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Leonard Eugene Dickson and his work in the Arithmetics of Algebras

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To my adviser, Karen V. H. Parshall, with warmest thanks.

Contents

Introduction

In 1900, four years after they awarded him one of their first doctorates in mathematics, the members of the University of Chicago mathematics department welcomed LEONARD EUGENE DICKSON as an assistant professor on their faculty. During his forty-year professional affiliation with his *alma mater*, DICKSON was to direct 67 Ph.D. students, write more than 300 publications, serve as editor of the *Monthly* and the *Transactions*, and guide the American Mathematical Society as its President from 1916–1918. By pursuing both his doctoral studies and the majority of his career at Chicago, DICKSON helped define and establish a strong and distinguished algebraic tradition at that institution. [PARSHALL & ROWE, 1994, 381].

Interestingly, just as DICKSON earned his doctorate and laid the foundations of his career, the International Congresses of Mathematicians emerged as important avenues for the distribution of mathematical ideas. The so-called zero-th and first Congresses of 1893 and 1897 in Chicago and Zürich, respectively, set the standards for their successors, which continue to the present day. By design, each Congress promoted the international exchange of scientific ideas through organized sectional (by mathematical field) meetings, social events, and plenary lectures. Relative to the latter, the organizing committee of each Congress selected a handful of distinguished mathematicians to present their current research to the entire constituency of mathematicians in attendance. An invitation to give such a talk validated the worth and importance of a given mathematician's research and ideas.

By the end of the second decade of the twentieth century, DICKSON's career had advanced to the point where he received an invitation to deliver a plenary address at the International Congress in Strasbourg in 1920. On this honorific occasion, he selected a topic which grew, in part, from his historical acquaintance with the theory of numbers and from his work in the theory of algebras [DICKSON, 1921e]. In particular, he demonstrated that the arithmetic of the complex numbers and the quaternions provide effective tools for the treatment of certain Diophantine equations. This result marked his point of entry into the arithmetics of algebras. In this paper, we highlight DICKSON's celebrated work in this field. In particular, we focus on the history of the arithmetics of algebras from the work of CARL FRIEDRICH GAUSS in 1801 through DICKSON's contributions in the 1920's.

In the early nineteenth century, Gauss defined a set of integers within the *complex numbers* and formed the associated arithmetic. Relative to the latter, he defined prime integral complex numbers, developed a process analogous to the Euclidean algorithm for determining the greatest common divisor of two complex integers, and established unique factorization of complex integers into complex primes. ERNST EDUARD KUMMER, RICHARD DEDEKIND, and LEOPOLD KRONECKER tackled similar questions for the *algebraic number fields*, with KUMMER's concept of an ideal prime number guaranteeing the unique factorization of algebraic integers into primes in every number field.

Reacting to the works of his eminent predecessors and especially to those of GAUSS, KUMMER, and DEDEKIND, RUDOLF LIPSCHITZ proposed an arithmetic of the *quaternions* in 1886. Since his concept of an integral quaternion yielded an arithmetic which lacked several of the key, desirable properties, ADOLF HURWITZ countered a decade later with a definition of an integral quaternion which led to an arithmetic analogous to that of the ordinary integers. HURWITZ's student, L. GUSTAVE DU PASQUIER, followed with a consideration of the arithmetic of rational (square) matrices and of small-dimensional and, eventually, *all* rational hypercomplex number systems. In the first two decades of the twentieth-century, DU PASQUIER built his theory of the arithmetics of algebras on his four-pronged definition of a set of integral elements and obtained unique factorization in some, *but not all* algebras. DICKSON finally solidified the arithmetics of algebras in the early 1920's by formulating a definition of a set of integral elements which yielded an arithmetic analogous to the ordinary integers in an *arbitrary rational algebra* and, ultimately, in an *arbitrary algebra*.

A glance at DICKSON's publications in the arithmetics of algebras make plain the mathematics he utilized as he made his contributions to the theory, the broad audience he intended to address, and the view he held as to how the theory had emerged. The general theory of algebras guided his work, while the quaternions and algebraic numbers motivated the theory and served to judge its success. As for his presentation of the theory, DICKSON often wrote for those with no "previous acquaintance with hypercomplex integers" [DICKSON, 1923b, 281; 1928b, 95] in an effort to make his ideas accessible to the widest mathematical audience possible, and he frequently lectured on his work in a variety of international settings. Moreover, he routinely included complete summaries of the theory's development in his publications. His particular version of the story, his apparent penchant for telling it, and his strong rhetoric, represent three of the salient features of this aspect of his work. Thus, as we begin to unravel DICKSON's role in the evolution of the arithmetics of algebras, we consider not only his mathematics but also his ideas regarding the emergence of the theory.

In placing this research in its historical context, a number of broader issues in the history of mathematics come to the fore. This paper, for example, gives a detailed technical analysis of the historical development of the concept of a set of integral elements, the crucial component in the evolution of the arithmetics of algebras. It thus explores the answers to questions such as "Was there a natural definition for a set of integral elements?," "What guided the above-mentioned mathematicians as they proposed their various definitions of an integral element?," and "What mathematics did they use in their associated theories of arithmetics?" In so doing, it highlights the importance of scientific exchange. With our focus on a man whose name is often associated with the arithmetics of algebras *today*, our twentieth-century eyes could blind us into thinking that the work in question originated solely in the mind of DICKSON. Nothing, however, could be further from the truth. The crucial component of the theory, the definition of an integral element, grew from the "international background of mathematical knowledge and the constant commerce of ideas between the old world and the new" [Bell, 8].

The current study also points to the important role of overarching aesthetic principles and to the use of rhetoric in mathematics. Relative to the former, it highlights the precepts which guided DICKSON toward the "best" definition of a set of integral elements. DICKSON's contributions to the formulation of a set of integral elements in an algebra provide further insight into the components of the algebraic research tradition at Chicago. In particular, DICKSON learned – and ultimately imparted – an emphasis on the breadth of applicability of mathematical concepts. As for the persuasive use of language, the presentation of mathematics is often only associated with definitions, theorems, and proofs. This study, however, analyzes DICKSON's use of rhetoric as he strove to convince his audience of the superiority of his theory. Moreover, the implementation of this effective tool by one of the more prominent American mathematicians of the early twentieth century can do nothing but dispel persistent myths regarding the strictly formal presentation of mathematics.

Finally, our examination of DICKSON's work in the theory of algebras introduces us to issues in the historiography of the history of mathematics. The "royal road to me" view of mathematics described by Ivor Grattan-Guinness in his article, "Does History of Science Treat of the History of Science: The Case of Mathematics," seems to characterize DICKSON's presentation of the emerging theory of the arithmetics of algebras. As the rather pejorative name implies, these types of accounts focus more on how older theories led to an individual version of a theory rather than on how the theory developed in its own right [GRATTAN-GUINNESS, 157].

Although the title of this manuscript may conjure up visions of one man's work in one particular field of mathematics, the development of this theory actually spanned oceans, centuries, and branches of mathematics. DICKSON primarily made his contributions from his home base at the University of Chicago and cultivated a flourishing research tradition at that institution in particular and in the United States in general. Thus, an investigation of DICKSON not only makes a valuable contribution to the history of mathematics from a technical point of view but also serves as a case study supporting the existence of a well-defined third stage of the consolidation and growth of a research tradition in the historical development of American mathematics [PARSHALL and ROWE, 1994, 427–428]. With these remarks by way of introduction, let us now begin our investigation of DICKSON's work in the arithmetics of algebras.

The Arithmetic of the Complex Numbers

DICKSON conducted the bulk of his research in the arithmetics of algebras between 1920 and 1924.[1] He gave his mainline treatment of this subject in four papers, "A New Simple Theory of Hypercomplex Integers" [1923b], "Algebras and their Arithmetics" [1924a], "Outline of the Theory to Date of the Arithmetics of Algebras" [1928b], and "Further Development of the Theory of the Arithmetic of Algebras" [1928a], and two books, *Algebras and Their Arithmetics* [1923a], and the revised German edition of this text, *Algebren und ihre Zahlentheorie* [1927]. In June of 1923, when DICKSON surveyed the theory of the arithmetics of algebras in the introduction to his *Algebras and Their Arithmetics*, he described the theory as one which "has been surprisingly slow in its evolution" [DICKSON, 1923a, viii]. "Quite naturally," he continued, "the arithmetic of quaternions received attention first" [DICKSON, 1923a, viii]. Although DICKSON made this comment relative to the general development of the theory, it seems a more fitting description of his individual pursuits in the area. His "Arithmetic of Quaternions," published in 1921, represented *his* first complete work in the theory of the arithmetics of algebras [DICKSON, 1921a].

By the time DICKSON's number-theoretic interests had led him to consider the arithmetic of quaternions in 1920, however, RUDOLPH LIPSCHITZ, professor of mathematics at Bonn University [SCHOENEBERG], and ADOLF HURWITZ, professor of mathematics and the successor of FROBENIUS at the Polytechnic Institute in Zürich, had each proposed their own theories for the arithmetic of quaternions. Prior to any of this work on the arithmetic of quaternions, some stellar figures in nineteenth-century mathematics had already considered the arithmetic of the complex numbers and the algebraic number fields.

GAUSS had planted the seeds for the arithmetic of the complex numbers, as for so much else, in his 1801 *Disquisitiones Arithmeticae* [GAUSS]. In this text, he introduced the concept of congruence [GAUSS, 1; BASHMAKOVA & RUDAKOV, 87]. He called two integers a and b congruent modulo an integer c if c divided $a - b$. He denoted congruence by the symbol $\equiv$ and expressed the relationship between $a,\ b$, and c as $a \equiv b(mod\ c)$. Just as the symbol for congruence suggested an analogy between congruence and equality, so

[1] The two papers DICKSON presented at the Toronto Congress (1924) did not appear in print until the 1928 publication of the Conference Proceedings. DICKSON's *Algebren und ihre Zahlentheorie* (Zurich: Orell Füssli, 1927), the German translation of a revised and expanded version of his *Algebras and Their Arithmetics* (Chicago: University of Chicago Press, 1923), also appeared in 1927.

the algebra of congruences was similar to that of the more familiar algebra of equality in GAUSS's formulation. He proved that most, but not all, of the ordinary rules of algebra held in his new algebra of congruences [GAUSS, 2–3, 9]. In particular, if $ax = ay$ and $a \neq 0$ in the algebra of equality, then $x = y$. In the algebra of congruences, however, this division property does not hold in general. Consider the congruence $(2 \times 3) \equiv (2 \times 5)(mod\ 4)$. Certainly $6 \equiv 10(mod\ 4)$ because 4 divides $(6 - 10)$. "Dividing" the congruence by 2 yields $3 \equiv 5(mod\ 4)$, or 4 divides $(3 - 5)$, which is obviously not the case. For $ax \equiv ay(mod\ c)$ to imply $x \equiv y(mod\ c)$, GAUSS found that a and c must be relatively prime [GAUSS, 9].

He also considered the analogous solutions for linear and quadratic equations. Whereas a linear equation of the form $ax = b$ has a unique solution when a and b are integers with $a \neq 0$, the congruence $ax \equiv b(mod\ c)$ has a number of possible solutions. As for quadratic congruences, GAUSS gave the first rigorous proof of the law of quadratic reciprocity in the *Disquistiones Arithmeticae* [GAUSS, 87ff., Boyer, 551]. This law relates the solvability of the congruences $x^2 \equiv p(mod\ q)$ and $x^2 \equiv q(mod\ p)$ where p and q are distinct odd primes [See, for example, NIVEN & ZUCKERMAN, 90]. In a later attempt to generalize the law of quadratic reciprocity, that is, to find similar theorems for congruences of the form $x^3 \equiv p(mod\ q)$ and $x^4 \equiv p(mod\ q)$, GAUSS found it necessary to extend the notion of an integer beyond the rational numbers [BASHMAKOVA & KUDAKOV, 94; BOYER, 551]. Specifically, he established the idea of an integer within the complex number system. He showed that complex integers, that is, numbers of the form $a + bi$ where $a, b \in \mathbf{Z}$, were closed under addition, subtraction and multiplication just as in the case of the ordinary integers. He designated ± 1 and $\pm i$ as the *units* of the system (analogous to the role of ± 1 in the rational integers) and defined *associates* as numbers obtainable from one another by multiplication by a unit. Corresponding to each $a = a + bi$ where $a, b \in \mathbf{Z}$, he defined the norm of a to be $a^2 + b^2$. (Thus, GAUSS linked the ordinary integers with his new complex integers.) He called a non-unit complex integer a *prime* if a could not be written as a product of two complex integers, neither of which was a unit.

What did the primes look like in this new system? Composite integers remained composite in the new system but 5, a prime integer, could be factored into the two non-unit complex integers $(1 + 2i)(1 - 2i)$. GAUSS found that prime integers of the form $4n + 3$ remained prime in this system of complex integers, but those of the form $4n + 1$ did not. In general, he determined all the primes in this system by considering the norm of a complex integer a. (A complex integer a is prime in the system of complex integers if and only if the norm of a is p or p^2 where p is a rational prime integer of the form $4n + 3$.) Finally, he proved unique factorization in this system of complex integers using congruences and properties of the norm [BASHMAKOVA & RUDAKOV, 96].

The Arithmetic of Algebraic Numbers

Unique factorization into primes would not come quite so easily in the arithmetic of the algebraic number fields, however. In his search of the higher reciprocity laws, ERNST EDUARD KUMMER discovered that some [hyper]complex integers of the form $a_0 + a_1a + \cdots + a_{n-1}a^{n-1}$, where $a_i \in \mathbf{Z}$ and a is a complex, non-real nth root of unity, lacked unique factorization into primes.[2] "KUMMER regretted this inconvenient fact," H. M. EDWARDS has written in his careful study of KUMMER's work, "because he appreciated how important and useful the property of unique factorization was in arithmetic" [Edwards, 1980, 324]. KUMMER introduced a new kind of [hyper]complex number, an "ideal prime number," to compensate for this lack of unique factorization in algebraic number fields defined by a root of unity [DICKSON, 1917, 169; EDWARDS, 1980, 324; 1977, 105].

