On Noetherian Spaces

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Abstract

A topological space is Noetherian *iff every open is compact. Our starting point is that this notion generalizes that of well-quasi order, in the sense that an Alexandroff-discrete space is Noetherian iff its specialization quasi-ordering is well. For more general spaces, this opens the way to verifying infinite transition systems based on non-well quasi ordered sets, but where the preimage operator satisfies an additional continuity assumption. The technical development rests heavily on techniques arising from topology and domain theory, including sobriety and the de Groot dual of a stably compact space. We show that the category* $\mathbf{Nthr}$ *of Noetherian spaces is finitely complete and finitely cocomplete. Finally, we note that if* X *is a Noetherian space, then the set of all (even infinite) subsets of* X *is again Noetherian, a result that fails for well-quasi orders.*

1. Introduction

A topological space X is *Noetherian* iff every open subset of X is compact [13, chapitre 0, § 2]. We shall explain how this generalizes the theory of well quasi-orders.

Recall that a well quasi-ordering is a quasi-ordering (a reflexive and transitive relation) that is not only well-founded, i.e., has no infinite descending chain, but also has no infinite antichain (a set of incomparable elements). One use of well quasi-orderings is in verifying *well-structured transition systems* [2, 4, 11, 14]. These are transition systems, usually infinite-state, with two ingredients.

First, a *well* quasi-ordering $\leq$ on the set X of states. Second, the transition relation δ commutes with $\leq$, i.e., if $x\ \delta\ y$ and $x \leq x'$, then there is a state y' such that $x'\ \delta\ y'$ and $y \leq y'$:

$$\begin{tikzcd} x \arrow[r, "\leq"] \arrow[d, "\delta"'] & x' \arrow[d, dotted, "\delta"] \\ y \arrow[r, dotted, "\leq"] & y' \end{tikzcd} \quad (1)$$

Examples include Petri nets, VASS [15], lossy channel systems [3], timed Petri nets [6] to cite a few.

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For any subset A of X, let $\mathrm{Pre}^{\exists}\delta(A)$ be the preimage $\{x \in X | \exists y \in A \cdot x\ \delta\ y\}$. The commutation property ensures that the preimage $\mathrm{Pre}^{\exists}\delta(V)$ of any upward-closed subset V is again upward-closed (V is upward-closed iff whenever $x \in V$ and $x \leq x'$, then $x' \in V$). Standard arguments then show that one may compute $\mathrm{Pre}^{\exists *}\ \delta(V)$, the set of states in X from which we can reach some state in V in finitely many steps: Compute the set V_i of states from which we can reach some state in V in at most i steps, backwards, by $V_0 = V$, $V_{i+1} = V_i \cup \mathrm{Pre}^{\exists}\ \delta(V_i)$: this stabilizes at some stage i, where $V_i = \mathrm{Pre}^{\exists *}\ \delta(V)$.

This provides an algorithm for coverability: given two states $x, x' \in X$, is there a trace $x = x_0\ \delta\ x_1\ \delta\ \ldots\ \delta\ x_k$ such that $x' \leq x_k$? Just check $x \in \mathrm{Pre}^{\exists *}(\uparrow x')$, where $\uparrow x'$ is the upward-closed set $\{y \in X | x' \leq y\}$.

Outline. We generalize this by replacing quasi-orderings by topologies. We shall definitely rest on the rich relationship between theories of order and topology. We recapitulate what we need in two sections, Section 2 for basic notions, and Section 5 for more advanced concepts such as Stone duality, sobriety, and stable compactness which we don't need in earlier sections. The Zariski topology on spectra of Noetherian rings was the first known Noetherian topology; we discuss it only in Section 8, in the light of the rest of our paper. Our contribution occupies the other sections. We first show the tight relationship between well-quasi orders and Noetherian spaces in Section 3, and show a few easy constructions of new Noetherian spaces from given Noetherian spaces in Section 4. This culminates in showing that the category $\mathbf{Nthr}$ of Noetherian spaces is finitely cocomplete. Section 6 is technically more challenging, and characterizes those Noetherian spaces that are also sober. This is the cornerstone of the theory. E.g., this is instrumental to show that $\mathbf{Nthr}$ is finitely complete, and that the Hoare space of a Noetherian space is again Noetherian. We show the latter in Section 7. We then prove the unexpected result that the set of *all* subsets of a Noetherian space X (even infinite ones) has a topology that makes it Noetherian. This would be wrong in a pure theory of orders; topology makes the difference. Finally, our theory of Noetherian sober spaces suggests an alternative algorithm for coverability based on computing downward-closed sets, which we describe in Section 9. We conclude in Section 10.

We stress that this paper is not specifically geared towards applications. Its aim is rather to lay the theoretical basis for Noetherian topological spaces.

Related Work. If $\leq$ is a quasi-ordering on X then let $\mathbb{P}_{fin}(X)$ be the set of finite subsets of X, and order it by $\leq^\sharp$, where $A \leq^\sharp B$ iff for every $y \in B$ there is an $x \in A$ such that $x \leq y$. It is well-known that $\leq^\sharp$ needs not be well even when $\leq$ is well. This is a shortcoming, among others, of the theory of well quasi-orderings. Such shortcomings led Nash-Williams [23] to invent better quasi-orderings (bqos). Bqos have a rather unintuitive definition but a wonderful theory, see [19]. The only application of bqos we know of to verification problems is by Abdulla and Nylén [5], where it is used to show the termination of the backward reachability iteration, using *disjunctive* constraints.

This paper is not on bqos, and in fact not specifically on well quasi-orderings. While bqos are *restrictions* of well quasi-orderings, Noetherian spaces *generalize* the latter. We hope that Noetherian spaces will be valuable in verification in the future. The fact that $\mathbb{P}_{fin}(X)$, with the upper topology of $\leq^\sharp$, and that $\mathbb{P}(X)$, with another topology, are Noetherian whenever X is (Section 7) is a promising result.

Our work is more connected to topology, and in particular to topology as it is practiced in domain theory. As we shall see later, the notions of specialization quasi-ordering of a topological space, of upper, Scott and Alexandroff topologies, of sober space, of sobrification of a space, and of stably compact spaces are central to our work. Topology and domain theory form another wonderful piece of mathematics, and one may consult [12, 7, 18, 21].

Last but not least, Noetherian spaces arise from algebraic geometry [13]: we discuss this briefly in Section 8.

2. Preliminaries I: Order and Topology

A *topology* $\mathcal{O}$ on a set X is a collection of subsets (the *opens*) of X that is closed under arbitrary unions and finite intersections. We say that X itself is a topological space, leaving $\mathcal{O}$ implicit. The complements of opens are *closed*. The largest open contained in A is its *interior*, the smallest closed subset $cl(A)$ containing it is its *closure*.

Every topology comes with a *specialization quasi-ordering* $\leq$, defined as $x \leq y$ iff every open that contains x also contains y. Equivalently, $x \in cl\{y\}$. It is easy to see that every open is upward-closed with respect to $\leq$. The converse need not hold. A subset A of X is *saturated* iff A equals the intersection of all opens U containing A, equivalently iff it is upward-closed with respect to $\leq$.

A subset K of X is *compact* iff every open cover $(U_i)_{i\in I}$ contains a finite subcover. Alternatively, K is compact iff, for every directed family $(U_i)_{i\in I}$ of opens such that $K \subseteq \bigcup_{i\in I} U_i$, then $K \subseteq U_i$ for some $i \in I$ already. (A family $(x_i)_{i\in I}$ of elements quasi-ordered by $\leq$ is a non-empty family such that for every $i, j \in I$ there is $k \in I$ such that $x_i \leq x_k$ and $x_j \leq x_k$.)

Write $\uparrow E = \{x \in X | \exists y \in E \cdot y \leq x\}$, $\downarrow E = \{x \in X | \exists y \in E \cdot x \leq y\}$. If K is compact, then $\uparrow K$ is, too, and is also saturated. We shall usually reserve the letter Q for saturated compacts. When E is finite, $\uparrow E$ is compact saturated: call these the *finitary compacts*. Similarly, $\downarrow E$ is closed: call these the *finitary closed subsets*.

We have gone one direction, from topology to quasi-orderings. There are in general many return paths. The finest topology having $\leq$ as specialization quasi-ordering is the *Alexandroff topology* of $\leq$. Its opens are the upward-closed subsets of X with respect to $\leq$. The coarsest is the *upper topology*, generated by the complements of sets $\downarrow \{x\}$, $x \in X$. Its closed sets are the unions of subsets of the form $\downarrow E$, E finite. An intermediate topology is the *Scott topology*, whose opens are those upward-closed subsets U such that every directed family $(x_i)_{i\in I}$ that has a least upper bound in U meets U. The latter crops up in domain theory, where a *cpo* is a partially ordered set where every directed family has a least upper bound.

