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Fiber polytopes and fractional power series

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Abstract

This paper explores the power series expansions of polynomial equations in N variables. Expansions considered have exponents lying in some convex conical region in $\mathbb{R}^N$. An N-variable analog of the Newton polygon construction for polynomials in two variables is used to construct such series expansions. The structure of these series is related to the theory of fiber polytopes as introduced by Billera and Sturmfels in [2], and this relationship is used to draw conclusions about certain ramification loci.

1. Introduction

Let $F(x_1, \ldots, x_{N+1}) = 0$ be an algebraic equation with complex coefficients. We are interested in the fractional power series expansions $\phi(x_1, \ldots, x_N)$ such that

$$F(x_1, \ldots, x_N, \phi) = 0.$$

For the case $N = 1$, such expansions can be obtained using a construction due to Newton involving the Newton polygon of F [11].

It may seem that the construction in the N-variable case could be obtained by iterating Newton's construction. However, such an iteration would produce series with increasing powers in all variables, and we are interested in more general series solutions whose exponents lie in some convex conical region in $\mathbb{Q}^N$. It turns out that the structure of such series expansions are related to the Newton polytope, $P(F)$, of F.

The fact that we have singled out x_{N+1} for the construction defines a projection $\psi: P(F) \to \mathbb{R}$. Our main result is that, under certain discriminantal conditions, the full

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systems of series solutions of $F = 0$ are in one-to-one correspondence with the vertices of the fiber polytope Σ_ψ of ψ introduced by Billera and Sturmfels [2]. In the case $N = 1$ the fiber polytope turns out to be a line segment, and the complete systems of solutions consist of expansions in increasing or decreasing powers of x.

As an application of the methods in this paper, we will look at the ramification locus of the projection $\pi: X \to (\mathbb{C}^*)^N$ where X is the hypersurface defined by $F = 0$ in $(\mathbb{C}^*)^{N+1}$. Under the same discriminantal conditions, such complete systems give a bound on the ramification locus of the projection.

2. Rings of fractional power series

2.1. Example: Rational functions. To motivate the ideas that follow, we recall the special case where x_{N+1} is a rational function in $x_1, \dots, x_N$. Let

$$x_{N+1} = \frac{1}{F(x_1, \dots, x_N)}. \tag{1}$$

For notational convenience if $I = (i_1, \dots, i_N) \in \mathbb{R}^N$ we will let $a_I x^I$ denote

$$a_{(i_1, \dots, i_N)} x_1^{i_1} \cdots x_N^{i_N}.$$

Definition 2.2. For a polynomial

$$F(x_1, \dots, x_m) = \sum a_I x^I$$

the Newton polytope of F is the convex hull of the set of exponents in F, i.e.

$$P(F) = \operatorname{conv}\{I \in \mathbb{Z}^m \colon a_I \neq 0\} \subset \mathbb{R}^m.$$

Consider the Newton polytope of $F(x)$, see Fig. 1. We can expand $1/F(x)$ as a power series in different ways using the geometric series expansion. Factor out of $f(x)$ a monomial that corresponds to one of the vertices of $P(F)$.

$$x_{N+1} = \frac{1}{a_{I_0} x^{I_0}} \cdot \frac{1}{1 + g(x)},$$

where $g(x) = \sum_{I \neq I_0} a_I x^I / a_{I_0} x^{I_0}$. By the geometric expansion we get

$$x_{N+1} = \frac{1}{a_{I_0} x^{I_0}} \sum_{i=0}^{\infty} (-1)^i g^i(x). \tag{2}$$

By factoring out one monomial we have shifted the exponents in the denominator so that the new denominator has the polytope shown in Fig. 2, a translate of the original polytope. We are taking the geometric progression of terms that lie in the cone C spanned by the faces containing the chosen vertex, so all monomials appearing in series (2) also lie in this cone.

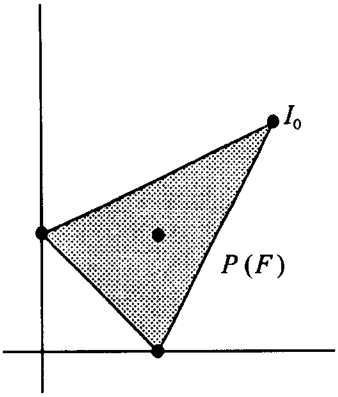

Fig. 1.

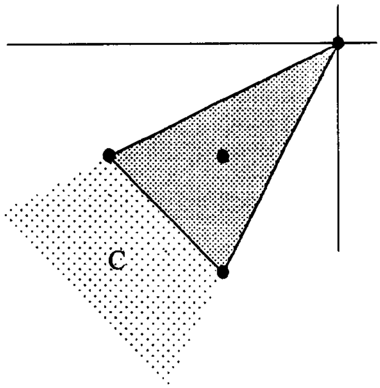

Fig. 2.

If we had tried to factor out a monomial corresponding to a point of $P(F)$ that was not a vertex we would have gotten a series in which finding a coefficient would involve summing an infinite number of terms. For obvious reasons, we avoid such expansions. The construction for rational functions can be summarized by the following theorem. For more information see [4, Chapter 6].

Theorem 2.3. *The series expansions of $x_{N+1} = 1/f(x_1, \dots, x_N)$ with monomials given by points in some convex cone correspond to the vertices of the Newton polytope of $f(x_1, \dots, x_N)$. If a vertex v of $P(f)$ corresponds to the series expansion $\phi(x_1, \dots, x_N)$, then the monomials in ϕ correspond to points lying in some translate of the cone spanned by the faces which contain v.*

2.4. Fractional power series and support cones. We will use the notation in [3] for the discussion of the cones. Consider a real vector space $V = \mathbb{R}^N$. A convex polyhedral cone in $\mathbb{R}^N$ is a set of the form

$$C = \{r_1 v_1 + \cdots + r_k v_k \colon r_i \in \mathbb{R},\ r_i \geq 0\},$$

where $v_1, \dots, v_k \in \mathbb{R}^N$ are vectors. A cone is called rational if v_i lies in $\mathbb{Q}^N$ for every i, and is called strongly convex if it contains no nontrivial linear subspaces.

We identify the dual space V^* of V with $\mathbb{R}^N$ by means of the usual pairing $\langle u, x \rangle = \sum u_i x_i$. Let C be a strongly convex rational polyhedral cone in $\mathbb{R}^N$. Define the dual cone, $C^* \subset V^*$, to be the set

$$C^* = \{u \in \mathbb{R}^N \colon \langle u, x \rangle \leq 0\ \forall x \in C\}.$$

Let w be any linear function on $\mathbb{R}^N$, we extend w trivially to a linear function of $\mathbb{R}^{N+1}$ by defining for $x \in \mathbb{R}^{N+1}$

$$\langle w, x \rangle := \langle w, (x_1, \dots, x_N) \rangle.$$

A hyperplane in H in $\mathbb{R}^{N+1}$ is called w-constant if for each $c \in \mathbb{R}$

$$\langle w, H \cap \{x_{N+1} = c\}\rangle := \{\langle w, x\rangle : x \in H \cap \{x_{N+1} = c\}\} = \{d_c\}$$

for some $d_c \in \mathbb{R}$. I.e. w is constant on each "vertical" section of H.

Note that through any line in $\mathbb{R}^{N+1}$ that is not parallel to the $x_1, \dots, x_N$-hyperplane there goes a unique w-constant hyperplane. Since we will be using the $x_1, \dots, x_N$-hyperplane frequently, we will call it the null hyperplane.

