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ELIMINATION OF UNKNOWNS FOR SYSTEMS OF ALGEBRAIC DIFFERENTIAL-DIFFERENCE EQUATIONS

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Abstract. We establish effective elimination theorems for ordinary differential-difference equations. Specifically, we find a computable function $B(r,s)$ of the natural number parameters r and s so that for any system of algebraic ordinary differential-difference equations in the variables $\boldsymbol{x} = x_1, \ldots, x_q$ and $\boldsymbol{y} = y_1, \ldots, y_r$, each of which has order and degree in $\boldsymbol{y}$ bounded by s over a differential-difference field, there is a nontrivial consequence of this system involving just the $\boldsymbol{x}$ variables if and only if such a consequence may be constructed algebraically by applying no more than $B(r,s)$ iterations of the basic difference and derivation operators to the equations in the system. We relate this finiteness theorem to the problem of finding solutions to such systems of differential-difference equations in rings of functions showing that a system of differential-difference equations over $\mathbb{C}$ is algebraically consistent if and only if it has solutions in a certain ring of germs of meromorphic functions.

1. Introduction

Differential-difference equations, or what are sometimes called delay differential equations, especially when the independent variable represents time, are ubiquitous in applications. See for instance [25] and the collection it introduces for a discussion of applications of delay differential equations in biology, the discussion of the follow-the-leader model in [20] for the use of differential-difference equations to model crowd behavior, and [1] for a thorough discussion of theory of delay differential equations and their applications to population dynamics and other fields. Much work has been undertaken in the analysis of the behavior of the solutions of these equations. We take up and solve parallel problems. First, we address the problem of determining the consistency of a system of algebraic differential-difference equations and, more generally, of eliminating variables for such a system of equations. Secondly, we ask and answer the question of what structures should serve as the universal differential-difference rings in which we seek our solutions to these equations.

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Our solution to the first problem, that is, of performing effective elimination for systems of differential-difference equations, is achieved by reducing the problem for differential-difference equations to one for ordinary polynomial equations to which standard methods in computational algebra may be applied. Let us state our main theorem, Theorem 3.1, now. Precise definitions are given in Section 2. We show that there is a computable function $B(r,s)$ of the natural number parameters r and s so that whenever one is given tuples of variables $\boldsymbol{x} = x_1, \dots, x_q$ and $\boldsymbol{y} = y_1, \dots, y_r$ and a set F of differential-difference polynomials in these variables each of which has order and degree in $\boldsymbol{y}$ bounded by s over some differential-difference field, then the differential-difference ideal generated by F, that is, the ideal generated by the elements of F and all of their transforms under iterated applications of the distinguished difference and derivation operators, contains a nontrivial differential-difference polynomial in just the $\mathbf{x}$ variables if and only if the ordinary ideal generated by the transforms of elements of F of order at most $B = B(r,s)$ already contains such a nontrivial differential-difference polynomial in $\boldsymbol{x}$. In particular, taking $q = 0$, this gives a procedure to test the consistency of a system of differential-difference equations.

The reader may rightly object that rather than giving a method for determining consistency of such a system of equations, what we have really done is to give a method for testing whether there is an explicit algebraic obstruction to the existence of a solution. In what sense must a solution actually exist if there is no such algebraic obstruction? This brings us to our second question of where to find the solutions. We address this problem in Section 5, in which we begin by proving an abstract Nullstellensatz theorem to the effect that solutions may always be found in differential-difference rings of sequences constructed from differential-difference fields. Of course, in practice, one might expect that the differential-difference equations describe *functions* for which the difference or delay operator takes the form $\sigma(f)(t) = f(t-\tau)$ (for some fixed parameter τ) and the derivation operator is given by usual differentiation so that $\delta f = \frac{df}{dt}$. We establish with Proposition 5.7 that certain rings of *germs* of meromorphic functions serve as universal differential-difference rings in the sense that every algebraically consistent system of differential-difference equations over $\mathbb{C}$ has solutions in these rings of germs. As a complement to this positive result, we show that there are algebraically consistent differential-difference equations that cannot be solved in any ring of meromorphic *functions* (as opposed to germs).

The method of proof of our main theorem is modeled on the approach taken by three of the present authors in [23] for algebraic difference equations in that we modify and extend the decomposition-elimination-prolongation (DEP) method. However, we encounter some very substantial obstacles in extending these arguments to the differential-difference context. First of all, we argue by reducing from differential-difference equations to *differential* equations and then complete the reduction to algebraic equations using methods in computational differential algebra. Differential algebra in the sense of Ritt and Kolchin, especially the Noetherianity of the Kolchin topology, substitutes for classical commutative algebra and properties of the Zariski topology, but there are essential distinctions preventing a smooth substitution. Most notably, in the computation of a bound for the length of a possible skew-periodic train in [23], one argues by induction on the *co*dimension of a certain subvariety. In that algebraic case, since the ambient dimension is finite, such an inductive argument is well-founded. This would fail in the case at hand with differential algebraic varieties. To deal with this difficulty, we must argue with a much subtler induction stepping through a decreasing (and hence finite) chain of Kolchin polynomials.

With other steps of the argument, we must invoke or prove delicate theorems on computational differential algebra for which the corresponding results for ordinary polynomial rings are fairly routine. For instance, a key step in the calculation of our bounds involves computing upper bounds on the number of irreducible components of a differential algebraic variety given bounds on such associated parameters as the degrees of defining equations and the dimensions of certain ambient varieties. In the algebraic case, these bounds are provided by Bézout-type theorems. Here, we work to establish such bounds with Proposition 6.12.

For the most part, the methods we employ could be used to prove analogous theorems for partial differential-difference equations. That is, we would work with a ring $\mathcal{R}$ equipped with a ring endomorphism $\sigma : \mathcal{R} \to \mathcal{R}$ and finitely many commuting derivations $\delta_1, \ldots, \delta_n : \mathcal{R} \to \mathcal{R}$ each of which commutes with σ. Our arguments go through verbatim in this case up to the point of the computation of bounds on the number of irreducible components of a differential variety, and, to our knowledge, it is an open problem whether such bounds exist, much less what the bounds might be. For the bounds we compute, it helps that for ordinary differential fields, the coefficients of the Kolchin polynomial are geometrically meaningful. This is not so for partial differential fields, but the bounds on these coefficients obtained in [15] should make it possible to extract explicit bounds analogous to $B(r, s)$ in the case of partial differential-difference equations once the issue of bounding the number of irreducible components has been resolved.

Interestingly, in contrast to our results, if one would like to extend the results of [23] in another direction, by adding one extra difference operator, the problem of deciding consistency of a system of equations becomes undecidable [24, Proposition 3.9]. It is shown in [24, Theorem 3.7] that the following problem is undecidable: given a difference polynomial and a system of difference equations, determine whether the polynomial vanishes on every solution of the system. This implies that the same problem for differential-difference equations is undecidable as well. Other potential generalizations present themselves, but they, too, lie outside the reach of our present methods. For example, one might wish for an elimination theorem for differential-difference equations in positive characteristic, especially as there is no restriction on the characteristic for the elimination theorem for algebraic difference equations, but many difficulties arise in positive characteristic, starting with the non-Noetherianity of the corresponding differential algebraic topology.

Finally, as a matter of proof technique, our approach is to reduce from equations in several operators to equations in fewer operators and so on until we reach purely algebraic equations. We expect that it may be possible to compute better bounds by making use of integrability conditions in the DEP method to reduce directly from equations with operators to algebraic equations. We do not pursue this idea here.

Some work on elimination theory for differential-difference equations appears in the literature, though the known results do not cover the problems we consider.

In [18] algorithms for computing analogues of Gröbner bases in certain differential-difference algebras are developed. These algebras are rings of linear differential-difference operators. So, the resulting elimination theorems are appropriate for linear equations, but not for the nonlinear differential-difference equations we consider. Gröbner bases of a different kind for rings of differential-difference polynomial rings are considered in [16, 29] with the aim of computing generalized Kolchin polynomials. In these papers, the invariants are computed in *fields*, but as one sees in applications and as we will show in Section 5, one must consider possible solutions in rings of sequences or of functions in which there are many zero divisors.

In the papers [5] and [6], characteristic set methods for differential-difference rings are developed. While one might imagine that these techniques may be relevant to the problems we consider, it is not clear how to apply them directly, as once again, generally, characteristic set methods are best adapted to studying solutions of such equations in fields. In [3] the model theory of differential-difference fields of characteristic zero is worked out. The results include a strong quantifier simplification theorem from which one could deduce an effective elimination theorem in our sense. In [21] such quantifier simplification theorems were proven for fields equipped with several operators. An overview of the model theory of fields with operators may be found in [4]. All of these quantifier elimination theorems for difference and differential fields very strongly use the hypothesis that the solutions are sought in a *field*. Already at the level of algebraic difference equations, the results of [9] show that if we allow for solving our equations in rings of sequences or their like, then the corresponding logical theory will be undecidable. In particular, effective no quantifier elimination theorem of the kind known for differential-difference fields can hold. This makes our effective elimination theorem all that more remarkable.

This paper is organized as follows. We start in Section 2 by introducing the technical definitions we require to state our main theorems. These theorems are then announced in Section 3. In Section 4 we give the definitions of the technical concepts used in our proofs. We deal with the question of where we should seek the solutions to our differential-difference equations in Section 5. The proof of our main theorem occupies Section 6.

2. Basic notation to state the main result

Definition 2.1 (Differential-difference rings)**.**

- A differential-difference ring $(\mathcal{R}, \delta, \sigma)$ is a commutative ring $\mathcal{R}$ endowed with a derivation δ and an endomorphism σ such that $\delta\sigma = \sigma\delta$.
- For simplicity of the notation, we say $\mathcal{R}$ is a δ-σ-ring.
- When $\mathcal{R}$ is additionally a field, it is called a δ-σ-field.
- If σ is an automorphism of $\mathcal{R}$, $\mathcal{R}$ is called an inversive δ-σ-ring, or simply a δ-σ^*-ring.
- If $\sigma = \mathrm{id}$, $\mathcal{R}$ is called a δ-ring or differential ring.
- Given two δ-σ-rings $\mathcal{R}_1$ and $\mathcal{R}_2$, a homomorphism $\phi : \mathcal{R}_1 \longrightarrow \mathcal{R}_2$ is called a δ-σ-homomorphism if ϕ commutes with δ and σ, i.e., $\phi\delta = \delta\phi$ and $\phi\sigma = \sigma\phi$.
- For a commutative ring R, the ideal generated by $F \subset R$ in R is denoted by $\langle F \rangle$.
- For a δ-ring R, the differential ideal generated by $F \subset R$ in R is denoted by $\langle F \rangle^{(\infty)}$; for a nonnegative integer B, the ideal in R generated by the set $\{\delta^i(F) \mid 0 \leqslant i \leqslant B\}$ in R is denoted by $\langle F \rangle^{(B)}$.

