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Exponential Numbers of Linear Operators in Normed Spaces*

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ABSTRACT

Let X be a real or complex normed space, A be a linear operator in the space X, and $x \in X$. We put $E(X, A, x) = \min\{l : l > 0, \|A^l x\| \neq \|x\|\}$, or 0 if $\|A^k x\| = \|x\|$ for all integer $k > 0$. Then let $E(X, A) = \sup_x E(X, A, x)$ and $E(X) = \sup_A E(X, A)$. If $\dim X \geq 2$ then $E(X) \geq \dim X + 1$. A space X is called E-finite if $E(X) < \infty$. In this case $\dim X < \infty$, and we set $\dim X = n$.

The main results are following. If X is polynomially normed of a degree p, then it is E-finite; moreover, $E(X) \leq C^p_{n+p-1}$ (over $\mathbf{R}$), and $E(X) \leq (C^{p/2}_{n+p/2-1})^2$ (over $\mathbf{C}$). If X is Euclidean complex, then $n^2 - n + 2 \leq E(X) \leq n^2 - 1$ for $n \geq 3$; in particular, $E(X) = 8$ if $n = 3$. Also, $E(X) = 4$ if $n = 2$. If X is Euclidean real, then $[n/2]^2 - [n/2] + 2 \leq E(X) \leq n(n+1)/2$, and $E(X) = 3$ if $n = 2$. Much more detailed information on E-numbers of individual operators in the complex Euclidean space is obtained. If A is not nilpotent, then $E(X, A) \leq 2ns - s^2$, where s is the number of nonzero eigenvalues. For any operator A we prove that $E(X, A) \leq n^2 - n + t$, where t is the number of distinct moduli of nonzero nonunitary eigenvalues. In some cases E-numbers are "small" and can be found exactly. For instance, $E(X, A) \leq 2$ if A is normal, and this bound is achieved. The topic is closely connected with some problems related to the number-theoretic trigonometric sums.

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1. INTRODUCTION

Let X be a real or complex normed space, A be a linear bounded operator in the space X, and $x \in X$. We define the *exponential number* or, briefly, *E-number* $E(X, A, x)$ as the integer $l \geq 1$ such that

$$\|A^k x\| = \|x\| \quad (0 \leq k \leq l-1), \qquad \|A^l x\| \neq \|x\|, \tag{1.1}$$

if the number l exists. If

$$\|A^k x\| = \|x\| \qquad (k \in \mathbf{N} = \{0, 1, 2, \ldots\}), \tag{1.2}$$

then we put $E(X, A, x) = 0$. Now we can introduce *E-numbers*

$$E(X, A) = \sup_x E(X, A, x) \tag{1.3}$$

and

$$E(X) = \sup_A E(X, A) = \sup_{A,\, x} E(X, A, x). \tag{1.4}$$

Finally, if $\mathcal{S}$ is a nonempty subset of the set of all operators in X, then $E(X, \mathcal{S})$ means the supremum of $E(X, A)$ when A runs over $\mathcal{S}$.

It is clear that the E-numbers are isometrically invariant. In particular, the E-numbers of n-dimensional real or complex Euclidean space depend only on n. We denote them by $E_{n,\,\mathbf{R}}$ or $E_{n,\,\mathbf{C}}$.

Notice that, if X is a complex space and $X_{\mathbf{R}}$ is the same X considered as a real one, then $E(X) \leq E(X_{\mathbf{R}})$. Indeed, every linear bounded operator A in X lives in $X_{\mathbf{R}}$. If $\dim X = n < \infty$ then $\dim X_{\mathbf{R}} = 2n$. If X is Euclidean, then $X_{\mathbf{R}}$ is also Euclidean with respect to the same norm. Thus, $E_{n,\,\mathbf{C}} \leq E_{2n,\,\mathbf{R}}$.

On the other hand, if X is a real space and $X_{\mathbf{C}} = X \otimes_{\mathbf{R}} \mathbf{C}$ is its complexification, then $E(X) \leq E(X_{\mathbf{C}})$ for any extension to $X_{\mathbf{C}}$ of the norm given on X. Indeed, every linear bounded operator A in X can be extended to a linear bounded operator $A_{\mathbf{C}}$ in $X_{\mathbf{C}}$, and obviously, $E(X, A) \leq E(X_{\mathbf{C}}, A_{\mathbf{C}})$. If $\dim X = n < \infty$, then $\dim X_{\mathbf{C}} = n$, and if X is Euclidean then $X_{\mathbf{R}}$ is also Euclidean with respect to the natural complexification of the original scalar product. Thus, $E_{n,\,\mathbf{R}} \leq E_{n,\,\mathbf{C}}$.

In the present work the following inequalities are established:

$$n^2 - n + 2 \leq E_{n,\,\mathbf{C}} \leq n^2. \tag{1.5}$$

In particular, this yields $E_{2,\,\mathbf{C}} = 4$. For $n \geq 3$ the above upper estimate can be sharpened, namely, $E_{n,\,\mathbf{C}} \leq n^2 - 1$. Therefore, $E_{3,\,\mathbf{C}} = 8$.

In the real case the upper and lower bounds are both less than they are in the complex case:

$$[n/2]^2 - [n/2] + 2 \le E_{n,\mathbf{R}} \le n(n+1)/2, \tag{1.6}$$

where $[\cdot]$ means the integer part. The lower bound follows directly from the corresponding side of (1.5) by the inequality $E_{n,\mathbf{R}} \le E_{[n/2],\mathbf{C}}$. [It is easy to show that $E(X) \ge E(X_1)$ for every subspace $X_1 \subset X$.]

It follows from (1.6) that $2 \le E_{2,\mathbf{R}} \le 3$, but actually we prove that $E_{2,\mathbf{R}} = 3$.

The upper bounds (1.5) and (1.6) are obtained in Section 5 in a more general context of polynomially normed spaces which were introduced in [9]. If p is the degree of the polynomial norm in the space X, then

$$E(X) \le C^p_{n+p-1} \quad (\text{over } \mathbf{R}); \qquad E(X) \le (C^{p/2}_{n+p/2-1})^2 \quad (\text{over } \mathbf{C}). \tag{1.7}$$

The Euclidean spaces are polynomially normed, and $p = 2$. The classical space l^n_p is polynomially normed of degree p if and only if the number p is an even integer. In that case we have over $\mathbf{R}$

$$[m/2]^2 - [m/2] + 2 \le E(l^n_p) \le C^p_{n+p-1}, \tag{1.8}$$

where m is maximal such that $C^{p/2}_{m+p/2-1} \le n$. Indeed, there exists an isometric embedding $l^m_2 \to l^n_p$ (see [4, 11]), and the lower bound (1.6) can be applied.

More detailed information on E-numbers of operators in n-dimensional complex Euclidean space is concentrated in Sections 7, 8. Let A_1 be the maximal nonsingular (i.e. invertible) part of an arbitrary operator A, $r = \operatorname{rank} A_1$, s be the number of nonzero points $\lambda \in \operatorname{spec} A$, t be the number of distinct moduli of them except unitary ones (i.e. such that $|\lambda| = 1$), and finally, m_0 be the maximal order of Jordan blocks for $\lambda = 0$. Then

$$E(X, A) \le m_0 + (2r-1)s - s^2 + t. \tag{1.9}$$

This general result implies many interesting consequences; for instance,

$$E(X, A) \le 2ns - s^2 \tag{1.10}$$

for A not nilpotent, and

$$E(X, A) \le n^2 - n + t \tag{1.11}$$

for every operator A. In particular,

$$E(X,\, A) \le n^2 - n \tag{1.12}$$

if $t = 0$, i.e. the spectrum of A is unitary. We prove that the inequalities (1.11) for $t = 0, 1, 2$ are exact for operators with corresponding spectral properties. The point is that though $E(X, A) = 0$ for every unitary operator A, the E-number can become big after a small perturbation of A.

The E-numbers turn out small not only for unitary A's. In Section 9 we show that $E(X, A) \le 2$ if A is normal (in particular, self-adjoint) and this estimate is exact. A quite different example is A annihilated by a trinomial $\lambda^m - \alpha\lambda - \beta$, $m \ge 2$. In this case $E(X, A) \le 2m$ and it is also an exact estimate for every $m \ge 3$.

In many situations like the last one we use so-called Frobenius bases for cyclic operators. Recall that an operator A is called *cyclic* if it has a *cyclic* vector x. The latter means that the invariant subspace $X_{A,x} = \mathrm{Lin}(A^k x)_{k\in\mathbf{N}}$ coincides with the whole space X. Notice that for every vector x the operator $A_x = A|X_{A,x}$ is cyclic with the cyclic vector x. If $d_x = \dim X_{A,x}$, then the system $(A^k x)_{k=0}^{d_x-1}$ is just a Frobenius basis in this subspace. In particular, if $(e_k)_{k=0}^{n-1}$ is a basis in X, then every operator A such that $Ae_k = e_{k+1}$ for $0 \le k \le n-2$ is cyclic with the cyclic vector e_0 and the given basis is Frobenius for A. If now $(\alpha_k)_{k=0}^{n-1}$ are the coordinates of the vector $e_n = Ae_{n-1}$, then the characteristic polynomial of A is

$$\chi_A(\lambda) = \lambda^n - \sum_{k=0}^{n-1} \alpha_k \lambda^k.$$

This is a well-known way to construct operators with prescribed characteristic and thus annihilating polynomials. In Section 2 this construction yields the inequality $E(X) \ge \dim X$ for any space X and, moreover, $E(X) \ge \dim X + 1$ in the case $\dim X \ge 2$. This shows that E-number of every infinite-dimensional space is infinity. It is just a motivation to restrict a further investigation to the finite-dimensional case.

An operator A in the space X is called *E-finite* if $E(X, A) < \infty$. Otherwise, it is called *E-infinite*. Similarly, the space X is called *E-finite* or *E-infinite* if $E(X) < \infty$ or $E(X) = \infty$ respectively. It is easy to see that all parts of an E-finite operator are E-finite and all subspaces of an E-finite space are E-finite as well. Moreover, we prove in Section 2 that if $X_1 \ne X$ and $E(X_1) < \infty$ then $E(X) > E(X_1)$. In particular, we find the sequences $(E_{n,\mathbf{R}})_{n=1}^{\infty}$ and $(E_{n,\mathbf{C}})_{n=1}^{\infty}$ are strictly increasing.

We say that an operator A in an E-finite space X is *optimal* if $E(X, A) = E(X)$. Correspondingly, for any given operator A a vector x

is called *A-optimal* if $E(X, A, x) = E(X, A)$. Obviously, if x is A-optimal then it is A_x-optimal, and the E-numbers of the operators A and A_x are equal to the E-number of the vector x. We show in Section 2 that every optimal operator A in a E-finite space X is cyclic. This fact explains why the Frobenius construction is natural in our context.

Section 3 contains some results on E-infinite operators. We show that if a space X is not strictly convex, then there exists an operator A in X such that $E(X, A) = \infty$. The converse is true if $\dim X = 2$.

In Section 4 we discuss a tight connection between E-numbers and so-called critical exponents for contractions. Critical exponents were introduced and first investigated in [7] and [5] (see for further information [1, Chapter 2; 8]). The E-numbers can be treated as a kind of "individual" critical exponents for arbitrary operators, not necessary contractions. We show that the "global" critical exponent coincides with the E-number of the set of all contractions with norm 1 and spectral radius less than 1.

