Full Abstraction for Probabilistic PCF

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We present a probabilistic version of PCF, a well-known simply typed universal functional language. The type hierarchy is based on a single ground type of natural numbers. Even if the language is globally call-by-name, we allow a call-by-value evaluation for ground-type arguments to provide the language with a suitable algorithmic expressiveness. We describe a denotational semantics based on probabilistic coherence spaces, a model of classical Linear Logic developed in previous works. We prove an adequacy and an equational full abstraction theorem showing that equality in the model coincides with a natural notion of observational equivalence.

CCS Concepts: • **Theory of computation → Lambda calculus**; **Probabilistic computation**; **Linear logic**; **Denotational semantics**; **Operational semantics**; **Categorical semantics**;

Additional Key Words and Phrases: Functional programming languages, full abstraction

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INTRODUCTION

Stochastic programming is an increasingly important tool in software design and all modern programming languages provide rich libraries of stochastic methods. As illustrated in Ramsey and Pfeffer (2002), functional programming languages can also be used efficiently for representing statistical models. The goal of this article is to explore the properties of a relatively new denotational interpretation of such languages, based on *probabilistic coherence spaces*, and where programs are interpreted as analytic functions.

From the viewpoint of denotational semantics, the language PCF introduced by Plotkin (1977) (based on earlier ideas of Dana Scott) and further studied by many authors plays a central role because of its conceptual simplicity combined with a full higher-order extended Turing completeness. PCF is a simply typed purely functional language, without any kind of *effects*: no references or mutable variables, no I/O primitives, no probabilistic or non-deterministic features, and so on; PCF should be considered as the purely functional and simply typed kernel of all functional programming languages. Because of this simplicity, PCF can be endowed with a very well-behaved

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Church-Rosser *rewriting system* on terms. Moreover, one can extract from this rewriting system a canonical set of rewriting rules called *weak head-reduction* that can be considered an *operational semantics* (it is a deterministic reduction strategy), turning PCF into a true programming language equipped with a call-by-name evaluation scheme that is complete in the sense that the evaluation of any closed term of ground type either diverges or converges to a uniquely determined natural number.

Since the seminal article (Scott 1976), most denotational models of PCF have been based on *domains* that are partially ordered sets enjoying suitable completeness properties (typically, all directed subsets have a least upper bound). In such models, PCF terms are interpreted as Scott continuous functions, that is, monotone functions that commute with all directed lubs: This property accounts abstractly for the finiteness of computation. Other similar models take into account in various ways the deterministic features of PCF (stable semantics (Berry 1978) that can be developed in the simplified framework of *coherence spaces* (Girard 1986), sequential algorithms (Berry and Curien 1982) that were later understood as deterministic strategies in games, strongly stable functions (Bucciarelli and Ehrhard 1994; Ehrhard 1993), etc.). All these models feature *soundness* (if the PCF term M rewrites to M' then M and M' have the same interpretation) and *computational adequacy*: If a closed term M of integer type is interpreted as the integer n in the model, then its weak head-reduction terminates with value n.[1]

Computational adequacy implies that if two PCF terms of the same type have the same interpretation in one of these models, then they are *observationally equivalent*, meaning that, when plugged in the same context of ground type, the resulting terms either both diverge or both converge to the same value by weak head-reduction. A model satisfying also the converse implication[2] is said to be *fully abstract*, and this is not the case of any of the models alluded to so far.

Usual Approach to Effects: Domains, Scott Continuous Maps, and Computational Monads. Moggi (1989) proposed a uniform way of interpreting effects in a denotational model of PCF (roughly speaking, a Cartesian closed category with additional structures accounting for basic arithmetic operations and fix-point constructs) that consists in encapsulating them into a *computational monad* that can be understood as an operation on types. For instance, extending the language with a global store (with write and read operations) can be represented as the computational monad that maps a type, or rather an object X of the model, to the object $S \Rightarrow X \times S$, where S is an object that represents the store: A program of type X returns a value of type X and, at the same time, modifies the store.

When one wants to extend PCF with probabilistic choice, the standard approach consists accordingly in defining a "probabilistic powerdomain" monad, that is, in associating with any object X of the category an object $T(X)$ whose purpose is to represent the "space of probability distributions" (or measures) on X. In that way, a morphism from X to Y in the associated Kleisli category—where terms of our extended PCF are to be interpreted—maps a value in X to a probabilistic choice of values in Y. Various constructions of this kind can be found in the literature, developing ideas initiated in Saheb-Djahromi (1980) and Kozen (1981). In this line of research, one should mention, notably, Jones and Plotkin (1989) and, of course, Jung and Tix (1998), which provides an interesting account of the difficulties encountered when trying to combine such monads with the higher-order features of functional programming languages.

This research direction led recently to new approaches, such as that in Battenfeld et al. (2007) based on a class of *topological domains* that form a Cartesian closed category with fix-point operators and probabilistic powerdomains. As explained in that article, the combination of

[1] From which, by the way, the above completeness of weak head-reduction property follows.

[2] That is, two terms are observationally equivalent iff they have the same interpretation in the model.

function space and powerdomain constructions can lead to fairly complicated objects to which the usual domain-theoretic intuitions do not apply anymore. Last, we should also mention Goubault-Larrecq (2015), which is based on a Cartesian closed category of bounded-complete domains and Scott-continuous functions and where full-abstraction is obtained for extensions of probabilistic PCF with some statistical termination testers or by combining probabilistic with nondeterministic choice.

The Probabilistic Coherence Space Approach. We develop here quite a different approach to the denotational semantics of probabilistic programs rooted in the idea of a *quantitative semantics* (Girard 1988), where *power series* are used for interpreting programs. The notion of *probabilistic coherence space* appears for the first time in Girard (2004) and is developed further in Danos and Ehrhard (2011), where this model is extended to the whole of Linear Logic, to PCF, and to the pure λ-calculus. In Ehrhard et al. (2014), we prove this model to be equationally fully abstract for probabilistic PCF for a notion of observational equivalence based on the equality of convergence probabilities. This model has some similarities with *coherence spaces* (Girard 1986, 1987), whence its name. A probabilistic coherence space X is a set $|X|$ together with a set of $\mathbb{R}_{\geq 0}$-valued functions on this set that we call *valuations*. This set of valuations is assumed to fulfill a closure property defined by means of a duality between valuations: u and u' are in duality if $\langle u, u' \rangle = \sum_{a \in |X|} u_a u'_a \leq 1$. The intuition behind this definition is clear: u should be considered as a program and u' as an observation, and $\langle u, u' \rangle$ is the probability of observing u' on u.

These probabilistic coherence spaces are the objects of a category **Pcoh** whose morphisms can be understood as matrices: a morphism from X to Y is a matrix indexed by $|X| \times |Y|$ and maps (by standard matrix application) any X-valuation to a Y-valuation. Such matrices can be composed by means of a standard (generally infinite dimensional) matrix product. This category can be endowed with a monoidal structure and is Cartesian; it is actually a model of (classical) Linear Logic and also allows to interpret arbitrary recursive types. On this category of probabilistic coherence spaces and matrices, we know currently only one exponential comonad !_, which is based on a simple multiset construction and that we proved recently to be the free one (Crubillé et al. 2017).[3]

This comonad gives rise to a Kleisli category whose morphisms can be seen as analytic functions defined on the sets of valuations of objects. This Kleisli category is Cartesian closed and admits fix-point operators for all objects, providing therefore a model of PCF. Moreover, since any barycentric combination of X-valuations is still an X-valuation, this category also provides a natural interpretation of the stochastic construction of probabilistic PCF.

To avoid confusions we stress that unlike Moggi's computational monads that typically act on Cartesian closed categories and induce Kleisli categories that model call-by-value languages, the Linear Logic resource comonads !_ act on symmetric monoidal closed categories that are typically not Cartesian closed and induce Cartesian closed Kleisli categories, modeling call-by-name languages. As in any model of Linear Logic, the functor !_ acts in turn as a computational monad on this latter Kleisli category, but, in the case of probabilistic coherence spaces, this monad is not specifically related with the interpretation of the probabilistic effect that, as explained above, is interpreted using directly the properties of probabilistic coherence spaces.

For the time being, the probabilistic coherence spaces interpretation is restricted to *discrete probabilities*; extending these ideas to a continuous setting is possible by considering a notion of stable functions acting on cones (Ehrhard et al. 2018).

Related Approaches. The second author of the present article has recently shown, in a joint work with Laird, Manzonetto, and McCusker (Laird et al. 2013), that constructions similar to those of

[3] Actually, after the present article has been submitted.

Pcoh (and actually simpler) can be applied to the more abstract setting of R-weighted relational categories, where the objects are sets and the morphisms are matrices taking coefficients in a complete semi-ring R. Depending on R, one can model different "quantitative" effects of a language. In particular, in the case where R is the order completion $\mathbb{R}_{\geq 0} \cup \{\infty\}$ of $\mathbb{R}_{\geq 0}$, one gets a model of linear logic strictly related with **Pcoh**. Precisely, the map associating a probabilistic coherence space to its web and acting as the identity on morphisms defines a forgetful functor from **Pcoh** to the category of sets and $\mathbb{R}_{\geq 0} \cup \{\infty\}$-weighted relations that respects the whole linear logic structure of the two categories.

This means in particular that the two models denote with exactly the same matrix any probabilistic PCF program. The additional information that we obtain thanks to the more constrained definition of probabilistic coherence spaces is that these denotations *never use infinite coefficients*, hence they can be seen as true power series with coefficients in $\mathbb{R}_{\geq 0}$. This last property is crucial in our proof of full-abstraction (precisely in the proof of Theorem 4.3). In turn, the full-abstraction theorem for **Pcoh** implies trivially also full-abstraction of the $\mathbb{R}_{\geq 0} \cup \{\infty\}$-weighted semantics, since all probabilistic PCF terms are interpreted by the same matrices in both models.

Ideas similar to ours have been considered in Scott (2014), where a graph model of the pure λ-calculus is extended with probabilistic choice, giving rise to a structure that seems to bear some similarities with the stochastic models of the pure λ-calculus introduced and studied in Danos and Ehrhard (2011) and Ehrhard et al. (2011), although Dana Scott considers general continuous functions and not analytic ones. Understanding the precise connection between the two settings deserves further investigation.

Last, we would also like to mention game-based models of stochastic programming languages, and, most notably, Danos and Harmer (2000), where full abstraction for a probabilistic extension of Idealized Algol is proven. It is one of our objectives to understand if the probabilistic coherence space semantics can be obtained as a kind of "extensional collapse" of such a probabilistic game model.

Soundness, Adequacy, and Full Abstraction. To give a more precise account of the connection between probabilistic coherence spaces and our probabilistic PCF target language, we have to explain better the operational semantics of this language. It is given by a rewriting system on terms that implements a call-by-name reduction. Due to the presence of a probabilistic primitive in the language, this rewriting system, though "deterministic" in the sense that any term has at most one redex for it, is given as a Markov chain whose states are terms, and we present this Markov chain as an infinite-dimensional stochastic matrix (indexed by terms) Red. Given two terms M and M', the coefficient $\mathrm{Red}_{M,M'} \in [0,1]$ is the probability of M to reduce to M' in one step.[4] The "transitive closure" of this reduction relation is obtained by taking powers of this matrix. The soundness of this interpretation is expressed as follows in Theorem 2.14: Given a term M of type σ (assume it is closed for the sake of simplicity), then, for all $n \in \mathbb{N}$,

$$[\![M]\!] = \sum_{\vdash M' : \sigma} \mathrm{Red}^n_{M,M'} [\![M']\!],$$

where the sum ranges over all closed terms M' of type σ (it has actually at most 2^n non-zero terms) and $[\![M]\!]$ is the denotational interpretation of M, which is a valuation in the probabilistic coherence space associated with the type σ.

Let now M' be a term in normal form. The probability of M to reduce to M' (in some finite number of steps) is obtained by taking an "infinite power" of the matrix Red or, more precisely,

[4]Since we use a "coin" stochastic basic operation that has only two possible outcomes, there are actually at most two terms M' such that $\mathrm{Red}_{M,M'} \neq 0$ for any given term M.

the least upper bound of the sequence of probabilities $\mathrm{Red}^n_{M,M'}$ which is monotone, because we require Red to satisfy $\mathrm{Red}_{M',M'} = 1$ when M' is in normal form. We denote this number as $\mathrm{Red}^\infty_{M,M'}$ (remember that this notation is meaningful only when M' is in normal form). Let M be closed of type ι (the type of natural numbers). The interpretation $[\![M]\!]$ is a sub-probability distribution on natural numbers. It results easily from soundness that $[\![M]\!]_n$ (the value of that valuation at integer n) is less than the probability $\mathrm{Red}^\infty_{M,\underline{n}}$ of M to reduce to the constant $\underline{n}$. We prove Theorem 3.5, which expresses that these two numbers are actually equal. The proof uses a fairly standard logical relation approach, borrowed to Amadio and Curien (1998). This allows to prove that if two terms M_1 and M_2 of type σ have the same interpretation (let us assume again that they are closed), then they are observationally equivalent in the sense that, for any closed term C of type $\sigma \Rightarrow \iota$, the terms $(C)\,M_1$ and $(C)\,M_2$ have the same probability to converge to $\underline{0}$ (or to any given constant $\underline{n}$). We actually give a more general statement allowing M_1 and M_2 to have free variables.

We finish the article by proving the converse statement: equational Full Abstraction, Theorem 4.4. This proof uses terms associated with points of the webs of probabilistic coherence spaces and has similarities with the proofs of Full Abstraction in Bucciarelli et al. (2011) and Nygaard and Winskel (2003). Its specificity is to rely fundamentally on the fact that the morphisms of the Kleisli category can be seen as analytic functions. It is worth noticing that, in contrast with most situations of full abstraction for PCF, our model does not satisfy any kind of *full completeness* (expressing that interpretations of terms form a dense set). In that sense also probabilistic coherence spaces are similar to ordinary coherence spaces: For instance, an analogue of the famous Gustave's function can be defined in the Kleisli category of probabilistic coherence spaces, without preventing, however, this model from being fully abstract. More details and intuitions on the proof are given in the introduction of Section 4.

An Improved Syntax for Probabilistic PCF: Call-by-Value for Integers. With respect to Ehrhard et al. (2014), our new presentation provides a major improvement concerning the syntax of the programming language under consideration.

Indeed, in this previous work we considered a *fully call-by-name* (CBN) version of probabilistic PCF. In this language, a closed term of type ι (the type of natural numbers) determines a sub-probability distribution on the natural numbers (with n we associate the probability that M reduces to the constant $\underline{n}$ of the language). A closed term P of type $\iota \Rightarrow \iota$ that receives M as argument will reduce M each time it needs its value (because the language is fully CBN) and will get different results each time unless the sub-probability distribution defined by M is concentrated on a single natural number. There are clearly cases where this is not a desirable behavior: Sometimes we need to roll a dice and to use the result several times!

As an example, consider the problem of writing a program that takes an array f of integers of length n and returns an index i such that $f(i) = 0$; we want to apply a "Las Vegas" random algorithm consisting in choosing i randomly (with a uniform probability on $\{0, \ldots, n-1\}$) repeatedly until we find an i such that $f(i) = 0$. This is implemented by means of a while loop (or, more precisely, of a recursively defined function, since we are in a functional setting) where at each step we choose i randomly, test the value of $f(i)$ (first use of i) and return i (second use) if $f(i) = 0$. It is intuitively clear that this basic algorithm cannot be implemented with the usual conditional of call-by-name PCF (it might be interesting and challenging to prove it).

To be able to write such an algorithm, we need to modify PCF a bit, allowing us to use ground terms in a CBV fashion (to simplify the presentation we use ι as single ground type).

Our choice has been to modify the conditional construct. The usual conditional construct $\mathrm{if}(M, P, Q)$ of PCF is operationally interpreted as follows: One first reduces M until one gets an integer n (or, more precisely, the corresponding term $\underline{n}$). If $n = 0$, then one evaluates P and,

otherwise, one evaluates Q, in the current context of course. Again, the trouble is that, in the second case, the value obtained for M, namely $\underline{n}$, is lost, whereas Q might need it. This problem can be easily solved by using M within Q each time this value is needed. Although clearly inefficient, this solution is perfectly correct in the usual deterministic version of PCF. It is absolutely inadequate in our probabilistic setting, since M should be considered as a *probabilistic process* whose reduction, or execution, will produce integer values with a sub-probability distribution depending on it. There is no reason for M to produce, within Q, the same result $\underline{n}$ that it reduced to during its first evaluation.

For these reasons, when M reduces to $\underline{n+1}$, our conditional construction $\mathsf{if}(M, P, z \cdot Q)$ allows to feed Q with $\underline{n}$ through the variable z (this has the positive side effect of making the predecessor function definable); in other words we have the reduction rules

$$\frac{}{\mathsf{if}(\underline{0}, P, z \cdot Q) \to P} \qquad \frac{}{\mathsf{if}(\underline{n+1}, P, z \cdot Q) \to Q\left[\underline{n}/z\right]}$$

$$\frac{M \to M'}{\mathsf{if}(M, P, z \cdot Q) \to \mathsf{if}(M', P, z \cdot Q)}.$$

This means that our conditional construct allows us to use a CBV reduction strategy, limited to the ground type of natural numbers.

From the point of view of Linear Logic and of its denotational models, this feature is completely justified by the fact that the object interpreting the type of natural numbers has a canonical structure of coalgebra for the !_ exponential functor. Intuitively, this means that *evaluated natural numbers* can be freely discarded and duplicated. Pushing this idea further leads to consider a calculus (Ehrhard 2016) close to Levy's Call-By-Push-Value (Levy 2006) whose probabilistic version is considered in Ehrhard and Tasson (2016).

Contents. In Section 1, we present the syntax of Probabilistic PCF (pPCF) and its weak-reduction relation, which we formalize as an infinite-dimensional stochastic matrix (indexed by pPCF terms). Based on this operational semantics, we define a notion of observational equivalence. Two terms of type σ in a typing context Γ are equivalent if, for any context $C^{\Gamma\vdash\sigma}$ of type ι in context Γ (with holes of type σ), the probability that $C[M]$ reduces to $\underline{0}$ (say) is equal to the probability that $C[M']$ reduces to $\underline{0}$.

Then we give various examples of programs written in this language; some of them will be essential in the proof of the Full Abstraction Theorem. In particular, we implement the above-mentioned simple Las Vegas algorithm.

Next, in Section 2, we introduce the model of Probabilistic Coherence Spaces (PCS), presented as a model of classical Linear Logic. We describe the interpretation of pPCF terms, presenting the semantics of terms as functions (this is possible because the Kleisli category of the !-comonad of this model is well pointed). In Section 3, we prove Theorem 3.5, which states that, for any closed term M of ground type ι and any $n \in \mathbb{N}$, the probability that M reduces to $\underline{n}$ is equal to the probability of n in the sub-probability distribution on $\mathbb{N}$, which is the semantics of M in the PCS model. This implies that any two closed terms of type σ that have the same interpretation in PCS are observationally equivalent.

Last, we prove the converse implication showing that PCS is a Fully Abstract model of pPCF. The proof uses strongly the fact that, in our model, morphisms are analytic functions (with real non-negative coefficients) and that the coefficients of the power series of two such functions are the same if the functions coincide on an open subset of their domain. Section 4 is devoted to this theorem (Theorem 4.4) and to its detailed proof; it starts with an accurate description of our proof method.

1 PROBABILISTIC PCF

There is only one ground type ι, types are defined by

$$\sigma, \tau, \ldots := \iota \mid \sigma \Rightarrow \tau.$$

The terms of pPCF are defined as follows:

$$M, N, \ldots := \underline{n} \mid x \mid \mathsf{succ}(M) \mid \mathsf{if}(M, P, z \cdot R) \mid \lambda x^{\sigma} M \mid (M)\, N \mid \mathsf{coin}(p) \mid \mathsf{fix}(M),$$

where $n \in \mathbb{N}$, $p \in [0,1] \cap \mathbb{Q}$ is a probability[5] and $x, y \ldots$ are variables.

A typing context is a sequence $\Gamma = (x_1 : \sigma_1, \ldots, x_n : \sigma_n)$, where the x_i's are pairwise distinct variables. A typing judgment is an expression $\Gamma \vdash M : \sigma$, where Γ is a typing context, M is a term, and σ is a type. The typing rules are as follows:

$$\frac{}{\Gamma \vdash \underline{n} : \iota} \qquad \frac{}{\Gamma, x : \sigma \vdash x : \sigma} \qquad \frac{\Gamma \vdash M : \iota}{\Gamma \vdash \mathsf{succ}(M) : \iota}$$

$$\frac{\Gamma \vdash M : \iota \qquad \Gamma \vdash P : \sigma \qquad \Gamma, z : \iota \vdash R : \sigma}{\Gamma \vdash \mathsf{if}(M, P, z \cdot R) : \sigma}$$

$$\frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x^{\sigma} M : \sigma \Rightarrow \tau} \qquad \frac{\Gamma \vdash M : \sigma \Rightarrow \tau \qquad \Gamma \vdash N : \sigma}{\Gamma \vdash (M)\, N : \tau} \qquad \frac{\Gamma \vdash M : \sigma \Rightarrow \sigma}{\Gamma \vdash \mathsf{fix}(M) : \sigma}$$

$$\frac{p \in [0,1] \cap \mathbb{Q}}{\Gamma \vdash \mathsf{coin}(p) : \iota}.$$

Usually in PCF, a closed term M of type ι (that is $\vdash M : \iota$) represents a program of integer type. The evaluation of such a program M can either diverge without producing any result, or terminate, providing a uniquely determined result $\underline{n}$. In pPCF, the outcome of such a computation will depend on the choices made at each evaluation of the term $\mathsf{coin}(p)$ required for producing a given result $\underline{n}$. So we can consider such a term M as a random variable from the set of choice sequences (which are simply infinite sequences of Booleans) to the set $\mathbb{N}$: This is the main idea underlying the semantic framework proposed in Goubault-Larrecq and Varacca (2011), which uses, moreover, the observation that these random variables are continuous (for the usual topology of the Cantor Space), because, when a program terminates, it has tossed the dice only finitely many times.

Remark: The typing rule for the $\mathsf{if}(M, P, z \cdot R)$ construct deserves a comment. It intuitively means that ι is recursively defined as a coproduct: $\iota = 1 \oplus \iota$, where 1 is a "unit type" with only one value. This will be actually the case also semantically in our probabilistic coherence space model, but *not in the Kleisli category* $\mathbf{Pcoh}_!$ that cannot have categorical coproducts, since it has fix-point operators. This description of ι as a coproduct will be possible in the Eilenberg-Moore category $\mathbf{Pcoh}^!$ that is Cartesian and co-Cartesian. Indeed, the "object of natural numbers" interpreting ι in our model is canonically a !-coalgebra, that is, an object of $\mathbf{Pcoh}^!$. This canonical coalgebra structure of the type ι allowing to see it as a coproduct has an operational byproduct: A value of type ι can be freely discarded and duplicated. This explains why it can be passed to the term R through the variable z whose linear type, in a standard CBN Girard translation of PCF into Linear Logic, would be $!\iota$ and not ι (the aforementioned coalgebra structure is precisely a linear morphism $\iota \multimap !\iota$, see Section 2.10). Passing the value of M to R through z requires of course first M to be evaluated to a value, that is, to a natural number constant $\underline{n}$ of the language. It is this specific value $\underline{n}$ that is then passed to R that can use it as many times as it wishes. This idea that data are freely copyable and discardable is also the main intuition behind *storage operators* associated with

[5] We restrict this probability to be rational to keep the set of terms countable. We could use any countable dense subset of [0, 1] instead of rationals, such as dyadic rationals.

datatypes in Krivine (1994) (in a lambda-calculus framework) whose purpose is exactly the same: evaluate a term and memorize the result for further uses.

