Modular Mix-and-Match Complementation of Büchi Automata (Technical Report)

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Abstract. Complementation of nondeterministic Büchi automata (BAs) is an important problem in automata theory with numerous applications in formal verification, such as termination analysis of programs, model checking, or in decision procedures of some logics. We build on ideas from a recent work on BA determinization by Li et al. and propose a new modular algorithm for BA complementation. Our algorithm allows to combine several BA complementation procedures together, with one procedure for a subset of the BA's strongly connected components (SCCs). In this way, one can exploit the structure of particular SCCs (such as when they are inherently weak or deterministic) and use more efficient specialized algorithms, regardless of the structure of the whole BA. We give a general framework into which partial complementation procedures can be plugged in, and its instantiation with several algorithms. The framework can, in general, produce a complement with an Emerson-Lei acceptance condition, which can often be more compact. Using the algorithm, we were able to establish an exponentially better new upper bound of $O(4^n)$ for complementation of the recently introduced class of elevator automata. We implemented the algorithm in a prototype and performed a comprehensive set of experiments on a large set of benchmarks, showing that our framework complements well the state of the art and that it can serve as a basis for future efficient BA complementation and inclusion checking algorithms.

1 Introduction

Nondeterministic Büchi automata (BAs) [8] are an elegant and conceptually simple framework to model infinite behaviors of systems and the properties they are expected to satisfy. BAs are widely used in many important verification tasks, such as termination analysis of programs [27], model checking [51], or as the underlying formal model of decision procedures for some logics (such as S1S [8] or a fragment of the first-order logic over Sturmian words [28]). Many of these applications require to perform *complementation* of BAs: For instance, in termination analysis of programs within Ultimate Automizer [27], complementation is used to keep track of the set of paths whose termination still needs to be proved. On the other hand, in model checking⁵ and decision

⁵ Here, we consider model checking w.r.t. a specification given in some more expressive logic, such as S1S [8], QPTL [47], or HyperLTL [12], rather than LTL [41], where negation is simple.

procedures of logics, complement is usually used to implement negation and quantifier alternation. Complementation is often the most difficult automata operation performed here; its worst-case state complexity is $O((0.76n)^n)$ [2,45] (which is tight [52]).

In these applications, efficiency of the complementation often determines the overall efficiency (or even feasibility) of the top-level application. For instance, the success of ULTIMATE AUTOMIZER in the Termination category of the International Competition on Software Verification (SV-COMP) [48] is to a large degree due to an efficient BA complementation algorithm [6,11] tailored for BAs with a special structure that it often encounters (as of the time of writing, it has won 6 gold medals in the years 2017–2022 and two silver medals in 2015 and 2016). The special structure in this case are the so-called *semi-deterministic BAs* (SDBAs), BAs consisting of two parts: (i) an initial part without accepting states/transitions and (ii) a deterministic part containing accepting states/transitions that cannot transition into the first part.

Complementation of SDBAs using one from the family of the so-called NCSB algorithms [5,6,11,25] has the worst-case complexity $O(4^n)$ (and usually also works much better in practice than general BA complementation procedures). Similarly, there are efficient complementation procedures for other subclasses of BAs, e.g., (i) *deterministic BAs* (DBAs) can be complemented into BAs with 2n states [32] (or into co-Büchi automata with n+1 states) or (ii) *inherently weak BAs* (BAs where in each *strongly connected component* (SCC), either all cycles are accepting or all cycles are rejecting) can be complemented into DBAs with $O(3^n)$ states using the Miyano-Hayashi algorithm [39].

For a long time, there has been no efficient algorithm for complementation of BAs that are highly structured but do not fall into one of the categories above, e.g., BAs containing inherently weak, deterministic, and some nondeterministic SCCs. For such BAs, one needed to use a general complementation algorithm with the $O((0.76n)^n)$ (or worse) complexity. To the best of our knowledge, only recently has there appeared works that exploit the structure of BAs to obtain a more efficient complementation algorithm: (i) The work of Havlena *et al.* [26], who introduce the class of *elevator automata* (BAs with an arbitrary mixture of inherently weak and deterministic SCCs) and give a $O(16^n)$ algorithm for them. (ii) The work of Li *et al.* [34], who propose a BA determinization procedure (into a deterministic Emerson-Lei automaton) that is based on decomposing the input BA into SCCs and using a different determinization procedure for different types of SCCs (inherently weak, deterministic, general) in a synchronous construction.

In this paper, we propose a new BA complementation algorithm inspired by [34], where we exploit the fact that complementation is, in a sense, more relaxed than determinization. In particular, we present a *framework* where one can plug-in different partial complementation procedures fine-tuned for SCCs with a specific structure. The procedures work only with the given SCCs, to some degree *independently* (thus reducing the potential state space explosion) from the rest of the BA. Our top-level algorithm then orchestrates runs of the different procedures in a *synchronous* manner (or completely independently in the so-called *postponed* strategy), obtaining a resulting automaton with potentially a more general acceptance condition (in general an Emerson-Lei condition), which can help keeping the result small. If the procedures satisfy given correctness requirements, our framework guarantees that its instantiation will also be correct. We also propose its optimizations by, e.g., using round-robin to decrease the amount of nondeter-

minism, using shared breakpoint to reduce the size and the number of colours for certain class of partial algorithms, and generalize simulation-based pruning of macrostates.

We provide a detailed description of partial complementation procedures for inherently weak, deterministic, and initial deterministic SCCs, which we use to obtain a *new* exponentially better upper bound of $O(4^n)$ for the class of elevator automata (i.e., the same upper bound as for its strict subclass of SDBAs). Furthermore, we also provide two partial procedures for general SCCs based on determinization (from [34]) and the rank-based construction. Using a prototype implementation, we then show our algorithm complements well existing approaches and significantly improves the state of the art.

2 Preliminaries

We fix a finite non-empty alphabet Σ and the first infinite ordinal ω . An (infinite) word w is a function $w: \omega \to \Sigma$ where the i-th symbol is denoted as w_i . Sometimes, we represent w as an infinite sequence $w = w_0 w_1 \dots$ We denote the set of all infinite words over Σ as Σ^{ω} ; an ω -language is a subset of Σ^{ω} .

Emerson-Lei Acceptance Conditions. Given a set $\Gamma = \{0, \dots, k-1\}$ of k colours (often depicted as 0, 1, etc.), we define the set of *Emerson-Lei acceptance conditions* $\mathbb{EL}(\Gamma)$ as the set of formulae constructed according to the following grammar:

$$\alpha ::= \mathsf{Inf}(c) \mid \mathsf{Fin}(c) \mid (\alpha \land \alpha) \mid (\alpha \lor \alpha) \tag{1}$$

for $c \in \Gamma$. The *satisfaction* relation \models for a set of colours $M \subseteq \Gamma$ and condition α is defined inductively as follows (for $c \in \Gamma$):

$$M \models \mathsf{Fin}(c) \text{ iff } c \notin M,$$
 $M \models \alpha_1 \lor \alpha_2 \text{ iff } M \models \alpha_1 \text{ or } M \models \alpha_2,$ $M \models \mathsf{Inf}(c) \text{ iff } c \in M,$ $M \models \alpha_1 \land \alpha_2 \text{ iff } M \models \alpha_1 \text{ and } M \models \alpha_2.$

Emerson-Lei Automata. A (nondeterministic transition-based⁶) Emerson-Lei automaton (TELA) over Σ is a tuple $\mathcal{A} = (Q, \delta, I, \Gamma, \mathsf{p}, \mathsf{Acc})$, where Q is a finite set of states, $\delta \subseteq Q \times \Sigma \times Q$ is a set of transitions⁷, $I \subseteq Q$ is the set of initial states, Γ is the set of colours, $\mathsf{p} \colon \delta \to 2^{\Gamma}$ is a colouring function of transitions, and $\mathsf{Acc} \in \mathbb{EL}(\Gamma)$. We use $p \stackrel{a}{\to} q$ to denote that $(p, a, q) \in \delta$ and sometimes also treat δ as a function with the signature $\delta \colon Q \times \Sigma \to 2^Q$. Moreover, we extend δ to sets of states $P \subseteq Q$ as $\delta(P, a) = \bigcup_{p \in P} \delta(p, a)$. We use $\mathcal{A}[q]$ for $q \in Q$ to denote the automaton $\mathcal{A}[q] = (Q, \delta, \{q\}, \Gamma, \mathsf{p}, \mathsf{Acc})$, i.e., the TELA obtained from \mathcal{A} by setting q as the only initial state. \mathcal{A} is called deterministic if $|I| \le 1$ and $|\delta(q, a)| \le 1$ for each $q \in Q$ and $a \in \Sigma$. If $\Gamma = \{\mathbf{0}\}$ and $Acc = \inf(\mathbf{0})$, we call \mathcal{A} a Büchi automaton (BA) and denote it as $\mathcal{A} = (Q, \delta, I, F)$ where F is the set of all transitions coloured by $\mathbf{0}$, i.e., $F = \mathsf{p}^{-1}(\{\mathbf{0}\})$. For a BA, we use $\delta_F(p, a) = \{q \in \delta(p, a) \mid \mathsf{p}(p \stackrel{a}{\to} q) = \{\mathbf{0}\}\}$ (and extend the notation to sets of states as for δ). A BA $\mathcal{A} = (Q, \delta, I, F)$ is called semi-deterministic (SDBA) if for every accepting transition $(p \stackrel{a}{\to} q) \in F$, the reachable fragment of $\mathcal{A}[q]$ is deterministic.

⁶ We only consider transition-based acceptance in order to avoid cluttering the paper by always dealing with accepting states *and* accepting transitions. Extending our approach to state/transition-based (or just state-based) automata is straightforward.

⁷ Note that some authors use a more general definition of TELAs with $\delta \subseteq Q \times \Sigma \times 2^{\Gamma} \times Q$; since we only use them on the output, we suffice with the simpler definition.

A run of \mathcal{A} from $q \in Q$ on an input word w is an infinite sequence $\rho \colon \omega \to Q$ that starts in q and respects δ , i.e., $\rho_0 = q$ and $\forall i \geq 0 \colon \rho_i \xrightarrow{w_i} \rho_{i+1} \in \delta$. Let $\inf_{\delta}(\rho) \subseteq \delta$ denote the set of transitions occurring in ρ infinitely often and $\inf_{\Gamma}(\rho) = \bigcup \{p(x) \mid x \in \inf_{\delta}(\rho)\}$ be the set of infinitely often occurring colours. A run ρ is accepting in \mathcal{A} iff $\inf_{\Gamma}(\rho) \models \mathsf{Acc}$ and the language of \mathcal{A} , denoted as $\mathcal{L}(\mathcal{A})$, is defined as the set of words $w \in \Sigma^{\omega}$ for which there exists an accepting run in \mathcal{A} starting with some state in I.

Consider a BA $\mathcal{A} = (Q, \delta, I, F)$. For a set of states $S \subseteq Q$ we use \mathcal{A}_S to denote the copy of \mathcal{A} where accepting transitions only occur between states from S, i.e., the BA $\mathcal{A}_S = (Q, \delta, I, F \cap \delta_{|S|})$ where $\delta_{|S|} = \{p \xrightarrow{a} q \in \delta \mid p, q \in S\}$. We say that a non-empty set of states $C \subseteq Q$ is a *strongly connected component* (SCC) if every pair of states of C can reach each other and C is a maximal such set. An SCC of \mathcal{A} is *trivial* if it consists of a single state that does not contain a self-loop and *non-trivial* otherwise. An SCC C is *accepting* if it contains at least one accepting transition and *inherently weak* iff either (i) every cycle in C contains a transition from F or (ii) no cycle in C contains any transitions from F. An SCC C is *deterministic* iff the BA $(C, \delta_{|C|}, \{q\}, \emptyset)$ for any $q \in C$ is deterministic. We denote inherently weak components as IWCs, accepting deterministic components that are not inherently weak as DACs (deterministic accepting), and the remaining accepting components as NACs (nondeterministic accepting). A BA \mathcal{A} is called an *elevator automaton* if it contains no NAC.

We assume that \mathcal{A} contains no accepting transition outside its SCCs (no run can cycle over such transitions). We use δ_{SCC} to denote the restriction of δ to transitions that do not leave their SCCs, formally, $\delta_{\mathrm{SCC}} = \{p \xrightarrow{a} q \in \delta \mid p \text{ and } q \text{ are in the same SCC}\}$. A partition block $P \subseteq Q$ of \mathcal{A} is a nonempty union of its accepting SCCs, and a partitioning of \mathcal{A} is a sequence P_1, \ldots, P_n of pairwise disjoint partition blocks of \mathcal{A} that contains all accepting SCCs of \mathcal{A} .

The complement (automaton) of a BA \mathcal{A} is a TELA that accepts the complement language $\Sigma^{\omega} \setminus \mathcal{L}(\mathcal{A})$ of $\mathcal{L}(\mathcal{A})$. In the paper, we call a state and a run of a complement automaton a *macrostate* and a *macrorun*, respectively.

3 A Modular Complementation Algorithm

In a nutshell, the main idea of our BA complementation algorithm is that we first decompose a BA \mathcal{A} into several partition blocks according to their properties, and then perform complementation for each of the partition blocks (potentially using a different algorithm) independently, using either a *synchronous* construction, synchronizing the complementation algorithms for all partition blocks in each step, or a *postponed* construction, which proceeds by complementing the partition blocks independently and combines the partial results later using automata product construction. The decomposition of \mathcal{A} into partition blocks can either be trivial—i.e., with one block for each accepting SCC—, or more elaborate, e.g., a partitioning where one partition block contains all accepting IWCs, another contains all DACs, and each NAC is given its own partition block.

In this way, one can avoid running a general complementation algorithm for unrestricted BAs with the state complexity upper bound $O((0.76n)^n)$ and, instead, apply the most suitable complementation procedure for each of the partition blocks. This comes with three main advantages:

1. The complementation algorithm for each partition block can be selected differently in order to exploit the properties of the block. For instance, for partition blocks

with IWCs, one can use complementation based on the breakpoint (the so-called Miyano-Hayashi) construction [39] with $O(3^n)$ macrostates (cf. Sec. 4.1), while for partition blocks with only DACs, one can use an algorithm with the state complexity $O(4^n)$ based on an adaptation of the NCSB construction [5, 6, 11, 25] for SDBAs (cf. Sec. 4.2). For NACs, one can choose between, e.g., rank- [10, 21, 24, 26, 31, 45] or determinization-based [40, 42, 43] algorithms, depending on the properties of the NACs (cf. Sec. 6).

- 2. The different complementation algorithms can focus only on the respective blocks and do not need to consider other parts of the BA. This is advantageous, e.g., for rank-based algorithms, which can use this restriction to obtain tighter bounds on the considered ranks (even tighter than using the refinement in [26]).
- 3. The obtained automaton can be more compact due to the use of a more general acceptance condition than Büchi [44]—in general, it can be a conjunction of any EL conditions (one condition for each partition block), depending on the output of the complementation procedures; this can allow a more compact encoding of the produced automaton allowed by using a mixture of conditions. E.g., a deterministic BA can be complemented with constant extra generated states when using a co-Büchi condition rather than a linear number of generated states for a Büchi condition (see Sec. 5.1).

Those partial complementation algorithms then need to be orchestrated by a top-level algorithm to produce the complement of \mathcal{A} .

One might regard our algorithm as an optimization of an approach that would for each partition block P obtain a BA \mathcal{A}_P , complement \mathcal{A}_P using the selected algorithm, and perform the intersection of all \mathcal{A}_P 's obtained in this way (which would, however, not be able to obtain the upper bound for elevator automata that we give in Sec. 4.3). Indeed, we also implemented the mentioned procedure (called the *postponed* approach, described in Sec. 5.2) and compared it to our main procedure (called the *synchronous* approach) described below.

3.1 Basic Synchronous Algorithm

In this section, we describe the basic *synchronous* top-level algorithm. Then, in Sec. 4, we provide its instantiation for elevator automata and give a new upper bound for their complementation; in Sec. 5, we discuss several optimizations of the algorithm; and in Sec. 6, we give a generalization for unrestricted BAs. Let us fix a BA $\mathcal{A} = (Q, \delta, I, F)$ and, w.l.o.g., assume that \mathcal{A} is *complete*, i.e., |I| > 0 and all states $q \in Q$ have an outgoing transition over all symbols $a \in \Sigma$.