A topological space is *Alexandroff-discrete* iff every intersection of opens is again open. Equivalently, iff its topology is the Alexandroff topology of its specialization quasi-ordering. While every finitary compact is compact saturated, the converse holds in Alexandroff-discrete spaces.

A map f from X to Y is *continuous* iff $f^{-1}(V)$ is open in X for every open V of Y. Any continuous function is monotonic with respect to the specialization quasi-orderings of X and Y. The converse holds when X is Alexandroff-discrete: while continuity is usually seen as stronger than monotonicity, continuity also *generalizes* monotonicity, in the sense that monotonicity is just continuity with respect to Alexandroff topologies.

When X and Y are equipped with Scott topologies, $f : X \to Y$ is continuous iff f is *Scott-continuous*, i.e., f is monotonic and, for every directed family $(x_i)_{i\in I}$ in X having a least upper bound x, the family $(f(x_i))_{i\in I}$ (which is directed in Y) admits $f(x)$ as least upper bound. Continuity notions extend to binary relations. A relation R from X to Y is a subset of $X \times Y$. It is *lower semi-continuous* iff $\mathrm{Pre}^\exists R(V) = \{x \in X | \exists y \in V \cdot x \; R \; y\}$ is open whenever V is. It is *upper semi-continuous* iff $\mathrm{Pre}^\forall R(V) = \{x \in X | \forall y \cdot x \; R \; y \Rightarrow y \in V\}$ is open whenever V is.

3. Well-Quasi Orders and Noetherian Spaces

We first relate well-quasi orders and Noetherian spaces.

Proposition 3.1 *Consider the following properties of a topological space X, with specialization quasi-ordering $\leq$:*

1. X *is Noetherian;*
2. X *is a space where every open is finitary compact;*
3. $\leq$ *is a well quasi-ordering.*

Then 3 *implies* 2, 2 *implies* 1, *and if* X *is Alexandroff-discrete then* 1 *implies* 3.

Proof. $3 \Rightarrow 2$: Every open is upward-closed, and every upward-closed subset is finitary compact by assumption. $2 \Rightarrow 1$ is obvious. Let us show $1 \Rightarrow 3$, assuming X Alexandroff-discrete. Each upward-closed subset is open, hence compact saturated by 1, hence finitary compact since X is Alexandroff-discrete. $\square$

This allows us to claim that Noetherian spaces are exactly the topological counterpart of the order-theoretic notion of well quasi-order. In particular, there are many Noetherian spaces: equip any well quasi-ordered set X with its Alexandroff topology. As we shall see, there are others.

The following well-known characterization of Noetherian spaces will be useful. Let $\Omega(X)$ be the set of all opens of X, ordered by inclusion. A set Y with a quasi-ordering $\sqsubseteq$ has the *ascending chain condition* iff every infinite ascending chain $y_0 \sqsubseteq y_1 \sqsubseteq \ldots \sqsubseteq y_k \sqsubseteq \ldots$ *stabilizes*, i.e., there is an integer N such that $y_k \sqsubseteq y_N$ for every $k \geq N$.

Proposition 3.2 *Let* X *be a topological space. Then* X *is Noetherian iff* $\Omega(X)$ *has the ascending chain condition.*

The backward computation of $\mathrm{Pre}^{\exists *}$ of the introduction extends easily, as follows. Let a *topological well-structured transition system* be a pair (X, δ), where X, the *state space*, is a Noetherian space, and δ, the *transition relation*, is lower semi-continuous. Then again, the sequence of backward iterates V_i terminates, by Proposition 3.2:

Proposition 3.3 *Let* (X, δ) *be a topological well-structured transition system. For any open subset* V*, let* $V_0 = V$, $V_{i+1} = V_i \cup \mathrm{Pre}^{\exists}\ \delta(V_i)$. *The sequence* $(V_i)_{i\in\mathbb{N}}$ *is an ascending chain, which stabilizes on* $\mathrm{Pre}^{\exists *}(V)$.

We retrieve that backwards iterations terminate on well-structured transition systems (a well-known fact), because:

Proposition 3.4 *Each well-structured transition system* (X, δ) *is a topological well-structured transition system. The converse holds when* X *is Alexandroff-discrete.*

Write $\overline{A}$ for the complement of A. Note that $\mathrm{Pre}^{\forall}\delta(\overline{A}) = \overline{\mathrm{Pre}^{\exists}\delta(A)}$, that complements of upward-closed sets are downward-closed, and conversely. So, again when X is Alexandroff-discrete, δ is upper semi-continuous iff the dual of Diagram (1) holds:

$$\begin{tikzcd} x \arrow[r, "\leq"] \arrow[d, dotted, "\delta"'] & x' \arrow[d, "\delta"] \\ y \arrow[r, dotted, "\leq"] & y' \end{tikzcd} \qquad (2)$$

One may then go a bit further than just reachability. Define the following negation-free fragment of the modal μ-calculus. Let $L = L_{must} \cup L_{may}$ be a finite set of *transition labels*, taken as the (not necessarily disjoint) union of two subsets of *must labels* and *may labels* respectively. Let $\mathcal{A}$ be a recursive set of so-called *atomic formulae*.

F	$::=$	A	atomic formula $(A \in \mathcal{A})$
	$\mid$	X	variable
	$\mid$	$\top$	true
	$\mid$	$F \wedge F$	conjunction
	$\mid$	$\bot$	false
	$\mid$	$F \vee F$	disjunction
	$\mid$	$[\ell]F$	box modality $(\ell \in L_{must})$
	$\mid$	$\langle\ell\rangle F$	diamond modality $(\ell \in L_{may})$
	$\mid$	$\mu X \cdot F$	least fixed point

Formulae are interpreted in a *Kripke structure* $I = (X, (\delta_\ell)_{\ell\in L}, (U_A)_{A\in\mathcal{A}})$, where X is a topological space, δ_ℓ is a binary relation on X, which is lower semi-continuous when $\ell \in L_{may}$ and upper semi-continuous when $\ell \in L_{must}$, and U_A is an open of X for every atomic formula A. An *environment* ρ maps variables X to opens of X. Define the satisfaction relation $x \models^I_\rho F$ as usual. In particular, $x \models^I_\rho [\ell]F$ iff for every state y such that $x\ \delta_\ell\ y$, $y \models^I_\rho F$; $x \models^I_\rho \langle\ell\rangle F$ iff for some state y such that $x\ \delta_\ell\ y$, $y \models^I_\rho F$; and $x \models^I_\rho \mu X \cdot F$ iff $x \in \bigcup_{i=0}^{+\infty} U_i$, where $U_0 = \emptyset$, $U_{i+1} = \{z \in X | z \models^I_{\rho[X:=U_i]} F\}$; $\rho[X := U]$ is the environment mapping X to U, and every $Y \neq X$ to $\rho(Y)$.

Let lfp be the least fixed point operator of Scott-continuous functions f: $\mathsf{lfp}(f) = \bigcup_{i=0}^{+\infty} f^i(\emptyset)$, and write $I\,[\![F]\!]_\delta\,\rho$ for the set of elements $z \in Z$ such that $z \models^I_\rho F$. The semantics of formulae is characterized by the clauses:

$$\begin{array}{ll} I\,[\![A]\!]_\delta\,\rho = U_A & I\,[\![X]\!]_\delta\,\rho = \rho(X) \\ I\,[\![\top]\!]_\delta\,\rho = X & I\,[\![F_1 \wedge F_2]\!]_\delta\,\rho = I\,[\![F_1]\!]_\delta\,\rho \cap I\,[\![F_2]\!]_\delta\,\rho \\ I\,[\![\bot]\!]_\delta\,\rho = \emptyset & I\,[\![F_1 \vee F_2]\!]_\delta\,\rho = I\,[\![F_1]\!]_\delta\,\rho \cup I\,[\![F_2]\!]_\delta\,\rho \end{array}$$
$$I\,[\![[\ell]F]\!]_\delta\,\rho = \mathrm{Pre}^{\forall}\delta_\ell(I\,[\![F]\!]_\delta\,\rho)$$
$$I\,[\![\langle\ell\rangle F]\!]_\delta\,\rho = \mathrm{Pre}^{\exists}\delta_\ell(I\,[\![F]\!]_\delta\,\rho)$$
$$I\,[\![\mu X \cdot F]\!]_\delta\,\rho = \mathsf{lfp}(\lambda U \in \Omega(X) \cdot I\,[\![F]\!]_\delta\,(\rho[X := U]))$$

An easy structural induction on F then shows that $I\,[\![F]\!]_\delta\,\rho$ is always open.

