Mathematics Magazine

ISSN: 0025-570X (Print) 1930-0980 (Online) Journal homepage: http://www.tandfonline.com/loi/umma20

A Primer on Bernoulli Numbers and Polynomials

Tom M. Apostol

To cite this article: Tom M. Apostol (2008) A Primer on Bernoulli Numbers and Polynomials, Mathematics Magazine, 81:3, 178-190, DOI: 10.1080/0025570X.2008.11953547

To link to this article: https://doi.org/10.1080/0025570X.2008.11953547

Published online: 16 Apr 2018.

A Primer on Bernoulli Numbers and Polynomials

TOM M. APOSTOL
Project MATHEMATICS!
California Institute of Technology
Pasadena, CA 91125
apostol@caltech.edu

A large literature exists on Bernoulli numbers and Bernoulli polynomials, much of it in widely scattered books and journals. This article serves as a brief primer on the subject, bringing together basic results (most of which are well known), together with proofs, in a manner readily accessible to those with a knowledge of elementary calculus. Some new formulas are also derived.

Background

Bernoulli numbers and polynomials are named after the Swiss mathematician Jakob Bernoulli (1654–1705), who introduced them in his book *Ars Conjectandi*, published posthumously (Basel, 1713). They first appeared in a list of formulas (reproduced in FIGURE 1) for summing the pth powers of n consecutive integers, for $p = 1$ to $p = 10$. Bernoulli uses the symbol $\int$, an elongated S, to indicate summation. In modern notation his first three examples are equivalent to the familiar relations

$$\sum_{k=1}^{n} k = \frac{1}{2}n(n+1),$$

$$\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1),$$

$$\sum_{k=1}^{n} k^3 = \frac{1}{4}n^2(n+1)^2.$$

A general formula for the sum of pth powers (not explicitly stated by Bernoulli), can be written as

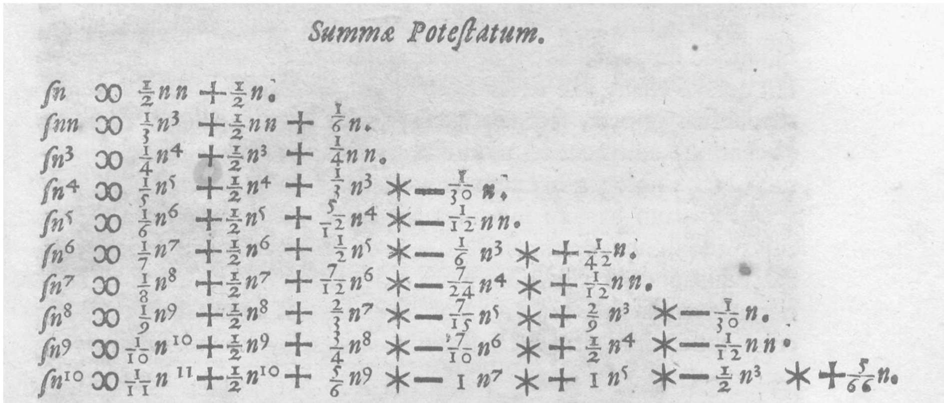
$$\sum_{k=1}^{m-1} k^p = \frac{B_{p+1}(m) - B_{p+1}}{p+1}, \quad p \geq 1,\ m \geq 2, \tag{1}$$

where $B_n(x)$ is a polynomial in x of degree n, now called a Bernoulli polynomial, given by

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}, \quad n \geq 0, \tag{2}$$

and where the B_k are rational numbers called Bernoulli numbers. They can be defined recursively as follows:

$$B_0 = 1, \quad \text{and} \quad \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0 \quad \text{for } n \geq 2. \tag{3}$$



Summæ Potestatum.