Proposition 1.1. *Let M be a term and Γ be a typing context. There is at most one type σ such that $\Gamma \vdash M : \sigma$.*

The proof is a simple inspection of the typing rules.

Given terms M and N and given a variable x, the term $M[N/x]$ stands for M, where x is substituted with N.

Lemma 1.2. *If $\Gamma, x : \sigma \vdash M : \tau$ and $\Gamma \vdash N : \sigma$, then $\Gamma \vdash M[N/x] : \tau$.*

The proof is a simple induction on the structure of M.

1.1 Reduction Rules

We provide now a *reduction strategy* for evaluating pPCF terms: For any pPCF term M, there will be at most one redex in M that will be reduced by this strategy. A reduction step will nevertheless lead to several outcomes by the stochastic nature of the calculus: This reduction strategy will be seen as a Markov chain whose states are terms.

Given two terms M, M' and a real number $p \in [0,1]$, we define $M \xrightarrow{p} M'$, meaning that M reduces in one step to M' with probability p, by the following deduction system.

We define first a deterministic reduction relation $\to_{\mathrm{d}}$ as follows:

$$\frac{}{(\lambda x^{\sigma}\, M)\, N \to_{\mathrm{d}} M[N/x]} \qquad \frac{}{\mathsf{fix}(M) \to_{\mathrm{d}} (M)\, \mathsf{fix}(M)}$$

$$\frac{}{\mathsf{succ}(\underline{n}) \to_{\mathrm{d}} \underline{n+1}} \qquad \frac{}{\mathsf{if}(\underline{0}, P, z \cdot R) \to_{\mathrm{d}} P} \qquad \frac{}{\mathsf{if}(\underline{n+1}, P, z \cdot R) \to_{\mathrm{d}} R\left[\underline{n}/z\right]}.$$

Then we define the probabilistic reduction by the following rules:

$$\frac{M \to_{\mathrm{d}} M'}{M \xrightarrow{1} M'} \qquad \frac{}{\mathsf{coin}(p) \xrightarrow{p} \underline{0}} \qquad \frac{}{\mathsf{coin}(p) \xrightarrow{1-p} \underline{1}}$$

$$\frac{M \xrightarrow{p} M'}{(M)\, N \xrightarrow{p} (M')\, N} \qquad \frac{M \xrightarrow{p} M'}{\mathsf{succ}(M) \xrightarrow{p} \mathsf{succ}(M')}$$

$$\frac{M \xrightarrow{p} M'}{\mathsf{if}(M, P, z \cdot R) \xrightarrow{p} \mathsf{if}(M', P, z \cdot R)}.$$

This reduction can be called *weak-head reduction* (or simply weak reduction), since it always reduces the leftmost outermost redex and never reduces redexes under abstractions. We say that M is *weak-normal* if there is no reduction $M \xrightarrow{p} M'$.

1.2 Observational Equivalence

Using this simple probabilistic reduction relation, we want now to define a notion of observational equivalence. For this purpose, we need first to describe as simply as possible the "transitive closure" of the probabilistic reduction relation defined in Section 1.1. We represent this relation as a matrix Red indexed by terms, the number $\mathrm{Red}_{M,M'}$ being the probability of M to reduce to M' in one step. We add also that $\mathrm{Red}_{M,M} = 1$ if M is weak-normal for the weak-reduction (that is, no reduction is possible from M); in all other cases, we have $\mathrm{Red}_{M,M'} = 0$. In other words, we consider the reduction as a discrete time Markov chain whose states are terms, stationary states are weak-normal terms and whose associated stochastic matrix is Red. Saying that Red is stochastic means that the coefficients of Red belong to $[0,1]$ and that, for any given term M, one has

$\sum_{M'} \mathrm{Red}_{M,M'} = 1$ (actually there are at most two terms M' such that $\mathrm{Red}_{M,M'} \neq 0$). Then if M' is normal, $\mathrm{Red}^k_{M,M'}$ (where $\mathrm{Red}^k = \overbrace{\mathrm{Red} \cdots \mathrm{Red}}^{k}$ is the kth power of Red for the matricial product) represents the probability of M to reduce to M' in *at most k steps*, and we obtain the probability of M to reduce to M' by taking the lub of these numbers; to obtain this effect, our assumption that M' is a stationary state is crucial. We explain this in more detail now, considering first the case of a general stochastic matrix S indexed by a countable set I of states.

Probability of Convergence to a Stationary State. Let I be a countable set and let $S \in [0,1]^{I \times I}$ to be understood as a matrix with I-indexed rows and columns. One says that S is stochastic if $\forall i \in I \ \ \sum_{j \in I} S_{i,j} = 1$. Given two such matrices T and S, their product TS is given by $\forall (i,j) \in I^2 \ (TS)_{i,j} = \sum_{k \in I} T_{i,k} S_{k,j}$ and is also a stochastic matrix.

Let I_1^S be the set of stationary states, $I_1^S = \{i \in I \mid S_{i,i} = 1\}$ (so that if $i \in I_1^S$ and $S_{i,j} \neq 0$ then $i = j$). Let $(i,j) \in I \times I_1^S$. Then the n-indexed sequence $(S^n)_{i,j} \in [0,1]$ is monotone. Indeed, for all n we have

$$\begin{aligned}(S^{n+1})_{i,j} &= \sum_{k \in I} (S^n)_{i,k} S_{k,j} \\ &\geq (S^n)_{i,j} S_{j,j} = (S^n)_{i,j}.\end{aligned}$$

So we can define a matrix $S^\infty \in [0,1]^{I \times I}$ as follows:

$$(S^\infty)_{i,j} = \begin{cases} \sup_{n \in \mathbb{N}} (S^n)_{i,j} & \text{if } (i,j) \in I \times I_1^S \\ 0 & \text{otherwise.} \end{cases}$$

The matrix S^∞ is a sub-stochastic matrix, because, given $i \in I$,

$$\begin{aligned}\sum_{j \in I} (S^\infty)_{i,j} &= \sum_{j \in I_1^S} \sup_{n \in \mathbb{N}} (S^n)_{i,j} \\ &= \sup_{n \in \mathbb{N}} \sum_{j \in I_1^S} (S^n)_{i,j} \quad \text{by the monotone convergence theorem} \\ &\leq \sup_{n \in \mathbb{N}} \sum_{j \in I} (S^n)_{i,j} = 1.\end{aligned}$$

Let $i, j \in I$. A *path* from i to j is a sequence $w = (i_1, \ldots, i_k)$ of elements of I (with $k \geq 1$) such that $i_1 = i$, $i_k = j$, and $i_k \neq i_l$ for all $l \in \{1, \ldots, k-1\}$. The *weight* of w is $\mathrm{p}(w) = \prod_{l=1}^{k-1} S_{i_l, i_{l+1}}$. The *length* of w is $k - 1$. We use $\mathrm{R}(i,j)$ to denote the set of all paths from i to j.

Lemma 1.3. *Let $(i,j) \in I \times I_1^S$. One has*

$$S^\infty_{i,j} = \sum_{w \in \mathrm{R}(i,j)} \mathrm{p}(w).$$

The proof is easy. To obtain this property, it is important in the definition of paths that the last element does not occur earlier.

The Stochastic Matrix of Terms. Let Γ be a typing context and σ be a type. Let Λ^σ_Γ be the set of all terms M such that $\Gamma \vdash M : \sigma$. In the case where Γ is empty, and so the elements of Λ^σ_Γ are closed, we use Λ^σ_0 to denote that set.

Let $\mathsf{Red}(\Gamma,\sigma) \in [0,1]^{\Lambda_\Gamma^\sigma \times \Lambda_\Gamma^\sigma}$ be the matrix (indexed by terms typable of type σ in context Γ) given by

$$\mathsf{Red}(\Gamma,\sigma)_{M,M'} = \begin{cases} p & \text{if } M \xrightarrow{p} M' \\ 1 & \text{if } M \text{ is weak-normal and } M' = M \\ 0 & \text{otherwise.} \end{cases}$$

This is a stochastic matrix. We also use the notation $\mathsf{Red}(\sigma)$ for the matrix $\mathsf{Red}(\Gamma,\sigma)$ when the typing context is empty.

When M' is weak-normal, the number $p = \mathsf{Red}(\Gamma,\sigma)^\infty_{M,M'}$ is the probability that M reduces to M' after a finite number of steps by Lemma 1.3. We write $M \downarrow^p M'$ if M' is weak-normal and $p = \mathsf{Red}(\Gamma,\sigma)^\infty_{M,M'}$.

Observation Contexts. We define a syntax for observation contexts with several typed holes, all holes having the same type. They are defined exactly as terms, adding a new "constant symbol" $[\ ]^{\Gamma\vdash\sigma}$, where Γ is a typing context and σ is a type, which represents a hole that can be filled with a term M such that $\Gamma \vdash M : \sigma$. Such an observation context will be denoted with letters $C, D \ldots$, adding $\Gamma \vdash \sigma$ as superscript for making explicit the typing judgment of the terms to be inserted in the holes of the context. So if C is an observation context with holes $[\ ]^{\Gamma\vdash\sigma}$, then this context will often be written $C^{\Gamma\vdash\sigma}$ and the term where all holes have been filled with the term M will be denoted $C[M]$: This is just an ordinary pPCF term. Notice that, in $C[M]$, some (possibly all) free variables of M can be bound by λ's of C. For instance, if $C = \lambda x^\sigma\ [\ ]^{x:\sigma\vdash\sigma}$, then $C[x] = \lambda x^\sigma\ x$.

More formally, we give now the typing rules for observation contexts:

$$\frac{}{\Gamma,\Delta \vdash [\ ]^{\Delta\vdash\tau} : \tau}$$

$$\frac{}{\Gamma \vdash \underline{n}^{\Delta\vdash\tau} : \iota} \qquad \frac{}{\Gamma, x:\sigma \vdash x^{\Delta\vdash\tau} : \sigma} \qquad \frac{\Gamma \vdash C^{\Delta\vdash\tau} : \iota}{\Gamma \vdash \mathsf{succ}(C)^{\Delta\vdash\tau} : \iota}$$

$$\frac{\Gamma \vdash C^{\Delta\vdash\tau} : \iota \qquad \Gamma \vdash D^{\Delta\vdash\tau} : \sigma \qquad \Gamma, z:\iota \vdash E^{\Delta\vdash\tau} : \sigma}{\Gamma \vdash \mathsf{if}(C,D,z\cdot E)^{\Delta\vdash\tau} : \sigma}$$

$$\frac{\Gamma, x:\sigma \vdash C^{\Delta\vdash\varphi} : \tau}{\Gamma \vdash (\lambda x^\sigma\ C)^{\Delta\vdash\varphi} : \sigma \Rightarrow \tau}$$

$$\frac{\Gamma \vdash C^{\Delta\vdash\varphi} : \sigma \Rightarrow \tau \qquad \Gamma \vdash D^{\Delta\vdash\varphi} : \sigma}{\Gamma \vdash (C)\,D^{\Delta\vdash\varphi} : \tau} \qquad \frac{\Gamma \vdash C^{\Delta\vdash\tau} : \sigma \Rightarrow \sigma}{\Gamma \vdash \mathsf{fix}(C)^{\Delta\vdash\tau} : \sigma}$$

$$\frac{p \in [0,1] \cap \mathbb{Q}}{\Gamma \vdash \mathsf{coin}(p)^{\Delta\vdash\tau} : \iota}.$$

Lemma 1.4. *If* $\Gamma \vdash C^{\Delta\vdash\tau} : \sigma$ *and* $\Delta \vdash M : \tau$, *then* $\Gamma \vdash C[M] : \sigma$.

The proof is a trivial induction on C.

Observational Equivalence. Let $M, M' \in \Lambda_\Gamma^\sigma$ (that is, both terms have type σ in the typing context Γ). We say that M and M' are observationally equivalent (notation $M \sim M'$) if, for all observation contexts $C^{\Gamma\vdash\sigma}$ such that $\vdash C^{\Gamma\vdash\sigma} : \iota$, one has

$$\mathsf{Red}(\iota)^\infty_{C[M],\underline{0}} = \mathsf{Red}(\iota)^\infty_{C[M'],\underline{0}}.$$

Remark: The choice of testing the probability of reducing to $\underline{0}$ in the definition above of observational equivalence is arbitrary. For instance, we would obtain the same notion of equivalence by stipulating that two terms M and M' typable of type σ in typing context Γ are observationally

equivalent if, for all observation context $C^{\Gamma\vdash\sigma}$, one has

$$\sum_{n\in\mathbb{N}} \mathsf{Red}(\iota)^{\infty}_{C[M],\underline{n}} = \sum_{n\in\mathbb{N}} \mathsf{Red}(\iota)^{\infty}_{C[M'],\underline{n}},$$

that is, the two closed terms $C[M]$ and $C[M']$ have the same probability of convergence to some value. This is due to the universal quantification on C.

1.3 Basic Examples

We give a series of example terms written in pPCF that implement natural simple algorithms to illustrate the expressive power of the language. We explain intuitively the behavior of these programs, and one can also have a look at Section 2.12 where the denotational interpretations of these terms in PCS are given, presented as functions.

Given a type σ, we set $\Omega^{\sigma} = \mathsf{fix}(\lambda x^{\sigma}\, x)$ so that $\vdash \Omega^{\sigma} : \sigma$, which is the ever-looping term of type σ.

Arithmetics. The predecessor function, which is usually a basic construction of PCF, is now definable as

$$\mathsf{pred} = \lambda x^{\iota}\, \mathsf{if}(x, \underline{0}, z \cdot z);$$

it is clear, then, that $(\mathsf{pred})\,\underline{0} \to_{\mathsf{d}}{}^{*}\, \underline{0}$ and that $(\mathsf{pred})\,\underline{n+1} \to_{\mathsf{d}}{}^{*}\, \underline{n}$.

The addition function can be defined as

$$\mathsf{add} = \lambda x^{\iota}\, \mathsf{fix}(\lambda a^{\iota\Rightarrow\iota}\, \lambda y^{\iota}\, \mathsf{if}(y, x, z \cdot \mathsf{succ}((a)\, z))),$$

and it is easily checked that $\vdash \mathsf{add} : \iota \Rightarrow \iota \Rightarrow \iota$. Given $k \in \mathbb{N}$, we set

$$\mathsf{shift}_k = (\mathsf{add})\,\underline{k}$$

so that $\vdash \mathsf{shift}_k : \iota \Rightarrow \iota$.

The exponential function can be defined as

$$\exp_2 = \mathsf{fix}(\lambda e^{\iota\Rightarrow\iota}\, \lambda x^{\iota}\, \mathsf{if}(x, 1, z \cdot (\mathsf{add})\,(e)\, z\ (e)\, z))$$

and satisfies $(\exp_2)\,\underline{n} \to_{\mathsf{d}}{}^{*}\, \underline{2^n}$.

Along the same line, one defines a comparison function cmp

$$\mathsf{cmp} = \mathsf{fix}(\lambda c^{\iota\Rightarrow\iota\Rightarrow\iota}\, \lambda x^{\iota}\, \lambda y^{\iota}\, \mathsf{if}(x, \underline{0}, z \cdot \mathsf{if}(y, \underline{1}, z' \cdot (c)\, z\, z')))$$

such that $(\mathsf{cmp})\,\underline{n}\,\underline{m}$ reduces to $\underline{0}$ if $n \leq m$ and to $\underline{1}$ otherwise.

More Tests. By induction on k, we define a family of terms prob_k such that $\vdash \mathsf{prob}_k : \iota \Rightarrow \iota$:

$$\mathsf{prob}_0 = \lambda x^{\iota}\, \mathsf{if}(x, \underline{0}, z \cdot \Omega^{\iota})$$
$$\mathsf{prob}_{k+1} = \lambda x^{\iota}\, \mathsf{if}(x, \Omega^{\iota}, z \cdot (\mathsf{prob}_k)\, z).$$

For M such that $\vdash M : \iota$, the term $(\mathsf{prob}_k)\, M$ reduces to $\underline{0}$ with a probability that is equal to the probability of M to reduce to $\underline{k}$ and diverges otherwise.

Similarly, we also define prod_k such that $\vdash \mathsf{prod}_k : \iota^k \Rightarrow \iota$:

$$\mathsf{prod}_0 = \underline{0}$$
$$\mathsf{prod}_{k+1} = \lambda x^{\iota}\, \mathsf{if}(x, \mathsf{prod}_k, z \cdot \Omega^{\iota^k \Rightarrow \iota}).$$

Given closed terms $M_1, \ldots, M_k$ such that $\vdash M_i : \iota$, the term $(\mathsf{prod}_k)\, M_1 \cdots M_k$ reduces to $\underline{0}$ with probability $\prod_{i=1}^{k} p_i$, where p_i is the probability of M_i to reduce to $\underline{0}$ and diverges otherwise.

Given a type σ and $k \in \mathbb{N}$, we also define a term choose_k such that $\vdash \mathsf{choose}_k : \iota \Rightarrow \sigma^k \Rightarrow \sigma$

$$\begin{aligned}\mathsf{choose}_0 &= \lambda\xi^{\iota}\, \Omega^{\sigma}\\ \mathsf{choose}_{k+1} &= \lambda\xi^{\iota}\, \lambda x_1^{\sigma}\, \cdots \lambda x_{k+1}^{\sigma}\, \mathsf{if}(\xi, x_1, \zeta \cdot (\mathsf{choose}_k)\, \zeta\, x_2\, \cdots\, x_{k+1}).\end{aligned}$$

Given a closed term M such that $\vdash M : \iota$ and terms $N_1, \ldots, N_k$ such that $\Gamma \vdash N_i : \sigma$ for each i, the term $(\mathsf{choose}_i)\, M\, N_1 \cdots N_k$ reduces to N_{i+1} with the probability that M reduces to $\underline{i}$ and diverges if M does not reduce to a constant $\underline{i}$ with $i \in \{0, \ldots, k-1\}$.

The let *Construct for Ground Type.* This version of PCF, which is globally CBN, offers, however, the possibility of handling integers in a CBV way. For instance, we can define the typical CBV "let" construction as follows:

$$\mathsf{let}\ x\ \mathsf{be}\ M\ \mathsf{in}\ N = \mathsf{if}(M, N\left[\underline{0}/x\right], z \cdot N\,[\mathsf{succ}(z)/x]),$$

and this construction is restricted to the type of natural numbers; it can be typed as

$$\frac{\Gamma \vdash M : \iota \qquad \Gamma, x : \iota \vdash N : \sigma}{\Gamma \vdash \mathsf{let}\ x\ \mathsf{be}\ M\ \mathsf{in}\ N : \sigma}.$$

The effect of this construction is that, before replacing x with M in N, M must be evaluated to a value $\underline{n}$. This is particularly important in the case where M is a probabilistic integer, since this construction allows to "roll the dice" only once and then provide N with as many copies of the result as needed.[6]

In accordance with this intuition, one can also check that the following reduction inference is derivable:

$$\frac{M \xrightarrow{p} M'}{\mathsf{let}\ x\ \mathsf{be}\ M\ \mathsf{in}\ N \xrightarrow{p} \mathsf{let}\ x\ \mathsf{be}\ M'\ \mathsf{in}\ N}.$$

whereas *it is not true* that

$$\frac{M \xrightarrow{p} M'}{N\,[M/x] \xrightarrow{p} N\,[M'/x]}.$$

(consider cases where x does not occur in N, or occurs twice ...). We have, of course,

$$\frac{}{\mathsf{let}\ x\ \mathsf{be}\ \underline{n}\ \mathsf{in}\ N \to_{\mathrm{d}} N\,[\theta(n)/x]}.$$

where $\theta(0) = \underline{0}$ and $\theta(n+1) = \mathsf{succ}(\underline{n})$ (which reduces to $\underline{n+1}$ in one deterministic step) by definition of this construction.

Random Generators. Using these constructions, we can define a closed term unif_2 of type $\iota \Rightarrow \iota$ that, given an integer n, yields a uniform probability distribution on the integers $0, \ldots, 2^n - 1$:

$$\mathsf{unif}_2 = \mathsf{fix}(\lambda u^{\iota \Rightarrow \iota}\, \lambda x^{\iota}\, \mathsf{if}(x, \underline{0}, z \cdot \mathsf{if}(\mathsf{coin}(1/2), (u)\, z, z' \cdot (\mathsf{add})\ (\mathsf{exp}_2)\ z\ (u)\ z))).$$

Observe that, when evaluating $(\mathsf{unif}_2)\, M$ (where $\vdash M : \iota$), the term M is evaluated only once thanks to the CBV feature of the conditional construct. Indeed, we do not want the upper bound of the interval on which we produce a probability distribution to change during the computation (the result would be unpredictable).

[6]Notice, however, that, since our language is CBN, this is not mandatory, and ordinary substitution (or application) allows to feed a function with an undetermined probabilistic integer, that is, with a probability distribution.

Using this construction, one can define a function unif that, given an integer n, yields a uniform probability distribution on the integers $0, \ldots, n$:

$$\mathsf{unif} = \lambda x^{\iota}\, \mathsf{let}\ y\ \mathsf{be}\ x\ \mathsf{in}\ \ \mathsf{fix}(\lambda u^{\iota}\, \mathsf{let}\ z\ \mathsf{be}\ \ (\mathsf{unif}_2)\, y\ \mathsf{in}\ \mathsf{if}((\mathsf{cmp})\, z\, y, z, w \cdot (u)\, y)).$$

One checks easily that $\vdash \mathsf{unif} : \iota \Rightarrow \iota$. Given $n \in \mathbb{N}$, this function applies iteratively unif_2 until the result is $\leq n$. It is not hard to check that the resulting distribution is uniform (with probability $\frac{1}{n+1}$ for each possible result).

Last, let $n \in \mathbb{N}$ and let $\vec{p} = (p_0, \ldots, p_n)$ be such that $p_i \in [0,1] \cap \mathbb{Q}$ and $p_0 + \cdots + p_n \leq 1$. Then one defines a closed term $\mathsf{ran}(\vec{p})$ that reduces to $\underline{i}$ with probability p_i for each $i \in \{0, \ldots, n\}$. The definition is by induction on n,

$$\mathsf{ran}(p_0, \ldots, p_n) = \begin{cases} \underline{0} & \text{if } p_0 = 1 \text{ whatever be the value of } n \\ \mathsf{if}(\mathsf{coin}(p_0), \underline{0}, z \cdot \Omega^{\iota}) & \text{if } n = 0 \\ \mathsf{if}(\mathsf{coin}(p_0), \underline{0}, z \cdot \mathsf{succ}(\mathsf{ran}(\frac{p_1}{1-p_0}, \ldots, \frac{p_n}{1-p_0}))) & \text{otherwise.} \end{cases}$$

Observe indeed that in the first case we must have $p_1 = \cdots = p_n = 0$.

A Simple Las Vegas Program. Given a function $f : \mathbb{N} \to \mathbb{N}$ and $n \in \mathbb{N}$, find a $k \in \{0, \ldots, n\}$ such that $f(k) = 0$. This can be done by iterating random choices of k until we get a value such that $f(k) = 0$: This is probably the simplest example of a Las Vegas algorithm. The following function does the job:

$$M = \lambda f^{\iota \Rightarrow \iota}\, \lambda x^{\iota}\, \mathsf{fix}(\lambda r^{\iota}\, \mathsf{let}\ y\ \mathsf{be}\ \ (\mathsf{unif})\, x\ \mathsf{in}\ \mathsf{if}((f)\, y, y, z \cdot r))$$

with $\vdash M : (\iota \Rightarrow \iota) \Rightarrow \iota \Rightarrow \iota$. Our CBV integers are crucial here, since without our version of the conditional, it would not be possible to get a random integer and use this value y both as an argument for f and as a result if the expected condition holds.

As mentioned in the Introduction, we strongly believe that this algorithm cannot be written with the usual version of the conditional (as in Ehrhard et al. (2014)), but we did not really try to prove this. A more precise statement of this conjecture could be the following, using the model **Pcoh** of probabilistic coherence spaces that we present in the sequel.

