WELL BEHAVED TRANSITION SYSTEMS

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ABSTRACT. The well-quasi-ordering (i.e., a well-founded quasi-ordering such that all antichains are finite) that defines well-structured transition systems (WSTS) is shown not to be the weakest hypothesis that implies decidability of the coverability problem. We show coverability decidable for monotone transition systems that only require the absence of infinite antichains and call *well behaved transitions systems* (WBTS) the new strict superclass of the class of WSTS that arises. By contrast, we confirm that boundedness and termination are undecidable for WBTS under the usual hypotheses, and show that stronger monotonicity conditions can enforce decidability. Proofs are similar or even identical to existing proofs but the surprising message is that a hypothesis implicitly assumed minimal for twenty years in the theory of WSTS can meaningfully be relaxed, allowing more orderings to be handled in an abstract way.

1. INTRODUCTION

The concept of a *well-structured transition system* (WSTS) arose thirty years ago, in 1987 precisely [Fin87, Fin90], where such systems were initially called *structured transition systems* and shown to have decidable termination and boundedness problems. WSTS were developed for the purpose of capturing properties common to a wide range of formal models used in model-checking, system verification and concurrent programming. The coverability for such systems was shown decidable in 1996 [ACJT96, ACJT00], thus generalizing the decidability of coverability for lossy channel systems [AJ93] but also generalizing a much older result by Arnold and Latteux [AL78, Theorem 5, p. 391], published in French and thus less accessible, stating that coverability for vector addition systems in the presence of

resets is decidable. It is interesting to note that the algorithm used by Arnold and Latteux in 1979 is an instance of the backward algorithm presented in [ACJT96] and applied to $\mathbb{N}^n$.

The usefulness of the WSTS stemmed from its clear abstract treatment of the properties responsible for the decidability of coverability, termination and boundedness. This provided the impetus for an intensive development of the theory of WSTS, begun in the year 2000 (see [FPS01, ACJT00] for surveys and [BDK$^+$12, KS12, WZH10, ZWH12, EFM99, KKW12, GHPR13] for a sample of recent applications of the WSTS). WSTS remain under development and are actively being investigated [FGL09a, FGL09b, GRB06, SS13, BS13, SS11].

At its core, a WSTS is simply an infinite set X (of states) with a transition relation $\rightarrow \subseteq X \times X$. The set X is quasi-ordered by $\leq$, and $\rightarrow$ fulfills one of various possible monotonicities, i.e. compatibilities with $\leq$. The quasi-ordering of X is further assumed to be well, i.e. well-founded and with no infinite antichains (see Section 2 for precise formal definitions).

Over the years, a number of strengthenings and weakenings of the notion of monotonicity (of $\rightarrow$ w.r.t. $\leq$) were introduced, with the goal of allowing WSTS to capture ever more models [FPS01]. But to the best of our knowledge, the wellness hypothesis attached to the quasi-ordering of X was never questioned, apparently under the assumption that wellness surely ought to be the weakest possible hypothesis that would allow deducing any form of decidability property.

Our main contribution is to prove the above assumption unjustified. Indeed, we show that the wellness assumption in the definition of WSTS can be relaxed while some decidabilities are retained. More precisely, wellness in a quasi-ordering is equivalent to the following two properties being fulfilled simultaneously:

- well-foundedness, i.e., the absence of an infinite descending sequence of elements, and
- finiteness of antichains, i.e., the absence of infinite sets of pairwise incomparable elements.

We show that dropping well-foundedness from the definition of a WSTS (resulting in a "WBTS") still allows deciding the coverability problem, even in the presence of infinite branching. Indeed, while the usual backward algorithm [ACJT96] for coverability relies on well-foundedness, the forward algorithm described here does not require that property!

For example, the set $\mathbb{Z}$ of integers with increment and decrement as its transitions defines a WBTS that is not a WSTS. Another example of a WBTS that is not a WSTS is that of a vector addition system with domain $\mathbb{Z}^d$ (hence without guards) rather than $\mathbb{N}^d$ and with d-tuples ordered by building on the usual $\mathbb{Z}$-ordering lexicographically rather than componentwise. Yet a less artificial example introduced in this paper is that of a *weighted vector addition system*, defined as a normal d-VASS (over $\mathbb{N}^d$) extended with a $\mathbb{Z}^w$-component ordered lexicographically (see Sect. 4 for precise definition and semantics).

Having defined WBTS, we argue that no general backward strategy would apply to determine coverability for WBTS. Our first contribution is to nonetheless show the coverability problem for WBTS decidable, by the use of a forward strategy. Coverability is thus decidable for each model mentioned in previous paragraph, sparing us the need for separate independent arguments.

Deciding any computational problem, for a general class of WBTS, naturally requires that the class verify a number of effectiveness conditions. One such condition in the case of coverability is the need to be able to manipulate downward-closed subsets of the system domain. Verifying this condition for weighted VASS requires an analysis of the subsets

of $\mathbb{Z}^d$ that are downward-closed under the lexicographical ordering. Elucidating the ideal structure of such downward-closed subsets of $\mathbb{Z}^d$ is our second contribution.

Our third contribution is to contrast WBTS and WSTS from the point of view of the termination and boundedness problems. As expected, under monotonicity conditions that ensure decidability of termination and boundedness for WSTS, we exhibit WBTS for which both problems are undecidable. By comparison, we investigate monotonicity conditions that, even in WBTS, allow one to decide termination (in the finitely branching case) and boundedness (in both the finitely and the infinitely branching cases).

The paper is organized as follows. Section 2 introduces terminology. Section 3 defines well-behaved transition systems, gives our first example of WBTS, defines effectiveness and studies downward-closed sets, including those of $\mathbb{Z}^d$ under the lexicographical ordering. Section 4 proves coverability decidable for WBTS and defines the weighted VASS model. Section 5 compares the WSTS and the WBTS from the point of view of the decidability of the termination and boundedness problems. Section 6 concludes with a discussion and future work.

2. Preliminaries

2.1. **Orderings.** Let X be a set and let $\leq \, \subseteq X \times X$. The relation $\leq$ is a *quasi-ordering* if it is reflexive and transitive. If $\leq$ is additionally antisymmetric, then $\leq$ is a *partial order*. The set X is *well-founded (under $\leq$)* if there is no infinite strictly decreasing sequence $x_0 > x_1 > \ldots$ of elements of X. An *antichain (under $\leq$)* is a subset $A \subseteq X$ of pairwise incomparable elements, i.e. for every $a, b \in A$, $a \not\leq b$ and $b \not\leq a$. We say that a quasi-ordering $\leq$ is a *well-quasi-ordering* for X if X is well-founded and contains no infinite antichain under $\leq$. Let $A \subseteq X$, we define the *downward closure* and *upward closure* of A respectively as $\uparrow A \stackrel{\text{def}}{=} \{x \in X : x \geq a \text{ for some } a \in A\}$ and $\downarrow A \stackrel{\text{def}}{=} \{x \in X : x \leq a \text{ for some } a \in A\}$. A subset $A \subseteq X$ is said to be *downward closed* if $A = \downarrow A$ and *upward closed* if $A = \uparrow A$. An *ideal* is a downward closed subset $I \subseteq X$ that is also *directed*, i.e. it is nonempty and for every $a, b \in I$, there exists $c \in I$ such that $a \leq c$ and $b \leq c$. The set of ideals of X is denoted $\mathsf{Ideals}(X) \stackrel{\text{def}}{=} \{I \subseteq X : I = \downarrow I \text{ and } I \text{ is directed}\}$.

2.2. **Transition systems and effectiveness.** A *transition system* is a pair $\mathcal{S} = (X, \rightarrow)$ such that X is a set whose elements are called the *states* of $\mathcal{S}$, and a *transition relation* $\rightarrow \, \subseteq X \times X$. We extend a transition relation $\rightarrow$ to

$$\rightarrow^k \stackrel{\text{def}}{=} \underbrace{\rightarrow \circ \rightarrow \circ \cdots \circ \rightarrow}_{k \text{ times}}, \quad \rightarrow^+ \stackrel{\text{def}}{=} \bigcup_{k \geq 1} \rightarrow^k \text{ and } \rightarrow^* \stackrel{\text{def}}{=} \mathsf{Id} \cup \rightarrow^+$$

where Id is the identity relation. For every $x \in X$, $\mathsf{Post}(x) \stackrel{\text{def}}{=} \{y \in X : x \rightarrow y\}$ and $\mathsf{Pre}(x) \stackrel{\text{def}}{=} \{y \in X : y \rightarrow x\}$ denote respectively the sets of *immediate successors* and *predecessors* of x. Similarly, for every $x \in X$, $\mathsf{Post}^*(x) \stackrel{\text{def}}{=} \{y \in X : x \rightarrow^* y\}$ and $\mathsf{Pre}^*(x) \stackrel{\text{def}}{=} \{y \in X : y \rightarrow^* x\}$ denote respectively the sets of *successors* and *predecessors* of x. A transition system is *finitely branching* if $\mathsf{Post}(x)$ is finite for every state x, otherwise it is *infinitely branching*. An *ordered transition system* $\mathcal{S} = (X, \rightarrow, \leq)$ is a transition system $(X, \rightarrow)$ equipped with a quasi-ordering $\leq \, \subseteq X \times X$. We naturally extend $\mathsf{Post}, \mathsf{Pre}, \mathsf{Post}^*$ and Pre^* to subsets of states, e.g. for $A \subseteq X$ we have $\mathsf{Post}^*(A) = \bigcup_{x \in A} \mathsf{Post}^*(x)$.

