JASS'07 - Polynomials: Their Power and How to Use Them

Differential Polynomials

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Abstract

This article gives an brief introduction into differential polynomials, ideals and manifolds and their correlations. Some examples for bad behaviour (in comparison to algebraic polynomials) are given.

Contents

1 Algebraic Aspects

1.1 Definitions

Definition 1.1 (Differential Ring) *A differential ring R is a ring with differential operators $\Delta = \{\delta_1, \dots, \delta_m\}$ and for all i, j:*

- $\delta_i(ab) = (\delta_i a)b + a(\delta_i b)$
- $\delta_i(a + b) = \delta_i a + \delta_i b$
- $\delta_i \delta_j = \delta_j \delta_i$

$\Theta = \Delta^*$ *is called the free abelian monoid of derivations.*

Example 1.2
- *Let R be an arbitrary ring and $\delta(x) = 0$ for $x \in R$. Then R is a differential ring with $\Delta = \{\delta\}$.*
- *Consider the polynomials over a ring R with variables θx_i for $\theta \in \Theta$ and $i \in \{1, \dots, n\}$. They form an differential ring denoted by $R\{X\} = R\{x_1, \dots, x_n\}$.*
  The degree $\deg(f)$ for a monomial $f = \prod_{i=1}^{s} v_i^{\alpha_i}$ is defined as in the algebraic case: $\deg(f) = \sum_{i=1}^{s} \alpha_i$ where $v_i = \theta_i x_i$ with $\theta_i \in \Theta, i \in \{1, \dots, n\}$
  On the other hand one defines the order of a variable v to be the number of differentiations contained in v, so $ord(\delta^\alpha x) = \sum_{i=1}^{n} \alpha_i$ with multiindex α.
  To combine these two, one defines the weight $wt(f) = \sum_{i=1}^{r} \beta_i ord(v_i)$.

Definition 1.3 (Differential Ideal) *An differential ideal I is a ideal of R with $\forall \delta \in \Delta : \delta I \subset I$.*
The differential ideal generated by a set G is denoted by $[G]$.

Example 1.4 (Differential Ideal) *The following polynomials are members of the differential ideal I generated by x^2 over $F\{x\}$ with $\Delta = \{d\}$ ($x_{(k)} := d^k x$):*
(You obtain them by differentiating x^2 (x^p for the general case) and then cancelling terms by linear combinations.)

1. fx^2 *for* $f \in F\{x\}$
2. $fx_{(1)}x$
3. $f(x_{(2)}x + (x_{(1)})^2)$ *and therefore* $f(x_{(1)})^2 x$
4. $f(2x_{(1)}x_{(2)}x + (x_{(1)})^3)$ *and therefore* $f(x_{(1)})^3$
5. $f(x_{(k)})^s$ *for some* $s > 1$
6. $\dots$

1.2 Nonrecursive Ideals

Example 1.5 *Consider over $\mathbb{Z}\{x\}$ with $\Delta = \{d\}$ the functions*
$f_i = (d^i x)^2$ for $i \geq 0$ and $I_k = [f_0, \dots, f_k]$ Then I claim: $I_0 \subsetneq I_1 \subsetneq \dots$

Proof First note that $\deg(f_i) = 2, \mathrm{wt}(f_i) = 2i$.
If we differentiate a monomial, all resulting terms have the same degree as the original monomial. Therefore $d^j f_i$ is homogeneous of degree 2. The weight of the terms increases by one per differentiation and therefore $d^j f_i$ is isobaric of weight $2i + j$.
Now assume $f_n \in I_{n-1}$, this means $f_n = \sum_{i=0}^{n-1} \sum_{j=0}^{k_i} \alpha_{i,j} d^j(f_i)$.
If $deg(\alpha_{i,j}) \geq 1$ for some i, j, these terms must cancel because $\deg(f_n) = 2$ and the derivatives of f_i are homogeneous. So we can assume that $\alpha_{i,j} \in \mathbb{Z}$.
Analogously we can assume $\alpha_{i,j} = 0$ for $j \neq 2n - 2i$ because $\mathrm{wt}(f_n) = 2n$ and $d^j f_i$ are isobaric

of weight $2i + j$.
So the equation simplifies to $f_n = c_0 d^{2n} f_0 + c_1 d^{(2n-2)} f_1 + \ldots + c_{n-1} d^2 f_{n-1}$ for $c_i \in \mathbb{Z}$.
$d^{2n} f_0$ contains the monomial $x_{(2n)} x$. No other term contains an x (that is not derivated). So c_0 must be zero. For analogous reasons also c_i must be 0. But $f_n \neq 0$, so we have a contradiction. $\square$

