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Contraction methods for nonlinear systems: a brief introduction and some open problems

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Abstract— **Contraction theory provides an elegant way to analyze the behaviors of certain nonlinear dynamical systems. Under sometimes easy to check hypotheses, systems can be shown to have the incremental stability property that trajectories converge to each other. The present paper provides a self-contained introduction to some of the basic concepts and results in contraction theory, discusses applications to synchronization and to reaction-diffusion partial differential equations, and poses several open questions.**

I. Introduction

Global stability is a central research topic in dynamical systems theory. Stability properties are typically defined in terms of attraction to an invariant set, for example to an equilibrium or a periodic orbit, often coupled with a Lyapunov stability requirement that trajectories that start near the attractor must stay close to the attractor for all future times.

A far stronger requirement than attraction to a pre-specified target set is to ask that any two trajectories should (exponentially, and with no overshoot) converge to each other, or, in more abstract mathematical terms, that the flow be a contraction in the state space. While this requirement will be less likely to be satisfied for a given system, it is sometimes comparatively easier to check. Indeed, checking stability properties often involves constructing an appropriate Lyapunov function, which, in turn, requires a priori knowledge of the attractor location. In contrast, contraction-based methods, discussed here, do not require the prior knowledge of attractors. Instead, one checks an infinitesimal property, that is to say, a property of the vector field defining the system, which guarantees exponential contractivity of the induced flow.

It is useful to first discuss the relatively trivial case of linear time-invariant systems of differential equations $\dot{x} = Ax$, with Euclidean norm. Since differences of solutions are also solutions, contractivity amounts simply to the requirement that there exists a positive number c such that, for all solutions, $|x(t)| \leq e^{-ct}|x(0)|$, where $|\cdot|$ refers to the Euclidean norm. This is clearly equivalent to the requirement that $A + A^T$ be a negative definite matrix. In Lyapunov-function terms, $x^T Px$ is a Lyapunov function for the system, when $P = I$.

This property is of course stronger than merely asymptotic stability of the zero equilibrium of $\dot{x} = Ax$, that is, that A be a Hurwitz matrix (all eigenvalues with negative real part). Of course, asymptotic stability is equivalent to the existence of some positive definite matrix P (but not necessarily the identity) so that $x^T Px$ is a Lyapunov function, and this can be interpreted, as remarked later, as a contractivity property with respect to a weighted Euclidean norm associated to P. This simple example with linear systems already illustrates why an appropriate choice of norms when defining "contractivity" is critical; even for linear systems, contractivity is not a topological, but is instead a metric property: it depends on the norm being used, in close analogy to the choice of an appropriate Lyapunov function.

Computing matrix measures, also called logarithmic norms (see e.g. [1], [2]) of the Jacobian of the vector field, evaluated at all possible states is an appropriate tool to characterize contractivity of nonlinear systems. This idea is a classical one, and can be traced back at least to work of D.C. Lewis in the 1940s, see [3], [4]. Dahlquist's 1958 thesis under Hörmander (see [5]) used logarithmic norms to show contractivity of differential equations, and more generally of differential inequalities, the latter applied to the analysis of convergence of numerical schemes for solving differential equations. The basic ideas have been rediscovered independently by, for example, Demidovič [6], [7] and Yoshizawa [8], [9]. In control theory, the field attracted much attention after the work of Lohmiller and Slotine [10], and follow-up papers by Slotine and collaborators, see for example [11], [12], [13], [14]. These papers showed the power of contraction techniques for the study of not only stability, but also observer problems, nonlinear regulation, and consensus problems in complex networks. (See also the work of Nijmejer and coworkers [15].) We refer the reader to the historical analysis given in [16], [17] and the survey [18].

In this paper, we first discuss the most basic results regarding contraction for ODE systems. We frame our discussion in the language of modern nonlinear functional analysis in the style of [19]. This language provides the natural concepts needed to understand abstract norms as well as extensions to infinite-dimensional spaces, including partial differential equations. We then turn to certain new developments regarding diffusive synchronization of ODE systems as well as uniform solutions of reaction-diffusion PDE systems. Given space constraints, the choice of these other topics is strictly a question of taste. We picked problems in which we have recently worked, the emphasis being on contractions with respect to non-Euclidean norms, and for which many problems remain open. We only consider deterministic systems; see [13] for

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applications of contraction theory to the analysis of certain stochastic systems. In addition, we restrict attention to norms that do not depend on state variables nor time; powerful tools that allow time-dependent and space-dependent norms have been developed by Slotine and others, see for example [11], [12], [13], [14]. These more general tools are especially useful when they are applied to certify the global exponential stability of an invariant manifold, for example to show that a periodic orbit is an attracting limit cycle. In a similar vein, problems involving synchronization (consensus) can be often formulated as involving analysis and design of an attractive invariant manifold. From a geometric point of view, space-dependent norms may be naturally interpreted as changes of coordinates on tangent spaces, and translate into "transverse contractions" or, in global coordinates, contractions with respect to a particular subset of variables. This more general view has been the subject of much recent research, see for example [20], [21], as well as the recent papers [22], [23]. Another paper in this tutorial session, [24], addresses this topic. We remark that contractivity can also be thought of as a very strong form of incremental stability [25]. Other weaker notions of contractivity have also been proposed, in which trajectories are required to start contracting only after a very small (time and/or magnitude) transient, see [26], [27].

Outline of paper: We first study dynamical systems described by possibly time-dependent systems of differential equations $\dot{x} = f(x,t)$, providing several basic results in contraction theory, including their proofs, and show by means of an example how even some very simple systems (in this case, an elementary biochemical model) can benefit from the use of non-L^2 norms.

We then use the contraction theory to show synchronization (or "consensus") in diffusively connected identical ODE systems. Synchronization results based on contraction-based techniques, have been developed mostly by using measures induced by L^2 or weighted L^2 norms, see for instance, [10], [28], [29], [30], [12]. For non L^2 norms, current results are partial, applying only to certain types of graphs, see [31].

The convergence to uniform solutions in reaction-diffusion partial differential equations $\partial u/\partial t = F(u,t) + D\Delta u$ where $u = u(\omega, t)$, is a formal analogue of the synchronization of ODE systems. Questions of convergence to uniform solutions in reaction-diffusion PDE's are also a classical topic of research. The "symmetry breaking" phenomenon of diffusion-induced, or Turing instability refers to the case where a dynamic equilibrium $\bar{u}$ of the non-diffusing ODE system $du/dt = F(u,t)$ is stable, but, at least for some diagonal positive matrices D, the corresponding uniform state $u(\omega) = \bar{u}$ is unstable for the PDE system $\partial u/\partial t = F(u,t) + D\Delta u$. This phenomenon has been studied at least since Turing's seminal work on pattern formation in morphogenesis [32], where he argued that chemicals might react and diffuse so as to give rise to heterogeneous spatial patterns. Subsequent work by Gierer and Meinhardt [33], [34] produced a molecularly plausible minimal model, using two substances that combine local autocatalysis and long-ranging inhibition. Since that early work, a variety of processes in physics, chemistry, biology, and many other areas have been studied from the point of view of diffusive instabilities, and the mathematics of the process has been extensively studied, see for instance [35], [36], [37]. Most past work has focused on local stability analysis, through the analysis of the instability of nonuniform spatial modes of the linearized PDE. Nonlinear, global, results are usually proved under strong constraints on diffusion constants as they compare to the growth of the reaction part. Contraction techniques add a useful set of tools to that analysis. As with synchronization, for non-Euclidean norms we only provide results in special cases, the general problem being open.

II. Contractions for ODE systems

In this section we study systems described by nonlinear deterministic systems of differential equations

$$\dot{x} \;=\; f(x,t)\,, \tag{1}$$

where $x(t) \in V \subseteq \mathbb{R}^n$ is an n dimensional vector corresponding to the state of the system, $t \in [0,\infty)$ is the time, and f is a nonlinear vector field which is differentiable on x. We assume that $f(x,t)$, as well as

$$J_f(x,t) = \frac{\partial f}{\partial x}(x,t)\,,$$

which denotes the Jacobian of f, are continuous in (x,t). The goal is to find a condition that guarantees that any two trajectories of (1) converge to each other exponentially.

As mentioned in the introduction, we focus here on conditions based on matrix measures. We recall (see for instance [1] or [2]) that, given a vector norm on Euclidean space ($|\cdot|$), with its induced matrix norm $\|A\|$, the associated *matrix measure* μ is defined as the directional derivative of the matrix norm in the direction of A and evaluated at the identity matrix, that is:

$$\mu(A) \;:=\; \lim_{h\to 0^+} \frac{1}{h}\left(\|I + hA\| - 1\right).$$

The limit is known to exist, and the convergence is monotonic, see [38], [5]. For example, if $|\cdot|$ is the standard Euclidean 2-norm, then $\mu(A)$ is the maximum eigenvalue of the symmetric part of A. Matrix measures, also known as "*logarithmic norms*", were independently introduced by Germund Dahlquist and Sergei Lozinskii in 1959, [5], [39]. More generally, it is useful, with a view to applications to infinite dimensional systems and in particular PDEs, to study generalizations to more general operators in Banach spaces.

A. Logarithmic Lipschitz constants

We now define and state elementary properties of logarithmic Lipschitz constants. (For applications to ODE's, we will always take $X = Y$ in the definitions to follow.)

