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QUINE'S 'LIMITS OF DECISION'

WILLIAM C. PURDY

Abstract. In a 1969 paper, Quine coined the term 'limits of decision'. This term evidently refers to limits on the logical vocabulary of a logic, beyond which satisfiability is no longer decidable. In the same paper, Quine showed that not only monadic formulas, but homogeneous k-adic formulas for arbitrary k lie on the decidable side of the limits of decision. But the precise location of the limits of decision has remained an open question. The present paper answers that question. It addresses the question of decidability of those sublogics of first-order logic that are defined in terms of their logical vocabularies. A complete answer is obtained, thus locating exactly Quine's limits of decision.

§1. Introduction. From an algebraic point of view, a logic can be presented as a free algebra of a type T on a set X of generators (Barnes and Mack [2]). The *logical vocabulary* consists of the set of operations in T and the logical generators, if any, in X. This paper considers sublogics of classical first-order logic from this algebraic point of view. It establishes the relationship between the logical vocabulary of a sublogic and the decidability of that sublogic.

Often variables are employed in the presentation of a logic, but this entails a tedious analysis of equivalence of formulas. This is avoided here by using Quine's predicate functors in the definition of the type T. Other means such as combinatory logic could be used. Predicate functors are preferred because they are more perspicuous and they do not introduce problems of overgeneration. But while predicate functors play a useful role, the results presented here are independent of Predicate Functor Logic.

The traditional approach to the question of decidability in first-order logic classifies formulas in terms of their quantifier prefixes (Dreben and Goldfarb [3], Lewis [9]). In this paper, logics are classified in terms of their logical vocabularies. Thus one might start with Boolean logic, with the logical operations $\wedge$ and $\neg$, and enrich this vocabulary in various directions ultimately arriving at first-order logic with identity. The logics lying between Boolean logic and first-order logic with identity can be visualized as a lattice, ordered by the sublogic relation.

In 1969, in 'On the Limits of Decision' [17], Quine showed that the method used by Herbrand to prove that monadic quantificational logic is decidable can be extended to show that 'homogeneous k-adic formulas' are decidable for any k. He went on to observe that

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> What evidently gives general quantification theory its escape velocity is the chance to switch or fuse the variable attached to a predicate letter, ...

The homogeneous k-adic formulas lack this capability. Quine later called these formulas (homogeneous) 'fluted formulas' [19]. The 'limits of decision' evidently refers to the boundary in the lattice of logics separating the decidable logics from the undecidable logics.

Noah [11] presents these findings in the framework of Quine's Predicate Functor Logic. (See Section 4 for a definition of Predicate Functor Logic.) The homogeneous fluted formulas are easily seen to be those formulas of Predicate Functor Logic that have no occurrence of a combinatory functor (i.e., are fluted), and conjoin only subformulas of the same arity (i.e., are homogeneous). Noah shows why the restriction to homogeneous conjunction is essential, and why Quine's method does not extend to nonhomogeneous formulas. While homogeneous fluted logic had been proved to lie on the decidable side of the limits of decision, the relative location of unrestricted fluted logic was a subject of debate for a number of years (Noah [12]).

Recently, fluted logic as well as fluted logic with identity were shown to be decidable (Purdy [16, 15]). This is not an unexpected result since ordinary natural language discourse clearly embeds these logics. It would seem strange indeed if ordinary natural language discourse were undecidable. This result places the logic whose vocabulary contains the logical entities $\wedge, \neg, \exists$, and the identity on the decidable side of the limits of decision. The present paper continues this line of investigation by considering addition of the following logical entities.

1. pad, which adds a vacuous argument,
2. ref, which fuses two arguments,
3. inv, which switches two adjacent arguments,
4. Inv, which rotates arguments,
5. binary converse, the restriction of inv and Inv to formulas with two free variables,
6. $\circ$, which composes relations, and
7. functions.

These entities, along with the identity, generate a lattice of logics bounded below by fluted logic, and above by first-order logic with identity. This paper demonstrates that in this lattice, the limits of decision are located precisely between the ideal generated by fluted logic with identity, binary converse, and functions, and the filter generated by fluted logic with relational composition. This ideal and filter exhaust the lattice. Therefore, this result constitutes a complete answer to the question of the decidability of sublogics of first-order logic defined in terms of their logical vocabularies. Thus the location of Quine's limits of decision has been determined exactly.

§2. **Preliminaries.** This paper assumes the usual definition of first-order predicate logic (FO). Typically the set of predicate symbols will be those that occur in some given finite set of formulas or *premises*. The finite set of predicate symbols will be referred to as the *lexicon*. Let L be a lexicon and $R \in L$. Then $\mathrm{ar}(R)$ denotes the arity of R.

A standard result from FO is the following.

THEOREM 1. (The Principle of Monotonicity) *Let θ be a subformula, in the scope of an even number of $\neg$, that occurs as a conjunct in formula ϕ. Then ϕ' can be inferred from ϕ, where ϕ' is obtained from ϕ by deleting θ.*

PROOF. See Andrews [1], Theorem 2105, Substitutivity of Implication. Note that the empty conjunction is defined to be equivalent to $\top$ (verum). ⊣

An *L-structure* $\mathscr{A}$ consists of a set A, the *domain*, and a mapping that assigns to each $R \in L$ a subset $R^{\mathscr{A}} \subseteq A^{\mathrm{ar}(R)}$.

The notions of satisfaction and truth are the standard ones. If ϕ is a formula over L with free variables among $\{x_1, \ldots, x_k\}$, and ϕ is satisfied in $\mathscr{A}$ by the assignment to variables $\{x_i \mapsto a_i\}_{1 \leq i \leq k}$, we write $a_1 \cdots a_k \models \phi$. If ϕ is a sentence and ϕ is true in $\mathscr{A}$, we write $\mathscr{A} \models \phi$.

§3. Fluted formulas. Let L be a finite set of predicate symbols. Let $X_m := \{x_1, \ldots, x_m\}$ be an ordered set of m variables where $m \geq 0$. An *atomic fluted formula* of L over X_m is $Rx_{m-n+1} \cdots x_m$, where $R \in L$ and $\mathrm{ar}(R) = n \leq m$. The set of all atomic fluted formulas of L over X_m will be denoted $Af_L(X_m)$. Define $Af_L(X_0) := \{\top\}$.

A *fluted formula* of L over X_m is defined inductively.

1. An atomic fluted formula of L over X_m is a fluted formula of L over X_m.
2. If ϕ is a fluted formula of L over X_{m+1}, then $\exists x_{m+1}\phi$ and $\forall x_{m+1}\phi$ are fluted formulas of L over X_m.
3. If ϕ and ψ are fluted formulas of L over X_m, then $\phi \wedge \psi$, $\phi \vee \psi$, $\phi \rightarrow \psi$, and $\neg\phi$ are fluted formulas of L over X_m.

The fluted formulas just defined will be referred to as *standard* fluted formulas. In addition, any alphabetic variant of a standard fluted formula is defined to be a fluted formula. Two formulas are alphabetic variants of one another if they differ only in an inessential renaming of variables (see Enderton [6], pp. 118-120 for a precise definition). No other formula is a fluted formula.

The fluted formulas of L form a proper subset of the formulas of pure predicate logic with predicate symbols L. The semantics of the fluted formulas of L coincides with the usual semantics of pure predicate logic. In connection with standard fluted formulas, $abc \cdots \models \phi$ will always mean that ϕ is satisfied (in the interpretation given by the context) by the assignment to variables $\{x_1 \mapsto a, x_2 \mapsto b, x_3 \mapsto c, \ldots\}$.

Predicate logic restricted to fluted formulas is called *fluted logic* (FL). In fluted logic it is possible to dispense with variables entirely, since the arity and position of a predicate symbol completely determine the sequence of variables that follow the predicate symbol. However, variables will be retained to make the presentation more explicit.

§4. Quine's Predicate Functor Logic. Predicate Functor Logic is a variable-free variant of predicate logic. For background see, for example, Quine [18, 19, 20, 21]. Variables play several distinct roles in predicate logic. In Predicate Functor Logic these roles are explicated by assigning each to a distinct functor.

The vocabulary of Predicate Functor Logic consists of predicate symbols and predicate functors. The logical identity relation may be included among the predicate symbols. The predicate functors are divided into alethic and combinatory functors. The alethic functors $\exists, \neg$, and $\wedge$ correspond directly to the operations denoted by the same symbols in predicate logic. The combinatory functors inv, Inv, pad, and ref replace the variables of predicate logic. If among the logical elements of the vocabulary only the alethic functors are permitted, the logic that results is a variant of fluted logic as defined in Section 3.

Predicate Functor Logic provides a natural and useful decomposition of predicate logic. It will be convenient to produce extensions of fluted logic by reintroducing the combinatory functors one at a time.

The syntax and semantics of Predicate Functor Logic is defined as follows. First a *predicate expression* is defined. Let $m \geq 0$.

1. If P is a m-ary predicate symbol, then P is a m-ary predicate expression.
2. If π is a $(m+2)$-ary predicate expression, then $(\text{inv } \pi)$ and $(\text{Inv } \pi)$ are $(m+2)$-ary predicate expressions.
3. If π is a m-ary predicate expression, then $(\text{pad } \pi)$ is a $(m+1)$-ary predicate expression.
4. If π is a $(m+2)$-ary predicate expression, then $(\text{ref } \pi)$ is a $(m+1)$-ary predicate expression.
5. If π is a m-ary predicate expression, then $(\neg\pi)$ is a m-ary predicate expression.
6. If π is a $(m+1)$-ary predicate expression, then $(\exists\ \pi)$ is a m-ary predicate expression.
7. If π and π' are m-ary and m'-ary predicate expressions, respectively, then $(\pi \wedge \pi')$ is a $\max(m, m')$-ary predicate expression.

Next the semantics of the predicate functors is defined. Let π be a predicate expression.

$$a_1 \cdots a_n \models \text{inv } \pi :\Leftrightarrow a_1 \cdots a_{n-2}a_n a_{n-1} \models \pi$$
$$a_1 \cdots a_n \models \text{Inv } \pi :\Leftrightarrow a_2 \cdots a_n a_1 \models \pi$$
$$a_1 \cdots a_n \models \text{pad } \pi :\Leftrightarrow a_1 \cdots a_{n-1} \models \pi$$
$$a_1 \cdots a_n \models \text{ref } \pi :\Leftrightarrow a_1 \cdots a_n a_n \models \pi$$
$$a_1 \cdots a_n \models \neg\pi :\Leftrightarrow a_1 \cdots a_n \not\models \pi$$
$$a_1 \cdots a_n \models \exists\, \pi :\Leftrightarrow \text{ there exists } a \text{ such that } a_1 \cdots a_n a \models \pi$$
$$a_1 \cdots a_n \models \pi \wedge \pi' :\Leftrightarrow (n \geq \max(m, m')) \text{ and } a_{n-m+1} \cdots a_n \models \pi \text{ and}$$
$$a_{n-m'+1} \cdots a_n \models \pi', \text{ where } m \text{ and } m' \text{ are the arities of } \pi \text{ and } \pi', \text{ respectively.}$$

It is remarked in passing that the predicate functors defined above act on the rightmost argument(s) of the predicate expression. In some other papers, the functors are defined to act on the leftmost argument(s). The results are independent of which convention is adopted.

§5. Fluted constituents. This section gives the main results of Hintikka's constituent theory. The results are given for fluted logic; however, they hold for first-order predicate logic. The constituents defined here are Hintikka's constituents of the second kind. A clear and concise review of constituent theory is given in Rantala [22]. The reader is directed to that source for background and for proofs of the main results.

A conjunction in which for each $\rho \in Af_L(X_m)$ either ρ or $\neg\rho$ (but not both) occurs as a conjunct will be called a *minimal conjunction over* $Af_L(X_m)$ (because it is an atom in the Boolean lattice generated by $Af_L(X_m)$). The set of minimal conjunctions over $Af_L(X_m)$ will be denoted $\Delta Af_L(X_m)$ (cf. [22]). Note that if $\Delta Af_L(X_m) = \{\theta_1, \dots, \theta_l\}$, and ϕ is any quantifier-free formula over $Af_L(X_m)$, then

1. $\neg(\theta_i \wedge \theta_j)$ for $i \neq j$,
2. $\theta_1 \vee \cdots \vee \theta_l$, and
3. either $\theta_i \rightarrow \phi$, or $\theta_i \rightarrow \neg\phi$, for $1 \leq i \leq l$,

are tautologies (see [22]).

If θ is a minimal conjunction over X_m and R is an n-ary predicate, then $R \in \theta$ abbreviates the assertion that $Rx_{m-n+1} \cdots x_m$ is a conjunct of θ, and similarly for $\neg R \in \theta$.

Let $\mathbf{P}$ be the positive integers, and $\mathbf{P}^*$ the set of finite strings over $\mathbf{P}$. String concatenation is denoted by juxtaposition. The empty string is ε. If $i_1, \dots, i_n \in \mathbf{P}$, and $\alpha = i_1 \cdots i_n$, then for $k \leq n$, $(k : \alpha) := i_1 \cdots i_k$ is the k-prefix of α.

A subset $\mathscr{T} \subseteq \mathbf{P}^*$ is a *tree domain* if

1. $\varepsilon \in \mathscr{T}$, and
2. if $\alpha i \in \mathscr{T}$, where $\alpha \in \mathbf{P}^*$ and $i \in \mathbf{P}$, then
   (a) $\alpha j \in \mathscr{T}$ for $0 < j < i$, and
   (b) $\alpha \in \mathscr{T}$.

