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On Convergence Rates in the Central Limit Theorems for Combinatorial Structures

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Flajolet and Soria established several central limit theorems for the parameter 'number of components' in a wide class of combinatorial structures. In this paper, we shall prove a simple theorem which applies to characterize the convergence rates in their central limit theorems. This theorem is also applicable to arithmetical functions. Moreover, asymptotic expressions are derived for moments of integral order. Many examples from different applications are discussed.

1. INTRODUCTION

This paper examines the occurrence of Gaussian laws in large random combinatorial structures. We are mainly concerned with the rate of normal approximation to discrete random distributions. Examples that we discuss include such diverse problems as cycles, inversions and involutions in permutations, the construction of heaps in the field of data structures, set partitions, and the problem of 'factorisatio numerorum' in number theory.

Problems in combinatorial enumeration and in number theory often lead to the decomposition of the underlying structure into more elementary ones by suitable algebraic operations. For instance, a permutation can be decomposed into a set of cycles, a binary tree can be defined to be either empty or the union of a root-node and two binary trees, and a factorization of a natural number n is a decomposition of n into a product of prime factors. It is well known that operations like *union*, *sequence*, *cycle*, *set* and *multiset* on the structural side correspond to explicitly expressible forms on the generating function side. Hence it is useful to establish general theorems from which one can conclude certain asymptotic results (especially, statistical properties) for parameters in the underlying structure by verifying some basic analytic properties of the generating function. Bender initiated this line of investigation; see [1–3, 7, 13–15, 19].

Applying singularity and probabilistic analysis on bivariate generating functions, Flajolet and Soria [13, 14] proved a series of central limit theorems for the parameter 'number of components' in a wide class of combinatorial structures issuing principally from combinatorial constructions. Briefly, they obtain their results by showing that the characteristic function $\varphi_n(t)$ of the normalized random variable Ω_n^* in question (to be explained below) tends, as n tends to infinity, to $e^{-\frac{1}{2}t^2}$; and this establishes the weak convergence of the distribution function $F_n(x)$ of Ω_n^* to the standard normal law by Levy's continuity theorem. With the aid of the Berry–Esseen inequality, we shall make explicit the convergence rates in their central limit theorems.

In the next section, we propose a simple theorem on convergence rate which turns out to have wide applications. In particular, it is applicable to the central limit theorems of Flajolet and Soria [13, 14] which we shall discuss in Section 3. Moreover, as a direct consequence of our general assumptions on the moment generating function, we derive an asymptotic expression for moments of integral order. Then, we state an effective version of the central limit theorem of Haigh [16] (by establishing the convergence rate). This theorem is borrowed from [21] where we applied it to derive the asymptotic normality of the cost of constructing a random heap. Examples from combinatorial enumerations, computer algorithms, probabilistic number

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theory, arithmetical semigroups and orthogonal polynomials will be discussed in the final section.

Throughout this paper, we denote by $\Phi(x)$ the standard normal distribution:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-\frac{1}{2}t^2}\,\mathrm{d}t \qquad (x \in \mathbb{R}).$$

All limits (including O, o and $\sim$), whenever unspecified, will be taken as $n \to \infty$. All generating functions (ordinary or exponential) will denote functions *analytic at* 0 with *non-negative coefficients*. The symbol $[z^n]f(z)$ represents the coefficient of z^n in the Taylor expansion of $f(z)$.

2. Main Results

Let $\{\Omega_n\}_{n\geq 1}$ be a sequence of random variables. Define $\mu_n = \mathrm{E}(\Omega_n)$, $\sigma_n^2 = \mathrm{Var}(\Omega_n)$ and $F_n(x) = \Pr\{\Omega_n < \mu_n + x\sigma_n\}$, $x \in \mathbb{R}$. If the distribution of Ω_n is asymptotically normal, then $F_n(x)$ satisfies

$$\sup_x |F_n(x) - \Phi(x)| \to 0, \tag{1}$$

the pointwise convergence being uniform with respect to x in any finite interval of $\mathbb{R}$. The first possible refinement to (1) is to determine the convergence rate, which, in most cases, is of order σ_n^{-1}. The most general method for establishing the convergence rate (not restricted to normal limiting distribution) is the Berry–Esseen inequality (see (2) below) which relates the estimate of the difference of two distribution functions to that of corresponding characteristic functions, the latter being usually more manageable, especially when $F_n(x)$ is a step-function.

Since in probability theory, this subject is almost thoroughly studied when Ω_n can be decomposed as a sum of (independent or dependent) random variables, we content ourselves here with presenting two simple theorems which, for most of our applications, turn out to be sufficient. Our object is not to search for results of the most general kind but for those which are easy to apply to combinatorial and number-theoretic problems, especially, when the probability generating function or characteristic function is available.

The following theorem is motivated by the observation that many asymptotically normal distributions have mean and variance of the same orders.

Let $\{\Omega_n\}_{n\geq 1}$ be a sequence of integral random variables.

Suppose that the moment generating function satisfies the asymptotic expression:

$$M_n(s) := \mathrm{E}(\mathrm{e}^{\Omega_n s}) = \sum_{m\geq 0} \Pr\{\Omega_n = m\}\mathrm{e}^{ms} = \mathrm{e}^{H_n(s)}(1 + O(\kappa_n^{-1})),$$

the O-term being uniform for $|s| \leq \tau$, $s \in \mathbb{C}$, $\tau > 0$, where

(i) $H_n(s) = u(s)\phi(n) + v(s)$, with $u(s)$ and $v(s)$ analytic for $|s| \leq \tau$ and independent of n; $u''(0) \neq 0$;
(ii) $\phi(n) \to \infty$;
(iii) $\kappa_n \to \infty$.

