A categorical semantics for causal structure

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***Abstract*—We present a categorical construction for modelling both definite and indefinite causal structures within a general class of process theories that include classical probability theory and quantum theory. Unlike prior constructions within categorical quantum mechanics, the objects of this theory encode fine-grained causal relationships between subsystems and give a new method for expressing and deriving consequences for a broad class of causal structures. To illustrate this point, we show that this framework admits processes with definite causal structures, namely one-way signalling processes, non-signalling processes, and quantum n-combs, as well as processes with indefinite causal structure, such as the quantum switch and the process matrices of Oreshkov, Costa, and Brukner. We furthermore give derivations of their operational behaviour using simple, diagrammatic axioms.**

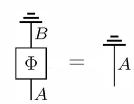
I. Introduction

Broadly, causal structures identify which events or processes taking place across space and time can, in principle, serve as causes or effects of one another. For instance, the causal structure of relativistic spacetime is given by demanding events can only have a causal influence on other events which can be reached without exceeding the speed of light.

In the context of quantum theory, causal relationships between inputs and outputs to quantum processes have been expressed in a variety of ways. Perhaps the simplest are in the form of non-signalling constraints, which guarantee that distant agents are not capable of sending information faster that the speed of light, e.g. to affect each other's measurement outcomes [4]. Quantum strategies [15] and more recently quantum combs [5] offer a means of expressing more intricate causal relationships, in the form of chains of causally ordered inputs and outputs. Furthermore, it has been shown recently that one can formulate a theory that is locally consistent with quantum theory yet assumes no fixed background causal structure [22]. Interestingly, such a theory admits *indefinite* causal structures. Namely, it allows one to express processes which inhabit a quantum superposition of causal orders. If physically realisable, such processes can outperform any theory that demands a fixed causal ordering in certain non-local games such as 'guess your neighbour's input' [22] and quantum computational tasks such as single-shot channel discrimination [8]. Perhaps even more surprisingly, it has been shown recently that, in the presence of three or more parties, causal bounds can be violated even within a theory that behaves locally like classical probability theory [3].

The key ingredient in studying (and varying) causal structures of processes is the development of a coherent theory of *higher order causal processes*, i.e. maps from causal processes to causal processes. For example, if we treat local agents (or events, laboratories, etc.) as first order causal processes, then the act of composing these processes in a particular causal order should be treated as a second-order process. In this paper, we develop a categorical framework for expressing and reasoning about such higher order processes. This starts from the general context of a *precausal category*, which is a compact closed category that satisfies a four extra axioms that provide enough additional structure to reason about causal relationships between systems and to prove simple no-go results such as *no time-travel*, i.e. the impossibility of process sending information into its own causal past.

Most importantly, precausal categories have a special *discarding process* defined on each object, which enables one to state the causality postulate for a process $\Phi : A \to B$:



This has a clear operational intuition: *if the output of a process is discarded, it doesn't matter which process occurred.* While seemingly obvious, this condition, originally given by [7] regarding operational probabilistic theories, is very powerful. For instance, it is equivalent to the non-signalling property for joint processes arising from shared correlations [12].

Starting from a precausal category $\mathcal{C}$, we give a construction of the $*$-autonomous category $\mathrm{Caus}[\mathcal{C}]$ of higher-order causal processes, into which the category of (first-order) causal processes satisfying the equation above embeds fully and faithfully. Our main examples start from the precausal categories of matrices of positive real numbers and completely positive maps, which will yield categories of higher-order classical stochastic processes and higher-order quantum channels, respectively.

While categorical quantum mechanics [2] has typically focussed on compact closed categories of quantum processes, we show that this $*$-autonomous structure yields a much richer type system for describing causal relationships between systems. A simple, yet striking example is in the use of $\otimes$ vs. $⅋$ to form the types of processes on a joint system:

$$(\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{B}') \quad \leftarrow \quad \text{non-signalling processes}$$
$$(\boldsymbol{A} \multimap \boldsymbol{A}') ⅋ (\boldsymbol{B} \multimap \boldsymbol{B}') \quad \leftarrow \quad \text{all processes}$$

Using this type system, we give logical characterisations of non-signalling and one-way signalling processes and then prove that these are equivalent to the operational characterisations in terms of their interactions with the discarding process. We then go on to characterise higher-order systems, notably bipartite, and n-partite second-order causal processes and give

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several examples which are known to exhibit indefinite causal structure, namely the OCB process from [22], and classical tripartite process from [3], and an abstract version of the quantum switch defined in [8]. Finally, we prove using just the structure of Caus[$\mathcal{C}$] that the switch does not admit a causal ordering by reducing to no time-travel.

Related work. This work was inspired by [23], which aims for a uniform description of higher-order *quantum* operations in terms of generalised Choi operators. However, rather than relying on the linear structure of spaces of operators, we work purely in terms of the $*$-autonomous structure and the pre-causal axioms, which concern the compositional behaviour of discarding processes. The construction of Caus[$\mathcal{C}$] is a variant of the *double gluing construction* used in [1] to construct models of linear logic. In the language of that paper, our construction consists of building the 'tight orthogonality category' induced by a focussed orthogonality on $\{1_I\} \subseteq \mathcal{C}(I, I)$, then restricting to objects satisfying the flatness condition in Definition IV.2. Since it is $*$-autonomous, Caus[$\mathcal{C}$] indeed gives a model of multiplicative linear logic, enabling us to enlist the aid of linear-logic based tools for proving theorems about causal types. We comment briefly on this in the conclusion.

An extended version of this paper with additional proofs can be found online. See [17].

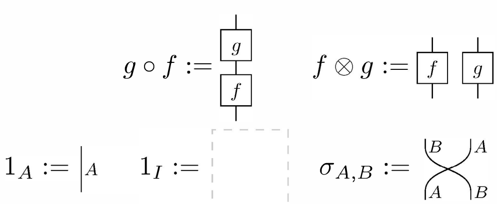
II. Preliminaries

We work in the context of *symmetric monoidal categories* (SMCs). An SMC consists of a collection of objects ob($\mathcal{C}$), for every pair of objects, $A, B \in$ ob($\mathcal{C}$) a set $\mathcal{C}(A, B)$ of morphisms, associative sequential composition '$\circ$' with units 1_A for all $A \in$ ob($\mathcal{C}$), associative (up to isomorphism) parallel composition '$\otimes$' for objects and morphisms with unit $I \in$ ob($\mathcal{C}$), and swap maps $\sigma_{A,B} : A \otimes B \to B \otimes A$, satisfying the usual equations one would expect for composition and tensor product. For simplicity, we furthermore assume $\mathcal{C}$ is *strict*, i.e.

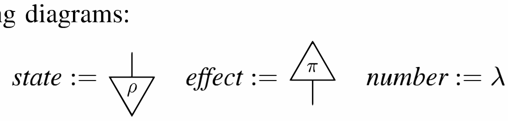
$$(A \otimes B) \otimes C = A \otimes (B \otimes C) \qquad\qquad A \otimes I = A = I \otimes A$$

This is no loss of generality since every SMC is equivalent to a strict one. For details, [21] is a standard reference.

We wish to treat SMCs as theories of physical processes, hence we often refer to objects as *systems* and morphisms a *processes*. We will also extensively use *string diagram* notation for SMCs, where systems are depicted as wires, processes and boxes, and:



Note that diagrams should be read from bottom-to-top. A process $\rho : I \to A$ is called a *state*, a process $\pi : A \to I$ is called an *effect*, and $\lambda : I \to I$ is called a *number*. Depicted as string diagrams:



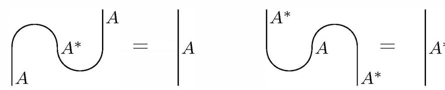
Numbers in an SMC always form a commutative monoid with 'multiplication' $\circ$ and unit the identity morphism 1_I. We typically write 1_I simply as 1.

We will begin with a category $\mathcal{C}$ and construct a new category Caus[$\mathcal{C}$] of *higher-order* causal processes. In order to make this construction, we first need a mechanism for expressing higher-order processes. *Compact closure* provides such a mechanism that is convenient within the graphical language and already familiar within the literature on quantum channels, in the guise of the Choi-Jamiołkowski isomorphism.

Definition II.1. An SMC $\mathcal{C}$ is called *compact closed* if every object A has a *dual* object A^*. That is, for every A there exists morphisms $\eta_A : I \to A^* \otimes A$ and $\epsilon_A : A \otimes A^* \to I$, satisfying:

$$(\epsilon_A \otimes 1_A) \circ (1_A \otimes \eta_A) = 1_A \quad (1_{A^*} \otimes \epsilon_A) \circ (\eta_A \otimes 1_{A^*}) = 1_{A^*}$$

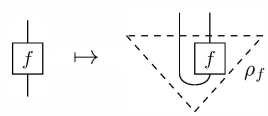
We refer to η_A and ϵ_A as a cup and a cap, denoted graphically as and , respectively. In this notation, the equations in Definition II.1 become:



It is always possible to choose cups and caps in such a way that the canonical isomorphisms $I^* \cong I$, $(A \otimes B)^* \cong A^* \otimes B^*$, and $A \cong A^{**}$ are all in fact equalities. We will assume this is the case throughout this paper.

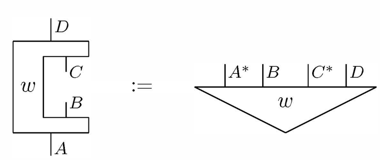
Crucially, two morphisms in a compact closed category are equal if and only if their string diagrams are the same. That is, if one diagram can be continuously deformed into the other while maintaining the connections between boxes. Hence, when we draw a string diagram, we mean *any* composition of boxes via cups, caps, and swaps which yields the given diagram, up to deformation. See [24] for an overview of string diagram languages for monoidal categories.

