Int J Game Theory (1998) 27:257–267

International Journal of Game Theory

Finitely additive stochastic games with Borel measurable payoffs*

A. Maitra, W. Sudderth

University of Minnesota, School of Statistics, 270 Vincent Hall, 206 Church Street S.E., Minneapolis, MN 55455-8868, USA

Received December 1996/Final version November 1997

Abstract. We prove that a two-person, zero-sum stochastic game with arbitrary state and action spaces, a finitely additive law of motion and a bounded Borel measurable payoff has a value.

Key words: Two-person, zero-sum stochastic game, finitely additive strategy, perfect information game, Borel measurable

1. Introduction

Let X, A, B be nonempty sets and let q be a function which assigns to each triple $(x, a, b) \in X \times A \times B$ a finitely additive probability measure defined on the power-set of X. Suppose that f is a bounded, Borel measurable function on $H = Z^N$, where N is the set of positive integers, $Z = A \times B \times X$ is endowed with the discrete topology, and Z^N the product topology. The stochastic game $\mathscr{S}(f)(x)$ starts at some initial state $x \in X$. Player I chooses an action $a_1 \in A$ and, simultaneously, player II chooses $b_1 \in B$. The players can choose their actions at random, which means that the players can choose their actions according to finitely additive probability measures on the power-sets of their respective action sets. The next state x_1 has distribution $q(\cdot|x, a_1, b_1)$ and is announced to the players along with their chosen actions. The procedure is iterated so as to generate a random sequence $h = ((a_1, b_1, x_1), (a_2, b_2, x_2), \ldots)$ and the payoff to II from I is $f(h)$.

A *strategy* σ for I is a sequence $\sigma_0, \sigma_1, \ldots$ such that $\sigma_0 \in P(A)$ and, for $n \geq 1$, σ_n is a mapping from Z^n into $P(A)$, where $P(A)$ is the set of all finitely

* Research supported by National Science Foundation Grants DMS-9423009 and DMS-9703285.

additive probability measures on the power-set of A. One defines a strategy τ for player II in an analogous manner, with $P(A)$ replaced by $P(B)$, the set of finitely additive probability measures on the power-set of B.

Strategies σ for player I and τ for player II together with an initial state x determine a finitely additive probability measure $P_{x,\sigma,\tau}$ on the Borel subsets of H. The construction of this probability measure is described in Maitra and Sudderth [7, section 2] and in the rest of this paper, we will follow the notation established in that article. With the probability measure $P_{x,\sigma,\tau}$ at hand, one can now define the expected payoff that accrues to I from II, when I uses σ and II uses τ, as

$$E_{x,\sigma,\tau}(f) = \int f \, dP_{x,\sigma,\tau}.$$

The upper and lower values $\overline{V}(x), \underline{V}(x)$ of the game $\mathscr{S}(f)(x)$ are defined in the usual manner:

$$\overline{V}(x) = \inf_{\tau} \sup_{\sigma} E_{x,\sigma,\tau}(f)$$

and

$$\underline{V}(x) = \sup_{\sigma} \inf_{\tau} E_{x,\sigma,\tau}(f),$$

where the sup is over all strategies σ of player I and the inf is over all strategies τ of player II.

The main result of this article is the following theorem of Zermelo-Fraenkel set theory with the axiom of choice (ZFC):

Theorem 1.1. *For every $x \in X$, the game $\mathscr{S}(f)(x)$ has a value, that is,*

$$\overline{V}(x) = \underline{V}(x).$$

Our proof of the theorem will follow very closely Martin's proof in [11] of the determinacy of Blackwell games played on finite sets. Some details will be different because there is the additional ingredient of changing states and furthermore our arguments take place in a finitely additive setting. But in essentials this proof is the same as Martin's. In particular, the proof will rely heavily on Martin's theorem [8, 9] in ZFC asserting the determinacy of Borel games of perfect information played on arbitrary sets.

