Solving Linear Recurrence Equations With Polynomial Coefficients

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Abstract

Summation is closely related to solving linear recurrence equations, since an indefinite sum satisfies a first-order linear recurrence with constant coefficients, and a definite proper-hypergeometric sum satisfies a linear recurrence with polynomial coefficients. Conversely, d'Alembertian solutions of linear recurrences can be expressed as nested indefinite sums with hypergeometric summands. We sketch the simplest algorithms for finding polynomial, rational, hypergeometric, d'Alembertian, and Liouvillian solutions of linear recurrences with polynomial coefficients, and refer to the relevant literature for state-of-the-art algorithms for these tasks. We outline an algorithm for finding the minimal annihilator of a given P-recursive sequence, prove the salient closure properties of d'Alembertian sequences, and present an alternative proof of a recent result of Reutenauer's that Liouvillian sequences are precisely the interlacings of d'Alembertian ones.

1 Introduction

Summation is related to solving linear recurrence equations in several ways. An indefinite sum

$$s(n) \; = \; \sum_{k=0}^{n-1} t(k)$$

satisfies the nonhomogeneous first-order recurrence equation

$$s(n+1) - s(n) = t(n); \quad s(0) = 0,$$

and also the homogeneous second-order recurrence equation

$$t(n)s(n+2) - (t(n)+t(n+1))s(n+1) + t(n+1)s(n) = 0; \quad s(0) = 0,\ s(1) = t(0).$$

A definite sum

$$s(n) \;=\; \sum_{k=0}^{n} F(n,k)$$

where the summand $F(n,k)$ is a proper hypergeometric term:

$$F(n,k) \;=\; P(n,k)\frac{\prod_{j=1}^{A}(\alpha_j)_{a_j n + \tilde{a}_j k}}{\prod_{j=1}^{B}(\beta_j)_{b_j n + \tilde{b}_j k}} z^k$$

with $P(n,k)$ a polynomial in both variables, $(z)_k$ the Pochhammer symbol, α_j, β_j, z commuting indeterminates, $a_j, b_j \in \mathbb{N}$, and $\tilde{a}_j, \tilde{b}_j \in \mathbb{Z}$, satisfies a linear recurrence equation with polynomial coefficients in n which can be computed with Zeilberger's algorithm (cf. [41], [42], [29], [11]). So the sum of interest may sometimes be found by solving a suitable recurrence equation.

The unknown object in a recurrence equation is a *sequence*, by which we mean a function mapping the nonnegative integers $\mathbb{N}$ to some algebraically closed field K of characteristic zero. Sequences can be represented in several different ways, among the most common of which are the following:

- *explicit* where a sequence $a : \mathbb{N} \to K$ is represented by an expression $e(x)$ such that $a(n) = e(n)$ for all $n \geq 0$,

- *recursive* where a sequence $a : \mathbb{N} \to K$ is represented by a function F and by some initial values $a(0), a(1), \ldots, a(d-1)$ such that

  $$a(n) \;=\; F(n, a(n-1), a(n-2), \ldots, a(0)) \tag{1}$$

  for all $n \geq d$,

- by *generating function* where a sequence $a : \mathbb{N} \to K$ is represented by the (formal) power series

  $$G_a(z) \;=\; \sum_{n=0}^{\infty} a(n) z^n.$$

Each of these representations has several variants and special cases. In particular, if $a(n) \;=\; F(n, a(n-1), a(n-2), \ldots, a(n-d))$ for all $n \geq d$, the recursive representation (1) is said to be *of order at most d*.

Example 1 (Fibonacci numbers)

- explicit representation: $a(n) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right)$
- recursive representation: $a(n) = a(n-1) + a(n-2)$ $(n \geq 2)$, $a(0) = a(1) = 1$
- generating function: $G_a(z) = \dfrac{1}{1 - z - z^2}$

From the viewpoint of representation of sequences, solving recurrence equations can be seen as the process of converting one (namely recursive) representation to another (explicit) representation.

In this paper we survey the properties of several important classes of sequences which satisfy linear recurrence equations with polynomial coefficients, and sketch algorithms for finding such solutions when they exist. In Sections 2 and 3 we review the main results about C-recursive and P-recursive sequences, then we describe algorithms for finding polynomial, rational and hypergeometric solutions in Sections 4 and 5. Difference rings and the Ore algebra of linear difference operators with rational coefficients, together with the outline of a factorization algorithm, are introduced in Sections 6 and 7. In Section 8 we define d'Alembertian sequences and prove their closure properties. Finally, in Section 9, we give an alternative proof of the recent result of Reutenauer [31] that Liouvillian sequences are precisely the interlacings of d'Alembertian sequences by showing that the latter enjoy all the closure properties of the former.

2 C-recursive sequences

C-recursive sequences satisfy homogeneous linear recurrences with constant coefficients. Typical examples are geometric sequences of the form $a(n) = c\,q^n$ with $c, q \in K^*$, polynomial sequences, their products, and their linear combinations (such as the Fibonacci numbers of Example 1).

Definition 1 *A sequence $a \in K^{\mathbb{N}}$ is C-recursive or C-finite*[1] *if there are $d \in \mathbb{N}$ and constants $c_1, c_2, \ldots, c_d \in K$, $c_d \neq 0$, such that*

$$a(n) = c_1\,a(n-1) + c_2\,a(n-2) + \cdots + c_d\,a(n-d)$$

for all $n \geq d$.

The following theorem describes the explicit and generating-function representations of C-recursive sequences. For a proof, see, e.g., [38].

Theorem 1 *Let $a \in K^{\mathbb{N}}$ and $G_a(z) = \sum_{n=0}^{\infty} a(n)z^n$. The following are equivalent:*

[1] C-recursive sequences are also called *linear recurrent (*or: *recurrence) sequences*. This neglects sequences satisfying linear recurrences with non-constant coefficients, and may lead to confusion.

1. *a is C-recursive,*

2. $a(n) = \sum_{i=1}^{r} P_i(n)\, \alpha_i^n$ *for all* $n \in \mathbb{N}$ *where* $P_i \in K[x]$ *and* $\alpha_i \in K$,

3. $G_a(z) = \frac{P(z)}{Q(z)}$ *where* $P, Q \in K[x]$, $\deg P < \deg Q$ *and* $Q(0) \neq 0$.

The next two theorems are easy corollaries of Theorem 1.

Theorem 2 *The set of C-recursive sequences is closed under the following binary operations* $(a, b) \mapsto c$*:*

1. *addition:* $c(n) = a(n) + b(n)$
2. *(Hadamard or termwise) multiplication:* $c(n) = a(n)b(n)$
3. *convolution (Cauchy multiplication):* $c(n) = \sum_{i=0}^{n} a(i)b(n-i)$
4. *interlacing:* $\langle c(0), c(1), c(2), c(3), \ldots \rangle = \langle a(0), b(0), a(1), b(1), \ldots \rangle$

Remark 1 *These operations extend naturally to any nonzero number of operands.*

Theorem 3 *The set of C-recursive sequences is closed under the following unary operations* $a \mapsto c$*:*

1. *scalar multiplication:* $c(n) = \lambda a(n)$ $(\lambda \in K)$
2. *(left) shift:* $c(n) = a(n+1)$
3. *indefinite summation:* $c(n) = \sum_{k=0}^{n} a(k)$
4. *multisection:* $c(n) = a(mn + r)$ $(m, r \in \mathbb{N},\ 0 \leq r < m)$

That (nonzero) C-recursive sequences are not closed under taking reciprocals is demonstrated, e.g., by $a(n) = n + 1$ which is C-recursive while its reciprocal $b(n) = 1/(n+1)$ is not, since its generating function $G_b(x) = -\ln(1-x)/x$ is not a rational function. Of course, there are C-recursive sequences whose reciprocals are C-recursive as well, such as all the geometric sequences.

Question 1. When are a and $1/a$ both C-recursive?

Theorem 4 *The sequences a and $1/a$ are both C-recursive iff a is the interlacing of one or more geometric sequences.*

For a proof, see [26].

