

The Complexity of Graph-Based Reductions for Reachability in Markov Decision Processes

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Abstract

We study the never-worse relation (NWR) for Markov decision processes with an infinite-horizon reachability objective. A state q is never worse than a state p if the maximal probability of reaching the target set of states from p is at most the same value from q , regardless of the probabilities labelling the transitions. Extremal-probability states, end components, and essential states are all special cases of the equivalence relation induced by the NWR. Using the NWR, states in the same equivalence class can be collapsed. Then, actions leading to sub-optimal states can be removed. We show that the natural decision problem associated to computing the NWR is coNP-complete. Finally, we extend a previously known incomplete polynomial-time iterative algorithm to under-approximate the NWR.

1 Introduction

Markov decision processes (MDPs) are a useful model for decision-making in the presence of a stochastic environment. They are used in several fields, including robotics, automated control, economics, manufacturing and in particular planning [21], model-based reinforcement learning [23], and formal verification [1]. We elaborate on the use of MDPs and the need for graph-based reductions thereof in verification and reinforcement learning applications below.

Several verification problems for MDPs reduce to reachability [1, 5]. For instance, MDPs can be model checked against linear-time objectives (expressed in, say, LTL) by constructing an omega-automaton recognizing the set of runs that satisfy the objective and considering the product of the automaton with the original MDP [6]. In this product MDP, accepting end components — a generalization of strongly connected components — are identified and selected

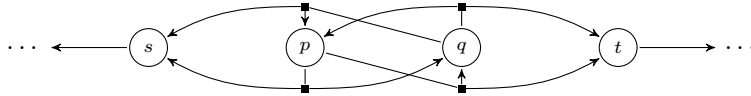


Figure 1: An MDP with states depicted as circles and distributions as squares. The maximal reachability probability values from p and q are the same since, from both, one can enforce to reach s with probability 1, or t with probability 1, using different strategies.

as target components. The question of maximizing the probability that the MDP behaviours satisfy the linear-time objective is thus reduced to maximizing the probability of reaching the target components.

The maximal reachability probability is computable in polynomial time by reduction to linear programming [6, 1]. In practice, however, most model checkers use value iteration to compute this value [18, 9]. The worst-case time complexity of value iteration is pseudo-polynomial. Hence, when implementing model checkers it is usual for a graph-based pre-processing step to remove as many unnecessary states and transitions as possible while preserving the maximal reachability probability. Well-known reductions include the identification of extremal-probability states and maximal end components [5, 1]. The intended outcome of this pre-processing step is a reduced amount of transition probability values that need to be considered when computing the number of iterations required by value iteration.

The main idea behind MDP reduction heuristics is to identify subsets of states from which the maximal probability of reaching the target set of states is the same. Such states are in fact redundant and can be “collapsed”. Figure 1 depicts an MDP with actions and probabilities omitted for clarity. From p and q there are strategies to ensure that s is reached with probability 1. The same holds for t . For instance, from p , to get to t almost surely, one plays to go to the distribution directly below q ; from q , to the distribution above q . Since from the state p , there is no strategy to ensure that q is reached with probability 1, p and q do not form an *end component*. In fact, to the best of our knowledge, no known MDP reduction heuristic captures this example (*i.e.*, recognizes that p and q have the same maximal reachability probability for all possible values of the transition probabilities).

In reinforcement learning the actual probabilities labelling the transitions of an MDP are not assumed to be known in advance. Thus, they have to be estimated by experimenting with different actions in different states and collecting statistics about the observed outcomes [15]. In order for the statistics to be good approximations, the number of experiments has to be high enough. In particular, when the approximations are required to be *probably approximately correct* [24], the necessary and sufficient number of experiments is pseudo-polynomial [13]. Furthermore, the expected number of steps before reaching a particular state even once may already be exponential (even if all the probabilities are fixed). The fact that an excessive amount of experiments is required is a known drawback of reinforcement learning [16, 20].

A natural and key question to ask in this context is whether the maximal reachability probability does indeed depend on the actual value of the probability labelling a particular transition of the MDP. If this is not the case, then it

need not be learnt. One natural way to remove transition probabilities which do not affect the maximal reachability value is to apply model checking MDP reduction techniques.

Contributions and structure of the paper. We view the directed graph underlying an MDP as a directed bipartite graph. Vertices in this graph are controlled by players *Protagonist* and *Nature*. Nature is only allowed to choose full-support probability distributions for each one of her vertices, thus instantiating an MDP from the graph; Protagonist has strategies just as he would in an MDP. Hence, we consider infinite families of MDPs with the same support. In the game played between Protagonist and Nature, and for vertices u and v , we are interested in knowing whether the maximal reachability probability from u is never (in any of the MDPs with the game as its underlying directed graph) worse than the same value from v .

In Section 2 we give the required definitions. We formalize the *never-worse relation* in Section 3. We also show that we can “collapse” sets of equivalent vertices with respect to the NWR (Theorem 1) and remove sub-optimal edges according to the NWR (Theorem 2). Finally, we also argue that the NWR generalizes most known heuristics to reduce MDP size before applying linear programming or value iteration. Then, in Section 4 we give a graph-based characterization of the relation (Theorem 3), which in turn gives us a CONP upper bound on its complexity. A matching lower bound is presented in Section 5 (Theorem 4). To conclude, we recall and extend an iterative algorithm to efficiently (in polynomial time) under-approximate the never-worse relation from [2].

Previous and related work. Reductions for MDP model checking were considered in [7] and [5]. From the reductions studied in both papers, extremal-probability states, essential states, and end components are computable using only graph-based algorithms. In [3], learning-based techniques are proposed to obtain approximations of the maximal reachability probability in MDPs. Their algorithms, however, do rely on the actual probability values of the MDP.

This work is also related to the widely studied model of interval MDPs, where the transition probabilities are given as intervals meant to model the uncertainty of the numerical values. Numberless MDPs [11] are a particular case of the latter in which values are only known to be zero or non-zero. In the context of numberless MDPs, a special case of the question we study can be simply rephrased as the comparison of the maximal reachability values of two given states.

In [2] a preliminary version of the iterative algorithm we give in Section 6 was described, implemented, and shown to be efficient in practice. Proposition 1 was first stated therein. In contrast with [2], we focus chiefly on characterizing the never-worse relation and determining its computational complexity.

2 Preliminaries

We use set-theoretic notation to indicate whether a letter $b \in \Sigma$ occurs in a word $\alpha = a_0 \dots a_k \in \Sigma^*$, i.e. $b \in \alpha$ if and only if $b = a_i$ for some $0 \leq i \leq k$.

Consider a directed graph $\mathcal{G} = (V, E)$ and a vertex $u \in V$. We write uE for the set of *successors* of u . That is to say, $uE := \{v \in V \mid (u, v) \in E\}$. We say

that a path $\pi = u_0 \dots u_k \in V^*$ in \mathcal{G} *visits* a vertex v if $v \in \pi$. We also say that π is a v - T path, for $T \subseteq V$, if $u_0 = v$ and $u_k \in T$.

2.1 Stochastic models

Let S be a finite set. We denote by $\mathbb{D}(S)$ the set of all (*rational*) *probabilistic distributions* on S , i.e. the set of all functions $f : S \rightarrow \mathbb{Q}_{\geq 0}$ such that $\sum_{s \in S} f(s) = 1$. A probabilistic distribution $f \in \mathbb{D}(S)$ has *full support* if $f(s) > 0$ for all $s \in S$.

Definition 1 (Markov chains). A *Markov chain* \mathcal{C} is a tuple (Q, δ) where Q is a finite set of states and δ is a probabilistic transition function $\delta : Q \rightarrow \mathbb{D}(Q)$.

A *run* of a Markov chain is a finite non-empty word $\varrho = p_0 \dots p_n$ over Q . We say ϱ *reaches* q if $q = p_i$ for some $0 \leq i \leq n$. The *probability of the run* is $\prod_{0 \leq i < n} \delta(p_i, p_{i+1})$.

Let $T \subseteq Q$ be a set of states. The *probability of (eventually) reaching* T in \mathcal{C} from q_0 , which will be denoted by $\mathbb{P}_{\mathcal{C}}^{q_0}[\Diamond T]$, is the measure of the runs of \mathcal{C} that start at q_0 and reach T . For convenience, let us first define the *probability of staying in states from* $S \subseteq Q$ *until* T *is reached*¹, written $\mathbb{P}_{\mathcal{C}}^{q_0}[S \cup T]$, as 1 if $q_0 \in T$ and otherwise

$$\sum \left\{ \prod_{0 \leq i < n} \delta(q_i, q_{i+1}) \mid q_0 \dots q_n \in (S \setminus T)^* T \text{ for } n \geq 1 \right\}.$$

We then define $\mathbb{P}_{\mathcal{C}}^{q_0}[\Diamond T] := \mathbb{P}_{\mathcal{C}}^{q_0}[Q \cup T]$.

When all runs from q_0 to T reach some set $U \subseteq Q$ before, the probability of reaching T can be decomposed into a finite sum as in the lemma below.

Lemma 1. Consider a Markov chain $\mathcal{C} = (Q, \delta)$, sets of states $U, T \subseteq Q$, and a state $q_0 \in Q \setminus U$. If $\mathbb{P}_{\mathcal{C}}^{q_0}[(Q \setminus U) \cup T] = 0$, then

$$\mathbb{P}_{\mathcal{C}}^{q_0}[\Diamond T] = \sum_{u \in U} \mathbb{P}_{\mathcal{C}}^{q_0}[(Q \setminus U) \cup u] \mathbb{P}_{\mathcal{C}}^u[\Diamond T].$$

Definition 2 (Markov decision processes). A (*finite, discrete-time*) *Markov decision process* \mathcal{M} , MDP for short, is a tuple (Q, A, δ, T) where Q is a finite set of states, A a finite set of actions, $\delta : Q \times A \rightarrow \mathbb{D}(Q)$ a *probabilistic transition function*, and $T \subseteq Q$ a set of *target* states.

For convenience, we write $\delta(q|p, a)$ instead of $\delta(p, a)(q)$.

