Theoretical Computer Science 562 (2015) 476–495

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

# On the commutative equivalence of semi-linear sets of $\mathbb{N}^k$

Flavio D'Alessandro [a,b,*], Benedetto Intrigila [c]

[a] Dipartimento di Matematica, Università di Roma "La Sapienza", Piazzale Aldo Moro 2, 00185 Roma, Italy
[b] Department of Mathematics, Boğaziçi University, 34342 Bebek, Istanbul, Turkey
[c] Dipartimento di Ingegneria dell'Impresa, Università di Roma "Tor Vergata", via del Politecnico 1, 00133 Roma, Italy

ARTICLE INFO

*Article history:*
Received 20 January 2014
Received in revised form 24 September 2014
Accepted 21 October 2014
Available online 27 October 2014
Communicated by G. Ausiello

*Keywords:*
Semilinear set
Recognizable set
Commutative equivalence

ABSTRACT

Given two subsets $S_1, S_2$ of $\mathbb{N}^k$, we say that $S_1$ is commutatively equivalent to $S_2$ if there exists a bijection $f : S_1 \longrightarrow S_2$ from $S_1$ onto $S_2$ such that, for every $\mathbf{v} \in S_1$, $|\mathbf{v}| = |f(\mathbf{v})|$, where $|\mathbf{v}|$ denotes the sum of the components of $\mathbf{v}$. We prove that every semi-linear set of $\mathbb{N}^k$ is commutatively equivalent to a recognizable subset of $\mathbb{N}^k$.



## 1. Introduction

This is the second paper of a group of three which provide the general solution of the following problem. Given two languages $L_1, L_2 \subseteq A^*$, we say that $L_1$ is commutatively equivalent to $L_2$ if there exists a bijection $f : L_1 \longrightarrow L_2$ from $L_1$ onto $L_2$ such that, for every $u \in L_1$, $u$ and $f(u)$ have the same Parikh vector. Now the problem is:

**Problem 1.** *Is every bounded context-free language commutatively equivalent to a regular language?*

We give a positive answer to Problem 1 by actually proving that every *bounded semi-linear language* is commutatively equivalent to a regular language (see [7,8]). The solution of Problem 1 was announced in [5] with a sketch of the proof. For the definition of bounded semi-linear language as well as for a general discussion of the problem and of related topics, the reader is referred to [7].

In this paper, as a part of our solution, we address and solve the following problem dealing with the commutative equivalence for sets of vectors. Given two subsets $S_1, S_2$ of $\mathbb{N}^k$, we say that $S_1$ is *commutatively equivalent to* $S_2$ if there exists a bijection $f$ from $S_1$ onto $S_2$ such that, for every $\mathbf{v} \in S_1$, $|\mathbf{v}| = |f(\mathbf{v})|$, where $|\mathbf{v}|$ denotes the sum of the components of $\mathbf{v}$. The main result of this paper is the following:

**Theorem 1.** *Given a semi-linear subset $S_1$ of $\mathbb{N}^k$, there exists a recognizable subset $S_2$ of $\mathbb{N}^k$ such that $S_2$ is commutatively equivalent to $S_1$.*

* Corresponding author.
*E-mail address:* dalessan@mat.uniroma1.it (F. D'Alessandro).

http://dx.doi.org/10.1016/j.tcs.2014.10.030
0304-3975/

Actually, we prove the following statement:

*If $S_1$ is a semi-linear subset of $\mathbb{N}^k$, there exist:*

1. *a partition of $S_1$ into $h$ simple, pairwise disjoint, subsets $S_{11}, \dots, S_{1h}$ of $\mathbb{N}^k$, where, for every $i = 1, \dots, h$, $S_{1i}$ has the form*

$$\mathbf{b}_0 + \{\mathbf{b}_1, \dots, \mathbf{b}_r\}^\oplus,$$

*where $\mathbf{b}_0, \mathbf{b}_1, \dots, \mathbf{b}_r$ are the vectors of the unambiguous representation of $S_{1i}$ and $r$ is the dimension of $S_{1i}$;*

2. *$h$ simple recognizable, pairwise disjoint, subsets $S'_{11}, \dots, S'_{1h}$ of $\mathbb{N}^k$, where, for every $i = 1, \dots, h$, $S'_{1i}$ has the form*

$$\mathbf{b}'_0 + \left\{\mathbf{b}'_1, \dots, \mathbf{b}'_r\right\}^\oplus,$$

*where, for every $j = 1, \dots, r$, exactly one component of $\mathbf{b}'_i$ is non-null and different vectors have a different non-null component;*

3. *For every $i = 1, \dots, h$, there exists a linear bijection $f_i$ from $S_{1i}$ onto $S'_{1i}$, that preserves the weight of vectors.*

We find useful to point out to the reader that by using Theorem 1 together with a refinement of the results of [7], we are able to positively solve Problem 1. Actually the solution of Problem 1 is presented in its full generality in [8].

However Theorem 1 is interesting in its own right and deserves some further remarks. First Theorem 1 is constructive. Indeed, given an effective presentation of $S_1$, one can effectively construct a presentation of $S_2$ as well as of a bijection $f$ from $S_1$ onto $S_2$ that preserves the weights of vectors. Moreover, by (3), one can define a bijection $f$ from $S_1$ onto $S_2$, preserving the weight of vectors, which is itself *semi-linear* as a subset of $\mathbb{N}^{2k}$. In view of the well-known equivalence of the notions of semilinear and Presburger definable subsets of $\mathbb{N}^{2k}$ [12], the latter can be seen as a result about the Presburger arithmetic. For a recent study of the expressiveness of Presburger arithmetic, see [1,2]. In our opinion, the notion of commutative equivalence between semilinear sets would deserve further investigations. As an example, we conjecture that similar results hold for semi-linear subsets of $\mathbb{Z}^k$.

The paper is structured as follows. In Section 2, all the basic notions are presented as well as some intermediate results required for the proof of the main theorem. In Section 3 a suitable standard decomposition of semi-simple sets is introduced. In Section 4, the proof of Theorem 1 is presented. To help the reading of the paper, some examples clarify the most significant steps of the proof.

## 2. Preliminaries

### 2.1. Basic notation

We find useful to recall the attention of the reader to some notation adopted in this paper.

The letter $k$ is always used to denote the dimension of the underlying working monoid $\mathbb{N}^k$.

A vector of $\mathbb{N}^k$ is denoted in **bold** as, for instance, for $\mathbf{v}$ which represents the vector $(v_1, \dots, v_k)$. Moreover if the vector is indexed, as for instance for $\mathbf{v}_j$, its components are denoted $(v_{j1}, v_{j2}, \dots, v_{jk})$.

A set of vectors of $\mathbb{N}^k$ is always denoted by using capital letters like, for instance, $X, Y, L$, etc.

An (indexed) family of sets of vectors is denoted by a calligraphic letter like, for instance, $\mathcal{X} = \{X_i\}_{i\geq 1}$ or $\mathcal{Y} = \{Y_i\}_{i\geq 1}$.

Given a set $S$, a family of $n$ pairwise disjoint sets $S_1, \dots, S_n$, such that $S = \bigcup_{i=1}^n S_i$, is called a *decomposition* of $S$. The number $n$ will be denoted $\sharp(S)$.

### 2.2. Semi-linear and recognizable sets of $\mathbb{N}^k$

We start by introducing some basic results concerning semi-linear sets of $\mathbb{N}^k$. The free abelian monoid on $k$ generators is identified with $\mathbb{N}^k$ with the usual additive structure. Let $B = \{\mathbf{b}_1, \dots, \mathbf{b}_m\}$ be a finite subset of $\mathbb{N}^k$. Then we denote by $B^\oplus$ the submonoid of $\mathbb{N}^k$ generated by $B$, that is

$$B^\oplus = \mathbf{b}_1^\oplus + \dots + \mathbf{b}_m^\oplus = \{n_1\mathbf{b}_1 + \dots + n_m\mathbf{b}_m \mid n_i \in \mathbb{N}\}.$$

In the sequel, in the formula above we will assume $B = \emptyset$ whenever $m = 0$.

**Definition 1.** Let $X$ be a subset of $\mathbb{N}^k$. Then

1. $X$ is *linear* in $\mathbb{N}^k$ if $X = \mathbf{b}_0 + \{\mathbf{b}_1, \dots, \mathbf{b}_m\}^\oplus$, where $\mathbf{b}_0, \mathbf{b}_1, \dots, \mathbf{b}_m$ are vectors of $\mathbb{N}^k$,
2. $X$ is *simple* in $\mathbb{N}^k$ if the vectors $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_m$ are linearly independent in $\mathbb{Q}^k$,
3. $X$ is *semi-linear* in $\mathbb{N}^k$ if $X$ is a finite union of linear sets in $\mathbb{N}^k$,
4. $X$ is *semi-simple* in $\mathbb{N}^k$ if $X$ is a finite disjoint union of simple sets in $\mathbb{N}^k$.

In the sequel, we will adopt the following terminology.

**Convention.** If $X = \mathbf{b}_0 + \{\mathbf{b}_1, \ldots, \mathbf{b}_m\}^{\oplus}$ is a simple set, then the vectors $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_m$, shall be called the *(unambiguous) representation* of $X$. Moreover, $\mathbf{b}_0$ will be called the *constant vector* and $\mathbf{b}_1, \ldots, \mathbf{b}_m$ will be called the *generators* of the representation, respectively.

The following theorem by Eilenberg and Schützenberger [10] provides a characterization of semi-linear sets. Also it is worth to remark that Theorem 2 was proved independently by Ito in [15].

**Theorem 2.** *Let $X$ be a subset of $\mathbb{N}^k$. Then $X$ is semi-linear in $\mathbb{N}^k$ if and only if $X$ is semi-simple in $\mathbb{N}^k$.*

Theorem 2 is effective. Indeed, one can effectively represent a semi-linear set $X$ as a semi-simple set. More precisely, one can effectively construct a finite family $\{V_i\}$ of finite sets of vectors such that the vectors in $V_i$ form a representation of a simple set $X_i$ and $X$ is the disjoint union of the sets $X_i$. The following notion of dimension will be useful in the sequel.

**Definition 2.** Let $X$ be a simple subset of $\mathbb{N}^k$ and let $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_m$ be the representation of $X$. Then the *dimension* of $X$ is $m$. If $X$ is a semi-simple set of $\mathbb{N}^k$, the *dimension* of $X$ is the maximal dimension of a simple set of a decomposition of $X$. The dimension of $X$ is denoted $\dim(X)$.

**Remark 1.** It is worth noticing that the uniqueness of the representation of a simple set is *folklore*. Thus the definition of dimension of a simple set is non-ambiguous. Moreover, by using a standard argument, one can prove that the dimension of a semi-simple set is unique. It is also useful to keep in mind that, according to Definition 2, the dimension of a finite non-empty set is null and, by convention, the dimension of the empty set is $-1$.

#### 2.2.1. Recognizable sets of $\mathbb{N}^k$

A well-known family of semi-linear sets of $\mathbb{N}^k$ is that of *recognizable sets*. We recall that, given a monoid $M$, a subset $X$ of $M$ is said to be recognizable if $X$ is a union of classes of a congruence of $M$ of finite index. In the case of the free commutative monoid $\mathbb{N}^k$, a classical theorem, attributed by Eilenberg to Mezei [11], provides a characterization of recognizable sets of $\mathbb{N}^k$, that we state here as a definition (see also [9]).

**Definition 3.** A subset $X$ of $\mathbb{N}^k$ is *recognizable* if it is union of finitely many linear sets of the form $\mathbf{b}_0 + \{\mathbf{b}_1, \ldots, \mathbf{b}_m\}^{\oplus}$, where $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_m$ are vectors of $\mathbb{N}^k$ and, for every $i = 1, \ldots, m$, one components of $\mathbf{b}_i$ is non-null at most.

It is also worth noticing that the definition of recognizable set in $\mathbb{N}^k$ is a special case of the definition of *set of stratified periods* introduced by Ginsburg and Spanier to study the properties of context-freeness and regularity of bounded semi-linear languages (cf. [13], Chap. 5). The following proposition states a basic result for recognizable sets.

**Proposition 1.** *The family of recognizable sets of $\mathbb{N}^k$ is closed under the Boolean set operations.*

### 2.3. On the commutative equivalence in $\mathbb{N}^k$

Given a vector $\mathbf{v} = (v_1, v_2, \ldots, v_k) \in \mathbb{N}^k$, the Lebesgue-norm $\ell_1$ of $\mathbf{v}$ is the non-negative integer $|\mathbf{v}| = v_1 + v_2 + \cdots + v_k$, that we call the *weight* of $\mathbf{v}$. For every $R \in \mathbb{N}$, we denote by $\mathbb{S}(\mathbf{0}, R)$ the subset of $\mathbb{N}^k$:

$$\left\{\mathbf{v} \in \mathbb{N}^k : |\mathbf{v}| = R\right\},$$

whose cardinality is given by the binomial coefficient $\binom{R+k-1}{k-1}$ (see [14]). Let us now give the notion of *commutative equivalence* for subsets of $\mathbb{N}^k$.

**Definition 4.** Let $S_1, S_2$ be two sets of $\mathbb{N}^k$. We say that $S_1$ is *commutatively equivalent to* $S_2$, or that $S_1$ and $S_2$ are *commutatively equivalent*, if there exists a bijection $f : S_1 \longrightarrow S_2$ such that, for every $\mathbf{v} \in S_1$, $|f(\mathbf{v})| = |\mathbf{v}|$.

In the sequel, if $S_1$ is commutatively equivalent to $S_2$, we write $S_1 \sim_{\mathbb{N}^k} S_2$, or, if no ambiguity arises, $S_1 \sim S_2$. The following lemma is immediate.

