



# Complexity of conflict-free colorings of graphs



Luisa Gargano, Adele A. Rescigno\*

Dipartimento di Informatica, University of Salerno, 84084 Fisciano (SA), Italy

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## ABSTRACT

We consider conflict-free colorings of graph neighborhoods: Each vertex of the graph must be assigned a color so that for each vertex  $v$  there is at least one color appearing exactly once in the neighborhood of  $v$ . The goal is to minimize the number of used colors. We consider both the case of closed neighborhoods, when the neighborhood of a node includes the node itself, and the case of open neighborhoods when a node does not belong to its neighborhood. In this paper, we study complexity aspects of the problem. We show that the problem of conflict-free coloring of closed neighborhoods is NP-complete. Moreover, we give non-approximability results for the conflict-free coloring of open neighborhoods. From a positive point of view, both problems become tractable if parameterized by the vertex cover number or the neighborhood diversity number of the graph. We present simple algorithms which improve on existing results.

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## 1. Introduction

Let  $\mathcal{H} = (V, \mathcal{E})$  be a hypergraph with vertex set  $V$  and edge set  $\mathcal{E}$ . A coloring of (the vertices of)  $\mathcal{H}$  is a function  $C: V \rightarrow \mathbb{Z}^+$ . A  $k$ -coloring of  $\mathcal{H}$  is a coloring  $C$  with  $|C(V)| \leq k$ , for some  $k \geq 1$ . A coloring is called proper if no edge  $e \in \mathcal{E}$  containing at least two vertices is monochromatic. The smallest number of colors needed to properly color the vertices of  $\mathcal{H}$  is called the chromatic number of  $\mathcal{H}$  and is denoted by  $\chi(\mathcal{H})$ . A coloring is said to be conflict-free if every hyperedge contains a vertex whose color is unique among those assigned to the vertices of the hyperedge.

**Definition 1** (*CF coloring*). Let  $C$  be a coloring of a hypergraph  $\mathcal{H} = (V, \mathcal{E})$ .  $C$  is a *conflict-free coloring* of  $\mathcal{H}$  if for each  $e \in \mathcal{E}$  there exists a vertex  $v \in e$  such that  $C(u) \neq C(v)$  for any  $u \in e$  with  $u \neq v$ .

The study of conflict-free colorings was initially motivated by a frequency assignment problem in cellular networks [10]. Such networks consist of fixed-position *base stations*, each assigned a fixed frequency, and roaming *clients*. Roaming clients have a range of communication and come under the influence of different subsets of base stations. This situation can be modeled by means of a hypergraph whose vertices correspond to the base stations. The range of communication of a mobile agent, that is, the set of base stations it can communicate with, is represented by a hyperedge  $e \in \mathcal{E}$ . A CF-coloring of such a hypergraph implies an assignment of frequencies, to the base stations, which enables clients to connect to a base station holding a unique frequency in the client's range, thus avoiding interferences. The goal is to minimize the number of assigned frequencies.

CF-coloring also finds application in RFID (Radio Frequency Identification) networks. RFID allows a reader device to sense the presence of a nearby object by reading a tag attached to the object itself. To improve coverage, multiple RFID readers

\* Corresponding author.

E-mail addresses: [lgargano@unisa.it](mailto:lgargano@unisa.it) (L. Gargano), [arescigno@unisa.it](mailto:arescigno@unisa.it) (A.A. Rescigno).

can be deployed in an area. However, two readers trying to access a tagged device simultaneously might cause mutual interference. It can be shown that CF-coloring of the readers can be used to assure that every possible tag will have a time slot and a single reader trying to access it in that time slot [23].

Due to both its practical motivations and its theoretical interest, conflict-free coloring has been the subject of several papers; a survey of results in the area is given in [23]. The theoretic study of the CF-chromatic number in general graphs and hypergraphs was initiated in [22] and has recently raised much interest due to the novel combinatorial and algorithmic questions it poses, see [2–4,6,15,16,18].

### 1.1. CF-colorings of neighborhoods

In this paper we study the conflict-free coloring of hypergraphs induced by the neighborhoods of the vertices of a graph [22].

Given a graph  $G = (V, E)$  and a vertex  $u \in V$ , the *open neighborhood*  $N_G(u)$  of  $u$  is defined as the set consisting of all the vertices in  $G$  connected to  $u$ . The set  $N_G[u] = N_G(u) \cup \{u\}$  is called the *closed neighborhood* of  $u$ . We will write  $N(u)$  and  $N[u]$  whenever  $G$  is clear from the context.

A conflict-free coloring with respect to the open (resp. closed) neighborhoods of  $G$  is defined as the conflict-free coloring of the hypergraph with vertex set  $V$  and edge set  $\{N_G(u) \mid u \in V\}$  (resp.  $\{N_G[u] \mid u \in V\}$ ). For the sake of simplicity, we now reformulate Definition 1 in terms of the graph  $G$ .

Given a graph  $G = (V, E)$  and a coloring  $C$ , we say that the set  $U \subseteq V$  has a *unique color* under  $C$  if there exists a color  $c$  such that  $|\{v \in U \mid C(v) = c\}| = 1$ . Equivalently, we say that  $c$  is unique for  $U$ . All the graphs considered in this paper are supposed to be connected.

**Definition 2.** Consider a graph  $G = (V, E)$ .

**CF-ON coloring:** A coloring  $C$  is called conflict-free with respect to the *open neighborhoods* of  $G$  if for each  $u \in V$  the set  $N(u)$  has a *unique color* under  $C$ .

**CF-CN coloring:** A coloring  $C$  is called conflict-free with respect to the *closed neighborhoods* of  $G$  if for each  $u \in V$  the set  $N[u]$  has a *unique color* under  $C$ .

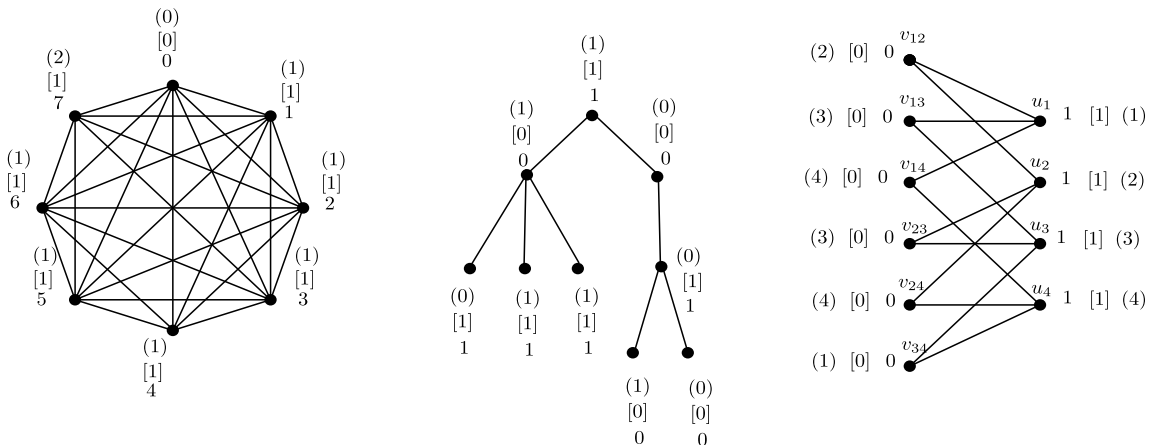
The smallest number of colors needed by any possible CF-ON (resp. CF-CN) coloring of  $G$  is called the CF-ON (resp. CF-CN) chromatic number of  $G$  and is denoted by  $\chi_{\text{CF}}(G)$  (resp.  $\chi_{\text{CF}}[G]$ ).