When X is Noetherian, the above formulae describe an obvious algorithm for computing $I\,[\![F]\!]_\delta\,\rho$. The only nontrivial case is for formulae of the form $\mu X \cdot F$. However, we may compute $\mathsf{lfp}(f)$ for any Scott-continuous function $f : \Omega(X) \to \Omega(X)$ (in fact for any monotonic f: when X is Noetherian, every monotonic $f : \Omega(X) \to \Omega(X)$ is Scott-continuous) by: $U_0 = \emptyset$, $U_{i+1} = f(U_i)$; this defines an ascending chain, which stabilizes by Proposition 3.2. We need to detect when this stabilizes, and so we require the inclusion relation to be decidable. Note that by Proposition 3.1, every open U can be represented as a finitary compact $\uparrow E$, that is, as a finite list of elements. Clearly, $\uparrow E \subseteq \uparrow E'$ iff $E' \leq^\sharp E$, i.e., for every $x \in E$, there is a $y \in E'$ such that $y \in x$. The quasi-ordering $\leq^\sharp$ is usually called the *Smyth* *quasi-ordering*, and is decidable as soon as $\leq$ is. (I.e., each element $x \in X$ has a unique code $\ulcorner x \urcorner \in \{0,1\}^*$, and $\leq$ is a computable binary predicate on codes.) Assume that U_A and $\rho(X)$ are specified by given finite sets E_A and E'_X, i.e., $U_A = \uparrow E_A$ and $\rho(X) = \uparrow E'_X$. We obtain:

Theorem 3.5 *Let X be a Noetherian space, and assume that its specialization quasi-ordering $\leq$ is recursive. Assume that δ_ℓ is recursive, in the sense that for any finite subset E of X, we can compute a finite subset E' of X such that $\mathrm{Pre}^\exists\delta(\uparrow E) = \uparrow E'$ ($\ell \in L_{may}$) and $\mathrm{Pre}^\forall\delta(\uparrow E) = \uparrow E'$ ($\ell \in L_{must}$). Let U_A, $\rho(X)$ be specified by given finite sets.*

Then there is an algorithm which, given a formula F, computes a finite set E of elements such that $I\,[\![F]\!]_\delta\,\rho = \uparrow E$. In particular, checking whether $x \models^I_\rho F$ is decidable.

When $\ell \in L_{may}$, computing $\mathrm{Pre}^{\exists *}\delta_\ell(V)$ is a special case of the above evaluation scheme for formulae: $\mathrm{Pre}^{\exists *}\delta_\ell(V) = I\,[\![\mu X \cdot A \vee \langle \ell \rangle X]\!]_\delta\,\rho$, where ρ is arbitrary and $U_A = V$. One may also evaluate some forms of monotonic games [1, 9]: reading δ_{ℓ_1} as the transition relation for player 1, and δ_{ℓ_2} as that for player 2, the formula $\mu X \cdot A \vee \langle \ell_1 \rangle (B \wedge [\ell_2] X)$ is true exactly at those states x_0 such that player 1 has a strategy to win (reach the open U_A while preventing player 2 from reaching U_B), whatever player 2's moves.

4. Easy Constructions of Noetherian Spaces

We know that every well quasi-ordering yields a Noetherian space, through its Alexandroff topology. For example, $\mathbb{N}$, $\mathbb{N}^k$ with the componentwise ordering (Dickson's Lemma), the set of finite words over a well-quasi-ordered alphabet, ordered by embedding (Higman's Lemma), the set of finite labelled trees over a well-quasi-ordered signature, ordered by embedding (Kruskal's Theorem). We describe more constructions of new Noetherian spaces from old, and start with some easy ones. We shall consider finite products in Section 6.2 only; this is harder.

The first observation is similar to the fact that, for every well quasi-ordering $\leq$, any quasi-ordering $\leq'$ such that $x \leq y$ implies $x \leq' y$ is also a well quasi-ordering.

Lemma 4.1 *Every topology coarser than a Noetherian topology is Noetherian.*

So for example, if $\leq$ is a well quasi-ordering, then its Scott topology and its upper topology are Noetherian topologies. Every topological space with finitely many opens is also trivially Noetherian. This includes the case of finite spaces.

Recall that a *subspace* Y of a topological space X is a subset of X whose topology is given by the intersections of opens of X with Y—the *induced* topology.

Lemma 4.2 *Every subspace of a Noetherian space is Noetherian.*

Proof. Let $U_1 \cap Y \subseteq U_2 \cap Y \subseteq \ldots \subseteq U_k \cap Y \subseteq \ldots$ be an ascending chain of opens in Y. The open subset $U = \bigcup_{i=1}^{+\infty} U_i$ of X is compact, since X is Noetherian. So for some K, $U \subseteq \bigcup_{i=1}^{K} U_i$. It follows that $U \cap Y \subseteq \bigcup_{i=1}^{K} U_i \cap Y = U_K \cap Y$, hence $U_k \cap Y \subseteq U_K \cap Y$ for every $k \geq 1$. We conclude by Proposition 3.2. $\square$

The *coproduct* $X_1 + \ldots + X_k$ of k spaces is their disjoint union. Its opens are disjoint unions of opens, one from each X_i, i.e., $\Omega(X_1 + \ldots + X_k) \cong \Omega(X_1) \times \ldots \times \Omega(X_k)$.

Lemma 4.3 *The coproduct of finitely many Noetherian spaces is Noetherian.*

Being Noetherian is also preserved under direct images:

Lemma 4.4 *Let $q : X \to Z$ be a surjective continuous map. If X is Noetherian then so is Z.*

Given an equivalence relation $\equiv$ on a topological space X, the *quotient space* $X/\!\equiv$ is the set of equivalence classes of $\equiv$, topologized by taking the finest topology that makes the *quotient map* $q_\equiv : X \to X/\!\equiv$ continuous, where $q_\equiv$ maps $x \in X$ to its equivalence class.

Corollary 4.5 *Let X be a Noetherian space, $\equiv$ an equivalence relation on X. Then $X/\!\equiv$ is Noetherian.*

Let $\mathbf{Nthr}$ be the category of Noetherian spaces and continuous maps. In other words, $\mathbf{Nthr}$ is the full subcategory of $\mathbf{Top}$ (the category of topological spaces) consisting of Noetherian spaces.

Corollary 4.6 $\mathbf{Nthr}$ *is finitely cocomplete.*

Proof. It is enough to show that it has all finite coproducts (Lemma 4.3), and all coequalizers of parallel pairs $f, f' : X \to Y$. Such a coequalizer exists in $\mathbf{Top}$, and is given by $Y/\!\equiv$, where $\equiv$ is the smallest equivalence relation such that $f(x) \equiv f'(x)$ for all $x \in X$. Now apply Corollary 4.5. $\square$

5. Preliminaries II: Sober Spaces

The material we shall now need is more involved, and can be found in [12, 7, 21] and in [18].

Stone Duality. For every topological space X, $\Omega(X)$ is a complete lattice. Every continuous map $f : X \to Y$ defines a function $\Omega(f) : \Omega(Y) \to \Omega(X)$, which maps every open subset V of Y to $\Omega(f)(V) = f^{-1}(V)$. The map $\Omega(f)$ preserves all least upper bounds (unions) and all finite greatest lower bounds (finite intersections), i.e. it is a *frame homomorphism*. Letting $\mathbf{CLat}$ be the category of complete lattices and frame homomorphisms, Ω defines a functor from $\mathbf{Top}$ to $\mathbf{CLat}^{op}$, the opposite category of $\mathbf{CLat}$.

A *frame* is any complete lattice that obeys the infinite distributivity law $x \wedge \bigvee_{i \in I} x_i = \bigvee_{i \in I}(x \wedge x_i)$. Let $\mathbf{Frm}$ be the category of frames. Its opposite category $\mathbf{Loc} = \mathbf{Frm}^{op}$ is the category of *locales*.

Going the other way around is known as *Stone duality*. A *filter* F on a complete lattice L is a non-empty upward-closed family of elements of L, such that whenever $x, y \in F$, the greatest lower bound $x \wedge y$ is also in F. F is *completely prime* iff for every family $M \subseteq L$ whose least upper bound $\bigvee M$ is in F, then some element of M is already in F. A *point* of L is by definition a completely prime filter of L. Let $\mathsf{pt}(L)$ be the set of points of L. Topologize it by defining its opens as the sets $\mathcal{O}_x = \{F \in \mathsf{pt}(L) | x \in F\}$, $x \in L$. One may check that this is indeed a topology [7, Proposition 7.1.13]. Moreover, pt defines a functor from $\mathbf{CLat}^{op}$ to $\mathbf{Top}$, and by restriction, from $\mathbf{Loc}$ to $\mathbf{Top}$.

