Synchronizable functions on integers

Didier Caucal and Chloé Rispal

CNRS, LIGM, University Gustave Eiffel, France
{didier.caucal,chloe.rispal}@univ-eiffel.fr

Abstract. For all natural numbers a, b and $d > 0$, we consider the function $f_{a,b,d}$ which associates $\frac{n}{d}$ to any integer n when it is a multiple of d, and $an + b$ otherwise; in particular $f_{3,1,2}$ is the Collatz function. Coding in base $a > 1$ with $b < a$, we realize these functions by input-deterministic letter-to-letter transducers with additional output final words. This particular form allows to explicit, for any integer n, the composition n times of such a transducer to compute $f_{a,b,d}^{n}$. We even realize the closure under composition $f_{a,b,d}^{*}$ by an infinite input-deterministic letter-to-letter transducer with a regular set of initial states and a length recurrent terminal function.

1 Introduction

Functions on integers have been studied as word functions using an integer base. With this coding, some functions on integers can be deterministically described by 2-automata [9]. In 1966, Ginsburg introduced the sequential transducers, having transitions labeled by an input letter and an output word, and computing functions deterministically in input [6]. Since 1977, Schutzenberger has extended sequential transducers with a terminal function associating an output word to each terminal state [11].

For all natural numbers a, b and $d > 0$, we consider the integer function $f_{a,b,d}$ which associates $\frac{n}{d}$ to any integer n when it is a multiple of d and $an + b$ otherwise. In particular, $f_{3,1,2}$ is the Collatz function [7]. To realize the function $f_{a,b,d}$, the transducer must compute the two operations of division by d and multiplication by a. The first natural approach is to take the base d with the least significant digit to the left to see right away if the input is a multiple of d and, if not, to realize the multiplication by a starting from the left. Another way is to choose the base a to solve the multiplication by a and to realize the division by d starting from the most significative digit to the left.
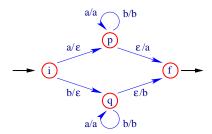
To realize the functions $f_{a,b,d}$ and its powers, we just need two particular forms of left-synchronized and right-synchronized [4] sequential transducers: the prefix and the suffix forms. The prefix sequential transducers are the letter-to-letter sequential transducers *i.e.* the left-synchronized sequential transducers without ε-output transition. The suffix sequential transducers are the right-synchronized sequential transducers without ε-input transition. Both families of functions realized by prefix and suffix sequential transducers are closed under composition, intersection and difference in a quadratic way.

To realize the function $f_{a,b,d}$, the choice of the base d gives a simple suffix sequential transducer but it is not suitable to specify, for all integers n, a generic transducer computing its composition n times. However, the base a turns out to be more revelant: in the case where $b < a$, which is the case of the Collatz function, it allows to define a prefix sequential transducer computing $f_{a,b,d}$ and to explicit, for any integer n, the composition n times of this transducer. Even better, still under the condition $b < a$, we construct an infinite input-deterministic prefix transducer realizing the iteration of the composition $f_{a,b,d}^*$. Its terminal function is not simply defined as the union, for all integers n of the previous terminal functions for the $f_{a,b,d}^n$, but we give a length recurrent definition of it. Finally, we give a geometric representation of this transducer in the form of a cone where for each integer n, the transducer of the division by d^n is represented by a circular section and the terminal function by transitions going from one section to its adjacent smaller section.

2   Transducers

A (finite) transducer [9,4,2] is a finite 2-automaton: it is labeled by pairs of words, or more simply by pairs made with letters or the empty word. We recall the composition of transducers and its iteration.

Let A be an *input alphabet* and B be an *output alphabet*. A *transducer* $\mathcal{T} = (Q, I, F, T)$ over (A, B), or over A when $B = A$, is defined by a set Q of *states*, two subsets I and F of Q of resp. *initial states* and *final states*, and a subset T of $Q \times (A \cup \{\varepsilon\}) \times (B \cup \{\varepsilon\}) \times Q$. Each $(p, a, b, q) \in T$, also denoted by $p \xrightarrow{a/b}_T q$ or $p \xrightarrow{a/b} q$ when T is clear from the context, is a *transition* from state p to state q labeled by (a, b), or more simply by a/b, of *input* a and of *output* b. We write $p \xrightarrow{a/\cdot} q$ and $p \xrightarrow{\cdot/b} q$ if there exists respectively b and a such that $p \xrightarrow{a/b} q$. By default, a transducer is finite: it has only a finite number of states. We represent $\mathcal{T}$ by its graph (Q, T) with incoming arrows to mark its initial states and outgoing arrows from its final states. Here is a representation of a transducer $\mathcal{T}_0$ over $\{a, b\}$:

A sequence $(p_0, a_1, b_1, p_1 \ldots, a_n, b_n, p_n)$ of n consecutive transitions $p_0 \xrightarrow{a_1/b_1} p_1$, $\ldots, p_{n-1} \xrightarrow{a_n/b_n} p_n$ is a *path* of *length* n of T and we write $p_0 \xRightarrow{u/v}_T p_n$ or

$p_0 \overset{u/v}{\Longrightarrow} p_n$ to only indicate the *source* p_0, the *goal* p_n, the *input* $u = a_1 \ldots a_n$ and the *output* $v = b_1 \ldots b_n$ of the path. The path (p_0) of length 0 is denoted by $p_0 \overset{\varepsilon/\varepsilon}{\Longrightarrow} p_0$. We also write $p \overset{u/.}{\Longrightarrow} q$ and $p \overset{./v}{\Longrightarrow} q$ if there exists respectively v and u such that $p \overset{u/v}{\Longrightarrow} q$. A path from an initial vertex to a final vertex is an *accepting path*. The transducer $\mathcal{T}$ *realizes* or *computes* the binary relation

$$<\mathcal{T}> \;=\; \{\, (u,v) \in A^* \times B^* \mid \exists\, i \in I \; \exists\, f \in F \; (i \overset{u/v}{\Longrightarrow}_T f) \,\}$$

of the labels of its accepting paths. The above transducer $\mathcal{T}_0$ realizes the function $xu \mapsto ux$ for any word u and letter x over $\{a, b\}$. By exchanging the input with the output of the transitions of a transducer $\mathcal{T} = (Q, I, F, T)$, we obtain the following *inverse transducer*:

$$\mathcal{T}^{-1} \;=\; (Q, I, F, T^{-1}) \quad \text{where} \quad T^{-1} = \{\, (p, b, a, q) \mid (p, a, b, q) \in T \,\}$$

which realizes the inverse relation $<\mathcal{T}>^{-1}$ of the relation computed by $\mathcal{T}$. We also associate to any transducer $\mathcal{T} = (Q, I, F, T)$ its *mirror transducer*:

$$\widetilde{\mathcal{T}} \;=\; (Q, F, I, \widetilde{T}) \quad \text{where} \quad \widetilde{T} = \{\, (q, a, b, p) \mid (p, a, b, q) \in T \,\}$$

which reverses the arrows of the transitions without changing the labels and realizes from F to I the mirror of the relation computed by $\mathcal{T}$: $<\widetilde{\mathcal{T}}>$ is equal to $\{\, (\tilde{u}, \tilde{v}) \mid (u, v) \in <\mathcal{T}> \,\}$ denoted by $\widetilde{<\mathcal{T}>}$.

The *composition* of a transducer $\mathcal{T} = (Q, I, F, T)$ over (A, B) with a transducer $\mathcal{T}' = (Q', I', F', T')$ over (B, C) is the following transducer over (A, C):

$$\mathcal{T} \circ \mathcal{T}' \;=\; (Q \times Q', I \times I', F \times F', T \circ T')$$

where $T \circ T' = \{\, (p, p') \overset{a/\varepsilon}{\longrightarrow} (q, p') \mid p \overset{a/\varepsilon}{\longrightarrow}_T q \wedge p' \in Q' \,\}$

$$\cup \; \{\, (p, p') \overset{\varepsilon/b}{\longrightarrow} (p, q') \mid p \in Q \wedge p' \overset{\varepsilon/b}{\longrightarrow}_{T'} q' \,\}$$

$$\cup \; \{\, (p, p') \overset{a/c}{\longrightarrow} (q, q') \mid \exists\, b \; (p \overset{a/b}{\longrightarrow}_T q \wedge p' \overset{b/c}{\longrightarrow}_{T'} q') \,\}$$

it realizes $<\mathcal{T}> \circ <\mathcal{T}'>$.

The composition $n \geq 0$ times of a transducer $\mathcal{T} = (Q, I, F, T)$ over A is the transducer $\mathcal{T}^n = (Q^n, I^n, F^n, T^n)$ where $T^0 = \{\, \varepsilon \overset{a/a}{\longrightarrow} \varepsilon \mid a \in A \,\}$ and

$$T^{n+1} = \{\, (p_1, \ldots, p_n, p_{n+1}) \overset{a/b}{\longrightarrow} (q_1, \ldots, q_n, q_{n+1}) \mid$$

$$((p_1, \ldots, p_n), p_{n+1}) \overset{a/b}{\longrightarrow}_{T^n \circ T} ((q_1, \ldots, q_n), q_{n+1}) \,\} \quad \text{for any } n \geq 0.$$

So $\mathcal{T}^n$ realizes the relational composition n times of $<\mathcal{T}>$: $<\mathcal{T}^n> \;=\; <\mathcal{T}>^n$. In particular $\mathcal{T}^0 = (\{\varepsilon\}, \{\varepsilon\}, \{\varepsilon\}, \{\, \varepsilon \overset{a/a}{\longrightarrow} \varepsilon \mid a \in A \,\}$ realizes the identity relation on A^*. For any $P \subseteq Q$, $P^* = \bigcup_{n \geq 0} P^n$ is the set of tuples of elements of P *i.e.* of words over P.

