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# Well-Quasi-Order of Relabel Functions

**Jean Daligault · Michael Rao · Stéphan Thomassé**

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**Abstract** We define $\mathrm{NLC}_k^{\mathcal{F}}$ to be the restriction of the class of graphs $\mathrm{NLC}_k$, where relabelling functions are exclusively taken from a set $\mathcal{F}$ of functions from $\{1, \ldots, k\}$ into $\{1, \ldots, k\}$. We characterize the sets of functions $\mathcal{F}$ for which $\mathrm{NLC}_k^{\mathcal{F}}$ is well-quasi-ordered by the induced subgraph relation $\leq_i$. Precisely, these sets $\mathcal{F}$ are those which satisfy that for every $f, g \in \mathcal{F}$, we have $Im(f \circ g) = Im(f)$ or $Im(g \circ f) = Im(g)$. To obtain this, we show that words (or trees) on $\mathcal{F}$ are well-quasi-ordered by a relation slightly more constrained than the usual subword (or subtree) relation. A class of graphs is *n-well-quasi-ordered* if the collection of its vertex-labellings into $n$ colors forms a well-quasi-order under $\leq_i$, where $\leq_i$ respects labels. Pouzet (C R Acad Sci, Paris Sér A–B 274:1677–1680, 1972) conjectured that a 2-well-quasi-ordered class closed under induced subgraph is in fact $n$-well-quasi-ordered for every $n$. A possible approach would be to characterize the 2-well-quasi-ordered classes of graphs. In this respect, we conjecture that such a class is always included in some well-quasi-ordered $\mathrm{NLC}_k^{\mathcal{F}}$ for some family $\mathcal{F}$. This would imply Pouzet's conjecture.

**Keywords** Well-quasi-order · Clique-width · Induced subgraph

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J. Daligault (✉) · S. Thomassé
LIRMM, Université Montpellier II, 161 rue Ada, 34392 Montpellier Cedex, France
e-mail: daligault@lirmm.fr, jean_daligault@yahoo.fr

S. Thomassé
e-mail: thomasse@lirmm.fr

M. Rao
LaBRI, Université Bordeaux 1, CNRS, 351 Cours de la Libération,
33405 Talence Cedex, France
e-mail: rao@labri.fr

## 1 Introduction

Let $S$ be a set and $\leq$ be a quasi-order on $S$, i.e. a reflexive and transitive relation. Given a sequence $(x_i)_{i\in\omega}$ of elements of $S$, a *good pair* consists of two elements $x_i \leq x_j$, with $i < j$. A sequence with no good pair is called a *bad sequence* of $(S, \leq)$. A quasi-order with no bad sequence is a *well-quasi-order*.

There are other equivalent presentations of the notion of well-quasi-ordering (see for instance [13]). A quasi-order is a well-quasi-order if and only if it has no infinite antichain and no infinite strictly decreasing sequence. Also, a quasi-order is a well-quasi-order if and only if every infinite sequence has an infinite increasing subsequence. Equivalently, a quasi-order is a well-quasi-order if and only if every nonempty subset of $S$ has a nonempty finite set of minimal elements (up to equivalence).

The theory of well-quasi-ordering has been flourishing. Higman's theorem states that the set of words over a well-quasi-ordered set is well-quasi-ordered by the subword relation [6], and this has been extended by Kruskal to trees [12]. Robertson and Seymour's celebrated graph minor theorem [16] asserts that the minor relation is a well-quasi-order on the set of finite graphs. It implies that every graph class closed under minor (or minor *ideal*) can be characterized by a finite list of excluded minors. This in turn implies that every minor ideal can be recognized in polynomial time.

The class of finite graphs is not well-quasi-ordered by the induced subgraph relation since the cycles form an antichain. The good properties of the minor ideals ensured by the minor theorem do not hold for induced subgraph ideals (for instance, the set of paths, which is well-quasi-ordered, does not have a finite set of forbidden induced subgraphs). This is a motivation for the stronger notion of 2-well-quasi-ordering.

In the following, we will be exclusively interested in the induced subgraph relation. Throughout this paper, we will abbreviate "well-quasi-ordered by the induced subgraph relation" by *well-quasi-ordered*, with the understanding that we are dealing with the induced subgraph relation.

An extension of the notion of well-quasi-order is the notion of $n$-well-quasi-order (see Kriz and Thomas [11] for a more general discussion in terms of the QO-category). A set $S$ of graphs is *$n$-well-quasi-ordered* if the class $\hat{S}$ consisting of all vertex $n$-colorings of graphs in $S$, is well-quasi-ordered by the colored induced subgraph relation $\leq_1$, where $G \leq_1 G'$ if there is an injection from $V(G)$ to $V(G')$ preserving adjacency and color. The set $S$ is *$\infty$-well-quasi-ordered* if $S$ is $n$-well-quasi-ordered, for any $n \geq 1$. Being 2-well-quasi-ordered is a stronger property than being well-quasi-ordered, for instance the set of paths is not 2-well-quasi-ordered.

The notion of 2-well-quasi-ordering is especially interesting in view of algorithmic properties, as induced subgraph ideals which are 2-well-quasi-ordered can be characterized by a finite list of forbidden induced subgraphs, and thus are polynomially recognizable. Our ultimate aim would be to characterize the 2-well-quasi-ordered ideals of graphs, in order to prove the following conjecture of Pouzet [15], also appearing in Fraïssé [3]:

**Conjecture 1** *An induced subgraph ideal is 2-well-quasi-ordered if and only if it is $\infty$-well-quasi-ordered.*

In the more general framework of categories, Pouzet's question has a negative answer, as shown by Kriz and Sgall [10].

We will come back to this topic in Section 5. Our main purpose here is to study a restriction of the hierarchy of graph classes NLC.

The class $\mathrm{NLC}_k$ of *k-node labelled controlled graphs* was introduced in [18]. A *k-labelled graph* is a graph in which every vertex has a label into $\{1, \ldots, k\}$. Let $\mathcal{F}$ be a set of functions from $\{1, \ldots, k\}$ into $\{1, \ldots, k\}$. The class $\mathrm{NLC}_k^{\mathcal{F}}$ is defined recursively on $k$-labelled graphs using three operators: $\bullet_i$, $\circ_f$ and $\chi_S$. For $i \in \{1, \ldots, k\}$, the operator $\bullet_i$ creates a single vertex labelled by $i$. The operator $\circ_f$, with $f \in \mathcal{F}$, applied to a $k$-labelled graph replaces every label $i$ with $f(i)$. The operator $\chi_S$, with $S \subseteq \{1, \ldots, k\} \times \{1, \ldots, k\}$, applied to two $k$-labelled graphs $G$ and $H$ in this order, creates the disjoint union of graphs $G$ and $H$, and for all $(i, j) \in S$ adds edges between every vertex of label $i$ in $G$ and every vertex of label $j$ in $H$. The class $\mathrm{NLC}_k$ is defined to be $\mathrm{NLC}_k^{\Phi}$ where $\Phi$ is the set of all functions from $\{1, \ldots, k\}$ into $\{1, \ldots, k\}$. The *NLC-width* of $G$ is the minimum $k$ for which some labelling of $G$ is in $\mathrm{NLC}_k$.

