

Pebble minimization: the last theorems

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Abstract Pebble transducers are nested two-way transducers which can drop marks (named “pebbles”) on their input word. Such machines can compute functions whose output size is polynomial in the size of their input. They can be seen as simple recursive programs whose recursion height is bounded. A natural problem is, given a pebble transducer, to compute an equivalent pebble transducer with minimal recursion height. This problem is open since the introduction of the model.

In this paper, we study two restrictions of pebble transducers, that cannot see the marks (“blind pebble transducers” introduced by Nguyễn et al.), or that can only see the last mark dropped (“last pebble transducers” introduced by Engelfriet et al.). For both models, we provide an effective algorithm for minimizing the recursion height. The key property used in both cases is that a function whose output size is linear (resp. quadratic, cubic, etc.) can always be computed by a machine whose recursion height is 1 (resp. 2, 3, etc.). We finally show that this key property fails as soon as we consider machines that can see more than one mark.

Keywords: Pebble transducers · Polyregular functions · Blind pebble transducers · Last pebble transducers · Factorization forests.

1 Introduction

Transducers are finite-state machines obtained by adding outputs to finite automata. They are very useful in a lot of areas like coding, computer arithmetic, language processing or program analysis, and more generally in data stream processing. In this paper, we consider deterministic transducers which compute functions from finite words to finite words. In particular, a **deterministic two-way transducer** is a two-way automaton with outputs. This model describes the class of **regular functions**, which is often considered as one of the functional counterparts of regular languages. It has been intensively studied for its properties such as closure under composition [5], equivalence with logical transductions [12] or regular expressions [7], decidable equivalence problem [14], etc.

Pebble transducers and polyregular functions. Two-way transducers can only describe functions whose output size is at most linear in the input size. A possible solution to overcome this limitation is to consider nested two-way

transducers. In particular, the model of **k -pebble transducer** has been studied for a long time [13]. For $k = 1$, a 1-pebble transducer is just a two-way transducer. For $k \geq 2$, a k -pebble transducer is a two-way transducer that, when on any position i of its input word, can call a $(k-1)$ -pebble transducer. The latter takes as input the original input where position i is marked by a “pebble”. The main two-way transducer then outputs the concatenation of all the outputs produced along its calls. The intuitive behavior of a 3-pebble transducer is depicted in fig. 1. It can be seen as recursive program whose recursion stack has height 3. The class of functions computed by pebble transducers is known as **polyregular functions**. It has been intensively studied due to its properties such as closure under composition [11], equivalence with logical interpretations [4], etc.

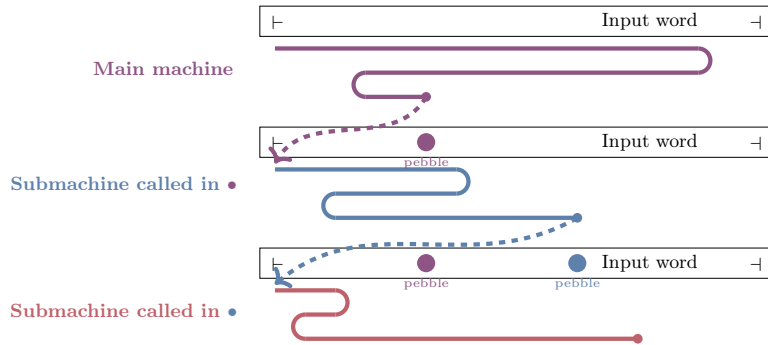


Figure 1: Behavior of a 3-pebble transducer.

Optimization of pebble transducers. Given a k -pebble transducer computing a function f , a very natural problem is to compute the least possible $1 \leq \ell \leq k$ such that f can be computed by an ℓ -pebble transducer. Furthermore, we can be interested in effectively building an ℓ -pebble transducer for f . Both questions are open, but they are meaningful in practice. Indeed, they ask whether we can optimize the recursion height (i.e. the running time) of a program.

It is easy to observe that if f is computed by a k -pebble transducer, then $|f(u)| = \mathcal{O}(|u|^k)$. It was first conjectured in the literature that the minimal recursion height ℓ of f (i.e. the least possible ℓ such that f can be computed by an ℓ -pebble transducer) was exactly the least possible ℓ such that $|f(u)| = \mathcal{O}(|u|^\ell)$. However, Bojańczyk recently disproved this statement in [3, Theorem 6.3]: the function **inner-squaring** : $u_1\# \cdots \# u_n \mapsto (u_1\#)^n \cdots (u_n\#)^n$ can be computed by a 3-pebble transducer and is such that $|\text{inner-squaring}(u)| = \mathcal{O}(|u|^2)$, but it cannot be computed by a 2-pebble transducer. Therefore, computing the minimal recursion height of f is believed to be hard, since this value not only depends on the output size of f , but also on the word combinatorics of this output.

Optimization of blind pebble transducers. A subclass of pebble transducers, named **blind pebble transducers**, was recently introduced in [16]. A blind k -pebble transducer is somehow a k -pebble transducer, with the difference that the positions are no longer marked when making recursive calls. The behavior of a blind 3-pebble transducer is depicted in fig. 2. The class of functions computed by blind pebble transducers is strictly included in polyregular functions [10,16]. The main result of [16] shows that for blind pebble transducers, the minimal recursion height for computing a function only depends on the growth of its output. More precisely, if f is computed by a blind k -pebble transducer, then the least possible $1 \leq \ell \leq k$ such that f can be computed by an blind ℓ -pebble transducer is the least possible ℓ such that $|f(u)| = \mathcal{O}(|u|^\ell)$.

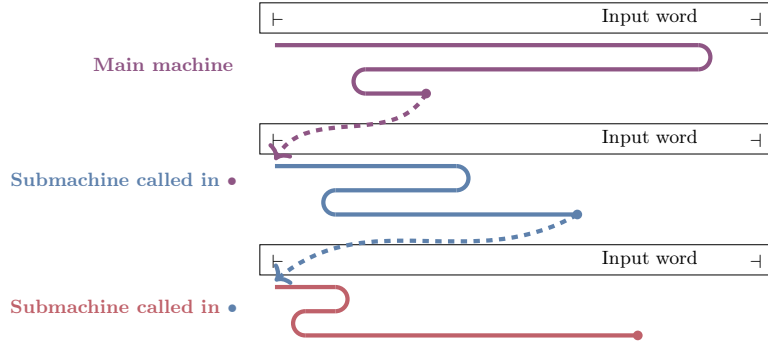


Figure 2: Behavior of a blind 3-pebble transducer.

Contributions. In this paper, we first give a new proof of the connection between minimal recursion height and growth of the output for blind pebble transducers. Furthermore, our proof provides an algorithm that, given a function computed by a blind k -pebble transducer, builds a blind ℓ -pebble transducer which computes it, for the least possible $1 \leq \ell \leq k$. This effective result is not claimed in [16], and our proof techniques significantly differ from theirs. Indeed, we make a heavy use of **factorization forests**, which have already been used as a powerful tool in the study of pebble transducers [2,8,10].

Secondly, the main contribution of this paper is to show that the (effective) connection between minimal recursion height and growth of the output also holds for the class of **last pebble transducers** (introduced in [13]). Intuitively, a last k -pebble transducer is a k -pebble transducer where a called submachine can only see the position of its call, but not the full stack of the former positions. The behavior of a last 3-pebble transducer is depicted in fig. 3. Observe that a blind k -pebble transducer is a restricted version of a last k -pebble transducer. Formally, we show that if f is computed by a last k -pebble transducer, then the least possible ℓ such that f can be computed by a last ℓ -pebble transducer is the least possible ℓ such that $|f(u)| = \mathcal{O}(|u|^\ell)$. Furthermore, our proof gives an algorithm that effectively builds a last ℓ -pebble transducer computing f .

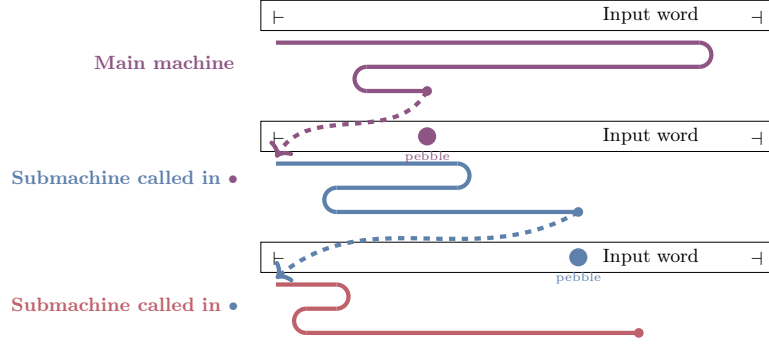


Figure 3: Behavior of a last 3-pebble transducer.

As a third theorem, we show that our result for last pebble transducers is tight, in the sense that the connection between minimal recursion height and growth of the output does not hold for more powerful models. More precisely, we define the model of **last-last k -pebble transducers**, which extends last k -pebble transducers by allowing them to see the two last positions of the calls (and not only the last one). We show that for all $k \geq 1$, there exists a function f such that $|f(u)| = \mathcal{O}(|u|^2)$ and that is computed by a last-last $(2k+1)$ -pebble transducer, but cannot be computed by a last-last $2k$ -pebble transducer. In this setting, minimizing the recursion height seems to be out of reach.

Outline. We introduce two-way transducers in section 2. In section 3 we describe blind pebble transducers and last pebble transducers. We also state our main results that connect the minimal recursion height of a function to the growth of its output. Their proof goes over sections 4 to 6. In section 7, we finally show that these results cannot be extended to two visible marks.

2 Preliminaries on two-way transducers

Capital letters A, B denote alphabets, i.e. a finite sets of letters. The empty word is denoted by ε . If $u \in A^*$, let $|u| \in \mathbb{N}$ be its length, and for $1 \leq i \leq |u|$ let $u[i]$ be its i -th letter. If $i \leq j$, we let $u[i:j]$ be $u[i]u[i+1] \cdots u[j]$ (empty if $j < i$). If $a \in A$, let $|u|_a$ be the number of letters a occurring in u . We assume that the reader is familiar with the basics of automata theory, in particular two-way automata and monoid morphisms. The type of total (resp. partial, i.e. possibly undefined on some inputs) functions is denoted $S \rightarrow T$ (resp. $S \rightharpoonup T$).

The machines described in this paper are always **deterministic**.

Definition 2.1. A **two-way transducer** $\mathcal{T} = (A, B, Q, q_0, F, \delta, \lambda)$ consists of:

- an input alphabet A and an output alphabet B ;
- a finite set of states Q with $q_0 \in Q$ initial and $F \subseteq Q$ final;
- a transition function $\delta : Q \times (A \uplus \{\vdash, \dashv\}) \rightarrow Q \times \{\triangleleft, \triangleright\}$;
- an output function $\lambda : Q \times (A \uplus \{\vdash, \dashv\}) \rightarrow B^*$ with same domain as δ .

