Europ. J. Combinatorics (2001) **22**, 225–241
doi:10.1006/eujc.2000.0458
Available online at http://www.idealibrary.com on IDEAL®

Orbits on *n*-tuples for Infinite Permutation Groups

FRANCESCA MEROLA

This paper presents a theorem on the growth rate of the orbit-counting sequences of a primitive oligomorphic group: if G is not a highly homogeneous group, then the growth rate for the sequence counting orbits on n-tuples of distinct elements is bounded below by $c^n n!$, where $c \approx 1.172$.

The previously known lower bounds concerned all not highly transitive groups, including highly homogeneous groups which are known to have roughly factorial growth rate. This paper shows that highly homogeneous groups are the only groups with such a growth rate, while for all other primitive groups the growth rate is faster and the bound is improved by an exponential factor.

1. INTRODUCTION

This paper deals with the growth rates of the orbit counting sequences for a primitive oligomorphic group; a permutation group G on an infinite set Ω is said to be oligomorphic if G has only finitely many orbits in its induced action on Ω^n for all n (see Cameron [5]). Let us set f_n to be the number of orbits on n-sets and F_n the number of orbits on n-tuples of distinct elements of Ω; it is easy to see that the sequence (F_n) is non-decreasing; the sequence (f_n) is also non-decreasing, though this result is not obvious (see Cameron [5]).

It becomes natural to study the growth rate of the sequences; lower bounds for these sequences have been studied since the 1980s, and the deepest result on the subject is the following theorem by H. D. Macpherson [10].

THEOREM 1.1 (MACPHERSON). *There is a constant $c > 1$ with the following property: let G be a primitive oligomorphic group. Then*

- *either $f_n = 1$ for all n or*
$$f_n > c^n$$
for all (sufficiently large) n;
- *either $F_n = 1$ for all n or*
$$F_n > \frac{n!}{p_G(n)}$$
for all (sufficiently large) n, where $p_G(n)$ is a polynomial depending on G.

Note that $F_1(G) = 1$ simply means that the group is transitive; the group G is called *n-transitive* if $F_n(G) = 1$, and it is *highly transitive* if it is n-transitive for each $n \in \mathbb{N}$; we say that G is *n-homogeneous* (or *n-set-transitive*) if it is transitive on unordered n-subsets of Ω, and *highly homogeneous* if it is n-homogeneous for all $n \in \mathbb{N}$. Cameron [1] has classified the (closed) highly homogeneous groups, and has thus shown that the sequence (F_n) associated with a highly homogeneous group has factorial growth rate.

Now the aim of this work is to prove a generalization of Macpherson's theorem which will distinguish the highly homogeneous groups, since it makes sense to suppose that the only primitive groups realizing the second bound in Theorem 1.1 are those with the slowest (f_n) sequence, that is the highly homogeneous ones.

The result proved in this paper is the following theorem.

THEOREM 1.2. *There is a constant $c > 1$ with the following property: let G be a primitive oligomorphic group. Then either $f_n = 1$ for all n or there is a constant $c > 1$ such that*

$$F_n(G) > \frac{c^n\, n!}{p_G(n)},$$

for all (sufficiently large) n, where $p_G(n)$ is a polynomial depending on G.

This theorem shows that if the group is not highly homogeneous, the bound in Macpherson's theorem can be strengthened by an exponential factor. Moreover, this new result also implies Macpherson's theorem as an easy corollary; I also improve slightly on the value of the constant in Macpherson's theorem.

Let us sketch how to prove Theorem 1.2: the proof will follow the argument used by Macpherson in [10].

First we will say that a sequence (a_n) is *almost bounded below* by c^n (respectively $c^n n!$) if there is a polynomial $p(n)$ such that $a_n > c^n/p(n)$ $\left(a_n > c^n \frac{n!}{p(n)}\right)$. The theorem is proved by showing that a primitive group whose (F_n) sequence is not almost bounded below by $c^n n!$ for any constant c is highly homogeneous; more precisely, we will see that it is n-homogeneous for each n in $\mathbb{N}$ by induction on n.

We will see that in order to prove the theorem it is enough to prove that a primitive group with 'slow' growth rate (that is with F_n not almost bounded below by $c^n\, n!$) must be at least 3-homogeneous (Theorem 7.1). Showing that Theorem 7.1 implies Theorem 1.2 is not very difficult, and relies on the fact that a group G has the property that the sequence $(F_n(G))$ is almost bounded below by $c^n\, n!$ if and only the sequence $(F_n(G_\alpha))$ of the pointwise stabilizer is almost bounded below by $c^n\, n!$ with the same constant c (Lemma 2.2); that is, a group with 'fast' growth rate and its pointwise stabilizer have the same growth rate (modulo a polynomial).

The hardest part of the proof is to show that primitivity and slow growth rate imply 3-homogeneity, that is verifying the first two inductive steps. This is done by considering first the case of a primitive, not 2-homogeneous group, then the case of a 2-homogeneous but not 2-transitive nor 3-homogeneous group and finally the case of a 2-transitive but not 3-homogeneous group. In each case, one uses the fact that there is a combinatorial structure on Ω that is G-invariant, so that it is possible to consider G as a subgroup of the automorphism group of this structure. In the different cases I either show directly that the number of orbits on n-tuples for the the automorphism group of such a structure is almost bounded below by $c^n\, n!$ or, in the case of a 2-homogeneous but not 2-transitive nor 3-homogeneous group and partly in the case of a 2-transitive group, I prove a slightly stronger result; namely, there are exponentially many n-subsets X of Ω with the property that the group induced on X by the setwise stabilizer G_X is 'small'; more precisely, there are functions f, g such that one has $f(n)$ subsets of size n belonging to different G-orbits and with the induced group of size bounded by $g(n)$, with $f(n)$ almost bounded below by $c^n\, g(n)$.

Let us note that, given the monotonicity of the sequence $(F_n(G))$, to get Theorem 1.2 it is enough to prove that the bound in the theorem holds for values of n forming an arithmetic progression with fixed modulus—for instance, in Section 3 we will use the odd values of n.

The paper is organized as follows: Section 2 contains some examples of growth rates for primitive oligomorphic groups and some preliminary results needed in what follows.

Then Theorem 7.1 is proved in Sections 3–6. Section 3 deals with the case of a primitive, not 2-homogeneous group; if G is not 2-homogeneous, then it leaves invariant a graph Γ (where the edges are a G-orbit on 2-sets), so that $G \leq \mathrm{Aut}(\Gamma)$; we then show that $F_n(\mathrm{Aut}(\Gamma))$ is almost bounded below by $c^n\, n!$.

Section 4 concerns 2-homogeneous, but not 3-homogeneous or 2-transitive groups; in this case the G-invariant structure turns out to be a tournament T, and we will similarly show that the growth rate for $\mathrm{Aut}(T)$ is almost bounded below by $c^n\, n!$.

Sections 5 and 6 deal with the case of a 2-transitive group; in Section 5 we study the case of a group preserving a Steiner system, and we will use the results obtained in this case to settle the general case of a 2-transitive, not 3-homogeneous group in Section 6. This will complete the proof of Theorem 7.1.

The arguments used to prove the results in Sections 3 and 4 require a slight modification of Macpherson's proofs: a more independent argument is used in Sections 5 and 6.

