Computing Moore-Penrose Inverses of Ore Polynomial Matrices

Yang Zhang

Department of Mathematics, University of Manitoba,
Winnipeg, MB R3T 2N2, Canada
yang.zhang@umanitoba.ca

Abstract. In this paper we define and discuss the generalized inverse and Moore-Penrose inverse for Ore polynomial matrices. Based on GCD computations and Leverrier-Faddeeva method, some fast algorithms for computing these inverses are constructed, and the corresponding Maple package including quaternion case is developed.

Keywords: Moore-Penrose inverse, Ore polynomial matrices, Quaternion.

1 Introduction

Ore polynomial matrices are matrices over Ore algebras (including differential operators and difference operators). It has a long research history, at least dated back to Jacobson's seminal work in 1940s. In past ten years, Ore matrices have attracted more and more people in computer algebra area, and many important properties of Ore matrices have been discussed by using symbolic computation methods, for example, various fast algorithms for computing Hermite forms and Smith forms, fraction-free algorithms for computing Popov forms (see, for example, [2] and [8]).

It is well-known that the generalized inverse of matrices over commutative rings (or fields), especially the Moore-Penrose inverse, play important roles in matrix theory and have applications in many areas: solving matrix equations, statistics, engineering, etc. This motivates us to consider the generalized inverse of Ore polynomial matrices.

First we define the generalized inverse and the Moore-Penrose inverse for Ore polynomial matrices, and prove some basic properties including uniqueness. Unlike the commutative case, the generalized inverse for a given Ore polynomial matrix may not exist. We use blocked matrices and greatest common right (left) divisor computations to give some sufficient and necessary conditions for Ore polynomial matrices to have the generalized inverses and the Moore-Penrose inverses. Moreover when these inverses exist, we construct algorithms to compute them. In quaternion case, we define generalized characteristic polynomials and give an analogy version of Leverrier-Faddeeva algorithm.

All our algorithms are implemented in the symbolic programming language Maple, and tested on examples. Our aim is to develop a Maple package for computing the generalized inverses and the Moore-Penrose inverse of Ore polynomial

H. Hong and C. Yap (Eds.): ICMS 2014, LNCS 8592, pp. 484–491, 2014.

matrices, in particular, for quaternion case. To our best knowledge, it is the first Maple package in this direction.

2 Definitions, Properties and Algorithms

Let D be a division ring (or called skew field) and $\sigma : D \to D$ be an automorphism of D. A σ-derivation $\delta : D \to D$ is a mapping satisfying: for any $a, b \in D$,

$$\delta(a+b) = \delta(a) + \delta(b), \quad \delta(ab) = \sigma(a)\delta(b) + \delta(a)b.$$

The Ore polynomial ring $R = D[x; \sigma, \delta]$ is defined as the set of usual polynomials in $D[x]$ under the usual addition, but with multiplication defined by

$$xa = \sigma(a)x + \delta(a), \quad \text{for any} \;\; a \in D.$$

The Ore polynomial matrices $R^{m\times n}$ will be the set of all $m \times n$ matrices with Ore polynomial entries. More properties can be found in, for example, Jacobson[12] and Lam[14].

To consider Moore-Penrose inverses, we assume that D has an involution "$*$", that is, "$*$" is an anti-automorphism of order 1 or 2 on D. It is easy to check that "$*$" can be extended to $D[x; \sigma, \delta]$ as an involution as follows. For any $f = \sum_{i=0}^{n} a_i x^i \in R = D[x; \sigma, \delta]$, we define

$$f^* = \sum_{i=0}^{n} a_i^* x^i.$$

Furthermore, "$*$" can be extended to Ore polynomial matrices $R^{m\times n}$ in a common way. Hence for any $A \in R^{m\times n}$ we can define the involution of transpose A^T of A as: $A^T = (A_{ij}^*)^T \in R^{n\times m}$.

Next we give the definition of Moore-Penrose inverses, and refer the reader to [3] and [21] for details in commutative case.

Definition 1. *A matrix $A^\dagger \in R^{n\times m}$ is called a Moore-Penrose inverse of $A \in R^{m\times n}$ if $A^\dagger$ satisfies:*

$$(i)\ AA^\dagger A = A \quad (ii)\ A^\dagger AA^\dagger = A^\dagger \quad (iii)\ (AA^\dagger)^T = AA^\dagger \quad (iv)\ (A^\dagger A)^T = A^\dagger A.$$

People are also interested in the matrices which only satisfy some of the above equations. If a matrix $A^{\{1\}}$ satisfies (i), then we say that $A^{\{1\}}$ is a $\{1\}$-inverse of A or a generalized inverse of A. Similarly, $\{1,2\}$-inverse (or more generally $\{i,j,k\}$-inverse) can be defined.