Theorem 1. *Under these assumptions, the distribution of Ω_n is asymptotically Gaussian:*

$$\Pr\left\{\frac{\Omega_n - u'(0)\phi(n)}{\sqrt{u''(0)\phi(n)}} < x\right\} = \Phi(x) + O\left(\frac{1}{\kappa_n} + \frac{1}{\sqrt{\phi(n)}}\right),$$

uniformly with respect to x, $x \in \mathbb{R}$.

PROOF. Let $\mu_n = u'(0)\phi(n)$ and $\sigma_n^2 = u''(0)\phi(n)$. Define the random variable $\Omega_n^* = (\Omega_n - \mu_n)/\sigma_n$ with distribution function (characteristic function) $F_n(x)$ ($\varphi_n(t)$) respectively.

To obtain an upper bound for the difference $|F_n(x) - \Phi(x)|$, we shall estimate the ratio $|(\varphi_n(t) - \mathrm{e}^{-t^2/2})/t|$ and apply the following Berry–Esseen inequality [29, p. 109]:

Let $F(x)$ be a non-decreasing function, $G(x)$ a differentiable function of bounded variation on the real line, $\varphi(t)$ and $\gamma(t)$ the corresponding Fourier–Stieltjes transforms:

$$\varphi(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx}\, \mathrm{d}F(x), \qquad \gamma(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx}\, \mathrm{d}G(x).$$

Suppose that $F(-\infty) = G(-\infty)$, $F(\infty) = G(\infty)$, T is an arbitrary positive number, and $|G'(x)| \le A$. Then for every $b > 1/(2\pi)$ we have

$$\sup_{-\infty<x<\infty} |F(x) - G(x)| \le b \int_{-T}^{T} \left| \frac{\varphi(t) - \gamma(t)}{t} \right| \mathrm{d}t + r(b)\frac{A}{T}, \tag{2}$$

where $r(b)$ is a positive constant depending only on b.

Take $G(x) = \Phi(x)$ (so that $A = 1/\sqrt{2\pi}$) and $T = T_n = c\sigma_n$, where $0 < c < \tau$ is a sufficiently small constant, we shall show that

$$J_n = \int_{-T_n}^{T_n} \left| \frac{\varphi_n(t) - \mathrm{e}^{-\frac{1}{2}t^2}}{t} \right| \mathrm{d}t = O(\sigma_n^{-1} + \kappa_n^{-1}),$$

so that Theorem 1 follows from inequality (2).

Let us write $M_n(s) = \mathrm{e}^{u(s)\phi(n)+v(s)}(1 + E_n(s))$, so that $E_n(0) = 0$ and $E_n(s) = O(\kappa_n^{-1})$ for $|s| \le \tau$. By taking a small circle around the origin we easily deduce that $E_n(s) = O(|s|\kappa_n^{-1})$ for $|s| \le c$. Since $|\varphi_n(t)| \le 1$, we have

$$\begin{aligned}\varphi_n(t) &= \exp\left(-\frac{\mu_n}{\sigma_n}\mathrm{i}t + u\left(\frac{\mathrm{i}t}{\sigma_n}\right)\phi(n) + v\left(\frac{\mathrm{i}t}{\sigma_n}\right)\right)\left(1 + O\left(\frac{|t|}{\sigma_n\kappa_n}\right)\right)\\ &= \exp\left(-\frac{t^2}{2} + O\left(\frac{|t| + |t|^3}{\sigma_n}\right)\right) + O\left(\frac{|t|}{\sigma_n\kappa_n}\right),\end{aligned}$$

for $|t| \le T_n$. Using the inequality $|\mathrm{e}^w - 1| \le |w|\mathrm{e}^{|w|}$ for all complex w, we obtain

$$\begin{aligned}\left| \frac{\varphi_n(t) - \mathrm{e}^{-\frac{1}{2}t^2}}{t} \right| &= O\left(\left(\frac{1+t^2}{\sigma_n}\right)\exp\left(-\frac{t^2}{2} + O\left(\frac{|t| + |t|^3}{\sigma_n}\right)\right) + \frac{1}{\sigma_n\kappa_n}\right)\\ &= O\left(\left(\frac{1+t^2}{\sigma_n}\right)\mathrm{e}^{-\frac{1}{4}t^2} + \frac{1}{\sigma_n\kappa_n}\right) \qquad (|t| \le T_n),\end{aligned}$$

for c sufficiently small. Hence

$$\begin{aligned}J_n &= \int_{-T_n}^{T_n} \left| \frac{\varphi_n(t) - \mathrm{e}^{-\frac{1}{2}t^2}}{t} \right| \mathrm{d}t\\ &= O\left(\frac{1}{\sigma_n}\int_{-T_n}^{T_n} (1+t^2)\mathrm{e}^{-\frac{1}{4}t^2}\, \mathrm{d}t + \frac{1}{\kappa_n}\right)\\ &= O\left(\frac{1}{\sigma_n} + \frac{1}{\kappa_n}\right).\end{aligned}$$

This completes the proof of Theorem 1. □

The assumption that the moment generating function $M_n(s)$ exists in the neighbourhood of 0 is not a necessary condition for the validity of Theorem 1. As is evident from the proof, a weaker sufficient condition is that $|u'''(\mathrm{i}t)|$ and $|v'(\mathrm{i}t)|$ are bounded for s close to 0. But the following theorem does require the analyticity of $M_n(s)$.

