-1.1)^4*t^4)/
       24 + ((-1.1)^5*t^5)/120+((-1.1)^6*t^6)/720+ erra
  and
expb = 1-1.3*t+((-1.3)^2*t^2)/2+((-1.3)^3*t^3)/6+((-1.3)^4*t^4)/
       24 + ((-1.3)^5*t^5)/120+((-1.3)^6*t^6)/720+errb
  and
erra <=((1.1)^7*(t^8+1))/5040
  and
erra >=-((1.1)^7*(t^8+1))/5040
  and
errb <=((1.3)^7*(t^8+1))/5040
  and
errb >=-((1.3)^7*(t^8+1))/5040
  and
ca = (vx1 + vy1 - vdx - vdy)/2
  and
cb = (vx1 - vy1 - vdx +  vdy)/2
  and
vx2 = ca*expa+cb*expb+vdx
  and
vy2=ca*expa-cb*expb+vdy
  and
cx = x1 - ca/(-1.1) - cb/(-1.3)
  and
cy = y1 - ca/(-1.1) + cb/(-1.3)
  and
x2 =(ca*expa)/(-1.1) + (cb*expb)/(-1.3) + vdx*t + cx
  and
y2 =(ca*expa)/(-1.1) - (cb*expb)/(-1.3) + vdy*t + cy
```

Fig. 6. The activity function in `redlog`.

Finally, for $k = 4, 5, 6$, `redlog` computes that from the cell in the top-right corner of the grid, the cell **A** is reached. Computation times and answers are summarized in Fig. 11.

The computation time requested for checking the behavior of the object in the second cell (top-right corner cell) is much greater than the computation time requested for checking the behavior in the starting cell. The reason is that we know that in the starting cell the object departs from a given fixed point with a given velocity, whereas in the second cell the departing point and the initial velocity are bounded in intervals but are not fixed.

Overall, we conclude that a relatively small k (i.e., $k = 4$) suffices to have a good approximation of the original trajectory.

In Fig. 12 we consider a second trajectory. Here, the initial position is $(x_0, y_0) = (0.5, 1.7)$ and the initial velocity is $(v_{0x}, v_{0y}) = (0.8, 1.0)$. This trajectory is more critical than the previous one, for two reasons. The first reason is that it passes near the border of the cell **B**, thus implying that the approximations of the trajectory should be quite precise to avoiding a fall in **B**. The second reason is that this trajectory crosses the central

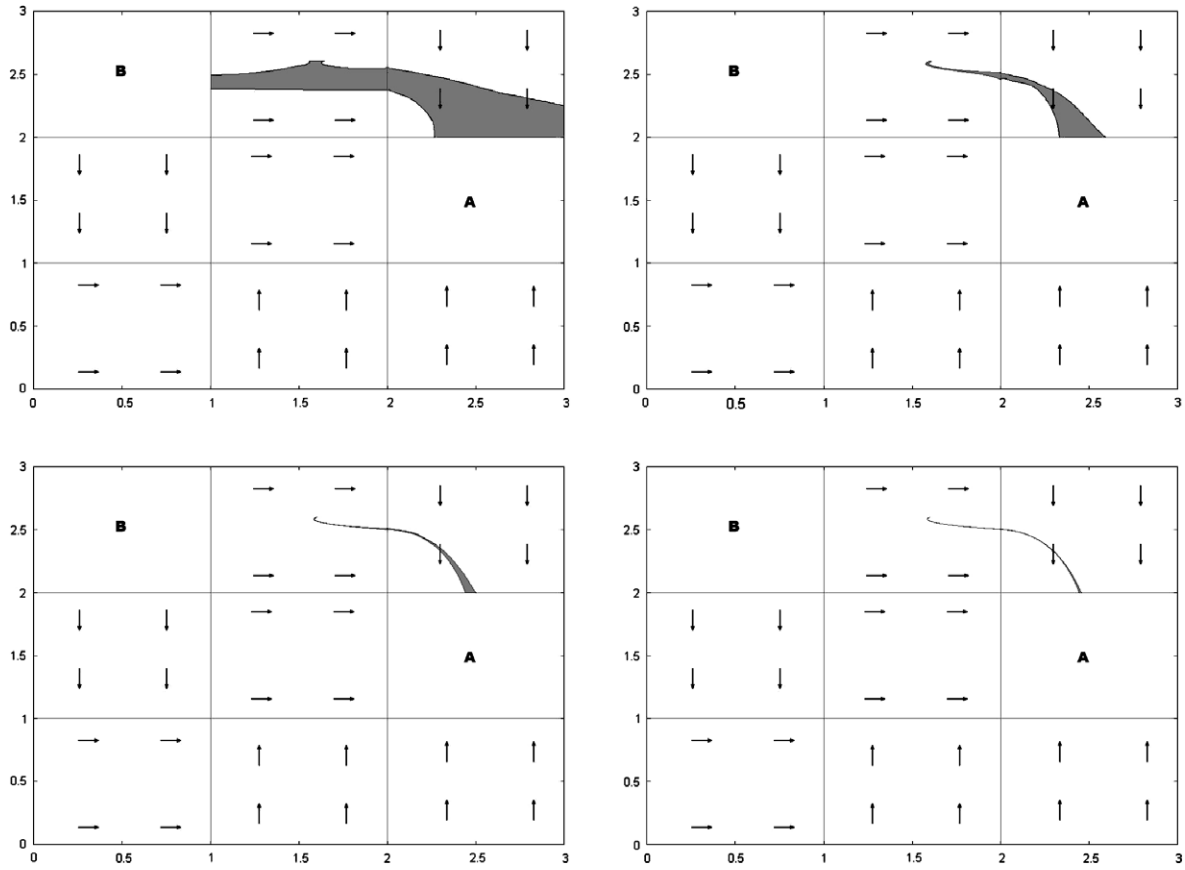

Fig. 7. Approximations of the first trajectory.

```
rlqe ex(
  {t, vx2, vy2, vdx, vdy, vx1, vy1, x1, y1, ca, cb, cx, cy,
