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Lee A. Rubel

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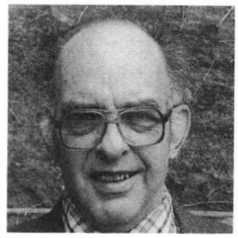
A Survey of Transcendentally Transcendental Functions

LEE A. RUBEL, *University of Illinois*

LEE A. RUBEL is Professor of Mathematics at the University of Illinois at Urbana-Champaign, where he has been for thirty years, except for scholarly leaves. He got his Ph.D. at the University of Wisconsin (Madison) under R. C. Buck's direction in 1954, then spent two years at Cornell University and two years at the Institute for Advanced Study before coming to Illinois. His earlier work was mostly in Complex Variables, but his recent and current work is mostly in Algebraic Differential Equations. He has published about 160 research articles and two books, and has lectured at many universities around the world. He has been an editor of several journals, and is currently an Associate Editor of the *Monthly*.



Everybody knows that a function f is called *algebraic* if it satisfies a polynomial equation $P(x, f(x)) = 0$, and is, otherwise, called transcendental.

A function (or a power series) f is called *transcendentally transcendental* (TT) if it satisfies no algebraic differential equation, that is, no differential equation of the form $P(x, y, y', \ldots, y^{(n)}) = 0$ where P is a nontrivial polynomial in its $n+2$ variables. (We call P a *differential polynomial* in this context.) If f does satisfy such an equation, it is called *differentially algebraic* (DA). Differentially algebraic functions abound in elementary mathematical analysis—witness polynomials, rational functions, algebraic functions, e^x, $\log x$, $\sin x$, $\arctan x$, $\sec x$, Bessel functions, and so on. Moreover (see **[OST-I]**), sums, products, differences, quotients, compositional inverses, and compositions of DA functions are again DA. So a function like

$$J_0\left(\sec^{-1}\left(\frac{e^{x^2}+\sqrt{\log x}}{e^{x^2}-\sqrt{\log x}}\right)^{x}\right)$$

is DA, where J_0 is Bessel's function. Thus, one begins to wonder if there even *exist* TT functions, and what they look like.

In this survey, we exhibit many of the known TT functions, and give some indications of *why* they are TT. Knowing that a function is DA (and hence *not* TT) is knowing that it is an output of a general-purpose analog computer—this is the content of the Shannon-Pour-El-Lipshitz-Rubel Theorem (see **[SHA]**, **[POE]**, **[LIR-II]**), so that a function being TT has more than incidental significance. Our list of TT functions is surely not complete, and we apologize in advance to those people whose favorite TT function has been left out.

Historically, the first function to be proved TT was Euler's gamma function

$$\Gamma(x) = \int_0^\infty t^x e^{-t}\, dt/t.$$

This was done in 1887 by Hölder (see **[HOL]**). We will give a proof of this later, but will begin with some more conceptually immediate examples. My interpretation of the phrase "transcendentally transcendental" (which has a marvelous ring to it!) is the following.

At first, functions were put into just two classes, "algebraic" (i.e., solutions $f(x)$ of $p(x, f(x)) = 0$, where p is a nontrivial polynomial in two variables) and "transcendental," for the remaining class. Then it was realized that there are degrees of transcendentality, and the class of transcendental functions was partitioned into the "algebraically transcendental" (what we here call "differentially algebraic," but transcendental) functions, which are, from the point of view of differential algebra, only half-bad, and the "transcendentally transcendental" functions, which are just terrible. Some alternative terminology is "hypertranscendental" for TT functions, and "hypotranscendental" for DA functions. There has recently begun a line of research (see, for example, **[BAK]**, **[BAN]**, **[LAI]**, **[MOK]**, and **[STR]**) that studies even finer shades of transcendentality of $f(x)$ by asking whether it satisfies an ADE, not over the ring of polynomials in x, but over certain rings of entire functions of slow growth. (For example, it is shown in **[BAK]** that $\Gamma(x)$ satisfies no ADE whose (entire) coefficients $C(z)$ satisfy $|C(z)| \leqslant A\exp(o(|z|))$. This is a revealing line of research and much remains to be done, but we won't discuss it further here. Another kind of question is whether $f(x)$ satisfies an "analytic" differential equation, or whether it satisfies an algebraic difference-differential equation. We will not pursue these notions here.

We note first that to say that an analytic function f (or a formal power series f) is DA is to say that there exists a field F of finite transcendence degree over the rational numbers $\mathbf{Q}$ that contains f and all its derivatives, as follows from the following result of Ritt and Gourin.

