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NOTE

A Simple Proof of Toda's Theorem*

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Abstract: Toda in his celebrated paper showed that the polynomial-time hierarchy is contained in $\mathrm{P}^{\#\mathrm{P}}$. We give a short and simple proof of the first half of Toda's Theorem that the polynomial-time hierarchy is contained in $\mathrm{BPP}^{\oplus\mathrm{P}}$. Our proof uses easy consequences of relativizable proofs of results that predate Toda.

For completeness we also include a proof of the second half of Toda's Theorem.

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1 Introduction

In 1991, Toda proved his celebrated theorem [7].

Theorem 1.1 (Toda)**.** $\mathrm{PH} \subseteq \mathrm{P}^{\#\mathrm{P}}$.

also, $\mathrm{PH} \subseteq \mathrm{PP}^{\mathrm{PP}}$

Here PH is the set of languages in the polynomial-time hierarchy.

The proof of Theorem 1.1 follows from the following two lemmas (since $\mathrm{BPP}^A \subseteq \mathrm{PP}^A$ for all A).

Lemma 1.2 (Toda)**.** $\mathrm{PH} \subseteq \mathrm{BPP}^{\oplus\mathrm{P}}$.

Lemma 1.3 (Toda)**.** $\mathrm{PP}^{\oplus\mathrm{P}} \subseteq \mathrm{P}^{\#\mathrm{P}}$.

In this paper we give a short proof of Lemma 1.2 using relativizable versions of results that predate Toda's Theorem. For completeness we will give a proof of Lemma 1.3 as well.

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formally, relativization of $\mathrm{BPP}^{\oplus \mathrm{P}}$ to A is $(\mathrm{BPP}^{\oplus \mathrm{P}^A})^A$

2 Preliminaries

The Complexity Zoo [1] and the Arora-Barak textbook [2] are good sources for descriptions of the complexity classes used in this note.

To relativize Satisfiability to an oracle A, we allow our CNF formulas to have predicates $A_0, A_1, A_2, \ldots$ where A_n is an n-ary predicate defined so $A_n(x_1, \ldots, x_n)$ is true exactly when $x_1 \ldots x_n$ is in A. For every A, SAT^A is NP^A-complete.

If $\mathcal{C}$ and $\mathcal{D}$ are relativizable classes, $\mathcal{C}^{\mathcal{D}} = \cup_{A \in \mathcal{D}} \mathcal{C}^A$. If $\mathcal{D}$ has a complete set D (such as $\mathcal{D} = \oplus\mathrm{P}$) then $\mathcal{C}^{\mathcal{D}} = \mathcal{C}^D$.

When we relativize a class like $\mathrm{BPP}^{\oplus\mathrm{P}}$ to an oracle A, both the BPP and the $\oplus$P machines should have access to the oracle A. The BPP machine can make its queries to A via the $\oplus\mathrm{P}^A$ oracle so we have $(\mathrm{BPP}^{\oplus\mathrm{P}})^A = \mathrm{BPP}^{(\oplus\mathrm{P}^A)}$ which we will write simply as $\mathrm{BPP}^{\oplus\mathrm{P}^A}$.

We define the polynomial-time hierarchy relative to A recursively:

- $\Sigma_0^A = \mathrm{P}^A$.
- $\Sigma_{i+1}^A = \mathrm{NP}^{\Sigma_i^A}$.
- $\mathrm{PH}^A = \cup_i \Sigma_i^A$.

The class GapP is the set of #P functions closed under subtraction. In particular GapP functions may take on negative values. Like #P, GapP functions are closed under uniform exponential-sized sums and polynomial-sized products and unlike #P, GapP functions are also closed under subtraction [3].

3 Proof of Toda's first lemma

We start with the following three results, all of which have proofs that easily relativize.

Theorem 3.1 (Valiant-Vazirani [8])**.** *There is a probabilistic polynomial-time procedure that, given a Boolean formula ϕ, will output formulas $\psi_1, \ldots, \psi_k$ such that*

- *if ϕ is not satisfiable then, for every i, ψ_i is not satisfiable;*
- *if ϕ is satisfiable then, with high probability, for some i, ψ_i has exactly one solution.*

Theorem 3.2 (Papadimitriou-Zachos [6])**.** $\oplus\mathrm{P}^{\oplus\mathrm{P}} = \oplus\mathrm{P}$.

Theorem 3.3 (Zachos [9])**.** *If* $\mathrm{NP} \subseteq \mathrm{BPP}$ *then* $\mathrm{PH} \subseteq \mathrm{BPP}$.

We first need the following easy consequence of Theorem 3.1 noted by Toda [7].

Lemma 3.4 (Valiant-Vazirani, Toda)**.** $\mathrm{NP} \subseteq \mathrm{BPP}^{\oplus\mathrm{P}}$.

Proof Sketch. Given a Boolean formula ϕ, randomly choose $\psi_1, \ldots, \psi_k$ (as in Theorem 3.1) and accept if any of the ψ_i have an odd number of satisfying assignments. Lemma 3.4 now follows from Theorem 3.1. □

Proof of Lemma 1.2.

1. By relativizing Lemma 3.4, we have
$$\mathrm{NP}^{\oplus\mathrm{P}} \subseteq \mathrm{BPP}^{\oplus\mathrm{P}^{\oplus\mathrm{P}}}.$$

2. Now apply Theorem 3.2 to get
$$\mathrm{NP}^{\oplus\mathrm{P}} \subseteq \mathrm{BPP}^{\oplus\mathrm{P}}.$$

3. By relativizing Theorem 3.3, we have
$$\mathrm{NP}^{\oplus\mathrm{P}} \subseteq \mathrm{BPP}^{\oplus\mathrm{P}} \Rightarrow \mathrm{PH}^{\oplus\mathrm{P}} \subseteq \mathrm{BPP}^{\oplus\mathrm{P}}.$$

4. Combining (2) and (3) we have
$$\mathrm{PH} \subseteq \mathrm{PH}^{\oplus\mathrm{P}} \subseteq \mathrm{BPP}^{\oplus\mathrm{P}}.$$

□

If we had replaced the use of Theorem 3.3 with the relativizable proof of it, we would essentially recover Toda's original proof.