Definition 3 (Strategies). A (*memoryless deterministic*) *strategy* σ in an MDP $\mathcal{M} = (Q, A, \delta, T)$ is a function $\sigma : Q \rightarrow A$.

Note that we have deliberately defined only memoryless deterministic strategies. This is at no loss of generality since, in this work, we focus on maximizing the probability of reaching a set of states. It is known that for this type of objective, memoryless deterministic strategies suffice [19].

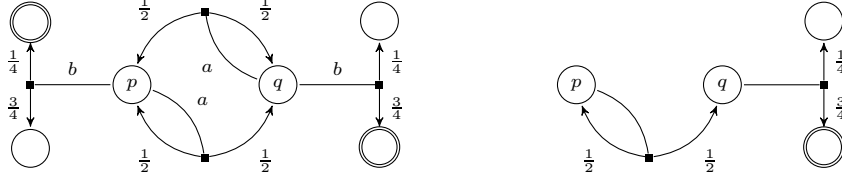


Figure 2: On the left we have an MDP with actions $\{a, b\}$. On the right we have the Markov chain induced by the left MDP and the strategy $\{p \mapsto a, q \mapsto b\}$.

From MDPs to chains. An MDP $\mathcal{M} = (Q, A, \delta, T)$ and a strategy σ induce the Markov chain $\mathcal{M}^\sigma = (Q, \mu)$ where $\mu(q) = \delta(q, \sigma(q))$ for all $q \in Q$.

Example 1. Figure 2 depicts an MDP on the left. Circles represent states; double-circles, target states; and squares, distributions. The labels on arrows from states to distributions are actions; those on arrows from distributions to states, probabilities.

Consider the strategy σ that plays from p the action a and from q the action b , i.e. $\sigma(p) = a$ and $\sigma(q) = b$. The Markov chain on the right is the chain induced by σ and the MDP on the left. Note that we no longer have action labels.

The probability of reaching a target state from q under σ is easily seen to be $3/4$. In other words, if we write \mathcal{M} for the MDP and T for the set of target states then $\mathbb{P}_{\mathcal{M}^\sigma}^q[\Diamond T] = \frac{3}{4}$.

2.2 Reachability games against Nature

We will speak about families of MDPs whose probabilistic transition functions have the same support. To do so, we abstract away the probabilities and focus on a game played on a graph. That is, given an MDP $\mathcal{M} = (Q, A, \delta, T)$ we consider its *underlying directed graph* $\mathcal{G}_{\mathcal{M}} = (V, E)$ where $V := Q \cup (Q \times A)$ and $E := \{(q, \langle q, a \rangle) \in Q \times (Q \times A)\} \cup \{(\langle p, a \rangle, q) \mid \delta(q|p, a) > 0\}$. In $\mathcal{G}_{\mathcal{M}}$, *Nature* controls the vertices $Q \times A$. We formalize the game and the *arena* it is played on below.

Definition 4 (Target arena). A *target arena* \mathcal{A} is a tuple (V, V_P, E, T) such that $(V_P, V_N := V \setminus V_P, E)$ is a bipartite directed graph, $T \subseteq V_P$ is a set of *target vertices*, and $uE \neq \emptyset$ for all $u \in V_N$.

Informally, there are two agents in a target arena: *Nature*, who controls the vertices in V_N , and *Protagonist*, who controls the vertices in V_P .

From arenas to MDPs. A target arena $\mathcal{A} = (V, V_P, E, T)$ together with a family of probability distributions $\mu = (\mu_u \in \mathbb{D}(uE))_{u \in V_N}$ induce an MDP. Formally, let \mathcal{A}_μ be the MDP (Q, A, δ, T) where $Q = V_P \uplus \{\perp\}$, $A = V_N$, $\delta(q|p, a)$ is $\mu_a(q)$ if $(p, a), (a, q) \in E$ and 0 otherwise, for all $p \in V_P \cup \{\perp\}$ and $a \in A$ we have $\delta(\perp|p, a) = 1$ if $(p, a) \notin E$.

¹ $S \cup T$ should be read as “ S until T ” and not understood as a set union.



Figure 3: Two target arenas with $T = \{fin\}$ are shown. Round vertices are elements from V_P ; square vertices, from V_N . In the left target arena we have that $p \not\leq \{q\}$ and $q \not\leq \{p\}$ since any path from either vertex visits t before T — see Lemma 1. In the right target arena we have that $t \leq \{p\}$ — see Proposition 1.

The value of a vertex. Consider a target arena $\mathcal{A} = (V, V_P, E, T)$ and a vertex $v \in V_P$. We define its (*maximal reachability probability*) value with respect to a family of full-support probability distributions μ as $\text{Val}^\mu(v) := \max_{\sigma} \mathbb{P}_{\mathcal{A}_\mu^\sigma}^v[\Diamond T]$. For $u \in V_N$ we set $\text{Val}^\mu(u) := \sum \{\mu_u(v) \text{Val}^\mu(v) \mid v \in uE\}$.

3 The never-worse relation

We are now in a position to define the relation that we study in this work. Let us fix a target arena $\mathcal{A} = (V, V_P, E, T)$.

Definition 5 (The never-worse relation (NWR)). A subset $W \subseteq V$ of vertices is *never worse* than a vertex $v \in V$, written $v \leq W$, if and only if

$$\forall \mu = (\mu_u \in \mathbb{D}(uE))_{u \in V_N}, \exists w \in W : \text{Val}^\mu(v) \leq \text{Val}^\mu(w)$$

where all the μ_u have **full support**. We write $v \sim w$ if $v \leq \{w\}$ and $w \leq \{v\}$.

It should be clear from the definition that \sim is an equivalence relation. For $u \in V$ let us denote by \tilde{u} the set of vertices that are \sim -equivalent and belong to the same owner, *i.e.* \tilde{u} is $\{v \in V_P \mid v \sim u\}$ if $u \in V_P$ and $\{v \in V_N \mid v \sim u\}$ otherwise.

Example 2. Consider the left target arena depicted in Figure 3. Using Lemma 1, it is easy to show that neither p nor q is ever worse than the other since t is visited before fin by all paths starting from p or q .

The literature contains various heuristics which consist in computing sets of states and “collapsing” them to reduce the size of the MDP without affecting the maximal reachability probability of the remaining states. We now show that we can collapse equivalence classes and, further, remove sub-optimal distributions using the NWR.

3.1 The usefulness of the NWR

We will now formalize the idea of “collapsing” equivalent vertices with respect to the NWR. For convenience, we will also remove self-loops while doing so.

Consider a target arena $\mathcal{A} = (V, V_P, E, T)$. We denote by $\mathcal{A}_{/\sim}$ its \sim -quotient. That is, $\mathcal{A}_{/\sim}$ is the target arena (S, S_P, R, U) where $S_P = \{\tilde{v} \mid \exists v \in V_P\}$, $S = \{\tilde{v} \mid \exists v \in V_N\} \cup S_P$, $U = \{\tilde{t} \mid \exists t \in T\}$, and

$$R = \{(\tilde{u}, \tilde{v}) \mid \exists (u, v) \in (V_P \times V_N) \cap E : vE \setminus \tilde{u} \neq \emptyset\} \\ \cup \{(\tilde{u}, \tilde{v}) \mid \exists (u, v) \in (V_N \times V_P) \cap E\}.$$

For a family $\mu = (\mu_u \in \mathbb{D}(uE))_{u \in V_N}$ of full-support distributions we denote by $\mu_{/\sim}$ the family $\nu = (\nu_{\tilde{u}} \in \mathbb{D}(\tilde{u}R))_{\tilde{u} \in S_N}$ defined as follows. For all $\tilde{u} \in S_N$ and all $\tilde{v} \in \tilde{u}R$ we have $\nu_{\tilde{u}}(\tilde{v}) = \sum_{w \in \tilde{v}} \mu_u(w)$, where u is any element of \tilde{u} .

The following property of the \sim -quotient follows from the fact that all the vertices in \tilde{v} have the same maximal probability of reaching the target vertices.

Theorem 1. Consider a target arena $\mathcal{A} = (V, V_P, E, T)$. For all families $\mu = (\mu_u \in \mathbb{D}(uE))_{u \in V_N}$ of full-support probability distributions and all $v \in V_P$ we have

$$\max_{\sigma} \mathbb{P}_{\mathcal{A}_{\mu}^{\sigma}}^v[\Diamond T] = \max_{\sigma'} \mathbb{P}_{\mathcal{B}_{\sigma'}^{\tilde{\nu}}}^{\tilde{v}}[\Diamond U],$$

where $\mathcal{B} = \mathcal{A}_{/\sim}$, $\nu = \mu_{/\sim}$, and $U = \{\tilde{t} \mid \exists t \in T\}$.

We can further remove edges that lead to sub-optimal Nature vertices. When this is done after \sim -quotienting the maximal reachability probabilities are preserved.

Theorem 2. Consider a target arena $\mathcal{A} = (V, V_P, E, T)$ such that $\mathcal{A}_{/\sim} = \mathcal{A}$. For all families $\mu = (\mu_u \in \mathbb{D}(uE))_{u \in V_N}$ of full-support probability distributions, for all $(w, x) \in E \cap (V_P \times V_N)$ such that $x \not\leq (wE \setminus \{x\})$, and all $v \in V_P$ we have

$$\max_{\sigma} \mathbb{P}_{\mathcal{A}_{\mu}^{\sigma}}^v[\Diamond T] = \max_{\sigma'} \mathbb{P}_{\mathcal{B}_{\mu}^{\sigma'}}^v[\Diamond T],$$

where $\mathcal{B} = (V, V_P, E \setminus \{(w, x)\}, T)$.

3.2 Known efficiently-computable special cases

We now recall the definitions of the set of extremal-probability states, end components, and essential states. Then, we observe that for all these sets of states their maximal probability reachability coincide and their definitions are independent of the probabilities labelling the transitions of the MDP. Hence, they are subsets of the set of the equivalence classes induced by \sim .

