PROOF FUNCTIONAL CONNECTIVES

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A characteristic of the classical propositional connectives is that they are truth-functional, that is, the truth of a sentence depends only on the truth of its prime components. On the other hand the intuitionistic connectives are supposed to be much less dependent on semantical notions. Consequently, one avoids saying that a "sentence is true", rather one tends to say:

"the construction c proves (or justifies) the sentence A".

Nevertheless, some aspects of truth-functionality can still be found in that:

(1) a construction c either proves a sentence A or it does not,
(2) the conditions for a construction to prove a compound sentence is given in terms of the conditions for the proof of the components; for example, c proves the conjunction $(A_0 \wedge A_1)$ iff c is a pair of constructions $\langle c_0, c_1 \rangle$ such that c_0 proves A_0 and c_1 proves A_1.

Thus for a conjunction to be provable it is necessary (and sufficient) that the conjuncts be provable. In other words, the fact that quite different constructions may prove a given formula is not exploited in traditional intuitionism.

G. Pottinger [1980] introduced a conjunction, which he called "strong conjunction", which requires more than the existence of constructions proving the conjuncts. According to Pottinger:

"The intuitive meaning of $\hat{\&}$ can be explained by saying that to assert $A \mathbin{\hat{\&}} B$ is to assert that one has a reason for asserting A which is also a reason for asserting B".

Hence $\hat{\&}$ is one of the first, if not the first, connective which is truly proof-functional. This paper is an introduction to the "logic" of a proof-functional connective; in fact for the most part we shall consider the extension of the positive intuitionistic calculus of implication obtained by the addition of strong-conjunction $\hat{\&}$.

As with any "new" logic, there are five questions that immediately come to mind:

(1) Is there a reasonable, intuitive concept of validity for the sentences of the language?
(2) Is there a formal concept of validity for the sentences of the language?
(3) Is there a formal concept of derivation for the sentences of the language?
(4) What are the relations between (1), (2) and (3)?
(5) How does the new connective affect familiar mathematical theories (for example : elementary number theory)?

The paper is broken down into 5 sections corresponding to the above 5 questions.

§1. INTUITIVE VALIDITY.

1.1 THE LANGUAGE $\cancel{L}$. The sentential language $\cancel{L}$ is to have the following symbols:

$p, q, r, \ldots$	for the sentential variables,
$\supset$	for the conditional connective,
$\&$	for Pottinger's strong conjunction,
(,)	for auxiliary symbols.

1.2 AN INTUITIVE INTERPRETATION FOR THE INTUITIONISTIC CONNECTIVES. Let $\pi_A(c)$ express the (decidable) predicate "the construction c proves the sentence A" and $\pi(c,\theta(x))$ express the (decidable) predicate "the construction c proves the free-variable formula $\theta(x)$". Finally if c,d are constructions then $d'c$ is result of applying d to c.

The Brouwer-Kreisel interpretation of the intuitionistic conditional is that:

$$(*) \qquad \pi_{A\supset B}(\langle c,d\rangle) \quad \text{iff} \quad \pi(c,(\pi_A(x) \supset \pi_B(d'x))).$$

And although (*) is not universally accepted, it certainly is, and was, a good point of departure for an intuitive interpretation for the intuitionistic conditional.

1.3 AN INTUITIVE INTERPRETATION FOR STRONG CONJUNCTION. The Brouwer-Kreisel interpretation for ordinary conjunction $\wedge$ is

$$\pi_{A\wedge B}(\langle c,d\rangle) \quad \text{iff} \quad \pi_A(c) \wedge \pi_B(d).$$

The latter suggests the following interpretation for strong conjunction $\&$:

$$\pi_{A\&B}(\langle c,c\rangle) \quad \text{iff} \quad \pi_A(c) \wedge \pi_B(c).$$

The following are equivalent interpretations (in an appropriate theory of constructions)

$$\pi_{A\&B}(c) \quad \text{iff} \quad \pi_A(c) \wedge \pi_B(c),$$

$$\pi_{A\&B}(\langle c,d\rangle) \quad \text{iff} \quad \pi_A(c) \wedge \pi_B(d) \wedge c \cong d,$$

where $\cong$ is an equality relation on constructions.

