Affine parts of monads

By

Harald Lindner

We introduce the notion of the affine part of a monad on a finitely complete category. This generalizes known constructions for algebraic theories. The inclusion-functor of the subcategory of affine monads is coadjoint. The affine part of a monad is characterized in terms of idempotent operations. The affine part of a (cartesian-) monoidal monad is monoidal, hence the corresponding Eilenberg-Moore situation is monoidal, too.

Let $\mathscr{V}$ be a finitely complete category. We choose a terminal object I of $\mathscr{V}$. For any $X \in |\mathscr{V}|$ we denote by $!_X$ the unique morphism from X to I.

1. Definition and proposition. *Let $\mathbb{T} = (T, \eta, \mu)$ be a monad (triple, triad, standard construction) on $\mathscr{V}$. There is a monad $\mathbb{T}' = (T', \eta', \mu')$ on $\mathscr{V}$, called the affine part of $\mathbb{T}$, together with a monad morphism $\tau\colon \mathbb{T}' \to \mathbb{T}$, determined both uniquely up to isomorphism, such that for every $X \in |\mathscr{V}|$ the diagram* (1) *is an equalizer diagram.*

(1)
$$\begin{tikzcd} T'X \arrow[r, "\tau_X"] & TX \arrow[rr, "T(!_X)"] \arrow[dr, "!_{TX}"'] & & TI \\ & & I \arrow[ur, "\eta_I"'] & \end{tikzcd}$$

Proof. For every $X \in |\mathscr{V}|$ we choose an equalizer $\tau_X\colon T'X \to TX$ of the pair $(T(!_X), \eta_I(!_{TX}))$. There is a unique way of extending the assignment $X \mapsto T'X$ to a functor $T'\colon \mathscr{V} \to \mathscr{V}$, such that $\tau = \{\tau_X \mid X \in |\mathscr{V}|\}$ becomes a natural transformation, i.e. the diagram (2) commutes for every $f\colon X \to Y$ in $\mathscr{V}$. This follows from the commutative diagrams (3), (4):

(2)
$$\begin{tikzcd} T'X \arrow[r, "\tau_X"] \arrow[d, "T'f"] & TX \arrow[d, "Tf"] \\ T'Y \arrow[r, "\tau_Y"] & TY \end{tikzcd}$$

(3)
$$\begin{tikzcd} TX \arrow[dr, "T(!_X)"] \arrow[dd, "Tf"] & \\ & TI \\ TY \arrow[ur, "T(!_Y)"'] & \end{tikzcd}$$

(4)
$$\begin{tikzcd} TX \arrow[dr, "!_{TX}"] \arrow[dd, "Tf"] & & \\ & I \arrow[r, "\eta_I"] & TI \\ TY \arrow[ur, "!_{TY}"'] & & \end{tikzcd}$$

If $\tau\colon T' \to T$ is going to be a morphism of monads $\mathbb{T}' \to \mathbb{T}$, the diagrams (5), (6) must commute for every $X \in |\mathscr{V}|$:

(5)
$$\begin{tikzcd} X \arrow[d, "\eta'_X"'] \arrow[dr, "\eta_X"] & \\ T'X \arrow[r, "\tau_X"] & TX \end{tikzcd}$$

(6)
$$\begin{tikzcd} T'T'X \arrow[r, "\tau\tau_X"] \arrow[d, "\mu'_X"] & TTX \arrow[d, "\mu_X"] \\ T'X \arrow[r, "\tau_X"] & TX \end{tikzcd}$$

If η' and μ' exist, they are uniquely determined by (5), (6), since τ is a pointwise monomorphism. On the other hand, the following two commutative diagrams imply the existence of η' and μ':

$$\begin{tikzcd} X \arrow[r, "!_X"] \arrow[d, "\eta_X"] & I \arrow[d, "\eta_I"] \\ TX \arrow[ur, "!_{TX}"] \arrow[r, "T(!_X)"'] & TI \end{tikzcd}$$

$$\begin{tikzcd} & TTX \arrow[dr, "\mu_X"] & \\ T'T'X \arrow[ur, "\tau\tau_X"] \arrow[r, "!_{T'T'X}"] \arrow[d, "\tau_{T'X}"'] & I \arrow[d, "\eta_I"] & TX \arrow[l, "!_{TX}"'] \\ TT'X \arrow[ur, "!_{TT'X}"] \arrow[r, "T(!_{T'X})"] \arrow[d, "T\tau_X"'] & TI \arrow[d, "T\eta_I"] \arrow[ddr, "1"] & \\ TTX \arrow[ur, "T(!_{TX})"] \arrow[r, "TT(!_X)"] \arrow[dr, "\mu_X"'] & TTI \arrow[dr, "\mu_I"] & \\ & TX \arrow[r, "T(!_X)"] & TI \end{tikzcd}$$

