Complexity of Computations with Pfaffian and Noetherian Functions

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Abstract

This paper is a survey of the upper bounds on the complexity of basic algebraic and geometric operations with Pfaffian and Noetherian functions, and with sets definable by these functions. Among other results, we consider bounds on Betti numbers of sub-Pfaffian sets, multiplicities of Pfaffian intersections, effective Łojasiewicz inequality for Pfaffian functions, computing frontier and closure of restricted semi-Pfaffian sets, constructing smooth stratifications and cylindrical cell decompositions (including an effective version of the complement theorem for restricted sub-Pfaffian sets), relative closures of non-restricted semi-Pfaffian sets and bounds on the number of their connected components, bounds on multiplicities of isolated solutions of systems of Noetherian equations.

1 Introduction

locally given by a convergent power series

Pfaffian functions, introduced by Khovanskii [24, 25] at the end of the 1970s, are analytic functions satisfying triangular systems of Pfaffian (first order partial differential) equations with polynomial coefficients. Over $\mathbb{R}$, these functions, and the corresponding semi- and sub-Pfaffian sets, are characterized by global finiteness properties similar to the properties of polynomials and semialgebraic sets. This allows one to establish efficient upper bounds on the complexity of different algebraic and geometric operations with these functions and sets.

One of important applications of the Pfaffian theory is in the real algebraic geometry of *fewnomials* — polynomials defined by simple formulas, possibly of a high degree. The complexity of operations with fewnomials in many cases allows upper bounds in terms of the complexity of the defining formulas, independent of their degree.

Over $\mathbb{C}$, Pfaffian functions, and more general Noetherian functions (satisfying the same kind of equations but without the triangularity condition) are characterized by local finiteness properties, sufficient for upper bounds on the complexity of stratification, frontier and closure.

This paper is a survey of results about the upper bounds for operations on Pfaffian and Noetherian functions, and on sets definable by these functions. All bounds are functions of a finite set, called *format*, of some natural parameters (like the degree or the number of variables) associated with a Pfaffian or Noetherian function or a definable set. The goal is to obtain as low upper bounds as possible.

The content of the paper is as follows. We start with definitions and examples of Pfaffian functions (including fewnomials), semi- and sub-Pfaffian sets. We discuss Khovanskii's bound on the number of isolated solutions of a system of Pfaffian equations, and its extensions for Betti numbers of semi- and sub-Pfaffian sets.

Next, we derive an upper bound for the multiplicity of a Pfaffian intersection. This bound is then used to obtain an effective version of Łojasiewicz inequality for a Pfaffian function, and an algorithm for constructing frontier and closure (in its domain of definition) of a semi-Pfaffian set. The bound for Pfaffian multiplicities is also used to construct an algorithm for a weak stratification of a semi-Pfaffian set, i.e., for a representation of the set as a disjoint union of smooth manifolds. The complexity (running time) of the stratification algorithm is explicitly estimated in terms of the format of the input semi-Pfaffian set. This also implies explicit upper bounds on the number of strata and their formats. Stratification is used for an effective proof of the following *complement theorem*: the complement of a projection of a restricted (relatively compact in its domain of definition) semi-Pfaffian set to a subspace is again a projection of a semi-Pfaffian set. The proof of the theorem uses an algorithm for a *cylindrical cell decomposition* of a restricted semi-Pfaffian set, a construction which is important in its own right.

For general (not necessarily restricted) semi-Pfaffian sets, the complement theorem is not known to be true. We describe a wider category of sets, called *limit sets*, which is a Boolean algebra and is also closed under projections to subspaces. We describe an explicit upper bound on the number of connected components of a limit set in terms of the format of the set.

Noetherian functions do not generally satisfy the global finiteness properties of Pfaffian functions. However, it is possible to establish some *local* finiteness properties. We describe in some detail a proof of an explicit upper bound on the multiplicity of an isolated solution of a system of Noetherian equations. This proof involves several stages. First, we give a brief introduction to integration over Euler characteristics. Next, we consider the univariate case, which implies, in particular, an upper bound on the vanishing order of a multivariate Noetherian function. As another application of the univariate result, we derive an upper bound for the degree of nonholonomy of a system of polynomial vector fields. The last stage of the proof for the multivariate case involves a lower bound on codimension of the set of intersections of high multiplicity.

2 Pfaffian functions and sub-Pfaffian sets

Pfaffian functions, introduced by Khovanskii at the end of the 1970s, are real or complex analytic functions satisfying triangular systems of Pfaffian (first order partial differential) equations with polynomial coefficients. We use the notation $\mathbb{K}^n$, where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$, in the statements relevant to both cases.

2.1 Definition [24, 25, 18] A *Pfaffian chain* of the order $r \geq 0$ and degree $\alpha \geq 1$ in an open domain $G \subset \mathbb{K}^n$ is a sequence of analytic functions $f_1, \dots , f_r$ in G satisfying differential equations

$$df_j(\mathbf{x}) = \sum_{1 \leq i \leq n} g_{ij}(\mathbf{x}, f_1(\mathbf{x}), \dots , f_j(\mathbf{x}))dx_i \tag{2.1}$$

for $1 \leq j \leq r$. Here $g_{ij}(\mathbf{x}, y_1, \dots , y_j)$ are polynomials in $\mathbf{x} = (x_1, \dots , x_n)$, $y_1, \dots , y_j$ of degrees not exceeding α. A function $f(\mathbf{x}) = P(\mathbf{x}, f_1(\mathbf{x}), \dots , f_r(\mathbf{x}))$, where $P(\mathbf{x}, y_1, \dots , y_r)$ is a polynomial of a degree not exceeding $\beta \geq 1$, is called a *Pfaffian function* of order r and degree (α, β). Note that the Pfaffian function f is defined only in the domain G where all functions $f_1, \dots , f_r$ are analytic, even if f itself can be extended as an analytic function to a larger domain.

2.2 Remark This definition is more restrictive than the definition from [25], where the Pfaffian chains are defined as sequences of nested integral manifolds of polynomial 1-forms. Both definitions lead to essentially the same class of Pfaffian functions, although the orders and degrees of Pfaffian chains for the same Pfaffian function can be different according to these two definitions. We found Definition 2.1 to be more convenient to trace the behaviour of parameters of Pfaffian functions under different operations.

More general definitions of Pfaffian functions, where the coefficients of (2.1) are not necessarily polynomial, were considered in [29, 32]. Most of our constructions can be adjusted to this more general definition, however upper bounds on the complexity may not be efficient enough in this case.

2.3 Example (a) Pfaffian functions of order 0 and degree $(1, \beta)$ are polynomials of degrees not exceeding β.

(b) The exponential function $f(x) = e^{ax}$ is a Pfaffian function of order 1 and degree $(1, 1)$ in $\mathbb{R}$, due to the equation $df(x) = af(x)dx$. More generally, for $i = 1, 2, \dots , r$, let $E_i(x) := e^{E_{i-1}(x)}$, $E_0(x) = ax$. Then $E_r(x)$ is a Pfaffian function of order r and degree $(r, 1)$, since $dE_r(x) = aE_1(x) \cdots E_r(x)dx$.

e^{e^x} is Pfaffian, but there is no power series expansion at $x=0$

(c) The function $f(x) = 1/x$ is a Pfaffian function of order 1 and degree $(2, 1)$ in the domain $\{x \in \mathbb{R} |\, x \neq 0\}$, due to the equation $df(x) = -f^2(x)dx$.

(d) The logarithmic function $f(x) = \ln(|x|)$ is a Pfaffian function of order 2 and degree $(2, 1)$ in the domain $\{x \in \mathbb{R} |\, x \neq 0\}$, due to equations $df(x) = g(x)dx$ and $dg(x) = -g^2(x)dx$, where $g(x) = 1/x$.

(e) The polynomial $f(x) = x^m$ can be viewed as a Pfaffian function of order 2 and degree $(2, 1)$ in the domain $\{x \in \mathbb{R} |\, x \neq 0\}$ (but not in $\mathbb{R}$), due to the equations $df(x) = mf(x)g(x)dx$ and $dg(x) = -g^2(x)dx$, where $g(x) = 1/x$. In some cases a better way to deal with x^m is to change the variable $x = e^u$ reducing this case to (b).

$e^{e^{e^x-1}-1}$

(f) The function $f(x) = \tan(x)$ is a Pfaffian function of order 1 and degree $(2, 1)$ in the domain $\bigcap_{k \in \mathbb{Z}} \{x \in \mathbb{R} |\, x \neq \pi/2 + k\pi\}$, due to the equation $df(x) = (1 + f^2(x))dx$.

e^{e^x-1}

(g) The function $f(x) = \arctan(x)$ is a Pfaffian function in $\mathbb{R}$ of order 2 and degree $(3, 1)$, due to equations $df(x) = g(x)dx$ and $dg(x) = -2xg^2(x)dx$, where $g(x) = (x^2 + 1)^{-1}$.

$e^{f(x)} = \sum_{n=0}^{\infty} \frac{f(x)^n}{n!}$ we need $f_0 = 0$ for convergence $\to$ even a power series

$\partial \underbrace{\frac{1}{a+x}}_{f} = -\frac{1}{(a+x)^2} = -f^2$, f is a power series iff $a \neq 0$.

(g) The function $\cos(x)$ is a Pfaffian function of order 2 and degree $(2,1)$ in the domain $\bigcap_{k\in\mathbb{Z}}\{x \in \mathbb{R}|\, x \neq \pi + 2k\pi\}$, due to equations $\cos(x) = 2f(x)-1$, $df(x) = -f(x)g(x)dx$, and $dg(x) = \frac{1}{2}(1+g^2(x))dx$, where $f(x) = \cos^2(x/2)$ and $g(x) = \tan(x/2)$. Also, since $\cos(x)$ is a polynomial of degree m of $\cos(x/m)$, the function $\cos(x)$ is Pfaffian of order 2 and degree $(2,m)$ in the domain $\bigcap_{k\in\mathbb{Z}}\{x \in \mathbb{R}|\, x \neq m\pi + 2km\pi\}$. The same is true, of course, for any shift of this domain by a multiple of π. However, $\cos(x)$ is not a Pfaffian function in the whole real line.

2.4 Lemma *The sum (resp. product) of two Pfaffian functions f_1 and f_2 of orders r_1 and r_2 and degrees (α_1,β_1) and (α_2,β_2) respectively, is a Pfaffian function of order r_1+r_2 and degree $(\alpha, \max\{\beta_1,\beta_2\})$ (resp. $(\alpha, \beta_1+\beta_2)$), where $\alpha = \max\{\alpha_1,\alpha_2\}$. If the two functions are defined by the same Pfaffian chain of order r, then the orders of the sum and of the product are both equal to r.*

Proof Combine Pfaffian chains for f_1 and f_2 into a Pfaffian chain for f_1+f_2 and f_1f_2. If a Pfaffian chain is common for the two functions, then it is also a Pfaffian chain for their sum and product. □

2.5 Lemma *A partial derivative of a Pfaffian function of order r and degree (α,β) is a Pfaffian function having the same Pfaffian chain of order r and degree $(\alpha, \alpha+\beta-1)$.*

Proof is straightforward. □

2.6 Example (*Fewnomials*) Generalizing Example 2.3 (e), we can view a polynomial $f \in \mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1,\dots,x_n]$ as a Pfaffian function in the following sense. Each monomial

$$f_{i_1\cdots i_n} := a_{i_1\cdots i_n} x_1^{i_1}\cdots x_n^{i_n}$$

of f with $a_{i_1\cdots i_n} \neq 0$ is a Pfaffian function in the domain $G := \{\mathbf{x} \in \mathbb{K}^n|\, x_1\cdots x_n \neq 0\}$, of order $n+1$ and degree $(2,1)$, due to equations

$$df_{i_1\cdots i_n} = \sum_{1\le j\le n} i_j f_{i_1\cdots i_n} g_j dx_j,$$

$$dg_j = -g_j^2 dx_j,$$

where $g_j = 1/x_j$. According to Lemma 2.4, f is a Pfaffian function in G of order $n+m$ and degree $(2,1)$, where m is the number of all monomials in f (with non-zero coefficients). Let $\mathcal{K}$ be a set of all monomials of f. Then f is called a *fewnomial* with the support $\mathcal{K}$.

A polynomial $F = P(x_1,\dots,x_n,u_1,\dots,u_m)$ of degree β in $x_1,\dots,x_n,u_1,\dots,u_m$, where $x_1,\dots,x_n$ are variables and $u_1,\dots,u_m \in \mathcal{K}$ are monomials, is called a *fewnomial of pseudodegree β* with support $\mathcal{K}$. Obviously F is a Pfaffian function of order $n+m$ and of degree $(2,\beta)$. Note that β may be different from the degree d of the polynomial P after the substitution of the monomials u_j. We call d the *degree* of F.

In the sequel we will reserve the term "polynomial" for Pfaffian functions of order 0 and degree $(1,\beta)$ (see Example 2.3 (a)).

We define specializations for the class of Pfaffian functions over $\mathbb{R}$ of more general concepts of semi- and subanalytic sets (see, e.g., [4]).

2.7 Definition (*Semi-Pfaffian sets*) A set $X \subset \mathbb{R}^n$ is called *semi-Pfaffian* in an open domain $G \subset \mathbb{R}^n$ if it consists of points in G satisfying a Boolean combination $\mathcal{F}$ of some atomic equations and inequalities $f = 0$, $g > 0$, where f, g are Pfaffian functions having a common Pfaffian chain defined in G. We will write $X = \{\mathcal{F}\}$. A semi-Pfaffian set X is *restricted in* G if its topological closure lies in G. A semi-Pfaffian set is called *basic* if the Boolean combination is just a conjunction of equations and strict inequalities.

2.8 Definition (*Sub-Pfaffian sets*) A set $X \subset \mathbb{R}^n$ is called *sub-Pfaffian* in an open domain $G \subset \mathbb{R}^n$ if it is an image of a semi-Pfaffian set under a projection into a subspace.

Our main object of study will be a following subclass of sub-Pfaffian sets.

2.9 Definition (*Restricted sub-Pfaffian sets*) Consider the closed cube $\mathcal{I}^{m+n} := [-1, 1]^{m+n}$ in an open domain $G \subset \mathbb{R}^{m+n}$ and the projection map $\pi : \mathbb{R}^{m+n} \to \mathbb{R}^n$. A subset $Y \subset \mathcal{I}^n$ is called *restricted sub-Pfaffian* if $Y = \pi(X)$ for a restricted semi-Pfaffian set $X \subset \mathcal{I}^{m+n}$.

A restricted sub-Pfaffian set need not be semi-Pfaffian as the following example, due to Osgood [35], shows. This fact is the most significant difference between the theories of semi- and sub-Pfaffian sets on one hand, and semialgebraic sets on another.

2.10 Example Let

$$Y := \{(x, y, z) \in \mathcal{I}^3 |\ \exists u \in [0, 1]\ (y = xu, z = xe^u)\} \subset \mathbb{R}^3.$$

Then Y is a two-dimensional restricted sub-Pfaffian set such that any real analytic function vanishing on Y in the neighbourhood of the origin is identically zero. Hence Y is not semi-Pfaffian.

Restricted sub-Pfaffian sets form a Boolean algebra. Finite unions and intersections of arbitrary sub-Pfaffian sets are clearly sub-Pfaffian. The fact that *complement* of a restricted sub-Pfaffian set in $\mathcal{I}^n$ is also restricted sub-Pfaffian is a particular case of Gabrielov's *complement theorem* [13]. We will consider an algorithmic version of this theorem in Section 7.

2.11 Definition (*Format*) Consider a semi-Pfaffian set

$$X := \bigcup_{1 \le i \le M} \{\mathbf{x} \in \mathbb{R}^s |\ f_{i1} = \cdots = f_{iI_i}, g_{i1} > 0, \dots , g_{iJ_i} > 0\} \subset G, \tag{2.2}$$

where f_{ij}, g_{ij} are Pfaffian functions with a common Pfaffian chain of order r and degree (α, β), defined in an open domain G. Its *format* is a tuple (r, N, α, β, s), where $N \ge \sum_{1 \le i \le M}(I_i + J_i)$. For $s = m + n$ and a sub-Pfaffian set $Y \subset \mathbb{R}^n$ such that $Y = \pi(X)$, its *format* is the format of X.

We will refer to the representation of a semi-Pfaffian set in the form (2.2) as to *disjunctive normal form (DNF)*.

2.12 Remark In this paper we are concerned with upper bounds of various characteristics of semi- and sub-Pfaffian sets and complexities of computations, as functions of the format. In fact, these characteristics and complexities also depend on the domain G in which the corresponding Pfaffian chains are defined. Note that Definition 2.1 imposes no restrictions on an open set G, thus allowing it to be arbitrarily complex and induce this complexity on the corresponding semi- and sub-Pfaffian sets. To avoid this, we will always assume in the sequel (unless explicitly stated otherwise) that G of a is “simple”, like $\mathbb{R}^n$, $\mathcal{I}^n$, $\{\mathbf{x}|\ x_1 > 0, \dots, x_n > 0\}$, or $\{\mathbf{x}|\ \|\mathbf{x}\|^2 < 1\}$. A more general approach allows G to be a semi-Pfaffian set defined by Pfaffian functions in a larger domain $G' \supset G$, which in turn is defined by Pfaffian functions in some $G'' \supset G'$, and so on (see details in [16]).

3 Betti numbers of sub-Pfaffian sets

3.1 Topological complexity of semi-Pfaffian sets

We start with the following fundamental result of Khovanskii which can be considered as an analogy of the Bezout's theorem for Pfaffian functions.

3.1 Theorem ([24, 25]) *Consider a system of equations $f_1 = \cdots = f_n = 0$, where f_i, $1 \le i \le n$ are Pfaffian functions in a domain $G \subset \mathbb{R}^n$, having a common Pfaffian chain of order r and degrees (α, β_i) respectively. Then the number of non-degenerate solutions of this system does not exceed*

$$\mathcal{M}(n, r, \alpha, \beta_1, \dots, \beta_n) := 2^{r(r-1)/2} \beta_1 \cdots \beta_n (\min\{n, r\}\alpha + \beta_1 + \cdots + \beta_n - n + 1)^r. \tag{3.1}$$

Applying directly this result to fewnomials (see Example 2.6) $f_1, \dots, f_n$ in the domain $\{\mathbf{x}|\ x_1 > 0, \dots, x_n > 0\} \subset \mathbb{R}^n$, we get the upper bound

$$2^{(n+m)(n+m-1)/2}(2n+1)^{n+m}$$

on the number of non-degenerate solutions of the system $f_1 = \cdots = f_n = 0$. Here m is the number of different monomials occurring in at least one of polynomials f_i. There is, however, a better upper bound.

