ZX-Rules for 2-Qubit Clifford+T Quantum Circuits

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Abstract. ZX-calculus is a high-level graphical formalism for qubit computation. In this paper we give the ZX-rules that enable one to derive all equations between 2-qubit Clifford+T quantum circuits. Our rule set is only a small extension of the rules of stabiliser ZX-calculus, and substantially less than those needed for the recently achieved universal completeness. One of our rules is new, and we expect it to also have other utilities.

These ZX-rules are much simpler than the complete of set Clifford+T circuit equations due to Selinger and Bian, which indicates that ZX-calculus provides a more convenient arena for quantum circuit rewriting than restricting oneself to circuit equations. The reason for this is that ZX-calculus is not constrained by a fixed unitary gate set for performing intermediate computations.

1 Introduction

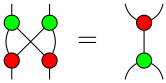
The ZX-calculus [9,10] is a universal graphical language for qubit theory, which comes equipped with simple rewriting rules that enable one to transform diagrams representing one quantum process into another quantum process. More broadly, it is part of categorical quantum mechanics which aims for a high-level formulation of quantum theory [1,13]. It has found applications both in quantum foundations [3,11,12] and quantum computation [6,7,17,20], and is subject to automation thanks to the Quantomatic software [24]. Recently ZX-calculus has been completed by Ng and Wang [25], that is, provided with sufficient additional rules so that any equation between matrices in Hilbert space can be derived in ZX-calculus. This followed earlier completions by Backens for stabiliser theory [2] and one-qubit Clifford+T circuits [4], and by Jeandel, Perdrix and Vilmart for general Clifford+T theory [21]. In Sect. 3 we present Backens' two theorems.

This paper concerns a sufficient set of ZX-rules for establishing all equations between 2-qubit Clifford+T quantum circuits, which again can be seen as a completeness result. We were motivated in two manners to seek this result:

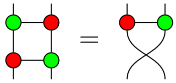
- Firstly, we wish to understand the utility of the ZX-rules. In the case of the full completion [22,25] these were added using a purely theoretical methodology which consisted of translating Hilbert space structure into diagrams, passing via another graphical calculus [18,19]. However, a natural question concerns

J. Kari and I. Ulidowski (Eds.): RC 2018, LNCS 11106, pp. 144–161, 2018.
https://doi.org/10.1007/978-3-319-99498-7_10

the actual practical use of each of these rules, as well as of other rules derived from them. As an example, one of the key ZX-rules:



is equivalent to the following well known circuit equation [10]:



involving CNOT gates (green $\simeq$ control). In this paper we are concerned with all such equations for 2-qubit Clifford+T quantum circuits.

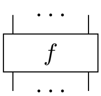
– Secondly, in quantum computing algorithms are converted into elementary gates making up circuits, and these circuits then have to be implemented on a computer. Currently the most considered universal set of elementary gates is the Clifford+T gate set. The high cost of implementing those gates makes any simplification of a circuit (cf. having less CNOT-gates and/or having less T-gates) highly desirable. We expect our result to be an important stepping stone towards efficient simplification of arbitrary n-qubit Clifford+T circuits, and that the quantomatic software will be a crucial part of this. The fact that a small set of rules suffices for us here raises the hope that general circuit simplification could already be done with a small set of ZX-rules.

Selinger and Bian derived a complete set of circuit equations for Clifford+T 2-qubit circuits [26]. However, these circuit equations are very large and rigid, and their method for producing these beyond two-qubits doesn't scale to more qubits. On the other hand, in the case of ZX-calculus we already have an overarching completeness results that carries over to circuits of arbitrary qubits. So the main question then concerns the rules needed specifically for efficient circuit rewriting.

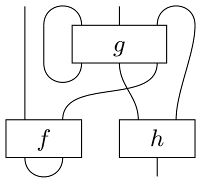
The advantage of ZX-rules is that they are not constrained by unitarity. Also, in the ZX computation at intermediate stages phase gates may not even be within Clifford+T, although their actual values play no roles, that is, they can be treated as variables. Note that going beyond the constraints of the formalism which one aims to prove something about is a standard practice in mathematics, e.g. complex analysis.

2 Background 1: ZX-calculus Language

A pedestrian introduction is [14]. There are two ways to present ZX-calculus, either as diagrams or as a category. Following [13], the 'language' of the ZX-calculus consists of certain special processes or boxes:



which can be wired together to form diagrams:

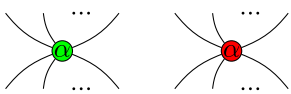


All the diagrams should be read from top to bottom. Note that the wiring of inputs to inputs and outputs to inputs, as well as feed-back loops is admitted. Equivalently, following [10], it consists of certain morphisms in a compact closed category, which has the natural numbers: $0, 1, 2, \cdots$ as objects, with the addition of numbers as the tensor:

$$m \otimes n = m + n$$

In diagrams n corresponds to n wires side-by-side.

The special processes/boxes/morphisms that we are concerned with in this paper are spiders of two colours:

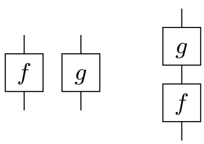


where $\alpha \in [0, 2\pi)$. Equivalently, one can only consider spiders of one colour as well as a colour changer (cf. rule (H2) below):

ZX-calculus can also be seen as a calculus of graphs, provided that one introduces special input and output nodes.