Since τ is a pointwise monomorphism, the naturality of η' and μ' follows from the naturality of η, μ, and τ. Finally, $\tau\colon \mathbb{T}' \to \mathbb{T}$ is a morphism of monads by the construction of η' and μ' (cf. (5), (6)).

The assignment $\mathbb{T} \mapsto \mathbb{T}'$ is natural: if $\vartheta\colon \mathbb{S} \to \mathbb{T}$ is a morphism of monads, there is a unique morphism of monads $\vartheta'\colon \mathbb{S}' \to \mathbb{T}'$ such that the diagram (7) commutes:

(7)
$$\begin{tikzcd} \mathbb{S}' \arrow[r, "\sigma"] \arrow[d, "\vartheta'"] & \mathbb{S} \arrow[d, "\vartheta"] \\ \mathbb{T}' \arrow[r, "\tau"] & \mathbb{T} \end{tikzcd}$$

A monad isomorphic to a monad of the form $\mathbb{T}'$ with $\mathbb{T}$ any monad is called *affine.*

2. Lemma. *Let $\mathbb{S}$ be a monad on $\mathscr{V}$. The following are equivalent*:

(i) *$\mathbb{S}$ is an affine monad,*

(ii) *the canonical morphism $\sigma\colon \mathbb{S}' \to \mathbb{S}$ is an isomorphism,*

(iii) *the diagram* (8) *is commutative for all $X \in |\mathscr{V}|$.*

(8)
$$\begin{tikzcd} SX \arrow[r, "!_{SX}"] \arrow[dr, "S(!_X)"'] & I \arrow[d, "\eta_I"] \\ & SI \end{tikzcd}$$

Proof. (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are trivial. To prove (i) $\Rightarrow$ (iii) we choose $\mathbb{T}$ with $\mathbb{S} \cong \mathbb{T}'$ and consider the following commutative diagram:

$$\begin{tikzcd}
T'X \arrow[rrrr, "!_{T'X}"] \arrow[drr, "\tau_X"] \arrow[dd, "T'(!_X)"'] & & & & I \arrow[dd, "\eta_I"] \arrow[drr, "\eta_I'"] & & \\
 & (\tau) & TX \arrow[drr, "T(!_X)"] & (1) & & (5) & T'I \arrow[dll, "\tau_I"] \\
T'I \arrow[rrrr, "\tau_I"] & & & & TI & &
\end{tikzcd}$$

In this way we obtain a functor G from the category $\mathscr{M}(\mathscr{V})$ of monads on $\mathscr{V}$ to the category $\mathscr{A}(\mathscr{V})$ of affine monads on $\mathscr{V}$ (these categories are in general illegitimate, i.e. they "live" in a higher universe).

3. Proposition. *The functor $G\colon \mathscr{M}(\mathscr{V}) \to \mathscr{A}(\mathscr{V})$ is adjoint to the inclusionfunctor $J\colon \mathscr{A}(\mathscr{V}) \to \mathscr{M}(\mathscr{V})$.*

Proof. The assignment $\mathbb{T} \mapsto (\tau\colon \mathbb{T}' \to \mathbb{T})$ defines a natural transformation $\varphi\colon JG \to 1_{\mathscr{M}(\mathscr{V})}$. Since J is fully faithful and φJ is an isomorphism (by lemma 2), there is a unique $\chi\colon 1_{\mathscr{A}(\mathscr{V})} \to GJ$ such that $(\varphi J)(J\chi) = 1_J$. The second equation $(G\varphi)(\chi G) = 1_G$ is a consequence of the following commutative diagram, because φ is a pointwise equalizer and J is faithful:

$$\begin{tikzcd}
JG \arrow[r, "J\chi G"] \arrow[dr, "1"'] & JGJG \arrow[r, "JG\varphi"] \arrow[d, "\varphi JG"] & JG \arrow[d, "\varphi"] \\
 & JG \arrow[r, "\varphi"] & 1
\end{tikzcd}$$

From now on we assume $\mathscr{V}$ to be a *cartesian closed category* $(\mathscr{V}, \otimes, I, \alpha, \lambda, \varrho, \gamma)$, i.e. $\otimes\colon \mathscr{V} \times \mathscr{V} \to \mathscr{V}$ is the productfunctor, I is a terminal object, $\alpha, \lambda, \varrho, \gamma$ are compatible with the projections, etc. We denote the internal Hom-functor of $\mathscr{V}$ by $\mathscr{V}(-, -)$. We proceed to show that our previous constructions can be lifted to the $\mathscr{V}$-enriched or symmetric monoidal closed level.

