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Canonical completeness of infinitary μ

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Abstract

This paper presents a new model construction for a natural cut-free infinitary version $\mathbf{K}_{\omega}^{+}(\mu)$ of the propositional modal μ-calculus. Based on that the completeness of $\mathbf{K}_{\omega}^{+}(\mu)$ and the related system $\mathbf{K}_{\omega}(\mu)$ can be established directly – no detour, for example through automata theory, is needed. As a side result we also obtain a finite, cut-free sound and complete system for the propositional modal μ-calculus.

1. Introduction

The propositional modal μ-calculus, introduced in Kozen [13], is a remarkable and well-established formalism which extends the usual (multi-)modal propositional logic by operators for least and greatest fixed points of positive operators. It is notoriously intricate, mainly due to the possibility of forming complicate nestings of least and greatest fixed points, and plays a central role in many logic-oriented approaches to computer science, in particular in connection with so-called programming logics and process calculi. The reader may consult Bradfield and Stirling [5] for a first overview and as a good guide to the literature.

During the previous two decades a lot of substantial research has been carried through in connection with the propositional modal μ-calculus, mainly focusing on its automata- and model-theoretic properties and its behavior with respect to model checking. There are also lines of research which consider the μ-calculus as an algebraic system rather than a logic. The relevant literature is affluent, and we confine ourselves to mentioning only a few typical articles which provide a good point of departure for further reading: Arnold and Niwiński [2], Bradfield [4], Emerson et al. [7], Grädel [9], Janin and Walukiewicz [12], Lenzi [16], Santocanale [18,19], Stirling and Walker [22], Streett and Emerson [23], Winskel [26].

Corresponding work on the proof theory of the propositional modal μ-calculus has been slower to be achieved. Kozen [13], among other results, presents a sound axiomatization and shows it to be complete for a restricted version of the μ-calculus. Walukiewicz [24] analyzes an interesting sound and complete deductive system for the μ-calculus, and Walukiewicz [25] deals with the completenes of Kozen's original axiomatization. However, automata- rather than proof-theoretic methods are at the core of this approach.

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There are also approaches to the propositional modal μ-calculus via tableaux calculi which provide some interesting proof-theoretic information and several useful deductive systems. Examples of those can be found, for example, in Stirling and Walker [22] and Winskel [26]. More recent work on the modal μ-calculus, also with some proof-theoertic flavor, is in Dam and Gurov [6] and Sprenger and Dam [21]. In these articles Gentzen-style sequent calculi for modal and first order μ-calculi with approximations are introduced and studied in connection with program verification and explicit inductions.

Kozen [14] exhibits an important connection between the finite model property of the propositional modal μ-calculus and the theory of well-quasi-orders. This paper also mentions an infinitary derivation rule, similar to those which we will introduce later, and proves soundness as well as completeness of a deduction system incorporating this rule.

In the focus of this article are two natural infinitary versions $\mathbf{K}_\omega(\mu)$ and $\mathbf{K}_\omega^+(\mu)$ of the propositional modal μ-calculus. Both are based on a sort of ω-rule for introducing greatest fixed points and the usual closure rule

$$\mathcal{A}[(\mu X)\mathcal{A}[X]] \implies (\mu X)\mathcal{A}[X] \qquad (\bigstar)$$

for least fixed points. Actually, in order to be precise, a Tait-style reformulation of $(\bigstar)$ will be used in the formulations of $\mathbf{K}_\omega(\mu)$ and $\mathbf{K}_\omega^+(\mu)$; see Section 4. By means of the small model property $\mathbf{K}_\omega(\mu)$ is later collapsed to a finite cut-free system $\mathbf{K}_{<\omega}(\mu)$. As it turns out, $\mathbf{K}_\omega(\mu)$ contains $\mathbf{K}_\omega^+(\mu)$ and is itself contained in $\mathbf{K}_{<\omega}(\mu)$; in addition, all three systems prove exactly those sentences which are valid with respect to the standard semantics of the μ-calculus.

Proving the completeness of $\mathbf{K}_\omega^+(\mu)$ is the technically challenging part of the present study. From Alberucci and Jäger [1] we adapt the notion of saturated set and use those sets to build a syntactic Kripke structure. Problems arise in connection with the rule $(\bigstar)$ which is inherently impredicative in the sense that the logical complexity of $\mathcal{A}[(\mu X)\mathcal{A}[X]]$ is greater than that of $(\mu X)\mathcal{A}[X]$. Hence direct proofs by induction on the lengths of formulas cannot be carried through. However, by a more careful assignment of ranks (finite sequences of ordinals rather than ordinals), combined with ideas from Streett and Emerson [23], we achieve our goal.

This paper presents an, as we think, new model construction which is a canonical extension of standard model constructions in modal logic. Based on that the completeness of natural infinitary versions of the propositional modal μ-calculus is obtained directly – no detour, for example through automata theory, is needed. As a side result we also obtain a cut-free and complete finite system for μ. Similar techniques have been exploited before, see Jäger et al. [10,11], in order to design cut-free, sound and complete deductive system for the logic of common knowledge and the stratified propositional modal μ-calculus.

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2. Monotone operators

The sole purpose of this section is to recapitulate some basic facts concerning least and greatest fixed points of monotone operators. While doing this, we also fix some notation which will be convenient for our later purposes. For further reading, proofs of the properties stated below and relevant background information we refer, for example, to the textbooks Barwise [3] and Moschovakis [17].

Given a set S, we write $Pow(S)$ for the power set of S. The collection of all ordinals is denoted by On, and ω is the least infinite ordinal, generally identified with the set of all natural numbers.

Definition 1. Let M be an arbitrary set. A *monotone operator* on M is a mapping Φ from $Pow(M)$ to $Pow(M)$, $\Phi : Pow(M) \to Pow(M)$, so that for all subsets S_0 and S_1 of M

$$S_0 \subset S_1 \implies \Phi(S_0) \subset \Phi(S_1).$$

If $\Phi(S) = S$ for some subset S of M, then S is called a *fixed point* of the operator Φ.

Starting with the empty set, iterated applications of the monotone operator Φ give us what we call the lower stages of Φ. Alternatively, it is also possible to begin with the whole domain, and then successive applications of Φ generate its upper stages.

Definition 2. Let M be an arbitrary set and Φ an arbitrary monotone operator on M.
1. By transfinite induction on the ordinals we define for each $\alpha \in On$ the lower and upper stages of Φ as follows:

$$I_\Phi^{<\alpha} := \bigcup_{\beta<\alpha} \Phi(I_\Phi^{<\beta}) \quad \text{and} \quad J_\Phi^{<\alpha} := \bigcap_{\beta<\alpha} \Phi(J_\Phi^{<\beta}).$$

2. Based on these stages we set

$$I_\Phi := \bigcup_{\alpha\in On} I_\Phi^{<\alpha} \quad \text{and} \quad J_\Phi := \bigcap_{\alpha\in On} J_\Phi^{<\alpha}.$$

Well-known classical results state that for any monotone operator Φ on a set M the sequence of its lower stages $(I_\Phi^{<\alpha} : \alpha \in On)$ is increasing and approximates its least fixed point I_Φ. What is more, to obtain I_Φ, not all ordinals are needed but only an initial segment whose cardinality is bound by the cardinality of M. By duality, we have the corresponding theorem concerning the greatest fixed point of a monotone operator.

Theorem 3 (Least and greatest fixed points). *Let M be an arbitrary set and Φ an arbitrary monotone operator on M. Then we have*:
1. *The lower stages of Φ are increasing and its upper stages decreasing, i.e. for all ordinals α and β*

$$\alpha \leq \beta \implies I_\Phi^{<\alpha} \subset I_\Phi^{<\beta} \text{ and } J_\Phi^{<\beta} \subset J_\Phi^{<\alpha}.$$

2. *I_Φ is the least fixed point of Φ and J_Φ its greatest fixed point; moreover*

$$I_\Phi = \bigcap\{S \subset M : \Phi(S) \subset S\} = \bigcap\{S \subset M : S = \Phi(S)\},$$
$$J_\Phi = \bigcup\{S \subset M : S \subset \Phi(S)\} = \bigcup\{S \subset M : \Phi(S) = S\}.$$

3. *There exist ordinals α and β of cardinality less than or equal to the cardinality of M so that $I_\Phi = I_\Phi^{<\alpha}$ and $J_\Phi = J_\Phi^{<\beta}$.*

3. Syntax and semantics of the propositional modal μ-calculus

We will formulate the propositional modal μ-calculus in a language $\mathcal{L}_\mu$ which comprises the following syntactically different basic symbols:
1. Arbitrarily many labels a, b, c and countably many atomic propositions P, Q, R (both possibly with subscripts);
2. Countably many free variables U, V, W and countably many bound variables X, Y, Z (both possibly with subscripts);
3. The propositional constants $\bot$ and $\top$, the propositional connectives $\vee$ and $\wedge$ plus the connective $\sim$ for forming the complements of atomic propositions and free variables;
4. For each label a the modal operators $\langle a\rangle$ and $[a]$;
5. The fixed point operators μ and ν.

As auxiliary symbols we allow parentheses, brackets and commas. Substitutions of formulas for free variables will be very important in the following and throughout the sequel. In order to be able to describe such manipulations in a convenient way, the following notation is introduced.

By an n-ary *nominal form* ($n \geq 1$) we mean a non-empty finite string of symbols which contains in addition to the basic symbols of $\mathcal{L}_\mu$ at most the nominal symbols $*_1, \dots, *_n$. These nominal symbols are supposed to be different from the basic symbols of $\mathcal{L}_\mu$. In an n-ary nominal form, the nominal symbols $*_1, \dots, *_n$ may occur arbitrarily often. We shall always denote nominal forms by the letters $\mathcal{A}, \mathcal{B}, \mathcal{C}$ (possibly with subscripts). If $\mathcal{A}$ is an n-ary nominal form ($n \geq 1$) and $\mathfrak{z}_1, \dots, \mathfrak{z}_n$ are non-empty finite strings of symbols, then $\mathcal{A}[\mathfrak{z}_1, \dots, \mathfrak{z}_n]$ denotes the string of symbols which is obtained from $\mathcal{A}$ by simultaneously replacing all nominal forms $*_1, \dots, *_n$ by $\mathfrak{z}_1, \dots, \mathfrak{z}_n$.

Definition 4. The *formulas* $A, B, C, \dots$ (possibly with subscripts) of $\mathcal{L}_\mu$ are inductively defined as follows:
1. All atomic propositions P and free variables U as well as their complements $\sim P$ and $\sim U$ are formulas of $\mathcal{L}_\mu$.

2. The propositional constants $\bot$ and $\top$ are formulas of $\mathcal{L}_\mu$.
3. If A and B are formulas of $\mathcal{L}_\mu$, then $(A \vee B)$ and $(A \wedge B)$ are formulas of $\mathcal{L}_\mu$.
4. If a is a label and B a formula of $\mathcal{L}_\mu$, then $\langle a \rangle B$ and $[a]B$ are formulas of $\mathcal{L}_\mu$.
5. If $\mathcal{A}[U]$ is a formula of $\mathcal{L}_\mu$ which does not contain occurrences of $\sim U$ and if the free variable U and the bound variable X do not occur in $\mathcal{A}$, then $(\mu X)\mathcal{A}[X]$ and $(\nu X)\mathcal{A}[X]$ are formulas of $\mathcal{L}_\mu$.

An $\mathcal{L}_\mu$ formula A is *positive* in U if it does not contain any occurrences of $\sim U$. Hence $\mathcal{L}_\mu$ formulas $(\mu X)\mathcal{A}[X]$ and $(\nu X)\mathcal{A}[X]$ can only be built if $\mathcal{A}[U]$ is positive in U.

The formulas $\sim P$ and $\sim U$ act as negations of P and U, respectively. For introducing the negations of general formulas we make use of the law of double negation, de Morgan's laws and specific dualities between the modal operators $\langle a \rangle/[a]$ and least/greatest fixed points.

Definition 5. The *negation* $\neg A$ of an $\mathcal{L}_\mu$ formula A is inductively defined as follows:
1. If A is the atomic proposition P, then $\neg A$ is $\sim P$; if A is the formula $\sim P$, the $\neg A$ is P.
2. If A is the free variable U, then $\neg A$ is $\sim U$; if A is the formula $\sim U$, the $\neg A$ is U.
3. If A is the propositional constant $\bot$, then $\neg A$ is $\top$; if A is the propositional constant $\top$, then $\neg A$ is $\bot$.
4. If A is the formula $(B \vee C)$ then $\neg A$ is $(\neg B \wedge \neg C)$; if A is the formula $(B \wedge C)$ then $\neg A$ is $(\neg B \vee \neg C)$.
5. If A is the formula $\langle a \rangle B$ then $\neg A$ is $[a]\neg B$; if A is the formula $[a]B$ then $\neg A$ is $\langle a \rangle \neg B$.
6. If A is the formula $(\mu X)\mathcal{A}[X]$, then $\neg A$ is $(\nu X)\widetilde{\mathcal{A}}[X]$; if A is the formula $(\nu X)\mathcal{A}[X]$, then $\neg A$ is $(\mu X)\widetilde{\mathcal{A}}[X]$. Here $\widetilde{\mathcal{A}}$ is the (uniquely determined) unary nominal form so that $\neg\mathcal{A}[\sim U]$ is $\widetilde{\mathcal{A}}[U]$ for all free variables U.

