Model Checking Existential Logic on Partially Ordered Sets

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We study the problem of checking whether an existential sentence (i.e., a first-order sentence in prefix form built using existential quantifiers and all Boolean connectives) is true in a finite partially ordered set (a poset). A poset is a reflexive, antisymmetric, and transitive digraph. The problem encompasses the fundamental embedding problem of finding an isomorphic copy of a poset as an induced substructure of another poset.

Model checking existential logic is already NP-hard on a fixed poset; thus, we investigate structural properties of posets yielding conditions for fixed-parameter tractability when the problem is parameterized by the sentence. We identify width as a central structural property (the width of a poset is the maximum size of a subset of pairwise incomparable elements); our main algorithmic result is that model checking existential logic on classes of finite posets of bounded width is fixed-parameter tractable. We observe a similar phenomenon in classical complexity, in which we prove that the isomorphism problem is polynomial-time tractable on classes of posets of bounded width; this settles an open problem in order theory.

We surround our main algorithmic result with complexity results on less restricted, natural neighboring classes of finite posets, establishing its tightness in this sense. We also relate our work with (and demonstrate its independence of) fundamental fixed-parameter tractability results for model checking on digraphs of bounded degree and bounded clique-width.

CCS Concepts: • **Mathematics of computing** → **Discrete mathematics**; • **Theory of computation** → **Finite Model Theory**

Additional Key Words and Phrases: Partially ordered sets, model checking, width, parameterized complexity

ACM Reference Format:
Simone Bova, Robert Ganian, and Stefan Szeider. 2015. Model checking existential logic on partially ordered sets. ACM Trans. Comput. Logic 17, 2, Article 10 (November 2015), 35 pages.
DOI: http://dx.doi.org/10.1145/2814937

1. INTRODUCTION

Motivation. The *model checking problem*, to decide whether a given logical sentence is true in a given structure, is a fundamental computational problem that appears in a variety of areas in computer science, including database theory, software verification, artificial intelligence, constraint satisfaction, and computational complexity. The problem is computationally intractable in its general version; hence, it is natural to seek restrictions of the class of structures or the class of sentences yielding sufficient or necessary conditions for computational tractability.

This research was supported by ERC Starting Grant (Complex Reason, 239962) and FWF Austrian Science Fund (Parameterized Compilation, P26200).
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DOI: http://dx.doi.org/10.1145/2814937

Here, as usual in the complexity investigation of the model-checking problem, computational tractability refers to *polynomial-time tractability* or, in cases in which polynomial-time tractability is unlikely, a relaxation known as *fixed-parameter tractability with the sentence as a parameter*. The latter guarantees a decision algorithm running in $f(k) \cdot n^c$ time on inputs of size n and sentences of size k, where f is a computable function and c is a constant. The complexity setup adopted here and its algorithmic motivations are further articulated in the literature [Grohe 2007; Flum and Grohe 2006].

The study of model checking first-order logic on restricted classes of finite *combinatorial structures* is an established line of research originating from the seminal work of Seese [1996]. Results in this area have provided very general conditions for computational tractability, and even exact characterizations in many relevant cases [Grohe et al. 2014]. As Grohe [2007] observes, though, it would also be interesting to investigate structural properties facilitating the model-checking problem in the realm of finite *algebraic structures*, for instance, groups or lattices.

In this article, we investigate the class of finite *partially ordered sets*. A partially ordered set (in short, a *poset*) is the structure obtained by equipping a nonempty set with a reflexive, antisymmetric, and transitive binary relation. In other words, the class of posets coincides with the class of directed graphs satisfying a certain universal first-order sentence (axiom), namely, the sentence that enforces reflexivity, antisymmetry, and transitivity of the edge relation. In this sense, from a logical perspective, posets form an intermediate case between combinatorial and algebraic structures; they can be viewed as being stronger than purely combinatorial structures, as the nonlogical vocabulary is presented by a first-order axiomatization; but weaker than genuinely algebraic structures, as the axiomatization is expressible in universal first-order logic (too weak of a fragment to define algebraic operations).

Posets are fundamental combinatorial objects [Graham et al. 1995, Chapter 8], with applications in many fields of computer science, ranging from software verification [Nielson et al. 2005] to computational biology [Rausch and Reinert 2010]. However, very little is known about the complexity of the model-checking problem on classes of finite posets; to the best of our knowledge, even the complexity of natural syntactic fragments of first-order logic on basic classes of finite posets is open.

A prominent logic in first-order model-checking is *primitive positive* logic, that is, first-order sentences built using existential quantification ($\exists$) and conjunction ($\wedge$); the problem of model checking primitive positive logic is equivalent to the *constraint satisfaction problem* and the *homomorphism problem* [Feder and Vardi 1998]. However, restricted to posets, the problem of model checking primitive positive logic and even *existential positive* logic, obtained from primitive positive logic by including disjunction ($\vee$) in the logical vocabulary, is trivial; because of reflexivity, every existential positive sentence is true on every poset!

As we observe in Proposition 3.2, the complexity scenario changes abruptly in *existential conjunctive* logic, that is, first-order sentences in prefix negation normal form built using $\exists$, $\wedge$, and negation ($\neg$). Here, the model-checking problem is NP-hard even on a certain fixed finite poset; in the complexity jargon, the *expression* complexity of existential conjunctive logic is NP-hard on finite posets. It follows that, as long as computational tractability is identified with polynomial-time tractability, structural properties of posets captured by numeric invariants are useless for algorithmic purposes (once the poset is fixed, these invariants act as constants in the runtime bound). There is then a natural quest for relaxations of polynomial-time tractability yielding *(i)* a nontrivial complexity analysis of the problem, and *(ii)* a refined perspective on the structural properties of posets underlying tamer algorithmic behaviors. In this article, we achieve *(i)* and *(ii)* through the glasses of fixed-parameter tractability.

Fig. 1. The left and right diagrams depict, respectively, the parameterized and classical complexity of the poset embedding problem. The gray region on the left (respectively, right) covers invariants such that, if a class of finite posets is bounded under the invariant, then the embedding problem is fixed-parameter tractable (respectively, polynomial-time tractable); the white region on the left (respectively, right) covers invariants such that there exists a class of finite posets bounded under the invariant where the problem is W[1]-hard (respectively, NP-hard). In classical complexity, as opposed to parameterized complexity, the tractability frontier of the poset embedding problem and the problem of model checking existential logic on posets are different (the latter is NP-hard on all invariants, since existential logic is already NP-hard on a fixed finite poset).

More precisely, as we discuss later, our contribution is a complete description of the parameterized complexity of model checking (all syntactic fragments of) existential first-order logic (first-order sentences in prefix normal form built using $\exists$, $\wedge$, $\vee$, and $\neg$), with respect to classes of finite posets in a hierarchy generated by fundamental poset invariants.[1]

Model checking existential logic encompasses as a special case the fundamental *embedding problem*, to decide whether a given structure contains an isomorphic copy of another given structure as an *induced* substructure; in fact, the embedding problem reduces in polynomial-time to the problem of model checking certain existential (even conjunctive) sentences. The aforementioned fact that existential conjunctive logic is already NP-hard on a fixed finite poset leaves open the existence of a nontrivial classical complexity classification of the embedding problem. We provide such a classification by giving a complete description of the classical complexity of the embedding problem in the introduced hierarchy of poset invariants.

We hope that the investigation of the existential fragment prepares the ground (and possibly provides basic tools) for understanding the model-checking problem for more expressive logics on posets.

Contribution. We refer the reader to Figure 1 for an overview of our contribution; the poset invariants and their relations are introduced in Section 3.

In contrast to the classical case, model checking existential logic on fixed structures is trivially fixed-parameter tractable; in fact, even the full first-order logic is trivially fixed-parameter tractable on any class of finite structures of bounded size because, in general, whether a first-order sentence of size k holds in a structure of size n is decidable in $O(n^k)$ time. On the other hand, there exist classes of finite posets for which existential logic is unlikely to be fixed-parameter tractable (in fact, there exist classes for which even the embedding problem is W[1]-hard), but the reduction class given by the natural hardness proof is rather wild; in particular, it has bounded depth but *unbounded width* (Proposition 3.4).

The *width* of a poset is the maximum size of a subset of pairwise incomparable elements (antichain); along with its *depth*, the maximum size of a subset of pairwise

[1]Existential *disjunctive* logic (first-order sentences in prefix negation normal form built using $\exists$, $\vee$, and $\neg$) is trivial on posets. Every sentence in the fragment reduces in linear time to a disjunction of sentences of the form $\exists x(x \leq x)$, $\exists x \exists y(x \leq y)$, $\exists x \exists y(\neg x \leq y)$, or $\exists x(\neg x \leq x)$. The first three forms are true in every (nontrivial) poset, and the fourth form is false in every (nontrivial) poset; hence, it is possible to check in linear time whether the given sentence is true on every (nontrivial) poset or false on every (nontrivial) poset.

comparable elements (chain), these two invariants form the basic and fundamental structural properties of a poset, arguably its most prominent and natural features. Our main result establishes that width helps algorithmically (in contrast to depth); specifically, we prove that *model checking existential logic on classes of finite posets of bounded width is fixed-parameter tractable* (Theorem 4.13). This, together with Seese's algorithm (plus a routine reduction described in Proposition 4.15), allows us to complete the parameterized complexity classification of the investigated poset invariants, as depicted in Figure 1.

We believe that our tractability result essentially enlightens the fundamental feature of posets of bounded width that can be exploited algorithmically; namely, *bounded width posets admit a polynomial-time compilation to certain semilattice structures*, which are algorithmically tamer than the original posets, but equally expressive with respect to the problem at hand. The proof proceeds in two stages. We first prove that, on any class of finite relational structures, model checking existential logic is fixed-parameter tractable if and only if the embedding problem is fixed-parameter tractable (Proposition 3.1). Next, we reduce the embedding problem on posets of bounded width to the homomorphism problem of certain semilattice structures, which is polynomial-time tractable by the classical results of Jeavons et al. [1997]. The reduction relies on perfect hash functions, already used by Alon et al. [1995] to derandomize color coding algorithms to attain, overall, a fixed-parameter tractable runtime.

Our approach is reminiscent of the fundamental theorem for finite distributive lattices by Birkhoff, putting finite distributive lattices and finite posets in a bijective correspondence [Birkhoff 1937]: Any finite distributive lattice is isomorphic to the lattice of downsets of the poset formed by its join irreducible elements (the correspondence is even stronger, in a sense that can be made precise in category-theoretical terms). Despite being well known in order theory, the algorithmic implications of this correspondence have been possibly overlooked. Indeed, using the correspondence and the known fact that the isomorphism problem is polynomial-time tractable on finite distributive lattices, we prove that *the isomorphism problem for posets of bounded width is polynomial-time tractable* (Theorem 5.7), which settles an open question in order theory [Caspard et al. 2012, p. 284].

Motivated by the equivalence (in parameterized complexity) between embedding and model checking existential conjunctive logic (Proposition 3.1) on one hand, and the fact that existential conjunctive logic is already NP-hard on a fixed finite poset (Proposition 3.2) on the other hand, we also revisit the classical complexity of the embedding problem for finite posets and classify it with respect to the poset invariants studied in the parameterized complexity setting. The outcome is pictured in Figure 1; here, polynomial-time tractability of the embedding problem on posets of bounded size is optimal with respect to the studied poset invariants. We remark that the hardness results are technically involved (Theorem 5.4 and Theorem 5.6); in particular, bounded width is a known obstruction for hardness proofs (e.g., the complexity of the dimension problem is unknown on bounded width posets).

We conclude mentioning that our work on posets relates with, but is independent of, general results by Seese [1996] and Courcelle et al. [2000], respectively, on model checking first-order logic on classes of finite graphs of bounded degree and bounded clique-width. Namely, the order relation of a poset has bounded degree if and only if the poset has bounded depth and bounded *cover degree* (i.e., its cover relation has bounded degree); moreover, if a poset has bounded width, then it has bounded cover-degree (Proposition 3.3). However, *there exist classes of bounded-width posets with unbounded degree* (e.g., chains), and *there exist classes of bounded-width posets with unbounded clique-width* (Proposition 3.5), which excludes the direct application of the aforementioned results.

2. PRELIMINARIES

For all integers $k \geq 1$, we let $[k]$ denote the set $\{1, \ldots, k\}$.

Logic. In this article, we focus on relational first-order logic. A *vocabulary* σ is a *finite* set of *relation symbols*, each of which is associated to a natural number called its *arity*; we let $\mathrm{ar}(R)$ denote the arity of $R \in \sigma$. An *atom* α (over vocabulary σ) is an equality of variables $(x = y)$ or is a predicate application $Rx_1 \ldots x_{\mathrm{ar}(R)}$, where $R \in \sigma$ and $x_1, \ldots, x_{\mathrm{ar}(R)}$ are variables. A *formula* (over vocabulary σ) is built from atoms (over σ), conjunction ($\wedge$), disjunction ($\vee$), negation ($\neg$), universal quantification ($\forall$), and existential quantification ($\exists$). A *sentence* is a formula having no free variables. We let $\mathcal{FO}$ denote the class of first-order sentences in *prefix negation normal form*, namely, each sentence in $\mathcal{FO}$ is formed by a prefix of quantifiers followed by a positive ($\wedge$ and $\vee$) Boolean combination of atoms and negated atoms (i.e., negations occur only in front of the atoms).

Let ρ be a subset of $\{\forall, \exists, \wedge, \vee, \neg\}$ containing at least one quantifier and at least one binary connective. We let $\mathcal{FO}(\rho) \subseteq \mathcal{FO}$ denote the *syntactic fragment* of $\mathcal{FO}$ sentences built using only logical symbols in ρ. We call $\mathcal{FO}(\exists, \wedge, \vee, \neg)$ the *existential* fragment, $\mathcal{FO}(\exists, \wedge, \neg)$ the *existential conjunctive* fragment, and $\mathcal{FO}(\exists, \wedge)$ the *existential conjunctive positive* (or *primitive positive*) fragment.

Structures. Let σ be a relational vocabulary. A *structure* $\mathbf{A}$ (over σ) is specified by a nonempty set A, called the *universe* of the structure, and a relation $R^{\mathbf{A}} \subseteq A^{\mathrm{ar}(R)}$ for each relation symbol $R \in \sigma$. A structure is *finite* if its universe is finite.

All structures considered in this article are finite.

Given a structure $\mathbf{A}$ and $B \subseteq A$, we denote by $\mathbf{A}|_B$ the substructure of $\mathbf{A}$ induced by B, namely, the universe of $\mathbf{A}|_B$ is B and $R^{\mathbf{A}|_B} = R^{\mathbf{A}} \cap B^{\mathrm{ar}(R)}$ for all $R \in \sigma$.

Let $\mathbf{A}$ and $\mathbf{B}$ be σ structures. A *homomorphism* from $\mathbf{A}$ to $\mathbf{B}$ is a function $h : A \to B$ such that $(a_1, \ldots, a_{\mathrm{ar}(R)}) \in R^{\mathbf{A}}$ implies $(h(a_1), \ldots, h(a_{\mathrm{ar}(R)})) \in R^{\mathbf{B}}$, for all $R \in \sigma$ and all $(a_1, \ldots, a_{\mathrm{ar}(R)}) \in A^{\mathrm{ar}(R)}$; a homomorphism from $\mathbf{A}$ to $\mathbf{B}$ is *strong* if $(a_1, \ldots, a_{\mathrm{ar}(R)}) \notin R^{\mathbf{A}}$ implies $(h(a_1), \ldots, h(a_{\mathrm{ar}(R)})) \notin R^{\mathbf{B}}$. An *embedding* from $\mathbf{A}$ to $\mathbf{B}$ is an injective strong homomorphism from $\mathbf{A}$ to $\mathbf{B}$. An *isomorphism* from $\mathbf{A}$ to $\mathbf{B}$ is a bijective embedding from $\mathbf{A}$ to $\mathbf{B}$.

In graph theory, an injective strong homomorphism is also called a "strong embedding," and the term "embedding" is used in the weaker sense of injective homomorphism. Here, we adopt the order-theoretic (and model-theoretic) terminology.

For a structure $\mathbf{A}$ and a sentence ϕ over the same vocabulary, we write $\mathbf{A} \models \phi$ if the sentence ϕ is *true* in the structure $\mathbf{A}$. When $\mathbf{A}$ is a structure, f is a mapping from variables to the universe of $\mathbf{A}$, and $\psi(x_1, \ldots, x_n)$ is a formula over the vocabulary of $\mathbf{A}$, we liberally write $\mathbf{A} \models \psi(f(x_1), \ldots, f(x_n))$ to indicate that ψ is satisfied by $\mathbf{A}$ and f.

A structure $\mathbf{G} = (V, E^{\mathbf{G}})$ with $\mathrm{ar}(E) = 2$ is called a *digraph*, and a *graph* if $E^{\mathbf{G}}$ is irreflexive and symmetric. We let $\mathcal{G}$ denote the class of all graphs. Let $\mathbf{G}$ be a digraph. The *degree* of $g \in G$, in symbols degree(g), is equal to $|\{(g', g) \in E^{\mathbf{G}} \mid g' \in G\} \cup \{(g, g') \in E^{\mathbf{G}} \mid g' \in G\}|$, and the *degree* of $\mathbf{G}$, in symbols degree$(\mathbf{G})$, is the maximum degree attained by the elements of $\mathbf{G}$.

A digraph $\mathbf{P} = (P, \leq^{\mathbf{P}})$ is a *poset* if $\leq^{\mathbf{P}}$ is a *reflexive*, *antisymmetric*, and *transitive* relation over P, that is, respectively, $\mathbf{P} \models \forall x(x \leq x)$, $\mathbf{P} \models \forall x \forall y((x \leq y \wedge y \leq x) \to x = y)$, and $\mathbf{P} \models \forall x \forall y \forall z((x \leq y \wedge y \leq z) \to x \leq z)$.

A *chain* in $\mathbf{P}$ is a subset $C \subseteq P$ such that $p \leq^{\mathbf{P}} q$ or $q \leq^{\mathbf{P}} p$ for all $p, q \in C$ (in particular, if P is a chain in $\mathbf{P}$, we call $\mathbf{P}$ itself a chain). We say that p and q are *incomparable* in $\mathbf{P}$ (denoted $p \parallel^{\mathbf{P}} q$) if $\mathbf{P} \not\models p \leq q \vee q \leq p$. An *antichain* in $\mathbf{P}$ is a subset $A \subseteq P$ such that $p \parallel^{\mathbf{P}} q$ for all $p, q \in A$ (in particular, if P is an antichain in $\mathbf{P}$, we call $\mathbf{P}$ itself an antichain).

Let $\mathbf{P}$ be a poset and let $p, q \in P$. We say that q *covers* p in $\mathbf{P}$ (denoted $p \prec^{\mathbf{P}} q$) if $p <^{\mathbf{P}} q$ and, for all $r \in P$, $p \leq^{\mathbf{P}} r <^{\mathbf{P}} q$ implies $p = r$. The *cover graph* of $\mathbf{P}$ is the digraph cover($\mathbf{P}$) with vertex set P and edge set $\{(p, q) \mid p \prec^{\mathbf{P}} q\}$. If $\mathcal{P}$ is a class of posets, we let $\text{cover}(\mathcal{P}) = \{\text{cover}(\mathbf{P}) \mid \mathbf{P} \in \mathcal{P}\}$. It is well known that computing the cover relation corresponding to a given order relation, and vice versa the order relation corresponding to a given cover relation, is feasible in polynomial time [Schröder 2003].

In the figures, posets are represented by their *Hasse diagrams*, that is, a diagram of their cover relation in which all edges are intended oriented upwards.

Let $\mathcal{P}$ be the class of all posets. A *poset invariant* is a mapping $\text{inv} : \mathcal{P} \to \mathbb{N}$ such that $\text{inv}(\mathbf{P}) = \text{inv}(\mathbf{Q})$ for all $\mathbf{P}, \mathbf{Q} \in \mathcal{P}$ such that $\mathbf{P}$ and $\mathbf{Q}$ are isomorphic. Let inv be any invariant over $\mathcal{P}$. Let $\mathcal{P}$ be any class of posets. We say that $\mathcal{P}$ is *bounded* with respect to inv if there exists $b \in \mathbb{N}$ such that $\text{inv}(\mathcal{P}) \leq b$ for all $\mathbf{P} \in \mathcal{P}$. Two poset invariants are *incomparable* if there exists a class of posets bounded under the first but unbounded under the second, and there exists a class of posets bounded under the second but unbounded under the first.

Problems. We refer the reader to a standard reference for the algorithmic setup of the model-checking problem (including the underlying computational model, encoding conventions for input structures and sentences, and the notion of *size* of a structure or sentence), and for further background in parameterized complexity theory (including the notion of *fpt many–one reduction* and *fpt Turing reduction*) [Flum and Grohe 2006].

