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Sister Celine's Technique and Its Generalizations

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Sister Celine Fasenmyer's technique for obtaining pure recurrence relations for hypergeometric polynomials is formalized and generalized in various directions. Applications include algorithms for verifying any given binomial coefficients identity and any identities involving sums and integrals of products of special functions. This is shown to lead to a new approach to the theory of special functions which allows a natural definition of special functions of several variables.

0. INTRODUCTION

About 35 years ago, Sister Mary Celine Fasenmyer developed a general method for obtaining pure recurrence relations for hypergeometric polynomials [6–8; 13, Chap. 14]. In this paper we hope to demonstrate some far-reaching implications stemming from Sister Celine's ideas. In particular, Sister Celine's technique enables one to "evaluate," either explicitly or inductively, any sum involving products of binomial coefficients. This simple fact was apparently overlooked by workers in combinatorics who developed various ad hoc methods for computing such sums. However, Sister Celine's method has a much wider scope than that. We shall generalize her method to give an algorithm for verifying any given identity involving sums and integrals of products of special functions, which will hopefully lead to a new approach to the theory of special functions.

One of the consequences of Sister Celine's technique is that if $F(n, k)$ is multi-hypergeometric (see Section 1 for definition), then

$$G(n) = \sum_{k=-\infty}^{\infty} F(n, k)$$

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satisfies a linear recurrence equation with polynomial coefficients. Stanley [14] was the first to consider such discrete functions as such, and even gave them a name which we adopted: **P**-recursive functions. Stanley [14] also considered real functions $f(x)$ satisfying a linear differential equation with polynomial coefficients, which he called "*D*-finite functions." Stanley [14] proved, among other results, that both the classes of *D*-finite functions and **P**-recursive functions are algebras under addition and multiplication.

Stanley's notions of **P**-recursiveness and *D*-finiteness, and their higher-dimensional counterparts, are the most fundamental concepts in the present paper. We chose to call these higher-dimensional analogs "multi-**P**-recursive" and "multi-*D*-finite." A discrete (continuous) function $F(m_1,..., m_n)$ ($f(x_1,..., x_n)$) is multi-**P**-recursive (multi-*D*-finite) if it satisfies a linear ordinary recurrence (differential) equation with polynomial coefficients in each of its variables. We prove that both multi-*D*-finiteness and multi-**P**-recursiveness are preserved under integration and summation.

BIG RESTRICTION !

The final synthesis is accomplished in Section 4, where we define a sequence of functions $(\mathbf{P}_n(x))$ to be special if there exist polynomials $a_r(n, x)$, $b_s(n, x)$ such that

$$\sum_{r=0}^{R} a_r(n, x)\, \mathbf{P}_n^{(r)}(x) \equiv 0,$$

$$\sum_{s=0}^{S} b_s(n, x)\, \mathbf{P}_{n+s}(x) \equiv 0.$$

Defining $\tilde{\mathbf{P}}(n, x) = \mathbf{P}_n(x)$, we see that $\tilde{\mathbf{P}}: N \times R \to \mathbb{C}$ is special if it satisfies ordinary equations in each of its variables. This definition immediately generalizes to functions of several discrete and continuous variables.

The reason **P**-recursiveness is so important is that in order to specify a linear recurrence equation with polynomial coefficients one only needs a finite number of parameters. Thus in order to encode a function satisfying, for $n \geqslant 2$, $(5n+3)\,a(n) + (4n-1)\,a(n-1) - (7n+11)\,a(n-2) \equiv 0$, we only need to "store" the numbers (5, 3; 4, −1; −7, −11) and the initial values $a(0)$, $a(1)$. Similar remarks hold for *D*-finiteness and their higher-dimensional analogs. This resembles the fact that an algebraic number is given by the coefficients of its minimal equation and that a polynomial is given by its coefficients.

0.1. *Nomenclature*

Z denotes the set of integers, N the set of positive integers. When we write $f: N \to \mathbb{C}$ we mean that f is defined on all of Z but supported in N (i.e., $0 = f(-1) = f(-2) = \cdots$). Thus if we say "$f: N \to \mathbb{C}$ satisfies the recurrence $a(n)\,f(n) + b(n)\,f(n-1) + c(n)\,f(n-2) = 0$," we mean $a(0)\,f(0) = 0$, $a(1)\,f(1) + b(1)\,f(0) = 0$, etc.

If $f: Z \to \mathbb{C}$, we define the shift operator $Xf(n) = f(n+1)$. A linear recurrence operator with polynomial coefficients is something of the form

$$\sum_{r=0}^{R} a_r(n) X^r,$$

where the $a_r(n)$'s are polynomials. It is easily checked that the set of linear recurrence operators with polynomial coefficients is an algebra:

$$\left(\sum_{r=0}^{R} a_r(n) X^r\right)\left(\sum_{s=0}^{S} b_s(n) X^s\right) = \sum_{r=0}^{R} \sum_{s=0}^{S} a_r(n)\, b_s(n+r)\, X^{r+s}.$$

Likewise, the set of linear differential operators with polynomial coefficients is also an algebra, a fact which follows easily from Leibnitz's rule. For a function of several discrete variables $f(m_1,\ldots, m_n)$, we set $X_i f(m_1,\ldots, m_i,\ldots, m_n) = f(m_1,\ldots, m_i+1,\ldots, m_n)$, $i = 1,\ldots, n$, and a general linear partial recurrence operator is written

$$\sum a_{\alpha_1 \cdots \alpha_n} X_1^{\alpha_1} \cdots X_n^{\alpha_n},$$

where the $a_{\alpha_1 \cdots \alpha_n}$'s are polynomials in $(m_1,\ldots, m_n)$.

For a detailed discussion of linear recurrence operators we refer the reader to [18], where the word "recurrence" is replaced by "difference."

1. Sister Celine's technique

cancelling polynomial for order 1

1.1.

Definition 1. $F: Z \to \mathbb{C}$ is hypergeometric if there exist polynomials p and q such that $p(n)\,F(n) - q(n)\,F(n-1) \equiv 0$.

Definition 2. F: $Z^2 \to \mathbb{C}$ is multi-hypergeometric if there exist polynomials in two variables, P, Q, P', Q', such that for all $(n, k) \in Z^2$

$$P(n, k)\,F(n, k) - Q(n, k)\,F(n-1, k) = 0,$$

$$P'(n, k)\,F(n, k) - Q'(n, k)\,F(n, k-1) = 0.$$

Remark. Every product of binomial coefficients is hypergeometric. For example $F(n, k) = \binom{n}{k}^r$ satisfies

$$n^r F(n, k) - (n-k)^r F(n-1, k) = 0,$$

$$k^r F(n, k) - (n-k+1)^r F(n, k-1) = 0.$$

DEFINITION 3 (Stanley [14]). $F: Z \to \mathbb{C}$ is *P-recursive* if it satisfies a recurrence equation with polynomial coefficients, namely, there exist polynomials $P_0, \dots, P_r$ such that $P_0(n) F(n) + P_1(n) F(n-1) + \cdots + P_r(n) F(n-r) \equiv 0$.

Of course every hypergeometric function is P-recursive, where the relevant recurrence is of first order.

Sister Celine (Fasenmyer [6, 7], Rainville [13]) described her algorithm in terms of example. A formal statement of her method is given by the following theorem and proof.

THEOREM 4. *Let $F(n, k)$ be multi-hypergeometric and assume that $\sum_{k=-\infty}^{\infty} F(n, k)$ converges for every n (in particular, if $F(n, \cdot)$ has finite support for all n). Then $G(n) = \sum_{k=-\infty}^{\infty} F(n, k)$ is P-recursive.*

Proof. For f: $Z^2 \to \mathbb{C}$ we introduce the negative shift operators $X^{-1}f(n, k) = f(n-1, k)$, $Y^{-1}f(n, k) = f(n, k-1)$. Of course $X^{-r}Y^{-s}f(n, k) = f(n-r, k-s)$. Since F is multi-hypergeometric, we have

$$X^{-1}F(n, k) = \frac{P(n, k)}{Q(n, k)} F(n, k), \tag{1.1a}$$

$$Y^{-1}F(n, k) = \frac{P'(n, k)}{Q'(n, k)} F(n, k), \tag{1.1b}$$

for some polynomials P, Q, P', Q'. Iterating (1.1) we get

$$X^{-r}Y^{-s}F(n, k) = \frac{P'(n-r, k-s+1)}{Q'(n-r, k-s+1)} \cdots \frac{P'(n-r, k)}{Q'(n-r, k)} \cdot$$

$$\frac{P(n-r+1, k)}{Q(n-r+1, k)} \frac{P(n-r+2, k)}{Q(n-r+2, k)} \cdots \frac{P(n, k)}{Q(n, k)} F(n, k) \stackrel{\text{def}}{=} \frac{A_{rs}(n, k)}{B_{rs}(n, k)} F(n, k).$$

From now on we shall consider all polynomials in (n, k) as polynomials in k whose coefficients are polynomials in n, i.e., we view $\mathbb{C}[n, k]$ as $C[n][k]$. Let

$$p = \max(\deg_k P, \deg_k Q),$$

$$p' = \max(\deg_k P', \deg_k Q').$$

Let us look for polynomials in n, $a_{rs}(n)$, such that

$$\sum_{r=0}^{M} \sum_{s=0}^{N} a_{rs}(n) X^{-r}Y^{-s}F(n, k) = 0, \tag{1.2}$$

where M and N are to be determined. This is true provided

$$\sum_{r=0}^{M} \sum_{s=0}^{N} a_{rs} \frac{A_{rs}(k)}{B_{rs}(k)} \equiv 0, \tag{1.2'}$$

where the dependence upon n is suppressed.

The common denominator is $B_{MN}(k)$, and multiplying by it yields

$$\sum_{r=0}^{M} \sum_{s=0}^{N} a_{rs} A_{rs}(k) \frac{B_{MN}(k)}{B_{rs}(k)} = 0.$$

The left-hand side is a polynomial of degree $Mp + Np'$ in k (check!) and setting each of the coefficients to 0 yields $Mp + Np' + 1$ homogeneous equations for the $(M+1)(N+1)$ unknowns a_{rs} ($r = 0,..., M$; $s = 0,..., N$). In order for such non-trivial a_{rs} to exist we must require that $(M+1)(N+1) > Mp + Np' + 1$. Certainly there exist such M and N. The least M by which we can get by is $M = p'$ and then $N = pp' - p' + 1$.