The proof of the theorem is finally completed in Section 7, which also contains some final remarks on the results obtained in the paper.

2. Examples and First Results

In this section we will see some examples of growth rates in oligomorphic permutation groups, and we will collect here some results used throughout the proof of the main theorem.

First of all the basic notation: we write permutations on the right, and compose from left to right (the image of $\alpha \in \Omega$ under $g \in G$ is αg, and $\alpha(gh) = (\alpha g)h$). If X is a subset of Ω, we denote by G_X the *setwise* stabilizer and by G_X^X the permutation group induced by G on X. Let G act transitively on Ω. Recall that G is said to be *primitive* if there are no nontrivial G-invariant equivalence relations. We also say that G is *n-primitive* if it is n-transitive and if the pointwise stabilizer $G_{\alpha_1,\ldots,\alpha_{n-1}}$ of $n-1$ points acts primitively on $\Omega \setminus \{\alpha_1, \ldots, \alpha_{n-1}\}$.

There is a connection between counting orbits for oligomorphic groups and counting finite substructures in a homogeneous relational structure: a *relational structure* X on a set Ω consists of a number of relations on Ω of various arities (the number of arguments). It is *homogeneous* if every isomorphism between finite substructures of X can be extended to an automorphism of the whole structure X. In the 1950s Fraïssé [12] gave a necessary and sufficient condition (discussed in detail in [5]) for a class $\mathcal{C}$ of finite structures to be *all* the finite substructures of a countable homogeneous structure (the *age* of X, in the terminology of Fraïssé). Now let X be a homogeneous structure and let $\mathcal{C}$ be the age of X. If G is the automorphism group of X, then G-orbits on n-sets correspond to isomorphism classes of n-element structures of $\mathcal{C}$ (unlabelled n-element substructures of X), while G-orbits on n-tuples of distinct elements correspond to the members of $\mathcal{C}$ with a fixed domain of cardinality n (labelled n-element substructures of X). So the problem of calculating the sequences f_n, F_n for an oligomorphic group G correspond to that of enumerating unlabelled and labelled structures in a class satisfying Fraïssé's condition.

Let us use this fact to see some examples of growth rates realized by primitive groups. We start with two examples of groups with slow growth rate for the sequence (F_n) to show the extent to which Theorem 1.2 is sharp.

Example 1. A *tournament* T is a directed complete graph, that is a digraph such that if α and β are vertices of T with $\alpha \neq \beta$, then exactly one of the edges $\alpha \to \beta$, $\beta \to \alpha$ exists. If α is a vertex of T, then we set $\alpha^+ := \{\beta \in VT : \alpha \to \beta\}$ and $\alpha^- := \{\beta \in VT : \beta \to \alpha\}$.

Now a *local order* is a tournament T with the property that no 4-point substructure consists of a 3-cycle dominating or dominated by a point: that is for any vertex α of T, the sets α^+ and α^- are linearly ordered by $\to$. Local orders satisfy Fraïssé's condition, so there is a unique countable homogeneous local order T; it can be shown (see Cameron [2]) that

$$f_n(\mathrm{Aut}(T)) \sim \frac{2^{n-1}}{n},$$

$$F_n(\mathrm{Aut}(T)) = 2^{n-1}(n-1)!.$$

If we consider the larger group G of automorphisms and anti-automorphisms of T (that is, orientation preserving and reversing permutations of T), we will have $f_n(G) \sim f_n(\mathrm{Aut}(T))/2$ and $F_n(G) \sim F_n(\mathrm{Aut}(T))/2$. This group has the slowest known growth rate realized by a primitive, not highly homogeneous group.

EXAMPLE 2. In the paper [3], Cameron constructed a 3-homogeneous, 2-transitive but not 2-primitive permutation group H of countable degree as the automorphism group of a given ternary relation. The point stabilizer H_α acts as H Wr A on the remaining points (where A is the highly homogeneous group $\mathrm{Aut}(\mathbb{Q}, \leq)$ of order-preserving permutations of the rational numbers). Cameron showed that

$$f_n(H) \sim c^n, \qquad c \approx 2.843,$$
$$F_n(H) = (2n-3)!! = 1 \cdot 3 \cdot 5 \ldots (2n-3);$$

this happens because orbits on n-sets (resp. n-tuples of distinct elements) correspond to unlabelled (resp. labelled) binary trees on n leaves—see Section 3.

Now

$$(2n-3)!! = \frac{1}{2n-1}\frac{(2n)!}{(2n)!!} = \frac{1}{2n-1}\frac{(2n)!}{2^n n!},$$

where $(2n)!! = 2 \cdot 4 \cdot 6 \ldots 2n$; since from Stirling's formula (see, for instance, [11, p. 1077]) one has

$$\frac{(2n)!}{n!} \geq \frac{2^{2n} n!}{n},$$

we have that $F_n(H)$ is almost bounded below by $2^n n!$.

These examples show that the bound in Theorem 1.2 is sharp and the best possible constant is not greater than 2.

In general, primitive groups with growth rate not faster than exponential times factorial are not very common; apart from local orders, the known examples seem to be related to trees (see, for instance, Cameron [4]). Let us see an example of faster growth rate.

EXAMPLE 3. Let $\mathcal{C}$ be the class of all finite graphs: $\mathcal{C}$ satisfies Fraïssé's condition, so let R be the unique countable homogeneous graph R (this is the well-known random graph of Erdős and Rényi [8]). Then $F_n(\mathrm{Aut}(R))$ is the number of labelled graphs on n vertices, and $f_n(\mathrm{Aut}(R))$ is the number of isomorphism types of n-vertex graphs:

$$F_n(G) = 2^{\binom{n}{2}},$$
$$f_n(G) \sim \frac{2^{\binom{n}{2}}}{n!}.$$

Let us now see some results we will need in order to prove the theorem.

An oligomorphic group may have the following property:

($\star$) There are two functions $f(n)$ and $g(n)$ such that for each n we can find $f(n)$ different sets of size n $\{X_i\}_{1\leq i\leq f(n)}$ belonging to different G-orbits, with the property that for each i

$$|G_{X_i}^{X_i}| \leq g(n),$$

and such that for some $c > 1$ one has

$$f(n) \quad \text{is almost bounded below by} \quad c^n g(n).$$

Note that a group satisfying $(\star)$ has growth rate almost bounded below by $c^n n!$:

LEMMA 2.1. *Let G be a group satisfying property $(\star)$ with constant c: then $F_n(G)$ is almost bounded below by $c^n n!$*

PROOF. It is clear that if $\mathcal{O}$ is an orbit on n-sets of elements of Ω, and if $X \in \mathcal{O}$, then there will be $\frac{n!}{|G_X^X|}$ orbits on n-tuples of distinct elements of Ω arising from the orbit $\mathcal{O}$. Also, if $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_k$ are distinct orbits on n-sets, with $X_i \in \mathcal{O}_i$, then

$$F_n(G) \geq \sum_{i=1}^{k} \frac{n!}{|G_{X_i}^{X_i}|}.$$

It follows that if G is a group satisfying property $(\star)$, then

$$F_n(G) \geq f(n)\frac{n!}{g(n)},$$

so $F_n(G)$ is almost bounded below by $c^n n!$, where c is the same constant appearing in $(\star)$. $\square$

LEMMA 2.2. *Let G be an oligomorphic group acting on a countable set Ω, and let $\alpha \in \Omega$. Then:*

(1) The sequence $F_n(G_\alpha)$ is almost bounded below by $c^n n!$ if and only if $F_n(G)$ is almost bounded below by $c^n n!$;
(2) the group G has the property $(\star)$ if and only if the stabilizer G_α has the property $(\star)$.