Define the *height of* $\alpha \in \mathscr{T}$, $h(\alpha) :=$ the length of string α. For all $\alpha, \beta \in \mathbf{P}^*$, $i \in \mathbf{P}$, if $\alpha i \beta \in \mathscr{T}$ then $\alpha i \beta$ is a *descendant of* α and αi is an *immediate descendant of* α. Define $w(\alpha) :=$ the number of immediate descendants of α. Thus $\alpha 1, \alpha 2, \dots, \alpha w(\alpha)$ are the immediate descendants of α. If $w(\alpha) = 0$, then α is *terminal* in $\mathscr{T}$. If all terminal elements of $\mathscr{T}$ have the same height, then $\mathscr{T}$ is *balanced*. In this case, $h(\mathscr{T}) := h(\alpha)$, where α is any terminal element in $\mathscr{T}$. Define the *depth of* $\alpha \in \mathscr{T}$, $d(\alpha) := h(\mathscr{T}) - h(\alpha)$. If $0 < h(\alpha) < h(\mathscr{T})$, then α is *internal* in $\mathscr{T}$. An element α together with all of its descendants is defined to be the *subtree rooted on* α, and is denoted $(\alpha]$.

Let $\mathscr{T}$ be a balanced tree domain. A *labeled tree domain* $\mathscr{T}_L$ is defined to be $\mathscr{T}$ with a formula $\theta_\alpha \in \Delta Af_L(X_{h(\alpha)})$ associated with each $\alpha \in \mathscr{T}$. The labeled subtree of $\mathscr{T}_L$ rooted on α will be denoted $(\theta_\alpha]$. The subtree $(\theta_\alpha]$ is given the following interpretation.

1. If α is terminal, then $(\theta_\alpha]$ denotes θ_α.
2. If α is nonterminal with height k, then $(\theta_\alpha]$ denotes $\theta_\alpha \wedge \exists x_{k+1}(\theta_{\alpha 1}] \wedge \cdots \wedge \exists x_{k+1}(\theta_{\alpha w(\alpha)}] \wedge \forall x_{k+1}((\theta_{\alpha 1}] \vee \cdots \vee (\theta_{\alpha w(\alpha)}])$.

The formula denoted by $(\theta_\alpha]$ is a *fluted constituent of L of height* $h(\mathscr{T}) - h(\alpha)$ *over the variables* $X_{h(\alpha)}$. If $h(\alpha) = 0$, the formula denoted by $(\theta_\alpha]$ is a *constituent sentence*. If $\theta_\varepsilon = \neg\top$, then $\mathscr{T}_L$ is *trivial*.

In the remainder of this paper, all tree domains will be nontrivial labeled balanced tree domains. Moreover, $(\theta_\alpha]$ will not be distinguished from the formula it denotes.

The following two theorems and corollary extend the results for atomic constituents given at the beginning of this section to nonatomic constituents.

THEOREM 2. (The Fundamental Property of Constituents)

1. *If ϕ and ψ are constituents of L of height k over the variables X_l, and $\phi \neq \psi$, then $\phi \wedge \psi$ is inconsistent.*
2. *The disjunction of all constituents of L of height k over the variables X_l is logically valid.*

PROOF. See [22], Theorem 3.10. ⊣

THEOREM 3. *Let ϕ be a formula of L containing variables X_m, where variables $X_k \subseteq X_m$ are free. Then ϕ is logically equivalent to a disjunction of constituents of height $h \geq m - k$ over X_k.*

PROOF. See [22], Theorem 4.1. ⊣

COROLLARY 4. *Let ϕ be a formula of L containing variables X_m, where variables $X_k \subseteq X_m$ are free. Let ψ be a constituent of height $h \geq m - k$ over X_k. Then either $\psi \rightarrow \phi$ or $\psi \rightarrow \neg\phi$ is logically valid.*

PROOF. See [22], Corollary 4.2. ⊣

If ϕ is a constituent, then define:

1. $\phi^{[-k]}$ to be ϕ with the last k variables eliminated;
2. $\phi_{[-k]}$ to be ϕ with the first k variables eliminated.

Here elimination of a variable is accomplished by removing all atomic formulas in which that variable occurs, as well as the quantifier, if any, associated with that variable, and any $\neg$-operators that thereby become idle.

If ϕ is a fluted formula whose variables are a subsequence of $x_2, \ldots, x_{k+1}$, then $\phi^\dagger := \phi\{x_2 \mapsto x_1, \ldots, x_{k+1} \mapsto x_k\}$. Similarly, if ϕ is a fluted formula whose variables are a subsequence of $x_1, \ldots, x_k$, then $\phi^\ddagger := \phi\{x_1 \mapsto x_2, \ldots, x_k \mapsto x_{k+1}\}$.

If $\mathscr{T}_L$ is a constituent sentence, there is a simple syntactic property that, if present, suffices to conclude that $\mathscr{T}_L$ is inconsistent.

THEOREM 5. *A constituent sentence $\mathscr{T}_L$ is inconsistent unless $\mathscr{T}_L^{[-1]} = (\mathscr{T}_{L[-1]})^\dagger$.*

PROOF. By the Principle of Monotonicity, $\mathscr{T}_L \rightarrow \mathscr{T}_L^{[-1]}$ and $\mathscr{T}_L \rightarrow (\mathscr{T}_{L[-1]})^\dagger$. Hence $\mathscr{T}_L \rightarrow (\mathscr{T}_L^{[-1]} \wedge (\mathscr{T}_{L[-1]})^\dagger)$. Moreover, $\mathscr{T}_L^{[-1]}$ and $(\mathscr{T}_{L[-1]})^\dagger$ are constituent sentences of the same height. It follows from the Fundamental Property of Constituents that either $\mathscr{T}_L^{[-1]}$ and $(\mathscr{T}_{L[-1]})^\dagger$ are identical or $\mathscr{T}_L$ is inconsistent. This completes the proof of the theorem. ⊣

It is remarkable that in FL, the converse of Theorem 5 holds [16]. That is, the condition of Theorem 5 is both necessary and sufficient for consistency. But to locate the limits of decision, extensions of FL must be investigated. In this connection, additional syntactic properties related to consistency will be formulated.

Define a relation $\searrow$ on $\mathscr{T}$ as follows.

1. For $1 \leq i \leq w(\varepsilon)$: $i \searrow \varepsilon$.
2. For $\alpha j, \gamma i \in \mathscr{T}$: $\alpha j \searrow \gamma i$ if and only if $\alpha \searrow \gamma$ and $((\theta_{\alpha j}]_{[-1]})^\dagger = (\theta_{\gamma i}]^{[-1]}$.

If a constituent $\mathscr{T}_L$ satisfies the condition of Theorem 5, then for any $\alpha \in \mathscr{T} - \{\varepsilon\}$, there exist $\gamma \in \mathscr{T}$ such that $\alpha \searrow \gamma$. That is, α and γ are matching elements in $(\mathscr{T}_{L[-1]})^\dagger$ and $\mathscr{T}_L^{[-1]}$, respectively. Similarly, if a constituent $\mathscr{T}_L$ satisfies the condition of Theorem 5, then for any nonterminal $\gamma \in \mathscr{T}$, there exist $\alpha \in \mathscr{T} - \{\varepsilon\}$, such that $\alpha \searrow \gamma$. Let $\searrow^n$, $\searrow^+$, and $\searrow^*$ be the n-fold composition, the transitive closure, and the reflexive transitive closure, respectively. Notice that $\alpha \searrow^n \gamma$ implies that $(\theta_\alpha]_{[-n]}$ and $(\theta_\gamma]^{[-n]}$ are identical modulo alphabetic equivalence, but the converse does not necessarily hold.

An important consequence of Theorem 5 is given by the following lemma.

LEMMA 6. *Let $\mathscr{T}_L$ be a fluted constituent sentence that satisfies the condition of Theorem* 5.

1. *For all $i, m \in \mathscr{T}$: $\mathscr{T}_L \rightarrow \forall x_1((\theta_m] \rightarrow \forall x_2(((\theta_i])^\ddagger \rightarrow \bigvee_{mq \searrow i} (\theta_{mq}]))$.*
2. *For all $m, mj \in \mathscr{T}$: $\mathscr{T}_L \rightarrow \forall x_1((\theta_m] \rightarrow \forall x_2((\theta_{mj}] \rightarrow \bigvee_{mj \searrow i} ((\theta_i])^\ddagger))$.*

PROOF. Let $\mathscr{T}_L = \bigwedge_{1 \leq r \leq w(\varepsilon)} \exists x_1(\theta_r] \wedge \forall x_1 \bigvee_{1 \leq r \leq w(\varepsilon)} (\theta_r]$. Then

$$\begin{aligned}
\mathscr{T}_L &\rightarrow \forall x_1((\theta_m] \rightarrow \forall x_2\left(\bigvee_{mq \searrow i} (\theta_{mq}] \vee \bigvee_{\neg(mq \searrow i)} (\theta_{mq}]\right)\right. \\
&\rightarrow \forall x_1\left((\theta_m] \rightarrow \forall x_2\left(((\theta_i]^{[-1]})^\ddagger \rightarrow \bigvee_{mq \searrow i} (\theta_{mq}]\right)\right) \\
&\qquad \text{since } (\theta_{mq}] \rightarrow (\theta_{mq}]_{[-1]} \text{ and } mq \searrow i \text{ if and only if } (\theta_{mq}]_{[-1]} = ((\theta_i]^{[-1]})^\ddagger \\
&\rightarrow \forall x_1((\theta_m] \rightarrow \forall x_2(((\theta_i])^\ddagger \rightarrow \bigvee_{mq \searrow i} (\theta_{mq}])) \\
&\qquad \text{since } (\theta_i] \rightarrow (\theta_i]^{[-1]}.
\end{aligned}$$

This establishes 6.1. Next suppose that 6.2 fails. Then

$$\begin{aligned}
\neg 6.2 &\rightarrow \exists x_1\left((\theta_m] \wedge \exists x_2((\theta_{mj}] \wedge \bigvee_{\neg(mj \searrow i)} (\theta_i]^\ddagger)\right) \\
&\rightarrow \exists x_1\left((\theta_m] \wedge \exists x_2\left((\theta_{mj}] \wedge \bigvee_{\neg(mj \searrow i)} ((\theta_i]^{[-1]})^\ddagger\right)\right) \\
&\qquad \text{since } (\theta_i] \rightarrow (\theta_i]^{[-1]} \\
&\rightarrow \exists x_1((\theta_m] \wedge \exists x_2((\theta_{mj}] \wedge \neg((\theta_{mj}]_{[-1]}))) \\
&\qquad \text{since } mj \searrow i \text{ if and only if } (\theta_{mj}]_{[-1]} = ((\theta_i]^{[-1]})^\ddagger \\
&\rightarrow \exists x_1((\theta_m] \wedge \exists x_2((\theta_{mj}] \wedge \neg((\theta_{mj}]))) \\
&\qquad \text{since } \neg((\theta_{mj}]_{[-1]}) \rightarrow \neg((\theta_{mj}]).
\end{aligned}$$

Thus the supposition that 6.2 fails results in a contradiction. This completes the proof of the lemma. ⊣

§6. Extension to include binary conversion. This section considers extension of the language of fluted formulas to include the binary converse operation.

If $\text{ar}(R) = 2$, then $\breve{R}$ denotes the *converse* of R. In general, if ψ is a fluted formula with two free variables, then $\breve{\psi}$ denotes the converse of ψ. If either R or $\breve{R}$ occurs in the premises, then both R and $\breve{R}$ are considered to be (distinct) elements of the lexicon.

The definition of an L-structure $\mathscr{A}$ is extended as follows. If $\operatorname{ar}(R) = 2$, then $\forall a, b \in A, ab \in R^{\mathscr{A}}$ if and only if $ba \in \breve{R}^{\mathscr{A}}$. The definition of the satisfaction relation is extended so that if ψ is a fluted formula with two free variables, $a_1a_2 \models \psi$ if and only if $a_2a_1 \models \breve{\psi}$.

It suffices to restrict consideration to the converse operation applied to binary predicates. This can be seen as follows. Suppose that ψ is a fluted formula with two free variables, x_k and x_{k+1}. Let Q be a new binary predicate symbol. Then

$$\phi = \forall x_k \forall x_{k+1}(Qx_kx_{k+1} \leftrightarrow \psi)$$

is a fluted sentence. It follows from the semantics of the converse operation that ϕ entails

$$\forall x_k \forall x_{k+1}(\breve{Q}x_kx_{k+1} \leftrightarrow \breve{\psi}).$$

That is, $\breve{Q}x_kx_{k+1}$ is logically equivalent to $\breve{\psi}$. In particular, $\breve{\breve{R}}x_1x_2$ is logically equivalent to Rx_1x_2, and $\neg\breve{R}x_1x_2$ is logically equivalent to $(\neg Rx_1x_2)\breve{}$.