See Fig. 1 for an example of an $\mathrm{NLC}_3$ expression of a small graph.

It is not known whether there exists a polynomial time algorithm computing a NLC decomposition using $k$ colors for graphs in $\mathrm{NLC}_k$. Only the cases $k = 1$ (which corresponds to cographs) and $k = 2$ [4, 7] have been solved so far. Computing the NLC-width is NP-hard [5].

The NLC-width has a strong link with another well-known parameter: the clique-width, introduced by Courcelle et al. [1]. NLC-width and clique-width indeed differ by a factor at most 2. More precisely, the clique-width of a graph is bounded below by its NLC-width, and above by twice its NLC-width. Moreover, transformations respecting these bounds between decompositions of the two types can be done in linear time.

The class of graphs $\mathrm{NLC}_1$ (cographs) is well-quasi-ordered, see [2] and [17] for the countable case. The class of graphs $\mathrm{NLC}_2$ is well-quasi-ordered (and even $\infty$-well-quasi-ordered), this easily follows from the results in [7]. Indeed, the $\mathrm{NLC}_2$ prime graphs for the modular decomposition are constructible in $\mathrm{NLC}_2$ without relabelling, and thus form a well-quasi-ordered family by Kruskal's tree theorem. However, the class $\mathrm{NLC}_3$ is not well-quasi-ordered, as it contains for every $i$ the graph depicted in Fig. 1 consisting of a path of length $i$ with two pendant vertices attached

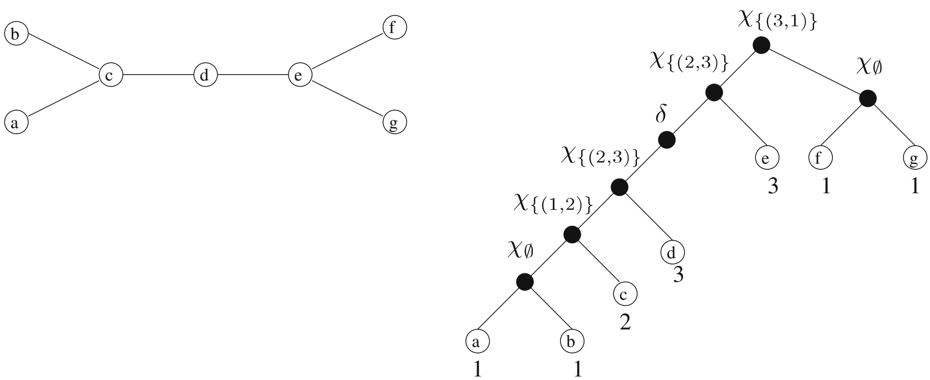


**Fig. 1** The graph $G_3$ and an associated NLC-expression, on one relabelling function, $\delta$: $3 \to 2 \to 1$

to each extremity, which we denote by $G_i$. These graphs indeed do not form a well-quasi-ordered family. Allowing all relabelling operators $\circ_f$ is too much to construct a well-quasi-ordered class of graphs if we have at least three colors. This is why we define a restriction of NLC, using only relabelling operators from a specified set of functions $\mathcal{F}$. Our main purpose is to characterize the sets $\mathcal{F}$ such that $\mathrm{NLC}_k^{\mathcal{F}}$ is well-quasi-ordered. We will see that $\mathrm{NLC}_k^{\mathcal{F}}$ is well-quasi-ordered (equivalently $\infty$-well-quasi-ordered) if and only if it does not contain arbitrarily large paths.

In Section 2, we introduce a binary relation $\preceq$ on a set of functions. In Section 3 we introduce a subword order $\leq$ on words labelled with a set of functions which is more constrained than Higman's order. In Section 4 we extend $\leq$ to trees, with the purpose of applications to $\mathrm{NLC}_k^{\mathcal{F}}$ expressions. In Section 5, we characterize the sets $\mathcal{F}$ for which $\mathrm{NLC}_k^{\mathcal{F}}$ is well-quasi-ordered. In the final section, we discuss Pouzet's conjecture on $n$-well-quasi-ordering, and further well-quasi-ordering problems.

Throughout this paper, we will obtain the following equivalent characterizations of $\mathcal{F}$ (with Theorems 1–3, Corollaries 1 and 2):

- The set $\mathcal{F}$ is totally quasi-ordered by $\preceq$.
- The set of words on $\mathcal{F}$ is well-quasi-ordered by $\leq$.
- The set of trees on $\mathcal{F}$ is well-quasi-ordered by $\leq$.
- The set of graphs $\mathrm{NLC}_k^{\mathcal{F}}$ is well-quasi-ordered.
- The set of graphs $\mathrm{NLC}_k^{\mathcal{F}}$ is $\infty$-well-quasi-ordered.
- The set of graphs $\mathrm{NLC}_k^{\mathcal{F}}$ does not contain arbitrarily large paths.

## 2 Totally Ordered Sets of Functions

Let $\mathcal{F}$ be a set of functions from $\{1, \dots, k\}$ into $\{1, \dots, k\}$ closed under composition (with the convention that the identity function $\varepsilon$ belongs to $\mathcal{F}$). Note that we can make these assumptions without loss of generality with respect to classes $\mathrm{NLC}_k^{\mathcal{F}}$. The key definition of this section is the following. Let us say that $f \preceq g$ whenever $Im(f \circ g) = Im(f)$.

Assume that $\preceq$ is total on $\mathcal{F}$, i.e. for every $f, g$ in $\mathcal{F}$, at least one of $f \preceq g$ and $g \preceq f$ holds. This implies in particular that $Im(f^2) = Im(f)$ for all $f \in \mathcal{F}$. Observe that $f \preceq g$ implies that $|Im(f)| \leq |Im(g)|$.

**Lemma 1** *If $\preceq$ is total on $\mathcal{F}$, then $\preceq$ is transitive.*

*Proof* Assume by way of contradiction that $Im(f \circ g) = Im(f)$, $Im(g \circ h) = Im(g)$ and $f \npreceq h$, i.e. $|Im(f \circ h)| < |Im(f)|$. Since $\preceq$ is total, we must have $h \preceq f$, and then $Im(h \circ f) = Im(h)$. Hence $f \preceq g \preceq h \preceq f$, and thus $|Im(f)| = |Im(h)|$. As $h \circ f \preceq h \circ f$ holds, we have $|Im(h \circ f \circ h \circ f)| = |Im(h \circ f)| = |Im(h)|$. Moreover $|Im(h \circ (f \circ h) \circ f)| \leq |Im(f \circ h)| < |Im(f)| = |Im(h)|$, which is a contradiction. □

Thus $\preceq$ is a reflexive and transitive relation, in other words $\preceq$ is a total quasi-order on $\mathcal{F}$. This is equivalent to the existence of a partition of $\mathcal{F}$ into $t$ equivalence classes $F_1$, ..., $F_t$ such that $f \in F_i$ and $g \in F_j$ satisfy $f \preceq g$ if and only if $i \leq j$.

