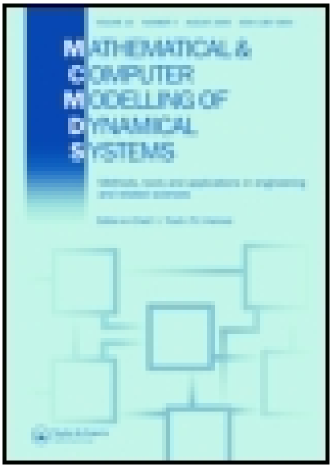

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Two Approaches for State Space Realization of NARMA Models: Bridging the Gap

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Two Approaches for State Space Realization of NARMA Models: Bridging the Gap

Ü. KOTTA[1] AND N. SADEGH[2]

ABSTRACT

This paper studies the necessary and sufficient conditions for observable realization of a general class of nonlinear high-order input-output difference equations. In particular, it proves the equivalence of the two seemingly different existing approaches in the literature. The paper also provides a subclass of NARMA input-output models that are guaranteed to have an observable realization. It is shown that this class covers several important subclasses of existing NARMA models.

Keywords: algebraic methods, discrete-time system, input-output systems, mathematical modeling, NARMA model, nonlinear control system, realizability.

1. INTRODUCTION

The state space realization of single-input single-output nonlinear input-output (i/o) difference equations has been the subject of two recent papers [1, 2]. In these papers two completely different approaches were provided to solve the problem. In [1], the intrinsic and coordinate-free generic necessary and sufficient realizability conditions were formulated in terms of integrability of certain subspaces of one-forms associated with the i/o model. One of the distinctive characteristics which makes this algebraic approach interesting and useful is its inherent simplicity and transparency. For characterizing the realizability conditions as well as for constructing the state coordinates, a single tool, based on elementary time shifting of a function (and a one-form), namely the notion of relative degree, provides the key. Despite the inherent simplicity of the approach, in order to find the state coordinates, one has to find the integrating factors and integrate the integrable one-forms of a certain subspace. Though it is easy to check the integrability property with the help of the Frobenius theorem, it can be extremely difficult to integrate the required one-forms to find the state coordinates, especially for complicated models.

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On the other hand, the local necessary and sufficient realizability conditions in [2] are formulated directly in terms of the partial derivatives of the i/o map of the NARMA model. Moreover, the state coordinates can be constructed based on the above i/o map. The most complicated task in finding the state coordinates is to solve a nonlinear algebraic equation, and not a set of nonlinear differential equations as in [1].

The main contributions of this paper is to prove that the two seemingly different approaches are actually equivalent and yield the same results. The algorithm in [2] can be understood as the method to compute the basis for a subspace of one-forms. Particularly, this equivalence allows to extend the results of [1] by providing an (almost) explicit choice of the state coordinates as in [2] without the need to integrate the one-forms and to solve the set of nonlinear differential equations. This is especially important for implementation of the realization algorithms via the symbolic computation packages such as *Mathematica* or *Maple*. Solving the set of nonlinear differential equations and integrating the one-forms are reported to be the most complicated tasks for computer algebra implementation; all the other operations related to the realization procedure, can be implemented rather straightforwardly [3].

Moreover, the equivalence of the two approaches provides a more general point of view for the results in [2]. In particular, it allows extension of the local results of [2] to global using the more sophisticated algebraic framework as in [1].

The second purpose of the paper is to provide a more general subclass of realizable NARMA models as the one suggested in [2].

2. EQUIVALENCE OF THE NECESSARY AND SUFFICIENT REALIZABILITY CONDITIONS

Consider a nonlinear system Σ described by

$$y(t+n) = \varphi(y(t), \ldots, y(t+n-1), u(t), \ldots, u(t+s)), \quad s \le n-1 \tag{1}$$

where $u \in \mathcal{U} \subset \mathbb{R}$ is the input variable, $y \in \mathcal{Y} \subset \mathbb{R}$ is the scalar output variable and φ is a real analytic function defined on $\mathcal{Y}^n \times \mathcal{U}^{s+1}$. Assume that either $\partial\varphi(\cdot)/\partial y(t)$ or $\partial\varphi(\cdot)/\partial u(t)$ is different from zero (Assumption 1). In the realization problem we are looking for transforming an i/o Equation (1) into the classical state-space form

$$\begin{aligned} x(t+1) &= f(x(t), u(t)) \\ y(t) &= h(x(t)). \end{aligned} \tag{2}$$

We will associate with the system Σ an extended state-space system Σ_e with input $v(t) = u(t+s+1)$ and state $z(t) = [y(t), \ldots, y(t+n-1), u(t), \ldots, u(t+s)]^T$ defined as

$$z(t+1) = f_e(z(t), v(t)) \tag{3}$$

where $f_e(\cdot) = [z_2, \ldots, z_n, \phi(z), z_{n+2}, \ldots, z_{n+s+1}, v]^T$. The system (3) will play a key role in the realizability conditions and the realization procedure of [1].

