# SOME RELATIONAL STRUCTURES WITH POLYNOMIAL GROWTH AND THEIR ASSOCIATED ALGEBRAS II. FINITE GENERATION.

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ABSTRACT. The *profile* of a relational structure R is the function  $\varphi_R$  which counts for every integer n the number, possibly infinite,  $\varphi_R(n)$  of substructures of R induced on the n-element subsets, isomorphic substructures being identified. If  $\varphi_R$  takes only finite values, this is the Hilbert function of a graded algebra associated with R, the  $age\ algebra\ \mathbb{K}.\mathcal{A}(R)$ , introduced by P. J. Cameron.

In a previous paper, we studied the relationship between the properties of a relational structure and those of their algebra, particularly when the relational structure R admits a finite monomorphic decomposition. This setting still encompasses well-studied graded commutative algebras like invariant rings of finite permutation groups, or the rings of quasi-symmetric polynomials.

In this paper, we investigate how far the well know algebraic properties of those rings extend to age algebras. The main result is a combinatorial characterization of when the age algebra is finitely generated. In the special case of tournaments, we show that the age algebra is finitely generated if and only if the profile is bounded. We explore the Cohen-Macaulay property in the special case of invariants of permutation groupoids. Finally, we exhibit sufficient conditions on the relational structure that make naturally the age algebra into a Hopf algebra.

**Keywords:** Relational structure, profile, polynomial growth, age, age algebra, graded commutative algebra, Hilbert function, invariants of permutation groups, quasi-symmetric polynomials,

# Introduction

In [Cam81, Section 2] Cameron defined the *orbit algebra* of a permutation group G acting on an infinite set E; by design, the Hilbert function  $h_{\mathbb{K},\mathcal{A}(G)}$  of this graded commutative algebra coincides with the *orbital profile* of G, namely the function that counts, for every n, the number  $\varphi_G(n)$  of orbits of G acting the finite subsets of size n of E. The main motivation was to study properties of orbital profiles, and in particular a phenomenon of jumps in the possible growth rates.

Similar phenomenon had been observed in the more general context of relational structures (permutation groups being in correspondence with homogeneous relational structures). There, the profile of a relational structure R on E counts, for every integer n, the number  $\varphi_R(n)$  of substructures of R induced on the n-element subsets of E, isomorphic substructures being identified. In [Cam97], Cameron proposed to generalize the approach, defining the age algebra of a relational structure. Familiar algebras like invariant rings of finite permutation

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groups, algebras of quasi-symmetric polynomials [Ges84] or the shuffle algebra over a finite alphabet can be realized as such age algebras.

As a follow up to [PT13], this paper investigates relationships between combinatorial properties of a relational structure R and algebraic properties of its age algebra  $\mathbb{K}.\mathcal{A}(R)$ . Specifically, we consider the following conditions:

Conditions 0.1. (BP) the profile is bounded above by a polynomial;

(QP) the profile is eventually a quasi-polynomial; equivalently its generating series is of the form:

where 
$$n_1 \leq \cdots \leq n_k$$
 and  $P(Z) \in \mathbb{Z}[Z]$ ;

- (QPP) same as (QP) with  $P \in \mathbb{N}[Z]$ ;
  - (AP) the profile is asymptotically equivalent to a polynomial;
  - (FG) the age algebra is finitely generated;
  - (CM) the age algebra is Cohen-Macaulay;

We also consider the following condition:

Condition 0.2. (H) the age algebra is a graded Hopf algebra; in particular it is free.

Let us review what is known, starting with the obvious or well know relations between those conditions:

- $(QP) \Longrightarrow (BP), (AP) \Longrightarrow (BP);$
- (QP)  $\Longrightarrow$  (AP), using that the profile is non decreasing (Pouzet, [Fra71, ex. 8 p. 113] for relational structures; Cameron, [Cam76, Theorem 2.2] for permutation groups);
- The two conditions of (QP) are equivalent as a straightforward consequence of [Sta97, Proposition 4.4.1];
- (FG)  $\Longrightarrow$  (QP) by a general property of graded commutative algebras (see e.g. [CLO97, Chapter 9,  $\S 2]$ );
- $(CM) \Longrightarrow (QPP);$
- $\bullet$  (CM)  $\Longrightarrow$  (FG).

For the examples mentioned earlier, all of Conditions 0.1 are equivalent. This is not an isolated phenomenon. Let us consider the case of a permutation group, or equivalently of an homogeneous relational structure. Cameron conjectured (BP)  $\Longrightarrow$  (AP) [Cam90, p. 69 and Macpherson asked whether (BP)  $\Longrightarrow$  (AP) [Mac85, p. 286]. Justine Falque and the second author recently provided a positive answer; in fact, all of Conditions 0.1 are equivalent [FT18].

In [PT13], we conjectured that, when the kernel of R is finite, the profile  $\varphi_R$  of a relational structure R is eventually a quasi-polynomial whenever  $\varphi_R$  is bounded by some polynomial (that is  $(BP) \Longrightarrow (AP)$ ). We then introduced the notion of monomorphic decomposition, restricted ourselves to the case of relational structure admitting a finite monomorphic decomposition and proved our conjecture there.

In this paper, we proceed in the same setting, and investigate the conditions (FG), (CM), and (H). This setting encompasses invariant rings of finite permutation groups and rings of quasi-symmetric polynomials for which conditions (FG) and (CM) hold; the later fact is a theorem of Garsia and Wallach [GW03]. For other examples, (FG) and (CM) fail. Our main result is a combinatorial characterization of when (FG) holds (Theorem 2.5).

In Section 1, we briefly review relational structures, their orbit algebras and monomorphic decompositions. We refer to [PT13] for a detailed approach. In addition, we mention there a relationship between the order properties of an age and properties of the ideals of the age algebra.

Section 2 is devoted to our main theorem (Theorem 2.5). We start by giving the key ideas on an example, and proceed with the general proof. With the help of [BP10], we show that the age algebra of a tournament is finitely generated if and only if the profile of the tournament is bounded (Theorem 2.7). Indeed, if an age algebra is finitely generated, the profile is bounded above by a polynomial and according to [BP10], tournaments with profile bounded by a polynomial have a finite monomorphic decomposition (meaning simply that these tournaments are lexicographical sums of acyclic tournaments indexed by a finite tournament) and our characterization applies.

In Section 3, we further restrict the setting to invariant rings of permutation groupoids, defined as age algebras of some appropriate relational structures (Section 3). This setting provides a very tight generalization of invariant rings of permutation groups which still includes quasi-symmetric polynomials. We analyze in details which properties of invariant rings of permutation groups carry over – or not – to permutation groupoids (cf. Propositions 3.17 and 3.9, and Theorems 2.5 and 3.16). To this end, we use in particular techniques from [GS84].

Finally, in Section 4, we give some sufficient conditions on the relational structure to endow the age algebra with a further structure of (coassociative) Hopf algebra, and recover several classical Hopf algebras. The age algebra is then a free algebra, which imposes a very rigid form for the Hilbert series and thus for the profile.

We conclude this introduction with general comments and perspectives.

For orbit algebras, our main theorem does not bring new insight; indeed, by Theorem 1.17, an orbit algebra whose homogeneous structure admits a finite monomorphic decomposition is isomorphic to the invariant rings of a finite permutation group (or straightforward quotient thereof); the latter is well known to be finitely generated.

There are other classes of structures for which polynomially bounded profile amounts to the existence of a finite monomorphic decomposition (e.g. permutations [MP08] and ordered graphs [BBM07, BBM06]) and for which we may use our characterization.

**Problem 0.3.** For relational structures admitting a finite monomorphic decomposition, characterize combinatorially when the age algebra is Cohen-Macaulay. This would provide an alternate proof of Garsia and Wallach's theorem for quasi-symmetric functions.

**Problem 0.4.** For general relational structures, characterize combinatorially when the age algebra is finitely generated. The remaining open case is when the minimal monomorphic decomposition has infinitely many blocs, a finite number of which being infinite. Examples A.9 and A.10 of [PT13] show that, in this case and even just for graphs, the age algebra can be finitely generated, or not.

Ultimately the notion of age algebra may just not be quite right, and should be adapted to ensure that all of Conditions 0.1 are equivalent:

**Problem 0.5.** Devise some *sensible* alternative graded algebra structure on  $\mathbb{K}.\mathcal{A}(R)$  which is finitely generated whenever the profile is bounded above by a polynomial.

**Problem 0.6.** Devise some *sensible* alternative graded algebra structure on  $\mathbb{K}.\mathcal{A}(R)$  which is Cohen-Macaulay whenever the Hilbert series has the appropriate form (by Proposition 4 of [BM04] such an algebra always exists).

Let R be a relational structure on a set E. It induces an equivalence relation on the finite subsets of E by setting  $A \sim_R B$  whenever the structures induced by R on A and B respectively are isomorphic. This equivalence relation is hereditary:

$$|A| = |B|$$
 and  $|\{X \subset A : X \sim_R C\}| = |\{X \subset B : X \sim_R C\}|$ 

whenever A, B, and C are finite subsets of E such that  $A \sim_R B$  (hereditary equivalence relations were introduced in [PR86]; see also [Buc09]. The definition of age algebra extends straightforwardly to hereditary equivalence relations.

**Problem 0.7.** Generalize the results of this paper to hereditary equivalence relations and the corresponding algebras.

#### 1. On the profile and age algebra of a relational structure

1.1. Relational structures and their monomorphic decompositions. A relational structure is a pair  $R := (E, (\rho_i)_{i \in I})$  made of a set E and a family of  $m_i$ -ary relations  $\rho_i$  on E. We denote by  $R_{\uparrow A}$ , that we call restriction of R to A, the substructure induced by R on a subset A of E. We consider these substructures up to isomorphism. If needed, we consider isomorphic types, objects associated to relational structures in such a way that the types  $\tau(S_1)$  and  $\tau(S_2)$  of two relational structures are equal if and only if the two relational structures are isomorphic. In our case, we may identify the isomorphic type  $\tau(R_{\uparrow A})$  of the substructure of R induced on a finite subset R of R to its orbit R is isomorphic to R.

Let R be a relational structure on a set E. A subset B of E is a monomorphic part of R if for every integer n and every pair A, A' of n-element subsets of E the induced structures on A and A' are isomorphic whenever  $A \setminus B = A' \setminus B$ . A partition of E into monomorphic parts is a monomorphic decomposition of R.

The crucial property of such a partition is given in Fact 1.1:

Let  $(E_x)_{x\in X}$  be a set partition of E. Set  $X_\infty := \{x \in X : |E_x| = \infty\}$ ; for a finite subset A of E, set  $d_x(A) := |A \cap E_x|$ , and denote by  $\mathbf{d}(A) := (d_x(A))_{x\in X}$  the statistics of intersection sizes.

**Fact 1.1.** The partition  $(E_x)_{x\in X}$  is a monomorphic decomposition of R if and only if the induced structures on two finite subsets A and A' of E are isomorphic whenever  $\mathbf{d}(A) = \mathbf{d}(A')$ .

As stated in [PT13][Proposition 2.12], each monomorphic part of E is included into a maximal one w.r.t. inclusion and these maximal parts form a monomorphic decomposition of R. Hence, every monomorphic decomposition is finer than the decomposition into maximal parts. We call this partition the *minimal monomorphic decomposition*. The *monomorphic dimension* of R is the number of infinite parts in its minimal monomorphic decomposition.

Revisiting these notions, Oudrar and Pouzet (see [Oud15, p. 168, § 7.2.5], [OP16, PSK16]) define this partition in a direct way as follows: Say that two elements x and y of E are equivalent and set  $x \simeq_R y$  if for every finite subset F of  $E \setminus \{x,y\}$ , the restrictions of R to  $\{x\} \cup F$  and  $\{y\} \cup F$  are isomorphic.

**Lemma 1.2.** The relation  $\simeq_R$  is an equivalence relation. Furthermore, the equivalence classes of  $\simeq_R$  are the maximal parts of R, hence they form the minimal monomorphic decomposition.

As an immediate consequence, we obtain the following result (occurring as a part of Corollary 2.13 of [PT13]).

Corollary 1.3. Every automorphism of R induces a permutation of the maximal parts of R.

1.2. Profile and age algebra of a relational structure. The age of a relational structure  $R := (E, (\rho_i)_{i \in I})$  is the set  $\mathcal{A} := \mathcal{A}(R)$  of finite substructures of R, isomorphic substructures being identified. This set was introduced by Fraïssé (see [Fra00]). The kernel of R is the set kernel $(R) := \{x \in E : \mathcal{A}(R_{\upharpoonright E}) \neq \mathcal{A}(R)\}$ . We say that  $\mathcal{A}(R)$  is inexhaustible and R is

age-inexhaustible if two members of  $\mathcal{A}(R)$  can be embedded in a third with disjoint images. This condition amounts to say that the kernel of R is empty.

The profile of R is the function  $\varphi_R$  which counts for every integer n the number  $\varphi_R(n)$  of substructures of R induced on the n-element subsets, isomorphic substructures being identified. Clearly, this function only depends upon the age of R. We recall that the profile of an infinite relational structure is non-decreasing. Furthermore, provided that some mild conditions hold, namely either the signature  $\mu := (m_i)_{i \in I}$  is bounded or the kernel kernel R of R is finite, there are jumps in the behavior of the profile: the growth of  $\varphi_R$  is either polynomial or as fast as every polynomial [Pou78, Pou06] and [BBM07, BBM06, Kla10] for independent developments on this theme.

