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POLYNOMIAL SPECIES AND CONNECTIONS AMONG BASES OF THE SYMMETRIC POLYNOMIALS

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1. INTRODUCTION

In two papers published in 1968 (see [8]) G.C. Rota, carrying to the limit the algebraic processes which Baxter and others introduced for the resolution of problems generated by theory of probability, shows that every identity in a Baxter algebra is equivalent to an identity between symmetric functions. In his second paper the Author proves,through combinatorial methods, classical identities between symmetric functions which translate identities in Baxter algebras of probability interest.

The concept of generating function of a function set, conveyed in these papers, is developed later by Doubilet, Rota and Stanley (see [4], [9]).These Authors introduce a process for the construction of algebras of generating functions, both classical and innovative, for the resolution of enumerative problems.

Using the concept of generating function of a function set and techniques involving the lattice of partition of a set, Doubilet (see [3]) derives many of the known results and new ones about symmetric functions. In many cases Doubilet utilizes the Möbius inversion formula, but he also succeds in giving bijective proofs of identities between symmetric functions. These proofs consist essentially in an interpretation of the functions that occur in the identities in terms of sets and in finding a bijections between them.

In this paper we use the theory of polynomial species (see [1]) which gives a systematic approach to this kind of proof. We prove with bijective arguments, some identities which occur among the classical bases of symmetric polynomials of degree n. The language is that of categories theory and this emphasizes the generality degree of the concept of species.

2. POLYNOMIAL SPECIES

Let Φ be the category of finite sets and bijections. A *finite species* (see [5]) is a functor M from Φ to Φ .

Let E, F $\in$ Ob(Φ), we shall denote by $M[E]$ the set of M-structures on E and by $M[u]$, $u \in \mathrm{Hom}(E,F)$, the bijection between

$M[E]$ and $M[F]$ obtained "transforming via u" every M-structure on E into an M-structure on F.

The concept of polynomial species is a generalization of that of finite species in the sense that the polynomial species functor, in addition to carrying the structures, defines a subset of functions from a finite set to a set of variables as specified in the following.

Let $\mathbf{X} = \{x_i : i \in \mathcal{I}\}$ be a family of variables with indices in a non empty and totally ordered set $\mathcal{I}$. Let I be the functor from to the category **Ens** of sets and functions defined by:

$$I[E] = \bigcup_{A \subseteq E} \mathrm{Hom}(A, \mathbf{X})$$

and, for each $A \subseteq E$ and $f: A \longrightarrow \mathbf{X}$,

$$I[u](f) = f\circ u^{-1} : u(A) \rightarrow \mathbf{X}.$$

Let M be a finite species. We shall denote by $\mathrm{Pol}(M)$ the functor from $\boldsymbol{\Phi}$ to **Ens** defined as follows:

$$\mathrm{Pol}(M)[E] = M[E] \times I[E]$$

$$\mathrm{Pol}(M)[u](s,f) = (M[u](s), I[u](f))$$

A *polynomial species* is any subfunctor $\boldsymbol{P}$ of $\mathrm{Pol}(M)$, i.e. for each $E \in \mathrm{Ob}(\boldsymbol{\Phi})$, $\boldsymbol{P}[E]$ is a subset of $\mathrm{Pol}(M)[E]$ such that if $u \in \mathrm{Hom}(E,F)$ and $(s,f) \in \boldsymbol{P}[E]$ then $(M[u](s), I[u](f)) \in \boldsymbol{P}[F]$. If $\boldsymbol{P}$ is a subfunctor of $\mathrm{Pol}(M)$ we shall write $\boldsymbol{P} \subseteq \mathrm{Pol}(M)$.