We add that the main estimate (1.9) is based on a uniqueness theorem for so-called quasipolynomials on $\mathbf{N}$. A related theory goes back to Euler, but for the reader's convenience we begin Section 6 with a short modern sketch. In particular, the trigonometric sums

$$f(k) = \sum_{j=1}^{n} a_j e^{2\pi i\theta_j k}$$

with arbitrary real θ_j, $0 \leq \theta_j < 1$ are quasipolynomials. Actually, we were stimulated to study the E-numbers by the following problem, which arises at the interface between harmonic analysis and number theory (cf. [2]): Given n, what is the minimal $M(n)$ such that for every trigonometric sum $f(k)$ with $|f(k)| = 1$ for $0 \leq k \leq M(n)-1$ one has $|f(k)| = 1$ for all $k \in \mathbf{N}$?

It is easy to show (see Theorem 6.4) that $M(n) \leq n^2 - n + 1$, and this result is a prototype of our general upper estimates [cf. (1.12)]. However, a stronger but similar conjecture can be expressed: $M(n) \leq Kn$, where K is an absolute constant.[1]

It is interesting to notice that a rougher estimate $M(n) \leq n^2$ follows from (1.12) in this way. Let us consider the sequence of vectors $x_k = (f(k+j-1))_{j=1}^n$, $k \in \mathbf{N}$, in the space $\mathbf{C}^n$ provided with the standard scalar product. Obviously, $x_k = A^k x_0$, where A is the diagonal operator with the eigenvectors $e_j = (e^{2\pi i\theta_j(m-1)})_{m=1}^n$ corresponding to the eigenvalues $e^{2\pi i\theta_j}$, $1 \leq j \leq n$. If $|f(k)| = 1$ for $0 \leq k \leq n^2 - 1$ then $\|A^k x_0\| = \sqrt{n}$ for $0 \leq k \leq n^2 - n$. Since the spectrum of A is unitary, (1.12) yields

[1] Addendum in proofs. This conjecture is not true in general. Recently we were informed by L. Lucht and C. Methfessel that $M(n) = n^2 - n + 1$ if n is a prime power.

$\|A^k x_0\| = \sqrt{n}$ for $k = n^2 - n + 1,\, n^2 - n + 2, \ldots$. Therefore, $|f(k)| = 1$ for all $k \in \mathbf{N}$.

Concluding in Section 10 we discuss some properties of trigonometric sums with constant modulus.

2. GENERAL PROPERTIES OF E-NUMBERS

Let us start with some simple examples and remarks.

EXAMPLE 2.1. Obviously, $E(X,\, A,\, 0) = 0$. Thus, $E(0) = 0$.

From now on we suppose that $X \neq 0$.

It is useful to notice that $E(X,\, \lambda A,\, \mu x) = E(X,\, A,\, x)$ if $|\lambda| = 1$, $\mu \neq 0$. Therefore, without loss of generality we can assume that $\|x\| = 1$ in (1.1). We also can "rotate" the operator A in this sense: $E(X,\, \lambda A) = E(X,\, A)$ if $|\lambda| = 1$.

EXAMPLE 2.2. $E(X,\, A) = 0 \Leftrightarrow E(X,\, A,\, x) = 0$ for all $x \in X \Leftrightarrow A$ is isometric, i.e., $\|Ax\| = \|x\|$ for all $x \in X$.

For every operator A one can consider its *isometric set* $\mathrm{Is}(A) = \{x : \|Ax\| = \|x\|\}$. Obviously, $0 \in \mathrm{Is}(A)$ and $x \in \mathrm{Is}(A) \Rightarrow \lambda x \in \mathrm{Is}(A)$ for all scalars λ. If $\mathrm{Is}(A)$ is nontrivial, i.e., $\mathrm{Is}(A) \neq 0$, then $\|A\| \geq 1$, and A is isometric if and only if $\mathrm{Is}(A) = X$. In terms of E-numbers $\mathrm{Is}(A) = \{x : E(X,\, A,\, x) \neq 1\}$. It is easy to see the following

PROPOSITION 2.3. *$E(X,\, A) = 1$ if and only if A is not isometric and $E(X,\, A,\, x) = 0$ for all $x \in Is(A)$ or, equivalently, the set $Is(A)$ is invariant for the operator A.*

EXAMPLE 2.4. $E(X,\, A) = 1$ for a strong contraction or dilation, i.e. if $\|Ax\| < \|x\|$ $(x \neq 0)$ or $\|Ax\| > \|x\|$ $(x \neq 0)$.

EXAMPLE 2.5. Let $A = \lambda I$, where I is the identity operator and λ is a scalar. If $|\lambda| \neq 1$, then A is a strong contraction or dilation, so $E(X,\, A) = 1$. If $|\lambda| = 1$, then A is an isometry, so $E(X,\, A) = 0$. In particular, $E(X,\, I) = 0$ and $E(X,\, 0) = 1$. The last equality implies

PROPOSITION 2.6. *$E(X) \geq 1$ for all X.*

Recall that $X \neq 0$. In the case $\dim X = 1$ all operators are of the form λI. Therefore, we have the following

PROPOSITION 2.7. *If* $\dim X = 1$ *then* $E(X) = 1$.

Now we notice that in (1.1) the points $A^k x (0 \leq k \leq l)$ must be pairwise distinct, since $A^l x$ cannot be equal to any of the above points. This implies that $E(X, A) \leq h + 1$ if $h = \mathrm{card}\{x : x \in \mathrm{Is}(A), \|x\| = 1\}$. Indeed, in (1.1) $A^k x \in \mathrm{Is}(A)$ for $0 \leq k \leq l-2$, and these points are pairwise distinct.

PROPOSITION 2.8. *Suppose that an operator A generates a finite semigroup with preperiod $r \geq 0$ and period $p \geq 1$, i.e., $A^{r+p} = A^r$. Then $E(X, A) \leq r + p - 1$.*

Proof. If $l \geq r + p$ then $A^l x = A^{l-p} x$. ■

COROLLARY 2.9. *Let A be periodic with period p, i.e. $A^p = I$. Then $E(X, A) \leq p - 1$.*

COROLLARY 2.10. *Let A be an involution, i.e., $A^2 = I$. Then $E(X, A) \leq 1$, and $E(X, A) = 1$ if and only if A is not isometric.*

COROLLARY 2.11. *Let A be a projection, i.e., $A^2 = A$. Then $E(X, A) \leq 1$, and $E(X, A) = 1$ if and only if $A \neq I$.*

Proof. A projection A is not isometric if and only if $A \neq I$. ■

COROLLARY 2.12. *Let A be a nilpotent operator of order $m \geq 1$, i.e., $A^m = 0$, $A^{m-1} \neq 0$. Then $E(X, A) \leq m$.*

Proof. In this case $A^{m+1} = A^m$. ■

Let us denote by $\mathcal{N}$ the set of all nilpotent operators, and let $\mathcal{N}_m$ be the subset of $\mathcal{N}$ consisting of the operators with order m. Notice that $m \leq \dim X$ and the set $\mathcal{N}_m$ is nonempty for every m even if $\dim X = \infty$. Indeed, in the last case we can take an m-dimensional subspace Y and a nilpotent operator A of the order m in Y. After that we can extend A to the whole of X, putting zero on a direct topological complement of Y.

PROPOSITION 2.13. $E(X, \mathcal{N}_m) = m$ *for all* m.

Proof. We know that $E(X, A) \leq m$ for $A \in \mathcal{N}_m$. On the other hand, one can construct a nilpotent A of order m such that $E(X,\, A) = m$. Indeed, taking m linearly independent normed vectors $e_0, \dots, e_{m-1}$, we can define the operator A by putting $Ae_k = e_{k+1}$ for $0 \le k \le m-2$ and $Ae_{m-1} = 0$. It remains to extend A as before. ∎

REMARK 2.14. In the inner-product space one can choose an orthonormal system $e_0, \dots, e_{m-1}$ and construct the corresponding nilpotent operator A. Then the additional property $\|A\| = 1$ is provided if $A \neq 0$.

COROLLARY 2.15. $E(X,\, \mathcal{N}) = \dim X$.

COROLLARY 2.16. $E(X) \ge \dim X$.

COROLLARY 2.17. *If* $\dim X = \infty$ *then* $E(X) = \infty$.

We also can settle the one-dimensional case: by virtue of Corollary 2.16 and Proposition 2.7 we get

COROLLARY 2.18. $E(X) = 1$ *if and only if* $\dim X = 1$.

The Corollary 2.17 is based on the presence of arbitrary big finite-dimensional subspaces in an infinite-dimensional space. One might hope to come to the same fact "directly" by answering the following question:

PROBLEM 2.19. *If* $\dim X = \infty$, *is there an E-infinite operator A?*

There is, at least, for the inner-product space. Indeed, if $(e_k)_{k=0}^{\infty}$ is an orthonormal system, then one can take the left shift A: $Ae_k = e_{k-1}$ for $k \ge 1$, and $Ae_0 = 0$. (By the way, $\|A\| = 1$. Are values $\|A\| > 1$ possible in this context?)

Now we notice that if $X_1 \subset X$ is an invariant subspace for an operator A and $A_1 = A|X_1$, then obviously

$$E(X_1,\, A_1) \le E(X,\, A). \tag{2.1}$$

Thus, if an operator A is E-finite, then all its parts are E-finite. Correspondingly, all subspaces of an E-finite space are E-finite. Moreover,

PROPOSITION 2.20. *If X_1 is a subspace of the space X, then* $E(X_1) \le E(X)$.

Proof. By Corollary 2.16 only $\dim X_1 < \infty$ should be considered.

Then the inequality (2.1) implies that $E(X_1) \le E(X)$, since X_1 is a complemented subspace of X. ■

From now on we restrict our subject to the finite-dimensional case: $\dim X = n$, $2 \le n < \infty$. The meaning of n will be fixed throughout the whole paper.

First of all we sharpen Proposition 2.20.

PROPOSITION 2.21. *If X_1 is a subspace of the space X, then*

$$E(X) \ge E(X_1) + \operatorname{codim} X_1. \tag{2.2}$$

Proof. If $E(X_1) = \infty$ then $E(X) = \infty$ by Proposition 2.20. Let $E(X_1) = l < \infty$, an operator A_1 in X_1 be optimal, and a normed vector $x \in X_1$ be A_1-optimal. This means that $\|A_1^k x\| = \|x\| = 1$ for $0 \le k \le l-1$ and $\|A_1^l x\| \ne 1$. Taking a complement X_2 to X_1 and a normed basis $e_0, \ldots, e_{d-1}$ in X_2, we can extend A_1 to an operator A on the whole space X by putting $Ae_k = e_{k+1}$ for $0 \le k \le d-2$, $Ae_{d-1} = x$. Obviously, $E(X, A, e_0) = l + d = E(X_1) + \operatorname{codim} X_1$. Therefore, the last sum does not exceed $E(X)$. ■

COROLLARY 2.22. *If $X_1 \ne X$ and $E(X_1) < \infty$ then $E(X) > E(X_1)$.*

COROLLARY 2.23. *Let the space X be E-finite. Every optimal operator A in X is cyclic. Moreover, every A-optimal vector x is cyclic for the operator A.*

Proof. In this situation the spaces $X_{A,x}$ and X have the same E-numbers; hence, $X_{A,x} = X$. ■

Let us denote by $\rho(A)$ the spectral radius of operator A, i.e. the maximum modulus of its eigenvalues (or the same for $A_{\mathbf{C}}$ if the space X is real). By the well-known Gelfand formula

$$\rho(A) = \lim_{k \to \infty} \|A^k\|^{1/k} = \inf_k \|A^k\|^{1/k}.$$

In particular, if A is a contraction, then $\rho(A) \le 1$, and $\rho(A) = 1 \Leftrightarrow \|A^k\| = 1\ (k \in \mathbf{N})$.

