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Lecture Notes on Hyperexponential Solutions of Linear Homogeneous Ode's

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Abstract. This note contains a classical algorithm for computing hyperexponential solutions of linear homogeneous ode's with rational function coefficients. This algorithm is fundamental for factoring linear homogeneous ode's. The structure of hyperexponential solutions of a linear ode is possibly new.

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1. Problem Statement and Preliminaries

Let $\mathbf{C}$ be the field of complex numbers and x be an indeterminate over $\mathbf{C}$. A linear homogeneous ordinary differential equation (abbreviated as: lode) over $\mathbf{C}(x)$ can be written as

$$A(y) = a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = 0, \qquad (1)$$

where $a_i \in \mathbf{C}[x]$, for $0 \le i \le n$, and $a_n \neq 0$. The integer n is called the order of (**??**) and denoted by ord (A). The usual differential operator $\frac{d}{dx}$ is also denoted by D or $'$. The set of all solutions of $A(y)$ is a $\mathbf{C}$-linear space of dimension n, which is denoted by $\mathcal{S}(A)$. A basis of $\mathcal{S}(A)$ is called a fundamental set of solutions of $A(y)$.

maybe a bit more

→ finite dimension: fixing sufficiently many init. val. determines $y(x)$ uniquely.

$\le \max_i \, i - \deg a_i$

A non-zero (complex meromorphic) function $u = u(x)$ is called a hyperexponential over $\mathbf{C}(x)$ if $\frac{u'}{u}$ belongs to $\mathbf{C}(x)$. By integration one can prove that u is a hyperexponential over $\mathbf{C}(x)$ if and only if $u = \exp(\int r)$, for some $r \in \mathbf{C}(x)$.

log. der.

Example 1.1 *All non-zero elements, i.e., non-zero rational functions are hyperexponentials (over $\mathbf{C}(x)$). Usual exponential functions are hyperexponentials. We now prove that root functions are hyperexponentials. Let u be a function such that $u^m - r \equiv 0$, for some nonzero $r \in \mathbf{C}(x)$. By differentiation we find $mu^{m-1}u' - r' \equiv 0$. It follows that*

$$\frac{u'}{u} = \frac{1}{m}\frac{r'}{r} \in \mathbf{C}(x).$$

However, the function $\log x$ is not a hyperexponential.

Exercise. Find an algebraic function that is not a hyperexponential.

$Ay = 0,\ Az = 0 \Rightarrow A(y+z) = 0$

$A(c \cdot y) = 0$

$\mathbb{C}[[x]]$ infinite dim. over $\mathbb{C}$.

The motivation to search for hyperexponential solutions of (**??**) is as follows. Assume that u is a hyperexponential solution of (**??**). Let

$$E(y) = y' - \frac{u'}{u} y.$$

Then $A(y)$ and $E(y)$ have non-zero common solution u. Since $\operatorname{ord}(E) = 1$, the differential remainder of $A(y)$ and $E(y)$ is equal to zero, that is,

$$A(y) = \underbrace{(b_{n-1}D^{n-1} + \cdots + b_1 D + b_0)}_{B} \circ E(y),$$

where $\circ$ means applying the operator B to E and all the b's are rational functions. Assume that $v_1, \ldots, v_k$ are k **C**-linearly independent solutions of $B(z) = 0$. Then we can easily check that

$$u,\ u \int \frac{v_1}{u}, \cdots, \int \frac{v_k}{u}$$

are $(k+1)$ **C**-linearly independent solutions of $A(y) = 0$. Thus, Solving $A(y) = 0$ is reduced to solving $B(z) = 0$ whose order is $(n-1)$ and whose coefficients are still in $\mathbf{C}(x)$.

Exercise. Let u_1 and u_2 be two distinct hyperexponential solutions of $A(y)$. Can we reduce the problem of solving $A(y) = 0$ to solving an lode whose order is $(n-2)$ and whose coefficients are still in $\mathbf{C}(x)$?

Recall some facts about solutions of (**??**). All solutions of (**??**) form an n-dimensional linear space over **C**, which is denoted by $\mathrm{sol}(A)$. Complex functions $u_1 = u_1(x), \ldots, u_m = u_m(x)$, are linearly independent over **C** if and only if

$$W(u_1, u_2, \ldots, u_m) = \begin{vmatrix} u_1 & u_2 & \cdots & u_m \\ u_1' & u_2' & \cdots & u_m' \\ \cdot & \cdot & \cdots & \cdot \\ u_1^{[m-1]} & u_2^{[m-1]} & \cdots & u_m^{[m-1]} \end{vmatrix} \neq 0.$$

The determinant $W(u_1, \ldots, u_m)$ is called the Wronskian of $u_1, \ldots, u_m$.

A possibly open problem is to decide whether $u_1, \ldots, u_m$ are linearly independent over $\mathbf{C}(x)$.

2. Polynomial solutions of a linear homogeneous ode

Let $\mathcal{P}(A)$ be the set of polynomial solutions of $A(y)$. The set $\mathcal{P}(A)$ is a **C**-linear subspace of ~~$A(y)$~~ $\mathrm{sol}(A)$. Computing polynomial solutions of $A(y)$ means computing a basis for $\mathcal{P}(A)$.

Let the polynomial

$$P = p_N x^N + p_{N-1} x^{N-1} + \cdots + p_0$$

be in $\mathbf{C}[x]$. If P is a solution of (**??**), then

$$\begin{aligned} & a_n \left(p_N \textstyle\prod_{i=0}^{n-1} (N-i) x^{N-n} + \text{lower terms} \right) \\ & + a_{n-1} \left(p_N \textstyle\prod_{i=0}^{n-2} (N-i) x^{N-n+1} + \text{lower terms} \right) \\ & + \cdots \\ & + a_1 \left(p_N N x^{N-1} + \text{lower terms} \right) \\ & + a_0 \left(p_N x^N + \text{lower terms} \right) = 0. \end{aligned} \tag{2}$$

$$\partial_x^i \cdot x^j = j(j-1)\cdots(j-i+1) \cdot x^{j-i}$$

does not depend on N !

Define the *order drop* of A at the infinity to be

$$\underbrace{\partial_x^n}\quad \underbrace{\partial_x^{n-1}} \quad \underbrace{\partial_x}$$

$$\mu_\infty = \max\{\deg a_n - n,\ \deg a_{n-1} - (n-1), \ldots, \deg a_1 - 1,\ \deg a_0\}$$

and the leading set of A at the infinity to be

$$\lambda_\infty(A) = \{j \mid 0 \le j \le n \text{ and } \deg a_j - j = \mu_\infty(A)\}.$$

indices of terms achieving the order drop

Coeff. of the $\deg a_j$ term of a_j

Hence, we have

$$\boxed{\sum_{j \in \lambda_\infty(A)} \mathrm{lc}(a_j) \overbrace{\prod_{i=0}^{j-1} (N - i)}^{N^{j-1}} = 0}$$

because the sum is the leading coefficient (w.r.t. x) of the left hand-side of (**??**). (any product with $\prod_{i=0}^{-1}$ is understood to be 1). Define the *indicial equation* of A at the infinity to be

$$E_\infty(A) := \sum_{j \in \lambda_\infty(A)} \mathrm{lc}(a_j) \prod_{i=0}^{j-1} (z - i), = 0$$

where z is a new indeterminate. We find that N is a integral root of $E_\infty(A)$. Thus, we have arrive at

Theorem 2.1 The degree of a polynomial solution of (**??**) is bounded by the maximal positive integral root of $E_\infty(A)$.