For the proof of this result, see the proof of Lemma 3.8 in [10]. See [10] also for the proof of the next lemma.

LEMMA 2.3. *Let G be transitive but not primitive on an infinite set Ω. Then any chain of blocks of imprimitivity contains at most $f_2(G) - 1$ proper blocks.*

We recall a result of J. P. J. McDermott (see, for instance, Dixon and Mortimer [7, p. 289])

THEOREM 2.1. *If G is a 3-homogeneous but not 2-transitive group, it preserves a linear order.*

Let H be the group of Example 2 (see Cameron [3]). We will need the following result (see [3, Theorem 3.3]).

THEOREM 2.2. *Suppose that G is 3-homogeneous and 2-transitive but not 2-primitive. Then either G preserves a betweenness relation, or G is a subgroup of H.*

3. NOT 2-HOMOGENEOUS GROUPS

The aim of this section is to prove the main theorem for the case of a not 2-homogeneous group G.

THEOREM 3.1. *Let G be a primitive but not 2-homogeneous oligomorphic group. Then there is a constant $c > 1$ such that $F_n(G)$ is almost bounded below by $c^n n!$.*

We will see in the proof of the theorem that $c \approx 2^{1/2}$. The idea here will be to show that there is a G-invariant graph Γ with vertex set Ω; we will use the properties of this graph, and of its n-vertex subgraphs, to prove the theorem. The argument in this section follows Macpherson—we also use the same terminology (see [10]).

We will start by recalling some combinatorial facts on binary trees that will be needed in the proof. A *binary tree* is a finite rooted tree in which the root has valency 2, and all the other vertices have valency 1 or 3—note that, unlike in the definition of binary trees used in computer science, these trees are not ordered.

We will need the asymptotics for the number of unlabelled and labelled binary trees on n vertices: let us call b_n the number of unlabelled and B_n the number of labelled binary trees on n vertices. Note that a binary trees with k leaves (as usual, we call *leaves* the vertices of valency 1) has $2k-1$ vertices; if we denote by β_k the number of unlabelled binary trees on k leaves, then $b_n = b_{2k-1} = \beta_k$. The numbers β_k satisfy the following recurrence relation:

$$\beta_k = \begin{cases} \beta_1\beta_{k-1} + \beta_2\beta_{k-2} + \cdots + \beta_{\frac{k-1}{2}}\beta_{\frac{k+1}{2}} & \text{for } k \text{ odd,} \\ \beta_1\beta_{k-1} + \beta_2\beta_{k-2} + \cdots + \frac{1}{2}\beta_{\frac{k}{2}}(\beta_{\frac{k}{2}} + 1) & \text{for } k \text{ even.} \end{cases}$$

There are results on the asymptotic behaviour of the sequence (β_k). One has

$$\beta_k \geq C^k,$$

where $C \approx 2.483$ (see, for instance, Comtet [9]), and from this we get for the number of binary trees on n vertices

$$b_n \geq c^n,$$

where $c \approx C^{1/2}$, for n odd (see also Macpherson [10]).

Now let us consider labelled binary trees. One has that the number of labelled binary trees on k leaves is $(2k-3)!! = 1 \cdot 3 \cdot 5 \ldots (2k-3)$ (see Cameron [3]); we need the number of labelled binary trees on n vertices. If we consider a binary tree on $n = 2k-1$ vertices, we have $\binom{2k-1}{k}$ ways of choosing the labels of the leaves, and $(2k-3)!!$ leaf-labelled trees. Now the labels for the remaining $k-1$ vertices (the 'internal' vertices) can be put in $(k-1)!$ ways, and these all give different labelled trees: if two of them were the same, there would be an isomorphism between the trees which fixes all the leaves, and this is impossible: working inductively from leaves to root, one sees that such an isomorphism would fix everything. Then

$$B_{2k-1} = \binom{2k-1}{k}(k-1)!(2k-3)!!,$$

and since (see Section 2, Example 2, and Odlyzko [11])

$$\binom{2k-1}{k} \geq 2^{2k-2}/k \quad \text{and} \quad (2k-3)!!(k-1)!\,2^{k-1} = (2k-2)!,$$

we have $B_{2k-1} \geq 2^{k-1}(2k-1)!$, that is

$$B_n \geq c^n n!,$$

where $c \approx 2^{1/2}$.

Now let us briefly view Macpherson's argument for the not 2-homogeneous case. Let G be a primitive but not 2-homogeneous permutation group on a countable set Ω. We can define a graph Γ with vertex set Ω, having as its edge set some orbit of G on the set of 2-subsets of Ω. Now G is a subgroup of $\mathrm{Aut}(\Gamma)$, so that if say γ_n is the number of non isomorphic n-vertex subgraphs of Γ, then $f_n(G) \geq \gamma_n$ for all n in $\mathbb{N}$. The graph Γ is not complete or null since G is not 2-homogeneous.

Macpherson shows that, since the group is primitive and oligomorphic, any two vertices of the graph have infinite symmetric difference; by repeatedly applying Ramsey's theorem he then shows that the graph Γ has a subgraph Δ for which many regularity properties hold. This enables him to prove that Δ has exponentially many non-isomorphic subgraphs: he shows that a binary tree T on n vertices can be encoded into a subgraph Δ_T of Δ in such a way that the isomorphism type of T can be almost completely recovered from the isomorphism type of the graph Δ_T. The only problem is that it is not possible with this encoding to recover the two neighbours in T of the root (labelled by (0),(1) in Macpherson's encoding); what he obtains, then, is that if T_1 and T_2 are nonisomorphic binary trees, then there is no isomorphism between Δ_{T_1} and Δ_{T_2} which fixes $\{(0), (1)\}$ setwise.

It follows from what we have just seen that the graph Δ has exponentially many non isomorphic n-vertices subgraphs for each n, corresponding to the exponentially many binary trees on n vertices. To allow for the fact that in the encoding we have to distinguish the subset $\{(0), (1)\}$, we must divide by $\binom{n}{2}$. Then

$$f_n(G) > \frac{2}{n(n-1)}\, b_n,$$

where b_n is the number of binary trees on n leaves (for n odd).

Let us prove Theorem 3.1; what we need is a lower bound on the sequence $(F_n(G))$ associated with a not 2-homogeneous group. The idea here is that, since there is an encoding of unlabelled binary trees into G-orbits on subsets, we have a correspondence between labelled binary trees and orbits on n-tuples.

PROOF OF THEOREM 3.1. Let n be an (odd) positive integer, and let $T_1, \ldots, T_{b_n}$ be the exponentially many binary trees on n vertices. Let $\Delta_{T_1}, \ldots, \Delta_{T_{b_n}}$ be the corresponding (according to Macpherson's encoding) subgraphs of the graph Δ. At least $a_n = b_n \frac{2}{n(n-1)}$ of these graphs are non-isomorphic subgraphs of Δ; now relabel the graphs so that $\Delta_1, \ldots, \Delta_{a_n}$ is the sequence of non-isomorphic subgraphs.