So far we have constructed a partial difference operator with polynomial coefficients (with k missing):

$$R(n, X^{-1}, Y^{-1}) = \sum_{r=0}^{M} \sum_{s=0}^{N} a_{rs}(n) X^{-r} Y^{-s},$$

$$\text{such that} \quad R(n, X^{-1}, Y^{-1}) F(n,k) \equiv 0.$$

We claim that $R(n, X^{-1}, I)\, G(n) \equiv 0$, i.e., that $G(n)$ is a solution of the recurrence equation with polynomial coefficients $\sum_{r=0}^{M} (\sum_{s=0}^{N} a_{rs}(n))\, G(n-r) \equiv 0$. This follows from

LEMMA 5. *Let $F(n,k)$ be a solution of the partial difference equation $R(X^{-1}, Y^{-1}, n)\, F(n, k) \equiv 0$, where k is missing from R. Then $G(n) = \sum_{k=-\infty}^{\infty} F(n,k)$ satisfies the ordinary difference equation $R(X^{-1}, I, n)\, G(n) \equiv 0$.*

Proof. Let $R(X^{-1}, Y^{-1}, n) = \sum_{i=0}^{I} R_i(X^{-1}, n)\, Y^{-i}$.

We have

$$\begin{aligned} 0 &= \sum_{k=-\infty}^{\infty} R(X^{-1}, Y^{-1}, n) F(n,k) = \sum_{k=-\infty}^{\infty} \sum_{i=0}^{I} R_i(X^{-1}, n)\, Y^{-i} F(n,k) \\ &= \sum_{i=0}^{I} R_i(X^{-1}, n) \sum_{k=-\infty}^{\infty} F(n, k-i) \\ &= \sum_{i=0}^{I} R_i(X^{-1}, n)\, G(n) = R(X^{-1}, I, n)\, G(n). \end{aligned}$$

Remark. Our notation is different from that of Sister Celine, who considered the polynomials $G(n, x) = \sum_{k=-\infty}^{\infty} F(n, k)\, x^k$. Since $xG(n, x) = \sum_{k=-\infty}^{\infty} F(n, k-1)\, x^k$, multiplication by x corresponds to the operator Y^{-1}. The above proof shows that $G(n, x)$ satisfies the pure recurrence relation

$$\sum_{r=0}^{M} \left(\sum_{s=0}^{N} a_{rs}(n)\, x^s \right) G(n-r, x) = 0.$$

Remark. Theorem 4 states that if $F(n, k)$ is multi-hypergeometric, then $\sum_k F(n, k)$ is P-recursive in the surviving variable n. Later on we shall generalize Sister Celine's method to show that even if $F(n, k)$ is multi-P-recursive (to be defined in Section 2), it is still true that $G(n) = \sum_{k=-\infty}^{\infty} F(n, k)$ is P-recursive.

1.2. *Examples*

Since every product of binomial coefficients is multi-hypergeometric, Sister Celine's technique gives a straightforward way to evaluate binomial sums (either "explicitly" if the resulting recurrence is of first order, or "inductively" if the recurrence is of higher order). This simple observation was apparently overlooked by combinatorists who dealt with binomial sums, probably because of the cultural gap between combinatorics and analysis (to which the theory of hypergeometric series belong, at least "officially").

We shall now illustrate the method by finding recurrences for some binomial sums.

EXAMPLE (i). $G(n) = \sum_{k=-\infty}^{\infty} \binom{n}{k}$. Here $X^{-1}F = [(n-k)/n]\, F$, $Y^{-1}F = [k/(n-k+1)]\, F$, so $X^{-1}Y^{-1} = (k/n)\, F$ and eliminating k yields $n(I - X^{-1} - X^{-1}Y^{-1})\, F(n, k) = 0$. Putting $Y^{-1} = I$, we get $n(I - 2X^{-1})\, G(n) = 0$, i.e., $G(n) = 2G(n-1)$ and so $G(n) = C \cdot 2^n$, for some constant C, which is found out to be 1, by plugging $n = 0$, i.e., $G(n) = 2^n$. In this trivial case $G(n)$ is much more than just P-recursive. Since the relevant recurrence is of the first order, it is hypergeometric, and as a matter of fact *geometric* (i.e., it satisfies a first-order recurrence relation with *constant* coefficients).

EXAMPLE (ii). $F(n, k) = \binom{n}{k}^2 = n!^2/[k!^2(n-k)!^2]$.
Applying Sister Celine's method yields the partial difference equation

$$\begin{aligned}[n - (2n-1)\, X^{-1} + (n-1)\, X^{-2} - (2n-1)\, X^{-1}Y^{-1} \\ - 2(n-1)\, X^{-2}Y^{-1} + (n-1)\, X^{-2}Y^{-2}]\, F(n, k) \equiv 0.\end{aligned}$$

Substituting $Y^{-1} = I$ shows that $G(n) = \sum_{k=0}^{n} \binom{n}{k}^2$ satisfies the recurrence $(n - (4n-2)\, X^{-1})\, G(n) = 0$, from which follows that $G(n) = \binom{2n}{n}$.

The above examples were given merely for pedagogical reasons, as they are much more easily handled by other methods. However, the next example, which is taken from Rainville [13, p. 234] is not as trivial.

EXAMPLE (iii). $F(n, k) = (-1)^k (n+k)!/[(k!)^2 (\frac{1}{2})_k (n-k)!]$ can be shown (using the routine method of Theorem 4) to satisfy the partial difference equation

$$[nI - (3n - 2 - 4Y^{-1}) X^{-1} \\ + (3n - 4 + 4Y^{-1}) X^{-2} - (n-2) X^{-3}] F(n, k) \equiv 0.$$

Plugging in $Y^{-1} = I$ yields

$$[nI - (3n-6) X^{-1} + 3nX^{-2} - (n-2) X^{-3}] G(n) = 0,$$

or more explicitly

$$nG(n) - (3n-6) G(n-1) + 3nG(n-2) - (n-2) G(n-3) \equiv 0.$$

Since $G(Z_-) = 0$, the recurrence enables us to compute $G(n)$ inductively, once $G(0)$ is given.

1.3. *The Method of Creative Telescoping*

One of the steps in Apery's proof of the irrationality of $\zeta(3)$ (Van der Poorten [15, Sect. 8]) was to prove that

$$u(n) = \sum_{k=0}^{n} \underbrace{\binom{n}{k}^2 \binom{n+k}{k}^2}_{\left(\frac{n!}{k!(n-k)!}\right)^2 \cdot \left(\frac{(n+k)!}{k!\,n!}\right)^2}$$

satisfies the recurrence

$$n^3 u(n) - (34n^3 - 51n^2 + 27n - 5) u(n-1) + (n-1)^3 u(n-2) = 0.$$

The way it is proved there is to "cleverly construct"

$$B(n, k) = 4(2n+1)[(2k+1) - (2n+1)^2] \binom{n}{k}^2 \binom{n+k}{k}^2$$

"with the motive that" (in our notation)

$$(1 - Y^{-1}) B(n, k) = P(n, X^{-1}) F(n, k), \qquad \text{where } F(n, k) = \binom{n}{k}^2 \binom{n+k}{k}^2,$$

$$P(n, X^{-1}) = n^3 I - (34n^3 - 51n^2 + 27n - 5) X^{-1} + (n-1)^3 X^{-2},$$

and then "0 mirabile dictu"

$$0 = \sum_{k=-\infty}^{\infty} (1 - Y^{-1})\, B(n, k) = \sum_{k=-\infty}^{\infty} P(n, X^{-1})\, F(n, k) = P(n, X^{-1})\, u(n).$$

Sister Celine's technique takes all the magic out of "creative telescoping." Indeed, we can use it to concoct short proofs to the fact that $G(n) = \sum_k F(n, k)$ indeed satisfies the particular recurrence obtained for it. (This resembles the fact that it is much easier to prove that a proposed function solves a given differential equation than to construct a solution from scratch.)

Given a binomial sum $G(n) = \sum_{k=-\infty}^{\infty} F(n, k)$ we use Sister Celine's method to find a recurrence equation $R(X^{-1}, Y^{-1}, n)\, F(n, k) = 0$. Now we write $R(X^{-1}, Y^{-1}, n) = R_0(X^{-1}, n) - (1 - Y^{-1})\, S(X^{-1}, Y^{-1}, n)$, where $R_0(X^{-1}, n) = R(X^{-1}, I, n)$. Next we compute $F'(n, k) = S(X^{-1}, Y^{-1}, n)$ $F(n, k)$ which is of the form $[a(n, k)/b(n, k)]\, F(n, k)$, for some polynomials a and b.

Once we have gone through the pain of finding $R_0(X^{-1}, n)$ and $F'(n, k)$ we can gracefully present a short proof to the fact that $R_0(X^{-1}, n)\, G(n) = 0$. All we have to do is urge the reader to verify that $R_0(X^{-1}, n)\, F(n, k) = (1 - Y^{-1})\, F'(n, k)$ and then conclude that

$$R_0(X^{-1}, n)\, G(n) = \sum_{k=-\infty}^{\infty} R_0(X^{-1}, n)\, F(n, k) = \sum_{k=-\infty}^{\infty} (1 - Y^{-1})\, F'(n, k) = 0.$$

Following the above recipe, let us present a short proof of the result obtained in Example (iii) of Section 1.2.

PROPOSITION.

$$G(n) = \sum_{k=-\infty}^{\infty} \frac{(-1)^k\, (n+k)!}{(k!)^2\, (\frac{1}{2})_k\, (n-k)!}$$

satisfies the recurrence

$$nG(n) - (3n-6)\, G(n-1) + 3nG(n-2) - (n-2)\, G(n-3) \equiv 0.$$

Proof. We cleverly construct

$$B(n, k) = (2n-2)(-1)^k\, (n+k-2)!/[(k!)^2 (\tfrac{1}{2})_k\, (n-k-1)!]$$

with the motivation that

$$\begin{aligned} B(n, k) - B(n, k-1) = {} & nF(n, k) - (3n-6)\, F(n-1, k) \\ & + 3nF(n-2, k) - (n-2)\, F(n-3, k) \end{aligned}$$

(check!). By telescoping,

$$nG(n)-(3n-6)\,G(n-1)+3nG(n-2)-(n-2)\,G(n-3)$$
$$=\sum_{k=-\infty}^{\infty}[B(n,k)-B(n,k-1)]=0.$$

1.4. *Form over Content*

We have already mentioned in the Introduction the fact that the knowledge that a sequence $G(n)$ satisfies some recurrence with polynomial coefficients is much more important than knowing the actual recurrence. Since it is possible to find, *from the outset*, upper bounds for the order of the recurrence satisfied by $G(n)$ and the degree of the coefficients, we are guaranteed that there exist constants C_{rs} such that $\sum_{r=0}^{R}\sum_{s=0}^{S} C_{rs}n^rG(n-s)\equiv 0$.

