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Report on the Dissertation

Uniformisation and Choice Questions for Regular Languages

by Vincent Michielini

The thesis of Vincent Michielini deals with uniformization questions over linear orders, in particular finite words. A uniformization of a binary relation is a function that maps the elements in the domain of the relation to one of their images in the relation. Uniformizations have been studied in set theory in order to understand the expressiveness of formalisms through questions like: "Does every relation definable in the formalism also have a uniformization that is definable in the formalism?". Variations of this question include the case where the formalisms for defining the relation and the uniformization can be different. In computer science, uniformizations play a role in the context of algorithmic synthesis of computable input-output functions from given specifications. In this setting, the specification is a relation that relates inputs with admissible outputs, and a uniformization is a function that is consistent with the specification. In this context, it is important to have computable uniformizations in order to be able to extract the uniformization from a given specification by an algorithm.

The thesis under review studies these kinds of uniformization questions for relations defined in monadic second-order logic (MSO) and fragments of MSO over labeled linear orders. The thesis starts with a clear and detailed introduction that explains the foundations and origins of uniformization questions, and places the results of the thesis in the context of existing literature. The first chapter introduces in detail the definitions and fundamental tools that are used throughout the thesis, in particular the basic logical and algebraic formalisms. The main contributions of the thesis are presented in Chapters 2–5.

In Chapter 2, the author presents a study of uniformization questions for various fragments of first-order logic (FO) over finite words. These fragments are obtained by varying the signature of the logic, by including the order on word positions $FO[<]$, only the successor relation $FO[s]$, or none of the two $FO[]$ (where equality is included in all the signatures). Further fragments are obtained by restricting the number of variables.

In order to define relations in logical formalisms, a pair of words is represented as a single word over a product alphabet. Defining a relation then is the same as defining a language, a set of labeled words over this product alphabet. The question that is studied in this chapter is, in which fragments of FO it is possible to define uniformizations of relations defined in some other fragment of FO. The following negative results presented in the thesis show that uniformization in these fragments of FO is not possible in most of the cases:

- $FO[<]$ does not uniformize $FO^2[s]$
- $FO[s]$ does not uniformize $FO^2[]$
- $FO^2[<, s]$ does not uniformize $FO^2[s] \cap FO^2[<]$

These results have be published in the conference DLT 2018 together with the first supervisor of the thesis.

Additionally, two new positive results are provided, namely that $FO[<]$ uniformizes $FO[]$, and that it is decidable whether a regular relation (defined in MSO logic) admits a uniformization in $FO[]$. These results are based on a characterization of the languages definable in $FO[]$ as those whose syntactic monoid is aperiodic and commutative.

Chapter 3 is devoted to the study of subclasses of regular languages of finite words that admit self-uniformization. It is a folklore result that all regular relations also have a regular uniformization. The result presented in Chapter 3 shows that there is no other non-empty subclass of regular languages that admits self-uniformization and that is a variety of languages, which means that the class has certain closure properties. The proof of this result uses semigroups as mechanism for defining regular languages, and shows that from the self-uniformization property and the closure properties of a variety, one can deduce that the language class must be able to define all regular languages. The results of Chapter 3 have been published in the conference MFCS 2019 together with the two supervisors of the thesis.

Chapters 4 and 5 go beyond finite words and study countable linear orders. Given a countable linear order, one can use MSO to define sets of labelings of this linear order. Relations can then be defined by using product alphabets, and identifying a labeling over this product alphabet as a pair of two labeled linear orders over the component alphabets. Chapter 4 provides a characterization of so-called finitary linear orders that admit MSO definable uniformizations of MSO-definable relations. Finitary linear orders are countable linear orders that can be produced from single elements by finitely many applications of concatenation, ω-iteration, inverse ω-iteration, and an operation for producing dense linear orders. The characterization theorem states that the finitary linear orders that admit MSO definable uniformizations are precisely those that do not have non-trivial automorphisms. Besides this characterization, the theorem also gives various other equivalent characterizations, namely the definability of a choice function, a well-ordering, or the definability of all the individual elements of the linear order. The main result in this chapter has been published in the conference MFCS 2020 together with the first supervisor of the thesis (in the thesis some more equivalent characterizations have been added to the theorem compared to the published version).

Chapter 5 continues Chapter 4 by looking at the equivalence theorem if the restriction on finitariness of the linear orders is dropped. It is shown that the theorem fails in this case, where the ordinal ω^ω can be used as counterexample.

In his thesis, Vincent Michielini has made several interesting and highly non-trivial technical contributions that lead to a much better understanding of uniformization questions over words and more general linear orders.

The thesis is well structured and written very carefully, with a clean presentation of the contributions and the mathematical tools for obtaining the results. The author has shown his deep knowledge of the subject by using various sophisticated techniques from algebra and logic for dealing with regular languages.

His work has already received high interest in the international research community, as documented by three publications in peer-reviewed proceedings of leading international conferences on logic and theoretical computer science.

In summary, I clearly recommend to accept this valuable work as doctoral dissertation.

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