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Phase transition phenomena in random discrete structures

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Abstract

The paper surveys most characteristic features of the phase transition phenomenon in various models of random discrete structures.

1. Introduction

In 1960 Erdős and Rényi, in a paper on the evolution of random graphs, noticed a spectacular phenomenon called 'the double jump': near the point when the expected average degree of a random graph is equal to one the 'typical' structure of such a graph rapidly changes and the giant component suddenly emerges. Their article became later a cornerstone of random graph theory, but, somewhat surprisingly, the problem of the abrupt transformation of a random graph was taken up only 24 years later by Bollobás, who initiated studies of the structure of a random graph near the 'critical point'. Since then, the critical behaviour of a random graph and other random discrete structure attracted much attention of combinatorists and probabilistists — in this paper we would like to sketch a general picture which emerges from their works. Thus, we will not survey all known results concerning the phase transition (although in the last chapter we give the reader some hints where to find them) but rather overview main models of random discrete sturctures trying to answer what each of them contributes to our understanding of this phenomenon.

The structure of this paper is the following. First we propose some terminology which can be used for a characterization of the phase transition in different models of random discrete structures. Then we give a brief description of this phenomenon in random graphs distinguishing three main features of the critical behaviour: the average degree heuristic, the fact that near the point of the phase transition the size of the second largest component attains its maximum, and the symmetry rule. Based on this characterization we examine various models of discrete structures trying to

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capture the nature of the phase transition in each particular case. Finally, in the last section, we sketch more specialized results concerning critical phenomena.

2. Generalities

Despite of numerous articles on the phase transitions in different kinds of random discrete structures, no consistent terminology concerning this phenomenon has been established yet. In this part of the paper, we try to generalize basic notions concerning random graphs in such a way that they can also be applied to other models of random discrete structures like for example, digraphs or hypergraphs.

A *discrete structure* of size n is a triple $D=(V,E,R)$, where V is the set of vertices (in most cases identified with the set $[n]=\{1,2,\dots,n\}$), the set of edges E is a family of subsets of V (where elements of an edge might be ordered) and R is an equivalence relation defined on V, whose abstract classes we shall call *components* of D. Since, clearly, relation R should be determined by the set of edges it is convenient to write R as $r_V(E)$, where r_V is a function which maps the set $\{E: E\subseteq 2^V\}$ into the set of all partitions of V.

A *randomized discrete structure* is a probability space whose underlying set consists of a finite family of discrete structures $\mathscr{D}=\{(V_i,E_i,R_i)\}_{i\in I}=\{(V_i,E_i,r_{V_i}(E_i))\}_{i\in I}$. Typically, a randomized discrete structure $\mathscr{D}$ is characterized by a set of parameters $(n,x^1,x^2,\dots,x^m)$, where n is related to the order of sets V_n, and other parameters $x^1,x^2,\dots,x^m$ may vary as functions of n (for example, in the case of random graphs, n denotes the number of vertices in a graph and as an additional parameter one typically takes either the number of edges M or the probability p that a given pair of vertices appear as an edge — for details see Section 3.1). Finally, a *random discrete structure* is defined as a sequence of randomized discrete structures

$$\{\mathscr{D}(n,x^1,x^2,\dots,x^m)\}_{n=0}^{\infty}=\{\mathscr{D}(n,\boldsymbol{x})\}_{n=0}^{\infty}.$$

Let $\mathscr{A}$ be a property of a discrete structure (e.g. the structure is connected, there exists precisely one largest component, etc). We say that the random structure $\{\mathscr{D}(n,x^1,x^2,\dots,x^m)\}_{n=0}^{\infty}$ has property $\mathscr{A}$ almost surely (a.s.) if

$$\lim_{n\to\infty}\operatorname{Prob}\{\mathscr{D}(n,x^1,x^2,\dots,x^m)\text{ has }\mathscr{A}\}=1.$$

(Note that we restrict ourselves to probabilistic structures which are large but finite. In particular, we shall say nothing about the exciting theory of transition phenomena in the infinite low-dimensional lattices developed in the percolation theory.)

For a random discrete structure $\{\mathscr{D}(n,\boldsymbol{x})\}_{n=0}^{\infty}$ and functions $s=s(n)$ and $s'=s'(n)$ define $P(n,\boldsymbol{x};s,s')$ as the limes superior of the probability that $\{\mathscr{D}(n,\boldsymbol{x})\}_{n=0}^{\infty}$ contains at least one component on $r(n)$ vertices, where $s(n)\leqslant r(n)\leqslant s(n)+s'(n)$. We say that for $\boldsymbol{x}=\boldsymbol{x}(n)$ the sequence of components of $\{\mathscr{D}(n,\boldsymbol{x})\}_{n=0}^{\infty}$ is *smooth* if there exists a constant A such that for every functions $s_1(n)\geqslant s_0(n)\geqslant A$ and $s'(n)$ we have

$P(n, x, s_1, s') \leqslant P(n, x, s_0, s')$, i.e. the probability of the existence of components 'essentially' decreases. On the other hand, we say that for $x = x(n)$ the sequence of components of $\{\mathscr{D}(n, x)\}_{n=0}^{\infty}$ has *gaps* if there exist functions $t_2(n) \geqslant t_1(n) \geqslant t_0(n)$ and $t'(n)$ for which $P(n, x, t_2, t') > 0$, $P(n, x, t_0, t') > 0$ but $P(n, x, t_1, t') = 0$. If the sequence of components of $\{\mathscr{D}(n, x)\}_{n=0}^{\infty}$ contains gaps, then each its component of the size larger than t_1 is called *giant* — note that in such a case, with probability bounded away from zero, $\{\mathscr{D}(n, x)\}_{n=0}^{\infty}$ contains at least one giant component.

We are now ready to define the point of the phase transition. For the sake of simplicity we restrict ourselves to the most common case when a random discrete structure has only two parameters, n and x.