3 P-recursive sequences

P-recursive sequences satisfy homogeneous linear recurrences with polynomial coefficients. While most of them lack a simple explicit representation, their generating functions do have a nice characterization in terms of differential equations. There exist also several important subclasses of P-recursive sequences such as *polynomial, rational, hypergeometric* (Sec. 4), *d'Alembertian* (Sec. 8), and *Liouvillian* (Sec. 9) sequences which have nice explicit representations. Figure 1 shows a hierarchy of these subclasses together with some examples. In the rest of the paper, we investigate their properties and sketch algorithms for finding such special solutions of linear recurrence equations with polynomial coefficients, whenever they exist.

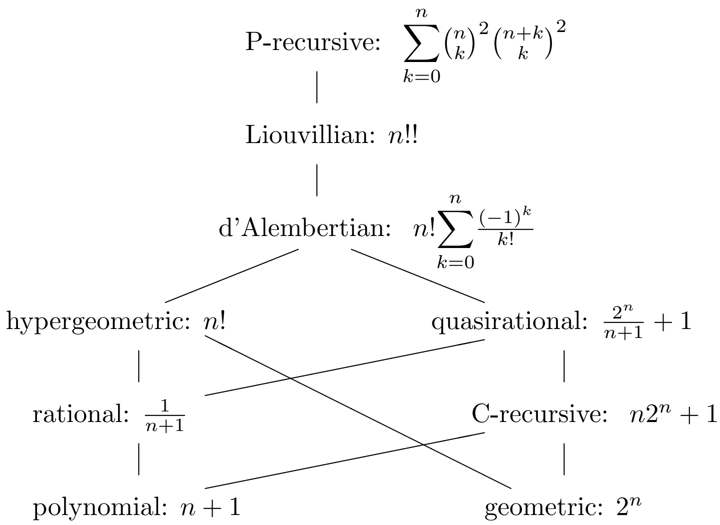

Figure 1: A hierarchy of P-recursive sequences (with examples)

Definition 2 *A sequence $a \in K^{\mathbb{N}}$ is P-recursive if there are $d \in \mathbb{N}$ and polynomials $p_0, p_1, \ldots, p_d \in K[x]$, $p_d \neq 0$, such that*

$$p_d(n)a(n+d) + p_{d-1}(n)a(n+d-1) + \cdots + p_0(n)a(n) \;=\; 0$$

for all $n \geq 0$.

Definition 3 *A formal power series $f(z) = \sum_{n=0}^{\infty} a(n)z^n \in K[[z]]$ is D-finite if there exist $d \in \mathbb{N}$ and polynomials $q_0, q_1, \ldots, q_d \in K[x]$, $q_d \neq 0$, such that*

$$q_d(z)f^{(d)}(z) + q_{d-1}(z)f^{(d-1)}(z) + \cdots + q_0(z)f(z) \;=\; 0.$$

Theorem 5 *Let $a \in K^{\mathbb{N}}$ and $G_a(z) = \sum_{n=0}^{\infty} a(n)z^n$. The following are equivalent:*

1. *a is P-recursive,*

2. *$G_a(z)$ is D-finite.*

For a proof, see [39] or [40].

Theorem 6 *P-recursive sequences are closed under the following operations:*

1. *addition,*

2. *multiplication,*

3. *convolution,*

4. *interlacing,*

5. *scalar multiplication,*

6. *shift,*

7. *indefinite summation,*

8. *multisection.*

For a proof, see [40].

Question 2. When are a and $1/a$ are both P-recursive?
The answer is given in Theorem 7.

Example 2 *The sequences $a(n) = n!$ and $b(n) = 1/n!$ are both P-recursive since $a(n+1) - (n+1)a(n) = 0$ and $(n+1)b(n+1) - b(n) = 0$.*

Example 3 *The sequence $a(n) = 2^n + 1$ is P-recursive (even C-recursive) while its reciprocal $b(n) = 1/(2^n + 1)$ is not P-recursive.*

Proof: We use the fact that a D-finite function can have at most finitely many singularities in the complex plane (see, e.g., [40]). The generating function

$$G_b(z) \;=\; \sum_{n=0}^{\infty} b(n) z^n \;=\; \sum_{n=0}^{\infty} \frac{z^n}{2^n + 1}$$

obviously has radius of convergence equal to two. We can rewrite

$$\begin{aligned} G_b(2z) \;&=\; \sum_{n=0}^{\infty} \frac{2^n}{2^n+1} z^n \;=\; \sum_{n=0}^{\infty} \left(1 - \frac{1}{2^n+1}\right) z^n \\ &=\; \frac{1}{1-z} - G_b(z). \end{aligned} \tag{2}$$

At $z = 1$ the function $1/(1-z)$ is singular, G_b is regular, so G_b is singular at $z = 2$. At $z = 2$ the function $1/(1-z)$ is regular, G_b is singular, so G_b is singular at $z = 4$. At $z = 4$ the function $1/(1-z)$ is regular, G_b is singular, so G_b is singular at $z = 8$, and so on. By induction on k it follows that $G_b(z)$ is singular at $z = 2^k$ for all $k \in \mathbb{N}$, $k \geq 1$, hence G_b is not D-finite, and b is not P-recursive.
□

4 Hypergeometric sequences

Hypergeometric sequences are P-recursive sequences which satisfy homogeneous linear recurrence equations with polynomial coefficients of order one. They can be represented explicitly as products of rational functions, Pochhammer symbols, and geometric sequences. The algorithm for finding hypergeometric solutions of linear recurrence equations with polynomial coefficients plays an important role in other, more involved computational tasks such as finding d'Alembertian or Liouvillian solutions, and factoring linear recurrence operators.

Definition 4 *A sequence $a \in K^{\mathbb{N}}$ is hypergeometric*[2] *if there is an $N \in \mathbb{N}$ such that $a(n) \neq 0$ for all $n \geq N$, and there are polynomials $p, q \in K[n] \setminus \{0\}$ such that*

$$p(n)\, a(n+1) \;=\; q(n)\, a(n) \tag{3}$$

for all $n \geq 0$. We denote by $\mathcal{H}(K)$ the set of all hypergeometric sequences in $K^{\mathbb{N}}$.

Clearly, each hypergeometric sequence is P-recursive.

Proposition 1 *The set $\mathcal{H}(K)$ is closed under the following operations:*

1. *multiplication,*
2. *reciprocation,*
3. *nonzero scalar multiplication,*
4. *shift,*
5. *multisection.*

Proof: For 1 – 4, see [29]. For multisection, let $a \in \mathcal{H}(K)$ satisfy (3) and let $b(n) = a(mn+r)$ where $m \in \mathbb{N}$, $m \geq 2$, and $0 \leq r < m$. For $i = 0, 1, \ldots, m-1$, substituting $mn + r + i$ for n in (3) yields

$$p(mn+r+i)\, a(mn+r+i+1) \;=\; q(mn+r+i)\, a(mn+r+i). \tag{4}$$

Multiply (4) by $\prod_{j=0}^{i-1} p(mn+r+j) \prod_{j=i+1}^{m-1} q(mn+r+j)$ on both sides to obtain $lhs_i = rhs_i$ for $i = 0, 1, \ldots, m-1$, where

$$\begin{aligned}
lhs_i &= \prod_{j=0}^{i} p(mn+r+j) \prod_{j=i+1}^{m-1} q(mn+r+j)\; a(mn+r+i+1), \\
rhs_i &= \prod_{j=0}^{i-1} p(mn+r+j) \prod_{j=i}^{m-1} q(mn+r+j)\; a(mn+r+i).
\end{aligned}$$

[2] A hypergeometric sequence is also called a *hypergeometric term*, because the nth term of a hypergeometric series, considered as a function of n, is a hypergeometric sequence in our sense.

Note that $lhs_i = rhs_{i+1}$ for $i = 0, 1, \ldots, m-2$, hence, by induction on i,

$$rhs_0 \ = \ lhs_i \quad \text{for } i = 0, 1, \ldots, m-1.$$

In particular, $lhs_{m-1} = rhs_0$, so

$$\prod_{j=0}^{m-1} p(mn + r + j) b(n+1) \ = \ \prod_{j=0}^{m-1} q(mn + r + j) b(n),$$

hence $b \in \mathcal{H}(K)$. □

Theorem 7 *The sequences a and $1/a$ are both P-recursive iff a is the interlacing of one or more hypergeometric sequences.*

For a proof, see [30].

5 Closed-form solutions

In this section, we sketch algorithms for finding polynomial, rational, and hypergeometric solutions of linear recurrence equations with polynomial coefficients.