So, to decipher the (or rather an) equation satisfied by G you need $(R+1)$ $(S+1)$ "bits" of information which can be obtained by plugging in values of G for $n=0, 1, 2,..., (R+1)(S+1)$, remembering that $G(Z_-)=0$. Many times, the resulting system of equations will have many solutions and it may turn out that $G(n)$ actually satisfies a recurrence of order less than S. (If this is the case we will get many possible recurrences, since once you know that $P(X^{-1}, n)\,G(n)=0$, then also $Q(X^{-1}, n)\,P(X^{-1}, n)\,G(n)=0$, for *every* operator Q).

Example (i). By a priori considerations, it can be shown that $G(n)=\sum\binom{n}{k}$ satisfies a recurrence of the form $(an+b)\,G(n)+(cn+d)$ $G(n-1)\equiv 0$. Since $G(-1)=0$, $G(0)=1$, $G(1)=2$, $G(2)=4$, $G(3)=8$, we have the following system of linear equations $b=0$, $2(a+b)+(c+d)=0$, $4(2a+b)+2(2c+d)=0$ and $8(3a+b)+4(3c+d)=0$, whose solution is $(a, b, c, d)=a(1, 0, -2, 0)$ and we obtain the expected recurrence $nG(n)-2nG(n-1)\equiv 0$.

In most cases the system of linear equation obtained is rather large. But if we have to *verify* that $G(n)$ satisfies a proposed recurrence life is much easier. All we have to do is plug in $n=0, 1,..., (R+1)(S+1)$ and verify that the proposed recurrence is satisfied for these values. Thus,

Proposition 6. *If it is known that $G(n)$ satisfies* some *recurrence of order R whose coefficients are polynomials of degree S, then in order to check that $G(n)$ satisfies a proposed recurrence (of the same or lower order) one only has to check it for a finite number of values of n. Namely, $n=0, 1,..., (R+1)(S+1)$.*

Corollary 6a. *If $F_1(n, k)$ and $F_2(n, k)$ are both multi-hypergeometric,* *then there exists a finite number L such that if* $\sum_k F_1(n,k) = \sum_k F_2(n,k)$ *is true for* $0 \leqslant n \leqslant L$, *it is true for every* $n \geqslant 0$.

EXAMPLE (ii). Prove that $\sum_{k=-n}^{n} (-1)^k \binom{2n}{n+k}^3 = (3n)!/(n)!^3$. By a priori considerations it is seen that $G(n) = \sum_{k=-n}^{n} (-1)^k \binom{2n}{n+k}^3$ satisfies a recurrence of the third order with coefficients of the third degree, i.e., $R = 3$, $S = 3$. We have to check that $G(n)$ satisfies the recurrence $n^2 G(n) - 3(3n-1)(3n-2)\,G(n-1) = 0$. All we have to do is let the computer check the above identity for $n = 0, \ldots, 16$.

Remark. The above resembles the fact that in order to check that two polynomials of degree $\leqslant N$ are equal, it is enough to check that they are equal at $N+1$ points.

1.5.

A considerable short-cut in Sister Celine's method is obtained in the case of binomial sums, as opposed to general hypergeometric sums. In this case we can write

$$F(n,k) = \prod_{i=1}^{L} \binom{a_i n + b_i k + c_i}{a_i' n + b_i' k + c_i'},$$

and then the polynomials $P(n,k)$, $P'(n,k)$, $Q(n,k)$, $Q'(n,k)$ of Theorem 4 can be factored with respect to k:

$$P'(n,k) = (k - m_1(n))(k - m_2(n)) \cdots (k - m_N(n)). \tag{1.3}$$

The main step in Sister Celine's method is finding the $(M+1)(N+1)$ unknowns $a_{rs}(n)$, $0 \leqslant r \leqslant n$, $0 \leqslant s \leqslant N$. Let us write (1.2′) more explicitly:

$$\begin{aligned}
&\sum_{s=0}^{N} a_{0s}(n) \left[\frac{P(n,k)}{Q(n,k)} \cdots \frac{P(n-s+1,k)}{Q(n-s+1,k)}\right] \\
&\quad + \frac{P'(n,k)}{Q'(n,k)} \sum_{s=0}^{N} a_{1s}(n) \left[\frac{P(n,k-1)}{Q(n,k-1)} \cdots \frac{P(n-s+1,k-1)}{Q(n-s+1,k-1)}\right] \\
&\quad + \frac{P'(n,k)\,P'(n,k-1)}{Q'(n,k)\,P'(n,k-1)} \sum_{s=0}^{N} a_{2s}(n) \left[\frac{P(n,k-2)}{Q(n,k-2)} \cdots \frac{P(n-s+1,k-2)}{Q(n-s+1,k-2)}\right] \\
&\quad \vdots \\
&\quad + \frac{P'(n,k) \cdots P'(n,k-M+1)}{Q'(n,k) \cdots P'(n,k-M+1)} \\
&\quad \times \sum_{s=0}^{N} a_{Ms}(n) \left[\frac{P(n,k-M)}{Q(n,k-M)} \cdots \frac{P(n-N+1,k-M)}{Q(n-N+1,k-M)}\right] = 0.
\end{aligned}$$

Plugging in $k = m_1(n), \ldots, m_N(n)$ yields N equations for the $N+1$ unknowns $a_{00}, \ldots, a_N$. Once we have found them we divide (1.4) by $P'(n, k)$ and substitute the zeros of $P'(n, k-1)$, viz., $k = m_1(n)+1, \ldots, m_N(n)+1$. This gives us a system for $a_{10}, \ldots, a_{1N}$. Repeating this process yields $M+1$ systems of equations each with $N+1$ unknowns, a considerable simplification over the initial system with $(M+1)(N+1)$ unknowns which was obtained by equating the coefficients of powers of k to zero.

Exercise. Find a recurrence equation for

$$G(n) = \sum_{k=-\infty}^{\infty} \binom{n}{k}\binom{n-1}{k+1}.$$

2. Generalizations of Sister Celine's Technique to the Class of Multi-P-Recursive Functions

2.1. $\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \cdot \left\{ {n-1 \atop k} \right\}$ not multi P-recursive!

Stanley's notion of P-recursiveness can be easily generalized to several variables.

Definition 7. $F\colon Z^n \to \mathbb{C}$ is *multi-P-recursive* if for $i = 1, \ldots, n$ there exist polynomials $P_j^i = P_j^i(m_1, \ldots, m_n)$, $j = 0, \ldots, r_i$, such that

"rectangular condition"

$$\sum_{j=0}^{r_i} P_j^i(\mathbf{m}) F(m_1, \ldots, m_{i-1}, m_i - j, m_{i+1}, \ldots, m_n) = 0. \tag{2.1}$$

In the shift operators notation, (2.1) can be written

$$\left(\sum_{j=0}^{r_i} P_j^i X_i^{-j} \right) F \equiv 0, \qquad i = 1, \ldots, n, \tag{2.1$'$}$$

where

$$X_i^{-1} F(\mathbf{m}) = F(\mathbf{m} - \mathbf{e}_i), \quad (\mathbf{e}_i = (0, \ldots, \underset{\overset{\cup}{i}}{1}, \ldots, 0)).$$

Loosely speaking F is multi-P-recursive if it satisfies an "ordinary" recurrence with polynomial coefficients in each of its variables. We are now ready for the following generalization of Theorem 4, which asserts that multi-P-recursiveness is preserved under the $\sum$ operation.

Theorem 8. *Let $F\colon Z^2 \to \mathbb{C}$ be multi-P-recursive and assume that $\sum_{k=-\infty}^{\infty} F(n, k)$ converges for every n. Then $G(n) = \sum_{k=-\infty}^{\infty} F(n, k)$ is P-recursive.*

Proof. Since $F(n,k)$ is multi-P-recursive, there exist polynomials $P_0,\ldots,P_r$, $Q_0,\ldots,Q_s$, in (n,k) such that

$$X^{-r}F=\left[\frac{P_0}{P_r}I+\frac{P_1}{P_r}X^{-1}+\cdots+\frac{P_{r-1}}{P_r}X^{-(r-1)}\right]F, \tag{2.2a}$$

$$Y^{-s}F=\left[\frac{Q_0}{Q_s}I+\frac{Q_1}{Q_s}Y^{-1}+\cdots+\frac{Q_{s-1}}{Q_s}Y^{-(s-1)}\right]F, \tag{2.2b}$$

where, as in the proof of Theorem 4,

$$X^{-1}f(n,k)=f(n-1,k), \qquad Y^{-1}f(n,k)=f(n,k-1).$$

For a rational function P/Q we define $\deg(P/Q)=\max(\deg P,\deg Q)$. Again we shall consider all polynomials in (n,k) as polynomials in k whose coefficients are polynomials in n. We need the following simple lemma.

LEMMA 8. *Let p (respectively q) be the maximal degree of the coefficients of* (2.2a) (2.2b). *If $L\geqslant r$, $K\geqslant s$, $X^{-L}Y^{-K}F(n,k)$ can be expressed as a linear combination of $\{X^{-i}Y^{-j}F(n,k);\ 0\leqslant i\leqslant r,\ 0\leqslant j<s\}$ with coefficients whose degree in k is at most $(L-r+1)\,p+(K-s+1)\,q$.*

Proof. The statement is certainly true for $(L,K)=(r,s-1)$ and $(L,K)=(r-1,s)$, by virtue of (2.2). We will presently show that the truth of the lemma for (L,K) implies its truth for $(L+1,K)$. The proof that (L,K) implies $(L,K+1)$ is similar, and the lemma would follow by double induction.

Indeed, assume that

$$X^{-L}Y^{-K}F=\sum_{i=0}^{r-1}\sum_{j=0}^{s-1}A_{ij}(X^{-i}Y^{-j}F),$$

where the A_{ij}'s are rational functions whose degree in k is at most $(L-r+1)\,P+(M-s+1)\,q$.