Here, we mention that a *parameterized problem* (Q, κ) is a *problem* $Q \subseteq \Sigma^*$ together with a *parameterization* $\kappa : \Sigma^* \to \mathbb{N}$, where Σ is a finite alphabet. A parameterized problem (Q, κ) is *fixed-parameter tractable (with respect to κ)*, in short *fpt*, if there exists a decision algorithm for Q, a computable function $f : \mathbb{N} \to \mathbb{N}$, and a polynomial function $p : \mathbb{N} \to \mathbb{N}$, such that for all $x \in \Sigma^*$, the runtime of the algorithm on x is at most $f(\kappa(x)) \cdot p(|x|)$. We provide evidence that a parameterized problem is not fixed-parameter tractable by proving that the problem is W[1]-hard under fpt many–one reductions; this holds unless the exponential time hypothesis fails [Flum and Grohe 2006].

The (parameterized) computational problems under consideration are the following. Let σ be a relational vocabulary, $\mathcal{C}$ be a class of σ-structures, and $\mathcal{L} \subseteq \mathcal{FO}$ be a class of σ-sentences. The *model-checking problem* for $\mathcal{C}$ and $\mathcal{L}$, in symbols $\text{MC}(\mathcal{C}, \mathcal{L})$, is the problem of deciding, given $(\mathbf{A}, \phi) \in \mathcal{C} \times \mathcal{L}$, whether $\mathbf{A} \models \phi$. The parameterization, given an instance $(\mathbf{A}, \phi)$, returns the size of the encoding of ϕ. The *embedding problem* for $\mathcal{C}$, in symbols $\textsc{Emb}(\mathcal{C})$, is the problem of deciding, given a pair $(\mathbf{A}, \mathbf{B})$, where $\mathbf{A}$ is a σ-structure and $\mathbf{B}$ is a σ-structure in $\mathcal{C}$, whether $\mathbf{A}$ embeds into $\mathbf{B}$. The parameterization, given an instance $(\mathbf{A}, \mathbf{B})$, returns the size of the encoding of $\mathbf{A}$. The problems $\textsc{Hom}(\mathcal{C})$ and $\textsc{Iso}(\mathcal{C})$ are defined similarly in terms of homomorphisms and isomorphisms, respectively.

3. BASIC RESULTS

In this section, we set the stage for our parameterized and classical complexity results in Section 4 and Section 5, respectively. We start observing some basic reducibilities between the problems under consideration.

Proposition 3.1. *Let $\mathcal{C}$ be a class of structures. The following are equivalent.*

(*i*) $\textsc{Emb}(\mathcal{C})$ *is fixed-parameter tractable.*
(*ii*) $\text{MC}(\mathcal{C}, \mathcal{FO}(\exists, \wedge, \neg))$ *is fixed-parameter tractable.*
(*iii*) $\text{MC}(\mathcal{C}, \mathcal{FO}(\exists, \wedge, \vee, \neg))$ *is fixed-parameter tractable.*

Moreover, $\textsc{Emb}(\mathcal{C})$ *polynomial-time (thus fpt) many–one reduces to* $\text{MC}(\mathcal{C}, \mathcal{FO}(\exists, \wedge, \vee, \neg))$.

Proof. Let $\mathcal{C}$ be a class of σ-structures.

We give a polynomial-time many–one reduction of $\textsc{Emb}(\mathcal{C})$ to $\mathrm{MC}(\mathcal{C}, \mathcal{FO}(\exists, \wedge, \neg))$. Note that embedding a σ-structure $\mathbf{A}$ into a σ structure $\mathbf{B} \in \mathcal{C}$ reduces to checking whether $\mathbf{B}$ verifies the existential closure of the $\mathcal{FO}(\wedge, \neg)$-formula

$$\bigwedge_{a,a' \in A, a \neq a'} a \neq a' \wedge \bigwedge_{R \in \sigma} \left(\bigwedge_{\mathbf{a} \in R^{\mathbf{A}}} R\mathbf{a} \wedge \bigwedge_{\mathbf{a} \notin R^{\mathbf{A}}} \neg R\mathbf{a} \right).$$

Clearly, $\mathrm{MC}(\mathcal{C}, \mathcal{FO}(\exists, \wedge, \neg))$ polynomial-time many–one reduces to $\mathrm{MC}(\mathcal{C}, \mathcal{FO}(\exists, \wedge, \vee, \neg))$. We conclude the proof giving an fpt Turing (in fact, even truth table) reduction, from $\mathrm{MC}(\mathcal{C}, \mathcal{FO}(\exists, \wedge, \vee, \neg))$ to $\textsc{Emb}(\mathcal{C})$.

Let $\phi \in \mathcal{FO}(\exists, \wedge, \vee, \neg)$. Say that ϕ is *disjunctive* if $\phi = \psi_1 \vee \cdots \vee \psi_l$ and $\psi_i \in \mathcal{FO}(\exists, \wedge, \neg)$ for all $i \in [l]$. Clearly, for every $\phi \in \mathcal{FO}(\exists, \wedge, \vee, \neg)$, a disjunctive $\phi' \in \mathcal{FO}(\exists, \wedge, \vee, \neg)$ such that $\phi \equiv \phi'$ is computable by (equivalence-preserving) syntactic replacements.

Let ψ be a σ-sentence in $\mathcal{FO}(\exists, \wedge, \neg)$. Say that the disjunctive σ-sentence $\psi' = \chi_1 \vee \cdots \vee \chi_l$ is a *completion* of ψ if $\psi' \equiv \psi$ and, for all $i \in [l]$, if the quantifier prefix of χ_i is $\exists x_1 \ldots \exists x_m$, then:

—for all $(y, y') \in \{x_1, \ldots, x_m\}^2$, it holds that $y = y'$ or $y \neq y'$ occur in the quantifier free part of χ_i;
—for all $R \in \sigma$ and all $(y_1, \ldots, y_{\mathrm{ar}(R)}) \in \{x_1, \ldots, x_m\}^{\mathrm{ar}(R)}$, it holds that $Ry_1 \ldots y_{\mathrm{ar}(R)}$ or $\neg Ry_1 \ldots y_{\mathrm{ar}(R)}$ occur in the quantifier free part of χ_i;

moreover, ψ' is said *reduced* if, for all $i \in [l]$, χ_i is satisfiable, χ_i does not contain dummy quantifiers, and χ_i does not contain atoms of the form $y = y'$.

Let $\psi' = \chi_1 \vee \cdots \vee \chi_l$ be a reduced completion of the σ-sentence $\psi \in \mathcal{FO}(\exists, \wedge, \neg)$. Clearly, ψ' is computable from ψ as follows. Let $\exists x_1 \ldots \exists x_m$ be the quantifier prefix of ψ.

—For all $(y, y') \in \{x_1, \ldots, x_m\}^2$ such that neither $y = y'$ nor $y \neq y'$ occur in the quantifier free part of ψ, conjoin $(y = y' \vee y \neq y')$ to the quantifier free part of ψ.
—For all $R \in \sigma$ and $(y_1, \ldots, y_{\mathrm{ar}(R)}) \in \{x_1, \ldots, x_l\}^{\mathrm{ar}(R)}$ such that neither $Ry_1 \ldots y_{\mathrm{ar}(R)}$ nor $\neg Ry_1 \ldots y_{\mathrm{ar}(R)}$ occur in the quantifier free part of ψ, conjoin $(Ry_1 \ldots y_{\mathrm{ar}(R)} \vee \neg Ry_1 \ldots y_{\mathrm{ar}(R)})$ to the quantifier free part of ψ.
—Compute a disjunctive form of the resulting sentence, eliminate equality atoms and dummy quantifiers from each disjunct, and finally eliminate unsatisfiable disjuncts (empty disjunctions are false on all structures).

We are now ready to describe the fpt Turing reduction from $\mathrm{MC}(\mathcal{C}, \mathcal{FO}(\exists, \wedge, \vee, \neg))$ to $\textsc{Emb}(\mathcal{C})$. Let $(\mathbf{B}, \phi)$ be an instance of $\mathrm{MC}(\mathcal{C}, \mathcal{FO}(\exists, \wedge, \vee, \neg))$. The algorithm first computes a disjunctive form logically equivalent to ϕ, say $\phi \equiv \psi_1 \vee \cdots \vee \psi_l$, and then, for each $i \in [l]$, computes a reduced completion ψ'_i logically equivalent to ψ_i, say $\psi'_i \equiv \chi'_{i,1} \vee \cdots \vee \chi'_{i,l_i}$. Note that

$$\phi \equiv \bigvee_{i \in [l]} \bigvee_{j \in [l_i]} \chi'_{i,j}.$$

For each $i \in [l]$ and $j \in [l_i]$, let $\mathbf{A}_{i,j}$ be the σ-structure naturally corresponding to $\chi'_{i,j}$; namely, if $\exists x_1 \ldots \exists x_m$ is the quantifier prefix of $\chi'_{i,j}$, then the universe $A_{i,j}$ is $\{x_1, \ldots, x_m\}$, and $(y_1, \ldots, y_{\mathrm{ar}(R)}) \in R^{\mathbf{A}_{i,j}}$ if and only if $Ry_1 \ldots y_{\mathrm{ar}(R)}$ occurs in the quantifier free part of $\chi'_{i,j}$. Note that $\mathbf{A}_{i,j}$ embeds into $\mathbf{B}$ if and only if $\mathbf{B} \models \chi'_{i,j}$. The algorithm poses the query $(\mathbf{A}_{i,j}, \mathbf{B})$ to the problem $\textsc{Emb}(\mathcal{C})$, and it accepts if and only if at least one query answers positively.

We check that the algorithm gives indeed the desired fpt Turing reduction. For the complexity, given ϕ, the algorithm effectively computes the structures $\mathbf{A}_{i,j}$, so that

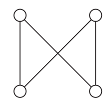

Fig. 2. The Hasse diagram of the bowtie poset **B** in Proposition 3.2.

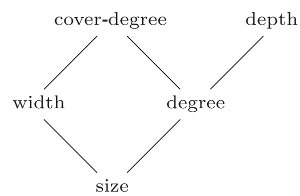

Fig. 3. The order of poset invariants induced by Proposition 3.3.

the size of each $\mathbf{A}_{i,j}$, as well as the number of $\mathbf{A}_{i,j}$'s, effectively depend on the size of ϕ only. Each query requires time polynomial in the size of $\mathbf{A}_{i,j}$ and $\mathbf{B}$. For the correctness, we claim that $\mathbf{B} \models \phi$ if and only if there exist $i \in [l]$ and $j \in [l_i]$ such that $\mathbf{A}_{i,j}$ embeds into $\mathbf{B}$. The backwards direction is clear. For the forwards direction, assume $\mathbf{B} \models \phi$. Then, there exist $i \in [l]$ and $j \in [l_i]$ such that $\mathbf{B} \models \chi'_{i,j}$. Then, $\mathbf{A}_{i,j}$ embeds into $\mathbf{B}$. □

The next observation is that model checking existential conjunctive logic (and thus the full existential logic) on posets is unlikely to be polynomial-time tractable, even if the poset is fixed. Let $\mathbf{B}$ be the bowtie poset defined by the universe $B = [4]$ and the covers $i \prec^{\mathbf{B}} j$ for all $i \in \{1, 2\}$ and $j \in \{3, 4\}$. See Figure 2.

PROPOSITION 3.2. $\text{MC}(\{\mathbf{B}\}, \mathcal{FO}(\exists, \wedge, \neg))$ *is* NP-*hard*.

PROOF. Let $\sigma = \{\leq, 1, 2, 3, 4\}$ be a relational vocabulary where $\text{ar}(\leq) = 2$ and $\text{ar}(i) = 1$ for all $i \in [4]$. Let $\mathbf{B}^*$ be the σ-structure such that $(B^*, \leq^{\mathbf{B}^*})$ is isomorphic to $\mathbf{B}$, say without loss of generality via the isomorphism $f(b) = b \in B^*$ for all $b \in B$, and where $b^{\mathbf{B}^*} = \{f(b)\} = \{b\}$ for all $b \in B$. Pratt and Tiuryn [1996, Theorem 2, $n = 2$] prove that $\text{Hom}(\{\mathbf{B}^*\})$ is NP-hard. We give a polynomial-time many–one reduction of $\text{Hom}(\{\mathbf{B}^*\})$ to $\text{MC}(\{\mathbf{B}\}, \mathcal{FO}(\exists, \wedge, \neg))$.

Let $\mathbf{A}$ be an instance of $\text{Hom}(\{\mathbf{B}^*\})$, and let ϕ be the existential closure of the conjunction of the following $\{\leq\}$-literals (thus, ϕ is an $\mathcal{FO}(\exists, \wedge, \neg)$-sentence on the vocabulary of $\mathbf{B}$):

—$z_i \neq z_j$, for all $1 \leq i < j \leq 4$;
—$z_i < z_j$, for all $i \in \{1, 2\}$ and $j \in \{3, 4\}$;
—$a = z_i$, for all $i \in [4]$ and $a \in i^{\mathbf{A}}$;
—$a \leq a'$, for all $a \leq^{\mathbf{A}} a'$.

It is easy to check that $\mathbf{A}$ maps homomorphically to $\mathbf{B}^*$ if and only if $\mathbf{B} \models \phi$. □

In contrast, model checking existential logic on any fixed poset $\mathbf{P}$ is trivially fixed-parameter tractable (the instance is a structure of constant size, and a sentence taken as a parameter). However, there are classes of posets in which the embedding problem, and hence, by Proposition 3.1, the problem of model checking existential logic, is unlikely to be fixed-parameter tractable, as we now show.

First, we introduce a family of poset invariants and relate them as in Figure 3. Let $\mathbf{P}$ be a poset.

—The *size* of $\mathbf{P}$ is the cardinality of its universe, $|P|$.
—The *width* of $\mathbf{P}$, in symbols width($\mathbf{P}$), is the maximum size attained by an antichain in $\mathbf{P}$.
—The *depth* of $\mathbf{P}$, in symbols depth($\mathbf{P}$), is the maximum size attained by a chain in $\mathbf{P}$.
—The *degree* of $\mathbf{P}$, in symbols degree($\mathbf{P}$), is the degree of the order relation of $\mathbf{P}$, that is, degree($\leq^{\mathbf{P}}$).
—The *cover-degree* of $\mathbf{P}$, in symbols cover-degree($\mathbf{P}$), is the degree of the cover relation of $\mathbf{P}$, that is, degree(cover($\mathbf{P}$)).

PROPOSITION 3.3. *Let $\mathcal{P}$ be a class of posets.*

(*i*) *$\mathcal{P}$ has bounded degree if and only if $\mathcal{P}$ has bounded depth and bounded cover-degree.*
(*ii*) *If $\mathcal{P}$ has bounded width, then $\mathcal{P}$ has bounded cover-degree.*
(*iii*) *$\mathcal{P}$ has bounded size if and only if $\mathcal{P}$ has bounded width and bounded degree.*

PROOF. We prove *(i)*. Assume that $\mathcal{P}$ has bounded degree. Let $\mathbf{P} \in \mathcal{P}$. Then cover-degree($\mathbf{P}$) $\leq$ degree($\mathbf{P}$) follows from the fact that cover($\mathbf{P}$) is contained in $\leq^{\mathbf{P}}$, while depth($\mathbf{P}$) $\leq$ degree($\mathbf{P}$) $+ 1$ follows from the fact that each chain forms a complete directed acyclic subgraph in $\mathbf{P}$. Conversely, let $d \in \mathbb{N}$ and $c \in \mathbb{N}$ be the largest depth and cover-degree attained by a poset in $\mathcal{P}$, respectively. Then, for every $\mathbf{P} \in \mathcal{P}$ and $p \in P$, it holds that degree(p) $\leq c^d$, hence $\mathcal{P}$ has bounded degree.

We prove *(ii)*. Let w be the largest width attained by a poset in $\mathcal{P}$. Then, for every $\mathbf{P} \in \mathcal{P}$ and $p \in P$, it holds that cover-degree(p) $\leq 2w$, because the lower covers of p and the upper covers of p form antichains in $\mathbf{P}$, hence p has at most $2w$ lower or upper covers. Hence, $\mathcal{P}$ has bounded cover-degree.

We prove *(iii)*. Assume that $\mathcal{P}$ has bounded size. Let s be the largest size attained by a poset in $\mathcal{P}$. Then, for every $\mathbf{P} \in \mathcal{P}$, it holds that width($\mathbf{P}$), degree($\mathbf{P}$) $\leq s$, that is, $\mathcal{P}$ has bounded width and bounded degree. Conversely, by *(i)*, $\mathcal{P}$ has bounded depth. Let d and w be the largest depth and width attained by a poset in $\mathcal{P}$, respectively. Let $\mathbf{P} \in \mathcal{P}$. By Dilworth's theorem, there exist w chains in $\mathbf{P}$ whose union is P, hence size($\mathbf{P}$) $\leq w \cdot d$. We conclude that $\mathcal{P}$ has bounded size. $\square$

The previous proposition, together with the observation that bounded width and bounded degree (bounded width and bounded depth, bounded cover-degree and bounded depth, respectively) are incomparable, justifies the order in Figure 3, whose interpretation is the following: invariant inv is below invariant inv$'$ if and only if, for every class $\mathcal{P}$ of posets, if $\mathcal{P}$ is bounded under inv, then $\mathcal{P}$ is bounded under inv$'$.

The emerging hierarchy of poset invariants will provide a measure of tightness for our positive algorithmic results, once we manage to surround them with complexity results on covering neighboring classes.

To this aim, we immediately observe that there exists a class of posets of bounded depth for which the embedding problem, and hence model-checking existential first-order logic, is W[1]-hard. Given any graph $\mathbf{G} \in \mathcal{G}$, construct a poset $r(\mathbf{G}) = \mathbf{P}$ by taking $|G|$ pairwise disjoint 3-element chains, and covering the bottom of the ith chain by the top of the jth chain if and only if i and j are adjacent in $\mathbf{G}$. Note that depth($\mathbf{P}$) ≤ 3. Hence, the class $\mathcal{P}_{\text{depth}} = \{r(\mathbf{G}) \mid \mathbf{G} \in \mathcal{G}\}$ has bounded depth. See Figure 4.

PROPOSITION 3.4. EMB($\mathcal{P}_{\text{depth}}$) *is* W[1]-*hard.*

PROOF. CLIQUE fpt many–one reduces to EMB($\mathcal{P}_{\text{depth}}$) by mapping ($\mathbf{G}$, k) to ($r(\mathbf{K}_k)$, $r(\mathbf{G})$), where $\mathbf{K}_k$ denotes the complete graph on k vertices. It is readily verified that $\mathbf{G}$ contains a clique on k vertices if and only if $r(\mathbf{K}_k)$ embeds into $r(\mathbf{G})$. $\square$

The goal of the technical part of this article is to establish the facts leading from Figure 3 to Figure 1:

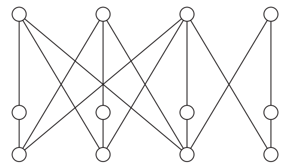

Fig. 4. The poset $r(\mathbf{G})$ in the construction of Proposition 3.4, where $\mathbf{G}$ is the graph whose universe is $G = [4]$ and whose edge relation $E^{\mathbf{G}}$ is the symmetric closure of $\{(1, 2), (1, 3), (2, 3), (3, 4)\}$.

—For the parameterized complexity of model checking existential logic, we have tractability on bounded degree classes by Seese's algorithm [Seese 1996], and hardness on (certain) bounded depth classes by Proposition 3.4. In Section 4, we establish tractability on bounded width classes by Theorem 4.13, and hardness on (certain) bounded cover-degree classes by Proposition 4.15.
—For the classical complexity of the embedding problem (Section 5), Proposition 5.1 establishes tractability on bounded size classes, Theorem 5.4 establishes hardness on (certain) bounded width classes, and Theorem 5.6 establishes hardness on (certain) bounded degree classes.

We conclude the section by relating our work on posets of bounded width with previous work on digraphs of bounded clique-width, and showing that our results are indeed independent.

Clique-width is a prominent invariant of undirected as well as directed graphs that generalizes treewidth [Courcelle and Olariu 2000]; in particular, it is known that monadic second-order logic (precisely, $\mathcal{MSO}_1$) is fixed-parameter tractable on digraphs of bounded clique-width [Courcelle et al. 2000], thus:

Observation 1. MC($\mathcal{P}$, $\mathcal{FO}$) *is fixed-parameter tractable for any class $\mathcal{P}$ of posets such that the clique width of $\mathcal{P}$ is bounded.*

Since it is possible to compute the cover relation from the order relation (and vice versa) in polynomial time, one might wonder whether using the clique-width of the cover graph would allow us to efficiently model check wider classes of posets. This turns out not to be the case. In particular, point (2) of Courcelle and Engelfriet [2012, Corollary 1.53] establishes that the image of a set of graphs of bounded clique-width under any *MSO transduction* operation has bounded clique-width, and Examples 1.32 and 1.33 in Courcelle and Engelfriet [2012] show that the transitive closure and transitive retract of a directed acyclic graph are examples of MSO transductions. We therefore have the following.

Observation 2. *For any class $\mathcal{P}$ of posets, the clique width of $\mathcal{P}$ is bounded if and only if the clique width of* cover($\mathcal{P}$) *is bounded.*

A natural class of posets that is easily observed having clique-width bounded by 2 (despite having unbounded treewidth) is the class of *series parallel posets*. However, we show that there exist classes of posets of bounded width which do not have bounded clique-width (if this were not the case, Theorem 4.13 would follow from Observation 1).

