Theoretical Computer Science 9 (1979) 73–81

THE TYPED λ-CALCULUS IS NOT ELEMENTARY RECURSIVE

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Communicated by A. Meyer
Received May 1977
Revised June 1978

Abstract. We prove that the problem of deciding for closed terms t_1, t_2 of the typed λ-calculus whether t_1 β-converts to t_2 is not elementary recursive.

1. Introduction

Historically, the principal interest in the typed λ-calculus is in connection with Gödel's functional ("Dialectica": see Gödel [4]) interpretation of intuitionistic arithmetic. However, since the early sixties interest has shifted to a wide variety of applications in diverse branches of logic, algebra and computer science. For example, in proof-theory (see for example, Tait [20]), in constructive logic (see for example, Lauchli [10]), in the theory of functionals (see for example, Friedman [3]), in cartesian closed categories (see for example, Mann [11]), in automatic theorem proving (see for example, Huet [8]), in the semantics of natural languages (see for example, Montague [14]), and in the semantics of programming languages (see for example, Milner [12]).

In almost all such applications there is a point at which one must ask, for closed terms t_1 and t_2, whether t_1 β-converts to t_2. We shall show that in general this question cannot be answered by a Turing machine in elementary time.

2. Type theory

The language of type theory, Ω, is the language of set-theory where each variable has a natural number type and there are two constants **0**, **1** of type 0. We require that prime formulae be "stratified", i.e., each prime formula has one of the forms $\mathbf{0} \in x^1$, $\mathbf{1} \in x^1$ and $y^n \in z^{n+1}$. Arbitrary formulae are built-up from prime ones by $\neg$, $\wedge$, and $\forall$. The intended interpretation of Ω has **0** denoting 0, **1** denoting 1 and x^n ranging over $\mathscr{D}_n$ where $\mathscr{D}_0 = \{0, 1\}$ and $\mathscr{D}_{n+1} = \text{powerset}\ (\mathscr{D}_n)$. If $A = A(x_1^{n_1} \cdots x_m^{n_m})$ and $\alpha_i \in \mathscr{D}_{n_i}$ for $1 \leq i \leq m$ we write $A[\alpha_1 \cdots \alpha_m]$ for A with $x_i^{n_i}$ denoting α_i.

The problem of deciding whether an arbitrary Ω-sentence is true is recursive. In fact there is a quantifier-elimination for Ω-sentences (see Henkin [6]). Briefly, if one extends the language by adding { , } and defines $x^k =_k y^k \Leftrightarrow \forall z^{k-1}(z^{k-1} \in x^k \leftrightarrow z^{k-1} \in y^k)$ for $k > 0$, each $\alpha \in \mathscr{D}_n$ can be defined by $\{x^{n-1}: t_1 =_{n-1} x^{n-1} \vee \cdots \vee t_l =_{n-1} x^{n-1}\}$, where $t_1 \cdots t_l$ define the elements of α and $t^{m-1} \in \{y^m: A(y^m)\} \Leftrightarrow_{\text{df}} A(t^{m-1})$, when $n > 0$. Thus $\forall x^n A(x^n) \leftrightarrow A(t_1) \wedge \cdots \wedge A(t_p)$ for $t_1 \cdots t_p$ definitions of the members of $\mathscr{D}_n$.

Proposition 1 (Fischer and Meyer, Statman). *The problem of determining if an arbitrary Ω-sentence is true cannot be solved in elementary time* (*see Meyer* [13, *p.* 479 *no.* 7]).

We shall use the above proposition together with a coding argument to prove our principal result (see below).

Let $V_0 = \emptyset$ and $V_{n+1} = \text{powerset}\ (V_n) \cup V_n$. We note in passing the following:

Corollary (for logicians). *Let $\mathscr{L}$ be the language of set theory supplemented by a constant for each V_n; then the problem of determining if an arbitrary Δ_0-sentence of $\mathscr{L}$ is true cannot be solved in elementary time.*

3. Typed λ-calculus

We consider the typed λ-calculus Λ with a single ground type 0, no constants, only power types ($\rightarrow$) and β-conversion. The reader not familiar with the typed λ-calculus should consult Hindley *et al.* [7].

We shall adopt the usual convention of ignoring α-conversion (change of bound variables) deleting type superscripts except where important and omitting parentheses selectively (association to the left). We shall also make use of the substitution prefix [/] both for substituting a term for a variable and for substituting a type for 0.

