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Fuzzy relation equations and reduction of fuzzy automata ☆,☆☆

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ABSTRACT

We show that the state reduction problem for fuzzy automata is related to the problem of finding a solution to a particular system of fuzzy relation equations in the set of all fuzzy equivalences on its set of states. This system may consist of infinitely many equations, and finding its non-trivial solutions may be a very difficult task. For that reason we aim our attention to some instances of this system which consist of finitely many equations and are easier to solve. First, we study right invariant fuzzy equivalences, and their duals, the left invariant ones. We prove that each fuzzy automaton possesses the greatest right (resp. left) invariant fuzzy equivalence, which provides the best reduction by means of fuzzy equivalences of this type, and we give an effective procedure for computing this fuzzy equivalence, which works if the underlying structure of truth values is a locally finite residuated lattice. Moreover, we show that even better reductions can be achieved alternating reductions by means of right and left invariant fuzzy equivalences. We also study strongly right and left invariant fuzzy equivalences, which give worse reductions than right and left invariant ones, but whose computing is much easier. We give an effective procedure for computing the greatest strongly right (resp. left) invariant fuzzy equivalence, which is applicable to fuzzy automata over an arbitrary complete residuated lattice.

1. Introduction

Unlike deterministic finite automata (DFA), whose minimization is efficiently possible, the state minimization problem for non-deterministic finite automata (NFA) is computationally hard (PSPACE-complete [34,67]) and known algorithms like in [9,35,45,46,61] cannot be used in practice. For that reason, many researchers aimed their attention to NFA state reduction methods which do not necessarily give a minimal one, but they give "reasonably" small NFAs which can be constructed efficiently. The basic idea of reducing number of states of NFAs by computing and merging indistinguishable states resembles the minimization algorithm for DFAs, but it is more complicated. That led to the concept of a right invariant equivalence on an NFA, studied first by Ilie and Yu in [29], and then in [8,11,30,32,33]. From another aspect, right invariant equivalences were studied by Calude et al. [7] under the name well-behaved equivalences. The same concept was studied under the name "bisimulation equivalence" in many areas of computer science and mathematics, such as modal logic, concurrency theory, set theory, formal verification, model checking, etc. Many algorithms have been proposed to compute the greatest bisimulation equivalence on a given labelled graph or a labelled transition system (cf. [43,47–49,52,58,60]). The faster algorithms are based on the crucial equivalence between the greatest bisimulation equivalence and the relational coarsest partition problem

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(see [20,22,36,51,57]). Bisimulations have been also studied in the context of weighted automata [5], and tree automata [23,24]. In terminology from [23,24], right and left invariant equivalences respectively correspond to forward and backward bisimulation equivalences.

Ilie and Yu in [29,30] gave an algorithm for computing the greatest right invariant equivalence on an NFA which runs in a low polynomial time, and in [32] they improved this algorithm employing an old very fast algorithm of Paige and Tarjan [51]. They also showed that right invariant equivalences can be used to construct small NFAs from regular expressions. In particular, both the partial derivative automaton and the follow automaton of a given regular expression are factor automata of the position automaton with respect to right invariant equivalences (cf. [12,13,28,30,31]).

Better results in state reduction of NFAs can be achieved in two ways. The first one was also proposed by Ilie and Yu in [29,30,32,33] who introduced the dual concept of a left invariant equivalence on an NFA and showed that even smaller NFAs can be obtained combining right invariant and left invariant equivalences. On the other hand, Champarnaud and Coulon in [10,11] proposed use of quasi-orders (preorders) instead of equivalences and showed that the method based on quasi-orders gives better reductions than the method based on equivalences. They gave an algorithm for computing the greatest right invariant and left invariant quasi-orders on an NFA working in a polynomial time, which was later improved in [32,33].

Fuzzy finite automata are generalizations of NFAs, and the mentioned problems concerning minimization and reduction of NFAs are also present in work with fuzzy automata. Reduction of number of states of fuzzy automata was studied in [1,14,41,44,50,53], and the algorithms given there were also based on the idea of computing and merging indistinguishable states. They were called minimization algorithms, but the term minimization is not adequate because it does not mean the usual construction of the minimal one in the set of all fuzzy automata recognizing a given fuzzy language, but just the procedure of computing and merging indistinguishable states. Therefore, these are just state reduction algorithms.

In contrast to the deterministic case, where we can effectively detect and merge indistinguishable states, in the non-deterministic case we have sets of states and it seems very difficult to decide whether two states are distinguishable or not. What we shall do is to find a superset, such that we are sure we do not merge state which we should not. There can always be states which could be merged but detecting those is too expensive. In the case of fuzzy automata this problem is even worse because we work with fuzzy sets of states. However, it turned out that in the non-deterministic case indistinguishability can be successfully modelled by equivalences and quasi-orders, and we will show that in the fuzzy case it can be modelled by fuzzy equivalences and fuzzy quasi-orders. As a matter of fact, here we will study state reduction of fuzzy automata by means of fuzzy equivalences, whereas state reduction by means of fuzzy quasi-orders will be a subject of our forthcomming paper.

In all previous papers dealing with reduction of fuzzy automata (cf. [1,14,41,44,50,53]) only reductions by means of crisp equivalences have been investigated. Here we show that better reductions can be achieved employing fuzzy equivalences. We start from a fuzzy equivalence E on a set of states A of a fuzzy automaton $\mathscr{A}$, and we turn the transition function on A into a related transition function on the factor set A_E, what results in the factor fuzzy automaton $\mathscr{A}_E$. If, in addition, the fuzzy automaton $\mathscr{A}$ has fuzzy sets of initial and terminal states, in a similar way we turn them into related fuzzy sets of initial and terminal states of the fuzzy factor automaton $\mathscr{A}_E$. But, if we do not impose any restriction on E then the fuzzy factor automaton $\mathscr{A}_E$ does not necessary behave like $\mathscr{A}$. For example, they do not necessary recognize the same fuzzy language. We show that $\mathscr{A}$ and $\mathscr{A}_E$ are equivalent if and only if E is a solution of a particular system of fuzzy relation equations including E, as an unknown fuzzy equivalence, the transition relations on $\mathscr{A}$ and the fuzzy sets of initial and terminal states. This system, called the *general system*, has at least one solution in the set $\mathscr{E}(A)$ of all fuzzy equivalences on A, the equality relation on A. To attain the best possible reduction of $\mathscr{A}$, we have to find the greatest solution to the general system in $\mathscr{E}(A)$, if it exists, or to find as big a solution as possible. However, the general system may consist of infinitely many equations, and finding its non-trivial solutions may be a very difficult task. For that reason we aim our attention to some instances of the general system. These instances have to be as general as possible, but they have to be easier to solve. From a practical point of view, these instances have to consist of finitely many equations.

In this paper we give solutions to several instances of the general system. In Section 4 we consider the system of the form $E \circ \delta_x \circ E = \delta_x \circ E$, whose solutions in $\mathscr{E}(A)$ are called right invariant fuzzy equivalences, and the dual system $E \circ \delta_x \circ E = E \circ \delta_x$, whose solutions in $\mathscr{E}(A)$ are called left invariant fuzzy equivalences. We show that right and left invariant fuzzy equivalences are immediate generalizations of the above mentioned right and left invariant equivalences on non-deterministic automata, and that congruences on fuzzy automata studied by Petković in [53] are just right invariant crisp equivalences on fuzzy automata. We prove that right invariant fuzzy equivalences on a fuzzy automaton $\mathscr{A}$ form a complete lattice (Theorem 4.2), and hence, there exists the greatest right invariant fuzzy equivalence on $\mathscr{A}$. This means that every fuzzy automaton $\mathscr{A}$ has the best reduction by means of right invariant fuzzy equivalences, and construction of this reduction is based on construction of the greatest right invariant fuzzy equivalence on $\mathscr{A}$. We give a procedure for computing the greatest right invariant fuzzy equivalence on $\mathscr{A}$ which works if the underlying structure $\mathscr{L}$ of truth values is a locally finite residuated lattice (Theorem 4.3), but it does not necessarily work if $\mathscr{L}$ is not locally finite (Example 4.1). In particular, it works in the case when $\mathscr{L}$ is the Gödel structure, that is, for classical fuzzy automata. This fact is not surprising if we have in mind recent results by Bělohlávek [2] and Li and Pedrycz [42], who found out that every finite fuzzy recognizer over $\mathscr{L}$ is equivalent to a deterministic fuzzy recognizer if and only if the semiring reduct $\mathscr{L}^* = (L, \vee, \otimes, 0, 1)$ of $\mathscr{L}$ is locally finite. By Theorem 4.5 we also characterize the greatest right invariant fuzzy equivalence on $\mathscr{A}$ in the case when $\mathscr{L}$ satisfies certain distributivity laws, which holds, for instance, for every continuous t-norm on the real unit interval, i.e., for every *BL*-algebra on the real unit interval. Consequently, Theorem 4.5 holds for Łukasiewicz, Goguen (product) and Gödel structures. By Example 4.2 we show that greatest right invariant fuzzy equivalences give better reductions of fuzzy automata than congruences on fuzzy automata studied by Petković in [53].

In Section 5 we show that the factor fuzzy automaton with respect to the greatest right invariant fuzzy equivalence cannot be further reduced by means of right invariant fuzzy equivalences, but in the general case, it can be reduced by left invariant ones. This leads to the concept of an alternate reduction, where we alternately perform right and left reductions, that is, reductions by means of the greatest right and left invariant fuzzy equivalences. It was proved in [30], in the case of non-deterministic automata, that alternate reductions can give exponentially smaller automata than right and left reductions. Alternate reductions which start with right reductions are called right–left alternate reductions, and those which start with left reductions are called left–right alternate reductions. In an alternate reduction of an arbitrary fuzzy finite automaton $\mathscr{A}$, after a finite number of steps, the number of states stops decreasing, and we obtain a fuzzy automaton called an alternate reduct of $\mathscr{A}$. In some cases we are able to recognize that we have reached the smallest number of states in an alternate reduction, but there is no yet a general procedure to recognize it. Moreover, by Example 5.2 we show that the right–left and left–right alternate reducts of a fuzzy automaton can have different number of states, and that the shortest right–left and left–right alternate reductions can have different lengths, and there is no a general procedure to decide which of these two alternate reductions would give better results in a given situation. In contrast to fuzzy automata, in the case of non-deterministic automata there is a general procedure to recognize whether we have reached the smallest number of states in an alternate reduction. Namely, if after two successive steps number of states did not changed, then we have reached the smallest number of states and this alternate reduction is finished. In other words, an alternate reduction finishes when we obtain a non-deterministic automaton which is both right and left reduced, and this automaton is an alternate reduct of the staring automaton. This does not hold for fuzzy automata because factorizing a fuzzy automaton by a fuzzy equality we change the fuzzy transition function and we obtain a fuzzy automaton which is not necessarily isomorphic to the original fuzzy automaton. As Example 5.1 shows, even if a fuzzy automaton $\mathscr{A}$ is both right and left reduced, factorizing it by its greatest right invariant fuzzy equivalence we can obtain a fuzzy automaton whose number of states can be further reduced factorizing it by its greatest left invariant fuzzy equivalence.

In Section 6 we study particular types of right and left invariant fuzzy equivalences, called strongly right invariant and strongly left invariant fuzzy equivalences. There are two main advantages of strongly right invariant fuzzy equivalences over right invariant ones. First, Theorem 6.2 gives an effective procedure for computing the greatest strongly right invariant fuzzy equivalence which is applicable to fuzzy automata over an arbitrary complete residuated lattice, whereas our procedure for computing the greatest right invariant one is applicable only to fuzzy automata over locally finite complete residuated lattices. Even for fuzzy automata over a locally finite complete residuated lattice, the greatest strongly right invariant fuzzy equivalence can be computed in a much simpler way than the greatest right invariant one. It can be computed directly, without forming a sequence of fuzzy equivalences which converges step by step toward the one we are searching for. However, although the greatest strongly right invariant fuzzy equivalence is easier to compute, it gives worse reduction than the greatest right invariant one (Example 6.1). Moreover, the fuzzy factor automaton w.r.t. the greatest right invariant fuzzy equivalence is right reduced, but the factor fuzzy automaton w.r.t. the greatest strongly right invariant one is not necessarily strongly right reduced, and we have to repeat factorization w.r.t. greatest strongly right invariant fuzzy equivalences many times until we obtain a strongly right reduced fuzzy automaton (Example 6.2). By Theorem 6.6 we give an effective procedure for deciding whether the strong right reduction of a fuzzy automaton has been finished.

2. Preliminaries

In this paper we will use complete residuated lattices as the structures of membership values. A *residuated lattice* is an algebra $\mathscr{L} = (L, \wedge, \vee, \otimes, \to, 0, 1)$ such that

(L1) $(L, \wedge, \vee, 0, 1)$ is a lattice with the least element 0 and the greatest element 1,
(L2) $(L, \otimes, 1)$ is a commutative monoid with the unit 1,
(L3) $\otimes$ and $\to$ form an *adjoint pair*, i.e., they satisfy the *adjunction property*: for all $x, y, z \in L$,

$$x \otimes y \leqslant z \quad \Leftrightarrow \quad x \leqslant y \to z. \tag{1}$$

Emphasizing their monoidal structure, in some sources residuated lattices are called integral, commutative, residuated ℓ-monoids [25]. If, in addition, $(L, \wedge, \vee, 0, 1)$ is a complete lattice, then $\mathscr{L}$ is called a *complete residuated lattice*. From now on we assume that $\mathscr{L}$ is a complete residuated lattice.

The operations $\otimes$ (called *multiplication*) and $\to$ (called *residuum*) are intended for modeling the conjunction and implication of the corresponding logical calculus, and supremum ($\bigvee$) and infimum ($\bigwedge$) are intended for modeling of the existential and general quantifier, respectively. An operation $\leftrightarrow$ defined by

$$x \leftrightarrow y = (x \to y) \wedge (y \to x), \tag{2}$$

called *biresiduum* (or *biimplication*), is used for modeling the equivalence of truth values. It can be easily verified that with respect to $\leqslant$, $\otimes$ is isotonic in both arguments, $\to$ is isotonic in the second and antitonic in the first argument, and for any $x, y, z \in L$ and any $\{x_i\}_{i\in I}, \{y_i\}_{i\in I} \subseteq L$, the following hold:

$$x \leftrightarrow y \leqslant x \otimes z \leftrightarrow y \otimes z, \tag{3}$$

$$\left(\bigvee_{i\in I} x_i\right) \otimes x = \bigvee_{i\in I}(x_i \otimes x), \tag{4}$$

$$\bigwedge_{i\in I}(x_1 \leftrightarrow y_i) \leqslant \left(\bigwedge_{i\in I} x_i\right) \leftrightarrow \left(\bigwedge_{i\in I} y_i\right), \tag{5}$$

$$\bigwedge_{i\in I}(x_i \leftrightarrow y_i) \leqslant \left(\bigvee_{i\in I} x_i\right) \leftrightarrow \left(\bigvee_{i\in I} y_i\right). \tag{6}$$

For other properties of complete residuated lattices we refer to [2,3].

The most studied and applied structures of truth values, defined on the real unit interval $[0, 1]$ with $x \wedge y = \min(x, y)$ and $x \vee y = \max(x, y)$, are the *Łukasiewicz structure* ($x \otimes y = \max(x + y - 1, 0)$, $x \to y = \min(1 - x + y, 1)$), the *Goguen* (*product*) *structure* ($x \otimes y = x \cdot y$, $x \to y = 1$ if $x \leqslant y$ and $= y/x$ otherwise) and the *Gödel structure* ($x \otimes y = \min(x, y)$, $x \to y = 1$ if $x \leqslant y$ and $= y$ otherwise). More generally, an algebra $([0, 1], \wedge, \vee, \otimes, \to, 0, 1)$ is a complete residuated lattice if and only if $\otimes$ is a left-continuous t-norm and the residuum is defined by $x \to y = \bigvee\{u \in [0, 1] \mid u \otimes x \leqslant y\}$. Another important set of truth values is the set $\{a_0, a_1, \ldots, a_n\}$, $0 = a_0 < \cdots < a_n = 1$, with $a_k \otimes a_l = a_{\max(k+l-n,0)}$ and $a_k \to a_l = a_{\min(n-k+l,n)}$. A special case of the latter algebras is the two-element Boolean algebra of classical logic with the support $\{0, 1\}$. This structure we call the *Boolean structure*. The only adjoint pair on the Boolean structure consists of the classical conjunction and implication operations. A residuated lattice $\mathscr{L}$ satisfying $x \otimes y = x \wedge y$ is called a *Heyting algebra*, whereas a Heyting algebra satisfying the prelinearity axiom $(x \to y) \vee (y \to x) = 1$ is called a *Gödel algebra*. If any finitely generated subalgebra of residuated lattice $\mathscr{L}$ is finite, then $\mathscr{L}$ is called *locally finite*. For example, every Gödel algebra, and hence, the Gödel structure, is locally finite, whereas the product structure is not locally finite.