3.2 Corollary *The number of non-degenerate solutions of a system of polynomial equations $f_1 = \cdots = f_n = 0$ belonging to the octant $\{\mathbf{x}|\ x_1 > 0, \dots, x_n > 0\}$ does not exceed*

$$2^{m(m-1)/2}(n+1)^m,$$

where m is the number of different monomials occurring in at least one of polynomials f_i.

Proof Making a change of variables $x_j = e^{y_j}$ we reduce the system $f_1 = \cdots = f_n = 0$ to a system of linear equations in m exponential functions of the kind $e^{i_1 y_1 + \cdots + i_n y_n}$. Left-hand sides of these equations are Pfaffian functions in $\mathbb{R}^n$ having a common Pfaffian chain of order m and degrees equal to $(1, 1)$. Now the bound follows directly from Theorem 3.1. □

Theorem 3.1 implies various upper bounds on the topological complexity of a semi-Pfaffian set as functions of the format by applying almost without change some well-developed techniques for semialgebraic sets. Here is an example.

3.3 Corollary *Consider a system of equations $f_1 = \cdots = f_k = 0$, where f_i, $1 \leq i \leq k$ are Pfaffian functions in a domain $G \subset \mathbb{R}^n$, having a common Pfaffian chain of order r and degrees (α, β_i) respectively. Then the number of connected components of $X := \{f_1 = \cdots = f_k = 0\}$ does not exceed*

$$2^{r(r-1)/2+1}\beta(\alpha + 2\beta - 1)^{n-1}((2n-1)(\alpha+\beta) - 2n + 2)^r, \tag{3.2}$$

where $\beta := \max_{1 \leq i \leq k}\{\beta_i\}$.

Proof (cf. [33]) Choose a large enough positive $R \in \mathbb{R}$ such that $\{\mathbf{x} \in X \,|\, \|\mathbf{x}\|^2 \leq R\}$ has the same number of connected components as X. For a sufficiently small real $\varepsilon > 0$ the number of connected components of X does not exceed the number of connected components of the smooth compact hypersurface $X_\varepsilon := \{F = 0\}$, where $F := f_1^2 + \cdots + f_k^2 + \varepsilon(\|\mathbf{x}\|^2 - R)$.

After a generic rotation of coordinates in $\mathbb{R}^n$, the projection of X_ε on any of them is a Morse function. In particular, all solutions of the system of equations

$$F = \partial F/\partial x_2 = \cdots = \partial F/\partial x_n = 0 \tag{3.3}$$

are non-degenerate and thus, by Theorem 3.1 and Lemma 2.5, their number does not exceed (3.2). Because X_ε is compact, each of its connected components contains a solution of system (3.3). This concludes the proof. $\square$

The most general (to our knowledge) upper bound on the topological complexity of semi-Pfaffian sets is provided by the following theorem (see [49]).

3.4 Theorem *Consider a semi-Pfaffian set $X := \{\mathcal{F}\} \subset G \subset \mathbb{R}^n$, where G is an open domain, $\mathcal{F}$ is either a conjunction of equations and strict inequalities or a Boolean combination (with no negations) of non-strict inequalities. Let $\mathcal{F}$ contain equations or inequalities of the kind $f * 0$, where $* \in \{=, >, \geq\}$, and there are s different Pfaffian functions f in G having a common Pfaffian chain of order r and degrees (α, β). Then the sum of Betti numbers of X does not exceed*

$$s^n 2^{r(r-1)/2} O(n\beta + \min\{n, r\}\alpha)^{n+r}. \tag{3.4}$$

This theorem is a direct analogy of the Basu [2] refinement of Petrovskii-Oleinik-Thom-Milnor [37, 34, 42, 33] bounds for semialgebraic sets, and can be proved in a similar way replacing references to Bezout's theorem by references to Khovanskii's Theorem 3.1.

3.5 Remark Corollary 3.3, Theorem 3.4, and most of other upper bounds for Pfaffian functions appearing in the sequel can be reformulated for a particular case of fewnomials similar to how it was done for Theorem 3.1. These specifications are straightforward and will be omitted.

3.2 Number of consistent sign assignments

3.6 Definition For a given finite family $h_1, \ldots, h_k$ of Pfaffian functions h_i in an open domain G define its *consistent sign assignment* as a non-empty semi-Pfaffian set in G of the kind

$$\{\mathbf{x} \in G \,|\, h_{i_1} = \cdots = h_{i_{k_1}} = 0, h_{i_{k_1}+1} > 0 \ldots, h_{i_{k_2}} > 0, h_{i_{k_2}+1} < 0, \ldots, h_{i_k} < 0\},$$

where $i_1, \ldots, i_{k_1}, \ldots, i_{k_2}, \ldots, i_k$ is a permutation of $1, \ldots, k$.

3.7 Theorem ([18]) *Let $h_1, \dots, h_k$ be Pfaffian functions in G having a common Pfaffian chain of order r and degrees $(\alpha, \beta_1), \dots, (\alpha, \beta_k)$ respectively. Then the number of distinct consistent sign assignments for $h_1, \dots, h_k$ does not exceed*

$$\min\{3^k, 2^{r(r-1)/2+1}(2n+1)^r(\alpha + 8k\beta)^{n+r+1}\},$$

where $\beta := \max_{1 \le i \le k}\{\beta_i\}$.

Proof The bound 3^k is trivial. Choose in every consistent sign assignment one arbitrary point. Let Λ be the set of all chosen points. There exists a positive $\varepsilon \in \mathbb{R}$ such that for every $\mathbf{x} \in \Lambda$ and every $i = 1, \dots, k$ the inequality $h_i(\mathbf{x}) > 0$ implies $h_i(\mathbf{x}) > \varepsilon$, and $h_i(\mathbf{x}) < 0$ implies $h_i(\mathbf{x}) < -\varepsilon$. It is easy to prove that the number of consistent sign assignments does not exceed the number of connected components of the semi-Pfaffian set

$$S := \Big\{\mathbf{x} \in G \,|\, h := \prod_{1 \le i \le k} (h_i + \varepsilon)^2 (h_i - \varepsilon)^2 > 0\Big\}.$$

For a small enough positive $\delta < \varepsilon$, the number of connected components of S does not exceed the number of connected components of $\{\mathbf{x} \in G \,|\, h = \delta\}$. It remains to apply Corollary 3.3 to $\{\mathbf{x} \in G \,|\, h = \delta\}$. □

3.3 Sub-Pfaffian sets defined by formulae with quantifiers

We now address the problem of estimating the topological complexity of restricted *sub-Pfaffian* sets. Until recently this question did not have a satisfactory solution even in the particular case of projections of *semialgebraic* sets defined by Boolean formulae. In Pfaffian category the situation is further complicated by the fact that the quantifier elimination process cannot be used.

In this section we describe a reduction of estimating of Betti numbers of sets defined by formulae with quantifiers to a similar problem for sets defined by quantifier-free formulae. More precisely, let X be a subset in $\mathcal{I}^{n_0} = [-1, 1]^{n_0} \subset \mathbb{R}^{n_0}$ defined by a formula

$$X = \{\mathbf{x}_0 |\, Q_1\mathbf{x}_1 Q_2\mathbf{x}_2 \cdots Q_\nu \mathbf{x}_\nu ((\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_\nu) \in X_\nu)\}, \tag{3.5}$$

where $Q_i \in \{\exists, \forall\}$, $Q_i \ne Q_{i+1}$, $\mathbf{x}_i \in \mathbb{R}^{n_i}$, and X_ν be either an open or a closed set in $\mathcal{I}^{n_0+\cdots+n_\nu}$ being a difference between a finite CW-complex and one of its subcomplexes. For instance, if $\nu = 1$ and $Q_1 = \exists$, then X is the projection of X_ν.

We express an upper bound on each Betti number of X via a sum of Betti numbers of some sets defined by quantifier-free formulae involving X_ν. In conjunction with Theorem 3.4 this implies an upper bound for restricted sub-Pfaffian sets defined by formulae with quantifiers.

Throughout this section each topological space is assumed to be a difference between a finite CW-complex and one of its subcomplexes.

3.8 Example The closure X of the interior of a compact set $Y \subset \mathcal{I}^n$ is homotopy equivalent to

$$X_{\varepsilon,\delta} = \{\mathbf{x} |\, \exists \mathbf{y}(\|\mathbf{x} - \mathbf{y}\| \le \delta)\, \forall \mathbf{z}(\|\mathbf{y} - \mathbf{z}\| < \varepsilon)\, (\mathbf{z} \in Y)\}$$

for small enough $\delta, \varepsilon > 0$ such that $\delta \gg \varepsilon$. Representing $X_{\varepsilon,\delta}$ in the form (3.5), we conclude that X is homotopy equivalent to $X_{\varepsilon,\delta} = \{\mathbf{x} | \exists \mathbf{y} \forall \mathbf{z}\, X_2\}$, where

$$X_2 = \{(\mathbf{x}, \mathbf{y}, \mathbf{z}) | (\|\mathbf{x} - \mathbf{y}\| \leq \delta \wedge (\|\mathbf{y} - \mathbf{z}\| \geq \varepsilon \vee \mathbf{z} \in Y))\}$$

is a closed set in $\mathcal{I}^{3n}$. Our results allow to bound from above Betti numbers of X in terms of Betti numbers of some sets defined by quantifier-free formulae involving X_2.

3.4 A spectral sequence associated with a surjective map

3.9 Definition A continuous map $f: X \to Y$ is *locally split* if for any $y \in Y$ there is an open neighbourhood U of y and a section $s: U \to X$ of f (i.e., s is continuous and $fs = \mathrm{Id}$). In particular, a projection of an open set in $\mathbb{R}^n$ on a subspace of $\mathbb{R}^n$ is always locally split.

For any two continuous surjective maps $f_1: X_1 \to Y$ and $f_2: X_2 \to Y$, define the operation $\times_Y$:

$$X_1 \times_Y X_2 := \bigcup_{\mathbf{y} \in Y} f_1^{-1}(\mathbf{y}) \times f_2^{-1}(\mathbf{y}).$$

Note that if Y is a singleton, then $X_1 \times_Y X_2$ coincides with the usual Cartesian product $X_1 \times X_2$.

3.10 Theorem ([20]) *Let $f: X \to Y$ be a surjective cellular map. Assume that f is either closed or locally split. Then for any Abelian group G, there exists a spectral sequence $E^r_{p,q}$ converging to $H_*(Y, G)$ with*

$$E^1_{p,q} = H_q(W_p, G) \tag{3.6}$$

where

$$W_p = \underbrace{X \times_Y \cdots \times_Y X}_{p+1 \text{ times}} \tag{3.7}$$

In particular,

$$\dim H_k(Y, G) \leq \sum_{p+q=k} \dim H_q(W_p, G), \tag{3.8}$$

for all k.

3.11 Remark Let $X, Y \subset \mathbb{R}^n$ and a surjective cellular map f satisfies the following property. For any convergent sequence in Y there is an infinite subsequence which is an f-image of a convergent sequence in X. This condition includes both the closed and the locally split cases and may be more convenient for applications. For such f Theorem 3.10 is also true.

3.5 Upper bounds for Betti numbers of sub-Pfaffian sets

Let $X = X_0 \subset \mathbb{R}^{n_0}$ be a sub-Pfaffian set defined by a formula

$$Q_1 \mathbf{x}_1 Q_2 \mathbf{x}_2 \cdots Q_\nu \mathbf{x}_\nu \mathcal{F}(\mathbf{x}_0, \mathbf{x}_1, \dots \mathbf{x}_\nu), \tag{3.9}$$

where $\mathcal{F}$ is a quantifier-free Boolean formula with no negations having s atoms of the kind $f > 0$. Let all f's be Pfaffian functions in an open domain G having a common Pfaffian chain of order r and degrees at most (α, β). Assume for definiteness that X is closed (the case of an open set is similar).

Consider first the case of a single quantifier $Q_1 = \exists$. Then $\nu = 1$ and (3.9) reduces to $\exists \mathbf{x}_1 X_1$, where $X_1 = \{\mathcal{F}(\mathbf{x}_0, \mathbf{x}_1)\}$. Note that in this case $X = f(X_1)$, where f is the projection map onto a subspace. According to Theorem 3.10,

$$\mathrm{b}_{q_0}(X) \le \sum_{p_1+q_1=q_0} \mathrm{b}_{q_1}(\underbrace{X_1 \times_X \cdots \times_X X_1}_{p_1+1 \text{ times}}), \tag{3.10}$$

where b_i stands for ith Betti number. Observe that $X_1 \times_X \cdots \times_X X_1$ is a closed set definable by a Boolean combination with no negations of $(p_1+1)s$ atoms of the kind $g > 0$, where g's are Pfaffian functions in an open domain $G \subset \mathbb{R}^{n_0+(p_1+1)n_1}$ having a common Pfaffian chain of order $(p_1+1)r$, degrees (α, β), and $n_0 + (p_1+1)n_1$ variables. Let $t_k := n_0 + (k+1)n_1$. According to Theorem 3.4, for any $q_1 \le \dim(X)$,

$$\mathrm{b}_{q_1}(X_1 \times_X \cdots \times_X X_1) \le ((p_1+1)s)^{t_{p_1}} 2^{(p_1+1)r((p_1+1)r-1)/2} O(t_{p_1}\beta + \min\{p_1 r, t_{p_1}\}\alpha)^{t_{p_1}+(p_1+1)r}.$$

Then, due to (3.10), for any $k \le \dim(X) \le n_0$,

$$\mathrm{b}_k(X) \le \sum_{p_1+q_1=k} \mathrm{b}_{q_1}(X_1 \times_X \cdots \times_X X_1) \le$$

$$\le k((k+1)s)^{t_k} 2^{(k+1)r((k+1)r-1)/2} O(t_k\beta + \min\{kr, t_k\}\alpha)^{t_k+(k+1)r}.$$

Relaxing the obtained bound, we get

$$\mathrm{b}_k(X) \le (ks)^{O(t_k)} 2^{(O(kr))^2} (t_k(\alpha+\beta))^{O(t_k+kr)}.$$

In [20] this bound is generalized for formulae with ν quantifiers. More precisely, if $X \subset \mathbb{R}^{n_0}$ is defined by (3.9), then for any $k \le \dim(X) \le n_0$,

$$\mathrm{b}_k(X) \le s^{O(u_\nu)} 2^{O(\nu u_\nu + r^2 v_\nu^2)} (u_\nu(\alpha+\beta))^{O(u_\nu + r v_\nu)},$$

where

$$u_\nu := 2^\nu n_0 n_1 \cdots n_\nu, \quad v_\nu := 2^{2\nu} n_0^2 n_1^2 \cdots n_{\nu-2}^2 n_{\nu-1}.$$

4 Multiplicities of Pfaffian intersections

4.1 Definition A *deformation* of a Pfaffian function $f(\mathbf{x})$ in $G \subset \mathbb{K}^n$ is an analytic function $\theta(\mathbf{x}, \varepsilon)$ in a domain $G' \subset \mathbb{K}^{n+1}$ such that $G = G' \cap \{\varepsilon = 0\}$, $\theta(\mathbf{x}, 0) = f(\mathbf{x})$, and, for a fixed ε, the function $\theta(\mathbf{x}, \varepsilon)$ is Pfaffian having the same Pfaffian chain and the same degree as $f(\mathbf{x})$.

4.2 Definition Let $f_1(\mathbf{x}), \dots, f_n(\mathbf{x})$ be Pfaffian functions in $G \subset \mathbb{K}^n$. The *multiplicity* at $\mathbf{y} \in G$ of the *Pfaffian intersection* $f_1 = \cdots = f_n = 0$ is a maximal number of isolated *complex* solutions, for a fixed $\varepsilon \neq 0$, of the system of equations $\theta_1(\mathbf{x}, \varepsilon) = \cdots = \theta_n(\mathbf{x}, \varepsilon) = 0$ converging to $\mathbf{y}$ as $\varepsilon \to 0$. Here $\theta_i(\mathbf{x}, \varepsilon)$ is any deformation of $f_i(\mathbf{x})$ for all $1 \leq i \leq n$.

4.3 Theorem ([11]) *Let $f_1(\mathbf{x}), \dots, f_n(\mathbf{x})$ be Pfaffian functions in $G \subset \mathbb{K}^n$ having a common Pfaffian chain of order r and degrees $(\alpha, \beta_1), \dots, (\alpha, \beta_n)$ respectively. Then the multiplicity of the Pfaffian intersection $f_1 = \cdots = f_n = 0$ at any point $\mathbf{y} \in G$ does not exceed (3.1).*

4.4 Corollary (Pfaffian Łojasiewicz inequality, [11]) *Let f be a Pfaffian function in an open domain $G \subset \mathbb{R}^n$ of order r and degree (α, β). Then there is a neighbourhood U of $\{f = 0\}$ in G such that for any $\mathbf{x} \in U$,*

$$|f(\mathbf{x})| \geq C(\operatorname{dist}(\mathbf{x}, \{f = 0\}))^q,$$

for a real $C > 0$ and a positive integer

$$q \leq 2^{r(r-1)+1} 4^{n-1} \beta(\alpha + \beta - 1)^{n-1} (\min\{n, r\}\alpha + (n-1)(4\alpha + 3\beta - 5) + \beta)^r \leq$$
$$\leq 2^{r(r-1)+1} O(n)^r O(\alpha + \beta)^{n+r}.$$

The following corollary will be needed in Section 5 for proving upper bounds on frontier and closure of a semi-Pfaffian set and for an algorithm which computes frontier and closure.

Let

$$\mathcal{K}(m, n, r, \alpha, \beta) := \mathcal{M}(n, r, \alpha, \underbrace{\beta, \dots, \beta}_{m+1}, \underbrace{\tau_m, \dots, \tau_m}_{n-m-1}),$$

where $\mathcal{M}(n, r, \alpha, \beta_1, \dots, \beta_n)$ is defined in (3.1) and $\tau_m = (m+2)(\alpha + \beta - 1)$. Let

$$\mathcal{K}(n, r, \alpha, \beta) := \max_{0 \leq m < n} \mathcal{K}(m, n, r, \alpha, \beta).$$

4.5 Corollary ([14]) *Let $f_1, \dots, f_I, g_1, \dots, g_J$ be Pfaffian functions in a domain $G \subset \mathbb{R}^n$ having a common Pfaffian chain of order r and degrees (α, β) each. Let $\kappa > \mathcal{K}(n, r, \alpha, \beta)$ be an integer number. For a point $\mathbf{y} \in G$, let $F_i(\mathbf{x}, \mathbf{y})$ be the Taylor expansion of $f_i(\mathbf{x})$ at $\mathbf{y}$ of order κ^2, and let $G_j(\mathbf{x}, \mathbf{y})$ be the Taylor expansion of $g_j(\mathbf{x})$ at $\mathbf{y}$ of order κ. Then the closure of the semi-Pfaffian set*

$$X := \{\mathbf{x} \in G |\ f_1(\mathbf{x}) = \cdots = f_I(\mathbf{x}) = 0, g_1(\mathbf{x}) > 0, \dots, g_J(\mathbf{x}) > 0\}$$

contains $\mathbf{y}$ if and only if the closure of the following semialgebraic *set $X_{\mathbf{y}}$ contains $\mathbf{y}$:*

$$X_{\mathbf{y}} := \{\mathbf{x} \in G |\ F_i(\mathbf{x}, \mathbf{y}) \leq |\mathbf{x} - \mathbf{y}|^{\kappa^2},\ for\ i = 1, \dots, I,\ G_j(\mathbf{x}, \mathbf{y}) > |\mathbf{x} - \mathbf{y}|^{\kappa},\ for\ j = 1, \dots, J\}.$$

The next corollary will be needed in Section 6 for a stratification algorithm.

