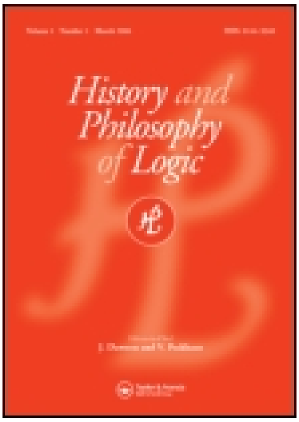

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The compactness of first-order logic:from gödel to lindström

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The Compactness of First-Order Logic: From Gödel to Lindström

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Though regarded today as one of the most important results in logic, the compactness theorem was largely ignored until nearly two decades after its discovery. This paper describes the vicissitudes of its evolution and transformation during the period 1930–1970, with special attention to the roles of Kurt Gödel, A. I. Maltsev, Leon Henkin, Abraham Robinson, and Alfred Tarski.

1. Exposition: The contributions of Gödel and Skolem

The history of mathematical disciplines often fails to reflect their logical structure. Especially striking are instances in which concepts or results now regarded as central lay fallow long after their enunciation and proof—ignored, forgotten, or simply not assimilated by the mathematical community. A case in point is the compactness theorem for first-order logic. In what follows I shall endeavour to sort out the sequence of events surrounding its discovery and application and address such questions as: What factors were responsible for its initial neglect? What factors led to its later revival? To what extent did the eventual recognition of its fundamental character require a conceptual reorientation? How did distinct formulations coalesce or ramify? And how are the roles of particular individuals to be understood?

At the outset, it is essential to distinguish terminologically among four fundamental theorems of first-order logic that are often conflated. In modern formulations they are:

The completeness theorem: Every valid sentence is provable. *The Skolem-Gödel theorem:*[1] A set of sentences is (syntactically) consistent if and only if it is satisfiable in some model. *The compactness theorem:* A set of sentences is satisfiable if and only if every finite subset of it is satisfiable (for short: if and only if the set of sentences is *finitely satisfiable*). *The* (downward) *Löwenheim-Skolem theorem:* If a set of sentences is satisfiable at all, it is satisfiable in a structure whose cardinality is at most that of the number of symbols in the underlying language.

The first two of these theorems establish correlations between certain *semantic* notions (validity, satisfiability, model) and corresponding *syntactic* ones (provability, consistency). The last two, on the other hand, make no reference to syntax.

Of the four theorems, that of Löwenheim and Skolem is the oldest. For the case of a single sentence in a denumerable language, a version of it was first stated by

1 Though well warranted historically (see, e.g., the remarks in Feferman and Tarski *1953*), this eponym no longer seems to be in common use. That it once was may be inferred from its appearance as a subject heading in the index to volumes 1–26 of *The Journal of Symbolic Logic* (vol. 26, 226). Herbrand's name, too, should perhaps be appended to the theorem. See van Heijenoort and Dreben *1986* for a discussion of his role.

Löwenheim in his *1915*; Löwenheim's proof, however, was faulty. A correct proof of Löwenheim's result—though stated in a form that at first *appears* to be the Skolem-Gödel theorem, because of the use of terms such as 'contradictory' and 'consistent' (interpreted, however, in terms of satisfiability)—was first given by Skolem in his *1920*. For the purpose, Skolem introduced the ∀∃ normal form for satisfiability that now bears his name. He also invoked the axiom of choice, and thereby obtained (as he noted only much later[2]) a somewhat stronger conclusion, to wit, that the countable model could be taken to be a substructure of the model whose existence was presupposed in the hypothesis. In the same paper Skolem also extended the result to the case of a denumerable set of sentences. Later, in his *1929*, Skolem presented a simplified form of the 1920 proof. In the meantime, however, he had given (in his *1923*) a proof of the weaker conclusion (without the substructure condition) that made no appeal to the axiom of choice.[3]

Explicit statements and proofs of the other three theorems, again only for denumerable languages, were first published in Gödel *1930*. The first two of them (but *not* the compactness theorem) were also included in Gödel's doctoral dissertation *1929*, of which his *1930* is a rewritten version. Gödel's proofs employed Skolem's methods; but, unlike Skolem, Gödel carefully distinguished between syntactic and semantic notions. The relation between the works of the two men has been examined by Vaught (*1974*, 157–159) and, in great detail, by van Heijenoort and Dreben *1986*. All three commentators agree that both the completeness and compactness theorems were implicit in Skolem *1923*, but that no one before Gödel drew them as conclusions, not even *after* Hilbert and Ackermann, in their 1928 book *Grundzüge der theoretischen Logik* singled out first-order logic for attention and explicitly posed the question of its completeness. Vaught attributes the delay in the enunciation of the completeness theorem to 'the lack of able logicians who knew and appreciated both the notion of model and the notion of logistic system', but he notes that such an excuse does not apply in the case of the compactness theorem, since it is a purely semantic statement. Rather, he opines that perhaps 'the compactness theorem was not [...] inferred by Skolem or others' at that time simply because, 'when viewed as a theorem of pure model theory [... it] appears wholly unlikely'.

Alternatively, Gödel himself attributed the 'blindness of logicians' toward the completeness theorem (and, by extension, the compactness theorem as well) to the 'widespread lack, at that time, of the required epistemological attitude', not only 'toward metamathematics' but 'toward non-finitary reasoning' (Wang *1974*, 8). Either explanation seems adequate to account for the delay in the *discovery* of the compactness theorem. But Gödel's may also help to explain (at least in part) why, even after he had stated and *proved* the theorem, logicians were so slow to exploit it.

It is that further delay in the application and extension of the theorem that is the subject of the present inquiry. For example, one particularly direct and important application of compactness, routinely presented in today's logic texts, is that of establishing the existence of countable non-standard models of arithmetic. Again, the techniques employed in the original construction of such models have turned out

2 During a discussion in 1938, not published until 1941; see Skolem *1941*. (Throughout this paper, citations refer to dates of publication, not presentation.)

3 For further details, the interested reader should consult the discussion in van Heijenoort *1967*, 228–232, 252–254 and 290–291, or Vaught *1974*, 155–158. Van Heijenoort's source book also includes English translations of Löwenheim *1915*, Skolem *1923*, and the first part of Skolem *1920*.

to be of central importance; and, once more, the principal players in the drama of discovery were Skolem and Gödel. In that case, however, their roles were reversed: Skolem established the existence of such models in his *1934*, by a direct method that foreshadowed the general technique now known as the ultraproduct construction; while Gödel, in his review *1934a* of Skolem's paper, noted that one of Skolem's results (the existence of non-isomorphic models of any recursive axiomatization of arithmetic) *could* also have been inferred from the first incompleteness theorem.[4] But Gödel had not in fact done so, and, in any case (to quote once more from commentary by Vaught *1986*, 377), 'the main result of Skolem's paper (as Skolem and Gödel both say) is certainly [...] that the set of *all* [...] sentences [that are true in the standard model ℕ] does not characterize ℕ'.[5]

To Vaught it seemed 'extraordinary' that neither Skolem nor Gödel saw that the latter result was 'a simple consequence of the compactness theorem'. But (as Gabriel Sabbagh has perceptively pointed out to me) that application is 'simple' *only* if one has also conceived the idea of expanding the underlying language by the adjunction of new constant symbols. It is *that* insight that seems to have eluded everyone prior to 1941. What *is* remarkable (and ironic), however, is that, much later, it was realized that the compactness theorem itself could be proved, without reference to syntax, by means of ultraproducts (see section 7 below).

To place later work in context, we note here a few salient details of the proofs in Gödel *1930*.

First, Gödel works within the formal framework of the 'restricted functional calculus' of Hilbert and Ackermann, a framework that subsumes both the propositional calculus and first-order logic. The proof of completeness (Theorem I) proceeds by induction on the complexity of the quantifier prefix of a normal formula belonging to a certain reduction class. It depends upon the completeness of the propositional calculus, which had been established earlier by Bernays *1926* and, independently, Post *1921*. Gödel cites the former, but not the latter.

