Arch. Math. Logic (1990) 30:193–199

Archive for
Mathematical
Logic

The undecidability of the second order predicate unification problem

Gilles Amiot

Equipe de Logique, CNRS UA 753, Université Paris 7, UFR de Mathématiques, 2 Place Jussieu, F-75251 Paris Cedex 05, France

Received October 26, 1989/in revised form March 19, 1990

Abstract. We prove that the second order predicate unification problem is undecidable by reducing the second order term unification problem to it.

Introduction

The unification problem for a formal language is the problem of determining whether any two formulas of the language possess a common instance or not. The problem for first order languages has long been known to be decidable (see [R]). On the other hand, for second order languages, which contain function variables of any arity, Goldfarb (see [G]) has shown that the unification problem is undecidable, if the language contains at least a n-ary function constant with $n \geqq 2$, by reducing Hilbert's Tenth Problem to it. Farmer (see [F, Chap. IV]) extend this result by allowing only one place function variables.

Our goal is to extend this result to second order languages without function variables but with predicate variables. We shall consider a simple second order predicate language L_p whose formulas are atomic formulas that may contain both individual and predicate variables, and a predicate constant of arity $\geqq 1$. The unification problem for L_p differs from first order in that, to obtain an instance of a formula of L_p, predicate variables as well as individual variables may be instanciated. Goldfarb and Farmer considered a language L_t whose formulas are terms that may contain both individual and function variables. The unification problem for L_p differs from L_t in that those languages have not the same syntactic properties. Function variables have the following property: you can compose them [if F_1 and F_2 are two one place function variables and t is a term then $F_1(F_2(t))$ is also a term], and this property is essential for the proof of undecidability. For predicate variables this property can be simulated. That's the essential task of the reduction we will describe below (see Sect. 3). We will reduce the unification problem for L_t to the problem for L_p and therefore prove it undecidable.

The reduction is a coding of second order terms of L_t into systems of pairs of atomic formulas of L_p. This encoding is linear in size, and proceeds in one simple bottom-up pass. The idea is rather simple: bring all function variables in front by introducing in their place new first order variables, and then associate them with predicate variables. The original term T is thus coded into a set of pairs $N(T)$ with distinguished variables $y_1, \ldots, y_n$ and all substitution instances of T and the subterms of T are closely related to the instances of $y_1, \ldots, y_n$ by a substitution which unifies $N(T)$.

Farmer (see [F]) gives a reduction from the predicate unification problem to the term unification problem. Using this reduction we can formulate our problem in an equivalent way: we consider only first order terms and terms of the form $F(t)$, with F a unary function variable and t a first order term. So we can see our problem as a restriction of the unification problem for L_t.

If the second order monadic predicate unification problem is undecidable (as we shall see) so is the second order predicate unification problem, we will therefore restrict ourselves to one place variables.

1 Languages

Let $\mathrm{IndVarx} = \{x_i / i \in \mathbb{N}\}$, $\mathrm{IndVary} = \{y^i_{(j,k)} / i \in \mathbb{N}, (j,k) \in \mathbb{N}^2\}$, be two countable sets of individual variables, also $\mathrm{IndVar} = \mathrm{IndVarx} \cup \mathrm{IndVary}$. Let $\mathrm{PredVar} = \{X_i / i \in \mathbb{N}\}$ be a countable set of one place predicate variables. Let Cons be a given set of individual and function constants. Let P be a one place predicate constant and $\mathrm{FnVar} = \{F_i / i \in \mathbb{N}\}$ a countable set of one place function variables.

$L_t = \mathrm{IndVarx} \cup \mathrm{FnVar} \cup \mathrm{Cons}$ is the language used by Farmer. The terms of L_t are defined inductively as usual.

$L_p = \mathrm{IndVar} \cup \mathrm{PredVar} \cup \mathrm{Cons} \cup \{P\}$ is the language we will use. The terms of L_p are defined in the usual way, and the formulas of L_p are defined as follows: if t is a term and R is a one place predicate constant or variable then $R(t)$ is a formula.