We say that the *phase transition* occurs for a random discrete structure $\{\mathscr{D}(n, x)\}_{n=0}^{\infty}$ if for some function $x(n)$ and every constant $\varepsilon > 0$ the sequence of components of $[\mathscr{D}(n, (1-\varepsilon)x)\}_{n=0}^{\infty}$ is smooth whereas the sequence of components of $\{\mathscr{D}(n, (1+\varepsilon)x)\}_{n=0}^{\infty}$ contains gaps. Note that the function $x(n)$ is determined only up to the factor of $1 + o(1)$, where $o(1)$ denotes the quantity which tends to 0 as $n \to \infty$. In order to be more precise let us define a *separating pair* as a pair of functions $(y_-, y_+) = (y_-(n), y_+(n))$ such that for some function $x = x(n) = x(n; y_-, y_+)$ and for every function $\omega(n) \to \infty$ the sequence of components of $\{\mathscr{D}(n, x - \omega y_-)\}_{n=0}^{\infty}$ is smooth but the sequence of components of $\{\mathscr{D}(n, x + y_+/\omega)\}_{n=0}^{\infty}$ contains gaps. A separating pair (y_-, y_+) is *minimal* if no pair of functions (z_-, z_+), with either $z_- = o(y_-)$ or $z_+ = o(y_+)$, is separating. Then the function $x_0 = x(n; y_-, y_+)$ is called *critical function* or the *point* (!) *of the phase transition*. Note that both y_- and y_+ are determined only up to a constant factor. On the other hand, as a critical function x_0 we may take any function $x'(n)$ for which $x_0 - x' = O(\min\{y_-, y_+\})$. Typically, in a separating pair we may set $y_- = y_+ = y$. In such a case we say that the phase transition is *symmetric* and call the ratio y/x its *sensibility*.

Thus, if $x_0 = x_0(n)$ is the point of the phase transition with a separating pair (y_-, y_+), we can distinguish three modes of the behaviour of the random discrete structure. In the *subcritical phase*, when $(x - x_0)/y_- \to -\infty$, the probability of the existence of a component decreases monotonically with its size, in the *critical phase*, when $-C_- y_- \leqslant x - x_0 \leqslant C_+ y_+$ for some constants $C_-, C_+ \geqslant 0$, first irregularities in the sequence of components appear, and finally, in the *supercritical phase*, when $(x - x_0)/y_+ \to \infty$, the giant component (or components) emerges. Unfortunately, in many cases, the *critical behaviour* of random discrete structure (i.e. its structure for $x(n) = (1 + o(1))x_0$) is very hard to analyse, and we can only characterize properties of the random structure in *early subcritical phase*, when $x(n) < (1-\varepsilon)x_0(n)$ for some positive constant ε and *late supercritical phase*, for $x(n) > (1+\varepsilon)x_0(n)$.

One can also give an analogous characterization of the phase transition for multiparameter random discrete structures. In such a case, the phase transition occurs if there exists a function $f(n, x^1, x^2, \ldots, x^m)$ which divides the space of all possible values of parameters into two parts: the subcritical region defined by the inequality $f(n, x^1, x^2, \ldots, x^m) < 1 - \varepsilon$ for some constant $\varepsilon > 0$, in which the sequence of components is smooth, and the supercritical one, with gaps in the sequence of components, for which $f(n, x^1, x^2, \ldots, x^m) > 1 + \varepsilon$.

3. The phase transition in random graphs

Random graphs $G(n,p)$ and $G(n,M)$ are the simplest and, by far, the most widely studied random discrete structures. In this section we give a brief description of the most characteristic features of the phase transition phenomenon in $G(n,p)$ and $G(n,M)$, which will later serve us a basis for the analysis of the critical behaviour of more sophisticated models of random discrete structures.

3.1. Models of random graphs

The two most often used models of random graphs are $G(n,p)$ and $G(n,M)$, or, more formally, $\{G(n,p(n))\}_{n=0}^{\infty}$ and $\{G(n,M(n))\}_{n=0}^{\infty}$. $G(n,p)$ is defined as a graph with vertex set $[n]=\{1,2,\ldots,n\}$ in which a pair of vertices appears as an edge with probability p, independently for each such pair. $G(n,M)$ can be viewed as a graph picked at random from the family of all

$$\binom{\binom{n}{2}}{M}$$

graphs with vertex set $[n]$ and M edges. More precisely, $G(n,p)$ is the probability space of all graphs on $[n]$ vertices, in which each graph G with $e(G)$ edges appears with probability

$$p^{e(G)}(1-p)^{\binom{n}{2}-e(G)},$$

while $G(n,M)$ denotes the probability space consisting of graphs with vertex set $[n]$ and M edges, in which each element is equally likely.

These two models are closely related to each other (as a matter of fact, $G(n,M)$ is the same as $G(n,p)$ under the condition that the number of edges of $G(n,p)$ is equal to M) and all results stated below (with two exceptions mentioned in Section 5) for one model remain true for the other one, provided $M=\lfloor p\binom{n}{2}\rfloor$.

Random graphs $G(n,p)$ and $G(n,M)$ can also be defined in a 'dynamic' way. A random graph process

$$\{G(n,M)\}_{M=0}^{\binom{n}{2}}$$

is the Markov chain which starts with the empty graph, and for $1\leqslant M\leqslant\binom{n}{2}$, $G(n,M)$ is a graph with M edges obtained from $G(n,M-1)$ by addition of a new edge, chosen randomly from all $\binom{n}{2}-M+1$ available pairs. Similarly, if $\{X_{i,j}\}_{i,j\in[n]}$ is a family of independent random variable with uniform distribution over an interval $(0,1)$, then $G(n,p)$ can be identified with the graph on $[n]$ with the set of edges $\{\{i,j\}: X_{i,j}<p\}$, i.e. we may consider $G(n,p)$ as a stage of certain Markov process $\{G(n,p)\}_{p\in(0,1)}$.

3.2. The phase transition in $G(n, M)$ — main results

In 1960 Erdős and Rényi, in their seminal work on the random graph theory [7], proved that in $G(n, M)$ the phase transition takes place when $M=(1+o(1))n/2$. More precisely, they showed that if $M=cn/2$ and c is a constant smaller than 1 then $G(n, M)$ almost surely consists of trees and unicyclic components of the size $O(\log n)$, whereas for $M=cn/2, c>1$-a.s. $G(n, M)$ contains a giant component with $(1+o(1))\alpha n$ vertices, where $\alpha=\alpha(c)$ is a constant defined as the positive root of the equation

$$\exp(-c\alpha)+\alpha=1. \qquad (*)$$

Furthermore, the giant component is unique and all other components are a.s. trees and unicyclic components with at most $O(\log n)$ vertices.

Erdős and Rényi proved also that the largest tree in $G(n, n/2)$ is a.s. of the order $n^{2/3}$ and called the phenomenon occurring near the point $M\sim n/2$ the 'double jump' (the size of the largest component 'jumps' from $\Theta(\log n)$ to $\Theta(n^{2/3})$ and then to $\Theta(n)$). But the addition of a single edge to a graph cannot increase the size of the largest component more than twice so the 'jump' phenomenon should disappear when one can analyse in detail the structure of $G(n, M)$ for $M\sim n/2$. Thus, we use rather the name 'phase transition', introduced by Stepanov [30] who first noticed striking similarities between the behaviour of components of $G(n, M)$ for $M\sim n/2$, and the phase transition phenomenon studied in the statistical mechanics.