5.1 Recurrence operators

Let $E : K^{\mathbb{N}} \to K^{\mathbb{N}}$ be the (left) shift operator acting on sequences by $(Ea)(n) = a(n+1)$, so that $(E^k a)(n) = a(n+k)$ for $k \in \mathbb{N}$. For a given $d \in \mathbb{N}$ and polynomials $p_0, p_1, \ldots, p_d \in K[n]$ such that $p_d \neq 0$, the operator $L : K^{\mathbb{N}} \to K^{\mathbb{N}}$ defined by

$$L \ = \ \sum_{k=0}^{d} p_k(n)\, E^k$$

is a *linear recurrence operator* of order d with polynomial coefficients, acting on a sequence a by $(La)(n) = \sum_{k=0}^{d} p_k(n)\, a(n+k)$. We denote by $K[n]\langle E\rangle$ the *algebra of linear recurrence operators with polynomial coefficients.* The commutation rule $E \cdot p(n) = p(n+1)\, E$ induces the rule for composition of operators:

$$\sum_{k=0}^{d} p_k(n) E^k \cdot \sum_{j=0}^{e} q_j(n) E^j = \sum_{k=0}^{d} \sum_{j=0}^{e} p_k(n) q_j(n+k) E^{j+k}.$$

5.2 Polynomial solutions

Given: $L \in K[n]\langle E\rangle$, $L \neq 0$
Find: a basis of the space $\{y \in K[n];\ \ Ly = 0\}$

Outline of algorithm

1. Find an upper bound for $\deg y$.
2. Use the method of undetermined coefficients.

For more details, see [3], [9], [29].

5.3 Rational solutions

Given: $L \in K[n]\langle E\rangle, \ L \neq 0$
Find: a basis of the space $\{y \in K(n); \ \ Ly = 0\}$

Outline of algorithm

1. Find a universal denominator for y.
2. Find polynomial solutions of the equation satisfied by the numerator of y.

For more details, see [4], [6], [22], and [7].

5.4 Hypergeometric solutions

Given: $L = \sum_{k=0}^{d} p_k\, E^k \in K[n]\langle E\rangle, \quad L \neq 0$
Find: a generating set for the linear hull of $\{y \in \mathcal{H}(K); \ Ly = 0\}$

Outline of algorithm

1. Construct the "Riccati equation" for $r = \frac{Ey}{y} \in K(n)$:

$$\sum_{k=0}^{d} p_k \prod_{j=0}^{k-1} E^j r \ = \ 0 \tag{5}$$

2. Use the ansatz

$$r \ = \ z\,\frac{a}{b}\,\frac{Ec}{c}$$

with $z \in K^*$, $a, b, c \in K[n]$ monic, a, c coprime, b, Ec coprime, $a, E^k b$ coprime for all $k \in \mathbb{N}$ to obtain

$$\sum_{k=0}^{d} z^k p_k \left(\prod_{j=0}^{k-1} E^j a\right)\left(\prod_{j=k}^{d-1} E^j b\right) E^k c \ = \ 0. \tag{6}$$

3. Construct a finite set of candidates for (a, b, z) using the following consequences of (6):

   - $a \,|\, p_0$,
   - $b \,|\, E^{1-d} p_d$,
   - $\displaystyle\sum_{\substack{0\le k\le d \\ \deg P_k = m}} \mathrm{lc}(P_k) z^k \ = \ 0$

   where $P_k = p_k \left(\prod_{j=0}^{k-1} E^j a\right)\left(\prod_{j=k}^{d-1} E^j b\right), \quad m = \max_{0\le k\le d} \deg P_k$.

4. For each candidate triple (a, b, z), find polynomial solutions c of the equation
$$\sum_{k=0}^{d} z^k P_k\, E^k c \;=\; 0.$$

For more details, see [28] or [29]. A much more efficient algorithm (although still exponential in $\deg p_0 + \deg p_d$ in the worst case) is given in [23] and [16].

Example 4 *(*Amer. Math. Monthly *problem no. 10375)* *Solve*

$$y(n+2) - 2(2n+3)^2 y(n+1) + 4(n+1)^2(2n+1)(2n+3)y(n) \;=\; 0. \tag{7}$$

Denote $p_2(n) = 1$, $p_1(n) = -2(2n+3)^2$, and $p_0(n) = 4(n+1)^2(2n+1)(2n+3)$. In search of hypergeometric solutions we follow the four steps described above:

1. Riccati equation:
$$p_2(n)\, r(n+1) r(n) \;+\; p_1(n)\, r(n) \;+\; p_0(n) \;=\; 0$$

2. plug in the ansatz:
$$\begin{array}{llllll} & z^2 & p_2(n) & a(n+1)\, a(n) & c(n+2) & \\ + & z & p_1(n) & a(n)\, b(n+1) & c(n+1) & \\ + & & p_0(n) & b(n+1)\, b(n) & c(n) & = \quad 0 \end{array}$$

3. candidates for (a, b, z):
   - $a(n) \mid 4(n+1)^2(2n+1)(2n+3)$
   - $b(n) \mid 1$
   - $z^2 - 8z + 16 \;=\; (z-4)^2 \;=\; 0$

   Take, e.g., $a(n) = (n+1)(n+\frac{1}{2})$, $b(n) = 1$, $z = 4$.

4. equation for c:
$$(n+2)c(n+2) - (2n+3)c(n+1) + (n+1)c(n) \;=\; 0$$

   Polynomial solution: $c(n) = 1$

We have found
$$\frac{y(n+1)}{y(n)} \;=\; r(n) \;=\; z\frac{a(n)}{b(n)}\,\frac{c(n+1)}{c(n)} \;=\; (2n+1)(2n+2),$$

therefore $y(n) = (2n)!$ is a hypergeometric solution of equation (7).

6 Difference rings

Definition 5 *A difference ring is a pair (K, σ) where K is a commutative ring with multiplicative identity and $\sigma : K \to K$ is a ring automorphism. If, in addition, K is a field, then (K, σ) is a* difference field.

Example 5

- *$(K[x], \sigma)$ with $\sigma x = x + 1$, $\sigma|_K = \mathrm{id}_K$ is a difference ring.*
- *$(K(x), \sigma)$ with $\sigma x = x + 1, \sigma|_K = \mathrm{id}_K$ is a difference field.*
- *$(K^{\mathbb{N}}, E)$ is* not *a difference ring since the shift operator E is not injective on $K^{\mathbb{N}}$.*

For $a, b \in K^{\mathbb{N}}$ define $a \sim b$ if there is an $N \in \mathbb{N}$ such that $a(n) = b(n)$ for all $n \geq N$. The ring $\mathcal{S}(K) = K^{\mathbb{N}}/\sim$ of equivalence classes is the ring of *germs of sequences* . Let $\varphi : K^{\mathbb{N}} \to \mathcal{S}(K)$ be the canonical projection, and $\sigma : \mathcal{S}(K) \to \mathcal{S}(K)$ the unique automorphism of $\mathcal{S}(K)$ s.t. $\sigma \circ \varphi = \varphi \circ E$. Then $(\mathcal{S}(K), \sigma)$ is a difference ring.

To avoid problems with sequences with some undefined terms (such as those given by rational functions with nonnegative integer poles), and to have the advantage of working in a difference ring, we will henceforth work in $(\mathcal{S}(K), \sigma)$ rather than in $K^{\mathbb{N}}$ (but will still call its elements just "sequences" for short). Consequently we identify sequences which agree from some point on, and our statements may have a finite set of exceptions. The sets $K[n]$, $K(n)$, $\mathcal{H}(K)$ all naturally embed into $\mathcal{S}(K)$ (e.g., by mapping $f \in K(n)$ to the germ of $\langle 0, 0, \ldots, 0, f(N), f(N + 1), \ldots \rangle$ where N is an integer larger than any integer pole of f).

7 An Ore algebra of operators

Instead of linear recurrence operators with polynomial coefficients from $K[n]\langle E \rangle$, we will henceforth use *linear difference operators with rational coefficients* from the algebra $K(n)\langle \sigma \rangle$. The rule for composition of these operators follows from the commutation rule $\sigma \cdot r(n) = r(n + 1)\, \sigma$ for all $r \in K(n)$.

