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POSITIVE RATIONAL SEQUENCES

Matti SOITTOLA
Mathematics Department, University of Turku, 20500 Turku 50, Finland

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0. Introduction

Rational sequences are encountered when dealing with regular and stochastic languages and when considering growths of DOL systems. In this paper positive rational sequences are the object of study. We shall see that a well-known theorem of Berstel which gives a necessary condition for positiveness is also essentially sufficient. Our characterization makes it possible to answer some problems appearing in Eilenberg [3] and to decide the **N**-rationality of a **Z**-rational sequence.

1. Preliminaries

Let A be a subring of $\mathbf{R}$ (we denote the field of real numbers with $\mathbf{R}$, the field of rational numbers with $\mathbf{Q}$, the ring of integers with $\mathbf{Z}$ and the semiring of natural numbers with $\mathbf{N}$) and let $A\langle\langle x\rangle\rangle$ be the ring of formal power series $f = \Sigma f_n x^n$, where $f_n \in A$. If $f_0 = 0$ then we define the quasi-inverse f^+ of f to be the series $f + f^2 + \dots$. We say that a series of $A\langle\langle x\rangle\rangle$ is A-rational if it can be formed of polynomials of $A\langle\langle x\rangle\rangle$ by applying the operations sum, product and quasi-inversion finitely many times. If the initial polynomials have their coefficients in $A_+ = A \cap [0,\infty)$ then we obtain an A_+-rational series. A sequence (r_n) is called A-rational (A_+-rational) if the series $\Sigma r_n x^n$ is A-rational (A_+-rational).

Because $f = (1-f)f^+$ we see that a series of $A\langle\langle x\rangle\rangle$ is A-rational iff it is the Maclaurin expansion of a rational function $p(x)/(1-q(x))$ where $p, q \in A[[x]]$ and $q(0) = 0$. Writing this generating function as a sum of partial fractions and a polynomial we note that if r is a rational series then for large values of n,

$$r_n = \sum_i P_i(n)\alpha_i^n,$$

where the numbers $1/\alpha_i$ are the poles of the generating function and each P_i is a polynomial whose degree is the multiplicity of $1/\alpha_i$ subtracted by one.

The reader unfamiliar with the following facts may consult e.g. Eilenberg [3].

Propositions. (1) *If r and s are A-rational (A_+-rational) series then their Hadamard product $\sum r_n s_n x^n$ is A-rational (A_+-rational), too.*

(2) *If A_1 is a subfield of A_2 and r is an A_2-rational series of $A_1\langle\langle x\rangle\rangle$ then r is A_1-rational.*

(3) *If r is a Q-rational series of $\mathbb{Z}\langle\langle x\rangle\rangle$ then r is $\mathbb{Z}$-rational.*

Berstel's Theorem. *If r is $\mathbb{R}_+$-rational then the poles of its generating function having minimal modulus (if there are any) are of the form $\rho\zeta$ where $\rho > 0$ and ζ is a root of unity.*

This theorem implies that if the series r is $\mathbb{R}_+$-rational then there are numbers m and p such that each of the sequences $(s_n^{(j)}) = (r_{m+j+np})$ $(j = 0, \ldots, p-1)$ has the form

$$\text{(B)} \qquad s_n^{(j)} = P(n)\alpha^n + \sum_i P_i(n)\alpha_i^n,$$

where $\alpha \geqslant 0$, $\alpha > \max|\alpha_i|$, P and the P_i's are polynomials and $P \neq 0$.

2. The main result

Theorem. *Let K be a subfield of $\mathbb{R}$ and let (r_n) be a K-rational sequence satisfying the following two conditions:*

(i) $r_n \geqslant 0$ *for every* n;

(ii) $r_n = P(n)\alpha^n + \sum_i P_i(n)\alpha_i^n$ ($\alpha > \max|\alpha_i|$, *P and P_i's are polynomials,* $P \neq 0$) *for large values of n.*

Then (r_n) is a K_+-rational sequence.

Proof. Choose a rational number q between $\max|\alpha_i|$ and α and form the K-rational sequence $(s_n) = (r_n/q^n)$. Let $F(x)/G(x)$ be its generating function. Denote $A = \alpha/q$, $A_i = \alpha_i/q$ and $l = \deg P$. Then the numbers A and A_i are the zeroes of $x^L G(1/x)$ ($L = \deg G$); the multiplicity of A is $l+1$.

Now we define $g(x) = x^k - g_1 x^{k-1} - \ldots - g_k$ as follows: If $l = 0$ then $g(x) = x^L G(1/x)$ and if $l > 0$ then g is the minimal polynomial of A over the field K. Thus A is a simple zero of g and all the other zeroes a_i have modulus smaller than one.

