Models of axiomatic theories admitting automorphisms

by

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The present paper is concerned with models of axiomatic theories based on the first order logic with identity and more specifically with automorphisms of such models. The main results of the paper are contained in section 5 and in particular in theorem 5.7 which says that if a theory possesses at least one infinite model, it also possesses a model with a "very large" automorphism group. It is a corollary to this theorem that axiomatic systems of arithmetic possess models which admit non-trivial automorphisms. This corollary solves a problem formulated by G. Hasenjaeger.

From the point of view of methods it may be interesting to note that the proofs of our fundamental results are not constructive and that for two reasons: First we use a theorem which states that if a theory is consistent, then the set of its axioms can be extended to a consistent and complete set. Secondly we use the so called ordering principle, *i. e.* an axiom stating that every set can be ordered. Since in the whole paper we are dealing with theories containing an arbitrary (not necessarily denumerable) number of constants, we see that the first non-constructive theorem mentioned above is equivalent to the so called fundamental theorem of the ideal theory in Boolean algebras (Henkin [2], especially p. 89 and Łoś [4]). Since the ordering principle is known to follow from that theorem (Łoś and Ryll-Nardzewski [6]), we conclude that the non-constructive tools used in the proofs of our principal theorems are all reducible to the fundamental theorem of the ideal theory in Boolean algebras.

It should also be mentioned that our proofs provide another instance of what has been called by Tarski [10] "the principle of condensation of singularities": The existence of a model admitting a large group of automorphisms is equivalent to the simultaneous satisfiability of an infinite number of sentences. We secure the satisfiability of these sentences by showing that the adjunction of an arbitrary finite number of them to the axioms does not render the theory inconsistent.

In order to make the paper self-contained we have collected in the introductory sections 1-3 all the notions and lemmas which are necessary to an exact formulation of the main theorems and to their proofs. None of these sections contain new results: in sections 1 and 2 we lay down the terminology and recall some well-known facts concerning models. In section 3 we expound the general method (due to Henkin [1], Novak [7], and Rasiowa [9]) of constructing models for arbitrary theories. In section 4 we recall some properties of automorphisms and prove a theorem stating that for each group G there is a theory some models of which possess an automorphism-group isomorphic with G (this is the only theorem in our paper in whose proof the full axiom of choice is used).

It seems to us that the automorphism-groups discussed in the present paper deserve a closer study. We intend publishing some of their applications in subsequent papers.

1. Axiomatic theories and their syntax. We consider axiomatic theories based on the functional calculus of the first order. Every such theory S is determined by three sets: 1° $F(S)$, the set of functors (symbols for functions), 2° $P(S)$, the set of predicates (symbols for relations (*i. e.*, for propositional functions)), 3° $A(S)$, the set of axioms. We make no assumptions as to the cardinal numbers of these sets, which may be finite or denumerable or even non-denumerable. We assume however that $P(S)$ contains at least one symbol, viz. the identity predicate ι. If φ is a functor or a predicate, then we denote by $a(\varphi)$ the number of arguments of φ. We do not exclude the case where $a(\varphi)=0$; in this case φ is called a *constant*. Of course we assume that $a(\iota)=2$. Finally we assume that all the theories which will be considered below contain the same individual variables and we denote these variables [1]) by $\xi_1, \xi_2, \xi_3, \ldots$

By $W(S)$ we denote the class of terms of S. Thus $W(S)$ is the smallest class that contains all the variables and contains the expression $\varphi(\omega_1, \ldots, \omega_{a(\varphi)})$ (where $\varphi \in F(S)$) whenever it contains $\omega_1, \omega_2, \ldots, \omega_{a(\varphi)}$.

By $Z(S)$ we denote the class of (sentential) matrices of S. Thus $Z(S)$ is the smallest class satisfying the following conditions: 1° If $\pi \in P(S)$ and $\omega_1, \ldots, \omega_{a(\pi)} \in W(S)$, then $\pi(\omega_1, \ldots, \omega_{a(\pi)}) \in Z(S)$; 2° if $\zeta_1, \zeta_2 \in Z(S)$, then $\sim\zeta_1$, $\zeta_1 \cdot \zeta_2 \in Z(S)$; if $\zeta \in Z(S)$, then $(\exists \xi_n)\zeta \in Z(S)$ for $n=1,2,\ldots$ [2]).

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[1]) The letters ξ_j are not variables but names for them. In a similar way we construe the symbols "$\sim$", "$\exists$" etc. which we shall use below as names for symbols actually occurring in S.

[2]) Other logical operations can be defined in an obvious way in terms of negation, conjunction, and existential quantifier.

An expression which results from an expression $\alpha \in W(S) \cup Z(S)$ by a substitution of a term ω_j for a variable ξ_j $(j=1,2,\ldots)$ will be denoted by

$$\operatorname{subst}\alpha\begin{pmatrix}\xi_1, \xi_2, \ldots\\ \omega_1, \omega_2, \ldots\end{pmatrix}$$

or, in cases where no misunderstanding is possible, more simply by $\operatorname{subst}\alpha(\omega_1, \omega_2, \ldots)$. We omit the explicit formulation of the well-known conditions which must be satisfied in order that the operation subst be performable.

A *matrix* $\zeta \in Z(S)$ is *open* or *closed* according as it contains no bound or no free variables. A *term* $\omega \in W(S)$ is called *constant* if it contains no variables. The set of constant terms will be denoted by $W^*(S)$.

The class of theorems of S will be denoted by $T(S)$. The following matrices are assumed to be contained in $T(S)$ for each S:

$$\xi_k \iota \xi_k, \quad \xi_k \iota \xi_l \supset \xi_l \iota \xi_k, \quad (\xi_k \iota \xi_l)(\xi_l \iota \xi_m) \supset (\xi_k \iota \xi_m),$$
$$\xi_k \iota \xi_l \supset \omega \iota \operatorname{subst}\omega\begin{pmatrix}\xi_k\\ \xi_l\end{pmatrix}, \quad \xi_k \iota \xi_l \supset \left[\zeta \equiv \operatorname{subst}\zeta\begin{pmatrix}\xi_k\\ \xi_l\end{pmatrix}\right],$$
$$(\omega \in W(S),\ \zeta \in Z(S),\ k,l,m=1,2,\ldots).$$

A theory S' is called an *extension* of S if $F(S) \subset F(S')$, $P(S) \subset P(S')$, and $T(S) \subset T(S')$. The extension is called *inessential* if $P(S')=P(S)$ and $T(S') \cap Z(S)=T(S)$.

A *theory* S is called *open* if all matrices that belong to $A(S)$ are open.

From the so called second ε-theorem (Hilbert and Bernays [3], p. 18-33) follows

THEOREM 1.1. *For every theory S there exists an open theory S' which is an inessential extension of S.*

2. Models of axiomatic theories. Let S be a theory and X a set. We consider a function M with the following properties: 1° M assigns a function M_φ (with $a(\varphi)$ arguments) defined in X and taking on values which are elements of X to each $\varphi \in F(S)$; 2° M assigns a relation M_π (with $a(\pi)$ arguments) defined in X to each $\pi \in P(S)$; 3° M assigns the relation of identity in X to the predicate ι. Every such function M we call a *pseudo-model of S over X.*

Let M be a pseudo-model of S over X. A function

$$f=\begin{pmatrix}\xi_1, \xi_2, \ldots\\ x_1, x_2, \ldots\end{pmatrix}$$

which assigns an element of X to each variable we call a *valuation.* We put $\operatorname{val}_{fM}\xi_j = x_j$ and extend this definition over the whole class $W(S)$ by assuming

$$\operatorname{val}_{fM}\varphi(\omega_1, \ldots, \omega_{a(\varphi)}) = M_\varphi(\operatorname{val}_{fM}\omega_1, \ldots, \operatorname{val}_{fM}\omega_{a(\varphi)}).$$

Instead of $\mathrm{val}_{fM}\omega$ we shall usually write

$$\mathrm{val}_M\omega\binom{\xi_1,\xi_2,\ldots}{x_1,x_2,\ldots}$$

or simpler $\mathrm{val}_M\omega(x_1,x_2,\ldots)$.

