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Abstraction-based solution of optimal stopping problems under uncertainty

Gunther Reissig and Matthias Rungger

Abstract— **In this paper we present novel results on the solution of optimal control problems with the help of finite-state approximations ("symbolic models") of infinite-state plants. We investigate optimal stopping problems in the minimax sense, with undiscounted running and terminal costs, for nonlinear discrete-time plants subject to perturbations and constraints. This problem class includes finite-horizon and exit-(entry-)time problems as well as pursuit-evasion and reach-avoid games as special cases. We utilize symbolic models of the plant to upper bound the value function, i.e., the achievable closed-loop performance, and to compute controllers realizing the bounds. The symbolic models are obtained from suitable discretizations of the state and input spaces, and we prove that the computed bounds converge to the value function as the discretization errors approach zero. The value function is in general discontinuous, and the convergence (in the hypographical sense) is uniform on every compact subset of the state space. We apply the proposed method to design an approximately optimal feedback controller that starts up a DC-DC converter and is robust against supply voltage as well as load fluctuations.**

I. Introduction

Discrete abstractions, which are finite state models as substitute of control systems, have been successfully applied to a variety of analysis and synthesis problems for continuous and hybrid systems, see [1], [2], [3], [4] and references therein. Central to all abstraction-based controller synthesis schemes are three steps: first, a discrete abstraction, i.e., a finite state automaton as substitute of the plant is computed; second, an auxiliary control synthesis problem is solved for the discrete abstraction at the place of the plant, using well-known tools from computer science, e.g. graph search algorithms or the like; third, the obtained controller is refined to the actual plant, which guarantees that the closed-loop composed of the original plant and the refined controller meets predefined performance specifications.

This very scheme, has been extended to approximate solutions of optimal control problems, i.e., to compute upper bounds on the achievable closed-loop performance and to design controllers realizing these bounds [5], [6], [7]. (For consistency, all optimal control problems are assumed to be posed as minimization problems.)

The abstraction-based solution of optimal control problems is closely related to the numerical approximation of the achievable closed-loop performance, i.e., of the *value function* (optimal cost-to-go function), which has a long standing history in control engineering. Some basic numerical approximation methods had already been developed by Bellman in the late nineteenfifties [8]. Later contributions provide convergence results for a broad variety of different settings, ranging from finite horizon [9], [7] over discounted infinite horizon [10], [11], undiscounted infinite horizon [12], [13], optimal stopping [14], to exit-(entry-)time [15], [16], [6], optimal control problems. From the viewpoint of control engineering, the results of [12], [7] are particularly useful as they provide upper bounds on the value function that approximate the latter to arbitrary precision, together with controllers realizing the bounds. Consequently, the performance of the closed loop can be made arbitrarily close to the achievable optimum. Unfortunately, the results of [12], [7] rely on the continuity of the value function, which rules out most of the optimal control problems with (hard) state or control constraints.

In this paper, we present novel results on the approximate solution of optimal control problems with the help of discrete abstractions, for nonlinear discrete-time systems of the form

$$x(t+1) \in F(x(t), u(t)). \tag{1}$$

Here, x represents a state signal with values in $\mathbb{R}^n$, and u is an input signal which is usually assumed to take its values in some compact or even finite set. The right-hand side F of (1) is a multifunction, i.e., $F(x, u)$ is a subset, as opposed to a single point, of the state space $\mathbb{R}^n$. This creates some non-determinism of the solutions of (1) which is used to model possible disturbances and uncertainties in the system's dynamics.

We investigate optimal stopping problems for the plant (1), in which the controller not only specifies the control symbol $u(t)$ at every instant t of time, but must also stop the evolution of the closed-loop at some finite time. At that point, the total cost is determined from a terminal cost added to the running cost accumulated prior to stopping. The problem is to design a controller that minimizes, or approximately minimizes, the total cost in the minimax (worst-case) sense.

We briefly discuss a potential application illustrated in Fig. 1. Assume that a controller for the system (1) is to be designed that enforces a predefined specification at minimum cost. Assume further that some controller which only partially solves the control problem, denoted as slave in Fig. 1, is already known. For example, if the system (1) is to be globally stabilized at an unstable equilibrium with a

G. Reissig is with University of the Federal Armed Forces Munich, Dept. Aerospace Eng., Chair of Control Eng. (LRT-15), D-85577 Neubiberg (Munich), Germany, http://www.reiszig.de/gunther/

M. Rungger is with the Electrical Engineering Department at the University of California, Los Angeles, CA 90095 USA.

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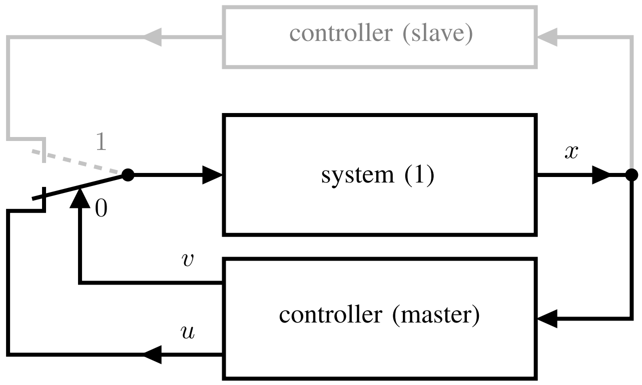

Fig. 1. Optimal stopping scenario. A controller (slave) which only partially solves a predefined control problem, at some cost, is assumed to be known. A master controller is to be designed that solves the remaining parts of the problem and then stops, i.e., hands over control to the slave, in such a way that the total cost incurred during the actions of master and slave is minimized.

minimum amount of control energy, the slave could stabilize the system (1) locally about the equilibrium at some, not necessarily minimum, cost. The problem then is to design a master controller that solves the remaining parts of the problem and eventually stops, i.e., hands over control to the slave. In the global stabilization example, the master should force the state of the system (1) into the stability region of the slave so as to minimize the sum of the terminal cost – the cost at which the slave stabilizes the plant from the time the master stops – and the cost caused by the action of the master prior to his stopping. Note that the master is required to stop at some finite time. This makes sense e.g. if the master has to serve several instances of the plant (1), each with its own slave, as is common in start-up scenarios for chemical plants, power supplies and the like.

