On Positivity and Divergence for Linear Recurrence Sequences*

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Abstract

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1 Introduction

2 Mathematical Tools

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Basis for multiplicative relationships

▶ **Theorem 1.** *Let m be fixed, and let $\lambda_1, \ldots, \lambda_m$ be complex algebraic numbers of modulus 1. Consider the abelian group L under addition given by*

$$L = \{(v_1, \ldots, v_m) \in \mathbb{Z}^m : \lambda_1^{v_1} \cdots \lambda_m^{v_m} = 1\}.$$

L has a basis $\{\boldsymbol{\ell_1}, \ldots, \boldsymbol{\ell_p}\} \subseteq \mathbb{Z}^m$ (with $p \leq m$), where the entries of each of the $\boldsymbol{\ell_j}$ are all polynomially bounded in $\|\lambda_1\| + \ldots + \|\lambda_m\|$. Moreover, such a a basis can be computed in time polynomial in $\|\lambda_1\| + \ldots + \|\lambda_m\|$.

The following lemma is due to Braverman [**?**].

▶ **Lemma 2** (Complex Unit Lemma)**.** *Let $\zeta_1, \zeta_2, \ldots, \zeta_m \in \mathbf{S}_1 \setminus \{1\}$ be distinct complex numbers, and let $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{C} \setminus \{0\}$. Set $z_n := \sum_{k=1}^{m} \alpha_k \zeta_k^n$. Then there exists $c < 0$ such that for infinitely many n, $Re\,(z_n) < c$.*

The following celebrated result is due to Baker and Wüstholz.

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▶ **Theorem 3** (Baker and Wüstholz). *Let $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$ be algebraic numbers different from 0 or 1, and let $b_1, \ldots, b_m \in \mathbb{Z}$ be integers. Write*

$$\Lambda = b_1 \log \alpha_1 + \ldots + b_m \log \alpha_m$$

Let $A_1, \ldots, A_m, B \geq e$ be real numbers such that, for each $j \in \{1, \ldots, m\}$, A_j is an upper bound for the height of α_j, and B is an upper bound for $|b_j|$. Let d be the degree of the extension field $\mathbb{Q}(\alpha_1, \ldots, \alpha_m)$ over $\mathbb{Q}$.

If $\Lambda \neq 0$, then $\log |\Lambda| > -(16md)^{2(m+2)} \log A_1 \cdots \log A_m \log B$.

A simple corollary of Theorem 3 is the following.

▶ **Corollary 4.** *There exists $D \in \mathbb{N}$ such that, for every algebraic numbers $\lambda, \zeta \in \mathbb{C}$ of modulus 1, and for all $n \geq 2$, if $\lambda^n \neq \zeta$, then $|\lambda^n - \zeta| > \frac{1}{n^{(\|\lambda\| + \|\zeta\|)^D}}$*

The following theorem is due to Everest, van der Poorten and Schlickewei [**?**].

▶ **Theorem 5.** *For any algebraic non-degenerate LRS $\langle u_n \rangle$ with spectral radius ρ, and for any $\epsilon > 0$, there exist $N \in \mathbb{N}$ such $|u_n| \geq \rho^{1-\epsilon} n$ for every $n > N$.*

The next theorem is due to Renegar.

▶ **Theorem 6** (Renegar). *Let $M \in \mathbb{N}$ be fixed. Let $\tau(\mathbf{y})$ be a formula of the first order theory of the reals. Assume that the number of (free and bound) variables in $\tau(\mathbf{y})$ is bounded by M. Denote the degree of $\tau(\mathbf{y})$ by d and the number of atomic predicates in $\tau(\mathbf{y})$ by n.*

There is a polynomial time (polynomial in $\|\tau(\mathbf{y})\|$) procedure which computes an equivalent quantifier-free formula

$$\chi(\mathbf{y}) = \bigvee_{i=1}^{I} \bigwedge_{j=1}^{J_i} h_{i,j}(y) \sim_{i,j} 0$$

where each $\sim_{i,j}$ is either $>$ or $=$, with the following properties:

1. *Each of I and J_i (for $1 \leq i \leq I$) is bounded by $(n + d)^{O(1)}$.*
2. *The degree of $\chi(\mathbf{y})$ is bounded by $(n + d)^{O(1)}$.*
3. *The height of $\chi(\mathbf{y})$ is bounded by $2^{\|\tau(\mathbf{y})\|(n+d)^{O(1)}}$.*

The following zero-dimensionality results are proved in [**?**].

▶ **Lemma 7.** *Let $a_1, \ldots, a_m \in \mathbb{R}$ and $\varphi_1, \ldots, \varphi_m \in \mathbb{R}$ be two collections of m real numbers, for $m \geq 1$, with each of the a_i non-zero, and let $l_1, \ldots l_m \in \mathbb{Z}$ be integers. Define $f, g : \mathbb{R}^m \to \mathbb{R}$ by setting $f(x_1, \ldots, x_m) = \sum_{i=1}^m a_i cos(x_i + \varphi_i)$ and $g(x_1, \ldots, x_m) = \sum_{i=1}^m l_i x_i$. Assume that $g(x_1, \ldots, x_m)$ is not of the form $l(x_i \pm x_j)$ for some non-zero $l \in \mathbb{Z}$ and indices $i \neq j$. Let $\psi \in \mathbb{R}$.*

Then the function f, subject to the constraint $g(x_1, \ldots, x_m) = \psi$, achieves its minimum only finitely many times over the domain $[0, 2\pi)^m$.

▶ **Lemma 8.** *Let $\langle u_n \rangle$ be a non-degenerate simple LRS with dominant characteristic roots $\rho \in \mathbb{R}$ and $\gamma_1, \bar{\gamma}_1, \ldots, \gamma_m, \bar{\gamma}_m \in \mathbb{C} \setminus \mathbb{R}$. Write $\lambda_i = \gamma_i / \rho$ for $1 \leq i \leq m$, and let $L = \{(v_1, \ldots, v_m) \in \mathbb{Z}^m : \lambda_1^{v_1} \cdots \lambda_m^{v_m} = 1\}$. Let $\{\boldsymbol{\ell}_\mathbf{1}, \ldots, \boldsymbol{\ell}_\mathbf{m-1}\}$ be a basis for L of cardinality $m - 1$, and write $\boldsymbol{\ell}_\mathbf{j} = (\ell_{j,1}, \ldots, \ell_{j,m})$ for $1 \leq j \leq m - 1$. Let*

$$M = \begin{pmatrix} l_{1,1} & l_{1,2} & \cdots & l_{1,m-1} & l_{1,m} \\ l_{2,1} & l_{2,2} & \cdots & l_{2,m-1} & l_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ l_{m-1,1} & l_{m-1,2} & \cdots & l_{m-1,m-1} & l_{m-1,m} \end{pmatrix}$$

Let $a_1, \ldots, a_m \in \mathbb{R}$ and $\varphi_1, \ldots, \varphi_m$ be two collections of m real numbers, with each of the a_i non-zero, and let $\mathbf{q} = (q_1, \ldots, q_{m-1}) \in \mathbb{Z}^{m-1}$ be a column vector of $m-1$ integers. Let us further write $\mathbf{x} = (x_1, \ldots, x_m)$ to denote a column vector of m real-valued variables.

Then the function $f(x_1, \ldots, x_m) = \sum_{i=1}^{m} a_i cos(x_i + \varphi_i)$, subject to the constraint $M\mathbf{x} = 2\pi\mathbf{q}$, achieves its minimum at only finitely many points over the domain $[0, 2\pi)^m$.

The following technical lemmas will enable us to obtain the effective bounds.

▶ **Lemma 9.** *Let $p \in \mathbb{Z}[x]$, and let $\epsilon > 0$ be algebraic of bounded degree. Then there exists $N \in \mathbb{N}$, computable in time polynomial in $\|p\|$ and $\|\epsilon\|$, such that for all $n > N$, $p(n) < (1+\epsilon)^n$.*

Proof. Let d and h denote the degree and height of p, respectively. Let $f \in \mathbb{Z}[x]$ be the defining polynomial of ϵ, and consider $g(x) = xf(x)$. Clearly 0 and ϵ are roots of g, and $\|g\| = O(\|f\|)$. Using the root separation bound in Theorem **??**, we can compute in polynomial time a rational number q such that $0 < q < \epsilon$, and an integer $k \geq 1/q$. Thus, $\epsilon \geq \frac{1}{k} > 0$. Let $N_0 = d^2 + d + 1 + h(d+1)^2 k^{d+1}(d+1)!$, and note that N_0 is computable in polynomial time in $\|\epsilon\|$ and $\|p\|$. It is easily checked that for $n > N$,

$$(1+\epsilon)^n \geq (1+\frac{1}{k})^n \geq \frac{\left(\frac{1}{k}\right)^{d+1}}{(d+1)!}(n-d)^{d+1} \geq h(d+1)(n-d)^d \geq (d+1)hn^d \geq p(n).$$

◀

▶ **Lemma 10.** *Let $\mathbf{u}$ be an LRS of bounded degree with spectral radius $\rho > 1$. Let $p \in \mathbb{Q}[x]$ and $N_0 \in \mathbb{N}$ such that for every $n > N_0$ we have $u_n > \frac{\rho^n}{p(n)}$, and let $\epsilon > 0$ be a rational number. Then we can compute in polynomial time $N \in \mathbb{N}$, such that for every $n > N$, we have $u_n > \rho^{n(1-\epsilon)}$.*

Proof. Let $\mathbf{u}$, ρ, p, N_0 and ϵ be as above. We assume that $\epsilon = \frac{1}{k}$ for some $k \in \mathbb{N}$ (indeed, otherwise we can compute in polynomial time $k \in \mathbb{N}$ such that $\epsilon \geq \frac{1}{k}$). Define $q \in \mathbb{Q}[x]$ by setting $q(x) = p((k+1)x)$. By Lemma 9, we can compute $N_1 \in \mathbb{N}$ in polynomial time such that $\rho^n > q(n)$ for $n \geq N_1$. By bounding the roots of $p'(x)$, we can also compute $N_2 \in \mathbb{N}$ in polynomial time such that p is monotonic from N_2. We assume that $p(x)$ is eventually positive, as otherwise the result is trivial.