Theorem 1.1 is a vast improvement on Theorem 2.1 of [7], which is the special case of Theorem 1.1 when f is of the form

$$f((a_1, b_1, x_1), (a_2, b_2, x_2), \ldots) = \limsup_n u(x_n),$$

where u is a bounded function on X.

The reader, unfamiliar with finitely additive probability theory, should assume that X is countable, A, B are finite and $q(\cdot|x, a, b)$ is countably additive, or that A, B are finite and $q(\cdot|x, a, b)$ is countably additive with countable support.

In the rest of this paper, we will follow the notation and terminology for stochastic games set down in section 3 of [7].

2. One-shot games

In a one-shot or one-move game $\mathscr{A}(\phi)(x)$ starting from x, where ϕ is a bounded function on $Z = A \times B \times X$, players I and II choose actions a, b simultaneously with $a \in A$ and $b \in B$ and the payoff from II to I is

$$\int \phi(a, b, x_1) q(dx_1 | x, a, b).$$

The expected payoff from II to I when I plays a random strategy $\mu \in P(A)$ and II plays $\nu \in P(B)$ is

$$\iiint \phi(a, b, x_1) q(dx_1 | x, a, b) \mu(da) \nu(db)$$

which we abbreviate by $\tilde{\phi}(\mu, \nu)$. Here the order of integration is important and we will always integrate in the above order. Also in the construction of $P_{x,\sigma,\tau}$, we use this order of integration (see section 2 of [7]).

Theorem 2.1. *For any bounded function ϕ on Z and any $x \in X$, the game $\mathscr{A}(\phi)(x)$ has value $(S\phi)(x)$, that is,*

$$\begin{aligned}(S\phi)(x) &= \inf_{\nu \in P(B)} \sup_{\mu \in P(A)} \tilde{\phi}(\mu, \nu) \\ &= \sup_{\mu \in P(A)} \inf_{\nu \in P(B)} \tilde{\phi}(\mu, \nu).\end{aligned}$$

Moreover, both players have optimal strategies.

Proof. The first assertion is by Theorem 3 of Heath and Sudderth [4]. (It also follows easily from an old result of Ky Fan [3].)

To see that player I has an optimal strategy, first choose, for each $n \in N$, a random strategy $\mu_n \in P(A)$ such that $\tilde{\phi}(\mu_n, b) \geq (S\phi)(x) - 1/n$ for all $b \in B$, where

$$\tilde{\phi}(\mu_n, b) = \iint \phi(a, b, x_1) q(dx_1 | x, a, b) \mu_n(da).$$

Next let γ be a finitely additive probability measure on the power set of N such that $\gamma(\{n\}) = 0$ for all $n \in N$ and define $\mu^* \in P(A)$ by

$$\int g(a) \mu^*(da) = \iint g(a) \mu_n(da) \gamma(dn)$$

for bounded, real-valued g on A. Then, for all $b \in B$,

$$\tilde{\phi}(\mu^*, b) = \iiint \phi(a, b, x_1) q(dx_1 | x, a, b) \mu_n(da) \gamma(dn) \geq (S\phi)(x),$$

and so $\tilde{\phi}(\mu^*, v) \geq (S\phi)(x)$ for all $v \in P(B)$. Thus μ^* is optimal for I. An optimal strategy for II can be found in a similar fashion. □

3. The game of perfect information

For this and the next section, assume without loss of generality that the Borel measurable function f of section 1 takes values in the unit interval.