It is possible to show (see also [22]) that, given a graph  $G$ , the same greedy upper bound  $\Delta_G + 1$  (where  $\Delta_G$  is the maximum degree of a vertex in  $G$ ) holds for the chromatic number, the CF-ON chromatic number, and the CF-CN chromatic number. However, as also noticed in [22], these values can be quite different and no ordering among them is valid for any graph. Examples are given in Fig. 1. Consider first the complete graph  $K_n$  on  $n$  vertices, it is not difficult to see that

$$\chi_{\text{CF}}[K_n] = 2 < \chi_{\text{CF}}(K_n) = 3 < \chi(K_n) = n.$$

Moreover, it is easy to show that for any tree  $T$

$$\chi_{\text{CF}}[T] = \chi_{\text{CF}}(T) = \chi(T) = 2.$$



**Fig. 1.** The graph  $K_8$ , a tree  $T$ , and the graph  $B_4$  with the corresponding colorings. For each node  $c$ ,  $[c']$ ,  $(c'')$  represent the colors assigned to the node in a proper, a CF-CN, and a CF-ON coloring, respectively.

Finally, consider the bipartite graph  $B_\ell = (U, V, E)$  such that  $|U| = \ell$ ,  $V = \{v_{u,w} \mid u, w \in U, u \neq w\}$ , and  $E$  contains the edges  $(v_{u,w}, u)$  and  $(v_{u,w}, w)$ , for each  $u, w \in U$ . It holds

$$\chi(B_\ell) = 2 = \chi_{\text{CF}}[B_\ell] < \chi_{\text{CF}}(B_\ell) = \ell = \Delta_{B_\ell} + 1 \approx \sqrt{2(|V| + |U|)}.$$

In this paper we study complexity aspects of the conflict-free colorings of graph open/closed neighborhoods. In Section 2, we show that the problem of determining the CF-CN chromatic number of a graph is NP-complete and give non-approximability results for the CF-ON coloring problem. On the positive side, in Section 3 we show that both the CF-CN and the CF-ON coloring problems are tractable when restricted to graphs with bounded vertex cover number or with bounded neighborhood diversity [13]; our results improve on the ones implied by the metatheorems in [20].

## 2. Computational complexity

In this section we study the computational complexity of computing optimal CF-colorings with respect to open and closed neighborhoods. We show that determining whether a graph has a CF-CN coloring of a given size is NP-complete. We prove the hardness of approximating the CF-ON chromatic number.

### 2.1. Conflict-free closed neighborhood colorings

To prove our NP-completeness result, let us consider the decision version of the CF-CN coloring problem.

**CF-CNC (CF Closed Neighborhood Coloring)**

*Instance:* A graph  $G = (V, E)$  and an integer bound  $k$ .

*Question:* Is there a function  $C: V \rightarrow \mathbb{Z}^+$  such that  $|C(V)| \leq k$ , and for each  $u \in V$ , the set  $N[u]$  has a unique color under  $C$ ?

CF-CNC is clearly in NP. We prove its NP-hardness by a reduction from the well known NP-complete 3-CNF SAT problem. Given any 3-CNF formula  $\Phi = K_0 \wedge K_1 \wedge \dots \wedge K_{\ell-1}$ , where  $K_i = (x_1^i \vee x_2^i \vee x_3^i)$  for  $i = 0, \dots, \ell - 1$ , we outline how to construct a graph  $G_\Phi = (V, E)$  which has a CF-CN coloring with 2 colors (CF-CN 2-coloring) if and only if  $\Phi$  is satisfiable.

Let  $G_\Phi(K_i)$  be the subgraph, given in Fig. 2(a), associated to the clause  $K_i$ , for any  $i = 0, \dots, \ell - 1$ , where each vertex  $v_h^i$  corresponds to the literal  $x_h^i$  in  $K_i$ , for  $h = 1, 2, 3$ . Furthermore, let  $AND_i$  be the gadget, given in Fig. 2(b), that we use to join vertex  $u^i$  in  $G_\Phi(K_i)$  with vertex  $u^{i+1}$  in  $G_\Phi(K_{i+1})$ ,<sup>1</sup> for  $i = 0, \dots, \ell - 1$ . The graph  $G_\Phi$  is then obtained as follows:

- Join  $G_\Phi(K_i)$  with  $G_\Phi(K_{i+1})$  using  $AND_i$ , for  $i = 0, \dots, \ell - 1$ ; in particular, connect  $u^i$  and  $w_1^i$  with an edge, and connect  $w_5^i$  and  $u_{i+1}$  with another edge.
- Connect with an edge, each vertex corresponding to a variable  $x$  with each vertex corresponding to the variable  $\bar{x}$ .

The construction of graph  $G_\Phi$  can obviously be done in polynomial time. An example is given in Fig. 2(c).

**Lemma 1.** Consider any CF-CN 2-coloring of  $G_\Phi$ . If there exists a variable  $x$  such that  $x$  appears in  $K_i$  and  $\bar{x}$  appears in  $K_j$ , for some  $i \neq j$ , then the color assigned to the vertex corresponding to  $x$  in  $G_\Phi(K_i)$  is different from the color assigned to the vertex corresponding to  $\bar{x}$  in  $G_\Phi(K_j)$ .

