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A SURVEY OF FACTORIZATION COUNTING FUNCTIONS

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The general field of additive number theory considers questions concerning representations of a given positive integer n as a *sum* of other integers. In particular, *partitions* treat the sums as unordered combinatorial objects, and *compositions* treat the sums as ordered. Sometimes the sums are restricted, so that, for example, the summands are distinct, or relatively prime, or all congruent to ± 1 modulo 5. In this paper we review work on analogous problems concerning representations of n as a *product* of positive integers. We survey techniques for enumerating product representations both in the unrestricted case and in the case when the factors are required to be distinct, and both when the product representations are considered as ordered objects and when they are unordered. We offer some new identities and observations for these and related counting functions and derive some new recursive algorithms to generate lists of factorizations with restrictions of various types.

Keywords: Factorization; multiplicative partition; *factorisatio numerorum*; branching factorization; factor perfect.

1. Introduction

There is a vast body of mathematical work concerning representations of a given positive integer n as a *sum* of other integers. Sometimes the sums are restricted, so that, for example, the summands are distinct, or relatively prime. Waring's problem restricts summands to be kth powers. In general, both ordered and unordered structures may be considered. Recent work by the authors on restricted compositions and partitions includes [25], [26], and [38].

This paper considers analogous problems concerning representations of n as a *product* of positive integers. Sometimes these problems appear in the literature as involving "*factorisatio numerorum*". We survey techniques for enumerating product representations both in the unrestricted case and in the case when the factors are required to be distinct, and both when the product representations are considered as ordered objects and when they are unordered. We offer some new identities and observations for these and related counting functions and derive some new recursive algorithms to generate lists of factorizations with restrictions of various types. The problem in the more general setting of arithmetical semigroups has been investigated in [24].

1.1. *Notation*

We use $H(n)$ to represent the number of factorizations of the positive integer n into factors in which the order of the factors distinguishes factorizations (in analogy with compositions for sums), and $P(n)$ to represent the number of factorization in which the order of factors is immaterial (in analogy with partitions). We would like to consider algorithms both for enumerating the number of factorizations and explicitly listing them, for a given n. Note that the easy way to enumerate or list structures in which order is immaterial is to impose a canonical order (for example, partitions are often listed with summands in decreasing order), so sometimes in the literature there are different conventions adopted on the meanings of ordered and unordered. Further refinement of the notation is possible. For instance, we will consider $H_d(n)$, the number of ordered factorizations into distinct parts, and $P_d(n)$, the number of unordered factorizations into distinct parts. Another variation that is sometimes useful is to explicitly keep track of the number of parts. We will introduce a second argument if this is desired, so that $H(n,k)$ will mean the number of ordered factorizations of n using exactly k factors. Of course, the convention is that all factors are integers greater than 1.

Write

$$n = p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}, \quad \text{with } 1 < p_1 < p_2 < \cdots < p_s, \tag{1.1}$$

to be the canonical factorization of n as a product of powers of distinct primes. Then many problems involving *factorisatio numerorum* depend only on the set of exponents in (1.1), $\{r_1, r_2, \ldots, r_s\}$. MacMahon [31] developed the theory of compositions of "multipartite numbers" from this perspective, and indeed considered these problems throughout his career [32], but Andrews suggests the more modern terminology "vector compositions" [1, p. 57]. We will extend the notation above, writing, for example, $H_d(\{r_1, r_2, \ldots, r_s\})$ in case we wish to suppress the particular primes in n's representation (1.1) as an ordered product of distinct factors. However, there are some properties we consider in which the particular primes are important.

There are two extreme situations to note. First, if $n = p^r$, we have that $H(n) = 2^{r-1}$ since each composition of the integer r corresponds to an ordered factorization of n. Similarly $P(n) = p(r)$, where $p(r)$ counts partitions of the integer r in this case. Second, if $n = p_1 p_2 \cdots p_s$ is square-free, then the number of unordered factorizations of n is given by the number of set partitions of the set of s prime factors of n. Set partitions are enumerated by the Bell numbers. For $H(p_1 p_2 \cdots p_s)$, there is the analogous idea of an ordered Bell number, counting set partitions in which the order of subsets (but not the order of elements in each subset) matters. Bell numbers have the exponential generating function e^{e^x-1}, and ordered Bell numbers have the exponential generating function $1/(2-e^z)$.

It is convenient to enumerate values of $H(n), H_d(n), P(n)$, and $P_d(n)$ in terms of the set of exponents $\{r_1, r_2, \ldots, r_s\}$ of (1.1). We will use the notation $\bar{H}(m)$ to indicate the sum over all partitions $m = r_1 + r_2 + \cdots + r_s, r_1 \leq r_2 \leq \cdots \leq r_s$ of values of $H(\{r_1, r_2, \ldots, r_s\})$, with a similar convention for the other counting functions.

Table 1 gives the values of the counting functions, indexed by partitions of the sum of exponents in the representation of n as a product of prime powers. Several numerical relations among the counting functions are suggested by the table. For example, $P - P_d$ for $\{2\}\{2,1\}\{2,1,1\}, \ldots$ or $\{3\}\{3,1\}\{3,1,1\}, \ldots$ forms the sequence of Bell numbers. We establish this later in the paper, using Eq. (2.22). One relation that does not continue is the following: for the values given, the partitions of m can be ordered so that all columns are monotone.

Table 1.