The *composition closure* of $\mathcal{T} = (Q, I, F, T)$ is the infinite transducer

$$\mathcal{T}^* \;=\; (Q^*, I^*, F^*, T^*) \quad \text{where} \quad T^* = \bigcup_{n \geq 0} T^n$$

which realizes the relation $<\mathcal{T}^*> \;=\; \bigcup_{n \geq 0} <\mathcal{T}>^n$.

For $\mathcal{T}_1 = (\{p\}, \{p\}, \{p\}, \{\, p \overset{a/b}{\longrightarrow} p, \; p \overset{b/a}{\longrightarrow} p \,\})$, its composition closure $\mathcal{T}_1^*$ is

$$(p^*, p^*, p^*, \{\, p^{2n} \overset{a/a}{\longrightarrow} p^{2n}, \; p^{2n} \overset{b/b}{\longrightarrow} p^{2n}, \; p^{2n+1} \overset{a/b}{\longrightarrow} p^{2n+1}, \; p^{2n+1} \overset{b/a}{\longrightarrow} p^{2n+1} \mid n \geq 0 \,\}).$$

The relations realized by the (finite) transducers are the *rational relations* namely those obtained from the finite binary relations by applying a finite number of times the *rational operations* of union, concatenation (componentwise) and its iteration (the Kleene star operation).

Proposition 1. [4] *The family of rational relations is closed under composition, inverse, mirror, union, concatenation and its iteration.*

From now on, we consider transducers realizing functions. We already notice that the intersection and the difference of two rational functions are generally not rational. For instance, the following functions are sequential:

$$f_1 \ = \ \{ \ (a^m b a^n, a^m) \mid m, n \geq 0 \ \} \quad \text{and} \quad f_2 \ = \ \{ \ (a^m b a^n, a^n) \mid m, n \geq 0 \ \}$$

but $f_1 \cap f_2 \ = \ \{ \ (a^n b a^n, a^n) \mid n \geq 0 \ \}$ and $f_1 - f_2 \ = \ \{ \ (a^m b a^n, a^m) \mid m \neq n \ \}$ are not rational since their domains are not regular languages.
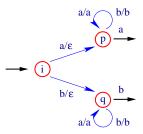
3   Sequential transducers

The sequential transducers have been defined [6,11] to compute functions in a deterministic way according to the inputs. The functions are realized from left to right reading one letter at a time on input and writing a word on output for each letter.

We use transitions of the form $\xrightarrow{a/v}$ where a is a letter and v is a word. A transducer over (A, B) can be defined in a *terminal form* $\mathcal{T} = (Q, I, \omega, T)$ with a finite transition set $T \subset Q \times A \times B^* \times Q$ and a *terminal function* ω from Q into the set $\mathrm{Reg}(B^*)$ of regular languages over B. It realizes the relation

$$<\mathcal{T}> \ = \ \{ \ (u, vw) \mid \exists \ i \in I \ \exists \ f \in \mathrm{dom}(\omega) \ (i \xRightarrow{u/v}_T f \ \wedge \ w \in \omega(f)) \ \}.$$

The domain of ω is the set of final states. We also represent $\mathcal{T}$ by its graph (Q, T) and for any $q \in \mathrm{dom}(\omega)$, we add an outgoing arrow from q labeled by $\omega(q)$ and this label can be omitted when $\omega(q) = \{\varepsilon\}$. Here is a terminal form representation of $\mathcal{T}_0$:

Note that any transducer $\mathcal{T} = (Q, I, F, T)$ has a terminal form $\mathcal{T}' = (Q, I, \omega', T')$ such that $<\mathcal{T}'> \ = \ <\mathcal{T}>$; it is defined by

$$T' = \{\ p \xrightarrow{a/vb} q \mid a \in A \wedge \exists\, r\, (p \stackrel{\varepsilon/v}{\Longrightarrow}_T r \xrightarrow{a/b}_T q)\ \}$$

$$\omega'(q) = \{\ v \mid \exists\, f \in F\ (q \stackrel{\varepsilon/v}{\Longrightarrow}_T f)\ \}.$$

The terminal form is well suited to define the determinism on input.

We say that a transition set $T \subset Q{\times}A{\times}B^*{\times}Q$ is *input-deterministic* if

$$(p \xrightarrow{a/u}_T q \wedge p \xrightarrow{a/v}_T r) \implies (u = v \wedge q = r).$$

We also say that T is *input-complete* if $q \xrightarrow{a/\cdot}$ for any $q \in Q$ and $a \in A$.

Such a general approach allows us to compute functions (resp. mappings) in a deterministic (and complete) way [11].

A *sequential transducer* over (A, B) is a transducer of terminal form (Q, i, ω, T) with a unique initial state i, a terminal function $\omega : Q \longrightarrow B^*$ and an input-deterministic transition set $T \subset Q{\times}A{\times}B^*{\times}Q$. In particular $\mathcal{T}_0$ is a sequential transducer. The relation realized by a sequential transducer is a *sequential function*. The composition of sequential transducers $\mathcal{T} = (Q, I, \omega, T)$ by $\mathcal{T}' = (Q', I', \omega', T')$ is then expressed by

$$\mathcal{T} \circ \mathcal{T}' = (Q{\times}Q', I{\times}I', \omega \circ \omega', T \circ T')$$

where $$T \circ T' = \{\ (p, p') \xrightarrow{a/v} (q, q') \mid \exists\, u\, (p \xrightarrow{a/u}_T q \wedge p' \stackrel{u/v}{\Longrightarrow}_{T'} q')\ \}$$

and $$\omega \circ \omega'((q, p')) = v.\omega'(q') \ \text{ for any } \ q \in \mathrm{dom}(\omega),\ q' \in \mathrm{dom}(\omega'),\ p' \stackrel{\omega(q)/v}{\Longrightarrow}_{T'} q'.$$

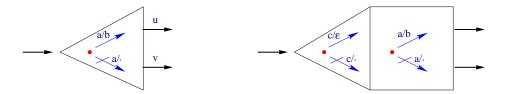**Proposition 2.** [11] *The sequential functions are preserved by composition.*

Note that the previous functions f_1 and f_2 are sequential whereas $f_1 \cap f_2$ and $f_1 - f_2$ are not rational.

4   Prefix and suffix sequential transducers

An easy way to realize functions is to use input-deterministic letter-to-letter transducers. In the following, we consider two particular forms of transducers composed by letter-to-letter sequential transducers with additional ε-input and ε-output transitions on the left or on the right.

We first recall letter-to-letter transducers. A *synchronous transducer* is a transducer $\mathcal{T} = (Q, I, F, T)$ over (A, B) whose transition set T is *letter-to-letter* i.e. $T \subseteq Q{\times}A{\times}B{\times}Q$; it realizes a *synchonous relation* which is *length-preserving*: for any $(u, v) \in <\mathcal{T}>$, $|u| = |v|$. A transducer which is both sequential and synchronous is a *synchronous sequential transducer* (or a Mealy machine [8]) i.e. of the form (Q, i, F, T) where T is letter-to-letter and input-deterministic. Note that the function $\{\ (aa, aa)\,, (ab, ba)\}$ is synchronous and sequential but it cannot be realized by a synchronous sequential transducer.

A *prefix sequential transducer* is an input-deterministic letter-to-letter transducer followed by ε-input transitions via the terminal function *i.e.* a sequential transducer $\mathcal{T} = (A, i, \omega, T)$ where T is letter-to-letter. It realizes a *prefix sequential function* which is *length-increasing*: $|u| \leq |v|$ for any $(u, v) \in <\mathcal{T}>$.

A *suffix sequential transducer* is an input-deterministic letter-to-letter transducer preceded by ε-output transitions *i.e.* a sequential transducer $\mathcal{T} =$

(A, i, F, T) having a terminal function reduced to a final state set F and a transition set $T \subseteq Q \times A \times (B \cup \{\varepsilon\}) \times Q$ which is input-deterministic and of *initial ε-output* meaning that *ε-output transitions* $\xrightarrow{\cdot/\varepsilon}$ can only be at the beginning of paths: $\xrightarrow{\cdot/b}_T \xrightarrow{\cdot/\varepsilon}_T \implies b = \varepsilon$. It realizes a *suffix sequential function* which is *length-decreasing*: $|u| \geq |v|$ for any $(u, v) \in \langle \mathcal{T} \rangle$. Here is an illustration of these two particular forms of synchronized sequential transducers.

prefix sequential transducer   suffix sequential transducer

The composition of sequential transducers preserves the prefixity and the suffixity.

Proposition 3. *The prefix and the suffix sequential functions are preserved in quadratic time and space under composition, intersection and difference.*

Encoding integers by words, we use these synchronized sequential transducers to describe functions on integers.

5   Automaticity of functions on integers

The purpose of automaticity is to describe objects by deterministic finite automata. Taking an integer base to code integers by words, we consider the automaticity of functions on integers using sequential transducers. Then we refine the automaticity using prefix or suffix sequential transducers.