3.2.1 Extremal-probability states.

The set of *extremal-probability states* of an MDP $\mathcal{M} = (Q, A, \delta, T)$ consists of the set of states with maximal probability reachability 0 and 1. Both sets can be computed in polynomial time [1, 4]. We give below a game-based definition of both sets inspired by the classical polynomial-time algorithm to compute them (see, e.g., [1]). Let us fix a target arena $\mathcal{A} = (V, V_P, E, T)$ for the sequel.

For a set $T \subseteq V$, let us write $\mathbf{Z}_T := \{v \in V \mid T \text{ is not reachable from } v\}$.

(Almost-surely winning) strategies. A strategy for Protagonist in a target arena is a function $\sigma : V_P \rightarrow V_N$. We then say that a path $v_0 \dots v_n \in V^*$ is *consistent with σ* if $v_i \in V_P \implies \sigma(v_i) = v_{i+1}$ for all $0 \leq i < n$. Let $\mathbf{Reach}(v_0, \sigma)$ denote the set of vertices reachable from v_0 under σ , i.e. $\mathbf{Reach}(v_0, \sigma) := \{v_k \mid v_0 \dots v_k \text{ is a path consistent with } \sigma\}$.

We say that a strategy σ for Protagonist is *almost-surely winning from $u_0 \in V$ to $T \subseteq V_P$* if, after modifying the arena to make all $t \in T$ into sinks, for all $v_0 \in \mathbf{Reach}(u_0, \sigma)$ we have $\mathbf{Reach}(v_0, \sigma) \cap T \neq \emptyset$. We denote the set of all such strategies by $\mathbf{Win}_T^{v_0}$.

The following properties regarding almost-surely winning strategies in a target arena follow from the correctness of the graph-based algorithm used to compute extremal-probability states in an MDP [1, Lemma 10.108].

Lemma 2 (From [1]). Consider a target arena $\mathcal{A} = (V, V_P, E, T)$. For all families $\mu = (\mu_u \in \mathbb{D}(uE))_{u \in V_N}$ of full-support probability distributions, for all $v \in V_P$ the following hold.

- (i) $\max_{\sigma} \mathbb{P}_{\mathcal{A}_{\mu}}^v[\Diamond T] = 0 \iff v \in \mathbf{Z}_T$
- (ii) $\forall \sigma : \sigma \in \mathbf{Win}_T^v \iff \mathbb{P}_{\mathcal{A}_{\mu}}^v[\Diamond T] = 1$

3.2.2 End components.

Let us consider an MDP $\mathcal{M} = (Q, A, \delta, T)$. A set $S \subseteq Q$ of states is an *end component* in \mathcal{M} if for all pairs of states $p, q \in S$ there exists a strategy σ such that $\mathbb{P}_{\mathcal{M}^{\sigma}}^p[S \cup q] = 1$.

Example 3. Let us consider the MDP shown on the left in Figure 2. The set $\{p, q\}$ is an end component since, by playing a from both states, one can ensure to reach either state from the other with probability 1.

It follows immediately from the definition of end component that the maximal probability of reaching T from states in the same end component is the same.

Lemma 3. Let $S \subseteq Q$ be an end component in \mathcal{M} . For all $p, q \in S$ we have that $\max_{\sigma} \mathbb{P}_{\mathcal{M}^{\sigma}}^p[\Diamond T] = \max_{\sigma} \mathbb{P}_{\mathcal{M}^{\sigma}}^q[\Diamond T]$.

We say an end component is *maximal* if it is maximal with respect to set inclusion. Furthermore, from the definition of end components in MDPs and Lemma 2 it follows that we can lift the notion of end component to target arenas. More precisely, a set $S \subseteq V_P$ is an end component in \mathcal{A} if and only if for some family of full-support probability distributions μ we have that S is an end component in \mathcal{A}_{μ} (if and only if for all μ' the set S is an end component in $\mathcal{A}_{\mu'}$).

The set of all maximal end components of a target arena can be computed in polynomial time using an algorithm based on the strongly connected components of the graph [8, 1].

3.2.3 Essential states.

Consider a target arena $\mathcal{A} = (V, V_P, E, T)$ and let \sqsubseteq be the smallest relation satisfying the following. For all $u \in V_P$ we have $u \sqsubseteq u$. For all $u_0, v \in V_P \setminus \mathbf{Z}_T$ such that $u_0 \neq v$ we have $u_0 \sqsubseteq v$ if for all paths $u_0 u_1 u_2$ we have that $u_2 \sqsubseteq v$ and there is at least one such path. Intuitively, $u \sqsubseteq v$ holds whenever all paths starting from u reach v . In [7], the maximal vertices according to \sqsubseteq are called *essential states*².

Lemma 4 (From [7]). Consider a target arena $\mathcal{A} = (V, V_P, E, T)$. For all families $\mu = (\mu_u \in \mathbb{D}(uE))_{u \in V_N}$ of full-support probability distributions, for all $v \in V_P$ and all essential states w , if $v \sqsubseteq w$ then $\max_{\sigma} \mathbb{P}_{\mathcal{A}_{\mu}}^v[\Diamond T] = \max_{\sigma} \mathbb{P}_{\mathcal{A}_{\mu}}^w[\Diamond T]$.

²This is not the usual notion of essential states from classical Markov chain theory

Note that, in the left arena in Figure 3, $p \sqsubseteq t$ does not hold since there is a cycle between p and q which does not visit t .

It was also shown in [7] that the \sqsubseteq relation is computable in polynomial time.

4 Graph-based characterization of the NWR

In this section we give a characterization of the NWR that is reminiscent of the topological-based value iteration proposed in [5]. The main intuition behind our characterization is as follows. If $v \sqsubseteq W$ does not hold, then for all $0 < \varepsilon < 1$ there is some family μ of full-support distributions such that $\text{Val}^\mu(v)$ is at least $1 - \varepsilon$, while $\text{Val}^\mu(w)$ is at most ε for all $w \in W$. In turn, this must mean that there is a path from v to T which can be assigned a high probability by μ while, from W , all paths go with high probability to \mathbf{Z}_T .

We capture the idea of separating a “good” v - T path from all paths starting from W by using partitioning of V into layers $S_i \subseteq V$. Intuitively, we would like it to be easy to construct a family μ of probability distributions such that from all vertices in S_{i+1} all paths going to vertices outside of S_{i+1} end up, with high probability, in lower layers, *i.e.* some S_k with $k < i$. A formal definition follows.

Definition 6 (Drift partition and vertices). Consider a target arena $\mathcal{A} = (V, V_P, E, T)$ and a partition $(S_i)_{0 \leq i \leq k}$ of V . For all $0 \leq i \leq k$, let $S_i^+ := \cup_{i < j} S_j$ and $S_i^- := \cup_{j < i} S_j$, and let $D_i := \{v \in S_i \cap V_N \mid vE \cap S_i^- \neq \emptyset\}$. We define the set $D := \cup_{0 < i < k} D_i$ of *drift vertices*. The partition is called a *drift partition* if the following hold.

- For all $i \leq k$ and all $v \in S_i \cap V_P$ we have $vE \cap S_i^+ = \emptyset$.
- For all $i \leq k$ and all $v \in S_i \cap V_N$ we have $vE \cap S_i^+ \neq \emptyset \implies v \in D$.

Using drift partitions, we can now formalize our characterization of the negation of the NWR.

Theorem 3. Consider a target arena $\mathcal{A} = (V, V_P, E, T)$, a non-empty set of vertices $W \subseteq V$, and a vertex $v \in V$. The following are equivalent

- (i) $\neg(v \sqsubseteq W)$
- (ii) There exists a drift partition $(S_i)_{0 \leq i \leq k}$ and a simple path π starting in v and ending in T such that $\pi \subseteq S_k$ and $W \subseteq S_k^-$.

Before proving Theorem 3 we need an additional definition and two intermediate results.

Definition 7 (Value-monotone paths). Let $\mathcal{A} = (V, V_P, E, T)$ be a target arena and consider a family of full-support probability distributions $\mu = (\mu_u \in \mathbb{D}(uE))_{u \in V_N}$. A path $v_0 \dots v_k$ is μ -*non-increasing* if and only if $\text{Val}^\mu(v_{i+1}) \leq \text{Val}^\mu(v_i)$ for all $0 \leq i < k$; it is μ -*non-decreasing* if and only if $\text{Val}^\mu(v_i) \leq \text{Val}^\mu(v_{i+1})$ for all $0 \leq i < k$.

It can be shown that from any path in a target arena ending in T one can obtain a simple non-decreasing one.

Lemma 5. Consider a target arena $\mathcal{A} = (V, V_P, E, T)$ and a family of full-support probability distributions $\mu = (\mu_u \in \mathbb{D}(uE))_{u \in V_N}$. If there is a path from some $v \in V$ to T , there is also a simple μ -non-decreasing one.

Additionally, we will make use of the following properties regarding vertex-values. They formalize the relation between the value of a vertex, its owner, and the values of its successors.

Lemma 6. Consider a target arena $\mathcal{A} = (V, V_P, E, T)$ and a family of full-support probability distributions $\mu = (\mu_u \in \mathbb{D}(uE))_{u \in V_N}$.

- (i) For all $u \in V_P$, for all successors $v \in uE$ it holds that $\text{Val}^\mu(v) \leq \text{Val}^\mu(u)$.
- (ii) For all $u \in V_N$ it holds that

$$(\exists v \in uE : \text{Val}^\mu(u) < \text{Val}^\mu(v)) \implies (\exists w \in uE : \text{Val}^\mu(w) < \text{Val}^\mu(u)).$$

Proof of Theorem 3. Recall that, by definition, (i) holds if and only if there exists a family $\mu = (\mu_u \in \mathbb{D}(uE))_{u \in V_N}$ of full-support probability distributions such that $\forall w \in W : \text{Val}^\mu(w) < \text{Val}^\mu(v)$.