The terms of *order 1* for the language L_t are the terms which don't contain any function variable. The other terms of L_t are said to be of *order 2*.

2 Substitutions

To specify the notion of an instance of a one place function or predicate variable we need an additional individual variable W, and to explain the reduction we must expand the language L_t with the set IndVary. Hence we consider expanded languages L_t^* and L_p^*. Let $L_t^* = L_t \cup \mathrm{IndVary} \cup \{W\}$ and $L_p^* = L_p \cup \{W\}$. The terms (respectively formulas) of L_t^* (resp. L_p^*) are defined in the same way as those of L_t (resp. L_p). By extension the terms of L_t^* which don't contain any function variable are said to be of *order 1* and the others to be of *order 2*.

A *substitution* σ for the language L_t^* (resp. L_p^*) is a finite set $\{t_1/v_1, \ldots, t_n/v_n\}$ of pairs such that $v_1, \ldots, v_n$ are distinct variables of L_t^* (resp. L_p^*), and, for each $i \leq n$, if v_i is an individual variable then t_i is a term which doesn't contain the variable W, and if v_i is a one place function (resp. predicate) variable then t_i is a term (resp. formula) of L_t^* (resp. L_p^*). A substitution for L_t (resp. L_p) is defined in the same way except that t_i must belong to $L_t \cup \{W\}$ (resp. $L_p \cup \{W\}$).

The result σs of applying a substitution $\sigma=\{t_1/v_1,\ldots,t_n/v_n\}$ to a term or formula s is inductively defined as follows:

(1) if s is an individual variable and $s=v_i$ for some $i\leqq n$, then $\sigma s=t_i$.

(2) if s is an individual variable or constant not among $v_1,\ldots,v_n$ then $\sigma s=s$.

(3) if $s=A(s')$ where A is a function (resp. predicate) variable and $A=v_i$ for some $i\leqq n$, then $\sigma s=\{\sigma s'/W\}t_i$.

(4) if $s=A(s')$ where A is a function (resp. predicate) variable not among $v_1,\ldots,v_n$ then $\sigma s=A(\sigma s')$.

If τ is an other substitution then the restriction on $\{v_1,\ldots,v_n\}$ of $\tau\circ\sigma$ is the substitution $\{\tau t_1/v_1,\ldots,\tau t_n/v_n\}$, and $\tau\circ\sigma v=\tau v$ for any variable v not among $v_1,\ldots,v_n$.

Note that if s is a term of L_t (resp. formula of L_p) then so is σs for every substitution σ. Notation: if Δ is a finite subset of $\mathbb{N}^3$ and σ is the substitution $\{\beta^j_{(p,q)}/y^j_{(p,q)};\ (j,p,q)\in\Delta\}$, then $[\beta^j_{(p,q)}]$ denotes σ.

An *instance* of a term or a formula s of the language L considered is simply any σs for some substitution σ of L. A substitution is a *unifier* of a (finite) set $\{\langle t^i;u^i\rangle/i\in\{1,\ldots,n\}\}$ of pairs of terms or formulas of L iff for every $i\leqq n$ $\sigma t^i=\sigma u^i$. The *unification problem* for L is the problem of determining, given any set $\{\langle t^i;u^i\rangle/i\in\{1,\ldots,n\}\}$, whether a unifier exists or not.

We have the following fact: if σ unifies $\{\langle t;u\rangle\}$ in L_t, there is a substitution τ which unifies $\{\langle t;u\rangle\}$ such that the terms substituted for the variables don't contain second order variables. To prove that fact, let θ be the substitution which substitutes for every function variable occurring in σt an individual variable x_i which appears anywhere, then $\theta\sigma t=\theta\sigma u$ (because $\sigma t=\sigma u$) and $\tau=\theta\sigma$ is a unifier of $\{\langle t;u\rangle\}$ and has the desired property. The same property holds for any pair of unifiable formulas of L_p.

