

# Separability and Non-Determinizability of WSTS

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## Abstract

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There is a recent separability result for the languages of well-structured transition systems (WSTS) that is surprisingly general: disjoint WSTS languages are always separated by a regular language. The result assumes that one of the languages is accepted by a deterministic WSTS, and it is not known whether this assumption is needed. There are two ways to get rid of the assumption, none of which has led to conclusions so far: (i) show that WSTS can be determinized or (ii) generalize the separability result to non-deterministic WSTS languages. Our contribution is to show that (i) does not work but (ii) does. As for (i), we give a non-deterministic WSTS language that we prove cannot be accepted by a deterministic WSTS. The proof relies on a novel characterization of the languages accepted by deterministic WSTS. As for (ii), we show how to find finitely represented inductive invariants without having the tool of ideal decompositions at hand. Instead, we work with closures under converging sequences. Our results hold for upward- and downward-compatible WSTS.

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## 1 Introduction

Czerwinski et al. [16, Theorems 6 and 7] have recently established a separability result for the languages of well-structured transition systems (WSTS) [20, 4, 2, 23] that is surprisingly general. Disjoint WSTS languages are always separated by a regular language: whenever we have  $L(U) \cap L(V) = \emptyset$ , then there is a regular language  $R$  with  $L(U) \subseteq R$  and  $R \cap L(V) = \emptyset$ . The result says that WSTS languages either intersect, or they are far apart in that a finite amount of information is sufficient to distinguish them. Applications abound, we elaborate on this in the related work. Unfortunately, the result comes with a grain of salt: it assumes that one of the WSTS,  $U$  or  $V$ , is deterministic. All attempts to remove the assumption have failed so far. The assumption is used for a central argument in the proof, namely that inductive invariants can be represented in a finite way. With determinism, these invariants are downward-closed sets in a WQO, and hence decompose into finitely many ideals [32, 21, 22]. This is precisely the finite amount of information needed for regularity.

A strategy to circumvent the assumption would be to show that WSTS can be determinized. Czerwinski et al. already argue in this direction. In [16, Theorem 5], they show that both finitely-branching WSTS and WSTS over so-called  $\omega^2$ -WQOs can be determinized. Unfortunately, this does not cover all WSTS. To sum up, it is still open whether the regular separability result holds for all WSTS languages, and we do not understand the impact of non-determinism on the expressiveness of the WSTS model.

Our first contribution is to prove the regular separability result for all WSTS languages, without the assumption of determinism. We accept the fact that determinizing a WSTS no



longer yields a WSTS, and carefully study the resulting class of transition systems. They are formed over a lattice in which sequences have subsequences that converge in a natural sense. This leads us to define the closure of a set by adding the limits of all converging sequences. The key insight is that the closure of an inductive invariant is again an inductive invariant. Together with the fact that closed sets have finitely many maximal elements, we arrive at the desired finite representation. In short, when moving from WQOs to converging lattices, maximal elements of closed sets form an alternative to ideal decompositions of downward-closed sets. We call the new transition systems converging.

Our second contribution is to show that WSTS cannot be determinized in general. We give a WSTS language  $T$  that we prove cannot be accepted by a deterministic WSTS. The proof relies on a novel characterization of the deterministic WSTS languages: they are precisely the languages whose Nerode (right) quasi order is a WQO. The characterization provides a first hint on how to construct  $T$ . The language should have an infinite antichain in the Nerode quasi order, for then this cannot be a WQO. The second hint stems from the determinizability result [16, Theorem 5]. The accepting WSTS should be infinitely branching and the WQO should be no  $\omega^2$ -WQO. Such WQOs embed the so-called Rado WQO [8, Section 2]. Moreover, the Rado WQO is known to have an infinite antichain when constructing downward-closed sets [22, Proposition 4.2]. The definition of  $T$  is thus guided by the idea of translating the Rado antichain into an antichain in the Nerode quasi order. Interestingly, the underlying WSTS is deterministic except for the choice of the initial state.

We develop these results for upward-compatible WSTS [23]. Our third contribution is to show that they also hold for downward-compatible WSTS. We achieve this by proving general relationships between the models. A key insight is that the complement of a deterministic upward-compatible WSTS is a deterministic downward-compatible WSTS. Moreover, the reversal of an upward-compatible WSTS language is a downward-compatible WSTS language.

**Related Work** The converging transition systems (CTS) we use to generalize the regular separability result [16] have a topological flavor, and indeed are inspired by Goubault-Larrecq's Noetherian transition systems [26, 27]. One difference is that we had to formulate CTS in lattice-theoretic terms to be able to import a theorem from [16] that links regular separability to the existence of finitely represented inductive invariants. Another difference is the study of such invariants (we prove stability under closure) that has no analogue in [26, 27].

We show that deterministic WSTS accept a strictly weaker class of languages than their non-deterministic counterparts. The work [3] also compares classes of WSTS languages, but for fixed models (extended Petri nets). We allow the determinization to freely select the WQO and the transitions, meaning we have considerably less syntactic constraints to work with. There are also pumping lemmas to distinguish WSTS languages from (among others) context-free languages [24]. Our characterization of the deterministic WSTS languages is stronger than the necessary conditions in pumping lemmas. Our language witnessing the weakness of deterministic WSTS is accepted by an infinitely-branching WSTS, a class of systems studied in [7]. That work concentrates on decidability results and pays attention to effectiveness, while we prove a statement of existence and do not need such assumptions.

There is recent interest in separability problems for infinite-state systems [17, 41, 14, 40, 12]. One reason is that standard algorithms rarely apply to separability problems, but these problems tend to call for new approaches. With the basic separator technique [18], Czerwinski and Zetsche have shown that there is hope for general methods that apply to a range of separability problems [10, 11, 15]. With the closure of inductive invariants under converging sequences, we hope to also have contributed a versatile tool.

Another reason for the popularity of separability problems is their usefulness in verification. In [1], separators act as interpolants in abstraction-guided verification [9]. In [6], separators are advocated as interfaces in rely-guarantee reasoning [31]. In this context, our result implies that regular interfaces yield a complete proof method, provided the system is well-structured.

## 2 Well-Structured Transition Systems

We recall well-structured transition systems (WSTS) with upward compatibility [20, 4, 2, 23]. Downward compatibility will be addressed in Section 5.

**Orders** Let  $(Q, \leq)$  be a quasi order and  $P \subseteq Q$ . We call  $P$  a chain, if  $\leq$  restricted to  $P$  is a total order. We call  $P$  an antichain, if the elements in  $P$  are pairwise incomparable. The upward closure of  $P$  is  $\uparrow P = \{q \in Q \mid \exists p \in P. p \leq q\}$ . We call  $P$  upward closed, if  $P = \uparrow P$ . The powerset of  $Q$  restricted to the upward-closed sets is  $\mathbb{U}(Q)$ . The downward closure is defined similarly and we use  $\mathbb{D}(Q)$  for the downward closed sets. We call  $(Q, \leq)$  a well quasi order (WQO), if for every infinite sequence  $[p_i]_{i \in \mathbb{N}}$  in  $Q$  there are indices  $i < j$  with  $p_i \leq p_j$ .

Let  $(Q, \leq)$  be a partially-ordered set. We write  $\max P$  for the set of maximal elements in the subset  $P \subseteq Q$ . They may not exist, in which case the set is empty. We call  $(Q, \leq)$  a complete lattice, if all  $P \subseteq Q$  have a greatest lower bound in  $Q$ , also called meet and denoted by  $\prod P \in Q$ , and a least upper bound in  $Q$ , also called join and denoted by  $\bigsqcup P \in Q$ . A function  $f : Q \rightarrow Q$  on the complete lattice is Scott continuous [39], if it distributes over arbitrary joins in that  $f(\bigsqcup P) = \bigsqcup f(P)$  for all  $P \subseteq Q$ , where  $f(P) = \{f(p) \mid p \in P\}$ . We call  $(Q, \leq)$  a completely distributive lattice, if it is a complete lattice where arbitrary meets distribute over arbitrary joins, and vice versa:

$$\prod_{a \in A} \bigsqcup_{b \in B_a} p_{a,b} = \bigsqcup_{f \in C_{A,B}} \prod_{a \in A} p_{a,f(a)} \quad \bigsqcup_{a \in A} \prod_{b \in B_a} p_{a,b} = \prod_{f \in C_{A,B}} \bigsqcup_{a \in A} p_{a,f(a)}.$$

The definition makes use of the Axiom of Choice:  $C_{A,B}$  denotes the set of choice functions that map each  $a \in A$  to a choice  $b \in B_a$ . It is also important to note that, for any set  $Q$ ,  $(\mathbb{D}(Q), \subseteq)$  is a completely distributive lattice.

**Labeled Transition Systems** A labeled transition system (LTS) is a tuple  $U = (Q, I, \Sigma, \delta, F)$  that consists of a set of states  $Q$ , in our setting typically infinite, a set of initial states  $I \subseteq Q$ , a set of final states  $F \subseteq Q$ , a finite alphabet  $\Sigma$ , and a set of labeled transitions  $\delta : Q \times \Sigma \rightarrow \mathbb{P}(Q)$ . The LTS is deterministic, if  $|I| = |\delta(p, a)| = 1$  for all  $p \in Q$  and  $a \in \Sigma$ . Its language is the set of words that can reach a final state from an initial state:

$$L(U) = \{w \in \Sigma^* \mid \delta(I, w) \cap F \neq \emptyset\}.$$

Here, we extend the transition relation to sets of states and words:  $\delta(P, w.a) = \delta(\delta(P, w), a)$  and  $\delta(P, a) = \bigcup_{p \in P} \delta(p, a)$ .

Let  $U_1$  and  $U_2$  be LTS with  $U_i = (Q_i, I_i, \Sigma, \delta_i, F_i)$ . We define their synchronized product to be the LTS  $U_1 \times U_2 = (Q_1 \times Q_2, I_1 \times I_2, \Sigma, \delta, F_1 \times F_2)$  where  $(q_1, q_2) \in \delta((p_1, p_2), a)$ , if  $q_1 \in \delta_1(p_1, a)$  and  $q_2 \in \delta_2(p_2, a)$ . Then  $L(U_1 \times U_2) = L(U_1) \cap L(U_2)$ .

**Compatibility** We work with LTS  $U = (Q, I, \Sigma, \delta, F)$  whose states form a quasi order  $(Q, \leq)$  that is compatible with the remaining components as follows. We have  $F = \uparrow F$ , the final

## XX:4 Separability and Non-Determinizability of WSTS

states are upward closed wrt.  $\leq$ . Moreover,  $\leq$  is a simulation relation [36]: for all pairs of related states  $p_1 \leq q_1$  and for all letters  $a \in \Sigma$  we have:

for all  $p_2 \in \delta(p_1, a)$  there is  $q_2 \in \delta(q_1, a)$  with  $p_2 \leq q_2$ .

We also make the quasi order explicit and call  $U = (Q, \leq, I, \Sigma, \delta, F)$  an *upward-compatible LTS* (ULTS).

ULTS can be determinized, in the case of  $U$  this yields

$$U^{det} = (\mathbb{D}(Q), \subseteq, \downarrow I, \Sigma, \delta^{det}, F^{det}).$$

The states are the downward-closed sets ordered by inclusion, the transition relation is defined by closing the result of the original transition relation downwards,  $\delta^{det}(D, a) = \downarrow \delta(D, a)$  for all  $D \in \mathbb{D}(Q)$  and  $a \in \Sigma$ , and final are all downward-closed sets that contain a final state in the original ULTS,  $F^{det} = \{D \in \mathbb{D}(Q) \mid D \cap F \neq \emptyset\}$ .

► **Lemma 1.** *Let  $U$  be an ULTS. Then  $U^{det}$  is a deterministic ULTS with  $L(U^{det}) = L(U)$ .*

We write  $\text{detULTS}$  for the class of deterministic ULTS. The synchronized product of ULTS is again an ULTS (with the product order).

**Well-Structuredness** An *upward-compatible well-structured transition system* (WSTS) is an ULTS  $U$  whose states  $(Q, \leq)$  form a WQO. The synchronized product of WSTS is again a WSTS. We are interested in  $L(\text{WSTS})$ , the class of all languages accepted by WSTS. We also study  $L(\text{detWSTS}) \subseteq L(\text{WSTS})$ , the class of languages accepted by deterministic WSTS.

We observe that we can focus on WSTS with a countable number of states.

► **Lemma 2.** *For every  $L \in L(\text{WSTS})$  there is a WSTS  $U$  with a countable number of states so that  $L = L(U)$ .*

The lemma needs two arguments: the language consists of a countable number of words, and we can assume the transition relation to yield downward-closed sets. The proof can be found in the appendix.

### 3 Regular Separability of WSTS Languages

Two languages  $L_1, L_2 \subseteq \Sigma^*$  are *separable by a regular language*, denoted by  $L_1 \mid L_2$ , if there is a regular language  $R \subseteq \Sigma^*$  with  $L_1 \subseteq R$  and  $R \cap L_2 = \emptyset$ . Our main result is that disjoint WSTS languages are always separable in this sense.

► **Theorem 3.** *For  $L_1, L_2 \in L(\text{WSTS})$ , we have  $L_1 \mid L_2$  if and only if  $L_1 \cap L_2 = \emptyset$ .*

This is the same as the main theorem in [16], but does not need the premise that one of the languages is accepted by a deterministic WSTS. The implication from left to right is trivial, the implication from right to left is our first contribution.

