A Note on Decision Problems for Three-Way Two-Dimensional Finite Automata

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Abstract

This note investigates some decision problems for three-way two-dimensional finite automata. It is shown, for example, that

(1) the emptiness problem for nondeterministic three-way two-dimensional finite automata over a one-letter alphabet is solvable,

(2) the universe problem for deterministic three-way two-dimensional finite automata over a one-letter alphabet is solvable,

and

(3) the universe, containment, and equivalence problems for non-deterministic three-way two-dimensional finite automata are unsolvable.

1. Introduction and Preliminaries

In [1], Rosenfeld introduced a new type of automaton on a two-dimensional tape, called the three-way two-dimensional finite automaton, and showed that three-way two-dimensional finite automata are less powerful than (four-way) two-dimensional finite automata [2, 3, 4].

In this note, we are mainly concerned with fundamental decision problems for three-way two-dimensional finite automata.

Let Σ be a finite set of symbols. A *two-dimensional tape* over Σ is a two-dimensional rectangular array of elements of Σ. The set of all two-dimensional tapes over Σ is denoted by $\Sigma^{(2)}$. Given a tape x in $\Sigma^{(2)}$, we let $l_1(x)$ be the number of rows of x and $l_2(x)$ be the number of columns of x. If $1 \le i \le l_1(x)$ and $1 \le j \le l_2(x)$, we let $x(i, j)$ denote the symbol in x with coordinates (i, j). Furthermore, we define

$$x[(i, j), (i', j')]$$

only when $1 \le i \le i' \le l_1(x)$ and $1 \le j \le j' \le l_2(x)$ as the two-dimensional tape z satisfying the following: (i) $l_1(z) = i' - i + 1$ and $l_2(z) = j' - j + 1$; (ii) for each k, r ($1 \le k \le l_1(z)$, $1 \le r \le l_2(z)$), $z(k, r) = x(k + i - 1, r + j - 1)$.

We denote a *non deterministic (deterministic) two-dimensional finite automaton* by "2–NA" ("2–DA"). (See [2, 3, 4] for definitions of 2–NA's and 2–DA's.) A 2–NA (2–DA) can move right, left, up, or down on a two-dimensional tape surrounded by the boundary symbol "#". A *three-way nondeterministic (deterministic) two-dimensional finite automaton* is a 2–NA (2–DA) whose input tape head can move right,

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left, or down, but not up. We denote a three-way nondeterministic (deterministic) two-dimensional finite automaton by "TR2–NA" ("TR2–DA"). We especially denote a 2–NA (2–DA, TR2–NA, TR2–DA) which operates on two-dimensional tapes over a one-letter alphabet, by "2–NA(O)" ("2–DA(O)", "TR2–NA(O)", "TR2–DA(O)"). A 2–NA (2–DA, TR2–NA, TR2–DA) M starts in its initial state, with the input head on the upper left-hand corner of an input tape x. We say that M *accepts* the tape x if M eventually halts in a specified state (accepting state) on the bottom boundary symbol $\sharp$ of the input. We denote the set of all two-dimensional tapes accepted by M by $T(M)$. Let $\mathscr{L}[2\text{–NA}] = \{T \mid T = T(M)$ for some 2–NA $M\}$. $\mathscr{L}[2\text{–DA}]$, $\mathscr{L}[\text{TR2–NA}]$, etc. have a similar meaning.

We first show that the emptiness problem for $\mathscr{L}[\text{TR2–NA(O)}]$ is decidable, and that the universe problem for $\mathscr{L}[\text{TR2–DA(O)}]$ is decidable. As a corollary of this result, we show that $\mathscr{L}[\text{TR2–NA(O)}] \subsetneqq \mathscr{L}[2\text{–NA(O)}]$ and $\mathscr{L}[\text{TR2–DA(O)}] \subsetneqq \mathscr{L}[2\text{–DA(O)}]$. We then show that the universe, containment, and equivalence problems for $\mathscr{L}[\text{TR2–NA}]$ are undecidable.

In the last section, we briefly state some problems which are related to decision problems for (three-way) two-dimensional finite automata.

In this note, we assume that the reader is familiar with fundamental knowledges of automata and formal languages theory.

