# *The Ritual Origin of Geometry*

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Contents



## 1. Introduction

If we wish to find the origins of geometry, we shall certainly go back to ancient Greece, but it is equally certain that we shall not stop there. We now know so much about Babylonian mathematics that no one any longer thinks that Greek geometry had a Greek origin.

In 1937, O. NEUGEBAUER wrote: "What is called Pythagorean in the Greek tradition had better be called Babylonian"; and in 1943, after a cuneiform text concerning "PYTHAGOREAN number triples" was discovered, he considered that his conjecture had been validated[1].

Yet long before 1937 people had suggested a non-Greek origin to Greek mathematics; and long before 1943 people had pointed out that sacred books of the East contain "PYTHAGOREAN numbers". Such numbers are mentioned

[1] *Mathematical Cuneiform Texts*, p. 41. See also B.L. VAN DER WAERDEN, *Science Awakening*, p. 5.

in the *Sulvasutras*, ancient Indian works on altar constructions. Moreover, some of the triples, for example (8, 15, 17), satisfy the basic property for Pythagorean triples, namely, the square of one of them is the sum of the squares of the other two ($17^2 = 8^2 + 15^2$), but are not amongst the triples ascribed to the Pythagoreans —the latter triples have the property that, after all common factors have been removed, the difference between the two largest numbers is 1 [2].

NEUGEBAUER does not mention the *Sulvasutras* in his book *Vorlesungen über Geschichte der Antiken Mathematischen Wissenschaften*, nor does B.L. VAN DER WAERDEN mention them in his book *Science Awakening*.

Why this omission? We are all subject to the ancient Greek rationalist ideology, which is apt to scorn priestly works, and this may have caused the general neglect of the *Sulvasutras*. It is true that those who maintained the priority of Indian geometry may have claimed too much when they said that Greek geometry came from India: what they should have said was that Indian geometry and Greek geometry derive from a common source.

We propose to show that geometry had a ritual origin. This proposal really should not cause surprise. Possibly, if one held that Greek geometry had a Greek origin, one might be justified in dismissing such a proposal beforehand as extravagant, but we have agreed that the origins go back to the more ancient East. Moreover, even confining oneself to Greek mathematics, there are many clues pointing toward a ritual origin: for example, the association of early Greek mathematics with cult is one of the commonplaces of Greek history.

The view that geometry had a ritual origin is not an isolated one, but is part of a wider view that civilization itself had a ritual origin.

Writing of India, G. THIBAUT says: "It is well known that not only Indian life with all its social and political institutions has been at all times under the mighty sway of religion, but that we are also led back to religious belief and worship when we try to account for the origin of research in those departments of knowledge which the Indians have cultivated with such remarkable success. At first sight, few traces of this origin may be visible in the S'ástras of later times, but looking closer we may always discern the connecting thread. The want of some norm by which to fix the right time for the sacrifices gave the first impulse to astronomical observations [3]; urged by this want, the priests remained watching night after night the advance of the moon through the circle of the nakshatras and day after day the alternate progress of the sun toward the north and the south. The laws of phonetics were investigated because the wrath of the gods followed the wrong pronunciation of a single letter of the sacrificial formulas; grammar and etymology had the task of securing the right understanding of the holy texts. The close connection of philosophy and theology—so close that it is often impossible to decide where the one ends and the other begins—is too well known to require any comment" [4].

[2] This was first noted by A. BÜRK, "Das Āpastamba-Śulva-Sūtra" (Einleitung), *Zeitschrift der Deutschen Morgenländischen Gesellschaft*, vol. **55** (1901), p. 564.

[3] A similar suggestion has been made for Egypt by FOUCART, *Encyclopaedia of Religion and Ethics*, vol. **3**, pp. 98—99. See also Lord RAGLAN, *The Origins of Religion*, p. 100.

[4] "On the Śulvasútras", *Journal of the Asiatic Society of Bengal*, vol. **44** (1875), p. 227.

Lord RAGLAN has stressed the importance of *comparative* studies in the search for the origins of civilization. Writing about the ancient world as a whole, he says: "We have seen that many of the principal discoveries and inventions upon which our civilization is based can be traced with considerable probability to an area with its focus near the head of the Persian Gulf, and such evidence as there is suggests that they were made by ingenious priests as a means of facilitating the performance of religious ritual. It is at least possible that animals were first domesticated for convenience in sacrifice and that the first use of the plough was as a method of symbolically fertilizing the soil; the first wheel may have been a labor-saving device for keeping the sun on its course, and metal-working may have started with the making of imitation suns in gold; the first bow and arrow may have ensured victory by symbolically destroying enemies at a distance; mummification kept the dead king ritually alive, and the kite conveyed his spirit to the sky. There is *some* evidence to support all these suggestions, and its cumulative effect strengthens the theory as a whole; the theory, that is, that civilization originated in ritual, though of course a great deal more evidence would be required to establish it. Alternative theories have no evidence to support them at all"[5].

## 2. The theorem of Pythagoras

When one hears that the ancient Indians had a work on altar constructions, one is apt to think (because one is dominated by the Greek rationalist ideology) that one will find in it a few "carpenter's rules", that is, a few applications of elementary geometry to the construction of material objects. One does, indeed, find there rules for constructing a right angle. This is sometimes accomplished using the Theorem of PYTHAGORAS: the ends of a cord of eight units' length are tied to two pegs hammered into the earth at points $A$, $B$ four units apart, a mark is made at a point three units from one end of the cord, say the end tied at $B$, the string is picked up at this point and stretched so that the mark touches the earth at $C$, whereupon the right angle $ABC$ is produced. Right angles are also constructed without recourse to the Theorem of PYTHAGORAS (see Appendix).

Yet a perusal of the *Sulvasutras* shows that matters are not quite so straightforward. The construction of altars of various shapes is described, the shape depending on the particular ritual. Thus there are square altars, circular altars, and altars of many other shapes: one, the falcon-shaped altar, was composed of rectangles and does to some extent look like a falcon—or, rather, "the shadow of a falcon about to take wing". "He who desires heaven may construct the falcon-shaped altar; for the falcon is the best flyer among the birds; thus he (the sacrificer) having become a falcon himself flies up to the heavenly world."

The altars were, for the most part, composed of five layers of bricks which reached together to the height of the knee; for some cases ten or fifteen layers and a corresponding increased height of the altar were prescribed. Most, though not all, of the altars had a level surface, and these were referred to in accordance with the shape and area of the top (or bottom) face. The basic falcon-shaped altar had an area of $7\frac{1}{2}$ square purushas: the word "purusha" means *man* and is, on the one hand, a linear measure, namely, the height of a man with his arms stretched upwards (about $7\frac{1}{2}$ feet, say), and, on the other, an areal measure

[5] Lord RAGLAN, *How Came Civilization?*, p. 176.

(about $56\frac{1}{4}$ square feet). Aside from secondary modifications or variations, the body of the falcon-shaped altar was a $2 \times 2$ square (4 square purushas), the wings and tail were one square purusha each; in order that the image might be a closer approach to the real shape of a bird, wings and tail were lengthened, the former by one-fifth of a purusha each, the latter by one-tenth (Fig. 1). This was the size and shape of the falcon-altar upon its first construction. On the second construction, one square purusha was to be added, that is, the area of the second altar constructed would then be $8\frac{1}{2}$ square purushas; on the next construction another square purusha is added, and so on, until one comes to the one-hundred-and-one (and a half)-fold altar. It is clear that the sacrificer is climbing a ladder, his sacrificial rank being determined by, or determining, the area.

In the construction of the larger altars ($8\frac{1}{2}$, $9\frac{1}{2}$, etc.), the same shape as the basic altar is required; the problem of finding a square equal in area to two given squares is actually and explicitly involved: the construction is carried out using the Theorem of PATHAGORAS. The problem of converting a rectangle into a square is also explicitly involved (see Appendix). This does not look like carpentry to us.

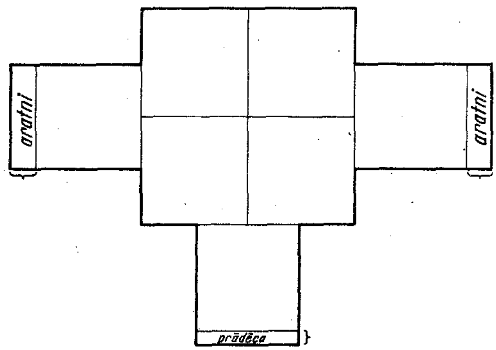

Fig. 1

The altar was the place of sacrifice and also a god himself: thus Agni is a Vedic god, but he is also the altar[6]. A.M. HOCART has well elucidated the ideas underlying the Vedic sacrifices, or altar rituals. In the course of the ritual, some object, say the sun, is identified with another object, say the eye. He warns us against supposing that this identification is made because the sacrificer was unable to tell the difference between analogy and equality. "It has frequently been represented", he says, "that primitive man mistook likeness for identity, that if he saw two things resembling one another he argued they must be the same, and whatever was done to one would react on the other. This doctrine is very unfair to primitive man and is arrived at by the study of survivals among savages who are unable to account for their proceedings because they have lost the theory... (If) we wish to know really how conclusions were arrived at in the distant past we must turn, not to those who carry out mechanically what they have inherited from their forefathers, but to those who were actually building up the doctrine or still remembered how it had been built up"[7]. He then goes on to explain that the basic idea of the Indian sacrifice was that of making an object equal to the altar, or to a part of it, by means of ritual action, and hence two things (distinct from the altar) equal to each other. "He who desires heaven, may construct the falcon-shaped altar." Why? The theory is that, by ritual action, we have:

$$\begin{aligned} \text{Falcon} &= \text{Altar} \\ \text{Sacrificer} &= \text{Altar} \end{aligned}$$

[6] The *agni* is the brick fire-altar; the *vedi* is the sacrificial ground: both words are translated as *altar*, sometimes giving rise to confusion.

[7] HOCART, *Kingship*, p. 198.

and therefore Sacrificer = Falcon, and hence he can fly to heaven. Of course, if the falcon is not well made, it will not fly.

In the successive augmentations of the falcon-shaped altar, not only is the Theorem of PYTHAGORAS used; but we see, in all likelihood, the motive of its invention. If we may paraphrase HOCART's idea, we would say that the Indian sacrifice was a kind of gematria[8], a geometric rather than a numerical gematria, the basic term of the equivalence being the area. Now the combination of gods into a single god is a familiar phenomenon in ancient religions, and if a god is assimilated to a square, this would lead to the problem of finding a square equal in area to the sum of two squares (more correctly: the observation that in a right triangle the square on the hypotenuse is the sum of the squares on the legs would have an immediate theological application). Certainly we see such ideas underlying the Hindu practice. In the *Satapatha Brâhmana* (VI, 1, 1, 1—3), we are explicitly told that "in the beginning" the Rishis (vital airs) created seven separate persons, who are assimilated to squares. After giving a reason (namely, that otherwise they could not generate), they say: "Let us make these seven persons one Person!", whereupon the seven are composed into the falcon-shaped altar. And in (X, 2, 3, 18), we read: "Sevenfold, indeed, Pragapati was created in the beginning. He went on constructing (developing) his body, and stopped at the one hundred and one-fold one."

According to PLUTARCH, the Egyptians knew the (3, 4, 5)-triangle and associated gods, male and female, with its sides, the perpendicular 3 with Osiris, the base 4 with Isis, and the hypotenuse 5 with Horus[9]. Thus here, too, the god of the hypotenuse is produced by the union of the gods of the legs.

## 3. Square vs. circle

As we have mentioned, the shape of the altar varied with the purpose of the sacrifice. One of these shapes was circular, another square, and THIBAUT[10] informs us that theological controversies arose over the question as to which shape was appropriate for the given occasion—according to HOCART[11], the same controversy as to the shape of the burial mound split Northern India into two schools; "the School of the Brahmanas, which considered itself orthodox, built square mounds in which to bury persons who had reached a certain degree in the curriculum of sacrifice. The heretics made their burial mounds circular." In these

[8] In gematria, a numerical value is assigned to each letter of the alphabet and the value of a word is the sum of the values of its letters; two words are counted as equivalent if they have the same value. See T. DANTZIG, *Number, the Language of Science*, p. 39. The Greeks and Hebrews had gematria. The Indians did not (so far as we know), but they did have the idea of identification through number.

[9] *De Iside et Osiride*, 56. See also G. J. ALLMAN, *Greek Geometry from Thales to Euclid*, p. 29; and T.L. HEATH, *Euclid's Elements*, vol. **1**, p. 417. The same notion is found in China, where "Heaven corresponds to the base of the right-angled triangle, earth to the height, man to the hypotenuse ..." (J. NEEDHAM, *Science and Civilization in China*, vol. **3**, p. 47). The Greeks also associated the Theorem with marriage, and echoes of this notion can still be heard today (see HEATH, *loc. cit.*).

[10] *Op. cit.*, p. 232. See also "The Śulvasútra of Baudhâyana", *The Pandit*, vol. **10** (1875), p. 145.

[11] *Op. cit.*, pp. 169, 177. See also *Satapatha Brahmana*, XIII, 8, 1, 5.

controversies, those concerning the altars at any rate, the area was understood to be constant, and this led, as we suggest, to the problems of squaring the circle and of turning the square into a circle. In the *Sulvasutras* there are attempts at solving these problems.

The problem of squaring the circle is one of the three famous (construction) problems of antiquity; the other two are (1) the duplication of the cube—the Delian problem, so called because the oracle at Delos is said to have given the doubling of Apollo's cubical altar as a means of ending a plague and (2) the trisection of an angle. A large part of Greek geometry actually built itself up around these problems. Writing on the problem of squaring the circle, VAN DER WAERDEN says[12]: "It has already been mentioned that, while he was in prison, Anaxagoras occupied himself with the quadrature of the circle. Altogether this problem was very popular towards the end of the fifth century. The comic poet Aristophanes even made a joke about it. In *The Birds*, he introduces the astronomer Meton, who says:

> 'With the straight ruler I set to work
> To make the circle four-cornered;
> In its center will be the market place,
> Into which all the streets will lead,
> Converging to its center like a star,
> Which, although only orbicular, sends
> Forth its rays to all sides in a straight line.'
> 'Verily, the man is a Thales!',

scoffs Pisthetaerus, the leader of the birds, and drives Meton away with blows... Is it not marvelous", continues VAN DER WAERDEN, "that a scholarly problem was so popular in Athens at that time, that it could be made a source of amusement in the theater?"