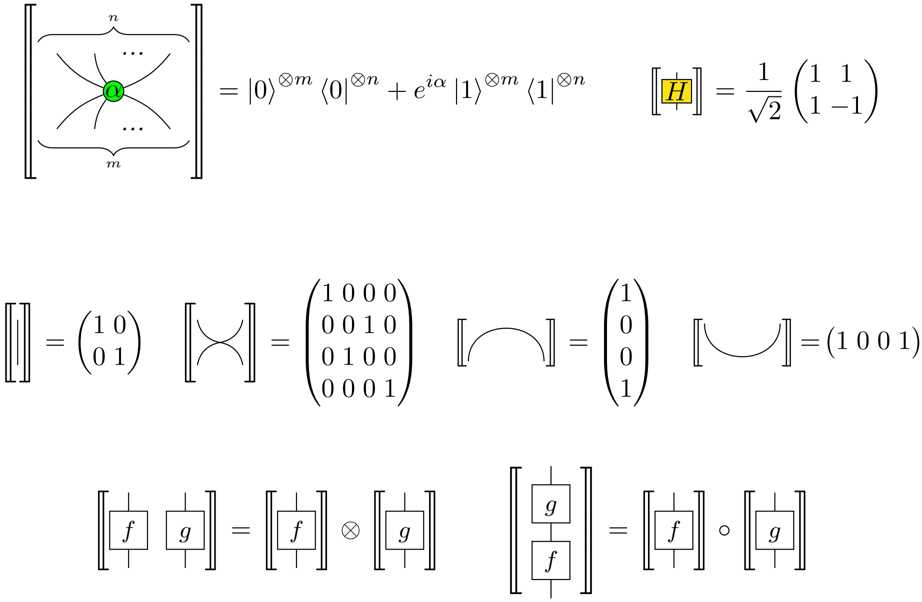
Sometimes it is useful to also think of wires appearing in the diagram as boxes, which can take the following forms:

In particular, then the full specification of what 'wiring boxes together' actually means can be reduced to what it means to put boxes side-by-side and connect the output of a box to the input of another box:



The following key property uses this fact:

Theorem 1 [8,10]**.** *The ZX language is* <u>*universal*</u> *for qubit quantum computing, when giving the following interpretation:*



That is, every linear map of type $\mathbb{C}^{2^n} \to \mathbb{C}^{2^m}$ can be written down as a ZX-diagram, and consequently, every qubit process can be written down as a ZX-diagram.

3 Background 2: ZX-calculus Rules

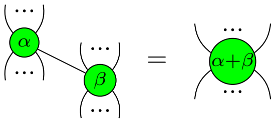
Above we specified the ingredients of the ZX-calculus as linear maps. Now, in quantum theory linear maps only matter up to a non-zero scalar multiple, i.e. a diagram with no inputs nor outputs. We will do so too here, since this makes that the rules of the ZX-calculus appear much simpler (see e.g. [5] for a presentation of the ZX-calculus rules with explicit scalars that make equations hold on-the-nose).

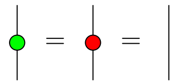
Due to the diagrammatic underpinning, in addition to the rules given below, there is one meta-rule that ZX-calculus obeys, namely:

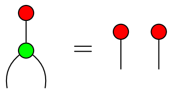
$\boxed{\textit{Only connectedness matters!}}$

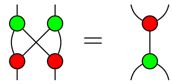
One could do without it by adding a few more rules, but it is entirely within the spirit of diagrammatic reasoning that it should all boil down to connectedness. We now give an overview of ZX-rule sets that have been considered.

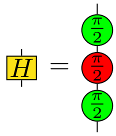
Stabiliser ZX-calculus is the restriction of ZX-calculus to $\alpha \in \{\frac{n\pi}{2} \mid n \in \mathbb{N}\}$. As shown in [2], the following rules make ZX-calculus complete for this fragment of quantum theory:

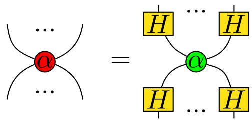

(S1)


(S2)


(B1)

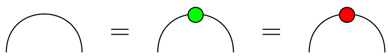

(B2)


(H1)

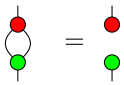

(H2)

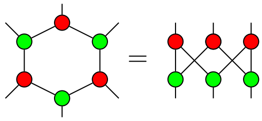
That is, any equation between stabiliser ZX-diagrams that can be proved using matrices can also be proved by using these rules.