Definition 1: [18] Let $(X, \|\cdot\|_X)$ be a normed space and $f\colon Y \to X$ be a function, where $Y \subseteq X$. The least upper bound (lub) Lipschitz constant of f induced by the norm

$\|\cdot\|_X$, on Y, is defined by

$$L_{Y,X}[f] = \sup_{u\neq v\in Y} \frac{\|f(u)-f(v)\|_X}{\|u-v\|_X}.$$

Note that $L_{Y,X}[f] < \infty$ if and only if f is Lipschitz on Y.

Definition 2: [18] Let $(X, \|\cdot\|_X)$ be a normed space and $f\colon Y \to X$ be a Lipschitz function. The least upper bound (lub) logarithmic Lipschitz constant of f induced by the norm $\|\cdot\|_X$, on $Y \subseteq X$, is defined by

$$M_{Y,X}[f] = \lim_{h\to 0+} \frac{1}{h}\left(L_{Y,X}[I+hf]-1\right),$$

or equivalently, it is equal to

$$\lim_{h\to 0^+} \sup_{u\neq v\in Y} \frac{1}{h}\left(\frac{\|u-v+h(f(u)-f(v))\|_X}{\|u-v\|_X}-1\right).$$

If $X = Y$, we write M_X instead of $M_{X,X}$. Whenever it is clear from the context, we drop the subscript and simply write M instead of $M_{X,Y}$.

Notation 1: Under the conditions of Definition 2, let $M^{\pm}_{Y,X}$ denote

$$\sup_{u\neq v\in Y} \lim_{h\to 0^\pm} \frac{1}{h}\left(\frac{\|u-v+h(f(u)-f(v))\|_X}{\|u-v\|_X}-1\right).$$

If $X = Y$, we write $M^{\pm}_X$ instead of $M^{\pm}_{X,X}$. Whenever it is clear from the context, we drop the subscript and simply write $M^{\pm}$ instead of $M^{\pm}_{X,Y}$.

Remark 1: [19], [18] Another way to define $M^{\pm}$ is by the concept of semi inner product which is in fact a generalization of inner product to non Hilbert spaces. Let $(X, \|\cdot\|_X)$ be a normed space. For $x_1, x_2 \in X$, the right and left semi inner products are defined by

$$(x_1, x_2)_{\pm} = \|x_1\|_X \lim_{h\to 0^\pm} \frac{1}{h}\left(\|x_1+hx_2\|_X - \|x_1\|_X\right).$$

In particular, when $\|\cdot\|_X$ is induced by a true inner product $(\cdot,\cdot)$, (for example when X is a Hilbert space), then $(\cdot,\cdot)_- = (\cdot,\cdot)_+ = (\cdot,\cdot)$.

Using this definition,

$$M^{\pm}_{Y,X}[f] = \sup_{u\neq v\in Y} \frac{(u-v, f(u)-f(v))_{\pm}}{\|u-v\|^2_X}.$$

The following elementary properties of semi inner products are consequences of the properties of norms. See [19], [18] for a proof.

Proposition 1: For $x, y, z \in X$ and $\alpha \geq 0$,

1) $(x, -y)_{\pm} = -(x, y)_{\mp}$;
2) $(x, \alpha y)_{\pm} = \alpha (x, y)_{\pm}$;
3) $(x, y)_- + (x, z)_{\pm} \leq (x, y+z)_{\pm} \leq (x, y)_+ + (x, z)_{\pm}$.

Remark 2: For any operator $f\colon Y \subset X \to X$:

$$M^-_{Y,X}[f] \leq M^+_{Y,X}[f] \leq M_{Y,X}[f].$$

However, $M^-[f] = M^+[f] = M[f]$ if the norm is induced by an inner product.

For linear f, one has the reverse of the second inequality as well, so $M^+_{Y,X}[f] = M_{Y,X}[f]$. See [40] for a detailed proof. When identifying a linear operator $f : \mathbb{R}^n \to \mathbb{R}^n$ with its matrix representation A with respect to the canonical basis, we write "$\mu(A)$" instead of $M^+_X[f]$, and call M or μ a "matrix measure".

Remark 3: For a linear operator f, M and M^+ can be written as follows:

$$M_{Y,X}[f] = \lim_{h\to 0^+} \sup_{u\neq 0\in Y} \frac{1}{h}\left(\frac{\|u+hf(u)\|_X}{\|u\|_X}-1\right) \quad (2)$$

and

$$M^+_{Y,X}[f] = \sup_{u\neq 0\in Y} \lim_{h\to 0^+} \frac{1}{h}\left(\frac{\|u+hf(u)\|_X}{\|u\|_X}-1\right). \quad (3)$$

Notation 2: In this work, for $(X, \|\cdot\|_X) = (\mathbb{R}^n, \|\cdot\|_p)$, where $\|\cdot\|_p$ is the L^p norm on $\mathbb{R}^n$, for some $1 \leq p \leq \infty$, we sometimes use the notation "M_p" instead of M_X for the least upper bound logarithmic Lipschitz constant, and by "$M_{p,Q}$" we denote the least upper bound logarithmic Lipschitz constant induced by the weighted L^p norm, $\|u\|_{p,Q} := \|Qu\|_p$ on $\mathbb{R}^n$, where Q is a fixed nonsingular matrix. Note that $M_{p,Q}[A] = M_p[QAQ^{-1}]$.

Remark 4: In Table I, the algebraic expression of the least upper bound logarithmic Lipschitz constant induced by the L^p norm for $p = 1, 2$, and ∞ are shown for matrices. For proofs, see for instance [41].

TABLE I: STANDARD MATRIX MEASURES FOR A REAL $n \times n$ MATRIX, $A = [a_{ij}]$.

vector norm, $\|\cdot\|$	induced matrix measure, $M[A]$
$\|x\|_1 = \sum_{i=1}^n \|x_i\|$	$M_1[A] = \max_j \left(a_{jj} + \sum_{i\neq j} \|a_{ij}\|\right)$
$\|x\|_2 = \left(\sum_{i=1}^n \|x_i\|^2\right)^{\frac{1}{2}}$	$M_2[A] = \max_{\lambda\in \text{spec}\frac{1}{2}(A+A^T)} \lambda$
$\|x\|_\infty = \max_{1\leq i\leq n} \|x_i\|$	$M_\infty[A] = \max_i \left(a_{ii} + \sum_{i\neq j} \|a_{ij}\|\right)$

The following subadditivity property is key to diffusive interconnection analysis.

Proposition 2: [18] Let $(X, \|\cdot\|_X)$ be a normed space. For any $f, g\colon Y \to X$ and any $Y \subseteq X$:

1) $M^+_{Y,X}[f+g] \leq M^+_{Y,X}[f] + M^+_{Y,X}[g]$;
2) $M^+_{Y,X}[\alpha f] = \alpha M_{Y,X}[f]$ for $\alpha \geq 0$.

The (lub) logarithmic Lipschitz constant makes sense even if f is not differentiable. However, the constant can be tightly estimated, for differentiable mappings on convex subsets of finite-dimensional spaces, by means of Jacobians.

Lemma 1: [42] For any given norm on $X = \mathbb{R}^n$, let M be the (lub) logarithmic Lipschitz constant induced by this norm. Let Y be a connected subset of $X = \mathbb{R}^n$. Then for any (globally) Lipschitz and continuously differentiable function $f\colon Y \to \mathbb{R}^n$,

$$\sup_{x\in Y} M_X[J_f(x)] \leq M_{Y,X}[f]$$

Moreover, if Y is convex, then

$$\sup_{x\in Y} M_X[J_f(x)] = M_{Y,X}[f]\,.$$

Note that for any $x \in Y$, $J_f(x)\colon X \to X$. Therefore, we use M_X instead of $M_{X,X}$, as we said in Definition 2.

We also recall a notion of generalized derivative, that can be used when taking derivatives of norms (which are not differentiable).

Definition 3: The upper left and right Dini derivatives for any continuous function, $\Psi\colon [0,\infty) \to \mathbb{R}$, are defined by

$$\left(D^{\pm}\Psi\right)(t) = \limsup_{h\to 0^{\pm}} \frac{1}{h}\left(\Psi(t+h) - \Psi(t)\right).$$

Note that $D^{+}\Psi$ and/or $D^{-}\Psi$ might be infinite.