Let $\pi\in P(S)$, $\omega_1,\ldots,\omega_{a(\pi)}\in W(S)$, and $\zeta=\pi(\omega_1,\ldots,\omega_{a(\pi)})$. We define [3])

$$\mathrm{stsf}_{fM}\zeta\equiv M_\pi(\mathrm{val}_{fM}\omega_1,\ldots,\mathrm{val}_{fM}\omega_{a(\pi)})$$

and extend this definition over the whole set $Z(S)$ by assuming [4])

$$\mathrm{stsf}_{fM}(\sim\zeta)\equiv\sim\mathrm{stsf}_{fM}\zeta,\qquad \mathrm{stsf}_{fM}(\zeta_1\cdot\zeta_2)\equiv\mathrm{stsf}_{fM}\zeta_1\cdot\mathrm{stsf}_{fM}\zeta_2,$$

$$\mathrm{stsf}_{fM}(\exists\xi_n)\zeta\equiv(\exists f')[(f'\sim_n f)\cdot\mathrm{stsf}_{f'M}\zeta],$$

where the formula $f'\sim_n f$ means that $f'(\xi_j)=f(\xi_j)$ for $j\neq n$.

Instead of $\mathrm{stsf}_{fM}\zeta$ we shall usually write

$$\mathrm{stsf}_M\zeta\binom{\xi_1,\xi_2,\ldots}{x_1,x_2,\ldots}$$

or simpler $\mathrm{stsf}_M\zeta(x_1,x_2,\ldots)$.

We denote by V_M the set of matrices ζ which are valid in M, *i. e.* are such that $\mathrm{stsf}_{fM}\zeta$ holds for all f. If $A(S)\subset V_M$, then we say that M is *a model of* S.

Let S' be an extension of S and let M' and M be pseudo-models of S' and S over the same set X. We call M' an extension [5]) of M if $M'_\varphi=M_\varphi$ for $\varphi\in F(S)\cup P(S)$.

The following theorem is an immediate consequence of the above definitions:

THEOREM 2.1. *If S' is an extension of S and M' an extension of M, then* $\mathrm{stsf}_{fM'}\zeta\equiv\mathrm{stsf}_{fM}\zeta$ *for each valuation f and each $\zeta\in Z(S)$.*

3. The construction of models. Let S be an open theory which possesses a model over an infinite set and let X be an arbitrary set. We assume that there is a one-to-one correspondence between the elements of X and certain symbols which do not occur in S. For simplicity we shall identify the elements of X with the corresponding symbols.

We extend the theory S to a theory $S^*(X)$ by adding the elements of X to the set $F(S)$ and the matrices $\sim(x'\iota x'')$ where $x',x''\in X$, $x'\neq x''$ to the set $A(S)$. We assume that $a(x)=0$ for $x\in X$, *i. e.* that each x is a constant term of the theory $S^*(X)$.

[3]) Here, as in many places below, we use the logical symbols as abbreviations of certain expressions of the informal language.

[4]) Note that in these formulas logical symbols have double meanings: they occur as names of symbols of S and as abbreviations of expressions in the informal language.

[5]) This meaning of the word "extension" is narrower than the meaning attributed to this word by Łoś. Cf. J. Łoś [5].

LEMMA 3.1. *The theory $S^*(X)$ is consistent.*

Proof. Let M be a model of S over an infinite set Y. Let us first assume that X is a finite set consisting of the elements $x_1, \ldots, x_n$. Let $y_1, \ldots, y_n$ be different elements of Y. We extend the model M of S over Y to a pseudo-model M^* of $S^*(X)$ over Y by putting

$$M^*_\varphi = M_\varphi \quad \text{for} \quad \varphi \in F(S), \qquad M^*_{x_j} = y_j \quad \text{for} \quad j = 1, 2, \ldots, n,$$
$$M^*_\pi = M_\pi \quad \text{for} \quad \pi \in P(S).$$

From theorem 2.1 it immediately follows that if $\zeta \in A(S)$, then $\zeta \in V_{M^*}$. Since the formula $\sim(x' \iota x'') \in V_{M^*}$ is evident, we conclude that M^* is a model of $S^*(X)$ over Y. Hence $S^*(X)$ is consistent.

The general case can be reduced to the case of a finite X by the observation that an inconsistency of $S^*(X)$ would entail the inconsistency of $S^*(X_1)$ where X_1 is a finite subset of X.

Now let I be an arbitrary consistent and complete subset of $Z(S^*(X))$ containing $A(S^*(X))$. The existence of I is secured by lemma 3.1. We denote by $S(X, I)$ a theory S' such that $F(S') = F(S^*(X))$, $Z(S') = Z(S^*(X))$ and $A(S') = I$. Two constant terms ω_1, ω_2 of the theory $S(X, I)$ will be called *equivalent* if $\omega_1 \iota \omega_2 \in I$. We write then $\omega_1 \approx \omega_2$. The following properties of the relation $\approx$ are obvious:

LEMMA 3.2. *$\approx$ is an equivalence relation and x_1 non $\approx x_2$ for $x_1, x_2 \in X$, $x_1 \neq x_2$.*

LEMMA 3.3. *If $\varphi \in F(S(X, I))$, $\pi \in P(S(X, I))$, $\omega_j, \omega'_j, \tau_k, \tau'_k$ are constant terms of the theory $S(X, I)$ $(j \leqslant a(\varphi), k \leqslant a(\pi))$, and if $\omega_j \approx \omega'_j$, $\tau_k \approx \tau'_k$ for $j \leqslant a(\varphi)$, $k \leqslant a(\pi)$, then*

$$\varphi(\omega_1, \ldots, \omega_{a(\varphi)}) \approx \varphi(\omega'_1, \ldots, \omega'_{a(\varphi)}), \qquad \pi(\tau_1, \ldots, \tau_{a(\pi)}) \equiv \pi(\tau'_1, \ldots, \tau'_{a(\pi)}) \in I.$$

We denote by $\mathfrak{X}_X$ the set of equivalence classes of $W^*(S(X, I))$ under the relation $\approx$. The equivalence class containing a constant term ω will be denoted by $[\omega]$.

We assign to a functor $\varphi \in F(S)$ a function $\mathfrak{M}_\varphi$ such that

$$\mathfrak{M}_\varphi([\omega_1], \ldots, [\omega_{a(\varphi)}]) = [\varphi(\omega_1, \ldots, \omega_{a(\varphi)})],$$

and to a predicate $\pi \in P(S)$ a relation $\mathfrak{M}_\pi$ such that

$$\mathfrak{M}_\pi([\omega_1], \ldots, [\omega_{a(\pi)}]) \equiv \pi(\omega_1, \ldots, \omega_{a(\pi)}) \in I.$$

It follows from 3.3 that the values of $\mathfrak{M}_\varphi$ and of $\mathfrak{M}_\pi$ do not depend on terms ω_j but on the equivalence classes $[\omega_j]$. Since

$$\mathfrak{M}_\iota([\omega_1], [\omega_2]) \equiv \omega_1 \iota \omega_2 \in I \equiv \omega_1 \approx \omega_2 \equiv [\omega_1] = [\omega_2],$$

we obtain

LEMMA 3.4. *The function $\mathfrak{M}$ is a pseudo-model of S over the set $\mathfrak{X}_X$.*

The pseudo-model $\mathfrak{M}$ depends on the sets X and I and will therefore be denoted by $\mathfrak{M}(X,I)$ if its dependence on X and I will have to be emphasized.

From the definitions we obtain by an easy induction

LEMMA 3.5. *If $\omega \in W(S)$ and $\tau_1,\tau_2,\ldots$ are constant terms of $S(X,I)$, then* $\mathrm{val}_{\mathfrak{M}}\,\omega([\tau_1],[\tau_2],\ldots)=[\mathrm{subst}\,\omega(\tau_1,\tau_2,\ldots)]$.

