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ORDINAL RECURSION, AND A REFINEMENT OF THE EXTENDED GRZEGORCZYK HIERARCHY

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It is well known that iteration of any number-theoretic function f, which grows at least exponentially, produces a new function f' such that f is elementary-recursive in f' (in the Csillag-Kalmar sense), but not conversely (since f' majorizes every function elementary-recursive in f). This device was first used by Grzegorczyk [3] in the construction of a properly expanding hierarchy $\{\mathscr{E}^n : n = 0, 1, 2, \cdots\}$ which provided a classification of the primitive recursive functions. More recently it was shown in [7] how, by iterating at successor stages and diagonalizing over fundamental sequences at limit stages, the Grzegorczyk hierarchy can be extended through Cantor's second number-class. A problem which immediately arises is that of classifying all recursive functions, and an answer to this problem is to be found in the general results of Feferman [1]. These results show that although hierarchies of various types (including the above extensions of Grzegorczyk's hierarchy) can be produced, which range over initial segments of the constructive ordinals and which do provide complete classifications of the recursive functions, these cannot be regarded as classifications "from below", since the method of assigning fundamental sequences at limit stages must be highly noneffective. We therefore adopt the more modest aim here (as in [7], [12], [14]) of characterising certain well-known (effectively generated) subclasses of the recursive functions, by means of hierarchies generated in a natural manner, "from below".

In [7] particular attention was paid to a hierarchy $\{\mathscr{F}_\beta\}_{\beta < \epsilon_0}$ whose construction is based on an obvious choice of fundamental sequences to the limit ordinals below ϵ_0. It is this hierarchy with which we are chiefly concerned here, although the results of this paper can be extended to much larger constructive initial segments of the ordinals (e.g., Γ_0, using the fundamental sequences given by Feferman [2]; see [14] for a preliminary discussion of the resulting hierarchy). Basically all that is required is a "natural" choice of fundamental sequences $\lambda i \cdot \sigma_i$ to the limits σ under consideration, together with a primitive recursive set of unique notations for the ordinals, on which the "predecessor" functions $\phi(\sigma, i) = \sigma_i$ and $\psi(\beta + 1) = \beta$ are primitive recursively representable.

It was proved in [12] that $\bigcup_{\beta < \epsilon_0} \mathscr{F}_\beta$ coincides with Kreisel's class of ordinal recursive functions of finite order [6]. We refine this result here by showing (in §3 below) that for every ordinal α such that $0 < \alpha < \epsilon_0$,

$$\bigcup_{\beta < \alpha \cdot \omega} \mathscr{F}_\beta = U(\omega^\alpha) = N(\omega \cdot \alpha),$$

where $U(\omega^\alpha)$ (resp. $N(\omega \cdot \alpha)$) is the class of all functions which can be defined from

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primitive recursive functions by means of explicit definitions and *unnested* (resp. *nested*) recursions over standard well-orderings of the natural numbers of order-type ω^α (resp., $\omega \cdot \alpha$). Thus in particular, with $\alpha = \omega^{k-1}$, we get the classification of Péter's k-recursive functions [8] previously obtained in [7] and [9].

The proof of the above result makes use of another hierarchy $\{\mathscr{H}_\beta\}_{\beta < \epsilon_0}$ which is shown in §2 to provide a convenient refinement of the classes $\mathscr{F}_\alpha$ as follows: $\bigcup_{\beta < \omega^{\alpha+1}} \mathscr{H}_\beta = \mathscr{F}_\alpha$ for each $\alpha > 1$. This refinement is based on functions h_β which grow more slowly than the functions F_β from which the classes $\mathscr{F}_\beta$ are defined. The idea behind the definition of the h_β's goes back to Hardy [4] who used it in order to exhibit a set of reals with cardinality $\aleph_1$. In §4 the definitions of h_β and F_β are used to obtain simple normal forms, involving *fixed* primitive recursive predecessor functions, for unnested and nested ordinal recursion.

I am indebted to the referee, for pointing out Lemma 6 below and suggesting several other improvements which have led to the present strengthened and more unified version of Theorem 5, and for considerably strengthening our original Corollary to Theorem 5.

§1. Preliminaries. We are concerned here with totally defined number-theoretic functions, and we use the letters f, g, h, p, q, r, s, t to denote them (capital letters will also sometimes be used to represent particular functions). All other lower case italics denote natural numbers, and we use boldface italics to denote arbitrary k-tuples of natural numbers. With the exception of the abstraction operator λ and the least-number operator μ, the lower case Greek letters denote ordinals $\leq \epsilon_0$.

For each 1-ary function f, f^n denotes the nth iterate of f, defined by $f^0(x) = x$, $f^{m+1}(x) = f(f^m(x))$.

A k-ary function g is said to be *bounded* by a 1-ary function f if there is a number c such that $g(\boldsymbol{x}) \leq f(\max(\boldsymbol{x}))$ whenever $\max(\boldsymbol{x}) \geq c$. If the inequality is *strict* whenever $\max(\boldsymbol{x}) \geq c$, then we say that g is *majorized* by f.

For each limit ordinal $\sigma < \epsilon_0$ we define a fixed fundamental sequence $\lambda n \cdot \{\sigma\}(n)$ by induction as follows:

(i) if σ is of the form $\omega^{\alpha+1} \cdot (\beta + 1)$ then $\{\sigma\}(n) = \omega^{\alpha+1} \cdot \beta + \omega^\alpha \cdot n$.

(ii) if σ is of the form $\omega^\alpha \cdot (\beta + 1)$ where α is a limit ordinal $< \sigma$ then $\{\sigma\}(n) = \omega^\alpha \cdot \beta + \omega^{\{\alpha\}(n)}$.

We also define a fundamental sequence for ϵ_0 by $\{\epsilon_0\}(n) = \omega_n$, where $\omega_0 = 1$ and $\omega_{m+1} = \omega^{\omega_m}$. (Note that these fundamental sequences are slightly simpler than those used in [7], [12]. However the results of [7], [12] go through in just the same way for the fundamental sequences defined above. See [13].)

The functions F_α can now be defined by the recursion:

$$\begin{aligned} F_0(x) &= x + 1, \\ F_1(x) &= (x + 1)^2, \\ F_{\beta+1}(x) &= F_\beta^{x+1}(x), && \text{if } \beta > 0, \\ F_\sigma(x) &= F_{\{\sigma\}(x)}(x), && \text{if } \sigma \text{ is a limit ordinal.} \end{aligned}$$

(For each α, F_α is just the function which was denoted by F_α^0 in [7], [12].)

Now, for each α, let $\mathscr{F}_\alpha$ be the smallest class of functions containing F_α, the zero, addition and projection functions, and closed under the operations of substitution and limited primitive recursion.[1,2]

Collecting together various results contained in [7], [12], [13] we have

LEMMA 1. *For every* $\alpha \leq \epsilon_0$,

(i) F_α *is strictly increasing.*

(ii) *If* α *is a limit ordinal and* $i < j \leq x$ *then* $F_{\{\alpha\}(i)}(x) \leq F_{\{\alpha\}(j)}(x)$.