**Lemma 1.** *Let $S_1, S_2, S'_1$ and $S'_2$ be subsets of $\mathbb{N}^k$. Suppose that $S_i \sim S'_i$ $(i = 1, 2)$ and $S_1 \cap S_2 = S'_1 \cap S'_2 = \emptyset$. Then $(S_1 \cup S_2) \sim (S'_1 \cup S'_2)$.*

The following lemma states a useful property of the commutative equivalence.

**Lemma 2.** *Let $X$ and $X'$ be simple sets of dimension $m$ and let*

$$\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_m, \quad \mathbf{b}'_0, \mathbf{b}'_1, \ldots, \mathbf{b}'_m,$$

*be the representations of $X$ and $X'$ respectively. If, for every $i = 0, \ldots, m$, $|\mathbf{b}_i| = |\mathbf{b}'_i|$, then $X$ and $X'$ are commutatively equivalent.*

**Proof.** If $m = 0$, the claim is trivial. Assume $m > 0$. Since $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_m$ is the representation of $X$, for every vector $\mathbf{v}$ of $X$, there exists a unique tuple $(x_1, \ldots, x_m) \in \mathbb{N}^m$ such that $\mathbf{v} = \mathbf{b}_0 + x_1\mathbf{b}_1 + \cdots + x_m\mathbf{b}_m$. This allows us to define without ambiguity the map $f : X \to X'$ as, for every $\mathbf{v} \in X$, $f(\mathbf{v}) = \mathbf{b}'_0 + x_1\mathbf{b}'_1 + x_2\mathbf{b}'_2 + \cdots + x_m\mathbf{b}'_m$. One easily checks that $f$ has the desired property and the claim is proved. $\square$

### 2.4. The counting function of a subset of $\mathbb{N}^k$

We now recall the definition of *counting function* of a subset of $\mathbb{N}^k$ (see [6]).

**Definition 5.** Let $S$ be a subset of $\mathbb{N}^k$. The *counting function of* $S$ is the function $c_S : \mathbb{N} \longrightarrow \mathbb{N}$ which maps every $n \in \mathbb{N}$ into the number $c_S(n)$ of vectors of $S$ of weight $n$.

The following lemma states some useful properties of the counting function of a subset of $\mathbb{N}^k$ and its proof is immediate.

**Lemma 3.** *Let $S_1, S_2$ be subsets of $\mathbb{N}^k$. The following properties hold:*

(i) *Suppose that $S_1 \sim S_2$. Then $c_{S_1} = c_{S_2}$.*
(ii) *Suppose that $S_1 \subseteq S_2$. Then, for every $n \in \mathbb{N}$, $c_{S_1}(n) \le c_{S_2}(n)$.*
(iii) *For every $n \in \mathbb{N}$, $c_{S_1 \cup S_2}(n) \le c_{S_1}(n) + c_{S_2}(n)$. If $S_1 \cap S_2 = \emptyset$, the latter inequality becomes an equality.*
(iv) *Then, for every $n \in \mathbb{N}$, $c_{S_1}(n) \le \mathrm{Card}(\mathbb{S}(\mathbf{0}, n))$.*

The following lemma provides an exact description of the counting function of a simple set under the assumption that all the generators of the set have the same weight. The proof relies upon well-known combinatorial arguments (see [3,4,14]).

**Lemma 4.** *Let $X = \mathbf{b}_0 + \{\mathbf{b}_1, \ldots, \mathbf{b}_m\}^{\oplus}$ be a simple subset of $\mathbb{N}^k$ of dimension $m$ with $1 \le m \le k$ where $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_m$ are the vectors of the representation of $X$. Assume that the vectors $\mathbf{b}_i$, with $i = 1, \ldots, m$, have the same weight $d$. Then the counting function of $X$ is given by: for every $n \in \mathbb{N}$*

$$c_S(n) = \begin{cases} p_m(n) & \text{if } n \equiv |\mathbf{b}_0| \bmod d, \\ 0 & \text{otherwise}, \end{cases}$$

*where $p_m(n)$ is the polynomial function of degree $m-1$ in the variable $n$*

$$\binom{(n - |\mathbf{b}_0|)/d + m - 1}{m - 1}.$$

### 2.5. Clustered sets

A notion that is relevant in our construction is that of *clustered set*. Informally speaking a clustered set of a simple set $X$ is a recognizable set canonically associated with $X$. In this section we will introduce such notion as well as some basic properties. For this purpose, for every $i = 1, \ldots, k$ let us denote $\mathbf{e}_i = (0, \ldots, 0, 1, \ldots, 0)$ the $i$-th vector of the canonical base for the Euclidean vector space $\mathbb{R}^k$.

**Definition 6.** Let $X = \mathbf{b}_0 + \{\mathbf{b}_1, \ldots, \mathbf{b}_m\}^{\oplus}$ be a simple subset of $\mathbb{N}^k$ of dimension $m$ with $1 \le m \le k$ where $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_m$, are the vectors of the representation of $X$. Let us consider the subset of $\mathbb{N}^k$ defined as:

$$\mathbf{b}_0 + \big(|\mathbf{b}_1|\mathbf{e}_1\big)^{\oplus} + \big(|\mathbf{b}_2|\mathbf{e}_2\big)^{\oplus} + \cdots + \big(|\mathbf{b}_m|\mathbf{e}_m\big)^{\oplus}. \tag{1}$$

This set is called *clustered set associated with* $X$ according to the enumeration $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_m$. It will be denoted $\mathrm{Clust}(X)$.

**Remark 2.** In the definition above, (1) depends upon the choice of an enumeration of the vectors of the representation of $X$. In the sequel, for the sake of simplicity, if no ambiguity arises, we will assume such an enumeration without explicitly defining it.

**Example 1.** In $\mathbb{N}^3$, if $X = \{x(2,1,0) + y(1,1,1) + z(0,0,3) : x, y, z \in \mathbb{N}\}$ then we have $\mathrm{Clust}(X) = \{x(3,0,0) + y(0,3,0) + z(0,0,3) : x, y, z \in \mathbb{N}\}$.

By Definition 6 and by applying Lemma 2 to $X$ and $\mathrm{Clust}(X)$ we get.

**Lemma 5.** *Let $X = \mathbf{b}_0 + \{\mathbf{b}_1, \ldots, \mathbf{b}_m\}^{\oplus}$ be a simple subset of $\mathbb{N}^k$ of dimension $m$ with $1 \le m \le k$ where $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_m$, are the vectors of the representation of $X$. Then $\mathrm{Clust}(X)$ satisfies the following properties:*

(i) $\mathrm{Clust}(X)$ *is a recognizable subset of $\mathbb{N}^k$ and its representation is given by the vectors $\mathbf{b}_0, |\mathbf{b}_1|\mathbf{e}_1, |\mathbf{b}_2|\mathbf{e}_2, \ldots, |\mathbf{b}_m|\mathbf{e}_m$.*
(ii) $\mathrm{Clust}(X)$ *is commutatively equivalent to $X$. In particular, the counting maps $c_X$ and $c_{\mathrm{Clust}(X)}$ of $X$ and $\mathrm{Clust}(X)$ respectively are equal.*

### 2.5.1. *Collisions over clustered sets*

We describe now a phenomenon of clustered sets that makes the study of the commutative equivalence a non-trivial task. For this purpose, we start by stating an immediate consequence of the results of the previous section.

**Corollary 1.** *Let $S = \bigcup_{i=1}^{\ell} S_i$ be a semi-simple set of $\mathbb{N}^k$ where the sets $S_i$, $i = 1, \ldots, \ell$ are pairwise disjoint and simple sets, and let $S' = \bigcup_{i=1}^{\ell} \mathrm{Clust}(S_i)$. The following two conditions hold:*

(i) *$S'$ is a recognizable subset of $\mathbb{N}^k$.*
(ii) *Assume that, for every $i, j$ with $1 \le i < j \le \ell$, $\mathrm{Clust}(S_i) \cap \mathrm{Clust}(S_j) = \emptyset$. Then $S$ is commutatively equivalent to $S'$.*

It is worth noticing that if Property (ii) of Corollary 1 holds, Theorem 1 holds. Nevertheless, in general, Property (ii) above is not satisfied since the clustered sets of two disjoint simple sets are not disjoint. We call this fact a *collision*. In the following example we show a very simple kind of collision.

**Example 2.** Let $X = X_1 \cup X_2 \cup X_3$ be the semi-simple set of $\mathbb{N}^2$ where

$$X_1 = \{(1,0) + x(1,0) + y(2,1) : x, y \in \mathbb{N}\},$$
$$X_2 = \{(0,0) + x(1,3) + y(1,1) : x, y \in \mathbb{N}\},$$
$$X_3 = \{(0,1) + x(1,3) + y(0,3) : x, y \in \mathbb{N}\}.$$

Then the corresponding clustered sets of $X_1, X_2$ and $X_3$ are

$$\mathrm{Clust}(X_1) = \{(1,0) + x(1,0) + y(0,3) : x, y \in \mathbb{N}\},$$
$$\mathrm{Clust}(X_2) = \{(0,0) + x(4,0) + y(0,2) : x, y \in \mathbb{N}\},$$
$$\mathrm{Clust}(X_3) = \{(0,1) + x(4,0) + y(0,3) : x, y \in \mathbb{N}\},$$

which have several collisions.

The existence of collision makes the study of commutative equivalence of semi-linear sets a non-trivial task. In the following, we exhibit a technique in order to cope with collisions when we cluster semi-linear sets of $\mathbb{N}^k$. By using this technique, we will be able to prove Theorem 1.

We conclude this section by proving some preliminary results that are needed in the proof of Theorem 1. Let $X = \mathbf{b}_0 + \{\mathbf{b}_1, \ldots, \mathbf{b}_m\}^{\oplus}$ be a simple subset of $\mathbb{N}^k$ of dimension $m$ with $1 \le m \le k$ where $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_m$ are the vectors of the representation of $X$.

Let $\alpha$ be a positive multiple of the number

$$\mathrm{lcm}\{|\mathbf{b}_i| : 1 \le i \le m\},$$

that is, of the least common multiple of the weights of all the generators of $X$ (we are excluding the constant vector $\mathbf{b}_0$).

**Remark 3.** Observe that $\alpha \neq 0$. Indeed, since the vectors $\mathbf{b}_i$, $1 \le i \le m$, belong to the representation of $X$, $|\mathbf{b}_i| > 0$, for every $i = 1, \ldots, m$.

For every $i = 1, \ldots, m$, let $k_i > 0$ such that $\alpha = k_i|\mathbf{b}_i|$. Let us denote by $\mathbf{R}$ the set of all the tuples

$$\mathbf{r} = (r_1, r_2, \ldots, r_m), \tag{2}$$

such that, for every $i = 1, \ldots, m$, one has $0 \le r_i < k_i$.

Let us associate with every $\mathbf{r} \in \mathbf{R}$, the following two sets of vectors

$$X_{\mathbf{r}} = \mathbf{b}_0 + \sum_{i=1}^{m} r_i \mathbf{b}_i + (k_1\mathbf{b}_1)^{\oplus} + (k_2\mathbf{b}_2)^{\oplus} + \cdots + (k_m\mathbf{b}_m)^{\oplus}, \tag{3}$$

and

$$Y_{\mathbf{r}} = \mathbf{b}_0 + \sum_{i=1}^{m} r_i |\mathbf{b}_i| \mathbf{e}_i + (\alpha\mathbf{e}_1)^{\oplus} + (\alpha\mathbf{e}_2)^{\oplus} + \ldots + (\alpha\mathbf{e}_m)^{\oplus}. \tag{4}$$

The proof of the following lemma is immediate.

**Lemma 6.** *The following conditions hold:*

(i) *The family of sets $\{X_{\mathbf{r}}\}_{\mathbf{r}\in\mathbf{R}}$ is a partition of the set $X$. For every $\mathbf{r} \in \mathbf{R}$, $X_{\mathbf{r}}$ is a simple set of dimension $m$ whose representation is given by the vectors $\mathbf{b}_0 + \sum_{i=1}^{m} r_i\mathbf{b}_i$, $k_1\mathbf{b}_1, \ldots, k_m\mathbf{b}_m$. Moreover there exists an integer $c_{\mathbf{r}}$ with $0 \le c_{\mathbf{r}} < \alpha$ such that, for every $\mathbf{v} \in X_{\mathbf{r}}$, $|\mathbf{v}| \equiv c_{\mathbf{r}} \bmod \alpha$.*

(ii) *The family of sets $\{Y_{\mathbf{r}}\}_{\mathbf{r}\in\mathbf{R}}$ is a partition of the set $\mathrm{Clust}(X)$. For every $\mathbf{r} \in \mathbf{R}$, $Y_{\mathbf{r}}$ is a recognizable simple set of dimension $m$ whose representation is given by the vectors $\mathbf{b}_0 + \sum_{i=1}^{m} r_i|\mathbf{b}_i|\mathbf{e}_i$, $\alpha\mathbf{e}_1, \ldots, \alpha\mathbf{e}_m$. Moreover there exists an integer $c_{\mathbf{r}}$ with $0 \le c_{\mathbf{r}} < \alpha$ such that, for every $\mathbf{v} \in Y_{\mathbf{r}}$, $|\mathbf{v}| \equiv c_{\mathbf{r}} \bmod \alpha$.*

The following example may be of some help to visualize the previous construction.