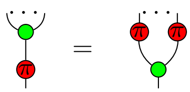
The ‘only connectivity matters rule’ means that we also have [5]:


(S2’)

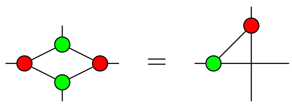
Some other derivable rules that we will use are:


(Hf)

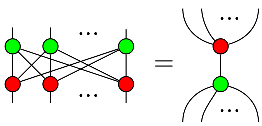

(Hex)


(Cy)

where the dots in (Cy) denote zero or more wires. The 1st and last rule are derived in [10] and the middle one in [16]. We also use the following variation form of (B2), to which we also refer as (B2):

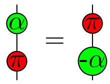

(B2)

The rules (S1) and (H) apply to spiders with an arbitrary number of input and output wires, including none, so (S1) and (H) appear to be an infinite set of rules. Firstly, these rules do have algebraic counterparts as Frobenius algebras, which constitute a finite set. Secondly, using the concept of bang-boxes [23], even in their present form these rules can be notationally reduced to a single rule, and the quantomatic-software accounts for rules in this form. Allowing for bang-boxes, one can also merge rules (B1) and (B2) into a single rule:



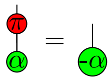
hence reducing the number of equations to be memorised to six.

Single-qubit Clifford+T ZX-calculus is the restriction of ZX-calculus to spiders with exactly one input and one output, and $\alpha \in \{\frac{n\pi}{4} \mid n \in \mathbb{N}\}$. As shown in [4], the rules (S1), (S2), (H1) and (H2) together with the rule:



(N)

make ZX-calculus complete for this fragment of quantum theory. We will also use the following special form of the (N) rule, to which we again refer as (N):



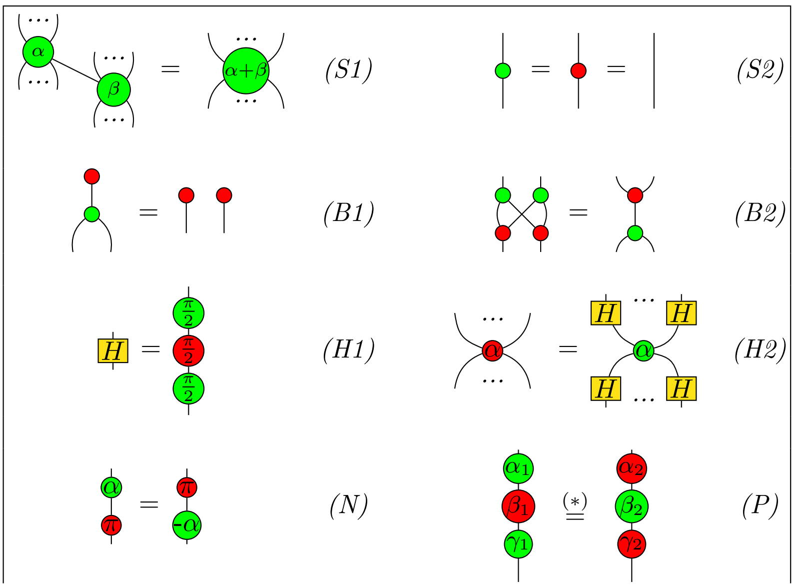
(N)

As single qubit circuits can be seen as a restriction of 2-qubit circuits, simply by letting the 2nd qubit unaltered, our result can also be seen as a completeness result for single-qubit Clifford+T ZX-calculus. However, it is weaker than Backens' as we employ more rules.

4 Result: ZX Rules vs. Circuit Equations

Recall that in this paper the ZX-rules hold up to a non-zero scalar.

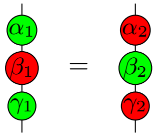
Theorem 2. *The rules (S1), (S2), (B1), (B2), (H1), (H2), (N) and (P) depicted below make ZX-calculus complete for 2-qubit Clifford+T circuits:*



where $\alpha_2 = \gamma_2$ if $\alpha_1 = \gamma_1$, and $\alpha_2 = \pi + \gamma_2$ if $\alpha_1 = -\gamma_1$; the equality (*) should be read as follows: for every diagram in LHS there exists α_2, β_2 and γ_2 such that LHS = RHS (and vice versa if conjugating by the Hadamard gate). In what follows we will see that we actually don't need to know the precise values of α_2, β_2 and γ_2.

So as compared to the rules that we saw in the previous section there is only one additional rule here, the (P) rule. This rule is a new rule that was not present as such in any previous presentation of the ZX-calculus. Of course, as the rules presented in [25] yield universal completeness, one should be able to derive it from these:

Lemma 1. *For $\alpha_1, \beta_1, \gamma_1 \in (0,\ 2\pi)$ we have:*



$$\text{with} \quad \begin{cases} \alpha_2 = \arg z + \arg z' \\ \beta_2 = 2\arg(|\frac{z}{z'}| + i) \\ \gamma_2 = \arg z - \arg z' \end{cases} \tag{1}$$

where:

$$z = \cos\tfrac{\beta_1}{2}\cos\tfrac{\alpha_1+\gamma_1}{2} + i\sin\tfrac{\beta_1}{2}\cos\tfrac{\alpha_1-\gamma_1}{2} \qquad z' = \cos\tfrac{\beta_1}{2}\sin\tfrac{\alpha_1+\gamma_1}{2} - i\sin\tfrac{\beta_1}{2}\sin\tfrac{\alpha_1-\gamma_1}{2}$$

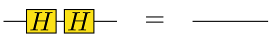
So if $\alpha_1 = \gamma_1$, then $\alpha_2 = \gamma_2$, and if $\alpha_1 = -\gamma_1$, then $\alpha_2 = \pi + \gamma_2$.

