Finitely Monotone Properties

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Abstract

A characterization of definability by positive first order formulas in terms of Fraïssé-Ehrenfeucht-like games is developed. Using this characterization, an elementary, purely combinatorial, proof of the failure of Lyndon's Lemma (that every monotone first order property is expressible positively) for finite models is given. The proof implies that first order logic is a bad candidate to the role of uniform version of positive Boolean circuits of constant depth and polynomial size.

Although Lyndon's Lemma fails for finite models, some similar characterization may be established for finitely monotone properties, and we formulate a particular open problem in this direction.

1 Introduction

In the Winter of 92/93, I taught a graduate course in Finite Model Theory at the Mathematics Department of UCLA. Although the experience was very satisfying, at least for me, I continued to feel somewhat unsatisfied about a few things, among them the most important for me was the result by Ajtai and Gurevich [2] that Lyndon's Lemma fails for finite models.

The classical result by Roger Lyndon [9] is that any first order formula, monotone in a certain predicate, is equivalent to some formula of the same signature that is positive in the predicate. Apart from the fact that the lemma is used in proofs of Lyndon Interpolation Theorem, it reflects a very interesting logical phenomenon and is hence extremely interesting by itself.

It is not hard to show that the class of monotone first order formulas is recursively enumerable, and not recursive. And then the fact that, however, a certain recursive subclass of the class of monotone formulas happens to semantically cover the whole class of monotone formulas is a sort of unexpected and nice. That the particular class of positive formulas happens to be such a subclass is then of a second importance.

One of important applications of Lyndon's Lemma is as follows. Monotonicity of a formula guarantees existence of the so called *least fixpoint.* One may consider the extension of first order logic by least fixpoints of monotone formulas then; according to Lyndon's Lemma, one may further restrict the class of monotone formulas to positive, and this makes the definition of the *least fixpoint logic* syntactic (see [11]). Notice, that any other *recursive* semantically full subclass of monotone formulas would do no worse.

And this was the original motivation for raising the question about the status of Lyndon's Lemma in the case of finite models (see [3]).

Fortunately, for this particular application, the solution was found: Gurevich and Shelah [8] showed that, although in finite models monotone formulas cannot be translated to positive, the least fixpoints of the two classes of formulas coincide.

But the use of monotone formulas is not limited to least fixpoint logic. For instance, under certain formalizations of the notion of knowledge, true monotone sentences characterize exactly the set of the consequences of an incomplete knowledge (first order expressible consequences, to be exact).

According to the well known classification by Kolaitis and Vardi, the result that Lyndon's Lemma fails for finite models should be attributed to the *negative* branch of Finite Model Theory that deals with (unfortunately, numerous) classical theorems that fail for finite models.

It should be mentioned that, unlike most other classical results that discontinue to hold for finite models, and whose refutation is kind of immediate (Compactness, Craig Interpolation, Gödel Completeness, etc.), Lyndon's Lemma resisted all attempts to refute it for many years. And the proof Ajtai and Gurevich [2] finally came up with is hairy indeed. Not only that the proof is hard and extremely involved (which one might expect, given that among immediate corollaries to the refutation was the result of Furst, Saxe, and Sipser [6] (see also Ajtai [1]) that, basically, constant-depth positive simulation of monotone constant-depth Boolean circuits leads to a non-polynomial explosive growth in the size of circuits), but it relies a great deal on deep results from foreign areas of Mathematics, like Probability Theory, Lattice Theory, and Analytic Number Theory, to name only a few. Moreover, even the structures of the counter-exampling class (in

*This work has been partially supported by NSF Grant CCR 9403809.

1043-6871/95 $4.00 © 1995 IEEE

which monotone formulas define generally not positive Borel point-sets) are somewhat hard to visualize.

In this paper we present a new refutation of Lyndon's Lemma for finite models, that is based on simple ideas, as outlined below.

First of all we explore an alternative characterization of expressibility by positive formulas in terms of Fraïssé-Ehrenfeucht-like games. It turns out that a minor modification of the classical Fraïssé-Ehrenfeucht games on pairs of models captures exactly the positive preservation.

Then we exemplify the use of these new games on the class of finite structures that we call *grids*. We show that any positive formula that holds in all grids does also hold in some *reduced* grid. Although the idea of the winning strategy for Duplicator seems in this case to be sort of straightforward, an accurate proof, as is usual with Fraïssé-Ehrenfeucht games, requires some efforts. The argument is, however, purely combinatorial.