Example 1.6 *Let $S \subset \mathbb{N}_0$ and $I_S = [\{f_i : i \in S\}]$. Then*

$$f_i \in I_S \Leftrightarrow i \in S$$

. (This follows from a proof similar to the one above.) So for a nonrecursive set $S \subset \mathbb{N}_0$ there is no algorithm to decide if a given differential polynomial g is in I_S.

This means that we have to consider nice ideals if we want to do calculaions, e.g. recursively generated or even finitely generated ideals.

1.3 Reduction

Definition 1.7 (Ranking) *Let $<$ be a total ordering on the set ΘX of differential variables which fulfills the following properties:*

- $v < w \Rightarrow \theta v < \theta w$ *for all* $v, w \in \Theta X, \theta \in \Theta$
- $v \leq \theta v$ *for* $v \in \Theta X$

Then $<$ is called ranking of ΘX.

Now let be $X = \{x_1, \ldots, x_n\}$ with $x_1 < \ldots < x_n$.

Example 1.8 (Lexicographic Ranking on ΘX) *Consider a monomial ordering $<$ on the differential operators Θ. Then the lexicographic ordering is given by $\theta x_i < \eta x_k$ iff $i < k$ or $i = k$ and $\theta < \eta$.*

Example 1.9 (Derivation Ranking on ΘX) *Consider a monomial ordering $<$ on the differential operators Θ. Then the derivation ordering is given by $\theta x_i < \eta x_k$ iff $\theta < \eta$ or $\theta = \eta$ and $i < k$.*

For $|X| = |\Delta| = 1$ there is only one ranking: $x_{(i)} < x_{(i+1)}$

Definition 1.10 (Admissible Ordering) *Let $<$ be a total ordering on the set M of monomials of $F\{X\}$. Then $<$ is called admissible iff*

1. *The restriction of $<$ to ΘX is a ranking.*
2. $1 \leq f$ *for all* $f \in M$
3. $f < g \Rightarrow hf < hg$ *for all* $f, g, h \in M$

Let be $f = \prod_{i=1}^{r} v_i^{\alpha_i}$ with $v_1 > \ldots > v_r$ and $g = \prod_{i=1}^{s} w_i^{\beta_i}$ with $w_1 > \ldots > w_s$.

Example 1.11 (Lexicographic Ordering on M) *Given an ranking on ΘX.*
$f <_{lex} g$ iff $\exists k \leq r, s : v_i = w_i$ for $i < k$ and $v_k < w_k$ or $v_k = w_k$ and $\alpha_i < \beta_i$ or $v_i = w_i$ for $i \leq r$ and $r < s$.

Example 1.12 (Graded (by Degree) Reverse Lexicographic Ordering on M) *Given an ranking on ΘX.*
$f <_{degrevlex} g$ iff $\deg(f) < \deg(g)$ or $\deg(f) = \deg(g)$ and $f <_{revlex} g$.

Definition 1.13 *Let $f \in R\{X\}$ be an differential polynomial and fix a monomial ordering. Then $lm(f)$ denotes the leading monomial of f with respect to the monomial ordering. $lc(f)$ denotes the leading coefficient of f and $lt(f) = lc(f)lm(f)$ the leading term of f.*

Definition 1.14 *f is reduced by g to h iff $\exists \theta \in \Theta, m \in M$ such that $lm(f) = mlm(\theta g)$ and $h = f - \frac{lc(f)}{lc(g)} m\theta g$.*
f is reducable by g, iff there is an h such that f is reduced by g to h.