Consider the subformula of a constituent

$$\psi = \exists x_1(\theta_m \wedge \exists x_2(\theta_{mj}^{(1)} \wedge \theta_{mj}^{(2)}))$$

where $\theta_{mj}^{(1)}$ is a minimal conjunction over unary predicates, $\theta_{mj}^{(2)}$ is a minimal conjunction over binary predicates, and $\theta_{mj} = \theta_{mj}^{(1)} \wedge \theta_{mj}^{(2)}$. In FO it is easy to show that ψ is logically equivalent to

$$\psi' = \exists x_1((\theta_{mj}^{(1)})^{\dagger} \wedge \exists x_2((\theta_m)^{\ddagger} \wedge (\theta_{mj}^{(2)})\breve{})).$$

By Theorem 5, if the constituent is consistent, then for some i, $mj \searrow i$ and so $(\theta_{mj}^{(1)})^{\dagger} = \theta_i$. If it is further the case that for some il, $il \searrow m$, $\theta_m = (\theta_{il}^{(1)})^{\dagger}$, and $(\theta_{mj}^{(2)})\breve{} = \theta_{il}^{(2)}$, we write $mj \bowtie il$. More precisely, $\bowtie$ is defined

$$mj \bowtie il :\Leftrightarrow (mj \searrow i \wedge il \searrow m \wedge (\theta_{mj}^{(2)})\breve{} = \theta_{il}^{(2)}).$$

Note that $\bowtie$ is a symmetric relation. In case $mj \bowtie il$, ψ also is logically equivalent to

$$\psi'' = \exists x_1(\theta_i \wedge \exists x_2(\theta_{il}^{(1)} \wedge \theta_{il}^{(2)})).$$

To simplify notation, in the following theorem it is assumed that quantification is restricted to $\mathscr{T}$. This practice will be continued in the remainder of the paper. Specifically, formulas such as

1. $\forall\alpha\exists\gamma(\alpha \searrow \gamma)$,
2. $\forall m\exists mj(mj \bowtie mj)$, and
3. $\{mj : \exists il(mj \bowtie il)\}$

are abbreviations for

1. $\forall\alpha(\alpha \in \mathscr{T} \rightarrow \exists\gamma(\gamma \in \mathscr{T} \wedge \alpha \searrow \gamma)$,
2. $\forall m(1 \leq m \leq w(\varepsilon) \rightarrow \exists mj(1 \leq j \leq w(m) \wedge mj \bowtie mj)$, and
3. $\{mj : 1 \leq j \leq w(m) \wedge \exists il(1 \leq l \leq w(i) \wedge mj \bowtie il)\}$, for $1 \leq i, m \leq w(\varepsilon)$.

THEOREM 7. *A constituent sentence $\mathscr{T}_L$ is inconsistent if any of the following conditions fails to hold.*

1. $\forall m\forall mj\exists i\exists il(mj \bowtie il)$.

2. $\forall m \forall i \exists mj \exists il (mj \bowtie il)$.
3. $\forall m \exists mj (mj \bowtie mj)$.

PROOF. Let
$$\mathscr{T}_L = \bigwedge_{1 \le r \le w(\varepsilon)} \exists x_1 (\theta_r] \wedge \forall x_1 \bigvee_{1 \le r \le w(\varepsilon)} (\theta_r].$$
It may be assumed that $\mathscr{T}_L$ satisfies the condition of Theorem 5, otherwise there is nothing to prove. Suppose Condition 7.1 fails. Then $\exists m \exists mj \forall i \forall il ((mj \searrow i \wedge il \searrow m) \rightarrow (\theta_{il}^{(2)} \rightarrow \neg(\theta_{mj}^{(2)})\breve{\ })).$ Hence

$$\begin{aligned}
\mathscr{T}_L &\rightarrow \forall x_1 \bigvee_{1 \le r \le w(\varepsilon)} (\theta_r] \\
&= \forall x_1 \left(\bigvee_{mj \searrow r} (\theta_r] \vee \bigvee_{\neg(mj \searrow r)} (\theta_r] \right) \\
&\rightarrow \forall x_1 \left(\bigvee_{mj \searrow r} \left((\theta_r] \wedge \forall x_2 \left(\bigvee_{rq \searrow m} (\theta_{rq}] \vee \bigvee_{\neg(rq \searrow m)} (\theta_{rq}] \right) \right) \vee \bigvee_{\neg(mj \searrow r)} (\theta_r] \right) \\
&\rightarrow \forall x_1 (((\theta_{mj}]_{[-1]})^{\dagger} \rightarrow \forall x_2 (((\theta_m]^{[-1]})^{\ddagger} \rightarrow \neg(\theta_{mj}^{(2)})\breve{\ }))
\end{aligned}$$

since $mj \searrow r$ if and only if $(\theta_r] \rightarrow (\theta_r]^{[-1]} = ((\theta_{mj}]_{[-1]})^{\dagger}$, and
$rq \searrow m$ if and only if $(\theta_{rq}] \rightarrow (\theta_{rq}]_{[-1]} = ((\theta_m]^{[-1]})^{\ddagger}$, and
$rq \searrow m$ implies $(\theta_{rq}] \rightarrow \neg(\theta_{mj}^{(2)})\breve{\ }$

$$\begin{aligned}
&\rightarrow \forall x_1 ((\theta_m]^{[-1]} \rightarrow \forall x_2 ((\theta_{mj}]_{[-1]} \rightarrow \neg(\theta_{mj}^{(2)}))) \\
&\rightarrow \forall x_1 ((\theta_m] \rightarrow \forall x_2 ((\theta_{mj}] \rightarrow \neg(\theta_{mj}^{(2)})))
\end{aligned}$$

since $(\theta_m] \rightarrow (\theta_m]^{[-1]}$ and $(\theta_{mj}] \rightarrow (\theta_{mj}]_{[-1]}$.

Since also
$$\mathscr{T}_L \rightarrow \exists x_1 ((\theta_m] \wedge \exists x_2 ((\theta_{mj}]))$$
$\mathscr{T}_L$ is inconsistent.

Suppose Condition 7.2 fails. Then $\exists m \exists i \forall mj \forall il ((mj \searrow i \wedge il \searrow m) \rightarrow (\theta_{il}^{(2)} \rightarrow \neg(\theta_{mj}^{(2)})\breve{\ }))$. By Lemma 6.1,

$$\begin{aligned}
\mathscr{T}_L &\rightarrow \forall x_1 \left((\theta_m] \rightarrow \forall x_2 \left(((\theta_i])^{\ddagger} \rightarrow \bigvee_{mq \searrow i} (\theta_{mq}] \right) \right) \\
&\rightarrow \forall x_1 \left((\theta_m] \rightarrow \forall x_2 \left(((\theta_i])^{\ddagger} \rightarrow \bigvee_{mq \searrow i} \theta_{mq}^{(2)} \right) \right) \\
&\rightarrow \forall x_1 \left((\theta_i] \rightarrow \forall x_2 \left(((\theta_m])^{\ddagger} \rightarrow \bigvee_{mq \searrow i} (\theta_{mq}^{(2)})\breve{\ } \right) \right) \\
&\rightarrow \forall x_1 \left((\theta_i] \rightarrow \forall x_2 \left(((\theta_m])^{\ddagger} \rightarrow \neg \bigvee_{il \searrow m} \theta_{il}^{(2)} \right) \right)
\end{aligned}$$

$$\rightarrow \forall x_1\left((\theta_i] \rightarrow \forall x_2\left(((\theta_m])^\ddagger \rightarrow \neg \bigvee_{il \searrow m} (\theta_{il}]\right)\right)$$

$$\rightarrow \forall x_1\left((\theta_i] \rightarrow \forall x_2\left(((\theta_m])^\ddagger \rightarrow \left(\bigvee_{il \searrow m} (\theta_{il}] \wedge \neg \bigvee_{il \searrow m} (\theta_{il}]\right)\right)\right)$$

by Lemma 6.1.

Since also

$$\mathscr{T}_L \rightarrow \exists x_1((\theta_i] \wedge \exists x_2(((\theta_m])^\ddagger))$$

$\mathscr{T}_L$ is inconsistent.

Suppose Condition 7.3 fails. Then $\exists m \forall mj(mj \searrow m \rightarrow (\theta^{(2)}_{mj} \rightarrow \neg(\theta^{(2)}_{mj})\breve{}))$. By Lemma 6.1,

$$\mathscr{T}_L \rightarrow \forall x_1\left((\theta_m] \rightarrow \forall x_2\left(((\theta_m])^\ddagger \rightarrow \bigvee_{mq \searrow m} (\theta_{mq}]\right)\right)$$

$$\rightarrow \forall x_1\left((\theta_m] \rightarrow \forall x_2\left(((\theta_m])^\ddagger \rightarrow \bigvee_{mq \searrow m} \theta^{(2)}_{mq}\right)\right)$$

$$\rightarrow \forall x_1\left((\theta_m] \rightarrow \forall x_2\left(((\theta_m])^\ddagger \rightarrow \bigvee_{mq \searrow m} (\theta^{(2)}_{mq})\breve{}\right)\right)$$

$$\rightarrow \forall x_1\left((\theta_m] \rightarrow \forall x_2\left(((\theta_m])^\ddagger \rightarrow \neg \bigvee_{mq \searrow m} \theta^{(2)}_{mq}\right)\right)$$

$$\rightarrow \forall x_1\left((\theta_m] \rightarrow \forall x_2\left(((\theta_m])^\ddagger \rightarrow \neg \bigvee_{mq \searrow m} (\theta_{mq}]\right)\right)$$

$$\rightarrow \forall x_1\left((\theta_m] \rightarrow \forall x_2\left(((\theta_m])^\ddagger \rightarrow \left(\bigvee_{mq \searrow m} (\theta_{mq}] \wedge \neg \bigvee_{mq \searrow m} (\theta_{mq}]\right)\right)\right)$$

by Lemma 6.1.

Since also

$$\mathscr{T}_L \rightarrow \exists x_1((\theta_m] \wedge \exists x_2(((\theta_m])^\ddagger))$$

$\mathscr{T}_L$ is inconsistent. This completes the proof of the theorem. ⊣

Theorem 7 shows that introduction of the binary converse operation imposes a further restriction on the syntax of consistent constituent sentences. The following lemma shows this more precisely. It refines Lemma 6 and shows the additional restriction that binary conversion brings to bear.

LEMMA 8. *Let $\mathscr{T}_L$ be a fluted constituent sentence with binary conversion that satisfies the conditions of Theorems* 5 *and* 7.

1. *For all $i, m \in \mathscr{T}$:*

$$\mathscr{T}_L \rightarrow \forall x_1\left((\theta_m] \rightarrow \forall x_2\left(((\theta_i])^\ddagger \rightarrow \bigvee\{(\theta_{mq}] : \exists il(mq \bowtie il)\}\right)\right).$$

2. *For all* $m, mj \in \mathscr{T}$:

$$\mathscr{T}_L \to \forall x_1\left((\theta_m] \to \forall x_2\left((\theta_{mj}] \to \bigvee\{((\theta_i])^\ddagger : \exists il(mj \bowtie il)\}\right)\right).$$

PROOF. Suppose that 8.1 fails. Then

$$\begin{aligned}
\neg 8.1 \ \to\ & \exists x_1\left((\theta_m] \wedge \exists x_2\left(((\theta_i])^\ddagger \wedge \bigvee\{(\theta_{mq}] : mq \searrow i \wedge \neg\exists il(mq \bowtie il)\}\right)\right)\\
& \text{by Lemma 6.1}\\
\to\ & \exists x_1\left((\theta_m] \wedge \exists x_2\left(((\theta_i])^\ddagger \wedge \bigvee\{\theta^{(2)}_{mq} : mq \searrow i \wedge \neg\exists il(mq \bowtie il)\}\right)\right)\\
\to\ & \exists x_1\left((\theta_m] \wedge \exists x_2\left(((\theta_i])^\ddagger \wedge \neg\bigvee\{(\theta^{(2)}_{il})\breve{} : il \searrow m\}\right)\right)\\
\to\ & \exists x_1\left((\theta_i] \wedge \exists x_2\left(((\theta_m])^\ddagger \wedge \neg\bigvee\{\theta^{(2)}_{il} : il \searrow m\}\right)\right)\\
\to\ & \exists x_1\left((\theta_i] \wedge \exists x_2\left(((\theta_m])^\ddagger \wedge \neg\bigvee\{(\theta_{il}] : il \searrow m\}\right)\right).
\end{aligned}$$

But this result contradicts Lemma 6.1.

Next suppose that 8.2 fails. Then

$$\begin{aligned}
\neg 8.2 \ \to\ & \exists x_1((\theta_m] \wedge \exists x_2((\theta_{mj}] \wedge \bigvee\{((\theta_i])^\ddagger : mj \searrow i \wedge \neg\exists il(mj \bowtie il)\}))\\
& \text{by Lemma 6.2}\\
\to\ & \bigvee\{\exists x_1((\theta_m] \wedge \exists x_2(((\theta_i])^\ddagger \wedge (\theta_{mj}])) : mj \searrow i \wedge \neg\exists il(mj \bowtie il)\}\\
\to\ & \bigvee\{\exists x_1((\theta_m] \wedge \exists x_2(((\theta_i])^\ddagger \wedge \theta^{(2)}_{mj})) : mj \searrow i \wedge \neg\exists il(mj \bowtie il)\}\\
\to\ & \bigvee\{\exists x_1((\theta_m] \wedge \exists x_2(((\theta_i])^\ddagger \wedge \neg\bigvee\{(\theta^{(2)}_{il})\breve{} : il \searrow m\})) : mj \searrow i\}\\
\to\ & \bigvee\{\exists x_1((\theta_i] \wedge \exists x_2(((\theta_m])^\ddagger \wedge \neg\bigvee\{\theta^{(2)}_{il} : il \searrow m\})) : mj \searrow i\}\\
\to\ & \bigvee\{\exists x_1((\theta_i] \wedge \exists x_2(((\theta_m])^\ddagger \wedge \neg\bigvee\{(\theta_{il}] : il \searrow m\})) : mj \searrow i\}.
\end{aligned}$$

This result contradicts Lemma 6.1. This completes the proof of the lemma. ⊣

In view of Lemma 8, a restriction of $\searrow$ is defined for fluted constituents with the converse operation as follows.

1. For elements mj at height 2, $mj \rightsquigarrow i :\Leftrightarrow \exists il(mj \bowtie il)$.
2. For elements α at height $\neq 2$, $\alpha \rightsquigarrow \gamma :\Leftrightarrow \alpha \searrow \gamma$.