N.B.: in the sequel, and for the sake of simplicity of exposition, we always make the assumption that  $\varphi_R(n)$  is finite for all n. This holds as soon as I is finite.

Let  $\mathbb{K}$  be a field of characteristic 0. Cameron associates a graded algebra  $\mathbb{K}.\mathcal{A}(R)$  to each relational structure R [Cam97]. This algebra  $\mathbb{K}.\mathcal{A}(R)$  depends only upon the age of R. Its main feature is that its Hilbert function coincides with the profile of R as long as it takes only finite values.

The graded algebra  $\mathbb{K}.\mathcal{A}(R)$  is the direct sum  $\bigoplus_{n<\omega} \mathbb{K}.\mathcal{A}(R)_n$ , where  $\mathbb{K}.\mathcal{A}(R)_n$  is the set of  $\mathbb{K}$ -linear combinations of elements of  $\mathcal{A}(R)_n$ , the set of substructures of R induced on the n-element subsets of E considered up to isomorphy.

Multiplication is defined by the rule that if  $\tau_1 \in \mathcal{A}(R)_n$ ,  $\tau_2 \in \mathcal{A}(R)_m$  then

(1) 
$$\tau_1.\tau_2 := \sum_{\tau \in \mathcal{A}(R)_{n+m}} c_{\tau_1,\tau_2}^{\tau} \tau$$

where

(2) 
$$c_{\tau_1,\tau_2}^{\tau} := |\{(A_1, A_2) : A_1 \uplus A_2 = A, \tau(A_1) = \tau_1, \tau(A_2) = \tau_2 \text{ and } \tau(A) = \tau\}|.$$

We recall two results:

Let e be the sum of isomorphic types of the one-element restrictions of R (we can identify it to  $\sum_{a \in E} \{a\}$ ). Let U be the graded free algebra  $\mathbb{K}[e] = \bigoplus_{n=0}^{\infty} \mathbb{K}e^n$ .

**Theorem 1.4.** [Cam97] If R is infinite then e is not a zero-divisor; namely for any  $u \in \mathbb{K}.\mathcal{A}(R)$ , eu = 0 if and only if u = 0.

This result implies that  $\varphi_R$  is non decreasing. Indeed, the image of a basis of the vector space  $\mathbb{K}.\mathcal{A}(R)_n$  under multiplication by e is a linearly independent subset of  $\mathbb{K}.\mathcal{A}(R)_{n+1}$ .

We recall the following result:

**Theorem 1.5.** [Pou08] If R is age-inexhaustible then the age algebra  $\mathbb{K}.\mathcal{A}(R)$  has no non-zero divisor.

In the sequel, we give some general properties relating relational structures and algebras.

1.3. Operations on relational structures and age algebras. In this section and in Section 1.6, a useful technical device is to embed the age algebra  $\mathbb{K}.\mathcal{A}(R)$  of a relational structure R in a larger algebra, the set algebra, whose definition we recall now.

Let  $\mathbb{K}$  be a field of characteristic 0, and E be a set. For  $n \geq 0$ , denote by  $[E]^n$  the set of the subsets of E of size n, and let  $\mathbb{K}^{[E]^n}$  be the vector space of maps  $f:[E]^n \to \mathbb{K}$ . The set algebra is the graded connected commutative algebra  $\mathbb{K}^{[E]^{<\omega}} := \bigoplus_n \mathbb{K}^{[E]^n}$ , where the product of  $f:[E]^m \to \mathbb{K}$  and  $g:[E]^n \to \mathbb{K}$  is defined as  $fg:[E]^{m+n} \to \mathbb{K}$  such that:

(3) 
$$(fg)(A) := \sum_{(A_1, A_2) : A = A_1 \uplus A_2} f(A_1)g(A_2).$$

Identifying a set S with its characteristic function  $\chi_S$ , elements of the set algebra can be thought as (possibly infinite but of bounded degree) linear combination of sets; the unit is the empty set, and the product of two sets is their disjoint union, or 0 if their intersection is non trivial.

The desired embedding of  $\mathbb{K}.\mathcal{A}(R)$  into  $\mathbb{K}^{[E]^{<\omega}}$  is obtained by mapping an isomorphism type  $\tau$  to its orbit sum  $\sum_{A: \tau(A)=\tau} A$ .

Remark 1.6. The original definition of the age algebra, which we used above, requires the assumption that the profile is finite. See e.g. Example A.9 for what can go wrong otherwise. The set algebra offers a natural frame to formulate an equivalent definition that extends beyond this assumption and further to hereditary equivalence relations (this is for example the presentation adopted in [Pou08]).

Namely, consider a relational structure R on a set E with profile not necessarily finite, or more generally a hereditary equivalence  $\equiv$  on the finite subsets of E. Say that a map  $f:[E]^n \to \mathbb{K}$  is invariant if it is constant on each equivalence class on  $[E]^n$  induced by R or  $\equiv$  (e.g. f(A) = f(A') whenever the restrictions  $R_{\uparrow A}$  and  $R_{\uparrow A'}$  are isomorphic). Consider the space  $\mathbb{K}.\mathcal{A} = \bigoplus_n \mathbb{K}.\mathcal{A}_n$ , where  $\mathbb{K}.\mathcal{A}_n$  is the set of all invariant maps for a given n. Observe that  $\mathbb{K}.\mathcal{A}$  is a subalgebra of the set algebra, and call it the age algebra.

- **Proposition 1.7.** (i) Let  $R := (E, (\rho_i)_{i \in I})$  be a relational structure, E' be a subset of E and  $R' := R_{\upharpoonright E'}$  be the structure induced by R on E'. Then,  $\mathbb{K}.\mathcal{A}(R')$  is a quotient of  $\mathbb{K}.\mathcal{A}(R)$ .
- (ii) Let  $R := (E, (\rho_i)_{i \in I})$  be a relational structure, I' be a subset of I, and  $R' := (E, (\rho_i)_{i \in I'})$ . Then,  $\mathbb{K}.\mathcal{A}(R')$  is a subalgebra of  $\mathbb{K}.\mathcal{A}(R)$ .
- (iii) Let  $R := (E, (\rho_i)_{i \in I})$  and  $R' := (E', (\rho'_i)_{i \in I'})$  be two relational structures on disjoint subsets, and define their direct sum as the relational structure  $R \oplus R' := (E \cup E', (\rho_i)_{i \in I}, (\rho'_i)_{i \in I'})$ . If needed, add an appropriate unary relation to ensure that singletons of E and E' are not isomorphic. Then,  $\mathbb{K}.\mathcal{A}(R \oplus R')$  is isomorphic to the tensor product  $\mathbb{K}.\mathcal{A}(R) \otimes \mathbb{K}.\mathcal{A}(R')$ . Furthermore, if  $(E_i)_i$  and  $(E'_j)_j$  are (minimal) monomorphic decompositions of respectively R and R', then  $(E_i)_i \cup (E'_j)_j$  form a (minimal) monomorphic decomposition of  $R \oplus R'$ .
- *Proof.* (i) At the level of the set algebra, the vector space spanned by the sets which are not subsets of E' is an ideal; so the linear map  $\phi$  which kills those sets is an algebra morphism. Furthermore looking at the image of basis elements shows that  $\mathbb{K}.\mathcal{A}(R')$  is the image by  $\phi$  of  $\mathbb{K}.\mathcal{A}(R)$ .
- (ii) Each isomorphism class for R splits into one or more isomorphism class(es) for R'; hence each basis element for  $\mathbb{K}.\mathcal{A}(R)$  is accordingly the sum of one or more basis element(s) for  $\mathbb{K}.\mathcal{A}(R')$ .
- (iii) Identify each subset A of  $E \cup E'$  with the element  $(A \cap E) \otimes (A \cap E')$  of the tensor product and check that a basis element for  $\mathbb{K}.\mathcal{A}(R \otimes R')$  is the tensor product of a basis element for  $\mathbb{K}.\mathcal{A}(R)$  by a basis element of  $\mathbb{K}.\mathcal{A}(R')$ . The resulting Hilbert series is the product of the Hilbert series for R and R'.
- Let  $R = (E, (\rho_i)_{i \in I})$  and  $R' = (E', (\rho'_i)_{i \in I'})$  be two relational structures. The wreath product of R and R' is the relational structure  $R \wr R' := (E \times E', (\tilde{\rho}_i)_{i \in I}, (\tilde{\rho}'_i)_{i \in I'})$ , where

$$\tilde{\rho}_i\left((e_1,e_1'),\ldots,(e_k,e_k')\right)$$
 if and only if  $\rho_i(e_1,\ldots,e_k)$  and  $e_1'=\cdots=e_k'$ ,

and

$$\tilde{\rho}_i\left((e_1,e_1'),\ldots,(e_k,e_k')\right)$$
 if and only if  $\rho_i'(e_1',\ldots,e_k')$ .

**Proposition 1.8.** If  $(E_i)_i$  and  $(E'_j)_j$  are monomorphic decompositions of respectively R and R', then  $(E_i \times E'_j)_{i,j}$  forms a monomorphic decomposition of  $R \wr R'$ . In particular, if R is

monomorphic, then the  $E_x := E \times \{x\}$  for  $x \in E'$  form a monomorphic decomposition of  $R \wr R'$ .

*Proof.* Direct consequence of Proposition 1.7 (ii) and (iii).

Let R be a relational structure. A local isomorphism of R is any isomorphism from a restriction of R to another restriction. We denote by Loc(R) the set of local isomorphisms of R. Endowed with the partial composition product, Loc(R) becomes a monoid (see e.g. Section 3.1). The wreath product of two such monoids can be defined in the same vein as for permutation groups: Loc(R) acts independently within each  $E_x$  while Loc(R') acts globally on the  $(E_x)_{x\in E'}$ .

Remark 1.9.  $Loc(R \wr R')$  includes the wreath product  $Loc(R) \wr Loc(R')$ ; this inclusion may be strict: take the wreath product of a 2-antichain with itself.

1.4. Initial segments of an age and ideals of a ring. Final segments play for posets the same role than ideals for rings. In this section, we order the age of a relational structure by embeddability: if  $\tau_1, \tau_2 \in \mathcal{A}(R)$ , we set  $\tau_1 \leq \tau_2$  if  $\tau_1$  is the type of a structure induced on some structure of type  $\tau_2$ . This order is up-directed: for any  $\tau_1, \tau_2 \in \mathcal{A}(R)$  there exists  $\tau \in \mathcal{A}(R)$  such that  $\tau_1, \tau_2 \leq \tau$ . An *initial segment* of  $\mathcal{A}(R)$  is any subset F of  $\mathcal{A}(R)$  such that  $\tau_2 \in F$  implies  $\tau_1 \in F$  whenever  $\tau_1 \leq \tau_2$ .

We describe briefly the correspondence between initial segments of an age and ideals of the age algebra.

**Theorem 1.10.** Let A := A(R) be the age of a relational structure R and K.A be its age algebra. Recall that we assume that R has finite profile. If A' is an initial segment of A then:

- (i) the vector subspace  $J := \mathbb{K}.(A \setminus A')$  spanned by  $A \setminus A'$  is an ideal of  $\mathbb{K}.A$ . Moreover, the quotient of  $\mathbb{K}.A$  by J is a ring isomorphic to the ring  $\mathbb{K}.A'$ .
- (ii) if J is irreducible then A' is a subage of A;
- (iii) I is a prime ideal if and only if A' is an inexhaustible age.

*Proof.* (i) Since J is a subspace of  $\mathbb{K}.\mathcal{A}$ , it suffices to show that:

(4) 
$$P \in \mathcal{A} \setminus \mathcal{A}' \text{ and } Q \in \mathcal{A} \text{ implies } P.Q \in J.$$

We have  $P.Q := \sum_{R=P \cup Q} R$ . Each member R of this sum embeds P; since  $\mathcal{A}'$  is an initial segment of  $\mathcal{A}$  and  $P \in \mathcal{A} \setminus \mathcal{A}'$ , it follows that  $R \in \mathcal{A} \setminus \mathcal{A}'$  proving that  $P.Q \in \mathcal{J}$ .

(ii) For convenience, we consider here complements of initial segments, that is final segments. Let F(A) be the set of final segments of A and  $Id(\mathbb{K}.A)$  be the set of ideals of  $\mathbb{K}.A$ , these sets being ordered by inclusion. Let  $\varphi: F(A) \to Id(\mathbb{K}.A)$  defined by setting  $\varphi(F) := \mathbb{K}.F$  if  $F \neq \emptyset$  and  $\varphi(F) := \{0\}$  otherwise. As shown in (i) this map is well-defined. It preserves arbitrary joins and meets that is:

(5) 
$$\mathbb{K}.\bigcup_{i\in I}F_i = \sum_{i\in I}\mathbb{K}.F_i;$$

(6) 
$$\mathbb{K}.\bigcap_{i\in I}F_i=\bigcap_{i\in I}\mathbb{K}.F_i.$$

The first equality is obvious. For the second equality, we have trivially  $\mathbb{K}.\bigcap_{i\in I}F_i\subseteq\bigcap_{i\in I}\mathbb{K}.F_i$ . For the converse, let  $P\in\bigcap_{i\in I}\mathbb{K}.F_i$ . We have  $P=\sum_{j\in I_i}P_{ij}$  with  $P_{ij}\in\mathbb{K}.F_i$  for each  $i\in I$ . By definition, members of  $\mathcal A$  are linearly independent, thus the decomposition of P into a linear sum of members of  $\mathcal A$  is unique. It follows that  $P_{ij}$  is independent of  $I_i$  thus belongs to  $\bigcap_{i\in I}F_i$ , hence  $P\in\mathbb{K}.\bigcap_{i\in I}F_i$ . This completes the proof that the second equality holds.