THEOREM 1. (**[RIG]**) *Let $y(x)$ be any differentially algebraic function. Then $y(x)$ satisfies an algebraic differential equation with integer coefficients.*

Proof. Representing by y_p the pth derivative of y, we write the ADE for y in the form

$$\sum c_{(i)} x^i y^{i_0} y_1^{i_1} \cdots y_n^{i_n} = 0. \tag{1}$$

Here, each c is a nonzero constant. It is understood that the expressions $x^i y^{i_0} \cdots y_n^{i_n}$ are distinct from one another.

Now (1) states that, for the given function $y(x)$, the expressions $x^i y^{i_0} \cdots y_n^{i_n}$ are linearly dependent over $\mathbf{C}$, the field of complex numbers. Thus, if we set the Wronskian of these expressions equal to zero, we shall have an ADE for $y(x)$, usually of order greater than n, with *integer* coefficients. All that is necessary to show is that the Wronskian does not vanish identically in $x, y, \ldots, y_n$. But if it did, every function with n derivatives would satisfy an equation like (1), that is, an equation with the same expressions $x^i y^{i_0} \cdots y_n^{i_n}$ that appear in (1), with constants c not all zero, because for *analytic* functions (see **[KAP]**), the vanishing of the Wronskian is necessary and sufficient for linear dependence over the constants. But because we can construct a polynomial with any given values for itself and for its first n derivatives at any number of points, it would follow easily that an equation like (1) exists, with coefficients c not all zero, which is an identity for $x, y, y_1, \ldots, y_n$

arbitrary real numbers, which is absurd. The theorem is thus proved by contradiction.

Now let $\alpha_0, \alpha_1, \alpha_2, \dots$ be any sequence of complex (or real) numbers that has infinite transcendence degree over $\mathbf{Q}$, and let $f(x) = \sum_{n=0}^{\infty}(\alpha_n/n!)x^n$. There is no difficulty in dividing the α_j by large integers to make the α_j so small (say $|\alpha_j| \leqslant 1$ for all j) that $f(z)$ is actually an *entire* function. Since $f^{(j)}(0) = \alpha_j$, we see that f is TT. Ritt and Gourin in **[RIG]** used a similar procedure, using a diagonal argument, to produce their example.

Maillet **[MAI]** and later Mahler **[MAH-II]** used recursion relations (see below) for the coefficients of a DA power series $f(z) = \sum_{n=0}^{\infty}a_n z^n$ to prove that

$$|a_n| \leqslant K[n!]^{\alpha} \quad \text{for some } \alpha < \infty, \qquad K < \infty. \tag{2}$$

Therefore, to produce a TT formal power series (but convergent only at 0), we need only write, say

$$f(z) = \sum [n!]^n z^n.$$

Note that the bound (2) is sharp, since $\sum_{n=0}^{\infty}n!z^n$ is easily shown to satisfy the ADE

$$z^2 f'(z) + (z-1)f(z) + 1 = 0,$$

with a similar differential equation for $\sum[n!]^k z^n$, for any positive integer k.

Popken, in his 1935 thesis **[POP]**, showed that if $f(z) = \sum a_n z^n$ is DA, and if the a_n are *algebraic* numbers, then, whenever $a_n \neq 0$,

$$|a_n| \geqslant \exp\left(-cn(\log n)^2\right),$$

where c is some constant that depends on f. Thus,

$$\sum_{n=0}^{\infty} \frac{x^n}{(n^n)!}$$

is TT, a result that was already known to Hurwitz (see **[HUR-I]**).

A different way to produce a TT function is to produce a power series with large gaps. This was done by Ostrowski (see **[OST-I]**), Maillet (see **[MAI]**), and Popken (see **[POP]**), and later by Lipshitz and Rubel (see **[LIR-I]**). Once this is done, we can take any DA power series

$$a_0 + a_1 z + a_2 z^2 + \cdots \tag{3}$$

and introduce changes of signs to get

$$\pm a_0 \pm a_1 z \pm a_2 z^2 \pm \cdots \tag{3'}$$

so that the series (3′) will be TT, by making the sum of the series (3) and (3′) have large gaps. We will end this article with a better result in this direction. The gap theorem in **[OST-I]** is the following:

THEOREM 2. **[OST-I]**. *If $\sum_{n=0}^{\infty}f_n z^n$ is differentially algebraic and if $f_n = 0$ for $m_k \leqslant n < n_k$, for $k = 1, 2, 3, \dots$ and if $\lim_{k\to\infty} n_k/m_k = \infty$, then $\sum_{n=0}^{\infty}f_n z^n$ is a polynomial.*