1.3 AN INTUITIVE CONCEPT OF VALIDITY FOR SENTENCES OF $\mathbb{L}$. A sentence A of $\mathbb{L}$ is intuitively valid iff there is a construction c such that $\pi_A(c)$.

The following sentences are easily seen to be intuitively valid:

(1) $A \,\&\, (A \supset B) \supset B$

(2) $(A \supset B) \,\&\, (A \supset C) \supset (A \supset B \& C)$

(3) $(A \supset B \& C) \supset (A \supset B) \,\&\, (A \supset C)$

(4) $(A \supset C) \supset (A \& B \supset C)$

(5) $A \,\&\, B \supset (A \supset B)$

(6) $A \supset (B \supset C) \supset (A \& B \supset C)$

(7) $A \,\&\, B \supset A$

(8) $A \supset A \,\&\, A$

(9) $A \,\&\, B \supset B \,\&\, A$

(10) $((A \& B) \& C) \supset A \,\&\, (B \& C)$

(11) $\{(A \supset B \& C) \supset (A \supset B) \,\&\, (A \supset C)\} \,\&\, \{(A \supset B) \,\&\, (A \supset C) \supset (A \supset B \& C)\}$

We let $IVAL_{\mathbb{L}}$ be the set of intuitively valid sentences of $\mathbb{L}$.

§2. FORMAL VALIDITY.

2.1 APPLICATIVE ALGEBRAS AND CURRY ALGEBRAS. The formal semantics for $\mathbb{L}$ is to be of the algebraic type. However instead of using the algebras of open sets, we plan to use algebras related to Curry's combinatory logic.

2.1.1 DEFINITION. An applicative algebra is an algebra $M = \langle M, \circ\rangle$, where $\circ$ is a binary operation on M.

Given an applicative algebra $\langle M,\circ\rangle$, then instead of writing "$\circ(a,b)$", we will write "(a,b)". A polynomial in $\langle M,\circ\rangle$ is a term built up from indeterminates $x,y,z,\ldots$, elements of M and the application operator.

2.1.2 DEFINITION. A combinatory algebra is an applicative algebra $\langle M,\circ\rangle$ which is non-trivial (i e. contains at least two elements), and which is combinatory complete; that is, to each polynomial $h(x,y,\dots,z)$ in $\langle M,\circ\rangle$ there corresponds an element $a \in M$ such that:

$$\forall x\forall y\dots\forall z[(\dots(a,x),y),\dots,z) = h(x,y,\dots,z)]$$

2.1.3 DEFINITION. A curry algebra is an algebra $C = \langle M,\circ,k,s\rangle$ such that $\langle M,\circ\rangle$ is an applicative algebra and

(1) $k \neq s$,

(2) $\forall x\forall y[((k,x),y) = x]$,

(3) $\forall x\forall y\forall z[(((s,x),y),z) = ((x,z),(y,z))]$.

It was one of the first theorems on combinatory logic that the Curry algebras are combinatory complete.

2.2 SATISFACTION IN CURRY ALGEBRAS.

Assume that $C = \langle M,\circ,k,s\rangle$ is a Curry algebra. A C-proof assignment is a function pa which maps the sentential variables to subsets of M. We extend pa to act on all sentences of $\underset{\sim}{L}$ by requiring that:

$$pa(A\supset B) = \{m \in M \mid \forall n[n \in pa(A) \rightarrow (m,n) \in pa(B)]\}$$

$$pa(A\&B) = pa(A) \cap pa(B).$$

The elements in pa(A) will be called the "pa-proofs of A (in the Curry algebra C)".

Loosely speaking, a sentence A of $\underset{\sim}{L}$ is to be formally valid iff $pa(A) \neq \emptyset$ for every Curry algebra C and C-proof assignment pa. However, because of the constructive (intuitionistic) character of the logic, a certain degree of uniformity is required. We achieve the required uniformity by using terms of a first order language $\underset{\sim}{L}_C$ suitable for both the Curry algebras and the proof-assignments. Or more specifically, $\underset{\sim}{L}_C$ is the first-order language (with equality: $\doteq$) which also contains

the individual constants: K, S,
the binary (infix) function symbol: $\cdot$,
the unary relation symbols: P,Q,R,...