A formal statement of the optimal stopping problem that we investigate, which includes reach-avoid games and finite-horizon problems as special cases, is given in Section III. The setup allows the consideration of (hard) state and control constraints, which are represented by discontinuous, extended real-valued cost functions and generally lead to discontinuous value functions. In Section III we also establish an optimality principle as well as semi-continuity of the value function. We present the main results of our contribution in Section IV, where we introduce the abstract optimal control problem, which represents a solvable substitute of the concrete optimal control problem from Section III. We ensure two properties of the abstraction: first, we show that the abstract value function provides an upper bound of the concrete value function; second, we show the convergence (in the hypographical sense) of the abstract value function to the concrete value function. In Section V, we apply the proposed method to design an approximately optimal feedback controller that starts up a DC-DC converter and is robust against supply voltage as well as load fluctuations. Due to the space limitations we omit the proofs of our results.

II. Preliminaries

A. Basic notation

$\mathbb{R}$ and $\mathbb{Z}$ denote the sets of real numbers and integers, respectively, $\mathbb{R}_+$ and $\mathbb{Z}_+$, their subsets of non-negative elements, and $\mathbb{N} = \mathbb{Z}_+ \setminus \{0\}$. $\max M$ and $\min M$ denote the maximum and the minimum, respectively, of the non-empty subset $M \subseteq \mathbb{R}$. $[a, b]$, $]a, b[$, $[a, b[$, and $]a, b]$ denote closed, open and half-open, respectively, intervals with end points a and b, e.g. $[0, \infty[\, = \mathbb{R}_+$. $[a; b]$, $]a; b[$, $[a; b[$, and $]a; b]$ stand for discrete intervals, e.g. $[a; b] = [a, b] \cap \mathbb{Z}$.

For any sets A and B, $\mathcal{P}(A)$ is the power set of A. $f\colon A \rightrightarrows B$ denotes a *set-valued map* of A into B, whereas $f\colon A \to B$ denotes an ordinary map of A into B; see [17], [18]. The set of maps $A \to B$ is denoted B^A, and these maps are sometimes identified with single-valued maps $A \rightrightarrows B$. If A is a discrete interval, a map $f\colon A \to B$ is also called a *sequence* and denoted by $(f_a)_{a \in A}$, where $f_a := f(a)$. If A is a singleton, $A = \{a\}$, we identify f with $f(a)$. If $f, g\colon A \to B$ and $f(a) \leq g(a)$ for all $a \in A$, we write $f \leq g$, and $f = b$, if $f(a) = b$ for all $a \in A$.

For maps $f\colon A \rightrightarrows B$, $\operatorname{dom} f$ and $\operatorname{im} f$ are the domain and the image, respectively, of f. We write $f \circ g$ for the composition of f and g, $(f \circ g)(x) = f(g(x))$, and denote $f^1 := f$ and $f^k := f^{k-1} \circ f$ for $k \in \mathbb{N}$.

We endow $\mathbb{R}^n$ with some norm $\| \cdot \|$, and product spaces $X \times Y$ are generally endowed with the norm defined by $\|(x, y)\| := \max\{\|x\|, \|y\|\}$ for $x \in X$ and $y \in Y$. We also use d to denote the distance in $\mathbb{R}^n$, i.e., $d(x, y) = \|x - y\|$ and define $d(x, N) = \inf \{d(x, y) \mid y \in N\}$ and $d(M, N) = \inf \{d(x, y) \mid x \in M, y \in N\}$ for all $x, y, \in \mathbb{R}^n$ and all compact subsets $M, N \subseteq \mathbb{R}^n$. The open and closed balls of radius r centered at x are denoted $B(x, r)$ and $\bar{B}(x, r)$, respectively.

An extended real-valued function $f\colon A \to \mathbb{R} \cup \{\infty\}$ is called *upper semi-continuous* (u.s.c.) if for every $x \in A$ and $\epsilon > 0$ there is a neighborhood U of x such that for all $x' \in U$ we have $f(x') \leq f(x) + \epsilon$. The set-valued map $H\colon X \rightrightarrows Y$ between metric spaces X and Y is *u.s.c.* if $H^{-1}(\Omega)$ is closed for every closed subset $\Omega \subseteq Y$, where $H^{-1}(\Omega) = \{x \in X \mid H(x) \cap \Omega \neq \varnothing\}$.

B. Systems and Solutions

In (1) with right hand side $F\colon X \times \mathbb{R}^m \rightrightarrows X$, the state x and the input u take their values in the state space X and in a *control set* $U \subseteq \mathbb{R}^m$, respectively. We will generally assume the following.

($\mathbf{A_1}$) U is non-empty, and the values of F are non-empty and compact.

Given $u\colon \mathbb{Z}_+ \to \mathbb{R}^m$, a sequence $x\colon \mathbb{Z}_+ \to X$ is a *solution* of (1) associated with the control u if (1) holds for all $t \in \mathbb{Z}_+$. The set of those solutions that *start* at $p \in X$, i.e., that satisfy $x(0) = p$, is denoted $\psi(p, u)$. When we consider a set $\psi(p,u)|_{[0;T]}$ for some $T \in \mathbb{Z}_+$, it is not necessary to specify u on the whole time axis, so we may write

$$\psi(p,u(0),\ldots,u(T-1)) := \psi(p,u|_{[0,T[}) := \psi(p,u)|_{[0;T]}.$$

III. The optimal stopping problem and its value function

In this section, we will formally define the optimal control problem to be investigated and establish an optimality principle, the principle of value iteration, as well as an extremal property and semi-continuity of the value function. These results will be useful later, when they save us the effort to investigate, e.g., the dependence of particular solutions on perturbations.