If n is an integer greater than zero, then the set

$$C_n = C \cap \frac{1}{n}\mathbb{Z}^N$$

forms a semigroup under addition. From such a semigroup we can form the semigroup ring $\mathbb{C}[C_n]$, i.e. the ring of all finite formal sums of the form $\sum a_\alpha x^\alpha$ where $\alpha \in C_n$. We regard elements of $\mathbb{C}[C_n]$ as fractional Laurent polynomials in the variables $x_1, \dots, x_n$. Let $\mathbb{C}[[C_n]]$ be the completion of the ring $\mathbb{C}[C_n]$, i.e. the ring of all formal fractional power series, $\sum_{\alpha \in C_n} a_\alpha x^\alpha$.

Definition 2.5. If C is a strongly convex rational polyhedral cone in $\mathbb{R}^N$, then the ring of fractional power series in the variables $x_1, \dots, x_N$ with support in C is defined by

$$\mathbb{C}[[C_\mathbb{Q}]] = \bigcup_{n=1}^{\infty} \mathbb{C}[[C_n]].$$

More generally, the ring of fractional power series with support in some translate of C is

$$\mathbb{C}((C_\mathbb{Q})) = \bigcup_{\alpha \in \mathbb{Q}^N} x^\alpha \mathbb{C}[[C_\mathbb{Q}]].$$

It is essential to require that C be strongly convex, otherwise the set $\mathbb{C}[[C_\mathbb{Q}]]$ does not have a well defined multiplicative structure, since finding a coefficient when multiplying two general series involves an infinite sum.

Let C be a strongly convex rational polyhedral cone. For any

$$f(x) = \sum_{\alpha \in \mathbb{Q}^N} a_\alpha x^\alpha$$

in $\mathbb{C}((C_\mathbb{Q}))$ we will define the support of f as the set of exponents which appear in f, i.e. $\mathrm{Supp}(f) = \{\alpha \in \mathbb{Q}^N : a_\alpha \neq 0\}$. Since $f \in x^\alpha \mathbb{C}[[C_n]]$ for some n, the support of f must lie in some lattice $(1/n)\mathbb{Z}$.

For example, if $N = 1$ and $C = \mathbb{R}_+$, then $\mathbb{C}[[C_1]]$ is the usual ring of Laurent power series, $\mathbb{C}[[x]]$, over the complex numbers, and $\mathbb{C}((C_\mathbb{Q})) = \bigcup_{\alpha,n} x^\alpha \mathbb{C}((x^{1/n}))$ is the ring of fractional Laurent series in one variable as in [11] where it is denoted $\mathbb{C}(x)^*$. In [11] the Newton polygon construction is used to show that $\mathbb{C}(x)^*$ is an algebraically closed field.

2.6. Convergence of elements of $\mathbb{C}((C_{\mathbb{Q}}))$. In order to speak of the convergence of fractional power series in $\mathbb{C}((C_{\mathbb{Q}}))$, we must define the manner in which these series act as functions on $\mathbb{C}^N$. More precisely, we must define the action of $x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N}$ on $(\mathbb{C}^*)^N$. To do this we only need to choose, in each variable, a sector in $\mathbb{C}^*$ and define a branch of the logarithm in this sector, e.g. the principal branch of the log: Let $\mathbb{C} \setminus \mathbb{R}_-$ be the chosen sector and define

$$x_i^{\alpha_i} = e^{\alpha_i \log x_i}$$

for each variable x_i. We are primarily interested in the regions for which an $f \in \mathbb{C}[[C]]$ is absolutely convergent (i.e. where $\sum |a_\alpha| |x|^\alpha$ converges).

Definition 2.7. If C is a convex rational polyhedral cone, then $\mathbb{C}\{\{C_{\mathbb{Q}}\}\}$ will denote the subset of $\mathbb{C}((C_{\mathbb{Q}}))$ consisting of all series which are convergent at some point of $(\mathbb{C}^*)^N$, i.e. if for $f \in \mathbb{C}((C_{\mathbb{Q}}))$, D_f is the domain of convergence of f, then

$$\mathbb{C}\{\{C_{\mathbb{Q}}\}\} = \{f \in \mathbb{C}((C_{\mathbb{Q}})) \colon D_f \neq \emptyset\}.$$

Note that $\mathbb{C}\{\{C_{\mathbb{Q}}\}\}$ consists only of convergent series whose exponents lie in $\mathbb{Z}[1/n]$ for some n.

It is convenient to pass to the logarithms of the $|x_i|$ when considering convergence, therefore we introduce the space $\mathbb{R}^N_{\log}$ called the logarithmic space of $(\mathbb{C}^*)^N$. This space is associated to $(\mathbb{C}^*)^N$ via the map

$$\mathrm{Log} \colon (\mathbb{C}^*)^N \to \mathbb{R}^N$$

given by

$$\mathrm{Log}(x_1, \ldots, x_N) = (\log(|x_1|), \ldots, \log(|x_N|)).$$

The usefulness of this notation is indicated by the following lemma.

Lemma 2.8. *For each $f \in \mathbb{C}\{\{C_{\mathbb{Q}}\}\}$ the domain of convergence of f has the form $\mathrm{Log}^{-1}(U)$, for some convex set $U \subset \mathbb{R}^N_{\log}$.*

Proof. For each such f there exists some $n \in \mathbb{Z}_+$ such that $f \in \mathbb{C}((C_n))$. Therefore, this lemma follows, by a change of variables, from the well known fact [6] that this is true for power series with integer exponents. □

Lemma 2.9. *Suppose $f = \sum a_\alpha x^\alpha$ is in $\mathbb{C}((C_{\mathbb{Q}}))$ and f has a nonempty domain of convergence D (i.e. $f \in \mathbb{C}\{\{C_{\mathbb{Q}}\}\}$), then there exists some $A \in (\mathbb{C}^*)^N$ such that $|a_\alpha| \leq |A^\alpha|$ for almost all α. Moreover, if x is any point in D, and C is any cone which contains the Newton polytope $P(f)$, then $C^* + \mathrm{Log}(x) \subset \mathrm{Log}(D)$.*

Proof. Suppose $x \in (\mathbb{C}^*)^N$ satisfies $\sum |a_\alpha| |x|^\alpha \leq \infty$. We may assume that $|a_\alpha| |x|^\alpha \leq 1$ for all α, since this must be true for all but a finite number of α. Also we can assume that $\mathrm{Supp}(f) \subset C$ since $f = x^\beta g$ for some β where g has support lying in C. Now if we rewrite the above inequality, we get

$$|a_\alpha| \leq \left|\frac{1}{x^\alpha}\right| = \left|\left(\frac{1}{x_1}, \dots, \frac{1}{x_N}\right)^\alpha\right|.$$

Suppose $x' \in (\mathbb{C}^*)^N$ such that $\mathrm{Log}(x') \in C^* + \mathrm{Log}(x)$. Then since $\mathrm{Log}(x') = w + \mathrm{Log}(x)$ for some $w \in C^*$ and $\langle w, \alpha \rangle \leq 0$, we have that for each $\alpha \in C$

$$\langle \mathrm{Log}(x'), \alpha \rangle \leq \langle \mathrm{Log}(x), \alpha \rangle$$

and so

$$\alpha_1 \log(|x_1'|) + \cdots + \alpha_N \log(|x_N'|) \leq \alpha_1 \log(|x_1|) + \cdots + \alpha_N \log(|x_N|)$$

which implies that

$$\log(|x_1'^{\alpha_1} \cdots x_N'^{\alpha_N}|) \leq \log(|x_1^{\alpha_1} \cdots x_N^{\alpha_N}|).$$