$\int n \infty \frac{1}{2}nn + \frac{1}{2}n.$
$\int nn \infty \frac{1}{3}n^3 + \frac{1}{2}nn + \frac{1}{6}n.$
$\int n^3 \infty \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}nn.$
$\int n^4 \infty \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 * - \frac{1}{30}n.$
$\int n^5 \infty \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 * - \frac{1}{12}nn.$
$\int n^6 \infty \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 * - \frac{1}{6}n^3 * + \frac{1}{42}n.$
$\int n^7 \infty \frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{12}n^6 * - \frac{7}{24}n^4 * + \frac{1}{12}nn.$
$\int n^8 \infty \frac{1}{9}n^9 + \frac{1}{2}n^8 + \frac{2}{3}n^7 * - \frac{7}{15}n^5 * + \frac{2}{9}n^3 * - \frac{1}{30}n.$
$\int n^9 \infty \frac{1}{10}n^{10} + \frac{1}{2}n^9 + \frac{3}{4}n^8 * - \frac{7}{10}n^6 * + \frac{1}{2}n^4 * - \frac{1}{12}nn.$
$\int n^{10} \infty \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \frac{5}{6}n^9 * - 1\,n^7 * + 1\,n^5 * - \frac{1}{2}n^3 * + \frac{5}{66}n.$

Figure 1 Reproduction of the list of formulas on p. 97 of *Ars Conjectandi,* in which Bernoulli numbers and Bernoulli polynomials first appeared in print. The open ∞ symbol in the second column was used in that era as an equals sign. The large asterisk in later columns indicates zero coefficients of missing powers. *Courtesy of the Archives, California Institute of Technology.*

Using this recursion with $n = 2, 3, \dots$, we quickly obtain the following:

$$B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}, \quad B_5 = 0, \quad B_6 = \frac{1}{42},$$
$$B_7 = 0, \quad B_8 = -\frac{1}{30}, \quad B_9 = 0, \quad B_{10} = \frac{5}{66}.$$

Equation (2) shows that $B_n = B_n(0)$ for $n \geq 0$. The sum in (3) can also be written in the form

$$B_n = \sum_{k=0}^{n} \binom{n}{k} B_k, \quad n \geq 2, \tag{4}$$

which, in view of (2), reveals that

$$B_n = B_n(1), \quad n \geq 2. \tag{5}$$

Bernoulli numbers with even subscripts ≥ 2 alternate in sign, and those with odd subscripts ≥ 3 are zero. A knowledge of Bernoulli numbers, in turn, quickly gives explicit formulas for the first few Bernoulli polynomials:

$$B_0(x) = 1, \quad B_1(x) = x - \frac{1}{2}, \quad B_2(x) = x^2 - x + \frac{1}{6},$$
$$B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \quad B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30},$$
$$B_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^2 - \frac{1}{6}x.$$

Bernoulli proudly announced [**5**, p. 90] that with the help of the last entry in his list of formulas in FIGURE 1 it took him less than half of a quarter of an hour to find that the tenth powers of the first 1000 numbers being added together will yield the sum

91,409,924,241,424,243,424,241,924,242,500.

Today Bernoulli numbers and polynomials play an important role in many diverse areas of mathematics, for example in the Euler-Maclaurin summation formula [**2**], in number theory [**1**], and in combinatorics [**4**]. One of the most remarkable connections is to the Riemann zeta function $\zeta(s)$, defined for $s > 1$ by the infinite series

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}. \tag{6}$$

Leonhard Euler (1707–1783) discovered that when s is an even integer, the sum can be expressed in terms of Bernoulli numbers by the formula

$$\zeta(2n) = (2\pi)^{2n} \frac{|B_{2n}|}{2(2n)!}. \tag{7}$$

In particular,

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945},$$
$$\zeta(8) = \frac{\pi^8}{9450}, \quad \zeta(10) = \frac{\pi^{10}}{93555}. \tag{8}$$

The Basel problem

The problem of evaluating $\zeta(2)$ in closed form has an interesting history, which apparently began in 1644 when Pietro Mengoli (1625–1686) asked for the sum of the reciprocals of the squares. The problem became widely known in 1689 when Jakob Bernoulli wrote, "If somebody should succeed in finding what till now withstood our efforts and communicate it to us we shall be much obliged to him." By the 1730s it was known as the Basel problem and had defied the best efforts of many leading mathematicians of that era. Euler solved the problem around 1735 in response to a challenge by Jakob's younger brother Johann Bernoulli (1667–1748), who was Euler's teacher and mentor. Euler soon obtained the more general formula in (7). Sadly, Jakob did not live to see young Euler's triumphant discovery and its surprising connection with Bernoulli numbers. For a proof of (7) see [**1**, p. 266]. To date, no simple closed form analogous to (7) is known for $\zeta(n)$ for any odd power $n \geq 3$.

