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On the properties of positive spanning sets and positive bases

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Abstract The concepts of positive span and positive basis are important in derivative-free optimization. In fact, a well-known result is that if the gradient of a continuously differentiable objective function on $\mathbb{R}^n$ is nonzero at a point, then one of the vectors in any positive basis (or any positive spanning set) of $\mathbb{R}^n$ is a descent direction for the objective function from that point. This article summarizes the basic results and explores additional properties of positive spanning sets, positively independent sets and positive bases that are potentially useful in the design of derivative-free optimization algorithms. In particular, it provides construction procedures for these special sets of vectors that were not previously mentioned in the literature. It also proves that invertible linear transformations preserve positive independence and the positive spanning property. Moreover, this article introduces the notion of linear equivalence between positive spanning sets and between positively independent sets to simplify the analysis of their structures. Linear equivalence turns out to be a generalization of the concept of structural equivalence between positive bases that was introduced by Coope and Price (SIAM J Optim 11:859–869, 2001). Furthermore, this article clarifies which properties of linearly independent sets, spanning sets and ordinary bases carry over to positively independent sets, positive spanning sets, and positive bases. For example, a linearly independent set can always be extended to a basis of a linear space but a positively independent set cannot always be extended to a positive basis. Also, the maximum size of a linearly independent set in R^n is n but there is no limit to the size of a positively independent set in $\mathbb{R}^n$ when $n \geq 3$. Whenever possible, the results are proved for the more general case of frames of convex cones instead of focusing only on positive bases of linear spaces. In addition, this article discusses some algorithms for determining whether a given set of vectors is positively independent or whether

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it positively spans a linear subspace of $\mathbb{R}^n$. Finally, it provides an algorithm for extending any finite set of vectors to a positive spanning set of $\mathbb{R}^n$ using only a relatively small number of additional vectors.

Keywords Positive independence · Positive spanning set · Positive basis · Derivative-free optimization · Direct search · Linear equivalence

1 Introduction

Positive spanning sets and positive bases have been studied for more than 50 years (Davis 1954; McKinney 1962) and interest on these topics was renewed relatively recently because of their importance in direct search methods of the directional type (Torczon 1997; Kolda et al. 2003; Audet and Dennis 2006). The importance of these sets of vectors in derivative-free optimization (DFO) stems from the well-known result that if the gradient of a continuously differentiable function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is nonzero at a point, then one of the vectors in any positive basis (or more generally, in any positive spanning set) of $\mathbb{R}^n$ is a descent direction from that point. Positive bases are used in several popular DFO algorithms, including pattern search (Torczon 1997; Custódio et al. 2008; Vaz and Vicente 2009), grid-based methods (Coope and Price 2001, 2002), generating set search (Kolda et al. 2003), Mesh Adaptive Direct Search (Audet and Dennis 2006; Abramson et al. 2009; Audet et al. 2014) and Implicit Filtering (Kelley 2011).

Although several papers have focused solely on positive bases (e.g., Davis 1954; McKinney 1962; Romanowicz 1987), most of them have focused on theoretical characterizations and not as much attention has been given on construction procedures and computational issues that are potentially useful in the design of DFO algorithms. The aim of this paper is to provide some basic background material and explore additional properties of positive spanning sets and positive bases, especially construction procedures, beyond what is available in the literature, papers on direct search methods and the DFO textbook by Conn et al. (2009). The hope is that some of this material might be useful for developing other provably convergent DFO algorithms in the future.

This article summarizes the basic properties and explores additional properties of positive spanning sets, positively independent sets and positive bases. In particular, it provides construction procedures for these special sets of vectors that were not previously mentioned in the literature. It also proves that invertible linear transformations preserve positive independence and the positive spanning property. These construction procedures are potentially useful in the design of derivative-free optimization algorithms. Moreover, this article introduces the notion of linear equivalence between positive spanning sets and between positively independent sets to simplify the analysis of the structures of these sets of vectors. The notion of linear equivalence turns out to be a generalization of the concept of structural equivalence between positive bases that was introduced by Coope and Price (2001). This article also compares the properties of linear span, linear independence, and basis with the properties of positive span, positive independence and positive basis. Whenever possible, the results are proved more generally for frames of convex cones instead of focusing on positive bases of $\mathbb{R}^n$. Some of these properties are hardly mentioned in the literature on direct search methods but they might be important for a good understanding of these topics. For example, linearly independent sets can always be extended to a basis of a linear space but positively independent sets cannot always be extended to a positive basis. Moreover, the maximum size of a linearly independent set in a finite-dimensional linear space is the dimension of the space. However, there is a positively independent set of any size in $\mathbb{R}^n$ when $n \geq 3$. Finally, this article discusses some algorithms for determining whether a set is positively independent and whether a set positively spans a linear subspace of $\mathbb{R}^n$ and provides an algorithm for extending any set of vectors to a positive spanning set using only a relatively small number of additional vectors.

This paper is organized as follows. Sections 2 and 3 provide the basic properties of the positive span and positive independence while Sect. 4 deals with the basic properties of frames of convex cones and positive bases of linear spaces. Section 5 presents some procedures for constructing positive spanning sets and positive bases of arbitrary sizes not previously mentioned in the literature. Section 6 deals with the structure of positive bases. Then Sect. 7 introduces the concept of linear equivalence between positive spanning sets and positive bases, explores its consequences, and shows its connection to structural equivalence. Section 8 deals with numerical algorithms identifying positively independent sets and positive spanning sets and for ensuring the positive spanning property. Finally, Sect. 9 provides a summary of the paper.

2 Positive span

This section provides the basic properties of the positive span of a set of vectors and of positive spanning sets of convex cones and linear subspaces of $\mathbb{R}^n$. One of the main results of this section is Theorem 2.3, which is Theorem 3.7 in Davis (1954) and is a stronger statement of Theorem 2.2 in Conn et al. (2009). This theorem states that removing any element from a positive spanning set of a subspace always yields a linear spanning set for that subspace thereby providing a lower bound on the size of a positive spanning set. The other main results are Theorems 2.5 and 2.6, which are also mostly from Davis (1954) and these provide characterizations of positive spanning sets. Theorem 2.5 is used in the numerical algorithms in Sect. 8 while Theorem 2.6 guarantees that there is a descent direction among the vectors in a positive spanning set when the gradient of the objective function is nonzero. We begin with the following definitions from Davis (1954) and from Conn et al. (2009):

Definition 2.1 A *positive combination* (or *conic combination*) of $v_1, \ldots, v_k \in \mathbb{R}^n$ is a linear combination $\lambda_1 v_1 + \ldots + \lambda_k v_k$ where $\lambda_i \geq 0$ for all $i = 1, \ldots, k$. If $\lambda_i > 0$ and $v_i \neq 0$ for all i, then $\lambda_1 v_1 + \ldots + \lambda_k v_k$ is called a *strictly positive combination* of $v_1, \ldots, v_k$.

In the previous definition, the term *nonnegative combination* is technically more appropriate than *positive combination*. However, this paper will follow tradition and use the original term from Davis (1954) as well as from other papers.

Definition 2.2 A set $C \subseteq \mathbb{R}^n$ is called a *convex cone* if for any $v_1, v_2 \in C$ and $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 v_1 + \lambda_2 v_2 \in C$.

Clearly, every linear subspace of $\mathbb{R}^n$ is a convex cone but the converse is not true. Moreover, it can be shown by a straightforward induction argument that a set $C \subseteq \mathbb{R}^n$ is a convex cone if and only if it is closed under positive combinations.

Definition 2.3 The *positive span* of a finite set of vectors $S = \{v_1, \ldots, v_k\} \subset \mathbb{R}^n$, denoted by $\text{pos}\,(S)$, is given by

$$\text{pos}\,(S) := \{\lambda_1 v_1 + \ldots + \lambda_k v_k \ : \ \lambda_i \geq 0 \ \text{ for all } i = 1, \ldots, k\}.$$

The positive span of an infinite set S of vectors in $\mathbb{R}^n$ can also be defined to be the set of all positive combinations of finite sets of vectors in S as was done in McKinney (1962). However, this paper will focus on positive spans of finite sets of vectors.

It is straightforward to verify that $\text{pos}\,(S)$ is a convex cone for any (finite or infinite) set $S \subseteq \mathbb{R}^n$.

Definition 2.4 A finite set $S \subset \mathbb{R}^n$ is a *positive spanning set* of a convex cone $C \subseteq \mathbb{R}^n$ if $\text{pos}\,(S) = C$. In this case, S is said to *positively span* the convex cone C. In particular, S positively spans $\mathbb{R}^n$ if $\text{pos}\,(S) = \mathbb{R}^n$.

The first proposition summarizes the basic properties of the positive span of a finite set of vectors in $\mathbb{R}^n$. Some of these properties are found in other papers. For example, Proposition 2.1(vii) can be found in Romanowicz (1987). The proofs of most of these properties are obvious so they are omitted.

Proposition 2.1 (Basic properties of the positive span) *Let S and T be finite sets of vectors in $\mathbb{R}^n$. Then the following statements hold.*

(i) $S \subseteq \text{conv}\,(S) \subseteq \text{pos}\,(S) \subseteq \text{span}\,(S)$, *where* $\text{conv}\,(S)$ *is the convex hull of S and* $\text{span}\,(S)$ *is the linear span of S.*
(ii) If $S \subseteq T$, *then* $\text{pos}\,(S) \subseteq \text{pos}\,(T)$.
(iii) If $S \subseteq \text{pos}\,(T)$, *then* $\text{pos}\,(S) \subseteq \text{pos}\,(T)$.
(iv) If $S = \{v_1, \ldots, v_k\}$ *and* $\alpha_1, \ldots, \alpha_k > 0$, *then* $\text{pos}\,(\{\alpha_1 v_1, \ldots, \alpha_k v_k\}) = \text{pos}\,(S)$.
(v) For any $\alpha \in \mathbb{R}$. $\text{pos}\,(\alpha \cdot S) = \text{sgn}\,(\alpha) \cdot \text{pos}\,(S)$, *where* sgn *is the sign function and* $\alpha \cdot S = \cup_{v \in S}\{\alpha v\}$.
(vi) $\text{pos}\,(S \cap T) \subseteq \text{pos}\,(S) \cap \text{pos}\,(T)$
(vii) If $S \neq \emptyset$ *and* $T \neq \emptyset$, *then* $\text{pos}\,(S \cup T) = \text{pos}\,(S) + \text{pos}\,(T)$.

Proof We only prove (iii) since the rest are more obvious. Let $S = \{v_1, \ldots, v_k\}$ and $T = \{u_1, \ldots, u_\ell\}$ and suppose $S \subseteq \text{pos}\,(T)$. We wish to show that $\text{pos}\,(S) \subseteq \text{pos}\,(T)$.

Let $v \in \text{ pos } (S)$. Then $v = \sum_{i=1}^{k} \lambda_i v_i$ for some $\lambda_i \geq 0,\ i = 1, \ldots, k$. Since $S \subseteq \text{ pos } (T)$, it follows that for each $i = 1, \ldots, k$, $v_i = \sum_{j=1}^{\ell} \eta_{i,j} u_j$ for some $\eta_{i,j} \geq 0,\ j = 1, \ldots, \ell$. Now

$$v = \sum_{i=1}^{k} \lambda_i v_i = \sum_{i=1}^{k} \lambda_i \left(\sum_{j=1}^{\ell} \eta_{i,j} u_j \right) = \sum_{j=1}^{\ell} \left(\sum_{i=1}^{k} \lambda_i \eta_{i,j} \right) u_j.$$

Since the coefficient of u_j in the previous expression is nonnegative for all $j = 1, \ldots, \ell$, it follows that $v \in \text{ pos } (T)$. □

Note that, just like for convex hulls, the reverse inclusion in Proposition 2.1(vi) does not hold. That is, $\text{pos } (S) \cap \text{ pos } (T) \not\subseteq \text{ pos } (S \cap T)$. For example, let $S = \{[1,0], [-1,1]\}$ and $T = \{[-1,0], [1,1]\}$. Clearly, $S \cap T = \emptyset$, and so, $\text{pos } (S \cap T) = \emptyset$. However, $[0,1] \in \text{ pos } (S) \cap \text{ pos } (T)$ since $[0,1] = [1,0] + [-1,1] = [-1,0] + [1,1]$.

Next, if the finite set S linearly spans a subspace V of $\mathbb{R}^n$ and $v \in S$ is a linear combination of the other elements of S, then $S \setminus \{v\}$ also linearly spans V. A similar property holds for positive spanning sets of convex cones.

Proposition 2.2 *Let $S = \{v_1, \ldots, v_k\}$ positively span the convex cone C in $\mathbb{R}^n$. Then some v_i is a positive combination of the other elements of S if and only if $S \setminus \{v_i\}$ positively spans C.*

Proof Suppose $v_i = \sum_{j=1,\, j \neq i}^{k} \lambda_j v_j$ for some $\lambda_j \geq 0$ for all $j = 1, \ldots, k,\ j \neq i$. Let $u \in C$. Since S positively spans C, $u = \sum_{j=1}^{k} \mu_j v_j$ for some $\mu_j \geq 0$ for all j. Now

$$\begin{aligned} u &= \sum_{j=1,\, j \neq i}^{k} \mu_j v_j + \mu_i v_i = \sum_{j=1,\, j \neq i}^{k} \mu_j v_j + \mu_i \left(\sum_{j=1,\, j \neq i}^{k} \lambda_j v_j \right) \\ &= \sum_{j=1,\, j \neq i}^{k} (\mu_j + \mu_i \lambda_j) v_j \in \text{ pos } (S \setminus \{v_i\}). \end{aligned}$$

Hence, $\text{pos } (S \setminus \{v_i\}) = C$.