(iii) *Hence, if* $\beta < \alpha$ *then* F_β *is majorized by* F_α.

(iv) *If* $\beta < \alpha$ *then* F_β *is elementary-recursive in* F_α.

THEOREM 1. *For every* $\alpha \leq \epsilon_0$,

(i) *If* $\beta < \alpha$ *then* $\mathscr{F}_\beta \subset \mathscr{F}_\alpha$ *and the inclusion is strict since every function in* $\mathscr{F}_\beta$ *is majorized by* F_α.

(ii) *If* $1 < \alpha$ *then* $\mathscr{F}_\alpha$ *is just the class of functions elementary-recursive in* F_α.

(iii) *If* $\alpha = n$ *(n a natural number) then* $\mathscr{F}_\alpha = \mathscr{E}^{n+1}$.

Hardy [4] associated a sequence of natural numbers with each countable ordinal by starting with the identity sequence, deleting the first number of any sequence to form its successor, and diagonalizing at limit stages. We use this method, allied to the fundamental sequences $\lambda n \cdot \{\sigma\}(n)$, in order to define a function h_α, for each $\alpha \leq \epsilon_0$, as follows:

$$\begin{aligned} h_0(x) &= x, \\ h_{\beta+1}(x) &= h_\beta(x+1), \\ h_\sigma(x) &= h_{\{\sigma\}(x)}(x) \quad \text{if } \sigma \text{ is a limit ordinal.} \end{aligned}$$

LEMMA 2. *For every* $\alpha \leq \epsilon_0$,

(i) h_α *is strictly increasing.*

(ii) *If* α *is a limit ordinal and* $i < j \leq x$ *then* $h_{\{\alpha\}(i)}(x) \leq h_{\{\alpha\}(j)}(x)$.

(iii) *Hence, if* $\beta < \alpha$ *then* h_β *is majorized by* h_α.

PROOF. The proof uses the same methods as those used in proving Lemma 1.

LEMMA 3. *If* γ *is of the form* $\omega^\alpha \cdot \beta$ *where* $\alpha, \beta < \epsilon_0$, *then for every* $\delta \leq \omega^\alpha$ *and all* x, $h_{\gamma+\delta}(x) = h_\gamma(h_\delta(x))$.

PROOF. The proof is by induction on δ, noting that if δ is a limit ordinal then $\{\gamma + \delta\}(x) = \gamma + \{\delta\}(x)$.

It follows from Lemma 3 and the definition of the fundamental sequences $\lambda n \cdot \{\sigma\}(n)$, that the functions h_{ω^α}, $\alpha < \epsilon_0$, can be defined by the recursion

$$\begin{aligned} h_{\omega^0}(x) &= x + 1, \\ h_{\omega^{\beta+1}}(x) &= h^x_{\omega^\beta}(x), \\ h_{\omega^\sigma}(x) &= h_{\omega^{\{\sigma\}(x)}}(x) \quad \text{if } \sigma \text{ is a limit ordinal;} \end{aligned}$$

which bears a striking resemblance to the definition of the functions F_α. It is this similarity which we exploit further, so as to obtain a refinement of the hierarchy $\{\mathscr{F}_\alpha : \alpha < \epsilon_0\}$, based on the functions h_α.

LEMMA 4. *For each* $\alpha \leq \epsilon_0$ *and all* x, $h_{\omega^\alpha}(x) \leq F_\alpha(x)$ *and* $F_\alpha(x) \leq h_{\omega^\alpha}(9^x)$.

PROOF. The proof is by induction on α.

[1] In this paper various classes of functions are generated by means of schemes which are non-effective as they stand. In all cases, however, these schemes can easily be replaced by equivalent versions which are effective, but often more complicated.

[2] In [7], [12], these classes were denoted by a Gothic "E", instead of the script $\mathscr{F}$ used here.

We next show that the rates of growth of the functions F_α, h_α reflect their computational complexity. (Robbin [9] calls such functions "honest".)

LEMMA 5. *For each* $\alpha \leq \epsilon_0$ *there are elementary-recursive functions* p_α, q_α, *and constants* b_α, c_α, *such that*

(i) $F_\alpha(x) = p_\alpha(x, s(x))$ *whenever* $s(x) \geq F_\alpha(x + b_\alpha)$.

(ii) $h_\alpha(x) = q_\alpha(x, t(x))$ *whenever* $t(x) \geq h_\alpha(x + c_\alpha)$.

PROOF. (i) It was proved in [12], [13] that there is a number b_α and a Turing machine Z_α which computes F_α in such a way that, for every x, the number of tape-squares used by Z_α in the computation of $F_\alpha(x)$ is no more than $F_\alpha(x + b_\alpha)$. Now there is an elementary-recursive function I_α such that if $s(x) \geq F_\alpha(x + b_\alpha)$, $I_\alpha(x, s(x))$ is an upper bound on the total number of possible instantaneous descriptions occurring in the computation of $F_\alpha(x)$ by Z_α. (No instantaneous description occurs more than once, for otherwise Z_α would go into a loop and never stop.) Also there is an elementary-recursive function V_α such that $V_\alpha(x, i)$ is the output produced by Z_α after i instantaneous descriptions in the computation starting with input x. Thus, defining $p_\alpha(x, y) = V_\alpha(x, I_\alpha(x, y))$, we have $F_\alpha(x) = p_\alpha(x, s(x))$.

(ii) Since the definition of the function h_{ω^β} is of basically the same form as that of the function F_α, the proof of (i) above can be easily adapted in order to provide a proof of (ii) in the case where α is of the form ω^β, $\beta \leq \epsilon_0$.

It is now easy to prove by induction, that (ii) holds for all $\alpha < \epsilon_0$. Obviously the result holds for $\alpha = 0$, and if $\alpha > 0$ there are ordinals $\alpha_0 < \alpha$, $\alpha_1 \leq \alpha$ such that α_1 is of the form ω^β, α_0 is either 0 or a polynomial of the form $\omega^{\beta_1} + \omega^{\beta_2} + \cdots + \omega^{\beta_k}$ where $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_k \geq \beta$, and $\alpha = \alpha_0 + \alpha_1$. Thus by the induction hypothesis and the above comment, the result (ii) holds for α_0 and α_1. Now put $c_\alpha = c_{\alpha_0} + c_{\alpha_1}$ and define $q_\alpha(x, y) = q_{\alpha_0}(q_{\alpha_1}(x, y), y)$. Then if $t(x) \geq h_\alpha(x + c_\alpha)$ we have $t(x) \geq h_{\alpha_0}(h_{\alpha_1}(x) + c_{\alpha_0})$ and $t(x) \geq h_{\alpha_1}(x + c_{\alpha_1})$ since by Lemmas 2 and 3,

$$h_\alpha(x + c_\alpha) = h_{\alpha_0}(h_{\alpha_1}(x + c_{\alpha_0} + c_{\alpha_1})) \geq h_{\alpha_0}(h_{\alpha_1}(x + c_{\alpha_1}) + c_{\alpha_0}).$$

So, again using Lemma 3, we have

$$\begin{aligned} h_\alpha(x) &= h_{\alpha_0}(h_{\alpha_1}(x)) \\ &= q_{\alpha_0}(h_{\alpha_1}(x), t(x)) \\ &= q_{\alpha_0}(q_{\alpha_1}(x, t(x)), t(x)) \\ &= q_\alpha(x, t(x)), \quad \text{if } t(x) \geq h_\alpha(x + c_\alpha). \end{aligned}$$

This completes the proof of (ii).

