SOME THEOREMS ON DEDUCIBILITY.*

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This paper is concerned with a form of the problem of categoricalness in connection with sets of defining properties for types of dense series. Sets of defining properties for serial relations are given on the base K, R_2. All of the properties which occur in the determination of dense series are first-order functions. A first-order function is a function whose values are first-order propositions, and a first-order proposition is a proposition which contains variable individuals but does not contain any variable functions.† Given any function $f(x, y, \cdots, n)$, such that when values are assigned to $f, x, y, \cdots, n$ the expression becomes a proposition containing no variable constituents, if we quantify the variable constituents $x, y, \cdots, n$ as in $(x)(\exists y) \cdots (n) \cdot f(x, y, \cdots, n)$, the result is a first-order function, and for any value of f this becomes a first-order proposition. Sets of postulates for three sorts of dense series are to be treated: (1) series having neither a first nor a last element, (2) series having a first but no last element, and (3) those having both a first and a last element.

The class of all first-order functions on the base K, R_2 is the class of all such functions which are formulated solely in terms of K and R. These functions may involve any finite number of variable constituents. It will be shown for each set of postulates that the truth-value of any first-order function on the base K, R_2 is determined. Given any first-order function $F(x, y, \cdots, n)$ on K, R_2, then any one of the sets is such that F follows from it or else the contradictory of F follows; no first-order function can be independent of the set.

It will be necessary, as a preliminary to the analysis to be undertaken, to exhibit in some detail the general properties of first-order functions.‡ We are concerned here with quantifications as applied to the variable constituents of a construct which denote individuals and not with variable predicates. In this case it will be shown that any proposition has either the form

$$(x, y, \cdots, t) :: (\exists z, w, \cdots, r) :. (\mu, \nu, \cdots, m) : \cdots \cdots . \varphi(x, y, \cdots, t; z, w, \cdots, r; \mu, \nu, \cdots, m; \cdots)$$

or the form

$$(\exists x, y, \cdots, t) :: (z, w, \cdots, r) :. (\mu, \nu, \cdots, m) : \cdots \cdots . \varphi(x, y, \cdots, t; z, w, \cdots, r; \mu, \nu, \cdots, m; \cdots).$$

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† Compare *Principia Mathematica*, second edition, vol. 1. p. xxiii.

‡ See *Principia Mathematica*, vol. 1, pp. 127–160.

In the simplest case such functions reduce to

$$(x, y, \cdots, t) . \varphi (x, y, \cdots, t) \quad \text{or} \quad (\exists x, y, \cdots, t) . \varphi (x, y, \cdots, t)$$

in which only a single quantifier "some" or "every" is applied. Propositions involving a single applicative "some" or "every" will be called singly-quantified propositions, whereas those involving more than one applicative will be said to be multiply quantified.* There are two factors in the structure of a general proposition which are of especial importance, namely, the applicative which attaches to a given variable and the scope of the variable. The scope of a variable is indicative of a mode of bracketing in the proposition. In a proposition of the form "For every x some y or other is such that $\varphi (x, y)$" which would be written $(x) : (\exists y) . \varphi (x, y)$, x is said to have a wider scope than y; whereas in "For some one and the same y every x is such that $\varphi (x, y)$," which would be written $(\exists y) : (x) . \varphi (x, y)$, y is said to have a wider scope than x. Wider to narrower scope is indicated by successive parantheses, those variables appearing in the first paranthesis having the widest scope. Variables which appear in the same paranthesis have the same scope. Thus

$$(x, y, \cdots, n) : (\exists z, w, \cdots, m) \cdot \varphi (x, y, \cdots, n; z, w, \cdots, m)$$

is to be read "For every $x, y, \cdots, n$, some or other $z, w, \cdots, m$ is such that $\varphi (x, y, \cdots, n; z, w, \cdots, m)$". In this point we are deviating from customary usage in which each variable is assigned a different scope. Thus we should have

$$(x) (y) \cdots (n) (\exists z) (\exists w) \cdots (\exists m) . \varphi (x, y, \cdots, n; z, w, \cdots, m)$$

which may be read "For every x every y, etc. is such that for some z some w, etc. is such that $\varphi (x, y, \cdots, n; z, w, \cdots, m)$". These two formūlations are clearly equivalent and the latter will be said to be reducible to the former. Such a reduction will be carried out frequently in the present treatment because the classification of general propositions to be used here does not depend upon the number of variables in the propositional form but upon the number of changes in applicative which occur. A singly-quantified proposition involves only one applicative, "some" or "every", though it may involve any number of variables. A proposition which involves a change from "some" to "every" or from "every" to "some" in the quantifier, will be said to be doubly-quantified, and a proposition which involves $n-1$ changes of applicative will be

* The terminology is due to Mr. W. E. Johnson who has used it in a slightly different sense. See *Mind*, N. S., vol. 17, 1892, p. 26 ff.

called an n-tuply quantified proposition. It will be seen subsequently that propositions having the same degree of quantification have properties very much alike while those differing as to degree of quantification differ in important respects.

The force of the "scope" of a variable is precisely the same as that of the "order" of a parameter, and a comparison here is of interest because in the parameter we have the factor of "scope" dissociated from the use of the applicatives "some" and "every". Roughly, the order of a parameter in a function is the "degree of fixity" of it as a variable constituent of the function. To use the simplest illustration, $y = kx + l$, as referred to the cartesian plane, will denote systems of parallel lines if k is fixed relatively to l, while if l is fixed relatively to k the expression denotes any pencil of lines on a point on the y-axis. If neither of these conditions is involved, then the expression denotes individual lines in the plane. In the latter case k and l are parameters of the same order. The three cases may be represented by $y_0 = k_2 x_0 + l_1$, $y_0 = k_1 x_0 + l_2$, and $y_0 = k_1 x_0 + l_1$ respectively, where the highest suffix denotes the parameter or parameters of highest order and a zero suffix the parameters of lowest order, or the variables. This situation, obvious geometrically, requires analytic interpretation and the following analysis seems to be what is needed. The use of parameters involves the notion of a denotative hierarchy. This is seen by asking for a typical value of a given expression. In the first place it is clear that any value of such an expression as $y = zx$, which involves no parameters, is a definite proposition such as $6 = 2 \cdot 3$. $y = zx$ denotes $6 = 2 \cdot 3$ as a value. But $y = kx$ does not denote propositions. It denotes, for example, $y = 2x$ which in turn denotes definite propositions. Again, $y = k_1 x + l_2$ has as a typical value $y = k_1 x + 5$, while $y = k_2 x + l_1$ has as a typical value $y = 4x + l_1$ and this in turn has for value $y = 4x + 5$ for example. But $y = k_1 x + l_1$ has $y = 4x + 5$ for a value though it is not a value of $y = k_2 x + l_1$, nor is $y = 4x + l_1$ a value of it. If we use $\rightarrow$ to denote "has as a typical value", then the following represents the significance of these expressions:

$$y = zx \rightarrow 6 = 2 \cdot 3;$$
$$y = kx \rightarrow y = 2x \rightarrow 6 = 2 \cdot 3;$$
$$y = k_1 x + l_1 \rightarrow y = 3x + 5 \rightarrow 11 = 3 \cdot 2 + 5;$$
$$y = k_1 x + l_2 \rightarrow y = k_1 x + 5 \rightarrow y = 3x + 5 \rightarrow 11 = 3 \cdot 2 + 5;$$
$$y = k_2 x + l_1 \rightarrow y = 3x + l_1 \rightarrow y = 3x + 5 \rightarrow 11 = 3 \cdot 2 + 5.$$

To return to the comparison with the notion of "scope"; following the notation just used, we may have $\varphi(x_3, y_2, z_1)$ in which the suffixes denote parametric order. Values of this function are $\varphi(a, y_2, z_1)$, $\varphi(b, y_2, z_1)$, and so on, while each of these latter expressions has a separate set of values, and so on. Now when an applicative, "some" or "every", is applied to $\varphi(x_3, y_2, z_1)$ it is to be applied to the parameter of highest order x. If the applicative is universal it makes an assertion about every one of the values of $\varphi(x_3, y_2, z_1)$ and if it is particular it makes an assertion about at least one of these values. The second applicative applies to the values of these values, while the third applicative applies to the values of these values of values. Thus $(x) :. (\exists y) : (z) \cdot \varphi(x, y, z)$ asserts that for any value of $\varphi(x_3, y_2, z_1)$, say $\varphi(a, y_2, z_1)$, $(\exists y) : (z) \cdot \varphi(a, y, z)$, and this in turn asserts for each such function that there is at least one value, say $\varphi(a, b, z_1)$, such that $(z) \cdot \varphi(a, b, z)$, and this latter function entails for any one of its values, say $\varphi(a, b, c)$, that $\varphi(a, b, c)$ holds. The scope of a variable indicates the point in the hierarchy of values at which substitutions for it are to be made.

In the foregoing illustrations the scopes of any two variable constituents of a proposition are given as either wider, narrower, or the same. In many general propositions, however, there are variables which have independent scope. For example, we may have $(x) :. (\exists y) : f_1(x, y) : (z) . f_2(x, z) . f_3(y, z) : (\exists w) . f_4(x, y, w)$. In this expression z and w have independent scope. (z) applies to $f_2(x, z) . f_3(y, z)$ alone while $(\exists w)$ applies to $f_4(x, y, w)$ alone. It has been pointed out by Russell* that it is always possible to give each variable a scope covering the entire function. This may be seen as follows. There are two ways in which two functions may be connected, conjunctively or disjunctively. All other modes of connection are defined in terms of these. We have accordingly six possible cases:

(1) $(x) . \varphi x . \mathbf{V} . (y) . \psi y;$

(2) $(x) . \varphi x . \mathbf{V} . (\exists y) . \psi y;$

(3) $(\exists x) . \varphi x . \mathbf{V} . (\exists y) . \psi y;$

(4) $(x) . \varphi x : (y) . \psi y;$

(5) $(x) . \varphi x : (\exists y) . \psi y;$

(6) $(\exists x) . \varphi x : (\exists y) . \psi y.$

(1) is clearly equivalent to $(x, y) : \varphi x . \mathbf{V} . \psi y$; (2) is equivalent to $(x) :. (\exists y) : \varphi x . \mathbf{V} . \psi y$ and to $(\exists y) :. (x) : \varphi x . \mathbf{V} . \psi y$;† (3) is equivalent to

* *Principia Mathematica*, vol. 1, p. 135 ff.

† Under the premise that there is at least one element within the range of significance of the variables. If this condition is not present a more careful formulation is required. These relations are to be used in connection with a condition for the existence of at least n elements, so that existence conditions will be presupposed here.