Since, on $\mathbb{R}_+$, the function log is monotone increasing, we get

$$|x_1'^{\alpha_1} \cdots x_N'^{\alpha_N}| \leq |x_1^{\alpha_1} \cdots x_N^{\alpha_N}|.$$

Therefore, for every $\alpha \in C$, $|x'|^\alpha \leq |x|^\alpha$, yielding that $\sum |a_\alpha| |x'|^\alpha \leq \sum |a_\alpha| |x|^\alpha$. □

We say that f converges at some point $y \in \mathbb{R}^N_{\log}$ if $\mathrm{Log}^{-1}(y) \subset D$ where D is the domain of convergence for f. The above lemma can be summarized by saying that if f converges at some point $y \in \mathbb{R}^N_{\log}$ then f converges on some translate of C^*. Using these two lemmas it is now possible for us to prove the following theorem.

Theorem 2.10. *If C is a cone in $\mathbb{R}^N$ and $f \in \mathbb{C}(C_\mathbb{Q}))$ is algebraic over $\mathbb{C}[x_1, \dots, x_N]$ then there is some translate of C^* on which f is convergent.*

Proof. The fact that there is a point at which f converges follows from [1]. Given this, the theorem then follows from Lemma 2.9. □

3. Fractional power series solutions to algebraic equations

3.1. Newton polytopes and normal cones. With these preliminaries we can now turn to the question posed in the introduction: How does one construct a fractional power series $x_{N+1} = \phi(x_1, \dots, x_N)$ which satisfies a given algebraic equation $F(x_1, \dots, x_{N+1}) = 0$? To answer this we will need to closely investigate the Newton polytope of F. Two constructions associated with the polytope will be of particular interest, normal cones and normal wedges.

Definition 3.2. Let v be any vertex of $P(F)$, the barrier cone of v is the cone

$$C(v) = \{\lambda(p - v) : \lambda \in \mathbb{R}_+,\ p \in P(F)\},$$

i.e. the cone spanned by the vectors from v to points in $P(F)$. We have already seen an example of this: the cone C in Fig. 2. The dual cone of $C(v)$ is called the normal cone of v and is denoted $C^*(v)$.

Let e be any edge of $P(F)$ which is not parallel to the null hyperplane. We will refer to edges with this property as admissible edges. The vertices of e with the largest and smallest x_{N+1} coordinates will be respectively called the major and minor vertices of e, and will be denoted by $m(e)$ and $M(e)$, respectively. Let $p = (p_1, \dots, p_{N+1})$ and $q = (q_1, \dots, q_{N+1})$ be the two endpoints of e. Define the slope, $S(e)$, of e with respect to x_{N+1} to be the vector

$$S(e) = \frac{1}{q_{N+1} - p_{N+1}}(q_1 - p_1, \dots, q_N - p_N).$$

For such an edge we define the barrier cone of e as the following subset of $\mathbb{R}^N$.

Definition 3.3. Let L be the line in $\mathbb{R}^{N+1}$ containing the line segment e, and let y be the point of intersection of L with the null hyperplane (such a point exists since e was assumed not parallel to this plane). Then define the barrier wedge of e in $\mathbb{R}^{N+1}$ by

$$W(e) = \{\lambda(p - x) + x : \lambda \in \mathbb{R}_+,\ p \in P(F),\ x \in L\}.$$

The intersection of this wedge with the null hyperplane is a convex rational polyhedral cone, $C(e) + y$, which has its vertex at y. Define the barrier cone of e to be $C(e)$, i.e. the translate of the cone based at the origin. The dual $C^*(e)$ of $C(e)$ will be called the normal cone of e.

Definition 3.4. Let P be a polytope in $\mathbb{R}^{N+1}$, and let $\psi : \mathbb{R}^N \to \mathbb{R}$ be the projection onto the last coordinate. A monotone edge path on P is a sequence $E = \{e_1, \dots, e_n\}$ such that for each i, $M(e_i) = m(e_{i+1})$ and e_i does not lie parallel to the $x_1, \dots, x_N$-plane. Therefore the edge path is increasing with respect to ψ.

A monotone edge path is called coherent if

$$\bigcap_{i=1}^{n} C^*(e_i) \neq \{0\}.$$

3.5. Solutions to algebraic equations. To state the next theorem we will need to refer to a special type of linear functional. A linear functional in $(\mathbb{R}^N)^*$ will be called irrational if its coordinates are linearly independent over $\mathbb{Q}$.

Theorem 3.6. *Let $F(x_1, \dots, x_{N+1})$ be a polynomial in $N+1$ variables. Let e be any admissible edge of the polytope $P(F)$. Let $C^* = C^*(e) \subset \mathbb{R}^N$ be its normal cone, and let k_e be the length of the projection of e onto the x_{N+1}-axis. Then*

(a) *For each irrational $w \in C^*$ there exists some strongly convex rational polyhedral cone C_w such that $w \in C_w^*$ and such that the ring $\mathbb{C}((C_{w\mathbb{Q}}))$ contains at least k_e series $\{\phi_i\}_i$ counted with multiplicity such that for each i*

$$F(x_1, \dots, x_N, \phi_i) = 0.$$

(b) *In fact, $\mathbb{C}((C_{w\mathbb{Q}}))$ contains, up to multiplicity, exactly k_e series that correspond to e.*

Proof. We will prove part (a) and defer part (b) until a later section. To prove (a) we inductively build a series of the form

$$\sum_{n=1}^{\infty} c_n x_1^{\alpha_{n,1}} \cdots x_N^{\alpha_{n,N}} = \sum_{n=1}^{\infty} \psi_n.$$

So we first build ϕ_1 and then move on to ϕ_n for any n. In most respects, the constructions will be identical.

Let

$$F_1(x_1, \dots, x_{N+1}) = F(x_1, \dots, x_{N+1}) = \sum_{I \in S_1} a_I x^I,$$

where $S_1 = \mathrm{Supp}(F)$. Let $e_1 = e$ be the edge of $P(F_1)$ chosen in the hypotheses, and let $N(e)$ and $C(e)$ be the barrier wedge and cone of e_1, respectively. The edge e_1 has a slope with respect to x_{N+1}, say $S(e_1) = (s_{1,1}, \dots, s_{1,N})$. Define

$$\phi_1(x_1, \dots, x_N) = c_1 x_1^{-s_{1,1}} \cdots x_N^{-s_{1,N}},$$

where c_1 is a solution of the equation

$$F_e(t) = \sum_{(i_1, \dots, i_{N+1}) \in e_1 \cap S_1} a_{i_1, \dots, i_{N+1}} t^{i_{N+1} - m(e)_{N+1}} = 0. \tag{3}$$

where $m(e)_{N+1}$ is the $(N+1)$st coordinate of the minor vertex. We will refer to this equation as the edge equation of e. This sum ranges over all of the terms of F_1 which correspond to points on the edge e_1. Such a solution exists since there must be at least two points of S on e_1.

The degree of Eq. (3) is equal to the difference between the largest and the smallest i_{N+1} appearing in $e_1 \cap S$. Therefore, the number of solutions to Eq. (3), counting multiplicity, is equal to the length, k_e, of the projection of the edge $e = e_1$ onto the x_{N+1}-axis.

