Separating regular languages by piecewise testable and unambiguous languages

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Separation problem

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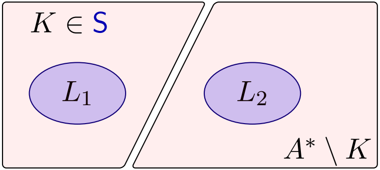

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- Can we compute a separator?
- What is the complexity of the decision problem?
  And of computing a separator?

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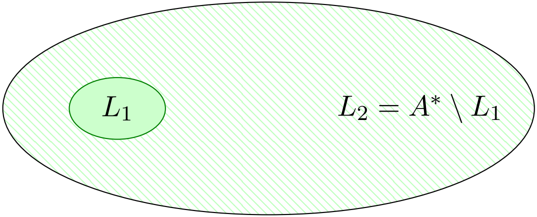

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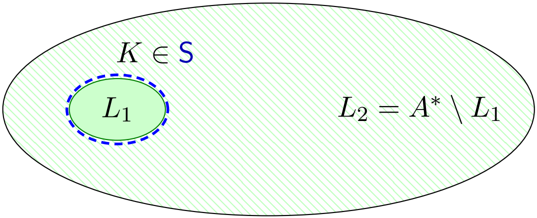

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- Piecewise testable languages: definable by $\mathcal{B}\Sigma_1(<)$ formulas (recall previous talk).
- Unambiguous languages: definable by $\mathsf{FO}^2(<)$ formulas.

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Membership is decidable for both $\mathcal{B}\Sigma_1(<)$ and $\mathsf{FO}^2(<)$ languages.

Simon '75
Schützenberger '76, Thérien, Wilke '98

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- Eg. $(a^2)^*$ and $b(b^2)^*$ have no smallest $\mathcal{B}\Sigma_1(<)$-separator, while $a^* \setminus \{a, a^3, a^5\}$ is the smallest $\mathcal{B}\Sigma_1(<)[6]$-separator.

Quantifier depth provides indexed equivalence relations for $\mathcal{B}\Sigma_1(<)$ resp. $\mathsf{FO}^2(<)$ languages:

Computing smallest separator for S$[k]$

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L is a $\mathcal{B}\Sigma_1(<)$ resp. $\mathsf{FO}^2(<)$ language

$$\Leftrightarrow$$

L is a union of $\sim_k$-classes for some $k \in \mathbb{N}$.

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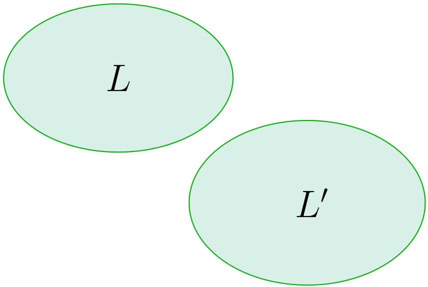
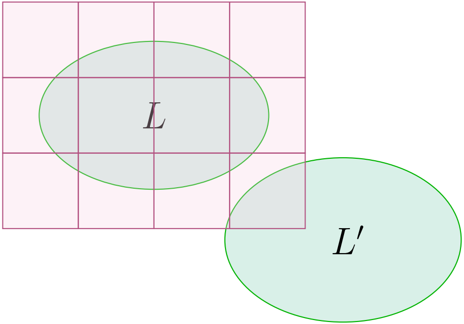


Increasing k refines the smallest potential separator:

$[L]_{\sim_1}$



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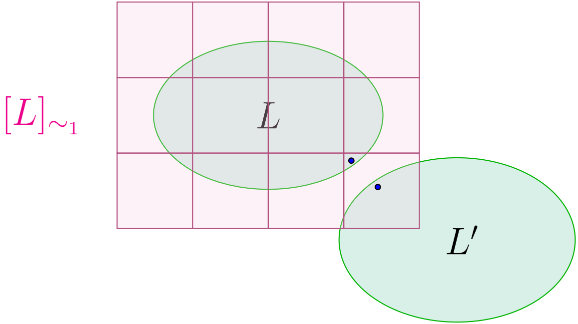