LEMMA 2.24. *If a vector $x \ne 0$ is such that $E(X, A, x) = 0$, then $\rho(A_x) = 1$.*

Proof. The subspace $X_{A,x}$ has a basis of a form $e_k = A^k x\,(0 \le k \le m-1;\, m = \dim X_{A,x})$. Since $\|A^j e_k\| = 1\,(0 \le k \le m-1;\, j \ge 1)$, the sequence of powers $A^j|X_{A,x}$ is bounded and does not tend to zero. Therefore, $\rho(A_x) = 1$. ∎

COROLLARY 2.25. *If there exists a vector $x \neq 0$ such that $E(X, A, x) = 0$, then* spec$A$ *intersects the unit circle* $|\lambda| = 1$.

Now we can sharpen the bound $E(X) \ge \dim X$. Recall that by our agreement $\dim X \ge 2$.

THEOREM 2.26. *The inequality*

$$E(X) \ge \dim X + 1 \tag{2.3}$$

holds for any space X.

Proof. It is sufficient to consider the two-dimensional case and after that to apply Proposition 2.21 with $\dim X_1 = 2$ and $E(X_1) \ge 3$.

Let us take a point $v \neq 0$, $\|v\| = \alpha < 1$ and a point e_0 on the unit circle $\|x\| = 1$ off the straight line which goes through v and 0. Then the ray directed from e_0 to v intersects the unit circle at a point $e_1 \neq -e_0$. We obtain a normed basis e_0, e_1 in X of which v is a convex combination, say, $v = p_0 e_0 + p_1 e_1$. Now we construct an operator A, putting $Ae_0 = e_1$ and $Ae_1 = v/\alpha$. There is the following alternative: $E(X, A, e_0) \ge 3$ or $E(X, A, e_0) = 0$. But the last case is impossible, as we now see.

The characteristic polynomial of the operator A is $\chi_A(\lambda) = \lambda^2 - \alpha^{-1}(p_0 + p_1\lambda)$. It has a root $\lambda_0 > 1$, since $\chi_A(1) = 1 - \alpha^{-1} < 0$. So $\rho(A) > 1$; meanwhile, Lemma 2.24 asserts $\rho(A) = 1$, since e_0 is cyclic. ∎

The following open problem is suggested by Theorem 2.26.

PROBLEM 2.27. Let $n \ge 2$. For which $m \ge n+1$ does there exist an n-dimensional space X such that $E(X) = m$?

The next open problem is

PROBLEM 2.28. Given a space X, what is its *E-spectrum*, i.e. the set of E-numbers of all its subspaces or of all subspaces of a given dimension?

Anyway, this set contains 0,1 and $E(X)$ and does not contain 2, by Theorem 2.26. If X is Euclidean, then its E-spectrum consists of just $n+1$

numbers, since in this case the E-number of a subspace depends only on its dimension m and the corresponding sequence $(E_m)_{m=0}^n$ is strictly increasing. (E-finiteness of Euclidean spaces will be proved later.) Generally, the E-spectrum of every E-infinite space is finite by Proposition 2.20. Can it happen for an E-finite space X that $\dim X \geq 3$?

REMARK 2.29. One can extend Problem 2.28 to the infinite-dimensional case. Then it make a sense to take into account only the finite-dimensional subspaces in order to exclude a trivial appearance of infinite E-numbers.

Similar problems can be posed for the operator or vector E-numbers. By the way, what are relations between E-numbers and norms of operators?

We are going to study various connections between spectral properties of operators and its E-numbers. One interesting example follows just below.

An operator A is called *weakly hyperbolic* if its spectrum contains a pair $\{\lambda, \mu\}$ such that $|\lambda| > 1$, $|\mu| < 1$. If A is weakly hyperbolic and its spectrum does not intersect the unit circle, then it is called *hyperbolic*. Every weakly hyperbolic operator has a hyperbolic part. Indeed, we can restrict A to the minimal invariant subspace L such that $\mathrm{spec}A|L$ contains the above mentioned pair $\{\lambda, \mu\}$. (In the complex case $\mathrm{spec}A|L = \{\lambda, \mu\}$ and $\dim L = 2$; in the real case $\mathrm{spec}A|L$ contains also $\{\overline{\lambda}, \overline{\mu}\}$ and $2 \leq \dim L \leq 4$).

LEMMA 2.30. *The isometric set of every weakly hyperbolic operator is nontrivial.*

Proof. We have $\max_{\|x\|=1} \|Ax\| = \|A\| \geq \rho(A) \geq |\lambda| > 1$. Moreover, if A is nonsingular then $\min_{\|x\|=1} \|Ax\| = \|A^{-1}\|^{-1} \leq \rho(A^{-1})^{-1} \leq |\mu| < 1$. This inequality extends to the singular case, since the minimum on the left is zero. In any case, the continuous function $\|Ax\|$ on the unit sphere must take the intermediate value 1. ∎

THEOREM 2.31. *If an operator A is weakly hyperbolic, then $E(X, A) \geq 2$.*

Proof. We can assume that A is hyperbolic. Since A is not isometric, we have $E(X, A) \neq 0$. If $E(X, A) = 1$, then by the previous lemma and Proposition 2.3 there exists a vector $x \neq 0$ such that $E(X, A, x) = 0$. This contradicts the hyperbolicity by Corollary 2.25. ∎

In Section 9 we will see that the above obtained estimate is achieved if

A is a normal weakly hyperbolic operator in an Euclidean space (complex or real).

3. E-INFINITE OPERATORS

In this section we investigate some geometrical reasons for operator E-infiniteness.

LEMMA 3.1. *If an operator A is E-infinite, then there exists a vector $x \neq 0$ such that $E(X, A, x) = 0$.*

Proof. There exists an increasing sequence of integer numbers $(l_j)_1^\infty$ and a sequence of vectors $(x_j)_1^\infty$ such that $E(X, A, x_j) = l_j (j = 1, 2, 3, \ldots)$. We can assume that x_j are normed and $x = \lim_{j\to\infty} x_j$ exists. If $l_j > k$ for a given k, then $\|A^k x_j\| = 1$, whence $\|A^k x\| = 1 (k \in \mathbf{N})$. ■

Combining Lemma 3.1 and Corollary 2.25, we obtain the following

COROLLARY 3.2. *Let an operator A be E-infinite. Then there exists an unitary point $\lambda \in$* spec$A$.

Now recall that a space X is called *strictly convex* if there are no distinct vectors e_0, e_1 such that the segment $[e_0, e_1]$ lies on the unit sphere. It means that $\|\tau e_0 + (1-\tau)e_1\| = 1$ for all τ, $0 \leq \tau \leq 1$. The classical examples are Euclidean spaces and, more generally, l_p^n $(1 < p < \infty)$ over $\mathbf{R}$ or $\mathbf{C}$. They are $\mathbf{R}^n$ or $\mathbf{C}^n$ provided with the norm

$$\|x\|_p = \left(\sum_{k=0}^{n-1} |\xi_k|^p\right)^{1/p},$$

where ξ_k $(0 \leq k \leq n-1)$ are the canonical coordinates of x. However, the "limit" spaces l_1^n and l_∞^n with the norms

$$\|x\|_1 = \sum_{k=1}^{n} |\xi_k|, \quad \|x\|_\infty = \max_{1\leq k\leq n} |\xi_k|$$

are not strictly convex.

THEOREM 3.3. *If a space X is not strictly convex, then there is an E-infinite linear operator in X.*

Proof. By (2.1) it is sufficient to consider the case $\dim X = 2$. Let X be real. We can identify the space X with $\mathbf{R}^2$ in such a way that the above-mentioned vectors e_0 and e_1 are $(1, -1)$ and $(1, 1)$ respectively. Now the segment $[e_0, e_1]$ is $\{x = (\xi_0, \xi_1) : \xi_0 = 1, |\xi_1| \leq 1\}$. One can assume that this segment is maximal on the unit circle $\|x\| = 1$, so that if $x = (1, \xi_2)$ with $|\xi_2| > 1$, then $\|x\| > 1$.

Let $A(\xi_1, \xi_2) = (\xi_1, 2\xi_2)$. If $x_l = (1, 2^{-(l-1)})$ for $l = 1, 2, 3, \ldots$, then $A^k x_l = (1, 2^{k-l+1})$ and $\|A^k x_l\| = 1$ for $0 \leq k \leq l-1$, $\|A^l x_l\| > 1$. We see that $E(X, A, x_l) = l$ for every l. Therefore, $E(X, A) = \infty$.

If X is complex, then it is the complexification of the real linear span Y of the vector e_0 and e_1. Since Y is not strictly convex, there is an E-infinite operator A in Y. Its complexification $A_{\mathbf{C}}$ is also E-infinite, since $E(X, A_{\mathbf{C}}) \geq E(Y, A) = \infty$. ■

COROLLARY 3.4. *If a space X is not strictly convex, then it is E-infinite.*

In the two-dimensional real case Theorem 3.3 has a converse.

THEOREM 3.5. *Let X be real,* $\dim X = 2$. *If there exists an E-infinite operator A in X, then X is not strictly convex.*

Proof. By Corollary 3.2 there exists $\lambda \in \operatorname{spec} A$, $|\lambda| = 1$. If λ is real, we can assume that $\lambda = 1$, since the case $\lambda = -1$ reduces to the previous one when $A \to -A$.

Let us take $u \neq 0$ such that $Au = u$. Let $\mu \neq 1$ be another eigenvalue and $v \neq 0$, $Av = \mu v$. The vectors u, v form a basis of the space X. Take a vector $x = \alpha u + \beta v$ such that $E(X, A, x) \geq 2$, and notice that $\alpha \neq 0$ [since $E(X, A, v) \leq 1$] and $\beta \neq 0$ [since $E(X, A, u) = 0$]. We also notice that the numbers 1, μ, μ^2 are pairwise distinct, since the semigroup of the powers A^k cannot be finite by Proposition 2.8. Therefore, we get three points $A^k x = \alpha u + \beta \mu^k v \, (k = 0, 1, 2)$ lying on the intersection of the unit circle with the straight line $\alpha u + \tau v \, (\tau \in \mathbf{R})$. This means that the space X is not strictly convex.

If $\operatorname{spec} A = \{1\}$, then one can take a vector v such that $Av = u + v$ and a vector x as above. The three points $A^k x = x + \beta k u \, (k = 0, 1, 2)$ play the same role. The case of a real spectrum is settled.