For the rest of this construction we will need to use the chosen element w of the normal cone of e_1. By assumption w is irrational. So, for any $\alpha \neq \alpha' \in \mathbb{Q}^N$ we have $\langle w, \alpha \rangle \neq \langle w, \alpha' \rangle$, and hence w induces a linear order on $\mathbb{Q}^N$. If $\psi = x_1^{i_1} \cdots x_N^{i_N}$ is a monomial in N variables, we will use the notation $\langle w, \psi \rangle$ to indicate the value $\langle w, (i_1, \ldots, i_N) \rangle$.

Assume that $F_{n-1}(x_1, \ldots, x_{N+1})$, e_{n-1} (an edge of $P(F_{n-1})$), and ϕ_{n-1} have been constructed. Let F_n be defined by

$$F_n(x_1, \ldots, x_{N+1}) = F_{n-1}(x_1, \ldots, x_N, \psi_{n-1} + x_{N+1}).$$

We assume that $\psi_n = 0$ is not a solution of $F_n = 0$. If it were we would have the desired solution of $F = 0$.

To construct ϕ_n we will choose an edge on the Newton polytope of F_n satisfying the conditions of the following lemma. For every $i = 1, \ldots, n-1$ we let k_i be the multiplicity of c_i as a root of the edge equation of e_i.

Lemma 3.7. *On the Newton polytope $P(F_n)$ there is a unique coherent edge path*

$$E = e_{1,n}, \ldots, e_{k,n}$$

such that

(a) *The major vertex $M(e_{k,n})$ lies on the line, L, through e_{n-1}.*
(b) *The minor vertex $m(e_{1,n})$ lies on the null hyperplane.*
(c) *The major vertex $M(e_{k,n})$ has x_{N+1} coordinate equal to k_{n-1}.*
(d) $\langle w, s(e_{n-1}) \rangle < \langle w, s(e_{k,n}) \rangle < \cdots < \langle w, s(e_{1,n}) \rangle$.
(e) $w \in \bigcap_{i=1}^{k} C^*(e_{i,n})$.

The last condition assures us that for each edge $e_{i,n}$ the unique w-constant hyperplane containing $e_{i,n}$ is a supporting hyperplane for the polytope $P(F_n)$.

Let e_n be any edge on the edge path of Lemma 3.7, and define

$$\psi_n = c_n x_1^{-s_{n,1}} \cdots x_N^{-s_{n,N}},$$

where the N-tuple $(s_{n,1}, \ldots, s_{n,N})$ is the slope of the edge e_n and c_n satisfies the edge equation

$$F_{e_n}(t) = \sum_{(i_1, \ldots, i_{N+1}) \in e_n \cap S_n} a_{i_1}, \ldots, i_{N+1}\, t^{i_{N+1} - m(e)_{N+1}} = 0, \tag{4}$$

where $S_n = \mathrm{Supp}(F_n)$.

The fact that

$$\langle w, s(e_{n+1})\rangle > \langle w, s(e_n)\rangle > \langle w, s(e_1)\rangle$$

assures us that the terms are linearly ordered under the order on $\mathbb{Q}^N$ induced by w. We will define $\phi_n = \phi_{n-1} + \psi_n$. Having inductively constructed ψ_n and ϕ_n for all n we define our candidate for a solution of $F = 0$ to be $\phi = \sum_{n=1}^{\infty} \psi_n$.

To complete this construction and show that ϕ satisfies the conditions of Theorem 3.6(a) we must

(1) prove Lemma 3.7.
(2) show that $\mathrm{Supp}(\phi)$ lies in some proper cone C_w of $\mathbb{R}^N$
(3) show that the exponents of ϕ lie in some lattice $(1/n)\mathbb{Z}$.
(4) show that ϕ satisfies the equation $F(x_1, \ldots, x_{N+1}) = 0$.
(5) show that the number of series, up to multiplicity, created by this process is greater than or equal to the length of the projection of e onto x_{N+1}.

3.8. Proof of Lemma 3.7. We will construct the required edge path and simultaneously prove parts (a), (b) and (e) of Lemma 3.7. Consider F_n and its Newton polytope $P(F_n)$. We will investigate its relationship to F_{n-1} and $P(F_{n-1})$. As above, let $S_{n-1} = \mathrm{Supp}(F_{n-1})$ and

$$F_{n-1}(x) = \sum_{I=(i_0,\ldots,i_{N+1})\in S_{n-1}} a_I x^I.$$

Therefore

$$\begin{aligned} F_n(x_1, \ldots, x_{N+1}) &= \sum_{I\in S_{n-1}} a_I x_1^{i_1} \cdots x_N^{i_N} (\psi_{n-1} + x_{N+1})^{i_{N+1}} \\ &= \sum_{I\in S_{n-1}} a_I x_1^{i_1} \cdots x_N^{i_N} \sum_{j=0}^{i_{N+1}} \binom{N+1}{j} \psi_{n-1}^j x_{N+1}^{i_{N+1}-j}. \end{aligned}$$

If we rearrange this second expression we get

$$F_n(x_1, \ldots, x_{N+1}) = \sum_{I\in S_{n-1}} \sum_{j=0}^{i_{N+1}} \binom{N+1}{j} a_I c_{n-1}^j x_1^{i_1 + j\alpha_{n-1,1}} \cdots x_N^{i_N + j\alpha_{n-1,N}} x_{N+1}^{i_{N+1}-j}. \tag{5}$$

Notice that, in these expressions, the exponent on x_{N+1} is always an integer. Examining these expressions, we can also see that for each term,

$$T = \binom{i_{N+1}}{j} a_I c_{n-1}^j \, x_1^{i_1 + j\alpha_{n-1,1}} \cdots x_N^{i_N + \alpha_{n-1,N}} x_{N+1}^{i_{N+1}-j},$$

the point on the Newton polytope, $P(F_n)$, which corresponds to T, lies on the line through I (a point of $\mathrm{Supp}(F_{n-1})$), with slope $(-\alpha_{n-1,1}, \ldots, -\alpha_{n-1,N})$. So each point on the Newton polytope $P(F_n)$ lies on a line through a point of $P(F_{n-1})$ and parallel to the edge e_{n-1}. One consequence of this is that the Newton polytope $P(F_n)$ is supported by the w-constant hyperplane P_w determined by e_{n-1}.

Consider the summand of F_n whose terms correspond to points of $P(F_n)$ which lie on the line L_{n-1} containing e_{n-1}. Let P_1 be the point $L_{n-1} \cap \{x_{N+1} = 0\}$, i.e. the point of intersection of L_{n-1} and the null hyperplane. Let $P_2 = M(e_{n-1})$ be the major vertex of e_{n-1}. We will examine the coefficients of the monomials in F_n which correspond P_1 and P_2. Both monomials have the possibility of occurring in F_n with nonzero coefficients, since both appear in Eq. (5).

For P_1, the coefficient of the corresponding monomial is

$$\sum_{I \in S_{n-1} \cap e_{n-1}} a_I c_{n-1}^{i_{N+1}}, \tag{6}$$

since all of the terms in expression (5) which contribute to expression (6) must correspond to points on the edge e_{n-1} of $P(F_{n-1})$ and must also have a vanishing x_{N+1}-exponent. By the construction of c_n this sum is equal to 0. Therefore P_1 is not in $P(F_n)$.