The category of polynomial species is defined as follows. Let M and N be finite species, and let φ be a natural transformation of M to N. We define a natural transformation $\bar{\varphi}$ of the polynomial species $\mathrm{Pol}(M)$ to the polynomial species $\mathrm{Pol}(N)$ as follows. The image of the set $\mathrm{Pol}(M)[E]$ is the set $(\varphi_{\mathbf{E}}(s), f)$, as (s,f) ranges over $\mathrm{Pol}(M)[E]$. If $\boldsymbol{P}$ and $\boldsymbol{Q}$ are polynomial species, we define $\mathrm{Hom}(\boldsymbol{P}, \boldsymbol{Q})$ to be the set of natural transformations of $\boldsymbol{P}$ to $\boldsymbol{Q}$ which are restrictions to $\boldsymbol{P}$ and $\boldsymbol{Q}$ of some natural transformation $\bar{\varphi} : \mathrm{Pol}(M) \longrightarrow \mathrm{Pol}(N)$, where $\boldsymbol{P} \subseteq \mathrm{Pol}(M)$ and $\boldsymbol{Q} \subseteq \mathrm{Pol}(N)$. In particular, we write $\boldsymbol{P} = \boldsymbol{Q}$, when $\boldsymbol{P}$ and $\boldsymbol{Q}$ are naturally equivalent in this category.

We associate to each polynomial species a generating function which considers both the structure and the subset of functions determined by the species.

Let C be a cofinite subset of $\mathbf{X}$. We write, if $p \in \mathbf{Z}[\mathbf{X}]$, $p/_{C=0}$ to denote the polynomial obtained from p by setting to 0 all $x \in C$. If $p \in \mathbf{Z}[\mathbf{X}]$ we let $N(p,C)$ be the set of all $q \in \mathbf{Z}[\mathbf{X}]$ such that $p/_{C=0} = q/_{C=0}$. This defines a topology on the ring $\mathbf{Z}[\mathbf{X}]$. Let $\mathbf{Z}[(\mathbf{X})]$ the completion of $\mathbf{Z}[\mathbf{X}]$ and $\mathrm{al}(\mathbf{X})$ the algebra over

$\mathbf{Z}$ generated by $\mathbf{X}$. A function $f: E \rightarrow X$ determines a function $\hat{f}: E \rightarrow al(X)$ defined by: $\hat{f}(e) = f(e)$ for each $e \in E$.

We shall call *generating monomial* of the function f the element of $Z[(X)]$ defined by:

$$gen(f) = \prod_{e \in E} \hat{f}(e) = \prod_i x_i^{|f^{-1}(x_i)|}$$

Let $P \subseteq Pol(M)$ be a polynomial species. We set:

$$gen(P[E]) = \sum_{(s,f) \in P[E]} gen(s,f)$$

where $gen(s,f) = gen(f)$. We note that $gen(P[E])$ depends on the cardinality of E alone and so we set: $gen(P[E]) = gen(P,n)$ for each set E such that $|E| = n$. We shall call $gen(P,n)$ *nth-coefficient polynomial of the species* P.

The *generating function* $Gen(P,z)$ of the polynomial species P is the formal power series

$$Gen(P,z) = \sum_{n \geq 0} gen(P,n) \frac{z^n}{n!}$$

with coefficients in $Z[(X)]$.

THEOREM 2.1. *Polynomial species isomorphic have the same generating function.*

3. THE SPECIES OF THE ASSEMBLIES

We introduce now the notion of assembly of finite species. Let N be a finite species without constant term, i.e. $N[\emptyset] = \emptyset$. We shall call *assembly of structures of species* N on the finite set E a partition of E on every block of which a structure of species N is defined. Frmally an assembly is a pair (π, S_π) where $\pi = \{B: B \subseteq E\}$ is a partition of E and $S_\pi = \{s_B : s_B \in N[B]\}$.

The *species* $Exp_k(N)$ *of the assemblies of structures of species* N *of order* k is defined as follows:

$Exp_k(N)[E] = \{$assemblies (π, S_π) on E with $|\pi| = k\}$.

The *species* $Exp(N)$ *of the assemblies of structures of species* N is

$Exp(N)[E] = \{$assemblies on $E\}$.

Let $\mathcal{P}(k)$ be the set of the partitions of E with k block and $P \subseteq Pol(M)$ a polynomial species without constant term (i.e. $P[\emptyset] = \emptyset$). An *assembly on* E *of order* k *of species* P is every pair (s,f) where:

i) $s = (\pi, S_\pi)$ is an assembly of species M with $\pi \in \mathcal{P}(k)$ and $S_\pi = \{s_B : B \in \pi, s_B \in M[B]\}$,

ii) there exist, for every $B \in \pi$, a function f_B such that $(s_B, f_B) \in P[B]$,

iii) if f_B is defined on a set $A \cap B$, then f is defined on $A = \bigcup_{B \in \pi} (A \cap B)$ and $f/_{A \cap B} = f_B$.