A. Problem statement

Before we formally define the optimal control problem we investigate in this paper, we would like to emphasize that our focus is on plants (1) with state space $X = \mathbb{R}^n$. This setting also covers the cases with hard state and control constraints, which will be represented by discontinuous, extended real-valued cost functions rather than by properties of the right hand side F of (1). However, we still need to consider the case of a more general state space for two reasons. Firstly, the computational method to approximately solve the optimal stopping problem, to be presented in Section IV, basically solves variants of the original problem whose state space is finite and not a subset of $\mathbb{R}^n$. Secondly, and more importantly, the convergence results we are going to present in later sections heavily rely on the application of the basic results obtained in the present section, applied to an auxiliary control problem for a plant whose state space is a suitably chosen abstract metric space.

The optimal control problems we consider are defined in terms of a *terminal cost* function G and a *running cost* function g,

$$G\colon X \to \mathbb{R}_+ \cup \{\infty\}, \tag{2a}$$

$$g\colon X \times X \times \mathbb{R}^m \to \mathbb{R}_+ \cup \{\infty\}, \tag{2b}$$

and the *cost functional* J given by

$$J(x,u,v) = G(x(T)) + \sum_{t=0}^{T-1} g(x(t),x(t+1),u(t)) \tag{3a}$$

whenever $v \neq 0$ and $T = \min v^{-1}(1)$, and

$$J(x,u,v) = \infty \tag{3b}$$

if $v = 0$. Here, $x\colon \mathbb{Z}_+ \to X$ and $u\colon \mathbb{Z}_+ \to \mathbb{R}^m$ represent the state and input signals, respectively, of the plant (1), and $v\colon \mathbb{Z}_+ \to \{0,1\}$ is a switching signal whose earliest $0-1$ edge indicates the time T at which the evolution is stopped. At that point, the total cost (3a) is determined from the terminal cost $G(x(T))$ and the running cost accumulated prior to stopping. Note that stopping is mandatory, i.e., the cost equals ∞ if v vanishes identically. See also Fig. 1.

In the closed-loop, the input signal u and the switching signal v are generated by a *controller*, by which we mean a non-empty-valued map

$$\mu\colon \bigcup_{T\in\mathbb{Z}_+} X^{[0;T]} \times (\mathbb{R}^m)^{[0;T[} \rightrightarrows \mathbb{R}^m \times \{0,1\}. \tag{4}$$

A triple (x,u,v) is a *solution* of the *closed-loop* composed of the system (1) and the controller (4) if $u\colon \mathbb{Z}_+ \to \mathbb{R}^m$, $v\colon \mathbb{Z}_+ \to \{0,1\}$, $x \in \psi(x(0),u)$, and

$$(u(t),v(t)) \in \mu(x|_{[0;t]}, u|_{[0;t[}) \tag{5}$$

for all $t \in \mathbb{Z}_+$. The set of those solutions (x,u,v) for which x starts at $p \in X$ is denoted $\Psi(p,\mu)$.

For given state space X, control set U, and cost functions (2), the *optimal stopping problem*

$$(X,U,F,G,g) \tag{6}$$

for the plant (1) is to design a controller (4) that maps into $U \times \{0,1\}$ and that, for each initial state $p \in X$, minimizes or approximately minimizes the worst case of the cost (3) over all closed-loop solutions (x,u,v) whose state component starts at p, $(x,u,v) \in \Psi(p,\mu)$. The achievable closed-loop performance for the problem (6) is thus given by the *value function* $V\colon X \to \mathbb{R}_+ \cup \{\infty\}$,

$$V(p) := \inf_{\mu\in\mathcal{F}(X,U)} \sup_{(x,u,v)\in\Psi(p,\mu)} J(x,u,v), \tag{7}$$

where $\mathcal{F}(X,U)$ denotes the set of controllers (4) that map into $U \times \{0,1\}$.

A particular instance of the OCP (6) is a constrained minimal time problem with respect to a perturbed control system $x(t+1) = f(x(t),u(t),w(t))$, $f\colon X \times U \times W \to X$, where w represents a perturbation, $w(t) \in W$. Let $Z \subseteq X$ denote the set of states that should be reached in minimal time without leaving the safe set $S \subseteq X$. We can cast this problem as (6) by defining $F(x,u) := f(x,u,W)$, $g(x,x',u) = 1$ for $x \in S$ and $g(x,x',u) = \infty$ otherwise, and $G(x) := 0$ for all $x \in Z$ and $G(x) := \infty$ otherwise.

B. Optimality principle and semi-continuity of the value function

In this paragraph we study the *dynamic programming operator* P of the OCP (6),

$$P(W)(p) = \min\left\{G(p), \inf_{u\in U} \sup_{q\in F(p,u)} (g(p,q,u) + W(q))\right\},$$

which maps the space of functions $W\colon X \to \mathbb{R}_+ \cup \{\infty\}$ into itself, to establish an optimality principle, the principle of value iteration, as well as an extremal property and semi-continuity of the value function.

III.1 Theorem. *Assume (A_1). Then the value function V of (6) is the maximal fixed point of the dynamic programming operator, i.e., $P(V) = V$, and $W \leq P(W)$ implies $W \leq V$.*

The following result shows that, under our standard assumptions, the value function is semi-continuous and equal to the monotone, pointwise limits of successive applications of the dynamic programming operator to the terminal cost function.

