

MathZero, the Classification Problem, and Set-Theoretic Type Theory

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Abstract

AlphaZero learns to play go, chess and shogi at a superhuman level through self play given only the rules of the game. This raises the question of whether a similar thing could be done for mathematics — a MathZero. MathZero would require a formal foundation and an objective. We propose the foundation of set-theoretic dependent type theory and an objective defined in terms of the classification problem — the problem of classifying concept instances up to isomorphism. The natural numbers arise as the solution to the classification problem for finite sets. Here we generalize classical Bourbaki set-theoretic isomorphism to set-theoretic dependent type theory. To our knowledge we give the first isomorphism inference rules for set-theoretic dependent type theory with propositional set-theoretic equality. The presentation is intended to be accessible to mathematicians with no prior exposure to type theory.

1 Introduction

AlphaZero learns to play go, chess and shogi at a superhuman level through self play given only the rules of the game. This raises the question of whether a similar thing could be done for mathematics — a MathZero. A necessary first step is a definition of some form of “mathematics game”. What are the rules and what is the objective? Presumably the rules correspond to some formal foundation and the objective somehow involves formulating and proving theorems. Here we propose a foundation of set-theoretic dependent type theory and an objective based on the classification problem — the problem of enumerating the instances of a given concept up to isomorphism.

Isomorphism is central to the structure of mathematics. Mathematics is organized around concepts such as graphs, groups, topological spaces and manifolds

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each of which is associated with a notion of isomorphism. Each concept is associated with a classification problem — the problem of enumerating the instances of a given concept up to isomorphism. We also have the related notions of symmetry and canonicity. There is no canonical point on a geometric circle — any point can be mapped to any other point by rotating the circle. A rotation of the circle is an isomorphism of the circle with itself — a symmetry or automorphism. Similarly, there is no canonical basis for a finite dimensional vector space. For any basis there is a symmetry (automorphism) of the vector space which moves the basis to a different basis. Isomorphism is also central to understanding representation. A group can be represented by a family of permutations. Different (non-isomorphic) families of permutation can represent the same group (up to isomorphism).

People have a strong intuition that isomorphic objects are “the same”. We talk about “the” complete graph on five nodes or “the” topological sphere. A graph-theoretic property must have the same truth value on any two isomorphic graphs. This can be viewed as a tautology — a property is by definition graph-theoretic if it respects graph isomorphism. But this tautological interpretation misses the fact that being a graph theoretic property can be guaranteed by syntactic well-formedness conditions. Under an appropriate type system syntactically well-formed expressions automatically respect isomorphism.

Martin-Löf dependent type theory [4] is capable of expressing statements and proofs of general mathematics [1] while placing strong syntactic well-formedness constraints on expressions. However, dependent type theory is typically formulated in terms of propositions as types with equality handled by axiom J. This formulation can be challenging for the uninitiated. Typically no meaning for the notation is specified with various interpretations (models) being possible.

From the beginning it has been clear that dependent type theory can be given a set-theoretic interpretation — a set-theoretic model. However, this interpretation has generally been eschewed by type theorists. Here we consider a version of set-theoretic type theory that avoids propositions as types and axiom J and that gives transparent set-theoretic meanings for all constructs. This semantically transparent type theory is adequate for formalizing general mathematics. We show that syntactic well-formedness constraints inherited from Martin-Löf type theory can guarantee that all expressions respect isomorphism.

A definition of structures and isomorphism was given by Bourbaki. As interpreted within type theory, Bourbaki defines a particular class of structures, such as the class of groups, by a type expression. Bourbaki-style structure types are restricted to have all set variables declared at the top level of the expression. The top level set variables declare “carrier sets”. Most familiar structure classes, such as groups, have only a single carrier set — the group elements. In addition to the carrier sets, a structure type specifies functions and predicates over the carrier sets — the structure imposed on the carriers. Two instances of a structure type are isomorphic if there exists a system of bijections between their carrier sets which identifies their structure — which carries the structure of the

first to the structure of the second. The classical definition of a structure type specifies structure by a simple (non-dependent) type. We generalize that here to handle dependent structure types.

Section 2.3 gives two inference rules for isomorphism. The first is the congruence rule which states that isomorphic objects are inter-substitutable in well-formed contexts. The second is the structure rule which states a generalization of Bourbaki isomorphism. The main result of this paper is a semantic definition of isomorphism applying to all types (not just structure types) and a proof of isomorphism congruence. We also show that the structure rule and the congruence rule together are complete for isomorphism — they allow any isomorphism relationship to be expressed directly in the base language without linguistic extensions for isomorphism.

Isomorphism was handled at an abstract level in the groupoid model of type theory [3]. However, a purely set-theoretic treatment with transparent set-theoretic semantics and set-theoretic propositional equality should make the formal treatment more accessible to mathematicians outside of the type theory community. To our knowledge inference rules for isomorphism in this setting have not previously been formulated.

Section 2 gives a high level presentation of set-theoretic dependent type theory including the inference rules for isomorphism. Section 3 more formally defines the semantics and well-formedness constraints. Section 4 gives a semantic definition of isomorphism covering all type expressions and proves isomorphism congruence.

2 An Overview of Set-Theoretic Type Theory

This section first lists the constructs of the type theory and their set-theoretic meanings. It then describes the well-formedness constraints at a high level with the precise specification given in section 3. It states the structure rule and the congruence rule and defines cryptomorphism. Cryptomorphism is used to show that the structure and congruence rules are complete for isomorphism.

2.1 The Constructs

We consider a set-theoretic dependent type theory consisting of the following constructs.

- The constant symbol **Set** denotes the class of all sets.
- The constant symbol **Bool** denotes the type containing the two truth values.
- A pair expression $\langle s, u \rangle$ denotes the pair of s and u and projections $\pi_1(e)$

and $\pi_2(e)$ denote the first and second component respectively of the pair e .

- A lambda expression $\lambda x:\sigma e[x]$ denotes the (set-theoretic) function mapping an element x in the set σ to the value $e[x]$.
- An application $f(e)$ denotes the value of the function f on argument e .
- The formula $e_1 = e_2$ is true if e_1 is the same as e_2 (set-theoretic equality).
- $\forall x:\sigma \Phi[x]$ is true if for every element x of the set or class σ we have that $\Phi[x]$ is true.
- The Boolean formulas $\neg\Phi$, $\Phi \vee \Psi$, $\Phi \Rightarrow \Psi$ and $\Phi \Leftrightarrow \Psi$ have their classical Boolean meaning.
- The dependent function type $\Pi_{x:\sigma} \tau[x]$ denotes the set of all set-theoretic functions f with domain set σ and such that for all $x \in \sigma$ we have that $f(x) \in \tau[x]$. If x does not occur in τ we abbreviate $\Pi_{x:\sigma} \tau$ as $\sigma \rightarrow \tau$.
- The dependent pair type $\sum_{x:\sigma} \tau[x]$ denotes the set or class of all pairs $\langle x, y \rangle$ with $x \in \sigma$ and $y \in \tau[x]$. If x does not occur in τ then we abbreviate $\sum_{x:\sigma} \tau$ as $\sigma \times \tau$.
- The subtype $S_{x:\sigma} \Phi[x]$ denotes the set or class of all $x \in \sigma$ such that $\Phi[x]$ is true.

We distinguish set-level expressions from class expressions both syntactically and semantically. An expression is syntactically set-level if occurrences of the constant **Set** are restricted to be inside formulas (Boolean expressions). A class expression is one in which the constant **Set** occurs outside of a formula. Set-level expressions denote set-level values while class expressions denote classes which are taken to be of a different semantic kind from sets. The set/class distinction is discussed in more detail in section 3.1.

As an example, the class of all groups can be defined as

$$\begin{aligned} \text{Magma} &\equiv \sum_{s:\mathbf{Set}} s \times s \rightarrow s \\ \text{Group} &\equiv S_{M:\text{Magma}} \Phi(M) \end{aligned}$$

where $\Phi(M)$ states the group axioms.

Under the meanings defined above the same value can be in different types. Most significantly, a pair $\langle u, v \rangle$ in the type $\sum_{x:\sigma} \tau[x]$ is also in the type $\sigma \times \tau[v]$. The well-formedness conditions do not prevent this. A pair $\langle s, f \rangle$ in the type **Magma** is also in the very different type $\mathbf{Set} \times ((s \times s) \rightarrow s)$.

As another example consider $\sum_{s:\mathbf{Set}} \sum_{w:\mathbf{Set}} s \rightarrow w$ and $\sum_{s:\mathbf{Set}} s \rightarrow w$ where w is a free set variable in the second expression. The first type allows isomorphisms defined by two bijections — bijections on both s and w — while the second type only allows isomorphisms defined by a single bijection on s . The classification problem for the second type gives rise to the concept of a bag of (or multiset of) w .

It is important that functions are restricted to be set-level (as opposed to being functors on proper classes). To see why consider a predicate $P:\mathbf{Group} \rightarrow \mathbf{Bool}$. In the presence of such a predicate we have that $P(G)$ is a meaningful formula for any group G . In this case we need that for two isomorphic groups G and G' we have $P(G) \Leftrightarrow P(G')$. This requires that the notion of isomorphism is built into the semantics of the type $\mathbf{Group} \rightarrow \mathbf{Bool}$. This is done in the groupoid model [3]. Here we take a different approach based on the observation that set-level function spaces require no such care — the fully naive definition of set-level function spaces just works. Functors then arise naturally as terms of the language. Isomorphism congruence implies that functors defined by terms of the language automatically respect isomorphism — they are automatically functors between groupoids.

2.2 Well-Formedness

While the set-theoretic meanings of the constructs are transparent, the well-formedness conditions are more subtle. Here we describe the well-formedness conditions at a high level and postpone a precise definition to section 3.