THEOREM 2. *For each* $\alpha \leq \epsilon_0$, F_α *and* h_{ω^α} *are of the same elementary-recursive degree* (*i.e. each is elementary-recursive in the other*).

PROOF. The proof is immediate from Lemmas 4 and 5.

Now let ξ be any ordinal such that $2 < \xi < \epsilon_0$, and suppose there is an elementary-recursive function r such that F_ξ is bounded by $\lambda x \cdot h_\xi(r(x))$. Then by Lemma 5, F_ξ is elementary-recursive in h_ξ. But there are ordinals $\xi_k \leq \xi_{k-1} \leq \cdots \leq \xi_1 < \xi$ such that $\xi = \omega^{\xi_1} + \cdots + \omega^{\xi_{k-1}} + \omega^{\xi_k}$, so by Lemma 3, h_ξ is built up by composition of the functions $h_{\omega^{\xi_i}}$, $1 \leq i \leq k$. By Theorem 2, each $h_{\omega^{\xi_i}}$ is elementary-recursive in F_{ξ_i}. So F_ξ is elementary-recursive in $F_{\xi_1}, \cdots, F_{\xi_k}$ which contradicts Theorem 1. Thus, by Lemmas 4 and 5, we have the following recursion-theoretic characterisations of ϵ_0:

$\epsilon_0 = \mu_{\xi<2}$ (there is an elementary-recursive function r such that F_ξ is bounded by $\lambda x\cdot h_\xi(r(x))$),
$= \mu_{\xi>2}$ (F_ξ is elementary-recursive in h_ξ).

A similar characterisation of the first strongly-critical ordinal Γ_0 is obtained in [14], by comparing the rates of growth of the functions F_ξ with functions G_ξ defined by the recursion:

$$G_0(x) = 0, \qquad G_{\alpha+1}(x) = 1 + G_\alpha(x), \qquad G_\sigma(x) = G_{\sigma_x}(x) \quad \text{for limits } \sigma,$$

where for each limit ordinal $\sigma \le \Gamma_0$, $\lambda x\cdot\sigma_x$ is the fundamental sequence defined by Feferman in [2].

§2. Refining the classes $\mathscr{F}_\alpha$. For each $\alpha < \epsilon_0$, let $\mathscr{H}_\alpha$ be the smallest class of functions containing h_α, the zero, successor and projection functions, and closed under limited primitive recursion and the following scheme of limited substitution:[1]

$$f(x) = g_0(g_1(x), \cdots, g_m(x)),$$
$$f \text{ is bounded by } g_{m+1}.$$

Clearly $\mathscr{H}_0 = \mathscr{H}_1$ and if $\alpha \ne 0$ then every function in $\mathscr{H}_\alpha$ is bounded by h_α.

THEOREM 3. *If* $0 < \alpha < \beta < \epsilon_0$ *then* $\mathscr{H}_\alpha \subsetneq \mathscr{H}_\beta$.

PROOF. Since h_α is majorized by h_β whenever $\alpha < \beta$ we immediately have $\mathscr{H}_\alpha \ne \mathscr{H}_\beta$. Thus it is sufficient to show that $h_\alpha \in \mathscr{H}_\beta$ if $\alpha < \beta$.

First suppose α is of the form $\omega^2\cdot m + \omega\cdot n + k$. Then by Lemma 3, $h_\alpha(x) = h^m_{\omega^2}(h^n_\omega(h^k_1(x)))$. But $h_1(x) = x + 1$, $h_\omega(x) = 2x$ and $h_{\omega^2}(x) = h_{\omega\cdot x}(x) = 2^x\cdot x$. So h_α is definable, from the zero and successor functions, by a sequence of substitutions and primitive recursions. But if $\alpha < \beta$, each application of these schemes can be bounded by h_β and hence $h_\alpha \in \mathscr{H}_\beta$.

Now suppose $\omega^3 \le \alpha < \beta < \alpha + \omega$, so $\beta = \alpha + k$ for some positive integer k. Thus if $i < k$ and $h_{\beta-i} \in \mathscr{H}_\beta$, $h_{\beta-(i+1)}$ can be defined from $h_{\beta-i}$ and the constant $a = h_{\beta-(i+1)}(0)$ by the limited primitive recursion:

$$h_{\beta-(i+1)}(0) = a, \qquad h_{\beta-(i+1)}(x+1) = h_{\beta-i}(x), \qquad h_{\beta-(i+1)}(x) < h_{\beta-i}(x),$$

and so $h_{\beta-(i+1)} \in \mathscr{H}_\beta$. Hence by induction on $i \le k$, $h_\alpha \in \mathscr{H}_\beta$.

Finally suppose $\omega^3 \le \alpha < \alpha + \omega \le \beta$. By Lemma 5 there is an elementary-recursive function q_α and a constant c_α such that $h_\alpha(x) = q_\alpha(x, t(x))$ whenever $t(x) \ge h_\alpha(x + c_\alpha)$. But for all but finitely-many x, $h_\beta(x) \ge h_{\alpha+c_\alpha}(x) = h_\alpha(x + c_\alpha)$ so $h_\alpha(x) = q_\alpha(x, h_\beta(x))$. Hence there is an elementary-recursive function q'_α such that $h_\alpha(x) = q'_\alpha(x, h_\beta(x))$ for all x. Now since $\beta > \omega^3$, h_β majorizes

$$\lambda x\cdot 2^{2^{\cdot^{\cdot^{2^x}}}}\Big\}x \text{ times},$$

so h_β majorizes every elementary-recursive function. It follows that every elementary-recursive function, and in particular q'_α, belongs to $\mathscr{H}_\beta$. Therefore h_α is definable from functions in $\mathscr{H}_\beta$ by limited substitution, and so $h_\alpha \in \mathscr{H}_\beta$.

This completes the proof.

THEOREM 4. *If* $1 < \alpha < \epsilon_0$ *then* $\bigcup_{\beta<\omega^{\alpha+1}}\mathscr{H}_\beta$ *is the class of all functions elementary-recursive in* h_{ω^α}.

PROOF. We first prove by induction on $\beta < \omega^{\alpha+1}$, that h_β is elementary-recursive in h_{ω^α}. Obviously h_0 is elementary-recursive. Suppose $\beta \ne 0$. Then $\beta = \beta_0 + \beta_1$

where β_0 is of the form $\omega^\gamma \cdot \delta < \beta$, and $\beta_1 \leq \omega^\gamma \leq \omega^\alpha$. By the induction hypothesis and Lemma 5, both h_{β_0} and h_{β_1} are elementary-recursive in h_{ω^α}. So h_β is elementary-recursive in h_{ω^α}, since $h_\beta(x) = h_{\beta_0}(h_{\beta_1}(x))$.