Let us consider the exponentially many subsets of the set Ω

$$X_1 = V\Delta_1, \ldots, X_{a_n} = V\Delta_{a_n}.$$

Since these subsets belong to different G-orbits, we have as in the proof of Lemma 2.1

$$F_n(G) \geq \sum_{i=1}^{a_n} \frac{n!}{|G_{X_i}^{X_i}|}.$$

Now $|G_{X_i}^{X_i}| \leq |\operatorname{Aut}(\Delta_i)|$; we consider the groups $\operatorname{Aut}(\Delta_i)$, $1 \leq i \leq a_i$, and set $H_i = \operatorname{Aut}(\Delta_i)$.

From Macpherson's encoding we have that the binary tree T_i corresponds completely to the graph Δ_i with a distinguished subset $\{(0), (1)\}$ of vertices: it follows that $|H_{i\{(0),(1)\}}| = |\operatorname{Aut}(T_i)|$; the only problems might appear when considering the vertices (0), (1), so we have to bound the size of the orbits of the set $\{(0), (1)\}$. The group H_i acts on a set of size n, so that the orbit of the 2-set $\{(0), (1)\}$ is bounded by $\binom{n}{2} < n^2$.

We then have $|H_i| \leq n^2 |\operatorname{Aut}(T_i)|$; now

$$F_n \geq \sum_{i=1}^{a_n} \frac{n!}{|H_i|} \geq \frac{1}{n^2} \sum_{i=1}^{a_n} \frac{n!}{|\operatorname{Aut}(T_i)|}.$$

Finally we recall that $a_n = b_n \frac{2}{n(n-1)}$: now

$$F_n \geq \frac{1}{n^2} \sum_{i=1}^{a_n} \frac{n!}{|\operatorname{Aut}(T_i)|} \geq \frac{1}{n^4} \sum_{i=1}^{b_n} \frac{n!}{|\operatorname{Aut}(T_i)|};$$

since we know that

$$\sum_{i=1}^{b_n} \frac{n!}{|\operatorname{Aut}(T_i)|} = B_n \sim c^n n!$$

where B_n is the number of labelled binary trees, we can conclude that the sequence $F_n(G)$ for a primitive, not 2-homogeneous group is almost bounded below by $c^n n!$ ($c \approx 2^{1/2}$).

4. 2-HOMOGENEOUS GROUPS

Let us consider the case of a group G which is 2-homogeneous but not 2-transitive or 3-homogeneous on Ω. In this section we will prove the following theorem.

THEOREM 4.1. *Let G be a 2-homogeneous, not 3-homogeneous nor 3-transitive group. Then there is a constant $c > 1$ such that $F_n(G)$ is almost bounded below by $c^n n!$.*

The value for the constant c will turn out to be $c \approx 1.174$. We will start by describing Macpherson's argument, then we will show how to modify his proof to obtain our result.

Let us first note that Theorem 4.1 is immediate for a group preserving a linear order, since we have the following result (see, for instance, Macpherson [10, Corollary 5.5])

LEMMA 4.1. *If the group G preserves a linear order on Ω, then $f_n(G)$ is bounded below exponentially with constant 2.*

It follows that if G is a group that preserves a linear order without being highly homogeneous, then is also satisfies the bound in the theorem, for any subset X of the linearly ordered set Ω clearly has the property that $|G_X^X| = 1$. We can therefore assume that G does not preserve a linear order.

We have recalled the definition of a tournament in Section 2. Now note that in the case of a 2-homogeneous, not 2-transitive group G there is a G-invariant tournament T with vertex set Ω, having as edge set one of the orbits of G on ordered pairs; choose any orbit, say $\mathcal{O}$, on ordered pairs, and if (α, β) is in $\mathcal{O}$ put an arc $\alpha \to \beta$. Now let $\{\gamma, \delta\}$ be any 2-subset of Ω; by 2-homogeneity there is a $g \in G$ with $\{\gamma, \delta\}g = \{\alpha, \beta\}$, and this implies $\delta \to \gamma$ or $\gamma \to \delta$.

Since G does not preserve a linear order, every arc $u \to v$ lies on a cycle $u \to v \to w \to u$.

Macpherson [10] has shown, by opportunely applying Ramsey's theorem, that in this case the tournament T has a subtournament $\mathcal{T}$ with the following regularity properties.

LEMMA 4.2. *The tournament T contains a subtournament $\mathcal{T}$ whose vertex set is the disjoint union of the infinite sets U, V, W and Z_i (for i in $\mathbb{N}^*$), where*

$$U = \{u_i \mid i \in \mathbb{N}^*\} \qquad V = \{v_i \mid i \in \mathbb{N}^*\}$$
$$W = \{w_i \mid i \in \mathbb{N}^*\} \qquad Z_i = \{z_{ij} \mid j \in \mathbb{N}^*\};$$

the tournament $\mathcal{T}$ has the following properties:

(1) the set $U \cup V \cup \bigcup_{i \in \mathbb{N}} Z_i$ is linearly ordered by $\to$, with

$$z_{11} \to z_{12} \to \cdots \to u_1 \to v_1 \to z_{21} \to z_{22} \to \cdots \to u_2 \to v_2 \to z_{31} \cdots$$

(2) *for all $i \in \mathbb{N}$, there is a cycle $u_i \to v_i \to w_i \to u_i$.*
(3) *Either*

(a) $Z_i \subseteq w_j^+$ *whenever* $i \leq j$, *or*
(b) $Z_i \subseteq w_j^-$ *whenever* $i \leq j$.

(4) *Either*

(a) $Z_i \subseteq w_j^+$ *whenever* $i > j$, *or*
(b) $Z_i \subseteq w_j^-$ *whenever* $i > j$.

(5) *The natural versions of* 3 *and* 4 *hold with the u_i replacing the Z_i (and with $\leq$ replaced by $<$ in* 3*).*
(6) *Case* 5 *holds with the v_i replacing the u_i.*
(7) *The set W is linearly ordered by $\to$, with*

$$w_1 \to w_2 \to w_3 \cdots \qquad \text{or} \qquad w_1 \leftarrow w_2 \leftarrow w_3 \cdots.$$

REMARK. In fact, Macpherson builds a tournament having the regularity properties 1 to 6. It is easily seen that this tournament has a subtournament $\mathcal{T}$ with the properties 1 to 7; just apply Ramsey's theorem to the index set of the set of vertices W, colouring the 2-subsets of $\mathbb{N}$; we colour the set $\{i, j\}$ red if $w_i \to w_j$ and blue if $w_i \leftarrow w_j$. Ramsey's theorem guarantees the existence of an infinite monochromatic subset M of $\mathbb{N}$; now taking M as the new index set for the sets of vertices U, V, W and Z_i, we will have a tournament $\mathcal{T}$ satisfying the properties 1 to 7.