§7. **Further extension to include the identity and functions.** This section considers extension of the language of fluted formulas with binary conversion to include n-ary functions, where $n \geq 0$. Nullary functions are constants. One unary function is singled out for special logical treatment: the diagonal function, denoted d.

With no loss in generality, the syntax of this extension will be considerably simplified by substituting *function predicates* for functions. If f is an n-ary function ($n \geq 0$), then the corresponding function predicate F is an $(n+1)$-ary predicate defining the graph of f, viz.,

$$Fx_1 \cdots x_n x_{n+1} :\Leftrightarrow fx_1 \cdots x_n = x_{n+1}.$$

In particular, unary function predicates correspond to constants. The *identity relation* I corresponds to the diagonal function. It is defined

$$Ix_1x_2 :\Leftrightarrow dx_1 = x_2 :\Leftrightarrow x_1 = x_2.$$

The definition of an L-structure is extended as follows. If F is an $(n+1)$-ary function predicate, then $\forall a_1, \dots, a_n \in A \;\; \exists! a_{n+1} \in A$ such that $a_1 \cdots a_n a_{n+1} \in F^{\mathscr{A}}$. Further, $I^{\mathscr{A}} := \{aa : a \in A\}$. This yields the following theorem.

THEOREM 9. *Let $F \in L$ be an $(n+1)$-ary function predicate, where $n \geq 0$. A constituent sentence $\mathscr{T}_L$ is inconsistent if either of the following conditions fails to hold.*

1. $\forall \alpha (h(\alpha) \geq n \rightarrow \exists! \alpha j (F \in \theta_{\alpha j}))$.
2. $\forall m \forall mj (I \in \theta_{mj} \rightarrow (mj \bowtie mj \wedge \forall il (mj \bowtie il \rightarrow il = mj)))$.

That there exists a function predicate F such that $F \in \theta_{\alpha j}$ will be denoted by $\mathrm{FCNL}(\alpha j)$.

In the following sections, fluted logic with binary conversion, identity, and functions will be referred to as *extended fluted logic* (EFL). It is this extension that defines Quine's limits of decision. The remainder of the paper is devoted to proving this statement.

§8. Valuation of a constituent. Throughout this section, let $\mathscr{T}_L$ be an EFL constituent sentence of height h. Let A be a domain (nonempty set). Sequences of elements of A will be written as strings, e.g., $a_1 \cdots a_k$. We use $a, b, c, \dots$ as metavariables ranging over elements of A; and $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \dots$ as metavariables ranging over sequences of elements of A. Concatenation of sequences will be denoted by juxtaposition. $\langle\rangle$ will denote the empty sequence.

Let $V \subseteq A^k$. Define

$V^{[-1]} := \{\boldsymbol{a} : \boldsymbol{a}a \in V\}$.
$V^{[-i-1]} := (V^{[-i]})^{[-1]}$.
$V_{[-1]} := \{\boldsymbol{a} : a\boldsymbol{a} \in V\}$.
$V_{[-i-1]} := (V_{[-i]})_{[-1]}$.
$V \times A := \{\boldsymbol{a}a : (\boldsymbol{a} \in V) \wedge (a \in A)\}$.
$A \times V := \{a\boldsymbol{a} : (\boldsymbol{a} \in V) \wedge (a \in A)\}$.

Notice that these are all monotone operations.

Now define a *valuation of $\mathscr{T}_L$ in A*:

$$\mathscr{V} := \{V_\alpha : \alpha \in \mathscr{T}\}, \text{ where for all } \alpha \in \mathscr{T} : \emptyset \neq V_\alpha \subseteq A^{h(\alpha)}.$$

Next some properties of valuations are defined.

$\mathscr{V}$ has the *prefix property* if for all nonterminal $\alpha \in \mathscr{T}$ and for all j such that $1 \leq j \leq w(\alpha)$: $V_{\alpha j}^{[-1]} \subseteq V_\alpha$.

$\mathscr{V}$ is *connected* if it has the prefix property, and for all nonterminal $\alpha \in \mathscr{T}$ and for all j such that $1 \leq j \leq w(\alpha)$: $V_\alpha \subseteq V_{\alpha j}^{[-1]}$.
$\mathscr{V}$ is *complete* if for all nonterminal $\alpha \in \mathscr{T}$: $V_\alpha \times A = \bigcup_{1 \leq j \leq w(\alpha)} V_{\alpha j}$.
$\mathscr{V}$ is *disjoint* if for all $\alpha, \gamma \in \mathscr{T}$: $V_\alpha \cap V_\gamma \neq \emptyset$ only if $\alpha = \gamma$.
$\mathscr{V}$ is *consistent* if

(a) for all $\alpha \in \mathscr{T} - \{\varepsilon\}$, $V_{\alpha[-1]} \subseteq \bigcup_{\alpha \rightsquigarrow \gamma} V_\gamma$,
(b) for all $il, mj \in \mathscr{T}$, for all $a, b \in A$, if $ab \in V_{mj}$ and $ba \in V_{il}$, then $mj \bowtie il$,
(c) for all $mj \in \mathscr{T}$ such that $I \in \theta_{mj}$, $V_{mj} = \mathrm{diag}(V_m^2)$, and
(d) for all $\alpha j \in \mathscr{T}$ such that $\theta_{\alpha j}$ contains a function predicate, for all $\boldsymbol{c} \in V_\alpha$, there exists exactly one $a \in A$ such that $\boldsymbol{c}a \in V_{\alpha j}$.

$\mathscr{V}$ is *adequate* if it is disjoint, connected, complete, and consistent.

LEMMA 10. *Let $\mathscr{V}$ be an adequate valuation of $\mathscr{T}_L$. Then the rule*

$$\boldsymbol{a} \models \theta_\alpha \textit{ if } \boldsymbol{a} \in V_\alpha$$

determines a well-defined L-structure with domain A.

PROOF. Suppose not. That is, suppose the rule in not consistent. There are five cases to consider.

1. For some n-ary predicate $P \in L$, there are $\alpha, \gamma \in \mathscr{T}$ and $\boldsymbol{a} \in A^n$ such that
   $\boldsymbol{ba} \in V_\alpha$,
   $\boldsymbol{ca} \in V_\gamma$,
   $P \in \theta_\alpha$, and
   $\neg P \in \theta_\gamma$.
2. $R \in L$ is a binary predicate, and there are $\alpha, \gamma \in \mathscr{T}$ and $a, b \in A$ such that
   $\boldsymbol{c}ab \in V_\alpha$,
   $\boldsymbol{d}ba \in V_\gamma$,
   $\neg R \in \theta_\alpha$, and
   $\check{R} \in \theta_\gamma$.
3. There is $\alpha \in \mathscr{T}$, $\boldsymbol{c} \in A^n$, and $a \in A$ such that
   $\boldsymbol{c}aa \in V_\alpha$, and
   $\neg I \in \theta_\alpha$.
4. There is $\alpha \in \mathscr{T}$, $\boldsymbol{c} \in A^n$, and $a, b \in A$ such that
   $\boldsymbol{c}ab \in V_\alpha$,
   $I \in \theta_\alpha$, and
   $a \neq b$.
5. For some $n+1$-ary function predicate $F \in L$, there are $\alpha, \gamma \in \mathscr{T}$ and $\boldsymbol{c} \in A^n$, and $a, b \in A$ such that
   $\boldsymbol{c}a \in V_\alpha$,
   $\boldsymbol{c}b \in V_\gamma$,
   $F \in \theta_\alpha$,
   $F \in \theta_\gamma$, and
   $a \neq b$.

1. In this case, $\boldsymbol{a} \in V_{\alpha[-r]} \cap V_{\gamma[-q]}$. Since $\mathscr{V}$ is consistent, $\boldsymbol{a} \in V_\delta \cap V_\beta$ for some δ and β such that $\alpha \leadsto^r \delta$ and $\gamma \leadsto^q \beta$. Thus $P \in \theta_\delta$ and $\neg P \in \theta_\beta$. But since $\mathscr{V}$ is disjoint, $\delta = \beta$, and so $\theta_\delta = \theta_\beta$, contradicting the supposition.

2. Since $\mathscr{V}$ is consistent, for some mj and il such that $\alpha \leadsto^r mj$ and $\gamma \leadsto^q il$, $ab \in V_{mj}$ and $ba \in V_{il}$. Further, since $\mathscr{V}$ is consistent, $mj \bowtie il$. But $\neg Rx_1x_2$ is a conjunct of θ_{mj}, and $\breve{R}x_1x_2$ is a conjunct of θ_{il}, contradicting $mj \bowtie il$.

The proofs for 3, 4, and 5 are similar. This completes the proof of the lemma. ⊣

The L-structure defined in Lemma 10 will be referred to as *the L-structure determined by $\mathscr{V}$*. The following theorem shows that it is a model of $\mathscr{T}_L$.

THEOREM 11. *Let $\mathscr{A}$ be the L-structure determined by an adequate valuation $\mathscr{V}$ of $\mathscr{T}_L$. Then $\mathscr{A} \models \mathscr{T}_L$.*

PROOF. It suffices to prove the claim: for all $\alpha \in \mathscr{T}$: if $\boldsymbol{a} \in V_\alpha$ then $\boldsymbol{a} \models (\theta_\alpha]$. The claim is proved by induction on $d = d(\alpha)$. For the basis, $d = 0$. Thus α is a terminal element. By definition, $(\theta_\alpha] = \theta_\alpha$, and $\boldsymbol{a} \models \theta_\alpha$ if $\boldsymbol{a} \in V_\alpha$. So the induction hypothesis holds at depth 0.

For the induction step, $d > 0$. Suppose that $\boldsymbol{a} \in V_\alpha$. Since $\mathscr{V}$ is connected, if $1 \leq j \leq w(\alpha)$, there exists $a \in A$ such that $\boldsymbol{a}a \in V_{\alpha j}$. By the induction hypothesis, if $\boldsymbol{a}a \in V_{\alpha j}$ then $\boldsymbol{a}a \models (\theta_{\alpha j}]$. Hence there exists $a \in A$ such that $\boldsymbol{a}a \models (\theta_{\alpha j}]$. Since $\mathscr{V}$ is complete, for all $a \in A$, $\boldsymbol{a}a \in \bigcup_{1 \leq j \leq w(\alpha)} V_{\alpha j}$. Applying the induction hypothesis as before yields: for all $a \in A$, $\boldsymbol{a}a \models (\theta_{\alpha 1}] \vee \cdots \vee (\theta_{\alpha w(\alpha)}]$. Therefore, $\boldsymbol{a} \models (\theta_\alpha]$, and so the induction hypothesis holds at depth d. This completes the proof of the theorem. ⊣

THEOREM 12. *If a constituent sentence $\mathscr{T}_L$ is consistent, then it has an adequate valuation in a countable domain.*

PROOF. Assume $\mathscr{T}_L$ is a consistent constituent. Then by the Löwenheim-Skolem Theorem, $\mathscr{T}_L$ has a countable model $\mathscr{A}$ with domain A. Let $((\theta_\alpha])^{\mathscr{A}}$ be the interpretation of $(\theta_\alpha]$ in $\mathscr{A}$. Define

1. $V_\varepsilon := \{\langle\rangle\}$, and
2. for $\alpha j \in \mathscr{T}$ and $h(\alpha j) = k > 0$: $V_{\alpha j} := \{\boldsymbol{a}b \in A^k : \boldsymbol{a} \in V_\alpha \wedge \boldsymbol{a}b \in ((\theta_{\alpha j}])^{\mathscr{A}}\}$.

We claim that $\mathscr{V} = \{V_\alpha : \alpha \in \mathscr{T}\}$ is an adequate valuation of $\mathscr{T}_L$ in A. First observe that

1. for $1 \leq j \leq w(\alpha)$: $\mathscr{T}_L \rightarrow \forall x_1((\theta_{1:\alpha}] \rightarrow \cdots \rightarrow \forall x_k(((\theta_\alpha] \rightarrow \exists x_{k+1}((\theta_{\alpha j}])\cdots)$, and
2. $\mathscr{T}_L \rightarrow \forall x_1((\theta_{1:\alpha}] \rightarrow \cdots \rightarrow \forall x_k(((\theta_\alpha] \rightarrow \forall x_{k+1}(\bigvee_{1 \leq j \leq w(\alpha)} (\theta_{\alpha j}])\cdots)$.

Therefore, in $\mathscr{A}$,

1. for $1 \leq j \leq w(\alpha)$: $\forall \boldsymbol{a} \in V_\alpha(\exists b \in A(\boldsymbol{a}b \in ((\theta_{\alpha j}])^{\mathscr{A}}))$, and
2. $\forall \boldsymbol{a} \in V_\alpha(\forall b \in A(\boldsymbol{a}b \in \bigcup_{1 \leq j \leq w(\alpha)}((\theta_{\alpha j}])^{\mathscr{A}}))$.

It follows that

1. for all $\alpha \in \mathscr{T} : V_\alpha \subseteq A^{h(\alpha)}$, by definition of V_α;
2. for all $\alpha \in \mathscr{T} : V_\alpha \neq \emptyset$, by observation 1;
3. $\mathscr{V}$ has the prefix property, by definition of V_α;
4. $\mathscr{V}$ is connected, by observation 1;

5. $\mathscr{V}$ is complete, by observation 2;
6. $\mathscr{V}$ is disjoint, by the Fundamental Property of Constituents;
7. $\mathscr{V}$ is consistent, since $\mathscr{T}_L$ is consistent.