Next, given a Curry algebra $C = \langle M,\circ,k,s\rangle$ and a C-proof assignment pa we form the first-order structure:

$C[pa] = \langle M,\circ,k,s,pa(p),pa(q),\dots\rangle$. Clearly $C[pa]$ is a structure associated to the language $\underset{\sim}{L}_C$.

In order to relate pa(A) to satisfaction in C[pa], we associate to each formula A of $\not{L}$ a formula $\underset{\sim}{r}_A[x]$ of $\underset{\sim}{L}_C$ as follows:

if A is the sentential variable p, then $\underset{\sim}{r}_A[x] = P(x), \ldots,$

if A = (B&C), then $\underset{\sim}{r}_A[x] = \underset{\sim}{r}_B[x] \wedge \underset{\sim}{r}_C[x]$,

if A = (B⊃C), then $\underset{\sim}{r}_A[x] = \forall y[\underset{\sim}{r}_B[y] \supset \underset{\sim}{r}_C[x\cdot y]]$.

A routine induction then gives us that for every sentence A of $\not{L}$:

(*) m satisfies $\underset{\sim}{r}_A[x]$ in the structure C[pa] iff $m \in pa(A)$.

Finally, given a closed term t of the first-order language $\underset{\sim}{L}_C$ and a Curry algebra C, t^C is the denotation of t in the algebra C.

We are now ready for the definition of formal validity:

2.2.1 <u>DEFINITION.</u> A sentence A of $\not{L}$ is formally valid iff there is a closed term t of the first-order language $\underset{\sim}{L}_C$ such that for all Curry algebras C and all proof-assignments pa: $t^C \in pa(A)$.

In view of the remark (*), the condition "$t^C \in pa(A)$" can be replaced by:

(**) The first-order sentence $\underset{\sim}{r}_A[t]$ is true in the structure C[pa].

We let $FVAL_{\not{L}}$ be the set of sentences of $\not{L}$ which are formally valid.

2.3 AXIOMATIZABILITY OF THE FORMALLY VALID SENTENCES OF $\not{L}$.

Let CA be the following set of sentences of $\underset{\sim}{L}_C$:

$\neg K \doteq S$,

$\forall x \forall y[((K\cdot x)\cdot y) \doteq x]$,

$\forall x \forall y \forall z[(((S\cdot x)\cdot y)\cdot z) \doteq ((x\cdot z)\cdot(y\cdot z))]$

and THM_{CA} be the set of logical consequences of CA.

Gödel'a completeness theorem and (**) gives us that:

2.3.1 <u>THEOREM.</u> A formula A of $\not{L}$ is formally valid iff there is a closed term t of $\underset{\sim}{L}_C$ such that $\underset{\sim}{r}_A[t] \in THM_{CA}$.

An immediate consequence of the above theorem is that $FVAL_{\not{L}}$ is a recursively enumerable set. A variant of Craig's lemma gives us the following:

2.3.3 <u>THEOREM.</u> There is a recursive axiomatization for $FVAL_{\not{L}}$.

<u>PROOF.</u> Let $S_0, S_1, \ldots, S_n, \ldots$ be an enumeration of $FVAL_{\not{L}}$ by a (primitive) recursive function. Then for axioms take the following

set of sentences of $\not{L}$:

$$\{(S_0 \& S_0), ((S_1 \& S_1) \& S_1), (((S_2 \& S_2) \& S_2) \& S_2), \ldots\}$$

Clearly the set is recursive. As rule of inference take the rule that allows one to conclude A from $((\cdots(A \& A) \& \ldots) \& A)$.

An application of the concept of formal validity is to show that strong conjunction is indeed different from conjunction. For example it can be shown that there are Curry algebras C and proof-assignments pa such that $pa(A)$ is empty for the following sentences A:

(1) $p \supset (q \supset p \& q)$

(2) $(p \supset q) \supset ((p \supset r) \supset (p \supset q \& r))$

(3) $(p \& q \supset r) \supset (p \supset (q \supset r))$

(4) $(p \supset p) \& (p \supset (q \supset p))$.

§3. FORMAL DERIVABILITY.