The coefficient of the term corresponding to P_2 is unchanged from what it was in F_{n-1}. Any term $a_I x^{i_1} \cdots x^{i_{N+1}}$ other than that corresponding to P_2 contributes only terms corresponding to points lying on the line through I, parallel to e_{n-1}, and lying to the *left* of I (i.e. their x_{N+1} coordinates are less than that of I). Since there are no terms in F_{n-1} corresponding to points on L_{n-1} to the right of P_2, the only contribution to P_2 in F_n comes from P_2 itself. So j in expression (5) is 0, and hence the coefficient remains unchanged.

Lastly we need to note that there are terms in F_n which correspond to points on the null hyperplane. Since $\psi_n = 0$ is not a root of $F_n = 0$, such points exist.

Putting all of this together, we get that there are points of $P(F_n)$ which lie on L_{n-1}. We know that all such points have strictly positive x_{N+1}-exponents, and that there are points of this polytope lying in the null hyperplane, i.e. strictly to the left of all points in $P(F_n)$ lying on L_{n-1}. Hence conditions (a) and (b) of the lemma are satisfied. Since $P(F_n)$ is the convex hull of a set containing these points and lying on one side of the w-constant plane P_w, there must be an edge path, $e_{1,n}, \ldots, e_{k,n}$, on $P(F_n)$ such that $M(e_{k,n})$ lies on L_{n-1} and $m(e_{1,n})$ lies on the null hyperplane, namely the edge path E that maximizes w on each vertical section (See Fig. 3). Hence condition *e*) is satisfied. Since w is irrational, this edge path is unique.

Next we establish (d) of Lemma 3.7 by showing that

$$\langle w, s(e_{k,n}) \rangle > \langle w, s(e_{n-1}) \rangle.$$

The other inequalities follow from the same argument.

Let P_w be the w-constant hyperplane containing e_{n-1}. Note that P_w is a supporting hyperplane for $P(F_{n-1})$ and $P(F_n)$. Define the vectors $u := (u_1, \ldots, u_N, 1)$ and $v_i := (v_1, \ldots, v_N, 1)$ associated to e_{n-1} and $e_{k,n}$ respectively as follows. If

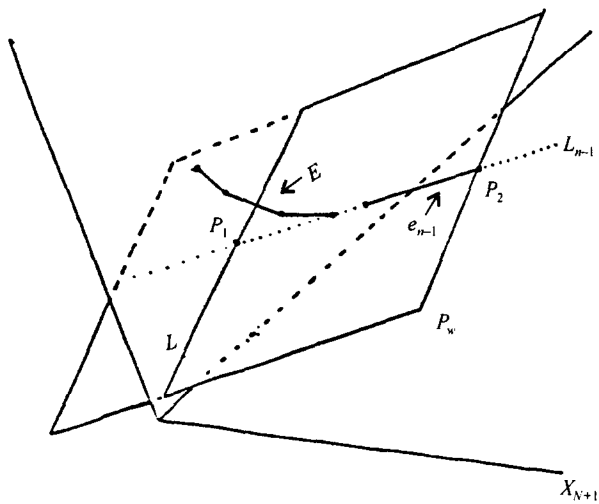

Fig. 3.

$M = (M_1, \dots, M_{N+1})$ and $m = (m_1, \dots, m_{N+1})$ are the major and minor vertices of e_{n-1}, then

$$u_i = \frac{M_i - m_i}{M_{N+1} - m_{N+1}}$$

and likewise for $e_{k,n}$ and v. Note that u and v are tangent to e_n and $e_{k,n}$, and $(u_1, \dots, u_N)$ and $(v_1, \dots, v_N)$ are the slopes of e_n and $e_{k,n}$ with respect to x_{N+1} respectively. If $I = (i_1, \dots, i_{N+1})$ is the point of intersection of $e_{k,n}$ and L_{n-1}, and L_n is the line containing $e_{k,n}$, then the points of intersection of L_{n-1} and L_n with the null hyperplane are

$$p_1 = (i_1 - i_{N+1}u_1, \dots, i_N - i_{N+1}u_N, 0),$$

$$p_2 = (i_1 - i_{N+1}v_1, \dots, i_N - i_{N+1}v_N, 0),$$

respectively. Let L be the line of intersection of the plane P_w with the null hyperplane. P_w contains L_{n-1}, so the point p_1 lies on L while P_2 lies in the interior of the half N-plane determined by P_w that contains $P(F_n) \cap \{x_{N+1} = 0\}$.

By the construction of the linear functional w and the edge path E, we have that $\langle w, p_1 \rangle > \langle w, p_2 \rangle$ and so

$$\langle w, \ -i_{N+1}u \rangle > \langle w, \ -i_{N+1}v \rangle$$

which implies, since w is linear, that $\langle w, u \rangle < \langle w, v \rangle$ and hence that $\langle w, s(e_{k,n}) \rangle > \langle w, s(e_{n-1}) \rangle$. This finishes the proof of condition (d) of Lemma 3.7.

Last, we need to prove condition (c). Let e_n be any edge on the edge path. Note that the normal cone of this edge does indeed intersect $C^*(e_{n-1})$, since in particular w is contained in both of these cones.

We need to show that the x_{N+1} coordinate of $M(e_{k,n})$ is k_n. Consider the derivatives with respect to x_{N+1} of both F_{n-1} and F_n. Recall that

$$F_n = F_{n-1}(x_1, \ldots, x_N, c_{n-1} x_1^{\alpha_{n-1,1}} \cdots x_N^{\alpha_{n-1,N}} + x_{N+1}).$$

Hence, $F_n' = F_{n-1}'(x_1, \ldots, x_N, \psi_n + x_{N+1})$, where F_n' denotes the derivative of F_n with respect to x_{N+1}. Let P and P' be the polytopes of F_{n-1} and F_{n-1}' respectively. We are taking the derivative of a polynomial in integer powers of x_{N+1}, so P' is obtained from P by removing the points of P lying on the null hyperplane and then shifting the rest of the polytope by -1 in the x_{N+1} coordinate.

Let L_{n-1} be the line through e_{n-1}, and let L_{n-1}' be the translate $L_{n-1} - (0, \ldots, 0, 1)$ of L_{n-1}. Finally, let Q_{n-1} and Q_{n-1}' be the intersection points of L_{n-1} and L_{n-1}' respectively with the null hyperplane. We define the coefficient restriction of F_{n-1} to the edge e_{n-1} by

$$F_{n-1}|_{e_{n-1}} = \sum_{I \in e_{n-1}} a_I x^{\alpha_I}.$$

Unless $(F_{n-1}|_{e_{n-1}})'$ is a constant, L_{n-1}' is the unique line which contains the terms of $(F_{n-1}|_{e_{n-1}})'$.