Partition	$H(n)$	$H_d(n)$	$P(n)$	$P_d(n)$
{1}	1	1	1	1
{2}	2	1	2	1
{1,1}	3	3	2	2
{3}	4	3	3	2
{2,1}	8	5	4	3
{1,1,1}	13	13	5	5
{4}	8	3	5	2
{3,1}	20	13	7	5
{2,2}	26	13	9	5
{2,1,1}	44	29	11	9
{1,1,1,1}	75	75	15	15
{5}	16	5	7	3
{4,1}	48	21	12	7
{3,2}	76	29	16	9
{3,1,1}	132	81	21	16
{2,2,1}	176	89	26	18
{2,1,1,1}	308	209	36	31
{1,1,1,1,1}	541	541	52	52
{6}	32	11	11	4
{5,1}	112	35	19	10
{4,2}	208	73	29	14
{4,1,1}	368	163	38	25
{3,3}	252	87	31	17
{3,2,1}	604	245	52	34
{3,1,1,1}	1076	637	74	59
{2,2,2}	818	313	66	40
{2,2,1,1}	1460	749	92	70
{2,1,1,1,1}	2612	1805	135	120
{1,1,1,1,1,1}	4683	4683	203	203

However, there are examples for every column for larger m in which the order is broken, for example $H(\{8,1,1,1\}) = 204032 < 222528 = H(\{7,3,1\})$ but $P(\{8,1,1,1\}) = 1106 > 1097 = P(\{7,3,1\})$. Nothing precludes different partitions giving equal values in a given column, although we have not observed this behavior through $n = 11$.

It is possible to represent the ordered factorizations of n graphically, with one spacial dimension allocated for each prime in the factorization of n. In Fig. 1, the 26 ordered factorizations of 36 are represented, with powers of 2 represented purely

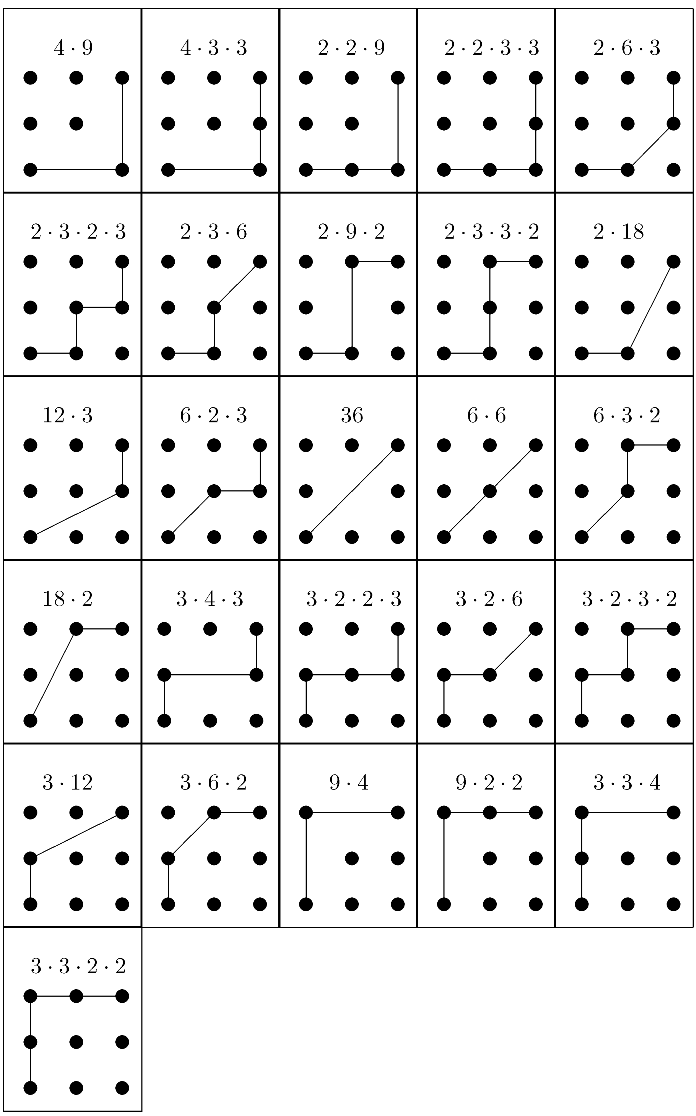

Fig. 1.

horizontally and powers of 3 represented purely vertically. The order is lexicographic in the factors, primarily putting "flatter" factors earlier, and secondarily "longer" factors before "shorter" ones.

2. Computational Techniques

The techniques described below have all been implemented in *Mathematica*, and a notebook [27] with working code for all of the counting functions and factorization generators is available from the authors. Applications to Dirichlet generating functions are available in [28].

2.1. $H(n)$

Perhaps the most direct route to a recursion for $H(n)$ is via the Dirichlet series representation. Summing over k the series

$$\sum_{n=2}^{\infty} H(n,k)n^{-s} = (\zeta(s)-1)^k$$

gives

$$\sum_{n=1}^{\infty} H(n)n^{-s} = (2-\zeta(s))^{-1}.$$

We obtain the basic recurrence

$$H(n) = 1 + \sum_{\substack{d|n \\ 1<d<n}} H(d). \tag{2.1}$$

This version of the recurrence can be implemented taking $H(1)=1$, although the Dirichlet series leaves $H(1)$ undefined. Alternatively, inverting the sum produces

$$H(n) = 2\left(\sum_{p_i} H(n/p_i) - \sum_{p_i,p_j} H(n/p_ip_j) + \cdots + (-1)^{k-1}H(n/p_1p_2\cdots p_k)\right). \tag{2.2}$$

This version demands $H(1)=1/2$, correcting a statement in [11]. The distinction between starting values can be resolved by carefully working through the Möbius inversion relating the two recursions for $H(n)$.