This theorem yields an algorithm for computing $\mathcal{P}(A)$ as follows: First, we compute a degree bound for the polynomial solutions of (**??**) by Theorem **??**. Let the bound be N. Then we make an ansatz

$$P = p_N x^N + p_{N-1} x^{N-1} + \cdots + p_0,$$

where the p_i's are unspecified constant coefficients. Substituting P for y in (**??**) yields a linear homogeneous system. The solution space of this linear system gives us all the polynomial solutions of (**??**).

Example 2.1 Let $A = \frac{d^2y}{dx^2}$. Find all the polynomial solutions of A. The polynomial $E_\infty(A) = z(z-1)$. Hence, Theorem **??** asserts that the degree of polynomial solutions of A will not exceed 2. Let a polynomial solution be

$$P = p_1 x + p_0$$

where $p_1, p_0 \in \mathbf{C}$. Substituting P for y in A, we find that the polynomial solutions of A are

$$P = p_1 x + p_0$$

where p_1 and p_0 are arbitrary elements in $\mathbf{C}$. In other words, $\mathcal{P}(A)$ is a $\mathbf{C}$-linear space generated by 1 and x.

Example 2.2 Compute $\mathcal{P}(A)$ where

$$A = x^2 \frac{d^2y}{dx^2} - 6y.$$

The indicial polynomial

$$E_\infty(A) = z(z-1) - 6 = (z-3)(z+2).$$

Hence, Theorem **??** asserts that the degree of polynomial solutions of A will not exceed 3. Let a polynomial solution be

$$P = p_3x^3 + p_2x^2 + p_1x + p_0$$

where $p_3, p_2, p_1, p_0 \in \mathbf{C}$. Substituting P for y in A, we get a linear system

$$\{p_2 = 0,\ p_1 = 0,\ p_0 = 0\}$$

Therefore, all the polynomial solutions of A are in the form

$$P = p_3x^3$$

$$q = (x-\alpha_1)^{d_1} \cdots (x-\alpha_m)^{d_m}$$

where p_3 is an arbitrary element in $\mathbf{C}$.

3. Rational function solutions of a linear homogeneous ode

Let $\mathcal{R}(A)$ be the set of rational solutions of $A(y) = 0$ given by (**??**). The set $\mathcal{R}(A)$ is a $\mathbf{C}$-linear subspace of sol(A). consequently of finite dimension

Write a rational function h in $\mathbf{C}(x)$ in its canonical form:

$h = \frac{a}{q}$, write $a = pq + r$ using Euclidean division

$$h = p + \frac{r}{q}$$

where $p, q, r \in \mathbf{C}[x]$, $\deg q > \deg r$, and $\gcd(p, r) = 1$. We call p and r/q the polynomial and fraction part of h, respectively. Let α be a pole of ~~h~~ r/q (root of q) with order $-N$, which we denote as $\nu_\alpha(h) = -N$ (valuation). Then h can be represented by Laurent's series as

$N = \deg q - \deg r$ — polynomial! $k_N(\alpha) \neq 0$

$$h = \frac{k_N}{(x-\alpha)^N} + \frac{k_{N-1}}{(x-\alpha)^{N-1}} + \cdots + \text{higher terms}. \tag{3}$$

Suppose that h is a solution of (**??**). Substituting h for y in (**??**) yields

$$\begin{aligned}
&\Big(\underbrace{k_N(-1)^n \textstyle\prod_{i=0}^{n-1}((N+i)(x-\alpha)^{-(N+n)}}_{a_n\cdot} + \text{higher terms}\Big) \\
&+ \tfrac{a_{n-1}}{\cancel{a_n}} \Big(k_N(-1)^{n-1} \textstyle\prod_{i=0}^{n-2}(N+i)(x-\alpha)^{-(N+n-1)} + \text{higher terms}\Big) \\
&+ \cdots \\
&+ \tfrac{a_1}{\cancel{a_n}} \Big(k_N(-N)(x-\alpha)^{-(N+1)} + \text{higher terms}\Big) \\
&+ \tfrac{a_0}{\cancel{a_n}} \Big(k_N(x-\alpha)^N + \text{higher terms}\Big) = 0.
\end{aligned} \tag{4}$$

It follows that α must be a root of a_n, for otherwise, the term in (**??**)

$$k_N(-1)^n \prod_{i=0}^{n-1} (N+i)(x-\alpha)^{-(N+n)}$$

$$\partial_x^i \left(\frac{1}{(x-\alpha)^m}\right) = (-m)(-m-1)\cdots(-m-i+1) \cdot \frac{1}{(x-\alpha)^{m+i}}$$

All poles α of the fractional part are roots of a_n.

$(a\partial + b)h = 0 \Rightarrow a \cdot \frac{K_N' \cdot (x-\alpha)^N - N \cdot K_N}{(x-\alpha)^{N+1}} + b \cdot \frac{K_N}{(x-\alpha)^N} \cdots = 0$

$\partial \frac{p}{q^k} = \partial(p \cdot q^{-k}) = p' \cdot q^{-k} + p(-k) q^{-k-1} = \frac{p'q - kp}{q^{k+1}}$

The highest order term is $-a \frac{N \cdot K_N}{(x-\alpha)^{N+1}}$, α not a root of K_N, α is a root of $a(x)$

would not be canceled.

Define the *order drop of B at the point α* to be

$$\mu_\alpha(A) = \max_{0 \le j \le n} \left(j - \nu_\alpha \left(\frac{a_j}{a_n} \right) \right)$$

$\mu_\alpha(a\partial + b) = \max(-\nu_\alpha(b) + \nu_\alpha(a), 1)$

and the *leading set of B at the point α* to be

$$\lambda_\alpha(A) = \left\{ j \,|\, 0 \le j \le n \text{ and } j - \nu_\alpha \left(\frac{a_j}{a_n} \right) = \mu_\alpha(A) \right\}.$$

Let $m = \mu_\alpha(A)$, and, for each $j \in \lambda_\alpha(A)$, write

$$\frac{a_j}{a_n} = \frac{k_{jm}}{(x-\alpha)^m} + \frac{k_{j,m-1}}{(x-\alpha)^{m-1}} + \text{higher terms}.$$

If (**??**) holds, then

$$\sum_{j \in \lambda_\alpha(A)} (-1)^j k_{mj} \prod_{i=0}^{j-1} (N+i) = 0.$$

So, we define the *indicial equation* of B at the point α to be

$$E_\alpha(A) = \sum_{j \in \lambda_\alpha(A)} (-1)^j k_{mj} \prod_{i=0}^{j-1} (z+i) = 0,$$

order of the pole α

where z is a new indeterminate. The above discussion implies that $|\nu_\alpha(h)|$ is not greater than the maximal integral root of $E_\alpha(A)$.

Using the same method as in Section **??**, we conclude that the degree bound given in Theorem **??** is also valid for the degree of the polynomial part p of the rational function h. The results of this section is summarized in

Theorem 3.1 Let a nonzero rational function in $\mathbf{C}(x)$ be

$$h = p + \frac{r}{q},$$

where $p, q, r \in \mathbf{C}[x]$ with $\deg q > \deg r$. If h is a solution of A given in (**??**), then the following hold.

- The degree of p is not greater than the maximal integral root of $E_\infty(A)$.
- If $r \neq 0$, then a pole of h is a root of a_n. Moreover, if α is a pole of h, then the order of h at the pole α is no less than the negative of the maximal integral root of $E_\alpha(A)$.

!