Thus $\mathscr{V}$ is an adequate valuation of $\mathscr{T}_L$ in A. This completes the proof of Theorem 12. ⊣

§9. **$\rightsquigarrow$-paths of a constituent.** In this section, $\rightsquigarrow$-paths are defined and certain properties derived. Let $\mathscr{T}_L$ be an EFL constituent sentence of height h. Suppose further that $\mathscr{T}_L$ satisfies the conditions of Theorems 5, 7, and 9. $\mathscr{T}_L$ defines a graph $G = (\mathscr{T}, \rightsquigarrow)$, which is weakly connected. Every point except ε has outdegree ≥ 1. Every point except the terminal elements of $\mathscr{T}$ has indegree ≥ 1. A *$\rightsquigarrow$-path* is a path in G, hereinafter called simply a path. If a path π is incident on α, we write $\alpha \in \pi$. If $\alpha \rightsquigarrow \gamma$ lies on a path π, we write $\alpha \rightsquigarrow \gamma \subseteq \pi$. A path of length $h-1$ incident on ε is a *penultimate path.*

Let $S_1 \subseteq \{\alpha 1, \dots, \alpha w(\alpha)\}$ and $S_2 \subseteq \{\gamma 1, \dots, \gamma w(\gamma)\}$. We write $S_1 \rightsquigarrow S_2$ if and only if

1. $(\alpha j \in S_1 \wedge \alpha j \rightsquigarrow \gamma i) \rightarrow \gamma i \in S_2$
2. $(\gamma i \in S_2 \wedge \alpha j \rightsquigarrow \gamma i) \rightarrow \alpha j \in S_1$.

Obviously,

$$\{\alpha 1, \dots, \alpha w(\alpha)\} \rightsquigarrow \{\gamma 1, \dots, \gamma w(\gamma)\}.$$

Further, if $\alpha j \rightsquigarrow \gamma i$ and FCNL(γi), then $\{\alpha j\} \rightsquigarrow \{\gamma i\}$. When $\alpha \rightsquigarrow \gamma$ is clear from the context and no confusion can result, $S_1 \subseteq \{\alpha 1, \dots, \alpha w(\alpha)\}$ and $S_2 \subseteq \{\gamma 1, \dots, \gamma w(\gamma)\}$ will be tacit. Let $S_1 \subseteq \{\alpha 1, \dots, \alpha w(\alpha)\}$. The *closure of* S_1, denoted cl(S_1), is defined to be the smallest subset of $\{\alpha 1, \dots, \alpha w(\alpha)\}$ such that $S_1 \subseteq \mathrm{cl}(S_1)$ and for some $S_2 \subseteq \{\gamma 1, \dots, \gamma w(\gamma)\}$, $\mathrm{cl}(S_1) \rightsquigarrow S_2$.

Using the observations in the proof of Theorem 12, it is easy to see that for $h(\alpha) = k < h$,

$$\mathscr{T}_L \rightarrow \forall x_1((\theta_{1:\alpha}] \rightarrow \cdots$$
$$\cdots \rightarrow \forall x_k\left((\theta_\alpha] \rightarrow \left(\bigwedge_{1\leq j\leq w(\alpha)} \exists x_{k+1}(\theta_{\alpha j}]\right) \wedge \forall x_{k+1}\left(\bigvee_{1\leq j\leq w(\alpha)} (\theta_{\alpha j}]\right)\cdots\right).$$

Further, if $\alpha \rightsquigarrow \gamma$, then

$$\begin{aligned}\mathscr{T}_L \rightarrow & \forall x_1((\theta_{1:\alpha}] \rightarrow \forall x_2(((\theta_{2:\alpha}] \wedge (\theta_{1:\gamma}]^\ddagger) \rightarrow \cdots \rightarrow \forall x_k(((\theta_\alpha] \wedge (\theta_\gamma]^\ddagger) \\ & \rightarrow \left(\bigwedge_{1\leq j\leq w(\alpha)} \exists x_{k+1}((\theta_{\alpha j}]) \wedge \bigwedge_{1\leq i\leq w(\gamma)} \exists x_{k+1}((\theta_{\gamma i}]^\ddagger)\right. \\ & \left.\wedge \forall x_{k+1}\left(\bigvee_{1\leq j\leq w(\alpha)} (\theta_{\alpha j}]\right) \wedge \forall x_{k+1}\left(\bigvee_{1\leq i\leq w(\gamma)} (\theta_{\gamma i}]^\ddagger\right)\cdots\right).\end{aligned}$$

If in addition $S_1 \rightsquigarrow S_2$, then since $((\theta_{\alpha j}] \wedge (\theta_{\gamma i}]^\ddagger) \rightarrow ((\theta_{\alpha j}]_{[-1]} \wedge ((\theta_{\gamma i}]^{[-1]})^\ddagger) \rightarrow \perp$ if $\neg(\alpha j \rightsquigarrow \gamma i)$,

$$\mathscr{T}_L \rightarrow \forall x_1((\theta_{1:\alpha}] \rightarrow \forall x_2(((\theta_{2:\alpha}] \wedge (\theta_{1:\gamma}]^\ddagger) \rightarrow \cdots \rightarrow \forall x_k(((\theta_\alpha] \wedge (\theta_\gamma]^\ddagger)$$

$$\to \left(\bigwedge_{1 \le j \le w(\alpha)} \exists x_{k+1}((\theta_{\alpha j}]) \right.$$
$$\left. \wedge \bigwedge_{1 \le i \le w(\gamma)} \exists x_{k+1}((\theta_{\gamma i}]^{\ddagger}) \wedge \forall x_{k+1} \left(\bigvee_{\alpha j \in S_1} (\theta_{\alpha j}] \leftrightarrow \bigvee_{\gamma i \in S_2} (\theta_{\gamma i}]^{\ddagger} \right) \cdots \right).$$

These results are generalized in the following lemma.

LEMMA 13. *Let $\mathscr{T}_L$ be an EFL constituent sentence that satisfies the conditions of Theorems* 5, 7, *and* 9. *Let π be the path $\gamma_1 \rightsquigarrow \gamma_2 \rightsquigarrow \cdots \rightsquigarrow \gamma_k \rightsquigarrow \varepsilon$. For $1 \le q \le k$, let $S_{1,q}, S_{2,q} \subseteq \{\gamma_q 1, \ldots, \gamma_q w(\gamma_q)\}$ such that $S_{1,q} \rightsquigarrow S_{2,q+1}$. Then*

$$\mathscr{T}_L \to \forall x_1((\theta_{1:\gamma_1}] \to \forall x_2(((\theta_{2:\gamma_1}] \wedge (\theta_{1:\gamma_2}]^{\ddagger}) \to \cdots$$
$$\to \forall x_k(((\theta_{\gamma_1}] \wedge (\theta_{\gamma_2}]^{\ddagger} \wedge \cdots \wedge (\theta_{\gamma_k}]^{\ddagger^{k-1}})$$
$$\to \left(\bigwedge_{1 \le q \le k} \bigwedge_{1 \le i_q \le w(\gamma_q)} \exists x_{k+1}((\theta_{\gamma_q i_q}]^{\ddagger^{q-1}}) \right.$$
$$\left. \wedge \bigwedge_{1 \le q \le k} \forall x_{k+1} \left(\bigvee_{\gamma_q i_q \in S_{1,q}} (\theta_{\gamma_q i_q}]^{\ddagger^{q-1}} \leftrightarrow \bigvee_{\gamma_{q+1} i_{q+1} \in S_{2,q+1}} (\theta_{\gamma_{q+1} i_{q+1}}]^{\ddagger^{q}} \right) \cdots \right).$$

PROOF. The proof is a straightforward generalization of the reasoning given above. ⊣

Lemma 13 can be rewritten in FO.

$$\mathscr{T}_L \to \forall x_1 \cdots \forall x_k \left(\phi_\pi \to \left(\bigwedge_{1 \le q \le k} \bigwedge_{1 \le i_q \le w(\gamma_q)} \exists x_{k+1}((\theta_{\gamma_q i_q}]^{\ddagger^{q-1}}) \right. \right.$$
$$\left. \left. \wedge \bigwedge_{1 \le q \le k} \forall x_{k+1} \left(\bigvee_{\gamma_q i_q \in S_{1,q}} (\theta_{\gamma_q i_q}]^{\ddagger^{q-1}} \leftrightarrow \bigvee_{\gamma_{q+1} i_{q+1} \in S_{2,q+1}} (\theta_{\gamma_{q+1} i_{q+1}}]^{\ddagger^{q}} \right) \right) \right),$$

where

$$\phi_\pi := (\theta_{1:\gamma_1}] \wedge (\theta_{2:\gamma_1}] \wedge (\theta_{1:\gamma_2}]^{\ddagger} \wedge \cdots \wedge (\theta_{\gamma_1}] \wedge (\theta_{\gamma_2}]^{\ddagger} \wedge \cdots \wedge (\theta_{\gamma_k}]^{\ddagger^{k-1}}.$$

If π is a path of length k incident on ε and $\mathscr{T}_L \to \exists x_1 \cdots \exists x_k \phi_\pi$, then π is called an *active* path. A set Π of such paths is active if every member is simultaneously active, i.e., $\mathscr{T}_L \to \bigwedge_{\pi \in \Pi} \exists x_1 \cdots \exists x_k \phi_\pi$. If Π is a set of paths of $\mathscr{T}_L$ and for every $\alpha j \in \mathscr{T}$ there exists $\pi \in \Pi$ such that $\alpha \in \pi$, then Π is called a *complete* set of paths of $\mathscr{T}_L$. By Theorem 5, if there exists a complete set of paths, there exists a complete set of penultimate paths.

LEMMA 14. *If $\mathscr{T}_L$ satisfies the conditions of Theorems* 5, 7, *and* 9, *there exists a complete active set of penultimate paths of $\mathscr{T}_L$.*

PROOF. The proof is by induction on $h = h(\mathscr{T})$. The induction hypothesis is that there exists an active set Π of penultimate paths of $\mathscr{T}_L$ and for every $\alpha j \in \mathscr{T}$ there exists a path $\pi \in \Pi$ such that $\alpha \in \pi$.

For the basis step, let $h = 2$. Define $\Pi := \{i \rightsquigarrow \varepsilon : 1 \leq i \leq w(\varepsilon)\}$. Then for every $il \in \mathscr{T}$ there is a $\pi \in \Pi$ such that $i \in \pi$. Moreover, every path $i \rightsquigarrow \varepsilon$ in Π is active since $\mathscr{T}_L \rightarrow \exists x_1(\theta_i]$.

For the induction step, $h > 2$. The induction hypothesis holds for $\mathscr{T}_L^{[-1]}$. Let Π be a complete active set of penultimate paths of $\mathscr{T}_L^{[-1]}$. Then for $\pi \in \Pi, \mathscr{T}_L \rightarrow \mathscr{T}_L^{[-1]} \rightarrow \exists x_1 \cdots \exists x_{h-2}\phi_\pi$. By Lemma 13, with $k = h - 2$ and $\pi \in \Pi$,

$$\mathscr{T}_L \rightarrow \forall x_1 \cdots \forall x_{h-2}\Bigg(\phi_\pi \rightarrow \Bigg(\bigwedge_{1\leq q\leq h-2}\ \bigwedge_{1\leq i_q\leq w(\gamma_q)} \exists x_{h-1}((\theta_{\gamma_q i_q}]^{\ddagger^{q-1}}\Bigg)$$
$$\wedge \bigwedge_{1\leq q\leq h-2} \forall x_{h-1}\Bigg(\bigvee_{\gamma_q i_q\in S_{1.q}} (\theta_{\gamma_q i_q}]^{\ddagger^{q-1}} \leftrightarrow \bigvee_{\gamma_{q+1}i_{q+1}\in S_{2.q+1}} (\theta_{\gamma_{q+1}i_{q+1}}]^{\ddagger^q}\Bigg)\Bigg)\Bigg).$$

Let j_1 be such that $1 \leq j_1 \leq w(\gamma_1)$ and $S_{1,1} = \mathrm{cl}(\{\gamma_1 j_1\})$. It follows that

$$\mathscr{T}_L \rightarrow \forall x_1 \cdots \forall x_{h-2}\Bigg(\phi_\pi \rightarrow \Bigg(\exists x_{h-1}\Bigg((\theta_{\gamma_1 j_1}])\wedge\forall x_{h-1}((\theta_{\gamma_1 j_1}] \rightarrow \bigvee_{\gamma_2 i_2\in S_{2.2}} (\theta_{\gamma_2 i_2}]^{\ddagger}\Bigg)\Bigg)\Bigg).$$

Then there exists $\gamma_2 j_2 \in S_{2.2}$ such that $\gamma_1 j_1 \rightsquigarrow \gamma_2 j_2$, and

$$\mathscr{T}_L \rightarrow \forall x_1 \cdots \forall x_{h-2}(\phi_\pi \rightarrow \exists x_{h-1}((\theta_{\gamma_1 j_1}] \wedge (\theta_{\gamma_2 j_2}]^{\ddagger})).$$

Moreover, by Theorem 5, for each j_2 such that $1 \leq j_2 \leq w(\gamma_2)$, there exists some j_1 such that this formula holds. Proceeding inductively, it follows that there are $j_1, j_2, \ldots, j_{h-1}$ such that $\gamma_1 j_1 \rightsquigarrow \gamma_2 j_2 \rightsquigarrow \cdots \rightsquigarrow \gamma_{h-2} j_{h-2} \rightsquigarrow j_{h-1} \rightsquigarrow \varepsilon$ and

$$\mathscr{T}_L \rightarrow \forall x_1 \cdots \forall x_{h-2}(\phi_\pi \rightarrow \exists x_{h-1}((\theta_{\gamma_1 j_1}] \wedge (\theta_{\gamma_2 j_2}]^{\ddagger} \wedge \cdots \wedge (\theta_{j_{h-1}}]^{\ddagger^{h-2}})).$$

Thus $\gamma_1 j_1 \rightsquigarrow \gamma_2 j_2 \rightsquigarrow \cdots \rightsquigarrow \gamma_{h-2} j_{h-2} \rightsquigarrow j_{h-1} \rightsquigarrow \varepsilon$ is an active path of $\mathscr{T}_L$, derived from $\pi \in \Pi$. Define $\Pi' := \{\pi' : \pi \in \Pi \wedge \pi' \text{ is derived from } \pi\}$. Then Π' satisfies the induction hypothesis for $\mathscr{T}_L$ of height h. This completes the proof of the lemma. ⊣

Further analysis of paths will be conducted in a more general logic, viz., first-order predicate logic with counting quantifiers (FO(C)). (Ebbinghaus and Flum [4] give a brief treatment of this logic.) FO(C) is a conservative extension of FO to include the counting quantifiers $\{\exists^{\geq l} : l \in \omega\}$. The formulas of FO(C) are those of FO and in addition $\exists^{\geq l} x\phi$ whenever ϕ is a formula. It is convenient also to introduce the abbreviations

1. $\exists^{=l} x\phi := \exists^{\geq l} x\phi \wedge \neg\exists^{\geq (l+1)} x\phi$,
2. $\exists^{=0} x\phi := \forall x\neg\phi$.