Let $\operatorname{spec} A = \{e^{\pi i \theta}, e^{-\pi i \theta}\}$, where $0 < \theta < 1$. The value θ is irrational, since the operator A cannot be periodic. By Lemma 3.1 one can choose a normed vector x such that $E(X, A, x) = 0$. Its orbit $(A^k x)_{k \in \mathbf{N}}$ lies on the unit circle. On the other hand, the closure of this orbit is a Euclidean circle. This means that the unit circle is Euclidean and A is an isometry.

But then $E(X, A) = 0$. ∎

PROBLEM 3.6. Does there exist a counterexample to Theorem 3.5 in the case when X is real but $\dim X = 3$ or X is complex and $\dim X = 2$?

It will be proved in Section 5 that every Euclidean space is E-finite. Is every strictly convex space E-finite? The answer is negative even in the two-dimensional real case.

EXAMPLE 3.7. It is easy to construct a non-Euclidean strictly convex unit circle in $\mathbf{R}^2$ which contains a Euclidean arc $\xi_1 = \cos\phi$, $\xi_2 = \sin\phi$ where $|\phi| \le \pi/4$. Let the operator A be Euclidean rotation through the angle $\pi/4m$, $m \ge 1$. Then E-number of A on the basic vector $(1, 0)$ is greater than m. Therefore, this strictly convex space is E-infinite.

However, in this example all of operators are E-finite by Theorem 3.5.

4. E-NUMBERS AND CRITICAL EXPONENTS

The critical exponent of a space X was defined in [7] as the minimal $q = q(X)$ such that $\|A\| = \|A^q\| = 1$ implies $\|A^k\| = 1$ for all $k > q$ [or equivalently, $\rho(A) = 1$]. There even exists a two-dimensional real space without a critical exponent [3]. It is convenient to write $q(X) = \infty$ if there is no critical exponent for a space X. In any case, $q(X) \ge 2$.

THEOREM 4.1. *For every space X the following equality holds:*

$$q(X) = E(X, \mathcal{C}_0), \tag{4.1}$$

where $\mathcal{C}_0$ is the set of all contractions with norm 1 *and spectral radius less than* 1.

Proof. If $q(X) = q < \infty$, then this is the critical exponent and, by definition, there exists an operator A such that $\|A\| = \|A^{q-1}\| = 1$ but $\|A^q\| < 1$. Obviously, $\rho(A) < 1$; hence $A \in \mathcal{C}_0$. Let us choose a normed vector x such that $\|A^{q-1}x\| = 1$. Then $\|A^k x\| = 1$ for $0 \le k \le q-1$, and $\|A^q x\| < 1$. Therefore, $E(X, A, x) = q$, whence $E(X, A) \ge q$ and $E(X, \mathcal{C}_0) \ge q$. If $q(X) = \infty$, we can apply the above argumentation with an arbitrarily big number playing the role of q. In any case $E(X, \mathcal{C}_0) \ge q(X)$.

Now let $E(X, \mathcal{C}_0) = l < \infty$. We can choose an operator $A \in \mathcal{C}_0$ and a normed vector x such that $\|A^k x\| = 1$ for $0 \le k \le l-1$. This yields

$\|A^k\| = 1$ for $1 \le k \le l-1$. If $l > q(X)$ then $\rho(A) = 1$, which contradicts the choice $A \in \mathcal{C}_0$. Therefore, $l \le q(X)$. If $E(X, \mathcal{C}_0) = \infty$, we can take an arbitrarily big number instead of l. In any case $E(X, \mathcal{C}_0) \le q(X)$. ■

COROLLARY 4.2. *The critical exponent of a space X exists if and only if $E(X, \mathcal{C}_0) < \infty$.*

COROLLARY 4.3. *If a space X is E-finite, then the critical exponent $q(X)$ exists and $q(X) \le E(X)$.*

Notice that E-finiteness does not follow from the existence of the critical exponent. Indeed, $q(l^n_{\infty,\mathbf{R}}) = n^2 - n + 1$ [5], but $E(l^n_{\infty,\mathbf{R}}) = \infty$ by Corollary 3.4.

COROLLARY 4.4. *For n-dimensional Euclidean space X,*

$$E(X, \mathcal{C}_0) = n. \tag{4.2}$$

Proof. In this case $q(X) = n$ [7]. ■

Accordingly [1], an operator is called an *extremal contraction* if its norm and spectral radius are both equal to 1. Let us denote the set of all extremal contractions in a space X by $\mathcal{C}_e$.

THEOREM 4.5. *For n-dimensional Euclidean space X*

$$E(X, \mathcal{C}_e) = n - 1. \tag{4.3}$$

Proof. Let A be an extremal contraction. Then the space X is an orthogonal sum of two invariant subspaces X_0 and X_1 such that $\rho(A_0) < 1$ and A_1 is an isometry, where $A_0 = A|X_0$ and $A_1 = A|X_1$ (see, for instance [1, Chapter 2]). If $x \in X$ and $x = x_0 + x_1$ where $x_0 \in X_0$ and $x_1 \in X_1$, then $\|A^k x\|^2 = \|A_0^k x_0\|^2 + \|x_1\|^2$ for all of $k \in \mathbf{N}$. Therefore, $E(X, A, x) = E(X_0, A_0, x_0)$, whence $E(X, A) = E(X_0, A_0) = \dim X_0$ by Corollary 4.4. Thus, $E(X, A) \le n - 1$, since $\dim X_0 < n$. [If $\dim X_0 = n$, i.e. $X_0 = X$, we have $\rho(A) < 1$.]

Now we can take an $(n-1)$-dimensional subspace X_0 and an operator A_0 of the class $\mathcal{C}_0$ in this subspace satisfying the condition $E(X_0, A_0) = n - 1$. Let A be A_0 on the subspace X_0, and the identity on its orthogonal complement. Then $A \in \mathcal{C}_e$ and $E(X, A) = n - 1$. ■

The set $\mathcal{C} = \{A : \|A\| \leq 1\}$ of all contractions is the union of $\mathcal{C}_e$, $\mathcal{C}_0$, and the set $\mathcal{C}_{00} = \{A : \|A\| < 1\}$. So we obtain the following

COROLLARY 4.6. *For n-dimensional Euclidean space X,*

$$E(X, \mathcal{C}) = n. \tag{4.4}$$

5. E-FINITENESS OF POLYNOMIALLY NORMED SPACES

A space X is called *polynomially normed* if there exists a number $p > 0$ such that the function $\varphi_{x,y}(\tau) = \|x + \tau y\|^p$ $(\tau \in \mathbf{R})$ is a polynomial for any $x, y \in X$, $y \neq 0$ [9]. Since $\varphi_{0,y}(\tau) = \|y\|^p |\tau|^p$, p must be an integer and even. The number p coincides with the degree of the polynomial $\varphi_{x,y}$ for all x, y, since $\varphi_{x,y}(\tau) \sim \|y\|^p \tau^p$ $(\tau \to \infty)$. Accordingly, this number is called the *degree* of the space X.

EXAMPLE 5.1. Every Euclidean space is polynomially normed of degree $p = 2$.

EXAMPLE 5.2. If p is an integer and even, then the spaces $l^n_{p,\mathbf{R}}$ and $l^n_{p,\mathbf{C}}$ are polynomially normed.

THEOREM 5.3. *If X is a n-dimensional polynomially normed space of degree p, then it is E-finite. Furthermore,*

$$E(X) \leq C^p_{n+p-1} \tag{5.1}$$

if X is real, and

$$E(X) \leq (C^{p/2}_{n+p/2-1})^2 \tag{5.2}$$

if X is complex.

Proof. Let us begin with the real case. We denote by $V(n, p)$ the space of all p-forms (i.e., homogeneous polynomials of degree p) of the vector variable $x \in X$. The function $\varphi(x) = \|x\|^p$ belongs to the space $V(n, p)$ [10]. For a given operator A we consider (cf. [3]) the decreasing sequence of sets (actually, of algebraic manifolds)

$$M_s = \{x : \varphi(A^k x) = \varphi(x)\ (1 \leq k \leq s)\} \qquad (s = 1, 2, 3, \ldots).$$

Denote by r the maximal s such that the p-forms $\varphi(A^k x) - \varphi(x)$ $(1 \le k \le s)$ are linearly independent. Obviously, $M_{r+1} = M_r$, since the equation $\varphi(A^{r+1}x) = \varphi(x)$ follows from the system $\varphi(A^k x) = \varphi(x)$ $(1 \le k \le r)$. But for any s

$$M_{s+1} = M_s \quad \Rightarrow \quad M_{s+2} = M_{s+1},$$

since $M_{s+2} \subset M_{s+1}$ and $x \in M_{s+1} \Rightarrow Ax \in M_s \Rightarrow Ax \in M_{s+1} \Rightarrow x \in M_{s+2}$. Therefore, $M_r = M_{r+1} = M_{r+2} = \cdots$. In this situation,

$$\|A^k x\| = \|x\| \quad (1 \le k \le r) \quad \Rightarrow \quad \|A^k x\| = \|x\| \quad (k > r).$$

We conclude that $E(X, A) \le r \le \dim V(n, p)$, whence

$$E(X) \le \dim V(n, p) = C^p_{n+p-1}.$$

In the complex case the proof is different from the above in only one point. Instead of $V(n, p)$ we should consider the real space $W(n, p)$ of all "Hermitian" forms of degree p (see [10]). Then $\dim W(n, p) = (C^{p/2}_{n+p/2-1})^2$. ■

By the way, if p is fixed and $n \to \infty$ then

$$C^p_{n+p-1} \sim \frac{n^p}{p!}, \qquad \left(C^{p/2}_{n+p/2-1}\right)^2 \sim \frac{n^p}{(p/2)!^2}.$$

COROLLARY 5.4. *If p is an even integer, then the spaces $l^n_{p,\mathbf{R}}$ and $l^n_{p,\mathbf{C}}$ are E-finite. Moreover,*

$$E(l^n_{p,\mathbf{R}}) \le C^p_{n+p-1}, \qquad E(l^n_{p,\mathbf{C}}) \le \left(C^{p/2}_{n+p/2-1}\right)^2. \tag{5.3}$$

COROLLARY 5.5. *If X is a n- dimensional Euclidean space, then it is E-finite. Moreover,*

$$E_{n,\mathbf{R}} \le \frac{n(n+1)}{2}, \quad E_{n,\mathbf{C}} \le n^2. \tag{5.4}$$

COROLLARY 5.6. $E_{2,\mathbf{R}} = 3$.

Proof. $E_{2,\mathbf{R}} \le 3$ by (5.4), and $E_{2,\mathbf{R}} \ge 3$ by Theorem 2.26. ■

In the complex case we have $3 \le E_{2,\mathbf{C}} \le 4$ from the same sources. Actually, $E_{2,\mathbf{C}} = 4$ (see Section 8).