Now for any $\beta < \omega^{\alpha+1}$, it is clear that all the functions in $\mathscr{H}_\beta$ are elementary-recursive in h_β, and hence elementary-recursive in h_{ω^α}.

Conversely, it is well known that a function is elementary-recursive in h_{ω^α} iff it can be defined from 0, $+$, 2^x, h_{ω^α} and the projection functions, by a sequence of substitutions and limited primitive recursions. But since $\alpha \geq 2$ and $h_{\omega^2}(x) = 2^x \cdot x$ it follows that $+$ and 2^x are bounded by h_{ω^α} and hence definable in $\mathscr{H}_{\omega^\alpha}$. Furthermore, every class $\mathscr{H}_\beta$ contains the zero and projection functions and is closed under limited primitive recursion. Thus it is sufficient to show that $\bigcup_{\beta<\omega^{\alpha+1}} \mathscr{H}_\beta$ is closed under substitution: $f(\mathbf{x}) = g_0(g_1(\mathbf{x}), \cdots, g_m(\mathbf{x}))$. Suppose $g_0, g_1, \cdots, g_m \in \bigcup_{\beta<\omega^{\alpha+1}} \mathscr{H}_\beta$. Then by Theorem 3 there is an ordinal γ, of the form $\omega^\alpha \cdot k$ for some integer k, such that $g_0, g_1, \cdots, g_m \in \mathscr{H}_\gamma$. Hence $g_0, g_1, \cdots, g_m$ are all bounded by h_γ and, in particular, there is a number m such that

$$g_0(y_1 \cdots y_m) \leq h_\gamma(\max(y_1 \cdots y_m) + m)$$

for all $y_1, \cdots, y_m$. So for all but finitely-many $\mathbf{x}$,

$$\begin{aligned} f(\mathbf{x}) &= g_0(g_1(\mathbf{x}), \cdots, g_m(\mathbf{x})) \\ &\leq h_\gamma(\max(g_1(\mathbf{x}), \cdots, g_m(\mathbf{x})) + m) \\ &\leq h_\gamma(h_\gamma(\max(\mathbf{x})) + m) \\ &\leq h_\gamma(h_\gamma(\max(\mathbf{x}) + m)) \\ &= h_{\gamma\cdot 2}(\max(\mathbf{x}) + m) \quad \text{by Lemma 3,} \\ &= h_{\gamma\cdot 2+m}(\max(\mathbf{x})) \quad \text{by definition.} \end{aligned}$$

It follows that $f \in \mathscr{H}_{\gamma\cdot 2+m}$, and so $\bigcup_{\beta<\omega^{\alpha+1}} \mathscr{H}_\beta$ is closed under full substitution, and contains all the functions elementary-recursive in h_{ω^α}. This completes the proof.

COROLLARY. *If* $1 < \alpha < \epsilon_0$ *then* $\bigcup_{\beta<\omega^{\alpha+1}} \mathscr{H}_\beta = \mathscr{F}_\alpha$.

If $i \leq 1$ *it can easily be shown that* $\bigcup_{\beta<\omega^{i+1}} \mathscr{H}_\beta = \mathscr{E}^i$.

§3. Ordinal Recursion. For each positive integer n define an elementary-recursive well-ordering $<_n$ of the natural numbers by

$$\begin{aligned} x <_1 y &\Leftrightarrow x < y, \\ x <_{n+1} y &\Leftrightarrow (Ei) \leq_y ((x+1)_i < (y+1)_i \\ &\qquad \& \ (j) \leq_{\max(x,y)} (i <_n j \rightarrow (x+1)_j = (y+1)_j)), \end{aligned}$$

where $(a)_k$ is the exponent of the $(k+1)$th prime p_k in the unique prime factorization of a. Then if $i_1, \cdots, i_k$ represent the ordinals $\alpha_1 > \alpha_2 > \cdots > \alpha_k$ in the well-ordering $<_n$, $(p_{i_1}^{a_1} p_{i_2}^{a_2} \cdots p_{i_k}^{a_k}) - 1$ represents the ordinal $\omega^{\alpha_1} \cdot a_1 + \omega^{\alpha_2} \cdot a_2 + \cdots + \omega^{\alpha_k} \cdot a_k$ in the well-ordering $<_{n+1}$. Thus for each n, $<_n$ is of order-type ω_n and has 0 as its least element. For each x let $|x|_n$ denote the ordinal represented by x in the well-ordering $<_n$, and for each $\alpha < \omega_n$ let $\text{num}_n(\alpha)$ be the unique x such that $\alpha = |x|_n$.

It is easy to see that for $n > 1$, $|x|_n$ is a successor ordinal iff x is odd, and in this case we have $|x|_n = |\frac{1}{2}(x-1)|_n + 1$. Furthermore for each $n > 1$ there is an elementary-recursive function s_n such that

$$\begin{aligned} s_n(x, i) &= \text{num}_n(\{|x|_n\}(i)) && \text{if } |x|_n \text{ is a limit,} \\ &= 0 && \text{if } |x|_n = 0 \text{ or a successor.} \end{aligned}$$

Now let α be any ordinal $<\epsilon_0$, and let n be the least positive integer such that $\alpha < \omega_n$. Then $U(\alpha)$ is the smallest class of functions which contains all primitive recursive functions and is closed under substitution and the following scheme of *unnested α-recursion*:[1]

$$f(0, \boldsymbol{u}) = g_0(\boldsymbol{u}),$$
$$f(x + 1, \boldsymbol{u}) = g_1(x + 1, \boldsymbol{u}, f(p(x + 1, \boldsymbol{u}), \boldsymbol{u})),$$

where $p(x, \boldsymbol{u}) <_n x$ whenever $0 <_n x <_n \mathrm{num}_n(\alpha)$, and $p(x, \boldsymbol{u}) = 0$ otherwise.

Thus in particular $U(\omega_n)$ is Kreisel's class of ordinal recursive functions of order n [6].

If, instead of the unnested α-recursion scheme in the above definition, we allow all *nested* recursions over the well-ordering $<_n$ restricted to just those x such that $x <_n \mathrm{num}_n(\alpha)$, then in general a larger class of functions is obtained, and we denote this class by $N(\alpha)$. (For further details concerning $U(\alpha)$, $N(\alpha)$ see Tait [11].)

We call any well-ordering of a subset of the natural numbers, with order-type $\alpha < \omega_n$ *standard* if it is primitive recursively isomorphic to the initial α-segment of $<_n$. It is not difficult to show (see Lemma 1 of Tait [11]) that $U(\alpha)$ ($N(\alpha)$) can be characterised more generally as the smallest class of functions containing all primitive recursive functions, and closed under substitution and unnested (nested) recursions over all standard well-orderings of order-type α. Also by Lemma 1 of [11] we have $U(\alpha) \subseteq U(\beta)$ and $N(\alpha) \subseteq N(\beta)$ whenever $\alpha < \beta$. Thus the way in which we have chosen to "code up" the ordinals below ϵ_0 has no real significance. However our choice of coding is a natural one and it proves to have certain combinatorial properties which are convenient for some of the proofs which follow.

