# Turn-based stochastic games with finitely and infinitely many states

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## Outline

- Preliminaries.
  - Game graphs can be infinite.
  - Strategies may be randomized and history-dependant.
- Stochastic games with reachability objectives.
  - The existence of a value.
  - The (non)existence of optimal strategies.
  - Algorithms for finite-state games.
  - Algorithms for infinite-state games.
- Stochastic games with branching-time objectives.

# Preliminaries

**We start by recalling...**

- Markov chains and the associated probability space.
- Turn-based stochastic games, strategies, and plays.
- Linear-time objectives.



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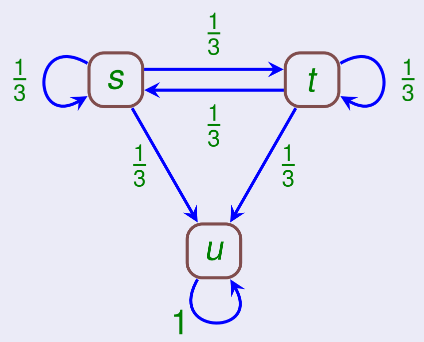
# Markov chains

**Definition 1 (Markov chain)**




$\mathcal{M} = (S, \rightarrow, Prob)$

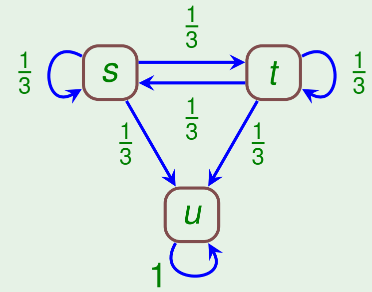
- $S$ is at most countable set of states;
- $\rightarrow \subseteq S \times S$ is a transition relation;
- $Prob$ is a probability assignment which to every transition assigns a positive probability s.t. $\sum_{s \xrightarrow{x} t} x = 1$ for every $s \in S$.

We want to measure the probability of certain subsets of $Run(s)$.

- For every finite path $w$ initiated in $s$, we define the probability of $Run(w)$ in the natural way.
- This assignment can be uniquely extended to the (Borel) $\sigma$-algebra $\mathcal{F}$ generated by all $Run(w)$.
- Thus, we obtain the probability space $(Run(s), \mathcal{F}, \mathcal{P})$.

# Markov chains (2)

## Example 2




- $Reach(u) = \{w \in Run(s) \mid w \text{ visits the vertex } u\}$.
- $Reach^i(u) = \{w \in Run(s) \mid w(i) = u, w(j) \neq u \text{ for every } j < i\}$.
- $Reach^i(u)$ is measurable, hence $Reach(u) = \biguplus_{i\in\mathbb{N}} Reach^i(u)$ is also measurable.

$$\mathcal{P}(Reach(u)) = \sum_{i\in\mathbb{N}} \mathcal{P}(Reach^i(u)) = \sum_{i\in\mathbb{N}} \left(\frac{2}{3}\right)^{i-1}\frac{1}{3} = 1$$

- Note that $Run(s) \setminus Reach(u)$ is not countable, and its probability is $0$.



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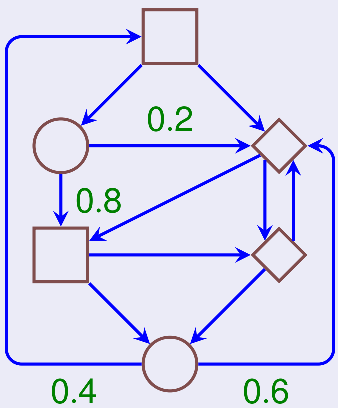
# Turn-based stochastic games

## Definition 3 (Turn-based stochastic game)




$G = (V, E, (V_\Box, V_\Diamond, V_\bigcirc), Prob)$

- *the set $V$ is at most countable;*
- *each vertex has a successor;*
- *$Prob$ is positive;*
- *$G$ is a Markov decision process (MDP) if $V_\Diamond = \emptyset$ or $V_\Box = \emptyset$.*

- A (linear-time) winning objective is a Borel set $\mathcal{W}$ of runs in $G$.
- The aim of player $\Box$ is to maximize the probability of $\mathcal{W}$, the aim of player $\Diamond$ is to minimize this probability.

# Strategies

**Definition 4 (Strategy)**

*Let $G = (V, E, (V_\Box, V_\Diamond, V_\bigcirc), Prob)$ be a game. A strategy for player $\Box$ is a function $\sigma$ which to every $wv \in V^* V_\Box$ assigns a probability distribution over the set of outgoing edges of $v$.*

- A strategy for player $\Diamond$ is defined analogously.
- We can classify strategies according to
  - memory requirements: history-dependent (H), finite-memory (F), memoryless (M)
  - randomization: randomized (R), deterministic (D)
- Thus, we obtain the classes of MD, MR, FD, FR, HD, and HR strategies.



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# Plays

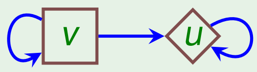
**Definition 5 (Play)**

*Let $G = (V, E, (V_\Box, V_\Diamond, V_\bigcirc), Prob)$ be a game. Each pair $(\sigma, \pi)$ of strategies for player $\Box$ and player $\Diamond$ determines a unique play $G^{(\sigma,\pi)}$, which is a Markov chain where $V^+$ is the set of states and transitions are defined accordingly.*

- Plays are infinite trees.
- Each run $w \in Run_{G^{(\sigma,\pi)}}(v)$ determines a unique run $w_G \in Run_G(v)$.
- If $\mathcal{W} \subseteq Run_G(v)$ is Borel, then $\mathcal{W}^{(\sigma,\pi)} = \{w \in Run_{G^{(\sigma,\pi)}}(v) \mid w_G \in \mathcal{W}\}$ is measurable for every pair of strategies $(\sigma, \pi)$.
- For a pair of memoryless strategies $(\sigma, \pi)$, the play $G^{(\sigma,\pi)}$ can be depicted as a Markov chain with the set of states $V$.