$\boldsymbol{\emptyset} =_{\text{df}} (0 \rightarrow 0) \rightarrow (0 \rightarrow 0)$ is the type of Λ-numbers. It is easy to verify that the closed (i.e., with no free variables) β-normal terms of type $\boldsymbol{\emptyset}$ are just λxx and for each n,

$$\lambda xy \underbrace{x(\cdots(x}_{n} y)\cdots).$$

Letting

$$\boldsymbol{n} =_{\text{df}} \lambda xy \underbrace{x(\cdots(x}_{n} y)\cdots),$$

if t is a closed term of type $\underbrace{\boldsymbol{\emptyset} \rightarrow (\cdots (\boldsymbol{\emptyset}}_{m} \rightarrow \boldsymbol{\emptyset})\cdots)$ for each $n_1 \cdots n_m$ there is a unique n such that $t\boldsymbol{n}_1 \cdots \boldsymbol{n}_m \beta\eta$-conv. $\boldsymbol{n}$. In this way t defines an m-ary number-theoretic function.

An extended polynomial is a polynomial built up from 0, 1, +, ·, sg and $\overline{\mathrm{sg}}$ (see Kleene [9, p. 223, no. 9 and no. 10).

Proposition 2 (Schwichtenberg [16], Statman). *The λ-definable m-ary number theoretic functions are just the extended polynomials.*

In particular, there are closed terms **+**, **·**, **sg** and $\overline{\mathbf{sg}}$ which λ-define resp. +, ·, sg, and $\overline{\mathrm{sg}}$.

There are some very short definitions of very large numbers in Λ. Set $s(0)=1$ and $s(n+1)=2^{s(n)}$ and set $a_1=\mathbf{2}$ and $a_{n+1}=([0\to 0/0]a_n)a_1$; by a computation of Church [2, p. 30] $a_n\beta$—conv. $\boldsymbol{s(n)}$.

The λ-definability of the extended polynomials allows us to code the Boolean operations into Λ. The short definitions of large numbers allow us to iterate λ-definable operations a very large (but fixed) number of times. These are precisely the conditions that permit us to simulate the quantifier-elimination for Ω by β-conversion.

The problem of determining for arbitrary closed terms t_1, t_2 of the same type whether t_1 β-conv. t_2 is decidable. By analyzing the normal form algorithm (see [7, p. 73]) it is easy to see that the problem can be solved in $\mathscr{E}^4$ time (here, $\mathscr{E}^4$ is the 5th level of the Grzegorczyk hierarchy; see Grzegorczyk [5]). Thus with respect to this crude classification our lower bound ($\mathscr{E}^3$ = elementary) is best possible.

4. Translation of Ω into Λ

We define recursively $\mathbf{N}_0=\emptyset$ and $\mathbf{N}_{n+1}=\mathbf{N}_n\to\emptyset$. The following definitions are central to what follows.

(1) $e_0 =_{\mathrm{df}} \lambda xy\,\mathbf{+}(\mathbf{\cdot}(\mathbf{sg}\,x)(\overline{\mathbf{sg}}\,y))(\mathbf{\cdot}(\mathbf{sg}\,y)(\overline{\mathbf{sg}}\,x))$.

For all n, m, $(e_0\,\boldsymbol{nm})\beta$-conv. $\boldsymbol{0} \Leftrightarrow n=0=m$ or $0<n$, m, and $(e\,\boldsymbol{nm})$ β-conv. $\boldsymbol{0}$ or $\boldsymbol{1}$. e_0 has type $\boldsymbol{\emptyset}\to(\boldsymbol{\emptyset}\to\boldsymbol{\emptyset})$.

(2) $\forall_0 =_{\mathrm{df}} \lambda h\,\mathbf{+}(h\boldsymbol{0})(h\boldsymbol{1})$,

$\forall_0$ has type $\mathbf{N}_1\to\boldsymbol{\emptyset}$.

(3) $C =_{\mathrm{df}} \lambda g\,\mathbf{+}(g(\lambda x\boldsymbol{1}))(g(\lambda xx))$,

C has type $\mathbf{N}_2\to\boldsymbol{\emptyset}$.

(4) $p_{n+1}(x,z) =_{\mathrm{df}} C(\lambda f(\forall_n(\lambda w(z(\lambda y\cdot(f(e_nwy);(xy)))))))$.

Here x has type $\mathbf{N}_n\to\boldsymbol{\emptyset}$, y has type $\mathbf{N}_n$, w has type $\mathbf{N}_n$, z has type $\mathbf{N}_n\to\boldsymbol{\emptyset}$, and f has type $\tilde{\boldsymbol{\emptyset}}\to\boldsymbol{\emptyset}$. We have $p_{n+1}(x,z)\beta$-conv. $\mathbf{+}(\forall_n(\lambda w(z(\lambda y\cdot(\lambda x\boldsymbol{1})(e_nwy)(xy)))))(\forall_n(\lambda w(z(\lambda y\cdot((\lambda xx)e_nwy)(xy)))))$. "$C$" stands for "choice (for f)". "p_{n+1}" stands for "prime constituent for building definitions of type $n+1$ objects".