We are now going to prove Theorem 4.1. The argument here closely follows Macpherson's. For n a positive integer let $\mathcal{T}'$ be a subtournament of $\mathcal{T}$ with vertex set $\{u_i, v_i, w_i : 1 \leq i \leq n-1\}$ and at least two points from Z_i for each i between 1 and n. Now Macpherson defines a relation R on $V\mathcal{T}'$: for a, b in $V\mathcal{T}'$ put aRb if for all the other vertices w of $\mathcal{T}'$ we have $w \to a$ if and only if $w \to b$. Then he considers the transitive closure $\mathcal{R}$ of R, that is a transitive relation on $V\mathcal{T}'$ defined as follows: put $a\mathcal{R}b$ whenever either one has aRb or there is a c in $V\mathcal{T}'$ with $a\mathcal{R}c$ and $c\mathcal{R}b$.

The relation $\mathcal{R}$ is an equivalence relation on $V\mathcal{T}'$, and each equivalence classes is linearly ordered by $\to$. From the properties of the tournament $\mathcal{T}$, one has that each set $Z_i \cap V\mathcal{T}'$ is contained in an $\mathcal{R}$-class, called $[Z_i]$. From the regularity properties of $\mathcal{T}$, it follows that for the 'extremal' classes $[Z_1]$, $[Z_n]$ one has $[Z_1] = Z_1 \cap V\mathcal{T}'$ or $[Z_1] = (Z_1 \cap V\mathcal{T}') \cup \{u_1\}$, and similarly $[Z_n] = Z_n \cap V\mathcal{T}'$ or $[Z_n] = (Z_n \cap V\mathcal{T}') \cup \{v_{n-1}\}$. For $2 \leq i \leq n-1$, one has that $[Z_i]$ is one of the sets $Z_i \cap V\mathcal{T}'$, $(Z_i \cap V\mathcal{T}') \cup \{u_i\}$, $(Z_i \cap V\mathcal{T}') \cup \{v_{i-1}\}$ or $(Z_i \cap V\mathcal{T}') \cup \{u_i, v_{i-1}\}$; moreover, the same case occurs for all such i. The only other possible $\mathcal{R}$-class of size greater than 1 is $\{w_1, \ldots, w_{n-1}\}$.

Now our argument starts to differ from Macpherson's: while he encodes ordered partitions of n into subtournaments of $\mathcal{T}$, we will need to encode n-tuples of positive integers summing up to cn.

First, let us remark that the number of n-tuples of positive integers with sum cn is equal to the number of n-tuples of non-negative integers with sum $(c-1)n$ which, as known, is $\binom{cn-1}{n-1}$. We will encode these n-tuples into subtournaments on $cn + n + 3(n-1)$ points as follows: if $\pi = (a_1, a_2, \ldots, a_n)$ is such an n-tuple, we take $a_i + 1$ integers from Z_i together with all u_i, v_i, w_i for $1 \leq i \leq n-1$. Let us call $\mathcal{T}_\pi$ this subtournament.

From what we saw above, if π_1 and π_2 are two distinct such n-tuples then $\mathcal{T}_{\pi_1}$ and $\mathcal{T}_{\pi_2}$ are nonisomorphic subtournaments of $\mathcal{T}$; the number $f(cn + 4n - 3)$ of nonisomorphic subtournaments of $\mathcal{T}$ is then at least $\binom{cn-1}{n-1}$.

Let us consider the possible automorphisms of any subtournament on $cn + 4n - 3$ vertices built as above. The subtournament might admit the $n - 1$ permutations $(u_i\ v_i\ w_i)$ for $1 \leq i \leq n - 1$ if the regularity conditions of Lemma 4.2 are.

$$\begin{array}{llll} Z_i \subset w_j^-, & i \leq j, & Z_i \subset w_j^+, & i > j \\ u_i, v_i \subset w_j^-, & i < j, & u_i, v_i \subset w_j^+, & i > j, \end{array}$$

and if the linear order on W is $w_1 \to w_2 \to w_3 \ldots$; apart from these disjoint 3-cycles, there is no other non-identical automorphism. Therefore, each of these subtournaments has induced automorphism group bounded by 3^{n-1}, that is $g(cn + 4n - 3) = 3^{n-1}$.

We can use the asymptotic estimate (which follows from Stirling's formula, see, for instance, Odlyzko [11, Eqn. (4.6)])

$$\log \binom{\alpha n}{\beta n} \sim \log (A^n), \qquad A \approx \frac{(\alpha)^\alpha}{\beta^\beta (\alpha - \beta)^{\alpha - \beta}}. \tag{1}$$

In our case, this becomes

$$\log \binom{cn - 1}{n - 1} \sim \log (k^n), \qquad k \approx \frac{c^c}{(c - 1)^{c-1}}.$$

Now since we are encoding in a set of size about $(c + 4)n = N$ we have

$$f(N) \quad \text{is almost bounded below by} \quad K^N g(N), \tag{2}$$

where

$$K \approx \left(\frac{c^c}{3(c - 1)^{c-1}} \right)^{\frac{1}{c+4}}.$$

It turns out that in order to maximize the constant K, we should choose $c = 7$: this gives

$$K \approx \left(\frac{7}{3}\right)^{\frac{7}{11}} \frac{1}{2}^{\frac{6}{11}} \approx 1.174.$$

This proves ($\star$) (see Lemma 2.1), and thus completes the proof of Theorem 4.1.

REMARK. Let us remark that if we are only interested in the number of non-isomorphic substructures of the tournament T, that is if we restrict our attention to the growth rate of the sequence f_n for a 2-homogeneous but not a 2-transitive nor 3-homogeneous group, our argument (Eqn. (2)) proves that f_n is almost bounded below by α^n, where now

$$\alpha \approx \left(\frac{c^c}{(c - 1)^{c-1}} \right)^{\frac{1}{c+4}}.$$

We maximize the constant α by taking $c = 4$; this gives $\alpha \approx (2/3)^{3/8} \approx 1.324$. In the same case, the constant in Macpherson [10] is $2^{1/5}$; since this constant α is roughly 1.324, which is bigger than $2^{1/3}$, one obtains a better bound.

5. GROUPS PRESERVING A STEINER SYSTEM

In this chapter we are going to consider a group G preserving a Steiner system.

A *Steiner system* $S = S(t, k, n)$ is an incidence structure with a set $\mathcal{V}$ of *n points* and a set $\mathcal{L}$ of *lines*, with each line containing k points, such that any t points lie on a unique line. We will also consider Steiner systems of type $S(t, k, \infty)$, where the set of points is countable, and of type $S(t, \infty, \infty)$, where both the set of points and the size of a line are countable. We are going to prove the following theorem.

THEOREM 5.1. *Let G be an oligomorphic group preserving a Steiner system S of type $S(t, k, \infty)$ or one of type $S(t, \infty, \infty)$ with more than one line. Then there is a constant $c > 1$ such that $F_n(G)$ is almost bounded below by $c^n n!$.*

Let us recall some combinatorial facts we will use in the proof: Macpherson [10] notes the following lemma.

LEMMA 5.1. *Suppose that S is a Steiner system of type $S(t, \infty, \infty)$ and has more than one line. Then there are infinitely many lines through each point.*

We will use Theorem 5.1 in the following situation. Let us consider the case of a 2-transitive but not 3-homogeneous group G, and suppose that $F_n(G)$ is not almost bounded below by $c^n n!$. First, G cannot be 2-primitive: for we have by the stabilizer lemma that G_α is not almost bounded below by $c^n n!$; then G_α cannot be primitive, or by Theorem 3.1 G_α would be 2-homogeneous, and G 3-homogeneous. Now note that by Lemma 2.3 if α is in Ω then G_α has a minimal block B. In what follows, we will suppose that the block B is finite; the case of an infinite minimal block will be dealt with in the next section.