The semantics of a two-way transducer \mathcal{T} is defined as follows. When given as input a word $u \in A^*$, \mathcal{T} disposes of a read-only input tape containing $\vdash u \dashv$. The marks \vdash and \dashv are used to detect the borders of the tape, by convention we denote them by positions 0 and $|u|+1$ of u . Formally, a configuration over $\vdash u \dashv$ is a tuple (q, i) where $q \in Q$ is the current state and $0 \leq i \leq |u|+1$ is the position of the reading head. The transition relation \rightarrow is defined as follows. Given a configuration (q, i) , let $(q', \star) := \delta(q, u[i])$. Then $(q, i) \rightarrow (q', i')$ whenever either $\star = \triangleleft$ and $i' = i-1$ (move left), or $\star = \triangleright$ and $i' = i+1$ (move right), with $0 \leq i' \leq |u|+1$. A run is a sequence of configurations $(q_1, i_1) \rightarrow \dots \rightarrow (q_n, i_n)$. Accepting runs are those that begin in $(q_0, 0)$ and end in a configuration of the form $(q, |u|+1)$ with $q \in F$ (and never visit such a configuration before).

The partial function $f : A^* \rightarrow B^*$ computed by the two-way transducer \mathcal{T} is defined as follows: for $u \in A^*$, if there exists an accepting run on $\vdash u \dashv$, then it is unique, and $f(u)$ is defined as $\lambda(q_1, (\vdash u \dashv)[i_1]) \dots \lambda(q_n, (\vdash u \dashv)[i_n]) \in B^*$. The class of functions computed by two-way transducers is called **regular functions**.

Example 2.2. Let \tilde{u} be the mirror image of $u \in A^*$. Let $\# \notin A$ be a fresh symbol. The function **map-reverse** : $u_1 \# \dots \# u_n \mapsto \tilde{u}_1 \# \dots \# \tilde{u}_n$ can be computed by a two-way transducer, that reads each factor u_j from right to left.

It is well-known that the domain of a regular function is always a **regular language** (see e.g. [17]). From now on, we assume without losing generalities that our two-way transducers only compute total functions (in other words, they have exactly one accepting run on each $\vdash u \dashv$). Furthermore, we assume that $\lambda(q, \vdash) = \lambda(q, \dashv) = \varepsilon$ for all $q \in Q$ (we only lose generalities for the image of ε).

In the rest of this section, \mathcal{T} denotes a two-way transducer with input alphabet A , output alphabet B and output function λ . Now, we define the **crossing sequence** in a position $1 \leq i \leq |u|$ of input $\vdash u \dashv$. Intuitively, it regroups the states of the accepting run which are visited in this position.

Definition 2.3. Let $u \in A^*$ and $1 \leq i \leq |u|$. Let $(q_1, i_1) \rightarrow \dots \rightarrow (q_n, i_n)$ be the accepting run of \mathcal{T} on $\vdash u \dashv$. The **crossing sequence** of \mathcal{T} in i , denoted $\text{cross}_{\mathcal{T}}^u(i)$, is defined as the sequence $(q_j)_{1 \leq j \leq n \text{ and } i_j = i}$.

If $\mu : A^* \rightarrow \mathbb{M}$ is a monoid morphism, we say that any $m, m' \in \mathbb{M}$ and $a \in A$ define a **μ -context** that we denote by $m \llbracket a \rrbracket m'$. It is well-known that the crossing sequence in a position of the input only depends on the context of this position, for a well-chosen monoid, as claimed in proposition 2.4 (see e.g. [7]).

Proposition 2.4. One can build a finite monoid \mathbb{T} and a monoid morphism $\mu : A^* \rightarrow \mathbb{T}$, called the **transition morphism** of \mathcal{T} , such that for all $u \in A^*$ and $1 \leq i \leq |u|$, $\text{cross}_{\mathcal{T}}^u(i)$ only depends on $\mu(u[1:i-1])$, $u[i]$ and $\mu(u[i+1:|u|])$. Thus we denote it $\text{cross}_{\mathcal{T}}(\mu(u[1:i-1]) \llbracket u[i] \rrbracket \mu(u[i+1:|u|]))$.

Finally, let us define “the output produced below position i ”.

Definition 2.5. Let $u \in A^*$ and $1 \leq i \leq |u|$ and $q_1 \dots q_n := \text{cross}_{\mathcal{T}}^u(i)$. We define the **production** of \mathcal{T} in i , denoted $\text{prod}_{\mathcal{T}}^u(i)$, as $\lambda(q_1, u[i]) \dots \lambda(q_n, u[i])$.

By proposition 2.4, it also makes sense to define $\text{prod}_{\mathcal{T}}(m \llbracket a \rrbracket m') \in B^*$ to be $\text{prod}_{\mathcal{T}}^u(i)$ whenever $m = \mu(u[1:i-1])$, $m' = \mu(u[i+1:|u|])$ and $a = u[i]$.

3 Blind and last pebble transducers

Now, we are ready to define formally the models of blind pebble transducers and last pebble transducers. Intuitively, they correspond to two-way transducers which make a tree of recursive calls to other two-way transducers.

Definition 3.1 (Blind pebble transducer [16]). For $k \geq 1$, a **blind k -pebble transducer** with input alphabet A and output alphabet B is:

- if $k = 1$, a two-way transducer with input alphabet A and output B ;
- if $k \geq 2$, a tree $\mathcal{T}\langle \mathcal{B}_1, \dots, \mathcal{B}_p \rangle$ where $\mathcal{B}_1, \dots, \mathcal{B}_p$ are blind $(k-1)$ -pebble transducers with input A and output B ; and \mathcal{T} is a two-way transducer with input A and output alphabet $\{\mathcal{B}_1, \dots, \mathcal{B}_p\}$.

The (total) function $f : A^* \rightarrow B^*$ computed by the blind k -pebble transducer of definition 3.1 is built in a recursive fashion, as follows:

- for $k = 1$, f is the function computed by the two-way transducer;
- for $k \geq 2$, let $u \in A^*$ and $(q_1, i_1) \rightarrow \dots \rightarrow (q_n, i_n)$ be the accepting run of $\mathcal{T} = (A, B, Q, q_0, F, \delta, \lambda)$ on $\vdash u \dashv$. For all $1 \leq j \leq n$, let $f_j : A^* \rightarrow B^*$ be the concatenation of the functions recursively computed by the sequence $\lambda(q_j, (\vdash u \dashv)[i_j]) \in \{\mathcal{B}_1, \dots, \mathcal{B}_p\}^*$. Then $f(u) := f_1(u) \dots f_n(u)$.

The behavior of a blind 3-pebble transducer is depicted in fig. 2.

Example 3.2. The function **unmarked-square** : $A^* \rightarrow A^* \uplus \{\#\}$, $u \mapsto (u\#)^{|u|}$ can be computed by a blind 2-pebble transducer. This machine has shape $\mathcal{T}\langle \mathcal{T}' \rangle$: \mathcal{T} calls \mathcal{T}' on each position $1 \leq i \leq |u|$ of its input u , and \mathcal{T}' outputs $u\#$.

The class of functions computed by a blind k -pebble transducer for some $k \geq 1$ is called **polyblind functions** [10]. They form a strict subclass of polyregular functions [8,10,16] which is closed under composition [16, Theorem 6.1].

Now, let us define last pebble transducers. They corresponds to blind pebble transducers enhanced with the ability to mark the current position of the input when doing a recursive call. Formally, this position is underlined and we define $u \bullet i := u[1] \dots u[i-1] \underline{u[i]} u[i+1] \dots u[|u|]$ for $u \in A^*$ and $1 \leq i \leq |u|$.

Definition 3.3 (Last pebble transducer [13]). For $k \geq 1$, a **last k -pebble transducer** with input alphabet A and output alphabet B is:

- if $k = 1$, a two-way transducer with input alphabet $A \uplus \underline{A}$ and output B ;
- if $k \geq 2$, a tree $\mathcal{T}\langle \mathcal{L}_1, \dots, \mathcal{L}_p \rangle$ where $\mathcal{L}_1, \dots, \mathcal{L}_p$ are last $(k-1)$ -pebble transducers with input A and output B ; and \mathcal{T} is a two-way transducer with input $A \uplus \underline{A}$ and output alphabet $\{\mathcal{L}_1, \dots, \mathcal{L}_p\}$.

The (total) function $f : (A \uplus \underline{A})^* \rightarrow B^*$ computed by the last k -pebble transducer of definition 3.3 is defined in a recursive fashion, as follows:

- for $k = 1$, f is the function computed by the two-way transducer;
- for $k \geq 2$, let $u \in A^*$ and $(q_1, i_1) \rightarrow \dots \rightarrow (q_n, i_n)$ be the accepting run of $\mathcal{T} = (A \uplus \underline{A}, B, Q, q_0, F, \delta, \lambda)$ on $\vdash u \dashv$. For all $1 \leq j \leq n$, let $f_j : A^* \rightarrow B^*$ be the concatenation of the functions recursively computed by $\lambda(q_j, (\vdash u \dashv)[i_j]) \in \{\mathcal{L}_1, \dots, \mathcal{L}_p\}^*$. Let $\tau : (A \uplus \underline{A})^* \rightarrow A^*$ be the morphism which erases the underlining (i.e. $\tau(\underline{a}) = a$), then $f(u) := f_1(\tau(u) \bullet i_1) \dots f_n(\tau(u) \bullet i_n)$.

The behavior of a last 3-pebble transducer is depicted in fig. 3. Observe that our definition builds a function of type $(A \uplus \underline{A})^* \rightarrow B^*$, but at shall in fact consider its restriction to A^* (the marks are only used within the induction step).

Example 3.4 ([1]). The function `square` : $u \mapsto (u \bullet 1) \# \cdots (u \bullet |u|) \#$ can be computed by a last 2-pebble transducer, which successively marks and makes recursive calls in positions 1, 2, etc. However this function is not polyblind [16].

We are ready to state our main result. Its proof goes over sections 4 to 6.

Theorem 3.5 (Minimization of the recursion height). *Let $1 \leq \ell \leq k$. Let $f : A^* \rightarrow B^*$ be computed by a blind k -pebble transducer (resp. by a last k -pebble transducer). Then f can be computed by a blind ℓ -pebble transducer (resp. by a last ℓ -pebble transducer) if and only if $|f(u)| = \mathcal{O}(|u|^\ell)$.*

This property is decidable and the construction is effective.

As an easy consequence, the class of functions computed by last pebble transducers form a strict subclass of the polyregular functions (because theorem 3.5 does not hold for the full model of pebble transducers [3, Theorem 6.3]) and therefore it is not closed under composition (because any polyregular function can be obtained as a composition of regular functions and squares [1]).

Even if a (non-effective) theorem 3.5 was already known for blind pebble transducers [16, Theorem 7.1], we shall first present our proof of this case. Indeed, it is a new proof (relying on factorization forests) which is simpler than the original one. Furthermore, understanding the techniques used is a key step for understanding the proof for last pebble transducers presented afterwards.

4 Factorization forests

In this section, we introduce the key tool of factorization forests. Given a monoid morphism $\mu : A^* \rightarrow \mathbb{M}$ and $u \in A^*$, a μ -factorization forest of u is an unranked tree structure defined as follows. We use the brackets $\langle \cdots \rangle$ to build a tree.