   expa, expb, erra, errb, x2, y2},
t > 0 and  t<1.2  and a and x2=1 );
```

Fig. 8. The command `rlqe ex`.

cell of the grid from the left border to the right one, thus implying that the object moves within this cell for a relatively long interval of time. This fact is critical since the Taylor polynomials are computed with respect to the instant in which the cell is entered (hence for $t = 0$), and the greater the time t elapsed in a cell, the bigger the error.

By exploiting `gnuplot` and `OpenOffice` we obtained a graphical representation of the spaces that approximate the trajectory for several values of k. In Fig. 13 we show these representations for $k = 4, 5, 6$.

Also in this case by applying `redlog` we checked whether our approximations behave like the original trajectory, namely they permit reaching the target cell **A** without touching the bad cell **B**, by crossing the same cells that are crossed by the original trajectory.

Since the second trajectory is more critical, we need a more accurate approximation ($k = 5$ instead of $k = 4$) to obtain a good approximation, but also in this case k is relatively small.

Computation times, answers and bounds are quite similar to that of the first trajectory and are summarized in tables of Figs. 14, 15 and 16.

value of k	Can $\mathbf{B}$ touched?	redlog time
1	true	0.070s
2	true	0.100s
3	true	0.100s
4	false	0.140s
5	false	0.140s
6	false	0.260s

Fig. 9. redlog computation times.

value of k	1^{st} cell border values				redlog time
	x	y	v_x	v_y	
4	2	$(2.47, 2.52)$	$(0.54, 0.80)$	$(-0.07, -0.03)$	1.763s
5	2	$(2.50, 2.52)$	$(0.67, 0.76)$	$(-0.07, -0.04)$	4.316s
6	2	$(2.50, 2.51)$	$(0.68, 0.71)$	$(-0.06, -0.05)$	5.018s

Fig. 10. redlog computation times.

value of k	is $\mathbf{A}$ correctly reached?	redlog time
4	true	06m 10.985s
5	true	07m 09.357s
6	true	21m 51.806s

Fig. 11. redlog computation times.

9. Future works

In this paper we have defined syntactical over-approximations for Hybrid Automata by means of Taylor polynomials, and we have studied their syntactical and semantical convergence w.r.t. the original specifications.

