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A CLASSIFICATION OF THE ORDINAL RECURSIVE FUNCTIONS*

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Introduction

In [7] a framework was developed, within which various hierarchies of number-theoretic functions can be generated.
Particular attention was paid to a hierarchy $\{\mathfrak{E}_\alpha\}$ obtained by restricting α appropriately to the ordinals below ε_0, and it was conjectured that this hierarchy provides a classification of the ordinal recursive functions.
The main purpose of this paper is to give an affirmative answer to this conjecture. In addition, a method of Robbin [9] is extended, in Section 4, in order to simplify further the definition of the classes $\mathfrak{E}_\alpha$, thus providing an affirmative answer to Problem A of [7]. An alternative characterization of the classes $\mathfrak{E}_\alpha$, in terms of computational complexity, is also obtained.
The notation is the same as that used in [7]. N denotes the set of non-negative integers, and lower-case italics $a, b, \ldots, x, y, z$, with or without subscripts, denote members of N. k-tuples $x_1, \ldots, x_k$ are denoted by $\underset{\sim}{x}$. With the exception of λ and μ, lower-case Greek letters denote ordinals below ε_0.

1. Preliminary Definitions and Results

Let α be a limit ordinal. Then a fundamental sequence for α is a strictly increasing ω-sequence of ordinals, whose limit is α.
For each limit ordinal $\alpha < \varepsilon_0$, a fundamental sequence $\{\alpha\}(n)$, $n \in N$, is provided by the following inductive definition:

(I) If $\alpha = \omega^{\alpha_1} \cdot a_1 + \omega^{\alpha_2} \cdot a_2 + \cdots + \omega^{\alpha_r} \cdot a_r + \omega^{k+1} \cdot (a_{r+1} + 1)$,
where $\alpha > \alpha_1 > \alpha_2 > \cdots > \alpha_r > k+1$, then for every $n \in N$,
$\{\alpha\}(n) = \omega^{\alpha_1} \cdot a_1 + \cdots + \omega^{\alpha_r} \cdot a_r + \omega^{k+1} \cdot a_{r+1} + \omega^k \cdot n + \cdots + \omega \cdot n + 2 \cdot n.$

(II) If $\alpha = \omega^{\alpha_1} \cdot a_1 + \omega^{\alpha_2} \cdot a_2 + \cdots + \omega^{\alpha_r} \cdot a_r + \omega^{\beta+1} \cdot (a_{r+1} + 1)$,
where $\alpha > \alpha_1 > \alpha_2 > \cdots > \alpha_r > \beta + 1 > \omega$, then for every $n \in N$,
$\{\alpha\}(n) = \omega^{\alpha_1} \cdot a_1 + \cdots + \omega^{\alpha_r} \cdot a_r + \omega^{\beta+1} \cdot a_{r+1} + \omega^\beta \cdot n$.

(III) If $\alpha = \omega^{\alpha_1} \cdot a_1 + \omega^{\alpha_2} \cdot a_2 + \cdots + \omega^{\alpha_r} \cdot a_r + \omega^\sigma \cdot (a_{r+1} + 1)$,
where $\alpha > \alpha_1 > \alpha_2 > \cdots > \alpha_r > \sigma$, and σ is a limit ordinal, then for every $n \in N$,
$\{\alpha\}(n) = \omega^{\alpha_1} \cdot a_1 + \cdots + \omega^{\alpha_r} \cdot a_r + \omega^\sigma \cdot a_{r+1} + \omega^{\{\sigma\}(n)}$.

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Now, for every ordinal $\alpha < \varepsilon_0$, define a function $F^n_\alpha(x)$ by recursion, as follows:

$$\begin{aligned} F^n_0(x) &= (n+1)\cdot(x+1)\,. \\ F^0_{\beta+1}(x) &= F^x_\beta(x)\,. \\ F^0_\sigma(x) &= F^0_{\{\sigma\}(x)}(x)\,,\ \sigma \text{ a limit ordinal.} \\ F^{n+1}_\gamma(x) &= F^0_\gamma(F^n_\gamma(x)),\ \gamma > 0\,. \end{aligned}$$

It is clear, from this definition, that for each $\alpha < \varepsilon_0$, $F^n_\alpha(x)$ is defined for every n and x.
In [7] various "monotonicity" properties of the functions $F^n_\alpha(x)$ were obtained. The following Theorem gives a summary of these results.

Theorem 1.1

(i) For each $\alpha < \varepsilon_0$ and all n, x, $F^n_\alpha(x) > \max(n, x)$.

(ii) For each $\alpha < \varepsilon_0$ and all n, x, y, if $x > y$ then $F^n_\alpha(x) > F^n_\alpha(y)$.

(iii) For each $\alpha < \varepsilon_0$ and all m, n, x, if $m > n$ then $F^m_\alpha(x) > F^n_\alpha(x)$.

(iv) For each $\alpha < \varepsilon_0$ and all n, x, $F^n_{\alpha+1}(x) \geqq F^n_\alpha(x)$, with equality holding only when $n = x = 0$.

(v) If $\alpha < \beta < \varepsilon_0$, then F^0_α is eventually majorized by F^0_β.

(vi) If σ is a limit ordinal $< \varepsilon_0$ and $x > 0$, then $F^0_{\{\sigma\}(x)}(x) > F^0_{\{\sigma\}(i)}(x)$ for every $i < x$.

These "monotonicity" results are of basic importance to the work contained in this paper, and they will often be used without explicit reference.
For each $\alpha < \varepsilon_0$, let $\mathfrak{E}_\alpha$ be the smallest class of functions which contains

$$\{\lambda x\,.\,0, \lambda xy\,.\,x+y, \lambda \underline{x}\,.\,x_i\} \cup \{\lambda x\,.\,F^0_\beta(x) | \beta \leqq \alpha\}\,,$$

and which is closed under the operations of Substitution and Limited Recursion.
Clearly, if $\alpha < \beta < \varepsilon_0$, then $\mathfrak{E}_\alpha \subseteqq \mathfrak{E}_\beta$.
The following results, concerning the hierarchy $\{\mathfrak{E}_\alpha\}_{\alpha<\varepsilon_0}$, are proved in [7].

Theorem 1.2

Let α be any ordinal such that $0 < \alpha < \varepsilon_0$.
Then for every function $f \in \mathfrak{E}_\alpha$ there is a number p such that, for all $\underline{x}$,

$$f(\underline{x}) < F^p_\alpha(\max(\underline{x}))\,.$$

Corollary

Suppose $0 < \alpha < \beta < \varepsilon_0$.
Then every function in $\mathfrak{E}_\alpha$ is eventually majorized by F^0_β, and hence $\mathfrak{E}_\alpha \subset \mathfrak{E}_\beta$.

Theorem 1.3

$\{\mathfrak{E}_\alpha\}_{\alpha<\varepsilon_0}$ is a proper extension of the Grzegorczyk hierarchy, and for each $k \in N$, $\bigcup_{\alpha<\omega^k} \mathfrak{E}_\alpha$ is the class of Péter's k-recursive functions.

Now for each $n \in N$, define an ordinal $\omega(n)$ by

$$\omega(0) = 1 \,,$$
$$\omega(n+1) = \omega^{\omega(n)} \,.$$

Clearly, $\{\omega(n) | \, n \in N\}$ is a fundamental sequence for ε_0.
For each $n > 0$, we construct a primitive recursive well-ordering $<_n$ of N, of order-type $\omega(n)$, as in Tait [10].
Functions ord_n and num_n will be defined along with $<_n$, such that $\mathrm{ord}_n(x)$ is the ordinal represented by x in the well-ordering $<_n$, and $\mathrm{num}_n(\alpha)$ is the number representing $\alpha < \omega(n)$ in the well-ordering $<_n$.

(i) $<_1$ is the natural well-ordering of N, and for every x, $\mathrm{ord}_1(x) = x$ and $\mathrm{num}_1(x) = x$.