Let an integer $a > 1$ and $\downarrow_a = \{0, \ldots, a-1\}$ be the alphabet of its *digits*. Any word $u \in \downarrow_a^*$ is a *representation in base a* of the integer $[u]_a$ defined by:

$$[c_n \ldots c_0]_a = \sum_{i=0}^{n} c_i a^i \quad \text{for any } n \geq 0 \text{ and } c_0, \ldots, c_n \in \downarrow_a$$

where the least significant digit is to the right. Any word $u \in \downarrow_a^*$ is also a *reverse representation in base a* of the integer $_a[u]$ defined by:

$$_a[c_0 \ldots c_n] = \sum_{i=0}^{n} c_i a^i \quad \text{for any } n \geq 0 \text{ and } c_0, \ldots, c_n \in \downarrow_a$$

where the least significant digit is to the left. Thus $_a[u] = [\widetilde{u}]_a$ for any $u \in \downarrow_a^*$. Representations of integers are extended to relations.

A relation $R \subseteq \downarrow_a^* \times \downarrow_a^*$ is a representation (resp. reverse representation) in base a of the binary relation $[R]_a$ (resp. $_a[R]$) on $\mathbb{N}$:

$$[R]_a = \{ ([u]_a, [v]_a) \mid (u, v) \in R \} \quad \text{and} \quad _a[R] = \{ (_a[u], _a[v]) \mid (u, v) \in R \}.$$

For $\widetilde{R} = \{ (\widetilde{u}, \widetilde{v}) \mid (u, v) \in R \}$ the mirror of R, we have $_a[R] = [\widetilde{R}]_a$.

Then, functions on integers can be realized by sequential transducers.

Definition 1. *A function* $f : \mathbb{N} \longrightarrow \mathbb{N}$ *is automatic (resp. reverse automatic) if there exists an integer base* $a > 1$ *and a sequential transducer* $\mathcal{T}$ *over* $\downarrow_a$ *such that* $f = [\mathrm{R}(\mathcal{T})]_a$ *(resp.* $f = {}_a[\mathrm{R}(\mathcal{T})]$ *).*

We refine the automaticity by using prefix or suffix sequential transducers. We say that a function $f : \mathbb{N} \longrightarrow \mathbb{N}$ is *prefix/suffix automatic* (resp. *reverse prefix/suffix automatic*) if there exists an integer base $a > 1$ and a prefix/suffix sequential transducer $\mathcal{T}$ over $\downarrow_a$ such that $f = [\mathrm{R}(\mathcal{T})]_a$ (resp. $f = {}_a[\mathrm{R}(\mathcal{T})]$).
In the following, we study the prefix and suffix automaticity of the mappings $f_{a,b,d} : \mathbb{N} \longrightarrow \mathbb{N}$ for all natural numbers a, b, d with $d \neq 0$ defined for any integer $n \geq 0$ by

$$f_{a,b,d}(n) \;=\; \begin{cases} \frac{n}{d} & \text{if} \quad n \text{ is a multiple of } d, \\ an + b & \text{otherwise.} \end{cases}$$
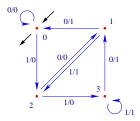
Note that $f_{3,1,2}$ is the Collatz function.

6 Suffix automaticity of functions $\mathbf{f_{a,b,d}}$

To realize the functions $f_{a,b,d}$ with sequential transducers, a natural and simple way is to take the base d to perform by shift the division by d. Taking the least significant digit to the left allows to test the multiplicity by d and, if not, to perform the multiplication by a and the addition of b. Simple suffix transducers will be sufficient to describe these operations. We therefore recall the realization of the multiplication by a by a synchronous sequential transducer.

The function $f_{a,b,1}$ is the identity mapping which can be realized in any base by the trivial synchronous sequential transducer with only one state and the labels of the loops are all pairs of identical digits. In this section, we assume that $d > 1$. For any natural number a, we realize the function $n \mapsto an$ in base d with the least significant digit to the left by the synchronous sequential transducer $_{d,a}* = (\downarrow_a, \{0\}, \{0\}, {}_{d,a}\times)$ where

$$i \xrightarrow{b/c}_{d,a\times} j \quad \text{if} \quad ab + i = c + dj \quad \text{for any} \quad i, j \in \downarrow_a \text{ and } b, c \in \downarrow_d.$$

Here is an illustration of the transducer $_{2,4}*$ realizing a reverse representation in base 2 of the multiplication by 4 :

The multiplication defined on digits by transitions extends to the multiplication of numbers by paths. This property holds even for transitions $i \xrightarrow{b/c}_{d,a\times} j$ where i and j are not necessarily lower than the multiplicator a.

Lemma 1. *For any $i, j \geq 0$ and $u, v \in \downarrow_d^*$, we have*

$$i \stackrel{u/v}{\Longrightarrow}_{d,a\times} j \quad \Longleftrightarrow \quad {}_d[u]a + i = {}_d[v] + j\,d^{|u|} \ \ and \ \ |u| = |v|.$$

Applying Lemma 1, the synchronous sequential transducer ${}_{d,a}*$ realizes the multiplication by a in base d:

$$<_{d,a}*> \ = \ \{ \ (u,v) \mid u, v \in \downarrow_d^* \ \wedge \ |u| = |v| \ \wedge \ {}_d[v] = {}_d[u]a \ \}$$

which is a reverse representation in base d of the function $n \mapsto an$.
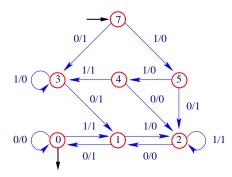Also by Lemma 1 and for any $b \geq 0$, the synchronous sequential transducer

$${}_{d,a,b}* \ = \ (\downarrow_{a,b}, \{b\}, \{0\}, {}_{d,a,b}\times) \ \text{ where}$$

${}_{d,a,b}\times$ is the restriction of ${}_{d,a}\times$ to $\downarrow_{a,b} = \{0, \ldots, \max(a-1, b)\}$

realizes the function $\{ \ (u,v) \mid u, v \in \downarrow_d^* \ \wedge \ |u| = |v| \ \wedge \ {}_d[v] = {}_d[u]a + b \ \}$ which multiplies by a and adds b.
We give a representation of the transducer ${}_{2,3,7}*$ realizing the multiplication by 3 and addition of 7 in base 2:

without the state 6 which is not accessible from the initial state 7.
We can now present a suffix sequential transducer realizing a reverse representation in base d of the function $f_{a,b,d}$.
First of all, the function $\{ \ (dn, n) \mid n \in \mathbb{N} \ \}$ has for reverse representation in base d the function $\{ \ (0u, u) \mid u \in \downarrow_d^* \ \}$ which is realized by the suffix sequential transducer

$$\mathcal{D} \ = \ (\{\alpha, \beta\}, \{\alpha\}, \{\beta\}, \{\alpha \stackrel{0/\varepsilon}{\longrightarrow} \beta\} \cup \{\beta \stackrel{c/c}{\longrightarrow} \beta \mid c \in \downarrow_d \}).$$

We now give a synchronous sequential transducer to compute a reverse representation in base d of $g_{a,b,d} \ = \ \{ \ (dn+c, \ a(dn+c)+b) \mid c, n \in \mathbb{N} \wedge 0 < c < d \ \}$. This is realized by the previous transducer ${}_{d,a,b}*$ except that an initial transition cannot be of input 0 *i.e.* of the form $b \stackrel{0/\cdot}{\longrightarrow}$. We just have to take a new initial state α and the following synchronous sequential transducer:

$${}_{d,a,b}\mathcal{M} = (\{\alpha\} \cup \downarrow_{a,b}, \{\alpha\}, \{0\}, {}_{d,a,b}T)$$

where ${}_{d,a,b}T = {}_{d,a,b}\times \ \cup \ \{ \alpha \stackrel{c/e}{\longrightarrow} j \mid c \neq 0 \wedge b \stackrel{c/e}{\longrightarrow}_{d,a,b\times} j \}$
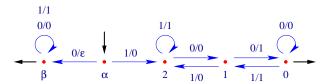
which realizes a reverse representation in base d of $g_{a,b,d}$. Finally, the suffix sequential transducer $\mathcal{D} \cup {}_{d,a,b}\mathcal{M}$ defined by

$$(\{\alpha, \beta\} \cup \downarrow_{a,b}, \{\alpha\}, \{0, \beta\}, {}_{d,a,b}T \cup \{\alpha \stackrel{0/\varepsilon}{\longrightarrow} \beta\} \cup \{\beta \stackrel{c/c}{\longrightarrow} \beta \mid c \in \downarrow_d \})$$

realizes a reverse representation in base d of $f_{a,b,d}$.

Proposition 4. *For all $a, b \geq 0$ and $d > 0$, $f_{a,b,d}$ is reverse suffix automatic.*

Here is an illustration of $\mathcal{D} \cup {}_{3,1,2}\mathcal{M}$ realizing a reverse representation in base 2 of the Collatz function:

Although this suffix sequential transducer is simple and easy to obtain from the definition of the Collatz function, it is tedious to compose it several times in order to realize the first powers of the Collatz function. Moreover, we are even less able to express in terms of n the composition n times of this transducer. We will see that this becomes possible for any function $f_{a,b,d}$ using prefix sequential transducers but under the restriction that $b < a$ which is a constraint satisfied by the Collatz function.

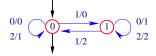## 7   Prefix automaticity of functions $f_{a,b,d}$

Another way to realize the functions $f_{a,b,d}$ with sequential transducers is to take the base a and the least significant digit to the right to compute deterministically the division by d. With this approach and for $b < a$, we realize $f_{a,b,d}$ by a prefix sequential transducer with d states.