2. Main Results

It is known [4] that the emptiness and universe problems for $\mathscr{L}[2\text{–DA(O)}]$ are undecidable.

We first show that a different situation emerges for $\mathscr{L}[\text{TR2–NA(O)}]$ and $\mathscr{L}[\text{TR2–DA(O)}]$.

THEOREM 1: The emptiness problem for $\mathscr{L}[\text{TR2–NA(O)}]$ and $\mathscr{L}[\text{TR2–DA(O)}]$ is decidable.

Proof. It is sufficient to show that the emptiness problem for $\mathscr{L}[\text{TR2–NA(O)}]$ is decidable. Let M be a TR2–NA(O). We consider the nondeterministic two-way finite automaton[+] with end markers, M', which operates on (one-dimensional) tapes over a one-letter alphabet as follows. Let $x \in \{O\}^{(2)}$ be an input tape to M. On the input string $w \in \{O\}^+$ of length $l_2(x)$, M' simulates the actions of M on x as follows. M' moves its head right (left) one square for every one right (left) move of the head of M. If the head of M moves down one square, M' chooses one of the following two actions.

(1) M' stays on the same square, and begins to simulate the next action of M.

(2) M' guesses that the head of M has reached the bottom boundary symbol $\sharp$, and begins to simulate the actions of M on the bottom boundary symbols $\sharp$'s.

After M' choose the action (2) above, M' enters an accepting state if it finds out that

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[+] See [5] for definitions of nondeterministic two-way finite automata.

M enters an accepting state.

It is obvious that M' accepts the set

$$T(M') = \{w \in \{0\}^+ \mid w = x[(1, 1), (1, l_2(x))] \text{ (i.e., } w \text{ is}$$

the top row of x) for some x in $T(M)\}$.

Clearly, $T(M')$ is empty if and only if $T(M)$ is empty. The theorem follows from the fact that the emptiness problem for the class of sets accepted by nondeterministic two-way finite automata with end markers (i.e., the class of regular languages) is decidable [5].   Q. E. D.

As stated before, the emptiness problem for $\mathscr{L}[2\text{–NA(O)}]$ is undecidable. From this fact and Theorem 1, we can get the following corollary.

COROLLARY 1:   (1) $\mathscr{L}[\text{TR2–NA(O)}] \subsetneq \mathscr{L}[2\text{–NA(O)}]$,   and
                   (2) $\mathscr{L}[\text{TR2–DA(O)}] \subsetneq \mathscr{L}[2\text{–DA(O)}]$.

REMARK 1:   Rosenfeld [1] showed that for the alphabet consisting of two letters, three-way finite automata are less powerful than four-way finite automata.

LEMMA 1:   If $T \subseteq \{0\}^{(2)}$ and $T \in \mathscr{L}[\text{TR2–DA(O)}]$, then $\overline{T} = \{0\}^{(2)} - T \in \mathscr{L}[\text{TR2–}$ DA(O)]. That is, $\mathscr{L}[\text{TR2–DA(O)}]$ is closed under complementation.

Proof. Let $T (\subseteq \{0\}^{(2)})$ be a set in $\mathscr{L}[\text{TR2–DA(O)}]$, and M be a TR2–DA(O) with q states accepting T.

We construct the TR2–DA(O) M' as follows. Let x be an input tape to M'. On each row of x, M' acts as follows.

(1) M' simulates M and if M ever halts, M' rejects the input x if M accepts and vice versa.

(2)   In the finite control, M' counts the number of times the input head of M hits the left and right boundary symbols #'s on the present row, respectively. If either of these counts grows larger than q, M' accepts the input x since M is cycling on the row.

(3)   In the finite control, M' counts the number of moves M has performed since its input head last encountered a boundary symbol # on the present row. If this count tries to grow larger than q, then M has entered a state "s" twice on the present row since encountering a boundary symbol and M either is heading for a boundary symbol (on the present row) or cycling near one end of the present row. M' now memorizes (in the finite control) the state "s" and counts the horizontal displacement of the input head of M from its present position until the memorized state "s" of M is repeated (on the present row). If the displacement is zero, then M is cycling and M' accepts the input; if the displacement is not zero, then the input head of M will eventually hit a boundary symbol, up the boundary symbol count, and the process starts all over.