Conjecture 1.5. *There is no term M' in the syntax of Ehrhard et al. (2014) such that $\vdash M' : (\iota \Rightarrow \iota) \Rightarrow \iota \Rightarrow \iota$ and that has the same semantics as M in* **Pcoh**.

Since the model is fully abstract for both syntaxes, this would mean that the syntax of Ehrhard et al. (2014) can define no term observationally equivalent to M.

2 PROBABILISTIC COHERENCE SPACES

We present shortly a model of probabilistic PCF that is actually a model of classical Linear Logic. For a longer and more detailed account, we refer to Danos and Ehrhard (2011).

2.1 An Informal Introduction to Probabilistic Coherence Spaces

The fact that Probabilistic Coherence Spaces are a model of Linear Logic, and not only of PCF or other λ-calculi, is a crucial feature in the sense that it would have been almost impossible to discover this mathematical interpretation of PCF without considering it at the first place as a model of Linear Logic. Indeed, the basic ingredient of the model is a fundamentally linear notion of duality between valuations that is very close in spirit to the standard duality of Linear Algebra between vectors and linear forms. See Section 2.2.

For that reason, the natural notion of morphism in this category is linear: morphisms can be seen as (generally infinite dimensional matrices) acting on valuations. Also, the most elementary operations on probabilistic coherence spaces are reminiscent of Linear Algebra: tensor product, linear

function space, linear dual space. Accordingly, the most natural setting for describing structure of this category is the theory of *monoidal categories* and more specifically of **-autonomous categories* (Barr 1979; Melliès 2009). These structures, called *multiplicative* in Linear Logic, are presented in Section 2.6.

Products and coproducts, called *additive* structures in Linear Logic, are also most naturally described in this linear setting where they satisfy the standard universal properties, whereas it is well known that coproducts cannot exist in categorical models of PCF due to the presence of fix-point operators. These structures are presented in Section 2.7.

However *-autonomous categories with products are not sufficient for interpreting PCF as they lack the basic structures allowing to duplicate and discard data: These operations are interpreted by means of the structural morphisms associated with a comonad on the linear category on which the main assumption is that it must transform Cartesian product into tensor products.[7] We describe this comonad and the associated structures in Section 2.8.

The Cartesian closed category where we interpret probabilistic PCF is the *Kleisli category* induced by this comonad. The morphisms of this category have *a priori* no reason to admit a description as functions characterized by simple preservation properties like Scott-continuity,[8] as it is usual in models based on domain theory. Here we can prove, however, that these morphisms are functions that are Scott continuous (with respect to a most natural domain structure that can be associated with any probabilistic coherence space). But the converse is very far from being true: Our Kleisli morphisms, being analytic functions, are much more regular than general Scott continuous functions, and we do not know any simple way of characterizing them by a preservation property.[9] It is certainly here that our model departs most strikingly from the usual domain-theoretic approach.

Nevertheless, Cartesian closeness of this category and the fact that morphisms are Scott continuous functions allow us to associate a least fix-point operator with any object, providing a simple interpretation of PCF general recursion. This Kleisli category construction is described in Section 2.9.

Last, in Section 2.10, we give a semantic counterpart to our observation that the ground type of natural numbers can be dealt with in a call-by-value way. This boils down to the fact that any morphism in the Kleisli category from our object N interpreting the type of natural numbers to some other object X can be turned into a *linear* morphism from N to X. This in turn boils down to the existence of a (well-behaved) linear morphism from N to $!\mathsf{N}$, a property that can be summarized by saying that N has a canonical structure of !-coalgebra. This is due to the definition of N as a coproduct of ω copies of 1 (the unit of the tensor product that can be thought of as a unit type and is canonically a !-coalgebra) and to the fact that a coproduct of !-coalgebras is canonically a !-coalgebra.

2.2 Basic Duality

Let I be a countable set. Given $u, u' \in (\mathbb{R}_{\geq 0})^I$ (such functions will sometimes be called *valuations*), we set

$$\langle u, u' \rangle = \sum_{i \in I} u_i u'_i \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

[7] Technically, this is a symmetric monoidal structure on the comonad consisting of the *Seely isomorphisms*.
[8] Or stability in the case of dI-domains or coherence spaces (Berry 1978; Girard 1986).
[9] Added in print. Using ideas of Bernstein of the 1930's, Crubille proved that the notion of stability as dened in (Ehrhard et al. 2018) characterizes indeed the Kleisli morphisms between probabilistic coherence spaces (Crubillé 2018).

Let $\mathcal{X} \subseteq (\mathbb{R}_{\geq 0})^I$, we define the *polar* $\mathcal{X}^\perp$ of $\mathcal{X}$

$$\mathcal{X}^\perp = \{u' \in (\mathbb{R}_{\geq 0})^I \mid \forall u \in \mathcal{X}\ \langle u, u' \rangle \leq 1\}.$$

We have as usual

- $\mathcal{X} \subseteq \mathcal{Y} \Rightarrow \mathcal{Y}^\perp \subseteq \mathcal{X}^\perp$
- $\mathcal{X} \subseteq \mathcal{X}^{\perp\perp}$

and it follows that $\mathcal{X}^{\perp\perp\perp} = \mathcal{X}^\perp$.

2.3 Definition and Basic Properties of Probabilistic Coherence Spaces

We present now the model of probabilistic coherence spaces.

A *probabilistic coherence space* (PCS) is a pair $X = (|X|, \mathrm{P}X)$, where $|X|$ is a countable set and $\mathrm{P}X \subseteq (\mathbb{R}_{\geq 0})^{|X|}$ satisfies

- $\mathrm{P}X^{\perp\perp} = \mathrm{P}X$ (equivalently, $\mathrm{P}X^{\perp\perp} \subseteq \mathrm{P}X$),
- for each $a \in |X|$ there exists $u \in \mathrm{P}X$ such that $u_a > 0$,
- for each $a \in |X|$ there exists $A > 0$ such that $\forall u \in \mathrm{P}X\ u_a \leq A$.

If only the first of these conditions holds, then we say that X is a *pre-probabilistic coherence space* (pre-PCS).

The purpose of the second and third conditions is to prevent infinite coefficients to appear in the semantics. This property in turn will be essential for guaranteeing the morphisms interpreting proofs to be analytic functions, which will be the key property to prove full abstraction. So these conditions, though cosmetics at first sight, are important for our ultimate goal.

Remark: The model of pre-PCS can be obtained by applying the general *double glueing* (Hyland and Schalk 2003) construction to the *weighted relational model* of Linear Logic (Laird et al. 2013) (with non-negative real numbers as coefficients). It is not completely clear, however, how to integrate naturally the second and third conditions of the definition of PCSs in the double-glueing setting.

Lemma 2.1. *Let X be a pre-PCS. The following conditions are equivalent:*

- *X is a PCS,*
- *$\forall a \in |X|\ \exists u \in \mathrm{P}X\ \exists u' \in \mathrm{P}X^\perp\ u_a > 0$ and $u'_a > 0$,*
- *$\forall a \in |X|\ \exists A > 0\ \forall u \in \mathrm{P}X\ \forall u' \in \mathrm{P}X^\perp\ u_a \leq A$ and $u'_a \leq 1/A$.*

The proof is straightforward.

We equip $\mathrm{P}X$ with the most obvious partial order relation: $u \leq v$ if $\forall a \in |X|\ u_a \leq v_a$ (using the usual order relation on $\mathbb{R}$).

Given $u \in (\mathbb{R}_{\geq 0})^{|X|}$ and $I \subseteq |X|$, we use $u|_I$ for the element v of $(\mathbb{R}_{\geq 0})^{|X|}$ such that $v_a = u_a$ if $a \in I$ and $v_a = 0$ otherwise. Of course, $u \in \mathrm{P}X \Rightarrow u|_I \in \mathrm{P}X$.

Theorem 2.2. *If X is a PCS, then $\mathrm{P}X$ is an ω-continuous domain. Given $u, v \in \mathrm{P}X$ and $\alpha, \beta \in \mathbb{R}^+$ such that $\alpha + \beta \leq 1$, one has $\alpha u + \beta v \in \mathrm{P}X$.*

Proof. Let us first prove that $\mathrm{P}X$ is complete. Let D be a directed subset of $\mathrm{P}X$. For any $a \in |X|$, the set $\{u_a \mid u \in D\}$ is bounded; let $v_a \in \mathbb{R}_{\geq 0}$ be the lub of that set. In that way, we define $v = (v_a)_{a \in |X|} \in (\mathbb{R}_{\geq 0})^{|X|}$.

We prove that $v \in \mathrm{P}X$. Let $u' \in \mathrm{P}X^\perp$, we must prove that $\langle v, u' \rangle \leq 1$. We know that $\{\langle u, u' \rangle \mid u \in D\} \subseteq [0, 1]$, and therefore this set has a lub $A \in [0, 1]$. Let $\varepsilon > 0$, we can find $u \in D$ such that

$\langle u, u'\rangle \geq A - \varepsilon$, and, since this holds for all ε, we have $\langle v, u'\rangle \geq A$. Let again $\varepsilon > 0$. We can find a finite set $I \subseteq |X|$ such that $\langle v|_I, u'\rangle \geq \langle v, u'\rangle - \frac{\varepsilon}{2}$. Since I is finite, we have $\langle v|_I, u'\rangle = \sup_{u\in D}\langle u|_I, u'\rangle$ (it is here that we use our hypothesis that D is directed), and hence we can find $u \in D$ such that $\langle u|_I, u'\rangle \geq \langle v|_I, u'\rangle - \frac{\varepsilon}{2}$ and hence $\langle u, u'\rangle \geq \langle u|_I, u'\rangle \geq \langle v|_I, u'\rangle - \frac{\varepsilon}{2} \geq \langle v, u'\rangle - \varepsilon$. It follows that $A = \sup_{u\in D}\langle u, u'\rangle \geq \langle v, u'\rangle$. So $\langle v, u'\rangle \in [0, 1]$, and hence $v \in \mathrm{P}X$.

It is clear that v is the lub of D in $\mathrm{P}X$ since the order relation is defined pointwise. Therefore, $\mathrm{P}X$ is a cpo, which has 0 as least element. Let R be the set of all the elements of $\mathrm{P}X$ that have a finite domain and take only rational values. Then it is clear that for each $u \in \mathrm{P}X$, the elements w of R that are way below u (in the present setting, this simply means that $w_a > 0 \Rightarrow w_a < u_a$) form a directed subset of $\mathrm{P}X$ whose lub is u. Therefore $\mathrm{P}X$ is an ω-continuous domain.

The last statement results from the linearity of the operation $u \mapsto \langle u, u'\rangle$. □

As a consequence, given a family $(u(i))_{i\in\mathbb{N}}$ of elements of $\mathrm{P}X$ and a family $(\alpha_i)_{i\in\mathbb{N}}$ of elements of $\mathbb{R}_{\geq 0}$ such that $\sum_{i\in\mathbb{N}} \alpha_i \leq 1$, one has $\sum_{i\in\mathbb{N}} \alpha_i u(i) \in \mathrm{P}X$.

2.4 Morphisms of PCSs

Let X and Y be PCSs. Let $t \in (\mathbb{R}_{\geq 0})^{|X|\times|Y|}$ (to be understood as a matrix). Given $u \in \mathrm{P}X$, we define $t\,u \in \overline{\mathbb{R}_{\geq 0}}^{|Y|}$ by $(t\,u)_b = \sum_{a\in|X|} t_{a,b}u_a$ (application of the matrix t to the vector u).[10] We say that t is a *(linear) morphism* from X to Y if $\forall u \in \mathrm{P}X\ t\,u \in \mathrm{P}Y$, that is,

$$\forall u \in \mathrm{P}X\, \forall v' \in \mathrm{P}Y^{\perp} \qquad \sum_{(a,b)\in|X|\times|Y|} t_{a,b}u_a v'_b \leq 1. \tag{1}$$

The diagonal matrix $\mathrm{Id}_X \in (\mathbb{R}_{\geq 0})^{|X|\times|X|}$ given by $\mathrm{Id}_{a,b} = 1$ if $a = b$ and $\mathrm{Id}_{a,b} = 0$ otherwise is a morphism. In that way, we have defined a category **Pcoh** whose objects are the PCSs and whose morphisms have just been defined. Composition of morphisms is defined as matrix multiplication: let $s \in \mathbf{Pcoh}(X, Y)$ and $t \in \mathbf{Pcoh}(Y, Z)$, we define $t\,s \in (\mathbb{R}_{\geq 0})^{|X|\times|Z|}$ by

$$(t\,s)_{a,c} = \sum_{b\in|Y|} s_{a,b}t_{b,c}$$

and a simple computation shows that $t\,s \in \mathbf{Pcoh}(X, Z)$. More precisely, we use the fact that, given $u \in \mathrm{P}X$, one has $(t\,s)\,u = t\,(s\,u)$. Associativity of composition holds, because matrix multiplication is associative. Id_X is the identity morphism at X.

Example 2.3. Consider the PCS N such that $|\mathrm{N}| = \mathbb{N}$ and $u \in \mathrm{P}\mathrm{N}$ iff $u \in (\mathbb{R}_{\geq 0})^{\mathbb{N}}$ and $\sum_{i=0}^{\infty} u_i \leq 1$ (more information on this object will be given in Section 2.10). Intuitively, an element of PN represents the law of a partial random variable with integer outcomes. We have seen in Section 1 that a term M such that $\vdash M : \iota$ can be seen as a (partial and continuous) integer valued random variable on choice sequences. Its interpretation in our model will be the law of this random variable for the usual probability measure on the Cantor space.[11]

The fact that N is indeed a PCS boils down to the observation that $\mathrm{P}\mathrm{N} = \{\vec{1}\}^{\perp}$, where $\vec{1} \in (\mathbb{R}_{\geq 0})^{\mathbb{N}}$ is defined by $\forall i \in \mathbb{N}\ \vec{1}_i = 1$.

A morphism $s \in \mathbf{Pcoh}(\mathrm{N}, \mathrm{N})$ is a matrix indexed by natural numbers such that $s\,u$ is a sub-probability distribution for all sub-probability distribution u. Equivalently, s is characterized by $\forall i \in \mathbb{N}\ \sum_{j=0}^{\infty} s_{i,j} \leq 1$, that is, s is a "sub-stochastic" matrix. Such a matrix should be understood as a "linear sub-probability distribution transformer" from PN to PN.

[10] This is an unordered sum, which is infinite in general. It makes sense, because all its terms are ≥ 0.

[11] This is true actually only if we have $p = 1/2$ for all terms $\mathrm{coin}(p)$ occurring in M. Otherwise, it is better to simply say that the distribution interpreting M maps $n \in \mathbb{N}$ to the probability that M reduces to $\underline{n}$ in the reduction strategy of Section 1.1, see Theorem 3.5.

Morphisms in **Pcoh** should intuitively be considered as generalized "linear transformers" acting on objects that are generally not sub-probability distributions anymore.

2.5 The Norm

Given $u \in \mathrm{P}X$, we define $\|u\|_X = \sup\{\langle u, u'\rangle \mid u' \in \mathrm{P}X^\perp\}$. By definition, we have $\|u\|_X \in [0,1]$.

2.6 Multiplicative Constructs

We start the description of the category **Pcoh** as a model of Linear Logic. For this purpose, we use Bierman's notion of Linear Category (Bierman 1995), as presented in Melliès (2009), which is our main reference for this topic.

One sets $X^\perp = (|X|, \mathrm{P}X^\perp)$. It results straightforwardly from the definition of PCSs that $X^\perp$ is a PCS. Given $t \in \mathbf{Pcoh}(X,Y)$, one has $t^\perp \in \mathbf{Pcoh}(Y^\perp, X^\perp)$ if $t^\perp$ is the transpose of t, that is, $(t^\perp)_{b,a} = t_{a,b}$.

We define now the tensor product $X \otimes Y$ of two probabilistic coherence spaces X and Y. For proving that $X \otimes Y$ is a PCS and not only a pre-PCS, we also need to define $X \multimap Y$ and prove that it is a PCS.

One defines $X \otimes Y$ by $|X \otimes Y| = |X| \times |Y|$ and

$$\mathrm{P}(X \otimes Y) = \{u \otimes v \mid u \in \mathrm{P}X \text{ and } v \in \mathrm{P}Y\}^{\perp\perp},$$

where $(u \otimes v)_{(a,b)} = u_a v_b$. Then $X \otimes Y$ is a pre-PCS.

We have

$$\mathrm{P}(X \otimes Y^\perp)^\perp = \{u \otimes v' \mid u \in \mathrm{P}X \text{ and } v' \in \mathrm{P}Y^\perp\}^\perp = \mathbf{Pcoh}(X,Y),$$

the latter equation resulting from Equation (1). We prove now that the pre-PCS $X \multimap Y = (X \otimes Y^\perp)^\perp$ is a PCS.

Let $(a,b) \in |X| \times |Y|$. Since X and $Y^\perp$ are PCSs, there is $A > 0$ such that $u_a v'_b < A$ for all $u \in \mathrm{P}X$ and $v' \in \mathrm{P}Y^\perp$. Let $t \in (\mathbb{R}_{\geq 0})^{|X \multimap Y|}$ be such that $t_{(a',b')} = 0$ for $(a',b') \neq (a,b)$ and $t_{(a,b)} = 1/A$, we have $t \in \mathrm{P}(X \multimap Y)$. This shows that $\exists t \in \mathrm{P}(X \multimap Y)$ such that $t_{(a,b)} > 0$. Similarly, we can find $u \in \mathrm{P}X$ and $v' \in \mathrm{P}Y^\perp$ such that $\varepsilon = u_a v'_b > 0$. It follows that $\forall t \in \mathrm{P}(X \multimap Y)$ one has $t_{(a,b)} \leq 1/\varepsilon$. We conclude that $X \multimap Y$ is a PCS.

Therefore, $X \otimes Y$ is also a PCS as contended.

Lemma 2.4. *Let X and Y be PCSs. One has $\mathrm{P}(X \multimap Y) = \mathbf{Pcoh}(X,Y)$. That is, given $t \in (\mathbb{R}_{\geq 0})^{|X| \times |Y|}$, one has $t \in \mathrm{P}(X \multimap Y)$ iff for all $u \in \mathrm{P}X$, one has $t\, u \in \mathrm{P}Y$.*

This results immediately from the definition above of $X \multimap Y$.

Lemma 2.5. *Let X_1, X_2, and Y be PCSs. Let $t \in (\mathbb{R}_{\geq 0})^{|X_1 \otimes X_2 \multimap Y|}$. One has $t \in \mathbf{Pcoh}(X_1 \otimes X_2, Y)$ iff for all $u_1 \in \mathrm{P}X_1$ and $u_2 \in \mathrm{P}X_2$ one has $t\,(u_1 \otimes u_2) \in \mathrm{P}Y$.*

Proof. The condition stated by the lemma is clearly necessary. Let us prove that it is sufficient: Under this condition, it suffices to prove that

$$t^\perp \in \mathbf{Pcoh}(Y^\perp, (X_1 \otimes X_2)^\perp).$$

Let $v' \in \mathrm{P}Y^\perp$; it suffices to prove that $t^\perp\, v' \in \mathrm{P}(X_1 \otimes X_2)^\perp$. So let $u_1 \in \mathrm{P}X_1$ and $u_2 \in \mathrm{P}X_2$; it suffices to prove that $\langle t^\perp\, v', u_1 \otimes u_2\rangle \leq 1$, that is, $\langle t\,(u_1 \otimes u_2), v'\rangle \leq 1$, which follows from our assumption. □

Let $s_i \in \mathbf{Pcoh}(X_i, Y_i)$ for $i = 1, 2$. Then one defines

$$s_1 \otimes s_2 \in (\mathbb{R}_{\geq 0})^{|X_1 \otimes X_2 \multimap Y_1 \otimes Y_2|}$$

by $(s_1 \otimes s_2)_{((a_1,a_2),(b_1,b_2))} = (s_1)_{(a_1,b_1)}(s_2)_{(a_2,b_2)}$, and one must check that $s_1 \otimes s_2 \in \mathbf{Pcoh}(X_1 \otimes X_2, Y_1 \otimes Y_2)$. This follows directly from Lemma 2.5. Let $1 = (\{*\}, [0,1])$. There are obvious choices of natural isomorphisms

$$\begin{aligned}\lambda_X &\in \mathbf{Pcoh}(1 \otimes X, X)\\ \rho_X &\in \mathbf{Pcoh}(X \otimes 1, X)\\ \alpha_{X_1,X_2,X_3} &\in \mathbf{Pcoh}((X_1 \otimes X_2) \otimes X_3, X_1 \otimes (X_2 \otimes X_3))\\ \gamma_{X_1,X_2} &\in \mathbf{Pcoh}(X_1 \otimes X_2, X_2 \otimes X_1),\end{aligned}$$

which satisfy the standard coherence properties. For example, $(\gamma_{X_1,X_2})_{(a_1,a_2),(a_2',a_1')} = \delta_{a_1,a_1'}\delta_{a_2,a_2'}$. This shows that the structure $(\mathbf{Pcoh}, 1, \lambda, \rho, \alpha, \gamma)$ is a symmetric monoidal category.

Internal Linear Hom. Given PCSs X and Y, let us define $\mathsf{ev} \in (\mathbb{R}_{\geq 0})^{|(X \multimap Y) \otimes X \multimap Y|}$ by

$$\mathsf{ev}_{(((a',b'),a),b)} = \begin{cases} 1 & \text{if } (a,b) = (a',b') \\ 0 & \text{otherwise.}\end{cases}$$

Then it is easy to see that $(X \multimap Y, \mathsf{ev})$ is an internal linear hom object in **Pcoh**, showing that this SMCC is closed. If $t \in \mathbf{Pcoh}(Z \otimes X, Y)$, then the corresponding linearly curryfied morphism $\mathrm{cur}(t) \in \mathbf{Pcoh}(Z, X \multimap Y)$ is given by $\mathrm{cur}(t)_{(c,(a,b))} = t_{((c,a),b)}$.

*-autonomy. Take $\bot = 1$, then one checks readily that the structure $(\mathbf{Pcoh}, 1, \lambda, \rho, \alpha, \gamma, \bot)$ is a *-autonomous category. The duality functor $X \mapsto (X \multimap \bot)$ can be identified with the strictly involutive contravariant functor $X \mapsto X^\bot$.

2.7 Additives

Let $(X_i)_{i\in I}$ be a countable family of PCSs. We define a PCS $\&_{i\in I} X_i$ by $|\&_{i\in I} X_i| = \bigcup_{i\in I}\{i\} \times |X_i|$ and $u \in \mathrm{P}(\&_{i\in I} X_i)$ if, for all $i \in I$, the family $u(i) \in (\mathbb{R}_{\geq 0})^{|X_i|}$ defined by $u(i)_a = u_{(i,a)}$ belongs to $\mathrm{P}X_i$.

Lemma 2.6. *Let $u' \in (\mathbb{R}_{\geq 0})^{|\&_{i\in I} X_i|}$. One has $u' \in \mathrm{P}(\&_{i\in I} X_i)^\bot$ iff*

- $\forall i \in I\ u'(i) \in \mathrm{P}X_i^\bot$
- *and* $\sum_{i\in I} \|u'(i)\|_{X_i^\bot} \leq 1$.