Compact closed categories exhibit *process-state duality*, that is, processes $f : A \to B$ are in 1-to-1 correspondence with states $\rho_f : I \to A^* \otimes B$:



(1)

Hence, we treat everything as a 'state' in $\mathcal{C}$ and write $f : X$ as shorthand for $f : I \to X$. In this notation, states are of the form $\rho : A$, effects $\pi : A^*$, and general processes $f : A^* \otimes B$. Furthermore, we won't require 'output' wires to exit upward, and we allow irregularly-shaped boxes. For example, we can write a process $w : A^* \otimes B \otimes C^* \otimes D$ as follows:

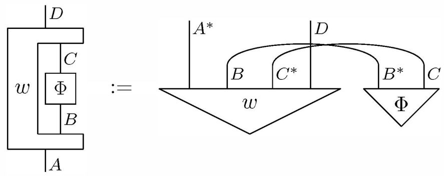


(2)

where we adopt the convention that an A-labelled 'input' wire is of the same type as an A^*-labelled 'output'.

While both the LHS and the RHS in equation (2) are notation for the same process w, the LHS is strongly suggestive of a *second-order mapping*, i.e., one that sends processes $B \to C$

to processes $A \to D$. Composition in this notation simply means applying the appropriate 'cap' processes to plug wires together:



Remark II.2. Since oddly-shaped boxes don't uniquely fix any ordering of systems with respect to $\otimes$, we will often 'name' each system by giving it a unique type and assume systems are permuted via σ-maps whenever necessary. This is a common practice e.g. in the quantum information literature.

Our key examples will be $\mathbf{Mat}(\mathbb{R}_+)$ and **CPM**, which contain stochastic matrices and quantum channels, respectively.

Example II.3. The category $\mathbf{Mat}(\mathbb{R}_+)$ has as objects natural numbers. Morphisms $f : m \to n$ are $n \times m$ matrices. Composition is given by matrix multiplication: $(g \circ f)_i^j := \sum_k f_i^k g_k^j$, $m \otimes n := mn$ and $f \otimes g$ is the Kronecker product of matrices:

$$(f \otimes g)_{ij}^{kl} := f_i^k g_j^l$$

Consequently, the tensor unit $I = 1$, so states are column vectors $\rho : 1 \to n$, effects are row vectors $\pi : n \to 1$, and numbers are $\lambda \in \mathbb{R}_+$. $\mathbf{Mat}(\mathbb{R}_+)$ is compact closed with $n = n^*$, where cups and caps are given by the Kronecker delta δ_{ij}:

$$\eta^{ij} := \delta_{ij} =: \epsilon_{ij}$$

Example II.4. The category **CPM** has as objects $\mathcal{B}(H)$, $\mathcal{B}(K), \ldots$, the bounded operators on finite dimensional Hilbert spaces $H, K, \ldots$, and as morphisms completely positive maps $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$. $\mathcal{B}(H) \otimes \mathcal{B}(K) = \mathcal{B}(H \otimes K)$, hence $I = \mathcal{B}(\mathbb{C}) \cong \mathbb{C}$. States are therefore positive operators and effects are (un-normalised) quantum effects, i.e. completely positive maps of the form: $\pi(\rho) := \mathrm{Tr}(\rho P)$, for some positive operator P. As with $\mathbf{Mat}(\mathbb{R}_+)$, numbers are $\mathbb{R}_+$. **CPM** is also compact closed, with cups and caps given by the (un-normalised) Bell state:

$$\eta = |\Phi_0\rangle\langle\Phi_0| \qquad \epsilon(\rho) = \mathrm{Tr}(\rho\, |\Phi_0\rangle\langle\Phi_0|)$$

where $|\Phi_0\rangle = \sum_i |i\rangle \otimes |i\rangle$. Consequentially, $\mathcal{B}(H)^* = \mathcal{B}(H)$ and equation (1) gives the basis-dependent version, i.e. 'Choi-style', of the Choi-Jamiołkowski isomorphism [19].

The biggest convenience of a compact closed structure is also its biggest drawback: all higher-order structure collapses to first-order structure! Indeed, if we let $A \Rightarrow B := A^* \otimes B$ represent the type of maps from A to B in a compact closed category, we have:

$$\begin{aligned}(A \Rightarrow B) \Rightarrow C &= (A^* \otimes B)^* \otimes C \\ &\cong (A \otimes B^*) \otimes C \\ &\cong B^* \otimes (A \otimes C) \\ &= B \Rightarrow (A \otimes C)\end{aligned}$$

which is just a first order expression again. As we will soon see, there is a pronounced difference between first-order causal processes, which we introduce in the next section (Definition III.1, and genuinely higher-order causal processes. Thus, while it is natural to take $\mathcal{C}$ to be compact closed, we expect Caus$[\mathcal{C}]$ to be a different kind of category, which allows this genuine higher-order structure.

Definition II.5. A $*$-*autonomous category* is a symmetric monoidal category equipped with a full and faithful functor $(-)^* : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ such that, by letting:

$$A \multimap B := (A \otimes B^*)^* \tag{3}$$

there exists a natural isomorphism:

$$\mathcal{C}(A \otimes B, C) \cong \mathcal{C}(A, B \multimap C) \tag{4}$$

As the notation and the isomorphism (4) suggest, $A \multimap B$ is the system whose states correspond to processes from A to B. We adopt the programmers' convention that $\otimes$ has precedence over $\multimap$ and $\multimap$ associates to the right:

$$A \otimes B \multimap C := (A \otimes B) \multimap C \qquad A \multimap B \multimap C := A \multimap (B \multimap C)$$

Either expression above represents the system whose states are processes with two inputs. Indeed (4) implies that $A \otimes B \multimap C \cong A \multimap B \multimap C$. Since $\mathcal{C}$ is symmetric, we can re-arrange the inputs at will, i.e.

$$A \multimap B \multimap C \cong B \multimap A \multimap C \tag{5}$$

Unlike in a compact closed category, the object $A \multimap B$ can be different from simply $A^* \otimes B$. Indeed any compact closed category is $*$-autonomous, where it additionally holds that:

$$A \otimes B \cong (A^* \otimes B^*)^* \tag{6}$$

in which case:

$$A \multimap B := (A \otimes B^*)^* \cong A^* \otimes B^{**} \cong A^* \otimes B$$

However, in a $*$-autonomous category, the RHS of (6) is not $A \otimes B$, but something new, called the 'par' of A and B:

$$A \parr B := (A^* \otimes B^*)^* \tag{7}$$

This new operation inherits its good behaviour from $\otimes$:

$$A \parr (B \parr C) \cong (A \parr B) \parr C \qquad\qquad A \parr B \cong B \parr A$$

So a compact closed category is just a $*$-autonomous category where $\otimes = \parr$. However, this little tweak yields a much richer structure of higher-order maps. We think of $A \otimes B$ as the joint state space of A and B, whereas $A \parr B$ is like taking the space of maps from A^* to B. For (first order) state spaces, these are basically the same, but as we go to higher order spaces, $A \parr B$ tends to be much bigger than $A \otimes B$.

III. Precausal categories

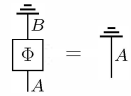
Precausal categories give a universe of all processes, and provide enough structure for us to identify which of those processes satisfy first-order and higher-order causality constraints.

As noted in [11], [13], [14], [7], the crucial ingredient for defining causality is a preferred *discarding* process $\bar{\bar{\top}}_A$ from every system A into I, satisfying

$$\bar{\bar{\top}}_{A \otimes B} := \bar{\bar{\top}}_A \; \bar{\bar{\top}}_B \qquad \bar{\bar{\top}}_I := 1 \tag{8}$$

Using this discard effect, we can define causality as follows:

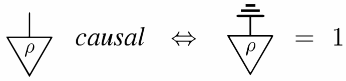
Definition III.1. For systems A and B with discarding, a process $\Phi : A \to B$ is said to be *(first-order) causal* if:



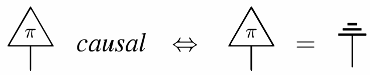
(9)

Intuitively, causality means that if we disregard the output of a process, it does not matter which process occurred.

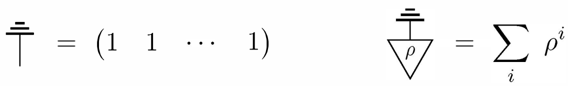
Hence *causal states* produce 1 when discarded:



Since discarding the 'output' of an effect $\pi : A \to I$ is the identity, there is a unique causal effect for any system, namely discarding itself:



Example III.2. For $\mathbf{Mat}(\mathbb{R}_+)$, the discarding process is a row vector consisting entirely of 1's; it sends a state to the sum over its vector entries:

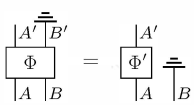


So, causality is precisely the statement that a vector of positive numbers sums to 1, i.e. forms a probability distribution. Consequentially, the causality equation (9) for a process Φ states that each column of Φ must sum to 1. That is, Φ is a stochastic map.

Example III.3. For **CPM**, discard is the trace. Hence causal states are density operators and causal processes are trace-preserving completely positive maps, a.k.a. quantum channels.

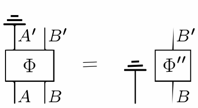
Discarding not only allows us to express when a process is causal, it also allows us to represent causal relationships between the systems involved. For example

Definition III.4. A causal process $\Phi : A \otimes B \to A' \otimes B'$ is *one-way signalling* with A before B (written $A \preceq B$) if there exists a process $\Phi' : A \to A'$ such that



(10)

It is one-way signalling with $B \preceq A$ if there exists Φ'' such that

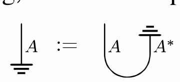


(11)

Such a process is called *non-signalling* if it is both one-way signalling with $A \preceq B$ and $B \preceq A$.

Note that $A \preceq B$ does not imply signalling from A to B, but rather that there *cannot* be signalling from B to A.

