Quasi-decidability of a Fragment of the First-order Theory of Real Numbers*

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Abstract

In this paper we consider a fragment of the first-order theory of the real numbers that includes systems of equations of continuous functions in bounded domains, and for which all functions are computable in the sense that it is possible to compute arbitrarily close piece-wise interval approximations. Even though this fragment is undecidable, we prove that there is a (possibly non-terminating) algorithm for checking satisfiability such that (1) whenever it terminates, it computes a correct answer, and (2) it always terminates when the input is robust. A formula is robust, if its satisfiability does not change under small perturbations. As a basic tool for our algorithm we use the notion of degree from the field of (differential) topology.

1 Introduction

It is well known that, while the theory of real numbers with addition and multiplication is decidable [42], any periodic function makes the problem undecidable, since it allows encoding of the integers. The root existence problem for a general class of computable continuous functions is also undecidable [43]. This even holds if we consider only functions on bounded domains, because an algorithm deciding it could be used to compute a fixed point of a continuous function

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from a ball to itself which is known to be non-computable for some computable functions [34].

Recently, several papers [19, 36, 37, 13] have argued, that in continuous domains (where we have notions of neighborhood, perturbation etc.) such undecidability results do not always have much practical relevance. The reason is, that real-world manifestations of abstract mathematical objects in such domains will always be exposed to perturbations (imprecision of production, engineering approximations, unpredictable influences of the environment etc.). Engineers take these perturbations into account by coming up with *robust* designs, that is, designs that do not change essentially under such perturbations. Hence, in this context, it is sufficient to come up with algorithms that are able to decide such robust problem instances. They are allowed to run forever in non-robust cases, but *must not* return incorrect results, in whatever case. In a recent paper we called problems possessing such an algorithm *quasi-decidable* [38].

In this paper we show quasi-decidability of a certain fragment of the first-order theory of the reals. The basic building blocks are existentially quantified disjunctions of systems of n equalities over n variables and arbitrarily many inequalities. Those blocks may be combined using universal quantifiers, conjunctions, and disjunctions. All variables are assumed to range over bounded intervals.

The allowed function symbols include addition, multiplication, exponentiation, and sine. More specifically, they have to be continuous, and for compact intervals $I_1, \ldots, I_n$, we need to be able to compute an interval $J \supseteq f(I_1, \ldots, I_n)$ such that the over-approximation of J over $f(I_1, \ldots, I_n)$ can be made arbitrarily small.

The main tool we use is the notion of the *degree of a continuous function* that comes from differential topology [25, 29, 33]. For continuous functions $f : [a, b] \to \mathbb{R}$, the degree $\deg(f, [a, b], 0)$ is 0 iff $f(a)$ and $f(b)$ have the same sign, otherwise the degree is either -1 or 1, depending on whether the sign changes from negative to positive or the other way round. Hence, in this case, the degree gives the information given by the intermediate value theorem plus some directional information. For higher dimensional functions, the degree is an integer whose value may be greater than 1, and that generalizes this information to higher dimensions. However, the degree is defined only when the dimensions of the domain and target space of f are equal.

If we can over-approximate the values of a function f arbitrarily precisely on intervals, then the degree is algorithmically computable. Our algorithm for checking the existence of a solution of $f = 0$ consists of approximating the connected components of the zero set of f by small neighborhoods and checking the degree of f in those neighborhoods. If for any neighborhood this degree is nonzero, then $f(x) = 0$ has a solution. Otherwise we show that there exists an arbitrarily small perturbation $\tilde{f}$ of f such that $\tilde{f}(x) = 0$ does not have a solution.

For handling inequalities, universal quantification, conjunction and disjunctions we use interval deductions. We show termination of the resulting algorithm for all robust cases based on robustness properties of the topological degree and convergence properties of the employed interval deductions.

Even though this work applies results from a quite distant field—topology—to automated reasoning, the paper is largely self-contained. Usage of results from topology that are not explicitly delineated in this paper is concentrated exclusively in Section 6.

The content of the paper is as follows: In Section 2, we define the notions of robustness and quasi-decidability, and state the main theorem of the paper. In Section 3, we provide the according quasi-decision procedure. In Section 4, we present the notion of topological degree and describe its main properties. In Section 5, we show that the quasi-decision procedure always returns a correct result. In Section 6 we show some non-algorithmic properties of the degree that will be the essential for showing termination of robust inputs in Section 7. In Section 8 we discuss related work, and in Section 9 we give an outlook on the difficulties involved in generalizing the result. Finally, in Section 10, we conclude the paper.

2 The Main Theorem

Let us assume a class $\mathcal{X}$ of first-order predicate logical formulas and let us fix a certain interpretation for all function and predicate symbols in $\mathcal{X}$. Moreover, assume a function d that, given two formulas ϕ and ψ in $\mathcal{X}$, returns a number $d(\phi, \psi)$ in $\mathbb{R}^{\geq 0} \cup \{\infty\}$. Intuitively, this function should model the distance between formulas in $\mathcal{X}$, but—at least for now—we do not require any specific properties.

Definition 1 *Let S be a sentence in $\mathcal{X}$ and $\varepsilon > 0$. We say that S is ε-*robust*, if for every sentence S', $d(S', S) < \varepsilon$ implies that S' and S are equi-satisfiable. We say that the sentence S is* robust*, if there is an $\varepsilon > 0$ such that S is ε-robust. We say that a sentence S is robustly true, if it is both robust and true (i.e., satisfiable). We say that a sentence S is robustly false, if it is both robust and false (i.e., unsatisfiable).*

Now, consider the following algorithm specification:

Input: A sentence S from $\mathcal{X}$,

Output: $a \in \{\mathbf{T}, \mathbf{F}\}$

such that

- if $a = \mathbf{F}$, then the sentence S is false, and
- if $a = \mathbf{T}$, then the sentence S is true,
- if the sentence S is robust then the algorithm terminates.

Definition 2 *Given a class $\mathcal{X}$ of first-order sentences and a function $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^{\geq 0} \cup \{\infty\}$, $\mathcal{X}$ is* quasi-decidable wrt. *d iff there exists an algorithm (a* quasi-decision procedure*) with the above specification.*

We will now concentrate on the real numbers. Our goal is to show quasi-decidability of a non-trivial class of first-order sentences that includes functions such as sin.

We define a *box* B in $\mathbb{R}^n$ to be the Cartesian product of n closed intervals of finite length (i.e., a hyper-rectangle) and the *width* of a box is the maximum of the width of its constituting intervals. For $x \in \mathbb{R}^n$, $|x|$ will refer to its *maximum norm* $|x| := \max\{|x_1|, \ldots, |x_n|\}$ and for a continuous function $f : \Omega \to \mathbb{R}^n$, we use the supremum norm $||f||_\Omega := \sup\{|f(x)|;\ x \in \Omega\}$. The zero set $\{f(x) = 0 \mid x \in \Omega\}$ will be denoted simply by $\{f = 0\}$. If $||f - g|| \leq \alpha$ for some $\alpha > 0$, we say that g is an *α-perturbation* of f. For a set $\Omega \subseteq \mathbb{R}^n$, $\bar{\Omega}$ is its closure, Ω° its interior and $\partial\Omega = \bar{\Omega} \backslash \Omega^\circ$ its boundary with respect to the Euclidean topology. We will call the closure $\bar{\Omega}$ of an open connected bounded set Ω a *closed region*.

Definition 3 *Let $\Omega \subseteq \mathbb{R}^n$ be a box with rational endpoints. We call a function $f : \Omega \to \mathbb{R}$* interval-computable *iff there exists a corresponding algorithm $I(f)$ that, for a given box $B \subseteq \Omega$ with rational endpoints and positive diameter, computes a closed (possibly degenerate) interval $I(f)(B)$ such that*

- *$I(f)(B) \supseteq \{f(x) \mid x \in B\}$, and*
- *for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every box B with $0 < diam(B) < \delta$, $I(f)(B) < \varepsilon$.*

We call a function $f = (f_1, \ldots, f_n) : \Omega \to \mathbb{R}^n$ interval-computable iff each f_i is interval-computable. In this case, the algorithm $I(f)$ returns a tuple of intervals, one for each f_i. We call a function $f : \mathbb{R}^n \to \mathbb{R}$ interval-computable iff every restriction of f to a box $\Omega \subseteq \mathbb{R}^n$ with rational endpoints is interval-computable.

Usually such functions are written in terms of symbolic expressions containing symbols denoting certain basic interval computable functions such as rational constants, the constant π, addition, multiplication, exponentiation, trigonometric function and square root. Then, $I(f)$ can be computed from the expression by interval arithmetic [32, 30]. The interval literature usually calls an interval function fulfilling the first property of Definition 3 "enclosure". Instead of the second property, it often uses a slightly stronger notion of an interval function being "Lipschitz continuous" [32, Section 2.1].

Each interval-computable function is continuous. Note that, in practice, interval arithmetic is often implemented in fixed-precision floating-point arithmetic which violates the second property above. However, the property can be fulfilled using some form of arbitrary-precision arithmetic. In any case, in this paper we require that the result is returned in the form of rational numbers. All further computations in this paper will be done based on rational number arithmetic.

We will use interval computable functions and expressions denoting them interchangeably and assume that for an expression denoting a function f, a corresponding algorithm $I(f)$ is given.

Definition 4 *Let $\mathcal{A}$ be the class of all first-order predicate logical formulas such that*

- *all quantified variables are real and bounded by closed intervals with rational endpoints,*
- *all terms are built from function symbols that denote interval-computable functions, and*
- *the allowed predicate symbols are $=, \leq, <$ with their usual interpretation over the real numbers.*

Let $\mathcal{B} \subseteq \mathcal{A}$ be the smallest subclass of formulas such that

(a) $\mathcal{B}$ contains all formulas of the form

$$\exists x \in B \ . \quad [f_1 = 0 \ \wedge \ f_2 = 0 \ \wedge \ \ldots \wedge \ f_n = 0 \ \wedge \ g_1 \geq 0 \ \wedge \ g_2 \geq 0 \ \wedge \ldots \wedge \ g_k \geq 0]$$

where $f_1, \ldots, f_n, g_1, \ldots, g_k$ are terms, B is an m-box (the expression $\exists x \in B$ denoting a block of m existential quantifiers) and either $n \geq m$ or $n = 0$. The integer k may be arbitrary and we also admit $k = 0$ (i.e., the case without inequalities).