The problem of the critical behaviour of $G(n, M)$ has remained basically untouched for almost a quarter of the century. A breakthrough came with the paper of Bollobás [3] (see also his monograph [4]), in which he showed that when $M=n/2+s$ and $s>n^{2/3}\sqrt{\log n}$ then a.s. $G(n, M)$ consists of a giant 'dense' component on $(4+o(1))s$ vertices and trees and unicyclic components of the size at most $n^{2/3}$ each. Later Łuczak [14] improved this result a little bit showing that for the existence of the giant component of size $(4+o(1))s$ it is enough to assume that $sn^{-2/3}\to\infty$, and that in such a case the size of the largest among 'small' sparse components is a.s.

$$l(n)=l(n; s)=(0.5+o(1))n^2s^{-2}\log s^3n^{-2}. \qquad (**)$$

The subcritical phase was somewhat easier to study — it was shown in papers of Bollobás [3], Kolchin [13] and Łuczak [14] that if $M=n/2-s$, where $sn^{-2/3}\to\infty$ then a.s. $G(n, M)$ consists of trees, the largest of which is of the size $l(n)$ given by (**), and unicyclic components with $O(n^2/s^2)$ vertices each. Furthermore, it can be shown that the sequence of components is smooth (although such a theorem has not being stated explicitly it follows straightforwardly from results of Bollobás [3, 4], Kolchin [14] or Łuczak [14]).

Finally, it was shown by Łuczak et al. [22] that if $M=n/2+O(n^{2/3})$ then the sequence of components of $G(n, M)$ contains no gaps. One can also prove that for such M the sequence of components is no longer smooth. Thus, at the point $M(n)=n/2$ the symmetric phase transition takes place, with sensibility equal to

$n^{-1/3}$ (one can set $x(n)=M(n)=n/2$ and $y_-(n)=y_+(n)=n^{2/3}$). The same result can be shown also for $G(n,p)$.

3.3. *Characteristic features of the phase transition*

In this section we shall examine more closely some of the results described above, seeking for general rules which, possibly, might be useful in the analysis of the phase transition phenomena in other models of the discrete structures.

First, let us try to identify the point of the phase transition. One can do it easily using the theory of branching processes. Indeed, let $p=c/n$ and w be a vertex of $G(n,p)$. Generate all edges of $G(n,p)$ adjacent to w finding all its neighbours $w_1, w_2, \ldots, w_m$. Mark these vertices as used. Among all unused vertices pick all neighbours $w_{1,1}, w_{1,2}, \ldots, w_{1,m_1}$ of w_1, mark them as used, generate all unused neighbours of w_2, those of w_3 and so on. After that examine vertices which belong to next generations, until the process dies, i.e. a spanning tree of the component of $G(n,p)$ containing w is constructed.

Clearly, in each step the number of new neighbours has a binomial distribution with parameters l,p, where l denotes the number of vertices which have not been used so far. Thus, if $c<1$, the expected size of the component can be bounded from above by the total number of descendents in a branching process in which the number of children has a binomial distribution with parameters n,p, and, thus, expectation $np=c<1$. Since for such a process the probability that the total number of descendants is K decreases exponentially with K, so the size of the largest component in $G(n,p)$ is bounded from above by $O(\log n)$.

On the other hand, if $c=np>1$, then the probability that the size of component containing w is less than, say, $\sqrt{n}$, is smaller than the probability that the total number of the descendants in the branching process in which the number of children of a single particle is binomially distributed with parameters $(n-\sqrt{n},p)$ is smaller than $\sqrt{n}$. The probability $\alpha=\alpha(c)$ that such a process does not die out is given by Eq. (*), so the expected number of vertices which belong to components having more than $\sqrt{n}$ vertices is αn (moreover, one can also rather easily show that such a component is unique).

It seems that the similar argument can be applied to other random discrete structures, provided that the number of neighbours of descendants of $w_{i,j}$ does not depend too much on the structure of the tree generated so far. Thus, we get the following *average degree heuristic* for identifying the point of the phase transitions in a random discrete structure:

The phase transition occurs when the expected number of neighbours of a given vertex is close to one.

In order to study the critical behaviour of a random graph it is better to view it as a stage of random graph process

$$\{G(n,M)\}_{M=0}^{\binom{n}{2}}.$$

Suppose that we add edges to the empty graph, one by one. Then, at the beginning the graph is very sparse and consists of a large number of small components. When we keep adding more edges, large components grow by 'absorbing' small ones. This process continues until, at some stage, largest components are so big that new edges can join them together, so they rapidly merge into one giant component which soon starts to absorb the largest from the remaining 'small' components. Thus, in the first period of the process, large components grow slow and smoothly acquiring vertices mainly from very small components. At the point of the phase transition all large components cluster together forming one component so the gap emerges, moreover, this giant component 'seizes' the largest from the remaining components before they have time to increase its size. Hence, the size of the second largest component decreases and the gap increases.

Since the above mechanism seems to work not only for $G(n, M)$ but for many similar situations when a random discrete structure is generated 'dynamically', we state it as the *maximization rule.*

The size of the second largest component attains maximum at the point of the phase transition.

Let us note that this rule suggests that, despite some pathological cases, a.s. in the supercritical phase there exists only one giant component.

If we compare the size of the largest component when $M=n/2-s$ with the size of the second largest component for $M=n/2+s$ we see that this two values are close to each other (both of them are given by $(**)$). One can easily see the reason behind that. If we delete from $G(n, M)$, $M=n/2+s$, the largest component, which has roughly $4s$ vertices and $4s$ edges (although it is 'dense' it is not very dense, see Section 5), we get the graph with $n' \sim n \sim 4s$ vertices and $M' \sim M-4s \sim n/2-3s=n'/2-s$ edges. But all such graphs are equiprobable, provided they do not contain components of size larger than $4s$ (which is quite unlikely). Thus, if we remove from $G(n, n/2+s)$ the giant component we get something very similar to $G(n', n'/2-s)$. Hence, we obtain the following *symmetry rule.*

If we remove the giant component from the random discrete structure in the supercritical phase the remaining structure has similar properties as the random discrete structure at some earlier moment of the subcritical phase.

The rigorous statement of this principle was given in [14] but the relation between sub- and supercritical phases has existed as a folklore for a long time, and some of its consequences were already used in earlier works (see e.g. Bollobás [3]).