$(\exists x, y) : \varphi x \,.\, \mathsf{V} \,.\, \psi y$; (4) is equivalent to $(x, y) \,.\, \varphi x \,.\, \psi y$; (5) is equivalent to $(x) : (\exists y) \,.\, \varphi x \,.\, \psi y$ and to $(\exists y) : (x) \,.\, \varphi x \,.\, \psi y$, and (6) is equivalent to $(\exists x, y) \,.\, \varphi x \,.\, \psi y$. Also,

$$\varphi x \,.\, \mathsf{V} \,.\, (\exists y) \,.\, \psi y : \equiv : (\exists y) \,.\, \varphi x \mathsf{V} \psi y,$$
$$\varphi x \,.\, \mathsf{V} \,.\, (y) \,.\, \psi y : \equiv : (y) \quad .\, \varphi x \mathsf{V} \psi y,$$
$$\varphi x \,.\, (\exists y) \,.\, \psi y : \equiv : (\exists y) \,.\, \varphi x \,.\, \psi y,$$
$$\varphi x \,.\, (y) \,.\, \psi y : \equiv : (y) \quad .\, \varphi x \,.\, \psi y.$$

Thus

$$(x) :. (\exists y) : f_1(x,y) : (z) \,.\, f_2(x,z) \,.\, f_3(y,z) : (\exists w) \,.\, f_4(x,y,w)$$
$$:: \equiv :: (x) :: (\exists y) :. f_1(x,y) :. (z) : (\exists w) \,.\, f_2(x,z) \,.\, f_3(y,z) \,.\, f_4(x,y,w)$$
$$:: \equiv :: (x) :: (\exists y) :. (z) : f_1(x,y) : (\exists w) \,.\, f_2(x,z) \,.\, f_3(y,z) \,.\, f_4(x,y,w)$$
$$:: \equiv :: (x) :: (\exists y) :. (z) : (\exists w) \,.\, f_1(x,y) \,.\, f_2(x,z) \,.\, f_3(y,z) \,.\, f_4(x,y,w).$$

$$(x) \,.\, \varphi x \,.\!\equiv\!.\, \sim (\exists x) \,.\, \sim \varphi x \quad \text{and} \quad (\exists x) \,.\, \varphi x \,.\!\equiv\!.\, \sim (x) \,.\, \sim \varphi x$$

Hence,

$$\sim (x) \,.\, \varphi x \,.\!\equiv\!. \quad (\exists x) \,.\, \sim \varphi x \quad \text{and} \quad \sim (\exists x) \,.\, \varphi x \,.\!\equiv\!. \quad (x) \,.\, \sim \varphi x.$$

Also,

$$\sim (\varphi x \mathsf{V} \psi y) \,.\!\equiv\!.\, \sim \varphi x \,.\, \sim \psi y \quad \text{and} \quad \sim (\varphi x \,.\, \psi y) \,.\!\equiv\!.\, \sim \varphi x \mathsf{V} \sim \psi y.$$

By the use of these equivalences the sign of negation can be removed from before any quantifier. Accordingly, any proposition can be expressed in terms of "some" and "every" alone, provided, of course, that $\sim$ be applied within the function itself. It follows that any proposition involving variables having independent scope can be expressed as a proposition all of whose variables have interdependent scopes.

To get the contradictory of a general proposition which involves a single complex quantifier change every universal variable into a particular and every particular into a universal and take the negative of the function. Thus $(x, \cdots, n) :. (\exists y, \cdots, m) : \cdots \,.\, \varphi(x, \cdots, n; y, \cdots, m)$ and $(\exists x, \cdots, n) :. (y, \cdots, m) : \cdots \,.\, \sim \varphi(x, \cdots, n; y, \cdots, m)$ are contradictories.

In formulating sets of defining properties for abstract systems in many cases we have a single class K and an n-adic relation R in terms of which all of the properties belonging to the set are formulated. Every postulate is a logical construct on K and R alone. Now the use of a class K is equivalent to the use of a predicate $\varphi \hat{x}$ and the use of an n-adic relation R is equivalent to the use of a predicate $\psi(\hat{x}, \hat{y}, \cdots, \hat{n})$. The set is then a logical function, $f(\varphi, \psi)$, of these two predicates, and if we are confined to this base no other predicates can be used except those definable in terms of φ and ψ. We are concerned in what follows with the class of all first-order functions which can be formulated in terms of a simple case of such a pair of predicates, namely, $\varphi \hat{x}$ and $\psi \hat{x}, \hat{y}$.

The simplest set of properties of any importance on the base K, R_2 is the set for a dense series without extreme elements:

(1) $(x) . \sim Rxx.$

(2) $(x, y) :. x \neq y . x, y \epsilon K : \supset : \quad Rxy . \vee . \quad Ryx.$

(3) $(x, y) :. x \neq y . x, y \epsilon K : \supset : \sim Rxy . \vee . \sim Ryx.$

(4) $(x, y, z) :. x \neq y . y \neq z . x \neq z . x, y, z \epsilon K : \supset : Rxy . Ryz . \supset . Rxz.$

(5) $(x) \quad :. (\exists y) : x \epsilon K . \supset . Rxy.$

(6) $(x) \quad :. (\exists y) : x \epsilon K . \supset . Ryx.$

(7) $(x, z) :. (\exists y) : Rxz . \supset . Rxy . Ryz.$

(8) $(\exists x) . x \epsilon K.$

These properties, with some modifications, are the ones usually assigned for this type of order. They are, however, with the exception of (1), confined to assertions relevant to elements in the class K, and it is customary to omit any mention of properties belonging to elements not in K. But it is necessary in the present case to consider such properties—otherwise some important theorems break down. The following properties are to be added to the list.

(9) $(x, y) :. x \neq y : \sim x \epsilon K . \vee . \sim y \epsilon K : \supset : \sim Rxy.$

(10) $(\exists x, y, \cdots, n) . x \neq y ., \cdots ,. x \neq n ., \cdots ,. y \neq n ., \cdots ,. \sim x \epsilon K . \sim y \epsilon K ., \cdots ,. \sim n \epsilon K.$*

If x and y do not both belong to K, then Rxy fails. Also, there are at least n elements not in K. (9) and (10) are properties which are usually tacitly understood. The ten properties may be classified as follows. (1), (2), (3), (4), (8), (9), and (10) are singly quantified functions. They involve the use of a single applicative, "some" or "every". (5), (6), and (7) are doubly quantified. They involve a single change from "every" to "some". Also, (1) and (8) are functions of one variable; (2), (3), (5), (6), and (9) are functions of two variables; (4) and (7) are functions of three variables, while (10) is a function of n variables. (1) and (7) are functions on R alone while (8) is a function on K alone. It is proposed to study the consequences of (1)–(10) in relation to the class of all first-order functions which can be formulated in terms of K and R_2.

Properties (1)–(4) have for values singly quantified propositions in which all of the variables are quantified universally. These four properties will be treated first. $R'(abc \cdots n)$ is to mean that any two terms x, y which occur in the order xy in R' are such that Rxy holds. Thus $R'(abc)$ means

* It is to be noted that existence propositions are always empirically significant and that the existence expressions which occur in logic are properties, not propositions.

$Rab \,.\, Rbc \,.\, Rac$. Also, $R''(abc \cdots lmn)$ is to mean that $R'(abc \cdots lmn)$ holds and that any two elements y, x which occur in R'' in the order from right to left are such that Ryx fails. This is equivalent to saying that $R''(abc \cdots lmn)$ means $R'(abc \cdots lmn)$ and R' fails for any other permutation of these elements. It will be shown that, as a consequence of (2)–(4), every set of n elements in K ($n > 1$ and finite) is such that for some permutation of the elements, $a, b, c, \cdots, n$, $R''(abc \cdots n)$ holds.

THEOREM I. $(x, y) :. x \neq y \,.\, x, y \epsilon K : \supset : R''xy \,.\, \mathbf{V} \,.\, R''yx$.

This follows from (2) and (3).

THEOREM II. $(x, y, z) :. x \neq y \,.\, x \neq z \,.\, y \neq z \,.\, x, y, z \epsilon K : \supset : R''xyz \,.\, \mathbf{V} \,.\, R''yzx \,.\, \mathbf{V} \,.\, R''zxy \,.\, \mathbf{V} \,.\, R''zyx \,.\, \mathbf{V} \,.\, R''xzy \,.\, \mathbf{V} \,.\, R''yzx$.

That is, for some permutation of any three elements in K, say abc, $R''abc$ holds. It is to be noted that if $R''abc$ holds the form of the elements with respect to R is completely determined for distinct dyads since R'' assigns a determinate validity value to every dyad of distinct elements. When for some permutation of a triad of elements R'' holds the elements will be said to have the form $R''(xyz)$. Similarly, when for some permutation of an n-ad of elements R'' holds the n elements will be said to have the form $R''(xyz \cdots n)$. We have to show that every three elements in K have the form $R''(xyz)$. Since $Rxy \,.\!\equiv\!.\, \sim Ryx$ for every distinct x, y in K, any three elements must be such that R holds for three of the six ordered dyads of distinct elements which can be formed of them. There are two possibilities: we may have the form $Rxy \,.\, Ryz \,.\, Rxz$ for some permutation of the three elements, or the form $Rxy \,.\, Ryz \,.\, Rzx$. The latter is contrary to postulate (4). Hence $R'xyz$, which implies $R''xyz$, by (3).

THEOREM III. *Any n distinct K-elements have the form $R''(abc \cdots n)$.*

If this property belongs to every subclass of m elemets ($m > 2$), then it belongs to every subclass of $m + 1$ elements. For consider any $m + 1$ elements. Some one of these elements, a, is such that Rxa holds for every other element x. Omitting a from the subclass, we have $R''(bc \cdots m)$ for some permutation of the remaining elements. But Rxa holds and Rax fails. Hence $R''(bc \cdots ma)$.

In view of (1) $\sim Rxx$ for every x in K. Accordingly, when $R''(abc \cdots n)$ holds we have also $\sim Raa \,.\, \sim Rbb \,.\, \sim Rcc \,.,\, \cdots, \,.\, \sim Rnn$, and, by (9), $a \epsilon K \,.\, b \epsilon K \,.\, c \epsilon K \,.,\, \cdots, \,.\, n \epsilon K$. We may denote $R''(abc \cdots n) \,.\, \sim Raa \,.\, \sim Rbb \,.\, \sim Rcc \,.,\, \cdots, \,.\, \sim Rnn \,.\, a \epsilon K \,.\, b \epsilon K \,.\, c \epsilon K \,.,\, \cdots,\, n \epsilon K$ by $R^0(abc \cdots n)$. Then for some permutation of every n elemets in K, R^0 holds. We may say that such a set of elements has the form $R^0(xyz \cdots n)$. Note that R^0 is significant for a single element. $R^0(a)$ means $\sim Raa \,.\, a \epsilon K$. Now every m elements not in K are such that R fails for every ordered dyad formed of them. Let $S(de \cdots m)$ mean $\sim Rde\,.\sim Red\,..\,\cdots\,.\sim Rdm\,.\sim Rmd\,.,\cdots,\,.\sim Rem\,.\sim Rme\,.,\cdots,\,.\sim Rdd\,.\sim Ree\,.,\cdots,\,.\sim Rmm\,.\sim d\epsilon K\,.\sim e\epsilon K\,.,\cdots,\,.\sim m\epsilon K$, and $d, e, \cdots m$ are all distinct. Then for every m elements not in K, S holds, and we may say that such a set of elements has the form $S(xyz \cdots m)$. S is of course significant for a single element. If $a, b, \cdots, n \epsilon K$ and $d, e, \cdots, m \epsilon$ not-K, then R fails for every ordered dyad which can be formed so as to consist of one element from each of these sets. Thus $\sim Rad\,.\sim Rda\,.\sim Rae$, etc. We may denote the failure of R for these dyads by writing $T(ab \cdots n;\ de \cdots m)$. As a consequence of (1)–(4) and (9) any $n+m$ elements are of the form $R^0(xy \cdots n)\,.\,S(zw \cdots m)\,.\,T(xy \cdots n;\ zw \cdots m)$. When m is zero this becomes $R^0(xy \cdots n)$ and when n is zero, $S(zw \cdots m)$.