The converse is obvious. □

The following theorem is essentially Theorem 3.7 of Davis (1954) except that it holds for any linear subspace of $\mathbb{R}^n$. The proof is not provided in Davis so it is included here for completeness. A related result is Theorem 2.2 of Conn et al. (2009), which states that if a finite set S positively spans $\mathbb{R}^n$, then S contains a subset with $|S| - 1$ elements that linearly spans $\mathbb{R}^n$. However, the next theorem is a stronger statement than Theorem 2.2 of Conn et al. (2009) because it says that removing *any* element of a positive spanning set of a linear subspace results in a set that linearly spans the subspace.

Theorem 2.3 (Removing an element from a positive spanning set) *If $S = \{v_1, \ldots, v_k\} \subset \mathbb{R}^n$ positively spans the linear subspace V, then $S \setminus \{v_i\}$ linearly spans V for any $i = 1, \ldots, k$.*

Proof Fix $1 \le i \le k$. Let $v \in V$. Since S positively spans V, we have $v = \lambda_1 v_1 + \ldots + \lambda_k v_k$, for some $\lambda_1, \ldots, \lambda_k \ge 0$. Since $v_i \in V$ and V is a linear subspace, it follows that $-v_i \in V$. Again, since S positively spans V, we get $-v_i = \mu_1 v_1 + \ldots + \mu_i v_i + \ldots + \mu_k v_k$, for some $\mu_1, \ldots, \mu_k \ge 0$. Note that $-1 - \mu_i \ne 0$. Hence, $v_i = \frac{1}{-1-\mu_i} \sum_{j=1,\, j\ne i}^{k} \mu_j v_j$, and so,

$$v = \sum_{j=1,\, j\ne i}^{k} \lambda_j v_j + \left(\frac{\lambda_i}{-1-\mu_i} \sum_{j=1,\, j\ne i}^{k} \mu_j v_j \right) = \sum_{j=1,\, j\ne i}^{k} \left(\lambda_j - \frac{\lambda_i}{1+\mu_i} \mu_j \right) v_j.$$

Thus, $v \in \text{ span } (S \setminus \{v_i\})$. □

The following result follows immediately from Theorem 2.3. It provides a lower bound on the size of a positive spanning set of linear subspace of $\mathbb{R}^n$.

Corollary 2.4 (Lower bound on the size of a positive spanning set) *Let S be any finite set in* $\mathbb{R}^n$ *that positively spans a linear subspace V. Then S properly contains a basis of V. Consequently*, $|S| \ge \dim(V) + 1$.

Corollary 2.4 says that a positive spanning set of a linear subspace of $\mathbb{R}^n$ always contains a basis of that subspace. This result suggests that construction of positive spanning sets of a subspace could begin with a basis of that subspace, and this in fact will be used in Sect. 5 to construct positive spanning sets and positive bases.

Next, the minimum size of a set that linearly spans a subspace V of $\mathbb{R}^n$ is $\dim(V)$. Corollary 2.4 also says that the minimum possible size of a positive spanning set of V is $\dim(V) + 1$.

The next theorem provides some characterizations of positive spanning sets that can be used to determine whether or not a given set of vectors positively spans a given linear subspace of $\mathbb{R}^n$. Parts (i)–(iii) of this result constitute Theorem 3.6 of Davis (1954) and Theorem 2.3(i)–(iii) of Conn et al. (2009) applied to any linear subspace V of $\mathbb{R}^n$.

Theorem 2.5 (Characterizations of positive spanning sets) *Suppose* $S = \{v_1, \ldots, v_k\} \subset \mathbb{R}^n$, *with* $v_i \ne 0$, *linearly spans the subspace V* of $\mathbb{R}^n$. *Then the following are equivalent:*

(i) $S = \{v_1, \ldots, v_k\}$ *positively spans V.*
(ii) *For every* $i = 1, \ldots, k$, $-v_i \in \text{ pos } (S \setminus \{v_i\})$.
(iii) *There exists* $\alpha_1, \ldots, \alpha_k > 0$ *such that* $\sum_{i=1}^{k} \alpha_i v_i = 0$.
(iv) *There exists* $\beta_1, \ldots, \beta_k \ge 1$ *such that* $\sum_{i=1}^{k} \beta_i v_i = 0$.
(v) *There exists* $\gamma_1, \ldots, \gamma_k \ge 0$ *such that* $\sum_{i=1}^{k} \gamma_i v_i = -\sum_{i=1}^{k} v_i$.

Proof The proofs for the equivalence of statements (i)–(iii) are omitted because they are essentially the same as the proofs in Conn et al. (2009) for the case where $V = \mathbb{R}^n$.

(iii) $\Longrightarrow$ (iv): Suppose $\exists \alpha_1, \ldots, \alpha_k > 0$ such that $\sum_{i=1}^{k} \alpha_i v_i = 0$. Clearly, $\min_{1 \le j \le k} \alpha_j > 0$, and define $\beta_i = \alpha_i / \min_{1 \le j \le k} \alpha_j$ for all $i = 1, \ldots, k$. Note that $\beta_i \ge 1$ for all i and

$$\sum_{i=1}^{k} \beta_i v_i = \frac{1}{\min_{1 \le j \le k} \alpha_j} \sum_{i=1}^{k} \alpha_i v_i = 0.$$

(iv) $\Longrightarrow$ (v): Suppose $\exists \beta_1, \ldots, \beta_k \geq 1$ such that $\sum_{i=1}^{k} \beta_i v_i = 0$. Define $\gamma_i = \beta_i - 1$, $i = 1, \ldots, k$. Clearly, $\gamma_i \geq 0$ for all i and

$$\sum_{i=1}^{k} \gamma_i v_i = \sum_{i=1}^{k} (\beta_i - 1) v_i = \sum_{i=1}^{k} \beta_i v_i - \sum_{i=1}^{k} v_i = -\sum_{i=1}^{k} v_i$$

(v) $\Longrightarrow$ (iii): Suppose $\gamma_1, \ldots, \gamma_k \geq 0$ such that $\sum_{i=1}^{k} \gamma_i v_i = -\sum_{i=1}^{k} v_i$. Define $\alpha_i = \gamma_i + 1$, $i = 1, \ldots, k$. Clearly, $\alpha_i > 0$ for all i and $\sum_{i=1}^{k} \alpha_i v_i = \sum_{i=1}^{k} (\gamma_i + 1) v_i = \sum_{i=1}^{k} \gamma_i v_i + \sum_{i=1}^{k} v_i = 0$. $\square$

Example 2.1 The set $S = \{[1,0],[0,1],[-1,-1]\}$ linearly spans $\mathbb{R}^2$ and $[1,0] + [0,1] + [-1,-1] = [0,0]$. By Theorem 2.5(iii), S positively spans $\mathbb{R}^2$.

The next result is Theorem 3.1 of Davis (1954), which is also Theorem 2.3(iv) of Conn et al. (2009). It states that for any nonzero vector in $\mathbb{R}^n$, there is always an element in any positive spanning set for $\mathbb{R}^n$ that makes an acute angle with this given vector. This result is relevant to derivative-free optimization since this guarantees the existence of a descent direction among the vectors in any positive spanning set of $\mathbb{R}^n$ when the gradient of the objective function is nonzero as explained below.

Theorem 2.6 (Another characterization of positive spanning sets) *Suppose $S = \{v_1, \ldots, v_k\}$ is a set of vectors in $\mathbb{R}^n$. Then S positively spans $\mathbb{R}^n$ if and only if the following condition holds: For every $w \in \mathbb{R}^n$, $w \neq 0$, there exists an index $i \in \{1, \ldots, k\}$ such that $w^T v_i > 0$.*

Suppose f is a continuously differentiable function in an open ball containing $\widetilde{x} \in \mathbb{R}^n$ and that $\nabla f(\widetilde{x}) \neq 0$. Consider a set $S = \{v_1, \ldots, v_k\}$ that positively spans $\mathbb{R}^n$ and let $w = -\nabla f(\widetilde{x})$. According to Theorem 2.6, $\exists v_i \in S$ such that $-\nabla f(\widetilde{x})^T v_i > 0$, which means v_i is a descent direction for f from $\widetilde{x}$.

A related result to the previous proposition is essentially Theorem 2.1 from Coope and Price (2001) except it is stated for positive spanning sets instead of positive bases. The proof is the same as that in Coope and Price (2001) so it is omitted.

Proposition 2.7 *Let S be a positive spanning set of $\mathbb{R}^n$. If $g \in \mathbb{R}^n$ satisfies $g^T v \geq 0$ for all $v \in S$, then $g = 0$.*

For computational purposes, we would like our positive spanning sets to be as small as possible. Hence, in Sect. 4, we discuss positive bases, which are minimal positive spanning sets.

3 Positive independence

This section defines the concepts of positive dependence and independence as in Davis (1954) and Conn et al. (2009). It explores the similarities and differences between linear independence and positive independence. One of the main results is Theorem 3.2, which provides a characterization and alternative definition for positive independence. Another result (Proposition 3.3) provides a sufficient, but not necessary, condition for extending a positively independent set to a larger positively independent set. In addition, this section shows that there is no limit to the size of a positively independent set in $\mathbb{R}^n$ when $n \geq 3$. In particular, Theorem 3.4 provides a procedure for constructing arbitrarily large positively independent sets in $\mathbb{R}^n$ when $n \geq 3$.

Definition 3.1 The set of vectors $S = \{v_1, \ldots, v_k\} \subset \mathbb{R}^n$ is *positively dependent* if some v_i is a positive combination of the others; otherwise, it is *positively independent*. That is, S is *positively independent* if $v_i \notin \text{pos}(S \setminus \{v_i\})$ for all $i = 1, \ldots, k$.

Definition 3.1 can also be extended to infinite sets as in McKinney (1962). That is, a (possibly infinite) set S of vectors in $\mathbb{R}^n$ is *positively dependent* if $\exists v \in S$ such that $v \in \text{pos}(S \setminus \{v\})$; otherwise, it is *positively independent*. As with positive spanning sets, this paper focuses on finite positively independent sets.

Example 3.1 It is easy to verify that every element of the set $S = \{[1, 0], [0, 1], [-1, 0], [0, -1]\}$ is *not* a positive combination of the other elements of S. Hence, S is positively independent set in $\mathbb{R}^2$.

From the previous definition, it is clear that any set containing the zero vector is positively dependent. Moreover, the following statements are obvious and are also given in Davis (1954).

Theorem 3.1 (Subsets of positively independent sets are positively independent) *Any finite set of vectors in $\mathbb{R}^n$ containing a positively dependent set is also positively dependent. Equivalently, any subset of a positively independent finite set of vectors in $\mathbb{R}^n$ is also positively independent.*

The first part of the next theorem is Theorem 3.2 in Davis (1954) and it follows from Proposition 2.2. The second part of this proposition is an alternative definition of positive independence used by Romanowicz (1987).

Theorem 3.2 (A characterization of positive independence) *Let S be a finite set of vectors in $\mathbb{R}^n$. Then S is positively independent if and only if no proper subset of S positively spans the convex cone* $\text{pos}(S)$. *Equivalently, S is positively independent if and only if* $\text{pos}(S \setminus \{v\}) \neq \text{pos}(S)$ *for every $v \in S$.*

Now, we explore the similarities and differences between linear independence and positive independence. Clearly, any linearly independent set is also positively independent but the converse is not true. For example, $S = \{[1, 0], [-1, 0]\}$ is positively independent but it is not linearly independent. Moreover, a finite ordered set $S = \langle v_1, \ldots, v_k \rangle \subset \mathbb{R}^n$ is linearly independent if and only if for all $i > 1$, v_i cannot be written as a linear combination of $v_1, \ldots, v_{i-1}$. However, this property does *not* hold for positively independent sets. For example, every element of the ordered set $S = \langle [1,0], [1,1], [0,1] \rangle$ is not a positive combination of the preceding elements in S but S is positively dependent because $[1,1] = [1,0] + [0,1]$.

Also, if the finite set S is a linearly independent and $v \notin \text{span}(S)$, then $S \cup \{v\}$ is also linearly independent. However, if S is positively independent and $v \notin \text{pos}(S)$, then $S \cup \{v\}$ is *not* necessarily positively independent. For example, $S = \{[1,0], [1,1]\}$ is positively independent and $[0,1] \notin \text{pos}(S)$ but $S \cup \{[0,1]\}$ is positively dependent as we have seen earlier. One way to guarantee that $S \cup \{v\}$ will be positively independent is given by the following proposition.

Proposition 3.3 (Sufficient condition for extending a positively independent set) *Let $S = \{v_1, \ldots, v_k\} \subset \mathbb{R}^n$ be positively independent. If $v \notin \text{span}(S)$, then $S \cup \{v\}$ is positively independent.*

Proof Suppose $v \notin \text{span}(S)$. Clearly, v is not a positive combination of the elements of S. Now suppose some v_i is a positive combination of the elements of $(S \setminus \{v_i\}) \cup \{v\}$, i.e.,

$$v_i = \sum_{j=1,\, j \neq i}^{k} \lambda_j v_j + \eta v,$$

for some $\eta \geq 0$ and $\lambda_j \geq 0$ for all $j \neq i$. Note that $\eta > 0$ since S is positively independent. Now

$$v = \frac{1}{\eta}\left(v_i - \sum_{j=1,\, j \neq i}^{k} \lambda_j v_j \right) \in \text{span}(S),$$

gives a contradiction. Hence, $S \cup \{v\}$ is positively independent. □

The condition in the previous proposition is *not* necessary. Consider the positively independent set $S = \{[1,0], [0,1]\}$ in $\mathbb{R}^2$. Clearly, $[-1,0] \in \text{span}(S)$ but $S \cup \{[-1,0]\}$ is positively independent.