Note that in the heuristic leading to the symmetry rule we treated $G(n, M)$ not as the stage of the random graph process but as the graph chosen at random from the family of all graphs with n vertices M edges. Thus, each of three principles governing the phase transition reflects a different aspect of random graph $G(n, M)$ — identifying the critical function we use the equivalence of $G(n, M)$ and $G(n, p)$, the maximization rule is based on the 'dynamic' interpretation of $G(n, M)$, while the symmetry rule is implied by the fact that we may view $G(n, M)$ 'statically'.

4. The phase transition in non-standard models of random discrete structure

In this section we try to examine different models of random discrete structures in order to capture their most characteristic features. We start with random hypergraphs and digraphs, where the phase transition looks very much the same as in the random graph case. Then we present results concerning the random cube, the model very important for the theory (as the intermediate stage between random graph theory and percolation theory) and widely applied in computer science. After that we mention a spectacular phenomenon which occurs when we look at k-edge connected components of $G(n, M)$ — the giant k-edge connected component is created in a single step of the random graph process. Other peculiarities can be observed for a random forest: depending on the model we may violate the average degree heuristic, the maximization rule or the symmetry rule. Rather special kind of the symmetry rule holds for a sharp non-symmetric phase transition in an evolutionary model of a random mappings. We conclude this section with the description of component structure of an unlabelled random graph — an example of a model where no phenomenon similar to the phase transition can be observed.

4.1. Random hypergraphs

A random hypergraph is a natural generalization of a random graph, where instead of edges we generate subsets of an underground set with more than two elements. Thus, $G(n; p_2, \dots, p_k)$ is a hypergraph with vertex set $[n]$ in which subsets with i elements, $2 \leqslant i \leqslant k$, appear as hyperedges with probability $p_i = p_i(n)$, independently for each subset (we assume that k which bounds from above the size of a hyperedge does not depend on n). Such a model was studied by Schmidt-Pruzan and Shamir [27], who showed that if $\sum_{i=2}^{k}(i-1)p_i = c < 1$ then a.s. all components of $G(n; p_2, \dots, p_k)$ are of size $O(\log n)$, whereas when $\sum_{i=2}^{k}(i-1)p_i = c > 1$ the giant component of size αn appears (with $\alpha = \alpha(c)$ given by $(*)$) and other components consist of $O(\log n)$ vertices each. Note that the expression $\sum_{i=2}^{k}(i-1)p_i$ follows from the average degree heuristic: if we fix a vertex w of a random hypergraph, $\sum_{i=2}^{k}(i-1)p_i$ is the expected number of vertices which belong to the same hyperedge as w.

The critical behaviour of $G(n; p_2, \dots, p_k)$ also resembles those of $G(n, M)$, although proofs are much more technical. In particular, Karoński and Łuczak [10] show that the phase transition in a random hypergraph is symmetric with sensibility $n^{-1/3}$ and that both maximization and symmetry rules apply.

Let us mention that the same remains true for a model of a random hypergraph analogous to $G(n, M)$. Furthermore, in many applications one can replace the hyperedge of a hypergraph by a clique so $G(n; p_1, p_2, \dots, p_k)$ can be also viewed as a special model of a random graph in which edges are generated randomly but not independently.

4.2. Random digraphs

Let $\vec{D}(n,p)$ denote a digraph with the vertex set $[n]$ in which probability that (i,j) is an arc equals p, independently for each pair $1 \leqslant i \leqslant j \leqslant n$. Clearly, in this case we shall be interested in the behaviour of *strongly connected* components of $\vec{D}(n,p)$. The existence of the phase transition in such a model was shown independently by Karp [11] and Łuczak [15] who noticed that if $pn=c<1$ then a.s. all non-trivial strongly connected components of $\vec{D}(n,p)$ are cycles, whose length is bounded in probability, whereas when $np=c>1$ digraph $\vec{D}(n,p)$ consists of the giant component of size larger than $\beta(c)n$ and some number of small components of bounded size. (Note that the underlying undirected graph of $\vec{D}(n,p)$ can be identified with $G(n,2p-p^2)$, so when $1/2<np<1$ all strongly connected components of $\vec{D}(n,p)$ are small but in the underlying graph the giant component already exists.) In [11] Karp studied also the critical behaviour of $\vec{D}(n,p)$ — it turns out to be somewhat similar to that of $G(n,p)$. He proved that if $pn=1-\varepsilon$ and $\varepsilon(n/\log^2 n)^{1/3}\to\infty$ then the sequence of components is smooth and all components are either vertices or cycles of length $O(1/\varepsilon)$, whereas for $pn=1+\varepsilon$, $\varepsilon(n/\log^2 n)^{1/3}\to\infty$ but $\varepsilon(n)\to 0$, a.s. the giant component has size $(4+o(1))\varepsilon^2 n$ and all other non-trivial components are cycles of size $O(1/\varepsilon)$. It seems natural to conjecture that this result remains true if we assumed only that $\varepsilon n^{1/3}\to\infty$, i.e. that the sensibility of the phase transition is, as it was in case of an undirected random graphs, $n^{-1/3}$. Since the critical behaviour of the model is quite similar to that of a random graph, both maximization and symmetry rules are expected to hold also in this case.

The fact that arcs are directed gives us a chance to introduce a multi-parameter model of random digraph. Let $\vec{D}(n;r,s,t)$ be a digraph on $[n]$ such that for a given pair i,j, $1 \leqslant i<j \leqslant n$, with probability r/n pair (i,j) is an arc of $\vec{D}(n;r,s,t)$, (j,i) is an arc with probability s/n, with probability t/n both arcs (i,j) belong to $\vec{D}(n;r,s,t)$ and, finally, the probability that there are no arcs between i and j is equal to $1-(r+s+t)/n$. Łuczak and Cohen [20] found such a function f that if $f(r,s,t)<1$ then a.s. all non-trivial components of $\vec{D}(n;r,s,t)$ are cycles of size $O(1)$, while for r,s,t such that $f(r,s,t)>1$ a.s. $\vec{D}(n;r,s,t)$ consists of the giant component which contains a positive fraction of all vertices and some number of components of bounded size. Nevertheless, the structure of $\vec{D}(n;r,s,t)$ near the critical surface seems to be rather hard to study.