Proof. Let $f(z) = \sum_{k=0}^{\infty} f_k z^k$ be DA. Then the transcendence degree of the field $\mathbf{C}(z, f, f', f'', \ldots)$ over $\mathbf{C}(z)$ is finite. Hence the transcendence degree of $\mathbf{C}(f, f', f'', \ldots)$ over $\mathbf{C}$ is also finite and thus f satisfies an ADE with coefficients from $\mathbf{C}$ (instead of $\mathbf{C}[z]$). This reduction will simplify our notation. Among all the ADE's with coefficients from $\mathbf{C}$, satisfied by f, let $F(\omega) = F(\omega, \omega', \ldots, \omega^{(m)}) = 0$ be of lowest possible order, m, and lowest possible total degree, n. $S(\omega) = \partial F/\partial \omega^{(m)}$ is the separant of F. By our choice of F, $S(f) \neq 0$, since S has either lower order or lower degree in $\omega^{(m)}$ than F. Let

$$F(\omega) = \sum_{\kappa} a_{(\kappa)} \omega^{(\kappa_1)} \cdots \omega^{(\kappa_N)}$$

where each $a_{(\kappa)} \in \mathbf{C}$ and the sum is over those systems of integers $(\kappa) = (\kappa_1, \ldots, \kappa_N)$ with $0 \leqslant \kappa_1 \leqslant \kappa_2 \leqslant \cdots \leqslant \kappa_N \leqslant m$, $N \leqslant n$, for which $a_{(\kappa)} \neq 0$. (Recall that m is the order of F and n the total degree of F.) Then, as Mahler showed in **[MAH-II]**, pp. 186–194, the f_k then satisfy a recursion formula of the form

$$\alpha(k) f_k = -B(k) \sum_{(\kappa)} \sum_{[\lambda]}{}^{*} a_{(\kappa)} \frac{(\kappa_1 + \lambda_1)!}{\lambda_1!} \cdots \frac{(\kappa_N + \lambda_N)!}{\lambda_N!} f_{\kappa_1 + \lambda_1} \cdots f_{\kappa_N + \lambda_N}, \quad (4)$$

where

(i) $\alpha(k)$ is a fixed nonzero polynomial in k, depending on F and f.

(ii) The first sum is over all the tuples (κ) described above.

(iii) The second sum $\sum_{[\lambda]}^{*}$ is over all N-tuples of integers $[\lambda] = [\lambda_1, \ldots, \lambda_N]$ with $0 \leqslant \lambda_i$, for $i = 1, \ldots, N$ and with $\sum_{i=1}^{N} \lambda_i = h = k - m + s$. Here m is the order of F and $s \geqslant 0$ is a fixed integer depending on f and F. The $*$ indicates that all the terms f_l for $l \geqslant k$ are to be omitted.

(iv) $B(k) = 1$ if $k \geqslant h$, while $B(k) = h!/k!$ if $h > k$.

Suppose now that $f_l = 0$ for all l with $m_j \leqslant l < n_j$, where $n_j/m_j \to \infty$. This means that we have extremely large blocks of zero coefficients, but says nothing about the nonzero coefficients. We will prove our result by induction by showing that for j large, $f_{n_j} = 0$ also.

Suppose that $\alpha(k) \neq 0$ as soon as $k \geqslant k_0$. Note that

$$k + A \geqslant \sum_{i=1}^{N} (\lambda_i + \kappa_i) \geqslant k - A$$

for a fixed finite number A. Suppose now that $k \geqslant k_0$ and that $f_l = 0$ for $l = m_j, m_j + 1, \ldots, n_j - 1$, but $f_{n_j} \neq 0$. We write $k = n_j$. Further, suppose $m_j/n_j < \varepsilon$. But $f_{\kappa_i + \lambda_i} = 0$ as soon as $k > \kappa_i + \lambda_i \geqslant \varepsilon k$. The only way then we can have all $\kappa_i + \lambda_i$ in $\sum^{*}$ satisfy $\kappa_i + \lambda_i < \varepsilon k$ is to have $N\varepsilon k > k - A$, which is impossible if ε is small enough. Thus, the theorem is proved.