Definition 2.2 (Differential-difference polynomials)**.**

- Let $\mathcal{R}$ be a δ-σ-ring. The differential-difference polynomial ring over $\mathcal{R}$ in $\boldsymbol{y} = y_1, \dots, y_n$, denoted by $\mathcal{R}[\boldsymbol{y}_\infty]$, is the δ-σ-ring
$$\big(\mathcal{R}[\delta^i\sigma^j y_k \mid i, j \geqslant 0;\ 1 \leqslant k \leqslant n], \delta, \sigma\big),$$
$$\sigma(\delta^i\sigma^j y_k) := \delta^i\sigma^{j+1}y_k, \quad \delta(\delta^i\sigma^j y_k) := \delta^{i+1}\sigma^j y_k.$$
A δ-σ-polynomial is an element of $\mathcal{R}[\boldsymbol{y}_\infty]$.
- Given $B \in \mathbb{N}$, let $\mathcal{R}[\boldsymbol{y}_B]$ denote the polynomial ring $\mathcal{R}[\delta^i\sigma^j y_k \mid 0 \leqslant i, j \leqslant B; 1 \leqslant k \leqslant n]$.
- Given $f \in \mathcal{R}[\boldsymbol{y}_\infty]$, the order of f is defined to be the maximal $i + j$ such that $\delta^i\sigma^j y_k$ effectively appears in f for some k, denoted by $\operatorname{ord}(f)$.
- The relative order of f with respect to δ (resp., σ), denoted by $\operatorname{ord}_\delta(f)$ (resp., $\operatorname{ord}_\sigma(f)$), is defined as the maximal i (resp., j) such that $\delta^i\sigma^j y_k$ effectively appears in f for some k.
- Let $\mathcal{R}$ be a δ-σ-ring containing a δ-σ-field k. Given a point $\boldsymbol{a} = (a_1, \dots, a_n) \in \mathcal{R}^n$, there exists a unique δ-σ-homomorphism over k,
$$\phi_{\boldsymbol{a}} : k[\boldsymbol{y}_\infty] \longrightarrow \mathcal{R} \quad \text{with} \quad \phi_{\boldsymbol{a}}(y_i) = a_i \quad \text{and} \quad \phi_{\boldsymbol{a}}|_k = \mathrm{id}\,.$$
Given $f \in k[\boldsymbol{y}_\infty]$, $\boldsymbol{a}$ is called a solution of f in $\mathcal{R}$ if $f \in \ker(\phi_{\boldsymbol{a}})$.

Definition 2.3 (Sequence rings and solutions)**.** For a δ-σ-k-algebra $\mathcal{R}$ and $I = \mathbb{N}$ or $\mathbb{Z}$, the sequence ring $\mathcal{R}^I$ has the following structure of a δ-σ-ring (δ-σ^*-ring for $I = \mathbb{Z}$) with σ and δ defined by
$$\sigma\big((x_i)_{i\in I}\big) := (x_{i+1})_{i\in I} \quad \text{and} \quad \delta\big((x_i)_{i\in I}\big) := (\delta(x_i))_{i\in I}.$$
For a k-δ-σ-algebra $\mathcal{R}$, $\mathcal{R}^I$ can be considered a k-δ-σ-algebra by embedding k into $\mathcal{R}^I$ in the following way:
$$a \mapsto (\sigma^i(a))_{i\in I}, \quad a \in k.$$
For $f \in k[\boldsymbol{y}_\infty]$, a solution of f with components in $\mathcal{R}^I$ is called a *sequence solution of f in $\mathcal{R}$.*

3. Main result

Theorem 3.1 (Effective elimination)**.** *For all nonnegative integers r, s, there exists a computable $B = B(r, s)$ such that, for all*

- *nonnegative integers q,*
- *δ-σ-fields k with $\operatorname{char} k = 0$, and*
- *sets of δ-σ-polynomials $F \subset k[\boldsymbol{x}_s, \boldsymbol{y}_s]$, where $\boldsymbol{x} = x_1, \dots, x_q$, $\boldsymbol{y} = y_1, \dots, y_r$, and $\deg_{\boldsymbol{y}} F \leqslant s$,*

we have
$$\big\langle \sigma^i(F) \mid i \in \mathbb{Z}_{\geqslant 0}\big\rangle^{(\infty)} \cap k[\boldsymbol{x}_\infty] = \{0\} \iff \big\langle \sigma^i(F) \mid i \in [0, B]\big\rangle^{(B)} \cap k[\boldsymbol{x}_B] = \{0\}.$$

By setting $q = 0$ in Theorem 3.1 and Remark 5.4, we obtain:

Corollary 3.2 (Effective Nullstellensatz)**.** *For all nonnegative integers r, s, there exists a computable $B = B(r, s)$ such that, for all*

- *δ-σ-fields k with $\operatorname{char} k = 0$, and*
- *sets of δ-σ-polynomials $F \subset k[\boldsymbol{y}_s]$, where $\boldsymbol{y} = y_1, \dots, y_r$ and $\deg_{\boldsymbol{y}} F \leqslant s$,*

the following statements are equivalent:

(1) *There exists a δ-σ^*-field L extending k such that $F = 0$ has a sequence solution in L.*
(2) $1 \notin \langle \sigma^i(F) \mid i \in [0, B] \rangle^{(B)}$.
(3) *There exists a field extension L of k such that the polynomial system $\{\sigma^i(F)^{(j)} = 0 \mid i, j \in [0, B]\}$ in the finitely many unknowns $\boldsymbol{x}_B$ has a solution in L.*

4. Definitions and notation used in the proofs

Let $\mathcal{R}$ be a δ-σ-ring.

- For $r, s \in \mathbb{N}$, let $\mathcal{R}[\boldsymbol{y}_{r,s}]$ and $\mathcal{R}[\boldsymbol{y}_{\infty,s}]$ denote the polynomial ring
$$\mathcal{R}[\delta^i \sigma^j y_k \mid 0 \leqslant i \leqslant r, 0 \leqslant j \leqslant s; 1 \leqslant k \leqslant n]$$
and the δ-ring
$$\mathcal{R}[\delta^i \sigma^j y_k \mid i \geqslant 0,\ 0 \leqslant j \leqslant s;\ 1 \leqslant k \leqslant n],$$
respectively. Additionally, $\mathcal{R}(\boldsymbol{y}_{r,s})$ and $\mathcal{R}(\boldsymbol{y}_{\infty,s})$ denote their fields of fractions.
- The radical of an ideal I in a commutative ring R is denoted by $\sqrt{I}$.
- Let $k \subset L$ be two δ-fields. A subset $S \subset L$ is said to be δ-independent over k if the set $\{\delta^k s \mid k \geqslant 0,\ s \in S\}$ is algebraically independent over k. The cardinality of any maximal subset of L that is δ-independent over k is denoted by δ-tr.deg L/k.
- In what follows, we will consider every δ-field (k, δ) as a δ-σ^*-field with respect to δ and the identity automorphism. From this standpoint, the ring of differential polynomials over k in $\boldsymbol{y}$ (see [12, Chapter I, §6]) can be realized as $k[\boldsymbol{y}_{\infty,0}] \subset k[\boldsymbol{y}_\infty]$. We use $k(\boldsymbol{a}_{\infty,0})$ to denote the differential field extension of k generated by a tuple $\boldsymbol{a}$.

Definition 4.1. A δ-field K is called *differentially closed* if, for all $F \subset K[\boldsymbol{y}_{\infty,0}]$ and δ-fields L containing K, the existence of a solution to $F = 0$ in L implies the existence of a solution to $F = 0$ in K.

Definition 4.2 (Differential varieties and diffspec)**.** Let (K, δ) be a differentially closed field containing a differential field (k, δ) and $\boldsymbol{y} = y_1, \ldots, y_n$.

- For $F \subset k[\boldsymbol{y}_{\infty,0}]$, we write
$$\mathbb{V}(F) = \{\boldsymbol{a} \in K^n \mid \forall f \in F\, f(\boldsymbol{a}) = 0\}.$$
- A subset $X \subset K^n$ is called a *differential variety* over k if there exists $F \subset k[\boldsymbol{y}_{\infty,0}]$ such that $X = \mathbb{V}(F)$.
- For a subset $X \subset K^n$, we also write $X = \operatorname{diffspec} R$ if there exists $F \subset K[\boldsymbol{y}_{\infty,0}]$ such that $X = \mathbb{V}(F)$ and $R = K[\boldsymbol{y}_{\infty,0}]/\langle F \rangle^{(\infty)}$ (note that R is not assumed to be reduced). We define $R_X := K[\boldsymbol{y}_{\infty,0}]/\sqrt{\langle F \rangle^{(\infty)}}$.
- A differential variety $\mathbb{V}(F)$ is called *irreducible* if $\sqrt{\langle F \rangle^{(\infty)}}$ is a prime ideal.
- The *generic point* $(a_1, \ldots, a_n)$ of an irreducible δ-variety $X = \mathbb{V}(F)$ is the image of the $\boldsymbol{y}$ under the homomorphism $K[\boldsymbol{y}_{\infty,0}] \to K[\boldsymbol{y}_{\infty,0}]/\sqrt{\langle F \rangle^{(\infty)}}$.
- Taking differential varieties as the basic closed sets, we define *the Kolchin topology* on K^n.

- For a subset $S \subset K^n$, we define *the Kolchin closure* of S (denoted by $\overline{S}^{\mathrm{Kol}}$) to be the intersection of all differential subvarieties of K^n containing S.

Let X be an irreducible δ-variety with the generic point $\boldsymbol{a} = (a_1, \ldots, a_n)$.

- The *differential dimension* of X, denoted by δ-dim(X), is defined as

$$\delta\text{-tr.deg}\, K(\boldsymbol{a}_{\infty,0})/K.$$

- A *parametric set* of X is a subset $\{y_i \mid i \in I\} \subset \{y_1, \ldots, y_n\}$ such that $\{a_i \mid i \in I\}$ is a differential transcendence basis of $K(\boldsymbol{a}_{\infty,0})$ over K.
- The *relative order* of X with respect to a parametric set $U = \{y_i \mid i \in I\}$, denoted by $\operatorname{ord}_U X$, is defined as

$$\operatorname{ord}_U X = \text{tr.deg}\, K(\boldsymbol{a}_{\infty,0})\big/K\big((a_i, i \in I)_{\infty,0}\big).$$

- The *order* of X is the maximum of all the relative orders of X ([7, Theorem 2.11]), that is,

$$\operatorname{ord}(X) = \max\{\operatorname{ord}_U X \mid U \text{ is a parametric set of } X\}.$$

5. What is a universal δ-σ-ring for solving equations?

This section is devoted to answering the following question:

Question. In what rings is it natural to look for solutions of differential-difference equations?

We will show that rings of sequences are universal solution rings in the abstract mathematical sense. More precisely, we prove an analogue of the Hilbert Nullstellensatz, Proposition 5.3. On the other hand, from the applications standpoint, it would be natural if solutions of delay-differential equations were functions defined on a subset of the complex plane or real line. It turns out that these two seemingly contradictory standpoints can be viewed as closely related via the construction described below.