One of the main results of [11] is that for every α, $N(\alpha) \subseteq U(\omega^\alpha)$. We are grateful to the referee for pointing out the following strengthened version of this result, and for suggesting the proof which we describe only briefly since it requires just a slight modification of Tait's proof.

LEMMA 6. *For every $\alpha < \epsilon_0$, $N(\omega \cdot \alpha) \subseteq U(\omega^\alpha)$.*

PROOF. We assume familiarity with the proof of Theorem 2 of Tait [11, pp. 243–246].

Suppose f is defined from certain given functions in $U(\omega^\alpha)$ by a nested $\omega \cdot \alpha$-recursion:

$$f(0, \boldsymbol{u}) = g(\boldsymbol{u}), \qquad f(x + 1, \boldsymbol{u}) = T,$$

where T is a numerical term built up from the variables x, $\boldsymbol{u}$, the given function letters and f, in such a way that for every f-term $f(s, \boldsymbol{t})$ occurring in T, s comes below $x + 1$ in the $\omega \cdot \alpha$-ordering. Let m be the number of occurrences of the function letter f in T. Then in the assignment of ordinals γ to f-terms on p. 244 of [11], the exponential base ω can be replaced by $m + 1$ without disturbing the rest of the proof. Thus the nested $\omega \cdot \alpha$-recursion is reduced to an unnested $(m + 1)^{\omega \cdot \alpha}$-recursion. But $(m + 1)^{\omega \cdot \alpha} = \omega^\alpha$ so f is definable within $U(\omega^\alpha)$. Hence $U(\omega^\alpha)$ is closed under nested $\omega \cdot \alpha$-recursion, and the lemma is proved.

In fact Tait also shows in [11] that $U(\omega^\alpha) \subseteq N(\omega \cdot \alpha)$, but we shall give an alternative proof of this result, using the hierarchies $\mathscr{H}_\beta$ and $\mathscr{F}_\beta$ (see Theorem 5 below).

Next we show that every function in $U(\omega^\alpha)$ lies in some $\mathscr{H}_\beta$ where $\beta < \omega^{\alpha \cdot \omega}$. We make use of the following characterisation of the classes $U(\alpha)$, given in Tait [11] and Robbin [9].

LEMMA 7. *Let α be any ordinal $<\epsilon_0$ and let n be the least integer such that $\alpha < \omega_n$. Then $U(\alpha)$ is the smallest class of functions which contains all primitive recursive functions and is closed under substitution, primitive recursion, and the following scheme*[1] *(called α-annihilation by Robbin* [9]*):*

$$f(0, \boldsymbol{u}) = 0, \qquad f(x+1, \boldsymbol{u}) = 1 + f(p(x+1, \boldsymbol{u}), \boldsymbol{u}),$$

where $p(x, \boldsymbol{u}) <_n x$ whenever $0 <_n x <_n \mathrm{num}_n(\alpha)$ and $p(x, \boldsymbol{u}) = 0$ otherwise.

LEMMA 8. *Let α be any ordinal $<\epsilon_0$ and let n be the least integer such that $\omega^\alpha < \omega_n$. Suppose $p \in \mathscr{H}_\beta$ for some $\beta < (\omega^\alpha)^\omega$, where $p(x, \boldsymbol{u}) <_n x$ whenever $0 <_n x <_n \mathrm{num}_n(\omega^\alpha)$ and $p(x, \boldsymbol{u}) = 0$ otherwise. Then if f is defined from p by ω^α-annihilation, there is a number k, and a function $r \in \mathscr{H}_{\omega^{\alpha\cdot k}}$ such that for all x, $\boldsymbol{u}$, $f(x, \boldsymbol{u}) < h_{\omega^{\alpha\cdot k}}(r(x, \boldsymbol{u}))$.*

PROOF. If $\alpha = 0$ then $f(x, \boldsymbol{u}) \leq 1 = h_1(0)$ for all x, $\boldsymbol{u}$.

So suppose $\alpha = \omega^{\alpha_1}\cdot m_1 + \cdots + \omega^{\alpha_l}\cdot m_l$ where $\alpha_1 > \cdots > a_l$ and $m_1 > 0$, and let γ be the ordinal $\omega^{\alpha_1}\cdot m_1$. Then if $p \in \mathscr{H}_\beta$ for some $\beta < (\omega^\alpha)^\omega = \omega^{\alpha\cdot\omega}$, there is an integer $m > 1$ such that $p \in \mathscr{H}_{\omega^{\gamma\cdot m}}$. Let δ be the ordinal $\omega^{\gamma\cdot(m+1)}$. Now the function $\max[p(x, u), 2p(p(x, \boldsymbol{u}), \boldsymbol{u}), \boldsymbol{u}, y] + y + 1$ is elementary-recursive in p, so by Theorems 3 and 4 it belongs to $\mathscr{H}_\delta$ and hence there is a number $b > 1$ such that, for all x, $\boldsymbol{u}$, y,

$$\max[p(x, \boldsymbol{u}), 2p(p(x, \boldsymbol{u}), \boldsymbol{u}), \boldsymbol{u}, y] + y + 1 \leq h_\delta(\max(x, \boldsymbol{u}, y)) + b.$$

Define $r(x, \boldsymbol{u}) = \max[x, 2p(x, \boldsymbol{u}), \boldsymbol{u}, b] + b$. Then r also belongs to $\mathscr{H}_\delta$ since it is elementary-recursive in p, and for all x, $\boldsymbol{u}$,

$$\begin{aligned} r(p(x, \boldsymbol{u}), \boldsymbol{u}) + 1 &= \max[p(x, \boldsymbol{u}), 2p(p(x, \boldsymbol{u}), \boldsymbol{u}), \boldsymbol{u}, b] + b + 1 \\ &\leq h_\delta(\max(x, \boldsymbol{u}, b)) + b \\ &\leq h_\delta(\max(x, \boldsymbol{u}, b) + b) \\ &\leq h_\delta(r(x, \boldsymbol{u})). \end{aligned}$$

Now suppose f is defined from p by ω^α-annihilation, where $p(x, \boldsymbol{u}) <_n x$ whenever $0 <_n x <_n \mathrm{num}_n(\omega^\alpha)$ and $p(x, \boldsymbol{u}) = 0$ otherwise. We prove, by induction over the well-ordering $<_n$ restricted to just those $x <_n \mathrm{num}_n(\omega^\alpha)$, that for all such x and all $\boldsymbol{u}$, $f(x, \boldsymbol{u}) \leq h_{\delta\cdot|x|_n}(r(x, \boldsymbol{u}))$.

Since n is fixed we let $|x| = |x|_n$ throughout the rest of this proof.