In the further text $\mathscr{L}$ will be a complete residuated lattice. A *fuzzy subset* of a set A *over* $\mathscr{L}$, or simply a *fuzzy subset* of A, is any mapping from A into L. Ordinary crisp subsets of A are considered as fuzzy subsets of A taking membership values in the set $\{0, 1\} \subseteq L$. Let f and g be two fuzzy subsets of A. The *equality* of f and g is defined as the usual equality of mappings, i.e., $f = g$ if and only if $f(x) = g(x)$, for every $x \in A$. The *inclusion* $f \leqslant g$ is also defined pointwise: $f \leqslant g$ if and only if $f(x) \leqslant g(x)$, for every $x \in A$. Endowed with this partial order the set $\mathscr{F}(A)$ of all fuzzy subsets of A forms a complete residuated lattice, in which the meet (intersection) $\bigwedge_{i\in I} f_i$ and the join (union) $\bigvee_{i\in I} f_i$ of an arbitrary family $\{f_i\}_{i\in I}$ of fuzzy subsets of A are mappings from A into L defined by

$$\left(\bigwedge_{i\in I} f_i\right)(x) = \bigwedge_{i\in I} f_i(x), \qquad \left(\bigvee_{i\in I} f_i\right)(x) = \bigvee_{i\in I} f_i(x),$$

and the *product* $f \otimes g$ is a fuzzy subset defined by $f \otimes g(x) = f(x) \otimes g(x)$, for every $x \in A$.

The *crisp part* of a fuzzy subset f of A is a crisp subset $\hat{f} = \{a \in A \mid f(a) = 1\}$ of A. We will also consider $\hat{f}$ as a mapping $\hat{f} : A \to L$ defined by $\hat{f}(a) = 1$, if $f(a) = 1$, and $\hat{f}(a) = 0$, if $f(a) < 1$.

A *fuzzy relation* on A is any mapping from $A \times A$ into L, that is to say, any fuzzy subset of $A \times A$, and the equality, inclusion, joins, meets and ordering of fuzzy relations are defined as for fuzzy sets.

For fuzzy relations R and S on A, their *composition* $R \circ S$ is a fuzzy relation on A defined by

$$(R \circ S)(a, b) = \bigvee_{c\in A} R(a, c) \otimes S(c, b), \tag{7}$$

for all $a, b \in A$, and for a fuzzy subset f of A and a fuzzy relation R on A, the *compositions* $f \circ R$ and $R \circ f$ are fuzzy subsets of A defined by

$$(f \circ R)(a) = \bigvee_{b\in A} f(b) \otimes R(b, a), \qquad (R \circ f)(a) = \bigvee_{b\in A} R(a, b) \otimes f(b), \tag{8}$$

for any $a \in A$. Finally, for fuzzy subsets f and g of A we write

$$f \circ g = \bigvee_{a\in A} f(a) \otimes g(a). \tag{9}$$

The value $f \circ g$ can be interpreted as the "degree of overlapping" of f and g. We know that the composition of fuzzy relations is associative, and we can also easily verify that

$$(f \circ R) \circ S = f \circ (R \circ S), \qquad (f \circ R) \circ g = f \circ (R \circ g), \tag{10}$$

for arbitrary fuzzy subsets f and g of A, and fuzzy relations R and S on A, and hence, the parentheses in (10) can be omitted. Note also that if A is a finite set with n elements, then R and S can be treated as $n \times n$ fuzzy matrices over $\mathscr{L}$ and $R \circ S$ is the matrix product, whereas $f \circ R$ can be treated as the product of a $1 \times n$ matrix f and an $n \times n$ matrix R, and $R \circ f$ as the product of an $n \times n$ matrix R and an $n \times 1$ matrix f^t (the transpose of f).

A fuzzy relation E on a set A is

(R) *reflexive* if $E(a,a) = 1$, for every $a \in A$;
(S) *symmetric* if $E(a,b) = E(b,a)$, for all $a, b \in A$;
(T) *transitive* if $E(a,b) \otimes E(b,c) \leqslant E(a,c)$, for all $a, b, c \in A$.

If E is reflexive and transitive, then $E \circ E = E$. A fuzzy relation on A which is reflexive, symmetric and transitive is called a *fuzzy equivalence*. With respect to the ordering of fuzzy relations, the set $\mathscr{E}(A)$ of all fuzzy equivalence relations on a set A is a complete lattice, in which the meet coincide with the ordinary intersection of fuzzy relations, but in the general case, the join in $\mathscr{E}(A)$ does not coincide with the ordinary union of fuzzy relations.

For a fuzzy equivalence E on A and $a \in A$ we define a fuzzy subset E_a of A by:

$$E_a(x) = E(a,x), \quad \text{for every } x \in A.$$

We call E_a an *equivalence class* of E determined by a. The set $A_E = \{E_a \mid a \in A\}$ is called the *factor set* of A w.r.t. E (cf. [2,15,16]). Cardinality of the factor set A_E, in notation $\operatorname{ind}(E)$, is called the *index* of E.

A fuzzy equivalence E on a set A is called a *fuzzy equality* if for all $x, y \in A$, $E(x,y) = 1$ implies $x = y$. In other words, E is a fuzzy equality if and only if its crisp part $\widehat{E}$ is a crisp equality. For a fuzzy equivalence E on a set A, a fuzzy relation $\widetilde{E}$ defined on the factor set A_E by

$$\widetilde{E}(E_x, E_y) = E(x,y),$$

for all $x, y \in A$, is well defined and it is a fuzzy equality on A_E.

A *crisp relation*, or just a *relation*, is a fuzzy relation which takes values only in the set $\{0,1\} \subseteq L$, and if R is a crisp relation on a set A, then expressions "$R(a,b) = 1$" and "$(a,b) \in R$" will have the same meaning. A relation π on a set A is an *equivalence* if it is reflexive, symmetric, and transitive, i.e., if $(a,a) \in \pi$, for every $a \in A$, if $(a,b) \in \pi$ implies $(b,a) \in \pi$, for all $a, b \in A$, and if $(a,b) \in \pi$ and $(b,c) \in \pi$ implies $(a,c) \in \pi$, for all $a, b, c \in A$. For an equivalence π on A, $\pi_a = \{b \in A \mid (a,b) \in \pi\}$ is the equivalence class of an element $a \in A$, $A/\pi = \{\pi_a \mid a \in A\}$ is the factor set of A w.r.t. π, and the cardinality of the factor set is called the index of π and denoted by $\operatorname{ind}(\pi)$.

The following properties of fuzzy equivalence relations will be useful in later work.

Lemma 2.1. *Let E be a fuzzy equivalence on a set A and let $\widehat{E}$ be its crisp part. Then $\widehat{E}$ is a crisp equivalence on A, and for every $a, b \in A$ the following conditions are equivalent*:

(i) $E(a,b) = 1$;
(ii) $E_a = E_b$;
(iii) $\widehat{E}_a = \widehat{E}_b$.

Consequently, $\operatorname{ind}(E) = \operatorname{ind}(\widehat{E})$.

Note that $\widehat{E}_a$ denotes the crisp equivalence class of $\widehat{E}$ determined by a.

By a *fuzzy automaton over* $\mathscr{L}$, or simply a *fuzzy automaton*, a triple $\mathscr{A} = (A, X, \delta)$ is meant, where A and X are sets, called the *set of states* and the *input alphabet*, and $\delta : A \times X \times A \to L$ is a fuzzy subset of $A \times X \times A$, called the *fuzzy transition function*. We can interpret $\delta(a,x,b)$ as the degree to which an input letter $x \in X$ causes a transition from a state $a \in A$ into a state $b \in A$. The input alphabet X will be always finite, but from methodological reasons we will allow the set of states A to be infinite. A fuzzy automaton whose set of states is finite is called a *fuzzy finite automaton*. Cardinality of a fuzzy automaton $\mathscr{A} = (A, X, \delta)$, in notation $|\mathscr{A}|$, is defined as cardinality of its set of states A.

Let X^* denote the free monoid over the alphabet X, and let $e \in X^*$ be the empty word. The mapping δ can be extended up to a mapping $\delta^* : A \times X^* \times A \to L$ as follows: If $a, b \in A$, then

$$\delta^*(a,e,b) = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{otherwise,} \end{cases} \tag{11}$$

and if $a, b \in A$, $u \in X^*$ and $x \in X$, then

$$\delta^*(a,ux,b) = \bigvee_{c \in A} \delta^*(a,u,c) \otimes \delta(c,x,b). \tag{12}$$

By (4) and Theorem 3.1 [42] (see also [26,27,54–56]), we have that

$$\delta^*(a,uv,b)=\bigvee_{c\in A}\delta^*(a,u,c)\otimes\delta^*(c,v,b), \tag{13}$$

for all $a,b\in A$ and $u,v\in X^*$, i.e., if $w=x_1\cdots x_n$, for $x_1,\dots,x_n\in X$, then

$$\delta^*(a,w,b)=\bigvee_{(c_1,\dots,c_{n-1})\in A^{n-1}}\delta(a,x_1,c_1)\otimes\delta(c_1,x_2,c_2)\otimes\cdots\otimes\delta(c_{n-1},x_n,b). \tag{14}$$

Intuitively, the product $\delta(a,x_1,c_1)\otimes\delta(c_1,x_2,c_2)\otimes\cdots\otimes\delta(c_{n-1},x_n,b)$ represents the degree to which the input word w causes a transition from a state a into a state b through the sequence of intermediate states $c_1,\dots,c_{n-1}\in A$, and $\delta^*(a,w,b)$ represents the supremum of degrees of all possible transitions from a into b caused by w. Also, we can visualize a fuzzy finite automaton $\mathscr{A}$ representing it as a labelled directed graph whose nodes are states of $\mathscr{A}$, and an edge from a node a into a node b is labelled by pairs of the form $x/\delta(a,x,b)$, for any $x\in X$, as we will do in examples given in this paper. If δ is a crisp subset of $A\times X\times A$, i.e., $\delta: A\times X\times A\to\{0,1\}$, then $\mathscr{A}$ is an ordinary crisp non-deterministic automaton, and if δ is a mapping of $A\times X$ into A, then $\mathscr{A}$ is an ordinary deterministic automaton. Evidently, in these two cases we have that δ^* is also a crisp subset of $A\times X^*\times A$, and a mapping of $A\times X^*$ into A, respectively.

If for any $u\in X^*$ we define a fuzzy relation δ_u on A by

$$\delta_u(a,b)=\delta^*(a,u,b), \tag{15}$$

for all $a,b\in A$, called the *fuzzy transition relation* determined by u, then (13) can be written as

$$\delta_{uv}=\delta_u\circ\delta_v, \tag{16}$$

for all $u,v\in X^*$.

An *initial fuzzy automaton* is a quadruple $\mathscr{A}=(A,\sigma,X,\delta)$, where (A,X,δ) is a fuzzy automaton and σ is a fuzzy subset of A, called the fuzzy set of *initial states*, and a *fuzzy recognizer* is defined as a five-tuple $\mathscr{A}=(A,\sigma,X,\delta,\tau)$, where (A,σ,X,δ) is as above, and τ is a fuzzy subset of A, called the fuzzy set of *terminal states*.

A *fuzzy language* in X^* over $\mathscr{L}$, or briefly a *fuzzy language*, is any fuzzy subset of X^*, i.e., any mapping from X^* into L. A fuzzy recognizer $\mathscr{A}=(A,\sigma,X,\delta,\tau)$ *recognizes* a fuzzy language $f\in\mathscr{F}(X^*)$ if for any $u\in X^*$ we have

$$f(u)=\bigvee_{a,b\in A}\sigma(a)\otimes\delta^*(a,u,b)\otimes\tau(b). \tag{17}$$

In other words, the equality (17) means that the membership degree of the word u to the fuzzy language f is equal to the degree to which $\mathscr{A}$ recognizes or accepts the word u. Using notation from (8), and the second equality in (10), we can state (17) as

$$f(u)=\sigma\circ\delta_u\circ\tau. \tag{18}$$

The unique fuzzy language recognized by a fuzzy recognizer $\mathscr{A}$ is denoted by $L(\mathscr{A})$. In fact, a fuzzy language in X^* over a complete residuated lattice $(L,\wedge,\vee,\otimes,\to,0,1)$ recognized by a fuzzy finite recognizer is a recognizable formal power series over the semiring $(L,\vee,\otimes,0,1)$ (cf. [4,21,40,59]).

The *reverse fuzzy automaton* of a fuzzy automaton $\mathscr{A}=(A,X,\delta)$ is a fuzzy automaton $\bar{\mathscr{A}}=(A,X,\bar{\delta})$, with the fuzzy transition function defined by $\bar{\delta}(a,x,b)=\delta(b,x,a)$, for all $a,b\in A$ and $x\in X$. Fuzzy automata $\mathscr{A}=(A,X,\delta)$ and $\mathscr{A}'=(A',X,\delta')$ are *isomorphic* if there exists a bijective mapping $\phi: A\to A'$ such that $\delta(a,x,b)=\delta'(\phi(a),x,\phi(b))$, for all $a,b\in A$ and $x\in X$. It is easy to check that in this case we also have that $\delta^*(a,u,b)=\delta'^*(\phi(a),u,\phi(b))$, for all $a,b\in A$ and $u\in X^*$.

It is worth noting that fuzzy automata with membership values in complete residuated lattices have been recently very intensively studied (cf. [17,26,27,54–56,63–66]). For undefined notions and notation we refer to [2,3,15,16,26,27,50,67].

3. Factor fuzzy automata and fuzzy relation equations

Let $\mathscr{A}=(A,X,\delta)$ be a fuzzy automaton and let E be a fuzzy equivalence on A. Without any restriction on the fuzzy equivalence E, we can define a fuzzy transition function $\delta^E: A_E\times X\times A_E\to L$ by

$$\delta^E(E_a,x,E_b)=\bigvee_{a',b'\in A}E\big(a,a'\big)\otimes\delta\big(a',x,b'\big)\otimes E\big(b',b\big) \tag{19}$$

or equivalently,

$$\delta^E(E_a,x,E_b)=(E\circ\delta_x\circ E)(a,b)=E_a\circ\delta_x\circ E_b, \tag{20}$$

for any $a,b\in A$ and $x\in X$. Evidently, δ^E is well defined, and $\mathscr{A}_E=(A_E,X,\delta^E)$ is a fuzzy automaton, called the *factor fuzzy automaton* of $\mathscr{A}$ w.r.t. E.

The following theorem can be conceived as a version, for fuzzy automata, of the well-known Second Isomorphism Theorem from universal algebra (cf. [6, II.§6]).

Theorem 3.1. *Let $\mathscr{A} = (A, X, \delta)$ be a fuzzy automaton and let E and F be fuzzy equivalences on A such that $E \leqslant F$. Then the fuzzy relation F_E on the factor fuzzy automaton $\mathscr{A}_E = (A_E, X, \delta^E)$ defined by*

$$F_E(E_a, E_b) = F(a,b), \quad \text{for all } a, b \in A, \tag{21}$$

is a fuzzy equivalence on A_E and the factor fuzzy automata $(\mathscr{A}_E)_{F_E}$ and $\mathscr{A}_F$ are isomorphic.

Proof. Let $a, a', b, b' \in A$ such that $E_a = E_{a'}$ and $E_b = E_{b'}$, i.e., $E(a, a') = E(b, b') = 1$. Since $E \leqslant F$, we have that $F(a, a') = F(b, b') = 1$, what implies $F(a, b) = F(a', b')$. Therefore, F_E is a well-defined fuzzy relation. It is easy to check that F_E is a fuzzy equivalence.