KUMMER never actually *defined* an ideal prime number but, rather, described it in terms of what it meant for such a number to divide a given algebraic integer. This test for divisibility depended on how the specified algebraic integer behaved under certain congruence relationships [DICKSON, 1917, 171; EDWARDS, 1980, 325–26]. Essentially, KUMMER's theory called upon these ideal prime numbers whenever unique factorization could not be achieved among the prime factors of the given algebra. DICKSON held this work in high regard. "It was the fundamental discovery of KUMMER," he remarked, "that the set of all complex integers defined by an nth root of unity could be so enlarged by the introduction of ideal numbers that unique factorization into primes would prevail in the enlarged set, which would therefore obey the laws of arithmetic as regards multiplication and division" [DICKSON, 1917, 170]. Moreover, DICKSON would not forget this strategy of enlargement when he considered the set of integral elements in an arbitrary algebra.

RICHARD DEDEKIND and LEOPOLD KRONECKER went on to obtain results similar to KUMMER's for general algebraic number fields Q(a), that is, algebraic extensions of the rational field Q, where a is the root of a polynomial with integer coefficients. These algebras consisted of numbers of the form $a_0 + a_1a + \cdots + a_{n-1}a^{n-1}$, where $a_i \in Q$. The "only serious obstacle" to overcome in generalizing KUMMER's theory to these algebraic number fields lay in the determination of an integral element [Edwards, 1980, 331]. As EDWARDS expressed it, "[a]lthough DEDEKIND and KRONECKER in other respects formulated their ideas in very different ways, they gave identical answers to this question: *An element of Q(a) is an integer if it is the root of a monic polynomial with integer coefficients*. (A polynomial is said to be monic if its leading coefficient is 1.) Unfortunately, neither of them gives a reason for this definition or explains how it was arrived at. Insofar as this is the crucial idea of the theory, the genesis of the theory appears, therefore, to be lost" [EDWARDS, 1980, 332]. The determination of integral elements would prove equally crucial in subsequent developments in the arithmetics of algebras as we shall soon see. Its genesis is not categorically lost, as the following pages reveal.

[2] We know these complex numbers as cyclotomic integers today. [EDWARDS, 1980, 324] advances the view that the study of higher reciprocity laws and *not* FERMAT's Last Theorem led KUMMER to his study of cyclotomic integers. EDWARDS developed this idea more thoroughly in his [1977, 79–81]. DICKSON felt that KUMMER's investigations of FERMAT's Last Theorem *and* the general law of higher reciprocity had led KUMMER to this work. See, for example, [DICKSON, 1917, 161, 170].

The Arithmetic of Quaternions: LIPSCHITZ

LIPSCHITZ proposed the first arithmetic of quaternions in 1886, a decade before DICKSON earned his doctorate from the University of Chicago and more than that long before the establishment of the definition of an algebra in general. In the opening paragraph of his "Recherches sur la Transformation, par des Substitutions réelles, d'une Somme de deux ou de trois Carrés en Elle-meme," LIPSCHITZ associated his arithmetic of quaternions with the arithmetic of the complex numbers formulated some fifty years earlier by GAUSS. As he put it, "[t]he theory, attributed to GAUSS, of complex integral numbers rests on the decomposition of these numbers into irreducible complex integral numbers. . . [I]n this study [of quaternions], the most important thing is the decomposition of integral quaternions into irreducible integral quaternions, in other words, prime quaternions" [Lipschitz, 373]. LIPSCHITZ also knew of GAUSS's method of factorization and outlined his strategy for implementing it: "[j]ust as the decomposition, into its prime factors, of an integral complex number depends on its norm, which is a sum of two full squares, so also the decomposition, into its prime factors, of an integral quaternion depends on its norm, which is a sum of four full squares" [LIPSCHITZ, 373–74].

In addition to drawing inspiration from the master, LIPSCHITZ considered the ideas of others, particularly those put forth by KUMMER and DEDEKIND for algebraic number fields. He outlined how all these ideas fitted together when he gave

> a general remark on the path which I have traced in my research. The decomposition of integral complex numbers into their prime factors is essentially linked, or so it seems, to the fact that the result of the multiplication of these numbers is independent of the order in which the operation is carried out. The multiplication of algebraic numbers has the same property, and once the appropriate notions are introduced, the laws of decomposition continue to hold....The demonstration of the existence of prime ideal factors in the theory, attributed to KUMMER, of integral complex numbers resulting from the division of the circle, are based, on the other hand, on certain congruences which appear in a simple form in the decomposition of integral complex numbers of the form $a+b\sqrt{(-1)}$. DEDEKIND has shown these corresponding congruences, referring to the theory of algebraic numbers. . . The reader will notice, in this paper, that the possibility of putting an integral quaternion in the form of the product of integral irreducible quaternions is based equally on the existence of certain congruences; but here the order in which the irreducible factors of the product succeed each other is most important. Thus, to the domain of integral complex numbers, as shown by GAUSS, should be added, on the one hand, the domain of all algebraic numbers, on the other hand, the domain of integral quaternions; but if one notices more than one analogy, one also notices more than one contrast between these different domains [LIPSCHITZ, 375–76].

Inasmuch as LIPSCHITZ had already linked his work with the integral complex numbers of GAUSS via norm properties, here he connected the development of the integral quaternions with that of the algebraic numbers by way of congruences. Moreover, he alerted the reader to differences in the various theories, namely, the importance of the order of the factors in his factorization of an integral quaternion.

With a desire to come to terms with these "analogies" and "contrasts," LIPSCHITZ began his development of the arithmetic of quaternions at the seemingly natural starting place, that is, with the concept of an integral quaternion which he defined as one with rational integer coefficients. He defined *units* as those integral quaternions with norm equal to 1. These are the analogues of ± 1 in the ordinary integers and $\pm 1,\ \pm i$ in the integral complex numbers. LIPSCHITZ's units, $\pm 1,\ \pm i,\ \pm j,\ \pm k$, do not mimic ± 1 in the integers precisely. LIPSCHITZ does not, for example, take into consideration any divisibility condition when defining his units. He went on to show that if A represents an arbitrary integral quaternion and I a unit, the eight resulting integral quaternions IA were indeed unequal. This, an accepted property of the quaternions today, would play an important role in LIPSCHITZ's factorization of integral quaternions.

Having established the concepts of an integer and a unit within the quaternions, LIPSCHITZ turned his attention to defining a prime integral quaternion. In this vein, he first commented that if an integral quaternion has prime norm and is a product of two integral quaternions, then one of its two factors has prime norm and the other is a unit. He subsequently defined a *prime quaternion* as one with norm equal to a rational prime number [LIPSCHITZ, 404–405]. In light of this set up, he needed a way to find integral quaternions with a given (prime) norm. If, for example, he wanted to factor the integral quaternion, $A = -1 + 3i + 1j + 2k$, with norm equal to fifteen into primes, he needed to find quaternions with norms equal to three and five. He laid out his solution to this type of problem in the following way [LIPSCHITZ, 1886, 416].

Given a *proper* integral quaternion (one with no common divisor among its coefficients) with norm m, LIPSCHITZ designated any of the prime odd factors of m by p, and formed a particular system of congruences modulo p^r where r is the highest power of p in m. (LIPSCHITZ treated the $p = 2$ case in a similar way [LIPSCHITZ, 414].) He reduced this more complicated original system of congruences to finding integral solutions of $x_1^2 + x_2^2 + x_3^2 \equiv 0\ (mod\ p^r)$ where $x_1,\ x_2,\ x_3$, are not all divisible by p. Out of the solutions to this congruence, LIPSCHITZ devised an arithmetic method to construct a proper integral quaternion with norm equal to the designated prime.[3] By multiplying this integral quaternion by each of the eight units, he formed what he termed a *class* of integral quaternions with norm p. Thus, each solution to $x_1^2 + x_2^2 + x_3^2 \equiv 0\ (mod\ p^r)$ ultimately generated eight integral quaternions with norm p. He established the number of classes of solutions to this congruence as $p^{r-1}(p+1)$, for p an odd prime.[4] With this collection

[3] DICKSON would later call LIPSCHITZ's theory "complicated" in his [1921a, 226]. By considering only a portion of LIPSCHITZ's method for constructing integral quaternions of norm 3 out of solutions to the congruence $x_1^2 + x_2^2 + x_3^2 \equiv 0$ (mod 3), we will see the accuracy of DICKSON's description. Taking the integers $x_1 \equiv 1,\ x_2 \equiv 2,\ x_3 \equiv 2$ (mod 3), which satisfy this congruence, LIPSCHITZ considered a new system of congruences $tp_1 \equiv x_1$ (mod 3), $tp_2 \equiv x_2$ (mod 3), $tp_3 \equiv x_3$ (mod 3) and $tp_4 \equiv 0$ (mod 3), for integers p_i less than $(p/2) = (3/2)$ and for t representing the greatest common divisor of the four integers. That is, in this case, $tp_1 \equiv 1$ (mod 3), $tp_2 \equiv 2$ (mod 3), $tp_3 \equiv 2$ (mod 3) and $tp_4 \equiv 0$ (mod 3), for integers p_i less than $(p/2) = (3/2)$. Thus, $tp_1 = 1,\ tp_2 = -1,\ tp_3 = -1$, and $tp_4 = 0$. With these values of the p_i's, LIPSCHITZ constructed a prime integral quaternion with norm 3 as $P = p_0 + p_1 i + p_2 j + p_3 k = 1 - i - j$.

[4] [LIPSCHITZ, 414–15]. Following the example in the previous note, the quaternion $P = 1 - i - j$ represents only one of $(p + 1) = (3 + 1) = 4$ possible classes of integral prime quaternions of

of (prime) integral quaternions, LIPSCHITZ determined (presumably by trial and error) the factorization of an integral quaternion. To factor, $A = -1 + 3i + 1j + 2k$, for example, he found triples satisfying $x_1^2 + x_2^2 + x_3^2 \equiv 0$ $(mod\ 3)$ and $x_1^2 + x_2^2 + x_3^2 \equiv 0$ $(mod\ 5)$, determined the associated integral quaternions with norms three and five, and obtained the two factorizations

$$-1 + 3i + 1j + 2k = (1 + 2i)(1 + i + j) = -(1 - i + j)(1 - 2k).$$

The above equation not only illustrates LIPSCHITZ's comment regarding the importance of the order of the factors but also shows the absence of unique factorization in his proposed system of integral quaternions. Switching the order of the factors yields a different quaternion. For example, $-(1 - 2k)(1 - i + j) = -1 - i - 3j + 2k \neq -1 + 3i + 1j + 2k = -(1 - i + j)(1 - 2k)$. The lack of unique factorization seems to grow worse when four divides the norm of the integral quaternion in question. In this case, with twenty-four renditions of an integral prime quaternion of norm 4, there are equally many factorizations. To his credit, however, this proposed system represented the first attempt at establishing an arithmetic for the quaternions. Moreover, as demonstrated in the remarks cited above, he studied the two most analogous examples available at the time – the arithmetics of the complex and the algebraic numbers – and drew pieces of his strategy from each of them. When ADOLF HURWITZ considered the problem of the arithmetic of the quaternions a decade later, he could remark that "Mr. LIPSCHITZ has already developed an arithmetic of quaternions" [HURWITZ, 1896b, 313].

The Arithmetic of Quaternions: HURWITZ

Although HURWITZ described LIPSCHITZ's paper as "interesting," he made it clear that he and his predecessor had "deeply rooted differences" in their approaches [HURWITZ, 1896b, 313]. In particular, HURWITZ noted that he and LIPSCHITZ grounded their research on integral quaternions differently [HURWITZ, 1896b, 313]. Whereas LIPSCHITZ depended on the congruence relationship to establish integral elements, HURWITZ relied on the concept of an integral domain [Integritätsbereich]. HURWITZ defined a system of infinitely many quaternions as an integral domain if it remained closed under the operations of addition, subtraction, and multiplication [HURWITZ, 1896b, 319]. Moreover, as HURWITZ noted, the arithmetic of quaternions, when developed along these latter lines, maintained a "similar simplicity" to elementary number theory [HURWITZ, 1896b, 313]. He seemed to use this final comment as a sort of "selling feature" of his arithmetic of quaternions since it preserved the structure of the arithmetic of rational integers. With LIPSCHITZ's arithmetic of quaternions lacking unique factorization (and a greatest common divisor

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norm three. The conjugate of $1 - i - j$, that is, $1 + i + j$, is another. LIPSCHITZ proved in general that an integral quaternion and its conjugate each belong to two different classes of solutions to the given congruence. Thus, once an integral prime quaternion is constructed according to this method, another class of integral prime quaternions of the same norm is easily obtained. In general, triples of numbers which satisfy $x_1^2 + x_2^2 + x_3^2 \equiv 0$ $(mod\ 3)$ and are not multiples of one another modulo 3, generate unique classes of solutions. The triples $(1, 2, 2)$, $(2, 2, 1)$, $(2, 1, 2)$, and $(1, 1, 1)$ form the four classes of prime quaternions of norm three.

process, although neither LIPSCHITZ nor HURWITZ mentioned this point), the alignment of HURWITZ's theory with the arithmetic of the rational integers brought a certain amount of authority to his work.