Applying X^{-1} to both sides yields

$$\begin{aligned}X^{-(L+1)}Y^{-K}F&=\sum_{i=0}^{r-1}\sum_{j=0}^{s-1}(X^{-1}A_{ij})(X^{-(i+1)}Y^{-j}F)\\&=\sum_{i=1}^{r-1}\sum_{j=0}^{s-1}(X^{-1}A_{i-1,j})\,X^{-i}Y^{-j}F\\&\quad+\sum_{j=0}^{s-1}(X^{-1}A_{r-1,j})\,Y^{-j}(X^{-r}F).\end{aligned}$$

Using (2.2a) we now express $X^{-r}F$ as a linear combination of $\{X^{-i}F; i=0,\ldots, r-1\}$, with coefficients of degree p. Thus, $X^{-(L+1)}Y^{-K}F$ can be expressed as a linear combination of $\{X^{-i}Y^{-j}F;\ 0 \leqslant i < r,\ 0 \leqslant j < s\}$ with coefficients whose degree does not exceed

$$(L-r+1)p+(K-s+1)q+p=[(L+1)-r+1]p+(K-s+1)q.$$

Completion of the proof of Theorem 8. Let us look for polynomials in n, $a_{ij}(n)$, such that $\sum_{i=0}^{M}\sum_{j=0}^{N} a_{ij}(n)\, X^{-i}Y^{-j}F(n,k) \equiv 0$, where M and N will be determined. As in the proof of Theorem 4, we would be able then to conclude that $G(n)$ satisfies a recurrence equation with polynomial coefficients. Clearing denominators and using the lemma yields

$$\sum_{i=0}^{M}\sum_{j=0}^{N} a_{ij}(n) \sum_{a=0}^{r-1}\sum_{b=0}^{s-1} A_{ab}^{ij} X^{-a}Y^{-b}F \equiv 0,$$

where A_{ab}^{ij} are polynomials, and changing the order of summation gives us

$$\sum_{a=0}^{r-1}\sum_{b=0}^{s-1}\left(\sum_{i=0}^{M}\sum_{j=0}^{N} a_{ij}(n)\, A_{ab}^{ij}\right) X^{-a}Y^{-b}F \equiv 0.$$

The expression in brackets is a polynomial of degree $(M-r+1)p+(N-s+1)q$ in k. Equating the coefficients of $k^{c}X^{-a}Y^{-b}F$ to zero,

$$(0 \leqslant c \leqslant (M-r+1)p+(N-s+1)q,\ 0 \leqslant a < r,\ 0 \leqslant b < s)$$

produces $rs[(M-r+1)p+(N-s+1)q+1]$ homogeneous equations for the $(M+1)(N+1)$ unknowns. In order to guarantee that there is a nontrivial solution we must require that

$$rs[(M-r+1)p+(N-s+1)q+1] < (M+1)(N+1).$$

Certainly there exist such M and N. This completes the proof of the theorem.

Consider $F\colon Z^n \to \mathbb{C}$ such that $\sum_{m_1=-\infty}^{\infty} \cdots \sum_{m_k=-\infty}^{\infty} F(m_1,\ldots, m_n)$ is convergent for all $(m_{k+1},\ldots, m_n)$. Summing over $m_1,\ldots, m_k$ kills the dependence on these variables and yields a function $G(m_{k+1},\ldots, m_n)$ in the surviving variables. The next theorem claims that if F is multi-P-recursive, so is G.

THEOREM 10. *Let $F\colon Z^n \to \mathbb{C}$ be multi-P-recursive and assume that*

$$G(m_{k+1},\ldots, m_n) = \sum_{m_1=-\infty}^{\infty} \cdots \sum_{m_k=-\infty}^{\infty} F(m_1,\ldots, m_k, m_{k+1},\ldots, m_n)$$

is defined for every $(m_{k+1},..., m_n) \in Z^{n-k}$. *Then* $G: Z^{n-k} \to \mathbb{C}$ *is also multi-P-recursive.*

Proof. The result would follow by iteration once we have proved it for $k = 1$. Since $F(\mathbf{m})$ is multi-P-recursive, it satisfies n recurrence equations of the following form:

$$\begin{aligned} &R_1(m_1,..., m_n; X_1)\, F = 0,\\ &R_2(m_1,..., m_n; X_2)\, F = 0,\\ &\quad\vdots\\ &R_n(m_1,..., m_n; X_n)\, F = 0. \end{aligned}$$

Viewing $\mathbb{C}[m_1,..., m_n]$ as $\mathbb{C}[m_2,..., m_n][m_1]$, we eliminate m_1 out of R_1 and R_2, R_1 and R_3,..., R_1 and R_n, like we did in Theorem 8. This produces equations of the following form:

$$\begin{aligned} &S_2(m_2,..., m_n; X_1, X_2)\, F = 0,\\ &\quad\vdots \qquad\qquad\quad \vdots\\ &S_n(m_2,..., m_n; X_1, X_2)\, F = 0, \end{aligned}$$

with m_1 missing. It follows, as in Lemma 5, that $G(m_2,..., m_n) = \sum_{m_1=-\infty}^{\infty} F(m_1,..., m_n)$ satisfies

$$\begin{aligned} &S_2(m_2,..., m_n); I, X_n)\, G = 0,\\ &\quad\vdots \qquad \vdots\\ &S_n(m_2,..., m_n); I, X_n)\, G = 0. \end{aligned}$$

So G is multi-P-recursive.

Remark. The above theorem and algorithm enables us to "evaluate" multi-binomial sums, either explicitly or inductively, in a parallel fashion to the ways indicated in Section 1.

2.2. *q-Binomial Identities*

All the foregoing has an immediate q-analog. A function $F: Z \to \mathbb{C}$ is q-hypergeometric if $F(n)/F(n-1)$ is a rational function of q^n. The notions of $q-P$-recursiveness and multi-$q-P$-recursiveness are similarly defined where $m_1,..., m_n$ are to be replaced by $q^{m_1},..., q^{m_n}$, respectively. It thus follows that every q-binomial identity can be verified in a finite number of steps.

EXAMPLE. Andrews [1] conjectured that the constant term of $\prod_{1 \leqslant i \neq j \leqslant n} (\varepsilon_{ij} X_i/X_j)_{a_i}$ (where $\varepsilon_{ij} = 1$ if $i < j$ and $= q$ if $i > j$, $a_1,..., a_n$ are positive integers, and $(y)_a = (1-y)(1-qy) \cdots (1-q^{a-1}y))$ is the multi-binomial coefficient $(q)_{a_1+\cdots+a_n}/[(q)_{a_1}\cdots(q)_{a_n}]$. This is known for $n \leqslant 3$ but still open for $n \geqslant 4$. Since the constant term of the above expression can be expressed as a multi-q-binomial sum, our method ensures that for every n there exists an integer $L(n)$, which can be explicitly computed, such that the truth of Andrews, conjecture for $0 \leqslant a_i \leqslant L(n)$, $i = 1,\ldots, n$, would imply its truth in general. However, $L(n)$ becomes very large with n and in any case there is no way our method can prove the conjecture for every n. We shall return to Andrews' conjecture later and present a line of attack which appears to be more promising.

2.3. *The Class of Multi-P-Recursive Functions*

Stanley [14] considered the class of P-recursive functions on Z and proved that they form an algebra with respect to addition and multiplication. His proofs can be easily generalized to show that the class of multi-P-recursive functions on Z^n is also an algebra.

Theorem 10 above answers some of the problems raised by Stanley in [14]. In particular, problems (c), (e), and (f) in [14] asked whether certain functions are P-recursive. Since these functions are expressible as binomial sums, the answer is affirmative.

2.4. *The Taylor Coefficients of a Rational Function*

The following theorem was conjectured by Stanley [14] and independently proved by Gessel [9].

THEOREM 11. *Let $P(x_1,\ldots, x_n)$ and $Q(x_1,\ldots, x_n)$ be two polynomials and assume that $Q(0,\ldots, 0) \neq 0$. Let $P/Q = \sum_{\mathbf{m}\in N^n} f(\mathbf{m})\,\mathbf{x}^{\mathbf{m}}$, then $f(\mathbf{m})$ is multi-P-recursive and in particular the diagonal $g(n) = f(n,\ldots, n)$ is P-recursive.*

Proof. Since the class of multi-P-recursive functions is an algebra, we can assume without loss of generality that $P = 1$. Let $Q = 1 - \sum_{i=1}^{K} C_i \mathbf{x}^{\boldsymbol{\beta}_i}$, then

$$Q^{-1} = \sum_{k_1,\ldots,k_K} \frac{(k_1+\cdots+k_K)!}{k_1!\cdots k_K!} C_1^{k_1}\cdots C_K^{k_K}\mathbf{x}^{k_1\boldsymbol{\beta}_1+\cdots+k_K\boldsymbol{\beta}_K}$$

and so

$$f(\mathbf{m}) = \sum \frac{(k_1+\cdots+k_K)!}{k_1!\cdots k_K!} C_1^{k_1}\cdots C_K^{k_K},$$

where the sum is taken over the set $\{(k_1,\ldots, k_K);\ k_1\boldsymbol{\beta}_1+\cdots+k_K\boldsymbol{\beta}_K = \mathbf{m}\}$. Elementary linear algebra shows the existence of constants (independent of $\mathbf{m}$) a_{ji} and b_{ij} such that a typical member of the above set can be written $k_j = \sum_{i=1}^{n} a_{ji}m_i + \sum_{i=1}^{l} b_{ij}S_i$, $j = 0,\ldots, K$, where $S_1,\ldots, S_l$ are running parameters. Substituting this expression in the above formula for $f(\mathbf{m})$ yields a multi-binomial sum with $l \sum$ signs. Now the theorem follows from Theorem 10.

It is easily seen that $f: Z^n \to \mathbb{C}$, defined in N^n by $Q^{-1} = \sum_{m \in N^n} f(\mathbf{m})\, \mathbf{x}^{\mathbf{m}}$ and $f = 0$ outside N^n, satisfies the inhomogeneous partial difference equation with constant coefficients $Q(X_1^{-1},..., X_n^{-1})\, f = \delta$, where δ is the discrete Dirac delta function: $\delta(\mathbf{0}) = 1$, $\delta(Z^n - \mathbf{0}) = 0$. Recall that a function g satisfying $Qg = \delta$ is called *a fundamental solution corresponding to Q*. In terms of this terminology we can rephrase the last theorem.