The synchronous algorithm works with partial complementation algorithms for BA's partition blocks. Each such algorithm Alg is provided with a structural condition φ_{Alg} characterizing properties of partition blocks that the algorithm is able to complement. For a BA \mathcal{B} , we abuse the notation and use $\mathcal{B} \models \varphi$ to denote that \mathcal{B} satisfies the condition φ . We say that Alg is a *partial complementation algorithm for a partition block P* if $\mathcal{A}_P \models \varphi_{\text{Alg}}$. We distinguish between Alg, a general algorithm able to complement a partition block of a given type, and Alg_P, its instantiation for the partition block P. We require each instance Alg_P to provide the following:

- T^{Alg} the type of the macrostates produced by the algorithm;
- Colours^{Alg_P} = $\{0, \dots, k^{Alg_P} 1\}$ the set of used colours;

- $\operatorname{Init}^{\operatorname{Alg}_P} \in 2^{\operatorname{T}^{\operatorname{Alg}_P}}$ the set of initial macrostates; $\operatorname{Succ}^{\operatorname{Alg}_P} : (2^Q \times \operatorname{T}^{\operatorname{Alg}_P} \times \Sigma) \to 2^{\operatorname{T}^{\operatorname{Alg}_P} \times \operatorname{Colours}^{\operatorname{Alg}_P}}$ a function returning the successors of a macrostate such that $Succ^{Alg_P}(H, M, a) = \{(M_1, \alpha_1), \dots, (M_k, \alpha_k)\},\$ where H is the set of all states of \mathcal{A} reached over the same word, M is the Alg_P's macrostate for the given partition block, a is the input symbol, and each (M_i, α_i) is a pair (macrostate, set of colours) such that M_i is a successor of M over a w.r.t. H and α_i is a set of colours on the edge from M to M_i (H helps to keep track of new runs coming into the partition block); and
- $Acc^{Alg_P} \in \mathbb{EL}(Colours^{Alg_P})$ the acceptance condition.

Let P_1, \ldots, P_n be a partitioning of \mathcal{A} (w.l.o.g., we assume that n > 0), and Alg^1, \ldots, Alg^n be a sequence of algorithms such that Alg^i is a partial complementation algorithm for P_i . Furthermore, let us define the following auxiliary renumbering function λ as $\lambda(c,j)=c+\sum_{i=1}^{j-1}|{\tt Colours}^{{\tt Alg}^i_{p_i}}|$, which is used to make the colours and acceptance conditions from the partial complementation algorithms disjoint. We also lift λ to sets of colours in the natural way, and also to \mathbb{EL} conditions such that $\lambda(\varphi, j)$ has the same structure as φ but each atom lnf(c) is substituted with the atom $lnf(\lambda(c, j))$ (and likewise for Fin atoms). The synchronous complementation algorithm then produces the TELA ModCompl($\mathrm{Alg}_{P_1}^1,\ldots,\mathrm{Alg}_{P_n}^n,\mathcal{A})=(Q^C,\delta^C,I^C,\Gamma^C,\mathsf{p}^C,\mathsf{Acc}^C)$ with components defined as follows (we use $[S_i]_{i=1}^n$ to abbreviate $S_1\times\cdots\times S_n$): $-Q^C=2^Q\times[\mathsf{T}^{\mathrm{Alg}_{P_i}^i}]_{i=1}^n,\qquad -\Gamma^C=\{0,\ldots,\lambda(k^{\mathrm{Alg}_{P_n}^n}-1,n)\},$

$$-Q^{C} = 2^{Q} \times [\mathbf{T}^{\mathrm{Alg}_{P_{i}}^{i}}]_{i=1}^{n}, \qquad -\Gamma^{C} = \{0, \dots, \lambda(k^{\mathrm{Alg}_{P_{n}}^{n}} - 1, n)\}, \\ -I^{C} = \{I\} \times [\mathrm{Init}^{\mathrm{Alg}_{P_{i}}^{i}}]_{i=1}^{n}, \qquad -\mathrm{Acc}^{C} = \bigwedge_{i=1}^{n} \lambda(\mathrm{Acc}^{\mathrm{Alg}_{P_{i}}^{i}}, i),^{8} \text{and}$$

- δ^C and p^C are defined such that if

$$((M_1',\alpha_1),\ldots,(M_n',\alpha_n)) \in [\operatorname{Succ}^{\operatorname{Alg}_{P_i}^i}(H,M_i,a)]_{i=1}^n,$$

then δ^C contains the transition $t: (H, M_1, \dots, M_n) \xrightarrow{a} (\delta(H, a), M'_1, \dots, M'_n)$, coloured by $p^C(t) = \bigcup \{\lambda(\alpha_i, i) \mid 1 \le i \le n\}$, and δ^C is the smallest such a set.

In order for ModCompl to be correct, the partial complementation algorithms need to satisfy certain properties, which we discuss below.

For a structural condition φ and a BA $\mathcal{B} = (Q, \delta, I, F)$, we define $\mathcal{B} \models_P \varphi$ iff $\mathcal{B} \models \varphi$, P is a partition block of \mathcal{B} , and \mathcal{B} contains no accepting transitions outside P. We can now provide the correctness condition on Alg.

Definition 1. We say that Alg is correct if for each \mathcal{B} such that $\mathcal{B} \models_P \varphi_{Alg}$ we have $\mathcal{L}(ModCompl(Alg_P, \mathcal{B})) = \Sigma^{\omega} \setminus \mathcal{L}(\mathcal{B}).$

The correctness of the synchronous algorithm (provided that each partial complementation algorithm is correct) is then established by Theorem 1.

Theorem 1. Let \mathcal{A} be a BA, P_1, \ldots, P_n be a partitioning of \mathcal{A} , and Alg^1, \ldots, Alg^n be a sequence of partial complementation algorithms such that Alg^i is correct for P_i . Then, we have $\mathcal{L}(\mathrm{ModCompl}(\mathrm{Alg}^1_{P_1},\ldots,\mathrm{Alg}^n_{P_n},\mathcal{A})) = \Sigma^\omega \setminus \mathcal{L}(\mathcal{A})$.

⁸ If we drop the condition that \mathcal{A} is complete, we also need to add an accepting sink state (representing the case for $H = \emptyset$) with self-loops over all symbols marked by a new colour \S , and enrich Acc^C with ... \vee Inf(\mathfrak{S}).

4 Modular Complementation of Elevator Automata

In this section, we first give partial algorithms to complement partition blocks with only accepting IWCs (Sec. 4.1) and partition blocks with only DACs (Sec. 4.2). Then, in Sec. 4.3, we show that using our algorithm, the upper bound on the size of the complement of elevator BAs is in $O(4^n)$, which is *exponentially better* than the known upper bound $O(16^n)$ established in [26].

4.1 Complementation of Inherently Weak Accepting Components

First, we introduce a partial algorithm MH with the condition φ_{MH} specifying that all SCCs in the partition block P are *accepting* IWCs. Let P be a partition block of \mathcal{A} such that $\mathcal{A}_P \models \varphi_{\text{MH}}$. Our proposed approach makes use of the Miyano-Hayashi construction [39]. Since in accepting IWCs, all runs are accepting, the idea of the construction is to accept words such that all runs over the words eventually leave P.

Therefore, we use a pair (C, B) of sets of states as a macrostate for complementing P. Intuitively, we use C to denote the set of all runs of \mathcal{A} that are in P (C for "check"). The set $B \subseteq C$ represents the runs being inspected whether they leave P at some point (B for "breakpoint"). Initially, we let $C = I \cap P$ and also sample into breakpoint all runs in P, i.e., set B = C. Along reading an ω -word w, if all runs that have entered P eventually leave P, i.e., P becomes empty infinitely often, the complement language of P should contain P0 (when P1 becomes empty, we sample P2 with all runs from the current P3. We formalize P4 as a partial procedure in the framework from Sec. 3.1 as follows:

$$\begin{split} &-\operatorname{T}^{\operatorname{MH}_P}=2^P\times 2^P, & \operatorname{Colours}^{\operatorname{MH}_P}=\{ \textcolor{red}{\mathbf{0}} \}, & \operatorname{Init}^{\operatorname{MH}_P}=\{ (I\cap P,I\cap P) \}, \\ &-\operatorname{Acc}^{\operatorname{MH}_P}=\operatorname{Inf}(\textcolor{red}{\mathbf{0}}), \text{ and} & \operatorname{Succ}^{\operatorname{MH}_P}(H,(C,B),a)=\{ ((C',B'),\alpha) \} \text{ where} \\ &\bullet C'=\delta(H,a)\cap P, \\ &\bullet B'= \begin{cases} C' & \text{if } B^\star=\emptyset \text{ for } B^\star=\delta(B,a)\cap C', \\ B^\star & \text{otherwise, and} \end{cases} &\bullet \alpha = \begin{cases} \{ \textcolor{red}{\mathbf{0}} \} & \text{if } B^\star=\emptyset \text{ and} \\ \emptyset & \text{otherwise.} \end{cases} \end{split}$$

We can see that checking whether w is accepted by the complement of P reduces to check whether B has been cleared infinitely often. Since every time when B becomes empty, we emit the color \mathbb{O} , we have that w is not accepted within P if and only if \mathbb{O} occurs infinitely often. Note that the transition function $Succ^{MH_P}$ is deterministic, i.e., there is exactly one successor.

Lemma 1. The partial algorithm MH is correct.

4.2 Complementation of Deterministic Accepting Components

In this section, we give a partial algorithm CSB with the condition φ_{CSB} specifying that a partition block P consists of DACs. Let P be a partition block of \mathcal{A} such that $\mathcal{A}_P \models \varphi_{\text{CSB}}$. Our approach is based on the NCSB family of algorithms [5, 6, 11, 25] for complementing SDBAs, in particular the NCSB-MaxRank construction [25]. The algorithm utilizes the fact that runs in DACs are deterministic, i.e., they do not branch into new runs. Therefore, one can check that a run is non-accepting if there is a time point from which the run does not see accepting transitions any more. We call such

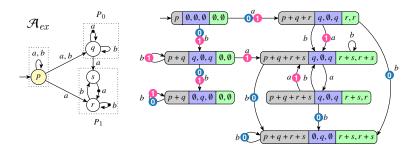


Fig. 1: Left: BA \mathcal{A}_{ex} (dots represent accepting transitions). Right: the outcome of ModCompl(CSB $_{P_0}$, MH $_{P_1}$, \mathcal{A}_{ex}) with Acc: Inf(\bigcirc) \land Inf(\bigcirc). States are given as $(H, (C_0, S_0, B_0), (C_1, B_1))$; to avoid too many braces, sets are given as sums.

a run that does not see accepting transitions any more *safe*. Then, an ω -word w is not accepted in P iff all runs over w in P either (i) leave P or (ii) eventually become safe.

For checking point (i), we can use a similar technique as in algorithm MH, i.e., use a pair (C, B). Moreover, to be able to check point (ii), we also use the set S that contains runs that are supposed to be safe, resulting in macrostates of the form $(C, S, B)^9$. To make sure that all runs are deterministic, we will use δ_{SCC} instead of δ when computing the successors of S and B since there may be nondeterministic jumps between different DACs in P; we will not miss any run in P since if a run moves between DACs of P, it can be seen as the run leaving P and a new run entering P. Since a run eventually stays in one SCC, this guarantees that the run will not be missed.

We formalize CSB_P in the top-level framework as follows:

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 \begin{split} &-\operatorname{T}^{\operatorname{CSB}_P} = 2^P \times 2^P \times 2^P, \operatorname{Init}^{\operatorname{CSB}_P} = \{(I \cap P, \emptyset, I \cap P)\}, \\ &-\operatorname{Colours}^{\operatorname{CSB}_P} = \{ \mathbf{0} \}, \operatorname{Acc}^{\operatorname{CSB}_P} = \operatorname{Inf}(\mathbf{0}), \operatorname{and} \\ &-\operatorname{Succ}^{\operatorname{CSB}_P}(H, (C, S, B), a) = U \operatorname{such} \operatorname{that} \\ &\bullet \operatorname{if} \delta_F(S, a) \neq \emptyset, \operatorname{then} U = \emptyset (\operatorname{Runs} \operatorname{in} S \operatorname{must} \operatorname{be} \operatorname{safe}), \\ &\bullet \operatorname{otherwise} U \operatorname{contains} ((C', S', B'), c) \operatorname{where} \\ &* S' = \delta_{\operatorname{SCC}}(S, a) \cap P, C' = (\delta(H, a) \cap P) \setminus S', \\ &* B' = \begin{cases} C' & \operatorname{if} B^* = \emptyset \operatorname{for} B^* = \delta_{\operatorname{SCC}}(B, a), \\ B^* & \operatorname{otherwise}, \operatorname{and} \end{cases} &* c = \begin{cases} \{\mathbf{0}\} & \operatorname{if} B^* = \emptyset, \\ \emptyset & \operatorname{otherwise}. \end{cases} \end{aligned}
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Moreover, in the case $\delta_F(B, a) \cap \delta_{SCC}(B, a) = \emptyset$, then U also contains $((C'', S'', C''), \{ \mathbf{0} \})$ where $S'' = S' \cup B'$ and $C'' = C' \setminus S''$.

Intuitively, when $\delta_F(B, a) \cap \delta_{\mathrm{SCC}}(B, a) = \emptyset$, we make two guesses: (i) either the runs in B all become safe (we move them to S) or (ii) there might be some unsafe runs (we keep them in B). Since the runs in B are deterministic, the number of tracked runs in B will not increase. Moreover, if all runs in B are eventually safe, we are guaranteed to move all of them to S at the right time point, e.g., the maximal time point where all runs are safe since the number of runs is finite.

⁹ In contrast to MH, here we use $C \cup S$ rather than C to keep track of all runs in P.

As mentioned above, w is not accepted within P iff all runs over w either (i) leave P or (ii) become safe. In the context of the presented algorithm, this corresponds to (i) B becoming empty infinitely often and (ii) $\delta_F(S,a)$ never seeing an accepting transition. Then we only need to check if there exists an infinite sequence of macrostates $\hat{\rho} = (C_0, S_0, B_0) \dots$ that emits 0 infinitely often.

Lemma 2. The partial algorithm CSB is correct.

It is worth noting that when the given partition block P contains all DACs of \mathcal{A} , we can still use the construction above, while the construction in [25] only works on SDBAs. *Example 1*. In Fig. 1, we give an example of the run of our algorithm on the BA \mathcal{A}_{ex} . The BA contains three SCCs, one of them (the one containing p) non-accepting (therefore, it does not need to occur in any partition block). The partition block P_0 contains a single DAC, so we can use algorithm CSB, and the partition block P_1 contains a single accepting IWC, so we can use MH. The resulting ModCompl(CSB $_{P_0}$, MH $_{P_1}$, \mathcal{A}_{ex}) uses two colours, $\mathbf{0}$ from CSB and $\mathbf{1}$ from MH. The acceptance condition is $\inf(\mathbf{0}) \land \inf(\mathbf{1})$.

4.3 Upper-bound for Elevator Automata Complementation

We now give an upper bound on the size of the complement generated by our algorithm for elevator automata, which significantly improves the best previously known upper bound of $O(16^n)$ [26] to $O(4^n)$, the same as for SDBAs, which are a strict subclass of elevator automata [6] (we note that this upper bound cannot be obtained by a determinization-based algorithm, since determinization of SDBAs is in $\Omega(n!)$ [17,37]). **Theorem 2.** Let \mathcal{A} be an elevator automaton with n states. Then there exists a BA with $O(4^n)$ states accepting the complement of $\mathcal{L}(\mathcal{A})$.

Proof (*Sketch*). Let Q_W be all states in accepting IWCs, Q_D be all states in DACs, and Q_N be the remaining states, i.e., $Q = Q_W \uplus Q_D \uplus Q_N$. We make two partition blocks: $P_0 = Q_W$ and $P_1 = Q_D$ and use MH and CSB respectively as the partial algorithms, with macrostates of the form $(H, (C_0, B_0), (C_1, S_1, B_1))$. For each state $q_N \in Q_N$, there are two options: either $q_N \notin H$ or $q_N \in H$. For each state $q_W \in Q_W$, there are three options: (i) $q_W \notin C_0$, (ii) $q_W \in C_0 \setminus B_0$, or (iii) $q_W \in C_0 \cap B_0$. Finally, for each $q_D \in Q_D$, there are four options: (i) $q_D \notin C_1 \cup S_1$, (ii) $q_D \in S_1$, (iii) $q_D \in C_1 \setminus B_1$, or (iv) $q_D \in C_1 \cap B_1$. Therefore, the total number of macrostates is $2 \cdot 2^{|Q_N|} \cdot 3^{|Q_M|} \cdot 4^{|Q_D|} \in O(4^n)$ where the initial factor 2 is due to degeneralization from two to one colour (the two colours can actually be avoided by using our shared breakpoint optimization from Sec. 5.4). □

5 Optimizations of the Modular Construction

In this section, we propose optimizations of the basic modular algorithm. In Sec. 5.1, we give a partial algorithm to complement initial partition blocks with DACs. Further, in Sec. 5.2, we propose the postponed construction allowing to use automata reduction on intermediate results. In Sec. 5.3, we propose the round-robin algorithm alleviating the problem with the explosion of the size of the Cartesian product of partial successors. In Sec. 5.4, we provide an optimization for partial algorithms that are based on the breakpoint construction, and, finally, in Sec. 5.5, we show how to employ simulation to decrease the size of macrostates in the synchronous construction.