4.3. Random cube

The n-dimensional cube Q_n is a graph whose set of vertices consists of 0–1 sequences of length n and two vertices are adjacent if the corresponding sequences differ on precisely one place. By a random cube $Q(n,p)$ we shall mean a graph obtained by removing edges from Q_n with probability $1-p$, independently for each edge. The fact that the phase transition occurs in $Q(n,p)$, anticipated already by Erdős and Spencer in 1979, was confirmed by Ajtai et al. [1] who showed that if $np=c<1$ then all components of $Q(n,p)$ are of size $O(n)$, whereas when $np=c>1$ the giant component

of size $\alpha 2^n$ emerges, where $\alpha=\alpha(c)$ is the root of an Eq. (*). The critical behaviour of $Q(n,p)$ was studied by Bollobás et al. [5]. They proved that if $pn=1-\varepsilon$ and $\varepsilon=\varepsilon(n)\geqslant\log^2 n/\sqrt{n^0}$ but $\varepsilon\to 0$ then the sequence of components is smooth and the largest component has a.s. $(\log 2+o(1))n/(-\varepsilon-\log(1-\varepsilon))$ vertices. If $pn=1+\varepsilon$ and $\varepsilon\geqslant 60\log^3 n/n$ but $\varepsilon\to 0$ then a.s. the giant component of $Q(n,p)$ and $(2+o(1))\varepsilon 2^n$ vertices whereas the second largest one has size $(\log 2+o(1))n/(-\varepsilon-\log(1-\varepsilon))$.

One can easily see that, despite of the above results, the picture of the phase transition in $Q(n,p)$ is far from being complete. Is the sensibility polynomially or exponentially small? What is the critical function? Does there exist only one giant component through the whole of the supercritical phase? These are just a few basic questions concerning this phenomenon that have remained open so far.

What is even more interesting, it is no clear whether the average degree heuristic, the maximization and symmetry rule hold in general (although they fit very well to the picture which emerges from known results). Indeed, the average degree heuristic is based on the assumption that the number of vertices reachable from a given vertex grows like the number of descendants in a branching process. But in a random cube the expected number of vertices which lies at the distance 2 from a fixed vertex is $n(n-1)p^2-0.5n(n-1)p^4$, while in the corresponding branching process the number of descendants in the second generation is just $n(n-1)p^2$. The additional factor $0.5n(n-1)p^4$ is negligible if the sensibility of the phase transition is larger than n^{-2}. Nevertheless, if the sensibility of the phase transition in $Q(n,p)$ is very small it would matter very much, so in such a case the precise equation for the critical function would depend not only on the fact that Q_n is n-regular but also on its cyclic structure.

Furthermore, it is possible (although a bit unlikely) that when p grows at some moment there are a few large components $Q(n,p)$ which absorb all medium-size components around creating a gap but nevertheless, they are spread over the cube so not all of them are merging together at once. Hence, some of them would exist some time after the moment when gap appeared violating the maximization rule. Let us observe also that the heuristic which led us to the symmetry rule can not be applied directly to the random cube. Indeed, if we delete the largest component from $Q(n,p)$ the remaining graph is not the random subgraph of Q_n but a random subgraph of some subgraph $\bar{Q}_n$ of Q_n. When we are far away from the critical point it turns out (see [1, 5]) that the vertices of the giant component are distributed 'uniformly' around Q_n and, consequently, graph $\bar{Q}_n$ is nearly regular. Furthermore, for the range of p which has been studied yet, the component structure of a $Q(n,p)$ depended mainly on the fact that it is a 'nearly regular', sparse graph, with average degree np — the fact that it was obtained from Q_n was not so crucial. Thus, the symmetry rule followed. Nevertheless, as we have already mentioned, very near the critical point properties of $Q(n,p)$ may (and probably do) depend on the structure of its underlying graph Q_n. Thus, possibly at some point this dependence become so important that $\bar{Q}_n$ can no longer simulate $Q_{n'}$ for some $n'<n$ and the symmetry between sub- and supercritical phases breaks down.

Finally, let us say few words about other models of a random cube. Random cube $Q(n, M)$ defined as a graph chosen at random from the family of all subgraphs of Q_n with M edges is expected to have properties very similar to $Q(n, p)$, and indeed, all we can prove on $Q(n, p)$ can easily be shown also for this model. Nevertheless, one can consider also a model $Q^s(n, p)$, in which we delete *vertices* from Q_n with probability $1-p$, independently from each other, or, more generally, a two-parameter mixed model $Q(n, p_v, p_e)$ obtained from Q_n by deleting vertices with probability $1-p_v$ and excluding remaining edges with probability $1-p_e$. It turns out that, roughly, all results we can prove for $Q(n, p)$ remain valid also for $Q^s(n, p)$ [6] and $Q(n, p_v, p_e)$, although, due to some technical reasons, estimates in general case are a bit weaker. However, it is quite possible that in the near-critical regions, non-penetrated yet, differences between these models will show up.

4.4. *Random graphs and k-connectivity*

In Section 3 we described the component structure of $G(n, M)$ (and $G(n, p)$) near the critical point; we now resume shortly the behaviour of k-edge connected components during the random graph process.

When $k=2$ the picture is not very different from those we had before. For $M=n/2-s$ and $sn^{-2/3}\to\infty$ the sequence of 2-components is smooth and a.s. all non-trivial 2-components are cycles of size $O(n/s)$. When $M=n/2+s$, $sn^{-2/3}\to\infty$, the sequence of 2-components contains gaps: a.s. the largest component has $8s^2/n$ vertices whereas all remaining ones are single vertices or cycles of size $O(n/s)$. Thus, the symmetric phase transition with sensibility $n^{-1/3}$ occurs at the point $M=n/2$ [18].

However, for $k\geqslant 3$ the critical behaviour of $G(n, M)$ becomes more interesting. It can be shown that with probability $1-o(1)$ the random graph process

$$\{G(n, M)\}_{M=0}^{\binom{n}{2}}$$

is such that for every $M, 0\leqslant M\leqslant\binom{n}{2}$, $G(n, M)$ contains at most one non-trivial k-component, which, if exists, has size larger than $\alpha_k n$ for some absolute constant $\alpha_k>0$ [16]. Thus, in the random graph process, adding one edge creates a giant k-component which grows gradually until the graph becomes k-edge connected. Unfortunately, although we know that such a critical edge exists we do not know in which moment of random graph process it appears, neither can we determine the precise value of the constant α_k.

4.5. *Random forests*

The probabilistic space of (labelled) forests can be introduced in various, non-equivalent, ways. Let us start with the model of random forest defined 'dynamically', by adding edges to the empty graph one by one, in such a way that in each step a new edge is chosen at random from all pairs which addition do not lead to a cycle. A graph obtained in Mth step of the process we call $\mathscr{F}^1(n, M)$.