HURWITZ originally presented his arithmetic of quaternions in 1896 in his "Über die Zahlentheorie der Quaternionen." He "amplified" his work in the 1919 book, *Vorlesungen über die Zahlentheorie der Quaternionen* [HURWITZ, 1919].[5] Although the discussion of the integral quaternions is essentially the same in both publications, HURWITZ could (obviously) offer a broader view of the theory in his book. In particular, he began his chapter entitled "The Integral Quaternions" by noting that "[t]he arithmetic of the rational quaternions is essentially dependent on what one understands as an integral quaternion" [HURWITZ, 1919, 17]. In this statement, he revealed not only the underlying cause of the problems in LIPSCHITZ's theory, but also the key to his own version of it. In other words, with his definition of an integral quaternion, HURWITZ could construct an arithmetic of quaternions which possessed, among other properties, unique factorization as well as appropriate conditions for divisibility in general and for the greatest common divisor process in particular. His arithmetic thus represented what mathematicians might call a "better" theory; it had properties analogous to those of the (well-established) arithmetic of the integers.

Since the concept of an integral domain lay at the heart of HURWITZ's definition of an integral quaternion, he began the development of his theory by defining an integral domain and stating what he called the "Integral domain theorem:" In an integral domain one can select quaternions $q_1,\ q_2, \ldots,\ q_n$, for $n \leqq 4$, in a variety of ways such that each quaternion q from the integral domain can be represented in one and only one way in the form $q = k_1q_1 + \cdots + k_nq_n$ where $k_1,\ k_2,\ k_3,\ k_4$ are rational integers [HURWITZ, 1896b, 319]. HURWITZ called $q_1,\ q_2,\ \ldots,\ q_n$ the basis of the integral domain (that is, the finite basis for the integral domain [of quaternions] over the integers).

He explained the need for this preliminary definition and theorem when he next remarked that "the question of which maximum integral domain contains the basis elements $1,\ i,\ j$, and k is of fundamental significance for the arithmetic of quaternions" [HURWITZ, 1896b, 319]. (Here, we use the familar form $i,\ j$, and k of the basis elements of the quaternions. HURWITZ used $i_1,\ i_2$, and i_3, respectively.) To answer this question, HURWITZ defined r as

$$r = (1/2)(1 + i + j + k)$$

and constructed quaternions of the form

$$g = k_0r + k_1i + k_2j + k_3k, \tag{1}$$

where $k_0,\ k_1,\ k_2,\ k_3 \in \mathbf{Z}$. He called the collection of such quaternions $\boldsymbol{J}$ and noted that $\boldsymbol{J}$ was an integral domain with basis (r, i, j, k) [over the integers]. Moreover, $\boldsymbol{J}$

[5] DICKSON used "amplified" to describe HURWITZ's presentation of the arithmetic of quaternions in his 1919 book relative to his 1896 paper. With the exception of the statement noted below, we will take this discussion from HURWITZ's 1896 article in order to preserve the chronological development of the theory.

represented the maximum integral domain with rational integer coefficients containing the basis elements $1,\ i,\ j$, and k [HURWITZ, 1896b, 319]. As for other integral domains with these properties, he wrote that "[t]he quaternions with rational integer coefficients likewise form an integral domain [with basis $1,\ i,\ j,\ k$], which I refer to as $\boldsymbol{J}_0$. Obviously $\boldsymbol{J}_0$ is entirely contained in the domain $\boldsymbol{J}$ [HURWITZ, 1896b, 319]. (Note that $\boldsymbol{J}_0$ is precisely the set of integral quaternions defined by LIPSCHITZ.) Aside from $\boldsymbol{J}$ and $\boldsymbol{J}_0$, HURWITZ continued, there exist no other integral domains which contain the basis elements $1,\ i,\ j$, and k. With these results on integral domains in hand, he made only one additional comment before stating his definition of an integral quaternion. "After analogy with the theory of finite number fields," he explained, "one has to expect simple divisibility laws in the domain $\boldsymbol{J}$ as in the domain $\boldsymbol{J}_0$, something further examination perfectly confirms" [HURWITZ, 1896b, 319–20]. Thus, like LIPSCHITZ, HURWITZ considered the arithmetic of algebraic number fields when designing his arithmetic of quaternions.

In light of these properties of $\boldsymbol{J}$, HURWITZ stated that "a quaternion is called integral when it belongs to the integral domain $\boldsymbol{J}$" [HURWITZ, 1896b, 320]. Using (1), he gave the general expression for an integral quaternion as

$$g = (1/2)(g_0 + g_1 i + g_2 j + g_3 k), \tag{2}$$

where the g_i's are all even or all odd. Like LIPSCHITZ, he next turned his attention to the definition of units within this system. Unlike his predecessor, however, who defined units solely in terms of the norm, HURWITZ conceived of units as those integral quaternions which divided every integral quaternion on the left and the right. He then noted the equivalence of this divisibility property with the condition that the norm equal one. In order to make such a definition, though, he had to come to terms with the general form of the norm of an integral quaternion, that is,

$$N(g) = (1/4)(g_0^2 + g_1^2 + g_2^2 + g_3^2) \tag{3}$$

and to establish the definitions for left- and right-hand divisibility [HURWITZ, 1896b, 320]. Setting (3) equal to one, he obtained the twenty-four units [HURWITZ, 1896b, 321]

$$\pm 1,\ \pm i,\ \pm j,\ \pm k, \quad \frac{(\pm 1 \pm i \pm j \pm k)}{2}.$$

At this point in his treatise, he had defined the notion of an "integer" within the quaternions and had constructed an arithmetic which, like the rational integers, possessed units and divisors (albeit left- and right-hand divisors because of the non-commutative nature of the quaternions). To maintain further a "similar simplicity" with the integers, he was ultimately interested in establishing unique factorization among these integral quaternions. This motivated him to determine the existence of a greatest common (left- or right-hand) divisor of two integral quaternions and to state the definition of a prime integral quaternion.

Similar to GAUSS's notion of a prime complex integer, HURWITZ's prime integral quaternion, which he denoted p, was defined as a non-unit integral quaternion expressible only as the product of two integral quaternions, one of which was a unit [HURWITZ, 1896b, 330–31]. With this definition and the lemma that there exist (rational) integer solutions to the congruence

$$1 + r^2 + s^2 \equiv 0 \ (mod\ m), \quad [m \in \mathbf{Z}^+,\ m \text{ odd}]$$

he proved two theorems which together provided a convenient way to classify prime integral quaternions: "[i]f the norm of p is a prime number, [then] p is a prime quaternion," and "[i]f p is a prime quaternion, [then] the norm of p is a prime number" [HURWITZ, 1896b, 331]. These theorems, like the method used by GAUSS for determining primes in the complex integers and the definition suggested by LIPSCHITZ for prime quaternions, associated the prime integral quaternions with the prime integers. This association, which we will also see in the work of DICKSON, demonstrates the effective strategy of linking old and new concepts in order to gain analogies with established results. To conclude this section of his paper, just as LIPSCHITZ had done at a similar juncture in his work, HURWITZ determined the number of what he called *primary* prime quaternions with norm equal to a given odd prime p. (LIPSCHITZ found the number of *proper* integral quaternions with norm a given prime ($\neq 2$). HURWITZ defined *primary* quaternions as those integral quaternions $\equiv 1$ or $(1 + 2r)(mod\ 2(1 + i))$.)

With these theorems in place, HURWITZ addressed the question of the factorization of an integral quaternion into prime quaternions [HURWITZ, 1896b, 333]. He based this work on the fact that he could express any integral quaternion a as

$$a = (1 + i)^r \times b \tag{4}$$

where b represented an odd integral quaternion, that is, one of odd norm [HURWITZ, 1896b, 325–27]. Moreover, he could treat b as a primary quaternion since he showed that each odd quaternion, when multiplied by some unit on the right, could be represented as a primary quaternion. In light of his result that the product of two primary quaternions was again primary, HURWITZ expressed b as the product mc, where m was the greatest common divisor of the (integer) coefficients of b and c was a primitive quaternion, that is, a primary quaternion with relatively prime coefficients. Thus, in light of (4), he could now express an arbitrary integral quaternion as $a = (1 + i)^r mc$. Since $m \in \mathbf{Z}$ and $(1 + i)^r$ is a power of a prime integral quaternion, the factorization of a depended on that of c. He handled this thanks to the following theorem: For c a primitive quaternion, let $\mathrm{N}(c) = pqr \ldots$, where $p, q, r, \ldots$ are the prime factors, not necessarily distinct, of $\mathrm{N}(c)$, arranged in an arbitrarily chosen, but fixed, order. Then c can be expresed in one and only one way as

$$c = PQR \ldots,$$

where $P, Q, R, \ldots,$ represent primitive prime quaternions with norms $p, q, r, \ldots,$ respectively [HURWITZ, 1896b, 334]. This established unique factorization in the arithmetic of quaternions based on the definition (2) of an integral element.

Thus, HURWITZ successfully stretched the concept of "integer" – already moved beyond the realm of the rational integers by GAUSS, KUMMER, DEDEKIND, and others – to the quaternions.[6] He accomplished this by enlarging the set of integral quaternions proposed by LIPSCHITZ a decade earlier to establish a more far-reaching, "better" theory.

[6] [GRAY] argues that the concept of the integer was stretched in the nineteenth-century and a new ontology emerged to support the expanded idea. See pp. 229–233, in particular.

Perhaps owing to his prior research on ideal theory applied to the theory of algebraic numbers [HURWITZ, 1894, 1895a, 1895b], HURWITZ began his theoretically more sophisticated view of the arithmetic of quaternions by defining integral quaternions according to whether or not they belonged to the maximal domain of finite basis [over the rational integers] which contained the basis elements of the quaternions; he *then* considered the resulting coefficients. Apparently, by taking his focus off of the coefficients of an integral quaternion, HURWITZ took an important step foward and constructed a rather *un*-integer-looking integral quaternion which led to an arithmetic nevertheless analogous to that of the integers.

DU PASQUIER: The Unknown Predecessor of DICKSON

HURWITZ imprinted his interest in the arithmetics of number systems, and, in particular, his emphasis on the role of an integral element in the development of these arithmetics, upon his sixth doctoral student at the Polytechnic Institute in Zürich, L. GUSTAVE DU PASQUIER [HURWITZ, 1962, 2: 754–55]. In his 1906 dissertation, "Zahlentheorie der Tettarionen," DU PASQUIER used ideal theory to develop the arithmetic of matrices with rational entries within the broader framework defined by his desire to apply the fundamental ideas and methods of the rational numbers to non-commutative number systems [DU PASQUIER, 1906]. He knew of the advances made in this area by LIPSCHITZ and HURWITZ with the non-commutative quaternions, and, apparently, he saw the (non-commutative) matrices as the next uncharted territory in which to extend the concept of integer and to develop the associated arithmetic. Like his adviser, DU PASQUIER emphasized the determination of the "integral" (and "fractional") matrices as the first necessary step in this investigation [DU PASQUIER, 1906, 73]. He defined an integral $n \times n$ matrix as one with integer coefficients [DU PASQUIER, 1906, 73]. Over the next fifty or so pages of this article, he utilized the theories of congruences and ideals to establish an arithmetic for such matrices [DU PASQUIER, 1906, 73–120].

In 1909, DU PASQUIER extended his work to more general settings. In "Über Holoïde Systeme von Düotettarionen," he put forth the first arithmetic for an arbitrary rational hypercomplex number system with four or fewer basis elements [DU PASQUIER, 1909].[7] He represented the elements as 2×2 matrices and, emulating the theory of algebraic numbers, suggested that an element was integral if it satisfied a monic polynomial of degree two with rational coefficients [DU PASQUIER, 1909, 119]. (Of course, DU PASQUIER used slightly less modern language.) This definition, however, did not yield a system (in DU PASQUIER's terminology) which remained closed under addition and multiplication [DU PASQUIER, 1909, 119]. Since DU PASQUIER considered it "only natural" to "demand" these properties in a system of integral elements, he found himself faced with the dilemma of attempting to "repair" the definition or to begin again [DU PASQUIER, 1909, 119]. He determined that he would not pursue this definition any further, but would instead base

[7] A *düotettarion* refers to a 2×2 matrix, or, equivalently, a hypercomplex number with 4 basis elements. DU PASQUIER does not use either of the terms hypercomplex number system or linear associative algebra in this paper (or in his earlier works).

his investigation on the analogy with the usual rational numbers [Du Pasquier, 1909, 119].

To this end, DU PASQUIER listed the properties of the rational integers and then constructed a system of integral elements with these same characteristics. This led him to define a rational domain of 2×2 matrices as "holoidal" if it contained the identity, posessed a finite basis, and remained closed under addition, subtraction, and multiplication [DU PASQUIER, 1909, 122]. In his language, the sets of integral elements were precisely the maximal "holoidal" domains. He proved the existence of infinitely many of these maximal and non-maximal "holoidal" domains and determined their general structure [DU PASQUIER, 1909, 132–33, 139–40]. In the final two pages of this work, however, he revealed a tiny problem with this definition. A "düotettarion" could belong to both a maximal and a non-maximal domain. To remedy this situation, he defined an element with this property as "integral with respect to" the maximal domain [DU PASQUIER, 1909, 147–48].

Thus, in 1909, guided by his desire to apply the theory of the rational integers to other number systems and his emphasis on the integral elements, DU PASQUIER made headway towards establishing the arithmetics of algebras with four (or fewer) basis elements. He had not entirely achieved his goal with these algebras, however. The introduction of a subsidiary definition of "integral" in the final two pages of the work indicated a kink in the theory which would have to be ironed out before he could claim a definition of an integral element analogous to that of the ordinary integers. From the modern viewpoint, we can see how DU PASQUIER's thinking ensnared him. By modeling his theory strictly on the rational numbers, where a unique (maximal) set of integers exist, he (naturally) looked for *the* set of integral elements in other hypercomplex number systems. A single set of integers does not often occur outside the rational numbers, however. Although he did not know it, DU PASQUIER began, in some sense, to press the bounds of the concept of a single set of integers to multiple sets of integers within one algebra. Moreover, at this point, he had only addressed the issue of the integral elements themselves. He had not yet considered the arithmetic – greatest common divisors, primes, and unique factorization – of these integral elements.