4 Proof of Toda's second lemma

For completeness we include a proof of Lemma 1.3 in this section. We give a GapP-based variant of Toda's original proof [7] originally given in a survey paper by the author [4].

We will use the following GapP characterization of $\oplus$P [3].

Lemma 4.1 (Fenner-Fortnow-Kurtz)**.** *A language B is in $\oplus$P if and only if there is a* GapP *function f such that*

- *if $x \in B$ then $f(x) \equiv 1 \pmod 2$;*
- *if $x \notin B$ then $f(x) \equiv 0 \pmod 2$.*

We can define PP^A using P^A predicates.

Lemma 4.2. *A language L is in PP^A if and only if there is a language $B \in \mathrm{P}^A$ and a polynomial q such that*

- *if $x \in L$ then*
$$\left|\left\{y \in \Sigma^{q(|x|)} \;\middle|\; (x,y) \in B\right\}\right| \geq \left|\left\{y \in \Sigma^{q(|x|)} \;\middle|\; (x,y) \notin B\right\}\right|.$$
- *if $x \notin L$ then*
$$\left|\left\{y \in \Sigma^{q(|x|)} \;\middle|\; (x,y) \in B\right\}\right| < \left|\left\{y \in \Sigma^{q(|x|)} \;\middle|\; (x,y) \notin B\right\}\right|.$$

Combining Lemmas 4.1 and 4.2 with Theorem 3.2 (which implies $\mathrm{P}^{\oplus\mathrm{P}} = \oplus\mathrm{P}$) we have the following characterization of $\mathrm{PP}^{\oplus\mathrm{P}}$.

Lemma 4.3. *A language L is in* $\mathrm{PP}^{\oplus\mathrm{P}}$ *if and only if there is a* GapP *function $f(x,y)$ and a polynomial q such that*

- *if $x \in L$ then*
$$\left|\left\{y \in \Sigma^{q(|x|)} \;\middle|\; f(x,y) \equiv 1 \pmod 2\right\}\right| \geq \left|\left\{y \in \Sigma^{q(|x|)} \;\middle|\; f(x,y) \equiv 0 \pmod 2\right\}\right|.$$
- *if $x \notin L$ then*
$$\left|\left\{y \in \Sigma^{q(|x|)} \;\middle|\; f(x,y) \equiv 1 \pmod 2\right\}\right| < \left|\left\{y \in \Sigma^{q(|x|)} \;\middle|\; f(x,y) \equiv 0 \pmod 2\right\}\right|.$$

We give an $\mathrm{FP}^{\mathrm{GapP}}$ algorithm to compute
$$\left|\left\{y \in \Sigma^{q(|x|)} \;\middle|\; f(x,y) \equiv 1 \pmod 2\right\}\right|$$
and
$$\left|\left\{y \in \Sigma^{q(|x|)} \;\middle|\; f(x,y) \equiv 0 \pmod 2\right\}\right|.$$
Lemma 1.3 follows since $\mathrm{FP}^{\mathrm{GapP}} = \mathrm{FP}^{\#\mathrm{P}}$ [3].

Consider the polynomial $g(m) = 3m^2 - 2m^3$. Let $g^{(k)}(m) = \overbrace{g(g(\ldots g}^{k}(m)\ldots))$.

Lemma 4.4. *For all m,*

1. *if $m \equiv 0 \pmod{2^j}$ then $g(m) \equiv 0 \pmod{2^{2j}}$,*
2. *if $m \equiv 1 \pmod{2^j}$ then $g(m) \equiv 1 \pmod{2^{2j}}$,*
3. *if $m \equiv 0 \pmod 2$ then $g^{(k)}(m) \equiv 0 \pmod{2^{2^k}}$, and*
4. *if $m \equiv 1 \pmod 2$ then $g^{(k)}(m) \equiv 1 \pmod{2^{2^k}}$.*

Proof. Items (1) and (2) follow from simple algebra, items (3) and (4) by induction using (1) and (2). □

Let $h(x,y) = g^{(1+\lceil \log q(|x|) \rceil)}(f(x,y))$. Since GapP functions are closed under uniform exponential-size sums and polynomial-size products, $h(x,y)$ is itself a GapP function and by Lemma 4.4

- if $f(x,y) \equiv 1 \pmod 2$ then $h(x,y) \equiv 1 \pmod{2^{q(|x|)+1}}$, and
- if $f(x,y) \equiv 0 \pmod 2$ then $h(x,y) \equiv 0 \pmod{2^{q(|x|)+1}}$.

Define $r(x)$ as
$$r(x) = \sum_{y \in \Sigma^{q(|x|)}} h(x,y),$$
also a GapP function. We then have
$$(r(x) \bmod 2^{q(|x|)+1}) = \left|\left\{y \in \Sigma^{q(|x|)} \;\middle|\; f(x,y) \equiv 1 \pmod 2\right\}\right|$$
and
$$2^{q(|x|)} - (r(x) \bmod 2^{q(|x|)+1}) = \left|\left\{y \in \Sigma^{q(|x|)} \;\middle|\; f(x,y) \equiv 0 \pmod 2\right\}\right|,$$
completing the proof. □

Remark 4.5. Toda uses #P functions and the polynomial $g(m) = 4m^3 + 3m^4$. Lemma 4.3 now holds with each occurrence of "1" replaced by "-1."

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