Observe that the definitions of $\neg(\mu X)\mathcal{A}[X]$ and $\neg(\nu X)\mathcal{A}[X]$ make sense since the U-positivity of $\mathcal{A}[U]$ implies the U-positivity of $\widetilde{\mathcal{A}}[U]$. We abbreviate the remaining connectives as usual,

$$(A \rightarrow B) := (\neg A \vee B),$$

$$(A \leftrightarrow B) := ((A \rightarrow B) \wedge (B \rightarrow A)),$$

and omit parentheses if there is no danger of confusion. Given a formula A, we write $fv(A)$ for the collection of all free variables occurring in A. A formula A is called *closed* or a *sentence* if $fv(A)$ is empty.

Definition 6. A *Kripke structure* for $\mathcal{L}_\mu$ is a triple $\mathfrak{M} = (M, H_0, H_1)$ satisfying the following three conditions:
(KS.1) M is a set, the so-called universe of $\mathfrak{M}$; the elements of M are the *worlds* of $\mathfrak{M}$.
(KS.2) H_0 is a mapping which assigns to any label a a binary relation $H_0(a)$ on M, i.e. $H_0(a) \subset M \times M$.
(KS.3) H_1 is a mapping which assigns to any atomic proposition P a subset $H_1(P)$ of M.

If $\mathfrak{M}$ is the Kripke structure (M, H_0, H_1), then we normally write $|\mathfrak{M}|$ for the universe M of $\mathfrak{M}$ as well as $\mathfrak{M}(a)$ and $\mathfrak{M}(P)$ for the interpretations $H_0(a)$ and $H_1(P)$ of the names a and atomic propositions P, respectively.

A *valuation* $\mathfrak{v}$ in a Kripke structure $\mathfrak{M}$ assigns to each free variable U a subset $\mathfrak{v}(U)$ of $|\mathfrak{M}|$. Now let $\mathfrak{v}$ be any valuation in $\mathfrak{M}$, U a free variable and S a subset of $|\mathfrak{M}|$. Then we write $\mathfrak{v}[U{:}S]$ for the valuation which maps U on S and otherwise agrees with $\mathfrak{v}$.

Definition 7. Consider a Kripke structure $\mathfrak{M}$. Then, for any valuation $\mathfrak{v}$ in $\mathfrak{M}$, the *truth set* $\|A\|_{(\mathfrak{M},\mathfrak{v})}$ of an $\mathcal{L}_\mu$ formula A is inductively defined as follows:
1. For atomic propositions, free variables and propositional constants:

$$\|P\|_{(\mathfrak{M},\mathfrak{v})} := \mathfrak{M}(P), \qquad \|\sim P\|_{(\mathfrak{M},\mathfrak{v})} := |\mathfrak{M}| \setminus \mathfrak{M}(P),$$

$$\|U\|_{(\mathfrak{M},\mathfrak{v})} := \mathfrak{v}(U), \qquad \|\sim U\|_{(\mathfrak{M},\mathfrak{v})} := |\mathfrak{M}| \setminus \mathfrak{v}(U),$$

$$\|\top\|_{(\mathfrak{M},\mathfrak{v})} := |\mathfrak{M}|, \qquad \|\bot\|_{(\mathfrak{M},\mathfrak{v})} := \emptyset.$$

2. For disjunctions and conjunctions:

$$\|A \vee B\|_{(\mathfrak{M},\mathfrak{v})} := \|A\|_{(\mathfrak{M},\mathfrak{v})} \cup \|B\|_{(\mathfrak{M},\mathfrak{v})},$$

$$\|A \wedge B\|_{(\mathfrak{M},\mathfrak{v})} := \|A\|_{(\mathfrak{M},\mathfrak{v})} \cap \|B\|_{(\mathfrak{M},\mathfrak{v})}.$$

3. For formulas prefixed by a modal operator:

$$\|\langle a\rangle B\|_{(\mathfrak{M},\mathfrak{v})} := \{x \in |\mathfrak{M}| : (\exists y)((x, y) \in \mathfrak{M}(a) \text{ and } y \in \|B\|_{(\mathfrak{M},\mathfrak{v})})\},$$

$$\|[a]B\|_{(\mathfrak{M},\mathfrak{v})} := \{x \in |\mathfrak{M}| : (\forall y)((x, y) \in \mathfrak{M}(a) \;\Rightarrow\; y \in \|B\|_{(\mathfrak{M},\mathfrak{v})})\}.$$

4. For fixed point formulas: Given a formula $\mathcal{A}[U]$ where U does not occur in $\mathcal{A}$ and $\mathcal{A}[U]$ is positive in U, we first introduce the monotone operator

$$\Phi : Pow(|\mathfrak{M}|) \to Pow(|\mathfrak{M}|), \quad \Phi(S) \;:=\; \|\mathcal{A}[U]\|_{(\mathfrak{M},\mathfrak{v}[U:S])}.$$

Based on this Φ, we now set

$$\|(\mu X)\mathcal{A}[X]\|_{(\mathfrak{M},\mathfrak{v})} \;:=\; I_\Phi \quad \text{and} \quad \|(\nu X)\mathcal{A}[X]\|_{(\mathfrak{M},\mathfrak{v})} \;:=\; J_\Phi.$$

With this definition in mind, we can easily introduce the notions of μ-validity and μ-satisfiability:

(μ-val) A formula A of $\mathcal{L}_\mu$ is said to be μ-*valid* if $|\mathfrak{M}| \subset \|A\|_{(\mathfrak{M},\mathfrak{v})}$ for all Kripke structures $\mathfrak{M}$ and all valuations $\mathfrak{v}$ in $\mathfrak{M}$; in this case we write $\mu \models A$.

(μ-sat) A formula A of $\mathcal{L}_\mu$ is said to be μ-*satisfiable* if there exists a Kripke structure $\mathfrak{M}$ and a valuation $\mathfrak{v}$ in $\mathfrak{M}$ so that $\|A\|_{(\mathfrak{M},\mathfrak{v})} \neq \emptyset$.

Following Kozen's paper [13], we now recall a Hilbert-style formalization $\mathbf{K}(\mu)$ of the propositional modal μ-calculus. The multi-modal version of the modal logic **K** is simply extended by closure axioms and induction rules for the least fixed point formulas $(\mu X)\mathcal{A}[X]$.

I. Logical axioms of $\mathbf{K}(\mu)$. For all propositional tautologies A of $\mathcal{L}_\mu$, all $\mathcal{L}_\mu$ formulas B and C and all labels a:

$$A, \tag{TAUT}$$

$$[a]B \wedge [a](B \to C) \;\to\; [a]C. \tag{K}$$

II. Logical rules of $\mathbf{K}(\mu)$. For all $\mathcal{L}_\mu$ formulas A and B and all labels a:

$$\frac{A \qquad A \to B}{B}, \tag{MP}$$

$$\frac{A}{[a]A}. \tag{NEC}$$

III. Closure axioms of $\mathbf{K}(\mu)$. For all $\mathcal{L}_\mu$ formulas $\mathcal{A}[U]$ so that the free variable U does not occur in $\mathcal{A}$ and $\mathcal{A}[U]$ is positive in U:

$$\mathcal{A}[(\mu X)\mathcal{A}[X]] \to (\mu X)\mathcal{A}[X]. \tag{μ-CLO}$$

IV. Induction rules of $\mathbf{K}(\mu)$. For all $\mathcal{L}_\mu$ formulas $\mathcal{A}[U]$ so that U does not occur in $\mathcal{A}$ and $\mathcal{A}[U]$ is positive in U and all $\mathcal{L}_\mu$ formulas B:

$$\frac{\mathcal{A}[B] \to B}{(\mu X)\mathcal{A}[X] \to B}. \tag{μ-IND}$$

Provability of a formula A in the Hilbert system $\mathbf{K}(\mu)$ is defined as usual and written as

$$\mathbf{K}(\mu) \vdash A.$$

It is easily checked that for greatest fixed point formulas $(\nu X)\mathcal{A}[X]$ the duals of (μ-AX) and (μ-IND) can be derived in $\mathbf{K}(\mu)$. The proof of the following lemma is left to the reader.

Lemma 8. *For all $\mathcal{L}_\mu$ formulas $\mathcal{A}[U]$ where U does not occur in $\mathcal{A}$ and $\mathcal{A}[U]$ is positive in U, and for all $\mathcal{L}_\mu$ formulas B we have*:

1. $\mathbf{K}(\mu) \vdash (\nu X)\mathcal{A}[X] \;\to\; \mathcal{A}[(\nu X)\mathcal{A}[X]]$.
2. $\mathbf{K}(\mu) \vdash B \to \mathcal{A}[B] \implies \mathbf{K}(\mu) \vdash B \to (\nu X)\mathcal{A}[X]$.

4. The infinitary μ-calculi $\mathbf{K}^+_\omega(\mu)$ and $\mathbf{K}_\omega(\mu)$

In this section we introduce two cut-free μ-calculi, the system $\mathbf{K}^+_\omega(\mu)$ and the system $\mathbf{K}_\omega(\mu)$. Both introduce formulas $(\nu X)\mathcal{A}[X]$ by a kind of ω-rule and therefore are infinitary deduction systems. We will later show that both are sound and complete.

$\mathbf{K}^+_\omega(\mu)$ is an auxiliary system, needed for some technical reasons, which is formulated in the extension $\mathcal{L}^+_\mu$ of $\mathcal{L}_\mu$. The language $\mathcal{L}^+_\mu$ is obtained from $\mathcal{L}_\mu$ by adding syntactic constructs $(\nu^n X)\mathcal{A}[X]$ for all natural numbers n greater than 0 to represent the finite approximations of $(\nu X)\mathcal{A}[X]$. More precisely: the definition of the formulas of $\mathcal{L}^+_\mu$ corresponds to Definition 4 with one additional clause:

6. If $\mathcal{A}[U]$ is a formula of $\mathcal{L}^+_\mu$ which does not contain occurrences of $\sim U$, if the free variable U and the bound variable X do not occur in $\mathcal{A}$ and if n is a natural number greater than 0, then $(\nu^n X)\mathcal{A}[X]$ is a formula of $\mathcal{L}^+_\mu$.

Furthermore, for any $\mathcal{L}^+_\mu$ formula A let A^- denote the $\mathcal{L}_\mu$ formula which is obtained from A by first replacing all subexpressions of the form $(\nu^n X)\mathcal{A}[X]$ by $(\nu X)\mathcal{A}[X]$ and afterwards all free variables by $\top$.

For measuring the complexities of $\mathcal{L}^+_\mu$ formulas and in connection with the truth lemma of Section 5 it turns out to be convenient to work with finite sequences of ordinals. If $\alpha_1, \dots, \alpha_n$ are ordinals, we write $\langle \alpha_1, \dots, \alpha_n \rangle$ for the sequence $\boldsymbol{\sigma}$ whose length $lh(\boldsymbol{\sigma})$ is n and whose ith component $(\boldsymbol{\sigma})_i$ is the ordinal α_i; i.e.

$$\boldsymbol{\sigma} = \langle \alpha_1, \dots, \alpha_n \rangle \implies lh(\boldsymbol{\sigma}) = n \quad \text{and} \quad (\boldsymbol{\sigma})_i = \alpha_i \quad \text{for } 1 \le i \le n.$$

The empty sequence is written as $\langle\rangle$, and $lh(\langle\rangle) = 0$. In the following we will often denote finite sequences of ordinals by the boldface Greek letters $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ (possibly with subscripts).

Let $<_{lex}$ be the strict lexicographical ordering of finite sequences of ordinals and $\le_{lex}$ its reflexive closure. Recall that $<_{lex}$ is a well-ordering on any set of sequences of bounded lengths, though not a well-ordering in general. We also need the component-wise ordering $\trianglelefteq$ of finite sequences of ordinals, given by

$$\boldsymbol{\sigma} \trianglelefteq \boldsymbol{\tau} \quad :\Leftrightarrow \quad lh(\boldsymbol{\sigma}) \le lh(\boldsymbol{\tau}) \quad \text{and} \quad (\boldsymbol{\sigma})_i \le (\boldsymbol{\tau})_i \quad \text{for } 1 \le i \le lh(\boldsymbol{\sigma}).$$

Clearly, the relation $\trianglelefteq$ is transitive. The concatenation $*$ of finite sequences of ordinals is as usual, and therefore we have

$$\boldsymbol{\sigma} * \boldsymbol{\tau} = \langle \alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n \rangle$$

whenever $\boldsymbol{\sigma} = \langle \alpha_1, \dots, \alpha_m \rangle$ and $\boldsymbol{\tau} = \langle \beta_1, \dots, \beta_n \rangle$. Ultimately, we introduce on the finite sequences of ordinals a specific maximum operation $\sqcup$ by setting: (i) $\boldsymbol{\sigma} \sqcup \langle\rangle := \langle\rangle \sqcup \boldsymbol{\sigma} := \boldsymbol{\sigma}$; (ii) if $\boldsymbol{\sigma} = \langle \alpha_1, \dots, \alpha_m \rangle$ and $\boldsymbol{\tau} = \langle \beta_1, \dots, \beta_n \rangle$, then

$$\boldsymbol{\sigma} \sqcup \boldsymbol{\tau} \;:=\; \begin{cases} \langle \max(\alpha_1, \beta_1), \dots, \max(\alpha_m, \beta_m), \beta_{m+1}, \dots, \beta_n \rangle & \text{if } m \le n, \\ \langle \max(\alpha_1, \beta_1), \dots, \max(\alpha_n, \beta_n), \alpha_{n+1}, \dots, \alpha_m \rangle & \text{if } n < m. \end{cases}$$

Now we are ready to turn to the ranks and lengths of all $\mathcal{L}^+_\mu$ formulas. As the following definition shows, the rank of every $\mathcal{L}^+_\mu$ formula A will be a finite sequence of ordinals less than or equal to ω, and the length of A simply is the length of this sequence, hence a natural number.