Recall that $A \in D^{m\times m}$ is unitary if $AA^T = A^T A = I_m$. One can prove the following elementary properties that will be often used throughout this paper.

Theorem 1. *Let $A \in R^{m\times n}$ and $B \in R^{n\times l}$. Then*

(i) $(AB)^T = B^T A^T$ *and* $AA^T = (AA^T)^T$.
(ii) *If A has a Moore-Penrose inverse $A^\dagger$, then* $(A^T)^\dagger = \left(A^\dagger\right)^T$, $A^\dagger \left(A^\dagger\right)^T A^T = A^\dagger = A^T \left(A^\dagger\right)^T A^\dagger$ *and* $A^\dagger A A^T = A^T = A^T A A^\dagger$.
(iii) *If A has a Moore-Penrose inverse $A^\dagger$, then $A^\dagger$ is unique.*
(iv) *Let A have the Moore-Penrose inverse $A^\dagger$. If $U \in D^{m\times m}$ is a unitary matrix, then* $(UA)^\dagger = A^\dagger U^T$.

Rao condition is a common assumption in commutative case. Now we can define it over division ring D with an involution "$*$": for any $a_1, \dots, a_s \in R$,

$$a_1 = a_1 \cdot a_1^* + \cdots + a_s \cdot a_s^* \quad \text{implies} \quad a_2 = \cdots = a_s = 0.$$

Clearly, Ore polynomial ring $R = D[x;\sigma,\delta]$ also satisfies Rao condition, in particular, when $D = \mathbb{C}(x)$ rational function field over complex number field $\mathbb{C}$ or $D = \mathbb{H}(x)$ quotient skew field over quaternion polynomial ring $\mathbb{H}[x]$.

Throughout this paper, we assume that D is a division ring with an involution "$*$" which satisfies Rao condition, and $R = D[x;\sigma,\delta]$ is an Ore polynomial ring over D.

Note that we require that $A^\dagger$ (and other generalized inverses) must be in $R^{n\times m}$, not in matrices over its quotient skew field $Q(R)$. Thus unlike the matrices over fields, the Moore-Penrose inverse for some Ore polynomial matrices might not exist.

In this paper, we first discuss the existence of the Moore-Penrose inverses including using Jacobson normal forms. Some interesting results in commutative case (see [3] and [22]) can be extended to Ore polynomial matrices. Here we just list two interesting results.

Theorem 2. *If $E \in R^{m\times m}$ is a symmetric projection, that is, $E = E^2 = E^T$, then $E \in D^{m\times m}$.*

Theorem 3. *Let $A \in R^{m\times n}$. Then A has the Moore-Penrose inverse $A^\dagger$ if and only if* $A = U \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}$ *with $U \in D^{m\times m}$ unitary and $A_1A_1^T + A_2A_2^T$ a unit in $R^{r\times r}$ with $r \le \min\{m, n\}$. Moreover,*

$$A^\dagger = \begin{bmatrix} A_1^T \left(A_1A_1^* + A_2A_2^T\right)^{-1} & 0 \\ A_2^T \left(A_1A_1^T + A_2A_2^T\right)^{-1} & 0 \end{bmatrix} U^T.$$

Next we outline how to use computing greatest common right (and left) divisors (gcrd) methods to find generalized inverses. First, from Section 3.7 of Jacobson[12], we know that for $a, b \in R$, not both zero, we can compute the

GCRD $g = \text{gcrd}(a,\ b)$, and $u, v \in R$ such that $ua + vb = g$, and $s, t \in R$ such that $sa = -tb = \text{lclm}(a, b)$. Furthermore we have

$$U = \begin{bmatrix} u & v \\ s & t \end{bmatrix} \in R^{2\times 2} \quad \text{such that} \quad U \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} g \\ 0 \end{bmatrix}.$$

Similarly, using greatest common left divisors and least common right multiplies, we can find a $V \in R^{2\times 2}$ such that $\begin{bmatrix} a & b \end{bmatrix} V = \begin{bmatrix} d & 0 \end{bmatrix}$. For general matrices, using the above fact, we can prove the following theorem by induction:

Theorem 4. *Given $A = (a_{ij}) \in R^{m\times n}$, let $r, c \in R$ be the gcrd of the entries on the first column and the gcld of the entries on the first row of A, respectively, that is, $r = gcrd(a_{11}, \ldots, a_{m1})$ and $c = gcld(a_{11},\ a_{12}, \ldots, a_{n1})$. Then there exist unimodular matrices $U \in R^{m\times m},\ V \in R^{n\times n}$ such that*

$$UA = \begin{bmatrix} r & * \\ \mathbf{0} & * \end{bmatrix}, \quad AV = \begin{bmatrix} c & \mathbf{0} \\ * & * \end{bmatrix}.$$

Note that R is noetherian, a right inverse of $A \in R^{n\times n}$ is also a left inverse of A. The following theorem provides a recursive method to compute $\{1\}$-inverse.

Theorem 5. *Suppose that $0 \neq a \in R, \boldsymbol{b} = (b_1, \ldots, b_n) \in R^{1\times n}$ and $A \in R^{m\times n}$.*

(i) If $\begin{bmatrix} a & \boldsymbol{b} \\ \mathbf{0} & A \end{bmatrix} \in R^{(m+1)\times(n+1)}$ has a $\{1\}$-inverse, then $gcld(a, b_1, \ldots, b_n) = 1$.

(ii) If $\begin{bmatrix} a & \mathbf{0} \\ \mathbf{0} & A \end{bmatrix} \in R^{(m+1)\times(n+1)}$ has a $\{1\}$-inverse, then $a \in D$ and A has a $\{1\}$-inverse.

Now we give an algorithm to compute $\{1\}$-inverse. The advantages of our algorithm is that we could use some well-known fast algorithms for computing gcrd, gcld, lclm and lcdm (see, for example, [4] and [15]).

Algorithm: Computing $\{1\}$-inverse

Input: $A \in R^{m\times n}$.

Output: $\{1\}$-inverse $A^{\{1\}}$ of A or no such $\{1\}$-inverse exists.

1. Compute a unimodular $U \in R^{m\times m}$ such that
$$UA = \begin{bmatrix} r & * \\ \mathbf{0} & * \end{bmatrix}.$$
2. Compute the gcld of the first row of UA. If gcld$\neq 1$, return "no such $\{1\}$-invere exists". Otherwise goto next step.
3. Compute a unimodular $V \in R^{n\times n}$ such that
$$UAV = \begin{bmatrix} 1 & \mathbf{0} \\ * & * \end{bmatrix}.$$

4. Do row transformations, find a unimodular $U_1 \in R^{m\times m}$ such that

$$U_1UAV = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & A_1 \end{bmatrix}.$$

5. Recursively apply Step 1 for A_1, and so on.
6. Finally we have two unimodular matrices U_0, V_0 such that

$$U_0AV_0 = A_0,$$

   where $(A_0)_{ii} = 1$ or 0, $i = 1, \ldots, \min\{m, n\}$ and other entries equal zero.
7. return $A^{\{1\}} = V_0A_0^TU_0$.

In fact, in Step 6, we could rearrange the rows and columns of A_0 such that $(A_0)_{ii} = 1$, $i = 1, \ldots, r \leq \min\{n, m\}$, and other entries are equal to zero. Moreover, we can prove that r is equal to the rank of A.

Using above algorithm, we can compute other kinds of inverses. For example, to compute $\{1, 2\}$-inverse of A, we first use above algorithm to find U_0, V_0 such that

$$U_0AV_0 = \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Then any $\{1, 2\}$-inverse of A is of the form $V_0 \begin{bmatrix} I & S \\ T & TS \end{bmatrix} U_0$, where S, T are arbitrary matrices with appropriate sizes.

As one of most important subclasses of Ore polynomials over division rings, we are interested in quaternion (skew) polynomial rings. The algebra $\mathbb{H}$ of real quaternion was discovered by Sir Rowan Hamilton in 1843, which is a four-dimensional non-commutative algebra over real number field $\mathbb{R}$ with canonical basis $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfying the conditions:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1,$$

that implies

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k},\ \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \text{and} \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

Elements in $\mathbb{H}$ can be written as a unique way $\alpha = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, where a, b, c and d are real numbers, that is, $\mathbb{H} = \{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \mid a, b, c, d \in \mathbb{R}\}$. The conjugate of α is defined as $\bar{\alpha} = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$.