A *class $\mathcal{C}$ of transition systems* is any countable set of transition systems. We denote the ith transition system of a class $\mathcal{C}$, for some fixed enumeration, by $\mathcal{C}(i)$. For every class $\mathcal{C}$ we require the existence of a set $\mathsf{Enc}_{\mathcal{C}} \subseteq \mathbb{N}$ and a surjective *representation map* $r : \mathsf{Enc}_{\mathcal{C}} \to \bigcup_i X_i$ where X_i is the set of states of $\mathcal{C}(i)$. Let $\mathsf{Enc}_{X_i} = \{e \in \mathsf{Enc}_{\mathcal{C}} : r(e) \in X_i\}$, we further require the set $\{(i,e) : i \in \mathbb{N}, e \in \mathsf{Enc}_{X_i}\}$ to be decidable. A Turing machine M over $\mathbb{N} \times \mathbb{N}$ is said to *compute* a relation $\rho \subseteq X_i \times X_i$ if M halts at least on $\mathsf{Enc}_{X_i} \times \mathsf{Enc}_{X_i}$ and for each $e, e' \in \mathsf{Enc}_{X_i}$, M accepts $(e, e') \iff (r(e), r(e')) \in \rho$.

A class $\mathcal{C}$ of ordered transition systems is *effective* if there exists a pair of Turing machines $(M_\to, M_\le)$ operating on $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that, for each $i \in \mathbb{N}$, $M_\to$ with first argument set to i computes the transition relation "$\to$" of $\mathcal{C}(i)$ and $M_\le$ with first argument set to i computes the ordering relation "$\le$" of $\mathcal{C}(i)$. We say that $\mathcal{C}$ is *post-effective* if it is effective, and if there exists an additional Turing machine that computes $|\mathsf{Post}_{\mathcal{C}(i)}(x)| \in \mathbb{N} \cup \{\infty\}$ on input (i, x), with $i \in \mathbb{N}$ and $x \in X_i$. Such a Turing machine, in combination with $M_\to$, allows computing $\mathsf{Post}(x)$ whenever the latter is finite. We say that $\mathcal{C}$ is *upward pre-effective* if it is effective, and if there exists an additional Turing machine that computes a minimal basis of $\uparrow\mathsf{Pre}_{\mathcal{C}(i)}(\uparrow x)$ on input (i, x), where $i \in \mathbb{N}$ and x is a state of $\mathcal{C}(i)$. By extension, we say that an ordered transition system $\mathcal{S}$ is *effective* (resp. *post-effective*, *upward pre-effective*) if the degenerate class $\{\mathcal{S}\}$ is effective (resp. post-effective, upward pre-effective).

Just as the states of an ordered transition system are encoded over the natural numbers, we assume the existence of a representation map for ideals, and that testing whether a natural number encodes an ideal under this map is decidable.

2.3. Monotone and well-structured transition systems.
Let $\mathcal{S} = (X, \to, \le)$ be an ordered transition system and $\bowtie\ \in \{\ge, \le\}$. We say that $\mathcal{S}$ is *(upward) monotone* if $\bowtie$ is $\ge$ (resp. *downward monotone* if $\bowtie$ is $\le$) and if for every $x, x', y \in X$,

$$x \to y \wedge x' \bowtie x \implies \exists y' \bowtie y \text{ s.t. } x' \to^* y' . \tag{2.1}$$

We will consider variants of monotonicity that were introduced in the literature by modifying (2.1) as follows:

transitive monotonicity: $x \to y \wedge x' \bowtie x \implies \exists y' \bowtie y$ s.t. $x' \to^+ y'$

strong monotonicity: $x \to y \wedge x' \bowtie x \implies \exists y' \bowtie y$ s.t. $x' \to y'$

Let $\bowtie\ \in \{>, <\}$ be the strict variant of $\bowtie$. For any one of the above monotonicities, an ordered transition system is said to be *strictly monotone* (with respect to the relevant monotonicity) if it *additionally* satisfies, for every $x, x', y \in X$,

$$x \to y \wedge x' \bowtie x \implies \exists y' \bowtie y \text{ s.t. } x' \to^\# y'$$

where $\# \in \{*, +, 1\}$ is in accord with the relevant monotonicity. Note that strong monotonicity implies transitive monotonicity which implies (standard) monotonicity.

Definition 2.1 ([Fin90])**.** A *well structured transition system (WSTS)* is a monotone transition system $S = (X, \to, \le)$ such that $(X, \le)$ is a wqo.

The notion of downward monotonicity, perhaps less known, has been introduced in [FPS01] to study so-called *downward WSTS* and has been used, for example, to analyze timed alternating automata in [OW07].

3. Beyond WSTS: Well Behaved Transition Systems

We generalize well-structured transition systems by weakening the well-quasi-ordering constraint. Instead, we consider monotone transition systems ordered by quasi-orderings with no infinite antichains. That is, we no longer require the ordering to be well-founded:

Definition 3.1. A *well behaved transition system (WBTS)* is a monotone transition system $S = (X, \rightarrow, \leq)$ such that $(X, \leq)$ contains no infinite antichain.

It is clear from the definition that every WSTS is a WBTS, however the converse is not true. For example, consider automata that can increase or decrease a single counter whose value ranges over $\mathbb{Z}$. Such integer one-counter automata are readily seen to be WBTS, however they are not WSTS since $\mathbb{Z}$ contains infinite strictly decreasing sequences. WBTS can, in particular, be built from the (classical) lexicographical ordering over finite words or integer tuples. These orderings cannot be used in the setting of WSTS since they are not well-founded, but are allowed in WBTS since these orderings do not induce infinite antichains. WBTS are also closed under ordering reversal, which is not the case of WSTS. More precisely, for an ordered transition system $\mathcal{S} = (X, \rightarrow, \leq)$, we define the *ordering reversal* of $\mathcal{S}$ as $\mathcal{S}^{\mathsf{ord\text{-}rev}} \stackrel{\mathrm{def}}{=} (X, \rightarrow, \geq)$. It is easily seen that $\mathcal{S}$ is a WBTS with upward monotonicity if, and only if, $\mathcal{S}^{\mathsf{ord\text{-}rev}}$ is a WBTS with downward monotonicity. In general, WSTS are not closed under ordering reversal since the well-foundedness of an ordering is not necessarily preserved when it is reversed, e.g. $\mathbb{N}$ is well-quasi-ordered by $\leq$, but $0 < 1 < 2 < \ldots$ is an infinite strictly decreasing sequence over $\geq$.