Thus, $p(n+1) \geq p(n)$ for every $n \geq N_2$. Let $N = \max\{N0, kN_1, N_2\}$, then for every $n > N$, we have $\rho^{n/k} \geq \rho^{\lfloor n/k \rfloor} > q(\lfloor n/k \rfloor) \geq q(n/k - 1) = p(n)$. Hence for every $n > N$, we have $u_n > \frac{\rho^n}{p(n)} > \frac{\rho^n}{\rho^{n/k}} = \rho^{n(1-\epsilon)}$, and we are done. ◀

Growth rate

▶ **Proposition 11.** *Let $a \geq 2$ and $\epsilon \in (0,1)$ be real numbers. Let $B \in \mathbb{Z}[x]$ have degree at most a^{D_1} and height at most $2^{a^{D_2}}$, and assume that $1/\epsilon \leq 2^{a^{D_3}}$ for some $D_1, D_2, D_3 \in \mathbb{N}$. Then there is $D_4 \in \mathbb{N}$ depending only on D_1, D_2, D_3 such that for all $n \geq 2^{a^{D_4}}$, $\frac{1}{B(n)} > (1-\epsilon)^n$.*

The following proposition from **??** allows us to bound the growth rate of the low-order terms in the exponential polynomial of an LRS.

▶ **Proposition 12.** *Consider an LRS $\mathbf{u} = \langle u_n \rangle_{n=1}^{\infty}$ of bounded order, with spectral radius ρ, and write*

$$\frac{u_n}{\rho^n} = A(n) + \sum_{i=1}^{m} \left(C_i(n)\lambda_i^n + \overline{C_i(n)\lambda_i}^n \right) + r(n)$$

where A is a real polynomial, C_i are non-zero complex polynomials, $\rho\lambda_i$ and $\rho\overline{\lambda_i}$ are conjugate pairs of non-real dominant roots of $\mathbf{u}$, and r is an exponentially decaying function.

We can compute in polynomial time $\epsilon \in (0,1)$ and $N \in \mathbb{N}$ such that

$$\frac{1}{\epsilon} = 2^{\|\mathbf{u}\|^{O(1)}} \tag{1}$$
$$N = 2^{\|\mathbf{u}\|^{O(1)}} \tag{2}$$
$$\text{For all } n > N, |r(n)| < (1-\epsilon)^n \tag{3}$$

2.1 Properties of LRS

2.2 Homogenization of Affine LRS

▶ **Theorem 13** (Homogenization). *Let $\mathbf{v} = \langle v_n \rangle_{n=1}^{\infty}$ be a non-homogeneous LRS of order k. We can compute in polynomial time a homogeneous LRS $\mathbf{u} = \langle u_n \rangle_{n=1}^{\infty}$ of order $k+1$, such that $v_n = u_n$ for every $n \in \mathbb{N}$.*

Proof. Let $v_{n+k} = a_1 v_{n+k-1} + \ldots a_k v_n + a_{k+1}$. Consider the homogeneous LRS $\mathbf{u}$ of order $k+1$ defined by

$$u_{n+k+1} = (1+a_1)u_{n+k} + (a_2-a_1)u_{n+k-1} + (a_3-a_2)u_{n+k-2} + \ldots + (a_k-a_{k-1})u_{n+1} + (-a_k)u_n$$

with the initial values $u_i = v_i$ for $1 \le i \le k+1$. We claim that $u_n = v_n$ for all $n \in \mathbb{N}$. Indeed, proceeding by induction, the claim holds up to $k+1$. Assume correctness up to $n+k$, we prove for $n+k+1$:

$$\begin{aligned} u_{n+k+1} =& a_1 u_{n+k} + \ldots + a_k u_{n+1} + u_{n+k} - (a_1 u_{n+k-1} + \ldots + a_k u_n) \\ =& a_1 v_{n+k} + \ldots + a_k v_{n+1} + v_{n+k} - (a_1 v_{n+k-1} + \ldots + a_k v_n) \\ =& v_{n+k+1} - a_{k+1} + v_{n+k} - (v_{n+k} - a_{k+1}) = v_{n+k+1} \end{aligned}$$

Concluding the proof. ◀

We refer to the LRS $\mathbf{u}$ obtained in the proof of Theorem 13 as the *homogenization* of $\mathbf{v}$, denoted $\textsc{hom}(\mathbf{v})$. We observe that the characteristic polynomial of $\mathbf{u} = \textsc{hom}(\mathbf{u})$ is $g(x) = (x-1)f(x)$, where $f(x)$ is the characteristic polynomial of $\mathbf{v}$. This gives us the following useful property.

▶ **Property 14.** *The eigenvalues of $\textsc{hom}(\mathbf{v})$ are the same as those of $\mathbf{v}$, with the same multiplicities, except for the eigenvalue 1, which always occurs in $\textsc{hom}(\mathbf{v})$, and has multiplicity either m or $m+1$, where m is the multiplicity of 1 in $\mathbf{v}$.*

▶ **Theorem 15** (De-Homogenization). *Let $\mathbf{u} = \langle u_n \rangle_{n=1}^{\infty}$ be a homogeneous LRS of order $k+1$ with a dominant real eigenvalue. We can compute in polynomial time a non-homogeneous LRS $\mathbf{v} = \langle v_n \rangle_{n=1}^{\infty}$ of order k, such that $v_n = \frac{u_n}{\rho^n}$ for every $n \in \mathbb{N}$, where ρ is the spectral radius of $\mathbf{u}$.*

Proof. Consider the homogenous LRS $\mathbf{w}$ such that $w_n = \frac{u_n}{\rho^n}$. Clearly, we can compute $\mathbf{w}$ in polynomial time, and $\mathbf{w}$ has spectral radius 1. It's enough to prove that there exists a non-homogeneous LRS $\mathbf{v}$ of order k such that $v_n = w_n$ for all $n \in \mathbb{N}$. Let

$$w_{n+k+1} = b_1 w_{n+k} + b_2 w_{n+k-1} + \cdots + b_k w_{n+1} + b_{k+1} w_n.$$

Thus, its characteristic polynomial is

$$f(x) = x^{k+1} - b_1 x^k - \ldots - b_k x - b_{k+1}$$

Since $\mathbf{w}$ has spectral radius 1, w.l.o.g. we assume $f(1) = 0$ (the case where $f(-1) = 0$ is similar). Thus, we have

$$b_{k+1} = 1 - b_1 - \ldots - b_k \tag{4}$$

In addition, $f(x)$ is divisible by $(x-1)$. Consider the polynomial $g(x) = f(x)/(x-1)$, and write

$$g(x) = x^k - a_1 x^{k-1} - \ldots - a_{k-1}x - a_k$$

We now define the non-homogeneous LRS $\mathbf{u}$ of order k by setting $u_i = v_i$ for all $1 \le i \le k$ and letting

$$u_{n+k} = a_1 u_{n+k-1} + \ldots a_k u_n + a_{k+1}$$

where a_{k+1} is defined such that

$$v_{k+1} = a_1 v_k + \ldots a_k v_1 + a_{k+1}$$

We now observe that since $f(x) = g(x) + 1$, then for every $1 \le i \le k$ we have that $a_i = \sum_{j=1}^{i} b_i - 1$. It now follows by induction, using Equation (4), that $u_n = v_n$ for all $n \in \mathbb{N}$ (with the base cases $1, \ldots, k$ holding by definition). ◀

3 Positivity and Ultimate Positivity

In this section we study the positivity and ultimate positivity problems for non-homogeneous LRS. These problems were studied in [?, ?, ?] for homogeneous LRS. Using Theorem 13 and some careful analysis, we extend the decidability results to the non-homogeneous case.

We start by citing some results from [?, ?, ?], split to upper and lower bounds.

▶ **Theorem 16** (Upper Bounds from [?, ?, ?]).