To realize the division by a synchronous sequential transducer, a nice prologue has been given in [10]. Let natural numbers $a > 1$, $d > 0$ and $0 \leq r < d$. We realize the function $dn + r \mapsto n$ in base a by just taking the inverse and mirror of the previous transducer ${}_{a,d,r}*$ for the multiplication. We define the synchronous sequential transducer $/_{a,d,r} = (\downarrow_d, \{0\}, \{r\}, :_{a,d})$ of the division by d in base a with remainder r where

$$i \xrightarrow{b/c}_{:_{a,d}} j \quad \text{if} \quad ia + b = cd + j \quad \text{for all } i, j \in \downarrow_d \text{ and } b, c \in \downarrow_a.$$

Here is a representation of the transducers $/_{2,3,2}$ and $/_{3,2,0}$:

Division by 3 in base 2 with remainder 2   Division by 2 in base 3 with remainder 0
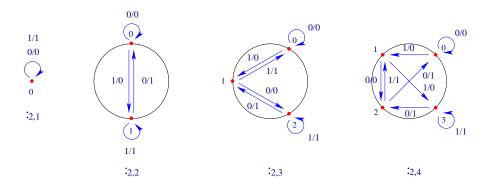
Let us apply Lemma 1.

Lemma 2. *For all $i, j \in \downarrow_d$ and $u, v \in \downarrow_a^*$, we have*

$$i \xRightarrow{u/v}_{:_{a,d}} j \quad \Longleftrightarrow \quad i\, a^{|u|} + [u]_a = [v]_a d + j \quad \text{and} \quad |u| = |v|.$$

Thus $/_{a,d,r}$ realizes $\{\ (u,v)\ |\ u,v \in \downarrow_a^*\ \wedge\ |u| = |v|\ \wedge\ [u]_a = [v]_a d + r\ \}$.

Let us propose a way to visualize these transducers to highlight basic symmetries. The d integers of the vertex set $\{0,\ldots,d-1\}$ of $:_{a,d}$ are equidistant on a counterclockwise circle in a way that the diameter between 0 and $d-1$ is horizontal with 0 at the top right. Here is a representation for respectively $d = 1,2,3,4$:
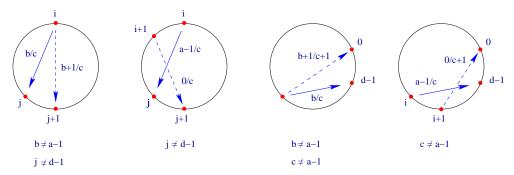

Here is a simple and natural algorithm to draw one by one the transitions of the transducer $:_{d,a}$ realizing the division by d in base a.

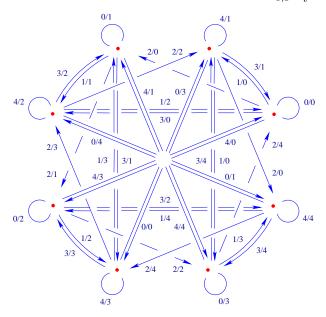From the Euclidean division of $n = dq + r$, the division of $n+1$ is

$n+1 = dq + (r+1)$ if $r < d-1$ and $n+1 = d(q+1)$ if $r = d-1$.

This allows us to incrementally determine $:_{a,d}$ as follows:

$E = \emptyset$
$output = 0$
$goal = 0$
For all $source$ from 0 to $d-1$ Do
    For all $input$ from 0 to $a-1$ Do
        add to E the transition $source \overset{input/output}{\longrightarrow} goal$
        add 1 to $goal$
        If $goal = d$ Then
            add 1 to $output$
            $goal = 0$
        EndIf
    EndFor
EndFor
return E

This algorithm processes the d vertices in increasing order. For each vertex i, it determines its a transitions by increasing input b. The goal j is the number $ia + b$ of transitions already given modulo d, and the output c is its quotient by d, which is justified by the equality $ia + b = cd + j$. Here is an illustration of this construction process.

We perform a single turn for the sources and a turns for the goals. We begin with the loop $0 \xrightarrow{0/0} 0$ and we end with the loop $d-1 \xrightarrow{a-1/a-1} d-1$.

It follows the illustration below of the Euclidean division $:_{5,8}$ by 8 in base 5:

The disposition of the vertices is useful to identify, among others, properties of symmetry for these transducers.

We have to give a prefix transducer realizing a representation in base a of the function $f_{a,b,d}$ for $b < a$. We start with $d = 2$ namely the functions $f_{a,b}$ defined for any integer $n \geq 0$ by

$$f_{a,b}(n) \;=\; \begin{cases} \frac{n}{2} & \text{if} \quad n \text{ is even,} \\ an + b & \text{otherwise.} \end{cases}$$

In the case where a and b are of the same parity, we also consider the function

$$f'_{a,b}(n) \;=\; \begin{cases} \frac{n}{2} & \text{if} \quad n \text{ is even,} \\ \frac{an+b}{2} & \text{otherwise} \end{cases}$$

which is an *acceleration* of $f_{a,b}$. A transducer realizing $f'_{a,b}$ can be obtained from the transducer $/_{a,2,0}$ of the division by 2 in base a with an additional terminal function. If an initial path ends to the state 0, the input is a representation in base a of an even integer n and the output represents $\frac{n}{2} = f'_{a,b}(n)$. If an initial path ends to the state 1, the input represents an odd integer n and the output represents $\frac{n-1}{2}$. So the digit of the terminal function must be $\frac{a+b}{2}$ to get $a\frac{n-1}{2} + \frac{a+b}{2} = \frac{an+b}{2} = f'_{a,b}(n)$.
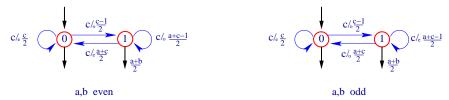
Proposition 5. *For all $0 \le b < a$ with $a > 1$ and a,b of the same parity,*
$$\mathcal{G}'_{a,b} = (\{0,1\},\{0\},\omega'_{a,b},:_{a,2}) \quad with \quad \omega'_{a,b}(0) = \varepsilon \ and \ \omega'_{a,b}(1) = \frac{a+b}{2}$$
is a prefix sequential transducer realizing a representation in base a of $f'_{a,b}$.

Proof. As $:_{a,2}$ is input-deterministic and input-complete, for all $u \in \downarrow_a^*$, there exists a unique $v \in \downarrow_a^*$ and $j \in \{0,1\}$ such that $0 \overset{u/v}{\Longrightarrow}_{:_{a,2}} j$.
By Lemma 2, we have $[u]_a = 2[v]_a + j$.
For $j = 0$, $[u]_a$ is even and $[v]_a = \frac{[u]_a}{2} = f'_{a,b}([u]_a)$.
For $j = 1$, $[u]_a$ is odd and as $1 \overset{b/c}{\longrightarrow}_{:_{a,2}} 0$ for $c = \frac{a+b}{2} < a$, $0 \overset{ub/vc}{\Longrightarrow}_{:_{a,2}} 0$ thus
$$a[u]_a + b = [ub]_a = 2[vc]_a \quad \text{hence} \quad [vc]_a = f'_{a,b}([u]_a).$$
◄
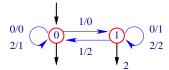
Denoting by $c/_{\mathrm{e}}$ when c is even, and by $c/_{\mathrm{o}}$ when c is odd, we can describe the transducer $\mathcal{G}'_{a,b}$ according to the parity of a as follows:

a,b even   a,b odd

So $\mathcal{G}'_{3,1}$ realizes in base 3 the acceleration $f'_{3,1}$ of the Collatz function:

Thus $\mathcal{G}'_{6,2}$ realizes in base 6 the Collatz function $f_{3,1} = f'_{6,2}$:

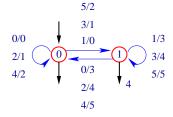More generally for any $0 \leq b < a$, the function $f_{a,b} = f'_{2a,2b}$ is represented in base $2a$ by the prefix sequential transducer $\mathcal{G}'_{2a,2b}$. This synchronizability extends for any $f_{a,b,d}$. We define a prefix sequential transducer realizing $f_{a,b,d}$ for $b < a$ from the transducer computing the division by d in base ad. If an initial path ends to the state j, the input represents an integer n multiple of d plus j and the output represents $\frac{n-j}{d}$. The final digit in $j \neq 0$ is $aj + b$ since $ad\frac{n-j}{d} + aj + b = an + b = f_{a,b,d}(n)$.

Theorem 1. *For all $0 \leq b < a \neq 1$ and $d > 0$, the following transducer $\mathcal{G}_{a,b,d}$*
$(\{0, \ldots, d-1\}, \{0\}, \omega_{a,b}, :_{ad,d})$ *with* $\omega_{a,b}(0) = \varepsilon$, $\omega_{a,b}(j) = aj + b \ \forall \ 0 < j < d$
is prefix sequential and realizes a representation in base ad of $f_{a,b,d}$.

The proof is similar to the one of Proposition 5. Let us apply Theorem 1.

Corollary 1. *For all integers $0 \leq b < a$ and $d > 0$, $f_{a,b,d}$ is prefix automatic.*

The generalization of Corollary 1 for all b remains open.

8   Prefix sequential transducers for $f_{a,b,d}^n$

For all natural numbers n and for $b < a \neq 1$, we can make explicit the composition n times $\mathcal{G}_{a,b,d}^n$ of $\mathcal{G}_{a,b,d}$.

First of all, we need to compose two Euclidean divisions in the same base a: dividing by d then by d' corresponds to divide by dd'. Precisely, the relation $:_{a,d} \circ :_{a,d'}$ is in bijection with the relation $:_{a,dd'}$ by coding any vertex (i, i') where $0 \leq i < d$ and $0 \leq i' < d'$ by the integer $_d[(i, i')] = i + i'd$.