Well-formedness is relative to a context declaring the types for variables and stating assumptions about those variables — a context Γ consists of variable declarations $x:\tau$ and assumptions Φ . The well-formedness conditions determine when an expression e is well-formed under a context Γ . Contexts themselves are subject to well-formedness constraints. For $\Gamma; x:\sigma$ to be well-formed σ must be a well-formed type expression under Γ and for $\Gamma; \Phi$ to be well-formed Φ must be a well-formed Boolean expression under Γ . All contexts are constructed in this way starting with the empty context ϵ and constants **Set** and **Bool**. We write $\Gamma \models e:\sigma$ to mean that σ is a well-formed type under Γ and that e is a well-formed expression under Γ and that for all variable interpretations satisfying the context Γ we have that the value of e is a member of the value of σ . We write $\Gamma \models \Phi$ to mean that Φ is a well-formed Boolean expression under Γ and that for every variable interpretation satisfying Γ we have that Φ is true.

While the details of well-formedness constraints are postponed to section 3 there are some cases that are particularly significant. For a set-theoretic equality $u = v$ to be well-formed we must first be able to derive $\Gamma \models u:\sigma$ and $\Gamma \models v:\sigma$ for some set expression σ . Without this constraint we would have

$$s:\mathbf{Set}; x:s; S:\left(\sum_{w:\mathbf{Set}} w\right) \models (x = \pi_2(S)):\mathbf{Bool}.$$

This would yield a counter-example to isomorphism congruence as isomorphic values of S yield different truth values for the formula.

Similarly, for an application $f(e)$ to be well-formed under Γ we must first be able to derive $\Gamma \models f : (\prod_{x:\sigma} \tau[x])$ and $\Gamma \models u : \sigma$ for some dependent function type $\prod_{x:\sigma} \tau[x]$. This restriction is used in the proof of the projectability lemma stated in section 4.3.

2.3 Isomorphism Inference Rules

We now extend the language with Boolean formulas of the form $u =_\sigma v$ which intuitively mean that u and v are isomorphic instances of the class σ . A precise semantics for this formula for general class expressions is given in section 4.3. We give two inference rules for isomorphism — the structure rule and the congruence rule. The congruence rule is simpler and we state it first.

$$\frac{\begin{array}{l} \Gamma; x:\sigma \models e[x] : \tau \quad x \text{ not free in } \tau \\ \Gamma \models u =_\sigma v \end{array}}{\Gamma \models e[u] =_\tau e[v]}$$

To state the structure rule we first define structure types.

Definition 1. *A structure type is either a set expression or a class expression of the form $\sum_{s_1:\mathbf{Set}} \cdots \sum_{s_n:\mathbf{Set}} \tau$ where τ is a set expression.*

The traditional (Bourbaki) notion of structure type requires τ to be simple — a type constructed from non-dependent pair and non-dependent function types over the set variables s_1, \dots, s_n . Here we allow τ to be any (dependent) set expression.

An example of a class expression that is not a structure type is a group-with-action type

$$\sum_{G:\mathbf{Group}} \sum_{s:\mathbf{Set}} S_{f:(\pi_1(G) \rightarrow (s \rightarrow s))} \Phi[G, s, f]$$

where $\Phi[G, s, f]$ states the group action conditions. Here the first component of a group-with-action is a group rather than a carrier set.

To state the structure rule we will use some additional notation. For set expressions s and w we let $\text{Bi}(s, w)$ abbreviate the set expression $S_{f:(s \rightarrow w)} \Phi[s, w, f]$ where $\Phi[s, w, f]$ states that f is a bijection. Also, for u an instance of the structure type $\sum_{s_1:\mathbf{Set}} \cdots \sum_{s_n:\mathbf{Set}} \tau$ we define \dot{u} to be the substitution mapping a set variable s_i to the value assigned to that variable in u . For example $\dot{u}(s_1)$ is $\pi_1(u)$ and $\dot{u}(s_2)$ is $\pi_1(\pi_2(u))$. For a substitution \dot{u} and an expression e we write $\dot{u}(e)$ for the result of applying substitution \dot{u} to expression e . We let $x(u)$ be the element of $\dot{u}(\tau)$ included in u . For example, if there are two carrier sets then $x(u)$ is $\pi_2(\pi_2(u))$. The structure rule is then

$$\begin{array}{l}
\sigma = \sum_{s_1:\mathbf{Set}} \cdots \sum_{s_n:\mathbf{Set}} \tau \quad \tau \text{ a set expression} \\
\Gamma \models u:\sigma \\
\Gamma \models v:\sigma \\
\hline
\Gamma \models (u =_\sigma v) \Leftrightarrow \exists f_1:\mathbf{Bi}(\dot{u}(s_1), \dot{v}(s_1)) \cdots f_n:\mathbf{Bi}(\dot{u}(s_n), \dot{v}(s_n)) \quad \mathbf{EQ}(\tau, \dot{u}, x(u), \dot{v}, x(v))
\end{array}$$

where the formula $\mathbf{EQ}(\tau, \dot{x}, x, \dot{y}, y)$ is defined below. In the formula $\mathbf{EQ}(\tau, \dot{x}, x, \dot{y}, y)$ we have that τ , x , and y are expressions and \dot{x} and \dot{y} are syntactic substitutions with $\Gamma \models x:\dot{x}(\tau)$ and $\Gamma \models y:\dot{y}(\tau)$. The formula $\mathbf{EQ}(\tau, \dot{x}, x, \dot{y}, y)$ is defined by the following clauses.

$$\begin{aligned}
\mathbf{EQ}(\tau, \dot{x}, x, \dot{y}, y) &\equiv x = y \text{ for } \tau \text{ not containing any } s_i \\
\mathbf{EQ}(s_i, \dot{x}, x, \dot{y}, y) &\equiv f_i(x) = y \\
\mathbf{EQ}((S_{z:\tau}\Phi), \dot{x}, x, \dot{y}, y) &\equiv \mathbf{EQ}(\tau, \dot{x}, x, \dot{y}, y) \\
\mathbf{EQ}\left(\left(\sum_{z:\tau_1} \tau_2\right), \dot{x}, x, \dot{y}, y\right) &\equiv \begin{cases} \mathbf{EQ}(\tau_1, \dot{x}, \pi_1(x), \dot{y}, \pi_1(y)) \\ \wedge \mathbf{EQ}(\tau_2, \dot{x}[z := \pi_1(x)], \pi_2(x), \dot{y}[z := \pi_1(y)], \pi_2(y)) \end{cases} \\
\mathbf{EQ}\left(\left(\prod_{z:\tau_1} \tau_2\right), \dot{f}, f, \dot{g}, g\right) &\equiv \begin{cases} \forall x:\dot{f}(\tau_1) \forall y:\dot{g}(\tau_1) \\ \mathbf{EQ}(\tau_1, \dot{f}, x, \dot{g}, y) \\ \Rightarrow \mathbf{EQ}(\tau_2, \dot{f}[z := x], f(x), \dot{g}[z := y], g(y)) \end{cases}
\end{aligned}$$

2.4 Functors, Cryptomorphism and Completeness

Assuming isomorphism congruence, if we have $\Gamma \ x:\sigma \models e[x]:\tau$ with x not free in τ then a lambda expression of the form $(\lambda x:\sigma e[x])$ where σ is a class expression can be assigned a semantics as a functor mapping elements of the class σ to elements of the class τ and respecting isomorphism. We show in section 4.6 that such a lambda expressions also defines a mapping of isomorphisms (groupoid morphisms) and hence defines a functor between groupoids.

Following the terminology of Birkoff [2] and Rota [5], two classes σ and τ that are well-formed under Γ will be called *cryptomorphic* (under Γ) if there exist functor-level lambda expressions $F:\sigma \rightarrow \tau$ and $G:\tau \rightarrow \sigma$ that are well-formed under Γ and where we have $\Gamma; x:\sigma \models G(F(x)) = x$ and $\Gamma; y:\tau \models F(G(y)) = y$.

For example a group can be defined either as a four-tuple of a set, a group operation, an inverse operation and an identity element or as a set and group operation such that inverses and an identity exist.

We now show that every class expression is cryptomorphic to a structure class. To determine if $u =_\sigma v$ it then suffices to show $F(u) =_\tau F(v)$ where F is a cryptomorphism from σ to a structure class τ . **Expressing isomorphism at structure classes then allows us to express isomorphism at every class.**

The proof that every class σ is cryptomorphic to a structure class is by induction on the class expression σ . Function types and the set **Bool** are set expressions and are by definition structure classes. The class **Set** is cryptomorphic to $\sum_{s:\mathbf{Set}} S_{P:\mathbf{Bool}} P$. For a subclass expression $S_{x:\sigma} \Phi$ we can assume by the induction hypothesis that σ is cryptomorphic to a structure class $\sum_{s_1:\mathbf{Set}} \cdots \sum_{s_n:\mathbf{Set}} \tau$ in which case $S_{x:\sigma} \Phi[x]$ is cryptomorphic to a class expression of the form $\sum_{s_1:\mathbf{Set}} \cdots \sum_{s_n:\mathbf{Set}} S_{x:\tau} \Phi'$. Finally we consider a dependent pair class $\sum_{x:\sigma} \tau$. By the induction hypothesis σ and τ are cryptomorphic to classes

$$\sum_{s_1:\mathbf{Set}} \cdots \sum_{s_n:\mathbf{Set}} \tau_1$$

and

$$\sum_{s_{n+1}:\mathbf{Set}} \cdots \sum_{s_{n+m}:\mathbf{Set}} \tau_2$$

respectively. In this case the class $\sum_{x:\sigma} \tau$ is cryptomorphic to a class of the form

$$\sum_{s_1:\mathbf{Set}} \cdots \sum_{s_{n+m}:\mathbf{Set}} \sum_{x:\tau_1} \tau_2'$$

3 A More Rigorous Treatment

This section gives a rigorous treatment of set-theoretic dependent type theory. To make the presentation more widely accessible we start by discussing the universe V of all sets and the set/class distinction. We then give a formal recursive definition of Tarskian semantic value functions which implicitly define well-formedness. Finally, we address some technical issues arising from the structure of the recursion defining the value functions.

3.1 The Universe of Sets

We assume a Grothendieck universe V . A Grothendieck universe is a set whose cardinality is inaccessible (definition omitted) and such that for $U \subseteq V$ we have $U \in V$ if and only if the cardinality of U is less than the cardinality of V . A Grothendieck universe is a model of ZFC set theory and by Gödel's incompleteness theorem the existence of a Grothendieck universe cannot be proved from ZFC. However, mathematicians generally accept that we can talk

about the class of all sets, the class of all groups, and so on. These classes are subsets of V that are too large to be elements of V .