Since $\mathbb{H}$ is a skew field, there are several forms of quaternion polynomials depending on the positions of coefficients. In our case, we will use one which put the coefficients on the left side of a variable x, which it is also called regular quaternion polynomials in [5] or quaternion simple polynomials in [17]. Furthermore, some Ore polynomials over $\mathbb{H}$ can be defined in a natural way, for example, $\mathbb{H}[x; -]$ and $\mathbb{H}(t)[x; \frac{\partial}{\partial t}]$. Some properties of quaternion (skew) polynomials and matrices over them have been discussed with many applications in control theory and physics (see, for example, [18] and [14]).

Based on the special properties of quaternion, we give another method to compute the Moore-Penrose inverse in quaternion case. First we define and discuss generalized characteristic polynomial for quaternion polynomials, and then consider the Leverrier-Faddeeva algorithm (see, [1], [6] and [7]) in quaternion case. Finally we explore the interpolation for quaternion polynomials and quaternion polynomial matrices and construct a fast algorithm. The detailed results will be included in a full paper after the conference.

3 Implementation

Our Maple package includes two parts: general Ore polynomial matrices and quaternion polynomial matrices.

In Part I, dedicated to general Ore polynomial matrices, all commands are compatible with OreTools and OreAlgebras in Maple 17. We use the same commands to set up Ore polynomials over fields and do basic computations including gcrd, gcld, lclm and lcdm. Here are a few key commands in our package:

- *OreMat*(A, m, n)*:* set up an $m \times n$ Ore matrix.
- *Rgcrd*$(A, 1, j)$*:* compute $\mathrm{gcrd}((A)_{11},\ (A)_{j1})$ and make row transformation such that $(A)_{11} = \mathrm{gcrd}((A)_{11},\ (A)_{j1})$ and $(A)_{j1} = 0$.
- *Cgcld*$(A, 1, j)$*:* compute $\mathrm{gcld}((A)_{11},\ (A)_{1j})$ and make column transformation such that $(A)_{11} = \mathrm{gcld}((A)_{11},\ (A)_{1j})$ and $(A)_{1j} = 0$.
- Iinverse(A) returns $\{1\}$-inverse of A.
- MPinverse(A) returns the Moore-Penrose inverse of A.

Note that our methods work for Ore polynomials over division rings, in particular, over quaternions. As we know that there are no quaternion package in Maple 17. Although there is a quaternion package available on Maple Help website, it only includes some basic operations. No quaternion polynomials and matrices are included. This motivates us to develop a Maple package for quaternion polynomials and matrices.

In Part II, dedicated to quaternion polynomial matrices, we develop this package from the beginning to keep consistence, that is, set up Maple commands for quaternion operations first, which include norm, conjugate, similar, etc.

Secondly we set up Maple commands for quaternion polynomial operations. Simple quaternion polynomials and quaternion skew polynomials with conjugate "$-$" are pre-defined. People can use the *SetQuaternionRing* command to define various quaternion (skew) polynomials.

We prove that the Extended Euclidean Algorithm also works for quaternion polynomials and use it to compute gcrd, lclm, etc. Some commands are as follows:

- *qGCRD(f, g)* returns the monic GCRD of quaternion polynomials of f and g.
- *qExtendedGCRD(f, g, A, 'u', 'v')* returns the monic GCRD of quaternion polynomials of f and g, and two pairs $\{u_1, v_1\}$ and $\{u_2, v_2\}$ such that

$$u_1 f + u_2 g = \mathrm{GCRD}(f, g), \quad v_1 f + v_2 g = 0.$$

  where $v_1 f$ is the LCLM of f and g. The parameter A presents a quaternion polynomial defined by the *SetQuaternionRing* function.

Regarding matrices over quaternion (skew) polynomials, we develop basic matrix operations and three kinds of row (column) transformations. Combing Part I and Part II, we can compute the generalized inverses for quaternion (skew) polynomial matrices.