Lemma 3. *For all $a > 1$ and $d, d' > 0$, $_d[:_{a,d} \circ :_{a,d'}]$ is equal to $:_{a,dd'}$.*

Thus and for any $n \geq 0$ and $x, y \in \{0, \ldots, d-1\}^n$, we get

$$x \xrightarrow{b/c}_{(:_{a,d})^n} y \quad \Longleftrightarrow \quad _d[x] \xrightarrow{b/c}_{:_{a,d^n}} {}_d[y]$$

and we can construct $(:_{a,d})^n$ from $:_{a,d^n}$ by renaming each integer vertex by the word of its reverse representation of length n in base d.
To realize the function $f'^n_{a,b}$, we first do the division by 2^n and then the numerator is performed by the terminal function: it multiplies by a and adds b each time an odd number is encountered in the orbit of length n. We define for all natural numbers a, b, n, q the number

$$\eta_{a,b,n}(q) = |\{ 0 \leq i < n \mid f'^i_{a,b}(q) \text{ odd } \}|$$

of odd integers among the first n numbers of the orbit from q of $f'_{a,b}$. Let us give a basic property [1] satisfied by the iterates of $f'_{a,b}$.

Lemma 4. *For all natural numbers a, b, n, p, q with a, b of same parity,*
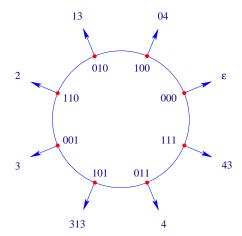$$f'^n_{a,b}(p2^n + q) = p\,a^{\eta_{a,b,n}(q)} + f'^n_{a,b}(q)$$
$$\eta_{a,b,n}(p2^n + q) = \eta_{a,b,n}(q).$$

The prefix sequential transducer $\mathcal{G}'^{\,n}_{a,b}$ realizing $f'^{\,n}_{a,b}$ is the synchronous sequential transducer of the division by 2^n in base a with a terminal function defined by the 2^n first values of $f'^{\,n}_{a,b}$. The length of the final words is needed to take into account the number of 0 to the left. For any vertex, the length of its final word is the number of odd numbers among the first n values of its orbit.

Proposition 6. *For all $n \geq 0$ and $0 \leq b < a \neq 1$ with a, b of the same parity,*
$$_2[\mathcal{G}'^{\,n}_{a,b}] = (\{0, \ldots, 2^n - 1\}, \{0\}, \omega'_n, :_{a,2^n}) \quad \text{with for any } 0 \leq i < 2^n,$$
$$\omega'_n(i) \in \{0, \ldots, a-1\}^* \quad \text{such that} \quad [\omega'_n(i)]_a = f'^{\,n}_{a,b}(i) \quad \text{and} \quad |\omega'_n(i)| = \eta_{a,b,n}(i).$$

Proof. By induction on $n \geq 0$.

$n = 0$: $\mathcal{G}'^{\,0}_{a,b} = (\{\varepsilon\}, \{\varepsilon\}, \omega, \{\varepsilon \xrightarrow{c/c} \varepsilon \mid c \in \{0, \ldots, a-1\}\})$ with $\omega(\varepsilon) = \varepsilon$.

$n \implies n+1$: we have $\mathcal{G}'^{\,n+1}_{a,b} = \mathcal{G}'_{a,b} \circ \mathcal{G}'^{\,n}_{a,b}$ for the composition of prefix sequential transducers. By Lemma 3, the relation $_2[:_{a,2} \circ :_{a,2^n}]$ is equal to $:_{a,2^{n+1}}$.
It remains to verify that ω'_{n+1} is the terminal function of $_2[\mathcal{G}'^{\,n+1}_{a,b}]$.
As $\omega'_{a,b}(0) = \varepsilon$, we get $\omega'_{n+1}(_2[0u]) = \omega'_n(_2[u])$ for any $u \in \{0,1\}^n$ i.e.
$\omega'_{n+1}(2i) = \omega'_n(i)$ for all $0 \leq i < 2^n$. By induction hypothesis, we get
$$[\omega'_{n+1}(2i)]_a = [\omega'_n(i)]_a = f'^{\,n}_{a,b}(i) = f'^{\,n+1}_{a,b}(2i)$$
$$|\omega'_{n+1}(2i)| = |\omega'_n(i)| = \eta_{a,b,n}(i) = \eta_{a,b,n+1}(2i).$$
Similarly $\omega'_{a,b}(1) = \frac{a+b}{2}$ and for any $0 \leq i < 2^n$, there exists unique j and c
such that $i \xrightarrow{\frac{a+b}{2}/c}_{:_{a,2^n}} j$ thus $\omega'_{n+1}(2i+1) = c.\omega'_n(j)$.
Moreover $f'_{a,b}(2i+1) = ia + \frac{a+b}{2} = c2^n + j$. By Lemma 4 and ind. hyp.,
$$[\omega'_{n+1}(2i+1)]_a = [c.\omega'_n(j)]_a = c\,a^{|\omega'_n(j)|} + [\omega'_n(j)]_a = c\,a^{\eta_{a,b,n}(j)} + f'^{\,n}_{a,b}(j)$$
$$= f'^{\,n}_{a,b}(c2^n + j) = f'^{\,n+1}_{a,b}(2i+1)$$
and
$$|\omega'_{n+1}(2i+1)| = 1 + |\omega'_n(j)| = 1 + \eta_{a,b,n}(j) = 1 + \eta_{a,b,n}(c2^n + j)$$
$$= 1 + \eta_{a,b,n}(f'_{a,b}(2i+1)) = \eta_{a,b,n+1}(2i+1).$$
◄

Let us apply this proposition to the acceleration $f'_{5,1}$ of the $5n + 1$ function. We realize a representation in base 5 of the third power $f'^{\,3}_{5,1}$ of $f'_{5,1}$ with the transducer $\mathcal{G}'^{\,3}_{5,1}$ of transition set $(:_{5,2})^3$ of the division $:_{5,8}$ by 8 in base 5 which was illustrated in the previous section. Here is a presentation of its terminal function:

Note that the final word of the state 100 is 04 and not 4, and the accepting path $\longrightarrow 000 \xrightarrow{4/0} 001 \xrightarrow{2/2} 011 \xrightarrow{3/4} 100 \xrightarrow{04}$ of input word 423 and output word 02404 represents $f_{5,1}'^3(113) = 354$ in base 5.

We can now give an explicit description of the prefix sequential transducer $\mathcal{G}_{a,b,d}^n$ realizing $f_{a,b,d}^n$ for all n. For all $a, b, n, q \geq 0$, $\eta_{a,b,n}(q)$ is generalized to

$$\mu_{a,b,d,n}(q) \;=\; |\{\, 0 \leq i < n \mid f_{a,b,d}^i(q) \text{ not multiple of } d \,\}|$$

the number of integers that are not multiples of d among the first n numbers of the orbit from q of $f_{a,b,d}$. Let us adapt Lemma 4 to the powers of $f_{a,b,d}$.

Lemma 5. *For all natural numbers a, b, d, n, p, q with $d > 0$, we have*
$$f_{a,b,d}^n(pd^n + q) \;=\; p\,(ad)^{\mu_{a,b,d,n}(q)} + f_{a,b,d}^n(q)$$
$$\mu_{a,b,d,n}(pd^n + q) \;=\; \mu_{a,b,d,n}(q).$$

Similarly to Proposition 6, we get an explicit description of $\mathcal{G}_{a,b,d}^n$.

Theorem 2. *For all integers $n \geq 0$ and $0 \leq b < a \neq 1$ and $d > 0$,*
$$_d[\mathcal{G}_{a,b,d}^n] \;=\; (\{0,\ldots,d^n-1\},\{0\},\omega_n,:_{ad,d^n})\ \text{ with for any }\ 0 \leq i < d^n,$$
$$\omega_n(i) \in \{0,\ldots,ad-1\}^*\ \text{ with }\ [\omega_n(i)]_{ad} = f_{a,b,d}^n(i)\ \text{ and }\ |\omega_n(i)| = \mu_{a,b,d,n}(i).$$

9   Prefix input-deterministic transducers for $f_{a,b,d}^*$

We present a simple infinite prefix input-deterministic transducer realizing the composition closure of $f_{a,b,d}$ for $b < a \neq 1$ and $d > 0$.

To give a description of a prefix input-deterministic transducer realizing the composition closure $f_{a,b}'^*$, we just take the union of the transition sets $(:_{a,2})^n$ of the division by 2^n in base a, plus the set of initial states 0^n, plus a terminal function defined according to b by length induction *i.e.* from the vertices of the division by 2^n to the vertices of the division by 2^{n-1}.