If A^* has discarding, we can also produce a state for A:



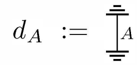
(12)

These ingredients allow us to make the following definition:

Definition III.5. A *precausal category* is a compact closed category $\mathcal{C}$ such that:

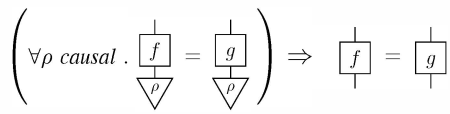
(C1) $\mathcal{C}$ has discarding processes for every system, compatible with the monoidal structure as in (8).

(C2) For every (non-zero) system A, the *dimension* of A:

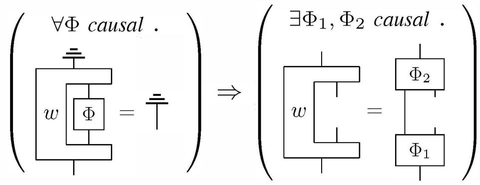


is an invertible scalar.

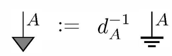
(C3) $\mathcal{C}$ has *enough causal states*:



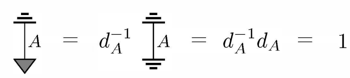
(C4) *Second-order causal* processes factorise:



(C1) enables one to talk about causal processes within $\mathcal{C}$. (C2) enables us to renormalise certain processes to produce causal ones. For example, every non-zero system has at least one causal state, called the *uniform state*. It is obtained by normalising (12):



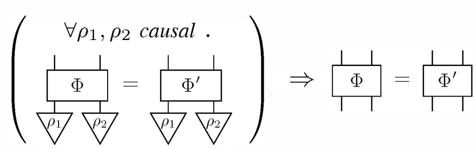
Then:



Note that we allow $\mathcal{C}$ to have a zero object, in which case $d_0 = 0$ is not required to be invertible.

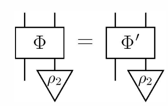
(C3) says that processes are characterised by their behaviour on causal states. Since $\mathcal{C}$ is compact closed, (C3) implies that it suffices to look only at product states to identify a process.

Proposition III.6. For any compact closed category $\mathcal{C}$, (C3) is equivalent to:

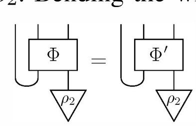


(13)

Proof. (C3) follows from (13) by taking one of the two systems involved to be trivial. Conversely, assume the premise of (13). Applying (C3) one time yields:



for all causal states ρ_2. Bending the wire yields:

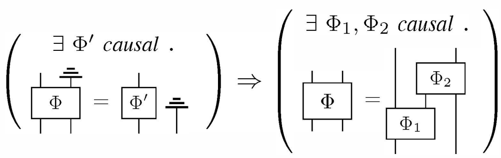


Hence we can apply (C3) a second time. Bending the wire back down gives the result. □

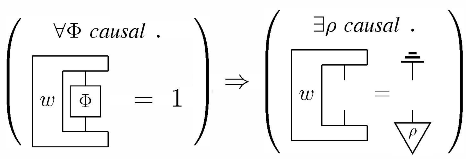
(C4) is perhaps the least transparent. It says that the only mappings from causal processes to causal processes are 'circuits with holes', i.e. those mappings which arise from plugging a causal process into a larger circuit of causal processes. This can equivalently be split into two smaller pieces, which may look more familiar to some readers.

Proposition III.7. For a compact closed category $\mathcal{C}$ satisfying (C1), (C2), and (C3), condition (C4) is equivalent to the following two conditions:

(C4′) Causal *one-way signalling* processes factorise:

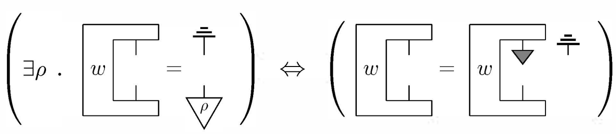


(C5′) For all $w : A \otimes B^*$:

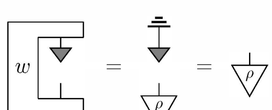


Proof. This follows essentially from diagram deformation, enough causal states, and the following Lemma III.8. A full proof can be found in [17]. □

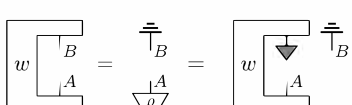
Lemma III.8. For any $w : A \otimes B^*$:



Proof. (⇐) immediately follows from diagram deformation. For (⇒), assume the leftmost equation above. Then we can obtain an expression for ρ by plugging the uniform state into w and applying causality of the uniform state:



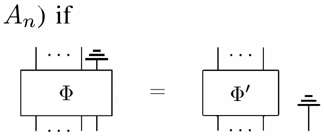
Substituting this expression back in for ρ yields:


□

Example III.9. **Mat**$(\mathbb{R}_+)$ and **CPM** are both precausal categories. See [17] for proofs of conditions (C1)-(C4).

Note that we can readily generalise one-way signalling from Definition (III.4) to n systems.

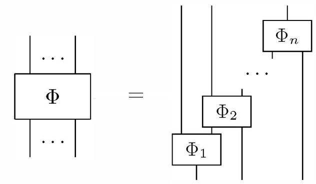
Definition III.10. A process Φ is *one-way signalling* ($A_1 \preceq \ldots \preceq A_{n-1} \preceq A_n$) if



with Φ' one-way signalling with $A_1 \preceq \ldots \preceq A_{n-1}$.

Then (C4′) implies an general n-fold version of itself:

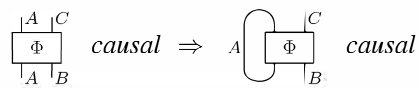
Proposition III.11. Let Φ be one-way signalling with $A_1 \preceq \ldots \preceq A_n$, then there exists $\Phi_1, \ldots, \Phi_n$ such that



Proof. This follows from induction, see [17]. □

While, as we shall soon see, precausal categories give us a source of processes exhibiting many varieties of definite and indefinite causal structure, the axioms rule out certain, paradoxical causal structures. To see this, we state our first no-go result for a precausal category $\mathcal{C}$.

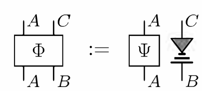
Theorem III.12 (No time-travel)**.** No non-trivial system A in a precausal category $\mathcal{C}$ admits *time travel*. That is, if there exist systems B and C such that:



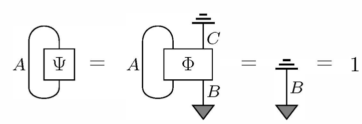
(14)

then $A \cong I$.

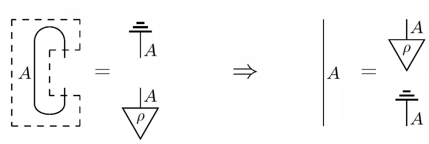
Proof. For any causal process $\Psi : A \to A$, we can define:



which is also a causal process. Then implication (14) gives:



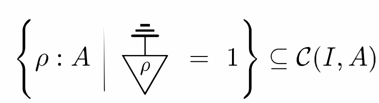
Applying (C5′), we have:



for some causal state $\rho : I \to A$. That is, $\rho \circ \bar{\top} = 1_A$, and by definition of causality for ρ, $\bar{\top} \circ \rho = 1_I$. Hence $\rho : I \cong A$. □

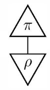
IV. Constructing Caus[$\mathcal{C}$]

We will now describe our main construction, the category Caus[$\mathcal{C}$] of higher-order causal processes for a precausal category $\mathcal{C}$. To motivate this construction, we begin by looking at the properties of the set of causal states for some system A:


(15)

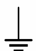
In the classical and quantum cases, these form convex subsets of real vector spaces (probability distributions in $\mathbb{R}^n$ and density matrices in the space of $n \times n$ self adjoint matrices, respectively). We would like to recapture the fact that this set is suitably closed without referring to convexity, so we appeal to duals instead.

Definition IV.1. For any set of states $c \subseteq \mathcal{C}(I, A)$, we can define the dual set $c^* \subseteq \mathcal{C}(I, A^*)$ as follows:

$$c^* \;:=\; \left\{ \pi : A^* \;\middle|\; \forall \rho \in c \;.\; \right.$$  $$\left. = \; 1 \right\}$$

Taking the dual again, we get back to a set of states in A. It immediately follows from the definition that $c \subseteq c^{**}$. For the set of all causal states (15), we see that furthermore $c = c^{**}$.

Definition IV.2. A set of states $c \subseteq \mathcal{C}(I, A)$ is *closed* if $c = c^{**}$ and *flat* if there exist invertible scalars λ, μ such that:

λ  $\in c$ $\qquad$ μ  $\in c^*$

Definition IV.3. For a precausal category $\mathcal{C}$, the category $\mathrm{Caus}[\mathcal{C}]$ has as objects pairs:

$$\boldsymbol{A} := (A, c_{\boldsymbol{A}} \subseteq \mathcal{C}(I, A))$$

where $c_{\boldsymbol{A}}$ is closed and flat. A morphism $f : \boldsymbol{A} \to \boldsymbol{B}$ is a morphism $f : A \to B$ in $\mathcal{C}$ such that:

$$\rho \in c_{\boldsymbol{A}} \;\Rightarrow\; f \circ \rho \in c_{\boldsymbol{B}} \tag{16}$$

While this condition on morphisms is in terms of states, closure allows us to also state it in terms of effects or numbers:

Proposition IV.4. For objects $\boldsymbol{A}, \boldsymbol{B}$ in $\mathrm{Caus}[\mathcal{C}]$ and a morphism $f : A \to B$ in $\mathcal{C}$, the following are equivalent:
(i) $\rho \in c_{\boldsymbol{A}} \Rightarrow f \circ \rho \in c_{\boldsymbol{B}}$
(ii) $\pi \in c^*_{\boldsymbol{B}} \Rightarrow \pi \circ f \in c^*_{\boldsymbol{A}}$
(iii) $\rho \in c_{\boldsymbol{A}}, \pi \in c^*_{\boldsymbol{B}} \Rightarrow \pi \circ f \circ \rho = 1$

Proof. (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) follow immediately from the definition of $(-)^*$, so assume (iii) and take any $\rho \in c_{\boldsymbol{A}}$. Then for all $\pi \in c^*_{\boldsymbol{B}}$, $\pi \circ (f \circ \rho) = 1$. Hence $f \circ \rho \in c^{**}_{\boldsymbol{B}} = c_{\boldsymbol{B}}$. $\square$

Since a set of states is closed when $c = c^{**}$, it is natural to ask if $(-)^{**}$ forms a closure operation, namely if it is idempotent. This is an immediate result of the following:

Lemma IV.5. For any set of states c we have $c^* = c^{***}$.