The proof is quite easy. It follows that $\&_{i\in I} X_i$ is a PCS. Moreover, we can define $\pi_i \in \mathbf{Pcoh}(\&_{j\in I} X_j, X_i)$ by

$$(\pi_i)_{(j,a),a'} = \begin{cases} 1 & \text{if } j = i \text{ and } a = a' \\ 0 & \text{otherwise.}\end{cases}$$

Then $(\&_{i\in I} X_i, (\pi_i)_{i\in I})$ is the Cartesian product of the family $(X_i)_{i\in I}$ in the category **Pcoh**. The coproduct $(\oplus_{i\in I} X_i, (\overline{\pi}_i)_{i\in I})$ is the dual operation, so that

$$|\bigoplus_{i\in I} X_i| = \bigcup_{i\in I}\{i\} \times |X_i|$$

and $u \in \mathrm{P}(\oplus_{i\in I} X_i)$ if $\forall i \in I\ u(i) \in \mathrm{P}X_i$ and $\sum_{i\in I} \|u(i)\|_{X_i} \leq 1$. The injections $\overline{\pi}_j \in \mathbf{Pcoh}(X_j, \oplus_{i\in I} X_i)$ are given by

$$(\overline{\pi}_i)_{a',(j,a)} = \begin{cases} 1 & \text{if } j = i \text{ and } a = a' \\ 0 & \text{otherwise.}\end{cases}$$

We define, in particular, $\mathbf{N} = \oplus_{i\in\mathbb{N}} 1$, that is, $|\mathbf{N}| = \mathbb{N}$ and $u \in (\mathbb{R}_{\geq 0})^{\mathbb{N}}$ belongs to $\mathrm{P}\mathbf{N}$ if $\sum_{n\in\mathbb{N}} u_n \leq 1$.

2.8 Exponentials

We present now a natural exponential comonad for PCSs. We could prove in Crubillé et al. (2017) that this exponential is actually the free one on **Pcoh** (we use the formula presented in Melliès et al. (2009) where more material about free exponentials can be found). For the time being, we do not know any other exponential comonad on **Pcoh**, and, in particular, no exponential comonad based on finite sets instead of finite multisets similar to the clique-based exponential of coherence spaces (Girard 1987). We give first the formal definitions and the functional intuitions underlying this exponential comonad will become more clear in Section 2.8.

Given a set I, a *finite multiset* of elements of I is a function $\mu : I \to \mathbb{N}$ whose *support* $\mathrm{supp}(\mu) = \{a \in I \mid \mu(a) \neq 0\}$ is finite. We use $\mathcal{M}_{\mathrm{fin}}(I)$ for the set of all finite multisets of elements of I. Given a finite family $a_1, \dots, a_n$ of elements of I, we use $[a_1, \dots, a_n]$ for the multiset μ such that $\mu(a) = \#\{i \mid a_i = a\}$. We use additive notations for multiset unions: $\sum_{i=1}^{k} \mu_i$ is the multiset μ such that $\mu(a) = \sum_{i=1}^{k} \mu_i(a)$. The empty multiset is denoted as 0 or $[\,]$. If $k \in \mathbb{N}$, then the multiset $k\mu$ maps a to $k\mu(a)$.

Let X be a PCS. Given $u \in \mathrm{P}X$ and $\mu \in \mathcal{M}_{\mathrm{fin}}(|X|)$, we define $u^\mu = \prod_{a \in |X|} u_a^{\mu(a)} \in \mathbb{R}_{\geq 0}$. Then we set $u^! = (u^\mu)_{\mu \in \mathcal{M}_{\mathrm{fin}}(|X|)}$ and, finally,

$$!X = (\mathcal{M}_{\mathrm{fin}}(|X|), \{u^! \mid u \in \mathrm{P}X\}^{\perp\perp}),$$

which is a pre-PCS.

We check quickly that $!X$ so defined is a PCS. Let $\mu = [a_1, \dots, a_n] \in \mathcal{M}_{\mathrm{fin}}(|X|)$. Because X is a PCS, and by Theorem 2.2, for each $i = 1, \dots, n$, there is $u(i) \in \mathrm{P}X$ such that $u(i)_{a_i} > 0$. Let $(\alpha_i)_{i=1}^n$ be a family of strictly positive real numbers such that $\sum_{i=1}^n \alpha_i \leq 1$. Then $u = \sum_{i=1}^n \alpha_i u(i) \in \mathrm{P}X$ satisfies $u_{a_i} > 0$ for each $i = 1, \dots, n$. Therefore, $u^!_\mu = u^\mu > 0$. This shows that there is $U \in \mathrm{P}(!X)$ such that $U_\mu > 0$.

Let now $A \in \mathbb{R}_{\geq 0}$ be such that $\forall u \in \mathrm{P}X\, \forall i \in \{1, \dots, n\}\, u_{a_i} \leq A$. For all $u \in \mathrm{P}X$, we have $u^\mu \leq A^n$. We have

$$(\mathrm{P}(!X))^\perp = \{u^! \mid u \in \mathrm{P}X\}^{\perp\perp\perp} = \{u^! \mid u \in \mathrm{P}X\}^\perp.$$

Let $t \in (\mathbb{R}_{\geq 0})^{|!X|}$ be defined by $t_\nu = 0$ if $\nu \neq \mu$ and $t_\mu = A^{-n} > 0$; we have $t \in (\mathrm{P}(!X))^\perp$. We have exhibited an element t of $(\mathrm{P}(!X))^\perp$ such that $t_\mu > 0$. By Lemma 2.1, it follows that $!X$ is a PCS.

Kleisli Morphisms as Functions. Before describing the action of the exponential $!\_$ on morphisms, we need to show that the object of $\mathbf{Pcoh}(!X, Y)$ (the Kleisli morphisms from X to Y, see in Section 2.9 the definition of the Kleisli category) can be considered as functions, this will make further considerations much easier.

Let $s \in (\mathbb{R}_{\geq 0})^{|!X \multimap Y|}$. We define a function $\widehat{s} : \mathrm{P}X \to \overline{\mathbb{R}_{\geq 0}}^{|Y|}$ as follows. Given $u \in \mathrm{P}X$, we set

$$\widehat{s}(u) = s\, u^! = \left(\sum_{\mu \in |!X|} s_{\mu,b} u^\mu \right)_{b \in |Y|}.$$

Proposition 2.7. *One has $s \in \mathrm{P}(!X \multimap Y)$ iff, for all $u \in \mathrm{P}X$, one has $\widehat{s}(u) \in \mathrm{P}Y$.*

Proof. By Lemma 2.4, the condition is necessary since $u \in \mathrm{P}X \Rightarrow u^! \in \mathrm{P}(!X)$; let us prove that it is sufficient. Given $v' \in \mathrm{P}Y^\perp$, it suffices to prove that $s^\perp\, v' \in \mathrm{P}(!X)^\perp$, that is, $\langle s^\perp\, v', u^! \rangle \leq 1$ for all $u \in \mathrm{P}X$. This results from the assumption because $\langle s^\perp\, v', u^! \rangle = \langle \widehat{s}(u), v' \rangle$. □

Theorem 2.8. *Let $s \in \mathbf{Pcoh}(!X, Y)$. The function $\widehat{s}$ is Scott-continuous. Moreover, given $s, s' \in \mathbf{Pcoh}(!X, Y)$, one has $s = s'$ (as matrices) iff $\widehat{s} = \widehat{s'}$ (as functions $\mathrm{P}X \to \mathrm{P}Y$).*

Proof. Let us first prove that $\widehat{s}$ is Scott continuous. It is clear that this function is monotone. Let D be a directed subset of $\mathrm{P}X$ and let w be its lub; we must prove that $\widehat{s}(w) = \sup_{u\in D}\widehat{s}(u)$. Let $b \in |Y|$. Since multiplication is a Scott-continuous function from $[0,1]^2$ to $[0,1]$, we have $\widehat{s}(w)_b = \sum_{\mu\in|!X|}\sup_{u\in D} s_{\mu,b}u^\mu$. The announced property follows by the monotone convergence theorem.

Let now $s, s' \in \mathbf{Pcoh}(!X, Y)$ be such that $\widehat{s}(u) = \widehat{s'}(u)$ for all $u \in \mathrm{P}X$. Let $\mu \in |!X|$ and $b \in |Y|$, we prove that $s_{\mu,b} = s'_{\mu,b}$. Let $I = \mathrm{supp}(\mu)$. Given $u \in (\mathbb{R}_{\geq 0})^I$, let $\eta(u) \in (\mathbb{R}_{\geq 0})^{|X|}$ be defined by $\eta(u)_a = 0$ for $a \notin I$ and $\eta(u)_a = u_a$ for $a \in I$. Let $A > 0$ be such that $\eta([0,A]^I) \subseteq \mathrm{P}X$ (such an A exists because I is finite and by our definition of PCS). Let $\pi_b : \mathrm{P}Y \to \mathbb{R}$ be defined by $\pi_b(v) = v_b$. Let $f = \pi_b \circ \widehat{s} \circ \eta : [0,A]^I \to \mathbb{R}$ and $f' = \pi_b \circ \widehat{s'} \circ \eta$. Then we have $f = f'$ by our assumption on s and s'. But f and f' are analytic functions and we have $f(u) = \sum_{v\in\mathcal{M}_{\mathrm{fin}}(I)} s_{v,b}\prod_{a\in I} u_a^{v(a)}$ and similarly for f' and s'. Since $[0,A]^I$ contains a non-empty open subset of $\mathbb{R}^I$, it follows that $s_{v,b} = s'_{v,b}$ for all $v \in \mathcal{M}_{\mathrm{fin}}(I)$. In particular, $s_{\mu,b} = s'_{\mu,b}$. □

So we can consider the elements of $\mathbf{Pcoh}_!(X,Y)$ (the morphisms of the Kleisli category of the comonad !_ on the category **Pcoh**) as particular Scott continuous functions $\mathrm{P}X \to \mathrm{P}Y$. Of course, not all Scott continuous function are morphisms in $\mathbf{Pcoh}_!$, see Section 2.9 for an example.

Proposition 2.9. *Let $s, s' \in \mathbf{Pcoh}_!(X,Y)$ be such that $s \leq s'$ (as elements of $\mathrm{P}(!X \multimap Y)$). Then $\forall u \in \mathrm{P}X\ \widehat{s}(u) \leq \widehat{s'}(u)$. Let $(s(i))_{i\in\mathbb{N}}$ be a monotone sequence of elements of $\mathbf{Pcoh}_!(X,Y)$, and let $s = \sup_{i\in\mathbb{N}} s(i)$. Then $\forall u \in \mathrm{P}X\ \widehat{s}(u) = \sup_{i\in I}\widehat{s_i}(u)$.*

The first statement is obvious. The second one results from the monotone convergence theorem.

Remark: We can have $s, s' \in \mathbf{Pcoh}_!(X,Y)$ such that $\forall u \in \mathrm{P}X\ \widehat{s}(u) \leq \widehat{s'}(u)$ but without having that $s \leq s'$. Take, for instance, $X = Y = 1$. As in the example above, we can see $\widehat{s}$ and $\widehat{s'}$ as functions $[0,1] \to [0,1]$ given by $\widehat{s}(u) = \sum_{n=0}^{\infty} s_n u^n$ and $\widehat{s'}(u) = \sum_{n=0}^{\infty} s'_n u^n$, and $s \leq s'$ means that $\forall n \in \mathbb{N}\ s_n \leq s'_n$. Then let s be defined by $s_n = 1$ if $n = 1$ and $s_n = 0$ otherwise and s' be defined by $s'_n = 1$ if $n = 0$ and $s'_n = 0$ otherwise. We have $\widehat{s}(u) = u \leq \widehat{s'}(u) = 1$ for all $u \in [0,1]$, whereas s and s' are not comparable in $\mathrm{P}(!1 \multimap 1)$.

2.8.1 Action of the Exponential on Morphisms. Given a multiset $\mu \in \mathcal{M}_{\mathrm{fin}}(I)$, we define its *factorial* $\mu! = \prod_{i\in I}\mu(i)!$ and its *multinomial coefficient* $\mathrm{mn}(\mu) = (\#\mu)!/\mu! \in \mathbb{N}^+$, where $\#\mu = \sum_{i\in I}\mu(i)$ is the cardinality of μ. Remember that, given an I-indexed family $a = (a_i)_{i\in I}$ of elements of a commutative semi-ring, one has the multinomial formula

$$\left(\sum_{i\in I} a_i\right)^n = \sum_{\mu\in\mathcal{M}_n(I)} \mathrm{mn}(\mu)a^\mu,$$

where $\mathcal{M}_n(I) = \{\mu \in \mathcal{M}_{\mathrm{fin}}(I) \mid \#\mu = n\}$.

Given $\mu \in |!X|$ and $\nu \in |!Y|$, we define $\mathsf{L}(\mu,\nu)$ as the set of all multisets ρ in $\mathcal{M}_{\mathrm{fin}}(|X| \times |Y|)$ such that

$$\forall a \in |X|\ \sum_{b\in|Y|}\rho(a,b) = \mu(a) \quad \text{and} \quad \forall b \in |Y|\ \sum_{a\in|X|}\rho(a,b) = \nu(b).$$

Let $t \in \mathbf{Pcoh}(X,Y)$; we define $!t \in (\mathbb{R}^+)^{!X\multimap !Y}$ by

$$(!t)_{\mu,\nu} = \sum_{\rho\in\mathsf{L}(\mu,\nu)} \frac{\nu!}{\rho!}t^\rho.$$

Observe that the coefficients in this sum are all non-negative integers.

Lemma 2.10. *For all $u \in \mathrm{P}X$, one has $!t\,u^! = (t\,u)^!$.*

Proof. Indeed, given $\nu \in |!Y|$, one has

$$\begin{aligned}
(t\,u)^!_\nu &= \prod_{b\in|Y|}\left(\sum_{a\in|X|} t_{a,b}u_a\right)^{\nu(b)}\\
&= \prod_{b\in|Y|}\left(\sum_{\substack{\mu\in|!X|\\ \#\mu=\nu(b)}} \mathsf{mn}(\mu)u^\mu \prod_{a\in|X|} t_{a,b}^{\mu(a)}\right)\\
&= \sum_{\substack{\theta\in|!X|^{|Y|}\\ \forall b\ \#\theta(b)=\nu(b)}} u^{\sum_{b\in|Y|}\theta(b)}\left(\prod_{b\in|Y|}\mathsf{mn}(\theta(b))\right)\left(\prod_{\substack{a\in|X|\\ b\in|Y|}} t_{a,b}^{\theta(b)(a)}\right)\\
&= \sum_{\mu\in|!X|} u^\mu \sum_{\rho\in\mathsf{L}(\mu,\nu)} t^\rho \prod_{b\in Y}\frac{\nu(b)!}{\prod_{a\in|X|}\rho(a,b)!}\\
&= \sum_{\mu\in|!X|}(!t)_{\mu,\nu}u^\mu,
\end{aligned}$$

since there is a bijective correspondence between the $\theta \in |!X|^{|Y|}$ such that $\forall b \in |Y|\ \#\theta(b) = \nu(b)$ and the $\rho \in \bigcup_{\mu\in|!X|}\mathsf{L}(\mu,\nu)$ (observe that this union of sets is actually a disjoint union): This bijection maps θ to the multiset ρ defined by $\rho(a,b) = \theta(b)(a)$. □

Proposition 2.11. *For all $t \in \mathbf{Pcoh}(X,Y)$, one has $!t \in \mathbf{Pcoh}(!X,!Y)$ and the operation $t \mapsto !t$ is functorial.*

Proof. Immediate consequences of Lemma 2.10 and Theorem 2.8. □

Comonad Structure of the Exponential. We equip now this functor with a structure of comonad: let $\mathrm{der}_X \in (\mathbb{R}_{\geq 0})^{|!X\multimap X|}$ be given by $(\mathrm{der}_X)_{\mu,a} = \delta_{[a],\mu}$ (the value of the Kronecker symbol $\delta_{i,j}$ is 1 if $i = j$ and 0 otherwise) and $\mathrm{dig}_X \in (\mathbb{R}_{\geq 0})^{|!X\multimap !!X|}$ be given by $(\mathrm{dig}_X)_{\mu,[\mu_1,\ldots,\mu_n]} = \delta_{\sum_{i=1}^n \mu_i,\mu}$. Then we have $\mathrm{der}_X \in \mathbf{Pcoh}(!X,X)$ and $\mathrm{dig}_X \in \mathbf{Pcoh}(!X,!!X)$ simply because

$$\widehat{\mathrm{der}_X}(u) = u \quad\text{and}\quad \widehat{\mathrm{dig}_X}(u) = (u^!)^!$$

for all $u \in \mathrm{P}X$, as easily checked. Using these equations, one also checks easily the naturality of these morphisms and the fact that $(!\_, \mathrm{der}, \mathrm{dig})$ is a comonad.

As to the monoidality of this comonad, we introduce $\mu^0 \in (\mathbb{R}_{\geq 0})^{|1\multimap !\top|}$ by $\mu^0_{*,[]} = 1$ and $\mu^2_{X,Y} \in (\mathbb{R}_{\geq 0})^{|!X\otimes !Y\multimap !(X\&Y)|}$ by $(\mu^2_{X,Y})_{\lambda,\rho,\mu} = \delta_{\mu,1\cdot\lambda+2\cdot\rho}$, where $i\cdot[a_1,\ldots,a_n] = [(i,a_1),\ldots,(i,a_n)]$. It is easily checked that the required commutations hold (again, we refer to Melliès (2009)).

It follows that we can define a lax symmetric monoidal structure for the functor $!\_$ from the symmetric monoidal category $(\mathbf{Pcoh},\otimes)$ to itself, that is, for each $n \in \mathbb{N}$, a natural morphism

$$\mathsf{m}^{(n)}_{X_1,\ldots,X_N} \in \mathbf{Pcoh}(!X_1\otimes\cdots\otimes !X_n, !(X_1\otimes\cdots\otimes X_n))$$

satisfying some coherence conditions.

Given $s \in \mathbf{Pcoh}(!X_1\otimes\cdots\otimes !X_n, Y)$, we define the *promotion* morphism $s^! \in \mathbf{Pcoh}(!X_1\otimes\cdots\otimes !X_n, !Y)$ as the following composition of morphisms in **Pcoh**:

$$\begin{array}{ccc}
!X_1\otimes\cdots\otimes !X_n & & !Y\\
{\scriptstyle \mathrm{dig}_{X_1}\otimes\cdots\otimes\mathrm{dig}_{X_n}}\downarrow & & \uparrow{\scriptstyle !s}\\
!!X_1\otimes\cdots\otimes !!X_n & \xrightarrow{\mathsf{m}^{(n)}_{!X_1,\ldots,!X_n}} & !(!X_1\otimes\cdots\otimes !X_n)
\end{array} \qquad (2)$$

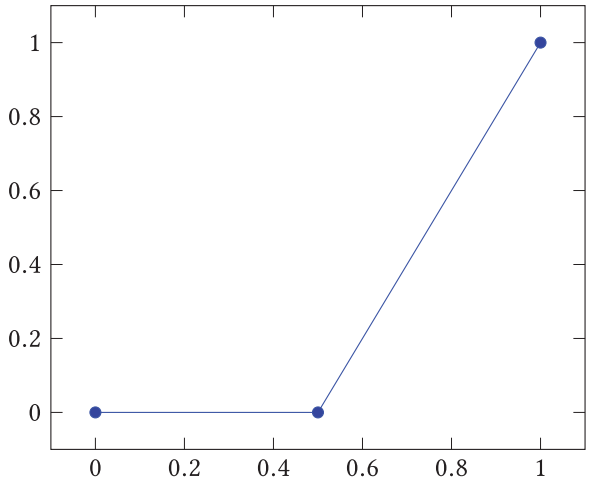

Fig. 1. A Scott continuous function that is not a morphism in $\mathbf{Pcoh}_!$.

2.9 The Kleisli Category

The Kleisli category $\mathbf{Pcoh}_!$ of the comonad !_ has the same objects as **Pcoh**, and $\mathbf{Pcoh}_!(X,Y) = \mathbf{Pcoh}(!X,Y)$. The identity morphism at object X is der_X and given $s \in \mathbf{Pcoh}_!(X,Y)$ and $t \in \mathbf{Pcoh}_!(Y,Z)$ the composition of s and t in $\mathbf{Pcoh}_!$, denoted as $t \circ s$, is given by

$$t \circ s = t\, s^! = t\, !s\ \mathrm{dig}_X.$$

Let $u \in \mathrm{P}X$. We have $\widehat{t \circ s}(u) = t\, !s\ \mathrm{dig}_X\ u^! = t\, !s\, (u^!)^! = t\, (s\, u^!)^! = \widehat{t}(\widehat{s}(u))$, which shows that $\widehat{t \circ s} = \widehat{t} \circ \widehat{s}$, that is, the Kleisli composition coincides with the usual notion of composition of functions through the $s \mapsto \widehat{s}$ correspondence. This standard observation together with the well-pointedness Theorem 2.8 allows us to consider the Kleisli category $\mathbf{Pcoh}_!$ as a category of spaces and functions.

This category is Cartesian closed: The terminal object is $\top$, the Cartesian product of two objects X and Y is $X \,\&\, Y$ (with projections defined in the obvious way, using $\mathrm{der}_{X\&Y}$ and the projections of the Cartesian product in **Pcoh**), and their internal hom object is $X \Rightarrow Y = !X \multimap Y$. The corresponding evaluation morphism $\mathrm{Ev} \in \mathbf{Pcoh}_!((X \Rightarrow Y) \,\&\, X, Y)$ is defined as the following composition of morphisms in **Pcoh**:

$$!((X \Rightarrow Y) \,\&\, X) \xrightarrow{(\mu^2)^{-1}} !(X \Rightarrow Y) \otimes !X \xrightarrow{\mathrm{der} \otimes !X} (X \Rightarrow Y) \otimes !X \xrightarrow{\mathrm{ev}} Y.$$

The curryfied version of a morphism $t \in \mathbf{Pcoh}_!(Z \,\&\, X, Y)$ is the morphism $\mathrm{Cur}(t) \in \mathbf{Pcoh}_!(Z, X \Rightarrow Y)$ defined as $\mathrm{Cur}(t) = \mathrm{cur}(t\, \mu^2)$.

Example of Morphisms and Non-morphisms in the Kleisli Category. We give now examples of morphisms in $\mathbf{Pcoh}_!$ providing in that way more intuitions on this category.

Take $X = Y = 1$. A morphism in $\mathbf{Pcoh}_!(1,1)$ can be seen as a function $f : [0,1] \to [0,1]$ such that $f(u) = \sum_{n=0}^{\infty} s_n u^n$, where the s_n's are ≥ 0 and satisfy $\sum_{n=0}^{\infty} s_n \leq 1$. Of course, not all Scott continuous functions $[0,1] \to [0,1]$ are of that particular shape. Take, for instance, the function $f : [0,1] \to [0,1]$ defined by $f(u) = 0$ if $u \leq \frac{1}{2}$ and $f(u) = 2u - 1$ if $u > \frac{1}{2}$; this function f whose graph is shown in Figure 1 is Scott continuous but has no derivative at $u = \frac{1}{2}$ and therefore cannot be expressed as a power series.

Take now $X = 1 \oplus 1$ and $Y = 1$. A morphism in $\mathbf{Pcoh}_!(1 \oplus 1, 1)$ can be seen as a function $f : [0,1]^2 \to [0,1]$ such that $f(u,v) = \sum_{m,n\in\mathbb{N}}^{\infty} s_{m,n} u^m v^n$, where the $s_{m,n}$'s are ≥ 0 and satisfy $\forall u \in [0,1]\ \sum_{m,n\in\mathbb{N}}^{\infty} s_{m,n} u^m (1-u)^n \leq 1$. For $m, n \in \mathbb{N}$, the function $u \mapsto u^m(1-u)^n$ from $[0,1]$ to $[0,1]$ takes its maximal value at $u = \frac{m}{m+n}$, and this value is $\frac{m^m n^n}{(m+n)^{m+n}}$. It follows that $s_{m,n} \leq \frac{(m+n)^{m+n}}{m^m n^n}$. On the other hand it can be proved that if f is definable in PCF (or rather in an adapted version of PCF with a type of Booleans interpreted as $1 \oplus 1$ and a unit type interpreted as 1), then the maximal possible value $s_{m,n}$ is the binomial coefficient $\frac{(m+n)!}{m!n!}$. Asymptotically, when n and m go to ∞, the ratio $\frac{(m+n)^{m+n}}{m^m n^n} / \frac{(m+n)!}{m!n!}$ goes to ∞ as $\sqrt{2\pi}\sqrt{\frac{nm}{n+m}}$ by the Stirling Formula $n! \sim (\frac{n}{e})^n \sqrt{2\pi n}$. This means that definable elements form a strict subset of $\mathbf{Pcoh}_!(X,Y)$ whose projections on "homogeneous subspaces" become smaller and smaller as the degree augments, tending to slowly vanish at the limit at a rate of $1/\sqrt{n}$ (with respect to the global space). A natural question is whether the same rate can be observed at all types, we do not know the answer yet.