Definition 5.1 (Rings of meromorphic functions)**.**

- Let $U \subset \mathbb{C}$ be an open nonempty set. We denote the ring of meromorphic functions on U by $\mathcal{M}(U)$. $\mathcal{M}(U)$ is a field if and only if U is connected.
- Let $D \subset \mathbb{C}$ be a nonempty discrete set. We define a ring $\mathcal{M}(D)$ of germs of meromorphic functions on D as the quotient

$$\mathcal{M}(D) := \{(f, U) \mid U \text{ is open such that } D \subset U,\ f \in \mathcal{M}(U)\}/\sim,$$

where the equivalence relation $\sim$ is defined by

$$\big((f_1, U_1) \sim (f_2, U_2)\big) \iff (\forall z \in U_1 \cap U_2,\ f_1(z) = f_2(z)).$$

- For every open nonempty $U \subset \mathbb{C}$, $\mathcal{M}(U)$ is a δ-ring with respect to the standard derivative. If $U = U + \{1\}$, then $\mathcal{M}(U)$ can be considered as a δ-σ^*-ring with respect to the shift automorphism $\sigma(f)(z) = f(z-1)$. Similarly, for a nonempty discrete $D \subset \mathbb{C}$, $\mathcal{M}(D)$ is a δ-ring. If additionally $D = D + \{1\}$, then $\mathcal{M}(D)$ is a δ-σ^*-ring with σ sending the equivalence class of $(f(z), U)$ to the equivalence class of $(f(z-1), U + \{1\})$.

Definition 5.2 (Transforms between functions and sequences)**.**

- We define $S := \{z \in \mathbb{C} \mid -0.5 < \operatorname{Re} z < 0.5\} \subset \mathbb{C}$.
- Consider a nonempty open subset $U \subset \mathbb{C}$ such that $U = U + \{1\}$. Then we define a map
$$\varphi_U \colon \mathcal{M}(U) \to (\mathcal{M}(U \cap S))^{\mathbb{Z}}$$
as follows. For every $f \in \mathcal{M}(U)$ and every $j \in \mathbb{Z}$, we define $f_j \in \mathcal{M}(U \cap S)$ by $f_j(z) := f(z + j)$. Then we set $\varphi_U(f) := (\ldots, f_{-1}, f_0, f_1, \ldots)$. One can check that φ_U defines an injective homomorphism of δ-σ^*-rings, where $(\mathcal{M}(U \cap S))^{\mathbb{Z}}$ bears a δ-σ^*-ring structure as descibed in Definition 2.3. The same can be done for a nonempty discrete $D \subset \mathbb{C}$ such that $D = D + \{1\}$ and $D \cap \partial S = \varnothing$.
- Consider a nonempty open subset $U_0 \subset S$. We define a map
$$\psi_{U_0} \colon (\mathcal{M}(U_0))^{\mathbb{Z}} \to \mathcal{M}(U_0 + \mathbb{Z})$$
as follows. For every $\{f_j\}_{j\in\mathbb{Z}} \in (\mathcal{M}(U_0))^{\mathbb{Z}}$, we define a function $f \in \mathcal{M}(U_0 + \mathbb{Z})$ by setting $f(z)|_{U_0+\{j\}} := f_j(z - j)$ for every $j \in \mathbb{Z}$. Then we define $\psi_{U_0}(\{f_j\}_{j\in\mathbb{Z}}) := f$. One can check that ψ_{U_0} defines an isomorphism of δ-σ^*-rings. The same can be done for a nonempty discrete $D \subset S$.

In Section 5.2, we show that $\mathcal{M}(\mathbb{Z})$ is a universal solution ring for δ-σ-equations over $\mathbb{C}$ (Proposition 5.7) and derive a version of our effective elimination theorem for this case (Corollary 5.8). Moreover, in Section 5.3, we show that there exists a system of δ-σ-equations that

- has a solution in $\mathcal{M}(\mathbb{Z})$ but,
- for every open $U \subset \mathbb{C}$ such that $U = U + \{1\}$, does not have a solution in $\mathcal{M}(U)$.

5.1. Solutions in sequences over δ-fields.

Proposition 5.3. *Let $n \in \mathbb{Z}_{\geqslant 0}$, k be a δ-σ^*-field. Then, for every $F \subset k[\boldsymbol{y}_\infty]$ and $f \in k[\boldsymbol{y}_\infty]$ with $\boldsymbol{y} = y_1, \ldots, y_n$, the following statements are equivalent:*

1. *For every δ-σ-field extension $k \subset K$, f vanishes on all solutions of $F = 0$ in $K^{\mathbb{Z}}$.*
2. *There exists $m \in \mathbb{N}$ such that*
$$\sigma^m(f^m) \in \langle \sigma^j(F) \mid j \in \mathbb{Z}_{\geqslant 0} \rangle^{(\infty)} \subset k[\boldsymbol{y}_\infty].$$

Moreover, if $\sigma = \mathrm{id}_k$, then (2) is equivalent to: for every δ-σ-field extension $k \subset K$ with $\sigma = \mathrm{id}_K$, f vanishes on all the sequence solutions of F in $K^{\mathbb{Z}}$.

Proof. The implication (2) $\implies$ (1) is straightforward because σ is injective. It remains to show (1) $\implies$ (2). Suppose that (2) does not hold. Let
$$\mathcal{I} := \sqrt{\langle \sigma^j(F) \mid j \in \mathbb{Z} \rangle^{(\infty)}} \subset k[\sigma^j(\boldsymbol{y}_{\infty,0}) \mid j \in \mathbb{Z}].$$
By [10, Theorem 2.1], $\mathcal{I}$ is an intersection of prime δ-ideals (maybe an infinite intersection). Assume that $f \in \mathcal{I}$. Then there exists $m \in \mathbb{N}$ such that
$$f^m \in \langle \sigma^j(F) \mid j \in [-m, m] \rangle^{(\infty)}.$$
Applying σ^m, we have $\sigma^m(f^m) \in \langle \sigma^j(F) \mid j \in \mathbb{Z}_{\geqslant 0} \rangle^{(\infty)}$, and this contradicts the assumption that (2) does not hold. Thus, $f \notin \mathcal{I}$, so there exists a prime δ-ideal P

with $\mathcal{I} \subseteq P$ and $f \notin P$. Let $\mathcal{U}_0$ be the quotient field of the δ-domain $k[\sigma^j(\boldsymbol{y}_{\infty,0}) \mid j \in \mathbb{Z}]/P$ that has a natural structure of a δ-field. Let $\mathcal{U}$ be a differentially closed field containing $\mathcal{U}_0$. [19, Lemma 2.3] together with Zorn's lemma implies that σ can be extended from k to $\mathcal{U}$ so that $\mathcal{U}$ is a δ-σ-field. Note that if $\sigma|_k = \mathrm{id}$, then we can set $\sigma|_{\mathcal{U}} = \mathrm{id}$. Let

$$\eta = \left((\overline{\sigma^j y_1})_{j\in\mathbb{Z}}, \ldots, (\overline{\sigma^j y_n})_{j\in\mathbb{Z}}\right) \in \mathcal{U}^{\mathbb{Z}} \times \cdots \times \mathcal{U}^{\mathbb{Z}},$$

where $\overline{\sigma^j y_k}$ is the canonical image of $\sigma^j y_k$. Clearly, η is a solution of $F = 0$ in $\mathcal{U}^{\mathbb{Z}}$, but f does not vanish at it. Thus, (1) does not hold. So, (1) implies (2). $\square$

Remark 5.4. The proof of Proposition 5.3 can be modified to show that the following conditions are also equivalent:

(1) For every δ-σ-field extension $k \subset K$, f vanishes on all solutions of $F = 0$ in $K^{\mathbb{N}}$.

(2) There exists $m \in \mathbb{Z}_{\geqslant 0}$ such that

$$f^m \in \langle \sigma^j(F) \mid j \in \mathbb{Z}_{\geqslant 0}\rangle^{(\infty)} \subset k[\boldsymbol{y}_\infty].$$

Remark 5.5. In the case $f = 1$ (so-called weak Nullstellensatz), the second condition of Proposition 5.3 is equivalent to the second condition in Remark 5.4. Thus, for $f = 1$, all the conditions of Proposition 5.3 and Remark 5.4 are equivalent. However, they are not equivalent for general f as the following example shows.

Example 5.6. Let $k = \mathbb{Q}$. Consider

$$F = \{y^2 - \sigma(y), y^2 - \sigma^2(y)\} \quad \text{and} \quad f = y(y-1).$$

Let $k \subset K$ be an extension of δ-σ-fields and let $\overline{a} = (\ldots, a_{-1}, a_0, a_1, a_2, \ldots) \in K^{\mathbb{Z}}$ be a solution of F. For every $i \in \mathbb{Z}$, we have

$$a_{i-1}^2 - a_i = a_{i-1}^2 - a_{i+1} = 0 \implies a_i = a_{i+1}.$$

Combining with $a_i^2 = a_{i+1}$, we have $a_i^2 = a_i$. Thus, f vanishes at $\overline{a}$. However, f does not vanish on the solution $(-1, 1, 1, \ldots)$ of $F = 0$ in $\mathbb{Q}^{\mathbb{N}}$.

5.2. **Solutions in germs.**

Proposition 5.7. *For every $n \in \mathbb{Z}_{\geqslant 0}$, $F \subset \mathbb{C}[\boldsymbol{y}_\infty]$, and $f \in \mathbb{C}[\boldsymbol{y}_\infty]$ with $\boldsymbol{y} = y_1, \ldots, y_n$, the following statements are equivalent:*

(1) *f vanishes on all the solutions of $F = 0$ in $\mathcal{M}(\mathbb{Z})$.*

(2) *There exists $m \in \mathbb{N}$ such that*

$$\sigma^m(f^m) \in \langle \sigma^j(F) \mid j \in \mathbb{Z}_{\geqslant 0}\rangle^{(\infty)} \subset \mathbb{C}[\boldsymbol{y}_\infty].$$

Proof. The implication (2) $\implies$ (1) is straightforward. It remains to show (1) $\implies$ (2). Suppose that (2) does not hold. Since $\mathbb{C}[\mathbf{y}_\infty]$ has countable dimension, we can replace F by at most a countable basis of the $\mathbb{C}$-span of F. This will not change the solution set of $F = 0$ and the ideal generated by F. Let E be the subfield of $\mathbb{C}$ generated by the coefficients of F and f over $\mathbb{Q}$. Proposition 5.3 implies that there exists a δ-field $K \supset E$ such that $F = 0$ has a solution $\overline{a} = \{a_j\}_{j\in\mathbb{Z}}$ in $K^{\mathbb{Z}}$ such that $f(\overline{a}) \neq 0$. Replacing K by its δ-subfield generated by E and $\{a_j\}_{j\in\mathbb{Z}}$, we can further assume that K is an at most countably generated δ-field extension of E. Hence K is at most countable. [19, Lemma A.1] implies that there exists a homomorphism of δ-fields $\theta\colon K \to \mathcal{M}(0)$ that maps $E \subset K$ isomorphically to $E \subset \mathbb{C} \subset \mathcal{M}(0)$. This homomorphism can be extended to an injective homomorphism $\theta\colon K^{\mathbb{Z}} \to (\mathcal{M}(0))^{\mathbb{Z}}$ of δ-σ^*-algebras over E. Then the composition of θ with the isomorphism $\psi_0\colon (\mathcal{M}(0))^{\mathbb{Z}} \to \mathcal{M}(\mathbb{Z})$ (see Definition 5.2) is an injective homomorphism of δ-σ^*-algebras over E.