LEMMA 3.6. *If ζ is an open matrix of S and $\tau_1,\tau_2,\ldots$ are constant terms of $S(X,I)$, then* $\mathrm{stsf}_{\mathfrak{M}}\zeta([\tau_1],[\tau_2],\ldots)\equiv\mathrm{subst}\,\zeta(\tau_1,\tau_2,\ldots)\in I$.

Since I contains the axioms of S and these axioms are open matrices, we obtain from lemma 3.6

THEOREM 3.7. *$\mathfrak{M}(X,I)$ is a model of S over $\mathfrak{X}_X$.*

Again let S be an arbitrary theory and X an arbitrary set. Let M be a pseudo-model of S over a set Y and let M' be its extension to a pseudo-model of $S^*(X)$ over Y. The following theorem will be needed in section 5:

THEOREM 3.8. *If $\omega\in W(S)$, $\zeta\in Z(S)$, ζ is open, and if $x_1,x_2,\ldots\in X$, then $\omega'=\mathrm{subst}\,\omega(x_1,x_2,\ldots)$ is a constant term of $S^*(X)$ and $\zeta'=\mathrm{subst}\,\zeta(x_1,x_2,\ldots)$ is a closed matrix of $S^*(X)$; moreover*

$$\mathrm{val}_{M'}\,\omega'=\mathrm{val}_M\,\omega(M'_{x_1},M'_{x_2},\ldots)\,, \tag{3.8.1}$$

$$\mathrm{stsf}_{M'}\zeta'\equiv\mathrm{stsf}_M\zeta(M'_{x_1},M'_{x_2},\ldots)\,. \tag{3.8.2}$$

Proof. If $\omega=\xi_j$, then both the left and the right hand sides of (3.8.1) are equal to M'_{x_j}. If $\omega=\varphi(\omega_1,\ldots,\omega_{a(\varphi)})$, then $\omega'=\varphi(\omega'_1,\ldots,\omega'_{a(\varphi)})$ where the accents denote the operation $\mathrm{subst}(x_1,x_2,\ldots)$. If (3.8.1) holds for the terms ω_j $(j\leqslant a(\varphi))$, then

$$\begin{aligned}\mathrm{val}_{M'}(\omega')&=M'_\varphi(\mathrm{val}_{M'}\,\omega'_1,\ldots,\mathrm{val}_{M'}\,\omega'_{a(\varphi)})\\&=M'_\varphi\big(\mathrm{val}_M\,\omega_1(M'_{x_1},M'_{x_2},\ldots),\ldots,\mathrm{val}_M\,\omega_{a(\varphi)}(M'_{x_1},M'_{x_2},\ldots)\big)\,.\end{aligned}$$

Since $M'_\varphi=M_\varphi$, we obtain (3.8.1) for the term ω.

Proof of (3.8.2) is similar.

4. Automorphisms of models. Let M be a model of S over X. A one-one mapping f of X onto itself is called an *automorphism* of M if the following equations are satisfied for arbitrary $\varphi\in F(S)$, $\pi\in P(S)$ and $x_1,x_2,\ldots\in X$:

$$f\big(M_\varphi(x_1,\ldots,x_{a(\varphi)})\big)=M_\varphi\big(f(x_1),\ldots,f(x_{a(\varphi)})\big)\,,$$

$$M_\pi(x_1,\ldots,x_{a(\pi)})\equiv M_\pi\big(f(x_1),\ldots,f(x_{a(\pi)})\big)\,.$$

The group of automorphisms of M is denoted by G_M.

In the following two lemmas we note some well-known properties of automorphisms:

LEMMA 4.1. *If* $f \in G_M$, $\omega \in W(S)$, $\zeta \in Z(S)$, *and* $x_1, x_2, \ldots \in X$, *then*

$$f\big(\mathrm{val}_M \omega(x_1, x_2, \ldots)\big) = \mathrm{val}_M \omega\big(f(x_1), f(x_2), \ldots\big),$$

$$\mathrm{stsf}_M \zeta(x_1, x_2, \ldots) \equiv \mathrm{stsf}_M \zeta\big(f(x_1), f(x_2), \ldots\big).$$

LEMMA 4.2. *If S' is an extension of S, M is a model of S over X and M' a model of S' which is an extension of M, then* $G_{M'} \subset G_M$.

We shall now show that each group can be represented as G_M for a suitably chosen model M of a suitable theory S.

THEOREM 4.3. *For each group G there is a theory S and a model M of S such that the groups G_M and G are isomorphic.*

Proof. We call, as usual, a left translation of G a mapping l of G onto itself defined by means of the formula $l(g) = g_0 g$, where g runs over G and g_0 is a fixed element of G.

We take as $F(S)$ the empty set and as $P(S)$ the set consisting of ι and of binary predicates π_f where f runs over one-one mappings of G onto itself that are not left translations of G. The set $A(S)$ is to consist exclusively of the axioms of identity enumerated on p. 52.

If f is a one-one mapping of G onto itself that is not a left translation of G, then there are two elements g_1, g_2 of G such that $f(g_1) \cdot g_1^{-1} \neq f(g_2) \cdot g_2^{-1}$. We select for each f a pair g_{1f}, g_{2f} of elements of G satisfying this condition and denote by M_{π_f} the binary relation defined in G such that

$$M_{\pi_f}(g', g'') \equiv (\exists g)[(g \in G) \cdot (g' = g g_{1f}) \cdot (g'' = g g_{2f})]. \tag{4.3.1}$$

Denoting by M_ι the relation of identity in G, we obtain a model of S over G.

Let l be a left translation of G, $l(g) = g_0 g$ where $g_0 \in G$. From (4.3.1) we immediately obtain

$$M_{\pi_f}(g', g'') \equiv M_{\pi_f}\big(l(g'), l(g'')\big) \quad \text{for} \quad \pi_f \in P(S) \tag{4.3.2}$$

and hence $l \in G_M$.

If l is not a left translation of G, then (4.3.2) does not hold for all $\pi \in P(S)$. Indeed, suppose that (4.3.2) is true for $f = l$. Since $M_{\pi_l}(g_{1l}, g_{2l})$, we obtain $M_{\pi_l}\big(l(g_{1l}), l(g_{2l})\big)$ and hence we infer that there is a $g \in G$ such that $l(g_{1l}) = g \cdot g_{1l}$ and $l(g_{2l}) = g \cdot g_{2l}$, *i. e.* $l(g_{1l}) \cdot g_{1l}^{-1} = l(g_{2l}) \cdot g_{2l}^{-1}$, which contradicts the choice of the elements g_{1l}, g_{2l}. Hence l is not an automorphism of M.

It follows that G_M is identical with the group of all left translations of G and hence isomorphic with G.

5. Models with non-trivial automorphism groups. The following theorem due to Ramsey ([8], theorem A on p. 384) is basic for all theorems given in this section:

THEOREM 5.1. *Let Y be an infinite set and Y^n the set of subsets of Y having exactly n elements. If $Y^n = C_1 \cup \ldots \cup C_k$ is a partition of Y^n into mutually disjoint sets, then there is a $j \leqslant k$ and an infinite set $Y_1 \subset Y$ such that $Y_1^n \subset C_j$.*

In the sequel we consider an open theory S and a model $\mathcal{M}(X, I)$ of S (cf. section 3).

LEMMA 5.2. *A one-one mapping h of X onto itself determines at most one automorphism f of $\mathcal{M}(X, I)$ satisfying the condition $f([x]) = [h(x)]$ for $x \in X$.*

Proof. A constant term τ of $S^*(X)$ has the form $\tau = \operatorname{subst}\omega(x_1, x_2, \ldots)$ where $\omega \in W(S)$ and $x_j \in X$ for $j = 1, 2, \ldots$ Hence by 3.5 and 4.1 we obtain the formula $f([\tau]) = \operatorname{val}_{\mathcal{M}}\omega\big(f([x_1]), f([x_2]), \ldots\big)$, which shows that the value of $f([\tau])$ is determined by the values of $f([x])$ for $x \in X$. This proves the lemma.

If h is a one-one mapping of X onto itself for which there exists an automorphism f with the properties described in lemma 5.2, then we shall say that h induces an automorphism. The automorphism induced by h will be denoted by f_h.