(b) Let $I \subseteq \mathbb{R}$ be a closed bounded interval. If U is in $\mathcal{B}$, then

$$\forall x \in I \ . \ U$$

is also in $\mathcal{B}$.

(c) If U, V are in $\mathcal{B}$, then

$$U \wedge V \qquad U \vee V$$

are also in $\mathcal{B}$.

The formulas corresponding to (a) represent systems of equations and inequalities. However, we assume that there are no more existential quantifiers than equations in (a), corresponding to the condition $n \geq m$.

The following sentence is an example of a formula in class $\mathcal{B}$:

$$\begin{aligned}
&\forall x \in [-1, 1] \\
&\quad \exists y \in [-1, 1] \ \exists z \in [-1, 1] \\
&\qquad [x^2 - y^2 - z^2 = 0 \ \wedge \ x^3 - y^3 - z^3 = 0].
\end{aligned}$$

The following sentence is an example of a sentence not in $\mathcal{B}$

$$\exists x \in [0, 1] \ \exists y \in [0, 1] \ . \ x - y = 0$$

because the domain of the particular function is a 2-dimensional box and there is only one equation, so the assumptions in (a) are violated.

Throughout we will use the convention that logical connectives bind stronger than quantifiers. Moreover, we use brackets to denote Boolean structure of formulas. Sometimes we will use line breaks instead of brackets for this purpose. We will use the symbol $\equiv$ to denote equality of first-order formulas.

Now let us define some notion of distance on $\mathcal{A}$.

Definition 5 *Let F, G be two sentences in the class $\mathcal{A}$. We say that F and G have the* same structure*, if they one can be obtained from the other by only exchanging terms (i.e., they have the same Boolean and quantification structure).*

We define the distance d on formulas as follows. If two formulas F and G do not have the same structure, then $d(F,G) := \infty$. In the case where they do have the same structure, assume that the sentence F contains terms denoting functions $f_1, \ldots, f_p$ and the sentence G contains on the corresponding places terms denoting the functions $g_1, \ldots, g_p$. We define the distance

$$d(F,G) := \max_{i \in \{1,\ldots,p\}} ||f_i - g_i||_{\Omega_i},$$

where Ω_i denotes the respective domain of those functions.[1]

For example the sentences

$$\exists x \in [0,1]\, \forall y \in [0,1] \,.\, x^2 - y = xy \wedge x = y$$

and

$$\exists x \in [0,1]\, \forall y \in [0,1] \,.\, x^2 - y = xy + 1 \wedge x = y^2$$

have the same structure, because the only difference is in the terms involved. If we assume that the domains of the involved functions is $[0,1]^2$, then $d(F,G) = 1$, because $d(x^2 - y, x^2 - y) = 0$, $d(xy, xy + 1) = 1$, $d(x,x) = 0$ and $d(y, y^2) = 1/4$.

If $f, g : K \to \mathbb{R}^n$ are continuous, then $d(\bigwedge_i (f_i = 0), \bigwedge_i (g_i = 0)) = \max_i ||f_i - g_i|| = ||f - g||$, which justifies the use of the max-norm in $\mathbb{R}^n$.

The main result of this paper is the following.

Theorem 1 *The class $\mathcal{B}$ is quasi-decidable wrt. the function d.*

If $\exists x \,.\, F_1$ and $\exists x \,.\, F_2$ are in the class $\mathcal{B}$, then $\exists x \in B \,.\, [F_1 \vee F_2]$ is robust if and only if the formula $[\exists x \in B \,.\, F_1] \vee [\exists x \in B \,.\, F_2]$ is robust and they are equi-satisfiable. Hence a quasi-decision procedure for $\mathcal{B}$ can handle disjunctions within existential quantification, too. Also, note that a formula with strict inequalities of the form $\exists x \in B \,.\, f = 0 \wedge g > 0$ is robust if and only if $\exists x \in B \,.\, f = 0 \wedge g \geq 0$ is robust and they are equi-satisfiable. In the following, however, we will restrict ourselves to the class $\mathcal{B}$.

We will show an argument in Section 9.1 that the constraint $m \leq n$ can probably be generalized to $m \leq 2n - 3$, and that without any constraint on m and n, there is not much hope to have a quasi-decision procedure for general systems of equations and inequalities.

3 The Quasi-decision Procedure

In this paper, we construct an algorithm that decides, whether or not a robust sentence in $\mathcal{B}$ is true.

[1] Usually, we suppose that the domain is the box defined by the quantification of all the variables.

For any formula $U \in \mathcal{B}$, variable x and $x_0 \in \mathbb{R}$ we denote by $U[x \leftarrow x_0]$ the formula derived from U by substituting x_0 for x in every free occurrence of x in U. We also allow x to be an n-tuple of variables, and $x_0 \in \mathbb{R}^n$, in which case $U[x \leftarrow x_0]$ denotes the parallel substitution of entries of x_0 with their corresponding entries of x.

In our algorithms, we use an alternative form of the Cartesian product that concatenates tuples from the argument sets, instead of forming pairs. That is, for sets $X \in \mathbb{R}^n$ and $Y \in \mathbb{R}^m$ it produces the set $\{(x_1, \ldots, x_n, y_1, \ldots, y_n) \mid (x_1, \ldots, x_n) \in X, (y_1, \ldots, y_m) \in Y\}$. Especially, for the set $\{()\}$ containing the 0-tuple, $\{()\} \times X$ will be X. The width of $\{()\}$, viewed as a box, is zero by definition.

For technical reasons, we construct an algorithm for the following, more general, specification:

Input:

- a formula S from $\mathcal{B}$ in l free variables p,
- an l-box P,
- $r \in \mathbb{R}^{>0}$,

such that the width of P is at most r.

Output: a nonempty subset of $\{\mathbf{T}, \mathbf{F}\}$

with the following two properties:

Correctness: If the algorithm terminates with $\{\mathbf{T}\}$ ($\{\mathbf{F}\}$), then for all $p_0 \in P$, $S[p \leftarrow p_0]$ is robustly true (robustly false).

Definiteness: If for a given l-box P_0, either for all $p_0 \in P_0$ the sentence $S[p \leftarrow p_0]$ is robustly true or for all $p_0 \in P_0$ the sentence $S[p \leftarrow p_0]$ is robustly false, then there exists an $\varepsilon > 0$ such that for every $r \le \varepsilon$ and every sub-box $P \subseteq P_0$ with width smaller than r, the algorithm terminates with $\{\mathbf{T}\}$ or $\{\mathbf{F}\}$ (as opposed to $\{\mathbf{T}, \mathbf{F}\}$).

We will denote this algorithm by CheckSat(S, P, r). If it fulfills its specification described above, then the algorithm below is clearly a quasi-decision procedure for $\mathcal{B}$.

$\varepsilon \leftarrow 1$
loop
  $R \leftarrow \text{CheckSat}(S, \{()\}, \varepsilon)$
  if $|R| = 1$ **then**
    return s s.t. $s \in R$
  else
    $\varepsilon \leftarrow \varepsilon/2$

Note that the above specification does not only result in a quasi-decision procedure, but also checks robustness of the input.

We will now define the algorithm CheckSat(S, P, r) in details. We will leave the proof that it fulfills the specification to Sections 5 (correctness) and 7 (definiteness). The algorithm is recursive, following the definition of class $\mathcal{B}$. We will describe the parts corresponding to the individual cases of this definition.

3.1 System of Equations and Inequalities

We first consider the case (a) of Definition 4, that is, a formula S of the form

$$\exists x \in B \,.\, [f_1 = 0 \wedge \ldots \wedge f_n = 0 \wedge g_1 \geq 0 \wedge \ldots \wedge g_k \geq 0]$$

where B is an m-box. In an abuse of notation we also use $f_1, \ldots, f_n$ and $g_1, \ldots, g_k$ for the functions denoted by those terms. They are functions in $\mathbb{R}^{l+m} \to \mathbb{R}$, where l is the number of free variables of S. We assume that the order of the arguments of those functions is the same as the order in which the respective variables are quantified in the overall formula. Finally, we denote by $f : \mathbb{R}^{l+m} \to \mathbb{R}^n$ the function defined by the components $(f_1, \ldots, f_n)$ and by $g : \mathbb{R}^{l+m} \to \mathbb{R}^k$ the function defined by the components $(g_1, \ldots, g_k)$. Since the composition of interval-computable functions is again interval-computable, f and g are interval-computable.

Disproving the formula is straight-forward using the information given by $I(f)$ and $I(g)$. However, in order to ensure that the computed over-approximation is not too big, instead of working with $I(f)(B)$ and $I(g)(B)$ we work with elements of a partition of B into small enough pieces, where "small enough" is determined by the parameter r (Line 2 of the algorithm SoEI below).

The core of the algorithm for proving the formula is a test whether a system of equations $f = 0$ has a solution in a bounded region. The test analyzes the boundary of the region and exploits continuity to deduce existence of a zero in the interior. In the one-dimensional case, a bounded region is simply a closed interval. If f has opposite sign on the two end-points of the interval, the intermediate value theorem tells us, that f has a solution in the interior. Here f has to be non-zero on both interval endpoints (since f is in general non-polynomial, we cannot verify that f is zero on an interval endpoint, we can only exclude this).

In general, we use the notion of the degree from the field of differential topology [25, 29, 33]. For a continuous function $f : \Omega \to \mathbb{R}^n$ where Ω is a bounded open set and $p \notin f(\partial\Omega)$, the degree of f with respect to Ω and a point $p \in \mathbb{R}^n$ is an integer denoted by $\deg(f, \Omega, p)$. If $\deg(f, \Omega, p) \neq 0$ then the equation $f = p$ has a solution in Ω. More details on the degree are given in Section 4 below.