**Example 3.** Let $X = \{(x_1 + 2x_2, 2x_1 + 2x_2) : x_1, x_2 \in \mathbb{N}\}$, so that $\mathrm{Clust}(X) = \{(3x_1, 4x_2) : x_1, x_2 \in \mathbb{N}\}$. Let $\alpha = 12$ so that $k_1 = 4$ and $k_2 = 3$. Then for every $(r_1, r_2)$ where $0 \le r_1 \le 3$ and $0 \le r_2 \le 2$, one has:

$$X_{\mathbf{r}} = \big\{(4x_1 + 6x_2 + r_1 + 2r_2, 8x_1 + 6x_2 + 2r_1 + 2r_2) : x_1, x_2 \in \mathbb{N}\big\},$$
$$Y_{\mathbf{r}} = \big\{(12x_1 + 3r_1, 12x_2 + 4r_2) : x_1, x_2 \in \mathbb{N}\big\}.$$

**Lemma 7.** *For every $\mathbf{r} \in \mathbf{R}$, $X_{\mathbf{r}}$ and $Y_{\mathbf{r}}$ are commutatively equivalent.*

**Proof.** The sets $X_{\mathbf{r}}$ and $Y_{\mathbf{r}}$ have the same dimension $m$. For every $i = 1, \ldots, m$, $|k_i\mathbf{b}_i| = \alpha = |\alpha\mathbf{e}_i|$, and the constant vectors of $X_{\mathbf{r}}$ and $Y_{\mathbf{r}}$ have the same weight. Then the claim follows by applying Lemma 2 to $X_{\mathbf{r}}$ and $Y_{\mathbf{r}}$. □

## 3. The Main Coding Procedure (MCP)

Let us first describe the goal of this section. Let $d$ and $P$ be integers with $d > 1$ and

$$P \ge (d-1)(k-1),$$

and let us consider the set $\mathbb{S}(\mathbf{0}, P) = \{\mathbf{v} \in \mathbb{N}^k : |\mathbf{v}| = P\}$.

Here, we prove that, with every point of $\mathbb{S}(\mathbf{0}, P)$, a canonical simple set – called (MCP)-set[1] – can be associated with, in order to get a partition in terms of (MCP)-sets of the set of points

$$\big\{\mathbf{v} \in \mathbb{N}^k : |\mathbf{v}| \ge P,\ |\mathbf{v}| \equiv P \bmod d\big\}.$$

This is shown in Proposition 2 and Proposition 3. It will be a crucial step in order to prove Theorem 1. Some preliminary notions and results are needed.

**Definition 7.** Let a vector $\mathbf{v} = (v_1, \ldots, v_k)$ be given, with entries in $\mathbb{N}$ and weight $|\mathbf{v}| = P$. Assume that $\mathbf{v} = (dq_1 + r_1, \ldots, dq_k + r_k)$, for some $q_1, \ldots, q_k$ in $\mathbb{N}$ and $r_1, \ldots, r_k$, with $0 \le r_i < d$, for every $i$, with $1 \le i \le k$. Then the *indicator* of $\mathbf{v}$ is the maximal index $i$ such that, for every $\ell = 1, \ldots, i-1$, $v_\ell = r_\ell$.

Equivalently, the *indicator* of $\mathbf{v}$ is the largest index $i$, with $1 \le i \le k$, such that:

$$\sum_{j=i}^{k} v_j = P - \sum_{t=1}^{i-1} r_t. \tag{5}$$

The indicator of $\mathbf{v}$ will be denoted $\mathrm{ind}(\mathbf{v})$.

[1] MCP stands for Main Coding Procedure.

Observe that, since $P \geq (d-1)(k-1)$, the indicator of a vector of weight $P$ is always well defined since (5) holds at least at index $i = 1$.

**Example 4.** Let $d = 2$ and $P = 4$. Then the subset $\mathbb{S}(\mathbf{0}, P)$ of $\mathbb{N}^3$ is:

$$\begin{array}{lllll}
(0,0,4)^{[\mathbf{3}]} & & & & \\
(1,0,3)^{[\mathbf{3}]} & (0,1,3)^{[\mathbf{3}]} & & & \\
(2,0,2)^{[\mathbf{1}]} & (0,2,2)^{[\mathbf{2}]} & (1,1,2)^{[\mathbf{3}]} & & \\
(3,0,1)^{[\mathbf{1}]} & (2,1,1)^{[\mathbf{1}]} & (1,2,1)^{[\mathbf{2}]} & (0,3,1)^{[\mathbf{2}]} & \\
(4,,0,0)^{[\mathbf{1}]} & (3,1,0)^{[\mathbf{1}]} & (2,2,0)^{[\mathbf{1}]} & (1,3,0)^{[\mathbf{2}]} & (0,4,0)^{[\mathbf{2}]},
\end{array}$$

where the symbol $(^{[\mathbf{i}]})$ denotes the indicator of the vector.

**Definition 8.** Let a vector $\mathbf{v} = (v_1, \ldots, v_k)$ be given, with entries in $\mathbb{N}$ and weight $|\mathbf{v}| = P$. Assume that $\mathbf{v} = (dq_1 + r_1, \ldots, dq_k + r_k)$, for some $q_1, \ldots, q_k$ in $\mathbb{N}$ and $r_1, \ldots, r_k$, with $0 \leq r_i < d$, for every $i$, with $1 \leq i \leq k$. Let us denote $i_0$ the indicator of $\mathbf{v}$. Then we associate with $\mathbf{v}$ the simple set $L_\mathbf{v}$ given by all the vectors:

$$(dx_1 + r_1, \ldots, dx_{i_0-1} + r_{i_0-1}, dx_{i_0} + v_{i_0}, v_{i_0+1}, \ldots, v_k), \tag{6}$$

where $x_1, \ldots, x_{i_0} \in \mathbb{N}$. The set $L_\mathbf{v}$ will be called the MCP-set associated with $\mathbf{v}$.

**Remark 4.** From Definition 8, we have that, for every vector $\mathbf{u} = (u_1, \ldots, u_k)$ of $L_\mathbf{v}$ the remainder mod $d$ of $u_i$, with $1 \leq i \leq k$, is $r_i$. Observe that, if $i_0 = k$ then (6) becomes $(dx_1 + r_1, \ldots, dx_{k-1} + r_{k-1}, dx_k + v_k)$.

The following lemma states some basic properties of the set $L_\mathbf{v}$.

**Lemma 8.** *Let $\mathbf{v} \in \mathbb{N}^k$ with $|\mathbf{v}| = P$. Then the following conditions hold:*

(i) *$L_\mathbf{v}$ is a simple set of dimension $i_0 = \mathrm{ind}(\mathbf{v})$. Moreover the generators of its representation have weight $d$ and the constant vector has weight $P$,*
(ii) *$L_\mathbf{v}$ is a recognizable set of $\mathbb{N}^k$,*
(iii) *There exists a polynomial $p$ of degree $\mathrm{ind}(\mathbf{v}) - 1$ such that, for every $N \in \mathbb{N}$ with $N \equiv P \bmod d$ the counting function of $L_\mathbf{v}$, computed at $N$, is equal to $p(N)$.*

**Proof.** By (6), $L_\mathbf{v}$ is a simple recognizable set of dimension $i_0 = \mathrm{ind}(\mathbf{v})$ whose representation is given by the vectors $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_{i_0}$, where $\mathbf{b}_0 = (r_1, r_2, \ldots, r_{i_0-1}, v_{i_0}, v_{i_0+1}, \ldots, v_k)$ and, for every $i = 1, \ldots, i_0$, $\mathbf{b}_i = d\mathbf{e}_i$. This implies (i) and (ii). (iii) follows by applying Lemma 4 to the set $L_\mathbf{v}$. $\square$

**Example 5.** Let us consider the points of the set $\mathbb{S}(\mathbf{0}, P)$ of Example 4. Then the MCP-sets of dimension 3 are:

$$L(0,0,4) = \{(2x, 2y, 2z+4) : x, y, z \in \mathbb{N}\}$$
$$L(1,0,3) = \{(2x+1, 2y, 2z+3) : x, y, z \in \mathbb{N}\}$$
$$L(0,1,3) = \{(2x, 2y+1, 2z+3) : x, y, z \in \mathbb{N}\}$$
$$L(1,1,2) = \{(2x+1, 2y+1, 2z+2) : x, y, z \in \mathbb{N}\};$$

the MCP-sets of dimension 2 are:

$$L(0,2,2) = \{(2x, 2y+2, 2) : x, y \in \mathbb{N}\}, \qquad L(1,2,1) = \{(2x+1, 2y+2, 1) : x, y \in \mathbb{N}\}$$
$$L(0,3,1) = \{(2x, 2y+3, 1) : x, y \in \mathbb{N}\}, \qquad L(0,4,0) = \{(2x, 2y+4, 0) : x, y \in \mathbb{N}\}$$
$$L(1,3,0) = \{(2x+1, 2y+3, 0) : x, y \in \mathbb{N}\};$$

and the MCP-sets of dimension 1 are:

$$L(2,0,2) = \{(2x, 0, 2) : x \in \mathbb{N}\}, \qquad L(3,0,1) = \{(2x+3, 0, 1) : x \in \mathbb{N}\}$$
$$L(2,1,1) = \{(2x, 1, 1) : x \in \mathbb{N}\}, \qquad L(4,0,0) = \{(2x+4, 0, 0) : x \in \mathbb{N}\}$$
$$L(3,1,0) = \{(2x+3, 1, 0) : x \in \mathbb{N}\}, \qquad L(2,2,0) = \{(2x+2, 2, 0) : x \in \mathbb{N}\}.$$

The following two propositions describe two crucial properties of the family of MCP-sets.

**Proposition 2.** *Let* $\mathbf{v}_1 = (v_{1,1}, \ldots, v_{1,k})$ *and* $\mathbf{v}_2 = (v_{2,1}, \ldots, v_{2,k})$ *be two distinct vectors of* $\mathbb{N}^k$ *with weight* $P$*. Then we have* $L_{\mathbf{v_1}} \cap L_{\mathbf{v_2}} = \emptyset$*.*

**Proof.** First observe that we can assume $\mathbf{v}_1 = (dq_{1,1} + r_1, \ldots, dq_{1,k} + r_k)$ and $\mathbf{v}_2 = (dq_{2,1} + r_1, \ldots, dq_{2,k} + r_k)$, for some $q_{1,1}, \ldots, q_{1,k}, q_{2,1}, \ldots, q_{2,k} \in \mathbb{N}$ and some $r_1, \ldots, r_k \in \mathbb{N}$, with $0 \le r_i < d$, for every $i$, with $1 \le i \le k$.

Indeed, if the latter assumption is not true, there exists an index $i$ with $1 \le i \le k$ such that $v_{1,i} \not\equiv v_{2,i} \bmod d$. Thus the remainder class mod $d$ of the $i$-th component of every vector of $L_{\mathbf{v}_1}$ is different from the $i$-th component of every vector of $L_{\mathbf{v}_2}$, and the thesis follows.

Now let us treat the case where $\mathrm{ind}(\mathbf{v}_1) = \mathrm{ind}(\mathbf{v}_2)$. If $\mathrm{ind}(\mathbf{v}_1) = \mathrm{ind}(\mathbf{v}_2) = k$, since $|\mathbf{v}_1| = |\mathbf{v}_2| = P$, one immediately has $\mathbf{v}_1 = \mathbf{v}_2$, which contradicts the hypothesis. Thus suppose that $\mathrm{ind}(\mathbf{v}_1) = \mathrm{ind}(\mathbf{v}_2) = i_0 < k$. Since, by hypothesis, $\mathbf{v}_1 \neq \mathbf{v}_2$, there exists an index $\ell$ with $1 \le \ell \le k$ such that $v_{1,\ell} \neq v_{2,\ell}$. Since $\mathrm{ind}(\mathbf{v}_1) = \mathrm{ind}(\mathbf{v}_2) = i_0$, the first $i_0 - 1$ components of both the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ coincide at every index, so that one has $\ell \ge i_0$. Recall that $L_{\mathbf{v}_1}$ is the set of all vectors

$$(dx_1 + r_1, \ldots, dx_{i_0-1} + r_{i_0-1}, dx_{i_0} + v_{1,i_0}, v_{1,i_0+1}, \ldots, v_{1,\ell} \ldots, v_{1,k}),$$

where $x_1, \ldots, x_{i_0} \in \mathbb{N}$, and $L_{\mathbf{v}_2}$ is the set of all vectors

$$(dx_1 + r_1, \ldots, dx_{i_0-1} + r_{i_0-1}, dx_{i_0} + v_{2,i_0}, v_{2,i_0+1}, \ldots, v_{2,\ell}, \ldots, v_{2,k}),$$

where $x_1, \ldots, x_{i_0} \in \mathbb{N}$. If $v_{1,\ell} \neq v_{2,\ell}$, with $i_0 < \ell \le k$, one has $L_{\mathbf{v}_1} \cap L_{\mathbf{v_2}} = \emptyset$. Thus assume $v_{1,\ell} = v_{2,\ell}$, with $i_0 < \ell \le k$ and $v_{1,i_0} \neq v_{2,i_0}$. Since $|\mathbf{v}_1| = |\mathbf{v}_2| = P$, we get a contradiction.