Definition 4.1 (Factorization forest [18]). *Given a morphism $\mu : A^* \rightarrow \mathbb{M}$ and $u \in A^*$, we say that \mathcal{F} is a μ -forest of u if:*

- either $u = \varepsilon$ and $\mathcal{F} = \varepsilon$; or $u = \langle a \rangle \in A$ and $\mathcal{F} = a$;
- or $\mathcal{F} = \langle \mathcal{F}_1, \dots, \mathcal{F}_n \rangle$, $u = u_1 \cdots u_n$, for all $1 \leq i \leq n$, \mathcal{F}_i is a μ -forest of $u_i \in A^+$, and if $n \geq 3$ then $\mu(u) = \mu(u_1) = \cdots = \mu(u_n)$ is idempotent.

We use the standard tree vocabulary of height, child, sibling, descendant and ancestor (a node being itself one of its ancestors/descendants), etc. We denote by $\text{Nodes}^{\mathcal{F}}$ the set of nodes of \mathcal{F} . In order to simplify the statements, we identify a node $t \in \text{Nodes}^{\mathcal{F}}$ with the subtree rooted in this node. Thus $\text{Nodes}^{\mathcal{F}}$ can also be seen as the set of subtrees of \mathcal{F} , and $\mathcal{F} \in \text{Nodes}^{\mathcal{F}}$. We say that a node is **idempotent** if it has at least 3 children. We denote by $\text{Forests}_\mu(u)$ (resp. $\text{Forests}_\mu^d(u)$) the set of μ -forests of $u \in A^*$ (resp. μ -forests of $u \in A^*$ of height at most d). We write Forests_μ and Forests_μ^d of all forests (of any word).

Building μ -forests of bounded height is especially useful for us, since it enables to decompose any word in a somehow bounded way. This decomposition will be guided by the following definitions, that have been introduced in [8,10]. First, we define *iterable nodes* as the middle children of *idempotent nodes*.

- if $\mathcal{F} = \langle a \rangle \in A$ or $\mathcal{F} = \varepsilon$, then $\text{Iter}^{\mathcal{F}} := \emptyset$;
- otherwise if $\mathcal{F} = \langle \mathcal{F}_1, \dots, \mathcal{F}_n \rangle$, then:

Now, we define the notion of *skeleton* of a node t , which contains all the descendants of t except those which are iterable.

– if $\mathbf{t} = \langle a \rangle \in A$ is a leaf, then $\text{Skel}^{\mathcal{F}}(\mathbf{t}) := \{\mathbf{t}\}$;
– otherwise if $\mathbf{t} = \langle \mathcal{F}_1, \dots, \mathcal{F}_n \rangle$, then $\text{Skel}^{\mathcal{F}}(\mathbf{t}) := \{\mathbf{t}\} \cup \text{Skel}^{\mathcal{F}}(\mathcal{F}_1) \cup \text{Skel}^{\mathcal{F}}(\mathcal{F}_n)$.
The **frontier** of \mathbf{t} is the set $\text{Fr}^{\mathcal{F}}(\mathbf{t}) \subseteq [1:|u|]$ containing the positions of u which belong to $\text{Skel}^{\mathcal{F}}(\mathbf{t})$ (when seen as leaves of the μ -forest \mathcal{F} over u).

It is easy to observe that for $\mathcal{F} \in \text{Forests}_\mu^d(u)$, the size of a skeleton, or of a frontier, is bounded independently from \mathcal{F} . Furthermore, the set of skeletons

$\{\text{Skel}^{\mathcal{F}}(\mathbf{t}) : \mathbf{t} \in \text{Iter}^{\mathcal{F}} \cup \{\mathcal{F}\}\}$ is a partition of $\text{Nodes}^{\mathcal{F}}$ [8, Lemma 33]. As a consequence, the set of **frontiers** $\{\text{Fr}^{\mathcal{F}}(\mathbf{t}) : \mathbf{t} \in \text{Iter}^{\mathcal{F}} \cup \{\mathcal{F}\}\}$ is a partition of $[1:|u|]$. Given a position $1 \leq i \leq |u|$, we can thus define the **origin** of i in \mathcal{F} , denoted $\text{origin}^{\mathcal{F}}(i)$, as the unique $\mathbf{t} \in \text{Iter}^{\mathcal{F}} \cup \{\mathcal{F}\}$ such that $i \in \text{Fr}^{\mathcal{F}}(\mathbf{t})$.

Definition 4.6 (Observation). Let $\mathcal{F} \in \text{Forests}_\mu$ and $\mathfrak{t}, \mathfrak{t}' \in \text{Nodes}^\mathcal{F}$. We say that $\mathfrak{t} \in \text{Nodes}^\mathcal{F}$ **observes** $\mathfrak{t}' \in \text{Nodes}^\mathcal{F}$ if either \mathfrak{t}' is an ancestor of \mathfrak{t} , or \mathfrak{t}' is the immediate right or left sibling of an ancestor of \mathfrak{t} .

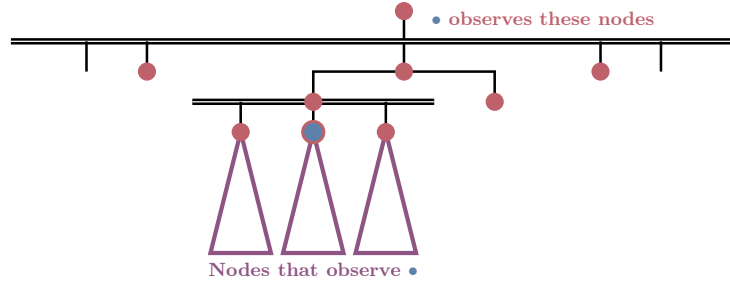


Figure 5: Nodes that observe \bullet and that \bullet observes

The intuition behind the notion of observation (which is *not* symmetrical) is depicted in fig. 5. Note that in a forest of bounded height, the number of nodes that some \mathbf{t} observes is bounded. This will be a key argument in the following. We say that \mathbf{t} and \mathbf{t}' are **dependent** if either \mathbf{t} observes \mathbf{t}' or the converse. Given \mathcal{F} , we can translate these notions to the positions of u : we say that i **observes** (resp. **depends on**) i' if $\text{origin}^{\mathcal{F}}(i)$ observes (resp. depends on) $\text{origin}^{\mathcal{F}}(i')$.

5 Height minimization of blind pebble transducers

In this section, we show theorem 3.5 for blind pebble transducers. We say that a two-way transducer \mathcal{T} is a **submachine** of a blind pebble transducer \mathcal{B} if \mathcal{T} labels a node in the tree description of \mathcal{B} . If $\mathcal{B} = \mathcal{T}(\mathcal{B}_1, \dots, \mathcal{B}_n)$, we say that the submachine \mathcal{T} is the **head** of \mathcal{B} . We let the **transition morphism** of \mathcal{B} be the cartesian product of all the transition morphisms of all the submachines of \mathcal{B} . Observe that it makes sense to consider the production of a submachine \mathcal{T} in a context defined using the transition morphism of \mathcal{B} .

5.1 Pumpability

We first give a sufficient condition, named pumpability, for a blind k -pebble transducer to compute a function f such that $|f(u)| \neq \mathcal{O}(|u|^{k-1})$. The behavior of a pumpable blind 2-pebble transducer is depicted in fig. 6 over a well-chosen

input: it has a factor in which the head \mathcal{T}_1 calls a submachine \mathcal{T}_2 , and a factor in which \mathcal{T}_2 produces a non-empty output. Furthermore both factors can be iterated without destroying the runs of these machines (due to idempotents).

Definition 5.1. Let \mathcal{B} be a blind k -pebble transducer whose transition morphism is $\mu : A^* \rightarrow \mathbb{T}$. We say that the transducer \mathcal{B} is **pumpable** if there exists:

- submachines $\mathcal{T}_1, \dots, \mathcal{T}_k$ of \mathcal{B} , such that \mathcal{T}_1 is the head of \mathcal{B} ;
- $m_0, \dots, m_k, \ell_1, \dots, \ell_k, r_1, \dots, r_k \in \mu(A^*)$;
- $a_1, \dots, a_k \in A$ such that for all $1 \leq j \leq k$, $e_i := \ell_j \mu(a_j) r_j$ is an idempotent;
- a permutation $\sigma : [1:k] \rightarrow [1:k]$;

such that if $\mathcal{M}_i^j := m_i e_{i+1} m_{i+1} \dots e_j m_j$ for all $0 \leq i \leq j \leq k$, and if we define the following context for all $1 \leq j \leq k$:

$$C_j := \mathcal{M}_0^{\sigma(j)-1} e_{\sigma(j)} \ell_{\sigma(j)} \llbracket a_{\sigma(j)} \rrbracket r_{\sigma(j)} e_{\sigma(j)} \mathcal{M}_{\sigma(j)}^k$$

then for all $1 \leq j \leq k-1$, $|\text{prod}_{\mathcal{T}_j}(C_j)|_{\mathcal{T}_{j+1}} \neq 0$, and $\text{prod}_{\mathcal{T}_k}(C_k) \neq \varepsilon$.

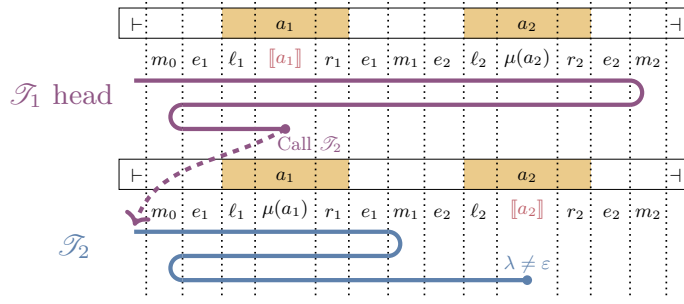


Figure 6: Pumpability in a blind 2-pebble transducer.

Lemma 5.2 follows by choosing inverse images in A^* for the m_i , ℓ_i and r_i .

Lemma 5.2. Let f be computed by a pumpable blind k -pebble transducer. There exists words $v_0, \dots, v_k, u_1, \dots, u_k$ such that $|f(v_0 u_1^X \dots u_k^X v_k)| = \theta(X^k)$.

Now, we use pumpability as a key ingredient for showing theorem 3.5, which directly follows by induction from the more precise theorem 5.3.

Theorem 5.3 (Removing one layer). Let $k \geq 2$ and $f : A^* \rightarrow B^*$ be computed by a blind k -pebble transducer \mathcal{B} . The following are equivalent:

1. $|f(u)| = \mathcal{O}(|u|^{k-1})$;
2. \mathcal{B} is not pumpable;
3. f can be computed by a blind $(k-1)$ -pebble transducer.

Furthermore, this property is decidable and the construction is effective.

Proof. Item 3 \Rightarrow item 1 is obvious. Item 1 \Rightarrow item 2 is lemma 5.2. Furthermore, pumpability can be tested by an enumeration of $\mu(A^*)$ and A . It remains to show item 2 \Rightarrow item 3 (in an effective fashion): this is the purpose of section 5.2.

5.2 Algorithm for removing a recursion layer

Let $k \geq 2$ and \mathcal{U} be a last k -pebble transducer that is not pumpable, and that computes $f : A^* \rightarrow B^*$. We build a blind $(k-1)$ -pebble transducer $\overline{\mathcal{U}}$ for f .