#### 3.1 Proof Principle for Regular Separability

To establish regular separability, we rely on a proof principle introduced in [16]. The notion of an inductive invariant will be recalled in a moment.

► **Theorem 4** (Proof principle for regular separability, [16, Theorem 11]). *Consider ULTS  $U, V$ , one deterministic. If  $U \times V$  has a finitely represented inductive invariant, then  $L(U) \mid L(V)$ .*

Interestingly, the proof principle does not need the WQO assumption of WSTS but holds for general ULTS. It does assume one of the ULTS to be deterministic, though. Recall that an *inductive invariant* for an ULTS  $(Q, \leq, I, \Sigma, \delta, F)$  is a downward-closed set of states  $S \subseteq Q$  that includes all initial states, excludes all final states, and is closed under taking transitions:

$$I \subseteq S \quad S \cap F = \emptyset \quad \delta(S, a) \subseteq S .$$

The inductive invariant is *finitely represented*, if there is a finite set  $C \subseteq S$  with  $S = \downarrow C$ . We refer to a set  $C$  that satisfies this as a *cover* of  $S$ .

When trying to invoke Theorem 4, finding an inductive invariant for  $U \times V$  is easy: the invariant is guaranteed to exist as soon as the language  $L(U \times V) = L(U) \cap L(V)$  is empty, which is precisely the hypothesis we start from.

► **Lemma 5** ([16, Lemma 10]). *An ULTS  $U$  admits an inductive invariant iff  $L(U) = \emptyset$ .*

The difficult part is to find an inductive invariant that can be represented in a finite way. In [16], this was addressed with ideal decompositions [32, 21, 22]. The ideal decompositions, however, needed the WQO assumption, which lead to the requirement in the main theorem that one WSTS had to be deterministic. As we show in Section 4, this is a real restriction: there are WSTS languages that cannot be accepted by a deterministic WSTS.

Our contribution is to find finitely represented inductive invariants without making use of ideal decompositions. Our approach is to determinize the given WSTS with the construction in Lemma 1, and accept that we can no longer guarantee the result to be a WSTS.

### 3.2 Converging Transition Systems: WSTS in Disguise

We propose converging transition systems (CTS), a new class of ULTS that is general enough to capture determinized WSTS and retains enough structure to establish the existence of finitely represented inductive invariants. Working with CTS rather than determinized WSTS leads to a cleaner development (we can avoid products and downward-closed sets) and allows us to highlight key arguments. CTS are inspired by Noetherian transition systems [26, 27], but are formulated in a lattice-theoretic rather than in a topological way.

Recall that determinized WSTS have a state space  $(\mathbb{D}(Q), \subseteq)$ , where  $(Q, \leq)$  is a WQO. In a WQO, every infinite sequence admits an increasing subsequence. It is well known [38] that this may not hold for  $(\mathbb{D}(Q), \subseteq)$ . However, a natural relaxation holds: every infinite sequence  $[X_i]_{i \in \mathbb{N}}$  admits an infinite subsequence  $[X_{\varphi(i)}]_{i \in \mathbb{N}}$ , where any element that is present in one set is present in almost every set. A similar property, defined for complete lattices, is called convergence in the literature [25]. Our definition differs from the citation in two ways. We restrict ourselves to sequences (as opposed to nets), and we require convergence to the join (as opposed to  $\limsup = \liminf$ ). This suffices for our setting.

► **Definition 6.** *A converging lattice  $(Q, \leq)$  is a completely distributive lattice, where every sequence  $[p_i]_{i \in \mathbb{N}}$  has a converging subsequence  $[p_{\varphi(i)}]_{i \in \mathbb{N}}$ . A converging sequence  $[q_i]_{i \in \mathbb{N}}$  is an infinite sequence with*

$$\bigsqcup_{i \in \mathbb{N}} \bigcap_{j \geq i} q_j = \bigsqcup_{i \in \mathbb{N}} q_i .$$

The equality formalizes our explanation from before. In the context of sets, where join and meet are respectively union and intersection, the right-hand side of the equation contains all elements that appear in any set in the sequence. The left side iterates over every finite

initial segment, and includes every element that appears in all sets outside of this segment. Every element that is missing in only finitely many sets will eventually be included.

Converging lattices not only generalize downward-closed subsets of WQOs, they are also a sufficient condition for them. The backward direction is by [38, Proof of Theorem 3]. The forward direction is by an application of the following fact [38], also [33, Fact III.3]:  $(\mathbb{D}(Q), \subseteq)$  is well-founded if and only if the order is a WQO. The details are in the appendix.

► **Lemma 7.**  $(\mathbb{D}(Q), \subseteq)$  is a converging lattice if and only if  $(Q, \leq)$  is a WQO.

The space of converging sequences is closed under the application of continuous functions as formulated in the next lemma. While we would expect this result to be known, we have not found a reference. The lemma is central to our argument, therefore we give the proof.

► **Lemma 8.** Let  $(Q, \leq)$  be a lattice,  $[p_i]_{i \in \mathbb{N}}$  a converging sequence in  $Q$ , and  $f : Q \rightarrow Q$  a Scott continuous function. Then also  $[f(p_i)]_{i \in \mathbb{N}}$  is converging.

**Proof.** Due to convergence of the given sequence, we have  $\bigsqcup_{i \in \mathbb{N}} \bigcap_{j \geq i} p_j = \bigsqcup_{i \in \mathbb{N}} p_i$ . This equality yields  $f(\bigsqcup_{i \in \mathbb{N}} \bigcap_{j \geq i} p_j) = f(\bigsqcup_{i \in \mathbb{N}} p_i)$ . By Scott continuity of  $f$ , we get

$$\bigsqcup_{i \in \mathbb{N}} f(\bigcap_{j \geq i} p_j) = \bigsqcup_{i \in \mathbb{N}} f(p_i) .$$

Function  $f$  is not assumed to be meet continuous. But we can show an inequality that is sufficient for our needs. For all  $S \subseteq Q$  and  $s \in S$ , we have  $f(\bigcap S) \leq f(s) \sqcup f(\bigcap S)$ . Scott continuity and the fact that  $s \in S$  yield  $f(s) \sqcup f(\bigcap S) = f(s \sqcup \bigcap S) = f(s)$ . We have thus shown  $f(\bigcap S) \leq f(s)$  for all  $s \in S$ . This means  $f(\bigcap S) \leq \bigcap_{s \in S} f(s)$ .

We apply this inequality to the previous equality:

$$\bigsqcup_{i \in \mathbb{N}} f(p_i) = \bigsqcup_{i \in \mathbb{N}} f(\bigcap_{j \geq i} p_j) \leq \bigsqcup_{i \in \mathbb{N}} \bigcap_{j \geq i} f(p_j) \leq \bigsqcup_{i \in \mathbb{N}} f(p_i) .$$

This is  $\bigsqcup_{i \in \mathbb{N}} \bigcap_{j \geq i} f(p_j) = \bigsqcup_{i \in \mathbb{N}} f(p_i)$ , as desired. ◀

We explain the considerations that lead us to the definition of CTS given below. In the light of Lemma 7, the states of a CTS should form a converging lattice. This, however, was not enough to guarantee the existence of finitely represented inductive invariants. One requirement of invariants is that they are closed under taking transitions. To understand which sets satisfy this, we had to restrict the transition relation. We define CTS only as a deterministic model. Then the transitions form a function  $\delta(-, a)$  for every letter  $a \in \Sigma$ . Upward compatibility of these functions is not very informative. Consider determinized WSTS: upward compatibility gives us  $\delta(S_0 \cup S_1, a) \supseteq \delta(S_0, a)$ , while we expect  $\delta(S_0 \cup S_1, a) = \delta(S_0, a) \cup \delta(S_1, a)$ . In lattice-theoretic terms, we expect the transition functions  $\delta(-, a)$  to be Scott continuous. A benefit of this requirement is of course that it makes Lemma 8 available. An invariant should also be disjoint from the final states so that we had to control this set as well. When determinizing WSTS, a set  $D \in \mathbb{D}(Q)$  is final as soon as it contains a single final state. Given the definition of convergence, we relax this to containing a finite set of final states.

► **Definition 9.** A converging transition system (CTS) is an ULTS  $U = (Q, \leq, y, \Sigma, \delta, F)$  that is deterministic, where  $(Q, \leq)$  is a converging lattice, the functions  $\delta(-, a)$  are Scott continuous for all  $a \in \Sigma$ , and the final states satisfy

*finite acceptance:* for every  $\bigsqcup K \in F$  there is a finite set  $N \subseteq K$  with  $\bigsqcup N \in F$ .



The determinization of a WSTS yields a CTS, as it was one of the goals of the CTS definition. Somewhat surprisingly, CTS do not add expressiveness but their languages are already accepted by (non-deterministic) WSTS. The construction is via join prime elements and can be found in the appendix. Together, the CTS languages are precisely the WSTS languages, and one may see Definition 9 as a reformulation of the WSTS model.

► **Proposition 10.** *If  $U$  is a WSTS, then  $U^{det}$  is a CTS. For every CTS  $V$ , there is a WSTS  $U$  with  $L(V) = L(U)$ . Together,  $L(WSTS) = L(CTS)$ .*

The correspondence allows us to import the countability assumption from Lemma 2. Indeed, if the WQO  $(Q, \leq)$  is countable, then there is only a countable number of downward-closed sets in  $(\mathbb{D}(Q), \subseteq)$ . This is by a standard argument for WSTS: each downward-closed set can be characterized by its complement, the complement is upward closed, and is therefore characterized by its finite set of minimal elements.

► **Lemma 11.** *For every  $L \in L(CTS)$ , there is a CTS  $U$  over a countable number of states so that  $L = L(U)$*

We will also need that CTS are closed under synchronized products.

► **Lemma 12.** *If  $U$  and  $V$  are CTS, so is  $U \times V$ .*

We summarize the findings so far. Given disjoint WSTS languages  $L(V_1) \cap L(V_2) = \emptyset$ , the goal is to show regular separability  $L(V_1) \mid L(V_2)$ . We first determinize both WSTS. By Proposition 10,  $V_1^{det}$  and  $V_2^{det}$  are CTS. Moreover, by Lemma 1, determinization preserves the language. We use Lemma 11 to obtain countable CTS  $U_1$  and  $U_2$  that accept the same languages. To show regular separability, we now intend to invoke Theorem 4 on  $U_1$  and  $U_2$ . CTS are already deterministic. It thus remains to show that  $U_1 \times U_2$  has a finitely represented inductive invariant. With Lemma 12,  $U_1 \times U_2$  is another CTS  $U$ . Moreover, the product corresponds to language intersection, so  $L(U) = \emptyset$ . By Lemma 5, we know that  $U$  has an inductive invariant. We now show how to turn this invariant into a finitely represented one.

### 3.3 Inductive Invariants in CTS

We show the following surprising property for countable CTS: every inductive invariant  $S$  can be generalized to an inductive invariant  $cl(S)$  that is finitely represented. The closure operator is defined by adding to  $S$  the joins of all converging sequences:

$$cl(S) = \left\{ \bigsqcup_{i \in \mathbb{N}} p_i \mid [p_i]_{i \in \mathbb{N}} \text{ a converging sequence in } S \right\}.$$

► **Proposition 13.** *Let  $U$  be a countable CTS and  $S$  an inductive invariant of  $U$ . Then also  $cl(S)$  is an inductive invariant of  $U$  and it is finitely represented.*

The proposition concludes the proof of Theorem 3. We simply invoke it on the inductive invariant that exists by Lemma 5 as discussed above. The rest of the section is devoted to the proof. We fix a countable CTS  $U = (Q, \leq, \gamma, \Sigma, \delta, F)$  and an inductive invariant  $S \subseteq Q$ .

The closure is expansive and idempotent, meaning sequences over joins of sequences do not add new elements. Here, we need the fact that we have a completely distributive lattice. Moreover, the closure yields a downward-closed set. The closure is also trivially monotonic, and hence an upper closure operator indeed [13], but we will not need monotonicity.

► **Lemma 14.**  $S \subseteq cl(S) = cl(cl(S)) = \downarrow cl(S)$ .

► **Lemma 15.**  $cl(S)$  is an inductive invariant.

**Proof.** To prove that  $cl(S)$  is an inductive invariant, we must show two properties for the joins  $\bigsqcup_{i \in \mathbb{N}} p_i = p$  of the converging sequences  $[p_i]_{i \in \mathbb{N}}$  in  $S$  that we added. First, we must show that we do not leave  $cl(S)$  when taking transitions,  $\delta(p, a) \in cl(S)$  for all  $a \in \Sigma$ . Second, we must show that the join is not a final state. We begin with the latter. Towards a contradiction, suppose  $p \in F$ . Convergence yields  $\bigsqcup_{i \in \mathbb{N}} \bigcap_{j \geq i} p_j \in F$ . By the finite acceptance property of CTS, there must be a finite set  $K \subseteq \mathbb{N}$  with  $k = \max K$  so that

$$\bigsqcup_{i \in K} \bigcap_{j \geq i} p_j = \bigcap_{j \geq k} p_j \in F .$$

Since  $\bigcap_{j \geq k} p_j \leq p_k$  and  $F$  is upward closed, we obtain  $p_k \in F$ . This is a contradiction:  $p_k$  belongs to the inductive invariant  $S$  and the invariant does not intersect the final states.