Generating functions

There are alternative methods for introducing Bernoulli numbers and polynomials. One of the most useful was conceived by Euler, who observed that they occur as coefficients in the following power series expansions:

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n, \quad |z| < 2\pi, \tag{9}$$

and

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n, \quad |z| < 2\pi. \tag{10}$$

The parameters x and z can be real or complex. The functions on the left are called generating functions for the Bernoulli numbers and polynomials.

Basic properties deduced from the generating functions

The use of generating functions leads to simple and direct proofs of many basic properties of Bernoulli numbers and polynomials, the most important of which are derived here. For example, to deduce (2) from (9) and (10), write

$$\sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n = \frac{z}{e^z - 1} \cdot e^{xz} = \left(\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n \right) \cdot \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} z^n \right). \tag{11}$$

Multiply the two power series on the right, using the fact that the product of two convergent power series

$$A(z) = \sum_{n=0}^{\infty} a(n) z^n \quad \text{and} \quad B(z) = \sum_{n=0}^{\infty} b(n) z^n,$$

is another power series given by

$$A(z)B(z) = \sum_{n=0}^{\infty} c(n) z^n, \quad \text{where} \quad c(n) = \sum_{k=0}^{n} a(k) b(n-k).$$

Equating coefficients of z^n in (11) we obtain

$$\frac{B_n(x)}{n!} = \sum_{k=0}^{n} \frac{B_k}{k!} \frac{x^{n-k}}{(n-k)!},$$

which is equivalent to (2). Incidentally, (9) and (10) also show that

$$B_n = B_n(0), \quad n \geq 0. \tag{12}$$

To deduce that

$$B_{2n+1} = 0 \quad \text{for} \quad n \geq 1, \tag{13}$$

rewrite (9) in the form

$$\frac{z}{e^z - 1} + \frac{z}{2} = 1 + \sum_{n=2}^{\infty} \frac{B_n}{n!} z^n.$$

Now observe that the left member is an even function of z (it is unchanged when z is replaced by $-z$), hence the right member is even and therefore contains no odd powers of z, and we get (13).

The power-sum formula (1) and its extension (16), are immediate consequences of the following:

Difference equation:

$$B_n(x+1) - B_n(x) = n x^{n-1}, \quad n \geq 1. \tag{14}$$

To prove (14), use (10) in the identity

$$z \frac{e^{(x+1)z}}{e^z - 1} - z \frac{e^{xz}}{e^z - 1} = z e^{xz}$$

to obtain

$$\sum_{n=0}^{\infty} \frac{B_n(x+1) - B_n(x)}{n!} z^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} z^{n+1}.$$

Equating coefficients of z^n gives (14). Taking $x = 0$ in (14) we find a companion to (12):

$$B_n(0) = B_n(1), \quad n \geq 2. \tag{15}$$

Replace x by $x + k$, and n by $p + 1$ in (14), then sum on k to obtain the following extension of (1):

$$\sum_{k=0}^{m-1} (x+k)^p = \frac{B_{p+1}(m+x) - B_{p+1}(x)}{p+1}, \quad p \geq 1,\ m \geq 1. \tag{16}$$

When $x = 0$, (16) reduces to (1), and when $x = a/d$, $d \neq 0$, it implies

$$\sum_{k=0}^{m-1} (a+dk)^p = d^p \frac{B_{p+1}(m+a/d) - B_{p+1}(a/d)}{p+1}, \quad p \geq 1,\ m \geq 1. \tag{17}$$

In particular, if a and d are integers, (17) provides a formula for the sum of the pth powers of m integers in arithmetic progression.