(ii) Suppose $<_n$, ord_n, num_n have been defined so that $<_n$ is of order-type $\omega(n)$, for every x, $\mathrm{num}_n(\mathrm{ord}_n(x)) = x$, and for every $\alpha < \omega(n)$, $\mathrm{ord}_n(\mathrm{num}_n(\alpha)) = \alpha$.
Then if $\beta = \omega^{\alpha_1} \cdot a_1 + \omega^{\alpha_2} \cdot a_2 + \cdots + \omega^{\alpha_r} \cdot a_r$, where $\omega(n) > \alpha_1 > \alpha_2 > \cdots \cdots > \alpha_r \geqq 0$, define

$$\mathrm{num}_{n+1}(\beta) = (\mathfrak{p}^{a_1}_{\mathrm{num}_n(\alpha_1)} \cdot \mathfrak{p}^{a_2}_{\mathrm{num}_n(\alpha_2)} \cdot \cdots \cdot \mathfrak{p}^{a_r}_{\mathrm{num}_n(\alpha_r)}) - 1$$

where $\mathfrak{p}_0, \mathfrak{p}_1, \mathfrak{p}_2, \ldots$ is the primitive recursive enumeration of prime numbers in increasing order.
Conversely, if $z_r <_n \cdots <_n z_2 <_n z_1$, and $z = (\mathfrak{p}^{a_1}_{z_1} \cdot \mathfrak{p}^{a_2}_{z_2} \cdots \mathfrak{p}^{a_r}_{z_r}) - 1$, define

$$\mathrm{ord}_{n+1}(z) = \omega^{\mathrm{ord}_n(z_1)} \cdot a_1 + \omega^{\mathrm{ord}_n(z_2)} \cdot a_2 + \cdots + \omega^{\mathrm{ord}_n(z_r)} \cdot a_r \,.$$

Finally, define $<_{n+1}$ by

$$x <_{n+1} y \equiv \mathrm{ord}_{n+1}(x) < \mathrm{ord}_{n+1}(y) \,.$$

It is clear that every ordinal $\beta < \omega(n+1)$ has a unique representation $\mathrm{num}_{n+1}(\beta)$, and that every number "represents" some ordinal $< \omega(n+1)$.
Hence $<_{n+1}$ is a well-ordering, of order-type $\omega(n+1)$.
Thus, for every $n > 0$; $<_n$ is a well-ordering of N, of order-type $\omega(n)$; $\mathrm{num}_n(\mathrm{ord}_n(x)) = x$ every x, and $\mathrm{ord}_n(\mathrm{num}_n(\alpha)) = \alpha$ for every $\alpha < \omega(n)$.
Notice also, that for every $n > 0$, 0 is the least element with respect to $<_n$.

Definition 1.4

For each $n > 0$, $U(<_n)$ is the smallest class of functions which contains the primitive recursive functions and which is closed under the operations of substitution, primitive recursion, and unnested ordinal recursion over the well-ordering $<_n$.
Functions belonging to $U(<_n)$ are called *$<_n$-recursive.*

Definition 1.5

A function f is said to be defined by *$<_n$-annihilation* from a function g if

$$\begin{cases} f(0, \underline{a}) = 0 \\ f(x+1, \underline{a}) = 1 + f(g(x+1, \underline{a}), \underline{a}) \end{cases}$$

where $g(0, \underline{a}) = 0$ and $g(x+1, \underline{a}) <_n x+1$ for all x.

Definition 1.6

For each $n > 0$, $A(<_n)$ is the smallest class of functions which contains the primitive recursive functions and which is closed under the operations of substitution, primitive recursion, and $<_n$-annihilation.

Theorem 1.7 (Robbin)

For each $n > 0$, $U(<_n) = A(<_n)$.

Definition 1.8

A function is called *ordinal recursive* if and only if it is $<_n$-recursive for some $n > 0$. The class of all ordinal recursive functions will be denoted by OR.
It follows immediately from Theorem 1.7 and Definition 1.8, that $\mathrm{OR} = \bigcup_{n \in N} A(<_n)$.

2. A Hierarchy of Ordinal Recursive Functions

The aim of this section is to show that, for every $\alpha < \varepsilon_0$, there is an r such that F^0_α is $<_r$-recursive. It then follows that every function belonging to $\mathfrak{E}_\alpha (\alpha < \varepsilon_0)$ is $<_r$-recursive, for some r.
First, suppose that $n > 0$ is fixed, and suppose that the definition of the functions $F^n_\alpha(x)$ is restricted to just those ordinals $\alpha < \omega(n)$.
Suppose also, that a function G is defined so that

$$G(x+1, a) = (m+1) \cdot (a+1) \text{ if } \mathrm{ord}_{n+1}(x+1) = \omega^0 + m = m+1\,.$$

$$G(x+1, a) = G(\mathrm{num}_{n+1}(\omega^\alpha + a), a) \text{ if } \mathrm{ord}_{n+1}(x+1) = \omega^{\alpha+1} + 0\,.$$

$$G(x+1, a) = G(\mathrm{num}_{n+1}(\omega^{\{\sigma\}(a)}), a) \text{ if } \mathrm{ord}_{n+1}(x+1) = \omega^\sigma + 0 \ (\sigma \text{ a limit})$$

$$G(x+1, a) = G\big(\mathrm{num}_{n+1}(\omega^\beta), G(\mathrm{num}_{n+1}(\omega^\beta + m), a)\big)$$
$$\text{if } \mathrm{ord}_{n+1}(x+1) = \omega^\beta + m + 1 \ (\beta > 0)\,.$$

Then it can easily be proved, by induction over the ordinals $< \omega(n)$, that for every $\alpha < \omega(n)$, and all m, a,

$$F^m_\alpha(a) = G(\mathrm{num}_{n+1}(\omega^\alpha + m), a)\,.$$

We now show that such a function G can be defined, from primitive recursive functions, by nested recursion over the well-ordering $<_{n+1}$.
Recall the definition of the fundamental sequences $\{\sigma\}(i)$, $i \in N$, for limit ordinals $\sigma < \varepsilon_0$.
If σ is a limit ordinal $< \omega^\omega$, then for each $i \in N$, $\{\sigma\}(i)$ is defined explicitly by clause (I).
Thus it is clear, from the way in which the well-ordering $<_2$ (of order-type ω^ω) is constructed, that there is a primitive recursive function fs_2 such that

$$\mathrm{fs}_2(z, i) = \mathrm{num}_2(\{\sigma\}(i)) \text{ if } \mathrm{ord}_2(z) \text{ is the limit } \sigma.$$

Now suppose that $m \geqq 2$ and that there is a primitive recursive function fs_m such that

$$\mathrm{fs}_m(z, i) = \mathrm{num}_m(\{\sigma\}(i)) \text{ if } \mathrm{ord}_m(z) \text{ is the limit } \sigma.$$

If δ is a limit ordinal $< \omega(m+1)$ then either δ is of the form $\omega^{\alpha_1} \cdot a_1 + \cdots + \omega^{\alpha_r+1} \cdot a_r$ where $\alpha_1 > \cdots > \alpha_r + 1$, in which case $\{\delta\}(i)$ is defined explicitly by clause (I) or clause (II); or else δ is of the form $\omega^{\alpha_1} \cdot a_1 + \cdots + \omega^{\alpha_s} \cdot a_s$ where $\omega(m) > \alpha_1 > \cdots > \alpha_s$ and where α_s is a limit ordinal, in which case $\{\delta\}(i)$ is defined (inductively) from $\{\alpha_s\}(i)$.

Hence, using the "arithmetization" of ordinals below $\omega(m+1)$ provided by the function num_{m+1}, it is possible to define a function fs_{m+1}, which is primitive recursive in fs_m, such that

$$\mathrm{fs}_{m+1}(z, i) = \mathrm{num}_{m+1}(\{\delta\}(i)) \text{ if } \mathrm{ord}_{m+1}(z) \text{ is the limit } \delta.$$

Since fs_m is primitive recursive, fs_{m+1} must be also.