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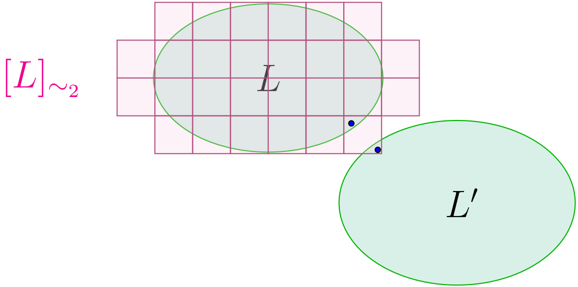

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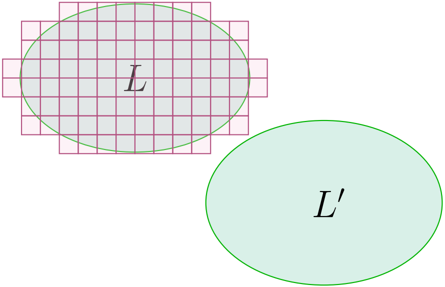

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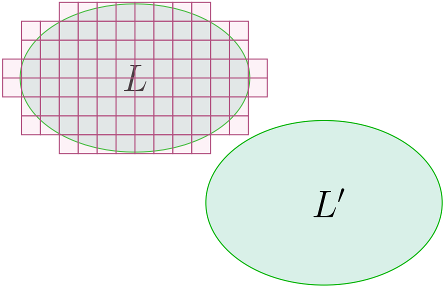

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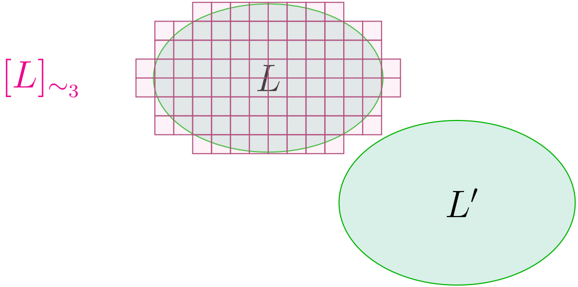

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From which level of refinement can we conclude non-separability?

Testing non-separability

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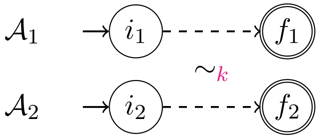
Abstraction: tuples (i_1, f_1, i_2, f_2)



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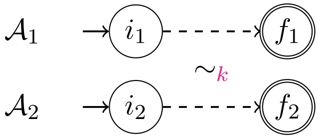

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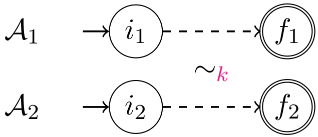

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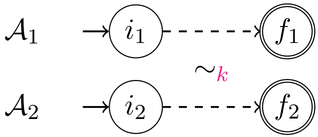

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- Limit behaviour?

  We can compute K such that

$$\text{limit} = \varnothing \quad \Leftrightarrow \quad \{K\text{-witnesses}\} = \varnothing$$

1. Verify that the languages are not separable by any language of a sufficient, computable, index K.
2. Find a pattern in the automata producing k-witnesses for arbitrarily large k. (For $\mathcal{B}\Sigma_1(<)$: same as in previous talk)

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   Patterns in the automata, independently of the bound K, provide

   - Better complexity
   - A yes/no answer for separability
   - But no separator

Main result

Theorem

For S $= \mathcal{B}\Sigma_1(<)$, $\mathsf{FO}^2(<)$, *we can compute* $K \in \mathbb{N}$ *such that TFAE*

i. L_1, L_2 *are* S*-separable.*
ii. L_1, L_2 *are* $\mathsf{S}[K]$*-separable.*
iii. $[L_1]_{\sim_K}$ *separates* L_1 *from* L_2.
iv. $\mathcal{A}_1, \mathcal{A}_2$ *do not contain a pattern witnessing non-separability.*

Main result

Condition iv. yields the following complexity results:

Theorem

Given NFAs $\mathcal{A}_1, \mathcal{A}_2$, one can determine whether the languages $L(\mathcal{A}_1)$ and $L(\mathcal{A}_2)$ are

- $\mathcal{B}\Sigma_1(<)$*-separable in* **PTIME**,
- $\mathsf{FO}^2(<)$*-separable in* **EXPTIME**,

with respect to $|Q_1|$,$|Q_2|$,$|A|$.