6. QUASIPOLYNOMIALS, TRIGONOMETRIC SUMS, AND E-NUMBERS

A complex function f on $\mathbf{N}$ is called a *quasipolynomial* if it has a form

$$f(k) = \sum_{\lambda\in\Lambda} p_\lambda(k)\lambda^k + f_0(k) \qquad (k \in \mathbf{N}), \tag{6.1}$$

where Λ is a finite subset of $\mathbf{C}^* = \mathbf{C}\setminus\{0\}$ (Λ may be empty; $\sum_{\lambda\in\emptyset} \equiv 0$), p_λ are polynomials of k, and f_0 is a *finite function*, i.e., there exists m_0 such that $f_0(k) = 0$ for $k \geq m_0$. By this definition the quasipolynomials are just linear combinations of the set of *elementary* scalar quasipolynomials $e_{\lambda,r}(k) = k^r\lambda^k$ ($\lambda \in \mathbf{C}^*$, $r \in \mathbf{N}$) and $e_{0,r}(k) = \delta_r(k) = 0$ for $k \neq r$ and $= 1$ for $k = r$.

Let us denote by T the operator of the left shift in the space of all complex functions on $\mathbf{N}$: $(Tg)(k) = g(k+1)$. Obviously, if $\lambda \neq 0$ then $(T - \lambda I)e_{\lambda,0} = 0$ and

$$(T - \lambda I)e_{\lambda,r} = \{(k+1)^r - k^r\}\lambda^{k+1} \in \operatorname{Lin}(e_{\lambda,s})_{s=0}^{r-1}.$$

In the case $\lambda = 0$ we have $Te_{0,r} = e_{0,r-1}$ for $r \geq 1$. Indeed, $\delta_r(k+1) = \delta_{r-1}(k)$. Thus, for every $\lambda \in \mathbf{C}$ the quasipolynomial $e_{\lambda,0}$ is an eigenfunction corresponding to the eigenvalue λ and the sequence $(e_{\lambda,r})_{r\in\mathbf{N}}$ is a corresponding Jordan chain, so

$$(T - \lambda I)^{r+1}e_{\lambda,r} = 0. \tag{6.2}$$

By virtue of a well-known fact from linear algebra we obtain the following

PROPOSITION 6.1. *The set of elementary quasipolynomials is linearly independent.*

In other words, this set is a basis of the linear space of all quasipolynomials. Therefore, in (6.1) the following things are uniquely determined under the condition $p_\lambda \neq 0$ for all of $\lambda \in \Lambda$: the set Λ, the system of the polynomials $\{p_\lambda\}_{\lambda\in\Lambda}$, and the finite function f_0. The set Λ is called the *spectrum* of the quasipolynomial f if $f_0 = 0$; otherwise the *spectrum* is $\Lambda \cup \{0\}$. In either case we use the notation $\operatorname{spec} f$. Obviously, $\operatorname{spec} f = \emptyset \Leftrightarrow f = 0$.

Now let us put $m_\lambda = \deg p_\lambda + 1$ for $\lambda \in \operatorname{spec} f$ and require m_0 to be minimal. The number m_λ is called the *order of the point* $\lambda \in \operatorname{spec} f$ and is denoted by $\operatorname{ord}\lambda$. The sum $m = m_0 + \sum_{\lambda\in\Lambda} m_\lambda$ is called the *order of the* *quasipolynomial* f and is denoted by $\mathrm{ord} f$. Thus,

$$\mathrm{ord} f = \sum_{\lambda \in \mathrm{spec} f} \mathrm{ord}\lambda.$$

The complex polynomial

$$\chi_f(\zeta) = \zeta^{m_0} \prod_{\lambda \in \mathrm{spec} f} (\zeta - \lambda)^{m_\lambda} = \sum_{j=m_0}^{m} \alpha_j \zeta^\lambda \qquad (\alpha_m = 1) \tag{6.3}$$

is called the *characteristic for* f. Its role is determined by the following

THEOREM 6.2. *Every quasipolynomial* f *satisfies the corresponding linear difference equation of the order* m:

$$\sum_{j=m_0}^{m} \alpha_j f(k+j) = 0 \qquad (k \in \mathbf{N}). \tag{6.4}$$

Proof. This equation can be written in the form $\chi_f(T)f = 0$, or in more detail,

$$T^{m_0} \prod_{\lambda \in \mathrm{spec} f} (T - \lambda I)^{m_\lambda} f = 0.$$

Rewrite (6.1) as

$$f = \sum_{\lambda \in \mathrm{spec} f} \sum_{r=0}^{m_\lambda - 1} c_{\lambda,r} e_{\lambda,r},$$

where $c_{\lambda,r}$ are constant complex coefficients. It remains to apply (6.2). ■

As an obvious consequence we obtain the following uniqueness theorem, which is the main tool for getting our upper estimates of E-numbers.

THEOREM 6.3. *If the quasipolynomial* f *is equal to zero on the segment* $0 \leq k \leq m-1$ *where* $m = \mathrm{ord} f$, *then* $f = 0$, *i.e.*, $f(k) = 0$ *for all* $k \in \mathbf{N}$.

The theory can be very much simplified by restriction to quasipolynomials whose spectra are *simple* in the following sense: all m_λ are equal to 1. The general form of a quasipolynomial with simple spectrum is

$$f(k) = \sum_{\lambda \in \Lambda} a_\lambda \lambda^k + \delta_0(k)$$

where a_λ are complex coefficients. In this situation everything follows directly because the corresponding Vandermond determinant is not zero.

If the spectrum of a quasipolynomial f is simple and unitary and ord$f = n$, then f can be written as a trigonometric sum

$$f(k) = \sum_{j=1}^{n} a_j e^{2\pi i \theta_j k}$$

with pairwise distinct exponents θ_j from the interval $[0, 1]$ and nonzero coefficients a_j. Let us consider the function

$$h(k) = |f(k)|^2 - 1 = \left(\sum_{j=1}^{n} |a_j|^2 - 1 \right) + \sum_{j \neq l} a_j \bar{a}_l e^{2\pi(\theta_j - \theta_l)k}.$$

It is also a quasipolynomial, and ord$h \leq n^2 - n + 1$. By the uniqueness theorem we obtain the following

THEOREM 6.4. *If f is a trigonometric sum of order n and $|f(k)| = 1$ for $0 \leq k \leq n^2 - n$, then $|f(k)| = 1$ for all $k \in \mathbf{N}$.*

Now let us pass from quasipolynomials to E-numbers. The following general lemma is a bridge on the way.

LEMMA 6.5. *Let A, B be arbitrary operators in finite-dimensional complex linear spaces X, Y respectively. Let a scalar product (x, y) on $X \times Y$ be given. Then for every fixed pair of vectors x, y the function*

$$g(k) = (A^k x, B^k y) \qquad (k \in \mathbf{N})$$

is a quasipolynomial of k, and

$$\operatorname{spec} g \subset \{\zeta : \zeta = \lambda\bar{\mu} \ (\lambda \in \operatorname{spec} A, \ \mu \in \operatorname{spec} B)\}. \tag{6.5}$$

Proof. By Jordan's theorem

$$A^k = \sum_{\lambda \in \operatorname{spec}^* A} P_\lambda(k)\lambda^k + A_0^k, \tag{6.6}$$

where $\operatorname{spec}^* A = \operatorname{spec} A \setminus \{0\}$, $P_\lambda(k)$ are polynomials of k with operator coefficients, and A_0 is the nilpotent component of A. Similarly,

$$B^k = \sum_{\mu \in \operatorname{spec}^* B} Q_\mu(k)\mu^k + B_0^k. \tag{6.7}$$

Therefore,

$$g(k) = \sum_{\lambda,\, \mu} (P_\lambda(k)x,\, Q_\mu(k)y)(\lambda\overline{\mu})^k + g_0(k) \tag{6.8}$$

where $g_0(k)$ is a finite function. ■

As the first application we can show that the equality $E(X,\, A,\, x) = 0$ in the Euclidean space yields more information than in a general normed one (cf. Lemma 2.24).

THEOREM 6.6. *If $E(X,\, A,\, x) = 0$ for an operator A and a vector $x \neq 0$ in a Euclidean space X, then the nonzero part of* spec$^*A_x$ *is unitary.*

Proof may be restricted to the complex case. Moreover, one can assume that $X_{A,\,x} = X$, i.e., the vector x is cyclic.

By Lemma 2.24 the space X is a direct sum of two invariant subspaces X_0, X_1 such that the spectrum of $A_0 = A|X_0$ lies inside of the unit disk and spectrum of $A_1 = A|X_1$ is unitary. We are going to prove that A_0 is a nilpotent.

Let us rewrite the condition $\|A^k x\| = 1$ using the decomposition $x = x_0 + x_1$ with $x_0 \in X_0$ and $x_1 \in X_1$. Namely,

$$1 - \|A_1^k x_1\|^2 = \|A_0^k x_0\|^2 + (A_0^k x_0,\, A_1^k x_1) + (A_1^k x_1,\, A_0^k x_0) \qquad (k \in \mathbf{N}).$$

By Lemma 6.5 both sides are quasipolynomials of k, and on the left the spectrum is unitary, while on the right it lies inside the unit disk. Therefore, both sides are zero; in particular,

$$\|A_0^k x_0\|^2 = -(A_0^k x_0,\, A_1^k x_1) - (A_1^k x_1,\, A_0^k x_0) \qquad (k \in \mathbf{N}). \tag{6.9}$$

Suppose that A_0 is not a nilpotent. Let $\rho_0 < 1$ be the minimal modulus of its nonzero eigenvalues. Then A_0 is a direct sum of three operators: $\rho_0 U$ with an unitary U, R with spectrum lying in the domain $D(\rho_0) = \{\lambda : |\lambda| > \rho_0\}$, and a nilpotent N (the last two summands may be absent). Accordingly, if k is so big that $N^k = 0$, then

$$\|A_0^k x_0\|^2 = c\rho_0^{2k} + f(k) \tag{6.10}$$

where $c = \|u\|^2$, u is the projection of x onto the subspace supporting U, and $f(k)$ is a quasipolynomial whose spectrum lies in $D(\rho_0^2)$ by Lemma 6.5. Similarly, the spectrum of the right side of the identity (6.9) lies in the domain $D(\rho_0) \subset D(\rho_0^2)$. It follows from (6.10) that $c = 0$ and then $u = 0$; but that is impossible, since the vector x is cyclic. ∎

In Theorem 6.6 the point 0 may be in specA_x.

EXAMPLE 6.7. Let $\dim X = 2$, and A be a nonorthogonal projection admitting a pair of normed vectors $x_0 \in \mathrm{Ker}A$, $x_1 \in \mathrm{Im}A$ such that $(x_0, x_1) = -\frac{1}{2}$. In this case the vector $x = x_0 + x_1$ is normed and cyclic, and $\|A^k x\| = 1$ for all of k.

One can ask the question: if a nonsingular operator A has a cyclic vector x such that $E(X, A, x) = 0$, must A be unitary? The answer is negative.

EXAMPLE 6.8. Let $\dim X = 3$, and $\{e_1, e_2, e_3\}$ be normalized linearly independent vectors with $(e_1, e_3) = 0$, $(e_1, e_2) = -(e_2, e_3) \neq 0$. Take them as eigenvectors of an operator A corresponding to the eigenvalues $1, \omega, \overline{\omega}$, where $\omega^3 = 1, \omega \neq 1$. This operator is not unitary, since e_1, e_2 are not orthogonal. But it is easy to check that the vector $x = e_1 + e_2 + e_3$ is cyclic and $E(X, A, x) = 0$.