(5) $e_{n+1} =_{\mathrm{df}} \lambda xy\forall_n(\lambda z(e_0(xz)(yz)))$,

e_{n+1}has type $\mathbf{N}_{n+1}\to(\mathbf{N}_{n+1}\to\boldsymbol{\emptyset})$.

(6) $\forall_{n+1} =_{\mathrm{df}} \lambda y(((([\mathbf{N}_{n+2}/0]a_{n+1})(\lambda zxp_{n+1}(x,z))y)\lambda w\mathbf{1})$,

$\forall_{n+1}$ has type $\mathbf{N}_{n+2}\to\boldsymbol{\emptyset}$.

We now define the translation $*$:

$$\begin{aligned}
&\mathbf{0}^* = \mathbf{0}\\
&\mathbf{1}^* = \mathbf{1}\\
&(x^n)^* = x^{\mathbf{N}_n}\\
&(t_1 \in t_2)^* = \mathrm{sg}(t_2^* t_1^*)\\
&(A \wedge B)^* = \mathrm{sg}(+A^*B^*)\\
&(\neg A)^* = \overline{\mathrm{sg}}A^*\\
&(\forall x^n A)^* = \mathrm{sg}(\forall_n \lambda x^{\mathbf{N}_n} A^*).
\end{aligned}$$

We shall show that for Ω-sentences A, A is true $\Leftrightarrow A^*\beta$-conv. $\mathbf{0}$ and $\neg A$ is true $\Leftrightarrow A^*\beta$-conv. $\mathbf{1}$. The key idea is that the β-reductions of $\forall_n$ simulate the quantifier-elimination for Ω-sentences. Here $+$ plays the role of $\wedge$ so $\cdot$ plays the role of $\vee$. In addition, e_n plays the role of equality between type n objects. This motivates the definitions below.

5. Verification that the translation is correct

We define the notion of a definition of an object of type n as follows.

(a) $\mathrm{def}^0(0) = \{\mathbf{0}\}$,

(b) $\mathrm{def}^0(1) = \{\mathbf{1}\}$,

(c) if $\alpha \in \mathscr{D}_{n+1}$, then $\mathrm{def}^{n+1}(\alpha) = \{\lambda y \cdot r_1(\cdots(\cdot\, r_{s(n+1)}((\lambda w\mathbf{1})y))\cdots)$: y, w have type $\mathbf{N}_n$, $r_i = \mathbf{1}$ or $r_i = e_n ty$ for $t \in \mathrm{def}^n(\beta)$ and $\beta \in \alpha$, for each $\beta \in \alpha$ for some $t \in \mathrm{def}^n(\beta)$ there is some i s.t. $r_i = e_n ty\}$. We set $\mathrm{def}_n = \bigcup_{\alpha\in\mathscr{D}_n} \mathrm{def}^n(\alpha)$.

Below we define sets N_n, orders $\underset{n}{\prec}$ and functions $d_n : N_n \to \mathrm{def}_n$. The members of N_n code various processes of constructing members of def_n and for $\eta \in N_n$, $d_n(\eta)$ is the member of def_n constructed by the process coded by η. The order $\underset{n}{\prec}$ describes a fixed process for generating the processes coded by members of N_n. First some set theoretic preliminaries.

If X and Y are sets then $X \otimes Y = \{(x, y): x \in X$ and $y \in Y\}$ and ${}^XY = \{\eta: \eta: X \to Y\}$. $\pi_1 : X \otimes Y \to X$ is defined by $\pi_1(x, y) = x$ and $\pi_2 : X \otimes Y \to Y$ is defined by $\pi_2(x, y) = y$. If ρ is an ordering of X and γ an ordering of Y, then $\delta = \rho \otimes \gamma$ is the ordering of $Z = X \otimes Y$ defined by $z_1\delta z_2$ if $\pi_2 z_1 \gamma \pi_2 z_2$ or $\pi_2 z_1 = \pi_2 z_2$ and $\pi_1 z_1 \rho \pi_1 z_2$. $[1, n] = \{k: 1 \le k \le n\}$. ρ^n is the ordering of ${}^{[1,n]}X$ defined by $\eta_1 \rho^n \eta_2$ if $\eta_1 \ne \eta_2$ and for $m = \max\{k: 1 \le k \le n$ and $\eta_1(k) \ne \eta_2(k)\}$, $\eta_1(m)\rho\eta_2(m)$.

Define for all n and $1 \le m \le s(n)$, N_n^m as follows; $N_0^1 = \{0, 1\}$ and $N_{n+1}^m = {}^{[1,m]}(N_n^{s(n)} \otimes N_0^1)$. Set $N_n = N_n^{s(n)}$ and define $\underset{n}{\overset{m}{\prec}}$ by $\underset{0}{\overset{1}{\prec}}$ is the natural order on N_0^1, $\underset{n+1}{\overset{m}{\prec}} = (\underset{n}{\overset{s(n)}{\prec}} \otimes \underset{0}{\overset{1}{\prec}})^m$. Set $\underset{n}{\prec} = \underset{n}{\overset{s(n)}{\prec}}$.