We now define what will be called the expanded form of a function $f(x, y, \cdots, n)$. Let $\psi(p, q, r, \cdots, w)$ be an elementary function of the propositions $p, q, r, \cdots, w$. Then ψ is a construct on $p, q, r, \cdots, w$ built up in terms of "and", "or", and "$\sim$". By a well known procedure in the logic of propositions, which will be given presently, ψ can be expressed as a disjunctive function of conjunctive functions of the constituents $p, q, r, \cdots, w$. For example, let ψ be $p\,.\supset.\,q \vee r$; $p\,.\supset.\,q\vee r :\equiv: \sim p \vee q \vee r :\equiv: \sim p\,.\,q\,.\,r\,.\vee.\sim p\,.\sim q\,.\,r\,.\vee.\sim p\,.\,q\,.\sim r\,.\vee.\sim p\,.\sim q\,.\sim r\,.\vee.\,p\,.\,q\,.\,r\,.\vee.\,p\,.\sim q\,.\,r\,.\vee.\,p\,.\,q\,.\sim r$. This is a disjunctive function of conjunctive functions of p, q, r. It excludes one possible alternative, *viz*, $p\,.\sim q\,.\sim r$. Note that every conjunctive function involves all of the propositional constituents p, q, r.

In an elementary function $f(x, y, \cdots, n)$ on K, R_2 the only elementary propositional constituents which occur are constructs of $x, y, \cdots, n$ on K, R, and "$=$" — such as Rxy or $\sim Rxy$, and $x \epsilon K$ or $\sim x \epsilon K$, and $x = y$ or $\sim x = y$. These constituents of $f(x, y, \cdots, n)$ correspond to $p, q, r, \cdots, w, \sim p, \sim q, \sim r, \cdots, \sim w$. It may happen that $f(x, y, \cdots, n)$ is a function on R but not explicitly on K. For example, $(x \neq y)\,.\supset.\,Rxy \vee Ryx$. This function is a case of the function $p\,.\supset.\,q\vee r$ in the example just cited. Let $(x \neq y) = p$ and $(Rxy) = q$ and $(Ryx) = r$. Then $x \neq y\,.\supset.\,Rxy \vee Ryx :\equiv: x = y\,.\,Rxy\,.\,Ryx\,.\vee.\,x = y\,.\sim Rxy\,.\,Ryx\,.\vee.\,x = y\,.\,Rxy\,.\sim Ryx\,.\vee.\,x = y\,.\sim Rxy\,.\sim Ryx\,.\vee.\,x \neq y\,.\sim Rxy\,.\,Ryx\,.\vee.\,x \neq y\,.\,Rxy\,.\sim Ryx\,.\vee.\,x \neq y\,.\,Rxy\,.\,Ryx$. In this case some of the alternatives, for example $x = y\,.\sim Rxy\,.\,Ryx$, are impossible and may be dropped from the disjunctive function. Note that in each alternative of the function any construct of x, y on R and "$=$" occurs or else its contradictory occurs. The function is in expanded form with respect to R and "$=$".

A function on R in expanded form involves the condition $x = y$ or $x \neq y$ for any two variables in any constituent conjunctive function. Any

R-function can be put in expanded form with respect to R and "$=$" by the following procedure, which also, of course, establishes the existence of an equivalent function in expanded form. Let $f(x, y, \cdots, n)$ be any function which is formulated on R alone. In general f is a function which involves "and", "or", negation, and identity ("$=$"). Some of these may, of course, be lacking. The constituent constructs connected by "and" and "or" and affected by $\sim$ will be such as Rxy or $\sim Rxy$ and $x = y$ or $\sim x = y \; (x \neq y)$. Since $\sim(p \vee q) \,.\equiv.\, \sim p \,.\, \sim q$ and $\sim(p \,.\, q) \,.\equiv.\, \sim p \vee \sim q$, the sign of negation can be removed from before any conjunctive or disjunctive constituent function, so that $\sim$ will appear before constituents of the form Rxy and $x = y$ only. To put $f(x, y, \cdots, n)$ in expanded form first remove $\sim$ from before every conjunctive or disjunctive constituent function in f. Now it may happen that some disjunctive function $p \vee q$ is subordinate to a conjunctive function, as in $p \vee q \,.\, s$. But $p \vee q \,.\, s :\equiv: p \,.\, s \vee . \, q \,.\, s$. Hence the function $f(x, y, \cdots, n)$ can be so expressed that no disjunctive function is subordinate to a conjunctive function. This is the second step. We have then a set of conjunctive functions $C_1, C_2, \cdots, C_s$ and f is a disjunctive function $C_1 \,.\vee.\, C_2 \,.\vee, \ldots, \vee.\, C_s$ of these conjunctive functions. For example, we may have $\sim(\sim p \,.\, q : \sim \overline{q} \cdot s) \,.\vee.\, s \,.\, t \,.\, u \,.\vee.\, p \,.\, q \,.\, r : v \,.\, w$, which is equivalent to $p \,.\, q \,.\vee.\, \overline{q} \,.\, s \,.\vee.\, s \,.\, t \,.\, u \,.\vee.\, p \,.\, q \,.\, r : v \,.\, w$, which is equivalent to $p \,.\, q \,.\, v \,.\, w \,.\vee.\, \overline{q} \,.\, s \,.\, v \,.\, w \,.\vee.\, s \,.\, t \,.\, u \,.\, v \,.\, w \,.\vee.\, p \,.\, q \,.\, r \,.\, v \,.\, w$.

Let $f(x, y, \cdots, n)$ be expressed as a disjunctive function of conjunctive functions of its propositional constituents, Rxy, $x = y$, and the like. Now it happens in general that for some of the conjunctive functions in such a set of disjunctions some propositional constituent is such that neither it nor its contradictory appears in these conjunctive functions. In the example just given there are three conjunctive functions in which neither r nor $\overline{r}$ appears. But $p \,.\vee.\, q :\equiv: (p \vee q)(r \vee \overline{r}) :\equiv: p \,.\, (r \vee \overline{r}) \,.\vee.\, q \,.\, (r \vee \overline{r}) :\equiv: p \,.\, r \,.\vee.\, p \,.\, \overline{r} \,.\vee.\, q \,.\, r \,.\vee.\, q \,.\, \overline{r}$, so that the function may be multiplied by $(r \vee \overline{r})$. We get $p \,.\, q \,.\, v \,.\, w \,.\, r \,.\vee.\, p \,.\, q \,.\, v \,.\, w \,.\, \overline{r} \,.\vee.\, \overline{q} \,.\, s \,.\, v \,.\, w \,.\, r \,.\vee.\, \overline{q} \,.\, s \,.\, v \,.\, w \,.\, \overline{r} \,.\vee.\, s \,.\, t \,.\, u \,.\, v \,.\, w \,.\, r \,.\vee.\, s \,.\, t \,.\, u \,.\, v \,.\, w \,.\, \overline{r} \,.\vee.\, p \,.\, q \,.\, r \,.\, v \,.\, w \,.\, r \,.\vee.\, p \,.\, q \,.\, r \,.\, v \,.\, \overline{r}$. Those conjunctive functions in which both r and $\overline{r}$ occur are impossible and may be dropped, so that we have $p \,.\, q \,.\, v \,.\, w \,.\, r \,.\vee.\, p \,.\, q \,.\, v \,.\, w \,.\, \overline{r} \,.\vee.\, \overline{q} \,.\, s \,.\, v \,.\, w \,.\, r \,.\vee.\, \overline{q} \,.\, s \,.\, v \,.\, w \,.\, \overline{r} \,.\vee.\, s \,.\, t \,.\, u \,.\, v \,.\, w \,.\, r \,.\vee.\, s \,.\, t \,.\, u \,.\, v \,.\, w \,.\, \overline{r}$, and in this function either r or $\overline{r}$ appears in every conjunctive constituent. Accordingly, in $f(x, y, \cdots, n)$, if some propositional constituent p does not appear in every conjunctive function, introduce p by multiplying by $(p \vee \overline{p})$.

With $f(x, y, \cdots, n)$ in this form it may still happen that some function of $x, y, \cdots, n$ on R or "$=$" does not occur in f at all. Consider, for example, the function $Rxy \,.\vee.\, Ryx$. Here neither $x = y$ nor $x \neq y$ occurs.

$Rxy . \mathbf{V} . Ryx :\equiv: Rxy . \sim Ryx . \mathbf{V} . Rxy . Ryx . \mathbf{V} . \sim Rxy . Ryx$. In this function we may introduce $x = y$ and $x \neq y$ by multiplying by ($x = y$ $\mathbf{V}\, x \neq y$). This gives $Rxy . \sim Ryx . x = y . \mathbf{V} . Rxy . \sim Ryx . x \neq y . \mathbf{V} . Rxy . Ryx . x = y . \mathbf{V} . Rxy . Ryx . x \neq y . \mathbf{V} . \sim Rxy . Ryx . x = y . \mathbf{V} . \sim Rxy . Ryx . x \neq y$. In this form the function is expressed in such a way that in each conjunctive function any propositional construct of x, y on R, "$=$" occurs or else its contradictory occurs.*

It has been shown that every general proposition can be so expressed that all of the variables have interdependent scopes. This means that any general proposition can be expressed by the use of a single complex quantifier applied to a function $f(x, \cdots, r)$. Thus every general proposition has either the form

$$(x, \cdots, l)\,(\exists y, \cdots, m)\,(z, \cdots, n), . f(x, \cdots, l;\ y, \cdots, m;\ z, \cdots, n;\ \cdots)$$

or the form

$$(\exists x, \cdots, l)\,(y, \cdots, m)\,(\exists z, \cdots, n), . f(x, \cdots, l;\ y, \cdots, m;\ z, \cdots, n;\ \cdots).$$

In general $f(x, \cdots r)$ in these functions is, of course, not a construct on R alone, but a construct on R and K. Such a construct will involve propositional constituents Rxy or $\sim Rxy$ and $x = y$ or $x \neq y$ and in addition $x \epsilon K$ or $\sim x \epsilon K$. Any function can be put in expanded form with respect to K, R, and "$=$" by the procedure which has been used to expand functions in terms of R and "$=$". For the argument there does not depend on what the elementary constituents are. As an example, consider $Rxy . \mathbf{V} . Ryx$ which has been expanded into $Rxy . \sim Ryx . x = y$

* And not both. If a constituent and its contradictory both occur in the same alternative, then clearly that alternative may be dropped without altering the truth-value of the function; and we shall consider a function in expanded form to be one from which all such alternatives have been discarded. This gives rise to a limiting case which we take account of here and dismiss from further consideration. It may happen that when a function is put in expanded form every alternative vanishes. (To express such a function positively we should have to write a function involving conjunctively the denial of every possible alternative.) Now any function from which every alternative vanishes and which demands existence (*e.g.*, $(\exists x) . x \epsilon K . \sim x \epsilon K$) is selfcontradictory. Its truth-value is determined; and since we are only concerned with showing determination of truth-value, these functions may be dismissed from further consideration. Any function from which every alternative vanishes and which does not demand existence (*e.g.*, $(x) . x \epsilon K . \sim x \epsilon K$) fails if there is at least one element within the range of significance of its variables. But (1)–(10) imply that there is at least one element in the system and therefore that every such function fails. We may, then, confine attention in what follows to functions which when put in expanded form involve at least one alternative.