Next, we look at the possible sizes of positively independent sets. A well-known fact is that the size of a linearly independent set in $\mathbb{R}^n$ cannot exceed n. However, it can be shown that there is no limit to the size of a positively independent set in $\mathbb{R}^n$ when $n \geq 3$. In fact, the example below (see Corollary 3.5) shows that there is a positively independent set of any size in $\mathbb{R}^n$ when $n \geq 3$.

Before we provide the example, we first prove the following proposition, which says that appending a vector of 1's to the extreme points of any polytope in at least two dimensions results in a positively independent set in higher dimensions.

Theorem 3.4 (Construction of positively independent sets of arbitrary sizes) *Let $E = \{v_1, \ldots, v_r\}$ be the set of extreme points of a polytope in $\mathbb{R}^m$, with $m \geq 2$, and let*

$n \geq m+1$. *Define* $S := \left\{ \begin{bmatrix} v_1 \\ 1_{n-m} \end{bmatrix}, \ldots, \begin{bmatrix} v_r \\ 1_{n-m} \end{bmatrix} \right\}$, *where* 1_{n-m} *is a column vector consisting of all 1's. Then* S *is a positively independent set in* $\mathbb{R}^n$.

Proof Argue by contradiction. Suppose S is positively dependent. Then there is an index $j \in \{1, \ldots, r\}$ such that

$$\begin{bmatrix} v_j \\ 1_{n-m} \end{bmatrix} = \sum_{k=1,\, k \neq j}^{r} \lambda_k \begin{bmatrix} v_k \\ 1_{n-m} \end{bmatrix}$$

for some $\lambda_k \geq 0,\ k \neq j$. The previous equation is equivalent to

$$v_j = \sum_{k=1,\, k \neq j}^{r} \lambda_k v_k$$

$$1 = \sum_{k=1,\, k \neq j}^{r} \lambda_k,$$

which implies that vertex v_j of the polytope is a convex combination of the other vertices, giving a contradiction. $\square$

Since it is possible to construct a polytope with any finite number of extreme points in $\mathbb{R}^m$, when $m \geq 2$, it follows that we can construct any positively independent set of any size in $\mathbb{R}^n$, with $n \geq 3$. In particular, we can use the extreme points of a regular polygon.

Corollary 3.5 *For any* $n \geq 3$ *and* $r \geq 1$, *the set* $S = \{[\cos(2\pi k/r), \sin(2\pi k/r), 1_{n-2}]^T\}_{k=0}^{r-1} \subset \mathbb{R}^n$ *is a positively independent set of size* r. *(Here,* 1_{n-2} *is a row vector of* $(n-2)$ *1's.)*

Proof The set $E = \{[\cos(2\pi k/r), \sin(2\pi k/r)]^T\}_{k=0}^{r-1}$ are the r vertices of a regular polygon on $\mathbb{R}^2$ centered at the origin. By Theorem 3.4, $S = \{[\cos(2\pi k/r), \sin(2\pi k/r), 1_{n-2}]^T\}_{k=0}^{r-1} \subset \mathbb{R}^n$ is positively independent. $\square$

4 Frames and positive bases

This section defines a frame of a convex cone, which is a more general concept than that of a positive basis, and presents some of its basic properties. When the convex cone is a linear subspace of $\mathbb{R}^n$, a frame of that cone is called a *positive basis*. Unlike ordinary bases, positive bases of a given linear subspace of $\mathbb{R}^n$ can have different sizes as will be shown below. One of the known main results of this section is Theorem 4.2, which characterizes a positive basis as a minimal positive spanning set of a subspace. Another main result is Theorem 4.3, which states that every positive spanning set contains a positive basis. Finally, Theorem 4.6 shows that injective linear transformations preserve positive independence and the positive spanning property.

Definition 4.1 Let C be a convex cone in $\mathbb{R}^n$. A finite set $\mathcal{F} \subset \mathbb{R}^n$ is a frame of C if it is a positively independent set whose positive span is C. If $C = \mathbb{R}^n$ or if C is a linear subspace of $\mathbb{R}^n$, then a frame of C is also called a positive basis of C.

An elementary result in linear algebra is that the size of a basis of a finite-dimensional linear space is constant and is equal to the dimension of the linear space. However, positive bases can have different sizes as shown below.

Example 4.1 It is easy to verify that the sets $B_1^+ = \{[1,0],[0,1],[-1,-1]\}$ and $B_2^+ = \{[1,0],[0,1],[-1,0],\ [0,-1]\}$ both positively span $\mathbb{R}^2$ and that they are both positively independent. Hence, B_1^+ and B_2^+ are positive bases of $\mathbb{R}^2$ of sizes 3 and 4, respectively.

From Corollary 2.4, a positive basis of a linear subspace V of $\mathbb{R}^n$ must have at least $\dim(V)+1$ elements. Moreover, it can be shown that a positive basis cannot have more than $2\dim(V)$ elements but this is far from obvious. There are several proofs of this result [e.g., Davis (1954); Shepard (1971); Romanowicz (1987)] but they are usually complicated. A short and elegant proof was discovered by Audet (2011) using a fundamental theorem in linear programming and we review this proof at the end of Sect. 6. Now given a linear subspace V of $\mathbb{R}^n$, we will prove that there is a positive basis of V of any size from $\dim(V)+1$ to $2\dim(V)$ by providing procedures for constructing them in Sect. 5.

Next, every element of a subspace V of $\mathbb{R}^n$ can be expressed uniquely as a linear combination of the elements of a basis of V. However, this property does not hold for positive bases. In fact, we have the following result.

Proposition 4.1 (Infinite representations of vectors as positive combinations of positive basis elements) *Let B^+ be a positive basis of a linear subspace V of $\mathbb{R}^n$. Then every element of V can be expressed as a strictly positive combination of the elements of B^+ in an infinite number of ways.*

Proof Consider a positive basis $B^+ = \{v_1,\ldots,v_k\}$ of the linear subspace V of $\mathbb{R}^n$. Let $u \in V$. Since B^+ positively spans V, $u = \sum_{i=1}^k \lambda_i v_i$ for some $\lambda_1,\ldots,\lambda_k \geq 0$, and it follows from Theorem 2.5(iii) that $\exists \alpha_1,\ldots,\alpha_k > 0$ such that $\sum_{i=1}^k \alpha_i v_i = 0$. Note that we also have $u = \sum_{i=1}^k (\lambda_i + \beta\alpha_i) v_i$, which is a strictly positive combination of elements of B^+, for any $\beta > 0$. $\square$

The following proposition says that a frame of a convex cone is a minimal positive spanning set for that cone. In fact, this result is sometimes used as an alternative way of defining a positive basis of a linear subspace of $\mathbb{R}^n$ (e.g., Kelley (2011)).

Theorem 4.2 (Minimal positive spanning property of frames) *A set of nonzero vectors forms a frame for a convex cone C in $\mathbb{R}^n$ if and only if they positively span C and no proper subset has this property. In particular, a positive basis of a linear subspace of $\mathbb{R}^n$ is a minimal positive spanning set of that subspace.*

Proof Let C be a convex cone in $\mathbb{R}^n$ and let $\mathcal{F} = \{v_1, \ldots, v_k\} \subset \mathbb{R}^n$. We wish to show that $\mathcal{F}$ is a frame of C if and only if $C = \text{pos}(\mathcal{F})$ and no proper subset of $\mathcal{F}$ positively spans C.

First, suppose that $\mathcal{F}$ is a frame of C. By definition, $C = \text{pos}(\mathcal{F})$ and $\mathcal{F}$ is positively independent. To show that no proper subset of $\mathcal{F}$ positively spans C, we argue by contradiction. That is, suppose there is a proper subset $\mathcal{F}' \subset \mathcal{F}$ such that $\text{pos}(\mathcal{F}') = C$. Let $v \in \mathcal{F} \setminus \mathcal{F}'$. Note that $v \in C = \text{pos}(\mathcal{F}')$, and so, the element v of $\mathcal{F}$ is a positive combination of other elements of $\mathcal{F}$. This contradicts the positive independence of $\mathcal{F}$.

Conversely, suppose $C = \text{pos}(\mathcal{F})$ and no proper subset of $\mathcal{F}$ positively spans C. To show that $\mathcal{F}$ is positively independent, we again argue by contradiction. Suppose $\mathcal{F}$ is positively dependent. Then some $v_i \in \mathcal{F}$ is a positive combination of the other elements of $\mathcal{F}$. By Proposition 2.2, $\mathcal{F} \setminus \{v_i\}$ will also positively span C, giving a contradiction. □

As noted above, positive bases of a linear subspace V of $\mathbb{R}^n$ can have different sizes. However, Theorem 4.2 implies that it is *not* possible for a positive basis of V to be a proper subset of another positive basis of V because this violates the minimality of the larger positive basis as a positive spanning set of V.

Next, if a set of vectors S linearly spans a subspace V of $\mathbb{R}^n$, then S contains a basis of V and finding this basis is straightforward. The next theorem is essentially Theorem 2.1 in McKinney (1962). It states that a positive spanning set for a convex cone (or linear subspace) in $\mathbb{R}^n$ contains a frame (or positive basis) of that cone (or subspace). However, finding this positive basis is not as simple as finding a basis in a set that linearly spans a subspace. Section 8 partially addresses this issue by providing a simple algorithm for this problem that follows the proof of Theorem 4.3 below. This procedure can be used when n is small and the size of S is not large. A more efficient algorithm is provided by Dulá et al. (1998).

Theorem 4.3 (Every positive spanning set contains a frame) *If the finite set S positively spans the convex cone C in $\mathbb{R}^n$, then S contains a frame of C. In particular, any finite set that positively spans a linear subspace of $\mathbb{R}^n$ contains a positive basis of that subspace.*

Proof Let the finite set S positively span the convex cone C in $\mathbb{R}^n$. If S is also positively independent, then it is a frame of C. Otherwise, $\exists v \in S$ that is a positive combination of the other elements of S. By Proposition 2.2, $S \setminus \{v\}$ positively spans C. Clearly, a single nonzero vector is a positively independent set. Hence, by successively removing elements of S that are positive combinations of the other elements of S, we will eventually get a positively independent set that will still positively span C and this resulting subset of S is a frame of C. □

In Sect. 3, it was shown that there is no limit to the size of a positively independent set in $\mathbb{R}^n$ when $n \geq 3$. In particular, Corollary 3.5 showed that there is a positively independent set of any size in $\mathbb{R}^n$ when $n \geq 3$. Combining this with the fact that a positive basis in $\mathbb{R}^n$ cannot have more than $2n$ elements (as mentioned earlier), we see that a positively independent set in $\mathbb{R}^n$ cannot always be extended to a positive basis of $\mathbb{R}^n$. A necessary and sufficient condition for a nonempty set of vectors to be extendable to a positive basis of $\mathbb{R}^n$ is provided by Romanowicz (1987). However, the condition is somewhat complicated and not straightforward to verify computationally.

Next, a finite set of vectors in $\mathbb{R}^n$ is a basis of $\mathbb{R}^n$ if and only if it is a maximal linearly independent set in $\mathbb{R}^n$. A positive basis of a linear space, or more generally, a frame of a convex cone is a maximal positively independent set as shown next.

Proposition 4.4 (Maximal positive independence property of frames) *A frame of a convex cone C in $\mathbb{R}^n$ is a maximal positively independent set in C.*

Proof First, suppose $\mathcal{F}$ is a frame of C. If $\mathcal{F}$ is not a maximal positively independent set in C, then $\exists v \in C \setminus \mathcal{F}$ such that $\mathcal{F} \cup \{v\}$ is positively independent. Since $\mathcal{F}$ positively spans C, it follows that $\mathcal{F} \cup \{v\}$ also positively spans C. This implies that $\mathcal{F} \cup \{v\}$ is also a frame of C. However, this contradicts Theorem 4.2 since $\mathcal{F}$ is a proper subset of $\mathcal{F} \cup \{v\}$ that positively spans C. Thus, $\mathcal{F}$ must be a maximal positively independent set in C. $\square$

The converse of Proposition 4.4 is *not* true when $n \geq 3$. As shown earlier, when $n \geq 3$, there is a positively independent set in $\mathbb{R}^n$ of any specified size. It can be shown via a standard argument using Zorn's lemma that any positively independent set in $\mathbb{R}^n$ can be extended to a (possibly infinite) maximal positively independent set. Since a positive basis of $\mathbb{R}^n$ has at most $2n$ elements, a maximal positively independent set in $\mathbb{R}^n$ is *not* necessarily a positive basis of $\mathbb{R}^n$ when $n \geq 3$. However, it will be shown in Sect. 7 that, when $n = 1$ or 2, a maximal positively independent set in $\mathbb{R}^n$ is a positive basis (Proposition 7.6).

Now although a maximal positively independent set in a linear subspace V of $\mathbb{R}^n$ is not necessarily a positive basis of that subspace, the following proposition guarantees that such a positively independent set contains a basis of V.