Let us prove (i) \implies (ii). Let $x_0 < x_1 < \dots$ be the finitely many (*i.e.* at most $|V|$) values that occur in the MDP \mathcal{A}_μ , and let k be such that $\text{Val}^\mu(v) = x_k$. For all $0 \leq i < k$ let $S_i := \{u \in V \mid \text{Val}^\mu(u) = x_i\}$, and let $S_k := V \setminus \cup_{i < k} S_i$. Let us show below that the S_i form a drift partition.

- $\forall i \leq k, \forall u \in S_i \cap S_P : uE \cap S_i^+ = \emptyset$ by Lemma 6.(i) (for $i < k$) and since $S_k^+ = \emptyset$.
- $\forall i \leq k, \forall u \in S_i \cap S_N : uE \cap S_i^+ \neq \emptyset \implies x \in D$ by Lemma 6.(ii) (for $i < k$) and since $S_k^+ = \emptyset$.

We have that $\text{Val}^\mu(w) < \text{Val}^\mu(v) = x_k$ for all $w \in W$, by assumption, so $W \subseteq S_k^-$ by construction. By Lemma 5 there exists a simple μ -non-decreasing path π from v to T , so all the vertices occurring in π have values at least $\text{Val}^\mu(v)$, so $\pi \subseteq S_k$.

We will prove (ii) \implies (i) by defining some full-support distribution family μ . The definition will be partial only, first on $\pi \cap V_N$, and then on the drift vertices in $V \setminus S_k$. Let $0 < \varepsilon < 1$, which is meant to be small enough. Let us write $\pi = v_0 \dots v_n$ so that $v_0 = v$ and $v_n \in T$. Let us define μ on $\pi \cap V_N$ as follows: for all $i < n$, if $v_i \in V_N$ let $\mu_{v_i}(v_{i+1}) := 1 - \varepsilon$. Let σ be an arbitrary Protagonist strategy such that for all $i < n$, if $v_i \in V_P$ then $\sigma(v_i) := v_{i+1}$. Therefore

$$\begin{aligned} (1 - \varepsilon)^{|V|} &\leq (1 - \varepsilon)^n && \text{since } \pi \text{ is simple} \\ &\leq \prod_{i < n, v_i \in S_N} \mu_{v_i}(v_{i+1}) && \text{by definition of } \mu \\ &\leq \mathbb{P}_{\mathcal{A}_\mu^\sigma}^v[\diamond T] \\ &\leq \max_{\sigma'} \mathbb{P}_{\mathcal{A}_\mu^{\sigma'}}^v[\diamond T] = \text{Val}^\mu(v). \end{aligned} \tag{1}$$

So, for $0 < \varepsilon < 1 - \frac{1}{|V|\sqrt{2}}$, we have $\frac{1}{2} < (1 - \varepsilon)^{|V|} \leq \text{Val}^\mu(v)$. Below we will further define μ such that $\text{Val}^\mu(w) \leq 1 - (1 - \varepsilon)^{|V|} < \frac{1}{2}$ for all $w \in W$ and all

$0 < \varepsilon < 1 - \frac{1}{\sqrt{|V|}}$, which will prove (ii) \implies (i). However, the last part of the proof is more difficult.

For all $1 \leq i \leq k$, for all drift vertices $u \in S_i$, let $\varrho(u)$ be a successor of u in S_i^- . Such a $\varrho(u)$ exists by definition of the drift vertices. Then let $\mu_u(\varrho(u)) := 1 - \varepsilon$. We then claim that

$$\forall u \in D : (1 - \varepsilon)(1 - \mathbb{P}_{\mathcal{A}_\mu^\sigma}^{\varrho(u)}[\Diamond T]) \leq 1 - \mathbb{P}_{\mathcal{A}_\mu^\sigma}^u[\Diamond T]. \quad (2)$$

Indeed, $1 - \mathbb{P}_{\mathcal{A}_\mu^\sigma}^u[\Diamond T]$ is the probability that, starting at u and following σ , T is never reached; and $(1 - \varepsilon)(1 - \mathbb{P}_{\mathcal{A}_\mu^\sigma}^{\varrho(u)}[\Diamond T])$ is the probability that, starting at u and following σ , the second vertex is $\varrho(u)$ and T is never reached.

Now let σ be an arbitrary strategy, and let us prove the following by induction on j .

$$\forall 0 \leq j < k, \forall w \in S_j \cup S_j^- : \mathbb{P}_{\mathcal{A}_\mu^\sigma}^w[\Diamond T] \leq 1 - (1 - \varepsilon)^j$$

Base case, $j = 0$: by assumption W is non-empty and included in S_k^- , so $0 < k$. Also by assumption $T \subseteq S_k$, so $T \cap S_0 = \emptyset$. By definition of a drift partition, there are no edges going out of S_0 , regardless of whether the starting vertex is in V_P or V_N . So there is no path from w to T , which implies $\text{Val}^\mu(w) = 0$ for all $w \in S_0$, and the claim holds for the base case. Inductive case, let $w \in S_j$, let $D' := D \cap (S_j \cup S_j^-)$ and let us argue that every path π from w to T must at some point leave $S_j \cup S_j^-$ to reach a vertex with higher index, *i.e.* there is some edge (π_i, π_{i+1}) from $\pi_i \in S_j \cup S_j^-$ to some $\pi_{i+1} \in S_\ell$ with $j < \ell$. By definition of a drift partition, π_i must also be a drift vertex, *i.e.* $\pi_i \in D'$. Thus, if we let $F := V_P \setminus D'$, Lemma 1 implies that $\mathbb{P}_{\mathcal{A}_\mu^\sigma}^w[\Diamond T] = \sum_{u \in D'} \mathbb{P}_{\mathcal{A}_\mu^\sigma}^w[F \cup u] \mathbb{P}_{\mathcal{A}_\mu^\sigma}^u[\Diamond T]$. Now, since

$$\begin{aligned} & \sum_{u \in D'} \mathbb{P}_{\mathcal{A}_\mu^\sigma}^u[\Diamond T] \\ &= \sum_{u \in D \cap S_j^-} \mathbb{P}_{\mathcal{A}_\mu^\sigma}^u[\Diamond T] + \sum_{u \in D_j} \mathbb{P}_{\mathcal{A}_\mu^\sigma}^u[\Diamond T] && \text{by splitting the sum} \\ &\leq \sum_{u \in D \cap S_j^-} \mathbb{P}_{\mathcal{A}_\mu^\sigma}^u[\Diamond T] + \sum_{u \in D_j} (1 - (1 - \varepsilon)(1 - \mathbb{P}_{\mathcal{A}_\mu^\sigma}^{\varrho(u)}[\Diamond T])) && \text{by (2)} \\ &\leq \sum_{u \in D \cap S_j^-} (1 - (1 - \varepsilon)^{j-1}) + && \text{by IH and since} \\ & \quad \sum_{u \in D_j} (1 - (1 - \varepsilon)(1 - \varepsilon)^{j-1}) && \forall x \in D_j : \varrho(x) \in S_j^- \\ &\leq \sum_{u \in D'} (1 - (1 - \varepsilon)^j) && (1 - \varepsilon)^j \leq (1 - \varepsilon)^{j-1} \end{aligned}$$

and $\sum_{u \in D'} \mathbb{P}_{\mathcal{A}_\mu^\sigma}^w[F \cup u] \leq 1$, we have that $\mathbb{P}_{\mathcal{A}_\mu^\sigma}^w[\Diamond T] \leq 1 - (1 - \varepsilon)^j$. The induction is thus complete. Since σ is arbitrary in the calculations above, and since $j < k \leq |V|$, we find that $\text{Val}^\mu(w) \leq 1 - (1 - \varepsilon)^{|V|}$ for all $w \in W \subseteq S_k^-$.

For $0 < \varepsilon < 1 - \frac{1}{\sqrt{|V|}}$ we have $\frac{1}{2} < (1 - \varepsilon)^{|V|}$, as mentioned after (1), so $\text{Val}^\mu(w) \leq 1 - (1 - \varepsilon)^{|V|} < \frac{1}{2}$. \square

5 Intractability of the NWR

It follows from Theorem 3 that we can decide whether a vertex is sometimes worse than a set of vertices by guessing a partition of the vertices and verifying that it is a drift partition. The verification can clearly be done in polynomial time.

Corollary 1. Given a target arena $\mathcal{A} = (V, V_P, E, T)$, a non-empty set $W \subseteq V$, and a vertex $v \in V$, determining whether $v \preceq W$ is decidable and in coNP.

We will now show that the problem is in fact coNP-complete already for Markov chains.

Theorem 4. Given a target arena $\mathcal{A} = (V, V_P, E, T)$, a non-empty vertex set $W \subseteq V$, and a vertex $v \in V$, determining whether $v \preceq W$ is coNP-complete even if $|uE| = 1$ for all $u \in V_P$.

The idea is to reduce the 2-DISJOINT PATHS PROBLEM (2DP) to the existence of a drift partition witnessing that $v \preceq \{w\}$ does not hold, for some $v \in V$. Recall that 2DP asks, given a directed graph $\mathcal{G} = (V, E)$ and vertex pairs $(s_1, t_1), (s_2, t_2) \in V \times V$, whether there exists an s_1 - t_1 path π_1 and an s_2 - t_2 path π_2 such that π_1 and π_2 are vertex disjoint, *i.e.* $\pi_1 \cap \pi_2 = \emptyset$. The problem is known to be NP-complete [12, 10]. In the sequel, we assume without loss of generality that (a) t_1 and t_2 are reachable from all $s \in V \setminus \{t_1, t_2\}$; and (b) t_1 and t_2 are the only sinks \mathcal{G} .

Proof of Theorem 4. From the 2DP input instance, we construct the target arena $\mathcal{A} = (S, S_P, R, T)$ with $S := V \cup E$, $R := \{(u, \langle u, v \rangle), (\langle u, v \rangle, v) \in S \times S \mid (u, v) \in E \text{ or } u = v \in \{t_1, t_2\}\}$, $S_P := V \times V$, and $T := \{\langle t_1, t_1 \rangle\}$. We will show there are vertex-disjoint s_1 - t_1 and s_2 - t_2 paths in \mathcal{G} if and only if there is a drift partition $(S_i)_{0 \leq i \leq k}$ and a simple s_1 - t_1 path π such that $\pi \subseteq S_k$ and $s_2 \in S_k^-$. The result will then follow from Theorem 3.