It is clear, therefore, that for each $k \geqq 2$, there is a primitive recursive function fs_k such that

$$\mathrm{fs}_k(z, i) = \mathrm{num}_k(\{\sigma\}(i)) \text{ if } \mathrm{ord}_k(z) \text{ is the limit } \sigma.$$

Hence if $\mathrm{ord}_{n+1}(x+1) = \omega^\sigma$ where σ is a limit, then we have

$$\mathrm{fs}_{n+1}(x+1, a) = \mathrm{num}_{n+1}(\{\omega^\sigma\}(a)) = \mathrm{num}_{n+1}(\omega^{\{\sigma\}(a)}) .$$

Now the predicates P_1, P_2, P_3, P_4 defined as follows:

$$P_1(y) \equiv \mathrm{ord}_{n+1}(y) \text{ is of the form } \omega^0 + m ,$$

$$P_2(y) \equiv \mathrm{ord}_{n+1}(y) \text{ is of the form } \omega^{\alpha+1} ,$$

$$P_3(y) \equiv \mathrm{ord}_{n+1}(y) \text{ is of the form } \omega^\sigma, \text{ where } \sigma \text{ is a limit} ,$$

$$P_4(y) \equiv \mathrm{ord}_{n+1}(y) \text{ is of the form } \omega^\beta + m + 1, \text{ where } \beta > 0$$

can all be decided primitive recursively, and furthermore, there are primitive recursive functions g_1, g_2, g_3, g_4 such that

$$g_1(x+1, a) = (m+1) \cdot (a+1) \text{ if } \mathrm{ord}_{n+1}(x+1) = \omega^0 + m .$$

$$g_2(x+1, a) = \mathrm{num}_{n+1}(\omega^\alpha + a) \text{ if } \mathrm{ord}_{n+1}(x+1) = \omega^{\alpha+1} + 0 .$$

$$g_3(x+1, a) = \mathrm{num}_{n+1}(\omega^\beta) \text{ if } \mathrm{ord}_{n+1}(x+1) = \omega^\beta + m + 1 .$$

$$g_4(x+1, a) = \mathrm{num}_{n+1}(\omega^\beta + m) \text{ if } \mathrm{ord}_{n+1}(x+1) = \omega^\beta + m + 1 .$$

Hence the function G can be defined as follows:

$$G(0, a) = 0$$

$$G(x+1, a) = \begin{cases} g_1(x+1, a) & \text{if } P_1(x+1) \\ G(g_2(x+1, a), a) & \text{if } P_2(x+1) \\ G(\mathrm{fs}_{n+1}(x+1, a), a) & \text{if } P_3(x+1) \\ G\big(g_3(x+1, a), G(g_4(x+1, a), a)\big) & \text{if } P_4(x+1) \\ 0 \text{ otherwise.} & \end{cases}$$

Since $\mathrm{fs}_{n+1}(x+1, a) <_{n+1} x+1$ for all x, and for each $i = 2, 3, 4$, $g_i(x+1, a) <_{n+1} x+1$, it is clear that G is defined, from primitive recursive functions, by a nested recursion over $<_{n+1}$.

Now Tait has shown in [10] that nested recursion over $<_{n+1}$ is reducible to unnested ordinal recursion over $<_{n+2}$.
Hence G is $<_{n+2}$ – recursive.
But for each $\alpha < \omega(n)$, and all a,

$$F^0_\alpha(a) = G(\mathrm{num}_{n+1}(\omega^\alpha), a)\,,$$

and so F^0_α is $<_{n+2}$ – recursive.
Thus, if $\alpha < \omega(n)$, all the initial functions of $\mathfrak{E}_\alpha$ are $<_{n+2}$-recursive, and hence every function belonging to $\mathfrak{E}_\alpha$ is $<_{n+2}$-recursive.
We have therefore proved

Theorem 2.1

For each $n > 0$, $\bigcup_{\alpha < \omega(n)} \mathfrak{E}_\alpha \subseteqq U(<_{n+2})$.

Corollary

$\bigcup_{\alpha < \varepsilon_0} \mathfrak{E}_\alpha \subseteqq \mathrm{OR}$.

3. A Complete Classification of the Ordinal Recursive Functions

In this section it is proved that every $<_n$-recursive function belongs to $\mathfrak{E}_{\omega(n)\cdot k}$ for some k. It follows that the hierarchy $\{\mathfrak{E}_\alpha\}_{\alpha<\varepsilon_0}$ exhausts the class of ordinal recursive functions, and so the following characterization of OR is obtained:

$$\mathrm{OR} = \bigcup_{\alpha < \varepsilon_0} \mathfrak{E}_\alpha\,.$$

These results depend on a strengthened form of the monotonicity property of the functions F^n_α stated in part (*V*) of Theorem 1.1, and this in turn depends upon a further analysis of the fundamental sequences $\{\sigma\}(i)$, $i \in N$, for limit ordinals $\sigma < \varepsilon_0$.

Lemma 3.1

For each $n > 0$, and every limit ordinal $\sigma < \omega(n)$, if $\alpha < \sigma$ and $\mathrm{num}_n(\alpha) < x$, then

$$\alpha < \{\sigma\}(x)\,.$$

Proof

We proceed by induction on n.
The result is vacuously true for $n = 1$.
Suppose that the result holds for any $n \geqq 1$, and let σ be a limit ordinal $< \omega(n+1)$.
If $\alpha < \sigma$ we can write α and σ in the forms

$$\alpha = \delta + \omega^{\alpha_1}\cdot a_1 + \omega^{\alpha_2}\cdot a_2 + \cdots + \omega^{\alpha_s}\cdot a_s\,,$$
$$\sigma = \delta + \omega^{\sigma_1}\cdot b_1 + \omega^{\sigma_2}\cdot b_2 + \cdots + \omega^{\sigma_r}\cdot b_r\,,$$

where

(i) $\alpha_1 > \alpha_2 > \cdots > \alpha_s \geqq 0$;

(ii) $\sigma_1 > \sigma_2 > \cdots > \sigma_r > 0$, and $b_1 > 0$;

(iii) either $\sigma_1 > \alpha_1$, or else $\sigma_1 = \alpha_1$ and $b_1 > a_1$;

(iv) δ is a polynomial in powers of ω greater than σ_1.

From the definition of $\{\sigma\}(x)$, it is clear that

$$\{\sigma\}(x) \geqq \delta + \omega^{\sigma_1} \cdot (b_1 - 1) + \{\omega^{\sigma_1}\}(x)\,.$$

We must now consider two cases:

(a) Suppose that $\sigma_1 > \alpha_1$.
Suppose also, that $x > \mathrm{num}_{n+1}(\alpha)$.
Now $\mathrm{num}_{n+1}(\alpha) \geqq \mathfrak{p}^{a_1}_{\mathrm{num}_n(\alpha_1)} - 1$.
Hence $x > a_1$, and $x > \mathrm{num}_n(\alpha_1)$ if $a_1 \neq 0$.
Thus, if σ_1 is a successor ordinal, we have

$$\begin{aligned}\{\sigma\}(x) &\geqq \delta + \omega^{\sigma_1} \cdot (b_1 - 1) + \{\omega^{\sigma_1}\}(x)\\ &\geqq \delta + \omega^{\sigma_1} \cdot (b_1 - 1) + \omega^{\sigma_1 - 1} \cdot x\\ &\geqq \delta + \omega^{\sigma_1} \cdot (b_1 - 1) + \omega^{\alpha_1} \cdot x \text{ since } \sigma_1 - 1 \geqq \alpha_1\\ &\geqq \delta + \omega^{\alpha_1} \cdot (b_1 - 1) + \omega^{\alpha_1} \cdot a_1 + \omega^{\alpha_1} \text{ since } x > a_1\\ &> \alpha.\end{aligned}$$

If σ_1 is a limit ordinal, and $a_1 = a_2 = \cdots = a_s = 0$, so that $\alpha = \delta$, we have

$$\{\sigma\}(x) > \{\sigma\}(0) \geqq \delta = \alpha, \text{ since } x > 0\,.$$

If σ_1 is a limit ordinal, and $\alpha > \delta$, then we can assume, without loss of generality, that $a_1 > 0$, and hence $x > \mathrm{num}_n(\alpha_1)$.
Therefore, by the induction hypothesis, we have $\{\sigma_1\}(x) > \alpha_1$, since $\sigma_1 > \alpha_1$; and so

$$\begin{aligned}\{\sigma\}(x) &\geqq \delta + \omega^{\sigma_1} \cdot (b_1 - 1) + \{\omega^{\sigma_1}\}(x)\\ &= \delta + \omega^{\sigma_1} \cdot (b_1 - 1) + \omega^{\{\sigma_1\}(x)}\\ &> \alpha\,.\end{aligned}$$

This completes case (*a*).