B. Single System of ODEs

Definition 4: [43] Given a norm $\|\cdot\|$, the system (1), or the time-dependent vector field f, is said to be *infinitesimally contracting with respect to this norm* on a set $V \subseteq \mathbb{R}^n$ if there exists some norm in V, with associated matrix measure M, such that, for some constant $c > 0$ (the *contraction rate*), it holds that:

$$M\left[J_f\left(x,t\right)\right] \le -c, \quad \forall x \in V, \quad \forall t \ge 0. \tag{4}$$

The key result is that infinitesimal contractivity implies global contractivity, see [3], [4], [10], [17], [44]:

Theorem 1: Suppose that V is a convex subset of $\mathbb{R}^n$ and that $f(x,t)$ is infinitesimally contracting with respect to a norm, $\|\cdot\|$, with contraction rate c. Then, for every two solutions $x(t)$ and $y(t)$ of (1), that remain in V, it holds that:

$$\|x(t) - y(t)\| \;\le\; e^{-ct}\|x(0) - y(0)\|, \qquad \forall\, t \ge 0\,. \tag{5}$$

To prove Theorem 1, we will use the following general result (with a proof as given in [45], and applied with "$-c$"), which estimates rates of contraction (or expansion, if $c > 0$) among any two functions, even functions that are not solutions of the same system of ODEs (see comment on observers to follow):

Lemma 2: Let $(X, \|\cdot\|_X)$ be a normed space and $G\colon Y \times [0,\infty) \to X$ be a C^1 function, where $Y \subseteq X$. Suppose $u, v\colon [0,\infty) \to Y$ satisfy

$$(\dot{u} - \dot{v})(t) \;=\; G_t(u(t)) - G_t(v(t)),$$

where $\dot{u} = \frac{du(t)}{dt}$ and $G_t(u) = G(u,t)$. Let

$$c := \sup_{t\in[0,\infty)} M_{Y,X}\left[G_t\right].$$

Then for all $t \in [0,\infty)$,

$$\|u(t) - v(t)\|_X \;\le\; e^{ct}\|u(0) - v(0)\|_X. \tag{6}$$

Remark 5: In the finite-dimensional case, Lemma 2 can be verified in terms of Jacobians. Indeed, suppose that $X = \mathbb{R}^n$, and that Y is a convex subset of $\mathbb{R}^n$. Then, by Lemma 1,

$$c \;=\; \tilde{c} := \sup_{(t,w)\in[0,\infty)\times Y} M_X\left[J_{G_t}(w)\right].$$

Therefore,

$$\|u(t) - v(t)\|_X \;\le\; e^{\tilde{c}t}\|u(0) - v(0)\|_X.$$

In fact, in the finite-dimensional case, a more direct proof of Lemma 2 can instead be given. We sketch it next. Let $z(t) = u(t) - v(t)$. We have that

$$\dot{z}(t) = A(t)z(t),$$

where $A(t) = \int_0^1 \frac{\partial f}{\partial x}\left(su(t) + (1-s)v(t)\right)\;ds$. Now, by subadditivity of matrix measures, which, by continuity, extends to integrals, we have:

$$M[A(t)] \le \sup_{w\in V} M\left[\frac{\partial f}{\partial x}(w)\right].$$

Applying Coppel's inequality, (see e. g. [46]), gives the result.

Proof of Theorem 1. Since f is infinitesimally contracting, i.e.,

$$\sup_{(x,t)} M\left[J_f\left(x,t\right)\right] = -c\,,$$

and

$$\dot{x} - \dot{y} \;=\; f(x,t) - f(y,t),$$

by Remark 5, (5) can be obtained. ∎

Note that we use the convexity of V to apply Remark 5 (or Lemma 1). One can prove Theorem 1 for any arbitrary V but for

$$M\left[f\left(x,t\right)\right] \le -c, \quad \forall x \in V, \quad \forall t \ge 0,$$

instead of

$$M\left[J_f\left(x,t\right)\right] \le -c, \quad \forall x \in V, \quad \forall t \ge 0.$$

In addition, one can prove the converse of Theorem 1 for any arbitrary V, but for

$$M^{+}\left[f\left(x,t\right)\right] \le -c, \quad \forall x \in V, \quad \forall t \ge 0,$$

see Proposition 3 below for more details.

Remark 6: The statement of Lemma 2 allows for considerably more generality than Theorem 1. Suppose for example that we consider a standard observer configuration:

$$\begin{aligned}\dot{x} &= f(x,u)\\ \dot{z} &= f(z,u) + K(h(z) - h(x))\end{aligned}$$

where h is an output function and K is an observer gain matrix. Let

$$G_t(y) \;:=\; f(y, u(t)) + Kh(y)$$

evaluated along any given solution with an input u. Then, $\dot{z} - \dot{x} = G_t(z) - G_t(x)$, and thus, if G_t has a contractivity property, it follows that the error $z - x$ between the estimate and the state converges exponentially to zero, by Lemma 2. (Theorem 1 does not apply, since x and z solve different equations.) This recovers the standard Luenberger observer construction for linear time-invariant systems.

Corollary 1: Under the assumptions of Theorem 1:

- If $\mathcal{A}$ is a non-empty forward-invariant set for the dynamics, then every solution must approach $\mathcal{A}$. Indeed, take any trajectory $x(t)$ and a trajectory $y(t)$ with $y(0) \in \mathcal{A}$. Then

$$\text{dist}\,(x(t), \mathcal{A}) \leq \|(x-y)(t)\| \;\leq\; e^{-ct}\|(x-y)(0)\| \to 0,$$

as $t \to \infty$.

- If an equilibrium exists, then it must be unique and globally asymptotically stable.

When contractive systems are forced by periodic signals, they are "entrained", in the sense that solutions converge to unique limit cycles. This property is very important in applications, see for example [26] and [44].

Definition 5: Given a number $T > 0$, we will say that system (1) is *T-periodic* if it holds that $f(x, t+T) = f(x,t) \quad \forall t \geq 0, x \in V$. Notice that a system $\dot{x} = f(x, u(t))$ with input $u(t)$ is T-periodic if $u(t)$ is itself a periodic function of period T.

The basic theoretical result about periodic orbits is as follows. For more details see [10], [11], [43].

Theorem 2: Suppose that:

- V is a closed convex subset of $\mathbb{R}^n$;
- f is infinitesimally contracting with contraction rate c;
- f is T-periodic.

Then, there is a unique periodic solution $\hat{x}(t) : [0,\infty) \to V$ of (1) of period T and, for every solution $x(t)$, it holds that $\|x(t) - \hat{x}(t)\| \to 0$ as $t \to \infty$.

Proof of Theorem 2. We denote by $\varphi(t,s,\xi)$ the value of the solution $x(t)$ at time t of the differential equation (1) with initial value $x(s) = \xi$. Define now $P(\xi) = \varphi(T,0,\xi)$, where $\xi = x(0) \in V$.

Claim. $P^k(\xi) = \varphi(kT,0,\xi)$ for all positive integers k and $\xi \in V$.

We will prove the claim by recursion. In particular, the statement is true by definition when $k = 1$. Inductively, assuming it is true for k, we have:

$$\begin{aligned} P^{k+1}(\xi) &= P(P^k(\xi)) = \varphi(T,0,P^k(\xi)) \\ &= \varphi(T,0,\varphi(kT,0,\xi)) = \varphi(kT+T,0,\xi). \end{aligned}$$

This proves the claim.

Observe that P is a contraction with factor $e^{-cT} < 1$: $\|P(\xi) - P(\zeta)\| \leq e^{-cT}\|\xi - \zeta\|$ for all $\xi, \zeta \in V$, as a consequence of Theorem 1. The set V is a closed subset of $\mathbb{R}^n$ and hence is complete as a metric space with respect to the distance induced by the norm being considered. Thus, by the contraction mapping theorem, there is a (unique) fixed point $\bar{\xi}$ of P. Let $\hat{x}(t) := \varphi(t,0,\bar{\xi})$. Since $\hat{x}(T) = P(\bar{\xi}) = \bar{\xi} = \hat{x}(0)$, $\hat{x}(t)$ is a periodic orbit of period T. Moreover, again by Theorem 1, we have that $\|x(t) - \hat{x}(t)\| \leq e^{-ct}\|\xi - \bar{\xi}\| \to 0$. Uniqueness is clear, since two different periodic orbits would be disjoint compact subsets, and hence at positive distance from each other, contradicting convergence. This completes the proof. ∎

The next result is for the special case of Euclidean norms.

Lemma 3: Suppose that P is a positive definite matrix and A is an arbitrary matrix.

1) If $M_{2,P}[A] = \mu$, then $QA + A^TQ \leq 2\mu Q$, where $Q = P^2$.
2) If for some positive definite matrix Q, $QA + A^TQ \leq 2\mu Q$, then there exists a positive definite matrix P such that $P^2 = Q$ and $M_{2,P}[A] \leq \mu$.

Proof: First suppose $M_{2,P}[A] = \mu$. By definition of μ:

$$\frac{1}{2}\left(PAP^{-1} + \left(PAP^{-1}\right)^T\right) \leq \mu I.$$

Since P is symmetric, so is P^{-1}, so

$$PAP^{-1} + P^{-1}A^TP \leq 2\mu I.$$

Now multiplying the last inequality by P on the right and the left, we get:

$$P^2A + A^TP^2 \leq 2\mu P^2.$$

This proves 1. Now assume that for some positive definite matrix Q, $QA + A^TQ \leq 2\mu Q$. Since $Q > 0$, there exists $P > 0$ such that $P^2 = Q$; moreover, because Q is symmetric, so is P. Hence we have:

$$P^2A + A^TP^2 \leq 2\mu P^2.$$

Multiplying the last inequality by P^{-1} on the right and the left, we conclude 2. ∎

Remark 7: Lemma 3 implies that, *for linear time-invariant systems $\dot{x} = Ax$, contractivity with respect to some weighted L^2 norm* (with a not necessarily diagonal weighting matrix) *is equivalent to A being a Hurwitz matrix.* One direction is clear, as contractivity obviously implies stability. Conversely, suppose that A is Hurwitz. Then, one may pick a quadratic Lyapunov function $V(x) = x^TQx$, where Q is a positive definite matrix. By definition of Lyapunov function, $QA + A^TQ \leq -\beta I$, for some $\beta > 0$. Letting $\gamma := \beta/\lambda_{\max}$, where $\lambda_{\max}$ is the largest eigenvalue of Q, we have that also $QA + A^TQ \leq -\gamma Q$. (Conversely, an inequality of the type $QA + A^TQ \leq -\mu Q$ implies that $QA + A^TQ \leq -\beta I$, if we define $\beta := \mu\lambda_{\min}$ and $\lambda_{\min}$ is the smallest eigenvalue of Q.) Thus, there is a positive definite P so that $M_{2,P}[A] \leq -\gamma/2 < 0$, showing contractivity with respect to the P-weighted L^2 norm. Of course, contractivity with respect to a *diagonally* weighted norm, a property which is required in the interconnection and PDE results mentioned later, imposes additional requirements even in the linear time invariant case, amounting to asking that a quadratic Lyapunov function's principal axes align with the natural coordinates in $\mathbb{R}^n$. Systems admitting such Lyapunov functions are often called "diagonally stable" [47], and the study of diagonal stability is closely related to passivity [48].