We set $b := \psi_0 \circ \theta(\overline{a}) \in \mathcal{M}(\mathbb{Z})$. Then b is a solution of $F = 0$, and, since $\psi_0 \circ \theta$ is injective, f does not vanish at b. This contradicts (1). ☐

Combining Proposition 5.7 with Theorem 6.22, we obtain:

Corollary 5.8. *For all nonnegative integers r, s, there exists a computable $B = B(r, s)$ such that, for all*

- *nonnegative integers q and t,*
- *a set of δ-σ-polynomials $F \subset \mathbb{C}[\boldsymbol{x}_t, \boldsymbol{y}_s]$, where $\boldsymbol{x} = x_1, \ldots, x_q$, $\boldsymbol{y} = y_1, \ldots, y_r$, and $\deg_{\boldsymbol{y}} F \leqslant s$,*

the following statements are equivalent:

- *there exists a nonzero $g \in \mathbb{C}[\boldsymbol{x}_\infty]$ that vanishes on every solution of $F = 0$ in $\mathcal{M}(\mathbb{Z})$;*
- $\langle \sigma^i(F) \mid i \in [0, B]\rangle^{(B)} \cap \mathbb{C}[\boldsymbol{x}_{B+t}] \neq \{0\}$.

5.3. **Solutions in meromorphic functions on open subsets of $\mathbb{C}$.** In this section, we will present a specific system of δ-σ-equations (3) that has a solution in $\mathcal{M}(\mathbb{Z})$ but, for any open $U \subset \mathbb{C}$ such that $U = U + \{1\}$, does not have a solution in $\mathcal{M}(U)$ (see Proposition 5.9). We recall some relevant facts about the Weierstrass $\wp$-function:

- Let $g_2, g_3 \in \mathbb{C}$ be the complex numbers such that the Weierstrass function $\wp(z)$ with periods 1 and i (the imaginary unit) is a solution of

$$(x')^2 = 4x^3 - g_2 x - g_3. \tag{1}$$

  We will use the fact that every nonconstant solution of (1) is of the form $\wp(z + z_0)$ for some $z_0 \in \mathbb{C}$; see [11, p. 39, Korollar F].
- Recall that the field of doubly periodic meromorphic functions on $\mathbb{C}$ with periods 1 and i is generated by $\wp(z)$ and $\wp'(z)$ [14, p. 8, Theorem 4]. Let $\omega := 1 + \sqrt{2}i$, and consider a rational function $R(x_1, x_2) \in \mathbb{C}(x_1, x_2)$ such that

$$\wp(z + \omega) = R(\wp(z), \wp'(z)). \tag{2}$$

Proposition 5.9. *Consider the following system of algebraic differential-difference equations in the unknowns x, y, w:*

$$\begin{cases} (x')^2 = 4x^3 - g_2 x - g_3, \\ \sigma(x) = R(x, x'), \\ y^3 = \frac{1}{x}, \\ x' w = 1. \end{cases} \tag{3}$$

1. *System (3) has a solution in $\mathcal{M}(\mathbb{Z})$.*
2. *For every nonempty open subset $U \subset \mathbb{C}$ such that $U = U + \{1\}$, system (3) does not have a solution in $\mathcal{M}(U)$.*

Proof. Proof of (1). Let K be the algebraic closure of the field $\mathcal{M}(\mathbb{C})$. We set

$$x_j = \wp(z + j\omega), \quad y_j = \sqrt[3]{\frac{1}{\wp(z + j\omega)}}, \quad \text{and} \quad w_j = \frac{1}{\wp'(z + j\omega)}.$$

The first equation in (3) has constant coefficients, so it holds for the above sequences because every shift of $\wp(z)$ is again a solution of the same equation. The second equation in (3) holds because

$$x_{j+1} = \wp(z + (j+1)\omega) = R(\wp(z + j\omega), \wp'(z + j\omega)) = R(x_j, x_j')$$

due to (2). A direct computation shows that the last two equations in (3) also hold. Thus, the system (3) has a solution in $K^{\mathbb{Z}}$. Combining Propositions 5.3 and 5.7, we see that (3) has a solution in $\mathcal{M}(\mathbb{Z})$.

Proof of (2). Assume the contrary. Let $U \subset \mathbb{C}$ be such a subset and let $(x(z), y(z), w(z))$ be such a solution. Since (3) is autonomous, we can assume that $0 \in U$ by shifting U and the solution if necessary. We denote the connected component of U containing 0 by U_0. The last equation of (3) implies that $x(z)|_{U_0}$ is nonconstant. Then the first equation of (3) implies that there exists $z_0 \in \mathbb{C}$ such that

$$x(z)|_{D_0} = \wp(z + z_0). \tag{4}$$

We will prove that, for every $z \in U_0$ and $s \in \mathbb{Z}_{\geqslant 0}$,

$$x(z + s) = \wp(z + z_0 + s\omega) \tag{5}$$

by induction on s. The base case $s = 0$ follows from (4). Assume that (5) holds for $s \geqslant 0$. Then, using the second equation in (3), the inductive hypothesis, and (2), we have

$$\begin{aligned} x(z + s + 1) &= R(x(z + s), x'(z + s)) \\ &= R(\wp(z + z_0 + s\omega), \wp'(z + z_0 + s\omega)) = \wp(z + z_0 + (s+1)\omega). \end{aligned}$$

This proves (5).

Let $\varepsilon > 0$ be a real number such that U contains the ε-neighbourhood of 0. Kronecker's theorem implies that there exist $s \in \mathbb{Z}_{\geqslant 0}$ and $m, n \in \mathbb{Z}$ (which we fix) such that

$$|z_0 + s\omega - n - mi| < \varepsilon.$$

We set $z_1 = n + mi - z_0 - s\omega$. Then, since $|z_1| < \varepsilon$, we have $z_1 \in U_0$, and so $z_1 + s \in U$. Equality (5) implies that

$$x(z_1 + s) = \wp(z_1 + z_0 + s\omega) = \wp(n + mi) = \infty.$$

Then $y(z_1 + s) = 0$. Let $d \geqslant 1$ be the order of zero of y at $z_1 + s$. Then $x = \frac{1}{y^3}$ has a pole of order $3d$ at $z_1 + s$. We arrive at a contradiction with the fact that all the poles of $\wp$ are of order two [14, p. 8]. □

6. Proof of the main result

The proofs are structured as follows. In Section 6.1, we embed the ground δ-σ-field k to a differentially closed δ-σ^*-field K. In Section 6.2, we extend the technique of trains (developed in [23] for difference equations) to the differential-difference case. Section 6.3 begins with Section 6.3.1, in which we establish a bound (Corollary 6.13) for the number of components of a differential-algebraic variety. This bound replaces the Bézout bound, extensively used in [23] but lacking in the differential-algebraic setting. In Section 6.3.2, we show that the existence of a sufficiently long train implies the existence of a solution in $K^{\mathbb{Z}}$. Finally, in Section 6.4, we use these ingredients to prove the main result, Theorem 3.1.

6.1. **Constructing big enough field $K \supset k$.** Throughout Section 6

- k is the δ-σ-field from Theorem 3.1,
- K is a fixed differentially closed δ-σ^*-field containing k. The existence of such a field follows from Lemma 6.1.

Lemma 6.1. *For every δ-σ-field k of characteristic zero, there exists an extension $k \subset K$ of δ-σ-fields, where K is a differentially closed δ-σ^*-field.*

Proof. We will show that there exists a δ-σ^*-field K_0 containing k. The proof of [17, Proposition 2.1.7] implies that one can build an ascending chain of σ-fields

$$k_0 \subset k_1 \subset k_2 \subset \cdots \tag{6}$$

such that, for every $i \in \mathbb{N}$, there exists an isomorphism $\varphi_i\colon k \to k_i$ of σ-fields, $\sigma(k_{i+1}) = k_i$, and $\varphi_i = \sigma \circ \varphi_{i+1}$ for every $i \in \mathbb{N}$. We transfer the δ-σ-structure from k to k_i's via φ_i's. Then $\varphi_i = \sigma \circ \varphi_{i+1}$ implies that the restriction of δ on k_{i+1} to k_i coincides with the action of δ on k_i. We set $K_0 := \bigcup_{i\in\mathbb{N}} k_i$. Since the action δ and σ is consistent with the ascending chain (6), K_0 is a δ-σ-extension of $k_0 \cong k$. It is shown in [17, Proposition 2.1.7] that the action of σ on K_0 is surjective. [3, Theorem 3.15] implies that K_1 can be embedded in a differentially closed δ-σ^*-field K. □

6.2. **Partial solutions and trains.**

Definition 6.2. Let $F = \{f_1, \ldots, f_N\} \subset k[\boldsymbol{y}_\infty]$, where $\boldsymbol{y} = y_1, \ldots, y_n$, be a set of δ-σ-polynomials. Suppose $h = \max\{\operatorname{ord}_\sigma(f) \mid f \in F\}$. A sequence of tuples $(\overline{a}_1, \ldots, \overline{a}_n) \in K^{\ell+h} \times \cdots \times K^{\ell+h}$ is called a *partial solution* of F of length ℓ if $(\overline{a}_1, \ldots, \overline{a}_n)$ is a δ-solution of the system in $\boldsymbol{y}_{\infty,\ell+h-1}$:

$$\{\sigma^i(F) = 0 \mid 0 \leqslant i \leqslant \ell - 1\}.$$

Example 6.3. Let $k = \mathbb{Q}(t)$ be the δ-σ^*-field with $\delta = \frac{d}{dt}$ and $\sigma(f(t)) = f(t+1)$ for each $f(t) \in \mathbb{Q}(t)$. Let $f = t \cdot \delta(y) + \sigma(y) \in k[y_\infty]$. So $h = 1$. A partial solution of f of length ℓ is a sequence $(a_0, a_1, \ldots, a_\ell) \in K^{\ell+1}$ such that

$$(t+i)\cdot\delta(a_i) + a_{i+1} = 0, \qquad i = 0, 1, \ldots, \ell - 1.$$

A solution of f in $K^{\mathbb{N}}$ is a sequence $(a_i)_{i\in\mathbb{N}} \in K^{\mathbb{N}}$ such that for each $i \in \mathbb{N}$,

$$(t+i)\cdot\delta(a_i) + a_{i+1} = 0.$$

With the above set F of δ-σ-polynomials, we associate the following geometric data analogously to [23]:

- the δ-variety $X \subset \mathbb{A}^H$ defined by $f_1 = 0, \ldots, f_N = 0$ regarded as δ-equations in $k[\boldsymbol{y}_{\infty,h}]$ with $H = n(h+1)$, and so
$$X = \operatorname{diffspec} R_X, \quad R_X := K[\boldsymbol{y}_{\infty,h}]/\sqrt{(f_1, \ldots, f_N)^{(\infty)}};$$
- two projections $\pi_1, \pi_2 : \mathbb{A}^H \longrightarrow \mathbb{A}^{H-n}$ defined by
$$\begin{aligned}&\pi_1(a_1, \ldots, \sigma^h(a_1); \ldots; a_n, \ldots, \sigma^h(a_n))\\ &\qquad := (a_1, \sigma(a_1), \ldots, \sigma^{h-1}(a_1); \ldots; a_n, \ldots, \sigma^{h-1}(a_n)),\\ &\pi_2(a_1, \ldots, \sigma^h(a_1); \ldots; a_n, \ldots, \sigma^h(a_n))\\ &\qquad := (\sigma(a_1), \ldots, \sigma^h(a_1); \ldots; \sigma(a_n), \ldots, \sigma^h(a_n)).\end{aligned}$$

By $\sigma(X)$, we mean the δ-variety in $\mathbb{A}^H$ defined by $f_1^\sigma, \dots, f_N^\sigma$, where f_i^σ is the result by applying σ to the coefficients of f_i.