(b) Suppose that $\sigma_1 = \alpha_1$ and $b_1 > a_1$.
Then we have

$$\begin{aligned}\{\sigma\}(x) &\geqq \delta + \omega^{\sigma_1} \cdot (b_1 - 1) + \{\omega^{\sigma_1}\}(x)\\ &\geqq \delta + \omega^{\sigma_1} \cdot a_1 + \{\omega^{\sigma_1}\}(x) \text{ since } b_1 > a_1\\ &= \delta + \omega^{\alpha_1} \cdot a_1 + \{\omega^{\alpha_1}\}(x) \text{ since } \sigma_1 = \alpha_1\,.\end{aligned}$$

Suppose also, that $x > \mathrm{num}_{n+1}(\alpha)$.
Now $\mathrm{num}_{n+1}(\alpha) \geqq (\mathfrak{p}^{a_2}_{\mathrm{num}_n(\alpha_2)}) - 1$.
Hence $x > a_2$ and $x > \mathrm{num}_n(\alpha_2)$ if $a_2 \neq 0$.
If α_1 is a successor ordinal, we have

$$\begin{aligned}\{\sigma\}(x) &\geqq \delta + \omega^{\alpha_1} \cdot a_1 + \{\omega^{\alpha_1}\}(x)\\ &\geqq \delta + \omega^{\alpha_1} \cdot a_1 + \omega^{\alpha_1 - 1} \cdot x\\ &\geqq \delta + \omega^{\alpha_1} \cdot a_1 + \omega^{\alpha_2} \cdot a_2 + \omega^{\alpha_2} \text{ since } \alpha_1 > \alpha_2,\ x > a_2\\ &> \alpha.\end{aligned}$$

If α_1 is a limit ordinal and $a_2 = a_3 = \cdots = a_s = 0$, then since $x > 0$, we have

$$\{\sigma\}(x) \geqq \delta + \omega^{\alpha_1} \cdot a_1 + \{\omega^{\alpha_1}\}(x) > \delta + \omega^{\alpha_1} \cdot a_1 = \alpha .$$

If α_1 is a limit ordinal and $\alpha > \delta + \omega^{\alpha_1} \cdot a_1$, we may assume, without loss of generality, that $a_2 \neq 0$, so that $x > \mathrm{num}_n(\alpha_2)$.
Then, by the induction hypothesis, $\{\alpha_1\}(x) > \alpha_2$, since $\omega(n) > \alpha_1 > \alpha_2$; and so

$$\begin{aligned}\{\sigma\}(x) &\geqq \delta + \omega^{\alpha_1} \cdot a_1 + \{\omega^{\alpha_1}\}(x) \\ &= \delta + \omega^{\alpha_1} \cdot a_1 + \omega^{\{\alpha_1\}(x)} \\ &> \alpha .\end{aligned}$$

This completes case (b).
(a) and (b) together show that if σ is a limit ordinal, $\alpha < \sigma < \omega(n+1)$, and $\mathrm{num}_{n+1}(\alpha) < x$, then $\alpha < \{\sigma\}(x)$.
This completes the induction step, and so Lemma 3.1 is proved.

Lemma 3.2

For each $n > 0$ and every $\beta < \omega(n)$, if $\alpha < \beta$ and $\mathrm{num}_n(\alpha) < x$, then for every k,

$$F^0_{\omega(n) \cdot k + \alpha}(x) < F^0_{\omega(n) \cdot k + \beta}(x) .$$

Proof

Suppose $n > 0$ is fixed.
We proceed by transfinite induction over the ordinals below $\omega(n)$.
The result is trivial when $\beta = 0$.
Suppose the result holds for all ordinals $< \beta$, where $\beta > 0$.
Let α be any ordinal $< \beta$, and suppose that $\mathrm{num}_n(\alpha) < x$.
Then if β is a successor ordinal, we have

$$\begin{aligned}F^0_{\omega(n) \cdot k + \beta}(x) &> F^0_{\omega(n) \cdot k + \beta - 1}(x) \text{ since } x > 0 \\ &\geqq F^0_{\omega(n) \cdot k + \alpha}(x) \text{ by induction hypothesis.}\end{aligned}$$

If β is a limit ordinal, $\alpha < \{\beta\}(x)$ by Lemma 3.1, and so

$$\begin{aligned}F^0_{\omega(n) \cdot k + \beta}(x) &= F^0_{\omega(n) \cdot k + \{\beta\}(x)}(x) \\ &> F^0_{\omega(n) \cdot k + \alpha}(x) \text{ by induction hypothesis.}\end{aligned}$$

Hence the result holds for β.
This completes the induction step, and so Lemma 3.2 is proved.
Now, for any $\alpha > 0$, $F^p_\alpha(x)$ is just the $(p+1)$th. iterate of F^0_α, applied to x.
Thus it is a simple matter to extend Lemma 3.2 in order to obtain.

Lemma 3.3

For each $n > 0$ and every $\beta < \omega(n)$, if $\alpha < \beta$ and $\mathrm{num}_n(\alpha) < x$, then for every k and every p,

$$F^p_{\omega(n) \cdot k + \alpha}(x) < F^p_{\omega(n) \cdot k + \beta}(x) .$$

Lemma 3.4

If $g \in \mathfrak{E}_{\omega(n) \cdot k}$ and f is defined from g by $<_n$-annihilation (as in Definition 1.5), then there is a function $h \in \mathfrak{E}_{\omega(n) \cdot k}$ such that for all x, $\underline{a}$,

$$f(x, \underline{a}) < F^0_{\omega(n) \cdot k + \mathrm{ord}_n(x)}(h(x, \underline{a})) .$$

Proof

If $g \in \mathfrak{E}_{\omega(n)\cdot k}$, then it is clear that the function $\max\big(g(x, \underline{a}),\ 2 \cdot g(g(x, \underline{a}), \underline{a}), \underline{a}\big)$ also belongs to $\mathfrak{E}_{\omega(n)\cdot k}$.
Hence, by Theorem 1.2, there is a number $p \geqq 2$ such that for all x, $\underline{a}$, $\max\big(g(x, \underline{a}),\ 2 \cdot g(g(x, \underline{a}), \underline{a}), \underline{a}\big) < F^{p}_{\omega(n)\cdot k}(\max(x, \underline{a}))$.
Define $h(x, \underline{a}) = \max(x, 2 \cdot g(x, \underline{a}), \underline{a}) + p$.
Then $h \in \mathfrak{E}_{\omega(n)\cdot k}$ and for all x, $\underline{a}$ we have

(i) $h(x, \underline{a}) > x$.

(ii) $h(x, \underline{a}) > 2 \cdot g(x, \underline{a}) + 1$.