# Plays (2)

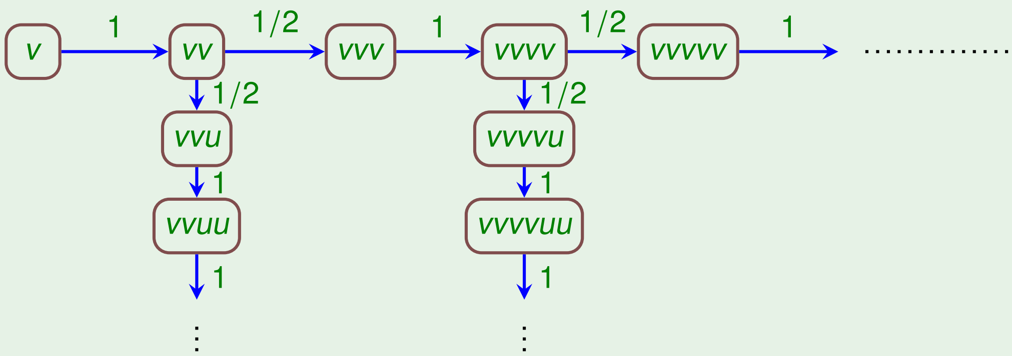
**Example 6 (A game and its play)**

- 

$\mathcal{W} = \{w \in Run(v) \mid w(i) = u \text{ for some } i \in \mathbb{N}\}$

- $\sigma(wv) = \begin{cases} v \xrightarrow{1} v & \text{if } |wv| \text{ is odd;} \\ v \xrightarrow{1/2} v,\ v \xrightarrow{1/2} u & \text{otherwise.} \end{cases} \qquad \pi(wu) = u \xrightarrow{1} u$

- $\mathcal{P}(\mathcal{W}^{(\sigma,\pi)}) = 1$








# Stochastic games have a value

**Theorem 7 (Donald Martin, 1998)**

*Let $G = (V, E, (V_\Box, V_\Diamond, V_\bigcirc), Prob)$ be a game, $v \in V$, and $\mathcal{W} \subseteq Run_G(v)$ a Borel set of runs. Then*

$$\sup_\sigma \inf_\pi \mathcal{P}(\mathcal{W}^{(\sigma,\pi)}) \quad = \quad \inf_\pi \sup_\sigma \mathcal{P}(\mathcal{W}^{(\sigma,\pi)})$$

- The equality of Thm. 7 defines the $\mathcal{W}$-value of $v$, denoted $val_{\mathcal{W}}(v)$.
- Thm. 7 does not impose any restrictions on $G$. The set of vertices and the branching degree of $G$ can be infinite.
- References:

  - D.A. Martin. *The Determinacy of Blackwell Games.* The Journal of Symbolic Logic, Vol. 63, No. 4 (Dec., 1998), pp. 1565–1581.
  - A. Maitra and W. Sudderth. *Finitely Additive Stochastic Games with Borel Measurable Payoffs.* International Journal of Game Theory, Vol. 27 (1998), pp. 257–267.

# Optimal strategies

## Definition 8

*Let $G = (V, E, (V_\Box, V_\Diamond, V_\bigcirc), Prob)$ be a game, $v \in V$, and $\mathcal{W} \subseteq Run_G(v)$ a Borel set of runs. Let $\varepsilon \in [0,1]$.*

- *An $\varepsilon$-optimal maximizing strategy is a strategy $\sigma$ for player $\Box$ such that for every strategy $\pi$ of player $\Diamond$ we have that $\mathcal{P}(\mathcal{W}^{(\sigma,\pi)}) \geq val_{\mathcal{W}}(v) - \varepsilon$.*
- *An $\varepsilon$-optimal minimizing strategy is a strategy $\pi$ for player $\Diamond$ such that for every strategy $\sigma$ of player $\Box$ we have that $\mathcal{P}(\mathcal{W}^{(\sigma,\pi)}) \leq val_{\mathcal{W}}(v) + \varepsilon$.*

*An optimal maximizing/minimizing strategy is a $0$-optimal maximizing/minimizing strategy.*

- According to Thm. 7, $\varepsilon$-optimal maximizing/minimizing strategies exist for every $\varepsilon > 0$.
- ...and we cannot say much more in the general setting.





# Reachability objectives

**Now we restrict ourselves to reachability objectives and show the following:**

- Properties of minimizing strategies:
  - optimal minimizing strategies do not necessarily exist and $\varepsilon$-optimal strategies may require infinite memory, even for MDPs;
  - in every finitely-branching game, there is an optimal minimizing MD strategy;
- Properties of maximizing strategies:
  - optimal maximizing strategies do not necessarily exist, even for finitely-branching MDPs;
  - in every finite-state game, there is an optimal maximizing MD strategy;

# Reachability games have a value (1)

**Theorem 9**

*Let $G = (V, E, (V_\Box, V_\Diamond, V_\bigcirc), Prob)$ be a game, $u \in V$ a target vertex. For every $v \in V$ we have that*

$$\sup_\sigma \inf_\pi \mathcal{P}(Reach(u)^{(\sigma,\pi)}) \quad = \quad \inf_\pi \sup_\sigma \mathcal{P}(Reach(u)^{(\sigma,\pi)})$$



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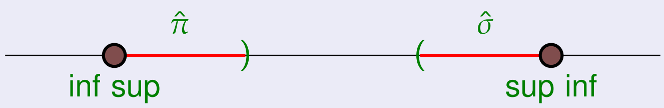
# Reachability games have a value (2)

**Proof sketch.**

- Let $\Gamma : [0,1]^{|V|} \to [0,1]^{|V|}$ be a (monotonic) function defined by

$$\Gamma(\alpha)(v) = \begin{cases} 1 & \text{if } v = u; \\ \sup\{\alpha(v') \mid (v,v') \in E\} & \text{if } v \neq u \text{ and } v \in V_\Box; \\ \inf\{\alpha(v') \mid (v,v') \in E\} & \text{if } v \neq u \text{ and } v \in V_\Diamond; \\ \sum_{(v,v')\in E} Prob(v,v') \cdot \alpha(v') & \text{if } v \neq u \text{ and } v \in V_\bigcirc. \end{cases}$$

- $\mu\Gamma(v) \ \leq \ \sup_\sigma \inf_\pi \mathcal{P}(Reach(u)^{(\sigma,\pi)}) \ \leq \ \inf_\pi \sup_\sigma \mathcal{P}(Reach(u)^{(\sigma,\pi)})$
  - the second inequality holds for all Borel objectives.