Another limiting case is that in which a function involves every possible alternative. All functions of this kind which do not demand existence hold. Any function of this kind which does demand existence holds if there is at least one element within the range of the variables. Hence all such functions hold in view of (1)–(10).

$. \vee . Rxy . \sim Ryx . x \neq y . \vee . Rxy . Ryx . x = y . \vee . Rxy . Ryx . x \neq y . \vee . \sim Rxy . Ryx . x = y . \vee . \sim Rxy . Ryx . x \neq y$. This is in terms of R and "$=$", and it is sufficient to multiply the function by $(x \epsilon K \vee \sim x \epsilon K)$ and by $(y \epsilon K \vee \sim y \epsilon K)$. In a function so expanded any conjunctive constituent involves, for any two variables x, y, Rxy or it involves $\sim Rxy$, and it involves Ryx or else $\sim Ryx$, and $x \epsilon K$ or $\sim x \epsilon K$, and $y \epsilon K$ or $\sim y \epsilon K$, and $x = y$ or $x \neq y$. Postulate (1), $(x) . \sim Rxx$, is a function of one variable on R alone. $\sim Rxx$ is equivalent to $x \epsilon K . \sim Rxx . \vee . \sim x \epsilon K . \sim Rxx$, so that the postulate becomes $(x) : x \epsilon K . \sim Rxx . \vee . \sim x \epsilon K . \sim Rxx$. It may, of course, be expanded as a function of two variables, since $(x) . \sim Rxx : \equiv : (x, y) : x = y . \supset . \sim Rxy$, or as a function of three variables, since $(x) . \sim Rxx : \equiv : (x, y, z) : x = y . y = z . x = z . \supset . \sim Rxy . \sim Ryz . \sim Rxz$, or as a function of n variables.

It is easily seen that any singly quantified function in two variables has its truth-value determined by (1)–(10). These functions are of two forms, $(x, y) . f(x, y)$ and $(\exists x, y) . f(x, y)$. Put any function of the form $(x, y) . f(x, y)$ in expanded form with respect to K, R, and "$=$". If this function is to hold $f(x, y)$ must involve the alternatives $R^0(xy)$, $R^0(yx)$, $R^0(x)$, $R^0(x) . S(y) . T(x; y)$, $R^0(y) . S(x) . T(y; x)$, $S(xy)$, and $S(x)$. The condition is necessary and sufficient. With regard to functions of the form $(\exists x, y) . f(x, y)$, it is necessary and sufficient that a function of this form involve at least one of these alternatives. Functions of one variable occur as special cases of these two types of function, so that the argument includes them.

A similar analysis is applicable to doubly quantified functions in two variables. Such functions are of the form $(x) : (\exists y) . f(x, y)$ or the form $(\exists y) : (x) . f(x, y)$.

Consider the first case first. Let $f(x, y)$ be put in expanded form. Such a function might be $(x) : (\exists y) . R^0(xy) \vee S(xy)$, for example. Consider the following alternatives which may be involved in $f(x, y)$: (1) $R^0(xy)$, $R^0(yx)$, $R^0(x)$, $R^0(x) . S(y) . T(x; y)$, and (2) $S(x) . R^0(y) . T(y; x)$, $S(xy)$, $S(x)$. Alternatives (1) have to do with values of x which belong to K and alternatives (2) have to do with values of x not in K. There are at least n elements in K and at least n elements not in K. From postulates (1), (5), (6), and (9) it follows that any function $(x) : (\exists y) . f(x, y)$ which involves at least one of the alternatives (1) and at least one of the alternatives (2) is true. Conversely, any function which does not involve at least one alternative from (1) and at least one from (2) is false. The condition is equivalent to the function.

In the case of functions $(\exists y) : (x) . f(x, y)$, for some one and the same y every x is such that $f(x, y)$, let $f(x, y)$ be expressed in expanded form.

Consider the possible alternatives (1) $R^0(xy)$, $R^0(yx)$, $R^0(y)$, $S(x).R^0(y).T(y;x)$, and (2) $R^0(x).S(y).T(x;y)$, $S(xy)$, $S(y)$. From postulates (1), (2), (3), and (9), if either all of the alternatives in (1) or all of the alternatives in (2) are involved in $f(x,y)$, then $(\exists y):(x).f(x,y)$ is true and if this is not the case the function is false. Since the truth-value of every singly quantified function in two variables is determined by (1)–(10), it follows that *every function in two variables is determined by the set.*

Let $F(x,\cdots,n)$ be any multiply quantified propositional function so expressed as to involve a single complex quantifier affecting the function $f(x,\cdots,n)$. Let f be put in expanded form. Then $f(x,\cdots,n)$ is a disjunctive function $f_1(x,\cdots,n).\vee.f_2(x,\cdots,n).\vee.\cdots$ etc, where $f_r(x,\cdots,n)$ is a conjunctive function of $x,\cdots,n$ on K, R, "$=$" and such that it involves the assertion or denial of every propositional construct of $x,\cdots,n$ on K, R, "$=$". Thus $f_r(x,\cdots,n)$ involves Rxn or it involves $\sim Rxn$; it involves $x\epsilon K$ or else $\sim x\epsilon K$; it involves $x=n$ or else $x\neq n$. Denote by $f'(x,\cdots,n)$ a function in expanded form which involves every possible alternative which can be formulated of the variables $x,\cdots,n$. As pointed out above, $F'(x,\cdots,n)$ is true whatever complex quantifier F' may involve. In general f differs from f' in that certain alternatives belonging to f' are excluded from f.

Suppose that $F(x.\cdots,n)$ is a function constructed by excluding certain alternatives from $F'(x,\cdots,n)$ in the following way. If x is any variable in F' to which the applicative "some" attaches, then every alternative in f' which involves the constituent $\sim x\epsilon K$ is excluded in forming f. Also, all other alternatives in f' for which at least one variable z is such that $\sim z\epsilon K$ is asserted are to be retained in f. Any remaining alternatives, i. e., those all of whose elements are assigned to K, may either be retained or excluded. Such propositions $F(x,\cdots,n)$ place no restriction on elements not in K and, in this sense, may be said to be about K-elements only.

There are two theorems which are important in connection with propositions involving a single or higher quantification.

THEOREM IV. $(a,b,c,\cdots,n)::R^0(abc\cdots n):.\supset:.(m):.m\epsilon K\cdot m\neq a,b,c,\cdots,n:\supset:R^0(mabc\cdots n).\vee.R^0(ambc\cdots n).\vee.R^0(abmc\cdots n).\vee.,\cdots.\vee.R^0(abc\cdots mn).\vee.R^0(abc\cdots nm)$.

If $R^0(abc\cdots n)$ holds, then every m in K other than $a,b,c,\cdots n$ is such that some permutation of it with the elements $a,b,c,\cdots n$ is such that R^0 holds. This follows immediately from Theorem III.

THEOREM V. $(a,b,c\cdots,n):.R^0(abc\cdots n):\supset:(\exists x).R^0(xabc\cdots n):(\exists y).R^0(aybc\cdots n):(\exists z).R^0(abzc\cdots n):,\cdots,:(\exists w).R^0(abc\cdots wn):(\exists t).R^0(abc\cdots nt)$.

If $R^0(abc \cdots n)$ holds, then there is at least one element x such that $R^0(xabc \cdots n)$ and at least one element y such that $R^0(aybc \cdots n)$, and so on. The theorem follows from (1), (5), (6), and (7).

We may determine in a particular case of doubly a quantified function, say $(x, y) :. (\exists z, w) : x, y \epsilon K \cdot x \neq y . \supset . R^0(xyzw) \vee R^0(yxzw)$, whether its truth-value is determined by (1)–(10). When put in expanded form this function may be written $(x, y) : (\exists z, w) . R^0(xyzw) \vee R^0(yxzw) \vee P(x, y, z, w)$, where $P(x, y, z, w)$ is a set of alternatives involving (a) all possible alternatives in which $z \epsilon K$ and $w \epsilon K$ occur and in which $\sim x \epsilon K$ or $\sim y \epsilon K$ occurs, and (b) all possible alternatives in which x, y, z, w are all assigned to K and in which $x = y$. No limitation is placed on these cases by the function. [1]: $(x, y) : (\exists z, w) . R^0(xyzw) \vee R^0(yxzw) \vee P(x, y, z, w)$ implies [2]: $(x, y) : (\exists z) . R^0(xyz) \vee R^0(yxz) \vee P(x, y, z)$. Here we simply drop the variable of narrowest scope w and consider those alternatives in the function which involve permutations of the remaining variables. By Theorem V, $R^0(xyz) . \supset . (\exists w) . R^0(xyzw)$ and $R^0(yxz) . \supset . (\exists w) . R^0(yxzw)$ and $P(x, y, z)$, of course, implies $(\exists w) . P(x, y, z, w)$. Hence [2] implies [1], so that [2] is equivalent to [1]. We have then a function in three variables equivalent to the original function in four variables. [2] implies [3]: $(x, y) . R^0(xy) \vee R^0(yx) \vee P(x, y)$, and [3] implies [2], by Theorem V. Hence [1] is equivalent to [3], and we have a function in two variables equivalent to the original function in four variables. But [3] is a singly quantified function and its truth-value is determined by (1)–(10). The truth-value of the original function is therefore determined.

The general procedure of the proofs which follow may be outlined: Every multiply quantified function in n variables is shown to be equivalent, in view of (1)–(10), to some function in two variables. This equivalence is, of course, material and not strict since it depends on (1)–(10). The function in two variables to which a function in n variables is shown to be materially equivalent is therefore in no sense another form of the same function. The functions have same truth-value solely in view of (1)–(10).

We consider any multiply quantified propositional function $F(x, \cdots, l, n)$ in expanded form, and the elementary function $f(x, \cdots, l, n)$ in F. As we have seen, $f(x, \cdots, l, n) = f_1(x, \cdots, l, n) \vee f_2(x, \cdots, l, n) \vee, \cdots, \vee f_t(x, \cdots, l, n)$, where each of these alternatives is a conjunctive function on K, R, and "$=$". Any alternative in f which does not have one of the forms $R^0(a \cdots j)$, $S(a \cdots j)$, or $R^0(a \cdots g) . S(h \cdots j) . T(a \cdots g; h \cdots j)$ cannot be satisfied by any set of n elements. Every alternative of this kind is to be discarded. Then, by dropping the variable of narrowest scope in $F(x, \cdots, l, n)$, say n, from $f(x, \cdots, l, n)$ we obtain a function $f'(x, \cdots, l), = f_1'(x, \cdots, l) \vee f_2'(x, \cdots, l) \vee, \cdots, \vee f_p'(x, \cdots, l)$, which is, of course, implied by $f(x, \cdots, l, n)$.