Since the degree only returns information that can be inferred from the function values on the boundary of Ω, we also need to work with a partition of B in this case, and we compute the degree of the individual pieces. However, for ensuring that f is non-zero on the boundary of the pieces, we merge those pieces for which we cannot prove that (Line 6).

Checking the inequalities is again straight-forward (Lines 10 to 13) using $I(g)$. In order to ensure that the used boxes are small enough, we undo the mergings before the check (Line 11) and apply $I(g)$ to the individual boxes (Line 12).

The algorithm looks as follows:

Algorithm SoEI(S, P, r) // System of equations and inequalities

1: Let B be the m-box for the domain of the quantified variables in S.
2: Let S_r by a grid of boxes in B of width at most r.
3: **if** for every box $A \in S_r$
    either $0 \notin I(f)(P \times A)$ or $I(g)(P \times A) \cap [0, \infty)^k = \emptyset$
4:         **return** $\{\mathbf{F}\}$ // $f = 0 \wedge g \geq 0$ has no solution
5: **if** m=n **then**
6:     Merge all boxes in S_r containing a common face C s.t. $0 \in I(f)(P \times C)$.
7:     Remove all grid elements in S_r containing a face C s.t. $C \subseteq \partial B$ and $0 \in I(f)(P \times C)$.
8:     Let p_0 be an arbitrary element of P
9:     **for** each grid element A
10:         **if** $\deg(f(p_0), A, 0) \neq 0$ **then** // equations hold, so check inequalities
11:             let $S_r(A)$ be a grid of boxes in A of width at most r
12:             **if** for all $E \in S_r(A)$, $I(g)(P \times E) \subseteq (0, \infty)^k$ **then**
13:                 **return** $\{\mathbf{T}\}$
14: **return** $\{\mathbf{T}, \mathbf{F}\}$ // no test succeeded, or $n > m$

Here we suppose that f is present in the formula (i.e., $n > 0$). The algorithm can be easily adapted to the case, where it is not. An illustration of the algorithm is shown in Figure 1.

3.2 Universal Quantifiers

The recursive call corresponding to Case (b) of Definition 4 looks as follows:

Algorithm Univ$(\forall x \in I \,.\, S, P, r)$:

Let I_r be a grid of sub-intervals I of width at most r
return $\tilde{\bigwedge}_{I' \in I_r} \mathrm{CheckSat}(S, P \times I', r)$

Here, in the return statement, the symbol $\tilde{\bigwedge}$ denotes the lifting of Boolean conjunction to sets of Boolean values:

$$U \tilde{\wedge} V := \{u \wedge v \mid u \in U \; v \in V\}.$$

See Section 9.1 on a discussion of why the dual approach to this algorithm does not work for existential quantification.

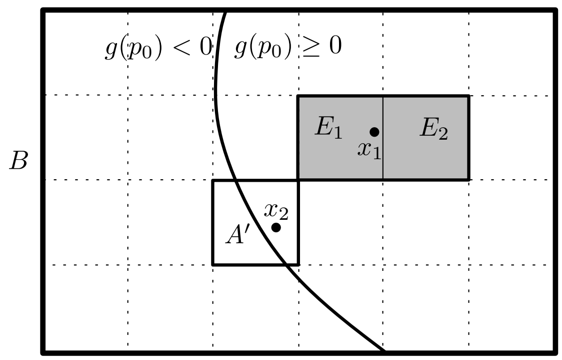

Figure 1: Illustration of the SoEI algorithm. There are two zeros x_1 and x_2 of $f(p_0)$ and $\{x \,|\, g(p_0, x) \geq 0\}$ is to the right of the thick curve. The algorithm creates a grid of boxes and possibly merges E_1, E_2 into one grid element A, if $f(p)$ is close to zero on their common boundary face for some p (line 6). If $\deg(f(p_0), A, 0) \neq 0$, it checks whether for each p, $g(p) \geq 0$ on E_1 and E_2 (line 12). If this is true as well, then $f = 0 \wedge g \geq 0$ is robustly satisfiable on B and the algorithm terminates with $\{\mathbf{T}\}$. In case of another box A' containing a robust zero of f, the given partition does not provide enough evidence for the claim that $g(p, x_2) \geq 0$ for each p (the condition on line 12 is not satisfied).

3.3 Conjunctions and Disjunctions

Finally, the recursive call corresponding to Case (c) of Definition 4 looks as follows:

Algorithm Conj($S \wedge T, P, r$)

return CheckSat(S, P_1, r) $\tilde{\wedge}$ CheckSat(T, P_2, r)
where P_1 (P_2) is the projection of P
to the free variables of S (T, respectively).

Here, in the return statement, the symbol $\tilde{\wedge}$ again denotes the lifting of conjunction to sets of Boolean values. The algorithm for disjunction is completely analogous, replacing $\tilde{\wedge}$ with $\tilde{\vee}$ (and its lifting to sets of Boolean values).

4 Degree of a Continuous Function

In this section we describe some basic properties of the topological degree. We have already mentioned in the introduction that in the one-dimensional case, that is, for continuous functions $f : [a, b] \to \mathbb{R}$, the degree $\deg(f, [a, b], 0)$ is 0 iff $f(a)$ and $f(b)$ have the same sign, otherwise the degree is either -1 or 1, depending on whether the sign changes from negative to positive or the other way round. Hence, in this case, the degree gives the information given by the intermediate value theorem plus some directional information.

In dimension two, the degree of a continuous function f from a disc to $\mathbb{R}^2$ is just the number of times $f(x)$ winds around the origin counter-clockwise as x follows the circle forming the boundary of the disc (i.e., the "winding number"). Again, a non-zero winding number implies that f has a zero.

There are several ways of defining the degree in general. We work with an axiomatic definition, that can be shown to be unique. Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded, $f : \bar{\Omega} \to \mathbb{R}^n$ continuous, and $p \notin f(\partial\Omega)$. Then $\deg(f, \Omega, p)$ is the unique integer satisfying the following properties [16, 33, e.g.]:

1. For the identity function I, $\deg(I, \Omega, p) = 1$ iff $p \in \Omega$
2. If $\deg(f, \Omega, p) \neq 0$ then $f(x) = p$ has a solution in Ω
3. If there is a continuous function (a "homotopy") $h : [0, 1] \times \bar{\Omega} \to \mathbb{R}^n$ such that $h(0) = f$, $h(1) = g$ and $p \notin h(t, \partial\Omega)$ for all t, then $\deg(f, \Omega, p) = \deg(g, \Omega, p)$
4. If $\Omega_1 \cap \Omega_2 = \emptyset$ and $p \notin f(\partial\Omega_1 \cup \partial\Omega_2)$, then $\deg(f, \Omega_1 \cup \Omega_2, p) = \deg(f, \Omega_1, p) + \deg(f, \Omega_2, p)$
5. $\deg(f, \Omega, p)$, as a function of p, is constant on any connected component of $\mathbb{R}^n \backslash f(\partial\Omega)$.

The first axiom says that for the identity function, the degree counts the zeros in Ω precisely. Due to the second axiom one can infer existence of a zero from a non-zero degree. Due to the third axiom, the degree is invariant under continuous deformations of the function that do not cause any essential change of the boundary information. From this it can be immediately seen that the degree depends only on the boundary $\partial\Omega$: for two functions f and g that agree on $\partial\Omega$, the function $h(t, x) = tf(x) + (1 - t)g(x)$ is a homotopy between f and g, as need by the premise of Axiom 3.

In the case where f is continuous and smooth (i.e., infinitely often differentiable) in Ω the degree can alternatively be defined based on the derivative of f. For regular values $p \in \mathbb{R}^n$ (i.e., values p such that for all y with $f(y) = p$, $\det f'(y) \neq 0$), a generalization of the directional information used in the one-dimensional case, is the sign of the determinant $\det f'(y)$. Adding up those signs results in the explicit definition [29] of $\deg(f, \Omega, p)$ by

$$\deg(f, \Omega, p) := \sum_{y \in f^{-1}(p)} \text{sign} \det f'(y).$$

See standard textbooks for a generalization to non-regular values [29].

Many algorithms for computing the degree have been proposed [15, 26, 5, 1] and it is known that it is computable for general interval-computable functions [17].

The axioms defining the degree only argue about zeros, but not about robustness. Still, a nonzero degree is closely connected with the existence of a robust root:

Lemma 1 *Let $\bar{\Omega} \subseteq \mathbb{R}^n$ be a closed region, $f : \bar{\Omega} \to \mathbb{R}^n$ be continuous, $0 \notin f(\partial\Omega)$ and let* $\deg(f, \Omega, 0) \neq 0$*. Then f has a robust zero in Ω.*

Proof. Let $\varepsilon < \min_{x \in \partial\Omega} |f|$. For any g such that $||g - f||_{\bar{\Omega}} < \varepsilon$, we define a homotopy $h(t, x) = tf(x) + (1-t)g(x)$ between f and g. We see that for $x \in \partial\Omega$ and $t \in [0, 1]$,

$$|h(t, x)| = |tf(x) + (1-t)g(x)| = |f(x) + (1-t)(g(x) - f(x))| \geq |f(x)| - \varepsilon > 0$$

so that $h(t, x) \neq 0$ for $x \in \partial\Omega$. From Properties 2 and 3, we see that $g(x) = 0$ has a solution. ■

For proving definiteness, we will need a partial converse of this statement which will be given in Theorem 5.

5 Proof of Correctness

We will prove here that the algorithm CheckSat proposed in Section 3 fulfills the first part of its specification, that is: it always returns a correct result. The proof will again be divided into the cases constituting the definition of class $\mathcal{B}$, from which correctness of the overall, recursive algorithm follows by induction.

Before that, we prove some technical results on the relationship between the class $\mathcal{B}$ and robustness.