Now f is defined by $<_n$-annihilation from g, so that

$$\begin{cases} f(0, \underline{a}) = 0 \\ f(x+1, \underline{a}) = 1 + f(g(x+1, \underline{a}), \underline{a}) \end{cases}$$

where $g(0, \underline{a}) = 0$ and $g(x+1, \underline{a}) <_n x + 1$ for every x.
We proceed by induction over the well-ordering $<_n$.
First, it is clear that

$$f(0, \underline{a}) < F^{0}_{\omega(n)\cdot k + \mathrm{ord}_n(0)}(h(0, \underline{a}))\,.$$

Assume, now, that

$$f(g(x+1, \underline{a}), \underline{a}) < F^{0}_{\omega(n)\cdot k + \mathrm{ord}_n(g(x+1, \underline{a}))}\big(h(g(x+1, \underline{a}), \underline{a})\big).$$

Then $f(x+1, \underline{a}) \leqq F^{0}_{\omega(n)\cdot k + \mathrm{ord}_n(g(x+1,\underline{a}))}\big(h(g(x+1, \underline{a}), \underline{a})\big)$.

$$\begin{aligned} \text{But } h(g(x+1, \underline{a}), \underline{a}) &= \max\big(g(x+1, \underline{a}), 2 \cdot g(g(x+1, \underline{a}), \underline{a}), \underline{a}\big) + p \\ &< F^{p}_{\omega(n)\cdot k}(\max(x+1, \underline{a})) + p \\ &\leqq F^{p}_{\omega(n)\cdot k}(\max(x+1, \underline{a}) + p) \\ &\leqq F^{p}_{\omega(n)\cdot k}(h(x+1, \underline{a}))\,. \end{aligned}$$

Also, $\mathrm{ord}_n(0) = 0 < h(x+1, \underline{a})$, and so by Lemma 3.3,

$$F^{p}_{\omega(n)\cdot k}(h(x+1, \underline{a})) \leqq F^{p}_{\omega(n)\cdot k + \mathrm{ord}_n(g(x+1,\underline{a}))}(h(x+1, \underline{a}))\,.$$

Therefore,

$$h(g(x+1, \underline{a}), \underline{a}) < F^{p}_{\omega(n)\cdot k + \mathrm{ord}_n(g(x+1,\underline{a}))}(h(x+1, \underline{a}))\,.$$

Thus we have the following:

$$\begin{aligned} f(x+1, \underline{a}) &\leqq F^{0}_{\omega(n)\cdot k + \mathrm{ord}_n(g(x+1,\underline{a}))}(h(g(x+1, \underline{a}), \underline{a})) \\ &< F^{0}_{\omega(n)\cdot k + \mathrm{ord}_n(g(x+1,\underline{a}))}\, F^{p}_{\omega(n)\cdot k + \mathrm{ord}_n(g(x+1,\underline{a}))}(h(x+1, \underline{a})) \\ &= F^{p+1}_{\omega(n)\cdot k + \mathrm{ord}_n(g(x+1,\underline{a}))}(h(x+1, \underline{a})) \\ &\leqq F^{h(x+1,\underline{a})}_{\omega(n)\cdot k + \mathrm{ord}_n(g(x+1,\underline{a}))}(h(x+1, \underline{a})) \text{ since } h(x+1, \underline{a}) > p \\ &= F^{0}_{\omega(n)\cdot k + \mathrm{ord}_n(g(x+1,\underline{a})) + 1}(h(x+1, \underline{a}))\,. \end{aligned}$$

Now $g(x+1, \underline{a}) <_n x + 1$, and so $\mathrm{ord}_n(g(x+1, \underline{a})) + 1 \leqq \mathrm{ord}_n(x+1)$.

Also, it follows, from the way in which the ordinals below $\omega(n)$ are arithmetized, that

$$\mathrm{num}_n\big(\mathrm{ord}_n(g(x+1,\underline{a}))+1\big) \leqq 2\cdot(g(x+1,\underline{a})+1)-1\,,$$

(with equality holding whenever $n > 1$).
Hence $\mathrm{num}_n\big(\mathrm{ord}_n(g(x+1,\underline{a}))+1\big) < h(x+1,\underline{a})$.
Therefore, by Lemma 3.2, we have
$F^0_{\omega(n)\cdot k+\mathrm{ord}_n(g(x+1,\underline{a}))+1}\big(h(x+1,\underline{a})\big) \leqq F^0_{\omega(n)\cdot k+\mathrm{ord}_n(x+1)}\big(h(x+1,\underline{a})\big)$.
Hence, $f(x+1,\underline{a}) < F^0_{\omega(n)\cdot k+\mathrm{ord}_n(x+1)}\big(h(x+1,\underline{a})\big)$.
This completes the induction step.
It follows that for all x, $\underline{a}$,

$$f(x,\underline{a}) < F^0_{\omega(n)\cdot k+\mathrm{ord}_n(x)}\big(h(x,\underline{a})\big)\,.$$

Lemma 3.5

For each $n > 0$, and every x,

$$\mathrm{ord}_n(x) \leqq \{\omega(n)\}(x)\,.$$

Proof

By induction on n.
First of all we have, for every x,

$$\mathrm{ord}_1(x) = x \leqq 2x = \{\omega\}(x) = \{\omega(1)\}(x)\,.$$

Now suppose that $n \geqq 1$, and that for every x,

$$\mathrm{ord}_n(x) \leqq \{\omega(n)\}(x)\,.$$

Clearly, $\mathrm{ord}_{n+1}(0) = 0 \leqq \{\omega(n+1)\}(0)$.
Suppose, then, that $x > 0$, and that

$$x = (\mathfrak{p}_{x_1}^{a_1}\cdot\mathfrak{p}_{x_2}^{a_2}\cdot\cdots\cdot\mathfrak{p}_{x_r}^{a_r}) - 1$$

where $x_r <_n \cdots <_n x_2 <_n x_1$, and $a_1 > 0$.
Then $x > x_1$, so that $\{\omega(n)\}(x) > \{\omega(n)\}(x_1)$.
But, by the induction hypothesis, $\{\omega(n)\}(x_1) \geqq \mathrm{ord}_n(x_1)$.
Hence $\{\omega(n)\}(x) > \mathrm{ord}_n(x_1)$, and we have

$$\mathrm{ord}_{n+1}(x) = \omega^{\mathrm{ord}_n(x_1)}\cdot a_1 + \omega^{\mathrm{ord}_n(x_2)}\cdot a_2 + \cdots + \omega^{\mathrm{ord}_n(x_r)}\cdot a_r\,,$$

where $\mathrm{ord}_n(x_1) > \mathrm{ord}_n(x_2) > \cdots > \mathrm{ord}_n(x_r)$.

$$\begin{aligned}\text{Thus, } \mathrm{ord}_{n+1}(x) &< \omega^{\{\omega(n)\}(x)}\\ &= \{\omega^{\omega(n)}\}(x)\\ &= \{\omega(n+1)\}(x)\,.\end{aligned}$$

This completes the induction step, and so Lemma 3.5 is proved.