Suppose we have a drift partition $(S_i)_{0 \leq i \leq k}$ with $s_2 \in S_k^-$ and a simple path $\pi = v_0 \langle v_0, v_1 \rangle \dots \langle v_{n-1}, v_n \rangle v_n$ with $v_0 = s_1, v_n = t_1$. Since the set $\{t_2, \langle t_2, t_2 \rangle\}$ is *trapping* in \mathcal{A} , *i.e.* all paths from vertices in the set visit only vertices from it, we can assume that $S_0 = \{t_2, \langle t_2, t_2 \rangle\}$. (Indeed, for any drift partition, one can obtain a new drift partition by moving any trapping set to a new lowest layer.) Now, using the assumption that t_2 is reachable from all $s \in V \setminus \{t_1, t_2\}$ one can show by induction that for all $0 \leq j < k$ and for all $\varrho = u_0 \in S_j$ there is a path $u_0 \dots u_m$ in \mathcal{G} with $u_m = t_2$ and $\varrho \subseteq S_{j+1}^-$. This implies that there is a s_2 - t_2 path π_2 in \mathcal{G} such that $\pi_2 \subseteq S_k^-$. It follows that π_2 is vertex disjoint with the s_1 - t_1 path $v_0 \dots v_n$ in \mathcal{G} .

Now, let us suppose that we have s_1 - t_1 and s_2 - t_2 vertex disjoint paths $\pi_1 = u_0 \dots u_n$ and $\pi_2 = v_0 \dots v_m$. Clearly, we can assume both π_1, π_2 are simple. We will construct a partition $(S_i)_{0 \leq i \leq m+1}$ and show that it is indeed a drift partition, that $u_0 \langle u_0, u_1 \rangle \dots \langle u_{n-1}, u_n \rangle u_n \subseteq S_{m+1}$, and $s_2 = v_0 \in S_{m+1}^-$. Let us set $S_0 := \{\langle v_{m-1}, v_m \rangle, v_m, \langle t_2, t_2 \rangle\}$, $S_i := \{\langle v_{m-i-1}, v_{m-i} \rangle, v_{m-i}\}$ for all $0 < i \leq m$, and $S_{m+1} := S \setminus \bigcup_{0 \leq i \leq m} S_i$. Since π_2 is simple, $(S_i)_{0 \leq i \leq m+1}$ is a partition of V . Furthermore, we have that $s_2 = v_0 \in S_{m+1}^-$, and $u_0 \langle u_0, u_1 \rangle \dots \langle u_{n-1}, u_n \rangle u_n \subseteq S_{m+1}$ since π_1 and π_2 are vertex disjoint. Thus, it only remains for us to argue that for all $0 \leq i \leq m+1$: for all $w \in S_i \cap S_N$ we have $wR \cap S_i^+ = \emptyset$, and for all $w \in S_i \cap V_N$ we have $wR \cap S_i^+ \neq \emptyset \implies wR \cap S_i^- \neq \emptyset$.

By construction of the S_i , we have that $eR \subseteq S_i$ for all $0 \leq i \leq m$ and all $e \in S_i \cap S_P$. Furthermore, for all $0 < i \leq m$, for all $x \in S_i \cap S_N = \{v_{m-i}\}$, there exists $y \in S_{i-1} \cap S_P = \{\langle v_{m-i}, v_{m-i+1} \rangle\}$ such that $(x, y) \in R$ — induced by $(v_{m-i}, v_{m-i+1}) \in E$ from π_2 . To conclude, we observe that since $S_0 = \{\langle v_{m-1}, v_m \rangle, v_m = t_2, \langle t_2, t_2 \rangle\}$ and $\{t_2, \langle t_2, t_2 \rangle\}$ is trapping in \mathcal{A} , the set t_2R is contained in S_0 . \square

6 Efficiently under-approximating the NWR

Although the full NWR cannot be efficiently computed for a given MDP, we can hope for “under-approximations” that are accurate and efficiently computable.

Definition 8 (Under-approximation of the NWR). Let $\mathcal{A} = (V, V_P, E, T)$ be a target arena and consider a relation $\preceq : V \times \mathcal{P}(V)$. The relation \preceq is an *under-approximation* of the NWR if and only if $\preceq \subseteq \preceq^*$.

We denote by \preceq^* the *pseudo transitive closure* of \preceq . That is, \preceq^* is the smallest relation such that $\preceq \subseteq \preceq^*$ and for all $u \in V, X \subseteq V$ if there exists $W \subseteq V$ such that $u \preceq W$ and $w \preceq^* X$ for all $w \in W$, then $u \preceq^* X$.

Remark 1. The empty set is an under-approximation of the NWR. For all under-approximations \preceq of the NWR, the pseudo transitive closure \preceq^* of \preceq is also an under-approximation of the NWR.

In [2], efficiently-decidable sufficient conditions for the NWR were given. In particular, those conditions suffice to infer relations such as those in the right MDP from Figure 3. We recall (Proposition 1) and extend (Proposition 2) these conditions below.

Proposition 1 (From [2]). Consider a target arena $\mathcal{A} = (V, V_P, E, T)$ and an under-approximation \preceq of the NWR. For all vertices $v_0 \in V$, and sets $W \subseteq V$ the following hold.

- (i) If there exists $S \subseteq \{s \in V \mid s \preceq W\}$ such that there exists no path $v_0 \dots v_n \in (V \setminus S)^*T$, then $v_0 \preceq W$.
- (ii) If $W = \{w\}$ and there exists $S \subseteq \{s \in V_P \mid w \preceq \{s\}\}$ such that $\mathbf{Win}_{S \cup T}^{v_0} \neq \emptyset$, then $w \preceq \{v_0\}$.

Proof Sketch. The main idea of the proof of item (i) is to note that S is visited before T . The desired result then follows from Lemma 1. For item (ii), we intuitively have that there is a strategy to visit T with some probability or visit W , where the chances of visiting T are worse than before. We then show that it is never worse to start from v_0 to have better odds of visiting T . \square

The above “rules” give an iterative algorithm to obtain increasingly better under-approximations of the NWR: from \preceq_i apply the rules and obtain a new under-approximation \preceq_{i+1} by adding the new pairs and taking the pseudo transitive closure; then repeat until convergence. Using the special cases from Section 3.2 we can obtain a nontrivial initial under-approximation \preceq_0 of the NWR in polynomial time.

The main problem is how to avoid testing all subsets $W \subseteq V$ in every iteration. One natural way to ensure we do not consider all subsets of vertices



Figure 4: Two target arenas with $T = \{fin\}$ are shown. Using Propositions 1 and 2 one can conclude that $p \sim q$ in both target arenas.

in every iteration is to apply the rules from Proposition 1 only on the successors of Protagonist vertices.

In the same spirit of the iterative algorithm described above, we now give two new rules to infer NWR pairs.

Proposition 2. Consider a target arena $\mathcal{A} = (V, V_P, E, T)$ and \preceq an under-approximation of the NWR.

- (i) For all $u \in V_N$, if for all $v, w \in uE$ we have $v \preceq \{w\}$ and $w \preceq \{v\}$, then $u \sim x$ for all $x \in uE$.
- (ii) For all $u, v \in V_P \setminus T$, if for all $w \in uE$ such that $w \preceq (uE \setminus \{w\})$ does not hold we have that $w \preceq vE$, then $u \preceq \{v\}$.

Proof Sketch. Item (i) follows immediately from the definition of Val. For item (ii) one can use the Bellman optimality equations for infinite-horizon reachability in MDPs to show that since the successors of v are never worse than the non-dominated successors of u , we must have $u \preceq \{v\}$. \square

The rules stated in Proposition 2 can be used to infer relations like those depicted in Figure 4 and are clearly seen to be computable in polynomial time as they speak only of successors of vertices.

7 Conclusions

We have shown that the never-worse relation is, unfortunately, not computable in polynomial time. On the bright side, we have extended the iterative polynomial-time algorithm from [2] to under-approximate the relation. In that paper, a prototype implementation of the algorithm was used to empirically show that interesting MDPs (from the set of benchmarks included in PRISM [18]) can be drastically reduced.

As future work, we believe it would be interesting to implement an exact algorithm to compute the NWR using SMT solvers. Symbolic implementations of the iterative algorithms should also be tested in practice. In a more theoretical direction, we observe that the planning community has also studied maximizing the probability of reaching a target set of states under the name of MAXPROB (see, *e.g.*, [17, 22]). There, online approximations of the NWR would make more sense than the under-approximation we have proposed here. Finally, one could define a notion of never-worse for finite-horizon or quantitative objectives.

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A Preliminaries

A.1 Definitions

Definition 9 (Memoryful strategies). A (*deterministic*) *strategy* σ in an MDP $\mathcal{M} = (Q, A, \delta, T)$ is a function $\sigma : (QA)^*Q \rightarrow A$.

A *finite-memory* strategy σ is a strategy that can be encoded as a deterministic *Mealy machine* $\mathcal{A} = (S, s_I, Q, A, \lambda_u, \lambda_o)$ where S is a finite set of (memory) states, s_I is the initial state, $\lambda_u : S \times Q \rightarrow S$ is the update function and $\lambda_o : S \times Q \rightarrow A$ is the output function. The machine encodes σ in the following sense: $\sigma(q_0 a_0 \dots q_n) = \lambda_o(s_n, q_n)$ where $s_0 = s_I$ and $s_{i+1} = \lambda_u(s_i, q_i)$ for all $0 \leq i < n$. We then say that \mathcal{A} *realizes* the strategy σ and that σ has *memory* $|S|$. In particular, strategies which have memory 1 are said to be *positional* (or *memoryless*).

A.2 Bellman equations

The following result [1, Theorem 10.100] about MDPs will be useful.