3.1. An example of WBTS.
As a proof of concept, and to build intuition, we exhibit a class of WBTS that satisfies the monotonicities presented. This class is based on integer vector addition systems with states that were recently studied in [HH14, CHH14, BFG$^+$15]. An *integer vector addition system with states* ($\mathbb{Z}^d$-*VASS*) is a pair $\mathcal{V} = (Q, T)$ such that Q and T are finite sets, and $T \subseteq Q \times \mathbb{Z}^d \times Q$ where $d > 0$. Sets Q and T are respectively called the *control states* and *transitions* of $\mathcal{V}$. Intuitively, a $\mathbb{Z}^d$-VASS is a vector addition systems with states (VASS), a model equivalent to Petri nets, but in which the counters of the VASS may drop below zero. Formally, a $\mathbb{Z}^d$-VASS induces a transition system $(Q \times \mathbb{Z}^d, \rightarrow)$ such that $(p, \boldsymbol{u}) \rightarrow (q, \boldsymbol{v}) \stackrel{\mathrm{def}}{\Longleftrightarrow} \exists (p, \boldsymbol{z}, q) \in T$ s.t. $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{z}$. The set of states $Q \times \mathbb{Z}^d$ of these systems is typically ordered by equality on Q and the usual componentwise ordering of $\mathbb{Z}^d$, and are therefore neither well-founded nor without infinite antichains. However, we show that $\mathbb{Z}^d$-VASS are WBTS when ordered lexicographically, i.e. under $\preceq_{\mathsf{lex}}$ where $(p, \boldsymbol{u}) \preceq_{\mathsf{lex}} (p', \boldsymbol{u}') \stackrel{\mathrm{def}}{\Longleftrightarrow} p = p' \wedge \boldsymbol{u} \leq_{\mathsf{lex}} \boldsymbol{u}'$ for $\leq_{\mathsf{lex}}$ the usual lexicographical ordering, i.e. $\boldsymbol{u} \leq_{\mathsf{lex}} \boldsymbol{u}' \stackrel{\mathrm{def}}{\Longleftrightarrow} \boldsymbol{u} = \boldsymbol{u}' \vee \exists i$ s.t. $\boldsymbol{u}(i) < \boldsymbol{u}'(i) \wedge \forall j < i, \boldsymbol{u}(j) = \boldsymbol{u}'(j)$.

Proposition 3.2. $\mathbb{Z}^d$-*VASS ordered by* $\preceq_{\mathsf{lex}}$ *are WBTS with upward, downward, strong and strict monotonicity.*

Proof. Let $\mathcal{V} = (Q, T)$ be a $\mathbb{Z}^d$-VASS. First note that any antichain of $Q \times \mathbb{Z}^d$ is of length at most $|Q|$. It remains to show that $\mathcal{V}$ is monotone. Let $(p, \boldsymbol{u}), (q, \boldsymbol{v}), (p', \boldsymbol{u}') \in Q \times \mathbb{Z}^d$ be such that $(p, \boldsymbol{u}) \rightarrow (q, \boldsymbol{v})$ and $(p, \boldsymbol{u}) <_{\mathsf{lex}} (p', \boldsymbol{u}')$. There exists $(p, \boldsymbol{z}, q) \in T$ such that $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{z}$. By definition of $\leq_{\mathsf{lex}}$, $p' = p$, hence $(p', \boldsymbol{u}') \rightarrow (q, \boldsymbol{v}')$ for $\boldsymbol{v}' \stackrel{\mathrm{def}}{=} \boldsymbol{u}' + \boldsymbol{z}$. Let $1 \leq i \leq d$ be the smallest component such that $\boldsymbol{u}(i) \neq \boldsymbol{u}'(i)$. Since $\boldsymbol{u} <_{\mathsf{lex}} \boldsymbol{u}'$, we have $\boldsymbol{u}(i) < \boldsymbol{u}'(i)$, hence $\boldsymbol{v}(j) = \boldsymbol{u}(j) + \boldsymbol{z}(j) = \boldsymbol{u}'(j) + \boldsymbol{z}(j) = \boldsymbol{v}'(j)$ for every $1 \leq j < i$, and $\boldsymbol{v}(j) = \boldsymbol{u}(j) + \boldsymbol{z}(j) <$

$\boldsymbol{u}'(j) + \boldsymbol{z}(j) = \boldsymbol{v}'(j)$. Therefore, $\boldsymbol{v} <_{\mathsf{lex}} \boldsymbol{v}'$ and consequently $(q, \boldsymbol{v}) \preceq_{\mathsf{lex}} (q, \boldsymbol{v}')$. Thus, $\mathcal{V}$ has upward, strong and strict monotonicity. Downward monotonicity follows symmetrically by considering $>_{\mathsf{lex}}$ instead of $<_{\mathsf{lex}}$. □

3.2. **Decomposition of downward closed sets into finite unions of ideals.** It was observed in [FGL09a, FGL16, BFM14, BFM16] that any downward closed subset of a well-quasi-ordered set is equal to a finite union of ideals, which led to further applications in the study of WSTS. Here we stress the fact that such finite decompositions also exist in quasi-ordered sets with no infinite antichain. The existence of such a decomposition has been proved numerous times (for partial orderings instead of quasi-orderings) in the order theory community [Bon75, Pou79, PZ85, Fra86, LMP87] under different terminologies, and is a particular case of a more general set theory result of Erdős & Tarski [ET43] on the existence of *limit numbers* between $\aleph_0$ and $2^{\aleph_0}$. We extract from Bonnet [Bon75] and Fraïssé [Fra86] a simple proof tailored to our situation. Specifically, our proof is based on the fact that such decompositions exist in well-quasi-ordered sets and is reminiscent of Fraïssé's proof strategy [Fra86, Sect. 4.7.2, p. 124], which is based on [Bon75, Lemma 2, p. 193].

Theorem 3.3 ([ET43, Bon75, Pou79, PZ85, Fra86, LMP87])**.** *A countable quasi-ordered set X contains no infinite antichain if, and only if, every downward closed subset of X is equal to a finite union of ideals.*

Proof. Let X be a countable set quasi-ordered by $\leq$.

Only if. Suppose that X contains no infinite antichain. Let D be a downward closed subset of X. We iteratively build a sequence of elements and of downward closed subsets of D. Let $D_0 \overset{\text{def}}{=} D$. For $i = 0, 1, \ldots$, we do the following: if $D_i = \emptyset$ we stop, otherwise we pick some $x_i \in D_i$ and we let $D_{i+1} \overset{\text{def}}{=} D_i \setminus \{y \in D_i : y < x_i\}$. Let $D' \overset{\text{def}}{=} \{x_i : i \in \mathbb{N}\}$, let $\leq'$ be the quasi-ordering $\leq$ restricted to D', and let $\downarrow'$ denote the downward closure under $\leq'$. We argue that D' is well-quasi-ordered by $\leq'$. If D' is finite we are done. Otherwise, by hypothesis on D, there is no infinite antichain in D'. Let us show that D' is well-founded under $\leq'$. Suppose that D' contains an infinite decreasing sequence $x_{i_0} > x_{i_1} > \ldots$ where each $i_j \in \mathbb{N}$. There must exist $j < k$ such that $i_j < i_k$, but then $x_{i_k} \notin D_{i_k}$ since $x_{i_k} < x_{i_j}$, which is a contradiction. Since D' is well-quasi-ordered, by [Fra86, FGL09a, BFM16], there exist $I_1, I_2, \ldots, I_k \in \mathsf{Ideals}(D')$ such that $\downarrow' D' = I_1 \cup I_2 \cup \cdots \cup I_k$.

We claim that $D \subseteq \downarrow D'$, hence $D = \downarrow D'$. Indeed, if $D = D'$, the claim is trivial, otherwise let $y \in D \setminus D'$, by construction of D', $y < x_i$ for some $i \in \mathbb{N}$, hence $y \in \downarrow x_i \subseteq \downarrow D'$. Therefore,
$$D = \downarrow D' = \downarrow (I_1 \cup I_2 \cup \cdots \cup I_k) = \downarrow I_1 \cup \downarrow I_2 \cup \ldots \downarrow I_k .$$
To conclude, it suffices to show that $\downarrow I_i \in \mathsf{Ideals}(X)$ for each $1 \leq i \leq k$. Obviously $\downarrow I_i$ is downward closed, hence it suffices to show that it is directed. Let $a, b \in \downarrow I_i$, there exist $a', b' \in I_i$ such that $a \leq a'$ and $b \leq b'$. Since $I_i \in \mathsf{Ideals}(D')$, there exists $c \in I_i$ such that $a' \leq' c$ and $b' \leq' c$. Thus, $a \leq a' \leq c$ and $b \leq b' \leq c$.

If. Conversely, suppose that there exists an infinite antichain $A \subseteq X$. We prove that there exists a downward closed subset $D \subseteq X$ that is not equal to a finite union of ideals. Let $D = \bigcup_{a \in A} \downarrow a$. Assume that there exist $I_1, I_2, \ldots, I_k \in \mathsf{Ideals}(X)$ such that $D = I_1 \cup I_2 \cup \cdots \cup I_k$. By the pigeonhole principle, there exists some $1 \leq i \leq k$ such that I_i contains infinitely many elements from A. Let $a, b \in I_i$ be distinct elements. Since I_i is

directed, there exists $c \in I_i$ such that $a \leq c$ and $b \leq c$. Moreover, since $I_i \subseteq D$, there exists some $a' \in A$ such that $c \leq a'$. Therefore, $a \leq a'$ and $b \leq a'$. Because a and b are distinct, at least two distinct elements of A are comparable, i.e. either a and a', or b and a'. Hence, A is not an antichain, which is a contradiction. □

Let X be a set quasi-ordered by an ordering $\leq$ having no infinite antichain. Theorem 3.3 allows us as in [BFM14] to define a canonical finite decomposition of a downward closed subset $D \subseteq X$, that is, the (finite) set $\mathsf{IdealDecomp}(D)$ of maximal ideals contained in D under inclusion.