Definition 9. The *rank* $rk(A)$ of an $\mathcal{L}^+_\mu$ formula A is inductively defined as follows:

1. If A is an atomic proposition, the negation of an atomic proposition, a free variable, the negation of a free variable or a propositional constant, then $rk(A) := \langle 0 \rangle$.

2. If A is a formula $(B \vee C)$ or a formula $(B \wedge C)$, then

$$rk(A) \ := \ (rk(B) \sqcup rk(C)) * \langle 0 \rangle.$$

3. If A is a formula $\langle a \rangle B$ or a formula $[a]B$, then

$$rk(A) \ := \ rk(B) * \langle 0 \rangle.$$

4. If A is a formula $(\mu X)\mathcal{A}[X]$, then

$$rk(A) \ := \ rk(\mathcal{A}[\top]) * \langle 0 \rangle.$$

5. If A is a formula $(\nu X)\mathcal{A}[X]$, then

$$rk(A) \ := \ rk(\mathcal{A}[\top]) * \langle \omega \rangle.$$

6. If A is a formula $(\nu^n X)\mathcal{A}[X]$ for some natural number n greater than 0, then

$$rk(A) \ := \ rk(\mathcal{A}[\top]) * \langle n \rangle.$$

The *length* $lh(A)$ of an $\mathcal{L}_\mu^+$ formula A is the length of the finite sequence $rk(A)$, i.e. $lh(A) := lh(rk(A))$.

The following two lemmas summarize some elementary properties of the rank and length functions. Their proofs are straightforward and will be omitted.

Lemma 10. *For all $\mathcal{L}_\mu^+$ formulas A we have*

$$rk(A) \trianglelefteq rk(A^-) \quad \textit{and} \quad lh(A) = lh(A^-).$$

Lemma 11. *For all $\mathcal{L}_\mu^+$ formulas A, B and $\mathcal{A}[\top]$, all labels a, all free variables U and all natural numbers n greater than* 0 *we have*:

1. $rk(A),\ rk(B) \ <_{lex} \ rk(A \vee B) \ = \ rk(A \wedge B)$.
2. $lh(A),\ lh(B) \ < \ lh(A \vee B) \ = \ lh(A \wedge B)$.
3. $rk(B) \ <_{lex} \ rk(\langle a \rangle B) \ = \ rk([a]B)$.
4. $lh(B) \ < \ lh(\langle a \rangle B) \ = \ lh([a]B)$.
5. $rk(\mathcal{A}[U]) \ = \ rk(\mathcal{A}[\bot]) \ = \ rk(\mathcal{A}[\top])$.
6. $lh(\mathcal{A}[U]) \ = \ lh(\mathcal{A}[\bot]) \ = \ lh(\mathcal{A}[\top])$.
7. $rk(\mathcal{A}[U]) \ <_{lex} \ rk((\mu X)\mathcal{A}[X]),\ rk((\nu X)\mathcal{A}[X]),\ rk((\nu^n X)\mathcal{A}[X])$.
8. $lh(\mathcal{A}[U]) \ < \ lh((\nu X)\mathcal{A}[X]) \ = \ lh((\nu^n X)\mathcal{A}[X])$.

The following lemma is more interesting and useful for establishing some connections between the ranks of formulas of the form $(\nu X)\mathcal{A}[X]$, $(\nu^{n+1} X)\mathcal{A}[X]$ and $\mathcal{A}[(\nu^n X)\mathcal{A}[X]]$; see Theorem 13 below.

Lemma 12. *Suppose that $\mathcal{A}[U]$ is an $\mathcal{L}_\mu^+$ formula positive in U with the free variable U occurring in $\mathcal{A}[U]$ but not in $\mathcal{A}$ and suppose that B is an $\mathcal{L}_\mu^+$ formula satisfying $rk(\mathcal{A}[U]) \trianglelefteq rk(B)$. Then there exists a finite* (*possibly empty*) *sequence of ordinals $\boldsymbol{\sigma}$ so that*

$$rk(\mathcal{A}[B]) \ = \ rk(B) * \boldsymbol{\sigma}.$$

Proof. We show this lemma by induction on $lh(\mathcal{A}[U])$ and distinguish the following cases:

1. $lh(\mathcal{A}[U]) = 1$. Then $\mathcal{A}[U]$ has to be the free variable U, and the assertion is trivially satisfied.
2. $\mathcal{A}[U]$ is a formula $(\mathcal{A}_1[U] \vee \mathcal{A}_2[U])$. Then we have

$$rk(\mathcal{A}_1[U]),\ rk(\mathcal{A}_2[U]) \ \trianglelefteq \ rk(\mathcal{A}[U]) \ \trianglelefteq \ rk(B). \tag{1}$$

If $\mathcal{A}_1[U]$ and $\mathcal{A}_2[U]$ contain the free variable U, we can apply the induction hypothesis to both formulas and obtain

$$rk(\mathcal{A}_1[B]) = rk(B) * \boldsymbol{\sigma}_1 \quad \text{and} \quad rk(\mathcal{A}_2[B]) = rk(B) * \boldsymbol{\sigma}_2 \tag{2}$$

for suitable $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$. Clearly, this implies

$$rk(\mathcal{A}[B]) = ((rk(B) * \boldsymbol{\sigma}_1) \sqcup (rk(B) * \boldsymbol{\sigma}_2)) * \langle 0 \rangle, \tag{3}$$

and we have our assertion for $\boldsymbol{\sigma}$ being the sequence $(\boldsymbol{\sigma}_1 \sqcup \boldsymbol{\sigma}_2) * \langle 0 \rangle$.

If only one of $\mathcal{A}_1[U]$ and $\mathcal{A}_2[U]$ – say $\mathcal{A}_1[U]$ – contains U, the induction hypothesis yields

$$rk(\mathcal{A}_1[B]) = rk(B) * \boldsymbol{\sigma}_1 \tag{4}$$

for a suitable $\boldsymbol{\sigma}_1$. But now we also know that $\mathcal{A}_2[B]$ is the formula $\mathcal{A}_2[U]$ and deduce from (1) that

$$rk(\mathcal{A}_2[B]) \trianglelefteq rk(B). \tag{5}$$

From (4) and (5) we conclude, for $\boldsymbol{\sigma}$ now being the sequence $\boldsymbol{\sigma}_1 * \langle 0 \rangle$,

$$rk(\mathcal{A}[B]) = ((rk(B) * \boldsymbol{\sigma}_1) \sqcup rk(\mathcal{A}_2[B])) * \langle 0 \rangle = rk(B) * \boldsymbol{\sigma}. \tag{6}$$

3. $\mathcal{A}[U]$ is a formula $(\mathcal{A}_1[U] \wedge \mathcal{A}_2[U])$. Then we proceed as in the previous case.

4. $\mathcal{A}[U]$ is a formula of a form not covered so far. Then the assertion immediately follows from the induction hypothesis. □

Theorem 13. *For all $\mathcal{L}_\mu^+$ formulas $(\nu X)\mathcal{A}[X]$ and all natural numbers n greater than 0 we have*:
1. $rk(\mathcal{A}[\top]) <_{lex} rk((\nu^1 X)\mathcal{A}[X])$.
2. $rk(\mathcal{A}[(\nu^n X)\mathcal{A}[X]]) <_{lex} rk((\nu^{n+1} X)\mathcal{A}[X])$.
3. $rk((\nu^n X)\mathcal{A}[X]) <_{lex} rk((\nu X)\mathcal{A}[X])$.

Proof. The first and the third assertion are immediate consequences of Definition 9. In order to prove the second assertion, pick a free variable U which does not occur in $\mathcal{A}$.

If this U does not even occur in $\mathcal{A}[U]$, then $\mathcal{A}[(\nu^n X)\mathcal{A}[X]]$ is identical to the formula $\mathcal{A}[\top]$, hence

$$rk(\mathcal{A}[(\nu^n X)\mathcal{A}[X]]) <_{lex} rk(\mathcal{A}[\top]) * \langle n+1 \rangle = rk((\nu^{n+1} X)\mathcal{A}[X]).$$

It remains to establish the second assertion for the case that U occurs in $\mathcal{A}[U]$. In view of Lemma 11 we know that $rk(\mathcal{A}[U]) \trianglelefteq rk((\nu^n X)\mathcal{A}[X])$. Hence the previous lemma and Definition 9 yield, for some $\boldsymbol{\sigma}$,

$$rk(\mathcal{A}[(\nu^n X)\mathcal{A}[X]]) = rk((\nu^n X)\mathcal{A}[X]) * \boldsymbol{\sigma} = (rk(\mathcal{A}[\top]) * \langle n \rangle) * \boldsymbol{\sigma}.$$

Together with $rk((\nu^{n+1} X)\mathcal{A}[X]) = rk(\mathcal{A}[\top]) * \langle n+1 \rangle$ this immediately gives $rk(\mathcal{A}[(\nu^n X)\mathcal{A}[X]]) <_{lex} rk((\nu^{n+1} X)\mathcal{A}[X])$, completing the proof of our theorem. □

The infinitary calculus $\mathbf{K}_\omega^+(\mu)$ is formulated as a Tait-style system which derives finite sets $\Gamma, \Delta, \Pi, \Sigma, \ldots$ (possibly with subscripts) of $\mathcal{L}_\mu^+$ formulas rather than individual $\mathcal{L}_\mu^+$ formulas. These finite sets of $\mathcal{L}_\mu^+$ formulas are interpreted disjunctively, and in general we write Γ, A for $\Gamma \cup \{A\}$; similarly for expressions of forms like Γ, Δ, A, B. In addition, if Γ is the set $\{A_1, \ldots, A_m\}$ of $\mathcal{L}_\mu^+$ formulas and a some label, then $\langle a \rangle \Gamma$ stands for the set $\{\langle a \rangle A_1, \ldots, \langle a \rangle A_m\}$.

$\mathbf{K}_\omega^+(\mu)$ contains the standard axioms and logical rules of the multi-modal version of the logic $\mathbf{K}$, the Tait-style analogues of the μ-closure-axioms plus rules for introducing $(\nu^n X)\mathcal{A}[X]$ and $(\nu X)\mathcal{A}[X]$.

I. Axioms of $\mathbf{K}_\omega^+(\mu)$. For all finite sets Γ of $\mathcal{L}_\mu^+$ formulas, all atomic propositions P and all free variables U:

$$\Gamma, \top, \tag{Ax1}$$

$$\Gamma, P, \sim P, \tag{Ax2}$$

$$\Gamma, U, \sim U. \tag{Ax3}$$

II. Logical rules of $\mathbf{K}^+_\omega(\mu)$. For all finite sets Γ, Δ of $\mathcal{L}^+_\mu$ formulas, all labels a and all $\mathcal{L}^+_\mu$ formulas A, B:

$$\frac{\Gamma,\ A,\ B}{\Gamma,\ A \vee B}, \tag{$\vee$}$$

$$\frac{\Gamma,\ A \qquad \Gamma,\ B}{\Gamma,\ A \wedge B}, \tag{$\wedge$}$$

$$\frac{\Gamma,\ A}{\langle a\rangle\Gamma,\ [a]A,\ \Delta}. \tag{K}$$

III. μ-rules of $\mathbf{K}^+_\omega(\mu)$. For all finite sets Γ of $\mathcal{L}^+_\mu$ formulas and all $\mathcal{L}^+_\mu$ formulas $\mathcal{A}[U]$ where U does not occur in $\mathcal{A}$ and $\mathcal{A}[U]$ is positive in U:

$$\frac{\Gamma,\ \mathcal{A}[(\mu X)\mathcal{A}[X]]}{\Gamma,\ (\mu X)\mathcal{A}[X]}. \tag{μ}$$

IV. ν-rules of $\mathbf{K}^+_\omega(\mu)$. For all finite sets Γ of $\mathcal{L}^+_\mu$ formulas and all $\mathcal{L}^+_\mu$ formulas $\mathcal{A}[U]$ where U does not occur in $\mathcal{A}$ and $\mathcal{A}[U]$ is positive in U:

$$\frac{\Gamma,\ \mathcal{A}[\top]}{\Gamma,\ (\nu^1 X)\mathcal{A}[X]}, \tag{$\nu.1$}$$

$$\frac{\Gamma,\ \mathcal{A}[(\nu^n X)\mathcal{A}[X]]}{\Gamma,\ (\nu^{n+1} X)\mathcal{A}[X]}, \tag{$\nu.n{+}1$}$$

$$\frac{\ldots\ \Gamma,\ (\nu^n X)\mathcal{A}[X]\ \ldots \quad (\text{for all } 0 < n < \omega)}{\Gamma,\ (\nu X)\mathcal{A}[X]}. \tag{$\nu.\omega$}$$

Provability of Γ in $\mathbf{K}^+_\omega(\mu)$ is defined as usual and denoted by $\mathbf{K}^+_\omega(\mu) \vdash \Gamma$. On account of the rule $(\nu.\omega)$ there are derivations in $\mathbf{K}^+_\omega(\mu)$ which are infinitely branching trees of infinite depths.

It is obvious from the formulation of these axioms and rules that $\mathbf{K}^+_\omega(\mu)$ satisfies weakening in the usual sense; that is, if Γ is provable in $\mathbf{K}^+_\omega(\mu)$ and if Γ is a subset of Δ, then Δ is also provable in $\mathbf{K}^+_\omega(\mu)$.

It is not at all obvious that $\mathbf{K}^+_\omega(\mu)$ is sound and complete. The completeness of $\mathbf{K}^+_\omega(\mu)$ will be established in the next section by means of specific saturated sets.

Concerning soundness, problems might occur in connection with the infinitary rule $(\nu.\omega)$. Its premises are exactly the finite stages of the greatest fixed point represented by $(\nu X)\mathcal{A}[X]$. However, in arbitrary Kripke structures a greatest fixed point may very well be composed of transfinite stages. Therefore, in the context of such models, the rule $(\nu.\omega)$ does not provide sufficiently many premises to be directly seen to be correct.