In order to define the left and right projection operations in section 4.1 we need to work with tagged values. We define basic values to be hereditarily tagged values over five tags.

Definition 2. *A basic value is either*

- *a tagged atom $\langle \text{"atom"}, x \rangle$ where x is an arbitrary element of V ,*
- *a tagged truth value $\langle \text{"Bool"}, x \rangle$ where x is one of the two values 1 or 0 (**True** or **False**),*
- *a tagged pair $\langle \text{"pair"}, \langle x, y \rangle \rangle$ where x and y are (recursively) basic values,*
- *a tagged function $\langle \text{"function"}, f \rangle$ where $f \in V$ is a function (a set of pairs) from basic values to basic values,*
- *or a tagged set $\langle \text{"set"}, s \rangle$ where $s \in V$ is a set of basic values.*

This definition is recursive but one can show by induction on set-theoretic containment (or rank) that there is a unique class satisfying this recursive specification.

3.2 Tarskian Value Functions

We now formally define two semantic value functions. The first assigns a meaning to well-formed contexts where a context is a sequence of variable declarations and assumptions. For a well-formed context Γ we have that $\mathcal{V}[\Gamma]$ is a set of variable interpretations. Each variable interpretation is mapping from a finite set of variables to basic values. For a variable interpretation $\gamma \in \mathcal{V}[\Gamma]$, and a well-formed expression e , we define $\mathcal{V}_\Gamma[e]\gamma$ to be the value (meaning) of e under the values assigned by γ to the variables in e . The meaning functions $\mathcal{V}[\Gamma]$ and $\mathcal{V}_\Gamma[e]\gamma$ are defined by mutual recursion. The definitions are first presented using simple naive recursion. This recursion is not just the standard recursion on the structure of expressions and the technical validity of the recursion is addressed in section 3.3.

It is important that the value functions are partial. The value function $\mathcal{V}_\Gamma[e]\gamma$ is only defined when e is well-formed under Γ . Here we equate well-formedness with definedness. A context Γ is well-formed if and only if $\mathcal{V}[\Gamma]$ is defined and an expression e is well-formed under a well-formed context Γ if and only if for all $\gamma \in \mathcal{V}[\Gamma]$ we have that $\mathcal{V}_\Gamma[e]\gamma$ is defined.

For a variable interpretation γ , a variable x not assigned a value in γ , and a value v we write $\gamma[x := v]$ for the variable interpretation γ extended to assign value v to variable x .

The semantic value function defined in this section has the property that well-formed set-level expressions always denote basic values in V while class expressions denote either the empty class, which we distinguish from the empty set, or a proper class — a subset of V too large to be an element of V . We note that the empty class can be written as $S_{s:\text{Set}} \mathbf{False}$.

The value functions are defined by following recursive clauses.

- (V1) $\mathcal{V}[\epsilon]$. $\mathcal{V}[\epsilon]$ is defined to be the set containing just the empty variable interpretation — the variable interpretation not assigning any value to any variable.
- (V2) $\mathcal{V}_\Gamma[\mathbf{Set}] \gamma$. If $\mathcal{V}[\Gamma]$ is defined and $\gamma \in \mathcal{V}[\Gamma]$ then we define $\mathcal{V}_\Gamma[\mathbf{Set}] \gamma$ to be the class of all sets.
- (V3) $\mathcal{V}_\Gamma[\mathbf{Bool}] \gamma$. If $\mathcal{V}[\Gamma]$ is defined and $\gamma \in \mathcal{V}[\Gamma]$ then we define $\mathcal{V}_\Gamma[\mathbf{Bool}] \gamma$ to be the set containing the two truth values **True** and **False**.
- (V4) We say that u is defined under Γ if $\mathcal{V}[\Gamma]$ is defined and for all $\gamma \in \mathcal{V}[\Gamma]$ we have that $\mathcal{V}_\Gamma[u] \gamma$ is defined.
- (V5) We write $\Gamma \models \sigma :: \mathbf{Class}$ to mean that σ is defined under Γ and for all $\gamma \in \mathcal{V}[\Gamma]$ we have that $\mathcal{V}_\Gamma[\sigma] \gamma$ is a class and similarly write $\Gamma \models \sigma :: \mathbf{Type}$ to mean that σ is defined under Γ and for all $\gamma \in \mathcal{V}[\Gamma]$ we have that $\mathcal{V}_\Gamma[\sigma] \gamma$ is either a set or a class.
- (V6) For a set-level expression u we write $\Gamma \models u : \sigma$ to mean that u is defined under Γ and $\Gamma \models \sigma :: \mathbf{Type}$ and for all $\gamma \in \mathcal{V}[\Gamma]$ we have $\tilde{u} \in \tilde{\sigma}$ where $\tilde{u} = \mathcal{V}_\Gamma[u] \gamma$ and $\tilde{\sigma} = \mathcal{V}_\Gamma[\sigma] \gamma$.
- (V7) For $\Gamma \models \Phi : \mathbf{Bool}$ we write $\Gamma \models \Phi$ to mean that for all $\gamma \in \mathcal{V}[\Gamma]$ we have $\mathcal{V}_\Gamma[\Phi] \gamma = \mathbf{True}$.
- (V8) $\mathcal{V}[\Gamma; x : \sigma]$. If $\Gamma \models \sigma :: \mathbf{Type}$, and x is a variable not declared in Γ , then $\mathcal{V}[\Gamma; x : \sigma]$ is defined to be the class of variable interpretations of the form $\gamma[x := \tilde{u}]$ for $\gamma \in \mathcal{V}[\Gamma]$ and $\tilde{u} \in \mathcal{V}_\Gamma[\sigma] \gamma$.
- (V9) $\mathcal{V}[\Gamma; \Phi]$. If $\Gamma \models \Phi : \mathbf{Bool}$ then $\mathcal{V}[\Gamma; \Phi]$ is defined to be the class of variable interpretations $\gamma \in \mathcal{V}[\Gamma]$ such that $\mathcal{V}_\Gamma[\Phi] \gamma = \mathbf{True}$.
- (V10) $\mathcal{V}_\Gamma[x] \gamma$. If x is declared in Γ , and $\gamma \in \mathcal{V}[\Gamma]$, then we define $\mathcal{V}_\Gamma[x] \gamma$ to be $\gamma(x)$.
- (V11) $\mathcal{V}_\Gamma[\sum_{x:\sigma} \tau[x]] \gamma$. If $\Gamma \models \sigma :: \mathbf{Type}$ and $\Gamma; x : \sigma \models \tau[x] :: \mathbf{Type}$ then $\mathcal{V}_\Gamma[\sum_{x:\sigma} \tau[x]] \gamma$ is defined to be the set or class of all pairs $\langle \tilde{u}, \tilde{v} \rangle$ with $\tilde{u} \in \mathcal{V}_\Gamma[\sigma] \gamma$ and $\tilde{v} \in \mathcal{V}_{\Gamma; x:\sigma}[\tau[x]] \gamma[x := \tilde{u}]$.
- (V12) $\mathcal{V}_\Gamma[S_{x:\sigma} \Phi[x]] \gamma$. If $\Gamma \models \sigma :: \mathbf{Type}$ and $\Gamma; x : \sigma \models \Phi[x] : \mathbf{Bool}$ then $\mathcal{V}_\Gamma[S_{x:\sigma} \Phi[x]] \gamma$ is defined to be the set or class of all $\tilde{u} \in \mathcal{V}_\Gamma[\sigma] \gamma$ such that $\mathcal{V}_{\Gamma; x:\sigma}[\Phi[x]] \gamma[x := \tilde{u}] = \mathbf{True}$.

- (V13) $\mathcal{V}_\Gamma [\prod_{x:\sigma} \tau[x]] \gamma$. If $\Gamma \models \sigma : \mathbf{Set}$ and $\Gamma; x:\sigma \models \tau[x] : \mathbf{Set}$ then $\mathcal{V}_\Gamma [\prod_{x:\sigma} \tau[x]] \gamma$ is defined to be the set of all functions \tilde{f} with domain $\mathcal{V}_\Gamma [\sigma] \gamma$ and such that for all $\tilde{u} \in \mathcal{V}_\Gamma [\sigma] \gamma$ we have $\tilde{f}(\tilde{u}) \in \mathcal{V}_{\Gamma; x:\sigma} [\tau[x]] \gamma[x := \tilde{u}]$.
- (V14) $\mathcal{V}_\Gamma [\Phi \vee \Psi] \gamma$. If $\Gamma \models \Phi : \mathbf{Bool}$ and $\Gamma \models \Psi : \mathbf{Bool}$ then $\mathcal{V}_\Gamma [\Phi \vee \Psi] \gamma$ is defined to be **True** if either $\mathcal{V}_\Gamma [\Phi] \gamma = \mathbf{True}$ or $\mathcal{V}_\Gamma [\Psi] \gamma = \mathbf{True}$. Similar definitions apply to the other Boolean operations.
- (V15) $\mathcal{V}_\Gamma [f(u)] \gamma$. If $\Gamma \models f : (\prod_{x:\sigma} \tau[x])$ and $\Gamma \models u : \sigma$ then $\mathcal{V}_\Gamma [f(u)] \gamma$ is defined to be $\tilde{f}(\tilde{u})$ for $\tilde{f} = \mathcal{V}_\Gamma [f] \gamma$ and $\tilde{u} = \mathcal{V}_\Gamma [u] \gamma$.
- (V16) $\mathcal{V}_\Gamma [u = v] \gamma$. If $\Gamma \models u : \sigma$ and $\Gamma \models v : \sigma$ and $\Gamma \models \sigma : \mathbf{Set}$, then $\mathcal{V}_\Gamma [u = v] \gamma$ is defined to be **True** if $\tilde{u} = \tilde{v}$ where $\tilde{u} = \mathcal{V}_\Gamma [u] \gamma$ and $\tilde{v} = \mathcal{V}_\Gamma [v] \gamma$.
- (V17) $\mathcal{V}_\Gamma [\forall x:\sigma \Phi[x]] \gamma$. If $\Gamma; x:\sigma \models \Phi[x] : \mathbf{Bool}$ then $\mathcal{V}_\Gamma [\forall x:\sigma \Phi[x]] \gamma$ is defined to be **True** if $\Gamma; x:\sigma \models \Phi[x]$ (and false otherwise).
- (V18) $\mathcal{V}_\Gamma [\langle u, v \rangle] \gamma$. If u and v are set-level and defined under Γ then $\mathcal{V}_\Gamma [\langle u, v \rangle] \gamma$ is defined to be $\langle \tilde{u}, \tilde{v} \rangle$ with $\tilde{u} = \mathcal{V}_\Gamma [u] \gamma$ and $\tilde{v} = \mathcal{V}_\Gamma [v] \gamma$.
- (V19) $\mathcal{V}_\Gamma [\pi_i(u)] \gamma$. If $\Gamma \models u : (\sum_{x:\sigma} \tau[x])$ then $\mathcal{V}_\Gamma [\pi_i(u)] \gamma$ is defined to be $\pi_i(\tilde{u})$ where $\tilde{u} = \mathcal{V}_\Gamma [u] \gamma$.
- (V20) $\mathcal{V}_\Gamma [\lambda x:\sigma e[x]] \gamma$. If $\Gamma \models \sigma : \mathbf{Set}$ and $\Gamma; x:\sigma \models e[x] : \tau[x]$ then $\mathcal{V}_\Gamma [\lambda x:\sigma e[x]] \gamma$ is defined to be the function \tilde{f} with domain $\mathcal{V}_\Gamma [\sigma] \gamma$ and satisfying $\tilde{f}(\tilde{u}) = \mathcal{V}_{\Gamma; x:\sigma} [e[x]] \gamma[x := \tilde{u}]$ for all $\tilde{u} \in \mathcal{V}_\Gamma [\sigma] \gamma$.