THEOREM 2. *The kth moment of Ω_n satisfies*

$$\frac{1}{k!}\sum_{m\ge 0} m^k \Pr\{\Omega_n = m\} = \pi_k(\phi(n)) + O\left(\frac{\phi(n)^k}{\kappa_n}\right) \qquad (k = 1, 2, 3, \dots),$$

where $\pi_k(X)$ is a polynomial in X of degree k define by

$$\pi_k(X) = [s^k]\mathrm{e}^{u(s)X+v(s)} \qquad (k = 1, 2, 3, \dots). \tag{3}$$

In particular, the mean and the variance of Ω_n satisfy

$$\mathrm{E}(\Omega_n) = u'(0)\phi(n) + v'(0) + O(\kappa_n^{-1}), \tag{4}$$
$$\mathrm{Var}(\Omega_n) = u''(0)\phi(n) + v''(0) + O(\kappa_n^{-1}). \tag{5}$$

PROOF. Since $M_n(s)$ is analytic for $|s| \le \tau$, by Cauchy's formula, we obtain

$$\sum_{m\ge 0} \Pr\{\Omega_n = m\}\frac{m^k}{k!} = \pi_k(\phi(n)) + R_k(n),$$

where the remainder term $R_n(n)$ satisfies

$$R_k(n) = O\left(\frac{1}{\kappa_n}\oint_{|s|=r} |s|^{-k-1}|\mathrm{e}^{u(s)\phi(n)}|\,|\mathrm{d}s|\right) \qquad (0 < r \le \tau).$$

Clearly, the degree of the polynomial $\pi_k(X)$ is k. Taking $r = \phi(n)^{-1}$, we obtain

$$R_k(n) = O\left(\frac{\phi(n)^k}{\kappa_n}\right),$$

for $k = 1, 2, 3, \dots$.

For the more refined results (4) and (5), let us consider the cumulant generating function of Ω_n:

$$\lambda_n(s) := \log M_n(s) = u(s)\phi(n) + v(s) + Y_n(s),$$

where $Y_n(s) = O(\kappa_n^{-1})$. By the analyticity of M_n and the relation $M_n(0) = 1$, the function $\lambda_n(s)$ is well-defined in a neighbourhood of the origin and is analytic there. Similar arguments as above yield the required results (4) and (5) since the mean and the variance of Ω_n are nothing but $\lambda_n'(0)$ and $\lambda_n''(0)$, respectively. This completes the proof. □

Writing $u_j := u^{(j)}(0)$ and $v_j := v^{(j)}(0)$, $j = 1, 2, 3, \dots$, we have

$$\pi_1(X) = u_1X + v_1, \quad \pi_2(X) = \frac{u_1^2}{2}X^2 + \frac{u_2 + 2u_1v_1}{2}X + \frac{v_2 + v_1^2}{2}.$$

In general, some of the terms in $\pi_k(\phi(n))$ may be absorbed by the remainder term $O(\phi(n)^k\kappa_n^{-1})$.

COROLLARY 1. *We have*

$$\sum_{m\ge 0} m^k \Pr\{\Omega_n = m\} = u'(0)^k\phi(n)^k\left(1 + O_k\left(\frac{1}{\phi(n)} + \frac{1}{\kappa_n}\right)\right),$$

for each fixed $k = 1, 2, 3, \dots$.

3. COMBINATORIAL SCHEMES OF FLAJOLET AND SORIA

We now apply Theorem 1 to establish effective versions of the central limit theorems considered in [13, 14]. The developments here are made possible since the singularity analysis provides uniform bounds on the coefficients of analytic functions.

3.1. Exp–log class. The exp–log class covers such problems as cycles in permutations, cycles in random mappings and random mapping patterns, irreducibles in polynomials over finite fields (and, more generally, over an additive arithmetic semigroup under Knopfmacher's axiom $A^{\#}$), profiles of increasing trees, etc. See [13, 19, 23].

Recall that a generating function $C(z)$ analytic at 0 is called *logarithmic* [13] if there exists a constant $a > 0$, such that for $z \sim \zeta$, $\zeta > 0$ being the radius of convergence of C,

$$C(z) = a \log \frac{1}{1 - z/\zeta} + H(z),$$

where the function $H(z)$ is analytic in the region

$$\Delta := \{z \,:\, |z| \le \zeta + \epsilon \text{ and } |\arg(z - \zeta)| \ge \delta\} \qquad \left(\epsilon > 0,\ 0 < \delta < \frac{\pi}{2}\right),$$

(with possibly a finite number of other branch cuts starting from the circle $|z| = \zeta$), and as $z \to \zeta$ in Δ,

$$H(z) = K + O\left(\log^{-\frac{1}{2}} \frac{1}{1 - z/\zeta}\right),$$

uniformly in z, where K is some constant. For brevity, we shall refer to as saying that $C(z)$ is logarithmic with parameters (ζ, a, K).

REMARK. The condition that we imposed here on $H(z)$ is weaker than that required in [14]. They need

$$H(z) = K + o\left(\log^{-1} \frac{1}{1 - z/\zeta}\right) \qquad (z \to \zeta, z \in \Delta),$$

to ensure that both the mean and the variance are $a \log n + O(1)$. But by our Theorem 2, only the weaker condition $H(z) = K + o(1)$ for $z \in \zeta$, $z \in \Delta$, suffices to guarantee the later. This was the original condition used in [13].

We first consider generating functions of the form

$$P(w, z) = \sum_{n,m \ge 0} p_{nm} w^m z^n = \exp(wC(z) + S(w, z)) \qquad (C(0) = 0),$$

where $C(z)$ is logarithmic with parameters (ζ, a, K) and the function $S(w, z)$ is analytic for $|z| < \zeta + \epsilon$ and $|w| < 1 + \epsilon'$, $\epsilon, \epsilon' > 0$. By singularity analysis [12, 14] (see [19] for details), we obtain

$$\begin{aligned} M_n(s) &= \sum_{m \ge 0} \Pr\{\Omega_n = m\} e^{ms} := \frac{[z^n]P(e^s, z)}{[z^n]P(1, z)} \\ &= \exp\left((e^s - 1)a \log n + K(e^s - 1) + S(e^s, \zeta) - S(1, \zeta) + \log \frac{\Gamma(a)}{\Gamma(ae^s)}\right) \\ &\quad \times \left(1 + O\left(\frac{1}{\sqrt{\log n}}\right)\right), \end{aligned} \tag{6}$$

uniformly for s in some complex neighbourhood of the origin.