Proposition 4.5 *If S is a finite maximal positively independent set in the linear subspace V of $\mathbb{R}^n$, then S contains a basis of V.*

Proof Suppose $S = \{v_1, \ldots, v_k\}$ is a maximal positively independent set in the subspace V and let T be a maximal linearly independent subset of S. We will show that T is a basis of V.

Since T is a maximal linearly independent subset of S, we must have $S \subseteq \operatorname{span}(T)$. By the same argument as in the proof of Proposition 2.1(iii), $\operatorname{span}(S) \subseteq \operatorname{span}(T)$. Clearly, $\operatorname{span}(T) \subseteq \operatorname{span}(S)$. Hence, $\operatorname{span}(T) = \operatorname{span}(S)$. Moreover, by Proposition 3.3, $\operatorname{span}(S) = V$. Now since T is linearly independent and $\operatorname{span}(T) = \operatorname{span}(S) = V$, it follows that T is a basis of V. $\square$

Given a positive spanning set, a positively independent set, a frame of a convex cone, or a positive basis of a linear space, it is possible to obtain other sets with the same property by means of linear transformations as shown in the next proposition.

Theorem 4.6 (Positive span and positive independence under a linear transformation) *Let C be a convex cone in $\mathbb{R}^n$ and let $\mathcal{L} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear* *transformation. Moreover, let S be a finite subset of C. Then the following statements hold:*

(i) *If S positively spans C, then $\mathcal{L}(S)$ positively spans the convex cone $\mathcal{L}(C)$ in $\mathbb{R}^m$.*
(ii) *If S is positively independent in $\mathbb{R}^n$ and $\mathcal{L}$ is an injective map, then $\mathcal{L}(S)$ is positively independent in $\mathbb{R}^m$.*
(iii) *If S is a frame of the convex cone C and $\mathcal{L}$ is an injective map, then $\mathcal{L}(S)$ is a frame of the convex cone $\mathcal{L}(C)$ in $\mathbb{R}^m$.*

Proof It is easy to check that $\mathcal{L}(C)$ is a convex cone in $\mathbb{R}^m$.

To prove (i), assume that $S = \{v_1, \ldots, v_k\}$ positively spans C. Let $w \in \mathcal{L}(C)$. Then $w = \mathcal{L}(v)$ for some $v \in C$. Since S positively spans C, it follows that $v = \sum_{i=1}^{k} \alpha_i v_i$ for some $\alpha_1, \ldots, \alpha_k \geq 0$, and so, $w = \mathcal{L}(v) = \sum_{i=1}^{k} \alpha_i \mathcal{L}(v_i)$. Hence, $\mathcal{L}(C) = \text{ pos } (\mathcal{L}(S))$.

To prove (ii), assume that $S = \{v_1, \ldots, v_k\}$ is positively independent in $\mathbb{R}^n$ and that $\mathcal{L}$ is an injective map. Argue by contradiction. That is, suppose $\mathcal{L}(S)$ is positively dependent. Then $\exists i \in \{1, \ldots, k\}$ such that $\mathcal{L}(v_i) = \sum_{j=1,\, j \neq i}^{k} \beta_j \mathcal{L}(v_j)$ for some $\beta_j \geq 0$ for all $j = 1, \ldots, k,\ j \neq i$. Since $\mathcal{L}$ has an inverse, this implies that $v_i = \sum_{j=1, j \neq i}^{k} \beta_j v_j$, contradicting the positive independence of S.

Finally, (iii) follows from (i) and (ii). □

Corollary 4.7 *Let C be a convex cone in $\mathbb{R}^n$ and let S be a finite subset of C. Then the following statements hold:*

(i) If $S = \{v_1, \ldots, v_k\}$ *positively spans C and $M \in \mathbb{R}^{m \times n}$ is any matrix with n columns, then $M \cdot S = \{Mv_1, \ldots, Mv_k\}$ also positively spans the convex cone $M \cdot C = \{Mv \ : \ v \in C\}$ in $\mathbb{R}^m$.*
(ii) If $S = \{v_1, \ldots, v_k\}$ *is positively independent and $M \in \mathbb{R}^{n \times n}$ is a nonsingular matrix, then $M \cdot S = \{Mv_1, \ldots, Mv_k\}$is also positively independent.*
(iii) If $S = \{v_1, \ldots, v_k\}$ *is a frame of C and $M \in \mathbb{R}^{n \times n}$ is a nonsingular matrix, then $M \cdot S = \{Mv_1, \ldots, Mv_k\}$ is a frame of the convex cone $M \cdot C = \{Mv \ : \ v \in C\}$ in $\mathbb{R}^n$.*

Proof To prove (i), note that the map $\mathcal{L} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ defined by $\mathcal{L}(v) = Mv$ is a linear transformation, and so, the result follows immediately from Theorem 4.6(i). Similarly, (ii) follows from Theorem 4.6(ii) by noting that the map $\mathcal{L} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by $\mathcal{L}(v) = Mv$, where M is nonsingular, is an injective linear transformation. Finally, as before, (iii) follows from (i) and (ii). □

Note that Corollary 4.7(i) applies to any matrix (possibly non-square) that can be pre-multiplied to the vectors in a positive spanning set. On the other hand, Corollary 4.7(ii), requires that the matrix used be nonsingular. Combining these two results yields Corollary 4.7(iii), which is the result by Lewis and Torczon (1999) extended to any convex cone in $\mathbb{R}^n$.

Another simple property of positive bases is given by the next proposition. The proofs are straightforward so they are omitted.

Proposition 4.8 *Let $S = \{v_1, \ldots, v_k\}$ be a set of vectors from a linear subspace V of $\mathbb{R}^n$, let $\alpha_1, \ldots, \alpha_k > 0$, and let $\widetilde{S} = \{\alpha_1 v_1, \ldots, \alpha_k v_k\}$. Then the following statements hold:*

(i) *If S positively spans V, then $\widetilde{S}$ and $-\widetilde{S}$ also positively span V.*
(ii) *If S is positively independent, then $\widetilde{S}$ and $-\widetilde{S}$ are also positively independent.*
(iii) *If S is a positive basis of V, then $\widetilde{S}$ and $-\widetilde{S}$ are also positive bases of V.*

5 Construction of positive spanning sets and positive bases

We now discuss procedures for constructing positive spanning sets and positive bases of a linear subspace of $\mathbb{R}^n$. To the best of my knowledge, these procedures have not yet appeared in the literature. Recall that the size of a positive basis of linear subspace V of $\mathbb{R}^n$ must be at least $\dim(V) + 1$ and at most $2\dim(V)$. Corollary 5.5 provides a procedure for constructing a positive basis of a linear subspace V any size between $\dim(V) + 1$ and $2\dim(V)$ inclusive. In the papers on direct search methods, only positive bases of $\mathbb{R}^n$ size $n + 1$ or $2n$ are usually considered. However, there may be situations in which a positive basis of size strictly between $n + 1$ and $2n$ might be useful as noted in Sect. 8. Now recall from Corollary 2.4 that any set that positively spans a linear subspace of $\mathbb{R}^n$ contains a basis of that subspace. Hence, it is natural to begin with a basis of a linear subspace when constructing positive spanning sets of that subspace.

Theorem 5.1 (Construction of positive spanning sets from a basis) *Let $B = \{v_1, \ldots, v_k\}$ be a basis of a linear subspace V of $\mathbb{R}^n$ and let $\mathcal{J} = \{J_1, \ldots, J_\ell\}$ be a collection of subsets of $K = \{1, \ldots, k\}$ such that $\cup_{i=1}^{\ell} J_i = K$. Then the set*

$$\widetilde{B} = B \bigcup \left\{ -\sum_{j \in J_1} \lambda_{1,j} v_j, \ldots, -\sum_{j \in J_\ell} \lambda_{\ell,j} v_j \right\},$$

where $\lambda_{i,j} > 0$ for all $i = 1, \ldots, \ell$, $j \in J_i$, positively spans V.

Proof Clearly, $\widetilde{B}$ linearly spans V since $\widetilde{B}$ contains a basis of V. Moreover, since B is linearly independent, every element of $\widetilde{B}$ is nonzero. Now for each $r = 1, \ldots, k$, define

$$\eta_r := \sum_{i : J_i \ni r} \lambda_{i,r}.$$

Since each $r \in K$ belongs to at least one J_i and $\lambda_{i,r} > 0$ for all i such that $J_i \ni r$, it follows that $\eta_r > 0$ for all $r \in K$. Now note that

$$
\begin{aligned}
&\sum_{r=1}^{k} \eta_r v_r + 1 \cdot \left(-\sum_{j \in J_1} \lambda_{1,j} v_j \right) + \ldots + 1 \cdot \left(-\sum_{j \in J_\ell} \lambda_{\ell,j} v_j \right) \\
&\quad = \sum_{r=1}^{k} \sum_{i:J_i \ni r} \lambda_{i,r} v_r - \sum_{j \in J_1} \lambda_{1,j} v_j - \ldots - \sum_{j \in J_\ell} \lambda_{\ell,j} v_j = \sum_{r=1}^{k} \sum_{i:J_i \ni r} \lambda_{i,r} v_r - \sum_{i=1}^{\ell} \sum_{r:J_i \ni r} \lambda_{i,r} v_r \\
&\quad = 0.
\end{aligned}
$$

By Theorem 2.5(iii), $\widetilde{B}$ positively spans V. $\square$

Example 5.1 Suppose $B = \{v_1, v_2, v_3, v_4\}$ is a basis of $\mathbb{R}^4$. Let $J_1 = \{1, 3\}, J_2 = \{2, 3\}$ and $J_3 = \{1, 2, 4\}$. Since $\cup_{i=1}^{3} J_i = \{1, 2, 3, 4\}$, it follows from Theorem 5.1 that

$$
\widetilde{B} = B \,\cup\, \{-\lambda_{1,1} v_1 - \lambda_{1,3} v_3,\ -\lambda_{2,2} v_2 - \lambda_{2,3} v_3,\ -\lambda_{3,1} v_1 - \lambda_{3,2} v_2 - \lambda_{3,4} v_4\}
$$

positively spans $\mathbb{R}^4$ for all $\lambda_{i,j} > 0,\ i = 1, 2, 3,\ j \in J_i$. $\square$

Clearly, a basis of a linear space is a positively independent set. The next theorem extends a basis to a larger positively independent set.

Theorem 5.2 (Construction of a positively independent set from a basis) *Let $B = \{v_1, \ldots, v_k\}$ be a basis of a linear subspace V of $\mathbb{R}^n$ and let $\mathcal{J} = \{J_1, \ldots, J_\ell\}$ be a collection of subsets of $K = \{1, \ldots, k\}$ such that $J_r \not\subseteq \cup_{i=1, i \neq r}^{\ell} J_i$ for all $r = 1, \ldots, \ell$. Then the set*

$$
\widetilde{B} = B \bigcup \left\{ -\sum_{j \in J_1} \lambda_{1,j} v_j, \ldots, -\sum_{j \in J_\ell} \lambda_{\ell,j} v_j \right\},
$$

where $\lambda_{i,j} > 0$ for all $i = 1, \ldots, \ell,\ j \in J_i$, is positively independent.

Proof Suppose that some $v_h, h \in K$, is positive combination of the other elements of $\widetilde{B}$. Then

$$
-v_h + \sum_{j=1,\, j \neq h}^{k} \mu_j v_j + \eta_1 \left(-\sum_{j \in J_1} \lambda_{1,j} v_j \right) + \ldots + \eta_\ell \left(-\sum_{j \in J_\ell} \lambda_{\ell,j} v_j \right) = 0,
$$

for some $\mu_j \geq 0,\ j = 1, \ldots, k,\ j \neq h$ and $\eta_i \geq 0$ for all $i = 1, \ldots, \ell$. Collecting all the terms involving v_h, and using the fact that B is linearly independent, we get the following

$$
-1 - \sum_{i:J_i \ni h} \eta_i \lambda_{i,h} = 0,
$$

which gives a contradiction since the terms inside the summation are all nonnegative.

Next, suppose that for some $r \in \{1, \ldots, \ell\}$,

$$-\sum_{j\in J_r}\lambda_{r,j}v_j=\sum_{j=1}^{k}\mu_j v_j+\sum_{i=1,\, i\neq r}^{\ell}\eta_i\left(-\sum_{j\in J_i}\lambda_{i,j}v_j\right),$$

for some $\mu_j \geq 0,\ j=1,\ldots,k$ and $\eta_i \geq 0$ for all $i=1,\ldots,\ell,\ i\neq r$. Then

$$\sum_{j=1}^{k}\mu_j v_j+\sum_{j\in J_r}\lambda_{r,j}v_j-\sum_{i=1,i\neq r}^{\ell}\eta_i\sum_{j\in J_i}\lambda_{i,j}v_j=0.$$

Since $J_r \not\subseteq \cup_{i=1,\, i\neq r}^{\ell} J_i$, let $h\in J_r\setminus \cup_{i=1,\, i\neq r}^{\ell} J_i$. Collect all the terms involving v_h and again use the fact that B is linearly independent, we obtain

$$\mu_h+\lambda_{r,h}=0,$$

which gives a contradiction since $\mu_h \geq 0$ and $\lambda_{r,h}>0$.