5.1 Complementation of Initial Deterministic Partition Blocks

Our first optimization is an optimized algorithm CoB for a subclass of partition blocks containing DACs. In particular, the condition φ_{CoB} specifies that the partition block P is deterministic and can be reached only deterministically in \mathcal{A} (i.e., \mathcal{A}_P after removing redundant states is deterministic). In that case we say that P is an *initial deterministic* partition block. The algorithm is based on complementation of deterministic BAs into co-Büchi automata.

The algorithm CoB_P is formalized below:

Intuitively, all runs reach P deterministically, which means that over a word w, at most one run can reach P. Thus, we have $|\delta(H,w_j)\cap P|=1$ for some $j\geq 0$ if there is a run over w to P, corresponding to $\delta(H,a)\cap P=\{r\}$ in the construction. To check whether w is not accepted in P, we only need to check whether the run from $r\in P$ over w visits accepting transitions only finitely often. We give an example of complementation of a BA containing an initial deterministic partition block in Fig. 5 in Appendix D. Notice that the use of the Fin condition helps to obtain a more concise automaton with only two states (even in this simple example, using CSB instead of CoB would yield a TELA with 4 states).

Lemma 3. The partial algorithm CoB is correct.

5.2 Postponed Construction

The modular synchronous construction from Sec. 3.1 utilizes the assumption that in the simultaneous construction of successors for each partition block over a, if one partial macrostate M_i does not have a successor over a, then there will be no successor of the (H, M_1, \ldots, M_n) macrostate in δ^C as well. This is useful, e.g., for inclusion testing, where it is not necessary to generate the whole complement. On the other hand, if we need to generate the whole automaton, a drawback of the proposed modular construction is that each partial complementation algorithm itself may generate a lot of useless states. In this section, we propose the *postponed construction*, which complements the partition blocks (with their surrounding) independently and later combines the intermediate results to obtain the complement automaton for \mathcal{A} . The main advantage of the postponed construction is that one can apply automata reduction (e.g., based on removing useless states or using simulation [1,9,13,18]) to decrease the size of the intermediate automata.

In the postponed construction, we use automata product operation implementing language intersection (i.e., for two TELAs \mathcal{B}_1 and \mathcal{B}_2 , a product automaton $\mathcal{B}_1 \cap \mathcal{B}_2$ satisfying $\mathcal{L}(\mathcal{B}_1 \cap \mathcal{B}_2) = \mathcal{L}(\mathcal{B}_1) \cap \mathcal{L}(\mathcal{B}_2)^{10}$). Further, we employ a function Red performing some language-preserving reduction of an input TELA. Then, the postponed

¹⁰ Alternatively, one might also avoid the product and generate linear-sized *alternating* TELA, but working with those is usually much harder and not used in practice.

construction for an elevator automaton \mathcal{A} with a partitioning P_1, \ldots, P_n and a sequence of algorithms $\mathrm{Alg}^1, \ldots, \mathrm{Alg}^n$ such that Alg^i is a partial complementation algorithm for P_i , is defined as follows:

$$\operatorname{PostpCompl}(\operatorname{Alg}_{P_1}^1,\ldots,\operatorname{Alg}_{P_n}^n,\mathcal{A}) = \bigcap_{i=1}^n \operatorname{Red}\left(\operatorname{ModCompl}(\operatorname{Alg}_{P_i}^i,\mathcal{A}_{P_i})\right). \tag{2}$$

The example of the postponed construction applied on the BA from Fig. 1 is shown in Appendix D. The correctness of the construction is then summarized by the following theorem.

Theorem 3. Let \mathcal{A} be a BA, P_1, \ldots, P_n be a partitioning of \mathcal{A} , and $\mathtt{Alg}^1, \ldots, \mathtt{Alg}^n$ be a sequence of partial complementation algorithms such that \mathtt{Alg}^i is correct for P_i . Then, $\mathcal{L}(PostpCompl(\mathtt{Alg}^1_{P_1}, \ldots, \mathtt{Alg}^n_{P_n}, \mathcal{A})) = \Sigma^{\omega} \setminus \mathcal{L}(\mathcal{A})$.

5.3 Round-Robin Algorithm

The proposed basic synchronous approach from Sec. 3.1 may suffer from the combinatorial explosion because the successors of a macrostate are given by the Cartesian product of all successors of the partial macrostates. To alleviate this explosion, we propose a *round-robin* top-level algorithm. Intuitively, the round-robin algorithm actively tracks runs in only one partial complementation algorithm at a time (while other algorithms stay passive). The algorithm periodically changes the active algorithm to avoid starvation (the decision to leave the active state is, however, fully directed by the partial complementation algorithm). This can alleviate an explosion in the number of successors for algorithms that generate more than one successor (e.g., for rank-based algorithms where one needs to make a nondeterministic choice of decreasing ranks of states in order to be able to accept [10, 21, 24, 26, 31, 45]; such a choice needs to be made only in the active phase while in the passive phase, the construction just needs to make sure that the run is consistent with the given ranking, which can be done deterministically).

The round-robin algorithm works on the level of partial complementation round-robin algorithms. Each instance of the partial algorithm provides passive types to represent partial macrostates that are passive and active types to represent currently active partial macrostates. In contrast to the basic partial complementation algorithms from Sec. 3.1, which provide only a single successor function, the round-robin partial algorithms provide several variants of them. In particular, SuccPass returns (passive) successors of a passive partial macrostate, Lift gives all possible active counterparts of a passive macrostate, and SuccAct returns successors of an active partial macrostate. If SuccAct returns a partial macrostate of the passive type, the round-robin algorithm promotes the next partial algorithm to be the active one. For instance, in the round-robin version of CSB, the passive type does not contain the breakpoint and only checks that safe runs stay safe, so it is deterministic. Due to space limitations, we give a formal definition and more details about the round-robin algorithm in Appendix A.

5.4 Shared Breakpoint

The partial complementation algorithms CSB and MH (and later RNK defined in Appendix C) use a breakpoint to check whether the runs under inspection are accepting

or not. As an optimization, we consider merging of breakpoints of several algorithms and keeping only a single breakpoint for all supported algorithms. The top-level algorithm then needs to manage only one breakpoint and emit a colour only if this sole breakpoint becomes empty. This may lead to a smaller number of generated macrostates since we synchronize the breakpoint sampling among several algorithms. The second benefit is that this allows us to generate fewer colours (in the case of elevator automata complemented using algorithms CSB and MH, we get only one colour).

5.5 Simulation Pruning

Our construction can be further optimized by a simulation (or other compatible) relation for pruning macrostates. A simulation is, broadly speaking, a relation $\leq Q \times Q$ implying language inclusion of states, i.e., $\forall p,q \in Q \colon p \leq q \Longrightarrow \mathcal{L}(\mathcal{R}[p]) \subseteq \mathcal{L}(\mathcal{R}[q])$. Intuitively, our optimization allows to remove a state p from a macrostate M if there is also a state q in M such that (i) $p \leq q$, (ii) p is not reachable from q, and (iii) p is smaller than q in an arbitrary total order over Q (this serves as a tie-breaker for simulation-equivalent mutually unreachable states). The reason why p can be removed is that its behaviour can be completely mimicked by q. In our construction, we can then, roughly speaking, replace each call to the functions $\delta(U,a)$ and $\delta_F(U,a)$, for a set of states U, by $pr(\delta(U,a))$ and $pr(\delta_F(U,a))$ respectively in each partial complementation algorithm, as well as in the top-level algorithm, where pr(S) is obtained from S by pruning all eligible states. The details are provided in Appendix B.

6 Modular Complementation of Non-Elevator Automata

A non-elevator automaton \mathcal{A} contains at least one NAC, besides possibly other IWCs or DACs. To complement \mathcal{A} in a modular way, we apply the techniques seen in Sec. 4 to its DACs and IWCs, while for its NACs we resort to a general complementation algorithm Alg. In theory, rank- [31], slice- [29], Ramsey- [47], subset-tuple- [2], and determinization- [43] based complementation algorithms adapted to work on a single partition block instead of the whole automaton are all valid instantiations of Alg. Below, we give a high-level description of two such algorithms: rank- and determinization-based.

Rank-based partial complementation algorithm. Working on each NAC independently benefits the complementation algorithm even if the input BA contains only NACs. For instance, in rank-based algorithms [10, 21, 24, 26, 30, 31, 45], the fact whether all runs of $\mathcal A$ over a given ω -word w are non-accepting is determined by ranks of states, given by the so-called ranking functions. A ranking function is a (partial) function from Q to ω . The main idea of rank-based algorithms is the following: (i) every run is initially nondeterministically assigned a rank, (ii) ranks can only decrease along a run, (iii) ranks need to be even every time a run visits an accepting transition, and (iv) the

¹¹ This optimization can be seen as a generalization of the simulation-based pruning techniques that appeared, e.g., in [25, 38] in the context of concrete determinization/complementation procedures. Here, we generalize the technique to all procedures that are based on run tracking.

complement automaton accepts iff all runs eventually get trapped in odd ranks¹². In the standard rank-based procedure, the initial assignment of ranks to states in (i) is a function $Q \rightarrow \{0, \ldots, 2n-1\}$ for n = |Q|. Using our framework, we can, however, significantly restrict the considered ranks in a partition block P to only $P \rightarrow \{0, \ldots, 2m-1\}$ for m = |P| (here, it makes sense to use partition blocks consisting of single SCCs). One can further reduce the considered ranks using the techniques introduced in, e.g., [24, 26].

In order to adapt the rank-based construction as a partial complementation algorithm RNK in our framework, we need to extend the ranking functions by a fresh "box state" representing states outside the partition block. The ranking function then uses to represent ranks of runs newly coming into the partition block. The box-extension also requires to change the transition in a way that always represents reachable states from the outside. We provide the details of the construction, which includes the MaxRank optimization from [24], in Appendix C.

Determinization-based partial complementation algorithm. In [26, 49] we can see that determinization-based complementation is also a good instantiation of A1g in practice, so, we also consider the standard Safra-Piterman determinization [40, 42, 43] as a choice of A1g for complementing NACs. Determinization-based algorithms use a layered subset construction to organize all runs over an ω -word w. The idea is to identify a subset $S \subseteq H$ of reachable states that occur infinitely often along reading w such that between every two occurrences of S, we have that (i) every state in the second occurrence of S can be reached by a state in the first occurrence of S and (ii) every state in the second occurrence is reached by a state in the first occurrence while seeing an accepting transition. According to König's lemma, there must then be an accepting run of $\mathcal A$ over w.

The construction initially maintains only one set H: the set of reachable states. Since S as defined does not necessarily need to be H, every time there are runs visiting accepting transitions, we create a new subset C for those runs and remember which subset C is coming from. This way, we actually organize the current states of all runs into a tree structure and do subset construction in parallel for the sets in each tree node. If we find a tree node whose labelled subset, say S', is equal to the union of states in its children, we know the set S' satisfies the condition above and we remove all its child nodes and emit a good event. If such good event happens infinitely often, it means that S' also occurs infinitely often. So in complementation, we only need to make sure those good events only happen for finitely many times. Working on each NAC separately also benefits the determinization-based approach since the number of possible trees will be less with smaller number of reachable states. Following the idea of [34], to adapt for the construction as the partial complementation algorithm, we put all the newly coming runs from other partition blocks in a newly created node without a parent node. In this way, we actually maintain a forest of trees for the partial complementation construction. We denote the determinization-based construction as DET; cf. [34] for details.

7 Experimental Evaluation

To evaluate the proposed approach, we implemented it in a prototype tool Kofola (written in C++) built on top of Spot [16] and compared it against COLA [34], RANKER [25]

¹² Since we focus on intuition here, we use runs rather than the directed acyclic graphs of runs.

Table 1: Statistics for our experiments. The column **unsolved** classifies unsolved instances by the form *timeouts: out of memory: other failures*. For the cases of VBS we provide just the number of unsolved cases. The columns **states** and **runtime** provide *mean: median* of the number of states and runtime, respectively.

tool	solved	unsolved	states	runtime	tool
$Kofola_S$	39,738	89:10:0	76: 3	0.32:0.03	COL
Kofola _P	39,750	76:11:0	86: 3	0.41:0.03	RAN
VBS ₊	39,834	3	78: 3	0.05:0.01	Semi
VBS_{-}	39,834	3	96: 3	0.05:0.01	Spot

tool	solved	unsolved	states	runtime
COLA	39,814	21: 0:2	80:3	0.17:0.02
RANKER	38,837	61:939:0	45:4	3.31:0.01
SEMINATOR	39,026	238:573:0	247:3	1.98:0.03
Spot	39,827	8: 0:2	160:4	0.08:0.02

(v. 2), Seminator [5] (v. 2.0), and Spot [15, 16] (v. 2.10.6), which are the state of the art in BA complementation [25, 26, 34]. Due to space restrictions, we give results for only two instantiations of our framework: Kofolas and Kofolas. Both instantiations use MH for IWCs, CSB for DACs, and DET for NACs. The partitioning selection algorithm merges all IWCs into one partition block, all DACs into one partition block, and keeps all NACs separate. Simulation-based pruning from Sec. 5.5 is turned on, and round-robin from Sec. 5.3 is turned off (since the selected algorithms are quite deterministic). Kofolas employs the *synchronous* and Kofolas employs the *postponed* strategy. We also consider the Virtual Best Solver (VBS), i.e., a virtual tool that would choose the best solver for each single benchmark among all tools (VBS₊) and among all tools except both versions of Kofola (VBS₋). We ran our experiments on an Ubuntu 20.04.4 LTS system running on a desktop machine with 16 GiB RAM and an Intel 3.6 GHz i7-4790 CPU. To constrain and collect statistics about the executions of the tools, we used Benchexec [3] and imposed a memory limit of 12 GiB and a timeout of 10 minutes; we used Spot to cross-validate the equivalence of the automata generated by the different tools.

As our data set, we used 39,837 BAs from the AUTOMATA-BENCHMARKS repository [33] (used before by, e.g., [25, 26, 34]), which contains BAs from the following sources: (i) randomly generated BAs used in [49] (21,876 BAs), (ii) BAs obtained from LTL formulae from the literature and randomly generated LTL formulae [5] (3,442 BAs), (iii) BAs obtained from Ultimate Automizer [11] (915 BAs), (iv) BAs obtained from the solver for first-order logic over Sturmian words Pecan [28] (13,216 BAs), (v) BAs obtained from an S1S solver [23] (370 BAs), and (vi) BAs from LTL to SDBA translation [46] (18 BAs). From these BAs, 23,850 are deterministic, 6,147 are SDBAs (but not deterministic), 4,105 are elevator (but not SDBAs), and 5,735 are the rest.

In Table 1 we present an overview of the outcomes. Despite being a prototype, Kofola is already able to complement a large portion of the input automata, with very few cases that can be complemented successfully only by Spot or COLA. Regarding the mean number of states, Kofolas has the **least mean value** from all tools (except Ranker, which, however, had 1,000 unsolved cases) Moreover, Kofola **significantly decreased the mean number of states** when included into the VBS: from 96 to 78! We consider this to be a strong validation of the usefulness of our approach. Regarding the running time, both versions of Kofola are rather similar; Kofola is just slightly slower than Spot and COLA but much faster than both Ranker and Seminator (the runtime

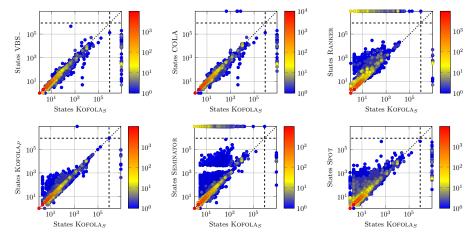
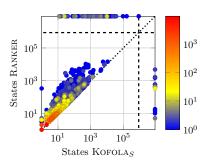


Fig. 2: Scatter plots comparing the numbers of states generated by the tools.

plot is in the appendix). Being a prototype, there are many engineering opportunities for speed-up.

In Fig. 2 we present a comparison of the number of states generated by Kofolas with those generated by the other tools; we omit VBS₊ since the corresponding plot can be derived from the one for VBS₋ (since Ranker and Seminator only output BAs, we compare the sizes of outputs transformed into BAs for all tools to be fair). In the plots, the number of benchmarks represented by each mark is given by its color; a mark above the diagonal means that Kofolas generated an automaton smaller than the other tool while a mark on the top border means that the other tool failed while Kofolas succeeded, and symmetrically for the bottom part and the right-hand border. Dashed lines represent the maximum number of states generated by one of the tools in the plot, axes are logarithmic.

From the results, Kofolas clearly dominates state-of-the-art tools that are not based on SCC decomposition (Ranker, Spot, Seminator). The outputs are quite comparable to COLA, which also uses SCC decomposition and can be seen as an instantiation of our framework. This supports our intuition that working on the single SCCs helps in reducing the size of the final automaton, confirming the validity of our modular mix-and-match Büchi comple-



mentation approach. Lastly, in the figure in the right, we compare our algorithm for elevator automata with the one in Ranker (the only other tool with a dedicated algorithm for this subclass). Our new algorithm clearly dominates the one in Ranker.