Example. Let Cons $=\{a,b,f,g\}$ where a and b are individual constants, f a one place function constant, and g a two place function constant. Let $\ell^1=F_1(f(x_1))$ and $\ell^2=g(f(F_2(a)),\ F_2(g(F_3(f(b)),\ F_4(x_1))))$. A unifier for $\{\langle\ell^1;\ell^2\rangle\}$ is $\sigma=\{g(f(f(a)),\ W)/F_1;\ f(W)/F_2;\ a/F_4;\ g(F_3(f(b)),\ a)/x_1\}$; we have $\sigma\ell^1=\sigma\ell^2=g(f(f(a)),f(g(F_3(f(b)),a)))$. Now $\theta=\{x_2/F_3\}$ and we get $\theta\sigma\ell^1=\theta\sigma\ell^2=g(f(f(a)),f(g(x_2,a)))$, and $\tau=\theta\sigma=\{g(f(f(a)),W)/F_1;f(W)/F_2;x_2/F_3;a/F_4;g(x_2,a)/x_1\}$ is a unifier for $\{\langle\ell^1;\ell^2\rangle\}$.

3 Reduction

To explain the reduction we need some preliminary definitions.

Definition 1. Let t be a term of L_t^*. The *basic elements* of t are all the subterms of the form $G(t^*)$ with G a function variable and t^* an order 1 term.

Example. The basic elements of ℓ^2 are: $F_2(a)$, $F_3(f(b))$ and $F_4(x_1)$. The only basic element of ℓ^1 is $\ell^1=F_1(f(x_1))$.

Definition 2. Let t be a term of L_t. We define three sequences $D_n(t)$, $T_n(t)$, $\Delta_n(t)$ as follows: $D_0(t)=\emptyset$, $T_0(t)=t$ and $\Delta_0(t)$ is the set of the basic elements of $T_0(t)$. Suppose $D_k(t)$, $T_k(t)$ and $\Delta_k(t)$ are defined. Then: If $\Delta_k(t)\neq\emptyset$, let $\Delta_k(t)=\{t^j_{(k,1)},\ldots,t^j_{(k,n)}\}$. Then $T_{k+1}(t)$ is the term of L_t^* obtained by replacing each occurrence of $t^j_{(k,i)}$ in $T_k(t)$ by $y^j_{(k,i)}$ for $i\in\{1,\ldots,n\}$. If $\Delta_k(t)=\emptyset$, then $T_{k+1}(t)=T_k(t)$. $D_{k+1}(t)=D_k(t)\cup\Delta_k(t)$, and $\Delta_{k+1}(t)$ is the set of the basic elements of $T_{k+1}(t)$.

Lemma 1. *There is a k such that: $\forall k' \geqq k\ \Delta_{k'}(t)=\emptyset$ (and hence $D_{k'}(t)=D_k(t)$ and $T_{k'}(t) = T_k(t)$).*

Proof. Use the fact that $T_{k+1}(t)$ has strictly less function variables than $T_k(t)$. $\square$

Example. For t^1 we get: $D_0(t^1)=\emptyset$, $T_0(t^1)=t^1$, $\Delta_0(t^1)=\{t^1\}=\{F_1(f(x_1))\}$; $D_1(t^1) =\{t^1\}$, $T_1(t^1)=y^1_{(0,1)}$, $\Delta_1(t^1)=\emptyset$.

For t^2 we get: $D_0(t^2)=\emptyset$, $T_0(t^2)=t^2$, $\Delta_0(t^2)=\{F_2(a), F_3(f(b)), F_4(x_1)\}$; $D_1(t^2) =\Delta_0(t^2)$, $T_1(t^2)=g(f(y^2_{(0,1)}), F_2(g(y^2_{(0,2)}, y^2_{(0,3)})))$, $\Delta_1(t^2)=\{F_2(g(y^2_{(0,2)}, y^2_{(0,3)}))\}$; $D_2(t^2)=\Delta_0(t^2)\cup\Delta_1(t^2)$, $T_2(t^2)=g(f(y^2_{(0,1)}), y^2_{(1,1)})$, $\Delta_2(t^2)=\emptyset$.