Let $\mathcal{K}$ denote the field of meromorphic functions in a finite number of variables $\{z(0), v(t), t \geq 0\}$. The forward-shift operator $\delta : \mathcal{K} \to \mathcal{K}$ is defined by

$$\delta\zeta(z(t), v(t)) = \zeta(f_e(z(t), v(t)), v(t+1))$$

Under Assumption 1, the pair $(\mathcal{K}, \delta)$ is a difference field, and up to an isomorphism, there exists a unique difference field $(\mathcal{K}^*, \delta^*)$ such that $\mathcal{K} \subset \mathcal{K}^*, \delta^* : \mathcal{K}^* \to \mathcal{K}^*$ is an automorphism and the restriction of δ^* to $\mathcal{K}$ equals δ [1]. By abuse of notation, hereinafter we assume that the inversive closure $(\mathcal{K}^*, \delta^*)$ is given and use the same symbol to denote the difference field $(\mathcal{K}, \delta)$ and its inversive closure.

We first define generic observability for system (2):

Definition 2.1. *We call system (2) locally generically observable if*

$$\operatorname{rank}_{\mathcal{K}} \frac{\partial(y(t), y(t+1), \ldots, y(t+n-1))}{\partial x(t)} = n. \tag{4}$$

Over the field $\mathcal{K}$ one can define a difference vector space $\mathcal{E} := \operatorname{span}_{\mathcal{K}}\{\mathrm{d}\varphi \mid \varphi \in \mathcal{K}\}$. The operator δ induces a forward-shift operator $\Delta : \mathcal{E} \to \mathcal{E}$ by

$$\sum_i a_i \mathrm{d}\varphi_i \mapsto \sum_i (\delta a_i)\mathrm{d}(\delta\varphi_i), \;\; a_i, \varphi_i \in \mathcal{K}.$$

The relative degree r of a one-form $\omega \in \mathcal{E}$ is defined to be the least integer such that $\Delta^r\omega \notin \operatorname{span}_{\mathcal{K}}\{\mathrm{d}z\}$. If such an integer does not exist, we set $r = \infty$. A sequence of subspaces $\{\mathcal{H}_k\}$ of $\mathcal{E}$ is defined by

$$\begin{aligned} \mathcal{H}_1 &= \operatorname{span}_{\mathcal{K}}\{\mathrm{d}z(0)\} = \operatorname{span}_{\mathcal{K}}\{\mathrm{d}y(0), \ldots, \mathrm{d}y(n-1), \mathrm{d}u(0), \ldots, \mathrm{d}u(s)\} \\ \mathcal{H}_{k+1} &= \operatorname{span}_{\mathcal{K}}\{\omega \in \mathcal{H}_k \mid \Delta\omega \in \mathcal{H}_k\}, \quad k \geq 1 \end{aligned} \tag{5}$$

It is clear that sequence (5) is decreasing. The subspaces are invariant under the (extended) state space diffeomorphism.

Theorem 2.2. *The nonlinear system described by the input-output difference equation (1) has a generically observable state space realization iff for $1 \leq k \leq s+2$ the subspaces $\mathcal{H}_k$ defined by (5) are completely integrable. Moreover, the state coordinates can be obtained by integrating the one-forms in $\mathcal{H}_{s+2}$.*

From now on, let us assume that $s = n-1$, as in [2]. Next, we will recall the realizability conditions, given in [2]. For that, we define the blocks of input and output by

$$\begin{aligned} \mathbf{u}(t) &= (u(t), \ldots, u(t+n-1)), \\ \mathbf{y}(t) &= (y(t), \ldots, y(t+n-1)). \end{aligned}$$

Evaluating $y(t), y(t+1), \ldots, y(t+n-1)$ recursively in terms of $\mathbf{y}(t-n), \mathbf{u}(t-n)$ and $\mathbf{u}(t)$ we obtain the block i/o map

$$\mathbf{y}(t) = \Phi(\mathbf{y}(t-n), \mathbf{u}(t-n), \mathbf{u}(t))$$

where $\Phi = (\varphi^1, \varphi^2 \ldots, \varphi^n)^T$, $\varphi^1(\mathbf{y}, \mathbf{u}, \mathbf{v}) = \varphi(y_1, \ldots, y_n, u_1, \ldots, u_n)$, and

$$\varphi^i(\mathbf{y}, \mathbf{u}, \mathbf{v}) = \varphi(y_i, \ldots, y_n, \varphi^1(\mathbf{y}, \mathbf{u}, \mathbf{v}), \ldots, \varphi^{i-1}(\mathbf{y}, \mathbf{u}, \mathbf{v}), u_i, \ldots, u_n, v_1, \ldots, v_{i-1})$$