Example 3.1 Let A be the same as that given in Example **??**. If

$$h = p + \frac{r}{q}$$

$A = x^2 \partial_x^2 - 6$

2.?

is a solution of A, then 3 is a bound for the degree of p, as seen in Example **??**. By Theorem **??** the denominator q must be x^k, for some non-negative integer k. To compute a bound for k, we compute

$$\mu_0(A) = 2 \quad \text{and} \quad \lambda_0(A) = \{2, 0\}.$$

So the indicial equation is

$$E_0(A) = z(z+1) - 6 = (z-2)(z+3).$$

It then follows from Theorem **??** that k is at most 2. Therefore, we make an ansatz that

$$h = p_3x^3 + p_2x^2 + p_1x + p_0 + \frac{r_1x + r_0}{x^2}.$$

Substituting h for y in A yields

$$\{p_2 = 0,\ p_1 = 0,\ p_0 = 0,\ r_1 = 0.\}$$

It follows that

$$h = p_3x^3 + \frac{r_0}{x^2},$$

where p_3 and r_0 are arbitrary elements in $\mathbf{C}$.

Example 3.2 Let

$$A = (x^2-1)\frac{d^2y}{dx^2} + 8x\frac{dy}{dx} + 12y = 0.$$

Compute

$$\mu_\infty(A) = 0 \quad \text{and} \quad \lambda_\infty(A) = \{2, 1, 0\}$$

Thus the indicial equation at infinity is

$$E_\infty(A) = z(z-1) + 8z + 12 = (z+4)(z+3).$$

There exists no positive integral root. Hence, any rational solutions of A must be ~~true~~ proper fractions

A rational function solution h of A may have two poles at 1 and -1. Let us now estimate the order of the pole $x = 1$. Compute

$$\mu_1(A) = 2 \quad \text{and} \quad \lambda_1(A) = \{2, 1\}.$$

So the indicial equation is

$$E_1(A) = z(z+1) - 4z = z(z-3).$$

Therefore, the order of h at the pole $x = 1$ is no less than -3, so is the order at $x = -1$ by the same reason. We then assume that

$$h = \frac{r_5x^5 + r_4x^4 + r_3x^3 + r_2x^2 + r_1x + r_0}{(x-1)^3(x+1)^3},$$

ansatz

where the r's are unspecified constants. Substituting h for y in $A(y)$, we get

$$6r_5x^5 + 2r_4x^4 - 20r_5x^3 - 12r_4x^2 + (-6r_3 + 2r_1)x - 2r_2 + 6r_0 = 0.$$

Thus, $r_5 = r_4 = 0$, $r_1 = 3r_3$ and $r_2 = 3r_0$. Hence

$$\mathcal{R}(A) = \left\{ \frac{r_3x^3 + 3r_0x^2 + 3r_3x + r_0}{(x-1)^3(x+1)^3} \,|\, r_3, r_0 \in \mathbf{C} \right\}.$$

$$\left((x^2-1)\partial^2 + 8x\partial + 12\right) = (A_1\partial + A_0)(B_1\partial + B_0)$$

$$= \underbrace{A_1B_1'}_{x^2-1}\partial^2 + \underbrace{(A_1\cancel{B_0'} + A_0B_1)}_{8x}\partial + \underbrace{A_0B_0}_{12}$$

$\deg A_1 + \deg B_1 = 3$

$Af = 0,\ Bf = 0 \Rightarrow \underbrace{GCRD(A,B)}_{C} f = 0$

indeed $C = \hat{A}A + \hat{B}B$, then $Cf = (\hat{A}A + \hat{B}B)f$

$= \hat{A}\underbrace{Af}_{=0} + \hat{B}\underbrace{Bf}_{=0} = 0$

4. Structure of Hyperexponential Solutions of Linear Homogeneous Ode's

We denote by $\mathcal{H}$ the union of the set of all hyperexponentials and $\{0\}$ and by $\mathcal{H}(A)$ the set of solutions of $A(y) = 0$ (given by (**??**)) that are in $\mathcal{H}$. It holds that

$$\mathcal{P}(A) \subset \mathcal{R}(A) \subset \mathcal{H}(A) \subset \mathcal{S}(A).$$

In this section we describe the structure of $\mathcal{H}(A)$.

Example 4.1 *Let $A(y) = y'' - y'$. It is easy to see that $1, \exp x \in \mathcal{H}(A)$, but $1 + \exp x$ is not in $\mathcal{H}(A)$, for otherwise*

$$\frac{(1 + \exp x)'}{(1 + \exp x)} \in \mathbf{C}(x)$$

so that $\exp x \in \mathbf{C}(x)$ too.

This example tells us that $\mathcal{H}(A)$ is not a linear space over $\mathbf{C}$.

Two nonzero elements f and g of $\mathcal{H}$ are said to be *similar* if $f/g \in \mathbf{C}(x)$. The element 0 is only similar to itself. This relation is denoted by "$\sim$".

Exercise Prove that "$\sim$ "is an equivalence relation on $\mathcal{H}$. Design an algorithm to decide whether two hyperexponentials are similar.

For nonzero $f \in \mathcal{H}$, denoted by $\mathcal{H}_f$ the union of the set of hyperexponentials similar to f and $\{0\}$. The next lemma is obvious.

Lemma 4.1 *For every nonzero $f \in \mathcal{H}$, $\mathcal{H}_f$ is a linear space over $\mathbf{C}$.*

A fundamental property of hyperexponentials is given in the next lemma whose proof is an easy induction.

Lemma 4.2 *If $f \in \mathcal{H}$, $f^{(n)} = rf$, for some $r \in \mathbf{C}(x)$.*

Lemma 4.3 *If nonzero $f_1, \ldots, f_k$ belong to $\mathcal{H}$, any two of which are not similar, then*

1. *$f_1, \ldots, f_n$ are linearly independent over $\mathbf{C}$.*
2. *$H_{f_1} + \cdots + H_{f_k}$ is direct.*

Proof We proceed by induction on k. If $k = 2$, then the lemma holds for $\mathcal{H}_{f_1} \cap \mathcal{H}_{f_2}$ is $\{0\}$. Assume that the lemma is true for $k - 1$. Suppose that, on the contrary,

$$f_k = c_1 f_1 + c_2 f_2 + \cdots + c_{k-1} f_{k-1}$$

where $c_1, c_2, \ldots, f_{k-1} \in \mathbf{C}$. By differentiation we get a linear system

$$\underbrace{\begin{pmatrix} f_1 & f_2 & \cdots & f_{k-1} \\ f_1' & f_2' & \cdots & f_{k-1}' \\ . & . & \cdots & . \\ f_1^{(k-2)} & f_2^{(k-2)} & \cdots & f_{k-1}^{(k-2)} \end{pmatrix}}_{W} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{k-1} \end{pmatrix} = \begin{pmatrix} f_k \\ f_k' \\ \vdots \\ f_k^{(k-2)} \end{pmatrix}. \qquad (5)$$

The determinant $\det(W)$ is nonzero because $f_1, \ldots, f_{k-1}$ are linearly independent over $\mathbf{C}$. By Lemma **??** we find that $\det(W) = r f_1 \cdots f_{k-1}$, where r is a nonzero element of $\mathbf{C}(x)$. Without loss of generality, suppose that $c_1 \neq 0$. Solving (**??**) to get c_1, we find

$$c_1 = \frac{\det \begin{pmatrix} f_k & f_2 & \cdots & f_{k-1} \\ f_k' & f_2' & \cdots & f_{k-1}' \\ \cdot & \cdot & \cdots & \cdot \\ f_k^{(k-2)} & f_2^{(k-2)} & \cdots & f_{k-1}^{(k-2)} \end{pmatrix}}{\det W} = \frac{r_1 f_k f_2 \cdots f_{k-1}}{r f_1 f_2 \cdots f_{k-1}} = \frac{r_1}{r} \frac{f_k}{f_1}$$

for some $r_1, r \in \mathbf{C}(x)$. Thus, $f_1 \sim f_k$, a contradiction. This proves the first assertion. The second assertion follows. □

Denote by $\mathcal{H}_f(A)$ the intersection $\mathcal{S}(A) \cap \mathcal{H}_f$, for nonzero $f \in \mathcal{H}$.