If the populace were really laughing simply over a scholarly problem, this would, indeed, be very difficult to understand. But if they were laughing at another man's religion, it would be easy.

## 4. Doubling the cube

Concerning the problem of the duplication of the cube, EUTOCIUS, the commentator of ARCHIMEDES, reproduces what he calls a letter from ERATOSTHENES to King PTOLEMY. This "letter" contains, amongst other things, two legends about the duplication problem. According to the first, "it is said that one of the ancient tragic poets brought Minos on the scene, who had a tomb built for Glaucus. When he heard that the tomb was a hundred feet long in every direction, he said: 'You have made the royal residence too small, it should be twice as great. Quickly double each side of the tomb, without spoiling the shape.' He seems to have made a mistake. For when the sides are doubled, the area is enlarged fourfold and the volume eightfold..." Afterwards, we are told, HIPPOCRATES of Chios worked on the problem. The "letter" continues: "It is further reported that, after some time, certain Delians, whom an oracle had given the task of doubling an altar, met the same difficulty. They sent emissaries to the geometers in Plato's academy to ask them for a solution. These took hold with great diligence of the problem..."

[12] *Op. cit.*, p. 130.

VAN DER WAERDEN[13], following HEATH and CANTOR, sees in this report a contradiction, as if the problem arose at the time of HIPPOCRATES, it could not have arisen at the time of PLATO, a half century after HIPPOCRATES (the letter does not quite say that the problem originated with the oracle of Delos, but perhaps this is its sense). VAN DER WAERDEN is therefore concerned with tracing the source, or sources, of the "letter". He finds that the second part derives from the *Platonicus* of ERATOSTHENES. He considers the *Platonicus* to be a dramatic story, hence not a historical source, though ERATOSTHENES may have made use of historical materials. The first part "probably derives from historical sources", and some credence can be given to the tradition that HIPPOCRATES of Chios worked on the problem. In any event, "the problem is a much older one, for it arose from the translation of the Babylonian cubic equation $x^3 = V$ into spatial geometric algebra".

Although we would like to know whether the oracle at Delos really did put the problem of the duplication of the cube, we do not consider this point to be of overriding importance. If the oracle had put the problem, the question, it seems to us, is: How did it ever occur to the oracle that doubling an altar was a way of fighting a plague? And if the oracle did not put the problem, the question is still the same: How did the person who made up the story get the idea that doubling the altar would defeat a plague?

The "letter" obviously gives us legends, and legends do not refer to history —that is, uniquely occurring events—but they do reflect custom. Historical facts may get mixed up with legends; and if we have independent historical information, it may be possible to separate off the historical facts. This still leaves us with the legends and the question as to what customs they reflect.

There is a famous legend that PYTHAGORAS, after discovering the Theorem of PYTHAGORAS, sacrificed an ox in honor of the discovery. "But the entire story is an impossible one", says VAN DER WAERDEN, "Pythagoras was strongly opposed to the killing and sacrificing of animals, of cattle especially". Yet it is a plain fact and not a reconstruction that many an ox fell victim to the Theorem of PYTHAGORAS.

Thus, the information that sacrifice and the Theorem of PYTHAGORAS are related is embedded in Greek history. This information cannot be read off from the legend by itself; but, as a comparison shows, evidence of the Indian practice is found in Greek history.

The Delian legend may be considered in the same light. In the legend, the temple "architects" are confronted with a geometrical problem: in India, the priests actually were involved with geometrical problems. In the legend, the altar of APOLLO was to be varied for a special purpose: in India, the altar actually was varied for special purposes. In the legend, the altar is varied, the form remaining constant: in India, this was the actual practice—in the one case, to be sure, it is the shape of a volume; in the other, of an area. In the legend, the altar is doubled: in India, it was also doubled—as we mentioned before, ordinarily the altar was built of five layers of bricks, but on some occasions ten, and on some fifteen, were required[14]

[13] *Op. cit.*, pp. 159ff.

[14] Multiplication of the area of the vedi by specified constants on specified occasions explicitly occurs. Thus "the vedi at the sautrámaní sacrifice was to be the third part of the vedi at the soma sacrifices". See THIBAUT, *On the Śulvasútras*, p. 232.

Where, then, did the oracle, or ERATOSTHENES, get the idea that doubling the altar is a way to fight a plague? Does it not seem likely that he got it from a once-existing rite, perhaps in the fossilized form of a legend?

## 5. Altars and plagues

So far, we have not considered the motives that led the Vedic priests to construct the ten-layered altar. If we take the Delian legend at its face value, then it says simply that *the way to fight a plague is to double the altar.* The question now is whether this conclusion is substantiated with reference to known practices. Although the *Sulvasutras* say something about the occasions of the various altars, their main concern is to give the geometric constructions. The *Sulvasutras* are associated with *Kalpasutras:* these "digest the teaching of the Veda and of the ancient Rishis regarding the performances of the sacrifices ..."[15]. The *Brahmanas* purport to give the symbolic meaning of the various rites[16]. Still these works say nothing about the meaning of the ten-layered altar; but the sacred writings presumably do not spell out what they consider well known. If only we knew that the ten-layered altar was used in fighting plagues, we would thereby come to the desired conclusion; namely, that it was the Vedic practice to double the basic altar of five layers in fighting plagues.

In default of this explicit information, we will attempt to indicate why it was precisely the double altar which was used to drive off a plague; that is, by entering into a discussion of the doctrine of the sacrifice. First, we remark that the altar is not simply a material object; it is consecrated just as a king is, and thereby obtains power: indeed, some altars are "described as thunderbolts which the sacrificer hurls on his enemies"[17]. We recall that sicknesses, plagues, *et cetera,* are frequently conceived of as due to demons[18].

From this we see that the altar was an instrument for fighting plagues; but the question remains: *How many layers?* To answer this we have to know the significance of the parts of the altar. This is given in the *Satapatha Brâhmana* (VIII), where we learn that the first layer is the earth, the third the atmosphere, and the fifth the sky. If we call the five-, ten-, and fifteen-layer altars respectively the Earth, Atmosphere, and Sky altars, as the facts suggest, the problem is to prove that the Atmosphere altar is the one to be used to fight plagues. Now, "he who has an adversary should sacrifice with the sacrifice of Indra, the Good Guardian. Thus he smites this sinful, hostile adversary and appropriates his strength, his vigour".[19] But Indra is the Wind, the Atmosphere, and the proof is complete. That is, the plague is an adversary to be fought by the Indra sacrifice, and the corresponding altar of course, the ten-layered Atmosphere altar.

We have been tempted to leave out these last considerations as they do not establish that the Vedic ritualists actually did use the double altar to fight

[15] GEORG BÜHLER, *Sacred Books of the East,* vol. **2**, p. xi.

[16] JULIUS EGGELING, *Sacred Books of the East,* vol. **12**, p. xxii.

[17] THIBAUT, *op. cit.*, p. 231.

[18] See, *e.g.*, HASTING'S *Encyclopaedia of Religion and Ethics,* vol. **4**, p. 763.

[19] *Satapatha Brahmana,* XII, 7, 3, 4, cited by HOCART, *Kingship,* p. 214. The passage speaks of the "Sautramani" sacrifice, which, according to XII, 8, 2, 24, "belongs to Indra"

plagues; and, moreover, they are not vital to the main point; namely, that the Delian legend reflects custom. THIBAUT wished, as do we, to know the motives behind the various altar shapes, but, finding no clear evidence, dropped the matter, adding—and here we cannot agree with him— "But the chief interest of the matter does not lie in the superstitious fancies in which the wish of varying the shapes of the altars may have originated, but in the geometrical operations without which these variations could not be accomplished." Those "superstitions" are of high importance because only by knowing them (or others like them) can we get an insight into the motivations for the constructions. If we are to understand the origin of civilization, we must be willing to enter into the labyrinth of archaic thought.

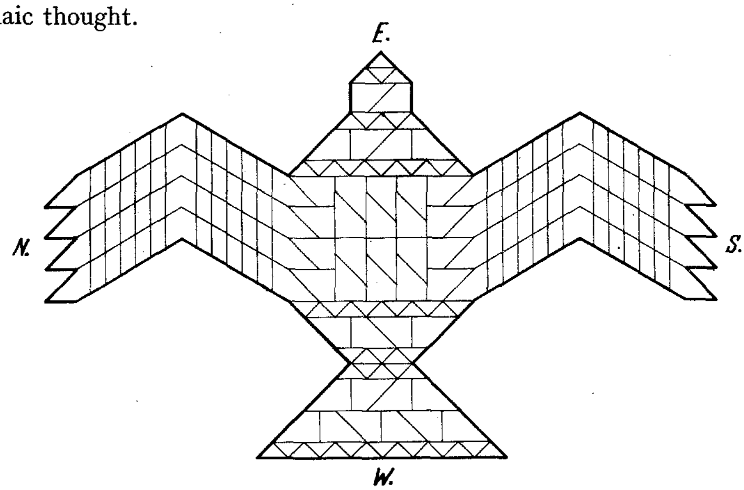

Fig. 2

Any suggestion, then, that the story of the Delian oracle is merely anecdotal would be not simply wrong: it would be just the opposite of the likely order of events. That is, the problem of the duplication of the cube starts as a problem in altar construction, with definite underlying theological motives, and ends, when the interest becomes purely mathematical, by dropping the "anecdote"

## 6. Building blocks

Each of the five layers of bricks of the Vedic altar consisted of 200 bricks of various shapes. (For an example, see Fig. 2.)[20] Layers one, three, and five had the same arrangement, as did layers two and four, but it was required that, except at the boundary, no edge of one brick lie above an edge of another. One can picture the ritualists playing with the bricks to get the various arrangements. THIBAUT writes: "It is impossible to retrace in all cases the steps by which the adhvaryus arrived at their results, which were most likely oftener obtained by repeated trying than by calculation. The commentator expresses in this case the area of

[20] Fig. 2 will be better understood if compared with Fig. 3. In Fig. 3, $HI = IP = 1$ purusha and $RK = \frac{1}{4}$ purusha; all other measurements are then what one might expect them to be except that $IK = 2\frac{1}{16}$ (and not 2) purushas. The figures are from BÜRK's work (following BURNELL).

the agni throughout by chaturthí bricks [a chaturthí brick has an area equal to $\frac{1}{16}$ square purushas], and it is possible that the priests took as many chaturthís as there are contained in the chaturasrasyenachiti and tried to form with them new figures."

The ancients had a toy called the *loculus Archimedius*, which was a sort of puzzle made of fourteen pieces of ivory of different shapes cut out of a square. T. L. HEATH says that it cannot be supposed to be ARCHIMEDES' invention[21]; but whether ARCHIMEDES' or not, we see here another trace of sacrificial "gematria".

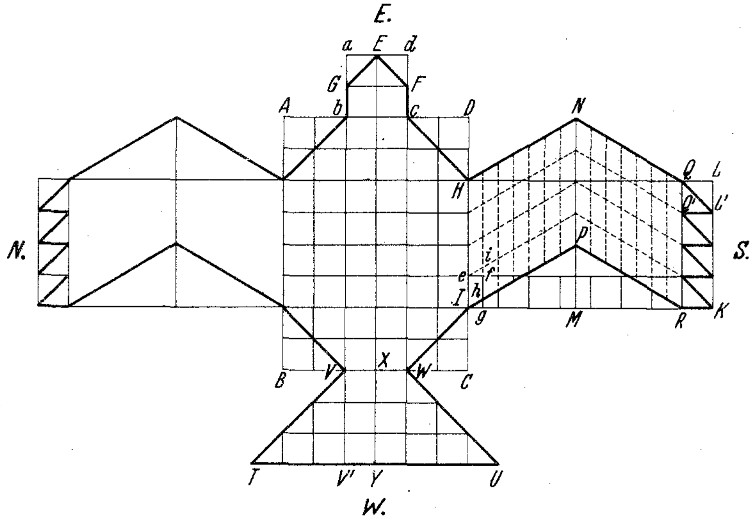


Fig. 3

## 7. The postulates of construction

In a previous paper[22], we have tried to show that a number of points in Greek geometry are illuminated by the hypothesis that it started from a tradition of peg-and-cord constructions of the kind we find with the Vedic ritualists. We recall that three of EUCLID's postulates are the so-called "postulates of construction": the first says that a straight line may be drawn from any point $A$ to any point $B$; the second, that such a line can be extended arbitrarily far beyond $B$; and the third, that a circle can be drawn with any point as center and any length as radius. With the ritualists, we see these constructions and, with minor exceptions, only these constructions. We view the postulates as arising by a process of abstracting from traditional and, in fact, ritual material.

As for the exceptions, an intersection of the circles with centers $A$, $B$ and radii $a$, $b$ respectively is found by taking a cord of length $a+b$, making a mark on it at a distance $a$ from one end, tying that end to a peg at $A$, tying the other end to a peg at $B$, and picking the cord up at the mark and stretching the cord: this construction can be immediately replaced by one allowed by EUCLID's postulates and, in fact, the alternative is actually performed sometimes by the ritualists

[21] HEATH, *The Works of Archimedes*, p. xxii.

[22] "Peg and Cord in Ancient Greek Geometry", *Scripta Mathematica*, vol. **24** (1959).

themselves. The only other exception—and this is a true exception—is that a segment is divided into $n$ equal parts by making a cord of the given length overlap itself $n$ times[23].

*Euclid*, Book I, Proposition II (Problem) is: *From a given point A to draw a straight line equal to a given straight line B C.* A ritualist would solve this by stretching a cord from $B$ to $C$, picking the cord up, placing one end at $A$, and stretching the cord again. EUCLID's solution is quite tricky. In view of the highly unsatisfactory method of superposition of figures that he uses in Book I, Proposition IV (Theorem), it is difficult to see what the point of Proposition II is unless it is an explicit desire to abstract from actual constructions. Control over this process of abstraction is obtained by referring all constructions back to the first three postulates.

It may be thought that the concern to refer the arguments back to the postulates is nothing but the famous axiomatic method. We have all been led to believe that EUCLID has such a method. As a result, when a mathematician actually looks at *Euclid*, Book I, he is scandalized. It becomes clear, however, that, in Book I, EUCLID had no such method in mind for the work as a whole, but that he was being meticulous, in the constructions, to abstract from the old "peg and cord" (or "straightedge and compass") constructions. If EUCLIDEAN geometry is what we define it to be, there are no faults with the constructions in *Euclid*, Book I. As far as the constructions are concerned, one can say there is an axiomatic method, but for the rest, it does not exist.