for $i = 2, \ldots n$. Note that in terms of the forward-shift operator δ, $\varphi^i(\mathbf{y}, \mathbf{u}, \mathbf{v}) = \delta^{i-1}\varphi(y_1, \ldots, y_m, u_1, \ldots, u_m)$. For $\mathbf{x} \in \mathbb{R}^n$, denoting $D_{\mathbf{x}} := [\,\partial/\partial x_1 \quad \cdots \quad \partial/\partial x_n\,]$, it can be easily seen that Assumption 1 implies that either $D_{\mathbf{y}}\Phi(\mathbf{y}, \mathbf{u}, \mathbf{v})$ or $D_{\mathbf{u}}\Phi(\mathbf{y}, \mathbf{u}, \mathbf{v})$ is invertible. To accommodate, these two cases we define the modified block i/o map $\hat{\Phi}(\mathbf{y}, \mathbf{u}, \mathbf{v}) := \Phi(\mathbf{y}, \mathbf{u}, \mathbf{v})$ if $\partial\varphi(\cdot)/\partial y(t)$ is nonzero and $\hat{\Phi}(\mathbf{u}, \mathbf{y}, \mathbf{v}) := \Phi(\mathbf{y}, \mathbf{u}, \mathbf{v})$ otherwise. Using the implicit function theorem, it can be seen that the map $\hat{\Phi}\ (\mathbf{y}, \mathbf{u}, \mathbf{v})$ is a local diffeomorphism with respect to $\mathbf{y}$. That is, there exists a smooth local function $\hat{\Phi}_y^{-1}$ such that $\mathbf{y} = \hat{\Phi}_y^{-1}(\mathbf{z}, \mathbf{u}, \mathbf{v}) \Longrightarrow \mathbf{z} = \hat{\Phi}(\mathbf{y}, \mathbf{u}, \mathbf{v})$. For future reference we need $D_{\mathbf{u}}\hat{\Phi}_y^{-1}(\mathbf{z}, \mathbf{u}, \mathbf{v})$. Taking the partial derivative of $\mathbf{z} = \hat{\Phi}(\hat{\Phi}_y^{-1}(\mathbf{z}, \mathbf{u}, \mathbf{v}), \mathbf{u}, \mathbf{v})$ with respect to $\mathbf{u}$ it follows that

$$D_{\mathbf{u}}\hat{\Phi}_y^{-1}(\mathbf{z}, \mathbf{u}, \mathbf{v}) = -\left(D_{\mathbf{y}}\hat{\Phi}(\mathbf{y}, \mathbf{u}, \mathbf{v})\right)^{-1} D_{\mathbf{u}}\hat{\Phi}(\mathbf{y}, \mathbf{u}, \mathbf{v}) \tag{6}$$

We now present the main result of the paper, the proof of which shows explicitly the equivalence of the realizability conditions in [1] and a generalization of those in [2].

Theorem 2.3. *The nonlinear system described by the input-output difference equation (1) has a generically observable state space realization iff* $D_{\mathbf{y}}\hat{\Phi}(\mathbf{y}, \mathbf{u}, \mathbf{v})^{-1}$ $D_{\mathbf{u}}\hat{\Phi}(\mathbf{y}, \mathbf{u}, \mathbf{v})$ *is independent of the third variable* $\mathbf{v}$. *Moreover, the state vector may be defined by* $x(t) = \hat{\Phi}(\mathbf{y}(t-n), \mathbf{u}(t-n), \mathbf{v})$ *for any constant vector* $\mathbf{v} \in \mathbb{R}^n$.

Proof. We need to show that integrability of $\mathcal{H}_{n+1}$ is equivalent to the necessary and sufficient condition of Theorem 2.3.

Let $\omega = A\mathrm{d}\mathbf{y}(0) + B\mathrm{d}\mathbf{u}(0) \in \mathcal{H}_1$, with $A, B \in \mathcal{K}^{1\times n}$. Then $\Delta^n\omega = \delta^n A\mathrm{d}\mathbf{y}(n) + \delta^n B\mathrm{d}\mathbf{u}(n)$. Since $\mathbf{y}(n) = \Phi(\mathbf{y}(0), \mathbf{u}(0), \mathbf{u}(n))$,

$$\begin{aligned}\mathrm{d}\mathbf{y}(n) &= D_{\mathbf{y}(0)}\Phi(\mathbf{y}(0), \mathbf{u}(0), \mathbf{u}(n))\mathrm{d}\mathbf{y}(0) + D_{\mathbf{u}(0)}\Phi(\mathbf{y}(0), \mathbf{u}(0), \mathbf{u}(n))\mathrm{d}\mathbf{u}(0)\\ &\quad + D_{\mathbf{u}(n)}\Phi(\mathbf{y}(0), \mathbf{u}(0), \mathbf{u}(n))\mathrm{d}\mathbf{u}(n).\end{aligned}$$