Lemma 7. Consider an MDP $\mathcal{M} = (Q, A, \delta, T)$. The vector $(x_p)_{p \in Q}$ with $x_p = \max_{\sigma} \mathbb{P}_{\mathcal{M}^{\sigma}}^p[\Diamond T]$ is the unique solution of the following equation system.

- If $p \in T$ then $x_p = 1$,
- if $p \in \mathbf{Z}_T$ then $x_p = 0$,
- otherwise

$$x_p = \max_{a \in A} \sum_{q \in Q} \delta(q|p, a) \cdot x_q.$$

B Proof of Lemma 1

Proof. Since $\mathbb{P}_{\mathcal{C}}^{q_0}[(Q \setminus U) \cup T] = 0$, then $q_0 \notin T$. Thus, by definition, we have

$$\mathbb{P}_{\mathcal{C}}^{q_0}[\Diamond T] = \sum_{n \geq 1} \left\{ \prod_{0 \leq i < n} \delta(q_i, q_{i+1}) \mid q_0 \dots q_n \in (Q \setminus T)^* T \right\}. \quad (3)$$

The set of runs that start at q_0 and reach T is equivalent to the union of: the set of runs that start at q_0 and stay in a set $S \subseteq Q$ until they reach T ; and the set of runs that start at q_0 and reach some state from $Q \setminus S$ before eventually reaching T . The measure of the runs from the first set is $\mathbb{P}_{\mathcal{C}}^{q_0}[S \cup T]$. Let us denote by $\mathbb{P}[\tau_{(Q \setminus S)} < \tau_T]$ the measure of the runs from the second set. Since in Markov chains we have that for all runs $p_0 \dots p_i \dots p_m$ the probabilities of the prefix $p_0 \dots p_i$ and the suffix $p_i \dots p_m$ are independent, we can rewrite (3) as follows.

$$\mathbb{P}_{\mathcal{C}}^{q_0}[\Diamond T] = \mathbb{P}_{\mathcal{C}}^{q_0}[(Q \setminus U) \cup T] + \mathbb{P}[\tau_U < \tau_T]$$

By assumption we have that the first summand is equal to 0, so

$$\mathbb{P}_{\mathcal{C}}^{q_0}[\Diamond T] = \mathbb{P}[\tau_U < \tau_T].$$

Also by assumption, we know that there are no runs of \mathcal{C} starting at q_0 and staying in $U \subseteq Q$ until reaching T . Hence, the set of runs from q_0 that reach some state in U and then eventually reach T is exactly the union, over all $u \in U$, of the sets of runs $q_0 \dots q_\ell \dots q_n \in Q^*$ that satisfy

- $\forall 0 \leq i < \ell : q_i \notin U$,
- $q_\ell = u$,
- $\forall \ell < j < n : q_j \notin T$, and
- $q_n \in T$.

Once more using the fact that events (*i.e.* transition probabilities) in a Markov chain are independent of the history (*i.e.* run prefixes) we can write the measure of the above set as follows.

$$\mathbb{P}[\tau_U < \tau_T] = \sum_{u \in U} \mathbb{P}_{\mathcal{C}}^{q_0} [(Q \setminus U) \cup u] \mathbb{P}_{\mathcal{C}}^u[\Diamond T]$$

This concludes the proof. \square

C Proof of Theorem 1

We will first prove that collapsing only the Protagonist vertices is safe.

Consider a target arena $\mathcal{A} = (V, V_P, E, T)$. Let \mathcal{C} be the target arena (S, S_P, R, U) where

- $S_P = \{\tilde{v} \mid \exists v \in V_P\}$,
- $S = V_N \cup S_P$,
- $U = \{\tilde{t} \mid \exists t \in T\}$, and
- $R = \{(\tilde{u}, v) \mid \exists u \in V_P \exists v \in V_N : (u, v) \in E \text{ and } vE \setminus \tilde{u} \neq \emptyset\} \cup \{u, \tilde{v}\} \mid \exists u \in V_N \exists v \in V_P : (u, v) \in E\}$.

For a family $\mu = (\mu_u \in \mathbb{D}(uE))_{u \in V_N}$ of full-support probability distributions we define $\lambda = (\lambda_u \in \mathbb{D}(uR))_{u \in V_N}$ as follows. For all $u \in V_N$ and all $\tilde{v} \in uR$ we have

$$\lambda_u(\tilde{v}) = \sum_{w \in \tilde{v}} \mu_u(w).$$

Proposition 3. For all families $\mu = (\mu_u \in \mathbb{D}(uE))_{u \in V_N}$ of full-support probability distributions and all $v \in V_P$ we have

$$\max_{\sigma} \mathbb{P}_{\mathcal{A}_{\mu}}^v[\Diamond T] = \max_{\sigma'} \mathbb{P}_{\mathcal{C}_{\lambda}}^{\tilde{v}}[\Diamond U],$$

where $U = \{\tilde{t} \mid t \in T\}$, and \mathcal{C} and λ are as defined above.

Proof. For convenience, let us assume that extremal-probability vertices have already been collapsed so that for all $u, v \in V_P$, if $u \sim v$ then u and v are not extremal-probability vertices. The argument for correctness of collapsing extremal-probability vertices is trivial. We will reduce correctness of collapsing NWR-equivalence classes to the correctness of collapsing end components.

Consider the following equation system defined for an arbitrary μ .

$$\text{For all } u \in V_P, x_u = \begin{cases} 1 & \text{if } u \in T \\ 0 & \text{if } u \in \mathbf{Z}_T \\ \max_{v \in uE} \sum_{w \in vE} \mu_v(w) \cdot x_w & \text{otherwise} \end{cases} \quad (4)$$

According to Lemma 7, the system has a unique solution assignment

$$x_u = \max_{\sigma} \mathbb{P}_{\mathcal{A}_{\mu}^{\sigma}}^u[\Diamond T] \quad (5)$$

for all $u \in V_P$. Now, by definition of the NWR we have that (5) is also a solution to the following system of equations. For all $u \in V_P$

$$\text{For all } u \in V_P, x_u = \begin{cases} 1 & \text{if } u \in T \\ 0 & \text{if } u \in \mathbf{Z}_T \\ \max(\{\sum_{w \in vE} \mu_v(w) \cdot x_w \mid v \in uE\} \cup \{x_y \mid y \in \tilde{u}\}) & \text{otherwise} \end{cases} \quad (6)$$

Observe that equation system (6) corresponds to the target arena \mathcal{C} we obtain by modifying \mathcal{A} as follows. For all $u \in V_P$ and for all $v \in \tilde{u}$ we add an edge from u to v (with an intermediate Nature vertex to preserve bipartiteness). It thus follows from Lemma 7 that the system has a unique solution assignment corresponding to the maximal reachability probability values of all Protagonist vertices from \mathcal{C} . We have already argued that (5) is a solution to the system (6). Hence, since μ was chosen arbitrarily, we conclude that for all $u \in V_P$

$$\max_{\sigma} \mathbb{P}_{\mathcal{A}_{\mu}^{\sigma}}^u[\Diamond T] = \max_{\sigma'} \mathbb{P}_{\mathcal{C}_{\mu'}^{\sigma'}}^u[\Diamond T],$$

where μ' is the same as μ for all Nature vertices from \mathcal{A} and for all newly added nature vertices w , μ'_w assigns 1 to the unique successor of w in \mathcal{C} .

We now remark that in \mathcal{C} we have that for all $u \in V_P$ the set \tilde{u} now forms an end component. Since collapsing end components preserves the desired value [8, 5] (and removing probability-1 self-loops clearly preserves this value too) the result follows. \square

We will now argue that collapsing NWR-equivalent Nature-owned vertices is also safe. The following result will be useful.

Lemma 8. Consider the target arena \mathcal{A} . For all $u, v \in V_N$ it holds that if $u \sim v$ and $u \neq v$ then $w \sim x$ for all $w, x \in (uE \cup vE)$.

Proof. We proceed by contradiction: suppose $u \sim v$, $u \neq v$, and that there exist $w, x \in (uE \cup vE)$ such that $w \not\sim x$. It follows from the latter that there are $w \in uE$ and $x \in vE$ such that $w \not\sim x$. Indeed, if this were not the case, then we would be able to conclude that $w \sim x$ for all $w, x \in (uE \cup vE)$ by using the transitivity of \sim . Without loss of generality, assume $\neg(w \preceq x)$.

We will now make use of the equation system (4) from the proof of Proposition 3 with μ the family that witnesses the fact that $\neg(w \preceq x)$, i.e. $\text{Val}^{\mu}(w) > \text{Val}^{\mu}(x)$. Recall that we have assumed $u \sim v$ and observe that

$$x_w = \max_{z \in wE} \text{Val}^{\mu}(z),$$

for all $w \notin (T \cup \mathbf{Z}_T)$ in the system (4). Now let μ' be a family of full-support probability distributions with $\mu'_u(w) = 1 - \varepsilon$, where $0 < \varepsilon < \frac{1}{2}(\text{Val}^\mu(w) - \text{Val}^\mu(x))$, and otherwise identical to μ . Since $u \sim v$ we have that $\text{Val}^{\mu'}(u) = \text{Val}^{\mu'}(v) = \text{Val}^\mu(v) = \text{Val}^\mu(u)$ and thus by, Lemma 7, the maximal reachability probability values for all states in \mathcal{A} are the same for both μ and μ' . Similarly, we can obtain a family μ'' such that $\mu''_v(x) = 1 - \varepsilon$ without changing the values of the states in \mathcal{A} . We thus have that

$$\begin{aligned} \text{Val}^{\mu''}(v) &\leq \text{Val}^{\mu''}(x) + \varepsilon \\ &< \text{Val}^{\mu''}(w) - \varepsilon \\ &\text{since } \text{Val}^{\mu''}(x) = \text{Val}^\mu(x) < \text{Val}^\mu(w) = \text{Val}^{\mu''}(w) \text{ and by choice of } \varepsilon \\ &\leq (1 - \varepsilon)\text{Val}^{\mu''}(w) \\ &\leq \text{Val}^{\mu''}(u) \end{aligned}$$

contradicting our initial assumption that $u \sim v$. \square

Let us now consider the following equation system — that corresponds to \mathcal{C} as defined above — for an arbitrary μ and corresponding λ as defined in the proof of Proposition 3.