8 Related Work

To the best of our knowledge, we provide the *first general framework* where one can plug-in different BA complementation algorithms while taking advantage of the specific structure of SCCs. We will discuss the difference between our work and the literature.

The breakpoint construction [39] was designed to complement BAs with only IWCs, while our construction treats it as a partial complementation procedure for IWCs and differs in the need to handle incoming states from other partition blocks. The NCSB family of algorithms [5, 6, 11, 25] for SDBAs do not work when there are nondeterministic jumps between DACs; they can, however, be adapted as partial procedures for complementing DACs in our framework, cf. Sec. 4.2. In [26], a deelevation-based procedure is applied to elevator automata to obtain BAs with a fixed maximum rank of 3, for which a rank-based construction produces a result of the size in $O(16^n)$. In our work, we exploit the structure of the SCCs much more to obtain an exponentially better upper bound of $O(4^n)$ (the same as for SDBAs). The upper bound $O(4^n)$ for complementing unambiguous BAs was established in [36], which is orthogonal to our work, but seems to be possible to incorporate into our framework in the future.

There is a huge body of work on complementation of general BAs [2, 5, 7, 8, 10, 19–22, 24, 26, 29, 31, 40, 42, 43, 45, 47, 49, 50]; all of them work on the whole graph structure of the input BAs. Our framework is general enough to allow including all of them as partial complementation procedures for NACs. On the contrary, our framework does not directly allow (at least in the synchronous strategy) to use algorithms that *do not* work on the structure of the input BA, such as the learning-based complementation algorithm from [35]. The recent determinization algorithm from [34], which serves as our inspiration, also handles SCCs separately (it can actually be seen as an instantiation of our framework). Our current algorithm is, however, more flexible, allowing to mix-and-match various constructions, keep SCCs separate or merge them into partition blocks, and allows to obtain the complexity $O(4^n)$, while [34] only allowed O(n!) (which is tight since SDBA determinization is in $\Omega(n!)$ [17, 37]).

Regarding the tool Spot [15, 16], it should not be perceived as a single complementation algorithm. Instead, Spot should be seen as a highly engineered platform utilizing breakpoint construction for inherently weak BAs, NCSB [6, 11] for SDBAs, and determinization-based complementation [40, 42, 43] for general BAs, while using many other heuristics along the way. Seminator uses semi-determinization [4, 5, 14] to make sure the input is an SDBA and then uses NCSB [6, 11] to compute the complement.

9 Conclusion and Future Work

We have proposed a general framework for BA complementation where one can plug-in different partial complementation procedures for SCCs by taking advantage of their specific structure. Our framework not only obtains exponentially better upper bound for elevator automata, but also complements existing approaches well. As shown by the experimental results (especially for the VBS), our framework significantly improves the current portfolio of complementation algorithms.

We believe that our framework is an ideal testbed for experimenting with different BA complementation algorithms, e.g., for the following two reasons: (i) One can develop an efficient complementation algorithm that only works for a quite restricted sub-class of

BAs (such as the algorithm for initial deterministic SCCs that we showed in Sec. 5.1) and the framework can leverage it for complementation of all BAs that contain such a substructure. (ii) When one tries to improve a general complementation algorithm, they can focus on complementation of the structurally hard SCCs (mainly the nondeterministic accepting SCCs) and do not need to look for heuristics that would improve the algorithm if there were some easier substructure present in the input BA (as was done, e.g., in [26]). From how the framework is defined, it immediately offers opportunities for being used for on-the-fly BA *language inclusion* testing, leveraging the partial complementation procedures present. Finally, we believe that the framework also enables new directions for future research by developing smart ways, probably based on machine learning, of selecting which partial complementation procedure should be used for which SCC, based on their features. In future, we want to incorporate other algorithms for complementation of NACs, and identify properties of SCCs that allow to use more efficient algorithms (such as unambiguous NACs [36]). Moreover, it seems that generalizing the Delayed optimization from [24] on the top-level algorithm could also help reduce the state space.

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References

- 1. Abdulla, P.A., Chen, Y., Holík, L., Vojnar, T.: Mediating for reduction (on minimizing alternating büchi automata). Theor. Comput. Sci. 552, 26–43 (2014). https://doi.org/10.1016/j.tcs.2014.08.003, https://doi.org/10.1016/j.tcs.2014.08.003
- Allred, J.D., Ultes-Nitsche, U.: A simple and optimal complementation algorithm for Büchi automata. In: Dawar, A., Grädel, E. (eds.) Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2018, Oxford, UK, July 09-12, 2018. pp. 46–55. ACM (2018). https://doi.org/10.1145/3209108.3209138, https://doi.org/10.1145/3209108.3209138
- 3. Beyer, D., Löwe, S., Wendler, P.: Reliable benchmarking: requirements and solutions. Int. J. Softw. Tools Technol. Transf. 21(1), 1–29 (2019). https://doi.org/10.1007/s10009-017-0469-y, https://doi.org/10.1007/s10009-017-0469-y
- 4. Blahoudek, F., Duret-Lutz, A., Klokocka, M., Kretínský, M., Strejcek, J.: Seminator: A tool for semi-determinization of omega-automata. In: Eiter, T., Sands, D. (eds.) LPAR-21, 21st International Conference on Logic for Programming, Artificial Intelligence and Reasoning, Maun, Botswana, May 7-12, 2017. EPiC Series in Computing, vol. 46, pp. 356–367. Easy-Chair (2017). https://doi.org/10.29007/k5nl, https://doi.org/10.29007/k5nl

- 5. Blahoudek, F., Duret-Lutz, A., Strejcek, J.: Seminator 2 can complement generalized Büchi automata via improved semi-determinization. In: Lahiri, S.K., Wang, C. (eds.) Computer Aided Verification 32nd International Conference, CAV 2020, Los Angeles, CA, USA, July 21-24, 2020, Proceedings, Part II. Lecture Notes in Computer Science, vol. 12225, pp. 15–27. Springer (2020). https://doi.org/10.1007/978-3-030-53291-8_2, https://doi.org/10.1007/978-3-030-53291-8_2
- 6. Blahoudek, F., Heizmann, M., Schewe, S., Strejček, J., Tsai, M.: Complementing semi-deterministic Büchi automata. In: Chechik, M., Raskin, J. (eds.) Tools and Algorithms for the Construction and Analysis of Systems 22nd International Conference, TACAS 2016, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2016, Eindhoven, The Netherlands, April 2-8, 2016, Proceedings. Lecture Notes in Computer Science, vol. 9636, pp. 770–787. Springer (2016). https://doi.org/10.1007/978-3-662-49674-9_49, https://doi.org/10.1007/978-3-662-49674-9_49
- Breuers, S., Löding, C., Olschewski, J.: Improved Ramsey-based Büchi complementation.
 In: Birkedal, L. (ed.) Foundations of Software Science and Computational Structures 15th
 International Conference, FOSSACS 2012, Held as Part of the European Joint Conferences
 on Theory and Practice of Software, ETAPS 2012, Tallinn, Estonia, March 24 April 1,
 2012. Proceedings. Lecture Notes in Computer Science, vol. 7213, pp. 150–164. Springer
 (2012).https://doi.org/10.1007/978-3-642-28729-9_10,https://doi.org/10.
 1007/978-3-642-28729-9_10
- Büchi, J.R.: On a decision method in restricted second order arithmetic. In: Mac Lane, S., Siefkes, D. (eds.) The Collected Works of J. Richard Büchi, pp. 425–435. Springer (1990). https://doi.org/10.1007/978-1-4613-8928-6_23, https://doi.org/10.1007/978-1-4613-8928-6_23
- Bustan, D., Grumberg, O.: Simulation-based Minimization. ACM Transactions on Computational Logic 4(2), 181–206 (2003)
- Chen, Y., Havlena, V., Lengál, O.: Simulations in rank-based Büchi automata complementation. In: Lin, A.W. (ed.) Programming Languages and Systems 17th Asian Symposium, APLAS 2019, Nusa Dua, Bali, Indonesia, December 1-4, 2019, Proceedings. Lecture Notes in Computer Science, vol. 11893, pp. 447–467. Springer (2019). https://doi.org/10.1007/978-3-030-34175-6_23, https://doi.org/10.1007/978-3-030-34175-6_23
- 11. Chen, Y., Heizmann, M., Lengál, O., Li, Y., Tsai, M., Turrini, A., Zhang, L.: Advanced automata-based algorithms for program termination checking. In: Foster, J.S., Grossman, D. (eds.) Proceedings of the 39th ACM SIGPLAN Conference on Programming Language Design and Implementation, PLDI 2018, Philadelphia, PA, USA, June 18-22, 2018. pp. 135–150. ACM (2018). https://doi.org/10.1145/3192366.3192405, https://doi.org/10.1145/3192366.3192405
- 12. Clarkson, M.R., Finkbeiner, B., Koleini, M., Micinski, K.K., Rabe, M.N., Sánchez, C.: Temporal logics for hyperproperties. In: Abadi, M., Kremer, S. (eds.) Principles of Security and Trust Third International Conference, POST 2014, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2014, Grenoble, France, April 5-13, 2014, Proceedings. Lecture Notes in Computer Science, vol. 8414, pp. 265–284. Springer (2014).https://doi.org/10.1007/978-3-642-54792-8_15, https://doi.org/10.1007/978-3-642-54792-8_15
- Clemente, L., Mayr, R.: Efficient reduction of nondeterministic automata with application to language inclusion testing. Log. Methods Comput. Sci. 15(1) (2019). https://doi.org/ 10.23638/LMCS-15(1:12)2019, https://doi.org/10.23638/LMCS-15(1:12)2019
- Courcoubetis, C., Yannakakis, M.: Verifying temporal properties of finite-state probabilistic programs. In: 29th Annual Symposium on Foundations of Computer Science, White Plains, New York, USA, 24-26 October 1988. pp. 338–345. IEEE Computer Society

- (1988). https://doi.org/10.1109/SFCS.1988.21950, https://doi.org/10.1109/SFCS.1988.21950
- Duret-Lutz, A., Lewkowicz, A., Fauchille, A., Michaud, T., Renault, E., Xu, L.: Spot 2.0 A framework for LTL and ω-automata manipulation. In: Artho, C., Legay, A., Peled, D. (eds.) Automated Technology for Verification and Analysis 14th International Symposium, ATVA 2016, Chiba, Japan, October 17-20, 2016, Proceedings. Lecture Notes in Computer Science, vol. 9938, pp. 122–129 (2016). https://doi.org/10.1007/978-3-319-46520-3_8, https://doi.org/10.1007/978-3-319-46520-3_8
- Duret-Lutz, A., Renault, E., Colange, M., Renkin, F., Aisse, A.G., Schlehuber-Caissier, P., Medioni, T., Martin, A., Dubois, J., Gillard, C., Lauko, H.: From Spot 2.0 to Spot 2.10: What's new? In: Shoham, S., Vizel, Y. (eds.) Computer Aided Verification 34th International Conference, CAV 2022, Haifa, Israel, August 7-10, 2022, Proceedings, Part II. Lecture Notes in Computer Science, vol. 13372, pp. 174–187. Springer (2022). https://doi.org/10.1007/978-3-031-13188-2_9, https://doi.org/10.1007/978-3-031-13188-2_9
- 17. Esparza, J., Kretínský, J., Raskin, J., Sickert, S.: From LTL and limit-deterministic Büchi automata to deterministic parity automata. In: Legay, A., Margaria, T. (eds.) Tools and Algorithms for the Construction and Analysis of Systems 23rd International Conference, TACAS 2017, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2017, Uppsala, Sweden, April 22-29, 2017, Proceedings, Part I. Lecture Notes in Computer Science, vol. 10205, pp. 426–442 (2017). https://doi.org/10.1007/978-3-662-54577-5_25, https://doi.org/10.1007/978-3-662-54577-5_25
- Etessami, K., Wilke, T., Schuller, R.A.: Fair simulation relations, parity games, and state space reduction for Büchi automata. SIAM J. Comput. 34(5), 1159– 1175 (2005). https://doi.org/10.1137/S0097539703420675, https://doi.org/ 10.1137/S0097539703420675
- 19. Fogarty, S., Kupferman, O., Vardi, M.Y., Wilke, T.: Profile trees for Büchi word automata, with application to determinization. Inf. Comput. **245**, 136–151 (2015). https://doi.org/10.1016/j.ic.2014.12.021
- Fogarty, S., Kupferman, O., Wilke, T., Vardi, M.Y.: Unifying Büchi complementation constructions. Log. Methods Comput. Sci. 9(1) (2013). https://doi.org/10.2168/LMCS-9(1:13)2013, https://doi.org/10.2168/LMCS-9(1:13)2013
- Friedgut, E., Kupferman, O., Vardi, M.Y.: Büchi complementation made tighter. Int. J. Found. Comput. Sci. 17(4), 851–868 (2006). https://doi.org/10.1142/S0129054106004145, https://doi.org/10.1142/S0129054106004145
- 22. Gurumurthy, S., Kupferman, O., Somenzi, F., Vardi, M.Y.: On complementing nondeterministic Büchi automata. In: Geist, D., Tronci, E. (eds.) Correct Hardware Design and Verification Methods, 12th IFIP WG 10.5 Advanced Research Working Conference, CHARME 2003, L'Aquila, Italy, October 21-24, 2003, Proceedings. Lecture Notes in Computer Science, vol. 2860, pp. 96–110. Springer (2003). https://doi.org/10.1007/978-3-540-39724-3_10, https://doi.org/10.1007/978-3-540-39724-3_10
- 23. Havlena, V., Lengál, O., Smahlíková, B.: Deciding S1S: down the rabbit hole and through the looking glass. In: Echihabi, K., Meyer, R. (eds.) Networked Systems 9th International Conference, NETYS 2021, Virtual Event, May 19-21, 2021, Proceedings. Lecture Notes in Computer Science, vol. 12754, pp. 215–222. Springer (2021). https://doi.org/10.1007/978-3-030-91014-3_15, https://doi.org/10.1007/978-3-030-91014-3_15
- Havlena, V., Lengál, O.: Reducing (to) the ranks: Efficient rank-based Büchi automata complementation. In: Haddad, S., Varacca, D. (eds.) 32nd International Conference on Concurrency Theory, CONCUR 2021, August 24-27, 2021, Virtual Conference. LIPIcs, vol. 203, pp. 2:1–2:19. Schloss Dagstuhl Leibniz-Zentrum für Informatik (2021). https://doi.org/10.4230/LIPIcs.CONCUR.2021.2, https://doi.org/10.4230/LIPIcs.CONCUR.2021.2

- 25. Havlena, V., Lengál, O., Šmahlíková, B.: Complementing Büchi automata with Ranker. In: Shoham, S., Vizel, Y. (eds.) Computer Aided Verification 34th International Conference, CAV 2022, Haifa, Israel, August 7-10, 2022, Proceedings, Part II. Lecture Notes in Computer Science, vol. 13372, pp. 188–201. Springer (2022). https://doi.org/10.1007/978-3-031-13188-2_10, https://doi.org/10.1007/978-3-031-13188-2_10
- 26. Havlena, V., Lengál, O., Šmahlíková, B.: Sky is not the limit tighter rank bounds for elevator automata in Büchi automata complementation. In: Fisman, D., Rosu, G. (eds.) Tools and Algorithms for the Construction and Analysis of Systems 28th International Conference, TACAS 2022, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2022, Munich, Germany, April 2-7, 2022, Proceedings, Part II. Lecture Notes in Computer Science, vol. 13244, pp. 118–136. Springer (2022). https://doi.org/10.1007/978-3-030-99527-0_7, https://doi.org/10.1007/978-3-030-99527-0_7
- 27. Heizmann, M., Hoenicke, J., Podelski, A.: Termination analysis by learning terminating programs. In: Biere, A., Bloem, R. (eds.) Computer Aided Verification 26th International Conference, CAV 2014, Held as Part of the Vienna Summer of Logic, VSL 2014, Vienna, Austria, July 18-22, 2014. Proceedings. Lecture Notes in Computer Science, vol. 8559, pp. 797–813. Springer (2014). https://doi.org/10.1007/978-3-319-08867-9_53, https://doi.org/10.1007/978-3-319-08867-9_53
- 28. Hieronymi, P., Ma, D., Oei, R., Schaeffer, L., Schulz, C., Shallit, J.O.: Decidability for Sturmian words. In: Manea, F., Simpson, A. (eds.) 30th EACSL Annual Conference on Computer Science Logic, CSL 2022, February 14-19, 2022, Göttingen, Germany (Virtual Conference). LIPIcs, vol. 216, pp. 24:1–24:23. Schloss Dagstuhl Leibniz-Zentrum für Informatik (2022). https://doi.org/10.4230/LIPIcs.CSL.2022.24, https://doi.org/10.4230/LIPIcs.CSL.2022.24
- Kähler, D., Wilke, T.: Complementation, disambiguation, and determinization of Büchi automata unified. In: Aceto, L., Damgård, I., Goldberg, L.A., Halldórsson, M.M., Ingólfsdóttir, A., Walukiewicz, I. (eds.) Automata, Languages and Programming, 35th International Colloquium, ICALP 2008, Reykjavik, Iceland, July 7-11, 2008, Proceedings, Part I: Tack A: Algorithms, Automata, Complexity, and Games. Lecture Notes in Computer Science, vol. 5125, pp. 724–735. Springer (2008). https://doi.org/10.1007/978-3-540-70575-8_59, https://doi.org/10.1007/978-3-540-70575-8_59
- 30. Karmarkar, H., Chakraborty, S.: On minimal odd rankings for Büchi complementation. In: Liu, Z., Ravn, A.P. (eds.) Automated Technology for Verification and Analysis, 7th International Symposium, ATVA 2009, Macao, China, October 14-16, 2009. Proceedings. Lecture Notes in Computer Science, vol. 5799, pp. 228–243. Springer (2009). https://doi.org/10.1007/978-3-642-04761-9_18, https://doi.org/10.1007/978-3-642-04761-9_18
- 31. Kupferman, O., Vardi, M.Y.: Weak alternating automata are not that weak. ACM Trans. Comput. Log. **2**(3), 408–429 (2001). https://doi.org/10.1145/377978.377993, https://doi.org/10.1145/377978.377993
- 32. Kurshan, R.P.: Complementing deterministic Büchi automata in polynomial time. J. Comput. Syst. Sci. **35**(1), 59–71 (1987). https://doi.org/10.1016/0022-0000(87)90036-5, https://doi.org/10.1016/0022-0000(87)90036-5
- 33. Lengál, O.: Automata benchmarks (2022), https://github.com/ondrik/automata-benchmarks
- Li, Y., Turrini, A., Feng, W., Vardi, M.Y., Zhang, L.: Divide-and-conquer determinization of Büchi automata based on SCC decomposition. In: Shoham, S., Vizel, Y. (eds.) Computer Aided Verification - 34th International Conference, CAV 2022, Haifa, Israel, August 7-10, 2022, Proceedings, Part II. Lecture Notes in Computer Science, vol. 13372, pp. 152–173.