Define d_n^m or N_n^m as follows: $d_0^1(0)=\boldsymbol{0}$, $d_0^1(1)=\boldsymbol{1}$ and for $\eta\in N_{n+1}^m$, $d_{n+1}^m(\eta)=\lambda y\cdot r_1(\cdots(\cdot r_m(xy))\cdots)$ where $r_i=\boldsymbol{1}$ if $\pi_2\eta(i)=0$ and $r_i=e_n([\lambda z\boldsymbol{1}/x]d_n^{s(n)}(\pi_1\eta(i)))y$ if $\pi_2\eta(i)=1$. Set $d_n=[\lambda z\boldsymbol{1}/x]d_n^{s(n)}$.

Now suppose that X is a set of occurrences of terms of type σ ordered by ρ, $|X|$ is a power of 2 and $X=X_1\cup X_2$ is a partition of X with $|X_1|=|X_2|$ and $t_1\in X_1$ and $t_2\in X_2\Rightarrow t_1\rho t_2$. Let z be a variable of type $\sigma\to\boldsymbol{\emptyset}$; we define the term $\sum_{t\in X}zt$ recursively by

$$\sum_{t\in X}zt=+\Big(\sum_{t_1\in X_1}zt_1\Big)\Big(\sum_{t_2\in X_2}zt_2\Big).$$

We shall prove the

Proposition 3. $\forall_n\beta$-conv. $\lambda y\sum_{\eta\in N_n}yd_n(\eta)$.

Think of

$$\sum_{\eta\in N_{n+1}}zd_{n+1}^m(\eta)$$

as a symmetric binary tree (branching upwards) with a member of N_{n+1}^m at each leaf. The order of the members from left to right is $\underset{n+1}{\overset{m}{\prec}}$. If we think of a member of N_{n+1}^m as a sequence of pairs then a member of N_{n+1}^{m+1} can be obtained by adding a member of N_{n+1}^1 at the end. Moreover if $\xi\in N_{n+1}^m$ and $\eta\in N_{n+1}^1$, then $[d_{n+1}^1(\eta)/x]d_{n+1}^m(\xi)\beta$-conv. $d_{n+1}^{m+1}(\widehat{\xi\eta})$. In addition if $\xi_1,\xi_2\in N_{n+1}^m$ and $\eta_1,\eta_2\in N_{n+1}^1$, then $\widehat{\xi_1\eta_1}\underset{n+1}{\overset{m+1}{\prec}}\widehat{\xi_2\eta_2}\Leftrightarrow\eta_1\underset{n+1}{\overset{1}{\prec}}\eta_2$ or $\eta_1=\eta_2$ and $\xi_1\underset{n+1}{\overset{m}{\prec}}\xi_2$. From these remarks it is easy to see the

Fact. $\displaystyle\sum_{\eta\in N_{n+1}}\Big(\sum_{\xi\in N_{n+1}^k}z[d_{n+1}^1(\eta)/x]d_n^m(\xi)\Big)\beta\text{-conv.}\sum_{\eta\in N_{n+1}^{k+1}}zd_{n+1}^{k+1}(\eta)$.

The members of N_1^1 are $(0,0)(1,0)(0,1)(1,1)$ in the $\underset{1}{\overset{1}{\prec}}$ ordering. We have $\lambda zxp_1(x,z)\beta$-conv. $\lambda zxC\lambda f(+(z(\lambda y\cdot f(e_0\boldsymbol{0}y)(xy)))(z(\lambda y\cdot f(e_0\boldsymbol{1}y)(xy))))$ β-conv. $\lambda zx+(+(z(\lambda y\cdot\boldsymbol{1}(xy)))(z(\lambda y\cdot\boldsymbol{1}(xy))))(+(z(\lambda y\cdot(e_0\boldsymbol{0}y)(xy))))(z(\lambda y\cdot(e_0\boldsymbol{1}y)\cdot(xy))))$. The last term is $\lambda zx\sum_{\eta\in N_1^1}zd_1^1(\eta)$ since $d_1^1((0,0))=\boldsymbol{1}$, $d_1^1((1,0))=\boldsymbol{1}$, $d_1^1((0,1))=e_0\boldsymbol{0}y$ and $d_1^1((1,1))=e_0\boldsymbol{1}y$. More generally we have the

Lemma. *For* $1\leqslant m\leqslant s(n+1)([\mathbf{N}_{n+2}/0]\boldsymbol{m})\lambda zx\rho_{n+1}(x,z)\beta$-conv.

$$\lambda zx\sum_{\eta\in N_{n+1}^m}zd_{n+1}^m(\eta).$$

Proof. By induction on (n,m) ordered lexicographically.