As future work we will study also under-approximations based on the same technique. The idea is to define the *under-approximation* of degree k of a formula ϕ, namely the set of formulae denoted $\mathbf{A}_U(\phi, k)$ defined inductively as follows:

(1) If $\phi \equiv f(\overrightarrow{x}) \sim b \cdot x_i + c$, then $\mathbf{A}_U(\phi, k)$ contains either ϕ, if f is a polynomial, or all formulae

$$\phi_{k,\overrightarrow{v}} \equiv P^k(f, \overrightarrow{x}, \overrightarrow{v}) + R^k(f, \overrightarrow{x}, \overrightarrow{v}, \phi) \sim b \cdot x_i + c$$

such that $\overrightarrow{v} \in V_f$, otherwise;
(2) If $\phi \equiv \phi^1 \wedge \phi^2$ then $\mathbf{A}_U(\phi, k) = \{\phi_k^1 \wedge \phi_k^2 \mid \phi_k^1 \in \mathbf{A}_U(\phi^1, k) \text{ and } \phi_k^2 \in \mathbf{A}_U(\phi^2, k)\}$;
(3) If $\phi \equiv \phi^1 \vee \phi^2$ then $\mathbf{A}_U(\phi, k) = \{\phi_k^1 \vee \phi_k^2 \mid \phi_k^1 \in \mathbf{A}_U(\phi^1, k) \text{ and } \phi_k^2 \in \mathbf{A}_U(\phi^2, k)\}$;
(4) If $\phi \equiv \exists y \in I_y.\, \phi'$ then $\mathbf{A}_U(\phi, k) = \{\exists y \in I_y.\, \phi'_k \mid \phi'_k \in \mathbf{A}_U(\phi', k)\}$;
(5) If $\phi \equiv \forall y \in I_y.\, \phi'$ then $\mathbf{A}_U(\phi, k) = \{\forall y \in I_y.\, \phi'_k \mid \phi'_k \in \mathbf{A}_U(\phi', k)\}$.

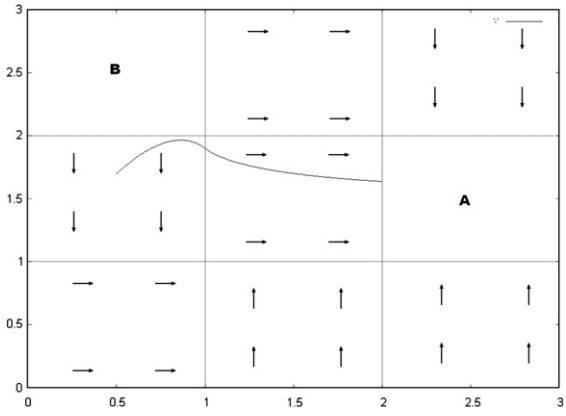

Fig. 12. The second trajectory.

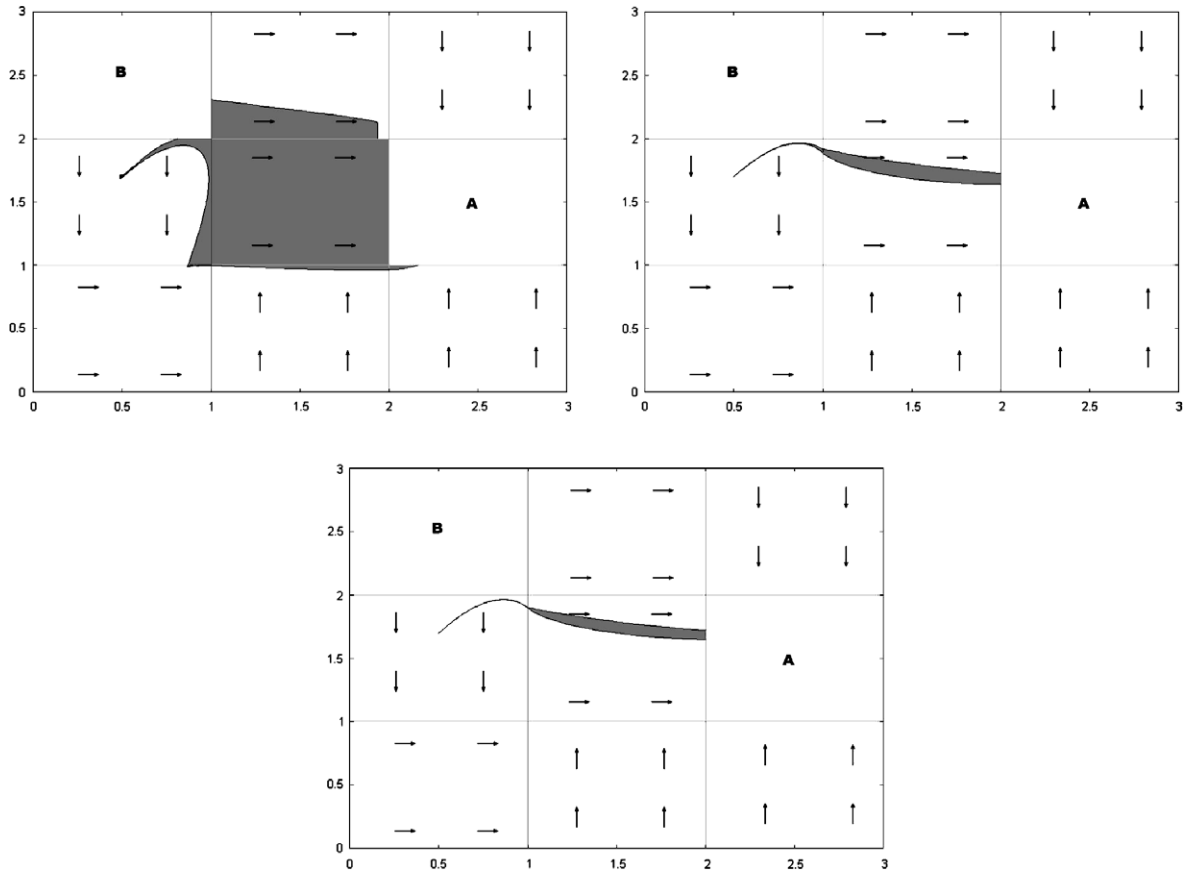

Fig. 13. Approximations of the second trajectory.