MacMahon [31] provided a closed form for $H(n)$ (this corrects the version provided in [30]). There is a generalized version developed by Carlitz [3], and carried forward by Evans [6] which incorporates another parameter λ. For $\lambda = 2$ we obtain information about our case. The formula uses $q=\sum r_i$, the sum of the exponents on the primes in the canonical representation (1.1) of n.

$$H(n) = \sum_{j=1}^{q}\sum_{i=0}^{j-1}(-1)^i\binom{j}{i}\prod_{h=1}^{s}\binom{r_h+j-i-1}{r_h}. \tag{2.3}$$

This formula provides the quickest value for $H(n)$.

In the special case that n has exactly two distinct factors (2.3) reduces to the simpler expression of Chor, Lemke, and Mador [4]

$$H(p_1^i p_2^j) = 2^{i+j-1} \sum_{k=0}^{j} \binom{i}{k} \binom{j}{k} 2^{-k}. \tag{2.4}$$

Long [30] was primarily interested not in ordered factorizations but in complementing subsets of the set $\{0, 1, 2, \ldots, n-1\}$. Ordered factorizations have many combinatorial associations. Riordan [39] points out that $H(n)$ is the number of perfect partitions of $n-1$, an insight that also goes back at least as far as MacMahon. Other early work on $H(n)$ is summarized in [41].

Carlitz [3] develops formulas for the extended Eulerian numbers $H(n, \lambda)$ given by

$$\frac{1-\lambda}{\zeta(s)-\lambda} = \sum_{n=1}^{\infty} \frac{H(n,\lambda)}{n^s}. \tag{2.5}$$

We are, of course, interested in the case $\lambda = 2$. In this case, the explicit formula developed for $H(n, \lambda)$ reduces to MacMahon's formula for $H(n)$ with a minor change of variable:

$$H(n) = \sum_{k=0}^{q} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \prod_{h=1}^{r} \binom{r_h + j - 1}{r_h}.$$

More interestingly, Carlitz obtains that for $H_s = H(p_1 p_2 \cdots p_s)$,

$$H(n) = \sum_{k=0}^{q} H_k \frac{\alpha_k(n)}{k!}, \tag{2.6}$$

where

$$\alpha_k(n) = \sum_{d_1 \cdots d_k = n} \alpha(d_1) \cdots \alpha(d_k), \tag{2.7}$$

and

$$\alpha(n) = \begin{cases} 1/r & (n = p^r, r \geq 1), \\ 0 & (\text{otherwise}). \end{cases} \tag{2.8}$$

This establishes that $H(n)$ is a linear combination of ordered Bell numbers. The next theorem establishes a bit more, that $H(n)$ is a weighted average of ordered Bell numbers.

Theorem 2.1. *For $\alpha(n)$, $\alpha_k(n)$, and q as defined above,* $\sum_{k=0}^{q} \frac{\alpha_k(n)}{k!} = 1$.

First we observe $\sum_k \frac{\alpha_k(n)}{k!} = 1$ if $n = p^r$. In this case,

$$\begin{aligned} \sum_{k=1}^{r} \frac{\alpha_k(p^r)}{k!} &= \frac{\sum_{a_1=r} \alpha(p^{a_1})}{1!} + \frac{\sum_{a_1+a_2=r} \alpha(p^{a_1})\alpha(p^{a_2})}{2!} + \cdots \\ &\quad + \frac{\sum_{a_1+a_2+\cdots+a_r=r} \alpha(p^{a_1})\alpha(p^{a_2}) \cdots \alpha(p^{a_r})}{r!} \\ &= \frac{\frac{1}{r}}{1!} + \frac{\sum_{a_1+a_2=r} \frac{1}{a_1}\frac{1}{a_2}}{2!} + \cdots + \frac{1}{r!}. \end{aligned}$$

A well-known relation for Stirling numbers of the first kind, written with positive values as $\left[{r \atop k}\right]$ in the notation of Knuth [29, p. 65 ff.], is that

$$\begin{bmatrix} r \\ k \end{bmatrix} = \frac{r!}{k!} \sum_{a_1+\cdots+a_k=r} \frac{1}{a_1 a_2 \cdots a_k}.$$

Hence the sum is

$$\sum_{k=1}^{r} \frac{\left[{r \atop k}\right]}{r!} = \frac{1}{r!} \sum_{k=1}^{r} \begin{bmatrix} r \\ k \end{bmatrix} = \frac{r!}{r!} = 1.$$

The Stirling number recurrence

$$\begin{bmatrix} r \\ k \end{bmatrix} = (r-1) \begin{bmatrix} r-1 \\ k \end{bmatrix} + \begin{bmatrix} r-1 \\ k-1 \end{bmatrix}$$

translates directly into the recurrence satisfied by $\alpha_k(p^r)$

$$r\alpha_k(p^r) = (r-1)\alpha_k(p^{r-1}) + k\alpha_{k-1}(p^{r-1}).$$

More generally, we can consider $n = p^r m$, where $\gcd(p, m) = 1$, and write

$$\begin{aligned} \alpha_k(p^r m) &= \sum_{d_1 d_2 \cdots d_k = p^r m} \alpha(d_1)\alpha(d_2)\cdots\alpha(d_k) \\ &= \sum_{i=1}^{k-1} \binom{k}{i} \sum_{\substack{d_1\cdots d_i = p^r \\ d_{i+1}\cdots d_k = m}} \alpha(d_1)\cdots\alpha(d_k) \\ &= \sum_{i=1}^{k-1} \binom{k}{i} \alpha_i(p^r)\alpha_{k-i}(m). \end{aligned}$$