Proposition 3.5. *There exists a class $\mathcal{P}$ of posets that has bounded width but does not have bounded clique-width.*

Proof. For each $i \in \mathbb{N}$, we define a poset $\mathbf{P}_i$ as follows; see Figure 5. The universe is $P_i = \{p_{a,b} \mid a, b \in [i]\}$ and the cover relation is defined by the following pairs:

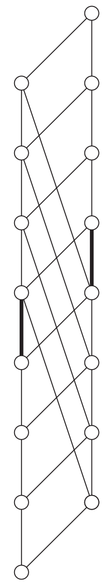

Fig. 5. The Hasse diagram of the poset $\mathbf{P}_4$ in Proposition 3.5. The chain on the left is $p_{1,1} \prec^{\mathbf{P}_4} \cdots \prec^{\mathbf{P}_4} p_{1,4} \prec^{\mathbf{P}_4} p_{3,1} \prec^{\mathbf{P}_4} \cdots \prec^{\mathbf{P}_4} p_{3,4}$. The chain on the right is $p_{2,1} \prec^{\mathbf{P}_4} \cdots \prec^{\mathbf{P}_4} p_{2,4} \prec^{\mathbf{P}_4} p_{4,1} \prec^{\mathbf{P}_4} \cdots \prec^{\mathbf{P}_4} p_{4,4}$. The second item in the construction creates edges between the left and right chains. A 4×4 grid appears upon removing the edges created by the third item of the construction (drawn thick in the diagram).

—$p_{a,1} \prec^{\mathbf{P}_i} \cdots \prec^{\mathbf{P}_i} p_{a,i}$ for all $a \in [i]$;
—$p_{a,b} \prec^{\mathbf{P}_i} p_{a+1,b}$ for all $a \in [i-1]$ and $b \in [i]$;
—$p_{a,i} \prec^{\mathbf{P}_i} p_{a+2,1}$ for all $a \in [i-2]$.

Note that $\mathbf{P}_i$ has width 2 and cover($\mathbf{P}_i$) has degree 4. We will prove that $\mathcal{P} = \{\mathbf{P}_i \mid i \in \mathbb{N}\}$ has unbounded clique-width.

Let $\mathcal{H}$ be the class of undirected (simple) graphs corresponding to the covers of $\mathcal{P}$ (that is, $\mathcal{H}$ contains the symmetric closure of cover($\mathbf{P}_i$) for all $\mathbf{P}_i \in \mathcal{P}$). Notice that $\mathcal{H}$ has bounded degree. Moreover, notice that the undirected graph corresponding to cover($\mathbf{P}_i$) contains an $i \times i$ grid as a subgraph. Indeed, one may define the jth row of the grid to consist of the (path generated by the) chain $p_{1,j} \prec^{\mathbf{P}_i} p_{2,j} \prec^{\mathbf{P}_i} \cdots \prec^{\mathbf{P}_i} p_{i,j}$ and, similarly, the jth column of the grid to consist of the (path generated by the) chain $p_{j,1} \prec^{\mathbf{P}_i} p_{j,2} \prec^{\mathbf{P}_i} \cdots \prec^{\mathbf{P}_i} p_{j,i}$. Hence, $\mathcal{H}$ has unbounded treewidth.

By point (3) of Courcelle and Engelfriet [2012, Corollary 1.53], a class of undirected (simple) graphs of bounded degree has bounded treewidth if and only if it has bounded clique-width. Since $\mathcal{H}$ has bounded degree and unbounded treewidth, it follows that $\mathcal{H}$ has unbounded clique-width. It is a folklore fact that, for any graph $\mathbf{G}$ and any orientation $\mathbf{G}'$ of $\mathbf{G}$, the clique-width of $\mathbf{G}$ is bounded by the clique-width of $\mathbf{G}'$ (one can use the same decomposition in this direction). Since cover($\mathcal{P}$) contains one orientation for each graph in $\mathcal{H}$ and since $\mathcal{H}$ has unbounded clique-width, we conclude that cover($\mathcal{P}$) has unbounded clique-width; therefore, $\mathcal{P}$ has unbounded clique-width by Observation 2. □

4. PARAMETERIZED COMPLEXITY

In this section, we study the parameterized complexity of the problems under consideration. The section is organized as follows.

—In Section 4.1, we develop a fixed-parameter tractable algorithm for the embedding problem on posets of bounded width (Theorem 4.12), which yields that model checking existential logic on such posets is fixed-parameter tractable (Theorem 4.13).
—In Section 4.2, we provide a reduction proving W[1]-hardness of model checking existential logic on posets of bounded cover-degree (Proposition 4.15).

4.1. Embedding is FPT on Bounded Width Posets

The idea of the proof is the following. Each poset $\mathbf{P}$ in a class $\mathcal{P}$ of posets of bounded width is associated with a family of relational structures, called "compilations," such that the following two key facts hold.

(i) Every compilation $\mathbf{P}^*$ of $\mathbf{P}$ is computable in polynomial time. Moreover, $\mathbf{P}^*$ has a semilattice polymorphism (Lemma 4.6), which implies that the homomorphism problem $\textsc{Hom}(\mathbf{P}^*)$ is polynomial-time tractable (Lemma 4.7).
(ii) Given posets $\mathbf{Q}$ and $\mathbf{P}$ in $\mathcal{P}$, the question whether $\mathbf{Q}$ embeds into $\mathbf{P}$ reduces to a search for "suitable" compilations $\mathbf{Q}^*$ and $\mathbf{P}^*$ such that $\mathbf{Q}^* \in \textsc{Hom}(\mathbf{P}^*)$ (Lemma 4.10).

It turns out that searching over all compilations, the brutal approach suggested by (ii) involves searching over all functions from P to Q and therefore takes exponential time in the worst case. The key observation is that, as far as embedding queries are concerned, the family of all functions from P to Q can be succinctly implemented by a family of hashing functions containing, for all subsets K of P having size $|Q|$, a function whose restriction to K is injective (Lemma 4.11). Such a representative family is computable in fixed-parameter tractable time (Theorem 4.2), from which fixed-parameter tractability of the embedding problem over $\mathcal{P}$ follows (Theorem 4.12).

The rest of the section is organized as follows. In Section 4.1.1, we recall some known facts about semilattice structures, hash functions, and chain partitions. In Section 4.1.2, we describe the compilation of a poset. In Section 4.1.3, we prove that poset compilations have tractable homomorphism problems. In Section 4.1.4, we give a characterization of embeddings between posets in terms of homomorphisms between compiled posets, which we then improve in Section 4.1.5 to a fixed-parameter tractable characterization using hashing functions.

4.1.1. Known Facts. Our proof relies on some known facts, collected here.

Semilattice Polymorphisms. Let σ be a finite relational vocabulary, and let $\mathbf{A}$ be a σ-structure. Let $f : A^m \to A$ be an m-ary function on A. We say that f is a *polymorphism* of $\mathbf{A}$ (or, $\mathbf{A}$ *admits* f) if f *preserves* all relations of $\mathbf{A}$, that is, for all $R \in \sigma$, where $\mathrm{ar}(R) = r$, if

$$(a_{1,1}, a_{1,2}, \ldots, a_{1,r}), \ldots, (a_{m,1}, a_{m,2}, \ldots, a_{m,r}) \in R^{\mathbf{A}},$$

then

$$(f(a_{1,1}, a_{2,1}, \ldots, a_{m,1}), \ldots, f(a_{1,r}, a_{2,r}, \ldots, a_{m,r})) \in R^{\mathbf{A}}.$$

We say that a function $f : A^2 \to A$ is a *semilattice* function over A if f is idempotent, associative, and commutative on A, that is, $f(a, a) = a$, $f(a, f(a', a'')) = f(f(a, a'), a'')$, and $f(a, a') = f(a', a)$ for all $a, a', a'' \in A$.

Theorem 4.1 [Jeavons et al. 1997]. *There exists a polynomial-time algorithm that, for all σ-structures $\mathbf{A}$, given a semilattice polymorphism f of $\mathbf{A}$ and a σ-structure $\mathbf{B}$, decides whether $\mathbf{B}$ maps homomorphically to $\mathbf{A}$.*

Hash Functions. Let M and N be sets, and let $k \in \mathbb{N}$. A *k-perfect family of hash functions* from M to N is a family Λ of functions from M to N such that, for every

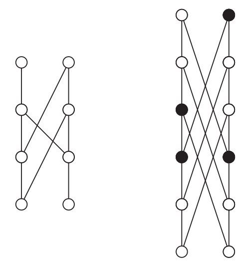

Fig. 6. The Hasse diagrams for the posets $\mathbf{Q}$ (left) and $\mathbf{P}$ (right) in Example 4.4. In the diagram for $\mathbf{Q}$, the chain $d_{11} \prec^{\mathbf{Q}} d_{12} \prec^{\mathbf{Q}} d_{13} \prec^{\mathbf{Q}} d_{14}$ sits on the left and the chain $d_{21} \prec^{\mathbf{Q}} d_{22} \prec^{\mathbf{Q}} d_{23} \prec^{\mathbf{Q}} d_{24}$ sits on the right. In the diagram for $\mathbf{P}$, the chain $c_{11} \prec^{\mathbf{Q}} c_{12} \prec^{\mathbf{Q}} \cdots \prec^{\mathbf{Q}} c_{16}$ sits on the left and the chain $c_{21} \prec^{\mathbf{Q}} c_{22} \prec^{\mathbf{Q}} \cdots \prec^{\mathbf{Q}} c_{26}$ sits on the right. The white points in P form the image of the embedding $e : Q \to P$ in Example 4.4.

subset $K \subseteq M$ of cardinality k, there exists $\lambda \in \Lambda$ such that λ restricted to K is injective.

THEOREM 4.2 [FLUM AND GROHE 2006, THEOREM 13.14]. *Let C be a finite set. There exists an algorithm that, given C and $k \in \mathbb{N}$, computes a k-perfect family $\Lambda_{C,k}$ of hash functions from C to $[k]$ of cardinality $2^{O(k)} \cdot \log^2 |C|$ in time $2^{O(k)} \cdot |C| \cdot \log^2 |C|$.*

Chain Partitions. Let $\mathbf{P}$ be a poset. A tuple $(\mathbf{C}_1, \ldots, \mathbf{C}_w)$ is a *chain partition* of $\mathbf{P}$ if there exist nonempty subsets $C_1, \ldots, C_w$ of P forming a partition of P (their union covers P and they are pairwise disjoint) such that $\mathbf{C}_i$, the substructure of $\mathbf{P}$ induced by C_i, is a chain ($i \in [w]$).

THEOREM 4.3 [FELSNER ET AL. 2003, THEOREM 1]. *Let $\mathbf{P}$ be a poset. Then, in time $O(\text{width}(\mathbf{P}) \cdot |P|^2)$, it is possible to compute both $\text{width}(\mathbf{P})$ and a chain partition of $\mathbf{P}$ of the form $(\mathbf{C}_1, \ldots, \mathbf{C}_{\text{width}(\mathbf{P})})$.*

The notions of chain partition and poset embedding are illustrated in the following example.

Example 4.4. Let $\mathbf{Q}$ be the poset with universe $Q = D_1 \cup D_2$, where $D_1 = \{d_{11}, d_{12}, d_{13}, d_{14}\}$ and $D_2 = \{d_{21}, d_{22}, d_{23}, d_{24}\}$, and cover relation $d_{11} \prec^{\mathbf{Q}} d_{12} \prec^{\mathbf{Q}} d_{13} \prec^{\mathbf{Q}} d_{14}$, $d_{21} \prec^{\mathbf{Q}} d_{22} \prec^{\mathbf{Q}} d_{23} \prec^{\mathbf{Q}} d_{24}$, $d_{11} \prec^{\mathbf{Q}} d_{23}$, $d_{12} \prec^{\mathbf{Q}} d_{24}$, and $d_{22} \prec^{\mathbf{Q}} d_{13}$. Then, $(\mathbf{D}_1, \mathbf{D}_2)$ is a chain partition of $\mathbf{Q}$. See Figure 6 (left).

Let $\mathbf{P}$ be the poset with universe $P = C_1 \cup C_2$, where $C_1 = \{c_{11}, \ldots, c_{16}\}$ and $C_2 = \{c_{21}, \ldots, c_{26}\}$, and cover relation $c_{11} \prec^{\mathbf{Q}} \cdots \prec^{\mathbf{Q}} c_{16}$, $c_{21} \prec^{\mathbf{Q}} \cdots \prec^{\mathbf{Q}} c_{26}$, $c_{11} \prec^{\mathbf{Q}} c_{24}$, $c_{12} \prec^{\mathbf{Q}} c_{25}$, $c_{13} \prec^{\mathbf{Q}} c_{26}$, $c_{21} \prec^{\mathbf{Q}} c_{14}$, $c_{22} \prec^{\mathbf{Q}} c_{15}$, and $c_{23} \prec^{\mathbf{Q}} c_{16}$. Then, $(\mathbf{C}_1, \mathbf{C}_2)$ is a chain partition of $\mathbf{P}$. See Figure 6 (right).

The mapping $e : Q \to P$ defined by $e(d_{11}) = c_{11}$, $e(d_{12}) = c_{12}$, $e(d_{13}) = c_{15}$, $e(d_{14}) = c_{16}$, $e(d_{21}) = c_{21}$, $e(d_{22}) = c_{22}$, $e(d_{23}) = c_{24}$, $e(d_{24}) = c_{25}$ embeds $\mathbf{Q}$ into $\mathbf{P}$.

4.1.2. Poset Compilation. We now define the "compilations" associated to a poset; the construction is illustrated in Example 4.5. Note that a compilation depends not only on the poset itself, but also on a "coordinatization" and a "coloring," as follows.

Let $\mathbf{P}$ be a poset of width w, let $(\mathbf{C}'_1, \ldots, \mathbf{C}'_w)$ be a chain partition of $\mathbf{P}$, and let $w' \leq w$. Relative to $(\mathbf{C}'_1, \ldots, \mathbf{C}'_w)$, a *coordinatization* of $\mathbf{P}$ is a subtuple $(\mathbf{C}_1, \ldots, \mathbf{C}_{w'})$ of $(\mathbf{C}'_1, \ldots, \mathbf{C}'_w)$ of $\mathbf{P}$; that is, $(\mathbf{C}_1, \ldots, \mathbf{C}_{w'})$ is obtained from $(\mathbf{C}'_1, \ldots, \mathbf{C}'_w)$ by deleting $w - w'$ coordinates. For all $j \in [w']$, let $k_j \in \mathbb{N}$ be such that $k_j \leq |C_j|$ and let Λ_j be a family of functions from C_j to $[k_j]$. Relative to $(\mathbf{C}_1, \ldots, \mathbf{C}_{w'})$, a *coloring* of $\mathbf{P}$ is a tuple $(\lambda_1, \ldots, \lambda_{w'}) \in \Lambda_1 \times \cdots \times \Lambda_{w'}$. We freely omit mentioning the chain partition underlying a coordinatization, as well as the coordinatization underlying a coloring, if they are clear from the context.

For a suitable relational vocabulary σ depending on w' and k_j for all $j \in [w']$, we define the σ-structure

$$\text{compil}(\mathbf{P}, \mathbf{C}_1, \dots, \mathbf{C}_{w'}, \lambda_1, \dots, \lambda_{w'}),$$

which we call the *compilation* of $\mathbf{P}$ with respect to the coordinatization $(\mathbf{C}_1, \dots, \mathbf{C}_{w'})$ and the coloring $(\lambda_1, \dots, \lambda_{w'})$, as follows. In the sequel, we use compil($\mathbf{P}$) as a shorthand if the intended coordinatization and coloring are clear.

The relational vocabulary σ of compil($\mathbf{P}$) consists of one binary relation symbol L, two unary relation symbols $I_{\{j,j'\}}$ and $O_{(j,j')}$ for each 2-element subset $\{j, j'\}$ of $[w']$, and one binary relation symbol $R_{(j,k)}$ for each $j \in [w']$ and $k \in [k_j]$.

The universe of compil($\mathbf{P}$) is

$$\text{compil}(P) = C_1 \times C_2 \times \cdots \times C_{w'},$$

that is, the Cartesian product of (the universes of) a family of chains $\mathbf{C}_1, \dots, \mathbf{C}_{w'}$ partitioning (the universe of) $\mathbf{P}$.

Let $\mathbf{c} = (c_1, \dots, c_{w'})$ and $\mathbf{c}' = (c'_1, \dots, c'_{w'})$ be elements of compil($\mathbf{P}$), and let $K_{(j,k)} = \{c \in C_j \mid \lambda_j(c) = k\}$. The interpretation of the vocabulary σ in compil($\mathbf{P}$) is the following:

(i) The interpretation of L is the set of all pairs $(\mathbf{c}, \mathbf{c}')$ such that $c_1 \leq^{\mathbf{P}} c'_1, \dots, c_{w'} \leq^{\mathbf{P}} c'_{w'}$, that is, the natural lattice order inherited by compil($\mathbf{P}$) from $\mathbf{C}_1, \dots, \mathbf{C}_{w'}$.

(ii) For each 2-element subset $\{j, j'\}$ of $[w']$, $I_{\{j,j'\}}$ and $O_{(j,j')}$ are interpreted, respectively, over $I_{\{j,j'\}} = \{\mathbf{c} \mid c_j \parallel^{\mathbf{P}} c_{j'}\}$ and $O_{(j,j')} = \{\mathbf{c} \mid c_j <^{\mathbf{P}} c_{j'}\}$. In other words, $I_{\{j,j'\}}$ and $O_{(j,j')}$ record, respectively, incomparabilities and comparabilities between the jth and j'th coordinate (corresponding to elements in the chains $\mathbf{C}_j$ and $\mathbf{C}_{j'}$, respectively) of the tuples in compil(P).

(iii) For each $j \in [w']$ and $k \in [k_{i_j}]$, $R_{(j,k)}$ is interpreted over the subset of the interpretation of L defined by

$$\{(\mathbf{c}, \mathbf{c}') \in L^{\text{compil}(\mathbf{P})} \mid c_j \in K_{(j,k)}, c_j = c'_j\},$$

that is, the restriction of the lattice order of compil($\mathbf{P}$) to those pairs of tuples in compil(P) such that their jth coordinate is colored k by λ_j.

The basic motivation underlying the compilation of a poset is to reduce poset embeddings to homomorphisms of poset compilations. We illustrate this point using the compilations $\mathbf{Q}^*$ and $\mathbf{P}^*$ of the posets $\mathbf{Q}$ and $\mathbf{P}$ in Example 4.5.

On the one hand, if h maps $\mathbf{Q}^*$ homomorphically to $\mathbf{P}^*$ then, roughly, the following happens: because of the L relation, the basic order structure (whose Hasse diagram is represented by the thin solid edges in the first picture in Figure 7) of $\mathbf{Q}^*$ is mimicked by the basic order structure (thin solid edges in the first picture in Figure 8) of $\mathbf{P}^*$; gray points in the first picture in Figure 7 go to gray points in the first picture in Figure 8, because of the $I_{\{j,j'\}}$ relations; light (respectively, dark) gray points in the second picture in Figure 7 go to light (respectively, dark) gray points in the second picture in Figure 8, because of the $O_{(j,j')}$ relations; chains of points linked by dotted (respectively, medium solid, thick solid, dashed) edges in the third (respectively, fourth) picture in Figure 7 go to chains of points linked by dotted (respectively, medium solid, thick solid, dashed) edges in the third (respectively, fourth) picture in Figure 8, because of the $R_{(j,k)}$ relations. Altogether, the structure of $\mathbf{Q}^*$ that h is able to find in $\mathbf{P}^*$ allows for deriving an embedding of $\mathbf{Q}$ into $\mathbf{P}$; this intuition is formally implemented by the implication (ii) $\Rightarrow$ (i) in Lemma 4.10.

On the other hand, if $\mathbf{Q}$ embeds into $\mathbf{P}$, say, as in Example 4.4, then it is easy to find suitable coordinatizations and colorings yielding compilations $\mathbf{Q}^*$ of $\mathbf{Q}$ and $\mathbf{P}^*$ of $\mathbf{P}$

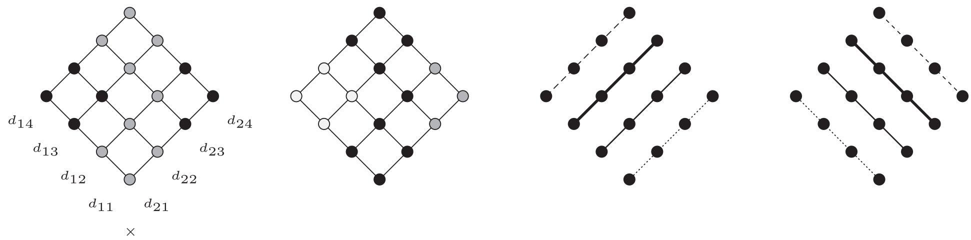

Fig. 7. Describing the structure $\mathbf{Q}^*$ in Example 4.5. Recall that $D_1 = \{d_{11}, d_{12}, d_{13}, d_{14}\}$ and $D_2 = \{d_{21}, d_{22}, d_{23}, d_{24}\}$. From left to right: The first picture displays the interpretation of L (thin solid edges) and $I_{\{1,2\}}$ (gray points) induced by *(i)* and *(ii)*. The second picture displays the interpretation of L (thin solid edges), $O_{(2,1)}$ (light gray points), and $O_{(1,2)}$ (dark gray points) induced by *(i)* and *(ii)*. The third picture displays the interpretation of $R_{(1,1)}$ (dotted edges), $R_{(1,2)}$ (medium solid edges), $R_{(1,3)}$ (thick solid edges), and $R_{(1,4)}$ (dashed edges), as induced by *(iii)* and λ_1. Similarly, the fourth picture displays the interpretation of $R_{(2,1)}$, $R_{(2,2)}$, $R_{(2,3)}$, and $R_{(2,4)}$ induced by *(iii)* and λ_2.