In contrast to this example we have

THEOREM 6.9. *Let $E(X, A, x) = 0$ for a nonsingular operator A and a vector $x \neq 0$ in a complex Euclidean space X. Let* spec $A_x = \{\lambda_1, \dots, \lambda_d\}$. *If the quotients λ_j/λ_l $(j \neq l)$ are pairwise distinct, then A_x is unitary.*

Proof. Theorem 6.6 guarantees that all $|\lambda_j| = 1$, so $\lambda_j/\lambda_l = \lambda_j\overline{\lambda}_l$. Notice that the eigenspaces of the operator A_x are one-dimensional, since it is cyclic. It remains to prove that they are mutually orthogonal. The considered operator is diagonalizable, since its powers are bounded. Therefore, $x = \sum_{j=1}^d x_j$, where x_j are corresponding eigenvectors. They are not zero, since x is cyclic for A_x. Now the conclusion $(x_j, x_l) = 0\,(j \neq l)$ follows from the identity

$$\sum_{j,\,l=1}^{d} (x_j, x_l)(\lambda_j\overline{\lambda}_l)^k = 1 \qquad (k \in \mathbf{N}).$$

∎

COROLLARY 6.10. *If A is a nonsingular operator and the unitary points of its spectrum have pairwise distinct quotients, then the set $\{x : E(X, A, x) = 0\}$ coincides with the union of all invariant subspaces L such that $A|L$ is unitary.*

Some important future constructions are based on the existence of such a system of points.

PROPOSITION 6.11. *Let $\lambda_j = \exp(2^j\theta_i)$ for $1 \le j \le n$ and $0 < \theta < 2^{-n}\pi$. Then the numbers $\lambda_j\overline{\lambda}_l$ $(j \ne l)$ are pairwise distinct (and, obviously, they are different from 1).*

Proof. The equality $\omega_{j_1, l_1} = \omega_{j_2, l_2}$ means that $2^{j_1} + 2^{l_2} = 2^{j_2} + 2^{l_1}$. By the uniqueness of binary decomposition the latter equality is possible only if $j_1 = j_2$, $l_1 = l_2$, since $j_1 \ne l_1$ and $j_2 \ne l_2$. ∎

7. UPPER BOUNDS OF E-NUMBERS OF OPERATORS IN COMPLEX EUCLIDEAN SPACE

Let us consider an arbitrary operator A in the n-dimensional complex Euclidean space X. Applying Lemma 6.5, we obtain

LEMMA 7.1. *For every vector $x \ne 0$ the function*

$$h_{A,x}(k) = \|A^k x\|^2 - \|x\|^2$$

is a quasipolynomial of k, and

$$\begin{aligned} \operatorname{spec} h_{A,x} \subset \{\zeta : \zeta = \lambda\overline{\mu}\,(\lambda,\, \mu \in \operatorname{spec} A,\ \lambda \ne \mu)\} \\ \cup\{\zeta : \zeta = |\lambda|^2\,(\lambda \in \operatorname{spec} A)\} \cup \{1\}. \end{aligned}$$

The summand 1 comes from the constant term $(-\|x\|^2)$.

There is a tight connection between $E(X, A, x)$ and $\operatorname{ord} h_{A,x}$ based on the same argumentation as for trigonometric sums in Section 6.

LEMMA 7.2. $E(X, A, x) \le \operatorname{ord} h_{A,x} - 1$.

Proof. Let $q = \operatorname{ord} h_{A,x} - 1$. By the uniqueness theorem, if $h_{A,x}(k) = 0$ for $0 \le k \le q$ then $h_{A,x} = 0$. In other words, if $\|A^k x\| = \|x\|$ for

$0 \leq k \leq q$, then the same equality takes place for $k \in \mathbf{N}$. It means that $E(X, A, x) \leq q$. ∎

Now we list the spectral parameters of the operator A which will be used in our estimations below. Let Λ be an arbitrary subset of the complex plane $\mathbf{C}$. We write Λ^* for $\Lambda \backslash \{0\}$ as above, and moreover let $\Lambda' = \{\lambda : \lambda \in \Lambda,\, \lambda \neq 0,\, |\lambda| \neq 1\}$ and $|\Lambda| = \{\rho : \rho = |\lambda|,\, \lambda \in \Lambda\}$. We put

$$s = \operatorname{card}(\operatorname{spec}^* A), \qquad t = \operatorname{card}|\operatorname{spec}' A|, \qquad t^* = \operatorname{card}|\operatorname{spec}^* A|. \tag{7.1}$$

Thus, s is the number of nonzero points of spec$A$, t^* is the number of distinct moduli of these points, and t is the number of these moduli other than 1. There is a relationship $t^* = t - \nu + 1$, where $\nu = 1$ if spec$A$ does not contain any unitary point and $\nu = 0$ in the opposite case. We denote by m_λ the maximal order of a Jordan block at a point $\lambda \in \operatorname{spec}A$, so that m_0 is the order of the nilpotent component A_0 of the operator A. Finally, let A_1 be the maximal nonsingular part of the operator A, so that A is a direct sum of A_0 and A_1. The last parameter we need is $r = \operatorname{rank}A_1$. Obviously, $0 \leq t \leq s \leq r \leq n - m_0$.

THEOREM 7.3. *For every operator A*

$$E(X, A) \leq m_0 + (2r - 1)s - s^2 + t. \tag{7.2}$$

Proof. We obtain a corresponding estimate of $\operatorname{ord} h_{A,x}$. It follows from (6.6) that

$$\begin{aligned} h_{A,x}(k) &= \sum_{\lambda,\, \mu \in \operatorname{spec}^* A,\, \lambda \neq \mu} (P_\lambda(k)x,\, P_\mu(k)x)(\lambda\bar{\mu})^k \\ &\quad + \sum_{\rho \in |\operatorname{spec}^* A|} \rho^{2k} \sum_{\lambda : |\lambda| = \rho} \|P_\lambda(k)x\|^2 - \|x\|^2 + f_0(k), \end{aligned} \tag{7.3}$$

where f_0 is a finite function vanishing for k such that $A_0^k x = 0$, so $\operatorname{ord} f_0 \leq m_0$. Then

$$\begin{aligned} \operatorname{ord} h_{A,x} &\leq m_0 + \sum_{\lambda,\, \mu \in \operatorname{spec}^* A,\, \lambda \neq \mu} (\deg P_\lambda + \deg P_\mu + 1) \\ &\quad + \sum_{\rho \in |\operatorname{spec}^* A|} \max_{|\lambda| = \rho} (2 \deg P_\lambda + 1) + \nu. \end{aligned}$$

Since $m_\lambda = \deg P_\lambda + 1$, we get

$$\begin{aligned}\operatorname{ord} h_{A,x} \;&\le\; m_0 + \sum_{\lambda,\,\mu\in\operatorname{spec}^* A,\,\lambda\neq\mu} (m_\lambda + m_\mu - 1)\\ &\quad + \sum_{\rho\in|\operatorname{spec}^* A|} \max_{|\lambda|=\rho}(2m_\lambda - 1) + \nu,\end{aligned}$$

whence

$$\begin{aligned}\operatorname{ord} h_{A,x} \;&\le\; m_0 + \left\{2(s-1)\sum_{\lambda\in\operatorname{spec}^* A} m_\lambda - s(s-1)\right\}\\ &\quad + \left\{2\sum_{\rho\in|\operatorname{spec}^* A|}\max_{|\lambda|=\rho} m_\lambda - t^*\right\} + \nu.\end{aligned}$$

Obviously,

$$\sum_{\lambda\in\operatorname{spec}^* A} m_\lambda \le r.$$

Now for every $\rho \in |\operatorname{spec}^* A|$ we can choose $\lambda(\rho) \in \operatorname{spec}^* A$ such that $m_{\lambda(\rho)} = \max_{|\lambda|=\rho} m_\lambda$. The function $\lambda(\rho)$ is injective; hence

$$\sum_{\rho\in|\operatorname{spec}^* A|}\max_{|\lambda|=\rho} m_\lambda = \sum_{\rho\in|\operatorname{spec}^* A|} m_{\lambda(\rho)} \le r - \sum_{\lambda\in R} m_\lambda$$

where $R = \operatorname{spec}^* A\setminus\{\zeta : \zeta = \lambda(\rho),\ \rho\in|\operatorname{spec}^* A|\}$. Since

$$\sum_{\lambda\in R} m_\lambda \ge \operatorname{card} R = s - t^*,$$

we obtain

$$\sum_{\rho\in|\operatorname{spec}^* A|}\max_{|\lambda|=\rho} m_\lambda \le r - s + t^*.$$

As a result

$$\begin{aligned}\operatorname{ord} h_{A,x} \;&\le\; m_0 + (2r-1)s - s^2 + t^* + \nu\\ &=\; m_0 + (2r-1)s - s^2 + t + 1.\end{aligned}$$

■

The above four-parameter inequality has many consequences. The following bound depends only on the parameters n, s.

COROLLARY 7.4. *If A is not a nilpotent operator, then*

$$E(X, A) \leq 2ns - s^2. \tag{7.4}$$

Proof. Applying the inequalities $t \leq s$ and $r \leq n - m_0$ to (7.2), we get

$$E(X, A) \leq 2ns - s^2 - (2s - 1)m_0 \leq 2ns - s^2,$$

since $s \geq 1$ if A is not nilpotent. ■

REMARK 7.5. If A is a nilpotent operator, then $t = s = r = 0$ in (7.2) and $E(X, A) \leq m_0$. But we know that already (Corollary 2.12).

Now let us maximize the bound (7.2) with respect to s, $s \leq r$. The maximum is achieved at $s = r$ (and at $s = r - 1$ as well). It yields

$$E(X, A) \leq m_0 + r^2 - r + t. \tag{7.5}$$

Since $m_0 \leq n - r$ and $r \leq n$, we obtain the following important result.

COROLLARY 7.6. *For every operator A*

$$E(X, A) \leq n^2 - n + t. \tag{7.6}$$

It is useful to formulate (7.6) in the cases $t = 0, 1, 2$ separately, since these special estimates are exact (see Section 8).

COROLLARY 7.7. *If the nonzero spectrum of A is unitary, then*

$$E(X, A) \leq n^2 - n. \tag{7.7}$$

COROLLARY 7.8. *If the nonzero nonunitary spectrum of an operator A lies on a circle $|\lambda| = c$, then*

$$E(X, A) \leq n^2 - n + 1. \tag{7.8}$$

COROLLARY 7.9. *If the nonzero nonunitary spectrum of an operator A lies on the union of two circles $|\lambda| = a$, $|\lambda| = b$, then*

$$E(X, A) \leq n^2 - n + 2. \tag{7.9}$$

Concluding this chain of corollaries, we take $t \le n-1$ in (7.6).

COROLLARY 7.10. *If the spectrum of an operator A contains zero or a unitary point, then*

$$E(X,\, A) \le n^2 - 1 \tag{7.10}$$

REMARK 7.11. In all the above estimates the value $n = \dim X$ can be replaced by the degree m of the minimal annihilated polynomial of A. Indeed, $E(X,\, A) = E(X_{A,\,x},\, A_x)$ and $\dim X_{A,\,x} \le m$ if x is an A-optimal vector.