3.3 The Recursion Index

Tarskian value functions are typically defined by structural induction on expressions. However, the recursion in clauses (V1) through (V20) is more subtle. Clauses (V1) through (V20) can be viewed as acting like inference rules. For example consider (V16) defining the meaning of set-theoretic equality. Here we must be able to “derive” $\Gamma \models u : \sigma$, $\Gamma \models v : \sigma$ and $\Gamma \models \sigma : \mathbf{Set}$ in order to derive that the equality $u = v$ is well-formed. Once an expression is determined to be well-formed its value is defined in the usual way by structural induction on the expression using the semantic values specified in section 2.1.

To formally treat the inference rule nature of clauses (V1) through (V20) we introduce a recursion index $i \geq 0$ and write $\mathcal{V}^i [\Gamma]$ and $\mathcal{V}_\Gamma^i [e] \gamma$. For $i = 0$ we take $\mathcal{V}^0 [\Gamma]$ and $\mathcal{V}_\Gamma^0 [e] \gamma$ to be undefined for all Γ and e . We then modify the clauses (V1) through (V20) so that they define $\mathcal{V}^{i+1} [\Gamma]$ and $\mathcal{V}_\Gamma^{i+1} [e] \gamma$ in terms of $\mathcal{V}^i [\Gamma]$ and $\mathcal{V}_\Gamma^i [e] \gamma$. This recursion index provides an induction principle — induction on the recursion index — needed in later proofs.

- (V1) $\mathcal{V}^{i+1} [\epsilon]$. $\mathcal{V}_\Gamma^{i+1} [\epsilon]$ is defined to be the set containing just the empty variable interpretation — the variable interpretation not assigning any value to any variable.

- (V2) $\mathcal{V}_\Gamma^{i+1} \llbracket \mathbf{Set} \rrbracket \gamma$. If $\mathcal{V}^i \llbracket \Gamma \rrbracket$ is defined and $\gamma \in \mathcal{V}^i \llbracket \Gamma \rrbracket$ then we define $\mathcal{V}_\Gamma^{i+1} \llbracket \mathbf{Set} \rrbracket \gamma$ to be the class of all sets.
- (V3) $\mathcal{V}_\Gamma^{i+1} \llbracket \mathbf{Bool} \rrbracket \gamma$. If $\mathcal{V}^i \llbracket \Gamma \rrbracket$ is defined and $\gamma \in \mathcal{V}^i \llbracket \Gamma \rrbracket$ then we define $\mathcal{V}_\Gamma^{i+1} \llbracket \mathbf{Bool} \rrbracket \gamma$ to be the set containing the two truth values **True** and **False**.
- (V4) We say that u is defined at i under Γ if $\mathcal{V}^i \llbracket \Gamma \rrbracket$ is defined and for all $\gamma \in \mathcal{V}^i \llbracket \Gamma \rrbracket$ we have that $\mathcal{V}_\Gamma^i \llbracket u \rrbracket \gamma$ is defined.
- (V5) We write $\Gamma \models^i \sigma :: \mathbf{Class}$ to mean that σ is defined at i under Γ and for all $\gamma \in \mathcal{V}^i \llbracket \Gamma \rrbracket$ we have that $\mathcal{V}_\Gamma^i \llbracket \sigma \rrbracket \gamma$ is a class and similarly write $\Gamma \models^i \sigma :: \mathbf{Type}$ to mean that σ is defined at i under Γ and for all $\gamma \in \mathcal{V}^i \llbracket \Gamma \rrbracket$ we have that $\mathcal{V}_\Gamma^i \llbracket \sigma \rrbracket \gamma$ is either a set or a class.
- (V6) For a set-level expression u we write $\Gamma \models^i u : \sigma$ to mean that u is defined at i under Γ and $\Gamma \models^i \sigma :: \mathbf{Type}$ and for all $\gamma \in \mathcal{V}^i \llbracket \Gamma \rrbracket$ we have $\tilde{u} \in \tilde{\sigma}$ where $\tilde{u} = \mathcal{V}_\Gamma^i \llbracket u \rrbracket \gamma$ and $\tilde{\sigma} = \mathcal{V}_\Gamma^i \llbracket \sigma \rrbracket \gamma$. We note that $\Gamma \models^i u : \sigma$ implies $\tilde{u} \in V$.
- (V7) For $\Gamma \models^i \Phi : \mathbf{Bool}$ we write $\Gamma \models^i \Phi$ to mean that for all $\gamma \in \mathcal{V}^i \llbracket \Gamma \rrbracket$ we have $\mathcal{V}_\Gamma^i \llbracket \Phi \rrbracket \gamma = \mathbf{True}$.
- (V8) $\mathcal{V}_\Gamma^{i+1} \llbracket \Gamma; x : \sigma \rrbracket$. If $\Gamma \models^i \sigma :: \mathbf{Type}$, and x is a variable not declared in Γ , then $\mathcal{V}_\Gamma^{i+1} \llbracket \Gamma; x : \sigma \rrbracket$ is defined to be the class of variable interpretations of the form $\gamma[x := \tilde{u}]$ for $\gamma \in \mathcal{V}^i \llbracket \Gamma \rrbracket$ and $\tilde{u} \in \mathcal{V}_\Gamma^i \llbracket \sigma \rrbracket \gamma$.
- (V9) $\mathcal{V}_\Gamma^{i+1} \llbracket \Gamma; \Phi \rrbracket$. If $\Gamma \models^i \Phi : \mathbf{Bool}$ then $\mathcal{V}_\Gamma^{i+1} \llbracket \Gamma; \Phi \rrbracket$ is defined to be the class of variable interpretations $\gamma \in \mathcal{V}^i \llbracket \Gamma \rrbracket$ such that $\mathcal{V}_\Gamma^i \llbracket \Phi \rrbracket \gamma = \mathbf{True}$.
- (V10) $\mathcal{V}_\Gamma^{i+1} \llbracket x \rrbracket \gamma$. If x is declared in Γ , and $\gamma \in \mathcal{V}^i \llbracket \Gamma \rrbracket$, then we define $\mathcal{V}_\Gamma^{i+1} \llbracket x \rrbracket \gamma$ to be $\gamma(x)$.
- (V11) $\mathcal{V}_\Gamma^{i+1} \llbracket \sum_{x:\sigma} \tau[x] \rrbracket \gamma$. If $\Gamma \models^i \sigma :: \mathbf{Type}$ and $\Gamma; x : \sigma \models^i \tau[x] :: \mathbf{Type}$ then $\mathcal{V}_\Gamma^{i+1} \llbracket \sum_{x:\sigma} \tau[x] \rrbracket \gamma$ is defined to be the set or class of all pairs $\langle \tilde{u}, \tilde{v} \rangle$ with $\tilde{u} \in \mathcal{V}_\Gamma^i \llbracket \sigma \rrbracket \gamma$ and $\tilde{v} \in \mathcal{V}_{\Gamma;x:\sigma}^i \llbracket \tau[x] \rrbracket \gamma[x := \tilde{u}]$.
- (V12) $\mathcal{V}_\Gamma^{i+1} \llbracket S_{x:\sigma} \Phi[x] \rrbracket \gamma$. If $\Gamma \models^i \sigma :: \mathbf{Type}$ and $\Gamma; x : \sigma \models^i \Phi[x] : \mathbf{Bool}$ then $\mathcal{V}_\Gamma^{i+1} \llbracket S_{x:\sigma} \Phi[x] \rrbracket \gamma$ is defined to be the set or class of all $\tilde{u} \in \mathcal{V}_\Gamma^i \llbracket \sigma \rrbracket \gamma$ such that $\mathcal{V}_{\Gamma;x:\sigma}^i \llbracket \Phi[x] \rrbracket \gamma[x := \tilde{u}] = \mathbf{True}$.
- (V13) $\mathcal{V}_\Gamma^{i+1} \llbracket \prod_{x:\sigma} \tau[x] \rrbracket \gamma$. If $\Gamma \models^i \sigma : \mathbf{Set}$ and $\Gamma; x : \sigma \models^i \tau[x] : \mathbf{Set}$ then $\mathcal{V}_\Gamma^{i+1} \llbracket \prod_{x:\sigma} \tau[x] \rrbracket \gamma$ is defined to be the set of all functions \tilde{f} with domain $\mathcal{V}_\Gamma^i \llbracket \sigma \rrbracket \gamma$ and such that for all $\tilde{u} \in \mathcal{V}_\Gamma^i \llbracket \sigma \rrbracket \gamma$ we have $\tilde{f}(\tilde{u}) \in \mathcal{V}_{\Gamma;x:\sigma}^i \llbracket \tau[x] \rrbracket \gamma[x := \tilde{u}]$.
- (V14) $\mathcal{V}_\Gamma^{i+1} \llbracket \Phi \vee \Psi \rrbracket \gamma$. If $\Gamma \models^i \Phi : \mathbf{Bool}$ and $\Gamma \models^i \Psi : \mathbf{Bool}$ then $\mathcal{V}_\Gamma^{i+1} \llbracket \Phi \vee \Psi \rrbracket \gamma$ is defined to be **True** if either $\mathcal{V}_\Gamma^i \llbracket \Phi \rrbracket \gamma = \mathbf{True}$ or $\mathcal{V}_\Gamma^i \llbracket \Psi \rrbracket \gamma = \mathbf{True}$. Similar definitions apply to the other Boolean operations.