The identity

$$r(n)\, \sigma^k = \left(\frac{r(n)}{s(n + k - j)} \sigma^{k-j} \right) \cdot s(n)\, \sigma^j$$

describes how to perform right division of $r(n)\, \sigma^k$ by $s(n)\, \sigma^j$. Hence there is an *algorithm for right division* in $K(n)\langle \sigma \rangle$:

Theorem 8 *For $L_1, L_2 \in K(n)\langle \sigma \rangle$, $L_2 \neq 0$, there are $Q, R \in K(n)\langle \sigma \rangle$ such that*

- *$L_1 = QL_2 + R$,*

- $\operatorname{ord} R \; < \; \operatorname{ord} L_2$.

As a consequence, the right extended Euclidean algorithm (REEA) can be used to compute a greatest common right divisor (gcrd) and a least common left multiple (lclm) of operators in $K(n)\langle\sigma\rangle$, which is therefore a left Ore algebra. In particular, given $L_1, L_2 \in K(n)\langle\sigma\rangle$, REEA yields $S, T, U, V \in K(n)\langle\sigma\rangle$ such that

- $SL_1 + TL_2 \; = \; \operatorname{gcrd}(L_1, L_2)$,
- $UL_1 \; = \; VL_2 \; = \; \operatorname{lclm}(L_1, L_2)$.

Definition 6 *Let a be P-recursive. The unique monic operator $M_a \in K(n)\langle\sigma\rangle \setminus \{0\}$ of least order such that $M_a a = 0$ is the* minimal operator *of a.*

Example 6 *Let $h \in \mathcal{H}(K)$ where $\sigma h/h = r \in K(n)^*$. Then $M_h = \sigma - r$.*

Question 3. How to compute M_a for a given P-recursive a?

The outline of an algorithm for solving this problem is given on page 13.

Proposition 2 *Let a be P-recursive, and $L \in K(n)\langle\sigma\rangle$ such that $La = 0$. Then L is right-divisible by M_a.*

Proof: Divide L by M_a. Then:

$$L = QM_a + R \quad \Longrightarrow \quad La = QM_a a + Ra \quad \Longrightarrow \quad 0 = Ra \quad \Longrightarrow \quad R = 0$$

$\square$

Corollary 1 *Let $L \in K(n)\langle\sigma\rangle$ and $h \in \mathcal{H}(K)$ be such that $Lh = 0$. Then there is $Q \in K(n)\langle\sigma\rangle$ such that $L = Q(\sigma - r)$ where $r = \sigma h/h \in K(n)^*$.*

Hence there is a one-to-one correspondence between hypergeometric solutions of $Ly = 0$ and first-order right factors of L having the form $\sigma - r$ with $r \neq 0$.

Example 7 *(*Amer. Math. Monthly *problem no. 10375 – continued from Example 4)*

$$L \;=\; \sigma^2 - 2(2n+3)^2\sigma + 4(n+1)^2(2n+1)(2n+3)$$

We saw in Example 4 that $Ly = 0$ is satisfied by $y(n) = (2n)!$. Hence $L = QL_1$ where

$$\begin{aligned} L_1 &= \sigma - (2n+1)(2n+2), \\ Q &= \sigma - (2n+2)(2n+3). \end{aligned}$$

Operator factorization problem

Given: $L \in K(n)\langle\sigma\rangle$ and $r \in \mathbb{N}$
Find: all $L_1 \in K(n)\langle\sigma\rangle$ s.t.

- ord $L_1 = r$,
- $L = QL_1$ for some $Q \in K(n)\langle\sigma\rangle$

Suppose such L_1 exists, and let $y^{(1)}, y^{(2)}, \ldots, y^{(r)}$ be linearly independent solutions of $L_1 y = 0$ in $\mathcal{S}(K)$. The *Casoratian* $\mathrm{Cas}(y^{(1)}, y^{(2)}, \ldots, y^{(r)})$ is defined as

$$\det \begin{bmatrix} y^{(1)} & y^{(2)} & \cdots & y^{(r)} \\ \sigma y^{(1)} & \sigma y^{(2)} & \cdots & \sigma y^{(r)} \\ \vdots & \vdots & & \vdots \\ \sigma^{r-1} y^{(1)} & \sigma^{r-1} y^{(2)} & \cdots & \sigma^{r-1} y^{(r)} \end{bmatrix}.$$

Then:

1. $\mathrm{Cas}(y^{(1)}, y^{(2)}, \ldots, y^{(r)}) \in \mathcal{H}(K)$,
2. $\mathrm{Cas}(y^{(1)}, y^{(2)}, \ldots, y^{(r)}) = \mathrm{Cas}(L_1)$,
3. from L and r one can construct a linear recurrence with polynomial coefficients satisfied by $\mathrm{Cas}(L_1)$,
4. from L and r one can construct linear recurrences with polynomial coefficients satisfied by the coefficients of L_1, multiplied by $\mathrm{Cas}(L_1)$.

Outline of an algorithm to solve the operator factorization problem:

1. Construct a recurrence satisfied by $\mathrm{Cas}(L_1)$.
2. Find all hypergeometric solutions of this recurrence.
3. Construct recurrences satisfied by the coefficients of L_1.
4. Find all rational solutions of these recurrences.
5. Select candidates for L_1 which right-divide L.

Outline of an algorithm to find the minimal operator of a P-recursive sequence:

Given: $L \in K(n)\langle\sigma\rangle$ and $a \in \mathcal{S}(K)$ s.t. $La = 0$
Find: minimal operator M_a of a

`for` $r = 1, 2, \ldots, \mathrm{ord}\, L$ `do:`

find all monic $L_1 \in K(n)\langle\sigma\rangle$ of order r s.t. $\exists Q \in K(n)\langle\sigma\rangle$: $L = QL_1$

`for` every such L_1 `do:`

`if` $(L_1\, a)(n) = 0$ for $\mathrm{ord}\, Q$ consecutive values of n

`then` return L_1.

In the last line, the $\mathrm{ord}\, Q$ consecutive values of n must be greater than any integer root of the leading coefficient of Q.

8 D'Alembertian solutions

Write $\Delta = \sigma - 1$ for the forward difference operator as usual. If $y = a$ satisfies $Ly = 0$, then substituting $y \leftarrow az$ where z is a new unknown sequence yields

$$L' \, \Delta z = 0$$

where $\operatorname{ord} L' = \operatorname{ord} L - 1$. This is known as *reduction of order* or *d'Alembert substitution* [5]. By using this substitution repeatedly we obtain a set of solutions which can be written as nested indefinite sums with hypergeometric summands. These so-called *d'Alembertian sequences* include harmonic numbers and their generalizations, and play an important role in the theory of Padé approximations (cf. [17], [18]), in combinatorics (cf. [27], [34]) and in particle physics (cf. [1], [12], [2]).

8.1 Definition and representation

Definition 7 *A sequence $a \in \mathcal{S}(K)$ is* d'Alembertian *if there are first-order operators $L_1, L_2, \ldots, L_d \in K(n)\langle\sigma\rangle$ such that*

$$L_d \cdots L_2 L_1 \, a \;=\; 0. \tag{8}$$

We denote by $\mathcal{A}(K)$ the set of all d'Alembertian elements of $\mathcal{S}(K)$, and write $\operatorname{nd}(a)$ *for the least $d \in \mathbb{N}$ for which (8) holds (the* nesting depth *of a).*

Remark 2 *Let $a \in \mathcal{A}(K)$. Then:*

1. $\operatorname{nd}(a) = 0$ *if and only if $a = 0$,*
2. $\operatorname{nd}(a) = 1$ *if and only if $a \in \mathcal{H}(K)$.*

Example 8

- *Harmonic numbers $H(n) = \sum_{k=1}^{n} \frac{1}{k}$ are d'Alembertian because*

$$\left(\sigma - \frac{n+1}{n+2}\right)(\sigma - 1)\, H(n) \;=\; \left(\sigma - \frac{n+1}{n+2}\right) \frac{1}{n+1} \;=\; 0.$$

- *Derangement numbers $d(n) = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$ are d'Alembertian because*

$$(\sigma{+}1)(\sigma{-}(n{+}1))\, d(n) \;=\; (\sigma{+}1)(n{+}1)!\frac{(-1)^{n+1}}{(n+1)!} \;=\; (\sigma{+}1)(-1)^{n+1} \;=\; 0.$$

Notation: For $a \in \mathcal{S}(K)$ and $A \subseteq \mathcal{S}(K)$ we write

$$\begin{aligned} \Sigma a &= \{b \in \mathcal{S}(K);\ \Delta b = a\} \;=\; \sum_{k=0}^{n-1} a(k) + C, \\ \Sigma A &= \{b \in \mathcal{S}(K);\ \Delta b \in A\} \end{aligned}$$

where $C \in K$ is an arbitrary constant.