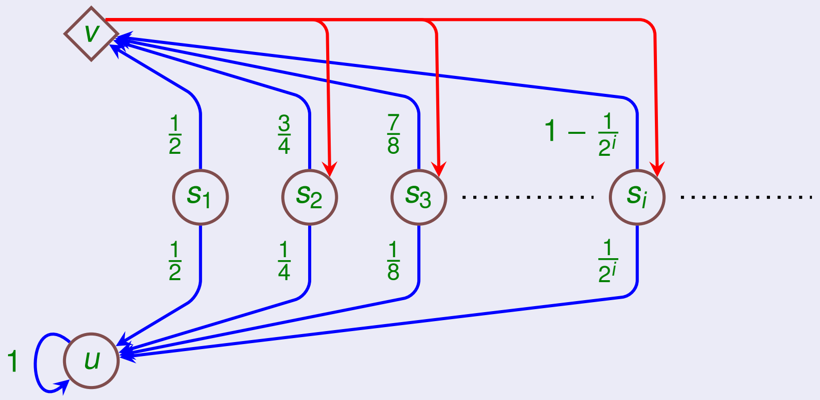

  - the tuple of all $\sup_\sigma \inf_\pi \mathcal{P}(Reach(u)^{(\sigma,\pi)})$ is a fixed-point of $\Gamma$;
- It cannot be that $\mu\Gamma(v) \ < \ \inf_\pi \sup_\sigma \ \mathcal{P}(Reach(u)^{(\sigma,\pi)})$
  - For all $\varepsilon > 0$ and $v \in V$, there is a strategy $\hat{\pi}$ such that $\sup_\sigma \mathcal{P}(Reach(u)^{(\sigma,\hat{\pi})}) \ \leq \ \mu\Gamma(v) + \varepsilon$. $\Box$

# Minimizing strategies (1)

## Theorem 10

*Optimal minimizing strategies do not necessarily exist, and $\varepsilon$-optimal minimizing strategies may require infinite memory (even for MDPs).*

## Proof.




□





# Minimizing strategies (2)

## Definition 11 (Locally optimal strategy)

*Let $G = (V, E, (V_\Box, V_\Diamond, V_\bigcirc), Prob)$ be a game.*

- *An edge $(v, v') \in E$ is value minimizing if*

$$val(v') = \min \left\{ val(\hat{v}) \in V \mid (v, \hat{v}) \in E \right\}$$

- *A locally optimal minimizing strategy is a strategy which in every play selects only value minimizing edges.*
- *Analogously, we define value maximizing edges and locally optimal maximizing strategies.*

## Observation 12

- *Every optimal maximizing/minimizing strategy is also a locally optimal maximizing/minimizing strategy.*
- *For every finitely-branching game, there is a locally optimal maximizing/minimizing MD strategy.*

# Minimizing strategies (3)

**Theorem 13**

*Every locally optimal min. strategy is an optimal min. strategy.*

**Proof.**

Let $v \in V$ be an initial vertex, and $u \in V$ a target vertex.

(1) After playing $k$ rounds according to a locally optimal minimizing strategy, player $\Diamond$ can switch to $\varepsilon$-optimal minimizing strategies in the current vertices of the play. Thus, we always (for every $k$ and $\varepsilon > 0$) obtain an $\varepsilon$-optimal minimizing strategy for $v$.

(2) Let $\pi$ be a locally optimal min. strategy which is not optimal.

- Then there is a strategy $\sigma$ of player $\Box$ such that $\mathcal{P}(Reach(u)^{(\sigma,\pi)}) = val(v) + \delta$, where $\delta > 0$.
- This means that there is $k \in \mathbb{N}$ such that $\mathcal{P}(Reach^k(u)^{(\sigma,\pi)}) > val(v) + \frac{\delta}{2}$.
- Hence, if player $\Diamond$ switches to $\frac{\delta}{2}$-optimal minimizing strategy after playing $k$ rounds according to $\pi$, we do not obtain a $\frac{\delta}{2}$-optimal minimizing strategy for $v$. $\Box$





# Minimizing strategies (4)

**Corollary 14 (Properties of minimizing strategies.)**

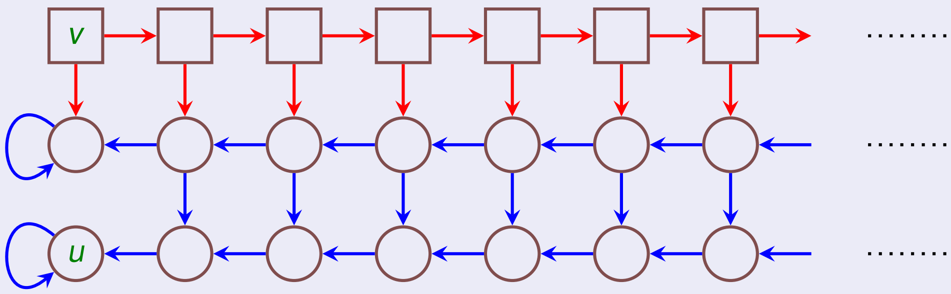
- *Optimal minimizing strategies do not necessarily exist and $\varepsilon$-optimal minimizing strategies may require infinite memory, even for MDPs.*
- *In every finitely-branching game, there is an optimal minimizing MD strategy.*
- *If there is some optimal minimizing strategy, then there is also an optimal minimizing MD strategy.*

# Maximizing strategies (1)

## Theorem 15

*Optimal maximizing strategies do not necessarily exist (even for finitely-branching MDPs).*

## Proof.




□



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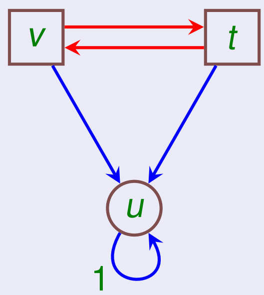
# Maximizing strategies (2)

## Observation 16

*A locally optimal maximizing strategy is not necessarily an optimal maximizing strategy. This holds even for finite-state MDPs.*

## Proof.

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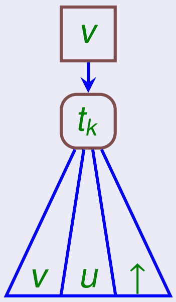
# Maximizing strategies (3)

### Theorem 17

*Let $v \in V_\Box$ be a vertex with finitely many successors $t_1, \ldots, t_n$. Then there is $1 \le i \le n$ such that $val(v)$ does not change if all edges $(v, t_j)$, where $i \neq j$, are deleted from the game.*

### Proof.




- $V_{t_k}^{(\sigma,\pi)} = \begin{cases} \frac{\mathcal{P}(u)}{\mathcal{P}(u)+\mathcal{P}(\uparrow)} & \text{if } \mathcal{P}(u)+\mathcal{P}(\uparrow) > 0; \\ 0 & \text{otherwise;} \end{cases}$
- $V_{t_k}^{\sigma} = \inf_\pi V_{t_k}^{(\sigma,\pi)}$
- $V_{t_k} = \sup_\sigma V_{t_k}^{\sigma}$
- There must be some $k$ such that $V_{t_k} = val(v)$.
- We put $i = k$.

$\square$





# Maximizing strategies (4)

### Corollary 18 (Properties of maximizing strategies.)

- *Optimal maximizing strategies do not necessarily exist, even for finitely-branching MDPs.*
- *In every finite-state game, there is an optimal maximizing MD strategy.*

References:


- M.L. Puterman. *Markov Decision Processes*, Wiley, 1994. (Theorem 7.2.11 implies the last claim of Corollary 18 for MDPs.)
- T. Brázdil, V. Brožek, V. Forejt, A. Kučera. *Reachability in recursive Markov decision processes.* Information and Computation, vol. 206, pp. 520–537, 2008.