Let $\mathbb{T} = (T, \eta, \mu)$ be a $\mathscr{V}$-monad on $\mathscr{V}$. The $\mathscr{V}$-naturality of a natural transformation $\tau\colon T' \to T$ is expressed by the commutativity of the diagram (9):

$$\begin{tikzcd}
\mathscr{V}(X, Y) \arrow[r, "T_{X,Y}"] \arrow[d, "T'_{X,Y}"'] & \mathscr{V}(TX, TY) \arrow[d, "\mathscr{V}(\tau_X, TY)"] \\
\mathscr{V}(T'X, T'Y) \arrow[r, "\mathscr{V}(T'X, \tau_Y)"] & \mathscr{V}(T'X, TY)
\end{tikzcd} \tag{9}$$

The $\mathscr{V}$-functor $\mathscr{V}(TX, -)\colon \mathscr{V} \to \mathscr{V}$ is adjoint, hence it preserves equalizers. Therefore, the (ordinary) functor $T'\colon \mathscr{V} \to \mathscr{V}$ can be equipped with the structure of a $\mathscr{V}$-functor making $\tau\colon T' \to T$ a $\mathscr{V}$-natural transformation, if and only if $\mathscr{V}(\tau_X, TY)\, T_{X,Y}$ equalizes $\mathscr{V}(T'X, T(!_Y))$ and $\mathscr{V}(T'X, \eta_I)\, \mathscr{V}(T'X, !_{TY})$ for all $X, Y \in |\mathscr{V}|$; but this is true, as the following two commutative diagrams prove:

$$\begin{tikzcd}
\mathscr{V}(X,Y) \arrow[r, "T_{X,Y}"] \arrow[d, "\mathscr{V}(X,!_Y)"] \arrow[dd, bend right=60, "!_{\mathscr{V}(X,Y)}"'] & \mathscr{V}(TX,TY) \arrow[r, "\mathscr{V}(\tau_X,TY)"] \arrow[d, "\mathscr{V}(TX,T(!_Y))"] & \mathscr{V}(T'X,TY) \arrow[d, "\mathscr{V}(T'X,T(!_Y))"] \\
\mathscr{V}(X,I) \arrow[r, "T_{X,I}"'] \arrow[d, "\ulcorner !_X \urcorner"] & \mathscr{V}(TX,TI) \arrow[r, "\mathscr{V}(\tau_X,TI)"'] & \mathscr{V}(T'X,TI) \arrow[d, "1"] \\
I \arrow[ur, "\ulcorner T(!_X) \urcorner"] \arrow[rr, shift left, "\ulcorner T(!_X)\tau_X \urcorner"] \arrow[rr, shift right, "\ulcorner \eta_I(!_{TX})\tau_X \urcorner"'] \arrow[d, "\ulcorner \tau_X \urcorner"] \arrow[dr, bend right=60, "\ulcorner !_{T'X} \urcorner"'] & & \mathscr{V}(T'X,TI) \arrow[d, "1"] \\
\mathscr{V}(T'X,TX) \arrow[r, "\mathscr{V}(T'X,!_{TX})"] & \mathscr{V}(T'X,I) \arrow[r, "\mathscr{V}(T'X,\eta_I)"] & \mathscr{V}(T'X,TI)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathscr{V}(X,Y) \arrow[r, "T_{X,Y}"] \arrow[d, "!_{\mathscr{V}(X,Y)}"] & \mathscr{V}(TX,TY) \arrow[r, "\mathscr{V}(\tau_X,TY)"] & \mathscr{V}(T'X,TY) \arrow[d, "\mathscr{V}(1,!_{TY})"'] & \\
I \arrow[rr, "\ulcorner !_{T'X} \urcorner"] & & \mathscr{V}(T'X,I) \arrow[r, "\mathscr{V}(1,\eta_I)"] & \mathscr{V}(T'X,TI)
\end{tikzcd}$$