3.1 MINIMAL REQUIREMENTS. Let $\Gamma \vdash A$ be a relation of formal derivability between a sentence A and a finite sequence Γ of sentences of $\not{L}$. Then the minimal requirements we place on $\vdash$ are:

(R) $A \vdash A$, for all sentences of $\not{L}$,

(T) if $\Gamma \vdash A$ and $A, \Gamma \vdash B$ then $\Gamma \vdash B$,

(M) if $\Gamma \vdash A$ then $\Delta \vdash A$ whenever every sentence that occurs in Γ also occurs in Δ.

Furthermore, since the conditional is the intuitionistic conditional we also require that

(D) $\Gamma, A \vdash B$ iff $\Gamma \vdash (A \supset B)$.

3.2 SEMANTICAL CONSEQUENCE. In view of the definition of formal validity (see 2.2.1), we propose the following definition for semantical consequence:

3.2.1 DEFINITION. The sentence A is a semantical consequence of the sequence $B_1, \ldots, B_r$ of sentences, in symbols: $B_1, \ldots, B_r \models A$, iff there is a term $t_{x_1, \ldots, x_r}$ of $\underset{\sim}{L}_C$ such that for all Curry algebras C and all C-proof assignments pa:

if $b_1 \in pa(B_1), \ldots, b_r \in pa(B_r)$, then $t^C_{b_1, \ldots, b_r} \in pa(A)$.

We shall say that the term $t_{x_1, \ldots, x_r}$ validates the pair $\langle\langle B_1, \ldots B_r\rangle, A\rangle$.

Call a pair $<\Gamma,A>$ a sequent. Define VALS = $\{<\Gamma,A> \mid \Gamma \models A\}$. Using the formulae $r_{\sim A}[x]$ of section 2.2 one can then show that:

3.2.2 <u>LEMMA.</u> VALS is a recursively enumerable set.

3.3 A RECURSIVE AXIOMATIZATION FOR DERIVABILITY.

Let $<<B_{00},\ldots,B_{0r_0}>,A_0>$, $<<B_{10},\ldots B_{1r}>,A_1>,\ldots$ be an enumeration of VALS by a (primitive) recursive function. Then let AXMS be the following set of sequents:

$$\{<<(B_{00}\& B_{00}),\ldots,(B_{0r_0}\& B_{0r_0})>, (A_0\& A_0)>,<<((B_{10}\& B_{10})\& B_{10}),\ldots$$
$$((B_{1r}\ \& B_{1r})\& B_{1r})>,((A_1\& A_1)\& A_1)>,\ldots\}$$

Again, it is clear that AXMS is a recursive set of sequents.

Then let $\vdash$ be the smallest relation containing AXMS $\cup$ $\{<<A>,A> \mid A$ a sentence of $\not{L}\}$ and closed under (T), (M),(D),$(\& \Rightarrow)^*$ and $(\Rightarrow\&)^*$, where the last two rules are:

$$(\& \Rightarrow)^* \qquad \frac{\Gamma,(\ldots(((B\& B)\& B)\ldots\& B)\ \vdash\ A}{\Gamma,\ B \vdash A,}$$

$$(\Rightarrow \&)^* \qquad \frac{\Gamma\vdash\ (\ldots(((A\& A)\& A)\ldots\& A)}{\Gamma \vdash A,}$$

respectively.

After verifying that the above mentioned rules preserve semantical consequences one easily obtains the following completeness theorem:

3.3.1 <u>THEOREM.</u> For any sequent $<\Gamma,A>$:

$$\Gamma \vdash A \text{ iff } \Gamma \models A.$$

3.4 A SOUND AND NATURAL AXIOMATIZATION FOR DERIVABILITY.

The set of axioms of the axiomatization given in §3.3, although recursive, is of no practical use. We now present another axiomatization which is closer to Gentzen's Sequent calculus for the intuitionistic sentential calculus. In fact, at first sight, it may appear as simply Gentzen's axiomatization for the intuitionistic sentential calculus of the conditional and conjunction. The modification (for the introduction of $\&$ in the succedent) is in the applicability of the rule, not in the schema for the rule. Some persons may find the restriction of interest since it is a global restriction involving all the nodes above the node of the inference, and not just those immediately above.