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Procedure: Reduce(f, g, r = 1)

if (deg(lm(f)) < deg(lm(g)) || wt(lm(f)) < wt(lm(g)))
        return f;
if (lm(g) | lm(f))
        return Reduce(f - (lt(f)/lt(g))*g, g);
for (i = r; i <= m; i++) {
        t = Reduce(f, delta(g, i), i);
        if (t != f) return Reduce(t, g);
}
return f;
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This procedure terminates. On every recursive call either f is reduced and therefore the leading monomial gets smaller or g is derivated (delta(g, i)) and therefore the weight of g increases. So after a finite number of calls Reduce terminates.
The returned polynomial cannot be reduced by g further because in Reduce the recuction with respect to all derivatives of g (which have no bigger weight or degree than f) is tried.
This process can - as in the algebraic case - be generalized to a reduction by several polynomials, but in general the remainder of the reduction is dependent on the order of these polynomials.

Definition 1.15 (Monoideal) *$E \subset M$ is called a monoideal iff $ME \subset E$ and $lm(\Delta E) \subset E$.*

Please note that in contrast to the algebraic case the definition of the monoideal needs an monomial ordering and is highly dependend on this (as we will see in the examples).

Definition 1.16 (Standard Basis) *$G \subset I$ is called a standard basis iff $lm(G)$ generates $lm(I)$ as monoideal.*

We now will investigate the monoideals generated by the polynomial x^2, for which we already considered the differential ideal.

Example 1.17 (Monoideal - Lexicographic Ordering) *The following monomials are members of the monoideal I generated by x^2 over $F\{x\}$ with $\Delta = \{d\}$ using lexicographic ordering ($x_{(k)} := d^k x$):*

1. *mx^2 for $m \in M$*
2. *$mx_{(1)}x$*
3. *$mx_{(2)}x$*
4. *$mx_{(k)}x$*
5. *BUT $(x_{(k)})^r \notin I$*

Example 1.18 (Monoideal - Graded Reverse Lexicographic Ordering) *The following monomials are members of the monoideal I generated by x^2 over $F\{x\}$ with $\Delta = \{'\}$ using graded lexicographic ordering ($x_{(k)} := d^k x$):*

1. mx^2 *for* $m \in M$
2. $mx_{(1)}x$
3. $m(x_{(1)})^2$
4. $mx_{(1)}x_{(2)}$
5. $m(x_{(2)})^2$
6. $mx_{(k)}x_{(k+1)}$
7. $mx_{(k)}^2$

Theorem 1 *Let G be a set of polynomials, I a differential ideal. Then the following propositions are equivalent:*

1. *G is a standard basis of I.*
2. *For $f \in F\{X\}$ yields: $f \in I \Leftrightarrow f$ is reduced to 0 by G.*

Proof

- $\Rightarrow$ Let $0 \neq f \in I$. Then f is reducible by G because the leading monomial of f is in the monoideal generated by I and therefore also in the monoideal generated by G. The reduction of f is $h \in I$. Therefore h is reducible again until $h = 0$. The process terminates because the leading monomial of the polynomial gets smaller in each reduction.
- $\Leftarrow$ Let $g \in G$. Then obviously g is reduced to 0 by G and therefore $G \subset I$.
  Let $f \in I$. Then f is reduced to 0 by G by definition and therefore $\mathrm{lm}I \subset \mathrm{lm}(M\Theta\mathrm{lm}(G))$ (otherwise reduction would fail).

$\square$

Example 1.19 *Remember $I = [x^2]$ over $F\{x\}$ with $\Delta = \{'\}$. Then for every $r \geq 0$ there is an $q > 1$ such that $(x_{(r)})^q \in I$.*

- *LEX: $lm(d(\prod_{i=1}^{r} v_i^{\alpha_i})) = d(v_1)v_1^{\alpha_1 - 1}\prod_{i=2}^{r} v_i^{\alpha_i}$ if $v_1 > \ldots > v_r$.*
  Therefore $(x_{(r)})^s$ for every $r \geq 0$ for some $s > 0$ is in every standard basis ($\rightarrow$ infinite).
- *DEGREVLEX: x^2 is a standard basis.*

Example 1.20 *Conjecture: There is no finite standard basis for $[xx']$ for no monomial ordering.*

Lemma 1.21 *The families of monomials*

1. $x^r x_{(r)}$ *for* $r \geq 1$
2. $x_{(r)}^{t_r}$ *for* $r \geq 1$ *and some* $t_r \geq r + 2$
3. $x_{(r)}^2 x_{(r+2)}^2 \cdots x_{(r+2k_r)}$ *for* $r \geq 0$ *and some* k_r
4. $x_{(r)}^2 x_{(r+3)}^2 \cdots x_{(r+3l_r)}$ *for* $r \geq 0$ *and some* $l_r \geq 2r - 1$

belong to the ideal $[xx']$.