THEOREM 3. *Let $F_n(x) = \Pr\{\Omega_n < a \log n + x\sqrt{a \log n}\}$, where Ω_n and a are defined by (6). Then $F_n(x)$ satisfies, uniformly with respect to x,*

$$F_n(x) = \Phi(x) + O\left(\frac{1}{\sqrt{\log n}}\right).$$

PROOF. The result follows from Theorem 1, since $1/\Gamma(ae^s) \neq 0$ for $|\Re(s)| < |\log a|$. $\square$

THEOREM 4. *The kth moment of Ω_n, $k \in \mathbb{N}^+$, satisfies*

$$\frac{1}{k!}\sum_{m\geq 0} m^k \Pr\{\Omega_n = m\} = \frac{(a\log n)^k}{k!}(1 + O((\log n)^{-1/2})),$$

In particular, the mean and the variance of Ω_n satisfy

$$\begin{aligned} \mathrm{E}(\Omega_n) &= a\log n - a\psi(a) + K + S_1 + O((\log n)^{-\frac{1}{2}}) \\ \mathrm{Var}(\Omega_n) &= a\log n - a\psi(a) - a^2\psi'(a) + K + S_2 + S_1 + O((\log n)^{-\frac{1}{2}}) \end{aligned}$$

where ψ denotes the logarithmic derivative of the gamma function and the two constants S_1 and S_2 are defined by

$$S_1 = \frac{\partial}{\partial w} S(w,\zeta)\Big|_{w=1}, \qquad S_2 = \frac{\partial^2}{\partial w^2} S(w,\zeta)\Big|_{w=1}.$$

PROOF. This follows from (6) and Theorem 2. $\square$

The above theorems apply to the following four classes of combinatorial structures when $C(z)$ is logarithmic:

1. *Partitional complex construction*: $P(w,z) = \exp(wC(z))$.
2. *Partitional complex construction in which no two components are order-isomorphic*:

$$P(w,z) = \prod_{k\geq 1}\left(1 + \frac{wz^k}{k!}\right)^{c_k} \qquad (c_k := k![z^k]C(z)).$$

3. *Multiset construction*: (i) total number of components (i.e. counted with multiplicity)

$$P(w,z) = \exp\left(\sum_{k\geq 1}\frac{w^k}{k}\, C(z^k)\right);$$

(ii) number of distinct components (counted without multiplicity)

$$P(w,z) = \prod_{k\geq 1}\left(1 + \frac{wz^k}{1-z^k}\right)^{c_k} \qquad (c_k := [z^k]C(z)),$$

when the dominant singularity of C is strictly inferior to 1.

4. *Set construction*:

$$P(w,z) = \exp\Big(\sum_{k\geq 1}\frac{(-1)^{k-1}}{k}\, w^k\, C(z^k)\Big),$$

when the dominant singularity of C is strictly less than 1.

Further refinement of Theorem 1 is possible under slightly stronger conditions on $H(z)$. This will be discussed in the companion paper [20]; cf. also [19, Chapter 2].

3.2. Algebraic–logarithmic class. Next, let us consider generating functions of the form

$$P(w,z)=\sum_{n,m\geq 0} p_{nm}w^m z^n=\frac{1}{(1-wC(z))^{\alpha}}\left(\log\frac{1}{1-wC(z)}\right)^{\beta}\qquad (C(0)=0),$$

where $\beta\in\mathbb{N}$, $\alpha\geq 0$, and $\alpha+\beta>0$. Define the random variable Ω_n by its moment generating function:

$$M_n(s)=\sum_{m\geq 0}\Pr\{\Omega_n=m\}\,\mathrm{e}^{ms}=\frac{[z^n]P(\mathrm{e}^s,z)}{[z^n]P(1,z)},$$

for sufficiently large n.

DEFINITION (1-REGULAR FUNCTION [14]). A generating function $C(z)\not\equiv z^q$ ($q=0,1,2,\dots$) analytic at $z=0$ is called 1-*regular* if (i) its Taylor expansion at $z=0$ involves only non-negative coefficients; (ii) there exists a positive number $\rho<\varrho$, ϱ being the radius of convergence of $C(z)$, such that $C(\rho)=1$. Assume, without loss of generality, that $C(z)$ is aperiodic, namely, $C(z)\not\equiv z^a D(z^b)$ for some generating function $D(z)$ and integers $a,b\geq 2$.

To start with, let us first suppose that $C(z)$ is 1-regular and consider the equation

$$\frac{1}{w}=C(z).$$

Since $C(z)$ is 1-regular, we have $C'(\rho)>0$, where ρ satisfies $C(\rho)=1$. By the inverse function theorem, the above equation is locally invertible, i.e. the equation has a unique solution $\rho(w)$ for $w\sim 1$ and $\rho(w)$ is analytic at $w=1$. Of course $\rho(1)=\rho$.

The following lemma is fundamental to further investigations.

LEMMA 1. *Let $P_n(w):=[z^n]P(w,z)$. The formula*

$$\begin{aligned}P_n(w)&=A^{\alpha}(w)\rho(w)^{-n}n^{\alpha-1}(\log n)^{\beta}\sum_{0\leq k\leq\beta}\binom{\beta}{k}(\log n)^{-k}\sum_{0\leq j\leq k}g_j(\alpha)(\log A(w))^{k-j}\\&\quad+O(|\rho(w)|^{-n}n^{\alpha-2}(\log n)^{\beta}),\end{aligned}$$

holds uniformly for $w\sim 1$, where

$$A(w)=\frac{1}{w\rho(w)C'(\rho(w))},\qquad g_j(\alpha)=\frac{\mathrm{d}^j}{\mathrm{d}x^j}\frac{1}{\Gamma(x)}\bigg|_{x=\alpha}.$$