Last, take $X = \mathsf{N} \,\&\, \mathsf{N} \,\&\, \mathsf{N}$. One defines a morphism $s \in \mathbf{Pcoh}_!(X, \mathsf{N})$ by setting

$$s_{(\mu,k)} = \begin{cases} 1 & \text{if } \mu \in \{[(2,0),(3,1)], [(3,0),(1,1)], [(1,0),(2,1)]\} \\ & \text{and } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, given $u, v, w \in \mathsf{PN}$, the corresponding function $\widehat{s} : \mathsf{P}X \to \mathsf{PN}$, or rather the $\mathbb{R}_{\geq 0}$-valued function f defined on PN^3 by $f(u,v,w) = \widehat{s}(u,v,w)_0$, satisfies

$$\begin{aligned} f(u,v,w) &= v_0 w_1 + w_0 u_1 + u_0 v_1 \\ &\leq v_0(1-w_0) + w_0(1-u_0) + u_0(1-v_0) \\ &\leq u_0 + v_0 + w_0 - (v_0 w_0 + w_0 u_0 + u_0 v_0) \\ &\leq u_0 + (v_0 + w_0)(1-u_0) - v_0 w_0 \\ &\leq \max(1, v_0 + w_0) - v_0 w_0 \leq 1, \end{aligned}$$

because $u_0 + v_0 - u_0 v_0 = u_0 + v_0(1-u_0) \leq \max(1, v_0) \leq 1$. This shows that $s \in \mathbf{Pcoh}_!(X, \mathsf{N})$ as contended (actually the maximal value taken by $\widehat{s}$ is $\frac{3}{4}$).

This is a version of the famous Gustave's function g that is a counter-example showing that there are stable functions (in Berry's dI-domains of Girard's coherence spaces) that are not PCF-definable and hence that the stable model is not fully abstract for ordinary PCF (see Amadio and Curien (1998)). The reason is that one can define a term M of type $(\iota \Rightarrow \iota \Rightarrow \iota \Rightarrow \iota) \Rightarrow \iota$ in ordinary PCF such that $[\![M]\!](g) \neq \bot$ (in the stable model under consideration; here $\bot$ denotes the least element of the flat domain of integers), although M is observationally equivalent to the completely undefined term $\Omega^{(\iota\Rightarrow\iota\Rightarrow\iota\Rightarrow\iota)\Rightarrow\iota}$.

And, indeed, s is not definable in pPCF either. Why does the presence of such functions not prevent the model **Pcoh** from being fully abstract? One possible answer relies on the observation that, although s is not pPCF-definable, $\frac{1}{2}s$ is as easily checked. And indeed, if the interpretation of a closed pPCF term M of type $(\iota \Rightarrow \iota \Rightarrow \iota \Rightarrow \iota) \Rightarrow \iota$ maps s to a non-0 value, then it cannot map $\frac{1}{2}s$ to a 0 value (because this interpretation is analytic) and therefore this term M is not observationally equivalent to Ω, since $\frac{1}{2}s$ is definable.

2.9.1 Least Fix-point Operators. Let X be an object of **Pcoh**. Let $\mathcal{F} \in \mathbf{Pcoh}_!((X \Rightarrow X) \Rightarrow X, (X \Rightarrow X) \Rightarrow X)$ be $\mathcal{F} = \mathsf{Cur}(\mathcal{F}_0)$, where $\mathcal{F}_0 \in \mathbf{Pcoh}_!(((X \Rightarrow X) \Rightarrow X) \,\&\, (X \Rightarrow X), X)$ is the following composition of morphisms in $\mathbf{Pcoh}_!$

$$\begin{array}{ccc} ((X \Rightarrow X) \Rightarrow X) \,\&\, (X \Rightarrow X) & & X \\ \downarrow{\scriptstyle \langle \pi_2, \pi_1, \pi_2 \rangle} & & \uparrow{\scriptstyle \mathsf{Ev}} \\ (X \Rightarrow X) \,\&\, ((X \Rightarrow X) \Rightarrow X) \,\&\, (X \Rightarrow X) & \xrightarrow{\langle \pi_1, \mathsf{Ev} \circ \langle \pi_2, \pi_3 \rangle \rangle} & (X \Rightarrow X) \,\&\, X. \end{array}$$

Then, given $F \in \mathsf{P}((X \Rightarrow X) \Rightarrow X)$, that is, $F \in \mathbf{Pcoh}_!(X \Rightarrow X, X)$, one has $\widehat{\mathcal{F}}(F) = \mathsf{Ev} \circ \langle \mathsf{Id}_{X \Rightarrow X}, F \rangle \in \mathbf{Pcoh}_!(X \Rightarrow X, X)$. Since $\mathcal{F}$ is a morphism in **Pcoh**, the function $\widehat{\mathcal{F}}$ is Scott continuous and therefore has a least fix-point $\mathsf{Y} \in \mathbf{Pcoh}_!(X \Rightarrow X, X)$, namely $\mathsf{Y} = \sup_{n \in \mathbb{N}} \widehat{\mathcal{F}}^n(0)$ (the sequence $(\widehat{\mathcal{F}}^n(0))_{n \in \mathbb{N}}$ is monotone in the cpo $\mathsf{P}((X \Rightarrow X) \Rightarrow X)$ because $\widehat{\mathcal{F}}$ is monotone).

If we set $\mathsf{Y}_n = \widehat{\mathcal{F}}^n(0) \in \mathbf{Pcoh}_!(X \Rightarrow X, X)$, then we have $\mathsf{Y}_0 = 0$ and $\mathsf{Y}_{n+1} = \mathsf{Ev} \circ \langle \mathsf{Id}, \mathsf{Y}_n \rangle$ so that, given $s \in \mathbf{Pcoh}_!(X, X)$, we have $\widehat{\mathsf{Y}_n}(s) = \widehat{s}^n(0)$ and $\widehat{\mathsf{Y}}(s) = \sup_{n \in \mathbb{N}} \widehat{s}^n(0)$. It follows that Y is the usual least fix-point operator, and this operation turns out to be a morphism in $\mathbf{Pcoh}_!$, namely $\mathsf{Y} \in \mathbf{Pcoh}_!(X \Rightarrow X, X)$. This means that this standard least fix-point operator can be described as a power series, which is not completely obvious at first sight.

2.10 Coalgebras

By definition, a coalgebra of the !_ comonad is a pair (X, h), where X is a PCS and $h \in \mathbf{Pcoh}(X, !X)$ satisfies the following commutations:

$$\begin{array}{ccc} X & \xrightarrow{h} & !X \\ & {\scriptstyle \mathsf{Id}_X} \searrow & \downarrow{\scriptstyle \mathsf{der}_X} \\ & & X \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{h} & !X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle !h} \\ !X & \xrightarrow[\mathsf{dig}_X]{} & !!X \end{array}$$

A morphism from a coalgebra (X_1, h_1) to a coalgebra (X_2, h_2) is an $f \in \mathbf{Pcoh}(X_1, X_2)$ such that the following diagram commutes:

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & X_2 \\ {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle h_2} \\ !X_1 & \xrightarrow[!f]{} & !X_2 \end{array}$$

Observe that 1 has a natural structure of !-coalgebra $\upsilon \in \mathbf{Pcoh}(1, !1)$ that is obtained as the following composition of morphisms

$$1 \xrightarrow{\mu^0} !\top \xrightarrow{\mathsf{dig}_\top} !!\top \xrightarrow{!(\mu^0)^{-1}} !1$$

Checking that $(1, \upsilon)$ is indeed a !-coalgebra boils down to a simple diagrammatic computation using the general axioms satisfied by the comonadic and monoidal structure of the !_ functor.

A simple computation shows that $\upsilon_{*,n} = 1$ for all $n \in |!1|$ (remember that $|!1| = \mathbb{N}$).

Let $(X_i, h_i)_{i \in I}$ be a countable family of coalgebras. Then we can endow $X = \bigoplus_{i \in I} X_i$ with a structure of coalgebra $h \in \mathbf{Pcoh}(X, !X)$. By the universal property of the coproduct, it suffices to define for each $i \in I$ a morphism $h'_i : X_i \to !X$. We set $h'_i = !\overline{\pi}_i \, h_i$, where we record that $\overline{\pi}_i : X_i \to X$ is the ith canonical injection into the coproduct. It is then quite easy to check that (X, h) so defined is a coalgebra using the fact that each (X_i, h_i) is a coalgebra.

Natural Numbers. Consider the case where $I = \mathbb{N}$, $X_i = 1$, and $h_i = \upsilon$ for each $i \in \mathbb{N}$. Then we use N to denote the corresponding object X and h_{N} for the corresponding coalgebra structure, $h_{\mathsf{N}} \in \mathbf{Pcoh}(\mathsf{N}, !\mathsf{N})$. We use $\overline{n} \in \mathbf{Pcoh}(1, \mathsf{N})$ for the nth injection that we consider also as the element of PN defined by $\overline{n}_k = \delta_{n,k}$.

An easy computation shows that

$$(h_{\mathsf{N}})_{n,\mu} = \begin{cases} 1 & \text{if } \mu = k[n] \text{ for some } k \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Let $t \in \mathbf{Pcoh}_!(\mathsf{N}, X)$ for some object X of **Pcoh**. Then $t\, h_{\mathsf{N}} \in \mathbf{Pcoh}(\mathsf{N}, X)$ is a linearized[12] version of t. Given $u \in \mathsf{PN}$, an easy computation shows that

$$t\, h_{\mathsf{N}}\, u = \sum_{n=0}^{\infty} u_n \widehat{t}(\overline{n}).$$

The objects N and $1 \oplus \mathsf{N}$ are obviously isomorphic, through the morphisms $p \in \mathbf{Pcoh}(\mathsf{N}, 1 \oplus \mathsf{N})$ and $s \in \mathbf{Pcoh}(1 \oplus \mathsf{N}, \mathsf{N})$ given by

$$p_{n,(1,*)} = s_{(1,*),n} = \delta_{n,0} \text{ and } p_{n,(2,n')} = s_{(2,n'),n} = \delta_{n,n'+1}.$$

We set $\overline{\mathsf{suc}} = s\, \overline{\pi}_2 \in \mathbf{Pcoh}(\mathsf{N}, \mathsf{N})$, so that $\overline{\mathsf{suc}}_{n,n'} = \delta_{n+1,n'}$ represents the successor function.

2.11 Conditional

Given an object X of **Pcoh**, we define a morphism

$$\overline{\mathsf{if}} \in \mathbf{Pcoh}(\mathsf{N} \otimes !X \otimes !(!\mathsf{N} \multimap X), X).$$

For this, we define first $\overline{\mathsf{if}}_0 \in \mathbf{Pcoh}(1 \otimes !X \otimes !(!\mathsf{N} \multimap X), X)$ as the following composition of morphisms (without mentioning the isomorphisms associated with the monoidality of $\otimes$):

$$!X \otimes !(!\mathsf{N} \multimap X) \xrightarrow{!X \otimes \mathsf{w}} !X \xrightarrow{\mathrm{der}_X} X$$

and next $\overline{\mathsf{if}}_+ \in \mathbf{Pcoh}(\mathsf{N} \otimes !X \otimes !(!\mathsf{N} \multimap X), X)$ (with the same conventions as above):

$$\mathsf{N} \otimes !X \otimes !(!\mathsf{N} \multimap X) \xrightarrow{h_{\mathsf{N}} \otimes \mathsf{w} \otimes \mathrm{der}} !\mathsf{N} \otimes (!\mathsf{N} \multimap X) \xrightarrow{\mathrm{ev}\ \gamma} X$$

where γ is the isomorphism associated with the symmetry of the functor $\otimes$, see Section 2.6.

The universal property of $\oplus$ and the fact that $\_ \otimes Y$ is a left adjoint for each object Y allows us therefore to define $\overline{\mathsf{if}}' \in \mathbf{Pcoh}((1 \oplus \mathsf{N}) \otimes !X \otimes !(!\mathsf{N} \multimap X), X)$. Finally, our conditional morphism is $\overline{\mathsf{if}} = \overline{\mathsf{if}}'\, (p \otimes !X \otimes !(!\mathsf{N} \multimap X)) \in \mathbf{Pcoh}(\mathsf{N} \otimes !X \otimes !(!\mathsf{N} \multimap X), X)$. The isomorphism $p \in \mathbf{Pcoh}(\mathsf{N}, 1 \oplus \mathsf{N})$ is defined at the end of Section 2.10.

It is important to notice that the two following diagrams commute:

$$\begin{array}{ccc} 1 \otimes !X \otimes !(!\mathsf{N} \multimap X) & \xrightarrow{\overline{0} \otimes \mathrm{Id}} & \mathsf{N} \otimes !X \otimes !(!\mathsf{N} \multimap X) \\ & \searrow^{\mathrm{der} \otimes \mathsf{w}} & \downarrow{\scriptstyle \overline{\mathsf{if}}} \\ & & X \end{array}$$

$$\begin{array}{ccc} 1 \otimes !X \otimes !(!\mathsf{N} \multimap X) & \xrightarrow{\overline{n+1} \otimes \mathrm{Id}} & \mathsf{N} \otimes !X \otimes !(!\mathsf{N} \multimap X) \\ \downarrow{\scriptstyle \overline{n}^! \otimes \mathsf{w}} & & \downarrow{\scriptstyle \overline{\mathsf{if}}} \\ !\mathsf{N} \otimes (!\mathsf{N} \multimap X) & \xrightarrow{\mathrm{ev}\ \gamma} & X \end{array}$$

This second commutation boils down to the following simple property: $\forall n \in \mathbb{N}\ h_{\mathsf{N}}\, \overline{n} = \overline{n}^!$. Observe that it is not true, however, that $\forall u \in \mathsf{PN}\ h_{\mathsf{N}}\, u = u^!$. This means that h_{N} allows us to duplicate and erase "true" natural numbers $\overline{n}$ but not general elements of PN that can be considered as "computations" and not as "values."

[12] This is not at all the same kind of linearization as the one introduced by Differential Linear Logic (Ehrhard 2017).

2.12 Interpreting Terms

Given a type σ, we define an object $[\![\sigma]\!]$ of **Pcoh** as follows: $[\![\iota]\!] = \mathsf{N}$ and $[\![\sigma \Rightarrow \tau]\!] = [\![\sigma]\!] \Rightarrow [\![\tau]\!]$.

Given a context $\Gamma = (x_1 : \sigma_1, \dots, x_k : \sigma_k)$, a type σ, and a term M such that $\Gamma \vdash M : \sigma$, we define a morphism $[\![M]\!]_\Gamma \in \mathbf{Pcoh}_!([\![\Gamma]\!], [\![\sigma]\!])$, where $[\![\Gamma]\!] = [\![\sigma_1]\!] \,\&\, \cdots \,\&\, [\![\sigma_k]\!]$. Equivalently, we can see $[\![M]\!]_\Gamma$ as a morphism in $\mathbf{Pcoh}([\![\Gamma]\!]^!, [\![\sigma]\!])$, where $[\![\Gamma]\!]^! = ![\![\sigma_1]\!] \otimes \cdots \otimes ![\![\sigma_k]\!]$. By Theorem 2.8, this morphism can be fully described as a function $\widehat{[\![M]\!]_\Gamma} : \prod_{i=1}^k \mathsf{P}[\![\sigma_i]\!] \to \mathsf{P}[\![\sigma]\!]$. The definition is by induction on the typing derivation of $\Gamma \vdash M : \sigma$, or, equivalently, on M.

If $M = x_i$, then $[\![M]\!]_\Gamma = \pi_i$, that is, $\widehat{[\![M]\!]_\Gamma}(u_1, \dots, u_k) = u_i$.

If $M = \underline{n}$, then $[\![M]\!]_\Gamma = \overline{n} \circ \tau$, where τ is the unique morphism in $\mathbf{Pcoh}_!([\![\Gamma]\!], \top)$. That is, $\widehat{[\![M]\!]_\Gamma}(\vec{u}) = \overline{n}$.

If $M = \mathrm{coin}(p)$ for some $p \in [0,1] \cap \mathbb{Q}$, then $[\![M]\!]_\Gamma = p\overline{0} + (1-p)\overline{1}$.

If $M = \mathrm{succ}(P)$ with $\Gamma \vdash P : \iota$, then we have $[\![P]\!]_\Gamma \in \mathbf{Pcoh}([\![\Gamma]\!]^!, \mathsf{N})$, and we set $[\![M]\!]_\Gamma = \overline{\mathrm{suc}}\,[\![P]\!]_\Gamma$, which is characterized by $\widehat{[\![M]\!]_\Gamma}(\vec{u}) = \sum_{n=0}^\infty (\widehat{[\![P]\!]_\Gamma}(\vec{u}))_n \overline{n+1}$.

If $M = \mathsf{if}(P, Q, z \cdot R)$, $\Gamma \vdash P : \iota$, $\Gamma \vdash Q : \sigma$ and $\Gamma, z : \iota \vdash R : \sigma$, then by inductive hypothesis we have $[\![P]\!]_\Gamma \in \mathbf{Pcoh}([\![\Gamma]\!]^!, \mathsf{N})$, $[\![Q]\!]_\Gamma \in \mathbf{Pcoh}([\![\Gamma]\!]^!, [\![\sigma]\!])$ and $[\![R]\!]_{\Gamma,z:\iota} \in \mathbf{Pcoh}([\![\Gamma]\!]^! \otimes !\mathsf{N}, [\![\sigma]\!])$. We have $\mathrm{cur}([\![R]\!]_{\Gamma,z:\iota}) \in \mathbf{Pcoh}([\![\Gamma]\!]^!, !\mathsf{N} \multimap [\![\sigma]\!])$, and hence we define $[\![M]\!]_\Gamma$ as the following composition of morphisms in **Pcoh**:

$$\begin{array}{ccc} [\![\Gamma]\!]^! & & [\![\sigma]\!] \\ \downarrow{\scriptstyle \mathrm{contr}_\Gamma} & & \uparrow{\scriptstyle \overline{\mathsf{if}}} \\ [\![\Gamma]\!]^! \otimes [\![\Gamma]\!]^! \otimes [\![\Gamma]\!]^! & \xrightarrow{[\![M]\!]_\Gamma \otimes [\![P]\!]^!_\Gamma \otimes \mathrm{cur}([\![R]\!]_{\Gamma,z:\iota})^!} & \mathsf{N} \otimes ![\![\sigma]\!] \otimes !(!\mathsf{N} \multimap [\![\sigma]\!]). \end{array}$$

where contr_Γ is an obvious composition of contraction morphisms and associativity and symmetry isomorphisms associated with the $\otimes$ functor (we also use promotion (2)). Seen as a function, this morphism is completely characterized by

$$\widehat{[\![M]\!]_\Gamma}(\vec{u}) = (\widehat{[\![P]\!]_\Gamma}(\vec{u}))_0 \widehat{[\![Q]\!]_\Gamma}(\vec{u}) + \sum_{n=0}^\infty (\widehat{[\![P]\!]_\Gamma}(\vec{u}))_{n+1} \widehat{[\![R]\!]_{\Gamma,z:\iota}}(\vec{u}, \overline{n}).$$

If $M = (P)\,Q$ with $\Gamma \vdash P : \sigma \Rightarrow \tau$ and $\Gamma \vdash Q : \sigma$, then we have $[\![P]\!]_\Gamma \in \mathbf{Pcoh}_!([\![\Gamma]\!]^!, ![\![\sigma]\!] \multimap [\![\tau]\!])$ and $[\![Q]\!]_\Gamma \in \mathbf{Pcoh}_!([\![\Gamma]\!]^!, [\![\sigma]\!])$, and we define $[\![M]\!]_\Gamma$ as the following composition of morphisms:

$$[\![\Gamma]\!]^! \xrightarrow{\mathrm{contr}_\Gamma} [\![\Gamma]\!]^! \otimes [\![\Gamma]\!]^! \xrightarrow{[\![P]\!]_\Gamma \otimes [\![Q]\!]^!_\Gamma} (![\![\sigma]\!] \multimap [\![\tau]\!]) \otimes ![\![\sigma]\!] \xrightarrow{\mathrm{ev}} [\![\tau]\!]$$

so that $[\![M]\!]_\Gamma$ is characterized by $\widehat{[\![M]\!]_\Gamma}(\vec{u}) = \widehat{[\![P]\!]_\Gamma}(\vec{u})(\widehat{[\![Q]\!]_\Gamma}(\vec{u}))$.

If $M = \lambda x^\sigma\, P$ with $\Gamma, x : \sigma \vdash P : \tau$, then we have $[\![P]\!]_{\Gamma,x:\sigma} \in \mathbf{Pcoh}([\![\Gamma]\!]^! \otimes ![\![\sigma]\!], [\![\tau]\!])$, and we set $[\![M]\!]_\Gamma = \mathrm{cur}([\![P]\!]_{\Gamma,x:\sigma}) \in \mathbf{Pcoh}([\![\Gamma]\!]^!, ![\![\sigma]\!] \multimap [\![\tau]\!])$ so that, given $\vec{u} \in \prod_{i=1}^k \mathsf{P}[\![\sigma_i]\!]$ (remember that $\Gamma = (x_1 : \sigma_1, \dots, x_k : \sigma_k)$), the semantics $[\![M]\!]_\Gamma(\vec{u})$ of M is the element of $\mathsf{P}(![\![\sigma]\!] \multimap [\![\tau]\!])$ which, as a function $\mathsf{P}[\![\sigma]\!] \to \mathsf{P}[\![\tau]\!]$, is characterized by $\widehat{[\![M]\!]_\Gamma}(\vec{u})(u) = \widehat{[\![P]\!]_{\Gamma,x:\sigma}}(\vec{u}, u)$.

If $M = \mathsf{fix}(P)$ with $\Gamma \vdash P : \sigma \Rightarrow \sigma$, then we have $[\![P]\!]_\Gamma \in \mathbf{Pcoh}([\![\Gamma]\!]^!, ![\![\sigma]\!] \multimap [\![\sigma]\!])$ and we set $[\![M]\!]_\Gamma = \mathsf{Y}\,[\![P]\!]^!_\Gamma$. This means that $\widehat{[\![M]\!]_\Gamma}(\vec{u}) = \sup_{n\in\mathbb{N}} f^n(0)$, where $f \in \mathbf{Pcoh}_!([\![\sigma]\!], [\![\sigma]\!])$ is given by $f(u) = \widehat{[\![P]\!]_\Gamma}(\vec{u})(u)$.

Lemma 2.12 (Substitution). *Assume that* $\Gamma, x : \sigma \vdash M : \tau$ *and that* $\Gamma \vdash P : \sigma$. *Then* $[\![M\,[P/x]]\!]_\Gamma = [\![M]\!]_{\Gamma,x:\sigma} \circ \langle \mathrm{Id}_{[\![\Gamma]\!]}, [\![P]\!]_\Gamma \rangle$ *in* $\mathbf{Pcoh}_!$. *In other words, for any* $\vec{u} \in \mathsf{P}[\![\Gamma]\!]$, *we have* $\widehat{[\![M\,[P/x]]\!]_\Gamma}(\vec{u}) = \widehat{[\![M]\!]_{\Gamma,x:\sigma}}(\vec{u}, \widehat{[\![P]\!]_\Gamma}(\vec{u}))$.