Proposition 7. *For all $0 \le b < a$ with $a > 1$ and a, b of the same parity,*

$$\mathcal{G}_{a,b}^{\prime *} = (\{0,1\}^*, 0^*, \omega_{a,b}', :_{a,2}^*) \quad \text{where for all } u \in \{0,1\}^*,$$

$$\omega_{a,b}'(0u) = \omega_{a,b}'(u) \quad \text{and} \quad \omega_{a,b}'(1u) = c.\omega_{a,b}'(v) \quad \text{for} \quad 1u \xrightarrow{b/c}_{:_{a,2}^*} 0v.$$

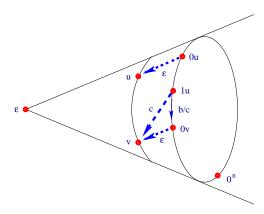*Proof.* We have seen in the proof of Proposition 6 that for all $n \ge 0$, the terminal function ω_{n+1}' of $\mathcal{G}_{a,b}^{\prime\, n+1}$ is defined recursively for all $0 \le i < 2^n$ by

$$\omega_{n+1}'(2i) = \omega_n'(i)$$
$$\omega_{n+1}'(2i+1) = c.\omega_n'(j) \text{ for } i \xrightarrow{\frac{a+b}{2}/c}_{:_{a,2^n}} j$$

and we have

$$i \xrightarrow{\frac{a+b}{2}/c}_{:_{a,2^n}} j \iff ia + \tfrac{a+b}{2} = c2^n + j \iff (2i+1)a + b = c2^{n+1} + 2j$$

$$\iff 2i+1 \xrightarrow{b/c}_{:_{a,2^{n+1}}} 2j.$$

◀

We visualize $\mathcal{G}_{a,b}^{\prime *}$ by a cone with ε at the tip and circular sections. The n-th section is the previously given representation of the Euclidean division $(:_{a,2})^n$ in base a by 2^n of initial state 0^n. The terminal function $\omega_{a,b}'$ is represented as follows:

with a transition $0u \xrightarrow{\varepsilon} u$ from any node starting by 0, and a transition $1u \xrightarrow{c} v$ from any node starting by 1 for the transition $1u \xrightarrow{b/c} 0v$ of the division by $2^{|u|+1}$ in base a. Note that these transitions of the terminal function can only be used at the end of an accepting path.

Similarly to Proposition 7, we get an explicit description of the prefix input-deterministic transducer $\mathcal{G}_{a,b,d}^*$.

Theorem 3. *For all integers $0 \le b < a \neq 1$ and $d > 0$,*

$$\mathcal{G}_{a,b,d}^* = (\downarrow_d^*, 0^*, \omega_{a,b,d}, :_{ad,d}^*) \quad \text{with for all } u \in \downarrow_d^* \text{ and } 0 < i < d,$$

$$\omega_{a,b,d}(0u) = \omega_{a,b,d}(u) \quad \text{and} \quad \omega_{a,b,d}(iu) = c.\omega_{a,b,d}(v) \quad \text{for} \quad iu \xrightarrow{bd/c}_{:_{ad,d}^*} 0v.$$

Theorem 3 states that under the condition $b < a \neq 1$, we realize the composition closure of $f_{a,b,d}$ by taking the union of the divisions $(:_{ad,d})^n$ by d^n in base ad of initial state 0^n, plus a recurrent terminal function on n dependent on ab.

Conclusion

For any natural numbers a, b, d with $b < a \neq 1$ and $d \neq 0$, we have given an explicit construction of an input-deterministic letter-to-letter transducer realizing the closure under composition of $f_{a,b,d}$. In its geometric representation, the disposition of the vertices is well appropriate for both the transitions of the Euclidean divisions and those of the terminal function. It might be a new approach to consider the circularity of the functions $f_{a,b,d}$ namely the existence of paths

$0^n \overset{uv/0^{|v|}u}{\Longrightarrow} x$ where v is the terminal word of the vertex x in the transducer of the division by d^n in base ad. However, the circularity of the Collatz function is already considered as a difficult subproblem of the Collatz conjecture. With this approach and for the acceleration $f'_{3,1,2}$ of the Collatz function, it comes down to a deeper understanding of the Euclidean divisions by 2^n in base 3 and its transitions of input 1.

In this paper, we studied the realizability of functions on integers by both sequential and synchronized transducers. It would be interesting to extend this paper to other functions.

References

1. J.-P. Allouche: *T. Tao et la conjecture de Syracuse*, Gazette de la SMF 168, 34–39 (2021).
2. J. Berstel: *Transductions and context-free languages*, Teubner Studienbücher Informatik (1979).
3. S. Eilenberg: *Automata, languages and machines*, Vol. A, Academic Press (1974).
4. C. Elgot and J. Mezei: *On relations defined by generalized finite automata*, IBM journal of research and development 9, 47–68 (1965)
5. C. Frougny and J. Sakarovitch: *Synchronized rational relations of finite and infinite words*, Theoretical Computer Science 108 (1), 45–82 (1993).
6. S. Ginsburg: *The mathematical theory of context-free languages*, McGraw-Hill (1966).
7. J. Lagarias: *The ultimate challenge: the 3x+1 problem*, American Mathematical Society (2010).
8. G. Mealy: *A Method for Synthesizing Sequential Circuits*, Bell System Technical Journal, 1045–1079 (1955).
9. M. Rabin and D. Scott: *Finite automata and their decision problems*, IBM journal of research and development 3(2), 125–144 (1959). Reprinted in *Sequential machines: Selected papers*, Addison Wesley (1965).
10. J. Sakarovitch: *Elements of automata theory*, Cambridge University Press (2009).
11. M.-P. Schützenberger: *Sur une variante des fonctions séquentielles*, Theoretical Computer Science 4 (1), 47–57 (1977).

A   Appendix

We give here in details all the missing proofs.

4   Prefix and suffix sequential transducers

The two families of prefix sequential and of suffix sequential transducers are closed under composition.

Proposition 3. *The prefix and the suffix sequential functions are preserved in quadratic time and space under composition, intersection and difference.*

Proof.
i) Let $\mathcal{T} = (Q, i, \omega, T)$ and $\mathcal{T}' = (Q', i', \omega', T')$ be prefix sequential transducers. We realize $<\mathcal{T}> \circ <\mathcal{T}'>$ by the composition of $\mathcal{T}$ by $\mathcal{T}'$ having this simpler form:

$$\mathcal{T} \circ \mathcal{T}' = (Q \times Q', (i, i'), \omega \circ \omega', T \circ T')$$

where $\qquad T \circ T' = \{\ (p, p') \xrightarrow{a/c} (q, q') \mid \exists\, b\ (p \xrightarrow{a/b}_T q \wedge p' \xrightarrow{b/c}_{T'} q')\ \}$

$$(\omega \circ \omega')(q, p') = v.\omega'(q') \quad \text{for} \quad p' \overset{\omega(q)/v}{\Longrightarrow}_{T'} q'$$

that is illustrated as follows:

$$\left. \begin{array}{l} \longrightarrow\ i \overset{u/v}{\Longrightarrow}_T q \overset{w}{\longrightarrow}_\omega \\[4pt] \longrightarrow\ i' \overset{v/x}{\Longrightarrow}_{T'} p' \overset{w/y}{\Longrightarrow}_{T'} \overset{z}{\longrightarrow}_{\omega'} \end{array} \right\} \ \longrightarrow\ (i, i') \overset{u/x}{\Longrightarrow}_{T \circ T'} (q, p') \overset{yz}{\longrightarrow}_{\omega \circ \omega'}$$

We realize $<\mathcal{T}> \cap <\mathcal{T}'>$ by the usual *synchronization product* of $\mathcal{T}$ and $\mathcal{T}'$:

$$\mathcal{T} \times \mathcal{T}' = (Q \times Q', (i, i'), \omega \times \omega', T \times T')$$

where $\qquad T \times T' = \{\ (p, p') \xrightarrow{a/b} (q, q') \mid p \xrightarrow{a/b}_T q \wedge p' \xrightarrow{a/b}_{T'} q'\ \}$

and $\quad (\omega \times \omega')(q, q') = \omega(q)$ when $\omega(q) = \omega'(q')$.

We also realize $<\mathcal{T}> - <\mathcal{T}'>$ by the following prefix sequential transducer:

$$\mathcal{T} - \mathcal{T}' = (Q \cup (Q \times Q'), (i, i'), \omega \backslash \omega', T \backslash T')$$

where $\qquad T \backslash T' = (T \times T') \cup T \cup \{\ (p, p') \xrightarrow{a/b} q \mid p \xrightarrow{a/b}_T q \wedge p' \overset{a/b}{\not\rightarrow}_{T'}\ \}$

and $\qquad (\omega \backslash \omega')(q) = \omega(q)$ for any $q \in \mathrm{dom}(\omega)$

$$(\omega \backslash \omega')(q, q') = \omega(q) \text{ for any } q \in \mathrm{dom}(\omega) \text{ and}$$

$$(q' \notin \mathrm{dom}(\omega') \vee \omega'(q') \neq \omega(q)).$$

The transducers $\mathcal{T} \circ \mathcal{T}'$, $\mathcal{T} \times \mathcal{T}'$, $\mathcal{T} - \mathcal{T}'$ can be constructed from $\mathcal{T}$ and $\mathcal{T}'$ in quadratic time and space.

ii) Let $\mathcal{T} = (Q, i, F, T)$ and $\mathcal{T}' = (Q', i', F', T')$ be suffix sequential transducers. The composition of $\mathcal{T}$ by $\mathcal{T}'$ has this simpler form:

$$\mathcal{T} \circ \mathcal{T}' = (Q \times Q', (i, i'), F \times F', T \circ T')$$

where $T \circ T' = \{ \ (p, p') \xrightarrow{a/\varepsilon} (q, p') \mid p \xrightarrow{a/\varepsilon}_T q \ \wedge \ p' \in Q' \ \}$

$$\cup \ \{ \ (p, p') \xrightarrow{a/b} (q, q') \mid \exists \, c \ (p \xrightarrow{a/c}_T q \ \wedge \ p' \xrightarrow{c/b}_{T'} q') \ \}.$$

The composition is illustrated as follows:

$$\left. \begin{array}{l} \longrightarrow \ i \overset{u/\varepsilon}{\Longrightarrow}_T \overset{v/x}{\Longrightarrow}_T p \overset{w/y}{\Longrightarrow}_T q \ \longrightarrow \\[4pt] \longrightarrow \ i' \overset{x/\varepsilon}{\Longrightarrow}_{T'} p' \overset{y/z}{\Longrightarrow}_{T'} q' \ \longrightarrow \\[4pt] |v| = |x| \ \wedge \ |w| = |y| = |z| \end{array} \right\} \ \longrightarrow \ (i, i') \overset{uv/\varepsilon}{\Longrightarrow}_{T \circ T'} (p, p') \overset{w/z}{\Longrightarrow}_{T \circ T'} (q, q') \ \longrightarrow$$

We have to show that $T \circ T'$ remains of initial ε-output.