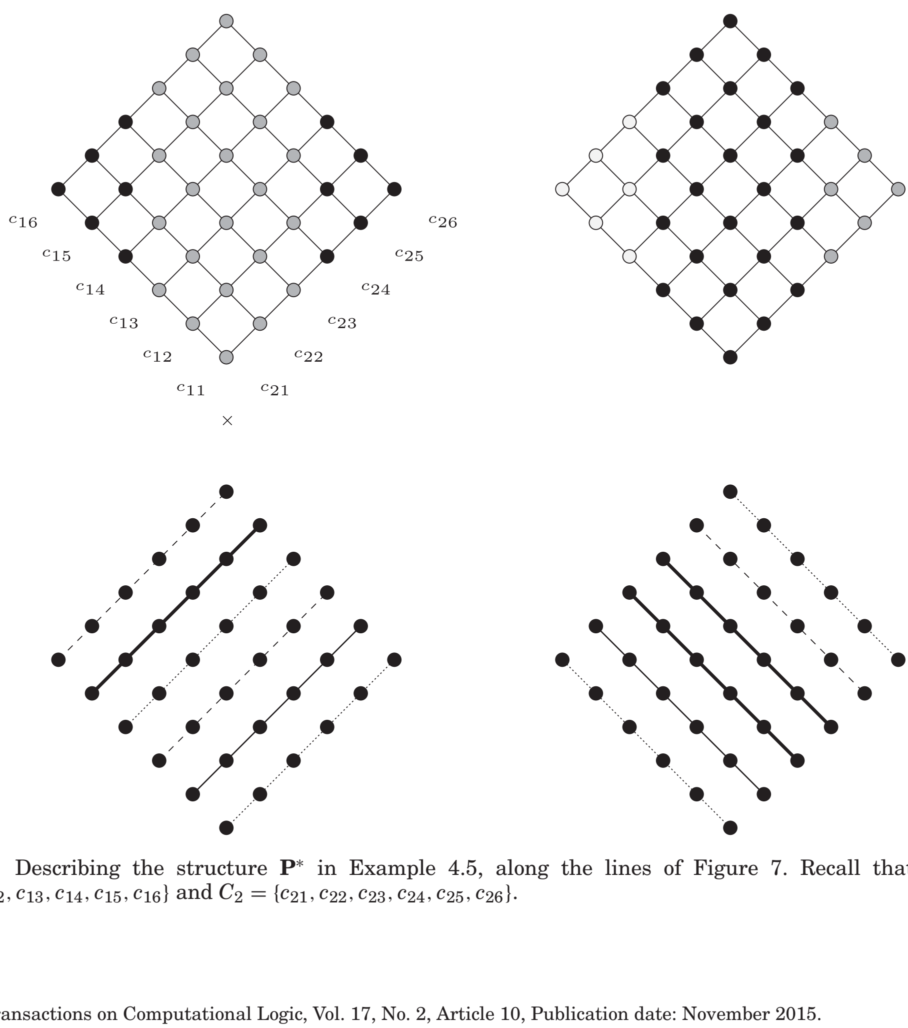

Fig. 8. Describing the structure $\mathbf{P}^*$ in Example 4.5, along the lines of Figure 7. Recall that $C_1 = \{c_{11}, c_{12}, c_{13}, c_{14}, c_{15}, c_{16}\}$ and $C_2 = \{c_{21}, c_{22}, c_{23}, c_{24}, c_{25}, c_{26}\}$.

such that $\mathbf{Q}^*$ maps homomorphically to $\mathbf{P}^*$; this intuition is formally implemented by the implication $(i) \Rightarrow (ii)$ in Lemma 4.10, and illustrated in Example 4.9.

Example 4.5. Let $\mathbf{Q}$ and $(\mathbf{D}_1, \mathbf{D}_2)$ be as in Example 4.4. Take the coordinatization $(\mathbf{D}_1, \mathbf{D}_2)$. Let $k_1 = k_2 = 4 = |D_1| = |D_2|$. Let $\mu_1 : D_1 \to [k_1]$ be defined by $\mu_1(d_{11}) = 1$, $\mu_1(d_{12}) = 2$, $\mu_1(d_{13}) = 3$, and $\mu_1(d_{14}) = 4$. Let $\mu_2 : D_2 \to [k_2]$ be defined by $\mu_2(c_{21}) = 1$, $\mu_2(c_{22}) = 2$, $\mu_2(c_{23}) = 3$, and $\mu_2(c_{24}) = 4$. Then, $\mathbf{Q}^* = \mathrm{compil}(\mathbf{Q}, \mathbf{D}_1, \mathbf{D}_2, \mu_1, \mu_2)$ is depicted in Figure 7.

Let $\mathbf{P}$ and $(\mathbf{C}_1, \mathbf{C}_2)$ be as in Example 4.4. Take the coordinatization $(\mathbf{C}_1, \mathbf{C}_2)$. Let $k_1 = k_2 = 4 \leq 6 = |C_1| = |C_2|$. Let $\lambda_1 : C_1 \to [k_1]$ be defined by $\lambda_1(c_{11}) = 1$, $\lambda_1(c_{12}) = 2$, $\lambda_1(c_{13}) = 4$, $\lambda_1(c_{14}) = 1$, $\lambda_1(c_{15}) = 3$, and $\lambda_1(c_{16}) = 4$. Let $\lambda_2 : C_2 \to [k_2]$ be defined by $\lambda_2(c_{21}) = 1$, $\lambda_2(c_{22}) = 2$, $\lambda_2(c_{23}) = 3$, $\lambda_2(c_{24}) = 3$, $\lambda_2(c_{25}) = 4$, and $\lambda_2(c_{26}) = 1$. Then, $\mathbf{P}^* = \mathrm{compil}(\mathbf{P}, \mathbf{C}_1, \mathbf{C}_2, \lambda_1, \lambda_2)$ is depicted in Figure 8.

4.1.3. Semilattice Polymorphisms on Compiled Posets. A key property of poset compilations is that they admit semilattice polymorphisms. Namely, let

$$s : \mathrm{compil}(P)^2 \to \mathrm{compil}(P)$$

be the binary function defined as follows. Let $\mathbf{c} = (c_1, \ldots, c_{w'})$ and $\mathbf{c}' = (c'_1, \ldots, c'_{w'})$ be elements in $\mathrm{compil}(P)$. Let $j \in [w']$. Recalling that $\mathbf{C}_{i_j}$ is a chain, let $d_j = \min^{\mathbf{C}_{i_j}}(c_j, c'_j)$. Define

$$s(\mathbf{c}, \mathbf{c}') = (d_1, \ldots, d_{w'}). \tag{1}$$

Clearly, s is idempotent, associative, and commutative, hence s is a semilattice function over $\mathrm{compil}(P)$. In view of the following lemma, we call the function s in Equation (1) the *canonical semilattice polymorphism* of the compilation $\mathrm{compil}(\mathbf{P})$.

LEMMA 4.6. *Let* $\mathbf{P}$ *be a poset,* $(\mathbf{C}_1, \ldots, \mathbf{C}_{w'})$ *be a coordinatization of* $\mathbf{P}$*, and* $(\lambda_1, \ldots, \lambda_{w'})$ *be a coloring of* $\mathbf{P}$. *The function* s *in Equation (1) is a polymorphism of* $\mathrm{compil}(\mathbf{P}, \mathbf{C}_1, \ldots, \mathbf{C}_{w'}, \lambda_1, \ldots, \lambda_{w'})$.

PROOF. We denote $\mathrm{compil}(\mathbf{P}, \mathbf{C}_1, \ldots, \mathbf{C}_{w'}, \lambda_1, \ldots, \lambda_{w'})$ by $\mathrm{compil}(\mathbf{P})$. We check that s preserves each relation in the vocabulary. In the rest of the proof, $\mathbf{c} = (c_1, \ldots, c_{w'})$, $\mathbf{c}' = (c'_1, \ldots, c'_{w'})$, $\mathbf{d} = (d_1, \ldots, d_{w'})$, and $\mathbf{d}' = (d'_1, \ldots, d'_{w'})$ are elements of $\mathrm{compil}(\mathbf{P})$.

Case L in σ. We claim that s preserves L. Let $(\mathbf{c}, \mathbf{c}'), (\mathbf{d}, \mathbf{d}') \in L$. It suffices to show that, for all $j \in [w']$, $\min^{\mathbf{P}}(c_j, d_j) \leq^{\mathbf{P}} \min^{\mathbf{P}}(c'_j, d'_j)$.

By hypothesis, $c_1 \leq^{\mathbf{P}} c'_1, \ldots, c_{w'} \leq^{\mathbf{P}} c'_{w'}$ and $d_1 \leq^{\mathbf{P}} d'_1, \ldots, d_{w'} \leq^{\mathbf{P}} d'_{w'}$ so that $c_1 \leq^{\mathbf{C}_1} c'_1, \ldots, c_{w'} \leq^{\mathbf{C}_{w'}} c'_{w'}$ and $d_1 \leq^{\mathbf{C}_1} d'_1, \ldots, d_{w'} \leq^{\mathbf{C}_{w'}} d'_{w'}$. For all $j \in [w']$, $c_j \leq^{\mathbf{C}_j} c'_j$ and $d_j \leq^{\mathbf{C}_j} d'_j$ implies that $\min^{\mathbf{C}_j}(c_j, d_j) \leq^{\mathbf{C}_j} \min^{\mathbf{C}_j}(c'_j, d'_j)$, which implies that $\min^{\mathbf{P}}(c_j, d_j) \leq^{\mathbf{P}} \min^{\mathbf{P}}(c'_j, d'_j)$, and we are done.

Case $I_{\{j,j'\}}$ in σ for $1 \leq j < j' \leq w'$. We claim that s preserves $I_{\{j,j'\}}$. Let $\mathbf{c}, \mathbf{d} \in I_{\{j,j'\}}$. It suffices to show that $\min^{\mathbf{C}_j}(c_j, d_j) \parallel^{\mathbf{P}} \min^{\mathbf{C}_{j'}}(c_{j'}, d_{j'})$.

Assume the contrary for a contradiction, say, $\min^{\mathbf{C}_j}(c_j, d_j) \leq^{\mathbf{P}} \min^{\mathbf{C}_{j'}}(c_{j'}, d_{j'})$ (the other case is similar). If $c_j \leq^{\mathbf{C}_j} d_j$ and $c_{j'} \leq^{\mathbf{C}_{j'}} d_{j'}$, then $c_j \leq^{\mathbf{P}} c_{j'}$, contradicting the hypothesis that $c_j \parallel^{\mathbf{P}} c_{j'}$. Similarly, it is impossible that $d_j \leq^{\mathbf{C}_j} c_j$ and $d_{j'} \leq^{\mathbf{C}_{j'}} c_{j'}$. So, assume that $c_j \leq^{\mathbf{C}_j} d_j$ and $d_{j'} \leq^{\mathbf{C}_{j'}} c_{j'}$. Then, $c_j \leq^{\mathbf{P}} d_{j'} \leq^{\mathbf{P}} c_{j'}$ by the absurdum hypothesis and the case distinction, a contradiction. The case $d_j \leq^{\mathbf{C}_j} c_j$ and $c_{j'} \leq^{\mathbf{C}_{j'}} d_{j'}$ is similar.

Case $O_{(j,j')}$ and $O_{(j',j)}$ in σ for $1 \leq j < j' \leq w'$. We claim that s preserves $O_{(j,j')}$ and $O_{(j',j)}$. The argument for $O_{(j,j')}$, and $O_{(j',j)}$ is similar. Let $\mathbf{c}, \mathbf{d} \in O_{(j,j')}$. It suffices to show that $\min^{\mathbf{C}_j}(c_j, d_j) \leq^{\mathbf{P}} \min^{\mathbf{C}_{j'}}(c_{j'}, d_{j'})$, since $\mathbf{C}_j \cap \mathbf{C}_{j'} = \emptyset$.

If $c_j \leq^{\mathbf{C}_j} d_j$ and $c_{j'} \leq^{\mathbf{C}_{j'}} d_{j'}$, then $c_j \leq^{\mathbf{P}} c_{j'}$ by hypothesis; similarly, if $d_j \leq^{\mathbf{C}_j} c_j$ and $d_{j'} \leq^{\mathbf{C}_{j'}} c_{j'}$. Thus, assume that $c_j \leq^{\mathbf{C}_j} d_j$ and $d_{j'} \leq^{\mathbf{C}_{j'}} c_{j'}$. Combining the main hypothesis and the case distinction, we have $c_j \leq^{\mathbf{C}_j} d_j \leq^{\mathbf{P}} d_{j'}$, that is, $c_j \leq^{\mathbf{P}} d_{j'}$. Similarly, $d_j \leq^{\mathbf{C}_j} c_j$ and $c_{j'} \leq^{\mathbf{C}_{j'}} d_{j'}$ implies that $d_j \leq^{\mathbf{P}} c_{j'}$.

Case $R_{(j,k)}$ for $j \in [w']$ and $k \in [k_j]$. We claim that s preserves $R_{(j,k)}$.

To prove the claim, let $(\mathbf{c}, \mathbf{d}), (\mathbf{c}', \mathbf{d}') \in R$. Let $b, b' \in K_{(j,k)}$ be such that $c_j = d_j = b$ and $c'_j = d'_j = b'$. Assume that $b \leq^{\mathbf{C}_j} b'$ (the other case is similar). Clearly, $\min^{\mathbf{C}_j}(c_j, c'_j) = \min^{\mathbf{C}_j}(d_j, d'_j) = b$. By hypothesis, $(\mathbf{c}, \mathbf{d}), (\mathbf{c}', \mathbf{d}') \in L$, so that, by the above,

$$(s(\mathbf{c}, \mathbf{c}'), s(\mathbf{d}, \mathbf{d}')) \in L,$$

and thus, by definition,

$$(s(\mathbf{c}, \mathbf{c}'), s(\mathbf{d}, \mathbf{d}')) \in R,$$

which completes the proof. □

It follows that the homomorphism problem, restricted to compilations of posets, is polynomial-time tractable.

LEMMA 4.7. *Let $\mathbf{P}$ be a poset. There exists a polynomial-time algorithm that, for all compilations $\mathbf{P}^*$ of $\mathbf{P}$, given the canonical semilattice polymorphism of $\mathbf{P}^*$ and a structure $\mathbf{Q}^*$ over the vocabulary of $\mathbf{P}^*$, decides whether $\mathbf{Q}^* \in$ HOM$(\mathbf{P}^*)$.*

PROOF. The statement follows from Lemma 4.6 and Theorem 4.1. □

4.1.4. Characterization of Embeddings via Homomorphisms. We now give the reduction of the problem of checking poset embeddings to the problem of checking homomorphisms over suitable poset compilations, informally described in Section 4.1.2.

We first prove that a poset embedding naturally yields a homomorphism between suitable poset compilations (Lemma 4.8 and Example 4.9). Let $\mathbf{Q}$ and $\mathbf{P}$ be posets, let $(\mathbf{C}'_1, \ldots, \mathbf{C}'_w)$ be a chain partition of $\mathbf{P}$, and let $e : Q \to P$ be an embedding of $\mathbf{Q}$ into $\mathbf{P}$. Let $e(Q) = \{e(q) \mid q \in Q\}$. Relative to $(\mathbf{C}'_1, \ldots, \mathbf{C}'_w)$, we define the notions of coordinatization of $\mathbf{P}$ *induced by e* and chain partition of $\mathbf{Q}$ *induced by e*, as follows:

—The coordinatization of $\mathbf{P}$ *induced by e* is the coordinatization $(\mathbf{C}_1, \ldots, \mathbf{C}_{w'})$ uniquely determined by deleting the coordinate $i \in [w]$ in $(\mathbf{C}'_1, \ldots, \mathbf{C}'_w)$ if and only if $e(Q) \cap C'_i = \emptyset$.
—The chain partition of $\mathbf{Q}$ *induced by e* is the tuple $(\mathbf{D}_1, \ldots, \mathbf{D}_{w'})$ obtained as follows. For all $j \in [w']$, let $D_j = e^{-1}(C_j)$, and let $\mathbf{D}_j$ be the substructure of $\mathbf{Q}$ induced by D_j. It follows from the properties of e that $(\mathbf{D}_1, \ldots, \mathbf{D}_{w'})$ is a chain partition of $\mathbf{Q}$, and that $|D_j| \leq |C_j|$.

Let $(\lambda_1, \ldots, \lambda_{w'})$ be such that λ_j is a function from C_j to $[|D_j|]$, and let $(\mu_1, \ldots, \mu_{w'})$ be such that μ_j is a bijection over $[|D_j|]$, $j \in [w']$. We say that $(\lambda_1, \ldots, \lambda_{w'})$ and $(\mu_1, \ldots, \mu_{w'})$ are *colorful* colorings of $\mathbf{P}$ and $\mathbf{Q}$ if the following holds:

—For all $j \in [w']$ and $d \in D_j$, it holds that $\mu_j(d) = k$ if and only if there exists $c \in C_j$ such that $e(d) = c$ and $\lambda_j(c) = k$.

We say that compil($\mathbf{Q}, \mathbf{D}_1, \ldots, \mathbf{D}_{w'}, \mu_1, \ldots, \mu_{w'}$) and compil($\mathbf{P}, \mathbf{C}_1, \ldots, \mathbf{C}_{w'}, \lambda_1, \ldots, \lambda_{w'}$) are compilations of $\mathbf{Q}$ and $\mathbf{P}$ induced by e if $(\mathbf{C}_1, \ldots, \mathbf{C}_{w'})$ is the coordinatization of $\mathbf{P}$ induced by e, $(\mathbf{D}_1, \ldots, \mathbf{D}_{w'})$ is the chain partition of $\mathbf{Q}$ induced by e, and $(\lambda_1, \ldots, \lambda_{w'})$ and $(\mu_1, \ldots, \mu_{w'})$ are colorful colorings of $\mathbf{P}$ and $\mathbf{Q}$.

LEMMA 4.8. *Let $\mathbf{Q}$ and $\mathbf{P}$ be posets, let $e : Q \to P$ be an embedding of $\mathbf{Q}$ into $\mathbf{P}$, and let $\mathbf{Q}^*$ and $\mathbf{P}^*$ be compilations of $\mathbf{Q}$ and $\mathbf{P}$ induced by e. Then $\mathbf{Q}^* \in \text{HOM}(\mathbf{P}^*)$.*

PROOF. Let $h : Q^* \to P^*$ be the function defined by

$$h((d_1, \ldots, d_{w'})) = (e(d_1), \ldots, e(d_{w'}))$$

for all $(d_1, \ldots, d_{w'}) \in Q^*$. We claim that h is a homomorphism from $\mathbf{Q}^*$ to $\mathbf{P}^*$.

First, note that $\mathbf{Q}^*$ and $\mathbf{P}^*$ have the same vocabulary. To prove the claim, we check that h preserves all relations in the vocabulary. In the following, $\mathbf{d} = (d_1, \ldots, d_{w'})$ and $\mathbf{d}' = (d'_1, \ldots, d'_{w'})$ are elements of $\mathbf{Q}^*$. Recall that $e : Q \to P$ is an embedding of $\mathbf{Q}$ into $\mathbf{P}$.

Case L. If $(\mathbf{d}, \mathbf{d}') \in L^{\mathbf{Q}^*}$, then $d_j \leq^{\mathbf{D}_j} d'_j$ for all $j \in [w']$, then $d_j \leq^{\mathbf{Q}} d'_j$ for all $j \in [w']$, then $e(d_j) \leq^{\mathbf{P}} e(d'_j)$ for all $j \in [w']$, then $e(d_j) \leq^{\mathbf{C}_j} e(d'_j)$ for all $j \in [w']$, then $((e(d_1), \ldots, e(d_{w'})), (e(d'_1), \ldots, e(d'_{w'}))) \in L^{\mathbf{P}^*}$. Altogether, this yields $(h(\mathbf{d}), h(\mathbf{d}')) \in L^{\mathbf{P}^*}$.

Case $I_{\{j,j'\}}$. If $\mathbf{d} \in I^{\mathbf{Q}^*}_{\{j,j'\}}$, then $d_j \parallel^{\mathbf{Q}} d_{j'}$, then $e(d_j) \parallel^{\mathbf{P}} e(d_{j'})$, then $(e(d_1), \ldots, e(d_{w'})) \in I^{\mathbf{P}^*}_{\{j,j'\}}$, that is, $h(\mathbf{d}) \in I^{\mathbf{P}^*}_{\{j,j'\}}$.