Unfortunately a price has to be paid for such naturalness and although we can show that the axiomatization is sound, we have not yet succeeded in showing it is complete.

In order to further emphasize the relation to Gentzen's systems we will write the sequents in the form "$\Gamma \Rightarrow A$".

3.4.1 AXIOMA SCHEMA:

$A \Rightarrow A$.

3.4.2 CUT RULE OF INFERENCE:

$$\frac{\Gamma, A \Rightarrow B \quad \Gamma \Rightarrow A}{\Gamma \Rightarrow B}$$

3.4.3 STRUCTURAL RULES OF INFERENCE

(MON) $$\frac{\Gamma \Rightarrow A}{\Delta \Rightarrow A},$$

provided every sentence occuring in Γ also occurs in Δ.

(REP) $$\frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow A}{\Gamma \Rightarrow A}$$

3.4.4 RULES OF INFERENCE FOR THE CONDITIONAL.

$(\Rightarrow \supset)$ $$\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B}$$ $(\supset \Rightarrow)$ $$\frac{\Gamma \Rightarrow A \quad \Gamma, B \Rightarrow C}{\Gamma, A \supset B \Rightarrow C}$$

3.4.5 RULES FOR STRONG CONJUNCTION.

$(\Rightarrow \&)$ $$\frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \& B}$$ $(\& \Rightarrow)$ $$\frac{\Gamma, B \Rightarrow C}{\Gamma, A \& B \Rightarrow C} \qquad \frac{\Gamma, B \Rightarrow C}{\Gamma, B \& A \Rightarrow C}$$

As already remarked the rule $(\Rightarrow \&)$ is not universally applicable. Loosely speaking $(\Rightarrow \&)$ may be applied when the (sub) derivations of $\Gamma \Rightarrow A$ and $\Gamma \Rightarrow B$ look the same.

We now proceed to make precise the latter statement.

3.4.7 PRE-DERIVATIONS. A pre-derivation consists of a finite tree T (of "nodes") and two functions $ and ₹ defined on T. $\$_N$ is the sequent occuring at the node N, while $₹_N$ is the name of the rule schema followed in obtaining $\$_N$; in other words a pre-derivation in the language Ⱡ is to all intent and purposes a derivation (with analysis) of the intuitionistic sentential calculus of the conditional and usual conjunction.

Given a pre-derivation P = <T,$,₹>, then <T,₹> is called the logical structure of P. The reduced logical structure of P is the pair

$\langle T, \hat{R} \rangle$ where $\hat{R}$ is the function defined on T such that $\hat{R}_N = \mathfrak{R}_N$ unless $\mathfrak{R}_N$ is either (MON), (REP), $(\& \Rightarrow)$ or $(\Rightarrow \&)$ in which case $\hat{R}_N = 0$.

Then two pre-derivations $P_1 = \langle T_1, \$_1, \mathfrak{R}_1 \rangle$ and $P_2 = \langle T_2, \$_2, \mathfrak{R}_2 \rangle$ are equivalent, in symbols: $P_1 \equiv P_2$, iff their reduced logical structures are isomorphic.

3.4.8 DERIVATIONS. A pre-derivation $\langle T, \$, \mathfrak{R} \rangle$ is a derivation when the following condition is met:

At each node N of T at which $\mathfrak{R}_N = (\Rightarrow \&)$, the two sub-pre-derivations immediately above N are equivalent.

If $D = \langle T, \$, \mathfrak{R} \rangle$ is a derivation and $\emptyset$ is the root of the tree T, then D is a derivation of the sequent $\$_\emptyset$. We write "$\Gamma \vdash A$" just in case that there is a derivation of the sequent $\Gamma \Rightarrow A$.

If Γ is empty and $\Gamma \vdash A$, then A is a (formally) derivable sentence of $\mathcal{L}$.