Second, Gödel extends his result to denumerably infinite sets of formulas by proving (Theorem IX, the Skolem-Gödel theorem) that every such set either is simultaneously satisfiable or else possesses a finite subset whose logical conjunction is refutable. In his dissertation, Gödel had sketched a direct proof of that fact. In *1930*, however, he obtains it as a *corollary* of compactness (stated and proved there as Theorem X)—*not vice-versa*, as has been claimed by several authors and as is customary today. Thus, despite its absence from *1929*, the compactness theorem appears in *1930* not as an *afterthought* to completeness, but rather as an important *lemma* in its proof. In essence, the proof in *1930* is a reorganization of that in *1929*. In both cases, the extension to an infinite set of formulas involves the definition of an ω-sequence of formulas such that the existential closure of the n^{th} formula in the sequence is a consequence of the first n formulas in the original set, and the simultaneous satisfiability of all the formulas in the sequence entails that of those in the set. (It is this argument that requires the denumerability of the underlying language.)

Finally, Gödel remarks that the proofs of Theorems IX and X 'can be extended without difficulty [...] to systems of formulas containing the equality symbol', and he

4 Together with Gödel's completeness theorem, as Kleene was later to point out.

5 Gödel's focus on recursive axiomatizations is perhaps understandable if we recall that at about the time his review of Skolem's paper appeared, Gödel was lecturing in Princeton on his incompleteness results—lectures *1934b* in which he introduced the notion '(general) recursive function'.

takes special note of the restriction of Theorem IX (and, implicitly, of compactness as well) to first-order logic proper (without propositional variables).

2. Development: The work of Maltsev

After 1934, neither Gödel nor Skolem figures again in our story. Indeed, for the next 13 years, the *only* person who seems to have recognized the importance of compactness was the Russian A. I. Maltsev. Beginning in 1936, he published 'a seminal run' of papers in what would now be called model-theoretic algebra, papers in which he 'obtained applications [of logic] to group theory of greater technical virtuosity than the possibly more basic applications to algebra later on found [...] by Henkin and Abraham Robinson' (Sabbagh *1991*). Three of those papers (*1936, 1940* and *1941* are of interest here.

In his first published work, written in German, Maltsev *1936* devoted his efforts to generalizing two theorems, one for the propositional calculus and the other for the restricted functional calculus. The theorems in question were Gödel's compactness theorem and Skolem's result that no denumerable set of formulas of first-order logic can completely characterize the structure of the natural numbers. The corresponding generalizations were, respectively, that compactness holds for sets of propositional formulas of arbitrary cardinality, and that for any system S of formulas in a first-order language with equality, every infinite model for S has at least one proper extension that is also a model for S. As an immediate consequence of the latter, Maltsev stated the result known today as the (upward) Löwenheim-Skolem-Tarski[6] theorem: Every system of formulas that has an infinite model has models of arbitrarily large cardinality. In a remark at the end of section 4 of the paper, as a corollary to the proof therein, Maltsev also stated the (downward) Löwenheim-Skolem theorem for languages of unrestricted cardinality.

The proof of the first theorem is by transfinite induction on the cardinality of the set of propositional formulas. The formulas having been well-ordered in a sequence of minimal order-type ω_α, for each initial segment of the sequence (necessarily of cardinality $< \aleph_\alpha$) there must, by the induction hypothesis, be some satisfying truth-valuation. Suppose that one such has been chosen, for each $\lambda < \alpha$. Then, since each formula in the sequence contains only finitely many new propositional variables, at least one of the finitely many distinct assignments of truth values to *those* variables must be common to $\aleph_\alpha$ of the chosen (total) valuations. It follows that at each stage $\lambda < \omega_\alpha$ a (total) truth-valuation, extending the partial truth-valuation already defined on all variables encountered at earlier stages, may be chosen so as to satisfy all formulas of order-index $\leqq \lambda$.

Having thus established the compactness of *propositional* logic for languages of arbitrary cardinality, Maltsev employed the result in the proof of his second theorem. To do so, in section 4 of the paper he endeavored 'to construct, for every system S of statements of the restricted functional calculus, a corresponding system $\mathfrak{B}$ of statements of the propositional calculus' that would be 'equivalent' to S; and, as

6 The addition of Tarski's name to the eponymy of the upward theorem is based on an editor's note appended to Skolem *1934*, cited by Maltsev, in which it is stated that Tarski had proved in his seminar in 1928 (two years *before* Gödel's statement and proof of compactness) that a set of sentences has a nondenumerable model whenever it has an infinite model. How Tarski did so is, as Vaught says, 'a fascinating question', since Tarski himself was unable to reconstruct the proof later. It *seems* that he somehow established the fact that every infinite structure has a proper elementary extension; but all *known* proofs of that fact 'have at least some connection with compactness' (Vaught *1974*, 160).

part of the preliminaries to that construction, he asserted in section 2 that it was 'well-known' that 'every expression of the restricted functional calculus can be replaced by an expression in [Skolem] normal form that is equivalent to it'.

Maltsev's arguments in section 4 effect a reduction of the compactness of first-order logic to that of propositional logic, so it is natural to presume that that was his intent. But if so, why did he neglect to *state* such an important generalization of his first theorem? And, more importantly: Were his arguments correct?

A brief review of Maltsev's paper appeared the following year in the *Journal of symbolic logic* as Rosser *1937*. He disputed Maltsev's claim that an arbitrary formula of the restricted functional calculus must be equivalent to one in $\forall\exists$-form, and so concluded that Maltsev's proof held 'only for sets of postulates all of which can be put in the specified form'. As we shall see, however, the issue is not clear cut; it hinges on just how the term 'equivalent' is to be understood. Later commentators would examine Maltsev's arguments more closely and offer more penetrating analyses (see section 6 below).

There the matter rested until, in his article in Russian Maltsev *1941* finally gave the first explicit formulation of the compactness theorem (or, as he called it, 'the general local theorem') for uncountable first-order languages.[7] Like Skolem, he used the term 'consistent' in the sense of 'satisfiable', but otherwise his formulation of the theorem was unremarkable. For the proof, however—as well as a 'precise formulation [. . .] of the proposition'—Maltsev directed the reader to his *1936*. Thus, in *1936* Maltsev 'proved' the theorem without ever stating it, while in *1941* he stated it without proof!

The balance of *1941* is devoted to applications to group theory, mostly corollaries of

> Theorem 1: Given a finite number $E_1, E_2, \ldots, E_k$ of elementary and hereditary properties of groups, a group $\mathbb{G}$ is said to be of type $[E_1, E_2, \ldots, E_k]$ iff it possesses a normal series of length k in which each factor group $\mathbb{G}_{i-1}/\mathbb{G}_i$ has property E_i. A necessary and sufficient condition that $\mathbb{G}$ be of type $[E_1, E_2, \ldots, E_k]$ is that every finitely generated subgroup of $\mathbb{G}$ be so.

By specializing to particular properties, $E_1, E_2, \ldots, E_k$, Maltsev extended two results of Černikov, proving that

(i) A group $\mathbb{G}$ has a solvable normal series of length k iff each of its finitely generated subgroups does;

and (ii) Locally special groups are direct products of their Sylow subgroups.

Similarly, he showed that a group $\mathbb{G}$ has an abelian normal divisor of index n iff each of its finitely generated subgroups does, and that if every finitely generated subgroup has a Sylow sequence, so does $\mathbb{G}$. As a corollary to the first of those facts, Maltsev also obtained Schur's theorem that every periodic group of matrices over a field of characteristic 0 has an abelian normal divisor of finite index.

7 As one who does not read Russian, I have had to rely on the English translation in Maltsev *1971* for the text of *1941*. Anyone who does so, however, should be mindful of the translator's *caveats* (p. vii) that the language has been translated 'freely' and that 'certain terms, such as [. . .] *compactness* [. . . and] *model* are used [. . .] before they appear in the Russian'. It is also difficult to distinguish Maltsev's original footnotes from the translator's paraphrases and commentary.