Lemma 3.6

If $g \in \mathfrak{E}_{\omega(n)\cdot k}$ and f is defined by $<_n$-annihilation from g, then there is a function $h \in \mathfrak{E}_{\omega(n)\cdot k}$ such that for all x, $\underline{a}$,

$$f(x,\underline{a}) < F^0_{\omega(n)\cdot(k+1)}\big(h(x,\underline{a})\big)\,.$$

Proof

Given $g \in \mathfrak{E}_{\omega(n)\cdot k}$, let h be the function defined in Lemma 3.4.
Then for all x, $\underline{a}$, we have

$$f(x, \underline{a}) < F^0_{\omega(n)\cdot k + \mathrm{ord}_n(x)}(h(x, \underline{a}))\,.$$

Now, by Lemma 3.5, $\mathrm{ord}_n(x) \leqq \{\omega(n)\}(x)$.
But $\mathrm{num}_n(\mathrm{ord}_n(x)) = x < h(x, \underline{a})$.
Hence, by Lemma 3.2,
$F^0_{\omega(n)\cdot k + \mathrm{ord}_n(x)}(h(x, \underline{a})) \leqq F^0_{\omega(n)\cdot k + \{\omega(n)\}(x)}(h(x, \underline{a})) = F^0_{\{\omega(n)\cdot(k+1)\}(x)}(h(x, \underline{a}))$.
Also, for every y, and each $i < y$,
$F^0_{\{\omega(n)\cdot(k+1)\}(i)}(y) < F^0_{\{\omega(n)\cdot(k+1)\}(y)}(y) = F^0_{\omega(n)\cdot(k+1)}(y)$.
Thus we have the following; for all x, $\underline{a}$,

$$\begin{aligned} f(x, \underline{a}) &< F^0_{\{\omega(n)\cdot(k+1)\}(x)}(h(x, \underline{a})) \\ &< F^0_{\omega(n)\cdot(k+1)}(h(x, \underline{a}))\,. \end{aligned}$$

This completes the proof.

Theorem 3.7

For each $n > 0$, $A(<_n) \subseteqq \bigcup_{k \in N} \mathfrak{E}_{\omega(n)\cdot k}$.

Proof

Clearly, all the primitive recursive functions belong to $\mathfrak{E}_{\omega(n)}$, and, by definition, each class $\mathfrak{E}_{\omega(n)\cdot k}$ is closed under substitution.
If a function f is defined by primitive recursion from functions belonging to $\mathfrak{E}_{\omega(n)\cdot k}$, then it can easily be shown that $f \in \mathfrak{E}_{\omega(n)\cdot k + \omega}$, and hence $f \in \mathfrak{E}_{\omega(n)\cdot(k+1)}$.
Finally ,suppose f is defined from $g \in \mathfrak{E}_{\omega(n)\cdot k}$ by $<_n$-annihilation.
Then, by Lemma 3.6, there is a function $h \in \mathfrak{E}_{\omega(n)\cdot k}$ such that for all x, $\underline{a}$,

$$f(x, \underline{a}) < F^0_{\omega(n)\cdot(k+1)}(h(x, \underline{a}))\,.$$

Define a function g' as follows:

$$\begin{cases} g'(0, x, \underline{a}) = x \\ g'(z+1, x, \underline{a}) = g(g'(z, x, \underline{a}), \underline{a})\,. \end{cases}$$

Then g' is primitive recursive in g, and so $g' \in \mathfrak{E}_{\omega(n)\cdot(k+1)}$.
Now it can easily be proved that for all x, $\underline{a}$,

$$f(x, \underline{a}) = \mu_z(g'(z, x, \underline{a}) = 0)\,.$$

Hence f can be defined as follows:

$$f(x, \underline{a}) = \mu_{z < F^0_{\omega(n)\cdot(k+1)}(h(x,\underline{a}))}[g'(z. x, \underline{a}) = 0]\,.$$

Thus, f is "elementary" in the functions $F^0_{\omega(n)\cdot(k+1)}$, h, and g', all three of which belong to $\mathfrak{E}_{\omega(n)\cdot(k+1)}$.
But, for every $\alpha \geqq 2$, $\mathfrak{E}_\alpha$ is closed under the "elementary" operations.
Hence $f \in \mathfrak{E}_{\omega(n)\cdot(k+1)}$.
It follows that $A(<_n) \subseteqq \bigcup_{k \in N} \mathfrak{E}_{\omega(n)\cdot k}$.
Now OR $= \bigcup_{n \in N} A(<_n)$ and so we have

Corollary

$\mathrm{OR} \subseteqq \bigcup_{\alpha<\varepsilon_0} \mathfrak{E}_\alpha$.

Hence, from the Corollary to Theorem 2.1, we get

Theorem 3.8

$\mathrm{OR} = \bigcup_{\alpha<\varepsilon_0} \mathfrak{E}_\alpha$.

Now let Prov R be the class of all functions which are provably recursive in (classical) first-order arithmetic.

Then it follows from the work of Kreisel in [5] that OR = Prov R. (For related results, see Kino [4]).

Let $\mathrm{PR}^{(0,0)}$ be the class of all primitive recursive functionals of type (0, 0), defined by Gödel in [2].

Gödel has proved that $\mathrm{OR} \subseteqq \mathrm{PR}^{(0,0)}$, and the converse, that $\mathrm{PR}^{(0,0)} \subseteqq \mathrm{OR}$, has been established by Kreisel (see [6]) and, more directly, by Tait [11].

Also, let NR be the smallest class of functions which contains the primitive recursive functions, and which is closed under the operations of substitution, primitive recursion, and nested recursion over $<_n$, for any n.

Then OR = NR, since for any $n > 0$, nested recursion over $<_n$ is reducible to unnested ordinal recursion over $<_{n+1}$ (Tait [10]).

Hence we have

Theorem 3.9

$\bigcup_{\alpha<\varepsilon_0} \mathfrak{E}_\alpha = \mathrm{OR} = \mathrm{Prov\ R} = \mathrm{PR}^{(0,0)} = \mathrm{NR}$.

Finally, suppose we define a function $F^0_{\varepsilon_0}$ by

$$F^0_{\varepsilon_0}(x) = F^0_{\omega(x)}(x) .$$

Then it can be shown that every function belonging to $\bigcup_{\alpha<\varepsilon_0} \mathfrak{E}_\alpha$ is eventually majorized by $F^0_{\varepsilon_0}$, and hence, $F^0_{\varepsilon_0} \notin \bigcup_{\alpha<\varepsilon_0} \mathfrak{E}_\alpha$.

4. A Simplified Definition of $\mathfrak{E}_\alpha$

In this section, Robbin's Honesty Lemma [9] is adapted in order to show that whenever $\alpha < \beta < \varepsilon_0$, F^0_α is "elementary" in F^0_β.

It follows from this result that if $2 \leqq \alpha < \varepsilon_0$, then $\mathfrak{E}_\alpha$ is the class of all functions which are "elementary" in F^0_α.

An alternative characterization of $\mathfrak{E}_\alpha (2 \leqq \alpha < \varepsilon_0)$, in terms of computational complexity, is also obtained.

The first step is to show that, for each $\alpha < \varepsilon_0$, there is a number k such that $F^n_\alpha(x)$ can be computed by a Turing machine in such a way that the number of tape-squares used in the computation is less than $F^n_\alpha(x + k)$.

For the basic results concerning Turing machines, see Davis [1].

Now let $n > 0$ be fixed.

Each ordinal $\alpha < \omega(n)$ will be represented, on the tape of a Turing machine, by a word $\boldsymbol{\alpha}$ which is built up from the tape-symbols

$$/, [1, 1], [2, 2], \ldots, [n-1, n-1],$$

in the following way:

(i) If $r \in N$ then $\boldsymbol{r}$ is the word consisting of $r + 1$ /'s.
Hence $\mathbf{0} = /, \mathbf{1} = //, \mathbf{2} = ///, \ldots, \boldsymbol{m} + \mathbf{1} = \boldsymbol{m}\ /$, etc.