Case $1 \leq j < j' \leq w'$, $O_{(j,j')}$. If $\mathbf{d} \in O^{\mathbf{Q}^*}_{(j,j')}$, then $d_j <^{\mathbf{Q}} d_{j'}$, then $e(d_j) <^{\mathbf{P}} e(d_{j'})$, then $(e(d_1), \ldots, e(d_{w'})) \in O^{\mathbf{P}^*}_{(j,j')}$, that is, $h(\mathbf{d}) \in O^{\mathbf{P}^*}_{(j,j')}$. The case $O_{(j',j)}$ is similar.

Case $j \in [w']$, $k \in [|D_j|]$, $R_{(j,k)}$. If $(\mathbf{d}, \mathbf{d}') \in R^{\mathbf{Q}^*}_{(j,k)}$, then first observe that $(\mathbf{d}, \mathbf{d}') \in L^{\mathbf{Q}^*}$, so that $(h(\mathbf{d}), h(\mathbf{d}')) \in L^{\mathbf{P}^*}$ by the earlier argument. We have that $d_j = d'_j = d$ for some $d \in D_j$ such that $\mu_j(d) = k$. By colorfulness, $\mu_j(d) = k$ if and only if there exists $c \in C_j$ such that $d = e^{-1}(c)$ and $\lambda_j(c) = k$. Therefore, $e(d_j) = e(d'_j) = e(d) = c$, so that $((e(d_1), \ldots, e(d_{w'})), (e(d'_1), \ldots, e(d'_{w'}))) \in R^{\mathbf{P}^*}_{(j,k)}$, that is, $(h(\mathbf{d}), h(\mathbf{d}')) \in R^{\mathbf{P}^*}_{(j,k)}$. $\square$

The lemma is illustrated by the following example.

Example 4.9. Let $\mathbf{Q}$ and $\mathbf{P}$ be the posets in Example 4.4, so that $\mathbf{Q}$ embeds into $\mathbf{P}$ via the map $e : Q \to P$ defined in the example (see Figure 6). Let $\mathbf{Q}^* = \text{compil}(\mathbf{Q}, \mathbf{D}_1, \mathbf{D}_2, \mu_1, \mu_2)$ and $\mathbf{P}^* = \text{compil}(\mathbf{P}, \mathbf{C}_1, \mathbf{C}_2, \lambda_1, \lambda_2)$ be the structures in Example 4.5, respectively, compiling $\mathbf{Q}$ and $\mathbf{P}$. The homomorphism $h : Q^* \to P^*$, corresponding to the embedding $e : Q \to P$ constructed in Lemma 4.8, is depicted in Figure 9.

We conclude the section proving the announced reduction, essentially establishing the converse of Lemma 4.8.

LEMMA 4.10. *Let $\mathbf{Q}$ and $\mathbf{P}$ be posets such that $\text{width}(\mathbf{Q}) \leq \text{width}(\mathbf{P}) = w$ and let $(\mathbf{C}'_1, \ldots, \mathbf{C}'_w)$ be a chain partition of $\mathbf{P}$. The following are equivalent.*

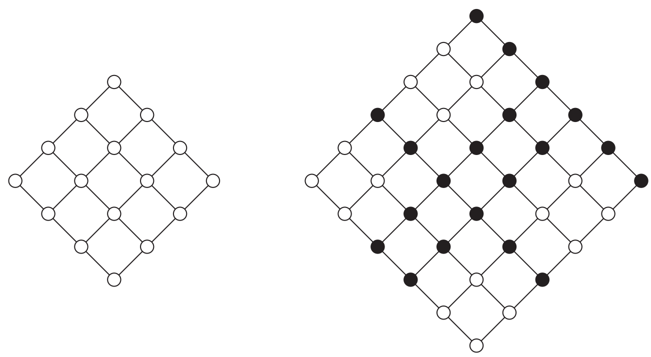

Fig. 9. The structures $\mathbf{Q}^*$ (left) and $\mathbf{P}^*$ (right) in Example 4.9. The white points in P^* form the image of the homomorphism $h : Q^* \to P^*$ in Example 4.9. It is possible to check that h is a homomorphism by direct inspection of Figure 7 and Figure 8.

(i) *$\mathbf{Q}$ embeds into $\mathbf{P}$.*
(ii) *There exist $w' \leq w$, a coordinatization $(\mathbf{C}_1, \ldots, \mathbf{C}_{w'})$ of $\mathbf{P}$ relative to $(\mathbf{C}'_1, \ldots, \mathbf{C}'_w)$, a chain partition $(\mathbf{D}_1, \ldots, \mathbf{D}_{w'})$ of $\mathbf{Q}$ such that $|D_j| \leq |C_j|$ for all $j \in [w']$, a tuple $(\lambda_1, \ldots, \lambda_{w'})$ of functions from C_j to $[|D_j|]$, and a tuple $(\mu_1, \ldots, \mu_{w'})$ of bijections from D_j to $[|D_j|]$ such that*

$$\text{compil}(\mathbf{Q}, \mathbf{D}_1, \ldots, \mathbf{D}_{w'}, \mu_1, \ldots, \mu_{w'}) \in \text{Hom}(\text{compil}(\mathbf{P}, \mathbf{C}_1, \ldots, \mathbf{C}_{w'}, \lambda_1, \ldots, \lambda_{w'}).$$

Notice that checking item (ii) directly requires exploring all functions from C_j to $[|D_j|]$ for $j \in [w']$, which might require time exponential in $|P|$. We tackle this complexity issue in Section 4.1.5.

Proof. We prove the two implications.

(i) ⇒ (ii) Let $e : Q \to P$ be an embedding of $\mathbf{Q}$ into $\mathbf{P}$, let $(\mathbf{C}_1, \ldots, \mathbf{C}_{w'})$ be the coordinatization of $\mathbf{P}$ induced by e, and let $(\mathbf{D}_1, \ldots, \mathbf{D}_{w'})$ be the chain partition of $\mathbf{Q}$ induced by e. For $j \in [w']$, let $D_j = \{d_{j,1}, \ldots, d_{j,|D_j|}\}$, and let $e(d_{j,i}) = c_{j,i} \in C_j$ for all $i \in [|D_j]|$.

We define colorful colorings $(\lambda_1, \ldots, \lambda_{w'})$ of $\mathbf{P}$ and $(\mu_1, \ldots, \mu_{w'})$ of $\mathbf{Q}$, as follows. Let $j \in [w']$. For all $i \in [|D_j|]$, put $\mu_j(d_{j,i}) = i$ and, for all $c \in C_j$,

$$\lambda_j(c) = \begin{cases} i & \text{if } c = c_{j,i} = e^{-1}(d_{j,i}) \text{ for some } i \in [|D_j|]. \\ 1 & \text{otherwise} \end{cases}$$

In other words, λ_j colors $c \in C_j$ arbitrarily (say, with color 1), unless c is the counterimage under e of an element $d \in D_j$, in which case c is colored as d. The implication $(i) \Rightarrow (ii)$ now follows directly from Lemma 4.8.

(ii) ⇒ (i). Let

$$\mathbf{Q}^* = \text{compil}(\mathbf{Q}, \mathbf{D}_1, \ldots, \mathbf{D}_{w'}, \mu_1, \ldots, \mu_{w'})$$

and

$$\mathbf{P}^* = \text{compil}(\mathbf{P}, \mathbf{C}_1, \ldots, \mathbf{C}_{w'}, \lambda_1, \ldots, \lambda_{w'})$$

be specified as in the statement of the lemma, and let $h : Q^* \to P^*$ be a homomorphism from $\mathbf{Q}^*$ to $\mathbf{P}^*$. We define a function $e : Q \to P$ as follows. Later, $\mathbf{c} = (c_1, \ldots, c_{w'})$, $\mathbf{c}' = (c'_1, \ldots, c'_{w'})$, $\mathbf{c}'' = (c''_1, \ldots, c''_{w'})$, $\mathbf{d} = (d_1, \ldots, d_{w'})$, and $\mathbf{d}' = (d'_1, \ldots, d'_{w'})$ are elements of $\mathbf{Q}^*$.

Let $q \in Q$. Let $j \in [w']$ be such that $q \in D_j$.

CLAIM 1. *There exists a unique* $p \in C_j \subseteq P$ *such that:*

—*if* $h(\mathbf{d}) = \mathbf{c}$ *and* $d_j = q$, *then* $c_j = p$;
—$\mu_j(q) = \lambda_j(p)$.

PROOF OF CLAIM 1. Let $\mu_j(q) = k$. Since $\{\mathbf{d} \in Q^* \mid d_j = q\}$ is nonempty, there exists at least one element $p \in C_j$ such that $\mathbf{c}$ is in the image of h in P^* and $c_j = p$. Let $p, p' \in C_j$ be such that, for some $\mathbf{d}, \mathbf{d}' \in Q^*$ with $d_j = d'_j = q$, $h(\mathbf{d}) = \mathbf{c}$ and $c_j = p$, and $h(\mathbf{d}') = \mathbf{c}'$ and $c'_j = p'$. We prove that $p = p'$ and $\lambda_j(p) = k$. We distinguish two cases.

Case 1. $(\mathbf{d}, \mathbf{d}') \in L^{\mathbf{Q}^*}$ or $(\mathbf{d}', \mathbf{d}) \in L^{\mathbf{Q}^*}$. Assume that $(\mathbf{d}, \mathbf{d}') \in L^{\mathbf{Q}^*}$. Then, $(\mathbf{d}, \mathbf{d}') \in R^{\mathbf{Q}^*}_{j,k}$. Then, $(\mathbf{c}, \mathbf{c}') \in R^{\mathbf{P}^*}_{j,k}$, so that $c_j = c'_j$ by definition of $R^{\mathbf{P}^*}_{j,k}$, that is, $p = p'$, and $\lambda_j(p) = k$. The argument is similar if $(\mathbf{d}', \mathbf{d}) \in L^{\mathbf{Q}^*}$.

Case 2. $(\mathbf{d}, \mathbf{d}') \notin L^{\mathbf{Q}^*}$ and $(\mathbf{d}', \mathbf{d}) \notin L^{\mathbf{Q}^*}$. It then holds that $\min^{\mathbf{D}_j}(d_j, d'_j) = q$ and $\min^{\mathbf{D}_{j'}}(d_{j'}, d'_{j'}) \leq^{\mathbf{D}_{j'}} d_{j'}, d'_{j'}$ for all $j' \in [w']$. Therefore,

$$((\min^{\mathbf{D}_1}(d_1, d'_1), \ldots, \min^{\mathbf{D}_{w'}}(d_{w'}, d'_{w'})), \mathbf{d}) \in R^{\mathbf{Q}^*}_{j,k},$$

and

$$((\min^{\mathbf{D}_1}(d_1, d'_1), \ldots, \min^{\mathbf{D}_{w'}}(d_{w'}, d'_{w'})), \mathbf{d}') \in R^{\mathbf{Q}^*}_{j,k}.$$

Let

$$h((\min^{\mathbf{D}_1}(d_1, d'_1), \ldots, \min^{\mathbf{D}_{w'}}(d_{w'}, d'_{w'}))) = \mathbf{c}''.$$

Then, $(\mathbf{c}'', \mathbf{c}) \in R^{\mathbf{P}^*}_{j,k}$ and $(\mathbf{c}'', \mathbf{c}') \in R^{\mathbf{P}^*}_{j,k}$, so that $c''_j = c_j = c'_j$, that is, $p = p'$, and $\lambda_j(p) = k$.

We define $e(q) = p$, where $p \in P$ is the unique element identified by Claim 1 relative to q. The following claim then settles $(ii) \Rightarrow (i)$.

CLAIM 2. e *embeds* $\mathbf{Q}$ *into* $\mathbf{P}$.

PROOF OF CLAIM 2. Let $q, q' \in Q$. It is sufficient to check that $q <^{\mathbf{Q}} q'$ implies $e(q) <^{\mathbf{P}} e(q')$, and $q \parallel^{\mathbf{Q}} q'$ implies $e(q) \parallel^{\mathbf{P}} e(q')$. Let $j, j' \in [w']$ be such that $q \in D_j$ and $q' \in D_{j'}$.

$q <^{\mathbf{Q}} q'$ *implies* $e(q) <^{\mathbf{P}} e(q')$. Assume that $q <^{\mathbf{Q}} q'$. We distinguish two cases.

Case 1. If $j = j'$, then let $\mu_j(q) = k$ and $\mu_j(q') = k'$. Since $q, q' \in D_j$ and μ_j is a bijection from D_j to $[|D_j|]$, we have that $k \neq k'$. Hence, if $e(q) = p \in C_j$ and $e(q') = p' \in C_j$, then by the definition of e, we have that $\lambda_j(p) = k \neq k' = \lambda_j(p')$, so that $p \neq p'$. We have that

$$\begin{aligned}&((\mathrm{bot}(\mathbf{D}_1), \ldots, q, \ldots, \mathrm{bot}(\mathbf{D}_{w'})),\\ &(\mathrm{bot}(\mathbf{D}_1), \ldots, q', \ldots, \mathrm{bot}(\mathbf{D}_{w'}))) \in L^{\mathbf{Q}^*},\end{aligned}$$

where q and q' occur at the jth coordinate, and $\mathrm{bot}(\mathbf{D}_{j''})$ is the bottom of chain $\mathbf{D}_{j''}$ for all $j'' \in [w']$. Let $h((\mathrm{bot}(\mathbf{D}_1), \ldots, q, \ldots, \mathrm{bot}(\mathbf{D}_{w'}))) = \mathbf{c} \in P^*$ and, similarly, $h((\mathrm{bot}(\mathbf{D}_1), \ldots, q', \ldots, \mathrm{bot}(\mathbf{D}_{w'}))) = \mathbf{c}' \in P^*$. Then

$$(\mathbf{c}, \mathbf{c}') \in L^{\mathbf{P}^*},$$

so that, in particular, $c_j \leq^{\mathbf{C}_j} c'_j$. We claim that $c_j = p$. Indeed, since h is a homomorphism, it is the case that $\mu_j(q) = \lambda_j(c_j) = k$, because there is a $R_{(j,k)}$ loop over the elements of $(\mathrm{bot}(\mathbf{D}_1), \ldots, q, \ldots, \mathrm{bot}(\mathbf{D}_{w'}))$ in $\mathbf{Q}^*$. By Claim 1, there exists a unique element in $\mathbf{C}_j$ having the same color of q and occurring at the jth coordinate of any $h((\ldots,q,\ldots)) \in P^*$; and this element is $e(q) = p$ by definition. Similarly, $c'_j = p'$. Thus, since we observed that $p \neq p'$, we have that $p <^{\mathbf{C}_j} p'$; therefore, $e(q) = p <^{\mathbf{P}} p' = e(q')$.

Case 2. If $j \neq j'$, then $e(q) = p \in C_j$ and $e(q') = p' \in C_{j'}$, so that $p \neq p'$ because $C_j \cap C_{j'} = \emptyset$. We have that

$$(\ldots,q,\ldots,q',\ldots) \in O^{\mathbf{Q}^*}_{(j,j')},$$

where q occurs at the jth coordinate and q' occurs at the j'th coordinate, so that, if $h((\ldots,q,\ldots,q',\ldots)) = \mathbf{c} \in P^*$, then

$$\mathbf{c} \in O^{\mathbf{P}^*}_{(j,j')},$$

that is, $c_j \leq^{\mathbf{P}} c_{j'}$. We claim that $c_j = p$ and $c_{j'} = p'$, which implies that $e(q) = p <^{\mathbf{P}} p' = e(q')$. Indeed, since h is a homomorphism, it is the case that $\mu_j(q) = \lambda_j(c_j) = k$ and $\mu_{j'}(q') = \lambda_{j'}(c_{j'}) = k'$, because there is both an $R_{(j,k)}$ loop and an $R_{(j',k')}$ loop over $(\ldots,q,\ldots,q',\ldots)$ in $\mathbf{Q}^*$. Then, by Claim 1 and the definition of e, it is the case that $c_j = e(q) = p$ and $c_{j'} = e(q') = p'$.

$q \parallel^{\mathbf{Q}} q'$ implies $e(q) \parallel^{\mathbf{P}} e(q')$. Let $\mu_j(q) = k$ and $\mu_{j'}(q') = k'$. We have that

$$(\ldots,q,\ldots,q',\ldots) \in I^{\mathbf{Q}^*}_{\{j,j'\}},$$

where q occurs at the jth coordinate and q' occurs at the j'th coordinate, so that, if $h((\ldots,q,\ldots,q',\ldots)) = \mathbf{c} \in P^*$, then

$$\mathbf{c} \in I^{\mathbf{P}^*}_{\{j,j'\}},$$

that is, $c_j \parallel^{\mathbf{P}} c_{j'}$. We claim that $c_j = e(q)$ and $c_{j'} = e(q')$, which implies that $e(q) \parallel^{\mathbf{P}} e(q')$. Indeed, since h is a homomorphism, it is the case that $\lambda_j(c_j) = k$ and $\lambda_{j'}(c_{j'}) = k'$, because there is both an $R_{(j,k)}$ loop and an $R_{(j',k')}$ loop over $(\ldots,q,\ldots,q',\ldots)$ in $\mathbf{Q}^*$. Then, by Claim 1 and the definition of e, it is the case that $c_j = e(q)$ and $c_{j'} = e(q')$.

The implication $(ii) \Rightarrow (i)$ is proved. □

4.1.5. FPT Characterization of Embeddings via Homomorphisms. We now appeal to the theory of hashing functions to provide a parsimonious, fixed-parameter tractable version of the characterization in Lemma 4.10.

LEMMA 4.11. *Let $\mathbf{Q}$ and $\mathbf{P}$ be posets such that* $\mathrm{width}(\mathbf{Q}) \leq \mathrm{width}(\mathbf{P}) = w$, *and let $(\mathbf{C}'_1,\ldots,\mathbf{C}'_w)$ be a chain partition of $\mathbf{P}$. The following are equivalent.*

- (i) *$\mathbf{Q}$ embeds into $\mathbf{P}$.*
- (ii) *There exist $w' \leq w$, a coordinatization $(\mathbf{C}_1,\ldots,\mathbf{C}_{w'})$ of $\mathbf{P}$ relative to $(\mathbf{C}'_1,\ldots,\mathbf{C}'_w)$, and a chain partition $(\mathbf{D}_1,\ldots,\mathbf{D}_{w'})$ of $\mathbf{Q}$ such that $|D_j| \leq |C_j|$ for all $j \in [w']$, satisfying the following. For all tuples $(\Lambda_1,\ldots,\Lambda_{w'})$ where Λ_j is a $|D_j|$-perfect family of hash functions from C_j to $[|D_j|]$, there exist a tuple $(\lambda_1,\ldots,\lambda_{w'}) \in \Lambda_1 \times \ldots \times \Lambda_{w'}$ and a tuple $(\mu_1,\ldots,\mu_{w'})$ of bijections from D_j to $[|D_j|]$ such that*

$$\mathrm{compil}(\mathbf{Q},\mathbf{D}_1,\ldots,\mathbf{D}_{w'},\mu_1,\ldots,\mu_{w'}) \in \mathrm{HOM}(\mathrm{compil}(\mathbf{P},\mathbf{C}_1,\ldots,\mathbf{C}_{w'},\lambda_1,\ldots,\lambda_{w'}).$$

PROOF. We prove the two implications.

(i) ⇒ (ii) Let $e : Q \to P$ be an embedding of $\mathbf{Q}$ into $\mathbf{P}$, let $(\mathbf{C}_1,\ldots,\mathbf{C}_{w'})$ be the coordinatization of $\mathbf{P}$ induced by e, and let $(\mathbf{D}_1,\ldots,\mathbf{D}_{w'})$ be the chain partition of $\mathbf{Q}$

induced by e. Note that $|D_j| \leq |C_j|$ for all $j \in [w']$. For $j \in [w']$, let $D_j = \{d_{j,1}, \ldots, d_{j,|D_j|}\}$, and let $e(d_{j,i}) = c_{j,i} \in C_j$ for all $i \in [|D_j|]$.

For each $j \in [w']$, let Λ_j be a $|D_{i_j}|$-perfect family of hash functions from C_{i_j} to $[|D_{i_j}|]$. We now define colorful colorings $(\lambda_1, \ldots, \lambda_{w'}) \in \Lambda_1 \times \cdots \times \Lambda_{w'}$ and $(\mu_1, \ldots, \mu_{w'})$ of $\mathbf{P}$ and $\mathbf{Q}$, respectively, as follows.

Let $j \in [w']$. Pick $\lambda_j \in \Lambda_j$ such that λ_j restricted to $e(D_j)$ is injective; such a λ_j exists, because $e(D_j)$ is a subset of C_j of cardinality $|Dj|$ (as e is injective), and Λ_j is a $|D_j|$-perfect family of hash functions from C_j to $[|D_j|]$. Moreover, for all $i \in [|D_j|]$, put

$$\mu_j(d_{j,i}) = \lambda_j(e(d_{j,i})) = \lambda_j(c_{j,i}).$$

In other words, μ_j colors $d \in D_j$ exactly as λ_j colors the image of d under e. The implication $(i) \Rightarrow (ii)$ now follows directly from Lemma 4.8.

