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A note on a theorem of Barrington, Straubing and Thérien

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Abstract

We show that the result of Barrington, Straubing and Thérien (1989) provides, as a direct corollary, an exponential lower bound for the size of depth-two MOD₆ circuits computing the AND function. This problem was solved, in a more general way, by Krause and Waack (1991). We point out that all known lower bounds rely on the special form of the MOD₆ gate occurring at the bottom of the circuits, so that in fact, proving a lower bound for "general" MOD₆ circuits of depth two is still an open question.

Keywords: Computational complexity; Lower bounds; Polynomials

Finding lower bounds for the size of constant depth circuits is a big challenge in complexity theory. The program over monoid model was developed in the hope of obtaining lower bounds using algebraic properties of monoids. The languages recognized by polynomial length programs over monoids were shown to be exactly those in the classes AC⁰, CC⁰, ACC⁰ and NC¹, depending on the algebraic properties of the underlying monoid [2,6]. In fact a super-linear lower bound for the length of such programs can be proved [4], but this does not give a super-linear lower bound for circuit size.

However, in their remarkable paper "Non-uniform automata over groups", Barrington, Straubing and Thérien obtain some exponential lower bounds for programs over some special kinds of groups. Roughly speaking, these lower bounds can be stated as lower

bounds for depth-two circuits with MOD_m gates at the inputs and a MOD_{p^k} gate at the output, where m is an integer, p is a prime number and k is an integer (Theorem 1 gives the full result).

Following the work of Razborov [9] (see also [1,10]), Barrington, Straubing and Thérien used multivariate polynomials over a finite field of characteristic p, to represent such circuits. When the output gate is a MOD_m gate, for composite m, it seems that finite fields are not powerful enough to represent these circuits, and finding a lower bound for the size of depth-two MOD₆ circuits was stated as an open question [11]. Krause and Waack used communication complexity techniques to solve this problem [8]. These techniques apply in particular to depth-two circuits having symmetric gates at the inputs and a MOD_m gate at the output. For such circuits Krause and Waack obtain an exponential lower bound for the SEQ_{2n} function whose value is 1 if and only if $x_i = x_{n+i}$ for all $i, 1 \le i \le n$.

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There are several possible definitions for "modulo" gates. Consider a subset S of $\{0, \ldots, m-1\}$. A general MOD_m gate is defined as follows:

$$MOD_m^S(x_1,\ldots,x_n) = \begin{cases} 1 & \text{if } \left(\sum_{i=1}^n x_i\right) \mod m \in S, \\ 0 & \text{otherwise.} \end{cases}$$

The MOD_m gate most often encountered is the special case in which $S = \{1, ..., m-1\}$. We show that the exponential lower bound for depth-two circuits of such MOD_m gates, obtained by Krause and Waack, is a direct consequence of the result of Barrington et al.

We denote by C_n a Boolean circuit with n inputs, this circuit computing a function from $\{0,1\}^n$ to $\{0,1\}$. The *size* of C_n is the number of gates in the circuit. The theorem of Barrington, Straubing and Thérien can be stated as follows in the circuit model framework:

Theorem 1 (see [5]). Let m_1, \ldots, m_u be a constant number of integers. Let p be a prime number and k_1, \ldots, k_i be a constant number of integers. Let B_n be a constant depth circuit, with general MOD_{m_i} gates at the input level and $MOD_{p^{k_i}}$ gates elsewhere. If B_n computes the AND function then its size must be greater than $2^{\alpha n}$ where α depends only on the m_i , p and k_i .

Let m be an integer. We say that a Boolean circuit C_n of size t(n) is a depth-two MOD_m circuit if it has t(n) general MOD_m in the first level and a MOD_m in the second (i.e. output) level. We shall prove the following theorem using the result of Barrington et al.

Theorem 2. Let m be an integer. Any depth-two MOD_m circuit with n inputs that computes the AND function has size $2^{\alpha n}$ where α depends only on m.

Proof. Let C_n be a depth-two MOD_m circuit of size t(n) that computes the AND function. Write $m = p_1^{k_1} \cdots p_l^{k_l}$ where the p_i are distinct prime numbers. Denote by M_i , $1 \le i \le t(n)$, the input gates. Because of the special definition of the MOD_m gate we have:

$$MOD_{m}(M_{1},...,M_{t(n)}) = OR(MOD_{p_{1}^{k_{1}}}(M_{1},...,M_{t(n)}), ..., MOD_{p_{t}^{k_{t}}}(M_{1},...,M_{t(n)})).$$

Now the simple observation which we require is that if the AND function is to be computed by a circuit of the form $OR(g_1, \ldots, g_l)$ then one of the sub-circuits g_i must by itself compute the AND function. So at least one of the

$$\mathrm{MOD}_{p_h^{k_h}}(M_1,\ldots,M_{t(n)})$$

computes the AND function. A lower bound for such a circuit is therefore obtained from Theorem 1. \Box

Note that if, as Smolensky [11], we take the MOD_m function to be the special case in which $S = \{0\}$, it is possible to derive a lower bound for the OR function using a dual argument. We stress that for general MOD_m gates neither this technique nor the technique in [8] works. In particular obtaining lower bounds for depth-two $MOD_6^{\{0,1\}}$ circuits is still an open question (papers related with similar questions are [3,7,12]).

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