Definition 6.4. A sequence $p_1, \dots, p_\ell \in \mathbb{A}^H$ is a *partial solution of the triple* (X, π_1, π_2) if

(1) for all i, $1 \leqslant i \leqslant \ell$, we have $p_i \in \sigma^{i-1}(X)$ and
(2) for all i, $1 \leqslant i < \ell$, we have $\pi_1(p_{i+1}) = \pi_2(p_i)$.

A two-sided infinite sequence with such a property is called a *solution of the triple* (X, π_1, π_2).

Lemma 6.5. *For every positive integer ℓ, F has a partial solution of length ℓ if and only if the triple (X, π_1, π_2) has a partial solution of length ℓ. System F has a solution in $K^{\mathbb{Z}}$ if and only if the triple (X, π_1, π_2) has a solution.*

Proof. It suffices to show that the first assertion holds. Suppose that $(a_0, a_1, \dots, a_{h+\ell-1})$ is a partial solution of F. Let $p_i = (a_{i-1}, a_i, \dots, a_{i-1+h})$ for $i = 1, \dots, \ell$. Then $p_1, \dots, p_\ell$ is a partial solution of (X, π_1, π_2) of length ℓ. For the other direction, let $p_1, \dots, p_\ell$ be a partial solution of (X, π_1, π_2) of length ℓ. Since $\pi_1(p_{i+1}) = \pi_2(p_i)$, there exist $a_0, \dots, a_{h+\ell-1} \in K$ such that, for each i, $p_i = (a_{i-1}, a_i, \dots, a_{i-1+h})$. Thus, $(a_0, a_1, \dots, a_{h+\ell-1})$ is a partial solution of F of length ℓ. $\square$

Definition 6.6 (Cf. [23])**.** For $\ell \in \mathbb{N}$ or $+\infty$, a sequence of irreducible δ-subvarieties $(Y_1, \dots, Y_\ell)$ in $\mathbb{A}^H$ is said to be *a train of length ℓ* in X if

(1) for all i, $1 \leqslant i \leqslant \ell$, we have $Y_i \subseteq \sigma^{i-1}(X)$ and
(2) for all i, $1 \leqslant i < \ell$, we have $\overline{\pi_1(Y_{i+1})}^{\mathrm{Kol}} = \overline{\pi_2(Y_i)}^{\mathrm{Kol}}$.

Lemma 6.7. *For every train $(Y_1, \dots, Y_\ell)$ in X, there exists a partial solution $p_1, \dots, p_\ell$ of (X, π_1, π_2) such that for all i, we have $p_i \in Y_i$. In particular, if there is an infinite train in X, then there is a solution of the triple (X, π_1, π_2).*

Proof. The proof is similar to that of [23, Lemma 6.8]. To make the paper self-contained, we will give the details below. To prove the existence of a partial solution of (X, π_1, π_2) with the desired property, it suffices to prove the following:

Claim. There exists a nonempty open (*in the sense of the Kolchin topology*) *subset $U \subseteq Y_\ell$ such that for each $p_\ell \in U$, p_ℓ can be extended to a partial solution $p_1, \dots, p_\ell$ of (X, π_1, π_2) with $p_i \in Y_i$ $(\forall i)$.*

We will prove the Claim by induction on ℓ. For $\ell = 1$, take $U = Y_1$. Since each point in Y_1 is a partial solution of (X, π_1, π_2) of length 1, the Claim holds for $\ell = 1$. Now suppose we have proved the Claim for $\ell - 1$. So there exists a nonempty open subset $U_0 \subseteq Y_{\ell-1}$ satisfying the desired property. Since $Y_{\ell-1}$ is irreducible, U_0 is dense in $Y_{\ell-1}$. So, $\pi_2(U_0)$ is dense in $\overline{\pi_2(Y_{\ell-1})}^{\mathrm{Kol}} = \overline{\pi_1(Y_\ell)}^{\mathrm{Kol}}$. Since U_0 is δ-constructible, $\pi_2(U_0)$ is δ-constructible too. So, $\pi_2(U_0)$ contains a nonempty open subset of $\overline{\pi_1(Y_\ell)}^{\mathrm{Kol}}$.

Since $\pi_1(Y_\ell)$ is δ-constructible and dense in $\overline{\pi_1(Y_\ell)}^{\mathrm{Kol}}$, $\pi_2(U_0) \cap \pi_1(Y_\ell) \neq \varnothing$ is δ-constructible and dense in $\overline{\pi_1(Y_\ell)}^{\mathrm{Kol}}$. Let U_1 be a nonempty open subset of $\overline{\pi_1(Y_\ell)}^{\mathrm{Kol}}$ contained in $\pi_2(U_0) \cap \pi_1(Y_\ell)$ and

$$U_2 = \pi_1^{-1}(U_1) \cap Y_\ell.$$

Then U_2 is a nonempty open subset of Y_ℓ. We will show that for each $p_\ell \in U_2$, there exists $p_i \in Y_i$ for $i = 1, \dots, \ell - 1$ such that $p_1, \dots, p_\ell$ is a partial solution of (X, π_1, π_2).

Since $\pi_1(p_\ell) \in U_1 \subset \pi_2(U_0)$, there exists $p_{\ell-1} \in U_0$ such that $\pi_1(p_\ell) = \pi_2(p_{\ell-1})$. Since $p_{\ell-1} \in U_0$, by the inductive hypothesis, there exists $p_i \in Y_i$ for $i = 1, \dots, \ell-1$ such that $p_1, \dots, p_{\ell-1}$ is a partial solution of (X, π_1, π_2) of length $\ell-1$. So $p_1, \dots, p_\ell$ is a partial solution of (X, π_1, π_2) of length ℓ. $\square$

For two trains $Y = (Y_1, \dots, Y_\ell)$ and $Y' = (Y'_1, \dots, Y'_\ell)$, denote $Y \subseteq Y'$ if $Y_i \subseteq Y'_i$ for each i. Given an increasing chain of trains $Y_i = (Y_{i,1}, \dots, Y_{i,\ell})$,

$$\big(\overline{\cup_i Y_{i,1}}^{\mathrm{Kol}}, \dots, \overline{\cup_i Y_{n,i}}^{\mathrm{Kol}}\big)$$

is a train in X which is an upper bound for this chain. (For each j, $\overline{\cup_i Y_{i,j}}^{\mathrm{Kol}}$ is an irreducible δ-variety in $\sigma^{j-1}(\mathrm{X})$.) So by Zorn's lemma, maximal trains of length ℓ always exist in X.

Fix an $\ell \in \mathbb{N}$. Consider the product

$$\mathbf{X}_\ell := X \times \sigma(X) \times \cdots \times \sigma^{\ell-1}(X),$$

and denote the projection of $\mathbf{X}_\ell$ onto $\sigma^{i-1}(X)$ by $\varphi_{\ell,i}$. Note that

$$\mathbf{X}_\ell = \operatorname{diffspec}\big(R_X \otimes_K R_{\sigma(X)} \otimes_K \cdots \otimes_K R_{\sigma^{\ell-1}(X)}\big).$$

Let

$$\mathbf{W}_\ell(X, \pi_1, \pi_2) := \{p \in \mathbf{X}_\ell : \pi_2(\varphi_{\ell,i}(p)) = \pi_1(\varphi_{\ell,i+1}(p)), i = 1, \dots, \ell - 1\}. \tag{7}$$

Note that

$$\mathbf{W}_\ell = \operatorname{diffspec}\big(R_X \otimes_{R_{\overline{\pi_2(X)}}} R_{\sigma(X)} \otimes_{R_{\overline{\pi_2(\sigma(X))}}} \cdots \otimes_{R_{\overline{\pi_2(\sigma^{\ell-2}(X))}}} R_{\sigma^{\ell-1}(X)}\big),$$

under the injective (K, δ)-algebra homomorphisms, for all i, $1 \leqslant i \leqslant \ell - 1$,

$$R_{\overline{\pi_2(\sigma^{i-1}(X))}} \to R_{\sigma^{i-1}(X)} \quad \text{and} \quad R_{\overline{\pi_2(\sigma^{i-1}(X))}} \to R_{\sigma^i(X)}$$

induced by π_2 and π_1, respectively.

Lemma 6.8. *For every irreducible δ-subvariety $W \subset \mathbf{W}_\ell$,*

$$\big(\overline{\varphi_{\ell,1}(W)}^{\mathrm{Kol}}, \dots, \overline{\varphi_{\ell,\ell}(W)}^{\mathrm{Kol}}\big)$$

is a train in X of length ℓ. Conversely, for each train $(Y_1, \dots, Y_\ell)$ in X, there exists an irreducible δ-subvariety $W \subseteq \mathbf{W}_\ell$ such that $Y_i = \overline{\varphi_{\ell,i}(W)}^{\mathrm{Kol}}$ for each $i = 1, \dots, \ell$.

Proof. The first assertion is straightforward. We will prove the second assertion by induction on ℓ. For $\ell = 1$, $\mathbf{W}_1 = X$, and we can set $W = Y_1$.

Let $\ell > 1$. Apply the inductive hypothesis to the train $(Y_1, \dots, Y_{\ell-1})$ and obtain an irreducible subvariety $Y' \subset \mathbf{W}_{\ell-1} \subset \mathbf{X}_{\ell-1}$. Then there is a natural embedding of $Y' \times Y_\ell$ into $\mathbf{X}_\ell$. Denote $(Y' \times Y_\ell) \cap \mathbf{W}_\ell$ by $\widetilde{Y}$. Since $Y' \subset \mathbf{W}_{\ell-1}$,

$$\widetilde{Y} = \{p \in Y' \times Y_\ell \mid \pi_2\left(\varphi_{\ell,\ell-1}(p)\right) = \pi_1\left(\varphi_{\ell,\ell}(p)\right)\}.$$

Let

$$Z := \overline{\pi_2\left(\varphi_{\ell-1,\ell-1}(Y')\right)} = \overline{\pi_1(Y_\ell)}. \tag{8}$$