We want to prove the following theorem.

THEOREM 5.2. *Let G be a 2-homogeneous, not 2-transitive, not 2-primitive group such that G_α has a nontrivial finite block. Then there is a constant $c > 1$ such that $F_n(G)$ is almost bounded below by $c^n n!$.*

If B is finite, we can use the following lemma (due to Cameron, see [10, p. 272]) to show that there is a G-invariant Steiner system.

LEMMA 5.2. *Let H be a s-homogeneous permutation group acting on Δ, with s a positive integer. Suppose that for fixed distinct $\delta_1, \ldots, \delta_s$ in Δ there are finitely many orbits of $H_{\delta_1,\ldots,\delta_s}$ on Δ; let B be the union of the finite orbits of $H_{\delta_1,\ldots,\delta_s}$ on Δ. Then $B^H := \{Bh : h \in H\}$ is the set of lines of a Steiner system $S(s, k, \infty)$ on Δ, where k is greater than $s - 1$.*

This lemma does apply in our case with $s = 2$, as G_α has a finite nontrivial block B on $\Omega\setminus\alpha$; for then if β is in B we have that $G_{\alpha\beta}$ has a finite orbit on $\Omega \setminus \{\alpha\beta\}$. By the last lemma, there is a G-invariant Steiner system $S = S(2, k, \infty)$ with $k > 2$ by construction. Then Theorem 5.1 will imply Theorem 5.2; Theorem 5.1 will also be needed in the next section.

We will now prove Theorem 5.1 by encoding 2-trees into tuples of points of the Steiner system. Let us introduce the combinatorial objects needed.

DEFINITION. A *t-tree* is a graph whose vertices can be ordered $x_1, x_2, \ldots, x_n$ in such a way that any vertex x_i is joined to precisely $\min\{i - 1, t\}$ of its predecessors x_j $(j < i)$.

REMARK. A 1-tree is the same as a tree.

LEMMA 5.3. *The number of labelled 2-trees on n vertices is at least*

$$\frac{(n-1)!(n-2)!}{2^{n-2}}.$$

PROOF OF LEMMA. For $i = 3, \ldots n$, there are $\binom{i-1}{2}$ choices for the points x_j $(j < i)$ joined to x_i. This gives

$$\prod_{i=3}^{n} \binom{i-1}{2} = \frac{(n-1)!(n-2)!}{2^{n-2}}.$$

There may be other labellings for which the defining condition does not hold. □

Now let us see how to encode 2-trees into the Steiner system: the idea here is to encode a 2-tree on n vertices (which has $2n - 3$ edges) into a $(3n - 3)$-tuple of points of S in such a way that the first n points will correspond to the vertices of the 2-tree, the other $2n - 3$ points will correspond to the edges and the point corresponding to an edge and the two points corresponding to the two vertices belonging to this edge will be collinear in the system S.

LEMMA 5.4. *Let S be a $S(2, k, \infty)$ Steiner system with an oligomorphic automorphism group. Then any 2-tree T on n vertices can be encoded into a $(3n - 3)$-tuple of points of S from which it can be recovered uniquely.*

PROOF OF LEMMA. We label the vertices of the tree with $1, 2, \dots, n$, and call ij the edge joining the vertex i to the vertex j.

Now we choose n points $x_1, \dots, x_n$ in S such that, if we call L_{ij} the line joining x_i and x_j, then $L_{ij} = L_{kl}$ if and only if $\{i, j\} = \{k, l\}$. The choice of such $x_1, \dots, x_n$ can be done inductively: if $x_1, \dots, x_k$ have been chosen, then choose x_{k+1} not on any of the lines L_{ij} nor on a line joining x_l to a point of L_{ij} for any $i, j, l \leq k$. Only finitely many points are thus excluded, and that leaves infinitely many choices for x_{k+1}. Now the n vertices of the 2-tree are identified with the points $x_1, \dots, x_n$ chosen as above, and the edge ij is identified with an arbitrarily chosen point y_{ij} of L_{ij} other than x_i and x_j.

The 2-tree is completely recoverable from such a $(3n - 3)$-tuple since the vertices are the first n entries of the tuple, the edges are the remaining entries, and each edge determines uniquely a pair of vertices. This proves the lemma. □

We can complete the proof of Theorem 5.1: the number of orbits on $(3n - 3)$-tuples is at least equal to the number of labelled 2-trees. Moreover, from a fixed n-vertex 2-tree we obtain $(2n - 3)!$ different orbits on $(3n - 3)$-tuples, corresponding to the permutations of the $2n - 3$ edge-points of the $(3n-3)$-tuple, since each edge uniquely specifies its two vertices, and since the group G preserves collinearity. Thus

$$F_{3n-3} \geq \frac{(n-1)!\,(n-2)!}{2^{n-2}}\,(2n-3)!,$$

and using Stirling's formula, we have

$$F_{3n-3} \text{ is almost bounded below by } (3n-3)!\,c^{3n-3}$$

for any constant c.

6. 2-TRANSITIVE GROUPS

In this section, we will prove the following theorem.

THEOREM 6.1. *Let G be a 2-transitive, not 3-homogeneous group. Then there is a constant $c > 1$ such that $(F_n(G))$ is almost bounded below by $c^n n!$.*

We will start by recalling some results by Macpherson that show that a group satisfying the hypothesis of the theorem but not the conclusion belongs to a special class of permutations groups. We will then use the properties of this class to establish our result.

Let G be a 2-transitive, not 3-homogeneous group, and suppose that $(F_n(G))$ is not almost bounded below by $c^n n!$. Now G cannot be 2-primitive, for then G_α would be a primitive group such that $(F_n(G_\alpha))$ is not almost bounded below by $c^n n!$, so that G_α would be 2-transitive and G would be 3-transitive.

We know by Lemma 2.3 that G_α has a minimal block that we call B: by Theorem 5.2, B must be infinite. The group $G^B_{\alpha B}$ is primitive and $(F_n(G^B_{\alpha B}))$ is not almost bounded below by $c^n n!$; therefore, from what we have proved so far, $G^B_{\alpha B}$ is 2-transitive or 3-homogeneous (or both). But we see from McDermott's Theorem 2.1 that $G^B_{\alpha B}$ must be 2-transitive, since by Lemma 4.1 it cannot preserve a linear order.

Let us consider the following relation: for distinct α, β, γ in Ω, we write $\alpha|\beta\gamma$ if β and γ lie in the same G_α-block. We say that the triple $\alpha\beta\gamma$ is good if at least one of the relations $\alpha|\beta\gamma$, $\beta|\alpha\gamma$, $\gamma|\alpha\beta$ holds. As G is 2-transitive and $G^B_{\alpha B}$ is 2-homogeneous, we see that G is transitive on the set of good triples in Ω; then there is a constant $t(= 1, 2, 3)$ such that any good triple $\alpha\beta\gamma$ has t relations of the form $\alpha|\beta\gamma$. Now since $G^B_{\alpha B}$ is 2-transitive, we cannot have $t = 2$. Also

LEMMA 6.1. *The constant t cannot take the value* 3*.*

PROOF. If $t = 3$ there is a G-invariant $(2, \infty, \infty)$ Steiner system with more than one line (see [10]). As $(F_n(G))$ is not almost bounded below by $c^n n!$, Theorem 5.1 gives a contradiction. □

Therefore, we must have $t = 1$. Then, if the triple $\alpha\beta\gamma$ is good with $\alpha|\beta\gamma$, we call α the *separating point* of the triple.