Some nine years passed before DU PASQUIER reentered the arena of the arithmetics of algebras. In 1918, he addressed the arithmetics of what he called the second and third types of complex numbers, $\{a + bj \mid j^2 = 1\}$ and $\{a + be \mid e^2 = 0\}$, respectively, where $a,\ b \in \mathbf{R}$ [DU PASQUIER, 1918]. For the first of these two sets, he initially suggested what he termed a "LIPSCHITZIAN" definition for the integral elements, namely, $\{a+bj \mid a, b \in \mathbf{Z}\}$. This definition, however, led to "curious anomalies" in the corresponding arithmetic like the fact that there existed no integers with norm two since $a^2 - b^2 = 2$ had no solutions in this proposed set of integers [DU PASQUIER, 1918, 452–3]. Moreover, the decomposition into prime factors, if possible, was not necessarily unique. To illustrate this aberration, DU PASQUIER cited the example [DU PASQUIER, 1918, 452]

$$(9 - 7j)(17 + 15j) = 2(24 + 8j) = 16(3 + j).$$

He compensated for these anomalies, by proposing what he referred to as a new definition for the integral elements in the system $\{a + bj \mid j^2 = 1\}$: "A complex number of the second kind will be called integral if it is of the form $a + (b/2) + (b/2)j$, where a and b represent any ordinary integers" [DU PASQUIER, 1918, 453]. He then showed that this definition led to an arithmetic which not only overcame the previous difficulties but also mimicked that of the ordinary rational numbers. What he did not show, however, proves equally interesting. DU PASQUIER did not mention that his proposed set of integral elements coincided precisely with those elements which satisfied monic polynomial equations with rational integer coefficients.[8] That is, DU PASQUIER may have put forth what he referred to as a "new" definition but, in actuality, it did not differ – in terms of the objects it selected – from the one proposed by DEDEKIND and KRONECKER for general algebraic number fields. This raises the question: did DU PASQUIER derive his set of integral elements for the complex numbers $\{a+bj \mid j^2 = 1\}$ using the standard definition for algebraic number fields? We simply cannot answer this question definitively. On the one hand, as we have seen, DU PASQUIER's 1909 work on arbitrary rational hypercomplex number systems with four or fewer elements certainly indicates his awareness of the monic polynomial definition for algebraic integers – and even his willingness to use it as a guide when looking for integral elements in a given hypercomplex number system. On the other hand, in this 1918 paper, DU PASQUIER suggests his definition grew out of his theory of maximal holoidal domains (see below).

Apparently, though, he knew this "new" definition would produce an arithmetic which outdistanced that generated by the "LIPSCHITZIAN" one. Just what compelled DU PASQUIER to use the LIPSCHITZ-style definition in this discussion, especially in light of its checkered past relative to the quaternions? He may simply have had a "royal road to me" agenda in his writing style. He noted GAUSS's successful use of the "LIPSCHITZIAN" definition in the development of the complex integers. He then asked why this definition worked in the case of the complex integers but not for the complex integers of the second type. To answer this question, he presented his theory of maximal "holoidal" domains and defined a rational hypercomplex number to be integral if it belonged to such a domain [DU PASQUIER, 1918, 455–56]. Perhaps to marshal support for his theory, he pointed out that the ordinary rational integers and GAUSS's complex integers satisfied the conditions for a maximal "holoidal" domain. More importantly, he emphasized that *his* definition of an integral element led to an arithmetic analogous to the ordinary integers for the complex integers of the second type. DU PASQUIER concluded this section of his work by noting that his "new" definition represented an enlargement of the LIPSCHITZ-style integers [DU PASQUIER, 1918, 457].

Thus, in this portion of his treatise, DU PASQUIER made three key points. First, he generalized LIPSCHITZ's definition of an integral quaternion to that of an integral element with integer coefficients. Secondly, he emphasized the theory of what he called a maximal "holoidal" domain as the more general approach to constructing integral elements in a system of rational [hyper]complex numbers. (Incidentally, he used somewhat of a postulational formulation to state the properties of these domains.) Thirdly, he mentioned the strategy of enlargement – already employed by his adviser, HURWITZ, in his work on the arithmetic of quaternions – in obtaining the definition of integral elements for

[8] DU PASQUIER's set of integers $\left(a+\frac{b}{2}\right)+\left(\frac{b}{2}\right)j$ (for $a,\ b\in\mathbf{Z}$) $=\frac{2a+b}{2}+\left(\frac{b}{2}\right)j=c+dj$ where $c=\frac{2a+b}{2},\ d=\frac{b}{2}j$. These elements satisfy monic polynomial equations with rational integer coefficients when $(2a+b)$ and b are either both even or both odd. And $(2a+b)$ is even (odd) if and only if b is even (odd).

complex integers of the second type which led to unique factorization and other properties of the ordinary integers. As an historical key point related to these latter two remarks, we again note that although DU PASQUIER may have situated his enlarging definition of an integral element in his theory of maximal holoidal domains, he actually derived the same set of integral elements as the monic polynomial definition given a few decades earlier.

The remainder of this work concerned establishing an arithmetic for the third type of complex numbers $\{a+be \mid a, b \in \mathbf{R}, e^2 = 0\}$. The arithmetic for this system, however, did not possess the "elegant simplicity" of the rational numbers [DU PASQUIER, 1918, 461]. To begin with, DU PASQUIER could not find a *single* maximal holoidal domain to contain his integral elements. He stated that the domain $D_1 = \{a + bse \mid a,\ b \in \mathbf{Z},\ s \in \mathbf{Q}$, arbitrarily chosen, but fixed$\}$ possessed closure under addition, subtraction and multiplication, contained the identity element, and had the finite basis $1, se$. Thus, D_1 represented a "holoidal" domain, not necessarily maximal, for each $s \in \mathbf{Q}$.[9] In light of this, DU PASQUIER abandoned the maximality condition and defined a complex number of the third type as integral if it belonged to any *one* of these domains. (This would prove to be a key decision for DU PASQUIER. He kept his finite basis requirement and gave up the maximality condition. To gain maximality, one has to relinquish the finite basis, and this, as we shall see, is precisely what DICKSON did.) He defined primes and units within the framework of this definition, but he could not always obtain a greatest common divisor for two integral elements [DU PASQUIER, 1918, 460–461]. Aware of these difficulties, he saw the need for more "profound methods" and proposed finding these in the theory of ideals [DU PASQUIER, 1918, 461].

Inasmuch as DU PASQUIER revealed the success of his general theory of "holoidal" domains in the first half of this paper, he unwittingly demonstrated its flaws in the work's last section. He could not apply it to the complex numbers of the third type, yet he remained convinced of his theory's potency. Still, in this paper and in the publication of 1909, DU PASQUIER was forced to abandon one of his guiding principles, that of mimicking the arithmetic of the rational integers. In 1909, he had to alter the notion of an integer; now he had to allow for multiple sets of integral elements.

Compromise remained in the air, albeit still subtly, over the next two years. In 1920, DU PASQUIER proposed three categories of hypercomplex numbers, including two types of integers, in his "Sur la Théorie des Nombres hypercomplexes à Coordonnées rationnelles" [DU PASQUIER, 1920]. The very title of this paper suggests other changes in the author's thinking. In particular, he had adopted the term "hypercomplex number," by then more than two decades old. Moreover, he had moved beyond matrices and specific algebras to a more broadly based investigation of the general theory of hypercomplex number systems.

He began this paper with a rather modern-looking introduction to rational hypercomplex number systems. He then set out "to build the arithmetic of this body [corps] of numbers" [DU PASQUIER, 1920, 113]. By way of background, he briefly recalled the obvious "LIPSCHITZIAN" definition and its "curious anomalies," the analogy he wanted to

[9] For example, as DICKSON noted in his [1923a, 145–46], $(1,\ se)$ is contained in the larger set $(1,\ \frac{1}{2}se)$, which is contained in the still larger set $(1,\ \frac{1}{4}se)$, etc. Hence there is no maximal set.

maintain between integral hypercomplex numbers and the rational integers, and the first satisfactory definition of an integral quaternion given by HURWITZ [DU PASQUIER, 1920, 113–14]. DU PASQUIER characterized the sets of integral numbers as maximal holoidal domains and defined a hypercomplex number as integral if it belonged to one particular domain [DU PASQUIER, 1920, 114–15]. By adopting this new definition, he declared, an arithmetic analogous to that of the ordinary rational integers results.

In the next paragraph, however, he introduced the idea of "integral with respect to the domain $\{M_p\}$" for hypercomplex numbers which belonged to the particular maximal domain $\{M_p\}$ but perhaps to no others [DU PASQUIER, 1920, 115–16]. This definition subsequently forced him to admit that he could not absolutely determine whether or not a given hypercomplex number was *integral* [DU PASQUIER, 1920, 116]. He thus divided a system of hypercomplex numbers into the three categories of *absolute integers, absolute fractions, and conditional integers*. Not surprisingly, these referred to numbers which belonged to all, no, or some but not all, of the maximal holoidal domains, respectively [DU PASQUIER, 1920, 116]. DU PASQUIER assumed that the arithmetic based on a system with multiple maximal holoidal domains would take on an "arbitrary" nature and declared that his "[e]xperience confirms this presumption" [DU PASQUIER, 1920, 116]. He supported this statement in the concluding section of this paper by determining, where possible, the maximal holoidal domains, or integral elements, of hypercomplex numbers in two and three variables. When he discussed the previously problematic hypercomplex number system $\{a + be \mid e^2 = 0\}$, for example, he concluded that the definition of a complex integer in this system was, in a certain sense, arbitrary [DU PASQUIER, 1920, 122].

DU PASQUIER chose to discuss his ideas on the arithmetics of hypercomplex number systems in his "Sur les Nombres complexes généraux" at the 1920 International Congress of Mathematicians in Strasbourg, where, along with DICKSON, he presided over the sessions on "Arithmetic. Algebra. Analysis" [DU PASQUIER, 1921]. Since both men maintained somewhat of a high profile at the Congress and since each included the arithmetics of algebras in his talk, it seems highly probable that they heard one another speak.

In Strasbourg, DU PASQUIER discussed the arithmetics of hypercomplex number systems in terms of three categories [Du Pasquier, 1921, 165–66]. As he saw it, an algebra possessed either an arithmetic like that of the rational integers, one which could be restored to this "classic" arithmetic by the theory of ideals, or one which, even after the introduction of ideals, did not reflect the properties of the ordinary integers. DU PASQUIER presented his maximal holoidal domain definition of an integral element as a way to overcome the anomalies in the third category of arithmetic [DU PASQUIER, 1921, 166]. As before, however, his definition led to difficulties when more than one – or no – such domain existed in the algebra [DU PASQUIER, 1921, 167–8]. (In the case where no maximal holoidal domain existed in the algebra, DU PASQUIER adopted the "LIPSCHITZIAN" definition of an integral element and accepted an "irregular" arithmetic.) He concluded his lecture by determining the arithmetics of all twenty-four possible hypercomplex number systems with four basis elements [DU PASQUIER, 1921, 169–175].

After two years of wrestling with these particular hypercomplex number systems, DU PASQUIER made no changes to his definition of a set of integral elements but, rather, attempted to alter the concept of integer within this system. Apparently, he could not determine a definition which yielded a unique set of integral elements in a hypercomplex number system like, say, the integers in the rational numbers. Thus, he could only conclude that certain hypercomplex number systems had an absolute and unique arithmetic, while others had an arbitrary arithmetic which depended on the chosen set of integral elements. This was not exactly what DU PASQUIER desired. In his 1924 summary of the development of an integral element, following his definitions dependent on whether one or infinitely many maximal holoidal domains existed in the hypercomplex number system, he remarked that "it is naturally preferable" to have one and the same definition apply to all hypercomplex number systems [DU PASQUIER, 1924, 201]. This very ambiguity indicated that the arithmetics of hypercomplex number systems remained unsettled. DU PASQUIER certainly brought the subject a long way from the arithmetization of the complex integers and the quaternions by GAUSS and HURWITZ. The final complete theory, however, along with its clear and concise presentation, lay in the hands of a mathematician more than an ocean away on an ordinary day, but one possibly among those in DU PASQUIER's audience in Strasbourg.

DICKSON's Early Work in the Arithmetics of Algebras

This mathematician, who axiomatized hypercomplex number systems [DICKSON, 1903, 1905] about the time DU PASQUIER wrote his dissertation on the arithmetic of matrices, came to the study of the arithmetics of algebras in the 1920's. With the integral quaternions a necessary component in LEONARD DICKSON's search for solutions to certain Diophantine equations, he found himself compelled to come to terms with the arithmetic of this specific algebra initially and with more general algebras later. Thus, the arithmetic of quaternions, the subject of DU PASQUIER's adviser's research and the avenue DICKSON followed to obtain solutions to outstanding number-theoretic problems, represented a common influence on the early work of both DU PASQUIER and DICKSON in the arithmetics of hypercomplex number systems.

DICKSON published his first complete work (as opposed to occasional results and comments) on the subject of quaternions in his 1921 paper, "Arithmetic of Quaternions" [1921a]. Here, like DU PASQUIER, he indicated an acquaintance with the existing theories of LIPSCHITZ and HURWITZ when he wrote

> R. LIPSCHITZ quite naturally called only those quaternions integral whose coordinates are integers (whole numbers). His complicated theory of integral quaternions was based upon the solutions of congruences
>
> $$e^2 + n^2 + c^2 = 0 \ (\mathrm{mod}\ p^k).$$
>
> He made no mention of a greatest divisor process, which in fact is not applicable in general.
>
> A. HURWITZ succeeded in developing a perfect arithmetic of quaternions by taking as his integral quaternions those whose coordinates are either all integers or all halves of odd integers. But the presence of the denominators 2 in certain integral quaternions was an inconvenience in the application which I recently made of HURWITZ's theory to the complete solution in integers of quadratic equations in several variables [the results presented in his Strasbourg paper] [DICKSON, 1921a, 226].