4.6 Definition For a set of differentiable functions $\mathbf{h} = (h_1, \dots, h_k)$, a set of distinct indices $\mathbf{i} = (i_1, \dots, i_k)$ with $1 \leq i_\nu \leq n$, and an index j, $1 \leq j \leq n$, different from all i_ν we define a partial differential operator

$$\partial_{\mathbf{h}, \mathbf{i}, j} := \det \begin{pmatrix} \partial h_1/\partial x_{i_1} & \cdots & \partial h_1/\partial x_{i_k} & \partial h_1/\partial x_j \\ \cdots & \cdots & \cdots & \cdots \\ \partial h_k/\partial x_{i_1} & \cdots & \partial h_k/\partial x_{i_k} & \partial h_k/\partial x_j \\ \partial/\partial x_{i_1} & \cdots & \partial/\partial x_{i_k} & \partial/\partial x_j \end{pmatrix}.$$

When $k = 0$, the corresponding operator is simply $\partial_j := \partial/\partial x_j$. For $m \geq 0$ we define $\partial^m_{\mathbf{h},\mathbf{i},j}$ (resp. ∂^m_j) as the mth iteration of $\partial_{\mathbf{h},\mathbf{i},j}$ (resp. ∂_j).

4.7 Corollary *Let* $\mathbf{i} = (i_1, \ldots, i_k)$ *be a set of distinct indices,* $1 \leq i_\nu \leq n$*. Let* f *be a Pfaffian function in an open neighbourhood* G *of a point* $\mathbf{x} \in \mathbb{R}^n$ *of order* r *and degree* (α, β_{k+1})*. Let* $\mathbf{h} = (h_1, \ldots, h_k)$ *be a set of Pfaffian functions in* G *of order* r *and degrees* $(\alpha, \beta_1), \ldots, (\alpha, \beta_k)$ *respectively, each having the same Pfaffian chain as* f*, and such that* $h_1(\mathbf{x}) = \cdots = h_k(\mathbf{x}) = 0$*,*

$$\det \begin{pmatrix} \partial h_1/\partial x_{i_1} & \cdots & \partial h_1/\partial x_{i_k} \\ \cdots & \cdots & \cdots \\ \partial h_k/\partial x_{i_1} & \cdots & \partial h_k/\partial x_{i_k} \end{pmatrix} (\mathbf{x}) \neq 0.$$

Suppose that $\partial^{m_1}_{\mathbf{h},\mathbf{i},1} \cdots \partial^{m_n}_{\mathbf{h},\mathbf{i},n} f(\mathbf{x}) = 0$ *for* $0 \leq m_1 + \cdots + m_n \leq \mathcal{M}(k+1, r, \alpha, \beta_1, \ldots, \beta_{k+1})$*,* $m_{i_1} = \cdots = m_{i_k} = 0$*. Then* f *vanishes identically on* $\{\mathbf{y} \in \mathbb{R}^n |\, h_1(\mathbf{y}) = \cdots = h_k(\mathbf{y}) = 0\}$ *in the neighbourhood of* $\mathbf{x}$*.*

5 Frontier and closure of a semi-Pfaffian set

5.1 Bounds on formats of frontier and closure

5.1 Definition The *closure* $\bar{X}$ of a set X in an open domain G is the intersection with G of the usual topological closure of X:

$$\bar{X} := \{\mathbf{x} \in G |\, \forall \varepsilon > 0\, \exists \mathbf{y} \in X\, (|\mathbf{x} - \mathbf{y}| < \varepsilon)\}.$$

The *frontier* ∂X of X in G is $\partial X := \bar{X} \setminus X$.

From the definition one could hope to infer that closure and frontier of a semi-Pfaffian set are sub-Pfaffian. It turns out that a much stronger statement is true: closure and frontier are actually semi-Pfaffian.

5.2 Theorem ([14]) *Consider a semi-Pfaffian set in disjunctive normal form (DNF)*

$$X := \bigcup_{1 \leq i \leq M} \{\mathbf{x} \in G |\, f_{i1} = \cdots = f_{iI_i} = 0, g_{i1} > 0, \ldots, g_{iJ_i} > 0\} \subset \mathbb{R}^n$$

having a format (r, N, α, β, n)*. Then the closure* $\bar{X}$ *and frontier* ∂X *of* X *in* G *are semi-Pfaffian sets which can be represented in DNF with the format of* $\bar{X}$ *being* $(r, N', \alpha, \beta', n)$*, where*

$$N' = (ND)^{(n+r+1)O(n)},$$

$\beta' = D^{O(n)}$*,* $D = \beta + \alpha(\mathcal{K}(n, r, \alpha, \beta) + 1)^2$*, and* $\mathcal{K}(n, r, \alpha, \beta)$ *is as defined in Section 4. The format of* ∂X *is* $(r, N'', \alpha, \beta', n)$*, where*

$$N'' = (ND)^{(n+r+1)^2 O(n)}.$$

Proof The idea is to reduce the problem of describing the closure to the *semialgebraic* case using Corollary 4.5.

Since the closure of the union of sets equals to the union of closures, it is sufficient to consider just the case of a basic semi-Pfaffian set

$$X := \{\mathbf{x} \in G |\ f_1 = \cdots = f_I = 0, g_1 > 0, \ldots , g_J > 0\}$$

with $I + J = N$. We let $\kappa := \mathcal{K}(n, r, \alpha, \beta) + 1$ and use the notations from Corollary 4.5. According to Corollary 4.5, the closure $\bar{X}$ contains $\mathbf{y} \in G$ if and only if $\mathbf{y}$ belongs to the closure $\bar{X}_{\mathbf{y}}$. Let $h_1(\mathbf{x}), \ldots , h_r(\mathbf{x})$ be the common Pfaffian chain for f_i, g_j, ($i = 1, \ldots , I$, $j = 1, \ldots , J$). The direct calculation shows that

$$F_i(\mathbf{x}, \mathbf{y}) = \Phi_i(\mathbf{x}, \mathbf{y}, h_1(\mathbf{y}), \ldots , h_r(\mathbf{y})) \quad \text{and} \quad G_j(\mathbf{x}, \mathbf{y}) = \Psi_j(\mathbf{x}, \mathbf{y}, h_1(\mathbf{y}), \ldots , h_r(\mathbf{y})),$$

where Φ_i and Ψ_j are polynomials in $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{h} = (h_1, \ldots , h_r)$, of degrees not exceeding $\beta + \alpha\kappa^2$ and $\beta + \alpha\kappa$, respectively.

Thus, $\mathbf{y} \in \bar{X}$ if and only if

$$(\mathbf{y}, h_1(\mathbf{y}), \ldots, h_r(\mathbf{y})) \in \{(\mathbf{z}, h_1, \ldots , h_r) \in G \times \mathbb{R}^r |\ \forall\varepsilon > 0 \exists \mathbf{x}(|\mathbf{x} - \mathbf{z}| \leq \varepsilon,$$

$$\Phi_i(\mathbf{x}, \mathbf{z}, \mathbf{h}) \leq |\mathbf{x} - \mathbf{z}|^{\kappa^2} \text{ for } i = 1, \ldots , I,\ \Psi_j(\mathbf{x}, \mathbf{z}, \mathbf{h}) > |\mathbf{x} - \mathbf{z}|^{\kappa} \text{ for } j = 1, \ldots , J)\}, \tag{5.1}$$

where $\mathbf{z} = (z_1, \ldots , z_n)$ are variables. Formula

$$\forall\varepsilon > 0 \exists \mathbf{x}(|\mathbf{x} - \mathbf{z}| \leq \varepsilon, \Phi_i(\mathbf{x}, \mathbf{z}, \mathbf{h}) \leq |\mathbf{x} - \mathbf{z}|^{\kappa^2} \text{ for } i = 1, \ldots , I,$$

$$\Psi_j(\mathbf{x}, \mathbf{z}, \mathbf{h}) > |\mathbf{x} - \mathbf{z}|^{\kappa} \text{ for } j = 1, \ldots , J) \tag{5.2}$$

of the first order theory of $\mathbb{R}$ contains two blocks of quantifiers of sizes 1 (for ε) and n (for x), $n+r$ free variables, and N polynomials of degrees at most $\beta+\alpha\kappa^2$. According to the efficient quantifier elimination algorithm [38] (see also [3]), there is an equivalent (defining the same set in $\mathbb{R}^{n+r}$) quantifier-free formula in DNF with format $(r, N', \alpha, \beta', n)$. Substituting $h_k(\mathbf{y})$ for h_k into this formula, we obtain a semi-Pfaffian set in DNF with the properties required in the theorem.

The statement for $\partial X = \bar{X} \setminus X$ follows from the statement for $\bar{X}$, using Theorem 3.7 to represent the difference of two sets in DNF as a set in DNF. □

5.2 Complexity of computing frontier and closure

The proof of Theorem 5.2 shows that closure and frontier of a semi-Pfaffian set can be efficiently computed. Indeed, for given input functions $f_{i1}, \ldots , f_{iI_i}, g_{i1}, \ldots , g_{iJ_i}$, $i = 1, \ldots M$, we can write out an explicit formula (5.2) with concrete polynomials $\Phi_i(\mathbf{x}, \mathbf{z}, \mathbf{h})$, $\Psi_j(\mathbf{x}, \mathbf{z}, \mathbf{h})$, and concrete integer κ. Then the quantifier elimination algorithm from [38, 3] is applied, which represents the closure $\bar{X}$ as a quantifier-free formula in DNF. To find the frontier ∂X, the algorithm first lists all consistent sign assignments for the family of all Pfaffian functions involved in DNF formulae for X and $\bar{X}$. Then it selects all the assignments A which lie in $\bar{X}$ and do not lie in X, by checking whether $A \cap \bar{X} \neq \emptyset$ and $A \cap X = \emptyset$. The union of the selected assignments coincides with ∂X.

In order to estimate the "efficiency" of a computation we need to specify more precisely a *model of computation*. As such we use a *real numbers machine* which is an analogy of a classical Turing machine but allows the exact arithmetic and comparisons on real numbers. Since we are interested only in upper complexity bounds for algorithms, we have no need in a formal definition of this model of computation (it can be found in [5]). In most of our computational problems we will need to modify the standard real numbers machine by equipping it with an *oracle* for deciding feasibility of any system of Pfaffian equations and inequalities. An oracle is a subroutine which can be used by a given algorithm any time the latter needs to check feasibility. We assume that this procedure always gives a correct answer ("true" or "false") though we do not specify how it actually works[1]. An *elementary step* of a real numbers machine is either an arithmetic operation, or a comparison (branching) operation, or an oracle call. The *complexity* of a real numbers machine is the number of elementary steps it makes in worst case until termination, as a function of the format of the input.

Using the complexity upper bound for the quantifier elimination procedure from [38, 3], we obtain the following statement.

5.3 Theorem *There are two algorithms which for an input semi-Pfaffian set X defined as in Theorem 5.2 produce the closure $\bar{X}$ and the frontier ∂X respectively, representing them as semi-Pfaffian sets in DNF with formats described in Theorem 5.2. The algorithm for $\bar{X}$ does not use the oracle, its complexity does not exceed $(ND)^{(n+r+1)O(n)}$. The algorithm for ∂X uses at most $(ND)^{(n+r+1)O(n)}$ calls of the oracle, its complexity does not exceed $(ND)^{(n+r+1)^2 O(n)}$.*

5.4 Remark An analysis of proofs of Theorems 5.2 and 5.3 easily shows that they are also true in a parametric form. More precisely, consider a set $X(\mathbf{t})$ defined by a formula in DNF, where all atomic Pfaffian functions f_{ij}, g_{ij} depend on variables $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{t} \in \mathbb{R}^s$. Then there is an algorithm which computes formulae $\Gamma(\mathbf{x}, \mathbf{t})$ and $\Delta(\mathbf{x}, \mathbf{t})$ in DNF such that for any fixed $\mathbf{t}_0 \in \mathbb{R}^s$ we have $\{\mathbf{x} \in \mathbb{R}^n |\ \Gamma(\mathbf{x}, \mathbf{t}_0)\} = \overline{X(\mathbf{t}_0)}$ and $\{\mathbf{x} \in \mathbb{R}^n |\ \Delta(\mathbf{x}, \mathbf{t}_0)\} = \partial X(\mathbf{t}_0)$. Upper bounds on formats of $\Gamma(\mathbf{x}, \mathbf{t})$, $\Delta(\mathbf{x}, \mathbf{t})$, and on the complexity of the algorithm are similar to the analogous bounds from Theorems 5.2 and 5.3.

5.3 Infinitesimal quantifiers

Theorems 5.2 and 5.3 can be interpreted using a language of "infinitesimal quantifiers" [39].

5.5 Definition Let $\mathcal{F}(\mathbf{x}, \mathbf{y})$ be a Boolean combination of some atomic equations and inequalities $f = 0$, $g > 0$, where f, g are Pfaffian functions in variable vectors $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^m$ in an open domain $G \subset \mathbb{R}^{n+m}$. Then

$$(\exists \mathbf{y} \sim 0)\mathcal{F} \text{ stands for } \forall \varepsilon > 0\, \exists \mathbf{y}(|\mathbf{y}| < \varepsilon \text{ and } \mathcal{F}),$$

$$(\forall \mathbf{y} \sim 0)\mathcal{F} \text{ stands for } \exists \varepsilon > 0\, \forall \mathbf{y}(\text{ if } |\mathbf{y}| < \varepsilon, \text{ then } \mathcal{F}).$$

[1]For some classes of Pfaffian functions the feasibility problem is decidable on standard real numbers machines or Turing machines. Apart from polynomials, such class is formed, for example, by terms of the kind $P(e^{x_1}, x_1, x_2, \dots, x_n)$, where P is a polynomial in variables $x_0, x_1, \dots, x_n$ (see [45]). For such classes the oracle can be replaced by a deciding procedure, and we get an algorithm in the usual sense.

Operators $(\exists \mathbf{y} \sim 0)$ and $(\forall \mathbf{y} \sim 0)$ are called *infinitesimal quantifiers*, and can be read "there exists arbitrarily small $\mathbf{y}$ such that $\mathcal{F}$ is true" and "for all sufficiently small $\mathbf{y}$, $\mathcal{F}$ is true", respectively.

It is easy to see that

$$\neg((\exists \mathbf{y} \sim 0)\mathcal{F}) \equiv (\forall \mathbf{y} \sim 0)\neg\mathcal{F} \quad \text{and} \quad \neg((\forall \mathbf{y} \sim 0)\mathcal{F}) \equiv (\exists \mathbf{y} \sim 0)\neg\mathcal{F}. \tag{5.3}$$

Infinitesimal quantifiers are convenient for describing some ε/δ-constructions.

5.6 Example A point $\mathbf{x}$ is a local maximum of a Pfaffian function f if and only if

$$(\forall \mathbf{y} \sim 0)((\mathbf{y} \neq 0) \to (f(\mathbf{x}+\mathbf{y}) \leq f(\mathbf{x}))).$$

It is not immediately obvious that the set of all local maxima of f is semi-Pfaffian.

Unlike ordinary quantifiers, the infinitesimal ones *can be eliminated* in the restricted case. Due to (5.3), it is sufficient to prove this just for the existential quantifier.

Consider in an open domain G a semi-Pfaffian set $X := \{(\mathbf{x}, \mathbf{y}) \in G | \mathcal{F}(\mathbf{x}, \mathbf{y})\} \subset \mathcal{I}^{n+m} \subset G$ and the projection map $\pi : X \to \mathbb{R}^n$ on the subspace of coordinates $\mathbf{x}$. Let $X(\mathbf{x}) := \pi^{-1}(\mathbf{x}) \cap X$.

5.7 Lemma *The set* $\{\mathbf{x} \in \mathcal{I}^n | \, (\exists \mathbf{y} \sim 0)\mathcal{F}(\mathbf{x}, \mathbf{y})\}$ *coincides with* $\{\mathbf{x} \in \mathcal{I}^n | \, (\mathbf{x}, \mathbf{0}) \in \overline{X(\mathbf{x})}\}$, *where* $\mathbf{0}$ *is the origin in* $\mathbb{R}^m$.

Proof If a semi-Pfaffian set Z is contained in the closed cube $\mathcal{I}^m$, then its topological closure coincides with $\bar{Z}$. Then the statement $(\exists \mathbf{y} \sim 0)(\mathbf{y} \in Z)$ is equivalent to $\mathbf{0} \in \bar{Z}$. It follows that for any fixed $\mathbf{x} \in \mathcal{I}^n$ the statement $(\exists \mathbf{y} \sim 0)\mathcal{F}(\mathbf{x}, \mathbf{y})$ is equivalent to $(\mathbf{x}, \mathbf{0}) \in \overline{X(\mathbf{x})}$. $\square$

Due to Remark 5.4, the set $\{\mathbf{x} \in \mathcal{I}^n | \, (\mathbf{x}, \mathbf{0}) \in \overline{X(\mathbf{x})}\}$ is semi-Pfaffian, with explicit upper bounds on the format and on the complexity of the algorithm for computing this set. Lemma 5.7 implies that the same is true for the set $\{\mathbf{x} \in \mathcal{I}^n | \, (\exists \mathbf{y} \sim 0)\mathcal{F}(\mathbf{x}, \mathbf{y})\}$.

In particular, the set of all points of local maxima (Example 5.6) is semi-Pfaffian. In [39] the singular locus of a semi-Pfaffian set in $\mathcal{I}^n$ is defined by a formula using only infinitesimal quantifiers, thus the singular locus is semi-Pfaffian with an explicit upper bound on the format.