LEMMA 5.3. *If $h_1 \neq h_2$ and the automorphisms f_{h_1}, f_{h_2} exist, then $f_{h_1} \neq f_{h_2}$.*

Proof follows immediately from lemma 3.2.

LEMMA 5.4. *A one-one mapping h of X onto itself induces an automorphism of $\mathcal{M}(X, I)$ if and only if the following condition is satisfied by each open matrix ζ and each assignment $\begin{pmatrix}\xi_1, \xi_2, \ldots \\ x_1, x_2, \ldots\end{pmatrix}$:*

$$(5.4.1)\qquad \operatorname{subst}\zeta\begin{pmatrix}\xi_1, \xi_2, \ldots \\ x_1, x_2, \ldots\end{pmatrix} \equiv \operatorname{subst}\begin{pmatrix}\xi_1, & \xi_2, \ldots \\ h(x_1), & h(x_2), \ldots\end{pmatrix} \in I\,.$$

Proof. From the completeness of I it follows that exactly one of the closed matrices $\operatorname{subst}\zeta(x_1, x_2, \ldots)$, $\sim\operatorname{subst}\zeta(x_1, x_2, \ldots)$ belongs to I. We can assume that it is the first.

By 3.6 we obtain the formula $\operatorname{stsf}_{\mathcal{M}}\zeta([x_1], [x_2], \ldots)$, whence we infer by lemma 4.1 that if h induces an automorphism of $\mathcal{M}(X, I)$, then $\operatorname{stsf}_{\mathcal{M}}\zeta\big([h(x_1)],\ [h(x_2)], \ldots\big)$, *i. e.*, by 3.6 $\operatorname{subst}\zeta\big(h(x_1), h(x_2), \ldots\big) \in I$. This proves the formula (5.4.1).

Let us now assume that (5.4.1) holds for each open matrix ζ and let τ be a constant term of the theory $S^*(X)$. We assign variables of S

to the elements of the set X occurring in τ in such a way that different variables are correlated with different elements. Let $\bar{x}$ denote the variable assigned to x and let $\bar{\tau}$ be a term of S obtained from τ by replacing each x by the corresponding variable $\bar{x}$.

Now let τ_1, τ_2 be two constant terms of the theory $S^*(X)$ and let $x_1, \dots, x_n$ be all the elements of X which occur in τ_1 or in τ_2 or in both of them. We shall show that

(5.4.2) *If* $\tau_1 \approx \tau_2$, *then* $\operatorname{subst} \bar{\tau}_1 \begin{pmatrix} \bar{x}_1, \dots, \bar{x}_n \\ h(x_1), \dots, h(x_n) \end{pmatrix} \approx \operatorname{subst} \bar{\tau}_2 \begin{pmatrix} \bar{x}_1, \dots, \bar{x}_n \\ h(x_1), \dots, h(x_n) \end{pmatrix}$.

Indeed, $\tau_1 \approx \tau_2$ means that $\tau_1 \iota \tau_2 \in I$, whence

$$\operatorname{subst} \bar{\tau}_1 \iota \bar{\tau}_2 \begin{pmatrix} \bar{x}_1, \dots, \bar{x}_n \\ x_1, \dots, x_n \end{pmatrix} \in I .$$

Now we use (5.4.1), in which we take $\zeta = \tau_1 \iota \tau_2$ and replace the variables $\xi_1, \xi_2, \dots$ by $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n$. In this way we obtain

$$\operatorname{subst} \bar{\tau}_1 \iota \bar{\tau}_2 \begin{pmatrix} \bar{x}_1, \bar{x}_2, \dots, \bar{x}_n \\ h(x_1), h(x_2), \dots, h(x_n) \end{pmatrix} \in I ,$$

which proves (5.4.2).

From (5.4.2) it follows that defining f_h by means of the formula

$$f_h([\tau]) = \left[\operatorname{subst} \bar{\tau} \begin{pmatrix} \bar{x}_1, \dots, \bar{x}_n \\ h(x_1), \dots, h(x_n) \end{pmatrix} \right]$$

(where $x_1, \dots, x_n$ are all the elements of X that occur in τ), we obtain a function defined on $\mathfrak{X}_X$.

Each element $[\tau]$ of $\mathfrak{X}_X$ is the value of f_h for a suitable argument. Indeed, if

$$\tau' = \operatorname{subst} \bar{\tau} \begin{pmatrix} \bar{x}_1, \dots, \bar{x}_n \\ h^{-1}(x_1), \dots, h^{-1}(x_n) \end{pmatrix},$$

then

$$\bar{\tau}' = \operatorname{subst} \bar{\tau} \begin{pmatrix} \bar{x}_1, \dots, \bar{x}_n \\ \overline{h^{-1}(x_1)}, \dots, \overline{h^{-1}(x_n)} \end{pmatrix},$$

and hence

$$f_h([\tau']) = \left[\operatorname{subst} \bar{\tau}' \begin{pmatrix} \overline{h^{-1}(x_1)}, \dots, \overline{h^{-1}(x_n)} \\ h\left(h^{-1}(x_1)\right), \dots, h\left(h^{-1}(x_n)\right) \end{pmatrix} \right]$$
$$= \left[\operatorname{subst} \bar{\tau} \begin{pmatrix} \bar{x}_1, \dots, \bar{x}_n \\ x_1, \dots, x_n \end{pmatrix} \right] = [\tau] .$$

In a similar way we show that the mapping f_h is one-one. Indeed, if $f_h([\tau_1]) = f_h([\tau_2])$ and $x_1, \dots, x_n$ have the same meaning as in (5.4.2), then

$$\operatorname{subst} \bar{\tau}_1 \begin{pmatrix} \bar{x}_1, \dots, \bar{x}_n \\ h(x_1), \dots, h(x_n) \end{pmatrix} \approx \operatorname{subst} \bar{\tau}_2 \begin{pmatrix} \bar{x}_1, \dots, \bar{x}_n \\ h(x_1), \dots, h(x_n) \end{pmatrix},$$

and hence

$$\operatorname{subst}\overline{\tau}_1 \iota \overline{\tau}_2 \begin{pmatrix} \overline{x}_1 & ,\ldots, & \overline{x}_n \\ h(x_1), & \ldots, & h(x_n) \end{pmatrix} \in I .$$

Using (5.4.1) we obtain

$$\operatorname{subst}\overline{\tau}_1 \iota \overline{\tau}_2 \begin{pmatrix} \overline{x}_1, \ldots, \overline{x}_n \\ x_1, \ldots, x_n \end{pmatrix} \in I$$

whence $\tau_1 \iota \tau_2 \in I$, $\tau_1 \approx \tau_2$, and $[\tau_1]=[\tau_2]$.

Finally we shall show that f_h is an automorphism of $\mathcal{M}(X, I)$. For $\varphi \in F(S)$ we have

$$f_h\big(\mathcal{M}_\varphi([\tau_1], \ldots, [\tau_{a(\varphi)}])\big) = f_h\big([\varphi(\tau_1, \ldots, \tau_{a(\varphi)})]\big) .$$

By putting $\omega = \varphi(\tau_1, \ldots, \tau_{a(\varphi)})$ and observing that $\overline{\omega} = \varphi(\overline{\tau}_1, \ldots, \overline{\tau}_{a(\varphi)})$ we obtain further

$$\begin{aligned} f_h\big(\mathcal{M}_\varphi([\tau_1], \ldots, [\tau_{a(\varphi)}])\big) &= f_h([\omega]) = \left[\operatorname{subst}\overline{\omega}\begin{pmatrix} \overline{x}_1 & ,\ldots, & \overline{x}_n \\ h(x_1), & \ldots, & h(x_n) \end{pmatrix}\right] \\ &= \left[\varphi\left(\operatorname{subst}\overline{\tau}_1\begin{pmatrix} \overline{x}_1 & ,\ldots, & \overline{x}_n \\ h(x_1), & \ldots, & h(x_n) \end{pmatrix}, \ldots, \operatorname{subst}\overline{\tau}_{a(\varphi)}\begin{pmatrix} \overline{x}_1 & ,\ldots, & \overline{x}_n \\ h(x_1), & \ldots, & h(x_n) \end{pmatrix}\right)\right] \\ &= \mathcal{M}_\varphi\left(\left[\operatorname{subst}\overline{\tau}_1\begin{pmatrix} \overline{x}_1 & ,\ldots, & \overline{x}_n \\ h(x_1), & \ldots, & h(x_n) \end{pmatrix}\right], \ldots, \left[\operatorname{subst}\overline{\tau}_{a(\varphi)}\begin{pmatrix} \overline{x}_1 & ,\ldots, & \overline{x}_n \\ h(x_1), & \ldots, & h(x_n) \end{pmatrix}\right]\right) \\ &= \mathcal{M}_\varphi\big(f_h([\tau_1]), \ldots, f_h([\tau_{a(\varphi)}])\big) . \end{aligned}$$

This is the required automorphism-property for $\varphi \in F(S)$.