Thus, in 1920, DICKSON found LIPSCHITZ's theory for the arithmetic of quaternions "complicated" and incomplete and HURWITZ's theory "perfect" in general but too inconvenient for applications to problems in number theory. In reaction to these deficiencies, his own strategy involved formulating

> a new theory of the arithmetic of quaternions in which, following LIPSCHITZ, the integral quaternions are those whose coordinates are integers exclusively. Call such a quaternion odd if its norm is odd. I shall prove that, if at least one of two integral quaternions is odd, they have a greatest common divisor which is expressible as a linear combination of them. It is then a simple matter to develop the theory of factorization of integral quaternions [DICKSON, 1921a, 226].

DICKSON, thus, took what he considered the best features of each of the previous theories to develop a "new," potentially more useful approach to the arithmetic of quaternions.

As his remarks indicate, DICKSON built his new arithmetic on the notion of an odd quaternion. He did not mention, however, that this idea originated with HURWITZ [HURWITZ, 1896b, 316]. Moreover, he failed to credit HURWITZ for the statements and proofs of many of his theorems and lemmas.[10] This lack of citation reflected DICKSON's general propensity for omitting references to mathematical publications other than his own. Although stingy with his (favorable or neutral) recognition of related works, he did not skimp on his mathematics. In this particular case, for example, DICKSON illustrated the necessity of the odd norm when determining the greatest common divisor of two integral quaternions in his theory. He wrote:

> [t]he limitation made in Theorem 3 [establishing the greatest common divisor of two integral quaternions and expressing it as a linear combination of them] that one of the quaternions be odd is essential. In fact, there exists no greatest common divisor of 2 and $q = 1 + i + j + k$, each of norm 4. If either 2 or q be a product of two integral quaternions not units, each factor is of norm 2. But $2 = (1 + i)^2(-i)$. Hence the only factors of 2 are the quaternions associated with $2,\ 1,\ 1+i,\ 1+j,\ 1+k$ (the last three being indecomposable); while those of q are associated with $q,\ 1,\ 1+i,\ 1+j,\ 1+k$. The only common factors are the last four, no one of which is divisible by all the others. Finally, 2 and q are not associated quaternions [DICKSON, 1921a, 227–28].

In this way, DICKSON also indirectly revealed the pitfalls of LIPSCHITZ's arithmetic of quaternions.

Finally, DICKSON justified his approach to the arithmetic of quaternions and situated it relative to the other existing theories when he claimed that

> [t]he limitation that one of the quaternions is odd causes no inconvenience for the applications [to number theory]. Moreover, it is of theoretical interest to know exactly to what extent we can meet the difficulties which arise in the arithmetic of quaternions in which the integral quaternions are defined naturally to be those

10 Compare, for example, Theorem 2 in DICKSON's [1921a] with HURWITZ's theorem on p. 313 in [1896b].

> with integral coordinates exclusively. Furthermore, the present theory is more direct and elementary than the earlier theories [DICKSON, 1921a, 226].

Thus, despite the fact that HURWITZ had already developed a "perfect" arithmetic of quaternions, DICKSON maintained a "theoretical interest" in integral quaternions defined as those with integer coefficients. In particular, since number theory is concerned with integers, the "natural" definition of an integral quaternion (as one with integer coefficients) intrigued DICKSON. A question along the lines of "what must be done to this set to yield the ordinary operations of arithmetic?" seemed to guide his work.

DICKSON presented his arithmetic of quaternions in the brief, terse writing style he tended to maintain throughout his mathematical career. Aside from motivating and justifying his definition, he included only the definitions, lemmas, and theorems essential to his arithmetic of quaternions. His presentation reads like that of the arithmetic of the integers in a modern number theory text. (See, for example, chapter one in [NIVEN & ZUCKERMAN]). In other words, he established: the greatest common divisor for two integral quaternions (one with odd norm); the definition of relatively prime integral quaternions; the relationship between relatively prime integral quaternions and relatively prime norms; the definition of prime integral quaternions; the association between prime numbers and integral prime quaternions; and the unique factorization of integral quaternions into primes. DICKSON's proofs relied on number-theoretic techniques – properties of division, factorization, and congruence, in particular – and not on the seemingly extraneous results used by his predecessors. (LIPSCHITZ's arithmetic of quaternions, recall, hinged on a complicated and involved theory of congruences.) Overall, his writing style indicates his emphasis on what he saw as clear and concise mathematics. His next work in the arithmetics of algebras reveals yet another characteristic of his mathematics: rigorous standards. By rigorous standards here, we mean that terms are adequately defined and statements are sufficiently proved. "Completeness," perhaps, is another accurate description of this aspect of DICKSON's mathematics.

In 1922, DICKSON wrote his "Impossibility of Restoring Unique Factorization in a Hypercomplex Arithmetic" in direct response to DU PASQUIER's claims regarding the hypercomplex number system $\{a + be \mid e^2 = 0\}$ [DICKSON, 1922, 438–442]. DICKSON explained in his opening paragraph that, in this system, most integral numbers

> admit of several factorizations into indecomposable numbers. It is proved... that we cannot restore unique factorization by defining hypercomplex ideals analogous to algebraic ideals, nor... by the introduction of any sort of ideals obeying the laws of arithmetic. L. G. DU PASQUIER had made statements, omitting proofs, concerning the failure of unique factorization after introducing ideals, apparently meaning those analogous to algebraic ideals [DICKSON, 1922, 438].

Thus, DICKSON took it upon himself to establish firmly (more than) DU PASQUIER's assertion regarding unique factorization in this algebra. In addition to this result, he presented a method for determining the factorizations of a hypercompex integer in this system [DICKSON, 1922, 442]. Whereas DU PASQUIER merely called the arithmetic "arbitrary" (a rather vague term in itself), DICKSON proved that ideal theory of *any sort* failed to assure unique factorization, and he provided a general method for finding the possible factorizations of a hypercomplex integer.

The Crucial Role of Integral Elements: DICKSON's Realization

These qualities became even more apparent in 1923 and 1924, the two banner years of DICKSON's work in the arithmetics of algebras. Gone were the articles concerned with the arithmetic of a *single* hypercomplex number system and in evidence were the books and papers on a *general* theory of the arithmetics of algebras. He began his mainline treatment of the subject in 1923 with his "A New Simple Theory of Hypercomplex Integers" and the book which would ultimately become so closely associated with his name, *Algebras and Their Arithmetics*. In 1924, he published an article with the same title as his book and presented both an "Outline of the Theory to Date of the Arithmetics of Algebras" and "Further Development of the Theory of the Arithmetics of Algebras" to the International Mathematical Congress in Toronto. The very titles of these articles reflected DICKSON's sense of progress made in the area.

Although we can characterize certain common aspects of these publications, DICKSON wrote each manuscript with a specific purpose which he briefly outlined in his introductions. In his "A New Simple Theory of Hypercomplex Integers," for example, he set out to

> present a new conception of hypercomplex integers which is entirely free from the fatal objections valid against the earlier conceptions of HURWITZ and DU PASQUIER. If their definitions are taken literally, there do not exist hypercomplex integers in the majority of algebras of hypercomplex numbers. If we discard a certain one of their assumptions, we obtain integers but are faced with the insurmountable difficulty that factorization into primes is not only not unique, but cannot be made unique by the introduction of ideals of any kind, a fact proved in this Memoir. These essential difficulties all disappear under the new definition proposed here [DICKSON, 1923b, 281].

DICKSON claimed that his definition was not only *new* in the sense that it differed from earlier theories, but also *simple* in the way that it avoided the previous "insurmountable difficulties." Thus, in keeping with his broader mathematical standards, which called for the most far-reaching and complete theories possible, he put forth a single definition of a hypercomplex integer which applied to an *arbitrary* algebra and led to an arithmetic with *all* the desired properties. As for his style of exposition, DICKSON would (of course) require a clear and concise expression of his mathematical ideas. As seen in the paragraph above, however, he conveyed more than just mathematical ideas. His use of words and phrases like "fatal objections," "insurmountable difficulty," and "essential difficulties," hardly disguises his bias.[11] This description presents in miniature the strategy DICKSON tended to employ when setting out his ideas on the development of integral elements in hypercomplex number systems: he stated the existing ("erroneous") theories of integral

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[11] Consider these remarks in light of DICKSON's view of the role of an historian as expressed in [DICKSON, 1919a, 2:xx]: "What is generally wanted is a full and correct statement of the facts, not an historian's personal explanation of those facts. The more completely the historian remains in the background or the less conscious the reader is of the historian's personality, the better the history." Specifically, in this quasi-historical account of the evolution of a hypercomplex integer, DICKSON hardly relegated his mathematical persona to the "background."

hypercomplex numbers, discussed the "difficulties" with these views, and introduced "new" ideas to overcome these inconsistencies.[12] Thus, DICKSON used a combination of mathematics *and* rhetoric to convince his audience of the superiority of his work. In this case, at least, the "proof" of his theory went beyond the traditionally recognized bounds associated with the formal presentation of mathematics.

Before DICKSON laid out the details of his "new simple theory," he gave what he called a "preliminary survey of hypercomplex integers" which he began with a brief reference to GAUSS and his definition of a complex integer [DICKSON, 1923b, 285–89]. He next turned his attention to the integral algebraic numbers and motivated the discussion by considering the integral elements in the algebraic number field $\mathbf{Q}(\sqrt{-3})$. In particular, he showed the lack of unique factorization into primes in $\mathbf{Q}(\sqrt{-3})$ if the definition of a quadratic (algebraic) integer referred to numbers of the form $a + b\sqrt{-3}$ for $a, b \in \mathbf{Z}$. A change in the definition of a quadratic integer, however, to one where a, b both represented rational integers *or* halves of odd integers, ensured unique factorization. (This result, as Dickson noted on the next page, came immediately from the "standard" monic polynomial definition of algebraic integer. See below.) On a broader scale, this alteration in the definition of an integral element demonstrated what DICKSON called the "wisdom of enlarging proposed systems of integers" [DICKSON, 1923b, 286]. With this comment, he rather clearly disclosed the value he placed on the process of "enlargement" by associating "wisdom" with the use of this technique. This would prove to be a (calculated?) wise move, for he would soon describe his own contributions to the development of hypercomplex integers in these terms.

Having presented an example involving integers in a specific algebraic number field, DICKSON gave the definition of an integral algebraic number in general. "An algebraic number is called integral," he explained, "if and only if it is a root of an equation having rational integral coefficients and having unity as the coefficient of the highest power of the unknown" [DICKSON, 1923b, 286]. He showed that for quadratic integers $a + b\sqrt{m}$, the value of m determined the coefficients for the integral elements. For $m = -1$, for example,

$$x^2 - 2ax + (a^2 - mb^2) = 0 \tag{5}$$

has integral coefficients precisely when a and b belong to the ordinary rational integers. Thus, the roots $x = a + b\sqrt{m}$ of Eq. (5) with integer coefficients take the form of the complex integers as defined by GAUSS. For $m = -3$, as Dickson himself noted, $x^2 - 2ax + (a^2 + 3b^2) = 0$ has rational integral coefficients when a and b are either both integers or both halves of odd integers. This definition did more than just produce the previously established complex integers in the field $\mathbf{Q}(\sqrt{-1})$, however. In the more general case, it led to the unique factorization of integral elements in an algebraic number field into primes, a result DICKSON described as "wholly satisfactory" and the reason for which he termed this definition "thoroughly satisfying" [DICKSON, 1923b, 287].

[12] DICKSON employed the same strategy in his presentation of algebraic numbers in his [1917, 168–170]. The three section headings tell it all: "Erroneous views on algebraic numbers prevalent in 1847;" "Difficulties presented by example;" and "Difficulties removed by use of KUMMER's ideal numbers."

DICKSON may have found this concept of an integer "satisfactory" for the algebraic number fields, but unfortunately, as he quickly pointed out, it "fails in general for hypercomplex numbers" [DICKSON, 1923b, 287]. To verify this statement, he constructed a set of integral 2×2 matrices according to this definition and showed that it lacked closure under multiplication and addition.[13] He also showed that "[w]e do not obtain satisfactory results by following the definition of integral algebraic numbers and calling a quaternion q integral if and only if its coordinates are rational and the coefficients of the quadratic equation satisfied by q and its conjugate… are rational integers" [DICKSON, 1923b, 287–88]. This definition may have failed in general when proposed as the *only* criterion for integers in the quaternions and other hypercomplex number systems, but DICKSON did not cast it aside entirely. In fact, it was ultimately his use of this idea which distinguished his concept of an integral element in an arbitrary algebra from earlier definitions crafted for more restricted settings.

Before stating his ideas, however, DICKSON gave his first of several glosses on the work of his predecessors, HURWITZ and DU PASQUIER. In what would ultimately become a rather trademark presentation of their theories, DICKSON asserted that,

> [a]lthough the definition [of a hypercomplex integer] by HURWITZ was stated only for quaternions, it may be expressed in general form as follows.
>
> Within a rational hypercomplex number system… a system of *integral* hypercomplex numbers shall have the following properties:
>
> B (basis): The system has a finite basis…
>
> C (closure): The system is closed under addition, subtraction, and multiplication;
>
> U (units): The system contains the basal units $e_0, \ldots, e_n$;
>
> M (maximal): The system is a maximal (i.e., it is not contained in a larger system having properties B, C, U).
>
> The only modification made by DU PASQUIER was to replace U by the weaker assumption U_1:
>
> U_1 (unit 1): The system contains the principal unit 1 [DICKSON, 1923b, 289].