Let $g^{(m)}(x) = x^k - g_1^{(m)} x^{k-1} - \ldots - g_k^{(m)}$ be the polynomial whose zeroes are the mth powers A^m and a_i^m. Because

$$g_1^{(m)} = A^m + \sum_i a_i^m, \qquad g_2^{(m)} = -\left(A^m \sum_i a_i^m + \sum_{i<j} a_i^m a_j^m\right), \ldots,$$

we can take a number M such that

$$g_1^{(M)} > 1, \qquad g_1^{(M)} + g_2^{(M)} > 1, \quad \dots, \quad g_1^{(M)} + \dots + g_k^{(M)} > 1.$$

Next we consider a sequence $(t_n) = (s_{h+nM})$ where h is large and define

$$\gamma_1 = g_1^{(M)} - 1, \quad \dots, \quad \gamma_k = g_1^{(M)} + \dots + g_k^{(M)} - 1,$$

$$\begin{aligned} T_n &= t_n - g_1^{(M)} t_{n-1} - \dots - g_k^{(M)} t_{n-k} \\ &= (t_n - t_{n-1}) - \gamma_1(t_{n-1} - t_{n-2}) - \dots - \gamma_{k-1}(t_{n-k+1} - t_{n-k}) - \gamma_k t_{n-k}. \end{aligned}$$

We observe that if $n > k$ then

$$T_n = (t_n - t_{n-1}) - \gamma_1(t_{n-1} - t_{n-2}) - \dots - \gamma_k(t_{n-k} - t_{n-k-1}) - \dots - \gamma_k(t_1 - t_0) - \gamma_k t_0,$$

and if $n > 2k$ then

$$T_n = (t_n - t_{n-1}) - \gamma_1(t_{n-1} - t_{n-2}) - \dots - \gamma_k(t_{n-k} - t_{n-k-1}) - \dots - \gamma_k(t_{k+1} - t_k) - \gamma_k t_k.$$

These formulas immediately imply that

$$\sum_{n>k} T_n x^n = (1 - \gamma_1 x - \dots - \gamma_k x^k - \gamma_k x^{k+1} - \dots) \sum_{n>k} (t_n - t_{n-1}) x^n - \gamma_k t_k \sum_{n>2k} x^n - p(x),$$

where p is a polynomial with coefficients in K_+. Hence

$$\sum_{n>k} (t_n - t_{n-1}) x^n = \frac{\sum_{n>k} T_n x^n + \gamma_k t_k x^{2k+1}/(1-x) + p(x)}{1 - (\gamma_1 x + \dots + \gamma_k x^k + \gamma_k x^{k+1}/(1-x))}.$$

If now $l = 0$ then $\Sigma\, T_n x^n = 0$ and the series

$$\sum_{n>k} (t_n - t_{n-1}) x^n$$

is seen to be K_+-rational. But then the series

$$\sum_{n\geq 0} t_n x^n = t_0 + \frac{1}{1-x} \sum_{n>0} (t_n - t_{n-1}) x^n$$

is K_+-rational, too. If $l > 0$ then the sequence (T_n) satisfies the conditions (i) and (ii) with $l-1$ in place of l and so we may utilize induction on l when proving the K_+-rationality of $\Sigma\, t_n x^n$. (Here it might be helpful to have a look at the theory of linear difference equations. See e.g. Milne–Thomson [5].)

We now know that for some numbers H and M the sequences $(t_n^{(j)}) = (s_{H+j+nM})$ $(j = 0, \dots, M-1)$ are K_+-rational. But as

$$\Sigma\, s_n x^n = Q(x) + \sum_{j=0}^{M-1} x^{H+j} \,\Sigma\, t_n^{(j)} (x^M)^n,$$

where Q is a polynomial with coefficients in K_+, the series $\Sigma\, r_n x^n = \Sigma\, s_n q^n x^n$ must be K_+-rational. $\square$

The above proof is greatly inspired b the constructions of Ruohonen in [6].

3. Consequences

Corollary 1. *Let K be a subfield of* $\mathbf{R}$ *and r a series of $K_+\langle\langle x\rangle\rangle$ which satisfies condition* (E). *Then r is K_+-rational.*

Here we may procede as at the end of the proof of our theorem.

Corollary 2. *Every* $\mathbf{R}_+$*-rational series of* $\mathbf{Q}_+\langle\langle x\rangle\rangle$ *is* $\mathbf{Q}_+$*-rational.*

This follows from Corollary 1.

