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A simple combinatorial proof for the small model property of two-variable logic

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ABSTRACT

We present another proof for the well-known *small model property* of two-variable logic. As far as we know, existing proofs of this property are based on a rather intricate model theoretic construction. In contrast, ours uses only simple combinatorial argument which we find more intuitive and direct.

1. Introduction

Two-variable logic (FO^2) is a well known fragment of first-order logic that uses only two variables and comes with decidable satisfiability problem. It was first proved to be decidable in double-exponential time by Mortimer [2]. The upper bound was later improved to single-exponential by Grädel, Kolaitis and Vardi [1]. Both upper bounds were achieved by showing that FO^2 has the so called *small model property*, i.e., if a formula is satisfiable, then it is satisfiable by a model with cardinality exponential in the length of the formula, or double-exponential in the case of Mortimer's. Both proofs use a rather intricate model theoretic construction.

In this note we present another proof for the FO^2 small model property. The bound we achieve is single-exponential, matching the one by Grädel, et al., although our precise bound is slightly larger. On the other hand, our proof is based only on a very simple combinatorial construction which in its core is just a counting argument, and arguably more intuitive and direct.

2. Small model property of two-variable logic

We consider FO^2 sentences with the equality predicate. A well known result of Scott [3] states that every FO^2 sentence can be transformed in linear time to another equi-satisfiable sentence in the following *Scott normal form*:

$$\Phi := \forall x \forall y\ \alpha(x,y)\ \wedge\ \bigwedge_{i=1}^{p} \forall x \exists y\ \big(\beta_i(x,y) \wedge x \neq y\big) \tag{1}$$

where $\alpha(x,y)$ and $\beta_i(x,y)$ are all quantifier free. Thus, it suffices to consider only sentences in this form. We refer interested reader to [1,3] for more details.

In the following, we fix a formula Φ as in (1) over a fixed vocabulary τ. Let n and m be the number of unary and binary predicate symbols in τ, respectively. For technical convenience, we assume Φ does not use any constant symbols, since they can be represented by unary predicates. We also assume that only unary and binary predicates are used. Atomic predicates of higher arity such as $R(x,y,x)$ can be viewed as binary predicates.

We recall a few terminology, most of which are adopted from [1]. A 1-*type* π (over vocabulary τ) is a maximally consistent set of atomic predicates or their negations using only variable x. Formally, π is a 1-type, if for every unary

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predicate symbol $U \in \tau$, either $U(x) \in \pi$ or $\neg U(x) \in \pi$ and for every binary predicate symbol $R \in \tau$, either $R(x,x) \in \pi$ or $\neg R(x,x) \in \pi$. Similarly, a *2-type* η (over τ) is a maximally consistent set that contains $x \neq y$ and atomic predicates or their negations that use both variables x and y. That is, η is a 2-type, if it contains $x \neq y$ and for every binary predicate $R \in \tau$, either $R(x,y) \in \eta$ or $\neg R(x,y) \in \eta$, and either $R(y,x) \in \eta$ or $\neg R(y,x) \in \eta$. The number of 1-types and 2-types are 2^{n+m} and 2^{2m}, respectively. We write π and η (possibly indexed) to denote 1-type and 2-type, respectively.

In the following we use calligraphic letters $\mathcal{A}$ and $\mathcal{B}$ to denote structures and capital letters A and B to denote their respective domains. We write $\mathcal{A}, x/a, y/b \models \varphi(x,y)$ to denote that the formula $\varphi(x,y)$ holds in the structure $\mathcal{A}$ when the free variables x and y are assigned with elements a and b, respectively.

Let $\mathcal{A}$ be a structure and A be its domain. The *1-type of an element $a \in A$* is the unique 1-type π that a satisfies in $\mathcal{A}$. We denote by A_π the set of elements in $\mathcal{A}$ with 1-type π. Similarly, the *2-type of a pair $(a,a') \in A \times A$*, where $a \neq a'$, is the unique 2-type that (a,a') satisfies in $\mathcal{A}$. We say that *a 1-type/2-type is realized in $\mathcal{A}$*, if there is an element/a pair of elements that satisfies it. For 1-types π, π', we denote by $D_{\pi,\pi'}(\mathcal{A})$ the set of 2-types η such that η is the 2-type of some $(a,a') \in A_\pi \times A_{\pi'}$.