Let us finally treat the case when $\mathrm{ind}(\mathbf{v}_1) \neq \mathrm{ind}(\mathbf{v}_2)$. Up to swapping $\mathbf{v}_1$ and $\mathbf{v}_2$, we may assume $\mathrm{ind}(\mathbf{v}_1) < \mathrm{ind}(\mathbf{v}_2)$. For the sake of simplicity, suppose $\mathrm{ind}(\mathbf{v}_1) = j$, and $\mathrm{ind}(\mathbf{v}_2) = j + 1$, with $1 \le j < k$, the general case being treated in the same way. In this case, we have that the set $L_{\mathbf{v}_1}$ associated with $\mathbf{v}_1$ is the set of all vectors

$$(dx_1 + r_1, \ldots, dx_{j-1} + r_{j-1}, dx_j + v_{1,j}, v_{1,j+1}, \ldots, v_{1,\ell}, \ldots, v_{1,k}), \tag{7}$$

where $x_1, \ldots, x_j \in \mathbb{N}$, and the set $L_{\mathbf{v_2}}$ associated with $\mathbf{v}_2$ is the set of all vectors

$$(dx_1 + r_1, \ldots, dx_j + r_j, dx_{j+1} + v_{2,j+1}, v_{2,j+2}, \ldots, v_{2,\ell}, \ldots, v_{2,k}), \tag{8}$$

where $x_1, \ldots, x_{j+1} \in \mathbb{N}$. By comparing (7) and (8), one easily checks that if there exists an index $\ell = j + 1, \ldots, k$, with $v_{1,\ell} < v_{2,\ell}$ one has $L_{\mathbf{v}_1} \cap L_{\mathbf{v_2}} = \emptyset$ and the thesis follows. Hence we can assume that, for every $\ell = j + 1, \ldots, k$, $v_{1,\ell} \ge v_{2,\ell}$, so that

$$\sum_{\ell=j+1}^{k} v_{1,\ell} \ge \sum_{\ell=j+1}^{k} v_{2,\ell}. \tag{9}$$

By hypothesis, $|\mathbf{v}_1| = |\mathbf{v}_2| = P$, so that by (5), one has:

$$\sum_{\ell=j}^{k} v_{1,\ell} = P - \sum_{\ell=1}^{j-1} r_\ell, \qquad \sum_{\ell=j+1}^{k} v_{2,\ell} = P - \sum_{\ell=1}^{j} r_\ell.$$

The latter, together with (9), yields $v_{1,j} \le r_j < d$. On the other hand, since $\mathrm{ind}(\mathbf{v}_1) = j$, one has $v_{1,j} \ge d$, which is a contradiction. Thus the latter case is not possible. This completes the proof. □

The following definition slightly generalizes Definition 7 and it will be used in the proof of the proposition below.

**Definition 9.** Let a vector $\mathbf{v} = (v_1, \ldots, v_k)$ be given, with entries in $\mathbb{N}$ and weight $|\mathbf{v}| \ge P$. Assume that $\mathbf{v} = (dq_1 + r_1, \ldots, dq_k + r_k)$, for some $q_1, \ldots, q_k$ in $\mathbb{N}$ and $r_1, \ldots, r_k$, with $0 \le r_i < d$, for every $i$, with $1 \le i \le k$. Then the $P$-*indicator* of $\mathbf{v}$ is the largest index $i$, with $1 \le i \le k$, such that:

$$\sum_{j=i}^{k} v_j \ge P - \sum_{t=1}^{i-1} r_t. \tag{10}$$

The $P$-indicator of $\mathbf{v}$ will be denoted $P$-$\mathrm{ind}(\mathbf{v})$.

Observe that the $P$-indicator of a vector is always well defined, as condition (10) holds at least at index $i = 1$.

**Remark 5.** It is easily checked that, when $|\mathbf{v}| = P$, the $P$-indicator of $\mathbf{v}$ coincides with its indicator.

**Example 6.** Let $d = 2$ and $P = 4$. Let $\mathbf{v}_1 = (0, 2, 8)$, $\mathbf{v}_2 = (3, 6, 1)$, and $\mathbf{v}_3 = (4, 0, 2)$. Then $P\text{-ind}(\mathbf{v}_1) = 3$, $P\text{-ind}(\mathbf{v}_2) = 2$, and $P\text{-ind}(\mathbf{v}_3) = 1$.

**Proposition 3.** *Let* $\mathbf{v}$ *be a vector of* $\mathbb{N}^k$ *such that* $|\mathbf{v}| \geq P$ *and* $|\mathbf{v}| \equiv P \bmod d$. *Then there exists a vector* $\tilde{\mathbf{v}}$, *with weight* $|\tilde{\mathbf{v}}| = P$ *such that* $\mathbf{v}$ *belongs to the MCP-set* $L_{\tilde{\mathbf{v}}}$ *associated with* $\tilde{\mathbf{v}}$.

**Proof.** Let us write $\mathbf{v} = (dq_1 + r_1, \dots, dq_k + r_k)$, for some $q_1, \dots, q_k$ of $\mathbb{N}$ and $r_1, \dots, r_k$ of $\mathbb{N}$, with $0 \leq r_i < d$, for every $i$, with $1 \leq i \leq k$. Let us denote by $i_0$ the $P$-indicator of $\mathbf{v}$. Observe that, by the definition of $P$-indicator, the following holds:

$$\sum_{j=i_0+1}^{k} v_j < P - \sum_{t=1}^{i_0} r_t, \tag{11}$$

where the left side member of (11) is 0 whenever $i_0 = k$. Let $\Delta$ be the positive integer defined by:

$$\Delta = P - \sum_{t=1}^{i_0} r_t - \sum_{j=i_0+1}^{k} v_j, \tag{12}$$

and let $\tilde{\mathbf{v}}$ be the vector defined as follows:

$$\tilde{\mathbf{v}} = (r_1, \dots, r_{i_0-1}, \tilde{v}_{i_0}, v_{i_0+1}, \dots, v_k),$$

where $\tilde{v}_{i_0} = \Delta + r_{i_0}$. First observe that from (12) one has $|\tilde{\mathbf{v}}| = P$. Moreover observe that, again from (12), one gets:

$$\tilde{v}_{i_0} + \sum_{j=i_0+1}^{k} v_j = P - \sum_{t=1}^{i_0-1} r_t. \tag{13}$$

It is easily checked that the indicator $\text{ind}(\tilde{\mathbf{v}})$ of $\tilde{\mathbf{v}}$ is $i_0$. Indeed, by (13), $\text{ind}(\tilde{\mathbf{v}}) \geq i_0$. From (11) and the fact that $r_{i_0}$ is the remainder mod $d$ of $\tilde{v}_{i_0}$, one gets $\text{ind}(\tilde{\mathbf{v}}) \leq i_0$. Since $|\tilde{\mathbf{v}}| = P$ and $\text{ind}(\tilde{\mathbf{v}}) = i_0$, the MCP-set $L_{\tilde{\mathbf{v}}}$ associated with $\tilde{\mathbf{v}}$ is:

$$(dx_1 + r_1, \dots, dx_{i_0-1} + r_{i_0-1}, dx_{i_0} + \tilde{v}_{i_0}, v_{i_0+1}, \dots, v_k).$$

Let us finally prove that $\mathbf{v}$ belongs to the set $L_{\tilde{\mathbf{v}}}$. For this purpose, taking into account that $\mathbf{v} = (v_1, \dots, v_k) = (dq_1 + r_1, \dots, dq_k + r_k)$, and the equation above of $L_{\tilde{\mathbf{v}}}$, we just need to prove that:

$$v_{i_0} \geq \tilde{v}_{i_0}, \qquad v_{i_0} \equiv \tilde{v}_{i_0} \bmod d.$$

Since $i_0$ is the $P$-indicator of $\mathbf{v}$ one has $\sum_{j=i_0}^{k} v_j \geq P - \sum_{t=1}^{i_0-1} r_t$, and together with (13), we get $v_{i_0} \geq \tilde{v}_{i_0}$. Moreover, since we have:

$$\tilde{v}_{i_0} = P - \sum_{t=1}^{i_0-1} r_t - \sum_{j=i_0+1}^{k} v_j, \qquad v_{i_0} = |\mathbf{v}| - \sum_{t=1}^{i_0-1} r_t - \sum_{j=i_0+1}^{k} v_j,$$

the fact that, by hypothesis, $P \equiv |\mathbf{v}| \bmod d$, yields $v_{i_0} \equiv \tilde{v}_{i_0} \bmod d$. The proof is thus complete. □

**Example 7.** Let $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ be the vectors of Example 6. Then $\mathbf{v}_1 \in L(0, 0, 4)$, $\mathbf{v}_2 \in L(1, 2, 1)$, and $\mathbf{v}_3 \in L(2, 0, 2)$.

### *3.1. The semi-simple sets $\mathcal{L}_\ell$*

We close this section by showing some straightforward corollaries of Proposition 2 and Proposition 3. Let us introduce a family of semi-simple sets of $\mathbb{N}^k$ that are obtained by grouping the MCP sets according to their dimension. More precisely, for every $\ell = 1, \dots, k$, we denote by $\mathcal{L}_\ell$ the semi-simple set

$$\mathcal{L}_\ell = \bigcup_{\mathbf{v} \in \mathbb{S}(\mathbf{0}, P), \text{ind}(\mathbf{v}) = \ell} L_{\mathbf{v}} \tag{14}$$

given by the union of all simple sets $L_{\mathbf{v}}$ where $\mathbf{v} \in \mathbb{S}(\mathbf{0}, P)$ and $\text{ind}(\mathbf{v}) = \ell$.

**Lemma 9.** *Let* $\mathcal{L}_\ell$, *with* $\ell = 1, \dots, k$, *be the set (14). Then the following conditions hold:*

(i) $\mathcal{L}_\ell$ *is a recognizable set of* $\mathbb{N}^k$,
(ii) *There exists a polynomial* $p$ *of degree* $\ell - 1$ *such that, for every* $N \in \mathbb{N}$ *with* $N \equiv P \bmod d$, *the counting function of* $\mathcal{L}_\ell$, *computed at* $N$, *is equal to* $p(N)$,

(iii) *The sets $\mathcal{L}_\ell$, with $\ell = 1, \ldots, k$, are pairwise disjoint,*
(iv) *The weight of every vector of $\mathcal{L}_\ell$, with $\ell = 1, \ldots, k$, is $P$ at least.*

**Proof.** i) By Lemma 8-(ii) every set $L_{\mathbf{v}}$, with $\mathbf{v} \in \mathbb{S}(\mathbf{0}, P)$, is recognizable. Then the claim follows from Proposition 1. ii) From (14), by Lemma 3-(iii), one has: $c_{\mathcal{L}_\ell} = \sum_{\mathbf{v}\in\mathbb{S}(\mathbf{0},P),\mathrm{ind}(\mathbf{v})=\ell} c_{L_{\mathbf{v}}}$. Then the claim follows from the latter equality by applying Lemma 8-(iii) to the counting function of every set $L_{\mathbf{v}}$, with $\mathrm{ind}(\mathbf{v}) = \ell$. iii) follows immediately from Proposition 2. iv) follows from Lemma 8-(i). $\square$

**Corollary 2.** *The following condition holds:*

$$\{\mathbf{v} \in \mathbb{N}^k : |\mathbf{v}| \geq P,\ |\mathbf{v}| \equiv P \bmod d\} = \bigcup_{\mathbf{v}\in\mathbb{S}(\mathbf{0},P)} L_{\mathbf{v}}. \tag{15}$$

*In particular, if we denote by $c_{L_{\mathbf{v}}}$ the counting function of the set $L_{\mathbf{v}}$, we have:*

$$\sum_{\mathbf{v}\in\mathbb{S}(\mathbf{0},P)} c_{L_{\mathbf{v}}}(N) = \mathrm{Card}\big(\mathbb{S}(\mathbf{0}, N)\big),$$

*for every $N$ such that $N \geq P$ and $N \equiv P \bmod d$.*

**Proof.** By Proposition 3, for every vector $\mathbf{w} \in \mathbb{N}^k$ such that $|\mathbf{w}| \geq P$ and $|\mathbf{w}| \equiv P \bmod d$, there exists a vector $\mathbf{v} \in \mathbb{S}(\mathbf{0}, P)$ such that $\mathbf{w} \in L_{\mathbf{v}}$. By Proposition 2 such set $L_{\mathbf{v}}$ is unique. This implies (15). For every integer $N$ with $N \geq P$ and $N \equiv P \bmod d$, from (15), we get

$$\mathbb{S}(\mathbf{0}, N) = \bigcup_{\mathbf{v}\in\mathbb{S}(\mathbf{0},P)} \{\mathbf{w} \in L_{\mathbf{v}} : |\mathbf{w}| = N\}$$

and thus $\mathrm{Card}(\mathbb{S}(\mathbf{0}, N)) = \sum_{\mathbf{v}\in\mathbb{S}(\mathbf{0},P)} c_{L_{\mathbf{v}}}(N)$. $\square$

## 4. The main theorem

In this section we give the proof of Theorem 1. More precisely, we provide an algorithm that, giving a semi-simple set of $\mathbb{N}^k$, computes a recognizable set $S'$ of $\mathbb{N}^k$ that is commutatively equivalent to $S$. For this purpose, let us consider a semi-simple set $S$ of $\mathbb{N}^k$ of dimension $\dim(S) > 0$

$$S = \bigcup_{i=1}^{\ell} S_i, \tag{16}$$

where, for every $i = 1, \ldots, \ell$, $S_i = \mathbf{b}_0^{(i)} + \{\mathbf{b}_1^{(i)}, \ldots, \mathbf{b}_{m_i}^{(i)}\}^{\oplus}$, is a simple set of dimension $m_i \geq 0$ and the vectors $\mathbf{b}_0^{(i)}, \mathbf{b}_1^{(i)}, \ldots, \mathbf{b}_{m_i}^{(i)}$, form the representation of $S_i$. The proof is structured into four steps that we describe in the sequel of the paper.