The issue here is obscured by the fact that there are two other postulates in some versions of *Euclid*. In the paper of ours referred to, we argue that these postulates are not genuine. But even if we go along with HEATH[24] and suppose these two postulates to be due to the great genius of EUCLID, this would imply that the pre-EUCLIDEAN versions of the *Elements* were founded on the postulates of construction alone.

Once the postulates of construction are put down, it is a merely logical matter to say that the only "instruments of construction" allowed by EUCLID are the "compass and straightedge". But, of course, it is not and cannot be a merely logical matter to put down the postulates. The postulates yield the restriction, but first the restriction yields the postulates. In the *Sulvasutras*, we find an actual restriction on the implements of construction: in fact, for one ceremony, canes were required to be used instead of cord[25].

## 8. Geometric algebra

Previously, we put forward what we consider to be the source of the Theorem of PYTHAGORAS (namely, the ritual identification of god and square), but it is also necessary to compare our views with views more generally held. This theorem, as given by EUCLID, says that the square on the hypotenuse of a right triangle is the sum of the squares on the other two sides. As thus stated, and also as proved

[23] See, for example, *Satapatha Brahmana*, X, 2, 3, 8, where a cord is divided into seven equal parts. The *Sulvasutras* do not speak of this at all. The method is really satisfactory only for dividing a cord into 2, and more generally $2^n$, equal parts.

[24] HEATH, *Euclid's Elements*, vol. **1**, p. 202.

[25] *The Pandit*, vol. **10**, p. 170.

in *Euclid* I, 47, the theorem involves areas and the theory of areas; it says that a certain area is the sum of two other areas. Nowadays, however, it would (or could) be understood in a somewhat different sense, namely, as giving a relation between the lengths of the sides of a right triangle: if $a$, $b$, $c$ are numbers representing the lengths of the two legs and hypotenuse of a right triangle, then $c^2 = a^2 + b^2$; here $c^2$, "$c$ squared", is the number resulting from multiplication of $c$ by itself. This second interpretation in no way leans on the notion of area. The connecting link between the two ways of looking at the theorem is that the area of a rectangle is the product of its sides.

According to the Pythagoreans, "the point is unity in position" (STECK, *Proklus/Euklid Kommentar*, p. 232). A conceivable construction on this is that PYTHAGORAS regarded a line segment as made up of points, like beads on a string, or "primary atoms" or "grains of sand" lying next to each other and filling up the line. Supposing this to be the case, every line would be associated with a number, and if $a$, $b$, $c$ were the numbers of points respectively in the legs and hypotenuse of a right triangle, then the relation $c^2 = a^2 + b^2$ would be a purely numerical proposition on right triangles. PYTHAGORAS's doctrine that "Number rules the universe" would have been vindicated, with respect to geometry at least, for his Theorem and theorems in general would be simply numerical propositions about the figures in question. Unfortunately, however, the Theorem itself contradicts this point of view: if $n$ were the number of points in the side of a square and $m$ the number of points in its diagonal, then the Theorem would yield $m^2 = 2n^2$. It is known that this is not a possible relation between (positive) integers $m, n$, a fact supposedly realized by PYTHAGORAS. He is said to have become considerably upset upon its discovery; and no wonder, because then, it would appear, Number doesn't even rule a line segment, let alone the universe.

The above is, of course, a reconstruction of what PYTHAGORAS might have thought, for we have no documents of his time setting forth his views explicitly. It corresponds largely with what is generally considered to have taken place. It might be historically quite wrong, but in this case it does not much matter, because the point is that, pending the discovery of incommensurable quantities, the Theorem might have been regarded as a numerical proposition. With the discovery of incommensurable quantities, a divorce between number and geometry was indicated.

The difficulty or fact encountered above, namely, that there is no segment that will go evenly both into the side and into the diagonal of a square, or in other words, that there exist line segments without common measure (so-called incommensurable quantities), prevented EUCLID, and the Greeks generally, from treating magnitude as number and from dealing arithmetically with geometric problems in the manner done today.

It looks as if EUCLID introduced the square areas simply because he had no other way to express himself, and that he regarded the fact that one square is the sum of two others as quite incidental. That is certainly the case with the modern use of the theorem. For us, the reason that the theorem is important is that it connects the notion of right angle with distance; typically, it permits us to compute the distance between the vertices of the acute angles of a right triangle. It (or its converse, rather) shows us, for example, that a triangle with sides 3, 4, 5 is a right triangle. This fact can be understood without any reference to the notion of area, and the modern treatment of the subject would (or might) deduce the theorem without reference to that notion. Moreover, there is no doubt that EUCLID had the same view of the theorem. He gives two proofs of the theorem, one involving the squares and the simplest facts about areas, and a second proof (in generalized form) involving the theory of proportion. In the second proof (VI, 31), it is shown that any leg is the mean proportional between its orthogonal projection on the hypotenuse and the hypotenuse; in the first proof, because of the organization of the work, this basic fact is expressed differently and, as a consequence, in somewhat obscure terms. The second proof is parallel to one we would give today using numbers: it is really much clearer than the first proof, and it would be simpler except for the preliminary difficulties encountered in the theory of proportion due to the existence of incommensurable quantities.

The Babylonian evidence fits in quite well with the above view—we see them assigning numbers to lengths and areas and dealing algebraically with these numbers much as we do today. VAN DER WAERDEN[26], building on the theory advanced by ZEUTHEN and NEUGEBAUER, considers that the Greeks inherited the Babylonian algebra but, because of the newly discovered incommensurables, were obliged to clothe the algebra in geometric language. For this reason, if we understand correctly, the PYTHAGOREANS invented their "geometric algebra". As the squares on the legs and hypotenuse of the right triangle are a basic part of this "geometric algebra", presumably they made their appearance for the same reason.

[By *geometric algebra* we mean algebra done by geometric means and without numbers (other than positive whole numbers), together with the associated geometry. What we find in EUCLID's *Elements* is geometric algebra; and, on the whole, the same can be said of the *Sulvasutras*—they do not compute the side of a square of area 3, but they do construct it. The Babylonians also had some composite of algebra and geometry: they interpret a product as the area of a rectangle and a number multiplied by itself is called a square; and there are other good reasons for supposing that Babylonian thinking contained a geometric component (see in particular P. HUBER, "Zu einem mathematischen Keilschrifttext (VAT 8512)" *Isis*, vol. 46, p. 104, or VAN DER WAERDEN, *op. cit.*, 2nd ed. (1961), pp. 72f.). But this composite is not geometric algebra. We can call it "algebraic geometry", perhaps, where by *algebraic geometry* we mean geometry done with numbers, together with the associated algebra. As we understand the current theory on geometric algebra, the Greeks took the algebraic geometry of the Babylonians and cleansed it of its arithmetical element, thereby producing geometric algebra.]

Let us now consider the age of the *Sulvasutras*. As might be expected, there is no general agreement on the date. Those who favor Greece tend, we may expect, to put the *Sulvasutras* late, and, correspondingly, those who favor India will put it early[27]. G.R. KAYE[28], after insisting that the dates are not known,

[26] *Op. cit.*, p. 125. For NEUGEBAUER's views see *Exact Sciences*, pp. 139—145 and pp. 41—47.

[27] Thus JOHN BURNET, *Greek Philosophy*, p. 9, n. 1, writes: "It is a pity that M. [G.] MILHAUD has been persuaded to accept an early date for the *Śulvasutras* in his *Nouvelle études* (1911), pp. 109sqq."

[28] KAYE, *Indian Mathematics*, pp. 4, 67.

puts a question mark after the *Sulvasutras*, but then places the question mark between ARCHIMEDES 250 and *The Nine Sections* 150 B.C., in ample time for Hellenism to reach India. He cautions us to bear in mind that "the contents of the *Śulvasūtras*, as known to us, are taken from quite modern manuscripts". He does not say what he means by "modern" (nor what he means by "manuscripts"), but anyway, he says, "priority of statement of a proposition does not necessarily imply its discovery"[29]. D. J. STRUIK writes: "The question of Greek and Babylonian influence determines profoundly the study of ancient Hindu and Chinese mathematics", and he adds that "the oldest Hindu texts are perhaps from the first centuries A.D."[30].

The date of the manuscript or text is, however, irrelevant. If we went out some fine day and caught a fish of a kind never before seen, would we try to fix its position in the evolutionary scale by the date on which it was caught?

As STRUIK justly writes, in studying Indian and Chinese mathematics we must bear Greece and Babylonia in mind, but it is equally true that in studying Greek and Babylonian mathematics, we should keep India and China in mind. The history of mathematics is not merely a matter of description, but is a theory. We have a few fragments and great gaps: the problem is to fit the pieces together. No theory of ancient mathematics can be considered adequate unless the pieces from India and China also fit.

If we accept the current theory of the origin of Greek "geometric algebra", then a consequence is that the Vedic ritualists received the Theorem of PYTHAGORAS from Greece, as (according to that theory) the squares arose in very special and highly technical circumstances. It would be interesting if someone who maintains this position would try to make plausible how the Vedic ritualists could seize on a technical point in mathematics and make of it a vital part of their most solemn rituals.

The reverse process, that is, the secularization of a ritual practice, is easy to understand. Ritual requires considerable social organization. If this organization breaks down for any reason, then the ritual ends, but parts of it can go on having a separate existence.

It may be suggested that perhaps the square on the hypotenuse should be conceded, but that the rest of the theory of "geometric algebra" can be maintained. But other things have to be conceded as well: the ritualists convert a rectangle into a square in a typically geometro-algebraic way. A rule for converting a square into a rectangle of prescribed width is given in such hermetic form that we cannot say they have a solution, but the very formulation is significant. In short, the basic facts of "geometric algebra" must also be conceded to the ritualists.

How does this affect the current theory on the origin of geometric algebra? As we see it, the Greeks inherited both the algebra and the geometric algebra, and when they ran into difficulties with incommensurables, they got around them by using the geometric algebra *that was already there*. To draw a parallel: KEPLER needed the ellipse to describe the paths of the planets around the sun; he did not,

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[29] "The Source of Hindu Mathematics", *Journal of the Royal Asiatic Society*, 1910, p. 750.

[30] STRUIK, *A Concise History of Mathematics*, p. 32.

however, invent the ellipse, but made use of a curve that had been lying around for nearly 2000 years. It is the distinction between *use* and *origin*.

There remains, of course, the historical problem of where the Greeks got their geometrical algebra. If they did not get it from the Babylonians (and, of course, there is no evidence that the Babylonians had it), then (assuming Babylonia and India are the only possible sources) they must have gotten it from India.

### 9. Greece vs. India: Cantor's view of 1877

Although the date of a manuscript or text cannot give us the age of the practices it discloses, nonetheless the evidence is contained in manuscripts. The examination of the manuscript, the attempted dating, the bringing of it into relation with other manuscripts, all bear on the validity of the testimony.

In his comparative work of 1875 on the Baudháyana, Ápastamba, and Katyáyana *Sulvasútras*, THIBAUT writes[31]: "Regarding the time in which the Sulvasutras may have been composed, it is impossible to give more accurate information than we are able to give about the date of the Kalpasútras. But whatever the period may have been during which Kalpasútras and Sulvasútras were composed in the form now before us, we must keep in view that they only give a systematically arranged description of sacrificial rites, which had been practiced during long preceding ages. The rules for the size of the various vedis, for the primitive shape and the variations of the agni, *etc.*, are given by the Brahmanas, although we cannot expect from this class of writings explanations of the manner in which the manifold measurements and transformations had to be managed. Many of the rules, which we find now in Baudháyana, Ápastamba, and Katyáyana, expressed in the same or almost the same words, must have formed the common property of all adhvaryus long before they were embodied in the Kalpasútras which have come down to us." As one sees, THIBAUT refrained from assigning an absolute date to the *Sulvasútras*.

In 1877, CANTOR, realizing the importance of THIBAUT's work, began a comparative study of Greek and Indian mathematics. He starts his paper[32] by reminding us that Greek studies were already about 400 years old, whereas Indian studies were only about 100. As a consequence, Greek dates could usually be given within a decade, whereas estimates of Indian dates varied by centuries. Yet even in 1907, in the third edition of his *History*, he postulates that HERON was about 100 B.C., emphasizing, however, that other opinions vary from 200 B.C. to 200 A.D. A recent estimate assigns 62 A.D. to HERON[33].

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[31] *Op. cit.*, p. 270. The *Sulvasutras* of Baudhayana and of Apastamba are the most important. The Baudhayana *Sulvasutra* has been translated by THIBAUT in *The Pandit*, vol. **9** (1874), vol. **10** (1875), n.s. vol. **1** (1876–77). BÜRK has translated the Apastamba *Sulvasutra* in the *Zeitschrift der Deutschen Morgenländischen Gesellschaft*, vol. **56** (1902). "The literature of the white Yajur Veda possesses a Śulvapariśishta [*i.e.*, supplement], ascribed to Kátyayána", which has been translated by THIBAUT in *The Pandit*, n.s. vol. **4** (1882). On p. 229 of his comparative work, THIBAUT mentions two further treatises.

[32] "Gräko-indische Studien", *Zeitschrift für Mathematik und Physik* (Historisch-literarische Abtheilung), vol. **22** (1877).

[33] See NEUGEBAUER, *Exact Sciences*, p. 171.

A part of the paper is devoted to the *Sulvasutras*: their contents are briefly described and comments thereon are made. CANTOR notes (following THIBAUT) that for the Indians the Theorem of PYTHAGORAS is not so much a theorem on triangles as a theorem on rectangles: "The cord stretched in the diagonal of an oblong", writes BAUDHAYANA, "produces both (areas) which the cords forming the longer and shorter side of an oblong produce separately". CANTOR compares this with the fact that HERON employs the Theorem of PYTHAGORAS to compute the diagonal of a rectangle before taking up the triangle. Moreover, the *Sulvasutras* give the theorem separately for a square and for an oblong; and HERON, in the place mentioned, gives two successive problems: one for the equal-sided rectangle, one for the unequal-sided rectangle. CANTOR considers that these coincidences cannot possibly be accidental.

[The oblong, by the way, was important for PYTHAGORAS: his first principles are ten in number and consist of pairs of opposites, *e.g.*, odd-even, male-female, *etc.*, and one of the pairs is square-oblong[34]. The Indians had this same duality: oblong bricks are human, square bricks divine[35].