With $v \in [0, 1]$ and initial state $x \in X$, define a game $G(v, x)$ of perfect information between players I and II as follows:

I chooses $\phi_0 \in [0, 1]^Z$ such that $(S\phi_0)(x) \geq v$, then II chooses $(a_1, b_1, x_1) \in Z$, after which I chooses $\phi_1 \in [0, 1]^Z$ such that $(S\phi_1)(x_1) \geq \phi_0((a_1, b_1, x_1))$, then II chooses $(a_2, b_2, x_2) \in Z$, and so on. In this way, a *run* is produced as follows:

$$\phi_0, (a_1, b_1, x_1), \phi_1, (a_2, b_2, x_2), \ldots, (a_n, b_n, x_n), \phi_n, \ldots$$

such that $(a_n, b_n, x_n) \in Z$ and $\phi_n \in [0, 1]^Z$ with $(S\phi_n)(x_n) \geq \phi_{n-1}((a_n, b_n, x_n))$ for all n. Note that I always has a move available, for he can play $\phi \equiv 1$ at any of his turns. The run is a *win* for I iff

$$f((a_1, b_1, x_1), (a_2, b_2, x_2), \ldots) \geq \liminf_n \phi_{n-1}((a_n, b_n, x_n)).$$

It is easily verified that the winning set is Borel measurable. More precisely, since player I's choices from $[0, 1]^Z$ are restricted, the game $G(v, x)$ can be formulated in terms of a pruned tree T, the nodes of which contain only legal positions of the players, on the set Y, which is the disjoint union of the sets $[0, 1]^Z$ and Z (for details, see Kechris [5, section 20]). The set $[T]$ of infinite branches of T is given the relative topology it inherits from Y^N, which is endowed with the product topology with Y discrete. The winning set for I in $G(v, x)$ is then a Borel subset of $[T]$ and, consequently, by a celebrated result of Martin [8, 9], the game $G(v, x)$ is determined, that is, either player I has a winning strategy or II has a winning strategy. (There is a beautiful exposition of this result in [5, section 20].)

4. Proof of theorem 1.1

Suppose that σ^* is a winning strategy for player I in the game $G(v, x)$. The strategy σ^* determines a sequence of moves of player I recursively along a history $h = ((a_1, b_1, x_1), (a_2, b_2, x_2), \ldots)$ in H which we denote as follows:

$$\sigma^*(e) = \phi_0$$

(e is the empty sequence.)

$$\sigma^*(\phi_0, (a_1, b_1, x_1), \ldots, \phi_{n-1}, (a_n, b_n, x_n)) = \phi_n.$$

Note that ϕ_n depends on the first n coordinates of h. We have suppressed this dependence on h to make the notation less tedious.

We now define a strategy σ for player I in the stochastic game $\mathscr{S}(f)(x)$ as follows:

$$\sigma_0 = \text{any optimal strategy for I in the game } \mathscr{A}(\phi_0)(x),$$

and for $n \geq 1$,

$$\begin{aligned}&\sigma_n((a_1, b_1, x_1), (a_2, b_2, x_2), \ldots, (a_n, b_n, x_n))\\ &\quad = \text{any optimal strategy for I in the game } \mathscr{A}(\phi_n)(x_n).\end{aligned}$$

Let τ be any strategy for II in $\mathscr{S}(f)(x)$. Abbreviate $P_{x,\sigma,\tau}$ by P and $E_{x,\sigma,\tau}$ by E. For each history h and positive integer n, let $p_n(h)$ be the first n coordinates (triples) of h.

Lemma 4.1. $E(f) \geq v$.

Proof. Define random variables Y_n on the history space $H = Z^N$ as follows:

$$Y_0(h) = v$$

and for $n \geq 1$,

$$Y_n(h) = \phi_{n-1}((a_n, b_n, x_n)),$$

where $h = ((a_1, b_1, x_1), (a_2, b_2, x_2), \ldots)$.

It now follows from the definition of σ that

$$E(Y_{n+1} | p_n(h)) \geq Y_n.$$

(This is a departure from the notation of [7]. In [7], we wrote $E_{\sigma[p_n(h)],\tau[p_n(h)]}(Y_{n+1}p_n(h))$ for $E(Y_{n+1}|p_n(h))$.) Consequently, $\{Y_n\}$ is a bounded submartingale. By the Optional Sampling Theorem (Dubins and Savage [2, Theorem 2.12.2]),