3.4.9 AN EXAMPLE OF A DERIVATION. First consider the following two derivations:

$$D_1 \qquad \dfrac{\dfrac{A \supset B \Rightarrow A \supset B}{(A\supset B) \;\&\; (A\supset C) \Rightarrow A \supset B} \qquad \dfrac{A \Rightarrow A \qquad B \Rightarrow B}{A, A \supset B \Rightarrow B}}{(A\supset B) \;\&\; (A\supset C), A \Rightarrow B}$$

$$D_2 \qquad \dfrac{\dfrac{A \supset C \Rightarrow A \supset C}{(A\supset B) \;\&\; (A\supset C) \Rightarrow A \supset C} \qquad \dfrac{A \Rightarrow A \qquad C \Rightarrow C}{A, A \supset C \Rightarrow C}}{(A\supset B) \;\&\; (A\supset C), A \Rightarrow C.}$$

A moments reflexion shows that they are equivalent. Thus the following is a derivation of $(A\supset B) \;\&\; (A\supset C) \supset (A\supset B \& C)$:

$$\dfrac{\dfrac{\dfrac{\begin{matrix} D_1 \\ (A\supset B) \;\&\; (A\supset C), A \Rightarrow B \end{matrix} \qquad \begin{matrix} D_2 \\ (A\supset B) \;\&\; (A\supset C), A \Rightarrow C \end{matrix}}{(A\supset B) \;\&\; (A\supset C), A \Rightarrow B \;\&\; C}}{(A\supset B) \;\&\; (A\supset C) \Rightarrow A \supset B \;\&\; C}}{\Rightarrow (A\supset B) \;\&\; (A\supset C) \supset (A\supset B \& C)}$$

3.4.10 SOUNDNESS THEOREM. An induction on the length of the derivation gives us that to each derivation D and each natural number n we can associate a term $t^D_{x_0, \ldots, x_{n-1}}$ such that :

(1) if D is a derivation of the sequent $S = B_0, \ldots, B_{n-1} \Rightarrow A$, then $t^D_{x_0,\ldots,x_{n-1}}$ validate S

(2) if D_1, D_2 are equivalent derivations, then

$$t^{D_1}_{x_0,\ldots,x_{n-1}} = t^{D_2}_{x_0,\ldots,x_{n-1}}.$$

The term t^D can then be used to show that:

THEOREM. If $\Gamma \vdash A$, then $\Gamma \models A$.

3.4.11 A NORMAL FORM THEOREM FOR DERIVABILITY. Another advantage of the natural axiomatization given in §3.4 is that we can prove a normal form (cut-elimination) theorem for it. An interesting aspect of the proof of normalization is the essential use of the rules of repetition (the structural rule (MON) also includes the usual rule of repetition).

Let us call a derivation D in which there are no cut-formulae of the form $(A\&B)$ and $\&$-cut-free derivation.

The first thing we show is how given a derivation $D = \langle T, \$, R\rangle$ and a cut-node N of T with cut-formula $(A\&B)$, one can transform D to an equivalent derivation with cut-formula either A or B. Thus assume that around the node N, the derivation D is as follows:

$$\dfrac{\begin{array}{c} D_1 \\ \Gamma, (A\&B) \Rightarrow C \end{array} \qquad \begin{array}{c} D_2 \\ \Gamma \Rightarrow (A\&B) \end{array}}{\begin{array}{c}\Gamma \Rightarrow C \\ \vdots \end{array}}$$

The critical case is when the last rule of inference applied in D_1 $[D_2]$ is $(\& \Rightarrow)$ $[(\Rightarrow \&)$ respec.$]$. In such a situation we would have (for example):

$$\dfrac{\dfrac{\begin{array}{c} D_1' \\ \Gamma, A \Rightarrow C \end{array}}{\Gamma, (A\&B) \Rightarrow C} \qquad \dfrac{\begin{array}{c} D_2^1 \\ \Gamma \Rightarrow A \end{array} \quad \begin{array}{c} D_2^2 \\ \Gamma \Rightarrow B \end{array}}{\Gamma \Rightarrow (A\&B)}}{\begin{array}{c}\Gamma \Rightarrow C \\ \vdots \end{array}}$$

Then we transform the above derivation into the derivation:

$$\dfrac{\dfrac{\begin{array}{c}D_1'\\ \Gamma, A \Rightarrow C\end{array}}{\Gamma, A \Rightarrow C} \qquad \dfrac{\begin{array}{cc}D_2^1 & D_2^1\\ \Gamma \Rightarrow A & \Gamma \Rightarrow A\end{array}}{\Gamma \Rightarrow A}}{\Gamma \Rightarrow C}$$
$$\vdots$$

Iterating the above procedure one can then transform a given derivation into a derivation which is $\mathcal{S}$-cut-free.