Clearly $f(0, \boldsymbol{u}) \leq h_{\delta\cdot|0|}(r(0, \boldsymbol{u}))$, so suppose $0 <_n x <_n \mathrm{num}_n(\omega^\alpha)$ and assume that for all $\boldsymbol{u}$, $f(p(x, \boldsymbol{u}), \boldsymbol{u}) \leq h_{\delta\cdot|p(x,\boldsymbol{u})|}(r(p(x, \boldsymbol{u}), \boldsymbol{u}))$. Then for all $\boldsymbol{u}$,

$$\begin{aligned} f(x, \boldsymbol{u}) &= 1 + f(p(x, \boldsymbol{u}), \boldsymbol{u}) \\ &\leq 1 + h_{\delta\cdot|p(x,\boldsymbol{u})|}(r(p(x, \boldsymbol{u}), \boldsymbol{u})) \\ &\leq h_{\delta\cdot|p(x,\boldsymbol{u})|}(r(p(x, \boldsymbol{u}), \boldsymbol{u}) + 1) \\ &\leq h_{\delta\cdot|p(x,\boldsymbol{u})|}(h_\delta(r(x, \boldsymbol{u}))) \\ &= h_{\delta\cdot(|p(x,\boldsymbol{u})|+1)}(r(x, \boldsymbol{u})) \quad \text{by Lemma 3.} \end{aligned}$$

Now it is proved in [12], [13] that if σ is any limit $<\omega_n$ and $\xi < \sigma$ then $\xi < \{\sigma\}(i)$ for every $i > \mathrm{num}_n(\xi)$. From this it follows that for every $\eta \leq \omega^\alpha$, if $\xi < \eta$ then

$h_{\delta\cdot\xi}(x) < h_{\delta\cdot\eta}(x)$ for every $x > \mathrm{num}_n(\xi)$. (The proof is by induction on $\eta \leq \omega^\alpha$. If η is a successor and $\xi < \eta$ then, by Lemma 3,

$$h_{\delta\cdot\eta}(x) = h_{\delta\cdot(\eta-1)}(h_\delta(x)) > h_{\delta\cdot(\eta-1)}(x) \geq h_{\delta\cdot\xi}(x)$$

for $x > \mathrm{num}_n(\xi)$. If η is a limit and $\xi < \eta$ then for $x > \mathrm{num}_n(\xi)$ we have $\xi < \{\eta\}(x)$ and hence $h_{\delta\cdot\xi}(x) < h_{\delta\cdot\{\eta\}(x)}(x)$ by hypothesis. But $\delta\cdot\{\eta\}(x) = \{\delta\cdot\eta\}(x)$ since $\eta \leq \omega^\alpha$, so $h_{\delta\cdot\xi}(x) < h_{\{\delta\cdot\eta\}(x)}(x) = h_{\delta\cdot\eta}(x)$ by definition.)

But $r(x, \boldsymbol{u}) > 2(p(x, \boldsymbol{u}) + 1) - 1 = \mathrm{num}_n(|p(x, \boldsymbol{u})| + 1)$. Hence

$$f(x, \boldsymbol{u}) \leq h_{\delta\cdot(|p(x,\boldsymbol{u})|+1)}(r(x, \boldsymbol{u})) \leq h_{\delta\cdot|x|}(r(x, \boldsymbol{u})).$$

So by induction, $f(x, \boldsymbol{u}) \leq h_{\delta\cdot|x|}(r(x, \boldsymbol{u}))$ for all $\boldsymbol{u}$ and all $x <_n \mathrm{num}_n(\omega^\alpha)$.

Furthermore, $r(x, \boldsymbol{u}) > x = \mathrm{num}_n(|x|)$ so if $x <_n \mathrm{num}_n(\omega^\alpha)$,

$$f(x, \boldsymbol{u}) \leq h_{\delta\cdot|x|}(r(x, \boldsymbol{u})) < h_{\delta\cdot\omega^\alpha}(r(x, \boldsymbol{u})).$$

If $\mathrm{num}_n(\omega^\alpha) \leq_n x$ then $f(x, \boldsymbol{u}) = 1 < h_{\delta\cdot\omega^\alpha}(r(x, \boldsymbol{u}))$ for all $\boldsymbol{u}$, since $r(x, \boldsymbol{u}) > 1$. Thus $f(x, \boldsymbol{u}) < h_{\delta\cdot\omega^\alpha}(r(x, \boldsymbol{u}))$ for all $x, \boldsymbol{u}$. But $\delta\cdot\omega^\alpha = \omega^{\gamma\cdot(m+1)+\alpha} = \omega^{\alpha\cdot(m+2)}$, so by putting $k = m + 2$ we have $f(x, \boldsymbol{u}) < h_{\omega^{\alpha\cdot k}}(r(x, \boldsymbol{u}))$ for all $x, \boldsymbol{u}$, and this completes the proof of Lemma 8.

LEMMA 9. *If $0 < \alpha < \epsilon_0$ then for every function $f \in U(\omega^\alpha)$ there is an ordinal $\beta < (\omega^\alpha)^\omega$ such that $f \in \mathscr{H}_\beta$.*

PROOF. We use the characterisation of $U(\omega^\alpha)$ provided by Lemma 7.

If f is primitive recursive then $f \in \mathscr{F}_m$ for some integer m, so by the Corollary to Theorem 4, $f \in \mathscr{H}_\beta$ for some $\beta < \omega^{m+1}$. But $\omega^{m+1} < (\omega^\alpha)^\omega$ since $\alpha \neq 0$.

If f is defined by substitution from functions $g_0, \cdots, g_m$ where for each $i \leq m$, $g_i \in \mathscr{H}_{\beta_i}$ with $\beta_i < (\omega^\alpha)^\omega$, then by Theorem 3, f is elementary-recursive in $h_{\omega^{\alpha\cdot k}}$ where k is the least positive integer such that $\max(\beta_0, \cdots, \beta_m) \leq \omega^{\alpha\cdot k}$. So by Theorem 4, there is a $\beta < \omega^{\alpha\cdot k+1}$ such that $f \in \mathscr{H}_\beta$.

Now suppose f is defined by a primitive recursion from functions $g_0 \in \mathscr{H}_{\beta_0}$ and $g_1 \in \mathscr{H}_{\beta_1}$ where $\beta_0, \beta_1 < (\omega^\alpha)^\omega$. Let k be the least positive integer such that $\beta_0, \beta_1 < \omega^{\alpha\cdot k}$. Then by the Corollary to Theorem 4, g_0 and g_1 belong to $\mathscr{F}_{\alpha\cdot k}$. But it is not difficult to show that a single primitive recursion, applied to functions $g_0, g_1 \in \mathscr{F}_{\alpha\cdot k}$, defines a function in $\mathscr{F}_{\alpha\cdot k+1}$ (see Theorem 2.12 of [13]). So $f \in \mathscr{F}_{\alpha\cdot k+1}$ and again by the Corollary to Theorem 4, there is a $\beta < \omega^{\alpha\cdot k+2}$ such that $f \in \mathscr{H}_\beta$.