For the sake of simplicity set $F_E = Q$. Define a mapping $\phi : A_F \to (A_E)_Q$ by

$$\phi(F_a) = Q_{E_a}, \quad \text{for every } a \in A.$$

According to Lemma 2.1, for arbitrary $a, b \in A$ we have that

$$F_a = F_b \quad \Leftrightarrow \quad F(a,b) = 1 \quad \Leftrightarrow \quad Q(E_a, E_b) = 1 \quad \Leftrightarrow \quad Q_{E_a} = Q_{E_b} \quad \Leftrightarrow \quad \phi(F_a) = \phi(F_b),$$

and hence, ϕ is a well defined and injective mapping. It is clear that ϕ is also a surjective mapping. Therefore, ϕ is a bijective mapping of A_F onto $(A_E)_Q$.

Since $E \leqslant F$ implies $E \circ F = F \circ E = F$, for arbitrary $a, b \in A$ and $x \in X$ we have that

$$\begin{aligned}
\delta_x^Q\big(\phi(F_a), \phi(F_b)\big) &= \delta_x^Q(Q_{E_a}, Q_{E_b}) = \big(Q \circ \delta_x^E \circ Q\big)(E_a, E_b) \\
&= \bigvee_{c,d \in A} Q(E_a, E_c) \otimes \delta_x^E(E_c, E_d) \otimes Q(E_d, E_b) \\
&= \bigvee_{c,d \in A} F(a,c) \otimes (E \circ \delta_x \circ E)(c,d) \otimes F(d,b) \\
&= (F \circ E \circ \delta_x \circ E \circ F)(a,b) = (F \circ \delta_x \circ F)(a,b) = \delta_x^F(F_a, F_b).
\end{aligned}$$

Therefore, ϕ is an isomorphism of the fuzzy automaton $\mathscr{A}_F$ onto the fuzzy automaton $(\mathscr{A}_E)_{F_E}$. $\square$

Let us note that if $\mathscr{A} = (A, X, \delta)$ is a fuzzy automaton and E, F and G are fuzzy equivalences on A such that $E \leqslant F$ and $E \leqslant G$, then

$$F \leqslant G \quad \Leftrightarrow \quad F_E \leqslant G_E, \tag{22}$$

and hence, a mapping $\Phi : \mathscr{E}_E(A) = \{F \in \mathscr{E}(A) \mid E \leqslant F\} \to \mathscr{E}(A_E)$, given by $\Phi : F \mapsto F_E$, is injective.

If, in addition, $\mathscr{A} = (A, \sigma, X, \delta, \tau)$ is a fuzzy recognizer, then without any restriction on E, we can also define a fuzzy set $\sigma^E \in \mathscr{F}(A_E)$ of initial states and a fuzzy set $\tau^E \in \mathscr{F}(A_E)$ of terminal states by

$$\sigma^E(E_a) = \bigvee_{a' \in A} \sigma\big(a'\big) \otimes E\big(a', a\big) = (\sigma \circ E)(a) = \sigma \circ E_a, \tag{23}$$

$$\tau^E(E_a) = \bigvee_{a' \in A} \tau\big(a'\big) \otimes E\big(a', a\big) = (\tau \circ E)(a) = \tau \circ E_a, \tag{24}$$

for any $a \in A$. Clearly, σ^E and τ^E are well defined and $\mathscr{A}_E = (A_E, \sigma^E, X, \delta^E, \tau^E)$ is a fuzzy recognizer, called the *factor fuzzy recognizer* of $\mathscr{A}$ w.r.t. E.

We illustrate factor fuzzy recognizers with the following example.

Example 3.1. Let $\mathscr{L}$ be the Boolean structure, and let $\mathscr{A} = (A, \sigma, X, \delta, \tau)$ be a fuzzy recognizer over $\mathscr{L}$ (i.e., a non-deterministic recognizer) with $|A| = 4$, $X = \{x\}$ and

$$\delta_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \sigma = [\,0 \quad 1 \quad 0 \quad 0\,], \qquad \tau = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix},$$

and let E and F be fuzzy equivalences on A given by

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \qquad F = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Then the factor fuzzy recognizers $\mathscr{A}_E = (A_E, \sigma^E, X, \delta^E, \tau^E)$ and $\mathscr{A}_F = (A_F, \sigma^F, X, \delta^F, \tau^F)$ are given as follows: $|A_E| = 3$, $|A_F| = 2$, and

$$\delta_x^E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \sigma^E = [\,0 \quad 1 \quad 0\,], \qquad \tau^E = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \delta_x^F = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \sigma^F = [\,1 \quad 0\,], \qquad \tau^F = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

For every $u \in X^*$ we have that

$$L(\mathscr{A})(u) = L(\mathscr{A}_E)(u) = \begin{cases} 1 & \text{if } u = x, \\ 0 & \text{if } u \neq x, \end{cases} \qquad L(\mathscr{A}_F)(u) = \begin{cases} 1 & \text{if } u \neq e, \\ 0 & \text{if } u = e, \end{cases}$$

and therefore, $\mathscr{A}_E$ is equivalent to $\mathscr{A}$, but $\mathscr{A}_F$ is not equivalent to $\mathscr{A}$.

Let us note that the fuzzy language $L(\mathscr{A}_E)$ recognized by the factor fuzzy recognizer $\mathscr{A}_E$ is given by

$$L(\mathscr{A}_E)(u) = \sigma \circ E \circ \delta_{x_1} \circ E \circ \delta_{x_2} \circ E \circ \cdots \circ E \circ \delta_{x_n} \circ E \circ \tau,$$

for any $u = x_1x_2\cdots x_n \in X^*$, where $x_1, x_2, \ldots, x_n \in X$, whereas the fuzzy language $L(\mathscr{A})$ recognized by the fuzzy recognizer $\mathscr{A}$ is given by

$$L(\mathscr{A})(u) = \sigma \circ \delta_{x_1} \circ \delta_{x_2} \circ \cdots \circ \delta_{x_n} \circ \tau,$$

for any $u = x_1x_2\cdots x_n \in X^*$, where $x_1, x_2, \ldots, x_n \in X$. Therefore, the fuzzy recognizer $\mathscr{A}$ and the factor fuzzy recognizer $\mathscr{A}_E$ are equivalent, i.e., they recognize the same fuzzy language, if and only if the fuzzy equivalence E is a solution to a system of fuzzy relation equations

$$\sigma \circ \delta_{x_1} \circ \delta_{x_2} \circ \cdots \circ \delta_{x_n} \circ \tau = \sigma \circ E \circ \delta_{x_1} \circ E \circ \delta_{x_2} \circ E \circ \cdots \circ E \circ \delta_{x_n} \circ E \circ \tau, \tag{25}$$

for all $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n \in X$. In the further text system (25) will be called the *general system*.

The general system has at least one solution in $\mathscr{E}(A)$, the equality relation on A. It will be called the *trivial solution*. To attain the best possible reduction of $\mathscr{A}$, we have to find the greatest solution to the general system in $\mathscr{E}(A)$, if it exists, or to find as big a solution as possible. However, the general system may consist of infinitely many equations, and finding its non-trivial solutions may be a very difficult task. For that reason we will aim our attention to some instances of the general system. These instances have to be as general as possible, but they have to be easier to solve. From a practical point of view, these instances have to consist of finitely many equations.

Recall that the fuzzy transition function δ^E is defined by $\delta^E(E_a, x, E_b) = (E \circ \delta_x \circ E)(a, b)$, for all $a, b \in A$ and $x \in X$. For all $a, b \in A$ and $u = x_1x_2\cdots x_n \in X^*$, where $x_1, x_2, \ldots, x_n \in X$, we have that

$$\delta^E(E_a, u, E_b) = (E \circ \delta_{x_1} \circ E \circ \delta_{x_2} \circ E \circ \cdots \circ \delta_{x_n} \circ E)(a, b), \tag{26}$$

but, in the general case, $\delta^E(E_a, u, E_b)$ is not necessary equal to $(E \circ \delta_u \circ E)(a, b)$. Namely, we have that $\delta^E(E_a, u, E_b) = (E \circ \delta_u \circ E)(a, b)$ for all $a, b \in A$ and $u = x_1x_2\cdots x_n \in X^*$, where $x_1, x_2, \ldots, x_n \in X$, if and only if E is a solution to a system of fuzzy relation equations

$$E \circ \delta_{x_1} \circ E \circ \delta_{x_2} \circ E \circ \cdots \circ \delta_{x_n} \circ E = E \circ \delta_{x_1} \circ \delta_{x_2} \circ \cdots \circ \delta_{x_n} \circ E, \tag{27}$$

for all $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n \in X$. In this case we say that the factor fuzzy automaton $\mathscr{A}_E$ is *compatible* with $\mathscr{A}$. This holds whether $\mathscr{A}$ is a fuzzy automaton or a fuzzy recognizer.

If $\mathscr{A}$ is a fuzzy recognizer, using system (27) we get a little less general instance of the general system

$$\begin{aligned} &E \circ \delta_{x_1} \circ E \circ \delta_{x_2} \circ E \circ \cdots \circ \delta_{x_n} \circ E = E \circ \delta_{x_1} \circ \delta_{x_2} \circ \cdots \circ \delta_{x_n} \circ E, \\ &\sigma \circ E = \sigma, \\ &\tau \circ E = \tau, \end{aligned} \tag{28}$$

for all $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n \in X$. In other words, any solution to (28) is a solution to the general system.

Let f be a fuzzy subset of a set A and let E be a fuzzy equivalence on A. Then f is said to be *extensional* with respect to E if for all $x, y \in A$ we have

$$f(x) \otimes E(x, y) \leqslant f(y). \tag{29}$$

According to (29), symmetry of E and the adjunction property, f is extensional w.r.t. E if and only if

$$E(x, y) \leqslant f(x) \leftrightarrow f(y), \tag{30}$$

for all $x, y \in A$ (cf. [18,19,37–39]). If a fuzzy equivalence E_f on A is defined by $E_f(x, y) = f(x) \leftrightarrow f(y)$, for all $x, y \in A$, then f is extensional w.r.t. E if and only if $E \leqslant E_f$, and hence, E_f is the greatest fuzzy equivalence on A such that f is extensional with respect to it. Let us note that condition (29) is equivalent to $f \circ E \leqslant f$, and by the reflexivity of E, it is equivalent to $f \circ E = f$.

Therefore, the equations $\sigma \circ E = \sigma$ and $\tau \circ E = \tau$ can be easily solved. Namely, we have that E is a solution to $\sigma \circ E = \sigma$ (resp. $\tau \circ E = \tau$) if and only if $E \leqslant E_\sigma$ (resp. $E \leqslant E_\tau$), and hence, E_σ (resp. E_τ) is the greatest solution to this equation. Hence, solutions to system (28) are exactly those solutions to system (27) which are contained in $E_\sigma \wedge E_\tau$. For that reason, in the further text we will aim our attention to system (27) and certain instances of this system.

Let us also note that the equations $\sigma \circ E = \sigma$ and $\tau \circ E = \tau$ have a natural interpretation. Roughly speaking, $E(a, b) \leqslant E_\sigma(a, b)$ and $E(a, b) \leqslant E_\tau(a, b)$, for all $a, b \in A$, mean that E does not merge initial and non-initial, and terminal and non-terminal states.

The system (27) still consists of infinitely many equations, so it is hard to solve and it does not have a practical importance. Because of that we consider the following two instances of system (27) consisting of finitely many equations:

$$\delta_x \circ E \circ \delta_y \circ E = \delta_x \circ \delta_y \circ E, \quad x, y \in X, \tag{31}$$

$$E \circ \delta_x \circ E \circ \delta_y = E \circ \delta_x \circ \delta_y, \quad x, y \in X. \tag{32}$$

Indeed, if a fuzzy equivalence E is a solution to system (31), then by induction we can easily prove that E is also a solution to any equation of the form

$$\delta_{x_1} \circ E \circ \delta_{x_2} \circ E \circ \cdots \circ \delta_{x_n} \circ E = \delta_{x_1} \circ \delta_{x_2} \circ \cdots \circ \delta_{x_n} \circ E, \tag{33}$$

for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$, and hence, it is a solution to system (27). Similarly we show that any solution to system (32) is also a solution to system (27).

4. Right invariant and left invariant fuzzy equivalences

We start the study of systems (31) and (32) considering some of the most interesting special cases of these systems. Let $\mathscr{A} = (A, X, \delta)$ be a fuzzy automaton. If a fuzzy equivalence E on A is a solution to system

$$E \circ \delta_x \circ E = \delta_x \circ E, \quad x \in X, \tag{34}$$

then it will be called a *right invariant* fuzzy equivalence on $\mathscr{A}$, and if it is a solution to system

$$E \circ \delta_x \circ E = E \circ \delta_x, \quad x \in X, \tag{35}$$

then it will be called a *left invariant* fuzzy equivalence on $\mathscr{A}$. A crisp equivalence on A which is a solution to (34) is called a *right invariant equivalence* on $\mathscr{A}$, and a crisp equivalence which is a solution to (35) is called a *left invariant equivalence* on $\mathscr{A}$.

In Remark 4.1 we will show that right invariant fuzzy equivalences are immediate generalizations of right invariant equivalences on non-deterministic automata, studied by Ilie, Yu and others [8,11,29,30,32,33], or well-behaved equivalences, studied by Calude et al. [7]. Moreover, in Remark 4.2 we will show that congruences on fuzzy automata, studied by Petković in [53], are just right invariant equivalences on fuzzy automata, in the terminology from this paper.

It can be easily verified that E is a left invariant fuzzy equivalence on a fuzzy automaton $\mathscr{A}$ if and only if E is a right invariant fuzzy equivalence on the reverse fuzzy automaton $\overline{\mathscr{A}}$ of $\mathscr{A}$. For that reason in the further text we consider only right invariant fuzzy equivalences. The corresponding results concerning left invariant fuzzy equivalences can be derived by symmetry from those concerning right invariant ones.

Right invariant fuzzy equivalences can be characterized as follows.

Theorem 4.1. *Let $\mathscr{A} = (A, X, \delta)$ be a fuzzy automaton and E a fuzzy equivalence on A. Then the following conditions are equivalent*:

(i) *E is a right invariant fuzzy equivalence*;
(ii) *$E \circ \delta_x \leqslant \delta_x \circ E$, for every $x \in X$*;
(iii) *for all $a, b \in A$ we have*

$$E(a, b) \leqslant \bigwedge_{x \in X} \bigwedge_{c \in A} (\delta_x \circ E)(a, c) \leftrightarrow (\delta_x \circ E)(b, c). \tag{36}$$

Proof. (i) $\Leftrightarrow$ (ii). Consider an arbitrary $x \in X$. If $E \circ \delta_x \circ E = \delta_x \circ E$, then $E \circ \delta_x \leqslant E \circ \delta_x \circ E = \delta_x \circ E$. Conversely, if $E \circ \delta_x \leqslant \delta_x \circ E$ then $E \circ \delta_x \circ E \leqslant \delta_x \circ E \circ E = \delta_x \circ E$, and since the opposite inequality always holds, we conclude that $E \circ \delta_x \circ E = \delta_x \circ E$.

(i) $\Rightarrow$ (iii). Let E be a right invariant fuzzy equivalence. Then for all $x \in X$ and $a, b, c \in A$ we have that

$$E(a, b) \otimes (\delta_x \circ E)(b, c) \leqslant (E \circ \delta_x \circ E)(a, c) = (\delta_x \circ E)(a, c),$$

and by the adjunction property we obtain that $E(a,b) \leqslant (\delta_x \circ E)(b,c) \to (\delta_x \circ E)(a,c)$. By symmetry, $E(a,b) = E(b,a) \leqslant (\delta_x \circ E)(a,c) \to (\delta_x \circ E)(b,c)$, and hence,

$$E(a,b) \leqslant (\delta_x \circ E)(a,c) \leftrightarrow (\delta_x \circ E)(b,c). \tag{37}$$

Since (37) is satisfied for every $c \in A$ and $x \in X$, we conclude that (36) holds.