Then traditional methods of reducing a cut-formula of the form $(A \supset B)$ can be modified so as to apply to our axiomatization. Intertwining the two reductions one can then obtain the normalization theorem.

§4. RELATION BETWEEN THE CONCEPTS INTRODUCED.

4.1 THE CLASSICAL CASE. Let $IVAL_C$, $SVAL_C$, THM_C be the sets of intuitively valid sentences, set-theoretically valid sentences and provable sentences (in some natural axiomatization) of classical logic respectively. Then one usually argues as follows:

(.1) $IVAL_C \subseteq SVAL_C$,

because if a sentence is valid in all possible structures then it certainly is valid in all set-theoretical structures.

(.2) $THM_C \subseteq IVAL_C$,

because the axioms and rules were chose so as to be correct.

Combining (.1) and (.2) one then obtains

(.3) $THM_C \subseteq IVAL_C \subseteq SVAL_C$.

Then Gödel's completeness theorem, using a rather weak set-theory, gives the mathematical result:

$$SVAL_C \subseteq THM_C.$$

Combining the latter with (.3) then gives us:

(.4) $THM_C = IVAL_C = SVAL_C$.

We remark once again that in order to derive (.4) certain existential assumptions in set-theory were required.

4.2 THE CASE FOR THE (CONSTRUCTIVE) LANGUAGE $\not{L}$. First of all, instead of 4.1.1 we have

(.1) $FVAL_{\not{L}} \subseteq IVAL_{\not{L}}$,

because if we have a term (of combinatory logic) which validates a sentence A, then we do have an intuitive construction that proves A. The converse is by no means obvious.

The soundness theorem, combined with (.1) then gives us

$$(.2) \quad \mathrm{THM}_{Ł} \subseteq \mathrm{FVAL}_{Ł} \subseteq \mathrm{IVAL}_{Ł}.$$

Now a completeness theorem for the axiomatization of §3.4 (we already know that $\mathrm{FVAL}_{Ł}$ is recursively axiomatizable) would give us only the mathematical result that

$$\mathrm{THM}_{Ł} = \mathrm{FVAL}_{Ł}.$$

Unfortunately the above result does not produce the analog of the classical 4.1.4. In order to obtain such a result (even in the presence of a mathematical completeness theorem) we need first to justify that

$$(.3) \quad \mathrm{IVAL}_{Ł} \subseteq \mathrm{FVAL}_{Ł}$$

The latter could be immediately obtained if one could show that any intuitive construction could be represented by a term of combinatory logic; in other words, 4.2.3 would be a consequence of (some form of) Church's thesis.

If Church's thesis is to be involved, then it is probably advisable to consider number theory in some more detail.

§5. NUMBER THEORY AND STRONG CONJUNCTION.

5.5 A CONCRETE MODEL FOR THE SENTENTIAL LANGUAGE Ł. Using the techniques of Troelstra [1979] one can show that there are primitive recursive terms ε, ρ, λ, primitive recursive predicates P, $\cong$ and for each sentence A of Ł a primitive recursive predicate P_A such that:

$$(.1) \quad \mathrm{PRA} \vdash P_{B\supset C}(\utilde{n}) \equiv P(\utilde{n}, \ulcorner P_B(x) \supset P_C(\varepsilon(\utilde{n},x))\urcorner),$$

$$(.2) \quad \mathrm{PRA} \vdash P_{B\&C}(\utilde{n}) \equiv P(\utilde{n}, \ulcorner P_B(\lambda\utilde{n}) \wedge P_C(\rho\utilde{n}) \wedge \lambda\utilde{n} \cong \rho\utilde{n}\urcorner),$$

$$(.3) \quad \mathrm{PRA} \vdash \utilde{n} \cong \utilde{n} \wedge (\utilde{n}\cong\utilde{m} \supset \utilde{m}\cong\utilde{n}) \wedge (\utilde{n}\cong\utilde{m} \wedge \utilde{m}\cong\utilde{r} \supset \utilde{n}\cong\utilde{r}),$$