The proof is a simple induction on M, the simplest way to write it is to use the functional characterization of the semantics.

For the notations Λ_Γ^σ and Λ_0^σ used below, we refer to Section 1.2. We formulate the invariance of the interpretation of terms under weak-reduction, using the stochastic reduction matrix introduced in Section 1.2.

Theorem 2.13. *Assume that* $\Gamma \vdash M : \sigma$. *One has*

$$[\![M]\!]_\Gamma = \sum_{M' \in \Lambda_\Gamma^\sigma} \mathsf{Red}(\Gamma, \sigma)_{M,M'} [\![M']\!]_\Gamma.$$

Proof. Simple case analysis, on the shape of M, and using the Substitution Lemma. □

As a corollary, we get the following inequality.

Theorem 2.14. *Let* M *be such that* $\vdash M : \iota$. *Then for all* $n \in \mathbb{N}$ *we have*

$$\mathsf{Red}(\iota)^\infty_{M,\underline{n}} \leq [\![M]\!]_n.$$

Proof. Iterating Theorem 2.13, we get, for all $k \in \mathbb{N}$:

$$[\![M]\!] = \sum_{M' \in \Lambda_0^\iota} \mathsf{Red}(\iota)^k_{M,M'} [\![M']\!].$$

Therefore, for all $k \in \mathbb{N}$, we have $[\![M]\!]_n \geq \mathsf{Red}(\iota)^k_{M,\underline{n}}$, and the result follows, since $\underline{n}$ is weak-normal. □

Examples. We refer to the various terms introduced in Section 1.3 and describe as functions the interpretation of some of them.

We have $\vdash \mathsf{pred} : \iota \Rightarrow \iota$ so $[\![\mathsf{pred}]\!] \in \mathsf{P}(\mathsf{N} \Rightarrow \mathsf{N})$, and one checks easily that $\widehat{[\![\mathsf{pred}]\!]}(u) = (u_0 + u_1)\overline{0} + \sum_{n=1}^\infty u_{n+1}\overline{n}$.

Similarly, we have

$$\widehat{[\![\mathsf{add}]\!]}(u)(v) = \sum_{n=0}^\infty \left(\sum_{i=0}^n u_i v_{n-i} \right) \overline{n}$$

$$\widehat{[\![\mathsf{exp}_2]\!]}(u) = \sum_{n=0}^\infty u_n \overline{2^n}$$

$$\widehat{[\![\mathsf{shift}_k]\!]}(u) = \sum_{n=0}^\infty u_n \overline{k+n}$$

$$\widehat{[\![\mathsf{cmp}]\!]}(u)(v) = \left(\sum_{i \leq j} u_i v_j \right) \overline{0} + \left(\sum_{i > j} u_i v_j \right) \overline{1}$$

$$\widehat{[\![\mathsf{prob}_k]\!]}(u) = u_k \overline{0}$$

$$\widehat{[\![\mathsf{prod}_k]\!]}(u^1, \ldots, u^k) = \left(\prod_{i=1}^k u_0^i \right) \overline{0}$$

$$\widehat{[\![\mathsf{choose}_k]\!]}(u)(w^1, \ldots, w^k) = \sum_{i=0}^{k-1} u_i w^{i+1}$$

$$\widehat{[\![\mathsf{unif}]\!]}(u) = \sum_{n=0}^\infty \frac{u_n}{n+1} \left(\sum_{i=0}^n \overline{i} \right) = \sum_{i=0}^\infty \left(\sum_{n=i}^\infty \frac{u_n}{n+1} \right) \overline{i}$$

$$[\![\mathsf{ran}\widehat{(p_0, \ldots, p_n)}]\!] = \sum_{i=0}^n p_i \overline{i}.$$

3 ADEQUACY

We want now to prove the converse inequality to that of Theorem 2.14. For this purpose, we define a logical relation between *closed* pPCF terms of type σ and elements of $\mathsf{P}[\![\sigma]\!]$. This proof follows a method introduced in Amadio and Curien (1998), simplifying the technique of Plotkin (1977).

For any type σ, we define a binary relation $\mathcal{R}^\sigma \subseteq \Lambda_0^\sigma \times \mathsf{P}[\![\sigma]\!]$ by induction on types as follows:

- $M\ \mathcal{R}^\iota\ u$ if $\forall n \in \mathbb{N}\ u_n \leq \mathsf{Red}(\iota)^\infty_{M,\underline{n}}$. In other words, for each $n \in \mathbb{N}$ the probability of M to reduce to $\underline{n}$ is larger than u_n (remember that, since $u \in \mathsf{PN}$, u is a sub-probability distribution).
- $M\ \mathcal{R}^{\sigma\Rightarrow\tau}\ t$ if $\forall P \in \Lambda_0^\sigma\ \forall u \in \mathsf{P}[\![\sigma]\!]\ P\ \mathcal{R}^\sigma\ u \Rightarrow (M)\,P\ \mathcal{R}^\tau\ \widehat{t}(u)$. Here we have $t \in \mathsf{P}[\![\sigma \Rightarrow \tau]\!]$ and hence $\widehat{t} : \mathsf{P}[\![\sigma]\!] \to \mathsf{P}[\![\tau]\!]$.

So $\mathcal{R}^\sigma$ is a logical relation.

Lemma 3.1. *If $M \in \Lambda_0^\sigma$, then $M\ \mathcal{R}^\sigma\ 0$. If $(u(i))_{i\in\mathbb{N}}$ is an increasing sequence in $\mathsf{P}[\![\sigma]\!]$ such that $\forall i \in \mathbb{N}\ M\ \mathcal{R}^\sigma\ u(i)$, then $M\ \mathcal{R}^\sigma\ \sup_{i\in\mathbb{N}} u(i)$.*

Proof. Simple induction on types, using Proposition 2.9. □

Lemma 3.2. *Assume that $\vdash M : \iota$, $\vdash P : \sigma$ and $z : \iota \vdash Q : \sigma$, where $\sigma = \sigma_1 \Rightarrow \cdots \sigma_k \Rightarrow \iota$. Let $N_1, \ldots, N_k$ be terms such that $\vdash N_i : \sigma_i$ for $i = 1, \ldots, k$.*

Then, for any $n \in \mathbb{N}$, we have

$$\mathsf{Red}(\iota)^\infty_{(\mathsf{if}(M,P,z\cdot Q))N_1\cdots N_k,\underline{n}} = \mathsf{Red}(\iota)^\infty_{M,\underline{0}}\mathsf{Red}(\iota)^\infty_{(P)N_1\cdots N_k,\underline{n}} + \sum_{k\in\mathbb{N}} \mathsf{Red}(\iota)^\infty_{M,\underline{k+1}}\mathsf{Red}(\iota)^\infty_{(Q[\underline{k}/z])N_1\cdots N_k,\underline{n}}.$$

This is a straightforward consequence of the definition of weak-reduction and of Lemma 1.3.

Lemma 3.3. *Let σ be a type. Let $M, M' \in \Lambda_0^\sigma$ and let $u \in \mathsf{P}[\![\sigma]\!]$. Then*

$$M'\ \mathcal{R}^\sigma\ u \Rightarrow M\ \mathcal{R}^\sigma\ \mathsf{Red}(\sigma)_{M,M'}u.$$

Proof. By induction on σ. Assume first that $\sigma = \iota$.

Assume that $M'\ \mathcal{R}^\iota\ u$. This means that, for all $n \in \mathbb{N}$, one has $u_n \leq \mathsf{Red}(\iota)^\infty_{M',\underline{n}}$. Let $n \in \mathbb{N}$; we want to prove that

$$\mathsf{Red}(\iota)^\infty_{M,\underline{n}} \geq \mathsf{Red}(\iota)_{M,M'}u_n.$$

This results from the fact that $\mathsf{Red}(\iota)^\infty_{M,\underline{n}} = \sum_{M''\in\Lambda_0^\iota} \mathsf{Red}(\iota)_{M,M''}\mathsf{Red}(\iota)^\infty_{M'',\underline{n}}$ and from our hypothesis about M'.

Assume now that $\sigma = \tau \Rightarrow \varphi$, and let $f \in \mathsf{P}[\![\sigma]\!]$. Assume that $M'\ \mathcal{R}^{\tau\Rightarrow\varphi}\ f$; we want to prove that

$$M\ \mathcal{R}^{\tau\Rightarrow\varphi}\ \mathsf{Red}(\tau \Rightarrow \varphi)_{M,M'}f.$$

If M is weak-normal, then either $M' = M$ and then $\mathsf{Red}(\tau \Rightarrow \varphi)_{M,M'} = 1$, and we can directly apply our hypothesis that $M'\ \mathcal{R}^{\tau\Rightarrow\varphi}\ f$, or $M' \neq M$ and then $\mathsf{Red}(\tau \Rightarrow \varphi)_{M,M'} = 0$, and we can apply Lemma 3.1. So assume that M is not weak-normal.

Let $P \in \Lambda_0^\tau$ and $u \in \mathsf{P}[\![\tau]\!]$ be such that $P\ \mathcal{R}^\tau\ u$. We need to prove that

$$(M)\,P\ \mathcal{R}^\varphi\ \mathsf{Red}(\varphi)_{M,M'}f(u).$$

This results from the inductive hypothesis and from the fact that, due to our definition of weak-reduction, it holds that $\mathsf{Red}(\tau \Rightarrow \varphi)_{M,M'} = \mathsf{Red}(\varphi)_{(M)P,(M')P}$, because M is not weak-normal. □

Remark: From now on, and for the purpose of avoiding too-heavy notations, we often consider implicitly morphisms of $\mathbf{Pcoh}_!$ as functions. Typically, if $f \in \mathbf{Pcoh}_!(X, Y)$ and $u \in \mathrm{P}X$, then we write as above $f(u)$ instead of $\widehat{f}(u)$.

Theorem 3.4. *Assume that* $\Gamma \vdash M : \sigma$, *where* $\Gamma = (x_1 : \sigma_1, \ldots, x_l : \sigma_l)$. *For all families* $(P_i)_{i=1}^l$ *and* $(u_i)_{i=1}^l$, *one has*

$$(\forall i\ P_i\ \mathcal{R}^{\sigma_i}\ u_i) \Rightarrow M\,[P_1/x_1, \ldots, P_l/x_l]\ \mathcal{R}^{\sigma}\ [\![M]\!]_\Gamma(u_1, \ldots, u_l).$$

Proof. By induction on the derivation of $\Gamma \vdash M : \sigma$ (that is, on M).

The cases $M = x_i$ and $M = \underline{n}$ are straightforward.

Assume that $M = \mathrm{coin}(p)$, where $p \in [0,1] \cap \mathbb{Q}$. Then $\sigma = \iota$ and $[\![M]\!]_\Gamma(u_1, \ldots, u_l) = p\overline{0} + (1-p)\overline{1}$. On the other hand,

$$\mathrm{Red}(\iota)_{M[P_1/x_1,\ldots,P_l/x_l],\underline{n}} = \begin{cases} p & \text{if } n = 0 \\ 1-p & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases},$$

and hence $(\forall i\ P_i\ \mathcal{R}^{\sigma_i}\ u_i) \Rightarrow M\,[P_1/x_1, \ldots, P_l/x_l]\ \mathcal{R}^{\sigma}\ [\![M]\!]_\Gamma(u_1, \ldots, u_l)$ by definition of $\mathcal{R}^\iota$.

Assume that $M = \mathrm{succ}(N)$. Assume that $\forall i\ P_i\ \mathcal{R}^{\sigma_i}\ u_i$. By inductive hypothesis, we have $N\,[P_1/x_1, \ldots, P_l/x_l]\ \mathcal{R}^{\iota}\ [\![N]\!]_\Gamma(u_1, \ldots, u_l)$. This means that, for all $n \in \mathbb{N}$, one has

$$[\![N]\!]_\Gamma(u_1, \ldots, u_l)_n \leq \mathrm{Red}(\iota)^\infty_{N[P_1/x_1,\ldots,P_l/x_l],\underline{n}}.$$

It follows that, for all $n \in \mathbb{N}$,

$$[\![\mathrm{succ}(N)]\!]_\Gamma(u_1, \ldots, u_l)_{n+1} \leq \mathrm{Red}(\iota)^\infty_{\mathrm{succ}(N)[P_1/x_1,\ldots,P_l/x_l],\underline{n+1}},$$

that is,

$$\forall n \in \mathbb{N} \quad [\![\mathrm{succ}(N)]\!]_\Gamma(u_1, \ldots, u_l)_n \leq \mathrm{Red}(\iota)^\infty_{\mathrm{succ}(N)[P_1/x_1,\ldots,P_l/x_l],\underline{n}},$$

since the inequality is obvious for $n = 0$.

Assume that $M = \mathrm{if}(P, Q, z \cdot R)$ with $\Gamma \vdash P : \iota$, $\Gamma \vdash Q : \sigma$ and $\Gamma, z : \iota \vdash R : \sigma$ with $\sigma = \tau_1 \Rightarrow \cdots \tau_h \Rightarrow \iota$. Assume that $\forall i\ P_i\ \mathcal{R}^{\sigma_i}\ u_i$. By inductive hypothesis, applying the definition of $\mathrm{Red}(\iota)$, we get

$$\forall n \in \mathbb{N} \quad \mathrm{Red}(\iota)^\infty_{P[P_1/x_1,\ldots,P_l/x_l],\underline{n}} \geq [\![P]\!]_\Gamma(u_1, \ldots, u_l)_n, \tag{3}$$

$$Q\,[P_1/x_1, \ldots, P_l/x_l]\ \mathcal{R}^{\sigma}\ [\![Q]\!]_\Gamma(u_1, \ldots, u_l), \tag{4}$$

$$\forall n \in \mathbb{N} \quad R\left[P_1/x_1, \ldots, P_l/x_l, \underline{n}/z\right]\ \mathcal{R}^{\sigma}\ [\![R]\!]_{\Gamma, z:\iota}(u_1, \ldots, u_l, \overline{n}). \tag{5}$$

Observe that, in the last equation, we use the inductive hypothesis with $l+1$ parameters, and we use the fact that, obviously, $\underline{k}\ \mathcal{R}^\iota\ \overline{k}$. On the other hand, we have

$$[\![M]\!]_\Gamma(u_1, \ldots, u_l) = [\![P]\!]_\Gamma(u_1, \ldots, u_l)_0 [\![Q]\!]_\Gamma(u_1, \ldots, u_l) + \sum_{k=0}^{\infty} [\![P]\!]_\Gamma(u_1, \ldots, u_l)_{k+1} [\![R]\!]_{\Gamma, z:\iota}(u_1, \ldots, u_l, \overline{k})$$

and we must prove that $M\,[P_1/x_1, \ldots, P_l/x_l]\ \mathcal{R}^{\sigma}\ [\![M]\!]_\Gamma(u_1, \ldots, u_l)$. So, for $j = 1, \ldots, h$, let R_j and v_j be such that $\vdash R_j : \tau_j$, $v_j \in \mathrm{P}[\![\tau_j]\!]$ and $R_j\ \mathcal{R}^{\tau_j}\ v_j$. We must prove that $(M[P_1/x_1, \ldots, P_l/x_l])$

$R_1 \cdots R_h\ \mathcal{R}^\iota\ [\![M]\!]_\Gamma(u_1,\dots,u_l)(v_1)\cdots(v_h)$. Let $n \in \mathbb{N}$. By Lemma 3.2, we have

$$\begin{aligned}
&\mathsf{Red}(\iota)^\infty_{(M[P_1/x_1,\dots,P_l/x_l])R_1\cdots R_h,\underline{n}}\\
&\quad = \mathsf{Red}(\iota)^\infty_{P[P_1/x_1,\dots,P_l/x_l],\underline{0}}\mathsf{Red}(\iota)^\infty_{(Q[P_1/x_1,\dots,P_l/x_l])R_1\cdots R_h,\underline{n}}\\
&\qquad + \sum_{k=0}^{\infty}\mathsf{Red}(\iota)^\infty_{P[P_1/x_1,\dots,P_l/x_l],\underline{k+1}}\mathsf{Red}(\iota)^\infty_{(R[P_1/x_1,\dots,P_l/x_l,\underline{k}/z])R_1\cdots R_h,\underline{n}}.
\end{aligned}$$

By Equations (3), (4), and (5), and by definition of $\mathcal{R}^\sigma$, we have therefore

$$\begin{aligned}
&\mathsf{Red}(\iota)^\infty_{(M[P_1/x_1,\dots,P_l/x_l])R_1\cdots R_h,\underline{n}}\\
&\quad \geq [\![P]\!]_\Gamma(u_1,\dots,u_l)_0[\![Q]\!]_\Gamma(u_1,\dots,u_l)(v_1)\cdots(v_h)_{\underline{n}}\\
&\qquad + \sum_{k=0}^{\infty}[\![P]\!]_\Gamma(u_1,\dots,u_l)_{\underline{k+1}}[\![R]\!]_\Gamma(u_1,\dots,u_l,\underline{k}/z)(v_1)\cdots(v_h)_{\underline{n}}\\
&\quad = [\![M]\!]_\Gamma(u_1,\dots,u_l)(v_1)\cdots(v_h)_{\underline{n}},
\end{aligned}$$

that is, $(M\,[P_1/x_1,\dots,P_l/x_l])\ R_1\cdots R_h\ \mathcal{R}^\iota\ [\![M]\!]_\Gamma(u_1,\dots,u_l)(v_1)\cdots(v_h)$, as contended.

Assume that $M = (P)\,Q$ with $\Gamma \vdash P : \tau \Rightarrow \sigma$ and $\Gamma \vdash Q : \tau$. Let $t = [\![P]\!]_\Gamma(u_1,\dots,u_l)$. Assume that $\forall i\ P_i\ \mathcal{R}^{\sigma_i}\ u_i$. By inductive hypothesis, we have

$$P\,[P_1/x_1,\dots,P_l/x_l]\ \mathcal{R}^{\tau\Rightarrow\sigma}\ t$$

and $Q\,[P_1/x_1,\dots,P_l/x_l]\ \mathcal{R}^\tau\ [\![Q]\!]_\Gamma(u_1,\dots,u_l)$. Hence, we have

$$((P)\,Q)\,[P_1/x_1,\dots,P_l/x_l]\ \mathcal{R}^\tau\ \widehat{t}([\![Q]\!]_\Gamma(u_1,\dots,u_l)),$$

which is the required property since $\widehat{t}([\![Q]\!]_\Gamma(u_1,\dots,u_l)) = [\![(P)\,Q]\!]_\Gamma(u_1,\dots,u_l)$ by definition of the interpretation of terms.

Assume that $\sigma = (\tau \Rightarrow \varphi)$, $M = \lambda x^\tau\, P$ with $\Gamma, x : \tau \vdash P : \varphi$. Let $t = [\![\lambda x^\tau\, P]\!]_\Gamma(u_1,\dots,u_l)$. Assume also that $\forall i\ P_i\ \mathcal{R}^{\sigma_i}\ u_i$. We must prove that

$$\lambda x^\tau\ (P\,[P_1/x_1,\dots,P_l/x_l])\ \mathcal{R}^{\tau\Rightarrow\varphi}\ t.$$

To this end, let Q be such that $\vdash Q : \tau$ and $v \in \mathsf{P}[\![\tau]\!]$ be such that $Q\ \mathcal{R}^\tau\ v$, and we have to make sure that

$$(\lambda x^\tau\ (P\,[P_1/x_1,\dots,P_l/x_l]))\,Q\ \mathcal{R}^\varphi\ \widehat{t}(v).$$

By Lemma 3.3, it suffices to prove that $P\,[P_1/x_1,\dots,P_l/x_l,Q/x]\ \mathcal{R}^\varphi\ \widehat{t}(v)$. This results from the inductive hypothesis, since we have $\widehat{t}(v) = [\![P]\!]_{\Gamma,x:\tau}(u_1,\dots,u_n,v)$ by Cartesian closeness.

Last assume that $M = \mathsf{fix}(P)$ with $\Gamma \vdash P : \sigma \Rightarrow \sigma$. Assume also that $\forall i\ P_i\ \mathcal{R}^{\sigma_i}\ u_i$. We must prove that

$$\mathsf{fix}(P\,[P_1/x_1,\dots,P_l/x_l])\ \mathcal{R}^\sigma\ [\![\mathsf{fix}(P)]\!]_\Gamma(u_1,\dots,u_l) = \sup_{k=0}^{\infty}\widehat{t}^k(0),$$

where $t = [\![P]\!]_\Gamma(u_1,\dots,u_l) \in \mathsf{P}([\![\sigma]\!] \Rightarrow [\![\sigma]\!])$. By Lemma 3.1, it suffices to prove that

$$\forall k \in \mathbb{N} \quad \mathsf{fix}(P\,[P_1/x_1,\dots,P_l/x_l])\ \mathcal{R}^\sigma\ \widehat{t}^k(0),$$

and we proceed by induction on k. The base case $k = 0$ results from Lemma 3.1. Assume now that $\mathsf{fix}(P\,[P_1/x_1,\dots,P_l/x_l])\ \mathcal{R}^\sigma\ \widehat{t}^k(0)$, and let us prove that

$$\mathsf{fix}(P\,[P_1/x_1,\dots,P_l/x_l])\ \mathcal{R}^\sigma\ \widehat{t}^{k+1}(0).$$

By Lemma 3.3, it suffices to prove that

$$(P\,[P_1/x_1,\dots,P_l/x_l])\,\mathsf{fix}(P\,[P_1/x_1,\dots,P_l/x_l])\ \mathcal{R}^\sigma\ \widehat{t}^{k+1}(0) = \widehat{t}(\widehat{t}^k(0)),$$

which results from the "internal" inductive hypothesis

$$\mathsf{fix}(P\,[P_1/x_1,\ldots,P_l/x_l])\ \mathcal{R}^{\sigma}\ \widehat{t}^{k}(0)$$

and from the "external" inductive hypothesis

$$P\,[P_1/x_1,\ldots,P_l/x_l]\ \mathcal{R}^{\sigma\Rightarrow\sigma}\ t. \qquad \square$$

In particular, if $\vdash M:\iota$, then we have $\forall n\in\mathbb{N}\ \mathsf{Red}(\iota)^{\infty}_{M,\underline{n}}\geq([\![M]\!])_n$. By Theorem 2.14, we have therefore the following operational interpretation of the semantics of ground type closed terms.

Theorem 3.5. *If $\vdash M:\iota$, then, for all $n\in\mathbb{N}$, we have $\forall n\in\mathbb{N}\ \mathsf{Red}(\iota)^{\infty}_{M,\underline{n}}=([\![M]\!])_n$.*

As usual, the Adequacy Theorem follows straightforwardly. The observational equivalence relation on terms is defined in Section 1.2.

Lemma 3.6. *Given an observation context $C^{\Gamma\vdash\iota}$, there is a function f_C such that, for any term $M\in\Lambda^{\sigma}_{\Gamma}$, one has $[\![C[M]]\!]_{=}f_C([\![M]\!]_\Gamma)$.*

The proof is a simple induction on C.

Theorem 3.7 (Adequacy). *Let $M,M'\in\Lambda^{\sigma}_{\Gamma}$ be terms of* pPCF. *If $[\![M]\!]_\Gamma=[\![M']\!]_\Gamma$, then $M\sim M'$.*

Proof. Assume that $[\![M]\!]_\Gamma=[\![M']\!]_\Gamma$. Let $C^{\Gamma\vdash\sigma}$ be an observation context such that $\vdash C^{\Gamma\vdash\sigma}:\iota$, and we have

$$\begin{aligned}\mathsf{Red}(\iota)^{\infty}_{C[M],\underline{0}}&=[\![C[M]]\!]_0 \quad \text{by Theorem 3.5}\\ &=f_C([\![M]\!]_\Gamma)_0 \quad \text{by Lemma 3.6}\\ &=f_C([\![M']\!]_\Gamma)_0\\ &=\mathsf{Red}(\iota)^{\infty}_{C[M'],\underline{0}}.\end{aligned}$$ $\square$

4 FULL ABSTRACTION

We want now to prove the converse of Theorem 3.7; that is, given two terms M and M' such that $\Gamma\vdash M:\sigma$ and $\Gamma\vdash M':\sigma$, if $M\sim M'$, then $[\![M]\!]_\Gamma=[\![M']\!]_\Gamma$. This means that **Pcoh** provides an equationally fully abstract model of pPCF.