The significance of Theorem 1 is that it is true for any norm. Different norms are appropriate to different problems, just as different Lyapunov functions have to be carefully chosen when analyzing a nonlinear system. The choice of norms is a key step in the application of contraction techniques. Non-

Euclidean (i.e., not weighted L^2) norms have been found to be useful in the study of many problems. To illustrate this fact, we next provide an example of a biochemical model which can be shown to be contractive by applying Theorem 1 when using a weighted L^1 norm, but which is not contractive in any weighted L^p norm, for any $p > 1$. The proof that L^1 norms suffice for this example is from [44], and the proof of non-contractivity in L^p, $p > 1$, is from [40].

C. Example: Biochemical Model

A typical biochemical reaction is one in which a molecule X (whose concentration is quantified by the non-negative variable $x = x(t)$) binds to a second molecule S (whose concentration is quantified by $s = s(t) \geq 0$), to produce a dimer Y (whose concentration is quantified by $y = y(t) \geq 0$), and the molecule X is subject to degradation and dilution (at rate δx, where $\delta > 0$) and production according to an external signal $z = z(t) \geq 0$. Examples of such reactions might be an enzyme binding to a substrate to produce a complex, or a transcription factor binding to an unoccupied promoter to make an active promoter, and the enzyme or the transcription factor is itself being continuously created and destroyed. The diagram for such a reaction is as follows:

$$0 \xrightarrow{z} X \xrightarrow{\delta} 0\,, \quad X + S \underset{k_1}{\overset{k_2}{\rightleftharpoons}} Y.$$

Using mass-action kinetics, and assuming a well-mixed reaction in a large volume, the system of chemical reactions is given by:

$$\begin{aligned}\dot{x} &= z(t) - \delta x + k_1 y - k_2 s x\\ \dot{y} &= -k_1 y + k_2 s x\\ \dot{s} &= k_1 y - k_2 s x.\end{aligned}$$

We observe that $y(t) + s(t) = S_Y$ remains constant along solutions. Thus we can study the following reduced system:

$$\begin{aligned}\dot{x} &= z(t) - \delta x + k_1 y - k_2(S_Y - y)x\\ \dot{y} &= -k_1 y + k_2(S_Y - y)x.\end{aligned}$$

Note that

$$(x(t), y(t)) \in V = [0,\infty) \times [0, S_Y]$$

for all $t \geq 0$ (V is convex and forward-invariant), and S_Y, k_1, k_2, δ, d_1, and d_2 are arbitrary positive constants.

Let J_F be the Jacobian of

$$F = (z - \delta x + k_1 y - k_2(S_Y - y)x, -k_1 y + k_2(S_Y - y)x)^T,$$

$$J_F \;:=\; \begin{pmatrix} -\delta - k_2(S_Y - y) & k_1 + k_2 x \\ k_2(S_Y - y) & -(k_1 + k_2 x)\end{pmatrix}.$$

Following [44], we show

$$\sup_{(x,y)\in V} M_{1,Q}\left[J_F(x,y)\right] < 0,$$

where

$$Q = \mathrm{diag}\left(1, 1 + \delta/(k_2 S_Y) - \zeta\right),$$

and we will pick a suitable $0 < \zeta < \frac{\delta}{k_2 S_Y}$. We will find a $q > 1$ such that the above holds with $Q = \mathrm{diag}\,(1,q)$. For any such q, we can always find ζ such that $q := 1 + \frac{\delta}{k_2 S_Y} - \zeta > 1$. With this form for Q,

$$QJ_FQ^{-1} \;=\; \begin{pmatrix} -\delta - a & \dfrac{b}{q} \\ aq & -b \end{pmatrix},$$

where $a = k_2(S_Y - y) \in [0, k_2 S_Y]$ and $b = k_1 + k_2 x \in [k_1, \infty)$. Since $a \geq 0$, $b > 0$, and assuming $q > 0$, by Table I, we have:

$$\begin{aligned} M_{1,Q}[J_F] &= M_1\left[QJ_FQ^{-1}\right] \\ &= \max\{-\delta - a + |aq|\,, -b + |b/q|\} \\ &= \max\{-\delta + a(q-1), b\,(1/q - 1)\}.\end{aligned}$$

So to show that $M_{1,Q}[J_F] < 0$, we need to find a range for the values of q such that:

$$-\delta + a(q-1) < 0, \tag{7}$$

and

$$b\left(\frac{1}{q} - 1\right) < 0. \tag{8}$$

Equation (8) holds if and only if $q > 1$. So we need to find an appropriate $q > 1$ such that Equation (7) holds:

$$\begin{aligned} -\delta + a(q-1) \;&<\; 0 \quad \text{iff} \\ q \;&<\; 1 + \frac{\delta}{a} \;=\; 1 + \frac{\delta}{k_2(S_Y - y)} \;<\; 1 + \frac{\delta}{k_2 S_Y}.\end{aligned}$$

Hence for $Q = \mathrm{diag}\,(1,q)$, with $1 < q < 1 + \dfrac{\delta}{k_2 S_Y}$, $M_{1,Q}[J_F] < 0$. Therefore, by Theorem 1, the system is contracting. Note that a *weighted* L^1 norm is necessary, since with $Q = I$ we obtain $M_1 = 0$.

We will show that for any $p > 1$ and any diagonal Q, it is not true that $M_{p,Q}[J_F(x,y)] < 0$ for all $(x,y) \in V$.

We first consider the case $p \neq \infty$. We will show that there exists $(x_0, y_0) \in V$ such that for any small $h > 0$, $\|I + hQJ_F(x_0,y_0)Q^{-1}\|_p > 1$. This will imply $M_{p,Q}\left[J_F(x_0,y_0)\right] \geq 0$. Computing explicitly, we have the following expression for $\|I + hQJ_FQ^{-1}\|_p$:

$$\begin{aligned} &\sup_{(\xi_1,\xi_2)\neq(0,0)} \frac{\left(\left|\xi_1 - h(\delta+a)\xi_1 + h\dfrac{b\xi_2}{q}\right|^p + |haq\xi_1 + \xi_2 - hb\xi_2|^p\right)^{\frac{1}{p}}}{(|\xi_1|^p + |\xi_2|^p)^{\frac{1}{p}}} \\ &\geq \frac{\left(\left|1 - h(\delta+a) + h\dfrac{b\lambda}{q}\right|^p + |haq + \lambda - hb\lambda|^p\right)^{\frac{1}{p}}}{(1 + |\lambda|^p)^{\frac{1}{p}}},\end{aligned}$$

where we take a point of the form $(\xi_1, \xi_2) = (1, \lambda)$, for a $\lambda > 0$ which will be determined later. To show

$$\frac{\left(\left|1 - h(\delta+a) + h\dfrac{b\lambda}{q}\right|^p + |haq + \lambda - hb\lambda|^p\right)^{\frac{1}{p}}}{(1 + |\lambda|^p)^{\frac{1}{p}}} > 1,$$

we will equivalently show that for any small enough $h > 0$:

$$\frac{1}{h}\left(\left|1 - h(\delta+a) + h\frac{b\lambda}{q}\right|^p + |haq + \lambda - hb\lambda|^p - 1 - |\lambda|^p\right) \tag{9}$$

is positive. Note that the $\lim\limits_{h\to 0^+}$ of the left hand side of the above inequality is $f'(0)$ where

$$f(h) \;=\; \left|1+h(\frac{b\lambda}{q}-(\delta+a))\right|^p + |\lambda+h(aq-b\lambda)|^p .$$

Therefore, it suffices to show that $f'(0) > 0$ for some value $(x_0, y_0) \in V$ (because $f'(0) > 0$ implies that there exists $h_0 > 0$ such that for $0 < h < h_0$, the expression in (9) is positive). Since $p > 1$, by assumption, f is differentiable and

$$\begin{aligned} f'(h) = p\left(\frac{b\lambda}{q}-(\delta+a)\right)&\left|1+h\left(\frac{b\lambda}{q}-(\delta+a)\right)\right|^{p-2} \\ &\left(1+h\left(\frac{b\lambda}{q}-(\delta+a)\right)\right) \\ &+p(aq-b\lambda)\left|\lambda+h(aq-b\lambda)\right|^{p-2}(\lambda+h(aq-b\lambda)) . \end{aligned}$$

(Note that $\frac{d}{dx}|u(x)|^p = |u(x)|^{p-2}u(x)\,\frac{du}{dx}(x)$.)
Hence, since $\lambda > 0$

$$\begin{aligned} f'(0) &= p\left(\frac{b\lambda}{q}-(\delta+a)\right)+p(aq-b\lambda)\lambda^{p-1} \\ &= p\left(\frac{b\lambda}{q}-a\right)(1-\lambda^{p-1}q)-p\delta. \end{aligned}$$

Choosing λ small enough such that $1-\lambda^{p-1}q > 0$ and choosing x, or equivalently b, large enough, we can make $f'(0) > 0$.
For $p = \infty$, using Table I,

$$M_p\left[QJ_FQ^{-1}\right] = \max\left\{-\delta-a+\left|\frac{b}{q}\right|, -b+|aq|\right\}.$$

For large enough x,

$$-\delta-a+|b/q| > 0$$

(and $-b+aq < 0$) and hence $M_\infty\left[QJ_FQ^{-1}\right] > 0$.