A 1-type is a *king 1-type* in $\mathcal{A}$, if there is only one element in A that satisfies it. The element whose 1-type is king is called a *king element*. We denote by $KT(\mathcal{A})$ the set of king 1-types in $\mathcal{A}$. Suppose $a_1,\ldots,a_t$ are the king elements in $\mathcal{A}$ with $\pi_1,\ldots,\pi_t$ being their respective 1-types. For a non-king element $a \in A$, *the profile of a* is the set $\{(\pi_1,\eta_1),\ldots,(\pi_t,\eta_t)\}$ where η_i is the 2-type of the pair (a_i,a). If there is no king element, the profile of a is the empty set. It is worth stating that the notion of *profile* is new and it does not appear in [1].

Lemma 1 below is a combinatorial lemma that will immediately imply the small model property of FO^2.

Lemma 1. *For every structure $\mathcal{A}$, there is a structure $\mathcal{B}$ (over the same vocabulary as $\mathcal{A}$) such that the following holds.*

- *(P1) For every 1-type π, π is realized in $\mathcal{A}$ if and only if π is realized in $\mathcal{B}$.*
- *(P2) $KT(\mathcal{A}) = KT(\mathcal{B})$.*
- *(P3) For every non-king 1-type π realized in $\mathcal{A}$, $|B_\pi| = \max(k\ell, 3\ell)$, where k and ℓ are the number of 1-types and 2-types realized in $\mathcal{A}$, respectively.*
- *(P4) For every 1-types π_1, π_2, $D_{\pi_1,\pi_2}(\mathcal{A}) = D_{\pi_1,\pi_2}(\mathcal{B})$.*
- *(P5) For every non-king element b in $\mathcal{B}$, there is a non-king element a in $\mathcal{A}$ with the same 1-type and profile as b.*
- *(P6) For every non-king 1-types π,π', for every 2-type $\eta \in D_{\pi,\pi'}(\mathcal{A})$, for every element $b \in B_\pi$, there is an element $b' \in B_{\pi'}$ such that the 2-type of (b,b') is η.*

Proof. Let $\mathcal{A}$ be a structure where k and ℓ are the number of 1-types and 2-types realized in $\mathcal{A}$, respectively. To construct $\mathcal{B}$, it suffices to define 1-type/2-type of all the elements/pairs of elements in $\mathcal{B}$. The domain of $\mathcal{B}$ and 1-types of the elements are as follows. For every 1-type π realized in $\mathcal{A}$, if π is a king 1-type in $\mathcal{A}$, B_π has only one element, and if π is a non-king 1-type in $\mathcal{A}$, B_π has exactly $\max(k\ell, 3\ell)$ elements. At this point (P1)–(P3) already hold for $\mathcal{B}$.

We set the 2-type of every pair of elements in $\mathcal{B}$ in four steps below. Note that the 2-type of (b,b') uniquely determines the 2-type of its reverse (b',b). Thus, when we set the 2-type of (b,b'), the 2-type of (b',b) is also set at the same time.

(Step 0) Setting the 2-type of pairs of king elements.

Let $a_1,\ldots,a_t$ and $b_1,\ldots,b_t$ be the king elements in $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $\pi_1,\ldots,\pi_t$ be their respective 1-types. For every $1 \leqslant i < j \leqslant t$, we set the 2-type of (b_i,b_j) in $\mathcal{B}$ the same as the 2-type of (a_i,a_j) in $\mathcal{A}$.

(Step 1) Setting the 2-type of pairs in $\{b_1,\ldots,b_t\} \times B_\pi$, for every non-king 1-type π.

We partition B_π into t sets $B_\pi^1 \cup \cdots \cup B_\pi^t$ such that each $|B_\pi^i| \geqslant \ell$. This is possible since $|B_\pi| = \max(k\ell, 3\ell)$ and $k \geqslant t$. For each $i \in \{1,\ldots,t\}$, we do the following.