Then Ω is left adjoint to pt, in notation $\Omega \dashv \mathsf{pt}$. This means that there are natural transformations $\eta_X : X \to \mathsf{pt}(\Omega(X))$ (the *unit* of the adjunction) and $\epsilon_L : \Omega(\mathsf{pt}(L)) \to L$ (the *counit* of the adjunction) such that $\epsilon_{\Omega(X)} \circ \Omega(\eta_X) = \mathrm{id}_{\Omega(X)}$ and $\mathsf{pt}(\epsilon_L) \circ \eta_{\mathsf{pt}(L)} = \mathrm{id}_{\mathsf{pt}(L)}$. Explicitly, $\eta_X(x)$ is the completely prime filter of all open neighborhoods of x in X, and ϵ_L maps $z \in L$ to the open set $\mathcal{O}_z$.

Sober Spaces. The space $\mathsf{pt}(\Omega(X))$ is called the *sobrification* of X. One may understand this as noticing that (at least if X is a T_0 space, i.e., when its specialization quasi-ordering is a partial ordering) η_X is an embedding of X into $\mathsf{pt}(\Omega(X))$, so that $\mathsf{pt}(\Omega(X))$ is obtained from X by adding elements, viz. those points of $\Omega(X)$ that are not of the form $\eta_X(x)$, $x \in X$. The space $\mathsf{pt}(\Omega(X))$ is then *sober*: a sober space is a T_0 space in which every irreducible closed set is the closure of a unique point. A closed set C is *irreducible* iff it is not empty, and if there are two closed sets C_1 and C_2 such that $C \subseteq C_1 \cup C_2$ then $C \subseteq C_1$ or $C \subseteq C_2$.

One shows that η_X is injective iff X is T_0, and additionally surjective iff X is sober. Equivalently, X is sober iff X is homeomorphic to $\mathsf{pt}(L)$, for some complete lattice L, iff $X \cong \mathsf{pt}(\Omega(X))$, iff η_X is bijective (in which case it is automatically a homeomorphism).

The explicit description of $\mathsf{pt}(\Omega(X))$ of X is relatively uninteresting. We have already said that $\mathsf{pt}(\Omega(X))$ was a form of completion of X, where we add elements. A crucial point is that this completion adds elements *but no new opens*: then opens of $\mathsf{pt}(\Omega(X))$ are of the form $\mathcal{O}_U$, one for each open subset U of X. Alternatively, the specialization quasi-ordering $\leq$ of a sober space turns it into a cpo. I.e., the $\mathsf{pt} \circ \Omega$ construction formally adds all missing directed least upper bounds. One may for example check that the sobrification of $\mathbb{N}$ (with the Alexandroff topology of its natural ordering) is, up to homeomorphism, $\mathbb{N} \cup \{+\infty\}$ with non-empty open subsets $\uparrow n$, $n \in \mathbb{N}$. (Exercise: This is both the Scott and the upper topology on $\mathbb{N} \cup \{+\infty\}$.)

Up to homeomorphism, there is a simpler way of describing the sobrification of X [12, Chapter V, Exercise 4.9], as the space $\mathcal{S}(X)$ of all irreducible closed subsets of X, with open subsets given by $\Diamond U = \{F$ irreducible closed $| F \cap U \neq \emptyset\}$, for each open subset U of X. Its specialization quasi-ordering is just inclusion. Up to this homeomorphism, the unit η_X can be seen as a function from X to $\mathcal{S}(X)$ that maps $x \in X$ to the irreducible closed set $\downarrow x$.

Stably Compact Spaces. Sober spaces are *well-filtered* [18, Definition 2.7]: for every open subset U, for every filtered family $(Q_i)_{i \in I}$ of saturated compacts such that $\bigcap_{i \in I} Q_i \subseteq U$, there is an $i \in I$ such that $Q_i \subseteq U$. (A family is *filtered* provided it is directed in the converse ordering $\supseteq$.) This is a consequence of the celebrated Hofmann-Mislove Theorem [7, Theorem 7.2.9].

Say that a topological space X is *coherent* iff the intersection of two saturated compacts is compact. X is *locally compact* iff every element has a basis of saturated compact neighborhoods. That is, whenever $x \in U$ with U open, there is a saturated compact Q such that x is in the interior of Q, and $Q \subseteq U$. A *stably compact space* is a sober, coherent, locally compact and compact space.

Let X be a stably compact space. One may show that the complements of saturated compacts of X form a new topology, the so-called *cocompact topology*. Write X^d for X under its cocompact topology: this is the *de Groot dual* of X. Then X^d is again stably compact, and $X^{dd} = X$ [18, Corollary 2.13]. Moreover, the specialization quasi-ordering of X^d is the converse $\geq$ of $\leq$.

6. More Constructions of Noetherian Spaces

Let $\mathbf{CCCLat}$ be the full subcategory of $\mathbf{CLat}^{op}$ consisting of complete lattices (resp., $\mathbf{Loccc}$ the full subcategory of $\mathbf{Loc}$ consisting of frames) that satisfy the ascending chain condition. Proposition 3.2 states that Ω induces a functor from $\mathbf{Nthr}$ to $\mathbf{CCCLat}$, and to $\mathbf{Loccc}$.

Lemma 6.1 *The functor* pt *induces a functor from* $\mathbf{CCCLat}$ *(resp.,* $\mathbf{Loccc}$*) to* $\mathbf{Nthr}$*, right adjoint to* Ω*.*

Proof. Let us show that pt is a functor from $\mathbf{CCCLat}$, resp. $\mathbf{Loccc}$, to $\mathbf{Nthr}$. This boils down to the fact that for every complete lattice L with the ascending chain condition, $\mathsf{pt}(L)$ is Noetherian. By Proposition 3.2, it suffices to show that every ascending chain $\mathcal{O}_{x_1} \subseteq \mathcal{O}_{x_2} \subseteq \ldots \subseteq \mathcal{O}_{x_k} \subseteq \ldots$ stabilizes. Note that $\mathcal{O}_x \subseteq \mathcal{O}_y$ iff $x \leq y$: the if direction is clear; conversely, if $\mathcal{O}_x \subseteq \mathcal{O}_y$, then the filter $\uparrow x$ is completely prime, belongs to $\mathcal{O}_x$, so it belongs to $\mathcal{O}_y$, i.e., $y \in \uparrow x$, that is, $x \leq y$. It follows that $x_1 \leq x_2 \leq \ldots \leq x_k \leq \ldots$ is an ascending chain in L. So it stabilizes. $\Box$

Sobrification preserves the property of being Noetherian:

Proposition 6.2 *A space X is Noetherian iff its sobrification* $\mathsf{pt}(\Omega(X)) \cong \mathcal{S}(X)$ *is Noetherian.*

Proof. If X is Noetherian, then so is $\mathsf{pt}(\Omega(X))$, because Ω is a functor from $\mathbf{Nthr}$ to $\mathbf{Loccc}$, and pt is one from $\mathbf{Loccc}$ to $\mathbf{Nthr}$ (Lemma 6.1). So $\mathcal{S}(X) \cong \mathsf{pt}(\Omega(X))$ is Noetherian, too. Conversely, assume $\mathcal{S}(X)$ is Noetherian, and let $U_1 \subseteq U_2 \subseteq \ldots \subseteq U_k \subseteq \ldots$ be an infinite ascending chain in X. Then $\Diamond U_1 \subseteq \Diamond U_2 \subseteq \ldots \subseteq \Diamond U_k \subseteq \ldots$ is an infinite ascending chain in $\mathcal{S}(X)$, so it stabilizes: for some $N \in \mathbb{N}$, for every $k \geq N$, $\Diamond U_k \subseteq \Diamond U_N$. For every $x \in U_k$, $\downarrow x$ is in $\Diamond U_k$, so it is in $\Diamond U_N$, therefore $x \in U_N$. So $U_k \subseteq U_N$, showing that the ascending chain $U_1 \subseteq U_2 \subseteq \ldots \subseteq U_k \subseteq \ldots$ also stabilizes. So X is Noetherian. $\square$

6.1. Noetherian Sober Spaces

Proposition 6.2 is crucial to our study. For the moment, it at least motivates a deeper study of those spaces that are both Noetherian and sober.