Then we have a (k,δ)-isomorphism

$$R_{Y'} \otimes_{R_Z} R_{Y_\ell} \to R_{\widetilde{Y}}$$

under the (k,δ)-algebra homomorphisms $i_1 : R_Z \to R_{Y'}$ and $i_2 : R_Z \to R_{Y_\ell}$ induced by $\pi_2 \circ \varphi_{\ell-1,\ell-1}$ and π_1, respectively. Equality (8) implies that i_1 and i_2 are injective. Denote the fields of fractions of $R_{Y'}$, R_{Y_ℓ}, and R_Z by E, F, and L, respectively. Let $\mathfrak{p}$ be any prime differential ideal in $E \otimes_L F$,

$$R := (E \otimes_L F)/\mathfrak{p},$$

and let $\pi\colon E \otimes_L F \to R$ be the canonical homomorphism. Consider the natural homomorphism $i\colon R_{Y'} \otimes_{R_Z} R_{Y_\ell} \to E \otimes_L F$. Since $1 \in i(R_{Y'} \otimes_{R_Z} R_{Y_\ell})$, the composition $\pi \circ i$ is a nonzero homomorphism. Since i_1 and i_2 are injective, the natural homomorphisms $i_{Y'}\colon R_{Y'} \to R_{Y'} \otimes_{R_Z} R_{Y_\ell}$ and $i_{Y_\ell}\colon R_{Y_\ell} \to R_{Y'} \otimes_{R_Z} R_{Y_\ell}$ are injective as well. We will show that the compositions

$$\pi \circ i \circ i_{Y'}\colon R_{Y'} \to R \quad \text{and} \quad \pi \circ i \circ i_{Y_\ell}\colon R_{Y_\ell} \to R$$

are injective. Introducing the natural embeddings $i_E\colon E \to E \otimes_L F$ and $j_{Y'}\colon R_{Y'} \to E$, we can rewrite

$$\pi \circ i \circ i_{Y'} = \pi \circ i_E \circ j_{Y'}.$$

The homomorphisms i_E and $j_{Y'}$ are injective. The restriction of π to $i_E(E)$ is also injective since E is a field. Hence, the whole composition $\pi \circ i_E \circ j_{Y'}$ is injective. The argument for $\pi \circ i \circ i_{Y_\ell}$ is analogous. Let

$$S := \big(R_{Y'} \otimes_{R_Z} R_{Y_\ell}\big)/\big(\mathfrak{p} \cap \big(R_{Y'} \otimes_{R_Z} R_{Y_\ell}\big)\big),$$

which is a domain, and the homomorphisms $\pi \circ i \circ i_{Y'} : R_{Y'} \to S$ and $\pi \circ i \circ i_{Y_\ell} : R_{Y_\ell} \to S$ are injective. We let

$$W := \operatorname{diffspec} S.$$

For every i, $1 \leqslant i < \ell$, the homomorphism

$$\varphi^\sharp_{\ell,i} = (\pi \circ i \circ i_{Y'}) \circ \varphi^\sharp_{\ell-1,i} : R_{Y_i} \to R_{Y'} \to S$$

is injective as a composition of two injective homomorphisms. Hence, the restriction $\varphi_{\ell,i}\colon W \to Y_i$ is dominant. ☐

6.3. **Technical bounds.**

6.3.1. *Number of prime components in differential varieties.* In this section, we fix a δ-field k and $\boldsymbol{x} = x_1, \ldots, x_n$. For a commutative ring R and subsets I and S of R, we let $I : S = \{r \in R \mid \exists\, s \in S : rs \in I\}$.

Lemma 6.9. *There exists a computable function $G(n,r,D)$ such that, for every $r \in \mathbb{Z}_{\geqslant 0}$ and a prime ideal $I \subset k[\boldsymbol{x}_{r,0}]$ such that $I = \sqrt{\langle I\rangle^{(\infty)}} \cap k[\boldsymbol{x}_{r,0}]$ and $\deg I \leqslant D$, there exists $f \in k[\boldsymbol{x}_{r,0}] \setminus I$ such that*

- $\langle I\rangle^{(\infty)} : f^\infty$ *is a prime differential ideal;*
- $\deg f \leqslant G(n,r,D)$.

Proof. Compute a regular decomposition of $\sqrt{\langle I\rangle^{(\infty)}}$ using the Rosenfeld-Gröbner algorithm (see, e.g., [2]) with an orderly ranking:

$$\sqrt{\langle I\rangle^{(\infty)}} = \big(\langle C_1\rangle^{(\infty)} : H_1^\infty\big) \cap \cdots \cap \big(\langle C_N\rangle^{(\infty)} : H_N^\infty\big).$$

Since the Rosenfeld-Gröbner algorithm with an orderly ranking does not increase the orders of the polynomials, $C_1, \ldots, C_N \subset k[\boldsymbol{x}_{r,0}]$. Since I is prime, there exists i, say $i = 1$, such that

$$k[\boldsymbol{x}_{r,0}] \cap \left(\langle C_1 \rangle^{(\infty)} : H_1^{\infty}\right) = I. \tag{9}$$

We show that $J := \langle C_1 \rangle^{(\infty)} : H_1^{\infty}$ is a prime differential ideal. Suppose $P_1, P_2 \in k[\boldsymbol{x}_{\infty,0}]$ with $P_1 P_2 \in J$. Let $\overline{P_i}\,(i = 1, 2)$ be the partial remainder of P_i with respect to C_1 [27, p. 396]. Then $\overline{P_1} \cdot \overline{P_2} \in J$. Due to Rosenfeld's lemma [27, p. 397],

$$\overline{P_1} \cdot \overline{P_2} \in k[\boldsymbol{x}_{\infty,0}] \cdot ((C_1) : H_1^{\infty}) \subseteq k[\boldsymbol{x}_{\infty,0}] \cdot (I : H_1^{\infty}) = k[\boldsymbol{x}_{\infty,0}] \cdot I.$$

Since $k[\boldsymbol{x}_{\infty,0}] \cdot I$ is prime, at least one of $\overline{P_i}$ belongs to $k[\boldsymbol{x}_{\infty,0}] \cdot I \subset J$. So $P_1 \in J$ or $P_2 \in J$. Thus, J is prime. Equality (9) together with $C_1 \subset k[\boldsymbol{x}_{r,0}]$ implies that

$$\langle C_1 \rangle^{(\infty)} : H_1^{\infty} = \langle I \rangle^{(\infty)} : H_1^{\infty},$$

so $\langle I \rangle^{(\infty)} : H_1^{\infty}$ is a prime differential ideal. Since the differential polynomials from C_1 together with some of their derivatives constitute a triangular set for I, [28, Theorem 1] implies that the degree of every initial and every separant of C_1 is bounded by

$$(2(n(r+1))^2 + 2)^{n(r+1)} D^{2n(r+1)+1} + D.$$

Since there are at most nr elements in C_1, setting

$G(n, r, D) := 2n(r+1)(2(n(r+1))^2 + 2)^{n(r+1)} D^{2n(r+1)+1} + 2n(r+1)D$ and $f := H_1$

finishes the proof of the lemma. ☐

Lemma 6.10. *For every differential ring R, subring $S \subset R$, and ideals $I, P_1, \ldots, P_\ell \subset S$ such that $I = P_1 \cap \cdots \cap P_\ell$, we have*

$$\sqrt{\langle I \rangle^{(\infty)}} = \sqrt{\langle P_1 \rangle^{(\infty)}} \cap \cdots \cap \sqrt{\langle P_\ell \rangle^{(\infty)}}.$$

Proof. [22, Lemma 8] implies that, for every $s > 0$,

$$P_1^{(s)} \cap \cdots \cap P_\ell^{(s)} \subset \sqrt{(P_1 \cap \cdots \cap P_\ell)^{(\ell s)}}.$$

Since taking the radical commutes with intersections, we have

$$\sqrt{P_1^{(s)}} \cap \cdots \cap \sqrt{P_\ell^{(s)}} \subset \sqrt{(P_1 \cap \cdots \cap P_\ell)^{(\ell s)}}.$$

We also have

$$\sqrt{(P_1 \cap \cdots \cap P_\ell)^{(\ell s)}} \subset \sqrt{P_1^{(\ell s)} \cap \cdots \cap P_\ell^{(\ell s)}} = \sqrt{P_1^{(\ell s)}} \cap \cdots \cap \sqrt{P_\ell^{(\ell s)}}.$$

Taking $s = \infty$, we obtain

$$\sqrt{\langle P_1 \rangle^{(\infty)}} \cap \cdots \cap \sqrt{\langle P_\ell \rangle^{(\infty)}} \subset \sqrt{\langle I \rangle^{(\infty)}} \subset \sqrt{\langle P_1 \rangle^{(\infty)}} \cap \cdots \cap \sqrt{\langle P_\ell \rangle^{(\infty)}}. \qquad ☐$$

Lemma 6.11. *There exists a computable function $F(n, r, m, D)$ such that, for every r, m, D, and radical ideal $J \subset k[\boldsymbol{x}_{r,0}]$ of dimension m and degree D,*

$$\deg\left(k[\boldsymbol{x}_{r,0}] \cap \sqrt{\langle J \rangle^{(\infty)}}\right) \leqslant F(n, r, m, D).$$

Proof. [22, Theorem 3] together with [8, Proposition 3] implies that

$$k[\boldsymbol{x}_{r,0}] \cap \sqrt{\langle J\rangle^{(\infty)}} = k[\boldsymbol{x}_{r,0}] \cap \sqrt{J^{(B)}},$$

where $B := D^{n(r+1)2^{m+1}}$. Thus, $\deg\left(k[\boldsymbol{x}_{r,0}] \cap \sqrt{\langle J\rangle^{(\infty)}}\right) \leqslant \deg\sqrt{J^{(B)}}$. The Bézout inequality implies that $\deg\sqrt{J^{(B)}} \leqslant D^{n(r+1)B}$. Thus, we can finish the proof of the lemma by setting $F(n,r,m,D) = D^{n(r+1)B}$. ☐

Proposition 6.12. *There is a computable function $C(n,r,m,D)$ such that, for every nonnegative integer r,m,D, and every radical ideal $I \subset k[\boldsymbol{x}_{r,0}]$ such that*

- $\deg I \leqslant D$,
- $\dim I \leqslant m$,
- $I = \sqrt{\langle I\rangle^{(\infty)}} \cap k[\boldsymbol{x}_{r,0}]$,

the number of prime components of $\sqrt{\langle I\rangle^{(\infty)}}$ does not exceed $C(n,r,m,D)$.

Proof. We fix r for the proof and will prove the proposition by constructing the function $C(r,m,D)$ by induction on a tuple (m,D) with respect to the lexicographic ordering.

Consider the base case $m=0$. Then there are at most D possible values for $(x_1,\ldots,x_n)$ and every prime component of $\sqrt{\langle I\rangle^{(\infty)}}$ is the maximal differential ideal corresponding to one of these values. Thus, the proposition is true for $C(n,r,0,D) = D$.

Consider $m>0$. *If I is not prime*, then Lemma 6.10 implies that the number of prime components of $\sqrt{\langle I\rangle^{(\infty)}}$ does not exceed

$$\max_{\ell} \max_{D_1+\cdots+D_\ell = D} \left(C(n,r,m,D_1) + \cdots + C(n,r,m,D_\ell)\right), \tag{10}$$

where all $C(n,r,m,D_i)$ are already defined by the inductive hypothesis.

Consider the case of prime I. Lemma 6.9 implies that there exists $f \in k[\boldsymbol{x}_{r,0}] \setminus I$ such that $\deg f \leqslant G(n,r,D)$ and $\langle I\rangle^{(\infty)} : f^\infty$ is a prime differential ideal. Every prime component of $\sqrt{\langle I\rangle^{(\infty)}}$ either is equal to $\langle I\rangle^{(\infty)} : f^\infty$ or contains f. In the latter case, the component is a component of $\sqrt{\langle I,f\rangle^{(\infty)}}$. Let $J := \sqrt{\langle I,f\rangle^{(\infty)}} \cap k[\boldsymbol{x}_{r,0}]$. Then $\dim J \leqslant m-1$, and Lemma 6.11 implies that $\deg J \leqslant F(r,m-1,G(n,r,D))$. Then the number of prime components of $\sqrt{\langle I\rangle^{(\infty)}}$ does not exceed

$$1 + C(n,r,m-1,F(r,m-1,G(n,r,D))). \tag{11}$$

Thus, one can define $C(n,r,m,D)$ to be the maximum of (10) and (11). ☐

Proposition 6.12 and Lemma 6.11 imply the following corollary.