$$E(Y_t) \geq E(Y_0) = v$$

for any stop rule t on H. Consequently,

$$\liminf_t E(Y_t) \geq v,$$

where the liminf is taken over the directed set of stop rules. It now follows from the Fatou equation (Sudderth [15] for the countably additive version of this result; Purves and Sudderth [13, Theorem 10.4] or Chen [1] for the finitely additive version used here) that

$$E(\liminf_n Y_n) \geq v.$$

(This inequality also follows directly from the submartingale convergence theorem and the dominated convergence theorem in the countably additive case.) But, because σ^* is a winning strategy for I in $G(v,x)$, $f \geq \liminf_n Y_n$. So

$$E(f) \geq v. \qquad \square$$

Corollary 4.2. *If I can win $G(v,x)$, then $\underline{V}(x) \geq v$.*

Suppose now that τ^* is a winning strategy for II in $G(v,x)$. Fix $\delta > 0$. We first define *consistency* (with respect to τ^* and δ) of a partial history $p = ((a_1,b_1,x_1),(a_2,b_2,x_2),\ldots,(a_n,b_n,x_n))$ of the stochastic game and at the same time define a function ψ on consistent sequences of positive length into $[0,1]^Z$. This will be done by induction on the length of p.

If p is inconsistent, all extensions of p are inconsistent by definition. The empty sequence is by definition consistent. Suppose now that p is consistent and that ψ has been defined on p and all its initial segments in such a way that

$$\psi((a_1,b_1,x_1)),(a_1,b_1,x_1),\psi(((a_1,b_1,x_1),(a_2,b_2,x_2))),(a_2,b_2,x_2),\ldots,$$
$$\psi(p),(a_n,b_n,x_n)$$

is a legal "partial run" in the game $G(v,x)$ in which II uses τ^*. Abbreviate the "partial run" above by $\alpha(p)$. (We define α at the empty sequence to be the empty sequence.)

Let r be an extension of p such that length$(r) = n+1$. We say that r is *consistent* if

$$\tau^*(\alpha(p),\phi) = l(r) \qquad (*)$$

for some legal move ϕ of I in $G(v,x)$ and $l(r)$ is the last coordinate (triple) of r. In order to define $\psi(r)$ for all consistent extensions r of p such that length$(r) = n+1$, we first set

$$u_p((a,b,y)) = \inf\{\phi((a,b,y)) : \phi \text{ is legal and satisfies } (*) \text{ with } r = p(a,b,y)\},$$

where $(a,b,y) \in Z$, and the infimum of the empty set is 1.

The next lemma follows from the assumptions made up to now on ψ.

Lemma 4.3. $S(u_p)(x_n) \leq \psi(p)((a_n,b_n,x_n))$.

Proof. Suppose not. Then there is $\varepsilon > 0$ such that

$$S(u_p)(x_n) \geq \psi(p)((a_n,b_n,x_n)) + \varepsilon.$$

Let

$$\phi^*((a,b,y)) = \begin{cases} u_p((a,b,y)) - \varepsilon, & \text{if } u_p((a,b,y)) > \varepsilon, \\ 0, & \text{otherwise.} \end{cases}$$

Plainly, $(S\phi^*)(x_n) \geq \psi(p)((a_n,b_n,x_n))$, so ϕ^* is a legal move with which I can extend the "partial run" to $\alpha(p)$, ϕ^* in $G(v,x)$. Let $\tau^*(\alpha(p),\phi^*) = (a^*,b^*,x^*)$.

If $\phi^*((a^*, b^*, x^*)) = 0$, then τ^* is easily defeated if I plays ϕ^* after $\alpha(p)$ and thereafter puts down the identically zero function on Z at each of his subsequent turns to play. So $\phi^*((a^*, b^*, x^*)) > 0$, hence $\phi^*((a^*, b^*, x^*)) = u_p((a^*, b^*, x^*)) - \varepsilon$, so that $\phi^*((a^*, b^*, x^*)) < u_p((a^*, b^*, x^*))$, contradicting the definition of u_p. This completes the proof of the lemma. □

If r is a consistent extension of p such that length$(r) = n + 1$, define $\psi(r)$ to be a legal move ϕ of I such that ϕ satisfies $(*)$ and

$$\phi(l(r)) \leq u_p(l(r)) + \frac{\delta}{2^{n+1}}.$$

Note that $\alpha(p)$, $\psi(r)$, $l(r)$ is an "initial run" of $G(v, x)$ in which II uses τ^*.