The proof of Theorem 1 displays Maltsev's sophistication in applying compactness to a set of axioms involving additional constant symbols;[8] yet his cavalier reference to the 1936 paper, without mention of Rosser's criticism of it, left the correctness of his results in doubt and, to some at least, has even suggested subterfuge. However, Maltsev may very well have been unaware of Rosser's review.[9] If so, he may have considered his proofs in *1936* to be unobjectionable; and since he *had* given a precise formulation and proof there of the compactness of propositional logic, together with arguments intended to justify the application of that theorem in first-order contexts, he may simply have forgotten that he had never actually *stated* the extended result.

Of course, since the compactness theorem does hold for uncountable languages, the results Maltsev obtained from it were in fact correct; they were also significant, and, as noted above, some of them were already well-known. Moreover, in the interim between *1936* and *1941* Maltsev had also obtained some fundamental results about 'local' properties—for example, that a group is linear of degree n over a field iff its finitely generated subgroups are—by purely algebraic means. He published those results in *1940*, a paper that, according to Sabbagh *1991*, 'does not contain a single word of logic' and that Maltsev 'never referred to [. . .] in his later model-theoretic papers'. (*1940* is neither included nor cited in the collection Maltsev *1971*.)

Taken together, the evidence suggests to me that Maltsev did not at first realize the full significance of what he had accomplished in *1936*. From *our* perspective, his arguments there appear to be an attempt to generalize compactness to arbitrary first-order languages. But Maltsev did not *then* claim to have done so; his aims at that time were much more limited. Only later, it seems—*after* he had obtained many similar results *without* appeal to compactness arguments—did he come to believe that he had established a truly *general* 'local theorem' in *1936*.

The parallel with Skolem—who 'essentially' proved completeness and compactness for countable languages without realizing he had done so—is striking. So, too, is the sloppiness common to both men's use of language. The question, in both cases, is: To what extent did the ambiguity in their terminology bespeak lack of understanding on their part?

3. Recapitulation: Henkin's proof and applications

A clear statement and proof of the compactness theorem for uncountable languages finally appeared in 1947, in the Princeton doctoral dissertation of Leon Henkin. That work, in which Henkin *1947* employed the so-called 'method of

8 Even in *1936*, Maltsev had constructed a universal set *U* syntactically, by substituting the elements of 'a set *B* of arbitrary objects . . . *viewed as individual constant symbols*' (my emphasis) for the variables in the elementary (atomic) predicates and their negations that belonged to a system S of first-order formulas. He called *B* a 'domain for S' if all the formulas of S could be satisfied in *B* by a suitable interpretation of the predicate symbols, and he noted that if *B* were a domain for S under some such interpretation, a corresponding 'complete and consistent configuration' (subset of *U*) could be obtained 'by taking the elementary predicates from S and their negations with those values of the variables that make them true' (Maltsev *1971*, 4–5). The method is essentially that of diagrams.

9 I do not know whether Maltsev read English nor, if so, whether Rosser's review would have been accessible to him. In any case, Maltsev *1936* was not reviewed in the German-language *Zentralblatt für Mathematik und ihre Grenzgebiete*, probably because of the turmoil and Nazi interference in its editorship at that time.

constants' to prove the completeness theorem for languages of arbitrary cardinality, is of course one of the cornerstones of modern logic.

The basic steps in Henkin's construction are well-known: Given a consistent set S of sentences in a first-order language L, expand L to a new language L' by adding as many new constant symbols as there are symbols in L. Add to the logical axioms of L' the set H of 'Henkin axioms' $(\exists x)\varphi_i(x) \rightarrow \varphi_i(b_i)$, where φ_i is the i^{th} formula (in some enumeration of all the formulas) of L' with one free variable and b_i is a constant, not appearing in any earlier axiom, chosen to witness the (possible) truth of $(\exists x)\varphi_i(x)$. Extend the set $S \cup H$ to a maximal consistent set T of sentences of L' ('Lindenbaum's lemma', first published in Tarski *1930*[10]) and construct a model for T whose domain consists of all terms of L'.

As Gödel had done in *his* dissertation, Henkin employed a language that includes propositional as well as individual variables; and, at first, he too worked in the context of a denumerable language. He obtained compactness for such languages on p. 18, as a corollary to the completeness theorem. The statement reads: 'A set of w.f.f.'s is simultaneously satisfiable if and only if every finite subset is simultaneously satisfiable'. Uncountable languages are introduced on p. 21, completeness for them is derived as Theorem II (p. 22), and compactness is once again stated as a corollary. Henkin cites Gödel *1930*, but does not mention Maltsev's work, of which he was then completely unaware.[11] He did, however, apply compactness, in the spirit of Maltsev, to obtain several important results in algebra, among them the Stone Representation Theorem for Boolean rings (p. 30, first proved in Stone *1936*), the fact that having characteristic zero is not an elementary property of a field (Corollary II, p. 35), and the implication that there must exist a field of characteristic zero having a given elementary property whenever there exist fields with that property of arbitrarily large prime characteristic (Theorem V, p. 32). Later, in part IV of the dissertation, Henkin also discussed non-standard models of arithmetic.

Henkin published the results of his dissertation in three separate papers (*1949b, 1950*, and *1953*). In the opening paragraph of the first of them he stated that his new method of proof possessed two advantages, the first being that 'an important property of formal systems [. . .] associated with completeness can now be generalized to systems containing a non-denumerable infinity of primitive symbols'. And, as if to explain why (as he seems to have thought) no one had made that generalization before, he went on to remark that 'While this is not of especial interest when formal systems are considered as *logics* [. . .] it leads to interesting applications in the field of abstract algebra, [a] matter [. . . to] be taken up in future papers'.

One would expect the 'important property' here to refer to compactness; but, surprisingly, that is *not* the case. Indeed, compactness as such is never mentioned in *1949b*! The main result of that paper (p. 162, designated simply as 'Theorem') is rather a hybrid of the Skolem-Gödel theorem and the downward Löwenheim-Skolem theorem. It reads: 'If Λ is a set of formulas of [a system] S_0 in which no member has any occurrence of a free individual variable, and if Λ is consistent, then Λ is simultaneously satisfiable in a domain of individuals having the same cardinal number as the set of primitive symbols of S_0'. Completeness and the (pure)

10 Two years after Tarski's work, in a little-known paper *1932*, Gödel in effect extended Lindenbaum's result to uncountable propositional calculi (see Vaught *1974*, 163; or Gödel *1986*, 238–239).

11 Compare footnote 13 below.

Löwenheim-Skolem theorem follow as corollaries on p. 164. The paper concludes with a single application: 'An axiom set which has models of arbitrarily large finite cardinality must also possess an infinite model'. The proof is in keeping with the 'Skolem-Gödel' perspective of the paper: Having made explicit (p. 161) that he is using 'consistent' in the syntactic sense (as distinct from 'satisfiable'), Henkin argues that 'Since by hypothesis any finite number of the statements C_i

$$[(\exists x_1)(\exists x_2)\cdots(\exists x_i).\ \sim(x_1 = x_2) \wedge \ldots \wedge \sim(x_{i-1} = x_i)]$$

are simultaneously satisfiable they are consistent. Hence all the C_i are consistent and so simultaneously satisfiable'.

What is remarkable here is Henkin's 'retreat' to the use of a syntactic argument, *after* having clearly stated compactness in purely semantic terms in his dissertation. In that he was by no means alone. Indeed, the resort to syntactic methods in applications of compactness recurs so often that I shall refer to the phenomenon as 'the syntactic detour'.[12]

As he had promised in *1949b*, Henkin gave many more applications in his subsequent papers. In *1953*, a paper addressed to mathematicians unfamiliar with formal logic, he stated his Skolem-Gödel hybrid in essentially the same form as in *1949b*, but went on to note that 'Actually, we shall only have to use the following corollary', another hybrid form that he referred to thereafter simply as '*our basic result from logic*':

> *Let Γ be any set of sentences of L. Suppose every finite subset of Γ is satisfiable. Then there exist interpretations satisfying Γ in which the domain D has cardinality at most equal to that of the set of symbols of L.*

After proving the corollary, Henkin *did* then give purely semantic arguments for all of the applications he presented. But in the meantime much had happened.