1. *Positivity and ultimate-positivity are decidable for homogeneous LRS of order 5 or less.*
2. *Positivity is decidable for simple homogeneous LRS or order 9 or less.*
3. *Ultimate-positivity is decidable for simple homogeneous LRS of any order.*
4. *Effective ultimate-positivity is solvable for simple homogeneous LRS of order 9 or less.*

TODO↑: Complexity? ⋆

▶ **Theorem 17** (Lower Bounds from [?, ?, ?]). *Positivity and ultimate positivity for LRS of order at least 6 are hard with respect to problems in number theory.*

TODO↑: Rephrase hardness ⋆

3.1 Upper Bounds

We proceed to prove analogous results to Theorem 16 for non-homogeneous LRS.

Theorem 16(1.) along with Theorem 13 readily give us the following:

▶ **Theorem 18.** *Positivity and ultimate positivity are decidable for non-homogeneous LRS of order 4 or less.*

For simple LRS, things become more involved, as the procedure in the proof of Theorem 13 does not preserve simplicity. However, Property 14 shows that simplicity is almost preserved, up to the multiplicity of eigenvalue 1. As we now show, this is sufficient to obtain upper bounds for non-homogeneous simple LRS.

We start by addressing effective ultimate positivity, which we then use for addressing positivity.

▶ **Theorem 19.** *Effective ultimate positivity is solvable for simple non-homogeneous LRS of order 8 or less.*

Proof. Let $\mathbf{v}$ be a simple, non-degenerate, non-homogeneous LRS or order 8 or less, and consider the homogeneous LRS $\mathbf{u} = \textsc{hom}(\mathbf{v})$. By Theorem 13, $\mathbf{u}$ is of order at most 9. If $\mathbf{u}$ is a simple LRS, then by [**?**] we can effectively decide its ultimate positivity. We hence assume that $\mathbf{u}$ is not simple.

By Property 14, it follows that the eigenvalues of $\mathbf{u}$ all have multiplicity 1, apart from the eigenvalue 1 which has multiplicity 2. Consider the spectral radius ρ of $\mathbf{u}$. If $\rho > 1$, then by writing the exponential polynomial of $\mathbf{u}$, we have

$$\frac{u_n}{\rho^n} = a + \sum_{i=1}^{m} (c_i \lambda_i^n + \overline{c_i}\overline{\lambda_i^n}) + r(n) \tag{5}$$

with $a \in \mathbb{R}$, $c_i \in \mathbb{C} \setminus \mathbb{R}$ and $|\lambda_i| = 1$ for every $1 \leq i \leq m$, and $|r(n)|$ exponentially decaying. Crucially, since 1 is not a dominating eigenvalue, its effect is only minor in $r(n)$. Specifically, we observe that the analysis of effective ultimate positivity in [**?**] only relies on Proposition 12. Since this holds in the case at hand, we can effectively decide ultimate positivity when 1 is not a dominant eigenvalue.

Finally, if 1 is a dominant eigenvalue, the exponential polynomial of $\mathbf{u}$ can be written as

$$u_n = A(n) + \sum_{i=1}^{m} (c_i \lambda_i^n + \overline{c_i}\overline{\lambda_i^n}) + r(n) \tag{6}$$

with $A(n) = an + b$ a linear polynomial, and $a \neq 0$. Let $C = \max_{1 \leq i \leq m} \{|c_i|\}$, and Observe that $|\sum_{i=1}^{m} (c_i \lambda_i^n + \overline{c_i}\overline{\lambda_i^n})| \leq 2mC \leq 8C$ (trivially $m \leq 4$).

If $a < 0$, then clearly $u_n \to -\infty$, and is thus not ultimately positive. If $a > 0$, by Proposition 12, we can compute in polynomial time $\epsilon \in (0, 1)$ and $N_0 \in \mathbb{N}$ such that $|r(n)| < (1 - \epsilon)^n < 1$ for all $n > N_0$. We can now easily compute $N_2 \in \mathbb{N}$ and $q > 0$ (depending on the coefficients of $A(n)$) such that for all $n > N_2$ we have $A(n) - C - 1 \geq qn > 0$. Taking $N = \max\{N_0, N_1, N_2\}$, we conclude that u_n is ultimately positive, and we can compute the relevant bound. Observe that by Proposition 12 we have $N = 2^{\mathbf{u}^{O(1)}}$.

This concludes the proof that ultimate positivity is effectively decidable for simple non-homogeneous LRS of order at most 8. ◀

▶ **Theorem 20.** *Ultimate positivity is decidable for simple non-homogeneous LRS of any order.*

Proof. Let $\mathbf{v}$ be a simple, non-degenerate, non-homogeneous LRS or order 8 or less, and consider the homogeneous LRS $\mathbf{u} = \textsc{hom}(\mathbf{v})$. By Theorem 13, $\mathbf{u}$ is of order at most 9. If $\mathbf{u}$ is a simple LRS, then by [**?**] we can effectively decide its ultimate positivity. We assume henceforth that $\mathbf{u}$ is not simple.

By Property 14, it follows that the eigenvalues of $\mathbf{u}$ all have multiplicity 1, apart from the eigenvalue 1 which has multiplicity 2. Consider the spectral radius ρ of $\mathbf{u}$. If $\rho > 1$, then by writing the exponential polynomial of $\mathbf{u}$, we have

$$\frac{u_n}{\rho^n} = a + \sum_{i=1}^{m}(c_i\lambda_i^n + \overline{c_i}\overline{\lambda_i^n}) + r(n) \tag{7}$$

with $a \in \mathbb{R}$, $c_i \in \mathbb{C} \setminus \mathbb{R}$ and $|\lambda_i| = 1$ for every $1 \leq i \leq m$, and $|r(n)|$ exponentially decaying. Crucially, since 1 is not a dominating eigenvalue, its effect is only minor in $r(n)$. Specifically, we observe that the analysis of effective ultimate positivity in [**?**] only relies on Proposition 12. Since this holds in the case at hand, we can effectively decide ultimate positivity when 1 is not a dominant eigenvalue.

Finally, if 1 is a dominant eigenvalue, the exponential polynomial of $\mathbf{u}$ can be written as

$$u_n = A(n) + \sum_{i=1}^{m}(c_i\lambda_i^n + \overline{c_i}\overline{\lambda_i^n}) + r(n) \tag{8}$$

with $A(n) = an + b$ a linear polynomial, and $a \neq 0$. Let $C = \max_{1\leq i\leq m}\{|c_i|\}$, and Observe that $|\sum_{i=1}^{m}(c_i\lambda_i^n + \overline{c_i}\overline{\lambda_i^n})| \leq 2mC$, which is independent of n.

If $a < 0$, then clearly $u_n \to -\infty$, and is thus not ultimately positive and if $a > 0$ then clearly $u_n \to \infty$, and is thus ultimately positive, and we are done. ◀

▶ **Theorem 21.** *Positivity is decidable for simple non-homogeneous LRS of order* 8 *or less.*

Proof. Given the proof of Theorem 19, positivity is now easily decidable: given a non-homogeneous simple LRS $\mathbf{u}$ of order at most 8, decide if its ultimately positive, and if so - compute the bound from which it is ultimately positive. Then, deciding positivity amounts to checking a finite number of elements.

Note that the bound computed in Theorem 19 is $N = 2^{\mathbf{u}^{O(1)}}$. This implies that checking whether an ultimately-positive LRS is *not* positive can be done using a *guess-and-check* procedure, and employing PoSSLP in order to compute double exponential numbers. This yields an NP^{PosSLP} algorithm. Thanks to [**?**], we obtain an upper bound of $coNP^{PP^{PP^{PP}}}$ for positivity (see [**?**] for details). ◀

3.2 Lower Bounds

We now turn to study lower bounds, proving analogous results to Theorem 17 for non-homogeneous LRS.

▶ **Theorem 22.** *Positivity and ultimate positivity are hard (w.r.t. number theoretic problems) for non-homogeneous LRS of order* 5 *or more.*

Proof. In [**?**], it is shown that deciding the positivity or ultimate positivity of the homogeneous LRS of order 6 given by

$$u_n = r\sin n\theta - n(1-\cos n\theta) \text{ and } v_n = -r\sin n\theta - n(1-\cos n\theta)$$

for every $r \in \mathbb{Q}$ such that $r > 0$ and $\theta \in (0, 2\pi)$ such that $(\cos\theta, \sin\theta) \in \mathbb{Q}^2$ would allow one to compute the Lagrange constant and type of certain real numbers, a task not known to be possible.

TODO↑: REPHRASE ⋆

We observe that both sequences u_n and v_n fall under the premise of Theorem 15. Thus, by applying Theorem 15, we obtain equivalent non-homogeneous LRS of order 5, concluding the proof. ◀

4 Divergence of LRS

We now turn our attention to divergence of LRS. Unlike Section 3, this problem has yet to have been studied even for homogeneous LRS. We thus tackle both the homogeneous and non-homogeneous cases.

Recall that divergence of LRS is the following decision problem: given an LRS $\langle u_n \rangle$ decide whether $u_n \to \infty$ as $n \to \infty$.