Distributing the p^r recurrence among the summands yields, with the canonical factorization of n explicitly given,

$$\alpha_k(p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}) = ((r_1-1)\alpha_k(p_1^{r_1-1} p_2^{r_2} \cdots p_s^{r_s}) + k\alpha_{k-1}(p_1^{r_1-1} p_2^{r_2} \cdots p_s^{r_s}))/r_1, \tag{2.9}$$

The calculation of α_k can be done without explicitly using the function $\alpha(n)$. The recursion (2.9) can be simplified if instead we work with a triangular integer array $a(k, \{r_1, r_2, \ldots, r_s\})$ whose entries can be interpreted as

$$\alpha_k(p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}) r_1! r_2! \cdots r_s!.$$

This array can be described in terms of repeatedly iterating the recurrence associated with Stirling numbers. In particular, for $\alpha_k(p_1^{r_1})$ we begin with zeroth row 1 and build through to row r_1 using the recursion

$$a(k, \{r+1\}) = a(k-1, \{r\}) + r a(k, \{r\}).$$

(We implicitly pad the rows of the array with zeros to the left and right of the exhibited entries.) Here r indexes the row, starting with the zeroth, and k indexes the diagonals, starting with the zeroth. This generates the familiar array that begins

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 0 & & 1 & & & & \\
 & & & 0 & & 1 & & 1 & & & \\
 & & 0 & & 2 & & 3 & & 1 & & \\
 & 0 & & 6 & & 11 & & 6 & & 1 & \\
0 & & 24 & & 50 & & 35 & & 10 & & 1.
\end{array}$$

Now to generate $\alpha_k(p_1^{r_1}p_2^{r_2})$, start with row r_1 of the array above and continue through to row $r_1 + r_2$ using the same Stirling number recursion as above, where the row multiplier for row $r_1 + i$ is i. This process is repeated as more prime powers are incorporated into n. For example, the array for $n = 2^3 \cdot 3 \cdot 5^2$ would be

$$\begin{array}{lccccccccccccc}
2^0: & & & & & & & 1 & & & & & & \\
2^1: & & & & & & 0 & & 1 & & & & & \\
2^2: & & & & & 0 & & 1 & & 1 & & & & \\
2^3: & & & & 0 & & 2 & & 3 & & 1 & & & \\
2^3 \cdot 3: & & & 0 & & 0 & & 2 & & 3 & & 1 & & \\
2^3 \cdot 3 \cdot 5: & & 0 & & 0 & & 0 & & 2 & & 3 & & 1 & \\
2^3 \cdot 3 \cdot 5^2: & 0 & & 0 & & 0 & & 2 & & 5 & & 4 & & 1
\end{array}$$

Dividing through the last row by $3!1!2!$ gives values of $\alpha_k(600)/k!, 0 \le k \le 6$, to be

$$\{0, 0, 0, 1/6, 5/12, 1/3, 1/12\}.$$

Since the last row of the a array sums to $3!1!2!$ with the factors row by row provided by the recurrence, we confirm that $\sum_k \alpha_k(n)/k! = 1$.

An explicit list of the ordered factorizations of n can be obtained by modifying the initial recurrence given in Eq. (2.1). Build up the factorizations one factor at time, starting with the basic fact that the only ordered factorization of a prime p is $\{p\}$, and iterating by appending factors, so that the set of ordered factorizations of n is $\{n\}$, together with a union over proper divisors d of n of the set of ordered factorizations of n/d, each with d appended.

The asymptotic behavior of $H(n)$ was first studied by Kalmár [18,19]. His work inspired papers by Ikehara [15–17] and others. More recently, Hwang [13] and Finch [7] have generalized Kalmár's work and studied the distribution of the number of factors in random ordered factorizations. Erdős [5] studied asymptotics for restricted factorizations in which factors belong to a given sequence. Other papers with useful information about $H(n)$ include [37] and [40].

Chor, Lemke, and Mador [4] considered values for the exponent t for which it can be guaranteed that for n in various special classes, $H(n) > n^t$. This complements the work of Hille [11], strengthened in [4], in which it is shown that $H(n) < n^\rho$, where $\rho = \zeta^{-1}(2) \approx 1.73$. There are open problems in [4] asking for an extension and/or strengthening of their results, from the case when n is a product of powers of two primes to more general n.

Klazar and Luca [23] provided a general upper bound for $H(n)$ and a lower bound that holds for infinitely many positive integers. Their bounds involve $\rho \sim 1.72864\cdots$ as an exponent, which is the real solution to $\zeta(\rho) = 2$.

2.2. $H_d(n)$

One way to count the ordered factorizations with distinct parts is to select from the set of all ordered factorizations only those with non-repeated factors. This can be very inefficient, depending on the relative sizes of $H(n)$ and $H_d(n)$.

The best algorithm we have found for calculating $H_d(n)$ is implicit in the work of Warlimont [43].