The map  $\varphi$  restricted to  $F(A) \setminus \{\emptyset\}$  is one-to-one. The inverse image of a meet-irreducible element of  $Id(\mathbb{K}.A)$  is a meet-irreducible element of F(A). As it is well-known, for an arbitrary poset  $\mathcal{P}$ , the meet-irreducible members of  $F(\mathcal{P})$  are exactly the complements of ideals of  $\mathcal{P}$  (that is up-directed initial segments of  $\mathcal{P}$ ); in the case  $\mathcal{P} = \mathcal{A}$  the ideals are the subages of  $\mathcal{A}$ . This completes the proof of (ii).

(iii) Suppose that  $\mathcal{A}'$  is not an inexhaustible age. Then there are  $S_1', S_2' \in \mathcal{A}'$  such that no  $R \in \mathcal{A}'$  extends  $S_1', S_2'$  on the disjoint union of their domains. But then the product  $S_1'.S_2'$  is in  $\mathbb{K}.(\mathcal{A} \setminus \mathcal{A}')$ , proving that  $\mathbb{K}(\mathcal{A} \setminus \mathcal{A}')$  is not prime. Conversely, let  $P, Q \in \mathbb{K}.\mathcal{A}$  such that  $P.Q \in J := \mathbb{K}.(\mathcal{A} \setminus \mathcal{A}')$ . We may write  $P = P_1 + P_2$ ,  $Q = Q_1 + Q_2$  with  $P_1, Q_1 \in J$ ,  $P_2, Q_2 \in \mathbb{K}.(\mathcal{A}')$ . We have  $P_2.Q_2 \in J$ . Since  $\mathbb{K}.\mathcal{A}'$  is isomorphic to the quotient  $\mathbb{K}.\mathcal{A}/J$ , the product  $P_2.Q_2$  in  $\mathbb{K}.\mathcal{A}'$  is 0. But since  $\mathcal{A}'$  is inexhaustible,  $\mathbb{K}.\mathcal{A}'$  has no non-zero divisor by Theorem 1.5, hence P or Q belongs to J, proving that J is prime.

**Problem 1.11.** Does the converse of (ii) hold. That is, is  $J := \mathbb{K}.(A \setminus A')$  meet-irreducible whenever A' is a subage of A?

Ages are special cases of hereditary classes of relational structures (see [Fra00, Oud15]). The definition of age algebra carries over straightforwardly to such classes, and the correspondence between initial segments and ideals would be best further explored in this context.

1.5. Well-quasi-ordering of an age. The age algebra of a relational structure being graded and connected, it is finitely generated if and only if it is a Noetherian ring. There are posets which play a role as important in the theory of ordered sets as noetherian rings in the theory of ring. These posets, studied first by Higman [Hig52], are said well-quasi-ordered, in brief wqo. Namely, a poset  $\mathcal{P}$  is wqo if the set  $F(\mathcal{P})$  of final segments of  $\mathcal{P}$  is Noetherian w.r.t. the inclusion order.

Consider an age  $\mathcal{A}(R)$  ordered by embeddability. By (i) of Theorem 1.10,  $F(\mathcal{A}(R))$  embeds into the collection of ideals of  $\mathbb{K}.\mathcal{A}(R)$ . Consequently:

**Proposition 1.12.** If the age algebra  $\mathbb{K}.\mathcal{A}(R)$  is finitely generated then the age of R is well-quasi-ordered by embeddability.

The reciprocal does not hold. Indeed, the age of a relational structure is well-quasi-ordered as soon as the age has polynomial growth and finite kernel [Pou78, Pou06] whereas we have seen that the age algebra is not necessarily finitely generated in this case. In fact, well-quasi-ordering of the age does not even imply that the profile is bounded above by a polynomial: indeed, any age with non polynomially bounded profile contains a well-quasi-ordered age with the same property (use Lemma 4.1 of [PT13] and the remark above it).

**Problem 1.13.** Is the profile of a relational structure R bounded by some exponential whenever the age R is well-quasi-ordered by embeddability?

## 1.6. The age algebra as a subring of a polynomial ring.

**Theorem 1.14.** If R has a monomorphic decomposition into finitely many infinite parts  $(E_i)_{i\in X}$ , then the age algebra  $\mathbb{K}.\mathcal{A}(R)$  is isomorphic to a subalgebra  $\mathbb{K}[X]^R$  of  $\mathbb{K}[X]$ . This may be generalized:

- If some parts are finite,  $\mathbb{K}.\mathcal{A}(R)$  is isomorphic to a subalgebra of the quotient ring  $\mathbb{K}[X]/(x^{|E_x|+1})$ .
- If there are infinitely many parts,  $\mathbb{K}.\mathcal{A}(R)$  is isomorphic to a subalgebra of  $\mathbb{K}[[X]]/(x^{|E_x|+1})$  made of series of bounded degree.

*Proof.* Let  $(E_x)_{x\in X}$  be a finite set partition of E into infinite subsets. We first use it to define an embedding of  $\mathbb{K}[X]$  into the set algebra  $\mathbb{K}^{[E]^{<\omega}}$ . For an exponent vector  $\mathbf{d} := (d_x)_{x\in X} \in$ 

 $\mathbb{N}^X$ , define the monomial  $X^{\mathbf{d}} := \prod_{x \in X} x^{d_x}$  and set  $\mathbf{d}! := \prod_{x \in X} d_x!$ . Set furthermore  $O(\mathbf{d}) := \{ A \subseteq E : \mathbf{d}(A) := \mathbf{d} \}$  and let  $\chi_{O(\mathbf{d})}$  be the characteristic map of  $O(\mathbf{d}) \in \mathbb{K}^{[E]^{<\omega}}$ which is best interpreted as the (possibly infinite) sum  $\sum_{\mathbf{d}(A)=\mathbf{d}} A$ . Let  $\phi: \mathbb{K}[X] \hookrightarrow \mathbb{K}^{[E]^{<\omega}}$ be defined by setting  $\phi(X^{\mathbf{d}}) := \mathbf{d}! \chi_{O(\mathbf{d})}$ .

**Claim.**  $\phi$  is a one-to-one morphism of algebras.

Indeed, we have  $\chi_{O(\mathbf{d})}\chi_{O(\mathbf{d}')} = \frac{(\mathbf{d}+\mathbf{d}')!}{\mathbf{d}!\mathbf{d}'!}\chi_{O(\mathbf{d}+\mathbf{d}')}$ . For an isomorphic type  $\tau$  in the age of R, viewed as a subset of E, set  $\mathbf{d}(\tau) := \{\mathbf{d}(A) :$  $A \in \tau$  and  $\mu(\tau) := \sum_{\mathbf{d} \in \mathbf{d}(\tau)} \frac{1}{\mathbf{d}!} X^{\mathbf{d}}$ . Note that any monomial  $X^{\mathbf{d}}$  in  $\mathbb{K}[X]$  appears in exactly one polynomial  $\mu(\tau)$ ; in particular, the later are linearly independent. Finally, let  $\mathbb{K}[X]^R$  be the subset of  $\mathbb{K}[X]$  made of finite linear combinations of polynomials of the form  $\mu(\tau)$ . **Claim.** The map  $\phi$  induces an isomorphism from  $\mathbb{K}[X]^R$  onto  $\mathbb{K}.\mathcal{A}(R)$ . Indeed, we have:

$$\phi(\mu(\tau)) = \sum_{\mathbf{d} \in \mathbf{d}(\tau)} \phi(\frac{1}{\mathbf{d}!} X^{\mathbf{d}}) = \sum_{\mathbf{d} \in \mathbf{d}(\tau)} \chi_{O(\mathbf{d})} = \chi_{\tau} \ .$$

The generalizations are straightforward

In the sequel, when the monomorphic decomposition of a relational structure is clear from the context, we often use polynomials as a convenient and compact way of writing elements of its age algebra.

At this stage, a natural question is to characterize the subalgebras of  $\mathbb{K}[X]$  that can be constructed as age algebras of relational structures with a finite monomorphic decomposition. As suggested in the introduction, the context can be generalized to algebras of hereditary equivalence relations. This makes the answer simpler: in the sequel, we obtain a one-to-one correspondence with subalgebras of  $\mathbb{K}[X]$  defined by "grouping monomials together".

Let  $\sim$  be an equivalence relation on  $E^{<\omega}$ , with equivalence classes  $(\tau_i)_{i\in I}$ . Consider the subspace  $\mathbb{K}.\mathcal{A}(\sim)$  of  $\mathbb{K}^{[E]^{<\omega}}$  spanned by the  $\chi_{\tau_i}$ , where  $\tau_i$  ranges through the finite equivalence classes. For simplicity, we assume that there are finitely equivalence classes for each size of subset.

**Lemma 1.15.**  $\mathbb{K}^{[E]^{<\omega}}$  is a subalgebra of  $\mathbb{K}^{[E]^{<\omega}}$  if and only if  $\sim$  is hereditary.

*Proof.* The if part is the same as for age algebras. For the reciprocal, consider a set C, its equivalence class  $\tau$ , and define  $e_d \in \mathbb{K}.\mathcal{A}$  as  $\sum \chi_{\tau_i}$  where i ranges through the finite equivalence classes of sets of size d. Equivalently,  $e_d$  is the (infinite) sum of all subsets of Eof size d.

$$\chi_{\tau} = \sum_{A : |A| = |C| + d} c_C^A A,$$

where  $c_C^A = |\{X \subset A : X \sim C\}|$ . One recognizes the coefficients appearing in the hereditary condition, and it follows that  $c_C^A = c_C^B$  whenever  $A \sim B$ .

**Proposition 1.16.** Let A be a graded subalgebra of  $\mathbb{K}[X]$  that admits a basis  $(B_i)_{i\in I}$  such that each monomial of  $\mathbb{K}[X]$  appears in exactly one  $B_i$ . Then, there exist a hereditary equivalence relation  $\sim$  whose algebra  $\mathbb{K}.\mathcal{A}(\sim)$  is isomorphic to A.

*Proof.* Write  $S_i$  the support of  $B_i$ . By construction,  $(S_i)_{i \in I}$  forms a partition of the monomials of  $\mathbb{K}[X]$ . Consider the induced equivalence relation of  $E^{<\omega}$ , with equivalence classes given by  $\tau_i := \mathbf{d}^{-1}(S_i)$  for  $i \in I$ . Let  $\mathbb{K}.\mathcal{A}(\sim)$  be the subspace of  $\mathbb{K}^{[E]^{<\omega}}$  spanned by the  $\chi_{\tau_i}$ .

We now prove that the basis elements  $B_i$  of the former are mapped one to one by  $\phi$  to the basis elements  $\chi_{\tau_i}$  of the latter.

Up to rescaling the variables in X once for all, we may assume without loss of generality that the basis elements of degree 1 are sums of variables. In particular, A contains e =  $x_1 + \dots, x_n$  and therefore also  $e^d = \sum_{\mathbf{d} : |\mathbf{d}| = d} \frac{x^{\mathbf{d}}}{\mathbf{d}!}$ . Up to rescaling the  $B_i$ 's, we may therefore assume without loss of generality that each  $B_i$  is of the form  $\sum_{\mathbf{d} \in S_i} x^{\frac{\mathbf{d}}{\mathbf{d}!}}$ , where  $S_i$  is the support of  $B_i$ . Therefore,  $\phi(B_i) = \chi_{\tau_i}$ , as desired.

It follows that  $\mathbb{K}.\mathcal{A}(\sim)$  is indeed an algebra, and therefore, using Lemma 1.15, that  $\sim$  is hereditary.

1.7. Case of orbital algebras. As pointed out by Cameron [Cam90], invariant rings of finite permutation groups are special cases of orbital algebras. They are also algebras of relational structures admitting a finite monomorphic decomposition (see Example A.16 of [PT13]). The converse holds, in the sense that, possibly up to some straightforward quotienting, invariant rings of finite permutation groups are exactly the orbital algebras of groups admitting a finite monomorphic decomposition.

Let us state this more precisely. Let G be a permutation group on a set E. Choose any relational structure R encoding the orbits of G. Note that the definition of a monomorphic decomposition depends only on the isomorphism relation between finite subsets of E, and thus is independent of the chosen relational structure. We can thus forget about the relational structure, and all the concepts of minimal monomorphic decomposition, monomorphic dimension, etc, are well defined for G itself.

**Theorem 1.17.** Let G be a permutation group on a set E, and assume that the minimal monomorphic decomposition  $(E_x)_{x\in X}$  of E is finite.

Then G induces a finite subgroup  $\tilde{G}$  of the symmetric group  $\mathfrak{S}_X$  on X. If the components are all infinite, then the orbital algebra is isomorphic to the invariant ring  $\mathbb{K}[X]^{\tilde{G}}$  of  $\tilde{G}$ . Otherwise, it is isomorphic to the quotient thereof obtained by setting  $x^{|E_x|+1}=0$  for all x such that  $E_x$  is finite.