In **[LIR-I]**, it was proved that if $f(z) = \sum_{k=1}^{\infty} f_{n_k} z^{n_k}$, where n_k approaches ∞ faster than $\exp((\log k)^2)$, say $n_k = \exp((\log k)^{2+\varepsilon})$ for some $\varepsilon > 0$, then f must be a polynomial. This includes the case of Hadamard gaps $n_{k+1}/n_k \geqslant \theta > 1$ for $k = 1, 2, 3, \ldots$. So, for example, $\sum_{k=0}^{\infty} a_k z^{2^k}$ is DA only if it is actually a polynomial. The idea of the proof was as follows. If $f(z) = \sum_{l=0}^{\infty} f_l z^l$ satisfies the gap condition (that $f_l = 0$ if $l \neq n_k$ for any k), so that most of the $f_l = 0$, then, for a large set of values of l, only one of the terms (in the sum in the Mahler recursion formula) $f_{\kappa_1+\lambda_1} \cdots f_{\kappa_N+\lambda_N}$ (up to a permutation) is nonzero. For these values of l, (4) then becomes

$$\alpha(l) f_l = P(\lambda_1, \ldots, \lambda_N) f_{\kappa_1+\lambda_1} \cdots f_{\kappa_N+\lambda_N},$$

where P is a fixed polynomial. Again using the fact that most of the f_l are zero, one sees that $P(t_1, \ldots, t_N)$ has so many zeroes that $P \equiv 0$. The details are complicated.

Another reason that a function may be TT is that it is *universal*. For example, an entire function f may have the property (see **[BIR]**, **[LUR]**, **[SEW]**, **[BLR]**, etc.) that its set of translates $f_t(z) = f(z - t)$, as t runs over $\mathbf{C}$ (or even over $\mathbf{N} = (0, 1, 2, \ldots)$) is dense in the space E of all entire functions, where the topology is that of uniform convergence on compact sets. This means that for any compact set K, any $\varepsilon > 0$, and any entire function g, there exists a $t \in \mathbf{C}$ such that $|f_t(z) - g(z)| < \varepsilon$ for all $z \in K$.

Let us see why such a universal f must be TT. For, as remarked in the proof of Theorem 2, if f satisfies an ADE, then it must satisfy an *autonomous* ADE, that is, one of the form

$$P\big(f(z), f'(z), \ldots, f^{(n)}(z)\big) = 0. \tag{5}$$

where P does not explicitly involve the independent variable z. But then every translate f_t of f also satisfies (5). From elementary complex-variables theory, if a sequence (F_n) of analytic functions converges uniformly on compact sets to a limit function g, then F_n' converges likewise to g', F_n'' to g'', etc. Therefore, g would have to satisfy (5) if it were a limit (in this topology) of translates of f. But it is easy, given any one differential polynomial, to find an entire function (even a polynomial), that does not annul it.

Heins, in **[HEI]**, has constructed a universal Blaschke product in the unit disk, that is, a convergent Blaschke product

$$B(z) = \prod_{n=1}^{\infty} - \frac{\bar{z}_n}{|z_n|} \frac{z - z_n}{1 - \bar{z}_n z}$$

whose non-Euclidean translates are dense in a suitable sense. Such a Blaschke product must then be TT, by an argument similar to the one just given.

In **[VOR]** and **[REI]** it is shown that the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is universal in the following sense.

THEOREM 3. **[VOR]** *Let* $0 < r < 1/4$; *let* $f(s)$ *be a function that is analytic inside the disc* $\{|s| \leqslant r\}$ *and continuous up to the boundary of the disc. If* $f(s)$ *has no zeroes inside the disc* $\{|s| \leqslant r\}$, *then for every* $\varepsilon > 0$ *there exists a real number* $T = T(\varepsilon)$ *such that* $\max_{|s| \leqslant r} |f(s) - \zeta(s + (\frac{3}{4} + iT))| < \varepsilon$.

This gives a contemporary proof that $\zeta(s)$ is TT. The first proof of this was given by Hilbert, and was written up by Stadigh (see **[STA]**) as part of his 1902 Helsinki thesis. A later proof was given by Ostrowski (see **[OST-I]**), where it was moreover shown that $\zeta(s)$ satisfies no algebraic difference-differential equation.

As we mentioned earlier, Euler's gamma function $\Gamma(x)$ was the first function to be proved TT. Hölder's proof in 1887 was based on the functional equation $\Gamma(z+1) = z\Gamma(z)$. We will give a brief proof by Ostrowski that $\Gamma(x)$ is TT. Later, Bank showed in **[BAN-II]** that $\Gamma(z)\Gamma(az)\Gamma(bz)$ is TT provided $ab \neq 0$. Other specific functions that have been shown to be TT are $\sum_{n=0}^{\infty} z^{2^n}$ (see **[MAH-II]**) and $\sum_{n=1}^{\infty}(z^n/1-2^n)$ (see **[CAR]**). I recall seeing somewhere (but where?) that the Bessel function $J_\nu(z)$ is TT (as a function of ν).