Let $H_d(n,k)$ represent the number of ordered factorizations of n with exactly k distinct parts, $n = n_1 n_2 \cdots n_k$, in which each factor $n_i > 1$ and $n_i \neq n_j$ for $i \neq j$. Put

$$G_k(s) = \sum_{n=1}^{\infty} H_d(n,k) n^{-s}. \tag{2.10}$$

For $G(s)$ the Dirichlet series for $H_d(n)$, we have

$$G(s) = \sum_{n=1}^{\infty} H_d(n) n^{-s} = \sum_{k=0}^{\infty} G_k(s). \tag{2.11}$$

We derive a recursion for $G_k(s)$. Put $Z(s) = \zeta(s) - 1$ Then

$$\begin{aligned}
G_{k+1}(s) = \sum_{\substack{n_j=2(1\le j\le k)\\ n_i\neq n_j}}^{\infty} (n_1 n_2 \cdots n_k n_{k+1})^{-s} &= \sum_{m=2}^{\infty} m^{-s} \sum_{\substack{n_j=2(1\le j\le k)\\ n_i\neq n_j\\ n_j\neq m}}^{\infty} (n_1 n_2 \cdots n_k)^{-s} \\
&= Z(s)G_k(s) - \sum_{m=2}^{\infty} m^{-s} \sum_{\substack{n_j=2\\ n_i\neq n_j\\ \exists r: n_r = m}}^{\infty} (n_1 n_2 \cdots n_k)^{-s} \\
&= Z(s)G_k(s) - k\sum_{m=2}^{\infty} m^{-2s} \sum_{\substack{n_j=2\\ n_i\neq n_j\\ n_j\neq m}}^{\infty} (n_1 n_2 \cdots n_{k-1})^{-s} \\
&= Z(s)G_k(s) - kZ(2s)G_{k-1}(s) + k\sum_{m=2}^{\infty} m^{-2s} \sum_{\substack{n_j=2\\ n_i\neq n_j\\ \exists r: n_r = m}}^{\infty} (n_1 n_2 \cdots n_{k-1})^{-s}.
\end{aligned}$$

By iterating this process we arrive at

$$G_{k+1}(s) = \sum_{j=0}^{k} (-1)^j \frac{k!}{(k-j)!} Z((j+1)s) G_{k-j}(s) \tag{2.12}$$

or

$$\sum_{n=1}^{\infty} H_d(n, k+1) n^{-s} = \sum_{j=0}^{k} (-1)^j \frac{k!}{(k-j)!} Z((j+1)s) \sum_{n=1}^{\infty} H_d(n, k-j) n^{-s}. \tag{2.13}$$

We wish to equate the coefficients of n^{-s} on each side of Eq. (2.13).

For $j = 0, 1, \ldots, k$,

$$\begin{aligned} Z((j+1)s) \sum_{n=1}^{\infty} H_d(n, k-j) n^{-s} &= \left(\sum_{m=2}^{\infty} m^{-(j+1)s} \right) \left(\sum_{n=1}^{\infty} H_d(n, k-j) n^{-s} \right) \\ &= \sum_{\substack{m \geq 2, n \geq 1 \\ m^{j+1} n = l}} l^{-s} H_d(n, k-j) \\ &= \sum_{l \geq 2^{j+1}} l^{-s} \sum_{\substack{d \mid l \\ d \text{ a } (j+1)\text{st power}}} H_d(l/d, k-j). \end{aligned}$$

So

$$H_d(n, k+1) = \sum_{j=0}^{k} (-1)^j \frac{k!}{(k-j)!} \sum_{\substack{d \mid n \\ d \text{ a } (j+1)\text{st power}, d \geq 2}} H_d(n/d, k-j). \tag{2.14}$$

As checks, we note that for n squarefree, $H_d(n, k) = H(n, k)$ and $H_d(n) = H(n)$.

2.3. $P(n)$

The analog to Euler's product representation of the zeta function,

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1},$$

is the Dirichlet series generating function for $P(n)$:

$$\sum_{n=1}^{\infty} P(n)/n^s = \prod_{n=2}^{\infty} (1 - n^{-s})^{-1}. \tag{2.15}$$

Harris and Subbarao [9] find a recursion for $P(n)$ without an explicit generation of factorizations. Using the auxiliary function d_i given as $d^{1/i}$ if i is an ith power, and 1 otherwise, with $\bar{d} = \prod_{i \geq 1} d_i$, they show

$$\prod_{d \mid n} \bar{d}^{P(n/d)} = n^{P(n)}. \tag{2.16}$$

This leads to a reasonable algorithm for calculating values of $P(n)$ for small values of n.

Hughes and Shallit [12] bound $P(n)$ above by $2n^{\sqrt{2}}$, and conjecture the stronger bounds $P(n) \leq n$, in fact $P(n) \leq n/\log n$ for all $n > 1$ except $n = 144$. The first conjecture was verified by Mattics and Dodd [33]. They also verified the second, and noted the improved bound that $P(n) \leq n(\log n)^{-\alpha}$ for each fixed $\alpha > 0$ and all sufficiently large n, in [34].

The technique that Hughes and Shallit used to calculate $P(n)$ approaches $P(n)$ via $P(n, m)$, the number of unordered factorizations of n with largest part at most m. Note the different role of the second argument of this function, compared to $H(n, k)$ introduced earlier. They noted that

$$P(n, m) = \sum_{\substack{d|n \\ d \leq m}} P(n/d, d). \tag{2.17}$$

Then $P(n) = P(n, n)$, and this provides a much faster algorithm than (2.16).

The Hughes–Shallit approach is also useful for generating a list of the unordered factorizations, recursively building up from the set of proper divisors d of n factorizations with largest part d, in each case appending d to each of the factorizations of n/d with largest part at most d.

Asymptotics for growth of $P(n)$ were first studied by Oppenheim [35, 36], and independently by Szekeres and Turán [42]. Hensley [10] discussed unordered factorizations in order to account for the distribution of the number of factors.