(ii) If $\omega(i) \leqq \beta < \omega(i+1) < \omega(n)$, and $\boldsymbol{\beta}$ has already been defined, denote by $\exp(\beta)$ the word

$$[i+1 \quad \boldsymbol{\beta} \quad i+1].$$

Suppose that $\alpha = \omega^{\alpha_1} \cdot a_1 + \omega^{\alpha_2} \cdot a_2 + \cdots + \omega^{\alpha_r} \cdot a_r + a_{r+1}$, where $0 < \alpha_r < \cdots < \alpha_2 < \alpha_1 < \omega(n-1)$, and where $\boldsymbol{\alpha}_r, \ldots, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_1$ have already been defined.
Then

$$\boldsymbol{\alpha} = \underbrace{\exp(\alpha_1) \ldots \exp(\alpha_1)}_{a_1 \text{ times}} \ \underbrace{\exp(\alpha_2) \ldots \exp(\alpha_2)}_{a_2 \text{ times}} \ \ldots \ \underbrace{\exp(\alpha_r) \ldots \exp(\alpha_r)}_{a_r \text{ times}} \boldsymbol{a}_{r+1}.$$

Thus, for example, the ordinal $\omega^{\omega \cdot 2} + \omega \cdot 3 + 1$ would be represented on tape by the word

$$[2\ [1\ //\ 1]\ [1\ //\ 1]\ /\ 2]\ [1\ //\ 1]\ [1\ //\ 1]\ [1\ //\ 1]\ //.$$

The triple α, m, x, where $\alpha < \omega(n)$, will be represented on the tape of a Turing machine by the word $\boldsymbol{\alpha} * \boldsymbol{m} * \boldsymbol{x}$.
We now construct a Turing machine Z which, when presented with a word $\boldsymbol{\alpha} * \boldsymbol{m} * \boldsymbol{x}$ $(\alpha < \omega(n))$, computes $F_\alpha^m(x)$.
Z has a tape which is infinite to the right.
The tape-symbols of Z include

$$/, [1, 1], [2, 2], \ldots, [n-1, n-1], *.$$

(Additional tape-symbols will also be required. These are to be used as markers in the course of a computation.)
In order to compute $F_\alpha^m(x)$, the word $\boldsymbol{\alpha} * \boldsymbol{m} * \boldsymbol{x}$ is written at the left-hand end of the tape, and the reading-head of Z is positioned at the right-hand end of this word.
Z then reacts according to the following scheme, where $W_1 \xrightarrow{Z} W_2$ means that Z converts word W_1 into word W_2, and positions its reading-head at the right-hand end of word W_2, in readiness for the next operation.

$$\mathbf{0} * \boldsymbol{m} * \boldsymbol{x} \xrightarrow{Z} (\boldsymbol{m} + \mathbf{1}) \cdot (\boldsymbol{x} + \mathbf{1})$$

$$\boldsymbol{\alpha} + \mathbf{1} * \mathbf{0} * \boldsymbol{x} \xrightarrow{Z} \boldsymbol{\alpha} * \boldsymbol{x} * \boldsymbol{x}$$

$$\boldsymbol{\sigma} * \mathbf{0} * \boldsymbol{x} \xrightarrow{Z} \{\boldsymbol{\sigma}\}(\boldsymbol{x}) * \mathbf{0} * \boldsymbol{x},\ \sigma \text{ a limit.}$$

$$\boldsymbol{\beta} * \boldsymbol{m} + \mathbf{1} * \boldsymbol{x} \xrightarrow{Z} \boldsymbol{\beta} * \mathbf{0} * \boldsymbol{\beta} * \boldsymbol{m} * \boldsymbol{x},\ \beta > 0.$$

The computation stops when there are no more occurrences of * left on the tape. Now let $\overline{Z}(\alpha, m, x)$ be the number of tape-squares used in the computation of $F^m_\alpha(x)$ by Z. Also, for any $\alpha < \omega(n)$, let $L(\alpha)$ be the length of the word $\boldsymbol{\alpha}$.
Then Z can be "programmed" in such a way that the following inequalities hold:

$$\overline{Z}(0, m, x) \leqq (m+1)\cdot(x+6)+1\,;$$

$$\overline{Z}(\alpha+1, 0, x) \leqq \max[L(\alpha+1)+(x+1)+5,\ 2+\overline{Z}(\alpha, x, x)]\,;$$

$$\overline{Z}(\sigma, 0, x) \leqq \max[L(\sigma)+(x+1)+5,\ 2+\overline{Z}(\{\sigma\}(x), 0, x)], \text{ if } \sigma \text{ is a limit ordinal};$$

$$\overline{Z}(\beta, m+1, x) \leqq \max[L(\beta)+\overline{Z}(\beta, m, x)+5,\ \overline{Z}(\beta, 0, F^m_\beta(x))], \text{ if } \beta > 0\,.$$

Now it can easily be proved, by induction, that for every limit ordinal $\sigma < \omega\ (n)$, and all x,

$$L(\{\sigma\}(x)) < L(\sigma) + x\cdot L(\sigma)^2\,.$$

Also, by methods similar to those used in the proof of Theorem 4.5 of [7], it is possible to obtain the following result, concerning the functions F^0_σ where σ is a limit ordinal $< \varepsilon_0$.

Lemma 4.1

For each limit ordinal $\sigma < \varepsilon_0$, all x, and all $y \geqq 2$,

$$F^0_{\{\sigma\}(x)+1}(x+y) \leqq F^0_\sigma(x+y)\,.$$

From these results, we obtain

Theorem 4.2

For each $\alpha < \omega(n)$, all m, and all x,

$$\overline{Z}(\alpha, m, x) \leqq F^m_\alpha(x + L(\alpha) + 5)\,.$$

Proof

We proceed by induction over the ordinals $< \omega(n)$.
First, it is clear that

$$\overline{Z}(0, m, x) \leqq (m+1)\cdot(x+6)+1 \leqq F^m_0(x+L(0)+5)\,.$$

Now suppose that $\alpha > 0$, and that for every $\delta < \alpha$,

$$\overline{Z}(\delta, m, x) \leqq F^m_\delta(x + L(\delta) + 5)\,.$$

We consider three cases:

(i) If α is a successor ordinal, then

$$\overline{Z}(\alpha, 0, x) \leqq \max[L(\alpha)+(x+1)+5,\ 2+\overline{Z}(\alpha-1, x, x)]\,.$$

But $L(\alpha) + (x+1) + 5 \leqq F^0_\alpha(x + L(\alpha) + 5)$, and by the induction hypothesis, we have

$$\begin{aligned} 2 + \overline{Z}(\alpha - 1, x, x) &\leqq 2 + F^x_{\alpha-1}(x + L(\alpha - 1) + 5) \\ &\leqq 2 + F^{x + L(\alpha) + 3}_{\alpha-1}(x + L(\alpha) + 5) \\ &\leqq F^{x + L(\alpha) + 5}_{\alpha-1}(x + L(\alpha) + 5) \\ &= F^0_\alpha(x + L(\alpha) + 5)\,. \end{aligned}$$

Hence $\overline{Z}(\alpha, 0, x) \leqq F^0_\alpha(x + L(\alpha) + 5)$.

(ii) If α is a limit ordinal, then
$\overline{Z}(\alpha, 0, x) \leqq \max[L(\alpha) + (x+1) + 5, 2 + \overline{Z}(\{\alpha\}(x), 0, x)]\,.$

Clearly, $L(\alpha) + (x+1) + 5 \leqq F^0_\alpha(x + L(\alpha) + 5)$.
If $\{\alpha\}(x) = 0$, then $x = 0$, and we have

$$2 + \overline{Z}(0, 0, 0) \leqq 9 \leqq F^0_\alpha(L(\alpha) + 5), \text{ since } L(\alpha) \geqq 3\,.$$

Now suppose that $\{\alpha\}(x) > 0$.
Then by the induction hypothesis, we have

$$2 + \overline{Z}(\{\alpha\}(x), 0, x) \leqq 2 + F^0_{\{\alpha\}(x)}\big(x + L(\{\alpha\}(x)) + 5\big)\,.$$

But $L(\{\alpha\}(x)) < L(\alpha) + x \cdot L(\alpha)^2$, and so

$$\begin{aligned} 2 + \overline{Z}(\{\alpha\}(x), 0, x) &\leqq 2 + F^0_{\{\alpha\}(x)}(x + L(\alpha) + x \cdot L(\alpha)^2 + 5) \\ &\leqq F^0_{\{\alpha\}(x)}(x + L(\alpha) + x \cdot L(\alpha)^2 + 7) \\ &\leqq F^0_{\{\alpha\}(x)}\big(((x + L(\alpha) + 5 + 1)^2 + 1)^2\big) \\ &= F^0_{\{\alpha\}(x)} F^0_1 F^0_1\,(x + L(\alpha) + 5) \\ &\leqq F^0_{\{\alpha\}(x)} F^0_{\{\alpha\}(x)} F^0_{\{\alpha\}(x)}(x + L(\alpha) + 5) \\ &= F^2_{\{\alpha\}(x)}(x + L(\alpha) + 5)\,. \\ &< F^0_{\{\alpha\}(x)+1}(x + L(\alpha) + 5) \\ &\leqq F^0_\alpha(x + L(\alpha) + 5) \text{ by Lemma 4.1}\,. \end{aligned}$$

Hence $\overline{Z}(\alpha, 0, x) \leqq F^0_\alpha(x + L(\alpha) + 5)$ if α is a limit ordinal.