Having presented these direct extensions of the existing theories to the more general case of the hypercomplex integers, DICKSON next showed that each of these definitions "fails completely" for the rational algebra $\{a + be \mid e^2 = 0\}$ [DICKSON, 1923b, 289]. He gave his results in the form of a theorem:

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[13] In the spirit of DICKSON's example, suppose $x = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is integral iff $a_{11}, a_{12}, a_{21}, a_{22} \in \mathbf{Q}$ and the coefficients of the quadratic equation it satisfies, $[*]\ x^2 - (a_{11} + a_{22})x + (a_{11}a_{22} - a_{21}a_{12}) \cdot I = 0$ where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, are rational integers. Then $M_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 0 & \frac{1}{2} \\ 2 & 0 \end{pmatrix}$ are integral matrices since they both satisfy $x^2 - 1 = 0$. But $M_2 \cdot M_1 = \begin{pmatrix} 0 & \frac{1}{2} \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}$ and $M_1 + M_2 = \begin{pmatrix} 0 & \frac{3}{2} \\ 3 & 0 \end{pmatrix}$, neither of which is an integral matrix since for $M_2 \cdot M_1$, the middle term of [*] is $(\frac{5}{2})$ and for $M_1 + M_2$, the constant term of [*] is $(\frac{-9}{2})$. Thus, M_1 and M_2 fail the closure axioms for addition and multiplication. See [DICKSON, 1923b, 287].

> *For the algebra with the basal units 1 and e, where $e^2 = 0$, the definitions of integers by both HURWITZ and DU PASQUIER fail since there is no maximal system. They fail also if we omit their requirement that a maximal system exists, since, if the integers are defined to be numbers of any chosen one of the infinitude of non-maximal systems, factorisation into indecomposable numbers is not unique and cannot be made unique by the introduction of ideals however defined* [DICKSON, 1923b, 291; DICKSON's emphasis].

Although DICKSON may have come to terms with this previously puzzling algebra, he did not have a realistic view of the earlier work in this area. Given that HURWITZ stated his definition of an integral element *only* for the quaternions, for example, it seems as though DICKSON misrepresented his work by extrapolating from the properties HURWITZ used to determine the set of integral elements in the quaternions to produce a definition applicable to arbitrary rational hypercomplex number systems. Granted, DICKSON gave a one sentence disclaimer before launching into his presentation of the theory, but he subsequently referred to HURWITZ's definition of a *hypercomplex integer*, a concept far more general than the one HURWITZ had actually defined.

In the statement of this theorem, DICKSON illustrated the difficulties in these definitions with the very algebra which had plagued DU PASQUIER. DICKSON generalized HURWITZ's work (although DICKSON suggested otherwise!) and mastered what had puzzled DU PASQUIER, namely, the construction of a suitable set of integral elements in this algebra. Moreover, he brought cohesion to a theory which had previously contained both ambiguous integers and arbitrary arithmetics by supplying a new, general definition of hypercomplex integers which overcame all of these difficulties and ensured unique factorization of integers into primes. DICKSON's definition must surely have represented a radical break from the work of HURWITZ and DU PASQUIER. Or did it?

As DICKSON put it when stating his new definition of hypercomplex integers "[w]e shall employ the assumptions C (closure), U_1 (unit 1), M (maximal) and R: R (rank equation): For every number of the system, the coefficients of the rank equation [minimum monic polynomial] are all rational integers" [DICKSON, 1923b, 292].[14] Thus, DICKSON's definition coincided with that proposed by DU PASQUIER on all but one of the axioms. Hence, DICKSON's definition, at least in part, grew out of the work of DU PASQUIER. The crucial points of departure involved his use of the rank equation [minimum monic polynomial] and his elimination of the finite basis property. These two dissimilarities made a significant difference in terms of results.[15]

In his paper, for example, with an eye towards collecting "material for an adequate comparison of the old and new definitions," DICKSON determined the integers for the

[14] As a testimony to DICKSON's mathematical finesse, he established a relationship between the coefficients of the rank and characteristic equations. DICKSON saw this as advantageous because "we know the explicit form of the characteristic equations for a general algebra, but not that of the rank equation" [DICKSON , 1923b, 290].

[15] DICKSON, however, noted that "for a semi-simple algebra (and no other algebra) of order *n* each set of integral elements of order *n* has a basis, so that the new definition of integral elements essentially coincides with the definitions by HURWITZ and DU PASQUIER for the case of semi-simple algebras and only in that case" [DICKSON, 1923a, 156–57].

"classic lists" of algebras with two, three and four basis elements, just as DU PASQUIER had done in his papers of 1918 and 1920 [DICKSON, 1923b, 281]. Unlike DU PASQUIER, however, DICKSON first developed a general theory for determining hypercomplex integers in reducible algebras[16] and then worked out the specific examples rather than simply applying his definition of a hypercomplex integer to the various algebras. In particular, he constructed the integral elements in a reducible algebra out of those of its irreducible components. With these results, he found a set of integral elements in each of the "classic algebras" and established the unique decomposition of integers into primes. Never one to overlook his accomplishments, DICKSON wrote that

> [t]he new definition has been tested by all the classic algebras in 2, 3 and 4 units, and found in every case to give wholly satisfactory results, as well as to explain serious difficulties arising under the earlier definitions. Moreover, the new theory is far simpler to apply than the old, and more readily lends itself to the proof of general theorems, which are wholly lacking in the writings based on the old definitions [DICKSON, 1923b, 282].

He, then, measured the success of his new definition in terms of how well it compared with the previous definition, the ease with which it could be applied, and the ability to develop a general theory from it. From his perspective, it represented a "winner" on all three counts.

DICKSON seemed to marshal more evidence in support of his new definition in the remainder of this paper when he applied it to the hypercomplex number systems of $n \times n$ square matrices, the quaternions, the octonions, and the linear associative algebras with complex coefficients. In the first instance, his definition of an integral element produced "infinitely many maximal systems of n-rowed square matrices with rational elements having properties C, [U], R," including the same integral $n \times n$ matrices as those given by DU PASQUIER in 1906 [DICKSON, 1923b, 310]. (The key difference here, of course, was that DICKSON recognized the existence of more than one set of integral elements in an algebra.) DICKSON went one step further, however, and established that the "arithmetic of all n-rowed square matrices with rational integral elements is associated with the arithmetic of the direct sum $(e_1) + (e_2) + \cdots + (e_n)$, and has unique factorization into primes" [DICKSON, 1923b, 311]. (DICKSON formed an integer in this direct sum out of the integers of the component algebras.) He dealt with the square matrices in less than four pages, as compared with DU PASQUIER's nearly fifty-page treatment of the subject in 1906. DICKSON indicated his awareness of the differences in these works with the remark that he saw "nothing simpler" than the arithmetic of integral square matrices "in spite of the very long discussion by ideals, etc., by DU PASQUIER in his Zürich thesis" [DICKSON, 1923b, 310]. DICKSON concluded his paper by mentioning that a "more general theory of hypercomplex integer" would appear in his upcoming book *Algebras and Their Arithmetics* [DICKSON, 1923b, 326].

[16] If an algebra A is the direct sum of two proper sub-algebras B and C, then A is said to be reducible into the components B and C.

A Brief Overview of DICKSON's Publications in the Arithmetics of Algebras

If his "A New Simple Theory of Hypercomplex Integers" marked DICKSON's entry into the arithmetics of arbitrary algebras, his *Algebras and Their Arithmetics* of 1923, and the briefer publications heralding aspects of this book's contents, indicated a commitment to the subject. In these writings he revealed a more far-reaching vision for the arithmetics of algebras than he had mentioned in his earlier work. As he explained in his 1923 treatise,

> [t]he chief purpose of this book is the development for the first time of a general theory of the arithmetics of algebras, which furnishes a direct generalization of the classic theory of algebraic numbers. The book should appeal not merely to those interested in either algebra or the theory of numbers, but also to those interested in the foundations of mathematics. Just as the final stage in the evolution of number was reached with the introduction of hypercomplex numbers (which make up a linear algebra), so also in arithmetic, which began with integers and was greatly enriched by the introduction of integral algebraic numbers, the final stage of its development is reached in the present new theory of arithmetics of linear algebras [DICKSON, 1923a, vii].

Inasmuch as his previous work on algebras formed part of "the final stage in the evolution of number," this book included the associated theory of arithmetics. Moreover, although the quaternions may have initially lured him into this subject, the generalization of the algebraic numbers represented a key component in the measurement of his theory's success.

In addition, he intended for the book to reach a wide audience and, as such, devoted the first eight chapters to the development of the general theory of algebras. In her review of the text for the *Bulletin of the American Mathematical Society*, OLIVE HAZLETT summed up this portion of DICKSON's manuscript when she wrote "in one hundred and twenty-seven pages, he gives the essence (the triple extract, as it were) of all that is interesting and important in the general theory of algebras" [HAZLETT, 1924, 265]. GAETANO SCORZA's *Corpi Numerici e Algebre* [SCORZA] provided "material assistance" to DICKSON in this regard but DICKSON owed his "chief obligations" to WEDDERBURN "both for his invention of the general theory of algebras and his cordial co-operation in the present attempt to perfect and simplify that theory and to render it readily accessible to general readers" [DICKSON, 1923a, vii]. Although DICKSON had expanded upon the work of ÉLIE CARTAN in his *Linear Algebras* of 1916, he chose to recast WEDDERBURN's work in his 1923 text highlighting the arithmetics of algebras.[17] (We discuss this work in some detail in the following subsection.)

As a testimony to the significance of these ideas, DICKSON received an invitation to present his theory of the arithmetics of algebras to the joint meetings of Section A of the American Association for the Advancement of Science (AAAS), the American Mathematical Society, and the Mathematical Association of America on 28 December,

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[17] [BIRKHOFF, 287] claims that the fundamental theorems of WEDDERBURN, put forth in 1907, remained "somewhat overlooked until DICKSON's book called attention to them."

1923. “Beginning with HAMILTON’s discovery of quaternions eighty years ago,” DICKSON opened,

> there has been a widespread interest in linear associative algebras, a subject known also under the name of hypercomplex numbers. The list of investigators in this field includes the following well known names: HAMILTON [actually in Ireland!], CAYLEY, CLIFFORD, and SYLVESTER in England; POINCARÉ and CARTAN in France; WEIERSTRASS, FROBENIUS, LIPSCHITZ, MOLIEN, SCHEFFERS, and STUDY in Germany; A. HURWITZ and DU PASQUIER in Switzerland; BENJAMIN PEIRCE, C. S. PEIRCE, TABER, WEDDERBURN, HAZLETT, and others in America [DICKSON, 1924a, 247].

For the very wide audience present at the joint meetings, DICKSON chose to set his work on the firm shoulders of some stellar figures in (mostly) nineteenth-century mathematics.

The published version of this lecture, “Algebras and Their Arithmetics,” together with two other papers presented the following day, “On the Theory of Numbers and Generalized Quaternions” [DICKSON, 1924c] and “Quadratic Fields In Which Factorization Is Always Unique” [DICKSON, 1924b] earned DICKSON the first award in the history of the AAAS for a paper presented at a meeting [LIVINGSTON, 77]. To commemorate the seventy-fifth anniversary of the AAAS, a member had donated $1000 “to be awarded as a prize to some person presenting at the... meeting a notable contribution to the advancement of science” [LIVINGSTON, 77]. In support of their choice for the award, the AAAS outlined DICKSON’s accomplishments. As they saw it,

> Professor DICKSON devised a new theory by which he was able to determine what should be the subject-matter of arithmetics and algebras, after which he found it possible to construct a highly developed science of arithmetics. The result is a rich array of fundamental results which mark great steps forward in the classic theory of algebraic numbers and in the development of HURWITZ’s integral quaternions. Professor DICKSON has been able to unify and greatly enlarge the whole subject of the theories of algebras. He has made a very notable contribution to the advancement of science... [LIVINGSTON, 77].

Since “[a]ll papers presented on the programs of the meeting were eligible for consideration” [LIVINGSTON, 77], the prize to DICKSON, for a paper in mathematics, gave even further indication of the significance of his new theory.

That this award went to a mathematician caught even E. H. MOORE by surprise. He expressed his thoughts on the matter when he wrote to his former University of Chicago colleague, HARRIS HANCOCK, on Christmas Eve in 1925. (HANCOCK served as an Assistant and Associate (Tutor) at the University of Chicago from 1892–1900. In 1900, he accepted a professorship at the Univerity of Cincinnatti [PARSHALL & ROWE, 1994, 364–365].) After thanking HANCOCK for his “thought of that award of the prize as a possibility,” he wrote, “I must confess that I had been thinking of Mathematics as being, by its natural abstractness and difficulty of comprehension, in fact practically excluded from consideration for any such award. Perhaps it is the advent of Relativity that has strongly impressed our non-mathematical scientific brethren” [MOORE to HANCOCK, 24

December, 1925, University of Chicago Archives, E. H. Moore Papers, Box 2, Folder 1]. HANCOCK replied two days later with an opinion of both the situation in mathematics relative to the other sciences and of DICKSON himself. In his words,

> I am always glad to do my little bit to help along any of us who are deserving. And DICKSON is certainly a great mathematician.
>
> I think that one cause of the lack of appreciation of the value of mathematics or better of mathematical thinking on the part of the ordinary citizen is that mathematicians themselves have been too modest both of their own merits and of their wares. Certainly modesty in itself is a comely and proper asset but it seems to me that mathematicians should emphasize more than they do, the value of mathematical training even at the risk of attracting a little attention to themselves in the doing of it.
>
> The Teachers College People and the pseudo educationalists have for years damned mathematics, the classics, and those real subjects that required study and the use of the intellect. They have scattered their propaganda throughout the country [HANCOCK to MOORE, 26 December, 1925, University of Chicago Archives, E. H. MOORE PAPERS, Box 2, Folder 1].

