Timed pushdown automata and branching vector addition systems

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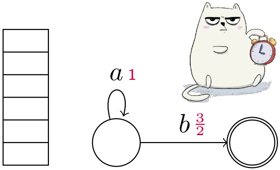
Outline

1. Three models

trPDA



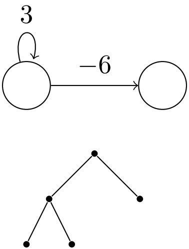
time registers x, y, z

Systems of equations

$$X_i \subseteq \mathbb{Z}$$

$$\begin{cases} X_1 \supseteq X_2 \cup X_3 \\ X_2 \supseteq X_1 + X_3 \\ X_3 \supseteq \{-1, 1\} \\ \vdots \end{cases}$$

1BVASS



2. Reductions between models

3. Decidability

trPDA

What is time?

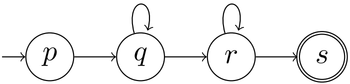
$(\mathbb{Q}, \leq, +1)$ or $\underline{(\mathbb{Z}, \leq, +1)}$

Input $A \times \mathbb{Z}$, $\quad A = \{a, b\}$ (finite in general)

Example

$L =$ "Palindromes such that $\#_a(w) = \#_b(w)$"

x ______



$(a, 3)(b, 2)(b, 2)(a, 3)$

- Non-monotonic time
- Only one register (or orbit-finiteness)

Strictly subsumes other models:

- [Bouajjani, Echahed, Robbana]
- [Abdulla, Atig, Stenman]

trPDA state of the art

Input: trPDA $\mathcal{A}$

Problem: non-emptiness of $L(\mathcal{A})$

(universality, equivalence, etc. undecidable)

Unrestricted – undecidable [Bojańczyk and Lasota, 2012]

(no stack, 3 registers)

Restrict to orbit-finite/one register

Timeless stack – EXPTIME-complete

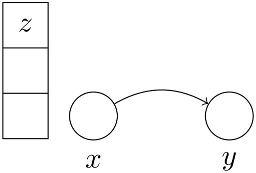
Orbit-finite time stack – in NEXPTIME [Clemente and Lasota, 2015]

Time stack – this paper

Transitions in trPDA

- Push and pop
- Only ϵ-transitions (no input to test non-emptiness)

3 time variables:



Example constraint:

$$(x = y + 1) \;\wedge\; (y \leq z + 1 + 1 + 1) \;\wedge\; (z \leq y + 1) \;\wedge\; (x \leq z)$$

$$x - y \in [1, 1] \qquad y - z \in [1, 3] \qquad x - z \in (-\infty, 0]$$

Transition: 3 intervals

Systems of equations over $\mathbb{Z}$

$X_1 \dots X_n \subseteq \mathbb{Z}$

Systems of equations $\mathcal{S}$ using: $\cup, \cap, +$ and $\{1\}, \{-1\}$

$$\begin{cases} X_0 \supseteq t_0 \\ \vdots \\ X_n \supseteq t_n \end{cases}$$

solution $\mu(X_i) \to \mathcal{P}(\mathbb{Z})$, $\mu(X_i) \supseteq \mu(t_i)$

goal: minimal solution of $\mathcal{S}$

Example: $X_0 \dots X_k$

$$\begin{aligned} X_0 &\supseteq \{1\} + \{-1\} \\ X_{2m} &\supseteq X_m + X_m \\ X_{2m+1} &\supseteq X_m + X_m + \{1\} \\ X_0 &\supseteq X_0 + X_k \end{aligned}$$

minimal solution: $\mu(X_i) = \{i\}$

$\mu(X_0) = k\mathbb{N}$

Systems of equations state of the art

Input: system $\mathcal{S}$, variable X

Problem: non-emptiness of $\mu(X)$

Unrestricted: undecidable [Jeż and Okhotin, 2010]

Restricting $\cap$

No intersections – in $\mathrm{PT{\scriptsize IME}}$

Intersections with $\{0\}$ – $\mathrm{NPT{\scriptsize IME}}$-complete [Clemente and Lasota, 2015]

(or any bounded intervals)

Intersections with $\mathbb{N}$ and $(-\mathbb{N})$ – this paper

(or any intervals)

trPDA to systems of equations

Non-emptiness: trPDA $\mathcal{A} \quad \rightarrow \quad$ system $(\mathcal{S}, X)$