The estimate (7.4) directly implies that $E_{n,\,\mathbf{C}} \le n^2$, which was firstly proved in Section 5. This "global" bound can be sharpened except in the case $n = 2$.

THEOREM 7.12. *If $n \ge 3$ then*

$$E_{n,\,\mathbf{C}} \le n^2 - 1. \tag{7.11}$$

In the next section we establish that in the case $n = 3$ this estimate is achieved.

Proof. Let $E_{n,\,\mathbf{C}} = n^2$, and A be an optimal operator. Then the left side of (7.6) is n^2, whence $t = n$ and then $s = r = n$. This means that $\mathrm{spec}A = \{\lambda_1,\, \dots,\, \lambda_n\}$, where $|\lambda_1| > \cdots > |\lambda_n| > 0$ and all the moduli are different from 1. Because the operator A is diagonalizable, we can decompose an A-optimal normed vector x as $x = \sum_{j=1}^{n} x_j$, where x_j is an eigenvector corresponding to the eigenvalue λ_j. By Corollary 2.23 the vector x must be cyclic; hence all $x_j \ne 0$. Taking the squares of norms of the vectors

$$A^k x = \sum_{j=1}^{n} \lambda_j^k x_j \qquad (k \in \mathbf{N}),$$

we get

$$\sum_{j=1}^{n} a_j^k g_{jj} + \sum_{j \ne l} (\lambda_j \overline{\lambda}_l)^k g_{jl} = 1 \qquad (k \in \mathbf{N}), \tag{7.12}$$

where $a_j = |\lambda_j|^2$ and $g_{jl} = (x_j,\, x_l)$ for all $j,\, l$. We show that if $n \ge 3$, then the inequalities $g_{jj} > 0$ contradict the previous system of linear equations with n^2 unknowns (g_{jl}).

Let us restrict this system to $k \leq n^2 - 1$. Then it has the unique solution (g_{jl}), since its determinant is Vandermond determined by the pairwise distinct numbers $\lambda_j \overline{\lambda}_l$ $(j,\, l = 1,\, 2,\, \ldots,\, n)$. The latter property is guaranteed, since otherwise $\operatorname{ord} h_{A,\,x} < n^2 + 1$ and $E_{n,\,\mathbf{C}} = E(X,\, A,\, x) < n^2$ by Lemma 7.2.

Using Cramer's rule we obtain, after cancellation of common factors in the corresponding Vandermond determinants,

$$\frac{g_{22}}{g_{11}} = -\frac{1 - a_1}{1 - a_2}, \qquad \frac{g_{33}}{g_{11}} = -\frac{1 - a_2}{1 - a_3}.$$

However, these fractions are both positive, so $a_2 < 1 < a_1$ and $a_3 < 1 < a_2$ at the same time. ■

8. LOWER BOUNDS OF E-NUMBERS OF OPERATORS IN COMPLEX EUCLIDEAN SPACES

First of all we prove that the bound (7.7) is exact.

THEOREM 8.1. *There exists an operator A with unitary spectrum and*

$$E(X,\, A) = n^2 - n. \tag{8.1}$$

Thus, $E(X, \mathcal{U}_0) = n^2 - n$, where $\mathcal{U}_0$ means the set of operators with unitary nonzero spectrum, i.e., it is just the case $t = 0$.

Proof. Let us consider the system of equations [cf. (7.12)]

$$\sum_{j=1}^{n} g_{jj} + \sum_{j \neq l} (\lambda_j \overline{\lambda}_l)^k g_{jl} = 1 \qquad (0 \leq k \leq n^2 - n - 1), \tag{8.2}$$

$$\sum_{j=1}^{n} g_{jj} + \sum_{j \neq l} (\lambda_j \overline{\lambda}_l)^{n^2 - n} g_{jl} = 1 + \epsilon, \tag{8.3}$$

where $\epsilon > 0$ is small enough and $\lambda_1,\, \ldots,\, \lambda_n$ are chosen on the unit circle according to Proposition 6.11. Then this linear system with $n^2 - n + 1$ unknowns $g_{jl}\,(j \neq l)$, $\gamma = \sum_{j=1}^{n} g_{jj}$, has a unique solution. The corresponding matrix (g_{jl}) with $g_{jj} = \gamma/n$ is Hermitian, since $(\lambda_j, \overline{\lambda}_l)$ is so. In the limit case $\epsilon = 0$ the solution is obvious: $\gamma = 1$, $(g_{jl}) = 0\,(j \neq l)$.

Therefore, if $\epsilon > 0$ is small enough, then $\gamma - 1$ and $(g_{jl})\,(j \neq l)$ are small, so that the matrix (g_{jl}) is positive definite.

Now a required operator A appears with $\operatorname{spec} A = \{\lambda_1, \ldots, \lambda_n\}$ and the corresponding eigenvectors $\{x_1, \ldots, x_n\}$ whose Gram matrix is (g_{jl}). The normed vector $x = \sum_{j=1}^{n} x_j$ is A-optimal. ∎

Developing the above "technique of small perturbations," we also establish that the bound (7.6) is also exact for $t = 1, 2$. In other words, if $\mathcal{U}_t$ is the set of operators with prescribed value of the parameter t, then for $t \leq 2$ we have the equality $E(X, \mathcal{U}_t) = n^2 - n + t$.

In the proofs below $\lambda_1, \ldots, \lambda_n$ will be the same as before.

THEOREM 8.2. *There exists an operator A with spectrum on a circle $|\lambda| = c$ $(c \neq 1)$ and*

$$E(X,\, A) = n^2 - n + 1. \tag{8.4}$$

Proof. Now we consider the system

$$c^{2k}\left(\sum_{j=1}^{n} g_{jj} + \sum_{j\neq l} (\lambda_j \overline{\lambda}_l)^k g_{jl}\right) = 1 \qquad (0 \leq k \leq n^2 - n). \tag{8.5}$$

The unique solution (g_{jl}) is Hermitian and positive definite if c is close to 1. The equality $E(X,\, A,\, x) = 0$ is impossible for the corresponding pair $A,\, x$, since the operator A has no unitary eigenvalues. ∎

THEOREM 8.3. *There exists an operator A with spectrum on the union of two circles $|\lambda| = a$, $|\lambda| = b\,(a \neq 1,\, b \neq 1,\, a \neq b)$ and*

$$E(X,\, A) = n^2 - n + 2. \tag{8.6}$$

Proof. Let us put $\operatorname{spec} A = \{\mu_1, \ldots, \mu_n\}$, where $\mu_1 = \sqrt{1 + 2\epsilon}\lambda_1$, $\mu_j = \sqrt{1 - \epsilon}\lambda_j$ for $j \geq 2$. There are just $n^2 - n + 2$ unknowns g_{11}, $\gamma_1 = \sum_{j=2}^{n} g_{jj}$, g_{jl} $(j \neq l)$ in the system

$$(1 + 2\epsilon)^k g_{11} + (1 - \epsilon)^k \sum_{j=2}^{n} g_{jj} + \sum_{j \neq l} (\mu_j \overline{\mu}_l)^k g_{jl} = 1 \qquad (0 \leq k \leq n^2 - n + 1) \tag{8.7}$$

whose determinant is non-zero Vandermond as before. By Cramer's rule,

$$g_{11} = \frac{1}{3}\prod_{j\neq l}\frac{1-\mu_j\overline{\mu}_l}{1+2\epsilon-\mu_j\overline{\mu}_l} = \frac{1}{3}\prod_{j<l}\left|\frac{1-\mu_j\overline{\mu}_l}{1+2\epsilon-\mu_j\overline{\mu}_l}\right|^2 > 0$$

and

$$\gamma_1 = \frac{2}{3}\prod_{j\neq l}\frac{1-\mu_j\overline{\mu}_l}{1-\epsilon-\mu_j\overline{\mu}_l} = \frac{1}{3}\prod_{j<l}\left|\frac{1-\mu_j\overline{\mu}_l}{1-\epsilon-\mu_j\overline{\mu}_l}\right|^2 > 0.$$

Moreover, we see that g_{11} and γ_1 tend to $\frac{1}{3}$ and $\frac{2}{3}$ respectively as ϵ tends to 0. The limit system restricted to $0 \leq k \leq n^2 - n - 1$ with respect to unknowns g_{jl} $(j \neq l)$ has only the trivial solution, all the $g_{jl} = 0$. This implies that g_{jl} $(j \neq l)$ from the system (8.7 are small. Putting $g_{jj} = \gamma_1/n-1$ for $j \geq 2$, we get a Hermitian positive definite matrix (g_{jl}) as required. ∎

As a result we have the following lower bound.

COROLLARY 8.4. $E_{n,\mathbf{C}} \geq n^2 - n + 2$.

Combination of this inequality with (5.6) and (7.11) yields

COROLLARY 8.5. $E_{2,\mathbf{C}} = 4$, $E_{3,\mathbf{C}} = 8$.

9. OPERATORS WITH SMALL E-NUMBERS

Certainly, there are many situations when the general Theorem 7.3 gives us only a very rough estimate. A remarkable example is the following.

THEOREM 9.1. *If A is a normal operator in a complex Euclidean space, then*

$$E(X, A) \leq 2. \tag{9.1}$$

The equality is achieved if and only if A is weakly hyperbolic.

Proof. We use the orthogonal decomposition

$$A = \sum_{\rho\in|\mathrm{spec}A|} \rho U(\rho),$$

where every operator $U(\rho)$ is unitary and concentrated on the corresponding spectral subspace $X(\rho)$: $U(0) = \mathrm{I}$ for definiteness. Respectively, for an arbitrary normed vector x

$$\|A^k x\|^2 = \sum_{\rho\in|\mathrm{spec}A|} p(x,\,\rho)\rho^{2k} \qquad (k \in \mathbf{N}) \tag{9.2}$$

where $p(x,\,\rho) = \|x(\rho)\|^2$ and $x(\rho)$ is the projection of x on the subspace $X(\rho)$. (We put $0^0 = 1$ for the case $\rho = 0,\, k = 0$). Obviously, $p(x,\,\rho) \geq 0$ and $\sum_\rho p(x,\,\rho) = 1$, i.e., $\{p(x,\,\rho)\}$ is a normed weight. Moreover, every normed weight $\{\pi(\rho)\}$ may appear there. Taking such a weight, we consider the function

$$f_\pi(\tau) = \sum_{\rho\in|\mathrm{spec}A|} \pi(\rho)\rho^\tau$$

on the whole real semiaxis $\tau \geq 0$. This function is convex for $\tau > 0$, and

$$f_\pi(+0) = \sum_{\rho\in|\mathrm{spec}^* A|} \pi(\rho) \leq \sum_{\rho\in|\mathrm{spec}A|} \pi(\rho) = f_\pi(0).$$

Therefore, f_π takes each its value no more than twice. For the choice $\pi(\rho) = p(x,\,\rho)$, this function interpolates the sequence (9.2) so that $f_\pi(2k) = \|A^k x\|^2$ $(k \in \mathbf{N})$. Thus, $E(X,\,A,\,x) \leq 2$ and then $E(X,\,A) \leq 2$.