At least among the members of the AAAS in the closing days of 1924, this negative "propaganda" apparently had little impact. The mathematical cause had finally received some positive press thanks to DICKSON's work in the arithmetics of algebras.

Just eight months later, DICKSON chose to present this prize-winning theory as the subject of his plenary address at the International Congress of Mathematics in Toronto. In spite of the title of his lecture, "Outline of the Theory to Date of the Arithmetics of Algebras," he maintained the basic content and overall style of presentation of his "Algebras and Their Arithmetics." In many ways, these papers served as (international) advertisements for the ideas contained in his 1923 book. DICKSON's remaining mainline treatment of the subject came in 1927 with the German edition of this text, *Algebren und ihre Zahlentheorie*.[18] Although largely a translation of his earlier English-language version, this new book also included some fresh results in the theory of division algebras and in the theory of numbers.

Since DICKSON explicitly wrote all of these works for an audience with no previous acquaintance with the concept of an algebra, he began each of them with an overview. He often did this by way of examples, representing the complex numbers, quaternions, and 2×2 matrices as algebras [DICKSON, 1924a, 1928b]. Less frequently, but certainly in the case of the two books, he also stated the definition of an algebra [DICKSON, 1923a, 9–13;

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[18] DICKSON received the first award of the FRANK NELSON COLE Prize in Algebra for this text. Members of the AMS donated the original monies for the prize to COLE himself as expressions of gratitude for his role as editor-in-chief of the *Bulletin of the American Mathematical Society* and his quarter-century of service as Secretary of the Society. COLE then donated the money to the Society, and the members established the COLE Prize in his honor. In 1925, the Council announced that the first award of the COLE Prize would be made at the end of 1927 for a significant contribution to the theory of linear algebras by an American or Canadian citizen. Interestingly, DICKSON was a member of the committee which established the first prize problem and determined the conditions of the award. See [RICHARDSON, 1923, 14] and [ARCHIBALD, 1938a, 38–39].

1927, 23–24] and defined concepts which would be "indispensable" in later discussions of the arithmetics of algebras [DICKSON, 1928b, 95]. The latter included terms like division algebras (every element has a multiplicative inverse in the algebra), invariant sub-algebras (equivalent to our modern-day two-sided ideals), simple algebras (those with no invariant sub-algebra other than itself), and semi-simple algebras (those with no nilpotent, invariant sub-algebra) [DICKSON, 1924a, 594–95; 1928b, 96–97; 1924a, chapters 1–7; and 1927, chapters 2–6].

He also included what he viewed as the fundamental structure theorems for rational algebras, namely, "[e]very semi-simple algebra is either simple or is a direct sum of simple algebras, and conversely" [DICKSON, 1928b, 97], "[the principal theorem:] every algebra which is neither nilpotent nor semi-simple is the sum $N+S$ of its unique maximal nilpotent invariant sub-algebra N and a semi-simple sub-algebra S" [DICKSON, 1928b, 97], and "every simple algebra A is a direct product of a simple matric algebra and a division algebra D; this may be understood to mean that all elements of A can be expressed as matrices whose elements belong to D" [DICKSON, 1924a, 250]. Having stated these definitions and results, he could then present the arithmetics of algebras.

True to his goal of a clear, elementary style of exposition, DICKSON first illustrated the "nature and properties" of the arithmetic of an algebra via a concrete example and then developed the "remarkable new theory for any algebra" [DICKSON, 1923a, 141]. He frequently, if not always, demonstrated his theory with the very algebra which had plagued DU PASQUIER. At the Toronto Mathematical Congress, which DU PASQUIER also attended, DICKSON presented his example in a compare-and-contrast setting. As he explained,

> [f]or $x = a + be$, we evidently have $(x - a)^2 = 0$, which is the rank equation when a and b are variables ranging independently over all rational numbers. Its coefficients are ordinary integers if and only if a is an integer. Evidently the unique maximal set of quantities having properties C, U, R is composed of all the $x = a + be$ in which a is an integer and b is a rational number. These quantities x are therefore the integral elements of the algebra. For any rational number k, the product of the integral quantities $u = 1 + ke$ and $1 - ke$ is 1, whence each is called a *unit*. [An integral element $u = a + be$ is called a unit if $N(u) = (a + be)(a - be) = 1$.] Let $a \neq 0$ and choose $k = -(b/a)$. Then $xu = a$. The product of x by any unit u is said to be associated with x. Associated quantities play equivalent roles in questions of divisibility. The integral quantities of our algebra are therefore associated with the ordinary integers a and may be replaced by the latter in questions of divisibility [DICKSON, 1928b, 99–100].

Thus, in establishing the arithmetic of $A = \{a + be \mid e^2 = 0\}$, DICKSON rid himself of the pathology of the algebra. Expressed in more modern terms, $A = S + N$ where the semi-simple component S consists of the integers and the radical N of the algebra consists of the set $\{be \mid b \in \mathbf{Q}\}$. For this example, he determined the integral elements according to his new, general definition and established divisibility conditions (directly) and unique factorization (indirectly) by associating x with a. In other words, he elaborated an arithmetic for this algebra. Armed with his theory, then, DICKSON associated the arithmetic of this once bewildering algebra with the arithmetic of ordinary integers. From a number-theoretic standpoint, his result could hardly have been better.

Even with DU PASQUIER a likely member of his audience, DICKSON chose to continue with this rather dogmatic comparison of the different results. "Contrast these simple and satisfactory results under the new conception," he remarked,

> with the unfortunate conclusion, under the conceptions of integral quantities held by HURWITZ and DU PASQUIER, that this algebra has no integral quantities. Faced with the dilemma that no maximal set exists under his definition, DU PASQUIER suggested that we omit the desirable postulate M, that the set be a maximal and hence define the integral quantities to be those of an arbitrarily chosen one of the infinitude of sets with the basal units 1 and se. But it has been definitely proven by the lecturer that factorization into indecomposable integral quantities is then not unique and cannot be made unique by the introduction of ideals however defined. These insurmountable difficulties are in marked contrast with the simple conclusion, under the new conception, that the integral quantities are uniquely determined and are associated with ordinary integers [DICKSON, 1928b, 100].

Where DU PASQUIER proposed the elimination of the maximality condition in his definition of an integral element in an arbitrary algebra, DICKSON dropped the finite basis condition and replaced it with the rank equation with integer coefficients. In modern terms, DICKSON did not demand that the maximal set M be finitely generated as a **Z**-module. Thus, first mathematically and then rhetorically, DICKSON emphasized, in an international setting, the "new" replacing the "old" and rectifying all inconsistencies in the process. In so doing, he more than hinted at the sort of jockeying involved in the development of this definition (and of the corresponding arithmetic), in particular, and in the development of new mathematical theories, in general.

Even with DICKSON's strong bias in presenting the development of an integral element, this episode in mathematics allows us more than a glimpse into the mind of the mathematician. This account can do nothing but dispel the widely held myth (among non-mathematicians, that is) that mathematical ideas seemingly emerge out of the rarefied air of a purely disinterested scientific atmosphere. The creation of mathematics involves individuals and their egos, thought processes full of give-and-take, and guiding aesthetic principles such as clarity and breadth of applicability.

DICKSON embodied all of these aspects of the mathematical endeavor. He had learned the latter especially at the feet of Moore, Bolza, and Maschke some twenty-five years before he first considered the arithmetics of algebras in the 1920's and incorporated them thoroughly and naturally into his work. It was perhaps these aesthetic qualities at which the reviewer of his *Algebren und ihre Zahlentheorie* more than hinted when he termed the book "[a]nother notable advance in the development of the subject of linear algebras.... The mathematical world naturally expects a high standard in the case of treatises by Professor DICKSON and the present one will not be found disappointing" [Mitchell, 261].

DICKSON's Arithmetics of Algebras

In what seems a very clever approach, DICKSON presented his two book-length treatments of the theory of the arithmetics of algebras, we could say, along the lines of WEDDERBURN's fundamental theorems on the structure of algebras. Specifically, DICKSON began by considering a rational algebra A with identity. According to WEDDERBURN's theory, A can be expressed as $A = S + N$, where N is the maximal nilpotent invariant sub-algebra of A and S is a semi-simple sub-algebra. DICKSON then proved that the arithmetic of A is associated with that of S "in the sense that every integral quantity (whose determinant is not zero) of A is the product of an integral quantity of S by a unit" [DICKSON, 1928b, 100]. In terms of the perennial example $A = \{a + be \mid e^2 = 0\}$, then, N consists of elements of the form *be* and S contains the rational components a. Thus, as DICKSON demonstrated in illustrating his theory earlier, the integral quantities of S are precisely the ordinary integers. He not only determined the integral elements of an arbitrary rational algebra from this association but also the properties of division. As he explained it, "[a]nother statement of this theorem is that in questions of divisibility we may suppress the bizarre nilpotent components belonging to N. This elimination of undesirable elements is fortunate both for the theory and its applications" [DICKSON, 1928b, 100]. Thus, since divisibility remained unchanged under multiplication by a unit, the division properties of the algebra, including unique factorization, mimicked those of its semi-simple component.

With his association of the arithmetic of an arbitrary rational algebra (with unity) with its semi-simple component, DICKSON "reduced the problem of the arithmetics of all algebras to that of semi-simple algebras S" [DICKSON, 1928b, 100]. But he could do even better than that. "We can further reduce the problem," he argued,

> to the case of simple algebras. For, we saw that S is a direct sum of simple algebras $S_1, S_2, \ldots$, so that each quantity s of S is a sum of components $s_1, s_2, \ldots$, belonging to $S_1, S_2, \ldots$, respectively. It is an important theorem that if each s_i is an integral quantity of S_i, then s is one of S, and conversely. Moreover, the divisibility properties for S follow at once from those of the component algebras S_i [DICKSON, 1928b, 100].

As he indicated by his choice of the word "important," this piece of his arithmetic hinged on his construction of integral elements, units, and properties of unique factorization of a direct sum of algebras out of the corresponding pieces of its component algebras. In particular, he proved that if an integral element $s = s_1 + s_2 + \cdots + s_n$ was a unit, then $s_1, s_2, \ldots, s_n$ were units in $S_1, S_2, \ldots, S_n$ and conversely [DICKSON, 1923a, 159]. "An integral element not a unit is called a *prime*," DICKSON stated, "if it admits only such representations as a product of two integral elements of the same algebra in which one of them is a unit" [DICKSON, 1923a, 159]. In a spirit similar to that of his proofs regarding the integral elements and units within an algebra expressed as a direct sum, he showed that if the integral elements of non-zero determinant of the component algebras $S_1, S_2, \ldots, S_n$ possessed unique factorization (apart from unit factors), then the same was true for integral elements of non-zero determinant of $S = \oplus S_i$ [DICKSON, 1923a, 159].

Not surprisingly, DICKSON next determined the arithmetic of simple algebras. According to WEDDERBURN's theory, any simple algebra S_i could be expressed as the tensor product of a matrix algebra M and a division algebra D. In other words, the elements of S_i could be written as matrices with coefficients in the division algebra D. DICKSON showed that the integral elements of a simple algebra S_i (together with, in modern terms, its representation) are those matrices with coefficients ranging over the integral elements of D and conversely. Thus, as he pointed out, "we know the integral quantities of $[S_i]$ as soon as we know those of D" [DICKSON, 1928b, 101]. Even with the integral elements in hand, however, he still needed to deduce the divisibility properties for these simple algebras from those of the division algebras.

DICKSON established properties of division for simple algebras built out of certain division algebras D. (In particular, the integral elements of these division algebras had the additional property that the process of division always yielded a remainder whose norm was numerically less than the norm of the divisor.) [DICKSON, 1923a, 168–174; 1928b, 101]. For this case, he developed a theory of "reduction and equivalence" of matrices with coefficients ranging over the integral elements of D. This amounted to the elementary transformations we know today as the GAUSS-JORDAN elimination, together with the multiplication by a unit factor before every element in a given row or after every element in a given column. This culminated in the proof that every matrix with coefficients from the integral elements of D "is a product of units and a matrix having zeros outside the diagonal" [DICKSON, 1928b, 101]. In other words, he associated an integral element in a simple algebra with a diagonal matrix. (The arithmetic of diagonal matrices, in turn, followed from the arithmetic of direct sums of algebras.) "The resulting theory," he summarized, "is a direct generalization of the classic theory of matrices whose elements are ordinary integers, and then factorization into prime matrices is unique apart from unit factors" [DICKSON, 1928b, 101]. With a tool box thus loaded with WEDDERBURN's structure theorems of algebras, associated arithmetics, and arithmetics of direct sums, DICKSON constructed the arithmetic of all rational algebras. Like WEDDERBURN, but in the context of the arithmetics of rational algebras, DICKSON successfully reduced his problem to the case of simple algebras, and, in large measure, to that of division algebras. But that was not all.

He took this theory and applied it to the study of Diophantine equations. Since he often used the very equation he had solved in his Strasbourg lecture, $x_1^2+x_2^2+\cdots+x_5^2 = x_6^2$, to highlight the application of the arithmetics of algebras to Diophantine equations, we could almost say he ended this theory where he had begun it. In DICKSON's words,

> [t]he theory of algebraic numbers finds applications only to problems involving forms which contain only two variables homogeneously and hence can be factored into linear forms. This serious limitation may often be removed by employing hypercomplex numbers. For example, $x^2 + y^2 + z^2 + w^2$ has as factors the quaternion $x + yi + zj + wk$ and its conjugate. Since the new theory of arithmetics of algebras finds applications to problems involving forms in any number of variables it furnishes us with an effective new tool for problems in algebra and number theory [DICKSON, 1924a, 257].

The production of this "effective new tool" thus represented not only the formulation of a new and potent mathematical theory but also a rich application of that theory in the surprising setting of Diophantine equations.