This Lemma, proved in the arXiv version of this paper [15], gives an analytic solution for the 'colour-swapping' property for arbitrary phases. The idea for the need for a rule of this kind was first suggested by Schröder de Witt and Zamdzhiev [27]. As already indicated in the introduction, it is also clear that this rule takes one out of the Clifford+T realm in the sense that the values of the angles in the RHS of (1) usually go beyond Clifford+T even if the LHS is inside of the realm.

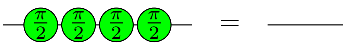
The proof of Theorem 2 draws from Selinger and Bian's [26] set of circuit equations that is complete for 2-qubit circuits. Here we rely on universality of ZX-language to write down these circuits, and in particular besides CNOT-gates these also involve symmetric CZ-gates:

In the statement of the following theorem we adopt the more usual left-to-right reading of circuits, although we still express it as ZX diagrams.

Theorem 3 [26]**.** *The following equations are complete for 2-qubit Clifford+T circuits:*



(2)



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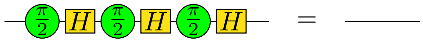
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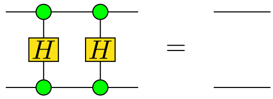
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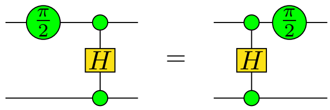
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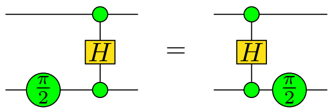
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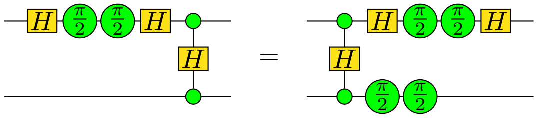
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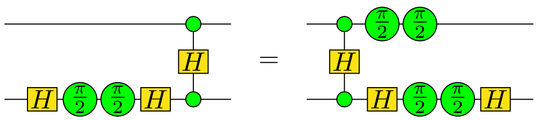
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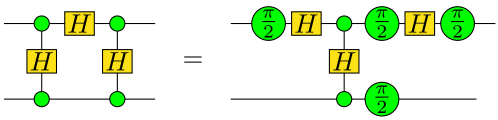
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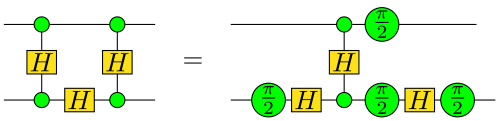
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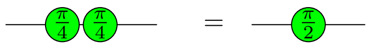
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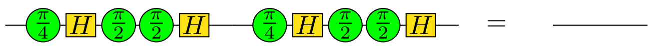
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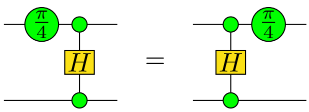
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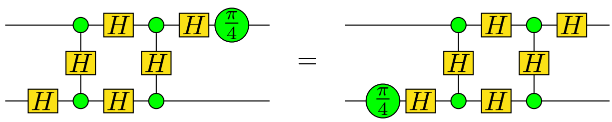
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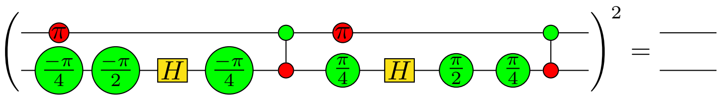
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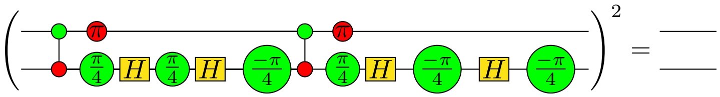
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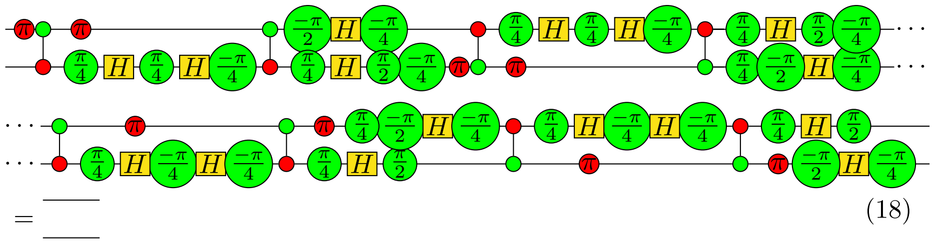
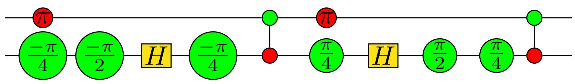

(18)

Not only does this Theorem serve as a stepping stone, it is also the main point of comparison of our result. The ZX-rules are clearly much simpler than the circuit equations, which, to say the least, are virtually impossible to memorise, let alone apply.

5 Proof

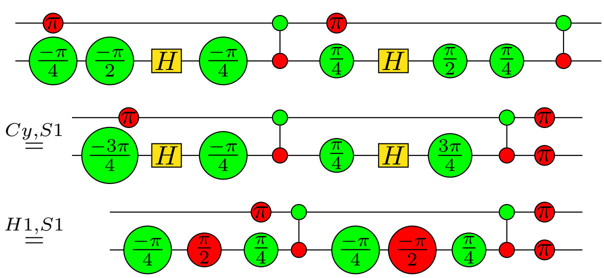
We need to show that the equations in Theorem 3 can be derived from those in Theorem 2. Doing so is a straightforward calculation for the first 14 ones. However, this is not the case for the remaining circuit relations (16), (17) and (18) each of which we prove as a lemma.