Proposition 6.3 *Every Noetherian sober space X is stably compact. Moreover, the upward-closed subsets of X coincide with its saturated compacts.*

Proof. X is trivially locally compact. Since X itself is open and Noetherian, X is compact. X is sober by assumption. It remains to show that X is coherent. This will be a trivial consequence of the second part of the proposition, since every intersection of upward-closed subsets is again upward-closed. Let therefore A be upward-closed in X. So A is saturated, i.e., A is the filtered intersection of the family $(U_i)_{i \in I}$ of all opens containing A. Since X is Noetherian, this is a family of saturated compacts. Since X is well-filtered, its intersection A is again saturated compact. $\square$

Corollary 6.4 *Let X be sober and Noetherian. Then the cocompact topology on X is the Alexandroff topology of $\geq$.*

In particular, the topology of X is entirely determined by its specialization quasi-ordering.

Corollary 6.5 *Let X be sober and Noetherian. The topology of X is the upper topology of its specialization quasi-ordering $\leq$. Moreover, every closed subset of X is finitary.*

Proof. The closed subsets of X, that is of X^{dd}, are the saturated compacts of X^d. By Corollary 6.4, the topology of X^d is the Alexandroff topology of $\geq$, so its saturated compacts are its finitary compacts. These are exactly the sets of the form $\downarrow E$, E finite. In other words, the closed subsets of X are exactly its finitary closed subsets.

Since all finitary closed subsets are closed in the upper topology, the topology of X is coarser than the upper topology. But the latter is the coarsest having $\leq$ as specialization quasi-ordering. So the two topologies coincide. $\square$

Recall that the sobrification of $\mathbb{N}$ is $\mathbb{N} \cup \{+\infty\}$, with opens $\uparrow n$, $n \in \mathbb{N}$. We have already noticed that this was the upper topology. Corollary 6.5 shows that this is no accident.

That X is *both* Noetherian and sober is essential. Note also that X itself is closed. Corollary 6.5 then implies that $X = \downarrow E$ for some finite E; that is, X has *property T*:

Definition 6.6 *The quasi-ordered set X has* property T *iff there is a finite subset E of X such that every element of X is less than or equal to some element of E.*

When X is Noetherian and sober, Corollary 6.5 also implies that for every $x, y \in X$, $\downarrow x \cap \downarrow y$, which is closed, is of the form $\downarrow E$, E finite. This is equivalent to the following property, a dual of Jung's property M [17, Definition, p.38]:

Definition 6.7 *The quasi-ordered set X has* property W *iff, for every $x, y \in X$, there is a finite subset E of maximal lower bounds of x and y, such that every lower bound of x and y is less than or equal to some element of E.*

Lemma 6.8 *Let X be a Noetherian space, $\leq$ its specialization quasi-ordering, $\geq$ its converse, and $\equiv$ be $\leq \cap \geq$. Then $\leq$ is well-founded: every infinite descending chain $\ldots \leq x_k \leq \ldots \leq x_2 \leq x_1$ stabilizes up to $\equiv$, i.e., there is an integer N such that $x_k \equiv x_N$ for every $k \geq N$.*

We prove the converse in Proposition 6.9 below. To this end, we need to define the *Hoare quasi-ordering* $\leq^\flat$ on the subsets of a set X quasi-ordered by $\leq$: $E \leq^\flat E'$ iff for every $x \in E$, there is an $x' \in E'$ such that $x \leq x'$. Equivalently, iff $\downarrow E \subseteq \downarrow E'$. Equating every finite subset with the obvious finite multiset, $\leq^\flat$ coincides with the multiset extension $\leq^{mul}$. It is well-known that (the strict part of) $\leq^{mul}$ if well-founded as soon as $\leq$ is.

Proposition 6.9 *Let $\leq$ be a quasi-ordering on X. If $\leq$ is well-founded and has property W, then X is Noetherian in its upper topology. Its closed subsets, except possibly X, are finitary.*

Proof. First, we show that: $(*)$ for every descending family $(\downarrow E_n)_{n \in \mathbb{N}}$, where each E_n is a finite subset of X, there is $k \in \mathbb{N}$ such that $\bigcap_{n \in N} \downarrow E_n = \downarrow E_k$. Note that, since $\downarrow E_{n+1} \subseteq \downarrow E_n$, for every $x \in E_{n+1}$, there is a $y \in E_n$ such that $x \leq y$, that is, $E_{n+1} \leq^\flat E_n$. Then $(*)$ follows since $\leq^\flat = \leq^{mul}$ is well-founded.

We obtain: $(**)$ for every filtered family $(\downarrow E_i)_{i \in I}$, where each E_i is finite, there is a finite subset $E \subseteq X$ such that $\bigcap_{i \in I} \downarrow E_i = \downarrow E$. Indeed, assume the contrary. We then build a descending sequence $\downarrow E'_n$, $n \in \mathbb{N}$, where each E'_n is some E_i, by induction on $n \in \mathbb{N}$. Let E'_0 be any E_i. Assuming E'_n has been built, for some $i \in I$ we must have $\downarrow E'_n \not\subseteq \downarrow E_i$, otherwise $\bigcap_{i \in I} \downarrow E_i = \downarrow E'_n$. Since $(\downarrow E_i)_{i \in I}$ is filtered, for some $j \in I$, $\downarrow E_j$ is contained in

$\downarrow E'_n$ and in $\downarrow E_i$: let $E'_{n+1} = E_j$. By construction, the chain $(\downarrow E'_n)_{n\in\mathbb{N}}$ is strictly decreasing, contradicting $(*)$.

The closed subsets in the upper topology are the (arbitrary) intersections of subsets of the form $\downarrow E_i$, $i \in I$, E_i finite. The empty intersection is X. Each non-empty intersection $\bigcap_{i\in I} A_i$ can be written as a filtered intersection of non-empty finite intersections: $\bigcap_{i\in I} A_i = \bigcap_{J\subseteq I, J\neq\emptyset \text{ finite}} \bigcap_{i\in J} A_i$. By property W, every non-empty finite intersection $\bigcap_{i\in J} \downarrow E_i$ is of the form $\downarrow E_J$ for some finite subset E_J. By $(**)$, every filtered intersection of subsets of the form $\downarrow E_J$ is again of the form $\downarrow E$, E finite.

So the closed subsets of the upper topology of X are exactly those of the form $\downarrow E$, E finite, plus the whole of X. Taking complements in $(*)$, every infinite ascending chain of opens stabilizes: by Proposition 3.2, X is Noetherian. $\square$

Lemma 6.10 *Let $\leq$ be a quasi-ordering on X. If $\leq$ is well-founded and has property W, then the irreducible closed subsets F of X are of the form $\downarrow x$, $x \in X$, plus possibly X itself. If X additionally has property T, then the only irreducible closed sets are of the first kind.*

This finally allows us to characterize the Noetherian sober spaces in terms of their specialization quasi-ordering:

Theorem 6.11 *The Noetherian sober spaces are exactly the spaces whose topology is the upper topology of a well-founded partial order that has properties W and T.*

When X is Alexandroff-discrete, $\mathcal{S}(X)$ is isomorphic to the ideal completion of X, with its Scott topology [20]. This shows first that, when X is well-quasi ordered, and equipped with its Alexandroff topology, then the upper topology on $\mathcal{S}(X)$ is just the familiar Scott topology. Second, this gives a more concrete description of $\mathcal{S}(X)$ in this case: the elements of $\mathcal{S}(X)$, i.e., the irreducible closed subsets F, are exactly the down-closed directed subsets of X.

6.2. Cartesian Products

The *product topology* on $X_1 \times X_2$ is the coarsest that makes the projections $\pi_i : X_1 \times X_2 \to X_i$ $(i = 1, 2)$ continuous. The *open rectangles* $U_1 \times U_2$, U_1 open in X_1, U_2 open in X_2, form a basis of this topology. Theorem 6.11 makes the following almost immediate.

Lemma 6.12 *The product $X_1 \times X_2$ of two Noetherian sober spaces is Noetherian and sober.*

Theorem 6.13 *The product $X_1 \times \ldots \times X_n$ of n Noetherian spaces $X_1, \ldots, X_n$ is Noetherian. Its opens are all* finite *unions of open rectangles $U_1 \times \ldots \times U_n$ ($U_i \in \Omega(X_i)$).*

Proof. By induction on n. The essential case is $n = 2$. First, note that for every open U of a space X, $\eta_X(x) \in \mathcal{O}_U$ iff $x \in U$. In particular: $(*)$ $\eta_X^{-1}(\mathcal{O}_U) = U$.