4.1 Intuition

Let us first convey some intuitions about our approach to Full Abstraction. The first thing to say is that the usual method, which consists in proving that the model contains a collection of definable elements that is "dense" in a topological sense, does not apply here, because definable elements are very sparse in **Pcoh**. For instance, in $\mathsf{P}[\![\iota\Rightarrow\iota]\!]$, there is an element t that is characterized by $\widehat{t}(u)=4u_0u_1\overline{0}$. We have $t\in\mathsf{P}[\![\iota\Rightarrow\iota]\!]$, because, for any $u\in\mathsf{PN}$, we have $u_0+u_1\leq 1$ and hence $u_0u_1\leq u_0(1-u_0)\leq 1/4$, and therefore $\widehat{t}(u)\in[0,1]$. It can be seen that t is not definable in pPCF[13].

The "best" definable approximation of t is obtained by means of the term

$$\lambda x^{\iota}\,\mathsf{if}(x,\mathsf{if}(x,\Omega^{\iota},z\cdot\mathsf{if}(z,\underline{0},z'\cdot\Omega^{\iota})),z'\cdot\mathsf{if}(x,\underline{0},z'\cdot\Omega^{\iota})),$$

whose interpretation s satisfies $\widehat{s}(u)=2u_0u_1\overline{0}$.

Let M and M' be terms (that we suppose closed for simplifying and without loss of generality) such that $\vdash M:\sigma$ and $\vdash M':\sigma$. Assume that $[\![M]\!]\neq[\![M']\!]$; we have to prove that $M\nsim M'$. Let

[13]Very roughly, for proving this, one observes that the normal forms (or, more precisely, the Böhm trees) of closed terms of this type can be described as barycentric linear combinations of binary decision trees and by computing the semantics of these trees.

$a \in |[\![\sigma]\!]|$ be such that $[\![M]\!]_a \neq [\![M']\!]_a$. We define a term F such that $\vdash F : \sigma \Rightarrow \iota$ and $[\![(F)\,M]\!]_0 \neq [\![(F)\,M']\!]_0$. Then we use the observation context $C = (F)\,[\ ]^{\vdash\sigma}$ to separate M and M'. For defining F, *independently of M and M'*, we associate with a a closed term a^- such that $\vdash a^- : \iota \Rightarrow \sigma \Rightarrow \iota$ and that has the following essential property:

> There is an $n \in \mathbb{N}$ – depending only on a – such that, given $w, w' \in \mathrm{P}[\![\sigma]\!]$ such that $w_a \neq w'_a$, there are rational numbers $p_0, \dots, p_{n-1} \in [0,1]$ such that $\widehat{[\![a^-]\!]}(u)(w)_0 \neq \widehat{[\![a^-]\!]}(u)(w')_0$, where $u = p_0\overline{0} + \cdots + p_{n-1}\overline{n-1}$.

Applying this property to $w = [\![M]\!]$ and $w' = [\![M']\!]$, we obtain the required term F by setting $F = (a^-)\,\mathrm{ran}(p_0, \dots, p_{n-1})$.

To prove this crucial property of a^-, we consider the map $\varphi_w : u \mapsto \widehat{[\![a^-]\!]}(u)(w)_0$ that is an analytic function depending only on the n first components $u_0, \dots, u_{n-1}$ of $u \in \mathrm{PN}$ (again, n is a non-negative integer that depends only on a).

> In Lemma 4.2, we prove that the coefficient in φ_w of the particular monomial $u_0 u_1 \dots u_{n-1}$ is w_a.

It follows that the functions φ_w and $\varphi_{w'}$ are different and therefore take different values on an argument of shape $p_0\overline{0} + \cdots + p_{n-1}\overline{n-1}$, where all p_is are rational, because φ_w and $\varphi_{w'}$ are continuous functions.

4.2 Useful Notions and Constructs

We introduce some elementary material used in the proof.

- First, for a morphism $t \in \mathbf{Pcoh}_!(\mathrm{N}, X)$, we explain what it means to depend on finitely many parameters, considering t as a function from a subset of $(\mathbb{R}_{\geq 0})^{\mathbb{N}}$ to $\mathrm{P}X$.
- Then we give the construction of the term a^- (testing term) and of the auxiliary term a^+. The interpretations of these terms are morphisms depending on a finite number of parameters; we define explicitly $|a|^-, |a|^+ \in \mathbb{N}$, which are the number of relevant parameters. We also give the interpretation of these morphisms as functions in the category $\mathbf{Pcoh}_!$. The construction of these terms bears some similarities with the "realizers" and "consumers" constructed in the proof of Full Abstraction of Nygaard and Winskel (2003).
- We introduce next useful notations that will be used in the proof of the main lemma.

Morphisms Depending on a Finite Number of Parameters. Let $k \in \mathbb{N}$. Let $\Delta_k = \overbrace{1 \oplus \cdots \oplus 1}^{k}$ so that $|\Delta_k| = \{0, \dots, k-1\}$ and $\mathrm{P}\Delta_k = \{x \in (\mathbb{R}_{\geq 0})^k \mid x_0 + \cdots + x_{k-1} \leq 1\}$. We have two morphisms $\eta^+(k) \in \mathbf{Pcoh}_!(\Delta_k, \mathbf{N})$ and $\eta^-(k) \in \mathbf{Pcoh}_!(\mathbf{N}, \Delta_k)$ defined by

$$\eta^+(k)_{m,j} = \eta^-(k)_{m,j} = \begin{cases} 1 & \text{if } m = [j] \text{ and } j < k \\ 0 & \text{otherwise} \end{cases}$$

for $j \in \mathbb{N}$ and $m \in \mathcal{M}_{\mathrm{fin}}(\mathbb{N})$ (both vectors $\eta^+(k)$ and $\eta^-(k)$ are indexed by pairs (m, j)).

Given $t \in \mathbf{Pcoh}_!(\mathbf{N}, X)$, the morphism $s = t \circ \eta^+(k) \circ \eta^-(k) \in \mathbf{Pcoh}_!(\mathbf{N}, X)$ satisfies

$$s(u) = t(u_0, \dots, u_{k-1}, 0, 0, \dots)$$

if we consider PN as a subset of $(\mathbb{R}_{\geq 0})^{\mathbb{N}}$. We say that t *depends on at most k parameters* if $t = t \circ \eta^+(k) \circ \eta^-(k)$, which simply means that, for any $(m, a) \in |!\mathbf{N} \multimap X| = \mathcal{M}_{\mathrm{fin}}(\mathbb{N}) \times |X|$, if $t_{m,a} \neq 0$ then $m \in \mathcal{M}_{\mathrm{fin}}(\{0, \dots, k-1\})$.

If $t \in \mathbf{Pcoh}_!(\mathbf{N}, X)$, then t is considered here as a function with infinitely many real parameters. Given $k \in \mathbb{N}$ and $u \in (\mathbb{R}_{\geq 0})^{\mathbb{N}}$, we define $u\{k\} \in (\mathbb{R}_{\geq 0})^{\mathbb{N}}$ by $u\{k\}_i = u_{i+k}$. Observe that $s = t \circ [\![\mathsf{shift}_k]\!]$ is characterized by $s(u) = t(u\{k\})$.

The term shift_k, as well as the other terms used below, is defined in Section 1.3.

Testing Term Associated with a Point of the Web. Given a type σ and an element a of $|[\![\sigma]\!]|$, we define two pPCF closed terms a^+ and a^- such that

$$\vdash a^+ : \iota \Rightarrow \sigma \quad \text{and} \quad \vdash a^- : \iota \Rightarrow \sigma \Rightarrow \iota.$$

The definition is by mutual induction on σ. We first associate with a two natural numbers $|a|^+$ and $|a|^-$.

If $\sigma = \iota$, and hence $a = n \in \mathbb{N}$, then we set $|a|^+ = |a|^- = 0$.

If $\sigma = (\varphi \Rightarrow \psi)$ so that $a = ([b_1, \dots, b_k], c)$ with $b_i \in |[\![\varphi]\!]|$ for each $i = 1, \dots, k$ and $c \in |[\![\psi]\!]|$, then we set

$$|a|^+ = |c|^+ + \sum_{i=1}^{k} |b_i|^-$$

$$|a|^- = |c|^- + k + \sum_{i=1}^{k} |b_i|^+.$$

We give now the definitions of the terms a^+ and a^-. These definitions use the following terms, which are defined in Section 1.3 and their semantics is described in Section 2.12.

- prob_n (for $n \in \mathbb{N}$) such that $\vdash \mathsf{prob}_n : \iota \Rightarrow \iota$, which returns $\underline{0}$ with a probability equal to the probability of its argument to reduce to $\underline{n}$ or diverges.
- prod_k (for $k \in \mathbb{N}$) such that $\vdash \mathsf{prod}_k : \iota^k \Rightarrow \iota$ and that returns $\underline{0}$ with a probability equal to the product of the probabilities of its k arguments to reduce to $\underline{0}$, or diverges.
- shift_k (for $n \in \mathbb{N}$) such that $\vdash \mathsf{shift}_k : \iota \Rightarrow \iota$, which is addition with the constant $\underline{k}$.
- choose_k such that $\vdash \mathsf{choose}_k : \iota \Rightarrow \sigma^k \Rightarrow \sigma$, which takes an argument M of type ι, considered as a sub-probability distribution on the natural numbers, k terms $N_1, \dots, N_k$ of type σ and returns N_i with the probability that M reduces to $\underline{i}$.

Assume that $\sigma = \iota$, then $a = n$ for some $n \in \mathbb{N}$. We set

$$a^+ = n^+ = \lambda\xi^\iota\, \underline{n} \quad \text{and} \quad a^- = n^- = \lambda\xi^\iota\, \mathsf{prob}_n$$

so that $[\![n^+]\!](u) = n$ and $[\![n^-]\!](u)(w) = w_n\overline{0}$. The term n^+ takes a sub-probability distribution ξ (on $\mathbb{N}$) and returns $\underline{n}$. The term n^- takes a sub-probability distribution ξ and a term M such that $\vdash M : \iota$ and returns $\underline{0}$ with the probability that M reduces to $\underline{n}$ and diverges otherwise.

Assume that $\sigma = (\varphi \Rightarrow \psi)$ so that $a = ([b_1, \dots, b_k], c)$ with $b_i \in |[\![\varphi]\!]|$ for each $i = 1, \dots, k$ and $c \in |[\![\psi]\!]|$. Then we define a^+ such that $\vdash a^+ : \iota \Rightarrow \varphi \Rightarrow \psi$ by

$$\begin{aligned} a^+ = \lambda\xi^\iota\, \lambda x^\varphi\, \mathsf{if}(\ & (\mathsf{prod}_k) \\ & (b_1^-)\ \xi\ x \\ & (b_2^-)\ (\mathsf{shift}_{|b_1|^-})\ \xi\ x \\ & \dots \\ & (b_k^-)\ \left(\mathsf{shift}_{|b_1|^- + \dots + |b_{k-1}|^-}\right)\ \xi\ x, \\ & (c^+)\ \left(\mathsf{shift}_{|b_1|^- + \dots + |b_k|^-}\right)\ \xi, \\ & z.\omega\psi. \end{aligned}$$

Therefore, we have, given $u \in \mathsf{PN}$ and $w \in \mathsf{P}[\![\varphi]\!]$,

$$[\![a^+]\!](u)(w) = \left(\prod_{i=1}^{k} [\![{b_i}^-]\!]\left(u\left\{\sum_{j=1}^{i-1}|b_j|^-\right\}\right)(w)\right)_0 [\![c^+]\!]\left(u\left\{\sum_{j=1}^{k}|b_j|^-\right\}\right).$$

Intuitively, given a sub-probability distribution ξ and a term M such that $\vdash M : \varphi$, the closed term $(a^+)\,\xi\,M$ of type ψ splits ξ into $k+1$ pieces $\xi(1),\dots,\xi(k),\xi(k+1)$ using the fact that the terms ${b_i}^-$ and c^+ depend only on a known finite number of parameters and returns $(c^+)\,\xi(k+1)$ with a probability equal to the product of the probabilities that the $({b_i}^-)\,\xi(i)$ M's converge and diverges otherwise.

The term a^- is such that $\vdash a^- : \iota \Rightarrow (\varphi \Rightarrow \psi) \Rightarrow \iota$ and is defined by

$$\begin{aligned} a^- = \lambda\xi^{\iota}\,\lambda f^{\varphi\Rightarrow\psi}\ & (c^-)\left(\mathsf{shift}_{k+|b_1|^+ +\cdots+|b_k|^+}\right)\xi \\ & (f)\,(\mathsf{choose}_k)\,\xi \\ & \qquad ({b_1}^+)\,(\mathsf{shift}_k)\,\xi \\ & \qquad \cdots \\ & \qquad ({b_k}^+)\left(\mathsf{shift}_{k+|b_1|^+ +\cdots+|b_{k-1}|^+}\right)\xi. \end{aligned}$$

Therefore, we have, given $u \in \mathsf{PN}$ and $t \in \mathsf{P}(\varphi \Rightarrow \psi)$,

$$[\![a^-]\!](u)(t) = [\![c^-]\!]\left(u\left\{k+\sum_{j=1}^{k}|b_j|^+\right\}\right)\left(t\left(\sum_{i=1}^{k}u_{i-1}[\![{b_i}^+]\!]\left(u\left\{k+\sum_{j=1}^{i-1}|b_j|^+\right\}\right)\right)\right).$$

Intuitively, given a sub-probability distribution ξ and a term M such that $\vdash M : \varphi \Rightarrow \psi$, the closed term $(a^-)\,\xi\,M$ of type ι splits ξ in $k+2$ pieces $\xi(0),\xi(1),\dots,\xi(k),\xi(k+1)$, the first one being of length k (again we use the fact that the terms ${b_i}^-$ and c^+ depend only on a known finite number of parameters). Then it applies M to a closed term of type φ that chooses with probabilities $\xi(0)_1,\dots,\xi(0)_k$ among $({b_1}^+)\,\xi(1),\dots,({b_k}^+)\,\xi(k)$ that are closed terms of type φ. Then it applies $(c^-)\,\xi(k+1)$ (a closed term of type $\psi \Rightarrow \iota$) to the resulting compound term. It is only this construction that creates a dependency with respect to the sub-probability distribution parameter ξ and allows us to observe M by testing its behavior when these probabilities are modified. The fact that this dependency is analytic is the basic ingredient of the proof.

Lemma 4.1. *Let σ be a type and $a \in |[\![\sigma]\!]|$. Seen as an element of $\mathbf{Pcoh}_!(\mathbf{N},[\![\sigma]\!])$ (respectively, of $\mathbf{Pcoh}_!(\mathbf{N},[\![\sigma \Rightarrow \iota]\!])$), $[\![a^+]\!]$ (respectively, $[\![a^-]\!]$) depends on at most $|a|^+$ (respectively, $|a|^-$) parameters.*

The proof is a simple induction on σ, based on an inspection of the expressions above for $[\![a^+]\!]$ and $[\![a^-]\!]$.

More Notations. Let $I = \{n_1 < \cdots < n_k\}$ be a finite subset of $\mathbb{N}$, and we use $\mathsf{o}(I)$ for the multiset $[n_1,\dots,n_k]$, where each element of I appears exactly once. Given $p,q \in \mathbb{N}$, we set

$$\begin{aligned} \mathsf{o}(p,q) &= \mathsf{o}(\{p,p+1,\dots,p+q-1\}) \\ \mathcal{M}_{\mathrm{fin}}(p,q) &= \mathcal{M}_{\mathrm{fin}}(\{p,p+1,\dots,p+q-1\}). \end{aligned}$$

These specific multisets, where each elements appears exactly once, play an essential role in Lemma 4.2.

Given $m \in \mathcal{M}_{\mathrm{fin}}(\mathbb{N})$ and p and q as above, we use the notation $\mathrm{W}(m,p,q)$ for the element m' of $\mathcal{M}_{\mathrm{fin}}(\mathbb{N})$ defined by

$$m'(i) = \begin{cases} m(i+p) & \text{if } 0 \leq i \leq q-1 \\ 0 & \text{otherwise} \end{cases}$$

and the notation $\mathrm{S}(m,p)$ for the element m' of $\mathcal{M}_{\mathrm{fin}}(\mathbb{N})$ defined by $m'(i) = m(i+p)$ for each $i \in \mathbb{N}$.

So $\mathrm{W}(m,p,q)$ is obtained by selecting in m a "window" starting at index p and ending at index $p+q-1$ and by shifting this window by p to the left. Similarly, $\mathrm{S}(m,p)$ is obtained by shifting m by p to the left.

Given a set I and an element i of I, we use e_i for the element of $(\mathbb{R}_{\geq 0})^I$ defined by $(\mathrm{e}_i)_j = \delta_{i,j}$.

Expression of the Semantics of Testing Terms. We write now the functions $[\![a^-]\!]$ and $[\![a^+]\!]$ in a form that makes explicit their dependency on their first argument $u \in \mathrm{PN}$. This also allows us to make explicit their dependency on a finite number of parameters.

Let σ be a type and let $a \in |[\![\sigma]\!]|$. By Lemma 4.1, for each $m \in \mathcal{M}_{\mathrm{fin}}(0, |a|^+)$, there are uniquely defined $\pi(a,m) \in (\mathbb{R}_{\geq 0})^{|[\![\sigma]\!]|}$ and $\mu(a,m) \in (\mathbb{R}_{\geq 0})^{|([\![\sigma]\!] \Rightarrow \perp)|}$ such that we can write

$$[\![a^+]\!](u) = \sum_{m \in \mathcal{M}_{\mathrm{fin}}(0,|a|^+)} u^m \pi(a,m) \tag{6}$$

for all $u \in \mathrm{PN}$ and, for each $w \in \mathrm{P}[\![\sigma]\!]$,

$$[\![a^-]\!](u)(w)_0 = \sum_{m \in \mathcal{M}_{\mathrm{fin}}(0,|a|^-)} u^m \mu(a,m)(w) \tag{7}$$

for all $u \in \mathrm{PN}$. Here we use the dualizing object $\perp$ of **Pcoh** introduced in Section 2.6 as output type for $\mu(a,m)$; remember that $|\perp|$ is a singleton and $\mathrm{P}\perp$ is the interval $[0,1]$ up to trivial iso.

Observe that, for any $w \in \mathrm{P}[\![\sigma]\!]$, we have

$$\mu(a,m)(w) = \sum_{h \in \mathcal{M}_{\mathrm{fin}}(|[\![\sigma]\!]|)} \mu(a,m)_{(h,*)} w^h. \tag{8}$$

4.3 Proof of Full Abstraction

We can now state and prove the main lemma in the proof of full abstraction. This lemma uses notations introduced in Section 4.2.

Lemma 4.2. *Let σ be a type and let $a \in |[\![\sigma]\!]|$. We have*

$$\pi(a, \mathrm{o}(0, |a|^+)) = \mathrm{e}_a$$
$$\mu(a, \mathrm{o}(0, |a|^-)) = \mathrm{e}_{([a],*)},$$

that is, $\mu(a, \mathrm{o}(0,|a|^-))(w) = w_a$ for each $w \in \mathrm{P}[\![\sigma]\!]$.

Proof. By induction on σ. Assume that $\sigma = \iota$ so that $a = n \in \mathbb{N}$ and we have $|n|^+ = |n|^- = 0$. We have $[\![n^+]\!](u) = \mathrm{e}_n$ and $[\![n^-]\!](u)(w) = w_n$ as expected.

Assume now that $\sigma = \varphi \Rightarrow \psi$ so that a can be written

$$a = ([b_1, \ldots, b_k], c)$$

for some $b_1, \ldots, b_k \in |[\![\varphi]\!]|$ and $c \in |[\![\psi]\!]|$.

For each $u \in \mathsf{PN}$ and $w \in \mathsf{P}[\![\varphi]\!]$, we have

$$\begin{aligned}
[\![a^+]\!](u)(w) &= \prod_{i=1}^{k} \left([\![b_i{}^-]\!] \left(u \left\{ \sum_{j=1}^{i-1} |b_j|^- \right\} \right)(w) \right)_! [\![c^+]\!] \left(u \left\{ \sum_{j=1}^{k} |b_j|^- \right\} \right) \quad \text{see Section 4.2} \\
&= \prod_{i=1}^{k} \left(\sum_{m \in \mathcal{M}_{\mathrm{fin}}(\sum_{j=1}^{i-1} |b_j|^-, |b_i|^-)} u^m \mu \left(b_i, \mathrm{S} \left(m, \sum_{j=1}^{i-1} |b_j|^- \right) \right)(w) \right) \\
&\qquad \left(\sum_{m \in \mathcal{M}_{\mathrm{fin}}(\sum_{j=1}^{k} |b_j|^-, |c|^+)} u^m \pi \left(c, \mathrm{S} \left(m, \sum_{i=1}^{k} |b_i|^- \right) \right) \right) \quad \text{see Section 4.2} \\
&= \sum_{m \in \mathcal{M}_{\mathrm{fin}}(0, |a|^+)} u^m \left(\prod_{i=1}^{k} \mu \left(b_i, \mathrm{W} \left(m, \sum_{j=1}^{i-1} |b_j|^-, |b_i|^- \right) \right)(w) \right) \\
&\qquad \pi \left(c, \mathrm{W} \left(m, \sum_{j=1}^{k} |b_j|^-, |c|^+ \right) \right),
\end{aligned}$$

using the fact that $|a|^+ = \sum_{j=1}^{k} |b_i|^- + |c|^+$ and distributing products over sums. We also use the fact that there is a bijection

$$\begin{aligned}
\mathcal{M}_{\mathrm{fin}}(0, |a|^+) &\to \left(\prod_{i=1}^{k} \mathcal{M}_{\mathrm{fin}} \left(\sum_{j=1}^{i-1} |b_j|^-, |b_i|^- \right) \right) \times \mathcal{M}_{\mathrm{fin}} \left(\sum_{j=1}^{k} |b_j|^-, |c|^+ \right) \\
m &\mapsto \left(\left(\mathrm{W} \left(m, \sum_{j=1}^{i-1} |b_j|^-, |b_i|^- \right) \right)_{i=1}^{k}, \mathrm{W} \left(m, \sum_{j=1}^{k} |b_j|^-, |c|^+ \right) \right).
\end{aligned}$$

Again we refer to Section 4.2 for the notations used in these expressions.

Therefore, given $m \in \mathcal{M}_{\mathrm{fin}}(0, |a|^+)$ and $w \in \mathsf{P}[\![\varphi]\!]$, the element $\pi(a, m)(w)$ of $\mathbb{R}_{\geq 0}$ satisfies

$$\pi(a, m)(w) = \left(\prod_{i=1}^{k} \mu \left(b_i, \mathrm{W} \left(m, \sum_{j=1}^{i-1} |b_j|^-, |b_i|^- \right) \right)(w) \right) \pi \left(c, \mathrm{W} \left(m, \sum_{j=1}^{k} |b_j|^-, |c|^+ \right) \right).$$

In this expression, we take now $m = \mathrm{o}(0, |a|^+)$. Since, clearly, $\mathrm{W}(m, p, q) = \mathrm{o}(0, q)$ for all $p, q \in \mathbb{N}$ such that $p + q \leq |a|^+$, we get, by inductive hypothesis,

$$\pi(a, \mathrm{o}(0, |a|^+))(w) = \left(\prod_{i=1}^{k} w_{b_i} \right) \mathrm{e}_c$$

and hence $\pi(a, \mathrm{o}(0, |a|^+)) = \mathrm{e}_a$ as contended.