Nevertheless, $\mathbf{K}^+_\omega(\mu)$ will turn out to be sound. But rather than showing its soundness directly, we will in Section 6 prove the soundness of a finite system $\mathbf{K}_{<\omega}(\mu)$ which contains $\mathbf{K}^+_\omega(\mu)$.

There is also a syntactic simplification $\mathbf{K}_\omega(\mu)$ of $\mathbf{K}^+_\omega(\mu)$ which stays within the language $\mathcal{L}_\mu$ and avoids the specific constructs $(\nu^n X)\mathcal{A}[X]$ which are not part of $\mathcal{L}_\mu$. In $\mathcal{L}_\mu$ the finite approximations of greatest fixed points $(\nu X)\mathcal{A}[X]$ are represented by the $\mathcal{L}_\mu$ formulas $(\nu X)^n\mathcal{A}[X]$ which are inductively defined, for each natural number $n > 0$, as follows:

$$(\nu X)^1\mathcal{A}[X] \ :=\ \mathcal{A}[\top] \quad \text{and} \quad (\nu X)^{n+1}\mathcal{A}[X] \ :=\ \mathcal{A}[(\nu X)^n\mathcal{A}[X]].$$

Recursively replacing all expressions $(\nu^n X)\mathcal{A}[X]$ of $\mathcal{L}^+_\mu$ by these $\mathcal{L}_\mu$ formulas $(\nu X)^n\mathcal{A}[X]$ provides a translation of $\mathcal{L}^+_\mu$ into $\mathcal{L}_\mu$.

Definition 14. The translation A^* of an $\mathcal{L}^+_\mu$ formula A is inductively defined as follows:

1. If A is an atomic proposition, the negation of an atomic proposition, a free variable, the negation of a free variable or a propositional constant, then $A^* := A$.

2. If A is a formula $(B \vee C)$, then $A^* := (B^* \vee C^*)$; if A is a formula $(B \wedge C)$, then $A^* := (B^* \wedge C^*)$.
3. If A is a formula $\langle a \rangle B$, then $A^* := \langle a \rangle B^*$; if A is a formula $[a]B$, then $A^* := [a]B^*$.
4. If A is a formula $(\mu X)\mathcal{A}[X]$, then $A^* := (\mu X)\mathcal{A}^*[X]$; if A is a formula $(\nu X)\mathcal{A}[X]$, then $A^* := (\nu X)\mathcal{A}^*[X]$.
5. If A is a formula $(\nu^n X)\mathcal{A}[X]$ for some natural number n greater than 0, then $A^* := (\nu X)^n \mathcal{A}^*[X]$.

This definition is extended to finite sets of $\mathcal{L}_\mu^+$ formulas in the obvious way: for $\Gamma = \{A_1, \dots, A_n\}$ we set $\Gamma^* := \{A_1^*, \dots, A_n^*\}$.

Lemma 15. *If A is a formula of $\mathcal{L}_\mu^+$, then A^* is a formula of $\mathcal{L}_\mu$. Moreover, if A is a formula of $\mathcal{L}_\mu$, then A^* and A are identical.*

The proof of this lemma is trivial. It is interesting to note that the $\mathcal{L}_\mu^+$ formula A and the $\mathcal{L}_\mu$ formula A have the same "meaning"; their ranks, on the other hand, can be completely different.

As $\mathbf{K}_\omega^+(\mu)$, the calculus $\mathbf{K}_\omega(\mu)$ is formulated in a Tait-style manner, now deriving finite sets of $\mathcal{L}_\mu$ formulas. The axioms, the logical rules and the μ-rules of $\mathbf{K}_\omega(\mu)$ correspond exactly to the axioms, logical rules and μ-rules of $\mathbf{K}_\omega^+(\mu)$; because of our syntactic simplification the rules $(\nu.1)$ and $(\nu.n{+}1)$ are not needed in $\mathbf{K}_\omega(\mu)$, and the rule $(\nu.\omega)$ is converted to the rule (ν) which works with the $\mathcal{L}_\mu$ formulas $(\nu X)^n\mathcal{A}[X]$ instead of the $\mathcal{L}_\mu^+$ formulas $(\nu^n X)\mathcal{A}[X]$.

I. Axioms of $\mathbf{K}_\omega(\mu)$. For all finite sets Γ of $\mathcal{L}_\mu$ formulas, all atomic propositions P and all free variables U:

$$\Gamma,\ \top, \tag{Ax1}$$

$$\Gamma,\ P,\ \sim P, \tag{Ax2}$$

$$\Gamma,\ U,\ \sim U. \tag{Ax3}$$

II. Logical rules of $\mathbf{K}_\omega(\mu)$. For all finite sets Γ, Δ of $\mathcal{L}_\mu$ formulas, all labels a and all $\mathcal{L}_\mu$ formulas A, B:

$$\frac{\Gamma,\ A,\ B}{\Gamma,\ A \vee B}, \tag{$\vee$}$$

$$\frac{\Gamma,\ A \qquad \Gamma,\ B}{\Gamma,\ A \wedge B}, \tag{$\wedge$}$$

$$\frac{\Gamma,\ A}{\langle a \rangle\Gamma,\ [a]A,\ \Delta}. \tag{K}$$

III. μ-rules of $\mathbf{K}_\omega(\mu)$. For all finite sets Γ of $\mathcal{L}_\mu$ formulas and all $\mathcal{L}_\mu$ formulas $\mathcal{A}[U]$ where U does not occur in $\mathcal{A}$ and $\mathcal{A}[U]$ is positive in U:

$$\frac{\Gamma,\ \mathcal{A}[(\mu X)\mathcal{A}[X]]}{\Gamma,\ (\mu X)\mathcal{A}[X]}. \tag{μ}$$

IV. ν-rules of $\mathbf{K}_\omega(\mu)$. For all finite sets Γ of $\mathcal{L}_\mu$ formulas and all $\mathcal{L}_\mu$ formulas $\mathcal{A}[U]$ where U does not occur in $\mathcal{A}$ and $\mathcal{A}[U]$ is positive in U:

$$\frac{\dots\ \Gamma,\ (\nu X)^n\mathcal{A}[X]\ \dots \quad (\text{for all } 0 < n < \omega)}{\Gamma,\ (\nu X)\mathcal{A}[X]}. \tag{ν}$$

In analogy to before, we write $\mathbf{K}_\omega(\mu) \vdash \Gamma$ to express that the finite set Γ of $\mathcal{L}_\mu$ formulas is provable in $\mathbf{K}_\omega(\mu)$.

We are free to regard $\mathbf{K}_\omega(\mu)$ as the $\mathcal{L}_\mu$ counterpart of the $\mathcal{L}_\mu^+$ calculus $\mathbf{K}_\omega^+(\mu)$. In particular, we have the following embedding result.

Theorem 16. *For all finite sets* Γ *of* $\mathcal{L}_\mu^+$ *formulas we have*

$$\mathbf{K}_\omega^+(\mu) \vdash \Gamma \implies \mathbf{K}_\omega(\mu) \vdash \Gamma^*.$$

In particular, if Δ *is a finite set of* $\mathcal{L}_\mu$ *formulas, then*

$$\mathbf{K}_\omega^+(\mu) \vdash \Delta \implies \mathbf{K}_\omega(\mu) \vdash \Delta.$$

Proof. To show the first assertion, we proceed by induction on the proof of Γ in $\mathbf{K}_\omega^+(\mu)$. Just observe that all applications of the rules $(\nu.1)$ and $(\nu.n{+}1)$ trivialize in view of the *-translation and that $(\nu.\omega)$ goes over into (ν). The second assertion is a direct consequence of the first and Lemma 15. □

5. Saturated sets

We will show the completeness of the calculus $\mathbf{K}_\omega^+(\mu)$ by extending the technique of saturated sets, cf. e.g. Alberucci and Jäger [1]. Before we go on, we need some additional notation.

Definition 17. The *Fischer–Ladner closure* $\mathbb{FL}(D)$ of an $\mathcal{L}_\mu$ sentence D is the set of $\mathcal{L}_\mu$ formulas which is inductively generated as follows:

1. D belongs to $\mathbb{FL}(D)$.
2. If $(A \vee B)$ belongs to $\mathbb{FL}(D)$, then A and B belong to $\mathbb{FL}(D)$.
3. If $(A \wedge B)$ belongs to $\mathbb{FL}(D)$, then A and B belong to $\mathbb{FL}(D)$.
4. If $\langle a\rangle B$ belongs to $\mathbb{FL}(D)$, then B belongs to $\mathbb{FL}(D)$.
5. If $[a]B$ belongs to $\mathbb{FL}(D)$, then B belongs to $\mathbb{FL}(D)$.
6. If $(\mu X)\mathcal{A}[X]$ belongs to $\mathbb{FL}(D)$, then $\mathcal{A}[\bot]$ and $\mathcal{A}[(\mu X)\mathcal{A}[X]]$ belong to $\mathbb{FL}(D)$.
7. If $(\nu X)\mathcal{A}[X]$ belongs to $\mathbb{FL}(D)$, then $\mathcal{A}[\top]$ and $\mathcal{A}[(\nu X)\mathcal{A}[X]]$ belong to $\mathbb{FL}(D)$.

The Fischer–Ladner closure is a standard concept in the realm of fixed-point and dynamic logics, going back to Fischer and Ladner [8]. An uncomplicated adaptation of the proof in this article yields the finiteness of $\mathbb{FL}(D)$.

Lemma 18. *The cardinality of the Fischer–Ladner closure* $\mathbb{FL}(D)$ *of an* $\mathcal{L}_\mu$ *formula D is linear in the length* $lh(D)$ *of D; in particular,* $\mathbb{FL}(D)$ *is finite.*

Unfortunately, the Fischer–Ladner closure does not provide a sufficiently rich framework for the model construction we plan to carry through. This will be provided by the notion of strong closure $\mathbb{SC}(D)$ of an $\mathcal{L}_\mu$ formula D which is defined now.

Definition 19. The *strong closure* $\mathbb{SC}(D)$ of an $\mathcal{L}_\mu$ sentence D is the set of $\mathcal{L}_\mu^+$ formulas which is inductively generated as follows:

1. D belongs to $\mathbb{SC}(D)$.
2. If $(A \vee B)$ belongs to $\mathbb{SC}(D)$, then A and B belong to $\mathbb{SC}(D)$.
3. If $(A \wedge B)$ belongs to $\mathbb{SC}(D)$, then A and B belong to $\mathbb{SC}(D)$.
4. If $\langle a\rangle B$ belongs to $\mathbb{SC}(D)$, then B belongs to $\mathbb{SC}(D)$.
5. If $[a]B$ belongs to $\mathbb{SC}(D)$, then B belongs to $\mathbb{SC}(D)$.
6. If $(\mu X)\mathcal{A}[X]$ belongs to $\mathbb{SC}(D)$, then $\mathcal{A}[\bot]$ and $\mathcal{A}[(\mu X)\mathcal{A}[X]]$ belong to $\mathbb{SC}(D)$.
7. If $(\nu X)\mathcal{A}[X]$ belongs to $\mathbb{SC}(D)$, then $\mathcal{A}[\top]$ and, for every natural number n greater than 0, $(\nu^n X)\mathcal{A}[X]$ belong to $\mathbb{SC}(D)$.
8. If $(\nu^1 X)\mathcal{A}[X]$ belongs to $\mathbb{SC}(D)$, then $\mathcal{A}[\top]$ belongs to $\mathbb{SC}(D)$.
9. If n is a natural number greater than 0 and $(\nu^{n+1} X)\mathcal{A}[X]$ belongs to $\mathbb{SC}(D)$, then $\mathcal{A}[(\nu^n X)\mathcal{A}[X]]$ belongs to $\mathbb{SC}(D)$.
10. If $\mathcal{A}[\top]$ belongs to $\mathbb{SC}(D)$, then, for every free variable U, $\mathcal{A}[U]$ belongs to $\mathbb{SC}(D)$.

The sets $\mathbb{SC}(D)$ are infinite in general. Nevertheless we have a decisive relationship between the sets $\mathbb{FL}(D)$ and $\mathbb{SC}(D)$ which is described in the following lemma and easily proved by induction on the generation of the set $\mathbb{SC}(D)$.

Lemma 20. *Let D be some $\mathcal{L}_\mu$ sentence. Then for all $\mathcal{L}_\mu^+$ formulas A we have*

$$A \in \mathbb{SC}(D) \implies A^- \in \mathbb{FL}(D).$$

It is a direct consequence of Lemma 10 that the sets $\{lh(A) : A \in \mathbb{SC}(D)\}$ and $\{lh(A^-) : A \in \mathbb{SC}(D)\}$ are identical. The previous lemma and Lemma 18, stating the finiteness of the Fischer-Ladner closure $\mathbb{FL}(D)$, thus imply a further finiteness result.

Lemma 21. *Let D be some $\mathcal{L}_\mu$ sentence. Then $\{lh(A) : A \in \mathbb{SC}(D)\}$ is a finite set of natural numbers.*

Therefore the ranks of all formulas in $\mathbb{SC}(D)$ are finite sequences of ordinals whose lengths are bounded by some natural number and, consequently, according to a standard result, well-ordered by their lexicographical ordering.