3.3. **Effectiveness of downward closed sets.** In this subsection, we describe effectiveness hypotheses that allow manipulating downward closed sets in ordered transition systems.

Definition 3.4. A class $\mathcal{C}$ of WBTS is *ideally effective* if, given $\mathcal{S} = (X, \rightarrow, \leq) \in \mathcal{C}$,
- the set of encodings of $\mathsf{Ideals}(X)$ is recursive,
- the function mapping the encoding of a state $x \in X$ to the encoding of the ideal $\downarrow x \in \mathsf{Ideals}(X)$ is computable;
- inclusion of ideals of X is decidable;
- the downward closure $\downarrow \mathsf{Post}(I)$ expressed as a finite union of ideals is computable from the ideal $I \in \mathsf{Ideals}(X)$.

Note that a class of WBTS is ideally effective if, and only if, the class of its so-called completions [BFM14, BFM16] is post-effective. The notion of completion naturally applies to WBTS, but we do not use the notion in this paper.

Enforcing WBTS to be ideally effective is not an issue for all the useful models of which we are aware. Indeed, a large scope of well structured transition systems, hence of WBTS, are ideally effective [FGL09a]: Petri nets, VASS and their extensions (with resets, transfers, affine functions), lossy channel systems and extensions with data.
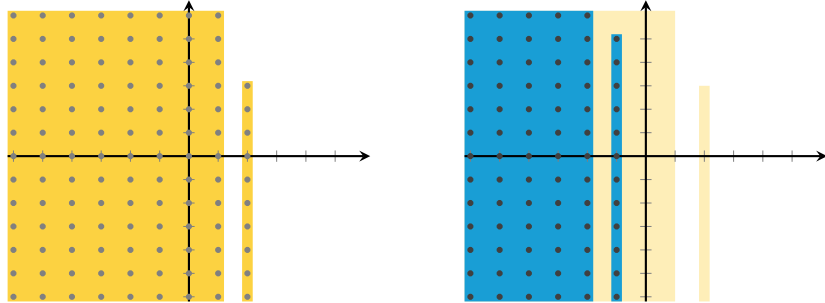

Figure 1: Left: $\downarrow_{\mathsf{lex}} (2,3)$. Right: $\downarrow_{\mathsf{lex}} (-1, 5) = \downarrow_{\mathsf{lex}} (2,3) + (-3, 2)$.

As an example, we argue that $\mathbb{Z}^d$-VASS introduced in Sect. 3.1 form an ideally effective class of WBTS. To do so, we need to investigate the downward and upward closed sets of $\mathbb{Z}^d$ under $\preceq_{\mathsf{lex}}$. Since the control states are ordered under equality, we may only consider $<_{\mathsf{lex}}$. Let $\boldsymbol{x} \in \mathbb{Z}^d$, we give descriptions of $\downarrow_{\mathsf{lex}} \boldsymbol{x}$ and $\uparrow_{\mathsf{lex}} \boldsymbol{x}$ where $\downarrow_{\mathsf{lex}}$ and $\uparrow_{\mathsf{lex}}$ denote respectively the downward and upward closures under $\leq_{\mathsf{lex}}$. Let $\mathsf{down}_d : \mathbb{Z}^d \rightarrow 2^{\mathbb{Z}^d}$ be defined as follows

$$\mathsf{down}_d(x_1, x_2, \ldots, x_d) = \begin{cases} \downarrow(x_1 - 1) \times \mathbb{Z}^{d-1} \cup \{x_1\} \times \mathsf{down}_{d-1}(x_2, x_3, \ldots, x_d) & \text{if } d > 1 \\ \downarrow x_1 & \text{if } d = 1 \end{cases}$$

and let $\mathsf{up}_d : \mathbb{Z}^d \to 2^{\mathbb{Z}^d}$ be defined in the same way by replacing $\downarrow$ with $\uparrow$. We have $\downarrow_{\mathsf{lex}} \boldsymbol{x} = \mathsf{down}_d(\boldsymbol{x})$ and $\uparrow_{\mathsf{lex}} \boldsymbol{x} = \mathsf{up}_d(\boldsymbol{x})$. For example, $\downarrow_{\mathsf{lex}}(2,3) = (\downarrow 1 \times \mathbb{Z}) \cup (\{2\} \times \downarrow 3)$ is depicted on the left of Fig. 1. More generally, the ideals of $\mathbb{Z}^d$ under $\leq_{\mathsf{lex}}$, denoted $\mathsf{Ideals}_{\mathsf{lex}}(\mathbb{Z}^d)$, can be described as follows:

Proposition 3.5. $\mathsf{Ideals}_{\mathsf{lex}}(\mathbb{Z}^d) = X_d$ *where*

$$X_d = \begin{cases} \{\mathbb{Z}^d\} \cup \{\downarrow(x-1) \times \mathbb{Z}^{d-1} \cup \{x\} \times I : x \in \mathbb{Z}, I \in X_{d-1}\} & \text{if } d > 1 \\ \{\mathbb{Z}\} \ \cup \{\downarrow x : x \in \mathbb{Z}\} & \text{if } d = 1 \end{cases}$$

Proof. We proceed by induction on d. The base case is immediate since $\leq_{\mathsf{lex}}$ coincides with $\leq$ for $d = 1$. Let $d > 1$, suppose the claim holds for $d - 1$. Recall that downward closed subsets under $\leq_{\mathsf{lex}}$ are ideals since $\leq_{\mathsf{lex}}$ is total.

"$\subseteq$": Let $I \in \mathsf{Ideals}_{\mathsf{lex}}(\mathbb{Z}^d)$ and let us show that $I \in X_d$. Let $F = \{\boldsymbol{v}(1) : \boldsymbol{v} \in I\}$. If F is unbounded from above, then $I = \mathbb{Z}^d$ and trivially $I \in X_d$. Otherwise, let x be the largest element of F. By downward closure of I under $\leq_{\mathsf{lex}}$ and by definition of $\leq_{\mathsf{lex}}$,

$$I = \{\boldsymbol{v} \in \mathbb{Z}^d : \boldsymbol{v}(1) < x\} \cup \{\boldsymbol{v} \in \mathbb{Z}^d : \exists \boldsymbol{u} \in \mathbb{Z}^d \text{ s.t. } [x \ \boldsymbol{u}] \in I\} . \tag{3.1}$$

Let us show that $I' \overset{\text{def}}{=} \{\boldsymbol{u} \in \mathbb{Z}^{d-1} : [x \ \boldsymbol{u}] \in I\}$ is downward closed under $\leq_{\mathsf{lex}}$, and hence in $\mathsf{Ideals}(\mathbb{Z}^{d-1})$. Let $\boldsymbol{u} \in I'$ and $\boldsymbol{u}' \leq_{\mathsf{lex}} \boldsymbol{u}$. We have $[x \ \boldsymbol{u}'] \leq_{\mathsf{lex}} [x \ \boldsymbol{u}]$, hence $[x \ \boldsymbol{u}'] \in I$ by downward closure under $\leq_{\mathsf{lex}}$ of I, and thus $\boldsymbol{u}' \in I'$. Therefore, by (3.1), we have

$$I = \{\boldsymbol{v} \in \mathbb{Z}^d : \boldsymbol{v}(1) < x\} \cup \{\boldsymbol{v} \in \mathbb{Z}^d : \exists \boldsymbol{u} \in I' \text{ s.t. } [x \ \boldsymbol{u}] \in I\}$$

$$= \downarrow(x-1) \times \mathbb{Z}^{d-1} \quad \cup \{x\} \times I' .$$

By induction hypothesis, $I' \in X_{d-1}$, and thus $I \in X_d$.