Let $f_r(x, \cdots, l, n)$ be any alternative in $f(x, \cdots, l, n)$ and let $f_r'(x, \cdots, l)$ be any alternative in $f'(x, \cdots, l)$. Every alternative f_r' is involved in at least one alternative f_r in $f(x, \cdots, l, n)$, and in general f_r' will occur in more than one alternative f_r. f_r' is implied by any alternative f_r in which it occurs since f_r involves the assertion of f_r'. But f_r' may be true and yet every alternative f_r in which it occurs be false. Now the variable n which is dropped in forming f' may be affected by the applicative "some" or it may be affected by the applicative "every". Suppose first that "some" attaches to n. Then it will be shown, by the aid of Theorem V and some supplementary theorems to be given presently, that if f_r gives rise to f_r' when n is dropped from f_r, then $f_r'(x, \cdots, l) . \supset . (\exists n) . f_r(x, \cdots, l, n)$. It follows that $f'(x, \cdots, l) \supset . (\exists n) . f(x, \cdots, l, n)$ and therefore that $F'(x, \cdots, l) . \equiv . F(x, \cdots, l, n)$.

Suppose that "every" attaches to n. Then, by the aid of Theorems IV and V together with certain supplementary theorems, it is always possible to show that the disjunction $f_{r_1} \vee f_{r_2} \vee, \cdots, \vee f_{r_m}$, formed of all those alternatives in which f_r' is involved, is such that $f_r' . \supset . (n) . f_{r_1} \vee f_{r_2} \vee, \cdots, \vee f_{r_m}$, or such that $f_r' . \supset . \sim (n) . f_{r_1} \vee f_{r_2} \vee, \cdots, \vee f_{r_m}$; $(n) . f_{r_1} \vee f_{r_2} \vee, \cdots, \vee f_{r_m}$ is never independent of f_r'. But if $f_r' . \supset . \sim (n) . f_{r_1} \vee f_{r_2} \vee, \cdots, \vee f_{r_m}$, then $(n) . f_{r_1} \vee f_{r_2} \vee, \cdots, \vee f_{r_m} . \supset . \sim f_r'$. But it is also true that $(n) . f_{r_1} \vee f_{r_2} \vee, \cdots, \vee f_{r_m} . \supset . f_r'$. So that when $f_r' \supset . \sim (n) . f_{r_1} \vee f_{r_2} \vee, \cdots, \vee f_{r_m}$, $(n) . f_{r_1} \vee f_{r_2} \vee, \cdots, \vee f_{r_m}$ is impossible. These alternatives may therefore be dropped from $f(x, \cdots, l, n)$ without altering the truth-value of $F(x, \cdots, l, n)$. When we discard those alternatives shown to be impossible there results an elementary function $f_0(x, \cdots, l, n)$ such that $(n) . f_0(x, \cdots, l, n) . \equiv . (n) . f(x, \cdots, l, n)$ and thus a function $F_0(x, \cdots, l, n)$ such that $F_0(x, \cdots, l, n) . \equiv . F(x, \cdots, l, n)$. We may now drop the variable of narrowest scope n from $f_0(x, \cdots, l, n)$ and obtain an elementary function $f''(x, \cdots, l)$. $(n) . f_0(x, \cdots, l, n)$ implies $f''(x, \cdots, l)$. But also $f''(x, \cdots, l)$ implies $(n) . f_0(x, \cdots, l, n)$, so that $F''(x, \cdots, l) . \equiv . F_0(x, \cdots, l, n) . \equiv . F(x, \cdots, l, n)$, and we have a function in $n-1$ variables with the same truth-value as $F(x, \cdots, l, n)$.

The following theorems are supplementary to IV and V.

THEOREM VI. $S(y \cdots n) . \supset . (\exists a) . S(y \cdots n\, a)$.

THEOREM VII. $S(y \cdots n) : \supset : (a) : \sim a \epsilon K . a \neq y, \cdots, n . \supset . S(y \cdots n\, a)$.

THEOREM VIII. $\Phi(x, \cdots, z, \cdots, n) . \supset . (\exists w) . w = z . \Phi(x, \cdots, w, \cdots, n)$.

THEOREM IX. $\Phi(x, \cdots, z, \cdots, n) . \supset . (w) . w = z \supset \Phi(x, \cdots, w, \cdots, n)$.

In particular, $R_0(x \cdots n) . \supset . (m) . m = n \supset R_0(x \cdots m)$, and $S(y \cdots n) . \supset . (m) . m = n \supset S(x \cdots m)$.

Consider any singly quantified function of the form $(\exists x, \cdots, l, n) . f(x, \cdots, l, n)$, Let f be put in expanded form. The alternatives in f fall into three classes: those in which every element is assigned to the class K, those in which every element is assigned to the class not-K, and those in which at least one element is assigned to K and at least one element is assigned to not-K. Each of these classes may again be subdivided into those alternatives in which all elements are asserted to be distinct and those which involve an identity condition for at least one pair of elements.

Any alternative in f in which every element is assigned to K and which does not have the form $R^0(x \cdots n)$, where $x, \cdots, n$ are the distinct elements in the alternative, cannot be satisfied by any set of n elements, since every n elements in K must have this form. Note that this applies to alternatives involving identical elements, as well as to those in which all elements are distinct, since identity conditions simply reduce the number of distinct elements. Any such alternative may be dropped from f since the function so obtained will be equivalent to the original one. Likewise, any alternative in which every element is assigned to not-K and which does not have the form $S(y \cdots n)$, where $y, \cdots, n$ are the distinct elements involved, will be false for every n elements and may be dropped from f. Any alternative involving $x, \cdots, m$ assigned to K, where $x, \cdots, m$ are distinct, and $y, \cdots, n$ assigned to not-K, where $y, \cdots, n$ are distinct, must have the form $R^0(x \cdots m) . S(y \cdots n) . T(x \cdots m; y \cdots n)$. Alternatives not having this form are to be discarded.

$(\exists x, \cdots, l, n) . f(x, \cdots, l, n) . \equiv . (\exists x, \cdots, l) \; (\exists n) . f(x, \cdots, l, n)$. Form the function $f'(x, \cdots, l)$ from f by dropping the variable of narrowest scope n and considering those conjunctive functions of $x, \cdots, l$ which remain. Consider the function f', and first any alternative in f' in which $x, \cdots, l$ are all assigned to K. Let $a, \cdots, g$ denote $x, \cdots, l$ in any such alternative such that $R^0(a \cdots g)$ holds. Consider those alternatives in f in which $R^0(a \cdots g)$ occurs. Let $P_1, P_2, \cdots, P_S$ be these alternatives. Each of the alternatives is a function of $a, \cdots, g, n$ and asserts $R^0(a \cdots g)$. There are three classes of alternatives in the set $P_1, P_2, \cdots, P_S$: those alternatives in which n is assigned to K and is not identified with any element $a, \cdots, g$, those alternatives in which n is identified with one of the elements $a, \cdots, g$, and those alternatives in which n is assigned to not-K.

Take the first case first, n is assigned to K and is not identical with any one of the set $a, \cdots, g$. Let P_x be any such alternative. Then $R^0(a \cdots g)$ implies that for some n in K, P_x holds, by Theorem V.

Suppose that n is identified with one of the set $a, \cdots, g$. If there is any such alternative P_y in f, P_y must assert simply $R^0(a \cdots g)$, by Theorem VIII. Any other possibility has already been dropped from f.

If P_z be any alternative in which n is assigned to not-K, then P_z must be $R^0(a \cdots g) . S(n) . T(a \cdots g; n)$. Alternatives of the kind not having this structure have already been dropped from f.

We may next take account of those alternatives in f' in which every element $x, \cdots, l$ is assigned to not-K Let P_x be any alternative in f in which $x, \cdots l$ are assigned to not-K and n is assigned to K. Then P_x must be $R^0(n) . S(x \cdots l) . T(n; x, \cdots, l)$. Alternatives in which this is not the case have already been excluded. When n is assigned to not-K and is distinct from each of the elements $x, \cdots, l$, we must have $S(x \ldots l\, n)$. When n is identical with some one of the elements $x, \cdots, l$, we must have simply $S(x \cdots l)$ in f.

Finally, there are those alternatives in f' in which at least one element is assigned to K and at least one element is assigned to not-K. Let $a, \cdots, g, h, \cdots, j$ be distinct elements in any such alternative, and such that $a, \cdots, g$ are assigned to K and $h, \cdots, j$ to not-K and $R^0(a \cdots g)$ holds. Then we have $R^0(a \cdots g) . S(h \cdots j) . T(a \cdots g; h \cdots j)$ since other possibilities have been excluded. Here there are three possibilities. (1) n is identified with one of the set $a, \cdots, g, h, \cdots, j$. Any such alternative in f must be identical with $R(a \cdots g) . S(h \cdots j)\, T(a \cdots g;\, h \ldots j)$. (2) n is assigned to K and is distinct from $a, \cdots, g$. In this case, for any P_x in f, Theorem V implies P_x. (3) n is assigned to not-K and is distinct from $h, \cdots, j$. In this case any alternative P_y in f must have the form $R^0(a \cdots g)$. $S(h \cdots j n) . T(a \cdots g;\, h \cdots j n)$, and this will in fact be the case.

It follows that, after eliminating from $f(x, \cdots, l, n)$ those alternatives the form of which it is impossible that any n elements should have, every alternative in f is entailed, for some n, by the corresponding alternative in f', and since f implies f', we have $(\exists x, \cdots, l, n) . f(x, \cdots, l, n) .\equiv. (\exists x, \cdots, l) . f'(x, \cdots. l)$. Here we have a function in $n-1$ variables equivalent to the original function in n variables. In precisely the same way we are able to find a function in $n-2$ variables which is equivalent to the function in $n-1$ variables and consequently to the function in n variables. This process may be continued until a function in two variables (whose truth-value is determined) is obtained. It follows that every function of the form $(\exists x, \cdots. l) . f(x, \cdots, l)$ has its truth-value determined by (1)-(10).

Let $F(x, \cdots, n)$ be any function of the form $(x, \cdots, l, n) . f(x, \cdots, l, n)$. It will be shown that $F(x, \cdots, n)$ follows from (1)-(10) or else $\sim F(x, \cdots, n)$ follows. As before, put f in expanded form and discard all those alternatives which do not have one of the forms $R^0(a \cdots g)$, $S(h \cdots j)$, or $R^0(a \cdots g) \cdot S(h \cdots j) \cdot T(a \cdots g;\, h \cdots j)$. Then form the function f' by dropping from f the variable of narrowest scope n.