THEOREM 4. *$\Gamma(x)$ is transcendentally transcendental.*

Ostrowski's Proof. (**[OST-II]**). Denote the unknown function by y and its derivatives $y', y'', \ldots, y^{(\nu)}, \ldots$ by $y_1, y_2, \ldots, y_\nu, \ldots$. For two different power-products in $y, y_1, y_2, \ldots$, say $A(x)y^{n_0}y_1^{n_1}y_2^{n_2}\cdots$, $\bar{A}(x)y^{\bar{n}_0}y_1^{\bar{n}_1}y_2^{\bar{n}_2}\cdots$, call the first *higher* than the second if the *last* of the nonvanishing ones among the differences $n_0 - \bar{n}_0$, $n_1 - \bar{n}_1, \ldots$ is strictly positive. This clearly defines a *transitive* ordering among the terms of a differential polynomial, so that one can talk about the *highest* term in a differential polynomial.

Among all the ADE's that $\Gamma(x)$ satisfies, supposing that it is DA, select that one whose highest term is lowest. (It is therefore of the lowest order, and of lowest degree, among those of that order (in the highest occurring derivative) and so on.) Say the equation is

$$f(y, y_1, y_2, \ldots; x) = 0$$

and the highest term is

$$A(x)y^{n_0}y_1^{n_1}y_2^{n_2}\cdots.$$

We may further assume that the degree of $A(x)$ is as small as possible and that $A(x)$ has leading coefficient 1. Then, surely, f is divisible neither by y nor by a linear factor $x - \alpha$.

If another ADE $\bar{f} = 0$ that is satisfied by $\Gamma(x)$ has the highest term $\bar{A}(x)y^{n_0}y_1^{n_1}y_2^{n_2}\cdots$, then $A(x)$ must be a divisor of $\bar{A}(x)$ and we have

$$\bar{f} = \frac{\bar{A}(x)}{A(x)}f.$$

For, by the Euclidean algorithm, we may write $\bar{A}(x) = QA(x) + P$, where P and Q are polynomials, and the degree of P is lower than that of $A(x)$. Then the differential polynomial $\bar{f} - Qf$ will have a lower highest term than f, or else the degree of the highest term of $\bar{f} - Qf$ with respect to x is smaller than the degree of $A(x)$, so that $\bar{f} - Qf$ must vanish identically.

On account of the functional equation $\Gamma(x+1) = x\Gamma(x)$, $\Gamma(x)$ also satisfies the ADE $f((x\Gamma(x)), (x\Gamma(x))', (x\Gamma(x))'', \ldots; x+1) = 0$. The left side comes from $f(y, y_1, y_2, \ldots; x)$ if one substitutes for $y, y_1, y_2, \ldots; x$ the quantities

$$\bar{y} = xy,\ \bar{y}_1 = xy_1 + y,\ \bar{y}_2 = xy_2 + 2y_1,\ \bar{y}_3 = xy_3 + 3y_1, \ldots;\ \bar{x} = x + 1. \quad (6)$$

Denote this substitution by S, and the result of its application to $g(y, y_1, y_2, \ldots; x)$ by Sg. If $B(x)y^{m_0}y_1^{m_1}\ldots$ is any power-product in $y, y_1, y_2, \ldots$, then it is clear that the highest term of $SB(x)y^{m_0}y_1^{m_1}\cdots$ is $x^mB(x+1)y^{m_0}y_1^{m_1}\cdots$, where $m = m_0 + m_1 + \cdots$. Hence, the highest term of Sf is $x^nA(x+1)y^{n_0}y_1^{n_1}\cdots$, where $n = n_0 + n_1 + \cdots$. By the above observation, $x^nA(x+1)/A(x)$ must be a polynomial $D(x) = x^n + \cdots$, and we thus have

$$Sf \equiv D(x)f. \tag{7}$$

To invert the process to get f from Sf, one must replace $x, y, y_1, \dots, y_\nu, \dots$ by

$$x-1, \frac{y}{x-1} \frac{(x-1)y_1 - y}{(x-1)^2}, \dots, \frac{g_\nu(y, y_1, \dots; x)}{(x-1)^{n_\nu}}, \dots,$$

where the g_ν are polynomials in $y, y_1, \dots, x$, and the n_ν are positive integers. If one makes this substitution in (7) and multiplies both sides by a suitable power of $x - 1$, then for a certain positive integer a, one obtains

$$(x-1)^a f(y, y_1, \dots; x) \equiv D(x-1)g(y, y_1, \dots; x),$$

where g is a polynomial in $y, y_1, \dots, x$.