In some sense, grids generalize segments of natural numbers with successor and $<$. In this sense, reduced grids generalize the segments in which successor behaves the same way, but $<$ is made undefined on some pairs.

Finally, we present a finitely monotone formula that holds in all grids, but does not hold in any reduced grid. This part is relatively simple, and generalizes the following idea. The difference between total linear order in the segments of natural numbers and its incomplete subset can be captured by the transitivity axiom. This axiom is not monotone. However, its nonmonotonicity applies only to the segments where $<$ is "over-defined". But these latter structures can be captured by a monotone formula false in all "under-defined" segments. So the disjunction of the transitivity axiom and the latter formula is (1) monotone, and (2) holds in all "standard" segments, but not in any "under-defined".

As a corollary, we get the refutation of Lyndon's Lemma.

Then, the results of the present paper allow us to negatively answer the question that was left open in [2], namely, *is every first order formula that defines positive Borel point-sets of a fixed rank necessarily (finitely) equivalent to a positive formula?* Indeed, it turns out that the monotone sentence that is shown in this paper to be not equivalent to any positive, defines however positive Borel point-sets of rank 4.

The generalization of the classical Borel hierarchy to finite topological spaces [7] is relevant to circuit complexity (see [12]). And now, combining the result of [2] and of the present paper, we can conclude that *first order logic is a bad candidate to the role of uniform version of positive Boolean circuits of constant depth and polynomial size,* because both monotone (Ajtai and Gurevich) and positive (present paper) first order formulas fail to adequately capture uniform classes of positive Boolean circuits of constant depth and polynomial size: the former express too much, and the latter too little.

These issues are being dealt with in Conclusion.

Now, from the viewpoint of the *positive* branch of Finite Model Theory, the question of characterization of finitely monotone formulas remains open.

For one thing, as we mentioned, any recursive semantically full subclass of monotone formulas would serve as well as positive formulas in many cases. So, it is an **open problem**, *does there exist a recursive subclass of finitely monotone formulas such that every finitely monotone formula is finitely equivalent to some in this subclass.* Herb Enderton noticed that, mathematically, this question is equivalent to the question of the existence of an r.e. subclass of such monotone formulas[1].

Similar questions can be asked about other interesting non-r.e. subclasses of formulas, for instance, distributive formulas, formulas preservable under substructures, etc.

2 Fraïssé-Ehrenfeucht Games for Positive Formulas

It is well-known that first order equivalence of models (elementary equivalence) can be alternatively characterized in terms of specific games of two players, commonly referred to as *Fraïssé-Ehrenfeucht games.* The technique was developed originally in [5] (see also [4]).

The goal of this section is to establish a similar alternative characterization of expressibility by positive formulas.

First of all, we need some definitions.

Definition 2.1 (Positive formula) *Let $\Omega = \Delta \bigcup \Theta$ be a signature, Δ and Θ be disjoint, and let Θ be relational.*

A first order formula in the signature Ω is called positive in Θ, iff in this formula no predicate symbol in Θ lies within the scope of a negation.

Normally, formulas positive in just one predicate symbol are considered.

Definition 2.2 (Quantifier Depth)
We consider first order formulas over a certain signature. The quantifier depth of a formula is defined by induction as follows.

1. *if the formula is quantifier-free, its quantifier depth is 0;*
2. *negations preserve the quantifier depth;*
3. *the quantifier depth of a conjunction, or a disjunction, of two formulas, is the maximum of the quantifier depths of these two formulas;*
4. *if the quantifier depth of φ is n, then the quantifier depth of $(\exists x)(\varphi)$ and of $(\forall x)(\varphi)$ is $n+1$.*

[1]The construction is similar to the one used to prove that any r.e.-axiomatizable theory is recursively axiomatizable.

Definition 2.3 (positive n-preservation) *Let A, B be two models of a certain relational signature $\Omega = \Delta \bigcup \Theta$, Δ and Θ be disjoint, and let $\vec{a} = a_1, a_2, \ldots, a_k$, $\vec{b} = b_1, b_2, \ldots, b_k$ be elements of $|A|$, $|B|$, respectively.*

We say that $(A, \vec{a})$ is Θ-positively n-preservable by $(B, \vec{b})$, iff any first order Θ-positive formula $\varphi(x_1, x_2, \ldots, x_k)$ of quantifier depth n with k free variables $x_1, x_2, \ldots, x_k$ that is true in A under $x_1, x_2, \ldots, x_k := a_1, a_2, \ldots, a_k$, is also true in B under $x_1, x_2, \ldots, x_k := b_1, b_2, \ldots, b_k$.