Consider the continuous map $i = \eta_{X_1} \times \eta_{X_2} : X_1 \times X_2 \to \mathsf{pt}(\Omega(X_1)) \times \mathsf{pt}(\Omega(X_2))$. Let U any open subset of $X_1 \times X_2$. Write U as $\bigcup_{i\in I} U_i^1 \times U_i^2$, where the U_i^1's are open in X_1 and the U_i^2's are open in X_2. By $(*)$, $U = \bigcup_{i\in I} \eta_{X_1}^{-1}(\mathcal{O}_{U_i^1}) \times \eta_{X_2}^{-1}(\mathcal{O}_{U_i^2}) = i^{-1}(\bigcup_{i\in I} \mathcal{O}_{U_i^1} \times \mathcal{O}_{U_i^2})$. Note that $\bigcup_{i\in I} \mathcal{O}_{U_i^1} \times \mathcal{O}_{U_i^2}$ is open in $\mathsf{pt}(\Omega(X_1)) \times \mathsf{pt}(\Omega(X_2))$. By Proposition 6.2, $\mathsf{pt}(\Omega(X_1))$ and $\mathsf{pt}(\Omega(X_2))$ are Noetherian. They are sober by construction. So by Lemma 6.12, $\mathsf{pt}(\Omega(X_1)) \times \mathsf{pt}(\Omega(X_2))$ is Noetherian, hence $\bigcup_{i\in I} \mathcal{O}_{U_i^1} \times \mathcal{O}_{U_i^2}$ is compact in $\mathsf{pt}(\Omega(X_1)) \times \mathsf{pt}(\Omega(X_2))$. The family $(\mathcal{O}_{U_i^1} \times \mathcal{O}_{U_i^2})_{i\in I}$ is an open cover of it. So there is a finite subset I_0 of I such that $\bigcup_{i\in I} \mathcal{O}_{U_i^1} \times \mathcal{O}_{U_i^2} = \bigcup_{i\in I_0} \mathcal{O}_{U_i^1} \times \mathcal{O}_{U_i^2}$. Then $U = i^{-1}(\bigcup_{i\in I_0} \mathcal{O}_{U_i^1} \times \mathcal{O}_{U_i^2}) = \bigcup_{i\in I_0} U_i^1 \times U_i^2$ is a finite union of open rectangles.

Since X_1 is Noetherian, U_i^1 is compact, and similarly for U_i^2, so $U_i^1 \times U_i^2$ is compact in $X_1 \times X_2$ by the finite case of Tychonoff's Theorem. It follows that U, qua finite union of compacts, is also compact. So $X_1 \times X_2$ is Noetherian. $\square$

Corollary 6.14 $\mathbf{Nthr}$ *is finitely complete.*

Proof. By Theorem 6.13, it has all finite products. We need only verify that it has all equalizers of parallel pairs $f, f' : X \to Y$. Their equalizer in $\mathbf{Top}$ is the subspace $Z = \{x \in X | f(x) = f'(x)\}$ of X. Z is Noetherian by Lemma 4.2, and clearly is an equalizer in $\mathbf{Nthr}$. $\square$

However, $\mathbf{Nthr}$ is not cartesian-closed. As a full subcategory of $\mathbf{Top}$, the exponential Y^X, if it exists in $\mathbf{Nthr}$, must be the space of continuous functions $[X \to Y]$ from X to Y. Take $X = \mathbb{N}$ and $Y = \{0, 1\}$ with the Alexandroff topologies of their natural orderings. Since application $app : Y^X \times X \to Y$ is continuous in its first argument, $U_i = \{f \in Y^X | f(i) = 1\}$ must be open in Y^X for any $i \in \mathbb{N}$. However, the chain $U_1 \subseteq U_2 \subseteq \ldots \subseteq U_i \subseteq \ldots$ is infinite. We may complete $\mathbf{Nthr}$ to a cartesian-closed category $\mathbf{Top}_{\mathbf{Nthr}}$ by standard constructions: by Day's Theorem [10, Theorem 3.6], the category $\mathbf{Top}_{\mathcal{C}}$ of so-called $\mathcal{C}$-generated spaces is cartesian-closed as soon as $\mathcal{C}$ is productive. The latter means that every space in $\mathcal{C}$ is exponentiable and the product in $\mathbf{Top}$ of two spaces in $\mathcal{C}$ is $\mathcal{C}$-generated. Taking $\mathcal{C} = \mathbf{Nthr}$ fits: first, every Noetherian space is locally compact, hence exponentiable; then the product in $\mathbf{Top}$ of two Noetherian spaces is Noetherian (Theorem 6.13), hence $\mathbf{Nthr}$-generated [10, Lemma 3.2 (i)]. By [10, Lemma 3.2 (v)], the $\mathbf{Nthr}$-generated spaces are exactly the (possibly infinite) colimits of Noetherian spaces.

7. A Noetherian Topology on $\mathbb{P}(X)$

Let us deal with the so-called Hoare powerdomain construction first. For each topological space X, let its *Hoare space* $\mathcal{H}(X)$ (resp., $\mathcal{H}_\emptyset(X)$) be the space of all non-empty closed subsets (resp., all closed subsets) of X with the upper topology of the $\subseteq$ ordering. It has subbasic open sets $\Diamond U = \{F \in \mathcal{H}(X) | F \cap U \neq \emptyset\}$, U open in X.

$\mathcal{H}(X)$ is used in denotational semantics to model angelic non-determinism. Note that the closure of an element $F \in \mathcal{H}(X)$ is $\Box F = \mathcal{H}(X) \setminus \Diamond(\overline{F}) = \{F' \in \mathcal{H}(X) | F' \subseteq F\}$, and similarly in $\mathcal{H}_\emptyset(X)$. On finitary closed sets, $\downarrow E \subseteq \downarrow E'$ iff $E \leq^\flat E'$. The following is then immediate.

Proposition 7.1 *For any Noetherian sober space X, $\mathcal{H}(X)$ and $\mathcal{H}_\emptyset(X)$ are Noetherian and sober.*

Theorem 7.2 *For any Noetherian space X, $\mathcal{H}(X)$ and $\mathcal{H}_\emptyset(X)$ are Noetherian.*

Proof. Call *basic* open set any finite intersection of subbasic opens $\Diamond U$. Every open $\mathcal{U}$ of $\mathcal{H}(X)$ is the union of the basic opens $\mathcal{V}$ contained in $\mathcal{U}$. Fix a way of writing each basic open $\mathcal{V}_i$, say $\mathcal{V}_i = \bigcap_{j \in J_i} \Diamond V_{ij}$, where J_i is finite. Let $\widehat{\mathcal{V}_i} = \bigcap_{j \in J_i} \Diamond \mathcal{O}_{V_{ij}}$, and finally $\widehat{\mathcal{U}}$ be the union of all $\widehat{\mathcal{V}_i}$, $\mathcal{V}_i$ basic open contained in $\mathcal{U}$. By construction, if $\mathcal{U} \subseteq \mathcal{U}'$, then $\widehat{\mathcal{U}} \subseteq \widehat{\mathcal{U}'}$. For every ascending chain $\mathcal{U}_1 \subseteq \mathcal{U}_2 \subseteq \ldots \subseteq \mathcal{U}_k \subseteq \ldots$ in $\mathcal{H}(X)$, $\widehat{\mathcal{U}_1} \subseteq \widehat{\mathcal{U}_2} \subseteq \ldots \subseteq \widehat{\mathcal{U}_k} \subseteq \ldots$ is an ascending chain in $\mathcal{H}(\mathsf{pt}(\Omega(X)))$. So it stabilizes, using Proposition 6.2, Proposition 7.1, and Proposition 3.2.

Recall that $\eta_X^{-1}(\mathcal{O}_U) = U$ for every open U of X. So $\Diamond V_{ij} = \{F \in \mathcal{H}(X) | F \cap \eta_X^{-1}(\mathcal{O}_{V_{ij}}) \neq \emptyset\} = \mathcal{H}(\eta_X)^{-1}(\Diamond \mathcal{O}_{V_{ij}})$, where $\mathcal{H}(\eta_X)$ maps $F \in \mathcal{H}(X)$ to $cl(\eta_X(F))$. Indeed, $\mathcal{H}(\eta_X)^{-1}(\Diamond \mathcal{O}_{V_{ij}}) = \{F \in \mathcal{H}(X) | cl(\eta_X(F)) \cap \mathcal{O}_{V_{ij}} \neq \emptyset\} = \{F \in \mathcal{H}(X) | \eta_X(F) \cap \mathcal{O}_{V_{ij}} \neq \emptyset\} = \{F \in \mathcal{H}(X) | F \cap \eta_X^{-1}(\mathcal{O}_{V_{ij}}) \neq \emptyset\}$. So $\mathcal{U} = \mathcal{H}(\eta_X)^{-1}(\widehat{\mathcal{U}})$ for every open subset $\mathcal{U}$ of $\mathcal{H}(X)$. In particular, the map $\mathcal{U} \mapsto \widehat{\mathcal{U}}$ is injective.