- Springer (2022). https://doi.org/10.1007/978-3-031-13188-2_8, https://doi.org/10.1007/978-3-031-13188-2_8
- Li, Y., Turrini, A., Zhang, L., Schewe, S.: Learning to complement Büchi automata. In: Dillig, I., Palsberg, J. (eds.) Verification, Model Checking, and Abstract Interpretation 19th International Conference, VMCAI 2018, Los Angeles, CA, USA, January 7-9, 2018, Proceedings. Lecture Notes in Computer Science, vol. 10747, pp. 313–335. Springer (2018).https://doi.org/10.1007/978-3-319-73721-8_15, https://doi.org/10.1007/978-3-319-73721-8_15
- 36. Li, Y., Vardi, M.Y., Zhang, L.: On the power of unambiguity in Büchi complementation. In: Raskin, J., Bresolin, D. (eds.) Proceedings 11th International Symposium on Games, Automata, Logics, and Formal Verification, GandALF 2020, Brussels, Belgium, September 21-22, 2020. EPTCS, vol. 326, pp. 182–198 (2020). https://doi.org/10.4204/EPTCS.326.12, https://doi.org/10.4204/EPTCS.326.12
- 37. Löding, C.: Optimal bounds for transformations of omega-automata. In: Rangan, C.P., Raman, V., Ramanujam, R. (eds.) Foundations of Software Technology and Theoretical Computer Science, 19th Conference, Chennai, India, December 13-15, 1999, Proceedings. Lecture Notes in Computer Science, vol. 1738, pp. 97–109. Springer (1999). https://doi.org/10.1007/3-540-46691-6_8, https://doi.org/10.1007/3-540-46691-6_8
- 38. Löding, C., Pirogov, A.: New optimizations and heuristics for determinization of Büchi automata. In: Chen, Y., Cheng, C., Esparza, J. (eds.) Automated Technology for Verification and Analysis 17th International Symposium, ATVA 2019, Taipei, Taiwan, October 28-31, 2019, Proceedings. Lecture Notes in Computer Science, vol. 11781, pp. 317–333. Springer (2019). https://doi.org/10.1007/978-3-030-31784-3_18, https://doi.org/10.1007/978-3-030-31784-3_18
- Miyano, S., Hayashi, T.: Alternating finite automata on omega-words. Theor. Comput. Sci. 32, 321–330 (1984). https://doi.org/10.1016/0304-3975(84)90049-5, https://doi.org/10.1016/0304-3975(84)90049-5
- Piterman, N.: From nondeterministic Büchi and Streett automata to deterministic parity automata. Log. Methods Comput. Sci. 3(3) (2007). https://doi.org/10.2168/LMCS-3(3:5)2007, https://doi.org/10.2168/LMCS-3(3:5)2007
- 41. Pnueli, A.: The temporal logic of programs. In: 18th Annual Symposium on Foundations of Computer Science, Providence, Rhode Island, USA, 31 October 1 November 1977. pp. 46–57. IEEE Computer Society (1977). https://doi.org/10.1109/SFCS.1977.32, https://doi.org/10.1109/SFCS.1977.32
- 42. Redziejowski, R.R.: An improved construction of deterministic omega-automaton using derivatives. Fundam. Informaticae 119(3-4), 393–406 (2012). https://doi.org/10.3233/FI-2012-744, https://doi.org/10.3233/FI-2012-744
- 43. Safra, S.: On the complexity of omega-automata. In: 29th Annual Symposium on Foundations of Computer Science, White Plains, New York, USA, 24-26 October 1988. pp. 319–327. IEEE Computer Society (1988). https://doi.org/10.1109/SFCS.1988.21948, https://doi.org/10.1109/SFCS.1988.21948
- 44. Safra, S., Vardi, M.Y.: On omega-automata and temporal logic (preliminary report). In: Johnson, D.S. (ed.) Proceedings of the 21st Annual ACM Symposium on Theory of Computing, May 14-17, 1989, Seattle, Washington, USA. pp. 127–137. ACM (1989). https://doi.org/10.1145/73007.73019, https://doi.org/10.1145/73007.73019
- 45. Schewe, S.: Büchi complementation made tight. In: Albers, S., Marion, J. (eds.) 26th International Symposium on Theoretical Aspects of Computer Science, STACS 2009, February 26-28, 2009, Freiburg, Germany, Proceedings. LIPIcs, vol. 3, pp. 661–672. Schloss Dagstuhl-Leibniz-Zentrum für Informatik, Germany (2009). https://doi.org/10.4230/LIPIcs.STACS.2009.1854, https://doi.org/10.4230/LIPIcs.STACS.2009.1854

- 46. Sickert, S., Esparza, J., Jaax, S., Kretínský, J.: Limit-deterministic Büchi automata for linear temporal logic. In: Chaudhuri, S., Farzan, A. (eds.) Computer Aided Verification 28th International Conference, CAV 2016, Toronto, ON, Canada, July 17-23, 2016, Proceedings, Part II. Lecture Notes in Computer Science, vol. 9780, pp. 312–332. Springer (2016). https://doi.org/10.1007/978-3-319-41540-6_17, https://doi.org/10.1007/978-3-319-41540-6_17
- 47. Sistla, A.P., Vardi, M.Y., Wolper, P.: The complementation problem for Büchi automata with applications to temporal logic. Theor. Comput. Sci. 49, 217-237 (1987). https://doi.org/10.1016/0304-3975(87)90008-9, https://doi.org/10.1016/0304-3975(87)90008-9
- 48. The SV-COMP Community: International competition on software verification (2022), https://sv-comp.sosy-lab.org/
- 49. Tsai, M., Fogarty, S., Vardi, M.Y., Tsay, Y.: State of Büchi complementation. Log. Methods Comput. Sci. 10(4) (2014). https://doi.org/10.2168/LMCS-10(4:13)2014, https://doi.org/10.2168/LMCS-10(4:13)2014
- Vardi, M.Y., Wilke, T.: Automata: from logics to algorithms. In: Flum, J., Grädel, E., Wilke, T. (eds.) Logic and Automata: History and Perspectives. Texts in Logic and Games, vol. 2, pp. 629–736. Amsterdam University Press (2008)
- Vardi, M.Y., Wolper, P.: An automata-theoretic approach to automatic program verification (preliminary report). In: Proceedings of the Symposium on Logic in Computer Science (LICS '86), Cambridge, Massachusetts, USA, June 16-18, 1986. pp. 332–344. IEEE Computer Society (1986)
- 52. Yan, Q.: Lower bounds for complementation of omega-automata via the full automata technique. Log. Methods Comput. Sci. 4(1) (2008). https://doi.org/10.2168/LMCS-4(1:5)2008, https://doi.org/10.2168/LMCS-4(1:5)2008

```
1 Function \Delta^{C}((H, M_1, \dots, M_n, \ell), a):
           \mathcal{M} := \emptyset;
 2
           Let AlgRR_i be AlgRR_{P_i}^i;
 3
           fSucc_i := SuccPass^{AlgRR_i} for each 1 \le i \le n;
           fSucc_{\ell} := SuccAct^{AlgRR_{\ell}};
           for ((M_1',c_1),\ldots,(M_n',c_n))\in[\mathtt{fSucc}_i(H,M_i,a)]_{i=1}^n do
                 L_i := \{M_i'\} for each 1 \le i \le n;
 8
                 if M'_{\ell} \in PT^{AlgRR_{\ell}} then
                     \ell' := (\ell \mod n) + 1;
L_{\ell'} := \text{Lift}^{\text{AlgRR}_{\ell'}}(M'_{\ell'});
10
11
                 \mathcal{M} := \mathcal{M} \cup \{ (\delta(H, a), (M''_1, c_1), \dots, (M''_n, c_n), \ell') \mid M''_i \in L_i, \forall 1 \le i \le n \};
12
13
```

Round-Robin Algorithm

In this section, we provide details related to the round-robin algorithm. The roundrobin algorithm works on the level of partial complementation round-robin algorithms AlgRR. We require an instance of the partial round-robin algorithm AlgRR_P to provide the following:

- $T^{AlgRR_P} = PT^{AlgRR_P} \cup AT^{AlgRR_P}$ the type of the macrostates produced by the algorithm consisting of the passive type and the active type;

- Colours $^{\mathrm{AlgRR}_P}$ the set of colours the algorithm produces; $\mathrm{Init}^{\mathrm{AlgRR}_P}$: $2^{\mathrm{PT}^{\mathrm{AlgRR}_P}}$ a function returning the set of initial macrostates; $\mathrm{SuccAct}^{\mathrm{AlgRR}_P}$: $(2^Q \times \mathrm{AT}^{\mathrm{AlgRR}_P} \times \Sigma) \to 2^{\mathrm{T}^{\mathrm{AlgRR}_P} \times \mathrm{Colours}^{\mathrm{AlgRR}_P}}$ transition function returning successors in the active phase;
- SuccPass^{AlgRR}_P: $(2^Q \times PT^{AlgRR}_P \times \Sigma) \rightarrow 2^{PT^{AlgRR}_P \times Colours^{AlgRR}_P}$ function performing lift from a passive state to a set of active states;

 - Lift^{AlgRR} $_P: PT^{AlgRR}_P \to 2^{AT^{AlgRR}}_P \longrightarrow transition function for switching from passive$
- Acc^{AlgRR_P} : $\mathbb{EL}(Colours^{AlgRR_P})$ a function that returns a formula for the acceptance condition.

Let $\mathcal{A} = (Q, \delta, I, F)$ be a BA, P_1, \ldots, P_n be a partitioning, and $AlgRR^1, \ldots, AlgRR^n$ be a sequence of algorithms such that AlgRRⁱ is a partial round-robin complementation algorithm for P_i . The complementation algorithm then produces the

TELA ModComplRR (AlgRR $_{P_1}^1, \ldots, \text{AlgRR}_{P_n}^n, \mathcal{A}$) = $(Q^C, \delta^C, I^C, \Gamma^C, p^C, \text{Acc}^C)$ whose components are defined as follows:

$$\begin{split} &-Q^C=2^Q\times[\mathtt{T}^{\mathtt{AlgRR}^i_{P_i}}]^n_{i=1}\times\{1,\ldots,n\},\\ &-I^C=\{I\}\times\mathtt{Lift}\big(\mathtt{Init}^{\mathtt{AlgRR}^i_{P_1}}\big)\times[\mathtt{Init}^{\mathtt{AlgRR}^i_{P_i}}]^n_{i=2}\times\{1\},\\ &-\Gamma^C=\{0,\ldots,\lambda(k^{\mathtt{AlgRR}^n_{P_n}}-1,n)\},\\ &-\mathtt{Acc}^C=\bigwedge^n_{i=1}\lambda(\mathtt{Acc}^{\mathtt{AlgRR}^i_{P_i}},i), \mathrm{and} \end{split}$$

- δ^C and p^C are defined such that if

$$(H', (M'_1, c_1), \dots, (M'_n, c_n), \ell') \in \Delta^C((H, M_1, \dots, M_n, \ell), a),$$

then δ^C contains the transition $t: (H, M_1, \dots, M_n, \ell) \xrightarrow{a} (H', M'_1, \dots, M'_n, \ell')$, whose colouring is set to $p^C(t) = \bigcup \{c_i \mid 1 \leq i \leq n\}$, and δ^C is the smallest such a set. We also set $p^C(q) = \emptyset$ for every $q \in Q^C$.

In the following, we focus on the correctness condition of our round-robin algorithm. For a run ρ , we use $\rho_{i:j}$ where $i \leq j$ to denote the sequence $\rho_i, \rho_{i+1}, \ldots, \rho_j$. Let \texttt{AlgRR}_P be an instance of partial round-robin complementation algorithm. We use $\texttt{Union}(\texttt{AlgRR}_P)$ to denote the partial complementation algorithm having the same type, the set of colors, the set of initial macrostates, and the acceptance condition as \texttt{AlgRR}_P . The successor function is given as

$$\mathtt{Succ}^{\mathtt{Union}(\mathtt{AlgRR}_P)}(H,M,a) = \begin{cases} \mathtt{SuccAct}^{\mathtt{AlgRR}_P}(H,M,a) & \text{if } M \in \mathtt{AT}^{\mathtt{AlgRR}_P}, \\ S \cup \bigcup_{(M',c) \in S} \mathtt{Lift}^{\mathtt{AlgRR}_P}(H,M') & \text{if } M \in \mathtt{PT}^{\mathtt{AlgRR}_P}, \end{cases}$$

where $S = SuccPass^{AlgRR_P}(H, M, a)$.

Moreover, for \mathcal{A} such that $\mathcal{A} \models_P \varphi_{\mathtt{AlgRR}}$ we say that \mathtt{AlgRR}_P is *consistent* if in the automaton ModCompl(Union(AlgRR_P), \mathcal{A}) the following holds:

- (C1) for each $w \in \mathcal{L}(\mathsf{ModCompl}(\mathsf{Union}(\mathsf{AlgRR}_P), \mathcal{A}))$ there is an accepting run ρ on w such that $\rho_i \in \mathsf{AT}^{\mathsf{AlgRR}_P}$ and $\rho_{i+1} \notin \mathsf{AT}^{\mathsf{AlgRR}_P}$ for infinitely many js;
- (C2) for each accepting run ρ and each $i \in \omega$, we have that there are accepting runs ρ' and ρ'' such that $i > 1 \Rightarrow \rho_{i-1} \in \operatorname{PT}^{\operatorname{AlgRR}_P}$ and $\rho'_{1:i-1} = \rho''_{1:i-1} = \rho_{1:i-1}$ and $\rho'_i \in \operatorname{PT}^{\operatorname{AlgRR}_P}$ and $\rho''_i \in \operatorname{AT}^{\operatorname{AlgRR}_P}$;
- (C3) let ρ be a run. If $\rho_j \in AT^{AlgRR_P}$ and $\rho_{j+1} \notin AT^{AlgRR_P}$ for infinitely many js, then ρ is accepting.

Intuitively, the first condition ensures that for an accepted word, there is a run containing infinitely many switches between the passive and active type. The second condition then expresses that the switch to the active phase can be postponed by a finite number of steps but still preserving the acceptance. The last one expresses that if a run encounters infinitely many switches between the active and passive, this run is accepting. The correctness condition on the partial round-robin algorithm is then given as follows:

Definition 2. We say that AlgRR is correct if for each \mathcal{A} such that $\mathcal{A} \models_P \varphi_{AlgRR}$ we have that AlgRR_P is consistent and $\mathcal{L}(ModCompl(Union(AlgRR_P), \mathcal{A})) = \Sigma^{\omega} \setminus \mathcal{L}(\mathcal{A})$.

Theorem 4. Let \mathcal{A} be a BA, P_1, \ldots, P_n be a partitioning of \mathcal{A} , and $\mathtt{AlgRR}^1, \ldots, \mathtt{AlgRR}^n$ be a sequence of partial round-robin complementation algorithms such that \mathtt{AlgRR}^i is correct for P_i . Then, $\mathcal{L}(ModComplRR(\mathtt{AlgRR}^1_{P_1}, \ldots, \mathtt{AlgRR}^n_{P_n}, \mathcal{A})) = \Sigma^\omega \setminus \mathcal{L}(\mathcal{A})$.