Symmetry relation:

$$B_n(1-x) = (-1)^n B_n(x), \quad n \geq 0. \tag{18}$$

This follows at once from the identity

$$z \frac{e^{(1-x)z}}{e^z - 1} = -z \frac{e^{-zx}}{e^{-z} - 1}.$$

Take $x = 0$ in (18) to find $B_{2n+1}(1) = -B_{2n+1}(0)$, so by (15) and (10) this gives another proof of (13). When $x = 1/2$ in (18) we get

$$B_{2n+1}\left(\frac{1}{2}\right) = 0, \quad n \geq 0. \tag{19}$$

Now replace x by $-x$ in (18) and use (14) to obtain

$$(-1)^n B_n(-x) = B_n(x) + nx^{n-1}, \quad n \geq 1. \tag{20}$$

Addition formula:

$$B_n(y+x) = \sum_{k=0}^{n} \binom{n}{k} B_k(y) x^{n-k}, \quad n \geq 0. \tag{21}$$

This is an immediate consequence of the identity

$$\frac{ze^{(y+x)z}}{e^z - 1} = \frac{ze^{yz}}{e^z - 1} \cdot e^{xz}.$$

Taking $y = 0$ in (21) gives (2).

Raabe's multiplication formula:

$$B_n(mx) = m^{n-1} \sum_{k=0}^{m-1} B_n\left(x + \frac{k}{m}\right), \quad n \geq 0, \; m \geq 1. \tag{22}$$

This states that a sum of Bernoulli polynomials of degree n at equally spaced values of the argument is another Bernoulli polynomial of degree n. To prove (22), equate coefficients of z^n in the identity

$$\sum_{n=0}^{\infty} \frac{m^{1-n} B_n(mx)}{n!} z^n = \sum_{n=0}^{\infty} \frac{\sum_{k=0}^{m-1} B_n(x + \frac{k}{m})}{n!} z^n, \tag{23}$$

which can be proved as follows. From (10) we have

$$\sum_{n=0}^{\infty} B_n\left(x + \frac{k}{m}\right) \frac{z^n}{n!} = \frac{z e^{(x+\frac{k}{m})z}}{e^z - 1} = \frac{z e^{xz}}{e^z - 1} e^{kz/m}.$$

Now sum both members on k for $k = 0, 1, \ldots, m-1$. The sum of exponentials is a geometric sum given by

$$\sum_{k=0}^{m-1} e^{kz/m} = \frac{e^z - 1}{e^{z/m} - 1},$$

if $z \neq 0$. (The restriction $z \neq 0$ is not serious because (23) holds trivially when $z = 0$.) Therefore

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\sum_{k=0}^{m-1} B_n(x + \frac{k}{m})}{n!} z^n &= \frac{z e^{xz}}{e^z - 1} \sum_{k=0}^{m-1} e^{kz/m} = \frac{z e^{xz}}{e^{z/m} - 1} \\ &= m \frac{(z/m) e^{(mx)(z/m)}}{e^{z/m} - 1} = m \sum_{n=0}^{\infty} \frac{B_n(mx)}{n!} \left(\frac{z}{m}\right)^n \\ &= \sum_{n=0}^{\infty} \frac{m^{1-n} B_n(mx)}{n!} z^n, \end{aligned}$$

which implies (23) and hence (22).

Another application of Bernoulli's power-sum formula

Equation (17) extends (1) to the sum of the pth powers of m integers in arithmetic progression. It is natural to ask if there is a formula similar to (1) when the sum on the left is extended over those integers relatively prime to m. We will show that

$$\sum_{k=1}^{m}{}' k^p = \sum_{d|m} \mu(d) d^p \frac{B_{p+1}(m/d) - B_{p+1}}{p+1}, \quad p \geq 1, \; m \geq 1, \tag{24}$$

where $\sum'$ indicates that the sum is extended over k relatively prime to m. In (24), $\mu(d)$ is the well known Möbius function of elementary number theory, which enters naturally because of the following formula [**1**, p. 25] that selects numbers relatively prime to m:

$$\sum_{d|(k,m)} \mu(d) = \begin{cases} 1 & \text{if } (k,m) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we have

$$\sum_{k=1}^{m}{}' k^p = \sum_{k=1}^{m-1} \sum_{d|m \text{ and } d|k} \mu(d) k^p.$$

Now write $k = rd$, and the foregoing equation becomes

$$\sum_{k=1}^{m}{}' k^p = \sum_{d|m} \mu(d) d^p \sum_{1 \le r < m/d} r^p = \sum_{d|m} \mu(d) d^p \frac{B_{p+1}(m/d) - B_{p+1}}{p+1},$$

where in the last step we used (1) with m replaced by m/d. This proves (24).