The only difference between $\mathscr{F}^{\mathrm{I}}(n, M)$ and random graph $G(n, M)$, viewed as the Mth stage of a random process is the fact that during the process leading to $\mathscr{F}^{\mathrm{I}}(n, M)$ we should avoid 'bad' edges which would create cycles. But such discarded edges do not affect size of components of a graph, so the component structure of $\mathscr{F}^{\mathrm{I}}(n, M)$ has the very same distribution as the component structure of $G(n, M')$, where M' is the moment of random graph process

$$\{G(n, M)\}_{M=0}^{\binom{n}{2}}$$

such that the number of components of $G(n, M')$ equals $n-M$. Thus, since near the point of the phase transition components of $G(n, M)$ are rather sparse, one can easily deduced the critical behaviour of $\mathscr{F}^{\mathrm{I}}(n, M)$ from results concerning $G(n, M)$ — at the point $M=n/2$ a symmetric phase transition with sensibility $n^{-1/3}$ takes place and all its quantitative parameters (e.g. asymptotic distributions of sizes of the largest components) are very similar to that of $G(n, M)$ (we should mention, however, that the symmetry equation relating early subcritical and later supercritical phases is slightly different in this case).

The 'static' equivalent of $G(n, M)$ is the random forest $\mathscr{F}^{\mathrm{II}}(n, M)$, defined as a graph chosen at random from the family of all forests with n vertices and M edges. Also in this model the phase transition with critical function $M=n/2$ and sensibility $n^{-1/3}$ can be observed (see [21]) but the critical behaviour of $\mathscr{F}^{\mathrm{II}}(n, M)$ is by no means identical with that of $\mathscr{F}^{\mathrm{I}}(n, M)$. Although for $M=n/2-s$, where $sn^{-2/3}\to\infty$, they behave similarly (in both models the size of the largest and second largest components is of the order $n^2/s^2\log(n^2s^{-3})$), in the supercritical phase, when $M=n/2+s$, $sn^{-2/3}\to\infty$, the size of the largest component in $\mathscr{F}^{\mathrm{I}}(n, M)$ grows roughly as $4s$, whereas in $\mathscr{F}^{\mathrm{II}}(n, M)$ it a.s. contains only $(2+o(1))s$ vertices. Hence, if we delete vertices of the giant component from $\mathscr{F}^{\mathrm{II}}(n, M)$, $M=n/2+s$, the remaining forest has $n'\sim n-2s$ vertices and $M'\sim n/2-s=n'/2$, i.e. it resembles the random forest in the critical, not the subcritical phase. Thus, while the size of the second largest component in $\mathscr{F}(n, n/2+s)$ is of the order $n^2/s^2\log(n^2s^{-3})$ and decreases with s, in $\mathscr{F}^{\mathrm{II}}(n, M)$ its size equals $n^{2/3}$ for most of the supercritical phase (as a matter of fact, the probability that the second largest component is smaller than $cn^{2/3}$, for a constant $c>0$, tends to the same limit for every M such that $(2M-n)/n^{-2/3}\to\infty$ but $2M-n=o(n)$.) This violation of the rule that near the point of the phase transition the second largest component attains its maximum size is, of course, a result of a static nature of our model which cannot be 'simulated' dynamically (i.e. unlike in the case of random graphs, there is no natural simple procedure which generates $\mathscr{F}^{\mathrm{II}}(n, M)$ from $\mathscr{F}^{\mathrm{II}}(n, M-1)$).

Two other models of random forests: a statistical and a dynamical one, naturally emerge in analysis of some algorithms. Thus, $\mathscr{F}^{\mathrm{III}}(n, M)$ is defined as a forest chosen at random from all forests with vertex set $[n]$, M edges, in which vertices $\{1, 2, \ldots, n-M\}$ belongs to different components, while $\mathscr{F}^{\mathrm{IV}}(n, M)$ is the forest obtained by deleting from a random tree on n vertices $n-M-1$ randomly chosen edges.

Somewhat surprisingly, as it was shown in [14], these two models turned out to be equivalent. More precisely, the probability of every graph property which does not depend on the labelling of vertices are the same for both $\mathscr{F}^{III}(n, M)$ and $\mathscr{F}^{IV}(n, M)$.

The behaviour of $\mathscr{F}^{III}(n, M)$ was studied by Pavlov in late seventies [23, 24]. He showed that if $(n-M)/\sqrt{n} \to \infty$ the sequence of components is smooth and the size of the kth largest components grows roughly as

$$\frac{2n^2}{(n-M)^2} \log \frac{(n-M)^2}{n},$$

while for $n-M=o(\sqrt{n})$ the largest component has a.s. $n-o(n)$ vertices whereas the size of the kth largest components for a fixed $k \geqslant 2$ decreases as $(n-M)^2 \log(n-M)$. Thus, although in this model, the phase transition, as defined in Section 2.2, cannot be observed, it seems natural to split the range of M into the subcritical, critical and supercritical phases when $(n-M)/\sqrt{n} \to \infty$, $M=n-O(\sqrt{n})$ and $M-n=o(\sqrt{n})$, respectively, which behave somewhat similar to that of typical phase transition phenomena. In particular, there exists a symmetry rule which relates the structure of graphs in sub- and supercritical phases (but, of course, the average degree heuristic fails in this case).

$\mathscr{F}^{i}(n, M)$, where $i=I, II, III, IV$, are by no means the only models of random forests studied in the literature (e.g. there exists weighted versions of models $\mathscr{F}^{II}(n, M)$ and $\mathscr{F}^{III}(n, M)$, a kind of $G(n, p)$ analog of $\mathscr{F}^{II}(n, M)$, etc.) but properties of others are analogous to that we described. The reader may refer to the last section of [21], which lists some other models of random forests together with their most characteristic features.

4.6. *Random mappings*

A random mapping is the mapping of a set $[n]$ into $[n]$ in which points pick their images uniformly and independently from each other. As a digraph, in which we join by an arc each point with its image, a random mapping is a special case of the well-known k-out model of a random digraph $\vec{G}_k=\vec{G}_k(n)$, defined as the digraph in which each vertex chooses uniformly and independently a set of k-out neighbours (we may or may not allow multiple edges and loops — this fact does not affect too much the structure of the graph and all results below remain true for both cases).

Properties of $\vec{G}_k$ and its underlying (multi-)graph have been studied for a long time (as a matter of fact, the Scottish Book contains Ulam's question concerning this model) but in order to see how the structure of these graphs is affected by small changes of their density one should find the way to define the analog of this graph with the number of edges different from kn. Thus, let $\{\vec{U}(n, M)\}_{M=0}^{n(n-1)}$ be a Markov chain of digraphs on the vertex set $[n]$ which starts with the empty digraph and for $M=1, 2, \ldots, n(n-1)$ digraph $\vec{U}(n, M)$ is obtained from $\vec{U}(n, M-1)$ by adding an arc which starts in a vertex v chosen randomly and uniformly from all vertices of

$\tilde{U}(n, M-1)$ of the minimum out-degree and ends a vertex $w, w \neq v$, randomly picked from $[n]$. If by $U(n, M)$ we denote the underlying multigraph of $\tilde{U}(n, M)$ then the resulting sequence $\{U(n, M)\}_{M=0}^{n(n-1)}$ we call the uniform random graph process.