(iii) ⇒ (i). If (iii) holds, then for arbitrary $x \in X$ and $a,b,c \in A$ we have that

$$E(a,b) \leqslant (\delta_x \circ E)(a,c) \leftrightarrow (\delta_x \circ E)(b,c) \leqslant (\delta_x \circ E)(b,c) \to (\delta_x \circ E)(a,c),$$

and by the adjunction property we obtain that $E(a,b) \otimes (\delta_x \circ E)(b,c) \leqslant (\delta_x \circ E)(a,c)$. Now,

$$(E \circ \delta_x \circ E)(a,c) = \bigvee_{b \in A} E(a,b) \otimes (\delta_x \circ E)(b,c) \leqslant (\delta_x \circ E)(a,c),$$

and hence, $E \circ \delta_x \circ E \leqslant \delta_x \circ E$. Since the opposite inequality always holds, we conclude that $E \circ \delta_x \circ E = \delta_x \circ E$, for every $x \in X$. □

Remark 4.1. Let $\mathscr{A} = (A, X, \delta)$ be a crisp non-deterministic automaton and E an equivalence on A. It is easy to check that $E \circ \delta_x \subseteq \delta_x \circ E$ is equivalent to

(P_2) $(\forall a, b \in A)(\forall x \in X)((a,b) \in E \Rightarrow (\forall b' \in \delta(b,x))(\exists a' \in \delta(a,x))(a',b') \in E)$,

what is the second of two conditions by which Ilie, Navarro and Yu in [32] defined the notion of a right invariant equivalence on a non-deterministic automaton (see also [33,11]). The first one, which requires that terminal and non-terminal states are not E-equivalent, can be written in the fuzzy case as $\tau \circ E = \tau$. Here we have excluded this condition from the definition of a right invariant fuzzy equivalence, and it will be considered separately.

Calude et al. in [7] called equivalences satisfying (P_2) well behaved. Note also that an equivalent form of condition (P_2), appearing in [29,30], corresponds to our condition (iii) in Theorem 4.1.

In the sense of Högberg, Maletti and May [24], an equivalence E satisfying $E \circ \delta_x \subseteq \delta_x \circ E$ is a forward bisimulation, and an equivalence E satisfying $\delta_x \circ E \subseteq E \circ \delta_x$ is a backward bisimulation (cf. Definitions 4.1 and 3.1 [24] with $k = 1$).

Remark 4.2. Let $\mathscr{A} = (A, X, \delta)$ be a fuzzy automaton and E a crisp equivalence on A.

By condition (iii) of Theorem 4.1 we have that E is a right invariant equivalence on $\mathscr{A}$ if and only if for any $a, b \in A$, by $(a,b) \in E$ it follows $(\delta_x \circ E)(a,c) = (\delta_x \circ E)(b,c)$, for all $x \in X$ and $c \in A$. But, $(\delta_x \circ E)(a,c) = (\delta_x \circ E)(b,c)$ is equivalent to

$$\bigvee_{b' \in E_b} \delta(a, x, b') = \bigvee_{b' \in E_b} \delta(b, x, b'),$$

and hence, right invariant crisp equivalences on fuzzy automata are nothing else than congruences on fuzzy automata studied by Petković in [53] (or partitions with substitution property from [1]).

Let $\mathscr{A} = (A, X, \delta)$ be a fuzzy automaton and E a fuzzy equivalence on A. Let us define fuzzy relations $E^x \in L^{A \times A}$, for any $x \in X$, and $E^r \in L^{A \times A}$ by

$$E^x(a,b) = \bigwedge_{c \in A} (\delta_x \circ E)(a,c) \leftrightarrow (\delta_x \circ E)(b,c), \qquad E^r(a,b) = \bigwedge_{x \in X} E^x(a,b), \tag{38}$$

for all $a, b \in A$. Thus, Eq. (36) could be stated as $E \leqslant E^r$. By the well-known Valverde's Representation Theorem [62] (see also [2,15,16]) we have that E^x, for each $x \in X$, and E^r are fuzzy equivalences. We also have the following

Lemma 4.1. *Let $\mathscr{A} = (A, X, \delta)$ be a fuzzy automaton, and let E and F be fuzzy equivalences on A.*

If $E \leqslant F$, then $E^r \leqslant F^r$.

Proof. Consider arbitrary $a, b \in A$ and $x \in X$. By $E \leqslant F$ it follows $E \circ F = F$, and by (3), for arbitrary $c, d \in A$ we have that

$$(\delta_x \circ E)(a,c) \leftrightarrow (\delta_x \circ E)(b,c) \leqslant (\delta_x \circ E)(a,c) \otimes F(c,d) \leftrightarrow (\delta_x \circ E)(b,c) \otimes F(c,d).$$

Now, by (6) we obtain that for every $x \in X$ and $d \in A$,

$$\begin{aligned} E^r(a,b) &\leqslant \bigwedge_{c \in A} (\delta_x \circ E)(a,c) \leftrightarrow (\delta_x \circ E)(b,c) \\ &\leqslant \bigwedge_{c \in A} \big[(\delta_x \circ E)(a,c) \otimes F(c,d) \leftrightarrow (\delta_x \circ E)(b,c) \otimes F(c,d)\big] \end{aligned}$$

$$\leqslant \left[\bigvee_{c\in A}(\delta_x\circ E)(a,c)\otimes F(c,d)\right]\leftrightarrow\left[\bigvee_{c\in A}(\delta_x\circ E)(b,c)\otimes F(c,d)\right]$$
$$=(\delta_x\circ E\circ F)(a,d)\leftrightarrow(\delta_x\circ E\circ F)(b,d)=(\delta_x\circ F)(a,d)\leftrightarrow(\delta_x\circ F)(b,d).$$

Thus, we conclude that $E^r\leqslant F^r$. □

Now we are ready to prove the following:

Theorem 4.2. *Let $\mathscr{A}=(A,X,\delta)$ be a fuzzy automaton.*

The set $\mathscr{E}^{\mathrm{ri}}(\mathscr{A})$ of all right invariant fuzzy equivalences on $\mathscr{A}$ forms a complete lattice. This lattice is a complete join-subsemilattice of the lattice $\mathscr{E}(A)$ of all fuzzy equivalences on A.

Proof. Since $\mathscr{E}^{\mathrm{ri}}(\mathscr{A})$ contains the least element of $\mathscr{E}(A)$, the equality relation on A, it is enough to prove that $\mathscr{E}^{\mathrm{ri}}(\mathscr{A})$ is a complete join-subsemilattice of $\mathscr{E}(A)$.

Let $\{E_i\}_{i\in I}$ be a family of right invariant fuzzy equivalences on $\mathscr{A}$, and let E be its join in $\mathscr{E}(A)$. Then for any $i\in I$, by $E_i\leqslant E$ and Lemma 4.1 we obtain that $E_i\leqslant E_i^r\leqslant E^r$, so $E\leqslant E^r$. Hence, by (iii) of Theorem 4.1 we obtain that E is a right invariant fuzzy equivalence. □

By Theorem 4.2 it follows that for any fuzzy equivalence E on $\mathscr{A}$ there exists the greatest right invariant fuzzy equivalence contained in E. It will be denoted by E^{ri}. In the next theorem we consider the problem how to construct it.

Theorem 4.3. *Let $\mathscr{A}=(A,X,\delta)$ be a fuzzy automaton, let E be a fuzzy equivalence on A and let E^{ri} be the greatest right invariant fuzzy equivalence contained in E.*

Define inductively a sequence $\{E_k\}_{k\in\mathbb{N}}$ of fuzzy equivalences on A as follows:

$$E_1=E,\qquad E_{k+1}=E_k\wedge E_k^r,\quad \text{for each } k\in\mathbb{N}. \tag{39}$$

Then

(a) $E^{\mathrm{ri}}\leqslant\cdots\leqslant E_{k+1}\leqslant E_k\leqslant\cdots\leqslant E_1=E$;
(b) *If $E_k=E_{k+m}$, for some $k,m\in\mathbb{N}$, then $E_k=E_{k+1}=E^{\mathrm{ri}}$;*
(c) *If $\mathscr{A}$ is finite and $\mathscr{L}$ is locally finite, then $E_k=E^{\mathrm{ri}}$ for some $k\in\mathbb{N}$.*

Proof. (a) Evidently, $E_{k+1}\leqslant E_k$, for each $k\in\mathbb{N}$, and $E^{\mathrm{ri}}\leqslant E_1$. Suppose that $E^{\mathrm{ri}}\leqslant E_k$, for some $k\in\mathbb{N}$. Then $E^{\mathrm{ri}}\leqslant(E^{\mathrm{ri}})^r\leqslant E_k^r$, so $E^{\mathrm{ri}}\leqslant E_k\wedge E_k^r=E_{k+1}$. Thus, by induction we get $E^{\mathrm{ri}}\leqslant E_k$, for all $k\in\mathbb{N}$.

(b) Let $E_k=E_{k+m}$, for some $k,m\in\mathbb{N}$. Then $E_k=E_{k+m}\leqslant E_{k+1}=E_k\wedge E_k^r\leqslant E_k^r$, what means that E_k is a right invariant fuzzy equivalence. Since E^{ri} is the greatest right invariant fuzzy equivalence contained in E, we conclude that $E_k=E_{k+1}=E^{\mathrm{ri}}$.

(c) Let $\mathscr{A}$ be a finite fuzzy automaton and $\mathscr{L}$ be a locally finite algebra. Let the carrier of the subalgebra of $\mathscr{L}$ generated by the set $\delta(A\times X\times A)\cup E(A\times A)$ be denoted by $L_{\mathscr{A}}$. This generating set is finite, so $L_{\mathscr{A}}$ is also finite, and hence, the set $L_{\mathscr{A}}^{A\times A}$ of all fuzzy relations on A with values in $L_{\mathscr{A}}$ is finite. By definitions of fuzzy relations E_k and E_k^r we have that $E_k\in L_{\mathscr{A}}^{A\times A}$, which implies that there exist $k,m\in\mathbb{N}$ such that $E_k=E_{k+m}$, and by (b) we conclude that $E_k=E^{\mathrm{ri}}$. □

The above theorem gives a procedure for computing the greatest right invariant fuzzy equivalence contained in a given fuzzy equivalence E on a finite fuzzy automaton, which works if $\mathscr{L}$ is locally finite. But, it does not necessarily work if $\mathscr{L}$ is not locally finite, what the following example shows:

Example 4.1. Let $\mathscr{L}$ be the product structure, $\mathscr{A}=(A,X,\delta)$ a fuzzy automaton over $\mathscr{L}$ with $|A|=2$, $X=\{x\}$, and the fuzzy transition relation δ_x given by

$$\delta_x=\begin{bmatrix}0 & 1\\ \frac{1}{2} & 0\end{bmatrix}.$$

Let E be the universal relation on A, i.e., for every $a,b\in A$ we have that $E(a,b)=1$. Applying to E the procedure from Theorem 4.3, we obtain a sequence $\{E_k\}_{k\in\mathbb{N}}$ of fuzzy equivalences given by

$$E_k=\begin{bmatrix}1 & \frac{1}{2^{k-1}}\\ \frac{1}{2^{k-1}} & 1\end{bmatrix},\quad k\in\mathbb{N},$$

whose all members are different. We have that E^{ri} is the equality relation on A, i.e.,

$$E^{\mathrm{ri}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

since the equality relation is the only right invariant fuzzy equivalence on $\mathscr{A}$.

Therefore, if the complete residuated lattice $\mathscr{L}$ is not locally finite, we know that there exists the greatest right invariant fuzzy equivalence E^{ri} contained in E, but the problem is how to construct it.

In some cases, to reduce the fuzzy automaton $\mathscr{A}$, we can try to find the greatest right invariant crisp equivalence E° contained in a fuzzy equivalence E (or in the crisp part of E). This equivalence can have the same index as E^{ri}, so the factor fuzzy automata $\mathscr{A}_{E^{\circ}}$ and $\mathscr{A}_{E^{\mathrm{ri}}}$ would have the same number of states. The equivalence E° can be constructed by a procedure given by Petković [53], which can be stated as follows:

Theorem 4.4. *(See [53].) Let $\mathscr{A} = (A, X, \delta)$ be a fuzzy automaton, let ϱ be an equivalence on A and let ϱ° be the greatest right invariant equivalence contained in ϱ.*

Define inductively a sequence $\{\varrho_k\}_{k\in\mathbb{N}}$ of equivalences on A as follows:

$$\varrho_1 = \varrho, \qquad \varrho_{k+1} = \varrho_k \cap \varrho_k^r, \quad \text{for each } k \in \mathbb{N},$$

where ϱ_k^r is defined by

$$(a,b) \in \varrho_k^r \quad \Leftrightarrow \quad (\forall x \in X)(\forall c \in A)(\delta_x \circ \varrho_k)(a,c) = (\delta_x \circ \varrho_k)(b,c),$$

for all $a, b \in A$. Then

(a) $\varrho^{\circ} \leqslant \cdots \leqslant \varrho_{k+1} \leqslant \varrho_k \leqslant \cdots \leqslant \varrho_1 = \varrho$;
(b) *If $\varrho_k = \varrho_{k+m}$, for some $k, m \in \mathbb{N}$, then $\varrho_k = \varrho_{k+1} = \varrho^{\circ}$*;
(c) *If $\mathscr{A}$ is finite, then $\varrho_k = \varrho^{\circ}$ for some $k \in \mathbb{N}$.*

However, E° can have a greater index than E^{ri}, and hence, $\mathscr{A}_{E^{\circ}}$ can have more states than $\mathscr{A}_{E^{\mathrm{ri}}}$, as the following example shows.

Example 4.2. Let $\mathscr{L}$ be the Gödel structure, and let $\mathscr{A} = (A, X, \delta)$ be a fuzzy automaton over $\mathscr{L}$ with $|A| = 4$, $X = \{x, y\}$, and the fuzzy transition relations given by

$$\delta_x = \begin{bmatrix} 1 & 0.8 & 0.6 & 0.8 \\ 0.8 & 1 & 0.8 & 0.6 \\ 0.2 & 0.3 & 0.8 & 0.9 \\ 0.2 & 0.3 & 0.8 & 0.9 \end{bmatrix}, \qquad \delta_y = \begin{bmatrix} 0.8 & 1 & 0.6 & 0.8 \\ 1 & 0.6 & 0.5 & 0.9 \\ 0.3 & 0.2 & 0.4 & 0.8 \\ 0.5 & 0.3 & 0.3 & 1 \end{bmatrix}.$$

Let E be the universal relation on A. Applying to E the procedures from Theorems 4.3 and 4.4, we obtain that $E_2 = E_3 = E^{\mathrm{ri}}$ and $\varrho_3 = \varrho_4 = E^{\circ}$, where

$$E^{\mathrm{ri}} = \begin{bmatrix} 1 & 1 & 0.8 & 0.9 \\ 1 & 1 & 0.8 & 0.9 \\ 0.8 & 0.8 & 1 & 0.8 \\ 0.9 & 0.9 & 0.8 & 1 \end{bmatrix}, \qquad E^{\circ} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Therefore, E° does not make a reduction of $\mathscr{A}$, whereas E^{ri} has three classes, and it reduces $\mathscr{A}$ to a fuzzy automaton $\mathscr{A}_{E^{\mathrm{ri}}}$ having three states and transition matrices

$$\delta_x^{E^{\mathrm{ri}}} = \begin{bmatrix} 1 & 0.8 & 0.9 \\ 0.9 & 0.8 & 0.9 \\ 0.9 & 0.8 & 0.9 \end{bmatrix}, \qquad \delta_y^{E^{\mathrm{ri}}} = \begin{bmatrix} 1 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.8 \\ 0.9 & 0.8 & 1 \end{bmatrix},$$

with entries taken from the matrices $\delta_x \circ E^{\mathrm{ri}} = \delta_x \circ E_2$ and $\delta_y \circ E^{\mathrm{ri}} = \delta_y \circ E_2$.

Remark 4.3. Let the complete residuated lattice $\mathscr{L}$ have the property that $\bigvee K = 1$ implies $1 \in K$, for any finite $K \subseteq L$. This property is satisfied, for example, whenever $\mathscr{L}$ is linearly ordered.