Proof. By Corollary 1.3, every permutation  $\sigma \in G$  induces a permutation  $\overline{\sigma}$  of the components  $(E_x)_{x \in X}$ , which we can identify with a permutation of X. Choose any relational structure encoding the orbits of G, and consider the isomorphism  $\Phi$  from  $K[X]^R$  to  $\mathbb{K}.\mathcal{A}(R)$  as in the proof of Theorem 1.14. If all components are infinite, it is easy to see that  $K[X]^R$  is nothing but the invariant ring  $K[X]^{\tilde{G}}$ . Otherwise, the same quotienting occurs as in Theorem 1.14.

#### 2. Finite Generation

This section is devoted to our main result: the combinatorial characterization of relational structures admitting a finite monomorphic decomposition whose algebra is finitely generated. We setup the ground with a special case and an example before proceeding to the general case. We conclude with the case of tournaments.

2.1. Finite generation for bounded profiles. We recall the following result.

**Theorem 2.1** (Theorem 1.5 of [PT13]). Let R be a relational structure with E infinite. Then, the following properties are equivalent:

- (a) The profile of R is bounded.
- (b) R is almost-monomorphic.
- (b') R has a monomorphic decomposition into finitely many parts, at most one being infinite.
- (c) R is almost-chainable.
- (d) The Hilbert series is of the following form, with  $P(Z) \in \mathbb{N}[Z]$  and  $P(1) \neq 0$ :

$$\mathcal{H}_R = \frac{P(Z)}{1 - Z} \ .$$

(e) The age algebra is a finite dimensional free-module over the free-algebra  $\mathbb{K}[e]$ , where  $e := \sum_{a \in E} \{a\}$ ; in particular it is finitely generated and Cohen-Macaulay.

See [PT13] for the definition of almost-monomorphy and almost-chainability. Note that, in [PT13], Theorem 1.5 is stated before the introduction of monomorphic decompositions, and thus does not mention (b'). The equivalence between (b) and (b') follows from Theorem 2.25 of [PT13].

- 2.2. Proof of non finite generation on a prototypical example. The age algebra of a relational structure R admitting a finite monomorphic decomposition is not necessarily finitely generated. A prototypical example is this:
- Example 2.2. Let G be the direct sum  $K_{(1,\omega)} \oplus \overline{K}_{\omega}$  of an infinite wheel and an infinite independent set. There are two infinite monomorphic parts,  $E_1$  the set of leaves of the wheel and  $E_2$  the independent set, and one finite,  $E_3$ , containing the center c of the wheel. Each isomorphism type consists of a wheel and an independent set, so the Hilbert series is  $\mathcal{H}_G(Z) = (1 + \frac{Z^2}{1-Z}) \frac{1}{1-Z} = \frac{1-Z+Z^2}{(1-Z)^2}$ .

What makes this relational structure special is that the monomorphic decomposition  $(E_1, E_2, E_3)$  is minimal, whereas  $(E_1, E_2)$  is not a minimal monomorphic decomposition of the restriction of R to  $E_1 \cup E_2$ . We now prove that this causes the age algebra  $\mathbb{K}.\mathcal{A}(G)$  not to be finitely generated. Consider the subalgebra  $U := \mathbb{K}[e]$ , where  $e := \sum_{a \in V(G)} \{a\}$ . In each degree d, it is spanned by the sum  $b_d$  of all subsets of size d of E, since  $e^d = d!b_d$ . Key fact: any element s of  $\mathbb{K}.\mathcal{A}(G)$  can be uniquely written as s := a(s) + b(s) where b(s) is in U, and all subsets in the support of a(s) contain the unique element c of  $E_3$ . Note in particular that a(s)a(s') = 0 for any s, s' homogeneous of positive degree.

Suppose that S is a finite generating set of  $\mathbb{K}.\mathcal{A}(G)$ ; we may suppose that S is made of homogeneous elements of positive degree. By the key observation above,  $\{a(s), s \in S\}$  generates  $\mathbb{K}.\mathcal{A}(G)$  as a U-module. It follows that the graded dimension of  $\mathbb{K}.\mathcal{A}(G)$  is bounded by |S|. But this graded dimension is the profile of G which grows linearly. This gives a contradiction.

- 2.3. Combinatorial characterization. The previous example suggests that the finite generation of the age algebra is related to the behavior of the minimal monomorphic decomposition with respect to restriction. This is indeed the case, and we get a complete characterization of when the age algebra is finitely generated.
- **Definition 2.3.** Let  $(E_x)_{x\in X}$  be the minimal monomorphic decomposition of R. Whenever restricting R to some union  $\bigcup_{x\in X'} E_x$  of infinite monomorphic parts,  $(E_x)_{x\in X'}$  remains a monomorphic decomposition. If it always remains minimal, the decomposition  $(E_x)_{x\in X}$  is called *hereditary minimal*.
- Remark 2.4. It is sufficient to check the condition on pairs of infinite parts. Namely, a monomorphic decomposition  $(E_x)_{x\in X}$  is hereditary minimal iff, for any two infinite monomorphic parts  $E_x$  and  $E_y$ , the monomorphic decomposition  $(E_x, E_y)$  of the restriction of R on  $E_x \cup E_y$  is minimal.
- **Theorem 2.5.** Let R be a relational structure admitting a finite monomorphic decomposition. Let  $(E_x)_{x\in X}$  be its minimal monomorphic decomposition, and  $X_{\infty}$  be the set of indices of the infinite monomorphic parts. Then, the following propositions are equivalent:
- (a) The monomorphic decomposition is hereditary minimal;
- (b) The age algebra  $\mathbb{K}.\mathcal{A}(R)$  is finitely generated;
- (c) For some large enough integer D, the age algebra  $\mathbb{K}.\mathcal{A}(R)$  contains (a triangular deformation of) the free subalgebra  $\mathrm{Sym}(x^D, x \in X_\infty)$  and is a module of finite type thereupon.

The implication (c)  $\Rightarrow$  (b) is immediate; we prove separately (a)  $\Rightarrow$  (c) and (b)  $\Rightarrow$  (a).

For the former implication, the proof is based on the Stanley-Reisner ring approach of [GS84] to construct generators of the invariant rings of a permutation group as a module over symmetric functions (also dubbed chain-product trick by the second author), together with the layer addition lemma used in [PT13] to prove that the Hilbert series is a rational fraction. For this, we need the tools and notations of Section 3 of [PT13], albeit with a small variant on the chosen total order on the monomials of  $\mathbb{K}[X]$ ; this variant is necessary to handle the fact that hereditary minimality is only about the infinite components  $E_x$  of the relational structure.

Say for short that a variable  $x \in X$  is finite if  $E_x$  is finite. Write  $\deg_{<\infty}(m)$  for the degree of a monomial  $m \in \mathbb{K}[X]$  in the finite variables. To compare two monomials m and m' in  $\mathbb{K}[X]$ , first compare their degree in the finite variables; e.g. set m > m' if  $\deg_{<\infty}(m) < \deg_{<\infty}(m')$ ; in case of tie, proceed as in [PT13] by comparing the shapes of m and m', and breaking ties with the usual lexicographic order. When there is no finite variable, nothing changes. Leading monomials, chain support, etc are defined as in [PT13]. With this order on monomials, Lemma 3.2 of [PT13] becomes:

**Lemma 2.6.** Let m be a leading monomial, and  $S \subseteq X$  be a layer of m with  $S \subset X_{\infty}$ . Then,  $mx_S$  is again a leading monomial.

*Proof.* Proceed as in the proof of Lemma 3.2 of [PT13]. Since we are considering only layers  $S \subset X_{\infty}$ , the tweak in the total order on monomials does not interfere.

Proof of  $(a) \Rightarrow (c)$  of Theorem 2.5. Consider the restriction R' of R on its infinite components. Since the monomorphic decomposition is hereditary minimal, the monomorphic decomposition  $(E_x)_{x \in X_\infty}$  remains minimal. Take d large enough as in Lemma 2.15 of [PT13]. By Corollary 2.16, for  $i = 1, \ldots, k$ , the collection C of all sets of size di and shape  $(d, \ldots, d)$  is closed under orbits. Identifying the elementary symmetric function  $e_i(x^d, x \in X_\infty)$  in  $\mathbb{K}[X]$  with (a constant factor of) the sum of all the elements of C in the set algebra, we derive that this symmetric function belongs to the age algebra of R'.

Let  $\overline{C}$  be the closure of C under orbits in R, and let  $\overline{e}_i \in \mathbb{K}.\mathcal{A}(R)$  be the sum of the elements in  $\overline{C}$ . Note that  $\overline{e}_i = e_i(x^d, x \in X_\infty)$ , up monomials containing variables x not in  $X_\infty$  and thus strictly smaller. It follows that  $\mathbb{K}[\overline{e}_i, i = 1, \dots, k]$  is a triangular deformation of the ring of symmetric functions  $\operatorname{Sym}(x^d, x \in X_\infty)$ . In particular, it's the free commutative algebra generated by  $(\overline{e}_i)_{i=1,\dots,k}$  in the age algebra.

**Claim 1:** Let m be a leading monomial with chain support C. Assume that  $m = x_S^d m'$  for some leading monomial m',  $S \in C$ , and  $d \ge 0$ . Then,

$$lm(o(m')\overline{e}_{|S|}) = m;$$

*Proof.* We say for short that a variable  $x \in X$  is finite if  $E_x$  is finite. Write  $\deg_{<\infty}(c)$  for the degree of a monomial c in the finite variables.

Take a monomial ab appearing in the product  $o(m')\overline{e}_{|S|}$ . If a=m' and  $b=x_S^d$ , then we recover m and we are done.

Note that  $\deg_{<\infty}(a) \ge \deg_{<\infty}(m')$ . If the comparison is strict or if  $\deg_{<\infty}(b) > 0$ , then  $\deg_{<\infty}(ab) > \deg_{<\infty}(m)$  and thus ab < m as desired.

Otherwise,  $\deg_{<\infty}(a) \geq \deg_{<\infty}(m')$ , and b is of the form  $x_{S'}^d$  for some  $S' \subseteq X^\infty$  with |S'| = |S|. Given that the shape of a is at most that of m', that the shape of  $x_{S'}^d$  coincide with that of  $x_{S}^d$ , and that S is in the chain support of m', the shape of ab is at most that of m. If equality does not holds, we are done. Otherwise, the shape of a coincides with that of m', and S' is in the chain support of m', and it follows that  $ab \leq_{\text{lex}} m$ , and therefore  $ab \leq m$ , as desired.

Fix a chain  $C := \emptyset \subset S_1 \subset \cdots \subset S_r \subseteq X$ , and let  $\operatorname{Im}_C$  be the set of leading monomials of the age algebra with this chain support. As in Section 3.2 of [PT13], consider  $\mathcal{J} := \mathbb{K}.\operatorname{Im}_C \oplus \mathcal{I}$  of  $\mathbb{K}[S_1,\ldots,S_l]$ ; By Lemma 2.6, this is a monomial ideal. Dickson's Lemma states that  $\mathcal{J}$  is finitely generated as an ideal, that is as a module over  $\mathbb{K}[S_1,\ldots,S_l]$ . It is in fact also finitely generated as a module over  $\mathbb{K}[S_1^d,\ldots,S_l^d]$ , with a canonical finite set  $G_C$  of monomials as generators (see e.g. Lemma 2.3.2 (a) of [Stu93]).

Let  $\mathcal{G}$  be the finite collection of the orbitsums whose leading monomial is in  $G_C$  for some chain C. We conclude by proving the following claim.

Claim 2:  $\mathcal{G}$  generates  $\mathbb{K}.\mathcal{A}$  as a module over  $\mathbb{K}[\overline{e}_i]$ .

Take an orbitsum o(m) in  $\mathbb{K}.\mathcal{A}$  with m its leading monomial, and let C be the chain support of m. Assume by induction that, for any leading monomial m' < m, the orbitsum o(m') is in the  $\mathbb{K}[\overline{e_i}]$ -module spanned by  $\mathcal{G}$ . We prove that this holds for o(m) too.

If m is in  $G_C$ , o is in  $\mathcal{G}$  and we are done. Otherwise, write m as  $m = s_S^d m'$  where  $S \subset X_\infty$ . Using Claim 1, o(m) is in the  $\mathbb{K}[\overline{e_i}]$ -module spanned by o(m') together with orbitsums of strictly smaller leading monomials. Therefore, by induction, o(m) is in the  $\mathbb{K}[\overline{e_i}]$ -module spanned by  $\mathcal{G}$  as desired.

Proof of  $(b) \Rightarrow (a)$  of Theorem 2.5. Assume that the monomorphic decomposition is not hereditary minimal but the age algebra  $\mathbb{K}.\mathcal{A} := \mathbb{K}.\mathcal{A}(R)$  is finitely generated.

The proof follows the same path as in Example 2.2: we first reduce the problem to the monomorphic dimension 2 case. Then, we construct a subalgebra  $\mathcal{B}$  of  $\mathbb{K}.\mathcal{A}$  which is "small" (dimension 1 in each degree) compared to  $\mathbb{K}.\mathcal{A}$  (dimension asymptotically equivalent to d in degree d). Using the fact that  $\mathbb{K}.\mathcal{A}$  is finitely generated as an algebra we prove that  $\mathbb{K}.\mathcal{A}$  is a finitely generated module over  $\mathcal{B}$ . This is impossible dimension-wise.