Let $\mu : A^* \rightarrow \mathbb{T}$ be the transition morphism of \mathcal{U} . We shall consider that, on input $u \in A^*$, the submachines of $\overline{\mathcal{U}}$ can in fact use $\text{forest}_\mu(u) \subseteq (\hat{A})^*$ as input. Indeed forest_μ is a rational function (by theorem 4.2), hence its information can be recovered by using a **lookaround**. Informally, the lookaround feature enables a two-way transducer to chose its transitions not only depending on its current state and current letter $u[i]$ in position $1 \leq i \leq |u|$, but also on a regular property of the prefix $u[1:i-1]$ and the suffix $u[i+1:|u|]$. It is well-known that given a two-way transducer \mathcal{T} with lookarounds, one can build an equivalent \mathcal{T}' that does not have this feature (see e.g. [15,12]). Furthermore, even if the accepting runs of \mathcal{T} and \mathcal{T}' may differ, they produce the same outputs from the same positions (this observation will be critical for last pebble transducers, in order to ensure that the marked positions of the recursive calls will be preserved).

Now, we describe the two-way transducers that are the submachines of $\overline{\mathcal{U}}$. First, it has submachines **old- \mathcal{T}** for \mathcal{T} a submachine of \mathcal{U} , which are described in algorithm 1. Intuitively, **old- \mathcal{T}** is just a copy of \mathcal{T} . It is clear that if \mathcal{T} is a submachine of \mathcal{U} , then **old- $\mathcal{T}(u)$** is the concatenation of the outputs produced by (the recursive calls of) \mathcal{T} along its accepting run on $\vdash u \dashv$.

Algorithm 1: Submachines that behave as the original ones

```

1 Submachine old- $\mathcal{T}(u)$ 
2    $\rho :=$  accepting run of  $\mathcal{T}$  over  $\vdash u \dashv$ ;  $\lambda :=$  output function of  $\mathcal{T}$ ;
3   for  $(q, i) \in \rho$  do
4     if  $\mathcal{T}$  is a leaf of  $\mathcal{U}$  then
5       Output  $\lambda(q, (\vdash u \dashv)[i])$ ; /*  $\mathcal{T}$  has output in  $B^*$ ; */
6     else
7       for  $\mathcal{B}' \in \lambda(q, (\vdash u \dashv)[i])$  do
8          $\mathcal{T}' :=$  head of  $\mathcal{B}'$ ;
9         Call old- $\mathcal{T}'(u)$ ; /*  $\mathcal{T}$  makes recursive calls; */
10      end
11    end
12  end

```

$\overline{\mathcal{U}}$ also has submachines **accelerate- \mathcal{T}** for \mathcal{T} a submachine of \mathcal{U} , which are described in algorithm 2. Intuitively, **accelerate- \mathcal{T}** simulates \mathcal{T} while trying to inline recursive calls in its own run. More precisely, let $u \in A^*$ be the input and $\mathcal{F} := \text{forest}_\mu(u)$. If \mathcal{T} calls \mathcal{B}' in $1 \leq i \leq |u|$ that belongs to the frontier of the root node \mathcal{F} of \mathcal{F} , then **accelerate- \mathcal{T}** inlines the behavior of the head of \mathcal{B}' . Otherwise it makes a recursive call, except if \mathcal{B}' is a leaf of \mathcal{U} . Hence if \mathcal{T} is a submachine of \mathcal{U} which is not a leaf, **accelerate- $\mathcal{T}(u)$** is the concatenation of the outputs produced by the calls of \mathcal{T} along its accepting run.

Algorithm 2: Submachines that try to simulate their recursive calls

```

1 Submachine accelerate- $\mathcal{T}$  ( $u$ )
2   /*  $\mathcal{T}$  is not a leaf of  $\mathcal{U}$  (i.e. it makes calls); */
3    $\rho :=$  accepting run of  $\mathcal{T}$  over  $\vdash u \dashv$ ;  $\mathcal{F} := \text{forest}_\mu(u)$ ;  $\lambda :=$  output of  $\mathcal{T}$ ;
4   for  $(q, i) \in \rho$  do
5     for  $\mathcal{B}' \in \lambda(q, (\vdash u \dashv)[i])$  do
6        $\mathcal{T}' :=$  head of  $\mathcal{B}'$ ;
7       if  $i \in \text{Fr}^\mathcal{F}(\mathcal{F})$  then
8         /* We can inline the call since  $|\text{Fr}^\mathcal{F}(\mathcal{F})|$  is bounded; */
9         Inline the code of old- $\mathcal{T}'$  ( $u$ );
10      else if  $\mathcal{B}'$  is a leaf of  $\mathcal{U}$  then
11        /* Then  $\mathcal{B}' = \mathcal{T}'$  and we can inline the call because
12           the output of  $\mathcal{T}'$  on input  $u$  is bounded; */
13        Inline the code of old- $\mathcal{T}'$  ( $u$ );
14      else
15        /* It is not possible to inline the call to  $\mathcal{B}'$ , so we
16           make a recursive call; */
17        Call accelerate- $\mathcal{T}'$  ( $u$ );
18      end
    end
  end
end

```

Finally, the transducer $\overline{\mathcal{U}}$ is obtained by defining **accelerate- \mathcal{T}** to be its head, where \mathcal{T} is the head of \mathcal{U} . Furthermore, we remove the submachines **old- \mathcal{T}** or **accelerate- \mathcal{T}** which are never called. Observe that $\overline{\mathcal{U}}$ indeed computes the function f . Furthermore, we observe that $\overline{\mathcal{U}}$ has recursion height (i.e. the number of nested **Call** instructions, plus 1 for the head) $k-1$, since each inlining of lines 9, 10 and 12 in algorithm 2 removes exactly one recursion layer of \mathcal{U} .

It remains to justify that each **accelerate- \mathcal{T}** can be implemented by a two-way transducer (i.e. with lookarounds but a bounded memory). We represent variable i by the current position of the transducer. Since it has access to \mathcal{F} , the lookahead can be used to check whether $i \in \text{Fr}^\mathcal{F}(\mathcal{F})$ or not (since the size of $\text{Fr}^\mathcal{F}(\mathcal{F})$ is bounded). It remains to explain how the inlinings are performed:

- if $i \in \text{Fr}^\mathcal{F}(\mathcal{F})$, the two-way transducer inlines **old- \mathcal{T}'** by executing the same moves and calls as \mathcal{T}' does. Once its computation is ended, it has to go back to position i . This is indeed possible since belonging to $\text{Fr}^\mathcal{F}(\mathcal{F})$ is a property that can be detected by using the lookahead, hence the machine only needs to remember that i was the ℓ -th position of $\text{Fr}^\mathcal{F}(\mathcal{F})$ (ℓ being bounded);
- else if $\mathcal{B}' = \mathcal{T}'$ is a blind 1-pebble transducer, we produce the output of \mathcal{T}' without moving. This is possible since for all $i' \notin \text{Fr}^\mathcal{F}(\mathcal{F})$, $\text{prod}_{\mathcal{T}'}^u(i') = \varepsilon$ (hence the output of \mathcal{T}' on u is bounded, and its value can be determined without moving, just by using the lookahead). Indeed, if $\text{prod}_{\mathcal{T}'}^u(i') \neq \varepsilon$ for such an $i' \notin \text{Fr}^\mathcal{F}(\mathcal{F})$ when reaching line 12 of algorithm 2, then the conditions of lemma 5.4 hold, which yields a contradiction. This lemma is the key argument of this proof, relying on the non-pumpability of \mathcal{U} .

Lemma 5.4 (Key lemma). *Let $u \in A^*$ and $\mathcal{F} \in \text{Forests}_\mu(u)$. Assume that there exists a sequence $\mathcal{T}_1, \dots, \mathcal{T}_k$ of submachines of \mathcal{U} and a sequence of positions $1 \leq i_1, \dots, i_k \leq |u|$ such that:*

- \mathcal{T}_1 is the head of \mathcal{U} ;
- for all $1 \leq j \leq k-1$, $|\text{prod}_{\mathcal{T}_j}^u(i_j)|_{\mathcal{T}_{j+1}} \neq 0$ and $\text{prod}_{\mathcal{T}_k}^u(i_k) \neq \varepsilon$;
- for all $1 \leq j \leq k$, $i_j \notin \text{Fr}^\mathcal{F}(\mathcal{F})$ (i.e. $\text{origin}^\mathcal{F}(i_j) \in \text{Iter}^\mathcal{F}$).

Then \mathcal{B} is pumpable.

Proof (idea). We first observe that pumpability follows as soon as the nodes $\text{origin}^\mathcal{F}(i_j)$ are two by two independent. We then show that this independence condition can always be obtained, up to duplicating some iterable subtrees of \mathcal{F} (and some factors of u), because the behavior of a submachine in a blind pebble transducer does not depend on the positions of the above recursive calls.

6 Height minimization of last pebble transducers

In this section, we show theorem 3.5 for last pebble transducers. The notions of **submachine**, **head** and **transition morphism** for a last pebble transducer are defined as in section 5. The transition morphism is now defined over $(A \uplus \underline{A})^*$.

6.1 Pumpability

The sketch of the proof is similar to section 5. We first give an equivalent of pumpability for last pebble transducers. The intuition behind this notion is depicted in fig. 7. The formal definition is however more cumbersome, since we need to keep track of the fact that the calling position is marked.

Definition 6.1. *Let \mathcal{L} be a last k -pebble transducer whose transition morphism is $\mu : (A \cup \underline{A})^* \rightarrow \mathbb{T}$. We say that the transducer \mathcal{L} is **pumpable** if there exists:*

- submachines $\mathcal{T}_1, \dots, \mathcal{T}_k$ of \mathcal{L} , such that \mathcal{T}_1 is the head of \mathcal{L} ;
- $m_0, \dots, m_k, \ell_1, \dots, \ell_k, r_1, \dots, r_k \in \mu(A^*)$;
- $a_1, \dots, a_k \in A$ such that for all $1 \leq j \leq k$ $e_j := \ell_j \mu(a_j) r_j$ is idempotent;
- a permutation $\sigma : [1:k] \rightarrow [1:k]$;

such that if we let $\mathcal{M}_i^j := m_i e_{i+1} m_{i+1} \dots e_j m_j$ for all $0 \leq i \leq j \leq k$, and if we define the following context:

$$\mathcal{C}_1 := \mathcal{M}_0^{\sigma(1)-1} e_{\sigma(1)} \ell_{\sigma(1)} \llbracket a_{\sigma(1)} \rrbracket r_1 e_{\sigma(1)} \mathcal{M}_{\sigma(1)}^k$$

and for all $1 \leq j \leq k-1$ the context:

$$\begin{aligned} \mathcal{C}_{j+1} &:= \mathcal{M}_0^{\sigma(j)-1} e_{\sigma(j)} \ell_{\sigma(j)} \mu(\underline{a_{\sigma(j)}}) r_{\sigma(j)} e_{\sigma(j)} \mathcal{M}_{\sigma(j)}^{\sigma(j+1)-1} \\ &\quad e_{\sigma(j+1)} \ell_{\sigma(j+1)} \llbracket a_{\sigma(j+1)} \rrbracket r_{\sigma(j+1)} e_{\sigma(j+1)} \mathcal{M}_{\sigma(j+1)}^k \quad \text{if } \sigma(j) < \sigma(j+1); \\ \mathcal{C}_{j+1} &:= \mathcal{M}_0^{\sigma(j)-1} e_{\sigma(j+1)} \ell_{\sigma(j+1)} \llbracket a_{\sigma(j+1)} \rrbracket r_{\sigma(j+1)} e_{\sigma(j+1)} \\ &\quad \mathcal{M}_{\sigma(j+1)}^{\sigma(j)-1} e_{\sigma(j)} \ell_{\sigma(j)} \mu(\underline{a_{\sigma(j)}}) r_{\sigma(j)} e_{\sigma(j)} \mathcal{M}_{\sigma(j)}^k \quad \text{otherwise;} \end{aligned}$$

then for all $1 \leq j \leq k-1$, $|\text{prod}_{\mathcal{T}_j}(\mathcal{C}_j)|_{\mathcal{T}_{j+1}} \neq 0$, and $\text{prod}_{\mathcal{T}_k}(\mathcal{C}_k) \neq \varepsilon$.