Proof. First, note that: $c \subseteq d \;\Rightarrow\; d^* \subseteq c^*$. Applying this to $c \subseteq c^{**}$ yields $c^{***} \subseteq c^*$. But then, it is already the case that c^* is contained in c^{***}, so $c^{***} = c^*$. $\square$

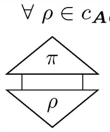
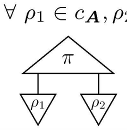
We will now show that $\mathrm{Caus}[\mathcal{C}]$ has the structure of a $*$-autonomous category. To do this, we will first define the tensor $\boldsymbol{A} \otimes \boldsymbol{B}$. For the sets of states $c_{\boldsymbol{A}}$ and $c_{\boldsymbol{B}}$, we denote the set of all product states as follows:

$$c_{\boldsymbol{A}} \otimes c_{\boldsymbol{B}} \;:=\; \{\rho_1 \otimes \rho_2 \mid \rho_1 \in c_{\boldsymbol{A}},\; \rho_2 \in c_{\boldsymbol{B}}\}$$

Then, $c_{\boldsymbol{A} \otimes \boldsymbol{B}}$ is the closure of the set of all product states:

$$c_{\boldsymbol{A} \otimes \boldsymbol{B}} := (c_{\boldsymbol{A}} \otimes c_{\boldsymbol{B}})^{**}$$

Lemma IV.6. For any effect $\pi : A^* \otimes B^*$ in $\mathcal{C}$:

$$\left(\forall \rho \in c_{\boldsymbol{A} \otimes \boldsymbol{B}} \;.\; \right.$$  $$\left. = \; 1 \right) \;\Leftrightarrow\; \left(\forall \rho_1 \in c_{\boldsymbol{A}}, \rho_2 \in c_{\boldsymbol{B}} \;.\; \right.$$  $$\left. = \; 1 \right) \tag{17}$$

Proof. The LHS of (17) states that

$$\pi \in c^*_{\boldsymbol{A} \otimes \boldsymbol{B}} := ((c_{\boldsymbol{A}} \otimes c_{\boldsymbol{B}})^{**})^* = (c_{\boldsymbol{A}} \otimes c_{\boldsymbol{B}})^{***}$$

whereas the RHS states that $\pi \in (c_{\boldsymbol{A}} \otimes c_{\boldsymbol{B}})^*$. Hence, (17) follows from Lemma IV.5. $\square$

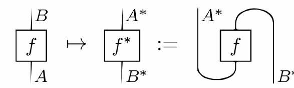
Theorem IV.7. $\mathrm{Caus}[\mathcal{C}]$ is an SMC, with tensor given by:

$$\boldsymbol{A} \otimes \boldsymbol{B} := (A \otimes B, c_{\boldsymbol{A} \otimes \boldsymbol{B}})$$

and tensor unit $\boldsymbol{I} := (I, \{1\})$.

Proof. The proof is [17]. $\square$

Now define objects $\boldsymbol{A}^* := (A^*, c_{\boldsymbol{A}^*})$ in the obvious way, by letting $c_{\boldsymbol{A}^*} := c^*_{\boldsymbol{A}}$. Since $\mathcal{C}$ is compact closed, we can define the *transposition functor* $(-)^* : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ via:

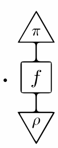
$$A \mapsto A^*$$  $$\tag{18}$$

Lemma IV.8. The transposition functor (18) lifts to a full and faithful functor $(-)^* : \mathrm{Caus}[\mathcal{C}]^{\mathrm{op}} \to \mathrm{Caus}[\mathcal{C}]$.

Proof. The main part is showing that f^* is again a morphism, but this follows from the definition of the star on sets of states. A full proof is in [17]. $\square$

We now have enough structure to define $\boldsymbol{A} \multimap \boldsymbol{B} := (\boldsymbol{A} \otimes \boldsymbol{B}^*)^*$. However, it is enlightening to give an explicit characterisation of the set $c_{\boldsymbol{A} \multimap \boldsymbol{B}}$. This will be no surprise:

Lemma IV.9. For objects $\boldsymbol{A}, \boldsymbol{B} \in \mathrm{Caus}[\mathcal{C}]$:

$$c_{\boldsymbol{A} \multimap \boldsymbol{B}} = \left\{ f : A^* \otimes B \;\middle|\; \forall \rho \in c_{\boldsymbol{A}}, \pi \in c^*_{\boldsymbol{B}} \;.\; \right.$$  $$\left. = \; 1 \right\}$$

Proof. This follows by simplifying:

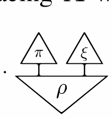
$$c_{(\boldsymbol{A} \otimes \boldsymbol{B}^*)^*} = c^*_{(\boldsymbol{A} \otimes \boldsymbol{B}^*)} = (c_{\boldsymbol{A}} \otimes c_{\boldsymbol{B}^*})^{***} = (c_{\boldsymbol{A}} \otimes c^*_{\boldsymbol{B}})^*$$

and noting that $f \in (c_{\boldsymbol{A}} \otimes c^*_{\boldsymbol{B}})^*$ is precisely the statement given in the lemma. $\square$

Theorem IV.10. For any precausal category $\mathcal{C}$, $\mathrm{Caus}[\mathcal{C}]$ is a $*$-autonomous category where $\boldsymbol{I} = \boldsymbol{I}^*$.

Proof. Since compact closed categories already admit an interpretation for $\multimap$ satisfying (4), it suffices to show that this isomorphism lifts to $\mathrm{Caus}[\mathcal{C}]$. This follows from the application of Lemma IV.6. The complete proof is [17]. $\square$

Since $\mathrm{Caus}[\mathcal{C}]$ is $*$-autonomous, we can also define the 'par' of two systems $\boldsymbol{A} \parr \boldsymbol{B}$. By 3 and 7, $\parr$ is related to $\multimap$ via $\boldsymbol{A} \parr \boldsymbol{B} \cong (\boldsymbol{A}^* \otimes \boldsymbol{B}^*)^* \cong \boldsymbol{A}^* \multimap \boldsymbol{B}$. Hence Lemma IV.9 also yields an explicit form for $c_{\boldsymbol{A} \parr \boldsymbol{B}}$ by replacing $\boldsymbol{A}$ with $\boldsymbol{A}^*$:

$$c_{\boldsymbol{A} \parr \boldsymbol{B}} = \left\{ \rho : A \otimes B \;\middle|\; \forall \pi \in c^*_{\boldsymbol{A}}, \xi \in c^*_{\boldsymbol{B}} \;.\; \right.$$  $$\left. = 1 \right\}$$

Note that the process $f : A^* \otimes B$ has become a bipartite state $\rho : A \otimes B$. That is, the states $\rho \in c_{\boldsymbol{A} \parr \boldsymbol{B}}$ are states which are normalised for all product effects. Symbolically, the two monoidal products are defined as follows:

$$c_{\boldsymbol{A} \otimes \boldsymbol{B}} = (c_{\boldsymbol{A}} \otimes c_{\boldsymbol{B}})^{**} \qquad c_{\boldsymbol{A} \parr \boldsymbol{B}} = (c^*_{\boldsymbol{A}} \otimes c^*_{\boldsymbol{B}})^*$$

One can easily check that $(c^*_{\boldsymbol{A}} \otimes c^*_{\boldsymbol{B}}) \subseteq (c_{\boldsymbol{A}} \otimes c_{\boldsymbol{B}})^*$. Thus, since $(-)^*$ reverses subset inclusions, that $c_{\boldsymbol{A} \otimes \boldsymbol{B}} \subseteq c_{\boldsymbol{A} \parr \boldsymbol{B}}$.

Consequently, the identity $1_{A\otimes B}$ in $\mathcal{C}$ lifts to a canonical embedding $\boldsymbol{A} \otimes \boldsymbol{B} \to \boldsymbol{A} \parr \boldsymbol{B}$ in Caus$[\mathcal{C}]$. This agrees with the intuition given in Section III that $\boldsymbol{A} \parr \boldsymbol{B}$ is the 'larger' of the two ways to combine A and B into a joint system.

Remark IV.11. A $*$-autonomous category with coherent isomorphism $I \cong I^*$, such as Caus$[\mathcal{C}]$, is also called an ISOMIX category [9]. This innocent-looking extra condition actually gives a great deal more structure. For instance, even though we showed it concretely, the existence of a canonical morphism $A \otimes B \to A \parr B$ is implied purely from this extra structure.

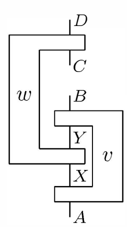
Rather than thinking of Caus$[\mathcal{C}]$ as a totally new category constructed from $\mathcal{C}$, it is useful to think of it as endowing the processes in $\mathcal{C}$ with a much richer type system. As in the compact closed case, it suffices to consider only processes out of $\boldsymbol{I}$ and we use $\rho : \boldsymbol{X}$ as shorthand for $\rho : \boldsymbol{I} \to \boldsymbol{X}$. However, unlike before, we will often use a statement of the form $\rho : \boldsymbol{X}$ as a *proposition* about a state $\rho \in \mathcal{C}(I, X)$.