**Lemma 2** *When $\mathcal{F}$ is totally quasi-ordered by $\preceq$, the equivalence classes $F_1$, ..., $F_t$ are exactly the classes of functions having an image of the same size, in increasing order of the image size.*

*Proof* If $|Im(f)| < |Im(g)|$ then $g \not\preceq f$, so we must have $f \preceq g$. Suppose now by contradiction that $|Im(f)| = |Im(g)|$, $f \preceq g$ and $g \not\preceq f$. This means that $|Im(f \circ (g \circ f) \circ g)| \leq |Im(g \circ f)| < |Im(g)|$. We also have that $|Im(f \circ g)| = |Im(f)| = |Im(g)|$, thus $|Im(f \circ g \circ f \circ g)| < |Im(f \circ g)|$. This contradicts $f \circ g \preceq f \circ g$. □

Observe that the top class $F_t$ contains $\varepsilon$, and contains only permutations.

**Lemma 3** *For all i, $F_i$ and $\cup_{k \geq i} F_k$ are closed under composition.*

*Proof* The first part of the statement follows by Lemma 2. To prove that $\cup_{k \geq i} F_k$ is closed under composition, consider $f \in F_i$ and $g \in F_j$ with $i < j$. Since $f \preceq g$, $f \circ g$ is in $F_i$. Assume now for contradiction that $g \circ f \in F_p$, with $p < i$. Then $|Im(f \circ g)| = |Im(f)| > |Im(g \circ f)|$. Moreover $|Im(g \circ f)| \geq |Im(f \circ (g \circ f) \circ g)|$. Thus $|Im(f \circ g)| \neq |Im((f \circ g)^2)|$, contradicting $f \circ g \preceq f \circ g$. □

**Lemma 4** *The functions of the bottom class $F_1$ verify a "left-cancellation" identity:*

$$\forall f \in F_1, \forall h, h' \in \mathcal{F},\ \textit{if}\ h \circ f \circ h' = h \circ f\ \textit{then}\ f \circ h' = f \tag{1}$$

*Proof* This identity actually holds whenever $f \preceq h$. Indeed, by Lemma 3, $f \preceq h \circ f$. Assume $h \circ f \circ h' = h \circ f$. If there is an $x \in \{1, \ldots, k\}$ such that $f \circ h'(x) \neq f(x)$, then these two distinct elements belong to the image of $f$ and have the same image under $h$. This means that $|Im(h \circ f)| < |Im(f)|$, and contradicts the fact that $f \preceq h \circ f$. □

Here is an example of a set of functions which is totally ordered by $\preceq$. An $(i, j)$*−cast*, with $i \leq j$, is a function $f$ from $\{1, \ldots, k\}$ into itself such that $f(l) = l$ for all $l < i$ and $f(l) = j$ whenever $i \leq l$. It is routine to check that the set of casts is indeed totally ordered by $\preceq$. We feel that the following problem would give some insight on the well-quasi-ordered $\text{NLC}_k^{\mathcal{F}}$ classes:

**Problem 2** Find a generic class of functions $\mathcal{G}$ (like casts for instance) such that for every totally ordered $\mathcal{F}$ and $k$, there exists some $k'$ for which $NLC_k^{\mathcal{F}}$ is included in $NLC_{k'}^{\mathcal{G}}$.

Such a class of function would describe much more precisely how to construct the well-quasi-ordered classes $NLC_k^{\mathcal{F}}$.

## 3 Words on Functions

An $\mathcal{F}$*-word* is a finite word on the alphabet $\mathcal{F}$, i.e. a finite sequence $f_1, \ldots, f_l$ of elements of $\mathcal{F}$, with $l \geq 0$. Let $\mathcal{W}^{\mathcal{F}}$ be the set of $\mathcal{F}$-words. Let $M = f_1, \ldots, f_l$ and $M' = f'_1, \ldots, f'_{l'}$ be two $\mathcal{F}$-words. We denote by $|M| = l$ the length of $M$. The word $M$ is a *subword* of $M'$ if there exists an increasing injection $\phi$ from $\{1, \ldots, l\}$ into $\{1, \ldots, l'\}$ such that $f_i = f'_{\phi(i)}$. Higman's theorem asserts that the subword partial order is a well-quasi-order when the alphabet is finite. In our more constrained partial order on $\mathcal{F}$-words, we have $M \leq M'$ if two conditions are satisfied:

- There is a function $\phi$ for which $M$ is a subword of $M'$.
- For all $1 \leq i < l$, we have $f_i = f'_{\phi(i)} \circ f'_{\phi(i)+1} \circ \cdots \circ f'_{\phi(i+1)-1}$.

Thus, when $M \leq M'$ and $i < j$, the composition of functions $f_i \circ f_{i+1} \circ \cdots \circ f_{j-1}$ is equal to the function $f'_{\phi(i)} \circ f'_{\phi(i)+1} \circ \cdots \circ f'_{\phi(j)-1}$. And since $f_j = f'_{\phi(j)}$, we also have $f_i \circ f_{i+1} \circ \cdots \circ f_j = f'_{\phi(i)} \circ f'_{\phi(i)+1} \circ \cdots \circ f'_{\phi(j)}$.

Our goal is to prove here that $\mathcal{W}^{\mathcal{F}}$ is well-quasi-ordered by $\leq$ if and only if $\preceq$ is total on $\mathcal{F}$. For this, we have to be more general and we need to consider $\mathcal{W}_Q^{\mathcal{F}}$, the set of words on the set $\mathcal{F} \times Q$, where $Q$ is a set endowed by a well-quasi-order $\leq_Q$.

We naturally extend the partial order $\leq$ on $\mathcal{W}_Q^{\mathcal{F}}$. For $w$ in $\mathcal{W}_Q^{\mathcal{F}}$ and $1 \leq x \leq |w|$ we denote by $(f_x^w, q_x^w)$ the $x^{\text{th}}$ letter of $w$. For any couple of indices $a, b$, with $1 \leq a < b \leq |w|$, we define $L^w(a, b)$ to be the composition $f_a^w \circ f_{a+1}^w \circ \ldots \circ f_{b-1}^w$. When $a = b$, we set $L^w(a, b) = \epsilon$. Let $\phi$ be an increasing injection from $\{1, \ldots, |w|\}$ into $\{1, \ldots, |w'|\}$. We say that $\phi$ *is compatible with labels* if $f_x^w = f_{\phi(x)}^{w'}$ and $q_x^w \leq_Q q_{\phi(x)}^{w'}$ for every $x \in 1, \ldots, |w|$. We say that $\phi$ *preserves path-composition* if for every $x < |w|$, we have that $L^w(x, x+1) = L^{w'}(\phi(x), \phi(x+1))$ (observe that by definition we have $L^w(x, x+1) = f_x^w$). We write $w \leq w'$ if there exists an increasing injection $\phi$ from $\{1, \ldots, |w|\}$ into $\{1, \ldots, |w'|\}$ compatible with labels and preserving path-composition. When $\phi$ is only compatible with labels, we simply say that $w$ is a *subword* of $w'$ and write $w \leq_0 w'$.

**Theorem 1** *The set of words $\mathcal{W}_Q^{\mathcal{F}}$, where $Q$ is a well-quasi-order, is well-quasi-ordered by $\leq$ if and only if $\preceq$ is a total quasi-order on $\mathcal{F}$.*

*Proof* Assume first that $\preceq$ is not a total quasi-order on $\mathcal{F}$, i.e. there exist two incomparable functions $f, g \in \mathcal{F}$. Depending if $f = g$ or not, let us show in both cases that $\mathcal{W}^{\mathcal{F}}$ is not well-quasi-ordered by $\leq$.