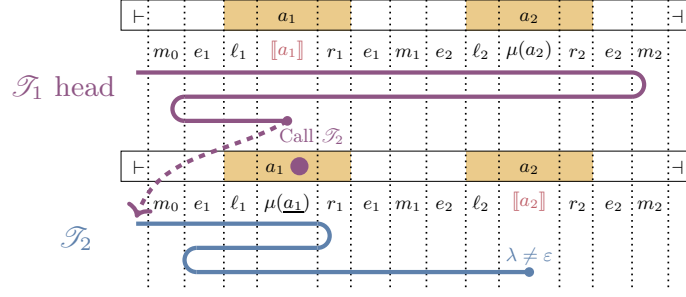


Figure 7: Pumpability in a last 2-pebble transducer.

We obtain lemma 6.2 by a proof which is similar to that of lemma 5.2.

Lemma 6.2. *Let f be computed by a pumpable last k -pebble transducer. There exists words $v_0, \dots, v_k, u_1, \dots, u_k$ such that $|f(v_0 u_1^X \dots u_k^X v_k)| = \theta(X^k)$.*

Theorem 6.3 (Removing one layer). *Let $k \geq 2$ and $f : A^* \rightarrow B^*$ be computed by a last k -pebble transducer \mathcal{L} . The following are equivalent:*

1. $|f(u)| = \mathcal{O}(|u|^{k-1})$;
2. \mathcal{L} is not pumpable;
3. f can be computed by a last $(k-1)$ -pebble transducer.

Furthermore, this property is decidable and the construction is effective.

Proof. Item 3 \Rightarrow item 1 is obvious. Item 1 \Rightarrow item 2 is lemma 6.2. Furthermore, pumpability can be tested by an enumeration of $\mu(A^*)$ and A . It remains to show item 2 \Rightarrow item 3 (in an effective fashion): this is the purpose of section 6.2.

6.2 Algorithm for removing a recursion layer

Let $k \geq 2$ and \mathcal{U} be a last k -pebble transducer that is not pumpable, and that computes $f : A^* \rightarrow B^*$. We build a last $(k-1)$ -pebble transducer $\overline{\mathcal{U}}$ for f . Let $\mu : (A \uplus \underline{A})^* \rightarrow \mathbb{T}$ be the transition morphism of \mathcal{U} . As before (using a lookaround), the submachines of $\overline{\mathcal{U}}$ will use $\text{forest}_\mu(u)$ as input instead of $u \in A^*$.

Now, we describe the submachines of $\overline{\mathcal{U}}$. It has submachines *old- \mathcal{T} -along- ρ* for \mathcal{T} a submachine of \mathcal{U} and ρ a run of \mathcal{T} , which are described in algorithm 1. Intuitively, these machines mimics the behavior of \mathcal{T} along the run ρ (which is not necessarily accepting) of \mathcal{T} over $\vdash v \dashv$ with $v \in (A \uplus \underline{A})^*$.

Since they are indexed by a run ρ , it may seem that we create an infinite number of submachines, but it will not be the case. Indeed, a run ρ will be represented by its first configuration (q_1, i_1) and last configuration (q_n, i_n) . This information is sufficient to simulate exactly the two-way moves of ρ , but there is still an unbounded information: the positions i_1 and i_n . In fact, the input will be of the form $v = u \bullet i$ and we shall guarantee that the i_1 and i_n can be detected by the lookaround if i is marked. Hence the run ρ will be represented in a bounded

way, independently from the input v , and so that its first and last configurations can be detected by the lookaround of the submachine.

By algorithm 1, it is clear that if \mathcal{T} is a submachine of \mathcal{U} , then for all $v \in (A \cup \underline{A})^*$ and ρ run of \mathcal{T} on $\vdash v \dashv$, $\text{old-}\mathcal{T}\text{-along-}\rho(v)$ is the concatenation of the outputs produced by (the recursive calls of) \mathcal{T} along ρ .

We also define a submachine **normal- \mathcal{T} -along- ρ -pebble- i** that is similar to $\text{old-}\mathcal{T}\text{-along-}\rho$, except that it ignores the mark of its input and acts as if it was in position i (as above for ρ , i will be encoded by a bounded information).

Algorithm 3: Submachines that behave like the original ones

```

1 Submachine  $\text{old-}\mathcal{T}\text{-along-}\rho(v)$ 
2   /*  $v \in (A \uplus \underline{A})^*$ ;  $\rho$  is a run of  $\mathcal{T}$  over  $\vdash v \dashv$ ; */
3    $\lambda :=$  output function of  $\mathcal{T}$ ;
4   for  $(q, i) \in \rho$  do
5     if  $\mathcal{T}$  is a leaf of  $\mathcal{U}$  then
6       | Output  $\lambda(q, (\vdash v \dashv)[i])$ ; /*  $\mathcal{T}$  has output in  $B^*$ ; */
7     else
8       | for  $\mathcal{L}' \in \lambda(q, (\vdash v \dashv)[i])$  do
9         | |  $\mathcal{T}' :=$  head of  $\mathcal{L}'$ ;  $\rho' :=$  accepting run of  $\mathcal{T}'$  on  $\vdash \tau(v) \bullet i \dashv$ ;
10        | | Call  $\text{old-}\mathcal{T}'\text{-along-}\rho'(\tau(v) \bullet i)$ ; /* Recursive call; */
11        | end
12      | end
13    end
14 Submachine normal- $\mathcal{T}$ -along- $\rho$ -pebble- $i$ ( $v$ )
15   /*  $v \in (A \uplus \underline{A})^*$ ;  $\rho$  is a run of  $\mathcal{T}$  over  $\vdash \tau(v) \bullet i \dashv$ ; */
16   Simulate  $\text{old-}\mathcal{T}\text{-along-}\rho(\tau(v) \bullet i)$ ;

```

$\overline{\mathcal{U}}$ also has submachines **accelerate- \mathcal{T} -along- ρ** for \mathcal{T} a submachine of \mathcal{U} , which are described in algorithm 4. Intuitively, **accelerate- \mathcal{T} -along- ρ** simulates \mathcal{T} along ρ while trying to inline some recursive calls. Whenever it is in position i and needs to call recursively \mathcal{L}' whose head is \mathcal{T}' , it first slices the accepting run ρ' of \mathcal{T}' on $\vdash u \bullet i \dashv$, with respect to $\text{forest}_\mu(u)$ and i , as explained in definition 6.4 and depicted in fig. 8. Intuitively, this operation splits ρ' into a bounded number of runs whose positions either all observe i , or i observes all of them, or none of these cases occur (the positions are either 0, $|u|+1$ or independent of i).

Definition 6.4 (Slicing). Let $u \in A^*$, $\mathcal{F} \in \text{Forests}_\mu(u)$ and $1 \leq i \leq |u|$. We let $\uparrow i$ (resp. $\downarrow i$) be the set of positions that i observes (resp. that observe i). Let $\rho = (q_1, i_1) \rightarrow \dots \rightarrow (q_n, i_n)$ be a run of a two-way transducer \mathcal{T} on $\vdash u \bullet i \dashv$. We build by induction a sequence $\ell_1, \dots, \ell_{N+1}$ with $\ell_1 := 1$ and:

- if $\ell_j = n+1$ then $j := N$ and the process ends;
- else if $\ell_j \in \uparrow i$ (resp. $\ell_j \in \downarrow i \setminus \uparrow i$, resp. $\ell_j \in [0:|u|+1] \setminus (\uparrow i \cup \downarrow i)$), then ℓ_{j+1} is the largest index such that for all $\ell_j \leq \ell \leq \ell_{j+1}-1$, $i_\ell \in \uparrow i$ (resp. $i_\ell \in \downarrow i \setminus \uparrow i$, resp. $i_\ell \in [0:|u|+1] \setminus (\uparrow i \cup \downarrow i)$).

Finally the **slicing** of ρ , with respect to \mathcal{F} and i , is the sequence of runs ρ_1, \dots, ρ_N where $\rho_j := (q_{\ell_j}, i_{\ell_j}) \rightarrow (q_{\ell_j+1}, i_{\ell_j+1}) \rightarrow \dots \rightarrow (q_{\ell_{j+1}-1}, i_{\ell_{j+1}-1})$.

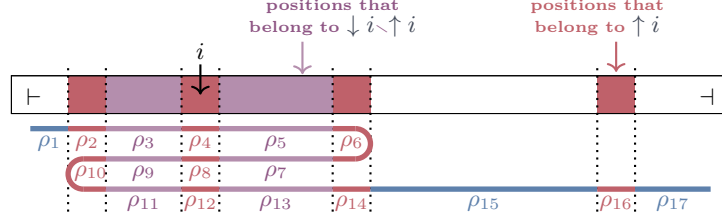


Figure 8: Slicing of a run ρ with respect to i and \mathcal{F} .

Now, let ρ'_1, \dots, ρ'_N be slicing of the run ρ' of \mathcal{T}' on the input $u \bullet i$. For all $1 \leq j \leq N$, there are mainly two cases. Either the positions of ρ'_j all are in $\uparrow i$ or $\downarrow i$. In this case, `accelerate- \mathcal{T} -along- ρ` directly inlines `old- \mathcal{T}' -along- ρ'_j` within its own run (i.e. without making a recursive call). Otherwise, it makes a recursive call to `accelerate- \mathcal{T}' -along- ρ'_j` , except if \mathcal{L}' is a leaf of \mathcal{U} (thus $\mathcal{L}' = \mathcal{T}'$).

Finally, $\overline{\mathcal{U}}$ is described as follows: on input $u \in A^*$, its head is the submachine `accelerate- \mathcal{T} -along- ρ` (u), where \mathcal{T} is the head of \mathcal{U} and ρ is the accepting run of \mathcal{T} on $\vdash u \vdash$ (represented by the bounded information that it is both initial and final). As before, we remove the submachines which are never called in $\overline{\mathcal{U}}$. Observe that we have created a machine with recursion height $k-1$ (because line 17 in algorithm 4 prevents from calling a k -th layer).