Proposition IV.12. For a system $\boldsymbol{X} = (X, c_{\boldsymbol{X}})$ in Caus$[\mathcal{C}]$ and a state $\rho \in \mathcal{C}(I, X)$, $\rho : \boldsymbol{X}$ if and only if $\rho \in c_{\boldsymbol{X}}$.

Proof. Since 1 is the unique state in $c_{\boldsymbol{I}}$, the result follows immediately from the definition of morphism in Caus$[\mathcal{C}]$:

$$\rho : \boldsymbol{X} \;\Leftrightarrow\; \rho \circ 1 \in c_{\boldsymbol{X}} \;\Leftrightarrow\; \rho \in c_{\boldsymbol{X}} \qquad \square$$

From now on, we will use $\rho : \boldsymbol{X}$ and $\rho \in c_{\boldsymbol{X}}$ interchangeably without further comment. We will also mix the graphical notation with the type-theoretic. So, for instance, if we write:



$$: \; \boldsymbol{A} \multimap (\boldsymbol{B} \multimap \boldsymbol{C}) \multimap \boldsymbol{D}$$

this should be interpreted as a morphism in $\mathcal{C}(I, A^* \otimes B \otimes C^* \otimes D)$, along with the assertion that this morphism has type $\boldsymbol{A} \multimap (\boldsymbol{B} \multimap \boldsymbol{C}) \multimap \boldsymbol{D}$. In particular, diagrams always depict $\mathcal{C}$-morphisms, as opposed to Caus$[\mathcal{C}]$-morphisms, so there is no ambiguity about whether parallel composition means $\otimes$ or $\parr$.

Furthermore, if we state that two types are isomorphic without giving the isomorphism explicitly, it should be understood that the underlying isomorphism in $\mathcal{C}$ is just the identity, up to a possible permutation of systems. In particular, $\boldsymbol{X} \cong \boldsymbol{Y}$ implies that $\rho : \boldsymbol{X}$ if and only if $\rho : \boldsymbol{Y}$.

V. First order systems

For any precausal category $\mathcal{C}$, we can always form the SMC of (first-order) causal processes $\mathcal{C}_c$ by restricting just to those processes satisfying the causality equation (9). Since Caus$[\mathcal{C}]$ is supposed to contain first and higher-order causal processes, one would naturally expect $\mathcal{C}_c$ to embed in Caus$[\mathcal{C}]$.

Definition V.1. A system $\boldsymbol{A} = (A, c_{\boldsymbol{A}})$ in Caus$[\mathcal{C}]$ is called *first order* if it is of the form $(A, \{\bar{\top}_A\}^*)$.

Note that $\{\bar{\top}_A\}^*$ is precisely the set (15) of causal states of type A. Clearly this set is flat and closed. Indeed, this was the motivation for these conditions in the first place. Now, we show that the processes between first-order systems in Caus$[\mathcal{C}]$ are exactly as expected.

Proposition V.2. Let $\boldsymbol{A}, \boldsymbol{B}$ be first-order systems. Then f is a morphism from $\boldsymbol{A}$ to $\boldsymbol{B}$ if and only if it is causal.

Proof. We first compute $c^*_{\boldsymbol{A}}$ for a first-order system. Suppose $\pi \in c^*_{\boldsymbol{A}}$. Then for all causal states ρ, $\pi \circ \rho = 1 = \bar{\top}_A \circ \rho$, so by (C3) $\pi = \bar{\top}_A$. Hence $c^*_{\boldsymbol{A}} = \{\bar{\top}_A\}$.

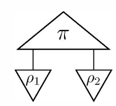
Now, by Proposition IV.4, $\Phi \in \mathcal{C}(A, B)$ is a morphism from $\boldsymbol{A}$ to $\boldsymbol{B}$ if and only if for every $\pi \in c^*_{\boldsymbol{B}}$, $\pi \circ \Phi \in c^*_{\boldsymbol{A}}$. Since both of these sets of effects only contain discarding, this reduces to the causality equation (9). $\square$

Furthermore, first-order systems are closed under $\otimes$.

Proposition V.3. For first order systems $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{A} \otimes \boldsymbol{B}$ is also a first-order system, given by:

$$\boldsymbol{A} \otimes \boldsymbol{B} = (A \otimes B, \{\bar{\top}_A \bar{\top}_B\}^*)$$

Proof. It suffices to show that $c^*_{\boldsymbol{A}\otimes\boldsymbol{B}} = \{\bar{\top}_A \bar{\top}_B\}$. Let $\pi \in c^*_{\boldsymbol{A}\otimes\boldsymbol{B}}$, then for all causal states $\rho_1 \in c_{\boldsymbol{A}}, \rho_2 \in c_{\boldsymbol{B}}$:



$$= \; 1$$

Hence, by Proposition III.6, $\pi = \bar{\top}_A \bar{\top}_B$. $\square$

Corollary V.4. There exists a full, faithful, monoidal embedding of the category $\mathcal{C}_c$ of causal processes into Caus$[\mathcal{C}]$ via:

$$A \mapsto (A, \{\bar{\top}_A\}^*) \qquad\qquad f \mapsto f$$

Hence, the full sub-category of first-order systems and processes behaves as expected; it is equivalent to $\mathcal{C}_c$. Perhaps a more surprising corollary to Proposition V.3 is the following.

Corollary V.5. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be first order systems, then:

$$\boldsymbol{A} \otimes \boldsymbol{B} \cong \boldsymbol{A} \parr \boldsymbol{B}$$

Proof. $c_{\boldsymbol{A}\parr\boldsymbol{B}} := (c^*_{\boldsymbol{A}} \otimes c^*_{\boldsymbol{B}})^* = \{\bar{\top}_A \bar{\top}_B\}^* = c_{\boldsymbol{A}\otimes\boldsymbol{B}}$. $\square$

So for first-order systems, there is really only one way to form the 'joint system'. However, we will now see that for higher-order systems, this is very much not the case.

VI. Higher-order systems

While it is important that the category of causal processes embeds fully and faithfully in Caus$[\mathcal{C}]$ when one restricts to first-order systems, the chief interest of Caus$[\mathcal{C}]$ are its higher-order systems. The goal of this section is to show that certain collections of maps fit nicely within the developed type theory.

The first non-trivial second-order system that it is natural to consider is $\boldsymbol{A} \multimap \boldsymbol{B}$ for first-order systems $\boldsymbol{A}$, $\boldsymbol{B}$. The isomorphism (4) for $*$-autonomous categories restricts to:

$$\text{Caus}[\mathcal{C}](\boldsymbol{A}, \boldsymbol{B}) \cong \text{Caus}[\mathcal{C}](\boldsymbol{I}, \boldsymbol{A} \multimap \boldsymbol{B})$$

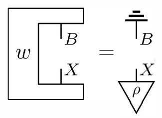
so 'states' $\Phi : \boldsymbol{A} \multimap \boldsymbol{B}$ are in bijective correspondence with morphisms from $\boldsymbol{A}$ to $\boldsymbol{B}$ in Caus$[\mathcal{C}]$. That is, they are precisely the causal processes from $\boldsymbol{A}$ to $\boldsymbol{B}$.

Now, starting from first-order systems $\boldsymbol{A}, \boldsymbol{A'}, \boldsymbol{B}, \boldsymbol{B'}$, we have two ways to form the 'joint system' from $\boldsymbol{A} \multimap \boldsymbol{A'}$ and

$\boldsymbol{B} \multimap \boldsymbol{B}'$, via $\otimes$ and $\parr$. Before we characterise these systems, we examine the dual of a second-order system.

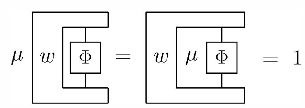
If we take a process $w : (\boldsymbol{A} \multimap \boldsymbol{B})^*$ in the dual system, we know from (C4) that that w must split into two pieces. In fact, using flatness, we can strengthen this condition by only requiring $\boldsymbol{B}$ to be first-order.

Lemma VI.1. For any system $\boldsymbol{X}$, first-order system $\boldsymbol{B}$, and process $w : (\boldsymbol{X} \multimap \boldsymbol{B})^*$, there exists $\rho : \boldsymbol{X}$ such that:

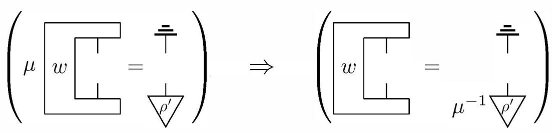


(19)

Proof. Since $\boldsymbol{B}$ is first-order, $c^*_{\boldsymbol{B}} = \{\bar{\top}_B\}$ and by flatness, for some μ, $\mu\bar{\top}_X \in c^*_{\boldsymbol{X}}$. Hence, for any (first-order) causal process $\Phi : X \to B$, we have $\bar{\top}_B \circ \mu\Phi = \mu\ \bar{\top}_X$. Hence $\mu\Phi$ is of type $\boldsymbol{X} \to \boldsymbol{B}$. It follows by definition of $(\boldsymbol{X} \multimap \boldsymbol{B})^*$ that:

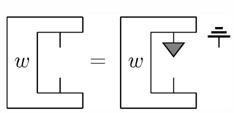


That is, μw sends every causal process Φ to 1, so by (C5′):



It is then straightforward to show that $\rho := \mu^{-1}\rho'$ is a state of $\boldsymbol{X}$, e.g. by plugging the uniform state for system $\boldsymbol{B}$ into both sides of the equation above. $\square$

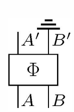
Equivalently, by Lemma III.8, $w : (\boldsymbol{X} \multimap \boldsymbol{B})^*$ implies:



(20)

Theorem VI.2. For first-order systems $\boldsymbol{A}, \boldsymbol{A}', \boldsymbol{B}, \boldsymbol{B}'$, a process Φ is of type $(\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{B}')$ if and only if it is causal and non-signalling.