4. Stretto, in four voices: Henkin, Tarski, Robinson, and Maltsev

A succinct summary of developments during the period 1946–1952 was given by Henkin himself, in a historical note, dated 15 December 1952, that was appended to his *1953* as footnote 34:

> Since this paper was written two works have appeared which overlap it in content: [Tarski *1952* and Robinson *1951* . . .]
>
> There is no doubt that credit for first envisaging the possibility of applications of 'our basic result from logic' is due to Tarski. Related ideas appear in his papers as early as 1931, and an announcement of the results was made in 1946 to the Princeton Bicentennial Conference, although without a description of methods. [. . .] A full account appears in the above-mentioned paper presented at the 1950 International Congress [. . .]. A further development of the theory was described

12 The seeming reluctance to apply compactness directly, without resort to the syntactic detour, calls to mind Gödel's remarks (in correspondence preserved in his *Nachlass*) that there was for long a prevailing 'prejudice, or whatever you may call it' among logicians against the use of semantic arguments. Indeed, the phenomenon may be observed in some well-known logic texts. A classic example is Mendelson's *Introduction to mathematical logic*, from which I myself first learned logic. The compactness theorem is absent from the first edition (*1964*) and is given scant attention even in the third. Yet many of the standard applications are there (see *1964*, 96–97), all proved via the syntactic detour.

by Tarski at the Colloquium Lectures of the American Mathematical Society in September, 1952.

The possibility of applying 'our basic result from logic' to problems of algebra was rediscovered by me in the Spring of 1947. Aside from detailed applications, two new basic ideas were added to those which had been developed by Tarski. One was the use of individual constants [. . .]. The other was the generalization to higher-order languages [. . .].

Finally, in the period September 1947–April 1949, the basic ideas were once again found in independent work by Abraham Robinson [for his dissertation at Birkbeck College, University of London, 1949, on which his *1951* is based]. Using both the 'basic result from logic' and the technique of individual constants, he obtained some of our results relating to the characteristic of models in an arithmetical class of commutative fields, adding important new examples. He also obtained further results in this direction, including the important theorem that an arithmetical class containing one algebraically closed field contains all others of the same characteristic (which was obtained by Tarski in a very different way). Results on non-Archimedean ordered fields also appear in Robinson's work, and further original material having no counterpart in our work.

Henkin's account is of interest for what it says about the relationships among his own work and that of Tarski and Robinson, but it also reflects the limits of Henkin's awareness at that time. For example, there is still no mention of Maltsev's applications; nor, despite the reference to Tarski's 1950 ICM address, is there any indication that Robinson *also* delivered an address there (Robinson *1952*).[13]

Conversely, we may ask, how aware were Robinson, Tarski, and Maltsev of each other's and of Henkin's work? In the published version of Robinson's ICM address *1952* there is no bibliography. In the text proper, Robinson remarks in passing that 'we may mention the names of K. Gödel, L. Henkin, and A. Tarski as representative of those who either directly or indirectly contributed towards the establishment of symbolic logic as an effective tool in mathematical research'; again, there is no mention of Maltsev, nor any specific citation of Gödel's, Henkin's or Tarski's works. Robinson does acknowledge that 'it is understood that the theorem on algebraically closed fields which is proved below [that all such fields of characteristic zero are elementarily equivalent] and which was stated by the present

13 In an earlier footnote, Henkin did acknowledge Maltsev *1936*, which he had apparently not yet examined with a critical eye. (In particular, he stated that 'A. Mal'cev [. . .] (1936) [. . .] showed that Gödel's proof could be extended to cover systems in which a nondenumerable number of constants were admitted. A *more direct* proof [sic—my emphasis] of the general result can be found in L. Henkin [*1949b*]'. But there is no reference to Maltsev *1941*, and indeed, in a recent conversation with the author, Henkin recalled that he first heard of Maltsev from Tarski, at about the time that he composed his historical note.

As to Henkin's ignorance of Robinson's address, two circumstances should be noted. First, Robinson was invited to speak at the last minute, after he sent the Congress organizers an abstract of his dissertation on 'The metamathematics of algebraic systems' (see Seligman *1979*, xxi). Second, even had Henkin known of Robinson's talk beforehand, he probably would not have been present to hear it; for at the time of the Congress he was on his way to Montreal to be married. (He stopped by for three hours en route.) No doubt he became aware of Robinson's work only with the publication of Robinson *1951*.

author in 1948[14] [. . .] has also been found independently by Professor Tarski'. However, neither in his 1950 address nor his 1951 book did Robinson reveal any awareness of Tarski's prior construction of non-Archimedean ordered fields. As to Tarski, *his* ICM address cites Maltsev *1936* (but not *1941*), as well as Henkin's dissertation and *1949b*: there is no mention of Robinson. Finally, when Maltsev returned once more to applications of compactness in his *1956*, he noted that his *1941* had fallen into obscurity and that 'several years ago the possibility of applying local theorems of mathematical logic to algebra was re-discovered by a series of authors [he cites Henkin *1953*], although the application to ordered groups and the above [compactness] theorem (but not the local theorems on solvable groups in [Maltsev *1941*]) were only recently rediscovered by B. Neumann [*1954*] and A. Robinson [*1955*]'.

It is strange that Maltsev did not note the appearance of the compactness theorem in Henkin *1953*. Also, Neumann's *1954* is purely algebraic; it contains no mention of logic whatever. Indeed, it is just the point of Robinson *1955* to show that Neumann's results can be obtained much more easily as a consequence of compactness.[15] But Robinson *1955* is also of interest for the present study for quite another reason, since only there did Robinson finally state (purely semantic) compactness. In all of his previous publications on the subject (including *1951*, *1952*, and *1953*) the applications had been obtained via the syntactic detour[16] from the Skolem-Gödel theorem (which he called the 'extended completeness theorem'). Remarkably, Robinson had also stated and proved the *strong* completeness theorem (that every logical consequence of a set S of first-order sentences is logically deducible from S) in his *1951* (apparently its first appearance in the literature; compare Corcoran *1969*, 177); yet, even so, he had failed to derive compactness as a corollary.

Nowhere does there appear to be any dispute among the principals involved as to the relative chronology of events or the independent nature of their discovery of certain basic applications of compactness. Except for its omission of Maltsev's contributions, Henkin's historical note quoted at the head of this section appears accurate in its apportionment of credit for the various results. It is also accurate, I believe, in placing Tarski's early contributions in proper methodological perspective. Of particular note is Henkin's reference to '*related* ideas' in Tarski's papers 'as early as 1931' and to Tarski's 'announcement of *results*' at the Princeton Bicentennial Conference '*without* a description of *methods*' (my emphases).

Similar remarks occur earlier in Henkin *1953*, especially in footnotes 11, 19, 20, and 25, where two of Tarski's early papers (*1931* and *1936*) are explicitly cited. Those papers were concerned, respectively, with the definability of real numbers and of dense orderings. In *1931* Tarski introduced the notion of an 'arithmetical class', upon which he was to discourse further in 1946 (at the Princeton Bicentennial

14 Robinson here refers to the abstract *1949* of his contributed paper 'On the metamathematics of algebra', which he presented to the eleventh annual meeting of the Association for Symbolic Logic, held on December 30–31 1948 at Ohio State University. It should be noted that 1948 was also the year in which Tarski's proof was finally published, long after its discovery (Vaught *1974*, 160).

15 The final sentence of Robinson *1955* states: 'The present paper arose out of a conversation between B. H. Neumann and the author, which took place during the Amsterdam Congress [of 1954]'.

16 *Contrary* to the statements in Henkin's reviews *1952* and *1955* of Robinson *1951* and *1953* in the *Journal of symbolic logic*, by this time he was apparently so accustomed to the semantic point of view that he failed to notice that it was foreign to Robinson's treatments.

Conference), 1949 (in a series of abstracts published in the *Bulletin of the American Mathematical Society*[17]), and 1950 (in his ICM remarks), while in *1936* he obtained, among others, the result that the notion of 'well-ordering [. . .] is not expressible in the elementary theory of ordering' (p. 301). But there is no hint of a compactness argument in those papers: to obtain the result just cited, Tarski employed the method of elimination of quantifiers, due originally to Langford.[18] He used the same method in establishing that algebraically closed fields of the same characteristic are elementarily equivalent. (It is this to which Henkin alluded in the parenthetical remark in the penultimate sentence of his historical note.)