- If $f = g$, we claim that $S = (w_k)_{k \geq 0}$, where $w = \epsilon(f)^k\epsilon$, is a bad sequence. Indeed, if $w_i \leq w_j$ with $i < j$, the identity functions must be mapped to identity functions. Moreover, to preserve path-composition, the first and the last $f$ of $w_i$ must be mapped to the first and the last $f$ of $w_j$. Hence, two consecutive $f$'s in $w_i$ must be mapped to nonconsecutive $f$'s in $w_j$, which contradicts path-composition since $f \neq f^l$ for every $l > 1$.
- If $f \neq g$, we claim that $S = (w_k)_{k \geq 0}$, where $w = \epsilon(fg)^k\epsilon$, is a bad sequence. Again, if $w_i \leq w_j$ with $i < j$, the identity functions must be mapped to identity functions. Moreover, to preserve path-composition, the first $f$ and the last $g$ of $w_i$ must be mapped to the first $f$ and to the last $g$ of $w_j$. Hence, two consecutive $f$ and $g$ in $w_i$ must be mapped to nonconsecutive $f$ and $g$ in $w_j$, which contradicts path-composition.

Assume now that $\preceq$ is a total quasi-order on $\mathcal{F}$. Our goal is to prove that $\mathcal{W}_Q^{\mathcal{F}}$ is well-quasi-ordered.

To achieve this, we need to enrich a little bit our structure, and consider words on $\mathcal{F} \times \mathcal{F} \times Q$ instead of $\mathcal{F} \times Q$. We just label every letter of our word by another function of $\mathcal{F}$. Precisely, we add to every letter $x$ of our word $w$ the function consisting of the composition of all functions of the prefix of $w$ before $x$. The reason for that is to keep track, in every letter of the word, of some information preceding this letter. Technically, this bit of information allows us to use a Nash-Williams' minimum bad sequence argument, cutting every word in a prefix-suffix way and applying the minimality to prove the well-quasi-order. The key idea is to cut the words on some letter which belongs to the bottom class of $\mathcal{F}$, since left-cancellation (thanks to the extra label) enables to glue back the prefix and the suffix.

Let us turn to technicalities. Instead of dealing with words $w = ((f_x^w, q_x^w))_{x=1\ldots|w|}$ on $\mathcal{F} \times Q$, we transform $w$ into the word $\tilde{w} = ((f_x^w, L^w(1,x), q_x^w))_{x=1\ldots|w|}$ on $\mathcal{F} \times \mathcal{F} \times Q$. To simplify notation, we still call this word $w$.

A word $((f_x, h_x, q_x))_{x=1\ldots k}$ on $\mathcal{F} \times \mathcal{F} \times Q$ is *coherent* if for every $1 \leq x < k$, we have $h_{x+1} = h_x \circ f_x$. Observe that $w$ is coherent by construction, and so is every factor of $w$. We denote by $\widetilde{\mathcal{W}}^{\mathcal{F}}_{\mathcal{F}\times Q}$ the set of coherent words on $\mathcal{F} \times \mathcal{F} \times Q$. The set $\mathcal{F} \times Q$ is equipped with the well-quasi-order $(f, q) \leq_{\mathcal{F}\times Q} (f', q')$ if $f = f'$ and $q \leq_Q q'$. Theorem 1 follows from the following result. □

**Lemma 5** *For any well-quasi-ordered set $Q$, the set $\widetilde{\mathcal{W}}^{\mathcal{F}}_{\mathcal{F}\times Q}$ is well-quasi-ordered by $\leq$.*

*Proof* By induction on $t$, the number of equivalence classes of $\mathcal{F}$.

Assume first that $t = 1$. Observe that $\{\varepsilon\} \subseteq \mathcal{F} \subseteq \mathfrak{S}_n$, where $\mathfrak{S}_n$ is the set of permutations of $\{1, \ldots, n\}$. By Higman's theorem, $\widetilde{\mathcal{W}}^{\mathcal{F}}_{\mathcal{F}\times Q}$ is well-quasi-ordered by $\leq_0$. Let us prove that $\leq$ and $\leq_0$ coincide in this case. Suppose that $w = ((f_x^w, h_x^w, q_x^w))_{x=1\ldots|w|}$ and $w' = ((f_x^{w'}, h_x^{w'}, q_x^{w'}))_{x=1\ldots|w'|}$ belong to $\widetilde{\mathcal{W}}^{\mathcal{F}}_{\mathcal{F}\times Q}$ and satisfy $w \leq_0 w'$. There exists an increasing injection $\phi$ from $\{1, \ldots, |w|\}$ into $\{1, \ldots, |w'|\}$ which is compatible with labels. Consider any letter $x < |w|$. Since $\phi$ preserves labels, we have $f_x^w = f_{\phi(x)}^{w'}$ (we call this function $f$) and $h_x^w = h_{\phi(x)}^{w'}$ (we call this function $h$). To prove that $\leq_0$ preserves path-composition, we just have to check that $f_x^w = f_{\phi(x)}^{w'} \circ f_{\phi(x)+1}^{w'} \circ \cdots \circ f_{\phi(x+1)-1}^{w'}$. Equivalently, by letting $h' = f_{\phi(x)+1}^{w'} \circ \cdots \circ f_{\phi(x+1)-1}^{w'}$ (the empty product being $\epsilon$), we want to prove that $f = f \circ h'$. Since $w$ is coherent, we have $h_{x+1}^w = h_x^w \circ f_x^w = h \circ f$. Since $w'$ is coherent, we have $h_{\phi(x+1)}^{w'} = h_{\phi(x)}^{w'} \circ f_{\phi(x)}^{w'} \circ f_{\phi(x)+1}^{w'} \circ \cdots \circ f_{\phi(x+1)-1}^{w'}$, hence $h_{\phi(x+1)}^{w'} = h \circ f \circ h'$. Since $\phi$ preserves labels, we have $h \circ f = h_{x+1}^w = h_{\phi(x+1)}^{w'} = h \circ f \circ h'$. We can then deduce that $f = f \circ h'$ by left-cancellation of the permutation $h$.

Let us assume that the induction hypothesis holds for $t-1$. Consider $\mathcal{F}$ with $t$ equivalence classes. We will prove that $\widetilde{\mathcal{W}}^{\mathcal{F}}_{\mathcal{F}\times Q}$ is well-quasi-ordered by $\leq$ using Nash-Williams' minimal bad sequence argument [14]. By contradiction, let $S = (w_i)_{i\in\omega}$ be an infinite bad sequence of $\widetilde{\mathcal{W}}^{\mathcal{F}}_{\mathcal{F}\times Q}$, minimal in the sense that for every $i \geq 1$, the $i^{\text{th}}$ word $w_i$ of this sequence is defined to be a word of minimal size such that there exists a bad sequence (with respect to $\leq$) starting with $w_1, \ldots, w_{i-1}, w_i$.