# Algorithms for finite-state MDPs and games

**We show how to compute the values and optimal strategies for reachability objectives in finite-state games and MDPs.**

- For finite-state MDPs we have that
  - the values and optimal strategies are computable in polynomial time;
- For finite-state games we have that
  - the values and optimal strategies are computable in polynomial space (for a fixed number of randomized vertices, the problem is in **P**);



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# Finite-state MDPs (1)

**Theorem 19**

*Let $G = (V, E, (V_\square, V_\bigcirc), Prob)$ be a finite-state MDP. Then*

- $\mathcal{V}^{=0} = \{v \in V \mid val(v) = 0\}$
- $\mathcal{V}^{=1} = \{v \in V \mid val(v) = 1\}$

*are computable in polynomial time.*

**Proof.**

It suffices to realize that $\mathcal{V}^{=1}$ is exactly the greatest $S \subseteq V$ satisfying the following conditions:

- If $v \in S$, then there is a finite path from $v$ to the target vertex $u$ which visits only the vertices of $S$.
- If $v \in S \cap V_\bigcirc$, then all successors of $v$ belong to $S$.

Hence, $\mathcal{V}^{=1}$ is computable in polynomial time. The set $\mathcal{V}^{=0}$ can be computed similarly. Note that the sets $\mathcal{V}^{=1}$ and $\mathcal{V}^{=0}$ depend only on the "topology" of $G$. □

# Finite-state MDPs (2)

**Theorem 20**

*Let $G = (V, E, (V_\Box, V_\bigcirc), Prob)$ be a finite-state MDP where $Prob$ is rational. The values $val(v)$, $v \in V$, are rational and computable in polynomial time. An optimal maximizing strategy is also constructible in polynomial time.*

**Proof.**

Let $V = \{v_1, \ldots, v_n\}$, where $v_n$ is the target vertex.

**minimize** $x_1 + \cdots + x_n$
subject to
$x_n = 1$
$x_i \geq x_j$ for all $(v_i, v_j) \in E$ where $v_i \in V_\Box$ and $i < n$
$x_i = \sum_{(v_i,v_j)\in E} Prob(v_i, v_j) \cdot x_j$ for all $v_i \in V_\bigcirc$, $i < n$
$x_i \geq 0$ for all $i \in \{1, \ldots, n\}$

An optimal strategy can be constructed by successively removing the ougoing edges of every $v \in V_\Box$ untill only one such edge is left. □



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# Finite-state MDPs (3)

Similarly, one can prove the following theorem about minimizing MDPs.

**Theorem 21**

*Let $G = (V, E, (V_\Diamond, V_\bigcirc), Prob)$ be a finite-state MDP. The sets*

- $\mathcal{V}^{=0} = \{v \in V \mid val(v) = 0\}$
- $\mathcal{V}^{=1} = \{v \in V \mid val(v) = 1\}$

*are computable in polynomial time. If $Prob$ is rational, then the values $val(v)$, $v \in V$, are rational and computable in polynomial time. An optimal minimizing strategy is also constructible in polynomial time.*

# Finite-state games (1)

### Theorem 22

*Let $G = (V, E, (V_\Box, V_\Diamond, V_\bigcirc), Prob)$ be a finite-state game. Then*

- $\mathcal{V}^{=0} = \{v \in V \mid val(v) = 0\}$
- $\mathcal{V}^{=1} = \{v \in V \mid val(v) = 1\}$

*are computable in polynomial time.*

### Proof.

- $\mathcal{V}^{>0} = \mu\Gamma$, where $\Gamma : 2^V \to 2^V$ is defined as follows:

$$\begin{aligned}\Gamma(A) &= \{u\} \cup \{v \in V_\Box \cup V_\bigcirc \mid \exists (v,v') \in E \text{ s.t. } v' \in A\}\\ &\cup \{v \in V_\Diamond \mid \forall (v,v') \in E \text{ we have that } v' \in A\}\end{aligned}$$

- $\mathcal{V}^{=0} = V \setminus \mathcal{V}^{>0}$
- $\mathcal{V}^{<1} = \mu\Gamma$, where $\Gamma : 2^V \to 2^V$ is defined as follows:

$$\begin{aligned}\Gamma(A) &= \mathcal{V}^{=0} \cup \{v \in V_\Diamond \cup V_\bigcirc \mid \exists (v,v') \in E \text{ s.t. } v' \in A\}\\ &\cup \{v \in V_\Box \mid \forall (v,v') \in E \text{ we have that } v' \in A\}\end{aligned}$$

- $\mathcal{V}^{=1} = V \setminus \mathcal{V}^{<1}$

□



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# Finite-state games (2)

### Theorem 23 (Anne Condon, 1992)

*Let $G = (V, E, (V_\Box, V_\Diamond, V_\bigcirc), Prob)$ be a finite-state game. The problem whether $val(v) > \frac{1}{2}$ for a given $v \in V$ is in* ***NP*** $\cap$ ***coNP****.*

### Proof.

Since both players have optimal MD strategies, it suffices to

- “guess” an optimal MD strategy for player □ (or player ◇);
- compute the value in the resulting MDP by solving the associated linear program.

□

Obviously, $val(v)$ and the optimal strategies for both players are computable in polynomial space.

# Finite-state games (3)

The original proof of Condon is somewhat different.

- It is assumed that randomized vertices have two successors, and the associted edges have probability $\frac{1}{2}$.
- It is shown that if $val(v) > \frac{1}{2}$, then $val(v) > \frac{1}{2} + \frac{1}{4^n}$, where $n = |V|$.
- The original game $G$ is efficiently transformed into another stopping game $G'$ such that $val(v) > \frac{1}{2}$ in $G$ iff $val(v) > \frac{1}{2}$ in $G'$.


- The main advantage of stopping games is that optimality equations have a unique solution. Thus, it is shown that both players have optimal MD strategies in stopping games.
- Many (not all) of the existing algorithms which compute the value and an optimal strategy in stochastic games assume that the game is stopping.



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# Finite-state games (4)

**Theorem 24 (Gimbert, Horn, 2008)**

*The values and MD optimal strategies in a finite-state game $G = (V, E, (V_\Box, V_\Diamond, V_\bigcirc), Prob)$ are computable in*

$$\mathcal{O}\Big(|V_\bigcirc|! \cdot (\log(|V|)\,|E| + |p|)\Big)$$

*time, where $|p|$ is the maximal bit-length of an edge probability.*

**Remark 25**

The question whether finite-state stochastic games are solvable in **P** is a longstanding open problem in algorithmic game theory.