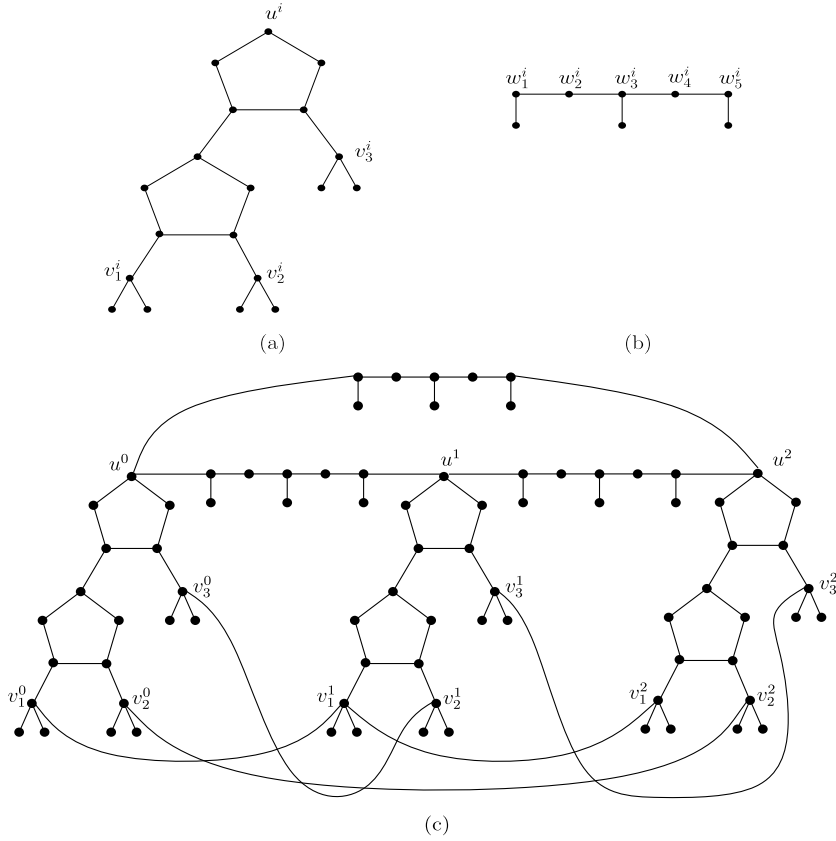
**Proof.** Let  $C$  be a CF-CN 2-coloring of  $G_\Phi$ . Let  $v_h^i$  and  $v_\ell^j$  be the vertices corresponding to  $x$  and  $\bar{x}$  in  $G_\Phi(K_i)$  and  $G_\Phi(K_j)$ , respectively. By contradiction, suppose that  $C(v_h^i) = C(v_\ell^j)$ . Since the neighbors of  $v_h^i$  include  $v_\ell^j$  and two leaves and since  $N[v_h^i]$  has a unique color (recall that the coloring uses 2 colors), then at most one neighbor of  $v_h^i$  has color different from  $C(v_h^i)$ . Hence, at least one of the two neighbor leaves of  $v_h^i$  has color  $C(v_h^i)$ . This implies that the closed neighborhood of such a leaf consists of two nodes of the same color. This contradicts the assumption of a CF-CN coloring.  $\square$

**Lemma 2.** For any CF-CN 2-coloring  $C$  of  $G_\Phi$ , we have  $C(u^0) = \dots = C(u^{\ell-1})$ .

**Proof.** For each  $i = 0, \dots, \ell - 1$ , the edge  $(u^i, w_1^i)$  connects  $G_\Phi(K_i)$  to the  $AND_i$  gadget and the edge  $(w_5^i, u^{i+1})$  connects the  $AND_i$  gadget to  $G_\Phi(K_{i+1})$ . We first prove that  $C(u^i) = C(w_2^i)$ . By contradiction, suppose that  $C(u^i) \neq C(w_2^i)$ . Recalling that  $N[w_1^i] = \{w_1^i, u^i, w_2^i, x\}$ , where  $x$  is a leaf, must have a unique color under  $C$ , we have that  $C(w_1^i) = C(x)$ . Therefore, we have a contradiction since  $N[x] = \{x, w_1^i\}$  would contain two nodes with the same color.

With a proof similar to the above one, we can prove that  $C(w_2^i) = C(w_4^i)$  and  $C(w_4^i) = C(u^{i+1})$ . Hence, the lemma follows.  $\square$

<sup>1</sup> The operation  $+$  refers to the addition modulo  $\ell$  when applied to the indices of the clauses in  $G_\Phi$ .



**Fig. 2.** (a)  $G_\Phi(K_i)$ . (b)  $AND_i$  gadget. (c)  $G_\Phi$  for  $\Phi = (x_1^0 \vee x_2^0 \vee x_3^0) \wedge (x_1^1 \vee x_2^1 \vee x_3^1) \wedge (x_1^2 \vee x_2^2 \vee x_3^2) = (a \vee b \vee \bar{c}) \wedge (\bar{a} \vee c \vee \bar{d}) \wedge (a \vee \bar{b} \vee d)$ .

**Lemma 3.** For each CF-CN 2-coloring  $C$  of  $G_\Phi$ , and for any  $i = 0, \dots, \ell - 1$ , if  $C(u^i) = 1$  then at least one of the nodes corresponding to the literals in  $K_i$  has color 1.

**Proof.** We show that any CF-CN 2-coloring  $C$  of  $G_\Phi$  that assigns 0 to the three nodes in  $G_\Phi(K_i)$  corresponding to the literals in  $K_i$  also assigns color 0 to  $u^i$ , that is  $C(v_1^i) = C(v_2^i) = C(v_3^i) = 0$  implies  $C(u^i) = 0$ . We first notice that  $N[v_h^i]$ , for  $h = 1, 2, 3$ , includes two leaves whose closed neighborhoods must have a unique color. Hence each of these leaves must have color 1, otherwise the closed neighborhood would consist of two nodes of the same color 0. It follows that all the neighbors of  $v_h^i$  must have color 1. This forces any CF-CN 2-coloring of the remaining vertices in  $G_\Phi(K_i)$  to be as in Fig. 3, thus proving the lemma.  $\square$

**Theorem 1.** CF-CNC is NP-complete.

**Proof.** We prove that  $G_\Phi$  has a CF-CN 2-coloring iff  $\Phi$  is satisfiable. Let  $C$  be any CF-CN 2-coloring of  $G_\Phi$ . By Lemma 2, w.l.o.g. we can assume that

$$C(u^0) = C(u^1) = \dots = C(u^{\ell-1}) = 1. \quad (1)$$

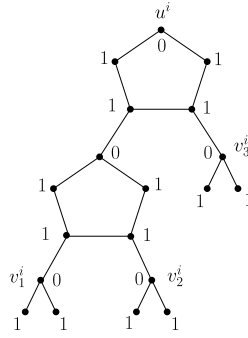
By (1) and Lemma 3, at least one of the vertices corresponding to the literals in  $K_i$  has color 1, for  $i = 0, \dots, \ell - 1$ . Hence, if we set

$$x_h^i = \text{TRUE} \quad \text{iff} \quad C(v_h^i) = 1, \quad \text{for } i = 0, \dots, \ell - 1 \text{ and } h = 1, 2, 3,$$

then by Lemma 1, the values assigned to the literals are consistent, and for each  $K_i$  there is at least one literal whose value is TRUE. It follows that  $\Phi$  is satisfiable.