Then for arbitrary fuzzy relations R and S on a finite set A we have that

$$\widehat{R \circ S} = \widehat{R} \circ \widehat{S}.$$

In this case, for any right invariant fuzzy equivalence E on a fuzzy automaton $\mathscr{A} = (A, X, \delta)$ we have that $\widehat{E}$ is a right invariant crisp equivalence on the non-deterministic automaton $\widehat{\mathscr{A}} = (A, X, \hat{\delta})$, the crisp part of $\mathscr{A}$. Also, for any fuzzy equivalence F on $\mathscr{A}$, the crisp part of F^{ri} is the greatest right invariant equivalence on $\widehat{\mathscr{A}}$ contained in $\widehat{F}$.

In Theorem 4.3 we have constructed a non-increasing sequence $\{E_k\}_{k\in\mathbb{N}}$ of fuzzy equivalences such that

$$E^{\mathrm{ri}} \leqslant \bigwedge_{k\in\mathbb{N}} E_k.$$

It is naturally to ask the question under which conditions the following is true

$$E^{\mathrm{ri}} = \bigwedge_{k\in\mathbb{N}} E_k.$$

This question will be considered in the further text. First we prove the following lemma.

Lemma 4.2. *Let $\mathscr{L}$ be a complete residuated lattice satisfying condition*

$$\bigwedge_{i\in I}(x \vee y_i) = x \vee \left(\bigwedge_{i\in I} y_i\right), \tag{40}$$

for all $x \in L$ and $\{y_i\}_{i\in I} \subseteq L$. Then for all non-increasing sequences $\{x_k\}_{k\in\mathbb{N}}, \{y_k\}_{k\in\mathbb{N}} \subseteq L$ we have

$$\bigwedge_{k\in\mathbb{N}}(x_k \vee y_k) = \left(\bigwedge_{k\in\mathbb{N}} x_k\right) \vee \left(\bigwedge_{k\in\mathbb{N}} y_k\right). \tag{41}$$

Proof. Consider arbitrary non-increasing sequences $\{x_k\}_{k\in\mathbb{N}}, \{y_k\}_{k\in\mathbb{N}} \subseteq L$ and arbitrary $m, n \in \mathbb{N}$. Since these sequences are non-increasing, for each $l \in \mathbb{N}$, $l \geqslant m, n$, we have that $x_l \vee y_l \leqslant x_m \vee y_n$, so

$$\bigwedge_{k\in\mathbb{N}}(x_k \vee y_k) \leqslant \bigwedge_{l\geqslant m,n}(x_l \vee y_l) \leqslant x_m \vee y_n.$$

This inequality holds for every $m, n \in \mathbb{N}$, and by (40) we obtain that

$$\bigwedge_{k\in\mathbb{N}}(x_k \vee y_k) \leqslant \bigwedge_{m\in\mathbb{N}}\left(\bigwedge_{n\in\mathbb{N}}(x_m \vee y_n)\right) = \bigwedge_{m\in\mathbb{N}}\left(x_m \vee \left(\bigwedge_{n\in\mathbb{N}} y_n\right)\right) = \left(\bigwedge_{m\in\mathbb{N}} x_m\right) \vee \left(\bigwedge_{n\in\mathbb{N}} y_n\right).$$

Since the opposite inequality is evident, we conclude that (41) is true. □

Now we are ready to state and prove the following theorem.

Theorem 4.5. *Let $\mathscr{L}$ be a complete residuated lattice satisfying condition* (40) *and*

$$\bigwedge_{i\in I}(x \otimes y_i) = x \otimes \left(\bigwedge_{i\in I} y_i\right), \tag{42}$$

for all $x \in L$ and $\{y_i\}_{i\in I} \subseteq L$. Let $\mathscr{A} = (A, X, \delta)$ be a fuzzy finite automaton over $\mathscr{L}$, let E be a fuzzy equivalence on A, let E^{ri} be the greatest right invariant fuzzy equivalence on $\mathscr{A}$ contained in E, and let $\{E_k\}_{k\in\mathbb{N}}$ be the sequence of fuzzy equivalences on A defined by (39)*. Then*

$$E^{\mathrm{ri}} = \bigwedge_{k\in\mathbb{N}} E_k. \tag{43}$$

Proof. For the sake of simplicity set

$$F = \bigwedge_{k\in\mathbb{N}} E_k.$$

Clearly, F is a fuzzy equivalence. To prove the equality (43) it is enough to prove that F is a right invariant fuzzy equivalence on $\mathscr{A}$ (because of Theorem 4.3). First, we have that

$$F(a,b) \leqslant E_{k+1}(a,b) \leqslant E_k^{\mathrm{r}}(a,b) \leqslant (\delta_x \circ E_k)(a,c) \leftrightarrow (\delta_x \circ E_k)(b,c), \tag{44}$$

for all $a, b, c \in A$, $x \in X$ and $k \in \mathbb{N}$. Now, by (44) and (5) we obtain that

$$F(a,b) \leqslant \bigwedge_{k\in\mathbb{N}}\big((\delta_x \circ E_k)(a,c) \leftrightarrow (\delta_x \circ E_k)(b,c)\big) \leqslant \bigwedge_{k\in\mathbb{N}}\big((\delta_x \circ E_k)(a,c)\big) \leftrightarrow \bigwedge_{k\in\mathbb{N}}\big((\delta_x \circ E_k)(b,c)\big), \tag{45}$$

for all $a, b, c \in A$ and $x \in X$. Next,

$$\begin{aligned}
\bigwedge_{k\in\mathbb{N}} \big((\delta_x \circ E_k)(a,c)\big) &= \bigwedge_{k\in\mathbb{N}} \Big(\bigvee_{d\in A} \big(\delta_x(a,d) \otimes E_k(d,c)\big)\Big) \\
&= \bigvee_{d\in A} \Big(\bigwedge_{k\in\mathbb{N}} \big(\delta_x(a,d) \otimes E_k(d,c)\big)\Big) \quad \big(\text{by } (41)\big) \\
&= \bigvee_{d\in A} \Big(\delta_x(a,d) \otimes \Big(\bigwedge_{k\in\mathbb{N}} E_k(d,c)\Big)\Big) \quad \big(\text{by } (42)\big) \\
&= \bigvee_{d\in A} \big(\delta_x(a,d) \otimes F(d,c)\big) = (\delta_x \circ F)(a,c). \qquad (46)
\end{aligned}$$

Using of condition (41) is justified by the facts that A is finite, and that $\{E_k(d,c)\}_{k\in\mathbb{N}}$ is a non-increasing sequence, so $\{\delta_x(a,d) \otimes E_k(d,c)\}_{k\in\mathbb{N}}$ is also a non-increasing sequence. In the same way we prove that

$$\bigwedge_{k\in\mathbb{N}} \big((\delta_x \circ E_k)(b,c)\big) = (\delta_x \circ F)(b,c). \qquad (47)$$

Therefore, by (45), (46) and (47) we obtain that

$$F(a,b) \leqslant (\delta_x \circ F)(a,c) \leftrightarrow (\delta_x \circ F)(b,c).$$

Since this inequality holds for all $x \in X$ and $c \in A$, we have that

$$F(a,b) \leqslant \bigwedge_{x\in X} \bigwedge_{c\in A} (\delta_x \circ F)(a,c) \leftrightarrow (\delta_x \circ F)(b,c),$$

and by (iii) of Theorem 4.1 we obtain that F is a right invariant fuzzy equivalence on $\mathscr{A}$. □

Remark 4.4. Let $\mathscr{L} = ([0,1], \wedge, \vee, \otimes, \to, 0, 1)$, where $[0,1]$ is the real unit interval and $\otimes$ is a left-continuous t-norm on $[0,1]$. Then $\mathscr{L}$ satisfies condition (40), because of the linearity.

Moreover, it is well known that $\mathscr{L}$ satisfies condition (42) if and only if $\otimes$ is a continuous t-norm, that is, if and only if $\mathscr{L}$ is a *BL*-algebra (cf. [2,3]). Therefore, the assertion of Theorem 4.5 is true for every *BL*-algebra on the real unit interval. In particular, the Łukasiewicz, Goguen (product) and Gödel structures fulfill the conditions of the theorem.

5. Right, left and alternate reductions

In this section we consider reductions of fuzzy automata by means of the greatest right invariant and greatest left invariant fuzzy equivalences, and we show that better reductions can be achieved alternating reductions by means of greatest right invariant and greatest left invariant fuzzy equivalences.

First we consider reductions by means of greatest right invariant fuzzy equivalences. Let $\mathscr{A}$ be a fuzzy automaton. A sequence $\mathscr{A}_1, \mathscr{A}_2, \dots, \mathscr{A}_n$ of fuzzy automata will be called a *right reduction* of $\mathscr{A}$ if $\mathscr{A}_1 = \mathscr{A}$ and for each $k \in \{1, 2, \dots, n-1\}$ we have that $\mathscr{A}_{k+1}$ is the factor fuzzy automaton of $\mathscr{A}_k$ w.r.t. the greatest right invariant fuzzy equivalence on $\mathscr{A}_k$. Analogously we define a *left reduction* of $\mathscr{A}$.

Let us note that for each fuzzy finite automaton $\mathscr{A}$ there exists a right reduction $\mathscr{A}_1, \mathscr{A}_2, \dots, \mathscr{A}_n$ of $\mathscr{A}$ such that for every right reduction $\mathscr{A}_1, \mathscr{A}_2, \dots, \mathscr{A}_n, \mathscr{A}_{n+1}, \dots, \mathscr{A}_{n+m}$ of $\mathscr{A}$ which is a continuation of this reduction we have that

$$|\mathscr{A}_n| = |\mathscr{A}_{n+1}| = \cdots = |\mathscr{A}_{n+m}|,$$

i.e., all fuzzy automata $\mathscr{A}_{n+1}, \dots, \mathscr{A}_{n+m}$ have the same number of states as $\mathscr{A}_n$. Also, there is a shortest right reduction $\mathscr{A}_1, \mathscr{A}_2, \dots, \mathscr{A}_n$ of $\mathscr{A}$ having this property, which will be called the *shortest right reduction* of $\mathscr{A}$, the fuzzy automaton $\mathscr{A}_n$ will be called a *right reduct* of $\mathscr{A}$, and the number n will be called the *length* of this shortest right reduction. If the fuzzy automaton $\mathscr{A}$ is its own right reduct, then it is called *right reduced*. Analogously we define a *shortest left reduction* of $\mathscr{A}$, a *left reduct* of $\mathscr{A}$, the length of the shortest left reduction and a *left reduced* fuzzy automaton.

We will show that the length of the shortest right and left reductions do not exceed 2. But, first we prove the following theorem.

Theorem 5.1. *Let $\mathscr{A} = (A, X, \delta)$ be a fuzzy automaton, let E be a right invariant fuzzy equivalence on $\mathscr{A}$ and let F be a fuzzy equivalence on A such that $E \leqslant F$. Then*

(a) *F is a right invariant fuzzy equivalence on $\mathscr{A}$ if and only if F_E is a right invariant fuzzy equivalence on $\mathscr{A}_E$;*
(b) *F is the greatest right invariant fuzzy equivalence on $\mathscr{A}$ if and only if F_E is the greatest right invariant fuzzy equivalence on $\mathscr{A}_E$;*

(c) *E is the greatest right invariant fuzzy equivalence on $\mathscr{A}$ if and only if $\widetilde{E}$ is the greatest right invariant fuzzy equivalence on the factor fuzzy automaton $\mathscr{A}_E$.*

Proof. (a) First we note that $E \leqslant F$ is equivalent to $E \circ F = F \circ E = F$. Next, consider arbitrary $a, b \in A$ and $x \in X$. By the proof of Theorem 3.1 we obtain that

$$\big(F_E \circ \delta_x^E \circ F_E\big)(E_a, E_b) = (F \circ \delta_x \circ F)(a, b),$$

and also,

$$\begin{aligned}\big(\delta_x^E \circ F_E\big)(E_a, E_b) &= \bigvee_{c \in A} \delta_x^E(E_a, E_c) \otimes F_E(E_c, E_b) \\ &= \bigvee_{c \in A} (E \circ \delta_x \circ E)(a, c) \otimes F(c, b) \\ &= (E \circ \delta_x \circ E \circ F)(a, b) = (\delta_x \circ E \circ F)(a, b) = (\delta_x \circ F)(a, b).\end{aligned}$$

Therefore,

$$F_E \circ \delta_x^E \circ F_E = \delta_x^E \circ F_E \quad \Leftrightarrow \quad F \circ \delta_x \circ F = \delta_x \circ F,$$

and we conclude that the assertion (a) holds.

(b) Let F be the greatest right invariant fuzzy equivalence on $\mathscr{A}$. According to (a), F_E is a right invariant fuzzy equivalence on $\mathscr{A}_E$. Assume that Q is the greatest right invariant fuzzy equivalence on $\mathscr{A}_E$. Define a fuzzy relation G on A by

$$G(a, b) = Q(E_a, E_b), \quad \text{for all } a, b \in A.$$

It is easy to verify that G is a fuzzy equivalence on $\mathscr{A}$. According to (a) of this theorem, $\widetilde{E}$ is a right invariant fuzzy equivalence on $\mathscr{A}_E$, what implies $\widetilde{E} \leqslant Q$, and for arbitrary $a, b \in A$ we obtain that

$$E(a, b) = \widetilde{E}(E_a, E_b) \leqslant Q(E_a, E_b) = G(a, b),$$

what means that $E \leqslant G$. Therefore, we have that $Q = G_E$, and by (a) we obtain that G is a right invariant fuzzy equivalence on $\mathscr{A}$, what implies $G \leqslant F$. Now, according to (22), we have that $Q = G_E \leqslant F_E$, and since F_E is a right invariant fuzzy equivalence on $\mathscr{A}_E$, we conclude that $Q = F_E$, i.e., F_E is the greatest right invariant fuzzy equivalence on $\mathscr{A}_E$.

Conversely, let F_E be the greatest right invariant fuzzy equivalence on $\mathscr{A}_E$. According to (a), F is a right invariant fuzzy equivalence on $\mathscr{A}$. Let G be the greatest right invariant fuzzy equivalence on $\mathscr{A}$. Then we have that $E \leqslant F \leqslant G$, and by (a) it follows that G_E is a right invariant fuzzy equivalence on $\mathscr{A}_E$, what yields $G_E \leqslant F_E$. Now, by (22) it follows that $G \leqslant F$, and hence, $G = F$, so we have proved that F is the greatest right invariant fuzzy equivalence on $\mathscr{A}$.

(c) This follows immediately by (b) and the fact that $E_E = \widetilde{E}$. $\square$

Now we prove the following:

Theorem 5.2. *A fuzzy automaton $\mathscr{A} = (A, X, \delta)$ is right reduced if and only if the greatest right invariant fuzzy equivalence on $\mathscr{A}$ is a fuzzy equality.*

Consequently, for every fuzzy automaton $\mathscr{A} = (A, X, \delta)$, its right reduct is the factor fuzzy automaton $\mathscr{A}_E$, where E is the greatest right invariant fuzzy equivalence on $\mathscr{A}$.

Proof. Let us denote by E the greatest right invariant fuzzy equivalence on $\mathscr{A}$.

Let the fuzzy automaton $\mathscr{A}$ be right reduced. If E is not a fuzzy equality, i.e., if $E(a, b) = 1$, for some $a, b \in A$, $a \neq b$, then $E_a = E_b$, with $a \neq b$, and by $a \mapsto E_a$ a surjective mapping of A onto A_E is defined, which is not injective, what yields $|\mathscr{A}_E| < |\mathscr{A}|$. But, this contradicts our starting hypothesis that $\mathscr{A}$ is right reduced. Thus, we conclude that E is a fuzzy equality.

Conversely, let E be a fuzzy equality. Consider an arbitrary right reduction $\mathscr{A}_1 = \mathscr{A}, \mathscr{A}_2, \dots, \mathscr{A}_n$ of $\mathscr{A}$. For each $k \in \{1, 2, \dots, n\}$ let E_k be the greatest right invariant fuzzy equivalence on $\mathscr{A}_k$. By Theorem 5.1, for each $k \in \{2, \dots, n\}$ we have that $E_k = \widetilde{E}_{k-1}$, so E_k is a fuzzy equality, and by the hypothesis, $E_1 = E$ is a fuzzy equality. Now, for every $k \in \{2, \dots, n\}$ we have that $|\mathscr{A}_k| = |(\mathscr{A}_{k-1})_{E_{k-1}}| = |\mathscr{A}_{k-1}|$, and hence, $|\mathscr{A}| = |\mathscr{A}_1| = |\mathscr{A}_2| = \cdots = |\mathscr{A}_n|$. Therefore, we conclude that the fuzzy automaton $\mathscr{A}$ is right reduced.