In his 1950 address, Tarski *did* explicitly state compactness, in several different (and quite new) forms (Theorems 13, 17, 19, and 24 of Tarski *1952*). We turn to them in the next section. But, contrary to the implication in footnote 18 of Tarski *1952* (and in agreement with Henkin's remarks, both in his *1953* and his review *1955*[19] of Robinson *1951*), it does *not* appear that Tarski made any mention of compactness arguments in his (still unpublished) Princeton Bicentennial remarks; none, at least, is to be found in the extant typescript of those remarks preserved in his *Nachlass*.[20]

5. Variations in a new key: the topological connection

Though Maltsev, Henkin, Tarski and Robinson shared an interest in applying logic to algebra and, in pursuing that interest, obtained many of the same results, Tarski's treatment of the subject in *1952* nonetheless marks a significant departure from all that had gone before. The difference is signaled by his very terminology: for there Henkin's 'basic result from logic', which Gödel and Robinson had left unnamed and Maltsev had called the 'general local theorem', is at last referred to as 'the compactness theorem'.

It may seem that Maltsev's notion of 'localization' differs but little from the concept of 'compactness', insofar as both terms suggest the idea of reducing an infinite set of mathematical entities to a finite subset. This somewhat naive view is perpetuated by most logic textbooks, few of which say anything at all about the origin of the term 'compactness'. (Presumably, their authors consider the usage 'obvious'.) But the word 'compactness' is fraught with much deeper, specifically *topological* connotations; and Tarski, well-grounded in the Polish tradition of analysis and descriptive set theory, undoubtedly had such connotations in mind, as his formulations and interpretations of the compactness theorem make clear.

Had the term 'compactness' been transferred to logic by simple analogy with the sort of 'finite reducibility' arguments invoked in point-set topology, it would seem odd that the logical usage did not develop long before 1950. But if we compare how compactness arguments are actually *used* in topology and logic, we observe a crucial difference. In topology, compactness is employed in a 'top-down' fashion: an infinite collection of open sets that *obviously* covers (say) a closed interval is

17 Vol. 55, pp. 63–65, Abstracts 74–78. One suspects that this flurry of abstracts may have been prompted in part by Tarski's concern for priority, once he became aware of Henkin's work. To judge from the unusually large number of errata to them published on p. 1192 of that same volume, they appear to have been composed in some haste.

18 Cf. the remarks by Vaught, in Gödel *1986*, 377.

19 Cited above in footnote 15.

20 I am grateful to Steve Givant of Mills College for allowing me to examine a copy of that typescript.

reduced, non-constructively, to a *particular* finite subcover whose structure is (usually) unknown. In logic, by contrast, the approach is 'bottom-up': the satisfiability of an infinite set of sentences, which is initially in *doubt*, is reduced to the satisfiability of *each* of its finite subsets, which is (generally) readily apparent.

The deeper reasons for adopting the name 'compactness' in logic are related to two distinct aspects of its use in topology:

(i) The idea of a compact *space* (one which has the finite intersection property: if the intersection of a family of closed sets is empty, so is the intersection of some finite subfamily);

and (ii) Tychonoff's product theorem (stated below).

The former is relevant to the *interpretation* of the compactness theorem, the latter to its *proof*. In order better to understand those connections, we digress briefly to review how the concept of compactness arose and developed in topology.[21]

In 1898, Emile Borel had proved the countable version of what has since become known as the Heine-Borel Theorem:

> Given a denumerably infinite collection of subintervals of a bounded segment of the real line, if every point of the segment lies within at least one of the subintervals, there is a finite subcollection of those subintervals having the same property.

Borel's result was extended to uncountable collections of intervals by Lebesgue in 1904.

The same year, in an address to the Paris Academy of Sciences, Maurice Fréchet proposed a generalized notion of limit, to wit:

> Given a collection C of distinguishable mathematical entities, let F be a function, defined on denumerable sequences from C, such that
>
> (i) $F(c, c, \ldots) = c$ for all c in C, and
> (ii) if $F(c_1, c_2, \ldots) = c$, then every subsequence of $c_1, c_2, \ldots$ yields the same F-value.
>
> The entity c is called a limit point of C if c occurs as a value of F.

In terms of this definition, he called 'closed' any set containing all of its limit points, and 'compact' any set for which the intersection of any decreasing sequence of non-empty closed subsets is itself non-empty; he then showed that a set is compact iff every infinite set of distinct elements from it has at least one limit point. A set E that was both compact and closed Fréchet called 'extremal', and he proved that E was extremal iff every (even uncountable) covering of it is reducible to a finite subcovering.

Later, in a paper submitted in 1923 but not published until 1929, Alexandroff and Urysohn generalized Fréchet's definition to arbitrary topological spaces by eliminating reference to denumerable sequences. They *defined* a space to be 'bicompact' if every open covering possesses a finite subcovering. Eventually this property came to be regarded as primary, and the prefix 'bi-' was dropped. In 1930 Andrei Tychonoff proved that the product of any number of copies of the interval

21 For a detailed history of compactness in topology the reader may refer to Pier *1980* or to section 4.6 of Moore *1982*. The overview presented here is based on those sources.

[0,1] is (bi) compact, and seven years later Eduard Čech extended the result to the product of arbitrary compact spaces (the form in which the 'Tychonoff' theorem is known today). Also in 1937, Henri Cartan *1937b* defined the notions of 'filter' and 'ultrafilter', and Chevalley and Weil pointed out (see Pier *1980*, 439) that to say a topological space is compact is to say that every filter on it has at least one accumulation point (an equivalence derived by recasting Borel's finite reducibility property as the finite intersection property).

In this same period, in a series of abstracts and papers extending from 1933 to 1937, Marshall Stone explored Boolean algebras from a topological point of view, concluding that 'the theory of Boolean algebras is co-extensive with the theory of totally disconnected bicompact [Hausdorff] spaces' (Stone *1934*, 198; proofs of his results were provided in his *1936* and *1937*). He showed how to associate a topological space to the Boolean algebra of propositions in a formalized theory, and he and Garrett Birkhoff, independently and almost simultaneously, proved forms of the isomorphism theorem now known as the Stone Representation Theorem. Stone's ideas were also taken up in Mostowski *1937*.

The notion of compactness in topology thus underwent its own, somewhat parallel evolution, culminating at about the time of Gödel's and Maltsev's work. Moreover, 'Birkhoff observed [in 1940] that his representation theorem for distributive lattices had been inspired by the researches of Tarski [. . .] on abstract measure theory' (Moore *1982*, 230), and J. L. Kelley proved the equivalence of the Tychonoff theorem with the axiom of choice[22] in the same year that Tarski delivered his ICM address.