6 Stratification of a semi-Pfaffian set

In [46] Whitney proved that an algebraic set in $\mathbb{R}^n$ can be represented as a finite disjoint union of smooth manifolds which are semialgebraic sets. Łojasiewicz [30, 31] extended Whitney's theorem to the class of real semianalytic sets. Later Gabrielov [13] showed, as a part of an elementary proof of his *complement theorem*, that the smooth strata of a semianalytic set X can be defined by functions belonging to the smallest extension of a family of functions defining X which is closed under additions, multiplications and taking partial derivatives. Important classes sharing this property are Pfaffian functions and their special subclasses: polynomials, fewnomials, exponential polynomials. Combined with estimates on multiplicities of Pfaffian intersection from Section 4, this result allows to construct an algorithm which produces a smooth stratification of a semi-Pfaffian set, to estimate its complexity, and to bound formats of the resulting strata [18]. Similar results are true for sets in $\mathbb{C}^n$ defined by Boolean combinations of atomic formulae of the kind $f = 0$ and $f \neq 0$, where f is a Pfaffian function over $\mathbb{C}$.

6.1 Definition A *weak stratification* of a semi-Pfaffian set X is partition of X into a disjoint union of smooth (i.e., nonsingular), not necessarily connected, possibly empty, semi-Pfaffian subsets X_i called *strata*. A stratification is *basic* if all strata are basic semi-Pfaffian sets which are *effectively nonsingular*, i.e., the system of equations and inequalities for each stratum X_i of codimension k includes a set of k Pfaffian functions $h_{i1}, \dots , h_{ik}$ such that the restriction $h_{ij}|_{X_i} \equiv 0$ for $j = 1, \dots , k$, and $dh_{i1} \wedge \cdots \wedge dh_{ik} \neq 0$ at every point of X_i. Note that we don't require the boundary of a stratum to coincide with a union of some other strata, let alone any regularity conditions.

Consider a semi-Pfaffian set X defined by (2.2) having a format (r, N, α, β, n). Let

$$\mathcal{B} := (\alpha + \beta + 1)^{(r+1)^{O(n)}}.$$

6.2 Theorem ([18]) *There is an algorithm (without an oracle) which produces a finite basic weak stratification of a semi-Pfaffian set X. The number of strata (some of which may be empty) is $N^{n+r}\mathcal{B}$, each having the format $(r, N\mathcal{B}, \alpha, \mathcal{B}, n)$. The complexity of the algorithm does not exceed $3^N N^{n+1}\mathcal{B}$. If the oracle is allowed, the algorithm produces only non-empty strata and its complexity does not exceed $N^{n+r}\mathcal{B}$.*

Outline of a proof (inspired by [46, 13]). Let $X = \{\mathbf{x} \in \mathbb{R}^2 |\, f(\mathbf{x}) = 0, g(\mathbf{x}) > 0\}$ be a basic semi-Pfaffian set (curve) in $\mathbb{R}^2$ and zero be a regular value of f. Then the sets $X^1 := \{\mathbf{x} \in X |\, (\partial_1 f, \partial_2 f)(\mathbf{x}) \neq 0\}$ and $X^2 := \{\mathbf{x} \in X |\, (\partial_1 f, \partial_2 f)(\mathbf{x}) = 0\}$ form a stratification of X. If zero is a singular value of f, then to define a stratification we will need to consider partial derivatives of f of higher orders, but, due to Corollary 4.7, not higher than $\mathcal{M}(1, r, \alpha, \beta)$. The proof of the theorem is a far-reaching generalization of this idea.

Assume that the use of the oracle is allowed. Note that Theorem 3.7 can be turned into an algorithm (with oracle) for listing all consistent sign assignments for a given family of Pfaffian functions. Thus, we can assume that X is a *basic* semi-Pfaffian set defined by a system of equations and strict inequalities:

$$X := \{\mathbf{x} |\, f_1(\mathbf{x}) = \cdots = f_\lambda(\mathbf{x}) = 0, g_1(\mathbf{x}) > 0, \dots , g_\mu(\mathbf{x}) > 0\}.$$

We now employ the notation for partial differential operators from Definition 4.6. Additionally, let $\mathcal{M}_1 := \mathcal{M}(1, r, \alpha, \beta)$, $\beta_k := \beta + (\mathcal{M}_k - 1)((\alpha - 1)k + \beta_1 + \cdots + \beta_{k-1})$ and $\mathcal{M}_{k+1} := \mathcal{M}(k + 1, r, \beta_k, \beta_1, \dots , \beta_k)$.

Consider a list of all partial derivatives $\partial_n^{q_n} \cdots \partial_1^{q_1}$ of functions f_j with $q_1 + \cdots + q_n \leq \mathcal{M}(0, r, \alpha, \beta)$, ordered lexicographically in $(q_n, \dots , q_1, j)$. Let $X^0 \subset X$ be a subset where all these derivatives vanish. According to Corollary 4.7, all the functions f_j are identically zero in a neighbourhood of each $\mathbf{x} \in X^0$, hence X^0, if nonempty, is a smooth open set in $\mathbb{R}^n$ coinciding with X.

Suppose now that $X^0 = \emptyset$, thus there can be found a function $f_j \not\equiv 0$. Then any $\mathbf{x} \in X$ belongs to one and only one of the sets

$$Z^1_{j_1,\mathbf{m}^1} \subset X, \quad \mathbf{m}^1 := (m^1_{i_1}, \dots , m^1_1), \quad m^1_1 + \cdots + m^1_{i_1} \leq \mathcal{M}_1, \quad m^1_{i_1} > 0,$$

at whose points the derivative $h'_1 := \partial_{i_1}^{m^1_{i_1}} \cdots \partial_1^{m^1_1} f_{j_1}$ is different from 0, while all derivatives in the lexicographically ordered list preceding h'_1 vanish. Let $h_1 := \partial_{i_1}^{m^1_{i_1}-1} \partial_{i_1}^{m^1_{i_1-1}} \cdots \partial_1^{m^1_1} f_{j_1}$, so that $h'_1 = \partial_{i_1} h_1$, and consider a smooth manifold $Y^1 := \{\mathbf{x} \in X |\, h_1(\mathbf{x}) = 0, h'_1(\mathbf{x}) \neq 0\} \supset Z^1_{j_1,\mathbf{m}^1}$ of codimension 1. Denote by F^1 the set of all functions that appear in equations defining $Z^1_{j_1,\mathbf{m}^1}$. Note that for any $i < i_1$ all functions from F^1, including h_1, do not depend on x_i, due to Corollary 4.7.

We now consider partial derivatives

$$\hat{\partial}_n^{q_n} \cdots \hat{\partial}_{i_1+1}^{q_{i_1+1}} := \partial_{h_1,i_1,n}^{q_n} \cdots \partial_{h_1,i_1,i_1+1}^{q_{i_1+1}}$$

of functions $f_\nu \in F^1$, $\nu = (q_{i_1}, \dots , q_1, j)$ "along" the manifold $Y^1 \cap \{x_i = 0 |\; i < i_1\}$ with $q_{i_1+1} + \cdots + q_n \leq \mathcal{M}_2$, ordered lexicographically in $(q_n, \dots , q_{i_1+1}, \nu)$. Let $X^1_{j_1,\mathbf{m}^1} \subset Y^1 \cap \{x_i = 0 |\; i < i_1\}$ be a subset of X where all these derivatives vanish. According to Corollary 4.7, all the functions f_ν are identically zero on $Y^1 \cap \{x_i = 0 |\; i < i_1\}$ in a neighbourhood of each $\mathbf{x} \in X^1_{j_1,\mathbf{m}^1}$, hence $X^1_{j_1,\mathbf{m}^1}$, if nonempty, is an open submanifold of $Y^1 \cap \{x_i = 0 | \,i < i_1\}$. Since all the functions in the equations defining $X^1_{j_1,\mathbf{m}^1}$ and Y^1 do not depend on x_i, for $i < i_1$, this implies that $X^1_{j_1,\mathbf{m}^1}$ is a smooth manifold of codimension 1.

If $\mathbf{x} \notin X^1_{j_1,\mathbf{m}^1}$, then $\mathbf{x}$ belongs to one and only one of the sets

$$Z^2_{j_1,j_2,\mathbf{m}^1,\mathbf{m}^2} \subset Z^1_{j_1,\mathbf{m}^1}, \quad \mathbf{m}^2 = (m^2_{i_2}, \dots , m^1_1), \quad m^2_{i_1+1} + \cdots + m^2_{i_2} \leq \mathcal{M}_2,$$

$$m^2_{i_2} > 0, \quad (m^2_{i_1}, \dots , m^2_1, j_2) \prec (m^1_{i_1}, \dots , m^1_1, j_1),$$

where $\prec$ is the lexicographic order, at whose points the derivative $h'_2 := \hat{\partial}_{i_2}^{m^2_{i_2}} \cdots \hat{\partial}_1^{m^2_{i_2}} f_{j_2}$ (where $\hat{\partial}_i := \partial_{h_1,i_1,i}$) is different from 0, while all derivatives in the lexicographically ordered list preceding h'_2 vanish. Let

$$h_2 := \hat{\partial}_{i_2}^{m^2_{i_2}-1} \partial_{i_2}^{m^2_{i_2-1}} \cdots \partial_1^{m^2_1} f_{j_2},$$

so that

$$h'_2 = \hat{\partial}_{i_2} h_2 = \partial_{i_1} h_1(\mathbf{x}) \partial_{i_2} h_2(\mathbf{x}) - \partial_{i_2} h_1(\mathbf{x}) \partial_{i_1} h_2(\mathbf{x}),$$

and consider a smooth manifold

$$Y^2 := \{\mathbf{x} \in X |\, h_1(\mathbf{x}) = h_2(\mathbf{x}) = 0, h'_2(\mathbf{x}) \neq 0\} \supset Z^2_{j_1,j_2,\mathbf{m}^1,\mathbf{m}^2}$$

of codimension 2.

The continuation of this procedure for $k = 2, \dots , n$ leads to the consecutive definition of the sets $X^k_{j_1,\dots,j_k,\mathbf{m}^1,\dots,\mathbf{m}^k}$, where

$$1 \leq j_t \leq \lambda, \quad 0 \leq m^t_1 + \cdots + m^t_{i_1} \leq \mathcal{M}_1, \dots , 0 \leq m^t_{i_{t-1}+1} + \cdots + m^t_{i_t} \leq \mathcal{M}_t, \quad m^t_{i_t} > 0,$$

$$(m^t_{i_s}, \dots , m^t_1, j_t) \prec (m^s_{i_s}, \dots , m^s_1, j_s) \quad \text{for} \quad 1 \leq s \leq t.$$

The same arguments as above show that sets $X^k_{j_1,\dots,j_k,\mathbf{m}^1,\dots,\mathbf{m}^k}$ form a stratification of X, i.e., they are disjoint smooth manifolds, and their union is X. The number of strata, their formats, and complexity of producing them can be estimated from the process of generating sets $X^k_{j_1,\dots,j_k,\mathbf{m}^1,\dots,\mathbf{m}^k}$. □

7 Cylindrical decompositions of sub-Pfaffian sets

In [10] by Gabrielov it was proved that the complement of any subanalytic set in a cube $\mathcal{I}^n$ is also subanalytic. This *complement theorem* plays a key role in real analytic geometry (see [4, 8]) and in model-theoretic study of o-minimality [9, 47].

The complement theorem immediately follows from the existence of a *cylindrical cell decomposition* of the ambient space compatible with a given subanalytic set. The existence was proved in [13] by means of a quasi-constructive process of manipulating with symbols of real analytic functions and their derivatives. In [19] the method from [13] was modified, so that being applied to a sub-Pfaffian set it yields an algorithm producing a cylindrical cell decomposition of this set. There is also an alternative algorithm for a cylindrical decomposition with a slightly better complexity bound [36].

For a special case of *semialgebraic sets* similar complexity results are known for a cylindrical cell decomposition problem [7, 48], and significantly better results are known for *quantifier elimination* problem (the latter is stronger than the complement theorem).

Simultaneously on m&K.

7.1 Definition *Cylindrical cell* is defined by induction as follows.

(1) Cylindrical 0-cell in $\mathbb{R}^n$ is an isolated point.

(2) Cylindrical 1-cell in $\mathbb{R}$ is an open interval $(a,b) \subset \mathbb{R}$.

(3) For $n \geq 2$ and $0 \leq k < n$ a cylindrical $(k+1)$-cell in $\mathbb{R}^n$ is either a graph of a continuous bounded function $f: C \to \mathbb{R}$, where C is a cylindrical ~~a cylindrical~~ $(k+1)$-cell in $\mathbb{R}^{n-1}$, or else a set of the form

$$\{(x_1,\dots,x_n) \in \mathbb{R}^n |\, (x_1,\dots,x_{n-1}) \in C \text{ and } f(x_1,\dots,x_{n-1}) < x_n < g(x_1,\dots,x_{n-1})\},$$

where C is a cylindrical k-cell in $\mathbb{R}^{n-1}$, and $f, g: C \to \mathbb{R}$ are continuous bounded functions such that $f(x_1,\dots,x_{n-1}) < g(x_1,\dots,x_{n-1})$ for all points $(x_1,\dots,x_{n-1}) \in C$.

The definition implies that any k-cell is homeomorphic to an open k-dimensional ball.

7.2 Definition *Cylindrical cell decomposition* $\mathcal{D}$ of a subset $A \subset \mathbb{R}^n$ is defined by induction as follows.

(1) If $n = 1$, then $\mathcal{D}$ is a finite family of pair-wise disjoint cylindrical cells (i.e., isolated points and intervals) whose union is A.

(2) If $n \geq 2$, then $\mathcal{D}$ is a finite family of pair-wise disjoint cylindrical cells in $\mathbb{R}^n$ whose union is A and there is a cylindrical cell decomposition of $\pi(A)$ such that $\pi(C)$ is its cell for each $C \in \mathcal{D}$, where $\pi: \mathbb{R}^n \to \mathbb{R}^{n-1}$ is the projection map onto the coordinate subspace of $x_1,\dots,x_{n-1}$.

7.3 Definition Let $B \subset A \subset \mathbb{R}^n$ and $\mathcal{D}$ be a cylindrical cell decomposition of A. Then $\mathcal{D}$ is *compatible* with B if for any $C \subset B$ we have either $C \subset B$ or $C \cap B = \emptyset$ (i.e., some subset $\mathcal{D}' \subset \mathcal{D}$ is a cylindrical cell decomposition of B).

7.4 Theorem ([19]) *Let X be a semi-Pfaffian set in an open domain $G \subset \mathbb{R}^{m+n}$ defined by (2.2) with $s = m+n$, format $(r, N, \alpha, \beta, m+n)$, and contained in an open cube $\hat{\mathcal{I}}^{m+n} := (-1,1)^{m+n}$ such that the closure $\mathcal{I}^{m+n} \subset G$. Let $\pi : \mathbb{R}^{m+n} \to \mathbb{R}^n$ be the projection function, $Y := \pi(X)$, and $d := \dim(Y)$. Then there is an algorithm (with the oracle) producing a cylindrical cell decomposition $\mathcal{D}$ of the image of $\hat{\mathcal{I}}^n = \pi(\hat{\mathcal{I}}^{m+n})$ under a linear coordinate change such that $\mathcal{D}$ is compatible with the image of Y.*

Each cell is described as a projection of a semi-Pfaffian set in DNF, i.e., by a formula of the type

$$\pi'\left(\bigcup_{1\le i\le M}\ \bigcap_{1\le j\le M_i} \{h_{ij} *_{ij} 0\}\right),$$

where h_{ij} are Pfaffian functions in $n' \ge m+n$ variables, $\pi' : \mathbb{R}^{n'} \to \mathbb{R}^n$ is the projection function, $*_{ij} \in \{=, >\}$, and M, M_i $(i = 1, \dots, M)$ are certain integers.

The number of cells in the decomposition is less than

$$\mathcal{N} := N^{(d!)^2(m+2n)^d(r+m+2n)^d}(\alpha+\beta)^{r^{O(d(m+dn))}},$$

the format of each cell is

$$\left(r, \mathcal{N}, (\alpha+\beta)^{r^{O(d(m+dn))}}, (\alpha+\beta)^{r^{O(d(m+dn))}}, \mathcal{N}\right).$$

The complexity of the algorithm is

$$N^{(r+m+n)^{O(d)}}(\alpha+\beta)^{(r+m+n)^{O(d(m+dn))}}.$$

7.5 Corollary *Under assumptions of Theorem 7.4 the complement $\widetilde{Y} := \hat{\mathcal{I}}^n \setminus Y$ is a sub-Pfaffian set. There is an algorithm for computing $\widetilde{Y}$ having the same complexity as the algorithm from Theorem 7.4. The complement $\widetilde{Y}$ is represented by the algorithm as a union of some cells of the cylindrical cell decomposition a linear image of $\hat{\mathcal{I}}^n$ described in Theorem 7.4.*

Outline of a proof of Theorem 7.4 Assume that points in $\mathbb{R}^{m+n}$ are of the kind $(\mathbf{y}, \mathbf{x}) = (x_1, \dots, x_n, x_{n+1}, \dots, x_{m+n})$ and $\pi(\mathbf{y}, \mathbf{x}) = \mathbf{y}$.

Computing the dimension $d = \dim(Y)$.

The algorithm computes a weak stratification of X using Theorem 6.2. Each stratum $X_\lambda = \{(\mathbf{y}, \mathbf{x}) \in \mathbb{R}^{m+n} |\ \mathbf{f} = 0, \mathbf{g} > 0\}$, where $\mathbf{f}$, $\mathbf{g}$ are vectors of Pfaffian functions and relations $=, >$ are understood component-wise, is effectively nonsingular (see Definition 6.1). In particular, the dimension $d'_\lambda := \dim(X_\lambda)$ is computed. To find $d_\lambda := \dim(\pi(X_\lambda))$, observe that by Sard's theorem rank $(\partial \mathbf{f}/\partial \mathbf{x})(\mathbf{y}, \mathbf{x}) \le m - d'_\lambda + d_\lambda$ for any $(\mathbf{y}, \mathbf{x}) \in X_\lambda$, while the equality is attained for *almost* any $(\mathbf{y}, \mathbf{x}) \in X_\lambda$. Using the oracle, choose the maximal a such that

$$\{(\mathbf{y}, \mathbf{x}) \in X_\lambda |\ \mathrm{rank}\, (\partial \mathbf{f}/\partial \mathbf{x})(\mathbf{y}, \mathbf{x}) = m - d'_\lambda + a\} \ne \emptyset,$$

then $a = d_\lambda$ and $d = \max_\lambda\{d_\lambda\}$.

An example.

To illustrate the idea of the algorithm, consider

$$X = \{\mathbf{x} = (x_1, x_2, x_3) |\, f := x_1^2 + x_2^2 + x_3^2 - 1/2 = 0\}, \quad Y = \{\mathbf{y} = (x_1, x_2) |\, x_1^2 + x_2^2 \leq 1/2\},$$

hence $n = d = 2$, $m = 1$.