Of course, M' moves its input head down one square for every one down move of the

input head of M. If the input head of M reaches the bottom boundary symbol ⌗, M' checks whether M eventually enters an accepting state, and M' rejects the input if M enters an accepting state and vice versa. (Clearly, M' can do this check in the finite control.) It will be obvious that $T(M') = \{0\}^{(2)} - T$.   Q. E. D.

From Theorem 1 and Lemma 1, the following result follows.

THEOREM 2: The universe problem for $\mathscr{L}[\text{TR2-DA(O)}]$ is decidable.

We next examine the universe, containment, and equivalence problems for $\mathscr{L}[\text{TR2-NA}]$.

THEOREM 3: The universe, containment, and equivalence problems for $\mathscr{L}[\text{TR2-NA}]$ are undecidable.

Proof. Given a single-tape (one-dimensional) Turing machine (see [5]) M, define a two-dimensional tape x_M representing a valid computation of M on an initially blank tape (if it exists) as follows:

(1) For each i $(1 \le i < k = l_1(x_M))$, there exists some integer j_i such that

$$x_M[(i, 1), (i, j_i)] = \alpha_i \quad \text{and} \quad x_M[(i, j_i+1), (i, l_2(x_M))] \in \{\mathcal{\not{c}}\}^{(2)},$$

and

(2) $x_M[(k, 1), (k, l_2(x_M))] = \alpha_k,$

where α_1 is the initial configuration of M, α_{i+1} is the configuration resulting from α_i after one move of M, α_k is a halting configuration, and $\mathcal{\not{c}}$ is a symbol not in $\Gamma \cup K$ (Γ is the tape alphabet of M and K is the state set of M)[‡].

Then x_M, if it exists, is in $(\Gamma \cup K \cup \{\mathcal{\not{c}}\})^{(2)}$. Now define the set

$$T_M = \begin{cases} (\Gamma \cup K \cup \{\mathcal{\not{c}}\})^{(2)} - \{x_M\} & \text{if } x_M \text{ exists,} \\ (\Gamma \cup K \cup \{\mathcal{\not{c}}\})^{(2)} & \text{otherwise.} \end{cases}$$

Then $T_M = \Sigma^{(2)}$ (where $\Sigma = \Gamma \cup K \cup \{\mathcal{\not{c}}\}$) if and only if M does not halt on an initially blank tape. We can easily construct a TR2-NA to accept T_M. (The construction is left to the reader.) The theorem now follows from the undecidability of the halting problem for Turing machines on blank tape.   Q. E. D.

It is unknown whether or not the emptiness, universe, containment, and equivalence problems for $\mathscr{L}[\text{TR2-DA}]$ are decidable. It is also unknown whether or not the containment problem for $\mathscr{L}[\text{TR2-DA(O)}]$ is decidable. (If $\mathscr{L}[\text{TR2-DA(O)}]$ were closed under intersection, we could show by using Theorem 1 and Lemma 1 that the containment problem for $\mathscr{L}[\text{TR2-DA(O)}]$ is decidable.)

[‡] Note that for each $i(1 \le i < k)$, the length of α_{i+1} is larger than or equal to the length of α_i since we may assume that M does not rewrite a non-blank symbol by the blank symbol, and so x_M is well defined.

3. Related Problems

We first demonstrate the relationship between (three-way or four-way) two-dimensional finite automata and two-tape finite automata over a one-letter alphabet. (See [6] for definitions of two-way two-tape finite automata.) A *semi-one-way two-tape finite automaton* (denoted by STFA) is a two-way two-tape finite automaton (denoted by TTFA) whose first input head can only move right and whose second input head can move right or left. We denote a deterministic STFA (nondeterministic STFA, deterministic TTFA, nondeterministic TTFA) over a one-letter alphabet by "DSTFA(O)" ("NSTFA(O)", "DTTFA(O)", "NTTFA(O)"). We denote the set of all pairs of tapes accepted by a two-tape finite automaton M by $T(M)$, and let, for example, $\mathscr{L}$[NSTFA(O)] denote the class of sets accepted by NSTFA(O)'s. Let T be a set of two-dimensional tapes over a one-letter alphabet (e.g., $\{0\}$), and let $R(T) = \{(t_1, t_2) \in \{0\}^+ \times \{0\}^+ \mid l(t_1) = l_1(x)$ and $l(t_2) = l_2(x)$ for some x in $T\}$ (where $l(t)$ is the length of t for each string t). Further, for each class $\mathscr{L}$ of sets of two-dimensional tapes over $\{0\}$ (a one-letter alphabet), let $R(\mathscr{L}) = \{R(T) \mid T$ in $\mathscr{L}\}$.