D. Some relations to dissipative operators and passivity

In this section, we show that the converse of Theorem 1 is true as well, and in fact that contractivity is equivalent to a number of other inequalities. After that, we review the definitions of accretive and dissipative operators on Banach spaces, and see how these are related to contractive operators. We also provide a remark about passivity.

The following result summarizes the basic equivalences.

Proposition 3: Consider

$$\dot{x} = f(x,t), \tag{10}$$

where $x(t) \in Y \subset X$, X is a Banach space with norm $\|\cdot\|$, and $t \in [0,\infty)$. We assume that

$$f\colon Y\times[0,\infty)\to X$$

is globally (uniformly) Lipschitz vector field in x and continuous in (x,t).

Then the following are equivalent:

1) For any two solutions x, y of (10), and all $t, s \geq 0$,
$$\|x(t+s)-y(t+s)\| \leq e^{ct}\|x(s)-y(s)\|.$$
2) For any $t \geq 0$,
$$M^+[f_t] \leq c,$$
where $f_t(x) = f(x,t)$.
3) For any two points x, y, and any $t \geq 0$
$$(x-y, f_t(x)-f_t(y))_+ \leq c\|x-y\|^2.$$
4) For any two solutions x, y of (10), and all $t \geq 0$,
$$D^+\|(x-y)(t)\| \leq c\|(x-y)(t)\|.$$

Proof:

- $1 \Rightarrow 2$. Fix $s \geq 0$ and let $a \neq b \in Y$ be arbitrary. For $t \geq s$, let $x(t), y(t)$ be the solutions of (10) with $x(s) = a$ and $y(s) = b$ respectively.
$$\begin{aligned} &\|x(s+h)-y(s+h)\| \\ &\quad= \|x(s)-y(s)+h(f_s(x(s))-f_s(y(s)))+o(h)\| \\ &\quad\leq e^{ch}\|x(s)-y(s)\| \end{aligned}$$
Therefore, by subtracting $\|x(s)-y(s)\|$ from both sides of the above inequality and dividing by $h > 0$, and taking the $\lim_{h\to0^+}$, we get: (we let $x(s) = a$ and $y(s) = b$ for simplicity)
$$\begin{aligned} &\lim_{h\to0^+}\frac{\|a-b+h(f_s(a)-f_s(b))+o(h)\|-\|a-b\|}{h} \\ &\quad\leq \lim_{h\to0^+}\frac{e^{ch}-1}{h}\,\|a-b\|, \end{aligned}$$
dividing by $\|a-b\|$, we get:
$$\lim_{h\to0^+}\frac{\|a-b+h(f_s(a)-f_s(b))\|-\|a-b\|}{h\|a-b\|} \quad\leq\quad c,$$
and now taking sup over all $a \neq b \in Y$, we get:
$$M^+[f_s] \leq c.$$
- $2 \Rightarrow 3$. For any fixed t, and any $x \neq y \in Y$
$$\begin{aligned} &(x-y, f_t(x)-f_t(y))_+ \\ &\quad= \|x-y\|\lim_{h\to0^+}\frac{\|x-y+h(f_t(x)-f_t(y))\|-\|x-y\|}{h} \\ &\quad= \|x-y\|^2\lim_{h\to0^+}\frac{\|x-y+h(f_t(x)-f_t(y))\|-\|x-y\|}{h\|x-y\|} \\ &\quad\leq M^+[f_t]\|x-y\|^2 \\ &\quad\leq c\|x-y\|^2. \end{aligned}$$
- $3 \Rightarrow 4$. Using the definition of upper Dini derivative, we have: (we drop the argument t for simplicity)
$$\begin{aligned} &D^+\|(x-y)(t)\| \\ &\quad= \limsup_{h\to0^+}\frac{1}{h}\,(\|(x-y)(t+h)\|-\|(x-y)(t)\|) \\ &\quad= \limsup_{h\to0^+}\frac{1}{h}\,(\|x-y+h(\dot{x}-\dot{y})\|-\|x-y\|) \\ &\quad= \lim_{h\to0^+}\frac{1}{h}\,(\|x-y+h(f_t(x)-f_t(y))\|-\|x-y\|) \end{aligned}$$
Note that if $(x-y)(t) = 0$, then using the above inequality $D^+\|(x-y)(t)\| = 0$, and 4 hold. Assume that $(x-y)(t) \neq 0$. Multiplying both sides of the above

inequality by $\|(x-y)(t)\|$, we get:

$$\begin{aligned}
&\|(x-y)(t)\| D^{+}\|(x-y)(t)\| \\
&= \|x-y\| \lim_{h \to 0^+} \frac{1}{h}\left(\|x-y+h(f_t(x)-f_t(y))\|-\|x-y\|\right) \\
&= ((x-y)(t), f_t(x(t))-f_t(y(t)))_+ \\
&\leq c\|(x-y)(t)\|^2 \quad \text{using 3.}
\end{aligned}$$

Dividing by $(x-y)(t) \neq 0$, we get 4.

- $4 \Rightarrow 1$. Let $\phi(t) := \|(x-y)(t)\|$. A simple calculation shows that
$$D^{+}\left(\phi(t) e^{-ct}\right) \leq D^{+}\phi(t) e^{-ct}+\phi(t)(-c e^{-ct}) \leq 0.$$
Applying Gronwall's Lemma in the form given for example in [49], Appendix A, we have that
$$\phi(t) \leq e^{ct}\phi(0)$$
for all t, as desired.

■

Note that even if Y is a convex subset of X, $1 \iff 2$, in Proposition 3 is not a generalization of Theorem 1, because $M^{+}[f] \leq c$ doesn't imply $M[f] \leq c$, in general.

Definition 6: [19] An $F\colon Y \subset X \to X$ satisfying
$$(x-y, F(x)-F(y))_+ \geq 0, \quad \text{for any } x, y \in Y$$
is said to be accretive (monotone when $(\cdot,\cdot)_+$ is a true inner product), while F is dissipative if $-F$ is accretive. Equivalently, by the definition of $M^{\pm}$, F is said to be accretive if $M^{+}[F] \geq 0$ and F is dissipative if $M^{+}[-F] \geq 0$, i.e. $M^{-}[F] \leq 0$, (by the definition of $M^{\pm}$ and Proposition 1, part 1).

Note that in Hilbert spaces, $M^{+}[F]=M^{-}[F]=M[F]$. Therefore, $F-cI$ is dissipative, if $M^{-}[F]=M^{+}[F] \leq c$. In particular, when $c<0$, F is dissipative if and only if F is infinitesimally contractive.

There is a connection with passivity, as well. Recall (see e.g. [50]) that, in control-theoretic terminology, an operator F between Hilbert spaces is said to be *passive* provided that $(u, F(u)) \geq 0$ for all inputs u. (More precisely, this property is defined for causal operators, and the inner product is taken for all truncations of the signal. The terminology is motivated by the special case where the input u and output $F(u)$ are a voltage and a current, respectively, and the energy absorbed by the dynamical system, which is the inner product of the input and output, is non-negative.) *Incremental* passivity is usually defined by asking that $(u-v, F(u)-F(v)) \geq 0$ for all inputs u and v (for linear systems this is of course the same as passivity). Thus, in Hilbert spaces (where that all variants of M coincide), the vector field F, seen as a memoryless mapping, is accretive if and only if it is incrementally passive.

III. DIFFUSIVE INTERCONNECTION OF ODE SYSTEMS

In this section, we study a network of identical ODE models which are diffusively interconnected.

The state of the system will be described by a vector x which one may interpret as a vector collecting the states x_i (each of them itself possibly a vector) of identical "agents" which tend to follow each other according to a diffusion rule, with interconnections specified by an undirected graph. Another interpretation, useful in the context of biological modeling, is a set of chemical reactions among species that evolve in separate compartments (e.g., nucleus, cytoplasm, membrane, in a cell); then the x_i's represent the vectors of concentrations of the species in each separate compartment.

In order to formally describe the interconnections, we use the following concepts in this section:

- For a fixed convex subset of $\mathbb{R}^n$, say V, $\tilde{F}\colon V^N \times [0,\infty) \to \mathbb{R}^{nN}$ is a function of the form:
$$\tilde{F}(x,t)=\left(F(x_1,t)^T, \ldots, F(x_N,t)^T\right)^T,$$
where $x=\left(x_1^T, \ldots, x_N^T\right)^T$, with $x_i \in V$ for each i, and $F(\cdot,t)=F_t\colon V \to \mathbb{R}^n$ is a C^1 function.
- For any $x \in V^N$ we define $\|x\|_{p,Q}$ as follows:
$$\|x\|_{p,Q}=\left\|\left(\|Qx_1\|_p, \cdots, \|Qx_N\|_p\right)^T\right\|_p,$$
for a positive diagonal matrix $Q=\operatorname{diag}(q_1, \ldots, q_n)$, and $1 \leq p \leq \infty$.
When $N=1$, we simply have a norm in $\mathbb{R}^n$:
$$\|x\|_{p,Q} := \|Qx\|_p.$$
- $D=\operatorname{diag}(d_1, \ldots, d_n)$ with $d_i \geq 0$, and $d_j>0$ for some j, which we call the diffusion matrix.
- $\mathcal{L} \in \mathbb{R}^{N \times N}$ is a symmetric matrix and $\mathcal{L}\mathbf{1}=0$, where $\mathbf{1}=(1, \ldots, 1)^T$. We think of $\mathcal{L}$ as the Laplacian of a graph that describes the interconnections among component subsystems.
- $\otimes$ denotes the Kronecker product of two matrices.