Lemma 22. *If D is a sentence of $\mathcal{L}_\mu$, then the restriction of the lexicographical ordering $<_{lex}$ to the set $\{rk(A) : A \in \mathbb{SC}(D)\}$ is a well-ordering.*

In other words, definitions and proofs by induction on the ranks of the formulas from $\mathbb{SC}(D)$ are legitimate. This justifies, for example, to canonically extend the semantics of $\mathcal{L}_\mu$ to $\mathcal{L}_\mu^+$. Given a sentence D of $\mathcal{L}_\mu$, a Kripke structure $\mathfrak{M}$ and a valuation $\mathfrak{v}$ in $\mathfrak{M}$, we simply add, for $n \geq 1$ and formulas $(\nu^1 X)\mathcal{A}[X]$ and $(\nu^{n+1} X)\mathcal{A}[X]$ from $\mathbb{SC}(D)$, the following clauses:

$$\|(\nu^1 X)\mathcal{A}[X]\|_{(\mathfrak{M},\mathfrak{v})} := \|\mathcal{A}[\top]\|_{(\mathfrak{M},\mathfrak{v})},$$

$$\|(\nu^{n+1} X)\mathcal{A}[X]\|_{(\mathfrak{M},\mathfrak{v})} := \|\mathcal{A}[(\nu^n X)\mathcal{A}[X]]\|_{(\mathfrak{M},\mathfrak{v})}.$$

We now come to the central concept of this section. The starting point is an arbitrary formula D of $\mathcal{L}_\mu$. Then we are interested in all finite subsets of $\mathbb{SC}(D)$ which are not derivable in $\mathbf{K}_\omega^+(\mu)$ and have the closure properties (S.2) and (S.3) below. These so-called D-saturated sets will form the elements of the Kripke structure $\mathfrak{S}_D$, cf. Definition 25, playing the crucial part in our proof of the completeness of $\mathbf{K}_\omega^+(\mu)$.

Definition 23. Let D be some $\mathcal{L}_\mu$ sentence. A finite subset Γ of $\mathbb{SC}(D)$ is called *D-saturated* (with respect to $\mathbf{K}_\omega^+(\mu)$) if the following conditions are satisfied:

(S.1) $\mathbf{K}_\omega^+(\mu) \nvdash \Gamma$.

(S.2) For all $\mathcal{L}_\mu^+$ formulas A and B we have

$$A \vee B \in \Gamma \implies A \in \Gamma \text{ and } B \in \Gamma,$$

$$A \wedge B \in \Gamma \implies A \in \Gamma \text{ or } B \in \Gamma.$$

(S.3) For all $\mathcal{L}_\mu^+$ formulas $\mathcal{A}[U]$ where U does not occur in $\mathcal{A}$ and $\mathcal{A}[U]$ is positive in U and all natural numbers n greater than 0 we have

$$(\mu X)\mathcal{A}[X] \in \Gamma \implies \mathcal{A}[(\mu X)\mathcal{A}[X]] \in \Gamma,$$

$$(\nu X)\mathcal{A}[X] \in \Gamma \implies (\nu^i X)\mathcal{A}[X] \in \Gamma \text{ for some } 0 < i < \omega,$$

$$(\nu^{n+1} X)\mathcal{A}[X] \in \Gamma \implies \mathcal{A}[(\nu^n X)\mathcal{A}[X]] \in \Gamma,$$

$$(\nu^1 X)\mathcal{A}[X] \in \Gamma \implies \mathcal{A}[\top] \in \Gamma.$$

Suppose that Γ is a finite subset of $\mathbb{SC}(D)$, not provable in $\mathbf{K}_\omega^+(\mu)$ and not D-saturated. Then one of the conditions (S.2), (S.3) has to be violated for Γ. By systematically correcting such deficiencies, we can extend this Γ to a D-saturated Δ.

Lemma 24. *Let D be some $\mathcal{L}_\mu$ sentence. For every finite subset Γ of $\mathbb{SC}(D)$ which is not provable in $\mathbf{K}^+_\omega(\mu)$ there exists a finite subset Δ of $\mathbb{SC}(D)$ which is D-saturated and contains Γ.*

Proof. We begin by fixing an arbitrary enumeration $F_0, F_1, \ldots$ of all formulas in $\mathbb{SC}(D)$ and call the least i such that the formula A from $\mathbb{SC}(D)$ is identical to F_i the *index* of A. Besides that, we introduce the following auxiliary notion:

Let N be a subset of $\mathbb{SC}(D)$. Then the N*-rank* $rk(N, A)$ of a formula A from $\mathbb{SC}(D)$ is set to be $\langle 0 \rangle$ provided that $A \in N$,

$$A \in N \implies rk(N, A) = \langle 0 \rangle,$$

and inductively defined according to Definition 9 for all A not from N. In analogy to the definition of $lh(A)$ we write $lh(N, A)$ for $lh(rk(N, A))$. Clearly, for all subsets N, N_1, N_2 of $\mathbb{SC}(D)$ and all elements A of $\mathbb{SC}(D)$, these modified ranks have the following properties:

$$lh(N, A) \leq lh(A), \tag{1}$$

$$A \in N \implies rk(N, \mathcal{B}[A]) = rk(N, \mathcal{B}[\top]), \tag{2}$$

$$N_1 \subset N_2 \implies rk(N_2, A) \leq_{lex} rk(N_1, A). \tag{3}$$

From (1) and Lemma 21 we obtain a strengthening of Lemma 22: even the restriction of $<_{lex}$ to the set $\{rk(N, A) : N \subset \mathbb{SC}(D) \text{ and } A \in \mathbb{SC}(D)\}$ is a well-ordering. Given a subset N of $\mathbb{SC}(D)$ and a formula A from $\mathbb{SC}(D)$, it therefore makes sense to write $ot(N, A)$ for the order type of $rk(N, A)$ with respect to this well-ordering.

Depending on the given finite subset Γ of $\mathbb{SC}(D)$ which, by assumption, is not provable in $\mathbf{K}^+_\omega(\mu)$ we now inductively define, for each natural number n, subsets Γ_n of $\mathbb{SC}(D)$ and auxiliary sets M_n; we convince ourselves during this process that these Γ_n are not provable in $\mathbf{K}^+_\omega(\mu)$:

1. $\Gamma_0 := \Gamma$ and $M_0 := \emptyset$.
2. If Γ_n is D-saturated, then $\Gamma_{n+1} := \Gamma_n$ and $M_{n+1} := M_n$.
3. If Γ_n is not D-saturated, we choose the formula A with least index that violates either of the conditions in (S.2) and (S.3); afterwards Γ_{n+1} and M_{n+1} are determined by distinguishing between the possible forms of A.

3.1. A is a formula $(B \vee C)$. Then we set

$$\Gamma_{n+1} := \Gamma_n \cup \{B, C\} \quad \text{and} \quad M_{n+1} := M_n.$$

3.2. A is a formula $(B \wedge C)$. Since Γ_n is not provable in $\mathbf{K}^+_\omega(\mu)$ we know that either

$$\mathbf{K}^+_\omega(\mu) \nvdash \Gamma_n, B \quad \text{or} \quad \mathbf{K}^+_\omega(\mu) \nvdash \Gamma_n, C.$$

Then we set

$$\Gamma_{n+1} := \begin{cases} \Gamma_n \cup \{B\} & \text{if } \mathbf{K}^+_\omega(\mu) \nvdash \Gamma_n, B \\ \Gamma_n \cup \{C\} & \text{otherwise} \end{cases} \quad \text{and} \quad M_{n+1} := M_n.$$

3.3. A is a formula $(\mu X)\mathcal{B}[X]$. Then we set

$$\Gamma_{n+1} := \Gamma_n \cup \{\mathcal{B}[(\mu X)\mathcal{B}[X]]\} \quad \text{and} \quad M_{n+1} := M_n \cup \{(\mu X)\mathcal{B}[X]\}.$$

3.4. A is a formula $(\nu X)\mathcal{B}[X]$. Since Γ_n is not provable in $\mathbf{K}^+_\omega(\mu)$ we know that

$$\mathbf{K}^+_\omega(\mu) \nvdash \Gamma_n, (\nu^i X)\mathcal{B}[X]$$

for some natural number i greater than 0. We choose the least such i and set

$$\Gamma_{n+1} := \Gamma_n \cup \{(\nu^i X)\mathcal{B}[X]\} \quad \text{and} \quad M_{n+1} := M_n.$$

3.5. A is a formula $(\nu^{i+1}X)\mathcal{B}[X]$ for some natural number i greater than 0. Then we set

$$\Gamma_{n+1} := \Gamma_n \cup \{\mathcal{B}[(\nu^i X)\mathcal{B}[X]]\} \quad \text{and} \quad M_{n+1} := M_n.$$

3.6. A is a formula $(\nu^1 X)\mathcal{B}[X]$. Then we set

$$\Gamma_{n+1} := \Gamma_n \cup \{\mathcal{B}[\top]\} \quad \text{and} \quad M_{n+1} := M_n.$$

What we have done so far guarantees that, for all natural numbers n,

$$\mathbf{K}^+_\omega(\mu) \nvdash \Gamma_n, \tag{4}$$

$$\Gamma \subset \Gamma_n \subset \Gamma_{n+1} \quad \text{and} \quad M_n \subset M_{n+1}, \tag{5}$$

$$(\mu X)\mathcal{B}[X] \in M_n \quad \Longrightarrow \quad \mathcal{B}[(\mu X)\mathcal{B}[X]] \in \Gamma_n. \tag{6}$$

Next we turn to two properties of this sequence $(\Gamma_0, M_0), (\Gamma_1, M_1), \ldots$ which will be crucial in proving that Γ_n will be D-saturated for some natural number n.

(i) If the formula $(\mu X)\mathcal{B}[X]$ belongs to M_{n+1} but not to M_n, then

$$ot(M_{n+1}, \mathcal{B}[(\mu X)\mathcal{B}[X]]) < ot(M_n, (\mu X)\mathcal{B}[X]).$$

(ii) If $(\mu X)\mathcal{B}[X]$ is the formula violating one of the conditions in (S.2) and (S.3) which is picked at step $n+1$ of the construction described above, then

$$ot(M_{n+1}, \mathcal{B}[(\mu X)\mathcal{B}[X]]) < ot(M_n, (\mu X)\mathcal{B}[X]).$$

To prove (i), assume that $(\mu X)\mathcal{B}[X]$ is an element of $M_{n+1} \setminus M_n$. By (2) and (3) this implies

$$rk(M_{n+1}, \mathcal{B}[(\mu X)\mathcal{B}[X]]) = rk(M_{n+1}, \mathcal{B}[\top]) \leq_{lex} rk(M_n, \mathcal{B}[\top]). \tag{7}$$

But we also have, since $(\mu X)\mathcal{B}[X] \notin M_n$,

$$rk(M_n, \mathcal{B}[\top]) <_{lex} rk(M_n, \mathcal{B}[\top]) * \langle 0 \rangle = rk(M_n, (\mu X)\mathcal{B}[X]). \tag{8}$$

Assertions (7) and (8) imply $rk(M_{n+1}, \mathcal{B}[(\mu X)\mathcal{B}[X]]) <_{lex} rk(M_n, (\mu X)\mathcal{B}[X])$, hence (i) is proved. Because of (6), (ii) is an immediate consequence of (i).

In a next step we assign to all finite subsets N and Π of $\mathbb{SC}(D)$, with Π not being provable in $\mathbf{K}^+_\omega(\mu)$, their *deficiency numbers* $dn(N, \Pi)$:

(D.1) If Π is D-saturated, then $dn(N, \Pi) := 0$.

(D.2) Otherwise, fix some enumeration $A_1, A_2, \ldots, A_m$ (without repetitions) of all elements of Π violating one of the conditions in (S.2) and (S.3) and set

$$dn(N, \Pi) := \omega^{ot(N,A_1)} \# \omega^{ot(N,A_2)} \# \cdots \# \omega^{ot(N,A_m)},$$

where # stands for the natural sum of ordinals as introduced, for example, in Schütte [20].

Coming to the end of this proof, we observe that (ii) together with the definition of the relativized rank function and (3) yields for all natural numbers n that

$$\Gamma_n \text{ is not } D\text{-saturated} \quad \Longrightarrow \quad dn(M_{n+1}, \Gamma_{n+1}) < dn(M_n, \Gamma_n). \tag{9}$$

Since there are no infinite decreasing sequences of ordinals, one of the sets Γ_n has to be D-saturated and is thus a possible candidate for the choice of Δ. $\square$

Our interest is in Kripke structures $\mathfrak{S}_D$, depending on $\mathcal{L}_\mu$ formulas D, whose universes are the D-saturated subsets of $\mathbb{SC}(D)$. We will eventually show that an $\mathcal{L}_\mu$ formula D is provable in $\mathbf{K}^+_\omega(\mu)$ if D is valid in $\mathfrak{S}_D$.

Definition 25. Let D be some $\mathcal{L}_\mu$ sentence. Then $\mathfrak{S}_D$ is the Kripke structure which is defined by the following three conditions:

($\mathfrak{S}_D$.1) The universe $|\mathfrak{S}_D|$ of $\mathfrak{S}_D$ consists exactly of the D-saturated sets.
($\mathfrak{S}_D$.2) For any label a, the binary relation $\mathfrak{S}_D(a)$ on $|\mathfrak{S}_D|$ is given by

$$(\Gamma, \Delta) \in \mathfrak{S}_D(a) \quad :\Longleftrightarrow \quad (\Gamma, \Delta) \in |\mathfrak{S}_D|^2 \text{ and } \{ B : \langle a \rangle B \in \Gamma \} \subset \Delta.$$

($\mathfrak{S}_D$.3) For any atomic proposition P, the subset $\mathfrak{S}_D(P)$ of $|\mathfrak{S}_D|$ is given by

$$\mathfrak{S}_D(P) := \{ \Gamma \in |\mathfrak{S}_D| : P \notin \Gamma \}.$$

Although we are finally interested in interpreting $\mathcal{L}_\mu$ and $\mathcal{L}_\mu^+$ formulas in Kripke structures $\mathfrak{S}_D$, technical reasons (see the proof of Lemma 33 below) compel us to work with intermediate structures in which formulas of the form $(\mu X)\mathcal{A}[X]$ are interpreted as stages of the least fixed points of the inductive definitions associated to (the interpretation of) $\mathcal{A}$. The depth of the nestings of the fixed point operator μ has to be taken into account as well.