5.1 Robustness and the Class $\mathcal{B}$

Lemma 2 *Let S be a formula from $\mathcal{B}$. If $S[p \leftarrow p_0]$ is a robust sentence, then there exists an open neighborhood U of p_0, such that for all $u \in U$, $S[p \leftarrow u]$ is robust, and equi-satisfiable to $S[p \leftarrow p_0]$.*

Proof. Assume that $S[p \leftarrow p_0]$ is robust. Then there is an $\varepsilon > 0$ such for all T with $d(S[p \leftarrow p_0], T) < \varepsilon$, T and $S[p \leftarrow p_0]$ are equi-satisfiable. Since the all quantified variables are from a compact box, there exists a δ s.t. $|u - p_0| < \delta$ implies $d(S[p \leftarrow p_0], S[p \leftarrow u]) < \varepsilon$. It follows that for $|u - p'| < \delta$, also $S[p \leftarrow u]$ is equi-satisfiable and robust. ■

Lemma 3 *Let S be a sentence from $\mathcal{B}$. If S is false, then it is robustly false.*

Proof. We proceed by induction, following the cases of Definition 4. Let S be the formula $\exists x \in B \, . \, f = 0 \wedge g \geq 0$, where $f = 0$, and $g \geq 0$ is the usual short-cut for conjunctions of equalities, and inequalities, respectively. Let S be false. If $f = 0$ have no solution in B, then $|f| > \varepsilon$ for some $\varepsilon > 0$ and $|\tilde{f}| > 0$ for small enough perturbations $\tilde{f}$ of f. Similarly, if $g < 0$ on B, then the same is true for small enough perturbations of g. Finally, if $f^{-1}(\{0\})$ and $g^{-1}[0, \infty)^k$ are both nonempty, then they are compact and disjoint, which implies that they have a positive distance. For small perturbations $\tilde{f}, \tilde{g}$ of f and g, $\tilde{f}^{-1}\{0\}$ and $\tilde{g}^{-1}[0,\infty)^k$ are still disjoint, which implies that S is robustly false.

Further, assume that $I \subseteq \mathbb{R}$ is a compact interval and $\forall x \in I \,.\, S$ is a false sentence. Then there exists an $x_0 \in I$ such that $S[x \leftarrow x_0]$ is false. From the induction hypothesis, it is robustly false. Let $\varepsilon > 0$ be such that $S[x \leftarrow x_0]$ is ε-robust and let S' be a formula such that $d(\forall x\, S', \forall x\, S) \leq \varepsilon$. Then $d(S'[x \leftarrow x_0], S[x \leftarrow x_0]) \leq \varepsilon$ and $S'[x \leftarrow x_0]$ is false. So, $\forall x \in I \,.\, S'$ is false and it follows that $\forall x \in I \,.\, S$ is robustly false.

Finally, let U and V be sentences in $\mathcal{B}$ and $U \wedge V$ be false. Then either U or V is false and the induction hypothesis says that it is robustly false. So, $U \wedge V$ is robustly false. Similarly, if $U \vee V$ is false, then both U and V are robustly false and $U \vee V$ is robustly false. ■

Lemma 4 *Let S be a formula containing a free variable x from a bounded closed interval I. Then the sentence $S[x \leftarrow x_0]$ is robustly true for all x_0 in I, if and only if the sentence $\forall x \in I \,.\, S$ is robustly true.*

Proof. Let $\forall x \in I \,.\, S$ be ε-robust and true, $x_0 \in I$ and let X be a sentence such that $d(X, S[x \leftarrow x_0]) < \varepsilon$. Consider the formula $U :\equiv (S+X-S[x \leftarrow x_0])$ where the subtraction is applied on each function involved in the formulas. Clearly, $d(\forall x \in I \,.\, S, \forall x \in I \,.\, U) = d(S,U) = d(X, S[x \leftarrow x_0]) < \varepsilon$ and $\forall x \in I \,.\, U$ is true. In particular, $U[x \leftarrow x_0] \equiv X$ is true and it follows that $S[x \leftarrow x_0]$ is ε-robust and true.

For the converse, assume that for all $x_0 \in I$, $S[x \leftarrow x_0]$ is robustly true. Let

$$\mu(x_0) := \sup\{\mu > 0;\, S[x \leftarrow x_0] \text{ is } \mu\text{-robust}\}.$$

Clearly, $\mu(x_0)$ is a continuous function and has strict lower bound $m > 0$ on the compact interval I. So, for each $x_0 \in I$, $S[x \leftarrow x_0]$ is m-robust. If $d(\forall x \in I \,.\, S, \forall x \in I \,.\, U) < m$, then for each $x_0 \in I$, $d(S[x \leftarrow x_0], U[x \leftarrow x_0]) < m$ and $U[x \leftarrow x_0]$ is true. So, $\forall x \in I \,.\, U$ is true and $\forall x \in I \,.\, S$ is robustly true. ■

5.2 System of Equations and Inequalities

Again we start with the case (a) of Definition 4 without disjunctions, that is, a formula S of the form

$$\exists x \in B \,.\, [f_1 = 0 \,\wedge\, f_2 = 0 \,\wedge\, \ldots \wedge\, f_n = 0 \,\wedge\, g_1 \geq 0 \,\wedge\, g_2 \geq 0 \,\wedge \ldots \wedge\, g_k \geq 0]$$

where B is an m-box. Assuming that the formula has l free variables, we again denote by $f : \mathbb{R}^{l+m} \to \mathbb{R}^n$ the function defined by the components $(f_1, \ldots, f_n)$ and $g : \mathbb{R}^{l+m} \to \mathbb{R}^k$ the function defined by the components $(g_1, \ldots, g_k)$.

Theorem 2 *The algorithm* SoEI(S, P, r) *fulfills the correctness property of the specification of* CheckSat(S, P, r).

Proof.

Assume first that the algorithm terminates with a negative result $\{\mathbf{F}\}$. It follows directly from Definition 3, that the input sentence $S[p \leftarrow p_0]$ is false for any $p_0 \in P$. Lemma 3 implies robustness.

Now assume that it terminates with a positive result $\{\mathbf{T}\}$. Then there exists a point $p_0 \in P \subseteq \mathbb{R}^l$ and a connected grid element $A \subseteq \mathbb{R}^m$ such that $\deg(f(p_0), A, 0) \neq 0$. For any $p \in P$, p and p_0 can be connected by a curve $\phi : [0,1] \to P$, and $f \circ \phi$ is then a homotopy between $f(p_0)$ and $f(p)$ nowhere zero on ∂A. So, $\deg(f(p), A, 0) \neq 0$ and it follows from Lemma 1 that $f(p) = 0$ has a robust solution in A. Moreover, the successful check whether for all $E \in S_r(A)$, $I(g)(P \times E) \subseteq (0, \infty)^k$ implies that for some small enough $d > 0$, for all $p \in P$, $x \in A$ and $j = 1, \ldots, k$, $g_j(p, x) > d$. It follows that the input formula is robustly true for all parameter values in P. ■

5.3 Universal Quantifiers

Theorem 3 *Let S be a formula containing free variables p. Let P be an l-box and I a closed interval. Assume that an algorithm* CheckSat *fulfilling the correctness property is given. Then also the algorithm* $\mathrm{Univ}(\forall x \in I \ . \ S, P, r)$ *fulfills the correctness property.*

Proof. If $\mathrm{Univ}(\forall x \in I \ . \ S, P, r)$ returns $\{\mathbf{F}\}$, then $\mathrm{CheckSat}(S, P \times I', r')$ returned $\{\mathbf{F}\}$ for some $I' \in I_r$ and it follows that for all $p_0 \in P$ and $x_0 \in I'$, $S[p \leftarrow p_0][x \leftarrow x_0]$ is robustly false. Then $\forall x \in I \ . \ S[p \leftarrow p_0]$ is false for each $p_0 \in P$ and it follows from Lemma 3 that it is robustly false.

If the algorithm returns $\{\mathbf{T}\}$, then $\mathrm{CheckSat}(S, P \times I', r')$ returned $\{\mathbf{T}\}$ for all $I' \in I_r$ and the sentence $S[p \leftarrow p_0][x \leftarrow x_0]$ is robustly true for all $x_0 \in I$ and $p_0 \in P$. It follows from Lemma 4 that for each $p_0 \in P$, $\forall x \in I \ . \ S[p \leftarrow p_0]$ is robustly true, so the result is correct. ■

5.4 Conjunction and Disjunction

Theorem 4 *Let S and T be two formulas in $\mathcal{B}$ and assume that* CheckSat *fulfills the correctness property when applied to S, or T. Then* $\mathrm{Conj}(S \wedge T, P, r)$ *also fulfills the correctness property.*

Proof. Let p_S, and p_T, respectively, be the function that projects any l-tuple corresponding to the free variables of $S \wedge T$ to those components corresponding to the free variables of S, and T, respectively.

If Conj returned $\{\mathbf{T}\}$ then the recursive calls for both S and T returned $\{\mathbf{T}\}$. Hence, by correctness of the result of the recursive calls, for all $p_0 \in P$, $S[p_S(p) \leftarrow p_S(p_0)]$ and $T[p_T(p) \leftarrow p_T(p_0)]$ are robustly true, and hence also $(S \wedge T)[p \leftarrow p_0]$.

If Conj returned $\{\mathbf{F}\}$ then the recursive calls for either S or T returned $\{\mathbf{F}\}$. Hence, by correctness of the result of the recursive calls, either for all $p_0 \in P$,

$S[p_S(p) \leftarrow p_S(p_0)]$ is robustly false, or for all $p_0 \in P$, $T[p_T(p) \leftarrow p_T(p_0)]$ is robustly false. Hence, also for all $p_0 \in P$, $(S \wedge T)[p \leftarrow p_0]$ is robustly false. ■

For disjunctions the situation is analogous.

6 From Robustness To Non-Zero Degree

For proving that the algorithm CheckSat fulfills the second part of its specification, definiteness, we need to prove that for a robust system of equations, the test provided by a non-zero topological degree eventually succeeds. While the algorithmic aspects of the proof are part of the next section, in this section we prove two properties of the degree necessary for this (Lemma 5 and Theorem 5). The first property, Lemma 5, simply says that in the case overdetermined system of n equations in $m < n$ variables, the input cannot be robust, and hence the implication (robust input implies succeeding test for non-zero degree) holds vacuously. The second property, Theorem 5, shows that robustness implies existence of a region for which the degree is non-zero. More precisely, we will show a partial converse to Lemma 1, that is, that a robust solution of $f = 0$ on Ω implies the existence of a region $U \subseteq \Omega$ s.t. $0 \notin f(\partial U)$ and $\deg(f, U, 0) \neq 0$.