PROOF. Let us fix a w, $w\sim 1$. Expanding $C(z)$ at $z=\rho(w)$, we get

$$C(z)=\frac{1}{w}-\rho(w)C'(\rho(w))\left(1-\frac{z}{\rho(w)}\right)+\frac{\rho^2(w)}{2}C''(\rho(w))\left(1-\frac{z}{\rho(w)}\right)^2+\cdots.$$

Thus

$$\begin{aligned}\frac{1}{1-wC(z)}&=A(w)\left(1-\frac{z}{\rho(w)}\right)^{-1}\\&\quad\times\left(1+A_1(w)\left(1-\frac{z}{\rho(w)}\right)+A_2(w)\left(1-\frac{z}{\rho(w)}\right)^2+\cdots\right),\end{aligned}$$

where

$$A_1(w)=\frac{\rho(w)C''(\rho(w))}{2C'(\rho(w))},\qquad A_2(w)=\rho^2(w)\left(\frac{C''(\rho(w))^2}{4C'(\rho(w))^2}-\frac{C'''(\rho(w))}{6C'(\rho(w))}\right).$$

For $P(w,z)$, we get

$$\begin{aligned}
&\left(\frac{1}{1-wC(z)}\right)^{\alpha}\left(\log\frac{1}{1-wC(z)}\right)^{\beta}\\
&= A^{\alpha}(w)\sum_{0\le j\le\beta}\binom{\beta}{j}(\log A(w))^{\beta-j}\left(1-\frac{z}{\rho(w)}\right)^{-\alpha}\left(\log\frac{1}{1-z/\rho(w)}\right)^{j}\\
&\quad+O\left(\left(1-\frac{z}{\rho(w)}\right)^{-\alpha+1}\left(\log\frac{1}{1-z/\rho(w)}\right)^{\beta}\right).
\end{aligned}$$

By the remark before the lemma, we can apply the singularity analysis [12]. Thus transferring to coefficients, we obtain

$$\begin{aligned}
P_n(w) = A^{\alpha}(w)&\sum_{0\le j\le\beta}\binom{\beta}{j}(\log A(w))^{\beta-j}\sum_{0\le k\le j} g_{j-k}(\alpha)(\log n)^{k}\\
&+O(|\rho(w)|^{-n}n^{\alpha-2}(\log n)^{\beta}).
\end{aligned}$$

The double sum can be rearranged in descending powers of $\log n$:

$$\begin{aligned}
&\sum_{0\le j\le\beta}\binom{\beta}{j}(\log A(w))^{\beta-j}\sum_{0\le k\le j} g_{j-k}(\alpha)(\log n)^{k}\\
&= (\log n)^{\beta}\sum_{0\le k\le\beta}\binom{\beta}{k}(\log n)^{-k}\sum_{0\le j\le k} g_{j}(\alpha)(\log A(w))^{k-j}.
\end{aligned}$$

This completes the proof. □

In particular, when $\beta = 0$ and $\alpha > 0$, we obtain

$$P_n(w) = A^{\alpha}(w)\rho(w)^{-n}n^{\alpha-1}(1+O(n^{-1}));$$

when $\alpha = 0$, $\beta = 1$,

$$P_n(w) = \rho(w)^{-n}n^{-1}(1+O(n^{-1}));$$

and when $\alpha = 0$ and $\beta > 0$,

$$\begin{aligned}
P_n(w) = \rho(w)^{-n}n^{-1}(\log n)^{\beta}&\sum_{1\le k\le\beta}\binom{\beta}{k}(\log n)^{-k}\sum_{1\le j\le k} g_{j}(0)(\log A(w))^{k-j} +\\
&+O(|\rho(w)|^{-n}n^{-2}(\log n)^{\beta}).
\end{aligned}$$

Note that the indices of the last two sums both begin at 1 since $g_0(0) = 1/\Gamma(0) = 0$.

REMARK. From a computational point of view, the following expressions for $g_k(0)$ are useful.

$$\begin{aligned}
g_k(0) &= k![x^k]\frac{1}{\Gamma(x)} = k!(-1)^{k-1}[x^{k-1}]\frac{1}{\Gamma(1-x)}\\
&= k!(-1)^{k-1}[x^{k-1}]\frac{\Gamma(x)\sin\pi x}{\pi}\\
&= k(-1)^{k-1}\sum_{0\le j\le\lfloor\frac{k-2}{2}\rfloor}(-1)^{j}\pi^{2j}\binom{k-1}{2j+1}\Gamma^{(k-2-2j)}(0)\\
&= \sum_{0\le j\le\lfloor\frac{k-1}{2}\rfloor}(-1)^{k-1+j}\pi^{2j}\binom{k}{2j+1}\Gamma^{(k-1-2j)}(1).
\end{aligned}$$

Thus, with $M_n(s) := P_n(\mathrm{e}^s)/P_n(1)$, we have, by Lemma 1,

$$M_n(s) = \left(\frac{\mathrm{e}^s \rho(\mathrm{e}^s) C'(\rho(\mathrm{e}^s))}{\rho C'(\rho)}\right)^{-\alpha} \frac{\rho(\mathrm{e}^s)^{-n}}{\rho^{-n}} (1 + \varepsilon_{\alpha,\beta}(n)), \tag{7}$$

where

$$\varepsilon_{\alpha,\beta}(n) = \begin{cases} O(n^{-1}), & \text{if } (\alpha > 0 \text{ and } \beta = 0) \text{ or } (\alpha = 0 \text{ and } \beta = 1); \\ O((\log n)^{-1}), & \text{if } (\alpha = 0 \text{ and } \beta \geq 2) \text{ or } (\alpha > 0 \text{ and } \beta > 0), \end{cases}$$

uniformly for $|s| \leq \tau$, $\tau > 0$.