Canfield, Erdős, and Pomerance [2] listed "highly factorable integers," which are the integers with champion values of $P(n)$. The factorizations were enumerated by imposing a tree structure on the set of partitions of a multiset, and then applying a standard tree traversal algorithm of computer science. Their paper also established that $P(n) = nL(n)^{-1+k(n)}$ for champions, where $L(n) = \exp(\log n \cdot \log\log\log n/\log\log n)$ and $k(n)$ is a function with $\lim_{n\to\infty} k(n) = 0$, which is stronger than the improved bound of [34]. A different approach to generating highly factorable numbers can be found in the *Mathematica* notebook [27], in which candidates are generated by pairing exponents gotten from a list of partitions (obtained from the first few primes), and filtered by applying a replacement rule. Kim [20] was able to resolve several conjectures of [2] dealing with the exponents in the prime factorization of a large highly factorable number and the limiting behavior of ratios involving consecutive highly factorable numbers, and has further work with Hahn [21] and Landman [22].

2.4. $P_d(n)$

One option for calculating $P_d(n)$ is to start with one of the more numerous classes of factorizations counted by $H(n)$, $H_d(n)$, or $P(n)$, and then filter to remove those factorizations with repeated parts, and choose only one representative from multiple representatives with the parts in different orders.

The approach embodied in Eq. (2.17) can be adapted to calculate the number of unordered factorizations into distinct parts. Let $P_d(n,m)$ be the number of unordered factorizations of n into distinct parts the largest of which is at most m. Then

$$P_d(n,m) = \sum_{\substack{d|n \\ d<m}} P(n/d,d). \tag{2.18}$$

We then note $P_d(n) = P_d(n,n)$.

This also leads to a recursion for explicitly generating the unordered factorizations. In fact, it is often more efficient to list the ordered factorizations of n (either into distinct parts, or unrestricted) by listing all permutations of the corresponding list of unordered factorizations.

Let us refine the counting function even more by writing $P_d(n,m,k)$ to be the number of distinct unordered factorizations into factors of size at most m, with exactly k factors in all. Then

$$H_d(n) = \sum_{k=1}^{n} k! P_d(n,n,k). \tag{2.19}$$

By sorting our lists of distinct unordered factorizations according to length we can compute the values of $P_d(n,n,k)$ and then use (2.19) to find $H_d(n)$. Although lists of factorizations must be generated as part of the counting process, this turns out to be the fastest method we have found to compute the values of $H_d(n)$.

We develop a more general relation between $P(n)$ and $P_d(n)$ as follows. Consider the two Dirichlet series $\sum_{n\geq 1} P(n)/n^s = \prod_{n=2}^{\infty}(1-n^{-s})^{-1}$ and $\sum_{n\geq 1} P_d(n)/n^s = \prod_{n=2}^{\infty}(1+n^{-s})$. Write

$$\frac{\prod_{n=2}^{\infty}(1+n^{-s})}{\prod_{n=2}^{\infty}(1-n^{-2s})} = \prod_{n=2}^{\infty} \frac{1}{1-n^{-s}}, \tag{2.20}$$

whence

$$\left(\sum_{n\geq 1} \frac{P_d(n)}{n^s}\right)\left(\sum_{m\geq 1} \frac{P(m)}{m^{2s}}\right) = \left(\sum_{n\geq 1} \frac{P(n)}{n^s}\right). \tag{2.21}$$

Equating coefficients of n^{-s},

$$P(n) = \sum_{m^2|n} P(m) P_d(n/m^2). \tag{2.22}$$

In particular, besides using this as the basis for a recursion, we can read off interesting special cases:

- If n is squarefree, $P(n) = P_d(n)$.
- If n has an exponent vector $\{2,1,1,\ldots\}$ or $\{3,1,1,\ldots\}$, with the prime p_1 having the higher exponent, $P(n) = P_d(n) + P(p_1)P_d(n/p_1^2) = P_d(n) + P(n/p_1^2)$. Since $P(n/p_1^2)$ is squarefree this proves the observation for $P - P_d$ from the table of values.

- If n has an exponent vector $\{4, 1, 1, \ldots\}$ or $\{5, 1, 1, \ldots\}$, with the prime p_1 having the higher exponent, $P(n) = P_d(n) + P(p_1)P_d(n/p_1^2) + P(p_1^2)P_d(n/p_1^4) = P_d(n) + P_d(n/p_1^2) + 2P_d(n/p_1^4)$.
- If n has an exponent vector $\{2, 2, 1, \ldots\}$, with the primes p_1 and p_2 having the higher exponents, $P(n) = P_d(n) + P(p_1)P_d(n/p_1^2) + P(p_2)P_d(n/p_2^2) + P(p_1p_2)P_d(n/p_1p_2) = P_d(n) + P_d(n/p_1^2) + P_d(n/p_2^2) + 2P_d(n/p_1^2p_2^2)$.

All of these results have immediate combinatorial proofs as well.

3. Factorizations with Relatively Prime Parts

Another interesting class of restricted factorizations are those in which the factors must be relatively prime to one another. Clearly this is a stronger restriction than that of requiring distinct factors. The asymptotic growth of the number of such factorizations has been studied by Warlimont [43]. We note that in the special case of squarefree integers, all factorizations are necessarily into relatively prime parts. Thus for squarefree integers the values tabulated for $H(n) = H_d(n)$ also count the number of factorizations into relatively prime parts.