Let $a, \cdots, g$ denote the elements in any alternative in f' in which all elements are assigned to K and such that $R^0(a \cdots g)$ holds. Consider the set of alternatives $P_1 . P_2, \cdots. P_s$ in f each of which involves the assertion $R^0(a \cdots g)$. Let $P_a, \cdots. P_d$ be those alternatives among $P_1, P_2. \cdots. P_s$

in which n is assigned to K and is distinct from every element $a, \cdots, g$. Then $R^0(a \cdots g)$ may imply, by Theorem IV, that $P_a \vee, \cdots, \vee P_d$ holds. But if this is not the case, then Theorem V implies that $P_a \vee, \cdots, \vee P_d$ is false for some value of n. But if $P_a \vee, \cdots, \vee P_d$ is false, then all the alternatives $P_1, P_2, \cdots, P_s$ may be dropped from f. For the function $F(x, \cdots, n)$ demands $P_1 \vee P_2 \vee, \cdots, \vee P_s$ for every value of n when $R^0(a \cdots g)$ is satisfied. But $P_a \vee, \cdots, \vee P_d$ fails for some value of n in K and different from $a, \cdots, g$. And none of the other alternatives can be satisfied by this value of n, so that $P_1 \vee P_2 \vee, \cdots, \vee P_s$ fails for this value. Hence $P_1 \vee P_2 \vee, \cdots, \vee P_s$ cannot be true for every value of n when $R^0(a \cdots g)$ holds, and it, of course, cannot be true for at least one value when $R^0(a \cdots g)$ fails. We omit for the present consideration of the case in which Theorem IV implies $P_a \vee, \cdots, \vee P_d$.

When n is identified with one of the elements $a, \cdots, g$ we must have $R^0(a \cdots g)$ as an alternative in f. If there is no such alternative, $P_1 \vee P_2 \vee, \cdots, \vee P_s$ cannot be true for every value of n, as demanded by $F(x, \cdots, n)$, and may accordingly be discarded.

When n is assigned to not-K, we must have $R^0(a \cdots g) . S(n) . T(a \cdots g; n)$ among $P_1, P_2, \cdots, P_s$. If no such alternative occurs, then $P_1 \vee P_2 \vee, \cdots, \vee P_s$ cannot hold for every value of n.

On the other hand, if, by Theorem IV, $R^0(a \cdots g)$ entails $P_a \vee, \cdots, \vee P_d$, and if there is an alternative among $P_1, P_2, \cdots, P_s$ which asserts simply $R^0(a \cdots g)$, and if there is an alternative of the form $R^0(a \cdots g) . S(n) . T(a \cdots g; n)$, then $R^0(a \cdots g) . \supset . (n) . P_1 \vee P_2 \vee, \cdots, \vee P_s$, and these alternatives are to be retained in f.

Let $h, \cdots, j$ denote the elements in any alternative in f' in which all of the elements are assigned to not-K. Then we have $S(h \cdots j)$ in f'. For every n not in K and distinct from $h, \cdots, j$, $S(h \cdots j)$ entails $S(h \cdots jn)$, so that we must have at least one alternative $S(h \cdots jn)$ in f. When n is identified with one of the elements $h, \cdots, j$, we must have an alternative $S(h \cdots j)$ in f. For the case in which n is assigned to K, we must have an alternative $R^0(n) . S(h \cdots j) . T(n; h \cdots j)$ in f. If all of these demands are satisfied, then all alternatives in f which involve $h, \cdots, j$, where $h, \cdots, j$ are assigned to not-K, are to be retained. If one of the demands fails, then every alternative in f involving $h, \cdots, j$ is to be dropped.

Let $a, \cdots, g, h, \cdots, j$ denote the elements in any alternative in f' in which $a, \cdots, g$ are assigned to K and such that $R^0(a \cdots g)$ holds and $h, \cdots, j$ are assigned to not-K. For any such alternative in f', let $P_1, P_2, \cdots, P_s$ be the set of alternatives in f which contain it. (1) When n is identified with one of the elements $a, \cdots, g; h, \cdots, j$, we must have simply $R^0(a \cdots g) . S(h \cdots j) . T(a \cdots g; h \cdots j)$ as an alternative in f. If no such alternative occurs, then all of the set $P_1, P_2, \cdots, P_s$ may be dropped, for F asserts $P_1 \vee P_2 \vee, \cdots, \vee P_s$ for every value of n. (2) Let $P_a, \cdots, P_d$ be those alternatives in which n is assigned to K and is distinct from $a, \cdots, g$. Then it may follow from Theorem IV that $P_a \vee, \cdots, \vee P_d$ holds for every such n. If not it follows from Theorem V that the function $P_a \vee, \cdots, \vee P_d$ fails for at least one value of n. (3) When n is assigned to the class not-K we must have an alternative $R^0(a \cdots g) . S(h \cdots jn) . T(a \cdots g; h \cdots jn)$ in f. If no such alternative occurs, then $P_1, P_2, \cdots, P_s$ may all be discarded. But if conditions (1), (2), and (3) are satisfied, $P_1, P_2, \cdots, P_s$ are to be retained in f.

Consider the function $F(x, \cdots, n)$ as reconstituted by the elimination of the alternatives indicated. $(x, \cdots, l, n) . f(x, \cdots, l, n) . \supset . (x, \cdots, l) f''(x \cdots, l)$, where the latter function is obtained from F by dropping the variable of narrowest scope n. But also, in view of the foregoing eliminations, $(x, \cdots, l) . f''(x, \cdots, l) . \supset . (x, \cdots, l, n) . f(x, \cdots, l, n)$, so that we have a function in $n-1$ variables which has the same truth-value as the original function in n variables. In like manner a function in $n-2$ variables can be found which has the same truth-value as the function in $n-1$ variables, and so on.

It may happen at some stage of the process of passing from a function $F_r(x, \cdots, m)$ in m variables to a function $F_r''(x, \cdots, m-1)$ in $m-1$ variables that every alternative in $f_r(x, \cdots, m)$ vanishes. But, whatever complex quantifier F_r may involve, $F_r(x, \cdots, m) . \supset . (\exists x, \cdots, m) . f_r(x, \cdots, m)$ And any function of the form $(\exists x, \cdots, m) . f_r(x, \cdots, m)$ must, if it is to be true, involve at least one alternative. So that $F_r(x, \cdots, m)$ must be false, and consequently the original function $F(x, \cdots, n)$, to which it is equivalent, must be false, and the truth-value of $F(x, \cdots, n)$ is therefore determined. But even here it is still possible to proceed to a function in two variables $F_w(x, y)$. For $F_r''(x, \cdots, m-1)$ is obtained from $F_r(x, \cdots, m)$ by dropping the variable of narrowest scope, and $F_r''(x, \cdots, m-1)$ will, therefore, involve no alternatives—i. e., it will involve the denial of every possible alternative. And finally, we obtain $F_w(x, y)$, in two variables, which involves no alternatives.

In any case therefore we reach a function in two variables whose truth-value is determined. It follows that *every singly quantified first-order function on the base K, R_2 has its truth-value determined by conditions* (1)–(10).

Strictly it is only necessary to carry out the proof of the theorem just given for at least one of the functions $(\exists x. \cdots, l, n) . f(x, \cdots, l, n)$ or $(x, \cdots, l, n) . f(x, \cdots, l, n)$. For suppose that the theorem has been established for functions of the first kind. The contradictory of any function of the second kind is of the form $(\exists x, \cdots, l, n) . \sim f(x, \cdots, l, n)$, which is a function of the first kind and its truth-value is, by hypothesis, determined. But if the truth-value of the contradictory of a function is determined, the truth-value of the function is, of course, determined. Similarly, the contradictory of any function of the first kind is a function of the second kind.

Let $F(x, \cdots, l, n)$ be any multiply quantified propositional function on the base K, R_2, and let $f(x, \cdots, l, n)$ be the elementary function in F. Put f in expanded form. Discard from f every alternative which does not have one of the forms $R^0(a \cdots j)$, $S(a \cdots j)$, or $R^0(a \cdots g) . S(h \cdots j) . T(a \cdots g; h \cdots j)$. This on the ground that the resulting function is equivalent to the original one since other forms of alternative are impossible.

Form the function f' by dropping the variable of narrowest scope n from f. Suppose that the applicative "some" attaches to n. Then, by an argument identical with that employed in the first part of the theorem just given, there exists a function $F''(x, \cdots, l)$ — which in this case is identical with $F'(x, \cdots, l)$—such that $F''(x, \cdots, l) . \equiv . F(x, \cdots, l, n)$. Suppose that the applicative "every" attaches to n. Then, by an argument identical with that of the second part of the theorem just given, there exists a function $F''(x, \cdots, l)$ such that $F''(x, \cdots, l) . \equiv . F(x, \cdots, l, n)$. In either case then there exists a function in $n-1$ variables having the same truth-value as the original function in n variables, and by a repetition of the argument, there exists such a function in $n-2$ variables, and finally, a function in two variables having the same truth-value as the original function. But every function in two variables has its truth-value determined. It follows that *every first-order function on the base* K, R_2 *has its truth-value determined by postulates* (1)–(10).

Dense Series with a First Element. For a dense series having a first but no last element we require an alteration in one postulate of the set (1)–(10). (6) is to be replaced by (6′) : $(\exists x) :. (y) : x \epsilon K : y \epsilon K . x \neq y . \supset . Rxy$. In connection with the set (1)–(6′)–(10) we wish to prove theorems analogous to those proved for the set (1)–(10). All of the results not dependent on (6) carry over unchanged. As a consequence of (1)–(4), every n elements in K are such that for some permutation, $x, \cdots, n$, $R^0(x \cdots n)$ holds, and from (1)–(4) together with (5) and (8) some n elements in K are such that $R^0(x \cdots n)$ holds. There are at least n elements not in K, from (10), and for every n elements not in K, $S(y \cdots n)$ holds. Every singly quantified first-order function in two variables on the base K, R_2 has its truth-value determined by (1)–(6′)–(10). None of these facts depends on (6), so that all of them carry over from previous proofs.

From (6′) it follows that $(\exists x)\ (y, \cdots n) : x \neq y, \cdots, n . R^0(y \cdots n) . \supset . R^0(xy \cdots n)$. The element x here is clearly unique; there can be only one first element. We may denote the element x, which is such that for every other K-element y, Rxy holds, by "a". Then $(y):y \epsilon K . y \neq a . \supset . Ray$. This function of "$a$" may be denoted by $g(a)$. For any other K-element b, $(\exists y) . y \epsilon \mathrm{K} . y \neq b . \sim Rby$ holds. This function of b is, of course, $\sim g(b)$. Accordingly, $g(a)$ holds and $(z): z \neq a . \supset . \sim g(z)$. Note that this latter function refers to values of z not in K as well — $\sim g(z)$ for every z other than a. Since there is only one element a such that $g(a)$ holds, the class K can be separated into two non-overlapping subclasses K_1 and K_2 such that K_1 has in it the single element a and K_2 has every other K-element.

If $R^0(bc \cdots e)$ holds and $b, c, \cdots, e \epsilon K_2$, then also $\sim g(b) . \sim g(c) ., \cdots, . \sim g(e)$ is true. But if $R^0(ab \cdots e)$ holds, where $a \epsilon K_1$, then $g(a) . \sim g(b) ., \ldots, . \sim g(e)$. Let us denote $R^0(bc \ldots e) . \sim g(b) . \sim g(c) ., \cdots . \sim g(e)$ by $R_0(bc \cdots e)$. But $R_0(ab \cdots e)$ is to mean $R^0(ab \cdots e) . g(a) . \sim g(b) ., \cdots, . \sim g(e)$: when the element a occurs in R_0 it is to be understood to assert $g(a)$. For any other element z it asserts $\sim g(z)$. Similarly, if $S(yz \cdots n)$ holds, then $\sim g(y) . \sim g(z) ., \cdots, . \sim g(n)$. We may denote $S(yz \cdots n) . \sim g(y) . \sim g(z) ., \cdots, . \sim g(n)$ by $S_0(yz \cdots n)$.