If we consider the details of Tarski's address, we find the topological perspective to be pervasive. Tarski is concerned with algebras (structures of a given similarity type) and their arithmetical properties (those that are expressible by formulas of a first-order language). He notes that to every formula Φ of such a language may be correlated a function $\mathfrak{F}$ whose domain is the totality of algebras of the corresponding similarity type and whose value at an algebra $\mathfrak{A}$ (= $\langle A, \ldots \rangle$) is the set of all sequences from A^{ω} that satisfy Φ. Any such function $\mathfrak{F}$ is called an 'arithmetical function', and the corresponding set of structures, $\mathfrak{CL}(\mathfrak{F})$ (= $\{\mathfrak{A} \mid \mathfrak{F}(\mathfrak{A}) = A^{\omega}\}$, the set of algebras whose sequences of elements all satisfy Φ), is called an 'arithmetical class'. Using AF and AC to denote the (proper) classes of arithmetical functions and classes, respectively, and $\mathfrak{z}$ to denote the arithmetical function whose range is $\varnothing$, Tarski goes on to state 'the *compactness theorem for arithmetical functions*' (Theorem 13)—

$$\text{If } \mathrm{K} \subseteq \mathrm{AF} \text{ and } \bigcap_{\mathfrak{F}\varepsilon \mathrm{K}} \mathfrak{F} = \mathfrak{z}, \text{ then } \bigcap_{\mathfrak{F}\varepsilon \mathrm{L}} \mathfrak{F} = \mathfrak{z} \text{ for some finite } \mathrm{L} \subseteq \mathrm{K}$$

—and 'the *compactness theorem for arithmetical classes*' (Theorem 17) (as above, with AF, $\mathfrak{F}$, and $\mathfrak{z}$ replaced, respectively, by AC, $\mathfrak{X}$, and $\varnothing$). By analogy with the Borel hierarchy, Tarski defines the classes AC_{σ}, AC_{δ}, $\mathrm{AC}_{\sigma\delta}$, and so on, and then notes that 'by means of . . . Stone's representation theory for Boolean algebras, the theory of arithmetical classes requires a topological interpretation'. Specifically, if 'with every set $\mathfrak{X}$ . . . we correlate the set $\mathfrak{C}(\mathfrak{X})$ defined by the formula $\mathfrak{C}(\mathfrak{X}) = \bigcap (\mathfrak{Y} \mid$

22 This chapter of the story is, however, not yet closed. In a recent paper Howard *1990* considers seven different definitions of compact that have appeared over the years in the topological literature and asks whether the equivalance of the Tychonoff theorem with the axiom of choice depends on the particular definition of compact selected. (Kelley's proof used the definition of compact couched in terms of the finite intersection property.) Despite some partial results, the question remains open.

$\mathfrak{Y} \subseteq \mathfrak{X}\ \varepsilon$ AC) [. . . then we obtain] a topological space with $\mathfrak{C}$ as closure operation'. (The space so obtained is not T_1, but can be converted into one that is 'by "identifying" two points $\mathfrak{A}$, $\mathfrak{B}$ . . . in case $\mathfrak{C}(\{\mathfrak{A}\}) = \mathfrak{C}(\{\mathfrak{B}\})$'.) Then '$AC_\delta$ and AC_σ are respectively the families of all closed and all open sets' in this 'arithmetical space', which 'is easily seen to be totally disconnected, [. . .] separable and by Theorem 17 [. . .] bicompact. (Thus, the bicompactness of the space [. . .] is a consequence of the completeness of elementary logic)'.[23]

Two points here are of special note. First, Tarski uses 'compactness' for the logical notion, but 'bicompactness' for its topological counterpart. Second, for Tarski the compactness theorem asserts the (bi)compactness of a space of *structures*. In contrast, in his article in the *Handbook of mathematical logic*, Keisler *1977* takes the set S of all *complete theories* (maximal consistent sets of formulas) in a first-order language L to constitute the points of the Stone space of L, and, for each sentence φ of L, defines the basic closed set $[\varphi]$ as $\{p\ \varepsilon\ S \mid \varphi\ \varepsilon\ p\}$. With this topology, S is readily shown to be a totally disconnected Hausdorff space, and Keisler observes that since 'a theory T is satisfiable just in case $\bigcap_{\varphi \varepsilon T} [\varphi] \neq \emptyset$ [. . .] the compactness theorem states that if every finite subset of a set of basic closed sets [. . .] has a non-empty intersection, [. . . so does] the whole set'. Thus, in choosing what to topologize, Tarski opts for *semantic* entities, Keisler for *syntactic* ones.[24]

To sum up the preceding discussion: Though—most surprisingly in view of his long-standing interest in applying logical methods to mathematical problems—Tarski recognized the power and utility of the compactness theorem no sooner than his contemporaries, when he finally did so it was with full understanding of the deeper (topological) significance of the result.

Tarski did not, however, provide a topological *proof* of either of his versions of the compactness theorem. He remarked that 'a mathematical proof of Theorem 13 is rather involved', and derived it instead from Gödel's completeness theorem. Nevertheless, topological proofs were not long in coming. For the propositional calculus, a proof of compactness based on Tychonoff's theorem was given by David Gale. Though apparently never published, it was mentioned by Henkin in a footnote to his *1949* and was probably very similar, if not identical, to the proofs

23 As noted earlier, few logic texts bother to explain the topological context of the compactness theorem at all. One that does is Monk *1976*, where, in exercise 11.59 (p. 218), we find:

> Let K be a nonempty set of L-structures. For each $L \subseteq K$ let $CL = \{\mathfrak{A}\ \varepsilon\ K : \mathfrak{A}$ is a model of every sentence which holds in all members of L$\}$. Show that with respect to C as a closure operator, K is a compact topological space.

The notation mimics that of Tarski, suggesting that Monk had Tarski's interpretation in mind. But, interestingly, Monk's statement is incorrect. For example, if K is taken to consist of all *finite* substructures of $\langle \mathbb{N}, < \rangle$ and if, for each $n < \omega$, φ_n denotes the sentence

$$\exists x_1 \ldots \exists x_n (x_1<x_2 \wedge x_2<x_3 \wedge \ldots \wedge x_{n-1}<x_n),$$

then each set $L(\varphi_n) = \{\mathfrak{A}\ \varepsilon\ K : \mathfrak{A} \models \varphi_n\}$ is closed, and every *finite* intersection of them is non-empty; but the intersection of all of them clearly is empty. Presumably, Monk intended to restrict K—for example, to classes that are elementary, or perhaps AC_δ.

24 Keisler has informed me that he adopted the syntactic approach 'in order to avoid the side issue that an elementary equivalence class is a proper class'. On the other hand, Solomon Feferman, who helped to draft Tarski's 1950 address, has commented that Tarski's aim in choosing a *non*-syntactic approach was to make the subject more attractive to mathematicians who were not logicians.

given in Barwise *1977*, 27–28, or Kreisel and Krivine *1971*, 8.[25] Already in 1948, in his address at the Tenth International Congress of Philosophy, Mostowski had broached the possibility of giving a topological proof of Gödel's completeness theorem[26]. Two years later, a proof of Gödel's theorem in Boolean-algebraic terms, in which a key lemma was proved using a Baire category argument, was given in Rasiowa and Sikorski *1950*. Then Rasiowa *1952* applied the same methods to prove Tarski's compactness theorem for arithmetical classes. In a footnote to that paper, she also sketched modifications necessary to prove the compactness theorem for arithmetical functions. Finally, Beth *1954* gave topological proofs of the Skolem-Gödel, Löwenheim-Skolem, and compactness theorems. A capstone to such efforts was provided that same year in Henkin *1954*, in which the compactness theorem for first-order logic was shown to be logically equivalent to the Boolean Prime Ideal Theorem (see Moore *1982*, 297).

6. Da Capo: Maltsev rediscovered

By the mid-1950s, the importance of the compactness theorem had at last become widely recognized among logicians, though it had yet to receive much notice in textbooks.[27] But Maltsev *1941* remained unknown in the West until his own allusion to it (quoted above) in his *1956*. Both papers were finally reviewed together in the *Journal of symbolic logic* in 1959—*18 years* after *1941* appeared.

25 It may seem strange that it took so long to discover this proof, since the idea is to observe that the space $\{0,1\}^P$ is compact (where P is the set of prime propositional formulas), and Tychonoff had shown that the product of any number of copies of the *interval* [0,1] is compact. But it must be remembered that topologists and logicians were often unacquainted with each other's work (there has never been a continuous tradition of applying topological methods in logic, despite some significant instances, notably in stability theory); and, from a topologist's point of view, discrete spaces such as $\{0,1\}$ are relatively uninteresting.

26 Mostowski *1949*. Long before that, in his *1937* (p. 35), he had stressed the 'interesting and quite unexpected connections' that obtained between such seemingly 'distant' fields as metamathematics and topology. He noted (p. 37) that to each deductive theory a corresponding Boolean field could be associated, so that every theorem about Boolean fields could be construed as a metamathematical theorem *and conversely*; and he cited Stone's work as evidence that topological methods could in fact lead 'more easily and quickly' to proofs of some metamathematical results (p. 35).