Furthermore, if $\mathscr{A}$ is an arbitrary fuzzy automaton and E is the greatest right invariant fuzzy equivalence on $\mathscr{A}$, then by Theorem 5.1 we have that $\widetilde{E}$ is the greatest right invariant fuzzy equivalence on the factor fuzzy automaton $\mathscr{A}_E$, and since it is a fuzzy equality, we obtain that $\mathscr{A}_E$ is right reduced and it is the right reduct of $\mathscr{A}$. $\square$

If a fuzzy automaton $\mathscr{A} = (A, X, \delta)$ is right reduced, i.e., if the greatest right invariant fuzzy equivalence E on $\mathscr{A}$ is a fuzzy equality, then the factor fuzzy automaton $\mathscr{A}_E$ has the same cardinality as $\mathscr{A}$, but it is not necessary isomorphic to $\mathscr{A}$ (see Example 5.1). If the factor automaton $\mathscr{A}_E$ is isomorphic to $\mathscr{A}$, then $\mathscr{A}$ is called *completely right reduced*. Analogously we define *completely left reduced* fuzzy automata.

Example 5.1. Let $\mathscr{L}$ be the Gödel structure, and let $\mathscr{A} = (A, X, \delta)$ be a fuzzy automaton over $\mathscr{L}$ with $|A| = 2$, $X = \{x, y\}$ and the fuzzy transition relations δ_x and δ_y given by

$$\delta_x = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0 \end{bmatrix}, \qquad \delta_y = \begin{bmatrix} 1 & 1 \\ 0 & 0.5 \end{bmatrix}.$$

The greatest right invariant fuzzy equivalence E^{ri} and the greatest left invariant fuzzy equivalence E^{li} on $\mathscr{A}$ are given by

$$E^{\mathrm{ri}} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}, \qquad E^{\mathrm{li}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, $\mathscr{A}$ is both right and left reduced.

The factor fuzzy automaton $\mathscr{A}_2 = \mathscr{A}_{E^{\mathrm{ri}}} = (A_2, X, \delta^2)$ has also two states and the fuzzy transition relations δ_x^2 and δ_y^2 are given by

$$\delta_x^2 = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}, \qquad \delta_y^2 = \begin{bmatrix} 1 & 1 \\ 0.5 & 0.5 \end{bmatrix}.$$

Evidently, the fuzzy automaton $\mathscr{A}_2$ is not isomorphic to $\mathscr{A}$, so $\mathscr{A}$ is not completely right reduced. On the other hand, the factor fuzzy automaton $\mathscr{A}_{E^{\mathrm{li}}}$ is isomorphic to $\mathscr{A}$, and $\mathscr{A}$ is completely left reduced.

According to Theorem 5.2, the fuzzy automaton $\mathscr{A}_2$ is right reduced. The greatest right invariant fuzzy equivalence E_2^{ri} and the greatest left invariant fuzzy equivalence E_2^{li} on $\mathscr{A}_2$ are given by

$$E_2^{\mathrm{ri}} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}, \qquad E_2^{\mathrm{li}} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

so the factor fuzzy automaton $(\mathscr{A}_2)_{E_2^{\mathrm{ri}}}$ is isomorphic to $\mathscr{A}_2$, and hence, $\mathscr{A}_2$ is completely right reduced, whereas the factor fuzzy automaton $\mathscr{A}_3 = (\mathscr{A}_2)_{E_2^{\mathrm{li}}} = (A_3, X, \delta^3)$ has one state and the fuzzy transition relations δ_x^3 and δ_y^3 are given by $\delta_x^3 = [\,0.5\,]$ and $\delta_y^3 = [\,1\,]$.

Therefore, although the fuzzy automaton $\mathscr{A}$ is both right reduced and left reduced, factorizing it by its greatest right invariant fuzzy equivalence we obtain the fuzzy automaton $\mathscr{A}_2$ whose number of states can be further reduced factorizing it by its greatest left invariant fuzzy equivalence.

The previous example shows that even if a fuzzy automaton is right of left reduced, we can continue the reduction of the number of states alternating reductions by means of greatest right invariant and greatest left invariant fuzzy equivalences. For that reason we introduce the following concepts.

Let $\mathscr{A}$ be a fuzzy automaton. A sequence $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_n$ of fuzzy automata will be called an *alternate reduction* of $\mathscr{A}$ if $\mathscr{A}_1 = \mathscr{A}$ and for every $k \in \{1, 2, \ldots, n-2\}$ the following is true:

(1) $\mathscr{A}_{k+1}$ is the factor fuzzy automaton of $\mathscr{A}_k$ w.r.t. the greatest right invariant or the greatest left invariant fuzzy equivalence on $\mathscr{A}_k$;
(2) If $\mathscr{A}_{k+1}$ is the factor fuzzy automaton of $\mathscr{A}_k$ w.r.t. the greatest right invariant fuzzy equivalence on $\mathscr{A}_k$, then $\mathscr{A}_{k+2}$ is the factor fuzzy automaton of $\mathscr{A}_{k+1}$ w.r.t. the greatest left invariant fuzzy equivalence on $\mathscr{A}_k$;
(3) If $\mathscr{A}_{k+1}$ is the factor fuzzy automaton of $\mathscr{A}_k$ w.r.t. the greatest left invariant fuzzy equivalence on $\mathscr{A}_k$, then $\mathscr{A}_{k+2}$ is the factor fuzzy automaton of $\mathscr{A}_{k+1}$ w.r.t. the greatest right invariant fuzzy equivalence on $\mathscr{A}_k$.

If $\mathscr{A}_2$ is the factor fuzzy automaton of $\mathscr{A}_1$ w.r.t. the greatest right invariant fuzzy equivalence on $\mathscr{A}_1$, then this alternate reduction is called a *right–left alternate reduction*, and if $\mathscr{A}_2$ is the factor fuzzy automaton of $\mathscr{A}_1$ w.r.t. the greatest left invariant fuzzy equivalence on $\mathscr{A}_1$, then this alternate reduction is called a *left–right alternate reduction*.

Note that for each fuzzy finite automaton $\mathscr{A}$ there exists a right–left alternate reduction $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_n$ of $\mathscr{A}$ such that for every right–left alternate reduction $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_n, \mathscr{A}_{n+1}, \ldots, \mathscr{A}_{n+m}$ which is a continuation of this reduction we have that

$$|\mathscr{A}_n| = |\mathscr{A}_{n+1}| = \cdots = |\mathscr{A}_{n+m}|,$$

i.e., all fuzzy automata $\mathscr{A}_{n+1}, \ldots, \mathscr{A}_{n+m}$ have the same number of states as $\mathscr{A}_n$. Also, there is a shortest right–left alternate reduction $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_n$ of $\mathscr{A}$ having this property, which will be called the *shortest right–left alternate reduction* of $\mathscr{A}$, and $\mathscr{A}_n$ will be called a *right–left alternate reduct* of $\mathscr{A}$, whereas the number n will be called the *length* of the shortest right–left alternate reduction of $\mathscr{A}$. Analogously we define a *shortest left–right alternate reduction*, its length, and a *left–right alternate reduct* of $\mathscr{A}$.

The next example shows that the right–left and the left–right alternate reducts of a fuzzy automaton can have different number of states, and that the shortest right–left and left–right alternate reductions can have different lengths.

Example 5.2. Let $\mathscr{L}$ be the Gödel structure, and let $\mathscr{A}_k = (A_k, X, \delta^k)$, $k \in \{1,2,3\}$, be a sequence of fuzzy automata over $\mathscr{L}$ such that $|\mathscr{A}_1| = 3$, $|\mathscr{A}_2| = 2$, $|\mathscr{A}_3| = 1$, $X = \{x\}$, and the fuzzy transition relations δ_x^k, $k \in \{1,2,3\}$, are given by

$$\delta_x^1 = \begin{bmatrix} 0.5 & 0.5 & 0.3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \delta_x^2 = \begin{bmatrix} 0.5 & 0.5 \\ 0 & 0 \end{bmatrix}, \qquad \delta_x^3 = [\,0.5\,].$$

Then the greatest right invariant fuzzy equivalence E_1^{ri} on $\mathscr{A}_1$ and the greatest left invariant fuzzy equivalence E_1^{li} on $\mathscr{A}_2$ are given by

$$E_1^{\mathrm{ri}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \qquad E_2^{\mathrm{li}} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

and the sequence $\mathscr{A}_1$, $\mathscr{A}_2$, $\mathscr{A}_3$ is a right–left alternate reduction of the fuzzy automaton $\mathscr{A}_1$, since we have that $\mathscr{A}_2 = (\mathscr{A}_1)_{E_1^{\mathrm{ri}}}$ and $\mathscr{A}_3 = (\mathscr{A}_2)_{E_2^{\mathrm{li}}}$. Clearly, the number of states of the fuzzy automaton $\mathscr{A}_3$ cannot be reduced, so the sequence $\mathscr{A}_1$, $\mathscr{A}_2$, $\mathscr{A}_3$ is the shortest right–left alternate reduction of $\mathscr{A}_1$, and hence, $\mathscr{A}_3$ is the right–left alternate reduct of $\mathscr{A}_1$.

On the other hand, let $\mathscr{A}'_m = (A'_m, X, \delta^{m\prime})$, $m \in \{1,2,3,4\}$, be a sequence of fuzzy automata such that $\mathscr{A}'_1 = \mathscr{A}_1$, $|\mathscr{A}_2| = |\mathscr{A}_3| = |\mathscr{A}_4| = 1$, $X = \{x\}$, and the fuzzy transition relations $\delta_x^{m\prime}$, $m \in \{2,3,4\}$, are given by

$$\delta_x^{2\prime} = \delta_x^{3\prime} = \delta_x^{4\prime} = \begin{bmatrix} 0.5 & 0.3 \\ 0.3 & 0.3 \end{bmatrix}.$$

Then the greatest left invariant fuzzy equivalences F_1^{li} on $\mathscr{A}'_1 = \mathscr{A}_1$ and F_3^{li} on $\mathscr{A}'_3$, and the greatest right invariant fuzzy equivalence F_2^{ri} on $\mathscr{A}'_2$ are given by

$$F_1^{\mathrm{li}} = \begin{bmatrix} 1 & 1 & 0.3 \\ 1 & 1 & 0.3 \\ 0.3 & 0.3 & 1 \end{bmatrix}, \qquad F_2^{\mathrm{ri}} = \begin{bmatrix} 1 & 0.3 \\ 0.3 & 1 \end{bmatrix}, \qquad F_3^{\mathrm{li}} = \begin{bmatrix} 1 & 0.3 \\ 0.3 & 1 \end{bmatrix},$$

and the sequence $\mathscr{A}_1$, $\mathscr{A}'_2$, $\mathscr{A}'_3$, $\mathscr{A}'_4$ is a left–right alternate reduction of the fuzzy automaton $\mathscr{A}_1$, since we have that $\mathscr{A}'_2 = (\mathscr{A}_1)_{F_1^{\mathrm{li}}}$, $\mathscr{A}'_3 = (\mathscr{A}'_2)_{F_2^{\mathrm{ri}}}$ and $\mathscr{A}'_4 = (\mathscr{A}'_3)_{F_2^{\mathrm{li}}}$. Also, we have that $\mathscr{A}'_2 \cong \mathscr{A}'_3 \cong \mathscr{A}'_4$, and we conclude that the number of states of $\mathscr{A}'_2$ cannot be reduced by means of right or left invariant fuzzy equivalences. Thus, the sequence $\mathscr{A}_1$, $\mathscr{A}'_2$ is the shortest left–right alternate reduction of $\mathscr{A}_1$, and $\mathscr{A}'_2$ is the left–right alternate reduct of $\mathscr{A}_1$.

Therefore, this example shows that the right–left and the left–right alternate reductions of a fuzzy automaton do not necessarily give fuzzy automata with the same number of states. Namely, the right–left alternate reduct of $\mathscr{A}_1$ has one state, and the left–right alternate reduct of $\mathscr{A}_1$ has two states. This example also shows that the shortest right–left and left–right alternate reductions do not necessarily have the same length, since the shortest right–left alternate reduction of $\mathscr{A}_1$ has length 3, whereas the shortest left–right alternate reduction of $\mathscr{A}_1$ has length 2.

In the alternate reductions considered in the previous example we were able to recognize that we have reached the smallest number of states for two reasons. First, in the right–left reduction we have obtained a fuzzy automaton with only one state, which cannot be further reduced. On the other hand, in the left–right reduction we have obtained three consecutive members which are isomorphic, and by this fact we have concluded that the number of states cannot be further reduced. However, we have not yet a general procedure to decide whether we have reached the smallest number of states in an alternate reduction.

In contrast to fuzzy automata, in the case of non-deterministic automata there is a general procedure to recognize whether we have reached the smallest number of states in an alternate reduction. Namely, if after two successive steps the number of states did not change, then we have reached the smallest number of states and this alternate reduction is finished. In other words, an alternate reduction finishes when we obtain a non-deterministic automaton which is both right and left reduced, and this automaton is an alternate reduct of the staring automaton. This does not hold for fuzzy automata because factorizing a fuzzy automaton by a fuzzy equality we change the fuzzy transition function and we obtain a fuzzy automaton which is not necessarily isomorphic to the original fuzzy automaton. As Example 5.1 shows, even if a fuzzy automaton $\mathscr{A}$ is both right and left reduced, factorizing it by its greatest right invariant fuzzy equivalence we can obtain a fuzzy automaton whose number of states can be further reduced factorizing it by its greatest left invariant fuzzy equivalence.

6. Strongly right invariant and strongly left invariant fuzzy equivalences

Let $\mathscr{A} = (A, X, \delta)$ be a fuzzy automaton. A particular type of right invariant fuzzy equivalences on $\mathscr{A}$ are fuzzy equivalences on A which are solutions to the following system of fuzzy relation equations

$$E \circ \delta_x = \delta_x, \quad x \in X. \tag{48}$$

They are called *strongly right invariant fuzzy equivalences* on $\mathscr{A}$. Similarly, a particular type of left invariant fuzzy equivalences on $\mathscr{A}$ are fuzzy equivalences on A which are solutions to system

$$\delta_x \circ E = \delta_x, \quad x \in X. \tag{49}$$

They are called *strongly left invariant fuzzy equivalences* on $\mathscr{A}$. Let us note that a fuzzy equivalence on $\mathscr{A}$ is both a strongly right invariant and a strongly left invariant fuzzy equivalence on $\mathscr{A}$ if and only if it is a solution to system

$$E \circ \delta_x \circ E = \delta_x, \quad x \in X. \tag{50}$$

In the further text we study strongly right invariant fuzzy equivalences. The corresponding results concerning strongly left invariant fuzzy equivalences can be derived by symmetry from those concerning strongly right invariant fuzzy equivalences.

Let E be a fuzzy equivalence on a set A and let R be a fuzzy relation on A. Then R is said to be *right extensional* w.r.t. E if

$$E(a, b) \otimes R(a, c) \leqslant R(b, c), \tag{51}$$

for all $a, b, c \in A$, and R is *left extensional* w.r.t. E if

$$E(b, a) \otimes R(c, a) \leqslant R(c, b), \tag{52}$$

for all $a, b, c \in A$. It is clear that R is right (resp. left) extensional w.r.t. E if and only if every foreset (resp. afterset) of R is extensional w.r.t. E.

The following theorem gives various characterizations of strongly right invariant fuzzy equivalences.

Theorem 6.1. *Let $\mathscr{A} = (A, X, \delta)$ be a fuzzy automaton and E a fuzzy equivalence on A. Then the following conditions are equivalent*:

(i) *E is a strongly right invariant fuzzy equivalence*;
(ii) *$E \circ \delta_x \leqslant \delta_x$, for every $x \in X$*;
(iii) *δ_x is right extensional w.r.t. E, for every $x \in X$*;
(iv) *for all $a, b \in A$ we have*

$$E(a, b) \leqslant \bigwedge_{x\in X} \bigwedge_{c\in A} \delta_x(a, c) \leftrightarrow \delta_x(b, c). \tag{53}$$

Proof. The inequality $\delta_x \leqslant E \circ \delta_x$ follows by the reflexivity of E, so (ii) $\Rightarrow$ (i), and the opposite implication (i) $\Rightarrow$ (ii) is obvious. According to (51), (ii) is equivalent to (iii).

