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RAMSEY'S THEOREM IN THE HIERARCHY OF CHOICE PRINCIPLES

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Ramsey's theorem [5] asserts that every infinite set X has the following partition property (RP): For every partition of the set $[X]^2$ of two-element subsets of X into two pieces, there is an infinite subset Y of X such that $[Y]^2$ is included in one of the pieces. Ramsey explicitly indicated that his proof of this theorem used the axiom of choice. Kleinberg [3] showed that every proof of Ramsey's theorem must use the axiom of choice, although rather weak forms of this axiom suffice. J. Dawson has raised the question of the position of Ramsey's theorem in the hierarchy of weak axioms of choice.

In this paper, we prove or refute the provability of each of the possible implications between Ramsey's theorem and the weak axioms of choice mentioned in Appendix A.3 of Jech's book [2]. Our results, along with some known facts which we include for completeness, may be summarized as follows (the notation being as in [2]):

A. The following principles do not (even jointly) imply Ramsey's theorem, nor does Ramsey's theorem imply any of them:

the Boolean prime ideal theorem,
the selection principle,
the order extension principle,
the ordering principle,
choice from wellordered sets (ACW),
choice from finite sets,
choice from pairs (C_2).

B. Each of the following principles implies Ramsey's theorem, but none of them follows from Ramsey's theorem:

the axiom of choice,
wellordered choice ($\forall \kappa \, AC_\kappa$),
dependent choice of any infinite length κ (DC_κ),
countable choice ($AC_{\aleph_0}$),
nonexistence of infinite Dedekind-finite sets ($W_{\aleph_0}$).

C. Ramsey's theorem implies that every infinite family of nonempty finite sets has an infinite subfamily with a choice function (which implies that the union of countably many finite sets is countable), but this principle does not imply Ramsey's theorem.

D. The Hahn–Banach theorem and the weak ultrafilter theorem do not

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(even jointly) imply Ramsey's theorem, nor does Ramsey's theorem imply either of them.

Turning to the proofs of these assertions, we first eliminate those that are already known or deducible from earlier ones. The positive part of C was proved by Kleinberg [3]; he used it to obtain the independence of Ramsey's theorem by constructing a model with a countably infinite family of pairs whose union is Dedekind-finite. The negative part of assertion C follows from A, since the partial choice principle in C is a trivial consequence of the axiom of choice from finite sets.

The partition property RP for countably infinite sets is provable without any use of choice; this follows from Shoenfield's absoluteness theorem (as Kleinberg observed in [3]) or, more simply, from an inspection of Ramsey's proof in [5]. Since RP for X obviously implies RP for all supersets of X, we see that no choice is needed to establish RP for Dedekind-infinite sets. Ramsey's theorem is therefore equivalent to the assertion RP for all infinite Dedekind-finite sets X.

This observation has three consequences that will be useful to us. First, it implies the positive part of B, for each of the principles listed in B implies that all Dedekind-finite sets are finite (see the diagram at the bottom of page 184 of [2]; by Theorem 8.2, there should also be an arrow from $\forall\kappa\, \mathrm{AC}_\kappa$ to DC in this diagram). Secondly, we obtain the second part of D by noting that Solovay's model [7], where all sets of reals are Lebesgue measurable, satisfies Ramsey's theorem (because it satisfies DC) but neither the Hahn–Banach theorem [7, p. 3] nor the weak ultrafilter theorem (a nonprincipal ultrafilter on ω yields a nonmeasurable set of reals [6]). Thirdly, by formulating Ramsey's theorem so that it refers only to Dedekind-finite sets, we see that it is injectively boundable in the sense of Pincus [4]; we shall return to this point later.

The remaining (first) part of D follows from A, because the Hahn–Banach theorem and the weak ultrafilter theorem follow from the Boolean prime ideal theorem.

The remaining assertions, A and the negative part of B, will be deduced from the following two theorems. (In the statement of these theorems, we use the terminology of Jech [2].)

THEOREM 1. *Ramsey's theorem is false in the basic Cohen model.*

THEOREM 2. *Ramsey's theorem is true in the basic Fraenkel model.*

Once we have these theorems, we can combine them with known facts to complete the proofs of A and B as follows. Halpern and Lévy [1] showed that the Boolean prime ideal theorem and the selection principle hold in the basic Cohen model; choice from wellorderable sets also holds there [2, Problem 5.22]. The other principles listed in A follow from the Boolean prime ideal theorem, so they also hold in this model. Thus, Theorem 1 implies the first part of A.