Definition 26. The *μ-height* $h_\mu(A)$ of an $\mathcal{L}_\mu^+$ formula A is inductively defined as follows:

1. If A is an atomic proposition, the negation of an atomic proposition, a free variable, the negation of a free variable or a propositional constant, then $h_\mu(A) := 0$.
2. If A is a formula $(B \vee C)$ or a formula $(B \wedge C)$, then
$$h_\mu(A) := \max(h_\mu(B), h_\mu(C)).$$
3. If A is a formula $\langle a \rangle B$ or a formula $[a]B$, then
$$h_\mu(A) := h_\mu(B).$$
4. If A is a formula $(\mu X)\mathcal{A}[X]$, then
$$h_\mu(A) := h_\mu(\mathcal{A}[\top]) + 1.$$
5. If A is a is a formula $(\nu X)\mathcal{A}[X]$ or $(\nu^n X)\mathcal{A}[X]$ for some natural number n greater than 0, then
$$h_\mu(A) := h_\mu(\mathcal{A}[\top]).$$

From this definition we immediately obtain the first assertion of the following lemma, and, together with Lemma 21, the second part.

Lemma 27

1. *For all $\mathcal{L}_\mu^+$ formulas A we have $h_\mu(A) = h_\mu(A^-)$ and $h_\mu(A) < lh(A)$.*
2. *If D is an $\mathcal{L}_\mu$ formula, then $\{h_\mu(A) : A \in \mathbb{SC}(D)\}$ is a finite set of natural numbers.*

These observations justify assigning to each $\mathcal{L}_\mu$ formula D a specific natural number, the μ-bound of D, majorizing the μ-heights of the formulas in $\mathbb{SC}(D)$.

Definition 28. The *μ-bound* $b_\mu(D)$ of an $\mathcal{L}_\mu$ sentence D is the least natural number n so that $h_\mu(A) \le n$ for all formulas A from $\mathbb{SC}(D)$.

The μ-heights and finite sequences of ordinals play an important role in the context of so-called signed truth sets. This concept has been introduced in Streett and Emerson [23] and is adjusted here to our needs.

Definition 29. Let D be some $\mathcal{L}_\mu$ sentence whose μ-bound $b_\mu(D)$ is the natural number n, and consider a sequence of ordinals $\boldsymbol{\sigma} = \langle \sigma_1, \dots, \sigma_n \rangle$. Then, for any formula A from the set $\mathbb{SC}(D)$ and any valuation $\mathfrak{v}$ in $\mathfrak{S}_D$, the *signed truth set* $\|A\|^{\boldsymbol{\sigma}}_{(D,\mathfrak{v})}$ is defined by induction on $rk(A)$ as follows:

1. For atomic propositions, free variables and propositional constants:

$$\|P\|^{\sigma}_{(D,\mathfrak{v})} := \mathfrak{S}_D(P), \qquad \|{\sim}P\|^{\sigma}_{(D,\mathfrak{v})} := |\mathfrak{S}_D| \setminus \mathfrak{S}_D(P),$$
$$\|U\|^{\sigma}_{(D,\mathfrak{v})} := \mathfrak{v}(U), \qquad \|{\sim}U\|^{\sigma}_{(D,\mathfrak{v})} := |\mathfrak{S}_D| \setminus \mathfrak{v}(U),$$
$$\|\top\|^{\sigma}_{(D,\mathfrak{v})} := |\mathfrak{S}_D|, \qquad \|\bot\|^{\sigma}_{(D,\mathfrak{v})} := \emptyset.$$

2. For disjunctions and conjunctions:

$$\|A \vee B\|^{\sigma}_{(D,\mathfrak{v})} := \|A\|^{\sigma}_{(D,\mathfrak{v})} \cup \|B\|^{\sigma}_{(D,\mathfrak{v})},$$
$$\|A \wedge B\|^{\sigma}_{(D,\mathfrak{v})} := \|A\|^{\sigma}_{(D,\mathfrak{v})} \cap \|B\|^{\sigma}_{(D,\mathfrak{v})}.$$

3. For formulas prefixed by a modal operator:

$$\|\langle a\rangle B\|^{\sigma}_{(D,\mathfrak{v})} := \{\Gamma \in |\mathfrak{S}_D| : (\exists\Delta)((\Gamma,\Delta) \in \mathfrak{S}_D(a) \;\&\; \Delta \in \|B\|^{\sigma}_{(D,\mathfrak{v})})\},$$
$$\|[a]B\|^{\sigma}_{(D,\mathfrak{v})} := \{\Gamma \in |\mathfrak{S}_D| : (\forall\Delta)((\Gamma,\Delta) \in \mathfrak{S}_D(a) \Rightarrow \Delta \in \|B\|^{\sigma}_{(D,\mathfrak{v})})\}.$$

4. For least fixed point formulas: Given a formula $\mathcal{A}[U]$ where U does not occur in $\mathcal{A}$ and $\mathcal{A}[U]$ is positive in U, we first introduce the monotone operator

$$\Phi : Pow(|\mathfrak{S}_D|) \to Pow(|\mathfrak{S}_D|), \quad \Phi(S) := \|\mathcal{A}[U]\|^{\sigma}_{(D,\mathfrak{v}[U:S])}.$$

Based on this Φ, we now set, for $m = h_\mu((\mu X)\mathcal{A}[X])$,

$$\|(\mu X)\mathcal{A}[X]\|^{\sigma}_{(D,\mathfrak{v})} := I_\Phi^{<\sigma_m}.$$

5. For greatest fixed point formulas: Given a formula $\mathcal{A}[U]$ where U does not occur in $\mathcal{A}$ and $\mathcal{A}[U]$ is positive in U and k is a natural number greater than 0:

$$\|(\nu^1 X)\mathcal{A}[X]\|^{\sigma}_{(D,\mathfrak{v})} := \|\mathcal{A}[\top]\|^{\sigma}_{(D,\mathfrak{v})},$$
$$\|(\nu^{k+1} X)\mathcal{A}[X]\|^{\sigma}_{(D,\mathfrak{v})} := \|\mathcal{A}[(\nu^k X)\mathcal{A}[X]]\|^{\sigma}_{(D,\mathfrak{v})},$$
$$\|(\nu X)\mathcal{A}[X]\|^{\sigma}_{(D,\mathfrak{v})} := \bigcap_{i<\omega} \|(\nu^i X)\mathcal{A}[X]\|^{\sigma}_{(D,\mathfrak{v})}.$$

In the special case of a finite sequence of ordinals consisting of identical components, a useful substitution property is available. Its proof is by induction on $rk(\mathcal{A}[U])$.

Lemma 30. *Let D be some $\mathcal{L}_\mu$ sentence, B some $\mathcal{L}_\mu^+$ formula and $\mathcal{A}[U]$ an $\mathcal{L}_\mu^+$ formula where U does not occur in $\mathcal{A}$ and $\mathcal{A}[U]$ is positive in U. Assume, in addition, that $\mathcal{A}[U]$ and $\mathcal{A}[B]$ belong to $\mathbb{SC}(D)$. For all ordinals σ and all sequences of ordinals $\boldsymbol{\sigma} = \langle\sigma,\ldots,\sigma\rangle$ of length $b_\mu(D)$, all valuations $\mathfrak{v}$ in $\mathfrak{S}_D$ and all subsets S of $|\mathfrak{S}_D|$ we then have*

$$S = \|B\|^{\sigma}_{(D,\mathfrak{v})} \implies \|\mathcal{A}[B]\|^{\sigma}_{(D,\mathfrak{v})} = \|\mathcal{A}[U]\|^{\sigma}_{(D,\mathfrak{v}[U:S])}.$$

Signed truth sets $\|A\|^{\sigma}_{(D,\mathfrak{v})}$ deviate in two fundamental aspects from the truth sets $\|A\|_{(\mathfrak{S}_D,\mathfrak{v})}$: (i) subformulas $(\mu X)\mathcal{B}[X]$ of A are not interpreted by the corresponding least fixed points but only by their approximations determined by the $h_\mu((\mu X)\mathcal{B}[X])$-th component of $\boldsymbol{\sigma}$; (ii) subformulas $(\nu X)\mathcal{B}[X]$ of A are interpreted as the intersection of the finite approximations of the corresponding greatest fixed points, meaning that, in general, their interpretations are proper supersets of the greatest fixed points. There is, however, an interesting relationship between signed truth sets and truth sets.

Lemma 31. *Let us assume that*

(A.1) *D is an $\mathcal{L}_\mu$ sentence,*
(A.2) *κ is the least cardinal greater than the cardinality of $|\mathfrak{S}_D|$,*
(A.3) *$\boldsymbol{\kappa}$ is the sequence of ordinals $\langle\kappa,\ldots,\kappa\rangle$ of length $b_\mu(D)$.*

Then for all formulas A from $\mathbb{SC}(D)$ and valuations $\mathfrak{v}$ in $\mathfrak{S}_D$ we have

$$\|A\|_{(\mathfrak{S}_D,\mathfrak{v})} \subset \|A\|^{\kappa}_{(D,\mathfrak{v})}.$$

Proof. This proceeds by induction on $rk(A)$, and we distinguish the following cases:

1. A is an atomic proposition, a free variable or a propositional constants. Then the assertion is obvious.

2. A is a disjunction or a conjunction, a formula prefixed by a modal operator or a formula $(\nu^n X)\mathcal{A}[X]$ for some natural number n greater than 0. Then the assertion follows directly from the induction hypothesis.

3. A is a formula $(\mu X)\mathcal{A}[X]$ or $(\nu X)\mathcal{A}[X]$. Then we first pick a free variable U which does not occur in $\mathcal{A}$ and consider the two monotone operators

$$\Phi : Pow(|\mathfrak{S}_D|) \to Pow(|\mathfrak{S}_D|), \quad \Phi(S) := \|\mathcal{A}[U]\|_{(\mathfrak{S}_D,\mathfrak{v}[U:S])},$$

$$\Psi : Pow(|\mathfrak{S}_D|) \to Pow(|\mathfrak{S}_D|), \quad \Psi(S) := \|\mathcal{A}[U]\|^{\kappa}_{(D,\mathfrak{v}[U:S])}.$$

By induction hypothesis we have $\Phi(S) \subset \Psi(S)$ for all $S \subset \mathbb{SC}(D)$; therefore $I_\Phi \subset I_\Psi$ and $J_\Phi \subset J_\Psi$. Consequently,

$$\|(\mu X)\mathcal{A}[X]\|_{(\mathfrak{S}_D,\mathfrak{v})} = I_\Phi \subset I_\Psi = I_\Psi^{<\kappa} = \|(\mu X)\mathcal{A}[X]\|^{\kappa}_{(D,\mathfrak{v})}, \tag{1}$$

$$\|(\nu X)\mathcal{A}[X]\|_{(\mathfrak{S}_D,\mathfrak{v})} = J_\Phi \subset J_\Psi \subset J_\Psi^{<\omega} = \|(\nu X)\mathcal{A}[X]\|^{\kappa}_{(D,\mathfrak{v})} \tag{2}$$

are easily obtained by recalling the definitions of the truth sets and signed truth sets of these fixed point formulas; for the last equality in (2) we also have to make use of the previous lemma. This completes our proof. □

Considering a formula $(\mu X)\mathcal{A}[X]$ and its unfolding $\mathcal{A}[(\mu X)\mathcal{A}[X]]$ we want to show that given a sequence of ordinals $\boldsymbol{\sigma}$ there exists a lexicographically smaller sequence $\boldsymbol{\tau}$ such that the signed truth set of $(\mu X)\mathcal{A}[X]$ under $\boldsymbol{\sigma}$ is a subset of the signed truth set of $\mathcal{A}[(\mu X)\mathcal{A}[X]]$ under $\boldsymbol{\tau}$. This property will be of crucial use to us when considering signed denotations in an induction on sequences of ordinals of bounded length. For technical reasons we deal with a more general version of this assertion.