Concerning a^-, for each $u \in \mathsf{PN}$ and $t \in \mathsf{P}[\![\varphi \Rightarrow \psi]\!]$, we have

$$[\![a^-]\!](u)(t)_0 = [\![c^-]\!]\left(u\left\{k + \sum_{i=1}^{k} |b_i|^+\right\}\right)\left(t\left(\sum_{i=1}^{k} u_{i-1}[\![b_i{}^+]\!]\left(u\left\{k + \sum_{j=1}^{i-1} |b_i|^+\right\}\right)\right)\right)_0$$

see Section 4.2

$$= [\![c^-]\!]\left(u\left\{k + \sum_{i=1}^{k} |b_i|^+\right\}\right)$$
$$\left(t\left(\sum_{i=1}^{k} u_{i-1} \sum_{r \in \mathcal{M}_{\mathrm{fin}}(k+\sum_{j=1}^{i-1}|b_j|^+, |b_i|^+)} u^r \pi\left(b_i, \mathrm{S}\left(r, k + \sum_{j=1}^{i-1} |b_j|^+\right)\right)\right)\right)_0$$

see Section 4.2

$$= \sum_{\substack{l \in \mathcal{M}_{\mathrm{fin}}(k+\sum_{i=1}^{k}|b_i|^+, |c|^-) \\ h \in \mathcal{M}_{\mathrm{fin}}(|[\![\psi]\!]|)}} u^l \mu\left(c, \mathrm{S}\left(l, k + \sum_{i=1}^{k} |b_i|^+\right)\right)_{(h,*)}$$
$$\left(\sum_{(m',c') \in |[\![\varphi \Rightarrow \psi]\!]|} t_{m',c'}\left(\sum_{i=1}^{k} u_{i-1} \sum_{r \in \mathcal{M}_{\mathrm{fin}}(k+\sum_{j=1}^{i-1}|b_j|^+, |b_i|^+)} u^r \pi\left(b_i, \mathrm{S}\left(r, k + \sum_{j=1}^{i-1} |b_j|^+\right)\right)\right)^{m'} \mathrm{e}_{c'}\right)^h$$

by Section 4.2 and Equations (7) and (8) and by definition of application in $\mathbf{Pcoh}_!$

$$= \sum_{\substack{l \in \mathcal{M}_{\mathrm{fin}}(k+\sum_{i=1}^{k}|b_i|^+, |c|^-) \\ h \in \mathcal{M}_{\mathrm{fin}}(|[\![\psi]\!]|)}} u^l \mu\left(c, \mathrm{S}\left(l, k + \sum_{i=1}^{k} |b_i|^+\right)\right)_{(h,*)} \prod_{c' \in |[\![\psi]\!]|} A(c')^{h(c')}$$

where, for each $c' \in |[\![\psi]\!]|$,

$$A(c') = \sum_{m' \in \mathcal{M}_{\mathrm{fin}}(|[\![\varphi]\!]|)} t_{m',c'} \prod_{b \in |[\![\varphi]\!]|}\left(\sum_{i=1}^{k} u^{[i-1]} \sum_{r \in \mathcal{M}_{\mathrm{fin}}(k+\sum_{j=1}^{i-1}|b_j|^+, |b_i|^+)} u^r \pi\left(b_i, \mathrm{S}\left(r, k + \sum_{j=1}^{i-1} |b_j|^+\right)\right)_b\right)^{m'(b)},$$

where we recall that $[i-1]$ is the multiset that has $i-1$ as a unique element. We can write $A(c') = \sum_{r \in \mathcal{M}_{\mathrm{fin}}(\mathbb{N})} u^r B(c')_r$, where u does not occur in the expression $B(c')_r$. For any $c' \in |[\![\psi]\!]|$, all the $r \in \mathcal{M}_{\mathrm{fin}}(\mathbb{N})$ such that $B(c')_r \neq 0$ satisfy $r \in \mathcal{M}_{\mathrm{fin}}(0, k + \sum_{i=1}^{k} |b_i|^+)$: This results from a simple inspection of the exponents of u in the expression $A(c')$. It follows that, for any $h \in \mathcal{M}_{\mathrm{fin}}(|[\![\psi]\!]|)$, we can write

$$\prod_{c' \in |[\![\psi]\!]|} A(c')^{h(c')} = \sum_{r \in \mathcal{M}_{\mathrm{fin}}(0, k+\sum_{i=1}^{k}|b_i|^+)} u^r D(r)_h, \tag{9}$$

where u does not occur in the expressions $D(r)_h$. With these notations, we have, therefore,

$$\begin{aligned}[\![a^-]\!](u)(t)_0 &= \sum_{\substack{l\in\mathcal{M}_{\mathrm{fin}}(k+\sum_{i=1}^k|b_i|^+,|c|^-)\\ h\in\mathcal{M}_{\mathrm{fin}}(|[\![\psi]\!]|)}} u^l\mu\left(c,\mathrm{S}\left(l,k+\sum_{i=1}^k|b_i|^+\right)\right)_{(h,*)}\prod_{c'\in|[\![\psi]\!]|}A(c')^{h(c')}\\ &= \sum_{\substack{m\in\mathcal{M}_{\mathrm{fin}}(0,|a|^-)\\ h\in\mathcal{M}_{\mathrm{fin}}(|[\![\psi]\!]|)}} u^m\mu\left(c,\mathrm{S}\left(m,k+\sum_{i=1}^k|b_i|^+\right)\right)_h D\left(\mathrm{W}\left(m,0,k+\sum_{i=1}^k|b_i|^+\right)\right)_h.\end{aligned}$$

In the second line, the u^m results from the product of the u^l of the first line with the u^r arising from Equation (9). Remember, indeed, that $|a|^- = k+\sum_{i=1}^k|b_i|^+ + |c|^-$. We are interested in the coefficient

$$\alpha = \mu(a,\mathrm{o}(0,|a|^-))(t) \tag{10}$$

of $u^{\mathrm{o}(0,|a|^-)}$ in the sum above. We have

$$\alpha = \sum_{h\in\mathcal{M}_{\mathrm{fin}}(|[\![\psi]\!]|)}\mu\left(c,\mathrm{S}\left(\mathrm{o}(0,|a|^-),k+\sum_{i=1}^k|b_i|^+\right)\right)_h D\left(\mathrm{W}\left(\mathrm{o}(0,|a|^-),0,k+\sum_{i=1}^k|b_i|^+\right)\right)_h.$$

But $\mathrm{S}(\mathrm{o}(0,|a|^-),k+\sum_{i=1}^k|b_i|^+) = \mathrm{o}(0,|c|^-)$, and, hence, applying the inductive hypothesis to c, we get

$$\alpha = D\left(\mathrm{o}\left(0,k+\sum_{i=1}^k|b_i|^+\right)\right)_{[c]}.$$

Coming back to Equation (9), we see that α is the coefficient of $u^{\mathrm{o}(0,k+\sum_{i=1}^k|b_i|^+)}$ in $A(c)$ (indeed, in the present situation $h=[c]$ and so the product that appears on the left side of Equation (9) has only one factor, namely $A(c)$).

So we focus our attention on $A(c)$; remember that

$$A(c) = \sum_{m'\in\mathcal{M}_{\mathrm{fin}}(|[\![\varphi]\!]|)} t_{m',c}\prod_{b\in|[\![\varphi]\!]|}\left(\sum_{i=1}^k u^{[i-1]}\sum_{r\in\mathcal{M}_{\mathrm{fin}}(k+\sum_{j=1}^{i-1}|b_j|^+,|b_i|^+)} u^r\pi\left(b_i,\mathrm{S}\left(r,k+\sum_{j=1}^{i-1}|b_j|^+\right)\right)_b\right)^{m'(b)}.$$

Let

$$J = \left\{(i,r)\mid i\in\{1,\dots,k\}\text{ and }r\in\mathcal{M}_{\mathrm{fin}}\left(k+\sum_{j=1}^{i-1}|b_j|^+,|b_i|^+\right)\right\}.$$

Observe that, given $(i,r),(i',r')\in J$, either $(i,r)=(i',r')$ or $i\neq i'$ and r and r' have disjoint supports.

Given $(i,r) \in J$, we set

$$\theta(i,r) = \pi\left(b_i, \mathrm{S}\left(r, k + \sum_{j=1}^{i-1} |b_j|^+\right)\right) \tag{11}$$

so that $\theta(i,r) \in (\mathbb{R}_{\geq 0})^{|[\![\varphi]\!]|}$ for each $(i,r) \in J$. With these notations, we have

$$\begin{aligned} A(c) &= \sum_{m' \in \mathcal{M}_{\mathrm{fin}}(|[\![\varphi]\!]|)} t_{m',c} \prod_{b \in |[\![\varphi]\!]|} \left(\sum_{(i,r) \in J} u^{[i-1]+r} \theta(i,r)_b \right)^{m'(b)} \\ &= \sum_{m' \in \mathcal{M}_{\mathrm{fin}}(|[\![\varphi]\!]|)} t_{m',c} \prod_{b \in |[\![\varphi]\!]|} \left(\sum_{\substack{p \in \mathcal{M}_{\mathrm{fin}}(J) \\ \#p = m'(b)}} u^{\sigma(p)} \mathrm{mn}(p) \theta_b^p \right), \end{aligned}$$

where we recall that $\mathrm{mn}(p) = (\#p)! / \prod_{b \in |[\![\varphi]\!]|} p(b)!$ is the multinomial coefficient associated with the finite multiset p by the multinomial formula. In this expression, for each $b \in |[\![\varphi]\!]|$, θ_b is the J-indexed family of real numbers defined by $\theta_b(i,r) = \theta(i,r)_b$ and $\sigma(p) \in \mathcal{M}_{\mathrm{fin}}(\mathbb{N})$ is defined as

$$\sigma(p) = \sum_{(i,r) \in J} p(i,r) \cdot ([i-1] + r). \tag{12}$$

Distributing the product over the sum and rearranging the sums, we get

$$\begin{aligned} A(c) &= \sum_{m' \in \mathcal{M}_{\mathrm{fin}}(|[\![\varphi]\!]|)} t_{m',c} \sum_{\substack{\rho \in \mathcal{M}_{\mathrm{fin}}(J)^{|[\![\varphi]\!]|} \\ \forall b \ \#\rho(b) = m'(b)}} u^{\sum_{b \in |[\![\varphi]\!]|} \sigma(\rho(b))} \prod_{b \in |[\![\varphi]\!]|} \mathrm{mn}(\rho(b)) \theta_b^{\rho(b)} \\ &= \sum_{m \in \mathcal{M}_{\mathrm{fin}}(0, k + \sum_{i=1}^k |b_i|^+)} u^m \sum_{\substack{\rho \in \mathcal{M}_{\mathrm{fin}}(J)^{|[\![\varphi]\!]|} \\ \sum_{b \in |[\![\varphi]\!]|} \sigma(\rho(b)) = m}} t_{\rho_1,c} \prod_{b \in |[\![\varphi]\!]|} \mathrm{mn}(\rho(b)) \theta_b^{\rho(b)}, \end{aligned}$$

where $\rho_1 \in \mathcal{M}_{\mathrm{fin}}(|[\![\varphi]\!]|)$ is defined by

$$\rho_1(b) = \#\rho(b) = \sum_{(i,r) \in J} \rho(b)(i,r) \tag{13}$$

for each $\rho \in \mathcal{M}_{\mathrm{fin}}(J)^{|[\![\varphi]\!]|}$. For $m \in \mathcal{M}_{\mathrm{fin}}(0, k + \sum_{i=1}^k |b_i|^+)$, let

$$\zeta(m) = \sum_{\substack{\rho \in \mathcal{M}_{\mathrm{fin}}(J)^{|[\![\varphi]\!]|} \\ \sum_{b \in |[\![\varphi]\!]|} \sigma(\rho(b)) = m}} t_{\rho_1,c} \prod_{b \in |[\![\varphi]\!]|} \mathrm{mn}(\rho(b)) \theta_b^{\rho(b)} \tag{14}$$

be the coefficient of u^m in $A(c)$.

Since we want to compute $\alpha = \zeta(\mathrm{o}(0, k + \sum_{i=1}^k |b_i|^+))$ defined in Equation (10), we consider the particular case where $m = \mathrm{o}(0, k + \sum_{i=1}^k |b_i|^+)$. The elements ρ of $\mathcal{M}_{\mathrm{fin}}(J)^{|[\![\varphi]\!]|}$ that index the sum (14) satisfy the condition $\sum_{b \in |[\![\varphi]\!]|} \sigma(\rho(b)) = \mathrm{o}(0, k + \sum_{i=1}^k |b_i|^+)$, that is, coming back to the definition (Equation (12)) of σ,

$$\sum_{\substack{(i,r) \in J \\ b \in |[\![\varphi]\!]|}} \rho(b)(i,r) \cdot ([i-1] + r) = \mathrm{o}\left(0, k + \sum_{i=1}^k |b_i|^+\right). \tag{15}$$

Since $\mathrm{o}(0, k + \sum_{i=1}^{k} |b_i|^+) = [0, \ldots, k + \sum_{i=1}^{k} |b_i|^+ - 1]$ (see Section 4.2), condition (15) implies that, for each $i \in \{1, \ldots, k\}$, there is exactly one $b_\rho(i) \in |[\![\varphi]\!]|$ and exactly one $r_\rho(i) \in \mathcal{M}_{\mathrm{fin}}(k + \sum_{j=1}^{i-1} |b_j|^+, |b_i|^+)$ such that

$$\rho(b_\rho(i))(i, r_\rho(i)) \neq 0,$$

and we know, moreover, that $\rho(b_\rho(i))(i, r_\rho(i)) = 1$, because $i - 1$ occurs exactly once in $[i - 1] + r_\rho(i)$ (since the multisets $[i - 1]$ and $r_\rho(i)$ have disjoint supports for $i = 1, \ldots, k$). Moreover, since $r_\rho(i)$ and $r_\rho(i')$ have disjoint supports when i and i' are distinct elements of $\{1, \ldots, k\}$, we must have

$$r_\rho(i) = \mathrm{o}\left(k + \sum_{j=1}^{i-1} |b_j|^+, |b_i|^+\right) \tag{16}$$

by Equation (15) again.

From the first part of these considerations (existence and uniqueness of $b_\rho(i)$ and $r_\rho(i)$), it follows that if $b \in |[\![\varphi]\!]|$ and $(i, r) \in J$ are such that $\rho(b)(i, r) \neq 0$, then we have $b = b_\rho(i)$ and $r = r_\rho(i)$, and hence $\rho(b)(i, r) = 1$. In particular, $\mathrm{mn}(\rho(b)) = 1$ for each b. It follows that

$$\begin{aligned}\prod_{b \in |[\![\varphi]\!]|} \mathrm{mn}(\rho(b))\theta_b^{\rho(b)} &= \prod_{b \in |[\![\varphi]\!]|} \prod_{\substack{(i,r) \in J \\ b_\rho(i) = b,\ r_\rho(i) = r}} \theta(i, r)_b \\ &= \prod_{i=1}^{k} \pi(b_i, \mathrm{S}(r_\rho(i), k + \sum_{j=1}^{i-1} |b_j|^+))_{b_\rho(i)}\end{aligned}$$

coming back to the definition of θ, see Equation (11). Let H be the set of all ρ's satisfying Equation (15); we have therefore

$$\begin{aligned}\alpha = \zeta\left(\mathrm{o}\left(0, k + \sum_{i=1}^{k} |b_i|^+\right)\right) &= \sum_{\rho \in H} t_{\rho_1, c} \prod_{b \in |[\![\varphi]\!]|} \mathrm{mn}(\rho(b))\theta_b^{\rho(b)} \\ &= \sum_{\rho \in H} t_{\rho_1, c} \prod_{i=1}^{k} \pi\left(b_i, \mathrm{S}\left(r_\rho(i), k + \sum_{j=1}^{i-1} |b_j|^+\right)\right)_{b_\rho(i)} \\ &\qquad \text{by the observations above} \\ &= \sum_{\rho \in H} t_{\rho_1, c} \prod_{i=1}^{k} \pi(b_i, \mathrm{o}(0, |b_i|^+))_{b_\rho(i)} \quad \text{by Equation (16).}\end{aligned}$$

By our inductive hypothesis about $\pi(b_i, \mathrm{o}(0, |b_i|^+))$, all the terms of this sum vanish, but the one corresponding to the unique element ρ of H such that $b_\rho(i) = b_i$ for $i = 1, \ldots, k$. For this specific ρ, coming back to the definition (Equation (13)) of ρ_1, we have $\rho_1 = [b_1, \ldots, b_k]$. It follows that

$$\alpha = \zeta\left(\mathrm{o}\left(0, k + \sum_{i=1}^{k} |b_i|^+\right)\right) = t_{[b_1, \ldots, b_k], c}$$

as contended, and this ends the proof of the lemma. □

Main Statements. We first state a separation theorem that seems interesting on its own right and expresses that our testing terms a^-, when fed with suitable rational probability distributions, are able to separate any two distinct elements of the interpretation of a type.

Theorem 4.3 (Separation). *Let σ be a type and let $a \in |[\![\sigma]\!]|$. Let $w, w' \in \mathsf{P}[\![\sigma]\!]$ be such that $w_a \neq w'_a$. Let $n = |a|^-$. There is a sequence $(q_i)_{i=0}^{n-1}$ of rational numbers such that the element $u = \sum_{i=0}^{n-1} q_i \mathsf{e}_i$ of* PN *satisfies $[\![a^-]\!](u)(w) \neq [\![a^-]\!](u)(w')$.*

Proof. With the notations of the statement of the proposition, we consider the functions $\varphi, \varphi' : \mathsf{PN} \to \mathbb{R}_{\geq 0}$ defined by $\varphi(u) = [\![a^-]\!](u)(w)_0$ and $\varphi'(u) = [\![a^-]\!](u)(w')_0$. By Lemma 4.1, the morphisms φ and φ' depend on at most $n = |a|^-$ parameters. In other words, there are $t, t' \in \mathbf{Pcoh}_!(\Delta_n, \bot)$ such that

$$\forall u \in \mathsf{PN} \quad \varphi(u) = t\left(\sum_{i=0}^{n-1} u_i \mathsf{e}_i\right) \text{ and } \varphi'(u) = t'\left(\sum_{i=0}^{n-1} u_i \mathsf{e}_i\right).$$

Coming back to Equation (7), we see that the coefficient of $u^{[0,\dots,n-1]}$ in the expression of $\varphi(u)$ is $\mu(a, [0, \dots, n-1])(w)$, whose value is w_a by Lemma 4.2. In other words $t_{[0,\dots,n-1],*} = w_a$ and, similarly, $t'_{[0,\dots,n-1],*} = w'_a$. From this, it results that the functions t and t' from $\mathsf{P}\Delta_n$ to $\mathbb{R}$ are distinct (because these are analytic functions with distinct power series, which are defined on the subset $\mathsf{P}\Delta_n$ of $(\mathbb{R}_{\geq 0})^n$, which contains a non-empty subset of $\mathbb{R}^n$ that is open for the usual topology). Since these functions are continuous (again, for the usual topology), there is an $u \in \mathsf{P}\Delta_n$ such that $u_0, \dots, u_{n-1} \in \mathbb{Q}$ and $t(u) \neq t'(u)$. □

Theorem 4.4 (Full Abstraction). *Let σ be a type, Γ be a typing context and let M and M' be terms such that $\Gamma \vdash M : \sigma$ and $\Gamma \vdash M' : \sigma$. If $M \sim M'$, then $[\![M]\!]_\Gamma = [\![M']\!]_\Gamma$.*

Proof. Assume that $[\![M]\!]_\Gamma \neq [\![M']\!]_\Gamma$.

Let $(x_1 : \sigma_1, \dots, x_k : \sigma_k)$ be the typing context Γ. Let $N = \lambda x_1^{\sigma_1} \cdots \lambda x_k^{\sigma_k} M$ and $N' = \lambda x_1^{\sigma_1} \cdots \lambda x_k^{\sigma_k} M'$ be closures of M and M'. Let $\tau = \sigma_1 \Rightarrow \cdots \sigma_k \Rightarrow \sigma$.

Let $w = [\![N]\!]$ and $w' = [\![N']\!]$, and we have $w \neq w'$ so there is $a \in |[\![\tau]\!]|$ such that $w_a \neq w'_a$. By Theorem 4.3, we can find a sequence $(q_i)_{i=0}^{n-1}$ of rational numbers such that for all $i \in \{0, \dots, n-1\}$ one has $q_i \geq 0$ and $\sum_{i=0}^{n-1} q_i \leq 1$, and $u = \sum_{i=0}^{n-1} q_i \mathsf{e}_i \in \mathsf{PN}$ satisfies $[\![a^-]\!](u)(w)_0 \neq [\![a^-]\!](u)(w')_0$.

Observe that $u = [\![\mathsf{ran}(q_0, \dots, q_{n-1})]\!]$.

Let C be the following observation context:

$$C^{\Gamma \vdash \sigma} = (a^-)\, \mathsf{ran}(q_0, \dots, q_{n-1})\, \lambda x_1^{\sigma_1} \cdots \lambda x_k^{\sigma_k} [\ ]^{\Gamma \vdash \sigma}$$

that satisfies $\vdash C^{\Gamma \vdash \sigma} : \iota$, $[\![C[M]]\!] = [\![a^-]\!](u)(w)$ and $[\![C[M']]\!] = [\![a^-]\!](u)(w')$.

Applying Theorem 3.5, we get that

$$\mathsf{Red}(\iota)^\infty_{C[M],\underline{0}} \neq \mathsf{Red}(\iota)^\infty_{C[M'],\underline{0}},$$

which shows that $M \not\sim M'$. □

Failure of Inequational Full Abstraction. We can define an observational preorder on closed terms: Given terms M and M' such that $\vdash M : \sigma$ and $\vdash M' : \sigma$, let us write $M \lesssim M'$ if, for all closed C such that $\vdash C : \sigma \Rightarrow \iota$, one has $\mathsf{Red}(\iota)^\infty_{(C)M,\underline{0}} \leq \mathsf{Red}(\iota)^\infty_{(C)M',\underline{0}}$. Then it is easy to see that $[\![M]\!] \leq [\![M']\!] \Rightarrow M \lesssim M'$ (just as in the proof of Theorem 3.7).

The converse implication, however, is far from being true. A typical counter-example (which is essentially the same as the example of the Remark following Proposition 2.9) is provided by the two terms

$$M_1 = \lambda x^\iota\, \underline{0}$$
$$M_2 = \lambda x^\iota\, \mathsf{if}(x, \underline{0}, z \cdot \Omega^\iota).$$

One has $\vdash M_i : \iota \Rightarrow \iota$ for $i = 1, 2$ and the functional behavior of the interpretations of these terms is given by

$$[\![M_1]\!](u) = \overline{0}$$
$$[\![M_2]\!](u) = u_0\overline{0}$$

for all $u \in \mathsf{PN}$ so $[\![M_1]\!]$ and $[\![M_2]\!]$ are not comparable in $\mathsf{P}[\![\iota \Rightarrow \iota]\!]$, and, nevertheless, one can check that $M_2 \lesssim M_1$. The proof boils down to the observation that, for each $u \in \mathsf{PN}$, one has $[\![M_2]\!](u) \leq [\![M_1]\!](u)$.

CONCLUSION

We have studied an operationally meaningful probabilistic extension of PCF, and, in particular, we have proven a full abstraction result for the probabilistic coherence spaces model of Linear Logic, with respect to a natural notion of observational equivalence on the terms of this language.

This observational equivalence can be considered as too restrictive, however, since it is based on a strict equality of probabilities of convergence. In the present probabilistic setting, a suitable *distance* on terms could certainly be more relevant and provide more interesting information on the behavior of programs, than our observational equivalence relation. We also plan to extend our adequacy and, if possible, full abstraction results to richer type structures in a call-by-push-value flavored setting.

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