- We set the 2-type of pairs in $\{b_i\} \times B_\pi^i$ so that all the 2-types in $D_{\pi_i,\pi}(\mathcal{A})$ are used. This is possible since $|D_{\pi_i,\pi}(\mathcal{A})| \leqslant \ell \leqslant |B_\pi^i|$.
- Note that at this point, for every $b \in B_\pi^i$, there is $a_b \in A_\pi$ such that the 2-type of (b,b_i) is the same as the 2-type of (a_b,a_i).
  Then, for every $b \in B_\pi^i$, we set the 2-type of all the pairs between b and the rest of the kings (i.e., kings that are not b_i) so that the profile of b in $\mathcal{B}$ is the same as the profile of a_b in $\mathcal{A}$.

We do this step for every non-king 1-type π. After this step (P5) is already established.

(Step 2) Setting the 2-type of pairs in $B_\pi \times B_\pi$, for every non-king 1-type π.

We re-partition B_π into three sets $B_\pi^0, B_\pi^1, B_\pi^2$ where each $|B_\pi^i| \geqslant \ell$. For each $i \in \{0,1,2\}$, for each element $b \in B_\pi^i$, we set the 2-types of pairs in $\{b\} \times B_\pi^{i+1 \pmod 3}$ so that all the 2-types in $D_{\pi,\pi}(\mathcal{A})$ are used. The 2-types of the rest of the pairs that are not yet defined are set with arbitrary 2-type from $D_{\pi,\pi}(\mathcal{A})$. The crux of the argument here is all 2-types in $D_{\pi,\pi}(\mathcal{A})$ can be exhausted on every element $b \in B_\pi$.

(Step 3) Setting the 2-type of pairs in $B_\pi \times B_{\pi'}$, for every non-king 1-types $\pi \neq \pi'$.

This step is similar to Step 2. We re-partition B_π and $B_{\pi'}$ into two sets $B_\pi^0 \cup B_\pi^1$ and $B_{\pi'}^0 \cup B_{\pi'}^1$, respectively, where each $|B_\pi^i|, |B_{\pi'}^i| \geqslant \ell$.

For each $i \in \{0,1\}$, for each element $b \in B_\pi^i$, we set the 2-type of pairs in $\{b\} \times B_{\pi'}^i$ such that all the 2-types in $D_{\pi,\pi'}(\mathcal{A})$ are used. Similarly, for each $i \in \{0,1\}$, for each element $b \in B_{\pi'}^i$, we set the 2-type of pairs in $\{b\} \times B_\pi^{1-i}$ such that all the 2-types in $D_{\pi',\pi}(\mathcal{A})$ are used. This marks the end of Step 3.

Note that after Step 2 and 3, it is straightforward that (P6) hold. Moreover, when we set the 2-types of pairs in $B_\pi \times B_{\pi'}$, we use only the 2-types in $D_{\pi,\pi'}(\mathcal{A})$. Thus, (P4) holds. It is also worth stating that when there is no king element in $\mathcal{A}$, we skip Steps 0 and 1. This completes our proof of Lemma 1. $\square$

We will now use Lemma 1 to prove the small model property of FO^2 as stated below.

Theorem 2. *Let Φ be an FO^2 sentence as in Eq. (1). If it is satisfiable, then it is satisfiable by a model with at most 2^{2n+4m} elements, where n and m are the number of unary and binary predicates in the vocabulary.*

Proof. Suppose $\mathcal{A} \models \Phi$. Adding redundant predicates, if necessary, we can assume that $n + m \geqslant 2$. Let $\mathcal{B}$ be the structure as constructed in Lemma 1. Note that if k and ℓ are as in Lemma 1 and t is the number of king elements, by (P3), the cardinality of $\mathcal{B}$ is $t + (k - t)\max(k\ell, 3\ell) \leqslant k\max(k\ell, 3\ell) \leqslant 2^{2n+4m}$. We now show that $\mathcal{B} \models \Phi$.

First, note that for every $b \in B$, whether $\mathcal{B}, x/b, y/b \models \alpha(x, y)$ depends only on the 1-type of b. Likewise, for every $b, b' \in B$, where $b \neq b'$, whether $\mathcal{B}, x/b, y/b' \models \alpha(x, y)$ depends only on the 1-types of b and b' and their 2-type. Since $\mathcal{A} \models \forall x\forall y\ \alpha(x, y)$, by (P1), (P2) and (P4), it is immediate that $\mathcal{B} \models \forall x\forall y\ \alpha(x, y)$.