The rest of the paper will only refer to the two mentioned properties, so a reader can safely skip this section after noting Lemma 5 and Theorem 5. The proofs in the section are the only place in the paper that use results from topology that are not explicitly delineated in this paper. The reader can find details on these in any standard textbook [31, e.g.,].

Lemma 5 *Let $\bar{\Omega}$ be a closed region in $\mathbb{R}^m$, $n > m$ and $f : \Omega \to \mathbb{R}^n$ be continuous. Then the sentence $S \equiv (\exists x \in \Omega \, . \, f = 0)$ is not robustly true.*

Proof. Assume that S is ε-robust and true. It follows from the Stone-Weierstrass theorem that the continuous function f may be approximated arbitrarily close with a smooth function $\tilde{f}$ (even with a polynomial). If $\tilde{f}$ has a robust zero, then $\tilde{f}(\Omega)$ contains an open neighborhood of $0 \in \mathbb{R}^n$. However, all values in $\tilde{f}(\Omega)$ are critical values and it follows from Sard's theorem that the set of critical values has zero measure in $\mathbb{R}^n$, so it cannot contain a neighborhood of $0 \in \mathbb{R}^n$.

The rest of the section considers the case of equal dimensions $m = n$. First we show that a zero degree of a function implies that any possible zero of the function can be removed by a change of the function only in the interior. Moreover, the result of the change will be small in a certain sense.

Lemma 6 *Let $\bar{\Omega}$ be a closed region in $\mathbb{R}^n$, $f : \bar{\Omega} \to \mathbb{R}^n$ continuous, $0 \notin f(\partial\Omega)$ and $\deg(f, \Omega, 0) = 0$. Then there exists a continuous nowhere zero function $g : \bar{\Omega} \to \mathbb{R}^n$ such that $g = f$ on $\partial\Omega$ and $||g||_{\bar{\Omega}} \leq ||f||_{\bar{\Omega}}$.*

Proof. If $0 \notin f(\Omega)$, we may take $g = f$. Otherwise, take a neighborhood $U \subseteq \Omega$ of $\{f = 0\}$ such that ∂U is an $(n-1)$-manifold (i.e. locally homeomorphic to

$\mathbb{R}^{n-1}$). Such neighborhood U might be constructed as a finite union of balls. It follows from the degree axioms that $\deg(f, U, 0) = \deg(f, \Omega, 0)$ and it is a well-known fact in differential topology that $f/|f| : \partial U \to S^{n-1}$ can be extended to a function $g_1 : U \to S^{n-1}$ iff the degree is zero [25, Theorem 8.1.]. Let $h : \bar{U} \to \mathbb{R}^+$ be an extension of $|f| : \partial U \to \mathbb{R}^+$ (such extension exists due to Tietze's Extension Theorem [8, Thm. 4.22]) and let $i : S^{n-1} \to \mathbb{R}^n \setminus \{0\}$ be the inclusion. Then $g_2 := h\,(i \circ g_1) : \bar{U} \to \mathbb{R}^n \setminus \{0\}$ is a nowhere zero extension of $f|_{\partial U}$. Define $g : \bar{\Omega} \to \mathbb{R}^n \setminus \{0\}$ by $g(x) = g_2(x)$ for $x \in U$ and $g(x) = f(x)$ for $x \notin U$. This function is continuous, nowhere zero and coincides with f on ∂U. Possibly multiplying g by a positive scalar valued function that equals 1 on ∂U and is small inside U°, we achieve that $||g||_\Omega \leq ||f||_\Omega$. ■

Now we show that for a smooth function f, we might change it within a small region N where the function is nonzero, to produce arbitrary many regular zero points, both orientation-preserving and orientation-reversing.

Lemma 7 *Let U be an open set in $\mathbb{R}^n$, $f : U \to \mathbb{R}^n$ be smooth. Let N be a neighborhood of $x^0 \in U$ such that $0 \notin f(N)$ and let $k \in \mathbb{N}$. Then there exists a function f_1 such that the following conditions are satisfied:*
(1) $f_1 = f$ on $U \setminus N$
(2) $||f_1|| \leq ||f||$
(3) 0 is a regular value of $f_1|_N$
(4) N contains $2k$ points $x_1, \ldots, x_k, y_1, \ldots, y_k$ such that $f_1(x_i) = f_1(y_i) = 0$, f_1 is orientation-preserving in the neighborhood of x_i and orientation-reversing in the neighborhood of y_i.

Proof. Choose $\delta > 0$ such that $x^0 + [-2\delta, 2\delta]^n \subseteq N$. We construct f_1 such that $f_1(x) = f(x)$ for $x \notin (x^0 + [-2\delta, 2\delta]^n)$. For $x \in (x^0 + [-\delta, \delta]^n)$ we set

$$(f_1)_i(x) = \left(\frac{|x_i - x_i^0|}{\delta} - \frac{1}{2} \right) f_i(x^0).$$

It is easy to see that $f_1^{-1}(0)$ contains in $(x^0 + [\pm\delta])$ 2^n points of the form $(x_1^0 \pm \delta/2, x_2^0 \pm \delta/2, \ldots, x_n^0 \pm \delta/2)$, half of them preserve orientation and half reverse orientation. Clearly, $|f_1(x)| \leq |f(x^0)| \leq ||f||$ on $x_0 + [\pm\delta]$. Because $\deg(f_1, x^0 + [\pm\delta], 0) = \deg(f, x^0 + [\pm 2\delta]) = 0$, it is easy to see that f_1 may be extended to $x^0 + [\pm 2\delta]$ so that $f_1 = f$ on $\partial(x^0 + [\pm 2\delta])$, f_1 is nonzero in $x_0 + [\pm 2\delta] \setminus (x_0 + [\pm\delta])$ and the norm $||f_1|| \leq ||f||$. The only zero points of f_1 in N are $(x_1^0 \pm \delta/2, \ldots, x_n^0 \pm \delta/2)$, so 0 is a regular value of $f_1|_N$. The details are left to the reader.

To produce more zeros we can choose any point $x_1 \in N$ s.t. $f_1(x_1) \neq 0$ and a small neighborhood of x_1 in N where f_1 is nonzero and continue in the same way. ■

Finally, we prove the following theorem that will be used in the proof of definiteness of the CheckSat procedure.

Theorem 5 *Let $\bar{\Omega}$ be a closed region in $\mathbb{R}^n$ with interior $\Omega \subseteq \mathbb{R}^n$ and $f : \bar{\Omega} \to \mathbb{R}^n$ be continuous. Then f has a robust zero in $\bar{\Omega}$ if and only if there exists an open set $U \subseteq \Omega$ such that $0 \notin f(\partial U)$ and $\deg(f, U, 0) \neq 0$.*

The assumption that Ω is the interior of $\bar{\Omega}$ is necessary to exclude some degenerate cases such as $\Omega = [-1, 1] \setminus \{0\}$ and $f(x) = x$; in this case, f has a robust zero in $\bar{\Omega}$ but for any $U \subseteq \Omega$ with $0 \notin f(\partial U)$, $\deg(f, U, 0) = 0$. **Proof.** If the dimension is $n = 1$, then $\bar{\Omega}$ is a compact interval and there exists a robust zero of f iff there exists $x, y \in \bar{\Omega}$ s.t. $f(x) < 0 < f(y)$, and the statement follows. In the rest of the proof we assume that $n \geq 2$.

If $\deg(f, U, 0) \neq 0$ for some U, then f has a robust zero in U by Lemma 1. For showing the other direction, we assume that for each open $U \subseteq \Omega$ s.t. $0 \notin f(\partial U)$, $\deg(f, U, 0) = 0$ and let $\epsilon > 0$. We will show that there exists a 4ϵ-perturbation of f with no root.

Let $\Omega_\epsilon := \{x \,|\, |f(x)| < \epsilon\}$. This is an open set in $\bar{\Omega}$. Let $x \in f^{-1}(0) \cap \Omega$. Then there exists a ball $U(x) \subseteq \mathbb{R}^n$ open in $\mathbb{R}^n$ such that $U(x) \subseteq \Omega_\epsilon$. For $y \in f^{-1}(0) \cap \partial\Omega$, we choose $U(y) \subseteq \mathbb{R}^n$ to be an open ball in $\mathbb{R}^n$ such that $U(y) \cap \bar{\Omega} \subseteq \Omega_\epsilon$. We assumed that Ω is the interior of $\bar{\Omega}$, which implies $\partial\Omega = \partial\bar{\Omega}$. So, for each such $U(y)$, the set $U(y) \setminus \bar{\Omega}$ is a nonempty open set in $\mathbb{R}^n$.

The set $\{U(x) \,|\, x \in f^{-1}(0)\}$ is an open cover of the compact set $f^{-1}(0)$, so we may take finitely many of these sets $U_1, \ldots, U_k$ that still cover $f^{-1}(0)$. Each U_i is either contained in Ω_ϵ, or has a nontrivial intersection with $\partial\Omega$. Let $V_1, \ldots, V_l, W_1, \ldots, W_m$ be the pairwise disjoint connected components of $\cup_i U_i$ such that $V_i \subseteq \Omega_\epsilon$ and $W_j \cap \partial\Omega \neq \emptyset$ for each i, j.

If $x \in \partial V_i$, then $f(x) \neq 0$, otherwise x would be contained in the interior of the same connected component V_i of $\cup_i U_i$. In particular, $0 \notin f(\partial V_i)$ and due to the assumption above $\deg(f, V_i, 0) = 0$. V_i is connected and it follows from Lemma 6 that we may change f inside V_i, without changing it on $\bar{\Omega} \setminus V_i$, to construct a function $f_1 : \bar{\Omega} \to \mathbb{R}^n$, $0 \notin f_1(V_i)$ and $||f_1||_{V_i} \leq ||f||_{V_i}$. The inequalities $||f_1|| \leq ||f|| \leq \epsilon$ imply that f_1 is a 2ϵ-perturbation of f. This can be done independently for each i, so we may assume that $0 \notin f_1(\cup_i V_i)$.