By using (2) to expand the Bernoulli polynomial in (24) in powers of m/d, we can also write

$$\sum_{k=1}^{m}{}' k^p = \frac{1}{p+1} \sum_{r=1}^{p+1} \binom{p+1}{r} m^r B_{p+1-r} \sum_{d|m} \mu(d) d^{p-r}. \tag{25}$$

The dependence on the Möbius function can be removed by invoking Theorem 2.8 of [**1**], which implies

$$\sum_{d|m} \mu(d) d^{p-r} = \prod_{q|m} (1 - q^{p-r}),$$

where the product is taken over all prime divisors q of m. Therefore we have an alternative form of (25):

$$\sum_{k=1}^{m}{}' k^p = \frac{1}{p+1} \sum_{r=1}^{p+1} \binom{p+1}{r} m^r B_{p+1-r} \prod_{q|m} (1 - q^{p-r}). \tag{26}$$

When $m = 1000$ and $p = 10$ the product contains only the primes 2 and 5, and (26) gives

36,366,968,829,066,536,008,898,579,270,000

for the sum of the tenth powers of those integers up to 1000 that are relatively prime to 1000. This sum is about 39.8% of Bernoulli's value mentioned earlier for the sum extended over all integers up to 1000, which is not too surprising because exactly 40% of the numbers less than 1000 are relatively prime to 1000. Like Bernoulli, the author did this calculation by hand, but unlike Bernoulli it took him more than a quarter of an hour.

Properties involving calculus

Differentiate each member of (10) with respect to x and equate coefficients of z^n to get the following:

Derivative formula:

$$B_n'(x) = n B_{n-1}(x), \quad n \ge 1, \tag{27}$$

which is also a consequence of (2). This leads to another proof of addition formula (21). Repeated differentiation of (27) gives

$$B_n''(x) = n(n-1)B_{n-2}(x), \dots, B_n^{(k)}(x) = k!\binom{n}{k}B_{n-k}(x). \tag{28}$$

On the other hand, for each fixed y the Taylor expansion of the polynomial $B_n(y+x)$ in powers of x is given by

$$B_n(y+x) = \sum_{k=0}^{n} \frac{1}{k!}B_n^{(k)}(y)x^k = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}(y)x^k = \sum_{k=0}^{n} \binom{n}{k} B_k(y)x^{n-k},$$

which is (21).

Now replace n by $n+1$ in (27) and integrate to obtain the following:

Integration formula:

$$\int_x^y B_n(t)\,dt = \frac{B_{n+1}(y) - B_{n+1}(x)}{n+1}, \quad n \geq 0. \tag{29}$$

This, together with (14), implies

$$\int_x^{x+1} B_n(t)\,dt = x^n, \quad n \geq 0, \tag{30}$$

which, when $x = 0$, gives

$$\int_0^1 B_n(t)\,dt = 0, \quad n \geq 1. \tag{31}$$

From (29) we find the recursion relation

$$B_n(x) = n\int_0^x B_{n-1}(t)\,dt + B_n(0), \quad n \geq 1. \tag{32}$$

Recursion formulas for Bernoulli polynomials

Equation (32) suggests another method for defining the Bernoulli polynomials and Bernoulli numbers recursively. Define $b_0 = 1$, and $b_0(x) = 1$. Guided by (32) and (31), define

$$b_n(x) = n\int_0^x b_{n-1}(t)\,dt + b_n, \tag{33}$$

where the constant b_n is chosen so that

$$\int_0^1 b_n(t)\,dt = 0. \tag{34}$$

It is easily verified that the functions $b_n(x)$ and constants b_n obtained in this manner are exactly the same as the Bernoulli polynomials $B_n(x)$ and the Bernoulli numbers B_n. For example, using (33) with $n = 1$ we find

$$b_1(x) = \int_0^x b_0(t)\,dt + b_1 = x + b_1,$$

whereas (34) requires

$$\int_0^1 b_1(t)\,dt = \frac{1}{2} + b_1 = 0.$$

This gives $b_1 = -\frac{1}{2} = B_1$, and $b_1(x) = x - \frac{1}{2} = B_1(x)$. Similarly, using (33) with $n = 2$ we find

$$b_2(x) = 2\int_0^x b_1(t)\,dt + b_2 = x^2 - x + b_2,$$

while (34) requires

$$\int_0^1 b_2(t)\,dt = \frac{1}{3} - \frac{1}{2} + b_2 = 0,$$

so $b_2 = \frac{1}{6} = B_2$, and $b_2(x) = x^2 - x + \frac{1}{6} = B_2(x)$.