Notice that the definition of $\mathbf{A}_U(\phi,k)$ differs w.r.t. the definition of $\mathbf{A}(\phi,k)$, since the lower bound $-R^k(f,\overrightarrow{x},\overrightarrow{v},\phi)$ of the error $r^k(f,\overrightarrow{x},\overrightarrow{v})$ employed in Definition 17.1 is replaced by the upper bound $R^k(f,\overrightarrow{x},\overrightarrow{v},\phi)$.

By mimicking the proof of Theorem 18 we can immediately show that $[\![\phi_k]\!] \subseteq [\![\phi]\!]$ for all $\phi_k \in \mathbf{A}_U(\phi,k)$.

value of k	Can $\mathbf{B}$ touched?	`redlog` time
1	true	0.070s
2	true	0.080s
3	true	0.090s
4	true	0.160s
5	false	0.240s
6	false	0.291s

Fig. 14. `redlog` computation times.

value of k	1^{st} cell border values				`redlog` time
	x	y	v_x	v_y	
5	1	$(1.87, 1.95)$	$(0.31, 0.32)$	$(-0.35, -0.34)$	2.073s
6	1	$(1.89, 1.91)$	$(0.31, 0.32)$	$(-0.35, -0.33)$	5.388s

Fig. 15. `redlog` computation times.

value of k	is $\mathbf{A}$ correctly reached?	`redlog` time
5	true	09m 45.291s
6	true	21m 27.663s

Fig. 16. `redlog` computation times.

Then, given any Hybrid Automaton H, the *under-approximation of degree k for H* is the set of the Polynomial Hybrid Automata denoted $\mathbf{A}_U(H,k)$ that are obtained from H by replacing each formula ϕ with some formula in $\mathbf{A}_U(\phi,k)$.

By mimicking the proof of Theorem 22 we can immediately prove that $Post^n(H) \supseteq Post^n(H_k)$, for all $H_k \in \mathbf{A}_U(H,k)$.

Whereas the over-approximation $\mathbf{A}(H,k)$ considered in this paper gives a sound technique for proving invariants, the under-approximations $\mathbf{A}_U(H,k)$ gives a sound technique for proving reachability.

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