Assume now that  $\Phi$  is satisfiable. We use the truth assignment of  $\Phi$  to get a CF-CN 2-coloring of  $G_\Phi$ . For each  $i = 0, \dots, \ell - 1$  and  $h = 1, 2, 3$ , we color the vertices corresponding to true literals with color 1 and the vertices corresponding to false literals with color 0, that is, we set

$$C(v_h^i) = \begin{cases} 0 & \text{if } x_h^i = \text{FALSE}, \\ 1 & \text{if } x_h^i = \text{TRUE}. \end{cases}$$



**Fig. 3.** The CF-CN 2-coloring of  $G_\phi(K_i)$  when  $v_1^i, v_2^i$  and  $v_3^i$  have color 0.

We then color all the neighbors of  $v_h^i$  with the color opposed to that of  $v_h^i$  itself, i.e., if  $v_h^i$  is colored with 1 then all its neighbors are colored with 0, and vice-versa. This implies that  $N[v_h^i]$  has a unique color under  $C$  as well as the closed neighborhood of each of the two neighbor leaves of  $v_h^i$ . Notice that if  $v_h^i$  has a neighbor  $v_{h'}^j$ , for some  $j \neq i$ , then  $v_h^i$  and  $v_{h'}^j$  have properly opposite colors since they correspond to a variable and its negation in two different clauses.

Finally, the coloring of vertices  $v_h^i$ , for  $i = 0, \dots, \ell - 1$  and  $h = 1, 2, 3$ , and the satisfiability of  $\Phi$  imply that at least one vertex  $v_h^i$ , in each  $G_\phi(K_i)$ , has color 1, i.e.,  $C(v_h^i) = 1$ . It is then easy to verify that any CF-CN 2-coloring of the remaining vertices of each  $G_\phi(K_i)$  assures that  $u^i$  has color 1, for  $i = 0, \dots, \ell - 1$ . The coloring of the vertices of  $G_\phi(K_i)$  uniquely specify the color that we can assign to the neighbors  $w_1^i$  and  $w_5^{i-1}$  of  $u^i$  and to the neighbors  $w_1^{i+1}$  and  $w_5^i$  of  $u^{i+1}$ . Now we color  $w_2^i$  and  $w_4^i$  with 1 and  $w_3^i$  with 0. Finally, we complete the CF-CN 2-coloring  $C$  of  $G_\phi$  by assigning to each leaf connected a vertex  $w_j^i$  the color opposite to  $C(w_j^i)$ , for each  $j = 1, \dots, 5$ .  $\square$

The NP-completeness of CF-CNC is proved for  $k = 2$  colors. Hence, [Theorem 1](#) implies the following inapproximability result [\[14\]](#).

**Corollary 1.** *CF-CNC chromatic number is hard to approximate within a factor less than  $3/2$  unless  $P = NP$ .*

## 2.2. Conflict-free open neighborhood colorings

Let us consider the decision version of the CF-ON coloring problem.

CF-ONC (CF Open Neighborhood Coloring)

*Instance:* A graph  $G = (V, E)$  and an integer bound  $k$ .

*Question:* Is there a function  $C: V \rightarrow \mathbb{Z}^+$  such that  $|C(V)| \leq k$ , and for each  $u \in V$ , the set  $N(u)$  has a unique color under  $C$ ?

It is easy to see that CF-ONC is in NP. We first prove that CF-ONC is NP-hard by considering  $k = 2$  and showing a reduction from the NP-complete problem Not-All-Equal-3-SAT [\[14\]](#). Moreover, when  $k \geq 3$ , we show a reduction from GRAPH COLORING. Since our reduction is gap-preserving [\[17\]](#), it implies corresponding inapproximability results for the problem of determining the CF-ONC chromatic number.

**Theorem 2.** *CF-ONC is NP-complete.*

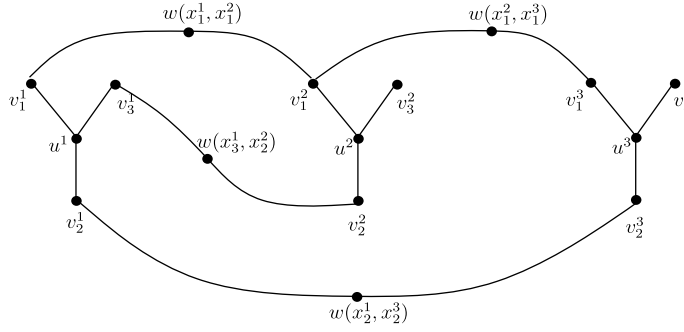
**Proof.** We give a reduction from the NP-complete Not-All-Equal-3-SAT problem [\[14\]](#).

NAE-3SAT (Not-All-Equal-3-SAT)

*Instance:* A boolean formula  $\Phi$  that is an instance of 3-SAT.

*Question:* Is there a truth-value assignment for the variables of  $\Phi$  such that each of its clauses has at least one true literal and at least one false literal?

Consider an instance of NAE-3SAT, that is, a formula  $\Phi = K_1 \wedge K_2 \wedge \dots \wedge K_\ell$ , where  $K_i = (x_1^i \vee x_2^i \vee x_3^i)$  for  $i = 1, \dots, \ell$ . We outline how to construct a graph  $G_\phi = (V, E)$  which has a CF-ON coloring with 2 colors (CF-ON 2-coloring) if and only if  $\Phi$  is a yes instance for NAE-3SAT. The graph  $G_\phi$  is obtained as follows:



**Fig. 4.**  $G_\Phi$  for  $\Phi = (x_1^1 \vee x_2^1 \vee x_3^1) \wedge (x_1^2 \vee x_2^2 \vee x_3^2) \wedge (x_1^3 \vee x_2^3 \vee x_3^3) = (a \vee b \vee \bar{c}) \wedge (\bar{a} \vee c \vee d) \wedge (a \vee \bar{b} \vee d)$ .

- For  $i = 1, \dots, \ell$ , the graph  $G_\Phi$  contains a vertex  $u_i$  and the three vertices  $v_1^i, v_2^i, v_3^i$ , where  $v_h^i$  corresponds to the literal  $x_h^i$  in  $K_i$ , for  $h = 1, 2, 3$ . The vertices  $u_i, v_1^i, v_2^i, v_3^i$  induce a star with vertex  $u_i$  and edges  $(u_i, v_1^i), (u_i, v_2^i), (u_i, v_3^i)$ .
- For each  $1 \leq i \leq j \leq \ell$  and  $1 \leq h, s \leq 3$  if  $x_h^i = \bar{x}_s^j$  then the graph  $G_\Phi$  contains the node  $w(x_h^i, x_s^j)$  and the two edges  $(v_h^i, w(x_h^i, x_s^j))$  and  $(w(x_h^i, x_s^j), v_s^j)$ .

The construction of graph  $G_\Phi$  can obviously be done in polynomial time. An example is given in Fig. 4.