Let us justify that each `accelerate- \mathcal{T} -along- ρ` can indeed be implemented by a two-way transducer. First, let us observe that since \mathcal{F} has bounded height, the number N of slices given in line 7 of algorithm 4 is bounded. Furthermore, we claim that the first and last positions of each ρ'_j belong to a given set of bounded size, which can be detected a lookahead which has access to i . For the ρ'_j whose positions are in $\uparrow i$, this is clear since $|\uparrow i|$ is bounded (because the frontier of any node is bounded). For $\downarrow i < \uparrow i$ we use lemma 6.5, which implies that this set is a bounded union of intervals. The last case is very similar.

Lemma 6.5. *Let $1 \leq i \leq |u|$, $\mathbf{t} := \text{origin}^{\mathcal{F}}(i)$ and \mathbf{t}_1 (resp. \mathbf{t}_2) be its immediate left (resp. right) sibling (they exist whenever $\mathbf{t} \in \text{Iter}^{\mathcal{F}}$, i.e. here $\mathbf{t} \neq \mathcal{F}$). Then:*

$$\downarrow i < \uparrow i = [\min(\text{Fr}^{\mathcal{F}}(\mathbf{t}_1)) : \max(\text{Fr}^{\mathcal{F}}(\mathbf{t}_2))] \setminus \{\text{Fr}^{\mathcal{F}}(\mathbf{t}_1), \text{Fr}^{\mathcal{F}}(\mathbf{t}), \text{Fr}^{\mathcal{F}}(\mathbf{t}_2)\}.$$

This analysis justifies why each ρ'_j can be encoded in a bounded way. Now, we show how to implement the inlinings while using i as the current position:

- if $i_1, \dots, i_n \in \uparrow i$, then n is bounded (because $|\uparrow i|$ is bounded). We can thus inline `old- \mathcal{T}' -along- ρ'_j` ($u \bullet i$) while staying in position i . However, when \mathcal{T}' calls some \mathcal{L}'' (of head \mathcal{T}'') on position i_ℓ , we would need to call `old- \mathcal{T}'' -along- ρ''` ($u \bullet i_\ell$) (where ρ'' is the accepting run of \mathcal{T}'' along $\vdash u \bullet i_\ell \vdash$). But we cannot do this operation, since we are in position i and not in i_ℓ . The solution is that the inlined code calls `normal- \mathcal{T}'' -along- ρ'' -pebble- i_ℓ` ($u \bullet i$)

Algorithm 4: Submachines that try to simulate their recursive calls

```

1 Submachine accelerate- $\mathcal{T}$ -along- $\rho$  ( $v$ )
2   /*  $\mathcal{T}$  is not a leaf of  $\mathcal{U}$  (i.e. it makes calls); */
3   /*  $v \in (A \uplus \underline{A})^*$ ;  $\rho$  is a run of  $\mathcal{T}$  over  $\vdash v \dashv$ ; */
4    $u := \tau(v)$ ;  $\mathcal{F} := \text{forest}_\mu(u)$ ;  $\lambda := \text{output function of } \mathcal{T}$ ;
5   for  $(q, i) \in \rho$  do
6     for  $\mathcal{L}' \in \lambda(q, (\vdash v \dashv)[i])$  do
7        $\mathcal{T}' := \text{head of } \mathcal{L}'$ ;  $\rho' := \text{accepting run of } \mathcal{T}' \text{ over } \vdash u \bullet i \dashv$ ;
8        $\rho'_1, \dots, \rho'_N := \text{slicing of } \rho' \text{ with respect to } \mathcal{F} \text{ and } i$ ;
9       for  $j = 1$  to  $N$  do
10         $(q_1, i_1) \rightarrow \dots (q_n, i_n) := \rho'_j$ 
11        if  $i_1, \dots, i_n \in \uparrow i$  then
12          /* We inline the call because  $n$  is bounded; */
13          Inline the code of old- $\mathcal{T}'$ -along- $\rho'_j$  ( $u \bullet i$ );
14        else if  $i_1, \dots, i_n \in \downarrow i$  then
15          /* We can inline the call because the positions
16              $i_1, \dots, i_n$  are “below”  $i$  in  $\mathcal{F}$ ; */
17          Inline the code of old- $\mathcal{T}'$ -along- $\rho'_j$  ( $u \bullet i$ );
18        else if  $\mathcal{L}'$  is a leaf of  $\mathcal{U}$  then
19          /* The output of  $\mathcal{L}' = \mathcal{T}'$  along  $\rho'_j$  is empty; */
20        else
21          /* It is not possible to inline the call to  $\mathcal{L}'$ , so
22             we make a recursive call; */
23          Call accelerate- $\mathcal{T}'$ -along- $\rho'_j$  ( $u \bullet i$ );
24        end
25      end
26    end
27  end

```

instead, which simulates an accepting run ρ'' of \mathcal{T} on $u \bullet i_\ell$, even if its input is $u \bullet i$. Note that i_ℓ can be represented as a bounded information and recovered by a lookaround given $u \bullet i$ as input, since i observes i_ℓ ;

- if $i_1, \dots, i_n \in \downarrow i \setminus \uparrow i$, then the nodes $\text{origin}^\mathcal{F}(i_1), \dots, \text{origin}^\mathcal{F}(i_n)$ are “below” $\text{origin}^\mathcal{F}(i)$ in \mathcal{F} (see fig. 5). We inline old- \mathcal{T}' -along- ρ'_j ($u \bullet i$), by moving along i_1, \dots, i_n as ρ'_j does. We can keep track of the height of $\text{origin}^\mathcal{F}(i)$ above the current $\text{origin}^\mathcal{F}(i_\ell)$ (it is a bounded information). With the lookaround, we can detect the end of ρ'_j , and go back to position i .

It remains to justify that $\overline{\mathcal{U}}$ is correct. For this, we only need to show that when it reaches line 18 in algorithm 4, the output of \mathcal{T}' along ρ'_j is indeed empty. Otherwise, the conditions of lemma 6.6 would hold (since we never execute two successive recursive calls in dependent positions). It provides a contradiction.

Lemma 6.6 (Key lemma). *Let $u \in A^*$ and $\mathcal{F} \in \text{Forests}_\mu(u)$. Assume that there exists a sequence $\mathcal{T}_1, \dots, \mathcal{T}_k$ of submachines of \mathcal{U} and a sequence of positions $1 \leq i_1, \dots, i_k \leq |u|$ such that:*

- \mathcal{T}_1 is the head of \mathcal{U} ;

- $|\text{prod}_{\mathcal{T}_1}^u(i_1)|_{\mathcal{T}_2} \neq 0$ and $\text{prod}_{\mathcal{T}_k}^{u \bullet i_{k-1}}(i_k) \neq \varepsilon$;
- for all $2 \leq j \leq k-1$, $|\text{prod}_{\mathcal{T}_j}^{u \bullet i_{j-1}}(i_j)|_{\mathcal{T}_{j+1}} \neq 0$;
- for all $1 \leq j \leq k-1$, $\text{origin}^{\mathcal{F}}(i_j)$ and $\text{origin}^{\mathcal{F}}(i_{j+1})$ are independent;

Then \mathcal{U} is pumpable.

Proof (idea). As for lemma 5.4, the key observation is that pumpability follows as soon as the nodes $\text{origin}^{\mathcal{F}}(i_j)$ are two by two independent. Furthermore, this condition can be obtained by duplicating some nodes in \mathcal{F} .

7 Making the two last pebbles visible

We can define a similar model to that of last k -pebble transducer, which sees the two last calling positions instead of only the previous one. Let us name this model a **last-last k -pebble transducer**. A very natural question is to know whether we can show an analog of theorem 3.5 for these machines.

Note that for $k = 1, 2$ and 3 , a last-last k -pebble transducer is exactly the same as a k -pebble transducer. Hence the function inner-squaring of page 2 is such that $|\text{inner-squaring}(u)| = \mathcal{O}(|u|^2)$ and can be computed by a last-last 3-pebble transducer, but it cannot be computed by a last-last 2-pebble transducer. It follows that the connection between minimal recursion height and growth of the output fails. However, this result is somehow artificial. Indeed, a last-last 2-pebble transducer is a degenerate case, since it can only see one last pebble. More interestingly, we show that the connection fails for arbitrary heights.

Theorem 7.1. *For all $k \geq 2$, there exists a function $f : A^* \rightarrow B^*$ such that $|f(u)| = \mathcal{O}(|u|^2)$ and that can be computed by a last-last $(2k+1)$ -pebble transducer, but not by a last-last $2k$ -pebble transducer.*

Proof (idea). We re-use a counterexample introduced by Bojańczyk in [2] to show a similar failure result for the model of k -pebble transducers.

8 Outlook

This paper somehow settles the discussion concerning the variants of pebble transducers for which the minimal recursion height only depends on the growth of the output. As soon as two marks are visible, the combinatorics of the output also has to be taken into account, hence minimizing the recursion height in this context seems to be out of reach (e.g. for last-last pebble transducers).

As observed in [13], one can extend last pebble transducers by allowing the recursion height to be unbounded (in the spirit of **marble transducers** [9]). This model enables to produce outputs whose size grows exponentially in the size of the input. A natural question is to know whether a function computed by this model, but whose output size is polynomial, can in fact be computed by a recursion stack of bounded height (i.e. a last k -pebble transducer).

Acknowledgements. The author is grateful to Tito Nguyen for suggesting the study of the recursion height for last pebble transducers.

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A Omitted proofs of section 5

A.1 Proof of lemma 5.2

Let $w_i \in A^*$ be such that $\mu(w_i) = \ell_i$, $w'_i \in A^*$ be such that $\mu(w'_i) = r_i$ and $u_i := w_i a_i w'_i$, for $1 \leq i \leq k$. Let $v_0, \dots, v_k \in A^*$ such that $\mu(v_i) = m_i$ for all $1 \leq i \leq k$. Let f_i be the function computed by \mathcal{T}_i for all $1 \leq i \leq k$.

It is easy to see that $|f_k(v_0 u_1^X \dots u_k^X v_k)| \geq (X-2)$. Indeed, \mathcal{T}_k must produce at least one letter when reading the letter $a_{\sigma(k)}$ of each factor $u_{\sigma(k)}$ (with possibly an exception for the borders, hence the -2). We then show by induction on $k \geq i \geq 1$ that $|f_i(v_0 u_1^X \dots u_k^X v_k)| \geq (X-2)^{k-i+1}$. The upper bound follows since \mathcal{B} is a blind k -pebble transducer, thus $|f(v_0 u_1^X \dots u_k^X v_k)| = \theta(X^k)$.

A.2 Additional arguments in section 5.2

We first justify that the construction of $\overline{\mathcal{W}}$ in section 5.2 indeed reduces the recursion height of \mathcal{W} by 1. This statement was claimed on page 12.

Claim. The machine $\overline{\mathcal{W}}$ described in section 5.2 has recursion height $k-1$.