To show  $\delta(p, a) \in cl(S)$ , we first note that  $\delta(p_i, a) \in S$  for all  $i \in \mathbb{N}$ . This holds as  $S$  is an invariant and  $p_i \in S$ . We now argue that not only the sequence  $[\delta(p_i, a)]_{i \in \mathbb{N}}$  is in  $S$ , but also its join is in the closure. We use that the transition function  $\delta(-, a)$  is Scott continuous. This allows us to apply Lemma 8 showing that  $[\delta(p_i, a)]_{i \in \mathbb{N}}$  converges. Since the sequence belongs to  $S$ , we obtain  $\bigsqcup_{i \in \mathbb{N}} \delta(p_i, a) \in cl(S)$ . We conclude by applying Scott continuity:

$$\delta(p, a) = \delta\left(\bigsqcup_{i \in \mathbb{N}} p_i, a\right) = \bigsqcup_{i \in \mathbb{N}} \delta(p_i, a) \in cl(S) .$$

◀

It only remains to show that  $cl(S)$  is finitely represented.

► **Proposition 16.** *There is a finite set  $C \subseteq cl(S)$  so that  $\downarrow C = cl(S)$ .*

We break down the proof of Proposition 16 into two steps. First, we show that  $cl(S)$  can be covered by an antichain. Then, we show that infinite antichain covers do not exist. This implies that there must be a finite antichain cover. The proofs reason over *closed* sets, sets that contain the limits of their converging sequences. We rely on the fact that closed sets have at least one maximal element.

► **Lemma 17.** *Consider  $G \subseteq Q$  closed and non-empty. Then  $\max G \neq \emptyset$ .*

Moreover, closedness remains intact after certain removals.

► **Lemma 18.** *Consider  $G, H \subseteq Q$  where  $G$  is closed. Then  $G \setminus \downarrow H$  is closed.*

We postpone the proofs of these lemmas until after the proof of Proposition 16.

► **Lemma 19.** *There is an antichain cover of  $cl(S)$ .*

**Proof.** We claim that the maximal elements  $\max cl(S)$  form an antichain cover of  $cl(S)$ . It is clear that  $\max cl(S)$  is an antichain. Since  $cl(S)$  is downward closed by Lemma 14, we also have  $\downarrow(\max cl(S)) \subseteq cl(S)$ . To see that  $\max cl(S)$  is a cover, let  $G = cl(S) \setminus \downarrow(\max cl(S))$  and suppose  $G \neq \emptyset$ . Lemma 18 tells us that  $G$  is closed. By Lemma 17, we get  $\max G \neq \emptyset$ . Consider  $p \in \max G$ . By the definition of  $G$ , we have  $p \notin \max cl(S)$ . Then, however, there must be  $q \in cl(S)$  with  $p \leq q$ . If  $q \in \downarrow(\max cl(S))$ , then  $p \in \downarrow(\max cl(S))$  as well, which is a contradiction to  $p \in G$ . If conversely  $q \in cl(S) \setminus \downarrow(\max cl(S)) = G$ , then we have a contradiction to  $p \in \max G$ . ◀



Now we prove the second part of Proposition 16, which states that there can be no infinite antichain cover.

► **Lemma 20.** *There is no infinite antichain cover of  $cl(S)$ .*

**Proof.** Suppose there is an infinite antichain cover  $C \subseteq cl(S)$ . Then, there is an infinite sequence  $[p_i]_{i \in \mathbb{N}}$  in  $C$ . By Definition 6, it has an infinite converging subsequence  $[p_{\varphi(i)}]_{i \in \mathbb{N}}$ . The closure operator adds  $\bigsqcup_{i \in \mathbb{N}} p_{\varphi(i)}$  to  $cl(S)$ . Since  $C$  is a cover of  $cl(S)$ , there must be  $q \in C$  with  $\bigsqcup_{i \in \mathbb{N}} p_{\varphi(i)} \leq q$ . So  $p_{\varphi(i)} \leq q$  for all  $i \in \mathbb{N}$ , while at the same time  $q, p_{\varphi(i)} \in C$ . This contradicts the antichain property. ◀

We conclude by showing Lemma 17 and 18.

**Proof of Lemma 17.** Let  $\emptyset \neq G \subseteq Q$  be closed. We prove  $G$  chain complete, meaning for every chain  $P \subseteq G$  the limit  $\bigsqcup P$  is again in  $G$ . Then Zorn's lemma [30] applies and yields  $\max G \neq \emptyset$ . We have Zorn's lemma, because we agreed on the Axiom of Choice. Towards chain completeness, consider an increasing sequence  $[p_i]_{i \in \mathbb{N}}$  in  $G$ . We prove that  $\bigsqcup_{i \in \mathbb{N}} p_i \in G$ . For any  $i \in \mathbb{N}$ , we have  $\bigsqcup_{j \geq i} p_j = p_i$ . Hence, replacing each meet with the smallest element shows convergence. Since  $[p_i]_{i \in \mathbb{N}}$  converges and  $G$  is closed, we have  $\bigsqcup_{i \in \mathbb{N}} p_i \in G$ .

Although we are in a countable setting, the argument for sequences does not yet cover all chains. The problem is that the counting process may not respect the order. To see this, consider a chain  $P \subseteq G$  of ordinal size  $|P| = \omega \cdot 2$ . The chain is countable, but no counting process can respect the order. We now argue that still  $\bigsqcup P \in G$ . By [34, Theorem 1], there is an (wrt. inclusion) increasing sequence of subsets  $[P_i]_{i \in \mathbb{N}}$  in  $\mathbb{P}(P)$ , where each  $P_i$  is finite and  $\bigcup_{i \in \mathbb{N}} P_i = P$ . Finite chains contain maximal elements, so let  $p_i = \max P_i = \bigsqcup P_i$ . Then

$$\bigsqcup P = \bigsqcup_{i \in \mathbb{N}} \bigsqcup P_i = \bigsqcup_{i \in \mathbb{N}} p_i = \bigsqcup_{i \in \mathbb{N}} p_i.$$

Since  $[P_i]_{i \in \mathbb{N}}$  is an increasing sequence, also  $[p_i]_{i \in \mathbb{N}}$  is an increasing sequence. As we have shown before,  $\bigsqcup_{i \in \mathbb{N}} p_i \in G$ . This concludes the proof. ◀

**Proof of Lemma 18.** Consider  $G, H \subseteq Q$  with  $G$  closed. We show that  $G \setminus \downarrow H$  is closed. Let  $[p_i]_{i \in \mathbb{N}}$  be a converging sequence in  $G \setminus \downarrow H$ . Let  $q = \bigsqcup_{i \in \mathbb{N}} p_i$  and suppose  $q \notin G \setminus \downarrow H$ . Since  $G$  is closed,  $q \in G$ . Then necessarily  $q \in \downarrow H$ . But by definition,  $p_i \leq q$  for all  $i \in \mathbb{N}$ . So  $p_i \in \downarrow H$  as well. This contradicts the fact that the sequence  $[p_i]_{i \in \mathbb{N}}$  lives in  $G \setminus \downarrow H$ . ◀

## 4 Non-Determinizability of WSTS

We show that the detWSTS languages form a strict subclass of the WSTS languages. To this end, we define a WSTS language  $T$  that we prove cannot be accepted by a detWSTS. The proof relies on a novel characterization of the detWSTS languages that may be of independent interest. In the following, we call  $T$  the *witness language*. This is our main result.

► **Theorem 21.**  $L(\det WSTS) \neq L(WSTS)$ .

Towards the definition of  $T$ , recall that finitely-branching WSTS and WSTS over so-called  $\omega^2$ -WQOs can be determinized [16, Theorem 5]. Moreover, it is known that  $\omega^2$ -WQOs are precisely the WQOs that do not embed the Rado WQO [8, Section 2]. This suggests we should accept the witness language  $T$  by an infinitely-branching WSTS over the Rado WQO. We begin with our characterization of detWSTS languages, as it will provide additional guidance in the definition of the witness language.

## 4.1 Characterization of the detWSTS Languages

Our characterization is based on a classical concept in formal languages [29, Theorem 3.9]. The *Nerode quasi order*  $\leq_L \subseteq \Sigma^* \times \Sigma^*$  of a language  $L \subseteq \Sigma^*$  is defined by  $w \leq_L v$ , if

for all  $u \in \Sigma^*$  we have that  $w.u \in L$  implies  $v.u \in L$ .

The characterization says that the detWSTS languages are precisely the languages whose Nerode quasi order is a WQO. Note that this is not the folklore result [5, Proposition 5.1] saying that a language is regular if and only if the syntactic quasi order is a WQO.

► **Lemma 22** (Characterization of  $L(\text{detWSTS})$ ).  $L \in L(\text{detWSTS})$  iff  $\leq_L$  is a WQO.

**Proof.**  $\Rightarrow$  Let  $L = L(U)$  with  $U = (Q, \leq, i, \Sigma, \delta, F)$  a detWSTS. We extend the order  $\leq \subseteq Q \times Q$  on the states to an order  $\leq_U \subseteq \Sigma^* \times \Sigma^*$  on words by setting  $w \leq_U v$ , if  $\delta(i, w) = p$  and  $\delta(i, v) = q$  with  $p \leq q$ . Since  $U$  is deterministic,  $p$  and  $q$  are guaranteed to exist and be unique. It is easy to see that  $\leq_U$  is a WQO. We now show that  $\leq_U$  is included in the Nerode quasi order, and so also  $\leq_L$  is a WQO. To this end, we consider  $w \leq_U v$  and  $u \in \Sigma^*$  with  $w.u \in L$ , and show that also  $v.u \in L$ . We have  $\delta(i, w.u) = \delta(p_1, u) = p_2$  and  $\delta(i, v.u) = \delta(q_1, u) = q_2$  with  $p_1 = \delta(i, w)$  and  $q_1 = \delta(i, v)$ . Since  $w \leq_U v$ , we have  $p_1 \leq q_1$ . With the simulation property of WSTS, this implies  $p_2 \leq q_2$ . Since  $w.u \in L$  and  $L = L(U)$ , we get  $p_2 \in F$ . Since  $F$  is upward closed, also  $q_2 \in F$ . Hence,  $v.u \in L(U) = L$  as desired.

$\Leftarrow$  Consider a language  $L \subseteq \Sigma^*$  whose Nerode quasi order  $\leq_L$  is a WQO. We define the trivial detWSTS  $U_L = (\Sigma^*, \leq_L, \varepsilon, \Sigma, \delta, L)$ . The states are all words ordered by the Nerode quasi order. The empty word is the initial state, the language  $L$  is the set of final states. Note that  $L$  is upward closed wrt.  $\leq_L$ . The transition relation is defined as expected,  $\delta(w, a) = w.a$ . It is readily checked that  $L(U_L) = L$ . ◀

The lemma gives a hint on how to construct the witness language  $T$ : we should make sure the associated Nerode quasi order  $\leq_T$  has an infinite antichain (then it cannot be a WQO). To obtain such an antichain, remember that  $T$  will be accepted by a WSTS over the Rado WQO  $(R, \leq_R)$  [38]. It is known that  $(\mathbb{D}(R), \subseteq)$  has an infinite antichain. Our strategy for the definition of  $T$  will therefore be to translate the infinite antichain in  $(\mathbb{D}(R), \subseteq)$  into an infinite antichain in  $(\Sigma^*, \leq_T)$ . We turn to the details, starting with the Rado WQO.

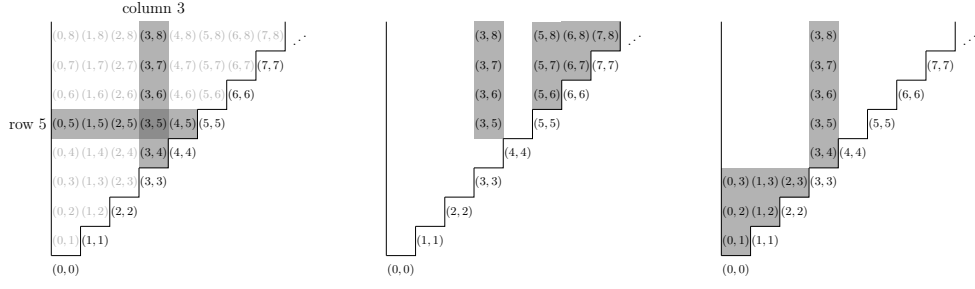
## 4.2 Witness Language

**Rado Order** Our presentation of the Rado WQO [38] follows [35]. The *Rado set* is the upper diagonal,  $R = \{(c, r) \mid c < r\} \subseteq \mathbb{N}^2$ . The Rado WQO  $\leq_R \subseteq R \times R$  is defined by:

$$(c_1, r_1) \leq_R (c_2, r_2), \quad \text{if} \quad r_1 \leq c_2 \vee (c_1 = c_2 \wedge r_1 \leq r_2).$$

Given an element  $(c, r)$ , we call  $c$  the *column* and  $r$  the *row*, as suggested by Figure 1(left). Columns will play an important role and we denote column  $i$  by  $C_i = \{(i, r) \mid i < r\} \subseteq R$ . To arrive at a larger element in the Rado WQO, one can increase the row while remaining in the same column, or move to the rightmost column of the current row, and select an element to the right, Figure 1(middle).