What can we say about components of $U(n, M)$? The structure of a random mapping, i.e. of $U(n, n)$, is well known (see, for example, Kolchin's monograph [12] — at this moment largest components are of the order n and the sequence of components is smooth. This suggests it should be smooth also for every $M < n$ and indeed, it is the case. Nevertheless, when we add to $U(n, n)$ even one edge, the sequence of components is not smooth anymore and one can show that the sequence of $U(n, n+\omega(n))$ contains gaps whenever $\omega(n) \to \infty$ [9]. Thus, in our terminology, the phase transition in this model is non-symmetric, with the critical function equal to $M(n) = n$ and separating pair $(0, 1)$.

This, rather peculiar, behaviour, can easily be explained. As long as $M \leqslant n$, the 'selective' way we add new edges prevents growing components from merging (note, for instance, that since all components of a random mapping are unicyclic no edge added in a uniform random process before the moment $M = n$ can join two unicyclic components). This constrain disappears after the process passes the point $n = M$ and the largest components quickly coalesce into the one giant one.

Is any symmetry rule applied to this non-symmetric phenomenon? At the first sight the answer seems to be negative — in the supercritical phase all small components of $U(n, M)$ must contain at least one cycle whereas $U(n, M)$ in the subcritical phase contains $M - n$ trees. Nevertheless, it turns out that some kind of the symmetry holds. Let us denote by $C(G)$ the subgraph of a graph G spanned by all its unicyclic components. Then, it can be shown that in the supercritical phase a.s. all small components are unicyclic (i.e. $C(U(n, M))$ is obtained from $U(n, M)$ by deleting the giant component), and, furthermore, for every $M, M - n \to \infty$, there exists $M' < n$ such that properties of $C(U(n, M))$ and $C(U(n, M'))$ are, roughly, identical [9].

4.7. Random unlabelled graphs

Finally, we give an example of a random structure where no phase transition occurs. Let $G^u(n, M)$ be a graph chosen at random from all *unlabelled* graphs with n vertices and M edges (one can also define $G^u(n, M)$ as the set of all labelled graphs with M edges but with suitable probability distribution, which is consistent with the notion of a randomized discrete structure given in Section 2.1). Note that, unlike $G(n, M)$, $G^u(n, M)$ does not 'react' on the addition of new isolated vertices. Indeed, the number of unlabelled graphs with n' vertices, M' edges and, say, k isolated vertices is precisely the same as the number of unlabelled graphs with $n' + l$ vertices, M' edges and $k + 1$ isolated vertices. Thus, if M is such that with large probability $G^u(n, M)$ contains a lot of isolated vertices, then the structure of the graph obtained from $G^u(n, M)$ by deleting all of them does not depend explicitly on n but only on $M = M(n)$. Hence one should not expect to have the phase transition in this model — any equation for the critical function would relate n and $M = M(n)$ while

n could affect the structure of $G^u(n, M)$ only by increasing the number of isolated vertices.

It turns out that this heuristic can be converted in a rigorous argument. Furthermore, Wright in series of paper (see [32] or [17]) showed that if $M \to \infty$ then a.s. $G^u(n, M)$ consists of one large component of size roughly $2M/\log M$ and some number of isolated vertices. Thus, the sequence of components is never smooth and the phase transition does not appear (Let us mention, however, that the apparently uninteresting evolution of $G^u(n, M)$ has one peculiar moment when $M \sim 0.5n \log n$, i.e. near the point where isolated vertices disappear $G'(n, M)$ becomes connected. Then, the size of the largest component of $G^u(n, M)$ becomes very 'sensitive' for the number of edges which somewhat resembles the critical behaviour taking place near the point of the phase transition (see [15] for details)).

5. Further developments

As we have already mentioned in the introduction, the primary purpose of this paper was to look at different random discrete structures in search of the characteristic features of the phase transition. Thus, we omitted many results concerning this phenomenon and others were presented in simplified form. In this section we would like to confess what kind of omissions and simplifications we made. Since we will be very sketchy, for details the reader should consult to the original articles.

The reader has probably noticed that almost nothing was said about the critical phase, the period when the giant component is actually born. It turns out however that the description of the component structure at this moment is rather complicated and quite technical. Due to the very nature of this process almost none of the parameters characterizing a random discrete structure at this moment is determined a.s. up to the factor of $1 + o(1)$, and their asymptotic distributions are given by rather non-elegant formulae, using infinite series or/and infinite integrals. Nevertheless, such limit distributions for the size of the largest components were found for a number of structures: for a random graph $G(n, M)$ by Łuczak et al. [22] and Janson et al. [8], for $\mathscr{F}^{II}(n, M)$ by Łuczak and Pittel [21], for the pseudo-critical phase of $\mathscr{F}^{III}(n, M)$, when the difference $M - n$ is of order $\sqrt{n}$, by Pavlov [24].

While in the critical period the limit distributions of the size of the largest components are rather hard to find, in the supercritical and particularly subcritical period this task is usually much easier. For $G(n, p)$ and $G(n, M)$ the asymptotic distributions of the size of largest components in the early subcritical and later supercritical phases was already determined by Erdős and Rényi [7], whose results were later strengthened by Pittel [25]. The fact that the size of the giant component in the later supercritical phase tends, after suitable standardization, to the normal distribution, was proved for $G(n, p)$ by Stepanov [29] and for $G(n, M)$ by Pittel [26] (this is one of rare cases where there exists small quantitative difference between these two models). For $np = 2M/n = 1 + o(1)$ the limit distributions of the size of the largest unicyclic

components and the largest unicyclic trees was determined by Kolchin [13] and Łuczak [14].

Pittel [26] showed that the limit distribution of the size of the giant 2-edge connected components of $G(n,p)$ and $G(n,M)$ tends also to the normal distribution in late supercritical phase (also here the difference between $G(n,M)$ and $G(n,p)$ shows up), and asymptotic normality of the size of the giant component of a random hypergraph at the beginning of the supercritical phase was shown by Karoński and Łuczak [10]. The paper of Łuczak and Pittel [21] contains rather detailed analysis of the behaviour of the size of the largest components in random forests $\mathscr{F}^{II}(n,M)$, but here in the supercritical phase the size of the giant component tends not to the normal but to the Holtsmark (3/2-stable) distribution.