"$\supseteq$": Let $I \in X_d$ and let us show that $I \in \mathsf{Ideals}_{\mathsf{lex}}(\mathbb{Z}^d)$. If $I = \mathbb{Z}^d$, then $I \in \mathsf{Ideals}_{\mathsf{lex}}(\mathbb{Z}^d)$. Assume that $I \neq \mathbb{Z}^d$, then by definition of X_d and by induction hypothesis

$$I = \downarrow(x-1) \times \mathbb{Z}^{d-1} \cup \{x\} \times I'$$

for some $x \in \mathbb{Z}$ and $I' \in \mathsf{Ideals}_{\mathsf{lex}}(\mathbb{Z}^{d-1})$. Let us show that I is downward closed under $\leq_{\mathsf{lex}}$, and hence in $\mathsf{Ideals}_{\mathsf{lex}}(\mathbb{Z}^d)$. Let $\boldsymbol{v} \in I$ and $\boldsymbol{v}' \leq_{\mathsf{lex}} \boldsymbol{v}$. If $\boldsymbol{v} \in \downarrow(x-1) \times \mathbb{Z}^{d-1}$, then $\boldsymbol{v}' \in \downarrow(x-1) \times \mathbb{Z}^{d-1} \subseteq I$ since $\boldsymbol{v}'(1) \leq \boldsymbol{v}(1)$. If $\boldsymbol{v} \in \{x\} \times I'$, then there are two cases to consider:

- If $\boldsymbol{v}'(1) < \boldsymbol{v}(1)$, then $\boldsymbol{v}'(1) \in \downarrow(x-1) \times \mathbb{Z}^{d-1} \subseteq I$.
- If $\boldsymbol{v}'(1) = \boldsymbol{v}(1)$, then there exist $\boldsymbol{u} \in I'$ and $\boldsymbol{u}' \in \mathbb{Z}^{d-1}$ such that $\boldsymbol{v} = [x \ \boldsymbol{u}]$, $\boldsymbol{v}' = [x \ \boldsymbol{u}']$ and $\boldsymbol{u}' \leq_{\mathsf{lex}} \boldsymbol{u}$. Since I' is downward closed under $\leq_{\mathsf{lex}}$, we have $\boldsymbol{u}' \in I'$. Therefore, $\boldsymbol{v}' = [x \ \boldsymbol{u}'] \in \{x\} \times I' \subseteq I$. $\qquad\square$

Ideals of $\mathbb{Z}^d$ can be categorized into $d+1$ types, as illustrated in Fig. 2 for $d = 2$. Since $\leq_{\mathsf{lex}}$ is total, every downward closed subset of $\mathbb{Z}^d$ is directed, hence ideals under $\leq_{\mathsf{lex}}$ are precisely the downward closed sets. Symmetrically, every upward closed subset of $\mathbb{Z}^d$ is directed under $>_{\mathsf{lex}}$ and can be described by replacing $\uparrow$ with $\downarrow$ in the above description of $\mathsf{Ideals}_{\mathsf{lex}}(\mathbb{Z}^d)$. Upward and downward closed subsets can thus be represented symbolically with disjoint finite unions of products of terms of the form $\{a\}$, $\downarrow a$ or $\uparrow a$, and $\mathbb{Z}$.

Inclusion between two downward (resp. upward) closed subsets is decidable, e.g. we may translate $I \subseteq J$ into a formula of the first-order theory of integers with addition, i.e. $\mathsf{FO}(\mathbb{Z}, +, <)$, which is decidable [Pre29]. For example, to test whether $(\downarrow 2 \times \mathbb{Z}) \subseteq$
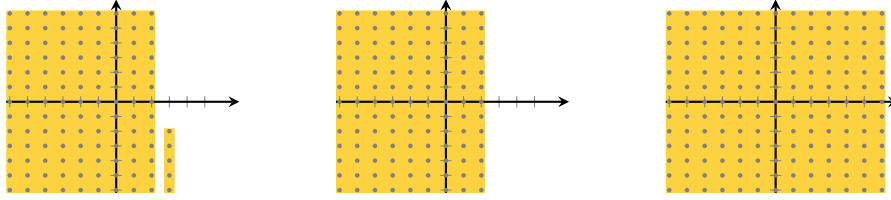
Figure 2: Example of each of the three types of ideals of $\mathbb{Z}^2$ under lexicographical ordering.

$(\downarrow 1 \times \mathbb{Z}) \cup (\{2\} \times \downarrow 3)$, we verify if the following formula is satisfiable: $\varphi \overset{\text{def}}{=} \forall x, y \in \mathbb{Z}, \ (x \leq 2) \implies ((x \leq 1) \vee (x = 2 \wedge y \leq 3))$.

Moreover, we can effectively add some $\boldsymbol{z} \in \mathbb{Z}^d$ to a downward (resp. upward) closed subset $A \subseteq \mathbb{Z}^d$ and obtain the representation of the result in polynomial time. For example, $\downarrow_{\text{lex}} (2, 3) + (-3, 2) = (\downarrow 1 \times \mathbb{Z}) \cup (\{2\} \times \downarrow 3) + (-3, 2) = (\downarrow (-2) \times \mathbb{Z}) \cup (\{-1\} \times \downarrow 5) = \downarrow_{\text{lex}} (-1, 5)$ as illustrated on the right of Fig. 1.

From these observations, we can encode and manipulate downward/upward closed subsets effectively, and thus:

Proposition 3.6. $\mathbb{Z}^d$-*VASS form a post-effective and ideally effective class of WBTS.*

4. Decidability of Coverability for Well Behaved Transition Systems

The coverability problem is defined as follows: on input an ordered transition system $\mathcal{S} = (X, \rightarrow, \leq)$ and two states $x, y \in X$, determine whether $x \rightarrow^* y'$ for some $y' \geq y$. In this section, we show coverability decidable for WBTS that enjoy a weak form of effectiveness. With all the effectiveness notions in place, we then define the (apparently new) notion of a (d, w)-VASS, i.e., *weighted d-VASS*, as a WBTS that fulfills the required effectiveness and thus has a decidable coverability problem.

The *backward algorithm* [ACJT96, AJ93, AL78] is perhaps the best known algorithm for deciding coverability in upward pre-effective WSTS. It proceeds by starting with $\uparrow y$ and computing iteratively the sequence $\uparrow \mathsf{Pre}(\uparrow y), \uparrow \mathsf{Pre}(\uparrow \mathsf{Pre}(\uparrow y)), \ldots$ until the union of this sequence stabilizes, which is guaranteed to happen by $\leq$ being a well-quasi-ordering. The finite union of this sequence yields $\uparrow \mathsf{Pre}^*(y)$, and hence it suffices to verify whether this contains x or not. When $\leq$ is not a well-quasi-ordering, this approach fails since the procedure may never halt. For example, consider the $\mathbb{Z}^2$-VASS $\mathcal{V} \overset{\text{def}}{=} (\{q\}, \{(q, (0, 1), q)\}$ ordered lexicographically. Since $\mathbb{Z}^d$-VASS are upward pre-effective, we may execute the backward algorithm on $\mathcal{V}$. To verify whether $\boldsymbol{y} = (1, 1)$ is coverable from $\boldsymbol{x} = (0, 0)$, the backward algorithm iteratively computes $(q, \uparrow_{\text{lex}} (1, 1)), (q, \uparrow_{\text{lex}} (1, 0)), (q, \uparrow_{\text{lex}} (1, -1)), (q, \uparrow_{\text{lex}} (1, -2)), \ldots$ as illustrated in Fig. 3. Since this sequence is strictly increasing and does not contain $\boldsymbol{x}$, the backward algorithm never halts.

By contrast, we show in this section that the forward approach for coverability, initially presented by Geeraerts, Raskin, and Van Begin [GRB04, GRB06] for WSTS[1] and simplified in [FGL09b, BFM14, BFM16], avoids this problem and actually works for WBTS under the same effectiveness hypothesis. The approach relies on decompositions of downward closed

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[1]The idea had also appeared in 1982; see [Pac82, Corollary 8.7] where it was applied to the reachability problem for communicating finite automata with FIFO channels.
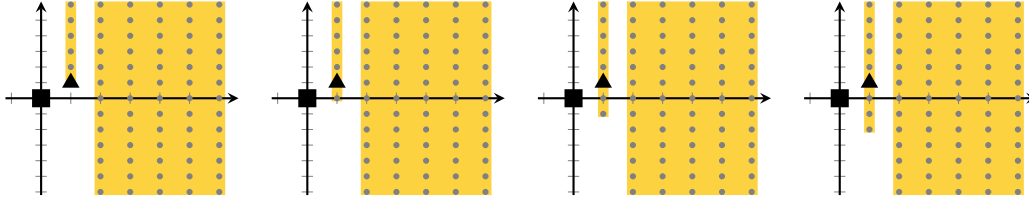
Figure 3: From left to right: first four iterations of the backward algorithm trying to determine whether $\boldsymbol{y} = (1,1)$ is coverable from $\boldsymbol{x} = (0,0)$ in the $\mathbb{Z}^2$-VASS $\mathcal{V} = (\{q\}, \{(q,(0,1),q)\})$. Vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are respectively marked as ■ and ▲.

sets into finitely many ideals. The proof that the forward approach of [BFM14, BFM16] is correct for WSTS requires no essential modification for WBTS, but we expand it in more details here.