For $\Delta^n\omega \in \mathcal{H}_1$, one must have $\delta^n A D_{\mathbf{u}(n)}\Phi(\mathbf{y}(0), \mathbf{u}(0), \mathbf{u}(n)) + \delta^n B = 0$ which yields $B = -AD_{\mathbf{u}(0)}\Phi(\mathbf{y}(-n), \mathbf{u}(-n), \mathbf{u}(0))$. From the modified block i/o map, there exists a smooth local function $\hat{\Phi}_y^{-1}$ such that, $\hat{\mathbf{y}}(-n) = \hat{\Phi}_y^{-1}(\mathbf{y}(0), \hat{\mathbf{u}}(-n), \mathbf{u}(0))$ where $(\hat{\mathbf{u}}, \hat{\mathbf{y}}) = (\mathbf{u}, \mathbf{y})$ if $\partial\varphi(\cdot)/\partial y(t)$ is nonzero and $(\hat{\mathbf{u}}, \hat{\mathbf{y}}) = (\mathbf{y}, \mathbf{u})$ otherwise. Using this in the preceding equation, gives

$$B = -AD_{\mathbf{w}}\hat{\Phi}(\hat{\Phi}_y^{-1}(\mathbf{y}(0), \hat{\mathbf{u}}(-n), \mathbf{u}(0)), \hat{\mathbf{u}}(-n), \mathbf{w})\ |_{\mathbf{w}=\mathbf{u}(0)}$$

Thus

$$\mathcal{H}_{n+1} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\mathbf{y}(0) - D_{\mathbf{w}}\hat{\Phi}(\hat{\Phi}_y^{-1}(\mathbf{y}(0), \hat{\mathbf{u}}(-n), \mathbf{u}(0)), \hat{\mathbf{u}}(-n), \mathbf{w}) \mid_{\mathbf{w}=\mathbf{u}(0)} \mathrm{d}\mathbf{u}(0)\}.$$

It can be seen that $\mathcal{H}_{n+1}$ is integrable iff $D_{\mathbf{w}}\hat{\Phi}(\hat{\Phi}_y^{-1}(\mathbf{z}, \mathbf{u}, \mathbf{v}), \mathbf{u}, \mathbf{w}) \mid_{\mathbf{w}=\mathbf{v}}$ is independent of $\mathbf{u}$ or equivalently

$$\frac{\partial}{\partial w_i} D_{\mathbf{u}}\hat{\Phi}(\hat{\Phi}_y^{-1}(\mathbf{z}, \mathbf{u}, \mathbf{v}), \mathbf{u}, \mathbf{w}) \mid_{\mathbf{w}=\mathbf{v}} = 0$$

for $i = 1, \ldots, n$ where w_i is the ith component of $\mathbf{w}$. Taking the derivative and using (6) yields

$$D_{\mathbf{u}}\hat{\Phi}(\hat{\Phi}_y^{-1}(\mathbf{z}, \mathbf{u}, \mathbf{v}), \mathbf{u}, \mathbf{w}) = D_{\mathbf{y}}\hat{\Phi}(\mathbf{y}, \mathbf{u}, \mathbf{w})(D_{\mathbf{u}}\hat{\Phi}_y^{-1}(\mathbf{z}, \mathbf{u}, \mathbf{v}) - D_{\mathbf{u}}\hat{\Phi}_y^{-1}(\mathbf{z}, \mathbf{u}, \mathbf{w}))$$

Thus

$$\frac{\partial}{\partial w_i} D_{\mathbf{u}}\hat{\Phi}(\hat{\Phi}_y^{-1}(\mathbf{z}, \mathbf{u}, \mathbf{v}), \mathbf{u}, \mathbf{w}) \mid_{\mathbf{w}=\mathbf{v}} = -D_{\mathbf{y}}\hat{\Phi}(\mathbf{y}, \mathbf{u}, \mathbf{w})\left(\frac{\partial}{\partial w_i} D_{\mathbf{u}}\hat{\Phi}_y^{-1}(\mathbf{z}, \mathbf{u}, \mathbf{w}) \mid_{\mathbf{w}=v}\right).$$

Since $D_{\mathbf{y}}\hat{\Phi}(\mathbf{y}, \mathbf{u}, \mathbf{w})$ is invertible, $\mathcal{H}_{n+1}$ is integrable iff $D_{\mathbf{u}}\hat{\Phi}_y^{-1}(\mathbf{z}, \mathbf{u}, \mathbf{v})$ is independent of $\mathbf{v}$ thereby proving the claim.

If $\mathcal{H}_{n+1}$ is integrable, then the same argument as the one in the sufficiency proof of Theorem 3.2 in [1] can be used to conclude integrability of all the $\mathcal{H}'_j$s.

Next we show that $\mathrm{d}x(0) = \mathrm{d}\hat{\Phi}(\hat{\Phi}_y^{-1}(\mathbf{y}(0), \mathbf{0}, \mathbf{u}(0)), \mathbf{0}, \mathbf{v}) \in \mathcal{H}_{n+1}$. Indeed,

$$\mathrm{d}x = D_{\mathbf{y}}\hat{\Phi}(\mathbf{z}, \mathbf{0}, \mathbf{v})(D_{\mathbf{y}}\hat{\Phi}(\mathbf{z}, \mathbf{0}, \mathbf{u}))^{-1}(\mathrm{d}\mathbf{y} - D_{\mathbf{u}}\hat{\Phi}(\mathbf{z}, \mathbf{0}, \mathbf{u})\mathrm{d}\mathbf{u})$$

where $\mathbf{z} = \hat{\Phi}_y^{-1}(\mathbf{y}, \mathbf{0}, \mathbf{u})$. Thus $\mathrm{d}x \in \mathcal{H}_{n+1}$.