Basis: $n=0$.

Case: $m=1$. $([\mathbf{N}_2/0]\mathbf{1})\lambda zxp_1(x,z)\beta$-conv. $\lambda zxp_1(x,z)$ so by the above computation $([\mathbf{N}_2/0]\mathbf{1})\lambda zxp_1(x,z)\beta$-conv. $\lambda zx\sum_{\eta\in N_1^1} zd_1^1(\eta)$.

Case: $m=2$. $([\mathbf{N}_2/0]\mathbf{2})\lambda zxp_1(x,z)\beta$-conv. $\lambda w\lambda zxp_1(x,z)(\lambda zxp_1(x,z)w)\beta$-conv. $\lambda xw\sum_{\eta\in N_1^1}(\lambda yp_1(y,w))d_1^1(\eta)$ by case $m=1$ β-conv.

$$\lambda zx\sum_{\eta\in N_1^1}\Big(\sum_{\xi\in N_1^1} z[d_1^1(\eta)/x]d_1^1(\xi)\Big)\beta\text{-conv. }\lambda zx\sum_{\eta\in N_1^2} zd_1^2(\eta)\text{ by the fact.}$$

Induction step: $n>0$.

Case: $m=1$. $([\mathbf{N}_{n+2}/0]\mathbf{1})\lambda zxp_{n+1}(x,z)\beta$-conv. $\lambda zxp_{n+1}(x,z)\beta$-conv. $\lambda zxC\lambda f$ $\sum_{\eta\in N_n}(\lambda wz(\lambda y\cdot(f(e_nwy))(xy)))d_n(\eta)$ by induction hypothesis β-conv.

$$\lambda zx+\Big(\sum_{\eta\in N_n} z(\lambda y\cdot\mathbf{1}(xy))\Big)\Big(\sum_{\eta\in N_n} z(\lambda y\cdot(e_nd_n(\eta)y)(xy))\Big)=\lambda zx\sum_{\eta\in N_{n+1}^1} zd_{n+1}^1(\eta).$$

Case: $m=k+1$. $([\mathbf{N}_{n+2}/0]\boldsymbol{m})\lambda zxp_{n+1}(x,z)\beta$-conv. $\lambda w_1\lambda zxp_{n+1}(x,z)$ $([\mathbf{N}_{n+2}/0]\boldsymbol{k})\lambda zxp_{n+1}(x,z)w_1)\beta$-conv. $\lambda w_1\lambda zxp_{n+1}(x,z)(\lambda x\sum_{\eta\in N_{n+1}^k} w_1d_{n+1}^k(\eta))$ by induction hypothesis β-conv.

$$\lambda zw_2+\Big(\sum_{\eta\in N_{n+1}^1}\Big(\lambda x\sum_{\xi\in N_{n+1}^k} zd_{n+1}^k(\xi)\Big)(\lambda y\cdot\mathbf{1}(w_2y))\Big)$$
$$\Big(\sum_{\eta\in N_{n+1}^1}\Big(\lambda x\sum_{\xi\in N_{n+1}^k} zd_{n+1}^k(\xi)\Big)(\lambda y\cdot(e_nd_{n+1}^1(\eta)y)(w_2y))\Big)$$

by case $m=1$ β-conv. $\lambda zx\sum_{\eta\in N_{n+1}^m} zd_{n+1}^m(\eta)$ by the fact.

Proof of Proposition 3.

$$\mathbf{V}_{n+1}\beta\text{-conv. }\lambda y\Big(\lambda zx\sum_{\eta\in N_{n+1}} zd_{n+1}^{s(n+1)}(\eta)\Big)y\lambda w\mathbf{1}$$

by the

$$\text{lemma }\beta\text{-conv. }\lambda y\sum_{\eta\in N_{n+1}} y[\lambda w\mathbf{1}/x]d_{n+1}^{s(n+1)}(\eta)=\lambda y\sum_{\eta\in N_{n+1}} yd_{n+1}(\eta).$$

The proposiition would be useless without the following easy

Observation. *If $\eta\in N_n$, then $d_n(\eta)\in\mathrm{def}^n$ and for each $\alpha\in\mathscr{D}_n$ there is an $\eta\in N_n$ such that $d_n(\eta)\in\mathrm{def}^n(\alpha)$.*