The *species of assemblies of P-structures of order* k is the polynomial species $\mathrm{Exp}_k(P) \subseteq \mathrm{Pol}(\mathrm{Exp}_k(M))$ defined as follows: $\mathrm{Exp}_k(P)[E] = \{$ assemblies of P-structures of order $k\}$.

THEOREM 3.1. *For any* $k \in \mathbf{N}$ *is:* $\mathrm{Gen}(\mathrm{Exp}_k(P), z) = \dfrac{\mathrm{Gen}(P, z)^k}{k!}$

The *species of the assemblies of species* P is the polynomial species $\mathrm{Exp}(P)$ defined by $\mathrm{Exp}(P)[E] = \bigcup_{k \geq 0} \mathrm{Exp}_k(P)[E]$.

THEOREM 3.2. $\mathrm{Gen}(\mathrm{Exp}(P), z) = e^{\mathrm{Gen}(P, z)}$.

PROPOSITION 3.1. *If* P *and* Q *are isomorphic, then*
$\mathrm{Exp}_t(P) = \mathrm{Exp}_t(Q)$.

Proof. Let ϱ an isomorphism between P and Q. An assembly $((\pi, S_\pi), f)$ of structures of species P is a partition π such that on each block $B \in \pi$ a structure $(s_B, f_B) \in P[B]$ is defined. The bijection $\bar{\varrho}_E : \mathrm{Exp}_t(P)[E] \longrightarrow \mathrm{Exp}_t(Q)[E]^B_B$ associates to $((\pi, S_\pi), f)$ the assembly of species Q related to partition π such that on each B the structure $\varrho_B(s_B, f_B)$ is defined. The bijection $\bar{\varrho}_E$ determines the requested natural isomorphism.

4. CONNECTIONS AMONG BASES OF SYMMETRIC POLYNOMIALS

Let n be an integer. A *partition* of n is any sequence $(\lambda) = (\lambda_1, \dots \lambda_q)$ of non negative integers in decreasing order $\lambda_1 \geqslant \dots \geqslant \lambda_q$ such that their sum is n. The non-zero λ_i are called the *parts* of (λ) and the number of the parts is the *length* of (λ). Sometime it is convenient to use a notation which indicates the numbers of time each integer occur as part: $(\lambda) = (1^{r_1}\ 2^{r_2} \dots)$ means that exactly r_i of the part of (λ) are equal to i.

We note that every partition π of a set E with $|E| = n$ determines the partition $(\pi) = (1^{r_1}\ 2^{r_2} \dots)$ of n, where r_i is the number of blocks of with i elements. We shall call (π) class of π.

The symmetric polynomials of degree n in the variables $x_1, \dots x_t$ with rational coefficients have four classical bases.

The elementary symmetric functions : a_λ

Let $p \in \mathbf{N}$ and $a_p = \sum x_{i_1} \dots x_{i_p}$ where the sum is over all the sequences $i_1, \dots i_p$ such that $i_1 < \dots < i_p \leq t$.

We set: $a_\lambda = a_{\lambda_1} a_{\lambda_2} \dots a_{\lambda_q}$

The monomial symmetric functions: k_λ

$$k_\lambda = \sum x_{i_1}^{\lambda_1} \ldots x_{i_q}^{\lambda_q}$$

where the sum is over all distinct monomials with distinct indices.

The homogeneus elementary functions: h_λ

Let $p \in \mathbf{N}$ and $h_p = \sum x_1^{i_1} \ldots x_t^{i_t}$ where the sum is over all the sequences $i_1, \ldots i_t$ such that $i_1 + \ldots + i_t = p$. We set:

$$h_\lambda = h_{\lambda_1} \ldots h_{\lambda_q}$$

The power sum functions: $\mathbf{s}_\lambda$

Let $p \in \mathbf{N}$ and $\mathbf{s}_p = \sum_{i=1}^{t} x_i^p$. We set:

$$\mathbf{s}_\lambda = \mathbf{s}_{\lambda_1} \ldots \mathbf{s}_{\lambda_q}$$

When (λ) ranges over all partitions of the integer n, the sets $\{a_\lambda\}$, $\{k_\lambda\}$, $\{h_\lambda\}$, $\{\mathbf{s}_\lambda\}$ are the classical bases of the symmetric polynomials of degree n.