Claim 1: we may assume without loss of generality that R is of monomorphic dimension 2. Otherwise, consider two monomorphic parts  $E_1$  and  $E_2$  such that R restricted to  $H = E_1 \cup E_2$  is monomorphic. By the minimality of the decomposition, H is not a monomorphic part of R: there exist two finite subsets A and A' of E such that A and A' are not isomorphic yet coincide outside H. Let G be the finite subset  $A \setminus H = A' \setminus H$ . Then, R restricted to  $H \cup G$  is of monomorphic dimension 2. The age algebra of the restriction is a quotient of the age algebra of R and is therefore still finitely generated, as desired.

Keeping the above notations, we now have  $E = H \cup G$  where  $H = E_1 \cup E_2$ . Consider the graded subalgebra  $\mathcal{B}$  of  $\mathbb{K}$ .  $\mathcal{A}$  spanned in each degree d by the sum  $e_d(E)$  of all subsets of size d ( $\mathcal{B}$  can be alternatively defined as the free graded commutative algebra generated by the sum of all points  $e_1(E)$ , or the age algebra of the trivial relational structure on E).

Claim 2:  $\mathbb{K}.\mathcal{A}$  is a finite-module over  $\mathcal{B}$ . This yields the desired contradiction because the graded dimension of such a module is bounded by a constant, whereas by Lemma 2.15 of [PT13], the graded dimension of  $\mathbb{K}.\mathcal{A}$  is asymptotically equivalent to d.

It remains to prove Claim 2.

Since the restriction of R to H is monomorphic, any orbitsum of degree d either contains all subsets of size d of H, or none. Therefore, any s in the age algebra decomposes uniquely as s =: a(s) + b(s), where b(s) is in  $\mathcal{B}$ , and all the subsets in the support of a(s) intersects G non trivially.

Key fact: any product  $a(s_1) \cdots a(s_k)$  of k > |G| homogeneous elements  $s_i$  of positive degree is zero (k > |G| subsets of E which intersect G non trivially cannot be disjoint).

Let S be a finite generating set of the age algebra made of homogeneous elements of positive degree. Then,  $\{a(s), s \in S\}$  generates  $\mathbb{K}.\mathcal{A}$  as a  $\mathcal{B}$ -algebra. By the above key fact the finite collection of all products  $a(s_1) \cdots a(s_k)$  of  $k \leq |G|$  elements of S generates  $\mathbb{K}.\mathcal{A}$  as a  $\mathcal{B}$ -module.

2.4. Finite generation for tournaments. The existence of a very simple tournament (Example A.8 of [PT13]) whose age algebra is not finitely generated is not an accident; in fact, the age algebra of a tournament is very seldom finitely generated.

**Theorem 2.7.** The age algebra of a tournament T is finitely generated if and only if the profile is bounded.

This is a consequence of Theorem 2.5 thanks to some simple remarks and a structural theorem on the monomorphic parts of a tournament.

Remarks 2.8. (a) A monomorphic part of size at least 4 is acyclic;

- (b) The union of two monomorphic parts of size at least 4 is acyclic;
- (c) A minimal monomorphic decomposition of a tournament with at least two infinite parts cannot be hereditary minimal.

**Theorem 2.9** (Boudabbous-Pouzet [BP10]). Let T be an infinite tournament whose profile is bounded above by a polynomial. Then, T is a lexicographical sum  $\sum_{i \in D} A_i$  of acyclic tournaments  $A_i$  indexed by a finite tournament D.

Proof of Theorem 2.7. Suppose that  $\varphi_T$  is bounded. Then, by Theorem 2.1,  $\mathbb{K}.\mathcal{A}(T)$  is finitely generated. Conversely, suppose that  $\mathbb{K}.\mathcal{A}(T)$  is finitely generated. Then, the profile  $\varphi_T$  has polynomial growth, and by Theorem 2.9 the minimal monomorphic decomposition of T is finite. Applying Theorem 2.5, this decomposition is hereditary minimal, and by Remark 2.8 (c), it has a single infinite monomorphic maximal part. Therefore, the profile is bounded.

Theorem 2.7 was stated as Theorem 3.5 in [Pou06]. The argument provided there expands on the idea given in Example 2.2 but, as stated, is incorrect. The correction is straightforward. Assume that the tournament T is a lexicographical sum  $T = \sum_{i \in D} A_i$  of acyclic tournaments – indexed by a finite tournament D – at least two of which,  $A_j$ ,  $A_k$ , are infinite. Then, according to Lemma 9 of [BP10] T contains a sub-tournament  $T' = \sum_{i \in D'} A'_i$  with the same property where D' has at most 5 elements and all  $A'_i$  but two, say  $A'_{j'}$  and  $A'_{k'}$ , are singletons. Supposing D' with minimum size (hence between 3 and 5), the union of the  $A'_{i'}$ 's, for  $i' \notin \{j', k'\}$ , is the kernel kernel(T') of T'. As stated in Theorem 3.5, if  $\mathbb{K}.\mathcal{A}(T)$  is finitely generated,  $\mathbb{K}.\mathcal{A}(T')$  is finitely generated. Then we may repeat the proof given in Example 2.2 with kernel(T') playing the role of  $\{c\}$  and obtain a contradiction (the proof of Theorem 3.5 deals only with D of size 3, in which case kernel(T') is a singleton).

## 3. Invariant rings of Permutation Groupoids

The common point of invariant rings of permutation groups and other interesting examples like the rings of quasi-symmetric polynomials (Example A.18 of [PT13]) is that they can be realized as age algebras of a relational structure R of the form  $K_{\infty} \wr (X, \rho_i)$ , where X is a finite set and  $K_{\infty}$  is an infinite clique (or similar monomorphic relational structure). In this section, we study further such age algebras which we call *invariant rings of permutation groupoids*.

Our motivations are twofold. On one hand, relate, in this simpler yet rich setting, the properties of the profile to algebraic properties of the invariant ring. In particular, find conditions under which the invariant ring is Cohen-Macaulay. On the other hand, generalize the theory, algorithms, and techniques of invariant rings of permutation groups to a larger class of subrings of  $\mathbb{K}[X]$ .

3.1. Permutation groupoids, inverse monoids and representations. In this section, we briefly review the groupoid and inverse monoid structures on the local isomorphism of a relational structure, and their representations. For details, see e.g. [Law98, Chapter 4], [Ste06].

Let  $G := \operatorname{Loc}(X, \rho_i)$  be the collection of local isomorphisms of a finite relational structure  $(X, \rho_i)$ . Recall that it can be endowed with a groupoid structure by composing two local isomorphisms f and g whenever the codomain of the first agrees with the domain domg of the second.

The properties of G are as follows:

- G contains all local identities of X;
- G is stable under composition, inverse, and restrictions.

We call *permutation groupoid* on a set X a collection of local bijections of X satisfying the above properties, endowed with the composition product.

**Proposition 3.1.** Any permutation groupoid on a finite set X can be realized as groupoid of local isomorphisms on some relational structure on X.

A permutation group on X induces a permutation groupoid on X by considering all the restrictions of its automorphisms.

Alternatively, G can be endowed with an inverse monoid structure, which we denote by  $\overline{G}$  by taking the partial composition as product:

$$fg: \begin{cases} g^{-1}(\operatorname{im} g \cap \operatorname{dom} f) & \hookrightarrow f(\operatorname{im} g \cap \operatorname{dom} g) \\ x \mapsto f(g(x)) \end{cases}$$
.

The groupoid and inverse monoid structures are tightly related through their algebras. Define as usual the groupoid algebra  $\mathbb{K}$ .G of G as the vector space of formal linear combinations of G, endowed with the product obtained by extending by bilinearity the groupoid product of G.

Define similarly the monoid algebra of G starting from its inverse monoid structure. The latter algebra is isomorphic to  $\mathbb{K}.G$ , by mapping a partial bijection f to  $\overline{f} = \sum_{A \subset \text{dom} f} f \upharpoonright_A$  in  $\mathbb{K}.G$ , where A is the domain of f. The inverse isomorphism can be defined by inclusion-exclusion. In the sequel, we identify both algebras, interpreting  $(\overline{f})_{f \in G}$  as an alternative basis of  $\mathbb{K}.G$ . The map  $f \mapsto \overline{f}$  also defines an embedding of  $\overline{G}$  in  $\mathbb{K}.G$ .

Because of this isomorphism, the representations of G and  $\overline{G}$  coincide. In particular, they are semi-simple.

3.2. Invariant rings of permutation groupoids. Take a relational structure R of the form  $K_{\infty} \wr (X, \rho_i)$ . The running example to keep in mind is that of quasi-symmetric functions (Example A.18 of [PT13]). The monomorphic decomposition  $(E_x := K_{\infty} \times \{x\})_{x \in X}$  of R is minimal, and even hereditary minimal. This makes the age algebra into a subring of  $\mathbb{K}[X]$ .

We extend the natural action of a permutation of X on  $\mathbb{K}[X]$  as follows. Take a monomial  $X^{\mathbf{d}}$  and a local bijection f. If the *support* of  $X^{\mathbf{d}}$  (that is:  $\{x \in X : d_x > 0\}$ ) coincides with the domain of f, set  $f.X^{\mathbf{d}} := \prod_{x,d_x>0} f(x)^{d_x}$ ; otherwise set  $f.X^{\mathbf{d}} := 0$ .

**Lemma 3.2.** Let A and B be two finite isomorphic subsets of R. Then, there exists a local isomorphism f of  $(X, \rho_i)$  such that  $X^B = f.X^A$ . In particular, the shape of A and B are identical.

*Proof.* Let g be a local isomorphism from A to B. Two points of A in the same monomorphic part must be sent by g to the same monomorphic part. Therefore, there exists a local isomorphism f of  $(X, \rho_i)$ , and local isomorphisms  $(f_x)_{x \in X}$  of  $K_\infty$  such that every point (j, x) of A is mapped to  $g(j, x) = (f_x(j), f(x))$ .

This proposition should really be interpreted as follows. The monomorphic parts are strings, and the finite subsets of E are sets of beads threaded on those strings and sliding freely on them. Furthermore some local permutations of the non empty strings are allowed, corresponding to the local isomorphisms of  $(X, \rho_i)$ . For example, for quasi-symmetric functions, one may move the string 1 to the string 3, and the string 2 to the string 4, assuming that all other strings contain no beads.

**Corollary 3.3.** The age and therefore the age algebra depend only on the groupoid G of the local isomorphisms of  $(X, \rho_i)$ . Reciprocally, the age algebra characterizes the groupoid.

By analogy with invariant rings of permutation group, we therefore write it  $\mathbb{K}[X]^G$ , and call it invariant ring of the permutation groupoid G. If G is induced by a permutation group, then both of their invariant rings coincide.

The basic notions of invariant rings of permutation groups, like *G-isomorphic monomials* and *G-orbits* extend straightforwardly; in particular one may define the *orbit sum*  $O(X^D)$  of a monomial  $X^D$  as the sum of all the monomials in its orbit.

**Proposition 3.4.** The orbitsums form a vector space basis of the invariant ring  $\mathbb{K}[X]^G$ . The later is a graded connected commutative algebra which contains symmetric functions in X.

Remark 3.5. The action being by local permutation, the isomorphism of two monomials does not depend on the actual values of the exponents, but only on the partition of X induced by them.

Formally: let f be a function from  $\mathbb{N}$  to  $\mathbb{N}$  such that f(0) = 0, and define  $f(X^D) := \prod_x x^{f(D_x)}$ . Then, if  $X^D$  and  $X^{D'}$  are isomorphic then so are  $f(X^D)$  and  $f(X^{D'})$ .

In this setting, Lemma 3.2 of [PT13] becomes immediate.

**Corollary 3.6.** Consider the lexicographic monomial order. If  $X^D$  is a leading monomial in  $\mathbb{K}[X]^G$ , and if  $S \subseteq X$  is a layer of  $X^D$ , then  $X^DX_S$  is again a leading monomial.

*Proof.* Duplicating a layer in  $X^D$  amounts to apply a strictly increasing function f. This function preserves both isomorphism and the monomial order.

- 3.3. Permutation groupoids versus permutation groups. Here, we discuss briefly how the action of a permutation groupoid on polynomials differs from that of a permutation group.
- 3.3.1. Restriction. The restriction  $G_{\uparrow X'}$  of a permutation groupoid G to a subset X' is the set of all local functions f in G such that  $\text{dom} f \subseteq X'$  and  $\text{im} f \subseteq X'$ , which is again a permutation groupoid. Furthermore, the orbits of monomials in  $\mathbb{K}[X']$  are unchanged by this restriction. In particular, the invariant ring of  $G_{\uparrow X'}$  is simply the quotient of the invariant ring of G obtained by killing all the variables  $x_i$  with  $i \notin X'$ . This simple fact is one of the motivations for considering permutation groupoids instead of just permutation groups (for which the restriction to a subset is not clearly defined). This may indeed give opportunities for induction techniques on the size of the underlying set.

**Proposition 3.7.** Any permutation groupoid on a finite set is the restriction of a permutation groupoid induced by a permutation group of some superset. However, this superset may need to be infinite.

*Proof.* This is an immediate consequence of the fact that every finite relational structure R embeds into a countable homogeneous structure (structure for which local isomorphisms of finite domain extend to automorphisms) [Fs54]. In several instances, the permutation groupoid may be chosen finite [HL00].

Examples 3.8. (a) The permutation groupoid on  $\{1,2,3\}$  generated by the rank 1 local bijection  $1 \mapsto 2$  is the restriction of the permutation group on  $\{1,2,3,4\}$  generated by the permutation (1,2)(3,4).