The diagrams for the $\mathscr{V}$-functoriality of T' and the $\mathscr{V}$-naturality of η', μ' follow from (5), (6) and the corresponding diagrams for T, η, μ since τ is a pointwise monomorphism. In this way we obtain a $\mathscr{V}$-monad $\mathbb{T}'$, together with a morphism of $\mathscr{V}$-monads $\tau\colon \mathbb{T}' \to \mathbb{T}$. The assignment $\mathbb{T} \mapsto (\tau\colon \mathbb{T}' \to \mathbb{T})$ belongs to a natural transformation $\tilde{\varphi}\colon \tilde{J}\tilde{G} \to 1_{\tilde{\mathscr{M}}(\mathscr{V})}$, the counit of an adjunction

(10) $\tilde{J} \dashv \tilde{G}$

where $\tilde{J}$ denotes the inclusion of the category $\tilde{\mathscr{A}}(\mathscr{V})$ of affine $\mathscr{V}$-monads into the category $\tilde{\mathscr{M}}(\mathscr{V})$ of $\mathscr{V}$-monads on $\mathscr{V}$.

We shall now describe the affine part of a $\mathscr{V}$-monad in terms of (idempotent) operations. If $\mathbb{T} = (T, \eta, \mu)$ is a $\mathscr{V}$-monad on $\mathscr{V}$, the object $\mathscr{V}(Y, TX)$ (for $X, Y \in |\mathscr{V}|$) of $\mathscr{V}$ can be considered as the $\mathscr{V}$-object of Y-tuples of X-ary operations of $\mathbb{T}$. Let (A, a) be a $\mathbb{T}$-algebra, i.e. $a\colon TA \to A$ satisfies $a\eta_A = 1_A$ and $a\mu_A = a(Ta)$. The applying of a Y-tuple of X-ary $\mathbb{T}$-operations to a X-tuple of elements of A, yielding a Y-tuple of elements of A, is an operation on the level of sets, which on the $\mathscr{V}$-level is described by an "applying"-morphism $W_{Y,X,a}$:

$$\text{(11)}\qquad \begin{tikzcd}
\mathscr{V}(X,A) \otimes \mathscr{V}(Y,TX) \arrow[r, "W_{Y,X,a}"] \arrow[d, "T_{X,A} \otimes 1"] & \mathscr{V}(Y,A) \\
\mathscr{V}(TX,TA) \otimes \mathscr{V}(Y,TX) \arrow[r, "\mu^{\mathscr{V}}_{Y,TX,TA}"] & \mathscr{V}(Y,TA) \arrow[u, phantom] \arrow[from=u, "\mathscr{V}(Y,a)", leftarrow]
\end{tikzcd}$$

The composing of operations is described on the $\mathscr{V}$-level by the "composition"-morphism $\varkappa_{Z,Y,X}$ (of the Kleisli-$\mathscr{V}$-category of $\mathbb{T}$):

$$\text{(12)}\qquad \begin{tikzcd}
\mathscr{V}(Y,TX) \otimes \mathscr{V}(Z,TY) \arrow[r, "\varkappa_{Z,Y,X}"] \arrow[d, "T_{Y,TX} \otimes 1"] & \mathscr{V}(Z,TX) \arrow[d, "\mathscr{V}(Z,\mu_X)"] \\
\mathscr{V}(TY,TTX) \otimes \mathscr{V}(Z,TY) \arrow[r, "\mu^{\mathscr{V}}_{Z,TY,TTX}"] & \mathscr{V}(Z,TTX)
\end{tikzcd}$$

The composing of operations is compatible with the applying of operations to

$\mathbb{T}$-algebras, i.e. the diagram (13) commutes for every $\mathbb{T}$-algebra (A, a) and for all $X, Y, Z \in |\mathscr{V}|$:

$$
\begin{tikzcd}
\mathscr{V}(X,A) \otimes (\mathscr{V}(Y,TX) \otimes \mathscr{V}(Z,TY)) \arrow[rr, "\alpha"] \arrow[d, "1 \otimes \varkappa_{Z,Y,X}"] & & (\mathscr{V}(X,A) \otimes \mathscr{V}(Y,TX)) \otimes \mathscr{V}(Z,TY) \arrow[d, "W_{Y,X,a}"] \\
\mathscr{V}(X,A) \otimes \mathscr{V}(Z,TX) \arrow[dr, "W_{Z,X,a}"'] & & \mathscr{V}(Y,A) \otimes \mathscr{V}(Z,TY) \arrow[dl, "W_{Z,Y,a}"] \\
& \mathscr{V}(Z,A) &
\end{tikzcd}
\tag{13}
$$

In fact, the composition morphism $\varkappa_{Z,Y,X}$ is a particular applying morphism, namely W_{Z,Y,μ_X}. The commutativity of the diagram (13) follows by an evident diagram chase.