Let  $C$  be any CF-ON 2-coloring of  $G_\Phi$ . Since the open neighborhood of each  $u_i$ , for  $i = 1, \dots, \ell$ , has a unique color then  $N(u_i)$  is not monochromatic, that is among the three vertices  $v_1^i, v_2^i, v_3^i$  corresponding to the literals in  $K_i$  at least one has color 1 and at least one has color 0. If we set

$$x_h^i = \text{TRUE} \quad \text{iff} \quad C(v_h^i) = 1, \quad \text{for } i = 1, \dots, \ell \text{ and } h = 1, 2, 3,$$

then every clause contains at least one true literal and at least one false literal. Moreover, the values of the literals are consistent since the colors of the two vertices in  $N(w(x_h^i, x_s^j))$  are different for each  $w(x_h^i, x_s^j)$  in  $G_\Phi$ , thus implying that the literals  $x_h^i$  and  $x_s^j$  get opposite values.

Assume now that  $\Phi$  is a yes instance for NAE-3SAT. We can use the truth assignment of  $\Phi$  to get a CF-ON 2-coloring of  $G_\Phi$ . For each  $i = 1, \dots, \ell$  and  $h = 1, 2, 3$ , set

$$C(v_h^i) = \begin{cases} 0 & \text{if } x_h^i = \text{FALSE}, \\ 1 & \text{if } x_h^i = \text{TRUE}. \end{cases} \quad (2)$$

Furthermore, we set  $C(u_i) = 1$  for each  $i = 1, \dots, \ell$ , and  $C(w(\cdot, \cdot)) = 0$  for each  $w(\cdot, \cdot)$ . The coloring of  $v_1^i, v_2^i, v_3^i$  assures that  $N(u_i)$  is not monochromatic. Moreover, the neighborhood of each vertex  $v_h^i$  consists of the vertex  $u_i$  with  $C(u_i) = 1$  and, of some  $w(x_h^i, \cdot)$ , if any; this assures that  $N(v_h^i)$  has 1 as a unique color. Finally, any vertex  $w(x_h^i, x_s^j)$  has exactly two neighbors corresponding to two literals  $x_h^i = \bar{x}_s^j$  and, by (2),  $C(v_h^i) \neq C(v_s^j)$ .  $\square$

**Theorem 3.** CF-ONC chromatic number is hard to approximate within a factor  $n^{1/2-\epsilon}$ , for each  $\epsilon > 0$ , unless  $P = NP$ .

**Proof.** Consider any graph  $G' = (V', E')$  and define the graph  $G = (V, E)$  as follows

$$V = V' \cup \{n_{uv} \mid (u, v) \in E'\} \quad \text{and} \quad E = \{(u, n_{uv}), (n_{uv}, v) \mid (u, v) \in E'\},$$

where,  $n_{uv}$  and  $n_{vu}$ , for  $(u, v) \in E'$ , denote the same vertex. The construction of graph  $G = (V, E)$  can obviously be done in polynomial time. We now show that for any  $k \geq 3$  the graph  $G'$  is  $k$ -colorable (i.e.,  $G'$  can be properly colored with  $k$  colors) if and only if  $G$  has a CF-ON  $k$ -coloring.

First, assume that  $G$  has a CF-ON  $k$ -coloring  $C$ . By construction, for any  $n_{uv} \in V$  we have  $N(n_{uv}) = \{u, v\}$ ; hence, the CF-ON  $k$ -coloring  $C$  assures that  $C(u) \neq C(v)$ . This allows to use the coloring  $C$ , restricted to the vertices in  $V' \subset V$ , as a  $k$ -coloring of  $G'$ .

Now, assume that  $G'$  is  $k$ -colorable. Let  $C'$  be a  $k$ -coloring of  $G'$ . The algorithm CF-Visit, given in Fig. 5, gives a coloring of the vertices in  $V$  that we will show to be a CF-ON  $k$ -coloring of  $G$ .

The algorithm CF-Visit makes a depth-first graph traversal of  $G'$ . It assigns to the vertices in  $V'$  the same color as in the coloring  $C'$ . Moreover, it assigns colors to the vertices in  $V \setminus V'$  so that for each  $u \in V'$  exactly one of the neighbors of  $u$  in  $G$  gets the color of  $u$ . The algorithm uses, for each node  $v \in V'$ , two sets  $T_v$  and  $B_v$  containing the tree edges and the back edges, respectively, visited starting from  $v$  during the depth first traversal of  $G'$ . The algorithm starts calling CF-Visit( $G', C', r$ ) where  $r$  is any node in  $V'$ .

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1. Set  $T_v = B_v = \emptyset$ , for each  $v \in V'$   $[T_v \text{ is the set of tree edges visited starting from } v]$   
 $[B_v \text{ is the set of back edges visited starting from } v]$ 
2. CF-Visit( $G', C', u$ )
3.    $C(u) = C'(u)$ 
4.   while there exists  $(u, v) \in E' \setminus (T_u \cup B_u)$  with  $v$  unvisited
5.      $T_u = T_u \cup \{(u, v)\}$ 
6.     if  $|T_u| = 1$  then  $C(n_{uv}) = C'(u)$ 
7.     else Set  $C(n_{uv})$  to be any color in  $\{0, \dots, k-1\} \setminus \{C'(u), C'(v)\}$ 
8.     CF-Visit( $G', C', v$ )
9.   while there exists  $(u, v) \in E' \setminus (T_u \cup B_u)$  with  $v$  visited
10.     $B_u = B_u \cup \{(u, v)\}$ 
11.    if  $|T_u| = 0$  and  $|B_u| = 1$  then  $C(n_{uv}) = C'(u)$ 
12.    else Set  $C(n_{uv})$  to be any color in  $\{0, \dots, k-1\} \setminus \{C'(u), C'(v)\}$ 

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Fig. 5. The CF-ON  $k$ -coloring of  $G$ .

We have  $C(u) = C'(u) \neq C'(v) = C(v)$  for each  $u, v \in V'$  such that  $(u, v) \in E'$ . Hence, the neighborhood  $N(n_{uv}) = \{u, v\}$  has a unique color, for each  $n_{uv} \in V \setminus V'$ .

To complete the proof we will prove that also the neighborhood  $N(u)$  has a unique color, for each  $u \in V'$ . Let  $u$  be any vertex in  $V'$ . The algorithm CF-Visit (excepting when  $u = r$  is the starting vertex of the algorithm) assures that there exists a vertex  $x$  such that  $(x, u) \in T_x$  (i.e.,  $(x, u)$  has been visited starting from  $x$ ). As effect of this visit we have either  $C(n_{ux}) = C'(x)$  or  $C(n_{ux})$  in  $\{0, \dots, k-1\} \setminus \{C'(u), C'(x)\}$ . In any case

$$C(n_{ux}) \neq C(u), \quad (3)$$

since  $C(u) = C'(u) \neq C'(x)$  (recall that  $u$  and  $x$  are neighbors in  $G'$  and  $C'$  is a  $k$ -coloring of  $G'$ ).

By algorithm CF-Visit, after  $u$  has been visited, each edge  $(u, v)$ , for  $v \in N(u) \setminus \{x\}$ , is visited and inserted either in  $T_u$  or in  $B_u$ . Then a color to the vertex  $n_{uv}$  is assigned. We now distinguish two cases according to the sizes of  $T_u$  and  $B_u$  and prove that in each case  $N(u)$  has a unique color.