Remark 3

1. $\Delta + 1 = \sigma$,
2. $\Delta\Sigma \;=\; 1$,
3. $\sigma\Sigma \;=\; \Sigma + 1$,
4. $\Sigma 0 \;=\; K$.

Proposition 3 *Let $r \in K(n)$, $\sigma h = rh$, and $f \in \mathcal{S}(K)$. Then*

$$\{y \in \mathcal{S}(K);\ (\sigma - r)y = f\} \quad = \quad h\,\Sigma\frac{f}{\sigma h}.$$

Proof: Assume that $(\sigma - r)y = f$ and write $y = h\,z$. Then

$$f = (\sigma - r)y = (\sigma - r)h\,z = \sigma h\,\sigma z - r\,h\,z = \sigma h\,\Delta z,$$

hence $\Delta z = \frac{f}{\sigma h}$, so $z \in \Sigma\frac{f}{\sigma h}$ and $y = h\,z \in h\,\Sigma\frac{f}{\sigma h}$. – Conversely,

$$(\sigma - r)h\,\Sigma\frac{f}{\sigma h} = \sigma h\,\sigma\Sigma\frac{f}{\sigma h} - rh\,\Sigma\frac{f}{\sigma h} = \sigma h\,\Delta\Sigma\frac{f}{\sigma h} = f.$$

□

Corollary 2

$$\mathrm{Ker}\ (\sigma - r_d)\cdots(\sigma - r_2)(\sigma - r_1) \;=\; h_1\,\Sigma\frac{h_2}{\sigma h_1}\,\Sigma\frac{h_3}{\sigma h_2}\cdots\Sigma\frac{h_d}{\sigma h_{d-1}}\,\Sigma 0 \tag{9}$$

where $\sigma h_i = r_i h_i$ for $i = 1, 2, \ldots, d$.

It turns out that for any $L \in K(n)\langle\sigma\rangle$, the space of all d'Alembertian solutions of $Ly = 0$ is of the form

$$h_1\,\Sigma h_2\,\Sigma h_3\cdots\Sigma h_d\,\Sigma 0 \tag{10}$$

for some $d \leq \mathrm{ord}\,L$ and $h_1, h_2, \ldots, h_d \in \mathcal{H}(K)$.

Example 9 *(*Amer. Math. Monthly *problem no. 10375 – continued from Example 7)*

$$\begin{array}{rcl} L &=& \sigma^2 - 2(2n+3)^2\sigma + 4(n+1)^2(2n+1)(2n+3), \\ L &=& L_2 L_1, \\ L_1 &=& \sigma - (2n+1)(2n+2), \\ L_2 &=& \sigma - (2n+2)(2n+3). \end{array}$$

Since $L_1(2n)! = 0$ and $L_2(2n+1)! = 0$, it follows from (9) that

$$\begin{array}{rcl} \mathrm{Ker}\,L &=& (2n)!\,\Sigma\dfrac{(2n+1)!}{(2n+2)!}\Sigma\,0 \;=\; (2n)!\,\Sigma\dfrac{C}{n+1} \\ &=& (2n)!\left(\displaystyle\sum_{k=0}^{n-1}\frac{C}{k+1} + D\right) \;=\; C\,(2n)!\,H(n) + D\,(2n)! \end{array}$$

where $C, D \in K$ are arbitrary constants.

8.2 Closure properties of $\mathcal{A}(K)$

Definition 8 *For operators $L, R \in K(n)\langle\sigma\rangle$, denote by L/R the right quotient of* $\mathrm{lclm}(L, R)$ *by R.*

Remark 4 *Clearly,* $(L/R)R \;=\; \mathrm{lclm}(L, R) \;=\; (R/L)L.$

Example 10 *Let $L_1 = \sigma - r_1$ and $L_2 = \sigma - r_2$ be first-order operators. If $r_1 = r_2$ then $L_1/L_2 = L_2/L_1 = 1$. If $r_1 \neq r_2$ it is straightforward to check that*

$$\left(\sigma - \frac{\sigma r_1 - \sigma r_2}{r_1 - r_2} r_1\right)(\sigma - r_2) \;=\; \left(\sigma - \frac{\sigma r_1 - \sigma r_2}{r_1 - r_2} r_2\right)(\sigma - r_1)\,,$$

hence

$$L_1/L_2 \;=\; \sigma - \frac{\sigma r_1 - \sigma r_2}{r_1 - r_2} r_1, \qquad L_2/L_1 \;=\; \sigma - \frac{\sigma r_1 - \sigma r_2}{r_1 - r_2} r_2.$$

Lemma 1 *Let $L_1, L_2, \ldots, L_k, R \in K(n)\langle\sigma\rangle$ be monic first-order operators. Then there are monic operators $N_1, N_2, \ldots, N_k, M \in K(n)\langle\sigma\rangle \setminus \{0\}$ of order ≤ 1 such that*

$$M L_k L_{k-1} \cdots L_1 \;=\; N_k N_{k-1} \cdots N_1 R.$$

Proof: By induction on k.

$k = 1$: Take $N_1 = L_1/R$, $M = R/L_1$. Then $ML_1 = (R/L_1)L_1 = (L_1/R)R = N_1 R$.

$k > 1$: By inductive hypothesis, there are monic operators $N_1, N_2, \ldots, N_{k-1}, \tilde{M} \in K(n)\langle\sigma\rangle \setminus \{0\}$ of order ≤ 1 such that

$$\tilde{M} L_{k-1} L_{k-2} \cdots L_1 \;=\; N_{k-1} N_{k-2} \cdots N_1 R. \tag{11}$$

Take $N_k = L_k/\tilde{M}$, $M = \tilde{M}/L_k$. Then, using (11) in the last line, we obtain

$$\begin{aligned} M L_k L_{k-1} \cdots L_1 &= (\tilde{M}/L_k) L_k L_{k-1} L_{k-2} \cdots L_1 \\ &= (L_k/\tilde{M}) \tilde{M} L_{k-1} L_{k-2} \cdots L_1 \\ &= N_k \tilde{M} L_{k-1} L_{k-2} \cdots L_1 \;=\; N_k N_{k-1} N_{k-2} \cdots N_1 R. \end{aligned}$$

$\square$

Lemma 2 *Let $L_1, L_2, \ldots, L_k, R_1, R_2, \ldots, R_m \in K(n)\langle\sigma\rangle$ be monic first-order operators. Then there are monic operators $M_1, M_2, \ldots, M_m, N_1, N_2, \ldots, N_k \in K(n)\langle\sigma\rangle \setminus \{0\}$ of order ≤ 1 such that*

$$M_m M_{m-1} \cdots M_1 L_k L_{k-1} \cdots L_1 \;=\; N_k N_{k-1} \cdots N_1 R_m R_{m-1} \cdots R_1.$$

Proof: By induction on m.

$m = 1$: By Lemma 1 applied to $L_1, L_2, \ldots, L_k, R_1$, there are $N_1, N_2, \ldots, N_k$ and M_1 such that $M_1 L_k L_{k-1} \cdots L_1 \;=\; N_k N_{k-1} \cdots N_1 R_1$.