Corollary 3. *Let (r_n) be a* $\mathbf{Z}$*-rational sequence which satisfies condition* (B) *and whose terms are non-negative. Then it is an* $\mathbf{N}$*-rational sequence.*

Here we need Corollary 1 and a theorem of Fliess [4].

Corollary 4. *Assume that (a_n) and (b_n) are* $\mathbf{N}$*-rational sequences, $b_n > 0$ for every n and (c_n) is an integer sequence. If $a_n = b_n c_n$ then the sequence (c_n) is* $\mathbf{N}$*-rational.*

It was noted by Berstel [1] that the sequence (c_n) is $\mathbf{Z}$-rational. Since (a_n) and (b_n) satisfy (B) also (c_n) must satisfy it and so Corollary 3 is applicable.

Among $\mathbf{N}$-rational sequences D0L and PD0L growth sequences are of special interest. In an earlier paper [7] we have shown that it is possible to decide when an $\mathbf{N}$-rational sequence is a D0L sequence. We also observed that the property of a sequence (r_n) being a PD0L sequence depends on the $\mathbf{N}$-rationality of the sequence $(r_{n+1} - r_n)$. Hence the following corollary gives as a byproduct a decision procedure for PD0L property.

Corollary 5. *It is possible to decide when a* $\mathbf{Z}$*-rational sequence is* $\mathbf{N}$*-rational.*

Suppose (r_n) is the given sequence and $r_n = \sum P_i(n)\alpha_i^n$ for large values of n. According to Berstel and Mignotte [2] we can find a natural number p such that no one of the numbers α_i^p is of the form $\rho\zeta$, where $\rho > 0$ and ζ is a root of unity different from one. Therefore we may assume that the numbers α_i themselves satisfy this condition. Then, by the results of this paper (r_n) is $\mathbf{N}$-rational iff the following conditions hold:

(1) the generating function is a polynomial or it has only one pole with minimum modulus (and that pole is positive);

(2) $r_n \geqslant 0$ for every n.

At first we describe some general decision methods.

If we have polynomials $q_1, \ldots, q_k \in \mathbf{Q}[[x]]$ it is possible to form a polynomial q_0 whose zeroes are exactly the differences of the zeroes of $q_1 \ldots q_k$. Hence we can find a positive rational number σ such that if ξ_1 and ξ_2 are zeroes of the q_i's then either $\xi_1 = \xi_2$ or $|\xi_1 - \xi_2| > \sigma$.

The polynomials q_i and $Q_i = q_i/\text{g.c.d.}(q_i, q_i')$ have the same zeroes but the zeroes of Q_i are simple. Thus we may compare the real zeroes of the q_i's by examining the signs of the numbers $Q_i(j\sigma)$ where $j \in \mathbf{Z}$.

By comparing the real zeroes of a polynomial $q \in \mathbf{Q}[[x]]$ and its derivatives it is possible to determine the multiplicities of the real zeroes of q.

Further, we can form a polynomial whose zeroes are the square roots of the pairwise products of the zeroes of q. By examining the real zeroes of this new polynomial we obtain knowledge of the moduli of the zeroes of q.

It should be clear that the above methods enable us to check the validity of condition (1).

In the presence of condition (1) we have

$$r_n = P_0(n)\alpha_0^n + \sum_{i>0} P_i(n)\alpha_i^n \qquad (\alpha_0 > 0, \alpha_0 > \max|\alpha_i|)$$

for large values of n and this makes it possible to decide the validity of (2) as follows:

By examining the sign of the generating function as $x \to 1/\alpha_0-$, we see the sign of the leading coefficient c_l of P_0. Of course this sign must be plus for (2) to hold. Assume that this is the case. Then we start by determining rational numbers a and b such that $\alpha_0 > a > b > \max|\alpha_i|$. Next we divide the denominator of the generating function into prime factors by using the method of undetermined coefficients. These factors show us the multiplicities of the poles. If m_i is the multiplicity of α_i then the generating function F has the partial fraction summands

$$\beta_{ij}/(x - \alpha_i)^j \qquad (j = 1, \ldots, m_i),$$

where

$$\beta_{ij} = \left[\frac{\mathrm{d}^{m_i - j}}{\mathrm{d}x}(x - \alpha_i)^{m_i}F(x)\frac{1}{(m_i - j)!}\right]_{x=\alpha_i}.$$

Using these formulas and the general methods described above we can find a rational number c such that all coefficients of the polynomials P_0 and P_i have a modulus smaller than c. Moreover, we can find a positive rational number C such that $c_l > C$.

The numbers a, b, c and C being available it is easy to find an index n_+ such that $r_n > 0$ when $n > n_+$. After that we have only to compute separately the first terms of the sequence (r_n).

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