- Let  $|T_u| \geq 1$  or  $|B_u| \geq 1$ . For exactly one vertex  $v \in N(u) - \{x\}$  (cf. lines 6 and 11 of the algorithm CF-Visit) we have

$$C(n_{uv}) = C'(u) = C(u). \quad (4)$$

For each  $w \in N(u) \setminus \{x, v\}$  we have  $C(n_{uw}) \in \{0, \dots, k-1\} \setminus \{C'(u), C'(w)\}$ . Recalling that  $C'(u) = C(u)$ , we have  $C(n_{uw}) \neq C(u)$ . By this, (3), and (4) we have

$$C(n_{uv}) = C(u) \quad \text{and} \quad C(n_{uw}) \neq C(u) \quad \text{for each } w \in N(u) \setminus \{v\}$$

and  $C(n_{uv})$  is a unique color in  $N(u)$ .

- Let  $|T_u| = |B_u| = 0$ . In this case  $N(u) = \{n_{ux}\}$  has obviously a unique color.

We conclude the proof, by noticing that the above reduction is a gap-preserving reduction [17] from the MINIMUM GRAPH COLORING, which is known to be NP-Hard to approximate within  $n^{1-\epsilon}$ , for any  $\epsilon > 0$  [24].  $\square$

### 3. Parameterized complexity

Some NP-hard problems can be solved by algorithms that are exponential only in the size of a fixed parameter while they are polynomial in the size of the input. Such algorithms are called fixed-parameter tractable, because the problem can be solved efficiently for small values of the fixed parameter [9,21]. Formally, a parameterized problem with input size  $n$  and parameter  $t$  is called *Fixed Parameter Tractable (FPT)* if it can be solved in time  $f(t) \cdot n^c$ , where  $f$  is a function only depending on  $t$  and  $c$  is a constant.

In this section, we study FPT algorithms for the CF-CNC and CF-ONC problems. Usually, a problem can be parameterized by a parameter that is associated to the problem, such as the number of colors  $k$  in CF-CNC and CF-ONC. However, this approach does not make the parameterized complexity theory applicable to CF-CNC and CF-ONC according to Theorems 1, 2, and 3. Courcelle's Theorem [8] shows that problems that can be expressed by an MSO formula are solvable in linear time on graphs of bounded treewidth. Since both CF-CN and CF-ON  $k$ -coloring problems can be expressed by MSO formulas for any fixed  $k$ , both problems are FPT when parameterized by treewidth.

An important quality of a parameter is its easy computability. Unfortunately there are parameters as clique-width whose computation is very hard; recently, it has been shown that determining clique-width is NP-hard [11], but the parameterized complexity of recognizing graphs of bounded clique-width is still an open problem [7]. Other parameters as treewidth, rankwidth, and vertex cover [5,12] are all computable in FPT time when their respective parameters are bounded. Here,



we devote our attention to *neighborhood diversity*, a parameter that was first introduced in [20]. It has recently received attention [13,19] since it is less restrictive than other parameters and, in particular, it has the nice property to be computable in polynomial time (see [20]). We show that both CF-CNC and CF-ONC become tractable if parameterized by the vertex cover or the neighborhood diversity of the graph. We present simple algorithms which improve on existing results.

**Definition 3.** Given a graph  $G = (V, E)$ , two vertices  $u, v \in V$  have the same *type* iff  $N(v) \setminus \{u\} = N(u) \setminus \{v\}$ .

The graph  $G$  has *neighborhood diversity*  $t$ , if there exists a partition of  $V$  into at most  $t$  sets,  $V_1, V_2, \dots, V_t$ , such that all the vertices in  $V_i$  have the same type, for  $i = 1, \dots, t$ .

The family  $\mathcal{V} = \{V_1, V_2, \dots, V_t\}$  is called the *type partition* of  $G$ .

The proof of the following theorem is given in Section 3.1.

**Theorem 4.** Let  $t$  be the neighborhood diversity of  $G$ . It is possible to decide the CF-CNC (resp. CF-ONC) question for a fixed number of colors  $k$  in time  $2^{O(kt \log k)}$  when  $k \leq 2t$  and in polynomial time when  $k > 2t$ .

Considering that the type partition  $\mathcal{V}$  can be obtained in polynomial time, one has that both the CF-CNC problem and the CF-ONC problem are in the class FPT when parameterized by the neighborhood diversity. We stress that the existence of FPT algorithms is also implied by the results presented in [20]. However, it is easy to verify that the running time of the Lampis' results is  $2^{O(k^2 \log k)}$ . Furthermore, our algorithm characterizes to be very simple.

The result of Theorem 4 can be used to have FPT linear time algorithms with vertex cover size as parameter for both CF-CNC and CF-ONC. Indeed, if a graph has vertex cover  $d$ , it cannot have a type partition with more than  $2^d + d$  sets, that is,  $t \leq 2^d + d$  [13]. We recall that while graphs of bounded vertex cover have bounded neighborhood diversity, the opposite is not true since large cliques have a neighborhood diversity 1 [13].

**Theorem 5.** Given a vertex cover of  $G$  of size at most  $d$  and a fixed number of colors  $k$ , it is possible to decide the CF-CNC (resp., CF-ONC) question in time  $2^{O(k2^d \log k)}$  when  $k \leq d$  (resp.,  $k \leq 2d$ ) and in polynomial time when  $k > d$  (resp.,  $k > 2d$ ).

Again Theorem 5 improves on the results implied by the general ones in [20], indeed the time to decide the CF-CNC (resp., CF-ONC) coloring question for a fixed number of colors  $k$ , applying the algorithm in [20], is doubly exponential in  $k$  and  $d$ , namely,  $2^{2^{O(k+d)}}$ .

We end by noticing that fast coloring exist when the number  $k$  of colors is large enough with respect to the vertex cover size  $d$ , namely if we are looking either for a CF-CN  $k$ -coloring with  $k \geq d + 1$  or for a CF-ON  $k$ -coloring with  $k \geq 2d + 1$ .

**Lemma 4.** If  $G$  has a vertex cover of size  $d$  then

$$\chi_{\text{CF}}[G] \leq d + 1 \quad \text{and} \quad \chi_{\text{CF}}(G) \leq 2d + 1.$$

**Proof.** Let  $W$  be a vertex cover of size  $d$  for  $G$  and let  $I = V(G) \setminus W$ . We recall that  $I$  is an independent set.

A CF-CN  $(d + 1)$ -coloring  $C$  of  $G$  can be obtained as follows:

1. To each  $u \in I$  assign the color  $C(u) = 0$ .
2. To each  $w \in W$  assign  $C(w) \in \{1, \dots, d\}$  so that  $C(w) \neq C(v)$  whenever  $w \neq v$ .

According to this coloring, the unique color in  $N[v]$  is  $C(v)$  if  $v \in W$  and 0 if  $v \in I$ .

