A Remark on Acceptable Sets of Numbers

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ABSTRACT. Two negative results concerning the so-called acceptable sets of numbers are extended to the case of arbitrary context-free languages with the help of conventional analytic techniques.

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Introduction

In what follows, X^* denotes the free monoid with neutral element e that is generated by a fixed finite nonempty set X, $\mathbf{N}$ denotes the nonnegative integers, and $\mathbf{L}$ is the family of all context-free languages on X [4, 7]. We consider a fixed crossed homomorphism ρ of X^* into the ring $\mathbf{Z}$ of rational integers; ρ is defined by its restriction to X and by the identity

$$\rho ff' = \rho f \cdot \alpha f' + \rho f', \qquad f, f' \in X^*, \tag{1}$$

where α is a homomorphism of X^* into the multiplicative structure of $\mathbf{Z}$. Thus $\rho e = 0$ by definition. We make the assumption that $| \alpha x | > 1$ for all $x \in X$. This condition is satisfied when $X = \{0, 1\}$, $\alpha 0 = \alpha 1 = 2$, $\rho 0 = 0$, and $\rho 1 = 1$, in which case ρf is the number whose binary expansion is f.

The problem of showing that certain remarkable subsets of $\mathbf{Z}$ cannot have the form $\rho L = \{\rho f : f \in L\}$ for $L \in \mathbf{L}$, or for L in some given subfamily of $\mathbf{L}$, was first attacked by Elgot [6] using metamathematical methods. Recently, Minsky and Papert [8] have considerably generalized these results by a delicate analysis of the asymptotic properties of the function **Card** $\{f \in L : | \rho f | < n\}$ of the nonnegative integer n. Being concerned with the subfamily of the so-called "regular sets," they indicated the possibility of extending their method to arbitrary languages $L \in \mathbf{L}$. (See also [2, 5, 10].) We show here two applications of the techniques of classical analysis to examples already discussed by other authors.

We rely on the following result [1]:

THEOREM [Bar-Hillel, Shamir, and Perles]. *Let $L \in \mathbf{L}$. Except for the members of a finite subset L_0 of L, every word $f \in L$ admits at least one factorization $f = g''h'g'hg$ such that $h' \neq e$ and that $H = \{h_n = g''h'^n g'h^n g : n \in \mathbf{N}\}$ is contained in L.*

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Without loss of generality we always assume $g = e$ when $h = e$. A straightforward computation gives

$$\rho h_n = b'' + b'(\alpha h)^n + b(\alpha hh')^n \tag{2}$$

where, setting $\beta f = \rho f(1 - \alpha f)^{-1}$ when $f \neq e$, and $\beta e = 0$, we have

$$b'' = \rho g + \beta h \cdot \alpha g;$$

$$b' = \beta h' \cdot \alpha g' g + \rho g' \cdot \alpha g - \beta h \cdot \alpha g;$$

$$b = \rho g'' \cdot \alpha g' g - \beta h' \cdot \alpha g' g.$$

In particular, $b'' = 0$ when $h = e$. Further, ρH is finite if and only if it reduces to $\{\rho h_0\} = \{\rho g'' g' g\}$.

First Example

Let $L \in \mathbf{L}$ and $k \in \mathbf{N}$ be such that no member of ρL has more than k different prime divisors. Then the set $\mathbf{Prm}(\rho L)$ of all prime divisors of the members of ρL is a finite set contained in $\mathbf{Prm}(\rho L_0 \cup \alpha X)$.