$$\text{For all } \tilde{u} \in V_P, x_{\tilde{u}} = \begin{cases} 1 & \text{if } \tilde{u} \in U \\ 0 & \text{if } \tilde{u} \in \mathbf{Z}_U \\ \max_{v \in \tilde{u}R} \sum_{\tilde{w} \in vR} \lambda_v(\tilde{w}) \cdot x_{\tilde{w}} & \text{otherwise} \end{cases} \quad (7)$$

It follows from Lemma 8 that for all $v \in V_N$ such that $|\tilde{v}| > 1$ the set vR is a singleton $\{\tilde{w}\}$ and $\lambda_v(\tilde{w}) = 1$. It is then easy to see that system (7) is the same equation system that corresponds to \mathcal{B} as constructed for Theorem 1. The desired result thus follows from Proposition 3 and Lemma 7. \square

D Proof of Theorem 2

The argument to show correctness for the removal of an edge to a sub-optimal vertex is very similar to the proof of Theorem 1. We first consider the equation system of \mathcal{A} with an arbitrary μ . Then, we remove the variable from the maximum corresponding to the edge we would remove in the target arena. The only way in which the latter would result in the system not being the one corresponding to the reduced target arena \mathcal{B} would be if by removing the edge, we make some vertex become an element of \mathbf{Z}_T . That is, we have removed all its paths to T . We will argue that this is not possible if \sim -quotienting and self-loop removal have already been applied.

We will need to formalize the intuition that if $\mathcal{A} = \mathcal{A}_{/\sim}$ there are no cycles with probability 1 in \mathcal{A} .

Lemma 9. Consider a target arena $\mathcal{A} = (V, V_P, E, T)$ such that $\mathcal{A}_{/\sim} = \mathcal{A}$. For all families $\mu = (\mu_u \in \mathbb{D}(uE))_{u \in V_N}$ of full-support distributions, and all $(v, w) \in E \cap (V_P \times V_N)$ we have

$$\max_{\sigma} \sum_{x \in wE} \mu_w(x) \mathbb{P}_{\mathcal{A}_{\mu}^{\sigma}}^x[\diamond v] < 1.$$

Proof. Towards a contradiction, suppose that there is some μ , an edge (v, w) , and a strategy σ for which

$$\sum_{x \in wE} \mu_w(x) \mathbb{P}_{\mathcal{A}_\mu^\sigma}^x[\Diamond v] = 1.$$

Observe that from v , Protagonist can play w and reach wE almost surely. Also, modifying σ , to play w from v does not affect the probability of reaching v from any $x \in wE$. Let σ' denote the modified strategy. It should be clear that σ' is such that for all $p, q \in \{v\} \cup wE$

$$\mathbb{P}_{\mathcal{A}_\mu^{\sigma'}}^p[\Diamond q] = 1.$$

Let $S := \{v\} \cup wE$. If $S = \{v\}$ then we must have a probability-1 self-loop on v , which contradicts the fact that $\mathcal{A}_{/\sim} = \mathcal{A}$. Thus, $|S| \geq 2$. It follows that S is a non-trivial end component and by Lemma 3, also an NWR-equivalence class. This, again, contradicts our initial assumptions. \square

We are now ready to prove our claim.

Proof of Theorem 2. We will only argue that w in \mathcal{B} is not an element of \mathbf{Z}_T .

Let $Y := wE \setminus \{x\}$. Observe wE is non-empty. In fact, it must include at least x and some other vertex so that $x \preceq Y$ holds. Thus, since $\mathcal{A}_{/\sim} = \mathcal{A}$, we have that $w \notin \mathbf{Z}_T$.

Towards a contradiction, let us suppose that w in $\mathcal{B} = (V, V_P, E \setminus (w, x), T)$ is an element of \mathbf{Z}_T . It follows that for all $s \in S := \{s \in V_P \mid \exists y \in Y : (y, s) \in E\}$ in \mathcal{A} every s - T path goes through w . Hence, by Lemma 1, for all μ and for all σ we have

$$\mathbb{P}_{\mathcal{A}_\mu^\sigma}^s[\Diamond T] = \mathbb{P}_{\mathcal{A}_\mu^\sigma}^s[(Q \setminus \{w\}) \cup w] \mathbb{P}_{\mathcal{A}_\mu^\sigma}^w[\Diamond T]. \quad (8)$$

Let us now fix arbitrary μ and $y \in Y$ such that

$$\text{Val}^\mu(w) = \sum_{z \in yE} \mu_y(z) \text{Val}^\mu(z).$$

(The existence of such a μ and y is guaranteed by Lemma 7 and the fact that $x \preceq Y$.) We remark that $\text{Val}^\mu(x) \leq \text{Val}^\mu(y)$. By definition of $\text{Val}^\mu(\cdot)$ we have that

$$\text{Val}^\mu(y) = \sum_{z \in yE} \mu_y(z) \text{Val}^\mu(z).$$

Together with Equation (8) we obtain

$$\text{Val}^\mu(y) = \max_{\sigma} \left(\mathbb{P}_{\mathcal{A}_\mu^\sigma}^w[\Diamond T] \cdot \sum_{z \in yE} \mu_y(z) \mathbb{P}_{\mathcal{A}_\mu^\sigma}^z[\Diamond w] \right).$$

Finally, using Lemma 9, we conclude that $\text{Val}^\mu(y) < \text{Val}^\mu(w)$, contradicting our assumptions. \square

E Proof of Lemma 3

Proof. Let σ be a strategy maximizing the value $\mathbb{P}_{\mathcal{M}\sigma}^q[\Diamond T]$. We can then construct a (finite-memory) strategy which, from p ensures $\mathbb{P}_{\mathcal{M}\sigma'}^p[S \cup q] = 1$ and from q onwards behaves as σ . If $q \in T$ then we are done. Otherwise, Lemma 1 implies that $\mathbb{P}_{\mathcal{M}\sigma'}^p[\Diamond T] = \mathbb{P}_{\mathcal{M}\sigma}^q[\Diamond T]$. Furthermore, we know that memoryless strategies suffice for reachability [19, 1], so σ' can be replaced by a memoryless strategy. \square

F Proof of Lemma 5

Let us denote by λ the empty word.

Definition 10 (Word simplification). For an alphabet Σ , we define the *word simplification function* $s : \Sigma^* \rightarrow \Sigma^*$ as follows.

- $s(\lambda) := \lambda$
- $\forall w \in \Sigma^*, \forall a \in \Sigma :$
 - If $a \notin s(w)$ then $s(wa) := s(w)a$,
 - If $a \in s(w)$ then $s(wa)$ is the (shortest) prefix of $s(w)$ that ends with a .

Lemma 10. Let w be a word. $s(w)$ is repetition-free; w and $s(w)$ have the same starting and ending letters; a letter occurs in $s(w)$ only if it occurs in w ; a two-letter word xy occurs as a factor in $s(w)$ only if it occurs in w .

Proof. By induction. The base case is clear. For the inductive case, let $w \in \Sigma^*$ and let $a \in \Sigma$.

First case, $a \notin s(w)$. So $s(wa) = s(w)a$ is repetition-free like $s(w)$ is by IH, and wa and $s(wa)$ share the same ending letter a . Since w and $s(w)$ share the same starting letter by IH, so do wa and $s(wa) = s(w)a$. (It is a if $w = \lambda$). Let $b \in s(wa)$. If $b = a$, the claim clearly holds; if $b \neq a$ then $b \in s(w)$, so $b \in w$ by IH, and $b \in wa$. Let xy occur as a two-letter factor in $s(wa)$. If $y \neq a$ then xy actually occurs in $s(w)$, so by IH it occurs in w , and thus in wa ; if $y = a$ then $s(w)$ ends with x , and by IH so do w , so wa ends with xy .

Second case, $a \in s(w)$. Since $s(wa)$ is a prefix of $s(w)$, it is also repetition-free, and $s(wa)$ ends with a by definition. As a prefix, $s(wa)$ starts with the same letter as $s(w)$, *i.e.* the same letter as w by IH, *i.e.* the same as wa . If $b \in s(wa)$ then $b \in s(w)$, so $b \in w$ by IH, and $b \in wa$. If xy occurs in $s(wa)$, it also occurs in its extension $s(w)$, and in w by IH, and finally in wa . \square

Lemma 11. For all paths γ , $s(\gamma)$ is a simple path starting (ending) with the same vertex as γ . It only visits vertices that are already visited in γ , and it only takes edges that are already taken in γ .

Proof. By Lemma 10 the word $s(\gamma)$ is repetition-free, and it starts and ends like γ . Again by Lemma 10, if two vertices xy occur consecutively in $s(\gamma)$, they also occur consecutively and in the same order in γ , *i.e.* $(x, y) \in E$. Therefore $s(\gamma)$ is a path, and moreover it takes only edges that are already taken in γ . \square

We can now proceed with the proof of the Lemma.

Proof of Lemma 5. Since there is a path from x to T by assumption, the set L of all the *simple* paths from x to T is non-empty. Let \mathcal{T} be the prefix closure of L , and let \mathcal{T}' be the set of the μ -non-decreasing paths in \mathcal{T} , so that $\mathcal{T}' \subseteq \mathcal{T}$. The set \mathcal{T} is a tree by prefix-closure construction, and \mathcal{T}' is a tree since the μ -non-decreasing paths are closed by taking prefixes. Moreover, the elements of \mathcal{T} (and thus \mathcal{T}') are simple paths since the prefixes of simple paths are again simple paths. Let L' be the prefix-wise maximal paths in \mathcal{T}' . The claimed lemma amounts to $L \cap L' \neq \emptyset$, equivalently $L \cap \mathcal{T}' \neq \emptyset$. Also note that a path in \mathcal{T} is in L if and only if it ends with T , so $L \cap \mathcal{T}' \neq \emptyset$ if and only if some path in \mathcal{T}' ends with T .

