NOTE ON LINEAR DIFFERENCE AND DIFFERENTIAL EQUATIONS

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The modern analytic theory of ordinary linear difference equations had its inception in an early paper of Henri Poincaré published in 1885. More than two decades later the work of N. E. Nörlund, R. D. Carmichael and myself led to what, in a certain limited sense, is a "general" theory. More recently the existence of a full complement of formal solutions has been established in all cases, and on this basis W. J. Trjitzinsky and I have given jointly a truly general theory.[1]

Much before this latest work, the very clear conviction had come to me that the analytic theory of linear difference equations provided a complete methodological pattern in the formulation of a comparable general theory for the essentially simpler but analogous field of linear differential equations. This point of view was altogether substantiated by Trjitzinsky in his subsequent development of the general theory of linear differential equations.[2]

What I desire to point out in the present Note is that the methodological relationship between the two fields is so close that the theory of linear differential equations falls out as an immediate corollary of the theory of linear difference equations. For the sake of simplicity, however, I shall establish this fact only for the typical case of a linear differential system (in matrix notation)

$$Y'(x) = A(x)Y(x) \tag{1}$$

with an irregular singular point of rank 0 at $x = \infty$, so that the nth order square matrix $A(x) \equiv (a_{ij}(x))$ is analytic at $x = \infty$. A graduate student at Harvard University, Mr. E. C. Gras, is undertaking to carry out systematically the extension of this program to the general case.

In the first place there is an immediately provable theorem[3] which allows us to conclude that any matrix solution $Y(x)$ of (1) satisfies a functional equation

$$Y(\varphi(x)) = M(x)Y(x)$$

where $M(x)$ is analytic at $x = \infty$, provided that $\varphi(x)$ is a function of the form (case $q = 0$ of the theorem)

$$\varphi(x) = x + l + \frac{m}{x} + \dots$$

For our purposes here it suffices merely to take

$$\varphi(x) = x + 1$$

and thus arrive at the difference equation

$$Y(x + 1) = M(x)Y(x) \tag{2}$$

where $M(x)$ is analytic at $x = \infty$.

Now in accordance with the general theory of the difference equation (2), there will exist a formal matrix solution $S(x)$ which has the form

$$S(x) \equiv (\rho_j^x x^{r_j'} s_{ij}(x)) \tag{3}$$

in case the characteristic equation

$$|\rho\delta_{ij} - m_{ij}(\infty)| = 0$$

has n distinct roots $\rho_1, \ldots \rho_n$; here the expressions $s_{ij}(x)$ stand for descending formal power series in $1/x$ such that the determinant of the leading coefficients $|s_{ij}(\infty)|$ is not zero. We confine attention to this case.[4]

But the general theory in this non-singular case[5] shows that there will then be two special "principal solutions" $Y_+(x)$ and $Y_-(x)$ such that the asymptotic formulae hold:

$$\left.\begin{aligned} Y_+(x) &\sim S(x), -\frac{\pi}{2} - \epsilon \leqq arg\ x \leqq \frac{\pi}{2} + \epsilon, \\ Y_-(x) &\sim S(x), \frac{\pi}{2} - \epsilon \leqq arg\ x \leqq \frac{3\pi}{2} + \epsilon. \end{aligned}\right\} \tag{4}$$

The most general solution of (2) may be written as $Y_0(x)P(x)$ where $Y_0(x)$ is any particular matrix solution and $P(x)$ is a periodic matrix of period 1. Hence any solution $Y(x)$ of (1) may be expressed in the first one of the specified sectors as

$$Y(x) = Y_+(x)P(x) \tag{5}$$

Here $P(x)$ is not only periodic of period 1 but analytic in the finite plane with $|P(x)| \neq 0$, since $|Y(x)| \neq 0$ and $|Y_+(x)| \neq 0$, for $|x|$ sufficiently large in the right half plane. Substituting this expression in (1) we readily obtain

$$Y_+^{-1}(x)[Y_+'(x) - A(x)Y_+(x)] = P'(x)P^{-1}(x), \tag{6}$$

which leads at once to an asymptotic relationship in this sector

$$P'(x)P^{-1}(x) \sim ((\rho_j/\rho_i)^x x^{r_j - r_i} t_{ij}(x)). \tag{7}$$

The matrix $P'(x)P^{-1}(x)$ appearing here on the left is of course analytic in the finite plane and periodic of period 1. The expressions $t_{ij}(x)$ appearing on the right stand for descending formal power series in $1/x$. The precise meaning of the asymptotic relationship written is obvious.

But if we transfer attention to the z-plane where

$$z = e^{2\pi i x},$$

this relationship shows that the typical element $q_{ij}(z)$ of the matrix on the left in (7) is not only single-valued and analytic for $z \neq 0, \infty$, but is of finite order at these excepted points. Thus $q_{ij}(z)$ must be polynomials in $1/z$ and z. The same relationship also shows this to be impossible unless $t_{ij}(x) \equiv 0$ for all i and j. It follows therefore from (7) that $S(x)$ is a formal matrix solution of the given differential equation (1), and that we may put $t_{ij}(x) \equiv 0$ in (7):

$$P'(x)P^{-1}(x) \sim ((\rho_{j/pi})^x(0)). \tag{8}$$

However it is then clear that the product $q_{ij}(z)q_{ij}(1/z)$ is not only analytic for $z \neq 0, \infty$, but vanishes at these points. Hence the matrix $P'(x)$-$P^{-1}(x)$ and so $P'(x)$ vanish identically. Consequently $P(x)$ is a constant matrix, of constants C; and, by (6), $Y_+(x)$ is a solution of the given differential equation (1), while $Y(x) = Y_+(x)C$ is the given solution of (1).

It is thus immediately established that the given linear differential system (1) admits of matrix solutions $Y(x)$ asymptotic to $S(x)$ in the two specified sectors covering the entire plane. From this follow at once all the usual facts about the asymptotic behavior of solutions in the complete neighborhood of $x = \infty$.

There can be no doubt that the intensification of this type of argument, based on the general theory of linear difference equations, will lead easily and directly to the asymptotic characterization of the solutions of linear differential equations at any regular or irregular singular point. Furthermore it is equally clear that in a similar manner the formal theory and the inverse Riemannian theory for differential equations follow directly from the corresponding results for difference equations.

[1] G. D. Birkhoff, "Formal Theory of Irregular Linear Difference Equations," *Acta Mathematica*, **54** (1930); G. D. Birkhoff and W. J. Trjitzinsky, "Analytic Theory of Singular Difference Equations," *ibid.*, **60**, 1–89 (1936).

[2] W. J. Trjitzinsky, "Analytic Theory of Linear Differential Equations," *ibid.*, **62**, 167–226 (1938).

[3] G. D. Birkhoff, "Theorem Concerning the Singular Points of Ordinary Differential Equations," these PROCEEDINGS, **1**, 578–581 (1915).

[4] If this restriction is not satisfied, more complicated types of series and a more complicated analytic situation may arise, but the general course of the argument is not affected.

[5] See, for instance, my paper, "General Theory of Linear Difference Equations," *Trans. Am. Math. Soc.*, **10**, 436–470 (1909).