Let $f \in L \backslash L_0$ and assume that $\mathbf{Prm}(\rho f') \subseteq \mathbf{Prm}(\rho L_0 \cup \alpha X)$ is already verified for every $f' \in L$ strictly shorter than f. Since $f \notin L_0$, we can write $f = h_1 = g'' h' g' h g$ as indicated in the Introduction; and the result is still true for f if ρH is finite since then we know that $\rho f = \rho h_0$ where $h_0 = g'' g' g$ is strictly shorter than f. Thus we can assume that ρH is infinite. According to (2), ρh_n is the coefficient of t^n in the Taylor series expansion of the rational function

$$r(t) = b'' \cdot (1 - t)^{-1} + b' \cdot (1 - t \cdot \alpha h)^{-1} + b \cdot (1 - t \cdot \alpha hh')^{-1}$$

of the variable t. Noting that $r(t)$ has a zero for $t = \infty$, a well-known theorem of Polyá [9, p. 14, Satz II] indicates that $\mathbf{Prm}(\rho H)$ is infinite unless $r(t)$ has the form

$$\sum_{0 \leq i < m} c_i t^i \cdot (1 - c_i t^m)^{-1}$$

for some finite m. Now this condition is satisfied only if $b'' = b'b = 0$, and then ρH has the form

$$\{b' \cdot (\alpha h)^n : n \in \mathbf{N}\} \quad \text{or} \quad \{b \cdot (\alpha hh')^n : n \in \mathbf{N}\}.$$

Furthermore, $\rho h_0 = b'$ or b; and since α is a homomorphism, $\mathbf{Prm}(\alpha h)$ and $\mathbf{Prm}(\alpha hh')$ are contained in $\mathbf{Prm}(\alpha X)$. Thus $\mathbf{Prm}(\alpha H)$ is contained in $\mathbf{Prm}(\rho h_0) \cup \mathbf{Prm}(\alpha X)$ and the verification is concluded.

Second Example

Let $L \in \mathbf{L}$ and the polynomial π be such that $\mathbf{Card}\ \rho L = \infty$ and $\rho L \subseteq \pi \mathbf{Z} (= \{\pi z : z \in \mathbf{Z}\}) \subseteq \mathbf{Z}$. Then π is a trinomial, i.e., $\pi t = c(t + s)^d + c'(t + s)^{d'} + c''$ for some constant s.

We can assume $\pi t = \sum_{0 \leq j \leq d} c_j t^{d-j}$ where the degree d of π is at least 3, since otherwise π is automatically a trinomial. Since ρL is infinite, L must contain a subset H of the type described in the introduction for which ρH is infinite. We set $a' = \alpha h, \quad a = \alpha hh'$.

The hypothesis $\rho L \subseteq \pi \mathbf{Z}$ implies the existence of a map, denoted by ζ_n, of $\mathbf{N}$ into $\mathbf{Z}$ such that $\pi\zeta_n = \rho h_{nd} = ba^{nd} + b'(a')^{nd} + b''$ identically.

Let ζ_n' satisfy $c_0\zeta_n' = \rho h_{nd} - b'' = ba^{nd} + b'a'^{nd}$. We have

$$\zeta_n' = a^n(r_0 + \sum_{0<i} r_i(a'^{dn}/a^{dn})^1)$$

where $r_0 = (bc_0^{-1})^{1/d}$. Thus letting $\zeta_n = \zeta_n'(1 + \epsilon_n')$, it follows from $\zeta_n = \rho h_{nd}$ that

$$(1 + \epsilon_n')^d + \sum_{0<j\leq d} \zeta'^{-j}_n(1 + \epsilon_n')^{d-j} c_j c_0^{-1} = 1 + b''\zeta'^{-d}_n c_0^{-1},$$

showing that $\epsilon_n' = r'\zeta'^{-1}_n + \epsilon_n'' \zeta'^{-2}_n$ where r' is a constant and ϵ_n'' has bounded modulus. Accordingly, if $| a'^d | \leq | a^{d-1} |$ we can write $\zeta_n = r_0a^n + r' + \epsilon_n$ where $| \epsilon_n |$ tends to zero at least as fast as $\max \{ | a^{-n} |, | a'^{dn} a^{-dn+n} | \}$. If $| a'^d | > | a^{d-1} |$ there exists a finite integer k such that $| a'^{kd}/a^{kd-1} | > 1 \geq | a'^{kd+d}/a^{kd+d-1} |$, and then we can write $\zeta_n = r_0a^n + \sum_{0<i\leq k} r_i a'^{idn} a^{-idn+n} + r' + \epsilon_n$ where $| \epsilon_n |$ tends to zero at least as fast as $| a'^{(kd+d)n} a^{-(kd+d)n+n} |$.