Lemma 2. *Let* $A =$

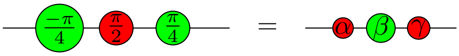


then $A^2 = I$.

Proof. First we have $A =$

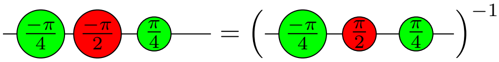

(19)

By the rule (P), we can assume that



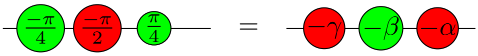
(20)

Since $e^{i\frac{-\pi}{4}}e^{i\frac{\pi}{4}} = 1$, we could let $\gamma = \alpha + \pi$. Also note that



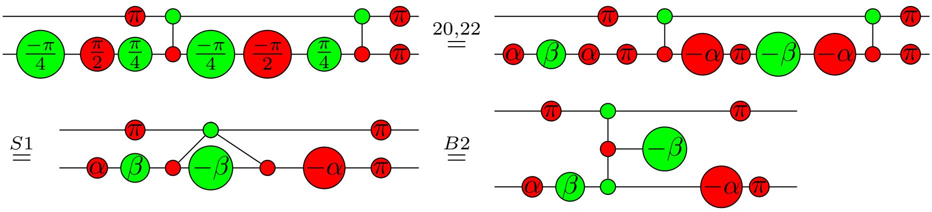
(21)

Thus:

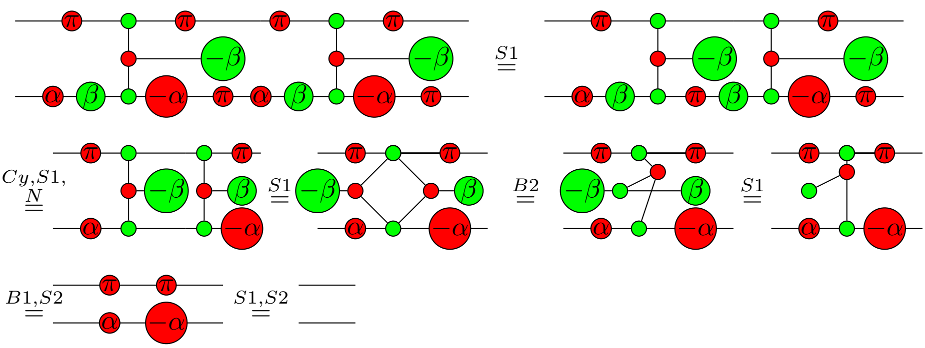


(22)

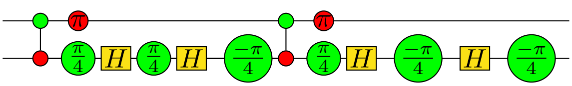
Therefore, $A =$



Finally, $A^2 =$

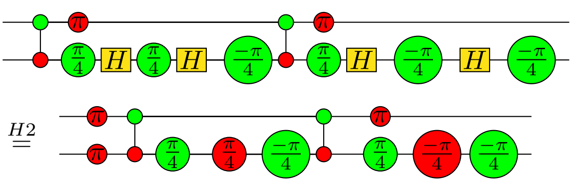


Lemma 3. *Let* $B =$

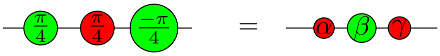


then $B^2 = I$.

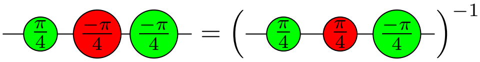
Proof. Firstly we have:



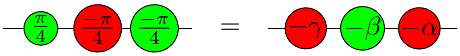
By the rule (P), we can assume that:

 (23)

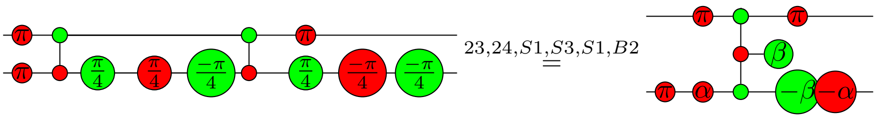
Since $e^{i\frac{-\pi}{4}}e^{i\frac{\pi}{4}} = 1$, we could let $\gamma = \alpha + \pi$. Also note that:



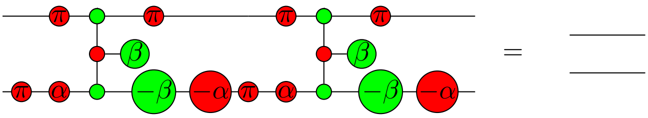
Thus:

 (24)

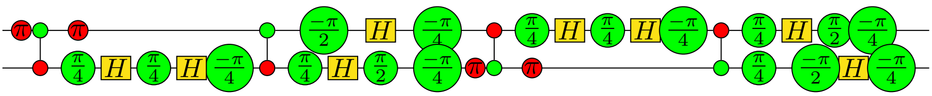
Using again the same technique as earlier we obtain:



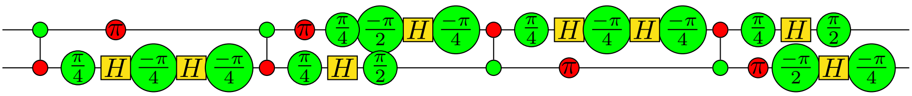
Finally, again following the previous lemma, $B^2 =$



Lemma 4. *Let* $C =$

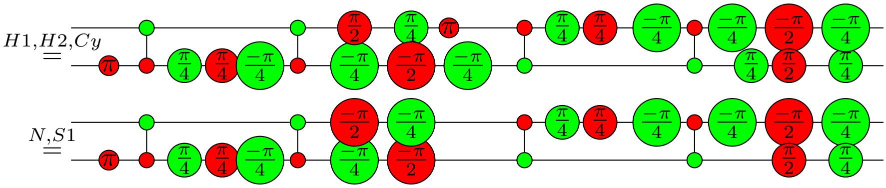


and $D =$

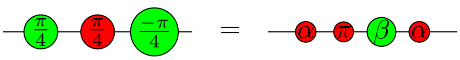


then $D \circ C = I$.

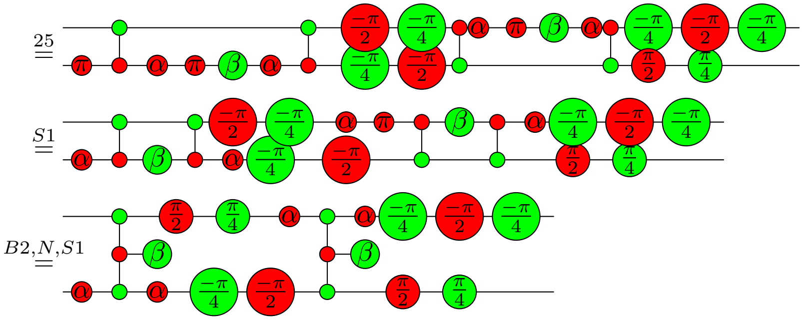
Proof. Firstly we simplify the circuit C as follows:



By the rule (P), we can assume that:

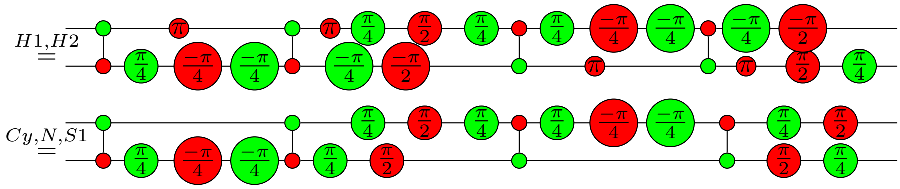
 (25)

Then we have for C:

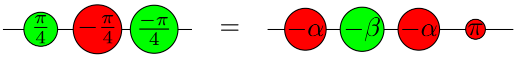


(26)

Secondly, we simplify the circuit D as follows:

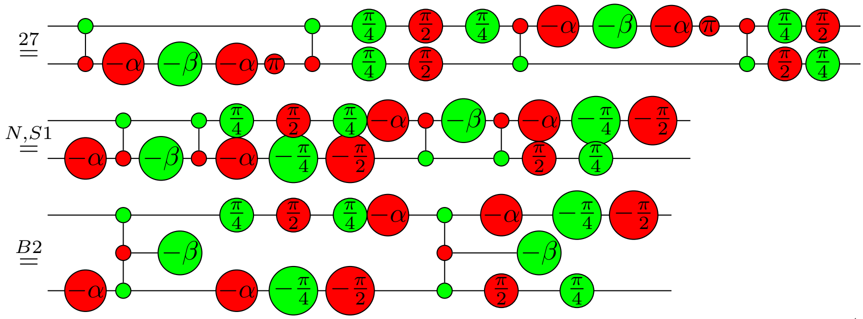


By the rule (P), we have



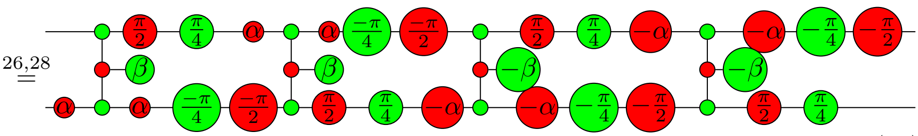
(27)

Therefore we have for D:



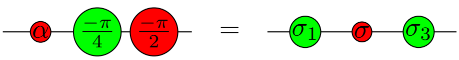
(28)

Then we obtain the composition for $D \circ C =$



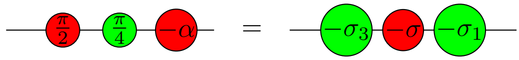
(29)

By the rule (P), we can assume that:



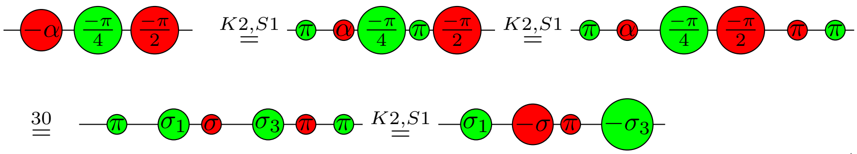
(30)