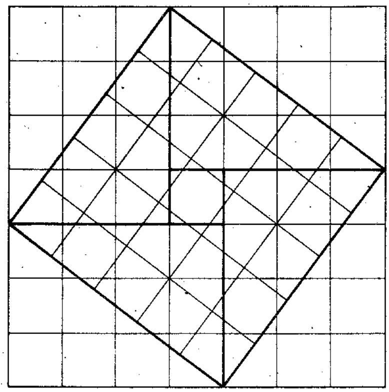

Fig. 4

According to CANTOR[36], the Theorem of PYTHAGORAS was also for the Chinese a theorem on a rectangle rather than on a right triangle. In the *Chou-pei*, an ancient Chinese work, one is told: "Make the breadth ... 3, ... the length ... 4. The king yu, that is, the way that joins the corners is 5." Accompanying the text is the following figure (given by MIKAMI[37]), called the "Figure of the Cord": The text continues: "Take the halves of the rectangles around the outside, there will be (left) a kuu." Here the (3, 4, 5)-triangle is not referred to as a "triangle" but as half a rectangle.

The earliest Babylonian problem in reference to the Theorem asks for the diagonal of a rectangle[38]; and the Old-Babylonian text "Plimpton 322", the one that gives the right triangles with rational sides, speaks of the "width" and the "diagonal".[39]]

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[34] ARISTOTLE, *Metaphysics*, i. 5; see M.C. NAHM, *Selections from Early Greek Philosophy*, pp. 75—76. EUCLID, by the way, defines *oblong*, but never uses the term. See *Euclid's Elements*, vol. **1**, p. 189.

[35] *The Pandit*, vol. **10**, p. 169. The duality is found as far away as Fiji: according to HOCART (*op. cit.*, p. 168), "the Fijians who dwelt round the Koro Sea built oblong houses, but their temples were usually square".

[36] *Geschichte der Mathematik* (Dritte Auflage), vol. **1**, p. 680.

[37] "The Development of Mathematics in China and Japan", *Abhandlungen zur Geschichte der Mathematischen Wissenschaften*, vol. **30**, p. 5.

[38] O. NEUGEBAUER, "Zur Geschichte des Pythagoräischen Lehrsatzes", *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen* (Math.-Phys. Klasse), 1928, p. 47. In this work NEUGEBAUER mentions the Apastamba *Sulvasutra* and allows that it was composed before 200 A.D.

[39] See VAN DER WAERDEN, *op. cit.*, p. 78.

In squaring the circle, the *Sulvasutras* proceed arithmetically and make use of fractions with unit numerator. This, according to CANTOR, gives the procedure an Egyptian look. We shall consider this point more minutely below.

CANTOR observes that HERON uses peg and cord for the construction of right angles. He also mentions the well-known (but at present disputed) statement of DEMOCRITUS, according to which peg and cord were used by the Egyptian harpenodaptai for geometric constructions (this, then, in the fifth century B.C.). In the temple mural paintings of Dendera, Thebes, Esne, and Edfu, one sees the king, as substitute for THOTH, engaged with the goddess SAFEKHABUI in a ceremony termed "stretching the cord". (See Fig. 5[40].) The founding of the temple of Edfu took place, according to CANTOR, following DÜMICHEN, on 23 August 237 B.C.[41].

Fig. 5

On the basis of these and similar considerations, CANTOR concludes that Indian geometry and Greek geometry, especially of Heron, are related; and the only remaining question is, Who borrowed from whom?. He expresses the opinion that the Indians were, in geometry, the pupils of the Greeks.

CANTOR did not observe (though he might have, and did in a later paper[42]) the significance of the fact that BAUDHAYANA knew the (8, 15, 17)-triangle. If he had, he would have argued that this fact, too, was evidence for a late date for the *Sulvasutras*, for a knowledge of this triangle is ascribed to PLATO but not to the Pythagoreans. Therefore, he would have argued, the *Sulvasutras*, or this part at any rate, were post-Platonic.

CANTOR was struck by the analogy of the Indian altar problems to the Greek duplication of the altar and grave problems. It seems to us that from this he should have derived a contradiction to his view that Indian geometry is a derivative of that of HERON of Alexandria. For according to that view HERON's geometry intruded, about 100 B.C., into India, where it was given a theological form. This theologic-geometry then left traces in Greece in poetry ascribed to EURIPIDES' (485—406)—a clear contradiction. One could get around this contradiction by supposing the Greek and Indian mathematics to derive from a common third source, but this does not correspond to CANTOR's point of view in 1877.

[40] BRUGSCH, *Demotisches Wörterbuch*, vol. 2, pp. 326—327. For further references to pictures, see *Gräko-indische Studien*, p. 23, n. 71.

[41] Cord-stretching can be referred back to the time of AMENEMHAT I, founder of the Twelfth Dynasty (T.L. HEATH, *Greek Mathematics*, vol. 1, p. 122); and even back to the Old Kingdom (A. GORDON CHILDE, *Man Makes Himself*, p. 218).

[42] "Über die älteste indische Mathematik", *Archiv der Mathematik und Physik*, vol. 8, p. 69.

### 10. When were the Sulvasutras composed?

We now propose to present some estimates on the date of composition of the *Sulvasutras*.

In 1879, G. BÜHLER published a translation of the *Dharma-Sutras* of the Apastamba school. These sutras form the twenty-eighth and twenty-ninth of the thirty sections of the *Kalpasutras* (of APASTAMBA), of which the thirtieth[43] is the *Sulvasutras*. In fixing the dates of the Sutra period, BÜHLER mentions M. MÜLLER'S estimate of nearly thirty years before, namely, 600—200 B.C. BÜHLER thought 600 B.C. too early for APASTAMBA and set the limits of the fifth to the third, and even fourth, centuries B.C. He was principally concerned with the *Dharma-Sutras*, but he mentions the *Sulvasutras* and says nothing to lead one to suppose that his estimates do not include them. He considers the BAUDHAYANA school to be older than the APASTAMBA school[44]. In 1900, A. A. MACDONELL in his history of Sanskrit literature puts the Sutra period circa 500—200 B.C. Throughout the book he concurs with BÜHLER'S judgments, and in particular he does this for the *Dharma-Sutras*. He mentions CANTOR'S opinion on Greek and Indian geometry, but considers the *Sulvasutras* to be far earlier than 100 B.C.[45]. In 1899, THIBAUT ventured to assign the fourth or the third centuries B.C. as the latest possible date for the composition of the *Sulvasutras*[46] (it being understood that this refers to a codification of far older material). In 1914, A. B. KEITH, though expressing some difference with BÜHLER, places APASTAMBA in the third or fourth century B.C.; and he considers the BAUDHAYANA school to be a good deal older[47]. In 1928, he estimated the *Sulvasutras* to be of the late Sutra period, "possibly of c. 200 B.C.", although insisting that this is a guess and that all guesses merely obscure knowledge[48].

According to A. C. BURNELL'S work of 1878, NEARCHUS (325 B.C.) expressly states that the Brahman laws were *not* written; on the other hand, NEARCHUS is also "represented as stating" that the Indians wrote letters on a sort of cotton cloth or paper. MEGASTHENES (c. 302 B.C.) mentions that they had no written books and that they did not know letters or use seals, but he also mentions milestones at a distance of ten stadia from one another, and, BURNELL adds, it is difficult, though not impossible, to believe that these indications were made by stones merely. The irregularities in the inscriptions of Asoka prove, according to BURNELL, that writing was in 250 B.C. a recent practice. From an examination of the changes in the Phoenician alphabet, he deduces that the source of the South Asokan character must be sought in the forms current in Phoenicia in or about the fifth century B.C. These facts (and others) led him to consider that writing was little used in India before 250 B.C.[49]. In the attempted chronologies, the date of the great grammarian PANINI is crucial. As PANINI (according to BURNELL) knew writing, the date of writing is important. In opposition to the

[43] The twenty-fourth in some versions. See BÜRK, *op. cit.*, p. 551.
[44] *Sacred Books of the East*, vol. **2**, pp. xviii, xliii, xii, xx.
[45] MACDONELL, *A History of Sanskrit Literature*, pp. 244, 259, and 424.
[46] "Astronomie, Astrologie und Mathematik", *Grundriss der Indo-Arischen Philologie und Altertumskunde*, vol. **3.9**, p. 78.
[47] "Taittiriya Samhita", *Harvard Oriental Series*, vol. **18**, pp. xlv—xlvi.
[48] KEITH, *A History of Sanskrit Literature*, pp. 517, xix.
[49] BURNELL, *Elements of South-Indian Palaeography*, pp. 1, 2, 9.

generally received opinion that PANINI lived in the fourth century B.C., BURNELL places him about a century later[50]. BÜHLER argues that APASTAMBA must be put, on linguistic grounds, before PANINI. Thus BURNELL's argument would allow a somewhat later date for APASTAMBA. BURNELL obscurely suggests that the Brahmanas and Sutras were not reduced to their present form before the Christian era, but he also, quite incidentally, makes the point that the geometric altar constructions found in BAUDHAYANA and APASTAMBA must be far earlier, as otherwise the differences between the ceremonies of the two schools could not be accounted for[51].

A. WEBER in 1884 expressed the opinion that there was "nothing of literary-historical nature standing in the way of the assumption of a use (on the part of the *Sulvasutras*) of the teachings of Hero of Alexandria" (of 215 B.C. according to WEBER)[52].

As one sees, there is a good deal of agreement in the above estimates, a great deal more than exists, or has existed, about HERON's date[53].

We have thought it well to give some of the estimates on the date of the *Sulvasutras*, but we repeat our insistence that the date is of no great importance. Greek geometry contains many aspects that can be explained on the assumption that it derives from ritual of the kind we find with the Indians. Whether Greek geometry derives from the Indian geometric ritual or from a different system very much like it, it is not essential to decide.

### 11. Relation of the Sulvasutras to the older sacred literature

We take up the question of the relation of the *Sulvasutras* to the rest of the sacred literature, as the connection has been neglected, and even disputed[54].

According to THIBAUT[55], "the earliest enumeration of [the different shapes for the agni] we find in the *Taittiriya Samhita*, V, 4, 11. Following this enumeration Baudhayana and Apastamba furnish us with full particulars about the shape of all these different chitis and the bricks which had to be employed for their construction." This indicates that the *Sulvasutras* are integrally related to the *Taittiriya Samhita*, generally considered to belong to a stratum of literature

[50] Similar estimates are still current. See *Journal of the Royal Asiatic Society*, 1959, p. 182, where A. L. BASHAM says: "... the most widely favoured date for PANINI, the latter half of the fourth century, is confirmed."

[51] *Op. cit.*, p. 12 and p. 1, n. 1.

[52] WEBER, Review of L. VON SCHROEDER's *Pythagoras und die Inder*, *Literarisches Zentralblat*, vol. **35** (1884), col. 1564.

[53] The above summary may be compared with KAYE's statement (*Indian Mathematics*, p. 4): "MAX MÜLLER gave the period [in which the *Sulvasutras* were composed] as lying between 500 and 200 B.C.; R.C. DUTT gave 800 B.C.; BÜHLER places the origin of the Apastamba school as probably somewhere within the last four centuries before the Christian era, and Baudhāyana somewhat earlier; MACDONNELL gives the limits as 500 B.C. and 200 A.D., and so on." R.C. DUTT (*Civilization in Ancient India*, vol. **1**, p. 14) does, indeed, assign the eighth century B.C. to the *Sulvasutras*, but he also says he takes this from THIBAUT, who nowhere gives such date. KAYE's work has often been cited.

[54] By KEITH in his review of VON SCHROEDER's *Kathaka Samhita*, *Journal of the Royal Asiatic Society*, 1910, pp. 519–521.

[55] THIBAUT, "On the Sulvasutras", p. 229, *J. Asiatic Soc. Bengal*, vol. **44**: **1** (1875).

older than the *Sulvasutras*; BÜRK places it not later than the eighth century B.C.[56], whereas KEITH places it not later than the sixth[57]. As the passage (*T.S.*, V, 4, 11) is especially significant, it would be well to have it before our eyes.

"... He should pile in hawk shape who desires the sky (*cf. Ap. S. S.*, XV, 1); the hawk is the best flier among the birds; verily becoming a hawk he flies to the world of heaven. He should pile in heron form who desires, 'May I be possessed of a head in yonder world' (*cf. Ap. S. S.*, XXI, 3); verily he becomes possessed of a head in yonder world. He should pile in the form of an Alaja bird, with four furrows, who desires support; there are four quarters; verily he finds support in the quarters (*cf. Ap. S. S.*, XXI, 1). He should pile in the form of a triangle who has foes (*cf. Ap. S. S.*, XII, 4); verily he repels his foes. He should pile in triangle form on both sides, who desires, 'May I repel the foes I have and those I shall have' (*cf. Ap. S. S.*, XII, 7); verily he repels the foes he has and those he will have. He should pile in the form of a chariot wheel, who has foes (*cf. Ap. S. S.*, XII, 11); the chariot is a thunderbolt; verily he hurls the thunderbolt at his foes. He should pile in the form of a wooden trough who desires food (*cf. Ap. S. S.*, XIII, 4); in a wooden trough food is kept; verily he wins food together with its place of birth. He should pile one that has to be collected together, who desires cattle (*cf. Ap. S. S.*, XIV, 1); verily he becomes rich in cattle. He should pile one in a circle, who desires a village (*cf. Ap. S. S.*, XIV, 4); verily he becomes possessed of a village. He should pile in the form of a cemetery, who desires, 'May I be successful in the world of the fathers' (*cf. Ap. S. S.*, XIV, 8); verily he is successful in the world of the fathers ..."

The seven- (and a half) fold bird altars, even the complicated ones involving bricks of many shapes, do not involve the Theorem of PYTHAGORAS (though the eight- (and a half) fold do). On the other hand, even the simple triangular altar involves the theorem. For, at least according to the *Sulvasutras*, the triangle had to have an area of $7\frac{1}{2}$ square purushas, *i.e.*, the same area as the basic falcon-shaped altar. This triangle is obtained from a square of area 15 square purushas. To construct such a square, the Theorem of PYTHAGORAS is applied. This indicates that the theorem was known and applied at the time of the *Taittiriya Samhita*.