(ii) $\Rightarrow$ (i). Note that Item (ii) in Lemma 4.10 is a special case of Item (ii) in the current lemma, namely, the case in which Λ_j is the family of all functions from C_j to $[|D_j|]$, for all $j \in [w']$. Hence, Item (ii) in the current lemma implies Item (ii) in Lemma 4.10, which in turn implies that $\mathbf{Q}$ embeds into $\mathbf{P}$ (by the implication $(ii) \Rightarrow (i)$ in Lemma 4.10), and we are done. □

We are now ready to list the pseudocode of our embedding algorithm, and prove that it has the desired properties. The input is a pair $(\mathbf{Q}, \mathbf{P})$ of posets.

ALGORITHM($\mathbf{Q}, \mathbf{P}$)
1 **if** ($|P| < |Q|$ **or** width($\mathbf{P}$) < width($\mathbf{Q}$)) **then reject**
2 $w \leftarrow$ width($\mathbf{P}$)
3 compute a chain partition $(\mathbf{C}'_1, \ldots, \mathbf{C}'_w)$ of $\mathbf{P}$
4 **foreach** $1 \leq w' \leq w$,
  coordinatization $(\mathbf{C}_1, \ldots, \mathbf{C}_{w'})$ of $\mathbf{P}$,
  chain partition $(\mathbf{D}_1, \ldots, \mathbf{D}_{w'})$ of $\mathbf{Q}$ such that $|D_j| \leq |C_j|$ for all $j \in [w']$ **do**
5   **foreach** $j \in [w']$ **do**
6     $\Lambda_j \leftarrow |D_j|$-perfect family of hashing functions from C_j to $[|D_j|]$
7   **foreach** $(\lambda_1, \ldots, \lambda_{w'}) \in \Lambda_1 \times \cdots \times \Lambda_{w'}$ **and** $(\mu_1, \ldots, \mu_{w'}) \in M_1 \times \cdots \times M_{w'}$ **do**
8     $\mathbf{P}^* \leftarrow$ compil($\mathbf{P}, \mathbf{C}_1, \ldots, \mathbf{C}_{w'}, \lambda_1, \ldots, \lambda_{w'}$)
9     $\mathbf{Q}^* \leftarrow$ compil($\mathbf{Q}, \mathbf{D}_1, \ldots, \mathbf{D}_{w'}, \mu_1, \ldots, \mu_{w'}$)
10     $s \leftarrow$ canonical semilattice polymorphism of $\mathbf{P}^*$
11   **if** $\mathbf{Q}^* \in$ HOM($\mathbf{P}^*$) **then accept**
12 **reject**

In Line 7, M_j is the set of all bijections from D_j to $[|D_j|]$, for all $j \in [w']$.

THEOREM 4.12. *Let $\mathcal{P}$ be a class of posets of bounded width. There exists an algorithm deciding any instance $(\mathbf{Q}, \mathbf{P})$ of* EMB($\mathcal{P}$) *in $2^{O(k \log k)} \cdot n^{O(1)}$ time, where $n = |P|$ and $k = |Q|$.*

PROOF. We prove that the algorithm is correct. If the algorithm accepts on Line 11, then Q embeds into P by $(ii) \Rightarrow (i)$ in Lemma 4.10. Conversely, if Q embeds into P, then by $(i) \Rightarrow (ii)$ in Lemma 4.11, letting $(\Lambda_j)_{j \in [w']}$ be the family of hashing functions fixed on Lines 5–6, there exist tuples $(\lambda_1, \ldots, \lambda_{w'}) \in \Lambda_1 \times \cdots \times \Lambda_{w'}$ and $(\mu_1, \ldots, \mu_{w'}) \in M_1 \times \cdots M_{w'}$ satisfying the condition in Line 7, where M_j is the set of all bijections from D_j to $[|D_j|]$. Hence, the algorithm accepts on Line 11.

We analyze the runtime. Lines 1 through 3 are feasible in time $n^{O(1)}$ by Theorem 4.3. The loop on Lines 4 through 11 executes at most $2^{O(k)}$ times, and the loop condition is tested in $n^{O(1)}$ time. The loop on Lines 5 and 6 is feasible in time $2^{O(k)} \cdot n^{O(1)}$ by Theorem 4.2. The loop on Lines 7 through 11 executes at most $2^{O(k \log k)} \cdot n^{O(1)}$ times. Lines 8 through 10 are feasible in $n^{O(1)}$ time. Line 11 runs in time $n^{O(1)}$ by Lemma 4.7

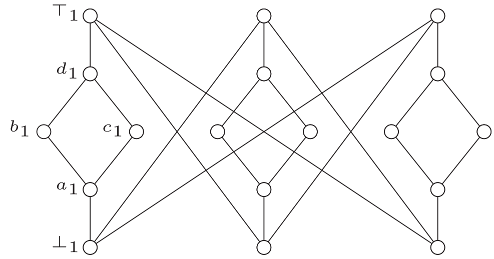

Fig. 10. The poset $\mathbf{Q}_k$ in the proof of Proposition 4.15, where $k = 3$, as in Example 4.14.

(a semilattice polymorphism is available from Line 10). Hence, the total runtime is bounded above by $2^{O(k \log k)} \cdot n^{O(1)}$. □

The main result follows.

THEOREM 4.13. *Let $\mathcal{P}$ be a class of posets of bounded width. Then,* $\mathrm{MC}(\mathcal{P}, \mathcal{FO}(\exists, \wedge, \vee, \neg))$ *is fixed-parameter tractable (with single exponential parameter dependence).*

PROOF. Directly from Proposition 3.1 and Theorem 4.12. □

4.2. Embedding is W[1]-hard on Bounded Cover-Degree Posets

We construct a class $\mathcal{P}_{\text{cover-degree}}$ of bounded cover-degree posets such that $\mathrm{EMB}(\mathcal{P}_{\text{cover-degree}})$ is W[1]-hard. By Proposition 3.1, it follows that $\mathrm{MC}(\mathcal{P}_{\text{cover-degree}}, \mathcal{FO}(\exists, \wedge, \vee, \neg))$ is W[1]-hard. The construction is illustrated in Example 4.14.

Let $\mathbf{G} = (V, E^{\mathbf{G}})$ be a graph and let $V = [n]$. Then $r(\mathbf{G}) = \mathbf{P}$ is the poset defined as follows. The universe of $\mathbf{P}$ is $P = \bigcup_{i \in [n]} P_i$ where, for all $i \in [n]$,

$$P_i = \{\bot_i, a_i, b_i, c_i, d_i, \top_i\} \cup \{l_{(i,j)}, u_{(i,j)} \mid j \in [n]\}.$$

The order is defined by the following. For each $i, j \in [n]$:

—$a_i \prec^{\mathbf{P}} b_i$, $a_i \prec^{\mathbf{P}} c_i$, $b_i \prec^{\mathbf{P}} d_i$, $c_i \prec^{\mathbf{P}} d_i$, and $b_i \parallel^{\mathbf{P}} c_i$;
—$\bot_i \prec^{\mathbf{P}} l_{(i,1)} \prec^{\mathbf{P}} \cdots \prec^{\mathbf{P}} l_{(i,n)} \prec^{\mathbf{P}} a_i$;
—$d_i \prec^{\mathbf{P}} u_{(i,1)} \prec^{\mathbf{P}} \cdots \prec^{\mathbf{P}} u_{(i,n)} \prec^{\mathbf{P}} \top_i$;
—$l_{(i,j)} \prec^{\mathbf{P}} u_{(j,i)}$ if and only if $(i, j) \in E^{\mathbf{G}}$.

The construction satisfies the following properties. Let $\mathbf{G} \in \mathcal{G}$:

(*i*) since cover-degree$(r(\mathbf{G})) \leq 3$, the class $\mathcal{P}_{\text{cover-degree}} = \{r(\mathbf{G}) \mid \mathbf{G} \in \mathcal{G}\}$ has bounded cover-degree;
(*ii*) $r(\mathbf{G})$ can be constructed in polynomial time;
(*iii*) for any $j, j' \in [n]$, $j \neq j'$, we have $\bot_j <^{\mathbf{P}} \top_{j'}$ if and only if $(j, j') \in E^{\mathbf{G}}$.

For $k \in \mathbb{N}$, let $\mathbf{Q}_k$ be the poset with universe $Q_k = \{\bot_i, a_i, b_i, c_i, d_i, \top_i \mid i \in [k]\}$, uniquely determined by the following relations:

—$a_i \prec^{\mathbf{Q}_k} b_i$, $a_i \prec^{\mathbf{Q}_k} c_i$, $b_i \prec^{\mathbf{Q}_k} d_i$, $c_i \prec^{\mathbf{Q}_k} d_i$, and $b_i \parallel^{\mathbf{Q}_k} c_i$;
—$\bot_i \prec^{\mathbf{Q}_k} a_i$ and $d_i \prec^{\mathbf{Q}_k} \top_i$ for all $i \in [k]$;
—$\bot_i \prec^{\mathbf{Q}_k} \top_{i'}$ for all $i, i' \in [k]$, $i \neq i'$.

Example 4.14. Let $(\mathbf{G}, k)$ be an instance of CLIQUE, where $\mathbf{G}$ is the graph whose universe is $G = [4]$ and whose edge relation $E^{\mathbf{G}}$ is the symmetric closure of $\{(1, 2), (1, 3), (2, 3), (2, 4), (3, 4)\}$, and $k = 3$. Then posets $\mathbf{Q}_k$ and $\mathbf{P}$ in the proof of Proposition 4.15 are depicted in Figure 10 and Figure 11, respectively.

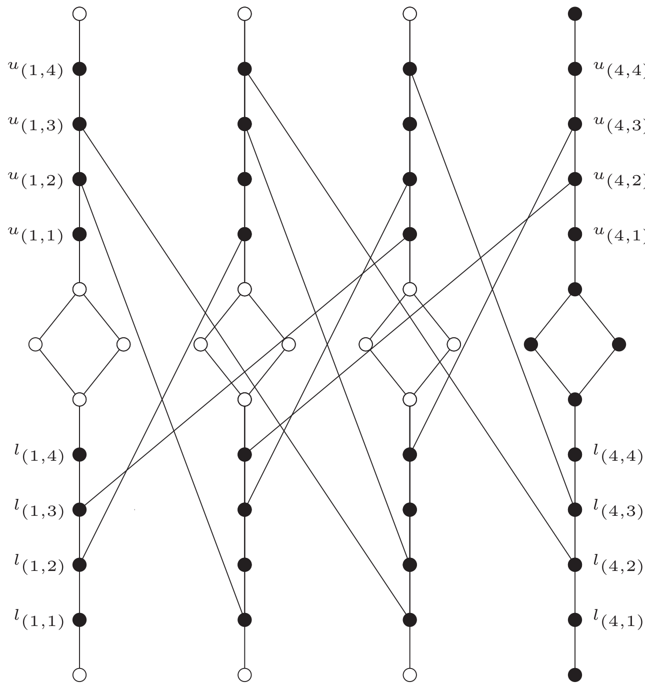

Fig. 11. The poset $\mathbf{P}$ in the proof of Proposition 4.15, where $\mathbf{G}$ is as in Example 4.14.

PROPOSITION 4.15. $\text{EMB}(\mathcal{P}_{\text{cover-degree}})$ *is* W[1]*-hard.*

PROOF. We give an fpt many–one reduction from the CLIQUE problem to $\text{EMB}(\mathcal{P}_{\text{cover-degree}})$, which suffices since CLIQUE is W[1]-hard. The reduction is illustrated in Example 4.14.

Let $(\mathbf{G}, k)$ be an instance of CLIQUE; the question is whether $\mathbf{G}$ contains a clique on $k \in \mathbb{N}$ vertices. The reduction maps $(\mathbf{G}, k)$ to the instance $(\mathbf{Q}_k, \mathbf{P})$ of $\text{EMB}(\mathcal{P}_{\text{cover-degree}})$, where $\mathbf{P} = r(\mathbf{G})$ and $\mathbf{Q}_k$ are as defined earlier. We argue correctness (the complexity of the reduction is clear).

If $\{j_1, \dots, j_k\} \subseteq G$ induces a clique of size k in $\mathbf{G}$, then $\mathbf{Q}_k$ embeds into $\mathbf{P}$ by $q_i \mapsto q_{j_i}$ for all $q \in \{\bot, a, b, c, d, \top\}$ and $i \in [k]$.

Conversely, assume that $\mathbf{Q}_k$ embeds into $\mathbf{P}$ via a mapping e. Let $i \in [k]$. We claim that there exists $j \in [n]$ such that $\{e(b_i), e(c_i)\} = \{b_j, c_j\}$. Indeed, by construction, $b_i \parallel^{\mathbf{Q}_k} c_i$ and $a_i \leq^{\mathbf{Q}_k} b_i, c_i \leq^{\mathbf{Q}_k} d_i$. Note that any two incomparable elements $p' \in P_{j'}$ and $p'' \in P_{j''}$ with $j', j'' \in [n]$, $j' \neq j''$, lack a common upper bound or a common lower bound. Hence, since e is an embedding, $\{e(b_i), e(c_i)\} \subseteq P_j$ for some $j \in [n]$, which forces $\{e(b_i), e(c_i)\} = \{b_j, c_j\}$ because b_j and c_j are the only two incomparable elements in P_j.

We claim that $C = \{j \mid \{b_j, c_j\} \cap e(Q_k) \neq \emptyset\} \subseteq V$ induces a clique of size k in $\mathbf{G}$. By the above, $|C| = k$. Hence, it suffices to show that $(j, j') \in E^{\mathbf{G}}$ for any $j, j' \in C$, $j \neq j'$. Let $i, i' \in [k]$, $i \neq i'$ be such that $\{e(b_i), e(c_i)\} = \{b_j, c_j\}$ and $\{e(b_{i'}), e(c_{i'})\} = \{b_{j'}, c_{j'}\}$. Since $e(\bot_i) <^{\mathbf{P}} e(b_i)$ and $e(\top_{i'}) >^{\mathbf{P}} e(b_{i'})$, we obtain that $e(\bot_i) \in P_j$ and $e(\top_{i'}) \in P_{j'}$ by construction. The embedding ensures $e(\bot_i) <^{\mathbf{P}} e(\top_{i'})$ and so $(j, j') \in E^{\mathbf{G}}$ by the properties listed before the statement, concluding the proof. □

5. CLASSICAL COMPLEXITY

In this section, we study the classical complexity of the embedding problem on the targeted classes of posets, and we prove a tractability result of independent interest on bounded-width posets. We first observe the following fact.

PROPOSITION 5.1. *Let $\mathcal{P}$ be a class of posets of bounded size. Then,* EMB($\mathcal{P}$) *is polynomial-time tractable.*

PROOF. Let $s \in \mathbb{N}$ be such that $|\mathbf{P}| \leq s$ for all $\mathbf{P} \in \mathcal{P}$. Let $(\mathbf{Q}, \mathbf{P})$ be an instance of EMB($\mathcal{P}$). If $|Q| > |P|$, reject. Otherwise, check whether one of the at most s^s many mappings from $\mathbf{Q}$ to $\mathbf{P}$ is an embedding. □

Note that this, together with Proposition 3.2, rules out a polynomial-time tractability analogue of Proposition 3.1. The section is organized as follows.

—In Sections 5.1 and 5.2, we prove that the embedding problem is NP-hard on bounded-width and bounded-degree posets, respectively. This implies that Proposition 5.1 is tight with respect to the studied invariants.
—In Section 5.3, we show how the ideas developed in Section 4 may be used to obtain a polynomial-time algorithm for the isomorphism of bounded-width posets, an open problem in order theory [Caspard et al. 2012, p. 284].

5.1. Embedding is NP-Hard on Bounded Width Posets

In this section, we construct a class $\mathcal{P}$ of posets of bounded width such that EMB($\mathcal{P}$) is NP-hard, which immediately implies NP-hardness of MC($\mathcal{P}$, $\mathcal{FO}(\exists, \wedge, \neg)$).

We give a reduction from a version of the Boolean satisfiability problem for propositional formulas in conjunctive form containing at least 3 clauses (3SAT); namely, given a 3SAT instance ϕ, we construct two posets $\mathbf{Q}_\phi$ and $\mathbf{P}_\phi$ (whose width is independent of ϕ) such that ϕ is satisfiable if and only if $\mathbf{Q}_\phi$ embeds into $\mathbf{P}_\phi$. The reduction is fairly involved.

Let $\mathcal{S}$ be the class of propositional formulas in conjunctive form with the following properties. Each formula in $\mathcal{S}$ contains at least 3 clauses, and each clause contains at most 3 literals; furthermore, no clause contains a pair of complementary literals, and each variable occurs in at least two clauses. See Example 5.2.

Example 5.2. Let $\phi(x_1, x_2, x_3, x_4) = \delta_1 \wedge \delta_2 \wedge \delta_3 \wedge \delta_4 \wedge \delta_5$, where $\delta_1 = x_4 \vee \neg x_2$, $\delta_2 = x_4 \vee \neg x_1$, $\delta_3 = x_1 \vee \neg x_2$, $\delta_4 = x_3 \vee \neg x_1$, and $\delta_5 = \neg x_3 \vee x_2$. Then $\phi \in \mathcal{S}$. Note that, for instance, ϕ is satisfied by $\{(x_1, 0), (x_2, 0), (x_3, 0), (x_4, 1)\}$.

Let $\phi(x_1, \ldots, x_n) = \delta_1 \wedge \cdots \wedge \delta_m$ be in $\mathcal{S}$. We describe informally the construction of $\mathbf{Q}_\phi$ in terms of three chains, left, central, and right; see Example 5.3 and Figure 12.

The central and right chains are formed by stacking, respectively, n and $n-1$ chains of m elements, so that they basically hold, for each variable, a dedicated copy of the clauses. For instance, the copy of clauses dedicated to variable x_2 in the central chain of $\mathbf{Q}_\phi$ in Figure 12 is realized by the 5 elements from the 6th to the 10th from the bottom in the central chain; call them $(\delta_1, 2), \ldots, (\delta_5, 2)$. Analogously, the 5 elements from the 6th to the 10th from the bottom in the right chain, call them $(\delta'_1, 2), \ldots, (\delta'_5, 2)$, record the copy of clauses dedicated to variable x_2 in the right chain.

The identity of each clause of ϕ among its many copies is maintained by zigzagging between the central chain and the right chain of $\mathbf{Q}_\phi$. For instance, clause δ_1 is maintained by zigzagging on the 1st element from the bottom in the central chain (the copy of δ_1 corresponding to variable x_1, namely, $(\delta_1, 1)$), the 1st element from the bottom in the right chain (the copy of δ_1 corresponding to variable x_1, namely, $(\delta'_1, 1)$), the 6th

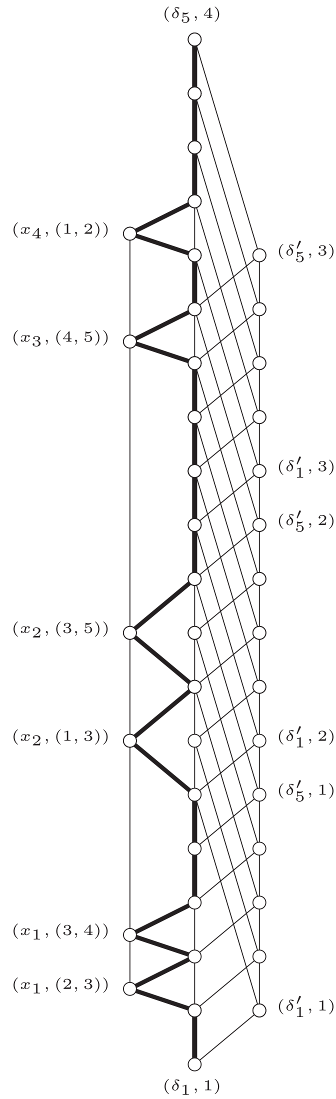

Fig. 12. The poset $\mathbf{Q}_\phi$ corresponding to $\phi \in \mathcal{S}$ in Example 5.2.

element from the bottom in the central chain (the copy of δ_1 corresponding to variable x_2, namely, $(\delta_1, 2)$), and so on.

In a similar fashion, occurrences of variables in clauses are recorded by zigzagging between the left chain and the central chain of $\mathbf{Q}_\phi$. For instance, variable x_2 occurring in the $t = 3$ clauses δ_1, δ_3, and δ_5 generates $t-1 = 2$ elements in the left chain of $\mathbf{Q}_\phi$; these are the 3rd and 4th element from the bottom; call them $(x_2, (1,3))$ and $(x_2, (3,5))$. The zigzag touches the 6th element $(\delta_1, 2)$ in the central chain, the 3rd element $(x_2, (1,3))$ in the left chain, the 8th element $(\delta_3, 2)$ in the central chain, the 5th element $(x_2, (3,5))$ in the left chain, and, finally, the 10th element $(\delta_5, 2)$ in the central chain.

Eventually, edges in the zigzags between the left chain and the central chain, together with a suitable selection of edges in the central chain, are transformed into long chains (the thick edges in Figure 12) to make sure that the maximum depth of $\mathbf{Q}_\phi$ is attained, uniquely, by the chain from the bottom to the top of $\mathbf{Q}_\phi$ that zigzags as much as possible between the left and central chains.