### 4.1. The first reduction step

The goal of this first reduction step is to reduce the study of our problem to the case of a semi-simple set such that the weight of all the vectors of the set have the same remainder with respect to a suitably defined positive integer. This is done in Proposition 4 described below (see also Remark 6 for a more precise comment). Let $d$ be the number

$$d = \mathrm{lcm}\big\{|\mathbf{b}_i^{(j)}| : 1 \leq j \leq m_i, 1 \leq i \leq \ell\big\}, \tag{17}$$

that is, $d$ is the least common multiple of the weights of all the generators of the simple sets $S_i$ of the decomposition (16) of $S$. Observe that, if the set $S_i, i = 1, \ldots, \ell$ has dimension $m_i = 0$, that is, $S_i = \{\mathbf{b}_0^{(i)}\}$ is a singleton, then it is excluded from the computation of the number $d$ above.

We recall that a simple set is called $d$-homogeneous if the weights of all its generators are $d$.

**Proposition 4.** *Let $S$ be the semi-simple set of $\mathbb{N}^k$ of dimension $> 0$ defined in (16) and let $d$ be the number defined in (17). Then there exists a partition of $S$ into $d$ (possibly empty) semi-simple sets:*

$$\mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_{d-1}, \tag{18}$$

*such that, for every $i = 0, \ldots, d-1$, $\mathcal{G}_i$ is a finite pairwise disjoint union of $d$-homogeneous simple sets and, for every $\mathbf{v} \in \mathcal{G}_i$, $|\mathbf{v}| \equiv i \bmod d$.*

**Proof.** By (16), one has $S = \bigcup_{i=1}^{\ell} S_i$. Let us first consider a simple set $S_i$ of dimension $\geq 1$ in the decomposition (16). By applying Lemma 6-(i) to $S_i$ with respect to $d$, we can write $S_i$ as a finite union of $\ell_i$ pairwise disjoint simple sets $S_j^{(i)}$:

$$S_i = \bigcup_{j=1}^{\ell_i} S_j^{(i)},$$

such that, for every $j = 1, \ldots, \ell_i$, the weight of the generators of $S_j^{(i)}$ is $d$, and there exists a remainder $c_j^{(i)}$ mod $d$, such that, for every $\mathbf{v} \in S_j^{(j)}$, $|\mathbf{v}| \equiv c_j^{(i)}$ mod $d$. By performing the latter operation to every set $S_i$, $i = 1, \ldots, \ell$, of dimension $> 0$ of the decomposition (16) of $S$, we get a new decomposition of $S$ into a finite family of pairwise disjoint simple sets:

$$S = \bigcup_{i=1}^{\ell} \bigcup_{j=1}^{\ell_i} S_j^{(i)} \cup T_0, \tag{19}$$

where $T_0$ denotes the family of simple sets of dimension 0 of $S$. By grouping the simple sets of (19) according to the remainders of the weights of the corresponding vectors, we can rewrite the decomposition (19) as desired. □

**Remark 6.** Assume that Theorem 1 holds for all the semi-simple sets $\mathcal{G}_i$ of (18). By Proposition 4, the family (18) gives a partition of $S$, so that, by Lemma 1, one has immediately that Theorem 1 holds for $S$ as well. Thus Proposition 4 allows us to reduce the proof of Theorem 1 to the case of a semi-simple set of $\mathbb{N}^k$ such that the remainder mod $d$ of the weights of its vectors is a fixed value.

### 4.2. The second reduction step

Let us first describe the goal of the second reduction step. Let $S$ be an arbitrary semi-simple set of $\mathbb{N}^k$ as in (16) and let $d$ be the number defined in (17). By Proposition 4, we can always assume the existence of a partition of $S$ into $d$ (possibly empty) semi-simple sets:

$$S = \mathcal{G}_0 \cup \mathcal{G}_1 \cup \cdots \cup \mathcal{G}_{d-1}, \tag{20}$$

such that, for every $i = 0, \ldots, d-1$, $\mathcal{G}_i$ is a finite union of pairwise disjoint $d$-homogeneous simple sets where the remainder mod $d$ of the weight of the constants is $i$. As the main result of this section, we prove that, for every simple set of dimension $> 0$ that appears in $\mathcal{G}_i$, we can always assume that the constant vector of its representation has the same weight and this weight is as large as we want. This task is done by Proposition 5.

**Proposition 5.** *Let us consider a semi-simple set $\mathcal{G}_i$ of dimension $> 0$ of the decomposition (20) of $S$. Let $P$ be an integer with $P \equiv i$ mod $d$ and larger than the maximal weight of all the vectors that belong to the representations of the simple sets of the decomposition of $\mathcal{G}_i$. Then there exists a decomposition of $\mathcal{G}_i$ into a finite family of $n_i \geq 1$ pairwise disjoint $d$-homogeneous simple sets $T_j$*

$$\mathcal{G}_i = \bigcup_{j=1}^{n_i} T_j,$$

*such that every simple set $T_j$ of dimension $m_j > 0$ can be written as*

$$T_j = \mathbf{b}_0^{(j)} + \{\mathbf{b}_1^{(j)}, \ldots, \mathbf{b}_{m_j}^{(j)}\}^{\oplus},$$

*where $\mathbf{b}_0^{(j)}, \mathbf{b}_1^{(j)}, \ldots, \mathbf{b}_{m_j}^{(j)}$ form the representation of $T_j$ and $|\mathbf{b}_0^{(j)}| = P$.*

**Proof.** Let $M$ be the dimension of $\mathcal{G}_i$. By hypothesis $M > 0$. Let us rearrange the simple sets of $\mathcal{G}_i$ according to their dimension so that $\mathcal{G}_i$ can be written as

$$\mathcal{G}_i = \bigcup_{\ell=0}^{M} \mathcal{D}_\ell, \tag{21}$$

where, for every $\ell = 0, \ldots, M$, $\mathcal{D}_\ell$ is the semi-simple set given by the union of all simple sets of $\mathcal{G}_i$ of dimension $\ell$. Let us first focus our attention on the semi-simple set $\mathcal{D}_M$. We can write $\mathcal{D}_M$ as a finite pairwise disjoint union:

$$\mathcal{D}_M = \bigcup_{\ell=1}^{h} X_\ell, \tag{22}$$

where, for every $\ell = 1, \ldots, h$, $X_\ell$ is a simple set of dimension $M$. Let $X_\ell$ be a simple set appearing in (22) and let us write $X_\ell$ as $X_\ell = \mathbf{b}_0 + \{\mathbf{b}_1, \ldots, \mathbf{b}_M\}^{\oplus}$ where $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_M$ are the vectors of the representation of $X_\ell$. If $q$ denotes a non-negative integer, we can write $X_\ell$ as

$$X_\ell = X'_\ell \cup Y_\ell, \tag{23}$$

where $X'_\ell$ is the simple set of dimension $M$

$$X'_\ell = \mathbf{b}_0 + \mathrm{q} \cdot \mathbf{b}_M + \{\mathbf{b}_1, \ldots, \mathbf{b}_M\}^{\oplus},$$

and $Y_\ell$ is a finite pairwise disjoint union of simple sets of dimension $M-1$

$$Y_\ell = \bigcup_{\ell=0}^{q-1} \left\{\mathbf{b}_0 + \ell \cdot \mathbf{b}_M + \{\mathbf{b}_1, \ldots, \mathbf{b}_{M-1}\}^{\oplus}\right\}.$$

Since $|\mathbf{b}_M| = d$ and $|\mathbf{b}_0| \equiv i \bmod d$, one has $\mathbf{b}_0, \mathbf{b}_0 + q \cdot \mathbf{b}_M \equiv i \bmod d$, and by choosing $q = \frac{P-|\mathbf{b}_0|}{|\mathbf{b}_M|}$, we get $|\mathbf{b}_0 + q \cdot \mathbf{b}_M| = P$, so that $X'_\ell$ satisfies the claim. Now let us perform the decomposition (23) for every $X_\ell$, with $\ell = 1, \ldots, h$, and let us set

$$\mathcal{D}'_M = \bigcup_{\ell=1}^{h} X'_\ell, \qquad \mathcal{D}'_{M-1} = \mathcal{D}_{M-1} \cup \bigcup_{\ell=1}^{h} Y_\ell,$$

where the sets $X'_\ell$ and $Y_\ell$ are defined, for every $\ell = 1, \ldots, h$, as in (23). Then (21) together with the latter two equations yields a new decomposition of $\mathcal{G}_i$ into a finite family of pairwise disjoint simple sets

$$\mathcal{G}_i = \mathcal{D}'_M \cup \mathcal{D}'_{M-1} \cup \bigcup_{\ell=0}^{M-2} \mathcal{D}_\ell, \tag{24}$$

where every simple set of dimension $M$ satisfies the claim. By applying the same argument to the simple sets of (24) of dimension $> 0$ but lower than $M$, we get the claim. $\square$

**Remark 7.** Observe that the family of simple sets of dimension 0 of the new decomposition of $\mathcal{G}_i$, obtained at the end of the proof, contains $\mathcal{D}_0$ and the latter inclusion will be, generally speaking, proper.

### *4.3. The third step: the commutative equivalence of the sets $\mathcal{G}_i$*

We first describe the goal of the third reduction step. For this purpose it is useful to recall the results we have gathered so far. Let $S$ be an arbitrary semi-simple set of $\mathbb{N}^k$ as in (16) and let $d$ be the number defined in (17). By the first reduction step, we can assume the existence of a partition of $S$ into $d$ (possibly empty) semi-simple sets (see Eq. (20)):

$$S = \mathcal{G}_0 \cup \mathcal{G}_1 \cup \cdots \cup \mathcal{G}_{d-1},$$

where, for every $i = 0, \ldots, d-1$, $\mathcal{G}_i$ is a finite union of pairwise disjoint simple sets such that, for every $\mathbf{v} \in \mathcal{G}_i$, $|\mathbf{v}| \equiv i \bmod d$. By the second reduction step, we can assume that every set $\mathcal{G}_i$ of dimension $> 0$ can be written as a union of finitely many pairwise disjoint simple sets:

$$\mathcal{G}_i = \bigcup_{j=1}^{n_i} X_j, \tag{25}$$

where, for every $j = 1, \ldots, n_i$, every $X_j$ is a simple set of dimension $m_j > 0$ whose representation is given by $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_{m_j}$ with $|\mathbf{b}_1| = |\mathbf{b}_2| = \cdots = |\mathbf{b}_{m_j}| = d$, $|\mathbf{b}_0| = P_i$ and $P_i$ is a fixed weight (not depending from $X_j$) satisfying the following three conditions:

- $P_i \equiv i \bmod d$;
- $P_i$ is larger than the maximal weight of all the vectors that belong to the representations of the simple sets of the decomposition of $\mathcal{G}_i$ given by Proposition 4;
- $P_i \geq (d-1)(k-1)$.

The main result of this section is the following.

**Theorem 3.** *For every $i = 0, \ldots, d-1$, there exists an effectively constructible recognizable set $\mathcal{G}'_i$ such that $\mathcal{G}_i$ is commutatively equivalent to $\mathcal{G}'_i$.*

It is useful to remark that Theorem 3 is trivial if $\mathcal{G}_i$ is finite since every finite set is recognizable. Therefore, in the sequel, we will suppose that the dimension of $\mathcal{G}_i$ is positive. For the sake of simplicity we will show the proof of Theorem 3 for the set $\mathcal{G}_i$ with $i = 0$, the other cases being treated exactly the same way. Therefore, in the sequel, we denote by $\mathcal{G}$ the set $\mathcal{G}_0$, by $M$ the dimension of $\mathcal{G}$, and by $P$ the weight $P_i$ of the constant vectors of the representations of all the sets (25) of dimension $> 0$. We will write $\mathcal{G}$ as:

$$\mathcal{G} = \bigcup_{\ell=0}^{M} \mathcal{D}_\ell, \tag{26}$$

where, for every $\ell = 0, \dots, M$, $\mathcal{D}_\ell$ is a semi-simple set of dimension $\ell$

$$\mathcal{D}_\ell = \bigcup_{j=1}^{a} X_j,$$

where $X_1, \dots, X_a$, are all the simple sets of dimension $\ell$ that appear in the decomposition (25) of $\mathcal{G}$.

Now let us consider the set $\mathbb{S}(\mathbf{0}, P) = \{\mathbf{v} \in \mathbb{N}^k : |\mathbf{v}| = P\}$ and let us consider the family of semi-simple sets $\mathcal{L}_\ell$, with $\ell = 1, \dots, k$, introduced in Section 3.1:

$$\mathcal{L}_\ell = \bigcup_{\mathbf{v} \in \mathbb{S}(\mathbf{0}, P), \mathrm{ind}(\mathbf{v}) = \ell} L_{\mathbf{v}}. \tag{27}$$

The set $\mathcal{L}_\ell$, with $\ell = 1, \dots, k$, is obtained by grouping all the simple sets $L_{\mathbf{v}}$, where $\mathbf{v} \in \mathbb{S}(\mathbf{0}, P)$, according to its indicator $\mathrm{ind}(\mathbf{v}) = \ell$.

For every $\ell = 1, \dots, k$, we denote by $\sharp(\mathcal{L}_\ell)$ the number of simple sets of the decomposition (27) for $\mathcal{L}_\ell$.

**Remark 8.** Note that $\dim(\mathcal{L}_\ell) = \ell$. Note also that, by Proposition 2, $\sharp(\mathcal{L}_\ell)$ equals the number of points $\mathbf{v}$ of $\mathbb{S}(\mathbf{0}, P)$ whose indicator is $\ell$.

The following lemma is crucial.