We are now ready to define a strategy τ for II in $\mathscr{S}(f)(x)$. For p consistent of length n, we define $\tau_n(p)$ to be any optimal strategy for II in the game $\mathscr{A}(u_p)(x_n)$, where $p = ((a_1, b_1, x_1), (a_2, b_2, x_2), \ldots, (a_n, b_n, x_n))$ and $x_0 = x$. For inconsistent p of length n, $\tau_n(p)$ can be taken to be an arbitrary finitely additive probability measure on the power-set of B.

Let σ be any strategy for I in $\mathscr{S}(f)(x)$. Abbreviate $P_{x,\sigma,\tau}$ by P and $E_{x,\sigma,\tau}$ by E.

Lemma 4.4. $E(f) \leq v + \delta$.

Proof. Define random variables Z_n on the history space H as follows:

$$Z_0(h) = v$$

and, for $n \geq 1$,

$$\begin{aligned} Z_n(h) &= \psi(p_n(h))(l(p_n(h))), \quad \text{if } p_n(h) \text{ is consistent,} \\ &= 1, \quad \text{otherwise.} \end{aligned}$$

Claim: $E(Z_{n+1} | p_n(h)) \leq Z_n + \delta/2^{n+1}$.

If $p_n(h)$ is inconsistent, $Z_n = 1$ and the inequality is clear. Suppose then that $p_n(h)$ is consistent. Then

$$\begin{aligned} &E(Z_{n+1} | p_n(h)) \\ &\quad = \iiint\limits_{\{(a,b,y)\,:\,p_n(h)(a,b,y) \text{ is consistent}\}} \psi(p_n(h)(a, b, y))((a, b, y)) q(dy | x_n, a, b) \sigma_n(p_n(h))(da) \tau_n(p_n(h))(db) \\ &\qquad + \iiint\limits_{\{(a,b,y)\,:\,p_n(h)(a,b,y) \text{ is inconsistent}\}} 1 q(dy | x_n, a, b) \sigma_n(p_n(h))(da) \tau_n(p_n(h))(db) \\ &\quad \leq \iiint u_{p_n(h)}((a, b, y)) q(dy \mid x_n, a, b) \sigma_n(p_n(h))(da) \tau_n(p_n(h))(db) + \frac{\delta}{2^{n+1}} \end{aligned}$$

$$\begin{aligned}
&\le S(u_{p_n(h)})(x_n) + \frac{\delta}{2^{n+1}}\\
&\le \psi(p_n(h))((a_n, b_n, x_n)) + \frac{\delta}{2^{n+1}}\\
&= Z_n + \frac{\delta}{2^{n+1}},
\end{aligned}$$

where the next-to-last inequality is by virtue of the definition of $\tau_n(p_n(h))$ and the last inequality is by virtue of Lemma 4.3.

Hence, if $W_n = Z_n + \delta(1 + 1/2^n)$, $n \geq 0$, then $\{W_n\}$ is a bounded supermartingale. Indeed,

$$\begin{aligned}
E(W_{n+1}|p_n(h)) &= E(Z_{n+1}|p_n(h)) + \delta\left(1 + \frac{1}{2^{n+1}}\right)\\
&\le Z_n + \frac{\delta}{2^{n+1}} + \delta\left(1 + \frac{1}{2^{n+1}}\right)\\
&= Z_n + \delta\left(1 + \frac{1}{2^n}\right)\\
&= W_n.
\end{aligned}$$