Definition 7: For any arbitrary graph $\mathcal{G}$ with the associated (graph) Laplacian matrix $\mathcal{L}$, any diagonal matrix D, and any $F\colon V \to \mathbb{R}^n$, the associated $\mathcal{G}-$compartment system, denoted by $(F, \mathcal{G}, D)$, is defined by
$$\dot{x}(t)=\tilde{F}(x(t),t)-(\mathcal{L} \otimes D)x(t), \tag{11}$$
where $x, \tilde{F}$, and D are as defined above.

The "symmetry breaking" phenomenon of diffusion-induced, or Turing, instability refers to the case where a dynamic equilibrium $\bar{u}$ of the non-diffusing ODE system $\dot{x}=F(x,t)$ is stable, but, at least for some diagonal positive matrices D, the corresponding interconnected system (11) is unstable.

The following theorem (from [40]), shows that, for contractive reaction part F, no diffusion instability will occur, no matter what is the size of the diffusion matrix D.

Theorem 3: Consider the system (11). Let
$$c=\sup_{t \in [0,\infty)} M_{p,Q}[F_t],$$
where $M_{p,Q}$ is the (lub) logarithmic Lipschitz constant induced by the norm $\|\cdot\|_{p,Q}$ on $\mathbb{R}^n$ defined by $\|x\|_{p,Q} := \|Qx\|_p$. Then for any two solutions x, y of (11), we have
$$\|x(t)-y(t)\|_{p,Q} \leq e^{ct}\|x(0)-y(0)\|_{p,Q}.$$

Remark 8: Under the assumptions of Theorem 3, by Remark 5, if for any $t \geq 0$ and any x, $M_{p,Q}[J_F(x,t)] \leq c$, then

$$\|x(t) - y(t)\|_{p,Q} \leq e^{ct} \|x(0) - y(0)\|_{p,Q}.$$

Remark 9: Consider the linear system $\dot{x} = Ax$, where $A = \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix}$. It is easy to see that A is Hurwitz. Therefore one may pick a quadratic Lyapunov function $V(x) = x^T Q x$, where Q is a positive definite matrix. By definition of Lyapunov function, $QA + A^T Q$ is negative definite. Next we show that Q must be a *non-diagonal*: Suppose that Q is a positive definite *diagonal* matrix $Q = \operatorname{diag}(q_1, q_2)$, and $V(x) = x^T Q x$. We observe that $QA + A^T Q$ cannot be negative definite. A simple calculation shows that

$$QA + A^T Q = \begin{pmatrix} 2q_1 & 2(q_1 - q_2) \\ 2(q_1 - q_2) & -6q_2 \end{pmatrix}.$$

The matrix $QA + A^T Q$ is negative definite if $q_1 < 0$ and $\det(QA + A^T Q) < 0$. But $q_1 < 0$ contradicts positive definiteness of Q.

Note that if $QA + A^T Q$ was negative definite for some *diagonal* matrix Q, then by Remark 7, for some diagonal positive definite matrix P, $M_{2,P}[A] < 0$ and by Theorem 3, $x = 0$ would remain a stable equilibrium for the interconnected system, with any diffusion matrix D, (D diagonal). On the other hand, for $D = \operatorname{diag}(1/4, 3)$, $x = 0$ loses its stability in the interconnected system:

$$\dot{x} = (A - D)x + Dx'$$
$$\dot{x}' = (A - D)x' + Dx$$

diagrammed in Figure 1.

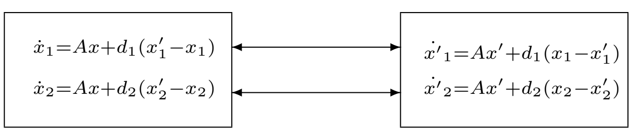

Fig. 1: Interconnection between two systems $\dot{x} = Ax$, for $x = (x_1, x_2)^T$

A simple calculation shows that

$$\begin{pmatrix} A - D & D \\ D & A - D \end{pmatrix}$$

has a positive eigenvalue, and therefore, $(0,0)^T$ cannot be a stable equilibrium. So diffusion de-stabilizes this system.

A. Synchronization

Definition 8: We say that the $\mathcal{G}-$compartment system (11) synchronizes, if for any solution $x = \left(x_1^T, \ldots, x_N^T\right)^T$ of (11), and for all $i, j \in \{1, \ldots, N\}$, $(x_i - x_j)(t) \to 0$ as $t \to \infty$.

An easy first result is as follows.

Proposition 4: Under the assumptions of Theorem 3, if $c < 0$, then the $\mathcal{G}-$compartment system (11) synchronizes.

Proof: Note that $z(t) := (z_1(t), \ldots, z_1(t))^T$ is a solution of (11), where $z_1(t)$ is a solution of $\dot{x} = F(x,t)$. By Theorem 3, if for any $t \geq 0$ and any x, $M_{p,Q}[J_F(x,t)] \leq c$, then for any solution $x(t)$ of (11),

$$\|x(t) - z(t)\|_{p,Q} \leq e^{ct} \|x(0) - z(0)\|_{p,Q}.$$

When $c < 0$, $(x_i - z_1)(t) \to 0$, hence $(x_i - x_j)(t) \to 0$ as $t \to \infty$. ■

In Proposition 4, we imposed a strong condition on F, which in turn leads to the very strong conclusion that all solutions should converge exponentially to a particular solution, no matter the strength of the interconnection (choice of diffusion matrix). A more interesting and challenging problem is to provide a condition that links the vector field, the graph structure, and the matrix D, so that interesting dynamical behaviors (such as oscillations in autonomous systems, which are impossible in contractive systems) can be exhibited by the individual systems, and yet the components synchronize.

In [31], we discuss several matrix measure based conditions that guarantee synchronization of ODE systems in special classes of graphs. As an example, we state a result for the case where systems interconnected through a complete graph.

Consider a $\mathcal{G}-$compartment system with an undirected complete graph $\mathcal{G}$. (Note that an undirected complete graph of N nodes has $m = \binom{N}{2} = \frac{N(N-1)}{2}$ vertices.) The following system of ODEs describes the evolution of the interconnected agents x_i's:

$$\dot{x}_i \;=\; F(x_i, t) + D \sum_{j=1}^{N} (x_j - x_i) \tag{12}$$

Proposition 5: Let $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^n$. Suppose x is a solution of Equation (12) and let

$$c := \sup_{(x,t)} M[J_F(x,t) - ND]$$

where M is the logarithmic norm induced by $\|\cdot\|$. Then

$$\sum_{i=1}^{m} \|e_i(t)\| \;\leq\; e^{ct} \sum_{i=1}^{m} \|e_i(0)\|, \tag{13}$$

where e_i, for $i = 1, \ldots, m$ are the edges of $\mathcal{G}$, meaning the differences $x_i(t) - x_j(t)$ for $i < j$. In particular, when $c < 0$, the system synchronizes.

Proof: For any fixed $i, j \in \{1, \ldots, N\}$, using Equation (12), we have:

$$\dot{x}_i - \dot{x}_j \;=\; G_t(x_i) - G_t(x_j),$$

where $G_t(x) := F(x,t) - NDx$. By Remark 5,

$$\|e_i(t)\| \leq e^{ct} \|e_i(0)\|,$$

where $c = \sup_{(x,t)} M[J_{G_t}(x)] = \sup_{(x,t)} M[J_F(x,t) - ND]$. By taking $\sum$ over all the edges (the differences $x_i - x_j$), we get Equation (13). ■

B. Open problems

Although synchronization of the interconnected system (11) in weighted L^2 norms is a well-understood problem, current results in non-L^2 norms so far only apply to certain special classes of graphs (line, complete, star graphs and the cartesian products of these graphs); good generalizations to arbitrary graphs are the subject of current research.