Corollary 6.13. *For every radical ideal $I \subset k[\boldsymbol{x}_{r,0}]$ of dimension at most m and degree at most D, the number of prime components of $\sqrt{\langle I\rangle^{(\infty)}}$ does not exceed*

$$C(n,r,m,F(n,r,m,D)).$$

6.3.2. *Bound for trains.* Now we try to give a bound so that the existence of a maximal train of certain length in X will definitely guarantee the existence of at least one infinite train in X.

Definition 6.14 (Kolchin polynomials for δ-varieties and trains)**.**

- The *Kolchin polynomial* of an irreducible δ-variety $V = \mathbb{V}(F)$, where $F \subset K[\boldsymbol{y}_{\infty,0}]$, where $\boldsymbol{y} = y_1, \dots, y_n$, is the unique numerical polynomial $\omega_V(t)$ such that there exists $t_0 \geqslant 0$ such that, for all $t \geqslant t_0$ and the generic point $\boldsymbol{a}$ of V (see Definition 4.2), $\omega_V(t) = \operatorname{tr.deg} K(\boldsymbol{a}_{t,0})/K$.
- Univariate polynomials over $\mathbb{R}$ are ordered as follows: $p(t)$ is said to be greater than $q(t)$ if the difference $p(t) - q(t)$ is eventually positive as $t \to +\infty$. The induced ordering on the set of Kolchin polynomials makes them well-ordered [13, Proposition 2.4.14].
- The Kolchin polynomial of a δ-variety is defined to be the maximal Kolchin polynomial of its irreducible components.
- An irreducible component X_1 of a δ-variety X is called a *generic component* if $\omega_{X_1}(t) = \omega_X(t)$.
- We define the *Kolchin polynomial of a train* $Y = (Y_1, \dots, Y_\ell)$ in X as
$$\omega_Y(t) := \min_i \omega_{Y_i}(t).$$

Remark 6.15. The Kolchin polynomial of an irreducible δ-variety V is of the form (see [13, formula (2.2.6)] and [12, Theorem II.12.6(d)])
$$\omega_V(t) = \delta\text{-}\dim(V) \cdot (t+1) + \operatorname{ord}(V).$$

The following lemma shows how the coefficients of a Kolchin polynomial change under a projection.

Lemma 6.16. *Let $V \subset \mathbb{A}^n$ be an irreducible δ-variety and let $\pi_1 : \mathbb{A}^n \longrightarrow \mathbb{A}^{n-1}$ be the projection to the first $n-1$ coordinates. Then we have*
$$\delta - \dim\left(\overline{\pi_1(V)}^{\text{Kol}}\right) \leqslant \delta - \dim(V) \quad \textit{and} \quad \operatorname{ord}\left(\overline{\pi_1(V)}^{\text{Kol}}\right) \leqslant \operatorname{ord}(V).$$

Proof. Let $\boldsymbol{a}$ be a generic point of V. Then $\pi_1(\boldsymbol{a})$ is a generic point of $W := \overline{\pi_1(V)}^{\text{Kol}}$. Clearly,
$$\omega_W(t) \leqslant \omega_V(t) \quad \text{and} \quad \delta\text{-}\dim(W) \leqslant \delta\text{-}\dim(V).$$
So, we have
$$\delta\text{-}\dim(W) = \delta\text{-}\dim(V) \implies \operatorname{ord}(W) \leqslant \operatorname{ord}(V).$$
It therefore suffices to show that
$$\delta\text{-}\dim(W) < \delta\text{-}\dim(V) \implies \operatorname{ord}(W) \leqslant \operatorname{ord}(V).$$
Suppose $\delta\text{-}\dim(W) < \delta\text{-}\dim(V) = d$. Then we have
$$\delta\text{-}\dim(W) = \delta\text{-}\dim(V) - 1 = d - 1.$$
Since the order of an irreducible δ-variety V is equal to the maximal relative order of V with respect to a parametric set, without loss of generality, suppose that
$$\operatorname{ord}(W) = \operatorname{tr.deg} K\big(\pi_1(\boldsymbol{a})_{\infty,0}\big)/K\big(\pi_{n-(d-1)}(\boldsymbol{a})_{\infty,0}\big),$$
where $\pi_{n-(d-1)} : \mathbb{A}^n \to \mathbb{A}^{d-1}$ is the projection to the first $d-1$ coordinates. Since
$$\delta\text{-}\operatorname{tr.deg} K\big(\boldsymbol{a}_{\infty,0}\big)/K = 1 + \delta\text{-}\operatorname{tr.deg} K\big(\pi_1(\boldsymbol{a})_{\infty,0}\big)/K,$$

a_n is δ-transcendental over $K(\pi_1(\boldsymbol{a})_{\infty,0})$, i.e., δ-tr.deg $K\big(\boldsymbol{a}_{\infty,0}\big)/K\big(\pi_1(\boldsymbol{a})_{\infty,0}\big) = 1$. Therefore, we have

$$\begin{aligned}\operatorname{ord}(W) &= \operatorname{tr.deg} K\big(\pi_1(\boldsymbol{a})_{\infty,0}\big)/K\big(\pi_{n-(d-1)}(\boldsymbol{a})_{\infty,0}\big)\\ &= \operatorname{tr.deg} K\big((a_n)_{\infty,0}\big)\big(\pi_1(\boldsymbol{a})_{\infty,0}\big)/K\big((a_n)_{\infty,0}\big)\big(\pi_{n-(d-1)}(\boldsymbol{a})_{\infty,0}\big)\\ &= \operatorname{ord}_{y_1,\ldots,y_{d-1},y_n}(V) \leqslant \operatorname{ord}(V). \qquad \square\end{aligned}$$

Lemma 6.17. *For all $s \in \mathbb{Z}_{\geqslant 0}$ and $F \subset k[\boldsymbol{y}_{s,0}]$, where $\boldsymbol{y} = y_1, \ldots, y_n$, the order of each component of $\mathbb{V}(F)$ is bounded by ns.*

Proof. It follows directly by [26, p. 135] and [7, Theorem 2.11]. □

Definition 6.18. For all

- nonnegative integers n, s, h, d,
- $\mathbb{Z}_{\geqslant 0}$-valued polynomials $\omega \in \mathbb{Z}[t]$,

we define $B(\omega, n, s, h, d)$ to be the smallest $M \in \mathbb{N} \cup \{\infty\}$ such that, for every triple (X, π_1, π_2) with

$$X = \mathbb{V}(F) \subseteq \mathbb{A}^{n(h+1)}, \quad F \subset k[\boldsymbol{y}_{s,h}],\ \boldsymbol{y} = y_1, \ldots, y_n, \quad \deg(F) \leqslant d,$$

if there exists a train in X of length M and with Kolchin polynomial at least ω, then there exists an infinite train in X.

Remark 6.19. For all nonnegative integers n, s, h, d and $\mathbb{Z}_{\geqslant 0}$-valued polynomials $\omega \in \mathbb{Z}[t]$,

$$B(\omega(t), n, s, h, d) \leqslant B(0, n, s, h, d).$$

In the following, we will show that $B(\omega(t), n, s, h, d)$ is finite for all $\omega(t) \geqslant 0$ and the numerical data n, s, h, d and is also bounded by a computable function in these numerical data. For ease of notation, we denote

$$L(n, r, d) := C\big(n, r, n(r+1), F(n, r, n(r+1), d)\big),$$

which is computable. So by Corollary 6.13, given a system S of δ-polynomials in n δ-variables of order bounded by r and degree bounded by d, the number of components of the δ-variety $\mathbb{V}(S)$ is bounded by $L(n, r, d)$. By Lemma 6.8, the number of maximal trains in X of length ℓ is bounded by $L(n(h+\ell), s, d)$. We now define two increasing sequences $(A_i(n, h, s, d))_{i\in\mathbb{N}}$ and $(\tau_i(n, h, s, d))_{i\in\mathbb{N}}$ as follows:

$$\begin{aligned}&A_0 = L(n(h+1), s, d) + 1,\ A_{i+1} = A_i + L(n(h+1)A_i, s, d) \text{ (for } i \geqslant 0),\\ &\tau_0 = ns(h+1),\ \tau_{i+1} = \tau_i + ns(h+1)A_{\tau_i} + 1 \text{ (for } i \geqslant 0).\end{aligned} \tag{12}$$

Lemma 6.20. *We have*

$$B(0, n, s, h, d) \leqslant A_{\tau_{n(h+1)}(n,h,s,d)}(n, h, s, d),$$

which is computable.

Proof. Temporarily, fix X. By Corollary 6.13, we know upper bounds for the number of irreducible components of X and for the number of maximal trains in X of any fixed length. The main idea of the proof is to construct a decreasing chain of Kolchin polynomials $\omega_0(t) > \omega_1(t) > \cdots$ and, for each $\omega_i(t)$, give an upper bound B_i for $B(\omega_i(t), n, s, h, d)$. Since the Kolchin polynomials are well-ordered, the decreasing chain will stop at some $\omega_J(t) = 0$.

Let $\omega_0(t) = \omega_X(t)$. Let B_0 be the number of generic components of the δ-variety X plus 1. Consider a train $(Y_1, \ldots, Y_{B_0})$ in X of Kolchin polynomial at least $\omega_0(t)$.

So for each i, $\sigma^{-i+1}(Y_i)$ is a δ-subvariety of X with Kolchin polynomial at least $\omega_X(t)$, so $\sigma^{-i+1}(Y_i)$ must be a generic component of X. Since X has only $B_0 - 1$ generic components, there exists $a < b \in \mathbb{N}$ such that $\sigma^{-a+1}(Y_a) = \sigma^{-b+1}(Y_b)$, which implies that $Y_b = \sigma^{b-a}(Y_a)$. Thus, we can construct an infinite train

$$(\ldots, Y_a, Y_{a+1}, \ldots, Y_{b-1}, \sigma^{b-a}(Y_a), \sigma^{b-a}(Y_{a+1}), \ldots, \sigma^{b-a}(Y_{b-1}), \ldots).$$

Suppose $\omega_i(t)$ and B_i have been constructed. We now try to do it for $i+1$. Let

$$B_{i+1} = B_i + D_i, \tag{13}$$

where D_i is the number of maximal trains in X of length B_i. Consider the fibered product $\mathbf{W}_{B_i}(X, \pi_1, \pi_2)$, as in (7), and, for each irreducible component W of $\mathbf{W}_{B_i}$, denote

$$Y_W = \big(\overline{\varphi_{B_i,1}(W)}^{\text{Kol}}, \ldots, \overline{\varphi_{B_i,B_i}(W)}^{\text{Kol}}\big)$$

to be the train corresponding to W. Let

$$\omega_{i+1}(t) := \max\big\{\omega_{Y_W}(t) \,\big|\, \omega_{Y_W}(t) < \omega_i(t), W \qquad \text{is a component of } \mathbf{W}_{B_i}\big\},$$

and set $\max \varnothing = 0$.