In the first case, we have $\zeta_{n+1} - a\zeta_n = r'(a - 1) + (\epsilon_{n+1} - a\epsilon_n)$. Since the left member of this relation is an integer and since $| \epsilon_{n+1} - a\epsilon_n |$ tends to zero for $n \to \infty$ we have in fact that, for all large enough $n \in \mathbf{N}$, $\epsilon_{n+1} - a\epsilon_n$ is equal to some fixed $r'' \in \mathbf{Z}$. Thus, for all large enough n, ζ_n satisfies a linear recurrence relation $\zeta_{n+1} - a\zeta_n = r'(a - 1) + r''$; hence $\zeta_n = sa^n + s'$ where s and s' are constant rational numbers. Bringing this expression into the relation $\pi\zeta_n = \rho h_{nd}$ and identifying terms, we see instantly that π must have the form $c(t + s'')^d + c'(t + s'')^{d'} + c''$, and further that a' and d' must be such that $a'^d = a^{d'}$. This concludes the verification in this case.

If $| a'^d/a^{d-1} | > 1 \geq | a^{2d}/a^{2d-1} |$ (i.e., if $k = 1$), we have

$$\zeta_n = r_0a^n + r_1 a'^{dn} a^{-dn+n} + r' + \epsilon_n .$$

Thus $a^{d-1}\zeta_{n+2} - (a^d + a'^d)\zeta_{n+1} + aa'^d \zeta_n$ is equal to a constant, plus a term whose modulus tends to zero when $n \to \infty$. As above we conclude that ζ_n satisfies a linear recurrence for all large enough n and, in fact, that $\zeta_n = sa^n + s'a'^{dn}a^{-dn+n} + s''$. More generally, for arbitrary $k > 1$, we replace the polynomial $\omega_1 = a^{d-1}t^2 - (a^d + a'^d)t + aa'^d$ used above by the polynomial ω_k of degree $k + 1$ whose roots are $\{a, a'^da^{-d+1}, a'^{2d}a^{-2d+1}, \ldots, a'^{kd}a^{-kd+1}\}$ and whose coefficient of t^{k+1} is the product $a^{d-1}a^{2d-1} \ldots a^{kd-1}$. Substituting ζ_{n+i} for t^i in ω_k we obtain an expression which is equal to a constant plus a term whose modulus tends to zero for $n \to \infty$, and we conclude that in all cases ζ_n can be expressed as a finite sum

$$s_0a^n + \sum_{0<i\leq k} s_i(a'^{id}/a^{id-1})^n + s_{k+1} .$$

We now show that this is incompatible with the hypothesis $\pi\zeta_n = \rho h_{nd}$. Indeed, bringing the expression of ζ_n which has been obtained into the equation $\pi\zeta_n = \rho h_{nd}$, we can identify terms. Noting that $ba^{nd} + b'a'^{nd}$ is equal to the sum of the first two terms in the expansion of $c_0\zeta_n^d$, we find that all the other nonconstant terms of $\pi\zeta_n$ must cancel between themselves. Let j be the largest index less than d such that $c_j \neq 0$, and let i be the largest index less than $k + 1$ such that $s_i \neq 0$. The term $(a'^{id}/a^{id-1})^n s_{k+1}^{j-1}$ (or the term $(a'^{id}/a^{id-1})^{nj}$ if $s_{k+1} = 0$) in $\zeta_n{}^j$ cannot cancel with any other term. Thus the equation $\pi\zeta_n = \rho h_{nd}$ with integral ζ_n is impossible when $k \geq 1$, and the verification is concluded.

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