Set $w_i = ((f_x^{w_i}, (h_x^{w_i}, q_x^{w_i})))_{x=1\ldots|w_i|}$, and consider the minimal $p_i$ such that $f_{p_i}^{w_i} \in F_1$, if such a $p_i$ exists. The subword $w_i' = ((f_x^{w_i}, (h_x^{w_i}, q_x^{w_i})))_{x=1\ldots p_i}$, or $w_i' = w_i$ if $p_i$ is undefined, is called the *beginning* of $w_i$. Likewise, the *end* of $w_i$ is the subword $w_i'' = ((f_x^{w_i}, (h_x^{w_i}, q_x^{w_i})))_{x=p_i+1\ldots|w_i|}$, or the empty word if $p_i$ is undefined.

Let $X$ be the set of all ends of words in $S$. We now prove that $X$ is well-quasi-ordered by $\leq$. Let $\sigma$ be an infinite sequence of $X$. Let $\alpha = (z_i)_{i\in\omega}$ be an infinite extracted subsequence of $\sigma$ such that for all $i$, if $z_i$ is the end of $w_j$ and $z_{i+1}$ is the end of $w_k$, then $j \leq k$. Let $w_n$ be the word of which $z_1$ is the end. The sequence

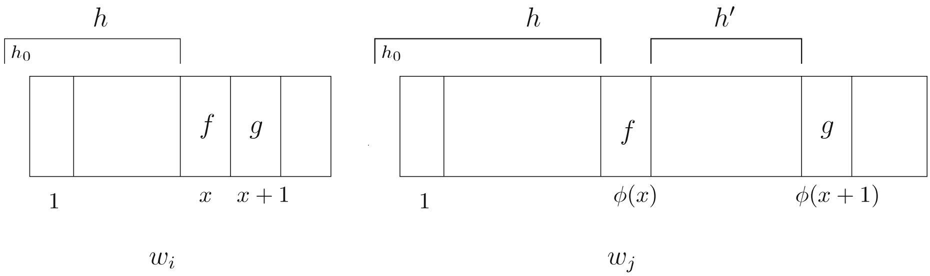


**Fig. 2** The words $w$ and $w'$

$(w_1, ..., w_{n-1}, z_1, z_2, ...)$ is good by the minimality of $S$. Since $S$ is bad, one cannot have $w_i \leq z_j$, or $w_i \leq w_j$ with $i < j$. Hence there exists a good pair $z_i \leq z_j$ with $i < j$, thus $X$ is well-quasi-ordered by $\leq$.

By Lemma 3, we know that $\mathcal{F} - F_1$ is closed under composition. Let $R = (F_1 \times X) \cup \{\triangle\}$ where $\triangle$ is a new marker. The set $R$ is ordered by $\leq_R$ as follows: $\triangle \leq_R \triangle$, and $(f, x) \leq_R (g, x')$ if $f = g$ and $x \leq x'$. Since $X$ is well-quasi-ordered, we have that $R$ and, hence $Q' = Q \times R$ are well-quasi-ordered. By our induction hypothesis, $\widetilde{\mathcal{W}}^{\mathcal{F}-F_1}_{(\mathcal{F}-F_1)\times Q'}$ is well-quasi-ordered by $\leq$, where the order $\leq_{Q'}$ on $Q'$ is defined by: $(q, r) \preceq_{Q'} (q', r')$ if $q \preceq_Q q'$ and $r \leq_R r'$.

Denote by $w_i'''$ the word of $\widetilde{\mathcal{W}}^{\mathcal{F}-F_1}_{(\mathcal{F}-F_1)\times Q'}$ which is the concatenation of the word $((f_x^{w_i}, (h_x^{w_i}, (q_x^{w_i}, \triangle))))_{x=1\ldots p_i-1}$ with the extra letter $(\varepsilon, (h_{p_i}^{w_i}, (q_{p_i}^{w_i}, (f_{p_i}^{w_i}, w_i''))))$ if $p_i$ exists, or $((f_x^{w_i}, (h_x^{w_i}, (q_x^{w_i}, \triangle))))_{x=1\ldots p_i-1}$ otherwise. Note that $w_i'''$ is coherent since $w_i$ is coherent.

As $\widetilde{\mathcal{W}}^{\mathcal{F}-F_1}_{(\mathcal{F}-F_1)\times Q'}$ is well-quasi-ordered by $\leq$, there exist $i < j$ such that $w_i''' \leq w_j'''$. We denote by $\phi$ the mapping from $w_i'''$ into $w_j'''$. Observe that if $w_i'''$ does not have an extra letter, we would directly have that $w_i \leq w_j$ which is impossible since $S$ is a bad sequence. Hence $p_i$ and $p_j$ exist. Let us now exhibit a mapping $\Phi$ from $w_i$ into $w_j$ which preserves labels and path-composition (see Fig. 2).

First, we set $\Phi$ to be the restriction of the function $\phi$ from $\{1, \ldots, p_i\}$ into $\{1, \ldots, p_j\}$. Observe that $\phi(p_i) = p_j$ since extra letters do not carry markers. Moreover, the extra letter $(\varepsilon, (h_{p_i}^{w_i}, (q_{p_i}^{w_i}, (f_{p_i}^{w_i}, w_i''))))$ is mapped to the extra letter $(\varepsilon, (h_{p_j}^{w_j}, (q_{p_j}^{w_j}, (f_{p_j}^{w_j}, w_j''))))$. In particular, we have $w_i'' \leq w_j''$. The mapping $\phi''$ from $w_i'' \leq w_j''$ which realizes $w_i'' \leq w_j''$ is our extension of $\Phi$ from $\{p_i + 1, \ldots, |w_i|\}$ into $\{p_j + 1, \ldots, |w_j|\}$.

Now $\Phi$ is completely defined. Moreover, it preserves labels by definition. We then just have to show that it also preserves path-composition, and more precisely path-composition exactly after the letter $p_i$, since the other cases are already taken into account by $\phi$ or $\phi'$.

We then have to show that $f_{p_i}^{w_i} = f_{\Phi(p_i)}^{w_j} \circ \cdots \circ f_{\Phi(p_i+1)-1}^{w_j}$ or equivalently $f_{p_i}^{w_i} = f_{p_j}^{w_j} \circ f_{p_j+1}^{w_j} \cdots \circ f_{\phi''(p_i+1)-1}^{w_j}$. Observe that this condition holds vacuously when $p_i = |w_i|$. Since the extra letter of $w_i'''$ is mapped to the extra letter of $w_j'''$, we have $f_{p_i}^{w_i} = f_{p_j}^{w_j}$ (we call this function $f$) and $h_{p_i}^{w_i} = h_{p_j}^{w_j}$ (we call this function $h$). We now let $h' = f_{p_j+1}^{w_j} \cdots \circ f_{\phi''(p_i+1)-1}^{w_j}$ (the empty product being $\epsilon$). Since the word $w_i$ is coherent, we have $h_{p_i+1}^{w_i} = h_{p_i}^{w_i} \circ f_{p_i}^{w_i} = h \circ f$. Since $w_j$ is coherent, we have $h_{\phi''(p_i+1)}^{w_j} = h_{p_j}^{w_j} \circ f_{p_j}^{w_j} \circ f_{p_j+1}^{w_j} \circ \cdots \circ f_{\phi''(p_i+1)-1}^{w_j}$. Hence $h_{\phi''(p_i+1)}^{w_j} = h \circ f \circ h'$. Since $\phi''$ preserves labels, we have $h \circ f = h^{w_i}_{p_i+1} = h^{w_j}_{\phi''(p_i+1)} = h \circ f \circ h'$. We now conclude by Lemma 4 that $f = f \circ h'$. This concludes the proof of Lemma 5 and Theorem 1. □

## 4 Trees on Functions

We extend in this section our results to trees. However, since the arguments are similar to the previous section, we will not give the same level of detail, especially concerning the verification of path-composition.