Theorem 4.4 *Let $A(y)$ be given by (**??**). Then $\mathcal{H}(A)$ is either $\{0\}$ or a union of linear spaces*

$$H_{f_1}(A) \cup \cdots \cup H_{f_k}(A)$$

for some nonzero $f_1, \ldots, f_k \in H$. Moreover

$$\dim H_{f_1}(A) + \cdots + \dim H_{f_1}(A) \leq n.$$

Proof Assume that $\mathcal{H}(A) \neq \{0\}$. Let F be a maximal element (w.r.t. set inclusion) of subsets of $\mathcal{H}(A)$ whose elements are not similar to each other. The set F must be finite by Lemma **??**. By the maximality of F, we find that

$$\mathcal{H}(A) = \mathcal{H}_{f_1}(A) \cup \cdots \cup \mathcal{H}_{f_k}(A).$$

Since $\mathcal{H}_{f_1}(A) + \cdots + \mathcal{H}_{f_k}(A)$ is contained in $\mathcal{S}(A)$,

$$\dim H_{f_1}(A) + \cdots + \dim H_{f_1}(A) \leq n$$

by the second assertion of Lemma (**??**). □

A set F of $\mathcal{H}$ is called a base set of $\mathcal{H}(A)$ if

1. any two elements of F are not similar.
2. If $g \in \mathcal{H}(A)$, then there exists $f \in F$ such that $g \sim f$.
3. If $f \in F$, then there exists $g \in \mathcal{H}(A)$ such that $f \sim g$.

Please notice that elements of F are not necessarily in $\mathcal{H}(A)$.

The algorithm for computing $\mathcal{H}(A)$ given in the next section has two steps:

1. Compute a finite set F of hyperexponentials containing a base set of $\mathcal{H}(A)$.
2. For each element f of F, compute a basis for $\mathcal{H}_f(A)$..

5. Hyperexponential solutions of linear ode's of order two

In this section we let

$$A(y) = y'' + ay' + b$$

where $a, b \in \mathbf{C}(x)$. By the substitution $y = \exp(\int z)$, we get

$$R_1(z) = z' + z^2 + az + b$$

which is called the Riccati equation of order one. Computing hyperexponential solutions of $A(y) = 0$ is equivalent to computing rational solutions of $R_1(z) = 0$. A function z is a rational solution of $R_1(z) = 0$ if and only if $\exp(\int z)$ is a hyperexponential solution of $A(y) = 0$.

Now we describe the structure of rational solutions of $R_1(z) = 0$.

Theorem 5.1 *The equation $R_1(z) = 0$ may have*

1. *no rational solutions; or*
2. *one rational solution; or*
3. *two distinct rational solutions; or*
4. *infinity many solutions depending on an unspecified constant.*

Proof By Theorem **??**, $\mathcal{H}(A)$ is either $\{0\}$, which corresponds to the first case, or has a base set F containing at most two elements. If F contains only one element f, then $\mathcal{H}_f(A)$ may be either of dimension 1 or 2. In the former case $R_1(z)$ has only one rational solution. In the latter case, $\dim \mathcal{H}_f(A) = 2$, so $\mathcal{H}_f(A)$ has a basis $\{r_1 f, r_2 f\}$ where $r_1, r_2 \in \mathbf{C}(x)$. Hence, all the hyperexponential solutions of $A(y) = 0$ can be written as $(c_1 r_1 + c_2 r_2) f$, where c_1 and c_2 are unspecified constants. Thus, all the rational solutions of $R_1(z) = 0$ can be written as

$$\frac{((c_1 r_1 + c_2 r_2) f)'}{(c_1 r_1 + c_2 r_3) f} = \frac{f'}{f} + \frac{c_1 r_1' + c_2 r_2'}{c_1 r_1 + c_2 r_2} = \frac{f'}{f} + \frac{c r_1' + r_2'}{c r_1 + r_2}$$

where $c = c_1/c_2$ is an unspecified constant in $\mathbf{C} \cup \{\infty\}$. This is the fourth case. If $F = \{f_1, f_2\}$, where f_1 and f_2 are not similar, then

$$\mathcal{H}(A) = \mathcal{H}_{f_1}(A) \cup \mathcal{H}_{f_2}(A).$$

Since $\dim H_{f_1}(A) = \dim H_{f_2}(A) = 1$, there are r_1 and r_2 in $\mathbf{C}(x)$ such that $H_{f_1}(A) = \{c r_1 f_1 \mid c \in \mathbf{C}\}$ and $H_{f_2}(A) = \{c r_2 f_2 \mid c \in \mathbf{C}\}$. Thus, $R_1(z)$ has only two rational solutions

$$\frac{(r_1 f_1)'}{r_1 f_1} \quad \text{and} \quad \frac{(r_2 f_2)'}{r_2 f_2}.$$

This corresponds to the third case. □

The next lemma will be used later.

Lemma 5.2 *For every $f \in \mathbf{C}(x)$, $R_1(z + f)$ is a Riccati equation of order* 1.

Proof The proof is done by a straightforward calculation. □

This lemma says that the structure of a Riccati equation (of order 1) is invariant under the group of translation.

We introduce the following notation. For nonzero $P \in \mathbf{C}[x]$ and $\alpha \in \mathbf{C}$, $\mathrm{mult}_\alpha(P)$ is the maximal nonnegative integer n such that $(x-\alpha)^n \mid P$.

Write rational functions a and b in their standard form

$$a = a_1 + \frac{a_2}{a_3} \quad \text{and} \quad b = b_1 + \frac{b_2}{b_3}$$

where the a's are polynomials such that $\deg a_2 < \deg a_3$ and $\gcd(a_2, a_2) = 1$, the same holds for the b's.

Lemma 5.3 *A rational solution of $R_1(z) = 0$ can be written as*

$$S = P + \frac{R}{Q} + \frac{G'}{G} \tag{6}$$

where

1. *$P \in \mathbf{C}[x]$ and $\deg P \leq \max(\deg a_1, \deg b_1/2)$;*
2. *$Q, R \in \mathbf{C}[x]$, $\deg R < \deg Q$, $\gcd(R, Q) = 1$, and if $Q(\alpha) = 0$, then $a_3 b_3(\alpha) = 0$ and $\mathrm{mult}_\alpha(Q) \leq \max(\mathrm{mult}_\alpha(a), \mathrm{mult}_\alpha(b)/2)$.*
3. *$G \in \mathbf{C}[x]$ and G has no common root with $a_3 b_3$.*

Proof Let a rational solution of $R_1(z)$ be

$$s = p + \frac{r}{q}$$

where $p, q, r \in \mathbf{C}[x]$, $\deg r < \deg q$ and $\gcd(r, q) = 1$. Substituting h into $R_1(z)$ yields

$$\begin{aligned} R_1(s) &= \left(p' + p^2 + a_1 p + b_1 + 2\frac{pr}{q} + \frac{a_2 p}{a_3} + \frac{a_1 r}{q}\right) + \left(\left(\frac{r}{q}\right)' + \frac{r^2}{q^2} + \frac{a_2}{a_3}\frac{r}{q} + \frac{b_2}{b_3}\right) \\ &= p^2 + a_1 p + b_1 + h \end{aligned} \tag{7}$$

where h is a rational function whose polynomial part has degree less than $\max(\deg p^2, \deg a_1 p, \deg b_1)$. In order to make $R_1(s) = 0$, we find that $\deg p^2$ cannot be greater than both $\deg p a_1$ and $\deg b_1$. It follows that $\deg p \leq \max(\deg a_1, \deg b_1/2)$.