So, by the Optional Sampling Theorem,

$$E(W_t) \le E(W_0) = v + 2\delta$$

for every stop rule t on H. It now follows that

$$E(Z_t) \le v + \delta$$

for every stop rule t. So, by the Fatou equation,

$$E(\liminf_n Z_n) \le v + \delta.$$

(In the countably additive case, the ordinary Fatou inequality would suffice to establish the inequality above.) But, because τ^* is a winning strategy for II in $G(v, x)$, $f \le \liminf_n Z_n$. So $E(f) \le v + \delta$. □

Corollary 4.5. *If II can win $G(v, x)$, then $\overline{V}(x) \le v$.*

In order to complete the proof of Theorem 1.1, note that if I can win $G(v, x)$ and $v' < v$, then I can win $G(v', x)$ as well. Indeed, any stragegy for I in $G(v, x)$ will produce only legal moves in $G(v', x)$, so will be a strategy for I in $G(v', x)$. Hence I can use a winning strategy in $G(v, x)$ to win $G(v', x)$. Consequently, by Corollary 4.2,

$$\sup\{v \in (0, 1] : \text{I can win } G(v, x)\} \le \underline{V}(x).$$

Similarly, if II can win $G(v, x)$ and $v < v'$, then II can win $G(v', x)$. Consequently, by Corollary 4.5,

$$\overline{V}(x) \leq \inf\{v \in (0,1] : \text{II can win } G(v,x)\}.$$

Since $G(v,x)$ is determined for every v, it follows that $\underline{V}(x) = \overline{V}(x)$. This completes the proof of Theorem 1.1. □

5. An approximation theorem

As remarked by Martin in [11], the proof that a Blackwell game has a value already contains in it the proof that the value of a Blackwell game with Borel measurable payoff function can be approximated by the value of a Blackwell game with payoff function belonging to a low Borel class. The same result holds for stochastic games. In order to state the result, we denote by $V(g)(x)$ the value of the stochastic game $\mathscr{S}(g)(x)$, where g is a bounded Borel measurable function on H. For a bounded function u on $Z^{<N} = \bigcup_{n \geq 1} Z^n$, define u_* and u^* on H as follows:

$$u_*(h) = \liminf_n \; u(p_n(h))$$

and

$$u^*(h) = \limsup_n \; u(p_n(h)).$$

Theorem 5.1. *For any bounded Borel measurable function f on H and $x \in X$,*

$$\begin{aligned} V(f)(x) &= \sup\{V(u_*)(x) : u \text{ is a bounded function on } Z^{<N} \text{ and } u_* \leq f\} \\ &= \inf\{V(u^*)(x) : u \text{ is a bounded function on } Z^{<N} \text{ and } f \leq u^*\}. \end{aligned}$$

Proof. Without loss of generality, assume $0 \leq f \leq 1$. Suppose that $v < V(f)(x)$. The random variables Y_n in the proof of Lemma 4.1 can be used to define a function u on $Z^{<N}$ such that $u_* \leq f$ and $V(u_*)(x) \geq v$.

The second inequality follows from the first by considering the stochastic game $\mathscr{S}(-f)(x)$. □

In [7], we proved that a stochastic game of the type $\mathscr{S}(u^*)(x)$ has a value. Theorem 5.1 states that the value of a general stochastic game $\mathscr{S}(f)(x)$, with f Borel measurable, can be approximated by the value of a special stochastic game of the type $\mathscr{S}(u^*)(x)$.

6. Concluding remarks

Theorem 1.1 actually holds for a wider class of functions than the class of bounded Borel measurable functions on H. In order to describe this class, we follow Martin [10] in saying that a subset C of H is *quasi-Borel* if both C and C^c can be obtained as results of operation $(\mathscr{A})$ on closed subsets of H. In the same article, Martin gives an inductive definition of these sets starting with the open subsets of H. Quasi-Borel sets form a σ-field containing the Borel σ-field and this inclusion can be a strict one. A real-valued function f on H is *quasi-*

Borel measurable if f is measurable with respect to the σ-field of quasi-Borel sets.