If $k = 0$, we will simply say that A is Θ-positively n-preservable by B.

The goal of this section is to characterize positive n-preservation in terms of some games of two players.

Let $\Omega = \Delta \bigcup \Theta$ be a relational signature, Δ and Θ be disjoint, and let A, B be two models of the signature. Let $\vec{a} = a_1, a_2, \ldots, a_k$, $\vec{b} = b_1, b_2, \ldots, b_k$ be elements of $|A|$, $|B|$, respectively. Consider the following game of two players on the pair $\left\langle (A, \vec{a}), (B, \vec{b}) \right\rangle$, called n-positive pebble game, for $n \geq 0$.

Each player initially has n pebbles, numbered $1, 2, \ldots, n$. In the first step of the game, the first player, whom we will call *Spoiler*, chooses a model among A, B, and places his pebble with number 1 onto some element of the model. Then the second player, whom we will call *Duplicator*, takes the other model, and places his pebble with number 1 onto some element of this model.

After $(r-1)$-th step of the game, for $1 < r \leq n$, each player retains the pebbles numbered $r, r+1, \ldots, n$, while the pebbles numbered $1, 2, \ldots, r-1$ are somehow placed onto elements of the models A, B. Then in the r-th step of the game, again, the first player chooses a model among A, B, and places his pebble r onto some element of the model. Then the second player takes the other model, and places his pebble r onto some element of this model.

The n-positive pebble game ends after its n-th step. After the game ends, each of the models A, B has pebbles numbered $1, 2, \ldots, n$ placed somehow onto its elements. Let $a_{k+1}, a_{k+2}, \ldots, a_{k+n}$ be the elements of A covered with, respectively, pebbles numbered $1, 2, \ldots, n$, and let $b_{k+1}, b_{k+2}, \ldots, b_{k+n}$ be the elements of B covered with, respectively, pebbles numbered $1, 2, \ldots, n$.

Let A' be the substructure of A generated with $a_1, a_2, \ldots, a_{k+n}$, and let B' be the substructure of B generated with $b_1, b_2, \ldots, b_{k+n}$.

By definition, the second player, Duplicator, wins the game, if the mapping $a_i \mapsto b_i$, for $i = 1, 2, \ldots, k+n$, is correctly defined and this mapping is:

1. an isomorphism of Δ-reducts of A' and B';
2. a homomorphism of Θ-reduct of A' onto Θ-reduct of B'.

Otherwise the first player, Spoiler, wins the game[2].

[2] After reading a draft version of the paper Peter Clote pointed out that a similar game is described in [10]. In the cited paper, Gregory McColm also proved the if-direction of Theorem 2.4.

Theorem 2.4 *Let A, B be two models of a finite relational signature $\Omega = \Delta \bigcup \Theta$ (with or without constants), Δ and Θ be disjoint, and let $\vec{a} = a_1, a_2, \ldots, a_k$, $\vec{b} = b_1, b_2, \ldots, b_k$ be elements of $|A|$, $|B|$, respectively.*

$(A, \vec{a})$ is Θ-positively n-preservable by $(B, \vec{b})$, iff Duplicator has a winning strategy in the n-positive pebble game on the pair $\left\langle (A, \vec{a}), (B, \vec{b}) \right\rangle$.

PROOF: *The proof will appear in the full version. Q.E.D.*

3 Grids

The goal of this section is to present pairs of finite structures, such that one element of the pair is positively n-preservable by the other (for some n), and still the structures are different. We call these structures *grids*.

Definition 3.1 *We consider the signature $\Omega = \langle H, < \rangle$, where H and $<$ are two binary predicate symbols. A* grid *is a structure of the domain*

$$\{(m, n) \mid m = 1, 2, \ldots w; n = 1, 2, \ldots h\},$$

where w and h are called, respectively, the width and the hight of the grid, with the two predicates defined as follows:

$$H((m, n), (m_1, n_1)) \Leftrightarrow \begin{cases} m = m_1 \text{ and } n_1 = n + 1 \\ \quad \text{or} \\ m_1 = m + 1 \text{ and } n = n_1, \end{cases}$$

while $<$ is the transitive closure of H.