In order to decide whether y is coverable from x, we execute two procedures in parallel, one looking for a coverability certificate and one looking for a non coverability certificate. Procedure 1 iteratively computes

$$\downarrow x, \ \downarrow \mathsf{Post}(\downarrow x), \ \downarrow \mathsf{Post}(\downarrow \mathsf{Post}(\downarrow x)), \ \ldots$$

until it finds y.

Procedure 1: searches for a coverability certificate of y from x

---
1 $D \leftarrow \downarrow x$
2 while $y \notin D$ **do**
3 $D \leftarrow D \cup \downarrow \mathsf{Post}(D)$
4 return *true*

The second procedure enumerates inductive invariants in some fixed order $D_1, D_2, \ldots$, i.e. downward closed subsets $D_i \subseteq X$ such that $\downarrow \mathsf{Post}(D_i) \subseteq D_i$. Any inductive invariant D_i such that $x \in D_i$ and $y \notin D_i$ is a certificate of non coverability. This is due to the fact that every inductive invariant D_i is an "over-approximation" of $\mathsf{Post}^*(x)$ if it contains x. Moreover, by standard monotonicity, $\downarrow \mathsf{Post}^*(x)$ is such an inductive invariant and may eventually be found.

Procedure 2: enumerates inductive invariants to find non coverability certificate of y from x.

---
1 $i \leftarrow 0$
2 while $\neg(\downarrow \mathsf{Post}(D_i) \subseteq D_i$ and $x \in D_i$ and $y \notin D_i)$ **do**
3 $i \leftarrow i + 1$
4 return *false*

We show that these two procedures are correct:

Theorem 4.1. *Let* $\mathcal{S} = (X, \rightarrow, \leq)$ *be a WBTS, and let* $x, y \in X$.

(1) *y is coverable from x if, and only if, Procedure 1 terminates.*
(2) *y is not coverable from x if, and only if, Procedure 2 terminates.*

Proof.

(1) Procedure 1 computes

$$D = \bigcup_{k=0} \underbrace{\downarrow\mathsf{Post}(\cdots \downarrow\mathsf{Post}(\downarrow x))}_{k \text{ times}} \ .$$

It suffices to show that $D = \downarrow\mathsf{Post}^*(x)$, since y is coverable from x if, and only if, $y \in \downarrow\mathsf{Post}^*(x)$.

The inclusion $\downarrow\mathsf{Post}^*(x) \subseteq D$ is immediate. Let us prove that $D \subseteq \downarrow\mathsf{Post}^*(x)$. Let $z \in D$. There exist $k \in \mathbb{N}$ and $x_0, x_0', x_1, x_1', \ldots, x_k, x_k'$ such that $x_0 = x$, $x_k' = z$, $x_i \geq x_i'$ for every $0 \leq i \leq k$, and $x_i' \to x_{i+1}$ for every $0 \leq i < k$. By applying monotonicity k times, we obtain $x \to^* z'$ for some $z' \geq z$. Thus, $z \in \downarrow\mathsf{Post}^*(x)$, whence $D \subseteq \downarrow\mathsf{Post}^*(x)$.

(2) By a simple induction, it can be shown that $\downarrow\mathsf{Post}^*(D) \subseteq D$ for every inductive invariant D. If Procedure 2 terminates, then $y \notin D \supseteq \downarrow\mathsf{Post}^*(D) \supseteq \downarrow\mathsf{Post}^*(x)$ which implies that y is not coverable from x.

It remains to show that Procedure 2 terminates whenever y is not coverable from x. To do so, it suffices to prove that $\downarrow\mathsf{Post}^*(x)$ is an inductive invariant. Indeed, this implies that $\downarrow\mathsf{Post}^*(x)$ is eventually found by Procedure 2 when y is not coverable from x. Formally, let us show that $\downarrow\mathsf{Post}(\downarrow\mathsf{Post}^*(x)) \subseteq \downarrow\mathsf{Post}^*(x)$. Let $b \in \downarrow\mathsf{Post}(\downarrow\mathsf{Post}^*(x))$, there exists a', a, b' such that $x \to^* a'$, $a' \geq a$, $a \to b'$ and $b' \geq b$. By monotonicity, there exists $b'' \geq b'$ such that $a' \to^* b''$. Therefore, $x \to^* b''$ and $b' \geq b$, hence $b \in \downarrow\mathsf{Post}^*(x)$. $\qquad\square$

In order to implement Procedure 1 and Procedure 2, some effectiveness hypotheses must be made. We argue that both procedures may be implemented for ideally effective classes of WBTS. We first need the following crucial proposition concerning inclusion of ideals, in particular for testing inclusion of downward closed sets. We include its proof for completeness:

Proposition 4.2 ([BFM14, BFM16])**.** *Let X be a quasi-ordered set. For every $I, J_1, J_2, \ldots, J_m \in \mathsf{Ideals}(X)$, $I \subseteq J_1 \cup J_2 \cup \cdots \cup J_m$ if, and only if, $I \subseteq J_j$ for some $1 \leq j \leq m$.*

Proof. We claim that if a directed set I is included in $J \cup K$ where J and K are downward closed, then either $I \subseteq J$ or $I \subseteq K$. The claim implies the proposition by a straightforward induction since an ideal is directed and any union of ideals is downward closed.

To see the claim, let $I \subseteq J \cup K$ under the conditions stated and suppose to the contrary that there exist $s \in I \setminus J$ and $t \in I \setminus K$. Since I is directed, there exists $u \in I$ such that $s \leq u$ and $t \leq u$. Since $u \in I$, either $u \in J$ or $u \in K$. By downward closures of J and K, either $s \in J$ or $t \in K$, a contradiction that proves the claim. $\qquad\square$

From the definition of ideally effective classes of WBTS and from Prop. 4.2, we can show that the elementary operations of Procedure 1 and Procedure 2 are computable. Formally:

Lemma 4.3. *Let $\mathcal{C}$ be an ideally effective class of WBTS. There exist Turing machines $(M_{\mathsf{down}}, M_{\subseteq}, M_{\downarrow\mathsf{Post}}, M_{\mathsf{memb}})$ such that, on input $\mathcal{S} = (X, \to, \leq) \in \mathcal{C}$,*

(1) M_{down} *enumerates every downward closed subsets of X by their ideal decomposition,*

(2) $M_{\subseteq}$ *decides inclusion between downward closed subsets of X prescribed by their ideal decomposition,*

(3) $M_{\downarrow\mathsf{Post}}$ *computes the ideal decomposition of $\downarrow\mathsf{Post}(D)$ for downward closed subsets D of X prescribed by their ideal decomposition,*

(4) M_{memb} *decides* $x \in D$, *given* $x \in X$ *and a downward closed subset* $D \subseteq X$ *prescribed by its ideal decomposition.*

Proof.

(1) By Theorem 3.3, every downward closed subset of X decomposes into finitely many ideals. Moreover, since $\mathcal{C}$ is ideally effective, ideals of X may be effectively enumerated. Thus, M_{down} enumerates downward closed subsets by enumerating finite subsets of ideals.

(2) Let $D, D' \subseteq X$ be the given downward closed subsets prescribed by their ideal decomposition. By Prop. 4.2, $D \subseteq D'$ if, and only if, for every $I \in \mathsf{IdealDecomp}(D)$ there exists $J \in \mathsf{IdealDecomp}D'$ such that $I \subseteq J$. Therefore, this test can be performed by $M_{\subseteq}$.

(3) Let D be the given downward closed subset prescribed by its ideal decomposition. Since $\mathcal{C}$ is ideally effective, $M_{\downarrow\mathsf{Post}}$ can compute $Y_I = \downarrow\mathsf{Post}(I)$ for every $I \in \mathsf{IdealDecomp}(D)$. We have,

$$\downarrow\mathsf{Post}(D) = \bigcup_{I \in \mathsf{IdealDecomp}(D)} \bigcup_{J \in Y_I} J . \tag{4.1}$$

In order to obtain precisely $\mathsf{IdealDecomp}(\downarrow\mathsf{Post}(D))$, $M_{\downarrow\mathsf{Post}}$ minimizes (4.1) by applying Prop. 4.2.