III.2 Corollary. *Assume (A_1), let X be a metric space, and additionally assume that F, g and G are upper semi-continuous (u.s.c.). Then the value function V of (6) is u.s.c.,*

$$V(p) = \lim_{T\to\infty} P^T(G)(p) \tag{8}$$

for all $p \in X$, and $P^{T+1}(G) \leq P^T(G)$ for all $T \in \mathbb{Z}_+$.

IV. The Abstract Optimal Control Problem

In this section we consider the optimal control problem (6) from Section III with state space $\mathbb{R}^n$ and input space $U \subseteq R^m$,

$$(\mathbb{R}^n, U, F, G, g), \tag{9}$$

whose value function we denote by $V : \mathbb{R}^n \to \mathbb{R}_+ \cup \{\infty\}$. We introduce an abstract OCP that can be solved automatically using known algorithms like value iteration or shortest path algorithms. The abstract OCP represents a substitute of the original OCP (9).

The main results of this section are formalized in Theorem IV.2 where we show two properties of the abstract value function $\widehat{V}$: first, we show that $\widehat{V}$ is an upper bound on V independent of the precision of the abstraction; second, we prove that $\widehat{V}$ uniformly converges to V on any compact subset of $\mathbb{R}^n$ in the hypographical sense (see [18]) as we increase the precision of the abstract OCP.

Note that it is not reasonable to assume that the points of discontinuity of $\widehat{V}$ and V coincide, and thus, we cannot hope for uniform convergence in the classical sense. Therefore, it is our believe, that the notion of graphical convergence is adequate for the problem at hand since it allows us to formulate uniform convergence statements on compact sets in presence of discontinuities. Moreover, it contains the pointwise convergence as well as the uniform convergence (in the classical sense) on compact sets where V is continuous as special cases, see Remark IV.3.

Intuitively, the convergence in the graphical sense implies, that for every $\epsilon > 0$ and every point x in a compact set $\Gamma \subseteq \mathbb{R}^n$ there is a point $x' \in B(x, \epsilon)$ such that $\widehat{V}(x) \leq V(x') + \epsilon$ holds for a sufficiently precise abstract OCP. This implies uniform convergence for the points away from any discontinuity and that the points of discontinuity of $\widehat{V}$ approach the points of discontinuity of V. This can also be observed in the conducted numerical experiments in Fig. 4 in Section V.

Let us now formalize our statements. Let $K(\mathbb{R}^n)$ denote the set of non-empty compact subsets of $\mathbb{R}^n$ endowed with the Hausdorff metric w.r.t. d, and for every $\rho \in \mathbb{R}_+$ we define

$$K_\rho(\mathbb{R}^n) = \{\Omega \in K(\mathbb{R}^n) \,|\, \operatorname{diam} \Omega \leq \rho\},$$

where $\operatorname{diam} \Omega$ denotes the diameter of Ω.

IV.1 Definition. *Assume that the OCP (9) satisfies (A_1), and let $\rho \in \mathbb{R}_+$. Then*

$$(C, E, \widehat{F}, \widehat{G}, \hat{g}) \tag{10}$$

is an **abstraction of precision ρ associated with the OCP (9)** *if the following holds: $C \subseteq K_\rho(\mathbb{R}^n)$ is a cover of $\mathbb{R}^n$, $E \subseteq U \subseteq \bar{B}(E, \rho)$, the right hand side $\widehat{F} \colon C \times \mathbb{R}^m \rightrightarrows C$ is compact-valued and u.s.c., the cost functions $\hat{g} \colon C \times C \times \mathbb{R}^m \to \mathbb{R}_+ \cup \{\infty\}$ and $\widehat{G} \colon C \to \mathbb{R}_+ \cup \{\infty\}$ are u.s.c., and the conditions*

$$F(p, e) \subseteq \bigcup_{\widetilde{\Omega} \in \widehat{F}(\Omega, e)} \widetilde{\Omega}, \tag{11a}$$

$$\widehat{F}(\Omega, e) \subseteq \{\Omega' \in K_\rho(\mathbb{R}^n) \,|\, d(F(\Omega, u), \Omega') \leq \rho\}, \tag{11b}$$

$$G(p) \leq \widehat{G}(\Omega) \leq \rho + \sup_{q \in \Omega} G(q), \tag{11c}$$

$$g(p, p', e) \leq \hat{g}(\Omega, \Omega', e) \leq \rho + \sup_{q \in \Omega} \sup_{q' \in \Omega'} g(q, q', e) \tag{11d}$$

are satisfied for all $\Omega, \Omega' \in C$, $p \in \Omega$, $p' \in \Omega'$ and $e \in E$. The abstraction (10) is **discrete** *if both C and E are so.*

We emphasize that our assumption of semi-continuity and compact-valuedness in Definition IV.1 is automatically satisfied if C and E are discrete.

Given the value function $\widehat{V}$ of an abstraction (10) of some precision associated with the OCP (9), its *pointwise upper bound* $\widehat{W}$ is defined by

$$\widehat{W}(p) = \sup \left\{ \widehat{V}(\Omega) \,\middle|\, p \in \Omega \in C \right\}$$

for all $p \in \mathbb{R}^n$. Moreover, the *hypograph* $\operatorname{hypo}(f)$ of a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is defined by

$$\operatorname{hypo}(f) := \{(x, \gamma) \in \mathbb{R}^n \times \mathbb{R} \,|\, \gamma \leq f\}.$$

The following result is the main result of the paper.

IV.2 Theorem. *Assume that the OCP (9) satisfies (A_1) and that F, G and g are u.s.c., and let V be the value function of (9).*

Then the pointwise upper bound on the value function of any abstraction associated with the OCP (9) (of arbitrary precision) is an upper bound on V. Moreover, for every $\varepsilon > 0$ and every compact subset $\Gamma \subseteq \mathbb{R}^n$ there exists $\rho > 0$ such that

$$(\Gamma \times \mathbb{R}) \cap \operatorname{hypo}(\widehat{W}) \subseteq B(\operatorname{hypo}(V), \varepsilon) \tag{12}$$

whenever $\widehat{W}$ is the pointwise upper bound on the value function of an abstraction of precision ρ associated with the OCP (9).