Lemma 32. *Let us assume that*

(A.1) *D is an $\mathcal{L}_\mu$ sentence, $\mathcal{A}[U]$ and $\mathcal{B}[U]$ are formulas from $\mathbb{SC}(D)$ positive in U, and U occurs neither in $\mathcal{A}$ nor in $\mathcal{B}$,*
(A.2) *$(\mu X)\mathcal{A}[X]$ and $\mathcal{B}[(\mu X)\mathcal{A}[X]]$ belong to $\mathbb{SC}(D)$,*
(A.3) *$h_\mu((\mu X)\mathcal{A}[X]) = m + 1$ and $h_\mu(\mathcal{B}[U]) \leq m$,*
(A.4) *κ is the least cardinal greater than the cardinality of $|\mathfrak{S}_D|$,*
(A.5) *$\boldsymbol{\sigma}$ is a sequence of ordinals $\langle \sigma_1, \ldots, \sigma_n\rangle$ of length $n = b_\mu(D)$,*
(A.6) *$\boldsymbol{\tau}$ the sequence of ordinals $\langle \sigma_1, \ldots, \sigma_m, \alpha, \kappa, \ldots, \kappa\rangle$ of length n.*

Then for every valuation $\mathfrak{v}$ in $\mathfrak{S}_D$ and the associated monotone operator

$$\Phi : Pow(|\mathfrak{S}_D|) \to Pow(|\mathfrak{S}_D|), \quad \Phi(S) := \|\mathcal{A}[U]\|^{\boldsymbol{\sigma}}_{(D,\mathfrak{v}[U:S])}$$

we have

$$\|\mathcal{B}[U]\|^{\boldsymbol{\sigma}}_{(D,\mathfrak{v}[U:I_\Phi^{<\alpha}])} \subset \|\mathcal{B}[(\mu X)\mathcal{A}[X]]\|^{\boldsymbol{\tau}}_{(D,\mathfrak{v})}.$$

Proof. We prove this assertion by induction on $rk(\mathcal{B}[U])$. Given a valuation $\mathfrak{v}$ in $\mathfrak{S}_D$, we distinguish the following cases:

1. U does not occur in $\mathcal{B}[U]$. Then, trivially,

$$\|\mathcal{B}[U]\|^{\boldsymbol{\sigma}}_{(D,\mathfrak{v}[U:I_\Phi^{<\alpha}])} = \|\mathcal{B}[U]\|^{\boldsymbol{\sigma}}_{(D,\mathfrak{v})} \quad \text{and} \quad \|\mathcal{B}[U]\|^{\boldsymbol{\tau}}_{(D,\mathfrak{v})} = \|\mathcal{B}[(\mu X)\mathcal{A}[X]]\|^{\boldsymbol{\tau}}_{(D,\mathfrak{v})}.$$

From $h_\mu(\mathcal{B}[U]) \leq m$ we further obtain $\|\mathcal{B}[U]\|^{\boldsymbol{\sigma}}_{(D,\mathfrak{v})} = \|\mathcal{B}[U]\|^{\boldsymbol{\tau}}_{(D,\mathfrak{v})}$, completing the discussion of this case.

2. $\mathcal{B}[U]$ is the formula U. In this case we first introduce the auxiliary monotone operator

$$\Psi : Pow(|\mathfrak{S}_D|) \to Pow(|\mathfrak{S}_D|), \quad \Psi(S) := \|\mathcal{A}[U]\|^{\boldsymbol{\tau}}_{(D,\mathfrak{v}[U:S])}$$

However, since $h_\mu(\mathcal{A}[U]) \leq m$, we have $\Phi(S) = \Psi(S)$ for all $S \subset |\mathfrak{S}_D|$, and this implies

$$\|\mathcal{B}[U]\|^{\boldsymbol{\sigma}}_{(D,\mathfrak{v}[U:I_\Phi^{<\alpha}])} = I_\Phi^{<\alpha} = I_\Psi^{<\alpha} = \|(\mu X)\mathcal{A}[X]\|^{\boldsymbol{\tau}}_{(D,\mathfrak{v})} = \|\mathcal{B}[(\mu X)\mathcal{A}[X]]\|^{\boldsymbol{\tau}}_{(D,\mathfrak{v})}.$$

3. $\mathcal{B}[U]$ is a disjunction or a conjunction, a formula prefixed by a modal operator, a formula $(\nu X)\mathcal{A}[X]$ or a formula $(\nu^n X)\mathcal{A}[X]$ for some natural number n greater than 0. Then the assertion immediately follows from the induction hypothesis.

4. $\mathcal{B}[U]$ is a formula $(\mu Y)\mathcal{C}[U, Y]$ with U occurring in $\mathcal{B}[U]$. We select a free variable V different from U which does not occur in $\mathcal{C}[(\mu X)\mathcal{A}[X], \top]$ and consider the monotone operators

$$\Psi : Pow(|\mathfrak{S}_D|) \to Pow(|\mathfrak{S}_D|), \quad \Psi(S) := \|\mathcal{C}[U, V]\|^{\boldsymbol{\sigma}}_{(D,\mathfrak{v}[U:I_\Phi^{<\alpha}][V:S])},$$

$$\Omega : Pow(|\mathfrak{S}_D|) \to Pow(|\mathfrak{S}_D|), \quad \Omega(S) := \|\mathcal{C}[(\mu X)\mathcal{A}[X], V]\|^{\boldsymbol{\tau}}_{(D,\mathfrak{v}[V:S])}.$$

We claim that

$$I_\Psi^{<\xi} \subset I_\Omega^{<\xi} \quad \text{for all ordinals } \xi. \tag{1}$$

This is proved by side induction on ξ. If Π is an element of $I_\Psi^{<\xi}$, we have $\Pi \in \Psi(I_\Psi^{<\zeta})$ for some $\zeta < \xi$. Consequently, the side induction hypothesis implies

$$\Pi \in \Psi(I_\Omega^{<\zeta}) = \|\mathcal{C}[U, V]\|^{\boldsymbol{\sigma}}_{(D,\mathfrak{v}[U:I_\Phi^{<\alpha}][V:I_\Omega^{<\zeta}])}.$$

By the choice of V we also know that the valuation $\mathfrak{v}[U:I_\Phi^{<\alpha}][V:I_\Omega^{<\zeta}]$ is identical to the valuation $\mathfrak{v}[V:I_\Omega^{<\zeta}][U:I_\Phi^{<\alpha}]$, and so

$$\Pi \in \|\mathcal{C}[U, V]\|^{\boldsymbol{\sigma}}_{(D,\mathfrak{v}[V:I_\Omega^{<\zeta}][U:I_\Phi^{<\alpha}])}.$$

Since $rk(\mathcal{C}[U, V]) < rk(\mathcal{B}[U])$, it is possible to apply the main induction hypothesis and to infer that

$$\Pi \in \|\mathcal{C}[(\mu X)\mathcal{A}[X], V]\|^{\boldsymbol{\tau}}_{(D,\mathfrak{v}[V:I_\Omega^{<\zeta}])} = \Omega(I_\Omega^{<\zeta}).$$

Together with the trivial fact $\Omega(I_\Omega^{<\zeta}) \subset I_\Omega^{<\xi}$, this makes clear that claim (1) holds.

From assumption (A.3) we obtain $h_\mu(\mathcal{B}[U]) = k$ for some $k \leq m$, and it is then seen that

$$\|\mathcal{B}[U]\|^{\boldsymbol{\sigma}}_{(D,\mathfrak{v}[U:I_\Phi^{<\alpha}])} = I_\Psi^{<\sigma_k}.$$

Combining this result with assertion (1) and the choice of κ, we may continue with

$$\|\mathcal{B}[U]\|^{\boldsymbol{\sigma}}_{(D,\mathfrak{v}[U:I_\Phi^{<\alpha}])} \subset I_\Omega^{<\sigma_k} \subset I_\Omega^{<\kappa}. \tag{2}$$

Recalling that U occurs in $\mathcal{B}[U]$, the inequality $m + 1 < h_\mu(\mathcal{B}[(\mu X)\mathcal{A}[X]])$ becomes obvious, and therefore

$$\|\mathcal{B}[(\mu X)\mathcal{A}[X]]\|^{\boldsymbol{\tau}}_{(D,\mathfrak{v})} = \|(\mu Y)\mathcal{C}[(\mu X)\mathcal{A}[X], Y]\|^{\boldsymbol{\tau}}_{(D,\mathfrak{v})} = I_\Omega^{<\kappa}. \tag{3}$$

By (2) and (3) our assertion is also proved for this case. □

Lemma 33 (Truth lemma). *Let D be some $\mathcal{L}_\mu$ sentence and n its μ-bound $b_\mu(D)$. Then for all sequences of ordinals $\boldsymbol{\sigma}$ of lengths less than or equal to n, all sentences A from $\mathbb{SC}(D)$, all D-saturated subsets Γ of $\mathbb{SC}(D)$ and all valuations $\mathfrak{v}$ in $\mathfrak{S}_D$ we have*

$$A \in \Gamma \implies \Gamma \notin \|A\|^{\boldsymbol{\sigma}}_{(D,\mathfrak{v})}.$$

Proof. We show this lemma by main induction on the sequences of ordinals of lengths less than or equal to n and side induction on $rk(A)$, and distinguish the following cases:

1. A is an atomic proposition, the negation of an atomic proposition or a propositional constant. Then the assertion is easily verified.

2. A is a disjunction or a conjunction, a formula $(\nu^n X)\mathcal{A}[X]$ for some natural number n greater than 0 or a formula $(\nu X)\mathcal{A}[X]$. Then the assertion follows directly from the side induction hypothesis; see Lemma 11 and Theorem 13.

3. A is a formula $\langle a\rangle B$. Then choose an arbitrary D-saturated Δ satisfying $(\Gamma, \Delta) \in \mathfrak{S}_D(a)$. Therefore $\{C : \langle a\rangle C \in \Gamma\} \subset \Delta$ and, since A is an element of Γ according to our assumption, this yields $B \in \Delta$. By the side induction hypothesis we obtain $\Delta \notin \|B\|^{\boldsymbol{\sigma}}_{(D,\mathfrak{v})}$. This implies that $\Gamma \notin \|A\|^{\boldsymbol{\sigma}}_{(D,\mathfrak{v})}$.

4. A is a formula $[a]B$. Since Γ is D-saturated, $\mathbf{K}^+_\omega(\mu)$ does not prove Γ. By the (K) rule of $\mathbf{K}^+_\omega(\mu)$ we infer that

$$\mathbf{K}^+_\omega(\mu) \nvdash \{C : \langle a\rangle C \in \Gamma\},\ B. \tag{1}$$

But now we know, because of Lemma 24, that there exists a D-saturated Δ with the properties

$$\{C : \langle a\rangle C \in \Gamma\} \subset \Delta, \tag{2}$$

$$B \in \Delta. \tag{3}$$

By (2) we have $(\Gamma, \Delta) \in \mathfrak{S}_D(a)$, by (3) the side induction hypothesis implies $\Delta \notin \|B\|^{\boldsymbol{\sigma}}_{(D,\mathfrak{v})}$. And together this means that $\Gamma \notin \|A\|^{\boldsymbol{\sigma}}_{(D,\mathfrak{v})}$.

5. A is a formula $(\mu X)\mathcal{A}[X]$. Since, by assumption, A is an element of the D-saturated Γ, we know

$$\mathcal{A}[(\mu X)\mathcal{A}[X]] \in \Gamma. \tag{4}$$

We pick a free variable U which does not occur in $\mathcal{A}$ plus some valuation $\mathfrak{v}$ in $\mathfrak{S}_D$ and consider the monotone operator

$$\Phi : Pow(|\mathfrak{S}_D|) \to Pow(|\mathfrak{S}_D|), \quad \Phi(S) := \|\mathcal{A}[U]\|^{\boldsymbol{\sigma}}_{(D,\mathfrak{v}[U:S])}.$$

The μ-height of $(\mu X)\mathcal{A}[X]$ is a natural number $m + 1$ with $h_\mu(\mathcal{A}[U]) \le m$, and $\boldsymbol{\sigma} = \langle\sigma_1, \dots, \sigma_n\rangle$ for suitable ordinals $\sigma_1, \dots, \sigma_n$. According to Definition 29 we thus have

$$\|A\|^{\boldsymbol{\sigma}}_{(D,\mathfrak{v})} = \|(\mu X)\mathcal{A}[X]\|^{\boldsymbol{\sigma}}_{(D,\mathfrak{v})} = I_\Phi^{<\sigma_{m+1}}. \tag{5}$$

To establish our claim, regard the statement

$$\Gamma \in \|A\|^{\boldsymbol{\sigma}}_{(D,\mathfrak{v})}. \tag{6}$$

In view of (5) there exists an ordinal $\alpha < \sigma_{m+1}$ so that $\Gamma \in \Phi(I_\Phi^{<\alpha})$, i.e.

$$\Gamma \in \|\mathcal{A}[U]\|^{\boldsymbol{\sigma}}_{(D,\mathfrak{v}[U:I_\Phi^{<\alpha})])}. \tag{7}$$

As above we choose κ to be the least cardinal greater than the cardinality of $Pow(|\mathbb{SC}(D)|)$ and set $\boldsymbol{\tau} := \langle\sigma_1, \dots, \sigma_m, \alpha, \kappa, \dots, \kappa\rangle$. The previous lemma therefore furnishes proof of

$$\Gamma \in \|\mathcal{A}[(\mu X)\mathcal{A}[X]]\|^{\boldsymbol{\tau}}_{(D,\mathfrak{v})}. \tag{8}$$

However, we also know that $\boldsymbol{\tau} <_{lex} \boldsymbol{\sigma}$, and, as a consequence of (4) and the main induction hypothesis,

$$\Gamma \notin \|\mathcal{A}[(\mu X)\mathcal{A}[X]]\|^{\boldsymbol{\tau}}_{(D,\mathfrak{v})}. \tag{9}$$

Lines (8) and (9) contradict each other, meaning that (6) is false. Hence Γ cannot be an element of $\|A\|^{\boldsymbol{\sigma}}_{(D,\mathfrak{v})}$, and our proof is complete. □

Theorem 34 (Truth theorem). *Let D be some $\mathcal{L}_\mu$ sentence and A a sentence from $\mathbb{SC}(D)$. Then for all D-saturated subsets Γ of $\mathbb{SC}(D)$ and all valuations $\mathfrak{v}$ in $\mathfrak{S}_D$ we have*

$$A \in \Gamma \quad \Longrightarrow \quad \Gamma \notin \|A\|_{(\mathfrak{S}_D,\mathfrak{v})}.$$

Proof. Again we take κ to be the least cardinal greater than the cardinality of $|\mathfrak{S}_D|$ and $\boldsymbol{\kappa}$ to be the sequence of ordinals $\langle\kappa, \dots, \kappa\rangle$ of length $b_\mu(D)$. Given a sentence A from $\mathbb{SC}(D)$, a D-saturated subset of $\mathfrak{S}(D)$ and a valuation $\mathfrak{v}$ in $\mathfrak{S}$, the truth lemma implies

$$\Gamma \notin \|A\|^{\boldsymbol{\kappa}}_{(D,\mathfrak{v})},$$

provided that A belongs to Γ. But now it only remains to apply Lemma 31 in order to deduce $\Gamma \notin \|A\|_{(\mathfrak{S}_D,\mathfrak{v})}$. This completes our argument. □

Theorem 35 (Completeness of $\mathbf{K}^+_\omega(\mu)$). *For all sentences A of $\mathcal{L}_\mu$ we have*

$$\mu \models A \quad \Longrightarrow \quad \mathbf{K}^+_\omega(\mu) \vdash A.$$

Proof. If A is not provable in $\mathbf{K}^+_\omega(\mu)$, then, by Lemma 24, there exists an A-saturated subset Γ of $\mathbb{SC}(A)$ which contains A as an element. Now we apply the truth theorem and obtain $\Gamma \notin \|A\|_{(\mathfrak{S}_A,\mathfrak{v})}$, with $\mathfrak{v}$ being any valuation in $\mathfrak{S}_A$. Consequently, A is not μ-valid. By contraposition we have the desired result. □

We close this section with the completeness result for the system $\mathbf{K}_\omega(\mu)$. Fortunately, the work is already done: by Theorem 16 the completeness of $\mathbf{K}^+_\omega(\mu)$ carries over to $\mathbf{K}_\omega(\mu)$.