Let us extend f_1 to a continuous function $f_2 : \bar{\Omega} \cup_j \bar{W}_j \to \mathbb{R}^n$ (such an extension exists by Tietze's Theorem). Possibly multiplying f_2 by a positive scalar valued function that equals 1 on $\bar{\Omega} \cap (\cup \bar{W}_j)$, we may assume that $||f_2||_{\cup_j \bar{W}_j} \leq \epsilon$. The zero set of f_2 is contained in $\cup_j \bar{W}_j$ and if $f_2(x) = 0$ for some $x \in \partial W_j$, then $x \notin \bar{\Omega}$ (otherwise, x would be contained in the same connected component of $\cup_i U_i$ as W_j, contradicting $x \in \partial W_j$). Therefore, f_2 is nowhere zero on the compact set $\bar{\Omega} \setminus \cup_j W_j$ and there exists some $0 < \epsilon_1 < \epsilon$ s.t. $|f(x)| > \epsilon_1$ for $x \in \bar{\Omega} \setminus \cup_j W_j$. Let f_3 be an ϵ_1-perturbation of f_2 that is smooth and 0 is a regular value of f_3 (such a perturbation exists by Stone-Weierstrass and Sard's theorems). The set $f_3^{-1}(0)$ is finite and contained in $\cup_j \bar{W}_j$. For each j and each $x \in f_3^{-1}(0) \cap \partial W_j$, we may find a small neighborhood O_x of x such that x is the only zero point of f_3 on $\bar{O}_x$, $\bar{O}_x \cap \bar{\Omega} = \emptyset$, $W_j \setminus \bar{O}_x$ is still connected, and replace W_j by $W_j \setminus \bar{O}_x$. So, we can assume that $0 \notin f_3(\partial W_j)$ for each j. Let $A^+(W_j) = \{x \in W_j \,|\, f_3(x) = 0,\, \det(f_3'(x)) > 0\}$ and $A^-(W_j) = \{x \in W_j \,|\, f_3(x) = 0,\, \det(f_3'(x)) < 0\}$.

$W_j \setminus \bar{\Omega}$ is open and nonempty, and we can use Lemma 7 to create at least $2||A^+(W_j)|-|A^-(W_j)||$ zeros in $W_j \setminus \bar{\Omega}$ of f_3 in which f_3 is orientation-preserving, resp. orientation-reversing, without changing f_3 in $W_j \cap \bar{\Omega}$. We can then pair all points in $A^+(W_j) \cap \bar{\Omega}$ with points in $A^-(W_j) \setminus \bar{\Omega}$ and points in $A^-(W_j) \cap \bar{\Omega}$ with $A^+(W_j) \setminus \bar{\Omega}$ (some zeros of f_3 outside $\bar{\Omega}$ may still remain unpaired). We suppose that the dimension $n \geq 2$, so we may connect each pair of points x_a^+ and x_a^- by a curve c_a so that the curves don't intersect themselves and the complement of these curves in W_j is still connected. Further, there exist connected and pairwise disjoint open neighborhoods N_a of these curves such that the only zero points of f_3 in N_a are x_a^+ and x_a^- for each a. The degree $\deg(f_3, N_a, 0) = 0$, so we may change f_3 inside N_a to a continuous function f_4 s.t. $||f_4||_{N_a} \leq ||f_3||_{N_a}$, and $0 \notin f_4(N_a)$. In this way, we destroy all zeros of f_3 in $\bar{\Omega}$ (although some zeros may still exists outside $\bar{\Omega}$). We assumed that $||f_2||_{W_j} \leq \epsilon$, so $||f_3||_{W_j} \leq \epsilon + \epsilon_1 \leq 2\epsilon$ and $f_4|_{W_j}$ is a 4ϵ-perturbation of $f|_{W_j}$. Changing f_3 independently in each N_a, the resulting function $f_4|_{\bar{\Omega}}$ is a nowhere zero 4ϵ-perturbation of f.

7 Proof of Definiteness

We will prove here that the algorithm CheckSat proposed in Section 3 fulfills the second part of its specification, that is, definiteness. This will complete the proof of the main theorem of this paper. The definiteness proof will again be divided into the cases constituting the definition of class $\mathcal{B}$, from which correctness of the overall, recursive algorithm follows by induction.

7.1 System of Equations and Inequalities

We again start with the case (a) of Definition 4, that is, a formula S of the form

$$\exists x \in B \,.\, [f_1 = 0 \,\wedge\, f_2 = 0 \,\wedge\, \ldots \wedge\, f_n = 0 \,\wedge\, g_1 \geq 0 \,\wedge\, g_2 \geq 0 \,\wedge \ldots \wedge\, g_k \geq 0]$$

where B is an m-box. Assuming that the formula has l free variables, we again denote by $f : \mathbb{R}^{l+m} \to \mathbb{R}^n$ the function defined by the components $(f_1, \ldots, f_n)$ and $g : \mathbb{R}^{l+m} \to \mathbb{R}^k$ the function defined by the components $(g_1, \ldots, g_k)$.

Theorem 6 *The algorithm* SoEI(S, P, r) *described in Section 3.1 fulfills the definiteness property of the specification of* CheckSat*.*

Proof. We divide the proof into two parts:

Negative case:

Let P_0 be an l-box and assume that $\exists x \in B \,.\, f(p_0) = 0 \,\wedge\, g(p_0) \geq 0$ is robustly false for each $p_0 \in P_0$. The sets $X = \{(p, x) \in P_0 \times B \mid f(p, x) = 0\}$ and $Y = \{(p, x) \in P_0 \times B \mid g(p, x) \geq 0\}$ are compact and disjoint, so they have a positive distance. For a small enough $\alpha > 0$, the sets $X' = \{(p, x) \in P_0 \times B \mid |f(p, x)| \leq \alpha\}$ and $Y' = \{(p, x) \in P_0 \times B \mid g(p, x) \geq (-\alpha, \ldots, -\alpha)\}$ are still disjoint and have a positive distance $d > 0$. If ε_0 is fine enough, any ε_0-box that has a nonempty intersection with Y' has an empty intersection with X'.

The second property of interval computability implies that if δ is small enough, then any δ-box $A \subseteq B$ and δ-box $P \subseteq P_0$ have the following properties:

- If $P \times A$ has empty intersection with Y', then $I(g)(P \times A) \cap [0, \infty)^k = \emptyset$,
- If $P \times A$ has empty intersection with X', then $0 \notin I(f)(P \times A)$

So, for every $A \subseteq B$ in the S_δ-grid, and every $P \subseteq P_0$ of width smaller then δ, either $P \times A$ has empty intersection with X' or it has empty intersection with Y' and due to the above properties, A satisfies that $0 \notin I(f)(P \times A)$ or $I(g)(P \times A) \cap [0, \infty)^k = \emptyset$. So the algorithm terminates with $\{\mathbf{F}\}$.

Positive Case:

Assume now that $\exists x \in B \;.\; f(p_0) = 0 \wedge g(p_0) \geq 0$ is robustly true for each $p_0 \in P_0$. Let $p_0 \in P_0$. Then $f(p_0) = 0$ has a robust solution on the set $\{x \in B \mid g(p_0) \geq 0\}$ and even on $\{x \in B \mid g(p_0) \geq \alpha\}$ for a small enough $\alpha > 0$. It follows from Lemma 5 that $m = n$. Let $\Omega \subseteq B$ be an open neighborhood of $\{g(p_0) \geq \alpha\}$ in B such that $\Omega \subseteq \{g(p_0) \geq \alpha/2\}$. Without loss of generality, we may assume that Ω is open in $\mathbb{R}^n$ and it is the interior of $\bar{\Omega}$ (otherwise we could replace Ω by the interior of $\bar{\Omega}$ in $\mathbb{R}^n$). The equation $f(p_0) = 0$ has a robust solution on Ω and it follows from Theorem 5 that there exists an open set $U \subseteq \Omega$ such that $0 \notin f(p_0)(\partial U)$ and $\deg(f(p_0), U, 0) \neq 0$. Let $U(p_0) \subseteq P_0$ be an open neighborhood of $\{p_0\}$ such that $(U(p_0) \times \Omega) \subseteq \{(p, x) \mid g(p, x) \geq \alpha/4\}$ and let $\varepsilon_1(p_0)$ be so small that for every box $K \subseteq U(p_0) \times \Omega$ of width less then $\varepsilon_1(p_0)$,

$$I(g)(K) \subseteq (0, \infty)^k. \tag{1}$$

Possibly making $U(p_0)$ smaller, we may assume that $0 \notin f(\overline{U(p_0)} \times \partial U)$. Let $V \subseteq \Omega$ be a neighborhood of ∂U open in B such that $0 \notin f(\overline{U(p_0)} \times \bar{V})$. The compactness of $\overline{U(p_0)} \times \bar{V}$ implies that $|f|$ has a positive minimum on this set and the second property of the definition of interval-computability implies that there exists a $\varepsilon_2(p_0) < \varepsilon_1(p_0)$ such that for every sub-box $K \subseteq U(p_0) \times V$ of width smaller then $\varepsilon_2(p_0)$,

$$0 \notin I(f)(K). \tag{2}$$

Let $\varepsilon_3(p_0) < \varepsilon_2(p_0)$ be so that each box of width less than $\varepsilon_3(p_0)$ that has a nonempty intersection with ∂U lies in V.