In the ordered case, factorizations of $n = p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}$ into relatively prime parts correspond to determining ordered partitions of a set with the s elements $\{p_1^{r_1}, p_2^{r_2}, \ldots, p_s^{r_s}\}$. The number of these is given by the so-called ordered Bell number, with exponential generating function $(2 - e^x)^{-1}$. These numbers are also given explicitly as

$$\sum_{k=0}^{s} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^s, \tag{3.1}$$

or as the infinite sum

$$\frac{1}{2} \sum_{m=0}^{\infty} m^s 2^{-m}. \tag{3.2}$$

In the unordered case, factorizations of n have a natural correspondence to (unordered) partitions of a set with the s elements $\{p_1^{r_1}, p_2^{r_2}, \ldots, p_s^{r_s}\}$. These are the ordinary Bell numbers, with exponential generating function $\exp(e^x - 1)$.

4. Factorizations into Non-Primes

Much of number theory is related to the fact that every positive integer has a unique (unordered) factorization into primes. By contrast consider now the number $N(n)$ of unordered factorizations into non-primes. For example, $N(12) = 1$ and $N(24) = 2$.

Any unordered factorization of n can be written as $m_1 m_2$ where m_1 is the product of all prime factors in the factorization and m_2 is the product of all composite factors. Since m_1 has a unique representation as a product of prime factors,

$$P(n) = \sum_{m_1 | n} N\left(\frac{n}{m_1}\right) = \sum_{d|n} N(d). \tag{4.1}$$

Thus by the Möbius inversion formula,

$$N(n) = \sum_{d|n} \mu(d) P\left(\frac{n}{d}\right), \tag{4.2}$$

where $\mu(d)$ is the Möbius function.

5. Branching Factorizations

These were introduced by Hwang in his PhD thesis [14]. Let $\mathcal{C}$ be a subset of the set of positive integers ≥ 2. Consider the following structure

$$\mathcal{B} = \mathcal{C} + \begin{array}{ccc} & \times & \\ / & & \backslash \\ \mathcal{B} & & \mathcal{B} \end{array}$$

which can be considered to be a *binary factorization tree*, or *binary branching factorization*. A binary factorization tree of a number n is either an element of $\mathcal{C}$ having size n or is the product of two binary trees.

In terms of Dirichlet series, the symbolic equation corresponds to

$$B(s) = \sum_{n \geq 2} b_n n^{-s} = C(s) + B^2(s),$$

where b_n counts the number of binary branching factorizations of n and $C(s) = \sum_{n \geq 2} c_n n^{-s} = \sum_{n \in \mathcal{C}} n^{-s}$.

For convenience, define $b_1 = 0$, then

$$b_n = c_n + \sum_{d|n} b_d b_{n/d}, \quad n \geq 2.$$

Solving the quadratic gives

$$B(s) = \frac{1 - \sqrt{1 - 4C(s)}}{2},$$

where the initial conditions $B(\infty) = C(\infty) = 0$ are used to determine the branch of the square-root, since both series have no constant terms.

Taking $C(s) = \zeta(s) - 1$,

$$B(s) = \frac{1 - \sqrt{1 - 4C(s)}}{2}, \quad \text{and} \quad b_n = 1 + \sum_{d|n} b_d b_{n/d}, \quad n \geq 2.$$

The b_n sequence begins as follows:

$$0, 1, 1, 2, 1, 3, 1, 5, 2, 3, 1, 11, 1, 3, 3, 15, 1, 11, 1, 11, 3, 3, 1, 45, 2, 3, 5, 11, 1, 19, 1, 51.$$

In Fig. 2 we illustrate the 11 different factorizations of $n = 12$.

If we now admit any number of branches at a node, we obtain a *planar branching factorization* with symbolic equation

$$\mathcal{B} = \mathcal{C} + \begin{array}{ccc} & \times & \\ / & & \backslash \\ \mathcal{B} & & \mathcal{B} \end{array} + \begin{array}{ccc} & \times & \\ / & | & \backslash \\ \mathcal{B} & \mathcal{B} & \mathcal{B} \end{array} + \cdots$$

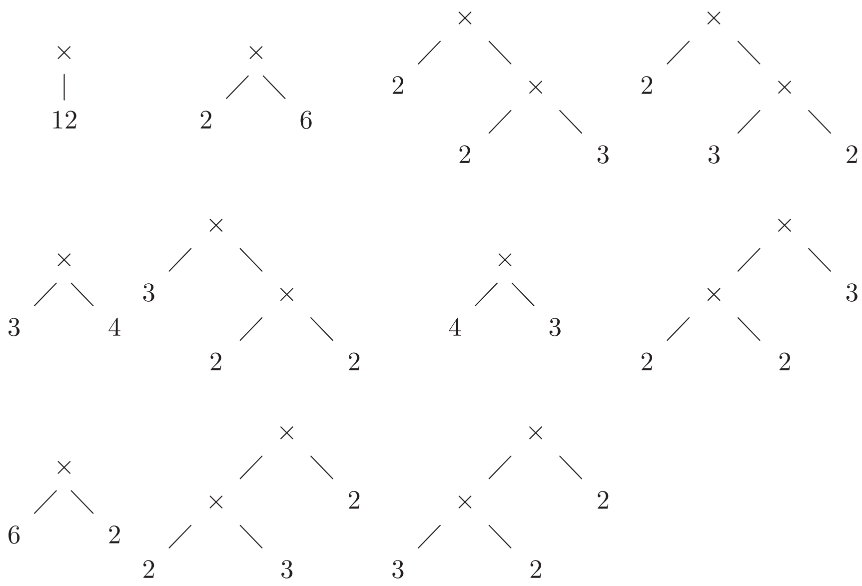

Fig. 2.

and corresponding Dirichlet series equation

$$P(s) = \sum_{n\geq 2} p_n n^{-s} = C(s) + P^2(s) + P^3(s) + P^4(s) + \cdots.$$

From this we obtain the following expression for the coefficients:

$$p_n = c_n + \sum_{d_1d_2=n} p_{d_1}p_{d_2} + \sum_{d_1d_2d_3=n} p_{d_1}p_{d_2}p_{d_3} + \cdots.$$