Thus, every element of $\widetilde{B}$ is not a positive combination of the other elements of $\widetilde{B}$. $\square$

Example 5.2 Suppose $B=\{v_1,v_2,v_3,v_4\}$ is a basis of $\mathbb{R}^4$. Let $J_1=\{1,2\}, J_2=\{2,3\}$ and $J_3=\{2,4\}$. Note that $J_r \not\subseteq \cup_{i=1,\, i\neq r}^{\ell} J_i$ for all $r=1,2,3$. By Theorem 5.2,

$$\widetilde{B}=B\ \cup\ \{-\lambda_{1,1}v_1-\lambda_{1,2}v_2,\ -\lambda_{2,2}v_2-\lambda_{2,3}v_3,\ -\lambda_{3,2}v_2-\lambda_{3,4}v_4\}$$

is positively independent for any choice of $\lambda_{i,j}>0,\ i=1,2,3,\ j\in J_i$. $\square$

The condition in the previous proposition is satisfied in the special case where the J_i's are pairwise disjoint so the next corollary is an immediate consequence of this proposition.

Corollary 5.3 *Let $B=\{v_1,\ldots,v_k\}$ be a basis of a linear subspace V of $\mathbb{R}^n$ and let $\mathcal{J}=\{J_1,\ldots,J_\ell\}$ be a collection of subsets of $K=\{1,\ldots,k\}$ that are pairwise disjoint. Then the set*

$$\widetilde{B}=B\bigcup\left\{-\sum_{j\in J_1}\lambda_{1,j}v_j,\ldots,-\sum_{j\in J_\ell}\lambda_{\ell,j}v_j\right\},$$

for any $\lambda_{i,j}>0,\ i=1,\ldots,\ell,\ j\in J_i$, is positively independent.

Combining the sufficient conditions for positive independence and for positively spanning a subspace, we obtain sufficient conditions for a positive basis of a linear subspace of $\mathbb{R}^n$.

Theorem 5.4 (Construction of a positive basis from a basis) *Let $B=\{v_1,\ldots,v_k\}$ be a basis of a linear subspace V of $\mathbb{R}^n$ and let $\mathcal{J}=\{J_1,\ldots,J_\ell\}$ be a collection of subsets of $K=\{1,\ldots,k\}$ such that $\cup_{i=1}^{\ell}J_i=K$ and such that $J_r\not\subseteq\cup_{i=1,i\neq r}^{\ell}J_i$ for all $r=1,\ldots,\ell$. Then the set*

$$\widetilde{B} = B \bigcup \left\{ -\sum_{j \in J_1} \lambda_{1,j} v_j, \ldots, -\sum_{j \in J_\ell} \lambda_{\ell,j} v_j \right\},$$

where $\lambda_{i,j} > 0$ for all $i = 1, \ldots, \ell$, $j \in J_i$, is a positive basis of V.

Example 5.3 Suppose $B = \{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$. Let $J_1 = \{1, 2\}$ and $J_2 = \{2, 3\}$. By Theorem 5.4,

$$\widetilde{B} = B \ \cup \ \{-2v_1 - v_2, -3v_2 - 5v_3\}$$

is a positive basis of $\mathbb{R}^3$. In particular,

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ -5 \end{bmatrix} \right\}$$

is a positive basis of $\mathbb{R}^3$. □

The condition in the previous proposition is clearly satisfied in the special case where the J_i's form a partition of the set K so the next corollary is an immediate consequence of this proposition.

Corollary 5.5 *Let $B = \{v_1, \ldots, v_k\}$ be a basis of a subspace V of $\mathbb{R}^n$ and let $\mathcal{J} = \{J_1, \ldots, J_\ell\}$ be a partition of $K = \{1, \ldots, k\}$, i.e., the J_i 's are pairwise disjoint and $\cup_{i=1}^{\ell} J_i = K$. Then the set*

$$\widetilde{B} = B \bigcup \left\{ -\sum_{j \in J_1} \lambda_{1,j} v_j, \ldots, -\sum_{j \in J_\ell} \lambda_{\ell,j} v_j \right\},$$

where $\lambda_{i,j} > 0$ for all $i = 1, \ldots, \ell$, $j \in J_i$, , is a positive basis. Consequently, there is a positive basis of V of any size between $\dim(V) + 1$ *and* $2\dim(V)$.

The previous corollary allows us to construct positive bases of a linear subspace V of $\mathbb{R}^n$ of any size between $\dim(V) + 1$ and $2\dim(V)$ by considering partitions of the indices of a basis of V. However, in the context of derivative-free optimization, positive bases with some additional properties such as having uniform angles between any two vectors (Alberto et al. 2004; Conn et al. 2009) are usually more desirable. Also, for convergence purposes, we need positive bases and positive spanning sets whose cosine measures (Kolda et al. 2003) are bounded away from 0, but this will be the subject of future work.

The next proposition deals with how to construct a positive basis for the internal direct sum of two subspaces of $\mathbb{R}^n$.

Proposition 5.6 (Positive basis for the internal direct sum of two linear subspaces) *Suppose V and W are linear subspaces of $\mathbb{R}^n$ such that $V \cap W = \{0\}$. Let $B^+ = \{v_1, \ldots, v_k\}$ and $D^+ = \{u_1, \ldots, u_\ell\}$ be positive bases for V and W, respectively. Then $B^+ \cup D^+$ is a positive basis for the subspace $V + W$, which is the (internal) direct sum of V and W.*

Proof Clearly, $B^+ \cup D^+$ linearly spans $V + W$. Since B^+ positively spans V, $\exists \alpha_1, \ldots, \alpha_k > 0$ such that $\sum_{i=1}^{k} \alpha_i v_i = 0$. Similarly, $\exists \beta_1, \ldots, \beta_\ell > 0$ such that $\sum_{j=1}^{\ell} \beta_j u_j = 0$. Hence, $\sum_{i=1}^{k} \alpha_i v_i + \sum_{j=1}^{\ell} \beta_j u_j = 0$, and so, by Theorem 2.5(iii), $B^+ \cup D^+$ positively spans $V + W$.

Next, we show that $B^+ \cup D^+$ is positively independent. Suppose v_i is a positive combination of the other elements of $B^+ \cup D^+$, i.e.,

$$v_i = \sum_{j=1,\, j\neq i}^{k} \lambda_j v_j + \sum_{j=1}^{\ell} \mu_j u_j,$$

for some $\lambda_j \geq 0, \mu_j \geq 0$. Then

$$v_i - \sum_{j=1, j\neq i}^{k} \lambda_j v_j = \sum_{j=1}^{\ell} \mu_j u_j \in V \cap W = \{0\},$$

and so, $v_i = \sum_{j=1,\, j\neq i}^{k} \lambda_j v_j$, contradicting the positive independence of B^+. By a similar argument, each u_j cannot be a positive combination of the other elements of $B^+ \cup D^+$. $\square$

The last result in this section provides a procedure for using a positive basis of $\mathbb{R}^n$ to construct a positive basis of $\mathbb{R}^{n+1}$.

Proposition 5.7 *If* $B^+ = \{v_1, \ldots, v_r\}$ *is a positive basis of* $\mathbb{R}^n$, *then* $\widetilde{B}^+ = \left\{ \begin{bmatrix} v_1 \\ \gamma_1 \end{bmatrix}, \ldots, \begin{bmatrix} v_r \\ \gamma_r \end{bmatrix}, \begin{bmatrix} 0_{n\times 1} \\ -\gamma_{r+1} \end{bmatrix} \right\}$, *where* $\gamma_i > 0$ *for* $i = 1, \ldots, r+1$, *is a positive basis of* $\mathbb{R}^{n+1}$.

Proof Let $B^+ = \{v_1, \ldots, v_r\}$ be a positive basis of $\mathbb{R}^n$ and define $\widetilde{B}^+ = \left\{ \begin{bmatrix} v_1 \\ \gamma_1 \end{bmatrix}, \ldots, \begin{bmatrix} v_r \\ \gamma_r \end{bmatrix}, \begin{bmatrix} 0_{n\times 1} \\ -\gamma_{r+1} \end{bmatrix} \right\}$, where $\gamma_i > 0$ for $i = 1, \ldots, r+1$.

First, show that $\widetilde{B}^+$ linearly spans $\mathbb{R}^{n+1}$. Let $\begin{bmatrix} v \\ \beta \end{bmatrix} \in \mathbb{R}^{n+1}$, where $v \in \mathbb{R}^n$. Since B^+ linearly spans $\mathbb{R}^n$, $v = \sum_{i=1}^{r} b_i v_i$ for some $b_i \in \mathbb{R}$, $i = 1, \ldots, r$. Note that

$$\begin{bmatrix} v \\ \beta \end{bmatrix} = b_1 \begin{bmatrix} v_1 \\ \gamma_1 \end{bmatrix} + \ldots + b_r \begin{bmatrix} v_r \\ \gamma_r \end{bmatrix} + \frac{1}{\gamma_{r+1}} \left(\sum_{i=1}^{r} b_i \gamma_i - \beta \right) \begin{bmatrix} 0_{n\times 1} \\ -\gamma_{r+1} \end{bmatrix}.$$

Hence, $\widetilde{B}^+$ linearly spans $\mathbb{R}^{n+1}$.

Next, since B^+ positively spans $\mathbb{R}^n$, it follows from Theorem 2.5(iii) that $\exists \alpha_1, \ldots, \alpha_r > 0$ such that $\sum_{i=1}^{r} \alpha_i v_i = 0$. Now observe that $\frac{1}{\gamma_{r+1}} \sum_{i=1}^{r} \alpha_i \gamma_i > 0$ and

$$\alpha_1 \begin{bmatrix} v_1 \\ \gamma_1 \end{bmatrix} + \ldots + \alpha_r \begin{bmatrix} v_r \\ \gamma_r \end{bmatrix} + \frac{1}{\gamma_{r+1}} \left(\sum_{i=1}^{r} \alpha_i \gamma_i \right) \begin{bmatrix} 0_{n\times 1} \\ -\gamma_{r+1} \end{bmatrix} = \begin{bmatrix} 0_{n\times 1} \\ 0 \end{bmatrix}.$$

Hence, by Theorem 2.5(iii), $\widetilde{B}^+$ positively spans $\mathbb{R}^{n+1}$.

Finally, show that $\widetilde{B}^+$ is positively independent. Since $\gamma_{r+1} > 0$, $\begin{bmatrix} 0_{n\times 1} \\ -\gamma_{r+1} \end{bmatrix}$ cannot be a positive combination of the other elements of $\widetilde{B}^+$. Now suppose that for some $1 \le i \le r$, $\begin{bmatrix} v_i \\ \gamma_i \end{bmatrix}$ is a positive combination of the other elements of $\widetilde{B}^+$, i.e.,

$$\begin{bmatrix} v_i \\ \gamma_i \end{bmatrix} = \sum_{j=1,\, j\neq i}^{r} c_j \begin{bmatrix} v_j \\ \gamma_j \end{bmatrix} + b \begin{bmatrix} 0_{n\times 1} \\ -\gamma_{r+1} \end{bmatrix},$$

where $c_j \ge 0$ for $j \neq i$ and $b \ge 0$. This leads to

$$v_i = \sum_{j=1,\, j\neq i}^{r} c_j v_j,$$

which contradicts the positive independence of B^+. Thus, $\widetilde{B}^+$ is positively independent. □

Example 5.4 Consider the following positive basis of $\mathbb{R}^2$:

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}.$$

By Proposition 5.7, it follows that

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right\}$$

is a positive basis of $\mathbb{R}^3$.

6 Structure of positive bases

This section provides the characterization theorems for minimal and maximal positive bases of a subspace (Theorems 6.1 and 6.3) and reviews the proof of the maximum size of a positive basis due to Audet (2011).

The next theorem characterizes the structure of any positive basis of a subspace V with $\dim(V) + 1$ elements. Such a set is called a *minimal positive basis* of V. This result was mentioned in Davis (1954) but the proof was omitted.

Theorem 6.1 (Structure of minimal positive bases) *Let V be a linear subspace of $\mathbb{R}^n$ of dimension $k \ge 1$. If $B = \{v_1, \ldots, v_k\}$ is a basis of V, then for any choice of $\lambda_1, \ldots, \lambda_k > 0$, the set $\widetilde{B} = \{v_1, \ldots, v_k, -\sum_{i=1}^{k} \lambda_i v_i\}$ is a (minimal) positive basis of V. Conversely, every minimal positive basis of V has this structure. In fact, if $B^+ = \{v_1, \ldots, v_{k+1}\}$ is a minimal positive basis of V, then for any $i = 1, \ldots, k+1$, $B^+ \setminus \{v_i\}$ is a basis of V and v_i is a strictly negative combination of the elements of $B^+ \setminus \{v_i\}$.*

Proof By Corollary 5.5, $\widetilde{B}$ is a positive basis of V.

To prove the second part of the proposition, define $B_i^+ := B^+ \setminus \{v_i\}$ for each $i = 1, \ldots, k+1$. By Theorem 2.3, each B_i^+ linearly spans V, and since B_i^+ has $\dim(V)$ elements, it must be a basis of V. Fix the index $i \in \{1, \ldots, k+1\}$. Since B^+ positively spans V, it follows from Theorem 2.5(ii) that

$$-v_i = \sum_{j=1,\, j\neq i}^{k+1} \lambda_j v_j \tag{6.1}$$

for some $\lambda_j \geq 0,\ j = 1, \ldots, k+1,\ j \neq i$.