A *structured tree* is a finite tree where the children of a node are ordered from left to right. Our trees have their nodes labelled by a well-quasi-ordered set $Q$. We denote by $\mathcal{T}_Q^{\mathcal{F}}$ the set of structured rooted trees with nodes labelled by $\mathcal{F} \times Q$, where $\mathcal{F}$ is as usual a set of functions. A node $x$ is then labelled by a pair $l(x) = (f(x), q(x))$. We simply write $\mathcal{T}^{\mathcal{F}}$ when there is no additional label $Q$. The set of nodes of $T$ is denoted by $V(T)$. We write $x \wedge y$ for the least common ancestor of $x$ and $y$. We say that $(x, y)$ is an *arc* of $T$ when $x$ is the father of $y$. A sequence of nodes $z_0, z_1, ..., z_n$ is a *downward path* in $T$ if $(z_i, z_{i+1})$ is an arc, for every $i = 0, \dots, n-1$, and we say that $z_0$ is an *ancestor* of $z_n$ and that $z_n$ is a *descendant* of $z_0$. For such a downward path $z_0, z_1, ..., z_n$, we denote by $L(z_0, z_n)$ the composition $f(z_0) \circ f(z_1) \circ ... \circ f(z_{n-1})$.

Let us define a partial order $\leq$ on $\mathcal{T}_Q^{\mathcal{F}}$ which extends the order $\leq$ on words. Precisely, let us write that $T \leq T'$ if there exists an injection $\phi$ from $V(T)$ into $V(T')$ such that:

- $\phi$ preserves descendants.
- $\phi$ preserves least common ancestors, i.e. $\phi(x \wedge y) = \phi(x) \wedge \phi(y)$.
- $\phi$ preserves the *left/right order*, i.e. if $x$ and $y$ are not in descendant relation, and the branch of $x \wedge y$ containing $x$ is to the left of the one containing $y$, the same holds for the branches of $\phi(x) \wedge \phi(y)$ containing $\phi(x)$ and $\phi(y)$.
- $\phi$ preserves labels, i.e. $f(x) = f(\phi(x))$ and $q(x) \leq_Q q(\phi(x))$.
- $\phi$ preserves *path-composition*, i.e. for any arc $(x, y)$ in $T$, we have that $L(x, y) = L(\phi(x), \phi(y))$, i.e. $f(x) = L(\phi(x), \phi(y))$.

When $\phi$ satisfies all these properties except possibly path-composition, we simply write $T \leq_0 T'$. Kruskal's Tree Theorem asserts that $\leq_0$ is a well-quasi-order on the set of trees.

This more constrained order relation $\leq$ presents some analogies with the so-called *gap-condition embedding* studied by Kriz in [9]. For instance, when the class of functions $\mathcal{F}$ is totally ordered, and hence partitioned into $F_1, \dots, F_t$, the path-composition property implies that if $y$ is a child of $x$ and $f(x)$ belongs to $F_i$, then every fonction of the product $L(\phi(x), \phi(y))$ belong to classes with height at least $i$. It could be interesting to state a common generalization of these results, possibly involving ordinal functions.

**Theorem 2** *The set $\mathcal{T}_Q^{\mathcal{F}}$, where $Q$ is a well-quasi-order, is well-quasi-ordered by $\leq$ if and only if $\preceq$ is total on $\mathcal{F}$.*

*Proof* If $\preceq$ is not total on $\mathcal{F}$, the set of words, hence of trees, is not well-quasi-ordered as we have seen in the previous section. □

Assume now that $\preceq$ is total on $\mathcal{F}$. As for words, we transform a tree $T$ in $\mathcal{T}_Q^{\mathcal{F}}$ into a tree in $\mathcal{T}_{\mathcal{F}\times Q}^{\mathcal{F}}$. To every vertex $x$ of $T$ distinct from the root $r$, we give the extra label $L(r, x)$, the extra label of the root being $\epsilon$. More generally, we say that a rooted tree $T$ with nodes labelled by $\mathcal{F} \times \mathcal{F} \times Q$ is *coherent* if for every arc $(x, y) \in T$, with $l(x) = (f, h, q)$ and $l(y) = (g, h', q')$, we have $h' = h \circ f$. We denote by $\widetilde{\mathcal{T}}_{\mathcal{F}\times Q}^{\mathcal{F}}$ the set of coherent structured rooted trees with nodes labelled by $\mathcal{F} \times \mathcal{F} \times Q$. Theorem 2 is a consequence of the following result.

**Lemma 6** *If $Q$ is well-quasi-ordered, the set $\widetilde{\mathcal{T}}_{\mathcal{F}\times Q}^{\mathcal{F}}$ is well-quasi-ordered by $\leq$.*

Let us prove this by induction on $t$, the number of equivalence classes of $\mathcal{F}$.

When $t = 1$, the proof goes according to the case of words: by Kruskal's Theorem, $\widetilde{\mathcal{T}}_{\mathcal{F}\times Q}^{\mathcal{F}}$ is well-quasi-ordered by $\leq_0$, which again coincides with $\leq$.

Let us assume that the induction hypothesis holds for $t - 1$. Consider $\mathcal{F}$ with $t$ equivalence classes. We prove that $\widetilde{\mathcal{T}}_{\mathcal{F}\times Q}^{\mathcal{F}}$ is well-quasi-ordered by $\leq$ using Nash-Williams' minimal bad sequence argument. By contradiction, let $S = (T_i)_{i\in\omega}$ be an infinite bad sequence of $\widetilde{\mathcal{T}}_{\mathcal{F}\times Q}^{\mathcal{F}}$, minimal in the sense that for $i \geq 1$, the $i^{\text{th}}$ tree $T_i$ of this sequence is defined to be a tree of minimal size for which there exists a bad sequence starting with $T_1, ..., T_{i-1}, T_i$.

A *branching vertex* is a node $x$ labelled by $(f, w, q)$, with $f \in F_1$, and with no ancestor having a label $(g, w', q')$ with $g$ in $F_1$. A *branch* is a subtree which is rooted in any child of a branching vertex. We denote by $T_i'$ the subtree of $T_i$ obtained by deleting all the branches of $T$ (see Figs. 3 and 4). Let $X$ be the set of all branches of the trees in $S$. As in the previous section, the minimality of $S$ ensures that $X$ is well-quasi-ordered by $\leq$.