Macpherson [10] then proves the following lemma.

LEMMA 6.2. *The group $G^B_{\alpha B}$ is not* 3*-homogeneous.*

It follows that as in [10], G belongs to the class $\mathcal{C}$ of groups defined as follows.

DEFINITION. A permutation group H acting on a set Δ is in $\mathcal{C}$ if the following holds: there are $\delta_i \in \Delta$, for all i in $\mathbb{N}$, and infinite subsets D_i of Δ, for all i in $\mathbb{N}$, with

$$\Delta = D_0 \supset D_1 \supset \cdots \supset D_i \supset D_{i+1} \supset \cdots$$

such that:

(1) for all i in $\mathbb{N}$, the set D_i is a minimal block of $H^{D_{i-1}}_{\delta_0,\ldots,\delta_{i-1}D_{i-1}}$, and contains $\{\delta_j,\ j \geq i\}$;
(2) for all i in $\mathbb{N}^*$ the action of $H^{D_i}_{\delta_0,\ldots,\delta_{i-1}D_i}$ is 2-transitive, not 2-primitive, not 3-homogeneous, and the constant which corresponds to t takes the value 1.

Let us see how to use this fact to prove the theorem. First we built a subset X of Ω recursively in the following way: choose a point x_1 in Ω at random; then choose two points x_2, y_2 belonging to *different* minimal blocks of G_{x_1}. Let us call B_1 the G_{x_1}-block that contains x_2, and let H^1 be the group $G^{B_1}_{x_1 B_1}$. We now choose two points x_3 and y_3 in B_1 belonging to different minimal $H^1_{x_2}$-blocks; we call B_2 the block containing x_3 and set $H^2 = H^{1B_2}_{x_2 B_2}$. This process is repeated recursively: at step k we choose points x_{k+1}, y_{k+1} in B_{k-1} belonging to different minimal blocks of $H^{k-1}_{x_k}$, call B_k the block containing x_{k+1} and set H^k to be the group $H^{(k-1)B_k}_{x_k B_k}$. Note that the groups $H^1, H^2, \ldots$ all belong to $\mathcal{C}$.

Now for each $n = m + k$ we consider the n-sets built by taking the points $\{x_1, x_2, \ldots, x_m\}$ and k points chosen from the set $\{y_2, \ldots, y_{m-2}\}$; for each such n we have $N = \binom{m-3}{k}$ choices for a set of this type, corresponding to the choice of a k-set out of the set $\{y_2, \ldots, y_{m-2}\}$ of size $m - 3$, giving us the n-sets $X_1, \ldots, X_N$.

From the action of G on a set X_i we can recover almost all of its structure: consider all the good triples amongst the 3-subsets of X_i and look at which points appear more often as the separating point of some triple. There will be one or possibly two such elements, that will correspond to x_1 and possibly y_2 (note that the roles of x_1 and y_2 can be interchanged); now delete these points and repeat this procedure recursively, thus recovering the points $x_2, \ldots, x_{m-2}$ and y_j for j in J_i, the set of the indices of the y that appear in X_i (we should note that at each step the roles of x_{j-1} and y_j could be interchanged); at the end we will be left with the points x_{m-1}, x_m, since it is not possible by construction to distinguish between them. Then one has $|G_{X_i}^{X_i}| \leq 2^{k+1}$, for this group is generated by at most $k+1$ disjoint transpositions; and, by construction, two different such sets will belong to different G-orbits.

Let us now choose k to be cn for some constant $c < 1$; then $m = (1-c)n$. For n big enough, we have $m - 3 = (1-c)n - 3 \sim (1-c)n$, and the number $f(n)$ of nonisomorphic n-sets becomes $\sim \binom{(1-c)n}{cn}$.

We can use the asymptotic estimate (which follows from Stirling's formula: see Eqn. (1) in Section 4)

$$\log \binom{\alpha n}{\beta n} \sim \log(A^n), \qquad A \approx \frac{(\alpha)^\alpha}{\beta^\beta(\alpha-\beta)^{\alpha-\beta}}.$$

In our case, this gives

$$\log \binom{(1-c)n}{cn} \sim \log(B^n), \qquad B \approx \frac{(1-c)^{1-c}}{c^c(1-2c)^{1-2c}}.$$

We want to choose c so that the number $f(n)$ will result exponentially bigger than the order of the induced group: if we take $c = 1/3$, we have $f(n) \sim (2^{2/3})^n$, while each such set has induced group of order bounded by $g(n) = 2^{k+1} = 2(2)^{cn} = 2(2^{1/3})^n$.

We thus have

$$f(n) \quad \text{is almost bounded below by} \quad (2^{1/3})^n g(n);$$

then a 2-transitive, not 3-homogeneous group satisfies $(\star)$, and Theorem 6.1 follows from Lemma 2.1.

7. Completion of the Proof of Theorem 1.2

In this section we will see how to complete the proof of the main result, and draw some concluding remarks on the theorem.

We have verified the first two inductive steps in the proof of Theorem 1.2 by proving the following result.

THEOREM 7.1. *Let G be primitive. If $(F_n(G))$ is* not *bounded below by $c^n n!$ (for any $c > 1$) then G is 3-homogeneous.*

The following lemma will furnish the inductive step needed to complete the proof of Theorem 1.2.

LEMMA 7.1. *Suppose that k is an integer greater than 2, and that every primitive but not k-homogeneous oligomorphic group (of countable degree) has $F_n(G)$ almost bounded below by $c^n n!$. Then also every primitive but not $(k+1)$-homogeneous oligomorphic group has F_n almost bounded below by $c^n n!$.*

PROOF. Let G be a primitive group with $F_n(G)$ not almost bounded below by $c^n n!$: then G is k-homogeneous. There are three possible cases.

Case 1: G is not 2-transitive. By Theorem 2.1, there is a G-invariant linear order $<$ on Ω. The stabilizer in G of a 3-set fixes it pointwise, so for all α in Ω, the group G_α has three orbits on 2-sets of $\Omega \setminus \alpha$, that is $\{\{\beta, \gamma\} : \beta, \gamma < \alpha\}$, $\{\{\beta, \gamma\} : \beta < \alpha < \gamma\}$ and $\{\{\beta, \gamma\} : \alpha < \beta, \gamma\}$. Thus G_α is 2-homogeneous (and hence primitive) on each of its orbits. Since $F_n(G_\alpha)$ is not almost bounded below by $c^n n!$, by assumption G_α is k-homogeneous on each of these orbits. This now implies that G is $(k+1)$-homogeneous, for suppose given two $(k+1)$-sets $\{\alpha_0, \ldots, \alpha_k\}$ and $\{\beta_0, \ldots, \beta_k\}$ (let us assume that these are in increasing order), we can take $g \in G$ such that $\alpha_0 g = \beta_0$ and use the k-homogeneity of G_{β_0}.