27 Two classic texts of that era are Kleene's *Introduction to metamathematics* (*1952*) and Church's *Introduction to mathematical logic* (*1956*). In the former, Kleene states the Skolem-Gödel theorem in the form (p. 397):

> Given an enumerably infinite (or finite) class of predicate letter formulas $F_0, F_1, F_2, \ldots$, if every conjunction of a finite number of them is irrefutable in the predicate calculus, then they are jointly satisfiable in the domain of the natural numbers . . . ,

On the following page, Kleene states as Corollary 2:

> If $F_0, F_1, F_2, \ldots$ are jointly satisfiable in some non-empty domain (or even if each conjunction of a finite number of $F_0, F_1, F_2, \ldots$ is satisfiable in a respective non-empty domain (Gödel 1930)), then $F_0, F_1, F_2, \ldots$ are jointly satisfiable in the domain of the natural numbers.

Note, in both statements, the restriction to the enumerable case. Maltsev is not mentioned, and the term 'compactness' is not used.

In Church's *1956*, 244, the Skolem-Gödel theorem is expressed as a pair of propositions:

> **453. Every consistent class of wffs is simultaneously satisfiable in an enumerably infinite domain. [. . .]
>
> **454. Every simultaneously satisfiable class of wffs is consistent.

There is no mention of compactness *per se*.

A third early text, Curry *1963*, does not even prove completeness (though it is discussed), because the proof is non-constructive.

Vaught *1974* regards the neglect of Maltsev's work as 'doubtlessly' due to the war—not just World War II, presumably, but the ensuing Cold War as well—but there was surely more to it than that. Even today there are few in the West who read Russian, and Maltsev's *1941* appeared in an obscure journal not among those regularly translated by the American Mathematical Society.[28] A few logicians might have turned away from Maltsev's work after reading Rosser's review *1937* of his *1936*, but many later authors have credited the proofs there, without qualification. It is also possible that disdain for the use of non-constructive methods, especially the axiom of choice, may have been a factor (even after Gödel's relative consistency proof for that axiom).

In any case, in 1959 Maltsev *1941* finally received the critical attention it deserved, in the review *1959* by Henkin and Mostowski. After describing the content of *1956* and noting that it was 'written rather carelessly', they devoted the remainder of the review to a 'historical note' concerning the details of *1941*. They noted Maltsev's proof in *1936* of propositional compactness, as well as his failure to 'explicitly formulate' there the corresponding result for non-denumerable sets of first-order sentences. But at the same time they acknowledged that Maltsev had 'discusse[d] the relation between first-order sentences and certain associated formulas of [the] propositional calculus' and that on the basis of that discussion 'it is possible to infer the desired theorem from [the propositional] result', even though 'the correctness of [Maltsev's own] proofs has been put in doubt by his use of Skolem normal forms'.

They noted that 'the nub of the matter' was Maltsev's claim that every first-order sentence is *equivalent* to one in Skolem normal form, which, if interpreted as *logical* equivalence is incorrect. But they observed further that 'a careful reading of [the 1936] paper shows that he several times uses the phrase that two formulas are "equivalent", and at one point states explicitly "in the sense that [. . .] the satisfiability of [either] one of the[m] follows [from that] of the other"'—under which interpretation his claim 'becomes a correct statement of Skolem's theorem'. However, they continued, 'to carry through a correct proof of 'the general local theorem' by Mal'cev's method appears to us to require something *stronger*', namely the fact that

> Any model which satisfies $\mathfrak{A}'$ [the Skolem formula associated to $\mathfrak{A}$] also satisfies $\mathfrak{A}$; and conversely, in any model which satisfies $\mathfrak{A}$ we can define additional sets and relations, corresponding to the new symbols $R_1, R_2, \ldots$ [adjoined in forming $\mathfrak{A}'$], so as to obtain a model satisfying $\mathfrak{A}'$.

This strengthened form of Skolem's theorem, they found, had apparently not been 'formulated explicitly in the literature, although it can be discerned by a careful reading of the usual proofs of Skolem's theorem'.

On the whole, this joint review seems remarkably fair and incisive, especially in view of the sensitive issues of priority that were involved. It is justly critical of

28 In contrast, Maltsev's purely algebraic *1940* appeared in the prominent journal *Mathematicheskie Sbornik*. That this may not have been purely incidental has been suggested by Yuri Gurevich, who knew Maltsev personally. Specifically, in recent correspondence Gurevich has stated that during the Stalin era Maltsev turned away from logic toward algebra, because at that time logic was viewed as allied with positivism, and so opposed to dialectical materialism.

Maltsev's carelessness—though of course, with 18 years' hindsight—but nevertheless serves in large measure to *vindicate* his methods.[29] There is, however, a certain irony in Henkin and Mostowski's criticism of Maltsev. For in the final sentence of their review they state that since they have 'not been able to find any formulation of the "general local theorem" in [Maltsev *1936*], and since the proofs there suffer from certain gaps, the reader in quest of a satisfactory proof of th[at] theorem must be referred to the later proofs of Henkin [*1949a*] and A. Robinson [1951]'; but, as we have seen, Maltsev's 'general local theorem'—that is, purely *semantic* compactness for non-denumerable languages—is *also* not explicitly formulated in either of *those* sources![30]

On balance, we may agree with Keisler's assessment (in recent correspondence) that Maltsev was 'perhaps . . . guilty of the minor crime of lazy writing and of presuming too much about what is well known, rather than the major crime of making gross errors in proofs'. He notes that Maltsev in *1936* 'prove[d] the compactness theorem at least for sets of $\forall\exists$ sentences' and that 'the full compactness theorem is a *direct* consequence' of that and either the strong form of Skolem's theorem stated by Henkin and Mostowski or the weaker fact that

> If, for each sentence A, we denote by A′ its Skolem normal form, then for any set S of first-order sentences, S is satisfiable in a domain D if and only if the set $\{A' | A \varepsilon S\}$ is satisfiable in D. (*)

Keisler notes that the latter 'follows from Skolem's construction of the normal form, provided that one uses different extra predicates for different A's', and he suggests that Maltsev's use of the term 'well known' suggests that he may have considered (*) to be a sort of 'folk theorem'.

Finally, it should be noted that a proof very much along Maltsev's lines (but employing a reduction to *universal* formulas with added *function* symbols, rather than $\forall\exists$ formulas with new predicates) is given in Kreisel and Krivine *1971*, 21–25.

7. Obbligato: The method of ultraproducts

In a footnote to his *1953*, Henkin had remarked that 'it is curious that although [the compactness theorem] deals only with sentences and models used to interpret them, any proof of [it] seems to bring in references to formal deductive systems, with their paraphernalia of axioms and operations of formal inference. It is true that we know of a systematic way to eliminate such reference, but the resulting proof then appears highly artificial and cumbersome'. It is not clear just what 'systematic' method Henkin had in mind, but it *is* true that the Boolean and topological approaches require a certain amount of 'paraphernalia' of their own.

Today a more direct and elegant model-theoretical proof of compactness is known, based on the notion of 'ultraproduct'. As noted above in section 1, that concept derives ultimately from Skolem's construction of a non-standard model of arithmetic. It emerged somewhat sporadically during the same decades as the main

29 It is certainly *not* accurate to cite this review, as Moore does in his *1982*, 258, as evidence that Maltsev's 'carelessness [. . .] earned him the opprobrium of several later logicians'.