IV.3 Remark. Note that by using $\Gamma = \{x\}$ we obtain pointwise convergence, and for any compact $\Gamma \subseteq \mathbb{R}^n$ on which the value function V is continuous we obtain uniform convergence in the classical sense. □

To prove Theorem IV.2 one shows that, roughly speaking, the value function of a suitable auxiliary OCP with state space $K(\mathbb{R}^n)$ is an upper bound on the value functions of

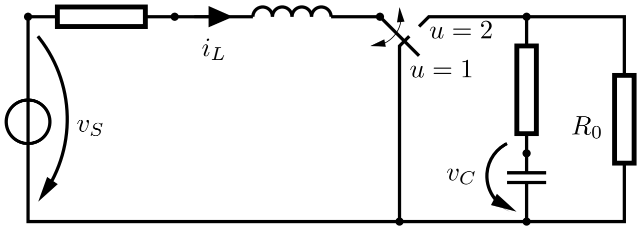

Fig. 2. DC-DC converter investigated in Section V.

all abstractions of prescribed precision and is additionally u.s.c. with respect to the precision. The technical result below guarantees that our results from Section III are applicable.

IV.4 Lemma. *Assume (A$_1$), let F be u.s.c., and define the maps* $\widetilde{F}\colon K(\mathbb{R}^n)\times\mathbb{R}^m\times\mathbb{R}_+ \rightrightarrows K(\mathbb{R}^n)$, $\widetilde{G}\colon K(\mathbb{R}^n)\times\mathbb{R}_+ \to \mathbb{R}_+\cup\{\infty\}$ *and* $\tilde{g}\colon K(\mathbb{R}^n)\times K(\mathbb{R}^n)\times\mathbb{R}^m\times\mathbb{R}_+ \to \mathbb{R}_+\cup\{\infty\}$ *by*

$$\begin{aligned}\widetilde{F}(\Omega,u,\rho) &= \left\{\Omega' \in K_\rho(\mathbb{R}^n) \,\middle|\, d(F(\Omega,\bar{B}(u,\rho)),\Omega') \le \rho\right\},\\ \widetilde{G}(\Omega,\rho) &= \rho + \sup_{q\in\Omega} G(q),\\ \tilde{g}(\Omega,\Omega',u,\rho) &= \rho + \sup_{q\in\Omega}\sup_{q'\in\Omega'}\sup_{v\in\bar{B}(u,\rho)} g(q,q',v).\end{aligned}$$

Then $\widetilde{F}$ *satisfies (A$_1$) and is u.s.c.. Moreover, if G (g, resp.) is u.s.c., then so is $\widetilde{G}$ ($\tilde{g}$, resp.).*

V. Numerical Experiments

In this section, we apply our results to design an approximately time-optimal feedback controller that starts up a DC-DC converter and is robust against supply voltage as well as load fluctuations. That example has been extensively investigated in the literature, and we use the same problem data as in [6], whenever possible.

A. Continuous-Time model of the DC-DC converter

As in [6] the boost DC-DC converter shown in Fig. 2 is modelled as a switched linear system $\dot{x}(t) = A_{u(t)}x(t) + B$, where $x = (i_L, 5v_C)$ and the input $u(t) \in \{1,2\}$ represents the position of the switch in Fig. 2. We generally use the parameter values from [6], but in contrast to [6], we consider the supply voltage v_S and the load resistance R_0 as disturbances that may vary over time within bounds that are defined later. This way we obtain the switched linear system

$$\dot{x}(t) = A_{u(t)}(t)x(t) + B(t), \tag{13}$$

where

$$\begin{aligned}A_1(t) &= \begin{pmatrix} -1/60 & 0 \\ 0 & \frac{-20}{7(200R_0(t)+1)}\end{pmatrix}, B(t) = \begin{pmatrix} v_S(t)/3 \\ 0\end{pmatrix},\\ A_2(t) &= \frac{1}{200R_0(t)+1}\begin{pmatrix} \frac{-(220R_0(t)+1)}{60} & -40R_0(t)/3 \\ 100R_0(t)/7 & -20/7\end{pmatrix}.\end{aligned}$$

We use $\varphi(t,x,u,v_s,R_0)$ to denote the value at time t of the solution of the system (13) passing through the initial value x at time 0.

B. Sampled system and optimal control problem

We sample the system with sampling time $h = 0.5$ to obtain a discrete-time system of the form (1) by the requirement that $q \in F(p,u)$ iff $q = \varphi(h,p,u,v_s,R_0)$ for some admissible perturbations v_s and R_0.

We aim at solving a reach-avoid problem for the sampled system in which the state should be forced into a target region Z in minimum time without leaving the safe set S,

$$\begin{aligned}Z &=]1.1, 1.6[\times]5.4, 5.9[\,,\\ S &=]0.65, 1.65[\times]4.95, 5.95[\,.\end{aligned}$$

This problem can be recast as an optimal stopping problem (6), where

$$\begin{aligned}U = \{1,2\}, \quad X &= \mathbb{R}^2\\ G(x) &= \begin{cases} 0, \text{if } x \in Z,\\ \infty, \text{otherwise},\end{cases}\\ g(x,x',u) &= \begin{cases} 1, \text{if } x \in S,\\ \infty, \text{otherwise}.\end{cases}\end{aligned}$$

With this formulation, we recover the problem statement for the nominal case $v_s = R_0 = 1$ in [6], with a slight difference: our target and safe sets are open, while the closures of these sets are used in [6].