In the final part of our package, we implement the Leverrier-Faddeeva algorithm for quaternion polynomials and use Lagrange interpolation in quaternion to construct a fast algorithm. Here are some key commands:

- *LFMPinverse*(A) returns the Moore-Penrose inverse of quaternion polynomial matrix A by using Leverrier-Faddeeva algorithm.
- *Linterpolation*$([a_1, .., a_n], [b_1, .., b_n])$ returns a quaternion polynomial f such that $f(a_i) = b_i$.
- *LLFMPinverse*(A) returns the Moore-Penrose inverse of quaternion polynomial matrix A by using Lagrange interpolation and Leverrier-Faddeeva algorithm.

Acknowledgment. This research was supported by the grants from the National Sciences and Engineering Research Council (NSERC) of Canada and URGP from the University of Manitoba.

References

1. Barnett, S.: Leverrier's algorithm: a new proof and extensions. SIAM J. Matrix Anal. Appl. 10, 551–556 (1989)
2. Beckermann, B., Cheng, H., Labahn, G.: Fraction-free Row Reduction of Matrices of Ore Polynomials. Journal of Symbolic Computation 41(5), 513–543 (2006)
3. Ben-Israel, A., Greville, T.N.E.: Generalized Inverses: Theory and Applications, 2nd edn. Springer, New York (2003)
4. Bostan, A., Chyzak, F., Li, Z., Salvy, B.: Fast computation of common left multiples of linear ordinary differential operators. In: Proceedings of the 2012 International Symposium on Symbolic and Algebraic Computation, pp. 99–106. ACM Press (2012)
5. Damiano, A., Gentili, G., Sreuppa, D.: Computations in the ring of quaternionic polynomials. J. of Symbolic Computation 45, 38–45 (2010)
6. Decell, H.P.: An application of the Cayley-Hamilton theorem to generalized matrix inversion. SIAM Rev. 7, 526–528 (1965)
7. Faddeeva, V.N.: Computational methods of linear algebra. Dover books on advanced mathematics. Dover Publications (1959)
8. Giesbrecht, M., Sub Kim, M.: Computation of the Hermite form of a Matrix of Ore Polynomials. Journal of Algebra 376, 341–362 (2013)
9. Gordon, B., Motzkin, T.S.: On the zeros of polynomials over division rings. Trans. Amer. Math. Soc. 116, 218–226 (1965)
10. Horn, R.A., Johnson, C.R.: Matrix analysis. Cambridge University Press (1990)
11. Huang, L., Zhang, Y.: Maple package for quaternion matrices, The University of Manitoba (2013)
12. Jacobson, N.: The Theory of Rings. Amer. Math. Soc., New York (1943)
13. Jones, J., Karampetakis, N.P., Pugh, A.C.: The computation and application of the generalized inverse via Maple. J. Symbolic Comput. 25, 99–124 (1998)

14. Lam, T.Y.: A first course in noncommutative rings. Graduate Texts in Mathematics, vol. 131. Springer (2001)
15. Liand, Z., Nemes, I.: A modular algorithm for computing greatest common right divisors of Ore polynomials. In: Proceedings of the 1997 International Symposium on Symbolic and Algebraic Computation, pp. 282–289. ACM Press (1997)
16. Niven, I.: The roots of a quaternion. Amer. Math. Monthly 49, 386–388 (1942)
17. Opfer, G.: Polynomials and Vandermonde matrices over the field of quaternions. Electronic Transactions on Numerical Analysis 36, 9–16 (2009)
18. Pereira, R., Rocha, P.: On the determinant of quaternionic polynomial matrices and its application to system stability. Math. Meth. Appl. Sci. 31, 99–122 (2008)
19. Pereira, R., Rocha, P., Vettori, P.: Algebraic tools for the study of quaternionic behavioral systems. Linear Algebra and its Applications 400, 121–140 (2005)
20. Pereira, R., Vettori, P.: Stability of Quaternionic linear systems. IEEE Transactions on Automatic Control 51, 518–523 (2006)
21. Penrose, R.: A generalized inverse for matrices. Proc. Cambridge Philos. Soc. 51, 406–413 (1955)
22. Puystjens, R.: Moore-Penrose inverses for matrices over some Noetherian rings. J. Pure Appl. Algebra 31(1-3), 191–198 (1984)
23. Zhang, F.: Quaternions and matrices of quaternions. Linear Algebra and its Applications 251, 21–57 (1997)
24. Ziegler, M.: Quasi-optimal arithmetic for quaternion polynomials. In: Ibaraki, T., Katoh, N., Ono, H. (eds.) ISAAC 2003. LNCS, vol. 2906, pp. 705–715. Springer, Heidelberg (2003)