This lemma (without proof) implies that all mentioned families of monomials have to be in the monoideal generated by the standard basis.

2 Geometric Ascpects

2.1 Manifolds

We choose e.g. F as set of all meromorphic function.

Definition 2.1 *Let Σ be a system of differential polynomials over $F\{x_1, \ldots, x_n\}$, F_1 an extension of F.*
If $Y = (y_1, \ldots, y_n) \in F_1^n$ such that for all $f \in \Sigma$ $f(y_1, \ldots, y_n) = 0$, then Y is a zero of Σ. The set of all zeros of Σ (for all possible extentions of F) is called manifold.

- Let M_1, M_2 be the manifolds of Σ_1, Σ_2. If $M_1 \cap M_2 \neq \emptyset$ then $M_1 \cap M_2$ is the manifold of $\Sigma_1 + \Sigma_2$. $M_1 \cup M_2$ is the manifold of $\{AB : A \in \Sigma_1, B \in \Sigma_2\}$.
- M is called *reducible* if it is union of two manifolds $M_1, M_2 \neq M$.
- Otherwise it is called *irreducible*.

Lemma 2.2 *M is irreducible $\Leftrightarrow$*
(AB vanishes over $M \Rightarrow A$ or B vanishes over M)

Proof

- $\Rightarrow$ Assume $\exists A, B$ such that AB vanishes over M, but A, B don't. Then the manifolds of $\Sigma + A$, $\Sigma + B$ are proper parts of M, their union is M.
- $\Leftarrow$ Let M be proper union of M_1, M_2 with systems Σ_1, Σ_2. Then $\exists A_i \in \Sigma_i$ be differential polynomials that do not vanish over M. $A_1 A_2$ vanishes over M.

$\square$

Theorem 2 *Every manifold is the union of a finite number of irreducible manifolds.*

Consider differential polynomials over $F\{x\}$ with $\Delta = \{d\}$ and F the meromorphic functions:

Example 2.3 *Let $\Sigma = \{f\}$ with $f = x_{(1)}^2 - 4x$. Then $df = 2x_{(1)}(x_{(2)} - 2)$.*

- *$x_{(1)} = 0$ has the solution $x(t) = c$. Looking at f, only $c = 0$ is valid.*
- *$x_{(2)} - 2 = 0$ has the solution $x(t) = (x + b)^2 + c$. Again $c = 0$.*
- *There are no other solutions.*

2.2 Algebraic Representation

Theorem 3 *Let $\Sigma = [f_1, \ldots, f_k]$ with manifold M. If g vanishes over M then $g^s \in \Sigma$ for some $s \in \mathbb{N}_0$.*

So the manifolds are represented by perfect ideals.

Theorem 4 *Every perfect differential ideal has a finite basis.*

Let Σ be a finite system of differential polynomials.
Question: Is $f \in \Sigma$?

- Resolve Σ into prime ideals.
- f must be member of each of these prime ideals.
- Test if the remainder of f with respect to the characteristic sets of the prime ideals is zero.

3 Conclusion

We have seen that differential polynomials can be used to model differential equation systems. There are many problems in contrast to algebraic polynomials. E.g. there are differential ideals that have no finite (even no recursive) basis and there are finitely generated ideals that have (presumably) no finite standard basis. The difficulties that arise when trying to find standard bases are also caused by the fact that differential monomial ideals depend on ordering.

We saw that manifolds (solutions of differential equation systems) correspond to perfect ideals, that are easier to handle than general differential ideals.

To conclude we remember that for some important problems finite algorithms exist, e.g. for the reduction with respect to a finite basis and the membership test for perfect ideals.

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