Next, we show that $\lambda_j > 0$ for all $j \neq i$. Argue by contradiction. Suppose $\lambda_r = 0$ for some $r \neq i$. Then (6.1) becomes

$$\left(\sum_{j=1,\, j\neq i,r}^{k+1} \lambda_j v_j \right) + v_i = 0.$$

Since $B_r^+ = B^+ \setminus \{v_r\}$ is also a basis of V, the previous equation yields a contradiction. Hence, v_i is a strictly negative combination of the elements of $B_i^+ = B^+ \setminus \{v_i\}$. $\square$

The following result follows immediately from the previous proposition.

Corollary 6.2 *Let $M \in \mathbb{R}^{n\times n}$ be a nonsingular matrix and let $\lambda \in \mathbb{R}^n$, $\lambda > 0$. Then the columns of $M[I_n, -\lambda] = [M, -M\lambda]$ form a minimal positive basis of $\mathbb{R}^n$. Moreover, any minimal positive basis of $\mathbb{R}^n$ has this structure.*

The next proposition characterizes the structure of a positive basis of a linear subspace V of $\mathbb{R}^n$ with $2\dim(V)$ elements. Such a set is called a *maximal positive basis* of V. The first part of this proposition follows from Corollary 5.5. The proof of the second part is more complicated and can be found in Romanowicz (1987).

Theorem 6.3 (Structure of maximal positive bases) *Suppose $B = \{v_1, \ldots, v_k\}$ is a basis of a linear subspace V of $\mathbb{R}^n$. Then for any choice of $\lambda_1, \ldots, \lambda_k > 0$, the set $\widetilde{B} = \{v_1, \ldots, v_k, -\lambda_1 v_1, \ldots, -\lambda_k v_k\}$ is a (maximal) positive basis of V. Conversely, every maximal positive basis of the subspace V has the form $B^+ = \{v_1, \ldots, v_k, -\lambda_1 v_1, \ldots, -\lambda_k v_k\}$, where $\{v_1, \ldots, v_k\}$ is a basis of V and $\lambda_1, \ldots, \lambda_k > 0$.*

Corollary 6.4 *Let $M \in \mathbb{R}^{n\times n}$ be a nonsingular matrix and let $\lambda_1, \ldots, \lambda_n > 0$. Then the columns of $M[I_n, -\operatorname{diag}(\lambda_1, \ldots, \lambda_n)] = [M, -\lambda_1 M e_1, \ldots, -\lambda_n M e_n]$ form a maximal positive basis of $\mathbb{R}^n$. Moreover, any maximal positive basis of $\mathbb{R}^n$ has this structure.*

Example The following statements from Conn et al. (2009) are trivial consequences of the above propositions.

(i) $\{e_1, \ldots, e_n, -\sum_{i=1}^n e_i\}$ is a minimal positive basis of $\mathbb{R}^n$
(ii) $\{e_1, \ldots, e_n, -e_1, \ldots, -e_n\}$ is a maximal positive basis of $\mathbb{R}^n$

(iii) If the points $v_0, v_1, \ldots, v_n$ are the vertices of a simplex in $\mathbb{R}^n$, then $\{v_1 - v_0, \ldots, v_n - v_0, \ -\sum_{i=1}^{n}(v_i - v_0)\}$ is a minimal positive basis and $\{v_1 - v_0, \ldots, v_n - v_0, \ -(v_1 - v_0), \ldots, -(v_n - v_0)\}$ is a maximal positive basis of $\mathbb{R}^n$.

A complete characterization of non-minimal and non-maximal positive bases of $\mathbb{R}^n$ is more complicated and is beyond the scope of this paper. The interested reader is referred to Reay (1965) (Theorem 2), Reay (1966) (Lemma 6), Shepard (1971) (Theorem 10) and Romanowicz (1987) (Theorem 1).

Finally, we review the proof by Audet (2011) that a positive basis of $\mathbb{R}^n$ cannot have more than $2n$ elements. Below is the proof by Audet (2011) extended to the case where V is any subspace of $\mathbb{R}^n$. It begins with the following lemma, which is also taken from the proof by Audet (2011).

Lemma 6.5 (Audet 2011) *Let S be a finite set of vectors that positively span the linear subspace V of $\mathbb{R}^n$. Then S contains a subset that positively spans V and that contains at most $2\dim(V)$ elements.*

Proof Let $S = \{v_1, \ldots, v_k\}$ positively span V and let M be the $n \times k$ matrix whose columns are the elements of S. By Corollary 2.4, S contains a basis of V, say $\mathcal{B} = \{v_j \ : \ j \in J\} \subseteq S$. Define $b := -\sum_{j \in J} v_j$ and

$$Y := \{y \in \mathbb{R}^k \ : \ My = b, y \geq 0\}. \tag{6.2}$$

Since $b \in C = \text{pos}(S)$, it follows that $Y \neq \emptyset$. Consider the reduced-row echelon form (RREF) of the matrix form of the linear system $My = b$ and let $\widetilde{M}y = \widetilde{b}$ be the corresponding linear system with *no* zero rows. Clearly, the two systems are equivalent, and so,

$$Y = \{y \in \mathbb{R}^k \ : \ \widetilde{M}y = \widetilde{b}, y \geq 0\}. \tag{6.3}$$

Now note that the number of (nonzero) rows of $\widetilde{M}$ is the dimension of the row space of M, which is equal to the dimension of the column space of M. Hence, the number of rows of $\widetilde{M}$ is $\dim(V)$. By a fundamental theorem in linear programming, the system $\widetilde{M}y = \widetilde{b}$ has a basic feasible solution (BFS) $\bar{y} \in Y$ with at most $\dim(V)$ nonzero components. Note that $\bar{y}$ is also a solution to the equivalent linear system $My = b$.

Next, let $J' \subseteq \{1, \ldots, k\}$ be the subset of indices corresponding to the nonzero components of $\bar{y}$ and consider $S' = \{v_j \ : \ j \in J \cup J'\} \subseteq S$. Clearly, $\mathcal{B} \subseteq S' \subseteq \text{pos}(S')$. Moreover, $b = M\bar{y} = \sum_{j \in J'} \bar{y}_j v_j \in \text{pos}(S')$, and so, $\mathcal{B} \cup \{b\} \subseteq \text{pos}(S') \subseteq \text{pos}(S) = V$. Since $\mathcal{B} \cup \{b\}$ is a (minimal) positive basis of V, it follows from Proposition 2.1(iii) that $V = \text{pos}(\mathcal{B} \cup \{b\}) \subseteq \text{pos}(S') \subseteq V$. Hence, S' positively spans V and $|S'| \leq |J| + |J'| \leq 2\dim(V)$. □

Theorem 6.6 (Maximum size of a positive basis) *The maximum number of elements in a positive basis of a linear subspace V of $\mathbb{R}^n$ is $2\dim(V)$.*

Proof Argue by contradiction. Suppose there is a positive basis B^+ of a subspace V of $\mathbb{R}^n$ with more than $2\dim(V)$ elements. Since B^+ is a positive basis, it follows from Theorem 4.2 that B^+ is a minimal positive spanning set for V. Moreover, since B^+ positively spans V, it follows from the previous lemma that B^+ contains a proper subset S' that also positively spans V and with at most $2\dim(V)$ elements, contradicting the minimality of B^+ as a positive spanning set for V. □

Finally, assuming the actual result proved by Audet (2011) that a positive basis of $\mathbb{R}^n$ cannot have more than $2n$ elements, there is a simpler way to extend this result to any linear subspace V of $\mathbb{R}^n$ by means of Theorem 4.6(iii). That is, given a nontrivial linear subspace V of $\mathbb{R}^n$, one can use an invertible linear transformation (i.e., an isomorphism) between V and $\mathbb{R}^{\dim(V)}$ to prove that a positive basis of V has at most $2\dim(V)$ elements.

7 Linear equivalence of positive spanning sets and positive bases

This section introduces the concept of *linear equivalence* between two finite sets of vectors of the same cardinality. This idea is meant to reduce the structure of positive spanning sets and positive bases into simpler forms that are easier to analyze. At the end of this section, we show that *linear equivalence* between positive bases is essentially the same as the notion of *structural equivalence* between positive bases introduced by Coope and Price (2001). However, linear equivalence is more general in the sense that it applies not just to positive bases but also to other special sets of vectors. We begin with the following definitions.

Definition 7.1 Let S and T be finite sets of vectors in $\mathbb{R}^n$ with the same number of elements. Then S is said to be *linearly equivalent* to T, written $S \sim_\ell T$, if there exists a nonsingular matrix $M \in \mathbb{R}^{n\times n}$ such that $S = M \cdot T = \cup_{v\in T}\{Mv\}$.

Example 7.1 Consider the set

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} \right\}.$$

The set S is a positive basis of $\mathbb{R}^3$ and it is linearly equivalent to the following positive basis of $\mathbb{R}^3$:

$$S' = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} \right\}.$$

This is because $S' = M^{-1} \cdot S$, where the columns of M are the 1st, 2nd and 5th elements of S:

$$M = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Definition 7.2 Let $\mathcal{T}$ be a (possibly infinite) collection of finite subsets of $\mathbb{R}^n$. We say that $\mathcal{T}$ is *closed under pre-multiplication by any nonsingular matrix in* $\mathbb{R}^{n\times n}$ if $M \cdot \mathcal{T} \subseteq \mathcal{T}$ for any nonsingular $M \in \mathbb{R}^{n\times n}$.

Define $\mathcal{I}^+(\mathbb{R}^n), \mathcal{S}^+(\mathbb{R}^n)$ and $\mathcal{B}^+(\mathbb{R}^n)$ to be the collection of all finite positively independent sets, positive spanning sets and positive bases of $\mathbb{R}^n$, respectively. By Corollary 4.7, $\mathcal{I}^+(\mathbb{R}^n), \mathcal{S}^+(\mathbb{R}^n)$ and $\mathcal{B}^+(\mathbb{R}^n)$ are all closed under pre-multiplication by any nonsingular matrix in $\mathbb{R}^{n\times n}$.

The next proposition says that linear equivalence is an equivalence relation on the collection of sets of vectors in $\mathbb{R}^n$ that are closed under pre-multiplication by any nonsingular matrix in $\mathbb{R}^{n\times n}$. In particular, it is an equivalence relation on $\mathcal{I}^+(\mathbb{R}^n), \mathcal{S}^+(\mathbb{R}^n)$ and $\mathcal{B}^+(\mathbb{R}^n)$. As a result, when $S \sim_\ell T$ in these collections of sets, we can simply say that S and T are linearly equivalent.

Proposition 7.1 *The relation* $\sim_\ell$ *is an equivalence relation on any collection of finite subsets of* $\mathbb{R}^n$ *that are closed under pre-multiplication by any nonsingular matrix in* $\mathbb{R}^{n\times n}$.

Proof Let S, T and U belong to a collection of finite subsets of $\mathbb{R}^n$ that are closed under pre-multiplication by any nonsingular matrix in $\mathbb{R}^{n\times n}$. Clearly, $S = I_n \cdot S$ so the relation $\sim_\ell$ is reflexive.

Next, suppose $S \sim_\ell T$. Then there exists a nonsingular matrix $M \in \mathbb{R}^{n\times n}$ such that $S = M \cdot T$. Note that $T = M^{-1} \cdot S$ so that $T \sim_\ell S$. Hence, the relation $\sim_\ell$ is symmetric.

Finally, suppose $S \sim_\ell T$ and $T \sim_\ell U$. Then there exists nonsingular matrices $M_1, M_2 \in \mathbb{R}^{n\times n}$ such that $S = M_1 \cdot T$ and $T = M_2 \cdot U$. Note that $S = M_1 \cdot (M_2 \cdot U) = (M_1 M_2) \cdot U$. Clearly, $M_1 M_2$ is nonsingular so $S \sim_\ell U$. Thus, the relation $\sim_\ell$ is also transitive. □

The next proposition says that any finite set of vectors in $\mathbb{R}^n$ that contains a basis is linearly equivalent to a set containing any specific basis of $\mathbb{R}^n$, including the natural basis.

Proposition 7.2 *Let S be a finite set of vectors in* $\mathbb{R}^n$ *that contains a basis of* $\mathbb{R}^n$ *(i.e., S linearly spans* $\mathbb{R}^n$*). Moreover, let* $B \subset \mathbb{R}^n$ *be a particular basis of* $\mathbb{R}^n$*. Then* $S \sim_\ell T$ *for some set of vectors T that contains B. In particular, S is linearly equivalent to a set of vectors that contains the natural basis* $\{e_1, \ldots, e_n\}$ *of* $\mathbb{R}^n$.

Proof Let $D = \{u_1, \ldots, u_n\}$ be a basis of $\mathbb{R}^n$ contained in S (note that there could be more than one basis contained in S) and let $B = \{v_1, \ldots, v_n\}$. Moreover, let $M(B)$ and $M(D)$ be the matrices whose columns are the vectors in B and D, respectively.

Define $T = (M(B)M(D)^{-1}) \cdot S$. Then S and T are linearly equivalent and $T = M(B) \cdot (M(D)^{-1} \cdot S)$ contains the basis B. $\square$

Since maximal positively independent sets, positive spanning sets and positive bases of $\mathbb{R}^n$ all contain a basis of $\mathbb{R}^n$, it follows from the previous proposition and Corollary 4.7 that these are all linearly equivalent to a set with the same properties and that contains the natural basis of $\mathbb{R}^n$. Hence, when analyzing the structure of maximal positively independent sets, positive spanning sets and positive bases, the previous proposition implies that there is no loss of generality in assuming that these special sets of vectors contain the much simpler natural basis of $\mathbb{R}^n$.