Proof. First assume that $\Phi : (\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{B}')$. Then, since discarding B' is causal, we can regard it as a morphism $\bar{\top}_{B'} : \boldsymbol{B}' \to \boldsymbol{I}$. Hence by functoriality of $\otimes$ and $\multimap$, we have:

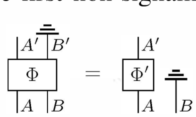


$: \ (\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{I})$

Then we can transform to an equivalent type as follows:

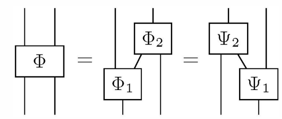
$$\begin{aligned}(\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{I}) &\cong (\boldsymbol{A} \multimap \boldsymbol{A}') \otimes \boldsymbol{B}^* \\ &\cong ((\boldsymbol{A} \multimap \boldsymbol{A}')^* \parr \boldsymbol{B})^* \\ &\cong ((\boldsymbol{A} \multimap \boldsymbol{A}') \multimap \boldsymbol{B})^*\end{aligned}$$

Hence, by equation 20, Φ splits as $\bar{\top}_B : \boldsymbol{B}$ and $\Phi' : \boldsymbol{A} \multimap \boldsymbol{A}'$. This gives exactly the first non-signalling equation:



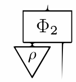
The other is shown similarly. Causality follows from discarding A' (or B') as well and Lemma IV.9.

Conversely, suppose that Φ is causal and non-signalling. Then it satisfies the two non-signalling equations in Definition III.4. Hence by (C4′), it can be factored in two ways:



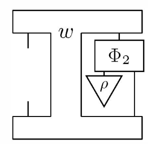
for causal processes Φ_i, Ψ_i.

Now, take any effect $w : ((\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{B}'))^* \cong (\boldsymbol{B} \multimap \boldsymbol{B}') \multimap (\boldsymbol{A} \multimap \boldsymbol{A}')^*$. For any causal state ρ,



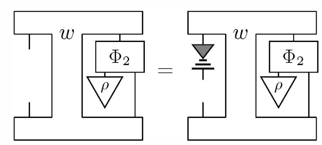
$: \ \boldsymbol{B} \multimap \boldsymbol{B}'$

Plugging this into one side of w gives:

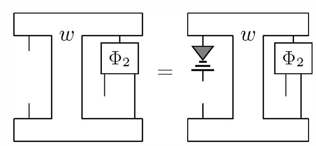


$: \ (\boldsymbol{A} \multimap \boldsymbol{A}')^*$

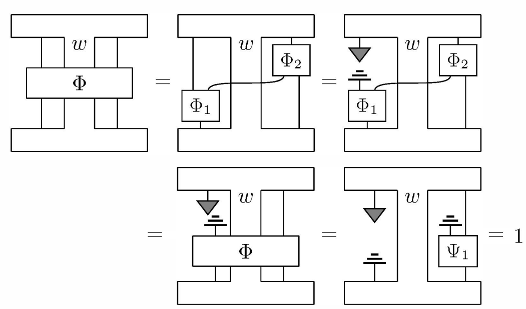
Applying equation (20) gives:



Hence by enough causal states we have

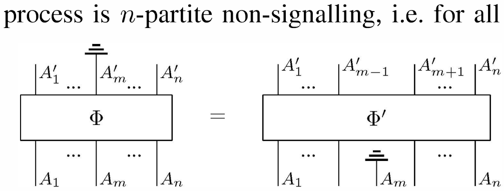


It then follows that



Therefore $\Phi : ((\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{B}'))^{**} = (\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{B}')$. $\square$

The proof above also generalises straightforwardly to show that a process is n-partite non-signalling, i.e. for all i:



if and only if:

$$\Phi : (\boldsymbol{A}_1 \multimap \boldsymbol{A}'_1) \otimes (\boldsymbol{A}_2 \multimap \boldsymbol{A}'_2) \otimes \ldots \otimes (\boldsymbol{A}_n \multimap \boldsymbol{A}'_n)$$

Theorem VI.3. For first-order systems $\boldsymbol{A}, \boldsymbol{A}', \boldsymbol{B}, \boldsymbol{B}'$, a process Φ is of type $(\boldsymbol{A} \multimap \boldsymbol{A}') \parr (\boldsymbol{B} \multimap \boldsymbol{B}')$ if and only if it is causal. That is:

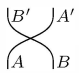
$$(\boldsymbol{A} \multimap \boldsymbol{A'}) \parr (\boldsymbol{B} \multimap \boldsymbol{B'}) \cong \boldsymbol{A} \otimes \boldsymbol{B} \multimap \boldsymbol{A'} \otimes \boldsymbol{B'}$$

Proof. We rely on the relationship between $\multimap$ and $\parr$:

$$\begin{aligned}(\boldsymbol{A} \multimap \boldsymbol{A'}) \parr (\boldsymbol{B} \multimap \boldsymbol{B'}) &\cong \boldsymbol{A}^* \parr \boldsymbol{A'} \parr \boldsymbol{B}^* \parr \boldsymbol{B'} \\ &\cong \boldsymbol{A}^* \parr \boldsymbol{B}^* \parr \boldsymbol{A'} \parr \boldsymbol{B'} \\ &\cong (\boldsymbol{A}^* \parr \boldsymbol{B}^*)^* \multimap \boldsymbol{A'} \parr \boldsymbol{B'} \\ &\cong \boldsymbol{A} \otimes \boldsymbol{B} \multimap \boldsymbol{A'} \parr \boldsymbol{B'}\end{aligned}$$

Then, since $\boldsymbol{A'}$ and $\boldsymbol{B'}$ are first-order, $\boldsymbol{A'} \parr \boldsymbol{B'} \cong \boldsymbol{A'} \otimes \boldsymbol{B'}$, which completes the proof. □

So $(\boldsymbol{A} \multimap \boldsymbol{A'}) \parr (\boldsymbol{B} \multimap \boldsymbol{B'})$ forms the joint system consisting of *all* causal processes from $A \otimes B$ to $A' \otimes B'$, including the signalling ones, like swap (where $A \cong A'$ cf. Remark II.2):

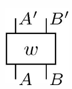


$: \ (\boldsymbol{A} \multimap \boldsymbol{B'}) \parr (\boldsymbol{B} \multimap \boldsymbol{A'})$

Hence, $\otimes$ and $\parr$ represent two extremes by which $\boldsymbol{A} \multimap \boldsymbol{A'}$ and $\boldsymbol{B} \multimap \boldsymbol{B'}$ can be combined, namely by requiring them to be non-signalling or imposing no non-signalling conditions. In the next section, we will see how to recover types for one-way signalling.

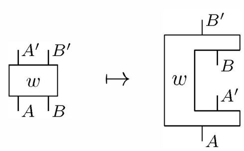
A. One-way signalling and combs

Theorem VI.4. For first order systems $\boldsymbol{A}, \boldsymbol{A'}, \boldsymbol{B}, \boldsymbol{B'}$, a process w is one-way signalling ($A \preceq B$) if and only if:

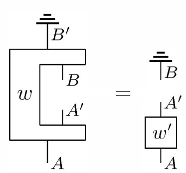


$: \ \boldsymbol{A} \multimap (\boldsymbol{A'} \multimap \boldsymbol{B}) \multimap \boldsymbol{B'}$

Proof. Suppose Φ is $A \preceq B$. First, we deform Φ to put the two A-labelled systems below the two B-labelled systems:



The one-way signalling equation (10) then becomes:

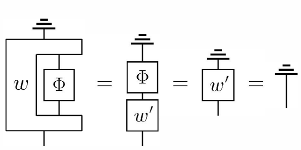


(21)

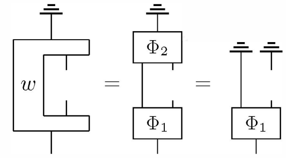
Now, by (5) we have:

$$\boldsymbol{A} \multimap (\boldsymbol{A'} \multimap \boldsymbol{B}) \multimap \boldsymbol{B'} \cong (\boldsymbol{A'} \multimap \boldsymbol{B}) \multimap \boldsymbol{A} \multimap \boldsymbol{B'}$$

So, for w to be of the above type, it must send all causal processes $\boldsymbol{A'} \multimap \boldsymbol{B}$ to causal processes $\boldsymbol{A} \multimap \boldsymbol{B'}$. This immediately follows from (21):



Conversely, if w sends causal processes to causal processes, it factorises as in (C4), which implies (21):



Hence w is one-way signalling. □

Hence, one-way signalling admits a more general class of processes than non-signalling.

Remark VI.5. One can show the embedding of non-signalling processes into one-way signalling processes purely at the level of types by relying on the *linear distributivity* property of $*$-autonomous categories [10]. Namely, in any $*$-autonomous category, there exists a canonical mapping:

$$(\boldsymbol{A} \parr \boldsymbol{B}) \otimes \boldsymbol{C} \to \boldsymbol{A} \parr (\boldsymbol{B} \otimes \boldsymbol{C})$$

We can use this to construct an embedding of two-way non-signalling processes into one-way:

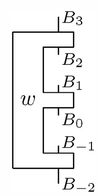
$$\begin{aligned}(\boldsymbol{A} \multimap \boldsymbol{A'}) \otimes (\boldsymbol{B} \multimap \boldsymbol{B'}) &\cong (\boldsymbol{A}^* \parr \boldsymbol{A'}) \otimes (\boldsymbol{B}^* \parr \boldsymbol{B'}) \\ &\to \boldsymbol{A}^* \parr (\boldsymbol{A'} \otimes (\boldsymbol{B}^* \parr \boldsymbol{B'})) \\ &\to \boldsymbol{A}^* \parr (\boldsymbol{A'} \otimes \boldsymbol{B}^*) \parr \boldsymbol{B'} \\ &\cong \boldsymbol{A} \multimap (\boldsymbol{A'} \multimap \boldsymbol{B}) \multimap \boldsymbol{B'}\end{aligned}$$

Following [23], we generalise from 2-party, one-way signalling processes to n-party processes by recursively defining the type of n*-combs*.