If $\pi \in P(S)$, then

$$\begin{aligned} \mathcal{M}_\pi([\tau_1], \ldots, [\tau_{a(\pi)}]) &\equiv \pi(\tau_1, \ldots, \tau_{a(\pi)}) \in I \\ &\equiv \operatorname{subst}\pi(\overline{\tau}_1, \ldots, \overline{\tau}_{a(\pi)}) \begin{pmatrix} \overline{x}_1, \ldots, \overline{x}_n \\ x_1, \ldots, x_n \end{pmatrix} \in I , \end{aligned}$$

where $x_1, \ldots, x_n$ are all the elements of X that occur in $\pi(\tau_1, \ldots, \tau_{a(\pi)})$. Using (5.4.1) for $\zeta = \pi(\overline{\tau}_1, \ldots, \overline{\tau}_{a(\pi)})$ we obtain therefore

$$\mathcal{M}_\pi([\tau_1], \ldots, [\tau_{a(\pi)}]) \equiv \operatorname{subst}\pi(\overline{\tau}_1, \ldots, \overline{\tau}_{a(\pi)}) \begin{pmatrix} \overline{x}_1 & ,\ldots, & \overline{x}_n \\ h(x_1), & \ldots, & h(x_n) \end{pmatrix} \in I .$$

The right-hand side of this equivalence means precisely the same as $\mathcal{M}_\pi\big(f_h([\tau_1]), \ldots, f_h([\tau_{a(\pi)}])\big)$. Lemma 5.4 is thus proved.

In order to express conveniently the content of lemmas 5.2, 5.3, and 5.4 we shall adopt the following

Definition. A group G_1 of transformations of a set X_1 *strongly contains* a group G of transformations of a set X if $X_1 \supset X$ and each $f \in G$ can be extended to at least one function $f_1 \in G_1$.

If G is a cyclic group generated by a transformation h, then instead of saying that G_1 strongly contains G we shall say that G_1 strongly contains h.

From lemmas 5.2, 5.3, and 5.4 we obtain

THEOREM 5.5. *Let S be an open theory and X a set. In order that there exist a model M of S over a set $X_1 \supset X$ such that G_M strongly contains a group G of transformations of X it is necessary and sufficient that the theory $S^*(X)$ remain consistent after the adjunction of all equivalences* (5.4.1) *to its axioms where h is an arbitrary element of G and ζ an arbitrary open matrix of S.*

Proof. If the condition is satisfied, we can extend the set $T(S^*(X))$ to a complete set I satisfying (5.4.1). On using lemma 5.4 we obtain a model $\mathfrak{M}(X,I)$ whose automorphism group strongly contains the group of transformations $[x] \to [h(x)]$ of the set $[X] = \underset{[x]}{E}[x \in X]$. Since there is a one-one correspondence between the elements of X and those of $[X]$, we can exchange the classes $[x]$ for the elements x and obtain thus from $\mathfrak{M}(X,I)$ (which is a model of S over $\mathfrak{X}_X$) a model M of S over a set $X_1 \supset X$ such that G_M strongly contains the group G.

Conversely, if there is a model M of S over a set $X_1 \supset X$ such that G_M strongly contains G, then we use 4.1 and find that formulas (5.4.1) belong to V_M for each open matrix $\zeta \in Z(S)$ and each $h \in G$. Since the axioms of $S^*(X)$ are evidently elements of V_M, we obtain the desired consistency.

THEOREM 5.6 [6]. *Each theory S (not necessarily open), which possesses at least one model over an infinite set, possesses a model M_0 such that the group G_{M_0} strongly contains an infinite cyclic group.*

Proof. Let us first assume that S is open and consider an infinite set

$$X = \{\ldots, x_{-n}, \ldots, x_{-1}, x_0, x_1, \ldots, x_n, \ldots\}$$

where $x_i \neq x_j$ for $i \neq j$. Let h be the transformation $h(x_j) = x_{j+1}$ $(j = 0, \pm 1, \pm 2, \ldots)$.

In order to prove our theorem we have to show that the adjunction of equivalences (5.4.1) (where $\zeta \in Z(S)$ and $x_1, \ldots, x_n$ are to be replaced by arbitrary elements of X) does not render theory $S^*(X)$ inconsistent. It will of course be sufficient to show that no inconsistency occurs if we adjoin an arbitrary finite number of equivalences (5.4.1) to the axioms of $S^*(X)$.

[6] Theorem 5.6 is contained as a special case in the theorem 5.7 which follows. Since however the proof of theorem 5.6 is much simpler than the proof of theorem 5.7 we thought it useful to give an independent proof of theorem 5.6.

Let us therefore consider s open matrices $\zeta_1,\dots,\zeta_s\in Z(S)$ and assume that no variable different from $\xi_1,\xi_2,\dots,\xi_t$ occurs in any of these matrices. We consider further s sequences of integers each containing exactly t (not necessarily different) terms:

$$i_1,j_1,\dots,m_1;\quad i_2,j_2,\dots,m_2;\quad \dots;\quad i_s,j_s,\dots,m_s\,.$$

We may assume that the terms of these s sequences lie in the interval $-n\leqslant x\leqslant n$.

We extend S to a theory S^* such that $P(S^*)=P(S)$, $F(S^*)=F(S)\cup\{x_{-n},\dots,x_{n+1}\}$ (where $a(x_j)=0$ for $-n\leqslant j\leqslant n+1$) and $A(S^*)$ is obtained from $A(S)$ by adjunction of the matrices

$$\text{(5.6.1)}\qquad \operatorname{subst}\zeta_l\begin{pmatrix}\xi_1,\xi_2,\dots,\xi_t\\ x_{i_l},x_{j_l},\dots,x_{m_l}\end{pmatrix}\equiv\operatorname{subst}\zeta_l\begin{pmatrix}\xi_1, & \xi_2, & \dots, & \xi_t\\ h(x_{i_l}), & h(x_{j_l}), & \dots, & h(x_{m_l})\end{pmatrix}$$
$$(l=1,2,\dots,s)\,,$$

$$\text{(5.6.2)}\qquad \sim(x_i\iota x_j)\qquad -n\leqslant i<j\leqslant n+1\,.$$

Note that the only symbols of S^* that do not occur in S are $x_{-n},\dots,x_n,x_{n+1}$.

In order to prove the consistency of S^* we shall construct a model for this theory. To this effect we first assign $2n+1$ different variables of S to the elements $x_{-n},\dots,x_{n+1}$ and denote by $\bar{x}_p$ the variable corresponding to x_p. We consider further 2^s matrices

$$\text{(5.6.3)}\qquad \psi_{\varepsilon_1\dots\varepsilon_s}=\operatorname{subst}\zeta_1^{\varepsilon_1}\begin{pmatrix}\xi_1,\xi_2,\dots,\xi_t\\ \bar{x}_{i_1},\bar{x}_{j_1},\dots,\bar{x}_{m_1}\end{pmatrix}\dots\operatorname{subst}\zeta_s^{\varepsilon_s}\begin{pmatrix}\xi_1,\xi_2,\dots,\xi_t\\ \bar{x}_{i_s},\bar{x}_{j_s},\dots,\bar{x}_{m_s}\end{pmatrix}$$

where $\varepsilon_p=\pm1$ for $p=1,2,\dots,s$ and ζ^ε stands for ζ or $\sim\zeta$ according as $\varepsilon=+1$ or $\varepsilon=-1$. These matrices evidently belong to $Z(S)$; their free variables are $\bar{x}_{-n},\dots,\bar{x}_n$ or some of these variables. It is also evident that the matrices (5.6.3) possess the following properties:

$$\text{(5.6.4)}\qquad \sim(\psi_{\varepsilon_1\dots\varepsilon_s}\cdot\psi_{\eta_1\dots\eta_s})\in T(S)\qquad\text{for}\qquad(\varepsilon_1\dots\varepsilon_s)\neq(\eta_1\dots\eta_s)\,,$$

(5.6.5) the alternation of 2^s matrices (5.6.3) belongs to $T(S)$.