**Lemma 10.** *Let $m$ be a dimension with $1 \le m \le k$. Assume that*

$$\forall \ell = m, \dots, k, \quad \sharp(\mathcal{D}_\ell) \ge \sharp(\mathcal{L}_\ell). \tag{28}$$

*Then, for every $\ell = m, \dots, k$, the following conditions hold:*

(i) $\sharp(\mathcal{D}_\ell) = \sharp(\mathcal{L}_\ell)$,
(ii) *$\mathcal{D}_\ell$ is commutatively equivalent to $\mathcal{L}_\ell$. In particular, $c_{\mathcal{D}_\ell} = c_{\mathcal{L}_\ell}$.*

**Proof.** First let us prove (i) for the dimension $\ell = k$. By contradiction, assume that $\sharp(\mathcal{D}_k) > \sharp(\mathcal{L}_k)$. To simplify the notation set $a = \sharp(\mathcal{D}_k)$ and $b = \sharp(\mathcal{L}_k)$. Let $\mathcal{D}_k = \bigcup_{i=1}^{a} X_i$, and $\mathcal{L}_k = \bigcup_{i=1}^{b} L_{\mathbf{v}_i}$ where

$$X_1, \dots, X_a, \qquad L_{\mathbf{v}_1}, \dots, L_{\mathbf{v}_b}, \tag{29}$$

are the simple sets that appear in the decompositions of $\mathcal{D}_k$ and $\mathcal{L}_k$, respectively. By Lemma 3-(iii) this implies

$$c_{\mathcal{D}_k} = \sum_{\ell=1}^{a} c_{X_\ell}, \qquad c_{\mathcal{L}_k} = \sum_{\ell=1}^{b} c_{L_{\mathbf{v}_\ell}}. \tag{30}$$

Let us now consider two simple sets $X_i$ and $L_{\mathbf{v}_j}$, with $1 \le i \le a$ and $1 \le j \le b$ in (29). Observe that $X_i$ and $L_{\mathbf{v}_j}$ have the same dimension $k$, the weights of the constant vectors of both $X_i$ and $L_{\mathbf{v}_j}$ are $P$ and the weights of the generators of the vectors of both $X_i$ and $L_{\mathbf{v}_j}$ are $d$. By Lemma 2 we have $X_i$ is commutatively equivalent to $L_{\mathbf{v}_j}$ and thus by Lemma 3-(i), one gets $c_{X_i} = c_{L_{\mathbf{v}_j}}$. By the latter condition and (30), one has:

$$c_{\mathcal{D}_k} = c_{\mathcal{L}_k} + \sum_{\ell=b+1}^{a} c_{X_\ell}. \tag{31}$$

On the other hand, by Lemma 3-(iv), for every $N \in \mathbb{N}$, one has $c_{\mathcal{D}_k}(N) \le \mathrm{Card}(\mathbb{S}(\mathbf{0}, N))$. The latter together with Corollary 2 gives

$$c_{\mathcal{D}_k}(N) \le \mathrm{Card}\big(\mathbb{S}(\mathbf{0}, N)\big) = \sum_{\mathbf{v} \in \mathbb{S}(\mathbf{0}, P)} c_{L_{\mathbf{v}}}(N), \tag{32}$$

for every integer $N$ with $N \ge P$ and $N \equiv P \bmod d$. Since $\sum_{\mathbf{v} \in \mathbb{S}(\mathbf{0}, P)} c_{L_{\mathbf{v}}} = \sum_{\ell=1}^{k} c_{\mathcal{L}_\ell}$, by (31) and (32), we have

$$\sum_{\ell=b+1}^{a} c_{X_\ell}(N) \leq \sum_{\ell=1}^{k-1} c_{\mathcal{L}_\ell}(N), \tag{33}$$

for every integer $N$ with $N \geq P$ and $N \equiv P \bmod d$. Now observe that, because of Lemma 9-(ii), the right-side member of (33) is a polynomial function of degree $k-2$ in the variable $N$. On the other hand, the left-side member of the latter inequality is a polynomial function of degree $k-1$ in the variable $N$. Indeed, since every simple set $X_\ell$ of (29) has dimension $k$ and the weights of its generator is $d$, by Lemma 4, its counting function $c_{X_\ell}(N)$ is a polynomial of degree $k-1$ in $N$. By the latter remark, (33) gives a contradiction for every sufficiently large $N$ with $N \equiv P \bmod d$. Hence $\sharp(\mathcal{D}_k) = \sharp(\mathcal{L}_k)$. We thus have proved (i) for the dimension $\ell = k$.

Let us now prove (ii) for the dimension $\ell = k$. Since $a = \sharp(\mathcal{D}_k) = \sharp(\mathcal{L}_k) = b$, the two lists (29) of simple sets appearing in the decompositions of $\mathcal{D}_k$ and $\mathcal{L}_k$ respectively, have the same length, so that we can define a bijection $X_i \longrightarrow L_{\mathbf{v}_i}$, which associates with every simple set $X_i$ of the decomposition of $\mathcal{D}_k$ exactly the simple set $L_{\mathbf{v}_i}$ of the decomposition of $\mathcal{L}_k$. Moreover, as already observed in the first part of the proof, for every $i = 1, \ldots, a$, $X_i$ is commutatively equivalent to $L_{\mathbf{v}_i}$. By Lemma 1 we have therefore that $\mathcal{D}_k$ is commutatively equivalent to $\mathcal{L}_k$. Thus, by applying Lemma 3-(i), one gets $c_{\mathcal{D}_k} = c_{\mathcal{L}_k}$. This proves (ii) for the dimension $\ell = k$.

Taking into account that $c_{\mathcal{D}_k} = c_{\mathcal{L}_k}$, exactly the same argument used to prove (i) and (ii) for the dimension $k$ allows one to show that (i) and (ii) hold for the dimension $k-1$ as well. Similarly, by taking into account that, for every dimension $\ell$ with $m < \ell \leq k$, conditions (i) and (ii) hold, by using exactly the previous arguments one easily proves (i) and (ii) for the dimension $\ell - 1 \geq m \geq 1$ as well. □

Now the proof of Theorem 3 is split into the following two cases that will be treated in Section 4.3.1 and 4.3.2 respectively.

**Full case.** $\forall \ell = 1, \ldots, k,\ \sharp(\mathcal{D}_\ell) \geq \sharp(\mathcal{L}_\ell)$,

**Gap case.** $\exists \ell = 1, \ldots, k,\ \sharp(\mathcal{D}_\ell) < \sharp(\mathcal{L}_\ell)$.

*4.3.1. The full case*

In this section, we prove Theorem 3 under the assumption of the **Full case**. An immediate consequence of Lemma 10 is the following.

**Corollary 3.** *For every $\ell = 1, \ldots, k$, the following conditions hold:*

(i) $\sharp(\mathcal{D}_\ell) = \sharp(\mathcal{L}_\ell)$,
(ii) *$\mathcal{D}_\ell$ is commutatively equivalent to $\mathcal{L}_\ell$. In particular, $c_{\mathcal{D}_\ell} = c_{\mathcal{L}_\ell}$.*

By Corollary 3, we get

$$\forall \ell = 1, \ldots, k, \quad \sharp(\mathcal{D}_\ell) = \sharp(\mathcal{L}_\ell), \tag{34}$$

which is equivalent to saying that, for every positive dimension $\ell$, the number $\sharp(\mathcal{D}_\ell)$ of simple sets of dimension $\ell$ of $\mathcal{G}$ is exactly equal to the number of vectors of weight $P$ whose indicators are $\ell$. This is the reason why this case has been called *full*.

**Lemma 11.** *Under the assumption (34), for every $\mathbf{v} \in \mathcal{D}_0$, $|\mathbf{v}| < P$.*

**Proof.** By contradiction, deny the claim. Hence there exists a vector $\mathbf{v} \in \mathcal{D}_0$ with $|\mathbf{v}| = N \geq P$. Observe that since $\mathbf{v} \in \mathcal{G}$, one has $N \equiv P \bmod d$. Let us compute the counting function $c_{\mathcal{G}}(N)$ of $\mathcal{G}$ at $N$. By (26) one has $\mathcal{G} = \bigcup_{\ell=0}^{k} \mathcal{D}_\ell$ where the sets $\mathcal{D}_\ell$ are pairwise disjoint. Thus, by Lemma 3-(iii), we have $c_{\mathcal{G}}(N) = \sum_{\ell=0}^{k} c_{\mathcal{D}_\ell}(N)$, and since $\mathbf{v} \in \mathcal{D}_0$, the latter implies $c_{\mathcal{G}}(N) \geq 1 + \sum_{\ell=1}^{k} c_{\mathcal{D}_\ell}(N)$. By Corollary 3-(ii), the latter inequality can be written as $c_{\mathcal{G}}(N) \geq 1 + \sum_{\ell=1}^{k} c_{\mathcal{L}_\ell}(N)$. Since $N \geq P$ and $N \equiv P \bmod d$, by Corollary 2, we have $\sum_{\ell=1}^{k} c_{\mathcal{L}_\ell}(N) = \mathrm{Card}(\mathbb{S}(\mathbf{0}, N))$, and thus $c_{\mathcal{G}}(N) \geq 1 + \mathrm{Card}(\mathbb{S}(\mathbf{0}, N))$. On the other hand, by Lemma 3-(iv), we have $c_{\mathcal{G}}(N) \leq \mathrm{Card}(\mathbb{S}(\mathbf{0}, N))$, which gives a contradiction. The claim is thus proved. □

Let us prove Theorem 3 under the assumption of the **Full case**.

**Proof of Theorem 3.** Let us define the set $\mathcal{G}'$ as

$$\mathcal{G}' = \bigcup_{\ell=1}^{k} \mathcal{L}_\ell \cup \mathcal{D}_0.$$

Let us prove that $\mathcal{G}'$ is a recognizable set of $\mathbb{N}^k$ and it is commutatively equivalent to $\mathcal{G}$. By Lemma 9-(i), one has that the set $\bigcup_{\ell=1}^{k}\mathcal{L}_\ell$ is a recognizable set of $\mathbb{N}^k$. Together with the fact that $\mathcal{D}_0$ is finite and thus a recognizable set of $\mathbb{N}^k$ one has that $\mathcal{G}'$ is recognizable as well. Let us now prove that $\mathcal{G}$ and $\mathcal{G}'$ are commutatively equivalent. For this purpose let us first observe that all the sets that appear in the union of $\mathcal{G}'$ are pairwise disjoint. Indeed, by Lemma 9-(iii), all the sets $\mathcal{L}_\ell$, with $\ell = 1,\dots,k$, are pairwise disjoint. Moreover, by Lemma 11, for every $\mathbf{v}\in\mathcal{D}_0$, one has $|\mathbf{v}| < P$, while, by Lemma 9-(iv), the weight of every vector of the set $\bigcup_{\ell=1}^{k}\mathcal{L}_\ell$ is at least $P$. This yields $\mathcal{D}_0\cap\bigcup_{\ell=1}^{k}\mathcal{L}_\ell = \emptyset$. On the other hand, by Corollary 3, for every $\ell = 1,\dots,k$, $\mathcal{D}_\ell$ is commutatively equivalent to $\mathcal{L}_\ell$. Since $\mathcal{G}' = \bigcup_{\ell=1}^{k}\mathcal{L}_\ell\cup\mathcal{D}_0$, and $\mathcal{G} = \bigcup_{\ell=0}^{k}\mathcal{D}_\ell$, the latter arguments, together with Lemma 1, yield that $\mathcal{G}$ and $\mathcal{G}'$ are commutatively equivalent. □

#### 4.3.2. *The case with gaps*

In this section, we prove Theorem 3 under the assumption of the **Gap case**:

$$\exists\ell = 1,\dots,k, \quad \sharp(\mathcal{D}_\ell) < \sharp(\mathcal{L}_\ell), \tag{35}$$

that is, for some $\ell$ with $0 < \ell \le k$, the number of simple sets of dimension $\ell$ that appear in the decomposition (26) of $\mathcal{G}$ is strictly less than the corresponding number of vectors of weight $P$ whose indicator is $\ell$. This is the reason why such case has been called *with gap*. Let us denote by $\ell_{gap}$ the largest integer $\ell$, with $1 \le \ell \le k$ such that (35) holds. Hence we have:

$$\forall\ell = \ell_{gap}+1,\dots,k, \quad \sharp(\mathcal{D}_\ell) \ge \sharp(\mathcal{L}_\ell) \quad \text{and} \quad \sharp(\mathcal{D}_{\ell_{gap}}) < \sharp(\mathcal{L}_{\ell_{gap}}),$$

so that, by Lemma 10-(i), we have:

$$\forall\ell = \ell_{gap}+1,\dots,k, \quad \sharp(\mathcal{D}_\ell) = \sharp(\mathcal{L}_\ell), \quad \text{and} \quad \sharp(\mathcal{D}_{\ell_{gap}}) < \sharp(\mathcal{L}_{\ell_{gap}}).$$

The following example shows that, in the Gap case, there exist also dimensions $m$ with $0 < m < \ell_{gap}$ such that $\sharp(\mathcal{D}_m) > \sharp(\mathcal{L}_m)$.

**Example 8.** Let $d = 2$ and $P = 4$. Let us consider the semi-simple set $S = \mathcal{D}_2 \cup \mathcal{D}_1$, where:

$$\mathcal{D}_2 = \{(x+y, x+y+4) : x, y \in \mathbb{N}\},$$
$$\mathcal{D}_1 = \bigcup_{r=0,\dots,3} S_r = \{(x+4-r, x+r) : x \in \mathbb{N}\}.$$

There is a gap at dimension $\ell = 2$ since $1 = \sharp(\mathcal{D}_2) < \sharp(\mathcal{L}_2) = 2$, and moreover $3 = \sharp(\mathcal{L}_1) < \sharp(\mathcal{D}_1) = 4$.