The details of the construction of $\mathbf{Q}_\phi$ follow. For $i \in [n]$ and $j \in [m]$, we write $x_i \in \delta_j$ if a literal on variable x_i occurs in clause δ_j, and we let $\mathrm{var}(\delta_j) = \{x_i \mid i \in [n], x_i \in \delta_j\}$. The universe Q_ϕ contains $Q^a_\phi = \{(\delta_i, j) \mid i \in [m], j \in [n]\}$, $Q^c_\phi = \{(\delta'_i, j) \mid i \in [m], j \in [n-1]\}$,

$$Q^v_\phi = \left\{ (x_i, (j, j')) \;\middle|\; \begin{array}{l} i \in [n], x_i \in \delta_j, x_i \in \delta_{j'}, j < j', \\ \text{and } x_i \notin \delta_{j''} \text{ for all } j < j'' < j' \end{array} \right\},$$

and a set Q^l_ϕ of auxiliary elements introduced here.

For $q, q' \in Q_\phi$, we let $\ll^{\mathbf{Q}_\phi}$ denote the fact that, in the order of $\mathbf{Q}_\phi$, there is a chain of $|Q^a_\phi|$ fresh auxiliary elements, contained in Q^l_ϕ, between q and q'. The order relation of $\mathbf{Q}_\phi$ is defined by the following cover relations:

(Q1) for all $(\delta_i, j), (\delta_{i+1}, j) \in Q^a_\phi$: if $i + 1 \leq i'$, where i' is the minimum in $[m]$ such that $x_j \in \delta_{i'}$, then $(\delta_i, j) \ll^{\mathbf{Q}_\phi} (\delta_{i+1}, j)$; if $i' \leq i$, where i' is the maximum in $[m]$ such that $x_j \in \delta_{i'}$, then $(\delta_i, j) \ll^{\mathbf{Q}_\phi} (\delta_{i+1}, j)$; otherwise, $(\delta_i, j) \prec^{\mathbf{Q}_\phi} (\delta_{i+1}, j)$;
(Q2) $(\delta_m, j) \ll^{\mathbf{Q}_\phi} (\delta_1, j+1)$, for $(\delta_m, j), (\delta_1, j+1) \in Q^a_\phi$;
(Q3) $(\delta'_i, j) \prec^{\mathbf{Q}_\phi} (\delta'_{i+1}, j)$, for $(\delta'_i, j), (\delta'_{i+1}, j) \in Q^c_\phi$;
(Q4) $(\delta'_m, j) \prec^{\mathbf{Q}_\phi} (\delta'_1, j+1)$, for $(\delta'_m, j), (\delta'_1, j+1) \in Q^c_\phi$;
(Q5) $(x_i, (j, j')) <^{\mathbf{Q}_\phi} (x_i, (j', j''))$, for $(x_i, (j, j')), (x_i, (j', j'')) \in Q^v_\phi$;
(Q6) $(x_i, (j, j')) <^{\mathbf{Q}_\phi} (x_{i+1}, (k, k'))$, for $(x_i, (j, j')), (x_i, (k, k')) \in Q^v_\phi$, where j' is maximum in $[m]$ such that $x_i \in \delta_{j'}$ and k is minimum in $[m]$ such that $x_{i+1} \in \delta_k$.
(Q7) $(\delta_i, j) \prec^{\mathbf{Q}_\phi} (\delta'_i, j) \prec^{\mathbf{Q}_\phi} (\delta_i, j+1)$, for all $(\delta_i, j), (\delta_i, j+1) \in Q^a_\phi$ and $(\delta'_i, j) \in Q^c_\phi$;
(Q8) $(\delta_i, j) \ll^{\mathbf{Q}_\phi} (x_j, (i, i')) \ll^{\mathbf{Q}_\phi} (\delta_{i'}, j)$, for all $(\delta_i, j), (\delta_{i'}, j) \in Q^a_\phi$ and $(x_j, (i, i')) \in Q^v_\phi$.

An example of the construction follows.

Example 5.3. For ϕ, as in Example 5.2, the poset $\mathbf{Q}_\phi$ is depicted in Figure 12, where the chain on the left is Q^v_ϕ, the chain in the middle contains Q^a_ϕ, and the chain on the right is Q^c_ϕ. Thick edges represent chains of $|Q^a_\phi|$ elements.

We now informally describe the construction of the poset $\mathbf{P}_\phi = r(\phi)$ in terms of the three blocks of elements displayed in Figure 13. The central and right blocks are analogous to the central and right chains in $\mathbf{Q}_\phi$, but instead of being formed by stacking n chains (variables of ϕ) of m elements (clauses of ϕ) as in $\mathbf{Q}_\phi$, they are formed by stacking n blocks (variables of ϕ), each having m layers (clauses of ϕ) of at most 7 elements (satisfying assignments of a clause on at most 3 literals). Again, each variable holds its own copy of satisfying assignments to clauses.

For instance, in Figure 13, the first 5 layers of the central block record the copy of satisfying assignments to clauses of ϕ held by variable x_1. The first layer of the central block, formed by the three points on the bottom of the diagram, corresponds to the satisfying assignments of δ_1: $f_{1,1} = \{(x_2, 0), (x_4, 0)\}$, $f_{1,2} = \{(x_2, 1), (x_4, 0)\}$, $f_{1,3} = \{(x_2, 1), (x_4, 1)\}$. The 2nd layer of the central block corresponds to the satisfying assignments of δ_2: $f_{2,1} = \{(x_1, 0), (x_4, 0)\}$, $f_{2,2} = \{(x_1, 1), (x_4, 0)\}$, $f_{2,3} = \{(x_1, 1), (x_4, 1)\}$. The 3rd layer of the central block corresponds to the satisfying assignments of δ_3: $f_{3,1} = \{(x_1, 0), (x_2, 0)\}$, $f_{3,2} = \{(x_1, 1), (x_2, 0)\}$, $f_{3,3} = \{(x_1, 1), (x_2, 1)\}$. The 4th layer of the central block corresponds to the satisfying assignments of δ_4: $\{(x_1, 0), (x_2, 0)\}$, $\{(x_1, 1), (x_2, 0)\}$, $f_{4,3} = \{(x_1, 1), (x_2, 1)\}$. The 5th layer of the central block corresponds to the satisfying assignments of δ_5: $f_{5,1} = \{(x_2, 0), (x_3, 0)\}$, $\{(x_2, 0), (x_3, 1)\}$, $f_{5,3} = \{(x_2, 1), (x_3, 1)\}$.

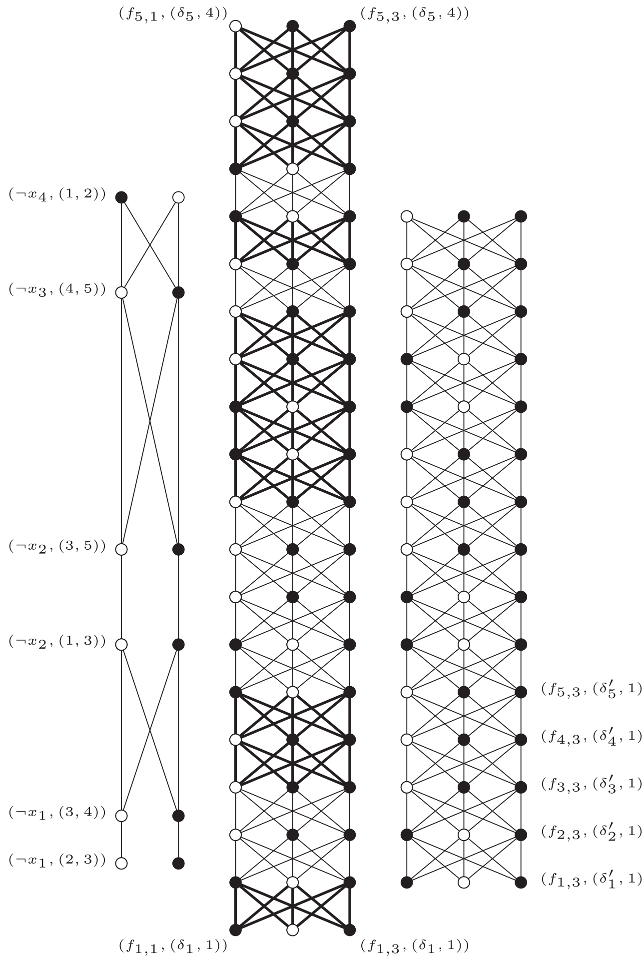

Fig. 13. Items (P1) through (P4) in the construction of poset $\mathbf{P}_\phi$, where $\phi \in \mathcal{S}$ is as in Example 5.2.

The identity of these satisfying assignments across their n copies is maintained by zigzagging between the central and right blocks of $\mathbf{P}_\phi$. For instance, let $(f_{1,1}, (\delta_1, 1))$ be the first element of the 1st layer of the central block, $(f_{1,1}, (\delta'_1, 1))$ be the first element of the 1st layer of the right block, $(f_{1,1}, (\delta_1, 2))$ be the first element of the 6th layer of the central block, and so on. Then a zigzag touches $(f_{1,1}, (\delta_1, 1))$, $(f_{1,1}, (\delta'_1, 1))$, and $(f_{1,1}, (\delta_1, 2))$, and so on, promoting the identity of $f_{1,1}$ upwards; see Figure 14.

For each element in the left chain of $\mathbf{Q}_\phi$ (corresponding to a variable), the left block of $\mathbf{P}_\phi$ has one layer of two elements (corresponding to the variable assignments). For instance, the two elements $(\neg x_1, (2, 3))$ and $(x_1, (2, 3))$ on the first layer of the left block of $\mathbf{P}_\phi$ correspond to the first element, $(x_1, (2, 3))$, in the left chain of $\mathbf{Q}_\phi$, and the two elements $(\neg x_1, (3, 4))$ and $(x_1, (3, 4))$ on the second layer of the left block of $\mathbf{P}_\phi$ correspond to the second element, $(x_1, (3, 4))$, in the left chain of $\mathbf{Q}_\phi$. Connections between the left block and the central block of $\mathbf{P}_\phi$ record which variable assignment is consistent with which satisfying assignment to a clause (in the copy corresponding to the variable). For instance: $(f_{2,1}, (\delta_2, 1))$ edges $(\neg x_1, (2, 3))$; $(f_{2,2}, (\delta_2, 1))$ edges $(\neg x_1, (2, 3))$; $(\neg x_1, (2, 3))$ edges $(f_{3,1}, (\delta_3, 1))$. See Figure 14.

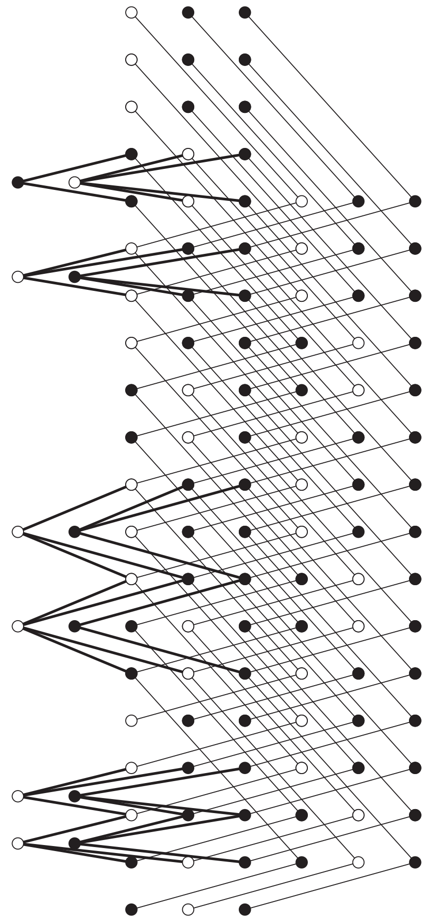

Fig. 14. Items (P5) and (P6) in the construction of poset $\mathbf{P}_\phi$, where $\phi \in \mathcal{S}$ is as in Example 5.2.

Finally, each edge in a suitable selection of edges in $\mathbf{P}_\phi$, corresponding to thick edges in $\mathbf{Q}_\phi$ as displayed in Figures 13 and 14, is replaced by a long chain making sure that the longest (thick) chain of $\mathbf{Q}_\phi$ can be embedded only within longest (thick) chains in $\mathbf{P}_\phi$.

The details of the construction of $\mathbf{P}_\phi$ follow. The universe P_ϕ is the union of

—$P^a_\phi = \bigcup_{(\delta_i, j)\in Q^a_\phi} \{(f, (\delta_i, j)) \mid f \in \{0,1\}^{\mathrm{var}(\delta_i)} \text{ satisfies } \delta_i\}$,
—$P^c_\phi = \bigcup_{(\delta'_i, j)\in Q^c_\phi} \{(f, (\delta'_i, j)) \mid f \in \{0,1\}^{\mathrm{var}(\delta_i)} \text{ satisfies } \delta_i\}$,
—$P^v_\phi = \bigcup_{(x_i,(j,j'))\in Q^v_\phi} \{(x_i, (j, j')), (\neg x_i, (j, j'))\}$,

and a set P^l_ϕ of auxiliary elements introduced next.

Again, for $p, p' \in P_\phi$, we let $\ll^{\mathbf{P}_\phi}$ denote the fact that, in the order of $\mathbf{P}_\phi$, there is a chain of $|Q^a_\phi|$ fresh auxiliary elements, contained in P^l_ϕ, between p and p'. The order relation of $\mathbf{P}_\phi$ is defined by the following cover relation:

(P1) for all $(f, (\delta_i, j)), (f', (\delta_{i'}, j')) \in P^a_\phi$, $(f, (\delta_i, j)) \prec^{\mathbf{P}_\phi} (f', (\delta_{i+1}, j))$ if and only if $(\delta_i, j) \prec^{\mathbf{Q}_\phi} (\delta_{i'}, j')$, and $(f, (\delta_i, j)) \ll^{\mathbf{P}_\phi} (f', (\delta_{i+1}, j))$ if and only if $(\delta_i, j) \ll^{\mathbf{Q}_\phi} (\delta_{i'}, j')$;

(P2) for all $(f, (\delta'_i, j)), (f', (\delta'_{i'}, j')) \in P^c_\phi$, $(f, (\delta'_i, j)) \prec^{\mathbf{P}_\phi} (f', (\delta'_{i+1}, j))$ if and only if $(\delta'_i, j) \prec^{\mathbf{Q}_\phi} (\delta'_{i'}, j')$;

(P3) for all $(x_i, (j, j'))$, $(\neg x_i, (j, j'))$, $(x_i, (j', j''))$, $(\neg x_i, (j', j''))$ in P^v_ϕ, $(x_i, (j, j')) \prec^{\mathbf{P}_\phi} (x_i, (j', j''))$ and $(\neg x_i, (j, j')) \prec^{\mathbf{P}_\phi} (\neg x_i, (j', j''))$ if and only if $(x_i, (j, j')) \prec^{\mathbf{Q}_\phi} (x_i, (j', j''))$;

(P4) for all $(x_i, (j, j'))$, $(\neg x_i, (j, j'))$, $(x_{i+1}, (k, k'))$, and $(\neg x_{i+1}, (k, k'))$ in P^v_ϕ, $(x_i, (j, j')) \prec^{\mathbf{P}_\phi} (x_{i+1}, (k, k'))$, $(x_i, (j, j')) \prec^{\mathbf{P}_\phi} (\neg x_{i+1}, (k, k'))$, $(\neg x_i, (j, j')) \prec^{\mathbf{P}_\phi} (x_{i+1}, (k, k'))$, and $(\neg x_i, (j, j')) \prec^{\mathbf{P}_\phi} (\neg x_{i+1}, (k, k'))$, if and only if $(x_i, (j, j')) \prec^{\mathbf{Q}_\phi} (x_{i+1}, (k, k'))$.

(P5) for all $(f, (\delta_i, j)), (f, (\delta_i, j+1)) \in P^a_\phi$ and $(f, (\delta'_i, j)) \in P^c_\phi$, $(f, (\delta_i, j)) \prec^{\mathbf{P}_\phi} (f, (\delta'_i, j)) \prec^{\mathbf{P}_\phi} (f, (\delta_i, j+1))$ if and only if $(\delta_i, j) \prec^{\mathbf{Q}_\phi} (\delta'_i, j) \prec^{\mathbf{Q}_\phi} (\delta_i, j+1)$;

(P6) for all $(f, (\delta_i, j)), (f', (\delta_{i'}, j)), (g, (\delta_i, j)), (g', (\delta_{i'}, j)) \in P^a_\phi$ and $(x_j, (i, i')), (\neg x_j, (i, i')) \in P^v_\phi$, it holds that $(f, (\delta_i, j)) \ll^{\mathbf{P}_\phi} (x_j, (i, i')) \ll^{\mathbf{P}_\phi} (f', (\delta_{i'}, j))$ and $(g, (\delta_i, j)) \ll^{\mathbf{P}_\phi} (\neg x_j, (i, i')) \ll^{\mathbf{P}_\phi} (g', (\delta_{i'}, j))$ if and only if $(\delta_i, j) \ll^{\mathbf{Q}_\phi} (x_j, (i, i')) \ll^{\mathbf{Q}_\phi} (\delta_{i'}, j)$, $f(x_j) = f'(x_j) = 1$, and $g(x_j) = g'(x_j) = 0$.

Note that $\mathrm{width}(\mathbf{Q}_\phi) \leq 4$ and $\mathrm{width}(\mathbf{P}_\phi) \leq 2^2 + 7^2 + 7^2 = 102$ for all $\phi \in \mathcal{S}$ (we remark that this width bound may be improved at the cost of a more complicated construction). Hence $\mathcal{P}_{\mathrm{width}} = \{r(\phi) \mid \phi \in \mathcal{S}\}$ has bounded width.

We are now ready to prove that this construction gives the desired reduction.

THEOREM 5.4. $\mathrm{EMB}(\mathcal{P}_{\mathrm{width}})$ *is* NP-*hard.*

PROOF. We give a polynomial-time many–one reduction from the satisfiability problem over $\mathcal{S}$ to the problem $\mathrm{EMB}(\mathcal{P}_{\mathrm{width}})$, which suffices since the source problem is NP-hard.

The reduction maps an instance $\phi \in \mathcal{S}$ of the satisfiability problem, say $\phi(x_1, \ldots, x_n) = \delta_1 \wedge \cdots \wedge \delta_m$, to the instance $(\mathbf{Q}_\phi, \mathbf{P}_\phi)$ of $\mathrm{EMB}(\mathcal{P}_{\mathrm{width}})$, where $\mathbf{Q}_\phi$ and $\mathbf{P}_\phi$ are constructed as delineated earlier. The reduction is clearly polynomial-time computable. We prove that the reduction is correct.

If ϕ is satisfiable, then let $g : \{x_1, \ldots, x_n\} \to \{0, 1\}$ be a satisfying assignment. We define a function $e : Q_\phi \to P_\phi$ as follows. Let $q \in Q_\phi$. Then:

—If $q = (\delta_i, j) \in Q^a_\phi$, then $e(q) = (f, (\delta_i, j)) \in P^a_\phi$ if and only if $g|_{\mathrm{var}(\delta_i)} = f$.

—If $q = (\delta'_i, j) \in Q^c_\phi$, then $e(q) = (f, (\delta'_i, j)) \in P^c_\phi$ if and only if $g|_{\mathrm{var}(\delta_i)} = f$.

—If $q = (x_i, (j, j')) \in Q^v_\phi$, then $e(q) = (x_i, (j, j')) \in P^v_\phi$ if $g(x_i) = 1$, and $e(q) = (\neg x_i, (j, j')) \in P^v_\phi$ if $g(x_i) = 0$.

—If $q \in Q^l_\phi$, then let $q', q'' \in Q_\phi$ and $q_1, \ldots, q_{|Q^a_\phi|} \in Q^l_\phi$ be such that $q' \prec^{\mathbf{Q}_\phi} q_1 \prec^{\mathbf{Q}_\phi} \cdots \prec^{\mathbf{Q}_\phi} q_{|Q^a_\phi|} \prec^{\mathbf{Q}_\phi} q''$ and $q = q_i$ for $i \in [|Q^a_\phi|]$. By construction, there exist $p_1, \ldots, p_{|Q^a_\phi|} \in P^l_\phi$ such that $e(q') \prec^{\mathbf{P}_\phi} p_1 \prec^{\mathbf{P}_\phi} \cdots \prec^{\mathbf{P}_\phi} p_{|Q^a_\phi|} \prec^{\mathbf{Q}_\phi} e(q'')$. Then, $e(q) = e(q_i) = p_i$.

It is easy to check that e embeds $\mathbf{Q}_\phi$ into $\mathbf{P}_\phi$.

Conversely, let $e : Q_\phi \to P_\phi$ be an embedding of $\mathbf{Q}_\phi$ into $\mathbf{P}_\phi$.

CLAIM 3. $e(Q^a_\phi) \subseteq P^a_\phi$, $e(Q^v_\phi) \subseteq P^v_\phi$, $e(Q^c_\phi) \subseteq P^c_\phi$.