Definition 3. Let t be a term of L_t. We define $k(t)$ to be the first natural number k such that $\Delta_k(t)=\emptyset$.

We can now explain the reduction. We will associate with each set $\{\langle t^{2i-1}; t^{2i}\rangle / i\in\{1,\dots,n\}\}$ of pairs of L_t terms a set N of pairs of L_p formulas such that: $\{\langle t^{2i-1}; t^{2i}\rangle / i\in\{1,\dots,n\}\}$ is unifiable in L_t iff N is unifiable in L_p.

Let $t^j, j\in\{1,\dots,2n\}$, be a term of L_p. We associate with t^j a formula ϕ^j and a set N^j.

Let $\phi^j=P(T_{k(t^j)}(t^j))$. Every $t^j_{(i,m)}$ belonging to $D_{k(t^j)}(t^j)$ is associated with a pair of formulas $\langle A^j_{(i,m)}, B^j_{(i,m)}\rangle$: By Defs. 1 and 2, we have $t^j_{(i,m)}=F_h(t^{j*}_{(i,m)})$ for some h and some order 1 term $t^{j*}_{(i,m)}$. We define $A^j_{(i,m)}=X_h(t^{j*}_{(i,m)})$ and $B^j_{(i,m)}=P(y^j_{(i,m)})$.

Let $N^j = \bigcup_{t^j_{(i,m)}\in D_{k(t^j)}(t^j)} \{\langle A^j_{(i,m)}; B^j_{(i,m)}\rangle\}$.

Finally, let $N=\bigcup_{1\leqq i\leqq n} (\{\langle\phi^{2i-1}; \phi^{2i}\rangle\}\cup N^{2i-1}\cup N^{2i})$.

Example. We get: $\phi^1=P(y^1_{(0,1)})$, $N^1=\{\langle X_1(f(x_1)); P(y^1_{(0,1)})\rangle\}$, and $\phi^2 =P(g(f(y^2_{(0,1)}), y^2_{(1,1)}))$; $N^2=\{\langle X_2(a); P(y^2_{(0,1)})\rangle, \langle X_3(f(b)); P(y^2_{(0,2)})\rangle, \langle X_4(x_1); P(y^2_{(0,3)})\rangle, \langle X_2(g(y^2_{(0,2)}, y^2_{(0,3)})); P(y^2_{(1,1)})\rangle\}$. And $N=\{\langle P(y^1_{(0,1)}); P(g(f(y^2_{(0,1)}), y^2_{(1,1)}))\rangle\}\cup N^1\cup N^2$.

4 Undecidability proof

Definition 4. For every $t^j_{(i,m)}$ belonging to $D_{k(t^j)}(t^j)$ we define a term $C^j_{(i,m)}$ of L_t by induction on i: $C^j_{(0,m)}=t^j_{(0,m)}$. For $i>0$, $C^j_{(i,m)}=[C^j_{(p,q)}]t^j_{(i,m)}$.

Note that the definition is correct because any $y^j_{(p,q)}$ occurring in $t^j_{(i,m)}$ verifies $p<i$.