Define two constants

$$\alpha_1 = \frac{1}{\rho C'(\rho)}, \qquad \text{and} \qquad \alpha_2 = \frac{1}{\rho^2 C'(\rho)^2} + \frac{C''(\rho)}{\rho C'(\rho)^3} - \frac{1}{\rho C'(\rho)}.$$

THEOREM 5. *Let $F_n(x) = \Pr\{\Omega_n < \alpha_1 n + x\sqrt{\alpha_2 n}\}$. Then $F_n(x)$ satisfies*

$$F_n(x) = \Phi(x) + O\left(\frac{1}{\sqrt{n}} + \varepsilon_{\alpha,\beta}(n)\right),$$

uniformly in x, $x \in \mathbb{R}$.

PROOF. This proved by using (7) and applying Theorem 1. □

THEOREM 6. *The kth moment of Ω_n satisfies*

$$\frac{1}{k!} \sum_{m \geq 0} m^k \Pr\{\Omega_n = m\} = \Pi_k(n) + O(n^k \varepsilon_{\alpha,\beta}(n)),$$

where $\varpi_k(n)$ is a polynomial in n of degree k defined by

$$\Pi_k(n) := [s^k] \exp\left(-n \log \frac{\rho(\mathrm{e}^s)}{\rho}\right) \left(\frac{\mathrm{e}^s \rho(\mathrm{e}^s) C'(\rho(\mathrm{e}^s))}{\rho C'(\rho)}\right)^{-\alpha},$$

for $k = 1, 2, 3, \ldots$ In particular, we have

$$\mathrm{E}(\Omega_n) = \alpha_1 n + \alpha L_1 + O(\varepsilon_{\alpha,\beta}(n)),$$
$$\mathrm{Var}(\Omega_n) = \alpha_2 n + \alpha L_2 + O(\varepsilon_{\alpha,\beta}(n)),$$

where the two constants L_1, L_2 are given by

$$L_1 := \frac{1}{\rho C'(\rho)} + \frac{C''(\rho)}{C'(\rho)^2} - 1$$

$$L_2 := \frac{2\alpha\rho C_1 C_2 + \alpha\rho^2 C_2^2 - \rho^2 C_1^2 C_2 - \rho C_1^3 + \rho C_1 C_2 - 2\alpha\rho C_1^3 + 2\rho^2 C_2^2 - \rho^2 C_1 C_3 + \alpha\rho^2 C_1^4 - 2\alpha\rho^2 C_1^2 C_2 + \alpha C_1^2 + C_1^2}{2\rho^2 C_1^4},$$

($C_j := C^{(j)}(\rho)$ for $j = 1, 2, 3$.)

PROOF. This follows from (7) and Theorem 2. □

These results apply to *sequence constructions* of both labelled and unlabelled structures

$$P(w, z) = \frac{1}{1 - wC(z)} \qquad (C(0) = 0);$$

and to *cycle constructions*

$$P(w, z) = \log \frac{1}{1 - wC(z)}, \qquad P(w, z) = \sum_{k \geq 1} \frac{\varphi(k)}{k} \log \frac{1}{1 - w^k C(z^k)},$$

provided that the radius of convergence of $C(z)$ in the last case is less than 1, where $\varphi(k)$ denotes Euler's totient function. It should be noted that for sequence constructions, we often have exponential convergence rate for $M_n(s)$.

REMARK. For Bender's central limit theorem [1, Theorem 1], the convergence rate is $O(n^{-\frac{1}{2}})$, provided that the function $A(s)$ in that theorem has bounded derivative for s near 0.

4. A Classical Central Limit Theorem

The following theorem is borrowed from [21] where we used it to derive the asymptotic normality of the cost of constructing a random heap. Since the proof has already been given there, it is omitted here. Without the explicit error term it is due to Haight [16].

The object of study here is sums of independent but not identically distributed random variables. This problem has been extensively studied in the probability literature; see, for instance, [6, Section 27] [25, Chapter VI], [17, 29].

THEOREM 7. *Let $\{\Omega_n\}_n$ be a sequence of random variables taking only non-negative integral values with mean μ_n and variance σ_n^2. Suppose that the probability generating function $P_n(z)$ of Ω_n can be decomposed as*

$$P_n(z) = \prod_{1\le j\le k_n} P_{nj}(z),$$

for some sequence $\{k_n\}_n$, where the $P_{nj}(z)$ are polynomials such that (i) each $P_{nj}(z)$ is itself a probability generating function of some random variable, say, Ω_{nj} $(1 \le j \le k_n)$; and (ii)

$$\frac{M_n}{\sigma_n} \to 0,$$

where $M_n = \max_{1\le j\le k_n} \deg P_{nj}(z)$. Then the distribution of Ω_n is asymptotically Gaussian:

$$\Pr\left\{\frac{\Omega_n - \mu_n}{\sigma_n} < x\right\} = \Phi(x) + O\left(\frac{M_n}{\sigma_n}\right),$$

uniformly with respect to x.

PROOF. See [21]. □

COROLLARY 2. *Let $P_n(z)$ be a probability generating function of the random variable Ω_n, $n \ge 1$. Suppose that, for each fixed $n \ge 1$, $P_n(z)$ is a polynomial whose roots are all located in and on the half-plane $\Re z \le 0$. If $\sigma_n \to \infty$, $\sigma_n^2 := \mathrm{Var}(\Omega_n)$, then Ω_n satisfies*

$$\Pr\left\{\frac{\Omega_n - \mu_n}{\sigma_n} < x\right\} = \Phi(x) + O\left(\frac{1}{\sigma_n}\right),$$

for $-\infty < x < \infty$, where $\mu_n = \mathrm{E}(\Omega_n)$.

PROOF. $M = 2$ in Theorem 7. □

In particular, if all roots of $P_n(z)$ are non-positive, then the results of the corollary hold, cf. [18].

5. Examples

Let us discuss some typical examples from different applications.