Finally suppose f is defined by ω^α-annihilation from a function $p \in \mathscr{H}_{\beta_1}$ where $\beta_1 < (\omega^\alpha)^\omega$, $p(x, \boldsymbol{u}) <_n x$ whenever $0 <_n x <_n \mathrm{num}_n(\omega^\alpha)$ and $p(x, \boldsymbol{u}) = 0$ otherwise (n being the least positive integer such that $\omega^\alpha < \omega_n$). Then it can easily be shown that for all $x <_n \mathrm{num}_n(\omega^\alpha)$ and all $\boldsymbol{u}$, $f(x, \boldsymbol{u}) = \mu y(q(y, x, \boldsymbol{u}) = 0)$ where q is defined from p by the primitive recursion $q(0, x, \boldsymbol{u}) = x$, $q(y + 1, x, \boldsymbol{u}) = p(q(y, x, \boldsymbol{u}), \boldsymbol{u})$. But by Lemma 8 there is a positive integer k and a function $r \in \mathscr{H}_{\omega^{\alpha\cdot k}}$ such that for all $x, \boldsymbol{u}$, $f(x, \boldsymbol{u}) < h_{\omega^{\alpha\cdot k}}(r(x, \boldsymbol{u}))$. Furthermore, by the previous comment concerning primitive recursion, and the way in which the integer k was obtained, it follows that the function q also belongs to $\mathscr{H}_{\omega^{\alpha\cdot k}}$. Hence we can define the function f as follows:

$$f(x, \boldsymbol{u}) = \mu y(y < h_{\omega^{\alpha\cdot k}}(r(x, \boldsymbol{u})) \,\&\, q(y, x, \boldsymbol{u}) = 0) \quad \text{if } x <_n \mathrm{num}_n(\omega^\alpha)$$

and $f(x, \boldsymbol{u}) = 1$ otherwise. So f is elementary-recursive in functions belonging to $\mathscr{H}_{\omega^{\alpha\cdot k}}$ and by Theorem 4, there is a $\beta < \omega^{\alpha\cdot k+1}$ such that $f \in \mathscr{H}_\beta$.

We have now considered all ways of generating functions in $U(\omega^\alpha)$ and so Lemma 9 is proved.

THEOREM 5. (i) *For every ordinal* α *such that* $0 < \alpha < \epsilon_0$,

$$N(\omega\cdot\alpha) = U(\omega^\alpha) = \bigcup_{\beta<\omega^{\alpha\cdot\omega}} \mathscr{H}_\beta = \bigcup_{\beta<\alpha\cdot\omega} \mathscr{F}_\beta.$$

(ii) *Thus if* $\omega^k \leq \alpha < \omega^{k+1}$ *for some positive integer* k *then* $N(\alpha) = U(\omega^{\omega^{k-1}}) = \bigcup_{\beta<\omega^k}\mathscr{F}_\beta$, *and if* $\omega^\omega \leq \alpha < \epsilon_0$ *then* $N(\alpha) = U(\omega^\alpha) = \bigcup_{\beta<\alpha\cdot\omega}\mathscr{F}_\beta$.

PROOF. (ii) follows easily from (i) since if $\alpha = \omega^{\alpha_1} + \omega^{\alpha_2} + \cdots + \omega^{\alpha_l}$ where $\epsilon_0 > \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_l$, then any nested α-recursion can be split up into l nested ω^{α_1}-recursions, so $N(\alpha) = N(\omega^{\alpha_1})$. If $\alpha_1 = k$, k a positive integer, we then have, by (i) $N(\alpha) = N(\omega\cdot\omega^{k-1}) = U(\omega^{\omega^{k-1}}) = \bigcup_{\beta<\omega^k}\mathscr{F}_\beta$. If $\alpha_1 \geq \omega$ then $N(\alpha) = N(\omega^{\alpha_1}) = N(\omega^{1+\alpha_1}) = N(\omega\cdot\alpha) = U(\omega^\alpha) = \bigcup_{\beta<\alpha\cdot\omega}\mathscr{F}_\beta$.

To prove (i) it is only necessary to show that $\mathscr{F}_\beta \subseteq N(\omega\cdot\alpha)$ for every $\beta < \alpha\cdot\omega$, since by Lemmas 6, 9, and the Corollary to Theorem 4, we already have $N(\omega\cdot\alpha) \subseteq U(\omega^\alpha) \subseteq \bigcup_{\beta<\omega^{\alpha\cdot\omega}}\mathscr{H}_\beta = \bigcup_{\beta<\alpha\cdot\omega}\mathscr{F}_\beta$.

Clearly it is sufficient to prove that for every $\beta < \alpha\cdot\omega$, $F_\beta \in N(\omega\cdot\alpha)$. Suppose then that $\alpha = \omega^{\alpha_1}\cdot m_1 + \cdots + \omega^{\alpha_l}\cdot m_l$ where $\alpha_1 > \cdots > \alpha_l$ and $m_1 \neq 0$. Let γ be the ordinal $\omega^{\alpha_1}\cdot m_1$ and let n be the least integer such that $\alpha < \omega_n$. Assume inductively that there is a $g_k \in N(\omega\cdot\alpha)$ such that for every $\delta < \alpha$, $F^{m+1}_{\gamma\cdot k+\delta}(u) = g_k(\langle m, \mathrm{num}_n(\delta)\rangle, u)$ where $\langle\,,\rangle$ is a primitive recursive pairing function with primitive recursive inverses. Define g_{k+1} as follows:

$$\begin{aligned}
&g_{k+1}(\langle 0, 0\rangle, u) = g_k(\langle 0, s_n(\mathrm{num}_n(\gamma), u)\rangle, u),\\
&g_{k+1}(\langle 0, x\rangle, u) = g_{k+1}(\langle u, [(x-1)/2]\rangle, u) \quad \text{if } x \text{ is odd and } x <_n \mathrm{num}_n(\alpha),\\
&g_{k+1}(\langle 0, x\rangle, u) = g_{k+1}(\langle 0, s_n(x, u)\rangle, u) \quad \text{if } x \text{ is even and } 0 <_n x <_n \mathrm{num}_n(\alpha),\\
&g_{k+1}(\langle m+1, x\rangle, u) = g_{k+1}(\langle 0, x\rangle, g_{k+1}(\langle m, x\rangle, u)) \quad \text{if } x <_n \mathrm{num}_n(\alpha),\\
&g_{k+1}(\langle m, x\rangle, u) = 0 \quad \text{if } \mathrm{num}_n(\alpha) \leq_n x.
\end{aligned}$$

This is a nested recursion over the well-ordering $\prec$ defined by

$$\langle m_1, x_1\rangle \prec \langle m_2, x_2\rangle \Leftrightarrow x_1 <_n \mathrm{num}_n(\alpha) \;\&\; x_2 <_n \mathrm{num}_n(\alpha) \;\&\; (x_1 <_n x_2 \vee (x_1 = x_2 \;\&\; m_1 < m_2)).$$

But $\prec$ is a standard well-ordering of order type $\omega\cdot\alpha$, and so $g_{k+1} \in N(\omega\cdot\alpha)$. Furthermore it is easily proved by induction on δ, that for every $\delta < \alpha$, $F^{m+1}_{\gamma\cdot(k+1)+\delta}(u) = g_{k+1}(\langle m, \mathrm{num}_n(\delta)\rangle, u)$. So by induction there is, for each k, a function $g_k \in N(\omega\cdot\alpha)$ such that $F^{m+1}_{\gamma\cdot k+\delta}(u) = g_k(\langle m, \mathrm{num}_n(\delta)\rangle, u)$ for every $\delta < \alpha$.