Now let M be a model of S over an infinite set Y. The existence of M is secured by the assumptions of the theorem. We assume Y to be ordered by an arbitrary relation $\leqslant$ which, in general, has nothing in common with relations definable in S. Let Y^{2n+1} be the set consisting of subsets of Y with exactly $2n+1$ elements and let $C_{\varepsilon_1\dots\varepsilon_s}$ be the set containing as elements all those sets $\{y_{-n},\dots,y_n\}\subset Y$ for which $y_{-n}\leqslant\dots\leqslant y_n$ and

$$\text{(5.6.6)}\qquad \operatorname{stsf}_M\psi_{\varepsilon_1\dots\varepsilon_s}\begin{pmatrix}\bar{x}_{-n},\dots,\bar{x}_n\\ y_{-n},\dots,y_n\end{pmatrix}.$$

From (5.6.4) and (5.6.5) it is clear that the sets $C_{\varepsilon_1\ldots\varepsilon_s}$ determine a partition of Y^{2n+1}. Applying theorem 5.1 we infer that there is a fixed system of indices $\varepsilon_1,\ldots,\varepsilon_s$ and an infinite set $Y_1\subset Y$ such that $Y_1^{2n+1}\subset C_{\varepsilon_1\ldots\varepsilon_s}$. We choose from Y_1 $2n+2$ elements $y_{-n},\ldots,y_n,y_{n+1}$ such that $y_{-n}\ll\ldots\ll y_n\ll y_{n+1}$. Hence we have the formula (5.6.6) and also the formula

$$(5.6.7)\qquad \mathrm{stsf}_M\varphi_{\varepsilon_1\ldots\varepsilon_s}\begin{pmatrix}\overline{x}_{-n}\ ,\ldots,\ \overline{x}_n\\ y_{-n+1},\ldots,y_{n+1}\end{pmatrix}.$$

We now define a pseudo-model M^* of S^* over Y by assuming

$$M^*_\varphi=M_\varphi\quad\text{for}\quad\varphi\in F(S),\qquad M^*_{x_j}=y_j\quad\text{for}\quad j=-n,\ldots,n,n+1,$$
$$M^*_\pi=M_\pi\quad\text{for}\quad\pi\in P(S).$$

If $\zeta\in A(S)$, then $\zeta\in V_M$ and hence $\zeta\in V_{M^*}$ (cf. theorem 2.1). Axioms (5.6.2) of S^* are evidently contained in V_{M^*} because $M^*_{x_i}=y_i\neq y_j=M^*_{x_j}$ for $i\neq j$. Formulas (5.6.6) and (5.6.7) prove that

$$\mathrm{stsf}_M\zeta_l^{\varepsilon_l}\begin{pmatrix}\overline{x}_{i_1},\ldots,\overline{x}_{m_l}\\ y_{i_1},\ldots,y_{m_l}\end{pmatrix}\quad\text{and}\quad\mathrm{stsf}_M\zeta_l^{\varepsilon_l}\begin{pmatrix}\overline{x}_{i_l}\ ,\ldots,\ \overline{x}_{m_l}\\ y_{i_l+1},\ldots,y_{m_l+1}\end{pmatrix}$$

for $l=1,2,\ldots,s$ and hence, in accordance with theorem 3.8,

$$\mathrm{stsf}_{M^*}\left(\mathrm{subst}\,\zeta_l^{\varepsilon_l}\begin{pmatrix}\xi_1,\ldots,\xi_t\\ x_{i_1},\ldots,x_{m_l}\end{pmatrix}\right)\quad\text{and}\quad\mathrm{stsf}_{M^*}\left(\mathrm{subst}\,\zeta_l^{\varepsilon_l}\begin{pmatrix}\xi_1\ ,\ldots,\ \xi_t\\ h(x_{i_l}),\ldots,h(x_{m_l})\end{pmatrix}\right).$$

From these two formulas it follows that axioms (5.6.1) are valid in M^*, *i. e.* belong to V_{M^*}. This proves the consistency of S^*.

Theorem 5.6 is thus proved for the case of an open theory. The general case can be reduced to the case of an open theory by means of theorems 1.1 and 4.2.

The following example shows that theorem 5.6 ceases to be true if we replace in it the words "infinite cyclic group" by the words "an arbitrary transformation group".

Assume that S is a consistent theory and that $P(S)$ contains a binary predicate π such that the matrices

$$(\xi_1\pi\xi_2)\cdot(\xi_2\pi\xi_3)\supset(\xi_1\pi\xi_3),\qquad\sim(\xi_1\pi\xi_1),\qquad(\xi_1\pi\xi_2)\vee(\xi_1\iota\xi_2)\vee(\xi_2\pi\xi_1)$$

belong to $T(S)$.

If M is an arbitrary model of S over an arbitrary set X_1, then G_M does not contain functions which, limited to a subset X of X_1, are transformations of finite order different from identity. For assume that $f\in G_M$, $f(x)\neq x$, and f, limited to a set $X\subset X_1$ containing x, is a transformation of order n. Since X is ordered by the relation M_π, we have either $M_\pi(x,f(x))$ or $M_\pi(f(x),x)$. It will be sufficient to consider only the first case. We have evidently $M_\pi(f^j(x),f^{j+1}(x))$ for $j=0,1,2,\dots,n-1$ because f is an automorphism of M. By the transitivity of M_π we obtain therefore $M_\pi(x,f^n(x))$, *i. e.* $M_\pi(x,x)$, which is a contradiction.

In connection with these remarks we shall introduce the following

Definition. For each set X ordered by a relation $\lessdot$ we denote by $G(X,\lessdot)$ the group of all transformations of X onto itself leaving invariant the relation $\lessdot$ (*i. e.* satisfying the condition $x_1 \lessdot x_2 \equiv f(x_1) \lessdot f(x_2)$ for $x_1,x_2 \in X$).

THEOREM 5.7. *If a theory S has at least one model over an infinite set, then for each ordered set X there is a model M_0 of S such that G_{M_0} strongly contains $G(X,\lessdot)$.*

Proof. As in the proof of theorem 5.6 we can limit ourselves to the case of an open theory S. According to theorem 5.5 we have only to show that the theory $S^*(X)$ remains consistent after the adjunction to its axioms of all matrices (5.4.1) where ζ is an open matrix of S and $h \in G(X,\lessdot)$. This again can be reduced to the proof that the theory S remains consistent after the adjunction of an arbitrary finite number of axioms of the form (5.4.1) and of a finite number of axioms of $S^*(X)$ which are not already contained in $A(S)$.