IV. Contractions for PDE systems

In this section, we study reaction-diffusion PDE systems of the general form:

$$\frac{\partial u_1}{\partial t}(\omega,t) = F_1(u(\omega,t),t) + d_1 \Delta u_1(\omega,t)$$
$$\vdots$$
$$\frac{\partial u_n}{\partial t}(\omega,t) = F_n(u(\omega,t),t) + d_n \Delta u_n(\omega,t)$$

subject to the Neumann boundary condition:

$$\frac{\partial u_i}{\partial \mathbf{n}}(\xi,t) = 0 \quad \forall \xi \in \partial\Omega, \ \ \forall t \in [0,\infty), \ \ \forall i = 1,\ldots,n,$$

which can be written as the following closed form:

$$\begin{aligned} &\frac{\partial u}{\partial t}(\omega,t) = F_t(u(\omega,t)) + D\Delta u(\omega,t) \\ &\frac{\partial u}{\partial \mathbf{n}}(\xi,t) = 0 \quad \forall \xi \in \partial\Omega, \ \ \forall t \in [0,\infty), \end{aligned} \tag{14}$$

where

- $F_t(x) = F(x,t)$ and $F\colon V \times [0,\infty) \to \mathbb{R}^n$ is a (globally) Lipschitz vector field with components F_i:
$$F(x,t) \ = \ (F_1(x,t),\cdots,F_n(x,t))^T,$$
for some functions $F_i\colon V \times [0,\infty) \to \mathbb{R}$, where V is a convex subset of $\mathbb{R}^n$.
- $D = \mathrm{diag}\,(d_1,\ldots,d_n)$ with $d_i \geq 0$, and $d_j > 0$ for some j, which we call the diffusion matrix.
- Ω is a bounded domain in $\mathbb{R}^m$ with smooth boundary $\partial\Omega$ and outward normal $\mathbf{n}$.
- $\dfrac{\partial u}{\partial \mathbf{n}} = \left(\dfrac{\partial u_1}{\partial \mathbf{n}},\ldots,\dfrac{\partial u_n}{\partial \mathbf{n}}\right)^T.$

In biology, a PDE system of this form describes individuals (particles, chemical species, etc.) of n different types, with respective abundances $u_i(\omega,t)$ at time t and location $\omega \in \Omega$, that can react instantaneously, guided by the interaction rules encoded into the vector field F, and can diffuse due to random motion.

We show that if the reaction system is "contractive" in the sense that trajectories globally and exponentially converge to each other with respect to a diagonally weighted L^p norm, then the same property is inherited by the PDE. In particular, if there is an equilibrium $\bar{u}$ of $du/dt = F(u,t)$, it will follow that this equilibrium is globally exponentially stable for the PDE system. When the time-dependence of F on t is periodic (as in the example below when $z(t)$ is periodic), there will be convergence to a (unique) globally asymptotically stable solution, uniform in space. This is because the corresponding ODE system admits a periodic limit cycle, which is also a solution of the associated PDE, as shown in Theorem 2.

Definition 9: By a solution of the PDE

$$\frac{\partial u}{\partial t}(\omega,t) = F_t(u(\omega,t)) + D\Delta u(\omega,t)$$
$$\frac{\partial u}{\partial \mathbf{n}}(\xi,t) = 0 \quad \forall \xi \in \partial\Omega, \ \ \forall t \in [0,\infty),$$

on an interval $[0,T)$, where $0 < T \leq \infty$, we mean a function $u = (u_1,\cdots,u_n)^T$, with $u\colon \bar{\Omega} \times [0,T) \to V$, such that:

1) for each $\omega \in \bar{\Omega}$, $u(\omega,\cdot)$ is continuously differentiable;
2) for each $t \in [0,T)$, $u(\cdot,t)$ is in $\mathbf{Y}$, where
$$\mathbf{Y} \ = \ \left\{ \begin{array}{cc} v\colon \bar{\Omega} \to V, & v = (v_1,\cdots,v_n), \\ v_i \in C^2_{\mathbb{R}}\left(\bar{\Omega}\right), & \frac{\partial v_i}{\partial \mathbf{n}}(\xi) = 0, \ \forall \xi \in \partial\Omega \ \ \forall i \end{array} \right\}$$
and $C^2_{\mathbb{R}}\left(\bar{\Omega}\right)$ is the set of twice continuously differentiable functions $\bar{\Omega} \to \mathbb{R}$; and
3) for each $\omega \in \bar{\Omega}$, and each $t \in [0,T)$, u satisfies the above PDE.

Under the additional assumptions that $F(x,t)$ is twice continuously differentiable with respect to x and continuous with respect to t, theorems on existence and uniqueness for PDEs such as (14) can be found in standard references, e.g. [51], [52]. One must impose appropriate conditions on the vector field, on the boundary of V, to insure invariance of V. Convexity of V insures that the Laplacian also preserves V. Since we are interested here in estimates relating pairs of solutions, we will not deal with existence and well-posedness. Our results will refer to solutions already assumed to exist.

Now we state the main theorem of this section, which is analogous to Theorem 3 in the discrete case. For a proof and more details, see [40].

Theorem 4: Consider the reaction diffusion PDE (14). Let $c = \sup\limits_{t\in[0,\infty)} M_{p,Q}[F_t]$ for some $1 \leq p \leq \infty$, and some positive diagonal matrix Q. Then for every two solutions u,v of the PDE (14) and all $t \in [0,T)$:

$$\|u(\cdot,t) - v(\cdot,t)\|_{p,Q} \leq e^{ct}\,\|u(\cdot,0) - v(\cdot,0)\|_{p,Q}\,.$$

A generalization of Theorem 4, to spatially-varying diffusion, for non diagonal matrices Q, but restricted to $p = 2$, is given in [53].

Back to the example in Section II-C, consider the same system in a spatial domain, (let's say $\Omega = (0,1)$) and assume that the species diffuse. We let the domain Ω represent the part of the cytoplasm where these chemicals are free to diffuse. Taking equal diffusion constants for S and Y (which is reasonable since typically S and Y have approximately the same size), a natural model is given by a reaction diffusion system (dropping the arguments (ω,t) for simplicity)

$$\frac{\partial x}{\partial t} = z(t) - \delta x + k_1 y - k_2 s x + d_1 \Delta x$$
$$\frac{\partial y}{\partial t} = -k_1 y + k_2 s x + d_2 \Delta y$$
$$\frac{\partial s}{\partial t} = k_1 y - k_2 s x + d_2 \Delta s.$$

If we assume that initially S and Y are uniformly distributed, it follows that $\frac{\partial}{\partial t}(y(\omega,t)+s(\omega,t)) = 0$, so $y(\omega,t) + s(\omega,t) = y(\omega,0)+s(\omega,0) = S_Y$ is a constant. Thus we can study the following reduced system:

$$\begin{aligned} \frac{\partial x}{\partial t} &= z(t) - \delta x + k_1 y - k_2(S_Y - y)x + d_1 \Delta x \\ \frac{\partial y}{\partial t} &= -k_1 y + k_2(S_Y - y)x + d_2 \Delta y. \end{aligned} \tag{15}$$

Note that

$$(x(\omega,t), y(\omega,t)) \in V = [0,\infty) \times [0, S_Y]$$

for all $t \geq 0$ and $\omega \in (0,1)$ (V is convex and forward-invariant), and S_Y, k_1, k_2, δ, d_1, and d_2 are arbitrary positive constants.

In Section II-C, we showed that the ODE system is contractive under a weighted L^1 norm. Now by Theorem 4, we conclude that (15) is also contractive, no matter what the d_i's are.

V. Uniform solutions of PDE systems

As in the discrete case, we are also interested in finding some conditions on F and D that guarantee the synchrony behavior of the solutions of the PDE (14). Note that under the conditions of Theorem 4, if $c < 0$, any solution u of the PDE (14) with $u(\omega,0) = u_0(\omega)$ exponentially converges to the spatially uniform solution $\bar{u}(t)$ which is itself the solution of the following ODE system:

$$\begin{aligned} \dot{x} &= F(x,t), \\ x(0) &= \frac{1}{|\Omega|}\int_\Omega u_0(\omega)\, d\omega. \end{aligned} \tag{16}$$

But, just as remarked for interconnections of ODE systems, the condition $c < 0$ rules out any interesting non-equilibrium behavior. So we look for a weaker condition than $c < 0$, to conclude spatial uniform convergence result (which is a weaker property than contraction).

Recall that for any bounded, open subset $\Omega \subset \mathbb{R}^m$, there exists a sequence of positive eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$ (going to ∞) and a sequence of corresponding orthonormal eigenfunctions: $\phi_1, \phi_2, \ldots$ (defining a Hilbert basis of $L^2(\Omega)$) satisfying the following Neumann eigenvalue problem:

$$\begin{aligned} -\Delta \phi_i &= \lambda_i \phi_i \quad \text{in } \Omega \\ \nabla \phi_i \cdot \mathbf{n} &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{17}$$

Note that the first eigenvalue is always zero, $\lambda_1 = 0$, and the corresponding eigenfunction is a nonzero constant ($\phi(\omega) = 1/\sqrt{|\Omega|}$).

The following re-phasing of a theorem from [28], provides a sufficient condition on F and D using the Jacobian matrix of the reaction term and the second Neumann eigenvalue of the Laplacian operator on the given spatial domain to insure the convergence of trajectories, in this case to their space averages in weighted L^2 norms. The proof is based on the use of a quadratic Lyapunov function, which is appropriate for Hilbert spaces. We have translated the result to the language of contractions. (Actually, the result in [28] is stronger, in that it allows for certain non-diagonal diffusion and also certain non-diagonal weighting matrices P, by substituting these assumptions by a commutativity type of condition.)

Theorem 5: Consider the reaction-diffusion system (14). Let

$$c := \sup_{(x,t)\in V\times[0,\infty)} M_{2,P}\left[J_F(x,t) - \lambda_2 D\right],$$

where P is a positive diagonal matrix. Then

$$\|u(\cdot,t) - \tilde{u}(t)\|_{2,P} \;\leq\; e^{ct}\|u(\cdot,0) - \tilde{u}(0)\|_{2,P}. \tag{18}$$

where $\tilde{u}(t) = \frac{1}{|\Omega|}\int_\Omega u(\omega,t)\, d\omega$.

We remark that the L^2 conditions given in [28] do not hold for the biochemical example discussed in Section II-C.