Towards a contradiction, let us assume that $L \cap L' = \emptyset$. We can thus let M consist of the one-vertex extensions in \mathcal{T} of the paths in L' . More formally, a path $\gamma \in \mathcal{T}$ is in M if and only if there exists a vertex y and a path $\beta \in L'$ such that $\beta y = \gamma$. Clearly, $\mathcal{T}' \cap M = \emptyset$ by prefix-wise maximality of L' within \mathcal{T}' . So, the elements in M are the least *not* μ -non-decreasing paths in \mathcal{T} . This implies that

$$\forall \gamma y z \in M, \quad \text{Val}^\mu(z) < \text{Val}^\mu(y) \quad (9)$$

(where γ is a path and y, z are vertices), since $\gamma y \in L'$. Let B' (resp. B) be the ending vertices of the paths in L' (resp. M). Let us proceed with a few remarks:

- Remark 2.*
1. x occurs as the first vertex of all paths in \mathcal{T} , and only as first vertex, otherwise the paths would not be simple. So $x \notin B$, since x (as a path) is clearly in \mathcal{T}' and since $\mathcal{T}' \cap M = \emptyset$.
 2. Every path in L has a prefix in L' (*i.e.* its longest μ -non-decreasing prefix), and it has a prefix in M too, since $L \cap L' = \emptyset$. So every path in L visits B .
 3. Every path from x to T visits B : indeed, if a path γ from x to T does not visit B , neither does $s(\gamma)$ by Lemma 11. Again by Lemma 11, $s(\gamma)$ is a simple path starting at x and ending at T , so $s(\gamma) \in L$, so it must visit B by Remark 2.2, contradiction

Let us now prove that for all paths $\gamma \in \mathcal{T}'$ starting with x , and for all paths β , if $\gamma\beta$ is a path ending with T , then β visits B (even if $\gamma\beta \notin \mathcal{T}$). We proceed by induction on γ . Base case, $\gamma = x$ is a one-vertex path: since $x \notin B$ by Remark 2.1, β visits B for all paths $x\beta$ ending with T , by Remark 2.3.

For the inductive case, let $\gamma'y \in \mathcal{T}'$ and let us assume that the claim holds for the prefixes of γ' . Let $y\beta'$ be a path where β' ends with T . By Lemma 11, $s(y\beta')$ can be decomposed as $y\beta$ for some β . By Lemma 11 again, every vertex occurring in β also occurs in β' , and $y\beta$ is a simple path ending in T . Let us make a case disjunction. First case, $\gamma'y\beta$ is simple, so it is in L , and it has a prefix in M by Remark 2.2. Since $\mathcal{T}' \cap M = \emptyset$, this prefix is a proper extension of $\gamma'y \in \mathcal{T}'$, so β visits B . Second case, $\gamma'y\beta$ is not simple. Since $\gamma'y$ and $y\beta$ are simple, some vertex z in $y\beta$ occurs already in γ' , hence the following decompositions: $\gamma' = \gamma_\ell z \gamma_r$ and $y\beta = \beta_\ell z \beta_r$. The path $\gamma_\ell z$ is in \mathcal{T}' , the path $\gamma_\ell z \beta_r$ ends with T , so by induction hypothesis β_r visits B , and so does β , and so does β' again by Lemma 11. The induction is thus complete.

On the one hand, let $\gamma y \in L'$ be such that $\text{Val}^\mu(y) = \max\{\text{Val}^\mu(z) \mid z \in B'\}$. So $\text{Val}^\mu(z) < \text{Val}^\mu(y)$ for all $z \in B$ by (9).

On the other hand, let σ_0 be an arbitrary positional strategy for Protagonist in the MDP. Every path from y to T visits B , as proved by induction above, which also holds in the graph restricted by σ_0 , so $\mathbb{P}_{\mathcal{A}_\mu^{\sigma_0}}^y[(V \setminus B) \cup T] = 0$. Then Lemma 1 implies

$$\begin{aligned} \mathbb{P}_{\mathcal{A}_\mu^{\sigma_0}}^y[\Diamond T] &= \sum_{z \in B} \mathbb{P}_{\mathcal{A}_\mu^{\sigma_0}}^y[(V \setminus B) \cup z] \mathbb{P}_{\mathcal{A}_\mu^{\sigma_0}}^z[\Diamond T] \\ &\leq \sum_{z \in B} \mathbb{P}_{\mathcal{A}_\mu^{\sigma_0}}^y[(V \setminus B) \cup z] \max_{\sigma} \mathbb{P}_{\mathcal{A}_\mu^{\sigma_0}}^z[\Diamond T] \\ &= \sum_{z \in B} \mathbb{P}_{\mathcal{A}_\mu^{\sigma_0}}^y[(V \setminus B) \cup z] \text{Val}^\mu(z) \\ &\leq \sum_{z \in B} \mathbb{P}_{\mathcal{A}_\mu^{\sigma_0}}^y[(V \setminus B) \cup z] \max_{b \in B} \text{Val}^\mu(b) \\ &\leq \max_{b \in B} \text{Val}^\mu(b), \text{ since } \sum_{z \in B} \mathbb{P}_{\mathcal{A}_\mu^{\sigma_0}}^y[(V \setminus B) \cup z] \leq 1. \end{aligned}$$

Therefore, $\text{Val}^\mu(y) = \max_{\sigma} \mathbb{P}_{\mathcal{A}_\mu^{\sigma}}^y[\Diamond T] \leq \max_{b \in B} \text{Val}^\mu(b)$. This contradicts the above claim that $\text{Val}^\mu(z) < \text{Val}^\mu(y)$ for all $z \in B$. \square

G Proof of Proposition 1

The following observation regarding convex combinations will be useful.

Lemma 12. Consider a finite set of values $N \subseteq \mathbb{Q}$ and a probability distribution $\delta \in \mathbb{D}(N)$. There exist $n, m \in N$ such that

- $\exists \underline{m} \in N : \underline{m} \leq \sum_{n \in N} \delta(n) \cdot n$
- $\exists \overline{m} \in N : \sum_{n \in N} \delta(n) \cdot n \leq \overline{m}$

Proof of Proposition 1. Let us start with item (i). From Lemma 1 we have that

$$\mathbb{P}_{\mathcal{A}_\mu^{\sigma}}^{v_0}[\Diamond T] = \sum_{s \in S} \mathbb{P}_{\mathcal{A}_\mu^{\sigma}}^{v_0}[(V_P \setminus S) \cup s] \mathbb{P}_{\mathcal{A}_\mu^{\sigma}}^s[\Diamond T]$$

for all families μ of full-support distributions and strategies σ . It follows from Lemma 12 that there is some $s \in S$ such that

$$\mathbb{P}_{\mathcal{A}_\mu^{\sigma}}^{v_0}[\Diamond T] \leq \mathbb{P}_{\mathcal{A}_\mu^{\sigma}}^s[\Diamond T].$$

By definition, this means that $v_0 \trianglelefteq S$. Hence, by choice of S and Remark 1 we have $v_0 \trianglelefteq W$.

For item (ii), let us assume, without loss of generality, that the vertices from $T \cup \mathbf{Z}_T$ are all sinks. We thus have that there is a strategy σ such that in there is no path from v_0 to \mathbf{Z}_T without first visiting $T \cup S$. We will also assume that σ , after visiting any vertex from $S \cup T$ starts playing optimally in order to maximize the probability of visiting T . (This may require memory.) We have from Lemma 1 that

$$\mathbb{P}_{\mathcal{A}_\mu^{\sigma}}^{v_0}[\Diamond \mathbf{Z}_T] = \sum_{s \in S \cup T} \mathbb{P}_{\mathcal{A}_\mu^{\sigma}}^{v_0}[V_P \setminus (T \cup S) \cup s] \mathbb{P}_{\mathcal{A}_\mu^{\sigma}}^s[\Diamond \mathbf{Z}_T]$$

for all full-support distribution families μ . Since all the vertices from T are sinks $\mathbb{P}_{\mathcal{A}_\mu^s}[\Diamond \mathbf{Z}_T] = 0$ for all $s \in T$. Therefore, we can rewrite the above as

$$\mathbb{P}_{\mathcal{A}_\mu^\sigma}^{v_0}[\Diamond \mathbf{Z}_T] = \sum_{s \in S} \mathbb{P}_{\mathcal{A}_\mu^\sigma}^{v_0}[V_P \setminus (T \cup S) \cup s] \mathbb{P}_{\mathcal{A}_\mu^\sigma}^s[\Diamond \mathbf{Z}_T].$$

From Lemma 12 we get that there is some $s \in S$ such that

$$\begin{aligned} \mathbb{P}_{\mathcal{A}_\mu^\sigma}^{v_0}[\Diamond \mathbf{Z}_T] &\leq \mathbb{P}_{\mathcal{A}_\mu^\sigma}^s[\Diamond \mathbf{Z}_T] \\ \iff 1 - \mathbb{P}_{\mathcal{A}_\mu^\sigma}^{v_0}[\Diamond T] &\leq 1 - \mathbb{P}_{\mathcal{A}_\mu^\sigma}^s[\Diamond T] \quad \sigma \text{ is maximizing the prob. of reaching } T \\ \iff \mathbb{P}_{\mathcal{A}_\mu^\sigma}^s[\Diamond T] &\leq \mathbb{P}_{\mathcal{A}_\mu^\sigma}^{v_0}[\Diamond T]. \end{aligned}$$

By choice of σ , this implies that $s \leq \{v_0\}$ (recall that memoryless strategies suffice to maximize the probability of reaching a target set of states). Then, by choice of $S \ni s$ we have that $w \leq \{v_0\}$. \square

H Proof of Proposition 2

Proof. Item (i) follows from the definition of Val .

For item (ii), we suppose the assumptions hold. Hence, by definition of \leq , we have that for all families μ of full-support distributions

$$\max_{w \in uE} \sum_{x \in wE} \mu_w(x) \text{Val}^\mu(x) \leq \max_{y \in vE} \sum_{z \in yE} \mu_y(z) \text{Val}^\mu(z).$$

The result thus follows from Lemma 7. \square