Let $(p, p') \xrightarrow{a/b}_{T \circ T'} (q, q') \xrightarrow{c/d}_{T \circ T'} (r, r')$ with $b \neq \varepsilon$.

We have to check that $d \neq \varepsilon$.

There exists e such that $p \xrightarrow{a/e}_T q$ and $p' \xrightarrow{e/b}_{T'} q'$.

As T' is without ε-input transition, $e \neq \varepsilon$.

As T is of initial ε-output, $q \overset{c/\varepsilon}{\not\to}_T r$. So $q' \xrightarrow{\cdot/d}_{T'} r'$.

As T' is of initial ε-output, $d \neq \varepsilon$.

We realize $<\mathcal{T}> \cap <\mathcal{T}'>$ by the synchronization product of $\mathcal{T}$ and $\mathcal{T}'$:
$$\mathcal{T} \times \mathcal{T}' = (Q \times Q', (i, i'), F \times F', T \times T')$$

where $T \times T' = \{ \ (p, p') \xrightarrow{a/b} (q, q') \mid p \xrightarrow{a/b}_T q \ \wedge \ p' \xrightarrow{a/b}_{T'} q' \ \}.$

We realize $<\mathcal{T}> - <\mathcal{T}'>$ by the following suffix sequential transducer:
$$\mathcal{T} - \mathcal{T}' = (Q \cup (Q \times Q'), (i, i'), F \cup (F \times (Q' - F')), T \setminus T').$$

These three transducers can be constructed in quadratic time and space according to $\mathcal{T}$ and $\mathcal{T}'$.

◄

6   Suffix automaticity of functions $\mathbf{f_{a,b,d}}$

Let us extend $_{d,a}\times$ to paths.

Lemma 1. *For any $i, j \geq 0$ and $u, v \in \downarrow_d^*$, we have*
$$i \overset{u/v}{\Longrightarrow}_{d,a\times} j \quad \Longleftrightarrow \quad {}_d[u]a + i = {}_d[v] + j \, d^{|u|} \ \text{ and } \ |u| = |v|.$$

Proof.

$\Longrightarrow$: As $_{d,a}\times$ is synchronous, $|u| = |v|$.

Let us check the equality by induction on $|u| \geq 0$.

$|u| = 0$: We have $u = \varepsilon = v$ and $i = j$ hence the equality.

Let $i \overset{bu/cv}{\Longrightarrow} j$ with $b, c \in \downarrow_d$ and the implication true for u.

There exists k such that $i \xrightarrow{b/c} k \overset{u/v}{\Longrightarrow} j$.

Thus $ab + i = c + dk$ and $_d[u]a + k = {}_d[v] + j \, d^{|u|}$. Hence

$$_d[cv] + j\,d^{|bu|} \;=\; c + (_d[v] + j\,d^{|u|})d \;=\; c + (_d[u]a + k)d$$
$$=\; _d[u]ad + c + dk \qquad =\; (_d[u]d + b)a + i$$
$$=\; _d[bu]a + i.$$

$\Longleftarrow$: by induction on $|u| \geq 0$.

$|u| = 0$: We have $u = \varepsilon = v$ and $i = j$ hence $i \overset{u/v}{\Longrightarrow} j$.

Suppose the implication true for $|u|$ and $_d[bu]a+i = \,_d[cv]+j\,d^{|bu|}$ with $|u| = |v|$ and $0 \leq b, c < d$. So, we have

$$(_d[u]a)d + ab + i \;=\; (_d[v] + j\,d^{|u|})d + c.$$

By Euclidean division of $ab + i$ by d, we have $ab + i = kd + c'$ with $c' < d$.
As $c < d$, we have $c = c'$ hence $_d[u]a + k = \,_d[v] + j\,d^{|u|}$.

As $|u| = |v|$ and by induction hypothesis, $k \overset{u/v}{\Longrightarrow} j$.

As $ab + i = kd + c$, we get $i \overset{b/c}{\longrightarrow} k$ hence $i \overset{bu/cv}{\Longrightarrow} j$.

◀

7 Prefix automaticity of functions $f_{a,b,d}$

We give a proof of the realization of $f_{a,b,d}$ by a prefix sequential transducer with d states.

Theorem 1. *For all* $0 \leq b < a \neq 1$ *and* $d > 0$*, the following transducer* $\mathcal{G}_{a,b,d}$
$(\{0, \ldots, d-1\}, \{0\}, \omega_{a,b}, :_{ad,d})$ *with* $\omega_{a,b}(0) = \varepsilon$, $\omega_{a,b}(j) = aj + b \;\forall\; 0 < j < d$
is prefix sequential and realizes a representation in base ad *of* $f_{a,b,d}$ *.*

Proof.
It suffices to generalize the proof of Proposition 5.
As $:_{ad,d}$ is input-deterministic and input-complete, for any $u \in \downarrow_{ad}^*$, there exists
a unique $v \in \downarrow_{ad}^*$ and a unique $j \in \downarrow_d$ such that $0 \overset{u/v}{\Longrightarrow}_{:ad,d} j$.
By Lemma 2, $[u]_{ad} = d\,[v]_{ad} + j$.

For $j = 0$, $[u]_{ad}$ is a multiple of d and $[v]_{ad} = \frac{[u]_{ad}}{d} = f_{a,b,d}([u]_{ad})$.
For $0 < j < d$, $[u]_{ad}$ is not multiple of d.
We have $j \overset{(bd)/\omega_{a,b}(j)}{\longrightarrow}_{:ad,d} 0$ hence $0 \overset{u(bd)/v.\omega_{a,b}(j)}{\Longrightarrow}_{:ad,d} 0$ thus

$$ad\,[u]_{ad} + bd \;=\; [u(bd)]_{ad} \;=\; d\,[v.\omega_{a,b}(j)]_{ad}$$

so

$$[v.\omega_{a,b}(j)]_{ad} \;=\; a\,[u]_{ad} + b \;=\; f_{a,b,d}([u]_{ad}).$$

◀

It follows the prefix automaticity of $f_{a,b,d}$.

Corollary 1. *For all integers* $0 \leq b < a$ *and* $d > 0$*,* $f_{a,b,d}$ *is prefix automatic.*

Proof.

It remains to check the case $a = 1$ and $b = 0$. For $d > 0$, the transducer
$$(\downarrow_d, 0, \omega, \{\, i \xrightarrow{j/i} j \mid i, j \in \downarrow_d \,\})\ \text{with}\ \omega(0) = \varepsilon\ \text{and}\ \omega(i) = i\ \text{for any}\ 0 < i < d$$
is prefix sequential and realizes
$$\{\, (u0, 0u) \mid u \in \downarrow_d^* \,\}\ \cup\ \{\, (ui, 0ui) \mid u \in \downarrow_d^* \wedge 0 < i < d \,\}$$
which is a representation of $f_{1,0,d}$ in base d (with the least significant digit to the right).

◀

8   Prefix sequential transducers for $\mathbf{f^{n}_{a,b,d}}$

Let us express the composition of two divisions in the same base.

Lemma 3. *For all* $a > 1$ *and* $d, d' > 0$, $_d[\,:_{a,d} \circ :_{a,d'}]$ *is equal to* $:_{a,dd'}$.

Proof.
For all $0 \le i, j < d$ and $0 \le i', j' < d'$, we have
$$(i, i') \xrightarrow{b/c}_{:_{a,d} \circ :_{a,d'}} (j, j')$$
$$\Longleftrightarrow\ \exists\, 0 \le e < a\ \text{such that}\ i \xrightarrow{b/e}_{:_{a,d}} j\ \text{and}\ i' \xrightarrow{e/c}_{:_{a,d'}} j'$$
$$\Longleftrightarrow\ \exists\, 0 \le e < a\ \text{such that}\ ia + b = ed + j\ \text{and}\ i'a + e = cd' + j'$$
$$\Longleftrightarrow\ ia + b = (cd' + j' - i'a)d + j$$
$$\Longleftrightarrow\ (i + i'd)a + b = cdd' + j + j'd$$
$$\Longleftrightarrow\ _d[(i, i')] \xrightarrow{b/c}_{:_{a,dd'}} {}_d[(j, j')]\,.$$
◀

To be complete, we verify a known property on the iterates of $f'_{a,b}$.