The members of def_1 are

$$\lambda y\cdot\mathbf{1}(\cdot\mathbf{1}((\lambda w1)y)),\ \lambda y\cdot\mathbf{1}(\cdot(e_0\mathbf{0}y)(\lambda w1)y)),$$
$$\lambda y\cdot\mathbf{1}(\cdot(e_0\mathbf{1}y)((\lambda w1)y)),\ \lambda y\cdot(e_0\mathbf{0}y)(\cdot\mathbf{1}((\lambda w1)y)),$$
$$\lambda y\cdot(e_0\mathbf{1}y)(\cdot\mathbf{1}((\lambda w1)y)),$$
$$\lambda y\cdot(e_0\mathbf{0}y)(\cdot(e_0\mathbf{0}y)((\lambda w1)y)),$$
$$\lambda y\cdot(e_0\mathbf{0}y)(\cdot(e_0\mathbf{1}y)((\lambda w1)y)),\ \lambda y\cdot(e_0\mathbf{1}y)(\cdot(e_0\mathbf{0}y)((\lambda w1)y))$$

and

$$\lambda y\cdot(e_0\boldsymbol{1}y)(\cdot(e_0\boldsymbol{1}y)((\lambda w1)y))$$

so if $\gamma\in\mathscr{D}_1\alpha, \beta\in\mathscr{D}_0 t_1\in\operatorname{def}^0(\beta) t_2\in\operatorname{def}^0(\alpha)$ and $t_3\in\operatorname{def}^1(\gamma)$, then $\beta\in\gamma\Leftrightarrow(t_3t_1\beta$-conv. $\boldsymbol{0})$ and $\alpha=\beta\Leftrightarrow(e_0t_1t_2\beta$-conv. $\boldsymbol{0})$. More generally we have the

Proposition 4. *Suppose* $\alpha,\beta\in\mathscr{D}_n$, $\gamma\in\mathscr{D}_{n+1}$, $t_1\in\operatorname{def}^n(\beta)$, $t_2\in\operatorname{def}^n(\alpha)$, *and* $t_3\in\operatorname{def}^{n+1}(\gamma)$, *then*

(a) $\beta=\alpha\Leftrightarrow(e_nt_1t_2\beta$-conv. $\boldsymbol{0})$, *and*

(b) $\beta\in\gamma\Leftrightarrow(t_3t_1\beta$-conv. $\boldsymbol{0})$.

Proof. By induction on n.

Basis: $n=0$. This is the preceding remark.

Induction step: $n=m+1$.

(a) We have $e_nt_1t_2\beta$-conv. $\sum_{\eta\in N_m}e_0(t_1d_m(\eta))(t_2d_m(\eta))$ by the previous proposition. If $\beta=\alpha$ and $d_m(\eta)\in\operatorname{def}^n(\beta)$ by hyp. ind. on (b) $e_0(t_1d_m(\eta))(t_2(d_n(\eta)))$ β-conv. $\boldsymbol{0}$ and if $d_m(\eta)\notin\operatorname{def}^n(\beta)$ by hyp. ind. on (b) $t_1d_m(\eta)$, $t_2d_m(\eta)\neg(\beta$-conv.$)$ $\boldsymbol{0}$ so $e_0(t_1d_m(\beta))(t_2d_n(\beta))$ β-conv. $\boldsymbol{0}$. If $\beta\neq\alpha$ w.l.o.g. assume $\delta\in\beta$ and $\delta\notin\alpha$. By the above observation there is an $\eta\in N_m$ such that $d_m(\eta)\in\operatorname{def}^m(\delta)$. By hyp. ind. on (b) $t_1d_m(\eta)$ β-conv. $\boldsymbol{0}$ and $t_2d_m(\eta)\neg(\beta$-conv.$)$ $\boldsymbol{0}$ so $e_0(t_1d_m(\eta))(t_2d_m(\eta))$ β-conv. $\boldsymbol{1}$. Thus in either case we have (a).

(b). Let $t_3=\lambda y\cdot r_1(\cdots(\cdot r_{s(n+1)}((\lambda w\boldsymbol{1})y))\cdots)$. If $\beta\in\gamma$, then for some $t_4\in\operatorname{def}^n(\beta)$ and some i, $r_i=e_nt_4y$. By case (a) $e_nt_4t_1\beta$-conv. $\boldsymbol{0}$ so t_3t_1 β-conv. $\boldsymbol{0}$. If $\beta\notin\gamma$, then for each $r_i=e_nt_4y$ $t_4\notin\operatorname{def}^n(\beta)$ so by case (a) $e_nt_4t_1\neg(\beta$-conv.$)\boldsymbol{0}$. Thus in either case we have (b).

The two propositions taken together tell us that our definitions of e_n and $\forall_n$ work correctly. This is summarized in the following

Theorem 1. *Suppose* $A=A(x_1^{n_1},\ldots,x_m^{n_m})$ *is an Ω-formula,* $\alpha_i\in\mathscr{D}_{n_i}$ *and* $t_i\in\operatorname{def}^{n_i}(\alpha_i)$ *for* $1\leq i\leq m$, *then* $A[\alpha_1,\ldots,\alpha_m]$ *is true* $\Leftrightarrow(\lambda x_1^{\mathrm{N}_n},\ldots,x_m^{\mathrm{N}_{n_m}}A^*)t_1\cdots t_m\beta$-conv. $\boldsymbol{0}$.