Let α be a pole of h, that is $h(\alpha) = 0$. Compute the Laurent expansion of h about α in $\mathbf{C}$, we get

$$h = \frac{h_N}{(x-\alpha)^N} + \text{higher terms},$$

where $N = \mathrm{mult}_\alpha(q) > 0$ and h_N is a non-zero element in $\mathbf{C}$. Assume that

$$a = \frac{a_u}{(x-\alpha)^u} + \text{higher terms} \quad \text{and} \quad b = \frac{b_v}{(x-\alpha)^v} + \text{higher terms},$$

where u and v are nonnegative integers. Substituting h into $R_1(z)$ yields

$$R_1(s) = \frac{-Nh_N}{(x-\alpha)^{N+1}} + \frac{h_N^2}{(x-\alpha)^{2N}} + \frac{a_u h_N}{(x-\alpha)^{u+N}} + \frac{b_v}{(x-\alpha)^v} + \text{higher terms.} \tag{8}$$

First, assume that $N > 1$. In order to force $R_1(z) = 0$, we find that $2N$ cannot be greater than both $N + u$ and v. Hence,

$$N \le \max\left(u, \frac{v}{2}\right). \tag{9}$$

As $N > 1$, either $u > 1$ or $v > 2$. It follows that If h has a pole with order greater than 1, the pole must be either a pole of a or a pole of b. Now we assume that $N = 1$.

Now let $N = 1$. Then (**??**) becomes

$$\frac{-h_1}{(x-\alpha)^2} + \frac{h_1^2}{(x-\alpha)^2} + \frac{a_u h_1}{(x-\alpha)^{u+1}} + \frac{b_v}{(x-\alpha)^v} + \text{higher terms} = 0. \tag{10}$$

If $h_1 \ne 1$, then α is a pole of either a or b. Moreover, either $u \ge 1$ or $v \ge 2$. Hence, (**??**) still holds. If $h_1 = 1$, then

$$\frac{-1}{(x-\alpha)^2} + \frac{1}{(x-\alpha)^2} + \frac{a_u h_1}{(x-\alpha)^{u+1}} + \frac{b_v}{(x-\alpha)^v} + \text{higher terms} = 0. \tag{11}$$

This shows that if α is a pole of h such that $a_3b_3(\alpha) \ne 0$ then the Laurent expansion of h is

$$\frac{1}{x-\alpha} + \text{higherterms.}$$

By partial fraction decomposition, we find that

$$s = P + \frac{R}{Q} + \sum_{i=1}^{m} \frac{1}{x - c_i}$$

where P, Q, and R are the same as those in (**??**), and $c_i \in \mathbf{C}$, $a_3b_3(c_i) \ne 0$, for $i = 1, \dots, m$. If $m = 0$, then the last sum is understood to be 0. Setting $G = \prod_{i=1}^{m}(x - c_i)$ proves (**??**). □

In view of Lemma **??**, a root of Q is called a normal pole and a root of G is called an extraneous pole. All normal poles are roots of $a_3b_3(x)$ and no extraneous pole is a root of $a_3b_3(x)$. Let $m_\infty = \max\left(\deg a_1, \frac{1}{2}\deg b_1\right)$ and $m_\alpha = \max\left(\text{mult}_\alpha(a_1), \text{mult}_\alpha(b_1)\right)$, where $a_3b_3(\alpha) = 0$. Decomposing **??** into its full partial fraction, we obtain

Theorem 5.4 *A rational solution of $R_1(z) = 0$ can be written as*

$$\sum_{i=0}^{m_\infty} p_i x^i + \sum_{\alpha,\,\text{any normal pole}} \sum_{j=1}^{m_\alpha} \frac{r_{\alpha j}}{(x-\alpha)^j} + \frac{G'}{G} \tag{12}$$

where $p_i, r_{\alpha j} \in \mathbf{C}$ and $G \in \mathbf{C}[x]$.

We study how to compute p_i, $r_{\alpha j}$, and G.

Lemma 5.5 *If s given in (**??**) is a solution of $R_1(z) = 0$, then we can compute polynomials I_{k_∞} and I_{k_α} in $\mathbf{C}[T]$ such that $I_{k_\infty}(p_{k_\infty}) = 0$ and $I_{k_\alpha}(r_{k_\alpha}) = 0$, where α is a root of $a_3b_3(x)$.*

Proof Substituting s into $R_1(z)$ yields a rational function whose polynomial part is

$$\begin{array}{rcl} R_1(s) & = & (p_{k_\infty}^2 x^{2k_\infty} + \text{lower terms}) \\ & & +(p_{k_\infty} \mathrm{lc}(a_1) x^{\deg a_1 + k_\infty} + \text{lower term}) \\ & & +(\mathrm{lc}(b_1) x^{\deg b_1} + \text{lower terms}). \end{array}$$

In order to make this expression zero, we find that p_{k_∞} is a root of a univariate polynomial whose degree less than 3.

Similarly, substituting s into $R_1(z)$ and looking at the powers of $(x-\alpha)$, we get

$$\begin{array}{rcl} R_1(s) & = & \left(\frac{-k_\alpha r_{\alpha k_\alpha}}{(x-\alpha)^{k_\alpha+1}} + \frac{r_{\alpha,k_\alpha}^2}{(x-\alpha)^{2k_\alpha}} + \text{higher terms} \right) \\ & & + \left(\frac{a_u r_{\alpha,k_\alpha}}{(x-\alpha)^{u+k_\alpha}} + \text{ higher terms} \right) \\ & & + \left(\frac{b_v}{(x-\alpha)^v} + \text{higher terms} \right) = 0, \end{array}$$

where $u = \mathrm{mult}_\alpha(a_3)$, $v = \mathrm{mult}_\alpha(b_3)$ and $a_u, b_v \in \mathbf{C}$. Thus r_{α,k_α} satisfies a univariate polynomial with degree ≤ 2. □

Before presenting a complete algorithm, we see some examples.

Example 5.1 *Compute the rational solutions of the equation*

$$R_2(z) = z' + z^2 + \frac{1}{x} z - \frac{1}{x^2} = 0. \tag{13}$$

Any polynomial part cannot exist by the first assertion of Lemma **??**. *The order of a rational solution at the pole $x = 0$ is bounded by -1. Therefore, we make an ansatz*

$$y = \frac{r_1}{x} + \frac{G'}{G}$$

*where $r_1 \in \mathbf{C}$ and $G \in \mathbf{C}[x]$. Substituting it into (**??**), we obtain*

$$r_1^2 = 1.$$

Substitute $\frac{-1}{x} + \frac{G'}{G}$ for z in $R_1(z)$ to get

$$G'' - \frac{1}{x} G' = 0$$

whose polynomial solutions are $(c_1 + c_2 x^2)$, where $c_1, c_2 \in \mathbf{C}$. Hence, $R_1(z)$ has rational solutions

$$\left\{ -\frac{1}{x} + \frac{2cx}{1 + cx^2} \mid c \in \mathbf{C} \cup \{\infty\} \right\}$$

which contains all rational solutions by Theorem **??**,

Example 5.2 *Find rational solutions of*

$$R_1(z) = z' + z^2 - \left(x^2 - 2x + 3 + \frac{1}{x} + \frac{7}{4x^2} - \frac{5}{x^3} + \frac{1}{x^4} \right).$$

By Lemma **??** *we get a rational solution of* $R_1(z)$ *can be written as:*

$$s = p_1 x + p_0 + \frac{r_2}{x^2} + \frac{r_1}{x} + \frac{G'}{G},$$

where $p_1, p_0, r_1, r_0 \in \mathbf{C}$ *and* $G \in \mathbf{C}[x]$. *By Theorem* **??** *we get*

$$p_1^2 - 1 = 0 \quad \text{and} \quad r_1^2 - 1 = 0.$$

Thus, we have four possibilities $p_1 = \pm 1$ *and* $r_1 = \pm 1$.