PROOF OF CLAIM 3. Let $Q^* = \{q \in Q^a_\phi \mid q \text{ is comparable to all elements in } Q^v_\phi\}$. Note that, by construction, $\mathrm{depth}(\mathbf{Q}_\phi) = |Q^v_\phi \cup Q^l_\phi \cup Q^*| = d$, and the chain $Q^v_\phi \cup Q^l_\phi \cup Q^*$ is the unique chain whose size equals d. In the poset $\mathbf{Q}_\phi$ depicted in Figure 12, Q^* contains exactly the elements of the middle chain hit by a thick edge, and the chain $Q^v_\phi \cup Q^l_\phi \cup Q^*$ is represented by the thick edges. Moreover, by construction again, $\mathrm{depth}(\mathbf{P}_\phi) = d$, and the only chains in $\mathbf{P}_\phi$ whose size equals d force the embedding to satisfy $e(Q^v_\phi) \subseteq P^v_\phi$ and $e(Q^*) \subseteq P^a_\phi$.

We now prove that $e(Q^a_\phi \setminus Q^*) \subseteq P^a_\phi$, which, together with the above, yields $e(Q^a_\phi) \subseteq P^a_\phi$. Let $q \in Q^a_\phi \setminus Q^*$. Let $q', q'' \in Q^*$ be such that $q' <^{\mathbf{Q}_\phi} q <^{\mathbf{Q}_\phi} q''$ and there do not exist $r', r'' \in Q^*$ such that $q' <^{\mathbf{Q}_\phi} r' <^{\mathbf{Q}_\phi} q$ or $q <^{\mathbf{Q}_\phi} r'' <^{\mathbf{Q}_\phi} q''$. In Figure 12, if, for instance, q is the 8th lowest element in the middle chain, then q' and q'' are, respectively, the 6th and 9th lowest elements in the middle chain. Let $S = \{p \in P_\phi \mid e(q') <^{\mathbf{P}_\phi} p <^{\mathbf{P}_\phi} e(q'')\}$, so that $e(q) \in S$, because e is an embedding. By the above, $S \cap (P^v_\phi \cup P^l_\phi) \subseteq e(Q^v_\phi \cup Q^l_\phi \cup Q^*)$, therefore $e(q) \in S \setminus (P^v_\phi \cup P^l_\phi)$. Moreover, the distance between $e(q')$ and $e(q'')$ in $\mathbf{P}_\phi$ is strictly less than m, therefore $S \cap P^c_\phi = \emptyset$. It follows that $e(q) \in S \setminus (P^v_\phi \cup P^l_\phi \cup P^c_\phi)$, that is, $e(q) \in P^a_\phi$.

Finally, we prove that $e(Q^c_\phi) \subseteq P^c_\phi$. Let $q \in Q^c_\phi$. By construction, there exist $m+1$ elements $q_0, \ldots, q_m \in Q^a_\phi$ such that $q_0 <^{\mathbf{Q}_\phi} \cdots <^{\mathbf{Q}_\phi} q_m$, $q_0 <^{\mathbf{Q}_\phi} q <^{\mathbf{Q}_\phi} q_m$, and q is incomparable to $q_1, \ldots, q_{m-1}$ in $\mathbf{Q}_\phi$. By the above, $e(q_0), \ldots, e(q_m) \in P^a_\phi$. As e is an embedding, $e(q_0) <^{\mathbf{P}_\phi} \cdots <^{\mathbf{P}_\phi} e(q_m)$, $e(q_0) <^{\mathbf{P}_\phi} e(q) <^{\mathbf{P}_\phi} e(q_m)$, and $e(q)$ is incomparable to $e(q_1), \ldots, e(q_{m-1})$ in $\mathbf{P}_\phi$. By inspection of the construction, we now prove that $e(q) \notin P^a_\phi \cup P^v_\phi \cup P^l_\phi$, which implies that $e(q) \in P^c_\phi$, as desired. If $e(q) \in P^a_\phi$, then $e(q)$ is incomparable to at most 1 element among $e(q_1), \ldots, e(q_{m-1})$, which implies that $e(q) \notin P^a_\phi$ since $m > 2$. If $e(q) \in P^v_\phi \cup P^l_\phi$, then $e(q)$ is incomparable to at most $m-2$ elements among $e(q_1), \ldots, e(q_{m-1})$, which implies $e(q) \notin P^v_\phi \cup P^l_\phi$.

The previous three properties uniquely determine the behavior of e over Q^l_ϕ. Next, we state two facts that follow from the embedding and specific properties of the construction of $\mathbf{Q}$ and $\mathbf{P}$.

—Items (Q1) through (Q4) and (Q7) on one hand, and (P1), (P2), and (P5) on the other hand, enforce the following: for all $i \in [m]$ and $j \in [n]$, there exists a unique $f \in \{0,1\}^{\mathrm{var}(\delta_i)}$ such that, for all $(\delta_i, j), (\delta'_i, j) \in Q^a_\phi \cup Q^c_\phi$, it holds that $e((\delta_i, j)) = (f, (\delta_i, j))$ and $e((\delta'_i, j)) = (f, (\delta'_i, j))$.

—Items (Q5), (Q6), and (Q8) on one hand, and (P3) through (P5) and (P6) on the other hand, enforce the following: for all $i, i' \in [m]$, $i \neq i'$, and $j \in [n]$ such that $x_j \in \mathrm{var}(\delta_i) \cap \mathrm{var}(\delta_{i'})$, it holds that if $e((\delta_i, j)) = (f, (\delta_i, j))$ and $e((\delta_{i'}, j)) = (f', (\delta_{i'}, j))$, then $f(x_j) = f'(x_j)$.

Therefore, the union of all the assignments f such that $e((\delta_i, \cdot)) = (f, (\delta_i, \cdot))$, taken over all $i \in [m]$, defines an assignment $g : \{x_1, \ldots, x_n\} \to \{0,1\}$; moreover, g satisfies ϕ. This concludes the proof. □

5.2. Embedding is NP-hard on Bounded Degree Posets

We reduce from the satisfiability problem. Let $\mathcal{S}$ be the class of propositional formulas in conjunctive form, where each clause contains exactly 3 pairwise noncomplementary literals (for notational convenience, but the construction works even if relaxed to at most 3 literals, which we use for illustration purposes in the examples).

The idea of the reduction is the following. We encode a formula in $\mathcal{S}$ by a poset $\mathbf{P}$, whose universe partitions into three blocks, P_0, P_1, and P_2. The set P_1 contains several groups of 7 elements, where each element corresponds to one possible satisfying assignment of a clause, and the embedding encodes an assignment for the whole formula by forcing us to choose one element out of each group. The set P_2 ensures that each assignment chosen by the embedding is consistent for each pair of clauses. To preserve

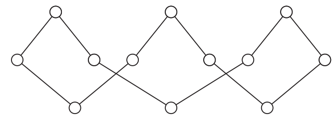

Fig. 15. The poset $\mathbf{Q}_\phi$ corresponding to $\phi \in \mathcal{S}$ in Example 5.5.

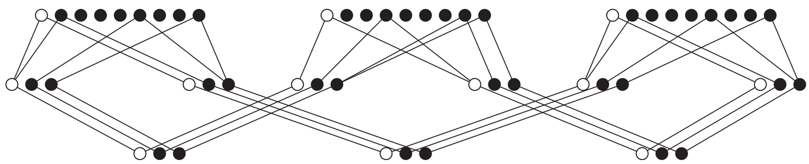

Fig. 16. The poset $\mathbf{P}_\phi$ corresponding to $\phi \in \mathcal{S}$ in Example 5.5.

bounded degree while ensuring the consistency of each pair of clauses, it is necessary to use many groups in P_1 for each clause. Finally, P_0 ensures that each choice made by the embedding for a given clause is consistent across all groups corresponding to that clause.

Example 5.5. Let $\phi(x_1, x_2, x_3) = \delta_1 \wedge \delta_2 \wedge \delta_3$, where $\delta_1 = x_1 \vee \neg x_2$, $\delta_2 = x_3 \vee \neg x_1$, and $\delta_3 = \neg x_3 \vee x_2$. Note that, for instance, ϕ is satisfied by $\{(x_1, 0), (x_2, 0), (x_3, 0)\}$.

The poset $\mathbf{Q}_\phi$ is depicted in Figure 15, where Q_0, Q_1, and Q_2 form, respectively, the bottom, middle, and top layers of the diagram; poset $\mathbf{P}_\phi$ is similarly displayed in Figure 16. The white points in $\mathbf{P}_\phi$ form the image of the embedding $e : Q_\phi \to P_\phi$ of $\mathbf{Q}_\phi$ into $\mathbf{P}_\phi$ corresponding to the satisfying assignment earlier as by (the easy direction of) Theorem 5.6.

We now formalize the ideas outlined earlier. Let $\phi(x_1, \ldots, x_n) = \delta_1 \wedge \cdots \wedge \delta_m$ be in $\mathcal{S}$. For $j \in [n]$ and $i \in [m]$, we write $x_j \in \delta_i$ if a literal on variable x_j occurs in clause δ_i, and we let $\mathrm{var}(\delta_i) = \{x_j \mid j \in [n], x_j \in \delta_i\}$. For all $i \in [m]$, let $(g_{i,1}, \ldots, g_{i,7})$ be a fixed ordering of the assignments in $\{0,1\}^{\mathrm{var}(\delta_i)}$ satisfying δ_i, and let $(i_1, i_2, \ldots, i_{m-1}) = (1, \ldots, i-1, i+1, \ldots, m)$. We define our two posets $\mathbf{Q}_\phi$ and $\mathbf{P}_\phi$ next.

The poset $\mathbf{Q}_\phi$ has universe $Q_\phi = Q_0 \cup Q_1 \cup Q_2$, where

$$\begin{aligned}
Q_0 &= \{c_{(i,j)}, c_{(i,m)}, c_{(m,j)} \mid i, j \in [m-1], i \neq j\},\\
Q_1 &= \{f_{(i,j)} \mid i, j \in [m], i \neq j\},\\
Q_2 &= \{d_{(i,j)} \mid 1 \leq i < j \leq m\},
\end{aligned}$$

and its cover relation is defined by the following:

(E1) $f_{(i,j)}, f_{(j,i)} \prec^{\mathbf{Q}_\phi} d_{(i,j)}$ for all $1 \leq i < j \leq m$.
(E2) For all $i \in [m]$,

$$f_{(i,i_1)} \succ^{\mathbf{Q}_\phi} c_{(i,i_1)} \prec^{\mathbf{Q}_\phi} f_{(i,i_2)} \succ^{\mathbf{Q}_\phi} \cdots \succ^{\mathbf{Q}_\phi} c_{(i,i_{m-1})} \prec^{\mathbf{Q}_\phi} f_{(i,i_{m-1})}.$$

The poset $\mathbf{P}_\phi$ has universe $P_\phi = P_0 \cup P_1 \cup P_2$ where

$$\begin{aligned}
P_0 &= \{c_{(i,j),a}, c_{(i,m),a}, c_{(m,j),a} \mid i, j \in [m-1], i \neq j, a \in [7]\},\\
P_1 &= \{f_{(i,j),a} \mid i, j \in [m], i \neq j, a \in [7]\},\\
P_2 &= \{d_{(i,j),(a,a')} \mid 1 \leq i < j \leq m, (a, a') \in [7]^2\},
\end{aligned}$$

and its cover relation is defined by the following:

(D1) For all $1 \leq i < j \leq m$, it holds that $f_{(i,j),a}, f_{(j,i),a'} \prec^{\mathbf{P}_\phi} d_{(i,j),(a,a')}$ if and only if $g_{i,a}(x) = g_{j,a'}(x)$ for all $x \in \mathrm{var}(\delta_i) \cap \mathrm{var}(\delta_j)$.

(D1) For all $i \in [m]$ and $a \in [7]$,

$$f_{(i,i_1),a} \succ^{\mathbf{P}_\phi} c_{(i,i_1),a} \prec^{\mathbf{P}_\phi} f_{(i,i_2),a} \succ^{\mathbf{P}_\phi} \cdots \succ^{\mathbf{P}_\phi} c_{(i,i_{m-1})} \prec^{\mathbf{P}_\phi} f_{(i,i_{m-1}),a}.$$

Since cover-degree$(\mathbf{P}_\phi) \leq 1 + 7 = 8$ and depth$(\mathbf{P}_\phi) \leq 3$, $\mathcal{P}_{\text{degree}} = \{\mathbf{P}_\phi \mid \phi \in \mathcal{S}\}$ has bounded degree by Proposition 3.3.

THEOREM 5.6. EMB$(\mathcal{P}_{\text{degree}})$ *is* NP-*hard.*

PROOF. We give a polynomial-time many–one reduction from the satisfiability problem over $\mathcal{S}$ to the problem EMB$(\mathcal{P}_{\text{degree}})$, which suffices since the source problem is NP-hard.

The reduction maps an instance $\phi \in \mathcal{S}$ of the satisfiability problem, say, $\phi(x_1, \ldots, x_n) = \delta_1 \wedge \cdots \wedge \delta_m$, to the instance $(\mathbf{Q}_\phi, \mathbf{P}_\phi)$ of EMB$(\mathcal{P}_{\text{degree}})$. The reduction is clearly polynomial-time computable.

For correctness, let $g : \{x_1, \ldots, x_n\} \to \{0, 1\}$ be an assignment satisfying ϕ. Recall that $(g_{i,1}, \ldots, g_{i,7})$ is a fixed ordering of the assignments in $\{0, 1\}^{\mathrm{var}(\delta_i)}$ satisfying δ_i, for all $i \in [m]$. Let $(a_1, \ldots, a_m) \in [7]^m$ be such that $g|_{\mathrm{var}(\delta_i)} = g_{i,a_i}$ for all $i \in [m]$. It is easy to check that the function $e : Q_\phi \to P_\phi$ defined by setting:

—$e(c_{(i,j)}) = c_{(i,j),a_i}$ for all $c_{(i,j)} \in Q_0$;
—$e(f_{(i,j)}) = f_{(i,j),a_i}$ for all $f_{(i,j)} \in Q_1$;
—$e(d_{(i,j)}) = d_{(i,j),(a_i,a_j)}$ for all $d_{(i,j)} \in Q_2$;

embeds $\mathbf{Q}_\phi$ into $\mathbf{P}_\phi$.

Conversely, let $e : Q_\phi \to P_\phi$ embed $\mathbf{Q}_\phi$ into $\mathbf{P}_\phi$. We show that ϕ is satisfiable. Note that $e(Q_i) \subseteq P_i$ for all $i \in \{0, 1, 2\}$ because e maps all 3-element chains in $\mathbf{Q}_\phi$ into 3-element chains in $\mathbf{P}_\phi$; all 3-element chains in $\mathbf{Q}_\phi$ link three elements in Q_0, Q_1, and Q_2 in this order; and all 3-element chains in $\mathbf{P}_\phi$ link three elements in P_0, P_1, and P_2 in this order.

We first claim that, for all $i \in [m]$, there exists exactly one $a \in [7]$ such that, for all $j \in [m] \setminus \{i\}$, it holds that $e(f_{(i,j)}) = f_{(i,j),a}$. Assume for a contradiction that $e(f_{(i,j)}) = f_{(i,j),a}$ and $e(f_{(i,j')}) = f_{(i,j'),a'}$ for some $i \in [m]$, $a \neq a' \in [7]$, and $j \neq j' \in [m] \setminus \{i\}$; without loss of generality, let $j < j'$. By (E2), $f_{(i,j)}$ reaches $f_{(i,j')}$ through a fence of length $2(j' - j)$, starting in Q_1 and alternating steps in Q_0 and Q_1; but, by (D2), $f_{(i,j),a}$ does not reach $f_{(i,j'),a'}$ through a fence of length $2(j' - j)$, starting in P_1 and alternating steps in P_0 and P_1, contradicting the assumption that e is an embedding.

Let $(a_1, \ldots, a_m) \in [7]^m$ be uniquely determined by the previous claim. We now claim that, for all $i, j \in [m]$ such that $i \neq j$, and all $x \in \mathrm{var}(\delta_i) \cap \mathrm{var}(\delta_j)$, it holds that $g_{i,a_i}(x) = g_{j,a_j}(x)$. Assume without loss of generality that $i < j$. By (E1), $f_{(i,j)}, f_{(j,i)} \prec^{\mathbf{Q}_\phi} d_{(i,j)}$. By hypothesis, $e(f_{(i,j)}) = f_{(i,j),a_i}$ and $e(f_{(j,i)}) = f_{(j,i),a_j}$. Therefore, since e is an embedding, $f_{(i,j),a_i}, f_{(j,i),a_j} \prec^{\mathbf{P}_\phi} e(d_{(i,j)})$; thus, by (D1), $e(d_{(i,j)}) = d_{(i,j),(a_i,a_j)}$, that is, $g_{i,a_i}(x) = g_{j,a_j}(x)$ for all $x \in \mathrm{var}(\delta_i) \cap \mathrm{var}(\delta_j)$.

By what has just been delineated, $g = g_{1,a_1} \cup \cdots \cup g_{m,a_m}$ is a function from $\{x_1, \ldots, x_n\}$ to $\{0, 1\}$. Since g_{i,a_i} satisfies δ_i for all $i \in [m]$, it follows that g satisfies ϕ, concluding the proof. □

5.3. Isomorphism in Polynomial Time on Bounded Width Posets

The insight on bounded width used to prove tractability of the embedding problem essentially scales to the isomorphism problem.

THEOREM 5.7. *Let $\mathcal{P}$ be a class of posets of bounded width. Then,* ISO($\mathcal{P}$) *is polynomial-time tractable.*

PROOF. Let $\mathbf{R}$ be any poset. For all $S \subseteq R$, let $(S]$ be *downset* generated by S in $\mathbf{R}$, that is, $(S] = \{r \in R \mid \exists s \in S \text{ such that } r \leq^{\mathbf{R}} s\}$. Let $l(\mathbf{R})$ be the order defined by equipping the universe of all antichains in $\mathbf{R}$ by the relation $A \leq^{l(\mathbf{R})} A'$ if and only if $(A] \subseteq (A']$.

We collect from the literature three known facts on which the proof is based. First, for any (finite) poset $\mathbf{R}$, the structure $l(\mathbf{R})$ is a (finite) distributive lattice [Schröder 2003, Proposition 5.5.5]. Second, the substructure of $l(\mathbf{R})$ generated by join irreducible elements is isomorphic to $\mathbf{R}$ [Schröder 2003, Theorem 5.5.6]; recall that, if $\mathbf{L} = (L, \leq)$ is a lattice, then $j \in L$ is *join irreducible* if, for all $l, l' \in L$, if j is the least upper bound of l and l', then $j = l$ or $j = l'$. Third, the isomorphism problem restricted to finite distributive lattices is polynomial-time tractable [Gorazd and Idziak 1995].

Using the previous facts, it is easy to design the desired polynomial-time algorithm. Let $(\mathbf{Q}, \mathbf{P})$ be an instance of ISO($\mathcal{P}$), and let $|P| = n$. If $|Q| \not\leq n$ or $\text{width}(\mathbf{Q}) \not\leq \text{width}(\mathbf{P})$, the algorithm rejects; noting that $\text{width}(\mathbf{Q}) \leq |Q|$, the condition is checkable in time $O(n^3)$ by Theorem 4.3. Otherwise, the algorithm computes $l(\mathbf{Q})$ and $l(\mathbf{P})$, and accept if and only if $l(\mathbf{Q})$ and $l(\mathbf{P})$ are isomorphic. Note that, if $\text{width}(\mathbf{P})$ is considered as a constant in the runtime bound, then the construction of $l(\mathbf{Q})$ and $l(\mathbf{P})$ is polynomial-time computable from $\mathbf{Q}$ and $\mathbf{P}$; moreover, checking whether $l(\mathbf{Q})$ and $l(\mathbf{P})$ are isomorphic is feasible in polynomial time by the first and third fact mentioned earlier. Hence, the algorithm runs in polynomial time.

We conclude observing that the algorithm is correct. It suffices to show that $\mathbf{Q}$ and $\mathbf{P}$ are isomorphic if and only if $l(\mathbf{Q})$ and $l(\mathbf{P})$ are isomorphic. For the nontrivial direction (backwards), if f is an isomorphism from $l(\mathbf{Q})$ to $l(\mathbf{P})$, then let f' be the restriction of f to the join irreducible elements of $l(\mathbf{Q})$. As f is an isomorphism, in particular, f' is a bijection between the join irreducible elements of $l(\mathbf{Q})$ and $l(\mathbf{P})$. It follows from the second fact mentioned earlier that f' is an isomorphism between $\mathbf{Q}$ and $\mathbf{P}$. □

6. CONCLUSION

We embarked on the study of the model-checking problem on posets; compared to graphs, the problem is largely unexplored, and we made a first contribution by studying basic syntactic fragments (existential logic) and fundamental poset invariants (including width, depth, and degree). Our complexity classification for existential logic also carries over to the *jump number* (between size and width in Figure 1); a future direction is to extend our study to *dimension* (above width [Caspard et al. 2012] and degree [Furedi and Kahn 1986] in Figure 1).

Our main algorithmic result, fixed-parameter tractability of existential logic on bounded-width posets, raises the natural question of whether model checking the full first-order logic is fixed-parameter tractable on classes of posets of bounded width. We propose this as a topic for future research. Another interesting direction is investigating the parameterized complexity of the model-checking problem on various classes of finite lattices.

ACKNOWLEDGMENTS

We would like to thank the anonymous reviewers for comments and suggestions, which greatly helped improve the presentation.

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Received September 2014; revised June 2016; accepted August 2015