Hence $D(x-1)$ can have no root α other than $\alpha = 1$, since otherwise f would vanish for $x = \alpha$ identically as a polynomial in $y, y_1, \dots$, and thus would have to be divisible by $x - \alpha$. Thus $D(x) = x^n$ and

$$f(xy, xy_1 + y, xy_2 + 2y_1, \dots; x+1) \equiv x^n f(y, y_1, y_2, \dots; x). \tag{8}$$

We set here $y = 0$ and equate the highest terms on both sides of the resulting equation

$$f(0, xy_1, xy_2 + 2y_1, \dots; x+1) \equiv x^n f(0, y_1, y_2, \dots; x).$$

If the highest term of $f(0, y_1, y_2, \dots; x)$ is, say, $C(x)y_1^{l_1}y_2^{l_2}\cdots$, then we have

$$x^{l_1+l_2+\cdots}C(x+1)y_1^{l_1}y_2^{l_2}\cdots \equiv x^nC(x)y_1^{l_1}y_2^{l_2}\cdots.$$

Consequently we must have $l_1 + l_2 + \cdots = n$ and $C(x+1) = C(x)$, so that $C(x)$ is a nonzero constant C. Consequently $f(0, y_1, y_2, \dots; x)$ is not divisible by $x - 1$ and $f(0, y_1, y_2, \dots; 1)$ does not vanish identically. But if we put $x = 0$ in (8), then

$$f(0, y, 2y_1, 3y_2, \dots; 1) \equiv 0,$$

or, replacing $y, 2y_1, 3y_2, \dots$ by $y_1, y_2, y_3, \dots$, respectively, we get

$$f(0, y_1, y_2, \dots; 1) \equiv 0,$$

which contradicts what we have just proved, and thus $\Gamma(x)$ must be transcendentally transcendental.

In **[RIT-II]**, Ritt found all DA solutions of the Poincaré functional equation (see **[POI]**)

$$y(mx) = R[y(x)], \tag{9}$$

with y a meromorphic function in $\mathbf{C}$ and $R(x)$ a rational and not linear function. He showed that if $y(x)$ is such a DA solution of (9), then it is obtained directly from one of the functions $e^{\alpha x}, \cos(\alpha x + \beta)$, or the Weierstrass p-function $p(x)$. This is one way to show that certain functions are TT.

Perhaps surprisingly, the Jacobi theta function $\theta(x) = \sum_{n=0}^{\infty} x^{n^2}$ is actually DA. This fact is proved periodically—see [**JAC**], [**HUR-II**], [**DRA**], [**RES**], [**CHO**], occurring with the dates 1847, 1889, 1938, 1966, and 1984, respectively, for a mean period of 35 years. This result shows, in conjunction with [**ALE**], that there are DA imbeddings of the unit disc in $\mathbf{C}^1$ into $\mathbf{C}^2$, which comes as a surprise, at least to me. Nobody seems to know whether $\theta_3(x) = \sum_{n=0}^{\infty} x^{n^3}$ is TT. In [**BOR**] and [**CRR**], solutions of certain Schröder's functional equations were shown to be TT.

In [**STE**], Steinmetz proved that if f and g are *entire* nonconstant functions, and if the composition $f \circ g$ is DA, then both f and g must be DA. Thus, $f(1/\Gamma(z))$ and $1/\Gamma(g(z))$ are always TT if f and g are non-constant entire functions. In [**BOS**], it was shown that any analytic (in a disc) function $\phi(z)$ such that $\phi(\phi(z)) = e^z$ must be TT.

Of course, one way to produce TT functions of a real variable is by insufficient differentiability. Weierstrass's nowhere-differentiable function is an extreme example. Less trivially, it was shown in [**BRR**] that if $f(x)$ satisfies an ADE (with the only differentiability requirement being that f have enough derivatives to enable it to be plugged into the ADE), then f must be analytic on a dense open subset of its domain. Using this, one can produce many TT functions.

Ostrowski, in [**OST-I**], produced real-analytic functions $f(x, y)$ that satisfy no algebraic *partial* differential equations (APDE). Here's an intriguing problem. If $u(x, y)$ satisfies an APDE and $v(x, y)$ satisfies an APDE, must $u(x, y) + v(x, y)$ also satisfy an APDE? (Suppose, say, that u and v are real-analytic.) NOTE: Since this was written, Wolfgang Schmidt has sent the author a lovely proof of the affirmative answer to this problem.

We close this survey with two "new" results whose gist is that "most" entire functions are TT.