A reduced grid *is a grid with the relation $<$ made undefined in some pairs on which it was defined in the grid.*

Lemma 3.2 *For any n there exist a grid P and a reduced grid R such that P is positively n-preservable by R.*

PROOF: So a natural number n, the number of steps in a game, is given. Let P be a grid of height 2^{n+1}, and of width 2^{n+2}. Further let R be a reduction of P such that $<^R = <^P \setminus \{((2^{n+1}, 1), (2^{n+1}, 2^{n+1}))\}$. In other words, R is P with the relation $<$ made undefined in exactly one pair of the middle-top, and middle-bottom, elements. For convenience, let's refer to these elements as a and b, thus, $a = (2^{n+1}, 1)$ and $b = (2^{n+1}, 2^{n+1})$.

By Theorem 2.4, it suffices to demonstrate that Duplicator has a winning strategy in the n-game on the pair of models.

The idea behind the strategy is as follows. Duplicator maintains a sufficiently large rectangle in the upper middle of the grid, such that b is covered by the rectangle. If Spoiler pebbles an element outside the rectangle, Duplicator answers by duplicating the move. If however Spoiler pebbles an element *inside* the rectangle, Duplicator answers by shifting the element, as follows. When answering in R, Spoiler shifts the element one step to the right (that is, answers $(x+1, y)$ to (x, y)), and when answering in P, Spoiler shifts the element one step to the left. So that, intuitively, inside the rectangle all elements of R represent the corresponding elements of P, shifted to the right.

Of course, in implementing such a strategy, it is important to not let the elements be too close to the borders of the rectangle. Because otherwise, we might be forced into a contradiction. So if an element close to a border of the rectangle is pebbled, Duplicator answers as described above, and then redraws the rectangle anew by moving the border in question further from the pebbled element.

This strategy is obviously winning. Indeed, pebbled elements within the rectangle satisfy exactly the same relations in both structures. And so do the elements outside the rectangle. Now, if one pebbled element $x \in P$ is outside the rectangle, and another pebbled element $y \in P$ is inside, and $x < y$, then their counterparts are in the same relation for the obvious reason that the right shift cannot spoil the relation.

The only technicality in implementing this strategy is to arrange the borders of the rectangle and to formalize the notion of being close to a border in such a way that the ends meet.

Duplicator initially sets the rectangle as the following subset of the elements of the grid:

$$\left\{(x, y) \mid 2^n < x < 3 \times 2^n, 2^n < y < 2^{n+1}\right\}.$$

By induction, at the step k, it is enforced that all pebbled elements are not within the distance 2^{n-k} of the borders.

Now suppose the new pebbled element sits close to a border (or to two borders). Then we obviously can move the border or the borders away from this element at the distance 2^{n-k}. Indeed, the distance from any old border to the closest old pebbled element is, by induction, at least 2^{n-k+1}, and even if we move the border all 2^{n-k} steps away from the newly pebbled element, it remains at least $2^{n-k+1} - 2^{n-k} = 2^{n-k}$ far from all the old pebbled elements.

This proves the lemma. *Q.E.D.*

4 Monotone Embracing of Grids

In this section we construct a finitely monotone (in $<$) first order sentence that holds in all grids, but does not hold in any reduced grid.

An immediate corollary will be that this sentence is finitely equivalent to no sentence positive in $<$, and that Lyndon's Lemma fails for finite models.

We will not try to axiomatize the class of grids, rather, we will axiomatize a larger class of finite models, that does not however include reduced grids.

Definition 4.1 (Monotone formula) *Let $\Omega = \Delta \bigcup \Theta$ be a signature, Δ and Θ be disjoint, and let Θ be relational.*

A first order formula $\phi(\vec{x})$ in the signature Ω is called (finitely) monotone in Θ, iff for any (finite) model M of the signature Δ and for any $\vec{a} \in |M|^{|\vec{x}|}$, if $\phi(\vec{a})$ holds in M under a certain interpretation of all symbols in Θ, it continues to hold under any extension of this interpretation.

Normally, formulas monotone in just one predicate are considered.