The following corollary provides a sufficient condition for the realizability of the input-output difference equation (1) giving rise to a special subclass of NARMA models to be discussed.

Corollary 2.4. *Consider a nonlinear system described by the input-output difference equation (1). Define*

$$\alpha_{i,j}(\mathbf{y}, \mathbf{u}, \mathbf{v}) := \delta^{i-1} D_{y_j}\varphi(y_1, \ldots, y_m, u_1, \ldots, u_m) \tag{7}$$

$$\beta_{i,j}(\mathbf{y}, \mathbf{u}, \mathbf{v}) := \delta^{i-1} D_{u_j}\varphi(y_1, \ldots, y_m, u_1, \ldots, u_m), \tag{8}$$

let $p = \min\{j : D_{y_j}\varphi(y_1, \ldots, y_m, u_1, \ldots, u_m) \neq 0\} - 1$ *and* $q = \min\{j : D_{u_j}\varphi(y_1, \ldots, y_m, u_1, \ldots, u_m) \neq 0\} - 1$, *where by Assumption (1)* $\min\{p, q\} = 0$. *Then the system has a generically observable state space realization if* $q = 0$ *and*

$$\alpha_{i,j}(\mathbf{y}, \mathbf{u}, \mathbf{v}),\ j = p+1, \ldots, n,\ i = 1, \ldots, n-j+1, \text{ and}$$
$$\beta_{i,j}(\mathbf{y}, \mathbf{u}, \mathbf{v}),\ j = 1, \ldots, n-p,\ i = 1, \ldots, n-p-j+1$$

or $p = 0$ *and*

$$\begin{aligned}&\alpha_{i,j}(\mathbf{y},\mathbf{u},\mathbf{v}),\ j = 1,\dots,n-q,\ i = 1,\dots,n-q-j+1\\&\beta_{i,j}(\mathbf{y},\mathbf{u},\mathbf{v}),\ j = q+1,\dots,n,\ i = 1,\dots,n-j+1\end{aligned}$$

are independent of $\mathbf{v}$.

Proof. We only prove the corollary for the case that $q = 0$ since the proof for the case of $p = 0$ is completely parallel to that of $q = 0$. It is shown in [2] that $D_{\mathbf{y}}\Phi = U^{-1}L_\alpha$ and $D_{\mathbf{u}}\Phi = U^{-1}L_\beta$ where $U(\mathbf{y},\mathbf{u},\mathbf{v})$ is an invertible lower triangular matrix, and $L_\alpha(\mathbf{y},\mathbf{u},\mathbf{v})$ and $L_\beta(\mathbf{y},\mathbf{u},\mathbf{v})$ are upper triangular matrices given by

$$L_\alpha = \begin{bmatrix}\alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,m}\\ 0 & \alpha_{2,1} & \cdots & \alpha_{2,m-1}\\ \vdots & \ddots & \ddots & \vdots\\ 0 & 0 & \cdots & \alpha_{m,1}\end{bmatrix},\quad L_\beta = \begin{bmatrix}\beta_{1,1} & \beta_{1,2} & \cdots & \beta_{1,m}\\ 0 & \beta_{2,1} & \cdots & \beta_{2,m-1}\\ \vdots & \ddots & \ddots & \vdots\\ 0 & 0 & \cdots & \beta_{m,1}\end{bmatrix}$$

By the hypothesis the above matrices can be partitioned as

$$L_\alpha = \begin{bmatrix}\mathbf{0} & L_{a,1}\\ \mathbf{0} & \mathbf{0}\end{bmatrix},\ L_\beta = \begin{bmatrix}L_{\beta,1} & L_{\beta,2}\\ \mathbf{0} & L_{\beta,3}\end{bmatrix}$$

where $L_{\alpha,1}$ and $L_{\beta,1}$ are $n-p\times n-p$ submatrices of L_α and L_β, respectively, and $L_{\beta,2}$ and $L_{\beta,3}$ are the remaining submatrices of L_β. The hypothesis implies that $L_\beta\ (\mathbf{u},\mathbf{y},\mathbf{v})$ is invertible and that

$$L_\beta^{-1}L_\alpha = \begin{bmatrix}\mathbf{0} & L_{\beta,1}^{-1}L_{a,1}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}$$

is independent of $\mathbf{v}$. Thus the matrix

$$\begin{aligned}D_{\mathbf{y}}\hat{\Phi}(\mathbf{y},\mathbf{u},\mathbf{v})^{-1}D_{\mathbf{u}}\hat{\Phi}(\mathbf{y},\mathbf{u},\mathbf{v}) &= D_{\mathbf{u}}\Phi(\mathbf{y},\mathbf{u},\mathbf{v})^{-1}D_{\mathbf{y}}\Phi(\mathbf{y},\mathbf{u},\mathbf{v})\\ &= L_\beta(\mathbf{y},\mathbf{u},\mathbf{v})^{-1}L_\alpha(\mathbf{y},\mathbf{u},\mathbf{v})\end{aligned}$$

is also independent of $\mathbf{v}$. The proof of the Corollary now follows from Theorem 2.3 as a special case.