References:

- A. Condon. *The Complexity of Stochastic Games.* Information and Computation, 96(2):203–224, 1992.
- L.S. Shapley. *Stochastic games.* Proceedings of the National Academy of Sciences USA, 39:1095–1100, 1953.
- H. Gimbert, F. Horn. *Simple Stochastic Games with Few Random Vertices Are Easy to Solve.* Proc. FoSSaCS 2008, pp. 5–19, LNCS 4962, Springer, 2008.

# Infinite-state games

- Interesting classes of infinite-state stochastic games are obtained by extending non-deterministic computational devices with randomized choice. So far, most of the results consider
  - pushdown automata (recursive state machines);
  - lossy channel systems.





# Stochastic BPA games (1)

### Definition 26

*A stochastic BPA game is a tuple* $\Delta = (\Gamma, \hookrightarrow, (\Gamma_\Box, \Gamma_\Diamond, \Gamma_\bigcirc), Prob)$ *where*

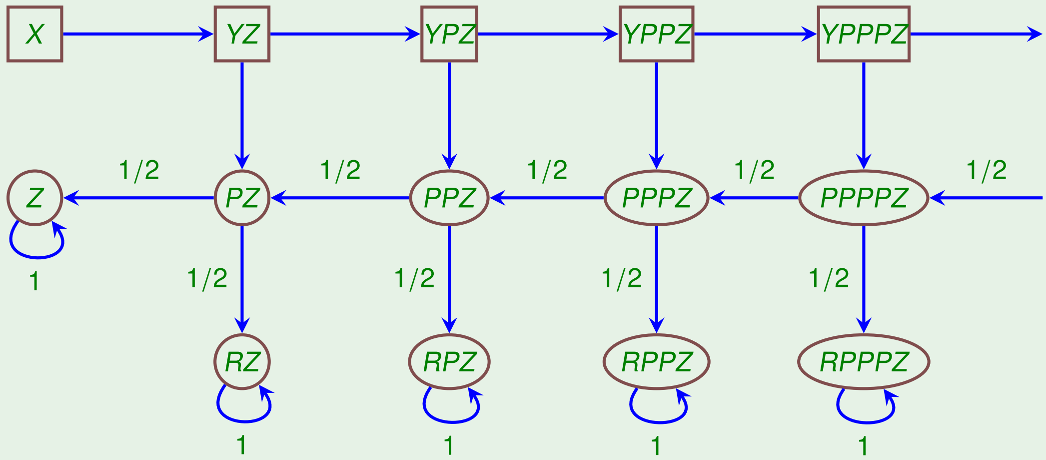
- $\Gamma$ *is a finite stack alphabet,*
- $\hookrightarrow \subseteq \Gamma \times \Gamma^{\leq 2}$ *is a finite set of rules,*
- $(\Gamma_\Box, \Gamma_\Diamond, \Gamma_\bigcirc)$ *is a partition of* $\Gamma$,
- $Prob$ *is a probability assignment which to each* $X \in \Gamma_\bigcirc$ *assigns a rational positive probability distribution on the set of all rules of the form* $X \hookrightarrow \alpha$.

# Stochastic BPA games (2)

## Example 27

Let $\Gamma = \{X, Y, Z, P, R\}$, where $\Gamma_\Box = \{X, Y\}$, $\Gamma_\Diamond = \emptyset$, $\Gamma_\bigcirc = \{P, R\}$, and

$$X \hookrightarrow YZ, \quad Y \hookrightarrow YP, \quad Y \hookrightarrow P, \quad P \overset{1/2}{\hookrightarrow} R, \quad P \overset{1/2}{\hookrightarrow} \varepsilon, \quad R \overset{1}{\hookrightarrow} R$$



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# BPA MDPs with reachability objectives (1)

- Let $\Delta = (\Gamma, \hookrightarrow, (\Gamma_\Box, \Gamma_\bigcirc), Prob)$ be a BPA Markov decision process, and $T \subseteq \Gamma^*$ a regular set of target configurations.
- We can safely assume that $T = \mathcal{R}\Gamma^*$, where $\mathcal{R} \subseteq \Gamma$.
- Consider the sets
  - $\mathcal{W}^{>0} = \{\alpha \in \Gamma^* \mid \exists \sigma : \mathcal{P}(Reach(\alpha \to T)^\sigma) > 0\}$
  - $\mathcal{W}^{=0} = \{\alpha \in \Gamma^* \mid \exists \sigma : \mathcal{P}(Reach(\alpha \to T)^\sigma) = 0\}$
  - $\mathcal{W}^{=1} = \{\alpha \in \Gamma^* \mid \exists \sigma : \mathcal{P}(Reach(\alpha \to T)^\sigma) = 1\}$
  - $\mathcal{W}^{<1} = \{\alpha \in \Gamma^* \mid \exists \sigma : \mathcal{P}(Reach(\alpha \to T)^\sigma) < 1\}$

  We show that these sets are regular and the associated finite-state automata are computable in polynomial time.

# BPA MDPs with reachability objectives (2)

**Theorem 28**

*The set $\mathcal{W}^{>0}$ is effectively regular.*

- Let us consider two sets of stack symbols:
  - $\mathcal{A} = \{X \in \Gamma \mid X \hookrightarrow^* \varepsilon\}$
  - $\mathcal{B} = \{X \in \Gamma \mid X \hookrightarrow^* R\beta, \text{ where } R \in \mathcal{R} \text{ and } \beta \in \Gamma^*\}$
- We have that $\mathcal{W}^{>0} = \mathcal{A}^*\mathcal{B}\Gamma^*$.
- The sets $\mathcal{A}$ and $\mathcal{B}$ are (easily) computable in polynomial time.





# BPA MDPs with reachability objectives (3)

**Theorem 29**

*The set $\mathcal{W}^{=0}$ is effectively regular.*

- Let us consider two sets of stack symbols:
  - $\mathcal{A} = \{X \in \Gamma \mid \exists \sigma : \mathcal{P}(Reach(X \rightarrow T_\varepsilon)^\sigma) = 0\}$
  - $\mathcal{B} = \{X \in \Gamma \mid \exists \sigma : \mathcal{P}(Reach(X \rightarrow T)^\sigma) = 0\}$
- We have that $\mathcal{W}^{=0} = \mathcal{B}^* \cup \mathcal{B}^*\mathcal{A}\Gamma^*$.
- The pair $(\mathcal{A}, \mathcal{B})$ is the greatest fixed-point of a suitably defined $\Theta : 2^\Gamma \times 2^\Gamma \rightarrow 2^\Gamma \times 2^\Gamma$.