A CF-ON  $(2d + 1)$ -coloring  $C$  of  $G$  can be obtained as follows:

1. To each  $u \in I$  assign the color  $C(u) = 0$ .
2. To each  $w \in W$  assign  $C(w) \in \{1, \dots, d\}$  so that  $C(w) \neq C(v)$  whenever  $w \neq v$ .
3. For each  $w \in W$  if  $C(N(w)) = \{0\}$ <sup>2</sup> choose one node  $u \in N(w)$  and recolor it with any  $C(u) \in \{d + 1, \dots, 2d\}$  that is not already used by  $C$ .

According to this coloring, each  $N(w)$ , for  $w \in W$ , has at least one unique color, that is, either the color of any of its neighbors in  $W$  or the color of the selected neighbor in  $I$  (cf. 3). Furthermore, since  $I$  is an independent set, any  $u \in I$  has so many unique colors in  $N(u)$  as the number of its neighbors in  $W$ .  $\square$

<sup>2</sup> Notice that if  $W$  is minimal then each vertex in  $W$  has at least one neighbor in  $I$ .



### 3.1. Parameterization with neighborhood diversity

Let  $G = (V, E)$  be a graph with type partition  $\mathcal{V} = \{V_1, V_2, \dots, V_t\}$ . By Definition 3, the vertices of a given type, i.e., the vertices of  $V_i$ , have the same neighborhood (excluding the vertices in  $V_i$  itself); furthermore, each  $V_i$  induces either a clique or an independent set in  $G$ . We will denote by  $cl(G)$  and  $ind(G)$  the number of sets in the type partition  $\mathcal{V} = \{V_1, V_2, \dots, V_t\}$  of  $G$  that are cliques and independent sets, respectively. For each  $V_i, V_j \in \mathcal{V}$ , we get that either each vertex in  $V_i$  is a neighbor of each vertex in  $V_j$  or no vertex in  $V_i$  is a neighbor of any vertex in  $V_j$ .

We can then define the *type graph* of  $G$  as the graph  $H = (\{1, \dots, t\}, E_H)$  where

$$E_H = \{(i, j) \mid 1 \leq i < j \leq t \text{ and each vertex in } V_i \text{ is a neighbor of each vertex in } V_j\}.$$

The vertex size of  $H$  is  $t$  and the degree of each vertex  $i$  in  $H$  is  $deg_H(i) \leq t - 1$ . Hence, if the neighborhood diversity  $t$  is bounded, then the size and the maximum degree of  $H$  have bounded values.

#### Lemma 5.

- (1)  $\chi_{CF}[G] \leq \chi_{CF}[H] + ind(G) + 1$ .
- (2)  $\chi_{CF}(G) \leq \chi_{CF}(H) + cl(G) + 1$ .

**Proof.** Let us first prove (1). Consider a CF-CN coloring  $C_H$  of  $H$  that uses  $\chi_{CF}[H]$  colors. Let  $B = \{0, b_1, \dots, b_{ind(G)}\}$  be a set of  $ind(G) + 1$  colors not used by  $C_H$ . We define a coloring  $C_G$  of the vertices of  $G$  by using  $C_H$  and the colors in  $B$ .

- For each  $i = 1, \dots, t$ , choose any vertex  $u_i \in V_i$  and assign each  $v \in V_i$  the color

$$C_G(v) = \begin{cases} C_H(i) & \text{if } v = u_i, \\ 0 & \text{if } v \in V_i \setminus \{u_i\} \text{ and } V_i \text{ is a clique,} \\ \text{a color in } B \setminus (\bigcup_{j=1}^{i-1} C(V_j) \cup \{0\}) & \text{otherwise.} \end{cases}$$

Now we prove that the above is a CF-CN coloring of  $G$ , that is, the closed neighborhood of each vertex in any  $V_i$  has a unique color. Consider first the vertex  $u_i \in V_i$  whose color is  $C_G(u_i) = C_H(i)$ . Let  $x_i \in N_H[i]$  be such that  $C_H(x_i)$  is a unique color in  $N_H[i]$  (i.e., either  $x_i = i$  or,  $x_i \neq i$  and  $(x_i, i) \in E_H$ ). By noticing that  $C_H(x_i)$  is used exactly once in  $V_{x_i}$  (eventually  $x_i = i$ ), and it is not used in any  $V_\ell$  such that  $(i, \ell) \in E_H$  and  $\ell \neq x_i$ , we get that  $C_H(x_i)$  is a unique color in  $N_G[u_i]$ . Consider now  $v \in V_i$  with  $v \neq u_i$ . If  $C_H(i)$  is not a unique color in  $N_H[i]$ , that is  $x_i \neq i$ , then by using the same argument used above we have that  $C_H(x_i)$  is unique in  $N_G[v]$ . Assume now that  $C_H(i)$  is a unique color in  $N_H[i]$ , that is  $x_i = i$ . If  $V_i$  is an independent set then  $C_G(v)$  is a color in  $B \setminus \{0\}$  that is not used in any  $V_\ell$  such that  $(i, \ell) \in E_H$ . This implies that  $C_G(v)$  is unique in  $N_G[v]$ . If  $V_i$  is a clique then  $C_G(v) = 0$ . Since  $u_i$  and  $v$  are neighbors in the clique  $V_i$  and, since  $C_G(u_i) = C_H(i)$  is unique in  $V_i$  and it is not used in any  $V_\ell$  such that  $(i, \ell) \in E_H$  we have that  $C_G(u_i)$  is the unique color in  $N_G[v]$ .

Now, we prove (2). Consider a CF-ON coloring  $C_H$  of  $H$  that uses  $\chi_{CF}(H)$  colors. Let  $B = \{0, b_1, \dots, b_{cl(G)}\}$  be a set of  $cl(G) + 1$  colors not used by  $C_H$ . Define a coloring  $C_G$  of the vertices of  $G$  as follows.

- For each  $i = 1, \dots, t$ , choose any vertex  $u_i \in V_i$  and assign each  $v \in V_i$  the color

$$C_G(v) = \begin{cases} C_H(i) & \text{if } v = u_i \text{ and } V_i \text{ is an independent set,} \\ \text{a color in } B \setminus (\bigcup_{j=1}^{i-1} C(V_j) \cup \{0\}) & \text{if } v = u_i \text{ and } V_i \text{ is a clique,} \\ 0 & \text{if } v \in V_i \setminus \{u_i\}. \end{cases}$$

Now we prove that  $C_G$  is a CF-ON coloring of  $G$ . In particular, we show that the (open) neighborhood of each vertex in any  $V_i$  has a unique color. Recall that by the coloring  $C_H$ , there exists  $x_i \neq i$  with  $(i, x_i) \in E_H$  such that  $C_H(x_i)$  is a unique color in  $N_H(i)$ . First consider  $u_i \in V_i$ . The coloring implies that  $C_G(u_{x_i})$  is unique in  $N_G(u_i)$  (recall that  $u_{x_i}$  has either color  $C_H(x_i)$  or a color never used in any other set of the partition type). Let  $v \in V_i$  and  $v \neq u_i$ . Recall that  $C_G(v) = 0$ . If  $V_i$  is an independent set then  $C_G(u_{x_i})$  is unique in  $N_G(v)$ . If  $V_i$  is a clique then at least one between  $C_G(u_i)$  and  $C_G(u_{x_i})$  is unique in  $N_G(v)$ .  $\square$

In the following we present an FPT-algorithm for the CF-CNC coloring problem with parameter  $t$ . To this aim we first need some definitions and a preliminary result.