$m > 1$: By inductive hypothesis applied to $R_1, R_2, \ldots, R_{m-1}$, there are monic operators $M_1, M_2, \ldots, M_{m-1}, \tilde{N}_1, \tilde{N}_2, \ldots, \tilde{N}_k \in K(n)\langle\sigma\rangle \setminus \{0\}$ of order ≤ 1 such that

$$M_{m-1}M_{m-2}\cdots M_1 L_k L_{k-1}\cdots L_1 \;=\; \tilde{N}_k\tilde{N}_{k-1}\cdots\tilde{N}_1 R_{m-1}R_{m-2}\cdots R_1. \quad (12)$$

By Lemma 1 applied to $\tilde{N}_1, \tilde{N}_2, \ldots, \tilde{N}_k, R_m$, there are $N_1, N_2, \ldots, N_k$ and M_m such that

$$M_m\tilde{N}_k\tilde{N}_{k-1}\cdots\tilde{N}_1 \;=\; N_kN_{k-1}\cdots N_1R_m,$$

hence, by multiplying (12) with M_m from the left, we obtain

$$\begin{aligned} M_mM_{m-1}\cdots M_1L_kL_{k-1}\cdots L_1 &= M_m\tilde{N}_k\tilde{N}_{k-1}\cdots\tilde{N}_1R_{m-1}R_{m-2}\cdots R_1 \\ &= N_kN_{k-1}\cdots N_1R_mR_{m-1}\cdots R_1. \end{aligned}$$

□

Proposition 4 *$\mathcal{A}(K)$ is closed under addition.*

Proof: Let $a, b \in \mathcal{A}(K)$. Then there are monic first-order operators $L_1, L_2, \ldots, L_k$, $R_1, R_2, \ldots, R_m \in K(n)\langle\sigma\rangle$ such that

$$L_kL_{k-1}\cdots L_1a \;=\; R_mR_{m-1}\cdots R_1b \;=\; 0.$$

By Lemma 2, there are monic operators $M_1, \ldots, M_m, N_1, \ldots, N_k \in K(n)\langle\sigma\rangle \setminus \{0\}$ of order ≤ 1 such that

$$L \;:=\; M_mM_{m-1}\cdots M_1L_kL_{k-1}\cdots L_1 \;=\; N_kN_{k-1}\cdots N_1R_mR_{m-1}\cdots R_1.$$

Then $La = Lb = 0$, so $L(a+b) = 0$ and $a + b \in \mathcal{A}(K)$. □

Proposition 5 *$\mathcal{A}(K)$ is closed under multiplication.*

Proof: Let $a, b \in \mathcal{A}(K)$. We show that $ab \in \mathcal{A}(K)$ by induction on the sum of their nesting depths $\mathrm{nd}(a) + \mathrm{nd}(b)$.

a) $\mathrm{nd}(a) = 0$ or $\mathrm{nd}(b) = 0$: In this case one of a, b is 0, hence $ab = 0 \in \mathcal{A}(K)$.

b) $\mathrm{nd}(a), \mathrm{nd}(b) \geq 1$: By (10) we can write $a \in h_1\,\Sigma h_2\,\Sigma h_3\cdots\Sigma h_d\,\Sigma 0$ and $b \in g_1\,\Sigma g_2\,\Sigma g_3\cdots\Sigma g_e\,\Sigma 0$ where $h_i, g_j \in \mathcal{H}(K)$, $d = \mathrm{nd}(a)$, and $e = \mathrm{nd}(b)$. Let $a_1 = h_2\,\Sigma h_3\cdots\Sigma h_d\,\Sigma 0$ and $b_1 = g_2\,\Sigma g_3\cdots\Sigma g_e\,\Sigma 0$, so that $a \in h_1\,\Sigma a_1$ and $b \in g_1\,\Sigma b_1$ with $a_1, b_1 \subseteq \mathcal{A}(K)$, $\mathrm{nd}(a_1) < \mathrm{nd}(a)$ and $\mathrm{nd}(b_1) < \mathrm{nd}(b)$. Clearly $ha \in \mathcal{A}(K)$ whenever $h \in \mathcal{H}(K)$ and $a \in \mathcal{A}(K)$, hence it suffices to show that $(\sum a_1)g_1\sum b_1 \subseteq \mathcal{A}(K)$. Using the product rule of difference calculus

$$\Delta uv \;=\; u\Delta v + \Delta u\,\sigma v$$

and Remark 3 repeatedly, we obtain

$$\begin{aligned} &\Delta\left((\textstyle\sum a_1)g_1\sum b_1\right) \\ &= (\textstyle\sum a_1)g_1\Delta\sum b_1 + \Delta((\sum a_1)g_1)\sigma\sum b_1 \\ &= (\textstyle\sum a_1)g_1b_1 + ((\sum a_1)\Delta g_1 + a_1\sigma g_1)(\sum b_1 + b_1) \\ &= \Delta g_1(\textstyle\sum a_1)\sum b_1 + (g_1 + \Delta g_1)b_1\sum a_1 + a_1\sigma g_1(\sum b_1 + b_1) \\ &= \Delta g_1(\textstyle\sum a_1)\sum b_1 + \sigma g_1(a_1\sum b_1 + b_1\sum a_1 + a_1b_1). \end{aligned}$$

Assume first that $g_1 = 1$. Then $\Delta\left((\sum a_1)\sum b_1\right) = a_1\sum b_1 + b_1\sum a_1 + a_1b_1$. By inductive hypothesis and from Proposition 4 it follows that $a_1\sum b_1 + b_1\sum a_1 + a_1b_1 \subseteq \mathcal{A}(K)$. Therefore there are first-order operators $L_1, L_2, \ldots, L_k \in K(n)\langle\sigma\rangle$ such that

$$L_k L_{k-1}\cdots L_1 \Delta\left((\textstyle\sum a_1)\sum b_1\right) = 0,$$

hence $(\sum a_1)\sum b_1 \subseteq \mathcal{A}(K)$. In the general case, $\Delta g_1, \sigma g_1 \in \mathcal{H}(K)$ now implies $\Delta\left((\sum a_1)g_1\sum b_1\right) \subseteq \mathcal{A}(K)$. Again we conclude that $(\sum a_1)g_1\sum b_1 \subseteq \mathcal{A}(K)$. $\square$

Proposition 6 *$\mathcal{A}(K)$ is closed under σ and σ^{-1}.*

Proof: Let $a \in \mathcal{A}(K)$. Then there are monic first-order operators $L_1, L_2, \ldots, L_k \in K(n)\langle\sigma\rangle$ such that $L_k L_{k-1}\cdots L_1 a = 0$.

By Lemma 1 with $R = \sigma$, there are monic operators $N_1, N_2, \ldots, N_k, M \in K(n)\langle\sigma\rangle \setminus \{0\}$ of order ≤ 1 such that $M L_k L_{k-1}\cdots L_1 = N_k N_{k-1}\cdots N_1\sigma$. Hence

$$N_k N_{k-1}\cdots N_1 \sigma a = M L_k L_{k-1}\cdots L_1 a = 0,$$

so $\sigma a \in \mathcal{A}(K)$.

From $L_k L_{k-1}\cdots L_1 a = 0$ it follows that $L_k L_{k-1}\cdots L_1\sigma(\sigma^{-1}a) = 0$, hence $\sigma^{-1}a \in \mathcal{A}(K)$ as well. $\square$

Theorem 9 *$\mathcal{A}(K)$ is a difference ring.*

Proof: This follows from Propositions 4, 5 and 6. $\square$

Corollary 3 *$\mathcal{A}(K)$ is the least subring of $\mathcal{S}(K)$ which contains $\mathcal{H}(K)$ and is closed under σ, σ^{-1}, and Σ.*

Proof: Denote by $HS(K)$ the least subring of $\mathcal{S}(K)$ which contains $\mathcal{H}(K)$ and is closed under σ, σ^{-1}, and Σ.

By Corollary 2, every $a \in \mathcal{A}(K)$ is obtained from 0 by using Σ and multiplication with elements from $\mathcal{H}(K)$. Hence $\mathcal{A}(K) \subseteq HS(K)$.

Conversely, $\mathcal{A}(K)$ is closed under σ and σ^{-1} by Proposition 6, and under Σ by Corollary 2. Since $\mathcal{A}(K)$ is a subring of $\mathcal{S}(K)$ containing $\mathcal{H}(K)$, it follows that $HS(K) \subseteq \mathcal{A}(K)$. $\square$

Proposition 7 *$\mathcal{A}(K)$ is closed under multisection.*

Proof: Let $a \in \mathcal{A}(K)$. We show that any multisection of a belongs to $\mathcal{A}(K)$ by induction on the nesting depth $\mathrm{nd}(a)$ of a.

a) $\mathrm{nd}(a) = 0$: In this case $a = 0$, so the assertion holds.

b) $\mathrm{nd}(a) \geq 1$: By (10) we can write $a \in h_1\,\Sigma h_2\,\Sigma h_3\cdots\Sigma h_d\,\Sigma 0$ where $d = \mathrm{nd}(a)$ and $h_1, h_2, \ldots, h_d \in \mathcal{H}(K)$. Let $h = h_1$ and $b = h_2\,\Sigma h_3\cdots\Sigma h_d\,\Sigma 0$, so that $a \in h\,\Sigma b$ where $b \subseteq \mathcal{A}(K)$ and $\mathrm{nd}(b) < \mathrm{nd}(a)$.