The existence of dimensions $m > 0$ for which $\sharp(\mathcal{D}_m) > \sharp(\mathcal{L}_m)$ shows that the technique used in the Full case seems not to work anymore. Indeed, we cannot put in one-to-one correspondence the simple sets of $\mathcal{D}_m$ with those of $\mathcal{L}_m$. In order to treat the Gap case, we will develop a suitable refinement of the previous technique. The idea underlying the proof is clarified in the following example.

**Example 9.** Let us construct a recognizable set $S'$ which is commutatively equivalent to the set $S$ of Example 8. Let us associate with $\mathcal{D}_2$ the MCP-set $L(0,4)$ associated with the tuple $(0,4)$. Thus $(1,3)$ is a tuple whose indicator is 2, which is free. Intuitively, this means that the set $2\mathbb{N}^2 + (1,1)$ can be used to construct pairwise disjoint recognizable sets to put into a 1-1 correspondence with those of $\mathcal{D}_1$. By enlarging the weights of the constants, for every $r = 0,\dots,3$ we rewrite $S_r$ as $S_r = \bigcup_{\ell=0,\dots,r-1}\{(\ell+4-r, \ell+r)\} \cup T_r$, where $T_r = \{(x+4, x+2r) : x \in \mathbb{N}\}$. For every $r = 0,\dots,3$, let $T'_r = \{(2r+1, 2x+3) : x \in \mathbb{N}\}$ and let $\mathcal{E}_1 = \bigcup_{r=0,\dots,3} T'_r$. To avoid collisions between $\mathcal{D}_0$ and $\mathcal{E}_1 \cup L(0,4)$, replace $\mathcal{D}_0$ with the set $\mathcal{E}_0 = \{(3,1), (2,2), (5,1), (4,0), (4,2), (6,2)\}$. One easily verifies that $S' = L(0,4) \cup \mathcal{E}_1 \cup \mathcal{E}_0$ is the desired set.

For the sake of simplicity, we will show the proof of Theorem 3 under the assumption that $\ell_{gap} = k-1$, that is,

$$\sharp(\mathcal{D}_k) = \sharp(\mathcal{L}_k), \quad \sharp(\mathcal{D}_{k-1}) < \sharp(\mathcal{L}_{k-1}). \tag{36}$$

Indeed the proof of Theorem 3 in the general case, that is, for every $\ell_{gap} = 1,\dots,k$, is obtained by using the very same argument. In order to prove Theorem 3 in the Gap case, we introduce another suitable decomposition for $\mathcal{G}$. This is done in the next proposition which is a slightly different version of Proposition 5. Its proof follows the very same scheme of Proposition 5 and it is omitted.

**Proposition 6.** *Let us consider the decomposition (26) of $\mathcal{G}$:*

$$\mathcal{G} = \bigcup_{\ell=0}^{k}\mathcal{D}_\ell.$$

*Then there exists a new decomposition of* $\mathcal{G}$

$$\mathcal{G} = \bigcup_{\ell=0}^{k} \mathcal{E}_\ell, \tag{37}$$

*such that*

$$\mathcal{E}_k = \mathcal{D}_k, \quad \mathcal{E}_{k-1} = \mathcal{D}_{k-1},$$

*and, for every* $\ell = 0, \ldots, k-2$, $\mathcal{E}_\ell$ *is a semi-simple set of dimension* $\ell$. *Moreover, for every* $\ell = 1, \ldots, k-2$, $\mathcal{E}_\ell$ *is a finite union of pairwise disjoint simple sets*

$$\mathcal{E}_\ell = \bigcup_{j=1}^{e_k} X_j,$$

*where, for every* $j = 1, \ldots, e_k$, $X_j = \mathbf{b}_0 + \{\mathbf{b}_1, \ldots, \mathbf{b}_\ell\}^\oplus$ *is a d-homogeneous simple set of dimension* $\ell$ *where* $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_\ell$ *form the representation of* $X_j$ *with*

$$|\mathbf{b}_0| = e_k d.$$

**Remark 9.** Observe that the family of simple sets of dimension 0 of the new decomposition of $\mathcal{G}$, obtained at the end of the proof, contains $\mathcal{D}_0$ and the latter inclusion will be, generally speaking, proper.

*4.3.3. The semi-simple sets* $\mathcal{M}_\ell$

From now on, in the sequel, we will consider the decomposition (37) of $\mathcal{G}$. In order to prove Theorem 3 in the Gap case, let us introduce a new family of semi-simple sets $\mathcal{M}_\ell$, with $\ell = 1, \ldots, k$.

If $\ell = k$ then we define

$$\mathcal{M}_k = \mathcal{L}_k, \tag{38}$$

that is, $\mathcal{M}_k$ is the same semi-simple set defined in (27) for the dimension $\ell = k$ (cf. also Section 3.1).

Let us define now the set $\mathcal{M}_\ell$, for $\ell = k-1$. For this purpose, observe that, by (36), we have

$$\sharp(\mathcal{D}_{k-1}) < \sharp(\mathcal{L}_{k-1}).$$

By Remark 8, the latter inequality is equivalent to saying that the number of simple sets of the decomposition of $\mathcal{D}_{k-1}$ is strictly less than the number of the vectors of weight $P$ whose indicator is $k-1$. Hence we can define without ambiguity the semi-simple set $\mathcal{M}_{k-1}$ as:

$$\mathcal{M}_{k-1} = \bigcup_{i=1}^{\sharp(\mathcal{D}_{k-1})} L_{\mathbf{v}_i}, \tag{39}$$

where, for every $i = 1, \ldots, \sharp(\mathcal{D}_{k-1})$, $L_{\mathbf{v}_i}$ is the MCP-set associated with a vector $\mathbf{v}_i$ that satisfies the following conditions:

$$\mathbf{v}_i \neq (\underbrace{0, \ldots, 0}_{(k-2)\text{-times}}, P, 0), \quad \mathrm{ind}(\mathbf{v}_i) = k-1, \quad |\mathbf{v}_i| = P.$$

For the definition of the sets of the family $\mathcal{M}_\ell$, for $\ell = 1, \ldots, k-2$, we need a preliminary result.

Let $\ell, j$ be two integers such that

$$1 \le \ell \le k-2, \quad 1 \le j \le \sharp(\mathcal{E}_\ell),$$

where $\mathcal{E}_\ell$ is the semi-simple of dimension $\ell$ that appears in the decomposition (37) of $\mathcal{G}$. Then let $M(\ell, j)$ be the set of all vectors

$$\big(dx_1, \ldots, dx_{\ell-1}, dx_\ell + d\sharp(\mathcal{E}_\ell) - dj, dj, \underbrace{0, \ldots, 0}_{k-(\ell+1)\text{-times}}\big), \tag{40}$$

where $x_1, \ldots, x_\ell \in \mathbb{N}$ and the right-most $k-(\ell+1)$ components of (40) are null. The following lemma holds.

**Lemma 12.** *Let* $M(\ell, j)$ *be a set of the family (40) with* $\ell = 1, \ldots, k-2$ *and* $j = 1, \ldots, \sharp(\mathcal{E}_\ell)$. *Then the following conditions hold:*

(i) $M(\ell, j)$ *is a simple set. Moreover the generators of its representation have all weight d and the constant vector has weight* $d\sharp(\mathcal{E}_\ell)$,

(ii) *$M(\ell, j)$ is a recognizable set of $\mathbb{N}^k$,*
(iii) *If $(\ell, j) \neq (\ell', j')$, $M(\ell, j) \cap M(\ell', j') = \emptyset$.*

**Proof.** (i) and (ii) follow immediately from (40). Let us prove (iii). Assume first that $\ell \neq \ell'$. For the sake of simplicity, let $\ell' > \ell$, the other case being similar. Then the $(\ell'+1)$-th component of every vector of $M(\ell, j)$ is null, while the $(\ell'+1)$-th component of every vector of $M(\ell', j')$ is $dj'$ and thus it is strictly positive. Hence $M(\ell, j) \cap M(\ell', j') = \emptyset$. Assume now that $\ell' = \ell$ and $j' \neq j$. Then the $(\ell+1)$-th component of every vector of $M(\ell, j')$ is $dj'$, while the $(\ell+1)$-th component of every vector of $M(\ell, j)$ is $dj$. Hence $M(\ell, j) \cap M(\ell', j') = \emptyset$. $\square$

For every $\ell = 1, \ldots, k-2$, let us define the semi-simple set $\mathcal{M}_\ell$, as:

$$\mathcal{M}_\ell = \bigcup_{j=1}^{\sharp(\mathcal{E}_\ell)} M(\ell, j). \tag{41}$$

**Lemma 13.** *For every $\ell = 1, \ldots, k$, the following conditions hold:*

(i) *$\mathcal{M}_\ell$ is a recognizable set of $\mathbb{N}^k$,*
(ii) *$\mathcal{M}_\ell$ is commutatively equivalent to $\mathcal{E}_\ell$. In particular, $c_{\mathcal{M}_\ell} = c_{\mathcal{E}_\ell}$,*
(iii) *The sets $\mathcal{M}_\ell$ are pairwise disjoint.*

**Proof.** Let us prove (i). If $\ell = k$, then, by (38), $\mathcal{M}_k = \mathcal{L}_k$ and the claim follows from Lemma 9-(i). Let $\ell = k-1$. By (39), we have $\mathcal{M}_{k-1} = \bigcup_{i=1}^{\sharp(\mathcal{D}_{k-1})} L_{\mathbf{v}_i}$, and the claim follows from Lemma 8-(ii). Finally, for every $\ell = 1, \ldots, k-2$, by (41), we have $\mathcal{M}_\ell = \bigcup_{j=1}^{\sharp(\mathcal{E}_\ell)} M(\ell, j)$, and the claim follows from Lemma 12-(ii).

Let us prove (ii). Let $\ell = k$. By (38), $\mathcal{M}_k = \mathcal{L}_k$ and, by (37), $\mathcal{E}_k = \mathcal{D}_k$. By (36) we have $\sharp(\mathcal{D}_k) = \sharp(\mathcal{L}_k)$, and hence, by Lemma 10-(ii), one has that $\mathcal{D}_k$ is commutatively equivalent to $\mathcal{L}_k$. By Lemma 3-(i), we get $c_{\mathcal{E}_k} = c_{\mathcal{M}_k}$.

Let us prove (ii) for $\ell = k-1$. By (37),

$$\mathcal{E}_{k-1} = \mathcal{D}_{k-1} = \bigcup_{i=1}^{\sharp(\mathcal{D}_{k-1})} X_i,$$

where the sets $X_i$, with $i = 1, \ldots, \sharp(\mathcal{D}_{k-1})$, are the pairwise disjoint simple sets of dimension $k-1$ that appear in the decomposition (26) of $\mathcal{G}$. By (39), we have

$$\mathcal{M}_{k-1} = \bigcup_{i=1}^{\sharp(\mathcal{D}_{k-1})} L_{\mathbf{v}_i},$$

where, for every $i = 1, \ldots, \sharp(\mathcal{D}_{k-1})$, $L_{\mathbf{v}_i}$ is the MCP-set associated with a vector $\mathbf{v}_i$ with $\mathrm{ind}(\mathbf{v}_i) = k-1$ and $|\mathbf{v}_i| = P$.

Observe that, for every $i = 1, \ldots, \sharp(\mathcal{D}_{k-1})$, $X_i$ and $L_{\mathbf{v}_i}$, have the same dimension $k-1$, the weights of the constant vectors of both $X_i$ and $L_{\mathbf{v}_i}$ are $P$ and the weights of the generators of the vectors of both $X_i$ and $L_{\mathbf{v}_i}$ are $d$. By Lemma 2 $X_i$ is commutatively equivalent to $L_{\mathbf{v}_i}$. By Lemma 1, $\mathcal{E}_{k-1}$ is commutatively equivalent to $\mathcal{M}_{k-1}$ and, by Lemma 3-(i), $c_{\mathcal{E}_{k-1}} = c_{\mathcal{M}_{k-1}}$.

Let us prove (ii) for every dimension $\ell$ with $1 \leq \ell \leq k-2$. For every $\ell = 1, \ldots, k-2$, let $\mathcal{E}_\ell$ be the semi-simple set of dimension $\ell$ of the decomposition (37) of $\mathcal{G}$ given by Proposition 6:

$$\mathcal{E}_\ell = \bigcup_{i=1}^{\sharp(\mathcal{E}_\ell)} X_i.$$

By (41), we have

$$\mathcal{M}_\ell = \bigcup_{i=1}^{\sharp(\mathcal{E}_\ell)} M(\ell, i),$$

where, for every $i = 1, \ldots, \sharp(\mathcal{E}_\ell)$, the set $M(\ell, i)$ is defined in (40).

Observe that, for every $i = 1, \ldots, \sharp(\mathcal{E}_\ell)$, $X_i$ and $M(\ell, i)$, have the same dimension $\ell$, the weights of the constant vectors of both $X_i$ and $M(\ell, i)$ are $d\sharp(\mathcal{E}_\ell)$ and the weights of the generators of the vectors of both $X_i$ and $M(\ell, i)$ are $d$. By Lemma 2, $X_i$ is commutatively equivalent to $M(\ell, i)$. By Lemma 1, $\mathcal{E}_\ell$ is commutatively equivalent to $\mathcal{M}_\ell$ and, by Lemma 3-(i), $c_{\mathcal{E}_\ell} = c_{\mathcal{M}_\ell}$.