Then for its inverse, we have



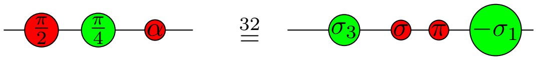
(31)

Also we can obtain that:



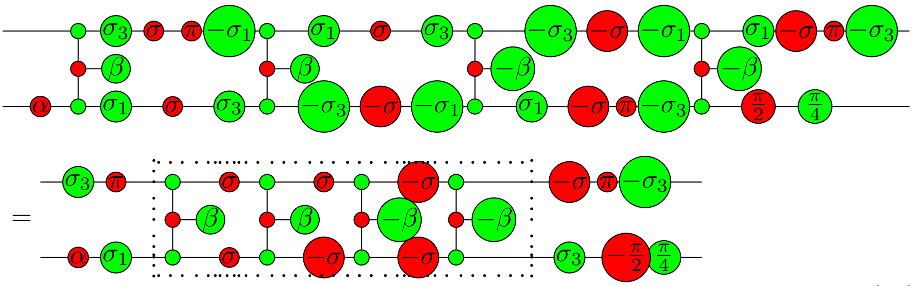
(32)

As a consequence, we have the inverse for both sides of (32):



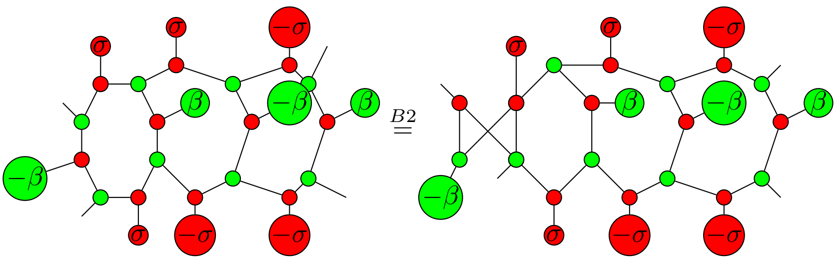
(33)

Now we can rewrite $D \circ C$ as:

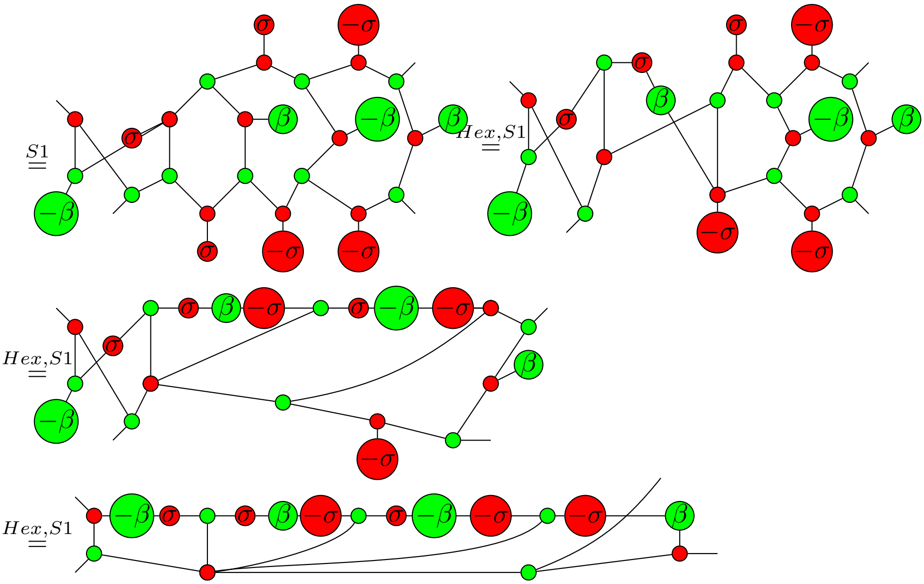


(34)

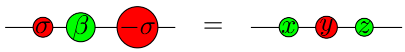
We can depict the dashed part of (34) in a form of connected octagons, and to deal with these octagons we use (Hex):



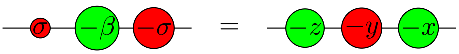
(35)



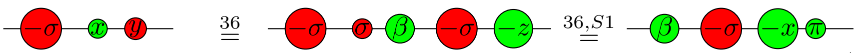
By the (P) rule, we have:


(36)

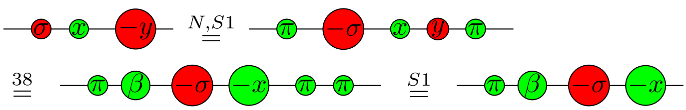
where $z = x + \pi$. Then we take inverse for each side of (36) and obtain that:


(37)

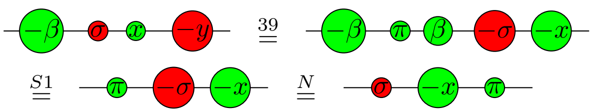
By rearranging the phases on both sides of (36), we have:


(38)

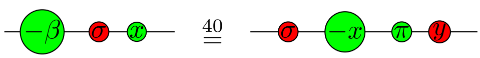
Thus:


(39)