The algorithm consists of two recursive procedures which we call **down** and **up**. Starting the first step of the **down** procedure observe that X is effectively non-singular. Let

$$X' := \{(x_1, x_2, x_3) \in X |\, \partial f / \partial x_3 \neq 0\},$$

$$V' := \{(x_1, x_2, x_3) \in X | \partial f / \partial x_3 = 0\} = \{(x_1, x_2, x_3) \in X |\, \hat{f}_1 := x_3 = 0, \hat{f}_2 := x_1^2 + x_2^2 - \frac{1}{2} = 0\}$$

be semi-Pfaffian sets of all regular and of all singular points respectively of the restriction of the projection map $\pi: (x_1, x_2, x_3) \mapsto (x_1, x_2)$ on X. Introduce new notation: $X_2 := V'$, $Y_2 := \pi(V')$, $d_2 := \dim(Y_2) = 1$. We have described the first recursive step of **down**.

Now we start the second step with X_2, Y_2 playing the role of X, Y. Observe that X_2 is effectively non-singular and all points are regular for the restriction $\pi|_{X_2}$. Consider the projection map $\rho_2 : (x_1, x_2) \mapsto x_1$. Let

$$S_2 := \left\{ (x_1, x_2, x_3) \in X_2 |\, \det \begin{pmatrix} \partial \hat{f}_1 / \partial x_2 & \partial \hat{f}_1 / \partial x_3 \\ \partial \hat{f}_2 / \partial x_2 & \partial \hat{f}_2 / \partial x_3 \end{pmatrix} = 0 \right\} =$$

$$= \{(x_1, x_2, x_3) \in \mathbb{R}^3 |\, x_1^2 = 1/2, x_2 = x_3 = 0\}$$

be the set of all critical points of $\rho_2 \pi|_{X_2}$. Let $Z_2 := \rho_2 \pi(S_2)$, then $\dim(Z_2) = \dim(Y_2 \cap \rho_2^{-1}(Z_2)) = 1 < d = 2$. Let $X_3 := S_2$, $Y_3 := Y_2 \cap \rho_2^{-1}(Z_2)$, $d_3 := \dim(Y_3) = 0$ We have completed the second recursive step of **down**.

On the last (degenerate) recursive step, X_3 is effectively non-singular and all points are regular for the restriction $\pi|_{X_3}$. For the projection map $\rho_3 : (x_1, x_2) \mapsto 0$, the set S_3 of all critical points of $\rho_3 \pi|_{X_3}$ is empty, thus the projection $Z_3 := \rho_2 \pi(S_3) = \emptyset$. The **down** procedure is completed.

Now the algorithm starts the recursive **up** procedure. On the first recursive step consider the sub-Pfaffian set Y_3 constructed on the last step of **down**. Since Y_3 consists of just two points, $(1/\sqrt{2}, 0)$ and $(-1/\sqrt{2}, 0)$, the construction of a cylindrical cell decomposition $\mathcal{D}_3$ of $\hat{\mathcal{I}}^2$ compatible with Y_3 is trivial.

On the second recursive step consider sub-Pfaffian sets Y_2 and Z_2. Decomposition $\mathcal{D}_3$, being cylindrical, induces a cell decomposition $\mathcal{D}_3'$ of $\hat{\mathcal{I}}^1 = \{x_1 |\, -1 < x_1 < 1\}$ into five cells:

- $C_1 := \{x_1 |\, -1 < x_1 < -1/\sqrt{2}\}$,
- $C_2 := \{x_1 |\, x_1 = -1/\sqrt{2}\}$,
- $C_3 := \{x_1 |\, -1/\sqrt{2} < x_1 < 1/\sqrt{2}\}$,
- $C_4 := \{x_1 |\, x_1 = 1/\sqrt{2}\}$,
- $C_5 := \{x_1 |\, 1/\sqrt{2} < x_1 < 1\}$.

For any $z \in C_3$ the cardinality of the fibre $\rho_2^{-1}(z) \cap Y_2$ is constant $(= 2)$. Moreover, the following five cells form a cylindrical cell decomposition of $\rho_2^{-1}(C_3) \cap \hat{\mathcal{I}}^2$ compatible with $\rho_2^{-1}(C_3) \cap Y$:

- $\{(x_1, x_2) \in \rho_2^{-1}(C_3) \cap \hat{\mathcal{I}}^2 |\ \exists (y_1, y_2) \in Y\ \exists (y_1', y_2') \in Y (y_1 = y_1', y_2 < y_2' < x_2)\}$
- $\{(x_1, x_2) \in \rho_2^{-1}(C_3) \cap \hat{\mathcal{I}}^2 |\ \exists (y_1, y_2) \in Y\ \exists (y_1', y_2') \in Y (y_1 = y_1', y_2 < y_2' = x_2)\}$
- $\{(x_1, x_2) \in \rho_2^{-1}(C_3) \cap \hat{\mathcal{I}}^2 |\ \exists (y_1, y_2) \in Y\ \exists (y_1', y_2') \in Y (y_1 = y_1', y_2 < x_2 < y_2')\}$
- $\{(x_1, x_2) \in \rho_2^{-1}(C_3) \cap \hat{\mathcal{I}}^2 |\ \exists (y_1, y_2) \in Y\ \exists (y_1', y_2') \in Y (y_1 = y_1', y_2 = x_2 < y_2')\}$
- $\{(x_1, x_2) \in \rho_2^{-1}(C_3) \cap \hat{\mathcal{I}}^2 |\ \exists (y_1, y_2) \in Y\ \exists (y_1', y_2') \in Y (y_1 = y_1', x_2 < y_2 < y_2')\}$.

Similar cylindrical cell decompositions of $\rho_2^{-1}(C_i) \cap \hat{\mathcal{I}}^2$ can be constructed for all other cells C_i, but in fact such decompositions have been already produced as parts of the cell decomposition $\mathcal{D}_3$. Combining all cell decomposition for $\rho_2^{-1}(C_i) \cap \hat{\mathcal{I}}^2$ with $\mathcal{D}_3$ we get a cylindrical cell decomposition of $\hat{\mathcal{I}}^2$ compatible with Y. This completes the **up** procedure and the whole construction.

General algorithm: the ***down*** *procedure.*

After computing d the algorithm uses one after another two recursive procedures, **down** and **up**. We start with **down** by conducting a descending recursion on d. We will describe in some detail only the first (typical) step of the recursion. For each stratum X_λ with $\dim(X_\lambda) \geq d$ the algorithm finds the semi-Pfaffian set X_λ' of all regular and the semi-Pfaffian set V_λ' of all critical points of the projection map $\pi|_{X_\lambda}$. Note that X_λ' and V_λ' can be described by explicit quantifier-free formulae and any of them may be empty. By Sard's theorem, $\dim(\pi(V_\lambda')) < d$. For any $y \in Y \setminus \pi(V_\lambda')$ the intersection $\pi^{-1}(y) \cap X_\lambda'$ is smooth.

Now the algorithm works with X_λ'. The next step is to select in this smooth manifold a semi-Pfaffian subset of the dimension d having the same π-projection as X_λ'. The algorithm finds a function $g : \mathbb{R}^{m+n} \to \mathbb{R}$ and a semi-Pfaffian set $V_\lambda'' \subset X_\lambda'$ such that $\dim(\pi(V_\lambda'')) < d$, and for any $y \in Y \setminus \pi(V_\lambda' \cup V_\lambda'')$ the critical points of $g|_{\pi^{-1}(y) \cap X_\lambda'}$ are non-degenerate, in particular isolated. Function g can be defined by an expression

$$(\prod_j h_j)(1 + (\mathbf{c}, \mathbf{x})),$$

where $\mathbf{c}$ is a vector of integers and the product of zero number of factors is assumed to be equal to 1 (see details in [19]). The set X_λ'' such that for any $y \in Y \setminus \pi(V_\lambda' \cup V_\lambda'')$ the intersection $X_\lambda'' \cap \pi^{-1}(y)$ is a finite set of all critical points of $g|_{\pi^{-1}(y) \cap X_\lambda'}$, can be described by an explicit quantifier-free formula. If $X_\lambda'' \neq \emptyset$, then $\dim(X_\lambda'') = d$.

The algorithm computes a weak stratification of X_λ'' using Theorem 6.2. For each stratum of $X_{\lambda\mu}$ the (maximal) dimension d the algorithm finds the semi-Pfaffian set $X_{\lambda\mu}'$ of all regular and the semi-Pfaffian set $V_{\lambda\mu}'$ of all critical points of the projection map $\pi|_{X_{\lambda\mu}}$. These sets can be described by explicit quantifier-free formulae. By Sard's theorem, $\dim(\pi(V_{\lambda\mu}'')) < d$. Let $V_{\lambda\mu} := V_\lambda' \cup V_\lambda'' \cup V_{\lambda\mu}'$, and T be the union of all strata of X of the dimension less than d.

If $d = n$, then define

$$X_{new} := \bigcup_{\lambda\mu} (\partial X_{\lambda\mu}' \cup V_{\lambda\mu}) \cup T,$$

$$Y_{new} := \bigcup_{\lambda,\mu} \pi(\partial X_{\lambda\mu}' \cup V_{\lambda\mu}) \cup \pi(T),$$

where the frontier $\partial X'_{\lambda\mu}$ can be computed using Theorem 5.3. Observe that $\dim(Y_{new}) < d$. The algorithm goes to the next recursive step with X_{new} and Y_{new} replacing X and Y respectively.

If $d < n$, then the algorithm continues the current recursive step. Consider the projection function $\rho:\ \mathbb{R}^n \to \mathbb{R}^d$, where $\mathbb{R}^n$ is equipped with coordinates $\mathbf{y}$. The algorithm performs a linear transformation of coordinates $\mathbf{y}$ such that in the new coordinates for each $\mathbf{y}$ in the closure $\bar{Y}$ the set $\rho^{-1}(\rho(\mathbf{y}))$ is finite[2]. In the sequel we assume that this condition is satisfied.

Consider the set $S_{\lambda\mu}$ of all critical points of the restriction of the composition $\rho\pi : \mathbb{R}^{m+n} \to \mathbb{R}^d$ to $X'_{\lambda\mu}$. This set can be described by an explicit quantifier-free formula. Observe that $\dim(\rho\pi(S_{\lambda\mu})) < d$ by Sard's theorem, hence $\dim(\pi(S_{\lambda\mu})) < d$ by the choice of the linear transformation.

The aim of the next action of the algorithm is to identify a subset in $\mathbb{R}^d$ of a positive codimension such that within each connected component of the complement of this set any two points have 0-dimensional ρ-fibers with the same lexicographic order. Introduce the following sets:

$$W_{i,\varepsilon} := \{(\mathbf{y}, \varepsilon)|\ \mathbf{y} = (z_1, \dots , z_{n-d}, y_{n-d+1}, \dots , y_n) \in Y,\ \varepsilon \in \mathbb{R},$$

$$\exists \mathbf{y}' = (z'_1, \dots , z'_{n-d}, y_{n-d+1}, \dots , y_n) \in Y,\ \rho(\mathbf{y}') = \rho(\mathbf{y}),$$
$$z'_1 = z_1, \dots , z'_{i-1} = z_{i-1}, z'_i \neq z_i,\ |z'_i - z_i| < \varepsilon\} \subset \mathbb{R}^{n+1};$$

$$W_i := \overline{W}_{i,\varepsilon} \cap \{(\mathbf{y}, \varepsilon)|\ \varepsilon = 0\} \subset \mathbb{R}^n;$$

$$W := \bigcup_{1 \le i \le n-d} W_i \subset \mathbb{R}^n;$$

$$Z := \rho\pi\left(\bigcup_{\lambda,\mu} (\partial X'_{\lambda\mu} \cup V_{\lambda\mu} \cup S_{\lambda\mu})\right) \cup \rho(W) \cup \rho\pi(T) \subset \mathbb{R}^d.$$

Then $\dim(Z) = \dim(Y \cap \rho^{-1}(Z)) < d$. Observe that W is a sub-Pfaffian set, more precisely, there exists an integer n_{new} such that $n + m \le n_{new} \le 2n + m$, and a semi-Pfaffian set $U \subset \mathbb{R}^{n_{new}}$ such that $\pi_{new}(U) = W$ for the projection map $\pi_{new} :\ \mathbb{R}^{n_{new}} \to \mathbb{R}^n$. There is an explicit quantifier-free formulae defining U.

Let U' denote the semi-Pfaffian set defined in $\mathbb{R}^{n_{new}}$ by the same quantifier-free formula as

$$\bigcup_{\lambda\mu} (\partial X'_{\lambda\mu} \cup V_{\lambda\mu} \cup S_{\lambda\mu}) \cup T \subset \mathbb{R}^{m+n}.$$

Let $Y_{new} := Y \cap \rho^{-1}(Z) = \pi_{new}(U \cup U')$, $X_{new} := \pi^{-1}_{new}(Y \cap \rho^{-1}(Z)) = U \cup U'$. Observe that X_{new} is defined by an explicit quantifier-free formula with Pfaffian functions in n_{new} variables. The algorithm determines $d_{new} := \dim(Y_{new}) = \dim(Z) < d$ using the method described in the beginning of the proof.

This completes the description of the recursive step. On the last recursive step (let it have number $l \le d$) the dimension $\dim(Y_{new}) = 0$ and $Z = \emptyset$.

[2] According to Koopman-Brown theorem such a transformation exists, for details on how to find it see [19].

*General algorithm: the **up** procedure.*

The algorithm then starts the **up** recursion procedure. An input data of the rth recursive step consists of:

- a pair $Y, Z \subset \mathbb{R}^n$ of sub-Pfaffian sets constructed on steps $l-r$ and $l-r+1$ respectively of the "down" procedure, and
- a cylindrical cell decomposition $\mathcal{D}$ of $\hat{\mathcal{I}}^n$ compatible with $Y \cap \rho^{-1}(Z)$.

The decomposition $\mathcal{D}$ induces a cylindrical cell decomposition $\mathcal{D}'$ of $\rho(\hat{\mathcal{I}}^n) = \hat{\mathcal{I}}^d$ compatible with Z, namely, the elements of $\mathcal{D}'$ are exactly the ρ-projections of the elements of $\mathcal{D}$. By the definition of Z, for any d-dimensional cell $C \in \mathcal{D}'$, for any $\mathbf{z} \in C$ the cardinality of $\rho^{-1}(\mathbf{z}) \cap Y$ is a constant, say $\mathcal{L}$. Moreover, the union

$$\bigcup_{1 \le \nu \le \mathcal{L}+1} \{\mathbf{y} \in \rho^{-1}(C) \cap \hat{\mathcal{I}}^n |\, \exists \mathbf{y}_1 \in Y \cdots \exists \mathbf{y}_{\mathcal{L}} \in Y (\mathbf{y}_i \neq \mathbf{y}_j \text{ for all } i, j, i \neq j,$$

$$\mathbf{y}_1 \prec \cdots \prec \mathbf{y}_{\nu-1} \prec \mathbf{y} \prec \mathbf{y}_\nu \prec \cdots \prec \mathbf{y}_{\mathcal{L}}, \rho(\mathbf{y}_1) = \cdots = \rho(\mathbf{y}_{\mathcal{L}}) = \rho(\mathbf{y}))\},$$

where the relation $\mathbf{u} \prec \mathbf{v}$ for $\mathbf{u} = (u_1, \ldots, u_n), \mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$ stands for the disjunction

$$\bigvee_{d+1 \le i \le n+1} (u_1 = v_1, \ldots, u_{i-1} = v_{i-1}, u_i < v_i),$$

represents a cylindrical cell decomposition of $\rho^{-1}(C) \cap \hat{\mathcal{I}}^n$ compatible with $\rho^{-1}(C) \cap Y$.[3]

Note that Corollary 3.3 provides an upper bound for $\mathcal{L}$ as an explicit function, say $\mathcal{M}$, of the format of Y. The algorithm finds $\mathcal{L}$ as the maximal l, $1 \le l \le \mathcal{M}$ such that, the statement

$$\exists \mathbf{y}_1 \in \rho^{-1}(C) \cdots \exists \mathbf{y}_l \in \rho^{-1}(C) (\mathbf{y}_i \neq \mathbf{y}_j \text{ for all } i, j,\ i \neq j,\ \rho(\mathbf{y}_1) = \cdots = \rho(\mathbf{y}_l))$$

is true. Then the algorithm computes the cell decomposition of $\rho^{-1}(C) \cap \hat{\mathcal{I}}^n$.

Combining the cylindrical cell decompositions for $\rho^{-1}(C) \cap \hat{\mathcal{I}}^n$ for all d-dimensional cells C of $\mathcal{D}'$, with the cell decomposition $\mathcal{D}$, the algorithm gets a cylindrical cell decomposition of $\hat{\mathcal{I}}^n$ compatible with Y. This finishes the description of the **up** procedure.

The formats of cells in the resulting cylindrical decomposition and the complexity of the algorithm can be estimated from the description of the algorithm using the upper bounds discussed in previous sections (see [19]). □

8 Limit sets

In this section we remove the condition on semi-Pfaffian sets to be *restricted.*

For arbitrary (including non-restricted) semi-Pfaffian sets a "closure at infinity" operation was introduced in [6] and [47]. The main theorem from [6, 47] (see also [23, 29, 41]) implies that the sets constructed from semi-Pfaffian sets by a finite sequence of projections on subspaces and closures at infinity constitute an o-minimal structure.

[3] A more detailed description of this decomposition can be found in [19].

In [16] Gabrielov introduced the "relative closure" operation for a one-parametric family of semi-Pfaffian sets, and the concept of a "limit set" as a finite union of relative closures of semi-Pfaffian families. Every semi-Pfaffian set is a limit set. The main results of [16] state that limit sets constitute an *effectively* o-minimal structure, i.e., any first-order formula with limit sets defines a limit set which admits an upper complexity bound in terms of the complexity of the formula. In [21] an explicit complexity bound was obtained for the number of connected components of a limit set. We now proceed to a more detailed descriptions of these results.

8.1 Exponential Łojasiewicz inequality

We start with another version of the Łojasiewicz inequality. Unlike the inequality from Corollary 4.4, it describes the rate of growth of a Pfaffian function not only in a neighbourhood of a point in the domain G but also in a neighbourhood of a point on the boundary of G. The price paid for this extension is a much weaker lower bound.

Let $cl(Z)$ denote the topological closure of a set Z. Introduce $fr(Z) := cl(Z) \setminus Z$.[4] We assume that the closure points of X at infinity are included in $cl(Z)$ and $fr(Z)$. To avoid the separate treatment of infinity, we assume that $\mathbb{R}^n$ is embedded in a projective space and all constructions are performed in an affine chart such that Z is relatively compact in that chart. To achieve this it may be necessary to subdivide Z into smaller pieces, each of them relatively compact in its own chart.