Previously: [Clemente and Lasota, 2015]

- $\mathcal{A}$ with timeless stack $\rightarrow (\mathcal{S}, X)$ with no $\cap$
- $\mathcal{A}$ with orbit-finite stack $\rightarrow (\mathcal{S}, X)$ with $\cap \{0\}$

This paper:

- $\mathcal{A}$ with stack $\rightarrow (\mathcal{S}, X)$ with $\cap\, \mathbb{N}$, $\cap\, (-\mathbb{N})$

trPDA to systems of equations

trPDA $\mathcal{A}$ with states Q, empty stack acceptance

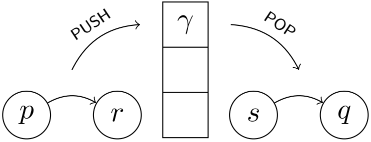
Variables:

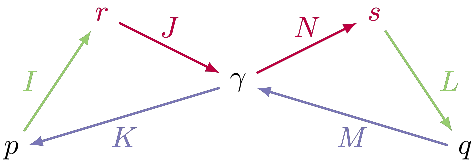
$X_{p,q}$ for every $p, q \in Q$

$t \in X_{p,q}$: "reach q from p (the same stack) changing time by t"

Inclusions:

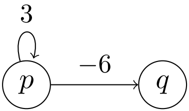
- $X_{p,p} \supseteq \{0\}$, for every p
- $X_{p,q} \supseteq X_{p,r} + X_{r,q}$, for all p, q, r
- $X_{p,q} \supseteq (I + (X_{r,s} \cap (J + N)) + L) \cap -(K + M)$





BVASS

Recall 1-VASS



Computations are words: $(p,0) \xrightarrow{3} (p,3) \xrightarrow{3} (p,6) \xrightarrow{-6} (q,0)$

States Q, transitions $T \subseteq Q \times \mathbb{Z} \times Q$, configurations $Q \times \mathbb{N}$

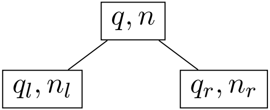
1-BVASS$^{\pm}$: states Q, <u>transitions $T \subseteq Q^3$</u>, configurations $Q \times \mathbb{N}$

Computations are binary trees:

1-BVASS

- leaves $(q_0, 1)$
- inner nodes

$(q, q_l, q_r) \in T$



$n = n_l + n_r$ if $q \in Q^+$

$n = n_l - n_r$ if $q \in Q^-$

BVASS state of the art

Input: BVASS $\mathcal{B}$, configuration (q, n)

Problem: reachability of (q, n)

1-BVASS (no subtraction):

Unary encoding – $\mathrm{PT\textsc{ime}}$-complete [Göller et al., 2016]

Binary encoding – $\mathrm{PS\textsc{pace}}$-complete [Figueira et al., 2017]

In higher dimensions – open

$1\text{-BVASS}^{\pm}$ unary/binary – this paper

In higher dimensions – undecidable $(d \geq 6)$ [Lazić, 2010]

Decidability BVASS

$1\text{-BVASS}^{\pm}$ $\mathcal{B}$, configuration (q, n)

Lemma

If (q, n) is reachable then there is a computation with all values bounded by $N = poly(n) \cdot exp(|B|)$.

Non-emptiness of tree-automaton, states $Q \times \{0 \ldots N\}$.

So in $\textsc{ExpTime}$

Results summary

Three models/problems

$$
\begin{tikzcd}[column sep=huge]
\begin{array}{c}\text{trPDA}\\\text{(non-emptiness)}\end{array} \arrow[r, bend left, "\text{exp}"] & \begin{array}{c}\text{Systems}\\\text{(non-emptiness)}\end{array} \arrow[l, bend left, "\text{poly}"] \arrow[r, bend left, "\text{poly}"] & \begin{array}{c}\text{1-BVASS}^{\pm}\\\text{(reachability)}\end{array} \arrow[l, bend left, "\text{poly}"]
\end{tikzcd}
$$

in 2ExpTime	in ExpTime	in ExpTime
ExpTime-hard	PSpace-hard	PSpace-hard

Conclusions

- Complexity gaps
- Reachability of BVASS?
- Reachability of $\mathrm{n}\text{-}\mathrm{BVASS}^{\pm}$ for $n < 6$