Proof. First, we propose the following auxiliary claim:

Claim 1: Let ρ be an accepting run in ModCompl (Union(AlgRR_P), \mathcal{A}_P) such that there are $i_1, i_2, i_1 \leq i_2$ and $\forall i_1 \leq \ell \leq i_2$ we have $\rho_\ell \in \operatorname{PT}^{\operatorname{AlgRR}_P}$ and $\rho_{i_2+1} \in \operatorname{AT}^{\operatorname{AlgRR}_P}$. Then,

for each $k \ge i_1$ there is an accepting run ρ' such that $\rho'_{k+1} \in \mathsf{AT^{AlgRR}_P}$, $\forall i_1 \le \ell \le k$ we have $\rho'_{\ell} \in \mathsf{PT^{AlgRR}_P}$, and $\rho'_{1:\min(k,i_2)} = \rho_{1:\min(k,i_2)}$.

Proof: Follows directly from a multiple application of (C2).

Since P_1, \ldots, P_n is a partitioning of \mathcal{A} , we have that

$$\bigcup_{i=1}^{n} \mathcal{L}(\mathcal{A}_{P_i}) = \mathcal{L}(\mathcal{A}). \tag{3}$$

Now, we proceed to the proof of the theorem.

- $(\subseteq) \text{ Let } \varrho = (M_1^1, \dots, M_n^1, i_1) \dots (M_1^k, \dots, M_n^k, i_k) \dots \text{ be an accepting run in } \\ \text{ModComplRR}(\text{AlgRR}_{P_1}^1, \dots, \text{AlgRR}_{P_n}^n, \mathcal{A}) \text{ on } w. \text{ From the construction, we have that } \\ M_\ell^1 \dots \text{ is an accepting run in } \\ \text{ModCompl}(\text{Union}(\text{AlgRR}_{P_\ell}^\ell), \mathcal{A}_{P_\ell})) = \bigcap_{\ell=1}^n \Sigma^\omega \setminus \mathcal{L}(\mathcal{A}_{P_\ell}) = \Sigma^\omega \setminus \mathcal{L}(\mathcal{A}). \\ \text{The latter follows from the correctness condition on } \\ \text{AlgRR}_P \text{ and from (3)}. \end{aligned}$
- (\supseteq) Consider a word $w \in \Sigma^\omega \setminus \mathcal{L}(\mathcal{A})$. We show by induction that there is also an accepting run $\varrho = (M_1^1, \dots, M_n^1, i_1) \dots (M_1^k, \dots, M_n^k, i_k) \dots$ in the automaton ModComplRR(AlgRR $_{P_1}^1, \dots, \text{AlgRR}_{P_n}^n, \mathcal{A})$ and moreover for each $j, M_\ell^1 \dots M_\ell^j$ is a prefix of an accepting run of w in ModCompl(Union(AlgRR $_{P_\ell}^\ell), \mathcal{A}_{P_\ell}$) for each ℓ . In the following, when we use (accepting) run, we implicitly mean on w.
 - Base case: Since ρ_1 satisfies the condition (C2), there is an accepting run ρ' from (C2) such that $\rho'(1) \in \operatorname{AT}^{\operatorname{AlgRR}_{p_1}^1}$. Moreover, there are also runs ρ'_2, \ldots, ρ'_n such that $\rho'_i(1) \in \operatorname{PT}^{\operatorname{AlgRR}_{p_1}^1}$. We hence set $\varrho_1 = ((\rho'_1)_1, \ldots, (\rho'_n)_1, 1)$.
 - Inductive case: Let $(M_1^1, \ldots, M_n^1, i_1) \ldots (M_1^k, \ldots, M_n^k, i_k)$ be a sequence of first k macrostates of ϱ . We prove that there is also (k+1)-th macrostate of ϱ . Since $M_{i_k}^k \in \operatorname{AT}^{\operatorname{AlgRR}_{P_{i_k}}^{i_k}}$ and moreover, $M_{i_k}^1 \ldots M_{i_k}^k$ is a prefix of an accepting run ρ' in the automaton ModCompl(Union(AlgRR_{P_{i_k}}^{i_k}), $\mathcal{A}_{P_{i_k}}$). Now assume that $(\rho'_{i_k})_{k+1} \in \operatorname{AT}^{\operatorname{AlgRR}_{P_{i_k}}^{i_k}}$. Since $p_\ell = M_\ell^1 \ldots M_\ell^k$ is the prefix of an accepting run in ModCompl(Union(AlgRR_{P_ℓ}^ℓ), \mathcal{A}_{P_ℓ}) for each $\ell \neq i_k$ and on top of that $M_\ell^k \in \operatorname{PT}^{\operatorname{AlgRR}_{P_\ell}^{\ell}}$. Therefore, from (C2) there are accepting runs ρ'_ℓ from (C2) satisfying that $(\rho'_\ell)_{k+1}$ is of the corresponding passive type. We hence set $((\rho'_1)_{k+1}, \ldots, (\rho'_n)_{k+1}, i_k)$ as the (k+1)-th macrostate of ϱ . Now assume that $(\rho'_{i_k})_{k+1} \notin \operatorname{AT}^{\operatorname{AlgRR}_{P_i}^{i_k}}$. Let $j = (i_k \mod n) + 1$. $p_j = M_j^1 \ldots M_j^k$ is the prefix of an accepting run in ModCompl(Union(AlgRR_{P_j}ⁱ), \mathcal{A}_{P_j}), from Claim 1 we obtain that there is also an accepting run ρ'_j extending p_j such that $(\rho'_j)_{k+1} \in \operatorname{AT}^{\operatorname{AlgRR}_{P_j}^{i_k}}$. Further, since $p_\ell = M_\ell^1 \ldots M_\ell^k$ is the prefix of an accepting run in ModCompl(Union(AlgRR_{P_ℓ}ⁱ), \mathcal{A}_{P_ℓ}) for each $\ell \notin \{j, i_k\}$ and on top of that $M_\ell^k \in \operatorname{PT}^{\operatorname{AlgRR}_{P_\ell}^{\ell}}$. Therefore, from (C2) there are accepting runs ρ'_ℓ from (C2) satisfying that $(\rho'_\ell)_{k+1}$ is of the corresponding passive type. We set $((\rho'_1)_{k+1}, \ldots, (\rho'_n)_{k+1}, j)$ as the (k+1)-th macrostate of ϱ .

It can be easily shown that ϱ is a run in ModCompler RR(Algraph_1, ..., Algraph_n, \mathcal{A}). Since each component contains infinitely many switches between the passive and the active phase, we have from (C3) that each partial run is accepting in the corresponding ModCompler (Union(Algraph_1)) and hence ϱ is accepting as well.

A.1 Complementation of Inherently Weak Components

In this section, we define the algorithm AlgRR = MHRR with a condition φ_{MHRR} specifying the condition that a partition P is inherently weak and *accepting*.

We formalize the instance $MHRR_P$ as follows:

```
\begin{split} &-\operatorname{T}^{\operatorname{MHRR}_P}=\operatorname{PT}^{\operatorname{MHRR}_P}\cup\operatorname{AT}^{\operatorname{MHRR}_P}\text{ where }\operatorname{PT}^{\operatorname{MHRR}_P}=2^P\text{ and }\operatorname{AT}^{\operatorname{MHRR}_P}=2^P\times 2^P\;,\\ &-\operatorname{Colours}^{\operatorname{MHRR}_P}=\{\mathbf{0}\}\;,\\ &-\operatorname{Init}^{\operatorname{MHRR}_P}=\{I\cap P\},\\ &-\operatorname{SuccAct}^{\operatorname{MHRR}_P}(H,(C,B),a)=\begin{cases} \{(\delta(H,a)\cap P,\{\mathbf{0}\})\} &\text{if }B'=\emptyset\text{ for }\\ B'=\delta(B,a)\cap P\\ \{((\delta(H,a)\cap P,B'),\emptyset)\} &\text{otherwise,} \end{cases}\\ &-\operatorname{SuccPass}^{\operatorname{MHRR}_P}(H,C,a)=\{(\delta(H,a)\cap P,\emptyset)\},\\ &-\operatorname{Lift}^{\operatorname{MHRR}_P}(C)=\{(C,C)\}\;,\\ &-\operatorname{Acc}^{\operatorname{MH}_P}=\operatorname{Inf}(\mathbf{0}). \end{split}
```

Lemma 4. The partial round-robin algorithm MHRR is correct.

Proof. (Sketch) Consider a BA \mathcal{A} such that $\mathcal{A} \models_P \varphi_{\text{MHRR}}$ and a word $w \in \mathcal{L}(\text{ModCompl}(\text{Union}(\text{MHRR}_P), \mathcal{A}))$. An accepting run φ must contain infinitely many accepting transitions labeled with $\mathbf{0}$. Such transitions go from the active to the passive state. We can therefore switch to the passive state after emptying B' and then after at least one step switch back to the active state. That satisfies the condition (C1). It is not important how many steps we make before switching back to the active state, but it has to be a finite number. That satisfies the condition (C2). The condition (C3) is also satisfied because we switch from active to passive phase only when B' is empty, i.e., only when an accepting transition is taken.

A.2 Complementation of Deterministic Accepting Components

In the following, we define the algorithm Alg = CSBRR with a condition φ_{CSBRR} specifying the condition that a partition P is deterministic within the SCCs.

We formalize the instance $CSBRR_P$ as below:

```
- \mathsf{T}^{\mathsf{CSBRR}_P} = \mathsf{PT}^{\mathsf{CSBRR}_P} \cup \mathsf{AT}^{\mathsf{CSBRR}_P} where \mathsf{PT}^{\mathsf{CSBRR}_P} = 2^P \times 2^P and \mathsf{AT}^{\mathsf{CSBRR}_P} = 2^P \times 2^P \times 2^P,
- \mathsf{Colours}^{\mathsf{CSB}_P} = \{ \mathbf{0} \},
- \mathsf{Init}^{\mathsf{CSBRR}_P} = \{ (I \cap P, \emptyset) \}
- \mathsf{SuccAct}^{\mathsf{CSBRR}_P} (H, (C, S, B), a) = U such that

• if \delta_F(S, a) \cap P \neq \emptyset, then U = \emptyset,
• otherwise U contains the pair (V, c) where
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* \ V = \begin{cases} (C',S') & \text{if } B^{\star} = \emptyset \text{ for } B^{\star} = \delta_{\operatorname{SCC}}(B,a) \\ (C',S',B^{\star}) & \text{otherwise} \end{cases}
* \ S' = \delta_{\operatorname{SCC}}(S,a) \cap P,
* \ C' = (\delta(H,a) \cap P) \setminus S',
* \ c = \begin{cases} \{\mathbf{0}\} & \text{if } B^{\star} = \emptyset \text{ and} \\ \emptyset & \text{otherwise.} \end{cases}
* \ and \ \text{if } \delta_F(B,a) \cap \delta_{\operatorname{SCC}}(B,a) = \emptyset, \text{ then } U \text{ also contains the pair } ((C'',S''),\{\mathbf{0}\}) \text{ where}
* \ S'' = S' \cup B' \text{ and}
* \ C'' = C' \setminus S''.
- \ \operatorname{SuccPass}^{\operatorname{CSBRR}_P}(H,(C,S),a) = U \text{ such that}
\bullet \ \text{if } \delta_F(S,a) \cap P \neq \emptyset, \text{ then } U = \emptyset,
\bullet \ \text{otherwise } U = \{((C',S'),\emptyset)\} \text{ where}
* \ S' = \delta(S,a) \cap P, \text{ and}
* \ C' = (\delta(C,a) \cap P) \setminus S',
- \ \operatorname{Lift}^{\operatorname{CSBRR}_P}(C,S) = \{(C,S,C)\}
- \ \operatorname{Acc}^{\operatorname{CSBRR}_P} = \operatorname{Inf}(\mathbf{0}).
```

Lemma 5. The partial round-robin algorithm CSBRR is correct.

Proof. (Sketch) Consider a BA \mathcal{A} such that $\mathcal{A} \models_P \varphi_{\text{CSBRR}}$ and a word $w \in \mathcal{L}(\text{ModCompl}(\text{Union}(\text{CSBR}_P), \mathcal{A}))$. An accepting run φ must contain infinitely many accepting transitions labeled with $\mathbf{0}$. Such transitions go from the active to the passive state. We can therefore switch to the passive state after emptying B^* and then after at least one step switch back to the active state. That satisfies the condition (C1). It is not important how many steps we make before switching back to the active state, but it has to be a finite number. That satisfies the condition (C2). The condition (C3) is also satisfied because we switch from active to passive phase only when B^* is empty, i.e., only when an accepting transition is taken.

B Simulation-based Optimizations

In the following, we, as in the main text, fix a BA $\mathcal{A} = (Q, \delta, I, F)$ with the set of maximal SCCs $\{C_1, \ldots, C_n\}$. In this section, we generalize the results introduced in [25]. We use $p \leadsto q$ to denote that q is reachable from p. Let $w \in \Sigma^{\omega}$ be a word. Let Π , Π' be sets of traces over w. We say that Π and Π' are acc-equivalent, denoted as $\Pi \sim \Pi'$ if $\exists \pi \in \Pi : \pi$ is accepting in \mathcal{A} if $\exists \pi' \in \Pi' : \pi'$ is accepting in \mathcal{A} .

We recall here the definition of the tie-breaking function and the pruning relation from Sec. 5.5. Let $\#\colon Q \to \omega$ be a function satisfying the following conditions for each $p,q\in Q$: (i) #(p)=#(q) iff $p,q\in C_i$ and (ii) if $p\leadsto q$ then $\#(p)\le \#(q)$. Let $\sqsubseteq\subseteq Q\times Q$ be a relation on the states of $\mathcal A$ defined as follows: $p\sqsubseteq q$ iff (i) $p\leqslant q$ and (ii) #(p)<#(q) The pruning function $pr\colon 2^Q\to 2^Q$ is then defined for each $S\subseteq Q$ as pr(S)=S' where $S'\subseteq S$ is the smallest set such that $\forall q\in S\exists q'\in S'\colon q\sqsubseteq q'$ Informally, pr removes simulation-smaller states.

Let $\rho = S_1 S_2 \dots$ be a sequence of sets of states and w be a word. We define Π_ρ to be a set of traces over w matching the sets of states. Formally, $\Pi_\rho = \{\pi \mid \pi \text{ over } w, \pi_i \in S_i \text{ for each } i\}$. Further, for a set of states B we use ρ_w^B to denote the sequence $S_1 S_2 \dots$ such that $S_1 = B$, $S_{i+1} = \delta(S_i, w_i)$ for each $i \in \omega$. We use ρ_w to denote ρ_w^I . Moreover, for a given mapping $\theta \colon 2^Q \to 2^Q$ and a sequence of sets of states ρ_w^B we define $\theta(\rho_w^B) = \theta(B)B_2 \dots$ where $B_{i+1} = \theta(\delta(B_i, w_i))$ for each $i \in \omega$. A trace π is eventually simulated by π' if there is some $i \in \omega$ such that $\pi_{i:\omega} \leqslant \pi'_{i:\omega}$.

Lemma 6. Let w be a word. Then, $\Pi_{\rho_w} \sim \Pi_{pr(\rho_w)}$.

Proof. First observe that $\Pi_{pr(\rho_w)}\subseteq \Pi_{\rho_w}$. Therefore, it suffices to show that if there is an accepting trace $\pi\in\Pi_{\rho_w}$, then there is also an accepting trace $\pi'\in\Pi_{pr(\rho_w)}$. We assume the former and we now show that there is $\pi'\in\Pi_{pr(\rho_w)}$ such that π is eventually simulated by π' . If $\pi\in\Pi_{pr(\rho_w)}$ we are done. Now, assume that this is not the case and that there is a maximum set of traces $P=\{\pi^1,\pi^2,\dots\}\subseteq\Pi_{\rho_w}$ with indices $\ell_1<\ell_2<\dots$ such that $p_i=\pi^i_{\ell_i}\sqsubseteq\pi^{i+1}_{\ell_i}=p'_i$ for each i, and moreover $\pi_1=\pi$. From the definition, we further have $\#(p_i)<\#(p'_i)$. Since the numbers are finite and you cannot reach a state with lower number, the sequence eventually stabilizes and hence P is finite. Since the set $P=\{\pi_1,\dots,\pi_n\}$ is maximum and finite, we have $\pi_n\in\Pi_{pr(\rho_w)}$. Moreover, $\pi'=\pi_n$ eventually \leq -simulates π (given by the step-wise property of simulation), which concludes the proof.