Let $\bar{I}$ be the finite species defined by $\bar{I}[E]=\{E\}$. We denote with $\mathbf{S} \subseteq \mathrm{Pol}(\bar{I})$ the *power sum species* defined by $\mathbf{S}[E] = \{(E,f)\colon f\colon E \to \mathbf{X} \text{ constant}\}$. The nth-coefficient of the generating function of $\mathbf{S}$ is:

$$\mathrm{gen}(\mathbf{S},n) = \mathbf{s}_n = \sum_{i \in \mathcal{I}} x_i^n, \text{ hence } \mathrm{Gen}(\mathbf{S},z) = \sum_{n \geq 0} \mathbf{s}_n \frac{z^n}{n!}$$

Let $t \in \mathbf{N}$ and P_t the finite species defined by $P_t[E] = \{\text{partitions of } E \text{ with } t \text{ blocks}\}$. We denote with $\mathbf{K}_t$ the polynomial species $\mathbf{K}_t[E] = \{(\pi,f)\colon |\pi| = t,\ f\colon E \to \mathbf{X},\ \ker f \geq \pi\} \subseteq \mathrm{Pol}(P_t)[E]$. The nth-coefficient of the generating functions of $\mathbf{K}_t$ is:

$$\mathrm{gen}(\mathbf{K}_t,n) = \sum_{\pi \in P_t[E]} \sum_{\ker f \geq \pi} \mathrm{gen}(\pi,f)$$

The other hand: $\displaystyle\sum_{\ker f \geq \pi} \mathrm{gen}(\pi,f) = \sum_{\sigma \geq \pi} s_1!\ s_2! \ldots k_{(\sigma)}$

where $(\sigma) = (1^{s_1} 2^{s_2} \ldots)$. The sum on the rigth depends on $(\lambda) = (\pi)$ alone, hence setting $b_\lambda = \sum_{\sigma \geq \pi} s_1!\ s_2! \ldots k_{(\sigma)}$, we have

$$\mathrm{gen}(\mathbf{K}_t,n) = \sum_{\pi \in P_t[E]} b_\lambda = \sum_{\lambda_1 + \ldots + \lambda_t = n} \binom{n}{\lambda_1 \ldots \lambda_t} \frac{b_\lambda}{t!}$$

from which

$$\mathrm{Gen}(\mathbf{K}_t,z) = \sum_{n \geq 0} \sum_{\lambda_1 + \ldots + \lambda_t = n} \binom{n}{\lambda_1 \ldots \lambda_t} \frac{b_\lambda}{t!} \frac{z^n}{n!}$$

THEOREM 4.1. $\mathrm{Exp}_t(\mathbf{S}) = \mathbf{K}_t$

Proof. The bijection that to any pair $((\pi, S_\pi), f) \in \mathrm{Exp}_t(\mathbf{S})[E]$ associates the pair $(\pi, f) \in \mathbf{K}_t[E]$ determines the requested isomorphism.

COROLLARY 4.1. *If we set* $(\pi)=(\lambda)$, *we have:*

(i) $$\mathbf{s}_\lambda = \sum_{\sigma \geqslant \pi} s_1!\, s_2! \dots k_{(\sigma)}$$

Proof. From the theorems 3.1, 4.1 and 2.1 we have:

$$\text{gen}(\text{Exp}_t(\mathbf{S}),n) = \frac{1}{t!}\sum_{\lambda_1+\dots+\lambda_t=n}\binom{n}{\lambda_1 \dots \lambda_t}\mathbf{s}_{\lambda_1}\dots = \frac{1}{t!}\sum_{\lambda_1+\dots+\lambda_t=n}\binom{n}{\lambda_1\dots\lambda_t}\mathbf{s}_\lambda =$$
$$\text{gen}(\mathbf{K}_t,n) = \frac{1}{t!}\sum_{\lambda_1+\dots+\lambda_t=n}\binom{n}{\lambda_1\dots\lambda_t} b_\lambda$$

The (i) gives the power sum functions in terms of the monomial symmetric functions.