So, consider the signature $\Omega' = \langle H, < \rangle$. Let $\alpha(x)$ be the following formula:

$$(\forall y)(y \leq x \longrightarrow$$
$$(\forall z)(H(z, y) \longrightarrow (\forall u)(u \leq z \longrightarrow u < y \wedge u < x)))$$

The intention of $\alpha(x)$ is to assert that $<$ is the transitive closure of H, on the set of H-predecessors of x. An *H-predecessor* of x is either a direct H-predecessor of x (that is an element y such that $H(y, x)$), or a predecessor of a direct predecessor of x. It is clear that $\alpha(x)$ cannot, generally, assert that, because H may be "wild". However, we only use $\alpha(x)$ when the "wild" cases are already ruled out, as we will see below.

Consider the following list of axioms:

$$(\exists x, y)(H(x, y) \wedge H(y, x)) \qquad H_1^-$$

$$(\exists x, y)(x < y \wedge y < x) \qquad <_1^-$$

$$(\exists x, y)((\forall z)(\neg H(z, y)) \wedge x < y) \qquad <_2^-$$

$$(\exists x, z)((\forall y)(H(y, x) \longrightarrow \alpha(y)) \wedge z < x \wedge$$
$$(\forall y)(H(y, x) \longrightarrow \neg z \leq y)) \qquad <_3^-$$

$$(\forall x, y)(H(x, y) \longrightarrow x < y) \qquad <_1^+$$

$$(\forall x, y, z)(x < y \wedge y < z \longrightarrow x < z) \qquad <_2^+$$

Let δ^- be the disjunction of all $-$-axioms from the above list, δ^+ be the conjunction of all $+$-axioms, and let δ be $\delta^- \vee \delta^+$.

Obviously, the axioms were coined with the grids in mind, and before we embark on the formal stuff, let us spend some time trying to build a good intuition about the axioms.

Intuitively, we want δ to be false in all reduced grids, and to hold in all other models. We can't quite achieve the goal, for the relation H may be too weak, but we are able to come very close to the desired effect.

The clear intention of the axioms H_1^-, $<_1^-$, and $<_2^-$ is to catch the cases when the model is *obviously not* a grid, whether reduced or not. Thus, H_1^- holds in all the models where H is not anti-symmetric, $<_1^-$ does the same with $<$, and $<_2^-$ captures the case when an element without an H-predecessor has a $<$-predecessor.

Together these axioms capture many cases when a model is not any close to a grid.

Now the idea is to take a model where all these three axioms are false, and see what may be wrong with such a model.

Observe that if the first three axioms are false, $\alpha(x)$ implies that on the set of H-predecessors of x, $<$ is indeed the transitive closure of H.

Then $<_3^-$ implies that there exist elements z, x, such that $z < x$ but z is not among the H-predecessors of x. Clearly, this rules out the possibility that $<$ can be extended exactly to the transitive closure of H, for $<$ already is "over-defined". It turns out that the property is monotone in $<$ (although in an extension it may be witnessed by another pair of x and z).

So much for the $-$-axioms. The $+$-axioms are more straightforward.

From the informal description it may be seen that δ^- is monotone. Let us now see how the formal proof is carried out.

Lemma 4.2 *Let M be a model of the signature $\Omega' = \langle H, <\rangle$ such that $M \not\models H_1^- \vee <_1^- \vee <_2^-$. Let b be an element of M such that $\alpha(b)$.*

Among the H-predecessors of b, together with b, there is no H-cycle.

PROOF: Suppose there is an H-cycle that involves only H-predecessors of b, and possibly b itself. Then there exists a sequence $a_1 H a_2 H \ldots H a_k = b$ of elements of this set, where two elements coincide.

By induction, since $\alpha(b)$, we can show that $a_i < b$, for all $i < k$. Then, using another induction, we can show that $a_i < a_j$ for $i < j$. But then, there exists an element a_i in the sequence such that $a_i < a_i$. Which makes $<_1^-$ true. *Q.E.D.*

Lemma 4.3 *δ^- is finitely $<$-monotone.*

PROOF: Suppose that δ^- is not monotone. Then there exist finite models M and N whose $\langle H\rangle$-reducts coincide, with $<^M \subset <^N$, such that $M \models \delta^-$, but $N \not\models \delta^-$. Since all $-$-axioms, except for $<_3^-$, are $<$-positive (and hence $<$-monotone), that means that $M \models <_3^-$, and $M \not\models H_1^- \vee <_1^- \vee <_2^-$.

But that means that in M, there exist elements a, b in the model M such that a is not among the H-predecessors of b, but $a < b$ in M. It is also guaranteed that among the H-predecessors of b there is no H-cycle, since $M \not\models H_1^- \vee <_1^- \vee <_2^-$ (Lemma 4.2).