Corollary 36 (Completeness of $\mathbf{K}_\omega(\mu)$). *For all sentences A of $\mathcal{L}_\mu$ we have*

$$\mu \models A \quad \Longrightarrow \quad \mathbf{K}_\omega(\mu) \vdash A.$$

As mentioned above, $\mathbf{K}_\omega(\mu)$ and $\mathbf{K}^+_\omega(\mu)$ are also sound with respect to the semantics introduced in Section 3. This follows immediately from Theorem 41 in the next section which states the soundness of the finite variant $\mathbf{K}_{<\omega}(\mu)$ of $\mathbf{K}_\omega(\mu)$.

6. The finitization $\mathbf{K}_{<\omega}(\mu)$ of $\mathbf{K}_\omega(\mu)$

It is only the rule (ν) which is responsible for possibly infinite derivations in $\mathbf{K}_\omega(\mu)$. All proofs will be completely finite if we succeed in restricting the infinitely many premises of each application of (ν) to a finite subset. Fortunately, this can be achieved by exploiting the small model property of the propositional modal μ-calculus. A similar approach for **PDL** appears in Leivant [15].

Theorem 37 (Small model property). *An exponential number-theoretic function f can be defined for which we have: if A is a μ-satisfiable $\mathcal{L}_\mu$ formula, then there exist a Kripke structure $\mathfrak{M}$ and a valuation $\mathfrak{v}$ in $\mathfrak{M}$ so that the cardinality of $|\mathfrak{M}|$ is smaller than $f(lh(A))$ and $\|A\|_{(\mathfrak{M},\mathfrak{v})} \neq \emptyset$.*

Hence, if A is satisfiable, then it is satisfiable in a finite Kripke structure whose number of worlds is exponentially bounded in the length of A. For more details about this important result we refer to Bradfield and Stirling [5] and Streett and Emerson [23].

Since the exact definition of the number-theoretic function f is not relevant for what we are doing now, we omit further details concerning f. In the following we simply write ℓ for the exponential function which assigns to any $\mathcal{L}_\mu$ formulas A the natural number $\ell(A) := f(lh(A))$. Moreover, for a finite set Γ of $\mathcal{L}_\mu$ formulas $\ell(\Gamma)$ is defined to be the number $\ell(A_1 \vee \dots \vee A_m)$, where $A_1, \dots, A_m$ is an enumeration of the elements of Γ without repetitions.

Utilizing ℓ to provide a finite bound for the number of premises of a rule (ν), the finite versions of the ν-rules are obtained. Observe that the number of premises of a finite ν-rule depends on the length of (the essential part of) its conclusion; the set Δ is added in the conclusions just to incorporate weakening.

V. Finite ν-rules. For all finite sets Γ, Δ of $\mathcal{L}_\mu$ formulas and all $\mathcal{L}_\mu$ formulas $\mathcal{A}[U]$ where U does not occur in $\mathcal{A}$ and $\mathcal{A}[U]$ is positive in U:

$$\frac{\ldots\ \Gamma,\ (\nu X)^n\mathcal{A}[X]\ \ldots \quad (\text{for all } 0 < n < \ell(\Gamma, (\nu X)\mathcal{A}[X]))}{\Gamma,\ (\nu X)\mathcal{A}[X],\ \Delta}. \tag{f-ν}$$

The system $\mathbf{K}_{<\omega}(\mu)$ is $\mathbf{K}_\omega(\mu)$ with the ν-rules (ν) replaced by their finite variants (f-ν), and the notion $\mathbf{K}_{<\omega}(\mu) \vdash \Gamma$ is introduced in analogy to $\mathbf{K}_\omega(\mu) \vdash \Gamma$.

Naturally, $\mathbf{K}_{<\omega}(\mu)$ is a finite system. Besides that, every derivation in $\mathbf{K}_\omega(\mu)$ collapses to a derivation in $\mathbf{K}_{<\omega}(\mu)$. The proof of this observation is by induction on the derivations in $\mathbf{K}_\omega(\mu)$, and one only has to observe that each application of a rule (ν) in $\mathbf{K}_\omega(\mu)$ may be replaced by the appropriate rule (f-ν) in $\mathbf{K}_{<\omega}(\mu)$.

Lemma 38. *For all finite sets Γ of $\mathcal{L}_\mu$ formulas we have*

$$\mathbf{K}_\omega(\mu) \vdash \Gamma \quad\Longrightarrow\quad \mathbf{K}_{<\omega}(\mu) \vdash \Gamma.$$

Of course, this means that the completeness of $\mathbf{K}_\omega(\mu)$ transfers to $\mathbf{K}_{<\omega}(\mu)$; just combine the previous lemma with Corollary 36.

Corollary 39 (Completeness of $\mathbf{K}_{<\omega}(\mu)$). *For all sentences A of $\mathcal{L}_\mu$ we have*

$$\mu \models A \quad\Longrightarrow\quad \mathbf{K}_{<\omega}(\mu) \vdash A.$$

What remains is to show the soundness of $\mathbf{K}_{<\omega}(\mu)$. To do so, we proceed by using the small model property, along with the following lemma about the denotation of greatest fixed point on finite Kripke structures. Its proof is standard and follows from more general results concerning approximations of least and greatest fixed points of inductive definitions; see Theorem 3 and, for example, Moschovakis [17].

Lemma 40. *Let $\mathfrak{M}$ be a Kripke structure so that the cardinality of its universe $|\mathfrak{M}|$ is less than or equal to the natural number n. Then for all $\mathcal{L}_\mu$ formulas $(\nu X)\mathcal{A}[X]$ and all valuations $\mathfrak{v}$ in $\mathfrak{M}$ we have*

$$\|(\nu X)\mathcal{A}[X]\|_{(\mathfrak{M},\mathfrak{v})} \;=\; \|(\nu X)^n\mathcal{A}[X]\|_{(\mathfrak{M},\mathfrak{v})}.$$

Summing up this lemma and the small model property of the μ-calculus, we can now easily establish the soundness of $\mathbf{K}_{<\omega}(\mu)$. Some additional notation is convenient: if Γ is the set $\{A_1, \ldots, A_m\}$ of $\mathcal{L}_\mu$ formulas, then $\Gamma^\vee$ stands for the $\mathcal{L}_\mu$ formula $(A_1, \vee \cdots \vee, A_m)$.

Theorem 41 (Soundness of $\mathbf{K}_{<\omega}(\mu)$). *For all finite sets Γ of $\mathcal{L}_\mu$ formulas we have*

$$\mathbf{K}_{<\omega}(\mu) \vdash \Gamma \;\Longrightarrow\; \mu \models \Gamma^\vee.$$

Proof. The proof proceeds by induction on the derivation of Γ, and we distinguish the following cases:

1. Γ is an axiom or the conclusion of a logical rule of $\mathbf{K}_{<\omega}(\mu)$. Then our assertion is obvious or an immediate consequence of the induction hypothesis.

2. Γ is the conclusion of a μ-rule of $\mathbf{K}_{<\omega}(\mu)$. Then there exist a set Δ of $\mathcal{L}_\mu$ formulas and an $\mathcal{L}_\mu$ formula $(\mu X)\mathcal{A}[X]$ so that Γ is the set $\Delta, (\mu X)\mathcal{A}[X]$ and this rule has the form

$$\frac{\Delta,\ \mathcal{A}[(\mu X)\mathcal{A}[X]]}{\Delta,\ (\mu X)\mathcal{A}[X]}.$$

Now the induction hypothesis yields

$$\mu \;\models\; \Delta^\vee \vee \mathcal{A}[(\mu X)\mathcal{A}[X]]. \tag{1}$$

But according to our semantics we also have

$$\mu \;\models\; \mathcal{A}[(\mu X)\mathcal{A}[X]] \to (\mu X)\mathcal{A}[X], \tag{2}$$

and therefore the desired μ-validity of $\Gamma^\vee$ is a trivial consequence of statements (1) and (2).

3. Γ is the conclusion of a finite ν-rule of $\mathbf{K}_{<\omega}(\mu)$. Then there exist a set Δ of $\mathcal{L}_\mu$ formulas and an $\mathcal{L}_\mu$ formula $(\nu X)\mathcal{A}[X]$ so that Γ is the set $\Delta, (\nu X)\mathcal{A}[X]$ and this rule has the form

$$\frac{\ldots\ \Delta,\ (\nu X)^n\mathcal{A}[X]\ \ldots \quad (\text{for all } 0 < n < \ell(\Delta, (\nu X)\mathcal{A}[X]))}{\Delta,\ (\nu X)\mathcal{A}[X],\ \Pi}$$

for some auxiliary set Π. In this case the induction hypothesis yields

$$\mu \models \Delta^\vee \vee (\nu X)^n\mathcal{A}[X] \tag{3}$$

for all natural numbers n such that $0 < n < \ell(\Delta, (\nu X)\mathcal{A}[X])$. Now assume that the formula $(\Delta^\vee \vee (\nu X)\mathcal{A}[X])$ is not μ-valid. Then $(\neg\Delta^\vee \wedge \neg(\nu X)\mathcal{A}[X])$ has to be μ-satisfiable, and we infer from the small model property, see Theorem 37, that there exist a Kripke structure $\mathfrak{M}$ and a valuation $\mathfrak{v}$ in $\mathfrak{M}$ so that the cardinality of $|\mathfrak{M}|$, we call it k, is smaller than $\ell(\Delta, (\nu X)\mathcal{A}[X])$ and

$$\|\neg\Delta^\vee \wedge \neg(\nu X)\mathcal{A}[X]\|_{(\mathfrak{M},\mathfrak{v})} \neq \emptyset. \tag{4}$$

In view of Lemma 40 this inequality can be rewritten as

$$\|\neg\Delta^\vee \wedge \neg(\nu X)^k\mathcal{A}[X]\|_{(\mathfrak{M},\mathfrak{v})} \neq \emptyset, \tag{5}$$

implying that the formula $(\Delta^\vee \vee (\nu X)^k\mathcal{A}[X])$ is not μ-valid. However, this is in contradiction to (3), and therefore $(\Delta^\vee \vee (\nu X)\mathcal{A}[X])$ has to be μ-valid. This completes the proof of our theorem. $\square$

Considering this theorem in the context of Lemma 38 and Theorem 16, it provides the soundness of the two infinitary calculi $\mathbf{K}_\omega^+(\mu)$ and $\mathbf{K}_\omega(\mu)$.

Corollary 42 (Soundness of $\mathbf{K}_\omega^+(\mu)$ and $\mathbf{K}_\omega(\mu)$). *For all finite sets Γ of $\mathcal{L}_\mu$ formulas we have*

$$\mathbf{K}_\omega^+(\mu) \vdash \Gamma \implies \mathbf{K}_\omega(\mu) \vdash \Gamma \implies \mu \models \Gamma^\vee.$$

What we have achieved are two very natural infinitary axiomatizations of the propositional modal μ-calculus which are both sound and complete. They are cut-free, but because of their completeness, cut rules could be added without changing their strength. In this sense we have semantic cut elimination for $\mathbf{K}_\omega^+(\mu)$ and $\mathbf{K}_\omega(\mu)$.

$\mathbf{K}_{<\omega}(\mu)$, on the other hand, is the finite collapse of $\mathbf{K}_\omega(\mu)$ and is also cut-free, sound and complete as such. One may argue how natural $\mathbf{K}_{<\omega}(\mu)$ is as a deductive system. However, the important purpose of this system is to provide an explicit proof that a cut-free adequate axiomatization of the propositional modal μ-calculus exists.

Corollary 43 (Summary). *The systems $\mathbf{K}_{<\omega}(\mu)$, $\mathbf{K}_\omega(\mu)$ and $\mathbf{K}_\omega^+(\mu)$ provide cut-free, sound and complete axiomatizations of the propositional modal μ-calculus.*

A natural research direction is to look for alternative cut-free, sound and complete axiomatizations of the propositional modal μ-calculus. It would also be interesting to see whether there is a syntactic procedure for transforming proofs in $\mathbf{K}_{<\omega}(\mu) + (\mathsf{Cut})$ into proofs in $\mathbf{K}_{<\omega}(\mu)$. As a preparatory step it might be reasonable to study the related question first for $\mathbf{K}_\omega(\mu) + (\mathsf{Cut})$ and $\mathbf{K}_\omega(\mu)$.

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