Let S be the finite species defined by: $S[E]=\{\text{permutations on } E\}$. We shall denote with $\mathbf{H} \subseteq \text{Pol}(\text{Exp}(S))$ the *disposition species* defined by:
$\mathbf{H}[E] = \{(s,f):\ s = (\pi, S_\pi) \in \text{Exp}(S)[E] \text{ and } f:E\to \mathbf{X} \text{ such that } \ker f = \pi\}$.
The nth-coefficient of the generating function of $\mathbf{H}$ is:

$$\text{gen}(\mathbf{H},n) = n!\, h_n \qquad \text{(see [1])}$$

hence $$\text{Gen}(\mathbf{H},z) = \sum_{n\geqslant 0} n!\, h_n \frac{z^n}{n!}$$

The *cyclic species* $\mathbf{C} \subseteq \text{Pol}(S)$ is defined by
$\mathbf{C}[E] = \{(\mu,f):\ \mu \text{ cyclic permutation of } E,\ f:E\to\mathbf{X} \text{ constant}\}$. The nth-coefficient of the generating function of species $\mathbf{C}$ is:

$$\text{gen}(\mathbf{C},n) = (n-1)! \sum_{i\in \mathcal{I}} x_i^n = (n-1)!\, \mathbf{s}_n,$$

hence: $$\text{Gen}(\mathbf{C},z) = \sum_{n\geqslant 0}(n-1)!\, \mathbf{s}_n \frac{z^n}{n!}$$

We shall calculate now the nth-coefficient of the species $\text{Exp}_t(\text{Exp}(\mathbf{C}))$ that for brevity we shall denote by $\bar{\mathbf{C}}$. From theorem 3.1 we have:

$$\text{gen}(\bar{\mathbf{C}},n) = \frac{1}{t!}\sum_{\lambda_1+\dots+\lambda_t=n}\binom{n}{\lambda_1\dots\lambda_t}\text{gen}(\text{Exp}(\mathbf{C}),\lambda_1)\dots\text{gen}(\text{Exp}(\mathbf{C}),\lambda_t)$$

For any partition π of E ($|E|=n$) such that $(\pi) = (\lambda_1,\dots\lambda_t)$ we have:

$$\text{gen}(\text{Exp}(\mathbf{C}),\lambda_1)\dots\text{gen}(\text{Exp}(\mathbf{C}),\lambda_t) = \sum_{((\pi,S_\pi),f)\in\bar{\mathbf{C}}[E]}\text{gen}((\pi,S_\pi),f) =$$
$$\sum_{\sigma\leqslant\pi}(\nu_1-1)!\,(\nu_2-1)!\dots\mathbf{s}_{(\sigma)}$$

where $(\sigma)=(\nu_1,\nu_2,\dots)$. The sum on the right depends on (π) alone,

hence setting: $c_\lambda = \sum_{\sigma\leqslant\pi}(\nu_1-1)!\,(\nu_2-1)!\dots\mathbf{s}_{(\sigma)}$, we have:

$$\text{Gen}(\text{Exp}_t(\text{Exp}(\mathbf{C})),z) = \frac{1}{t!}\sum_{n\geqslant 0}\ \sum_{\lambda_1\dots+\lambda_t=n}\binom{n}{\lambda_1\dots\lambda_t} c_\lambda.$$

THEOREM 4.2. $\text{Exp}(\mathbf{C}) = \mathbf{H}$ (see [1], [2]).

THEOREM 4.3. *If* $(\pi) = (\lambda_1,\dots\lambda_t)$

(ii) $$\lambda_1!\ \lambda_2!\ldots h_\lambda = \sum_{\sigma\le\pi} (\nu_1-1)!\ (\nu_2-1)!\ldots \mathbf{s}_{(\sigma)}$$

Proof. From the theorems 4.2 and 3.3, we have: $\mathrm{Exp}_t(\mathrm{Exp}(\mathbf{C}))=\mathrm{Exp}_t(\mathbf{H})$ thus, using the theorem 2.1, we obtain:

$$\mathrm{gen}(\mathrm{Exp}_t(\mathrm{Exp}(\mathbf{C})),n)\ \frac{1}{t!}\sum_{\lambda_1+\ldots+\lambda_t=n}\binom{n}{\lambda_1\ldots\lambda_t} c_\lambda=\mathrm{gen}(\mathrm{Exp}_t(\mathbf{H}),n) =$$
$$\frac{1}{t!}\sum_{\lambda_1+\ldots+\lambda_t=n}\binom{n}{\lambda_1\ldots\lambda_t}\lambda_1!h_{\lambda_1}\ \lambda_2!h_{\lambda_2}\ldots = \frac{1}{t!}\sum_{\lambda_1+\ldots+\lambda_t=n}\binom{n}{\lambda_1\ldots\lambda_t}\lambda_1!\ \lambda_2!\ldots h_\lambda$$

The (ii) gives the homogeneus elementary functions in terms of the power-sum functions.