(iii) $\Leftrightarrow$ (iv). According to (51), we have that (iii) is equivalent to

$$E(a, b) \leqslant \delta_x(a, c) \leftrightarrow \delta_x(b, c),$$

for all $a, b, c \in A$ and $x \in X$, what is evidently equivalent to (53). Therefore, we have proved that (iii) is equivalent to (iv). $\square$

By the following theorem we give characterizations of the greatest strongly right invariant fuzzy equivalence on a fuzzy automaton, as well as of the lattice of all strongly right invariant fuzzy equivalences on a fuzzy automaton.

Theorem 6.2. *Let $\mathscr{A} = (A, X, \delta)$ be a fuzzy automaton. Then the following statements hold.*

(a) *A fuzzy equivalence E^{sri} on A defined by*

$$E^{\text{sri}}(a, b) = \bigwedge_{x\in X} \bigwedge_{c\in A} \delta_x(a, c) \leftrightarrow \delta_x(b, c), \tag{54}$$

for all $a, b \in A$, is the greatest strongly right invariant fuzzy equivalence on $\mathscr{A}$.

(b) *The set $\mathscr{E}^{\text{sri}}(A)$ of all strongly right invariant fuzzy equivalences on $\mathscr{A}$ is the principal ideal of the lattice $\mathscr{E}(A)$ of all fuzzy equivalences on A generated by the fuzzy equivalence E^{sri}.*

(c) *For an arbitrary fuzzy equivalence E on A we have that $E \wedge E^{\text{sri}}$ is the greatest strongly right invariant fuzzy equivalence on $\mathscr{A}$ contained in E.*

Proof. (a) By the Valverde's Representation Theorem [62] (see also [2,15,16]) we have that E^{sri} is a fuzzy equivalence on A, and by (iv) of Theorem 6.1 we obtain that E^{sri} is a strongly right invariant fuzzy equivalence, and that it is the greatest such fuzzy equivalence on $\mathscr{A}$.

The assertions (b) and (c) are also immediate consequences of condition (iv) of Theorem 6.1. □

Let $\mathscr{A} = (A, X, \delta)$ be a fuzzy automaton and let E be a fuzzy equivalence on A. Then we construct a new fuzzy automaton $\mathscr{A}^{(E)} = (A, X, \delta^{(E)})$, with the same set of states and the same input alphabet as $\mathscr{A}$, and with fuzzy transition relations defined by:

$$\delta_x^{(E)} = E \circ \delta_x \circ E, \quad \text{for every } x \in X. \tag{55}$$

We have the following:

Lemma 6.1. *Let $\mathscr{A} = (A, X, \delta)$ be a fuzzy automaton and let E be a fuzzy equivalence on A. Then*

(a) *E is both a strongly right invariant and a strongly left invariant fuzzy equivalence on $\mathscr{A}^{(E)}$;*
(b) *$\widetilde{E}$ is both a strongly right invariant and a strongly left invariant fuzzy equivalence on $\mathscr{A}_E$.*

Proof. (a) For each $x \in X$ we have that

$$E \circ \delta_x^{(E)} \circ E = E \circ E \circ \delta_x \circ E \circ E = E \circ \delta_x \circ E = \delta_x^{(E)},$$

and hence, E is both a strongly right invariant and a strongly left invariant fuzzy equivalence on $\mathscr{A}^{(E)}$.

(b) For arbitrary $a, b \in A$ and $x \in X$ we have that

$$\begin{aligned}\big(\widetilde{E} \circ \delta_x^E \circ \widetilde{E}\big)(E_a, E_b) &= \bigvee_{c,d \in A} \widetilde{E}(E_a, E_c) \otimes \delta_x^E(E_c, E_d) \otimes \widetilde{E}(E_d, E_b) \\ &= \bigvee_{c,d \in A} E(a, c) \otimes (E \circ \delta_x \circ E)(c, d) \otimes E(d, b) \\ &= (E \circ E \circ \delta_x \circ E \circ E)(a, b) = (E \circ \delta_x \circ E)(a, b) = \delta_x^E(E_a, E_b).\end{aligned}$$

Therefore, $\widetilde{E} \circ \delta_x^E \circ \widetilde{E} = \delta_x^E$, so $\widetilde{E}$ is both a strongly right invariant and a strongly left invariant fuzzy equivalence on the factor fuzzy automaton $\mathscr{A}_E$. □

The next example shows that the greatest strongly right invariant fuzzy equivalence on a fuzzy automaton may be strictly smaller than the greatest right invariant fuzzy equivalence, and consequently, the greatest right invariant fuzzy equivalence can give better reduction of a fuzzy automaton than the greatest strongly right invariant fuzzy equivalence.

Example 6.1. Let $\mathscr{L}$ be the Gödel structure, and let $\mathscr{A} = (A, X, \delta)$ be a fuzzy automaton over $\mathscr{L}$ with $|A| = 4$, $X = \{x\}$, and the fuzzy transition relation δ_x given by

$$\delta_x = \begin{bmatrix} 1 & 0.8 & 0.6 & 0.8 \\ 0.8 & 1 & 0.8 & 0.6 \\ 0.2 & 0.3 & 0.8 & 0.9 \\ 0.2 & 0.3 & 0.8 & 0.9 \end{bmatrix}.$$

Then the greatest right invariant fuzzy equivalence E^{ri} and the greatest strongly right invariant fuzzy equivalence E^{sri} on $\mathscr{A}$ are given by

$$E^{\mathrm{ri}} = \begin{bmatrix} 1 & 1 & 0.9 & 0.9 \\ 1 & 1 & 0.9 & 0.9 \\ 0.9 & 0.9 & 1 & 1 \\ 0.9 & 0.9 & 1 & 1 \end{bmatrix}, \qquad E^{\mathrm{sri}} = \begin{bmatrix} 1 & 0.6 & 0.2 & 0.2 \\ 0.6 & 1 & 0.2 & 0.2 \\ 0.2 & 0.2 & 1 & 1 \\ 0.2 & 0.2 & 1 & 1 \end{bmatrix}.$$

Therefore, the factor fuzzy automaton $\mathscr{A}_2 = \mathscr{A}_{E^{\mathrm{ri}}} = (A_2, X, \delta^2)$ has two states, whereas the factor fuzzy automaton $\mathscr{A}_3 = \mathscr{A}_{E^{\mathrm{sri}}} = (A_3, X, \delta^3)$ has three states. The fuzzy transition relations δ_x^2 and δ_x^3 are given by

$$\delta_x^2 = \begin{bmatrix} 1 & 0.9 \\ 0.9 & 0.9 \end{bmatrix}, \qquad \delta_x^3 = \begin{bmatrix} 1 & 0.8 & 0.8 \\ 0.8 & 1 & 0.8 \\ 0.3 & 0.3 & 0.9 \end{bmatrix}.$$

Let $\mathscr{A}$ be a fuzzy automaton. A sequence $\mathscr{A}_1, \mathscr{A}_2, \dots, \mathscr{A}_n$ of fuzzy automata will be called a *strong right reduction* of $\mathscr{A}$ if $\mathscr{A}_1 = \mathscr{A}$ and for each $k \in \{1, 2, \dots, n-1\}$ we have that $\mathscr{A}_{k+1}$ is the factor fuzzy automaton of $\mathscr{A}_k$ w.r.t. the greatest strongly right invariant fuzzy equivalence on $\mathscr{A}_k$. Analogously we define a *strong left reduction* of $\mathscr{A}$.

For every fuzzy finite automaton $\mathscr{A}$ there exists a strong right reduction $\mathscr{A}_1, \mathscr{A}_2, \dots, \mathscr{A}_n$ of $\mathscr{A}$ such that for every strong right reduction $\mathscr{A}_1, \mathscr{A}_2, \dots, \mathscr{A}_n, \mathscr{A}_{n+1}, \dots, \mathscr{A}_{n+m}$ of $\mathscr{A}$ which is a continuation of this reduction we have that

$$|\mathscr{A}_n| = |\mathscr{A}_{n+1}| = \cdots = |\mathscr{A}_{n+m}|,$$

i.e., all fuzzy automata $\mathscr{A}_{n+1}, \dots, \mathscr{A}_{n+m}$ have the same number of states as $\mathscr{A}_n$. Also, there is a shortest strong right reduction $\mathscr{A}_1, \mathscr{A}_2, \dots, \mathscr{A}_n$ of $\mathscr{A}$ having this property, which will be called the *shortest strong right reduction* of $\mathscr{A}$, the fuzzy automaton $\mathscr{A}_n$ will be called a *strong right reduct* of $\mathscr{A}$, and the number n will be called the *length* of this shortest strong right reduction. If the fuzzy automaton $\mathscr{A}$ is its own strong right reduct, then it is called *strongly right reduced*. Analogously we define a *shortest strong left reduction* of $\mathscr{A}$, a *strong left reduct* of $\mathscr{A}$, the length of the shortest strongly left reduction and a *strongly left reduced* fuzzy automaton.

In contrast to right reductions, which stop immediately after the first factorization by means of the greatest right invariant fuzzy equivalence, strong right reductions do not necessarily stop after the first factorization by means of the greatest strongly right invariant fuzzy equivalence, and we can apply such factorization many times until we obtain a fuzzy automaton which cannot be reduced by means of the greatest strongly right invariant fuzzy equivalence. This is demonstrated by the following example.

Example 6.2. Let $\mathscr{L}$ be the Gödel structure, and let $\mathscr{A} = (A, X, \delta)$ be a fuzzy automaton over $\mathscr{L}$ with $|A| = 4$, $X = \{x\}$, and the fuzzy transition relation δ_x given by

$$\delta_x = \begin{bmatrix} 1 & 0.8 & 0.6 & 0.8 \\ 0.8 & 1 & 0.8 & 0.6 \\ 0.2 & 0.3 & 0.2 & 0.9 \\ 0.2 & 0.3 & 0.1 & 0.9 \end{bmatrix}.$$

Then the greatest strongly right invariant fuzzy equivalence E^{sri} on $\mathscr{A}$ is given by

$$E^{\mathrm{sri}} = \begin{bmatrix} 1 & 0.6 & 0.2 & 0.1 \\ 0.6 & 1 & 0.2 & 0.1 \\ 0.2 & 0.2 & 1 & 0.1 \\ 0.2 & 0.2 & 0.1 & 1 \end{bmatrix}.$$

The factor fuzzy automaton $\mathscr{A}_2 = \mathscr{A}_{E^{\mathrm{sri}}} = (A_2, X, \delta^2)$ has also four states, and the fuzzy transition relation δ^2_x is given by

$$\delta^2_x = \begin{bmatrix} 1 & 0.8 & 0.6 & 0.8 \\ 0.8 & 1 & 0.8 & 0.6 \\ 0.3 & 0.3 & 0.2 & 0.9 \\ 0.3 & 0.3 & 0.2 & 0.9 \end{bmatrix}.$$

The fuzzy equality $\widetilde{E}^{\mathrm{sri}}$ and the greatest strongly right invariant fuzzy equivalence E^{sri}_2 on $\mathscr{A}_2$ are given by

$$\widetilde{E}^{\mathrm{sri}} = \begin{bmatrix} 1 & 0.6 & 0.2 & 0.1 \\ 0.6 & 1 & 0.2 & 0.1 \\ 0.2 & 0.2 & 1 & 0.1 \\ 0.2 & 0.2 & 0.1 & 1 \end{bmatrix}, \qquad E^{\mathrm{sri}}_2 = \begin{bmatrix} 1 & 0.6 & 0.2 & 0.2 \\ 0.6 & 1 & 0.2 & 0.2 \\ 0.2 & 0.2 & 1 & 1 \\ 0.2 & 0.2 & 1 & 1 \end{bmatrix},$$

and hence, $\widetilde{E}^{\mathrm{sri}}$ is not the greatest strongly right invariant fuzzy equivalence on $\mathscr{A}_2$, and the number of states of the factor fuzzy automaton $\mathscr{A}_2 = \mathscr{A}_{E^{\mathrm{sri}}}$ can be further reduced by means of the greatest strongly right invariant fuzzy equivalence on $\mathscr{A}_2$.

The factor fuzzy automaton $\mathscr{A}_3 = (\mathscr{A}_2)_{E^{\mathrm{sri}}_2} = (A_3, X, \delta^3)$ has three states, and the fuzzy transition relation δ^3_x is given by

$$\delta^3_x = \begin{bmatrix} 1 & 0.8 & 0.8 \\ 0.8 & 1 & 0.8 \\ 0.3 & 0.3 & 0.9 \end{bmatrix},$$

and the greatest strongly right invariant fuzzy equivalence on $\mathscr{A}_2$ is given by

$$E^{\mathrm{sri}}_3 = \begin{bmatrix} 1 & 0.8 & 0.3 \\ 0.8 & 1 & 0.3 \\ 0.3 & 0.3 & 1 \end{bmatrix}.$$

Since $\delta^3_x \circ E^{\mathrm{sri}}_3 = \delta^3_x$, i.e., E^{sri}_3 is both a strongly right invariant and a strongly left invariant fuzzy equivalence, we have that the factor fuzzy automaton $\mathscr{A}_4 = (\mathscr{A}_3)_{E^{\mathrm{sri}}_3}$ is isomorphic to $\mathscr{A}_3$, and hence, the number of states of the fuzzy automaton $\mathscr{A}_3$ cannot be further reduced by means of the greatest strongly right invariant fuzzy equivalences.

The strong right reduction from the previous example has stopped when we have obtained two consecutive members which are isomorphic. In the sequel we study conditions under which strong right reductions stop.

First we prove the following:

Lemma 6.2. *Let $\mathscr{A} = (A, X, \delta)$ be a fuzzy automaton and let E and F be fuzzy equivalences on A such that $E \leqslant F$. Then the factor fuzzy automata $\mathscr{A}_F$, $(\mathscr{A}^{(E)})_F$, and $(\mathscr{A}_E)_{F_E}$ are isomorphic.*

Proof. According to Theorem 3.1, fuzzy automata $(\mathscr{A}_E)_{F_E}$ and $\mathscr{A}_F$ are isomorphic. We will show that the fuzzy automaton $(\mathscr{A}^{(E)})_F$ is also isomorphic to $\mathscr{A}_F$. These two fuzzy automata have the same set of states, the factor set A_F, and for all $a, b \in A$ and $x \in X$, by $F \circ E = E \circ F = F$ it follows that

$$\begin{aligned}\big(\delta^{(E)}\big)^F(F_a, x, F_b) &= \big(F \circ \delta_x^{(E)} \circ F\big)(a, b) = (F \circ E \circ \delta_x \circ E \circ F)(a, b)\\ &= (F \circ \delta_x \circ F)(a, b) = \delta^F(F_a, x, F_b).\end{aligned}$$

This means that fuzzy automata $(\mathscr{A}^{(E)})_F$ and $\mathscr{A}_F$ are isomorphic. $\square$

Another significant difference between right invariant and strongly right invariant fuzzy equivalences is that the assertions of Theorem 5.1 do not necessarily hold if we replace right invariant fuzzy equivalences by strongly right invariant ones. However, for strongly right invariant fuzzy equivalences a similar theorem holds.