3. A SPECIAL SUBCLASS OF NARMA MODELS

In many situations the NARMA model is obtained from experimental data using the identification procedures or neural networks. It is clear from Theorems 2.2 and 2.3 that an arbitrarily structured NARMA model does not necessarily have a state space realization. Using such a model is highly undesirable in further stability analysis and/or control design since practically all existing control theory for nonlinear systems bases on a state space description. Motivated by the above and relaying on the necessary and sufficient realizability conditions stated in Theorems 2.2 and 2.3, we now introduce a subclass of NARMA models, each of which is guaranteed to have a classical state space description.

The subclass provided by us is described by the following equations:

$$\begin{aligned} y(t+n) = \sum_{i=0}^{n-k-2} & \varphi_{n-k-i}(y(t+p+i),\ldots,y(t+k+i+1),u(t+i),\\ & \ldots,u(t+p+i)) + \varphi_1(y(t+p+\bar{\imath}),\ldots,y(t+n-1),\\ & u(t+\bar{\imath}),\ldots,u(t+p+\bar{\imath})) \end{aligned} \tag{9}$$

for $k=0,1,\ldots,n-1$, $p=0,1,\ldots,k$, $\bar{\imath}=n-k-1$, where φ_j's are smooth functions of their arguments. Noting that

$$\begin{aligned} \varphi(y_1,\ldots,y_n,u_1,\ldots,u_n) = & \sum_{i=0}^{n-k-2} \varphi_{n-k-i}(y_{p+i+1},\ldots,y_{k+i+2},u_{i+1},\ldots,u_{i+p+1}) \\ & + \varphi_1(y_{n-k+p},\ldots,y_n,u_{n-k},\ldots,u_{n-k+p}) \end{aligned}$$

then it can be easily checked that this NARMA satisfies the sufficient conditions of Corollary (2.4). Also, computing the subspace

$$\begin{aligned} \mathcal{H}_{s+2} = \mathcal{H}_{n-k+p+1} = \mathrm{sp}_{\mathcal{K}}\{ & \mathrm{d}y(t),\ldots,\mathrm{d}y(t+k-p),\mathrm{d}y_1(t+k-p+1),\ldots,\\ & \mathrm{d}y_{n-k-1}(t+n-p-1),\mathrm{d}u(t-p),\ldots,\mathrm{d}u(t-1)\} \end{aligned}$$

where (note that n is treated as a constant here)

$$\begin{aligned} y_j(t+n) = \sum_{i=0}^{n-k-j-1} & \varphi_{n-k-i}(y(t+p+i),\ldots,y(t+k+i+1),\\ & u(t+i),\ldots,u(t+p+i)), \quad j=1,\ldots,n-k-1 \end{aligned}$$

shows that the NARMA model (9) is realizable in the classical state space form.

The subclass given by Equation (9) is more general than those introduced in the earlier works [2, 4] and covers several important subclasses of NARMA models as a special case. For example, for $p=0$, one obtains the i/o model

$$\begin{aligned} y(t+n) = \sum_{i=0}^{n-k-2} & \varphi_{n-k-i}(y(t+i),\ldots,y(t+k+i+1),u(t+i)) \\ & + \varphi_1(y(t+n-k-1),\ldots,y(t+n-1),u(t+n-k-1)),\\ k=0,\ldots,n-1 & \end{aligned}$$

introduced in reference [4], which excludes any couplings among the input terms. For $k=0$ it reduces to the special subclass, presented in [2] and proved to be realizable in the Kalmanian form.

When studying the control problems for NARMA models, majority of the researchers typically assume the system to be in the controller (Brunovsky) canonical form. This form corresponds to the case $k = n - 1$, $p = 0$.

On the other hand, for $p = k$ yields

$$\begin{aligned} y(t+n) = \sum_{i=0}^{n-k-2} &\varphi_{n-k-i}(y(t+k+i), y(t+k+i+1), u(t+i),\\ &\ldots, u(t+i+k)) + \varphi_1(y(t+n-1), u(t+n-k-1),\\ &\ldots, u(t+n-1)) \end{aligned}$$

it results in an input-output model with a minimal amount of output couplings.