We have already quoted THIBAUT as saying: "The rules for the size of the various vedis, for the primitive shape and the variations of the agni, *etc.*, are given by the Brahmanas, although we cannot expect from this class of writings explanations of the manner in which the manifold measurements and transformations had to be managed." Unfortunately, THIBAUT gives no explicit references, but it is not difficult with a good translation and index to the *Satapatha Brahmana*—EGGELING'S—to supply some. Thus, we have already referred to VI, 1, 1, 3, which gives the primitive form of the falcon-shaped altar. In X, 2, 1, 1—8, the variation in the wings is spoken of. Several times we read: "He [the sacrificer] thus expands it [the wing] by as much as he contracts it; and thus, indeed, he neither exceeds (its proper size) nor does he make it too small." This passage shows that the *Satapatha Brahmana* is concerned with exact geometrical constructions. In X, 2, 2, 7, we learn that the wings are to be augmented, and

[56] *Op. cit.*, vol. **55**, p. 553.

[57] *The Taittiriya Samhita*, vol. **18**, p. xlvi.

similarly, in X, 2, 2, 8, for the tail. In III, 5, 1, 1—6, the shape of the vedi, the sacrificial ground, is given: fifteen steps south from the "intermediate peg" $F$ (see Fig. 6), the sacrificer drives in peg $C$; fifteen steps to the north, peg $B$; thirty-six steps east, peg $E$; from E twelve steps to the right (south), peg $D$; twelve to the north, the peg A. The vedi is the trapezoid $ABCD$. The distance $EC$ is, by the Theorem of PYTHAGORAS, 39, and this fact is utilized in the *Sulvasutras* for the construction of the trapezoid. In X, 2, 3, 8—10, cords of length 36, 30, and 24 are considered (thus peg and cord are mentioned in the Brahmanas).

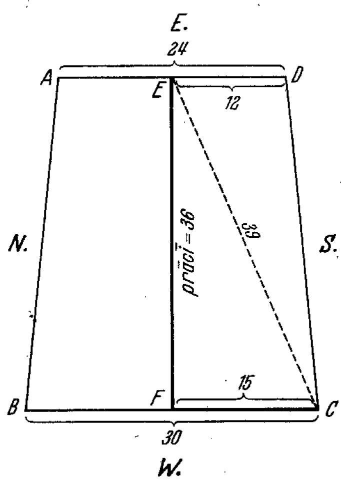

Fig. 6

The *Satapatha Brahmana* also speaks about the augmentation of the altar. Unfortunately, EGGELING has confused linear and areal measure (see S.B.E., vol. 43, p. 310, n.1, p. 306, n. 3, and p. 1, n. 1). This confusion has affected his translation in one vital place. We will give his translation and our correction simultaneously. EGGELING's error consists in translating *purusha* as "man's length", a meaning it can have, but in the passage in question it means an area of one square purusha. The passage, X, 2, 3, 11, runs:

"Now as to the (other) forms of the fire-altar. Twenty-eight

{ man's lengths long (from west to east) / (square) purushas } and twenty-eight

{ man's lengths across / (square) purushas } is the body (of the altar), fourteen

{ man's lengths / (square) purushas } the right, and fourteen the left wing,

and fourteen the tail. Fourteen cubits (aratni) he covers (with bricks) on the right, and fourteen on the left wing, and fourteen spans (vistasti) on the tail. Such is the measure of (an altar of) ninety-eight

{ man's lengths / (square) purushas } with the additional space for wings and tail."

If EGGELING's translation is taken, the 98-fold bird altar has a wing span, not counting the aratnis, of $56 \times 7\frac{1}{2}$ feet, or 420 feet; the 101-fold square altar he speaks of in a footnote would be $757\frac{1}{2} \times 757\frac{1}{2}$ feet. The altars were to be built in a year. EGGELING does not comment on the tremendous size of the supposed constructions, and we think he must have absent-mindedly chosen the linear meaning for "purusha".

Moreover, it is clear that if "purusha" is taken in its linear sense, then the passage does not describe a 98-fold altar, but if taken in its areal sense, then the passage does.

There is some difficulty about the words "aratni" and "vistasti". These are, as far as we know, linear measurements, being respectively $\frac{1}{5}$ and $\frac{1}{10}$ of a purusha. But the addition to a wing of the basic bird altar is always spoken of as an aratni, and in context could be taken to mean $\frac{1}{5}$ square purushas. If so, then "fourteen aratnis" are the correct amount for a proportionate increase of a wing.

According to our understanding of the passage, the $n$-fold bird altar is obtained by expanding the 7-fold altar without the additions for the wings and tail up to an area of $n$ square purushas and then adding proportionate areas for the wings and tail. This is different from the rule previously given, according to which the basic altar is expanded from $7\frac{1}{2}$ to $n+\frac{1}{2}$ square purushas. This is no difficulty, however, since the various schools had various doctrines and both the rules mentioned actually obtained[58]

The construction of a square of area 14 square purushas cannot be obtained by juxtaposition of squares of area one square purusha. The construction requires, in effect, the Theorem of PYTHAGORAS.

In 1901—02, A. BÜRK, besides translating the Apastamba *Sulvasutra*, brought in further evidence and arguments concerning the age of Vedic geometrical knowledge. The beginnings of the Vedic sacrificial system goes back (at least) to the time of the *Rig-Veda* (a collection of hymns). The *Rig-Veda* knows not only the *vedi*, the sacrificial ground, but also the threefold disposition of the *agni*, the fire altar. As described by the *Sulvasutras*, the disposition of the three fires involves the construction of straight lines (*i.e.*, a series of collinear points), triangles (of prescribed shape), circles, and squares. To suppose from this that the same constructions were made at the time of the *Rig-Veda* is, of course, to make a reconstruction; but we learn at least that "skillful men measure out the seat of the agni" (*Rig-Veda*, I, 67, 5). The *Rig-Veda* is generally considered to belong to the oldest stratum of Vedic literature. The *Rig-Veda* has been dated 2000—1500 B.C. by WHITNEY, and even earlier by JACOBI[59].

[Let us recall that the falcon-shaped altar has five layers, of two hundred bricks each. *Rig-Veda* (X, 90) says: "Purusha was thousand-headed, thousand-eyed, thousand-footed. ..." This means either that at the time of the *Rig-Veda* there already existed the thousand-brick altar; or, possibly, that the altar was, for some reason we do not know, conceived as thousandfold, and this led to the thousand bricks[60].]

BÜRK correctly emphasized the significance of the (8, 15, 17)-triangle found in the *Sulvasutras.* He also pointed out that they know the gnomon (the gnomon is the L-shaped figure obtained by subtracting from one square a second square having with the first a common vertex). THIBAUT had been puzzled by the prescription for building up one of the agnis: first a small square of four bricks is made, then one of nine by adding five bricks, then one of sixteen by adding seven bricks, and so on. According to ARISTOTLE, this way of building up squares by the adjunction of gnomons was known to the Pythagoreans; and there is a

[58] THIBAUT, *The Pandit*, n.s. vol. 1, p. 769.
[59] BÜRK, *op. cit.*, vol. 55.
[60] BÜRK does not mention *Rig-Veda* X, 90.

fragment of PHILOLAUS (c. 400 B.C.) indicating that he already knew it[61]. Another instance of the gnomon occurs in Apastamba *Sulvasutra* (III, 9), which gives a "general rule" for augmenting the square on $a$ to the square on $a+b$: it analyzes the adjoined gnomon into two rectangles $a$ by $b$ and one square $b$ by $b$, and amounts to the rule $(a+b)^2=a^2+2ab+b^2$. EUCLID in Book II of the *Elements* defines the gnomon; and, in II, 4, in proving the proposition $(a+b)^2=a^2+2ab+b^2$, makes use of a gnomon. Because of the importance of the gnomon to the Pythagoreans, HEATH does not hesitate to attribute Book II to them[62]. For a similar reason, the occurrence in the *Sulvasutras* of the gnomon links Indian and Pythagorean geometry.

Besides giving new evidence and arguments, BÜRK expanded some of THIBAUT'S; for example, he gave explicit references to the *Satapatha Brahmana*. In particular, he noted that the trapezoidal vedi we described before is mentioned in the *Taittiriya Samhita* (VI, 2, 4, 5).

To recapitulate: evidence of geometric altar constructions occurs in all strata of Vedic literature, and there is reason to believe that all of them, except possibly the oldest, know the Theorem of PYTHAGORAS.

## 12. Cantor's view of 1904

In 1904, CANTOR, after holding for over twenty years the opinion that the geometry of the *Sulvasutras* was a derivative of Alexandrian knowledge, finally renounced it[63]. It is not easy to understand why he changed his mind. The chronology may have bothered him a little. He says that it was BÜRK'S papers of 1901—02 that brought about an essential shift in the situation. This is to give BÜRK too much credit. BÜRK'S papers are excellent and he does make original points, but the argument occurs in all its essential aspects already in THIBAUT'S paper. Of course we do not deny that, as a personal matter, it really was BÜRK'S papers that led CANTOR to change his mind.

For some reason, CANTOR preferred Egypt to India as a source of Greek geometry: he "confesses [himself not] charmed" by the idea that PYTHAGORAS was a pupil of the Indians rather than of the Egyptians. He does not dispute, in fact he is willing to accept, VON SCHROEDER'S opinion that the Indian peg-and-cord constructions go back to the tenth century B.C.; but he can point to the time of King AMENEMHAT, considerably before 2000 B.C., for which cord-stretching is established. In a manuscript of about that time (found at Kahun) he finds the problem: to divide an area of 100 into two squares whose sides are in the ratio of 1 to $\frac{3}{4}$. This involves the sum of two squares, just as the Theorem of PYTHAGORAS does, and the striking relation $(3c)^2+(4c)^2=(5c)^2$. From this, CANTOR concludes—with justice, in our opinion—that the Theorem of PYTHAGORAS, at least for the (3,4,5)-triangle, was known at that time: therefore PYTHAGORAS *could* have obtained his initial intellectual capital from the Egyptians. More-

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[61] HEATH, *Euclid's Elements*, vol. **1**, p. 351. Professor VAN DER WAERDEN has kindly supplied us with some references for the date: DIELS, *Hermes* **28**, p. 217, CORSEN, *Philol.* **71**, p. 346; see also ZELLER, *Philos. der Griechen*, 6th ed., p. 424 and DIOGENES LAERTIOS (**8**, 84).

[62] *Ibid.*, p. 351.

[63] *Archiv der Mathematik und Physik*, vol. **8**.

over—this is a strange argument—the fact that the Indians knew the (8, 15, 17)-triangle and that PYTHAGORAS did not confirms for him the idea that PYTHAGORAS did not get his triples from India[64]. On the other hand, he was not able to find (nor do we have) any example of the gnomon from Egypt. CANTOR leaves this difficulty unresolved.

In the midst of these perplexities, it occurred to CANTOR that perhaps in very ancient times (roughly speaking, three or four thousand years ago) there already existed a not altogether insignificant mathematical knowledge common to the whole cultured area of those times, which was further developed, here in one direction, there in another. Below we will add some remarks on this idea, especially as the information available to CANTOR was trifling in comparison with our present knowledge.

CANTOR deserves great credit for never having had recourse to the idea that the various peoples of the ancient world independently invented elementary geometry. Even THIBAUT does this. "There is nothing striking in the independent development of a limited amount of practical geometrical knowledge by two different peoples", writes THIBAUT. Gnomons and all, presumably! It is as though one were to argue that not only is the concept of the deity a simple and necessary idea, but that it is also necessary to come upon the word *god* to describe him.

THIBAUT tried this way out in 1899[65]. Perhaps it was only a polite gesture to CANTOR, who a few years later changed his opinion.

The Theorem of PYTHAGORAS is, by the way, not so much practical as symbolic knowledge. The reason that the theorem appeals to us is its character of exactness—its theoretical character. It was very much the same with the ritualists. When the question came up: What should be done about the shrinkage that takes place in the baking of the bricks?, the answer was: Nothing[66].

## 13. Egypt and Babylonia

Let us now look at ancient geometry, of the earlier part of the second millenium B.C., and first at that of Egypt.

VAN DER WAERDEN writes: "We are going to show that Egyptian geometry is ... merely applied arithmetic"[67]. Yet it seems to us that the very evidence VAN DER WAERDEN brings forward proves the opposite.

[64] The argument has no logical force, since the Indians could have passed on some of their information, even if not all. Still some historical weight can be attached to the argument, since it requires us to visualize a situation in which the part, but not the whole, was transmitted to the Greeks. How much more does this apply to the Babylonians? But here another point enters. It is not simply that the Pythagoreans knew less, but they knew this as part of a theory. The early Pythagoreans, according to PROCLUS (410—485 A.D.), had the Pythagorean number triples $(m, \frac{1}{2}(m^2-1), \frac{1}{2}(m^2+1))$; and had obtained them (as has been suggested) using the gnomom (see *Greek Mathematics*, vol. **1**, p. 80). The Babylonians of 1700 B.C. were way beyond the gnomom, but the Indians had it. Hence in number theory also, and not only in geometry, Pythagorean mathematics has more of an Indian than a Babylonian look.

[65] *Astronomie, Astrologie und Mathematik*, p. 78.

[66] *The Pandit*, vol. **10**, p. 145. The fact that APASTAMBA is willing to use an approximation to $\sqrt{2}$ in constructing a square, shows that he is losing the meaning of what he is doing. (See *On the Śulvasútras*, p. 249.)

[67] *Op. cit.*, p. 31.

Let us first consider the circle. "To determine the area of a circle, the Egyptians square $\frac{8}{9}$ of the diameter. This corresponds to a very good approximation: $\pi \sim 4 \cdot (\frac{8}{9})^2 = 3.1605 \ldots$"

When one summarizes the Egyptian procedure of finding the area by saying that: "The Egyptians took $\pi = 4 \cdot (\frac{8}{9})^2$", as is frequently done, one is, of course, using a convenient shorthand. With appropriate precautions, there is nothing wrong in expressing (and in fact it is helpful to express) oneself in this way[68]. Yet it is possible easily to be misleading, and to be misled. In the Bible (I, Kings 7, 23), we read: "And he made a molten sea, ten cubits from one brim to the other: it was round all about ...: and a line of thirty cubits did compass it round about." This one summarizes by saying that: "The Hebrews took $\pi = 3$." And then one says: "The Egyptians had a better approximation to $\pi$ than the Hebrews did." This is an incorrect inference from the preceding facts, because the Egyptians and the Hebrews are speaking of essentially different aspects of the circle.