Lemma 2. *For every $k\in\mathbb{N}$, $[C^j_{(p,q)}]T_k(t^j)=t^j$. In particular $[C^j_{(p,q)}]T_{k(t^j)}(t^j)=t^j$.*

Proof. By induction on k. If $k=0$, then $T_0(t^j)=t^j$. The property is obviously true. We suppose the property for k. If $\Delta_k(t^j)=\emptyset$, then $T_{k+1}(t^j)=T_k(t^j)$ and $[C^j_{(p,q)}]T_k(t^j) =t^j$ by induction hypothesis. Hence $[C^j_{(p,q)}]T_{k+1}(t^j)=t^j$. If $\Delta_k(t^j)\neq\emptyset$, let $\Delta_k(t^j) =\{t^j_{(k,1)},\dots,t^j_{(k,n)}\}$ and let $\sigma_k=\{t^j_{(k,1)}/y^j_{(k,1)},\dots,t^j_{(k,n)}/y^j_{(k,n)}\}$. By definition: $\sigma_k(T_{k+1}(t^j)) =T_k(t^j)$. By induction hypothesis: $[C^j_{(p,q)}]T_k(t^j)=t^j$. So: $[C^j_{(p,q)}](\sigma_k T_{k+1}(t^j))=t^j$. By Def. 4, $[C^j_{(p,q)}]t^j_{(k,r)}=C^j_{(k,r)}$. So: $[C^j_{(p,q)}]\circ\sigma_k=[C^j_{(p,q)}]$. Finally $[C^j_{(p,q)}]T_{k+1}(t^j) =t^j$. $\square$

Example. $C^1_{(0,1)}=F_1(f(x_1))$; $C^2_{(0,1)}=F_2(a)$; $C^2_{(0,2)}=F_3(f(b))$; $C^2_{(0,3)}=F_4(x_1)$; $C^2_{(1,1)} =F_2(g(F_3(f(b)), F_4(x_1)))$.

Let $\{\langle t^{2i-1}; t^{2i}\rangle / i \in \{1, \ldots, n\}\}$ be a set of pairs of L_t terms, and let N be defined as above. Let

$$\varepsilon = (\cup\{f_h(W)/F_h\}) \bigcup (\cup\{s_h/x_h\})$$

be any substitution for L_t such that $f_h(W)$ are order 1 terms of $L_t \cup \{W\}$ (i.e. terms of IndVarx$\cup$Cons$\cup\{W\}$). We define

$$\varepsilon' = (\cup\{P(f_h(W))/X_h\}) \bigcup (\cup\{s_h/x_h\}) \bigcup (\cup\{\varepsilon C^j_{(i,m)}/y^j_{(i,m)}\})$$

(ε' is a substitution of L_p). Then we have:

Lemma 3. *For every $j \in \{1, \ldots, 2n\}$, $\varepsilon'\phi^j = P(\varepsilon t^j)$; and for every (i, m), $\varepsilon' A^j_{(i,m)} = \varepsilon' B^j_{(i,m)}$.*

N.B.: For any term t, $f_h(t)$ denotes $\{t/W\} f_h(W)$.

Remark 1. ε and ε' have the same restriction on IndVarx; hence for every order 1 term t of L_t, $\varepsilon' t = \varepsilon t$.

Remark 2. ε' and $\varepsilon \circ [C^j_{(p,q)}]$ have the same restriction on the set IndVar (= IndVarx $\cup$IndVary), by Remark 1 and the definition of ε'. Hence if t is a term of L_p, $\varepsilon' t = \varepsilon \circ [C^j_{(p,q)}] t$.

Proof. Let $j \in \{1, \ldots, 2n\}$. We will prove $\varepsilon'\phi^j = P(\varepsilon t^j)$ by induction on $k(t^j)$. If $k(t^j) = 0$ then $N^j = \emptyset$, $\Delta_0(t^j) = \emptyset$ and t^j is an order 1 term of L_t. So $\phi^j = P(t^j)$ and $\varepsilon'\phi^j = \varepsilon' P(t^j) = P(\varepsilon' t^j) = P(\varepsilon t^j)$ by Remark 1.