Let $\{a_i x^{\alpha_i}\}$ be the monomials of $F_{n-1}|_{e_{n-1}}$. By construction, c_{n-1} is a root of the edge equation

$$\sum a_i t^{\alpha_{N+1,i} - i_0} = 0. \tag{7}$$

So the coefficient in F_n of the monomial corresponding to Q_{n-1} is

$$\left(\sum a_i t^{\alpha_{N+1,i} - i_0} \right) \bigg|_{t = c_{n-1}} = 0.$$

To determine the coefficient on the monomial corresponding to Q_{n-1}' in F_n', consider the terms of $(F_{n-1}|_{e_{n-1}})'$. They are

$$\alpha_{N+1,i} a_i x_1^{\alpha_{1,i}} \cdots x_N^{\alpha_{N,i}} x_{N+1}^{\alpha_{N+1,i} - 1} = \frac{\mathrm{d}}{\mathrm{d}x_{N+1}} (a_i x^{\alpha_i}).$$

By the chain rule we get

$$\frac{\mathrm{d}}{\mathrm{d}x_{N+1}} (F_n) = \left(\frac{\mathrm{d}}{\mathrm{d}x_{N+1}} (F_{n-1}) \right) (x_1, \ldots, x_N, \phi_{n-1} + x_{N+1})$$

So in the same way as we showed that (7) was the coefficient of the monomial corresponding to Q_{n-1} in F_n we see that

$$\sum \frac{\mathrm{d}}{\mathrm{d}t} (a_i t^{\alpha_{N+1,i}}) \bigg|_{t = c_{n-1}} = \frac{\mathrm{d}}{\mathrm{d}t} \left(\sum a_i t^{\alpha_{N+1,i}} \right) \bigg|_{t = c_{n-1}} \tag{8}$$

is the coefficient of the monomial corresponding to Q_{n-1}' in F_n'.

If $k_{n-1} = 1$ then c_{n-1} is not a root of Eq. (8). Therefore the coefficient of Q'_{n-1} in F'_n is nonzero, and so the x_{N+1} coordinate of $M(e_{k,n})$ is 1. The case where $k_{n-1} > 1$ follows by applying the same argument as above with higher-order derivatives. This completes the proof of part (c) and therefore the proof of Lemma 3.7.

3.9. The exponents of ϕ lie in a lattice. This argument is almost identical to its single variable counterpart [11]. Let (a_j, j) and $(a_0, 0)$ be the major and minor vertices of e_n respectively, where $a_i \in \mathbb{R}^N$, and $j = k_{n-1}$. Since we have only carried out a finite number of steps, F_n lies in some lattice $1/m\mathbb{Z}^N$ and so there are vectors m_0, m_j such that

$$\frac{m_0}{m} = a_0 \quad \text{and} \quad \frac{m_j}{m} = a_j.$$

Since e_i determines a line with slope $\alpha_n = (\alpha_{1,n}, \ldots, \alpha_{N,n})$

$$a_0 = a_j + j\alpha_n$$

and hence

$$\alpha_n = \frac{a_0 - a_j}{j} = \frac{m_0 - m_j}{mj} = \frac{p}{mq}$$

where $p \in \mathbb{Z}^N$ and $q \in \mathbb{Z}$ such that there is some i_0 for which p_{i_0} and q are relatively prime.

If $(a_h, h) \in e_n$ and $a_h = m_h/m$ then

$$\frac{p}{mq} = \alpha_n = \frac{a_0 - a_h}{h} = \frac{m_0 - m_h}{mh}.$$

Therefore $q(m_0 - m_h)_{i_0} = p_{i_0} h$, and since $(q, p_{i_0}) = 1$, we know that q divides h. Therefore the edge equation of e_n has the form

$$F_{e_n}(t) = g(t^q).$$

But by above for n sufficiently large c_n is a root of F_{e_n} of multiplicity $k_{n-1} = \deg_{x_{N+1}}(F_{e_n})$. Therefore

$$F_{e_n} = d(t - c_n)^{k_{n-1}}.$$

Since k_{n-1}, d and c_n are all nonzero, $F_{e_n}(t)$ must have a nonzero coefficient at t^1. Hence $q = 1$ which implies that $\alpha_n \in 1/m\mathbb{Z}^N$ for all n sufficiently large.

3.10. Proof that the support of ϕ lies in a cone

Lemma 3.11. *There exists some $n_0 > 0$ such that for all $n \geq n_0$ the major vertices of e_n are the same.*

Proof. By Lemma 3.7 we know that the major vertex of e_n has x_{N+1}-coordinate equal to k_{n-1} which is a decreasing sequence. Since $k_n \geq 1$ for all n there must be some $n_0 > 0$ such that $k_{n_0-1} > k_{n_0}$ but $k_{n'} = k_{n_0}$ for all $n' > n_0$.

We now proceed by induction on $n \geq n_0$. Since the claim is obvious for e_{n_0}, let $n > n_0$. The length of e_n is equal to the length of e_{n-1}. Therefore the edge path constructed in Lemma 3.7 consists entirely of one edge, e_n. Therefore, the major vertex lies on the line L_{n-1} that contains e_{n-1} and has an x_{N+1} coordinate less than or equal to that of the major vertex of e_{n-1}. Hence these two points are equal. $\square$

With this lemma we can prove that the support of ϕ lies in some translate of $C(e_{n_0})$. Let w be any element of $C^*(e_{n_0})$. Now, F_{n_0} lies in the barrier wedge of e_{n_0}, and since the minor vertex $m(e_{n_0+1})$ lies inside this barrier wedge, we can conclude that $P(F_{n_0+2})$ also lies in this barrier wedge. Using this argument recursively shows that for every $n > n_0$, $P(F_n)$ lies in the barrier wedge of e_{n_0}. This fact implies that the minor vertex of e_n lies in the barrier wedge of e_{n_0} but not on the line through e_{n_0}.

Using the same argument as was used in Lemma 3.7, we can show that

$$\langle w, s(e_n) \rangle > \langle w, s(e_{n_0}) \rangle$$

which shows, since w was arbitrary, that $\alpha_n = -S(e_n) \in C(e_{n_0})$ for all $n > n_0$. Since all but a finite terms are in this cone, the entire series must be contained in a translate of $C(e_{n_0})$.

3.12. Proof that ϕ satisfies the equation. Consider F as an element of the polynomial ring over $\mathbb{C}((C_w))$. Since ϕ is itself a member of this ring, we have a well defined notion of ϕ satisfying $F(\phi) = 0$, where $F(\phi) = F(x_1, \ldots, x_N, \phi)$.

We must show that for each $r > 0$ there is an $n_0 > 0$ such that if $n > n_0$ and if $B(0; r)$ is the ball of radius r about 0, then

$$\operatorname{Supp}(F(x_1, \ldots, x_N, \phi_n)) \subset \mathbb{R}^N \backslash B(0; r),$$

i.e. all the exponents of Supp $(F(x_1, \ldots, x_N, \phi_n))$ lie beyond the ball of radius r.

Let v_1 be the intersection of the line L_1 containing e_1 with the null hyperplane. Since $\operatorname{Supp}(\phi)$ is contained in a half-plane determined by w, and w is a linear functional, it is sufficient to show the following. If, under the ordering induced by w, p_1 and p_2 are the largest points of $\operatorname{Supp}(F(\phi_n))$ and $\operatorname{Supp}(F(\phi_{n+1}))$ respectively, then $p_1 > p_2$. (Recall that w gets smaller as we move out along the terms of ϕ.)

We need to compare the two series $F(x_1, \ldots, x_N, \phi_n)$ and $F(x_1, \ldots, x_N, \phi_{n+1})$. To do this we will consider

$$F_n = F(x_1, \ldots, x_N, \phi_n + x_{N+1}) \quad \text{and} \quad F_{n+1} = F(x_1, \ldots, x_N, \phi_{n+1} + x_{N+1}).$$

The key point to notice here is that the former can be obtained by substituting $x_{N+1} = 0$ into the latter. Assume that neither $F(x_1, \ldots, x_N, \phi_n)$ nor $F(x_1, \ldots, x_N, \phi_{n+1})$ is 0, since, if either were true, we would trivially have the desired result.