Look at this set P_b of the H-predecessors of b in the model N. Let $P_{b,0}$ be the subset of P_b consisting of elements with no H-predecessors at all. If there exists $c \in P_{b,0}$ that is greater than some element (not necessarily from $P_{b,0}$), then $N \models <_2^-$—a contradiction.

Let $P_{b,k+1}$ be the set of all elements $c \in P_b$, all of whose H-predecessors are in the set $\bigcup_{i=0}^k P_{b,i}$. By induction, we may assume that for any $d \in \bigcup_{i=0}^k P_{b,i}$, $\alpha(d)$, and, if $e < d$, then e is among the H-predecessors of d.

If $b \in P_{b,k+1}$, then $<_3^-$ is obviously satisfied (as witnessed by a and b)—a contradiction.

Otherwise, if there existed d and $c \in P_{b,k+1}$ such that, in N, $d < c$ but d were not among the H-predecessors of c, then $N \models <_3^-$ (witnessed by d and c)—a contradiction. But if this is not the case, then for any $c \in P_{b,k+1}$, $\alpha(c)$ (obvious), and the induction hypothesis holds, and we can proceed to the next level.

However, we cannot proceed further than the level of b, which is a finite level, so sooner or later we will get into a contradiction.

This proves the lemma. *Q.E.D.*

Lemma 4.4 *δ is finitely $<$-monotone.*

PROOF: The proof is as follows. If, in some model M, δ^- holds, then it is going to continue to hold in any extension (Lemma 4.3).

If δ^- does not hold, but δ^+ holds, then

1. the $\langle H\rangle$-reduct of M is without cycles, and
2. $<^M$ extends (maybe, trivially) the transitive closure of H.

It remains to show that δ^- holds in any proper extension of an H-acyclic model where $<$ properly extends the transitive closure of H. The nontrivial case is when $H_1^- \vee <_1^- \vee <_2^-$ is not satisfied, but then $<_3^-$ holds, as can be proved by induction similar to the one from the proof of Lemma 4.3. *Q.E.D.*

Lemma 4.5 *δ holds in any grid, but does not hold in any reduced grid.*

PROOF: It can be instantly verified that in any grid, δ^+ holds, and δ^- does not.

Hence, in any reduced grid, δ^- does not hold either (Lemma 4.3). While the fact that δ^+ does not hold in any reduced grid is obvious. *Q.E.D.*

The following lemma is a corollary to Lemma 4.5 and to the main lemma from the previous section:

Lemma 4.6 *δ is finitely equivalent to no $<$-positive sentence.*

Corollary 4.7 *Lyndon's Lemma fails for finite models.*

5 Conclusion

So we refuted Lyndon's Lemma for finite case, by showing that there exists a first order formula of a certain signature finitely monotone in a (binary) predicate and not finitely equivalent to any first order formula positive in this predicate.

To achieve this result, we introduced a new characterization of expressibility by positive formulas, namely, positive version of Fraïssé-Ehrenfeucht games, and showed that these games exactly characterize pairs of (finite or infinite) models that preserve truth of positive first order formulas.

Then we came up with an example of a sequence of pairs of finite models, that we call grids and reduced grids, respectively, and that preserve truth of positive sentences (as demonstrated by the proof of existence of a winning strategy in positive games), and showed that, however, a monotone sentence is capable of telling grids from reduced grids.

It turns out that the technical results of this paper provide for a much finer distinction between positive and monotone formulas, as compared to the technique in [2], and imply amazing consequences for uniform circuit complexity.

Ajtai and Gurevich [2] used quite a different criterion in separating positive and monotone formulas, namely, the so called *positive Borel sets*—a generalization of the classical Borel hierarchy to finite topological spaces [7].

From the forthcoming definition it is easy to see that positive formulas can define only positive Borel sets whose ranks don't exceed the logical depths of the corresponding formulas. Then the main technical part of their paper was to show that there exists a finitely monotone formula whose point-sets are not positive Borel of any fixed rank.

This left open, however, the following intriguing question that we answer here negatively: *is every first order formula that defines positive Borel point-sets of a fixed rank necessarily (finitely) equivalent to a positive formula?*

We will end this paper by showing that the formula δ from the previous section defines in fact a positive Borel point-set.