Denote by $F(x, y, \cdots, n)$ any propositional function on K, R_2 and let $f(x, y, \cdots, n)$ be the elementary function in F. Put $f(x, y, \cdots, n)$ in expanded form with respect to K, R, and "$=$". On the variables $x, y, \cdots, n$ form the functions $g(x) \vee \sim g(x)$; $g(y) \vee \sim g(y)$; $\cdots$; $g(n) \vee \sim g(n)$. Multiply $f(x, y, \cdots, n)$ by each of these functions. This puts $f(x, y, \cdots, n)$ in expanded form with respect K, R, "$=$", and g. Every alternative in f involves, for any variable z, $g(z)$ or else it involves $\sim g(z)$.

The following theorems are of importance.

Theorem X. $R_0(ab \cdots n) : \supset : (z) : z \neq a, b, \cdots, n . z \epsilon K . \supset . R_0(azb \cdots n) \vee R_0(abz \cdots n) \vee, \cdots, \vee R_0(ab \cdots zn) \vee R_0(ab \cdots nz)$.

The only possibility excluded here is $R_0(zab \cdots n)$.

Theorem XI. $R_0(ab \cdots n) : \supset : (\exists z) . R_0(azb \cdots n) : (\exists t) . R_0(abt \cdots n) :, \cdots, : (\exists u) . R_0(ab \cdots un) : (\exists w) . R_0(ab \cdots nw)$.

The only possibility omitted is $(\exists s) . R_0(sab \cdots n)$. In these theorems $a \epsilon K_1$. For elements in K_2 theorems analogous to IV and V are clearly entailed.

Theorem XII. $S_0(xy \cdots n) : \supset : (w) : \sim w \epsilon K . w \neq x, y, \cdots, n . \supset . S_0(xy \cdots nw)$.

Theorem XIII. $S_0(xy \cdots n) : \supset : (\exists w) . S_0(xy \cdots nw)$.

It will be shown that every doubly quantified first order function in two variables has its truth-value determined by (1)–(6′)–(10). Any such function has one of the forms $(x) : (\exists y) . \varphi(x, y)$ or $(\exists x) : (y) . \varphi(x, y)$. Consider any function of the form $(x) : (\exists y) . \varphi(x, y)$. Put $\varphi(x, y)$ in expanded form with respect to K, R, "$=$", and g. Then in accordance with (1)–(6′)–(10), if the function is to be true, $\varphi(x, y)$ must contain at least one alternative from each of the following three sets: (1) When x takes the value a in K_1 we must have $R_0(a)$, or $R_0(ay)$, or $R_0(a) . S_0(y) . T(a;y)$. (2) when x takes values in K_2 we must have $R_0(ax)$, or $R_0(xy)$, or $R_0(yx)$, or $R_0(x)$, or $R_0(x) . S_0(y) . T(x;y)$. (3) when x takes values in not-K, $R_0(a) . S_0(x) . T(a:x)$, or $R_0(y) . S_0(x) . T(y;x)$, or $S_0(xy)$, or $S_0(x)$. For a function $(x):(\exists y) . \varphi(x, y)$ to be true it is clearly necessary and sufficient that it contain at least one alternative from each of these sets. With regard to functions of the form $(\exists x):(y) . \varphi(x, y)$, note that $(\exists x):(y) . \varphi(x, y) :\equiv: \sim (x):(\exists y) . \sim \varphi(x, y)$, and that $(x):(\exists y) . \sim \varphi(x, y)$ has its truth-value determined.

Since every singly quantified function in two variables is determined by (1)–(6′)–(10), it follows that every function in two variables has its truth-value determined by the set.

Let $F(x, \cdots, l, n)$ be any multiply quantified function on K, R_2 such that $f(x, \cdots, l, n)$ is in expanded form with respect to K, R, "=", and g. It is clear that any alternative in f, if it is ever to be satisfied, must have one of the forms $R_0(a)$, $R_0(b \cdots e)$, $S_0(h \cdots j)$, $R_0(ab \cdots e)$, $R_0(a) . S_0(h \cdots j) . T(a; h \cdots j)$, $R_0(b \cdots e) . S_0(h \cdots j) . T(b \cdots e; h \cdots j)$, or $R_0(ab \cdots e) . S_0(h \cdots j) . T(ab \cdots e; h \cdots j)$. Every alternative not having one or the other of these forms is to be dropped from f. Any alternative which involves the element a and elements belonging to K_2, if it is to be possible, must be such that a is asserted to have the relation R to every element assigned to K_2. Also, no alternative can hold which involves $g(x) . g(y) . x \neq y$; there cannot be more than one element x such that $g(x)$ holds in any alternative.

Form $f'(x, \cdots, l)$ by dropping from f the variable of narrowest scope n. The alternatives in f' fall into seven classes corresponding to the seven types of alternative given above. One or more of these classes may, of course, be null.

Consider f' in relation to f, and suppose first that n is a variable to which the applicative "some" attaches. Now any alternative in f' must have one of the seven forms given above, as must any alternative in f. (1) Suppose that $R_0(a)$ appears in f'. Then it must have been derived from one or more of the functions $R_0(a)$, $R_0(an)$, $R_0(a) . S_0(n) . T(a; n)$. But $R_0(a) : \supset : (\exists n) . n = a . R_0(a) : (\exists n) . R_0(an) : (\exists n) . R_0(a) . S_0(n) . T(a;n)$. So that whatever alternatives in f may have given rise to $R_0(a)$, $R_0(a)$ implies any one of them and therefore their disjunction. (2) Suppose that $R_0(b \cdots e)$ occurs in f'. Then it may have arisen from $R_0(b \cdots e)$, or $R_0(ab \cdots e)$, where $n = a$, or $R_0(nb \cdots e)$, where $n \neq a$, or $. R_0(b \cdots n \cdots e)$, or $R_0(b \cdots en)$, or $R_0(b \cdots e) . S_0(n) . T(b \cdots e; n)$, or from any combination of these. But $R_0(b \cdots e)$ implies, for some n, any one of these alternatives. (3) If $S_0(h \cdots j)$ occurs in f', then it must have arisen from some of the set $S_0(h\cdots j)$, $S_0(h\cdots jn)$, $R_0(a).S_0(h\cdots j).T(a;h\cdots j)$, where $n=a$, or $R_0(n).S_0(h\cdots j).T(n;h\cdots j)$, where $n\neq a$. But $S_0(h\cdots j)$ implies any one of these for some n. (4) If $R_0(ab\cdots e)$ occurs in f', then it must have arisen from $R_0(ab\cdots e)$, or $R_0(anb\cdots e)$, or $R_0(ab\cdots n\cdots e)$, or $R_0(ab\cdots en)$, or $R_0(ab\cdots e).S_0(n).T(ab\cdots e;n)$. (5) If $R_0(a).S_0(h\cdots j).T(a;h\cdots j)$ occurs in f', then it must have arisen from $R_0(a).S_0(h\cdots j).T(a;h\cdots j)$, or $R_0(an).S_0(h\cdots j).T(an;h\cdots j)$, or $R_0(a).S_0(h\cdots jn).T(a;h\cdots jn)$. (6) If an alternative $R_0(b\cdots e).S_0(h\cdots j).T(b\cdots e;h\cdots j)$ occurs in f', then it must have arisen from $R_0(b\cdots e).S_0(h\cdots j).T(b\cdots e;h\cdots j)$, or $R_0(ab\cdots e).S_0(h\cdots j).T(ab\cdots e;h\cdots j)$, where $a=n$, or $R_0(nb\cdots e).S_0(h\cdots j).T(nb\cdots e;h\cdots j)$, or $R_0(b\cdots n\cdots e).S_0(h\cdots j).T(b\cdots n\cdots e;h\cdots j)$, or $R_0(b\cdots en).S_0(h\cdots j).T(b\cdots en;h\cdots j)$, or $R_0(b\cdots e).S_0(h\cdots jn).T(b\cdots e;h\cdots jn)$. (7) If $R_0(ab\cdots e).S_0(h\cdots j).T(ab\cdots e;h\cdots j)$ occurs, then it arises from $R_0(ab\cdots e).S_0(h\cdots j).T(ab\cdots e;h\cdots j)$, or $R_0(anb\cdots e).S_0(h\cdots j).T(anb\cdots e;h\cdots j)$, or $R_0(ab\cdots n\cdots e).S_0(h\cdots j).T(ab\cdots n\cdots e;h\cdots j)$, or $R_0(ab\cdots en).S_0(h\cdots j).T(ab\cdots en;h\cdots j)$, or $R_0(ab\cdots e).S_0(h\cdots jn).T(ab\cdots e;h\cdots jn)$.

It follows that $F'(x,\cdots,l).\supset.F(x,\cdots,l,n)$, and since $F(x,\cdots,l,n).\supset.F'(x,\cdots,l)$, that $F'(x,\cdots,l).\equiv.F(x,\cdots,l,n)$.

Suppose that the variable of narrowest scope n in f is such that the applicative "every" attaches to it, that is, that n is quantified universally. As before there are seven classes of alternatives in f', any one of which may be null. We have pointed out in the first part of this theorem that to each alternative in f', there corresponds a set of alternatives in f, and we have specified under seven headings precisely what the alternatives are from which a given alternative in f' may arise. In view of Theorems VIII, IX, X, XI, XII and XIII, if a given alternative in f' is satisfied, a disjunctive function involving all of the corresponding alternatives in f is implied, and moreover any disjunctive function lacking at least one of these alternatives is implied to be false. For example, $R_0(a):\supset:(n):n=a.R_0(a).\vee.R_0(an).\vee.R_0(a).S_0(n).T(a;n)$. But $R_0(a):\supset:(\exists n).R_0(a).n=a:(\exists n).R_0(an):(\exists n).R_0(a).S_0(n).T(a;n)$. So that if any alternative is lacking in the first implication, it is false that the implication holds for every n. But if this is so, the alternatives in f can never by satisfied, since they cannot be satisfied when $R_0(a)$ is false. Accordingly, we may discard every set of alternatives which does not follow, for every n, from the corresponding alternative in f', and retain all those sets which do follow.

Consider $F(x,\cdots,l,n)$ thus reconstituted. Form $F''(x,\cdots,l)$ by dropping n from F. $F(x,\cdots l,n).\equiv.F''(x,\cdots l)$. Whether "some" applies to n or "every" applies to n, we have a function in $n-1$ variables equivalent to $F(x, \cdots, l, n)$. It follows that *every first-order function on* K, R_2 *has its truth-value determined by* (1)-(6′)-(10).