THEOREM 12. *Every partial difference operator with constant coefficients $Q(X_1^{-1},..., X_n^{-1})$ such that $Q(0,..., 0) \neq 0$ has a multi-P-recursive fundamental solution.*

3. THE CONTINUOUS ANALOG OF SISTER CELINE'S TECHNIQUE AND MULTI-D-FINITE FUNCTIONS

3.1.

One of the methods for evaluating definite integrals is to differentiate with respect to a parameter and then integrate by parts, thus getting a certain differential equation with respect to the parameter. For example (Gillespie [10, p. 99]), to evaluate $I(b) = \int^{\infty} e^{-x^2} \cos 2bx\, dx$. We have

$$I'(b) = \int_{-\infty}^{\infty} -2xe^{-x^2} \sin 2bx\, dx = \int_{-\infty}^{\infty} -2be^{-x^2} \cos 2bx = -2bI(b),$$

obtaining the differential equation $I'(b) + 2bI(b) = 0$. Solving it gives $I(b) = KE^{-b^2}$ and since $I(0) = \sqrt{\pi}$, $I(b) = \sqrt{\pi}\, e^{-b^2}$.

The continuous analog of Sister Celine's technique offers a uniform setting for what appeared to be a collection of tricks. But before describing it we need to introduce the continuous counterpart of the notions "hypergeometric," "P-recursive," and "multi-P-recursive." Recall that $f: Z \to \mathbb{C}$ is hypergeometric (P-recursive) if it is a solution of a first order (any order) linear recurrence equation with polynomial coefficients.

DEFINITION 13. Let F: $R \to \mathbb{C}$ be a function, distribution, or a formal power series. F is hyperexponential if these exist polynomials $p(x)$ and $q(x)$ such that $p(x)\, F'(x) + q(x)\, F(x) \equiv 0$. In other words $F(x)$ is hyperexponential if its logarithmic derivative is a rational function. It is clear that the product of 2 hyperexponential functions is again hyperexponential.

The following definition is due to Stanley [14].

DEFINITION 14. $F: R \to \mathbb{C}$ is D-finite if it satisfies a differential equation with polynomial coefficients. Namely, there exist polynomials $p_0, \ldots, p_r$ such that $(p_0 + P_1 D + \cdots + p_r D^r) F \equiv 0$.

This definition is immediately generalizable to several variables.

DEFINITION 15. $F: R^n \to \mathbb{C}$ is multi-D-finite if for $i = 1, \ldots, n$ there exist polynomials $P_j^i = P_j^i(x_1, \ldots, x_n)$, $j = 0, \ldots, r_i$, such that i.e. D-finite in each component separately

$$\left(\sum_{j=0}^{r_i} P_j^i D_j^i \right) F \equiv 0. \tag{3.1}$$

(Here $D_i = \partial/\partial x_i$, $i = 1, \ldots, n$).

We are now ready for the continuous analog of Sister Celine's technique.

THEOREM 16. *Let $F(x, y)$ be multi-hyperexponential such that $G(y) = \int_{-\infty}^{\infty} F(x, y)\, dx$ converges for every y. Then $G(y)$ is D-finite.*

Proof. There exist polynomials P, P', Q, Q' in (x, y) such that $D_x F = (P/Q) F$, $D_y F = (P'/Q') F$. If the degrees in x of P/Q and P'/Q' are p, p', respectively, then it is easily seen, using Leibnitz's rule inductively, that $D_x^i D_y^j F = (A_{ij}/B_{ij}) F$, where the degree of A_{ij}/B_{ij} is $ip + jp'$. Now we eliminate x in the same way as we eliminated k in Theorem 4, getting an operator $R(D_y, D_x, y)$ such that $RF \equiv 0$. Now write $R(D_y, D_x, y) = R_0(D_y, y) - D_x S(D_y, D_x, y)$, where $R_0(D_y, y) = R(D_y, 0, y)$. Finally, it is seen that

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} R(D_y, D_x, y) F(x, y) \\ &= \int_{-\infty}^{\infty} R_0(D_y, y) F(x, y)\, dx + \int_{-\infty}^{\infty} D_x[S(D_y, D_x, y) F]\, dx \\ &= R_0(D_y, y)\, G(y) + 0. \end{aligned}$$

Remark. The continuous counterpart of the method of creative telescoping would be to first find $F'(x, y) = S(D_y, D_x, y) F(x, y)$ and then ask the reader to verify (or believe) the formula $R_0(D_y, y) F(x, y) = D_x F'(x, y)$, from which $R_0(D_y, y)\, G(y) = 0$ readily follows.

Exercise. Find a differential equation with polynomial coefficients satisfied by

$$G(y) = \int_{-y}^{y} e^{1/(x^2 - y^2)}\, dx.$$

Hint. Consider the function

$$F(x, y) = e^{(x^2-y^2)-1}, \qquad |x| < y$$
$$= 0, \qquad |x| \geqslant y.$$

We have

$$D_x F = \frac{-2x}{(x^2 - y^2)^2} F; \qquad D_y F = \frac{2y}{(x^2 - y^2)^2} F.$$

The proof of Theorem 10 can be easily translated to continuous language to yield.

THEOREM 17. *Let F: $R^n \to \mathbb{C}$ be multi-D-finite and assume that $G(x_{k+1},..., x_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} F_1(x_1,..., x_n)\, dx_1,..., dx_k$ is defined for every $(x_{k+1},..., x_n) \in R^{n-k}$, then G is also multi-D-finite.*

3.2.

The class of multi-D-finite tempered distributions has been considered extensively by I. N. Bernstein who gave them a different definition. A very clear exposition of I. N. Bernstein's deep theory can be found in the recent monograph of Björk [4]. It follows from Bernstein's theory that f is multi-D-finite if the $\mathbb{C}^{2n}$ subvariety $\{(\mathbf{x}, \boldsymbol{\xi}) \in \mathbb{C}^{2n};\ \sigma(P)(\mathbf{x}, \boldsymbol{\xi}) = 0\ \ \forall P \in I(f)\}$ has dimension n. Here

$$I(f) = \{P = \sum C_{\alpha\beta} \mathbf{x}^{\boldsymbol{\alpha}} \mathbf{D}^{\boldsymbol{\beta}};\ Pf = 0\}$$

and

$$\sigma\Bigg(\sum_{\substack{|\alpha| \leqslant M \\ |\beta| \leqslant N}} C_{\alpha\beta} \mathbf{X}^{\boldsymbol{\alpha}} \mathbf{D}^{\boldsymbol{\beta}}\Bigg) = \sum_{\substack{|\boldsymbol{\alpha}| \leqslant M \\ |\boldsymbol{\beta}| = N}} C_{\alpha\beta} \mathbf{x}^{\boldsymbol{\alpha}} \boldsymbol{\xi}^{\boldsymbol{\beta}}.$$

(I.N. Bernstein proved that $\dim V_f \geqslant n$ for every f, and $\dim V_f = n$ if f is multi-D-finite).

One of I. N. Bernstein's major achievements was the result that every linear partial differential operator with constant coefficients has a multi-D-finite fundamental solution. Since multi-P-recursiveness is the discrete analog of multi-D-finiteness, Theorem 2 can be viewed as a partial discrete analog of Bernstein's theorem. It is possible to imitate Bernstein's proof to show that Theorem 12 is still true even if $Q(0,..., 0) = 0$, but the proof is much more complicated.

3.3. *The Isomorphism between Multi-D-Finite Formal Power Series and Multi-P-Recursive Functions*

Stanley proved that $f\colon N \to \mathbb{C}$ is P-recursive iff the formal power series $\sum_{n=0}^{\infty} f(n)\, x^n$ is D-finite. A slightly more complicated proof shows that $f\colon N^n \to \mathbb{C}$ is multi-P-recursive iff the formal power series $\sum_{\mathbf{m}\in N^n} f(\mathbf{m})\, \mathbf{x}^{\mathbf{m}}$ is multi-D-finite. This furnishes another proof to theorem 11 since it is easily verified that $Q(x_1,..., x_n)^{-1}$ is multi-D-finite.

4. The Ultimate Generalization: Toward a New Approach to the Theory of Special Functions

4.1.

Askey [3, p. xxiv] defines a special function as "a function which occurs often enough that it gets a name." This very apt "meta" mathematical definition explains why the more than two centuries old theory of special functions was so reluctant to be confined to a narrow theoretical framework. The first attempt at a unified theory was undertaken by Truesdell [15] who implicitly defined a special function as one which can be transformed to a solution of what he called "the F-equation," namely, the partial differential recurrence equation

$$F(z, \alpha + 1) - F(z, \alpha) = \frac{\partial F}{\partial Z}(z, \alpha).$$

Another line of attack, which employed the deep theory of Lie groups and algebras, was started by Wigner, continued by Weisner and carried into perfection by Miller [12] and others. Here we shall present an approach based on the observation that all known special functions are both D-finite and P-recursive, in a sense to be explained.

Indeed, all families of special functions have some things in common. In particular, they satisfy a *differential equation* and a *recurrence equation*. For example, the Legendre polynomials satisfy [13, pp. 160, 161].

$$nP_n(x) - (2n-1)\, xP_{n-1}(x) + (n-1)\, P_{n-2}(x) = 0, \tag{4.1}$$

$$(1-x^2)\, P_n''(x) - 2xP_n'(x) + n(n+1)\, P_n(x) = 0. \tag{4.2}$$

Notice that all these equations have coefficients which are polynomials in n and x.

The reader might have noticed that in previous sections we refrained from using the terms "sequence" and the notation $\{a_n\}$. Instead we chose to upgrade n from a subscript to a variable and say "the function $f(n)$ defined on Z."

This "discrete lib" put discrete and continuous functions on the same footing and enabled us to realize the close analogy between P-recursiveness and D-finiteness (first noticed by Stanley [14]). Accordingly, instead of the phrase "the sequence of functions $\{P_n(x)\}_0^\infty$, we shall say, "the function $\tilde{P}: N \times R \to \mathbb{C}$ defined by $P(n, x) = \tilde{P}_n(x)$. Writing $X\tilde{P}(n, x) = \tilde{\mathbf{P}}(n+1, x)$, $D = d/dx$, Eq. (4.1) and (4.2) become

$$[nI - (2n-1)\,xX^{-1} + (n-1)\,X^{-2}]\,\tilde{\mathbf{P}} \equiv 0, \tag{4.1'}$$

$$[(1-x^2)\,D^2 - 2xD + n(n+1)\,I]\,\tilde{\mathbf{P}} \equiv 0. \tag{4.2'}$$

Thus $\tilde{\mathbf{P}}$ satisfies "ordinary" equations with polynomial coefficients in each of its variables. The same is true for the Bessel, Laguerre, Hermite, Jacobi and all other known special functions. It is seen that being "special" is nothing but the analog on $Z \times R$ of both D-finiteness and P-recursiveness, and consequently the next definition should not come as a surprise.