Definition VI.6. The n*-combs* C_n are defined by

- $C_0 = \boldsymbol{I}$,
- $C_{i+1} = \boldsymbol{B}_{-i} \multimap C_i \multimap \boldsymbol{B}_{i+1}$, for first order $\boldsymbol{B}_j$.

A 1-comb has type $\boldsymbol{B}_0 \multimap \boldsymbol{I} \multimap \boldsymbol{B}_1 \cong \boldsymbol{B}_0 \multimap \boldsymbol{B}_1$, so it is just a causal process. For higher combs, the '$-i$' is employed to maintain the left-to-right order of indices. For example, a 3-comb has type:

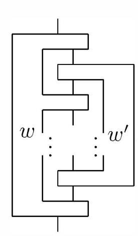


$: \boldsymbol{B}_{-2} \multimap (\boldsymbol{B}_{-1} \multimap (\boldsymbol{B}_0 \multimap \boldsymbol{B}_1) \multimap \boldsymbol{B}_2) \multimap \boldsymbol{B}_3$

When necessary, we rename $\boldsymbol{A}_i := \boldsymbol{B}_{2i-n-1}$ and $\boldsymbol{A}'_i := \boldsymbol{B}_{2i-n}$ to obtain e.g.

$$\boldsymbol{A}_1 \multimap (\boldsymbol{A}'_1 \multimap (\boldsymbol{A}_2 \multimap \boldsymbol{A}'_2) \multimap \boldsymbol{A}_3) \multimap \boldsymbol{A}'_3$$

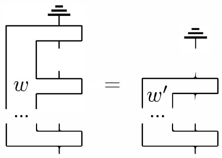
We can interpret the recursive definition as follows. Think of an n-comb as an n-step communication protocol which takes an input and an $(n-1)$-step protocol and where the output of the one protocol serves as an input of the other. Hence, we can represent the overall process of two agents running a communication protocol by plugging together Alice's n-comb and Bob's $(n-1)$-comb:



$: \quad \boldsymbol{A}_1 \multimap \boldsymbol{A}'_n$

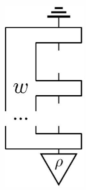
We give an alternative characterisation for combs in Caus[$\mathcal{C}$], which will relate to one-way signalling processes.

Lemma VI.7. For any n-comb $w : C_n$, discarding the output $\boldsymbol{A}'_n$ separates as follows, for some w':



(22)

Proof. Plugging any causal state into the first input of w and discarding the last output yields:

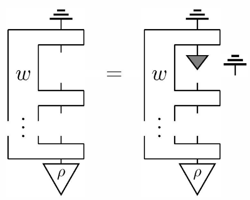


$: \ C_{n-1} \multimap \boldsymbol{I}$

Then:

$$C_{n-1} \multimap \boldsymbol{I} \cong C_{n-1}^* \cong (\boldsymbol{B}_{-(n-2)} \multimap C_{n-2} \multimap \boldsymbol{B}_{n-1})^*$$
$$\cong (\boldsymbol{B}_{-(n-2)} \otimes C_{n-2} \multimap \boldsymbol{B}_{n-1})^*$$

Hence by Lemma VI.1, in particular equation (20), we obtain:

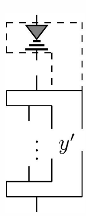


The result then follows from enough causal states. ☐

Note that we haven't actually said that w' is itself an $(n-1)$-comb. We will show this now.

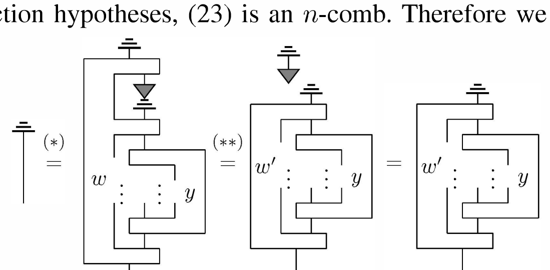
Theorem VI.8. w is an n-comb, i.e. $w : C_n$, if and only if it separates as in equation (22) for some $w' : C_{n-1}$.

Proof. By induction. For $n = 1$ the theorem is true because a 0-comb is always $\boldsymbol{I}$. Suppose the theorem is true for n. Let w be an $(n+1)$-comb. We need to show that w' is an n-comb. So let y be any $(n-1)$-comb. Then, if we form the process:



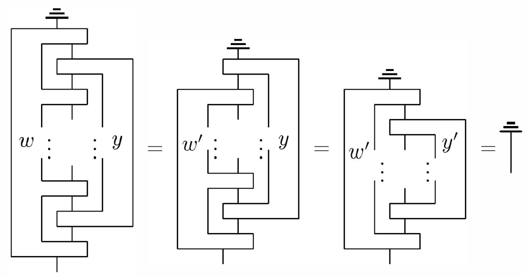
(23)

then clearly discarding the top output results in an $(n-1)$ comb (namely y) and a discard on the top input. So by the induction hypotheses, (23) is an n-comb. Therefore we have



where $(*)$ follows from the definition of $(n+1)$-comb and $(**)$ is Lemma VI.7. Hence w' sends any $(n-1)$-comb to a causal map, so w' is itself an n-comb.

Conversely, let w' in equation (22) be an n-comb, and take any n-comb y. Then by the induction hypothesis, discarding the top output of y separates as discarding and an $(n-1)$-comb y'. Hence:

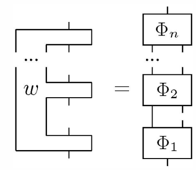


so w is an $(n+1)$-comb. ☐

Hence, n-combs can be characterised inductively in exactly the same way as n-party one-way signalling processes. Since 1-combs are just causal processes, the following is immediate.

Corollary VI.9. For first order systems $\boldsymbol{A}_1, \boldsymbol{A}'_1, \ldots, \boldsymbol{A}_n, \boldsymbol{A}'_n$, a map $w : A_1 \otimes \ldots \otimes A_n \to A'_1 \otimes \ldots \otimes A'_n$ is one-way signalling $(A_1 \preceq \ldots \preceq A_n)$ if and only if it is of type $\boldsymbol{A}_1 \multimap (\boldsymbol{A}'_1 \multimap (\ldots) \multimap \boldsymbol{A}_n) \multimap \boldsymbol{A}'_n$. That is, it is an n-comb.

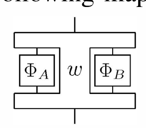
Proposition III.11, then generalises the characterisation theorem for quantum combs in [6] to Caus[$\mathcal{C}$] for any precausal $\mathcal{C}$: an n-comb always factors as a sequence of 'memory channels', i.e. a composition of causal processes of the form:



B. SOC$_2$ and SOC$_n$ processes

In this section we shall take a look at *process matrices*, introduced in [22], to investigate processes which do not have a definite causal order. Such processes were called bipartite second-order causal in [18].

Definition VI.10. A process $w : (A^* \otimes A') \otimes (B^* \otimes B') \to C^* \otimes C$ is called *bipartite second-order causal* (SOC$_2$) if for all causal Φ_A, Φ_B the following map is causal:



So SOC$_2$ maps send products of causal processes to a causal process. The following shows that SOC$_2$ processes are actually normalized on *all* non-signalling maps, not just product maps.

Theorem VI.11. For first order systems $\boldsymbol{A}, \boldsymbol{A}', \boldsymbol{B}, \boldsymbol{B}', \boldsymbol{C}, \boldsymbol{C}'$, a process w is SOC$_2$ if and only if it is of type $(\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{B}') \multimap (\boldsymbol{C} \multimap \boldsymbol{C}')$.

Proof. Since products of causal processes are non-signalling, they are in $(\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{B}')$, so any process of the above type is indeed SOC$_2$.

For the converse, let π be an effect of type $(\boldsymbol{C} \multimap \boldsymbol{C}')^*$. Then $\pi \circ w$ is an effect on products of causal processes. Now Lemma IV.6 states that $\pi \circ w$ yields 1 for product states if and only if it yields 1 for any state in the tensor product. Hence it is an effect for $(\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{B}')$, which by Theorem VI.2 are precisely the non-signalling maps. By Proposition IV.4 this means $w : (\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{B}') \multimap (\boldsymbol{C} \multimap \boldsymbol{C}')$ □

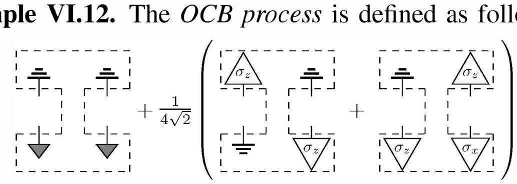
This represents a significant strengthening of the result in [18], which was only able to show that SOC_2 extends to all so-called *strongly non-signalling* processes, which are a special case of non-signalling processes.

Special cases of SOC_2 processes are 3-combs which arise from fixing a causal ordering between $\boldsymbol{A}$ and $\boldsymbol{B}$:

$$\boldsymbol{C} \multimap (\boldsymbol{A} \multimap (\boldsymbol{A}' \multimap \boldsymbol{B}) \multimap \boldsymbol{B}') \multimap \boldsymbol{C}'$$
$$\boldsymbol{C} \multimap (\boldsymbol{B} \multimap (\boldsymbol{B}' \multimap \boldsymbol{A}) \multimap \boldsymbol{A}') \multimap \boldsymbol{C}'$$

Indeed one can show the containment of either of these types into the type of SOC_2 processes using a simple calculation on types much like in Remark VI.5. However, the most interesting SOC_2 processes are those which do not arise from combs.