Let $c \in \mathcal{S}(K)$, defined by $c(n) = a(mn + r)$ for all $n \in \mathbb{N}$, where $m, r \in \mathbb{N}$, $m \geq 2$, $0 \leq r < m$, be a multisection of a. Then for all $n \in \mathbb{N}$

$$\begin{aligned}
c(n) &= a(mn + r) \;=\; h(mn + r) \sum_{k=0}^{mn+r-1} b(k) \\
&= h(mn + r) \left(\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} b(mj + i) + \sum_{i=0}^{r-1} b(mn + i) \right) \\
&= h_{m,r}(n) \left(\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} b_{m,i}(j) + \sum_{i=0}^{r-1} b_{m,i}(n) \right)
\end{aligned}$$

where $h_{m,r}(n) = h(mn + r)$ and $b_{m,i}(n) = b(mn + i)$ for $0 \leq i < m$. Hence

$$c \;=\; h_{m,r} \left(\sum_{i=0}^{m-1} \Sigma b_{m,i} + \sum_{i=0}^{r-1} b_{m,i} \right)$$

where $h_{m,r} \in \mathcal{H}(K) \subseteq \mathcal{A}(K)$ by Proposition 1, and $b_{m,i} \in \mathcal{A}(K)$ by inductive hypothesis as a multisection of b. Since $\mathcal{A}(K)$ is closed under Σ, addition and multiplication, it follows that $c \in \mathcal{A}(K)$. $\square$

8.3 Finding d'Alembertian solutions

The following theorem provides a way to find d'Alembertian solutions of $Ly = 0$.

Theorem 10 *$Ly = 0$ has a nonzero d'Alembertian solution if and only if $Ly = 0$ has a hypergeometric solution.*

For a proof, see [10].

Outline of an algorithm for finding the space of all d'Alembertian solutions:

1. Find a hypergeometric solution h_1 of $Ly = 0$.
   If none exists then return 0 and stop.
2. Let $L_1 = \sigma - \frac{\sigma h_1}{h_1}$. Right-divide L by L_1 to obtain $L = QL_1$.
3. Recursively use the algorithm on $Qy = 0$. Let the output be a.
4. Return $h_1 \, \Sigma \frac{a}{\sigma h_1}$ and stop.

A much more general algorithm which finds solutions in $\Pi\Sigma^*$-difference extension fields of $(K(n), \sigma)$ is presented in [32]. For the relevant theory, see [33], [35], [36], [37].

9 Liouvillian solutions

Definition 9 *$\mathcal{L}(K)$ is the least subring of $\mathcal{S}(K)$ containing $\mathcal{H}(K)$, closed under*

- *σ, σ^{-1},*
- *Σ,*
- *interlacing of an arbitrary number of sequences.*

The elements of $\mathcal{L}(K)$ are Liouvillian sequences.

Example 11 *The sequence*

$$n!! \;=\; \begin{cases} 2^k k!, & n = 2k, \\ \frac{(2k+1)!}{2^k k!}, & n = 2k+1 \end{cases}$$

is Liouvillian (as an interlacing of two hypergeometric sequences).

The following theorem provides a way to find Liouvillian solutions of $Ly = 0$.

Theorem 11 *$Ly = 0$ has a nonzero Liouvillian solution if and only if $Ly = 0$ has a solution which is an interlacing of at most* $\operatorname{ord} L$ *hypergeometric sequences.*

For a proof, see [21]. For algorithms to find Liouvillian solutions, see [13], [25], [8], [14], [15], [24], [19], [20].

Theorem 12 *A sequence in $\mathcal{S}(K)$ is Liouvillian if and only if it is an interlacing of d'Alembertian sequences.*

This is proved in [31] as a corollary of the results of [21] obtained by means of Galois theory of difference equations. Here we give a self-contained proof based on closure properties of interlacings of d'Alembertian sequences.

Let $\Lambda(a_0, a_1, \ldots, a_{k-1})$, or $\Lambda_{j=0}^{k-1} a_j$, denote the interlacing of $a_0, a_1, \ldots, a_{k-1}$. By definition of interlacing we have

$$\left(\Lambda_{j=0}^{k-1} a_j\right)(n) \;=\; \Lambda(a_0, a_1, \ldots, a_{k-1})(n) \;=\; a_{n \bmod k}(n \operatorname{div} k)$$

for all $n \in \mathbb{N}$, where

$$n \operatorname{div} k \;=\; \left\lfloor \frac{n}{k} \right\rfloor, \quad n \bmod k \;=\; n - \left\lfloor \frac{n}{k} \right\rfloor k.$$

Denote temporarily the set of all interlacings of (one or more) d'Alembertian sequences by $AL(K)$. The goal is to prove that $AL(K) = \mathcal{L}(K)$.

Proposition 8 *$AL(K) \subseteq \mathcal{L}(K)$.*

Proof: Since $\mathcal{H}(K) \subseteq \mathcal{L}(K)$ and $\mathcal{L}(K)$ is a ring closed under Σ, we have $\mathcal{A}(K) \subseteq \mathcal{L}(K)$. Since $\mathcal{L}(K)$ is closed under interlacing, $AL(K) \subseteq \mathcal{L}(K)$. $\square$

Lemma 3 *$AL(K)$ is closed under addition and multiplication.*

Proof: Let $\odot$ denote either addition or multiplication in K and $\mathcal{S}(K)$. We claim that, for $k, m \in \mathbb{N}$ and $a_0, a_1, \ldots, a_{k-1}, b_0, b_1, \ldots, b_{m-1} \in \mathcal{A}(K)$, we have

$$\left(\Lambda_{j=0}^{k-1} a_j\right) \odot \left(\Lambda_{j=0}^{m-1} b_j\right) \;=\; \Lambda_{\ell=0}^{km-1} (a_{\ell,k,m} \odot b_{\ell,k,m}) \tag{13}$$

where for all $n \in \mathbb{N}$,

$$\begin{aligned} a_{\ell,k,m}(n) &= a_{\ell \bmod k}(mn + \ell \text{ div } k), \\ b_{\ell,k,m}(n) &= b_{\ell \bmod m}(kn + \ell \text{ div } m). \end{aligned}$$

Indeed,

$$\begin{aligned} &\left(\Lambda_{\ell=0}^{km-1} (a_{\ell,k,m} \odot b_{\ell,k,m})\right)(n) \\ &\qquad = \; a_{n \bmod km,k,m}(n \text{ div } km) \odot b_{n \bmod km,k,m}(n \text{ div } km) \;=\; u \odot v \end{aligned}$$

where

$$\begin{aligned} u &= a_{(n \bmod km) \bmod k}(m(n \text{ div } km) + (n \bmod km) \text{ div } k), \\ v &= b_{(n \bmod km) \bmod m}(k(n \text{ div } km) + (n \bmod km) \text{ div } m). \end{aligned}$$

From

$$\begin{aligned} (n \bmod km) \bmod k &= \left(n - \left\lfloor \frac{n}{km} \right\rfloor km\right) \bmod k \;=\; n \bmod k, \\ (n \bmod km) \bmod m &= \left(n - \left\lfloor \frac{n}{km} \right\rfloor km\right) \bmod m \;=\; n \bmod m, \end{aligned}$$

$$\begin{aligned} m(n \text{ div } km) + (n \bmod km) \text{ div } k &= m \left\lfloor \frac{n}{km} \right\rfloor + \left\lfloor \frac{n - \left\lfloor \frac{n}{km} \right\rfloor km}{k} \right\rfloor \;=\; \left\lfloor \frac{n}{k} \right\rfloor \\ &= n \text{ div } k, \\ k(n \text{ div } km) + (n \bmod km) \text{ div } m &= k \left\lfloor \frac{n}{km} \right\rfloor + \left\lfloor \frac{n - \left\lfloor \frac{n}{km} \right\rfloor km}{m} \right\rfloor \;=\; \left\lfloor \frac{n}{m} \right\rfloor \\ &= n \text{ div } m \end{aligned}$$

it follows that

$$\begin{aligned} u \odot v &= a_{n \bmod k}(n \text{ div } k) \odot b_{n \bmod m}(n \text{ div } m) \\ &= \left(\Lambda_{j=0}^{k-1} a_j\right)(n) \odot \left(\Lambda_{j=0}^{m-1} b_j\right)(n) \;=\; \left(\left(\Lambda_{j=0}^{k-1} a_j\right) \odot \left(\Lambda_{j=0}^{m-1} b_j\right)\right)(n), \end{aligned}$$

proving (13). By Proposition 7, the sequences $a_{\ell,k,m}$ and $b_{\ell,k,m}$ belong to $\mathcal{A}(K)$. Since $\mathcal{A}(K)$ is a ring, the right-hand side of (13) is an interlacing of d'Alembertian sequences, and hence so is the left-hand side. $\Box$