Since $\widehat{\mathcal{U}_1} \subseteq \widehat{\mathcal{U}_2} \subseteq \ldots \subseteq \widehat{\mathcal{U}_k} \subseteq \ldots$ stabilizes, $\mathcal{U}_1 \subseteq \mathcal{U}_2 \subseteq \ldots \subseteq \mathcal{U}_k \subseteq \ldots$ stabilizes, too. We conclude by Theorem 3.2. The argument is similar for $\mathcal{H}_\emptyset(X)$. ☐

This has the following surprising consequence:

Proposition 7.3 *Let X be a topological space, with specialization quasi-ordering $\leq$. Let $\mathbb{P}(X)$ be the set of all subsets (resp. $\mathbb{P}^*(X)$ of all non-empty subsets) of X, quasi-ordered by the* topological Hoare quasi-ordering $\leq^{\flat *}$, *defined as: $A \leq^{\flat *} B$ iff $cl(A) \subseteq cl(B)$. Equip $\mathbb{P}(X)$, resp. $\mathbb{P}^*(X)$, with the corresponding upper topology.*

If X is Noetherian, then so are $\mathbb{P}(X)$ and $\mathbb{P}^*(X)$.

Proof. Let $\equiv^{\flat *}$ be the equivalence relation induced by $\leq^{\flat *}$, and q the quotient map. Up to homeomorphism, $\mathbb{P}(X)/\equiv^{\flat *}$ is exactly $\mathcal{H}(X)$, and q maps each subset A to $cl(A)$. Note that q is continuous: the inverse image $q^{-1}(\Diamond U)$ is the set of all subsets A such that $cl(A) \cap U \neq \emptyset$, equivalently $cl(A) \subseteq \overline{U}$, equivalently $A \leq^{\flat *} \overline{U}$, since $\overline{U}$ is closed. So $q^{-1}(\Diamond O)$ is open in the upper topology.

We have actually just shown that $\Omega(q) : \Omega(\mathcal{H}(X)) \to \Omega(\mathbb{P}(X))$ maps $\Diamond O$ to $\downarrow^{\flat *} \overline{O}$. Recall that $\Omega(q)$ is a frame homomorphism, and is therefore entirely determined by this property. This is clearly a bijection, whose inverse is the unique frame homomorphism mapping $\downarrow^{\flat *} F$ to $\Diamond \overline{F}$, for each closed subset F of X. Every ascending chain of opens $O_1 \subseteq O_2 \subseteq \ldots \subseteq O_k \subseteq \ldots$ of $\mathbb{P}(X)$ then induces an ascending chain of opens of $\mathcal{H}(X)$ through $\Omega(q)^{-1}$. By Theorem 7.2 and Proposition 3.2, the latter stabilizes. So the former stabilizes, too. Hence $\mathbb{P}(X)$ is Noetherian. ☐

Corollary 7.4 *Let $\leq$ be a well quasi-ordering on X. $\mathbb{P}(X)$ and $\mathbb{P}^*(X)$, with the upper topology of $\leq^\flat$, are Noetherian.*

This is remarkable: in general $\leq^\flat$ is *not* a well quasi-ordering on $\mathbb{P}(X)$. The standard counterexample is Rado's example [24]. Let X_{Rado} be the set $\{(m, n) \in \mathbb{N}^2 | m < n\}$, ordered by $\leq_{Rado}$: $(m, n) \leq_{Rado} (m', n')$ iff $m = m'$ and $n \leq n'$, or $n < m'$. It is well-known that $\leq_{Rado}$ is a well quasi-ordering. However, $\mathcal{H}(X_{Rado}) \cong \mathbb{P}(X_{Rado})$ is not well quasi-ordered by $\leq^\flat_{Rado}$ [5, Example 3.2].

A trivial consequence of this is that $\mathcal{H}(X)$ and $\mathcal{H}_\emptyset(X)$ are in general not Alexandroff-discrete, even when X is. A more important observation is that choosing the right topology (here, the upper topology) matters.

The *Smyth space* $\mathcal{Q}(X)$ usually models demonic non-determinism; this is the set of non-empty compact saturated subsets of X, ordered by reverse inclusion $\supseteq$, and equipped with the corresponding Scott topology. The latter is generated by basic opens $\Box U = \{Q \in \mathcal{Q}(X) | Q \subseteq U\}$, U open in X, as soon as X is well-filtered and locally compact. Contrarily to $\mathcal{H}(X)$, $\mathcal{Q}(X)$ is in general not Noetherian: in $X = \mathcal{H}(X_{Rado})$, which is not well quasi-ordered, there is an infinite sequence of elements $F_1, F_2, \ldots$ such that $F_i \subseteq F_j$ for no $i < j$; then $\uparrow F_1 \subseteq \uparrow \{F_1, F_2\} \subseteq \ldots \subseteq \uparrow \{F_1, \ldots, F_k\} \subseteq \ldots$ is an infinite $\supseteq$-descending sequence of elements of $\mathcal{Q}(\mathcal{H}(X_{Rado}))$. This would be impossible if the latter were Noetherian, by Lemma 6.8.

But consider the smaller set $\mathcal{O}(X)$ of opens (remember that every open is compact), equipped with the upper topology of $\supseteq$. Clearly, $\mathcal{O}(X) \cong \mathcal{H}_\emptyset(X)$, by the homeomorphism sending an open set to its complement. So by Corollary 7.4, $\mathcal{O}(X)$ is Noetherian. Now note that if X is Alexandroff-discrete, then opens coincide with finitary compacts $\uparrow E$. Equip $\mathbb{P}_{fin}(X)$ with the upper topology of the Smyth quasi-ordering $\leq^\sharp$. It is easy to check that, when X is equipped with the Alexandroff topology of a well quasi-ordering $\leq$, the map $\uparrow$ that sends $E \in \mathbb{P}_{fin}(X)$ to $\uparrow E \in \mathcal{O}(X)$ is a homeomorphism. Using this, the fact that $\mathcal{O}(X) \cong \mathcal{H}_\emptyset(X)$, and Theorem 7.2, we get:

Proposition 7.5 *Let $\leq$ be a well quasi-ordering on X, and equip X with its Alexandroff topology. Then $\mathbb{P}_{fin}(X)$, with the upper topology of $\leq^\sharp$, is Noetherian.*

Again, this contrasts with the theory of well-quasi orderings. Rado's example shows that $\leq^\sharp$ is in general not a well quasi-ordering on $\mathbb{P}_{fin}(X)$. It is when $\leq$ is ω^2-wqo [16].

8. Ring Spectra, and the Zariski Topology

Let R be a commutative ring. The *spectrum* $\mathrm{Spec}(R)$ of R is the set of all prime ideals of R. It is equipped with the *Zariski topology*, whose closed subsets are $F_I = \{p \in \mathrm{Spec}(R) | I \subseteq p\}$, where I ranges over the ideals of R. When R is the ring of polynomials $\overline{K}[X_1, \ldots, X_k]$ over an algebraically closed field $\overline{K}$ (e.g., $\mathbb{C}$), there is a canonical one-to-one mapping from $\overline{K}^k$ to $\mathrm{Spec}(R)$, which equates the point $(v_1, \ldots, v_k)$ of $\overline{K}^k$ with the prime ideal $\{P \in R | P(v_1, \ldots, v_k) = 0\}$. In this case, elements of $\mathrm{Spec}(R)$ are points of the space $\overline{K}^k$. In general, it is useful to think of $\mathrm{Spec}(R)$ as a general notion of space of points.

A ring R is *Noetherian* iff every $\subseteq$-increasing sequence of ideals in R is stationary; e.g., $K[X_1, \ldots, X_k]$ is Noetherian for any field K. It is well-known that, for any Noetherian ring R, $\mathrm{Spec}(R)$ is a Noetherian topological space [13, corollaire 1.1.6]. $\mathrm{Spec}(R)$ is always sober [13, corollaire 1.1.4, (ii)], with $\supseteq$ as specialization ordering. By [13, proposition 1.1.10, (i)], the sets $\mathrm{Spec}(R) \setminus \downarrow (x)$ form a basis of the Zariski topology, where (x) is the (prime) ideal generated by $x \in R$, so that $\downarrow (x) = \{p | p \supseteq (x)\} = \{p | x \in p\}$. In particular, the Zariski topology coincides with the upper topology, even when R is not Noetherian.