By induction we find $b_n = B_n$ and $b_n(x) = B_n(x)$ for all $n \geq 0$. In other words, the recursion formulas

$$B_n(x) = n\int_0^x B_{n-1}(t)\,dt + B_n, \qquad \int_0^1 B_n(t)\,dt = 0, \quad n \geq 1, \tag{35}$$

together with $B_0 = 1$, $B_0(x) = 1$, provide an alternative method for defining Bernoulli numbers and polynomials recursively.

The following further recursion formula for Bernoulli polynomials

$$nB_n(x) - xnB_{n-1}(x) = \sum_{k=1}^{n} \binom{n}{k} kB_k x^{n-k}, \quad n \geq 1, \tag{36}$$

is a simple consequence of the familiar Pascal triangle property of binomial coefficients,

$$\binom{n}{k} - \binom{n-1}{k} = \binom{n-1}{k-1}.$$

In fact, from (2) we have

$$\begin{aligned} B_n(x) - xB_{n-1}(x) &= \sum_{k=0}^{n} \left[\binom{n}{k} - \binom{n-1}{k}\right] B_k x^{n-k} \\ &= \sum_{k=1}^{n} \binom{n-1}{k-1} B_k x^{n-k} = \sum_{k=1}^{n} \binom{n}{k} \frac{k}{n} x^{n-k}, \end{aligned}$$

which gives (36) after multiplication by n.

Further recursion formulas for Bernoulli numbers

The defining recursion in (3) and its variation in (4) can be written in other equivalent forms. First we show that (4) is equivalent to

$$\sum_{k=1}^{n} \binom{n}{k-1} B_k = -B_n, \tag{37}$$

which is valid for $n \geq 2$ (but not for $n = 1$). Write

$$\binom{n}{k-1} = \binom{n+1}{k} - \binom{n}{k},$$

to obtain

$$\sum_{k=1}^{n} \binom{n}{k-1} B_k = \sum_{k=0}^{n} \binom{n+1}{k} B_k - \sum_{k=0}^{n} \binom{n}{k} B_k,$$

because the terms with $k = 0$ cancel. The first sum on the right is 0 by (3) and we see that (4) is equivalent to (37).

Next we show that (37), in turn, is equivalent to the following recursion which we believe is new:

$$\sum_{k=2}^{n} \binom{n}{k-2} \frac{B_k}{k} = \frac{1}{(n+1)(n+2)} - B_{n+1}, \quad n \geq 2. \tag{38}$$

To derive this, use the relation

$$\binom{n}{k-2} \frac{1}{k} = \frac{k-1}{(n+1)(n+2)} \binom{n+2}{k},$$

multiply both members of (38) by $(n+1)(n+2)$, then add the summand terms for $k = 0$, 1, and $n+1$ to both sides to get the equivalent formula

$$\sum_{k=0}^{n+1} (k-1) \binom{n+2}{k} B_k = -(n+2) B_{n+1}.$$

Because of (3), this equation, in turn, is equivalent to

$$\sum_{k=1}^{n+1} k \binom{n+2}{k} B_k = -(n+2) B_{n+1}. \tag{39}$$

Now write $k\binom{n+2}{k} = (n+2)\binom{n+1}{k-1}$, cancel the common factor $(n+2)$, and replace n by $n-1$, and (39) becomes (37). Hence (38) is equivalent to (4).

A special case of (38), obtained by Horata [**3**] states that

$$\sum_{k=0}^{n-1} \binom{2n}{2k} \frac{B_{2k+2}}{2k+2} = \frac{1}{(2n+1)(2n+2)}, \quad n \geq 1. \tag{40}$$

To get this from (38), replace n by $2n$ in (38) and use the fact that $B_k = 0$ for odd $k \geq 3$.