If $p_1 = 1$ *and* $r_1 = 1$, *then*

$$s = x + \frac{1}{x^2} + p_0 + \frac{r_1}{x} + \frac{G'}{G}.$$

If s *is a solution of* $R_1(z)$, *for some* $p_0, r_1 \in \mathbf{C}$, *then*

$$R_1\left(u + x + \frac{1}{x^2}\right) = u' + u^2 - \left(2 + \frac{2}{x^2}\right)u + 2x - 2 + \frac{1}{x} - \frac{7}{4x^2} + \frac{3}{x^3}$$

has a solution in the form

$$s_1 = p_0 + \frac{r_1}{x} + \frac{G'}{G}.$$

Substituting s_1 *into* $R_1\left(z - x - \frac{1}{x^2}\right)$, *we find*

$$2p_0 + 2 = 0 \quad \text{and} \quad 2r_1 + 3 = 0.$$

If

$$s = \underbrace{x - 1 + \frac{1}{x^2} - \frac{3}{2x}}_{f} + \frac{G'}{G}$$

is a solution of $R_1(z) = 0$, *then* $\frac{G'}{G}$ *is a solution of* $R_1\left(f + \frac{G'}{G}\right)$, *which is simplified to be a second-order linear ode*

$$L = G'' + \left(2x - 2 - \frac{3}{x} + \frac{2}{x^2}\right)G' + \left(\frac{-4+4}{x}\right)G = 0.$$

$\mathcal{P}(L) = \{c(x^2 - 1)\}$. *Thus,*

$$x - 1 + \frac{1}{x^2} - \frac{3}{2x} + \frac{2x}{x^2 - 1}$$

is a solution of $R_1(z)$. *In other words, a hyperexponential solution of*

$$y'' - \left(x^2 - 2x + 3 + \frac{1}{x} + \frac{7}{4x^2} - \frac{5}{x^3} + \frac{1}{x^4}\right)y = 0$$

is

$$\exp\left(\int x^2 - 2x + 3 + \frac{1}{x} + \frac{7}{4x^2} - \frac{5}{x^3} + \frac{1}{x^4}\right) = x^{-3/2}(x^2 - 1)\exp\left(-\frac{1}{x} + \frac{x^2}{2} - x\right).$$

If $p_1 = 1$ *and* $r_2 = -1$, *then*

$$R_1\left(x + \frac{-1}{x^2} + p_0 + \frac{r_1}{x} + \frac{G'}{G}\right) = 0$$

gives

$$2p_0 + 2 = 0 \quad \text{and} - 2r_1 + 7 = 0.$$

$$R_1\left(x + \frac{-1}{x^2} - 1 + \frac{7}{2x} + \frac{G'}{G}\right) = 0$$

gives

$$L = G'' + \left(2x - 2 - \frac{7}{x} - \frac{2}{x^2}\right)G' + \left(-8 + \frac{4}{x} + \frac{16}{x^2} + \frac{14}{x^3}\right).$$

$\mathcal{P}(L) = \{0\}$. *Hence, there is no rational solution with* $p_1 = 1$ *and* $r_2 = -1$.

Similarly, we can check that there are no rational solutions with $p_1 = -1, r_2 = 1$ *and* $p_1 = -1, r_2 = -1$. *Hence,* $R_1(z)$ *has only one rational solution.*

Example 5.3 *Find hyperexponential solutions of Bessel's equation*

$$y'' = \underbrace{\left(\frac{4n^2 - 1}{4x^2} - 1\right)}_{b} y, \quad n \in \mathbf{C}.$$

The corresponding Riccati equation is

$$R_1(z) = z' + z^2 - b = 0.$$

Our ansatz is

$$s = p + \frac{r}{x} + \frac{G'}{G},$$

where $p, r \in \mathbf{C}$ *and* $G \in \mathbf{C}[x]$. *Substituting* s *into* $R_1(z)$ *yields*

$$p^2 + 1 = 0 \quad \text{and} \quad \left(r - \frac{1}{2}\right)^2 - n^2 = 0.$$

Hence,

$$s = \pm i_1 + \frac{\frac{1}{2} \pm n}{x} + \frac{G'}{G}.$$

Consider the case

$$s = i + \frac{\frac{1}{2} + n}{x} + \frac{G'}{G}.$$

we get

$$L = G'' + 2\left(\frac{\frac{1}{2} + n}{x} + i\right)G' + 2i\frac{\frac{1}{2} + n}{x}G.$$

If a polynomial G *is a solution, then* $\deg G = -\frac{1}{2} - n = m$. *Hence,* m *is a nonnegative integer if* L *has nontrivial polynomial solutions. Under this assumption, we find that*

$$G = \sum_{j=0}^{m} \frac{1}{(-2i)^{m-j}} \frac{(2m-j)!}{j!(m-j)!} x^j.$$

Exercise. Find other hyperexponential solutions of Bessel's equation.

Denote by N_1 the set of normal poles of $R_1(z)$. The algorithm for computing rational solutions of $R_1(z)$ is outlined in the following steps:

1. Compute a degree bound m for the polynomial part, and an order bound m_α for $\alpha \in N_1$ (see Lemma **??**).

2. Make the following ansatz for the rational solutions

$$s = \sum_{i=0}^{m} p_i x^i + \sum_{\alpha \in N_1} \sum_{j=1}^{m_\alpha} \frac{r_{\alpha j}}{(x-\alpha)^j} + \frac{G'}{G},$$

where $p_i, r_{\alpha j} \in \mathbf{C}$ and $G \in \mathbf{C}[x]$ are to be determined (see Theorem **??**).

3. Substitute s for z in $R_1(z)$ to determine p_m and $r_{\alpha j}$, for $\alpha \in N_1$, up to finitely many possibilities by solving univariate polynomials of degree two.

4. For each possibility $\tilde{p}_m$ and $\tilde{r}_{\alpha m_\alpha}$ in N_1, substitute $s(\tilde{p}_m, \cdots, \tilde{r}_{\alpha m_{alpha}}, \cdots)$ for z into $R_1(z)$ to determine p_{m-1}, $r_{\alpha, m_\alpha - 1}$ up to finitely many possibilities by solving univariate polynomial equations of degrees no greater than 2. Repeat this process until all the p_i's and $r_{\alpha j}$'s are determined up to finitely many possibilities (see Lemmas **??** and **??**).

5. For each possibility $\tilde{p}_i$ for p_i $(0 \le i \le m)$ and $\tilde{r}_{\alpha j}$ for $r_{\alpha j}$ $(\alpha \in N_1$ and $1 \le j \le m_\alpha)$, substitute

$$s(\tilde{p}_m, \ldots, p_0, \ldots, \tilde{r}_{\alpha, m_\alpha}, \ldots, r_{\alpha 1}, \ldots) \sum_{i=0}^{m} \tilde{p}_i x^i + \sum_{\alpha \in N_1} \sum_{j=1}^{m_\alpha} \frac{\tilde{r}_{\alpha j}}{(x-\alpha)^j} + \frac{G'}{G}$$

into $R_1(z)$ to get a second-order linear ode $L(G)$. Find $\mathcal{P}(L)$. Each nonzero $g \in \mathcal{P}(L)$ leads to a rational solution

$$\sum_{i=0}^{m} \tilde{p}_i x^i + \sum_{\alpha \in N_1} \sum_{j=1}^{m_\alpha} \frac{\tilde{r}_{\alpha j}}{(x-\alpha)^j} + \frac{g'}{g}.$$

(*NOTE: The last step is interrupted if we have found either two distinct rational solutions or a family of rational solutions depending on one constant, according to Theorem* **??**).