Proof. By induction on A.

Basis: A is atomic. This is just the previous proposition case (b)

Induction step. Cases: $A=B\wedge C$, $A=\neg B$. Immediate by hyp. ind.

Case: $A=\forall x^nB$. We have $A[\alpha_1,\ldots,\alpha_m]\Leftrightarrow\forall\beta\in\mathscr{D}_nB[\alpha_1,\ldots,\alpha_m\beta]\Leftrightarrow$ $\forall t\in\operatorname{def}_n(\lambda x_1^{\mathrm{N}_{n_1}}\cdots x_m^{\mathrm{N}_{n_m}}x^{\mathrm{N}_n}B^*)t_1\cdots t_mt$ β-conv. $\boldsymbol{0}$ by hyp. ind. $\Leftrightarrow\mathrm{sg}\sum_{\eta\in N_n}$ $(\lambda x_1^{\mathrm{N}_{n_1}}\cdots x_m^{\mathrm{N}_{n_m}}x^{\mathrm{N}_n}B^*)t_1\cdots t_md_n(\eta)$ β-conv. $\boldsymbol{0}$ by the observation

$$(\lambda x_1^{\mathrm{N}_{n_1}}\cdots x^{\mathrm{N}_{n_m}}\mathrm{sg}\sum_{\eta\in N_m}(\lambda x^{\mathrm{N}_n}B^*)d_m(\eta))t_1\cdots t_m\beta\text{-conv. }\boldsymbol{0}$$

$$(\lambda x_1^{\mathrm{N}_{n_1}}\cdots x_m^{\mathrm{N}_{n_m}}A^*)t_1\cdots t_m\beta\text{-conv. }\boldsymbol{0}.$$

Corollary. *For each type $\sigma \neq 0 \to 0$ which is the type of a closed term there is a closed term t^σ such that the problem of determining for arbitrary closed terms r of type σ whether $r\beta$-conv. t^σ, $r\beta$-red t^σ, or t^σ is the β-normal form of r cannot be solved in elementary time.* ($0 \to 0$ *is anomalous because it contains only one β-normal closed term, viz. λxx.*)

Proof. The above theorem establishes the corollary for $\sigma = \boldsymbol{0}$ with $t^\sigma = \boldsymbol{0}$. Note that for Ω-sentences A, $\neg A$ is true $\Leftrightarrow A^*\beta$-conv. $\boldsymbol{1}$.

Case: $\sigma = \underset{m}{0 \to (\cdots(0 \to 0)\cdots)}$ for $m > 1$. We have for closed r of type $\boldsymbol{0}$, $r\beta$-conv. $\boldsymbol{0} \Leftrightarrow r(\lambda v_0^0 v_1^0)v_2^0\beta$-conv. $v_2^2 \Leftrightarrow \lambda v_1^0 \cdots v_m r(\lambda v_0^0 v_1^0)v_2^0\beta$-conv. $\lambda v_1^0 \cdots v_m^0 v_2^0$ so we can set $t^\sigma = \lambda v_1^0 \cdots v_m^0 v_2^0$.

Case: otherwise. We say that σ contains a splinter if there is a closed term t of type σ and a closed term s of type $\sigma \to \sigma$ such that the β-normal forms of $t, st, \ldots, s(\cdots(st)\cdots), \ldots$ are all distinct. It is easy to prove that σ contains a splinter $\Leftrightarrow \sigma$ contains a closed term and σ does not have the form $0 \to (\cdots(0 \to 0)\cdots)$. Suppose σ contains a splinter generated by t and s; we have for closed r of type $\boldsymbol{0}$, $r\beta$-conv. $\boldsymbol{0} \Leftrightarrow [\sigma/0]r\beta$-conv. $[\sigma/0]\boldsymbol{0} \Leftrightarrow ([\sigma/0]r)st\,\beta$-conv. t so we can set $t^\sigma = t$.

6. Extensions and refinements

By a consistent extension Λ^+ of Λ we mean an extension of Λ with a model whose ground domain has ≥ 2 elements (note that Λ^+ need not be closed under the inductive definition of β-conversion and the model need not be extensional). If Λ^+ is an extension of Λ and $\Lambda^+ \vdash \boldsymbol{0} = \boldsymbol{1}$, then $\Lambda^+ \vdash \mathrm{v}_1^0 = \mathrm{v}_2^0$ so Λ^+ is not consistent. Thus if Λ^+ is a consistent extension of Λ, for Ω-sentences A, A is true $\Leftrightarrow \Lambda^+ \vdash A^* = \boldsymbol{0}$. More generally we have the

Theorem 2. *If σ is the type of a closed term and σ contains no positive occurrence of a subtype of the form $\sigma_1 \to (\sigma_2 \to \sigma_3)$ (see Prawitz [15, p. 43 and read $\to$ for $\supset$]), then there is a closed term t^σ of type σ such that the problem of determining for an arbitrary closed term r of type σ whether r β-conv. t^σ, r β-red t^σ or t^σ is the β-normal form of r cannot be solved in elementary time.*

Our proof of this theorem uses the model theory of Statman [18] and is proved there.