Recall that a positive spanning set for $\mathbb{R}^n$ is linearly equivalent to a positive spanning set for $\mathbb{R}^n$ that contains the natural basis $S = \{e_1, \ldots, e_n\}$. Hence, to construct any positive spanning set for $\mathbb{R}^n$, one can begin with the natural basis and then augment it with additional vectors. The following proposition characterizes the structure of any positive spanning set for $\mathbb{R}^n$ that contains the natural basis.

Proposition 7.3 *Let $S = \{e_1, \ldots, e_n, u_1, \ldots, u_\ell\}$, where $\ell \geq 1$, be a set of vectors in $\mathbb{R}^n$ that contains the natural basis. Then S positively spans $\mathbb{R}^n$ if and only if $\exists \eta_1, \ldots, \eta_\ell > 0$ such that $\sum_{j=1}^{\ell} \eta_j u_j < 0$.*

Proof Clearly, S linearly spans $\mathbb{R}^n$ since it contains a basis of $\mathbb{R}^n$. First, suppose $\exists \eta_1, \ldots, \eta_\ell > 0$ such that $\sum_{j=1}^{\ell} \eta_j u_j < 0$. Let $v = [v^{(1)} \ldots v^{(n)}]^T := -\sum_{j=1}^{\ell} \eta_j u_j$. Note that $v^{(i)} > 0$ for all $i = 1, \ldots, n$ and

$$\sum_{i=1}^{n} v^{(i)} e_i + \sum_{j=1}^{\ell} \eta_j u_j = v + (-v) = 0.$$

By Theorem 2.5(iii), S positively spans $\mathbb{R}^n$.

Conversely, suppose S positively spans $\mathbb{R}^n$. By Theorem 2.5(iii), $\exists \lambda_1, \ldots, \lambda_n, \eta_1, \ldots, \eta_\ell > 0$ such that

$$\sum_{i=1}^{n} \lambda_i e_i + \sum_{j=1}^{\ell} \eta_j u_j = 0,$$

and so,

$$\sum_{j=1}^{\ell} \eta_j u_j = -\sum_{i=1}^{n} \lambda_i e_i < 0.$$

$\square$

An easy consequence of the previous proposition is that if $S = \{e_1, \ldots, e_n, u_1, \ldots, u_\ell\}$ positively spans $\mathbb{R}^n$, then every row of the matrix $[u_1 \ldots u_\ell]$ must contain at least one strictly negative entry.

Example 7.2 The set

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -8 \\ -5 \end{bmatrix} \right\}.$$

positively spans $\mathbb{R}^3$ since

$$2 \cdot \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -8 \\ -5 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix}$$

consists only of negative entries. □

Characterizing positively independent sets that contain the natural basis does not seem as straightforward. However, the next proposition provides a sufficient, but not necessary, condition for a set $S = \{e_1, \ldots, e_n, u_1, \ldots, u_\ell\} \subset \mathbb{R}^n$ to become positively *dependent*.

Proposition 7.4 *Let $S = \{e_1, \ldots, e_n, u_1, \ldots, u_\ell\}$, where $\ell \geq 1$, be a set of vectors in $\mathbb{R}^n$ that contains the natural basis. Then the natural basis vector e_r is a positive combination of the other elements of S if and only if $\exists \eta_1, \ldots, \eta_\ell \geq 0$ such that $\sum_{j=1}^{\ell} \eta_j u_j$ has a strictly positive entry in the rth position and has nonpositive entries in the other positions.*

Proof First, assume $\exists \eta_1, \ldots, \eta_\ell \geq 0$ such that $\sum_{j=1}^{\ell} \eta_j u_j$ has exactly one strictly positive entry, which is in the rth position. Let $v = [v^{(1)} \ldots v^{(n)}]^T := \sum_{j=1}^{\ell} \eta_j u_j$. Then $v^{(r)} > 0$ and $v^{(i)} \leq 0$ for all $i \neq r$. Define $\lambda_i := -v^{(i)}$, $i = 1, \ldots, n$, $i \neq r$. Clearly, $\lambda_i \geq 0$, $i = 1, \ldots, n$, $i \neq r$ and

$$v + \sum_{i=1,\, i \neq r}^{n} \lambda_i e_i = v^{(r)} e_r,$$

and so,

$$\sum_{j=1}^{\ell} \eta_j u_j + \sum_{i=1,\, i \neq r}^{n} \lambda_i e_i = v^{(r)} e_r.$$

Hence,

$$e_r = \sum_{i=1,\, i \neq r}^{n} \frac{\lambda_i}{v^{(r)}} e_i + \sum_{j=1}^{\ell} \frac{\eta_j}{v^{(r)}} u_j \in \text{ pos } (S \setminus \{e_r\}).$$

Conversely, suppose e_r $(1 \leq r \leq n)$ is a positive combination of the other elements of S. That is,

$$e_r = \sum_{i=1,\, i \neq r}^{n} \lambda_i e_i + \sum_{j=1}^{\ell} \eta_j u_j,$$

for some $\lambda_i \geq 0,\ i = 1, \ldots, n,\ i \neq r$ and $\eta_j \geq 0,\ j = 1, \ldots, \ell$. Note that

$$\sum_{j=1}^{\ell} \eta_j u_j = e_r - \sum_{i=1,\ i \neq r}^{n} \lambda_i e_i = [-\lambda_1 \ldots - \lambda_{r-1}\ 1\ \ - \lambda_{r+1} \ldots - \lambda_n]^T$$

has exactly one strictly positive entry, which is in the rth position. □

Example 7.3 Consider the set

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

Note that $e_2 = [0\ 1\ 0]^T$ is a positive combination of the other elements of S since

$$\begin{bmatrix} 2 \\ -4 \\ -3 \end{bmatrix} + 3 \cdot \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

has exactly one positive entry, which is in the 2nd position. Hence, S is positively dependent. □

Theorems 6.1 and 6.3 characterize the structure of all minimal and maximal positive bases of $\mathbb{R}^n$. The next proposition follows immediately from these theorems.

Proposition 7.5 *Any minimal positive basis of $\mathbb{R}^n$ is linearly equivalent to a minimal positive basis of the form $\{e_1, \ldots, e_n, -\sum_{i=1}^{n} \lambda_i e_i\}$, where $\lambda_1, \ldots, \lambda_n > 0$. Any maximal positive basis of $\mathbb{R}^n$ is linearly equivalent to a maximal positive basis of the form $\{e_1, \ldots, e_n, -\lambda_1 e_1, \ldots, -\lambda_n e_n\}$, where $\lambda_1, \ldots, \lambda_n > 0$.*

In Sect. 3, we showed that, when $n \geq 3$, there is a positively independent set of any size in $\mathbb{R}^n$. The next proposition characterizes the maximal positively independent sets in $\mathbb{R}^1$ and $\mathbb{R}^2$.

Proposition 7.6 *When $n = 1$ or 2, any maximal positively independent set in $\mathbb{R}^n$ is a positive basis, and so, it has at most 2n elements.*

Proof When $n = 1$, any maximal positively independent set has the form $\{a, -a\}$, where $a > 0$, and this is a positive basis of $\mathbb{R}^1$. Now suppose $n = 2$ and let S be a maximal positively independent set in $\mathbb{R}^2$. By Corollary 4.7(ii) and Proposition 7.2, S is linearly equivalent to a maximal positively independent set T containing $\{e_1, e_2\}$. Note that $\{e_1, e_2\}$ is a positively independent but it is not maximal since it can be easily extended to a larger positively independent set (e.g., $\{e_1, e_2, -e_1 - e_2\}$).

Let $u \in T \setminus \{e_1, e_2\}$. We consider the possibilities for $u = [u^{(1)}, u^{(2)}]^T$. Clearly, $u^{(1)}$ and $u^{(2)}$ cannot be both strictly positive for otherwise u can be written as a strictly positive combination of e_1 and e_2. Next, suppose one of the components of u

is strictly positive. Without loss of generality we may assume that $u = [-a, b]^T$, where $a \geq 0$ and $b > 0$. (The argument is similar if $u = [a, -b]^T$ where $a > 0, b \geq 0$.) Note that

$$e_2 = \frac{a}{b} e_1 + \frac{1}{b} \begin{bmatrix} -a \\ b \end{bmatrix},$$

and so $\{e_1, e_2, u\} \subseteq T$ is positively dependent, which gives a contradiction. Hence, neither component of u is strictly positive (i.e., the components of u are 0 or strictly negative).

If both components of u are strictly negative, then $u = [-a, -b]^T = -ae_1 - be_2$ for some $a, b > 0$. By Theorem 6.1, $\{e_1, e_2, u\} \subseteq T$ is a positive basis of $\mathbb{R}^2$, and by Proposition 4.4, $\{e_1, e_2, u\}$ is a maximal positively independent set in $\mathbb{R}^2$ so it must be the set T.

The other possibility is that one of the components of u is strictly negative and the other is 0. So assume without loss of generality that $u = [-a, 0]^T$, where $a > 0$. Then $\{e_1, e_2, u\} \subseteq T$ is positively independent and it is not maximal since it can be extended to the positive basis $\{e_1, e_2, [-a, 0]^T, [0, -b]^T\}$, where $b > 0$. Let $v \in T \setminus \{e_1, e_2, [-a, 0]^T\}$. By the same reasoning as above, one of the components of v is strictly negative and the other 0. Note that v cannot have the form $[-c, 0]^T$ for some other $c > 0$ since this will be a positive combination of $[-a, 0]^T$. Hence, v must have the form $[0, -b]^T$, where $b > 0$, and so the only possible extension of $\{e_1, e_2, [-a, 0]^T\}$ to a larger positively independent set has the form $\{e_1, e_2, [-a, 0]^T, [0, -b]^T\}$, which is a positive basis of $\mathbb{R}^2$, and hence, this is the set T.

Thus, maximal positively independent sets in $\mathbb{R}^2$ have the form $\{e_1, e_2, -ae_1 - be_2\}$ or $\{e_1, e_2, -ae_1, -be_2\}$, where $a, b > 0$, and these are positive bases of $\mathbb{R}^2$. □

Finally, we explore the relationship between linear equivalence and the concept of structural equivalence of ordered positive bases introduced by Coope and Price (2001). They focus on ordered positive bases of $\mathbb{R}^n$ where the first n elements form a basis of $\mathbb{R}^n$ and the remaining positive basis elements are integer linear combinations of the ordinary basis elements. We will show that structural equivalence between ordered positive bases is essentially the same as linear equivalence between those positive bases. However, note that linear equivalence is more general in that it applies to collections of sets of vectors that are closed under pre-multiplication by a nonsingular matrix. Below is the definition of structural equivalence from Coope and Price (2001) modified slightly to allow non-integer coefficients for the linear combinations defining the positive basis elements that are not among the n basis elements.

Definition 7.3 Let $B^+ = \langle v_1, \ldots, v_k \rangle$ and $D^+ = \langle w_1, \ldots, w_k \rangle$ be two ordered positive bases in $\mathbb{R}^n$ (so that $k \geq n + 1$) such that the first n elements of these ordered sets form a basis of $\mathbb{R}^n$. Then B^+ and D^+ are said to be *structurally equivalent* if the following statement holds:

$$\forall j = n+1, \ldots, k \qquad v_j = \sum_{i=1}^{n} \zeta_{i,j} v_i \quad \Longleftrightarrow \quad w_j = \sum_{i=1}^{n} \zeta_{i,j} w_i,$$

where $\zeta_{i,j} \in \mathbb{R}$ for $i = 1, \ldots, n$.

Theorem 7.7 (Structural and linear equivalence of ordered positive bases) *Let $B^+ = \langle v_1, \ldots, v_k \rangle$ and $D^+ = \langle w_1, \ldots, w_k \rangle$ be two ordered positive bases in $\mathbb{R}^n$ whose first n elements form a basis of $\mathbb{R}^n$. Then B^+ and D^+ are structurally equivalent if and only if B^+ and D^+ are linearly equivalent.*

Proof First, observe that each v_j $(j = n+1, \ldots, k)$ is a linear combination of the basis elements $v_1, \ldots, v_n$:

$$\forall\, j = n+1, \ldots, k, \qquad v_j = \sum_{i=1}^{n} \zeta_{i,j} v_i,$$

for some $\zeta_{i,j} \in \mathbb{R}$ for $i = 1, \ldots, n$. For convenience, define the nonsingular matrices $M(B^+) := [v_1 \ldots v_n]$, $M(D^+) := [w_1 \ldots w_n] \in \mathbb{R}^{n \times n}$ and the vectors $z_j := [\zeta_{1,j}, \zeta_{2,j}, \ldots, \zeta_{n,j}]^T \in \mathbb{R}^n$ for $j = n+1, \ldots, k$. Note that $v_j = M(B^+) z_j$ for all $j = n+1, \ldots, k$, and so,

$$B^+ = \{v_1, \ldots, v_n, M(B^+) z_{n+1}, \ldots, M(B^+) z_k\}.$$

Now suppose B^+ and D^+ are structurally equivalent. Then

$$\forall j = n+1, \ldots, k \qquad w_j = \sum_{i=1}^{n} \zeta_{i,j} w_i,$$

and so,

$$D^+ = \{w_1, \ldots, w_n, M(D^+) z_{n+1}, \ldots, M(D^+) z_k\}.$$

Note that

$$M(B^+)^{-1} \cdot B^+ = \{e_1, \ldots, e_n, z_{n+1}, \ldots, z_k\} = M(D^+)^{-1} \cdot D^+.$$

Hence, $B^+ = (M(B^+) M(D^+)^{-1}) \cdot D^+$, and so, B^+ and D^+ are linearly equivalent.