DEFINITION 18. F: $Z \times R \to \mathbb{C}$ is a *special function* if these exist polynomials $\mathbf{P}_0(n, x),..., \mathbf{P}_r(n, x)$, $Q_0(n, x),..., Q_s(n, x)$ that that

$$(\mathbf{P}_0 + \mathbf{P}_1 D + \cdots + \mathbf{P}_r D^r)\,F \equiv 0,$$

$$(Q_0 + Q_1 X^{-1} + \cdots + Q_s X^{-s})\,F \equiv 0.$$

The class of special functions on $Z \times R$ will be denoted $S(1, 1)$.

The above definition immediately suggests a general definition for special functions of several variables, possibly paving the way to a general theory of special functions of several variables.

DEFINITION 19. F: $Z^k \times R^l \to \mathbb{C}$ is *special* if there exist polynomials in $(m_1,..., m_k; x_1,..., x_l)$, $\mathbf{P}^i_0,..., \mathbf{P}^i_{r_i}$, $i = 1,..., k$; $Q^j_0,..., Q^j_{r_j}$, $j = 1,..., l$, such that

$$(\mathbf{P}^i_0 + \mathbf{P}^i_1 X_i^{-1} + \cdots + \mathbf{P}^i_{r_i} X_i^{r_i})\,F \equiv 0, \qquad i = 1,..., k$$

$$(Q^j_0 + Q^j_1 D_j + \cdots + Q^j_{r_j} D_j^{r_j})\,F \equiv 0, \qquad j = 1,..., l.$$

The class of special functions on $Z^k \times R^l$ will be denoted $L(k, l)$. It is clear that if $F(m_1,..., m_k;\ x_1,..., x_l)$ is in $L(k, l)$, then for every integer a, $F(a, m_2,..., m_k; x_1,..., x_l)$ is in $L(k-1, l)$ and for every real y_0, $F(m_1,..., m_k; y_0, x_2,..., x_l)$ is in $L(k, l-1)$. Note that $L(n, 0)$ and $L(0, n)$ are the classes of multi-P-recursive and multi-D-finite functions, respectively. The method of proof of Theorems 10 and 11 yields immediately.

THEOREM 20. *Let F: $Z^k \times R^l \to \mathbb{C}$ belong to $L(k, l)$, then if*

$$G(m_2, ..., m_k, x_1, ..., x_l) = \sum_{m_1=-\infty}^{\infty} F(m_1, ..., m_k, x_1, ..., x_l)$$

is defined on $Z^{k-1} \times R^l$, then it belongs to $L(k-1, l)$. Similarly, if

$$H(m_1, ..., m_k, x_2, ..., x_k) = \int_{-\infty}^{\infty} F(m_1, ..., m_k, x_1, ..., x_k)\, dx_1$$

is defined on $Z^k \times R^{l-1}$, it belongs to $L(k, l-1)$.

As a matter of fact, the limits of integration or summation need not be $(-\infty, \infty)$, and we have the following more general result.

THEOREM 21. *Let F: $Z^k \times R^l \to \mathbb{C}$ belong to $L(k, l)$; then if a_0, a_0' are constant integers (possibly $\pm\infty$) and a, b are real constants (possibly $\pm\infty$), then*

$$G(m_2, ..., m_k, x_1, ..., x_k) = \sum_{m_1=a_0}^{a_0'} F(m_1, ..., m_k, x_1, ..., x_k)$$

and

$$H(m_1, ..., m_k, x_2, ..., x_k) = \int_a^b F(m_1, ..., m_k; x_1, ..., x_k)\, dx_1$$

belongs to $L(k-1, l)$ and $L(k, l-1)$, respectively, provided they are defined.

Proof. We shall prove that if $F \in L(1, 1)$, then $G(n) = \int_a^b F(n, x)\, dx$ is P-recursive (i.e., belongs to $L(1, 0)$). The proof of the general result is similar. Once again we eliminate x, getting an operator $R(n, E, D)$ such that $R(n, E, D) F \equiv 0$. Write $R(n, E, D) = R_0(n, E) + DS(n, E, D)$; then

$$\begin{aligned} 0 &= \int_a^b R(n, E, D) F(n, x)\, dx \\ &= \int_a^b R_0(n, E) F(n, x)\, dx + \int_a^b D[S(n, E, D) F(n, x)]\, dx. \end{aligned}$$

By the fundamental theorem of calculus,

$$R_0(n, E) G(n) = S(n, E, D) F(n, a) - S(n, E, D) F(n, b). \qquad (*)$$

But since $F(n, x)$ is special so is $S(n, E, D) F(n, x)$ and therefore both $S(n, E, D) F(n, a)$ and $S(n, E, D) F(n, b)$ are P-recursive, i.e., belong to

$L(1, 0)$. By Stanley's theorem the difference is also P-recursive, and therefore there exists an operator $R_1(n, E)$ annihilating the right-hand side of (*). Consequently $R_1(n, E)\, R_0(n, E)\, G(n) = 0$ and G is P-recursive.

The last theorem explains why so many special functions are defined either by a sum or an integral.

Example (i).

$$_pF_q(\alpha_1, \alpha_2, ..., \alpha_p, \beta_1, ..., \beta_q, x) = 1 + \sum_{m=1}^{\infty} \frac{\prod_{i=1}^{p} (\alpha_i)_n}{\prod_{j=1}^{\infty} (\beta_j)_n} \frac{x^n}{n!}$$

satisfies a differential equation with respect to x and pure recurrence relations, with polynomial coefficients, in each of the discrete variables $\alpha_1, ..., \alpha_p, \beta_1, ..., \beta_q$. The summand belongs to $L(p+q+1, 1)$ and the sum belongs to $L(p+q, 1)$. Of course $_pF_q$ also satisfies many mixed relations involving simultaneous shifts in several α_i's and β_j's, called continguous relations. Wilson [7] found all linearly independent contiguous relations satisfied by $_3F_2(1)$.

EXAMPLE (ii). $F(m, n, x) = x^m(1-x)^n$ belongs to $L(2, 1)$, and indeed

$$F(m+1, n, x) - xF(m, n, x) = 0,$$

$$F(m, n+1, x) - (1-x)\, F(m, n, x) = 0,$$

$$x(1-x)\, DF - [m - (n+n)\, x]\, F = 0.$$

It follows from Theorem 21 that

$$B(m, n) = \int_0^1 x^m(1-x)^m\, dx$$

belongs to $L(2, 0)$, which is hardly surprising considering the fact that $B(m, n) = (m-1)!\, (m-1)!/(m+m-1)!$, the very famous beta function which is not just multi-P-recursive but multi-hypergeometric:

$$(m+m)\, B(m+1, n) - mB(m, n) = 0,$$

$$(m+m)\, B(m, n+1) - nB(m, n) = 0.$$

Carlson [5] uses generalization of $\int_0^1 x^m(1-x)^n\, dx$, which he calls "Dirichlet Averages," as a basis of his approach to special functions. Such integrals enable him to express all the known special functions as certain integrals of elementary functions. In this respect, his method is a special case of ours. In general, if $P_1(x),\ldots, P_k(x)$ are rational functions, then

$$F(m_1,\ldots, m_k) = \int_0^1 P_1(x)^{m_1} \cdots P_k(x)^{m_k}\, dx$$

is multi-P-recursive.

EXAMPLE (iii). $f(k, \theta, z) = \cos(2k\theta) \cos(z \sin \theta)$ is easily seen to belong to $L(1, 2)$, and so $\int_0^\pi f(k, \theta, z)\, d\theta$ must belong to $L(1, 1)$, i.e., is special. It turns out [13, p. 120] that it is the Bessel function $J_{2k}(z)$.

4.2. *All Identities Involving Special Functionss Can Be Checked in a Finite Number of Steps*

Like polynomials and multi-P-recursive functions, special functions are completely characterized by a finite number of parameters, namely, the coefficients of P_l^j and the initial values in the "characteristic set" (the set of common zeros of $P_{r_i}^i, Q_{r_j}^j$) which is a finite set in virtue of I. N. Bernstein's theory. Thus, given any identity involving sums or integrals, we can use the generalized Sister Celine technique to find the appropriate differential and recurrence equations satisfied by both sides and see whether they match, and then compare initial values. Alternatively, we can plug in enough special values and check for them and then deduce equality in general, like we did in Section 1.

EXAMPLE.

$$x^n = 2^{-n} n! \sum_{k=0}^{[n/2]} \frac{(2n-4k+1)\, P_{n-2k}(x)}{k!\, (3/2)_{n-k}}.$$

The summand on the right-hand side belongs to $L(2, 1)$ and the sum therefore belongs to $L(1, 1)$. Calling the right-hand side $G(n, x)$, we should obtain, using the above methods,

$$G(n, x) - xG(n-1, x) = 0,$$
$$(XD - n)\, G = 0$$

from which follows $G(n, x) = x^n$.

4.3. *Generating Functions*

It follows that if $F \in L(k, l)$, then

$$G(m_1,\ldots, m_{k-1}; S, x_1,\ldots, x_k) = \sum_{m_k=0}^{\infty} F(m_1,\ldots, m_k, x_1,\ldots, x_l)\, S^{m_k}$$

belongs to $L(k-1, l+1)$. Conversely $H(m_1, \ldots, m_{k+1}; x_2, \ldots, x_e) =$ coeff. of $x_1^{m_{k+1}}$ in $G(m_1, \ldots, x_e)$ belongs to $L(k+1, l-1)$.

EXAMPLE (i) [13, p. 165]. For the Legendre polynomials

$$\sum_{n=0}^{\infty} \mathbf{P}_n(x)\, t^n = (1 - 2xt + t^2)^{-1/2},$$

which certainly belongs to $L(0, 2)$.

EXAMPLE (ii). For the Laguerre polynomials

$$\sum_{n=0}^{\infty} L_n^{(a)}(x)\, t^n = (1-t)^{-(1+a)} \exp(-xt/(1-t)),$$

which is clearly in $L(0, 2)$.