Let E be the space of all entire functions, in the topology of uniform convergence on compact sets. It is known (see [**LUR**]) that E is a complete metric space.

THEOREM 5. *The differentially algebraic entire functions form a set of the first Baire category in E.*

Remark. There is no trouble in extending this result (and its proof) from E to $H(G)$, the space of all holomorphic functions on a region G in $\mathbf{C}$.

Proof of Theorem 5. By Theorem 1 and an early remark, if $f \in$ DA, then f must annul one of $P_1, P_2, P_3, \ldots,$ which are the *autonomous* nontrivial differential polynomials with integer coefficients, written out in a countable list. So it is enough to prove that each $P_j^{\perp} = \{f \in E : P_j \text{ annuls } f\}$ is nowhere dense. But it is clear that each $P_j^{\perp}$ is closed in E (since if $f_j \to f$ in E, then $f_j^{(m)} \to f^{(m)}$ in E for $m = 0, 1, 2, \ldots$.) Hence, it is enough to prove that each $P_j^{\perp}$ contains no neighborhood in E. But a basic neighborhood, $N(f : K, \varepsilon)$ of the function $f \in E$ is indexed by the compact set $K \subseteq \mathbf{C}$ and the positive number ε, and is given by

$$N(f : K, \varepsilon) = \{g \in E : \sup\{|f(z) - g(z)| : z \in K\} < \varepsilon\}.$$

Given f, K, ε, look at

$$g(z) = f(z) + \sum_{k=0}^{M} \frac{\varepsilon_k}{k!} z^k. \tag{10}$$

If all the ε_k are chosen to be sufficiently small complex numbers, say $|\varepsilon_k| < \varepsilon'$, $k = 0, 1, \dots, M$, then $g \in N(f: K, \varepsilon)$. Now let $P(\not\equiv 0)$ be an autonomous differential polynomial. We claim that $P^\perp$ contains no $N(f: K, \varepsilon)$. Otherwise, we would have $P(g) = 0$ for every g of the form (10), with the ε_k small enough. But $g^{(l)}(0) = f^{(l)}(0) + \varepsilon_l$, for $l = 0, 1, \dots, M$, so that $P(f(0) + \varepsilon_0, f'(0) + \varepsilon_1, \dots, f^{(n)}(0) + \varepsilon_n) = 0$. Thus $P(w_0, w_1, \dots, w_n)$ would vanish identically over a whole open ball in $\mathbf{C}^{n+1}$, and thus $P \equiv 0$, contradictory to our hypothesis.

In an earlier version of this survey, we stated and proved the following (true) result.

THEOREM 6. *Given a power series $\sum_{n=0}^\infty a_n z^n$ that is not a polynomial, the probability is* 1 *that the randomized series $\sum_{n=0}^\infty \pm a_n z^n$ is transcendentally transcendental.*

Discussion. The assertion means that the set of $\pm$ for which $\sum_{n=0}^\infty \pm a_n z^n$ is DA has measure 0, where to a sequence of $\pm$ signs, say $((-1)^{\varepsilon_n})$, $\varepsilon_n = 0$ or 1, we assign the real number $\sum_{n=0}^\infty \varepsilon_n 2^{-(n+1)}$ in the interval $[0, 1]$, and the measure is ordinary Lebesgue measure. But much more than this is true—namely that there are at most *countably* many choices of $\pm$ for which $\sum_{n=0}^\infty \pm a_n z^n$ is differentially algebraic. This is a consequence of the next result, which is essentially equivalent to Proposition 6.1 of **[LIR-I]**, that there are only countably many *spectra* for the totality of differentially algebraic power series. (The spectrum of $\sum_{k=0}^\infty f_k z^k$ is the set of k where $f_k \neq 0$.) That result is due to Laohakosol, Lipshitz, Richman, and the author, jointly.

THEOREM 6*. *Given a* (*finite or*) *countable set S, there are at most countably many differentially algebraic power series all of whose coefficients are elements of S.*

One gets the first-mentioned result from Theorem 6* by choosing $S = \{a_0, -a_0, a_1, -a_1, a_2, -a_2, \dots\}$. Our proof depends on the Hurwitz recursion formula, which we now state in detail. (For a proof, see **[DEL]**, specifically the proofs of Lemmas 2.3 and 2.4.)