# BPA MDPs with reachability objectives (4)

**Theorem 30**

*The set $\mathcal{W}^{=1}$ is effectively regular.*

- Let us consider three sets of stack symbols:
  - $\mathcal{A} = \{X \in \Gamma \mid \exists\sigma : \mathcal{P}(Reach(X \rightarrow T)^\sigma) = 1\}$
  - $\mathcal{B} = \{X \in \Gamma \mid \exists\sigma : \mathcal{P}(Reach(X \rightarrow T_\varepsilon)^\sigma) = 1$ and $\mathcal{P}(Reach(X \rightarrow T)^\sigma) > 0\}$
  - $\mathcal{C} = \{X \in \Gamma \mid \exists\sigma : \mathcal{P}(Reach(X \rightarrow \varepsilon)^\sigma) = 1\}$
- We have that $\mathcal{W}^{=1} = (\mathcal{B} \cup \mathcal{C})^* \mathcal{A} \Gamma^*$.
- Moreover, the set $\mathcal{W}_\varepsilon^{=1} = \{\alpha \in \Gamma^* \mid \exists\sigma : \mathcal{P}(Reach(\alpha \rightarrow T_\varepsilon)^\sigma) = 1\}$ is equal to $(\mathcal{B} \cup \mathcal{C})^* \cup (\mathcal{B} \cup \mathcal{C})^* \mathcal{A} \Gamma^*$.
- The set $\mathcal{C}$ is computable in polynomial time by the results of Etessami & Yannakakis (STACS 2006).
- The sets $\mathcal{A}, \mathcal{B}$ are again computable as the greatest fixed-point of a suitably defined $\Theta : 2^\Gamma \times 2^\Gamma \rightarrow 2^\Gamma \times 2^\Gamma$.







# BPA MDPs with reachability objectives (5)

**Theorem 31**

*The set $\mathcal{W}^{<1}$ is effectively regular.*

- Let us consider two sets of stack symbols:
  - $\mathcal{A} = \{X \in \Gamma \mid \exists\sigma : \mathcal{P}(Reach(X \rightarrow \varepsilon)^\sigma) > 0\}$
  - $\mathcal{B} = \{X \in \Gamma \mid \exists\sigma : \mathcal{P}(Reach(X \rightarrow T_\varepsilon)^\sigma) < 1\}$
- We have that $\mathcal{W}^{<1} = \mathcal{A}^* \cup (\mathcal{A}^* \mathcal{B} \Gamma^*)$.
- The set $\mathcal{A}$ is computable by the previous results.
- The challenge is to compute the set $\mathcal{B}$. The membership $X \in \mathcal{B}$ is witnessed in two ways.
  - There is strategy $\sigma$ such that $\mathcal{P}(Reach(X \rightarrow \mathcal{W}_\varepsilon^{=0})^\sigma) > 0)$.
  - There is a "closed" family of stack symbols disjoint with $\mathcal{R}$ (this family forms a BPA MDP $\Delta'$) such that $X$ can be forced to enter $\Delta'$ with a positive probability, and there is a strategy $\sigma$ in $\Delta'$ such that $\mathcal{P}(Reach(X \rightarrow \varepsilon)^\sigma) < 1$.

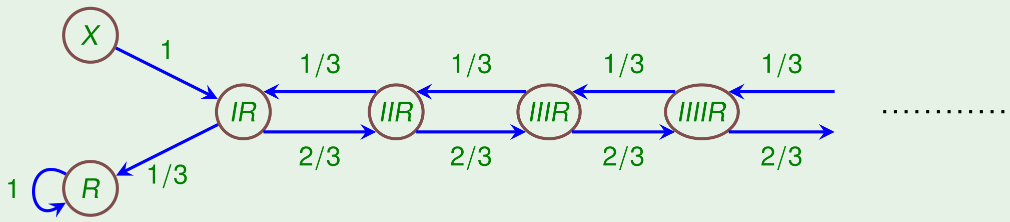
# BPA MDPs with reachability objectives (6)

## Example 32

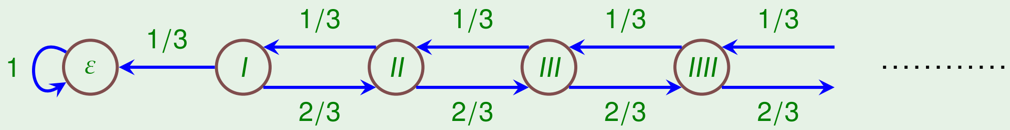
- Let $\Gamma = \{X, R, I\}$, where $\Gamma_\Box = \emptyset$, $\Gamma_\bigcirc = \{X, R, I\}$, and

$$X \overset{1}{\hookrightarrow} IR,\ R \overset{1}{\hookrightarrow} R,\ I \overset{2/3}{\hookrightarrow} II,\ I \overset{1/3}{\hookrightarrow} \varepsilon$$

- Let $\mathcal{R} = \{R\}$. Then $\mathcal{A} = \emptyset$, $\mathcal{B} = \{X, I\}$, hence $\mathcal{W}^{<1} = \{X, I\}\Gamma^*$.




- The membership $I \in \mathcal{B}$ is witnesses by a BPA MDP $\Delta'$ where $\Gamma' = \Gamma'_\bigcirc = \{I\}$ and $I \overset{2/3}{\hookrightarrow} II$, $I \overset{1/3}{\hookrightarrow} \varepsilon$.








# BPA games with termination objectives

## Theorem 33 (Etessami, Yannakakis, 2006)

*Let $\Delta = (\Gamma, \hookrightarrow, (\Gamma_\Box, \Gamma_\Diamond, \Gamma_\bigcirc), Prob)$ be a BPA game. For every $\alpha \in \Gamma^*$, let $val(\alpha) = val_{Reach(\varepsilon)}(\alpha)$. Then*

- *for all $\beta, \gamma \in \Gamma^*$ we have that $val(\beta\gamma) = val(\beta) \cdot val(\gamma)$;*
- *the tuple of all $val(X)$, where $X \in \Gamma$, forms the least solution of a system of recursive equations constructed as follows:*
  - *if $X \hookrightarrow \varepsilon$ and $X \in \Gamma_\Box$, we put $V_X = 1$;*
  - *otherwise, we put*
    - $V_X = \max_{X \hookrightarrow Y,\ X \hookrightarrow YZ} \{V_Y, V_Y \cdot V_Z\}$
    - $V_X = \min_{X \hookrightarrow Y,\ X \hookrightarrow YZ} \{V_Y, V_Y \cdot V_Z\}$
    - $V_X = \sum_{X \overset{p}{\hookrightarrow} \varepsilon} p + \sum_{X \overset{p}{\hookrightarrow} Y} p \cdot V_Y + \sum_{X \overset{p}{\hookrightarrow} YZ} p \cdot V_Y \cdot V_Z$