Now take any ordinal $\beta < \alpha\cdot\omega$. Then there is a number k and an ordinal $\delta < \alpha$ such that $\beta = \gamma\cdot k + \delta$, and so $F_\beta(u) = g_k(\langle 0, \mathrm{num}_n(\delta)\rangle, u)$ for all u. Hence $F_\beta \in N(\omega\cdot\alpha)$.

COROLLARY. $\bigcup_{\beta<\sigma}\mathscr{F}_\beta = \bigcup_{\beta<\sigma}N(\beta)$ *iff* σ *is of the form* ω^δ *where* $\delta = 1$ *or* $\delta \geq \omega$.

PROOF. If $\sigma = \omega$ the two classes coincide with the primitive recursive functions. If $\sigma = \omega^\delta$, $\delta \geq \omega$, then for every $\beta < \sigma$, $\beta\cdot\omega \leq \sigma$ and $\omega\cdot\beta < \sigma$, so by Theorem 5(i), $N(\beta) \subseteq N(\omega\cdot\beta) \subseteq \bigcup_{\gamma<\sigma}\mathscr{F}_\gamma$ and $\mathscr{F}_\beta \subseteq N(\omega\cdot\beta) \subseteq \bigcup_{\gamma<\sigma}N(\gamma)$.

Conversely suppose $\bigcup_{\beta<\sigma}\mathscr{F}_\beta = \bigcup_{\beta<\sigma}N(\beta)$ and let $\sigma = \omega^\delta\cdot m + \gamma$ where $m \neq 0$ and $\gamma < \omega^\delta$. Assume that $\omega^\delta < \sigma$. Then $\mathscr{F}_{\omega^\delta} \subseteq \bigcup_{\beta<\sigma}N(\beta) = N(\omega^\delta) \subsetneqq \mathscr{F}_\sigma$. But if δ is finite $\mathscr{F}_{\omega^\delta} \subseteq N(\omega^\delta)$ contradicts the first part of Theorem 5(ii), and if δ is infinite $N(\omega^\delta) \subsetneqq \mathscr{F}_\sigma$ contradicts the second part of Theorem 5(ii) since $\sigma < \omega^\delta\cdot\omega$. Therefore $\sigma = \omega^\delta$. Finally if $1 < \delta < \omega$ then $\bigcup_{\beta<\sigma}N(\beta) = N(\omega^{\delta-1}) = \bigcup_{\beta<\sigma}\mathscr{F}_\beta$ which contradicts Theorem 5(ii), so $\delta = 1$ or $\delta \geq \omega$.

Throughout this paper we have been chiefly concerned with hierarchies based on "majorization". Another obvious way of extending an effectively generated class of functions is by "enumeration", and subrecursive hierarchies based on this principle have been developed by Kleene [5] as follows.

First construct an elementary-recursive "indexing" of the functions which are elementary-recursive in a binary function g. Then define $e^g(i, x)$ so that $e^g(i, x) = f((x)_0, \cdots, (x)_{k-1})$ if i is an index of the k-ary function f, and $e^g(i, x) = 0$ if i is not an index. Clearly $\lambda g \cdot e^g$ is an "elementary" analogue of the jump operator and (transfinite) iteration of it leads to an ascending sequence of elementary-recursive degrees. Using this method in conjunction with the fundamental sequences $\lambda i \cdot \{\sigma\}(i)$, a hierarchy $\{\mathscr{L}_\alpha\}_{\alpha<\epsilon_0}$ is obtained, where $\mathscr{L}_\alpha$ is the class of functions elementary-recursive in E_α, which is defined by the recursion: $E_0(i, x) = 0$, $E_{\beta+1}(i, x) = e^{E_\beta}(i, x)$, $E_\sigma(i, x) = E_{\{\sigma\}((i)_0)}((i)_1, x)$ if σ is a limit. Now Schwichtenberg [10] has proved that $\mathscr{L}_\alpha = \mathscr{F}_{2+\alpha}$ for every $\alpha < \epsilon_0$. Hence the results of §§2 and 3 above hold not only for the extended Grzegorczyk hierarchy $\{\mathscr{F}_\alpha\}$, but also for the Kleene hierarchy $\{\mathscr{L}_\alpha\}$.

The extended Grzegorczyk hierarchy however has the advantage that the functions F_β on which it is based are defined by a very simple recursion (whereas the definition of the E_β's involves a rather complicated course of values recursion at successor stages) and this provides a corresponding simple normal form for nested recursion, since every function in $N(\omega \cdot \alpha)$ is elementary-recursive in some F_β where $\beta < \alpha \cdot \omega$.

§4. Normal Forms for Ordinal Recursion. A recursion scheme S is said to be a *normal form* for unnested (nested) α-recursion if $U(\alpha)$ ($N(\alpha)$) is closed under applications of S, and if every function in $U(\alpha)$ ($N(\alpha)$) is definable from primitive recursive functions by successive applications of S and substitution.

By Theorem 5(i), in order to obtain a normal form for unnested ω^α-recursion (resp. nested $\omega \cdot \alpha$-recursion) it is only necessary to provide the means for defining the functions $h_{\omega^\alpha}, h_{\omega^{\alpha \cdot 2}}, h_{\omega^{\alpha \cdot 3}}, \cdots$ (resp. $F_\alpha, F_{\alpha \cdot 2}, F_{\alpha \cdot 3}, \cdots$).

Thus the "essentially-unnested" scheme

$$\begin{aligned} f(0, u) &= u, \\ f(x+1, u) &= f([x/2], g(u)) \quad \text{if } x+1 \text{ is odd and } x+1 <_n \mathrm{num}_n(\omega^\alpha), \\ f(x+1, u) &= f(s_n(x+1, u), u) \quad \text{if } x+1 \text{ is even and } x+1 <_n \mathrm{num}_n(\omega^\alpha), \\ f(x+1, u) &= 0 \quad \text{otherwise}, \end{aligned}$$

where n is the least integer such that $\omega^\alpha < \omega_n$, is a normal form for unnested ω^α-recursion.

Similarly the "single-nested" scheme

$$\begin{aligned} f(m, 0, u) &= g(m, u), \\ f(0, x+1, u) &= f(u, [x/2], u) \quad \text{if } x+1 \text{ is odd and } x+1 <_n \mathrm{num}_n(\alpha), \\ f(0, x+1, u) &= f(0, s_n(x+1, u), u) \quad \text{if } x+1 \text{ is even and } x+1 <_n \mathrm{num}_n(\alpha), \\ f(m+1, x+1, u) &= f(0, x+1, f(m, x+1, u)) \quad \text{if } x+1 <_n \mathrm{num}_n(\alpha), \\ f(m, x+1, u) &= 0 \quad \text{otherwise}, \end{aligned}$$

where n is the least integer such that $\alpha < \omega_n$, is a normal form for nested $\omega \cdot \alpha$-recursion.

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