Horata proved (40) by a complicated method that used formulas expressing Bernoulli numbers in terms of Stirling numbers of the second kind, and showed that both members of (40) are congruent modulo p for all primes p. Our direct proof of the more general result (38) does not depend on Stirling numbers or congruences.

An alternative form of (37) can be obtained by taking $x = 1$ in (36) and using (5):

$$n^2 B_{n-1} = -\sum_{k=1}^{n-2} \binom{n}{k} k B_k, \quad n \geq 3. \tag{41}$$

To see that this is equivalent to (37), divide by n in (41), then subtract $(n-1)B_{n-1}$ from both sides to get (37) with $n-1$ in place of n.

Yet another recursion can be obtained by integrating the product $xB_n(x)$ in two ways. On the one hand, from (2) we have

$$\int_0^x tB_n(t)\,dt = \sum_{k=0}^{n}\binom{n}{k}B_k\int_0^x t^{n+1-k}dx = \sum_{k=0}^{n}\binom{n}{k}\frac{B_k}{n+2-k}x^{n+2-k}.$$

Now calculate the same integral using integration by parts and (27) to obtain

$$\begin{aligned}\int_0^x tB_n(t)\,dt &= \frac{1}{n+1}\int_0^x tB'_{n+1}(t)\,dt = \frac{1}{n+1}\left\{xB_{n+1}(x) - \int_0^x B_{n+1}(t)\,dt\right\}\\ &= \frac{1}{n+1}\left\{xB_{n+1}(x) - \frac{B_{n+2}(x)-B_{n+2}(0)}{n+2}\right\}.\end{aligned}$$

Equating the two results we find the polynomial identity

$$\sum_{k=0}^{n}\binom{n}{k}\frac{B_k}{n+2-k}x^{n+2-k} = \frac{1}{n+1}\left\{xB_{n+1}(x) - \frac{B_{n+2}(x)-B_{n+2}(0)}{n+2}\right\}.$$

When $x = 1$ this simplifies to

$$\sum_{k=0}^{n}\binom{n}{k}\frac{B_k}{n+2-k} = \frac{B_{n+1}}{n+1}, \quad n \geq 1. \tag{42}$$

We leave it as a challenge to the reader to find another proof of (42) as a direct consequence of (3) without the use of integration.

Concluding remarks

In the three centuries after Jakob Bernoulli introduced his power-sum formulas, the polynomials and numbers that bear his name have been generalized in many different directions and have spawned hundreds of papers. You can get an idea of the many important areas of mathematics that have been influenced by these elementary topics by searching for *Bernoulli numbers* or *Bernoulli polynomials* on the world wide web.

The author is grateful to an anonymous referee who pointed out that that an English translation of Bernoulli's *Ars Conjectandi* was published in 2005 by The Johns Hopkins University Press under the title *The Art of Conjecturing, together with Letter to a Friend on Sets in Court Tennis*, by Dudley Sylla, and that collateral material related to the early history of (1) can be found in a paper by D. E. Knuth, *Johann Faulhaber and Sums of Powers*, in *Mathematics of Computation* **61** (1993), 277–294, and in another Johns Hopkins publication, *Pascals Arithmetic Triangle*, by A. W. F. Edwards (2002), especially Chapter 10.

Summary of basic formulas (as numbered in the text)

Defining relations:

$$B_n(x) = \sum_{k=0}^{n}\binom{n}{k}B_k x^{n-k}, \quad n \geq 0 \tag{2}$$

$$B_0 = 1, \quad B_n = \sum_{k=0}^{n}\binom{n}{k}B_k, \quad n \geq 2 \tag{4}$$

Special values:

$$B_n = B_n(1), \quad n \ge 2, \qquad B_n = B_n(0), \quad n \ge 0 \tag{12}$$

$$B_{2n+1} = 0, \quad n \ge 1 \tag{13}$$

Generating functions:

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n, \quad |z| < 2\pi \tag{9}$$

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n, \quad |z| < 2\pi \tag{10}$$