Dense Series with Both Extreme Elements. The properties of dense series having both extreme elements differ from those of the set (1)-(10) in that (5) is replaced by (5′) : $(\exists x) :. (y) : x \epsilon K : y \neq x . y \epsilon K . \supset . Rxy$, and in that (6) is replaced by (6′) : $(\exists x) :. (y) : x \epsilon K : y \neq x . y \epsilon K . \supset . Ryx$. It is also necessary to replace (8) by (8′) : $(\exists x, y) . x \neq y . x \epsilon K . y \epsilon K$. All of the remaining properties are the same. In view of (7) and (8′) there are at least n elements in K. That every n elements in K are such that for some permutation, $x, y, \cdots, n$, $R^0(x, y, \cdots, n)$ holds carries over from the properties of the set (1)-(10). We may denote the set obtained by altering these three properties of (1)-(10) by (1′), $\cdots$, (10′). Then the truth-value of every singly quantified function in two variable on K, R_2 is determined by the set (1′)-(10′). It follows immediately that

$$(\exists x, y) ;. (z) : z \epsilon K . z \neq x . z \neq y . \supset . R^0(xzy)$$

and that

$$(\exists x, y) :. (z, \cdots, n) : x \neq z, \cdots, n . y \neq z, \cdots, n . R^0(z \cdots n) . \supset . R^0(xz \cdots ny).$$

x and y are clearly unique and distinct. And since this is the case, the class K can be divided into three non-overlapping subclasses, K_1, K_2 and K_3, where K_1 comprises the single element x and K_3 the single element y and K_2 comprises all other elements which belong to K. Denote the element in K_1 by "a" and the element in K_3 by "c". Then $(x) : x \epsilon K . x \neq a . \supset Rax$, and "$a$" is the only element which has this property. As before, denote this function by g, so that $g(a)$ holds and $(z) : z \neq a . \supset . \sim g(z)$. $(x) : x \epsilon K . x \neq c . \supset . Rxc$, and "$c$" is the only element which has this property. Denote this function by q, so that $q(c)$ holds and $(z) : z \neq c . \supset . \sim q(z)$.

Let $F(x, \cdots, l, n)$ be any function on K, R_2. Then $f(x, \cdots, l, n)$ can be expressed in expanded form with respect to K, R, "$=$", g, and q. R_0 and S_0 will be used in this section in senses very much like their previous ones. $R_0(b \cdots e)$ is to mean $R^0(b \cdots e) . \sim g(b) ., \cdots, . \sim g(e) . \sim q(b) ., \cdots, . \sim q(e)$. But $R_0(ab \cdots e)$ is to mean $R^0(ab \cdots e) . g(a) . \sim g(b) ., \cdots, . \sim g(e) . \sim q(a) . \sim q(b) ., \cdots, . \sim q(e)$, and $R_0(b \cdots ec)$ is to mean $R^0(b \cdots ec) . \sim g(b) ., \cdots, . \sim g(e) . \sim g(c) . \sim q(b) ., \cdots, . \sim q(e) . q(c)$, while $R_0(ab \cdots ec)$ is to mean $R^0(ab \cdots ec) . g(a) . \sim g(b) ., \cdots, . \sim g(e) . \sim g(c) . \sim q(a) . \sim q(b) ., \cdots, . \sim q(e) . q(c)$. $S_0(h \cdots j)$ is to mean $S(h \cdots j) . \sim g(h) ., \cdots, . \sim g(j) . \sim q(h) ., \cdots, . \sim q(j)$.

THEOREM XIV. $R_0(ab \cdots nc) : \supset : (z) . z \epsilon K . z \neq a, b, \cdots, n, c . \supset . R_0(azb \cdots nc) \vee R_0(abz \cdots nc) \vee, \cdots, \vee R_0(ab \cdots znc) \vee R_0(ab \cdots nzc)$.

The only possibilities excluded are $R_0(zab \cdots nc)$ and $R_0(ab \cdots ncz)$.

THEOREM XV. $R_0(ab\cdots nc) : \supset : (\exists z) . R_0(azb\cdots nc) : (\exists t) . R_0(abt\cdots nc) :, \cdots, : (\exists u) . R_0(ab\cdots unc) : (\exists w) . R_0(ab\cdots nwc)$.

This omits the possibilities $(\exists r) . R_0(rab\cdots nc)$ and $(\exists s) . R_0(ab\cdots ncs)$. They would be incompatible with XIV.

THEOREM XVI. $S_0(xy\cdots n) : \supset : (z) : \sim z\epsilon K . z \neq x, y, \cdots, n . \supset . S_0(xy\cdots nz)$.

THEOREM XVII. $S_0(xy\cdots n) . \supset . (\exists z) . S_0(xy\cdots nz)$.

It will be shown that every doubly quantified first-order function in two variables has its truth-value determined by (1′)–(10′). If every function of the form $(x) : (\exists y) . \varphi(x, y)$ is dependent on (1′)–(10′), then every function $(\exists x) : (y) . \varphi(x, y)$ is dependent, since $(\exists x) : (y) . \varphi(x, y) : \equiv : \sim (x) : (\exists y) . \sim \varphi(x, y)$. Consider any function $(x) : (\exists y) . \varphi(x, y)$ in which $\varphi(x, y)$ is in expanded form with respect to K, R, "$=$", g, and q. Then when x takes the value a in K_1, if the function is to hold, we must have at least one of the following alternatives in $\varphi(x, y)$. (1) $R_0(a)$, where $y = a$, or $R_0(ay)$, or $R_0(ac)$, where $c\epsilon K_3$, or $R_0(a) . S_0(y) . T(a; y)$. When x takes values in K_2 we must have (2) $R_0(x)$, where $y = x$, or $R_0(xy)$, or $R_0(yx)$, or $R_0(ax)$, where $y = a$, or $R_0(xc)$, where $y = c$, or $R_0(x) . S_0(y) . T(x; y)$. When x takes the value c we must have (3) $R_0(c)$, where $y = c$, or $R_0(ac)$, or $R_0(yc)$, or $R_0(c) . S_0(y) . T(c; y)$. For values of x in not-K we must have (4) $S_0(x)$, where $y = x$, or $S_0(xy)$, or $R_0(a) . S_0(x) . T(a; x)$ or $R_0(y) . S_0(x) . T(y; x)$, or $R_0(c) . S_0(x) . T(c; x)$. To contain at least one alternative from each of these sets is necessary and sufficient for the truth of $(x) : (\exists y) . \varphi(x, y)$. Since every singly quantified function is dependent on (1′)–(10′), it follows that every first-order function in two variables on K, R_2 is dependent on conditions (1′)–(10′).

To return to functions of n variables; consider $F(x, \cdots, l, n)$ where $f(x, \cdots, l, n)$ is in expanded form with respect to K, R, "$=$", g, and q. Every alternative in f must have one of the forms $R_0(a)$, $R_0(c)$, $R_0(b\cdots e)$, $R_0(ab\cdots e)$, $R_0(b\cdots ec)$, $R_0(ab\cdots ec)$, $R_0(ac)$, $S_0(h\cdots j)$, $R_0(a) . S_0(h\cdots j) . T(a; h\cdots j)$, $R_0(c) . S_0(h\cdots j) . T(c; h\cdots j)$, $R_0(b\cdots e) . S_0(h\cdots j) . T(b\cdots e; h\cdots j)$, $R_0(ab\cdots e) . S_0(h\cdots j) . T(ab\cdots e; h\cdots j)$, $R_0(b\cdots ec) . S_0(h\cdots j) . T(b\cdots ec; h\cdots j)$, $R_0(ab\cdots ec) . S_0(h\cdots j) . T(ab\cdots ec; h\cdots j)$, or $R_0(ac) . S_0(h\cdots j) . T(ac; h\cdots j)$. Any alternative in f not having one of these forms is to be discarded. The resulting function will have the same truth-value as the original one. Form $f'(x, \cdots, l)$ from $f(x, \cdots, l, n)$ by dropping the variable of narrowest scope n. Let $P(x, \cdots, l, n)$ represent any alternative in f, and let $P'(x, \cdots, l)$ be the alternative obtained from P by dropping n. In view of theorems VIII, XV, and XVII, it is clear that $P'(x, \cdots, l) . \supset . (\exists n) . P(x, \cdots, l, n)$, and, of course, $(\exists n) \cdot P(x, \cdots, l, n) . \supset . P'(x, \cdots, l)$. It follows that $F'(x, \cdots, l) . \equiv . F(x, \cdots, l, n)$ if the applicative "some" attaches to n.

Suppose that the applicative "every" attaches to n, and consider the following cases. (1) Let $R_0(a)$ be an alternative in f', and let $P_1, P_2, \cdots, P_s$ be the corresponding alternatives in f. Then for $n = a$ we must have $R_0(a)$ among these alternatives. For $n \,\epsilon\, K_2$ we must have $R_0(an)$, and for $n \,\epsilon\, K_3$ we must have $R_0(ac)$, where $n = c$. For $n \,\epsilon$ not-K we must have $R_0(a) \,.\, S_0(n) \,.\, T(a; n)$. If all of these alternatives occur among $P_1, P_2, \cdots, P_s$, $P_1 \vee P_2 \vee, \cdots, \vee P_s$ holds for every n when $R_0(a)$ holds. But if one of these alternatives is lacking, $P_1 \vee P_2 \vee, \cdots, \vee P_s$ fails for some n when $R_0(a)$ holds; and, of course, it fails when $R_0(a)$ fails, so that in this case these alternatives may be discarded. Similarly, (2) let $R_0(b \cdots e)$ be an alternative in f' and let $P_1, P_2, \cdots, P_s$ be the corresponding alternatives in f. When n takes the value a in K_1 we must have $R_0(ab \cdots e)$ among these alternatives. When $n \,\epsilon\, K_3$ we must have $R_0(b \cdots ec)$. When $n \,\epsilon\, K_2$ and $n \neq b, \cdots, e$ we must have the set of alternatives entailed by Theorem XIV. When n is identified with one of the set $b, \cdots, e$ we must have $R_0(b \cdots e)$. When $n \,\epsilon$ not-K we must have $R_0(b \cdots e) \,.\, S_0(n) \,.\, T(b \cdots e; n)$. (3) Let $R_0(c)$ be an alternative in f'. Then this implies $R_0(ac)$ in f when $n = a$. When $n \,\epsilon\, K_2$ it implies $R_0(nc)$, and when $n = c$, $R_0(c)$. When $n \,\epsilon$ not-K we require $R_0(c) \,.\, S_0(n) \,.\, T(c; n)$. (4) Let $S_0(h \cdots j)$ be an alternative in f'. When n is identified with one of the terms $h, \cdots, j$ this alternative requires $S_0(h \cdots j)$ in f. When $n \,\epsilon$ not-K and $n \neq h, \cdots, j$ it requires $S_0(h \cdots jn)$. When $n = a$ it requires $R_0(a) \,.\, S(h \cdots j) \,.\, T(a; h \cdots j)$, and when $n \,\epsilon\, K_2$, $R_0(n) \,.\, S_0(h \cdots j) \,.\, T(n; h \cdots j)$, and when $n = c$, $R_0(c) \,.\, S_0(h \cdots j) \,.\, T(c; h \cdots j)$. Every other case is a simple combination of these four cases.

Accordingly, $f''(x, \cdots, l)$ exists such that $F''(x, \cdots, l) \,.\!\equiv\!.\, F(x, \cdots, l, n)$. It follows that *every first-order function on the base K, R_2 has its truth-value determined by conditions* (1′)–(10′).

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