- (V15) $\mathcal{V}_\Gamma^{i+1} \llbracket f(u) \rrbracket \gamma$. If $\Gamma \models^i f : (\prod_{x:\sigma} \tau[x])$ and $\Gamma \models^i u : \sigma$ then $\mathcal{V}_\Gamma^{i+1} \llbracket f(u) \rrbracket \gamma$ is defined to be $\tilde{f}(\tilde{u})$ for $\tilde{f} = \mathcal{V}_\Gamma^i \llbracket f \rrbracket \gamma$ and $\tilde{u} = \mathcal{V}_\Gamma^i \llbracket u \rrbracket \gamma$.
- (V16) $\mathcal{V}_\Gamma^{i+1} \llbracket u = v \rrbracket \gamma$. If $\Gamma \models^i u : \sigma$ and $\Gamma \models^i v : \sigma$ and $\Gamma \models^i \sigma : \mathbf{Set}$, then $\mathcal{V}_\Gamma^{i+1} \llbracket u = v \rrbracket \gamma$ is defined to be **True** if $\tilde{u} = \tilde{v}$ where $\tilde{u} = \mathcal{V}_\Gamma^i \llbracket u \rrbracket \gamma$ and $\tilde{v} = \mathcal{V}_\Gamma^i \llbracket v \rrbracket \gamma$.
- (V17) $\mathcal{V}_\Gamma^{i+1} \llbracket \forall x:\sigma \Phi[x] \rrbracket \gamma$. If $\Gamma; x:\sigma \models^i \Phi[x] : \mathbf{Bool}$ then $\mathcal{V}_\Gamma^{i+1} \llbracket \forall x:\sigma \Phi[x] \rrbracket \gamma$ is defined to be **True** if $\Gamma; x:\sigma \models^i \Phi[x]$ (and false otherwise).
- (V18) $\mathcal{V}_\Gamma^{i+1} \llbracket \langle u, v \rangle \rrbracket \gamma$. If u and v are set-level and defined at i under Γ then $\mathcal{V}_\Gamma^{i+1} \llbracket \langle u, v \rangle \rrbracket \gamma$ is defined to be $\langle \tilde{u}, \tilde{v} \rangle$ with $\tilde{u} = \mathcal{V}_\Gamma^i \llbracket u \rrbracket \gamma$ and $\tilde{v} = \mathcal{V}_\Gamma^i \llbracket v \rrbracket \gamma$.
- (V19) $\mathcal{V}_\Gamma^{i+1} \llbracket \pi_i(u) \rrbracket \gamma$. If $\Gamma \models^i u : (\sum_{x:\sigma} \tau[x])$ then $\mathcal{V}_\Gamma^{i+1} \llbracket \pi_i(u) \rrbracket \gamma$ is defined to be $\pi_i(\tilde{u})$ where $\tilde{u} = \mathcal{V}_\Gamma^i \llbracket u \rrbracket \gamma$.
- (V20) $\mathcal{V}_\Gamma^{i+1} \llbracket \lambda x:\sigma e[x] \rrbracket \gamma$. If $\Gamma \models^i \sigma : \mathbf{Set}$ and $\Gamma; x:\sigma \models^i e[x] : \tau[x]$ then $\mathcal{V}_\Gamma^{i+1} \llbracket \lambda x:\sigma e[x] \rrbracket \gamma$ is defined to be the function \tilde{f} with domain $\mathcal{V}_\Gamma^i \llbracket \sigma \rrbracket \gamma$ and satisfying $\tilde{f}(\tilde{u}) = \mathcal{V}_{\Gamma;x:\sigma}^i \llbracket e[x] \rrbracket \gamma[x := \tilde{u}]$ for all $\tilde{u} \in \mathcal{V}_\Gamma^i \llbracket \sigma \rrbracket \gamma$.

If e is defined at i for Γ then the value of $\mathcal{V}_\Gamma^i \llbracket e \rrbracket \gamma$ is defined compositionally on the structure of the expression e as is normally done in Tarskian semantics. This implies that for values of i for which $\mathcal{V}_\Gamma^i \llbracket e \rrbracket \gamma$ is defined, this value is independent of the choice of i . This also holds for $\mathcal{V}^i \llbracket \Gamma \rrbracket$. So we can define $\mathcal{V}_\Gamma \llbracket e \rrbracket \gamma$ to be $\mathcal{V}^i \llbracket \Gamma \rrbracket \gamma$ for any i where this is defined, and to be undefined if no such i exists. $\mathcal{V} \llbracket \Gamma \rrbracket$ is defined similarly.

4 Isomorphism and Isomorphism Congruence

Section 4.1 introduces “projectable values”. A projectable value is one that has both a left and a right projection where the value and its two projections are all isomorphic. To define projectable values we add a new tag to the value space to get values of the form $\langle \text{"isopair"}, \langle x, y \rangle \rangle$ where x and y are basic values (values that do not contain iso-pairs). The left projection of this iso-pair is x and the right projection is y . A bijective set of iso-pairs is a projectable set — a set that defines a bijection between its left and right projections.

Section 4.2 introduces a new constant symbol **Bij** into the language which denotes all projectable sets. As far as the type system is concerned the constant **Bij** is indistinguishable from the constant **Set**. Section 4.2 also proves that all sets definable in the language extended with **Bij** define bijections between their left and right projections.

Section 4.3 gives a definition of isomorphism covering all class expressions (not just structure classes).

Section 4.4 gives a proof of isomorphism congruence.

4.1 Projectable Values

We first define a class of extended values similar to the class of basic values of section 3.1 but extended with iso-pairs.

Definition 3. *An extended value is either*

- a tagged atom $\langle \text{"atom"}, x \rangle$ where x is an arbitrary element of V ,
- a tagged truth value $\langle \text{"Bool"}, x \rangle$ where x is one of the two values 1 or 0 (**True** or **False**),
- a tagged pair $\langle \text{"pair"}, \langle x, y \rangle \rangle$ where x and y are (recursively) extended values,
- a tagged function $\langle \text{"function"}, f \rangle$ where f is a function (a set of pairs) from extended values to extended values,
- a tagged set $\langle \text{"set"}, s \rangle$ where $s \in V$ is a set of extended values,
- or a tagged iso-pair $\langle \text{"isopair"}, \langle x, y \rangle \rangle$ where x and y are **basic values**.

We define the projectable values to be a subclass of the extended values. More specifically we define partial left and right projection functions. An extended value is projectable if its left and right projections are defined.

Definition 4 (Left and Right Projections). *We define a partial left projection function L on extended values by the following clauses.*

- $L(\langle \text{"atom"}, x \rangle) = \langle \text{"atom"}, x \rangle$.
- $L(\langle \text{"Bool"}, x \rangle) = \langle \text{"Bool"}, x \rangle$.
- $L(\langle \text{"pair"}, \langle u, w \rangle \rangle) = \langle \text{"pair"}, \langle L(u), L(w) \rangle \rangle$
- For $L(\langle \text{"set"}, s \rangle)$ to be defined we require that $L(u)$ is defined for all $u \in s$ and that the mappings $u \mapsto L(u)$ for $u \in s$ form a bijection in which case

$$L(\langle \text{"set"}, s \rangle) = \langle \text{"set"}, \{L(x) : x \in s\} \rangle.$$

- For $L(\langle \text{"function"}, f \rangle)$ to be defined we require that for all u in the domain of f both $L(u)$ and $L(f(u))$ are defined and that the mapping $u \mapsto L(u)$ defines a bijection on the domain of f , in which case

$$L(\langle \text{"function"}, f \rangle) = \langle \text{"function"}, g \rangle$$

where g is the function containing the mappings $L(u) \mapsto L(f(u))$ for u in the domain of f .

- $L(\langle \text{"isopair"}, \langle u, w \rangle \rangle) = u$.

The partial right projection function R is defined similarly but with

$$R(\langle \text{"isopair"}, \langle u, w \rangle \rangle) = w.$$

Definition 5. A projectable value is an extended value v such that $L(v)$ and $R(v)$ are both defined.

For any basic value v we have that $L(v) = R(v) = v$ and v is projectable. Also, for any bijection f between two basic sets we have that the set of iso-pairs of the form $\langle \text{"isopair"}, \langle x, f(x) \rangle \rangle$ for x in the domain of f is a projectable set. Two groups G and G' are isomorphic if there exists a group \tilde{G} whose set of group elements is projectable and where $L(\tilde{G}) = G$ and $R(\tilde{G}) = G'$.

4.2 The Constant Bij and the Projectability Lemma

We now introduce a new constant symbol **Bij** defined as follows where we repeat the definition of the constant **Set** for contrast and clarity.

Definition 6.

- $\mathcal{V}_\Gamma \llbracket \mathbf{Set} \rrbracket \gamma$ is defined to be the class of all basic sets.
- $\mathcal{V}_\Gamma \llbracket \mathbf{Bij} \rrbracket \gamma$ is defined to be the class of all projectable sets.

Since all basic sets are projectable, we immediately have that the meaning of **Set** is a subset of the meaning of **Bij**.

The semantics of expressions in the language extended with the constant **Bij** is given by the clauses of section 3.2, and their semantic clarification in section 3.3, but extended with the above definition of the meaning of **Bij**.