There are a number of theorems involved here:

(i) *The areas of two circles are to each other as the squares on their diameters.*—We can rephrase this by saying: $A = \frac{\pi}{4} d^2$, where $\frac{\pi}{4}$ is the ratio of any circle to the square on its diameter. The Egyptians say $A = (\frac{8}{9} d)^2$, and in comparing, this gives $\pi = 4 \cdot (\frac{8}{9})^2$. It will be helpful, for a reason that will appear in a moment, to write $A = \frac{\pi_1}{4} d^2$, where $\frac{\pi_1}{4}$ = ratio of any circle to the square on its diameter.

(ii) *The circumferences of two circles are to each other as their diameters.*—As before, we rephrase this, saying: $C = \pi d$, where $\pi$ is the ratio of the circumference of any circle to its diameter. The Hebrews say $C = 3d$, and in comparing, this gives $\pi = 3$. Here it will be helpful to write $C = \pi_2 d$, where $\pi_2$ = the ratio of the circumference of any circle to its diameter.

The following theorem is well known to us:

(iii) $\pi_1 = \pi_2$.

But unless it was well known to either the Egyptians or the Hebrews, it makes no sense to say that the Egyptians had a better approximation to $\pi$ than the Hebrews did.

From the above, it is clear that we have to allow the Egyptians a knowledge of (i) and the Hebrews a knowledge of (ii). If one wishes, one can say that this knowledge is intuitive and that the approximations are guesses.

Now VAN DER WAERDEN presents another piece of evidence ("when you are told a basket (of $4\frac{1}{2}$) in diameter by $4\frac{1}{2}$ in depth, then tell me the area") from which it appears that the Egyptians knew theorem (ii) with $\pi_2 = 4(\frac{8}{9})^2$, *i.e.*, $\pi_1 = \pi_2$. NEUGEBAUER[69] and PEET (whom VAN DER WAERDEN follows) have different interpretations of the problem, but both interpretations require $\pi_2 = 4(\frac{8}{9})^2$.

We may allow the $\pi_1 = 4(\frac{8}{9})^2$ of the Egyptians to be a guess but it is difficult to suppose that $\pi_2 = 4(\frac{8}{9})^2$ is a guess. It is easier to suppose that the Egyptians (or their forerunners) realized that $\pi_1 = \pi_2$. And how would they come to such a

[68] E. T. BELL (*Development of Mathematics*, p. 38) facetiously writes this approximation as $(4/3)^4$.

[69] "Die Geometrie der ägyptischen mathematischen Texte", *Quellen und Studien zur Geschichte der Mathematik*, Abt. B, vol. 1 (1930), p. 427.

realization? Surely by an application of intelligence. If this was done through a geometric analysis—and we see no other way—then this analysis, no matter how crude, shows that Egyptian geometrical knowledge was not merely arithmetical.

The Babylonians used the formulas $A = C^2/12$ and $C = 3d$, from which it would appear that they knew that $\pi_1 = \pi_2$, though here it is a bit more difficult to judge[70].

Next let us consider the truncated pyramid, with square bases $a \times a$ and $b \times b$ and altitude $h$. Here the Egyptians use the correct formula $V = \frac{h}{3}(a^2 + a\,b + b^2)$ in computing the volume. For a pyramid of square base $a \times a$ and altitude $h$, this yields $V = \frac{h}{3} a^2$. We may perhaps suppose that the Egyptians guessed that $V = \text{constant} \times h\,a^2$, in analogy with the formula $A = \frac{1}{2} b\,h$ for the area of a triangle, and even that they guessed: $\text{constant} = \frac{1}{3}$. But the formula $V = \frac{h}{3}(a^2 + a\,b + b^2)$ could not very well be a guess. It could be obtained only by a geometrical analysis or by algebra $\left(\text{from the formula } V = \frac{a^2 h}{3}\right)$. If by algebra, then either the Egyptians knew algebra, or they got this formula from someone who did. The Babylonians knew algebra, and, as VAN DER WAERDEN suggests, possibly the Egyptians got the formula from them. The Babylonians also knew a formula for the volume of a truncated pyramid, but it is different $\left(V = h\left[\left(\frac{a+b}{2}\right)^2 + \frac{1}{3}\left(\frac{a-b}{2}\right)^2\right]\right)$. This suggests divergence from a common source, although there is not enough evidence to judge. In any event, we cannot consider the Egyptian knowledge of the pyramid to be merely arithmetical.

All of the Egyptian manuscripts we are considering for the present date from the Middle Kingdom, 2000—1800 B.C. They appear to be purely arithmetical works, taking geometric material as the background for the arithmetical computation[71]. (A like situation holds for Babylonia of about the same time.) That is why we are not disturbed by the absence of the Theorem of PYTHAGORAS itself from the KAHUN fragment: the arithmetical aspect of the theorem is detached from it and presented in an arithmetical way.

The Egyptians had the concept of slope and to this extent realized the basic property of similar triangles.

We give one more example (not presented by VAN DER WAERDEN, but of course known to him). The problem is: Given the area $A$ of a rectangle and the ratio $b:l$ of the breadth to the length, to find the breadth and length[72]. (The Babylonians have the same problem[73].) The solution is given by: $l = \sqrt{F/(b:l)}$, $b = l \cdot (b:l)$. This is a problem with two equations and two unknowns. It requires the extraction of a square root. Here, as elsewhere in the manuscripts, all square roots

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[70] O. NEUGEBAUER & W. STRUVE, "Über die Geometrie des Kreises in Babylonien" *ibid.*, p. 85.

[71] The statement that the papyri are "purely arithmetical works" is, of course, an abstraction. It may be helpful here in emphasizing one aspect of them, but in other contexts may be misleading. In fact, our view is that the arithmetic grew up on geometry as a basis, hence almost necessarily treats of geometrical material.

[72] *Die Geometrie der ägyptischen mathematischen Texte*, p. 418.

[73] O. NEUGEBAUER & A. SACHS (eds.), *Mathematical Cuneiform Texts*, p. 105.

come out even. We suspect that the problems were arranged so that this would happen; and hence we suspect, too, that the Egyptians made approximations, something which is not explicit in the texts (except for the value of $\pi$, which is a different matter). All this requires a high degree of sophistication and is not merely arithmetic.

VAN DER WAERDEN writes: "... the general character of the mathematics which a people has at its command, remains the same, whether one considers elementary texts or more advanced ones. Literal calculation, decimal fractions, differential calculus and coordinate geometry, which are characteristic of Western mathematics, could be abstracted by a competent historian from a handbook for engineers, just as well as from a volume of the *Mathematische Annalen*"[74]. It seems to us that this is just the point! The Egyptian texts really are elementary arithmetical texts, but we can conclude from them that the Egyptians of 2000 B.C. (or their forerunners) could analyze geometrical problems and (with great likelihood) that there must have been, at the time we are speaking of or earlier, a knowledge of algebra sufficient to derive the formula for a truncated pyramid from the one for the whole pyramid.

Let us now glance at Babylonia. The story is to an extent the same, but the picture is clearer: what can be at best extracted from the Egyptian papyri with difficulty, can be read off explicitly from the cuneiform texts.

The Babylonians, too, are concerned with rectangles, triangles, trapezoids, circles, and pyramids, and, as with the Egyptians, the geometry is a secondary matter. They know the Theorem of PYTHAGORAS and even all right triangles with rational sides. They know that in a triangle, a line parallel to the base divides the other sides proportionately. With the possible exception of the theorem that the angle in a semicircle is right, they do not know any basic geometry unknown to the Egyptians.

The Babylonians are experts in algebra. They can solve complicated simultaneous equations that lead to quadratic equations. We are thrilled at their virtuosity, and even our guesses only place the Egyptians a low second in this field.

It cannot be denied that Egyptian and Babylonian mathematics have a great deal in common. This and the differences—the different values of $\pi$ and the different formulae for the truncated pyramid—point to a divergence from a common source. Seventeen hundred B.C. may have been a flowering time for algebra in Babylonia and some of this may have gotten over to Egypt, but elementary geometry already has an ancient look in Old-Babylonian times.

Let us return to the *Sulvasutras*. Although we have had to struggle to get them back of 0 A.D., we have seen good reason for supposing that the main part of their contents goes back at least to the seventh century B.C. and perhaps even to the second millenium B.C. For the sake of the argument, let us suppose that the Vedic rituals described, including the application of the Theorem of Pythagoras, go back even to 1500 B.C. Since the Babylonians already had the theorem in 1700 B.C., does this not create a difficulty for the theory that it had a ritual origin?

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[74] *Op. cit.*, p. 36

The problem is the same as that encountered before with Greece. Whatever the difficulty there may here be, it is small in comparison with the difficulty of deriving the Vedic ritual application of the theorem from Babylonia. (The reverse derivation is easy.) Recall, too, that the application involves geometric algebra, and there is no evidence of geometric algebra from Babylonia. And the geometry of Babylonia is already secondary whereas in India it is primary. Hence we do not hesitate to place the Vedic altar rituals, or, more exactly, rituals exactly like them, far back of 1700 B.C.

To summarize the argument: the elements of ancient geometry found in Egypt and Babylonia stem from a ritual system of the kind described in the *Sulvasutras*.

### 14. Quadrature of the circle and circulature of the square

We return now, as promised, to the problem of squaring the circle and to the solution as given in the *Sulvasutras*, as it was in particular in relation to this solution that CANTOR judged the work to be a derivative of Heronic knowledge.

First, the facts. In the *Sulvasutras* one finds not only the squaring of the circle, but the reverse problem, the turning of a square into a circle ("circulature of the square"). This is done as follows (see Fig. 7). In square $ABCD$, let $M$ be the intersection of the diagonals. Draw the circle with $M$ as center and $MA$ as radius; and let $ME$ be a radius of this circle perpendicular to $AD$ and cutting $AD$ in $G$. Let $GN=\frac{1}{3}GE$. Then $MN$ is the radius of a circle having area equal to the square $ABCD$. (Taking $MG=1$), this comes to saying that $\pi\left(1+\frac{\sqrt{2}-1}{3}\right)^2=4$ or $\pi=\left(\frac{6}{2+\sqrt{2}}\right)^2$. For the reverse problem, that of squaring the circle, one is given the rule:

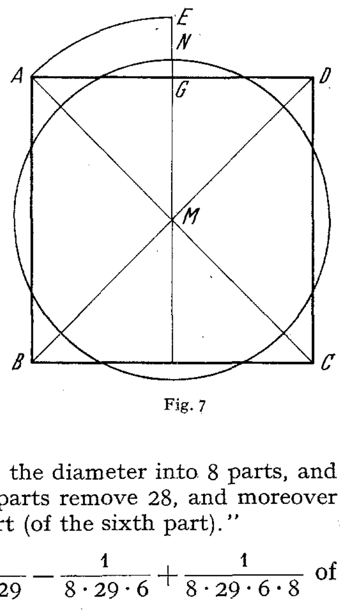


Fig. 7

"If you wish to turn a circle into a square, divide the diameter into 8 parts, and again one of these 8 parts into 29 parts; of these 29 parts remove 28, and moreover the sixth part (of the one part left) less the eighth part (of the sixth part)."

The meaning is: side of required square $=\frac{7}{8}+\frac{1}{8\cdot 29}-\frac{1}{8\cdot 29\cdot 6}+\frac{1}{8\cdot 29\cdot 6\cdot 8}$ of the diameter of given circle.

One also finds the approximation: $\sqrt{2}=1+\frac{1}{3}+\frac{1}{3\cdot 4}-\frac{1}{3\cdot 4\cdot 34}$ (more precisely: the diagonal of a square $=1+\frac{1}{3}+\frac{1}{3\cdot 4}-\frac{1}{3\cdot 4\cdot 34}$ of a side).

CANTOR was struck by the use here of unit fractions, *i.e.*, fractions of the form $1/n$, where $n$ is an integer. Unit fractions are famous from Egyptian mathematics. Therefore, CANTOR argued, the *Sulvasutras* were composed under Egyptian influence.

In looking at the *Sulvasutras* as a whole, one notes that the squaring of the circle differs in character, in several respects, from the rest of the work. The work has definitely a geometric and not an arithmetic character. There is, to be sure, some arithmetic (not counting the squaring of the circle and the approximation to $\sqrt{2}$). For example, it is realized (Apastamba S., III, 7) that the square of $n$ units in length has area $n^2$; from this it is deduced that $\frac{1}{2}$ the side of a square produces $\frac{1}{4}$ the area of the square, and $\frac{1}{3}$, the ninth. It is also explicitly stated that $1\frac{1}{2}$ purushas produces $2\frac{1}{4}$ square purushas; and even the general rule $(a+b)^2 = a^2+2ab+b^2$ is set up. Fractions thus enter, but there is little arithmetic involved with them. The elaborate (seven- (and a half) fold) bird altars involve several types of bricks, but most of them have an integral number of sixteenths of a square purusha as area; as already noted, the commentators make all their computations in terms of chaturthi-bricks ($=\frac{1}{16}$ square purusha). Thus most of the arithmetic is with integers, and there is nothing in the remainder (with the exceptions noted) having an Egyptian look.

The geometric character of the work is also seen in comparing the text with the remarks of the commentators and, quite generally, with the procedures of the latter-day Indian mathematicians. These always bring in arithmetic when they can. Consider, for example, the two statements:

(1) "The cord stretched in the diagonal of an oblong produces both (areas) which the cords forming the longer and shorter sides of an oblong produce separately."

(2) "The square root of the sum of the squares of these (of the two shorter sides of a rectangular triangle) is the diagonal."

The first of these statements is from the BAUDHAYANA *Sulvasutra*, the second is by BHASKARA (not a commentator). The first refers to a geometric construction; the second is expressed with a view to calculation. THIBAUT has pointed out several instances in which the commentators go wrong because they replace the geometry of the text by a computation[75].

As THIBAUT has pointed out, the squaring of the circle is "nothing but the reverse of the rule for turning a square into a circle"; that is, if $d$ = diameter of a circle, $s$ = side of an equal square, then the circulature of the square gives

$$\frac{d}{s} = \frac{2+\sqrt{2}}{3}.$$

After replacing $\sqrt{2}$ by the rational approximation $1+\frac{1}{3}+\frac{1}{3\cdot 4}-\frac{1}{3\cdot 4\cdot 34}$, it is easy, by simple arithmetic, to find the reciprocal $s/d$. This gives

$$\frac{s}{d} = \frac{7}{8} + \frac{1}{8\cdot 29} - \frac{1}{8\cdot 29\cdot 6} + \frac{1}{8\cdot 29\cdot 6\cdot 8} - \frac{41}{8\cdot 9\cdot 6\cdot 8\cdot 1393},$$

which, neglecting the last term as explained by THIBAUT[76], is the expression in the *Sulvasutras*.