Recall that $\mathcal{F} - F_1$ is closed under composition. We denote by $Seq(X)$ the set of sequences of $X$. The usual order $\leq_{seq}$ on $Seq(X)$ is defined as follows: for any two sequences $(x_1, ..., x_m)$ and $(x_1', ..., x_n')$ of $X$, we have $(x_1, ..., x_m) \leq_{seq} (x_1', ..., x_n')$ if there exists an increasing injection $\sigma$ from $\{1, \ldots, m\}$ to $\{1, \ldots, n\}$ such that $\forall i \in \{1, \ldots, m\}\ x_i \leq x'_{\sigma(i)}$. Let $R = ((F_1 \times Seq(X)) \cup \{\triangle\})$ with $\triangle$ a new marker. Let $\leq_R$ be the following order on $R$: $\triangle \leq_R \triangle$, and $(f, x) \leq_R (g, x')$ if $f = g$ and $x \leq_{seq} x'$. By Higman's Theorem $\leq_{seq}$ is a well-quasi-order on $Seq(X)$, hence $\leq_R$ is a well-quasi-order on $R$. By our induction hypothesis $\widetilde{\mathcal{T}}_{(\mathcal{F}-F_1)\times Q'}^{\mathcal{F}-F_1}$, with $Q' = Q \times R$, is well-quasi-ordered by $\leq$.

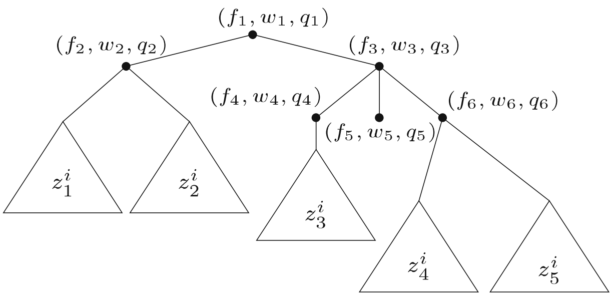


**Fig. 3** The tree $T_i$ ($f_2$, $f_4$, $f_5$ and $f_6$ are in $F_1$, while $f_1$ and $f_3$ are not)

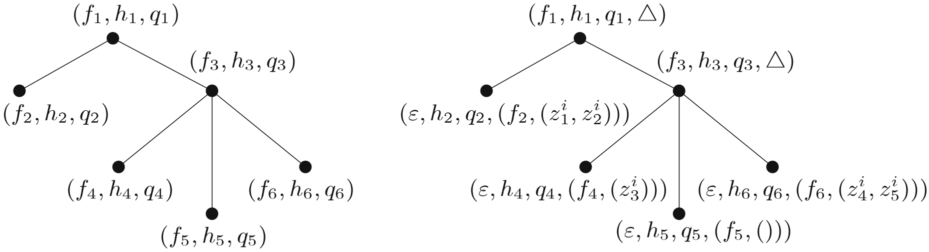


**Fig. 4** The trees $T_i'$ (*left*) and $T_i''$ (*right*)

Denote by $T_i''$ the tree obtained from $T_i'$ by modifying the labels of its nodes as follows:

- Labels of internal nodes are modified from $(f, h, q)$ into $(f, h, (q, \triangle))$.
- Labels of leaves $x$ are modified from $(f, h, q)$ into $(\epsilon, h, (q, s))$, where $s$ is the sequence of branches of the branching vertex $x$ (see Fig. 4).

Observe that the trees $T_i''$ are coherent, and since $\widetilde{\mathcal{T}}^{\mathcal{F}-F_1}_{(\mathcal{F}-F_1)\times Q'}$ is well-quasi-ordered by $\leq$, there exist $i < j$ such that $T_i'' \leq T_j''$. The node injection $\phi$ from $T_i''$ into $T_j''$ can be extended, via the labels $s$, to the vertices of the branches of $T_i$ and $T_j$. By construction, $\phi$ directly gives $T_i \leq_0 T_j$, and coherence property, as for words, gives path-composition. Thus $T_i \leq T_j$, contradicting the badness of $S$. This completes the proof of Lemma 6 and Theorem 2.

## 5 NLC with Restricted Relabelling Functions

We can see $\mathrm{NLC}_k^{\mathcal{F}}$ expressions as binary trees, where the leaves are labelled by $\bullet_i$, the nodes of degree 1 by $\circ_f$, and the nodes of degree 2 by $\chi_S$. To fit in the framework of the previous section, we add an extra label to every node of such an $\mathrm{NLC}_k^{\mathcal{F}}$ construction tree, to see $\chi_S$ and $\bullet_i$ as identity relabelling functions. For this, replace $\bullet_i$ with $(\varepsilon, \bullet_i)$, $\circ_f$ with $(f, \circ_f)$ and $\chi_S$ with $(\varepsilon, \chi_S)$. Such a tree is a *construction tree* for the graph corresponding to this $\mathrm{NLC}_k^{\mathcal{F}}$ expression. Let $T_G$ be a construction tree for a graph $G$. Let $x$ be a vertex of $G$ which corresponds to the leaf $x'$ of $T_G$ of label $\bullet_i$ and $y$ be an ancestor of $x'$ in $T_G$. When we apply the operation corresponding to the node $y$ of $T_G$ to the vertex $x$, the color of $x$, denoted by $c_x(y)$ is exactly $L(y, x')(i)$.

**Lemma 7** *Let $G$ and $H$ be two $NLC_k$ graphs together with their $NLC_k^{\mathcal{F}}$ construction trees $T_G$ and $T_H$. If $T_G \leq T_H$, then $G \leq_i H$.*

*Proof* Let $\phi$ be an injection from $V(T_G)$ into $V(T_H)$. The restriction of $\phi$ on the leaves of $T_G$ can be seen as an injection from $V(G)$ into $V(H)$. Let $x, y$ be two vertices of $G$, with $x$ on the left of $y$ in $T$. Then $x$ and $y$ are neighbours in $G$ if and only if their least common ancestor in $V(T_G)$ is a node labelled by $\chi_S$ with $(c_x(x \wedge y), c_y(x \wedge y)) \in S$. As $\phi$ preserves labels, path composition and right/left order, this is the case if and only if $\phi(x)$ and $\phi(y)$ are neighbours in $H$. So $G \leq_i H$. □

Theorem 2 and Lemma 7 immediately give that if $\preceq$ is total on $\mathcal{F}$, then $\mathrm{NLC}_k^{\mathcal{F}}$ is well-quasi-ordered by $\leq_i$. Moreover, since we can always add some extra vertex-labels, we obtain that:

**Corollary 1** *$NLC_k^{\mathcal{F}}$ is $\infty$-well-quasi-ordered when $\preceq$ is total on $\mathcal{F}$.*

Let us now characterize the sets $\mathrm{NLC}_k^{\mathcal{F}}$ which are well-quasi-ordered:

**Theorem 3** *$NLC_k^{\mathcal{F}}$ is well-quasi-ordered by $\leq_i$ if and only if $\preceq$ is total on $\mathcal{F}$.*

*Proof* Assume that $\preceq$ is not total on $\mathcal{F}$, and let $(f, g)$ be an incomparable pair for the relation $\preceq$. Let us show that for any $n \geq 1$, the graph $G_n$ consisting of a path of length $n$ with two pending vertices attached to each extremity is in $\mathrm{NLC}_k^{\mathcal{F}}$. The set $\{G_n | n \in \omega\}$ is clearly not well-quasi-ordered.