Now let $\mathbb{T}'$ be the affine part of the monad $\mathbb{T}$. We will show that for all $Z, Y \in |\mathscr{V}|$, $\mathscr{V}(Z, T'Y)$ is the subobject of all *idempotent* Z-tuples of Y-ary $\mathbb{T}$-operations (prop. 4 below). A Z-tuple of Y-ary operations $\omega: I \to \mathscr{V}(Z, TY)$ is called *idempotent* iff it equalizes the pair of morphisms (14) (a not necessarily commutative diagram), where $\ulcorner\delta_Y\urcorner$ denotes the *name* of δ_Y (diagram (15)):

$$
\begin{tikzcd}
\mathscr{V}(Z,TY) \arrow[rr, "!"] \arrow[dr, "\lambda^{-1}"'] & & I \arrow[rr, "\ulcorner\delta_Z\urcorner"] & & \mathscr{V}(Z,TI) \\
& I \otimes \mathscr{V}(Z,TY) \arrow[rr, "\ulcorner\delta_Y\urcorner \otimes 1"] & & \mathscr{V}(Y,TI) \otimes \mathscr{V}(Z,TY) \arrow[ur, "\varkappa_{Z,Y,I}"'] &
\end{tikzcd}
\tag{14}
$$

$$
\begin{tikzcd}
Y \arrow[rr, "\delta_Y"] \arrow[dr, "!_Y"'] & & TI \\
& I \arrow[ur, "\eta_I"'] &
\end{tikzcd}
\tag{15}
$$

4. Proposition. *Let $\mathbb{T} = (T, \eta, \mu)$ be a $\mathscr{V}$-monad on $\mathscr{V}$. For all $Z, Y \in |\mathscr{V}|$, $\mathscr{V}(Z, \tau_Y): \mathscr{V}(Z, T'Y) \to \mathscr{V}(Z, TY)$ is an equalizer of the pair of morphisms* (14), *i.e. $\mathscr{V}(Z, T'Y)$ is the subobject of $\mathscr{V}(Z, TY)$ of idempotent operations.*

Since $\mathscr{V}(Z, -): \mathscr{V} \to \mathscr{V}$ preserves equalizers, this follows from (1) and the following two commutative diagrams (16), (17), the proof of which we leave to the reader:

$$
\begin{tikzcd}
\mathscr{V}(Z,TY) \arrow[r, "\mathscr{V}(Z, !_{TY})"] \arrow[d, "!"] & \mathscr{V}(Z,I) \arrow[d, "\mathscr{V}(Z,\eta_I)"] \\
I \arrow[r, "\ulcorner\delta_Z\urcorner"] & \mathscr{V}(Z,TI)
\end{tikzcd}
\tag{16}
$$

$$
\begin{tikzcd}
\mathscr{V}(Z,TY) \arrow[r, "\mathscr{V}(Z, T(!_Y))"] \arrow[d, "\lambda^{-1}"'] & \mathscr{V}(Z,TI) \\
I \otimes \mathscr{V}(Z,TY) \arrow[r, "\ulcorner\delta_Y\urcorner \otimes 1"] & \mathscr{V}(Y,TI) \otimes \mathscr{V}(Z,TY) \arrow[u, "\varkappa_{Z,Y,I}"']
\end{tikzcd}
\tag{17}
$$

The inclusion of idempotent operations $\mathscr{V}(Z, \tau_Y)$ is natural with respect to Z: if $f: X \to Z$ is a morphism, the diagram (18) commutes:

$$
(18)\qquad
\begin{tikzcd}
\mathscr{V}(Z, T'Y) \arrow[r, "\mathscr{V}(Z,\tau_Y)"] \arrow[d, "\mathscr{V}(f,T'Y)"'] & \mathscr{V}(Z, TY) \arrow[d, "\mathscr{V}(f,TY)"] \\
\mathscr{V}(X, T'Y) \arrow[r, "\mathscr{V}(X,\tau_Y)"] & \mathscr{V}(X, TY)
\end{tikzcd}
$$

If, in particular, $X = I$, then $\mathscr{V}(f, T'Y)$ can be interpreted as the fth projection, mapping an operation to its fth component. Hence every component of an idempotent operation is idempotent. In the case $\mathscr{V} = Ens$, (the category of sets) the converse of this statement also holds (cf. [2]).