- (b) The local automorphism permutation groupoid of the chain a < b is the restriction of the cyclic group  $C_3$  on  $\{a, b, c\}$ .
- (c) Consider a relational structure R such that there exists three elements a,b,c and a binary relation < which restricts on  $\{a,b,c\}$  to the chain a < b < c. Typically, R is a chain of length at least 3 (giving  $\operatorname{QSym}(X)$  as invariants) or a poset of height at least 3. Then, there exists no relational structure  $\overline{R}$  on a finite superset where all local isomorphisms extend to global isomorphisms.

Consider indeed the local isomorphism  $a \mapsto a, c \mapsto b$ , and extend it to a global isomorphism  $\sigma$  of  $\overline{R}$ . It is easy to check that  $a < \sigma(b) < b < c$  is again a chain, which implies that  $\sigma(b) \notin \{a,b,c\}$ . By induction,  $a < \sigma^k(b) < \cdots < \sigma(b) < c$  is again a chain, which proves that all  $\sigma^k(b)$  are distinct. Hence R' is infinite.

3.3.2. Multiplicativity. As for a permutation group, the groupoid algebra  $\mathbb{K}.G$  is semi-simple, and the action of G extends to a linear representation of G. However, for a permutation groupoid, the action is not multiplicative on polynomials. Take for example  $f:=\mathrm{id}_{\{1,2\}}$ ,  $P:=x_1$  and  $Q:=x_2$ . This requires a bit of care in the upcoming generalization of the Reynolds operator and explains why it is not any more a Sym-module morphism.

Multiplicativity can be partially recovered by considering the inverse monoid  $\overline{G}$ , whose on polynomials is given by  $\overline{f}.X^D:=\prod_{x,D_x>0}f(x)^{D_x}$  if  $\{x,D_x>0\}\subseteq \mathrm{dom} f$  and  $f.X^D:=0$  otherwise. This action is multiplicative.

3.4. The Reynolds operator. The first essential feature of invariant rings is the so-called Reynolds operator  $\mathcal{R}$ , a projector on the invariant ring. The following proposition states that this operator still exists for invariants of permutation groupoids, albeit missing the important property of being a  $\mathbb{K}[X]^G$ -module morphism. In particular, although  $\mathbb{K}[X]^G$  still contains the ring of symmetric polynomials  $\mathrm{Sym}(X)$ , R is not anymore a  $\mathrm{Sym}(X)$ -module morphism. Recall also that, for a permutation group,  $\mathbb{K}[X]^G$  is the isotopic component for the trivial representation of  $\mathbb{K}.G$  in  $\mathbb{K}[X]$ ; furthermore, the Reynolds operator is the unique central idempotent of  $\mathbb{K}.G$  projecting on the trivial representation. These properties fail for a permutation groupoid; in fact  $\mathbb{K}[X]$  is not even stable under the action by  $\mathbb{K}.G$ .

**Proposition 3.9.** There exists an idempotent  $\mathcal{R}$  in the groupoid algebra  $\mathbb{K}.G$  which projects  $\mathbb{K}[X]$  onto the invariant ring  $\mathbb{K}[X]^G$ :

$$\mathcal{R} := \sum_{A \subseteq X} \frac{1}{|\{g \in G : \operatorname{dom} g = A\}|} \sum_{g \in G : \operatorname{dom} g = A} g.$$

Furthermore, the four following conditions are equivalent:

- (i) G is induced by a permutation group.
- (ii)  $\mathcal{R}$  is a  $\mathbb{K}[X]^G$ -module morphism;
- (iii)  $\mathcal{R}$  is a Sym(X)-module morphism;
- (iv)  $\ker \mathcal{R}$  is a Sym-module;

*Proof.* By construction,  $\mathcal{R}$  is in the groupoid algebra. One easily checks that, up to a scalar factor, the image of a monomial by  $\mathcal{R}$  is the orbitsum of that monomial, and the image of an orbitsum is itself. Therefore, it projects onto the invariant ring.

- $(i) \Rightarrow (ii), (ii) \Rightarrow (iii), (iii) \Rightarrow (iv)$  are well known or obvious.
- (iv)  $\Rightarrow$  (i): Assume that ker R is a Sym-module and that G does not come from a permutation group, and let  $f: A \mapsto B$  be a local bijection which does not extend to a permutation of X. Let  $x_X$  be the product of the variables, and m be a monomial with support A with

all exponents distinct. Then, f.m is in the orbit of m, whereas  $x_X f.m$  is not in the orbit of  $x_X m$ . Therefore,  $\mathcal{R}(m-f.m)=0$  whereas  $x_X \mathcal{R}(m-f.m)=\mathcal{R}(x_X m-x_X f.m)\neq 0$ , a contradiction.

3.5. Fine grading, the chain product, and degree bounds. Recall that the degree bound  $\beta(A)$  of a finitely generated graded algebra A is the smallest integer such that A is generated by its elements of degree at most  $\beta(A)$ . In [GS84], Garsia and Stanton construct an associated finely-graded algebra of the invariant ring  $\mathbb{K}[X]^G$  of a permutation group G by letting G act on the Stanley-Reisner ring of the boolean lattice. They use it to exhibit  $\mathrm{Sym}(X)$ -module generators in some special cases and prove, in general, the quadratic degree bound  $\beta(\mathbb{K}[X]^G) \leq \binom{|X|}{2}$ . A notable feature is that this approach is combinatorial and thus characteristic free!

In this section, we show that this construction generalizes essentially straightforwardly to permutation groupoids, and derive some properties of the invariant ring: degree bounds and finite generation; in the following section we further derive a necessary conditions for being Cohen-Macaulay.

As in [Thi00], we follow the basic approach of realizing the Stanley-Reisner ring as  $\mathbb{K}[X]$  endowed with another product  $\star$ , called the *chain product*. The chain product preserves a finer grading, and many algebraic properties of the invariant ring w.r.t. the chain product transfer back to the usual product. We refer to [Eis95] for background on filtrations and associated graded algebras.

Given a subset S of X, set  $x_S := \prod_{i \in S} x_i$ . By square-free decomposition, any monomial  $x^d$  can be identified uniquely with a multichain  $S_1 \subseteq \cdots \subseteq S_k$  of nested subsets of X, so that:

$$x^d = x_{S_1} \cdots x_{S_k}$$
.

We call each  $S_k$  a layer of x.

The fine degree of the monomial  $x^d$  is the integer vector  $(r_1, \ldots, r_n)$  where each  $r_i$  counts the (possibly null) number of repetitions of the layer of size i of  $x^d$ . Orbit sums are homogeneous w.r.t. the fine degree; the invariant ring is therefore graded as a vector space.

One may compute the fine Hilbert series of the invariant ring as follows: take a representative Y for each orbit of subsets of X; consider the group  $G_Y$  of local automorphisms of domain Y; use Pólya enumeration to compute the fine generating series for monomials with full support Y. Sum all the results.

The fine grading is not preserved by multiplication; however it still defines a filtration on  $\mathbb{K}[X]$ . The *chain product*  $\star$  of two monomials  $x^d := x_{S_1} \cdots x_{S_k}$  and  $x^{d'} := x_{S_1'} \cdots x_{S_k'}$  is defined by:

$$x^d \star x^{d'} := \begin{cases} x^d x^{d'} & \text{if } \{S_1, \dots, S_k, S_1', \dots, S_k'\} \text{ is again a multichain of subsets,} \\ 0 & \text{otherwise.} \end{cases}$$

For example,  $x_1 \star x_1 = x_1^2$ ,  $x_1 \star x_2 = 0$ ,  $x_1 x_3^2 \star x_1 x_2 x_3^2 = x_1^2 x_2 x_3^4$ , and  $x_1 x_3^2 \star x_1 x_2 = 0$ .

The chain product endows  $\mathbb{K}[X]$  with a second algebra structure  $(\mathbb{K}[X],\star)$ , isomorphic to the quotient

$$\mathbb{K}[x_S, S \subseteq X]/\{x_S x_{S'} = 0 : S \not\subseteq S' \text{ and } S' \not\subseteq S\}.$$

 $(\mathbb{K}[X], \star)$  is also finely graded, fine degrees being added term-by-term. In fact,  $(\mathbb{K}[X], \star)$  is exactly the associated graded algebra of  $\mathbb{K}[X]$  w.r.t. the fine degree filtration. Beware however that  $(\mathbb{K}[X], \star)$  is not an integral domain.

The elementary symmetric functions

$$e_d := \sum_{S \subseteq X, |S| = d} x_S$$

are still algebraically independent and generate  $(\operatorname{Sym}(X)_n, \star)$ . This is no longer true for, say, the symmetric powersums. The following simple fact turns out to be an essential key:

Remark 3.10. Consider the chain product of a monomial  $x_{S_1} \cdots x_{S_k}$  by the elementary symmetric function  $e_d$ . It is the sum of all monomials  $x_{S_1} \cdots x_S \cdots x_{S_k}$ , where S is of size k, and fits in the multichain  $S_1 \subseteq \cdots \subseteq S \subseteq \cdots \subseteq S_k$ . In particular, if  $x_{S_1} \cdots x_{S_k}$  contains a layer S of size k, then  $x_{S_1} \cdots x_{S_k} \star e_k$  is the unique monomial obtained by replicating this layer.

More generally,  $(\mathbb{K}[X]^G, \star)$  is a subring of  $(\mathbb{K}[X], \star)$ . In particular,  $(\mathbb{K}[X]^G, \star)$  is a Sym(X)-module. Furthermore, we may transfer the following algebraic properties from  $(\mathbb{K}[X], \star)$  to  $\mathbb{K}[X]^G$ , as in the case of permutation groups [GS84].

**Proposition 3.11.** (a) A family F of finely homogeneous invariants of positive degree which generates  $(\mathbb{K}[X]^G, \star)$ , also generates  $\mathbb{K}[X]^G$ ;

- (b)  $\beta(\mathbb{K}[X]^G, \star) \ge \beta(\mathbb{K}[X]^G);$
- (c) A family F of finely homogeneous invariants which generates  $(\mathbb{K}[X]^G, \star)$  as a  $\operatorname{Sym}(X)$ module also generates  $\mathbb{K}[X]^G$  as a  $\operatorname{Sym}(X)$ -module;
- (d) If  $(\mathbb{K}[X]^G, \star)$  is a free  $\operatorname{Sym}(X)$ -module, then so is  $\mathbb{K}[X]^G$ .

*Proof.* This is a standard fact about filtrations and associated graded connected algebras. The key of the proof is that, if p and q are finely homogeneous, the maximal finely homogeneous component of pq is exactly  $p \star q$ . (a) and (c) follow by induction over the fine grading. Then, (b) follows straightaway from (a), and (d) from (c) by a simple Hilbert series argument.

The converse of (a) and (b) do not hold. In fact, with most permutation groups, the degree bound  $\beta(\mathbb{K}[X]^G, \star)$  is much larger than  $\beta(\mathbb{K}[X]^G)$ . We conjecture that the converse of (c) and (d) hold. However (d) does not hold anymore in a slightly larger setting which includes the r-quasi-symmetric polynomials of F. Hivert [Hiv05b], a counter example being  $\operatorname{QSym}^2(X_3)$  (there is an obstruction in the fine Hilbert series).

We now generalize the quadratic degree bound of Garsia and Stanton to permutation groupoids.

**Theorem 3.12.** Let G be a permutation groupoid acting on X and n = |X|. Then, the invariant ring  $\mathbb{K}[X]^G$  is a finitely generated algebra and  $\mathrm{Sym}(X)$ -module, in degree at most  $\frac{|X|(|X|+1)}{2}$ . This degree bound is tight.

Note that, as usual, when G does not act transitively on the variables, the degree bound can be greatly improved by considering the symmetric polynomials on each transitive component instead.

*Proof.* The set of orbit sums  $o(x_{S_1}\cdots x_{S_k})$ , where  $S_1\subsetneq\cdots\subsetneq S_k$  is a chain, generates  $(\mathbb{K}[X]^G,\star)$  as a  $(\operatorname{Sym},\star)$ -module. This transfers back to  $\mathbb{K}[X]^G$  and  $\operatorname{Sym}$ . Note that we may need to consider chains with  $S_k=X$ ; hence the degree bound of

Note that we may need to consider chains with  $S_k = X$ ; hence the degree bound of  $\frac{|X|(|X|+1)}{2}$  instead of  $\binom{|X|}{2}$  for permutation groups. For an example where the bound is achieved, consider the group G made of the identity together with all the local bijections of  $X = \{1, \ldots, n\}$  whose domain is of size at most |X| - 1; then,  $\mathbb{K}[X]^G$  is freely generated as a Sym-module by 1 and the *staircase* monomials  $x_1^{d_1} \cdots x_n^{d_n}$  with  $1 \le d_i \le i$ .

3.6. The Cohen-Macaulay property. Invariant rings of permutation groups are always Cohen-Macaulay, and in fact free  $\operatorname{Sym}(X)$ -modules. The key ingredients are that  $\mathbb{K}[X]$  is a free  $\operatorname{Sym}(X)$ -module and the Reynolds operator a  $\operatorname{Sym}(X)$ -module morphism. A more involved result is that, for all n,  $\operatorname{QSym}(X_n)$  is also a free  $\operatorname{Sym}(X_n)$ -module [GW03].

As Example A.19 of [PT13] shows, this property does not hold for all permutation groupoids G. Still,  $\mathbb{K}[X]^G$  and  $(\mathbb{K}[X]^G, \star)$  being finitely generated over  $\mathrm{Sym}(X)$ , they are Cohen-Macaulay if and only if they are free  $\mathrm{Sym}(X)$ -modules.