LEMMA H. (*The Hurwitz recurrence relation.*) *Let $P(x, y, y', \dots, y^{(n)})$ be any differential polynomial with coefficients in $\mathbf{Q}$ in the differential indeterminate y, of order n. Then for any $k \in \mathbf{N}$, $P^{(2k+2)} = y^{(n+2k+2)} f_n + y^{(n+2k+1)} f_{n+1} + y^{(n+2k)} f_{n+2} + \cdots + y^{(n+k+2)} f_{n+k} + f_{n+k+1}$, where the f_j are differential polynomials* (*with rational coefficients*) *in y of order at most j, for $j = n, n+1, \dots, n+k+1$, and $f_n = \partial P / \partial y^{(n)}$. Note that $f_{n+1}, f_{n+2}, \dots$ depend on k. Let $\bar{y} \in \mathbf{C}[[x]]$, the formal power series in x, satisfy $P(x, \bar{y}, \bar{y}', \dots, \bar{y}^{(n)}) = 0$, and suppose that*

$$\frac{\partial P}{\partial y^{(n)}}(x, \bar{y}, \bar{y}', \dots, \bar{y}^{(n)}) = C_0 x^k + C_1 x^{k+1} + \cdots \tag{11}$$

with $C_0 \neq 0$. Then there exists a least $r \in \mathbb{N}$, $0 \leqslant r \leqslant k$, such that

$$\left[f_{n+r} + qf'_{n+r-1} + \cdots + \binom{q}{r} f_n^{(r)}\right](0, \bar{y}(0), \bar{y}'(0), \ldots) \tag{12}$$

is a nonzero polynomial in q. Let $\gamma \in \mathbb{N}$ be bigger than any root $q \in \mathbb{N}$ of polynomial (12). Then for all $q \geqslant \gamma + r$,

$$\bar{y}^{(n+2k+2+q-r)}(0) = \frac{-H_{n+2k+1+q-r}(0, \bar{y}_0, \bar{y}'_0, \ldots)}{A(0, \bar{y}_0, \bar{y}'_0, \ldots, q)} \tag{13}$$

where $A(x, y, y', \ldots, q) = f_{n+r} + qf'_{n+r-1} + \cdots + \binom{q}{r} f_n^{(r)}$, and $H_{n+2k+1+q-r}$ is a rational-coefficient differential polynomial in y of order at most $n + 2k + 1 + q - r$, whose definition depends only on P, k, γ, r, and q.

We remark that the ADE $y(xy)'' - (xy)'y' = 0$ has the solution $y = ax^l$ for any positive integer l and any constant a. This example shows that the recurrence relation for the coefficients of the solutions of an ADE may well depend on the solution as well as on the ADE.

Proof of Theorem 6*. If $\phi(z) = \sum_{n=0}^{\infty} s_n z^n$ is DA for uncountably many choices of $s = (s_n)$, where all the $s_n \in S$, then by Theorem 1, there must be a differential polynomial P with *integer* coefficients that annuls uncountably many of the ϕ. On taking the successive separants $S_1, S_2, S_3, \ldots$ of P, we eventually arrive at a non-zero constant. (Here, $S_1 = \partial P/\partial y^{(n)}$, where n is the order of P, and $S_{j+1} = \partial S_j/\partial y^{(n_j)}$, where n_j is the order of S_j.)

Thus, we would have, for an uncountable set of ϕ, that there is a fixed differential polynomial P that annuls ϕ, such that its separant S_p does not annul ϕ. Turning now to Lemma H, we will have an uncountable subset of ϕ that shares the same k. Then there will be an uncountable subset of that set that shares the same r. Then there will be an uncountable subset of that set, where the biggest integer root of the polynomial (12) is $\leqslant L$ for some fixed integer L. Here, k and r are the quantities appearing in Lemma H, and we take $\bar{y} = \phi$. On this last set of ϕ's we have, for $N \geqslant N_0$,

$$\frac{\phi^{(N+1)}(0)}{(N+1)!} = \frac{-H_N(0, \phi(0), \phi'(0), \ldots)}{A(0, \phi(0), \phi'(0), \ldots, q)}.$$

This says, that for these ϕ's, the coefficients are determined once we know the first $N_0 + 1$ of them. Since there are only countably many choices of the initial segment of coefficients, our supposedly uncountable set turns out to be countable after all. This contradiction proves the Theorem.

We end the paper with an

OUTRAGEOUS CONJECTURE. *Let $\varepsilon_n = 0, 1$ for $n = 0, 1, 2, \ldots$. Then $\sum_{n=0}^{\infty} \varepsilon_n z^n$ is a differentially algebraic function if and only if $\sum_{n=0}^{\infty} \varepsilon_n 2^{-n}$ is an algebraic number.*

Since the conjecture is officially labelled as "outrageous" we don't have to justify it except to say that we haven't found any obvious counterexamples.

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