The circulature of the square involves no arithmetic. One may imagine an ancient ritualist starting from the square, observing that the inscribed circle is too small, the circumscribed circle too large, and guessing that one should take $GN = \frac{1}{3}GE$. (See Fig. 7.) The line of thought, though approximative, is geometric. We may suppose that this solution of the circulature of the square, having

[75] *The Pandit*, vol. **10**, pp. 46, 73; *J. Asiatic Soc. Bengal*, vol. 44: 1, pp. 271–274.
[76] THIBAUT, "On the Śulvasútras", *J. Asiatic Soc. Bengal*, p. 254.

become fixed in tradition, became the starting point for squaring the circle. This reverse problem, though an easy exercise for us, may well have baffled the Vedic ritualists (as, indeed, it appears to have baffled THIBAUT): How, given the circle of radius $MN$, is one to get hold of $NG$ and thereby reverse the steps in the circulature of the square? Not being able to solve this problem geometrically, the ancients went over to an arithmetic solution. Here they needed a rational expression for $\sqrt{2}$; of course, they might have rationalized the denominator of $\frac{3}{2+\sqrt{2}}$, *i.e.*, brought $\frac{3}{2+\sqrt{2}}$ to the form $\frac{3}{2}(2-\sqrt{2})$, but presumably they did not know enough algebra, either.

Note that the problem of finding a rational expression for $\sqrt{2}$ arose from the problem of squaring the circle.

We note, too, that the circulature of the square is an integral part of the *Sulvasutras*—one has to construct a circular altar of $7\frac{1}{2}$ square purushas, and this is done by converting the $7\frac{1}{2}$ square purushas into a square and then this square into a circle; but the squaring of the circle is never *applied* in the *Sulvasutras*.

These considerations show, or tend to show, that the squaring of the circle (not the circulature of the square) was interpolated into the *Sulvasutras*. But, of course, we do not know when it was interpolated.

Although we are thus ready to concede an Egyptian influence on this part of the *Sulvasutras* (even if we do not concede a late date for it), we may observe that the expression $\frac{7}{8}+\frac{1}{8\cdot 29}-\frac{1}{8\cdot 29\cdot 6}+\frac{1}{8\cdot 29\cdot 6\cdot 8}$ is not quite Egyptian, since it involves a minus sign and a non-unit fraction. But this remark may well be irrelevant, as the expression does contain elements that are usually attributed to Egypt as a source.

## 15. Application of areas; proofs

It will be convenient to take up here again a point mentioned previously. This is the question whether the Vedic ritualist could turn a square into a rectangle with given side.

The problem is posed and the solution reads[77]:

"In order to turn a square into an oblong, make a side as long as you wish the oblong to be (*i.e.*, cut off from the square an oblong one side of which is equal to one side of the desired oblong); then join to that the remaining portion as it fits."

This is not intelligible. Turning to the commentators for help, we learn:

"Given, for instance, a square the side of which is five, and required an oblong one side of which is equal to three. Cut off from the square an oblong the sides of which are five and three. There remains an oblong the sides of which are five and two; from this we cut off an oblong of three by two, and join it to the oblong of five by three. There remains a square of two by two, instead of which we take an oblong of 3 by $1\frac{1}{3}$. Joining this oblong to the two oblongs joined previously we get altogether an oblong of 3 by $8\frac{1}{3}$, the area of which is equal to the area of the square 5 by 5."

[77] *Ibid.*, p. 246. See also *The Pandit*, vol. **10**, pp. 19–20.

But this is ridiculous! The hermetic character of the *Sulvasutras* often forces us to rely on the commentators; on the other hand, the commentators are not always reliable. The arithmetic form of the commentary shows that the commentators have not gotten at the meaning of the sutra.

The commentary involves, incidentally, a begging of the question, but the commentators are led into this by following the first part of the sutra. They themselves would simply have given the other side at once to be $8\frac{1}{3}$.

One commentator does, indeed, give a correct solution to the problem. On the side $BC$ of the given square $ABCD$, lay off the given (shorter) side $CH$ of the proposed rectangle. (See Fig. 8.) Complete the rectangle $HCDG$ and let $CG$ meet $BA$ in $E$. Then $BE$ is the other side of the required rectangle, and $HCFJ$ is one such rectangle. In fact, subtracting the congruent triangles $CHG$, $CDG$ and the congruent triangles $GAE$, $GJE$ from the congruent triangles $CBE$, $CFE$ (much as in the *Elements*, I, 44), one sees that rectangle $BHGA$ = rectangle $GDFJ$ and hence square $ABCD$ = rectangle $HCFJ$.

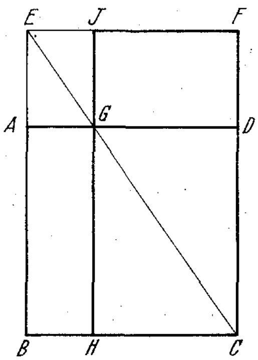

Fig. 8

Of course, we cannot without more ado ascribe the commentator's solution to the Vedic ritualists, but if we are obliged to give some *geometric* meaning to the sutra, if we take into account that the sutra to some extent coincides with the commentator's solution, if we observe that this solution is in character with the rest of the *Sulvasutras*, and if we follow the rule generally followed in such reconstructions, namely, to give the ancients the benefit of the doubt, a rule justified by the fragmentary nature of the evidence, then we shall be tempted to credit the ritualists with a correct solution.

A solution, whether correct or not, is nowhere applied in the *Sulvasutras*, but there is one point where it *might* have been applied, and it is noteworthy that there were different doctrines in the various schools on just this point. As we remarked, the bird-altar upon being given a larger area was nonetheless to retain its shape; but this was variously interpreted, some schools holding that only the body and the wings and tail before the addition of the aratnis and the pradesa (*i.e.*, the "seven", but not the "half") were to be proportionately increased[78]. In one school, the aratnis and pradesa maintain their width[79]. (We suppose these controversies referred to the primitive shape, as the complicated bird-altars would not easily accommodate themselves to a difference of opinion.) This, of course, would give the $n$-fold altar an area greater than $n+\frac{1}{2}$. If the problem were to increase the "7" proportionately to "$n$" and then add the "$\frac{1}{2}$", as it may well at one time have been, then there would be a use for turning the square into a rectangle. Even then it would not be necessary. For example, in the 8-fold altar, a wing, before lengthening, would be a square of area $1\frac{1}{7}$; the wing would then be lengthened by $\frac{1}{5}$ of $\frac{7}{8}$ of the square's side. But if the problem is

[78] *The Pandit*, n.s., vol. 1, p. 769.

[79] At least this was so for the "21"-fold altar. This brought the area up to $21+\sqrt{3}\cdot\frac{1}{2}$, which is less than 22.

formulated in general terms, then one comes to the problem of turning a square into a rectangle with given side[80].

Many writers who refer to the *Sulvasutras* say that there are no proofs there. We can only suppose that these writers have not bothered to examine the work. Thus in the Apastamba *Sulvasutra* (V, 7), the *vedi*, which is an isosceles trapezoid 36 units wide with parallel sides of 30 and 24, is said to have area 972 square units. "(To establish this), one draws (a line) from the southern *amsa* ($D$ in Fig. 9) toward the southern *śrōni* ($C$), (namely) to (the point $E$ which is) 12 (padas from the point $L$ of the *pṛsthya*). Thereupon one turns the piece cut off (*i.e.*, the triangle $DEC$) around and carries it to the other side (*i.e.*, to the north). Thus the *vedi* obtains the form of a rectangle. In this form ($FBED$) one computes its area." This is a proof. And there are others.

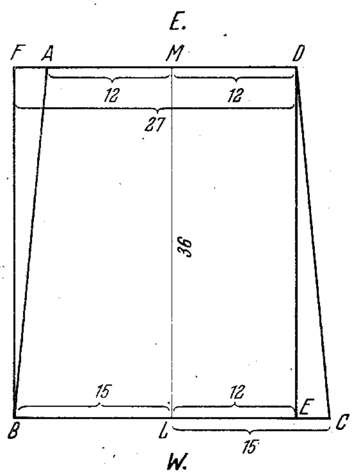


Fig. 9

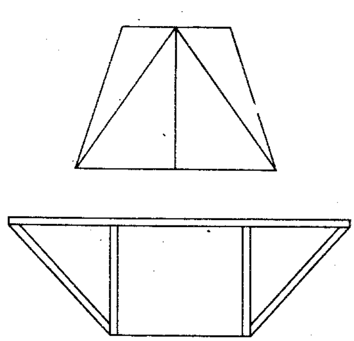

Fig. 10

## 16. The trapezoid

In speaking of the value for the history of geometry of the figures found on Egyptian monuments, CANTOR says that the isosceles trapezoid occurs, in variants depending on their decomposition into other figures, at "practically all times"[81]. The following two figures (Fig. 10), taken from PRISSE D'AVENNES' *Histoire de l'art Egyptian d'apres les monuments*, are given by CANTOR: How can one understand the agelong interest in such figures?

The figures remind us of the Indian vedi. It is not merely their trapezoidal shapes which impress us, but their subdivisions. The subdivision of the first figure occurs in the *Sulvasutras*, and the second calls to mind the computation of the area in Apastamba *Sulvasutra* (V, 7). If these figures occurred on Indian monuments, we could understand the Indian interest in them: all the hopes of the Indian for health and wealth were tied up with a trapezoid. Now we have seen reason to suppose that the Egyptians inherited the geometry of the *Sulvasutras*; and we suppose that with it came the above figures.

[80] Let $R$ = rational number field, $x, y, \ldots$ indeterminate lengths. Since the Theorem of PYTHAGORAS applied to $R[x, y, \ldots]$ leads only to quantities integral over this ring and since $1/x$ is not integral over it, the problem of turning a square into a rectangle with given side involves an essentially new point.

[81] *Geschichte der Mathematik*, vol. **1**, p. 108.

The hypotenuses in the first figure suggest that the first one who drew this figure knew the Theorem of PYTHAGORAS. For how else are we to account for their being drawn in? Recall that the *Sulvasutras* introduce the hypotenuse in the construction making the base perpendicular to the *prachi* (the axis of symmetry).

The above considerations, if correct, have a bearing on the chronology of the theorem. According to them, the theorem was known at "practically all times" in Egyptian history.

By the way, it would be of interest to know why the vedi was taken in trapezoidal form. The delimitation of the ritual scene is one of the basic elements of ancient ritual[82], and we may supose the trapezoid to be a variant of a simpler figure (a square is our guess), but this still does not explain the trapezoid. According to the *Satapatha Brahmana* (I, 2, 5, 15—16), the vedi is female (and the agni, male) and hence "it should be broader on the west side ...". The explanations of the *Satapatha Brahmana* are not necessarily correct, but as the vedi is also spoken of as a womb and as the ritual is a creation ritual, it may this time be close to the truth.

The Sanskrit word "vedi", or its root, means "earth". *Satapatha Brahmana* (I, 2, 5, 7) says: "... By [the sacrifice] they obtained (sam-vid) this entire earth, therefore it (the sacrificial ground) is called vedi (the altar). For this reason they say, 'As great as the altar is, so great is the earth'; for by it (the altar) they obtained this entire (earth) ..." If only PLATO had known that those who first spoke of geometry as earth-measurement meant measurement of the ritual scene, he would not have thought them foolish.

## 17. Traces of theologic-geometry

The main reason for putting theologic-geometry before the Babylonian and Egyptian geometry is that we can understand how the latter may have resulted from the former, but we cannot understand the reverse process. Then there are the religious traces: these are found in Egypt, Babylonia, India, China, and Greece, and point to a common source. Here are a few more of these traces.

In the *Chou-pei*, the ancient Chinese work mentioned before, SHANG KAO is represented as saying in reference to the (3, 4, 5)-triangle: "The means, therefore, which Yü had used in governing the world arise from these numbers", *i.e.*, Yü used the Theorem of PYTHAGORAS to govern the world. These words are readily intelligible in the context of the Vedic rituals.

In the *de Vita Contemplativa*, a work commonly attributed to PHILO, the therapeutae are represented as holding their great feast on the fiftieth day, "because *fifty* is the most holy and natural number, through the influence of the right-angled triangle, which is the first principle of the origin and existence of the world"[83]. Here "fifty" looks as if it were $3^2+4^2+5^2$ and the triangle as if it were the (3, 4, 5)-triangle.

The Babylonians performed augury by means of geometrical figures[84]. The tying of one's fate to geometrical figures is familiar to us from Vedic ritual. The

[82] Lord RAGLAN, *The Origins of Religion*, p. 56.

[83] A. H. SAYCE, "Babylonian Augury by Means of Geometrical Figures", *Transactions of the Society of Biblical Archeology*, vol. **4** (1875), p. 303.

[84] *Ibid.*

augury is clearly derivative, as the Vedic ritual is close to the sacrifice, the central feature of ancient ritual, whereas the augury has an independent existence.

### 18. How far back do geometric rituals go?

No matter how far back we put the Theorem of PYTHAGORAS, we cannot suppose it was there from the beginning. What, then, were the first geometrical constructions? And did they arise on a ritual basis?

"In the Orient", according to V. G. CHILDE, "temples can be traced back to 4000 B.C"[85]. "The construction of a temple", he says elsewhere[86], "was a cooperative task. The labor of hundreds of participants must be coordinated and directed. The whole must be planned accurately in advance. The outlines of the temple were in fact laid out with strings before the walls were begun. The ground plan of a temple, marked out on the bitumen floor by the thin red lines left by a colored string, has actually been found on the summit of the artificial mountain at Erech ..." It is not generally realized that, because of those red lines, we know more about the geometry of the times in which they were made than we do about the geometry of PYTHAGORAS. "From other cities and later times", the passage continues, "we have temple plans drawn to scale on clay tablets. The Sumerians believed that such plans were designed by the gods themselves and revealed in dreams. But the real architects were presumably the priests." Temple building was a ritual and stretching cord was a ritual, so we have geometrical rituals going back to times "rather earlier than ... the end of the Uruk phase at Erech", *i.e.*, about 3500 B.C.[87].

In short, no matter how far we go back in "history", we find geometrical rituals.