Let us prove (iii). One immediately checks that

$$\mathcal{M}_k \cap \mathcal{M}_{k-1} = \emptyset.$$

Indeed, by (38), $\mathcal{M}_k = \mathcal{L}_k$, and, by (39), $\mathcal{M}_{k-1} \subseteq \mathcal{L}_{k-1}$. Thus the claim follows by applying Lemma 9-(iii). Let us now check that, for every $\ell$ with $1 \le \ell \le k-2$,

$$\mathcal{M}_k \cap \mathcal{M}_\ell = \emptyset. \tag{42}$$

Let $\mathbf{v} = (v_1, \ldots, v_k)$ be a vector of $\mathbb{N}^k$ and let $\mathbf{r_v} = (r_1, \ldots, r_k)$ be the vector where, for every $i = 1, \ldots, k$, $r_i$ is the remainder of $v_i$ mod $d$. Since $1 \le \ell \le k-2$, from (40) and (41), it follows that the rightmost component of every vector $\mathbf{w}$ of the set $\mathcal{M}_\ell$ is null and $\mathbf{r_w} = (0, \ldots, 0)$. On the other side, taking into account that, by (38), $\mathcal{M}_k = \mathcal{L}_k$, and that the unique vector $\mathbf{v}$ of weight $P$, $\mathrm{ind}(\mathbf{v}) = k$, and $\mathbf{r_v} = (0, \ldots, 0)$ is

$$\mathbf{v} = (\underbrace{0, \ldots, 0}_{(k-1)\text{-}times}, P),$$

one has that the rightmost component of every vector $\mathbf{w}$ of the set $\mathcal{M}_k$ is not null. The latter implies (42).

Let us now check that, for every $\ell$ with $1 \le \ell \le k-2$,

$$\mathcal{M}_{k-1} \cap \mathcal{M}_\ell = \emptyset. \tag{43}$$

If $\mathbf{w} \in \mathcal{M}_\ell$, with $1 \le \ell \le k-2$, from (40) and (41), one has $\mathbf{r_w} = (0, \ldots, 0)$.

On the other hand, the unique vector $\mathbf{v}$ of weight $P$, $\mathrm{ind}(\mathbf{v}) = k-1$, and $\mathbf{r_v} = (0, \ldots, 0)$ is

$$\mathbf{v} = (\underbrace{0, \ldots, 0}_{(k-2)\text{-}times}, P, 0).$$

By (39) and by Remark 4, for every $\mathbf{w} \in \mathcal{M}_{k-1}$, one has $\mathbf{r_w} \ne (0, \ldots, 0)$ thus implying (43). Finally, for every $i, j$ with $1 \le i < j \le k-2$, $\mathcal{M}_i \cap \mathcal{M}_j = \emptyset$, follows immediately from (41) by applying Lemma 12-(iii). □

The next proposition plays a crucial role in our construction. It allows us to re-arrange all the sets of dimension 0 of the decomposition (37) of $\mathcal{G}$ in a way that these sets are all disjoint (as singletons) from all the sets $\mathcal{M}_i$.

**Proposition 7.** *There exists a finite subset $\mathcal{E}'_0$ of $\mathbb{N}^k$ such that*

$$\mathcal{E}'_0 \cap \bigcup_{i=1}^{k} \mathcal{M}_i = \emptyset,$$

*and $\mathcal{E}'_0$ is commutatively equivalent to $\mathcal{E}_0$.*

**Proof.** Let $P_{max} = \max_{\mathbf{v} \in \mathcal{E}_0} |\mathbf{v}|$. Then $\mathcal{E}_0$ can be written as

$$\mathcal{E}_0 = \bigcup_{n=0}^{P_{max}} \mathcal{E}_{0,n},$$

where, for every $n = 0, \ldots, P_{max}$, the (possibly empty) set $\mathcal{E}_{0,n}$ is defined as $\mathcal{E}_{0,n} = \{\mathbf{v} \in \mathcal{E}_0 : |\mathbf{v}| = n\}$. We now prove the following claim: for every $n = 0, \ldots, P_{max}$, if $\mathcal{E}_{0,n}$ is not empty, there exists a set $\mathcal{E}'_{0,n}$ which is commutatively equivalent to $\mathcal{E}_{0,n}$ and such that

$$\mathcal{E}'_{0,n} \cap \bigcup_{i=1}^{k} \mathcal{M}_i = \emptyset. \tag{44}$$

It is worth to remark that, if the latter claim holds, one easily concludes the proof. Indeed, let us consider the set $\mathcal{E}'_0 = \bigcup_{n=0}^{P_{max}} \mathcal{E}'_{0,n}$. Observe that all the sets $\mathcal{E}'_{0,n}$, with $n = 0, \ldots, P_{max}$, are pairwise disjoint. Then, by Lemma 1, one has that $\mathcal{E}'_0$ is commutatively equivalent to $\mathcal{E}_0$. By (44), one has that also $\mathcal{E}'_0$ is disjoint from the set $\bigcup_{i=1}^{k} \mathcal{M}_i$.

Let us prove the claim. For this purpose, let $n$ be a non-negative integer with $n \le P_{max}$ and $\mathcal{E}_{0,n} \ne \emptyset$. Let us denote by $m$ the number

$$m = \mathrm{Card}\left(\left\{\mathbf{v} \in \mathbb{N}^k : \mathbf{v} \in \bigcup_{i=1}^{k} \mathcal{M}_i,\ |\mathbf{v}| = n\right\}\right), \tag{45}$$

and let us prove the following inequality

$$m + \mathrm{Card}(\mathcal{E}_{0,n}) \le \mathrm{Card}\big(\mathbb{S}(\mathbf{0}, n)\big). \tag{46}$$

By Lemma 3-(iv), one has $c_{\mathcal{G}}(n) \le \text{Card}(\mathbb{S}(\mathbf{0}, n))$. On the other hand, by (37), we have $\mathcal{G} = \bigcup_{i=0}^{k} \mathcal{E}_i$, which gives $c_{\mathcal{G}}(n) = \sum_{i=1}^{k} c_{\mathcal{E}_i}(n) + \text{Card}(\mathcal{E}_{0,n})$, and, by Lemma 13-(ii), the latter becomes $c_{\mathcal{G}}(n) = \sum_{i=1}^{k} c_{\mathcal{M}_i}(n) + \text{Card}(\mathcal{E}_{0,n})$. Taking into account that, by Lemma 13-(iii), all the sets $\mathcal{M}_i$, $i = 1, \ldots, k$, are pairwise disjoint, from (45) we have: $\sum_{i=1}^{k} c_{\mathcal{M}_i}(n) = m$, which, together with the previous equations, gives (46). By (46), the number of vectors of weight $n$ not lying in the set $\bigcup_{i=1}^{k} \mathcal{M}_i$ is at least $\text{Card}(\mathcal{E}_{0,n})$.

This implies that every vector $\mathbf{v}$ of $\mathcal{E}_{0,n}$ can be mapped into a suitable vector – that we denote $f(\mathbf{v})$ – with $|f(\mathbf{v})| = n$, and $f(\mathbf{v}) \notin \bigcup_{i=1}^{k} \mathcal{M}_i$, in such a way that $\mathcal{E}_{0,n}$ and $f(\mathcal{E}_{0,n})$ have the same cardinality. Therefore, the set $f(\mathcal{E}_{0,n})$ is commutatively equivalent to $\mathcal{E}_{0,n}$ and $f(\mathcal{E}_{0,n}) \cap \bigcup_{i=1}^{k} \mathcal{M}_i = \emptyset$. Taking $f(\mathcal{E}_{0,n}) = \mathcal{E}'_{0,n}$, the proof of the claim is thus complete. This concludes the proof. $\square$

We now prove Theorem 3 under the assumption of the **Gap case**.

**Proof of Theorem 3.** Let us define the set $\mathcal{G}'$ as:

$$\mathcal{G}' = \bigcup_{\ell=1}^{k} \mathcal{M}_\ell \cup \mathcal{E}'_0, \tag{47}$$

where $\mathcal{E}'_0$ is the set defined in Proposition 7. Let us prove that $\mathcal{G}'$ is a recognizable set of $\mathbb{N}^k$ and it is commutatively equivalent to $\mathcal{G}$. By Lemma 13-(i), one has that the set $\bigcup_{\ell=1}^{k} \mathcal{M}_\ell$ is a recognizable set of $\mathbb{N}^k$. Together with the fact that $\mathcal{E}'_0$ is finite and thus a recognizable set of $\mathbb{N}^k$ one has that $\mathcal{G}'$ is recognizable as well.

Let us now prove that $\mathcal{G}$ and $\mathcal{G}'$ are commutatively equivalent. For this purpose let us first observe that all the sets that appear in the union of $\mathcal{G}'$ are pairwise disjoint. Indeed, by Lemma 13-(iii), all the sets $\mathcal{M}_\ell$, with $\ell = 1, \ldots, k$, are pairwise disjoint. Moreover, by Proposition 7, one gets $\mathcal{E}'_0 \cap \bigcup_{\ell=1}^{k} \mathcal{M}_\ell = \emptyset$. On the other hand, by Lemma 13-(ii), for every $\ell = 1, \ldots, k$, $\mathcal{E}_\ell$ is commutatively equivalent to $\mathcal{M}_\ell$. Since, by (47) and by (37), we respectively have

$$\mathcal{G}' = \bigcup_{\ell=1}^{k} \mathcal{M}_\ell \cup \mathcal{E}'_0, \qquad \mathcal{G} = \bigcup_{\ell=0}^{k} \mathcal{E}_\ell,$$

the latter arguments, together with Lemma 1, yield that $\mathcal{G}$ and $\mathcal{G}'$ are commutatively equivalent. $\square$

### 4.4. *The fourth step: the proof of Theorem 1*

We are now in position to prove Theorem 1:

Let $S$ be a semi-simple set of $\mathbb{N}^k$ defined as in (16) and let $d$ be the positive integer defined as in (17). If $S$ is finite then $S$ is recognizable and there is nothing to prove. Assume that $S$ is not finite. By Proposition 4, $S$ can be written as a union of finitely many pairwise disjoint sets $S = \bigcup_{i=0}^{d-1} \mathcal{G}_i$, where, for every $i = 0, \ldots, d-1$, $\mathcal{G}_i$ is a semi-simple set such that, for every $\mathbf{v} \in \mathcal{G}_i$, $|\mathbf{v}| \equiv i \bmod d$. Let us now consider the set $S' = \bigcup_{i=0}^{d-1} \mathcal{G}'_i$, where, for every $i = 0, \ldots, d-1$, $\mathcal{G}'_i$ is equal to $\mathcal{G}_i$ if $\mathcal{G}_i$ is finite, otherwise, $\mathcal{G}'_i$ is obtained from $\mathcal{G}_i$ by applying Theorem 3. Observe that, for every $i = 0, \ldots, d-1$, $\mathcal{G}'_i$ is a recognizable subset of $\mathbb{N}^k$ and it is commutatively equivalent to $\mathcal{G}_i$. Hence the $S'$ is a recognizable set as well. Moreover, for every $i, j$ with $0 \le i < j \le d-1$ one obviously has $\mathcal{G}'_i \cap \mathcal{G}'_j = \emptyset$. Finally, by Lemma 1, one has that $\mathcal{G}'$ is commutatively equivalent to $\mathcal{G}$. As the reader can easily check, the proof is constructive. $\square$

## Acknowledgements

We thank the anonymous referee for her (his) help in improving a previous version of this paper.

## References

[1] C. Choffrut, A. Frigeri, Definable sets in weak Presburger arithmetic, in: Proc. 10th Italian Conference on Theoretical Computer Science, ICTCS 2007, World Scientific Pub. Co., Singapore, 2007, pp. 175–186.
[2] C. Choffrut, A. Frigeri, Deciding whether the ordering is necessary in a Presburger formula, Discrete Math. Theor. Comput. Sci. 12 (2010) 21–38.
[3] F. D'Alessandro, B. Intrigila, S. Varricchio, On the structure of the counting function of context-free languages, Theoret. Comput. Sci. 356 (2006) 104–117.
[4] F. D'Alessandro, B. Intrigila, S. Varricchio, The Parikh counting functions of sparse context-free languages are quasi-polynomials, Theoret. Comput. Sci. 410 (2009) 5158–5181.
[5] F. D'Alessandro, B. Intrigila, The commutative equivalence of bounded context-free and regular languages, in: International Conference on Words and Formal Languages, Words 2011, in: Electronic Proceedings in Theoretical Computer Science, 2011, pp. 1–21.
[6] F. D'Alessandro, B. Intrigila, S. Varricchio, Quasi-polynomials, semi-linear set, and linear Diophantine equations, Theoret. Comput. Sci. 416 (2012) 1–16.
[7] F. D'Alessandro, B. Intrigila, On the commutative equivalence of bounded context-free and regular languages: the code case, Theoret. Comput. Sci. 562 (2015) 304–319, this issue.

[8] F. D'Alessandro, B. Intrigila, On the commutative equivalence of bounded context-free and regular languages: the semi-linear case, preprint (2013), to appear in Theoret. Comput. Sci. (2015).
[9] A. de Luca, F. D'Alessandro, Teoria degli Automi Finiti, Springer Italia, Milan, 2013.
[10] S. Eilenberg, M.-P. Schützenberger, Rational sets in commutative monoids, J. Algebra 13 (1969) 173–191.
[11] S. Eilenberg, Automata, Languages and Machines, vol. A, Academic Press, New York, 1974.
[12] S. Ginsburg, E.H. Spanier, Semigroups, Presburger formulas, and languages, Pacific J. Math. 16 (1966) 285–296.
[13] S. Ginsburg, The Mathematical Theory of Context-Free Languages, McGraw–Hill, New York, 1966.
[14] R.L. Graham, D.E. Knuth, O. Patashnik, Concrete Mathematics: A Foundation for Computer Science, Addison–Wesley, Reading, Mass, 1989.
[15] R. Ito, Every semi-linear set is a finite union of disjoint linear sets, J. Comput. System Sci. 3 (1969) 295–342.