If $k(t^j) \neq 0$ then $N^j \neq \emptyset$. We remark that $T_{k(t^j)}(t^j)$ doesn't contain any function variable ($\Delta_{k(t^j)}(t^j) = \emptyset$) and doesn't contain the variable W (by construction), so $T_{k(t^j)}(t^j)$ is also a term of L_p. Then:

$$\varepsilon'\phi^j = P(\varepsilon' T_{k(t^j)}(t^j)) = P(\varepsilon([C^j_{(p,q)}]\,(T_{k(t^j)}(t^j)))) = P(\varepsilon t^j)$$

(Def., Remark 2 and Lemma 2).

For $i < k(t^j)$, we have $\Delta_i(t^j) \neq \emptyset$; and $t^j_{(i,m)} = F_h(t^{j*}_{(i,m)})$ for some h. To prove $\varepsilon' A^j_{(i,m)} = \varepsilon' B^j_{(i,m)}$, we first prove, by induction on i, that $f_h(\varepsilon' t^{j*}_{(i,m)}) = \varepsilon C^j_{(i,m)}$. For $i = 0$, $t^{j*}_{(0,m)}$ is an order 1 term of L_t. So $f_h(\varepsilon' t^{j*}_{(0,m)}) = f_h(\varepsilon t^{j*}_{(0,m)})$, by Remark 1, and

$$f_h(\varepsilon t^{j*}_{(0,m)}) = \varepsilon(F_h(t^{j*}_{(0,m)})) = \varepsilon t^j_{(0,m)} = \varepsilon C^j_{(0,m)}$$

(Defs.).

For $i \geq 1$, $t^{j*}_{(i,m)}$ is an order 1 term of L^*_t and doesn't contain the variable W, hence it is also a term of L_p; the equality holds by Remark 2. So:

$$\begin{aligned} f_h(\varepsilon' t^{j*}_{(i,m)}) &= f_h(\varepsilon([C^j_{(p,q)}]\,(t^{j*}_{(i,m)}))) = \varepsilon(F_h([C^j_{(p,q)}] t^{j*}_{(i,m)})) \\ &= \varepsilon([C^j_{(p,q)}]\,(F_h(t^{j*}_{(i,m)}))) = \varepsilon([C^j_{(p,q)}] t^j_{(i,m)}) \end{aligned}$$

(Remark 2, Defs. and the fact that $F_h \notin$ IndVary). So $\varepsilon' t^{j*}_{(i,m)} = \varepsilon \circ [C^j_{(p,q)}]\,(t^{j*}_{(i,m)})$. Hence:

$$\begin{aligned} \varepsilon' A^j_{(i,m)} &= \varepsilon' X_h(t^{j*}_{(i,m)}) = P(f_h(\varepsilon' t^{j*}_{(i,m)})) = P(\varepsilon C^j_{(i,m)}) \\ &= P(\varepsilon' y^j_{(i,m)}) = \varepsilon' B^j_{(i,m)} \end{aligned}$$

(Defs. and the proof above). $\square$

Lemma 4. *$\{\langle t^{2i-1}; t^{2i}\rangle / i \in \{1, \ldots, n\}\}$ is unifiable in L_t iff N is unifiable in L_p.*

Example. $\varepsilon=\{(f(f(a)), W)/F_1; f(W)/F_2; x_2/F_3; a/F_4; g(x_2, a)/x_1\}$ is a unifier for $\{\langle t^1; t^2\rangle\}$; and $\varepsilon'=\{P(g(f(f(a)), W))/X_1; P(f(W))/X_2; P(x_2)/X_3; P(a)/X_4; g(x_2, a)/x_1; g(f(f(a)), f(g(x_2, a)))/y^1_{(0,1)}; f(a)/y^2_{(0,1)}; x_2/y^2_{(0,2)}; a/y^2_{(0,3)}; f(g(x_2, a))/y^2_{(1,1)}\}$ is a unifier for N.

Proof. Suppose that $\{\langle t^{2i-1}; t^{2i}\rangle/i\in\{1,\dots,n\}\}$ is unifiable in L_t. Then it is unifiable by substituting only order 1 terms of L_t for variables (order 1 terms are common to both languages).