A generalization of Theorem 5 to spatially-varying diffusion is given in [53].

We next prove an analogous result to Theorem 5 for any norm but restricted to the linear operators F, $F(u,t) = A(t)u$, where for any t, $A(t) \in \mathbb{R}^{n\times n}$.

Theorem 6: For a given norm $\|\cdot\|$ in $\mathbb{R}^n$, consider the reaction-diffusion system (14), for a linear operator F. Let

$$c := \sup_{(x,t)\in V\times[0,\infty)} M\left[J_F(x,t) - \lambda_2 D\right],$$

where M is the logarithmic norm induced by $\|\cdot\|$. Then for any $\omega \in \Omega$ and any $t \geq 0$,

$$\begin{aligned} \|u(\omega,t) - \bar{u}(t)\| &\leq \sum_{i\geq 2}\|\alpha_i(t)\phi_i(\omega)\| \\ &\leq e^{ct}\sum_{i\geq 2}\|\alpha_i(0)\phi_i(\omega)\|. \end{aligned}$$

where $\bar{u}(t)$ is the solution of the system (16) with $u_0(\omega) = u(\omega,0)$, and $\alpha_i(t) = \int_\Omega u(\omega,t)\phi_i(\omega)\, d\omega$. In particular, when $c < 0$,

$$\|u(\omega,t) - \bar{u}(t)\| \to 0 \quad \text{exponentially, as } t \to \infty.$$

Proof: We first show that the solution of Equation (16), $\bar{u}$, is equal to $\tilde{u}(t) = \frac{1}{|\Omega|}\int_\Omega u(\omega,t)\, d\omega$. Note that both $\bar{u}$ and $\tilde{u}$ satisfy $\dot{x} = A(t)x$. In addition, by the definition, $\bar{u}(0) = \tilde{u}(0) = \frac{1}{|\Omega|}\int_\Omega u(\omega,0)\, d\omega$. Therefore, by uniqueness of the solutions of ODEs, $\bar{u}(t) = \tilde{u}(t)$. The solution $u(\omega,t)$ can be written as follows:

$$u(\omega,t) \;=\; \sum_{i\geq 1}\phi_i(\omega)\alpha_i(t) \tag{19}$$

where for any t, $\alpha_i(t) = \int_\Omega u(\omega,t)\phi_i(\omega)\, d\omega \in \mathbb{R}^n$ and ϕ_i's are the eigenfunctions of (17).

Claim 1**.**

$$u(\omega,t) - \bar{u}(t) \;=\; \sum_{i\geq 2}\alpha_i(t)\phi_i(\omega). \tag{20}$$

Using the expansion of u as in (19), we have

$$u(\omega,t) - \bar{u}(t) \;=\; \alpha_1(t)\phi_1(\omega) - \bar{u}(t) + \sum_{i\geq 2}\phi_i(\omega)\alpha_i(t). \tag{21}$$

Multiplying both sides of the above equality by the constant eigenfunction ϕ_1 and taking integral over Ω, by orthonormality of ϕ_i's, we get:

$$\int_\Omega (u(\omega,t) - \bar{u}(t))\ d\omega = \alpha_1(t).$$

We showed that $\bar{u}(t) = \frac{1}{|\Omega|}\int_\Omega u(\omega,t)\ d\omega$, hence $\alpha_1(t) = 0$. This proves Claim 1.

Claim 2. Fix $\omega \in \Omega$. Then for any $i \geq 1$,

$$\dot{\alpha}_i(t) \ = \ (A(t) - \lambda_i D)\alpha_i(t). \tag{22}$$

Using the expansion of u as in (19) and after omitting the arguments ω, t for simplicity, we have:

$$\begin{aligned}\sum_{i\geq 1} \dot{\alpha}_i\phi_i = \dot{u} = A(t)u + D\Delta u \\ &= A(t)\left(\sum_{i\geq 1}\alpha_i\phi_i\right) + D\Delta\left(\sum_{i\geq 1}\alpha_i\phi_i\right) \\ &= \sum_{i\geq 1}(A(t) - \lambda_i D)\,\alpha_i\phi_i.\end{aligned}$$

Multiplying both sides of the above equality by ϕ_i and taking integral over Ω, by orthonormality of ϕ_i's we get:

$$\dot{\alpha}_i(t) \ = \ (A(t) - \lambda_i D)\alpha_i(t).$$

This proves Claim 2.

If for any t, $M\left[A(t) - \lambda_2 D\right] \leq c$, then for any t and any $i > 2$, $M\left[A(t) - \lambda_i D\right] \leq c$ too. Then by Theorem 1 and Claim 2:

$$\|\alpha_i(t)\| \leq e^{ct}\|\alpha_i(0)\|. \tag{23}$$

Using the above inequality and triangle inequality in Equation (20), for any $\omega \in \Omega$ and any t, we get the following inequality:

$$\begin{aligned}\|u(\omega,t) - \bar{u}(t)\| &\leq \sum_{i\geq 2}\|\alpha_i(t)\phi_i(\omega)\| \\ &\leq e^{ct}\sum_{i\geq 2}\|\alpha_i(0)\phi_i(\omega)\|.\end{aligned}$$

Specifically, when $c < 0$, $\|u(\omega,t) - \bar{u}(t)\| \to 0$, exponentially as $t \to \infty$. ■

The following theorem from [45] presents a condition which guarantees spatial uniformity for the asymptotic behavior of the solutions of a nonlinear reaction-diffusion PDE with Neumann boundary conditions in one dimension, using the Jacobian matrix of the reaction term and the first Dirichlet eigenvalue of the Laplacian operator (which in one dimensional space, it is equal to the second Neumann eigenvalue of the Laplacian operator) on the given spatial domain. Very different techniques, from nonlinear functional analysis for normed spaces, than the quadratic Lyapunov function approaches, appropriate for Hilbert spaces, are used. Also, the following theorem is an analogous result to ([31], Proposition 1) in the discrete case.

Theorem 7: Let $u(\omega,t)$ be a solution of (14), defined for all $t \in [0,T)$ for some $0 < T \leq \infty$, where $\Omega = (0,L)$. In addition, assume that $u(\cdot,t) \in C^3(\Omega)$, for all $t \in [0,T)$. Let

$$c \ = \ \sup_{(x,t)} M_{1,Q,\phi}\left[J_{F_t}(x) - \frac{\pi^2}{L^2}D\right],$$

where $M_{1,Q,\phi}$ is the least upper bound logarithmic Lipschitz constant induced by the following norm:

$$\|\cdot\|_{1,Q,\phi} := \|\sin(\pi\omega/L)(\cdot)\|_{1,Q}.$$

Then for all $t \in [0,T)$:

$$\left\|\frac{\partial u}{\partial \omega}(\cdot,t)\right\|_{1,Q,\phi} \ \leq \ e^{ct}\ \left\|\frac{\partial u}{\partial \omega}(\cdot,0)\right\|_{1,Q,\phi}.$$

Note that $-\pi^2/L^2$ is equal to the second Neumann eigenvalue of the Laplacian operator on $(0,L)$.

The significance of Theorem 7 lies in the fact that $\sin(\pi\omega/L)$ is nonzero everywhere in the domain (except at the boundary). In that sense, we have exponential convergence to uniform solutions in a weighted L^1 norm. We remark that, more generally, the extension to general domains of non-L^2 results is still an open problem, just as with the analogous question of synchronization.

We now revisit the Goodwin oscillator example, assuming a continuous model where species diffuse in space. This example has been studied in [28]. The following system of PDEs, subject to Neumann boundary conditions, describe the evolution of X, Y, and Z on $(0,1)\times[0,\infty)$:

$$\begin{aligned}\frac{\partial x}{\partial t} &= \frac{a}{k+z} - b\ x + d_1\Delta x \\ \frac{\partial y}{\partial t} &= \alpha\ x - \beta\ y + d_2\Delta y \\ \frac{\partial z}{\partial t} &= \gamma\ y - \frac{\delta z}{k_M + z} + d_3\Delta z\end{aligned} \tag{24}$$

In ([28], Equation 55), the following sufficient condition is given for synchronization, where $\lambda = \pi^2$:

$$\frac{\alpha\gamma a}{k(b+\lambda d_1)(\beta+\lambda d_2)\lambda d_3} \ < \ 4. \tag{25}$$

In [45], it has been shown that for the weighted matrix $Q = \operatorname{diag}(1,12,11)$, and for $2.2/\pi^2 < d_1$, and $d_2 = d_3 = 0$,

$$\sup_{w=(x,y,z)^T} M_{1,Q}[J_F(w) - \pi^2 D] < 0.$$

Applying Theorem 7, the authors concluded that for $2.2/\pi^2 < d_1$, and $d_2 = d_3 = 0$, (24) synchronizes, meaning that solutions tend to uniform solutions. Note that when $d_3 = 0$, one cannot apply (25) directly to get synchronization.

In ([54], Equation 3), Othmer provides a sufficient condition for uniform behavior of the solutions of the reaction-diffusion (14) on $(0,L)$, subject to Neumann boundary conditions:

$$\sup_w \|J_F(w)\| < \pi^2/L^2 \min\{d_i\}. \tag{26}$$

In Goodwin's example (24), $\sup_w \|J_F(w)\|$ is positive and finite (the sup is taken at $z = 0$), and $\min\{d_i\} = 0$, hence (26) doesn't hold and this condition is not applicable for this example.

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