It is not difficult to see that  $(R, \leq_R)$  is a WQO [38]. In an infinite sequence, either the columns eventually plateau out, in which case the rows lead to comparable elements, or the columns grow unboundedly, in which case they eventually exceed the row in the initial pair. The interest in the Rado WQO is that the WQO property is lost when moving to  $(\mathbb{D}(R), \subseteq)$ . This failure is due to the following well-known fact.



■ **Figure 1** Rado order with the column and row of (3, 5) marked (left), with the elements larger than (3, 5) marked (middle), and with the downward closure of column 3 marked (right).

► **Lemma 23** ([22], Proposition 4.2).  $\{\downarrow C_i \mid i \in \mathbb{N}\}$  is an infinite antichain in  $(\mathbb{D}(R), \subseteq)$ .

To see the lemma, we illustrate the downward closure of a column in Figure 1(right). Inclusion fails to be a WQO as each column  $C_i$  forms an infinite set that the downward closure  $\downarrow C_j$  with  $j > i$  cannot cover. Indeed,  $\downarrow C_j$  only has the triangle to the bottom-left of column  $C_j$  available to cover  $C_i$ , and the triangle is a finite set. We will use exactly this difference between infinite and finite sets in our witness language. It will become clearer as we proceed.

**Definition of  $T$**  The witness language is the language accepted by  $U_R = (R, \leq_R, C_0, \Sigma, \delta, R)$ . The set of states is the Rado set, the set of initial states is the first column, and the set of final states is again the entire Rado set. The latter means that a word is accepted as long as it admits a run. The letters in  $\Sigma = \{a, \bar{a}, \text{zero}\}$  reflect the operation that the transitions  $\delta \subseteq R \times \Sigma \times R$  perform on the states:

$$\begin{aligned} \delta((c, r), a) &= (c + 1, r + 1) & \delta((c + 1, r + 1), \bar{a}) &= (c, r) \\ \delta((c + 1, r), \text{zero}) &= (0, c) & \delta((0, r + 1), \text{zero}) &= (0, r) . \end{aligned}$$

We will explain the transitions in a moment, but remark that they are designed in a way that makes  $\leq_R$  a simulation relation and hence  $U_R$  a WSTS.

► **Lemma 24.**  $T \in L(WSTS)$ .

To prove  $T \notin L(\det WSTS)$ , we associate with each column  $C_i$  the *column language*  $L_i = \{w \in \Sigma^* \mid \delta(C_0, w) = C_i\}$ . It consists of the words that reach *all* states in  $C_i$  from the initial column  $C_0$ . The column languages are non-empty.

► **Lemma 25.**  $L_i \neq \emptyset$  for all  $i \in \mathbb{N}$ .

We start from the entire initial column, meaning  $\varepsilon \in L_0$ . The transitions labeled by  $a$  move from all states in one column to all states in the next column,  $L_i.a \subseteq L_{i+1}$ . This already proves the lemma. The  $\bar{a}$ -labeled transitions undo the effect of the  $a$ -labeled transitions and decrement the column,  $L_{i+1}.\bar{a} \subseteq L_i$ . In the initial column, this is impossible,  $\delta(C_0, \bar{a}) = \emptyset$ . We illustrate the behaviour of  $a$  and  $\bar{a}$  in Figure 2(left)

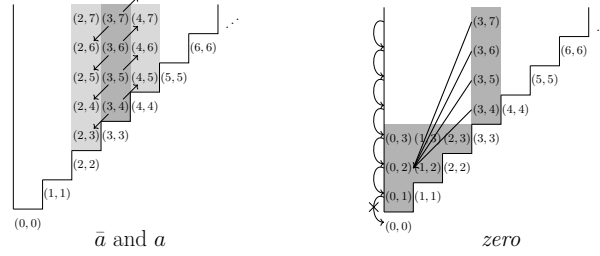
By Lemma 23, the columns form an antichain in  $(\mathbb{D}(R), \subseteq)$ . The languages  $L_i$  we have just defined translate this antichain into (actually several) antichains of the form we need.

► **Lemma 26.** Every set  $\{w_i \mid w_i \in L_i\}$  is an antichain in  $(\Sigma^*, \leq_T)$ .

Combined, Lemmas 25, 26, and 22 conclude the proof of Theorem 21.

► **Proposition 27.**  $T \notin L(\det WSTS)$

In the rest of the section, we prove Lemma 26. The lemma claims that entire column languages are incomparable in the Nerode quasi order, so we write  $L \not\sim_T K$  if for all  $w \in L$  and all  $v \in K$  we have  $w \not\leq_T v$  and  $v \not\leq_T w$ . Difficult is the incomparability with  $L_0$  stated in the next lemma. The proof will explain the purpose of the *zero*-labeled transitions.



■ **Figure 2** The effect of  $a$  and  $\bar{a}$ -labeled transitions on column 3 (left) and the effect of *zero*-labeled transitions on columns 0 and 3 (right).

► **Lemma 28.**  $L_0 \not\sim_T L_k$  for all  $k > 0$ .

**Proof.** Let  $w \in L_0$  and  $v \in L_k$ , meaning  $w$  leads to all states in column 0 while  $v$  leads to all states in column  $k > 0$ . It is easy to find a suffix that shows the  $v \not\leq_T w$ , namely  $\bar{a}$ . Appending  $\bar{a}$  to  $v$  leads to column  $C_{k-1}$ , and so  $v.\bar{a} \in T$ , while there is no transition on  $\bar{a}$  from  $C_0$ , and so  $w.\bar{a} \notin T$ .

For  $w \not\leq_T v$ , we need the *zero* transitions. The idea is to make them fail in  $C_k$  for  $k > 0$ , and have no effect in  $C_0$ . The problem is that the states in  $C_k$  must simulate  $(0, r)$  for  $r \leq k$ . The trick is to fail with a delay. Instead of having no effect in  $C_0$ , we let the *zero* transitions decrement the row. Instead of failing in  $C_k$ , we let the *zero* transitions imitate the behavior from  $(0, k)$  and move to  $(0, k - 1)$ . This is illustrated in Figure 2(right).

By working with column languages, the *zero* transitions fail in  $C_k$  with a delay as follows. We have  $L_0.zero \subseteq L_0$  but  $L_k.zero \not\subseteq L_0$ , meaning from  $C_0$  we again reach the entire column  $C_0$ , while from  $C_k$  we only reach the state  $(0, k - 1)$ . The decrement behavior in the initial column allows us to distinguish the cases by exhausting the rows. Certainly,  $zero^{k-1}$  is enabled in large enough states of  $C_0$ , meaning  $w.zero^k \in T$ . The state  $(0, k - 1)$  reached by  $v.zero$ , however, does not enable corresponding transitions,  $v.zero^k \notin T$ . ◀

When executed in  $C_k$  with  $k > 0$ , the *zero* transitions resemble reset transitions [19]. An analogue of leaving  $C_0$  unchanged despite decrements does not exist in the classical model. Moreover, reset nets are defined over  $\mathbb{N}^k$  (an  $\omega^2$ -WQO) as opposed to the Rado set. To conclude the proof of Lemma 26, we lift the previous result to arbitrary column languages.

► **Lemma 29.**  $L_i \not\sim_T L_j$  for all  $i \neq j$ .

**Proof.** Let  $i < j$  and consider  $w \in L_i$  and  $v \in L_j$ . For  $v \not\leq_T w$ , we append  $\bar{a}^j$ , which is possible only from the larger column:  $v.\bar{a}^j \in T$  but  $w.\bar{a}^j \notin T$ . For  $w \not\leq_T v$ , we append  $\bar{a}^i$ . Then  $w.\bar{a}^i \in L_0$  while  $v.\bar{a}^i \in L_k$  with  $k > 0$ . Now Lemma 28 applies and yields a suffix  $u$  so that  $w.\bar{a}^i.u \in T$  but  $v.\bar{a}^i.u \notin T$ . ◀

The WSTS accepting the witness language  $T$  only uses non-determinism in the choice of the initial state. The transitions are deterministic. Moreover, the Rado WQO is embedded in every non- $\omega^2$ -WQO [8, Section 2]. Given the determinizability results from [16, Theorem 5], language  $T$  thus shows non-determinizability of WSTS with minimal requirements.



**Non-Determinizability of DWSTS** To show that DWSTS cannot be determinized, recall our witness language  $T$  from Section 4. Surprisingly, we have the following.

► **Lemma 33.**  $T^{rev} \in L(detDWSTS)$  and  $\bar{T}^{rev} \in L(detWSTS)$ .

For the first claim, recall that the witness language is accepted by the WSTS  $U_R$ . The DWSTS  $U_R^{rev}$  has one minimal initial state, and transition images  $\delta^{rev}(p, b)$  with one minimal element for all  $p \in R$  and  $b \in \Sigma$ . Removing simulated states yields a deterministic DWSTS. The details are in the appendix. For the second claim,  $\bar{T}^{rev} \in L(detWSTS)$  by Lemma 31. But  $\overline{T^{rev}} = \bar{T}^{rev}$ , and so  $\bar{T}^{rev} \in L(detWSTS)$ . Behind this is the fact that bijections commute with complements, and reversal is a bijection.

The lemma allows us to prove non-determinizability for DWSTS. Notably, we do not need a characterization for the languages of deterministic DWSTS.

► **Theorem 34.**  $\bar{T} \in L(DWSTS) \setminus L(detDWSTS)$  and so  $L(DWSTS) \neq L(detDWSTS)$ .

**Proof.** By Lemma 33,  $\bar{T}^{rev} \in L(detWSTS)$ . Lemma 30 yields  $\bar{T} \in L(DWSTS)$ . Suppose  $\bar{T} \in L(detDWSTS)$ . Then  $T \in L(detWSTS)$  by Lemma 31. This contradicts Proposition 27. ◀

## 5.2 Consequences

We have shown that neither upward- nor downward-compatible WSTS can be determinized. This does not yet rule out the possibility of determinizing an upward-compatible WSTS into a downward-compatible one, and vice versa. Given the correspondence in Lemma 30, we should allow the determinization to reverse the language. We now show that also this form of reverse-determinization is impossible: there are even deterministic languages that cannot be reverse-determinized. This is by Lemma 33, Proposition 27, and Theorem 34.

► **Lemma 35.**  $T^{rev} \in L(detDWSTS)$  but  $T \notin L(detWSTS)$ . Similarly,  $\bar{T}^{rev} \in L(detWSTS)$  but  $\bar{T} \notin L(detDWSTS)$ .

After reversal, both witness languages  $T$  and  $\bar{T}$  can be accepted by a deterministic WSTS. When it comes to separability, this means the results from [16] apply to them. A consequence of Lemma 35, however, is that there are WSTS languages that can neither be determinized nor reverse-determinized. An instance is  $K = T.\#\bar{T}^{rev}$  with  $\#$  a fresh letter.

► **Lemma 36.**  $K \in L(WSTS)$ ,  $K \notin L(detWSTS)$ , and  $K^{rev} \notin L(detDWSTS)$ .

When considering  $L \cap K = \emptyset$ , the separability result from [16] does not apply. Theorem 3 is stronger and yields  $L \mid K$ . The situation is similar for downward-compatible WSTS and  $K^{rev}$ .

## 6 Conclusion and Future Work

We have shown that disjoint WSTS languages are always separated by a regular language. This strengthens the popular separability result from [16] by showing that the premise in that work (one language had to be accepted by a deterministic WSTS) is not needed. We have also show that deterministic WSTS accept a strictly weaker class of languages than their non-deterministic counterparts, meaning the premise was a real restriction.

Behind our separability result is a closure of inductive invariants that adds limits of converging sequences, and the fact that the transition relation is compatible with limits. It would be interesting to formulate this in a topological setting [26, 27]. It would also be interesting to apply our invariant closure in settings where separability does not coincide with intersection emptiness and the complexity is open [6]. Finally, it would be interesting to develop compositional verification technology based on separability.



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## A

 Proofs Missing From Section 3

In all of the following sections, for any LTS  $U = (Q, I, \Sigma, \delta, F)$ ,  $X \subseteq Q$ , and  $w, v \in \Sigma^*$  we use the fact that  $\delta(X, w.v) = \delta(\delta(X, w), v)$  without explicitly mentioning it. The correctness of this statement is clear from the definition of the extended transition relation.

**Details on Lemma 2.** Let  $U = (Q, \leq, I, \delta, F)$  be an arbitrary WSTS. We show below that there is a countable  $P$  where the WSTS  $V = (P, \leq, I \cap Q, \gamma, F \cap P)$  accepts  $L(U)$ , with  $\gamma = \delta \cap (P \times \Sigma \times P)$ .