From the following proposition, we can say that the emptiness, universe, containment, and equivalence problems for $\mathscr{L}$[TR2–XA(O)] and $\mathscr{L}$[2–XA(O)] ($X \in \{D, N\}$) are equivalent to those for $\mathscr{L}$[XSTFA(O)] and $\mathscr{L}$[XTTFA(O)], respectively.

PROPOSITION 1: For each $X \in \{D, N\}$,
(1) $R(\mathscr{L}[TR2–XA(O)]) = \mathscr{L}[XSTFA(O)]$, and
(2) $R(\mathscr{L}[2–XA(O)]) = \mathscr{L}[XTTFA(O)]$.

Proof. We only show that $R(\mathscr{L}[TR2–XA(O)]) \subseteq \mathscr{L}[XSTFA(O)]$. (The other cases are left to the reader.) Let M be a TR2–XA(O). It suffices to show that there exists some XSTFA(O) M' accepting $R(T(M))$. Consider a XSTFA M' which acts as follows. (Let the first and second heads of M' be H_1 and H_2, respectively.) First, M' keeps the initial state of M in the finite control with both H_1 and H_2 positioned on the right square immediately of the left endmarkers. After making this situation, M' simulates one move of M as follows. If the input head H of M moves down one square, M moves H_1 right one square without moving H_2, and if H moves right (left) one square, M' moves H_2 right (left) one square without moving H_1. (If H_1 reaches the right endmarker, M' sees that H reaches the bottom boundary symbol, and if H_2 reaches the right (left) endmarker, M' sees that H reaches the right (left) boundary symbol.) Of course, M' enters an accepting state only if it finds out that M enters an accepting state. It is obvious that $T(M') = R(T(M))$. 　　　　Q. E. D.

COROLLARY 2: (1) The emptiness problem for $\mathscr{L}$[NSTFA(O)] is decidable. (2) The universe problem for $\mathscr{L}$[DSTFA(O)] is decidable. (3) The emptiness, universe, containment, and equivalence problems for $\mathscr{L}$[DTTFA(O)] are undecidable.

Proof. The result (1) ((2)) follows from Proposition 1 and Theorem 1 (from Proposition 1 and Theorem 2). The result (3) follows from Proposition 1 and the

undecidabilities [4] of the emptiness, universe, containment, and equivalence problems for [2–DA(O)]. Q. E. D.

In fact, we can get the following result which strengthens part (1) of Corollary 2.

FACT 1: The emptiness problem for the class of sets accepted by nondeterministic semi-one-way two-tape finite automata is decidable.

Proof. Let M be an nondeterministic semi-one-way two-tape finite automaton. Consider an nondeterministic one-way stack automaton M' which acts as follows. (See [7] for definitions of nondeterministic one-way stack automata.) When an input string t_1 is presented to M', M' first nondeterministically writes down a string t_2 on the stack tape with the input head on the left endmarker. M' then simulates the actions of M on the pair of tapes (t_1, t_2), and accepts the input string t_1 only if M' finds out that M accepts the pair (t_1, t_2). (To describe the details of the actions of M' is left to the reader.) Clearly, $T(M)$ is empty if and only if the set accepted by M' is empty. The result now follows from the decidability [7] of the emptiness problem for nondeterministic one-way stack automata. Q. E. D.

Bird [8] showed that the equivalence problem for deterministic one-way two-tape finite automata is decidable. As far as we know, it is unknown whether the equivalence problem for deterministic semi-one-way two-tape finite automata is decidable. If the equivalence problem for $\mathscr{L}$[TR2–DA(O)] were decidable, then it would follow from Proposition 1 that the equivalence problem for $\mathscr{L}$[DSTFA(O)] is decidable.

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