30 For his part, Robinson *1962* paid homage to Maltsev at length in his survey 'Recent developments in model theory', presented to the 1960 International Congress of Logic, Methodology and Philosophy of Science at Stanford.

line of development that we have been discussing, but its lineage is distinct and independent. I shall trace the highlights of that development here.[31]

The modern definition of ultraproduct, due to Frayne, Morel, Scott and Tarski (announced in 1958 in three joint abstracts of the *Notices* of the American Mathematical Society (pp. 673–675), and published in Frayne, Morel and Scott *1962*), is couched in terms of filters, a topological notion which, as we have already noted, was introduced in Henri Cartan *1937a*. Specifically, he defined a filter to be a non-empty family F of subsets of a set I such that $\emptyset \notin F$ and F is closed under supersets and finite intersections. He called a maximal such family an ultrafilter, and in his *1937b* he proved (using Zermelo's Well-Ordering Theorem) that every filter can be extended to an ultrafilter (see Moore *1982*, 240 for further details). Using this framework (though eschewing Cartan's terminology[32]), Frayne, Morel and Scott considered a set $\{\mathfrak{A}_i\}_{i \varepsilon I}$ of structures for a first-order language L, and, given a filter F on I, defined the 'reduced direct product' of the $\mathfrak{A}_i$ with respect to F to be the quotient structure $\prod_{i \varepsilon I} \mathfrak{A}_i/F$; that is, a relation was defined to hold at a point f in the domain $\prod_{i \varepsilon I} A_i/F$ of the product structure if and only if it held at f(i) in $\mathfrak{A}_i$ for 'almost all' $i \varepsilon I$. In the special case of an ultraproduct (when F was taken to be an ultrafilter), they noted that an equivalent definition (though 'in a quite different formulation' that now seems rather roundabout) had been given earlier by Łoś (*1955*), who had also stated (in effect)—but not proved—the fundamental theorem that a sentence of L holds in the ultraproduct iff it holds in almost all the $\mathfrak{A}_i$. Frayne, Morel and Scott *1962* provided a proof of Łoś's theorem and then applied it to give a very simple proof of the compactness theorem. (Given a finitely satisfiable set $S = \{\Phi_i | i \varepsilon I\}$ of sentences and a model $\mathfrak{A}_i$ of each Φ_i, the ultraproduct $\prod_{i \varepsilon I} \mathfrak{A}_i/F$ is a model of S whenever F is a ultrafilter extending each set $J_j = \{i \varepsilon I | \mathfrak{A}_i \models \Phi_j\}$.) In the introduction to their paper, Tarski is credited with having been the first to realize that ultraproducts could be used to prove compactness. In an appendix, his priority in recognizing the topological significance of compactness is also acknowledged, and the relationship between Boolean spaces (totally disconnected, compact Hausdorff spaces) and spaces with an 'ultralimit' operation is spelled out.

As Keisler has remarked, 'The beauty of the ultraproduct construction lies in the fact that it is defined using purely set-theoretical considerations which are independent of the particular logic or structures [. . .] involved' (*1965*, 117). That abstract character is likely also the reason, however, why ultraproducts were not discovered earlier. For as Vaught noted in his *1974*, the non-standard model of arithmetic that Skolem constructed in his *1934* was in fact 'the (countable, elementary) submodel of the ultrapower $(\omega,+,\cdot)/D$ formed by the functions definable in $(\omega,+,\cdot)$', where D denotes a non-principal ultrafilter on ω. But Skolem did not construct the full ultrapower nor identify D as having the properties of a filter, and '[his] exact construction can only be carried out on structures having a rich set of definable functions'. (For further discussion of Skolem's construction, see

31 For more detailed historical discussion, see Zygmunt *1973*; for extensive overviews of the theory and applications of ultraproducts, see Bell and Slomson *1974*, Keisler *1965*, Chang and Keisler *1973* (ch. 4), or Eklof *1977*.

32 In adopting the 'dual' terminology (filter = dual ideal, ultrafilter = prime dual ideal, ultraproduct = prime reduced direct product) the authors confess that 'the choice of terminology and notation has not been easy [. . .] and no doubt will not please everyone'.

Zygmunt *1973*, 60–64; or Scott *1961*, 245–246.) Similarly, what amounted to an ultraproduct construction in the context of fields had also appeared in Hewitt *1948*, but was only recognized as such much later.

8. Coda: Abstract logics and Lindström's characterization theorem

Ultraproducts can be used to obtain compactness results not only for first-order logic, but for certain extensions of it as well; for example, Fuhrken *1964* used ultraproducts to prove the countable compactness of logic with the cardinality quantifier 'there exist uncountable many' (an application for which they appear to be essential: see Kaufmann *1985* for further details).

The study of such 'generalized' quantifiers, sparked by Mostowski *1957*, marked a major turning-point in 20th century logic, in a direction that was, in some respects, a return to an earlier point of view. For as Moore has described in *1988*, logics more general than first-order logic, including some infinitary and second-order logics, were considered by several logicians prior to the 1920s, while first-order logic emerged only gradually as a focus of study. When it did so, however, first-order logic became entrenched for decades as *the* logic—no doubt in part because Gödel's proofs of completeness and compactness showed it to be a particularly tractable fragment for investigation.

The re-emergence of more general logics had first to await the ascendence of the model-theoretic point of view, which occurred only in the late 1950s and early 1960s.[33] Toward the end of the latter decade, several logicians attempted to extend the methods and results of first-order logic to languages with more complex syntax, and eventually it came to be realized that different logics could be studied from an abstract point of view that allowed their relative strengths to be compared. Specifically (compare Ebbinghaus *1985*), a logic L may be regarded as a pair $(\Sigma, \models)$, consisting of the set Σ of *sentences* of an underlying language together with a *satisfaction relation* between those sentences and corresponding *structures*. (To ensure that the logics are at least as expressive as first-order logic, certain closure conditions involving Σ and $\models$ are usually assumed.) Within this general framework, a class of structures is an *elementary class* with respect to L if it consists of all the structures for L that satisfy some particular $\sigma \in \Sigma$; and a logic L' is *as strong as* L, denoted $L \leqq L'$, if every elementary class with respect to L is also an elementary class with respect to L'.

The relation $\leqq$ gives a partial ordering of logics under which L and L' are regarded as *equivalent* ($L \equiv L'$) if both $L \leqq L'$ and $L' \leqq L$. In terms of that ordering, Lindström *1969* proved a fundamental *characterization theorem* that may be taken as the culmination of our story:

If L denotes first-order logic and L' is such that
(i) $L \leqq L'$,
(ii) the countable compactness theorem holds for L', and

33 The time may then have been particularly ripe for such a development, since, prior to the creation of complexity theory and Cohen's introduction of the method of forcing (both in the mid 1960s), recursion theory had become temporarily 'pot bound', as it were, as a result of its luxuriant (over?) growth, while set theory had exhausted its methods for attacking its most outstanding open problems.

(iii) the downward Löwenheim-Skolem theorem holds for L' (that is, whenever $\sigma' \varepsilon \Sigma'$ has an infinite model, it has a countable model),

then $L \boxminus L'$.

Thus first-order logic is the strongest logic that satisfies both the countable compactness theorem and the downward Löwenheim-Skolem theorem.

One might expect Lindström's theorem to have stifled further interest in general logics, but the reverse has been the case (see Ebbinghaus *1985* or Flum *1985*). The reason, as Hao Wang has perceptively observed (*1974*, 154; quoted in Flum *1985*, 77–78), is that from the point of view of 'set theory or classical analysis, the Löwenheim theorem is usually taken as a sort of *defect* [my emphasis] [. . . of . . .] first order logic. Therefore what [Lindström] established is not that first order logic is the only possible logic, but rather that it is [such only . . .] when we in a sense deny reality to the concept of uncountability'. Compactness, in other words, is not the problem; the Löwenheim-Skolem property is.

Keisler *1965* offered an assessment that has since prevailed: 'The most useful theorem in model theory is probably the compactness theorem'. As we have seen, however, that view was a long time in coming. Recognition of the importance of compactness—and indeed of model theory itself—required a shift away from a concern for foundations (as exemplified by Gödel) toward a concern for applications (as exemplified by Maltsev, Tarski and Robinson). Of special importance was the appearance in 1966 of Robinson's *Non-standard analysis*, which awoke interest in logical methods among historians and 'working' mathematicians and provided analysts with a new methodology that remains to this day the most important and familiar 'non-logical' application of the compactness theorem.

The shift toward applications brought with it an infusion of ideas from algebra and topology and, most importantly, an emphasis on *particular* structures—and that, at last, caused logicians to recognize the legitimacy, and even primacy, of semantic considerations. So long as qualms about semantic methods had prevailed, applications of compactness were retarded; in overcoming those qualms, the power of compactness arguments was itself perhaps the most important factor.

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