Lemma 4 *$AL(K)$ is closed under σ and σ^{-1}.*

Proof: Let $a_0, a_1, \ldots, a_{k-1}$ be d'Alembertian sequences. Then:

$$\begin{aligned}
\left(\sigma\left(\Lambda_{j=0}^{k-1} a_j\right)\right)(n) &= \left(\Lambda_{j=0}^{k-1} a_j\right)(n+1) \\
&= a_{(n+1) \bmod k}((n+1) \text{ div } k) \\
&= \begin{cases} a_{n \bmod k+1}(n \text{ div } k), & n \bmod k \neq k-1, \\ a_0(n \text{ div } k+1), & n \bmod k = k-1 \end{cases} \\
&= \begin{cases} a_{n \bmod k+1}(n \text{ div } k), & n \bmod k \neq k-1, \\ (\sigma a_0)(n \text{ div } k), & n \bmod k = k-1 \end{cases} \\
&= \left(\Lambda_{j=0}^{k-1} b_j\right)(n)
\end{aligned}$$

where

$$b_j = \begin{cases} a_{j+1}, & j \neq k-1, \\ \sigma a_0, & j = k-1. \end{cases}$$

By Proposition 6, $b_0, b_1, \ldots, b_{k-1}$ are d'Alembertian. So $\sigma\left(\Lambda_{j=0}^{k-1} a_j\right) = \Lambda_{j=0}^{k-1} b_j$ is an interlacing of d'Alembertian sequences.

Similarly,

$$\begin{aligned}
\left(\sigma^{-1}\left(\Lambda_{j=0}^{k-1} a_j\right)\right)(n) &= \left(\Lambda_{j=0}^{k-1} a_j\right)(n-1) \\
&= a_{(n-1) \bmod k}((n-1) \text{ div } k) \\
&= \begin{cases} a_{n \bmod k-1}(n \text{ div } k), & n \bmod k \neq 0, \\ a_{k-1}(n \text{ div } k-1), & n \bmod k = 0 \end{cases} \\
&= \begin{cases} a_{n \bmod k-1}(n \text{ div } k), & n \bmod k \neq 0, \\ (\sigma^{-1} a_{k-1})(n \text{ div } k), & n \bmod k = 0 \end{cases} \\
&= \left(\Lambda_{j=0}^{k-1} c_j\right)(n)
\end{aligned}$$

where

$$c_j = \begin{cases} a_{j-1}, & j \neq 0, \\ \sigma^{-1} a_{k-1}, & j = 0. \end{cases}$$

By Proposition 6, $c_0, c_1, \ldots, c_{k-1}$ are d'Alembertian. So $\sigma^{-1}\left(\Lambda_{j=0}^{k-1} a_j\right) = \Lambda_{j=0}^{k-1} c_j$ is an interlacing of d'Alembertian sequences. □

Lemma 5 *$AL(K)$ is closed under Σ.*

Proof: Let $a_0, a_1, \ldots, a_{k-1}$ be d'Alembertian sequences. We claim that

$$\Sigma\left(\Lambda_{j=0}^{k-1} a_j\right) = \Lambda_{j=0}^{k-1}\left(\sum_{i=0}^{j-1} \sigma \Sigma a_i + \sum_{i=j}^{k-1} \Sigma a_i\right). \tag{14}$$

Indeed, for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\left(\Sigma\left(\Lambda_{j=0}^{k-1} a_j\right)\right)(n) \\
&= \sum_{\ell=0}^{n-1}\left(\Lambda_{j=0}^{k-1} a_j\right)(\ell) \quad = \quad \sum_{\ell=0}^{n-1} a_{\ell \bmod k}(\ell \operatorname{div} k) \\
&= \sum_{i=0}^{(n-1) \bmod k} \sum_{j=0}^{\left\lfloor \frac{n-1}{k} \right\rfloor} a_i(j) \quad + \sum_{i=(n-1) \bmod k+1}^{k-1} \sum_{j=0}^{\left\lfloor \frac{n-1}{k} \right\rfloor - 1} a_i(j) \qquad (15) \\
&= \sum_{i=0}^{n \bmod k-1} \sum_{j=0}^{\left\lfloor \frac{n}{k} \right\rfloor} a_i(j) \quad + \sum_{i=n \bmod k}^{k-1} \sum_{j=0}^{\left\lfloor \frac{n}{k} \right\rfloor - 1} a_i(j) \qquad (16) \\
&= \sum_{i=0}^{n \bmod k-1} (\sigma \Sigma a_i)(n \operatorname{div} k) \quad + \sum_{i=n \bmod k}^{k-1} (\Sigma a_i)(n \operatorname{div} k) \\
&= \left(\Lambda_{j=0}^{k-1}\left(\sum_{i=0}^{j-1} \sigma \Sigma a_i + \sum_{i=j}^{k-1} \Sigma a_i\right)\right)(n),
\end{aligned}$$

proving (14). Here equality in (15) follows by mapping each $\ell \in \{0, 1, \ldots, n-1\}$ to the pair $(i, j) = (\ell \bmod k,\ \ell \operatorname{div} k)$ and summing over all the resulting pairs, and equality in (16) follows by noting that when $n \bmod k \neq 0$, we have

$$\begin{aligned}
(n-1) \bmod k &= n \bmod k - 1, \\
(n-1) \operatorname{div} k &= n \operatorname{div} k,
\end{aligned}$$

while for $n \bmod k = 0$, both (15) and (16) are equal to $\sum_{i=0}^{k-1} \sum_{j=0}^{\frac{n}{k}-1} a_i(j)$.

Since $\mathcal{A}(K)$ is closed under Σ, σ and addition, the right-hand side of (14) is an interlacing of d'Alembertian sequences, and hence so is the left-hand side. □

Lemma 6 *$AL(K)$ is closed under interlacing.*

Proof: An arbitrary interlacing can be obtained by using addition, shifts, and interlacing of zero sequences with a single non-zero sequence by the formula

$$\Lambda(a_0, a_1, \ldots, a_{k-1}) \;=\; \sum_{i=0}^{k-1} \sigma^i \Lambda(0, 0, \ldots, 0, a_{k-1-i}).$$

Hence, by Propositions 3 and 4, it suffices to show that $AL(K)$ is closed under interlacing of zero sequences with a single non-zero sequence from $AL(K)$. But this is immediate: Let $a_0, a_1, \ldots, a_{k-1}$ be d'Alembertian sequences. Then the interlacing of m zero sequences with $\Lambda(a_0, a_1, \ldots, a_{k-1})$

$$\begin{aligned}
&\Lambda(0, 0, \ldots, 0, \Lambda(a_0, a_1, \ldots, a_{k-1})) \\
&\quad = \; \Lambda(0, 0, \ldots, 0, a_0, 0, 0, \ldots, 0, a_1, \ldots, 0, 0, \ldots, 0, a_{k-1})
\end{aligned}$$

is an interlacing of $mk + k$ d'Alembertian sequences. $\square$

Proof of Theorem 12. By Proposition 8, it suffices to show that $\mathcal{L}(K) \subseteq AL(K)$. This is true since by Lemmas 3 – 6, $AL(K)$ is a subring of $\mathcal{S}(K)$ containing $\mathcal{H}(K)$ and closed under σ, σ^{-1}, Σ and interlacing, while $\mathcal{L}(K)$ is the least such ring. $\square$

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