The following proposition is a consequence of the semantics of counting quantifiers.

Proposition 15. *In FO(C) all universal closures of the following formulas hold.*

1. $\exists x\phi \leftrightarrow \exists^{\geq 1} x\phi$.
2. $(\exists^{\geq l} x\phi \wedge \forall x(\phi \rightarrow \psi)) \rightarrow \exists^{\geq l} x\psi$.
3. $(\exists^{\geq l} x\phi \wedge \exists^{\geq m} x\psi \wedge \neg\exists x(\phi \wedge \psi)) \rightarrow \exists^{\geq (l+m)} x(\phi \vee \psi)$.
4. $(\exists^{\geq l} x\phi \wedge \exists^{=m} x\psi \wedge \neg\exists x(\phi \wedge \psi)) \rightarrow \exists^{\geq (l+m)} x(\phi \vee \psi)$.
5. $(\exists^{=l} x\phi \wedge \exists^{=m} x\psi \wedge \neg\exists x(\phi \wedge \psi)) \rightarrow \exists^{=(l+m)} x(\phi \vee \psi)$.

Define a *count predicate* to be either $\geq l$ or $= l$, where $l \in \omega$. Define 'addition' of count predicates

1. $\sum\{\geq l, \geq m\} := \ \geq (l+m)$.
2. $\sum\{\geq l, = m\} := \ \geq (l+m)$.
3. $\sum\{= l, = m\} := \ = (l+m)$.

Define 'conjunction' of count predicates

1. $\bigwedge\{\geq l, \geq m\} := \ \geq \max(l, m)$.
2. $\bigwedge\{\geq l, = m\} :=$ if $l \leq m$ then $= m$, otherwise $\perp$.
3. $\bigwedge\{= l, = m\} :=$ if $l = m$ then $= m$, otherwise $\perp$.

Define the *argument* of a count predicate $|\geq l| := l$ and $|= l| := l$. If $|p| > 0$, the count predicate p is *positive*.

Let $\mathscr{T}_L$ be a constituent sentence that satisfies the conditions of Theorems 5, 7, and 9. Let Π be a complete active set of penultimate paths of $\mathscr{T}_L$. For each $\pi \in \Pi$ and each γi such that $\gamma \in \pi$, associate a count predicate $p(\gamma i, \pi)$. Starting with the terminal elements $\gamma_1 i_1$, with $\gamma_1 \in \pi$, let $p(\gamma_1 i_1, \pi)$ be the count predicate $= 1$ if $\mathrm{FCNL}(\gamma_1 i_1)$, and the count predicate $\geq l$ for some $l \geq 1$ otherwise. Thus

$$\mathscr{T}_L \to \forall x_1 \cdots \forall x_{h-1}\left(\phi_\pi \to \left(\bigwedge_{1 \leq i_1 \leq w(\gamma_1)} \exists^{p(\gamma_1 i_1, \pi)} x_h ((\theta_{\gamma_1 i_1}]) \right.\right.$$
$$\left.\left.\wedge \forall x_h \left(\bigvee_{\gamma_1 i_1 \in S_{1.1}} (\theta_{\gamma_1 i_1}] \leftrightarrow \bigvee_{\gamma_2 i_2 \in S_{2.2}} (\theta_{\gamma_2 i_2}]^\ddagger\right)\right)\right).$$

Hence by Proposition 15,

$$\mathscr{T}_L \to \forall x_1 \cdots \forall x_{h-1}\left(\phi_\pi \to \bigwedge_{1 \leq i_2 \leq w(\gamma_2)} \exists^{p(\gamma_2 i_2, \pi)} x_h ((\theta_{\gamma_2 i_2}]^\ddagger)\right)$$

where $\sum_{\gamma_1 i_1 \in S_{1.1}} p(\gamma_1 i_1, \pi) = \sum_{\gamma_2 i_2 \in S_{2.2}} p(\gamma_2 i_2, \pi)$. Proceeding in this manner, for $2 < q < h$,

$$\mathscr{T}_L \to \forall x_1 \cdots \forall x_{h-1}\left(\phi_\pi \to \bigwedge_{1 \leq i_q \leq w(\gamma_q)} \exists^{p(\gamma_q i_q, \pi)} x_h ((\theta_{\gamma_q i_q}]^{\ddagger^{q-1}})\right),$$

where

$$\sum_{\gamma_{q-1} i_{q-1} \in S_{1,q-1}} p(\gamma_{q-1} i_{q-1,\pi}) = \sum_{\gamma_q i_q \in S_{2.q}} p(\gamma_q i_q, \pi).$$

Since π is a penultimate path, $\gamma_h = \varepsilon$. Since

$$\forall x_1 \cdots \forall x_{h-1}(\phi_\pi \to \exists^{p(i_h, \pi)} x_h (\theta_{i_h}]^{\ddagger^{h-1}}) \leftrightarrow ((\exists x_1 \cdots \exists x_{h-1} \phi_\pi) \to \exists^{p(i_h, \pi)} x_h (\theta_{i_h}]^{\ddagger^{h-1}})$$

and

$$\mathscr{T}_L \to \exists x_1 \cdots \exists x_{h-1} \phi_\pi$$

it follows that

$$\mathscr{T}_L \to \exists^{p(i_h, \pi)} x_h (\theta_{i_h}]^{\ddagger^{h-1}}.$$

Hence $\bigwedge_{\pi \in \Pi} \exists^{p(i_h, \pi)} x_h (\theta_{i_h}]^{\ddagger^{h-1}}$ must be consistent if $\mathscr{T}_L$ is. That is, $\bigwedge_{\pi \in \Pi} p(i_h, \pi) \neq \perp$. Indeed, there must be values for the $p(\gamma i, \pi)$ such that $p(i_h, \pi)$ is constant for all $\pi \in \Pi$, say p_i.

Moreover for $i \in \pi$,

$$\mathcal{T}_L \to \forall x_1 \cdots \forall x_{h-1}(\phi_\pi \to \exists^{p(il,\pi)} x_h(\theta_{il}]^{\ddagger^{h-2}}).$$

Defining $\psi_\pi := \exists x_1 \cdots \exists x_{h-2}\phi_\pi$, this can be rewritten

$$\mathcal{T}_L \to \forall x_{h-1}(\psi_\pi \to \exists^{p(il,\pi)} x_h(\theta_{il}]^{\ddagger^{h-2}}).$$

The latter formula shows that in any model of $\mathcal{T}_L$, ψ_π denotes part or all of the denotation of $(\theta_i]$. Suppose $i \in \pi'$ also. Then the denotations of ψ_π and $\psi_{\pi'}$ are disjoint if for some il, $p(il,\pi) \wedge p(il,\pi') = \bot$. Thus the ψ_π refine the partition of the domain of interpretation induced by the $(\theta_i]$. In view of this, it is appropriate to examine the conjunctions of the ψ_π. Define $\Pi_i := \{\pi \in \Pi : i \in \pi\}$, and $\Sigma_i := \Delta\Pi_i$, the set of minimal conjunctions over Π_i. For $\sigma \in \Sigma_i$, define

$$\psi_\sigma := \bigwedge\{\psi_\pi : \pi \in \sigma\} \wedge \bigwedge\{\neg\psi_\pi : \neg\pi \in \sigma\} \wedge (\theta_i]^{\ddagger^{h-2}}$$

and $\Sigma := \bigcup\{\Sigma_i : 1 \leq i \leq w(\varepsilon)\}$. Then in a model of $\mathcal{T}_L$, ψ_σ denotes a subset of the denotation of $(\theta_i]$. If this subset is nonempty, i.e., if $\mathcal{T}_L \to \exists x_{h-1}\psi_\sigma$, σ will be said to be *nonvacuous*.

Now for $1 \leq i \leq w(\varepsilon), 1 \leq l \leq w(i), \sigma \in \Sigma_i, \sigma' \in \Sigma$, and σ, σ' nonvacuous, define the count predicate $q(il,\sigma,\sigma')$ as follows.

$$\mathcal{T}_L \to \forall x_{h-1}(\psi_\sigma \to \exists^{q(il,\sigma,\sigma')} x_h(\psi^\ddagger_{\sigma'} \wedge (\theta_{il}]^{\ddagger^{h-2}})),$$

with the constraint that

$$\sum_{\sigma' \in \Sigma} q(il,\sigma,\sigma') = \bigwedge_{\pi \in \sigma} p(il,\pi).$$

Intuitively, in a model of $\mathcal{T}_L$, for each element in the denotation of ψ_σ there are $q(il,\sigma,\sigma')$ images by the denotation of $(\theta_{il}]$ lying in the denotation of $\psi_{\sigma'}$. Note that $q(il,\sigma,\sigma')$ is not necessarily positive.

Clearly, $\mathcal{T}_L$ implies the following. Assume σ, σ' nonvacuous.

1. For $1 \leq i \leq w(\varepsilon), \sigma' \in \Sigma_m$, $\sum_{il \rightsquigarrow m} q(il,\sigma,\sigma')$ is constant, say $q_{\sigma'}$, for all $\sigma \in \Sigma_i$.
2. $\sum_{\sigma \in \Sigma_i} q_\sigma = p_i$.
3. If $1 \leq l \leq w(i), 1 \leq j \leq w(m), \sigma \in \Sigma_i, \sigma' \in \Sigma_m$,
$$q_\sigma \sum_{il \rightsquigarrow m} q(il,\sigma,\sigma') = q_{\sigma'} \sum_{mj \rightsquigarrow i} q(mj,\sigma',\sigma).$$
4. If $I \in \theta_{il}$, then $q(il,\sigma,\sigma') > 0 \to \sigma' = \sigma$.

The following theorem has been proved.

THEOREM 16. *Let $\mathcal{T}_L$ be a constituent sentence that satisfies the conditions of Theorems* 5, 7, *and* 9. *Then $\mathcal{T}_L$ is inconsistent unless*

1. *there exists a complete subset Π of the penultimate paths of $\mathcal{T}_L$, where*
   $\Pi_i := \{\pi \in \Pi : i \in \pi\}$,
   $\Sigma_i :\subseteq \Delta\Pi_i$,
   $\Sigma := \bigcup_{1 \leq i \leq w(\varepsilon)} \Sigma_i$,
   such that

2. *the following system has a solution for the count predicates* $\{p(\alpha j, \pi) : \pi \in \Pi \wedge \alpha \in \pi\}$ *and* $\{q(il, \sigma, \sigma') : 1 \leq i \leq w(\varepsilon) \wedge 1 \leq l \leq w(i) \wedge \sigma \in \Sigma_i \wedge \sigma' \in \Sigma\}$.
   (a) *If* αj *is terminal,* $\pi \in \Pi$, *and* $\alpha \in \pi$,
   if $\mathrm{FCNL}(\alpha j)$, *then* $p(\alpha j, \pi)$ *is* $= 1$;
   otherwise, $p(\alpha j, \pi)$ *is* $\geq l$ *for some* $l \geq 1$.
   (b) *If* α *is internal,* $\alpha \rightsquigarrow \gamma \subseteq \pi$, *and* $S_1 \rightsquigarrow S_2$,
   $$\sum_{\alpha j \in S_1} p(\alpha j, \pi) = \sum_{\gamma i \in S_2} p(\gamma i, \pi).$$
   (c) *If* $1 \leq i \leq w(\varepsilon)$, *then for all* $\pi \in \Pi$, $p(i, \pi)$ *is constant, say* p_i.
   (d) *Every* $p(\alpha j, \pi)$ *is positive.*
   (e) *If* $1 \leq i \leq w(\varepsilon), 1 \leq l \leq w(i), \sigma \in \Sigma_i, \sigma' \in \Sigma$,
   $$\sum_{\sigma' \in \Sigma} q(il, \sigma, \sigma') = \bigwedge_{\pi \in \sigma} p(il, \pi) \text{ if } \bigwedge_{\pi \in \sigma} p(il, \pi) \neq \perp;$$
   $$\sum_{1 \leq l \leq w(i)} q(il, \sigma, \sigma') \text{ is constant for all } \sigma \in \Sigma_i, \text{ say } q_{\sigma'};$$
   $$\sum_{\sigma \in \Sigma_i} q_\sigma = p_i.$$
   (f) *If* $1 \leq l \leq w(i), 1 \leq j \leq w(m), \sigma \in \Sigma_i, \sigma' \in \Sigma_m$,
   $$q_\sigma \sum_{il \rightsquigarrow m} q(il, \sigma, \sigma') = q_{\sigma'} \sum_{mj \rightsquigarrow i} q(mj, \sigma', \sigma),$$
   (g) *If* $I \in \theta_{il}$, *then* $q(il, \sigma, \sigma') > 0 \rightarrow \sigma' = \sigma$.