    *depending on whether $X \in \Gamma_\Box$, $X \in \Gamma_\Diamond$, or $X \in \Gamma_\bigcirc$, respectively.*
- *both players have optimal SMD strategies constructible in polynomial space;*
- *the problem whether $val(X) = 1$ and $val(X) \leq \varrho$ is in **NP** $\cap$ **coNP** and **PSPACE**, respectively.*

# BPA games with reachability objectives

**Theorem 34 (Brázdil, Brožek, K., Obdržálek, 2009)**

*Let $\Delta = (\Gamma, \hookrightarrow, (\Gamma_\Box, \Gamma_\Diamond, \Gamma_\bigcirc), Prob)$ be a BPA game, and $T \subseteq \Gamma^*$ a regular set of target configurations. For every $\alpha \in \Gamma^*$, let $val(\alpha) = val_{Reach(T)}(\alpha)$. Then*

- *The sets*
  - $\mathcal{V}^{=0} = \{v \in V \mid val(v) = 0\}$
  - $\mathcal{W}^{=1} = \{v \in V \mid val(v) = 1 \text{ and player } \Box \text{ has an optimal max. strategy}\}$

  *are regular. The associated finite-state automata are computable by a deterministic polynomial-time algorithm with* **NP** $\cap$ **coNP** *oracle.*
- *The membership to $\mathcal{V}^{=0}$ and $\mathcal{W}^{=1}$ is in* **NP** $\cap$ **coNP**.
- *Optimal strategies for both players are not necessarily SMD and it does not hold that $val(\beta\gamma) = val(\beta) \cdot val(\gamma)$.*

References:


- K. Etessami, M. Yannakakis. *Efficient Qualitative Analysis of Classes of Recursive Markov Decision Processes and Simple Stochastic Games.* Proc. STACS 2006, pp. 634–645, LNCS 3884, Springer 2006.
- T. Brázdil, V. Brožek, A. Kučera, and J. Obdržálek. *Qualitative Reachability in Stochastic BPA Games.* Proc. STACS 2009, pp. 207–218, 2009.






# A taxonomy of objectives in stoch. games

- Linear-time objectives.
  - Specified by Borel sets of runs in stochastic games.
  - Büchi, parity, Rabin, Street, Muller winning objectives.
  - In finite-state games, optimal strategies exist and are either memoryless (Büchi, parity) or require a finite memory.
- Long-run average objectives.
  - Specified as certain “limit” random variables defined over runs.
  - Mean payoff: $MP(w) = \lim_{n\to\infty} \frac{\sum_{i=0}^{n} rew(w(i))}{n}$
  - Discounted payoff: $DP(w) = \sum_{i=0}^{\infty} \lambda^i \cdot rew(w(i))$
  - The players aim at maximizing/minimizing the expected value of $MP$, $DP$.
- Multi-objectives (studied mainly for MDPs).
  - Specified by a vector of characteristics that are to be maximized/minimized. Generally, there is no “best” vector of values. The algorithms aim at computing/approximating the associated Pareto curve.
- Branching-time objectives.
  - Specified by formulae of branching-time logics that are interpreted over Markov chains (such as PCTL or PCTL*).

# Branching-time winning objectives

- Specified by formulae of branching-time logics that are interpreted over Markov chains (such as PCTL or PCTL*).
- $\mathcal{G}^{=1}(p \Rightarrow \mathcal{F}^{\geq 0.1} q)$
- The aim of player □ and player ◇ is to satisfy and falsify a given formula, respectively.
- Properties of stochastic games with branching-time objectives are quite different from the ones with linear-time objectives.



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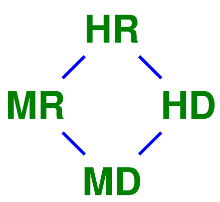
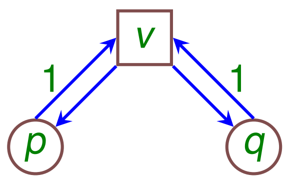
# Properties of games with b.-t. objectives (I)

- Memory and randomization help:



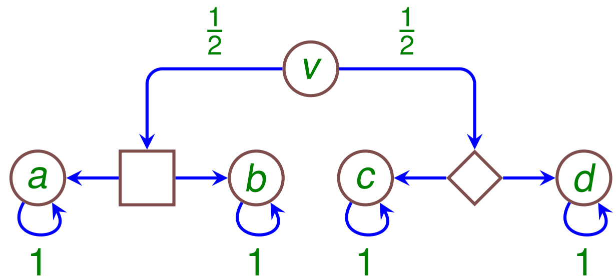

- Consider the following game:




- $\mathcal{X}^{=1} p \wedge \mathcal{F}^{=1} q$. Requires memory.
- $\mathcal{X}^{>0} p \wedge \mathcal{X}^{>0} q$. Requires randomization.
- $\mathcal{X}^{>0} p \wedge \mathcal{X}^{>0} q \wedge \mathcal{F}^{=1}\mathcal{G}^{=1} q$. Requires both memory and randomization.
- In some cases, infinite memory is required.

# Properties of games with b.-t. objectives (II)

- The games are not determined (for any strategy type).
- $\mathcal{F}^{=1}(a \vee c) \;\vee\; \mathcal{F}^{=1}(b \vee d) \;\vee\; \left(\mathcal{F}^{>0}c \wedge \mathcal{F}^{>0}d\right)$








# Who wins the game (MD strategies) ?

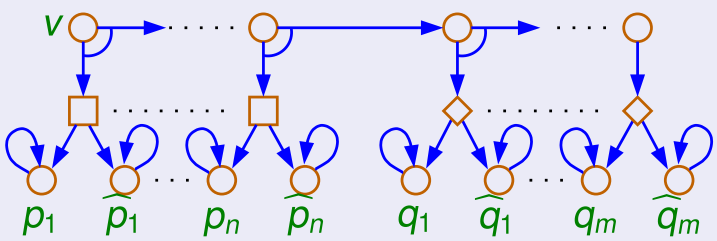
**Theorem 35 (Brázdil, Brožek, Forejt, K., 2006)**

*The existence of a winning MD strategy for player $\Box$ is $\Sigma_2 = \mathbf{NP}^{\mathbf{NP}}$ complete.*

**Proof.**

The membership to $\Sigma_2$ follows easily. The $\Sigma_2$-hardness can be established as follows:

- Let $\exists x_1, \cdots, x_n \forall y_1, \cdots, y_m\, B$ be a $\Sigma_2$ formula.
- Consider the following game:




- Let $\varphi$ be the PCTL formula obtained from $B$ by substituting each occurrence of $x_i$, $\neg x_i$, $y_j$, and $\neg y_j$ with $\mathcal{F}^{>0}p_i$, $\mathcal{F}^{>0}\widehat{p}_i$, $\mathcal{F}^{>0}q_j$, and $\mathcal{F}^{>0}\widehat{q}_j$, respectively. $\Box$

# Who wins the game (MR strategies) ?

**Theorem 36 (Brázdil, Brožek, Forejt, K., 2006)**

*The existence of a winning MR strategy for player $\Box$ is $\Sigma_2$-hard and in **EXPTIME**. For the qualitative fragment of PCTL, the problem is $\Sigma_2$-complete.*

**Proof.**

- The $\Sigma_2$-hardness is established similarly as for MD strategies.
- The membership to **EXPTIME** is obtained by encoding the condition into Tarski algebra.
- The membership to $\Sigma_2$ for the qualitative PCTL follows easily.