Another trend in the literature dealing with critical phenomena in random discrete structures is the problem of the 'internal' structure of components near the phase transition point. It was shown by Łuczak [14] and Janson et al. [8] that when $M=n/2+s, s^3n^{-2}\to\infty$ but $s=o(n)$, then a.s. the giant component of $G(n,M)$ has $(16/3+o(1))s^3n^{-2}$ more edges than vertices. In [8] the distribution of the number of these extra edges was shown to be asymptotically normal. In this paper the internal structure of components in the critical period is also studied — the authors determine the joined distribution of the number components sorted by their co-cyclic number (a similar result was proved independently by Łuczak et al. [22]). Łuczak [18] studied the cyclic structure of $G(n,M)$ in the supercritical phase showing, among others, that near the point of the phase transition the giant component contains no short cycles.

The internal structure of the giant component of a random digraph is not very well studied. In random hypergraphs its density was estimated by Karoński and Łuczak [10]. They strengthened earlier results of Schmidt-Pruzan and Shamir [27] showing that in sub- and supercritical phases a.s. all components of a random hypergraph, except the giant one, contain at most one cycle. The structure of the giant 2-edge connected components was given by Pittel [26] and Łuczak [18] (see also [19], where a more general model of a random graph is considered). The internal structure of a giant component of a random forest $\mathscr{F}^i(n,M)$ is easy to study when $i=$ II, III, IV — in these cases it is the same as well-studied structure of a random tree. Not too much is known about the structure of the giant component of $\mathscr{F}^I(n,M)$ — the degree sequence of $\mathscr{F}^I(n,n-1)$ was analysed by Aldous [2].

Instead studying the critical behaviour of a random graph one may consider properties of the random graph process near the point of the phase transition. Łuczak [14] showed that when we view the process as a kind of the race of components, the last edge which merges two small components into a component larger than the largest one is added somewhere in the critical phase. The process of creating a giant component was studied in detail by Janson et al. [8]. Finally, let us mention that in the random graph $G(n,p)$ the point of the phase transition $p=1/n$ in 'critical' also for properties of $G(n,p)$ viewed as a model of some language. Shelah and Spencer [28] showed that if $np=c<1$ then the probability of each sentence of the second monadic language of graphs converges. On the other hand, they gave an example of a sentence

of the second monadic language of graphs whose probability oscillates between 0 and 1 as n grows, whenever $np=c>1$.

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References

[1] M. Ajtai, J. Komlós and E. Szemerédi, Largest random component of k-cube, Combinatorica 2 (1982) 1–7.
[2] D. Aldous, A random tree model associated with random graphs, Random Struct. Algorithms 1 (1990) 383–402.
[3] B. Bollobás, The evolution of random graphs, Trans. Amer. Math. Soc. 286 (1984) 257–274.
[4] B. Bollobás, Random Graphs (Academic Press, London, 1985).
[5] B. Bollobás, Y. Kohayakawa and T. Łuczak, The evolution of random subgraphs of the cube, Random Struct. Algorithms 3 (1992) 55–90.
[6] B. Bollobás, Y. Kohayakawa and T. Łuczak, The evolution of random Boolean functions, to appear.
[7] P. Erdős and A. Rényi, On the evolution of random graphs, Magyar Tud. Akad. Mat. Kutató Int. Közl 5 (1960) 17–61.
[8] S. Janson, D.E. Knuth, T. Łuczak and B. Pittel, The birth of the giant component, Random Struct. Algorithms 4 (1993) 233–358.
[9] J. Jaworski and T. Łuczak, Cycles in a uniform graph process, Combin. Probab. Comput. 1 (1992) 223–239.
[10] M. Karoński and T. Łuczak, The phase transition in random hypergraphs, in preparation.
[11] R.M. Karp, The transitive closure of a random digraph, Random Struct. Algorithms 1 (1990) 73–94.
[12] V.F. Kolchin, Random Mappings (Optimization Softward, New York, 1986).
[13] V.F. Kolchin, On the behavior of a random graph near a critical point, Theory Prob. Appl. 31 (1987) 439–451.
[14] T. Łuczak, Component behaviour near the critical point of the random graph process, Random Struct. Algorithms 1 (1990) 287–310.
[15] T. Łuczak, The phase transition in the evolution of a random digraph, J. Graph Theory 14 (1990) 217–223.
[16] T. Łuczak, Size and connectivity of the k-core of a random graph, Discrete Math. 91 (1991) 61–68.
[17] T. Łuczak, How to deal with unlabelled random graphs, J. Grah Theory 15 (1991) 303–316.
[18] T. Łuczak, Cycles in a random graph near the critical point, Random Struct. Algorithms 2 (1991) 421–440.
[19] T. Łuczak, Sparse random graphs with a given degree sequence, in: A. Frieze and T. Łuczak, eds., Random Graphs, Vol. 2 (Wiley, New York, 1992) 215–232.
[20] T. Łuczak and J.E. Cohen, Giant components in three-parameter random directed graphs, Adv. Appl. Probab 24 (1992) 845–857.
[21] T. Łuczak and B. Pittel, Components of random forests, Combin. Probab. Comput. 1 (1992) 35–52.
[22] T. Łuczak, B. Pittel and J.C. Wierman, The structure of a random graph near the point of the phase transition, Trans. Amer. Math. Soc. 341 (1994) 721–748.
[23] Yu.L. Pavlov, The asymptotic distribution of maximum tree size in a random forest, Theory Probab. Appl. 22 (1977) 509–520.
[24] Yu.L. Pavlov, A case of limit distribution of the maximal volume on a tree in a random forest, Math. Notes 25 (1979) 387–392.
[25] B. Pittel, A random graph with a subcritical number of edges, Trans. Amer. Math. Soc. 309 (1988) 51–75.

[26] B. Pittel, On tree census and the giant component in sparse random graphs, Random Struct. Algorithms 1 (1990) 311–342.
[27] J. Schmidt-Pruzan and E. Shamir, Component structure in the evolution of random hypergraphs, Combinatorica 5 (1985) 81–94.
[28] S. Shelah and J. Spencer, Can you feel the double jump?, Random Struct. Algorithms 5 (1994) 191–204.
[29] V.E. Stepanov, On the probability of connectedness of a random graph $G_m(t)$, Theory Probab. Appl. 15 (1970) 55–67.
[30] V.E. Stepanov, Phase transitions in random graphs, Theory Probab. Appl. 15 (1970) 187–203.
[31] V.E. Stepanov, On some features of the structure of a random graph near a critical point, Theory Probab. Appl. 32 (1988) 573–594.
[32] E.M. Wright, The evolution of unlabelled graphs, J. London Math. Soc. 14 (1976) 554–558.