Example VI.12. The *OCB process* is defined as follows:

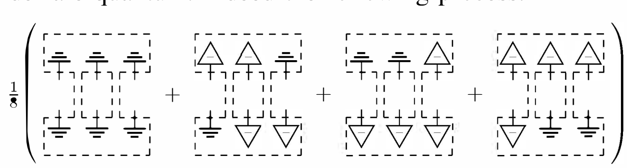


where σ_x, σ_z are Pauli matrices and associated effects and '+' is the sum of linear maps. Note that, while the individual summands are not positive, the result is, yielding a process in **CPM**. The fact that it is an SOC_2 process in Caus[**CPM**] follows straightforwardly from the fact that the Pauli matrices are trace-free. Furthermore, it was shown in [22] that it can be used to win certain non-local games with higher probability than any causally-ordered process, due to the fact that Bob can, to some extent, choose a causal ordering between himself and Alice *a posteriori* by his choice of quantum measurement.

Theorem VI.11 extends naturally to a characterisation of n-partite second-order causal processes (SOC_n) via:

$$(\boldsymbol{A}_1 \multimap \boldsymbol{A}_1') \otimes \ldots \otimes (\boldsymbol{A}_n \multimap \boldsymbol{A}_n') \multimap (\boldsymbol{C} \multimap \boldsymbol{C}')$$

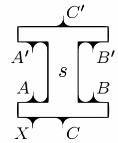
Example VI.13. Not all processes exhibiting indefinite causal order are quantum. Indeed the following process:



is an SOC_3 process in Caus[$\mathbf{Mat}(\mathbb{R}_+)$], where the '−' labelled states and effects with values $(1, -1)$, given as column vectors and row vectors, respectively. It was shown in [3] that this process, as well as a generalisation to an SOC_n process for odd n, was incompatible with any pre-defined causal order.

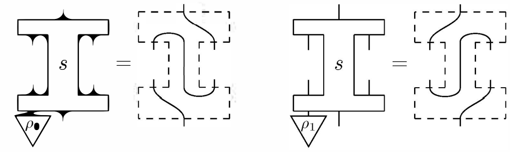
An interesting family of SOC_2 processes are *switches*, where an auxiliary system is used to control the causal ordering of the input processes.

Definition VI.14. For first-order systems $\boldsymbol{X}$ and $\boldsymbol{A} = \boldsymbol{A}' = \boldsymbol{B} = \boldsymbol{B}' = \boldsymbol{C} = \boldsymbol{C}'$, a *switch* is a process of type:



$$: \boldsymbol{X} \otimes \boldsymbol{C} \multimap (\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{B}') \multimap \boldsymbol{C}' \quad (24)$$

in Caus[$\mathcal{C}$], such that for distinct states $\rho_0, \rho_1 : \boldsymbol{X}$, we have:



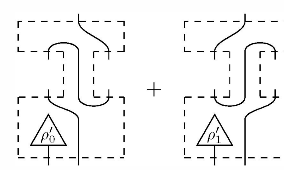
(25)

We now see some concrete examples of the switch, as a higher-order stochastic map and as a quantum channel.

Example VI.15. For $\mathcal{C} = \mathbf{Mat}(\mathbb{R}_+)$, the *classical switch* process is uniquely fixed by (25) if we let $\boldsymbol{X} = 2$ and:

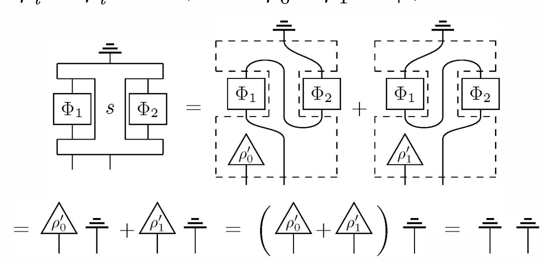
$$\rho_0 := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \rho_1 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Indeed, s is given by:



(26)

where $\rho_i' := \rho_i^T$. Then, since $\rho_0' + \rho_1' = \bar{\top}$, we have:



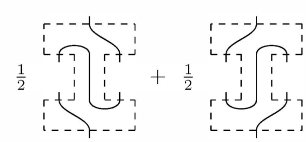
Hence s has the correct type shown in (24).

Example VI.16. For $\mathcal{C} = \mathbf{CPM}$, a switch process can be defined just as in (26), by letting $\boldsymbol{X} = \mathcal{B}(\mathbb{C}^2)$ and replacing ρ_i and ρ_i^T with the appropriate qubit projections and their associated quantum effects:

$$\rho_i := |i\rangle\langle i| \qquad \rho_i'(\mu) := \mathrm{Tr}(|i\rangle\langle i|\,\mu)$$

This is precisely the $\mathcal{Z}$ superoperator defined in [8], which defines a (decoherent) switch for quantum channels.

However, unlike Example VI.15, this channel is *not* uniquely fixed by (25), since ρ_0, ρ_1 do not form a basis for $\mathcal{B}(\mathbb{C}^2)$. For instance, plugging $\rho := |+\rangle\langle+|$ into $\boldsymbol{X}$ of this process yields a classical mixture of the two possible wirings:



One can also define a *coherent* quantum switch satisfying (25), where inputting the state $|+\rangle\langle+|$ into $\boldsymbol{X}$ yields a quantum superposition of causal orderings. See [8] for details.

Theorem VI.17. A switch cannot be causally ordered. That is, the type of s does not restrict to one of the following:

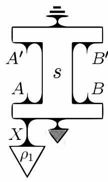
$$\boldsymbol{X} \otimes \boldsymbol{C} \multimap (\boldsymbol{A} \multimap (\boldsymbol{A}' \multimap \boldsymbol{B}) \multimap \boldsymbol{B}') \multimap \boldsymbol{C}'$$
$$\boldsymbol{X} \otimes \boldsymbol{C} \multimap (\boldsymbol{B} \multimap (\boldsymbol{B}' \multimap \boldsymbol{A}) \multimap \boldsymbol{A}') \multimap \boldsymbol{C}'$$

unless $\boldsymbol{A} \cong \boldsymbol{I}$.

Proof. Suppose s is causally ordered with $A \preceq B$. That is:

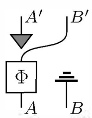
$$s : \boldsymbol{X} \otimes \boldsymbol{C} \multimap (\boldsymbol{A} \multimap (\boldsymbol{A}' \multimap \boldsymbol{B}) \multimap \boldsymbol{B}') \multimap \boldsymbol{C}'$$

Plugging states and effects into s yields a simpler type:



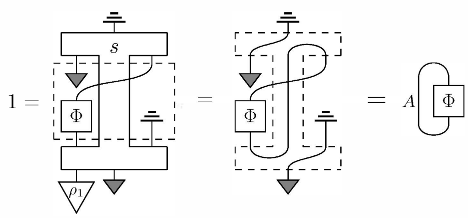
$$: \ (\boldsymbol{A} \multimap (\boldsymbol{A}' \multimap \boldsymbol{B}) \multimap \boldsymbol{B}')^* \qquad (27)$$

By Theorem VI.4 characterising one-way signalling processes, one can verify that for any $\Phi : \boldsymbol{A} \multimap \boldsymbol{B}'$, we have:



$$: \ \boldsymbol{A} \multimap (\boldsymbol{A}' \multimap \boldsymbol{B}) \multimap \boldsymbol{B}'$$

Since this is the dual type of (27), composing the two yields



which violates no time-travel, Theorem III.12. Hence $\boldsymbol{A} \cong \boldsymbol{I}$. The second causal ordering can be ruled out symmetrically, by plugging the state ρ_0 into s. □

VII. Conclusion and Future Work

In order to study higher order processes, we have created a categorical construction which sends certain compact closed categories $\mathcal{C}$ to a new category Caus$[\mathcal{C}]$. There is a fully faithful embedding of the category of first order causal processes of $\mathcal{C}$ into Caus$[\mathcal{C}]$, but we are also able to talk about genuine higher order causal processes. This new category also has a richer structure which allows us to develop a type theory for its objects. We classify certain kinds of processes in this type theory, such as non-signalling and one-way signalling processes, combs and bipartite second order causal processes and show that the type theoretic characterisation of these processes coincides with the operational one involving discarding.

The construction of Caus$[\mathcal{C}]$ can be generalised straightforwardly to encompass 'sub-causal' processes as well, by replacing the definition of $(-)^*$ with:

$$c^* = \{\pi : A^* \mid \forall x \in c, \pi \circ x \in \mathcal{M}\}$$

for a suitable sub-monoid $\mathcal{M}$ of $\mathcal{I} := \mathcal{C}(I, I)$ to get Caus$_{\mathcal{M}}[\mathcal{C}]$. Then, we recover Caus$[\mathcal{C}]$ as Caus$_{\{1\}}[\mathcal{C}]$. However, we obtain other interesting examples by varying $\mathcal{M}$. In the case of **CPM**, $\mathcal{I} = \mathbb{R}_+$. Taking $\mathcal{M}$ to be the unit interval, Caus$_{[0,1]}[\mathcal{C}]$ has trace non-increasing CP maps as its first-order processes, and generalisations thereof at higher orders. Alternatively, we can build a category of 'causal processes with failure' by letting $\mathcal{M}$ be $\{0, 1\}$. Exploring the properties of these categories, and how they relate to Caus$[\mathcal{C}]$ is a subject of future work. Another subject for future research is the relation between the types of causal systems and multiplicative linear logic (MLL). Since $*$-autonomous categories are a model of MLL, MLL provides a (decidable) fragment of the logic of type containment in Caus$[\mathcal{C}]$. This opens up possibilities to automate many proofs using existing automated linear logic provers. Indeed many of the type relationships in this paper were discovered with the help of such a tool, called `llprover` [25].

A third direction for future work is to generalise from linear causal orderings to causal orderings provided by an arbitrary directed acyclic graph and hence explore connections with other dag-based approaches for modelling causal structures for quantum (or more general) processes [20], [16].

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