Solving the functional equation for $P(s)$ leads to

$$\begin{aligned} P(s) &= \frac{C(s)+1-\sqrt{C(s)^2-6C(s)+1}}{4} \\ &= \frac{C(s)+1-\sqrt{3-\sqrt{8}-C(s)}\sqrt{3+\sqrt{8}-C(s)}}{4}. \end{aligned}$$

The case $C(s) = \zeta(s) - 1$ leads to

$$P(s) = \frac{\zeta(s) - \sqrt{4-\sqrt{8}-\zeta(s)}\sqrt{4+\sqrt{8}-\zeta(s)}}{4}.$$

The p_n sequence begins as follows for $n \geq 0$,

$$\begin{aligned} &0, 1, 1, 2, 1, 3, 1, 6, 2, 3, 1, 14, 1, 3, 3, 24, 1, 14, 1, 14, 3, 3, 1, 78, 2, 3, \\ &6, 14, 1, 25, 1, 112, 3, 3, 3, 110, 1, 3, 3, 78, 1, 25, 1, 14, 14, 3, \\ &1, 464, 2, 14, 3, 14, 1, 78, 3, 78, 3, 3, 1, 206, 1, 3, 14, 568. \end{aligned}$$

For $n = 12$ we have in addition to the 11 binary branching factorizations illustrated above, a further 3 planar branching factorizations given by

$$\begin{array}{ccc} & \times & \\ / & | & \backslash \\ 2 & 2 & 3 \end{array} \qquad \begin{array}{ccc} & \times & \\ / & | & \backslash \\ 2 & 3 & 2 \end{array} \qquad \begin{array}{ccc} & \times & \\ / & | & \backslash \\ 3 & 2 & 2 \end{array}$$

Hwang [14] also considers branching factorizations corresponding to other combinatorial tree structures. In addition, he obtains asymptotic estimates for the number of such branching factorizations as well as for the average number of parts and other parameters in the trees.

6. Factor-Perfect Numbers

When studying divisibility, one notes that the sum of the proper divisors of n may exceed, equal, or fall short of n. Perfect numbers are those that are the sum of their proper divisors. Motivated by this, we investigate the size of our factorization counting functions relative to n. The Hughes–Shallit bound for $P(n)$ makes the question uninteresting for $P(n)$ or $P_d(n)$, but for $H(n)$ values are large enough to allow the possibility that $n = H(n)$.

Definition 6.1. A factor-perfect number n is a positive integer $n > 1$ with the property that $n = H(n)$.

Theorem 6.2. *If $k = 2^{n-1}(n+2)$ and $n+2$ is prime, then k is factor-perfect.*

Corollary 6.3. *There are infinitely many factor-perfect numbers.*

Theorem 6.4. *If $k = 2^{n-2}(n^2+4n+1)$ and n^2+4n+1 is a product of 2 distinct primes, then k is factor-perfect.*

Corollary 6.5. *If $n = 6j$, then $(n^2+6n+6)/2$ is 3 times a quadratic. If the quadratic represents a prime, then $2^{n-1}(n^2+6n+6)$ is factor-perfect.*

There are other solutions as well, not of these special forms, such as $2^4 \cdot 3^4 \cdot 23$. There are also other classes of factor-perfect numbers, but deciding whether or not the classes are infinite awaits knowledge of whether polynomials of degree 2 or higher infinitely often represent primes or certain products of primes.

Finally, we consider ordered factorizations into primes. Let $F(n)$ denote the number of ordered factorizations of n into primes. Then we have the Dirichlet generating function

$$\sum_{n=1}^{\infty} F(n)n^{-s} = \frac{1}{1 - \sum_{p \text{ prime}} p^{-s}}.$$

In this case there is an explicit formula for $F(n)$ in terms of the prime factorization of n.

If $n = p_1^{r_1} p_1^{r_1} \cdots p_k^{r_k}$ then

$$F(n) = \frac{(r_1 + r_2 + \cdots + r_k)!}{r_1! r_2! \cdots r_k!}.$$

Definition 6.6. A prime-factor-perfect number n is a positive integer $n > 1$ with the property that $n = F(n)$.

We can use the following approach to search for examples of prime-factor-perfect numbers. Since $F(n)$ depends only the prime exponents $r_1, \ldots, r_k$ we compute the value of all multinomial coefficients with parameters corresponding to the parts in all partitions of a number n, $n = 1, 2, 3, \ldots, M$ for some bound M. We then factorize each of these numbers to test for the "perfect" property. Testing all partitions of numbers $n \leq 65$ yields only 5 examples of prime-factor-perfect numbers:

$$2^9 3^5 5^2 7^2 11^2 13 \cdot 17 \cdot 19 \cdot 23,\ 2^9 3^4 5^3 7^2 11^2 13 \cdot 17 \cdot 19 \cdot 23,\ 2^8 3^6 5^2 7^2 11^2 13 \cdot 17 \cdot 19 \cdot 23,$$
$$2^6 3^8 5^4 7^2 11^2 13^2 17 \cdot 19 \cdot 23,\ 2^7 3^8 5^4 7 \cdot 11^2 13^2 17 \cdot 19 \cdot 23.$$

Multiple prime-factor-perfect numbers, i.e. with $F(n) = kn$ for some $k \geq 2$, seem to be far more common.

The question of whether infinitely many prime-factor-perfect numbers exist remains unsettled.

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