Accordingly we consider a finite number of open matrices $\zeta_1,\zeta_2,\dots,\zeta_s$ of S and assume that no variable different from $\xi_1,\xi_2,\dots,\xi_t$ occurs in any of these matrices. We further consider s sequences each containing t elements of X

$$(5.7.1)\qquad x_{(i-1)t+1},\dots,x_{it},\qquad i=1,2,\dots,s$$

(we do not assume that $x_j \neq x_k$ for $j \neq k$). Finally we consider s functions $g_1,\dots,g_s \in G(X,\lessdot)$ and denote by X^* the set containing all the elements (5.7.1) and all the elements $g_i(x_j)$ where $i=1,2,\dots,s$ and $j=1,2,\dots,st$. We extend S to a theory S^* assuming that $F(S^*)=F(S) \cup X^*$, $P(S^*)=P(S)$ and letting $A(S^*)$ to consist of $A(S)$ and of matrices

$$(5.7.2)\qquad \sim(x' \iota x'')\qquad x',x'' \in X^*,\qquad x' \neq x'',$$

$$(5.7.3)\qquad \operatorname{subst}\zeta_i\begin{pmatrix}\xi_1 & ,\dots, & \xi_t\\ x_{(i-1)t+1}, & \dots, & x_{it}\end{pmatrix} \equiv \operatorname{subst}\zeta_i\begin{pmatrix}\xi_1 & ,\dots, & \xi_t\\ g_i(x_{(i-1)t+1}), & \dots, & g_i(x_{it})\end{pmatrix},$$
$$i=1,2,\dots,s.$$

In order to prove the theorem it will be sufficient to define a model of S^*.

We begin by assigning a variable $\bar{x}$ to each element x of X^* in such a way that $\bar{x}'\neq\bar{x}''$ for $x'\neq x''$. We further put $p=st$ and introduce the following matrices:

$$\psi_i=\mathrm{subst}\,\zeta_i\begin{pmatrix}\xi_1 & ,\dots, \xi_t\\ \bar{x}_{(i-1)t+1}, & \dots, \bar{x}_{it}\end{pmatrix},\qquad i=1,2,\dots,s,$$

$$\bar{\psi}_i=\mathrm{subst}\,\zeta_i\left(\frac{\xi_1}{g_i(x_{(i-1)t+1})},\dots,\frac{\xi_t}{g_i(x_{it})}\right),\qquad i=1,2,\dots,s.$$

Since the assignment $x\to\bar{x}$ is one-one, we easily see that axioms (5.7.3) can be written in the form

$$\mathrm{subst}\,\psi_i\begin{pmatrix}\bar{x}_1,\dots,\bar{x}_p\\ x_1,\dots,x_p\end{pmatrix}=\mathrm{subst}\,\psi_i\begin{pmatrix}\bar{x}_1 & ,\dots, & \bar{x}_p\\ g_i(x_1), & \dots, & g_i(x_p)\end{pmatrix},\qquad i=1,2,\dots,s. \tag{5.7.4}$$

We have noted above that the elements (5.7.1) need not be distinct; let us assume that they form a set with n elements

$$X_0=\{x_1,\dots,x_p\}=\{x_1^0,\dots,x_n^0\} \tag{5.7.5}$$

where $x_i^0\neq x_j^0$ for $i\neq j$. Each of the sets

$$X_i=\{g_i(x_1^0),\dots,g_i(x_n^0)\}\qquad i=1,2,\dots,s$$

has exactly n elements and is ordered similarly to X_0. The set X^* is the union of the sets $X_0,X_1,\dots,X_s$:

$$X^*=X_0\cup X_1\cup\dots\cup X_s=\{x_1^0,\dots,x_n^0,\dots,x_m^0\}.$$

Let M be a model of S over an infinite set Y. We can assume that the set Y is ordered and denote by $\leqslant$ the ordering relation.

Let U be an element of Y^n, *i. e.* a subset of Y with exactly n elements. A sequence $(u_1,\dots,u_p)$ with p (not necessarily distinct) terms $u_j\in U$ will be called a distinguished ordering of U if $u_h\leqslant u_j\equiv x_h\precsim x_j$ for $h,j\leqslant p$. It is evident that for each $U\in Y^n$ there exists exactly one distinguished ordering.

We now define a partition of Y^n into 2^s sets $C_{\varepsilon_1\dots\varepsilon_s}$ where $\varepsilon_i=\pm 1$ for $i=1,2,\dots,s$ by including a set $U\in Y^n$ to $C_{\varepsilon_1\dots\varepsilon_s}$ if the distinguished ordering $(u_1,\dots,u_p)$ of U satisfies the condition

$$\mathrm{stsf}_M\psi_i^{\varepsilon_i}\begin{pmatrix}\bar{x}_1,\dots,\bar{x}_p\\ u_1,\dots,u_p\end{pmatrix}\quad\text{for}\quad i=1,2,\dots,s. \tag{5.7.6}$$

It is evident that the union of all sets $C_{\varepsilon_1\dots\varepsilon_s}$ is Y^n and that two different sets $C_{\varepsilon_1\dots\varepsilon_s}$ are disjoint. By theorem 5.1 there is a fixed system $\varepsilon_1,\dots,\varepsilon_s$ of indices ± 1 and an infinite set $Y_1\subset Y$ such that (5.7.6) holds for each $U \in Y_1^n$. We select from Y_1 a set $\{y_1, \dots, y_m\}$ with m elements ordered (by the relation $\leqslant$) similarly to X^*:

(5.7.7) $$y_i \leqslant y_j \equiv x_i^0 \precsim x_j^0 \quad i,j \leqslant m.$$

We can now define a model M^* of S^* over Y by taking

$$M^*_\varphi = M_\varphi \quad \text{for} \quad \varphi \in F(S), \qquad M^*_{x_j^0} = y_j \quad \text{for} \quad j = 1, 2, \dots, m,$$
$$M^*_\pi = M_\pi \quad \text{for} \quad \pi \in P(S).$$

It is evident that axioms of S and axioms (5.7.2) are valid in M^*. It remains therefore to prove that axioms (5.7.4) are valid in M^*. We first prove the following auxiliary statements:

(5.7.8) The sequence $(M^*_{x_1}, \dots, M^*_{x_p})$ is a distinguished ordering of the set $\{M^*_{x_1}, \dots, M^*_{x_p}\}$,

(5.7.9) The sequence $(M^*_{g_i(x_1)}, \dots, M^*_{g_i(x_p)})$ is a distinguished ordering of the set $\{M^*_{g_i(x_1)}, \dots, M^*_{g_i(x_p)}\}$.

(Note that both sets, $\{M^*_{x_1}, \dots, M^*_{x_p}\}$ and $\{M^*_{g_i(x_1)}, \dots, M^*_{g_i(x_p)}\}$, have exactly n elements).

P r o o f of (5.7.8). Each x_h $(h \leqslant p)$ is identical with x_u^0 where $u \leqslant n$ (cf. (5.7.5)). Assume that $h, j \leqslant p$ and $x_h = x_u^0$, $x_j = x_v^0$. Hence we have the equivalence

$$M^*_{x_h} \leqslant M^*_{x_j} \equiv M^*_{x_u^0} \leqslant M^*_{x_v^0} \equiv y_u \leqslant y_v$$

which together with (5.7.7) yields

$$M^*_{x_h} \leqslant M^*_{x_j} \equiv x_u^0 \precsim x_v^0 \equiv x_h \precsim x_j, \quad \text{q. e. d.}$$

P r o o f of (5.7.9). Each $g_i(x_h)$ $(h \leqslant p)$ is an element of X_i and hence identical with an element of the form $g_i(x_u^0)$ where $u \leqslant n$. Assume that $h, j \leqslant p$ and $g_i(x_h) = g_i(x_u^0)$, $g_i(x_j) = g_i(x_v^0)$. Since $g_i(x_u^0)$ and $g_i(x_v^0)$ belong to X^*, they are identical with elements x_w^0, x_z^0 where $w, z \leqslant m$. Hence, on account of (5.7.7), we obtain

$$M^*_{g_i(x_h)} \leqslant M^*_{g_i(x_j)} \equiv M^*_{g_i(x_u^0)} \leqslant M^*_{g_i(x_v^0)} \equiv M^*_{x_w^0} \leqslant M^*_{x_z^0}$$
$$\equiv y_w \leqslant_z y \equiv x_w^0 \precsim x_z^0 \equiv g_i(x_u^0) \precsim g_i(x_v^0) \equiv g_i(x_h) \precsim g_i(x_j).$$

Since $g \in G(X, \precsim)$, it preserves the ordering relation $\precsim$ and hence the last part of the above formula is equivalent to $x_h \precsim x_j$, q. e. d.