As will become evident in this section, when an LRS is divergent, the rate of divergence depends asymptotically on the spectral radius ρ. Specifically, if $\rho < 1$, the sequence does not diverge. If $\rho > 1$, the sequence diverges roughly at the rate of ρ^n (see Theorem 23 for a precise formulation), and if $\rho = 1$ the sequences may diverge only if the multiplicity of the eigenvalue 1 is at least two, in which case the rate of divergence is polynomial.

Thus, we formulate the functional version of the divergence problem as follows. Consider a divergent LRS with spectral radius $\rho \geq 1$. Given some rational $\epsilon > 0$, if $\rho > 1$, compute $N \in \mathbb{N}$ such that for every $n > N$ it holds that $u_n > \rho^{(1-\epsilon)n}$, and if $\rho = 1$, compute $N, k \in \mathbb{N}$ and $a \in \mathbb{Q}$ such that for every $n > N$ it holds that $u_n > an^k$.

We say that divergence is *effectively decidable* if it is decidable and the functional version is solvable.

4.1 Effective Decidability of Divergence

We start by showing the effective decidability of divergence. We show that unlike positivity and ultimate-positivity, divergence is decidable up to order 5 for non-homogeneous LRS (and in particular for homogeneous LRS), and up to order 8 for simple non-homogeneous LRS. We prove Theorems 23 and 24 in the remainder of this section.

▶ **Theorem 23.** *Divergence is effectively decidable for non-homogeneous LRS of order 5 oe less in polynomial time.*

▶ **Theorem 24.** *Divergence is effectively decidable for simple non-homogeneous LRS of order 8 or less in polynomial time.*

Before going into the proof, we introduce some mathematical tools that will be used throughout this section.

4.2 Proof of Theorem 23

We initially prove Theorem 23 for homogeneous LRS. We then show how to handle the non-homogeneous case, using Property 14.

Consider an LRS $\mathbf{u} = \langle u_n \rangle_{n=1}^{\infty}$ of order $d \leq 5$ with spectral radius ρ, and let $\epsilon > 0$ be a rational number. First, we note that without loss of generality, we can assume $\mathbf{u}$ is non-degenerate, as we may decompose a degenerate sequence and recast analysis at lower orders. We also note that if $\rho < 1$, then $|u_n| \to 0$ as $n \to \infty$, and in particular the sequence does not diverge. Thus, we may assume $\rho \geq 1$.

By Lemma 2, if $\mathbf{u}$ does not have a real positive dominant root, then $u_n \not\to \infty$. Thus, we may assume a real positive dominant root. Note that all other dominant roots must be complex, and come in conjugate pairs, since if $-\rho$ were a root, then $\mathbf{u}$ would be degenerate.

Writing u_n as an exponential polynomial and dividing by ρ^n, we have

$$\frac{u_n}{\rho^n} = A(n) + \sum_{i=1}^{m} \left(C_i(n)\lambda_i^n + \overline{C_i(n)\lambda_i^n} \right) + r(n) \tag{9}$$

where A is a real polynomial, C_i are non-zero complex polynomials, $\rho\lambda_i$ and $\rho\overline{\lambda}_i$ are conjugate pairs of non-real dominant roots of $\mathbf{u}$, and r is an exponentially decaying function (possibly identically zero). We can assume that either $A \not\equiv 0$ or $m \neq 0$. Indeed, otherwise we can consider the LRS $\langle \rho^n r(n) \rangle_{n=1}^{\infty}$, which is of lower order than $\mathbf{u}$.

We proceed to decide divergence by a case analysis of Equation (9).

Case 1: $\rho = 1$**.**

Note that in this case, $\frac{u_n}{\rho^n} = u_n$. If $A(n)$ is a constant, then it does not affect the divergence of $\mathbf{u}$. We claim that $u_n \not\to \infty$. Indeed, by Lemma 2, the expression $\sum_{i=1}^{m}\left(C_i(n)\lambda_i^n + \overline{C_i(n)}\overline{\lambda_i}^n\right)$ becomes negative infinitely often (regardless of whether C_i are constants or polynomials), whereas the effect of $r(n)$ is exponentially decreasing. Thus, $\mathbf{u}$ does not diverge.

If $A(n)$ is not a constant, then $m \leq 1$. If $m = 0$, then clearly $u_n \to \infty$ iff the leading coefficient of $A(n)$ is positive. Otherwise, if $m = 1$, then C_1 is a constant, and thus $|C_1\lambda_1^n + \overline{C_1\lambda_1}^n| \leq 2|C_1|$, and again $u_n \to \infty$ iff the leading coefficient of $A(n)$ is positive.

Recall that since $\rho = 1$, then if $\mathbf{u}$ diverges, there exist $N, k \in \mathbb{N}$ such that $u_n \geq n^k$ for all $n > N$. We now show how to effectively compute N and k.

From Proposition 12, we can compute in polynomial time $\epsilon \in (0,1)$ and $N_1 \in \mathbb{N}$ such that $r(n) < (1-\epsilon)^n < 1$ for all $n > N_1$. We thus have that $u_n \geq A(n) - |C_1| - 1$, and we can easily compute $N_2 \in \mathbb{N}$ and $a \in \mathbb{Q}$ (depending on the coefficients of $A(n)$) such that for all $n > N_2$ we have $A(n) - |C_1| - 1 \geq an^k$, where k is the degree of $A(n)$, namely 1 or 2. Taking $N = \max\{N_1, N_2\}$, we conclude this case.

Case 2: $\rho > 1$ **and there exists a non-constant** C_i**.**

In this case, $m = 1$, C_1 is linear, and $A(n)$ is constant. Let C_1 have leading coefficient $b \neq 0$. By Lemma 2, there exists $\epsilon > 0$ such that $b\lambda^n + \overline{b\lambda}^n < -\epsilon$ infinitely often. Then $C_1(n)\lambda_1^n + \overline{C_1(n)\lambda_1}^n$ (and hence u_n) is unbounded below, so $u_n \not\to \infty$.

Case 3: $\rho > 1$ **and every** C_i **is a nonzero constant.**

In this case, $m \leq 2$. In the following, we set $m = 2$, as the cases where $m < 2$ are similar and slightly simpler.[1]

Let $L = \left\{(v_1, v_2) \in \mathbb{Z}^2 : \lambda^{v_1}\lambda^{v_2} = 1\right\}$, and let $\{\boldsymbol{\ell}_{\mathbf{1}}, \ldots, \boldsymbol{\ell}_{\mathbf{p}}\}$ be a basis for L of cardinality p. Write $\boldsymbol{\ell}_{\mathbf{q}} = (\ell_{q,1}, \ell_{q,2})$ for $1 \leq q \leq p$. From Theorem 1, such a basis can be computed in polynomial time, and moreover – each $\ell_{q,j}$ may be assumed to have magnitude polynomial in $\|\mathbf{u}\|$.

Consider the set $\mathbb{T} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| = |z_2| = 1 \text{ and for each } 1 \leq q \leq p,\ z_1^{\ell_{q,1}} z_2^{\ell_{q,2}} = 1\}$.

Define $h : \mathbb{T} \to \mathbb{R}$ by setting $h(z_1, z_2) = \sum_{i=1}^{2}(C_i z_i + \overline{C_i}\overline{z_i})$, so that for every $n \in \mathbb{N}$, $\frac{u_n}{\rho^n} = A(n) + h(\lambda_1^n, \lambda_2^n) + r(n)$. Recall that the set $\{(\lambda_1^n, \lambda_2^n) : n \in \mathbb{N}\}$ is a dense subset of $\mathbb{T}$. Since h is continuous, it follows that $\inf\{h(\lambda_1^n, \lambda_2^n) : n \in \mathbb{N}\} = \min h|_{\mathbb{T}} = \mu$ for some $\mu \in \mathbb{R}$.

We now claim that μ is an algebraic number, computable in polynomial time, with $\|\mu\| = \|\mathbf{u}\|^{O(1)}$. We can represent μ via the following formula $\tau(y)$:

$$\exists(\zeta_1, \zeta_2) \in \mathbb{T} :\ [h(\zeta_1, \zeta_2) = y \wedge \forall (z_1, z_2) \in \mathbb{T}, y \leq h(z_1, z_2)].$$

[1] One may notice that taking $m = 2$ means that some of the cases we handle actually require order 6, e.g. when $A(n)$ is linear and $m = 2$. Still, the analysis covers all possible cases of order 5.