8.1 Theorem (Exponential Łojasiewicz inequality, [16, 22, 27, 28]) *Let X be a semi-Pfaffian set in a domain $G \subset \mathbb{R}^n$ defined by a formula with Pfaffian functions of order r, and let $f(\mathbf{x})$ be a Pfaffian function in $\mathbb{R}^n$. Suppose that $\mathbf{0} \in cl(X \cap \{\mathbf{x} \in \mathbb{R}^n |\ f(\mathbf{x}) > 0\})$. Then*

$$\mathbf{0} \in cl(\{\mathbf{x} \in X |\ f(\mathbf{x}) \geq 1/E_r(\|\mathbf{x}\|^{-q})\}),$$

for some integer $q > 0$. Here E_r is the iterated exponential function from Example 2.3 (b).

A proof of this theorem can be found in [16], it uses an iterated exponential upper bound from [40] on the asymptotic growth of a function in a Hardy field .

8.2 Relative closure and limit sets

Let the space $\mathbb{R}^n \times \mathbb{R}$ have coordinates $(\mathbf{x}, \lambda)$. For a set $X \subset \mathbb{R}^n \times \mathbb{R}$ we define: $X_+ := X \cap \{\lambda > 0\}$, $X_\lambda := X \cap \{\lambda = \text{const}\}$, and $\check{X} := cl(X_+) \cap \{\lambda = 0\}$. Coordinate λ is considered as a *parameter*, and X is considered as a *family* of sets X_λ in $\mathbb{R}^n$.

8.2 Definition (*Semi-Pfaffian family*) A semi-Pfaffian set X in DNF in a domain G is called a *semi-Pfaffian family* if for any $\varepsilon > 0$ the intersection $X \cap \{\lambda > \varepsilon\}$ is restricted in G. The *format* of X is defined as the format of a semi-Pfaffian set X_λ for a small $\lambda > 0$.

8.3 Lemma *Let X be a semi-Pfaffian family. Then $cl(X)_+$ and $fr(X)_+$ are semi-Pfaffian families. The formats of these families admit upper bounds in terms of the format of X.*

[4] For a semi-Pfaffian set Z in a domain G the topological closure $cl(Z)$ may be different from the closure $\bar{Z} = cl(Z) \cap G$, and $fr(Z)$ may be different from ∂Z (see Definition 5.1).

Proof The set $cl(X)_+$ is contained in G since $X \cap \{\lambda > \varepsilon\}$ is restricted in G for any $\varepsilon > 0$. Hence, also $fr(X)_+ \subset G$. According to Theorem 5.2, the sets $cl(X)_+$ and $fr(X)_+$ are semi-Pfaffian in G. The sets $cl(X)_+ \cap \{\lambda > \varepsilon\}$ and $fr(X)_+ \cap \{\lambda > \varepsilon\}$ are restricted in G for any $\varepsilon > 0$, since this is true for X.

The statement on formats follows from Theorem 5.2, since $cl(X)_\lambda = cl(X_\lambda)$ and $fr(X)_\lambda = fr(X_\lambda)$ for a generic $\lambda > 0$. These equalities can be derived from the existence of a weak stratification (Theorem 6.2), Sard's theorem, and the finiteness properties of semi-Pfaffian sets. □

8.4 Definition (*Semi-Pfaffian couple*) Two semi-Pfaffian families X and Y with a common domain $G \subset \mathbb{R}^n \times \mathbb{R}$ form a *semi-Pfaffian couple* (X, Y) in G if the set Y is relatively closed in $\{\lambda > 0\}$ (i.e., $cl(Y)_+ = Y_+$) and contains $fr(X)_+$. The format of a couple (X, Y) is defined as a component-wise maximum of the formats of X and Y.

8.5 Definition (*Relative closure*) Let (X, Y) be a semi-Pfaffian couple in $G \subset \mathbb{R}^n \times \mathbb{R}$. The *relative closure* of (X, Y) is defined as

$$(X, Y)_0 := \check{X} \setminus \check{Y} \subset \check{G} \subset \mathbb{R}^n.$$

The format of $(X, Y)_0$ is defined as the format of the couple (X, Y).

8.6 Definition (*Limit set*) A *limit set* in $\Omega \subset \mathbb{R}^n$ is a finite union of relative closures $(X_i, Y_i)_0$ of semi-Pfaffian couples (X_i, Y_i) in $G_i \subset \mathbb{R}^n \times \mathbb{R}$, such that $\check{G}_i = \Omega$ for all i. The format of a limit set is defined as $(K, r, N, \alpha, \beta, s)$, where (r, N, α, β, s) is a component-wise maximum of the formats of couples (X_i, Y_i), and K is the number of these couples.

8.7 Example Any (not necessarily restricted) semi-Pfaffian set is a limit set. If is sufficient to show that a basic set

$$X := \{\mathbf{x} \in G |\ f_1(\mathbf{x}) = \cdots = f_I(\mathbf{x}) = 0, g_1(\mathbf{x}) > 0, \cdots, g_J(\mathbf{x}) > 0\}$$

in a domain $G := \{\mathbf{x} \in \mathbb{R}^n |\ h_1(\mathbf{x}) > 0, \ldots, h_\ell(\mathbf{x}) > 0\}$ (see Remark 2.12) is the relative closure of a semi-Pfaffian couple. Let $g := g_1 \cdots g_J$, $h := h_1 \cdots h_\ell$. Define sets

$$W := \{(\mathbf{x}, \lambda) \in X \times (0, 1] |\ h(\mathbf{x}) > \lambda, \|\mathbf{x}\| < \lambda^{-1}\};$$

$$Y_1 := \{(\mathbf{x}, \lambda) \in G \times (0, 1] |\ f_1(\mathbf{x}) = \cdots = f_I(\mathbf{x}) = 0, g(\mathbf{x}) = 0, h(\mathbf{x}) \geq \lambda, \|\mathbf{x}\| \leq \lambda^{-1}\};$$

$$Y_2 := \{(\mathbf{x}, \lambda) \in G \times (0, 1] |\ f_1(\mathbf{x}) = \cdots = f_I(\mathbf{x}) = 0, h(\mathbf{x}) = \lambda, \|\mathbf{x}\| \leq \lambda^{-1}\};$$

$$Y_3 := \{(\mathbf{x}, \lambda) \in G \times (0, 1] |\ f_1(\mathbf{x}) = \cdots = f_I(\mathbf{x}) = 0, h(\mathbf{x}) \geq \lambda, \|\mathbf{x}\| = \lambda^{-1}\}.$$

Observe that $(W, Y_1 \cup Y_2 \cup Y_3)$ is a semi-Pfaffian couple. Its relative closure is X.

8.3 Boolean and projection operations over limit sets

8.8 Lemma *Let (X,Y) be a semi-Pfaffian couple in a domain $G \subset \mathbb{R}^n \times \mathbb{R}$. Then the complement $\check{G} \setminus (X,Y)_0$ of $(X,Y)_0$ in $\check{G}$ is a limit set with the format admitting an upper bound in terms of the format of (X,Y).*

Proof Let $G = \{(\mathbf{x},\lambda) \in \mathbb{R}^n \times \mathbb{R} \mid h_1(\mathbf{x},\lambda) > 0, \dots, h_\ell(\mathbf{x},\lambda) > 0\}$ (see Remark 2.12) and introduce $h := h_1 \cdots h_\ell$. Then h is positive in G and vanishes on $fr(G)$. Let

$$G' := \{(\mathbf{x},\lambda) \in G | \, \lambda > 0, h(\mathbf{x},\lambda) \geq 1/E_r(\lambda^{-q})\},$$

where r is the order of Pfaffian functions in the formula defining X and q is a positive integer. Let $Z := G \setminus X$ and $Z' := Z \cap G'$. It is clear that Z' is a semi-Pfaffian family in G (while Z may be not). It follows from the exponential Łojasiewicz inequality (Theorem 8.1) that $\check{Z} = \check{Z}'$ for large q. We now prove that

$$\check{G} \setminus (X,Y)_0 = (Z', cl(X)_+)_0 \cup (Y, \emptyset)_0. \tag{8.1}$$

Indeed, by the definition of the relative closure, the right-hand side of (8.1) coincides with

$$(\check{Z}' \setminus \check{X}) \cup \check{Y} = (\check{Z} \setminus \check{X}) \cup \check{Y}.$$

Since $(X,Y)_0 \cap (\check{Z} \setminus \check{X}) = \emptyset$ and $(X,Y)_0 \cap \check{Y} = \emptyset$, the left-hand side of (8.1) contains its right-hand side. Let now $\mathbf{x} \in \check{G} \setminus (X,Y)_0$. Note that $x \in \check{X} \cup \check{Z}$. If $x \in \check{X}$, then $x \in \check{Y}$, else $x \in \check{Z} \setminus \check{X}$. Thus, the right-hand side of (8.1) contains its left-hand side.

The right-hand side of (8.1) is a limit set. A proof of the statement on its format is straightforward. □

8.9 Lemma *Let (X,Y) and (X',Y') be two semi-Pfaffian couples in domains $G \subset \mathbb{R}^n \times \mathbb{R}$ and $G' \subset \mathbb{R}^n \times \mathbb{R}$ respectively. Then $(X,Y)_0 \cap (X',Y')_0$ is a limit set with the format admitting an upper bound in terms of formats of (X,Y) and (X',Y').*

A proof of this lemma, technically similar to the proof of Lemma 8.8 above, can be found in [16].

8.10 Theorem *Limit sets form a Boolean algebra. The format of a limit set defined by a Boolean combination of limit sets $X_1, \dots, X_\ell$ admits an upper bound in terms of the complexity of the formula and the formats of $X_1, \dots, X_\ell$.*

Proof This immediately follows from Lemmas 8.8 and 8.9. □

8.11 Theorem *Let (X,Y) be a semi-Pfaffian couple in a domain $G \subset \mathbb{R}^{m+n} \times \mathbb{R}$, and $\pi : \mathbb{R}^{m+n} \to \mathbb{R}^n$ be the projection map. Then $\pi((X,Y)_0)$ is a limit set in $\pi(\check{G}) \subset \mathbb{R}^n$, and its format admits an upper bound in terms of the format of (X,Y).*

A proof of this theorem can be found in [16] and is difficult.

8.12 Remark Theorems 8.10 and 8.11 imply that limit sets form an *o-minimal structure* [9]. Moreover, it's an *effectively* o-minimal structure, i.e., the formats of the results of Boolean and projection operations over limit sets admit upper bounds in terms of the formats of these sets. However, no explicit expressions for the bounds have yet been obtained. This is a promising direction for a future research.

8.4 Connected components of limit sets

We now establish an explicit upper bound on the number of connected components of the relative closure of a semi-Pfaffian couple (X, Y).

Note that if $Y = \emptyset$, then $fr(X)_+ = \emptyset$, i.e., X_λ is compact for any $\lambda > 0$. In this case the relative closure $(X, Y)_0 = \check{X}$ is also compact, and therefore the number of its connected components does not exceed the number of the connected components of X_λ, for all sufficiently small $\lambda > 0$. An upper bound for X_λ follows from Theorem 3.4.

Suppose now that $Y \neq \emptyset$. For $\mathbf{x} \in \mathbb{R}^n$ and $\lambda > 0$, let $\Psi_\lambda(\mathbf{x}) := \min_{\mathbf{y} \in Y_\lambda} \|\mathbf{x} - \mathbf{y}\|^2$ be the (squared) distance from $\mathbf{x}$ to Y_λ, let $\Psi(\mathbf{x}) := \min_{\mathbf{y} \in \check{Y}} \|\mathbf{x} - \mathbf{y}\|^2$ be the distance from $\mathbf{x}$ to $\check{Y}$. Note that these two functions are well defined (minima exist) since Y_λ and $\check{Y}$ are both closed. Let Z_λ be the set of local maxima of $\Psi_\lambda(\mathbf{x})$ on X_λ.

8.13 Lemma *The number of connected components of $(X, Y)_0$ does not exceed the number of connected components of Z_λ for all sufficiently small $\lambda > 0$.*

Proof Let C be a connected component of $(X, Y)_0$. By the definition of relative closure, if $\mathbf{x} \in C$, then $\mathbf{x} \notin \check{Y}$. Thus, $\Psi(\mathbf{x}) > 0$. Since $fr(C) \subset \check{Y}$, function Ψ vanishes on $fr(C)$. It follows that Ψ has a local maximum, say $\mathbf{x}_0$, in C. There exists $\mathbf{x}_\lambda \in X_\lambda$ such that $\|\mathbf{x}_\lambda - \mathbf{x}_0\| \to 0$ as $\lambda \searrow 0$. It follows that $\lim_{\lambda \searrow 0} \Psi_\lambda(\mathbf{x}_\lambda) = \Psi(\mathbf{x}_0) > 0$. In particular, there exists a constant $\varepsilon > 0$ such that $\Psi_\lambda(\mathbf{x}_\lambda) > \varepsilon$ for all sufficiently small $\lambda > 0$. Let $W_{\lambda,\varepsilon} := \{\mathbf{x} \in X_\lambda |\ \Psi_\lambda(\mathbf{x}) > \varepsilon\}$ and let C_λ be a connected component of $W_{\lambda,\varepsilon}$ which contains $\mathbf{x}_\lambda$. Since $\Psi_\lambda(\mathbf{x}) > \varepsilon$ for any $\mathbf{x} \in C_\lambda$, the sets C_λ are "close" to C for sufficiently small $\lambda > 0$, i.e., $cl\big(\bigcup_{\lambda > 0} C_\lambda\big) \cap \{(\mathbf{x}, \lambda) |\ \lambda = 0\}$ is a connected subset of $(X, Y)_0$ containing $\mathbf{x}_0$, hence a subset of C. From the definition of C_λ, there exists a local maximum $\mathbf{z}_\lambda$ of Ψ_λ on C_λ, and a connected component V_λ of Z_λ containing $\mathbf{z}_\lambda$ lies in C_λ. It follows that V_λ is "close" to C for all sufficiently small $\lambda > 0$. This implies the lemma. □

8.14 Theorem *Let (X, Y) be a semi-Pfaffian couple. The number of connected components of $(X, Y)_0$ is finite and admits an explicit upper bound in terms of the format of (X, Y).*

Proof According to Lemma 8.13, it is sufficient to bound from above the number of connected components of Z_λ. Since Z_λ is a *restricted sub-Pfaffian* set, the number of its connected components is finite. Moreover, due to Theorem 7.4, Z_λ admits a cylindrical cell decomposition with the number of cells explicitly bounded from above in terms of the format of (X, Y). The number of connected components of Z_λ does not exceed the number of cells in the decomposition, which implies the second statement of the theorem. □

The upper bound (via a cylindrical cell decomposition) which can be extracted from the proof of Theorem 8.14 is doubly exponential in the number n of variables, and is not the best possible. A better bound can be obtained by applying the results of Section 3.5 to a formula with quantifiers describing Z_λ. In [21] a much more specialized method was used to prove the following, currently the best, upper bound.

8.15 Theorem ([21]) *Let the format of a semi-Pfaffian couple (X, Y) be (r, N, α, β, n). Then the number of connected components of $(X, Y)_0$ does not exceed*

$$N^2 2^{O(n^2 r^2)} (n(\alpha + \beta))^{O(n^2 + nr)}.$$

A proof of the theorem and a slightly more precise bound (not using the O-notation) can be found in [21]. Further results on upper bounds for higher Betti numbers of limit sets were recently obtained in [50].

9 Noetherian functions

Noetherian functions are analytic functions in $G \subset \mathbb{K}^n$ defined by equations similar to (2.1) but without triangularity condition. Given a *Noetherian chain*, i.e., a sequence of analytic in G functions $f_1(\mathbf{x}), \dots , f_r(\mathbf{x})$ such that

multivariate

$$df_j(\mathbf{x}) = \sum_{1 \le i \le n} g_{ij}(\mathbf{x}, f_1(\mathbf{x}), \dots , f_r(\mathbf{x}))dx_i \text{ for all } j, \tag{9.1}$$

with g_{ij} polynomials in $n + r$ variables of degree at most α, a *Noetherian function* of order r and degree (α, β) is a function $\phi(\mathbf{x}) = P(\mathbf{x}, f_1(\mathbf{x}), \dots , f_r(\mathbf{x}))$, where $P(\mathbf{x}, \mathbf{y})$ is a polynomial of degree β in $n + r$ variables.

Alternatively, a Noetherian chain can be defined as an integral manifold $\Lambda = \{(z_1 = f_1(\mathbf{x}), \dots , z_r = f_r(\mathbf{x}))\}$ of a n-dimensional distribution in $\mathbb{K}^{n+r}$:

$$dy_j = \sum_{i=1}^{n} g_{ij}(\mathbf{x}, \mathbf{y})dx_i, \text{ for } j = 1, \dots , r, \tag{9.2}$$

and a Noetherian function as a restriction of $P(\mathbf{x}, \mathbf{y})$ to Λ.

All Noetherian functions defined by the same chain constitute a subring $\mathfrak{R}$ of the ring of analytic functions in G, finitely generated over the polynomial ring $\mathbb{K}[\mathbf{x}]$ and closed under differentiation, i.e., for any $\phi \in \mathfrak{R}$, all partial derivatives $\partial\phi/\partial x_i$ are in $\mathfrak{R}$. Conversely, any such ring is a ring of Noetherian functions. Any set of its generators can be taken as a Noetherian chain. The name "Noetherian function" introduced by Tougeron [43] refers to the Noetherian property of that ring.

!

For the univariate case $\mathbf{x} = t \in \mathbb{K}$, Noetherian functions are simply polynomials restricted to a solution $y_j = f_j(t)$ of a system $dy_j/dt = g_j(t, y_1, \dots , y_r)$ of ordinary differential equations with polynomial coefficients.

The simplest example of a Noetherian chain $f_1 = \sin t$, $f_2 = \cos t$ in $\mathbb{R}$ shows that the global finiteness properties of Pfaffian functions do not hold for Noetherian functions. In the early eighties, Khovanskii conjectured that Noetherian functions (also in the complex domain) satisfy *local* finiteness properties. Assume $0 \in G$, and consider a germ X_0 at 0 of a set X defined by equations and inequalities between Noetherian functions (a *semi-Noetherian set*), or an intersection X_δ of the set X with a ball of radius δ centered at 0. Khovanskii's conjecture states that the topological and geometric complexity of X_0 (or of X_δ, for a small $\delta > 0$) can be bounded from above by an explicit function of its *format* (see 2.11).