Let $P' \subseteq U(p_0)$ be a box of width at most $\varepsilon_3(p_0)$. We will show that $\mathrm{SoEI}(S, P', \varepsilon_3(p_0))$ terminates with $\{\mathbf{T}\}$. The algorithm creates a grid of boxes S_ε, where ε is at most $\varepsilon_3(p_0)$. It merges boxes containing a face C such that $0 \in I(f)(P' \times C)$ and removes grid elements containing a face $C \subseteq \partial B$ such that $0 \in I(f)(P' \times C)$. Let M be the smallest union of grid element in S_ε containing U. M consists of boxes in S_ε that are either contained in U or intersect ∂U and hence are contained in V. It follows that $M \subseteq \Omega$. Further, $\partial M \subseteq V$, $0 \notin I(f)(P' \times C)$ for any boundary box $C \subseteq \partial M$ (due to (2)) and

$$\deg(f(p'), M, 0) = \deg(f(p_0), M, 0) = \deg(f(p_0), U, 0) \neq 0$$

for any $p' \in P'$. Let $p' \in P'$ be chosen in the algorithm. There exists a subset $M' \subseteq M$, merging of grid elements in S_ε where the algorithm finds that $\deg(f(p'), M', 0) \neq 0$ (otherwise, M would be a union of subsets on which $f(p')$ has zero degree, contradicting $\deg(f(p'), M, 0) \neq 0$). Then it checks the condition whether for all $E \in S_\epsilon(M')$, $I(g)(P' \times E) \subseteq (0, \infty)^k$. This is satisfied due to (1) and the algorithm terminates with $\{\mathbf{T}\}$.

All this can be done for any $p_0 \in P_0$. So, we have a covering $\{U(p_0) \mid p_0 \in P_0\}$ of the compact set P_0 and can choose a finite sub-covering $\{U(p_1), \ldots, U(p_s)\}$. There exists an ε' such that each box $P \subseteq P_0$ of width smaller then ε' is contained in some $U(p_j)$. Let ε be the minimum of ε' and all the $\varepsilon_3(p_j)$. For any $P \subseteq P_0$ of width at most ε, SoEI(S, P, ε) terminates with a positive result $\{\mathbf{T}\}$. ■

7.2 Universal quantifiers

Theorem 7 *Let S be a formula containing l free variables $p = (p_1, \ldots, p_l)$. Let P be an l-box and I a closed interval. Assume that an algorithm* CheckSat *fulfilling the definiteness property is given. Then also the algorithm* Univ$(\forall x \in I \,.\, S, P, r)$ *described in Section 3.2 fulfills the definiteness property.*

Proof. Assume that for all $p_0 \in P_0$, the sentence $\forall x \in I \,.\, S[p \leftarrow p_0]$ is robustly true. Then, by Lemma 4, for all $p_0 \in P_0$ and all $x_0 \in I$, $S[p \leftarrow p_0][x \leftarrow x_0]$ is robustly true and the property follows directly from the assumption on CheckSat.

Assume now that for all $p_0 \in P_0$, $\forall x \in I \,.\, S[p \leftarrow p_0]$ is robustly false. Let $p_0 \in P_0$. Then there exists a $x_0 \in I$ such that $S[p \leftarrow p_0][x \leftarrow x_0]$ is false, and hence, due to Lemma 3, it is also robustly false. From this, Lemma 2 implies that there is a neighborhood $P(p_0)$ of p_0 and I_0 of x_0 such that for all $p_0' \in P(p_0)$ and $x_0' \in I_0$, $S[p \leftarrow p_0'][x \leftarrow x_0']$ is false. It follows from the assumption on CheckSat that there exists an ε_{p_0} such that for all $P' \subseteq P(p_0), I' \subseteq I$ of width at most ε_{p_0}, CheckSat$(P' \times I', S, \varepsilon_{p_0})$ terminates with $\{\mathbf{F}\}$.

Because P_0 is compact, we can cover it by $\{P(p_0);\, p_0 \in \Lambda\}$ for a finite set Λ. It is easy to see that there exists a ε' such that any P'-box of side-length smaller then ε' is in at least one of these $P(p_0)$. Now, choose ε to be smaller then ε' and smaller then ε_{p_0} for all $p_0 \in \Lambda$. For any box P of side-length at most ε, the algorithm Univ$(\forall x \in I \,.\, S, P, \varepsilon)$ terminates with $\{\mathbf{F}\}$. ■

7.3 Conjunction and Disjunction

Theorem 8 *Let S and T be two formulas in $\mathcal{B}$ and assume that* CheckSat *fulfills the definiteness property when applied to S, or T. Then* Conj$(S \wedge T, P, r)$ *(described in Section 3.3) also fulfills the definiteness property.*

Proof.

Let p_S, and p_T, respectively, be the function that projects any l-tuple corresponding to the free variables of $S \wedge T$ to those components corresponding to the free variables of S, and T, respectively.

We first assume that for all $p_0 \in P_0$ the sentence $(S \wedge T)[p \leftarrow p_0]$ is robustly true. Then for all $p_0 \in P_0$, $S[p_S(p) \leftarrow p_S(p_0)]$ is robustly true and for all $p_0 \in P_0$, $T[p_T(p) \leftarrow p_T(p_0)]$ is robustly true. So, by definiteness of the recursive call, there exists an $\varepsilon_1 > 0$ such that if $r \leq \varepsilon_1$ and the width of $P_1 \subseteq P_0$ is less than r, then $\mathrm{CheckSat}(S, p_S(P_1), r)$ terminates with $\{\mathbf{T}\}$. An analogous ε_2 exists for T. For $\varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$, $r \leq \varepsilon$ and $P \subseteq P_0$ of width less than r, $\mathrm{Conj}(S \wedge T, P, r)$ terminates with $\{\mathbf{T}\}$.

Suppose that for all $p_0 \in P_0$, $(S \wedge T)[p \leftarrow p_0]$ is robustly false. Then, for any $p_0 \in P_0$, either $S[p_S(p) \leftarrow p_S(p_0)]$ or $T[p_T(p) \leftarrow p_T(p_0)]$ is robustly false. Let $p_0 \in P_0$. Assume, without loss of generality, that $S[p_S(p)) \leftarrow p_S(p_0)]$ is robustly false. By Lemma 2 there exists a neighborhood U of p_0 such that for every $u \in U$, $S[p_S(p) \leftarrow p_S(u)]$ is robustly false. Let $P(p_0) \subseteq P$ be a box neighborhood of p_0 contained in the interior of U (note that $B(p_0)$ may have dimension lower than l, if $p_0 \in \partial P$). By assumption, there exists an $\varepsilon_{p_0} > 0$ such that if $r \leq \varepsilon_{p_0}$ and $P' \subseteq P(p_0)$ has width at most r, then $\mathrm{CheckSat}(S, p_S(P'), r)$ terminates with $\{\mathbf{F}\}$, hence $\mathrm{Conj}(S \wedge T, P', r)$ terminates with $\{\mathbf{F}\}$ as well.

This can be done for each $p_0 \in P_0$. Let $P(p')^\circ$ be the interior of $P(p')$ in the topology of the box P. Then $\{P(p')^\circ \mid p' \in P\}$ is an open cover of the compact space P and there exists a finite subcovering $\{P(p_1)^\circ, \ldots, P(p_m)^\circ\}$ of P. Take ε to be so small that each box $P \subseteq P_0$ of width at most ε is contained in some $P(p_j)$ and $\varepsilon < \min_i \epsilon_{p_i}$. Then $\mathrm{Conj}(S \wedge T, P, r)$ terminates with $\{\mathbf{F}\}$ for any $r \leq \varepsilon$ and any box P of width at most r.

For disjunctions the situation is analogous.

Together with the correctness proof from Section 5 this concludes the proof of the main theorem of the paper.

8 Related Work

From the very beginning of engineering the notion of robustness has played a key role. This is being recognized more and more in several scientific fields: For example, the field of robust control [46, 4] is now considered as a central subject of control engineering. Robustness also plays an increasingly important role in applied and computational mathematics, as shown by the emerging fields of robust optimization [3] and uncertainty quantification (with a journal of the same name recently having been launched by SIAM).

Also in the computing field, robustness has been a core issue from the very beginning. In computer systems design this is usually captured by the keyword "fault-tolerance" and for numerical algorithms "stability". Robustness also plays an important role in computational geometry [45].

In the complexity analysis of algorithms, the notion of perturbation has help to explain the good practical behavior of algorithms with exponential worst-case complexity [40]. The present paper in analogy applies Spielman and Deng's [41] motivation "The basic idea is to identify typical properties of practical data, define an input model that captures these properties, and then rigorously analyze the performance of algorithms assuming their inputs have these properties" to undecidable problems, where the main goal then is not performance analysis but finding a terminating algorithm.

Apparently, the first paper that follows this approach of exploiting robustness to find a terminating algorithm for an undecidable problem (in this case safety verification of hybrid systems) is due to Fränzle [19]. Since then, a similar approach has been applied several times [19, 20, 36, 37, 13, e.g.] to problems in formal verification.

To the best of our knowledge, the first paper to apply such a method to decision procedures for the real numbers is by one of the co-authors [35, Theorem 5], based on an analysis of robustness of first-order formulas [36]. The main difference to the present paper and—at the same time—main weakness is, that it expresses equalities of the form $f(x) = 0$ as a conjunction of two equalities of the form $f(x) \leq 0 \wedge -f(x) \leq 0$ which—in general—loses robustness, since the two occurrences of f can be perturbed independently and a solution of $f(x) = 0$ can vanish under perturbations of $f(x) \leq 0 \wedge -f(x) \leq 0$.

Recently, Gao et. al. [22] took a similar approach, but instead of allowing non-termination in non-robust cases, they use the notion of δ-decidability that requires an algorithm to terminate always, but allows an incorrect result in cases where a perturbation smaller than δ changes the input from being satisfiable to unsatisfiable. Due to this, δ-decidability does not imply quasi-decidability, in general. However, it does imply quasi-decidability for classes of formulas that are closed under negation, running the corresponding algorithm in parallel on both the input formula and its negation. It would be an easy extension of the algorithm in this paper to return also quantitative information on the robustness (i.e., a value $\varepsilon \in \mathbb{R}$ s.t. the input is ε-robust).

The paper by Gao et. al. in addition also studies complexity of such algorithms in some model of computable analysis [44]. In contrast to the present paper, and in a similar way as described above, also in this approach, equalities are expressed as a conjunction of two inequalities. Hence—in contrast to the present paper—it cannot prove satisfiability of equalities such as $\exists x \in [-10, 10] \,.\, x = 0$: This formula would be handled in the same way as the formula $\exists x \in [-10, 10] \,.\, [x \leq 0 \wedge -x \leq 0]$ which becomes unsatisfiable under small perturbation. Hence, an algorithm following such an approach, is allowed to return "unsatisfiable" for such a formula.