6. Hyperexponential solutions of linear ode's

This section presents an algorithm for computing hyperexponential solutions of linear homogeneous ode's.

Lemma 6.1 *If $y = \exp\left(\int z\right)$, then*

$$y^{(n)} = B_n(z) y$$

where $B_n(z)$ is a polynomial in $z, z', \ldots, z^{(n-1)}$. Moreover, $B_n(z)$ satisfies the recursion

$$B_0(z) = 1, \quad B_n(z) = B_{n-1}(z)' + z B_{n-1}(z).$$

Proof The lemma is true for $n = 1$. Assume that it is true for $n - 1$. Then

$$y^{(n)} = (B_{n-1}(z)y)' = B_{n-1}(z)'y + zB_{n-1}(z)y.$$

The proof is completed. $\square$

We call B_n *the* $(n-1)$*th Riccati operator* since $B_n(z$ is a polynomial in $z, \ldots, z^{(n-1)}$.

Finding hyperexponential solutions of $A(y)$ given in (**??**) is equivalent to finding rational solutions of $A\left(exp\left(\int z\right)\right) = 0$, which is

$$R_{n-1}(z) = a_n B_n(z) + a_{n-1}B_{n-1}(z) + \cdots + a_0 B_0(z) = 0 \tag{14}$$

where $a_n, a_{n-1}, \ldots, a_0 \in \mathbf{C}[x]$. The equation $R_{n-1}(z)$ is called the $(n-1)$th Riccati equation associated with $A(y)$. An easy induction shows that $R_{n-1}(f+z) = 0$ is also an $(n-1)$th-order Riccati equation, where $f \in \mathbf{C}(x)$.

Next, we analyze leading powers of $B_n(u)$, where u is either a polynomial or a Laurent series.

Lemma 6.2 *1. If* $P = p_m x^m +$ lower terms, *where* $m \geq 0$, p_m *and other coefficients are in* $\mathbf{C}$, *then*

$$B_n(P) = p_m^n x^{mn} + \text{lower terms}.$$

2. If $Q = \frac{r_m}{(x-\alpha)^m} +$ higher terms, *where* $m > 1$, r_m *and other coefficients are in* $\mathbf{C}$, *then*

$$B_n(Q) = \frac{r_m^n}{x^{mn}} + \text{higher terms}.$$

3. If $Q = \frac{r_1}{x-\alpha} +$ higher terms, *then*

$$B_n(Q) = \frac{\prod_{i=0}^{n-1}(r_1 - i)}{(x-\alpha)^n} + \text{higher terms}.$$

Proof The first assertion holds for $n = 1$. Assume that it holds for $n - 1$. We compute

$$\begin{array}{rcl} B_n(P) & = & (B_{n-1}(P))' + PB_{n-1}(P) \quad \text{by Lemma } \mathbf{??} \\ & = & \left(p_m^{n-1}x^{m(n-1)} + \cdots\right)' + (p_m x^m + \cdots)(p_m^{n-1}x^{m(n-1)} + \cdots) \\ & = & p_m^n x^{mn} + \cdots. \end{array}$$

The second assertion is proved in the same way.

The last assertion is true for $n = 1$. Assume that it is true for $n - 1$. We compute

$$\begin{array}{rcl} B_n(Q) & = & (B_{n-1}(Q))' + PB_{n-1}(Q) \quad \text{by Lemma } \mathbf{??} \\ & = & \left(\frac{\prod_{i=0}^{n-2}(r_1-i)}{(x-\alpha)^{n-1}} + \cdots\right)' + \left(\frac{r_1}{x-\alpha} + \cdots\right)\left(\frac{\prod_{i=0}^{n-1}(r_1-i)}{(x-\alpha)^{n-1}} + \cdots\right) \\ & = & \frac{\prod_{i=0}^{n-1}(r_1-i)}{(x-\alpha)^n} + \cdots. \end{array}$$

The proof is complete. $\square$

A root of $a_n(x)$ in (**??**) is called a normal pole of $R_{n-1}(z)$. If a rational solution of $R_{n-1}(z) = 0$ has a pole which is not normal, then the pole is an extraneous one. The set of normal poles of R_{n-1} is denoted by $N_{R_{n-1}}$. The next theorem describes rational solutions of $R_{n-1}(z) = 0$.

Theorem 6.3 *Let a rational function S be a solution of $R_{n-1}(z)$. Then S can be written as*

$$S = \sum_{i=0}^{k_\infty} p_i x^i + \sum_{\alpha \in N_{R_{n-1}}} \sum_{j=1}^{k_\alpha} \frac{r_{\alpha j}}{(x-\alpha)^j} + \frac{G'}{G}, \tag{15}$$

where the p's and r's are in $\mathbf{C}$, and G is in $\mathbf{C}[x]$ whose roots are not normal poles. Furthermore,

$$k_\infty \le \max\left(\max_{1\le i<j\le n} \left(\frac{\deg a_i - \deg a_j}{j-i} \right), 0 \right)$$

and

$$k_\alpha \le \max\left(\max_{1\le i<j\le n} \left(\frac{\mathrm{mult}_\alpha a_i - \mathrm{mult}_\alpha a_j}{i-j} \right), 1 \right).$$

For each r_{α,k_α} or p_{k_∞}, we may construct a univariate polynomial I_α or I_∞ with degree at most n such that

$$I_\alpha(r_{\alpha k_\alpha}) = 0 \quad \text{or} \quad I_\infty(p_{k_\infty}) = 0.$$

Proof Assume that S has a pole β, which is not normal. Expanding S around β yields

$$S = \frac{r_m}{(x-\beta)^m} + \text{higher terms},$$

where $r_m \neq 0$. Substituting S for z into $R_{n-1}(z)$, we find, by Lemma **??**, that the lowest term w.r.t. $(x-\beta)$ in $R_{n-1}(S)$ is either

$$\mathrm{lc}(a_n) \frac{r_m^n}{(x-\beta)^{mn}}$$

if $m > 1$, or, otherwise

$$\mathrm{lc}(a_n) \frac{\prod_{i=0}^{n-1}(r_1 - i)}{(x-\beta)^n}.$$

Since $R_{n-1}(S) = 0$, $r_m^n = 0$ if $m > 1$, or $\prod_{i=0}^{n-1}(r_1 - i) = 0$, otherwise. Hence, m must be 1 and r_1 must be an integer between 1 and $n-1$. Let extraneous poles of S be $\beta_1, \beta_2, \ldots, \beta_e$ with respective residues $m_1, m_2, \ldots, m_e$, which are positive integers. Letting $G = \prod_{i=1}^{e}(x-\beta_i)^e$, we prove (**??**) by partial fraction decomposition.