The rank of a type is defined as follows: $\mathrm{rnk}(0) = 0$ and $\mathrm{rnk}(\sigma \to \tau) = \max\{\mathrm{rnk}(\sigma) + 1, \mathrm{rnk}(\tau)\}$. Set $T_n = \{t \in \Lambda : \text{each subterm of } t \text{ has type with rnk} \leq n\}$. It is easy to see (by analysis of the normal form algorithm) that the problem for arbitrary closed terms $t_1, t_2 \in T_n$ of the same type of whether $t_1\beta$-conv. t_2 can be solved in elementary time. By modifying the above construction (using Meyer's result for the monadic predicate calculus instead of Ω; see Meyer [13, p. 478]) it is easy to find an n such that

Proposition 5. *The problem for arbitrary closed $t \in T_n$ of whether t β-conv. **0** cannot be solved in polynomial time.*

If F is a finite set of types let $T_F = \{t \in \Lambda$: each subterm of t has type $\in F\}$. By modifying the above construction (using the Meyer–Stockmeyer result for B_ω instead of Ω; see Stockmeyer [19, p. 12]) it is easy to find an F such that

Proposition 6. *The problem for arbitrary closed $t \in T_F$ of whether t β-conv. **0** is polynomial-space hard.*

References

[1] A.V. Aho, J.E. Hopcroft and J.D. Ullman, *The Design and Analysis of Computer Algorithms* (*Addison-Wesley, Reading, MA*, 1974).

[2] *A. Church, The Calculi of Lambda-Conversion*, Annals of Math. Studies No. 6 (Princeton, 1941).

[3] H. Friedman, Equality between functionals, in: R. Parikh, ed., *Lecture Notes in Math. 453* (Springer-Verlag, Berlin, 1974).

[4] K. Gödel, On an extension of finitary mathematics which has not yet been used, *Dialectica* **12** (1958).

[5] A. Grzegorczyk, Some classes of recursive functions, *Rozprawy Mat.* **4** (1953) 46.

[6] L. Henkin, A theory of propositional types, *Fund. Math.* **52** (1963) 323–344.

[7] J.R. Hindley, B. Lercher and J.P. Seldin, *Introduction to Combinatory Logic*, London Math. Soc. Lecture Note No. 7 (Cambridge University Press, 1972).

[8] G. Huet, A unification algorithm for typed λ-calculus, Theoret. Comput. Sci. **1** (1975) 27–57.

[9] S.L. Kleene, Introduction to Mathematics (Van Nostrand, New York, 1952).

[10] H. Lauchli, An abstract notion of realizability for which the intuitionistic predicate calculus is complete, in: A. Kino, J. Myhill and P.E. Vesley, eds., *Intuitionism and Proof Theory* (North-Holland, Amsterdam, 1968).

[11] C. Mann, The connection between equivalence of proofs and cartesian closed categories, *Proc. London Math. Soc.* (3) **31** (3) (1975).

[12] R. Milner, Fully abstract models of typed λ-calculi, *Theoret. Comput. Sci.* **4** (1977) 1–22.

[13] A.R. Meyer, The inherent computational complexity of theories of ordered sets, in: *Proc. Int. Congr. of Math.* (C.M.C., 1974).

[14] R. Montague, The proper treatment of quantification in ordinary English, in: J. Hintikka, J. Moravcsik and P. Suppes, eds., *Approaches to Natural Language* (Reidel, 1973).

[15] D. Prawitz, *Natural Deduction* (Almqvist and Wiksell, 1965).

[16] H. Schwichtenberg, Definierbare Functionem im λ-Kalkul mit Typen. *Arch. Math. Logic* **17** (3–4) (1975–76).

[17] R. Statman, Intuitionistic propositional logic is polynomial-space complete, *Theoret. Comput. Sci.* **9** (1979) 67–72.

[18] R. Statman, Completeness, invariance and λ-definability, *J. Symbolic Logic*, to appear.

[19] L. V. Stockmeyer, The polynomial-time hierarchy, *Theoret. Comput. Sci.* **3** (1976) 1–22.

[20] W.W. Tait, A realizability interpretation of the theory of species, in: R. Parikh, ed., *Lecture Notes in Math.* 453 (Springer-Verlag, Berlin, 1974).