To show that $\mathcal{B} \models \bigwedge_{i=1}^{p} \forall x\exists y\big(\beta_i(x, y) \wedge x \neq y\big)$, let $b \in B$ and π be its 1-type. Let $1 \leqslant i \leqslant p$. We will show that there is $b' \neq b$ such that $\mathcal{B}, x/b, y/b' \models \beta_i(x, y)$. There are two cases.

- Case 1: b is a king element.
  By (P2), there is a king element a in $\mathcal{A}$ with 1-type π, i.e., the same 1-type as b. Since $\mathcal{A} \models \Phi$, there is $a' \neq a$ such that $\mathcal{A}, x/a, y/a' \models \beta_i(x, y)$. Let π' be the 1-type of a'. By (P1), π' is also realized in $\mathcal{B}$. Since a is a king element, $\pi \neq \pi'$. By (P4), $D_{\pi,\pi'}(\mathcal{A}) = D_{\pi,\pi'}(\mathcal{B})$. Thus, there is $b' \neq b$ such that the 1-type of b' is π' and the 2-type of (b, b') is the same as the 2-type of (a, a'). Therefore, $\mathcal{B}, x/b, y/b' \models \beta_i(x, y)$.
- Case 2: b is not a king element.
  Let a be a (non-king) element in $\mathcal{A}$ with the same 1-type and profile as b. By (P5), such element a exists. Since $\mathcal{A} \models \Phi$, there is $a' \neq a$ such that $\mathcal{A}, x/a, y/a' \models \beta_i(x, y)$. Let π' be the 1-type of a'. There are two cases: (i) π' is a king 1-type and (ii) π' is not a king 1-type.
  For (i), since a and b have the same profile, there is a king element $b' \neq b$ with 1-type π' and the 2-type of (b, b') is the same as the 2-type of (a, a'). Thus, $\mathcal{B}, x/b, y/b' \models \beta_i(x, y)$.
  For (ii), by (P6), there is $b' \neq b$ with 1-type π' and the 2-type of (b, b') is the same as the 2-type of (a, a'). Thus, $\mathcal{B}, x/b, y/b' \models \beta_i(x, y)$.

This completes our proof of Theorem 2. □

To conclude, it may be worth comparing our proof with the one in [1]. The one in [1] starts by rewriting the sentence Φ into the following form, where $g_1, \ldots, g_p$ are function symbols called *Skolem* functions.

$$\Phi := \forall x\forall y\ \alpha(x, y)\ \wedge\ \bigwedge_{i=1}^{p} \forall x\ \beta_i(x, g_i(x)) \tag{2}$$

Suppose $\mathcal{A} \models \Phi$. For an element $a \in A$, the element $g_i(a)$ is called a *Skolem witness* of a.

Let $\mathcal{A}_0$ be the substructure $\mathcal{A}$ that contains only the king elements. Let $\mathcal{A}_1$ be the extension of $\mathcal{A}_0$ with the Skolem witnesses of elements in $\mathcal{A}_0$. ($\mathcal{A}_1$ is called *royal court* in [1]) We can then construct a structure $\mathcal{B} \models \Phi$ by extending $\mathcal{A}_1$ with three disjoint sets D, E, F of fresh elements such that the following holds.

- The skolem witnesses of elements in A_1 are all contained in $A_0 \cup D$.
- The skolem witnesses of elements in D are all contained in $A_0 \cup E$.
- The skolem witnesses of elements in E are all contained in $A_0 \cup F$.
- The skolem witnesses of elements in F are all contained in $A_0 \cup D$.

It can be argued that the sizes of D, E, F are small enough so that the total size of $\mathcal{B}$ is at most $3p2^{n+m}$, which is smaller than our bound 2^{2n+4m}. Note, however, the "circular" direction of the skolem functions in $\mathcal{B}$, i.e., from D to E to F and back to D. This is where much of the intricacy of the argument arises. In our proof we manage to circumvent it due to (P5) and (P6) in Lemma 1, which we achieve by simply ensuring profiles in $\mathcal{B}$ are also profiles in $\mathcal{A}$ (Step 1) and by exhausting all possible 2-types on every element in $\mathcal{B}$ (Steps 2 and 3).

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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