Lemma 4. *For all natural numbers* a, b, n, p, q *with* a, b *of same parity,*
$$f'^{\,n}_{a,b}(p2^n + q) = p\, a^{\eta_{a,b,n}(q)} + f'^{\,n}_{a,b}(q)$$
$$\eta_{a,b,n}(p2^n + q) = \eta_{a,b,n}(q).$$

Proof.
By induction on $n \ge 0$.
$n = 0$: $\eta_{a,b,0}$ is the constant mapping 0 and $f'^{\,0}_{a,b}$ is the identity.
$n \Longrightarrow n+1$: For q even, we have
$$f'^{\,n+1}_{a,b}(p2^{n+1} + q) = f'^{\,n}_{a,b}(f'_{a,b}(p2^{n+1} + q)) = f'^{\,n}_{a,b}(p2^n + \tfrac{q}{2})$$
$$= p\, a^{\eta_{a,b,n}(\frac{q}{2})} + f'^{\,n}_{a,b}(\tfrac{q}{2}) = p\, a^{\eta_{a,b,n+1}(q)} + f'^{\,n+1}_{a,b}(q)$$
and $\eta_{a,b,n+1}(p2^{n+1} + q) = \eta_{a,b,n}(p2^n + \tfrac{q}{2}) = \eta_{a,b,n}(\tfrac{q}{2}) = \eta_{a,b,n+1}(q).$
For q odd, we have

$$f_{a,b}'^{\,n+1}(p2^{n+1}+q) = f_{a,b}'^{\,n}(f_{a,b}'(p2^{n+1}+q)) \qquad = f_{a,b}'^{\,n}(ap2^n + \tfrac{aq+b}{2})$$
$$= f_{a,b}'^{\,n}(ap2^n + f_{a,b}'(q)) \qquad = p\,a^{1+\eta_{a,b,n}(f_{a,b}'(q))} + f_{a,b}'^{\,n}(f_{a,b}'(q))$$
$$= p\,a^{\eta_{a,b,n+1}(q)} + f_{a,b}'^{\,n+1}(q)$$

and

$$\eta_{a,b,n+1}(p2^{n+1}+q) = 1 + \eta_{a,b,n}(f_{a,b}'(p2^{n+1}+q)) = 1 + \eta_{a,b,n}(ap2^n + f_{a,b}'(q))$$
$$= 1 + \eta_{a,b,n}(f_{a,b}'(q)) \qquad = \eta_{a,b,n+1}(q).$$

◀

Similarly to Lemma 4, we give a basic property satisfied by the powers of $f_{a,b,d}$.

Lemma 5. *For all natural numbers a, b, d, n, p, q with $d > 0$, we have*
$$f_{a,b,d}^{\,n}(pd^n + q) = p\,(ad)^{\mu_{a,b,d,n}(q)} + f_{a,b,d}^{\,n}(q)$$
$$\mu_{a,b,d,n}(pd^n + q) = \mu_{a,b,d,n}(q).$$

Proof.
By induction on $n \geq 0$.
$n = 0$: immediate because $\mu_{a,b,d,0}$ is the constant mapping 0 and $f_{a,b,d}^{\,0}$ is the identity.
$n \implies n+1$: For q multiple of d, we have
$$f_{a,b,d}^{\,n+1}(pd^{n+1}+q) = f_{a,b,d}^{\,n}(f_{a,b,d}(pd^{n+1}+q)) \qquad = f_{a,b,d}^{\,n}(pd^n + \tfrac{q}{d})$$
$$= p\,(ad)^{\mu_{a,b,d,n}(\frac{q}{d})} + f_{a,b,d}^{\,n}(\tfrac{q}{d}) = p\,(ad)^{\mu_{a,b,d,n+1}(q)} + f_{a,b,d}^{\,n+1}(q)$$
and $\mu_{a,b,d,n+1}(pd^{n+1}+q) = \mu_{a,b,d,n}(pd^n + \tfrac{q}{d}) = \mu_{a,b,d,n}(\tfrac{q}{d}) = \mu_{a,b,d,n+1}(q)$.
For q not multiple of d, we have
$$f_{a,b,d}^{\,n+1}(pd^{n+1}+q)$$
$$= f_{a,b,d}^{\,n}(f_{a,b,d}(pd^{n+1}+q)) \qquad = f_{a,b,d}^{\,n}(apd^{n+1} + aq + b)$$
$$= f_{a,b,d}^{\,n}((pad)d^n + f_{a,b,d}(q)) \qquad = pad\,(ad)^{\mu_{a,b,d,n}(f_{a,b,d}(q))} + f_{a,b,d}^{\,n}(f_{a,b,d}(q))$$
$$= p\,(ad)^{\mu_{a,b,d,n+1}(q)} + f_{a,b,d}^{\,n+1}(q)$$

and

$$\mu_{a,b,d,n+1}(pd^{n+1}+q)$$
$$= 1 + \mu_{a,b,d,n}(apd^{n+1} + aq + b) = 1 + \mu_{a,b,d,n}((pad)d^n + f_{a,b,d}(q))$$
$$= 1 + \mu_{a,b,d,n}(f_{a,b,d}(q)) \qquad = \mu_{a,b,d,n+1}(q).$$

◀

We explicit the composition n times of the transducer $\mathcal{G}_{a,b,d}$.

Theorem 2. *For all integers $n \geq 0$ and $0 \leq b < a \neq 1$ and $d > 0$,*
$$_d[\mathcal{G}_{a,b,d}^{\,n}] = (\{0, \dots, d^n - 1\}, \{0\}, \omega_n, :_{ad,d^n}) \quad \text{with for any } 0 \leq i < d^n,$$
$$\omega_n(i) \in \{0, \dots, ad - 1\}^* \quad \text{with } [\omega_n(i)]_{ad} = f_{a,b,d}^{\,n}(i) \quad \text{and } |\omega_n(i)| = \mu_{a,b,d,n}(i).$$

Proof.
By induction on $n \geq 0$.

$n = 0$: $\mathcal{G}^0_{a,b,d} = (\{\varepsilon\}, \{\varepsilon\}, \omega, \{\varepsilon \xrightarrow{c/c} \varepsilon \mid c \in \{0, \ldots, ad-1\}\})$ with $\omega(\varepsilon) = \varepsilon$.

$n \implies n+1$: we have $\mathcal{G}^{n+1}_{a,b,d} = \mathcal{G}_{a,b,d} \circ \mathcal{G}^n_{a,b,d}$ for the composition of prefix sequential transducers.

By Lemma 3, the transition relation $_d[:_{ad,d} \circ :_{ad,d^n}]$ is equal to $:_{ad,d^{n+1}}$.

It remains to verify that ω_{n+1} is the terminal function of $_d[\mathcal{G}^{n+1}_{a,b,d}]$.

As $\omega_{a,b}(0) = \varepsilon$, we get $\omega_{n+1}(_d[0u]) = \omega_n(_d[u])$ for any $u \in \downarrow^n_d$

i.e. $\omega_{n+1}(di) = \omega_n(i)$ for all $0 \le i < d^n$. By induction hypothesis, we get

$$[\omega_{n+1}(di)]_{ad} = [\omega_n(i)]_{ad} = f^n_{a,b,d}(i) \quad = f^{n+1}_{a,b,d}(di)$$
$$|\omega_{n+1}(di)| = |\omega_n(i)| \quad = \mu_{a,b,d,n}(i) = \mu_{a,b,d,n+1}(di).$$

Let $0 \le i < d^n$ and $0 < j < d$. So $\omega_{a,b}(j) = aj + b \le a(d-1) + b < ad$.

There exists unique k and c such that $i \xrightarrow{aj+b/c}_{:_{ad,d^n}} k$

thus $\omega_{n+1}(di + j) = c.\omega_n(k)$.

Moreover $f_{a,b,d}(di + j) = iad + aj + b = cd^n + k$. By Lemma 5 and ind. hyp.,

$$[\omega_{n+1}(di+j)]_{ad} = [c.\omega_n(k)]_{ad} \quad = c\,(ad)^{|\omega_n(k)|} + [\omega_n(k)]_{ad} = c\,(ad)^{\mu_{a,b,d,n}(k)} + f^n_{a,b,d}(k)$$
$$= f^n_{a,b,d}(cd^n + k) = f^{n+1}_{a,b,d}(di + j)$$

and

$$|\omega_{n+1}(di+j)| = 1 + |\omega_n(k)| \qquad\qquad = 1 + \mu_{a,b,d,n}(k) \quad = 1 + \mu_{a,b,d,n}(cd^n + k)$$
$$= 1 + \mu_{a,b,d,n}(f_{a,b,d}(di+j)) = \mu_{a,b,d,n+1}(di + j).$$

◀

9   Prefix input-deterministic transducers for $\mathbf{f^*_{a,b,d}}$

We explicit the closure under composition of the transducer $\mathcal{G}_{a,b,d}$.

Theorem 3. *For all integers $0 \le b < a \ne 1$ and $d > 0$,*

$$\mathcal{G}^*_{a,b,d} = (\downarrow^*_d, 0^*, \omega_{a,b,d}, :^*_{ad,d}) \text{ with for all } u \in \downarrow^*_d \text{ and } 0 < i < d,$$

$$\omega_{a,b,d}(0u) = \omega_{a,b,d}(u) \text{ and } \omega_{a,b,d}(iu) = c.\omega_{a,b,d}(v) \text{ for } iu \xrightarrow{bd/c}_{:^*_{ad,d}} 0v.$$

Proof.

We have seen in the proof of Theorem 2 that for all $n \ge 0$, the terminal function ω_{n+1} of $\mathcal{G}^{n+1}_{a,b,d}$ is defined recursively for all $0 \le i < d^n$ and $0 < j < d$ by

$$\omega_{n+1}(di) = \omega_n(i)$$
$$\omega_{n+1}(di + j) = c.\omega_n(k) \text{ for } i \xrightarrow{aj+b/c}_{:_{ad,d^n}} k$$

and we have

$$i \xrightarrow{aj+b/c}_{:_{ad,d^n}} k \iff iad + aj + b = cd^n + k \iff (di + j)ad + bd = cd^{n+1} + dk$$

$$\iff di + j \xrightarrow{bd/c}_{:_{ad,d^{n+1}}} dk .$$

◀