Note that $\tau(y)$ is not a formula in the first-order theory of the reals, as it involves complex numbers. However, we can rewrite it as a sentence in the first-order theory of the reals by representing the real and imaginary parts of each complex quantity and combine them using real arithmetic (see [**?**, ICALP]or details). In addition, the obtained formula $\tau'(y)$ is of size polynomial in $\|\mathbf{u}\|$. By Theorem 6, we can then compute in polynomial time an equivalent quantifier-free formula

$$\chi(x) = \bigvee_{i=1}^{I} \bigwedge_{j=1}^{J_i} h_{i,j} \sim_{i,j} 0.$$

Recall that each $\sim_{i,j}$ is either $>$ or $=$. Now $\chi(x)$ must have a satisfiable disjunct, and since the satisfying assignment to y is unique (namely $y = \mu$), this disjunct must comprise at least one equality predicate. Since Theorem 6 guarantees that the degree and height of each $h_{i,j}$ are bounded by $\|\mathbf{u}\|^{O(1)}$ and $2^{\|\mathbf{u}\|^{O(1)}}$ respectively, we immediately conclude that μ is an algebraic number and with $\|\mu\| = \|\mathbf{u}\|^{O(1)}$.

We now split the analysis into several cases.

• If $A(n)$ is linear with negative leading coefficient, or if A is a constant and $A + \mu < 0$, then u_n is unbounded from below, and in particular $u_n \not\to \infty$.

• If $A(n)$ is linear with positive leading coefficient, or if A is a constant and $A + \mu > 0$, we can compute in polynomial time $N_0 \in \mathbb{N}$ and a rational $\epsilon_0 > 0$ such that $A(n) + \mu > 2\epsilon_0$ for all $n > N_0$. By Proposition 12, we can also compute in polynomial time $N_1 \in \mathbb{N}$ and $\epsilon_1 \in (0, 1)$ such that $|r(n)| < (1 - \epsilon_1)^n$ for all $n > N_1$. Taking $N_2 \geq \log_{1-\epsilon_1} \epsilon_0$, we have that for all $n > \max\{N_0, N_1, N_2\}$, $|r(n)| < \epsilon_0$, and thus

$$\frac{u_n}{\rho^n} = A(n) + h(\lambda_1, \ldots, \lambda_m) + r(n) \geq A(n) + \mu - \epsilon_0 > 2\epsilon_0 - \epsilon_0 = \epsilon_0$$

The constant ϵ_0 can be thought of as an inverse polynomial for the purpose of applying Lemma 10, so we conclude the effective decidability of divergence in this case.

• If A is a constant and $A + \mu = 0$, things are more involved. Let $\lambda_j = e^{i\theta_j}$ and $C_j = |C_j|e^{i\varphi_j}$ for $1 \leq j \leq 2$. From Equation (9) we have

$$\frac{u_n}{\rho^n} = A + \sum_{j=1}^{2} 2|C_j| \cos(n\theta_j + \varphi_j) + r(n)$$

We further assume that all the C_j are non-zero. Indeed, if this does not hold, we can recast our analysis in lower dimension.

We now claim that h achieves its minimum μ only finitely many times over $\mathbb{T}$. To establish this claim, we proceed according to the cardinality p of the basis $\{\boldsymbol{\ell}_\mathbf{1}, \ldots, \boldsymbol{\ell}_\mathbf{p}\}$ of L:

(i) We first consider the case in which $p = 1$, and handle the case $p = 0$ immediately afterwards. Let $\boldsymbol{\ell}_\mathbf{1} = (\ell_{1,1}, \ell_{1,2}) \in \mathbb{Z}^2$ be the sole vector spanning L. For $x \in \mathbb{R}$, recall that we denote by $[x]_{2\pi}$ the distance from x to the closest integer multiple of 2π

⋆ TODO↑: ADD THIS NOTATION

Write

$$R = \left\{(x_1, x_2) \in [0, 2\pi)^2 : [\ell_{1,1}x_1 + \ell_{1,2}x_2]_{2\pi} = 0\right\}.$$

Clearly, for any $(x_1, x_2) \in [0, 2\pi)^2$, we have $(x_1, x_2) \in R$ iff $(e^{ix_1}, e^{ix_2}) \in \mathbb{T}$. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by setting

$$f(x_1, x_2) = \sum_{j=1^2} 2|C_j| \cos(x_j + \varphi_j).$$

Clearly, for all $(x_1, x_2) \in [0, 2\pi)^2$ we have $f(x_1, x_2) = h(e^{ix_1}, e^{ix_2})$, and therefore the minimal of f over $\mathbb{R}$ are in one-to-one correspondence with those of h over $\mathbb{T}$.

Define $g : \mathbb{R}^2 \to \mathbb{R}$ by setting

$$g(x_1, x_2) = \ell_{1,1}x_1 + \ell_{1,2}x_2.$$

Note that $g(x_1, x_2)$ cannot be of the form $\ell(x_i - x_j)$, for nonzero $\ell \in \mathbb{Z}$ and $i \neq j$, otherwise $\lambda_i^{\ell}\lambda_j^{-\ell} = 1$, i.e. λ_i/λ_j would be a root of unity, contradicting the non-degeneracy of $\mathbf{u}$. Likewise, g cannot be of the form $\ell(x_i + x_j)$, otherwise $\lambda_i/\overline{\lambda}_j$ would be a root of unity.

Finally, observe that for $(x_1, x_2) \in [0, 2\pi)^2$, we have $(x_1, x_2) \in R$ iff $\ell_{1,1}x_1 + \ell_{1,2}x_2 = 2\pi q$ for some $q \in \mathbb{Z}$ with $|q| \leq |\ell_{1,1}| + |\ell_{1,2}|$. For each of these finitely many q, we can invoke Lemma 7 with f, g, and $\psi = 2\pi q$, to conclude that f achieves its minimum μ finitely many times over R, and therefore that h achieves the same minimum finitely many times over $\mathbb{T}$.

The case $p = 0$, i.e. in which there are no non-trivial integer multiplicative relationships among λ_1, λ_2, is now a special case of the above, where we have $\ell_{1,1} = \ell_{1,2}$.

(ii) We observe that the case $p = 2$ cannot occur: indeed, a basis for L of dimension 2 would immediately entail that every λ_j is a root of unity.

This concludes the proof of the claim that h achieves its minimum at a finite number of points $Z = \{(\zeta_1, \zeta_2) \in \mathbb{T} : h(\zeta_1, \zeta_2) = \mu\}$.

We concentrate on the set Z_1 of first coordinates of Z. Write

$$\tau_1(x) = \exists z_1(\mathrm{Re}(z_1) = x \wedge z_1 \in Z_1)$$
$$\tau_2(y) = \exists z_1(\mathrm{Im}(z_1) = y \wedge z_1 \in Z_1)$$

Similarly to our earlier construction, $\tau_1(x)$ is equivalent to a formula $t_1'(x)$ in the in the first-order theory of the reals, over a bounded number of real variables, with $\|\tau_1'(x)\| = \|\mathbf{u}\|^{O(1)}$. Thanks to Theorem 6, we then obtain an equivalent quantifier-free formula

$$\chi_1(x) = \bigvee_{i=1}^{I} \bigwedge_{j=1}^{J_i} h_{i,j} \sim_{i,j} 0.$$

Note that since there can only be finitely many $\hat{x} \in \mathbb{R}$ such that $\chi_1(\hat{x})$ holds, each disjunct of $\chi_1(\hat{x})$ must comprise at least one equality predicate, or can otherwise be entirely discarded as having no solution. A similar exercise can be carried out with $\tau_2(y)$ to obtain $\chi_2(y)$. The bounds on the degree and height of each $h_{i,j}$ in $\chi_1(x)$ and $\chi_2(y)$ then enables us to conclude that any $\zeta_1 = \hat{x} + i\hat{y} \in Z_1$ is algebraic, and moreover satisfies $\|\zeta_1\| = \|\mathbf{u}\|^{O(1)}$. In addition, bounds on I and J_i guarantee that the cardinality of Z_1 is at most polynomial in $\|\mathbf{u}\|$.

Since λ_1 is not a root of unity, for each $\zeta_1 \in Z_1$ there is at most one value of n such that $\lambda_1^n = \zeta_1$. Theorem 1 then entails that this value (if it exists) is at most $M = \|\mathbf{u}\|^{O(1)}$, which we can take to be uniform across all $\zeta_1 \in Z_1$. We can now invoke Corollary 4 to conclude that, for $n > M$, and for all $\zeta_1 \in Z_1$, we have

$$|\lambda_1^n - \zeta_1| > \frac{1}{n^{\|\mathbf{u}\|^D}} \tag{10}$$

where $D \in \mathbb{N}$ is some absolute constant.

Let $b > 0$ be minimal such that the set

$$\left\{ z_1 \in \mathbb{C} : |z_1| = 1 \text{ and, for all } \zeta_1 \in Z_1, |z_1 - \zeta_1| \geq \frac{1}{b} \right\}$$

is non empty. Thanks to our bounds on the cardinality of Z_1, we can use the first-order theory of the reals, together with Theorem 6, to conclude that b is algebraic and $\|b\| = \|\mathbf{u}\|^{O(1)}$.