We can now prove that axioms (5.7.4) are valid in M^*. From (5.7.6), (5.7.8), and the remark that $M^*_{x_1}, \dots, M^*_{x_p}$ are elements of Y_1 we obtain the formulas

$$\mathrm{stsf}_M \psi_i^{g_i} \begin{pmatrix} \overline{x}_1, \dots, \overline{x}_p \\ M^*_{x_1}, \dots, M^*_{x_p} \end{pmatrix}, \qquad i = 1, 2, \dots, s$$

whence, on account of theorem 3.8, we further obtain

$$\text{stsf}_{M^*}\text{subst}\,\psi_i^{g_i}\begin{pmatrix}\overline{x}_1,\ldots,\overline{x}_p\\ x_1,\ldots,x_p\end{pmatrix},\quad i=1,2,\ldots,s. \tag{5.7.10}$$

From (5.7.9) we obtain in the same manner

$$\text{stsf}_M\psi_i^{g_i}\begin{pmatrix}\overline{x}_1 & ,\ldots, & \overline{x}_p\\ M^*_{g_i(x_1)}, & \ldots, & M^*_{g_i(x_p)}\end{pmatrix},\quad i=1,2,\ldots,s.$$

Since ψ_i contains only the variables $\overline{x}_{(i-1)t+1},\ldots,\overline{x}_{it}$, the last formula can be written in the form

$$\text{stsf}_M\psi_i^{g_i}\begin{pmatrix}\overline{x}_{(i-1)t+1} & ,\ldots, & \overline{x}_{it}\\ M^*_{g_i(x_{(i-1)t+1})}, & \ldots, & M^*_{g_i(x_{it})}\end{pmatrix},\quad i=1,2,\ldots,s.$$

We now remark that $\overline{\psi}_i$ results from ψ_i by a substitution of variables $\overline{g_i(x_{(i-1)t+1})},\ldots,\overline{g_i(x_{it})}$ for the variables $\overline{x}_{(i-1)t+1},\ldots,\overline{x}_{it}$. Hence we can write the last formula in the form

$$\text{stsf}_M\overline{\psi}_i^{g_i}\begin{pmatrix}\overline{g_i(x_{(i-1)t+1})}, & \ldots, & \overline{g_i(x_{it})}\\ M^*_{g_i(x_{(i-1)t+1})}, & \ldots, & M^*_{g_i(x_{it})}\end{pmatrix},\quad i=1,2,\ldots,s.$$

We simplify this formula by inserting the "fictitious" variables $\overline{g_i(x_{jt+k})}$ $(j\neq i,\ k=1,2,\ldots,t)$ in the upper row. The validity of the formula is unaffected since these variables do not occur in $\overline{\psi}_i$. We thus obtain

$$\text{stsf}_M\overline{\psi}_i^{g_i}\begin{pmatrix}\overline{g_i(x_1)} & ,\ldots, & \overline{g_i(x_p)}\\ M^*_{g_i(x_1)}, & \ldots, & M^*_{g_i(x_p)}\end{pmatrix},\quad i=1,2,\ldots,s$$

or, what amounts to the same,

$$\text{stsf}_M\psi_i^{g_i}\begin{pmatrix}\overline{x}_1 & ,\ldots, & \overline{x}_p\\ M^*_{g_i(x_1)}, & \ldots, & M^*_{g_i(x_p)}\end{pmatrix},\quad i=1,2,\ldots,s.$$

Using theorem 3.8 we finally obtain the formula

$$\text{stsf}_{M^*}\text{subst}\,\psi_i^{g_i}\begin{pmatrix}\overline{x}_1 & ,\ldots, & \overline{x}_p\\ M^*_{g_i(x_1)}, & \ldots, & M^*_{g_i(x_p)}\end{pmatrix},\quad i=1,2,\ldots,s,$$

which together with (5.7.10) proves that the matrix (5.7.4) is valid in M^*.

Theorem 5.7 is thus proved.

6. We shall conclude by proving one more theorem, which is not directly connected with the subject-matter of the present paper but which will be needed in one of the subsequent papers mentioned at the end of the introduction. It seems appropriate to include the proof here because the method of proof is very close to that used in the proof of theorem 5.7.

THEOREM 6.1. *Let X be a set ordered by a relation $\precsim$ and S an open theory which possesses a model M over an infinite set Y ordered by a relation $\leqslant$. Further let Y^* be an infinite set contained in Y and η an open matrix of S with the free variables $\xi_1,\dots,\xi_q$ such that*

$$\mathrm{stsf}_M \eta\begin{pmatrix}\xi_1,\dots,\xi_q\\ y_1,\dots,y_q\end{pmatrix}$$

for each sequence $(y_1,\dots,y_q)$ of elements of Y^* satisfying the conditions $y_1 \leqslant y_2 \leqslant \dots \leqslant y_q$. Under these assumptions there exists a model M_0 of S over a set $X_1 \supset X$ such that

(6.1.1) *G_{M_0} strongly contains the group $G(X,\precsim)$,*

(6.1.2) $\mathrm{stsf}_{M_0}\eta\begin{pmatrix}\xi_1,\dots,\xi_q\\ x_1,\dots,x_q\end{pmatrix}$ *holds for each sequence $(x_1,\dots,x_q)$ such that $x_1 \precsim x_2 \precsim \dots \precsim x_q$.*

Proof. We first show that $S^*(X)$ remains consistent if we add to its axioms 1° all formulas (5.4.1) where $h \in G(X,\precsim)$, ζ is an open matrix of S and $x_1, x_2, \dots$ are arbitrary elements of X, 2° all matrices

$$\mathrm{subst}\,\eta\begin{pmatrix}\xi_1,\dots,\xi_q\\ x_1,\dots,x_q\end{pmatrix} \tag{6.1.3}$$

where $x_1,\dots,x_q \in X$ and $x_1 \precsim x_2 \precsim \dots \precsim x_q$. As before it is sufficient to exhibit for each finite subset X^* of X a model of a theory S^* such that $F(S^*) = F(S) \cup X^*$, $P(S^*) = P(S)$, and $A(S^*)$ consists of $A(S)$ and of those matrices (5.7.2), (5.7.3), and (6.1.3) which contain no x from the outside of X^*.

To achieve this result we repeat word for word the construction carried out in the proof of theorem 5.7 with the only change that we construct the partition not of the whole set Y^n but of its part Y^{*n}. In this way we obtain a pseudo-model M^* of S^* over Y in which $M^*_x \in Y^*$ for $x \in X^*$ and in which axioms belonging $A(S)$ as well as the axioms (5.7.2) and (5.7.3) are valid. If $x_1,\dots,x_q \in X^*$ and $x_1 \precsim \dots \precsim x_q$, then $M^*_{x_1} \leqslant M^*_{x_2} \leqslant \dots \leqslant M^*_{x_q}$ (cf. (5.7.7)) and, since $M^*_{x_1},\dots,M^*_{x_q}$ belong to Y^*, the assumptions of the theorem yield

$$\mathrm{stsf}_{M^*}\mathrm{subst}\,\eta\begin{pmatrix}\xi_1,\dots,\xi_q\\ x_1,\dots,x_q\end{pmatrix}.$$

The consistency of $S^*(X)$ extended as indicated above is thus proved.

We now select a complete set I which contains $A\big(S^*(X)\big)$ as well as matrices (5.4.1), and (6.1.3), and consider the model $\mathcal{M}(X,I)$ of S

over $\mathfrak{X}_X$. Each function $h \in G(X, \precsim)$ determines an automorphism f_h of $\mathfrak{M}(X, I)$ (cf. lemma 5.4), and the formula

$$\mathrm{stsf}_{\mathfrak{M}(X,I)}\,\eta\begin{pmatrix}\xi_1, \ldots, \xi_q \\ [x_1], \ldots, [x_q]\end{pmatrix}$$

holds for each sequence $(x_1, \ldots, x_q)$ such that $x_1 \precsim x_2 \precsim \ldots \precsim x_q$ (cf. lemma 3.6). Owing to the fact that $[x'] \neq [x'']$ for $x' \neq x''$, we can identify the classes $[x]$ where $x \in X$ with the elements x, and obtain thus a model M_0 satisfying (6.1.1) and (6.1.2).

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