Proof. Recall that the recursion height corresponds to the number of nested **Call** instructions, plus 1 (due to the head). We first show by (decreasing) induction on $1 \leq \ell \leq k$ that if \mathcal{T} is the root of a subtree of \mathcal{W} whose recursion height is $1 \leq \ell \leq k$, then $\text{old-}\mathcal{T}$ has recursion height ℓ as well.

Then, we show by (decreasing) induction on $1 \leq \ell \leq k$ that if \mathcal{T} is the root of a subtree of \mathcal{W} whose recursion height is ℓ , then $\text{accelerate-}\mathcal{T}$ has recursion height $\ell-1$. Indeed, the base case $\ell = 2$ is justified by line 12 in algorithm 2 (there are no calls since we inline all the computations). For $\ell > 2$ the function $\text{accelerate-}\mathcal{T}$ inlines $\text{old-}\mathcal{T}'$ of recursion height $\ell-1$ and makes a recursive call to $\text{accelerate-}\mathcal{T}'$ whose height is $\ell-2$ by induction hypothesis.

The result follows since the head of \mathcal{W} is the root of the tree describing \mathcal{W} , which has recursion height k by definition of a blind k -pebble transducer.

Since we have justified in the main paper how each function of $\overline{\mathcal{W}}$ can be implemented by a two-way transducer, then $\overline{\mathcal{W}}$ is indeed a blind $(k-1)$ -pebble transducer. Now, we justify a claim of page 12, that is used to show that the output in line 12 of algorithm 2 must be bounded.

Claim. Assume that, in the execution of $\overline{\mathcal{W}}$ on input $u \in A^*$, we reach line 12 in algorithm 2 and that $\text{prod}_{\mathcal{T}'}^u(i') \neq \varepsilon$ for $i' \notin \text{Fr}^{\mathcal{F}}(\mathcal{F})$. Then the conditions of lemma 5.4 hold, that is there exists $\mathcal{F} \in \text{Forests}_{\mu}(u)$, a sequence $\mathcal{T}_1, \dots, \mathcal{T}_k$ of submachines of \mathcal{W} and a sequence of positions $1 \leq i_1, \dots, i_k \leq |u|$ such that:

- \mathcal{T}_1 is the head of \mathcal{W} ;
- for all $1 \leq j \leq k-1$, $|\text{prod}_{\mathcal{T}_j}^u(i_j)|_{\mathcal{T}_{j+1}} \neq 0$ and $\text{prod}_{\mathcal{T}_k}^u(i_k) \neq \varepsilon$;
- for all $1 \leq j \leq k$, $i_j \notin \text{Fr}^{\mathcal{F}}(\mathcal{F})$ (i.e. $\text{origin}^{\mathcal{F}}(i_j) \in \text{Iter}^{\mathcal{F}}$).

Proof. Let \mathcal{T} be a submachine of \mathcal{U} which is not a leaf (i.e. it labels the root of a subtree of height > 1). We claim that for $\text{accelerate-}\mathcal{T}(u)$ to be called when executing $\overline{\mathcal{U}}$ on input $u \in A^*$, there must exist a sequence $\mathcal{T}_1, \dots, \mathcal{T}_\ell$ of submachines of \mathcal{U} and a sequence of positions $1 \leq i_1, \dots, i_{\ell-1} \leq |u|$ such that:

- \mathcal{T}_1 is the head of \mathcal{U} and $\mathcal{T}_\ell = \mathcal{T}$;
- for all $1 \leq j \leq \ell-1$, $|\text{prod}_{\mathcal{T}_j}^u(i_j)|_{\mathcal{T}_{j+1}} \neq 0$;
- for all $1 \leq j \leq \ell-1$, $i_j \notin \text{Fr}^{\mathcal{F}}(\mathcal{F})$ where $\mathcal{F} := \text{forest}_\mu(u)$.

This result can be checked by induction. Intuitively, it means that to systematically avoid inlinings, we have to make recursive calls in a sequence of positions which are never in the frontier on the root.

Finally, by considering \mathcal{T} the root of a subtree of height 2, we see that the conditions of lemma 5.4 must hold if we reach we reach line 12 in $\text{accelerate-}\mathcal{T}$ and if $\text{prod}_{\mathcal{T}}^u(i') \neq \varepsilon$ for $i' \notin \text{Fr}^{\mathcal{F}}(\mathcal{F})$.

To conclude about the omitted proofs in section 5.2, it remains to show lemma 5.4. This is the purpose of appendix A.3.

A.3 Proof of lemma 5.4

Assume that the conditions of lemma 5.4 hold and let $\mathbf{t}_j := \text{origin}^{\mathcal{F}}(i_j)$ for all $1 \leq j \leq k$. If the \mathbf{t}_j are two by two independent, then each \mathbf{t}_j is surrounded by two nodes whose frontiers cannot contain a position $i_{j'}$ for some $1 \leq j' \leq k$. The image of the factor of u which is below these nodes provides an idempotent e_j . It can easily be concluded that \mathcal{U} is pumpable (see also [10, Lemma E.5]).

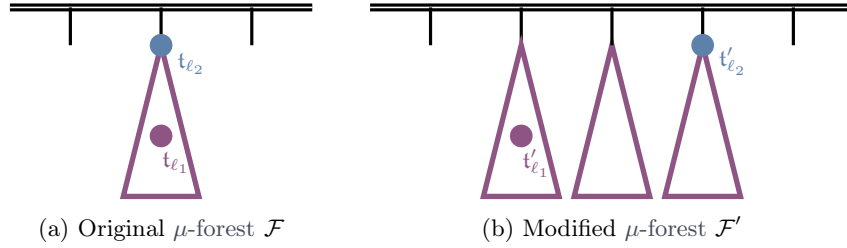


Figure 9: Duplicating a subtree in \mathcal{F} so that \mathbf{t}'_{ℓ_1} and \mathbf{t}'_{ℓ_2} are not dependent.

Now, we suppose that the \mathbf{t}_j are not necessarily two by two independent. Let us show how to make the number of dependent couples of $(\mathbf{t}_{j_1}, \mathbf{t}_{j_2})$ decrease strictly, while preserving the properties of lemma 5.4. Indeed, repeating this process will enable us to make all the nodes two by two independent. Assume that \mathbf{t}_{ℓ_1} observes \mathbf{t}_{ℓ_2} for some $1 \leq \ell_1 \neq \ell_2 \leq k$. To simplify the proof, we assume that \mathbf{t}_{ℓ_2} is an ancestor of \mathbf{t}_{ℓ_1} (the case of the immediate sibling of an ancestor is similar). Let \mathcal{F}' be \mathcal{F} in which the subtree \mathbf{t}_{ℓ_2} has been copied 3 times (since \mathbf{t}_{ℓ_2} is an iterable node, then \mathcal{F}' still a μ -forest), see fig. 9. We define for $1 \leq j \leq k$ the nodes $\mathbf{t}'_j \in \text{Nodes}^{\mathcal{F}'}$ as follows:

- if $j = \ell_2$, then t'_j is (the root of) the third copy of t_j ;
- else if j is such that t_j was a descendant of t_{ℓ_2} (including t_{ℓ_1}), then we let t'_j be the corresponding node in the first copy of t_j ;
- else t_j was in the rest of \mathcal{F} , and we let t'_j be the corresponding node in \mathcal{F}' .

Observe that now, t'_{ℓ_1} and t'_{ℓ_2} are not dependent. Furthermore if t_{j_1} and t_{j_2} were independent, then t'_{j_1} and t'_{j_2} are also independent. Let $u' \in A^*$ be the word such that $\mathcal{F}' \in \text{Forests}_\mu(u')$. We also define $1 \leq i'_1, \dots, i'_k \leq |u'|$ as the positions which correspond to the former $1 \leq i_1, \dots, i_k \leq |u|$ in the frontiers of t'_1, \dots, t'_k in the new μ -forest \mathcal{F}' . The conditions of lemma 5.4 still hold, because $\text{prod}_{\mathcal{T}_j}^u(i_j) = \text{prod}_{\mathcal{T}'_j}^{u'}(i'_j)$ (indeed, we have only duplicated an iterable node, which does neither modify the context around i'_j nor its crossing sequence).

B Omitted proofs of section 6

B.1 Proof of lemma 6.2

The proof is similar to that of lemma 5.2. Let $w_i \in A^*$ be such that $\mu(w_i) = \ell_i$, $w'_i \in A^*$ be such that $\mu(w'_i) = r_i$, $u_i := w_i a_i w'_i$ and $\underline{u}_i := w_i a_i w'_i$, for $1 \leq i \leq k$. Let $v_0, \dots, v_k \in A^*$ such that $\mu(v_i) = m_i$ for all $1 \leq i \leq k$. Let f_i be the function computed by \mathcal{T}_i for all $1 \leq i \leq k$.

To simply the proof, we assume that $\sigma : [1:k] \rightarrow [1:k]$ is the identity function. We then observe that for all $X \geq 2$, for all $1 \leq Y \leq X-2$:

$$f_k(v_0 u_1^X \cdots v_{k-2} (u_{k-1}^Y \underline{u_{k-1}} u_{k-1}^{X-Y-1}) v_{k-1} u_k^X) \geq (X-2).$$

Observe that the use of $u_{k-1}^Y \underline{u_{k-1}} u_{k-1}^{X-Y-1}$ means that the result holds independently from the factor in which the call (i.e. the mark) to \mathcal{T}_k was done. Finally, we conclude by induction in a similar way to lemma 5.2.

B.2 Additional arguments in section 6.2

We first justify that the construction of $\overline{\mathcal{U}}$ in section 5.2 indeed reduces the recursion height of \mathcal{U} by 1. This statement was claimed on page 16.

Claim. The machine $\overline{\mathcal{U}}$ described in section 6.2 has recursion height $k-1$.

Proof. Recall that the recursion height corresponds to the number of nested **Call** instructions, plus 1 (due to the head). We first show by (decreasing) induction on $1 \leq \ell \leq k$ that if \mathcal{T} is the root of a subtree of \mathcal{U} whose recursion height is $1 \leq \ell \leq k$, then $\text{old-}\mathcal{T}\text{-along-}\rho$ has recursion height ℓ as well.

Then, we show by (decreasing) induction on $2 \leq \ell \leq k$ that if \mathcal{T} is the root of a subtree of \mathcal{U} whose recursion height is ℓ , then $\text{accelerate-}\mathcal{T}\text{-along-}\rho$ has recursion height $\ell-1$. Indeed, the base case $\ell = 2$ is justified by line 18 in algorithm 4 (there are no calls). For $\ell > 2$ the function $\text{accelerate-}\mathcal{T}\text{-along-}\rho$ inlines some $\text{old-}\mathcal{T}'\text{-along-}\rho'_j$ of recursion height $\ell-1$ and makes recursive calls to $\text{accelerate-}\mathcal{T}'\text{-along-}\rho_j$ whose height is $\ell-2$ by induction hypothesis.

The result follows since the head of \mathcal{U} is the root of the tree describing \mathcal{U} , which has recursion height k by definition of a last k -pebble transducer.

Now, let us justify a claim of page 17, that is used to show that the output of line 18 in algorithm 4 is indeed empty.