Conversely, suppose B^+ and D^+ are linearly equivalent. Then there exists a nonsingular matrix $P \in \mathbb{R}^{n \times n}$ such that $B^+ = P \cdot D^+$. Note that for each $j = n+1, \ldots, k$, $w_j = P^{-1} v_j$. Moreover, $M(B^+) = P M(D^+)$, and so for each $j = n+1, \ldots, k$,

$$\begin{aligned} w_j &= (M(B^+) M(D^+)^{-1})^{-1} v_j = M(D^+) M(B^+)^{-1} v_j \\ &= M(D^+) M(B^+)^{-1} (M(B^+) z_j) = M(D^+) z_j = \sum_{i=1}^{n} \zeta_{i,j} w_i. \end{aligned}$$

Thus, B^+ and D^+ are structurally equivalent. $\square$

8 Algorithms for determining positively independent sets and positive spanning sets

This section discusses some algorithms for determining if a set of vectors in $\mathbb{R}^n$ is positively independent or if it positively spans a given linear subspace of $\mathbb{R}^n$. It will also provide an algorithm for extending any set of vectors to a positive spanning set for $\mathbb{R}^n$ using only a relatively small number of additional vectors.

First, we present three numerical problems involving positive spanning sets and positively independent sets that all involve finding nonnegative solutions to a system of linear equations.

Problem 1 Given $S = \{v_1, \ldots, v_k\} \subset \mathbb{R}^n$ and $u \in \mathbb{R}^n$, determine if u belongs to pos (S).

The most direct way to solve Problem 1 is to check if the following system has a solution:

$$\begin{aligned} u &= \sum_{j=1}^{k} \lambda_j v_j \\ \lambda_j &\geq 0, \ \forall j = 1, \ldots, k \end{aligned} \tag{8.1}$$

Problem 2 Given a set of nonzero vectors $S = \{v_1, \ldots, v_k\} \subset \mathbb{R}^n$, with $k \geq n + 1$, determine if S positively spans $\mathbb{R}^n$.

By Theorem 2.5, Problem 2 can be solved using the following steps:

Step 1 Check that S linearly spans $\mathbb{R}^n$ by verifying that the matrix $[v_1 \ldots v_k]$ has full (row) rank. If S does *not* linearly span $\mathbb{R}^n$, then it cannot positively span $\mathbb{R}^n$ so stop.

Step 2 For each $i = 1, \ldots, k$, verify that the system below has a solution:

$$\begin{aligned} -v_i &= \sum_{j=1,\, j \neq i}^{k} \lambda_j v_j \\ \lambda_j &\geq 0, \ \forall j = 1, \ldots, k, \ j \neq i \end{aligned} \tag{8.2}$$

If the system (8.2) has a solution for each $i = 1, \ldots, k$, then S positively spans $\mathbb{R}^n$; otherwise, S does *not* positively span $\mathbb{R}^n$.

By Theorem 2.5(v), Step 2 can be greatly simplified by replacing it with the following:

Step 2′ Verify that the system below has a solution:

$$\begin{aligned} \sum_{i=1}^{k} \gamma_i v_i &= -\sum_{i=1}^{k} v_i \\ \gamma_i &\geq 0, \ \forall i = 1, \ldots, k \end{aligned} \tag{8.3}$$

If (8.3) has a solution, then S positively spans $\mathbb{R}^n$; otherwise, S does *not* positively span $\mathbb{R}^n$.

However, the advantage of using Step 2 over Step 2′ is that when S does *not* positively span $\mathbb{R}^n$, Step 2 identifies a $-v_i$ that cannot be written as a positive combination of the elements of S while Step 2′ gives no such example.

Note that if S linearly spans $\mathbb{R}^n$ in Step 1, then the above algorithm can be simplified using the idea of linear equivalence. That is, we can reduce S to another set containing the natural basis of $\mathbb{R}^n$ before proceeding with either Step 2 or Step 2′.

Problem 3 Given a set of vectors $S = \{v_1, \ldots, v_k\} \subset \mathbb{R}^n$, determine if S is positively independent.

The simplest, not necessarily the most efficient, way to solve Problem 3 is to verify that each element of S is not a positive combination of the other elements of S. That is, for each $i = 1, \ldots, k$, we verify that the system below has no solution:

$$\begin{gathered} v_i = \sum_{j=1, j\neq i}^{k} \lambda_j v_j \\ \lambda_j \geq 0, \ \forall j = 1, \ldots, k, \ j \neq i \end{gathered} \tag{8.4}$$

Next, note that the systems (8.1)–(8.4) in Problems 1–3 are all of the form:

$$\begin{gathered} A^T \lambda = c \\ \lambda \geq 0 \end{gathered} \tag{8.5}$$

where $A \in \mathbb{R}^{m \times n}, \lambda \in \mathbb{R}^m$ ($m = k$ for Problem 1 and $m = k - 1$ for Problems 2 and 3). By viewing the general system (8.5) as the feasible region of a linear program (LP) with constant objective function $z = 0$, we can check if the system has a solution by determining whether or not the LP is feasible. However, a more efficient procedure for solving (8.5) is provided by Huang and Pardalos (2001) and uses the dual of this LP. They note that it follows from Farkas' Lemma that (8.5) has a feasible solution if and only if the optimal value of the following LP is zero:

$$\begin{aligned} & \max \ c^T x \\ & \text{s.t.} \\ & \quad Ax \leq 0 \\ & \quad x \in \mathbb{R}^n \end{aligned} \tag{8.6}$$

Huang and Pardalos (2001) developed an algorithm for solving (8.6) that is efficient especially when the number of variables or constraints is large.

As mentioned earlier, positive bases and positive spanning sets are of interest in derivative-free optimization since they are guaranteed to produce a descent direction if the gradient at the current iterate is nonzero. Hence, an important problem that could be useful in the design of derivative-free optimization methods is the following.

Problem 4 Given a set of nonzero vectors S in $\mathbb{R}^n$ that does *not* positively span $\mathbb{R}^n$, extend S to a larger set that positively spans $\mathbb{R}^n$ using only a few additional vectors.

One way to solve Problem 4 is provided by the following algorithm:

Step 1 Find a linearly independent subset of S of maximum size and call this subset $\widetilde{S} = \{v_1, \ldots, v_k\}$, where $k \leq n$. (For example, this subset can be obtained by row reducing the matrix whose columns are the elements of S and selecting the elements of S corresponding to the leading 1's in the RREF form of the matrix.)
Step 2 Extend $\widetilde{S}$ to a basis $B = \{v_1, \ldots, v_n\}$ of $\mathbb{R}^n$ and let $\widetilde{B} = B \setminus \widetilde{S}$.
Step 3 Check if one of the elements of $S \setminus \widetilde{S}$ is a negative linear combination of the elements of $\widetilde{S}$ by solving systems of linear equations in nonnegative variables of the form:

$$\begin{aligned} M(\widetilde{S})\lambda &= -v \\ \lambda \geq 0,\ \ \lambda &\in \mathbb{R}^k \end{aligned} \tag{8.7}$$

where $M(\widetilde{S}) = [v_1, \ldots, v_k]$ and $v \in S \setminus \widetilde{S}$.
Step 4 If the system (8.7) has a solution for some $v \in S \setminus \widetilde{S}$, then let $v = -\sum_{i=1}^{k} \lambda_i v_i$, where $\lambda_i \geq 0$ and define $J(v)$ to be the subset of indices $i \in \{1, \ldots, k\}$ such that $\lambda_i > 0$. Now, the set $S \cup \widetilde{B} \cup \{-\sum_{i \in \{1,\ldots,n\} \setminus J(v)} \eta_i v_i\}$, where $\eta_i > 0$ for all $i \in \{1, \ldots, n\} \setminus J(v)$, contains the positive basis $B \cup \{v\} \cup \{-\sum_{i \in \{1,\ldots,n\} \setminus J(v)} \eta_i v_i\}$ of $\mathbb{R}^n$ of size $n+2$.
Step 5 If the system (8.7) has no solution for all $v \in S \setminus \widetilde{S}$, then the set $S \cup \widetilde{B} \cup \{-\sum_{i=1}^{n} \lambda_i v_i\}$, where $\lambda_i > 0$ for all i, contains the minimal positive basis of $\mathbb{R}^n$ given by $B \cup \{-\sum_{i=1}^{n} \lambda_i v_i\}$.
Step 6 The set containing a positive basis of $\mathbb{R}^n$ in either Step 4 or Step 5 will positively span $\mathbb{R}^n$.

A few remarks are in order. First, it is possible to go directly from Step 2 to Step 5 and create a positive spanning set of $\mathbb{R}^n$. Steps 3 and 4 check if it is possible to use one of the vectors in $S \setminus \widetilde{S}$ to get a positive spanning set of $\mathbb{R}^n$ before including any new vectors and these steps are helpful when the black-box objective function to be optimized is computationally expensive. Note that the system (8.7) does not involve any vectors from $\widetilde{B}$ (i.e., it only involves the vectors in $\widetilde{S}$ from the basis B) since $S \setminus \widetilde{S} \subseteq \operatorname{span}(\widetilde{S})$. In Step 4, note that $J(v) \neq \emptyset$ since $0 \notin S$. Also, if $\widetilde{B} \neq \emptyset$, then $J(v) \neq \{1, \ldots, n\}$. Moreover, if $\widetilde{B} = \emptyset$ (i.e., if S contains a basis of $\mathbb{R}^n$), then again $J(v) \neq \{1, \ldots, n\}$ since S does *not* positively span $\mathbb{R}^n$ by assumption. Finally, note that Steps 4 and 5 can be modified to accommodate a larger positive basis if needed using the construction method provided by Theorem 5.4.

Next, below is another important problem about positive bases that has applications in computational geometry, stochastic programming and Data Envelopment Analysis (Dulá et al. 1998):

Problem 5 (*Conical hull problem*) Given a set of vectors $S = \{v_1, \ldots, v_k\} \subset \mathbb{R}^n$ that positively spans a convex cone C, determine a subset of S that is a frame of C. (If C is a linear subspace, then this is the problem of determining a subset of S that is a positive basis of C.)

Note that Theorem 4.3 guarantees the existence of a subset of S that is a frame of the convex cone C in Problem 5. One way to solve Problem 5 is given by the proof of Theorem 4.3:

Step 1 Set $i = 1$ and $\mathcal{F} = S$.
Step 2 Determine if v_i is a positive combination of the elements of $\mathcal{F} \setminus \{v_i\}$ (Problem 1).
Step 3 If $v_i \in \text{ pos } (\mathcal{F} \setminus \{v_i\})$, then replace $\mathcal{F} \leftarrow \mathcal{F} \setminus \{v_i\}$.
Step 4 Increment $i \leftarrow i + 1$. If $i \leq k$, then go back to Step 2; else, stop.

Note that the resulting set $\mathcal{F}$ will be a frame of C.

The above algorithm would only be practical if the set S is small. When S is large, a more efficient approach was developed by Dulá et al. (1998).

If V is a linear subspace of $\mathbb{R}^n$, recall that a positive basis of V properly contains an ordinary basis of V. Hence, one might be tempted to consider solving Problem 5 by finding a subset of S that is a basis of V and then extending it to a positive basis of V. However, this procedure might fail if one starts with the wrong basis of V as the following example shows.

Example 8.1 The set $S = \{[1,0],[1,1/2],[0,1],[-1,-1]\}$ positively spans $\mathbb{R}^2$. If one starts with the basis $\{[1,0],[1,1/2]\}$, it cannot be extended to a positive basis of $\mathbb{R}^2$ by including either [0, 1] or $[-1,-1]$ or both.

9 Summary

This article reviewed the basic properties and explored additional properties of positive spanning sets, positively independent sets, frames of convex cones, and positive bases of linear subspaces of $\mathbb{R}^n$. In particular, it provided procedures for constructing these special sets of vectors that were not previously mentioned in the literature. This includes procedures for constructing positive bases of a linear subspace V of $\mathbb{R}^n$ of arbitrary sizes between $\dim(V)+1$ and $2\dim(V)$. It also proved that linear transformations preserve the positive spanning property and that invertible linear transformations preserve positive independence, generalizing a theorem by Lewis and Torczon (1999). This article also introduced the notion of linear equivalence between sets of vectors that are closed under pre-multiplication by any nonsingular matrix of suitable size, and this includes positive spanning sets, positively independent sets and positive bases. This concept is meant to simplify the analysis of the structures of these special sets of vectors. It was also shown that linear equivalence generalizes the concept of structural equivalence introduced by Coope and Price (2001). This paper also explored some similarities and differences between the linear span and positive span, between linear and positive independence, and between bases and positive bases of linear spaces. For example, a linearly independent set can always be extended to a basis of a linear space but a positively independent set cannot always be extended to a positive basis. Moreover, the size of a linearly independent set in $\mathbb{R}^n$ is at most n but there is a positively independent set of any size in R^n when $n \geq 3$ and a procedure for constructing such a set was provided. In addition, this article discussed algorithms for determining if a set positively spans a linear subspace of $\mathbb{R}^n$ and for determining if a set of vectors is positively independent. Finally, it also presented an algorithm for extending any set of vectors in a linear subspace to a positive spanning set of that subspace using a relatively small number of additional vectors. Since positive spanning sets and positive bases are important in black-box and derivative-free optimization, the ideas presented here are potentially helpful in the design of future DFO algorithms.

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