Definition 5.1 (positive Borel point-set) *Subsets of a set S will be called points over S.*

Sets of points over S will be called point-sets over S.

A point-set M will be called positive Borel of rank 0 if M is empty, or M contains all points over S, or $M = \{X \subseteq S \mid a \in X\}$ for some $a \in S$.

M will be called positive Borel of rank $i+1$ iff it is the union or an intersection of at most $|S|$ positive Borel point-sets over S of level at most i.

Definition 5.2 (definable point-set) *Let $\phi(P)$ be a sentence of a signature $\Omega \bigcup \{P\}$, where P is a k-ary predicate symbol not in Ω, and let $\mathcal{M}$ be a structure of the signature Ω. $\phi(P)$ defines a point-set $\left\{P \subseteq |\mathcal{M}|^k \mid \mathcal{M} \models \phi(P)\right\}$ over the set $|\mathcal{M}|^k$.*

The following is a simple observation from [2][3].

Claim 5.3 *Let $\phi(P)$ be a positive sentence of a signature $\Omega \bigcup \{P\}$, where P is a k-ary predicate symbol not in Ω, and let d be the logical depth of the sentence.*

For any finite model $\mathcal{M}$ of the signature Ω, the point-set over $|\mathcal{M}|^k$ that is definable by the sentence is positive Borel of rank d.

Intuitively, the notion of positive Borel point-sets generalizes that of first order positive definability by throwing away uniformity.

As we mentioned, the main instrument of refuting Lyndon's Lemma in [2] was the theorem that a certain finitely monotone formula defines a point-set that is not positive Borel of any uniform (in the set of finite models) finite rank.

The following theorem, however, shows that the distinction between positive and monotone formulas is much finer than the above result suggests.

Theorem 5.4 *Let $\delta(<)$ be the sentence from the last section, and let $\mathcal{M}$ be a finite model of the signature $\langle H \rangle$.*

The point-set over $|\mathcal{M}|^2$ that δ defines is positive Borel of rank 4.

IDEA OF PROOF: The nontrivial case is related to the axiom $<_3^-$.

Take x, z such that z is not among H-predecessors of x. Form the point-set $S_{x,z}$ in which any $<$-point contains $z < x$ and extends the transitive closure of H for H-predecessors of x. This guarantees the truth of one of the $-$-axioms (although maybe not $<_3^-$), and all $<$'s that satisfy $<_3^-$ and not the other $-$-axioms can be generated this way.

Finally, take the union of $S_{x,z}$ for all x, z as above. This makes the rank 4.

The correctness proof is similar to the technique used in the proof of Lemma 4.3 from the last section.

Notice that $+$-axioms contribute the single point: the transitive closure of H. *Q.E.D.*

Corollary 5.5 *There exists a finitely monotone first order sentence that defines a positive Borel point-set and is not finitely equivalent to any positive first order sentence.*

This settles the question that was left open in [2].

Sipser [12] explored connections between circuit complexity and Borel hierarchy, in particular, he showed that positive Boolean circuits of fixed depth and polynomial size can recognize only positive Borel sets of fixed ranks.

[3] The definition and the claim here are generalized to the case of k-ary predicates though.

Then an immediate corollary to the main result in [2] was the theorem of Furst, Saxe, and Sipser [6] and Ajtai [1] that *there is a uniform sequence of polynomial size constant depth monotone Boolean circuits that is not equivalent to whatever non-uniform sequence of polynomial size constant depth positive Boolean circuits.* The thing is, we can simply translate the point-sets definable in finite models by a monotone first-order formula into monotone Boolean circuits (whose depth is basically the logical depth of the formula). This provides for uniformity in a very strong sense.

Theorem 5.4 above shows that positive first order logic fails however to capture even strongly uniform (first order definable) positive Boolean circuits of constant depth and polynomial size, and therefore *first order logic seems to be a bad candidate to the role of uniform version of positive Boolean circuits of constant depth and polynomial size.*

Indeed, both monotone (Ajtai and Gurevich) and positive (present paper) first order formulas fail to adequately capture uniform classes of positive Boolean circuits of constant depth and polynomial size: the former express too much, and the latter too little.

Acknowledgments

Peter Clote, Herb Enderton, Tony Martin, and Michael Taitslin made some very valuable comments to earlier versions of the paper, and I would like to express my sincere thanks to them.

I am especially grateful to an anonymous referee who, among other things, suggested a simplification of the proof of Lemma 3.2.

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