Another paper by Gao et.al. [23] studies algorithms for δ-decidability, calling them δ-complete algorithms. The paper restricts itself to the existential fragment and perturbs equalities $f(x) = 0$ to $|f(x)| \leq \varepsilon$ which results in behavior that is equivalent to the handling of an equality as two inequalities.

The form of perturbations in those approaches [37, 22, 23] results in algorithms that do not need to, and in fact do not exploit continuity of the involved functions. In contrast to that, in the present paper we use the topological degree as the notion that captures the essential information about the roots of continuous functions under continuous perturbations.

Collins [12] presents similar result to ours for the special case of systems of n equalities in n variables, formulated in the language of computable analysis [44]. However, the paper unfortunately contains only very rough proof sketches, that we were not able to complete into full proofs.

Verification of zeros of systems of equations is a major topic in the interval computation community [32, 39, 27, 21]. However, here people are usually not interested in some form of completeness of their methods, but in usability within numerical solvers for systems of equations or global optimization.

Basic existence theorems that are commonly used for proving that an equation $f = 0$ has a solution in B are Kantorovich, Miranda's and Borsuk's theorem. Among these Borsuk's theorem is the strongest [2, 21], that is, if the assumptions of the other theorems are fulfilled, then the assumptions of Borsuk's theorem are fulfilled as well.

We will now remind the Borsuk's theorem and then compare its power for proving existence of a zero with the degree:

Theorem 9 (Borsuk's theorem) *If B is open, bounded, convex and symmetric with respect to an interior point x, $f : \bar{B} \to \mathbb{R}^n$ is continuous such that $f(x) \neq 0$ on ∂B and for any $x + y \in \partial B$ and $\lambda > 0$,*

$$f(x+y) \neq \lambda f(x-y),$$

then $f = 0$ has a solution in B.

It can be shown that if the assumption of Miranda's theorem are satisfied, then the degree has to be 1 or -1 and if the assumption of Borsuk's theorem are satisfied, then the degree $\deg(f, B, 0)$ has to be an odd number. On the other hand, if f has an isolated zero of even degree, then one cannot prove that using Borsuk's theorem. A simple illustration of this is the complex function $f(z) = z^2$ from $\mathbb{C} \simeq \mathbb{R}^2$ to itself, defined in a symmetric and convex neighborhood B of $0 \simeq (0, 0)$. This function has a robust zero in B and $\deg(f, B, 0) = 2$, so the assumptions of Borsuk's theorem are not fulfilled in any such neighborhood B.

Some interest in properties of continuous functions that are invariant with respect to perturbations can be found in the computational topology community. In [14], the authors analyze robustness of topological invariants of the intersection $f(X) \cap A$, where $f : X \to Y$ is continuous and $A \subseteq Y$. They show that if the intersection is transversal, then small enough changes of the function f do not change the topology of $f(X) \cap A$. In the case of a one-point space $A = \{0\}$, transversal intersection immediately implies a robust solution of the system $f = 0$.

Nielsen theory is another well-established concept in the topological community. The *Nielsen root number* $N(f; 0)$ is a computable number defined for maps between manifolds of the same dimension that approximates a lower bound for the number of different solutions of $f(x) = 0$ [6, 7, 24]. Like the topological degree, it is a homotopy invariant and does not change under small perturbations of f. Recently, many papers appeared describing the *coincidence set* of two functions $f_1, f_2 : X \to Y$ using elements of Nielsen theory [28]. For $f_2 = 0$, this again reduces to the robust satisfiability of systems of equations.

An essential ingredience of our algorithm is the computation of the topological degree. Many papers deal with the question of an effective implementation, e.g. [15, 26, 5, 1, 17]. Our online package `TopDeg`[2] computes $\deg(f, B, 0)$ for a function f defined as an expression containing symbols such as polynomials and sin, and a low-dimensional box B. The degree can also be computed by the use of packages for *simplicial homology* computations, such as Chomp[3], GAP homology packes[4], or a collection of MATLAB routines PLEX [5]. However, to compute the degree with the use of these programs, one has to create first a simplicial approximation of $f/|f| : \partial\Omega \to S^{n-1}$, which can be done by means of interval arithmetic.

9 Outlook: Possible Generalizations

9.1 Systems of Equations

The main restriction of class $\mathcal{B}$ in case of a system of equations and inequalities $\exists x \in B \,.\, f = 0 \wedge g \geq 0$ is the condition $m \leq n$, where B is an m-box and f is $\mathbb{R}^n$-valued. Robust satisfiability of such system can be reduced to robust satisfiability of $\exists x \in \bar{\Omega} \cdot f(x) = 0$, where $\bar{\Omega} = \{x \,|\, g(x) \geq \alpha\}$ for small enough $\alpha > 0$. We showed in Lemma 5 that for an overdetermined system of equations ($m < n$) we never have a robust solution. For $n = m$, a system of equations has a robust solution iff $\deg(f, U, 0) \neq 0$ for some $U \subseteq \Omega$. Consider a system of underdetermined equations $f = 0$ for $f : \bar{\Omega} \to \mathbb{R}^n$, Ω an open region in $\mathbb{R}^m$ and $m > n$. In some cases, we could fix $m - n$ input variables in f to constants $a \in \mathbb{R}^{m-n}$ and try to analyze the formula $\exists x \in \bar{\Omega}^a \,.\, f(a, x) = 0$, where $\bar{\Omega}^a = \{x \in \mathbb{R}^m \,|\, (a, x) \in \bar{\Omega}\}$. If $f(a, \cdot)$ has a robust zero in $\bar{\Omega}^a$, then f has a robust zero in $\bar{\Omega}$. However, the converse is not true. If $f(a, \cdot)$ does not have a robust zero in $\bar{\Omega}^a$ for any fixed choice of $a \in \mathbb{R}^{m-n}$ (the components of a ranging over all $(m - n)$-subsets of the total number of m variables), f still may have a robust zero in $\bar{\Omega}$.

The ideas presented in Section 6 may be generalized to underdetermined systems as follows. We may find a small neighborhood U of some component of the zero set of f s.t. $0 \notin f(\partial U)$ and analyze, whether $f|_{\partial U}$ can be extended to a nowhere zero function on U. The non-existence of such extension indicates a robust zero of f in U. Such extension exists iff the map $f/|f|$ from ∂U to the sphere S^{n-1} can be continuously extended to a map from $\bar{U}$ to S^{n-1}. This is the *topological extension problem* in computational homotopy theory. If U is a connected region in $\mathbb{R}^n$, then such extension exists if and only if the degree is zero, but there is not an analogous simple invariant in case of an m-dimensional

[2] `http://topdeg.sourceforge.net`
[3] `http://chomp.rutgers.edu`
[4] `http://www.linalg.org/gap.html`
[5] `http://comptop.stanford.edu/u/programs/plex/`

region U and $m > n$. It has been recently proved that if $A \subseteq X$ are finite simplicial complexes, $f : A \to S^{n-1}$ is simplicial, and $\dim X \le 2n - 3$, then it can be algorithmically checked whether there exists a continuous extension $F : X \to S^{n-1}$ of f or not [9]. If we could find simplicial approximations of $\partial U, U$ and $f/|f| : \partial U \to S^{n-1}$, we could solve the extension problem, whereas the nonexistence of an extension indicates a robust root; therefore, we believe that it is possible to generalize our result to a larger class of formulas, including systems of n equations in m existentially quantified variables for $n < m \le 2n - 3$. However, the algorithm for the solution of the extension problem is very complicated and includes computation of the homotopy groups of spheres (among other things), which is believed to have a very high complexity. It has also been proved that the topological extension problem is undecidable for $\dim X \ge 2n - 2$ [10].

If the domain is not a box but a more general compact set $\bar{\Omega}$ that we may represent combinatorially, then $\bar{\Omega} = \{x \in B \,|\, g(x) \ge 0\}$ for some interval-computable scalar-valued function g and a box $B \supseteq \Omega$. A system of equations and inequalities on $\bar{\Omega}$ can be reduced to a system on B with an additional inequality $g(x) \ge 0$.

The quasi-decidability result of this paper cannot be generalized to a general class of *unbounded* domains. If $f : \mathbb{R}^m \to \mathbb{R}^n$ has a robust zero, then it has a robust zero in some compact box $B_k := [-k, k]^m$. This can be checked for $B_1, B_2, \ldots$ and we would eventually find a B_k where it has a robust zero. However, if f has not a root in $\mathbb{R}^n$ at all, then there is no algorithm that would check this and terminate for all robust inputs. This follows from [43], where a polynomial P with integer coefficient is converted into an equations $f_P = 0$ on unbounded domain containing real polynomials and the sin function, such that it is satisfiable iff P has an integral root. A careful analysis of the paper shows that the sentence $\exists x \cdot f_P(x) = 0$ is robust, so a quasi-decision procedure on unbounded domain would imply the decidability of the satisfiability of integral polynomial equations which is known to be undecidable.

9.2 Problem with Existential Quantifiers

The main ingredient is of the generalization of the algorithm SoEI to universal quantifiers is the fact that $\forall x \in I \,.\, S$ is robustly true if and only if for each $x_0 \in I$, the sentence $S[x \leftarrow x_0]$ is robustly true (Lemma 4). However, nothing like that holds for existential quantifiers. The sentence $\exists x \in [-1, 1] \,.\, x = 0$ is robustly true but for any $x_0 \in [-1, 1]$, the sentence $x_0 = 0$ (x_0 is considered to be a constant function here) is not robustly true. A topological reformulation of adding an existence quantifier to the beginning of a formula would be desirable and could be a subject of future research.

10 Conclusion

Motivated by the fact that in many application domains robustness is an essential property of formal models, we showed that for an undecidable class of first-order formulas over the real numbers one can algorithmically check satisfiability in all robust cases (under the additional assumption that all variables range over bound intervals).

Related work suggests that generalization to the full first-order case is difficult or even undecidable.

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