Of course if $\mathbf{P}_n(x)$ is special and C_n is $\mathbf{P}$-recursive, then $C_n \mathbf{P}_n(x)\, t^n$ is in $L(1, 2)$ and

$$\sum_{n=0}^{\infty} C_n \mathbf{P}_n(x)\, t^n$$

is in $L(0, 2)$, i.e., is multi-D-finite in x and t.

4.4. *Connection Coefficients*

Most of the known special functions are also orthogonal and hence satisfy a linear recurrence of the form

$$x\mathbf{P}_n(x) = a(n)\, \mathbf{P}_{n+1}(x) + b(n)\, \mathbf{P}_n(x) + c(n)\, \mathbf{P}_{n-1}(x). \qquad (*)^n$$

Any family of polynomials satisfying $(*)$ for general $a(n)$, $b(n)$, $c(n)$ is called *orthogonal*. Given two families of orthogonal polynomials, we are interested in the connection coefficients

$$C(n, k) = \int_a^b \mathbf{P}_n(x)\, Q_k(x)\, w(x)\, dx,$$

where it is assumed that $\{Q_k(x)\}$ is orthogonal with respect to the measure $w(x)\, dx$ over the interval (a, b). It so happens that in all the classical cases $w(x)$ is algebraic and therefore (Stanley [14]) D-finite. If both $\{\mathbf{P}_n(x)\}$ and $\{Q_k(x)\}$ belong to $L(1, 1)$ and $w(x)$ is D-finite, then $F(n, k, x) = \mathbf{P}_n(x)\, Q_k(x)\, w(x)$ is in $L(2, 1)$ and by Theorem 21, $C(n, k)$ is in $L(2, 0)$. Thus, PROPOSITION 22. *If* $\{\mathbf{P}_n(x)\}$ and $\{Q_k(x)\}$ *are two families of orthogonal special functions and* $\{Q_k(x)\}$ *is orthogonal over* (a, b) *with respect to* $w(x)\,dx$, *where* $w(x)$ *is D-finite, then the connection coefficients function*

$$C(n, k) = \int_a^b \mathbf{P}_n(x)\, Q_k(x)\, w(x)\, dx$$

is multi-P-recursive.

Sometimes we are lucky and the relevant recurrences turn out to be first order, in which case $C(n, k)$ can be expressed as a single product, as is the case between Gegenbauer polynomials with different indices (Asley [2, p. 59]). However in most cases the resulting recurrences are of higher order and the best that can be done is to express $C(n, k)$ as a sum of products (e.g., [2, p. 62, (7.28)]).

Remark. If you already have a guess what $C(n, k)$ might be, it is very *easy* to verify (or falsify) your guess. If $\mathbf{P}_n(x)\, Q_k(x)$ satisfy

(1) $x\mathbf{P}_n(x) = a(n)\,\mathbf{P}_{n+1}(x) + b(n)\,\mathbf{P}_n(x) + C(n)\,\mathbf{P}_{n-1}(x)$.

(2) $xQ_k(x) = a'(k)\, Q_{k+1}(x) + b'(k)\, Q_k(x) + c'(k)\, Q_{k-1}(x)$,

then $F(n, k, x) = \mathbf{P}_n(x)\, Q_k(x)\, w(x)$ satisfies

(3) $$\begin{aligned} &a(n)\,F(n+1, k, x) + b(n)\,F(n, k, x) + c(n)\,F(n-1, k, x) \\ &\quad = a'(k)\,F(n, k+1, x) + b'(k)\,F(n, k, x) + c'(k)\,F(n, k-1, x). \end{aligned}$$

Since the coefficients are independent of x the same partial difference equation is satisfied by

$$C(n, k) = \int_a^b F(n, k, x)\, dx:$$

(4) $$\begin{aligned} a(n)\,C(n+1, k) &= a'(k)\,C(n, k+1) + (b'(k) - b(n))\,C(n, k) \\ &\quad + c'(k)\,C(n, k-1) - c(n)\,C(n-1, k). \end{aligned}$$

All you have to do is verify (4) and check $C(0, k)$ and $C(1, k)$.

4.5. *Linearization Coefficients*

PROPOSITION 23. *Let* $\{\mathbf{P}_n(x)\}$ *be an orthogonal family of special functions (i.e.,* $\tilde{\mathbf{P}}(n, x) = \mathbf{P}_n(x) \in L(1, 1)$) *which is orthogonal over* (a, b) *with respect to the measure* $w(x)\,dx$. *Assume that* $w(x)$ *is D-finite. Then the linearization coefficients function*

$$b(n, k, l) = \int_a^b \mathbf{P}_n(x)\,\mathbf{P}_k(x)\,\mathbf{P}_l(x)\, w(x)\, dx$$

belongs to $L(3, 0)$, *i.e., is multi-P-recursive.*

Proof. $F(n,k,l,x) = \mathbf{P}_n(x)\,\mathbf{P}_k(x)\,\mathbf{P}_l(x)\,w(x)$ belongs to $L(3,1)$, being the product of special functions. The result follows from Theorem 21.

Once again the appropriate ordinary recurrences satisfied by $b(n,k,l)$ may turn out to be of the first order, in which case $b(n,k,l)$ is expressible as a single product. This is the case with the Legendre, ultraspherical, and Hermite polynomials [2, pp. 39, 42]. But in most cases we have higher-order recurrences. In the case of the Jacobi polynomials, Hylleraas [2, p. 40] found second-order recurrence relations satisfied by $b(n,k,l)$.

Askey [2, p. 40] is unhappy about "guessing the solution for small n and then proving it by induction." The generalized Sister Celine elimination algorithm eliminates all the "guess work" from computing the linearization coefficients (or rather of finding the appropriate recurrences) but is very hard to carry out. Once again, the significance of Proposition 20 rests on the fact that indeed *there are* ordinary recurrence equations satisfied by $b(n,k,l)$. So by computing $b(n,k,l)$ for small values of (n,k,l) we can plug them in and let the computer solve the resulting linear system of equations. To check that the recurrences obtained are true, all one has to do is verify that the following partial recurrence equation is satisfied

$$\begin{aligned}\alpha(n)\,b(n-1,k,l)&+\beta(n)\,b(n,k,l)+\gamma(n)\,b(n+1,k,l)\\&=\alpha(k)\,b(n,k-1,l)+\beta(k)\,b(n,k,l)+\gamma(k)\,b(n,k+1,l)\qquad(*)\end{aligned}$$

(where $x\mathbf{P}_n(x)=\alpha(n)\,\mathbf{P}_{n-1}(x)+\beta(n)\,\mathbf{P}_n(x)+\gamma(n)\,\mathbf{P}_{n+1}(x)$). The proof of $(*)$ is similar to the one indicated in the previous subsection. Of course one also has to check the boundary values $b(0,k,l)$ and $b(1,k,l)$.

5. Coefficients of Rational Functions and Their q-Analog

5.1.

Dyson's ex-conjencture (see [18]) states that the constant term of

$$\prod_{1\leqslant i\neq j\leqslant m}\left(1-\frac{x_i}{x_j}\right)^{a_i}$$

is $(a_1+\cdots+a_n)!/[a_1!\cdots a_n!]$. This result was proved independently by Wilson, Gunson, and Good (see [18] for a detailed discussion of Good's elegant proof). Certainly $(a_1+\cdots+a_n)!/(a_1!\cdots a_n!)$ is multi-hypergeometric. If $\phi_1(x_1,\dots,x_m),\dots,\phi_n(x_1,\dots,x_m)$ are general rational functions, we can hardly expect the constant term of $\phi_1^{a_1}\cdots\phi_n^{a_n}$ to be multi-hypergeometric, but the following is true.

Proposition 24. *Let $F_0(a_1,\dots,a_n)$ be the constant term of $\phi_1^{a_1}\cdots\phi_n^{a_n}$, where $\phi_1,\dots,\phi_n$ are rational functions of $x_1,\dots,x_m$. Then F_0 is multi-P-recursive.*

Proof. $F(x_1,\dots,x_m,a_1,\dots,a_n)=\phi_1^{a_1}\cdots\phi_n^{a_n}$ is in $L(n,m)$ (check!) so by the comments in Section 4.3,

$$\begin{aligned}&G(b_1,\dots,b_m,a_1,\dots,a_n)\\&\qquad=\text{coefficient of } x_1^{b_1}\cdots x_m^{b_m} \text{ in } F(x_1,\dots,x_m,a_1,\dots,a_n)\end{aligned}$$

is in $L(m+n,0)$ and by the remarks following Definition 19, $F_0(a_1,\dots,a_n)=G(0,\dots,0;a_1,\dots,a_n)$ is in $L(n,0)$.

We shall now describe how to actually obtain the ordinary recurrences satisfied by $F_0(a_1,\dots,a_n)$. We have

$$\begin{aligned}E_1F/F&=\mathbf{P}_1/Q_1,\\ \vdots\quad&\quad\vdots\\ E_nF/F&=\mathbf{P}_n/Q_n\\ (x_1D_1)\,F/F&=\mathbf{P}_1'/Q_1',\\ \vdots\quad&\quad\vdots\\ (x_mD_m)\,F/F&=P_m'/Q_m',\end{aligned}$$

where $\mathbf{P}_1,\dots,\mathbf{P}_n$, $Q_1,\dots,Q_n$, $\mathbf{P}_1',\dots,P_m'$, $Q_1',\dots,Q_m'$ are polynomials in $(x_1,\dots,x_m)$. By iterating the above equations, using the continuous and discrete Leibnitz rules, we form $E_1^{\alpha_1}(x_1D_1)^{\beta_1}\cdots(x_nD_n)^{\beta_n}$ for sufficiently large $\alpha_1,\beta_1,\dots,\beta_m$. Then we look for polynomials in $(a_1,\dots,a_n)$, $C_{\alpha_1\beta_1,\dots,\beta_m}(a_1,\dots,a_n)$, such that

$$\left[\sum C_{\alpha_1\beta_1\cdots\beta_m}E_1^{\alpha_1}(x_1D_1)^{\beta_1}\cdots(x_mD_m)^{\beta_m}\right]F\equiv 0 \tag{*}$$

by equating all the coefficients of $x_1^{\alpha_1}\cdots x_m^{\alpha_m}$ to zero. But since $(xD)(\sum_{-\infty}^{\infty}a_nx^n)=\sum_{-\infty}^{\infty}na_nx^n$, comparing the constant term on both sides of (*) yields

$$\left(\sum C_{\alpha_10\cdots0}E_1^{\alpha_1}\right)F_0\equiv 0.$$

The ordinary recurrence equations in $E_2,\dots,E_n$ are obtained similarly.