Theorem 6.3. *Let $\mathscr{A} = (A, X, \delta)$ be a fuzzy automaton, let E be a strongly right invariant fuzzy equivalence on $\mathscr{A}$ and let F be a fuzzy equivalence on A such that $E \leqslant F$. Then the following statements hold.*

(a) *If F is a strongly right invariant fuzzy equivalence on $\mathscr{A}$ then F_E is a strongly right invariant fuzzy equivalence on $\mathscr{A}_E$.*
(b) *F is a strongly right invariant fuzzy equivalence on $\mathscr{A}^{(E)}$ if and only if F_E is a strongly right invariant fuzzy equivalence on $\mathscr{A}_E$.*
(c) *F is the greatest strongly right invariant fuzzy equivalence on $\mathscr{A}^{(E)}$ if and only if F_E is the greatest strongly right invariant fuzzy equivalence on $\mathscr{A}_E$.*

Proof. (b) For arbitrary $a, b \in A$ and $x \in X$ we have that

$$\begin{aligned}\big(F_E \circ \delta_x^E\big)(E_a, E_b) &= \bigvee_{c \in A} F_E(E_a, E_c) \otimes \delta_x^E(E_c, E_b) = \bigvee_{c \in A} F(a, c) \otimes (E \circ \delta_x \circ E)(c, b)\\ &= \bigvee_{c \in A} F(a, c) \otimes \delta_x^{(E)}(c, b) = \big(F \circ \delta_x^{(E)}\big)(a, b),\end{aligned}$$

and

$$\delta_x^E(E_a, E_b) = (E \circ \delta_x \circ E)(a, b) = \delta_x^{(E)}(a, b),$$

so we have that

$$F_E \circ \delta_x^E = \delta_x^E \quad \Leftrightarrow \quad F \circ \delta_x^{(E)} = \delta_x^{(E)}.$$

Therefore, F is a strongly right invariant fuzzy equivalence on $\mathscr{A}^{(E)}$ if and only if F_E is a strongly right invariant fuzzy equivalence on $\mathscr{A}_E$.

(a) If F is a strongly right invariant fuzzy equivalence on $\mathscr{A}$, then

$$F \circ \delta_x^{(E)} = F \circ E \circ \delta_x \circ E = F \circ \delta_x \circ E = \delta_x \circ E = E \circ \delta_x \circ E = \delta_x^{(E)},$$

i.e., F is also a strongly right invariant fuzzy equivalence on $\mathscr{A}^{(E)}$, and by (b) we obtain that F_E is a strongly right invariant fuzzy equivalence on $\mathscr{A}_E$.

(c) Let F be the greatest strongly right invariant fuzzy equivalence on $\mathscr{A}^{(E)}$, and let Q be the greatest strongly right invariant fuzzy equivalence on $\mathscr{A}_E$. Define a fuzzy relation G on A by

$$G(a, b) = Q(E_a, E_b), \quad \text{for all } a, b \in A.$$

Then G is a fuzzy equivalence on A. By Lemma 6.1, $\widetilde{E}$ is a strongly right invariant fuzzy equivalence on $\mathscr{A}_E$, so $\widetilde{E} \leqslant Q$, and for all $a, b \in A$ we have that

$$E(a, b) = \widetilde{E}(E_a, E_b) \leqslant Q(E_a, E_b) = G(a, b).$$

Thus, $E \leqslant G$ and $Q = G_E$, and according to (a), G is a strongly right invariant fuzzy equivalence on $\mathscr{A}^{(E)}$. Now we have that $G \leqslant F$, what implies $Q = G_E \leqslant F_E$. By the assertion (b), F_E is a strongly right invariant fuzzy equivalence on $\mathscr{A}_E$, and we obtain that $Q = F_E$. Therefore, we have proved that F_E is the greatest strongly right invariant fuzzy equivalence on $\mathscr{A}_E$.

Conversely, let F_E be the greatest strongly right invariant fuzzy equivalence on $\mathscr{A}_E$. According to (b), F is a strongly right invariant fuzzy equivalence on $\mathscr{A}^{(E)}$. Let G be the greatest strongly right invariant fuzzy equivalence on $\mathscr{A}^{(E)}$. Then by (b) we obtain that G_E is a strongly right invariant fuzzy equivalence on $\mathscr{A}_E$, so $G_E \leqslant F_E$. On the other hand, by $F \leqslant G$ it follows that $F_E \leqslant G_E$, and hence, $F_E = G_E$, what yields $F = G$. Therefore, F is the greatest strongly right invariant fuzzy equivalence on $\mathscr{A}^{(E)}$. □

According to the previous theorem, there is a correspondence between strongly right invariant fuzzy equivalences on the factor fuzzy automaton $\mathscr{A}_E$ and those on the fuzzy automaton $\mathscr{A}^{(E)}$. Using this correspondence we will study a strong right reduction of a fuzzy automaton through a sequence $\{\mathscr{A}_k\}_{k\in\mathbb{N}}$ of fuzzy automata whose members are defined by $\mathscr{A}_{k+1} = \mathscr{A}_k^{(E_k)}$, where $\mathscr{A}_1 = \mathscr{A}$ and E_k is the greatest strongly right invariant fuzzy equivalence on $\mathscr{A}_k$.

First we prove the following result.

Theorem 6.4. *Let $\mathscr{A} = (A, X, \delta)$ be a fuzzy automaton. Let a sequence $\{\mathscr{A}_k\}_{k\in\mathbb{N}}$ of fuzzy automata be defined inductively by*:

$$\mathscr{A}_1 = \mathscr{A}, \qquad \mathscr{A}_{k+1} = \mathscr{A}_k^{(E_k)}, \quad \text{for each } k \in \mathbb{N}, \tag{56}$$

where E_k is the greatest strongly right invariant fuzzy equivalence on $\mathscr{A}_k$.

If $\mathscr{A}_k = (A, X, \delta^k)$, for each $k \in \mathbb{N}$, then the following is true:

(a) *$E_k \leqslant E_{k+1}$ and $\delta_x^k \leqslant \delta_x^{k+1}$, for each $k \in \mathbb{N}$ and $x \in X$*;
(b) *If $E_k = E_{k+m}$, for some $k, m \in \mathbb{N}$, then $E_k = E_{k+1}$*;
(c) *If $\delta^k = \delta^{k+m}$, for some $k, m \in \mathbb{N}$, then $\delta^k = \delta^{k+1}$*;
(d) *If $E_k = E_{k+1}$, for some $k \in \mathbb{N}$, then $\delta^{k+1} = \delta^{k+2}$*;
(e) *If $\delta^k = \delta^{k+1}$, for some $k \in \mathbb{N}$, then $E_k = E_{k+1}$.*

Proof. First, let us observe that sequences $\{\delta_x^k\}_{k\in\mathbb{N}}$ of fuzzy relations on A, for every $x \in X$, and $\{E_k\}_{k\in\mathbb{N}}$ of fuzzy equivalences on A, are defined by

$$\delta_x^1 = \delta_x, \qquad \delta_x^{k+1} = \delta_x^k \circ E_k, \quad \text{for each } x \in X \text{ and } k \in \mathbb{N}, \tag{57}$$

$$E_k(a,b) = \bigwedge_{x\in X} \bigwedge_{c\in A} \delta_x^k(a,c) \leftrightarrow \delta_x^k(b,c), \quad \text{for all } k \in \mathbb{N} \text{ and } a, b \in A. \tag{58}$$

(a) For all $x \in X$ and $k \in \mathbb{N}$ we have that E_k is a strongly right invariant fuzzy equivalence on $\mathscr{A}_k$, so

$$E_k \circ \delta_x^{k+1} = E_k \circ \delta_x^k \circ E_k = \delta_x^k \circ E_k = \delta_x^{k+1},$$

and hence, E_k is also a strongly right invariant fuzzy equivalence on $\mathscr{A}_{k+1}$. Since E_{k+1} is the greatest strongly right invariant fuzzy equivalence on $\mathscr{A}_{k+1}$, we conclude that $E_k \leqslant E_{k+1}$.

Moreover, for arbitrary $x \in X$, $k \in \mathbb{N}$ and $a, b \in A$, we have that

$$\delta_x^k(a,b) = \delta_x^k(a,b) \otimes E_k(b,b) \leqslant \big(\delta_x^k \circ E_k\big)(a,b) = \delta_x^{k+1}(a,b),$$

and hence, $\delta_x^k \leqslant \delta_x^{k+1}$.

(b) Let $E_k = E_{k+m}$, for some $k, m \in \mathbb{N}$. Then

$$E_k \leqslant E_{k+1} \leqslant E_{k+m} = E_k,$$

whence $E_k = E_{k+1}$. Similarly we prove (c).

(d) Let $E_k = E_{k+1}$, for some $k \in \mathbb{N}$. Then for each $x \in X$ we have that

$$\delta_x^{k+2} = \delta_x^{k+1} \circ E_{k+1} = \delta_x^{k+1} \circ E_k = \delta_x^k \circ E_k \circ E_k = \delta_x^k \circ E_k = \delta_x^{k+1},$$

so $\delta^{k+2} = \delta^{k+1}$.

(e) Let $\delta^k = \delta^{k+1}$, for some $k \in \mathbb{N}$, i.e., let $\delta_x^k = \delta_x^{k+1}$, for every $x \in X$. Then by (58) we obtain that $E_k = E_{k+1}$. □

By the next theorem we establish a correspondence between the sequence of fuzzy automata defined by (59) and a strong right reduction of a fuzzy automaton.

Theorem 6.5. *Let $\mathscr{A} = (A, X, \delta)$ be a fuzzy automaton. Let the sequences $\{\mathscr{A}_k\}_{k\in\mathbb{N}}$ and $\{\mathscr{B}_k\}_{k\in\mathbb{N}}$ of fuzzy automata be defined inductively by*:

$$\mathscr{A}_1 = \mathscr{A}, \qquad \mathscr{A}_{k+1} = \mathscr{A}_k^{(E_k)}, \quad \text{for each } k \in \mathbb{N}, \tag{59}$$

where E_k is the greatest strongly right invariant fuzzy equivalence on $\mathscr{A}_k$, and

$$\mathscr{B}_1 = \mathscr{A}, \qquad \mathscr{B}_{k+1} = (\mathscr{B}_k)_{F_k}, \quad \text{for each } k \in \mathbb{N}, \tag{60}$$

where F_k is the greatest strongly right invariant fuzzy equivalence on $\mathscr{B}_k$.

Then for every $k \in \mathbb{N}$ we have that the fuzzy automata $\mathscr{B}_{k+1}$, $(\mathscr{A}_k)_{E_k}$ and $\mathscr{A}_{E_k}$ are isomorphic.

Proof. First we prove that $(\mathscr{A}_k)_{E_k} \cong \mathscr{A}_{E_k}$, for every $k \in \mathbb{N}$. Clearly, this is true for $k = 1$. Suppose that $k \geqslant 2$ and that $(\mathscr{A}_m)_{E_k} \cong \mathscr{A}_{E_k}$, for some $m \in \mathbb{N}$, $m \leqslant k-1$. Since $E_m \leqslant E_k$, by Lemma 6.2 we have that

$$(\mathscr{A}_{m+1})_{E_k} = \left(\mathscr{A}_m^{(E_m)}\right)_{E_k} \cong (\mathscr{A}_m)_{E_k} \cong \mathscr{A}_{E_k},$$

and by induction we obtain that $(\mathscr{A}_m)_{E_k} \cong \mathscr{A}_{E_k}$, for every $m \in \mathbb{N}$, $1 \leqslant m \leqslant k$, and hence, $(\mathscr{A}_k)_{E_k} \cong \mathscr{A}_{E_k}$.

Next we prove that $\mathscr{B}_{k+1} \cong \mathscr{A}_{E_k}$, for every $k \in \mathbb{N}$. This will be proved by induction on k. It is evident that this assertion holds for $k = 1$.

Suppose that this assertion holds for some $k \in \mathbb{N}$. Since E_{k+1} is the greatest strongly right invariant fuzzy equivalence on $\mathscr{A}_{k+1} = \mathscr{A}_k^{(E_k)}$, by Theorem 6.3 it follows that $(E_{k+1})_{E_k}$ is the greatest strongly right invariant fuzzy equivalence on $(\mathscr{A}_k)_{E_k} \cong \mathscr{A}_{E_k} \cong \mathscr{B}_{k+1}$. Now, by Theorem 3.1 we obtain that

$$\mathscr{B}_{k+2} = (\mathscr{B}_{k+1})_{F_{k+1}} \cong (\mathscr{A}_{E_k})_{(E_{k+1})_{E_k}} \cong \mathscr{A}_{E_{k+1}}.$$

Therefore, by induction we conclude that $\mathscr{B}_{k+1} \cong \mathscr{A}_{E_k}$, for every $k \in \mathbb{N}$. This completes the proof of the theorem. □

Now we show that a strong right reduction of a fuzzy automaton finishes when the sequence of fuzzy transition functions defined by (57) stops, and we also show that this must happen if the considered fuzzy automaton is finite and the underlying structure of truth values is locally finite.

Theorem 6.6. *Let $\mathscr{A} = (A, X, \delta)$ be a fuzzy automaton and let sequences $\{\mathscr{A}_k\}_{k\in\mathbb{N}}$ and $\{\mathscr{B}_k\}_{k\in\mathbb{N}}$ of fuzzy automata be defined by* (59) *and* (60)*, where $\mathscr{A}_k = (A, X, \delta^k)$, for each $k \in \mathbb{N}$. Then*

(a) *if $k \in \mathbb{N}$ is the least number such that $\delta^k = \delta^{k+1}$, then $\mathscr{B}_{k+1} \cong \mathscr{B}_{k+m+1}$, for every $m \in \mathbb{N}$, and $\mathscr{B}_{k+1}$ is a strong right reduct of $\mathscr{A}$;*
(b) *if $\mathscr{A}$ is finite and $\mathscr{L}$ is locally finite, then there exists $k \in \mathbb{N}$ such that $\delta^k = \delta^{k+1}$.*

Proof. (a) Let $k \in \mathbb{N}$ be the least number such that $\delta^k = \delta^{k+1}$. By Theorem 6.4 we have that $E_k = E_{k+m}$, for each $m \in \mathbb{N}$, and by Theorem 6.5 we obtain that $\mathscr{B}_{k+1} \cong \mathscr{A}_{E_k} = \mathscr{A}_{E_{k+m}} \cong \mathscr{B}_{k+m+1}$. From that reason, $\mathscr{B}_{k+1}$ cannot be further reduced by means of strongly right invariant fuzzy equivalences, so it is a strong right reduct of $\mathscr{A}$.

(b) Let $\mathscr{A}$ be a fuzzy finite automaton and let $\mathscr{L}$ be a locally finite algebra. Let the carrier of the subalgebra of $\mathscr{L}$ generated by the set $\delta(A \times X \times A)$ be denoted by $L_{\mathscr{A}}$. This generating set is finite, so $L_{\mathscr{A}}$ is also finite, and hence, the set $L_{\mathscr{A}}^{A\times A}$ of all fuzzy relations on A with membership values in $L_{\mathscr{A}}$ is finite. Since $E_k \in L_{\mathscr{A}}^{A\times A}$, for every $k \in \mathbb{N}$, and $L_{\mathscr{A}}^{A\times A}$ is a finite set, we conclude that there exist $k, m \in \mathbb{N}$ such that $E_k = E_{k+m}$, and by Theorem 6.4 we obtain that $\delta^{k+1} = \delta^{k+2}$. This completes the proof of the theorem. □

7. Concluding remarks

In this paper we have shown that the state reduction problem for fuzzy automata is related to the problem of finding a solution to a particular system of fuzzy relation equations in the set of all fuzzy equivalences on its set of states. This system consists of infinitely many equations and finding its non-trivial solutions is a very difficult task. For that reason we have aimed our attention to some instances of this system which consist of finitely many equations and are easier to solve. First, we have studied right invariant fuzzy equivalences, and their duals, the left invariant ones. These fuzzy equivalences are immediate generalizations of right and left invariant equivalences used by Ilie, Yu and others [29,30,32,33,10,11] in state reduction of non-deterministic automata, whereas crisp versions of right invariant fuzzy equivalences are just congruences of fuzzy automata studied by Petković in [53]. We have proved that each fuzzy automaton possesses the greatest right (resp. left) invariant fuzzy equivalence, which provides the best reduction by means of fuzzy equivalences of this type, and we have given an effective procedure for computing this fuzzy equivalence, which works if the underlying structure of truth values is a locally finite residuated lattice. Moreover, we have shown that even better reductions can be achieved alternating reductions by means of right and left invariant fuzzy equivalences. We have also studied strongly right and left invariant fuzzy equivalences, which give worse reductions than right and left invariant ones, but whose computing is much easier. We have given an effective procedure for computing the greatest strongly right (resp. left) invariant fuzzy equivalence, which is applicable to fuzzy automata over an arbitrary complete residuated lattice.

In our further work we will show that better reductions can be achieved if we replace fuzzy equivalences by fuzzy quasi-orders (preorders), and our results concerning fuzzy automata will be applied to non-deterministic ones. We will also search for better algorithms for computing the greatest right and left invariant fuzzy equivalences, as well as for computing the greatest strongly right and left invariant ones.

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