**Proof of Lemma 2.** We acquire a function  $f : L(U) \rightarrow \mathbb{P}(Q)$  that chooses an accepting run of a word, and maps the word to the states that appear in this run. Formally,  $f : L(U) \rightarrow \mathbb{P}(Q)$  such that  $f(a_0 \dots a_{n-1}) = \{p_0, \dots, p_n\}$  where  $p_i \in \delta(I, a_0 \dots a_{i-1})$  for all  $i < n$  and  $p_n \in F$ . Assume wlog. that  $\delta(p, a)$  is downward closed for all  $p \in Q$  and  $a \in \Sigma$ . We let  $P = \bigcup_{w \in L(U)} f(w)$ . Since  $P \subseteq Q$ ,  $I \cap Q \subseteq I$ ,  $\gamma \subseteq \delta$ , and  $F \cap P \subseteq F$ , it is clear that  $L(V) \subseteq L(U)$ . This is because all states reachable by some word in  $V$  are also reachable by it in  $U$ . We now prove  $L(U) \subseteq L(V)$ . Let  $w = a_0 \dots a_{n-1} \in L(U)$ . Then, we have  $f(w) = \{p_0, \dots, p_n\}$  where  $(p_0, a_0, p_1), \dots, (p_{n-1}, a_{n-1}, p_n) \in \delta$ ,  $p_0 \in I$ , and  $p_n \in F$ . Since  $f(w) \subseteq P$ , we have  $f(w) = \{p_0, \dots, p_n\}$  where  $(p_0, a_0, p_1), \dots, (p_{n-1}, a_{n-1}, p_n) \in \delta$ ,  $p_0 \in I$ , and  $p_n \in F$ . The same sequence of transitions are also present in  $\gamma$ , and we have  $p_0 \in I \cap P$  and  $p_n \in F \cap P$ . Then,  $w \in L(V)$ . We also know that a WQO restricted to a subset is also a WQO. So we only need to show upward-compatibility. The condition on final states is clear, since  $F$  was upward-closed in  $(Q, \leq)$ ,  $F \cap P$  is upward-closed in  $(P, \leq)$ . Towards showing  $\gamma$  is upward-compatible, let  $p, q \in P$ ,  $p \leq q$ ,  $a \in \Sigma$ , and  $p' \in \gamma(p, a)$ . Since  $\gamma \subseteq \delta$ ,  $p' \in \delta(p, a)$  as well. We already know that  $\delta$  is upwards compatible so there is some  $q' \in \delta(q, a)$  with  $p' \leq q'$ . Since  $\delta(q, a)$  is downward closed, we have  $p' \in \delta(q, a)$  as well. We already know  $p' \in P$ , so  $p' \in \gamma(q, a)$ . This concludes the proof.  $\blacktriangleleft$

**Details on Lemma 7.** In  $(\mathbb{D}(Q), \subseteq)$  where  $(Q, \leq)$  is a WQO, all descending chains stabilize [33, Fact III.3]. This condition holds for converging lattices as well. We expect the property to be well known, but we could not find a suitable reference for it. For the sake of completeness, we include a proof below.

► **Lemma 37.** Let  $(Q, \leq)$  be a converging lattice and  $[p_i]_{i \in \mathbb{N}} \in Q$  a descending sequence. Then, there is a  $n \in \mathbb{N}$  where for all  $k \geq n$ ,  $p_n = p_k$ .

**Proof.** Suppose this is not the case. Then, there is an infinite  $[p_i]_{i \in \mathbb{N}} \in Q$  that is strictly descending. This means we have  $p_{i+1} \not\geq p_i$  for all  $i \in \mathbb{N}$ . Per definition of converging lattice there must be a subsequence  $[p_{\phi(i)}]_{i \in \mathbb{N}}$  that is converging. Transitivity tells that  $[p_{\phi(i)}]_{i \in \mathbb{N}}$  is also a strictly descending. Since  $[p_{\phi(i)}]_{i \in \mathbb{N}}$  converges, we must have

$$\bigcup_{i \in \mathbb{N}} \bigcap_{j \geq i} p_{\phi(j)} = \bigcup_{i \in \mathbb{N}} p_{\phi(i)}.$$

We know that  $\bigcap_{j \geq 0} p_{\phi(j)} \leq \bigcap_{j \geq 1} p_{\phi(j)}$ , which means that we can remove  $\bigcap_{j \geq 0} p_{\phi(j)}$  from the join on the left without changing the result. We acquire

$$p_{\phi(0)} \leq \bigcup_{i \in \mathbb{N}} p_{\phi(i)} = \bigcup_{i \in \mathbb{N}} \bigcap_{j \geq i} p_{\phi(j)} = \bigcup_{i \geq 1} \bigcap_{j \geq i} p_{\phi(j)} \leq p_{\phi(1)}.$$

Final inequality follows from  $\bigcap_{j \geq 1} p_{\phi(j)} \leq p_{\phi(1)}$  for all  $i \in \mathbb{N}$ . We get  $p_{\phi(0)} \leq p_{\phi(1)}$ , which is a contradiction to strictness.  $\blacktriangleleft$

## XX:18 Separability and Non-Determinizability of WSTS

**Proof of Lemma 7.** Let  $(Q, \leq)$  be a quasi order. The forward direction is given in [38]. For the backward direction, we use the observation in Lemma 37. It says that there are no strictly descending infinite sequences in  $(\mathbb{D}(Q), \subseteq)$ . This is a characterization of WQO's [33, Fact III.3].  $\blacktriangleleft$

**Details on Proposition 10.** We first show the forward direction. Namely, for any  $U = (Q, \subseteq, \downarrow I, \Sigma, \delta, F) \in \text{WSTS}$ ,  $U^{det} = (\mathbb{D}(Q), \subseteq, \downarrow I, \Sigma, \delta^{det}, F^{det}) \in \text{CTS}$ .

**Proof of Proposition 10, Forward Direction.** It is clear per Lemma 7 that  $(\mathbb{D}(Q), \subseteq)$  is a converging lattice. To see that transitions are Scott-continuous, let  $[X_i]_{i \in A} \in \mathbb{D}(Q)$  with  $\delta^{det}(\bigcup_{i \in A} X_i, a) = Y$ . This is defined as  $\delta^{det}(\bigcup_{i \in A} X_i, a) = \bigcup_{i \in A} \delta^{det}(X_i, a)$ . It only remains to show finite acceptance. Let  $\bigcup_{i \in A} X_i \in F^{det}$ . Definition of  $F^{det}$ , tells us that there is a  $p \in F$  with  $p \in \bigcup_{i \in A} X_i$ . Therefore,  $p \in X_i$  for some  $i \in \mathbb{N}$ . We get  $X_i \in F^{det}$ . This is a trivial finite join of a single element,  $X_i$ .  $\blacktriangleleft$

The reverse direction is more involved. Let  $U = (Q, \leq, I, \Sigma, \delta, F)$  be a CTS. We define the set of Join Prime Elements  $\text{JP}(Q)$ . We let  $p \in \text{JP}(Q) \subseteq Q$  if and only if

$$\text{for all } \{q_i \mid i \in A\} \subseteq Q \text{ with } p \leq \bigsqcup_{i \in A} q_i \text{ there is } i \in A \text{ with } p \leq q_i.$$

Each element in the lattice can be acquired by a join of elements in  $\text{JP}(Q)$ .

► **Lemma 38.** *For all  $p \in Q$ , there is a  $\{q_i \mid i \in A\} \subseteq Q$  such that  $p = \bigsqcup_{i \in A} q_i$ .*

**Proof.** Suppose there is a  $p \notin \text{JP}(Q)$  that is not expressible by a join of  $\text{JP}(Q)$  elements. We will inductively build a strictly descending infinite sequence  $[p_i]_{i \in \mathbb{N}}$ . This is a contradiction to Lemma 37. As our invariant, we require for the strictly descending finite segment  $p_0, \dots, p_n$ , that  $p_n \notin \text{JP}(Q)$  can not be expressed by a join of elements in  $\text{JP}(Q)$ . For the base case, we let  $p_0 = p \notin \text{JP}(Q)$  and we already have that  $p$  inexpressible by a join in  $\text{JP}(Q)$ . For the inductive case, let  $p_0, \dots, p_n$  a strictly descending sequence with  $p_n \notin \text{JP}(Q)$  inexpressible by a join of elements in  $\text{JP}(Q)$ . Since  $p_n \notin \text{JP}(Q)$ , there must be a join  $p_n \leq \bigsqcup_{i \in A} q_i$  where for all  $i \in A$ , we have  $p_n \not\leq q_i$ . We can apply a meet with  $p_n$  on both sides and get  $p_n \leq (\bigsqcup_{i \in A} q_i) \sqcap p_n = \bigsqcup_{i \in A} (q_i \sqcap p_n) = p_n$ . The last equality makes use of the fact  $p_n \leq \bigsqcup_{i \in A} q_i$ . For any  $i \in \mathbb{N}$ , we have  $p_n \not\leq q_i$ , so also  $p_n \sqcap q_i \neq p_n$ . In particular, we observe  $p_n \sqcap q_i < p_n$  for all  $i \in \mathbb{N}$ . If  $q_i \sqcap p_n$  were to be expressible by a join in  $\text{JP}(Q)$  for all  $i \in A$ , then so would be  $p_n$ . Therefore, there must be a  $q_i \sqcap p_n \notin \text{JP}(Q)$  that is not expressible by a join in  $\text{JP}(Q)$ . Letting  $p_{n+1} = q_i \sqcap p_n < p_n$  concludes the proof.  $\blacktriangleleft$

We show that  $\text{JP}(Q)$  is a well quasi order. This is essential to constructing the WSTS equivalent to  $U$ .

► **Lemma 39.**  *$(\text{JP}(Q), \leq)$  is a well quasi ordering.*

**Proof.** Let  $[p_i]_{i \in \mathbb{N}} \in \text{JP}(Q)$  be a sequence of join irreducible elements. We will derive two indices  $a < b$  with  $p_a \leq p_b$ . Per convergence, we have a subsequence  $[p_{\phi(i)}]_{i \in \mathbb{N}}$  that converges. We get

$$\bigsqcup_{i \in \mathbb{N}} \bigsqcap_{j \geq i} p_{\phi(j)} = \bigsqcup_{i \in \mathbb{N}} p_{\phi(i)}.$$

For  $a \in \mathbb{N}$ , we get the meet of both sides with  $p_{\phi(a)}$ . This results in

$$\left( \bigsqcup_{i \in \mathbb{N}} \bigsqcap_{j \geq i} p_{\phi(j)} \right) \sqcap p_{\phi(a)} = \left( \bigsqcup_{i \in \mathbb{N}} p_{\phi(i)} \right) \sqcap p_{\phi(a)}.$$

We apply lattice absorption and distribution to get

$$\bigsqcup_{i \in \mathbb{N}} (p_{\phi(a)} \sqcap \bigsqcap_{j \geq i} p_{\phi(j)}) = p_{\phi(a)}.$$

Since  $p_{\phi(a)}$  is join prime, there must be a  $i \in \mathbb{N}$  with  $p_{\phi(a)} \leq p_{\phi(a)} \sqcap \bigsqcap_{j \geq i} p_{\phi(j)}$ . Then, per lattice rules,  $p_{\phi(a)} \leq \bigsqcap_{j \geq i} p_{\phi(j)} \leq p_{\phi(b)}$  for any  $b \geq i$ . Then we choose  $b \geq i$  with  $b > a$ , and get  $p_{\phi(a)} \leq p_{\phi(b)}$  with  $\phi(a) \leq \phi(b)$ . This concludes the proof.  $\blacktriangleleft$

**Proof of Proposition 10, Backward Direction.** We now construct a WSTS that accepts the same language as  $U$ . We claim that  $V = (\text{FJP}(Q), \sqsubseteq, I, \Sigma, \gamma, G)$  is a WSTS with  $L(V) = L(U)$ , where  $\text{FJP}(Q) = \{X \subseteq \text{JP}(Q) \mid X \text{ finite}\}$ , and

$$\begin{aligned} X \sqsubseteq Y &\text{ iff } \bigsqcup X \leq \bigsqcup Y \\ I &= \{X \in \text{FJP}(Q) \mid \bigsqcup X \leq y\} \\ G &= \{X \in \text{FJP}(Q) \mid \bigsqcup X \in F\} \\ \gamma(X, a) &= \{Y \in \text{FJP}(Q) \mid \bigsqcup Y \leq \delta(\bigsqcup X, a)\}. \end{aligned}$$

We first show that  $V$  is a WSTS. Per Lemma 39, we know that  $(\text{JP}(Q), \leq)$  is a WQO. The order  $(\text{FJP}(Q), \sqsubseteq)$  is isomorphic to  $\subseteq$  on the downward closed finite subsets of  $\text{JP}(Q)$ . It is well known that this is also WQO [28]. We need to show upward-compatibility. First, we show that  $G$  is upward closed. Let  $X \in G$  and  $X \sqsubseteq Y$ . Then,  $\bigsqcup X \leq \bigsqcup Y$  and  $\bigsqcup X \in F$ . Since  $F$  is upward-closed,  $\bigsqcup Y \in F$  as well. So  $Y \in G$ . For the transitions, let  $X \sqsubseteq Y$  and  $X' \in \gamma(X, a)$ . Then,  $\bigsqcup X' \leq \delta(\bigsqcup X, a)$ . Since  $X \sqsubseteq Y$ , then  $\bigsqcup X \leq \bigsqcup Y$ . Per upward-compatibility of  $U$  and determinicity, we have  $\delta(\bigsqcup X, a) \leq \delta(\bigsqcup Y, a)$ . Combining this with the previous inequality we get  $\bigsqcup X' \leq \delta(\bigsqcup Y, a)$ . Therefore  $X' \in \gamma(Y, a)$ .