$$(.4) \quad \mathrm{PRA} \vdash P_A(\utilde{n}) \wedge \utilde{n} \cong \utilde{m} \supset P_A(\utilde{m}),$$

$$(.5) \quad \text{if } A \in \mathrm{FVAL}_{Ł}, \text{ then for some } n, \ \mathrm{PRA} \vdash P_A(\utilde{n}),$$

where PRA is primitive recursive arithmetic and $\ulcorner\ \urcorner$ gives the numeral corresponding to the Gödel number.

5.2 A FORMAL SYSTEM OF NUMBER THEORY FOR STRONG CONJUNCTION. Let $\utilde{\mathrm{HA}}$ be intuitionistic number theory formulated in a sequent calculus. Then $\utilde{\mathrm{HA}}(\&)$ is the extension of $\utilde{\mathrm{HA}}$ obtained by

(1) Enlarging the class of formulae so as to include $\&$.

(2) Add the inference schemas corresponding to $(\& \Rightarrow)$ and $(\Rightarrow \&)$,

(3) Add the (binary) rule of repetition,

(4) Define pre-derivations and equivalent pre-derivations analogously to §3.4.7.

(5) Define derivations in $\underset{\sim\sim}{HA}(\&)$ analogously to §3.4.8.

5.2.1 FORMAL REALIZABILITY FOR $\underset{\sim\sim}{HA}(\&)$. Using an induction on the logical complexity of the formula A of $\underset{\sim\sim}{HA}(\&)$ one can show that there is a formula $x\underset{\sim}{r}A$ of $\underset{\sim\sim}{HA}$ which formally expresses the recursive realizability of A. The only addition required to Troelstra [1973] is the clause:

$$x\underset{\sim}{r}(A\&B) \;=\; x\underset{\sim}{r}A \wedge x\underset{\sim}{r}B$$

For each formula A of $\underset{\sim\sim}{HA}(\&)$, $x\underset{\sim}{r}A$ is an almost-negative formula of $\underset{\sim\sim}{HA}$. (see page 193 of Troelstra [1973]).

The usual techniques then give us that:

(.1) if $\underset{\sim\sim}{HA}(\&) \vdash A$, then $\underset{\sim\sim}{HA} \vdash \exists x[x\underset{\sim}{r}A]$

Now let ECT_0 be the schema of the "extended Church's thesis". In Troelstra [1973] , page 196 it is shown that for formulae B of $\underset{\sim\sim}{HA}$

(.2) $\underset{\sim\sim}{HA} \vdash \exists x[x\underset{\sim}{r}B]$ iff $\underset{\sim\sim}{HA} + ECT_0 \vdash B$.

Combining (.1) and (.2) we then obtain that $\underset{\sim\sim}{HA}(\&)$ is conservative over $\underset{\sim\sim}{HA} + ECT_0$ (with respect to the formulae of $\underset{\sim\sim}{HA}$). However, of more interest would be to show that $\underset{\sim\sim}{HA}(\&)$ is conservative over $\underset{\sim\sim}{HA}$ (which in our opinion is a minimum requirement for $\&$ to be considered as an "intuitionistic connective"). In any case, the above conservative extension result further supports the contention that the completeness of the axiomatization of §3.4 will involve some form of Church's thesis.

§6. CONCLUSION.

Although we feel confident that the results obtained in this note show that proof-functional connectives are viable concepts, we do not find that strong conjunction is, per se, of great interest. Of much more interest would be some kind of strong equivalence; two possible candidates would be given by:

(1) $A \Leftrightarrow B \;=\; (A \supset B) \;\&\; (B \supset A)$,

(2) $\pi_{A\sim B}(c) \;\equiv\; \pi(c, \ulcorner \pi_A(x) \equiv \pi_B(x) \urcorner)$.

But of course, it was strong conjunction that led us to the concept of a proof-functional connective.

It perhaps should be remarked that accepting proof-functional connectives, such as strong conjunction, requires rejecting the assumption that a construction proves a unique sentence and thus forces us to distinguish between a construction as an object and a construction as a method.

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