(4) Testing $x \in D$ is equivalent to testing $\downarrow x \subseteq D$. M_{memb} obtains the encoding of $\downarrow x$ and tests $\downarrow x \subseteq D$ by using $M_{\subseteq}$. □

From Theorem 4.1 and Lemma 4.3, we obtain the following result:

Corollary 4.4. *Coverability is decidable for any ideally effective class of WBTS.*

We recall that coverability is undecidable for a large class of WSTS (hence for WBTS) when computations on ideals are not effective. It was shown in [BFM14, BFM16] that coverability is undecidable even for some post-effective classes of finitely branching WSTS with strong and strict monotonicity.

As an application of Corollary 4.4, we now argue that vector addition systems with states, a model computationally equivalent to Petri nets and thus a WSTS, can be extended in a non articifial way to yield a WBTS that we will call a *weighted VASS*. Recall that a *vector addition addition with states with d counters (d-VASS)* is defined as a $\mathbb{Z}^d$-VASS (see Sect. 3.1), but where the counters are not allowed to drop below zero, and where the values of counters are ordered by the usual componentwise ordering on $\mathbb{N}^d$. We propose to extend VASS with weights, i.e. with additional counters over $\mathbb{Z}$. These counters may represent, e.g., energy, fuel, time, money, or items of an inventory, where positive amounts correspond to production or availability, and negative amounts correspond to consumption or deficits [DG07, BCHK11, EFLQ13, BGM14, JLS15]. To the best of our knowledge, such an extension has never been studied nor introduced. Formally, this new model is defined as follows.

Definition 4.5. A *weighted (d, w)-VASS*, where $d, w \in \mathbb{N}$, is a pair $\mathcal{V} = (Q, T)$ such that Q is a finite set of *control states* and $T \subseteq Q \times \mathbb{Z}^d \times \mathbb{Z}^w \times Q$ is a finite set of *transitions*. A weighted (d, w)-VASS induces a transition system $(Q \times \mathbb{N}^d \times \mathbb{Z}^w, \to)$ such that $(p, \boldsymbol{u}) \to (q, \boldsymbol{v}) \overset{\mathrm{def}}{\iff} \exists (p, \boldsymbol{z}, q) \in T$ s.t. $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{z}$ and $\boldsymbol{v}[1..d] \geq \boldsymbol{0}$.

By definition, d-VASS and the $\mathbb{Z}^w$-VASS of Sect. 3.1 are special cases of weighted VASS. Weighted VASS ordered with the usual componentwise ordering, are not well-quasi-ordered even with a unique weight counter, and are not WBTS as soon as they have more than one weight counter. However, weighted VASS are WBTS when the weight counters are ordered lexicographically, i.e. when the states are ordered by $= \times \leq^d \times \leq^w_{\mathsf{lex}}$. Such an ordering allows modelling priorities among types of weights. Moreover, with a single weight counter, the lexicographical ordering is precisely the usual ordering over $\mathbb{Z}$.

Since d-VASS and $\mathbb{Z}^w$-VASS are both ideally effective, weighted (d, w)-VASS are also ideally effective. This is due to the fact that $\mathsf{Ideals}(X \times Y) = \mathsf{Ideals}(X) \times \mathsf{Ideals}(Y)$ for every quasi-ordered sets X and Y when the ordering on $X \times Y$ is defined componentwise from the ordering on X and the ordering on Y. Hence, weighted VASS form a post-effective and ideally effective class of WBTS, and by Corollary 4.4, coverability is decidable for this model.

5. TERMINATION AND BOUNDEDNESS

The termination and boundedness problems are respectively defined as follows: on input an ordered transition system $\mathcal{S} = (X, \rightarrow, \leq)$ and a state x, determine respectively whether

- $\mathcal{S}$ terminates from x, i.e. there is no infinite sequence $x_1, x_2, \cdots \in X$ such that $x \rightarrow x_1 \rightarrow x_2 \rightarrow \dots$;
- $\mathcal{S}$ is bounded from x, i.e. $\mathsf{Post}^*(x)$ is finite.

These two problems are undecidable in general, even for some classes of finitely branching (non effective) WSTS. However, they are decidable under reasonable monotonicity and effectiveness hypotheses (see e.g. [FPS01]). We observe that under these hypotheses, termination and boundedness do not remain decidable for WBTS. Hence WSTS and WBTS behave differently with respect to the decidabilities of their termination and boundedness problems.

Lemma 5.1. *There exists a post-effective class of finitely branching WBTS, with strong and strict monotonicity, and partial ordering, for which termination and boundedness are undecidable.*

Proof. We give a reduction from the halting problem. Let Turing_i be the ith Turing machine in a classical enumeration. Let $\mathcal{S}_i \stackrel{\text{def}}{=} (X, \rightarrow, \leq)$ be the ordered transition system defined by $X \stackrel{\text{def}}{=} \{0\} \cup (\mathbb{Z} - \mathbb{N}) = \{0, -1, -2, \dots\}$ and

$$x \rightarrow x - 1 \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad \mathsf{Turing}_i \text{ does not halt on its encoding in } |x| \text{ steps or less .}$$

Let $\mathcal{C} \stackrel{\text{def}}{=} \{\mathcal{S}_i : i \in \mathbb{N}\}$. Since $|\mathsf{Post}_{\mathcal{S}_i}(x)| \leq 1$ for every $x \in X$, $\mathcal{S}_i$ is finitely branching. Moreover, $\mathcal{C}$ is post-effective since testing $x \rightarrow y$ only requires executing a Turing machine for a finite number of steps. Because X is a partially ordered set without any infinite antichain, it remains to prove strong and strict monotonicity. Let $x, y, x' \in X$ be such that $x \rightarrow y$ in $\mathcal{S}_i$ and $x' > x$. By definition of $\rightarrow$, $y = x - 1$ and Turing_i does not halt in $|x|$ steps or less. Therefore, by $|x'| < |x|$, Turing_i does not halt in $|x'|$ steps or less, hence $x' \rightarrow y'$ where $y' = x' - 1 > x - 1 = y$.

Now, we note that there exists an infinite sequence $0, x_1, x_2, \cdots \in X$ such that $0 \rightarrow x_1 \rightarrow x_2 \rightarrow \dots$ in $\mathcal{S}_i$ if, and only if, Turing_i does not halt, if, and only if, $\mathsf{Post}^*(0)$ is infinite. Therefore, we conclude that termination and boundedness are both undecidable. $\qquad\square$

Despite these negative results, we may exhibit a subclass of WBTS for which termination and boundedness are decidable. Recall that the reachability tree from an initial state x_0 in a transition system $\mathcal{S}$ is a tree rooted at x_0 and having an edge (x, y) for each pair of states x, y such that $x \to y$. Analogous to the finite reachability tree for WSTS [FPS01], which is obtained from truncation of the reachability tree, we define the antichain tree that will provide algorithms for termination and boundedness. Informally, whereas the criterion for truncating a branch at a node labelled x_j in the reachability tree is the occurrence of an ancestor labelled x_i with $x_i \leq x_j$, the criterion for truncation in the antichain tree will be the weaker condition on x_i that *either $x_i \leq x_j$ or $x_j \leq x_i$*:

Definition 5.2 (Antichain tree)**.** Let $\mathcal{S} = (X, \to, \leq)$ be a WBTS, and let $x_0 \in X$. The *antichain tree* of $\mathcal{S}$ from the initial state x_0 is a partial reachability tree $\mathsf{AT}(\mathcal{S}, x_0)$ with root c_0 labelled x_0 that is defined and built as follows. For every $x \in \mathsf{Post}(x_0)$ we add a child labelled x to c_0. The tree is then built iteratively in the following way. Only an unmarked node c labelled x is picked:

- if c has an ancestor c' labelled x' such that $x' \leq x$ or $x \leq x'$, we mark c
- otherwise, we mark c and for every $y \in \mathsf{Post}(x)$ we add a child labelled y to c.

We observe that each path of the antichain tree is a prefix of a path of the finite reachability tree, which is finite [FPS01, Lemma 4.2], hence the antichain tree is also finite. More formally:

Lemma 5.3. *The antichain tree is finite and computable for finitely branching and post-effective WBTS.*

Proof. Suppose that $\mathsf{AT}(\mathcal{S}, x_0)$ is infinite. As $\mathcal{S}$ is finitely branching, by König's Lemma, there is an infinite branch $c_0 \to_{\mathsf{AT}} c_1 \to_{\mathsf{AT}} \ldots$ in this tree labelled by the following infinite sequence: $x_0, x_1, \ldots$ Since $\leq$ is without infinite antichains, there is a least j for which some $i < j$ satisfies $x_i \leq x_j$ or $x_j \leq x_i$. But then the branch would have been truncated at c_j or at c_i and this is a contradiction, hence $\mathsf{AT}(\mathcal{S}, x_0)$ is finite. The tree is computable since $\mathcal{S}$ is post-effective. $\qquad\square$

Let us state a useful lemma.