Fix the number  $k$  of colors. Let  $h_i = \min\{|V_i|, k + 1\}$ . For each  $V_i \in \mathcal{V}$ , consider any subset  $V'_i \subseteq V_i$  such that  $|V'_i| = h_i$  and denote by  $G'$  the subgraph of  $G$  induced by the set  $\bigcup_{i=1}^t V'_i$ . The sets  $V'_1, \dots, V'_t$  are disjoint and the number of vertices of  $G'$  is upper bounded by  $t(k + 1)$ .

**Lemma 6.** *If there exists a CF-CN  $k$ -coloring of  $G$  then  $G'$  admits a CF-CN  $k$ -coloring.*

**Proof.** We produce a coloring  $C'$  of the vertices in each  $V'_i$  by using the colors of a CF-CN  $k$ -coloring  $C$  of  $G$ . If  $V'_i = V_i$  then  $C'(v) = C(v)$  for each vertex  $v \in V'_i$ . If  $V'_i \subset V_i$  then  $|V'_i| = k + 1$  and  $V_i$  contains at least one color that is used twice; hence, if  $o_i$  colors are used exactly once in  $V_i$  under  $C$ , then  $C'$  assigns these colors to  $o_i$  vertices in  $V'_i$ ; moreover  $C'$  assigns to the remaining vertices of  $V'_i$  some other color in  $C(V_i)$ .

We now prove that  $C'$  is a CF-CN coloring of  $G'$ . By definition of  $C'$ , if a color is unique in  $V_i$  then it is also unique in  $V'_i$ , for  $i = 1, \dots, t$ . Moreover, for each  $v \in V'_i$  there exists  $w \in V_i$  (eventually,  $v = w$ ) such that  $C'(v) = C(w)$ . Recalling that  $N_{G'}[v] \subseteq N_G[w]$  and that  $N_G[w]$  has a unique color, we have that  $N_{G'}[v]$  has a unique color.  $\square$

**Lemma 5** gives an algorithm that obtains a CF-CN  $k$ -coloring of  $G$  when  $k \geq \chi_{CF}[H] + \text{ind}(G) + 1$ . Now we present an FPT-algorithm with parameter  $t$  that decides the CF-CNC coloring problem for a fixed number of colors  $k$ , where  $k \leq \chi_{CF}[H] + \text{ind}(G)$ .

Our algorithm considers all the possible  $k$ -colorings of vertices of  $G'$ . If none of these colorings is a CF-CN  $k$ -coloring of  $G'$  then we answer no to the CF-CNC question for  $G$ . Otherwise, we answer YES and use any of the CF-CN  $k$ -coloring of  $G'$  to color the vertices of  $G$  as follows: We use the colors of the vertices in  $V'_i$  to color any  $h_i$  vertices in  $V_i$ ; if  $h_i = k + 1$  then we select any color that is used at least twice in  $V'_i$  and we assign it to the remaining  $|V_i| - (k + 1)$  vertices in  $V_i$ .

**Lemma 7.** *Let  $t$  be the neighborhood diversity of  $G$ . It is possible to decide the CF-CNC question for a fixed number of colors  $k$  in time  $O(k^{(k+1)t}(k+1)^2t^2)$  whenever  $k \leq \chi_{CF}[H] + \text{ind}(G)$ .*

**Proof.** **Lemma 6** proves that, when the above algorithm answers no to the CF-CN  $k$ -coloring question for  $G$ , then it is not possible to have a CF-CN  $k$ -coloring of  $G$ .

We prove now that if the answers is YES then we obtain a CF-CN  $k$ -coloring of  $G$ . To this aim we show that  $N_G[v]$  has a unique color for any  $v \in V_j$  and  $V_j \in \mathcal{V}$ . If  $v \in V'_j \subseteq V_j$  then  $N_G[v]$  has a unique color since  $N_{G'}[v]$  has a unique color. If, otherwise,  $v \in V_j \setminus V'_j$  then  $v$  has the same color of some  $w \in V'_j$  and, since  $N_G[v] = N_G[w]$  and  $N_{G'}[w]$  has a unique color, we have that also  $N_G[v]$  has a unique color.

The number of all possible colorings of the vertices of  $G'$  is  $O(k^{(k+1)t})$ . Moreover, one needs  $O(((k+1)t)^2)$  to check whether a given coloring is a CF-CN  $k$ -coloring of  $G'$ . Hence, the running time for deciding if a CF-CN  $k$ -coloring of  $G$  exists is

$$O(k^{(k+1)t}((k+1)t)^2).$$

Notice that the time to extend the CF-CN  $k$ -coloring to  $G$  is included in the above bound.  $\square$

In **Lemma 5** we have presented an algorithm that obtains a CF-ON  $k$ -coloring of  $G$  when  $k \geq \chi_{CF}(H) + \text{cl}(G) + 1$ . Using an algorithm similar to that presented above we can obtain an FPT algorithm with parameter  $t$  for the CF-ONC problem when the fixed number of colors is  $k \leq \chi_{CF}(H) + \text{cl}(G)$ . The time required by this algorithm can be proved (as in the proof of **Lemma 7**) to be  $O(k^{(k+1)t}(k+1)^2t^2)$ . Summarizing, we get the desired **Theorem 4**.

### 3.1.1. Conclusion and open problems

We have studied some complexity questions concerning colorings of the vertices of a graph that are conflict-free with respect to the open/closed neighborhoods of the graph. We have shown that the closed neighborhoods conflict-free coloring problem is NP-complete when two color are used; it implies that it is not possible to have approximation factor lower than  $3/2$ . In the case of open neighborhoods, we have obtained inapproximability results. On the positive side, both closed and open neighborhood colorings are fixed parameter tractable problems, when parameterized by the vertex cover number or the neighborhood diversity of the graph. With regard to future work, many questions remain open. In particular, it would be interesting extend the NP-hardness of the closed neighborhoods conflict-free coloring problem for any  $k \geq 3$  as well as to further study the (in)approximability of CF-CNC chromatic number. It would also be interesting to improve and/or extend our results to other important parameters. Furthermore, it would be intriguing to consider the fault-tolerant version of the problem, where one requires that each neighborhood contains at least  $t$  different unique colors, for some fixed  $t$  [1,4,23].

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