Define the function $g : [b, \infty) \to \mathbb{R}$ as follows:

$$g(x) = \min\{h(z_1, z_2) - \mu : (z_1, z_2) \in \mathbb{T} \text{ and for all } \zeta_1 \in Z_1, |z_1 - \zeta_1| \geq \frac{1}{x}\}.$$

It is clear that g is continuous and $g(x) > 0$ for all $x \in [b, \infty)$. Moreover, g can be translated in polynomial time into a function in the first-order theory of the reals over a bounded number of variables. It follows from Proposition 2.6.2 of [**?**, **?**] (invoked with the function $1/g$) that there is a polynomial $P \in \mathbb{Z}[x]$ such that, for all $x \in [b, \infty)$,

$$g(x) \geq \frac{1}{P(x)} \tag{11}$$

Moreover, and examination of the proof of [**?**, **?**] reveals that P is obtained through a process which hinges on quantifier elimination. By Theorem 6, we are therefore able to conclude that $\|P\| = \|\mathbf{u}\|^{O(1)}$, a fact which relies among others on our upper bounds for $\|b\|$.

By Proposition 12 we can find $\epsilon \in (0, 1)$ and $N = 2^{\|\mathbf{u}\|^{O(1)}}$ such that for all $n > N$, we have $|r(n)| < (1-\epsilon)^n$, and moreover $1/\epsilon = 2^{\|\mathbf{u}\|^{O(1)}}$. In addition, by Proposition 11, there is $N' = 2^{\|\mathbf{u}\|^{O(1)}}$ such that for every $n \geq N'$

$$\frac{1}{2P(n^{\|\mathbf{u}\|^D})} > (1-\epsilon)^n. \tag{12}$$

Combining Equations (9)–(12), we get

$$\begin{aligned}
\frac{u_n}{\rho^n} &= A + h(\lambda_1^n, \lambda_2^n) + r(n) \\
&\geq -\mu + h(\lambda_1^n, \lambda_2^n) - (1-\epsilon)^n \\
&\geq g(n^{\|\mathbf{u}\|^D}) - (1-\epsilon)^n \\
&\geq \frac{1}{P(n^{\|\mathbf{u}\|^D})} - (1-\epsilon)^n \\
&= \frac{1}{2P(n^{\|\mathbf{u}\|^D})} + \frac{1}{2P(n^{\|\mathbf{u}\|^D})} - (1-\epsilon)^n \\
&\geq \frac{1}{2P(n^{\|\mathbf{u}\|^D})}
\end{aligned}$$

provided $n > \max\{M, N, N'\}$. We thus have that $\frac{u_n}{\rho^n}$ is eventually lower bounded by an inverse polynomial, and by Lemma 10 we conclude that $\mathbf{u}$ diverges and the bounds are effectively computable.

This concludes the effective decidability of divergence for homogeneous LRS of order at most 5.

It remains to show how to handle non-homogeneous LRS of order at most[2] 5. Consider a non-homogeneous LRS $\mathbf{v} = \langle v_n \rangle_{n=1}^{\infty}$ of order 5, and let $\mathbf{u} = \textsc{hom}(\mathbf{v})$. Consider the spectral radius ρ of u_n. If $\rho > 1$, then by property 14 the exponential polynomial of $\frac{u_n}{\rho^n}$ is the same as that in Equation (9). Thus, we can proceed with the case analysis of Case 2 and Case 3 without change. If $\rho = 1$, things become more involved. Consider the exponential polynomial

$$u_n = A(n) + \sum_{i=1}^{m} \left(C_i(n)(\lambda_i^n) + \overline{C_i(n)}(\overline{\lambda_i}^n) \right) + r(n) \tag{13}$$

[2] In fact, by property 14, LRS of order at most 4 can be handled by homogenization. Thus, it is enough to handle exactly order 5.

where $|r(n)|$ is exponentially decaying and the λ_i are eigenvalues of modulus 1.

If $A(n)$ is constant, or if $A(n)$ is not a constant and all the C_i are constants (if there are any), then the same analysis of Case 1 applies here, *mutatis-mutandis*. Otherwise, the only possible case is where $A(n)$ is linear, $m = 1$, $C_1(n)$ is linear, and $r(n) \equiv 0$. Indeed, this corresponds to the case where the eigenvalues of u_n are $1, \lambda, \overline{\lambda}$, each with multiplicity 2. Let $A(n) = a_1 n + b_1$ and $C_1(n) = a_2 n + b_2$, then we can write

$$u_n = a_1 n + b_1 + (a_2 n + b_2)\lambda^n + (\overline{a_2} n + \overline{b_2})\overline{\lambda}^n = n(a_1 + a_2\lambda^n + \overline{a_2}\overline{\lambda^n}) + (b_1 + b_2\lambda^n + \overline{b_2\lambda^n})$$

Since $|(b_1 + b_2\lambda^n + \overline{b_2\lambda^n})|$ is bounded, then u_n diverges iff $n(a_1 + a_2\lambda^n + \overline{a_2}\overline{\lambda^n})$ diverges. Let $\theta = \arg\lambda$ and $\varphi = \arg a_2$. We have $n(a_1 + a_2\lambda^n + \overline{a_2}\overline{\lambda^n}) = n(a_1 + 2|a_2|\cos(n\theta + \varphi))$.

Again, we split into cases.

• If $a_1 > 2|a_2|$, we have that u_n diverges. Then, we can compute in polynomial time a rational $\epsilon > 0$ and $N \in \mathbb{N}$ such that $a_1 - 2|a_2| > \epsilon$ and $n(a_1 + 2|a_2|) - (b_1 - 2|b_2|) > \epsilon n$ for all $n > N$. We then have that $u_n > \epsilon n$ for all $n > N$, thus concluding effective decidability of divergence in this case.

• If $a_1 < 2|a_2|$, then u_n does not diverge, as it becomes negative infinitely often.

• The remaining case is when $a_1 = 2|a_2|$, where the expression above becomes $na_1(1 + \cos(n\theta + \varphi))$. We show that in this case, u_n does not diverge. By Taylor approximation, for every $x \in (-\pi, \pi]$ it holds that $1 - \cos(x) \leq \frac{x^2}{2}$. For $n \in \mathbb{N}$, write $\Lambda(n) = n\theta + \varphi - (2j+1)\pi$, where $j \in \mathbb{Z}$ is the unique integer such that $-\pi < \Lambda(n) \leq \pi$. We now have that

$$na_1(1 + \cos(n\theta + \varphi)) = na_1(1 - \cos(n\theta + \varphi + \pi)) = na_1(1 - \cos(\Lambda(n))) < na_1\frac{\Lambda(n)^2}{2}$$

By Dirichlet's Approximation Theorem, we have that $|\Lambda(n)| < \frac{t}{n}$ for infinitely many values of n, where t is a constant depending on φ. Thus, we have $na_1\frac{\Lambda(n)^2}{2} < \frac{a_1 t^2}{2n}$. It follows that u_n is infinitely often bounded by a constant, and does not diverge.

4.3 Proof of Theorem 24

As in the proof of Theorem 23, we start by considering the homogeneous case, and we let $\langle u_n \rangle$ be a non-degenerate simple LRS of order $d \leq 8$ with a real positive dominant eigenvalue $\rho \geq 1$.

As before, we write

$$\frac{u_n}{\rho^n} = a + \sum_{i=1}^{m}(c_i\lambda_i^n + \overline{c_i}\overline{\lambda_i^n}) + r(n) \tag{14}$$

with $a \in \mathbb{R}$, $c_i \in \mathbb{C} \setminus \mathbb{R}$ for every $1 \leq i \leq m$, and $|r(n)|$ exponentially decaying. Note that since $d \leq 8$ and $a \in \mathbb{R}$, it follows that $0 \leq m \leq 3$. In the following, we consider the case where $m = 3$. The cases where $m < 3$ are very similar and slightly simpler, and are therefore omitted.

Observe that if $\rho = 1$, the sequence u_n is bounded, and therefore does not diverge. We hence assume $\rho > 1$.

Let $L = \{(v_1, \ldots, v_m) \in \mathbb{Z}^m : \lambda^{v_1} \cdots \lambda^{v_m} = 1\}$, and let $\{\boldsymbol{\ell}_\mathbf{1}, \ldots, \boldsymbol{\ell}_\mathbf{p}\}$ be a basis for L of cardinality p. Write $\boldsymbol{\ell}_\mathbf{q} = (\ell_{q,1}, \ldots, \ell_{q,m})$ for $1 \leq q \leq p$. From Theorem 1, such a basis can be computed in polynomial time, and moreover – each $\ell_{q,j}$ may be assumed to have magnitude polynomial in $\|u\|$.

Consider the set $\mathbb{T} = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1| = |z_2| = |z_3| = 1$ and for each $1 \leq q \leq p$, $z_1^{\ell_{q,1}} z_2^{\ell_{q,2}} z_3^{\ell_{q,3}} = 1\}$.

Define $h : \mathbb{T} \to \mathbb{R}$ by setting $h(z_1, z_2, z_3) = \sum_{i=1}^{3}(c_i z_i + \overline{c_i z_i})$, so that for every $n \in \mathbb{N}$, $\frac{u_n}{\rho^n} = a + h(\lambda_1^n, \lambda_2^n, \lambda_3^n) + r(n)$. Recall that the set $\{\lambda_1^n, \lambda_2^n, \lambda_3^n : n \in \mathbb{N}\}$ is a dense subset of $\mathbb{T}$. Since h is continuous, it follows that $\inf\{h(\lambda_1^n, \lambda_2^n, \lambda_3^n) : n \in \mathbb{N}\} = \min h|_{\mathbb{T}} = \mu$ for some $\mu \in \mathbb{R}$.