Let $\varepsilon=(\cup\{f_h(W)/F_h\})\bigcup(\cup\{s_h/x_h\})$ be such a substitution, where $f_h(W)$ are order 1 terms of $L_t\cup\{W\}$ and s_h order 1 terms of L_t. Let

$$\varepsilon'=(\cup\{P(f_h(W))/X_h\})\bigcup(\cup\{s_h/x_h\})\bigcup(\cup\{\varepsilon C^j_{(i,m)}/y^j_{(i,m)}\}).$$

By Lemma 3, ε' unifies the N^j for $j\in\{1,\dots,2n\}$. By hypothesis we get $\varepsilon t^{2i-1}=\varepsilon t^{2i}$ for $i\in\{1,\dots,n\}$, and by Lemma 3 $\varepsilon'\phi^j=P(\varepsilon t^j)$ for $j\in\{1,\dots,2n\}$; hence we have $\varepsilon'\phi^{2i-1}=\varepsilon'\phi^{2i}$ for $i\in\{1,\dots,n\}$. So ε' unifies N.

Conversely: If N is unifiable in L_p then N is unifiable by a substituion

$$\varepsilon'=(\cup\{P(f_h(W))/X_h\})\bigcup(\cup\{s_h/x_h\})\bigcup(\cup\{\beta^j_{(i,m)}/y^j_{(i,m)}\}),$$

such that $f_h(W)$ is an order 1 term of $L_p\cup\{W\}$, s_h and $\beta^j_{(i,m)}$ are order 1 terms.

Let $\varepsilon=(\cup\{f_h(W)/F_h\})\bigcup(\cup\{s_h/x_h\})$. Then ε unifies $\{\langle t^{2i-1}; t^{2i}\rangle/i\in\{1,\dots,n\}\}$ ($\beta^j_{(i,m)}=\varepsilon C^j_{(i,m)}$, and Lemma 3).

Remark 3. ε and ε' have the same restriction on Indarx; hence for every order 1 terms t of L_t, $\varepsilon' t=\varepsilon t$.

Remark 4. On the set IndVar, $\varepsilon'=\varepsilon\circ[\beta^j_{(p,q)}]$ by Remark 3 and the definition of ε'. Hence, if t is a term of L_p, $\varepsilon' t=\varepsilon\circ[\beta^j_{(p,q)}]t$.

We prove $\beta^j_{(i,m)}=\varepsilon C^j_{(i,m)}$ by induction on i:

If $i=0$, from $\varepsilon' A^j_{(0,m)}=\varepsilon' B^j_{(0,m)}$ (because ε' unifies N) we get: $\varepsilon' P(y^j_{(0,m)})=\varepsilon' X_h(t^{j*}_{(0,m)})$. So

$$\begin{aligned}P(\beta^j_{(0,m)})&=\varepsilon' X_h(t^{j*}_{(0,m)})=P(f_h(\varepsilon' t^{j*}_{(0,m)}))=P(f_h(\varepsilon t^{j*}_{(0,m)}))\\&=P(\varepsilon(F_h(t^{j*}_{(0,m)})))=P(\varepsilon t^j_{(0,m)})=P(\varepsilon C^j_{(0,m)})\end{aligned}$$

(Defs. and Remark 1). So $\beta^j_{(0,m)}=\varepsilon C^j_{(0,m)}$.

Suppose that the property is true for $i-1$. We have: $\varepsilon' A^j_{(i,m)}=\varepsilon P(X_h(t^{j*}_{(i,m)}))=P(f_h(\varepsilon' t^{j*}_{(i,m)}))$ (Defs.).