§10. Trivial inconsistency. Theorems 5, 7, 9, and 16 show that certain simple syntactic properties of a constituent suffice to conclude that the constituent is inconsistent. The conditions of all but Theorem 16 can be ascertained by observation. The condition of Theorem 16 can be ascertained by well-known effective algorithms that compute the feasibility of linear constraint systems (see Papadimitriou and Steiglitz [13]). When a constituent is inconsistent by virtue of these simple properties, it will be said to be *trivially inconsistent*.

A similar situation holds in predicate logic (Hintikka [7, 8]). But in predicate logic, the converse fails. That is, absence of these or any other simple syntactic properties does not permit the conclusion that the constituent is consistent. Rather it is in general necessary to expand the constituent into a disjunction of constituents (called its *distributive normal form*) of greater and greater height. It is a reflection of the undecidability of predicate logic that one is assured only that if the constituent is inconsistent, then at some height, trivial inconsistency will manifest itself in every constituent in the expansion at that height. This is called by Hintikka the *completeness theorem* of the theory of distributive normal forms.

It is the objective of the next section to show that in contrast to predicate logic, in EFL the converse does hold.

§11. Satisfiability of constituents. Let $\mathscr{T}_L$ be an EFL constituent sentence of height h. The objective of this section is to show that if $\mathscr{T}_L$ is not trivially inconsistent, then it is consistent. By Theorem 11, it suffices to show that an adequate valuation of $\mathscr{T}_L$ can be constructed.

THEOREM 17. *Let* $\mathscr{T}_L$ *be an EFL constituent sentence. If* $\mathscr{T}_L$ *is not trivially inconsistent, then an adequate valuation* $\mathscr{V}$ *of* $\mathscr{T}_L$ *can be constructed.*

PROOF. Since $\mathscr{T}_L$ is not trivially inconsistent, by Theorem 16, there exists a solution for the linear constraint system defined there. This solution will be used to define an adequate valuation $\mathscr{V}$ of $\mathscr{T}_L$.

Three steps are required: two basis steps at levels 1 and 2, and one inductive step, at level $k+1$ assuming completion of level k. Let $\mathscr{V}^{(k)}$ be $\mathscr{V}$ restricted to level k and below.

Step 1. Define the domain A of $\mathscr{V}$, $A := \boldsymbol{n} := \{0, 1, \dots, n-1\}$, where $n := \sum_{1 \leq i \leq w(\varepsilon)} n_i$ and $n_i := |p_i|$. Assign $V_\varepsilon := \{\langle\rangle\}$. Assign $V_1 := \boldsymbol{n_1}, V_2 := \boldsymbol{n_2} - V_1, \dots, V_{w(\varepsilon)} := \boldsymbol{n} - (V_1 \cup \dots \cup V_{w(\varepsilon)-1})$. For $1 \leq i \leq w(\varepsilon)$, let Σ_i be ordered, say $\Sigma_i = \{\sigma_{i1}, \dots, \sigma_{ik_i}\}$, and $n_\sigma := |q_\sigma|$. By Theorem 16, $n_{\sigma_{i1}} + \dots + n_{\sigma_{ik_i}} = n_i$. Define $V_{\sigma_{i1}} :=$ the first $n_{\sigma_{i1}}$ elements of $V_i, \dots, V_{\sigma_{ik_i}} :=$ the last $n_{\sigma_{ik_i}}$ elements of V_i.

Step 2. Let il be the unique element such that $1 \leq l \leq w(i)$ and $I \in \theta_{il}$. Assign $V_{il} := \operatorname{diag}(V_i^2)$. For each remaining $il, \sigma \in \Sigma_i, \sigma' \in \Sigma$, for each $a \in V_\sigma$, choose $|q(il, \sigma, \sigma')|$ elements from $\{a\} \times V_{\sigma'}$, not already assigned, and assign them to V_{il}. Assign their converses to $\bigcup_{mj \rhd\lhd il} V_{mj}$, restricted only to not exceed $|q(mj, \sigma', \sigma)|$ in total assigned to V_{mj}. By Theorem 16,

$$n_\sigma \sum_{il \rightsquigarrow m} |q(il, \sigma, \sigma')| = n_{\sigma'} \sum_{mj \rightsquigarrow i} |q(mj, \sigma', \sigma)|$$

so there are exactly enough elements.

CLAIM. $\mathscr{V}^{(2)}$ is adequate.

PROOF. $\mathscr{V}^{(2)}$ has the prefix property, disjointness, and consistency by construction, and the fact that $\mathrm{FCNL}(il) \rightarrow |p(il, \pi)| = 1$. By Theorem 16,

$$\sum_{\sigma' \in \Sigma} q(il, \sigma, \sigma') = \bigwedge_{\pi \in \sigma} p(il, \pi)$$

since $\bigwedge_{\pi \in \sigma} p(il, \pi) \neq \perp$ is positive. Hence $\mathscr{V}^{(2)}$ is connected. Completeness follows from the observations that $\sum_{1 \leq l \leq w(i)} q(il, \sigma, \sigma') = q_{\sigma'}$, $\sum_{\sigma' \in \Sigma_i} q_{\sigma'} = p_i$, and $n := \sum_{1 \leq i \leq w(\varepsilon)} n_i$. This completes the proof of the claim. ⊣

Step 3. Inductively, assume that $\mathscr{V}^{(k)}$ has been constructed. For each α at height k, for each $\pi \in \Pi$ such that $\alpha \rightsquigarrow \gamma \subseteq \pi$, for each $\boldsymbol{a} \in V_\alpha \cap (A \times V_\gamma)$, for $1 \leq j \leq w(\alpha)$, assign $|p(\alpha j, \pi)|$ elements from $(\{\boldsymbol{a}\} \times A) \cap (A \times \bigcup_{\alpha j \rightsquigarrow \gamma i} V_{\gamma i})$, not already assigned, to $V_{\alpha j}$. By Theorem 16, if α is internal, $\alpha \rightsquigarrow \gamma \subseteq \pi$, and $S_1 \rightsquigarrow S_2$,

$$\sum_{\alpha j \in S_1} |p(\alpha j, \pi)| = \sum_{\gamma i \in S_2} |p(\gamma i, \pi)|$$

so there are exactly enough elements.

CLAIM. $\mathscr{V}^{(k+1)}$ is adequate if $\mathscr{V}^{(k)}$ is.

PROOF. The proof is similar to that for the previous claim. $\mathscr{V}^{(k+1)}$ has the prefix property, disjointness, and consistency by construction, and the fact that $\mathrm{FCNL}(\alpha j) \rightarrow |p(\alpha j, \pi)| = 1$. Since $p(\alpha j, \pi)$ is positive, $\mathscr{V}^{(k+1)}$ is connected. Completeness follows from Theorem 16, viz., if α is internal, $\alpha \rightsquigarrow \gamma \subseteq \pi$, $S_1 = \{\alpha 1, \dots, \alpha w(\alpha)\}$, and $S_2 = \{\gamma 1, \dots, \gamma w(\gamma)\}$,

$$\sum_{\alpha j \in S_1} p(\alpha j, \pi) = \sum_{\gamma i \in S_2} p(\gamma i, \pi),$$

and the induction hypothesis. This completes the proof of the claim. ⊣

This completes the proof of Theorem 17. ⊣

§12. **Decidability of EFL.** A logic is *decidable* if the satisfiability (equivalently, the validity) of any sentence of the logic is decidable. If ϕ is an EFL sentence, Theorem 3 states that ϕ is equivalent to the disjunction of its constituents. Moreover, the proof of Theorem 3 provides an effective method of transforming ϕ into the disjunction of its constituents. Obviously ϕ is satisfiable if and only if one of its constituents is satisfiable. Theorem 17 entails that a constituent is satisfiable if and only if it is not trivially inconsistent. Trivial inconsistency can be decided by a finite number of tests on the syntax of the constituent. Theorems 3 and 17 therefore yield the following conclusions.

THEOREM 18. *The satisfiability of a sentence of EFL is decidable.*

COROLLARY 19. *EFL has the finite model property.*

COROLLARY 20. *EFL is decidable.*

FO^2 is the sublogic of FO that is restricted to two variable letters. The well-known properties of FO^2 also are corollaries of the results obtained above.

COROLLARY 21. *FO^2 has the finite model property.*

COROLLARY 22. *FO^2 is decidable.*

Turning next to the combinatory functors of Predicate Functor Logic, it will be shown that pad and ref add no additional expressiveness to EFL. Then in the following section it will be shown that the remaining functors, inv and Inv, bring greater expressiveness and also undecidability.

THEOREM 23. *The addition of* pad *to FL is a conservative extension.*

PROOF. Let ϕ be a sentence of FL with pad. It suffices to show that there exists a sentence ψ of FL such that ψ is logically equivalent to ϕ. Without loss of generality, assume that ϕ is a standard FL sentence with pad.

Proceeding inductively, suppose that ϕ contains n occurrences of pad. Let $(\text{pad } \theta)$ be a subformula of ϕ with k free variables, such that θ is a formula of FL. Let Q be a k-ary predicate symbol having no occurrence in ϕ. Define

$$\psi' := \phi' \wedge \forall x_1 \cdots \forall x_{k-1}((\theta \rightarrow \forall x_k Q x_1 \cdots x_k) \wedge (\neg\theta \rightarrow \forall x_k \neg Q x_1 \cdots x_k))$$

where ϕ' is obtained from ϕ by replacing that occurrence of $(\text{pad } \theta)$ with $Q x_1 \cdots x_k$. Now by the induction hypothesis, there exists a sentence ψ of FL such that ψ is logically equivalent to ψ'. It remains to prove that ψ' is logically equivalent to ϕ.

Let $\mathscr{A}$ be any interpretation of FL in domain A. Let $a_1, \dots, a_{k-1}, a \in A$. Then $a_1 \cdots a_{k-1} a \models (\text{pad } \theta)$ implies $a_1 \cdots a_{k-1} \models \theta$ implies $a_1 \cdots a_{k-1} \models \forall x_k Q x_1 \cdots x_k$

implies $a_1 \cdots a_{k-1}a \models Qx_1 \cdots x_k$. Conversely, $a_1 \cdots a_{k-1}a \models Qx_1 \cdots x_k$ implies $a_1 \cdots a_{k-1} \models \exists x_k Qx_1 \cdots x_k$ implies $a_1 \cdots a_{k-1} \models \theta$ implies $a_1 \cdots a_{k-1}a \models (\text{pad } \theta)$. Thus ψ' is logically equivalent to ϕ. This completes the proof of the theorem. ⊣

THEOREM 24. *The addition of* ref *to EFL is a conservative extension.*

PROOF. Let ϕ be a sentence of EFL with ref. It suffices to show that there exists a sentence ψ of EFL such that ψ is logically equivalent to ϕ. Again, without loss of generality, assume that ϕ is a standard EFL sentence with ref.

Proceeding inductively, suppose that ϕ contains n occurrences of ref. Let $(\text{ref } \theta)$ be a subformula of ϕ with k free variables, such that θ is a formula of EFL. Let Q be a k-ary predicate symbol having no occurrence in ϕ. Define

$$\psi' := \phi' \wedge \forall x_1 \cdots \forall x_k (Qx_1 \cdots x_k \leftrightarrow \exists x_{k+1}(Ix_k x_{k+1} \wedge \theta))$$

where ϕ' is obtained from ϕ by replacing that occurrence of $(\text{ref } \theta)$ with $Qx_1 \cdots x_k$. Now by the induction hypothesis, there exists a sentence ψ of EFL such that ψ is logically equivalent to ψ'. It remains to prove that ψ' is logically equivalent to ϕ.

Let $\mathscr{A}$ be any interpretation of EFL in domain A. Let $a_1, \dots, a_k \in A$. Then $a_1 \cdots a_k \models (\text{ref } \theta)$ if and only if $a_1 \cdots a_k a_k \models \theta$ if and only if $a_1 \cdots a_k a_k \models Ix_k x_{k+1} \wedge \theta$ if and only if $a_1 \cdots a_k \models \exists x_{k+1}(Ix_k x_{k+1} \wedge \theta)$ if and only if $a_1 \cdots a_k \models Qx_1 \cdots x_k$. Thus ψ' is logically equivalent to ϕ. This completes the proof of the theorem. ⊣

In the remaining sections, it will be shown that any significant additions to EFL must sacrifice decidability.

§13. **Permutation of variables beyond binary conversion.** Undecidability of a logic will be established by showing that a problem known to be undecidable, viz., the tiling problem (TP), can be expressed in that logic. The form of the tiling problem most useful for present purposes is the form without function symbols and with arbitrary origin. Lewis and Papadimitriou [10] can be consulted for details and further references. The problem is expressed as the conjunction of the following sentences.

$$\exists x_1 \exists x_2 P_0 x_1 x_2$$

$$\bigwedge_{i \neq j} \forall x_1 \forall x_2 \neg(P_i x_1 x_2 \wedge P_j x_1 x_2)$$

$$\forall x_1 \exists x_2 \forall x_3 \left(\bigvee_{(i,j)\in H} (P_i x_1 x_3 \wedge P_j x_2 x_3) \wedge \bigvee_{(i,j)\in V} (P_i x_3 x_1 \wedge P_j x_3 x_2) \right).$$

It can be shown that the conjunction of these sentences is satisfiable if and only if there exists a tiling of a quadrant of the plane with oriented tiles of finitely many distinct kinds such that abutting edges match. Specifically,

there is a location (the lower left corner) containing a tile of kind 0;
no location contains two distinct kinds of tile;
every pair of horizontally adjacent locations contains tiles with matching edges; and

every pair of vertically adjacent locations contains tiles with matching edges.