Once again, it is much easier to verify that F_0 equals a conjectured function than to construct F_0 from scratch. If G_0 is a known function of $a_1,\dots, a_n$ and it is conjectured that $F_0 = G_0$, then it is enough to find *one partial* recurrence equation rather than *n ordinary* recurrence equations. This was how Good proved Dyson's conjecture.

We are looking for polynomials in $(a_1,\dots, a_n)$, $C_{\alpha_1\cdots\alpha_n\beta_1\cdots\beta_m}(a_1,\dots, a_n)$ such that

$$\left(\sum C_{\alpha_1,\dots,\beta_m} E_1^{\alpha_1}\cdots E_n^{\alpha_n}(x_1 D_1)^{\beta_1}\cdots(x_m D_m)^{\beta_m}\right) F = 0$$

There are many possibilities and we are interested in making $\alpha_1,\dots, \alpha_n$ as small as possible, while we do not care how big the $\beta_1,\dots, \beta_m$ are. Now

$$\left(\sum C_{\alpha_1\cdots\alpha_n 0\cdots 0} E^{\alpha_1}\cdots E_n^{\alpha_n}\right) F_0 \equiv 0 \qquad (**)$$

is such a partial recurrence equation. In order to verify that $F_0 = G_0$ all that need be done is check that G_0 satisfies (**) and that $F_0 = G_0$ on and near the boundary of N^n.

Exercise. Find a linear recurrence equation satisfied by $\phi(m) =$ const. term of $(x + 1 + x^{-1})^m$.

Hint. Put $F(m, x) = (x + 1 + x^{-1})^m$, then

$$EF/F = x + 1 + x^{-1} = (x^2 + x + 1)/x,$$

$$(XD)\,F/F = m(x - x^{-1})/(x + 1 + x^{-1})$$

$$= m(x^2 - 1)/(x^2 + x + 1).$$

5.2. *Andrews' Conjecture*

We shall now describe how to obtain the n ordinary recurrence equations satisfied by the constant term of $\prod_{1<i\neq j<n} (\varepsilon_{ij} x_i/x_j)\, a_i$ (see Section 2.2 for definitions of the symbols). Let

$$E_i f(a_1,\dots, a_i,\dots, a_m; x_1,\dots, x_m)$$
$$= f(a_1,\dots, a_i + 1,\dots, a_n; x_1,\dots, x_m)$$

and

$$Q_j f(a_1,\dots, a_n; x_1,\dots, x_j,\dots, x_m)$$
$$= f(a_1,\dots, a_n; x_1,\dots, qx_j,\dots, x_m).$$

For simplicity, only the case $n = 3$ will be considered. The procedure for general n is similar. Consider thus

$$F(x_1, x_2, x_3, a_1, a_2, a_3)$$
$$= \left(1-\frac{x_1}{x_2}\right)\cdots\left(1-q^{a_1-1}\frac{x_1}{x_2}\right)\left(1-\frac{x_1}{x_3}\right)\cdots\left(1-q^{a_1-1}\frac{x_1}{x_3}\right)$$
$$\times\left(1-q\frac{x_2}{x_1}\right)\cdots\left(1-q^{a_2}\frac{x_2}{x_1}\right)\left(1-\frac{x_2}{x_3}\right)\cdots\left(1-q^{a_2-1}\frac{x_2}{x_3}\right)$$
$$\times\left(1-q\frac{x_3}{x_1}\right)\cdots\left(1-q^{a_3}\frac{x_3}{x_1}\right)\left(1-q\frac{x_3}{x_2}\right)\cdots\left(1-q^{a_3}\frac{x_3}{x_2}\right).$$

Now

$$E_1F/F = \left(1-q^{a_1}\frac{x_1}{x_2}\right)\left(1-q^{a_1}\frac{x_1}{x_3}\right),$$
$$E_2F/F = \left(1-q^{a_2+1}\frac{x_2}{x_1}\right)\left(1-q^{a_2}\frac{x_2}{x_3}\right),$$
$$E_3F/F = \left(1-q^{a_3+1}\frac{x_3}{x_1}\right)\left(1-q^{a_3+1}\frac{x_3}{x_2}\right),$$
$$Q_1F/F = \frac{x_2x_3}{x_1^2}\frac{(1-q^{a_1}x_1/x_2)(1-q^{a_1}x_1/x_3)}{(1-q^{a_2}x_2/x_1)(1-q^{a_3}x_3/x_1)}$$
$$Q_2F/F = \frac{x_1x_3}{x_2^2}\frac{(1-q^{a_2+1}x_2/x_1)(1-q^{a_2}x_2/x_3)}{(1-q^{a_1+1}x_1/x_2)(1-q^{a_3}x_3/x_2)}.$$

Now we eliminate x_1, x_2, x_3 and obtain equations

$$\mathbf{P}_1(q^{a_1}, q^{a_2}, q^{a_3}; E_1, Q_1, Q_2)\,F \equiv 0,$$
$$\mathbf{P}_2(q^{a_1}, q^{a_2}, q^{a_3}; E_2, Q_1, Q_2)\,F \equiv 0,$$
$$\mathbf{P}_3(q^{a_1}, q^{a_2}, q^{a_3}; E_3, Q_1, Q_2)\,F \equiv 0.$$

If $F_0(a_1, a_2, a_3)$ is the constant term of F, then

$$\mathbf{P}_1(q^{a_1}, q^{a_2}, q^{a_3}; E_1, I, I)\,F_0 \equiv 0, \text{ etc.}$$

This is so because $Q_i(\sum a_n x_i^n) = \sum a_n q^n x_i^n$ and therefore Q_i acts like the identity on the constant term.

The drawback of the above method is that it is only good for one specific n at a time, besides being very complicated. However, we believe that MACSYMA can be used to find a simple and symmetric equation of the form

$$\left(\sum C_{\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3} E_1^{\alpha_1}E_2^{\alpha_2}E_3^{\alpha_3}Q_1^{\beta_1}Q_2^{\beta_2}Q_3^{\beta_3}\right)\prod_{1\leqslant i\neq j\leqslant 3}\left(\frac{\varepsilon_{ij}x_i}{x_j}\right)_{a_i} = 0,$$

which will enable us to conjecture an equation of the form

$$\left(\sum C_{\alpha_1\cdots\alpha_n\beta_n\cdots\beta_n} E_1^{\alpha_1}\cdots E_n^{\alpha_n} Q_1^{\beta_1}\cdots Q_n^{\beta_n}\right) \sum_{1\leqslant i=j\leqslant n} \left(\frac{\varepsilon_{ij}x_i}{x_j}\right)_{a_i} = 0.$$

Once this has been verified, we check that

$$\left(\sum C_{\alpha_1\cdots\alpha_n\beta_1\cdots\beta_n} E_1^{\alpha_1}\cdots E_n^{\alpha_n}\right)[(q)_{a_1+\cdots+a_n}/(q)_{a_1}\cdots(q)_{a_n}] = 0$$

and the equality of the boundary values of F_0 and $(q)_{a_1+\cdots+a_n}/(q)_{a_1}\cdots(q)_{a_n}$ will hopefully follow by induction.

6. Final Comments

Every P-recursive function satisfies many linear recurrence equations with polynomial coefficients. Indeed, if $\mathbf{P}(n, X)f = 0$, then for every operator $Q(n, X)$, $Q(n, X)\,\mathbf{P}(n, X)f = 0$. For example $f(n) = n!$ satisfies the first-order recurrence equation $(X - (n+1))f(n) = 0$ and therefore $(X+n)(X-(n+1))f(n) = 0$, i.e., $(X^2 - 2X - n(n+1))f(n) = 0$. Sister Celine's technique enables us to find *one* recurrence satisfied by $G(n) = \sum_k F(n, k)$, but there is no guarantee that the recurrence found is the one of minimal order.

Gosper [11] developed a decision procedure which is equivalent to the problem of determining whether $(a(n)X + b(n))(I - X)$ possesses a different factorization. It would be very interesting if one could generalize Gosper's algorithm to determine whether a linear recurrence operator $\mathbf{P}(n, X)$ can be factorized or whether it is irreducible. If $\mathbf{P}(n, X) = Q(n, X)\,R(n, X)$, then some solutions of $\mathbf{P}(n, x)f = 0$ are also solutions of he lower-order recurrence equation $R(n, X)f = 0$.

In Corollary 6a we proved that in order to check

$$\sum_k F_1(n, k) = \sum_k F_2(n, k) \qquad (*)$$

it is enough to check for $0 \leqslant n \leqslant L$, for some integer L. Sometimes it might be more efficient to actually find the recurrence equations satisfied by both sides of $(*)$. Let us call the left and right sides of $(*)$ $G_1(n)$ and $G_2(n)$, respectively, and suppose that the recurrences found for $G_1(n)$ and $G_2(n)$ are

$$(1) \quad \sum_{r=0}^{R} c_r(n)\, G_1(n+r) \equiv 0,$$

$$(2) \quad \sum_{s=0}^{S} b_s(n)\, G_2(n+s) \equiv 0.$$

If $R = S$ then both equations should be equal, up to a polynomial multiple (i.e., c_r/b_r is the same polynomial for all r). However if $R < S$, we write

$$(4) \quad G_1(n+R) = -\sum_{r=0}^{R-1} \frac{c_r(n)}{c_R(n)} G(n+r)$$

and then

$$\begin{aligned}(5) \quad G_1(n+R+1) &= -\sum_{r=0}^{R-1} \frac{c_r(n+1)}{c_R(n+1)} G_1(n+r+1) \\ &= -\frac{c_{R-1}(n+1)}{c_R(n+1)} G_1(n+R) \\ &\quad -\sum_{r=1}^{R-1} \frac{c_{r-1}(n+1)}{c_R(n+1)} G_1(n+r).\end{aligned}$$

Now we plug (4) in (5), thus expressing $G_1(n+R+1)$ as a linear combination with rational coefficients of $G_1(n)$, $G_1(n+1), \ldots, G_1(n+R-1)$. By iterating the process, we similarly express $G_1(n+R+2), \ldots, G_1(n+S)$ and finally plug them in (2). If it turns out that

$$\sum_{s=0}^{S} b_s(n)\, G_1(n+s) = 0,$$

then $G_1(n) \equiv G_2(n)$ provided they are equal on the "characteristic set" $\{n; b_S(n) = 0\}$.

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