**Problem 3.13.** Characterize the permutation groupoids G whose invariant ring  $\mathbb{K}[X]^G$  (or  $(\mathbb{K}[X]^G, \star)$ ) is Cohen-Macaulay.

In practice, a first test is to compute the (fine) degrees of tentative free generators of  $\mathbb{K}[X]^G$  by dividing the (fine) Hilbert series of  $\mathbb{K}[X]^G$  by that of  $\mathrm{Sym}(X)$ .

The following theorem is a straightforward extension of Theorem 6.1 of [GS84] applied to the quotient of the Stanley Reiner ring of the boolean lattice by a permutation group.

**Theorem 3.14.**  $(\mathbb{K}[X]^G, \star)$  is a free  $\operatorname{Sym}(X)$ -module if and only if the incidence matrix between generators and orbits of maximal chains is invertible. In particular, for a set F of finely homogeneous invariants whose fine degrees are given by the Hilbert series of  $\mathbb{K}[X]^G$ , the three following conditions are equivalent: F spans  $\mathbb{K}[X]^G$  as a  $\operatorname{Sym}(X)$ -module, F is a free  $\operatorname{Sym}(X)$ -family, and F is a  $\operatorname{Sym}(X)$ -basis of  $\mathbb{K}[X]^G$ .

This immediately gives us a necessary condition on the number of generators.

**Corollary 3.15.** If  $(K[X]^G, \star)$  is a free Sym(X)-module, then it is of rank  $\frac{|X|!}{|G(X,X)|}$ , where G(X,X) is the underlying permutation group of G.

As a consequence, we recover that the ring QSym(X) of quasi symmetric polynomials in X has to be of rank |X|! over Sym(X).

3.7. **SAGBI bases.** SAGBI bases (Subalgebra Analog of a Gröbner Bases for Ideals) were introduced in [KM89, RS90] to develop an elimination theory in subalgebras of polynomial rings. Unlike Gröbner bases, not all subalgebras have a finite SAGBI basis, and it remains a long open problem to characterize those subalgebras which have a one. The following theorem states that, as in the case of permutation groups, invariant rings of permutation groupoids seldom have finite SAGBI bases. The proof follows the short proof given by the second author in [TT04] for permutation groups, with some adaptations. For example QSym( $X_n$ ), represented as a subring of  $\mathbb{K}[X]$ , has no finite SAGBI basis whenever n > 1. In particular, QSym( $X_2$ ) becomes the smallest example of finitely generated algebra which has no finite SAGBI bases (the standard example being the invariant ring of the alternating group  $A_3$ ). Still, SAGBI bases and SAGBI-Gröbner bases provide a useful device in the computational study of invariant rings of permutation groups [Thi01], and should play the same role with permutation groupoids.

**Theorem 3.16.** Let G be a permutation groupoid, and < be any admissible term order on  $\mathbb{K}[X]$ . Then, the invariant ring  $\mathbb{K}[X]^G$  has a finite SAGBI basis w.r.t. < if, and only if, G is induced by a permutation group generated by reflections (that is transpositions).

The following proof is a close variant on the short proof given by the second author in [TT04] in the special case of permutation groups. For the sake of readability and completeness, we include it in full here. The key fact is that a submonoid M of  $\mathbb{N}^n$  is finitely generated if, and only if, the convex cone  $C := \mathbb{R}_+ M$  it spans in  $\mathbb{R}^n_+$  is finitely generated (that is C is a polyhedral cone). For details, see for example [BG09, Corollary 2.10]. In particular C must be the intersection of finitely many half spaces, and thus closed for the euclidean topology.

*Proof.* The if-part is easy, a finite SAGBI basis being given by the elementary symmetric polynomials in the variables in each G-transitive components.

Without loss of generality, we may assume  $X = \{1, ..., n\}$  with  $x_1 > \cdots > x_n$ . Let M be the monoid of initial monomials in  $\mathbb{K}[X]^G$ , seen as a submonoid of  $\mathbb{N}^n$ , and  $C := \mathbb{R}_+ M$  be the convex cone it spans in  $\mathbb{R}^n_+$ .

At this stage, we cannot give an explicit description of C, but we can construct a convex cone C' which approximates it closely enough for our purposes. By the standard characterization of admissible term orders on  $\mathbb{K}[X]$ , there exists a family of n linear forms  $l=(l_1,\ldots,l_n)$  such that  $x^d>x^{d'}$  if and only if  $l(d)>_{\mathrm{lex}} l(d')$ , where we denote by l(d) the n-uple  $l_1(d_1,\ldots,d_n),\ldots,l_n(d_1,\ldots,d_n)$ . Given two vectors v and v' in  $\mathbb{R}^n_+$ , we write v>v' if  $l(v)>_{\mathrm{lex}} l(v')$ . The partial action of G on monomials extends naturally to a partial action on  $\mathbb{R}^n_+$ : whenever the support of  $v=(v_1,\ldots,v_n)$  in contained in the domain of a local bijection  $f\in G$ , f.v is the vector obtained by permuting the non zero entries of v according to f. Let f' be the subset of all vectors f' of f' such that f' for all f in the f'-orbit of f'. In fact, f' is a convex cone with non empty interior (it contains the f' linearly independent vectors f' with integer coordinates. It follows that f' is the topological closure of f' with integer coordinates. It follows that f' is the topological closure of f'.

Assume now that M is finitely generated. Then, C is a closed convex cone, and C and C' simply coincide.

Assume further that G is not generated by transpositions. Then, there exists a < b such that the transposition (a,b) is not in G, while a is in the G-orbit of b. Choose such a pair a < b with b minimal. We claim that there is no transposition (a',b) in G with a' < b. Otherwise, a and a' are in the same G-orbit, and by minimality of b,  $(a,a') \in G$ ; thus,  $(a,b) = (a,a')(a',b)(a,a') \in G$ . Pick  $g \in G$  such that g.b = a, and for  $t \geq 0$ , define the vector in  $\mathbb{R}^n_+$ :

$$u_t := (nt, (n-1)t, \ldots, (n-b+2)t, n-b+1, (n-b)t, \ldots, t, 1)$$
.

Note that  $u_1 = (n, ..., 1)$  is in C, whereas  $u_0 = (0, ..., 0, n - b + 1, 0, ... 0)$  is not in C because  $g.u_0 > u_0$ .

Take t such that  $0 < t \le 1$ . Then, the vector  $u_t$  has no zero coefficients, and in particular its G-orbit coincides with its orbit w.r.t. the underlying permutation group G(X,X). Furthermore, the entries of  $u_t$  are all distinct, except when  $t = \frac{n-b}{n-a'}$  for some a' < b, in which case the a'-th and b-th entries are equal. Since  $(a',b) \notin G$ , the orbit of  $u_t$  is of size |G(X,X)|, and there exists a unique permutation  $f_t \in G(X,X)$  such that  $f_t.u_t$  is in C.

Let  $t_0 = \inf\{t \geq 0, u_t \in C\}$ . If  $u_{t_0} \notin C$ , then  $u_{t_0}$  is in the closure of C, but not in C, a contradiction. Otherwise,  $u_{t_0} \in C$ , and  $t_0 > 0$  because  $u_0 \notin C$ . For any permutation f,  $\{f.u_t, t \geq 0\}$  is a half-line; so, C being convex and closed,  $I_f := \{t, f.u_t \in C\}$  is a closed interval  $[x_f, y_f]$ . For example,  $I_{\text{id}} = [t_0, 1] \subseteq [0, 1]$ . Since the interval [0, 1] is the union of all the  $I_f$ , there exists  $f \neq \text{id}$  such that  $t_0 \in I_f$ . This contradicts the uniqueness of  $f_{t_0}$ .  $\square$ 

3.8. Stability by derivation. We denote by  $\partial_i$  the derivative w.r.t. the variable  $x_i$ , and consider the derivation  $D := \sum_{i \in X} \partial_i$  on  $\mathbb{K}[X]$ .

**Proposition 3.17.** Let G be a permutation groupoid. Then,  $\mathbb{K}[X]^G$  is stable by the derivation D if and only if G is induced by a permutation group. On the other hand,  $\mathbb{K}[X]^G$  is always stable w.r.t. the action of the rational Steenrod operators  $S_k := \sum_i x_i^{k+1} \partial_i$  for  $k \geq 0$  (see [HT04] for details on the rational Steenrod operators).

*Proof.* The if-part is trivial, since D commutes with the action of the symmetric group  $\mathfrak{S}_X$  on  $\mathbb{K}[X]$ . Similarly, the rational Steenrod operators always stabilize  $\mathbb{K}[X]^G$  because they commute with the action of any local bijection on  $\mathbb{K}[X]^G$ .

Assume now that  $\mathbb{K}[X]^G$  is stable by derivation. Let  $f: A \mapsto B$  be a local bijection such that  $A \subsetneq X$ , and take i in  $X \setminus A$ . We just need to prove that f extends to a local bijection

g in G with domain  $A \cup \{i\}$ . Applying induction, any local bijection in G will then extend to a permutation, as desired.

Take a monomial m whose support is A and whose exponents are all distinct and at least 2, and consider the derivation  $p = D(o(mx_i))$  of the orbitsum of the monomial  $mx_i$  in  $\mathbb{K}[X]^G$ . The monomial m occurs in p; hence, by invariance of p, f(m) also occurs in p, as the derivative of some monomial  $g(mx_i)$  in the orbit of  $mx_i$ . By the choice of the exponents of m, f and g must coincide on A, while at the same time i belongs to the domain of g.  $\square$ 

Example 3.18.  $QSym(X_2)$  has no graded derivation of degree -1.

#### 4. Hopf algebra structure

Let R be a relational structure on a set E. In this section, we propose one approach to try to endow its age algebra with a Hopf algebra structure by looking at copies of R within R

Assume that there exists two disjoint subsets  $E_1$  and  $E_2$  of E such that:

- (a) R restricted to  $E_i$ , i = 1, 2 is isomorphic to R, or at least has the same age as R.
- (b) The isomorphism type of a set  $A \subseteq E_1 \cup E_2$  is entirely determined by the isomorphism types of  $A \cap E_1$  and  $A \cap E_2$ .

Define the following graded algebra morphism on the set algebra:

(7) 
$$\Delta_E^{E_1, E_2} : \begin{cases} \mathbb{K}^{[E]^{<\omega}} & \twoheadrightarrow \mathbb{K}^{[E_1]^{<\omega}} \otimes \mathbb{K}^{[E_2]^{<\omega}} \\ A & \mapsto \begin{cases} A \cap E_1 \otimes A \cap E_2 & \text{if } A \subseteq E_1 \cup E_2, \\ 0 & \text{otherwise.} \end{cases}$$

**Lemma 4.1.**  $\Delta_E^{E_1,E_2}$  induces a coproduct  $\Delta$  on  $\mathbb{K}.\mathcal{A}(R)$ , that is a graded algebra morphism from  $\mathbb{K}.\mathcal{A}(R)$  to  $\mathbb{K}.\mathcal{A}(R) \otimes \mathbb{K}.\mathcal{A}(R)$ .

Proof. All we have to check is that  $\Delta_E^{E_1,E_2}$  indeed maps  $\mathbb{K}.\mathcal{A}(E)$  to  $\mathbb{K}.\mathcal{A}(E_1)\otimes\mathbb{K}.\mathcal{A}(E_2)$ . Things are easier in the graded dual which is the quotient of the set algebra by the isomorphism equivalence relation. There, we need to check that the dual product is well defined; that is, given two types  $\tau_1 \in \mathcal{A}(E_1)$  and  $\tau_2 \in \mathcal{A}(E_2)$ , the type of the product  $A_1 \cup A_2$  of two representatives  $A_1$  and  $A_2$  of the types  $\tau_1$  and  $\tau_2$  respectively shall be independent of that choice of representatives. This is exactly condition (b). By condition (a),  $\Delta$  can then be interpreted as going from  $\mathbb{K}.\mathcal{A}(R)$  to  $\mathbb{K}.\mathcal{A}(R)\otimes\mathbb{K}.\mathcal{A}(R)$  and is, by construction, an algebra morphism.

Let in addition  $\phi_1, \phi_2$  be isomorphisms from E to  $E_1, E_2$  respectively, and denote by  $E_{i,j} := \phi_i(E_j)$  the induced copy of  $E_j$  inside  $E_i$ . Consider the induced bijection

$$\Phi: \begin{cases} E_{1,1} \times E_{1,2} \times E_2 & \hookrightarrow E_1 \times E_{2,1} \times E_{2,2} \\ (a,b,c) & \mapsto (\phi_1^{-1}(a), \phi_1 \circ \phi_2 \circ \phi_1^{-1} \circ \phi_2^{-1}(b), \phi_2(c)) \end{cases},$$

and assume further

(c)  $\Phi$  is an isomorphism: namely, if A, B, C are three subsets of  $E_{1,1}, E_{1,2}, E_2$  respectively and  $A', B', C' = \Phi(A, B, C)$ , then  $A \cup B \cup C \subset E$  and  $A' \cup B' \cup C' \subset E$  have the same type in  $\mathcal{A}(R)$ .

**Proposition 4.2.** Let R,  $E_1$ ,  $E_2$ ,  $\Delta$ ,  $\phi_1$ ,  $\phi_2$ , as above satisfying (a) and (b) and (c). Then the coproduct is coassociative, turning  $\mathbb{K}.\mathcal{A}(R)$  into a freely generated graded connected commutative Hopf algebra. In particular, its Hilbert series is of the form:

(8) 
$$\mathcal{H}_R(Z) = \prod_i \frac{1}{(1 - Z^{d_i})},$$

where the product may be infinite.