For the other part of A and the negative part of B, we need a model in which all the principles listed in A and B fail but Ramsey's theorem holds. Since all the principles in A imply choice from pairs and since all those in B imply that all Dedekind-finite sets are finite, we need only find a model in which

(a) there is an infinite Dedekind-finite set,
(b) choice from pairs fails, and
(c) Ramsey's theorem holds.

Conditions (a) and (b) are evidently boundable, and we have already noted that (c) is injectively boundable. Therefore, by Theorem 2A6 of [4], we need only find a Fraenkel–Mostowski model for (a), (b), and (c). Since (a) and (b) are clearly satisfied by the basic Fraenkel model, Theorem 2 completes the proof of our assertions.

All that remains is to prove the two theorems.

Proof of Theorem 1. We use the notation of the theory SP of [1, pp. 90–91]. Partition the two-element subsets $\{x, y\}$ of b (the generic set of generic subsets of ω) according to whether the first element of the symmetric difference $x \Delta y$ is an even or an odd number. Suppose $c \subseteq b$ is homogeneous for this partition, and let c be the denotation of a term u with respect to an assignment f (see axiom (vi) of SP). Homogeneity of c and density of b clearly prevent c from containing an absolute interval of b. So, by the continuity schema of SP, all the elements of c are in the range of f, and thus c is finite. □

Proof of Theorem 2. We first prove RP for infinite subsets X of the set A of atoms. In this case, if E is a support of a partition of $[X]^2$ into two pieces (so E is a finite subset of A), then $X - E$ is a homogeneous set for that partition. Indeed, $[X - E]^2$ is permuted transitively by permutations that fix E pointwise and therefore fix the partition.

Since we have already noted that RP holds for wellorderable sets and is preserved upward, from sets to their supersets, we see that Theorem 2 is a consequence of the following lemma.

Lemma. *In the basic Fraenkel model, every nonwellorderable set has an infinite subset in one-to-one correspondence with a subset of A.*

Proof. Let s be a nonwellorderable set with support E. Some $x \in s$ is not supported by E [2, p. 47]; fix such an x, and find a support for it which has the fewest possible members outside E. (Actually, x has a least support, but we do not need this fact.) Write this support as $F \cup \{a\}$ with $a \in A - (E \cup F)$. In the following, π is always a permutation of A that fixes $E \cup F$ pointwise; we use the same symbol π for the corresponding automorphism of the universe over A.

For such π, we have $\pi(x) \in s$ as E supports s. Also, $\pi(x)$ depends only on $\pi(a)$, not on the rest of π, as $F \cup \{a\}$ supports x. Thus,

$$f = \{(\pi(a), \pi(x)) \mid \pi \text{ fixes } E \cup F \text{ pointwise}\}$$

is a function from $A - (E \cup F)$ into s. It is in the model, because $E \cup F$ supports it; indeed, if σ fixes $E \cup F$ pointwise, then

$$\sigma(f) = \{(\sigma\pi(a), \sigma\pi(x)) \mid \pi \text{ fixes } E \cup F \text{ pointwise}\} = f$$

because, as π varies over all permutations fixing $E \cup F$ pointwise, so does $\sigma\pi$.

It remains to show that f is one-to-one. Suppose not; say $\pi_1(x) = \pi_2(x)$ but $\pi_1(a) \neq \pi_2(a)$. Setting $\pi = \pi_1^{-1}\pi_2$, we have $\pi(x) = x$ but $\pi(a) = b \neq a$. For any $c \in A - (E \cup F \cup \{a\})$, let σ fix $E \cup F \cup \{a\}$ pointwise and map b to c. Then, by definition, f sends $\sigma\pi(a) = \sigma(b) = c$ to $\sigma\pi(x) = x$ (because both π and σ fix x). Therefore, f is constant with value x. But this means that $E \cup F$ supports x, contrary to the assumption that $F \cup \{a\}$ is a support of x with the fewest members outside E. This contradiction completes the proof of the lemma and the theorem. $\square$

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