Considerable progress towards proving this conjecture was made in [17]:

9.1 Theorem *Let $\phi_1, \dots , \phi_n$ be Noetherian functions of order r and degree (α, β) in $G \subset \mathbb{C}^n$, with the same Noetherian chain. Then the multiplicity of any isolated solution of the system of equations $\phi_1 = \dots = \phi_n = 0$ does not exceed the maximum of the following two numbers:*

$$\begin{aligned} &\tfrac{1}{2}Q\big((r+1)(\alpha-1)[2\alpha(n+r+2)-2r-2]^{2r+2}+2\alpha(n+2)-2\big)^{2(r+n)}, \\ &\tfrac{1}{2}Q\big(2(Q+n)^n(\beta+Q(\alpha-1))\big)^{2(r+n)}, \quad \textit{where} \quad Q = en\left(\tfrac{e(n+r)}{\sqrt{n}}\right)^{\ln n+1}\left(\tfrac{n}{e^2}\right)^n. \end{aligned} \tag{9.3}$$

9.2 Remark One can show (see [43]) that, for any given integers n, r, α, and β, there exists an integer $M(n,r,\alpha,\beta)$ such that the multiplicity μ of any isolated intersection $\phi_1 = \cdots = \phi_n = 0$, where $\phi_j(\mathbf{y}) = P_j(\mathbf{y}, f_1(\mathbf{y}), \dots, f_r(\mathbf{y}))$ are Noetherian functions in $G \subset \mathbb{C}^n$ of degree at most β, with a Noetherian chain $f_1, \dots, f_r$ of order r and degree α, does not exceed $M(n,r,\alpha,\beta)$.

To prove this, recall first that the condition $\mu \geq M$, for any analytic functions $\phi_1, \dots, \phi_n$, can be formulated as a system of polynomial equations on the values of the functions ϕ_j and their partial derivatives of the order not exceeding M. For Noetherian functions, the values of their partial derivatives can be expressed as polynomials of the variables y_i, values of ϕ_j, and the coefficients of polynomials g_{ij} and P_j in their definition. Consider now the ring $\mathfrak{S}$ of polynomials in all these variables. Condition $\mu \geq M$ is represented by an ideal I_M in $\mathfrak{S}$. As I_M is an increasing sequence and $\mathfrak{S}$ is a Noetherian ring, the sequence I_M stabilizes at some $M = M(n,r,\alpha,\beta)$. This means that any intersection with the multiplicity $\mu \geq M(n,r,\alpha,\beta)$ has infinite multiplicity. Theorem 9.1 can be interpreted as an explicit upper bound for the number $M(n,r,\alpha,\beta)$

A sketch of the proof of Theorem 9.1 is given below, based on several preliminary results. First, we give a brief introduction to integration over Euler characteristics (see [44]). Next, we consider the univariate case $n = 1$. The univariate result implies, in particular, an upper bound on the vanishing order of a multivariate Noetherian function. As an application of the univariate result, we derive an upper bound for the degree of nonholonomy of a system of polynomial vector fields. In the multivariate case, we have to take care of possible non-integrability of the distribution (9.2). Finally, we need a lower bound on the codimension of the set of intersections of high multiplicity.

9.1 Integration over Euler characteristics

Assume that we are working in a category of "tame" sets (such as semialgebraic or global subanalytic sets). This means that our category is closed under Boolean operations and projections, and each set is homotopy equivalent to a finite simplicial complex. In the language of model theory, this is called an *o-minimal structure* (see [9] for the definitions and properties of sets definable in an o-minimal structure). In particular, Betti numbers $\mathrm{b}_i(X)$ and Euler characteristics $\chi(X) = \sum_i (-1)^i \mathrm{b}_i(X)$ are finite for any set X. The key properties of the Euler characteristics of compact sets are its additivity and multiplicativity:

$$\chi(X \cup Y) + \chi(X \cap Y) = \chi(X) + \chi(Y) \tag{9.4}$$

$$\chi(X \times Y) = \chi(X)\,\chi(Y). \tag{9.5}$$

These properties allow one to extend χ as an additive and multiplicative function to all (not necessarily compact) sets. Of course, $\chi(X)$ for a non-compact set may be different from the topological Euler characteristics. For example, if B^n is a closed ball in $\mathbb{R}^n$ and $X = B^n \setminus S^{n-1}$ is an open ball, then $\chi(X) = \chi(B^n) - \chi(S^{n-1})$ is $(-1)^n$.

A *constructible* function $f(\mathbf{x})$ in $\mathbb{K}^n$ is a function with finitely many values y_i such that all its level sets $X_i = \{\mathbf{x} : f(\mathbf{x}) = y_i\}$ are "tame." Its *integral over Euler characteristics* over $U \subset \mathbb{K}^n$ is defined as

$$\int_U f(\mathbf{x})\, d\chi := \sum_i y_i\, \chi(X_i \cap U). \tag{9.6}$$

The properties (9.4) and (9.5) imply finite additivity and Fubini theorem for this integral:

9.3 Theorem *For two constructible functions, f and g,*

$$\int_U f(\mathbf{x})\, d\chi + \int_U g(\mathbf{x})\, d\chi = \int_U (f(\mathbf{x}) + g(\mathbf{x}))\, d\chi.$$

9.4 Theorem *Let $f(\mathbf{x}, \mathbf{y})$ be a constructible function in $\mathbb{K}^{n+m}$. For $U \subset \mathbb{K}^n$ and $V \subset \mathbb{K}^m$, the function*

$$g(\mathbf{x}) = \int_V f(\mathbf{x}, \mathbf{y})|_{\mathbf{x}=\mathrm{const}}\, d\chi$$

is constructible and

$$\int_U g(\mathbf{x})\, d\chi = \int_{U\times V} f(\mathbf{x}, \mathbf{y})\, d\chi.$$

9.2 Univariate case

When $n = 1$, Noetherian chains are trajectories of vector fields with polynomial coefficients, and Noetherian functions are polynomials restricted to trajectories of such vector fields.

Let $t \in \mathbb{C}$, $\mathbf{y} = (y_1, \dots, y_r) \in \mathbb{C}^r$, and let $\gamma = \{\mathbf{y} = \mathbf{y}(t)\}$ be a germ of a trajectory through $0 \in \mathbb{C}^{r+1}$ of a vector field $\xi = g_0(t, \mathbf{y})\partial/\partial t + \sum_i g_i(t, \mathbf{y})\partial/\partial y_i$, where g_i are germs of analytic functions at $0 \in \mathbb{C}^{r+1}$, $g_0(0) \neq 0$. Let $P(t, \mathbf{y})$ be a germ of an analytic function at $0 \in \mathbb{C}^{r+1}$, and let $\phi(t) = P(t, \mathbf{y}(t))$ be the restriction of $P(t, \mathbf{y})$ to γ. Suppose that $\phi(t) \not\equiv 0$, and let μ be the order of a zero of ϕ at $t = 0$. Let $S(t, \mathbf{y}, \varepsilon)$ be a one-parametric deformation of P, i.e., a germ of an analytic function at $0 \in \mathbb{C}^{r+2}$ such that $S(t, \mathbf{y}, 0) = P(t, \mathbf{y})$. We write $S_\varepsilon(t, \mathbf{y})$ for $S(t, \mathbf{y}, \varepsilon)|_{\varepsilon=\mathrm{const}}$ considered as a function in $\mathbb{C}^{r+1}$.

9.5 Definition For a positive integer q, the *Milnor fiber* $Z_q(\xi, S)$ of the deformation S with respect to a vector field ξ is the intersection of a ball $\|(t, \mathbf{y})\| \leq \delta$ in $\mathbb{C}^{r+1}$ with a set $S_\varepsilon = \xi S_\varepsilon = \cdots = \xi^{q-1} S_\varepsilon = 0$, for a small positive δ and a complex nonzero ε much smaller than δ.

One can show (see [26]) that the homotopy type of $Z_q(\xi, S)$ does not depend on ε and δ, as long as $|\varepsilon| \ll \delta \ll 1$. Unless P has an isolated singularity, this homotopy type does depend on the deformation S.

9.6 Theorem *Let S be a one-parametric deformation of an analytic function P. For positive integer q, let $Z_q = Z_q(\xi, S)$ be the Milnor fibers of S with respect to an analytic vector field ξ, and let $\chi(Z_q)$ be the Euler characteristics of Z_q. Suppose that P restricted to a trajectory of ξ through 0 has a zero of order $\mu < \infty$ at 0. Let $Q := \max\{q : Z_q \neq \emptyset\}$. Then*

$$\mu = \sum_{q=1}^{Q} \chi(Z_q). \tag{9.7}$$

Proof One can assume, after a change of coordinates in $(\mathbb{C}^{r+1}, 0)$, that $\xi = \partial/\partial t$. It follows from [26] that the homotopy type of Z_q does not depend on the coordinate system. Let π be the projection $\mathbb{C}^{r+1} \to \mathbb{C}^r$ along the t axis. Let $B_\eta = \{\|\mathbf{y}\| \leq \eta\}$ be a ball of radius η in $\mathbb{C}^r$. We can replace the ball $\{\|(t, \mathbf{y})\| \leq \delta\}$ in Definition 9.5 by $D_{\delta,\eta} = \{|t| \leq \delta,\ \mathbf{y} \in B_\eta\}$, where $0 < \eta \ll \delta$, so that the projection $\pi : \{P = 0\} \cap D_{\delta,\eta} \to B_\eta$ is a finite μ-fold ramified covering (counting the multiplicities). This also would not change the homotopy type of Z_q, as long as $|\varepsilon| \ll \eta$.

For $\mathbf{y} \in B_\eta$, each set $\pi^{-1}(\mathbf{y}) \cap Z_q$ is finite, and its Euler characteristics $\zeta_q(\mathbf{y})$ equals the number of points in it (not counting multiplicities). From Theorem 9.4,

$$\int_{B_\eta} \zeta_q(\mathbf{y})\, d\chi = \chi(Z_q). \tag{9.8}$$

Note that each point $(t, \mathbf{y}) \in \{S_\varepsilon = 0\} \cap D_{\delta,\eta}$ belongs to exactly k sets $Z_1, \dots, Z_k$, where k equals the multiplicity of t in $\pi^{-1}(\mathbf{y}) \cap \{S_\varepsilon = 0\}$. Hence $\sum_{q=1}^{Q} \zeta_q(\mathbf{y}) \equiv \mu$ does not depend on $\mathbf{y}$. From (9.8) and Theorem 9.3,

$$\sum_{q=1}^{Q} \chi(Z_q) = \int_{B_\eta} \sum_{q=1}^{Q} \zeta_q(\mathbf{y})\, d\chi = \int_{B_\eta} \mu\, d\chi = \mu\chi(B_\eta) = \mu.$$

□

The following is a special case of Thom's transversality theorem (see Lemma 1 in [15]).

9.7 Lemma *Let $\xi = g_0(t, \mathbf{y})\partial/\partial t + \sum_{i=1}^{r} g_i(t, \mathbf{y})\partial/\partial y_i$ be a germ at $0 \in \mathbb{K}^{r+1}$ of an analytic vector field, $g_0(0) \neq 0$, and $P(t, \mathbf{y})$ a germ of an analytic function. Let $c = (c_0, \dots, c_r) \in \mathbb{K}^{r+1}$ and*

$$S_c(t, \mathbf{y}, \varepsilon) = P(t, \mathbf{y}) + \varepsilon \sum_{i=0}^{r} c_i t^i. \tag{9.9}$$

For a generic c, the sets $Z_q = Z_q(\xi, S_c)$ are nonsingular of codimension q for $q = 1, \dots, r+1$, and empty for $q > r+1$.

9.8 Corollary *Let ξ and P be as in Lemma 9.7, and let $\mu < \infty$ be the multiplicity of P on the trajectory of ξ through 0. Let $Z_q = Z_q(\xi, S_c)$ be the Milnor fibers of (9.9) with respect to ξ. For a generic c,*

$$\mu = \chi(Z_1) + \cdots + \chi(Z_{r+1}). \tag{9.10}$$

9.9 Theorem *Let $\xi = g_0(t, \mathbf{y})\partial/\partial t + \sum_i g_i(t, \mathbf{y})\partial/\partial y_i$ where g_i are polynomials of degree not exceeding $\alpha \geq 1$, $g_0(0) \neq 0$, and let P be a polynomial of degree not exceeding $\beta \geq r$. Let $\mu < \infty$ be the multiplicity of P on the trajectory of ξ through 0. Then μ does not exceed*

$$\frac{1}{2} \sum_{k=0}^{r} [2\beta + 2k(\alpha - 1)]^{2r+2}. \tag{9.11}$$

Proof This follows from (9.10) and from an estimate [33] of the Euler characteristics of a set Z_q defined by polynomial equations of degree not exceeding $\beta + (q-1)(\alpha - 1)$. □

9.10 Corollary *Let $\phi \not\equiv 0$ be a Noetherian function of order r and degree (α, β) in a neighborhood of $0 \subset \mathbb{K}^n$. Then the vanishing order of ϕ at 0 does not exceed (9.11).*

Proof This follows from Theorem 9.9 after restricting ϕ to a generic line through 0. □

9.3 Degree of nonholonomy of a system of vector fields

9.11 Definition Let Ξ be a system of analytic vector fields ξ_i in $G \subset \mathbb{K}^n$. Let $\mathcal{L}_1(\Xi)$ be the space of all linear combinations of ξ_i with coefficients in $\mathbb{K}$. For $k \geq 2$, define $\mathcal{L}_k(\Xi) := \mathcal{L}_{k-1}(\Xi) + [\mathcal{L}_1(\Xi), \mathcal{L}_{k-1}(\Xi)]$. Then $\mathcal{L}(\Xi) := \cup_k \mathcal{L}_k(\Xi)$ is the Lie algebra generated by the vector fields ξ_i. For $z \in \mathbb{K}^n$, let $d_k(z)$ be dimension of the subspace generated by the values at z of the vector fields from $\mathcal{L}_k(\Xi)$, and let $d(\Xi, z) := \max_k d_k(z)$ be dimension of the subspace generated by the values at z of the vector fields from $\mathcal{L}(\Xi)$. In particular, when $d(\Xi, z) = n$, the system Ξ is called totally nonholonomic (controllable). The minimal k such that $d_k(z) = d(\Xi, z)$ is called *degree of nonholonomy* of Ξ at z. It is easy to check that the values d_k, and degree of nonholonomy, do not change if we allow linear combinations of vector fields with analytic (instead of constant) coefficients.

by Gabrielov

It is shown in [12] that an upper bound for the multiplicity of a zero of a polynomial on a trajectory of a polynomial vector field implies an upper bound on degree of nonholonomy of a system of polynomial vector fields. In particular, the upper bound in Theorem 9.9 implies the following upper bound for degree of nonholonomy:

9.12 Theorem *Let $\Xi := \{\xi_i\}$ be a system of vector fields in $\mathbb{K}^n$ with polynomial coefficients of degree not exceeding $p \geq 1$. For $z \in \mathbb{K}^n$, let $d = d(\Xi, z) > 1$. Then the degree of nonholonomy of Ξ at z does not exceed*

$$F_d + \frac{F_{d-1}}{2} \sum_{k=0}^{n-1} [2p(F_{d+2} - 1) + 2k(pF_d - 1)]^{2n} \tag{9.12}$$

where F_i are the Fibonacci numbers.

Proof According to Proposition 1 of [12], there exist vector fields $\chi_0, \dots, \chi_{d-1}$ such that (a) χ_0 and χ_1 are some of ξ_i, and $\chi_0(z) \neq 0$; (b) for $k > 1$, χ_k is either one of ξ_i or a linear combination of brackets $[\chi_i, f\chi_j]$ where $i, j < k$ and f is a linear function; (c) for a generic small $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{d-2})$, $Q = \chi_0 \wedge \cdots \wedge \chi_{d-1}$ does not vanish identically on a trajectory γ of $\chi_\varepsilon = \chi_0 + \varepsilon_1\chi_1 + \cdots + \varepsilon_{d-2}\chi_{d-2}$ through z. Taking into account that $[\chi, \chi] = 0$ for any vector field χ, the arguments in the proof of Proposition 1 of [12] can be modified to replace (b) by (b') for $k > 1$, χ_k is either one of ξ_i or a linear combination of brackets $[\chi_i, f\chi_j]$ where $j < i < k$ and f is a linear function. In particular, each χ_k is a vector field with polynomial coefficients of degree not exceeding pF_{k+1}, where F_i are the Fibonacci numbers: $F_1 = 1,\ F_2 = 1,\ F_{i+1} = F_i + F_{i-1}$.

Let $x_1, \dots, x_n$ be linear coordinates in $\mathbb{K}^n$. Then

$$Q = \sum_{i_1 < \cdots < i_d} Q_{i_1 \dots i_d} \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_d}},$$

where $Q_{i_1 \dots i_d}$ are polynomials of degrees not exceeding $\beta = p(F_1 + \cdots + F_d) = p(F_{d+2} - 1)$. Due to (c), some of these polynomials do not vanish identically on the trajectory γ of a vector field χ_ε with polynomial coefficients of degree not exceeding $\alpha = pF_d$. Due to Theorem 9.9, the multiplicity μ of a zero of such a polynomial restricted to γ does not exceed

$$\frac{1}{2} \sum_{k=0}^{n-1} [2\beta + 2k(\alpha - 1)]^{2n} = \frac{1}{2} \sum_{k=0}^{n-1} [2p(F_{d+2} - 1) + 2k(pF_d - 1)]^{2n}.$$

Each derivation of Q along χ_ε decreases this multiplicity by 1. Hence the result of μ consecutive derivations of Q along χ_ε does not vanish at z. From (b'), each χ_k is a linear combination with polynomial coefficients of brackets of ξ_i of order at most F_{k+1}, and χ_ε is a combination of brackets of ξ_i of order at most F_{d-1}. Taking into account a formula for a derivation along χ_ε:

$$\partial_{\chi_\varepsilon}(\chi_0 \wedge \cdots \wedge \chi_{d-1}) = \sum_{i=0}^{d-1} \chi_0 \wedge \cdots \wedge [\chi_\varepsilon, \chi_i] \wedge \ldots \wedge \chi_{d-1},$$

we see that the result of μ derivations of Q along χ_ε is a linear combination, with polynomial coefficients, of wedge-products of vector fields which are brackets of ξ_i of order not exceeding $F_d + F_{d-1}\mu$, which is equal to (9.12). Since the result of μ derivations of Q along χ_ε does not vanish at z, there exist $d = d(\Xi, z)$ brackets of ξ_i of order not exceeding (9.12) which are linearly independent at z. □

9.4 Multivariate case: integrability

For a system (9.1) of partial differential equations in $\mathbb{K}^n$, when $n > 1$, analytic solutions $f_1(\mathbf{x}), \ldots, f_r(\mathbf{x})$, in general, do not exist due to non-integrability. When such solutions do exist, their union is an analytic set (Lemma 9.13 below). When the coefficients g_{ij} are polynomials, this set is algebraic.