EXAMPLE 1 (CYCLES IN PERMUTATIONS). A permutation is a set of cycles. The number of permutations having m cycles is enumerated by the (signless) Stirling number of the first kind $s(n, m)$:

$$s(n,m) := [w^m z^n] \exp\left(w \log \frac{1}{1-z}\right) = [w^m](w(w+1)\cdots(w+n-1)) \qquad (1 \le m \le n,\ n \ge 1).$$

Both Theorems 3 and 7 apply to the random variable Ω_n, defined as the number of cycles in a random permutation of $\{1, 2, \ldots, n\}$, where each of the $n!$ permutations is equally likely. We obtain

$$\Pr\left\{\frac{\Omega_n - \log n}{\sqrt{\log n}} < x\right\} = \Phi(x) + O\left(\frac{1}{\sqrt{\log n}}\right),$$

a result originally due to Goncharov (without error term).

Many other examples in this same (exp–log) class can be found in [5, 13, 23, 33] [26, Sections 2.4 and 2.5] and [19, Chapter 5].

EXAMPLE 2 (STIRLING NUMBERS OF THE SECOND KIND). The generating function for the number of blocks (marked by w) in set partitions satisfies

$$\sum_{n,m\geq 0} S(n,m)\, w^m \frac{z^n}{n!} = e^{w(e^z-1)},$$

where $S(n,m)$ denotes Stirling numbers of the second kind. Theorem 1 is not applicable since, as we shall see, the mean and the variance are not of the same order. So we turn to Theorem 7. Harper [18] showed that the polynomials

$$P_n(z) := \sum_{0\leq m\leq n} S(n,m)\, z^m \qquad (n = 1, 2, 3, \ldots),$$

have only simple non-positive roots. It remains to show that $\sigma_n \to \infty$, namely,

$$\sigma_n^2 := \mathrm{Var}(\Omega_n) = \frac{b_{n+2}}{b_n} - \frac{b_{n+1}}{b_n} - 1 \to \infty,$$

where $b_n := n!\,[z^n]e^{e^z-1}$ are the Bell numbers. Using the asymptotic expression [9, Chapter 6],

$$b_n = \frac{e^{n(\rho+\rho^{-1}-1)-1}}{\sqrt{\rho+1}}\left(1 - \frac{\rho^2(2\rho^2+7\rho+10)}{24n(\rho+1)^3} + O\left(\frac{\rho^2}{n^2}\right)\right),$$

where $\rho e^\rho = n$, it is not hard to show that

$$\sigma_n^2 = \frac{n}{\rho(\rho+1)} + O(1) \sim \frac{n}{(\log n)^2} \to \infty;$$

so that Theorem 7 gives, defining $\Pr\{\Omega_n = m\} := b_n^{-1} S(n,m)$,

$$\Pr\left\{\frac{\Omega_n - \mu_n}{\sigma_n} < x\right\} = \Phi(x) + O\left(\frac{\log n}{\sqrt{n}}\right),$$

uniformly with respect to x, where

$$\mu_n = \mathrm{E}(\Omega_n) = \frac{b_{n+1}}{b_n} - 1 = \frac{n}{\rho} + O(1) \sim \frac{n}{\log n}.$$

The asymptotic normality is due to Harper [18]. Alternatively, we can write

$$\Pr\left\{\frac{\Omega_n - n/\rho}{\sqrt{n/(\rho(\rho+1))}} < x\right\} = \Phi(x) + O\left(\frac{\log n}{\sqrt{n}}\right),$$

but this formula cannot be further simplified by using the asymptotics of ρ.

For other examples, see [28, 31].

EXAMPLE 3 (COST OF HEAP CONSTRUCTION). Doberkat [11] shows that the probability generating functions for the number of exchanges and for the number of comparisons used by Floyd's algorithm to construct a heap of size n have the form $\prod_{1\le j\le n} p_{n,j}(z)$, where $p_{n,j}(z)$ are polynomials of degree $\le \log_2 n$. Moreover, the mean and the variance are all linear. Thus Theorem 7 applies with $M_n = \log n$ to both cases [21]. More precisely, given a random permutation of size n, defining Ω_n as the number of exchanges used by Floyd's algorithm to construct a heap from this permutation, we have [21, Theorem 4]

$$\Pr\left\{\frac{\Omega_n - c_1 n}{\sqrt{c_2 n}} < x\right\} = \Phi(x) + O\left(\frac{\log n}{\sqrt{n}}\right),$$

uniformly with respect to x, where $c_1 = -2 + \sum_{j\ge 1} j(2^j-1)^{-1} = 0.744\,033\ldots$ and $c_2 = 2 - \sum_{j\ge 1} j^2(2^j-1)^2 = 0.261\,217\ldots$. A similar result holds for the number of comparisons and other construction algorithms preserving randomness in each step, cf. [27].

For other examples on algorithms, see [19, Chapter 1].

EXAMPLE 4 (ORDERED SET-PARTITIONS). The generating function for the number of blocks (marked by w) in an ordered set-partition is $(1 - w(e^z - 1))^{-1}$. Obviously, $e^z - 1$ is 1-regular. Theorem 5 applies. We can also consider other constraints like parity on the size of each block and on the number of blocks, cf. [8, pp. 225–6]. Many other examples can be found in [1, 14, 22] and [19, Chapter 7].

EXAMPLE 5 (INVERSIONS IN PERMUTATIONS). Given a permutation $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ of $\{1, 2, 3, \ldots, n\}$, a pair π_i, π_j is called an *inversion* if $i < j$ and $\pi_i > \pi_j$. Let Ω_n denote the number of inversions in a random permutation of size n. Then it is easily seen that [24, Section 5.1.1],

$$\Pr\{\Omega_n = m\} = \frac{1}{n!}[z^m]((1+z)(1+z+z^2)\cdots(1+z+z^2+\cdots+z^{n-1})).$$

From which we obtain

$$\mathrm{E}(\Omega_n) = \frac{n(n-1)}{4}, \qquad \text{and} \qquad \mathrm{Var}(\Omega_n) = \frac{n(n-1)(2n+5)}{72}, \tag{8}$$

for $n \ge 1$. Theorem 7 applies and we get, in view of (8),

$$\Pr\left\{\frac{\Omega_n - n^2/4}{n^{3/2}/6} < x\right\} = \Phi(x) + O\left(\frac{1}{\sqrt{n}}\right),$$

for all x.