It was shown in Section 12 that addition of the padding functor to FL is conservative. Therefore, without loss of generality, it can be assumed that FL contains the padding functor. Suppose that the minor inversion functor inv is added. Then it is possible to rewrite TP as follows.

$$\exists x_1 \exists x_2 P_0 x_1 x_2$$

$$\bigwedge_{i \neq j} \forall x_1 \forall x_2 \neg(P_i x_1 x_2 \wedge P_j x_1 x_2)$$

$$\forall x_1 \exists x_2 \forall x_3 \left(\bigvee_{(i,j) \in H} ((\text{inv pad } P_i) x_1 x_2 x_3 \wedge P_j x_2 x_3) \right.$$

$$\left. \wedge \bigvee_{(i,j) \in V} ((\text{inv pad inv } P_i) x_1 x_2 x_3 \wedge (\text{inv } P_j) x_2 x_3) \right).$$

Since these are sentences of FL with functors pad and inv, the following theorem has been established.

THEOREM 25. *The addition of* inv *to FL introduces undecidability.*

Next suppose that the major inversion functor Inv is added to FL. Then it is possible to rewrite TP as follows.

$$\exists x_1 \exists x_2 P_0 x_1 x_2$$

$$\bigwedge_{i \neq j} \forall x_1 \forall x_2 \neg(P_i x_1 x_2 \wedge P_j x_1 x_2)$$

$$\forall x_1 \exists x_2 \forall x_3 \left(\bigvee_{(i,j) \in H} ((\text{Inv Inv pad Inv } P_i) x_1 x_2 x_3 \wedge P_j x_2 x_3) \right.$$

$$\left. \wedge \bigvee_{(i,j) \in V} ((\text{Inv Inv pad } P_i) x_1 x_2 x_3 \wedge (\text{Inv } P_j) x_2 x_3) \right).$$

Since these are sentences of FL with functors pad and Inv, the following theorem has been established.

THEOREM 26. *The addition of* Inv *to FL introduces undecidability.*

Thus the principal filters generated by the logic FL with inv and the logic FL with Inv together constitute an undecidable region in the lattice of logics. It is easy to see that these two filters together with the ideal generated by the logic EFL exhaust this lattice of logics. Thus Quine's limits of decision are located exactly in this lattice.

§14. Composition of relations. This section considers the implications of adding a predicate functor that is frequently encountered, viz., the relative product or composition of relations. In so doing, the lattice of logics will be refined and the limits of decision located more precisely. The composition functor will be denoted $\circ$. First $\circ$ is shown to be definable in FL with inv and in FL with Inv.

THEOREM 27. *The addition of* $\circ$ *to FL with* inv *is a conservative extension. Therefore, FL with* $\circ$ *is a sublogic of FL with* inv.

PROOF. Composition can be defined in FO as follows.

$$\forall x_1 \forall x_2((R \circ Q)x_1x_2 \leftrightarrow \exists x_3(Rx_1x_3 \wedge Qx_3x_2)).$$

Therefore, it suffices to present the following translation of this definition into the language of FL with inv.

$$\forall x_1 \forall x_2((R \circ Q)x_1x_2 \leftrightarrow \exists x_3((\text{inv pad } R)x_1x_2x_3 \wedge (\text{inv } Q)x_2x_3)). \qquad \dashv$$

THEOREM 28. *The addition of* $\circ$ *to FL with* Inv *is a conservative extension. Therefore, FL with* $\circ$ *is a sublogic of FL with* Inv.

PROOF. As before, it suffices to present the following translation of the FO definition of composition into the language of FL with Inv.

$$\forall x_1 \forall x_2((R \circ Q)x_1x_2 \leftrightarrow \exists x_3((\text{Inv Inv pad Inv } R)x_1x_2x_3 \wedge (\text{Inv } Q)x_2x_3)). \qquad \dashv$$

Next, by showing that FL with $\circ$ is a *proper* sublogic of both FL with inv and FL with Inv, the distance between decidable and undecidable will be narrowed. This is the import of the next theorem.

THEOREM 29. *Let* $L = \{R, Q\}$, *where* $\text{ar}(R) = \text{ar}(Q) = 2$. *There is no sentence* ϕ *of FL with* $\circ$ *over lexicon* L *such that for all interpretations* $\mathscr{A}$ *of* L, $\mathscr{A} \models \phi$ *implies that* $Q^{\mathscr{A}}$ *is the converse of* $R^{\mathscr{A}}$.

PROOF. The proof employs Padoa's principle. Let ϕ be any sentence of FL with $\circ$ over L and let $\text{qr}(\phi) = r$. Two interpretations $\mathscr{A}$ and $\mathscr{B}$ of L are defined so that they differ only in the interpretation of Q, and are such that $Q^{\mathscr{A}}$ is the converse of $R^{\mathscr{A}}$, but $Q^{\mathscr{B}}$ is not the converse of $R^{\mathscr{B}}$. It is then shown that $\mathscr{A} \models \phi$ if and only if $\mathscr{B} \models \phi$. This is done by proving that $\mathscr{A}$ and $\mathscr{B}$ are r-isomorphic ($\mathscr{A} \cong_r \mathscr{B}$). Then by a corollary of Fraissé's Theorem (Ebbinghaus et al. [5]), it follows that for any sentence ϕ with $\text{qr}(\phi) \le r$, $\mathscr{A} \models \phi$ if and only if $\mathscr{B} \models \phi$.

$\mathscr{A}$ and $\mathscr{B}$ are defined over the domain of natural numbers as follows.

$$\begin{aligned} R^{\mathscr{A}} &:= \{(i, j) : 0 \le i \le r\}, \\ Q^{\mathscr{A}} &:= \{(i, j) : 0 \le j \le r\}, \\ R^{\mathscr{B}} &:= \{(i, j) : 0 \le i \le r\}, \\ Q^{\mathscr{B}} &:= \{(i, j) : 0 \le j < r\}. \end{aligned}$$

It is easy to see that the following sentences are true in both $\mathscr{A}$ and $\mathscr{B}$.

$$\begin{aligned} &\forall x_1 \forall x_2((R \circ R)x_1x_2 \leftrightarrow Rx_1x_2), \\ &\forall x_1 \forall x_2((Q \circ Q)x_1x_2 \leftrightarrow Qx_1x_2), \\ &\forall x_1 \forall x_2((R \circ Q)x_1x_2 \leftrightarrow (Rx_1x_2 \wedge Qx_1x_2)), \\ &\forall x_1 \forall x_2((Q \circ R)x_1x_2 \leftrightarrow (Rx_1x_2 \vee \neg Rx_1x_2)). \end{aligned}$$

Hence, in $\mathscr{A}$ and $\mathscr{B}$, satisfaction of an arbitrary formula of FL with $\circ$ coincides with satisfaction of some fluted formula. Therefore, to define partial isomorphism from $\mathscr{A}$ to $\mathscr{B}$ it suffices to consider only fluted atomic formulas.

To prove $\mathscr{A} \cong_r \mathscr{B}$ it must be shown that there exists a sequence $I_0, \dots, I_r$ of nonempty sets of partial isomorphisms from $\mathscr{A}$ to $\mathscr{B}$ with the back-and-forth property. Define I_r to contain only $p_\emptyset$, the empty isomorphism. Suppose that $p \in I_{r-n}$ is a partial isomorphism from $\mathscr{A}$ to $\mathscr{B}$ such that $p : i_k \mapsto j_k$ for $1 \le k \le n < r$. Define $q \in I_{r-n-1}$ such that q extends p as follows.

Forth: If i_{n+1} is given, define j_{n+1} such that $j_{n+1} \notin rg(p)$ and

$j_{n+1} < r$ if $i_{n+1} \leq r$, and
$j_{n+1} > r$ if $i_{n+1} > r$.

Back: If j_{n+1} is given, define i_{n+1} such that $i_{n+1} \notin \mathrm{dom}(p)$ and
$i_{n+1} < r$ if $j_{n+1} < r$, and
$i_{n+1} > r$ if $j_{n+1} \geq r$.

It is easy to verify that q preserves satisfaction of fluted atomic formulas if p does. It is easy to verify also that the definitions of $\mathscr{A}$ and $\mathscr{B}$ permit up to r such extensions of $p_\emptyset$. Thus $\mathscr{A} \cong_r \mathscr{B}$. This completes the proof of the theorem. ⊣

Undecidability of FL with $\circ$ will be established by showing that a problem known to be undecidable, viz., the word problem for semi-Thue systems (WP), can be expressed in that logic. The undecidability of the word problem for Thue systems is proved in [10]. Since any Thue system is also a semi-Thue system, undecidability of WP follows.

A semi-Thue system consists of a finite alphabet $\Sigma = \{a_1, \ldots, a_n\}$, and a finite set of rewrite rules $\mathscr{R} \subseteq (\Sigma^+ \times \Sigma^+)$, where Σ^+ is the set of finite nonempty strings over Σ. An element of $\mathscr{R}$ will be written $\lambda \to \rho$. If $\alpha = \gamma\lambda\delta$, $\beta = \gamma\rho\delta$, and $(\lambda \to \rho) \in \mathscr{R}$, then β is directly generated by α, written $\alpha \Rightarrow \beta$. As usual, $\Rightarrow^*$ is the reflexive transitive closure of $\Rightarrow$. The word problem for semi-Thue systems is

$$\alpha \Rightarrow^* \beta\ ?$$

WP can be translated into the language of FL with $\circ$ by the mapping τ defined as follows. Let the lexicon $L = \{R_1, \ldots, R_n\}$, where the R_i are binary relations.

(i) For $a_i \in \Sigma : \tau(a_i) := R_i$.
(ii) For $a_i \in \Sigma, \alpha \in \Sigma^+ : \tau(a_i\alpha) := \tau(a_i) \circ \tau(\alpha)$.
(iii) For $(\lambda \to \rho) \in \mathscr{R} : \tau(\lambda \to \rho) := \forall x_1 \forall x_2 (\tau(\lambda) x_1 x_2 \to \tau(\rho) x_1 x_2)$.
(iv) $\tau(\alpha \Rightarrow \beta) := \forall x_1 \forall x_2 (\tau(\alpha) x_1 x_2) \to \forall x_1 \forall x_2 (\tau(\beta) x_1 x_2)$.

Let θ_i denote a composition of relations. If

$$\phi = \forall x_1 \forall x_2 ((\theta_3 \circ \theta_1 \circ \theta_4) x_1 x_2)$$

$$\psi = \forall x_1 \forall x_2 ((\theta_3 \circ \theta_2 \circ \theta_4) x_1 x_2)$$

and

$$\models \forall x_1 \forall x_2 (\theta_1 x_1 x_2 \to \theta_2 x_1 x_2)$$

then

$$\models \phi \to \psi$$

by the logical Principle of Monotonicity. Thus the relation 'is directly generated by' translates to the relation 'is a logical consequence of'. Because logical implication is reflexive and transitive,

$$\alpha \Rightarrow^* \beta\ ?$$

translates to

$$\forall x_1 \forall x_2 (\tau(\alpha) x_1 x_2) \to \forall x_1 \forall x_2 (\tau(\beta) x_1 x_2) \text{ valid?}$$

or equivalently,

$$\forall x_1 \forall x_2 (\tau(\alpha) x_1 x_2) \wedge \exists x_1 \exists x_2 (\neg \tau(\beta) x_1 x_2) \text{ unsatisfiable?}$$

Since WP can be represented by sentences of FL with functor $\circ$, the following theorem has been established.

THEOREM 30. *The addition of $\circ$ to FL introduces undecidability.*

In this refined lattice, the ideal generated by the logic EFL again contains only decidable logics. But now the filter generated by the logic FL with $\circ$ comprises the undecidable logics. This ideal and filter exhaust the lattice. Thus Quine's limits of decision have been determined more precisely.

§15. **Discussion.** This section offers two observations regarding the significance of these results. First, the lattice of logics defined in Section 14 is complete with respect to the logical operations and operands normally encountered in logic. They are either already present or can be introduced as simple abbreviations. This is not to deny that the lattice could be refined further by introducing unusual logical operations that would affect the limits of decision. An example is nonassociative composition of relations. Fluted logic extended to include this operation is probably decidable. If this is the case, introduction of this operation would further narrow the distance between decidable and undecidable. Excepting such contrived operations however, Quine's limits of decision have been located precisely.

Second, the limits of decision derived here may have importance for the philosophy of language. A natural language such as English does not contain variables explicitly, but English does incorporate operations that are surrogates for some operations mediated by variables. These include

1. The passive transformation.
2. The reflexive transformation.
3. Coindexing by proper nouns.
4. Coindexing by E-type pronouns and related anaphora.

I have conjectured that there are no other surrogates for variables in *ordinary* English (Purdy [14]). If one accepts this premise, then it follows that logical discourse in ordinary English can be modeled in EFL. Thus logical discourse in ordinary English is decidable. Of course, the universal character of natural language entails that undecidable sentences can be formed. Therefore, the above argument must turn on the definition of ordinary English discourse.

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DEPARTMENT OF ELECTRICAL ENGINEERING AND COMPUTER SCIENCE
SYRACUSE UNIVERSITY
SYRACUSE, NY 13244-4100
E-mail: wcpurdy@ecs.syr.edu