Now by the discussion presented in the proof of Lemma 3.7, we know that $P(F_n)$ lies above the w-constant plane determined by e_{n-1} and likewise $P(F_{n+1})$ lies above the plane determined by e_n. With this, the inequality,

$$\langle w, s(e_{n-1})\rangle < \langle w, s(e_n)\rangle$$

yields the desired result.

3.13. Proof of the lower bound on the number of solutions. To conclude the proof of Theorem 3.6(a), we need to show that the number of solutions obtained from an edge is at least the length of the projection of that edge onto the x_{N+1}-axis.

The first coefficient of any solution series corresponding to the chosen edge e is a root of a polynomial whose degree is equal to the length, k_e, of the projection of that edge. So there are, counting multiplicity, k_e possibilities for this first coefficient. For each distinct root we get a different solution series and hence we get at least as many series as the number of distinct roots of this equation.

For each multiple root we need to consider what happens with later coefficients. So assume that c_n is a multiple root of the edge equation of e_n. Since the length of the edge path constructed in Lemma 3.7 for e_{n+1} is equal to the multiplicity of c_n as a root of the edge equation of e_n, we see that the total number of possible choices for the $(n+1)$st coefficient is at least equal to the multiplicity of c_n. Again, distinct coefficients will yield distinct series expansions.

Let ϕ be built as above. By Lemma 3.7, for all n sufficiently large, c_n has some fixed multiplicity k_ϕ as a root of the edge equation $F_{e_n}(t)$. We claim that the multiplicity of ϕ as a root of $F = 0$ is at least k_ϕ. Suppose $k_\phi > 1$ then by the proof of Lemma 3.7 each c_n is a root of $\mathrm{d}^{(k)}F_{e_n}(t)/\mathrm{d}t^k$ for all n and all $1 \le k \le k_\phi$. Therefore ϕ is a root of $\mathrm{d}^kF/\mathrm{d}x_{N+1}^k$ and hence has multiplicity at least k_ϕ. Since the k_ϕ's must add up to at least the degree of the edge equation of e_1 we see that up to multiplicity the number of series solutions is greater than or equal to the degree of F_{e_1} and hence the length of the projection of e_1 onto the x_{N+1}-axis.

Since all of these solutions lie in strongly convex rational cones whose duals contain w, and since w is irrational, there must be some strongly convex rational cone C_w containing the supports of all of these series. This finishes the proof of Theorem 3.6. □

3.14. Simple roots – a special case. For a polynomial $F(x_1, \ldots, x_{N+1})$ we will let $\Delta_{x_{N+1}}(F)$ denote the classical discriminant of F with respect to x_{N+1}. Recall that the discriminant of a polynomial $p(x)$ in one variable is a polynomial in the coefficients of $p(x)$ that vanishes precisely when $p(x)$ has a multiple root. (For a thorough discussion of classical discriminants see [4, Chapter 12] or [5].) Likewise for a polynomial $g(t)$ of one variable, $\Delta_t(g)$ will denote the discriminant of g with respect to t.

Corollary 3.15. *If c_1 is a simple root of $F_e(t)$, and ϕ is the series built by the above algorithm, then* $\mathrm{Supp}(\phi) \subset C(e)$. *More generally, if the discriminant $\Delta_t(F_e(t)) \neq 0$ then all series generated from this edge have support in $C(e)$.*

Proof. Since $k_1 = 1$, $n_0 = 1$ and so the result follows from the proof in Section 3.10. $\square$

Remark 3.16. Recall that in Section 2.1 we considered the case of a rational function

$$F(x) = x_{N+1} f(x_1, \dots, x_N) - 1 = 0.$$

Note that the Newton polytope $P(F)$ is a cone over the Newton polytope of f. Therefore, the vertices of $P(f)$ correspond to the admissible edges of $P(F)$. For $(v_1, \dots, v_N)$ a vertex of $P(f)$, the slope of the corresponding edge in $P(F)$ is $(-v_1, \dots, -v_N)$.

4. Fiber polytopes and full sets of solution series

If d is the degree of F in the variable x_{N+1}, then we would like to find rings of fractional power series which contain a full set of d solutions to the equation $F = 0$. The answer involves the notion of the fiber polytope of the projection $P \xrightarrow{\psi} Q$ as defined in [2]. Let us recall the definitions.

Let $P \subset \mathbb{R}^N$ be a convex polytope. Let $\psi: \mathbb{R}^N \to \mathbb{R}^M$ be a surjective linear map and let $Q = \psi(P)$. The fiber polytope $\Sigma_\psi(P, Q)$ is defined to be the set of vector integrals

$$\int_Q \gamma(x)\,\mathrm{d}x$$

where γ ranges over all continuous sections of ψ, i.e. maps $\gamma: Q \to P$ such that $\psi \circ \gamma = id_Q$.

We apply this construction to the following situation. Let $P = P(F)$ be the Newton polytope of F, let $Q = [0, d]$ where $d = \deg_{x_{N+1}}(F)$ and let ψ be the projection onto the last coordinate.

If E is a monotone edge path (see Section 3.1) which is maximal, i.e. $\psi(E) = Q$, then E defines a section γ_E of ψ. From now on we consider only maximal edge paths. It was shown in [2] that the vertices of $\Sigma_\psi(P, Q)$ have the form $\int_{[0,d]} \gamma_E(x)\,\mathrm{d}x$, where γ_E is the section corresponding to a coherent edge path E. Moreover, if v is a vertex $\int \gamma_E(x)\,\mathrm{d}x$ of $\Sigma_\psi(P, Q)$, then the barrier cone $C(v)$ is the union of the barrier cones of the edges in E.

With these definitions we may formulate the following corollary to Theorem 3.6(a).

Corollary 4.1. *Let $F(x_1, \dots, x_{N+1}) = 0$ be an algebraic equation such that each edge of $P(F)$ satisfies $\Delta_t(F_e) \neq 0$. Let $E = \{e_1, \dots, e_n\}$ be a coherent edge path on $P(F)$. Let $d = \deg_{x_{N+1}}(F)$ be the degree of F with respect to x_{N+1} and let*

$$C = \bigcup_{i=1}^{n} C(e_i)$$

be the union of the barrier cones of the edges in E. Then the ring $\mathbb{C}((C_{\mathbb{Q}}))$ *contains d solutions to* $F(x_1, \ldots, x_{N+1}) = 0$, *counting multiplicity. Therefore complete systems of solutions are in one-to-one correspondence with the vertices of* $\Sigma(P(F), [0, d])$.

Proof. By proof of Theorem 3.6, if k_i is the length of the projection $\psi(e_i)$, then the number of solutions, counting multiplicity, that correspond to e_i, is at least k_i. Again, as in the proof of Theorem 3.6, we can order $C_{\mathbb{Q}}$ via an element of $C_{\mathbb{Q}}^*$, to get that if ϕ corresponds to e_i and ϕ' corresponds to $e_{i'}$, then the lowest-order monomial of ϕ and ϕ' must differ. Hence, in this case $\phi \neq \phi'$.

By these two facts we see that the number of solutions of $F(x_1, \ldots, x_{N+1}) = 0$ in $\mathbb{C}((C_{\mathbb{Q}}))$ is at least $k_1 + \cdots + k_n = d$. Since this equation can have at most d solutions in any integral domain, we have that the number of such solutions is exactly d. □

Remark 4.2. In the case where the edge discriminants are not necessarily 0 the same proof will apply to show that some cone C_w with $w \in C_w^*$ has a full set of d solutions, where w is chosen as in Theorem 3.6.