The following projectability lemma implies that every definable set is bijective — the left and right projection functions are each bijections on that set. This is a strong invariant on the definable sets and is central to proving isomorphism congruence.

Lemma 1 (Projectability Lemma). *In the language extended with the constant **Bij** we have that if $\Gamma \models \sigma :: \mathbf{Class}$ then every member of $\mathcal{V}_\Gamma \llbracket \tau \rrbracket \gamma$ is projectable and for every set-level expression e defined under Γ we have that $\mathcal{V}_\Gamma \llbracket e \rrbracket \gamma$ is projectable.*

Proof. We can prove the projectability lemma by induction on the recursion index i in the clauses of section 3.3. For this we note that the lemma is trivial for $i = 0$ where nothing is defined. We then assume that the lemma holds for i and prove it for $i + 1$. We need to consider each clause in section 3.3 that defines a value $\mathcal{V}_\Gamma^{i+1} \llbracket e \rrbracket \gamma$.

Clauses (V1) and (V4) through (V9) do not define expression values.

Clauses (V2) and (V3) define the constants **Set** and **Bool**. In this case the result follows from the fact that elements of these types are basic values. We need to add a clause (V2.5) for the constant **Bij** defining $\mathcal{V}_\Gamma^{i+1} \llbracket \mathbf{Bij} \rrbracket \gamma$ but the lemma holds for **Bij** since the elements of **Bij** are defined to be projectable sets.

Clause (V10) defines the variable value $\mathcal{V}_\Gamma^{i+1} \llbracket x \rrbracket \gamma$ to be $\gamma(x)$. In this case we have that $\gamma(x) \in \mathcal{V}_\Gamma^i \llbracket \tau \rrbracket \gamma$ where τ is the type expression declared for x in Γ . By the induction hypothesis every element of $\mathcal{V}_\Gamma^i \llbracket \tau \rrbracket \gamma$ is projectable.

Clause (V11) defines the dependent pair type $\mathcal{V}_\Gamma^{i+1} \llbracket \sum_{x:\sigma} \tau[x] \rrbracket$. By the induction hypothesis the projectability lemma applies to σ under Γ and to $\tau[x]$ under $\Gamma; x:\sigma$. We then immediately have that every element of the dependent pair type is a pair of projectable values and every pair of projectable values is projectable. If σ and $\tau[x]$ are set level we must also show that the pair type is bijective. In this case the induction hypothesis gives us that $\mathcal{V}_\Gamma \llbracket \sigma \rrbracket \gamma$ is bijective and for $v \in \mathcal{V}_\Gamma \llbracket \sigma \rrbracket \gamma$ we have that $\mathcal{V}_{\Gamma;x:\sigma} \llbracket \tau[x] \rrbracket \gamma[x := v]$ is bijective. These two statements together imply that $\mathcal{V}_\Gamma \llbracket \sum_{x:\sigma} \tau[x] \rrbracket$ is bijective.

Clause (V12) defines the subtype $\mathcal{V}_\Gamma^{i+1} \llbracket S_{x:\sigma} \Phi[x] \rrbracket$. In this case we have that σ is defined at i for Γ and hence satisfies the induction hypothesis. This gives that every element of the subtype is projectable. If σ is set-level we must also show that the subtype is bijective. But this follows from the observation that any subset of a bijective set is bijective.

Clause (V13) defines the dependent function type $\mathcal{V}_\Gamma^{i+1} \llbracket \prod_{x:\sigma} \tau[x] \rrbracket$. Let $\tilde{\sigma}$ denote $\mathcal{V}_\Gamma^i \llbracket \sigma \rrbracket \rho$. By the induction hypothesis for σ we get that $\tilde{\sigma}$ is projectable and hence every function in the dependent function type is projectable. To show that the function type is bijective consider a function f in the bijective type and consider $\tilde{x} \in \tilde{\sigma}$. Let $L_{\tilde{\sigma}}$ be the left operation restricted to $\tilde{\sigma}$. Let $\tilde{\tau}[\tilde{x}]$ denote $\mathcal{V}_{\Gamma;x:\sigma}^i \llbracket \tau[x] \rrbracket \gamma[x := \tilde{x}]$ and let $L_{\tilde{\tau}[\tilde{x}]}$ be the left operation restricted to $\tilde{\tau}[\tilde{x}]$. By the definition of the left operation on functions we have $L(f)(L(\tilde{x})) = L(f(\tilde{x}))$. By the definition of the dependent function type we have $f(\tilde{x}) \in \tilde{\tau}[\tilde{x}]$ which gives $L(f(\tilde{x})) \in L_{\tilde{\tau}[\tilde{x}]}$. By the induction hypothesis we have $L_{\tilde{\tau}[\tilde{x}]}$ is bijective and we get $f(\tilde{x}) = L_{\tilde{\tau}[\tilde{x}]}^{-1}(L(f)(L(\tilde{x})))$. So the function f can be recovered from its projection $L(f)$ and the function type is bijective.

Clauses (V14), (V16) and (V17) define formulas — expressions whose value is Boolean. Boolean values are basic and hence projectable.

Clause (V15) defines $\mathcal{V}_\Gamma^{i+1} \llbracket f(e) \rrbracket \gamma$. We are given $\Gamma \models^i f : (\prod_{x:\sigma} \tau[x])$ and $\Gamma \models^i e : \sigma$. This implies that the value of $f(e)$ is in $\mathcal{V}_\Gamma^i \llbracket \tau[x] \rrbracket \gamma[x := \tilde{e}]$ where \tilde{e} is the value of e . By the induction hypothesis for $\tau[x]$ we get that $f(e)$ is projectable.

Clauses (V18) and (V19) define pairing and the π_1 and π_2 projection functions on pairs. In each case the projectability of the value of the defined expression follows immediately from the induction hypothesis.

Clause (V20) defines $\mathcal{V}_\Gamma^{i+1} \llbracket \lambda x:\sigma e[x] \rrbracket \gamma$. By the induction hypothesis we have that the value of σ is projectable and hence the value of the lambda expression

is projectable. □

4.3 Isomorphism

Definition 7. For a class expression τ not including the constant **Bij** we define τ^* to be the result of replacing all occurrences of the constant **Set** in τ that occur outside of Boolean expressions with the constant **Bij**.

Section 4.6 proves that if τ is defined under Γ then so is τ^* . We can now give the following definition of isomorphism.

Definition 8 (Isomorphism). For $\Gamma \models u : \tau$ and $\Gamma \models v : \tau$, where Γ , u , v and τ do not contain **Bij**, we define $\mathcal{V}_\Gamma \llbracket u =_\tau v \rrbracket \gamma$ to be true if there exists $\vec{u} \in \mathcal{V}_\Gamma \llbracket \tau^* \rrbracket \gamma$ with $L(\vec{u}) = \mathcal{V}_\Gamma \llbracket u \rrbracket \gamma$ and $R(\vec{u}) = \mathcal{V}_\Gamma \llbracket v \rrbracket \gamma$.

Recall that we can write an expression for the class **Group** as follows.

$$\begin{aligned} \mathbf{Group} &\equiv S_{G:\mathbf{Magma}} \Phi[G] \\ \mathbf{Magma} &\equiv \sum_{u:\mathbf{Set}} (u \times u) \rightarrow u \end{aligned}$$

The “morphism class” **Group**^{*} is then

$$\begin{aligned} \mathbf{Group}^* &\equiv S_{G:\mathbf{Magma}^*} \Phi[G] \\ \mathbf{Magma}^* &\equiv \sum_{u:\mathbf{Bij}} (u \times u) \rightarrow u \end{aligned}$$

It seems intuitively clear that if we consider a group G in **Group**^{*} we have that G is group isomorphic to both $L(G)$ and $R(G)$.

4.4 Isomorphism Congruence

The following theorem captures the sense in which well-formed expressions respect isomorphism.

Theorem 1 (Isomorphism Congruence). If $\Gamma; x:\sigma \models e[x]:\tau$ where Γ , σ , τ and $e[x]$ do not contain **Bij**, and x does not occur in τ , then $\Gamma \models u =_\sigma v$ implies $\Gamma \models e[u] =_\tau e[v]$.

Isomorphism congruence is implied fairly directly by the following two lemmas proved in sections 4.5 and 4.6 respectively.

Lemma 2 (Commutation Lemma). The following conditions hold where **Bij** is allowed to occur in all expressions involved.

- (a) For $\gamma \in \mathcal{V} \llbracket \Gamma \rrbracket$ we have $L(\gamma) \in \mathcal{V} \llbracket \Gamma \rrbracket$.

- (b) If $\Gamma \models \sigma :: \mathbf{Type}$ then for $\gamma \in \mathcal{V}[\Gamma]$ and $\tilde{u} \in \mathcal{V}_\Gamma[\sigma] \gamma$ we have $L(\tilde{u}) \in \mathcal{V}_\Gamma[\sigma] L(\gamma)$.
- (c) If u is a set-level expression defined under Γ then for $\gamma \in \mathcal{V}[\Gamma]$ we have $L(\mathcal{V}_\Gamma[u] \gamma) = \mathcal{V}_\Gamma[u] L(\gamma)$

And (a), (b) and (c) hold similarly for the right projection R .

Lemma 3 (Functor Lemma). *If $\Gamma; x : \sigma \models e[x] : \tau$ where Γ , σ and τ do not contain **Bij** and x does not appear free in τ , then $\Gamma; x : \sigma^* \models e[x] : \tau^*$*

The functor lemma can be interpreted as saying that the function mapping $x \in \sigma$ to $e[x] \in \tau$, in addition to mapping objects in σ to objects in τ , map morphisms in σ^* to morphisms in τ^* . Before giving the proofs of these lemmas we show that they imply isomorphism congruence.