*Proof.* Working in the graded dual as in the proof of Lemma 4.1, we need to check the dual product is associative. Take A, B, C representatives of three orbits; without loss of generality they may be chosen in  $E_{1,1}, E_{1,2}, E_2$  respectively. Take  $A', B', C' = \Phi(A, B, C)$ , and note that A', B', C' are representatives of the same orbit. Condition (c) guarantees exactly that the product  $(A \cup B) \cup C$  and  $A' \cup (B' \cup C')$  are in the same orbit, as desired.

Since  $\mathbb{K}.\mathcal{A}(R)$  is graded connected, most axioms of Hopf algebras (in particular concerning the antipode) are satisfied for free. The age algebra is then a graded commutative Hopf algebra. Using the classical Milnor-Moore theorem, it must be a free commutative algebra. The form of the Hilbert series follows.

#### APPENDIX A. MORE EXAMPLES OF RELATIONAL STRUCTURES AND AGE ALGEBRAS

## A.1. Examples with or without a Hopf age algebra structure.

Example A.1. Take the set (-1,1) equipped with either its natural antichain or chain structure, and some relational structure S on some set F. Construct the relational structure  $R := \mathbb{Q} \wr S$  on  $E := \mathbb{Q} \times F$  by substituting each rational number by a copy of F. Define  $E_1 = (-1,0) \times F$  and  $E_2 = (0,1) \times F$ . Consider the affine isomorphisms  $\phi_1$  and  $\phi_2$  mapping E on  $E_1$  and  $E_2$  respectively, so that, e.g.,  $E_{2,1} = (0,\frac{1}{2})$ .

Then, the axioms (a), (b), (c) of Proposition 4.2 are satisfied, and  $\mathbb{K}.\mathcal{A}(R)$  is a Hopf algebra. When S is an infinite chain and  $\mathbb{Q}$  is equipped respectively with its natural antichain or chain structure, one get respectively the Hopf algebras of symmetric and quasi-symmetric functions on the monomial basis.

Example A.9 of [PT13] is obtained by taking a finite graph G for S and the antichain on  $\mathbb{Q}$ . We recover the free algebra generated by connected finite restrictions of S. This generalizes immediately for any relational structure S.

In the examples above, the construction mimics the standard trick of doubling the alphabet to construct Hopf algebras (see e.g. [DHT02, Section, 3.2] or [Hiv05a, Pri13]) which is our original inspiration. The existence of a coassociative coproduct is a very strong constraint, and it is not clear whether there exist interesting coassociative examples where the construction really goes beyond this trick.

Example A.2. Let  $X := x_1, \ldots, x_n$ , and consider the polynomial rings  $\mathbb{K}[X]$  realized as invariant ring of the trivial permutation groupoid  $\mathbb{K}[X] = \mathbb{K}[X]^{\mathrm{id}}$ . One can take as relational structure  $R := X \times \mathbb{Q}$  where each piece  $\{x_i\} \times \mathbb{Q}$  is colored differently by a unary predicate. Taking  $E_1 := X \times (-1,0)$  and  $E_2 := X \times (0,1)$ , Proposition 4.2 endows  $\mathbb{K}[X]$  with its usual Hopf algebra structure where the generators  $x_i$  are primitive.

Example A.3. The age algebra of a direct sum  $R := K_{\omega} \oplus \cdots \oplus K_{\omega}$  of k infinite cliques is the algebra of symmetric polynomials on k variables (see [PT13, Example A.2]). Since it is a free algebra, it can be endowed with a Hopf algebra structure (for example by making its generators group-like). Yet, we have not found a way to achieve this using Proposition 4.2 on this particular relational structure R.

Example A.4 (The Planar Shuffle Algebra). In [PT13, Section A.4], we realized the Planar Shuffle Algebra of Gerritzen [DG04, Ger04b, Ger04a] as an age algebra.

Consider the infinite tree T depicted in Figure 1 of [PT13, Section A.4]. Recall that the relational structure consists of the infix total order and three ternary relations on the set E of leaves of T. Choose two non-leaf children  $x_1$  and  $x_2$  of the root of T, and define  $E_1$  and  $E_2$  respectively as the leaves of the subtrees  $E_1$  and  $E_2$  of  $E_3$  dandling from  $E_3$  and  $E_4$  respectively as the leaves of the subtrees  $E_3$  and  $E_4$  dandling from  $E_3$  and  $E_4$  respectively as the leaves of the subtrees  $E_3$  and  $E_4$  dandling from  $E_4$  and  $E_5$  dandling from  $E_4$  and  $E_5$  dandling from  $E_4$  and  $E_5$  dandling from  $E_5$  dandling

respectively. Choose isomorphisms  $\phi_1$  and  $\phi_2$  from E to  $E_1$  and  $E_2$  respectively. Define  $E_{i,j}$ accordingly.

Then, conditions (a) and (b) of Proposition 4.2 are satisfied, but not condition (c). Take indeed a, b, c in  $E_{1,1}, E_{1,2}, E_{2,2}$  respectively, and define  $a', b', c' := \Phi(a, b, c)$  in  $E_1, E_{2,1}, E_{2,2}$ respectively. The set  $\{a,b,c\}$  has type  $((\circ,\circ),\circ)$  whereas the set  $\{a',b',c'\}$  has type  $(\circ,(\circ,\circ))$ . We recover the non-coassociative coproduct of the Planar Shuffle Algebra which splits the children of the root nodes in two consecutive ranges in all possible ways, and reduces the two resulting trees. For example:

$$\Delta(\circ) = \circ \otimes 1 + 1 \otimes \circ$$

$$\Delta((\circ, \circ)) = (\circ, \circ) \otimes 1 + \circ \otimes \circ + 1 \otimes (\circ, \circ)$$

$$\Delta(((\circ, \circ), \circ)) = ((\circ, \circ), \circ) \otimes 1 + (\circ, \circ) \otimes \circ + 1 \otimes ((\circ, \circ), \circ).$$

Iterating the above,  $\circ \otimes \circ \otimes \circ$  appears in  $(\Delta \otimes id)(\Delta(((\circ, \circ), \circ)))$  but not in  $(id \otimes \Delta)(\Delta(((\circ, \circ), \circ)))$ ; that's the dual of the aforementioned counter example to (c).

As far as we know, there currently is no known coassociative coproduct for this algebra, though it's likely to exist.

## A.2. Examples with a finite monomorphic decomposition.

Example A.5. This example features another age algebra that is not finitely generated.

Consider the relational structure  $R := (E, \rho)$ , where  $E := \mathbb{N} \times \{1, 2, 3\}$  is endowed with the ternary relation  $\rho := \{((i,1),(j,2),(k,3)),\ i,j,k\in\mathbb{N}\}.$  The minimal monomorphic decomposition is given by  $(E_i := \mathbb{N} \times \{i\})_{i \in X := \{1,2,3\}}$ . A basis of the age algebra  $\mathbb{K}.\mathcal{A} =$  $\mathbb{K}.\mathcal{A}(R)$  in degree d is given by

$$x^{\mathbf{d}}$$
, for  $d_i > 0$ ,  $d_1 + d_2 + d_3 = d$ 

together with

$$\sum_{\mathbf{d} \ : \ d_1 \, = \, 0 \text{ or } d_2 \, = \, 0 \text{ or } d_3 \, = \, 0, \, d_1 \, + \, d_2 \, + \, d_3 \, = \, d} x^{\mathbf{d}}.$$

The profile is given by  $\phi_R(d) = {d-1 \choose 2} + 1$  and the Hilbert series is  $(\frac{x}{1-x})^3 + \frac{1}{1-x} = \frac{x^3 + x^2 - 2x + 1}{(1-x)^3}$ . By construction, the restriction of E on  $E_1 \cup E_2$  is monomorphic. Therefore, the minimal

monomorphic decomposition is not hereditary minimal and the age algebra is not finitely generated.

Example A.6. A variation of Example A.19 of [PT13] featuring a finitely generated age algebra with a Hilbert series whose numerator cannot be chosen with non-negative coefficients. This one has only two monomorphic parts which are both infinite.

Let  $R := (E, \rho)$ , where  $E := \mathbb{N} \times \{0, 1\}, \ \rho := [\mathbb{N} \times \{0\}]^3 \cup [\mathbb{N} \times \{1\}]^3$ . Then R has two monomorphic parts, namely  $\mathbb{N} \times \{0\}$  and  $\mathbb{N} \times \{1\}$ . Each type of n-element restriction has a representative made of a m+k element subset of  $\mathbb{N} \times \{0\}$  and of a m-element subset of  $\mathbb{N} \times \{1\}$  such that n = 2m + k; these representatives are non-isomorphic, except if n = 2(in the later case, all 2 -element restrictions are isomorphic, hence we may eliminate the representative corresponding to m=1, k=0). With this observation, a straightforward computation shows that  $\varphi_R(0) = \varphi_R(1) = \varphi_R(2) = 1$  and  $\varphi_R(n) = \lfloor \frac{n}{2} \rfloor + 1$  for  $n \geq 3$ . Hence the generating series  $H_{\varphi_R} = \frac{1}{(1-Z)(1-Z^2)} - Z^2 = \frac{1-Z^2+Z^3+Z^4-Z^5}{(1-Z)(1-Z^2)}$ .

But, then  $H_{\varphi_R}$  cannot be written as a quotient of the form  $\frac{P}{(1-Z)(1-Z^k)}$  where P is a polynomial with non-negative integer coefficients. Suppose indeed that  $H_{\varphi_R}$  is of this form.

We may assume k even (otherwise, multiply P and  $(1-Z)(1-Z^k)$  by  $(1+Z^k)$ . Set  $k':=\frac{k}{2}$ . Multiplying  $1-Z^2+Z^3+Z^4-Z^5$  and  $(1-Z)(1-Z^2)$  by  $1+Z^2+\cdots+Z^{2(k'-1)}$ , we get

 $P = (1 - Z^2 + Z^3 + Z^4 - Z^5)(1 + Z^2 + \cdots + Z^{2(k'-1)})$ . Hence, the term of largest degree has a negative coefficient, a contradiction.

Example A.7. Another variation on Example A.19 of [PT13], with four variables; now the numerator can take either positive or negative coefficients.

Let  $R := (E, (\rho, U_2, U_3))$ , where  $E := \mathbb{N} \times \{0, 1, 2, 3\}$ ,  $\rho := \{((n, i), (m, j)) : i = 0, j \in \{1, 2\}$  or  $i = 1, j = 3\}$ ;  $U_i := \mathbb{N} \times \{i\}$  for  $i \in \{2, 3\}$ . Then R has four monomorphic components, namely  $\mathbb{N} \times \{0\}$ ,  $\mathbb{N} \times \{1\}$ ,  $\mathbb{N} \times \{2\}$ ,  $\mathbb{N} \times \{3\}$ . Let S be the induced structure on four elements of the form  $(x_i, i), i \in \{0, 1, 2, 3\}$ . A crucial property is that S has only two non-trivial local isomorphisms, namely the map sending  $(x_0, 0)$  onto  $(x_1, 1)$  and its inverse. From this follows that the induced substructures on two n-element subsets E are isomorphic if either they have the same number of elements on each  $\mathbb{N} \times \{i\}$  or one subset is included into  $\mathbb{N} \times \{0\}$ , the other into  $\mathbb{N} \times \{1\}$ . Hence, the generating series  $\mathcal{H}_{\varphi_R}$  is  $\frac{1}{(1-Z)^4} - \frac{Z}{1-Z} = \frac{1-Z+3Z^2-3Z^3+Z^4}{(1-Z)^4}$ . We may write it  $\mathcal{H}_{\varphi_R} = \frac{Q_1}{(1-Z)(1-Z^4)(1-Z^5)(1-Z^5)}$  where  $Q_1 := 1 + 2Z + 6Z^2 + 10Z^3 + 14Z^4 + 17Z^5 + 18Z^6 + 14Z^7 + 10Z^8 + 6Z^9 + Z^{10}$ , as well as  $\mathcal{H}_{\varphi_R} = \frac{Q}{(1-Z)(1-Z^5)^3}$  where  $Q_2 := 1 + 2Z + 6Z^2 + 10Z^3 + 15Z^4 + 18Z^5 + 22Z^6 + 18Z^7 + 15Z^8 + 10Z^9 + 6Z^{10} + Z^{12} + Z^{16}$ .

## A.3. Example with polynomial growth but infinitely many monomorphic parts.

Examples A.8. Consider the direct sum  $R' := \overline{K}_{\infty} \wr K_{1,1} \oplus R$  of the infinite matching and the relational structure R of Example 2.2. The profile has polynomial growth, but R has infinitely many monomorphic parts and the age algebra is not finitely generated.

## A.4. Miscellaneous example.

Example A.9. This example illustrates why some care needs to be taken when defining the age algebra for a relational structure with non finite profile.

Take an infinite set E, with one binary relation for each couple (i, j) of distinct elements of E, which holds just on (i, j). In degree 1, there is a single type; let  $e_1$  be the corresponding element in the age algebra. Then,  $e_1^2$  shall be the sum of all the types of degree 2, of which there are infinitely many. Hence, for the age algebra to be indeed stable by multiplication, one shall consider infinite linear combinations of types.

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