$\Box$



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# Who wins the game (HD, HR, FD, FR) ?

**Theorem 37 (Brázdil, Brožek, Forejt, K., 2006)**

*The existence of a winning HD (or HR) strategy for player $\Box$ in MDPs is highly undecidable (and $\Sigma_1^1$-complete). Moreover, the existence of a winning FD (or FR) strategy is also undecidable.*

- The result holds for the $\mathcal{L}(\mathcal{F}^{=1/2}, \mathcal{F}^{=1}, \mathcal{F}^{>0}, \mathcal{G}^{=1})$ fragment of PCTL (the role of $\mathcal{F}^{=1/2}$ is crucial).
- The proof is obtained by reduction of the problem whether a given non-deterministic Minsky machine has an infinite recurrent computation.

# The undecidability proof

- A non-deterministic Minsky machine $\mathcal{M}$ with two counters $c_1, c_2$:

$$1 : ins_1, \cdots, n : ins_n$$

  where each $ins_i$ takes one of the following forms:
  - $c_j := c_j + 1;\ goto\ k$
  - $if\ c_j{=}0\ then\ goto\ k\ else\ c_j := c_j - 1;\ goto\ m$
  - $goto\ \{k\ or\ m\}$
- The problem whether a given non-deterministic Minsky machine with two counters initialized to zero has an infinite computation that executes $ins_1$ infinitely often is $\Sigma_1^1$-complete.
- For a given machine $\mathcal{M}$, we construct a finite-state MDP $G(\mathcal{M})$ and a formula $\varphi \in \mathcal{L}(\mathcal{F}^{=1/2}, \mathcal{F}^{=1}, \mathcal{F}^{>0}, \mathcal{G}^{=1})$ such that $\mathcal{M}$ has an infinite recurrent computation iff player $\Box$ has a winning HD (or HR) strategy for $\varphi$ in a distinguished vertex $v$ of $G(\mathcal{M})$.



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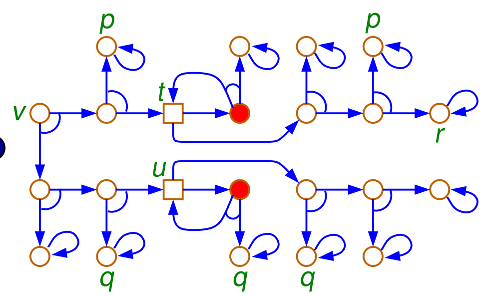
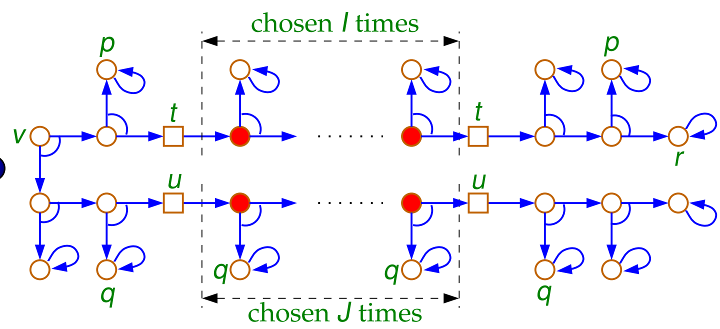
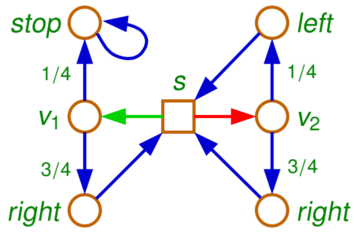
# The construction of $G(\mathcal{M})$ and $\varphi$

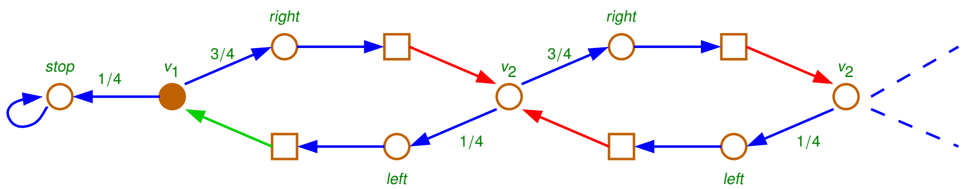
- 


- 


- $I = J < \omega$ iff $v \models \mathcal{F}^{>0} r \wedge \mathcal{F}^{=1/2}(p \vee q)$
- The probability of $\mathcal{F}(p \vee q)$: $0.01\underbrace{0\cdots0}_{I}01 + 0.001\underbrace{1\cdots1}_{J}1$

# Positive results (1)

- We restrict ourselves to qualitative fragments of probabilistic branching time logics.
- Even MDPs with qualitative PCTL objectives may require infinite memory.
- 


  $\mathcal{G}^{>0}(\neg stop \wedge \mathcal{F}^{>0} stop)$
  $\wedge\ \mathcal{G}^{=1}(s \Rightarrow (X^{=1} v_1 \vee X^{=1} v_2))$
- A winning strategy: if $\#left < \#right$ use the red transition, otherwise use the green one.
- 

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# Positive result (2)

### Theorem 38 (Brázdil, Forejt, K., 2008)

- *The existence of a winning HD (or HR) strategy for player □ in MDPs with qualitative PECTL* objectives is decidable in time which is polynomial in the size of MDP and doubly exponential in the size of the formula. The problem is* **2-EXPTIME***-hard.*
- *Moreover, iff there is a winning HD (or HR) strategy, there is also a one-counter winning strategy and one can effectively construct a one-counter automaton which implements this strategy (the associated complexity bounds are the same as above).*

References:

- T. Brázdil, V. Brožek, V. Forejt, and A. Kučera. *Stochastic Games with Branching-Time Winning Objectives.* Proc. of LICS 2006, pp. 349-358, 2006.
- T. Brázdil, V. Forejt, and A. Kučera. *Controller Synthesis and Verification for Markov Decision Processes with Qualitative Branching Time Objectives.* Proc. of ICALP 2008, pp. 148-159, volume 5126 of LNCS. Springer, 2008.