### 19. Some anthropological evidence

We will now present some evidence from the anthropological literature. Our view here is that "'savage' practices are merely ancient civilized practices which have not been displaced by more recent civilized practices ... so that 'savage' practices yield living documents of archaic civilization"[88].

In Imerina, Madagascar, the "mpanandro", the maker of days, lays out the foundations of a house, square in shape. Using ropes, he finds the center as intersection of the diagonals. Thus we see a ritual personage engaged in geometrical constructions. Unfortunately, we learn little about the geometry, but at least it is clear that something more than orientation—for there is orientation—is involved. The "mpanandro" and the square house with its special subdivisions are part of a complex, evidence of which is also contained in certain Scandinavian and Chinese texts[89].

The Kwakiutl of Vancouver Island have a way of laying out the lines for a square house. As we understand the construction, the description of which by F. BOAS is obscure in one small point, the Kwakiutl start by putting in stakes at the points $A$, $B$ that are to be the centers of the front and rear of the house

[85] CHILDE, *Progress and Archeology*, p. 101.

[86] CHILDE, *What Happened in History*, p. 86.

[87] *Ibid.*, pp. 86, 84; and (for the chronology) *Progress and Archeology*, p. 10.

[88] A. SEIDENBERG, "The Diffusion of Counting Practices", *University of California Publications in Mathematics*, vol. **3** (1960), p. 219.

[89] MARY DANIELLI, "The State Concept of Imerina", *Folk-Lore*, vol. **61** (1950), p. 191.

(see Fig. 11). A rope of length $AB$ is bisected and placed in the position $CD$, with its mid-point at $A$. Another rope is used to see whether $BC = BD$, and if not, then the position of rope $CD$ is adjusted. The other corners of the house are found similarly[90].

On first thought, the construction seems disappointing, as it is not of the exact kind we expect or hope for, only achieving its end by successive approximations. But on second thought, it strikes one as rather clever. We are far from regarding this construction as degenerate.

The construction reminds us very much of the *Sulvasutras*, and not merely because of the peg and cord. There, too, the constructions of the square—there are several—start (in most cases) from the *prachi*, the (east-west) axis of symmetry. In the *Sulvasutras*, the various figures, the trapezoidal vedi, the falcon-shaped agni, *etc.*, are considered as animals, and hence they have an axis of symmetry, the *prsthya* or "backbone". Starting from $AB$ is there a part of a system of ideas.

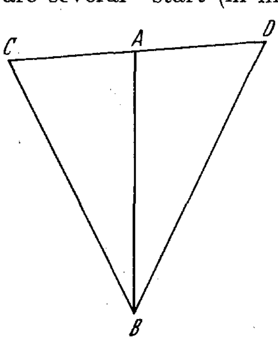

Fig. 11

With the Omaha Indians, "when the location [of the earth lodge] was chosen, a stick was thrust in the spot where the fireplace was to be, one end of a rawhide rope was fastened to the stick and a circle 20 to 60 feet in diameter was drawn on the earth to mark where the wall was to be erected". The figure of the earth lodge, mainly a circle, was considered sacred[91].

The Aztec god Tezcatlipoca (Ursa Major) had had one of his feet bitten off. In a hieroglyph, Tezcatlipoca is seen with the ankle of this foot held in the mouth of a tecpatl, symbol of the North[92]. On the basis of this, one may attribute to the Aztecs the knowledge that a circle is the locus of points at a given distance from a given point.

The ancient Maya divided their fields into squares. "When the field has been selected, the farmer divides it into *mecates*, or squares measuring $65\frac{3}{5}$ feet (20 meters) on each side, making piles of loose stones as markers at the four corners of each *mecate*. In measuring his land, the Maya farmer uses a rope which is invariably a little longer than the regulation 20 meters; indeed, in northern Yucatan these measuring ropes average 21.5 meters, or 70 feet instead of $65\frac{3}{5}$ feet in length. The Maya say the *mecates* have to be measured a little longer "because of what the birds take"[93]. We recall that according to HERODOTUS, the Egyptians divided their fields into quadrangles of equal area[94].

The Chavante Indians of Brazil have villages in the form of a perfect circle.[95] As this shape could have no useful purpose, its origin in ritual is indicated.

[90] F. BOAS, *The Kwakiutl of Vancouver Island*, p. 412. There is some further geometrical information on p. 411.

[91] A.C. FLETCHER & F. LA FLESCHE, "The Omaha Tribe", *27th Annual Report, Bureau of American Ethnology* (1908), p. 97.

[92] Z. NUTTAL, "Fundamental Principles of Old and New World Civilizations" *Peabody Museum of American Archeology and Ethnology*, vol. 2 (1901), p. 10.

[93] S.G. MORLEY, *The Ancient Maya*, p. 143.

[94] F. CAJORI, *A History of Mathematics*, p. 9.

[95] E. WEYER jr., *Primitive People Today*, p. 10.

These few examples from the anthropological literature do not constitute much evidence, it is true, but still there is more here (relative to geometric constructions) than we have from Babylonia and Egypt combined. They result from a sampling of the literature. If someone who knows the literature better would extract the material on geometry, he would do a useful thing[96].

We believe there is enough in the above evidence to connect it with the geometric rituals of the Ancient East. It is not given with chronological intent, but merely to round out the picture we already have of ancient geometrical rituals.

The circle and square go back to prehistoric times. We believe it is possible to say how they arose, but this would require the presentation of a new range of evidence and a somewhat theoretical discussion of it. Therefore we prefer to postpone consideration of the origin of the circle and square to another paper. Meanwhile, we remark that we have been led to the view that: (*i*) the circle and the square arise from ritual activities, (*ii*) these activities impose these shapes on, and other circular and square artifacts derive their shape from, the ritual scene, and (*iii*) the circle and square are dual objects (just as later square and oblong are dual). We mention these points in order to indicate continuity with prehistoric times.

## Summary

Let us sum up the history of geometry from its beginnings in peg-and-cord constructions for circles and squares.

The circle and square were sacred figures and were studied by the priests for the same reason they studied the stars, namely, to know their gods better. The observation that the square on the diagonal of a rectangle was the sum of the squares on the sides found an immediate ritual application. Its elaboration in the sacrificial ritual gave it a dominant position in ancient thought and ensured its conservation for thousands of years. This initial elaboration took place well before 2000 B.C. By 2000 B.C., it was already old and had diffused parts of itself into Egypt and Babylonia (unless, indeed, one of these places was the homeland of the elaboration). These parts became the basis of a new development in these centers[97]. The new, big development was the solution of the quadratic. A thousand years and more later, Greece inherited algebra from Babylonia, but its geometry has more of an Indian than a Babylonian look. It inherited geometric algebra, the problem of squaring the circle, the problem of expressing $\sqrt{2}$ rationally, and some notions of proof.

*Acknowledgement.* The author thanks Professor VAN DER WAERDEN for several comments, most of which have been incorporated in the present version.

[96] We are referring to explicit constructions — there are ample references to the circle and square; see, for example, A. H. ALLCROFT, *The Circle and the Cross*, and T. T. WATERMAN, "The Architecture of the American Indian", *American Anthropologist*, vol. **29** (1927).

[97] In saying this, we are merely going along with current opinion. Of the Old Babylonian mathematics no one can say with any confidence, or at any rate with reference to the evidence, that it was not known a thousand years earlier to the Sumerians. As NEUGEBAUER says (*Exact Sciences*, p. 28f.): "For the Old Babylonian texts, no prehistory can be given ... All that will be described in the subsequent sections is fully developed in the earliest texts known." It would, therefore, be quite false to say that one can see practical roots in the Babylonian texts, because in these texts one can see no roots of any kind. On the other hand, notice that one can trace a development in the *Sulvasutras*.

## Appendix

*Pandit* = THIBAUT, "Śulvasútra of Baudháyana"
*ZDMG* = BÜRK, "Das Āpastamba-Śulba-S tra"
*JASB* = THIBAUT, "On the Śulvasútras".

For the construction of right angles using the Theorem of PYTHAGORAS see *Pandit*, vol. 9, pp. 295–297; *ZDMG*, vol. 56, pp. 327f.; *JASB*, vol. 44: 1, pp. 235 to 238; for without, *Pandit*, vol. **9**, p. 296; *JASB*, pp. 249–251.

For the construction of squares, see *Pandit*, vol. **9**, pp. 294–296; *ZDMG*, vol. **56**, pp. 330f., pp. 352f.; *JASB*, pp. 247–251. One construction for a square, and hence incidentally a right angle, is as follows (Ap. SS. 1, 7, *ZDMG*, vol. **56**, p. 330; *JASB*, p. 249):

Take a cord of the length of the desired square. Make a mark at its mid-point and at the mid-points of its halves. Stretch the cord on the ground (assumed plane) and fix pegs at its end-points $A$, $B$, its mid-point $C$, and at the two other marks $D$, $E$. The points $A$, $B$ will be the mid-points of two opposite sides of the proposed square. Attach the ends of the cord to $D$ and $E$, stretch the cord to one side having taken it by the middle mark, and let this mark touch the ground at $F$. Then $ACF$ is a right angle. Now attach both ends of the cord to $C$ and bring the stretched doubled cord over $F$, the mid-point falling at $G$. Then $G$ will be the mid-point of another side of the square. Fix a peg at $G$ and tie the ends of the cord to $A$ and $G$. Pick the cord up again at its mid-point and stretch it, the mark falling at $H$ with $H \neq C$. Then $H$ is a vertex of the desired square; and the other vertices are found similarly.

Baud. SS. 1, 45 says: "The cord stretched across a square (*i.e.*, in the diagonal) produces an area of the double size." The next sutra says to build an oblong with the side of the square as width, the above diagonal as length; then "the diagonal of that oblong is the side of a square, the area of which is three times the area of the square". *Cf.* Ap. SS. I, 5 and II, 2. (*Pandit*, vol. **9**, p. 297; *ZDMG*, vol. **56**, pp. 329, 332; *JASB*, pp. 233, 242.)

The Theorem of PYTHAGORAS is stated in full generality (except for isoceles right triangles, or squares, rather) in Baud. SS. I, 48; *cf.* Ap. SS. I, 4 (see above, Section 14). In I, 50 and I, 51 the Theorem is applied in constructing the sum and difference of two squares; *cf.* Ap. SS. II 3, 4 (*Pandit*, vol. **9**, p. 298, vol. **10**, pp. 18–19; *ZDMG*, vol. **56**, pp. 328, 332f.; *JASB*, p. 234).

Baud. SS. I, 54 (*cf.* Ap. SS. II, 7) reads: "If you wish to turn an oblong into a square, take the tiryaṅmání, *i.e.*, the shorter side of the oblong, for the side of a square, divide the remainder (that part of the oblong which remains after the square has been cut off) into two parts and inverting (their places) join those two parts to two sides of the square. (We get thus a large square out of one corner of which a small square is cut out as it were.) Fill the empty place (in the corner) by adding a piece (a small square). It has been taught how to deduct it (the added piece)." (*Pandit*, vol. **10**, p. 19; *ZDMG*, vol. **56**, p. 333; *JASB*, p. 245.)

Baud. SS. II, 64 (*Pandit*, vol. **10**, p. 145; *cf.* Ap. SS. VII, 7, *ZDMG*, vol. **56**, p. 348) speaks of dividing a square into 21 congruent rectangles: "Having divided the square ... into seven parts (by lines running from east to west) one has to divide its breadth into three parts." As remarked above (Section 7), the Sulvasutras say nothing about how to divide a segment into $n$ equal parts (except for $n=2$).

Baud. I, 47 (*Pandit*, vol. **9**, p. 297; *cf.* Ap. SS. II, 3, *ZDMG*, vol. **56**, p. 332) explains how to construct a square equal to one-third of a given square (see our footnote [14]): one first constructs a square of area three times the given square and then divides this into 9 little squares. An alternate way would be to divide the given square into 3 congruent rectangles and then convert one of them into a square.

Baud. II, 1—5 (*Pandit*, vol. **10**, p. 72) speak about the augmentation one square purusha at a time from the 7 (and a half)-fold altar to the 101 (and a half)-fold altar. *Cf.* Ap. SS VIII, 4 (*ZDMG*, vol. **56**, p. 352), which, however, differs in that it does not speak of the "half".

Baud. II, 12 (*op. cit.*, p. 73; *JASB*, p. 272) tells how the larger altars are to be constructed: "That which is different from the original form of the agni (*i.e.*, that area which has to be added to the area of the saptavidha agni) is to be divided into fifteen parts and two of these parts are to be added to each vidhá (to each of the seven purushas; the one remaining part has consequently to be added to the remaining half purusha); with seven and a half of these (increased vidhás the agni has to be constructed)." Ap. SS. VIII, 6 gives a similar rule.

The Sulvasutras do not explain that the "two parts", which we may suppose to be in the form of a rectangle, are to be converted into a square as explained and then added, as explained, to the smaller vidha (*i.e.*, $1\times 1$ square) to get the increased vidha; nor should this have been expected, as the Sulvasutras are as brief as possible. Nor do the commentators supply the details. In fact, "the commentary instead of showing how the desired end could be obtained by making use of geometrical constructions taught in the parisháshá-sutras, employs arithmetical calculation; but this was of course not the method of the sutrakaras" (*Pandit*, vol. **10**, p. 73).

Baud. II, 26 (p. 139) says: "He who constructs the agni for the first time has to construct it with one thousand bricks", and **II**, 81 (p. 167) gives instruction to the sacrificer "after having constructed the agni consisting of three thousand bricks", but we know nothing about the circumstances for these multiple altars. (See also *JASB*, pp. 230, 269.)

For agnis with non-level surface, see *JASB*, pp. 269f. and Plate XVIII; and *The Pandit*, n.s. vol. **1** (1877), p. 695.

(For a method of fixing an east-west line, see THIBAUT, "Katyayana's Sulba-pariśishta," *The Pandit*, n.s. vol. **4** (1882), p. 86.)

There are a great many other points of interest in the Sulvasutras; we content ourselves by mentioning only one of them. BAUDHAYANA, after constructing an isosceles triangle (of base $c$ and height $3c$), has a sutra (*Pandit*, n.s. vol. **1**, p. 640; *JASB*, p. 266) that says: "This triangle is divided into ten parts." The commentator explains why this must mean the following: The base and the sides are each divided by three marks into four equal parts. The first mark on the base is joined to the nearest mark on the adjacent side, the second with the next, and the third with the remaining mark; and similarly with the base and the other side. In this way the triangle is broken up into 4 congruent triangles and 6 congruent "double triangles".

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