C. The Abstract Optimal Control Problem

We define the abstract optimal control problem

$$(C, \{1,2\}, F_C, g_C, G_C) \tag{14}$$

of (6) following Section IV. The covering C of $\mathbb{R}^2$ consists of rectangles

$$H(x,l,r) := \{x' \in \mathbb{R}^2 \mid \forall_{i\in\{1,2\}} : -l_i \le x'_i - x_i \le r_i\}$$

centered at grid points $x \in [\mathbb{R}^2]_{\eta_1,\eta_2} := 2\eta_1\mathbb{Z} \times 2\eta_2\mathbb{Z}$,

$$C = \left\{H(x,(\eta_1,\eta_2),(\eta_1,\eta_2)) \,\middle|\, x \in [\mathbb{R}^2]_{\eta_1,\eta_2}\right\}.$$

The parameters $\eta_1, \eta_2 > 0$ of the rectangular grid will be specified later. For notational convenience, we will sometimes write $H(x,r)$ for $H(x,r,r)$.

The cost functions g_C and G_C are defined by

$$\begin{aligned}G_C(\Omega) = \max G(\Omega) &= \begin{cases} 0, \text{if } \Omega \subseteq Z,\\ \infty, \text{otherwise},\end{cases}\\ g_C(\Omega,\Omega',u) = \max g(\Omega,\Omega',u) &= \begin{cases} 1, \text{if } \Omega \subseteq S,\\ \infty, \text{otherwise}.\end{cases}\end{aligned}$$

In other words, the safety and target regions of the abstract problem consist of the cells in C that are subsets of S and Z, respectively.

In order to overestimate the right hand side F of (6), the right hand side $F_C\colon C \times \{1,2\} \rightrightarrows C$ of the abstract problem is defined by

$$F_C(H(x,(\eta_1,\eta_2)),u) = \left\{\Omega \in C \,\middle|\, \Omega \cap H(\bar{x},\bar{l},\bar{r}) \neq \varnothing\right\}, \tag{15}$$

where $\bar{x} = \varphi(h,x,u,1,1)$, and the left $\bar{l} \in \mathbb{R}^2$ and right $\bar{r} \in \mathbb{R}^2$ radius are determined by the components $c, \Delta l, \Delta r$,

$$\bar{l} = c + \Delta l, \quad \text{and} \quad \bar{r} = c + \Delta r.$$

Here, $c \in \mathbb{R}^2$ accounts for the nominal system behavior, and $\Delta l, \Delta r \in \mathbb{R}^2$, for the perturbations. These parameters have to be chosen such that the elements of $F_C(H(x,(\eta_1,\eta_2)),u)$ cover the reachable set $F(H(x,(\eta_1,\eta_2)),u)$.

Given a matrix $A \in \mathbb{R}^{n\times n}$, let $M(A)$ denote the Metzler part of A, $M(A)_{i,j} = |A_{i,j}|$ for $i,j, \in [1;n]$, $i \neq j$ and $M(A)_{i,i} = A_{i,i}$. It is well-known that $e^{At} \leq e^{M(A)t}$ holds component-wise for all $t \geq 0$. Therefore, the choice $c = e^{M(A)h}(\eta_1,\eta_2)$ implies that $\varphi(h,H(x,(\eta_1,\eta_2)),u,1,1) \subseteq H(\bar{x},c)$ holds.

Now we focus on the calculation of Δl and Δr with nominal $R_0(t) = 2$, $t \in [0,h]$. In the presence of perturbations, if $u = 1$, due to the simple dynamics, we can directly compute the bounds $\Delta l = 0$, $\Delta r_1 = 20\Delta v_S(1-\mathrm{e}^{-h/60})$ and $\Delta r_2 = (x_2+\eta_2)(1-\mathrm{e}^{-20h/2807})$. Then $\varphi(h,H(x,(\eta_1,\eta_2)),1,v_s,R_0) \subseteq H(\bar{x},\bar{l},\bar{r})$ with $\bar{x} = \varphi(h,x,u,1,2)$ as long as

$$v_s(t) \in [1, 1+\Delta v_S], \quad R_0(t) \in [2,\infty[\text{ for all } t \in [0,h]. \tag{16}$$

The case $u = 2$ is more difficult. Using matrix measures and well-known estimates [20] one can show that (16) implies the estimate

$$\|\varphi(t,x,u,1,2) - \varphi(t,x,u,v_s,R_0)\|_\infty \leq$$
$$e^{\tilde{\mu}h}\left(\frac{\delta(\|x\|_\infty + \frac{\beta}{\mu_+})}{\mu_+ - \tilde{\mu}_+}(e^{(\mu_+ - \tilde{\mu}_+)h} - 1) + \frac{\delta\beta/\mu_+ - \gamma}{\tilde{\mu}_+}(e^{-\tilde{\mu}_+ h} - 1)\right)$$

where $\beta = 1/3$, $\gamma = \Delta v_S/3$, $\delta = 41/5614$, $\mu_+ = 80/1407$, and $\tilde{\mu}_+ = 1/14$. This determines Δl_1, Δr_1 and Δr_2. One can further show that the perturbations v_S and R_0 never decrease the second component of the solution, we may thus define $\Delta l_2 = 0$. Overall, our choices for c, Δl and Δr guarantee that the elements of $F_C(H(x,(\eta_1,\eta_2)),u)$ cover the reachable set $F(H(x,(\eta_1,\eta_2)),u)$, so the abstract system satisfies the requirements in Section IV.

For the estimation of reachable sets in the case of nonlinear system dynamics, see e.g. [21], [2], [22], [23].

D. Computational Results

We consider three instances of the optimal control problem defined above, namely, the nominal case $R_0 = 1$, $v_S = 1$ of [6], and two perturbed cases in which

$$R_0(t) \in [2,\infty[$$

and

$$v_S(t) \in [1, 1.05] \quad \text{and} \quad v_S(t) \in [1, 1.2],$$

respectively.