We now claim that μ is an algebraic number, computable in polynomial time, with $\|\mu\| = \|\mathbf{u}\|^{O(1)}$. We can represent μ via the following formula $\tau(y)$:

$$\exists(\zeta_1, \zeta_2, \zeta_3) \in \mathbb{T}[h(\zeta_1, \zeta_2, \zeta_3) = y \wedge \forall(z_1, z_2, z_3) \in \mathbb{T}, y \leq h(z_1, z_2, z_3)].$$

Note that $\tau(y)$ is not a formula in the first-order theory of the reals, as it involves complex numbers. However, we can rewrite it as a sentence in the first-order theory of the reals by representing the real and imaginary parts of each complex quantity and combine them using real arithmetic (see [**?**, ICALP]or details). In addition, the obtained formula $\tau'(y)$ is of size polynomial in $\|\mathbf{u}\|$. By Theorem 6, we can then compute in polynomial time an equivalent quantifier-free formula

$$\chi(x) = \bigvee_{i=1}^{I} \bigwedge_{j=1}^{J_i} h_{i,j} \sim_{i,j} 0.$$

Recall that each $\sim_{i,j}$ is either $>$ or $=$. Now $\chi(x)$ must have a satisfiable disjunct, and since the satisfying assignment to y is unique (namely $y = \mu$), this disjunct must comprise at least one equality predicate. Since Theorem 6 guarantees that the degree and height of each $h_{i,j}$ are bounded by $\|\mathbf{u}\|^{O(1)}$ and $2^{\|\mathbf{u}\|^{O(1)}}$ respectively, we immediately conclude that μ is an algebraic number and with $\|\mu\| = \|\mathbf{u}\|^{O(1)}$.

We now split to cases according to the sign of $a + \mu$.

- If $a + \mu < 0$, then $\mathbf{u}$ is infinitely often negative, and does not diverge.
- If $a + \mu > 0$, then $\mathbf{u}$ diverges, and it remains to show an effective bound. We can compute in polynomial time a rational $\epsilon_0 > 0$ such that $a + \mu > 2\epsilon_0$. By Proposition 12, we can also compute in polynomial time $N_1 \in \mathbb{N}$ and $\epsilon_1 \in (0, 1)$ such that $|r(n)| < (1 - \epsilon_1)^n$ for all $n > N_1$. Taking $N_2 \geq \log_{1-\epsilon_1} \epsilon_0$, we have that for all $n > \max\{N_1, N_2\}$, $|r(n)| < \epsilon_0$, and thus

$$\frac{u_n}{\rho^n} = A(n) + h(\lambda_1, \ldots, \lambda_m) + r(n) \geq A(n) + \mu - \epsilon_0 > 2\epsilon_0 - \epsilon_0 = \epsilon_0$$

The constant ϵ_0 can be thought of as an inverse polynomial for the purpose of applying Lemma 10, so we conclude the effective decidability of divergence in this case.

- It remains to analyze the case where $a + \mu = 0$. To this end, let $\lambda_j = e^{i\theta_j}$ and $c_j = |c_j|e^{i\varphi_j}$ for $1 \leq j \leq 3$. From Equation (14) we have

$$\frac{u_n}{\rho^n} = a + \sum_{j=1}^{3} 2|c_j| \cos(n\theta_j + \varphi_j) + r(n)$$

We further assume that all the c_j are non-zero. Indeed, if this does not hold, we can recast our analysis in lower dimension.

We now claim that h achieves its minimum μ only finitely many times over $\mathbb{T}$. To establish this claim, we proceed according to the cardinality p of the basis $\{\boldsymbol{\ell_1}, \ldots, \boldsymbol{\ell_p}\}$ of L:

(i) We first consider the case in which $p = 1$, and handle the case $p = 0$ immediately afterwards. Let $\boldsymbol{\ell_1} = (\ell_{1,1}, \ell_{1,2}, \ell_{1,3}) \in \mathbb{Z}^3$ be the sole vector spanning L. For $x \in \mathbb{R}$, recall that we denote by $[x]_{2\pi}$ the distance from x to the closest integer multiple of 2π

⋆ TODO↑: ADD THIS NOTATION

Write

$$R = \left\{(x_1, x_2, x_3) \in [0, 2\pi)^3 : [\ell_{1,1}x_1 + \ell_{1,2}x_2 + \ell_{1,3}x_3]_{2\pi} = 0\right\}.$$

Clearly, for any $(x_1, x_2, x_3) \in [0, 2\pi)^3$, we have $(x_1, x_2, x_3) \in R$ iff $(e^{ix_1}, e^{ix_2}, e^{ix_3}) \in \mathbb{T}$. Define $f : \mathbb{R}^3 \to \mathbb{R}$ by setting

$$f(x_1, x_2, x_3) = \sum_{j=1^3} 2|c_j| \cos(x_j + \varphi_j).$$

Clearly, for all $(x_1, x_2, x_3) \in [0, 2\pi)^3$ we have $f(x_1, x_2, x_3) = h(e^{ix_1}, e^{ix_2}, e^{ix_3})$, and therefore the minimal of f over $\mathbb{R}$ are in one-to-one correspondence with those of h over $\mathbb{T}$.

Define $g : \mathbb{R}^3 \to \mathbb{R}$ by setting

$$g(x_1, x_2, x_3) = \ell_{1,1}x_1 + \ell_{1,2}x_2 + \ell_{1,3}x_3.$$

Note that $g(x_1, x_2, x_3)$ cannot be of the form $\ell(x_i - x_j)$, for nonzero $\ell \in \mathbb{Z}$ and $i \neq j$, otherwise $\lambda_i^\ell \lambda_j^{-\ell} = 1$, i.e. λ_i/λ_j would be a root of unity, contradicting the non-degeneracy of $\mathbf{u}$. Likewise, g cannot be of the form $\ell(x_i + x_j)$, otherwise $\lambda_i/\overline{\lambda}_j$ would be a root of unity.

Finally, observe that for $(x_1, x_2, x_3) \in [0, 2\pi)^3$, we have $(x_1, x_2, x_3) \in R$ iff $\ell_{1,1}x_1 + \ell_{1,2}x_2 + \ell_{1,3}x_3 = 2\pi q$ for some $q \in \mathbb{Z}$ with $|q| \leq |\ell_{1,1}| + |\ell_{1,2}| + |\ell_{1,3}|$. For each of these finitely many q, we can invoke Lemma 7 with f, g, and $\psi = 2\pi q$, to conclude that f achieves its minimum μ finitely many times over R, and therefore that h achieves the same minimum finitely many times over $\mathbb{T}$.

The case $p = 0$, i.e. in which there are no non-trivial integer multiplicative relationships among $\lambda_1, \lambda_2, \lambda_3$, is now a special case of the above, where we have $\ell_{1,1} = \ell_{1,2} = \ell_{1,3} = 0$.

(ii)We now turn to the case $p = 2$. We have $\boldsymbol{\ell_1} = (\ell_{1,1}, \ell_{1,2}, \ell_{1,3}) \in \mathbb{Z}^3$ and $\boldsymbol{\ell_2} = (\ell_{2,1}, \ell_{2,2}, \ell_{2,3}) \in \mathbb{Z}^3$ spanning L. Let $\mathbf{x}$ denote the column vector (x_1, x_2, x_3), and write

$$R = \left\{(x_1, x_2, x_3) \in [0, 2\pi)^3 : [\boldsymbol{\ell_1} \cdot \mathbf{x}]_{2\pi} = 0 \text{ and } [\boldsymbol{\ell_2} \cdot \mathbf{x}]_{2\pi} = 0\right\}.$$

Define $f : \mathbb{R}^3 \to \mathbb{R}$ by setting $f(x_1, x_2, x_3) = \sum_{j=1^3} 2|c_j| \cos(x_j + \varphi_j)$. As before, the minima of f over R are in one-to-one correspondence with those of h over $\mathbb{T}$.

For $(x_1, x_2, x_3) \in [0, 2\pi)^3$, we have $[\boldsymbol{\ell_1} \cdot \mathbf{x}]_{2\pi} = 0$ and $[\boldsymbol{\ell_2} \cdot \mathbf{x}]_{2\pi} = 0$ iff there exist $q_1, q_2 \in \mathbb{Z}$, with $|q_1| \leq |\ell_{1,1}| + |\ell_{1,2}| + |\ell_{1,3}|$ and $|q_2| \leq |\ell_{2,1}| + |\ell_{2,2}| + |\ell_{2,3}|$ such that $\boldsymbol{\ell_1} \cdot \mathbf{x} = 2\pi q_1$ and $\boldsymbol{\ell_2} \cdot \mathbf{x} = 2\pi q_2$. For each of theses finitely many $\mathbf{q} = (q_1, q_2)$, we can invoke Lemma 8 with f, $M = \begin{pmatrix} \ell_{1,1} & \ell_{1,2} & \ell_{1,3} \\ \ell_{2,1} & \ell_{2,2} & \ell_{2,3} \end{pmatrix}$, and $\mathbf{q}$, to conclude that f achieves its minimum μ finitely many times over R, and therefore that h achieves the same minimum finitely many times over $\mathbb{T}$.