Now, we estimate order (degree) bounds for S. Plugging S given in (**??**) into $R_1(z)$, we find, by Lemma **??**, that

$$\begin{aligned} R_1(S) \quad = \quad & a_n \left(p_{k_\infty}^n x^{k_\infty n} + \cdots \right) \\ & + a_{n-1} \left(p_{k_\infty}^{n-1} x^{k_\infty (n-1)} + \cdots \right) \\ & + \cdots \\ & + a_1 \left(p_{k_\infty} x^{k_\infty} + \cdots \right) \\ & + a_0 \end{aligned} \tag{16}$$

Hence, the highest power of x in the right-hand side of (**??**) is equal to $\max_{0\le i\le n}(\deg a_i + ik_\infty) =: m_\infty$. In order to make $R_1(S) = 0$, we find there are at least $i, j \in \{0, \ldots, n\}$ such that

$$\deg a_i + ik_\infty = \deg a_j + jk_\infty = m_\infty.$$

Hence, $k_\infty = (\deg a_i - \deg a_j)/(j-i)$. The bound for k_∞ given in the statement of the theorem is correct. Moreover,

$$I_\infty(p_{k_\infty}) = \sum_{ik_\infty + \deg a_i = m_\infty} \mathrm{lc}(a_i) p_{k_\infty}^i.$$

The part for normal poles can be proved similarly. □

From these conclusions we see that rational solutions of general Riccati equations can be computed in a similar way as we compute rational solutions of first-order Riccati equation. Two minor differences are

- It is more complicated to compute order and degree bounds
- To determine p_{k_∞} and $r_{\alpha,k_{alpha}}$, we need to solve univariate polynomial equations with degree at most n (instead of 2).

Example 6.1 Find rational functions solutions of the second-order Riccati equation

$$R_2(z) = B_3(z) + aB_2(z) + bB_1(z) + cB_0(z) = 0, \tag{17}$$

where

$$a = -2x^3 + 1 - \frac{1}{x-1} - \frac{2+16x-6x^2-4x^3-12x^5+7x^6}{x^7-2x^6-x^4-2x^3+8x^2+2x+2},$$

$$b = x^6 - x^3 + 6x^2 + 4x + 6 + \frac{8x^6+39x^5-62x^4-25x^3-62x^2-38x-12}{x^7-2x^6-x^4-2x^3+8x^2+2x+2},$$

and

$$c = -x^5 - x^4 - x^3 - 6x - 10 + \frac{3}{x-1} - \frac{-26-26x-116x^2-53x^3+28x^4-11x^5+19x^6}{x^7-2x^6-x^4-2x^3+8x^2+2x+2}.$$

We make an ansatz that

$$y = p + \frac{r}{q} + \frac{h'}{h},$$

where $p, q, r, h \in \mathbf{C}[x]$ with the properties that $\deg q \geq \deg r$, the roots of q are the poles of **??**, and h is squarefree and has no root equal to any pole of (**??**). It is clear that $\deg p \leq 3$. First, we let

$$p = p_3x^3 + p_2x^2 + p_1x + p_0$$

and decide the candidates for the p_i's. Substituting the ansatz for y in (**??**) yields

$$\begin{cases} p_3(p_3-1)^2 = 0 \\ (p_3-1)\left((3p_3-1)p_2 - 3p_3^2 + 3p_3\right) = 0 \\ (p_3-1)(3p_3-1)p_1 + 12p_3p_2 - 2p_2^2 + 2p_3^3 - 4p_3^2 - 3p_2 + 2p_3 + 3p_3p_2^2 - 9p_3^2p_2 = 0 \\ (p_3-1)(3p_3-1)p_0 - 8p_3p_2 - 3p_1 + 6p_3p_2p_1 + 6p_2^2 - 4p_2p_1 \\ +3p_3^2 - 2p_3 + 12p_3p_1 + 2p_2 - p_3^3 + p_2^3 + 6p_3^2p_2 - 9p_3^2p_1 - 9p_3p_2^2 = 0 \end{cases} \tag{18}$$

This system has two solutions

$$\{p_3 = p_2 = p_1 = p_0 = 0\} \quad \text{and} \quad \{p_3 = 1, p_2 = 0\}.$$

Thus we have

$$p = 0 \quad \text{or} \quad p = x^3 + p_1 x + p_0.$$

In order to fix p_1 and p_0, we transform (**??**) by the transformation $y = z + x^3$ to

$$z'' + 3zz' + z^3 + a_1(z' + z^2) + b_1 z + c_1 = 0, \tag{19}$$

where

$$a_1 = x^3 + 1 - \frac{1}{x-1} - \frac{2 + 16x - 6x^2 - 4x^3 - 12x^5 + 7x^6}{x^7 - 2x^6 - x^4 - 2x^3 + 8x^2 + 2x + 2},$$

$$b_1 = x^3 - x^2 - 2x - 4 - \frac{2}{x-1} - \frac{-4 + 14x - 38x^2 - 15x^3 - 2x^4 - 19x^5 + 14x^6}{x^7 - 2x^6 - x^4 - 2x^3 + 8x^2 + 2x + 2}$$

and

$$c_1 = -x^2 - 2x - 4 - \frac{1}{x-1} - \frac{-6 - 2x - 32x^2 - 11x^3 - 2x^4 - 7x^5 + 7x^6}{x^7 - 2x^6 - x^4 - 2x^3 + 8x^2 + 2x + 2}.$$

We look for the rational function solutions of (**??**) whose polynomial parts are in the form

$$y = p_1 x + p_0 + \frac{r}{q} + \frac{H'}{H}.$$

Substituting the above ansatz for y in (**??**) yields the system

$$\{p_1^2 = 0,\ 2p_1p_0 - 3p_1^2 + p_1 = 0,\ p_0^2 + (-6p_1 + 1)p_0 + 2p_1^2 - 3p_1 + p_1^3 = 0\}.$$

Thus $p_1 = 0$ and $p_0 = -1$ or 0. Therefore the polynomial parts of the solution candidates of (**??**) are

$$p = 0,\ p = x^3,\ \text{and}\ p = x^3 - 1.$$

Second, we handle the pole $x = 1$. It's clear that the order of the pole of any rational function solutions is less than 2. Hence, our ansatz is

$$y = \frac{r_1}{(x-1)} + f$$

where $f \in \mathbf{C}[[(x-1)]]$. Substituting this ansatz for y in (**??**) yields

$$r_1(r_1^2 - 4r_1 + 3) = 0,$$

that is, $r_1 = 0$ or $r_1 = 1$ or $r_1 = 3$. Third, we handle the pole $x = \alpha$ where α is a root of the irreducible polynomial

$$x^7 - 2x^6 - x^4 - 2x^3 + 8x^2 + 2x + 2.$$

Once again the order of the pole of any rational function solutions is less than 2. Hence, our ansatz is

$$y = \frac{r_\alpha}{(x-\alpha)} + g$$

where $g \in \mathbf{C}[[x-\alpha]]$. The same substitution leads to

$$r_\alpha^3 - 3r_\alpha^2 + 2r_\alpha = 0,$$

that is $r_\alpha = 0$ or $r_\alpha = 1$ or $r_\alpha = 2$. Fix $r_\alpha = 0$. Then our solution candidates are in the form

$$p_3 x^3 + p_2 x^2 + p_1 x^1 + p_0 + \frac{r_1}{x-1},$$

where the p's and r_1 have been computed. The last task is to find extraneous poles for each solution candidate. We find three solutions

$$y_1 = x^3 + \frac{1}{x-3}, \; y_2 = x^3 - 1 \text{ and } y_3 = \frac{1}{x-1}.$$

Since $\exp\left(\int y_1\right)$, $\exp\left(\int y_1\right)$, *and* $\exp\left(\int y_1\right)$ *are unsimilar to each other,* y_1, y_2 *and* y_3 *are all rational solutions by Theorem* **??**.

7. Bibliographic notes

The algorithms given in Sections **??** and **??** can be found in [**?**], in which the authors used balanced factorization instead of partial fraction decompositionsΦ Φ to analyze normal poles. Their approach may be more efficient in practice. A more general algorithm for computing "in-field" solutions is presented in [**?**]. The structure theorem (Theorem **??**) in Section **??** is a special case of a theorem in [**?**]. The materials of the last two sections are essentially from [**?**, **?**, **?**]. The algorithm in [**?**] is most suitable for implementation.

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