This bound is achieved if and only if there exists a normed weight $\{\pi(\rho)\}$ such that $f_\pi(0) = f_\pi(2) \neq f_\pi(4)$. It is so in just two cases: (1) $f_\pi(\tau)$ is nonmonotone for $\tau > 0$; (2) $f_\pi(\tau)$ is increasing for $\tau > 0$, and $0 \in \mathrm{spec}A$. Finally, this means that the family of exponential functions $\{\rho^\tau : \rho \in |\mathrm{spec}A|\}$ contains an increasing member (i.e. such that $\rho > 1$) jointly with a decreasing one (i.e. such that $0 < \rho < 1$) or 0^τ. ■

REMARK 9.2. The second part of Theorem 9.1 can be also proved in the following way. Combining the estimate (9.1) with the general Theorem 2.31, we see that $E(X,\,A) = 2$ if A is normal weakly hyperbolic. On the other hand, if a normal operator A is not weakly hyperbolic, then it is a contraction or dilation. In this case $E(X,\,A) = 0$ if A is unitary, and $E(X,\,A) = 1$ otherwise.

REMARK 9.3. Theorem 9.1 can be extended to the real case using complexification.

In conclusion we estimate the E-numbers of operators with a given annihilating trinomial $\lambda^m - \alpha\lambda - \beta,\ m \geq 2$. We denote this class of operators in a real n-dimensional Euclidean space X by $\mathcal{A}_n(m,\,\alpha,\,\beta)$. Let $E_n(m,\,\alpha,\,\beta)$ be the E-number of this class. It is trivial that $E_n(m,\,0,\,\beta) \leq m$ and $E_n(m,\,\alpha,0) \leq m$. Further we assume that $\alpha \neq 0,\, \beta \neq 0$ and put

$\gamma = (1 - \alpha^2 - \beta^2)/2\alpha\beta$, which is just the value of the cosine of the angle between two sides $|\alpha|$, $|\beta|$ in a triangle whose third side is 1. In particular, if $A \in \mathcal{A}_n(m, \alpha, \beta)$ and x is a vector with $\|x\| = \|Ax\| = 1$, then we have such a triangle with the vertices βx, αAx, $A^m x$, so $\|A^m x\| = 1 \Leftrightarrow (Ax, x) = \gamma$. For every $l \geq m+1$ we can use not only x but also the vectors $Ax, \ldots, A^{l-m-1}x$ if $\|A^k x\| = 1$ for $0 \leq k \leq l-m$. Under this condition

$$\begin{aligned} &\|A^k x\| = 1 \quad \text{for } m \leq k \leq l-1 \\ &\quad \Leftrightarrow \quad (A^{k+1}x, A^k x) = \gamma \quad \text{for } 0 \leq k \leq l-m-1. \end{aligned} \tag{9.3}$$

The last system of equalities means that the points $(A^k x)_{k=0}^{l-m}$ are the vertices of a regular broken line on the unit sphere. These geometrical observations lead us to the following upper bound.

THEOREM 9.4. *For any n, m and α, β*

$$E_n(m, \alpha, \beta) \leq 2m. \tag{9.4}$$

Moreover, $E_n(2, \alpha, \beta) \leq 3$.

Proof. Let us suppose that $A \in \mathcal{A}_n(m, \alpha, \beta)$ and $E(X, A) \geq 2m+1$. If x is an A-optimal vector, then (9.3) works for $l = 2m+1$; in particular, $(A^{m+1}x, A^m x) = \gamma$. By substituting $A^m x = \alpha Ax + \beta x$ and $A^{m+1}x = \alpha A^2 x + \beta Ax$ we get $(A^2 x, x) = 2\gamma^2 - 1$. Therefore, the Gram determinant of the vectors x, Ax, A^2x is zero. So these vectors are linearly dependent. Then $\dim X_{A,x} \leq 2$ and, accordingly, the E-number of the operator A_x does not exceed 3. But that number equals $E(X, A)$ because of the optimality of the vector x. We have got a contradiction, since $m \geq 2$.

Now we consider the case $m = 2$ separately. Take an operator $A \in \mathcal{A}_n(2, \alpha, \beta)$ and an A-optimal vector x. Then $E(X, A) = E(X_{A,x}, A_x) \leq 3$, since the vectors x, Ax, A^2x are linearly dependent in the case $m = 2$. ∎

COROLLARY 9.5. *There are no annihilating trinomials of degree less than $\frac{1}{2}E(X, A)$ for any operator A in a real Euclidean space.*

REMARK 9.6. Let $A \in \mathcal{A}_n(m, \alpha, \beta)$ and $E(X, A) \geq m+1$. Applying (9.3) for $l = m+1$ we get $(Ax, x) = \gamma$ for any A-optimal vector x. So $|\gamma| < 1$, since the vectors x, Ax are not collinear. Therefore,

$$||\alpha| - |\beta|| < 1 < |\alpha| + |\beta|. \tag{9.5}$$

Now we establish that Theorem 9.4 is exact.

THEOREM 9.7. *Let* $\alpha > 0$, $\beta > 0$, $\alpha^2 + \beta^2 = 1$. *Then* $E_n(m, \alpha, \beta) = 2m$ *if* $3 \le m \le n$, *and* $E_n(2, \alpha, \beta) = 3$.

Proof. We start with the main case when $m = n$. Let $(e_k)_{k=0}^{n-1}$ be an orthonormal basis in the space X and, as usual, an operator A be defined by the formulas $Ae_k = e_{k+1}$ for $0 \le k \le n-2$ and $Ae_{n-1} = \alpha e_1 + \beta e_0$, so that $A^k e_0 = e_k$ $(0 \le k \le n-1)$ and $A^n e_0 = \alpha A e_0 + \beta e_0$. As e_0 is a cyclic vector, we have eventually $A^n x = \alpha A x + \beta x$ for all of $x \in X$. This means that $A \in \mathcal{A}_n(m, \alpha, \beta)$.

Obviously, $\|A^k e_0\| = 1$ for $0 \le k \le n$, and $(A^{k+1}e_0, A^k e_0) = 0$ for $0 \le k \le n-2$. Moreover, $(A^n e_0, A^{n-1}e_0) = 0$ if $n \ge 3$. In the last case we can use (9.3) for $l = 2n$, $m = n$, and $\gamma = 0$. This yields $\|A^k e_0\| = 1$ for $0 \le k \le 2n-1$. If $n = 2$ then $\|A^k e_0\| = 1$ for $0 \le k \le 2$.

By Theorem 9.4 it remains to show that $E(X, A, e_0) \ne 0$. Otherwise, the equation $\lambda^n = \alpha\lambda + \beta$ has a unitary root, which must be equal to i or $-i$, since $|\alpha\lambda + \beta| = 1$ and our conditions on α, β are satisfied. However, the equality $i^n = \alpha i + \beta$ is also impossible for the real nonzero numbers α, β.

To finish the proof we suppose that $3 \le m < n$ and observe that the equation $\lambda^m = \alpha\lambda + \beta$ has a root $\rho > 0$, since $\alpha + \beta > 1$. Let us decompose the space X into a direct sum of an m-dimensional subspace Y and its complement Z. As we have proved, there exists an operator B in Y annihilated by the trinomial $\lambda^m - \alpha\lambda - \beta$ and such that $E(X, B) = 2m$. On the other hand, the operator ρI in Z is also annihilated by the same trinomial. The direct sum A of these operators belongs to the class $\mathcal{A}_n(m, \alpha, \beta)$, and $E(X, A) = 2m$. ∎

10. TRIGONOMETRIC SUMS OF CONSTANT MODULUS

The following lemma is a key to an understanding of some properties of trigonometric sums under the above-mentioned condition. (Without loss of generality one can assume that the modulus is equal to 1.)

LEMMA 10.1. *Let* $p(\zeta_1, \dots, \zeta_d)$ *be a polynomial of* d *complex variables and its modulus be equal to* 1 *everywhere on the torus* $|\zeta_1| = \dots = |\zeta_d| = 1$. *Then* p *is monomial.*

Proof. In the case $d = 1$ we have a polynomial $p(\zeta)$ in a complex variable ζ. It is proportional to the Blashke product

$$\prod_{k=1}^{m} \frac{\zeta - z_k}{1 - \bar{z}_k \zeta}$$

corresponding to the roots z_k of $p(\zeta)$ inside the unit disk, the multiplicities of the roots being taken into account (see [6, Part 3, # 296]). However, in this case all z_k are zeros, since the polynomial $p(\zeta)$ has no poles. Therefore, $p(\zeta) = a\zeta^m$, $a = \text{const}$.

If now our assertion is true for $d - 1$, then $p(\zeta_1, \dots, \zeta_d) = P(\zeta_1) \zeta_2^{m_2} \cdots \zeta_d^{m_d}$. Since the coefficient $P(\zeta_1)$ also is of modulus 1 on the unit circle, it has the form $a\zeta_1^{m_1}$, $a = \text{const}$. ∎

THEOREM 10.2. *Let*

$$f(k) = \sum_{j=1}^{n} a_j e^{2\pi i \theta_j k}, \quad |f(k)| = 1 \qquad (k \in \mathbf{N})$$

and all $a_j \neq 0$. *Then all differences* $\theta_j - \theta_j'$ *are rational.*

Proof. Let us consider the real field $\mathbf{R}$ as a linear space over the rational field $\mathbf{Q}$. Taking a basis $1, \omega_1, \dots, \omega_d$ in the linear span of the set $\{1, \theta_1, \dots, \theta_n\}$, we get the decompositions

$$q\theta_j = \sum_{l=1}^{d} p_{jl}\omega_l + c_j \tag{10.1}$$

with some integer p_{jl}, c_j, $q\,(q \geq 1)$. Because $|f(kq)| = 1\,(k \in \mathbf{N})$ we obtain

$$\left| \sum_{j=1}^{n} a_j \prod_{l=1}^{d} (e^{2\pi i \omega_l k})^{p_{jl}} \right| = 1.$$

for all of k. By the classic Kronecker theorem one can approximate any given point on the torus $|\zeta_1| = \cdots = |\zeta_d| = 1$ by points of the form $(e^{2\pi i \omega_1 k}, \dots, e^{2\pi i \omega_d k})$. Therefore,

$$\left| \sum_{j=1}^{n} a_j \prod_{l=1}^{d} \zeta_l^{p_{jl}} \right| = 1.$$

By Lemma 10.1 we conclude that p_{jl} does not depend on j. It follows by subtracting one instance of (10.1) from another that $q(\theta_j - \theta_{j'}) = c_j - c_{j'}$. ■

COROLLARY 10.3. *Under the conditions of Theorem* 10.2, $f(k)$ *has a form* $e^{2\pi i\theta k}g(k)$ *where* $g(k)$ *is a periodic trigonometric sum and* $|g(k)| = 1$ $(k \in \mathbf{N})$.

An open problem is to estimate the minimal period of g by a function of n. We have the following conjecture: this period does not exceed Ln, where L is an absolute constant.[2] It is closely related to the following problem.

Let p be a prime number. What is the smallest integer $N = N(p) > 1$ such that there exists a trigonometric sum of constant modulus, period p, and length N? (The *length* of the sum is the number of its nonzero summands.)

At present we do not even know whether $N(p) \geq p^{\epsilon}$ with $\epsilon > 0$.

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[2] *Addendum in proofs.* This conjecture is not true in general (see footnote on page 229.)

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