(iii) Finally suppose α is any ordinal > 0.

Then we have

$$\overline{Z}(\alpha, m+1, x) \leqq \max[L(\alpha) + \overline{Z}(\alpha, m, x) + 5, \overline{Z}(\alpha, 0, F^m_\alpha(x))]$$

Now, by (i) and (ii) we have

$$\begin{aligned} \overline{Z}(\alpha, 0, F^m_\alpha(x)) &\leqq F^0_\alpha(F^m_\alpha(x) + L(\alpha) + 5) \\ &\leqq F^0_\alpha F^m_\alpha(x + L(\alpha) + 5) \\ &= F^{m+1}(x + L(\alpha) + 5)\,. \end{aligned}$$

Also, if we assume (inductively) that

$$\overline{Z}(\alpha, m, x) \leqq F^m_\alpha(x + L(\alpha) + 5)\,,$$

then we have the following:

$$\begin{aligned} L(\alpha) + \overline{Z}(\alpha, m, x) + 5 &\leqq L(\alpha) + F_\alpha^m(x + L(\alpha) + 5) + 5 \\ &\leqq 2 \cdot F_\alpha^m(x + L(\alpha) + 5) \\ &\leqq F_\alpha^0 F_\alpha^m(x + L(\alpha) + 5) \text{ since } \alpha > 0 . \\ &= F_\alpha^{m+1}(x + L(\alpha) + 5) . \end{aligned}$$

Hence, $\overline{Z}(\alpha, m + 1, x) \leqq F_\alpha^{m+1}(x + L(\alpha) + 5)$.
This completes the induction step, and so for every $\alpha < \omega(n)$, and all m, x,

$$\overline{Z}(\alpha, m, x) \leqq F_\alpha^m(x + L(\alpha) + 5) .$$

Now it is well-known that the functions used in the arithmetization of Turing machine computations are elementary.
Thus it can be shown that if f is computable by a Turing machine in such a way that, for every $\underline{x}$, the number of tape-squares used in the computation of $f(\underline{x})$ is less than $g(\underline{x})$, then f is elementary in g.
From this result we obtain the following:

Theorem 4.3

If $\alpha < \beta < \varepsilon_0$, then F_α^0 is elementary in F_β^0.

Proof

First we construct a Turing machine Z_α, which computes F_α^0.
Suppose that $\alpha < \omega(n)$, where $n > 0$.
Z_α has an infinite tape (to the right), and the same tape-symbols as Z.
Given any "input" x, Z_α writes the word $\boldsymbol{\alpha} * \mathbf{0} * \boldsymbol{x}$ at the left-hand end of its tape, and positions its reading-head at the right-hand end of this word. It then computes in exactly the same way as the machine Z.
It is clear that Z_α computes F_α^0.
Now let $\overline{Z}_\alpha(x)$ be the number of tape-squares used in the computation of $F_\alpha^0(x)$ by Z_α.
Then $\overline{Z}_\alpha(x) = \overline{Z}(\alpha, 0, x)$ for x, and so, by Theorem 4.2 $\overline{Z}_\alpha(x) \leqq F_\alpha^0(x + L(\alpha) + 5)$ for all x.
But if $\alpha < \beta$, F_α^0 is eventually majorized by F_β^0, and so there is a number p such that, for every x,

$$\overline{Z}_\alpha(x) \leqq F_\alpha^0(x + L(\alpha) + 5) < F_\beta^0(x + L(\alpha) + 5 + p) .$$

Hence F_α^0 is elementary in F_β^0.

Corollary

For each α such that $2 \leqq \alpha < \varepsilon_0$, $\mathfrak{E}_\alpha$ is the class of all functions which are elementary in F_α^0.

Proof

From Theorem 4.3, it follows that every function in $\mathfrak{E}_\alpha$ is elementary in F^0_α.
Now it can easily be proved that the function $\lambda x y \cdot x^y$ belongs to $\mathfrak{E}_\alpha$, for every $\alpha \geqq 2$.

Therefore, since $\mathfrak{E}_\alpha$ contains
$\{\lambda x.\, 0, \lambda x\, y.\, x + y, \lambda \underline{x}.\, x_i, \lambda x\, y.\ x^y, \lambda x.\, F^0_\alpha(x)\}$ and is closed under substitution and limited recursion, it is clear that $\mathfrak{E}_\alpha$ contains all the functions which are elementary in F^0_α.
This completes the proof.
The number of tape-squares used in the course of a Turing machine computation can be regarded as a measure of the complexity of the computation.
Theorem 4.2 leads us to a characterization of $\mathfrak{E}_\alpha$ in terms of computational complexity, as follows:
For each $\alpha < \varepsilon_0$, call a function f *α-complex* if there is a number p such that f is computable by a Turing machine Z_f in such a way that for every $\underline{x}$, the number of tape-squares used in the computation of $f(\underline{x})$ by Z_f is less than $F^p_\alpha(\max(x))$.
For each $\alpha < \varepsilon_0$, let $\mathfrak{C}_\alpha$ be the class of all α-complex functions.
Clearly, if $\alpha \leqq \beta < \varepsilon_0$, then $\mathfrak{C}_\alpha \subseteqq \mathfrak{C}_\beta$.

Theorem 4.4

For each α such that $2 \leqq \alpha < \varepsilon_0$, $\mathfrak{C}_\alpha = \mathfrak{E}_\alpha$.

Proof

Suppose f is α-complex, where $2 \leqq \alpha < \varepsilon_0$.
Then, by the remarks preceding Theorem 4.3, f is elementary in F^p_α, for some fixed p.
But F^p_α is defined by substitution from F^0_α, and so f is elementary in F^0_α.
Hence, by the Corollary to Theorem 4.3, $f \in \mathfrak{E}_\alpha$.
Conversely, suppose that $f \in \mathfrak{E}_\alpha$ $(2 \leqq \alpha < \varepsilon_0)$, so that f is elementary in F^0_α.
Let Z_α be the Turing machine, constructed in the proof of Theorem 4.3, which computes F^0_α, and let $\overline{Z}_\alpha(x)$ be the number of tape-squares used in the computation of $F^0_\alpha(x)$ by Z_α.
Then there is a number k such that, for all x, $\overline{Z}_\alpha(x) < F^0_\alpha(x + k)$.
Now, since f is elementary in F^0_α, it can be shown that there is a Turing machine Z_f, and a function g which is elementary in F^0_α, such that for all $\underline{x}$, the number of tape-squares used in the computation of $f(\underline{x})$ by Z_f is less than $g(\underline{x})$.
Since g is elementary in F^0_α, $g \in \mathfrak{E}_\alpha$, and so, by Theorem 1.2, there is a number p such for all $\underline{x}$, $g(\underline{x}) < F^p_\alpha(\max(\underline{x}))$.
Hence f is α-complex.
This completes the proof.

Corollary

$\mathrm{OR} = \bigcup_{\alpha < \varepsilon_0} \mathfrak{C}_\alpha\,.$

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