We vary the number of states $\#C \in \{8192, 32\,768, 131\,072, 524\,288\}$ of the discrete abstraction throughout the computations. Tab. I shows the time to compute the discrete abstraction plus the time to solve the abstract optimal control problem. Moreover, we denote by $\#V_C$ the ratio of the numbers of abstract states in $\operatorname{dom} V_C$ and $\operatorname{dom} g_C$, i.e., $\#\operatorname{dom} V_C/\#\operatorname{dom} g_C \cdot 100$. We illustrate the value function V_C of the abstract optimal control problem for $\#C = 8192$ (coarsest cover) and $\#C = 524288$ (finest cover) in Fig. 3.

TABLE I

COMPUTATIONAL RESULTS.

	$R_0 = 1$, $v_S = 1$ "p_0"		$R_0(t) \in [2,\infty[$, $v_s(t) \in [1, 1.05]$ "p_1"		$v_s(t) \in [1, 1.2]$ "p_2"	
$\#C$	*cpu*	$\#V_C$	*cpu*	$\#V_C$	*cpu*	$\#V_C$
8192	0.4 s	92 %	0.4 s	82 %	0.4 s	65 %
$32 \cdot 10^3$	0.8 s	94 %	0.8 s	88 %	1.2 s	73 %
$131 \cdot 10^3$	2.3 s	95 %	3.5 s	91 %	5.6 s	77 %
$524 \cdot 10^3$	8.5 s	96 %	25 s	92 %	61 s	78 %

We can clearly observe that the performance improves in terms of both a larger domain $\operatorname{dom} V_C$ (see Tab. I) and lower values of V_C (see Fig. 3) with a finer cover C. The improved performance comes at the price of higher memory demand as well as larger computation times, see Tab. I. The increase of the computation times to solve the perturbed abstract OCP in comparison with the unperturbed OCP results from the higher effort to compute the intersection in (15) for a larger $H(\bar{x},\bar{l},\bar{r})$, since we use binary trees to represent the cover C.

Note that the improved performance for finer covers is also visible in Fig. 4, where we show the profile of the value function for the parameter set p_2 for fixed $x_1 = 1$ and $x_2 \in [4.95, 5.4]$. It is easy to verify, that the value function V is discontinuous at $\bar{x} \approx (1, 5.4193)$. For $x = (1, x_2)$ with $x_2 > \bar{x}_2$ the system $F(x,1)$ reaches the target set Z in one step and is allowed to stop. However, for $\bar{x}$ we have $F(\bar{x},1) \not\subseteq Z$, so more than one step is required to reach Z. We illustrate the point of discontinuity by the black bar in Fig. 4. In the plot in the upper right corner of Fig. 4, we can see, as predicted by the hypographical convergence, that the discontinuity of the abstract value function V_C approaches the discontinuity of the true value function V as the cover gets finer.

All the computations were performed on an 1.8 GHz Intel Core i7 with 4GB of memory.

REFERENCES

[1] P. Tabuada, *Verification and control of hybrid systems*. Springer 2009.

[2] G. Reißig, "Computing abstractions of nonlinear systems," *IEEE Trans. Automat. Control*, vol. 56, no. 11, pp. 2583–2598, Nov. 2011, arXiv:0910.2187. http://dx.doi.org/10.1109/TAC.2011.2118950

[3] ——, "Abstraction based solution of complex attainability problems for decomposable continuous plants," in *Proc. 49th IEEE Conf. Decision and Control (CDC), Atlanta, GA, U.S.A., 15-17 Dec. 2010*, pp. 5911–5917. http://dx.doi.org/10.1109/CDC.2010.5718125