We know that $t^{j*}_{(i,m)}$ is also a term of L_p, therefore:

$\varepsilon'(t^{j*}_{(i,m)})=\varepsilon\circ[\beta^j_{(p,q)}]t^{j*}_{(i,m)}$ (Remark 4)

ag1 $=\varepsilon\circ[C^j_{(p,q)}]t^{j*}_{(i,m)}$ (induction hypothesis and if $y^j_{(p,q)}$ appears in $t^{j*}_{(i,m)}$ then $p<i$).

Hence:

$$\begin{aligned}\varepsilon' A^j_{(i,m)}&=P(f_h(\varepsilon([C^j_{(p,q)}]t^{j*}_{(i,m)})))=P(\varepsilon(F_h([C^j_{(p,q)}]t^{j*}_{(i,m)})))\\&=P(\varepsilon([C^j_{(p,q)}]F_h(t^{j*}_{(i,m)})))\\&=P(\varepsilon([C^j_{(p,q)}]t^j_{(i,m)}))=P(\varepsilon C^j_{(i,m)})\end{aligned}$$

(Defs., the proof above and the fact that $F_h\notin$ IndVary).

By hypothesis ε' unifies N, so $\varepsilon' A^j_{(i,m)}=\varepsilon' B^j_{(i,m)}$. By definition we have $\varepsilon' B^j_{(i,m)}=P(\beta^j_{(i,m)})$. So $P(\varepsilon C^j_{(i,m)})=P(\beta^j_{(i,m)})$. Hence $\beta^j_{(i,m)}=\varepsilon C^j_{(i,m)}$. Now by Lemma 3, $\varepsilon'\phi^j=P(\varepsilon t^j)$ for $j\in\{1,\dots,2n\}$. But ε' unifies N, so $\varepsilon'\phi^{2i-1}=\varepsilon'\phi^{2i}$ for $i\in\{1,\dots,n\}$; and therefore $P(\varepsilon t^{2i-1})=P(\varepsilon t^{2i})$ for $i\in\{1,\dots,n\}$; i.e. $\varepsilon t^{2i-1}=\varepsilon t^{2i}$ for $i\in\{1,\dots,n\}$. So ε unifies $\{\langle t^{2i-1}; t^{2i}\rangle / i\in\{1,\dots,n\}\}$. □

Theorem 1. *The unification problem for the language L_t is undecidable if* Cons *contains a function constant of arity* $\geqq 2$.

Proof. See [F, Chap. IV]. □

Theorem 2. *The unification problem for the language L_p is undecidable if* Cons *contains a function constant of arity* $\geqq 2$.

Proof. The reduction is obviously recursive, and we have reduced an undecidable problem (Theorem 1) to this problem. Hence the problem for L_p is undecidable. □

Remark. The second order predicate unification problem for a lagnuage L is undecidable if L contains a predicate constant of arity $\geqq 1$ (the P we used) and a function constant of arity $\geqq 2$. This condition can't be weakened. Farmer gives an (obvious) reduction from the unification predicate problem to the unification term problem, and if the language contains only function constants of arity $\leqq 1$ the unification problem can easily be reduced to the monoïd problem which was shown decidable by Makanin (see [M]). Farmer's reduction doesn't change the set of function constants; therefore, if L contains a predicate constant of arity $\geqq 1$ and no function constant of arity $\geqq 2$, the unification problem for L becomes decidable. If L doesn't contain any predicate constant of arity $\geqq 1$ the unification problem for such a predicate language is obviously decidable; we can reduce it to the first order unification problem.

References

[F] Farmer, W.M.: Length of proofs and unification theory. Ph.D. Thesis, University of Wisconsin, Madison, 1984

[G] Goldfarb, W.D.: The undecidability of the second-order unification problem. Theor. Comput. Sci. **13**, 225–230 (1981)

[M] Makanin, G.S.: The problem of solvability of equations in a free semigroup. Sov. Math. Dokl. **18**, 330–334 (1977)

[R] Robinson, J.A.: A machine-oriented logic based on the resolution principle. J. Assoc. Comput. Mach. **12**, 23–41 (1965)