Assume first that $f = g$, that is $|Im(f^2)| < |Im(f)|$. Hence there exist $x, y \notin Im(f^2)$, such that $f(x) = y$. To construct $G_n$, start from two vertices labelled by $y$ and one vertex labelled by $x$, and apply $\chi_{\{(x,y)\}}$ to form a path of length 2. Relabel by $f$. Observe that the two extremities of this path will never again be labelled by $x$ or $y$ since their labels will stay within $Im(f^2)$. Add a vertex labelled by $x$, apply again $\chi_{\{(x,y)\}}$. This adds an edge between the middle vertex of the path and the new one. Then relabel by $f$, and keep on building the path up to the desired length. The point is that after any step, the extremity of the path is distinguished by its label from the other vertices. When the last vertex of the path has been added (with label $x$ as usual), add two isolated vertices with label $x$ for instance, and apply $\chi_{\{(x,x)\}}$, completing the graph $G_n$.

We can generalize this when $f$ and $g$ are distinct. An *$f$-class* is a subset $S$ of $\{1, \ldots, k\}$ such that $|f(S)| = 1$ and which is maximum with respect to inclusion. Since $f \npreceq g$, there exists an $f$-class disjoint from $Im(g)$. Let $x$ be one of its elements. Similarly, let $y$ be in a $g$-class disjoint from $Im(f)$. Let us prove by induction that for every $n$, we can build paths of length $2n$ where the last vertex is labelled by $y$ and the other vertices are labelled in the set $Im(f)$. We will therefore be able to build graphs $G_n$ for arbitrarily large $n$ then, adding two pending nodes on each extremity as in the previous case.

To start with, take a vertex $z \in Im(f)$, add a vertex $y$, and apply $\chi_{\{(z,y)\}}$. Now assume that we have a path of length $2n$ where the last vertex is labelled by $y$ and the other vertices by some elements of $Im(f)$. Relabel by $g$. Observe that the last vertex is still distinguished from the rest. Add a vertex $x$. At this point, no other vertex has label $x$, since $x$ is not in $Im(g)$. Apply $\chi_{\{(x,y)\}}$. This constructs a path of length $2n + 1$. Now relabel by $f$, add a vertex $y$ and apply $\chi_{\{(y,x)\}}$ in order to get a path of length $2(n + 1)$ which satisfies the induction hypothesis. □

This proof actually shows the following corollary:

**Corollary 2** *If $\preceq$ is not total on $\mathcal{F}$, then $NLC_k^{\mathcal{F}}$ contains arbitrarily large paths.*

Also, if $\mathrm{NLC}_k^{\mathcal{F}}$ contains arbitrarily large paths then $\mathrm{NLC}_k^{\mathcal{F}}$ is not 2-well-quasi-ordered, which completes the proof of the equivalences claimed in the introduction.

## 6 Further Well-Quasi-Ordering Problems

As we have mentioned before, one important motivation for the notion of the 2-well-quasi-ordered class is that it can be described by a finite set of bounds [15].

**Proposition 3** *Let $I$ be a 2-well-quasi-ordered induced subgraph ideal. There are finitely many graphs in the complement $\bar{I}$ of $I$ which are minimal with respect to the induced subgraph relation.*

*Proof* By contradiction, we assume that the set $B$ of minimal graphs in $\bar{I}$ is infinite. For every graph $G$ in $B$, choose a vertex, color its neighbours red and its non-neighbours black, and delete it. Call the resulting bicolored graph $G'$. The set $B' = \{G' | G \in B\}$ is infinite, and consists of graphs whose underlying graphs are in $I$, by minimality of the graphs in $B$. As $I$ is 2-well-quasi-ordered, there exist two graphs $G'_1$ and $G'_2$ in $B'$, such that $G'_1$ is a colored induced subgraph of $G'_2$. Hence $G_1$ is an induced subgraph of $G_2$, contradicting the fact that $G_2$ is in $B$. □

We call these graphs the *forbidden graphs of $I$*, and denote by $F(I)$ the set of these graphs. The same argument shows that $I \cup F(I)$ is well-quasi-ordered when $I$ is 2-well-quasi-ordered.

Proposition 3 implies that any 2-well-quasi-ordered induced subgraph ideal is polynomially recognizable. This means in particular that for a set $\mathcal{F}$ totally quasi-ordered by $\preceq$, the class $\mathrm{NLC}_k^{\mathcal{F}}$ is polynomially recognizable.

The following question would give an answer to Pouzet's conjecture.

**Conjecture 4** *If $\mathcal{G}$ is a 2-well-quasi-ordered induced subgraph ideal, there exists an $\infty$-well-quasi-ordered set $NLC_k^{\mathcal{F}}$ which contains $\mathcal{G}$.*

This problem seems hard. One first step would be to show the following:

**Conjecture 5** *No class of graphs closed under induced subgraph, and of unbounded clique-width, is 2-well-quasi-ordered.*

The next step would be to show that if indeed a subclass of $NLC_k$ is 2-well-quasi-ordered, then it is contained in some $\infty$-well-quasi-ordered set $\mathrm{NLC}_{k'}^{\mathcal{F}}$.

Recall that clique-width and NLC-width are equivalent from the point of view of boundedness. We present the conjectures of this final section in terms of clique-width, the more familiar of the two, while we used NLC-width in our proofs for technical convenience.

Concerning classes of unbounded clique-width, the following problem (which generalizes Conjecture 5) is of independant interest:

**Problem 6** Is it true that every class of graphs closed under induced subgraph, and of unbounded clique-width, is not well-quasi-ordered by the induced subgraph relation?

Let us say that an ideal $I' \subseteq I$ is a *sub-ideal* of an ideal $I$, and a *strict sub-ideal* if $I' \subsetneq I$. To prove Conjecture 5, one can actually restrict to minimal such classes:

**Proposition 7** *Let I be a* 2*-well-quasi-ordered induced subgraph ideal of unbounded clique-width. There exists a sub-ideal $I'$ of I of unbounded clique-width such that every strict sub-ideal of $I'$ is of bounded clique-width.*

*Proof* Since $I$ is well-quasi-ordered, every non empty collection of sub-ideals of $I$ has a minimal element by Higman's Theorem. This fact applied to the collection of sub-ideals of $I$ with unbounded clique-width yields the result. □

Thus the following conjecture implies Conjecture 5:

**Conjecture 8** *No ideal $\mathcal{G}$ verifies all of the following properties:*

- *$\mathcal{G}$ has unbounded clique-width*
- *$\mathcal{G}$ has a finite number of forbidden graphs*
- *Every strict sub-ideal of $\mathcal{G}$ has bounded clique-width*

For further discussion on minimal ideals of unbounded clique-width, see [8] for instance.

Finally, let us mention a question which would extend further Pouzet's conjecture. The answer for cographs can be found in [17].

A quasi-order $Q$ is a *better-quasi-order* if the class of countable ordinals labelled by $Q$ is a well-quasi-order [14].

**Conjecture 9** *Let $\mathcal{G}$ be a class of countable graphs. If the class of finite induced subgraphs $\mathcal{G}_F$ of $\mathcal{G}$ is* 2*-well-quasi-ordered, then $\mathcal{G}$ is better-quasi-ordered for every better-quasi-ordered vertex-labelling.*

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