We denote with $\mathbf{A}\subseteq \mathrm{Pol}(\bar{1})$ the *elementary symmetric species* defined by: $\mathbf{A}[E] = \{(E,f) : f: E\to X \text{ monomorphism}\}$. The coefficient of the generating function of the species $\mathbf{A}$ is: $\mathrm{gen}(\mathbf{A},n) = n!\ a_n$ hence:

$$\mathrm{Gen}(\mathbf{A},z) = \sum_{n\ge 0} n!\ a_n \frac{z^n}{n!}$$

Let $t\in\mathbf{N}$. We denote by $\mathbf{A_t} \subseteq \mathrm{Pol}(P_t)$ the species defined by: $\mathbf{A_t}[E] = \{(\pi,f) : |\pi| = t,\ f: E\to X\}^t$ and $\ker f\wedge\pi=\hat{0}$. The coefficient of $\mathbf{A_t}$ is:

$$\mathrm{gen}(\mathbf{A_t},n) = \sum_{\pi\in P_t[E]}\ \sum_{\ker f\wedge\pi=\hat{0}} \mathrm{gen}(\pi,f).$$ The other hand:
$$\sum_{\ker f\wedge\pi=\hat{0}}\mathrm{gen}(\pi,f) = \sum_{\sigma\wedge\pi=\hat{0}} s_1!\ s_2!\ldots k_{(\sigma)}$$ where $(\sigma) = (1^{s_1}2^{s_2}\ldots)$.

The sum on the right depends on (π) alone. Hence setting $d_\lambda = \sum_{\sigma\wedge\pi=\hat{0}} s_1!\ s_2!\ldots k_{(\sigma)}$ we have:

$$\mathrm{gen}(\mathbf{A_t},n) = \sum_{\pi\in P_t[E]} d_\lambda = \sum_{\lambda_1+\ldots+\lambda_t=n}\binom{n}{\lambda_1\ldots\lambda_t}\frac{d_\lambda}{t!}$$

from which

$$\mathrm{Gen}(\mathbf{A_t},z) = \sum_{n\ge 0}\ \sum_{\lambda_1+\ldots+\lambda_t=n}\binom{n}{\lambda_1\ldots\lambda_t}\frac{d_\lambda}{t!}\frac{z^n}{n!}$$

THEOREM 4.4. $$\mathrm{Exp}_t(\mathbf{A}) = \mathbf{A_t}$$

Proof. The bijection that to any pair $((\sigma,S_\sigma),f)\in\mathrm{Exp}_t(\mathbf{A})[E]$ associates the pair $(\sigma,f)\in\mathbf{A_t}[E]$ determinates the requested isomorphism.

COROLLARY 4.2. *If* $(\pi)=(\lambda)$, *we have:*

(iii) $$\lambda_1!\ \lambda_2!\ldots a_\lambda = \sum_{\sigma\wedge\pi=\hat{0}} s_1!\ s_2!\ldots k_{(\sigma)}$$

Poof. From the theorem 3.1, 4.4 and 2.1 follows:

$$\mathrm{gen}(\mathrm{Exp}_t(\mathbf{A}),n) = \frac{1}{t!}\sum_{\lambda_1+\ldots+\lambda_t=n}\binom{n}{\lambda_1\ldots\lambda_t}\lambda_1!\ a_{\lambda_1}\ \lambda_2!\ a_{\lambda_2}\ldots =$$
$$\frac{1}{t!}\sum_{\lambda_1+\ldots+\lambda_t=n}\binom{n}{\lambda_1\ldots\lambda_t}\lambda_1!\ \lambda_2!\ldots a_\lambda = \mathrm{gen}(\mathbf{A_t},n) = \frac{1}{t!}\sum_{\lambda_1+\ldots+\lambda_t=n}\binom{n}{\lambda_1\ldots\lambda_t}d_\lambda$$

The (iii) gives the elementary symmetric functions in terms of the monomial symmetric functions.

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