Consider a function $\theta\colon 2^Q\to 2^Q$ and let the $\operatorname{run} DAG$ of $\mathcal A$ over a word w wrt. θ be a DAG (directed acyclic graph) $\mathcal G_w^\theta=(V,E)$ containing vertices V and edges E such that

```
- V \subseteq Q \times \omega such that (q, i) \in V iff q \in \tau_i where \tau = \theta(\rho_w^I),

- E \subseteq V \times V such that ((q, i), (q', i')) \in E iff i' = i + 1 and q' \in \delta(q, w_i).
```

We use \mathcal{G}_w to denote \mathcal{G}_w^{id} . We say that \mathcal{G}_w^{θ} is accepting if there is a path in the graph encountering infinitely many times a vertex corresponding to accepting state/transition.

Lemma 7. Let w be a word. Then, \mathcal{G}_{w}^{pr} is accepting iff \mathcal{G}_{w} is accepting.

Proof. Follows directly from Lemma 6.

All of the MH, CSB, CoB, and RNK complementation algorithms can be seen as procedures taking a run DAG as an input and checking whether this graph is accepting or not. Therefore, we can change the DAG in an "arbitrary" way, if we ensure that the modified DAG is accepting iff the original one is. Therefore, we can modify each of the algorithms to taking into account the pruned run DAG. This means that we can change each call of the functions $\delta(W,a)$ and $\delta_F(W,a)$ for $pr(\delta(W,a))$ and $pr(\delta_F(W,a))$ respectively. It is due to the fact, that each of the algorithms work on the level of run DAGs, which are created by these transition functions.

C Rank-based Complementation of a NAC

In this section, we define the algorithm RNK with the condition φ_{RNK} specifying that a partition block P is a general nondeterministic accepting component. Let P be a partition of \mathcal{A} such that $\mathcal{A}_P \models \varphi_{\text{RNK}}$. Here, we consider an instantiation of the framework for a rank-based complementation procedure for P (we use a modification of Schewe's optimal algorithms from [45]) and its optimizations from [24,26]. Let us start with some definitions.

For $i \in \omega$ we use $[\![i]\!]$ to denote the largest even number smaller or equal to i, e.g., $[\![42]\!] = [\![43]\!] = 42$. Let $f \colon X \to Y$ be a function. We use dom(f) to denote the domain of f. For a function g, we use $f \lhd g$ to denote the function such that for each $x \in X$ if $x \in dom(g)$ it returns g(x), otherwise f(x). Let $Q^{\blacksquare}_P = P \cup \{\blacksquare\}$ where \blacksquare is a fresh symbol (which is used to represent ranks of runs outside P; we pronounce \blacksquare as "box") and reach(q) denote the set of states reachable from q in \mathcal{A} . Given a set of states $T \subseteq Q$, we define δ^P_P as follows:

$$- \ \delta_P^T(\blacksquare, a) = (\delta(T \setminus P, a) \cap P) \cup A \quad \text{where } A = \begin{cases} \{\blacksquare\} & \text{if } reach(\delta(T, a) \setminus P) \cap P \neq \emptyset, \\ \emptyset & \text{otherwise.} \end{cases}$$
$$- \ \delta_P^T(q, a) = \delta(q, a) \cap P \quad \text{for } q \neq \blacksquare.$$

Intuitively, δ_P^T is used to take into account runs outside of P (represented collectively by \blacksquare). We also extend δ_P^T to sets of states as usual.

Now, we proceed to the definition of rankings for the modified rank-based procedure. A Q_P^\bullet -ranking is a (partial) function $f\colon Q_P^\bullet \to \{0,\dots,2|Q_P^\bullet|\}$. The rank of f is the value $rank(f) = \max\{f(q) \mid q \in Q_P^\bullet\}$. For a set $S \subseteq Q_P^\bullet$, a ranking f is called S-tight if (i) r = rank(f) is an odd number, (ii) f is onto $\{1,3,\dots,r\}$, and (iii) dom(f) = S. Note that because in our definition, a ranking is a partial function, the ranking's domain tells us which states are active; therefore, we do not need to keep a separate set for this (the set S used in [45]). Furthermore, we say that a Q_P^\bullet -ranking f is \bullet -tight iff the following holds:

- (i) f is dom(f)-tight and
- (ii) if $\blacksquare \in dom(f)$ then $f(\blacksquare) = rank(f)$, and $rank(f \setminus \{\blacksquare \mapsto f(\blacksquare)\}) < rank(f)$.

Intuitively, a \blacksquare -tight ranking is tight over its domain and if the domain contains \blacksquare , the rank of \blacksquare is strictly larger than the rank of any state from P.

For a pair of Q_P^{\blacksquare} -rankings f and g, we define $f \Leftrightarrow_T^a g$ iff the following hold:

- (i) $dom(g) = \delta_P^T(dom(f), a),$
- (ii) for each $q \in dom(f)$ and $q' \in \delta_P^T(q, a)$ we have $g(q') \leq f(q)$, and
- (iii) for each $q \in dom(f) \setminus \{\blacksquare\}$ and $q' \in \delta_F(q, a) \cap P$ it holds that $g(q') \leq \|f(q)\|$.

We further define $f_1 \leq^{\bullet} f_2$ iff $rank(f_1) = rank(f_2)$ and for each $q \in dom(f_1)$ we have $f_1(q) \leq f_2(q)$. We also define

$$maxrank_T(f, a) = \begin{cases} g_{\text{max}} & \text{if } g_{\text{max}} = \max_{\leq \bullet} \{g \mid f \Leftrightarrow_T^a g\} \text{ is tight} \\ \bot & \text{otherwise} \end{cases}$$
 (4)

Finally, we are ready to give an instantiation of the rank-based complementation procedure for the decomposition-based construction. We start with the definition of types:

- 1. $PT^{RNK_P} = PT^{wait}_{P} \cup PT^{tight}_{P}$ where
 - $PT_{P}^{wait} = 2^{Q_{P}^{\bullet}}$. This part represents the waiting part of the complemented BA. Note that we use $2^{\mathbb{Q}_{p}^{\blacksquare}}$ instead of 2^{P} ; this is because we need to keep track whether there is some run that can reach P (represented using \blacksquare).
 - $-\operatorname{PT}_P^{tight}=\{f\mid f\text{ is a }\blacksquare\text{-tight }Q_P^\blacksquare\text{-ranking}\}.$
- 2. $AT^{RNK_P} = AT^{wait}_P \cup AT^{tight}_P$ where

 - $\operatorname{AT}_P^{wait} = 2^{Q_P^{\bullet}} \text{ and }$ $\operatorname{AT}_P^{tight} = \{(f, O, i) \mid f \text{ is a \bullet-tight } Q_P^{\bullet}\text{-ranking, } O \subseteq dom(f) \cap f^{-1}(i),$ $i \in \{0, 2, \dots, rank(f) - 1\}\}.$

The instance RNK_P implements the MaxRank construction (the version from the paper [24]) without η_4 (η_4 is responsible for nondeterministically decreasing ranks) for SuccPass^{RNK}P. Then, for SuccAct^{RNK}P we use the standard MaxRank, where a macrostate may have two nondeterministic successors. Formally, the instance RNKP provides the following functions:

- $$\begin{split} & \ \, \operatorname{Init}^{\mathsf{RNK}_P} = \big\{ (I \cap P) \cup A \mid \text{if } reach(I) \cap P \neq \emptyset \text{ then } A = \{\blacksquare\} \text{ else } A = \emptyset \big\}. \\ & \ \, \operatorname{SuccPass}^{\mathsf{RNK}_P} \text{ is defined as follows based on the type of the partial macrostate:} \\ \bullet & \ \, \operatorname{SuccPass}^{\mathsf{RNK}_P}(T,S,a) = \big\{ (\delta_P^T(S,a),\emptyset) \big\} \qquad \text{if } S \in \operatorname{PT}_P^{wait} \text{ else} \\ \bullet & \ \, \operatorname{SuccPass}^{\mathsf{RNK}_P}(T,f,a) = \begin{cases} \{(f',\emptyset)\} & \text{if } f' = maxrank_T(f,a) \neq \bot, \\ \emptyset & \text{otherwise.} \end{cases} \end{split}$$
- Lift^{RNK}P also takes into account the type of the partial macrostate:
 - Lift^{RNK} $_{P}(S) = \{S\} \cup \{(g, g^{-1}(0), 0) \mid g \in \max_{\leq \bullet} \{f \mid f \text{ is } \blacksquare \text{-tight,} \}$
 - $\operatorname{Lift}^{\operatorname{RNK}_P}(f) = \{(f, O', 0)\}$ where
 - $* O' = f^{-1}(0).$
 - * Note that it can happen that $\operatorname{Lift}^{\operatorname{RNK}_P}(f) = \emptyset$, e.g., when $SuccPass^{RNK_P}(T, f, a) = \emptyset.$
- $SuccAct^{RNK_P}$ is defined as follows based on the type of the partial macrostate:
 - $\bullet \ \operatorname{SuccAct}^{\operatorname{RNK}_P}(T,S,a) = \begin{cases} \left\{ (U,\{\mathbf{0}\}) \right\} \text{ with } U = \delta_P^T(S,a) \in \operatorname{PT}^{\operatorname{RNK}_P} & \text{if } S \subseteq \{\blacksquare\} \\ \operatorname{Lift}^{\operatorname{RNK}_P}(U) \in \operatorname{AT}^{\operatorname{RNK}_P} & \text{otherwise} \end{cases}$

 $dom(f) = S\}$

- SuccAct^{RNK} $_P(T,(f,O,i),a)$: let us first define functions η_3 and η_4 as follows (for a $Q_{\mathbf{p}}^{\blacksquare}$ -ranking g and $i \in \omega$, we use $i++g=(i+2) \mod (rank(g)+1)$):
 - * if SuccPass^{RNK} $_P(T, f, a) = \{g\}$ then

$$\eta_{3} = \begin{cases} \{(g, \delta_{P}^{T}(O, a) \cap g^{-1}(i), i)\} & \text{if } O \neq \emptyset \\ \{(g, \delta_{P}^{T}(dom(f), a) \cap g^{-1}(i++g), i++g)\} & \text{if } O = \emptyset \end{cases}$$

$$\cdot \eta_{4} = \{(g', M, i) \mid g' = g \triangleleft \{q \mapsto g(q) - 1 \mid q \in M, i \neq 0\}\} \text{ where } M = \delta_{f}^{T}(O, a) \cap g^{-1}(i)$$

- * else $\eta_3(T, (f, O, i), a) = \eta_4(T, (f, O, i), a) = \emptyset$.

Let
$$U = \{(f', O', i') \in \eta_3 \cup \eta_4 \mid O' = \emptyset \land i' = rank(f') - 1\}$$
. We then define SuccAct^{RNK} $_P(T, (f, O, i), a) =$

$$\{((f', O', i'), \emptyset) \mid (f', O', i') \in \eta_3 \cup \eta_4 \setminus U\} \cup \{(f', \{0\}) \mid (f', O', i') \in U\}$$

Note that in the previous, the first set is from AT^{RNK_P} and the other is from PT^{RNK_P} .

The correctness is then summarized by the following lemmas.

Lemma 8. The partial algorithm Union(RNK) is correct.

Proof. (Sketch) In this proof, we use MaxRank to denote the MaxRank algorithm from [24] and InitDet to denote the procedure performing subset construction of the initial part of the automaton with no accepting transitions. Consider a BA \mathcal{B} such that $\mathcal{B} \models_P \varphi_{\text{RNK}}$. Moreover, we assume that \mathcal{B} has not redundant states. Consider a run $\rho = (H_1, M_1)(H_2, M_2) \dots$ over a word w in ModCompl(Union(RNK_P), \mathcal{B}). We can construct a run ρ' over w in InitDet(MaxRank(\mathcal{B})) such that ρ'_i is obtained from ρ_i by replacing ■ by $H_i \setminus dom(f_i)$ where f_i is the ranking function of the macrostate M_i . It can be quite easily shown that ρ is accepting iff ρ' is accepting. □

Lemma 9. The partial round-robin algorithm RNK is correct.

Proof. (*Sketch*) We start with the condition (C1). For an arbitrary word $w \in \mathcal{L}(\mathsf{ModCompl}(\mathsf{ENK}_P), \mathcal{B}))$ we can construct an accepting run ρ such that after we flush the O-set ($O = \emptyset$), we can switch for a single step to the passive state and then back to the active in the following step. Since we need to empty the O-set infinitely often, it does not matter when we make a new sample (we must just ensure that we do a finite number of steps in the passive phase)—which also fulfills the condition (C2). The condition (C3) follows from the fact that the switch from the active to passive phase is done only if the O-set becames empty (hence infinitely many switches mean that the run is accepting). The rest of the correctness follows from Lemma 8.

D Additional Examples

In this section, we provide additional examples to the optimization. The example of the postponed construction depicting also intermediate steps of the construction is shown in Fig. 3. The example of the round-robin algorithm is shown in Fig. 4.

The example of the complementation of initial deterministic partition block is shown in Fig. 5:

E Missing Proofs from the Main Text

E.1 Proofs of Sec. 3

Theorem 1. Let \mathcal{A} be a BA, P_1, \ldots, P_n be a partitioning of \mathcal{A} , and $\mathtt{Alg}^1, \ldots, \mathtt{Alg}^n$ be a sequence of partial complementation algorithms such that \mathtt{Alg}^i is correct for P_i . Then, we have $\mathcal{L}(ModCompl(\mathtt{Alg}^1_{P_1}, \ldots, \mathtt{Alg}^n_{P_n}, \mathcal{A})) = \Sigma^{\omega} \setminus \mathcal{L}(\mathcal{A})$.

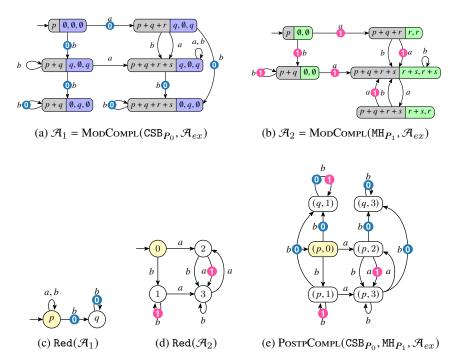


Fig. 3: Example of the postponed construction applied on \mathcal{A}_{ex} with the result's accepting condition Acc: $Inf(0) \wedge Inf(1)$.

Proof. Let $\mathcal{A} = (Q, \delta, I, F)$ be a BA. Moreover, since P_1, \ldots, P_n is partitioning of \mathcal{A} , we have that

$$\bigcup_{i=1}^{n} \mathcal{L}(\mathcal{A}_{P_i}) = \mathcal{L}(\mathcal{A}) \tag{5}$$

In the first part of the proof, we prove the following claim

Claim 2:

$$\bigcap_{i=1}^{n} \mathcal{L}(ModCompl(\mathsf{Alg}_{P_i}^i, \mathcal{A}_{P_i})) = \mathcal{L}(ModCompl(\mathsf{Alg}_{P_1}^1, \dots, \mathsf{Alg}_{P_n}^n, \mathcal{A})) \tag{6}$$

 $\begin{array}{l} \underline{\operatorname{Proof:}} \; (\subseteq) \; \operatorname{Consider} \; \operatorname{a} \; \operatorname{word} \; \alpha \; \in \; \bigcap_{i=1}^n \mathcal{L}(\operatorname{ModCompl}(\operatorname{Alg}_{P_i}^i, \mathcal{A}_{P_i})). \; \text{Then, there are accepting runs} \; \rho_i \; \operatorname{of the form} \; \rho_i = (H_i^1, M_i^1)(H_i^2, M_i^2) \ldots \; \operatorname{such that} \; H_i^{\ell+1} = \delta(H_i^\ell, \alpha_\ell) \; \operatorname{and} \; M_i^{\ell+1} \; \in \; \operatorname{Succ}^{\operatorname{Alg}_{P_i}^i}(H_i^\ell, M_i^\ell, \alpha_\ell) \; \operatorname{for \; each} \; 1 \; \leq \; i \; \leq \; n. \; \operatorname{From \; the \; definition \; of } \; \operatorname{these \; runs} \; \operatorname{we \; have \; that} \; H_i^\ell = H_j^\ell \; \operatorname{for \; each} \; 1 \; \leq \; i, j \; \leq \; n. \; \operatorname{Therefore, \; there \; is \; also } \; \operatorname{a \; run} \; \varrho = (H_1^1, M_1^1, M_2^1, \ldots, M_n^1)(H_1^2, M_1^2, M_2^2, \ldots, M_n^2) \ldots \; \operatorname{over} \; \alpha \; \operatorname{in \; the \; automaton \; ModCompl}(\operatorname{Alg}_{P_1}^1, \ldots, \operatorname{Alg}_{P_n}^n, \mathcal{A}). \; \operatorname{Since} \; \rho_i \; \models \; \operatorname{Acc}^{\operatorname{Alg}_{P_i}^i} \; \operatorname{for \; each} \; i, \; \operatorname{we \; also \; have} \; \varrho \; \models \; \bigwedge_{i=1}^n \operatorname{Acc}^{\operatorname{Alg}_{P_i}^i} \; \operatorname{implying \; that} \; \varrho \; \operatorname{is \; accepting \; in \; ModCompl}(\operatorname{Alg}_{P_1}^1, \ldots, \operatorname{Alg}_{P_n}^n, \mathcal{A}). \end{array}$

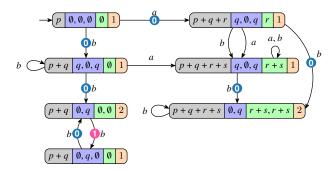


Fig. 4: The outcome of ModComplRR(CSBRR $_{P_0}$, MHRR $_{P_1}$, \mathcal{A}_{ex}) with Acc: Inf(\bigcirc) \land Inf(\bigcirc) applied on the BA from Fig. 1.