The state coordinates can be obtained from Theorem 2.3 or by integrating the one-forms in $\mathcal{H}_{n-k+p+1}$. So, we choose

$$\begin{aligned} x_1(t) &= y(t)\\ &\vdots\\ x_{k-p+1}(t) &= y(t+k-p)\\ x_{k-p+2}(t) &= y_1(t+k-p+1)\\ &\vdots\\ x_{n-p}(t) &= y_{n-k-1}(t+n-p-1)\\ x_{n-p+1}(t) &= u(t-p)\\ &\vdots\\ x_n(t) &= u(t-1) \end{aligned}$$

and obtain the state equations as follows

$$\begin{aligned} x_1^+ &= x_2\\ &\vdots\\ x_{k-p}^+ &= x_{k-p+1}\\ x_{k-p+1}^+ &= x_{k-p+2} + \varphi_1(x_1, \ldots, x_{k-p+1}, x_{n-p+1}, \ldots, x_n, u)\\ x_{k-p+2}^+ &= x_{k-p+3} + \varphi_2(x_1, \ldots, x_{k-p+1}, x_{k-p+1}^+, x_{n-p+1}, \ldots, x_n, u)\\ &\vdots\\ x_{n-p-1}^+ &= x_{n-p} + \varphi_{n-k-1}(x_1, \ldots, x_{k-p+1}, x_{k-p+1}^+, x_{n-p+1}, \ldots, x_n, u)\\ x_{n-p}^+ &= \varphi_{n-k}(x_1, \ldots, x_{k-p+1}, x_{k-p+1}^+, x_{n-p+1}, \ldots, x_n, u) \end{aligned}$$

$$
\begin{aligned}
x^+_{n-p+1} &= x_{n-p+2} \\
&\vdots \\
x^+_n &= u \\
y &= x_1
\end{aligned}
\tag{10}
$$

where $x_i^+(t) = x_i(t+1)$.

4. EXAMPLES

4.1. Example 1

We explain the relationship of the two seemingly different approaches on the example of the 3rd order bilinear input-output equation

$$
\begin{aligned}
y(t+3) = a_1 y(t+2) + a_2 y(t+1) + a_3 y(t) + b_1 u(t+2) + c_{11} y(t+2) u(t+2) + \\
+ c_{21} y(t+1) u(t+2) + c_{31} y(t) u(t+2)
\end{aligned}
$$

that does not belong to the subclass (9) described in Section 3 but is realizable in the classical state space form.

First compute the subspace

$$
\mathcal{H}_{s+2} = \mathcal{H}_5 = \mathrm{sp}_{\mathcal{K}}\{\omega_1(t), \omega_2(t), \omega_3(t)\},
$$

where

$$
\begin{aligned}
\omega_1(t) &= \mathrm{d}y(t) \\
\omega_2(t) &= \mathrm{d}[y(t+1) - b_1 u(t) - a_1 y(t) - c_{11} y(t) u(t) - c_{21} y(t-1) u(t) \\
&\quad - c_{31} y(t-2) u(t)] \\
\omega_3(t) &= \mathrm{d}[y(t+2) - b_1 u(t+1) - a_1 y(t+1) - c_{11} y(t+1) u(t+1) \\
&\quad - c_{21} y(t) u(t+1) - c_{31} y(t-1) u(t+1) - a_2 y(t)].
\end{aligned}
$$

Note that the subspace is completely integrable and both $y(t-1)$ and $y(t-2)$ can be expressed as functions of $y(t+2), y(t+1), y(t), u(t+1)$ and $y(t+2), y(t+1), y(t), u(t+1), u(t)$, respectively. To keep the notation simple we do not change $y(t-1)$ and $y(t-2)$ in $\mathcal{H}_5$ for those functions. We get the state components by integrating the one forms in $\mathcal{H}_5$:

$$
\begin{aligned}
x_1(t) &= y(t) \\
x_2(t) &= y(t+1) - b_1 u(t) - c_{11} y(t) u(t) - c_{21} y(t-1) u(t) \\
&\quad - c_{31} y(t-2) u(t) - a_1 y(t) \\
x_3(t) &= y(t+2) - b_1 u(t+1) - c_{11} y(t+1) u(t+1) - c_{21} y(t) u(t+1) - \\
&\quad - c_{31} y(t-1) u(t+1) - a_1 y(t+1) - a_2 y(t)
\end{aligned}
$$

and shifting the above equations one step forward we obtain

$$\begin{aligned} x_1(t+1) &= x_2(t) + b_1u(t) + a_1x_1(t) + c_{11}x_1(t)u(t) + c_{21}y(t-1)u(t) \\ &\quad + c_{31}y(t-2)u(t) \\ x_2(t+1) &= x_3(t) + a_2x_1(t) \\ x_3(t+1) &= a_3x_1(t) \\ y(t) &= x_1(t) \end{aligned} \tag{11}$$

From the 2nd and the 3rd state equations we get

$$\begin{aligned} y(t-1) &= \frac{x_3(t)}{a_3} \\ y(t-2) &= \frac{1}{a_3}x_3(t-1) = \frac{1}{a_3}(x_2(t) - a_2y(t-1)) = \frac{1}{a_3}(x_2(t) - \frac{a_2}{a_3}x_3(t)) \end{aligned}$$

After substituting $y(t-1)$ and $y(t-2)$ from the above equations into (11) we reach the state equations.

