# D-finiteness: Algorithms and Applications

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## ABSTRACT

Differentially finite series are solutions of linear differential equations with polynomial coefficients. P-recursive sequences are solutions of linear recurrences with polynomial coefficients. Corresponding notions are obtained by replacing classical differentiation or difference operators by their $q$-analogues. All these objects share numerous properties that are described in the framework of "D-finiteness". Our aim in this area is to enable computer algebra systems to deal in an algorithmic way with a large number of special functions and sequences. Indeed, it can be estimated that approximately 60% of the functions described in Abramowitz & Stegun's handbook [1] fall into this category, as well as 25% of the sequences in Sloane's encyclopedia [20, 21].

In a way, D-finite sequences or series are non-commutative analogues of algebraic numbers: the role of the minimal polynomial is played by a linear operator. Ore [14] described a non-commutative version of Euclidean division and extended Euclid algorithm for these linear operators (known as Ore polynomials). In the same way as in the commutative case, these algorithms make several closure properties effective (see [22]). It follows that identities between these functions or sequences can be proved or computed automatically. Part of the success of the GFUN package [17] comes from an implementation of these operations. Another part comes from the possibility of *discovering* such identities empirically, with Padé-Hermite approximants on power series [2] taking the place of the LLL algorithm on floating-point numbers.

The discovery that a series is D-finite is also important from the complexity point of view: several operations can be performed on D-finite series at a lower cost than on arbitrary power series. This includes multiplication, but also evaluation at rational points by binary splitting [4]. A typical application is the numerical evaluation of $\pi$ in computer algebra systems; we give another one in these proceedings [3].

Also, the local behaviour of solutions of linear differential equations in the neighbourhood of their singularities is well understood [9] and implementations of algorithms computing the corresponding expansions are available [24, 13]. This gives access to the asymptotics of numerous sequences or to analytic proofs that sequences or functions cannot satisfy such equations [10]. Results of a more algebraic nature are obtained by differential Galois theory [18, 19], which naturally shares many subroutines with algorithms for D-finite series.

The truly spectacular applications of D-finiteness come from the multivariate case: instead of series or sequences, one works with multivariate series or sequences, or with sequences of series or polynomials,.... They obey systems of linear operators that may be of differential, difference, $q$-difference or mixed types, with the extra constraint that a finite number of initial conditions are sufficient to specify the solution. This is a non-commutative analogue of polynomial systems with a finite number of solutions. It turns out that, as in the polynomial case, Gröbner bases give algorithmic answers to many decision questions, by providing normal forms in a finite dimensional vector space. This has been observed first in the differential case [11, 23] and then extended to the more general multivariate Ore case [8].

A crucial insight of Zeilberger [27, 15] is that elimination in this non-commutative setting computes definite integrals or sums. This is known as *creative telescoping*. In the *hypergeometric* setting (when the quotient is a vector space of dimension 1), a fast algorithm for this operation is known as Zeilberger's fast algorithm [26]. In the more general case, Gröbner bases are of help in this elimination. This is true in the differential case [16, 25] and to a large extent in the more general multivariate case [8]. Also, Zeilberger's fast algorithm has been generalized to the multivariate Ore case by Chyzak [5, 6]. Still, various efficiency issues remain and phenomena of non-minimality of the eliminated operators are not completely understood.

A further generalization of D-finite series is due to Gessel [12] who developed a theory of symmetric series. These series are such than when all but a finite number of their variables (in a certain basis) are specialized to 0, the resulting series is D-finite in the previous sense. Closure properties under scalar product lead to proofs of D-finiteness (in the classical sense) for various combinatorial sequences. Again, algorithms based on Gröbner bases make these operations effective [7].

The talk will survey the nicest of these algorithms and their applications. I will also indicate where current work is in progress, or where more work is needed.


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Algorithms

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