5. Proposition. *Let $\mathbb{T} = (T, \psi, \eta, \mu)$ be a monoidal monad on $\mathscr{V}$. There is a unique monoidal structure ψ' on the affine part of (T, η, μ), such that $\mathbb{T}' = (T', \psi', \eta', \mu')$ is a monoidal monad and $\tau\colon \mathbb{T}' \to \mathbb{T}$ is a monoidal monad transformation. Furthermore, if $\mathbb{T}$ is symmetric monoidal then so is $\mathbb{T}'$.*

Proof. If τ is going to be a monoidal transformation, the diagram (19) must commute for all $X, Y \in |\mathscr{V}|$:

$$
(19)\qquad
\begin{tikzcd}
T'X \otimes T'Y \arrow[r, "\psi'_{X,Y}"] \arrow[d, "\tau_X \otimes \tau_Y"'] & T'(X \otimes Y) \arrow[d, "\tau_{X \otimes Y}"] \\
TX \otimes TY \arrow[r, "\psi_{X,Y}"] & T(X \otimes Y)
\end{tikzcd}
$$

Since $\tau_{X \otimes Y}$ is an equalizer of $T(!_{X \otimes Y})$, $\eta_I(!_{T(X \otimes Y)})$, the uniqueness and existence of a morphism $\psi'_{X,Y}\colon T'X \otimes T'Y \to T'(X \otimes Y)$ making (19) commute is implied by the exterior of the following commutative diagram:

$$
\begin{tikzcd}
T'X \otimes T'Y \arrow[r, "\tau_X \otimes \tau_Y"] \arrow[d, "\tau_X \otimes \tau_Y"'] & TX \otimes TY \arrow[rr, "\psi_{X,Y}"] \arrow[dr, "T(!_X) \otimes T(!_Y)"] & & T(X \otimes Y) \arrow[d, "T(!_X \otimes !_Y)"'] \arrow[dd, bend left=60, "T(!_{X \otimes Y})"] \\
TX \otimes TY \arrow[r, "!_X \otimes !_Y"] \arrow[d, "\psi_{X,Y}"'] & I \otimes I \arrow[r, "\eta_I \otimes \eta_I"] \arrow[d, "\lambda_I"] \arrow[rr, bend right, "\eta_{I \otimes I}"'] & TI \otimes TI \arrow[r, "\psi_{I,I}"] & T(I \otimes I) \arrow[d, "T(\lambda_I)"'] \\
T(X \otimes Y) \arrow[r, "!_{X \otimes Y}"] & I \arrow[rr, "\eta_I"] & & TI
\end{tikzcd}
$$

Since τ is a pointwise monomorphism, the required diagrams for ψ' follow from the corresponding diagrams for ψ. This completes the proof of the proposition 5.

As an application we mention that the Eilenberg-Moore situation of a monad inherits from it the pleasant properties of being (symmetric) monoidal or closed, respectively (cf. [1, 3, 4, 5, 7]).

References

[1] B. J. DAY, Note on monoidal monads. J. Austral. Math. Soc. **23**, 292–311 (1977).

[2] J. R. ISBELL, M. I. KLUN and S. H. SCHANUEL, Affine parts of algebraic theories I. J. Algebra **44**, 1–8 (1977).

[3] A. KOCK, Closed categories generated by commutative monads. J. Austral. Math. Soc. **12**, 405–424 (1971).

[4] H. LINDNER, A characterization of symmetric monoidal closed Eilenberg-Moore adjunctions. manuscript, Universität Düsseldorf 1977.
[5] H. LINDNER, Commutative monads. Cahiers de Topo. et Géo. diff. **16**, 283–288 (1975).
[6] H. LINDNER, Monoidale und geschlossene Kategorien. Habilitationsschrift, Universität Düsseldorf 1976.
[7] F. E. J. LINTON, Autonomous equational categories. J. Math. Mech. **15**, 637–642 (1966).

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Anschrift des Autors:

Harald Lindner
Universität Düsseldorf
Mathematisches Institut II
Universitätsstraße 1
D-4000 Düsseldorf