The approach presented in Theorem 2.3 suggests the state vector to be

$$\begin{aligned} \tilde{x}_1(t) &= a_1y(t-1) + a_2y(t-2) + a_3y(t-3) - b_1u(t-1) \\ &\quad - c_{11}y(t-1)u(t-1) - c_{21}y(t-2)u(t-1) - c_{31}y(t-3)u(t-1) \\ \tilde{x}_2(t) &= a_1y(t) + a_2y(t-1) + a_3y(t-2) \\ \tilde{x}_3(t) &= a_1\tilde{x}_2 + a_2y(t) + a_3y(t-1) \end{aligned} \tag{12}$$

which can be rewritten as

$$\begin{aligned} \tilde{x}_1(t) &= y(t) = x_1(t) \\ \tilde{x}_2(t) &= y(t+1) - b_1u(t) - c_{11}y(t)u(t) - c_{21}y(t-1)u(t) - c_{31}y(t-2)u(t) \\ &= x_2(t) + a_1x_1(t) \\ \tilde{x}_3(t) &= y(t+2) - b_1u(t+1) - c_{11}y(t+1)u(t+1) - c_{21}y(t)u(t+1) \\ &\quad - c_{31}y(t-1)u(t+1) - a_1y(t+1) + a_1(x_2(t) + a_1x_1(t)) \\ &= x_3(t) + a_2x_1(t) + a_1(x_2(t) + a_1x_1(t)) \end{aligned}$$

So, both methods will result in the equivalent state equations. Moreover, note that $\mathcal{H}_5$ can be rewritten as

$$\mathcal{H}_5 = \mathrm{sp}_{\mathcal{K}}\{\omega_1(t), \omega_2(t) + a_1\omega_1(t), \omega_3(t) + a_1(\omega_2(t) + a_1\omega_1(t)) + a_2\omega_1(t)\}$$

which will result exactly in state coordinates defined by (12).

4.2. Example 2

The model below describes the relationship between the varying part of the current u and the frequency y of the generated voltage [5]:

$$
\begin{aligned}
y(t+4) = {} & -0.00113 - 0.0628u(t+2) - 0.0675u(t+1) - 0.0215u(t) \\
& + 0.84y(t+3) - 0.0526u(t+1)y(t+2) - 0.053u(t+2)y(t+3) \\
& + 0.0613y^2(t+3) - 0.0071u(t+2)u(t+1) - 0.0234u^2(t+2)y(t+3) \\
& - 0.044u^2(t+1)y(t+3) + 0.0573u(t+2)y^2(t+3) - 0.02y^2(t+1)
\end{aligned}
$$

Direct observation reveals that $y(t)$ and $u(t+3)$ are missing in the above equations. So, the natural choice in accomodating those equation into (9) is $k=2$ and $p=1$. Obviously, there is some freedom for several terms to be incorporated either in φ_1 or φ_2. Since in (10) φ_1 has to be substituted in φ_2 we have chosen φ_1 as simple as possible

$$
\begin{aligned}
& \varphi_1(y(t+2), y(t+3), u(t+1), u(t+2)) = \\
& \quad = -0.0628u(t+2) - 0.053u(t+2)y(t+3) - 0.0071u(t+2)u(t+1) \\
& \qquad - 0.0234u^2(t+2)y(t+3) + 0.0573u(t+2)y^2(t+3) \\
& \quad \varphi_2(y(t+1), y(t+2), y(t+3), u(t), u(t+1)) = \\
& \quad = -0.00113 - 0.0675u(t+1) - 0.0215u(t) + 0.84y(t+3) \\
& \qquad - 0.0526u(t+1)y(t+2) - 0.0613y^2(t+3) - 0.044u^2(t+1)y(t+3) \\
& \qquad - 0.02y^2(t+1)
\end{aligned}
$$

which results in state equations

$$
\begin{aligned}
x_1^+ &= x_2 \\
x_2^+ &= x_3 + \varphi_1(x_1, x_2, x_4, u) \\
x_3^+ &= \varphi_2(x_1, x_2, x_3 + \varphi_1(x_1, x_2, x_4, u), x_4, u) \\
x_4^+ &= u \\
y &= x_1
\end{aligned}
$$

with

$$
\begin{aligned}
x_2 &= y(t+1) \\
x_3 &= y(t+2) - \varphi_1(y(t), y(t), u(t-1), u(t)) \\
x_4 &= u(t-1)
\end{aligned}
$$

5. CONCLUSIONS

This paper explicitly proved the equivalence of the necessary and sufficient conditions in papers [1] and [2] for observable realization of a general class of nonlinear high-order input-output difference equations. The paper also formulated a general subclass of realizable NARMA models, which covers several important subclasses of existing NARMA models.

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