We now proceed with the proof of  $L(U) = L(V)$  under two assumptions. First, (i) for all  $w \in \Sigma^*$  and  $X_0, X_1 \in \gamma(I, w)$ , we assume  $X_0 \cup X_1 \in \gamma(I, w)$ . Second, (ii) for all  $w \in \Sigma^*$ , we assume  $\bigsqcup \{\bigsqcup X \mid X \in \gamma(I, w)\} = \delta(y, w)$ . We will prove these assumptions once we have shown  $L(U) = L(V)$ . Towards showing the forward inclusion, let  $w \in L(U)$ . Then  $\delta(y, w) \cap F \neq \emptyset$ . Since CTS are deterministic,  $\delta(y, w) \in F$ . Per assumption (ii), we have  $\bigsqcup \{\bigsqcup X \mid X \in \gamma(I, w)\} = \delta(y, w) \in F$ . By finite acceptance in  $U$ , there must be  $X_0, \dots, X_k \in \gamma(I, w)$  with  $\bigsqcup \{\bigsqcup X_i \mid i \leq k\} = \bigsqcup \bigcup_{i \leq k} X_i \in F$ . We apply assumption (i) and get  $\bigcup_{i \leq k} X_i \in \gamma(I, w)$ . Thus  $\bigcup_{i \leq k} X_i \in G$  as well. Then  $\gamma(I, w) \cap G \neq \emptyset$ . This means  $w \in L(V)$ . Towards showing the backward inclusion, let  $w \in L(V)$ . Then,  $\gamma(I, w) \cap G \neq \emptyset$ . There must be some  $X \in \gamma(I, w)$  with  $X \in G$  and thus  $\bigsqcup X \in F$ . We show that  $\delta(y, w) \in F$ , which implies  $w \in L(U)$ . We apply assumption (ii) and get  $\delta(y, w) = \bigsqcup \{\bigsqcup Y \mid Y \in \gamma(I, w)\} \geq \bigsqcup X \in F$ . Since  $F$  is upward closed, we observe  $\delta(y, w) \in F$ .

Now, we show our assumptions. We proceed by assumption (i) and prove it by induction on  $|w|$ . For the base case we have  $w = \varepsilon$ . Let  $X_0, X_1 \in \gamma(I, \varepsilon) = I$ . Then  $\bigsqcup X_0, \bigsqcup X_1 \leq y$ . So we have  $\bigsqcup X_0 \cup X_1 \leq y$  as well. Per definition of  $I$ ,  $X_0 \cup X_1 \in I$  holds. For the inductive case, we handle  $w = v.a \in \Sigma^*$ . Let  $X_0, X_1 \in \gamma(I, v.a)$ . Then, there are  $Y_0, Y_1 \in \gamma(I, v)$  with  $(X_0, a, Y_0), (X_1, a, Y_1) \in \gamma$ . We also apply the induction hypothesis and get  $Y_0 \cup Y_1 \in \gamma(I, v)$ . By definition, we have  $\bigsqcup X_i \leq \delta(\bigsqcup Y_i, a)$  for  $i \in \{0, 1\}$ . So,  $\bigsqcup (X_0 \cup X_1) = \bigsqcup X_0 \sqcup \bigsqcup X_1 \leq \delta(\bigsqcup Y_0, a) \sqcup \delta(\bigsqcup Y_1, a)$ . Per Scott continuity,  $\delta(\bigsqcup Y_0, a) \sqcup \delta(\bigsqcup Y_1, a) = \delta(\bigsqcup Y_0 \sqcup \bigsqcup Y_1, a) = \delta(\bigsqcup (Y_0 \cup Y_1), a)$ . Combining these inequalities with the definition of  $\gamma$ , we get  $X_0 \cup X_1 \in \gamma(Y_0 \cup Y_1, a)$ . Since  $Y_0 \cup Y_1 \in \gamma(I, v)$ , we have  $X_0 \cup X_1 \in \gamma(I, v.a)$  as well.

Now we prove assumption (ii) by induction on  $|w|$ . For the base case, we have  $w = \varepsilon$ . Then,  $p = \bigsqcup \{\bigsqcup X \mid X \in \gamma(I, \varepsilon)\} = \bigsqcup \{\bigsqcup X \mid X \in I\}$ . Since  $\bigsqcup X \leq y$  for all  $X \in I$ ,

we also have  $p \leq y$ . Now let  $y = \bigsqcup_{i \in A} s_i$  with  $\{s_i \mid i \in A\} \subseteq \text{JP}(Q)$ , per Lemma 38. Then,  $s_i \leq y$  and thus  $\{s_i\} \in I$  for all  $i \in A$ . The join  $p = \bigsqcup\{\bigsqcup X \mid X \in I\}$ , includes  $\bigsqcup_{i \in A} s_i$ , so we have  $p \geq y$  and thus  $p = y$ . For the inductive case, let  $w = v.a$ . We write  $\delta(I, v.a) = \delta(\delta(I, v), a)$ . We apply the induction hypothesis and get  $\delta(I, v) = \bigsqcup\{\bigsqcup X \mid X \in \gamma(I, w)\}$ . Then,  $\delta(\delta(I, v), a) = \delta(\bigsqcup\{\bigsqcup X \mid X \in \gamma(I, w)\}, a)$ . Under the assumption  $\delta(\bigsqcup X, a) = \bigsqcup\{\bigsqcup Y \mid Y \in \gamma(Y, a)\}$ , which we will show shortly, we observe:

$$\begin{aligned} \delta(\bigsqcup\{\bigsqcup X \mid X \in \gamma(I, w)\}, a) &= \bigsqcup\{\delta(\bigsqcup X, a) \mid X \in \gamma(I, w)\} \\ &= \bigsqcup\{\bigsqcup\{\bigsqcup Y \mid Y \in \gamma(X, a)\} \mid X \in \gamma(I, w)\} \\ &= \bigsqcup\{\bigsqcup Y \mid Y \in \gamma(I, v.a)\}. \end{aligned}$$

To prove our assumption  $\delta(\bigsqcup X, a) = \bigsqcup\{\bigsqcup Y \mid Y \in \gamma(Y, a)\}$ , we derive

$$\begin{aligned} \bigsqcup\{\bigsqcup Y \mid Y \in \gamma(X, a)\} &\leq \delta(\bigsqcup X, a) \\ &= \bigsqcup\{s \in \text{JP}(Q) \mid s \leq \delta(X, a)\} \leq \bigsqcup\{\bigsqcup Y \mid Y \in \gamma(X, a)\}. \end{aligned}$$

The initial inequality is by the definition of  $\gamma(X, a)$ . It consists of  $Y \in \text{FJP}(Q)$  where  $\bigsqcup Y \leq \delta(X, a)$ . Therefore, the join is also smaller. The equality that follows it acquires the join of  $\text{JP}(Q)$  elements that expresses  $\delta(\bigsqcup X, a)$ , guaranteed per Lemma 38. The final inequality follows from the observation  $\{s\} \in \gamma(X, a)$  for all  $s \in \text{JP}(Q)$  with  $s \leq \delta(X, a)$ . ◀

**Details on Lemma 12.** We now show that CTS are closed under product construction. Let  $U = (Q, \leq, I, \Sigma, \delta, F)$  and  $V = (P, \sqsubseteq, I, \Sigma, \gamma, G)$  be CTS. We show that  $U \times V$  is a CTS.

**Proof of Lemma 12.** We start by showing that the product order  $(Q \times P, \leq_\times)$  is a converging lattice. Usual lattice properties carry over, with the join  $\bigsqcup_{i \in K} (p_i, q_i) = (\bigsqcup_{i \in K} p_i, \bigsqcup_{i \in K} q_i)$  and the meet  $\sqcap_{i \in K} (p_i, q_i) = (\sqcap_{i \in K} p_i, \sqcap_{i \in K} q_i)$ . The symbols  $\bigsqcup$  and  $\sqcap$  refer to respective operations in  $(Q, \leq)$  when used on elements from  $Q$ , and to those in  $(P, \sqsubseteq)$  when used on elements from  $P$ . Let  $[(p_i, q_i)]_{i \in \mathbb{N}} \in Q \times P$  be a sequence. Then, since  $(Q, \leq)$  is a converging lattice, there is a subsequence  $[p_{\phi(i)}]_{i \in \mathbb{N}}$  that converges in  $(Q, \leq)$ . We also apply this to the other side, and acquire a subsequence  $[q_{\psi(i)}]_{i \in \mathbb{N}}$  of  $[q_{\phi(i)}]_{i \in \mathbb{N}}$  that converges in  $(P, \sqsubseteq)$ . Subsequences of converging sequences also converge. Then,  $[p_{\psi(i)}]_{i \in \mathbb{N}}$  also converges. We get the below equalities. The second equality is acquired by the convergence of component sequences.

$$\begin{aligned} \bigsqcup_{i \in \mathbb{N}} \sqcap_{j \geq i} (p_{\phi(\psi(j))}, q_{\phi(\psi(j))}) &= (\bigsqcup_{i \in \mathbb{N}} \sqcap_{j \geq i} p_{\phi(\psi(j))}, \bigsqcup_{i \in \mathbb{N}} \sqcap_{j \geq i} q_{\phi(\psi(j))}) \\ &= (\bigsqcup_{i \in \mathbb{N}} p_{\phi(\psi(i))}, \bigsqcup_{i \in \mathbb{N}} q_{\phi(\psi(i))}) = \bigsqcup_{i \in \mathbb{N}} (p_i, q_i) \end{aligned}$$

We now show Scott continuity. Let  $\bigsqcup_{i \in K} (p_i, q_i) \in Q \times P$ . The Scott continuity follows from the continuity of the individual components. Transition relations in the product  $U \times V$  is expressed by the symbol  $\delta_\times$ .

$$\begin{aligned} \delta_\times(\bigsqcup_{i \in K} (p_i, q_i), a) &= (\delta_\times((\bigsqcup_{i \in K} p_i, \bigsqcup_{i \in K} q_i), a) = (\delta(\bigsqcup_{i \in K} p_i, a), \gamma(\bigsqcup_{i \in K} q_i, a)) \\ &= (\bigsqcup_{i \in K} \delta(p_i, a), \bigsqcup_{i \in K} \gamma(q_i, a)) = \bigsqcup_{i \in K} (\delta(p_i, a), \gamma(q_i, a)) \\ &= \bigsqcup_{i \in K} \delta_\times((p_i, q_i), a) \end{aligned}$$

We conclude the proof by the finite acceptance. Let  $\bigsqcup_{i \in K} (p_i, q_i) = (\bigsqcup_{i \in K} p_i, \bigsqcup_{i \in K} q_i) \in F \times G$ . Then  $\bigsqcup_{i \in K} p_i \in F$  and  $\bigsqcup_{i \in K} q_i \in G$ . By finite acceptance of  $U$  and  $V$ , there must be finite  $K', L' \subseteq K$  with  $\bigsqcup_{i \in K'} p_i \in F$  and  $\bigsqcup_{i \in L'} q_i \in G$ . Then,  $\bigsqcup_{i \in K' \cup L'} p_i \in F$  and  $\bigsqcup_{i \in K' \cup L'} q_i \in G$  by upward-compatibility. Therefore,  $(\bigsqcup_{i \in K' \cup L'} p_i, \bigsqcup_{i \in K' \cup L'} q_i) \in F \times G$ . This is equal to  $\bigsqcup_{i \in K' \cup L'} (p_i, q_i) \in F \times G$ . The set  $K' \cup L'$  is also finite, which concludes the proof.  $\blacktriangleleft$

**Details on Lemma 14:** The result  $S \subseteq cl(S)$  is clear, since any sequence where every element is equal to  $p \in Q$  converges to  $p$ . We show  $cl(S) = cl(cl(S)) = \downarrow cl(S)$  below.

**Proof of Lemma 14.** We proceed by showing  $cl(S) = \downarrow cl(S)$ . Let  $p \in cl(S)$  and  $q \in Q$  with  $q \leq p$ . Since  $p \in cl(S)$ , there is a converging sequence  $[p_i]_{i \in \mathbb{N}} \in S$  with  $\bigsqcup_{i \in \mathbb{N}} p_i = p$ . Note that  $S$  is downward closed and  $p_i \sqcap q \leq p_i$ , so for all  $i \in \mathbb{N}$ , we get  $p_i \sqcap q \in S$  for all  $i \in \mathbb{N}$ . We now show that  $[p_i \sqcap q]_{i \in \mathbb{N}} \in S$  converges to  $q$  and therefore  $q \in cl(S)$ . We apply a meet with  $q$  on both sides and get

$$\begin{aligned} \bigsqcup_{i \in \mathbb{N}} \bigsqcap_{j \geq i} p_j &= \bigsqcup_{i \in \mathbb{N}} p_i = p \\ \left( \bigsqcup_{i \in \mathbb{N}} \bigsqcap_{j \geq i} p_j \right) \sqcap q &= \left( \bigsqcup_{i \in \mathbb{N}} p_i \right) \sqcap q = p \sqcap q = q \\ \bigsqcup_{i \in \mathbb{N}} \left( \bigsqcap_{j \geq i} p_j \sqcap q \right) &= \bigsqcup_{i \in \mathbb{N}} (p_i \sqcap q) = q. \end{aligned}$$

Finally, we show  $cl(S) = cl(cl(S))$ . Here, we need the countability of  $Q$ , or in particular, of  $\text{JP}(Q)$ . It is clear that  $cl(S) \subseteq cl(cl(S))$  by an argument similar to  $S \subseteq cl(S)$ . To see  $cl(cl(S)) \subseteq cl(S)$ , let  $p \in cl(cl(S))$ . Then, there is a converging  $[p_i]_{i \in \mathbb{N}} \in cl(S)$ . Since  $p_i \in cl(S)$ , there is a converging sequence  $[p_{i,j}]_{j \in \mathbb{N}} \in S$ . We apply join prime decomposition by Lemma 38. We acquire  $p = \bigsqcup_{i \in I} s_i$  for some  $\{s_i \mid i \in I\} \subseteq \text{JP}(Q)$ . Since  $Q$  is countable, then so is  $\text{JP}(Q)$ . Wlog. we exchange the index set  $I$  with  $\mathbb{N}$  and write  $p = \bigsqcup_{i \in \mathbb{N}} s_i$ .