Difference equation:

$$B_n(x+1) - B_n(x) = nx^{n-1}, \quad n \ge 1 \tag{14}$$

Symmetry relation:

$$B_n(1-x) = (-1)^n B_n(x), \quad n \ge 0 \tag{18}$$

Addition formula:

$$B_n(y+x) = \sum_{k=0}^{n} \binom{n}{k} B_k(y) x^{n-k}, \quad n \ge 0 \tag{21}$$

Raabe's multiplication formula:

$$B_n(mx) = m^{n-1} \sum_{k=0}^{m-1} B_n\left(x + \frac{k}{m}\right), \quad n \ge 0,\ m \ge 1 \tag{22}$$

Derivative formulas:

$$B_n'(x) = nB_{n-1}(x), \quad n \ge 2 \tag{27}$$

$$B_n''(x) = n(n-1)B_{n-2}(x), \ldots, B_n^{(k)}(x) = k!\binom{n}{k} B_{n-k}(x) \tag{28}$$

Integration formulas:

$$\int_x^y B_n(t)\,dt = \frac{B_{n+1}(y) - B_{n+1}(x)}{n+1}, \quad n \ge 1 \tag{29}$$

$$\int_x^{x+1} B_n(t)\,dt = x^n, \quad n \ge 1 \tag{30}$$

$$\int_0^1 B_n(t)\,dt = 0, \quad n \ge 1 \tag{31}$$

Alternative recursion formulas:

$$B_n(x) = n\int_0^x B_{n-1}(t)\,dt + B_n, \quad \int_0^1 B_n(t)\,dt = 0, \quad n \ge 1 \tag{35}$$

with $B_0 = 1$, $B_0(x) = 1$,

$$\sum_{k=2}^{n} \binom{n}{k-2} \frac{B_k}{k} = \frac{1}{(n+1)(n+2)} - B_{n+1}, \quad n \geq 2 \tag{38}$$

$$n^2 B_{n-1} = -\sum_{k=1}^{n-2} \binom{n}{k} k B_k, \quad n \geq 3 \tag{41}$$

$$\sum_{k=0}^{n} \binom{n}{k} \frac{B_k}{n+2-k} = \frac{B_{n+1}}{n+1}, \quad n \geq 0 \tag{42}$$

Power-sum formulas:

$$\sum_{k=1}^{m-1} k^p = \frac{B_{p+1}(m) - B_{p+1}}{p+1}, \quad p \geq 1,\ m \geq 2 \tag{1}$$

$$\sum_{k=0}^{m-1} (a+dk)^p = d^p \frac{B_{p+1}(m+a/d) - B_{p+1}(a/d)}{p+1}, \quad p \geq 1,\ m \geq 2 \tag{17}$$

$$\sum_{k=1}^{m}{}' k^p = \sum_{d|m} \mu(d) d^p \frac{B_{p+1}(m/d) - B_{p+1}}{p+1}, \quad p \geq 1,\ m \geq 1 \tag{24}$$

$$\sum_{k=1}^{m}{}' k^p = \frac{1}{p+1} \sum_{r=1}^{p+1} \binom{p+1}{r} m^r B_{p+1-r} \prod_{q|m} (1 - q^{p-r}) \quad p \geq 1,\ m \geq 1 \tag{26}$$

REFERENCES

1. T. M. Apostol, *Introduction to Analytic Number Theory*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1976.
2. T. M. Apostol, An elementary view of Euler's summation formula, *American Mathematical Monthly* **106** (1999) 409–418.
3. K. Horata, On congruences involving Bernoulli numbers and irregular primes, II, *Report Fac. Sci. and Technol. Meijo Univ.* **31** (1991) 1–8.
4. J. Riordan, *Combinatorial Identities*, Robert E. Krieger, Huntington, NY, 1979. Reprint of the 1968 original with corrections.
5. D. E. Smith, *A Source Book in Mathematics*, McGraw-Hill, New York, 1929.

Erratum

Authors Chungwu Ho and Seth Zimmerman have written to point out an error on page 14 of their paper "On Infinitely Nested Radicals," this MAGAZINE, Vol. 81, February 2008: The gaps mentioned for the set S_2 do not exist. Gaps exist only for sets S_a with $a \geq 3$.