(iii) Finally, we observe that the case $p = 3$ cannot occur: indeed, a basis for L of dimension 3 would immediately entail that every λ_j is a root of unity.

This concludes the proof of the claim that h achieves its minimum at a finite number of points $Z = \{(\zeta_1, \zeta_2, \zeta_3) \in \mathbb{T} : h(\zeta_1, \zeta_2, \zeta_3) = \mu\}$.

We concentrate on the set Z_1 of first coordinates of Z. Write

$$\tau_1(x) = \exists z_1(\mathrm{Re}(z_1) = x \wedge z_1 \in Z_1)$$
$$\tau_2(y) = \exists z_1(\mathrm{Im}(z_1) = y \wedge z_1 \in Z_1)$$

Similarly to our earlier constructions $\tau_1(x)$ is equivalent to a formula $t'_1(x)$ in the in the first-order theory of the reals, over a bounded number of real variables, with $\|\tau'_1(x)\| = \|\mathbf{u}\|^{O(1)}$. Thanks to Theorem 6, we then obtain an equivalent quantifier-free formula

$$\chi_1(x) = \bigvee_{i=1}^{I} \bigwedge_{j=1}^{J_i} h_{i,j} \sim_{i,j} 0.$$

Note that since there can only be finitely many $\hat{x} \in \mathbb{R}$ such that $\chi_1(\hat{x})$ holds, each disjunct of $\chi_1(\hat{x})$ must comprise at least one equality predicate, or can otherwise be entirely discarded as having no solution. A similar exercise can be carried out with $\tau_2(x)$. The bounds on the degree and height of each $h_{i,j}$ in $\chi_1(x)$ and $\chi_2(y)$ then enables us to conclude that any $\zeta_1 = \hat{x} + i\hat{y} \in Z_1$ is algebraic, and moreover satisfies $\|\zeta_1\| = \|\mathbf{u}\|^{O(1)}$. In addition, bounds on I and J_i guarantee that the cardinality of Z_1 is at most polynomial in $\|\mathbf{u}\|$.

Since λ_1 is not a root of unity, for each $\zeta_1 \in Z_1$ there is at most one value of n such that $\lambda_1^n = \zeta_1$. Theorem 1 then entails that this value (if it exists) is at most $M = \|\mathbf{u}\|^{O(1)}$, which we can take to be uniform across all $\zeta_1 \in Z_1$. We can now invoke Corollary 4 to conclude that, for $n > M$, and for all $\zeta_1 \in Z_1$, we have

$$|\lambda_1^n - \zeta_1| > \frac{1}{n^{\|\mathbf{u}\|^D}} \tag{15}$$

where $D \in \mathbb{N}$ is some absolute constant.

Let $b > 0$ be minimal such that the set

$$\left\{ z_1 \in \mathbb{C} : |z_1| = 1 \text{ and, for all } \zeta_1 \in Z_1, |z_1 - \zeta_1| \geq \frac{1}{b} \right\}$$

is non empty. Thanks to our bounds on the cardinality of Z_1, we can use the first-order theory of the reals, together with Theorem 6, to conclude that b is algebraic and $\|b\| = \|\mathbf{u}\|^{O(1)}$.

Define the function $g : [b, \infty) \to \mathbb{R}$ as follows:

$$g(x) = \min\{h(z_1, z_2, z_3) - \mu : (z_1, z_2, z_3) \in \mathbb{T} \text{ and for all } \zeta_1 \in Z_1, |z_1 - \zeta_1| \geq \frac{1}{x}\}.$$

It is clear that g is continuous and $g(x) > 0$ for all $x \in [b, \infty)$. Moreover, g can be translated in polynomial time into a function in the first-order theory of the reals over a bounded number of variables. It follows from Proposition 2.6.2 of [**?**, **?**] (invoked with the function $1/g$) that there is a polynomial $P \in \mathbb{Z}[x]$ such that, for all $x \in [b, \infty)$,

$$g(x) \geq \frac{1}{P(x)} \tag{16}$$

Moreover, and examination of the proof of [**?**, **?**] reveals that P is obtained through a process which hinges on quantifier elimination. By Theorem 6, we are therefore able to conclude that $\|P\| = \|\mathbf{u}\|^{O(1)}$, a fact which relies among others on our upper bounds for $\|b\|$.

By Proposition 12 we can find $\epsilon \in (0, 1)$ and $N = 2^{\|\mathbf{u}\|^{O(1)}}$ such that for all $n > N$, we have $|r(n)| < (1 - \epsilon)^n$, and moreover $1/\epsilon = 2^{\|\mathbf{u}\|^{O(1)}}$. In addition, by Proposition 11, there is $N' = 2^{\|\mathbf{u}\|^{O(1)}}$ such that for every $n \geq N'$

$$\frac{1}{2P(n^{\|\mathbf{u}\|^D})} > (1 - \epsilon)^n. \tag{17}$$

Combining Equations (14)–(17), we get

$$\begin{aligned}
\frac{u_n}{\rho^n} &= a + h(\lambda_1^n, \lambda_2^n, \lambda_3^n) + r(n) \\
&\geq -\mu + h(\lambda_1^n, \lambda_2^n, \lambda_3^n) - (1-\epsilon)^n \\
&\geq g(n^{\|\mathbf{u}\|^D}) - (1-\epsilon)^n \\
&\geq \frac{1}{P(n^{\|\mathbf{u}\|^D})} - (1-\epsilon)^n \\
&= \frac{1}{2P(n^{\|\mathbf{u}\|^D})} + \frac{1}{2P(n^{\|\mathbf{u}\|^D})} - (1-\epsilon)^n \\
&\geq \frac{1}{2P(n^{\|\mathbf{u}\|^D})}
\end{aligned}$$

provided $n > \max\{M, N, N'\}$. We thus have that $\frac{u_n}{\rho^n}$ is eventually lower bounded by an inverse polynomial, and by Lemma 10 we conclude that $\mathbf{u}$ diverges and the bounds are effectively computable.

It remains to show how to handle non-homogeneous LRS of order at most 8. Consider a non-homogeneous LRS $\langle v_n \rangle$ of order at most 8, and let $u_n = \text{HOM}(v_n)$. Observe that by Property 14, u_n might not be a simple LRS. However, all its eigenvalues have multiplicity 1, apart from, possibly, the eigenvalue 1.

Consider the spectral radius ρ of u_n. If $\rho > 1$, then by property 14 the exponential polynomial of $\frac{u_n}{\rho^n}$ is of the same form as that in Equation (9), in the sense that $r(n)$ is still exponentially decaying. Thus, we can proceed with the analysis above without change. If $\rho = 1$, things become slightly more involved. Consider the exponential polynomial

$$u_n = A(n) + \sum_{i=1}^{m} (c_i \lambda_i^n + \overline{c_i}\overline{\lambda_i^n}) + r(n) \tag{18}$$

where $A(n)$ is either a constant or a polynomial, $c_i \in \mathbb{C} \setminus \mathbb{R}$ for every $1 \leq i \leq m$, $|r(n)|$ is exponentially decaying, and $0 \leq m \leq 4$. Observe that if $A(n)$ is a constant, then $|u_n|$ is bounded, and so does not diverge. In particular, if $m = 4$ then it has to be the case that $A(n)$ is constant. Thus, it suffices to consider the case where $m \leq 3$ and $A(n)$ is a polynomial.

In this case, similarly to Case 1 in the proof of Theorem 23, we have that $\mathbf{u}$ diverges iff the leading coefficient of $A(n)$ is positive, and in this case the bounds are effectively computable.

This completes the proof of Theorem 24.

4.4 Lower Bounds

▶ **Theorem 25.** *Divergence is hard (w.r.t. number theoretic problems) for homogeneous LRS of order* 6*.*

Proof. We show a reduction from the ultimate-positivity problem for non-degenerate LRS of order 6, shown to be hard in [**?**]. The key ingredient in the reduction is Theorem 5.

Consider a non-degenerate homogeneous LRS $\langle u_n \rangle$ of order 6 with spectral radius ρ, and let $\mu = \max\left\{2, \frac{2}{\rho}\right\}$, then the sequence $v_n = \mu^n u_n$ is a non-degenerate homogeneous LRS of order 6 with spectral radius $\mu\rho \geq 2$. By Theorem 5, taking $\epsilon = \frac{1}{2}$, it follows that there exists $N \in \mathbb{N}$ such $|v_n| \geq 2^{n/2}$ for every $n > N$. It immediately follows that v_n is ultimately positive iff $v_n \to \infty$. Clearly, however, v_n and u_n have the same sign, and therefore u_n is ultimately positive iff $v_n \to \infty$, and we are done. ◀

References