Lemma 5.4. *Any path in the reachability tree of a WBTS $\mathcal{S}$ from x_0 has a finite prefix labelling a maximal path in the antichain tree $\mathsf{AT}(\mathcal{S}, x_0)$.*

The proof of Lemma 5.5 is a (self-contained) adaptation of the proof of [FPS01, Prop. 4.5].

Lemma 5.5. *Let $\mathcal{S} = (X, \to, \leq)$ be a finitely branching WBTS with upward and downward transitive monotonicity. Then $\mathcal{S}$ does not terminate from x_0 if, and only if, there is a path $c_0 \to_{\mathsf{AT}} c_1 \to_{\mathsf{AT}} \ldots \to_{\mathsf{AT}} c_i \to_{\mathsf{AT}} \ldots \to_{\mathsf{AT}} c_j$ in $\mathsf{AT}(\mathcal{S}, x_0)$ with labels $x_0, x_1, \ldots, x_j$ such that $x_i \leq x_j$ or $x_j \leq x_i$.*

Proof. Only if. Suppose that an infinite run $x_0 \to x_1 \to \ldots$ exists in $\mathcal{S}$. By Lemma 5.4, a maximal path $c_0 \to_{\mathsf{AT}} c_1 \to_{\mathsf{AT}} \ldots \to_{\mathsf{AT}} c_j$ with labels $x_0, x_1, \ldots, x_j$ exists in $\mathsf{AT}(\mathcal{S}, x_0)$. Since this path is maximal, it ought to have been the presence of some $i < j$ with $x_i \leq x_j$ or $x_j \leq x_i$ that caused the truncation.

If. Suppose that a path $c_0 \to_{\mathsf{AT}} c_1 \to_{\mathsf{AT}} \ldots \to_{\mathsf{AT}} c_i \to_{\mathsf{AT}} \ldots \to_{\mathsf{AT}} c_j$ with comparable labels x_i and x_j exists in $\mathsf{AT}(\mathcal{S}, x_0)$. Then a run $x_0 \to^* x_i \to x_{i+1} \to^* x_j$ is possible in $\mathcal{S}$. If $x_j \leq x_i$, then by downward transitive monotonicity, there exists $x_{j+1} \leq x_{i+1}$ such that $x_j \to^+ x_{j+1}$. By induction, for every $m > j$, there exist $x_{j+1}, x_{j+2}, \ldots, x_{j+m}$ such that

$x_0 \to^* x_i \to^+ x_j \to^+ x_{j+1} \to^+ x_{j+2} \to^+ \cdots \to^+ x_{j+m}$. But then, by applying König's Lemma to the finitely branching reachability tree of $\mathcal{S}$, we note that $\mathcal{S}$ does not terminate from x_0. The case $x_i \leq x_j$ is treated similarly, using the upward transitive monotonicity. $\square$

The proof of Lemma 5.6 adapts [FPS01, Prop. 4.10] and strengthens it in that both transitive and strict monotonicity are required there, while only strict monotonicity is required here.

Lemma 5.6. *Let $\mathcal{S} = (X, \to, \leq)$ be a finitely branching WBTS with upward and downward strict monotonicity and such that $\leq$ is a partial ordering. Then $\mathcal{S}$ is not bounded from x_0 if, and only if, there is a path $c_0 \to_{AT} c_1 \to_{AT} \cdots \to_{AT} c_i \to_{AT} \cdots \to_{AT} c_j$ in $\mathsf{AT}(\mathcal{S}, x_0)$ with labels $x_0, x_1, \ldots, x_j$ such that $x_i < x_j$ or $x_j < x_i$.*

Proof. Only if. Suppose that $\mathsf{Post}^*(x_0)$ is infinite. Consider the reachability tree defined from cycle-free runs (hence runs with no repeated states) from x_0 in $\mathcal{S}$. By König's lemma applied to this finitely branching tree, some such run $x_0 \to x_1 \to \ldots$ in $\mathcal{S}$ is infinite. As in the proof of Lemma 5.5, Lemma 5.4 implies the existence in $\mathsf{AT}(S, x_0)$ of a path $c_0 \to_{AT} c_1 \to_{AT} \cdots \to_{AT} c_i \to_{AT} \cdots \to_{AT} c_j$ with labels $x_0, x_1, \ldots, x_j$ such that either $x_i \leq x_j$ or $x_j \leq x_i$. Being distinct and comparable in a partial order, x_i and x_j satisfy $x_i < x_j$ or $x_j < x_i$, as required.

If. Suppose that there exists a path $c_0 \to_{AT} c_1 \to_{AT} \cdots \to_{AT} c_i \to_{AT} \cdots \to_{AT} c_j$ such that $x_i < x_j$ or $x_j < x_i$ exists in $\mathsf{AT}(\mathcal{S}, x_0)$. Then a run

$$x_0 \to^* x_i \to^{k_0} x_j$$

is possible in $\mathcal{S}$ for the appropriate $k_0 > 0$. If $x_j < x_i$, then by k_0 applications of strict downward monotonicity, there exists $y_1 < x_j$ such that

$$y_0 \overset{\text{def}}{=} x_j \to^* y_1.$$

Since $y_1 < y_0$, $y_0 \to^{k_1} y_1$ for some $k_1 > 0$. Hence the argument can be repeated to exhibit an infinite descending chain $y_0 > y_1 > y_2 > \ldots$ such that

$$x_0 \to^* x_i \to^* y_0 \to^* y_1 \to^* y_2 \cdots .$$

Hence $\mathsf{Post}^*(x_0)$ is infinite. The case $x_i < x_j$ is treated similarly, using upward strict monotonicity. $\square$

The following holds:

Theorem 5.7.
- *Termination is decidable for any post-effective class of finitely branching WBTS with upward and downward transitive monotonicity.*
- *Boundedness is decidable for any post-effective class of finitely branching WBTS with upward and downward strict monotonicity and partial ordering.*

Proof. Given a WBTS $\mathcal{S} = (X, \to, \leq)$ and $x_0 \in X$, both the termination and the boundedness algorithm begin with the computation of $\mathsf{AT}(\mathcal{S}, x_0)$, doable by Lemma 5.3. Since $\mathsf{AT}(\mathcal{S}, x_0)$ is finite, the algorithm for termination can proceed to test the condition of Lemma 5.5 and the algorithm for boundedness the condition of Lemma 5.6. $\square$

Remark 5.8. Under the hypotheses of Theorem 5.7, boundedness is decidable even when WBTS are infinitely branching. Indeed, it suffices in this case to add to the construction of the antichain tree the rule that a branch is further truncated when a node x such that $|\mathsf{Post}(x)| = \infty$ is encountered. Such an occurrence can be detected by post-effectiveness, and any such occurrence in the antichain tree implies unboundedness.

6. Conclusion

In this work we have noted that well-foundedness of the quasi-ordering traditionally used to define a WSTS is not required for the purpose of deciding coverability. Accordingly, we have defined WBTS by relaxing the conditions on the ordering so as to only require the absence of infinite antichains.

As proof of concept, we have introduced an extension of vector addition systems called weighted (d, w)-VASS. Weighted (d, w)-VASS operate on their $\mathbb{N}^d$ component as normal VASS and they operate without guards on a new $\mathbb{Z}^w$ component ordered by lexicographically extending the usual order on $\mathbb{Z}$. The resulting model is a WBTS that is not a WSTS. From studying the ideal structure of downward-closed subsets of $\mathbb{Z}^w$ under the latter ordering, we deduced that all necessary effectiveness conditions hold for a forward algorithm to be able to decide coverability for weighted (d, w)-VASS. More generally, this forward algorithm was shown able to decide coverability for any WBTS that possesses the "ideally effective" property.

To delimit the picture, we have further shown that, unlike in the well-studied case of WSTS, the termination and the boundedness problems for WBTS become undecidable. On the other hand, appropriate downward and upward monotonicity conditions were shown to bring back decidability for these problems.

As future work directions, other WBTS and orderings could be studied. For example, the lexicographical ordering on words over a finite alphabet could be used in lieu of $\mathbb{Z}^w$ as the weight domain of WBTS and weighted VASS. Beyond studying the ideal structure of Σ^* under this ordering for its own sake, it is conceivable that models of practical use in verification might use such an ordering for the purpose of modelling priorities. Given the recent focus on the complexity of VASS problems [Sch16], investigating complexity questions for weighted VASS and specific WBTS would certainly be worthwhile.

But the final take-home message of this paper might be that, as the need arises, new models weaker than the WSTS can now be defined with some hope for usability.

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