Future work

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- Consider other classes S

Thank you

$L(\mathcal{A}_1)$ and $L(\mathcal{A}_2)$ are not $\mathcal{B}\Sigma_1(<)$-separable
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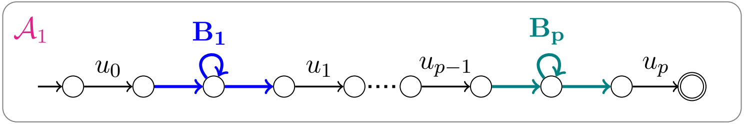

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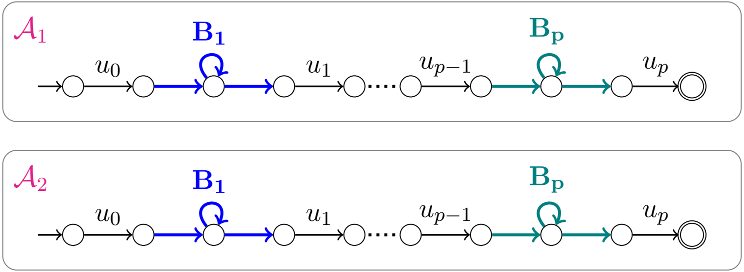

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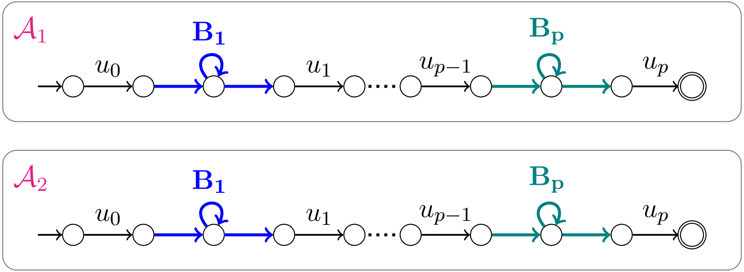

This can be determined in PTIME$(|Q_1|, |Q_2|, |A|)$.

Detecting forbidden patterns

One can determine in $\mathsf{PTIME}(|Q_1|, |Q_2|, |A|)$ whether $\exists\ (\vec{u}, \vec{B})$ such that both $\mathcal{A}_1$ and $\mathcal{A}_2$ have a $(\vec{u}, \vec{B})$-path.

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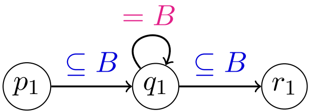

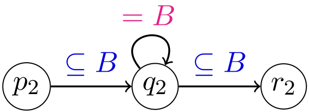

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- This is in PTIME: iteratively use Tarjan's algorithm.

Detecting forbidden patterns

- If B exists, $B \subseteq C_1 \stackrel{\text{def}}{=} \mathsf{alph\_scc}(q_1, \mathcal{A}_1) \cap \mathsf{alph\_scc}(q_2, \mathcal{A}_2)$ (LINEAR)
- Restrict the automata to alphabet C_1, and repeat the process:

$$C_{i+1} \stackrel{\text{def}}{=} \mathsf{alph\_scc}(q_1, \mathcal{A}_1 \restriction_{C_i}) \cap \mathsf{alph\_scc}(q_2, \mathcal{A}_2 \restriction_{C_i}).$$

- After $n \leqslant |A|$ iterations, $C_n = C_{n+1}$.
  - If $C_n = \varnothing$, the answer is no.
  - If $C_n \neq \varnothing$, it is the maximal possible B with $(= B)$-loops around q_1, q_2.
- Then, determine the remaining paths. (LINEAR)
- Overall LINEAR algorithm.