Case 2: G is 2-primitive. Then G_α is primitive and $F_n(G_\alpha)$ is not bounded below exponentially by $c^n n!$ by Lemma 2.2, so G_α is k-homogeneous by assumption. Then G is $(k+1)$-homogeneous.

Case 3: G is 2-transitive but not 2-primitive. Now G satisfies the hypotheses of Theorem 2.2: then either G satisfies a betweenness relation, or it is a subgroup of the group H of Cameron [3] (see Section 2, Example 2); the latter case cannot hold, since we have recalled in Section 2 that the sequence $(F_n(H))$ is almost bounded below by $2^n n!$; then G preserves a betweenness relation, and we may reason as in case 1. $\square$

Now we can prove the main theorem: a primitive group whose (F_n) sequence is not almost bounded below by $c^n n!$ for any constant c is k-homogeneous for each k in $\mathbb{N}$. The base of the induction, $k = 1$, is verified trivially, for any primitive group is transitive. Now Theorem 7.1 and Lemma 7.1 guarantee the inductive steps, and this proves Theorem 1.2.

Note that Theorem 1.2 implies Macpherson's Theorem 1.1: this follows from the easy inequality

$$f_n(G) \leq F_n(G) \leq n!\, f_n(G),$$

which holds for any oligomorphic group.

Let us make some remarks on the value of the constants appearing in the statement of Theorem 1.2 and in Macpherson's Theorem 1.1. One of the reasons why we are interested in the possible values of this constant c is that the number $1/c$ bounds the radius of convergence for the generating functions $F(G)$, $f(G)$ for any primitive but not homogeneous group: Macpherson's Theorem 1.1 shows that the radius of convergence of f_G (as a power series) is less than 1 (more precisely, it is at most $1/c$), while now Theorem 1.2 shows that the same holds for F_G (where for the sequence (F_n) one considers the exponential generating function); moreover we remarked in Section 2 that primitive groups with finite radius c are 'special' amongst primitive not highly homogeneous groups.

Let us then point out the constants appearing in our bound in the various stages of the proof of Theorem 1.2. For groups that are not 2-homogeneous, we obtained the value $2^{1/2}$; for 2-homogeneous groups we have that the value of the constant is roughly 1.174: for groups preserving a Steiner system have c arbitrarily big; finally, 2-transitive groups have $c = 2^{1/3}$. Thus the value of the constant in the statement of Theorem 1.2 is roughly 1.174.

It is worth pointing out that the methods applied in the proof of Theorem 1.2 can also be used to improve the value of the constant of Macpherson's Theorem 1.1. Indeed, very often in the proof of Theorem 1.2 we proved a relation of the type (see Section 2)

($\star$) There are two functions $f(n)$ and $g(n)$ such that for each n we can find 'many' sets of size n $\{X_i\}_{1 \leq i \leq f(n)}$ belonging to different G-orbits such that

$$|G_{X_i}^{X_i}| \leq g(n)$$

and such that for some $c > 1$ one has

$$f(n) \quad \text{is almost bounded below by} \quad c^n g(n).$$

If, say, the above function $g(n)$ is roughly k^n (this happens, for instance, in the case of a 2-homogeneous, not 2-transitive nor 3-homogeneous group), we have that the number of orbits on n-sets, which is clearly bounded below by $f(n)$, grows exponentially with constant $c \cdot k$. The constant appearing in the bound in our results concerning the growth rate of the sequence (F_n) will then be c while the one appearing in results dealing with the growth rate of the sequence (f_n) counting orbits on n-sets will turn out to be the bigger constant $c \cdot k$.

Let us see the constants obtained case by case, as above; for the case of a not 2-homogeneous group, we do not improve on Macpherson's $(2.48)^{1/2} \approx 1.57$; for 2-homogeneous groups we have that our constant is roughly 1.324, which is bigger that $2^{1/3}$ (see the remark at the end of Section 4); this improves on Macpherson's $2^{1/5}$. We have c arbitrarily big for the case of groups preserving a Steiner system, improving on Macpherson's $(2.955)^{1/2} \approx 1.719$, and 2-transitive groups have constant $2^{2/3}$, improving on Macpherson's $2^{1/3}$. We thus managed to improve the value of the constant appearing in Macpherson's Theorem 1.1 from $2^{1/5} \approx 1.148$ to roughly $\frac{2}{3^{3/8}} \approx 1.324$.

As we saw from the examples given in Section 2, the best we can hope to obtain is $c = 2$. These examples suggested to Macpherson the following conjecture.

CONJECTURE 7.1 (MACPHERSON). *Let G be primitive but not highly homogeneous. Then $f_n(G)$ has growth almost bounded below by c^n, where $c = 2$.*

It is then completely natural to formulate the following analogous conjecture.

CONJECTURE 7.2. *Let G be primitive but not highly homogeneous. Then $F_n(G)$ has growth almost bounded below by $c^n\, n!$, where $c = 2$.*

ACKNOWLEDGEMENTS

The results of this paper form a good part of my Doctorate Thesis at the Università di Palermo. I am very grateful to Peter Cameron for proposing the problem, for many helpful hints and suggestions, for his patience in supervising my work; I also thank the School of Mathematical Sciences, Queen Mary and Westfield College, London, for its hospitality while I was working on the Thesis.

REFERENCES

1. P. J. Cameron, Transitivity of permutation groups on unordered sets, *Math. Z.*, **148** (1976), 127–139.
2. P. J. Cameron, Orbits of permutation groups on unordered sets, II, *J. London Math. Soc. (2)*, **23** (1981), 249–264.
3. P. J. Cameron, Orbits of permutation groups on unordered sets, IV, *J. London Math. Soc. (2)*, **27** (1983), 238–247.
4. P. J. Cameron, Some treelike objects, *Quart. J. Math. Oxford (2)*, **38** (1987), 155–183.
5. P. J. Cameron, *Oligomorphic Permutation Groups*, *London Mathematical Society Lecture Note Series*, **152**, Cambridge University Press, Cambridge, 1990.
6. P. J. Cameron, *Permutation Groups*, *London Mathematical Society Student Texts*, **45**, Cambridge University Press, Cambridge, 1999.
7. J. D. Dixon and Brian Mortimer, *Permutation Groups*, Springer, Berlin, 1996.
8. P. Erdős and A. Rényi, Asymmetric graphs, *Acta Math. Acad. Sci. Hungar.*, **14** (1963), 295–315.
9. Louis Comtet, *Advanced Combinatorics*, D. Reidel, Dordrecht, 1970.
10. H. D. Macpherson, Orbits of infinite permutation groups, *Proc. London Math. Soc. (3)*, **51** (1985), 246–284.

11. A. M. Odlyzko, Asymptotic enumerative combinatorics, in: *Handbook of Combinatorics*, R. L. Graham, M. Grötschel and L. Lovász (eds), Elsevier, Amsterdam, 1995.
12. Roland Fraïssé, Sur certaines relations qui généralisent l'ordre des nombres rationnels, *C. R. Acad. Sci. Paris*, **237** (1953), 540–542.

Received 14 February 2000 and accepted 28 September 2000

FRANCESCA MEROLA
Dipartimento di Matematica,
Università di Roma 'La Sapienza',
P.le Aldo Moro 2–00185,
Roma,
Italy
E-mail: merola@mat.uniroma1.it