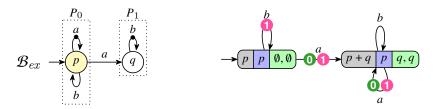


Fig. 5: Left: \mathcal{B}_{ex} . Right: ModCompl(CoB $_{P_0}$, MH $_{P_1}$, \mathcal{B}_{ex}) with Acc: Fin($\mathbf{0}$) \wedge Inf($\mathbf{1}$).

 $(\supseteq) \text{ Consider a word } \alpha \in \mathcal{L}(\text{ModCompl}(\text{Alg}_{P_1}^1, \dots, \text{Alg}_{P_n}^n, \mathcal{A})). \text{ Then, there is an accepting run } \varrho = (H^1, M_1^1, M_2^1, \dots, M_n^1)(H^2, M_1^2, M_2^2, \dots, M_n^2) \dots \text{ over } \alpha. \text{ From the definition of ModCompl, we have that there are runs } \varrho_i = (H^1, M_i^1)(H^2, M_i^2) \dots \text{ on } \alpha \text{ in ModCompl}(\text{Alg}_{P_i}^i, \mathcal{A}_{P_i}) \text{ for each } 1 \leq i \leq n. \text{ Since, } \varrho \models \bigwedge_{i=1}^n \operatorname{Acc}^{\operatorname{Alg}_{P_i}^i}, \text{ we have that each } \varrho_i \text{ is accepting as well.}$

Then we proceed as follows. From Definition 1 we get

$$\mathcal{L}(\mathsf{ModCompl}(\mathsf{Alg}_{P_i}^i,\mathcal{A}_{P_i})) = \Sigma^{\omega} \setminus \mathcal{L}(\mathcal{A}_{P_i})$$

and hence using (5)

$$\bigcap_{i=1}^n \mathcal{L}(\mathsf{ModCompl}(\mathsf{Alg}_{P_i}^i,\mathcal{A}_{P_i})) = \Sigma^\omega \setminus \left(\bigcup_{i=1}^n \mathcal{L}(\mathcal{A}_{P_i})\right) = \Sigma^\omega \setminus \mathcal{L}(\mathcal{A}),$$

which, together with (6), concludes the proof.

E.2 Proofs of Sec. 4

Lemma 1. The partial algorithm MH is correct.

Proof. Let w be an ω -word. Our goal is to prove that w is not accepted within P if and only if $\mathbf{0}$ occurs infinitely often.

Assume that there exists a sequence $\hat{\rho} = (C_0, B_0) \cdots (C_i, B_i) \cdots$ of macrostates over w that emits infinitely often color $\mathbf{0}$. Our goal is to prove that w is not accepted within P. Note that \mathcal{A} is complete, so each run ρ of \mathcal{A} over w is an infinite run. Since all SCCs are accepting and inherently weak in P, we only need to prove that every run ρ entering P will eventually exit P. First, we let ρ enter P at some point, say k > 0. That is, we have $\rho_k \in C_k$. Since $\hat{\rho}$ emits infinitely often the color $\mathbf{0}$, there must be an integer $\ell \geq k$ such that $\delta(B_\ell, w_\ell) \cap C'_{\ell+1} = \emptyset$. It follows that $\rho'_{\ell+1} \in B_{\ell+1} = C'_{\ell+1}$ for all runs branching from ρ will be present in the $B_{\ell+1}$ -set. Again, by assumption, there must be an integer $\ell' \geq \ell$ such that $\delta(B_{\ell'}, w_{\ell'}) \cap C'_{\ell'+1} = \emptyset$. It follows that all runs branching from ρ must have left the B_j -set for all $j > \ell'$. Since $\mathbf{0}$ occurs infinitely often, i.e., there are infinitely many empty B-sets along $\hat{\rho}$, all runs entering P must eventually exit P. Thus, w is not accepted within the partition block P.

Now we assume that w is not accepted within P and show that $\mathbf{0}$ occurs infinitely often. We prove it by contradiction. Suppose that $\mathbf{0}$ occurs only for a finite number of times along the sequence $\hat{\rho} = (C_0, B_0) \cdots (C_i, B_i) \cdots$ of macrostates over w. Then there exists an integer k > 0 such that $B_j \neq \emptyset$ for all $j \geq k$. It follows that $B_{j+1} = \delta(B_j, w_j) \cap C'_{j+1} = \delta(B_j, w_j) \cap \delta(H_j, w_j) \cap P$ for all $j \geq k$, i.e., $B_{j+1} \subseteq \delta(B_j, w_j) \subseteq P$ for all $j \geq k$. By König's lemma, there must be an infinite run ρ within $B \subseteq P$. Since all SCCs in P are accepting and inherently weak, we know that ρ must be accepting, which contradicts the assumption that w is not accepted within P. Thus, we have proved that $\mathbf{0}$ must occur infinitely often.

Therefore, we have proved that w is not accepted within P if and only if $\mathbf{0}$ occurs infinitely often.

Lemma 2. The partial algorithm CSB is correct.

Proof. Let w be an ω -word. Our goal is to prove that w is not accepted within P if and only if there exists an infinite sequence of macrostates $\hat{\rho} = (C_0, S_0, B_1) \cdots$ that emits color $\mathbf{0}$ infinitely often.

First, assume that there exists an infinite sequence of macrostates $\hat{\rho} = (C_0, S_0, B_0) \cdots$ over w that emits infinitely often the color $\mathbf{0}$. We then need to prove that a run ρ of \mathcal{A} over w that enters P will either leave at some time or not be accepting. For simplicity, we let h > 0 be an integer such that ρ_j belongs to the same SCC in P for all $j \ge h$. When a state s on a run transitions from an SCC to a state t in another SCC, we say this run dies out and there is a new run entering P from state t. Assume that the run ρ is present in P at time $k \ge h$. Once ρ is in P, we know that ρ is deterministic, i.e., no branching runs will be derived from ρ . Therefore, we only need to focus on the deterministic run. Let $\rho_k \in C_k$ (the cases when $\rho_k \in B_k$ and $\rho_k \in S_k$ are easier and will be discussed later). There must be an integer $\ell \ge k$ such that $B_{\ell+1}^* = \delta_{\text{SCC}}(B_\ell, w_\ell) = \emptyset$ since $\mathbf{0}$ occurs infinitely often. Thus $(C_{\ell+1}, S_{\ell+1}, B_{\ell+1})$ has two possibilities: (1) $(C'_{\ell+1}, S'_{\ell+1}, B'_{\ell+1} = C'_{\ell+1})$, i.e., ρ is moved to to B-set and (2) $(C''_{\ell+1}, S''_{\ell+1}, B''_{\ell+1} = C''_{\ell+1})$, i.e., ρ is moved to the S-set. Since $\rho_{\ell+1} \in C'_{\ell+1}$, we either have $\rho_{\ell+1} \in B_{\ell+1}$ or $\rho_{\ell+1} \in S_{\ell+1}$. Assume that $\rho_{\ell+1} \in S_{\ell+1}$. Since ρ emits infinitely often $\mathbf{0}$, $\hat{\rho}$ must be infinite. That is, $\delta_F(B_j, w_j) \cap P = \emptyset$ for all $j \ge \ell+1$. Since $\rho_j \in \delta_{\text{SCC}}(B_j, w_j) \cap P$ for all $j \ge \ell+1$, by definition, we have $\delta_F(B_j, w_j) = \emptyset$ for all $j \ge \ell+1$. That is, ρ must not visit accepting transitions any more after $j \ge \ell+1$;

otherwise $\hat{\rho}$ will be finite if there is some j such that $\delta_F(B_j, w_j) \neq \emptyset$. If $\rho_{\ell+1} \in B_{\ell+1}$, we know that there must exist an integer $\ell' \geq \ell$ such that $\delta_{\text{SCC}}(B_{\ell'}, w_{\ell'}) = \emptyset$. That is, we have either $\rho_{\ell'+1} \notin P$ or $\rho_{\ell'+1} \in P$ but $\rho_{\ell'+1}$ and $\rho_{\ell'}$ are not in the same SCC. In the latter case, we treat ρ as died out and there will be a new run in C. Since ρ will eventually stay in an SCC forever, it is easy to see that ρ will either be in B or leave P. Therefore we have proved that all runs of $\mathcal A$ over W that enter P will either leave at some time or not be accepting.

Now, assume that w is not accepted within P. Our goal is to prove that there exists an infinite sequence of macrostates $\hat{\rho} = (C_0, S_0, B_0) \cdots$ over w that emits infinitely often the color $\mathbf{0}$. Since every run ρ of \mathcal{A} over w will either leave P or become safe, we can construct such an infinite sequence $\hat{\rho}$. First, we need $\hat{\rho}$ to be infinite and we only need to be careful about the condition $\delta_F(S, a) \neq \emptyset$. All runs in S can be seen as coming from B (including (C'', S'', C'')) as it still needs to first compute B'). We only need to resolve the nondeterministic choices when constructing $\hat{\rho}$. If B is empty all the time, we are done. Otherwise let k be the smallest integer when $B \neq \emptyset$. That is, the current macrostate is (C_k, S_k, B_k) . Since all runs in B are deterministic, we can do standard construction (by following the successor (C', S', B')) until either we reach a point where all runs in the B-set die out or become safe. If B becomes empty, we still follow the successor (C', S', B') and the construction will emit $\mathbf{0}$. It can happen that all runs in B become safe since the number of runs in B is finite and they will be safe eventually by assumption. In such a case, it is easy to see that $\delta_F(B, a) \cap \delta_{SCC}(B, a) = \emptyset$. Then we follow the successor (C'', S'', C'') this time and emit 0. Since all the runs we move to S are safe, so $\delta_F(S,a) \neq \emptyset$ will not be satisfied in future. In this way, we obtain a macrostate $(C_{k+\ell}, S_{k+\ell}, B_{k+\ell})$ for some $\ell \geq 1$. We can repeat the above procedure and construct an infinite sequence of macrostates $\hat{\rho}$ over w that emits infinitely often the color **0**.

Lemma 3. The partial algorithm CoB is correct.

Proof. Let w be an ω -word. We need to prove that w is not accepted in P if and only if we receive only finitely many times the color $\mathbf{0}$.

First, we prove that direction from right to left by contraposition. By assumption, we have finitely many occurrences of the color $\mathbf{0}$ along the word w. Suppose that w is accepted in P. There must exist an accepting run ρ that eventually stays in P. It is easy to see that ρ is accepted by the reduced deterministic BA \mathcal{A}_P . Let $k \geq 0$ be the smallest integer such that $\rho_k \in P$. Therefore, we have $\operatorname{Succ}^{\operatorname{CoB}_P}(H_j, \rho_j, w_j) = \{(\rho_{j+1}, \alpha_{j+1})\}$ for all $j \geq k$. Since ρ will visit infinitely many accepting transitions, we will also see infinitely often the color $\mathbf{0}$. This leads to a contradiction to our assumption. Thus, w cannot be accepted in P.

Second, we prove the other direction also by contraposition. By assumption, w is not accepted in P. Assume that we see infinitely many $\mathbf{0}$ and the sequence of macrostates over w is $\hat{\rho}$. Then there must be infinitely many integers k>0 such that $\hat{\rho}_k, \hat{\rho}_{k+1} \in P$ and $\hat{\rho}_k \stackrel{w_k}{\to} \hat{\rho}_{k+1} \in F$. If $\hat{\rho}_j \in P$ for all $j \geq k$, we must have an accepting run in P, which contradicts the assumption that w is not accepted in P. So there must be some integer $\ell > k$ such that $\delta(H_\ell, w_\ell) \cap P = \emptyset$. This indicates that every run starting from $\hat{\rho}_k$ is finite. Since P is deterministic, it follows that every run over w that enters P is finite,

therefore w is not accepted in P. Contradiction. Thus, we have proved that if w is not accepted in P, we only can see finitely many times the color $\mathbf{0}$.

Theorem 2. Let \mathcal{A} be an elevator automaton with n states. Then there exists a BA with $O(4^n)$ states accepting the complement of $\mathcal{L}(\mathcal{A})$.

Proof. Assume that Q_D is the union of all SCCs of \mathcal{A} satisfying φ_{CSB} , Q_W is the union of all SCCs satisfying φ_{MH} and Q_N is the union of all nonaccepting SCCs; moreover $Q_D \cap Q_W = \emptyset$, $Q_N \cap Q_W = \emptyset$ and $Q_N \cap Q_D = \emptyset$. Since \mathcal{A} is elevator, $Q_D \cup Q_W \cup Q_N$ is the set of all states in \mathcal{A} and $Q_D \cup Q_W$ is the union of all partition blocks of \mathcal{A} . From Theorem 1, Lemma 1, and Lemma 2 we have that $\mathcal{L}(\text{ModCompl}(\text{CSB}_{Q_D}, \text{MH}_{Q_W}, \mathcal{A})) = \Sigma^\omega \setminus \mathcal{L}(\mathcal{A})$. We now compute the number of states of ModCompl(CSB $_{Q_D}, \text{MH}_{Q_W}, \mathcal{A})$. For a state $q \in Q_D$ there are 4 possibilities of distributing q within $T^{\text{CSB}_{Q_D}}$: (i) $q \notin C \cup S$, (ii) $q \in C$, (iii) $q \in C$, (iii) $q \in C \cap B$, (iv) $q \in S$. For a state $q \in Q_W$ there are 3 possibilities of distributing q within $T^{\text{MH}_{Q_W}}$: (i) $q \notin C$, (ii) $q \in C \cap B$. Lastly, for a state $q \in Q_N$ there are 2 possibilities of distributing q within the reachable states $H: q \in H$ or $q \notin H$. Therefore, the number of macrostates is given as $4^{|Q_D|} \cdot 3^{|Q_W|} \cdot 2^{|Q_N|} \in O(4^n)$.

E.3 Proofs of Sec. 5

Theorem 3. Let \mathcal{A} be a BA, P_1, \ldots, P_n be a partitioning of \mathcal{A} , and $\mathtt{Alg}^1, \ldots, \mathtt{Alg}^n$ be a sequence of partial complementation algorithms such that \mathtt{Alg}^i is correct for P_i . Then, $\mathcal{L}(PostpCompl(\mathtt{Alg}^1_{P_1}, \ldots, \mathtt{Alg}^n_{P_n}, \mathcal{A})) = \Sigma^{\omega} \setminus \mathcal{L}(\mathcal{A})$.

Proof. From Claim 2 and Theorem 1 we have that

$$\bigcap_{i=1}^{n} \mathcal{L}(\mathsf{ModCompl}(\mathsf{Alg}_{P_{i}}^{i},\mathcal{A}_{P_{i}})) = \Sigma^{\omega} \setminus \mathcal{L}(\mathcal{A}).$$

Since reduction Red preserves the language, we have $\mathcal{L}(\mathsf{ModCompl}(\mathsf{Alg}_{P_i}^i, \mathcal{A}_{P_i})) = \mathcal{L}\left(\mathsf{Red}\left(\mathsf{ModCompl}(\mathsf{Alg}_{P_i}^i, \mathcal{A}_{P_i})\right)\right)$ for each i, which concludes the proof.

F Additional Plots from the Experiments

In this section we present more plots about the outcomes of the experiments.

In Fig. 6 we provide a cactus plot presenting for each tool, including the virtual best solvers, the number of benchmarks (on the x axis) successfully complemented within the time given on the y axis; the more the plot is near the right border, the better the tool behaves. Fig. 7 provides a clearer view of the part of the plot in Fig. 6 above 39,000 states. As we can see from the plots, Spot is the clear winner when considering the time needed to complement the input TBA, since its plot is almost superimposed to the one of both VBS; this confirms the high quality and maturity of Spot and the several techniques it implements to manage at the best Büchi automata operations. Kofolap is slightly better than Kofolas and very close to COLA on the automata requiring short time to be complemented; then both versions of Kofola behave similarly with COLA being a bit faster in producing larger automata.

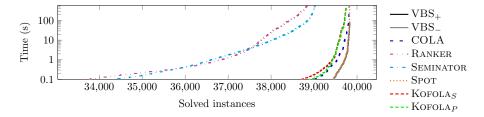


Fig. 6: Cactus plot showing the number of instances solved by each tool within the time on the y axis.

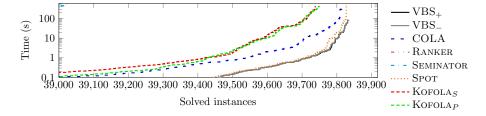


Fig. 7: Particular of the cactus plot in Fig. 6.