9.13 Lemma *Let $\mathbf{x} \in \mathbb{K}^n$, $\mathbf{y} \in \mathbb{K}^r$, and let $g_{ij}(\mathbf{x}, \mathbf{y})$ be analytic functions in $U \subset \mathbb{K}^{n+r}$, for $i = 1, \ldots, n$ and $j = 1, \ldots, r$. The union of all integral manifolds of (9.2) is an analytic subset of U.*

Proof The statement can be proved locally, in a neighborhood of a point $(\mathbf{x}, \mathbf{y}) \in U$. Let ξ_i, $i = 1, \ldots, n$ be the following vector fields tangent to the distribution (9.2):

$$\xi_i = \frac{\partial}{\partial x_i} + \sum_{j=1}^{r} g_{ij}(\mathbf{x}, \mathbf{y}) \frac{\partial}{\partial y_j}. \tag{9.13}$$

For a nonzero $c = (c_1, \ldots, c_n) \in \mathbb{K}^n$, let $\xi_c = c_1\xi_1 + \cdots + c_n\xi_n$, and let γ_c be a germ of a trajectory of ξ_c through $(\mathbf{x}, \mathbf{y})$. Let Λ be the union of all γ_c. Then Λ is a germ of an n-dimensional manifold.

If there exists an integral manifold of (9.2) through $(\mathbf{x}, \mathbf{y})$, it contains all trajectories γ_c and its germ at $(\mathbf{x}, \mathbf{y})$ coincides with Λ. Since all vector fields ξ_i are tangent to Λ, any bracket $[\xi_j, \xi_k]$ at each point of Λ belongs to the subspace generated by ξ_i.

Due to Lemmas 4 and 5 in [12], the opposite is also true: if any bracket $[\xi_j, \xi_k]$ at each point of Λ belongs to the subspace generated by ξ_i, then Λ is an integral manifold of (9.2).

Let $Q_{jk} = [\xi_j, \xi_k] \wedge \xi_1 \cdots \wedge \xi_n$. If Λ is an integral manifold of (9.2) then all Q_{jk} vanish identically on γ_c. In particular,

$$\xi_c^\nu Q_{jk}(\mathbf{x}, \mathbf{y}), \tag{9.14}$$

the value at $(\mathbf{x}, \mathbf{y})$ of the ν-th derivative of Q_{jk} along ξ_c, is zero for each j, k, and ν. In the opposite case, when Q_{jk} does not vanish identically on γ_c for some j, k, and c, there exists a nonzero derivative (9.14).

Hence, $(\mathbf{x},\mathbf{y})$ belongs to an integral manifold of (9.2) iff all derivatives (9.14) vanish. As these derivatives are analytic in $(\mathbf{x},\mathbf{y})$, their common zeros constitute an analytic set. $\square$

9.14 Proposition *Let g_{ij} in (9.2) be polynomials of degree not exceeding $\alpha \geq 1$, and let Y be the union of all integral manifolds of (9.2). Then Y can be defined by a system of algebraic equations of degree not exceeding*

$$d_Y(n,r,\alpha) = \frac{1}{2}(r+1)(\alpha-1)[2\alpha(n+r+2)-2r-2]^{2r+2} + 2\alpha(n+2) - 2. \tag{9.15}$$

Proof From the proof of Lemma 9.13, the set Y is defined by zeros of derivatives (9.14) of Q_{ij} along ξ_c, for all $i,\ j$, and c. Here ξ_c is a linear combination of the vector fields ξ_i defined in (9.13), and $Q_{jk} = [\xi_j, \xi_k] \wedge \xi_1 \cdots \wedge \xi_n$.

As the coefficients of ξ_i are polynomials of degree not exceeding α, the coefficients of Q_{jk} are polynomials of degree not exceeding $B = \alpha(n+2)-1$. If $(\mathbf{x},\mathbf{y})$ does not belong to Y, some of these coefficients do not vanish identically on the trajectory γ_c of ξ_c through $(\mathbf{x},\mathbf{y})$. Due to Theorem 9.9, the multiplicity of zero at $(\mathbf{x},\mathbf{y})$ of these coefficients restricted to γ_c cannot exceed

$$N = \frac{1}{2}\sum_{k=0}^{r}[2B + 2k(\alpha-1)]^{2r+2} = \frac{1}{2}\sum_{k=0}^{r}[2\alpha(n+k+2)-2k-2]^{2r+2}.$$

This means that Y can be defined by zeros of derivatives (9.14) with $\nu \leq N$. But these derivatives are polynomials in $(\mathbf{x},\mathbf{y})$ of degree at most $B + N(\alpha-1)$. Hence Y is defined by a system of algebraic equations of degree at most

$$\frac{\alpha-1}{2}\sum_{k=0}^{r}[2\alpha(n+k+2)-2k-2]^{2r+2} + \alpha(n+2) - 1.$$

As $\alpha \geq 1$, this does not exceed the right side of (9.15). $\square$

9.5 Multivariate case: Milnor fibers

Let $g_{ij}(\mathbf{x},\mathbf{y})$ be germs of analytic functions at $0 \in \mathbb{C}^{n+r}$, and let $\{\mathbf{y} = f(\mathbf{x})\}$ be a germ of an integral manifold of (9.2) through 0. Let $P(\mathbf{x},\mathbf{y}) = (P_1(\mathbf{x},\mathbf{y}), \ldots, P_n(\mathbf{x},\mathbf{y}))$ be a germ of an analytic vector-function at 0, and let $\phi_i(\mathbf{x}) = P_i(\mathbf{x}, f(\mathbf{x}))$. Let $S(\mathbf{x},\mathbf{y},\varepsilon)$ be a one-parametric deformation of $P(\mathbf{x},\mathbf{y})$, i.e., a germ of an analytic vector-function at $0 \in \mathbb{C}^{n+r+1}$ such that $S(\mathbf{x},\mathbf{y},0) = P(\mathbf{x},\mathbf{y})$. For a fixed ε, we write $S_\varepsilon(\mathbf{x},\mathbf{y})$ for $S(\mathbf{x},\mathbf{y},\varepsilon)$ considered as a function in $\mathbb{C}^{n+r}$.

Let Y be the union of all integral manifolds of (9.2). Due to Lemma 9.13, Y is a germ of an analytic set. For $(\mathbf{x},\mathbf{y}) \in Y$, let $\mu_\varepsilon(\mathbf{x},\mathbf{y})$ be the multiplicity of the intersection $S_{1,\varepsilon}|_\Lambda = \cdots = S_{n,\varepsilon}|_\Lambda = 0$ at $(\mathbf{x},\mathbf{y})$, where Λ is an integral manifold of (9.2) through $(\mathbf{x},\mathbf{y})$. Let W_q be the set of those $(\mathbf{x},\mathbf{y},\varepsilon)$ where $(\mathbf{x},\mathbf{y}) \in Y$ and $\mu_\varepsilon(\mathbf{x},\mathbf{y}) \geq q$.

9.15 Lemma *W_q is an analytic set.*

Proof One can choose a system of coordinates $(\mathbf{x}, \mathbf{z})$ in the neighborhood of $0 \in \mathbb{C}^{n+r}$ so that each integral manifold of (9.2) is defined by $\mathbf{z} = \text{const}$.

Due to Lemma 5.5 in [1], the condition $\mu_\varepsilon(\mathbf{x}_0, \mathbf{z}_0) \geq q$ depends only on the Taylor expansion $\check{S}_i$ in $\mathbf{x}$ of S_i at $(\mathbf{x}_0, \mathbf{z}_0, \varepsilon)$ of order $q-1$. The coefficients of $\check{S}_i$ are

$$\frac{\partial^{|\nu|} S_i}{\partial x^\nu}(\mathbf{x}_0, \mathbf{z}_0, \varepsilon). \tag{9.16}$$

Let $K = \binom{q+n-1}{n}$ be the number of monomials in n variables of degree less than q. For fixed $\mathbf{z}_0$ and ε, consider $\check{S}_i$ as a vector in $\mathbb{C}^K$. For any multi-index $\nu = (\nu_1, \ldots, \nu_n)$, with $|\nu| = \nu_1 + \cdots + \nu_n < q$, consider $(\mathbf{x} - \mathbf{x}_0)^\nu \check{S}_i$ as a vector in $\mathbb{C}^K$, disregarding terms of order q and higher in $\mathbf{x} - \mathbf{x}_0$. Condition $\mu_\varepsilon(\mathbf{x}_0, \mathbf{z}_0) \geq q$ means that rank of the set of Kn vectors $(\mathbf{x} - \mathbf{x}_0)^\nu \check{S}_i$ in $\mathbb{C}^K$ is at most $K - q$. This means vanishing of all $(K - q + 1)$-minors of a $(K \times Kn)$-matrix composed of these vectors. As the elements of this matrix are the partial derivatives (9.16), which are analytic in $(\mathbf{x}_0, \mathbf{z}_0, \varepsilon)$, this, in combination with equations for Y, provides a system of analytic equations for W_q. □

9.16 Definition For a positive integer q, the *Milnor fiber* Z_q of the deformation S with respect to the distribution (9.2) is the intersection of W_q with a closed ball $\{\|(\mathbf{x}, \mathbf{y})\| \leq \delta\}$ in the space $\{\varepsilon = \text{const}\} \subset \mathbb{C}^{n+r+1}$, for a small positive δ and a complex nonzero ε much smaller than δ. According to [26], the homotopy type of Z_q does not depend on ε and δ. Let $\chi(Z_q)$ be the Euler characteristics of Z_q.

9.17 Theorem *Let $P = (P_1(\mathbf{x}, \mathbf{y}), \ldots, P_n(\mathbf{x}, \mathbf{y}))$ be a germ of an analytic function at 0 in $\mathbb{C}^{n+r}$. Let $f = (f_1(\mathbf{x}), \ldots, f_r(\mathbf{x}))$ be a germ of an analytic function at $0 \in \mathbb{C}^n$ satisfying (9.1) and let $\phi_i(\mathbf{x}) = P_i(\mathbf{x}, f(\mathbf{x}))$. Suppose that the intersection $\phi_1(\mathbf{x}) = \cdots = \phi_n(\mathbf{x}) = 0$ is isolated at $\mathbf{x} = 0$, with the multiplicity μ. Let $S = S(\mathbf{x}, \mathbf{y}, \varepsilon)$ be a one-parametric deformation of P, let Z_q be the Milnor fibers of S with respect to the distribution (9.2), and let $Q = \max\{q : Z_q \neq \emptyset\}$. Then*

$$\mu = \sum_{q=1}^{Q} \chi(Z_q). \tag{9.17}$$

Proof The arguments, based on integration over Euler characteristics, essentially repeat those in the proof of Theorem 9.6, except everything should be restricted to the set Y of all integral manifolds of (9.2).

Let us choose a system of coordinates $(\mathbf{x}, \mathbf{z})$ in $\mathbb{C}^{n+r}$, as in the proof of Lemma 9.15, so that each integral manifold of (9.2) through $(\mathbf{x}_0, \mathbf{z}_0) \in Y$ is defined by $\mathbf{z} = \mathbf{z}_0$. Let π be projection $\mathbb{C}^{n+r} \to \mathbb{C}^r$. Let $B_\eta = \{\|\mathbf{z}\| \leq \eta\}$ be a ball of radius η in $\mathbb{C}^r$. We can replace the ball $\{\|(\mathbf{x}, \mathbf{y})\| \leq \delta\}$ in Definition 9.16 by $D_{\delta,\eta} = \{\|\mathbf{x}\| \leq \delta,\ \mathbf{z} \in B_\eta\}$, where $0 < \eta \ll \delta$, so that the projection $\pi : \{P = 0\} \cap D_{\delta,\eta} \to B_\eta$ is a finite covering. This would not change the homotopy type of Z_q, as long as $|\varepsilon| \ll \eta$. Then $\pi : \{D_{\delta,\eta} = 0\} \cap Y \to B_\eta \cap \pi(Y)$ is a finite μ-fold ramified covering (counting the multiplicities). Let $\zeta_q(\mathbf{z}) = \chi(\pi^{-1}(\mathbf{z}) \cap Z_q)$ be the number of preimages of a point $\mathbf{z}$ in Z_q, not counting multiplicities. Then $\sum_{q=1}^{Q} \zeta_q(\mathbf{z}) \equiv \mu$ does not depend on $\mathbf{z}$. We have

$$\int_{B_\eta \cap \pi(Y)} \zeta_q \, d\chi = \chi(Z_q), \quad \text{and} \quad \int_{B_\eta \cap \pi(Y)} \sum_{q=1}^{Q} \zeta_q \, d\chi = \int_{B_\eta \cap \pi(Y)} \mu d\chi = \mu.$$

The last equality holds because $B_\eta \cap \pi(Y)$ is a compact contractible set, hence its Euler characteristics equals 1. $\Box$

9.18 Proposition *Let g_{ij} in the expression (9.2) be polynomials of degree not exceeding α, and let $S_i(\mathbf{x}, \mathbf{y}, \varepsilon)$ be polynomials of degree at most β in $(\mathbf{x}, \mathbf{y})$, with coefficients analytic in ε. Then each Milnor fiber Z_q of S with respect to (9.2) can be defined by polynomial equations of degree not exceeding maximum of (9.15) and*

$$d(n, q, \alpha, \beta) = (K - q + 1)[\beta + (q-1)(\alpha-1)], \text{ where } K = \binom{q+n-1}{n}. \tag{9.18}$$

Proof Let us fix a small nonzero ε. According to the arguments in the proof of Lemma 9.15, condition $(\mathbf{x}_0, \mathbf{y}_0) \in Z_q$ is equivalent to vanishing of all $(K-q+1)$-minors of a matrix composed of the partial derivatives (9.16) of S_ε of order $\nu < q$, in a system of coordinates $(\mathbf{x}, \mathbf{z})$ where integral manifolds of (9.2) are rectified.

Let us define a germ of the integral manifold Λ of (9.2) through $(\mathbf{x}_0, \mathbf{y}_0)$ by a function $f(\mathbf{x}) = (f_1(\mathbf{x}), \dots, f_r(\mathbf{x}))$ satisfying (9.1). Equations (9.1) allow one to represent a partial derivative (9.16) as a polynomial in $(\mathbf{x}_0, \mathbf{y}_0)$ of degree not exceeding $\beta + \nu(\alpha - 1)$. Hence the elements of our matrix are polynomials in $(\mathbf{x}_0, \mathbf{y}_0)$ of degree not exceeding $\beta + (q-1)(\alpha-1)$, and its $(K-q+1)$-minors are polynomials in $(\mathbf{x}_0, \mathbf{y}_0)$ of degree not exceeding $(K-q+1)[\beta + (q-1)(\alpha-1)$. These polynomials, in combination with equations for Y, provide a system of equations for Z_q. $\Box$

9.19 Corollary *Under conditions of Proposition 9.18, the absolute value of the Euler characteristics of Z_q does not exceed*

$$\frac{1}{2}\max\big(2d_Y(n, r, \alpha), 2d(n, q, \alpha, \beta)\big)^{2(n+r)}, \tag{9.19}$$

where $d_Y(n, r, \alpha)$ and $d(n, q, \alpha, \beta)$ are defined in (9.15) and (9.18), respectively.

Proof This follows from an estimate [33] of the Euler characteristics of the set Z_q defined by polynomial equations of degree not exceeding maximum of (9.15) and (9.18). $\Box$

9.6 Proof of Theorem 9.1

To complete the proof of Theorem 9.1, we need to find a low-degree deformation S of a polynomial P such that Milnor fibers Z_q with large q would be empty, i.e., to derive an upper bound on the value of Q in Theorem 9.17. Such upper bound can be derived from the following theorem (see Theorem 7 in [17]):

9.20 Theorem *Let $\nu = (\nu_1, \dots, \nu_n)$ be a sequence of nonnegative integers, $|\nu| = \nu_1 + \cdots + \nu_n$, and $\mathbf{x}^\nu = x_1^{\nu_1} \cdots x_n^{\nu_n}$. For a nonnegative integer r, an analytic map $P = (P_1, \dots, P_n) : \mathbb{C}^n \to \mathbb{C}^n$, and a vector $c = (c_1, \dots, c_n)$ where $c_j = c_{j,\nu}$, $|\nu| \le r$, let*

$$P^c = (P_1^c, \dots, P_n^c) \text{ where } P_j^c(\mathbf{x}) = P_j(\mathbf{x}) + \sum_{\nu : |\nu| \le r} c_{\nu,j} x^\nu. \tag{9.20}$$

The set of those $(\mathbf{x}, c)$ *where the multiplicity of* P^c *at* $\mathbf{x}$ *exceeds*

$$Q(n,r) = \left(\frac{n+r}{1+\cdots+1/n} \right)^{1+\cdots+1/n} \prod_{k=1}^{n} \left(\frac{(k-1)!}{k} \right)^{1/k}. \tag{9.21}$$

has codimension greater than $n+r$*.*

9.21 Corollary *Let* $P_{\mathbf{y}}(\mathbf{x}) = (P_{\mathbf{y},1}(\mathbf{x}), \ldots, P_{\mathbf{y},n}(\mathbf{x}))$ *be a generic family of analytic maps* $\mathbb{C}^n \to \mathbb{C}^n$ *depending on parameters* $\mathbf{y} \in \mathbb{C}^r$*. Then the multiplicity of* $P_{\mathbf{y}}$*, at any point* $\mathbf{x} \in \mathbb{C}^n$ *and for any* $\mathbf{y}$*, is less than (9.21).*

Consider now a deformation $S(\mathbf{x}, \mathbf{y}, \varepsilon)$ of the polynomial $P(\mathbf{x}, \mathbf{y})$ defined by

$$S_j(\mathbf{x}, \mathbf{y}, \varepsilon) = P_j(\mathbf{x}, \mathbf{y}) + \varepsilon \sum_{\nu : |\nu| \leq r} c_{\nu,j} x^{\nu},$$

where $c_{\nu,j}$ are generic complex numbers. From Theorem 9.20, the Milnor fibers Z_q of this deformation are empty for $q \geq Q(n,r)$, where $Q(n,r)$ is defined in (9.20).

According to (9.17),

$$\mu = \sum_{q=1}^{Q(n,r)} \chi(Z_q) \leq Q(n,r) \max_{q \leq Q(n,r)} |\chi(Z_q)|. \tag{9.22}$$

From (9.19), the right side of (9.22) does not exceed

$$\frac{1}{2} Q(n,r) \max(2d_Y(n,r,\alpha), 2d(n, Q(n,r), \alpha, \beta))^{2(m+n)},$$

where d_Y and d are defined in (9.15) and (9.18), respectively. The value of $d(n,q,\alpha,\beta)$ in (9.18) does not exceed

$$(q+n-1)^n(\beta + (q-1)(\alpha-1)) < (q+n)^n(\beta + q(\alpha-1)).$$

The statement of Theorem 1 follows now from the following estimate for $Q(n,r)$ (see Proposition 1 in [17]):

9.22 Proposition *The value of* $Q(n,r)$ *in (9.20) does not exceed*

$$en \left(\frac{e(n+r)}{\sqrt{n}} \right)^{\ln n + 1} \left(\frac{n}{e^2} \right)^n.$$

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