By induction, we construct a sequence  $[q_i]_{i \in \mathbb{N}} \in S$  where for all  $i \in \mathbb{N}$ ,  $\bigsqcup_{j \leq i} s_j \leq q_i$  holds. As an inductive invariant, we require that the finite segment  $q_0, \dots, q_{n-1} \in \{p_{i,j} \mid i, j \in \mathbb{N}\}$  has (i) for all  $i < n$ ,  $\bigsqcup_{j \leq i} q_j \leq p_i$ , (ii) there is  $\{r_{i,j} \mid i, j \in \mathbb{N}\} \subseteq \{p_{i,j} \mid i, j \in \mathbb{N}\}$  with  $[r_{k,j}]_{j \in \mathbb{N}}$  converging to some  $r_k$ , where  $[r_i]_{i \in \mathbb{N}}$  converges to  $p$ , (iii)  $(\bigsqcup_{k \leq n-1} s_k) \leq r_{i,j}$  for all  $i, j \in \mathbb{N}$ .

We consider  $n = 0$  as the base case, where  $r_{i,j} = p_{i,j}$  and the empty sequence satisfy all of the requirements. For the inductive case, let  $q_0, \dots, q_{n-1}$  be a sequence that satisfies (i), (ii), and (iii) with  $\{r_{i,j} \mid i, j \in \mathbb{N}\}$  being the sequence of sequences from (ii) and (iii). Note that  $[r_{i,j}]_{j \in \mathbb{N}}$  converge to some  $r_i$  for each  $i \in \mathbb{N}$ , where  $[r_i]_{i \in \mathbb{N}}$  converges to  $p$ . We now find a  $q_n \in \{r_{i,j} \mid i, j \in \mathbb{N}\}$  with  $\bigsqcup_{i \leq n} s_i \leq q_n$ , along with  $\{r'_{i,j} \mid i, j \in \mathbb{N}\} \subseteq \{r_{i,j} \mid i, j \in \mathbb{N}\} \subseteq \{p_{i,j} \mid i, j \in \mathbb{N}\}$  that satisfies (ii) and (iii). We have  $s_n \leq p = \bigsqcup_{i \in \mathbb{N}} \bigsqcup_{j \geq i} r_j$  since  $[r_i]_{i \in \mathbb{N}}$  converges to  $p$ . By join primeness, we have an  $i \in \mathbb{N}$  with  $s_n \leq \bigsqcap_{k \geq i} r_k$ . Then, for all  $k \geq i$ ,  $s_n \leq r_k$ . Let  $k \geq i$ . Per convergence of  $[r_{k,j}]_{j \in \mathbb{N}}$  we have  $s_n \leq \bigsqcup_{j \in \mathbb{N}} \bigsqcap_{l \in \mathbb{N}} r_{j,l}$ . Join primeness again gives  $s_n \leq \bigsqcap_{l \geq j_k} r_{k,l}$  for some  $j_k \in \mathbb{N}$ . Then, for all  $k \geq i$ , and  $l \geq j_k$ , we have  $s_n \leq r_{k,l}$ . Since  $\bigsqcup_{m \leq n-1} s_m \leq r_{k,l}$  for all such  $k$  and  $l$ , we also have  $\bigsqcup_{m \leq n} s_m \leq r_{k,l}$ . Then, in order to get a set that satisfies (iii), we can let  $r'_{a,b} = r_{k,j_k+b}$ , where  $k = i + a$ . To see that (ii) is also satisfied, consider that infinite subsequences of convergent sequences also converge to the same state. Then,  $[r'_{a,b}]_{b \in \mathbb{N}}$  converges to  $r_{i+a} = r'_a$ , and  $[r'_a]_{a \in \mathbb{N}}$  converges to  $p$ . We choose  $q_{n+1} = r'_{0,0} \geq \bigsqcup_{m \leq n} s_m$ . This satisfies (i) and concludes the inductive case.

We claim that the infinite sequence  $[q_i]_{i \in \mathbb{N}}$  converges to  $p = \bigsqcup_{i \in \mathbb{N}} s_i$ . First, observe that  $\bigsqcup_{i \in \mathbb{N}} q_i \leq p$ . This is because  $[p_{i,j}]_{j \in \mathbb{N}}$  converges to  $p_i$  for all  $i \in \mathbb{N}$ , and  $[p_i]_{i \in \mathbb{N}}$  converges

## XX:22 Separability and Non-Determinizability of WSTS

to  $p$ . Then,  $p = \bigsqcup_{i,j \in \mathbb{N}} p_{i,j}$  where we have  $\{q_i \mid i \in \mathbb{N}\} \subseteq \{p_{i,j} \mid i, j \in \mathbb{N}\}$ . We now show that  $p \leq \bigsqcup_{i \in \mathbb{N}} \bigcap_{j \geq i} q_j$ . This is sufficient for showing  $\bigsqcup_{i \in \mathbb{N}} \bigcap_{j \geq i} q_j = \bigsqcup_{i \in \mathbb{N}} q_i = p$ , because  $\bigsqcup_{i \in \mathbb{N}} q_i \geq \bigsqcup_{i \in \mathbb{N}} \bigcap_{j \geq i} q_j$  always holds. The latter fact follows from  $q_i \leq \bigcap_{j \geq i} q_j$  for all  $i \in \mathbb{N}$ . By applying the properties of  $[q_i]_{i \in \mathbb{N}}$  we get

$$\bigsqcup_{i \in \mathbb{N}} \bigcap_{j \geq i} q_j \geq \bigsqcup_{i \in \mathbb{N}} \bigcap_{j \geq i} \bigcup_{k \leq j} s_k = \bigsqcup_{i \in \mathbb{N}} \bigcup_{k \leq i} s_k = \bigsqcup_{i \in \mathbb{N}} s_i = p.$$

For the first inequality, we employ  $\bigcup_{k \leq j} s_k \leq q_j$  for all  $j \in \mathbb{N}$ . For the following equality, we observe that  $\bigcup_{k \leq i} s_i$  is the smallest element in the meet. Then, we reorganize the joins. This results in the join of  $\text{JP}(Q)$  elements that represent  $p$ . ◀

### B Proofs Missing From Section 4

**Details on Lemma 24.** We show the upward-compatibility of  $U_R$ . For states  $x, y \in R$  with  $x \leq_R y$ , letter  $b$ , and  $x$ -successor  $z \in \delta(x, b)$ , we show there is a  $y$ -successor  $z' \in \delta(y, b)$  with  $z \leq_R z'$ . During the proof, we let  $x = (c, r)$  and  $y = (c', r')$ .

**Proof of Lemma 24.** Since  $x \leq_R y$ , we have  $r \leq c'$  or ( $c = c'$  and  $r \leq r'$ ). We do a case distinction along the letters.

**Case  $a$ .** Then  $\delta((c, r), a) = (c + 1, r + 1)$  and  $\delta((c', r'), a) = (c' + 1, r' + 1)$ . If  $r \leq r'$  and  $c = c'$ , then  $r + 1 \leq r' + 1$  and  $c + 1 = c' + 1$ . If  $r \leq c'$ , then  $r + 1 \leq c' + 1$ .

**Case  $\bar{a}$ .** If  $c = 0$ , there is no transition from  $(c, r)$ . So consider  $c > 0$ . The transition is now allowed, and we see  $\delta((c, r), \bar{a}) = (c - 1, r - 1)$ . If  $c = c'$ , we also have  $c' > 0$ . This also holds if  $r \leq c'$ , since  $r > c > 0$ . Then,  $\delta((c', r'), \bar{a}) = (c' - 1, r' - 1)$ . The argument for  $(c - 1, r - 1) \leq_R (c' - 1, r' - 1)$  is like in the previous case.

**Case  $zero$ .** Transitions labeled *zero* are not activated from row or column 1. Therefore, we can assume  $c \neq 1$  and  $r > 1$ . Let  $c' = 0$ . Assume  $c = c' = 0$  and  $r \leq r'$ . Then we have  $\delta((c, r), zero) = (0, r - 1)$  and  $\delta((c', r'), zero) = (0, r' - 1)$ , with  $r \leq r'$ . Therefore  $(0, r - 1) \leq_R (0, r' - 1)$  as well. The case  $r \leq c'$  can not happen as  $r > 1$  per definition of  $R$ . Let  $c = 0$  and assume  $c < c'$ , since  $c' = 0$  has already been covered. Then,  $\delta((c, r), zero) = (0, r - 1)$  and  $\delta((c', r'), zero) = (0, c' - 1)$ . Since  $c \neq c'$ ,  $r \leq c'$  must hold, and so  $(0, r - 1) \leq_R (0, c' - 1)$ . Finally, let  $c, c' > 0$ . Note that  $c \neq 1$  and  $c, c' > 0$ , necessarily gives  $c, c' \geq 2$ . Then, the *zero* transition is activated from both of  $(c, r)$  and  $(c', r')$ . So we get  $\delta((c, r), zero) = (0, c - 1)$  and  $\delta((c', r'), zero) = (0, c' - 1)$ . Let  $c = c'$  and  $r \leq r'$ . Then we have  $(0, c - 1) = (0, c' - 1)$ , satisfying the condition. Let  $r \leq c'$ . Then, we have  $c < r \leq c'$  and therefore  $(0, c - 1) \leq_R (0, c' - 1)$ . ◀

### C Proofs Missing From Section 5

**Details on Lemma 30.** Let  $U = (Q, \leq, I, \delta, F)$  and  $U^{rev} = (Q, \leq, F, \delta^{rev}, \downarrow I)$ . We recall the definition

$$\delta^{rev} = \{(p, a, q) \mid (q, a, p') \in \delta, p \leq p'\}.$$

We now show  $L(U)^{rev} = L(U^{rev})$ .



**Proof of Lemma 30.** We first show that  $U^{rev}$  is downward-compatible. The final states,  $\downarrow I$ , are downward-closed by definition. To show that the condition on transitions is also satisfied, let  $p, q \in Q$  and  $a \in \Sigma$  with  $p \leq q$ ,  $(q, a, q') \in \delta^{rev}$ . Per definition, there must be a  $r \geq q$  with  $(q', a, r) \in \delta$ . Then, since  $p \leq q \leq r$ , we also have  $(p, a, q') \in \delta^{rev}$ . This concludes the proof of upward-compatibility. We do a case distinction along the letters.

Now we show  $L(U)^{rev} = L(U^{rev})$ . First, we make an assumption that we prove later:  $\downarrow \delta(X, w) \cap Y \neq \emptyset$  if and only if  $\uparrow \delta^{rev}(Y, w^{rev}) \cap X \neq \emptyset$  for all  $X, Y \subseteq Q$  and  $w \in \Sigma^*$ . Now we prove  $L(U^{rev}) = L(U)^{rev}$ . Let  $w \in L(U)^{rev}$ . Then  $\delta(I, w^{rev}) \cap F \neq \emptyset$ . We see  $\uparrow \delta^{rev}(F, w) \cap I \neq \emptyset$  and thus  $\delta^{rev}(F, w) \cap \downarrow I \neq \emptyset$ . So  $w \in L(U^{rev})$ . Now let  $w \in L(U^{rev})$ . Then  $\delta^{rev}(F, w) \cap \downarrow I \neq \emptyset$ . We get  $\downarrow \delta(\downarrow I, w^{rev}) \cap F \neq \emptyset$ . Since  $F$  is upward-closed, then  $\delta(\downarrow I, w^{rev}) \cap F \neq \emptyset$ . This means that there must be  $p \in I$  with  $p' \leq p$ ,  $q' \in \delta(p', w^{rev})$ , and  $q' \in F$ . By standard induction on word length, we see that there must be a  $q \in \delta(p, w^{rev})$  with  $q' \leq q$ . Since  $F$  is upward-closed,  $q \in F$  and thus  $w^{rev} \in L(U)$ . Therefore  $w \in L(U)^{rev}$ .

Finally we prove our assumption by induction on  $|w|$ . The base case is  $w = w^{rev} = \varepsilon$ . Let  $X, Y \subseteq Q$ . For the forward direction, we have  $\downarrow \delta(X, \varepsilon) = \downarrow X$  and  $\uparrow \delta^{rev}(Y, \varepsilon) = \uparrow Y$ . We get  $\downarrow \delta(X, \varepsilon) \cap Y = \downarrow X \cap Y \neq \emptyset$ . This implies  $X \cap \uparrow Y = \uparrow \delta(Y, \varepsilon) \cap X \neq \emptyset$ . The backward direction is handled similarly.