From a computer science perspective, the case $R = \overline{K}[X_1, \ldots, X_k]$ is probably the most interesting, with $\overline{K}$ an algebraically closed extension of the field K (e.g., $K = \mathbb{Q}$, $\overline{K} = \mathbb{C}$). Recall that $\mathrm{Spec}(R)$ is Noetherian in this case. The elements of $\mathrm{Spec}(R)$ can be equated with elements of $\overline{K}^k$. The closed subsets F_I can be represented by providing a Gröbner basis for the polynomial ideal I [8, Section 11]. This Gröbner basis is not unique: for any given I, it depends on the choice of a so-called admissible ordering of monomials, and I itself is not determined uniquely; however F_I only depends on the radical ideal $\sqrt{I} = \{P \in R | \exists k \geq 1 \cdot P^k \in I\}$ of I. Note that radical ideals, hence also the closed subsets F_I, are in one-to-one correspondence with affine varieties, i.e., with sets of common zeroes of polynomials over $\overline{K}$: this is Hilbert's Nullstellensatz [25].

Now consider *polynomial automata* over K, i.e., pairs $\mathcal{A} = (Q, \delta)$, where Q is a finite set of *states*, $R = K[X_1, \ldots, X_k]$, and $\delta \subseteq Q \times \mathbb{P}_{fin}(R) \times R^k \times Q$ is the transition relation. The intent is to model programs over k variables taking values in $\overline{K}$, and with polynomial operations. The semantics of $\mathcal{A}$ is an infinite state transition system, whose *configurations* are tuples $(q, v_1, \ldots, v_k) \in Q \times \overline{K}^k$, and valid transitions are of the form $(q, v_1, \ldots, v_k) \to (q', P_1(v_1, \ldots, v_k), \ldots, P_k(v_1, \ldots, v_k))$, where $(q, E, (P_1, \ldots, P_k), q') \in \delta$ and $P(v_1, \ldots, v_k) = 0$ for every $P \in E$. The E part models guards, testing equality between polynomial expressions, and the $(P_1, \ldots, P_k)$ part models variable updates. E.g., the program fragment `if` $3X_1^2 = X_2 - 1 \wedge X_1X_2 = X_3$ `then` $(X_1 := X_1^2 - 2X_3 + X_2; X_2 := X_3 + 1)$, in ML-like syntax, would be described with $E = \{3X_1^2 - X_2 + 1, X_1X_2 - X_3\}$, $P_1 = X_1^2 - 2X_3 + X_2$, $P_2 = X_3 + 1$, $P_3 = X_3$. Equip $\overline{K}^k$ with the Zariski topology, obtained through the one-to-one correspondence between $\overline{K}^k$ and $\mathrm{Spec}(R)$. It is easy to see that the binary relation $\to$ is upper semi-continuous. This is because $\mathrm{Pre}^\exists(\to)(F_I) = F_{I'}$, where $I' = \bigcup_{(q,E,(P_1,\ldots,P_k),q') \in \delta} (E) \cap \{P \circ (P_1, \ldots, P_k) | P \in I\}$, and (E) is the ideal generated by E. This is easily computed using Gröbner bases when $K = \mathbb{Q}$. This was explored by Müller-Olm and Seidl in a very nice paper [22], and we refer the reader to this for missing details and a gentler introduction. We leave it as future research to explore the application of other Noetherian rings to computer verification problems; also, the use of more complex Noetherian spaces such as $\mathrm{Spec}(\overline{K}[X_1, \ldots, X_k]) \times \mathbb{N}^p$, modeling programs with k variables in $\overline{K}$ and p integer variables.

9. A New Data Structure for Coverability?

Consider the following argument. Start from a Noetherian space X, e.g., $\mathbb{N}^k$ with its Alexandroff topology (this is the space of markings of a Petri net). By Proposition 6.2, $\mathcal{S}(X)$ is Noetherian. By Corollary 6.5, its opens are exactly the finitary closed subsets $\downarrow E$, $E \subseteq \mathcal{S}(X)$. We equate X with a subspace of $\mathcal{S}(X)$, i.e., we equate $x \in X$ with $\eta_X(x) \in \mathcal{S}(X)$. For instance, $\mathcal{S}(\mathbb{N}^k) = (\mathbb{N} \cup \{+\infty\})^k$, as we have seen. Now the topology of X is exactly the topology induced on X by that of $\mathcal{S}(X)$, so we have a way of representing all opens of X using a finite set E of elements of $\mathcal{S}(X)$, as the complement in X of $X \cap \downarrow E$. This is clear on $\mathbb{N}^k$: for example, with $k = 3$, we may represent the upward-closed set (open in $\mathbb{N}^3$) $\uparrow \{(2, 3, 5)\}$ as the complement of $X \cap \downarrow \{(1, +\infty, +\infty), (+\infty, 2, +\infty), (+\infty, +\infty, 4)\}$.

This is completely general: we can *always* represent opens of Noetherian spaces X as complements of sets of the form $X \cap \downarrow E$, E a finite subset of $\mathcal{S}(X)$. This is important for those Noetherian spaces, e.g., $\mathbb{P}(X)$, that do not arise from well quasi-orderings, and where opens cannot be represented as $\uparrow E$ (E finite $\subseteq X$). Even on well quasi-ordered spaces such as $\mathbb{N}^k$, this may provide an alternate representation of sets of the form $\mathrm{Pre}^{\exists *}\delta(V)$ or $I\,[\![F]\!]_\delta\,\rho$. E.g., on $\mathbb{N}^k$, when δ is the transition relation of a Petri net (taken as a finite set of rewrite rules $\vec{m}_i \to \vec{n}_i$, $1 \leq i \leq \ell$, of vectors in $\mathbb{N}^k$, where $\vec{m}_i, \vec{n}_i \in \mathbb{N}^k$), we may compute $\mathrm{Pre}^\exists\delta(\overline{X \cap \downarrow E}) = \overline{\mathrm{Pre}^\forall\delta(X \cap \downarrow E)}$ and

$\mathrm{Pre}^{\forall}\delta(\overline{X \cap \downarrow E}) = \overline{\mathrm{Pre}^{\exists}\delta(X \cap \downarrow E)}$ by the formulae:

$$\begin{aligned}\mathrm{Pre}^{\exists}\delta(X \cap \downarrow E) &= X \cap \downarrow \{\vec{p} + \vec{m}_i - \vec{n}_i \\ &\qquad \mid \vec{p} \in E, 1 \leq i \leq \ell \cdot \vec{p} \geq \vec{n}_i\} \\ \mathrm{Pre}^{\forall}\delta(X \cap \downarrow E) &= X \cap \bigcap_{i=1}^{\ell} \Big(\complement\uparrow \vec{m}_i \cup \downarrow \{\vec{p} + \vec{m}_i - \vec{n}_i \\ &\qquad \mid \vec{p} \in E \cdot \vec{p} \geq \vec{n}_i\}\Big)\end{aligned}$$

(We leave the computation of finite unions and intersections of sets of the form $\downarrow E$, and of the complement $\complement\uparrow \vec{m}_i$ of $\uparrow \vec{m}_i$ in $\mathcal{S}(X)$, as an exercise to the reader.)

10. Conclusion

We have laid down the first steps towards a theory of Noetherian spaces as generalized well quasi-orderings. Noetherian spaces enjoy many nice properties. Every finite product, equalizer, subspace, finite coproduct, coequalizer, quotient, retract of Noetherian topological spaces is again Noetherian. We have also characterized those Noetherian topologies that are sober, as the upper topologies of well-founded quasi-orderings with properties W and T. We have shown that a space is Noetherian iff its sobrification is Noetherian. The Hoare space of a Noetherian space is Noetherian, which implies the surprising property that the set $\mathbb{P}(X)$ of all subsets of a Noetherian space X, even infinite ones, under the upper topology of the Hoare quasi-ordering $\leq^\flat$, is Noetherian, although $\leq^\flat$ is not in general a well quasi-ordering. (Similarly with $\mathbb{P}_{fin}(X)$ and $\leq^\sharp$.) Our Noetherian space approach to model-checking negation-free μ-calculus formulae, finally, allows one to verify infinite transition systems that are more general than well-structured transition systems, including e.g. polynomially definable transitions, as we have argued using Zariski topologies on ring spectra.

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