Proof of Isomorphism Congruence. To show $\Gamma \models e[u] =_\tau e[v]$ consider $\gamma \in \mathcal{V}[\Gamma]$. Let \tilde{u} be $\mathcal{V}_\Gamma[u] \gamma$ and let \tilde{v} be $\mathcal{V}_\Gamma[v] \gamma$. We are given $\Gamma \models u =_\sigma v$ and by the definition of $=_\sigma$ we then have that there exists a projectable value $\vec{u} \in \mathcal{V}_\Gamma[\sigma^*] \gamma$ with $L(\vec{u}) = \tilde{u}$ and $R(\vec{u}) = \tilde{v}$. We then have $\rho[x := \vec{u}] \in \mathcal{V}[\Gamma; x : \sigma^*]$. Let $\tilde{e}[\tilde{u}]$ be $\mathcal{V}_{\Gamma; x : \sigma^*}[e[x]] \gamma[x := \vec{u}]$. By the functor lemma we then have $\tilde{e}[\tilde{u}] \in \mathcal{V}_\Gamma[\tau^*] \gamma$. Since Γ does not contain **Bij** every value in γ is basic and we have $L(\gamma) = \gamma$. By part (c) of the commutation lemma we then have

$$\begin{aligned} L(\tilde{e}[\tilde{u}]) &= \mathcal{V}_{\Gamma; x : \sigma^*}[e[x]] \gamma[x := L(\vec{u})] = \mathcal{V}_\Gamma[e[u]] \gamma \\ R(\tilde{e}[\tilde{u}]) &= \mathcal{V}_{\Gamma; x : \sigma^*}[e[x]] \gamma[x := R(\vec{u})] = \mathcal{V}_\Gamma[e[v]] \gamma \end{aligned}$$

which implies the result. \square

4.5 Proof of the commutation lemma.

We first note that the converse of (a) fails — it is possible that $L(\gamma) \in \mathcal{V}[\Gamma]$ but $\gamma \notin \mathcal{V}[\Gamma]$. A trivial example is that for $\gamma(s)$ a bijective set of iso-pairs we have $L(\gamma) \in \mathcal{V}[s : \mathbf{Set}]$ but $\gamma \notin \mathcal{V}[s : \mathbf{Set}]$. But there are also counter examples not involving the constant **Set**. Consider the contexts

$$\begin{aligned} \Gamma_1 &\equiv s : \mathbf{Bij}; u : \mathbf{Bij}; P : u \\ \Gamma_2 &\equiv s : \mathbf{Bij}; u : \mathbf{Bij}; P : s \rightarrow \mathbf{Bool}. \end{aligned}$$

Now consider $\gamma \in \mathcal{V}[\Gamma_1]$ where $\gamma(s)$ is a basic set, $\gamma(u)$ is bijective set of iso-pairs, and $L(\gamma(u))$ is the set $s \rightarrow \mathbf{Bool}$. We then have $L(\gamma) \in \mathcal{V}[\Gamma_2]$ but $\gamma \notin \mathcal{V}[\Gamma_2]$. In general the left and right projection operations expose representation structure not visible when the iso-pairs are treated as atoms.

The proof is by induction on the recursion index. We must consider each clause of section 3.3 that defined either a value for $\mathcal{V}^{i+1}[\Gamma]$ or a value for $\mathcal{V}_\Gamma^{i+1}[e] \gamma$.

All three parts of the commutation lemma are immediate for clauses (V1), (V2) and (V3) defining the constants ϵ , **Set** and **Bool** respectively. We only note that for (V2) and (V3) and $\gamma \in \mathcal{V}^i \llbracket \Gamma \rrbracket$ the induction hypothesis for i gives $L(\gamma) \in \mathcal{V}^i \llbracket \Gamma \rrbracket$.

Clauses (V4) through (V7) define notation in terms of $\mathcal{V}^i \llbracket \Gamma \rrbracket$ and $\mathcal{V}_\Gamma^i \llbracket e \rrbracket \gamma$ without defining new values.

Clauses (V8) and (V9) define the meaning of nonempty contexts and we must verify part (a) of the lemma.

For (V8) defining $\mathcal{V}^{i+1} \llbracket \Gamma; x:\sigma \rrbracket$ consider $\gamma \in \mathcal{V}^i \llbracket \Gamma \rrbracket$ and $\tilde{u} \in \mathcal{V}_\Gamma^i \llbracket \sigma \rrbracket \gamma$. By part (a) of the induction hypothesis we have $L(\gamma) \in \mathcal{V}^i \llbracket \Gamma \rrbracket$. By part (b) of the induction hypothesis we have $L(\tilde{u}) \in \mathcal{V}_\Gamma^i \llbracket \sigma \rrbracket L(\gamma)$. These together give $L(\gamma[x := \tilde{u}]) \in \mathcal{V}^{i+1} \llbracket \Gamma; x:\sigma \rrbracket$.

For (V9) defining $\mathcal{V}^{i+1} \llbracket \Gamma; \Phi \rrbracket$ consider $\gamma \in \mathcal{V}^i \llbracket \Gamma \rrbracket$ such that $\mathcal{V}_\Gamma^i \llbracket \Phi \rrbracket \gamma$ is true. By part (a) of the induction hypothesis we have $L(\gamma) \in \mathcal{V}^i \llbracket \Gamma \rrbracket$. By part (c) of the induction hypothesis we have $\mathcal{V}_\Gamma^i \llbracket \Phi \rrbracket \gamma = L(\mathcal{V}_\Gamma^i \llbracket \Phi \rrbracket \gamma) = \mathcal{V}_\Gamma^i \llbracket \Phi \rrbracket L(\gamma) = \mathbf{True}$. This gives $L(\gamma) \in \mathcal{V}^{i+1} \llbracket \Gamma; \Phi \rrbracket$ as desired.

Clauses (V10) through (V20) define $\mathcal{V}_\Gamma^{i+1} \llbracket e \rrbracket \gamma$. In each of these clauses we are assuming $\gamma \in \mathcal{V}^i \llbracket \Gamma \rrbracket$ and by part (a) of the induction hypothesis we have $L(\gamma) \in \mathcal{V}^i \llbracket \Gamma \rrbracket$. For each clause we need to show parts (b) and (c).

For (V10) defining the value of a variable condition (b) applies when a set variable s is declared in Γ by either $s:\mathbf{Set}$ or $s:\mathbf{Bij}$. In this case condition (b) reduces to $L(\tilde{u}) \in L(\gamma(s))$ for $\tilde{u} \in \gamma(s)$ which follows from the definition of the left operation. Part (c) is also immediate for this case.

For (V11) defining dependent pair types we first consider part (b). Consider $\langle \tilde{u}, \tilde{v} \rangle \in \mathcal{V}_\Gamma^{i+1} \llbracket \sum_{x:s} \tau[x] \rrbracket \gamma$. We must show

$$\langle L(\tilde{u}), L(\tilde{v}) \rangle \in \mathcal{V}_\Gamma^{i+1} \left[\left[\sum_{x:\sigma} \tau[x] \right] \right] L(\gamma).$$

In this case we have $\tilde{u} \in \mathcal{V}_\Gamma^i \llbracket \sigma \rrbracket \gamma$ and $\tilde{v} \in \mathcal{V}_{\Gamma; x:\sigma}^i \llbracket \tau[x] \rrbracket \gamma[x := \tilde{u}]$. The induction hypothesis gives $L(\tilde{u}) \in \mathcal{V}_\Gamma^i \llbracket \sigma \rrbracket L(\gamma)$ and $L(\tilde{v}) \in \mathcal{V}_{\Gamma; x:\sigma}^i \llbracket \tau[x] \rrbracket L(\gamma)[x := L(\tilde{u})]$ which proves the result.

Part (c) applies when σ and $\tau[x]$ are both set expressions. In this case we must show

$$L \left(\mathcal{V}_\Gamma^{i+1} \left[\left[\sum_{x:\sigma} \tau[x] \right] \right] \right) \gamma = \mathcal{V}_\Gamma^{i+1} \left[\left[\sum_{x:\sigma} \tau[x] \right] \right] L(\gamma).$$

This follows straightforwardly from the induction hypothesis for part (c) applied to set expressions σ and $\tau[x]$.

For (V12) defining subtypes we again first consider part (b). Consider $\tilde{x} \in \mathcal{V}_\Gamma^{i+1} \llbracket S_{x:\sigma} \Phi[x] \rrbracket \gamma$. We must show

$$L(\tilde{x}) \in \mathcal{V}_\Gamma^{i+1} \llbracket S_{x:\sigma} \Phi[x] \rrbracket L(\gamma).$$

We are given $\tilde{x} \in \mathcal{V}_\Gamma^i \llbracket \sigma \rrbracket \gamma$ and $\mathcal{V}_{\Gamma; x:\sigma}^i \llbracket \Phi[x] \rrbracket \gamma[x := \tilde{x}] = \mathbf{True}$. By the induction hypothesis for part (b) we have $L(\tilde{x}) \in \mathcal{V}_\Gamma^i \llbracket \sigma \rrbracket L(\gamma)$ and by part (c) of the induction hypothesis we have

$$L(\mathcal{V}_{\Gamma; x:\sigma}^i \llbracket \Phi[x] \rrbracket L(\gamma)[x := \tilde{x}]) = \mathcal{V}_{\Gamma; x:\sigma}^i \llbracket \Phi[x] \rrbracket L(\gamma)[x := L(\tilde{x})] = \mathbf{True}$$

which proves the result.

Part (c) applies when σ is a set expression in which case we must show

$$L(\mathcal{V}_\Gamma^{i+1} \llbracket S_{x:\sigma} \Phi[x] \rrbracket \gamma) = \mathcal{V}_\Gamma^{i+1} \llbracket S_{x:\sigma} \Phi[x] \rrbracket L(\gamma)$$

Again this follows straightforwardly from part (c) of the induction hypothesis applied to σ and $\Phi[x]$.