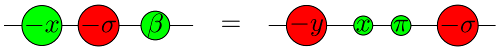
Therefore:


(40)

It then follows that:

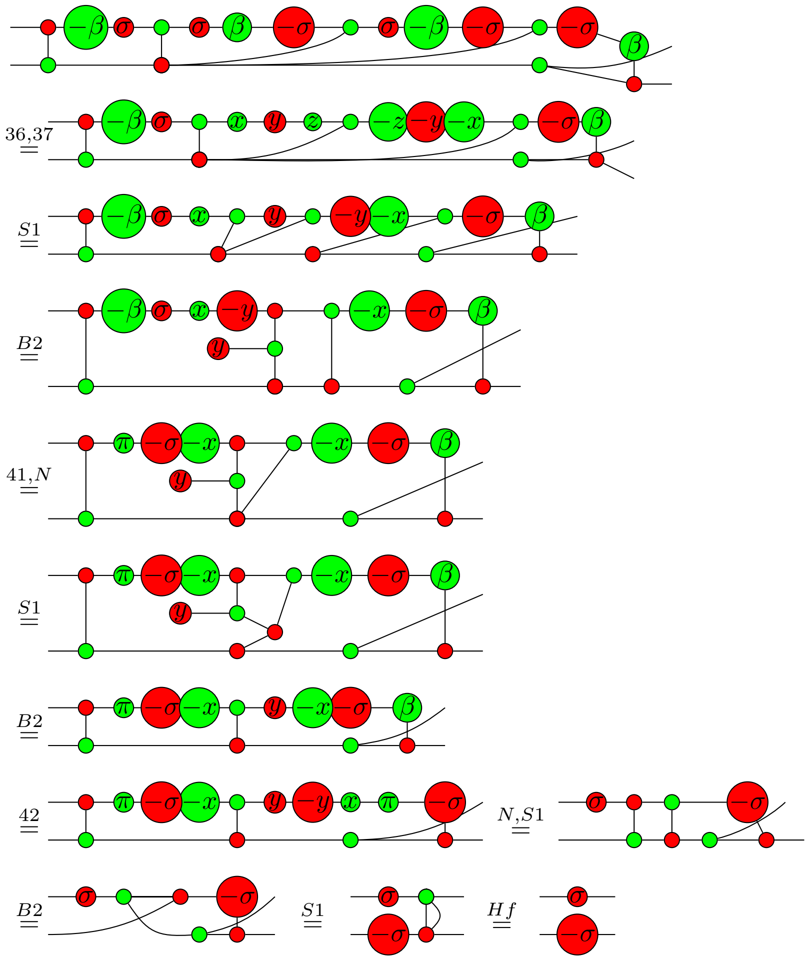

(41)

If we take the inverse of the left-hand-side of (41), then we have:



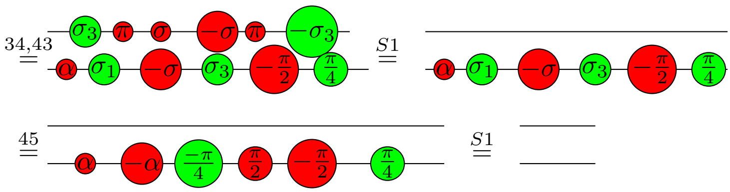
(42)

Now we can further simplify the final diagram in (35) as follows:



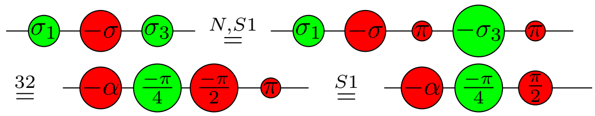
(43)

Finally, the composite circuit $D \circ C$ as can be simplified as follows:



(44)

where we used the following property:



(45)

6 Conclusion and Further Work

We gave a set of ZX-rules that allows one to establish all equations between 2-qubit circuits, and these ZX-rules are remarkably simpler than the relations between unitary gates from which they were derived. The key to this simplicity is: (i) abandoning unitarity at intermediate stages, and (ii) abandoning the T-restriction, which comes about when applying rule (P). In the case of the latter, it is important to stress again that the actual values of the phases in the RHS of (P) don't have to be known.

Also, while the techniques used to establish the relations between two-qubit unitary gates don't scale to more than two qubits, the ZX-calculus, by being complete, already provides us with such a set. It is just a matter to figure out if all of those rules are actually needed for the case of circuits. Automation is moreover also possible thanks to the quantomatic software. Although we don't yet have a general strategy for simplifying quantum circuits by the ZX-calculus, it is possible at least in some cases. In fact, in ongoing work in collaboration with Niel de Beaudrap, using similar techniques as some of the ones in this paper, we have shown that using ZX-calculus we can outperform the state-of-the-art for quantum circuit simplification. A paper on this is forthcoming.

We expect the new rule (P) to have many more utilities within the domain of quantum computation and information. The same question remains for other rules that emerged as part of the completion of ZX-calculus.

A natural challenge of interest to the Reversible Computing community is whether the classical fragment of ZX-calculus can be used for deriving similar completeness results for classical circuits.

Acknowledgments. This work was sponsored by Cambridge Quantum Computing Inc. for which we are grateful. QW also thanks Kang Feng Ng for useful discussions.

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