For the inductive case, let  $w \in \Sigma^*$  with  $w = a.v = u.b$  for some  $a, b \in \Sigma$  and  $X, Y \subseteq Q$ . To prove the forward direction, assume  $\downarrow \delta(X, a.v) \cap Y \neq \emptyset$ . This is the same as  $\downarrow \delta(\delta(X, a), v) \cap Y \neq \emptyset$ . Induction hypothesis, tells us  $\uparrow \delta^{rev}(Y, v^{rev}) \cap \delta(X, a) \neq \emptyset$ . So, there is  $p \in X$ ,  $(p, a, q) \in \delta$ , along with a  $q' \in \delta^{rev}(Y, v^{rev})$  with  $q' \leq q$ . By definition of  $\delta^{rev}$ , we also observe  $(q', a, p) \in \delta^{rev}$ . Since we know  $\uparrow \delta^{rev}(Y, v^{rev}.a) = \uparrow \delta^{rev}(\delta^{rev}(Y, v^{rev}), a)$ ,  $q' \in \delta^{rev}(Y, v^{rev})$ , and  $(q', a, p) \in \delta^{rev}$ , we see  $p \in \delta^{rev}(\delta^{rev}(Y, v^{rev}), a)$ . We also had  $p \in X$ , so  $\uparrow \delta^{rev}(Y, v^{rev}.a) \cap X \neq \emptyset$ .

To prove the backward direction, assume  $\uparrow \delta^{rev}(Y, b.u^{rev}) \cap X \neq \emptyset$ . This is the same as  $\uparrow \delta^{rev}(\delta^{rev}(Y, b), u^{rev}) \cap X \neq \emptyset$ . Applying the induction hypothesis we observe  $\downarrow \delta(X, u) \cap \delta^{rev}(Y, b) \neq \emptyset$ . Therefore, there must be  $p \in Y$ ,  $(p, b, q) \in \delta^{rev}$ , and a  $q' \in \delta(X, u)$  with  $q \leq q'$ . We show  $p \in \downarrow \delta(X, u.b)$ . Because  $(p, b, q) \in \delta^{rev}$ , there must be a  $p' \geq p$  with  $(q, b, p') \in \delta$  per definition of  $\delta^{rev}$ . Here, we use upwards compatibility. Since  $q' \geq q$ , there must be a  $p'' \geq p'$  and  $(q', b, p'') \in \delta$ . Recall that  $q' \in \delta(X, u)$ . Then,  $p'' \in \downarrow \delta(\delta(X, u), b) = \downarrow \delta(X, u.b)$ . Because  $p \leq p' \leq p''$ , we get  $p \in \downarrow \delta(X, u.b)$ . We already have  $p \in Y$ , so  $\downarrow \delta(X, u.b) \cap Y \neq \emptyset$ . ◀

**Details on Lemma 31.** Let  $U = (Q, \leq, I, \Sigma, \delta, F)$  be a deterministic ULTS. We show that  $\overline{U} = (Q, \leq, I, \Sigma, \delta, \overline{F})$  is a deterministic DLTS with  $\overline{L(U)} = L(\overline{U})$ .

**Proof of Lemma 31.** That  $\overline{U}$  accepts the complement language is due to Rabin and Scott. To see that  $\geq$  is a simulation, consider  $p_1, q_1, q_2 \in Q$ , and  $a \in \Sigma$  with  $p_1 \leq q_1$  and  $q_2 \in \delta(q_1, a)$ . By determinism, there is  $p_2 \in \delta(p_1, a)$ . As  $\leq$  is a simulation, we obtain  $q_3 \in \delta(q_1, a)$  with  $p_2 \leq q_3$ . By determinism,  $q_3 = q_2$  has to hold. ◀

**Details on Lemma 33.** Let  $U_R^{rev} = (R, \leq, R, \Sigma, \delta^{rev}, \downarrow I)$ . We claim that  $R$  and  $\delta(p, a)$  are represented by one minimal state, for all  $p \in R$  and  $a \in \Sigma$ .

**Proof of Lemma 33.** It is easy to see that  $\uparrow \{(0, 1)\} = R$ . For any  $(c, r) \in R$ , either  $c = 0$ , in which case  $r \geq 1$ , or  $c > 0$ , in which case  $c \geq 1$ . For  $\delta^{rev}$ , we do a case distinction along the letters. We recall the definition  $\delta^{rev}(p, a) = \{q \in R \mid p' \in \delta(q, a), p \leq p'\}$ . We use the fact that  $|\delta(q, a)| = 1$  for all  $q \in Q$  and  $a \in \Sigma$  without explicitly mentioning it.

**Case zero:** We claim  $\delta^{rev}((c, r), zero) = \emptyset$  if  $c > 0$ , and  $\delta^{rev}((0, r), zero) = \uparrow\{(0, r+1)\}$ . The former is easy to see. Transitions labeled *zero* only result in elements in column 0, none of which dominate an element in other columns. Now we show the latter case. For the direction  $\uparrow\{(0, r+1)\} \subseteq \delta^{rev}((0, r), zero)$ , let  $(c', r') \geq_R (0, r+1)$ . We have  $c' = 0$  and  $r' \geq r+1$ , or  $1 \leq r+1 \leq c'$ . If  $c' = 0$  and  $r' \geq r+1$ , we have  $\delta((c', r'), zero) = (0, r'-1) \geq_R (0, r)$ . So  $(c', r') \in \delta^{rev}((0, r), zero)$ . If  $1 \leq r+1 \leq c'$  holds, then  $\delta((c', r'), zero) = (0, c'-1) \geq_R (0, r)$ , so  $(c', r') \in \delta^{rev}((0, r), zero)$  as well. For the direction  $\delta^{rev}((0, r), zero) \subseteq \uparrow\{(0, r+1)\}$ , let  $(c', r') \in \delta^{rev}((0, r), zero)$ . Then,  $\delta((c', r'), zero) = (0, c'-1) \geq_R (0, r)$ . We deduce  $c'-1 \geq r$  and  $c' \geq r+1$ . Then,  $(c', r') \in \uparrow\{(0, r+1)\}$ .

**Case a:** We claim  $\delta^{rev}((c, r), a) = \uparrow\{(c-1, r-1)\}$  in the case of  $c > 0$ , and  $\delta^{rev}((0, r), a) = \uparrow\{(r-1, r)\}$ . Note that since  $r \geq 1$  is guaranteed by  $R$ ,  $r-1 \geq 0$  holds.

We start with the former case. To see the inclusion  $\uparrow\{(c-1, r-1)\} \subseteq \delta^{rev}((c, r), a)$ , let  $(c', r') \geq_R (c-1, r-1)$ . Then,  $c' = c-1$  and  $r' \geq r-1$ , or  $c' \geq r-1$ . We have  $\delta((c', r'), a) = (c'+1, r'+1)$ . In the first case we observe  $c'+1 = c$  and  $r'+1 \geq r$ , so  $(c'+1, r'+1) \geq_R (c, r)$ . In the second case, we observe  $c'+1 \geq r$ , so again  $(c'+1, r'+1) \geq_R (c, r)$ . Then,  $(c', r') \in \delta^{rev}((c, r), a)$ . For the inclusion  $\delta^{rev}((c, r), a) \subseteq \uparrow\{(c-1, r-1)\}$ , let  $(c', r') \in \delta^{rev}((c, r), a)$ . Then,  $\delta((c', r'), a) = (c'+1, r'+1) \geq_R (c, r)$ . So,  $c'+1 = c$  and  $r'+1 \geq r$ , or  $c'+1 \geq r$ . Then,  $c' = c-1$  and  $r' \geq r-1$  or  $c' \geq r-1$ , meaning  $(c', r') \geq_R (c-1, r-1)$ .

Now we handle the latter case. For the inclusion  $\uparrow\{(r-1, r)\} \subseteq \delta^{rev}((0, r), a)$ , let  $(c', r') \geq_R (r-1, r)$ . Then,  $c' = r-1$  and  $r' \geq r$ , or  $c' \geq r$ . In any case,  $c'+1 \geq r$ . We have  $\delta((c', r'), a) = (c'+1, r'+1)$ . Since  $c'+1 \geq r$ ,  $(c'+1, r'+1) \geq_R (0, r)$ . For the inclusion  $\delta^{rev}((0, r), a) \subseteq \uparrow\{(r-1, r)\}$ , let  $(c', r') \in \delta^{rev}((0, r), a)$ . Then we have  $\delta((c', r'), a) = (c'+1, r'+1) \geq_R (0, r)$ . Since  $c' \geq 0$ , we have  $c'+1 \neq 0$ . So  $c'+1 \geq r$  must hold. We argue that this implies  $(c', r') \geq_R (r-1, r)$ . If  $c'+1 > r$ , then  $c' \geq r$ , so  $(c', r') \geq_R (r-1, r)$ . If  $c'+1 = r$ , then we have  $r' > c' = r-1$  and thus  $r' \geq r$ . So  $(c', r') \geq_R (r-1, r)$  as well.

**Case  $\bar{a}$ :** We claim  $\delta^{rev}((c, r), \bar{a}) = \uparrow\{(c+1, r+1)\}$ . Toward the inclusion  $\uparrow\{(c+1, r+1)\} \subseteq \delta^{rev}((c, r), \bar{a})$ , let  $(c', r') \geq_R (c+1, r+1)$ . Then,  $c' = c+1$  and  $r' \geq r+1$ , or  $c' \geq r+1$ . We have  $\delta((c', r'), \bar{a}) = (c'-1, r'-1)$ . If the first condition holds, we have  $c' = c+1$  and  $r' \geq r+1$  so  $c'-1 = c$  and  $r'-1 \geq r$ . Therefore  $(c'-1, r'-1) \geq_R (c, r)$ . If the second condition holds, we have  $c' \geq r+1$  and  $c'-1 \geq r$ . So,  $(c'-1, r'-1) \geq_R (c, r)$  as well. This implies  $(c', r') \in \delta^{rev}((c, r), \bar{a})$ . Towards the inclusion  $\delta^{rev}((c, r), \bar{a}) \subseteq \uparrow\{(c+1, r+1)\}$ , let  $(c', r') \in \delta^{rev}((c, r), \bar{a})$ . The  $\bar{a}$  transitions are only active from columns greater than 0, so we can say  $\delta((c', r'), \bar{a}) = (c'-1, r'-1) \geq_R (c, r)$ . This implies  $c'-1 = c$  and  $r'-1 \geq r$ , or  $c'-1 \geq r$ . So, we see  $c' = c+1$  and  $r' \geq r+1$ , or  $c' \geq r+1$ . Therefore,  $(c', r') \geq_R (c+1, r+1)$ . This concludes the proof.  $\blacktriangleleft$

**Details on Lemma 36.** For  $K = T.\#.\bar{T}^{rev}$ , we prove that we have  $K \in L(\text{WSTS})$ ,  $K \notin L(\text{detWSTS})$ , and  $K \notin L(\text{detDWSTS})$ .

**Proof of Lemma 36.** Since  $\bar{T}^{rev}, T \in L(\text{WSTS})$ , and  $L(\text{WSTS})$  is closed under concatenation [24, Theorem 2], we get  $T.\#.\bar{T}^{rev} \in L(\text{WSTS})$ . We only show that  $K \notin L(\text{detWSTS})$ , as the proof of  $K^{rev} \notin L(\text{detDWSTS})$  is similar. Suppose that  $K \in L(\text{detWSTS})$ . Then, there is a  $\text{detWSTS}$   $U = (Q, \leq, y, \Sigma, \delta, F)$  with  $L(U) = K$ . Let  $v \in \bar{T}^{rev}$ . This is guaranteed to exist, since  $T$  is neither empty nor universal, per  $T \notin L(\text{detWSTS})$ . Let  $U' = (Q, \leq, y, \Sigma, \delta, G)$  with  $G = \{p \in Q \mid \delta(p, \#.\bar{v}) \in F\}$ . We claim that  $U'$  is a  $\text{detWSTS}$  with  $L(U') = T$ , which contradicts Proposition 27. First we show that  $U'$  is indeed a  $\text{detWSTS}$ . Since we

only modified the final states, we only need to verify that  $G$  is upward-closed. Let  $p \in G$  and  $p \leq q$ . By the definition of  $G$ , we get  $\delta(p, \#v) \in F$ . By standard induction on word length, the simulation property delivers  $\delta(q, \#v) \geq \delta(p, \#v)$ . Since  $F$  is upward-closed and  $\delta(p, \#v) \in F$ , we have  $\delta(q, \#v) \in F$  and thus  $q \in G$ .

Now we show  $L(U') = T$ . To see  $L(T) \subseteq L(U')$ , let  $w \in T$ . Then,  $\delta(y, w.\#v) \in F$  since  $w.\#v \in K$ . We have  $\delta(y, w.\#v) = \delta(\delta(y, w), \#v) \in F$ , so  $\delta(y, w) \in G$ . So  $w \in L(U')$ . To see  $L(U') \subseteq T$ , let  $w \in L(U')$ . Then  $\delta(y, w) \in G$ . This implies  $\delta(\delta(y, w), \#v) = \delta(y, w.\#v) \in F$ . Then  $w.\#v \in K$ . Neither words in  $T$ , nor words in  $\overline{T}^{rev}$  contain  $\#$ . This is because  $\overline{T}$  refers to the complement of  $T$  with respect to  $(\Sigma \setminus \{\#\})^*$ . Therefore, words in  $K$  contain exactly one  $\#$ . So, for  $w.\#v \in K$  to hold,  $w \in T$  and  $v \in \overline{T}^{rev}$  must also hold. This concludes the proof.  $\blacktriangleleft$