For (V13) defining dependent function types we first consider part (b). For $f \in \mathcal{V}_\Gamma^{i+1} \llbracket \prod_{x:\sigma} \tau[x] \rrbracket \gamma$ we must show $L(f) \in \mathcal{V}_\Gamma^{i+1} \llbracket \prod_{x:\sigma} \tau[x] \rrbracket L(\gamma)$. Let $\tilde{\sigma}$ be $\mathcal{V}_\Gamma^i \llbracket \sigma \rrbracket \gamma$. By the definition of the left operation the domain of the function $L(f)$ is $L(\tilde{\sigma})$. By the projectability lemma we have that the left operation is a bijection on $\tilde{\sigma}$. Therefore every element of $L(\tilde{\sigma})$ can be written as $L(\tilde{x})$ for $\tilde{x} \in \tilde{\sigma}$. So it now suffices to show that

$$L(f)(L(\tilde{x})) \in \mathcal{V}_{\Gamma; x:\sigma}^i \llbracket \tau[x] \rrbracket L(\gamma)[x := L(\tilde{x})].$$

By the definition of the left operation on functions we have that $L(f)(L(\tilde{x}))$ equals $L(f(\tilde{x}))$. Now let $\tilde{\tau}[\tilde{x}]$ denote $\mathcal{V}_{\Gamma; x:\sigma}^i \llbracket \tau[x] \rrbracket \gamma[x := \tilde{x}]$. By the definition of dependent function types we have $f(\tilde{x}) \in \tilde{\tau}[\tilde{x}]$. Part (c) of the induction hypothesis gives $L(f(\tilde{x})) \in L(\tilde{\tau}[\tilde{x}]) = \mathcal{V}_{\Gamma; x:\sigma}^i \llbracket \tau[x] \rrbracket L(\gamma)[x := L(\tilde{x})]$ which proves part (b).

For dependent function types we have that σ and $\tau[x]$ are set expressions and we must show

$$L\left(\mathcal{V}_\Gamma^{i+1} \left\llbracket \prod_{x:\sigma} \tau[x] \right\rrbracket \gamma\right) = \mathcal{V}_\Gamma^{i+1} \left\llbracket \prod_{x:\sigma} \tau[x] \right\rrbracket L(\gamma)$$

Again this follows from the induction hypothesis applied to σ and $\tau[x]$.

For the remaining clauses (V14) through (V20) we need only consider part (c). Clauses (V14) for boolean connectives, (V15) for function application, (V18) for pairing, and (V19) for π_i are simple compositional definitions and the result follows immediately from the induction hypothesis. Clause (V20) is for lambda expressions $\lambda x:\sigma e[x]$. All lambda expressions must be set level and in this case the proof is similarly straightforward by applying the induction hypothesis to σ and $e[x]$. We explicitly consider the two remaining clauses (V16) and (V17).

For (V16) defining equality we must show

$$\mathcal{V}_\Gamma^{i+1} \llbracket u = v \rrbracket \gamma = \mathcal{V}_\Gamma^{i+1} \llbracket u = v \rrbracket L(\gamma).$$

In this case we are given that there exists a set expression σ such that $\Gamma \models^i u:\sigma$ and $\Gamma \models^i v:\sigma$. Let \tilde{u} , \tilde{v} and $\tilde{\sigma}$ denote $\mathcal{V}_\gamma \llbracket u \rrbracket \gamma$, $\mathcal{V}_\gamma \llbracket v \rrbracket \gamma$ and $\mathcal{V}_\gamma \llbracket \sigma \rrbracket \gamma$ respectively.

By the projection lemma of section 4.2 we have that $\tilde{\sigma}$ is bijective. So we have $\tilde{u} = \tilde{v}$ if and only if $L(\tilde{u}) = L(\tilde{v})$. By the induction hypothesis we have $L(\tilde{u}) = \mathcal{V}_\Gamma^i \llbracket u \rrbracket L(\gamma)$ and $L(\tilde{v}) = \mathcal{V}_\Gamma^i \llbracket v \rrbracket L(\gamma)$ which proves the result.

Clause (V17) handles quantified formulas $\forall x:\sigma \Phi[x]$. Here we have allowed σ to be a class expression — class expressions within Boolean formulas are allowed. For this case we have

$$\mathcal{V}_\Gamma^{i+1} \llbracket \forall x:\sigma \Phi[x] \rrbracket \gamma = \mathbf{True} \quad \text{iff} \quad \Gamma; x:\sigma \models^i \Phi[x].$$

Applying the induction hypothesis to the expression $\Phi[x]$ gives that for $\gamma \in \mathcal{V}_\Gamma^i \llbracket \Gamma; x:\sigma \rrbracket$ we have $\mathcal{V}_\Gamma^i \llbracket \Phi[x] \rrbracket \gamma = \mathcal{V}_\Gamma^i \llbracket \Phi[x] \rrbracket L(\gamma)$ from which the result follows.

4.6 Proof of the Functor Lemma.

To prove the functor lemma we introduce two new functions `pack` and `unpack` on projectable values defined as follows.

Definition 9 (`pack` and `unpack`).

- $\text{pack}(\langle \text{"isopair"}, \langle u, w \rangle \rangle) = \langle \text{"atom"}, \langle \text{"isopair"}, \langle u, w \rangle \rangle$
- $\text{pack}(\langle \text{"atom"}, x \rangle) = \langle \text{"atom"}, \langle \text{"atom"}, x \rangle \rangle$
- $\text{pack}(\langle \text{"Bool"}, x \rangle) = \langle \text{"Bool"}, x \rangle$.
- $\text{pack}(\langle \text{"pair"}, \langle u, w \rangle \rangle) = \langle \text{"pair"}, \langle \text{pack}(u), \text{pack}(w) \rangle \rangle$
- $\text{pack}(\langle \text{"set"}, s \rangle) = \langle \text{"set"}, \{ \text{pack}(x) : x \in s \} \rangle$
- $\text{pack}(\langle \text{"function"}, f \rangle) = \langle \text{"function"}, g \rangle$ where g is the function containing the mappings $\text{pack}(u) \mapsto \text{pack}(f(u))$ for u in the domain of f .
- $\text{unpack}(\langle \text{"atom"}, x \rangle) = x$
- $\text{unpack}(\langle \text{"Bool"}, x \rangle) = \langle \text{"Bool"}, x \rangle$.
- $\text{unpack}(\langle \text{"pair"}, \langle u, w \rangle \rangle) = \langle \text{"pair"}, \langle \text{unpack}(u), \text{unpack}(w) \rangle \rangle$
- $\text{unpack}(\langle \text{"set"}, s \rangle) = \langle \text{"set"}, \{ \text{unpack}(x) : x \in s \} \rangle$
- $\text{unpack}(\langle \text{"function"}, f \rangle) = \langle \text{"function"}, g \rangle$ where g is the function containing the mappings $\text{unpack}(u) \mapsto \text{unpack}(f(u))$ for u in the domain of f .

Lemma 4. For any extended value x we have $\text{unpack}(\text{pack}(x)) = x$.

The functor lemma is proved from the from the following pack lemma.

Lemma 5 (Pack Lemma). *The pack operation satisfies the following conditions where, unless otherwise specified, all expressions are allowed to contain the constant **Bij** and where we define $\text{pack}(\gamma)$ by $\text{pack}(\gamma)(x) = \text{pack}(\gamma(x))$.*

- (a) *For $\gamma \in \mathcal{V} \llbracket \Gamma \rrbracket$ we have $\text{pack}(\gamma) \in \mathcal{V} \llbracket \Gamma \rrbracket$.*
- (b) *For any set-level expression e we have that $\mathcal{V}_\Gamma \llbracket e \rrbracket \gamma$ is defined if and only if $\mathcal{V}_\Gamma \llbracket e \rrbracket \text{pack}(\gamma)$ is defined and $\text{pack}(\mathcal{V}_\Gamma \llbracket e \rrbracket \gamma) = \mathcal{V}_\Gamma \llbracket e \rrbracket \text{pack}(\gamma)$.*
- (c) *For $\mathcal{V}_\Gamma \llbracket \tau \rrbracket \gamma$ defined we have that $\mathcal{V}_\Gamma \llbracket \tau^* \rrbracket \gamma$ is also defined and for any projectable value \tilde{u} we have $\text{pack}(\tilde{u}) \in \mathcal{V}_\Gamma \llbracket \tau \rrbracket \text{pack}(\gamma)$ if and only if $\tilde{u} \in \mathcal{V}_\Gamma \llbracket \tau^* \rrbracket \gamma$.*

The pack lemma can be proved by induction on the recursion index. We omit the proof. However, we note that the restriction that e be set-level in (b) (that e not contain the constants **Set** or **Bij**) is needed. In particular we have

$$\text{pack}(\mathcal{V}_\epsilon \llbracket \mathbf{Set} \rrbracket \epsilon) \neq \mathcal{V}_\epsilon \llbracket \mathbf{Set} \rrbracket \epsilon$$

Proof of the Functor Lemma. We must show that under the conditions of the lemma we have $\Gamma; x : \sigma^* \models e[x] : \tau^*$. Consider $\gamma \in \mathcal{V} \llbracket \Gamma \rrbracket$ and $\tilde{u} \in \mathcal{V}_\Gamma \llbracket \sigma^* \rrbracket \gamma$. By condition (a) of the pack lemma we have $\text{pack}(\gamma) \in \mathcal{V} \llbracket \Gamma \rrbracket$. By condition (c) of the pack lemma we have $\text{pack}(\tilde{u}) \in \mathcal{V}_\Gamma \llbracket \sigma \rrbracket \text{pack}(\gamma)$. These together give $\text{pack}(\gamma[x := \tilde{u}]) \in \mathcal{V} \llbracket \Gamma; x : \sigma \rrbracket$. We then have

$$\mathcal{V}_\Gamma \llbracket e[x] \rrbracket \text{pack}(\gamma[x := \tilde{u}]) \in \mathcal{V}_\Gamma \llbracket \tau \rrbracket \text{pack}(\gamma[x := \tilde{u}])$$

By condition (b) of the pack lemma we then have

$$\text{pack}(\mathcal{V}_\Gamma \llbracket e[x] \rrbracket \gamma[x := \tilde{u}]) \in \mathcal{V}_\Gamma \llbracket \tau \rrbracket \text{pack}(\gamma[x := \tilde{u}])$$

By condition (c) of the pack lemma we have

$$\mathcal{V}_\Gamma \llbracket e[x] \rrbracket \gamma[x := \tilde{u}] \in \mathcal{V}_\Gamma \llbracket \tau^* \rrbracket \gamma[x := \tilde{u}]$$

which proves the result. \square

5 Summary

We have presented well-formedness conditions on purely set-theoretic notation guaranteeing that the expressions built from these notations respect isomorphism as commonly understood in mathematics. Isomorphism, symmetry, canonicity, functors, natural transformations and cryptomorphism all emerge from these well-formedness conditions on set-theoretic language.

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