Consider a maximal train $(Y_1, \ldots, Y_{B_{i+1}})$ in X with Kolchin polynomial at least $\omega_{i+1}(t)$. We will show this B_{i+1} works. Introduce $D_i + 1$ trains $Z^{(1)}, \ldots, Z^{(D_i+1)}$ of length B_i in $X, \sigma(X), \ldots, \sigma^{D_i}(X)$, respectively, such that for each j,

$$Z^{(j)} = \big(Z_1^{(j)}, \ldots, Z_{B_i}^{(j)}\big) := (Y_j, \ldots, Y_{j+B_i-1}).$$

Then for each j, consider a maximal train $\tilde{Z}^{(j)}$ of length B_i containing $Z^{(j)}$. So $\sigma^{-j+1}(\tilde{Z}^{(j)})$ is a maximal train of length B_i in X. There are two cases to consider:

(Case 1) $\quad \big\{\omega_{Y_W}(t) \,\big|\, \omega_{Y_W}(t) < \omega_i(t), \quad W \text{ is a component of } \mathbf{W}_{B_i}\big\} = \varnothing.$

In this case, $\omega_{i+1}(t) = 0$, and for each j,

$$\omega_{\sigma^{-j+1}(\tilde{Z}^{(j)})}(t) \geqslant \omega_i(t).$$

By the construction of B_i, we could construct an infinite train through each $\sigma^{-j+1}(\tilde{Z}^{(j)})$.

(Case 2) $\quad \big\{\omega_{Y_W}(t) \,\big|\, \omega_{Y_W}(t) < \omega_i(t), \quad W \text{ is a component of } \mathbf{W}_{B_i}\big\} \neq \varnothing.$

If there exists some j_0 such that $\omega_{\sigma^{-j_0+1}(\tilde{Z}^{(j_0)})}(t) \geqslant \omega_i(t)$, then by the construction of B_i, we could construct an infinite train through this $\sigma^{-j_0+1}(\tilde{Z}^{(j_0)})$. Suppose now that, for each j,

$$\omega_{\sigma^{-j+1}(\tilde{Z}^{(j)})}(t) = \omega_{i+1}(t).$$

Since there are only a D_i number of maximal trains in X of length B_i, there exist $a < b$ such that

$$\sigma^{-a+1}(\tilde{Z}^{(a)}) = \sigma^{-b+1}(\tilde{Z}^{(b)}).$$

Since $\omega_{\sigma^{-a+1}(\tilde{Z}^{(a)})}(t) = \omega_{i+1}(t)$, there exists l such that

$$\omega_{\sigma^{-a+1}(\tilde{Z}_l^{(a)})}(t) = \omega_{i+1}(t).$$

Then

$$\omega_{\sigma^{-a+1}(Z_l^{(a)})}(t) = \omega_{i+1}(t),$$

for $\sigma^{-a+1}(Z_l^{(a)}) \subseteq \sigma^{-a+1}(\tilde{Z}_l^{(a)})$, and the Kolchin polynomial of $(Y_1, \ldots, Y_{B_{i+1}})$ is at least $\omega_{i+1}(t)$. So we have

$$\sigma^{-a+1}(Z_l^{(a)}) = \sigma^{-a+1}(\tilde{Z}_l^{(a)}).$$

Similarly, we can show that

$$\sigma^{-b+1}(Z_l^{(b)}) = \sigma^{-b+1}(\tilde{Z}_l^{(b)}).$$

So

$$\begin{aligned}\sigma^{-a+1}(Y_{a+l-1}) &= \sigma^{-a+1}(Z_l^{(a)}) = \sigma^{-a+1}(\tilde{Z}_l^{(a)} \\ &= \sigma^{-b+1}(\tilde{Z}_l^{(b)}) = \sigma^{-b+1}(Z_l^{(b)}) = \sigma^{-b+1}(Y_{b+l-1}).\end{aligned}$$

Thus, we have

$$Y_{b+l-1} = \sigma^{b-a}(Y_{a+l-1}).$$

Therefore, we can construct an infinite sequence

$$\Big(Y_1, \ldots, Y_{a+l-1}, , \ldots, Y_{b+l-2}, \sigma^{b-a}(Y_{a+l-1}), \ldots, \sigma^{b-a}(Y_{b+l-2}), \ldots\Big).$$

As we described in the first paragraph, as the process goes on, we have constructed a decreasing chain of Kolchin polynomials

$$\omega_0(t) = \omega_X(t) > \omega_1(t) > \omega_2(t) > \cdots.$$

Since the Kolchin polynomials are well-ordered, this chain is finite, so the above process will stop at step J, at which we could get $\omega_J(t) = 0$, either in the case in which

$$\big\{\omega_{Y_W}(t) \,\big|\, \omega_{Y_W}(t) < \omega_{J-1}(t), W \text{ is a component of } \mathbf{W}_{B_{J-1}}\big\} = \varnothing$$

or in the case in which the set is nonempty and the maximal Kolchin polynomial in the set is 0.

By Lemma 6.8 and Corollary 6.13, for the number D_i of maximal trains in X of length B_i, we have

$$D_i \leqslant L(n(h+1)B_i, s, d), \quad \text{so} \quad B_{i+1} \leqslant B_i + L(n(h+1)B_i, s, d). \tag{14}$$

By Corollary 6.13, we have

$$B_0 \leqslant L(n(h+1), s, d) + 1.$$

For each i, $0 \leqslant i \leqslant J$, let a_i and b_i be such that

$$\omega_i(t) = a_i(t+1) + b_i.$$

For $i = 0$, we have $a_0 = \delta\text{-dim}(X)$ and $b_0 = \operatorname{ord}(X)$. For every j, $0 \leqslant j \leqslant a_0$, we define i_j to be the largest integer in $[0, J]$ such that $a_0 - a_{i_j} \leqslant j$. Then $J = i_{a_0}$. The decreasing of the Kolchin polynomials implies that, for all j, $0 \leqslant j < a_0$:

- we have

$$i_0 \leqslant b_0 \quad \text{and} \quad i_{j+1} \leqslant i_j + b_{i_j+1} + 1, \tag{15}$$

- by the definition of $\omega_{i_j+1}(t)$ and Lemma 6.16, b_{i_j+1} is bounded by the maximal order of the components of $W_{B_{i_j}}$, so
- by Lemma 6.17,

$$b_{i_j+1} \leqslant ns(h+1)B_{i_j}. \tag{16}$$

Comparing the recursive formulas (12) with inequalities (14), (15), and (16), we see that

- $B_i \leqslant A_i$ for every i, $0 \leqslant i \leqslant J$;
- $i_j \leqslant \tau_j$ for every j, $0 \leqslant j \leqslant a_0 = \delta\text{-dim}(X)$.

Thus,

$$B_J = B_{i_{\delta\text{-dim}(X)}} \leqslant A_{i_{\delta\text{-dim}(X)}} \leqslant A_{\tau_{\delta\text{-dim}(X)}} \leqslant A_{\tau_{n(h+1)}}.$$

□

As a consequence, we have the following result.

Corollary 6.21. *For all $s, h \in \mathbb{Z}_{\geqslant 0}$ and $F \subset k[\boldsymbol{y}_{s,h}]$ with $\deg F \leqslant d$, $F = 0$ has a solution in $K^{\mathbb{Z}}$ if and only if $F = 0$ has a partial solution of length $D := A_{\tau_{n(h+1)}(n,h,s,d)}(n, h, s, d)$.*

Proof. Let $X \subset \mathbb{A}^H$ be the δ-variety defined by $F = 0$ regarded as a system of δ-equations in $\boldsymbol{y}, \sigma(\boldsymbol{y}), \ldots, \sigma^h(\boldsymbol{y})$, where $H = n(h+1)$. By Lemmas 6.5 and 6.7, $F = 0$ has a partial solution of length D (resp., $F = 0$ has a solution in $K^{\mathbb{Z}}$) if and only if there exists a train of length D in X (resp., there exists an infinite train in X). By Lemma 6.20, if there exists a train of length D in X, then there exists a infinite train in X. So the assertion holds. □

6.4. **Proof of Theorem 3.1.** We will prove a more refined version of Theorem 3.1. Theorem 3.1 follows from Theorem 6.22 if one sets $B(r, s) := B_0(r, s) + s$.

Theorem 6.22. *For all nonnegative integers r, s, there exists a computable $B_0 = B_0(r, s)$ such that, for all*

- *nonnegative integers q and t,*
- *δ-σ-fields k with $\operatorname{char} k = 0$,*
- *sets of δ-σ-polynomials $F \subset k[\boldsymbol{x}_t, \boldsymbol{y}_s]$, where $\boldsymbol{x} = x_1, \ldots, x_q$, $\boldsymbol{y} = y_1, \ldots, y_r$, and $\deg_{\boldsymbol{y}} F \leqslant s$,*

we have

$$\left\langle \sigma^i(F) \mid i \in \mathbb{Z}_{\geqslant 0} \right\rangle^{(\infty)} \cap k[\boldsymbol{x}_\infty] \neq \{0\} \iff \left\langle \sigma^i(F) \mid i \in [0, B_0] \right\rangle^{(B_0)} \cap k[\boldsymbol{x}_{B_0+t}] \neq \{0\}.$$

Proof. The " $\Longleftarrow$ " implication is straightforward. We will prove the " $\Longrightarrow$ " implication. For this, let $A := A_{\tau_{r(s+1)}(r,s,s,s)}(r, s, s, s)$, and let B_0 be the bound from [22, Theorem 1] with

$$|\boldsymbol{\alpha}| \leftarrow r(s+1)(A+s+1), \quad m \leftarrow r(s+1)(A+s+1), \quad \text{and} \quad d \leftarrow s.$$

By assumption,

$$1 \in \left\langle \sigma^i(F) \mid i \in \mathbb{Z}_{\geqslant 0} \right\rangle^{(\infty)} \cdot k(\boldsymbol{x}_\infty)[\boldsymbol{y}_\infty]. \tag{17}$$

Suppose that

$$\left\langle \sigma^i(F) \mid i \in [0, A] \right\rangle^{(B_0)} \cap k[\boldsymbol{x}_{B_0+t}] = \{0\}. \tag{18}$$

If

$$1 \in \left\langle \sigma^i(F) \mid i \in [0, A] \right\rangle^{(B_0)} \cdot k(\boldsymbol{x}_{B_0+t})[\boldsymbol{y}_{\infty,A+s}],$$

then there would exist $c_{i,j} \in k(\boldsymbol{x}_{B_0+t})[\boldsymbol{y}_{\infty,A+s}]$ such that

$$1 = \sum_{i=0}^{B_0} \sum_{j=0}^{A} \sum_{f \in F} c_{i,j} \delta^i \sigma^j(f). \tag{19}$$

Multiplying equation (19) by the common denominator in the variables $\boldsymbol{x}_{B_0+t}$, we obtain a contradiction to (18). Hence, by [22, Theorem 1],

$$1 \notin \left\langle \sigma^i(F) \mid i \in [0, A] \right\rangle^{(\infty)} \cdot k(\boldsymbol{x}_{B_0+t})[\boldsymbol{y}_{\infty,A+s}].$$

By Lemma 6.1, there exists a differentially closed δ-σ^*-field extension $L \supset k(\boldsymbol{x}_\infty) \supset k(\boldsymbol{x}_{B_0+t})$. Then differential Nullstellensatz implies that the system of differential equations

$$\{\sigma^i(F) = 0 \mid i \in [0, A]\}$$

in the unknowns $\boldsymbol{y}_{\infty, A+s}$ has a solution in L. Then the system $F = 0$ has a partial solution of length $A+1$ in L. However, by (17), the system $F = 0$ has no solutions in $L^{\mathbb{Z}}$. Together with the existence of a partial solution of length $A+1$, this contradicts Corollary 6.21. □

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