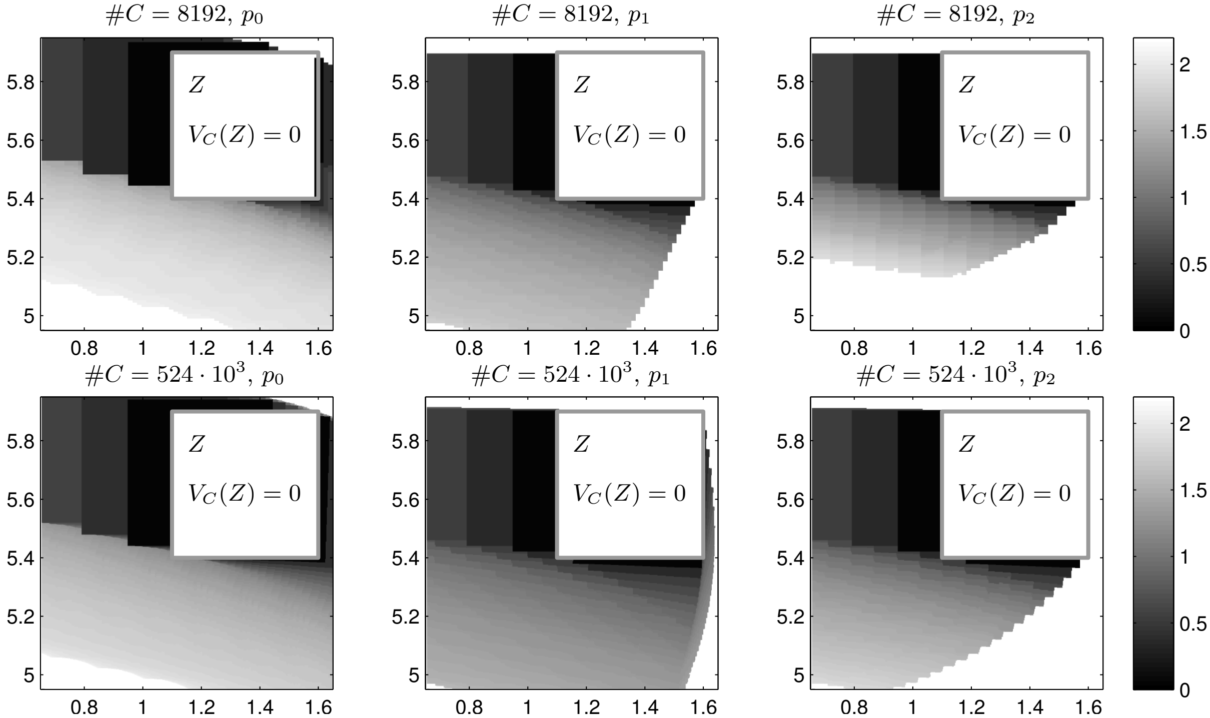

Fig. 3. Logarithm ($\log_{10}$) of V_C for $\#C = 8192$ and $\#C = 524\,288$ and parameter configurations p_0, p_1 and p_2.

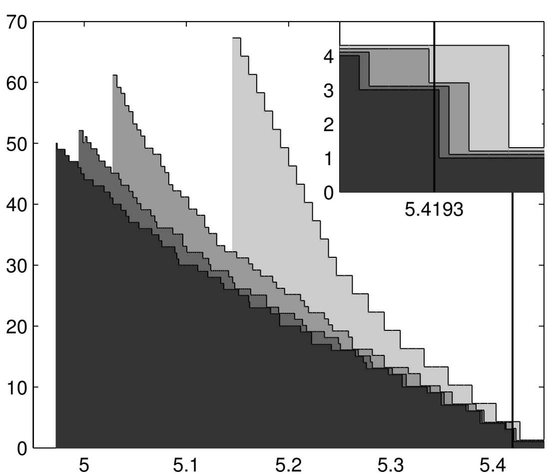

Fig. 4. Hypograph of V_C restricted to $\operatorname{dom} V_C$ for parameter set p_2. (V_C is of course integer-valued. For the purpose of better illustration, a slightly shifted profile is displayed.)

[4] M. Rungger, M. Mazo, and P. Tabuada. Specification-Guided Controller Synthesis for Linear Systems and Safe Linear-Time Temporal Logic. In *Proc. of the 16th Int. Conf. on Hybrid Systems: Computation and Control*. ACM, 2013.

[5] M. Mazo, Jr. and P. Tabuada, "Symbolic approximate time-optimal control," *Systems Control Lett.*, vol. 60, no. 4, pp. 256–263, 2011.

[6] A. Girard, "Controller synthesis for safety and reachability via approximate bisimulation," *Automatica*, vol. 48, no. 5, pp. 947–953, 2012.

[7] Y. Tazaki and J.-i. Imura, "Discrete abstractions of nonlinear systems based on error propagation analysis," *IEEE Trans. Automat. Control*, vol. 57, no. 3, pp. 550–564, 2012.

[8] R. Bellman, *Dynamic Programming*. Princeton Univ. Press, 1957.

[9] D. Bertsekas, "Convergence of discretization procedures in dynamic programming," *IEEE Trans. Automat. Control*, vol. 20, pp. 415–419, 1975.

[10] M. Falcone, "A numerical approach to the infinite horizon problem of deterministic control theory," *Applied Mathematics and Optimization*, vol. 15, pp. 1–13, 1987.

[11] R. L. V. González and M. M. Tidball, "On the rates of convergence of fully discrete solutions of hamilton-jacobi equations," INRIA, Tech. Rep. 1376, 1991.

[12] L. Grüne and O. Junge, "Approximately optimal nonlinear stabilization with preservation of the Lyapunov function property," in *Proc. 46th IEEE Conf. Decision and Control (CDC), 2007*, pp. 702–707.

[13] ——, "Global optimal control of perturbed systems," *J. Optim. Theory Appl.*, vol. 136, no. 3, pp. 411–429, 2008.

[14] G. Kreisselmeier and T. Birkhölzer, "Numerical nonlinear regulator design," *IEEE Trans. Automat. Control*, vol. 39, pp. 33–46, 1994.

[15] P. Cardaliaguet, M. Quincampoix, and P. Saint-Pierre, "Numerical schemes for discontinuous value functions of optimal control," *Set-Valued Analysis*, vol. 8, pp. 111–126, 2000.

[16] M. Broucke, M. D. Di Benedetto, S. Di Gennaro, and A. Sangiovanni-Vincentelli, "Efficient solution of optimal control problems using hybrid systems," *SIAM J. Control Optim.*, vol. 43, no. 6, pp. 1923–1952 (electronic), 2005.

[17] G. Beer, *Topologies on closed and closed convex sets*, ser. Mathematics and its Applications. Kluwer!, 1993, vol. 268.

[18] R.T. Rockafellar and R.J.-B. Wets, *Variational analysis*. Springer 2009

[19] P. Kosmol and D. Müller-Wichards, *Optimization in function spaces*, De Gruyter, 2011

[20] M. Vidyasagar, *Nonlinear systems analysis*, SIAM, 2002.

[21] G. Reißig, "Convexity of reachable sets of nonlinear ordinary differential equations," *Automat. Remote Control*, vol. 68, no. 9, pp. 1527–1543, Sep. 2007, arXiv:1211.6080. [Online]. Available: http://dx.doi.org/10.1134/S000511790709007X

[22] A. Weber and G. Reißig, "Local characterization of strongly convex sets," *J. Math. Anal. Appl.*, vol. 400, no. 2, pp. 743–750, Apr. 2013, arXiv:1207.4347. [Online]. Available: http://dx.doi.org/10.1016/j.jmaa.2012.10.071

[23] A. Weber and G. Reissig, "Strongly convex attainable sets and low complexity finite-state controllers," in *Proc. Australian Control Conf. (AUCC), Perth, Australia, 4-5 Nov. 2013*, 2013, accepted.