5. Proof of Theorem 3.6(b)

We can use Remark 4.2 to prove Theorem 3.6(b). Let e be an edge of the polytope $P(F)$, and let k be the length of $\psi(e)$. Let $w \in C^*(e)$ be a linear functional with coordinates that are linearly independent over $\mathbb{Q}$. Then as described above, w defines a coherent edge path $E = \{e_1, \ldots, e_n\}$ on $P(F)$. Since w is maximized on e, we get that $e \in E$.

By Remark 4.2, there is some cone, C, such that for each i, $C(e_i) \subset C$, and such that the ring $\mathbb{C}((C_{\mathbb{Q}}))$ contains d solutions, counting multiplicity, to $F(x_1, \ldots, x_{N+1}) = 0$, k_i of which correspond to each e_i. Since $\mathbb{C}((C_{\mathbb{Q}}))$ can contain no more than d solutions and $k_1 + \cdots + k_n = d$, we have that the number of solutions corresponding to each edge, and in particular, e, is exactly k_i, the length of $\psi(e)$. This completes the proof of Theorem 3.6.

6. Remarks and examples

6.1. Fields other than $\mathbb{C}$. Note that the proof above works for any algebraically closed field of characteristic 0. Also, as in the single variable case, a version of this construction works for fields of arbitrary characteristic. For fields of characteristic p, the inductive construction must be carried out transfinitely and so we may have solution series which have supports containing limit points, as in [7, 9, 10]. Such solution series will always have well ordered supports in $\mathbb{Q}^N$ with respect to the chosen linear functional $-w$. The statement that the support of the solution series lies in a lattice and the proof that the constructed series lies in some cone no longer hold.

6.2. Estimate for the ramification locus. Let $X \subset (\mathbb{C}^*)^{N+1}$ be the variety given by the equation $F(x_1, \ldots, x_{N+1}) = 0$. Assume that X is smooth and that F satisfies the discriminantal condition of Corollary 4.1. Consider the projection $\Pi: X \to (\mathbb{C}^*)^N$ defined by

$$\Pi(x_1, \ldots, x_{N+1}) = (x_1, \ldots, x_N).$$

A point $(\mathbb{C}^*)^N$ is called ramified under Π if the number of inverse images of y is less than the degree d of the mapping f, $d = \deg_{x_{N+1}}(F)$. A point that has n inverse images is called unramified, see [8]. Note that the locus D thus defined also includes points where x_{N+1} becomes infinite. Therefore this locus is a subvariety given by the equation

$$P_d(x_1, \ldots, x_N)\Delta_{x_{N+1}}(F) = 0$$

where P_d is the coefficient of x_{N+1}^d in $F(x_1, \ldots, x_{N+1})$ and $\Delta_{x_{N+1}}(F)$ is the discriminant of F with respect to the x_{N+1}.

Suppose that F has a complete set of d fractional power series expansions of x_{N+1} through $x_1, \ldots, x_{N+1}$ in some ring $\mathbb{C}((C_{\mathrm{Q}}))$. There exists some translate C'_{Q} of C_{Q} such that the inverse image, under Π, of $\mathrm{Log}^{-1}(C'_{\mathrm{Q}})$ is a union of graphs of analytic functions given by convergent fractional power series in $(\mathbb{C}^*)^{N+1}$. Since X is smooth, these graphs do not intersect. Therefore, the number of inverse images of any $y \in \mathrm{Log}^{-1}(C_{\mathrm{Q}})$ is precisely d. Hence, none of the points of this set are ramified.

Suppose that F is chosen so that $\Delta_t(F_e) \neq 0$ for all admissible edges $e \subset P(F)$. Let $\{C_i \subset \mathbb{R}^N\}$ be the normal cones of all vertices of the fiber polytope $\Sigma(P(F), [0, d])$.

By Corollary 4.1 there are translates C'_i of C_i for each i such that all points of $\mathrm{Log}^{-1}(C'_i)$ are unramified points of Π. Hence

$$\mathrm{Log}(D) \cap \bigcup_{i=1}^{m} (C'_i) = \emptyset$$

where m is the number of vertices of $\Sigma(p(F), [0, d])$. So through the above methods we get a bound on the image, under the map Log, of the ramification locus of Π as indicated in Fig. 4.

6.3. Necessity of the discriminantal condition. The following example will show that the conclusion of Corollary 3.15 is not necessarily true for edges that do not satisfy the discriminant condition of that corollary. Let

$$F(x, y, z) = 1 + 2xz + 2yz + 2z^2 + x^2z^2 + y^2z^2.$$

We want to express z as a series in x and y. The Newton polytope of F is a simplex with vertices (0, 0, 0), (2, 0, 2), and (0, 2, 2). Let e be the edge with end points (0, 0, 0)

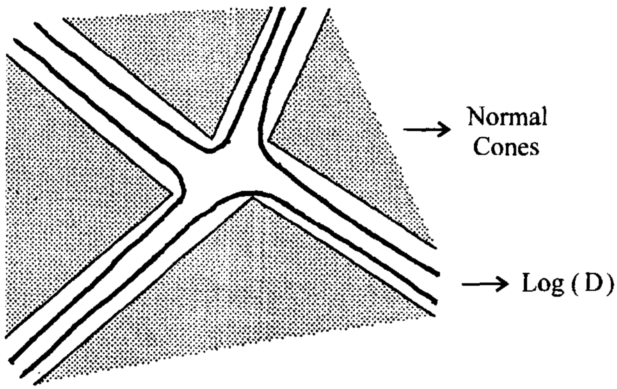

Fig. 4.

and (2, 0, 2). Then e itself is a maximal edge path and

$$F|_e = 1 + 2xz + x^2z^2 = (xz + 1)^2.$$

Therefore $F_e(t) = 1 + 2t + t^2$ and so $\Delta_t F_e(t) = 0$. Let $C(e)$ and $C^*(e)$ be the barrier and normal cones of e respectively. Assume that $\mathbb{C}((C(e)_{\mathbb{Q}}))$ contains a full set of 2 convergent series expansions for $F = 0$. There is then some translate $C^*(e) + \gamma$ of $C^*(e)$ such that there exist two series expansions on $C^*(e) + \gamma$. Note that $C^*(e)$ is the cone generated by $(0, -1)$ and $(1, 1)$.

Calculating the discriminant $\Delta_z(F)$ we see that it is equal to $8(xy - 1)$ and so the zero locus of $\Delta_x(F)$ is the irreducible variety $xy - 1 = 0$. The ramification locus $R(F)$ of the projection $\pi: \{F = 0\} \to \mathbb{R}$ must be a subvariety of the zero locus of $\Delta_z(F)$. Therefore $R(F) = \{xy - 1 = 0\}$. Taking $\mathrm{Log}(R(F))$ we get $\log|y| = -\log|x|$. Therefore

$$R(F) = \{(u, v) \in \mathbb{R}^2_{\log} : u = -v\}.$$

This implies that, regardless of what γ is, $R(F) \cap C^*(e) + \gamma \neq \emptyset$. By the remarks in Section 6.2, this is a contradiction. Therefore $\mathbb{C}((C(e)_{\mathbb{Q}}))$ cannot contain 2 series solutions for $F = 0$.

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