Claim. Assume that, in the execution of $\overline{\mathcal{W}}$ on input $u \in A^*$, we reach line 18 in algorithm 4 and that the output of \mathcal{T}' along ρ'_j is not empty. Then the conditions of lemma 6.6 hold, that is there exists $\mathcal{F} \in \text{Forests}_\mu(u)$, a sequence $\mathcal{T}_1, \dots, \mathcal{T}_k$ of submachines of \mathcal{W} and a sequence of positions $1 \leq i_1, \dots, i_k \leq |u|$ such that:

- \mathcal{T}_1 is the head of \mathcal{W} ;
- $|\text{prod}_{\mathcal{T}_1}^u(i_1)|_{\mathcal{T}_2} \neq 0$ and $\text{prod}_{\mathcal{T}_k}^{u \bullet i_{k-1}}(i_k) \neq \varepsilon$;
- for all $2 \leq j \leq k-1$, $|\text{prod}_{\mathcal{T}_j}^{u \bullet i_{j-1}}(i_j)|_{\mathcal{T}_{j+1}} \neq 0$;
- for all $1 \leq j \leq k-1$, $\text{origin}^{\mathcal{F}}(i_j)$ and $\text{origin}^{\mathcal{F}}(i_{j+1})$ are independent;

Proof. Let \mathcal{T} be a submachine of \mathcal{W} which is neither a leaf (i.e. it labels the root of a subtree of height > 1) nor the head of \mathcal{W} (i.e. not the root of \mathcal{W}). We claim that for $\text{accelerate-}\mathcal{T}\text{-along-}\rho(v)$ to be called within the execution of $\overline{\mathcal{W}}$ on input $u \in A^*$, there must exist a sequence $\mathcal{T}_1, \dots, \mathcal{T}_\ell$ of submachines of \mathcal{W} and a sequence of positions $1 \leq i_1, \dots, i_{\ell-1} \leq |u|$ such that:

- \mathcal{T}_1 is the head of \mathcal{W} ;
- $v = u \bullet i_{\ell-1}$;
- $|\text{prod}_{\mathcal{T}_1}^u(i_1)|_{\mathcal{T}_2} \neq 0$ and for all $2 \leq j \leq \ell-1$, $|\text{prod}_{\mathcal{T}_j}^{u \bullet i_{j-1}}(i_j)|_{\mathcal{T}_{j+1}} \neq 0$;
- for all $1 \leq j \leq \ell-2$, $\text{origin}^{\mathcal{F}}(i_j)$ and $\text{origin}^{\mathcal{F}}(i_{j+1})$ are independent, where we define $\mathcal{F} := \text{forest}_\mu(u)$;
- for all $(i, q) \in \rho$, $\text{origin}^{\mathcal{F}}(i_{\ell-1})$ and $\text{origin}^{\mathcal{F}}(i)$ are independent

This result can be checked by induction. Intuitively, the two crucial last point follows from the fact that we only make recursive calls in portions of runs whose positions are not dependent on the calling position.

Finally, by considering \mathcal{T} the root of a subtree of height 2, we see that the conditions of lemma 6.6 must hold if we reach line 18 in some $\text{accelerate-}\mathcal{T}\text{-along-}\rho$ called in $\overline{\mathcal{W}}$ and if the output of \mathcal{T}' along ρ'_j is not empty.

To conclude about the omitted proofs in section 6.2, it remains to show lemmas 6.5 and 6.6. This is the purpose of appendices B.3 and B.4.

B.3 Proof of lemma 6.5

Let $1 \leq i \leq |u|$, $\mathbf{t} := \text{origin}^{\mathcal{F}}(i)$ and \mathbf{t}_1 (resp. \mathbf{t}_2) be its immediate left (resp. right) sibling (they exist whenever $\mathbf{t} \in \text{Iter}^{\mathcal{F}}$, i.e. here $\mathbf{t} \neq \mathcal{F}$). We show that:

$$\downarrow i \searrow \uparrow i = [\min(\text{Fr}^{\mathcal{F}}(\mathbf{t}_1)) : \max(\text{Fr}^{\mathcal{F}}(\mathbf{t}_2))] \setminus \{\text{Fr}^{\mathcal{F}}(\mathbf{t}_1), \text{Fr}^{\mathcal{F}}(\mathbf{t}), \text{Fr}^{\mathcal{F}}(\mathbf{t}_2)\}.$$

Let us assume that \mathbf{t}_1 and \mathbf{t}_2 are iterable nodes of \mathcal{F} (the other cases are similar). By considering the forest of fig. 5, it can be noted that $\downarrow i$ is the interval $[\min(\text{Fr}^{\mathcal{F}}(\mathbf{t}_1)) : \max(\text{Fr}^{\mathcal{F}}(\mathbf{t}_2))]$. We conclude since \mathbf{t}, \mathbf{t}_1 and \mathbf{t}_2 are the only iterable nodes that both observe \mathbf{t} and that \mathbf{t} observes.

Therefore $\downarrow i \searrow \uparrow i$ is the union of a bounded number of intervals (since the frontiers have bounded size). It is easy to observe that the “borders” of these intervals can easily be recovered by a lookaround, if \mathbf{t} (or i) is given.

B.4 Proof of lemma 6.6

The proof is similar to that of lemma 5.4. The goal is to show that the for $1 \leq j \leq k$, the $t_j := \text{origin}^{\mathcal{F}}(i_j)$ can be chosen two by two independent (in the hypothesis, it is only assumed for the consecutive pairs (t_j, t_{j+1})).

For this, we show once more how to make the number of dependent nodes decrease strictly, while preserving the properties of lemma 6.6. Assume that t_{ℓ_1} observes t_{ℓ_2} for some $1 \leq \ell_1 \neq \ell_2 \leq k$ (note that ℓ_1 and ℓ_2 are not consecutive). To simplify the proof, we assume that t_{ℓ_2} is an ancestor of t_{ℓ_1} (the case of the immediate sibling of an ancestor is similar). We build $\mathcal{F}' \in \text{Forests}_{\mu}(u')$ as in the proof of lemma 5.4 (see fig. 9), and define the new nodes $t'_1, \dots, t'_k \in \text{Nodes}^{\mathcal{F}'}$ in the same way. We also define $1 \leq i'_1, \dots, i'_k \leq |u'|$ as the positions which correspond to the former $1 \leq i_1, \dots, i_k \leq |u|$ adapted to the new nodes t'_1, \dots, t'_k .

Now, we justify that $\text{prod}_{\mathcal{G}_j}^{u \bullet i_{j-1}}(i_j) = \text{prod}_{\mathcal{G}_j}^{u' \bullet i'_{j-1}}(i'_j)$ for all $2 \leq j \leq k$. This is the only difference with the proof of lemma 5.4 that we need to treat:

- if both i_{j-1} and i_j belong to the subtree rooted in i_{ℓ_2} , then $j \neq \ell_2$ (since otherwise i_j and i_{j-1} would be dependent) and similarly $j-1 \neq \ell_2$. The result holds because we only iterate an iterable node;
- if both i_{j-1} and i_j do not belong to this subtree, the argument is similar;
- if i_{j-1} is in the subtree but not i_j (the converse is similar), we use the fact that these two nodes are independent. Indeed, it implies that i_j cannot be “below” an immediate sibling of t_{ℓ_2} . Hence duplicating this iterable node will not change the monoid value between positions i_{j-1} and i_j .

C Proof of theorem 7.1

Let A be an alphabet and $k \geq 1$. We define the tree language T_A^k as the set of trees such that all root-to-leaf branches have exactly k nodes (hence the tree has height k), whose leaves are labelled by words of A^* and whose inner nodes have no labels. As observed for factorization forests, T_A^k can be seen as a regular word language over the alphabet $\hat{A} := A \uplus \{\langle, \rangle\}$. Given such a tree, we say that the root has *height* 1, its children height 2, etc. and the leaves have height k .

Now, we describe for all $k \geq 1$ a function **zebra**^k : $T_A^{k+1} \rightarrow T_{A \uplus \#}^{2k+1}$ which goes from words to words (i.e. it works on the word representation of the trees). This function was introduced by Bojańczyk in [2]. Intuitively, it produces a tree whose leaves labels are tuples $u \# v$ for u, v labels of the original tree, but the ordering of these tuples is very specific.

Let us first describe the function **zebra**¹. It takes as input a tree of height 2 of shape $\langle \langle u_1 \rangle, \langle u_2 \rangle, \dots, \langle u_n \rangle \rangle$ and it produces a tree of height 3 whose i -th child of the root is $\langle \langle u_i \# u_1 \rangle, \dots, \langle u_i \# u_n \rangle \rangle$ (i.e. the tuples are ordered lexicographically). This function be implemented by a 3-pebble transducer (i.e. a last-last 3-pebble transducer) which uses its two first pebble to see which leaves have to be produced, and the last layer to indeed output these leaves. Observe that **zebra**¹ can be seen as a variant of unmarked-square (see example 3.2).

Now, the function **zebra**² is described formally in algorithm 5.

Algorithm 5: Computing the zebra² function

```

1 Function zebra2( $u$ )
2    $u \in A^*$  represents a tree of depth  $k+1$ ;
3    $i_0 := j_0 :=$  the root of  $u$ ;
4   for  $i_1$  ranging from left to right on the children of  $i_0$  do
5     Output  $\langle$ ;
6     for  $j_1$  ranging from left to right on the children of  $j_0$  do
7       Output  $\langle$ ;
8       for  $i_2$  ranging from left to right on the children of  $i_1$  do
9         Output  $\langle$ ;
10        for  $j_2$  ranging from left to right on the children of  $j_1$  do
11          /* Depth 3:  $i_2$  and  $j_2$  are leaves; */
12           $u :=$  label of  $i_2$ ;  $v :=$  label of  $j_2$ ;
13          Output  $\langle u\#v \rangle$ 
14        end
15      Output  $\rangle$ ;
16    end
17  Output  $\rangle$ ;
18 end
19 Output  $\rangle$ ;
20 end

```

Observe that zebra² no longer produces the tuples $u\#v$ in a lexicographic ordering. Indeed, it corresponds to two ranges over the leaves of the original tree (one with i_1, i_2 and one with j_1, j_2) which are highly entangled. It is easy to guess how to extend algorithm 5, in order to define zebra ^{k} using $2k$ nested loops. Originally, the zebra ^{k} functions were used in order to show that the minimal number of layers and the growth of the output do not coincide for pebble transducers, as claimed in theorem C.1.

Theorem C.1 ([2, section 3]). *For all $k \geq 1$, the function zebra ^{k} is such that $|\text{zebra}^k(u)| = \mathcal{O}(|u|^2)$. Furthermore, it can be computed by a $(2k+1)$ -pebble transducer but not by a $2k$ -pebble transducer.*

As a consequence, zebra ^{k} cannot be computed by a last-last $2k$ -pebble transducer. To show theorem 7.1, we only need to justify that zebra ^{k} can be computed by last-last $(2k+1)$ -pebble transducer. This is indeed the case: it uses its $2k$ first layers to describe the nested loops on $i_1, j_1, i_2, j_2, \dots, i_k, j_k$ and the last one to range over the labels of the tuple of leaves and output them (see algorithm 5). The key observation is that it only needs to see the two last loop indexes, since this information is sufficient to find their children.