EXAMPLE 6 (ERDŐS–KAC THEOREM). Define a random variable ξ_n by its probability distribution:

$$\Pr\{\xi_n = m\} = \frac{1}{n}|\{k : 1 \le k \le n,\ \omega(k) = m\}| = \frac{1}{n}[z^m]\sum_{1\le k\le n} z^{\omega(k)},$$

where $\omega(n)$ denotes the number of distinct prime divisors of n, e.g., $\omega(2\cdot 3^4\cdot 5^6\cdot 7^{8910}\cdot 11^{12}\cdot 13^{141516}) = 6$. It is known that $\mathrm{E}(\xi_n) = \log\log n + O(1)$, and $\mathrm{Var}(\xi_n) = \log\log n + O(1)$. The Erdős–Kac Theorem states that

$$\Pr\left\{\frac{\xi_n - \log\log n}{\sqrt{\log\log n}} < x\right\} \to \Phi(x).$$

The convergence rate was first conjectured by Le Veque and then proved by Rényi and Turán [30] to be $O((\log\log n)^{-\frac{1}{2}})$. Let us apply Theorem 1 to establish this fact.

Starting with the well-known Selberg's formula [32]

$$M_n(z) := \mathrm{E}(\mathrm{e}^{\xi_n z}) = V(\mathrm{e}^z)(\log n)^{\mathrm{e}^z-1}\left(1 + O\left(\frac{1}{\log n}\right)\right),$$

uniformly for $|\Im z| \leq \pi$, where $V(t)$ is an entire function in the t-plane, we readily obtain

$$\Pr\left\{\frac{\xi_n - \log\log n}{\sqrt{\log\log n}} < x\right\} = \Phi(x) + O\left(\frac{1}{\sqrt{\log\log n}}\right),$$

uniformly with respect to x.

For many other examples, see [10] and [19, Chapter 9].

EXAMPLE 7 (NUMBER OF FACTORS IN SQUARE-FREE FACTORIZATIONS). Let $f_{n,k}$ denote the number of factorizations of n into k distinct factors greater than 1, $n \geq 2$, $k \geq 1$. Define $f_{1,k} = \delta_{0,k}$, where $\delta_{a,b}$ is Kronecker's symbol. Consider the bivariate generating function:

$$P(w, s) := 1 + \sum_{n\geq 2} n^{-s} \sum_{m\geq 1} f_{n,m} w^m,$$

which equals to

$$P(w, s) = \prod_{n\geq 2}\left(1 + \frac{w}{n^s}\right) \qquad (\Re s > 1,\ w \in \mathbb{C}).$$

Define a polynomial $P_n(w) := 1 + \sum_{2\leq k\leq n} \sum_{m\geq 1} f_{k,m} w^m$ and a random variable ξ_n for which

$$\Pr\{\xi_n = m\} = \frac{[w^m]\, P_n(w)}{P_n(1)}.$$

We show in [19, Chapter 10] that

$$P_n(w) = \frac{n\mathrm{e}^{2\sqrt{w\log n}}}{2\sqrt{\pi}(\log n)^{3/4} w^{3/4}(1+w)\Gamma(w)}\left(1 + O\left(\frac{1}{\sqrt{\log n}}\right)\right),$$

the error term being uniform with respect to $w \in \mathbb{C} \setminus (-\infty, 0]$. Thus we obtain

$$M_n(z) := \mathrm{E}(\mathrm{e}^{\xi_n z}) = \frac{2\mathrm{e}^{2\sqrt{\log n}(\mathrm{e}^{z/2}-1)-3z/4}}{\Gamma(\mathrm{e}^z)(1+\mathrm{e}^z)}\left(1 + O\left(\frac{1}{\sqrt{\log n}}\right)\right),$$

and Theorem 1 gives

$$\Pr\left\{\frac{\xi_n - \sqrt{\log n}}{\sqrt[4]{\frac{1}{4}\log n}} < x\right\} = \Phi(x) + O\left(\frac{1}{\sqrt[4]{\log n}}\right),$$

for all x.

For other examples, see [19, Chapter 10].

EXAMPLE 8 (HERMITE POLYNOMIALS). The number of fixed points (cycles of length 1), marked by w, in the involutions is enumerated by

$$\exp\left(wz + \frac{z^2}{2}\right) = \sum_{n\geq 0} \frac{h_n(w)}{n!}\ , z^n,$$

where $h_n(w)$ are Hermite polynomials. The asymptotic behaviour of $h_n(w)$ is well known [34, p. 200]:

$$h_n(w) = 2^{-\frac{1}{2}} n^{n/2} \mathrm{e}^{-n/2+\sqrt{n+1}\,w-w^2/4}\left(1+O\left(\frac{1}{\sqrt{n}}\right)\right),$$

uniformly for all finite w. Defining Ω_n as the number of fixed points in a random involution, we obtain

$$M_n(s) = \mathrm{E}(\mathrm{e}^{\Omega_n s}) = \mathrm{e}^{\sqrt{n+1}(\mathrm{e}^s-1)-\frac{1}{4}(\mathrm{e}^{2s}-1)}\left(1+O\left(\frac{1}{\sqrt{n}}\right)\right),$$

uniformly for $|\Im s| \leq \pi$. We obtain

$$\Pr\left\{\frac{\Omega_n-\sqrt{n+1}}{\sqrt[4]{n+1}} < x\right\} = \Phi(x) + O\left(\frac{1}{\sqrt[4]{n}}\right),$$

the O being uniform with respect to x, $x \in \mathbb{R}$.

Asymptotic expansions in the central and local limit theorems are discussed in the companion paper [20].

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