Justification for DICKSON's "New" Definition

As should now be clear from the discussion above, much of the history of the development of the arithmetics of algebras lies in the development and acceptance of the concept of an integral element. What made DICKSON's definition the best of the various proposals? Perhaps the answer to this lies in his arguments for its adoption, arguments which reflect his mathematical aesthetics. DICKSON himself seemed to recognize that his definition needed further validation when he stated that "[i]t is obviously more difficult to justify a new determination of the proper subject matter of an embryo science than to compare different foundations of an established science" [DICKSON, 1924a, 255]. Let us analyze the justification DICKSON offered for his definition of a set of integral elements as the maximal set of elements with properties C, R, and U. We note that although DICKSON gave this justification for his definition of a set of integral elements in a rational algebra, it applies equally well to the case of algebras over an arbitrary field.

One of the key ingredients in the different formulations of the definition rested in DU PASQUIER's decision to let the properties of the integers (solely) serve as his guide and in DICKSON's apparent willingness to consider an application of the integral algebraic numbers. Moreover, while DU PASQUIER held tightly to the finite basis condition for a set of integral elements, DICKSON opted for the maximality requirement. DICKSON's definition, for example, not only freed him from the problems faced by DU PASQUIER, but also led him to establish far-reaching results in the arithmetics of algebras and number theory. DICKSON sold his definition to the mathematical world on the basis of such convincing evidence.

"As a first justification of our definition," he often began his sales pitch, the set of elements in an algebraic number field possessing postulates C, U, R, and M coincided with the integral algebraic numbers [DICKSON, 1924a, 254]. "In other words," he added, "the new theory is a direct generalization of the classic theory of algebraic numbers" [DICKSON, 1924a, 254]. Since DICKSON insisted that the definition of an algebra include the algebraic number fields, he naturally desired the definition of an integral element to produce the already established integral algebraic numbers. Similarly, he argued, when the algebra under consideration was the rational quaternions, the new definition should lead to the integral quaternions of HURWITZ.

Secondly, when compared to the previous notions of an integral element, DICKSON's definition produced a set of integral elements in *every* rational algebra, including the troublesome $A = \{a + be \mid e^2 = 0\}$ whose arithmetic DU PASQUIER was forced to describe as "arbitrary" [DICKSON, 1928a, 174–175]. As we have seen, DICKSON routinely contrasted the "old" and "new" conceptions of an integral element in a rational algebra by illustrating the "failure" and "serious difficulties" of earlier definitions and the "satisfactory" nature of his set. (These adjectives abound in DICKSON's work on the arithmetics of algebras. See, for example, [1923a, 145–146], [1924a, 254], [1928b, 99–100].) Thus, he repeatedly intermingled strong rhetoric and mathematics to highlight the superiority of his theory.

"The final justification of the new conception of integral quantities of an algebra," DICKSON would write, "lies in the rich array of fundamental general theorems which have been developed under the new conception . . . whereas under the earlier conceptions no general theorem had been obtained" [DICKSON, 1928b, 99]. In some ways, given his rhetorical style, this represented a rather tame announcement of some fairly remarkable results. DICKSON had certainly accomplished far more than HURWITZ, for example, who had merely addressed the arithmetic of a specific algebra. Relative to DU PASQUIER, DICKSON had not only arrived at a suitable definition of a set of integral elements in an arbitrary algebra, but he had also created the corresponding arithmetic. DICKSON's theory, in a nutshell, reduced the problem of the arithmetics of algebras to the case of simple algebras and, in many ways, to the case of division algebras. Finally, using this theory as a tool, he established a new method of solving previously unsolvable Diophantine equations.

DICKSON may have termed the theory which resulted from his definition of an integral element as its "final justification," but, in fact, he gave another indirect validation of his framework. Throughout his work in the arithmetics of algebras, he commented on the "many instances in the history of mathematics where success has been achieved by the principle of enlargement" [DICKSON, 1923a, 255]. He frequently cited the "growth of our number system" and the "introduction of ideals in the theory of algebraic numbers" as examples [DICKSON, 1923a, 255]. In Toronto, in concluding his "Outline of the Theory to Date of the Arithmetics of Algebras," he placed his work in this tradition when he said, "[t]he gradual enlargement of the conception of number from the primitive numbers used in counting to the system of all complex numbers and finally to its culmination in hypercomplex numbers (or quantities of any algebra) has its parallel in the growth of the concept integer, which was first restricted to the counting numbers, was greatly enriched in the last century by the study of integral algebraic numbers, and now finds its culmination in the integral quantities of any algebra" [DICKSON, 1928b, 102]. By situating the concept of an integral element at the "culmination" of the development of the notion of an integer – a status achieved, moreover, by the "wisdom" of enlargement – DICKSON associated his contribution with a long line of "successful" mathematics.

DICKSON's defense of his new definition lacked little, if anything. The additions he made to the theory in the next few years only enhanced and confirmed his original ideas. In particular, by the time of his Toronto address in 1924, he and his former student, OLIVE HAZLETT, had each independently extended the arithmetics of rational algebras to algebras over an *arbitrary field* [DICKSON, 1928a, 1:173–184], [HAZLETT, 1928, 1:185–191]. His "remarkable theory" for rational algebras now held true for algebras over any field. He also proved that every rational algebra contained a set of integral elements [DICKSON, 1928a, 173–174]. This result quite naturally marshaled even more evidence for *his* postulational formulation of the definition of an integral element.[19]

[19] Not every mathematician had quite the same enthusiasm for DICKSON's definition. DU PASQUIER, for example, presented his own view of the development of the concept of an hypercomplex number at the Toronto Congress in his [1928]. Specifically, DU PASQUIER described the "evolution" as a six-stage process which, not surprisingly, emphasized *his* first proposed definition of an integral hypercomplex number and presented DICKSON's definition as one which (merely) introduced the property N, a "norm" postulate which required every element in the system to possess an (ordinary) integral norm. DICKSON proposed this as a weaker assumption than R in [DICKSON, 1923b, 292]. He did not make use of it in his version of the theory of the arithmetic of algebras as given in [1923a].

The setting of the Toronto Congress must have been a bittersweet moment for DU PASQUIER. In particular, while he chose to give his view of the development of the concept of an integral hypercomplex number, DICKSON presented his arithmetic of (rational) algebras and its extension to algebras over an arbitrary field.

Conclusion

The theory of the arithmetics of algebras hinged on the determination of a set of integral elements which led to an arithmetic analogous to that of the ordinary integers. Although GAUSS, KUMMER, DEDEKIND, KRONECKER, and LIPSCHITZ had considered the arithmetic of complex integers, algebraic number fields, and quaternions, it was HURWITZ who first articulated this precise goal. By that time, GAUSS's "natural" complex integer and KUMMER's inclusion of ideal numbers in his set of algebraic integers had already brought both the notion of an obvious definition (integer coefficients) and the "wise" strategy of enlargement to the formulations of a concept of an integral element. LIPSCHITZ's difficulties apparently prompted HURWITZ to recognize that the set of integral quaternions held the key to an arithmetic like that of the integers. HURWITZ thus shifted his focus from the coefficients of an integral quaternion to the structure of a set of integral quaternions. From this vantage point, he obtained an *un*-integer-like integral quaternion, but one which yielded the desired arithmetic.

Drawing from the work of his adviser HURWITZ, DU PASQUIER considered the arithmetic of non-commutative algebras. He formulated his definition of a set of integral elements based on the structure of the set of ordinary integers. Both the strengths and weaknesses of his theory resulted from this particular construction. On the one hand, he had so much confidence in his definition that he applied it to *all* rational hypercomplex number systems. Prior to DU PASQUIER, mathematicians had only considered the arithmetic of *specific* algebras. On the other hand, he clung so tightly to his definition that he seemingly lost sight of his objective. When he could not obtain unique factorization into primes, for example, he attempted to alter the concept of integer to include absolute, fractional, and conditional integers. With the ordinary integers supposedly serving as his guide, this idea diverted him from his intended course. Still, DU PASQUIER persisted. He never reconsidered his definition, but he did renegotiate its consequences.

DICKSON more than likely heard DU PASQUIER present this work at the Strasbourg Congress. Soon after, DICKSON established DU PASQUIER's unsubstantiated claims regarding the use of ideals in certain algebras to obtain unique factorization. In so doing, the American not only demonstrated his commitment to rigorous mathematics but also categorically asserted the superiority of his methods. He did not discreetly build the arguments DU PASQUIER lacked. He, rather, proclaimed both the incompleteness of DU PASQUIER's ideas and the absence of parts of his proofs. He went on to construct a set of integral elements, first in a rational algebra and later in an arbitrary algebra, which led to a rich *theory* (as opposed to case-by-case studies) of the arithmetics of algebras.

In tracing the evolution of this concept, we (again) see the crucial role of international scientific exchange in the dynamic development of mathematics. Before DICKSON ever brought the arithmetics of algebras to America, the subject had already advanced and retreated across Swiss and German borders for more than 100 years. Each mathematician who joined this pursuit drew, for better (HURWITZ) or for worse (LIPSCHITZ), from the work of his predecessors. Naturally, published works served to transmit ideas from one mathematician to another. The International Congress of 1920 may have also aided the progress of this arithmetic for it more than likely brought DICKSON and DU PASQUIER within earshot of one another. Moreover, HURWITZ inspired DU PASQUIER's research in this area from his role as adviser.

This episode in mathematics allows us more than a glimpse into the mind of the mathematician. The historical details of DU PASQUIER's involvement in the subject, for example, highlight the tension of knowing the desired mathematical goal and, yet, not knowing precisely how to attain it. His publications represent what one might call "work in progress." In spite of the ambiguity and impreciseness of his results, however, DICKSON incorporated some of DU PASQUIER's ideas into his successful theory. DICKSON's contributions, as it were, involved sifting the chaff from the grain and, once sifted, making the proper additions.

Moreover, although DICKSON had a seemingly sound mathematical theory, he apparently felt compelled to justify further his ideas to his colleagues through his rather pointed rhetoric. In so doing, he made manifest the mathematical aesthetics which served to guide him in this (as well as other) mathematical endeavors. In particular, DICKSON valued a definition of a set of integral elements which coincided with exisiting concepts for specific algebras like the quaternions and the algebraic number fields. Rather than considering a case-by-case study of the associated arithmetic, he desired (and determined) a general theory of arithmetic which applied to *all* algebras. Thus, DICKSON strove for the most far-reaching mathematical concepts and theories possible. He also apparently looked to previous effective mathematical strategies when launching into uncharted territory himself. In this case, he found the principle of enlargement as used by KUMMER and HURWITZ especially beneficial in his developing notion of a set of integral elements. Finally, DICKSON naturally esteemed a theory with wide application, especially in unexpected realms. DICKSON asked a lot of a theory and, in this case, at least, it delivered.

All in all, the mathematics and rhetoric of DICKSON's theory of the arithmetics of algebras support the view of PHILIP J. DAVIS and REUBEN HERSH that "[t]he myth of totally rigorous, totally formalized mathematics is indeed a myth. Mathematics in real life is a form of social interaction where "proof" is a complex of the formal and the informal, of calculations and casual comments, of convincing argument and appeals to the imagination and the intuition" [DAVIS & HERSH, 68].

Moreover, DICKSON's presentation of the development of the arithmetics of algebras suggests an extension of another twentieth-century reflection on mathematics. Specifically, in his book *The Structure of Scientific Revolutions* [KUHN], THOMAS KUHN suggests that we tend to draw our images of science largely from the "study of finished scientific achievements as these are recorded in the classics and, more recently, in the textbooks from which each new scientific generation learns to practice its trade. Inevitably, however, the aim of such books is persuasive and pedagogic; a concept of science drawn from them is no more likely to fit the enterprise that produced them than an image of a national culture drawn from a tourist brochure" [KUHN, 1]. While KUHN is right to argue that scientific texts have become tidy packages which propagate completed scientific ideas rather than the often convoluted process of their development, DICKSON's research monographs and the two texts in the arithmetics of algebras indicate that there are other sources for the novice.

The former expositions, as KUHN rightly suggests, portray a distorted view of the scientific enterprise to the subsequent generations who learn from them. DICKSON's presentation of the emergence of the best definition of an integral element in an arbitrary rational algebra, however, provided the general audience for whom he wrote a much more realistic view of the development of mathematics. Although he certainly outlined the development of this concept in a "royal road to me" fashion [GRATTAN-GUINESS, 157], DICKSON *included* the initial (incorrect) propositions and the subsequent attempts to correct any resulting errors in his classic writings on the subject. In this way, he illustrated the exchange of ideas from one mathematician to another, demonstrated the sort of jockeying that goes on as properties are included and excluded according to their more far-reaching contributions, and asserted the superiority of his own formulation, all in one fell swoop. Thus, for this particular episode in the history of mathematics, DICKSON gave a description, which, following KUHN's analogy, drew his readers away from the shiny gloss of the tourist brochure into the less traveled paths of the countryside.

With our focus on DICKSON's tour through the backroads of the theory of the arithmetics of algebras, we must not lose sight of the fact that he pursued this research while a professor at the University of Chicago. As his ideas came together in this work, they became a part of the more general algebraic tradition at Chicago. DICKSON's emphasis on far-reaching theories, for example, further entrenched this characteristic (originally initiated by his adviser E. H. MOORE) in the Chicago algebraic school. His intolerance for unsubstantiated claims also permeated the institution's algebraic thinking. His worldwide recognition brought the same to Chicago's algebraic heritage. In other words, what DICKSON gave to algebra through his work in the arithmetics of algebras (and in other areas, for that matter) was worth emulating, and many students came to Chicago to do just that [FENSTER]. Both through his ideas *per se* and through his efforts at training future algebraists, DICKSON consolidated the theory of the arithmetics of algebras and the Chicago algebraic research tradition while at the same time contributing to their growth.

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