In [10], Martin proved in ZFC that quasi-Borel games of perfect information are determined. So if f is a bounded quasi-Borel measurable function on H, then the game $G(v, x)$ of section 3 is a quasi-Borel game of perfect information and hence determined. So the proof of Theorem 1.1 goes through exactly as before by noting that, by virtue of Theorem 5.3 in Purves and Sudderth [14], the "strategic measure" $P_{x,\sigma,\tau}$ is actually defined on the quasi-Borel sets and formulas (3.1) and (3.2) in [7] for expectations continue to hold for bounded, quasi-Borel measurable functions. Consequently, Theorem 1.1 extends to stochastic games $\mathscr{S}(f)(x)$ with f bounded, quasi-Borel measurable.

There are technical difficulties in implementing the proof of Theorem 1.1 presented here in the case of measurable stochastic games. More precisely, suppose that X is Polish, A, B are finite, $q(\cdot|x, a, b)$ is a Borel measurable transition function which is countably additive on the Borel σ-field of X and f is a bounded, Borel measurable function on H. The main technical difficulty arises because now the strategies in the stochastic game have to be measurable. But since good strategies in the stochastic game are defined from winning strategies in the game of perfect information, their measurability depends on the existence of measurable winning strategies in the game of perfect information. But this latter is a hard problem and is related to Moschovakis's Third Periodicity Theorem [12, Theorem 6E.1], which establishes the definability of winning strategies where the players make choices from a countable set.

However, the following fact may not be without interest. Extend each probability measure $q(\cdot|x, a, b)$ to a finitely additive probability measure on the power-set of X. Then the resulting finitely additive stochastic game has a value (by Theorem 1.1), say, $V(x)$. Denote the upper and lower values of the measurable stochastic game by $\overline{V}_c(x)$ and $\underline{V}_c(x)$, respectively. Then it follows by virtue of Theorem 1.2 in Maitra, Purves and Sudderth [6] that

$$\underline{V}_c(x) \leq V(x) \leq \overline{V}_c(x).$$

Acknowledgement. We want to thank Sylvain Sorin and two anonymous referees for their helpful comments.

References

1. Chen R (1975) A note on the Dubins-Savage utility of a strategy. Israel J. Math. 21:1–6
2. Dubins LE, Savage LJ (1976) Inequalities for stochastic processes. Dover, New York
3. Fan K (1953) Minimax theorems. Proc. Nat. Acad. Sci. USA 39:42–47
4. Heath D, Sudderth W (1972) On a theorem of de Finetti, oddsmaking and game theory. Ann. Math. Statist. 43:2072–2077
5. Kechris AS (1995) Classical descriptive set theory. Springer-Verlag, New York
6. Maitra A, Purves R, Sudderth W (1992) A capacitability theorem in measurable gambling theory. Trans. Amer. Math. Soc. 333:221–249
7. Maitra A, Sudderth W (1993) Finitely additive and measurable stochastic games. Internat. J. Game Theory 22:201–223
8. Martin DA (1975) Borel determinacy. Annals of Mathematics 102:363–371
9. Martin DA (1985) A purely inductive proof of Borel determinacy. In: Nerode A, Shore RA (eds.) Recursion theory, Proc. Symposia Pure Math. 42 Amer. Math. Soc., Providence, pp. 303–308

10. Martin DA (1990) An extension of Borel determinacy. Annals of Pure and Applied Logic 49: 279–293
11. Martin DA (1996) The determinacy of Blackwell games. J. Symbolic Logic, to appear
12. Moschovakis YN (1980) Descriptive set theory. North-Holland, Amsterdam
13. Purves R, Sudderth W (1973) Some finitely additive probability. University of Minnesota Technical Report 220
14. Purves R, Sudderth W (1976) Some finitely additive probability. Ann. Probab. 4:259–276
15. Sudderth W (1971) A "Fatou equation" for randomly stopped variables. Ann. Math. Statist. 42:2143–2146