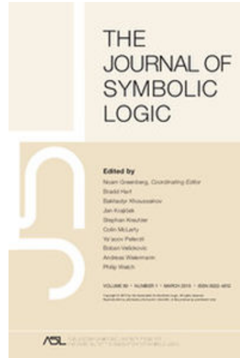

On the decidability of implicational ticket entailment

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ON THE DECIDABILITY OF IMPLICATIONAL TICKET ENTAILMENT

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Abstract. The implicational fragment of the logic of relevant implication, $R_{\rightarrow}$ is known to be decidable. We show that the implicational fragment of the logic of ticket entailment, $T_{\rightarrow}$ is *decidable*. Our proof is based on the *consecution calculus* that we introduced specifically to solve this 50-year old open problem. We reduce the decidability problem of $T_{\rightarrow}$ to the decidability problem of $R_{\rightarrow}$. The decidability of $T_{\rightarrow}$ is equivalent to the *decidability* of the *inhabitation* problem of implicational types by *combinators* over the base $\{\mathsf{B}, \mathsf{B}', \mathsf{I}, \mathsf{W}\}$.

Introduction and history. The pure implicational fragment of Ticket Entailment, which we denote by $T_{\rightarrow}$, has a deceptive simplicity. Its formulas are made up from atomic sentences using the single (binary) connective $\rightarrow$, it has just one rule of inference

(MP) $\mathscr{A} \rightarrow \mathscr{B}, \mathscr{A} \vdash \mathscr{B}$,

and it has just the following four axiom schemas:

1. $\mathscr{A} \rightarrow \mathscr{A}$ (Identity)
2. $(\mathscr{A} \rightarrow \mathscr{B}) \rightarrow ((\mathscr{B} \rightarrow \mathscr{C}) \rightarrow (\mathscr{A} \rightarrow \mathscr{C}))$, (Suffixing)
3. $(\mathscr{A} \rightarrow \mathscr{B}) \rightarrow ((\mathscr{C} \rightarrow \mathscr{A}) \rightarrow (\mathscr{C} \rightarrow \mathscr{B}))$, (Prefixing)
4. $(\mathscr{A} \rightarrow (\mathscr{A} \rightarrow \mathscr{B})) \rightarrow (\mathscr{A} \rightarrow \mathscr{B})$. (Contraction)

Contraction causes potential problems for decidability since one can imagine a proof search for $\mathscr{A} \rightarrow \mathscr{B}$ that keeps adding further occurrences of $\mathscr{A}$ in the antecedent. So it seems plausible that one might get a decidable system if one simply drops Contraction, and indeed, Giambrone (1985) was able to show the decidability of not just the implicational fragment of TW (sometimes referred to in the literature as $T\text{–}W$), but of the system with conjunction and disjunction as well (TW_{+}).

However, Contraction does not always get in the way of decidability, since Kripke (1959) was able to show that the implicational fragments of both the relevance logics R and E are decidable. One gets the system $R_{\rightarrow}$ by adding 5 to the axioms of $T_{\rightarrow}$.

5. $(\mathscr{A} \rightarrow (\mathscr{B} \rightarrow \mathscr{C})) \rightarrow (\mathscr{B} \rightarrow (\mathscr{A} \rightarrow \mathscr{C}))$. (Permutation)

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The addition of 6 to the axioms of $T_{\rightarrow}$ gives $E_{\rightarrow}$. $\overrightarrow{\mathscr{B}}$ means that $\mathscr{B}$ must be an implicational formula.

6. $(\mathscr{A} \rightarrow (\overrightarrow{\mathscr{B}} \rightarrow \mathscr{C})) \rightarrow (\overrightarrow{\mathscr{B}} \rightarrow (\mathscr{A} \rightarrow \mathscr{C}))$. (Restricted Permutation)

So $T_{\rightarrow}$ sits between decidable systems, and the question then is: Does Contraction hurt decidability in the absence of (Restricted) Permutation? Kripke's argument for the decidability of $R_{\rightarrow}/E_{\rightarrow}$, certainly, uses Permutation/Restricted Permutation, but until our paper it has not been clear how to extend Kripke's argument to $T_{\rightarrow}$.[1]

The TLCA (Typed Lambda Calculi and Applications) List of Open Problems contains the decidability of the implicational fragment of Ticket Entailment $T_{\rightarrow}$ as Problem #2 (posted by Roger Hindley). See `tlca.di.unito.it/opltlca/`.[2] Hindley (private communication) says that the problem was first brought to the attention of the TLCA community by Robert Meyer in 1973. It is not clear that there was a publication with this date. But Anderson and Belnap (1975, p. 69), definitely has this as an open problem. The system $T_{\rightarrow}$ first appeared in Anderson (1960). The abstract was received on October 20, 1961 and the paper was presented at the 26th Annual Meeting of the Association for Symbolic Logic held on December 27, 1961. The conference report was published in the volume of ***The Journal of Symbolic Logic*** that was dated 1960. Hindley mentions this publication but adds that the problem "dates back at least to work of Belnap in 1957." Nuel Belnap in a personal communication to us remembers trying to prove its decidability in the context of the "merge" sequent calculi that he developed in 1957 when he was working with Feys.[3] And it would seem Anderson was surely aware of the problem when he first mentioned the system $T_{\rightarrow}$, and even more so when in Anderson (1963) he mentioned the decidability of the system E of entailment as the first among the open problems for that system. Riche and Meyer (1999, fn. 12, p. 229) say that "Having been around since circa 1960, this [the decision problem for $\mathbf{T}_{\rightarrow}$] is the most venerable problem in all of relevant logic. And a clue would appear to lie in the principle of commutation of antecedents."

We learned from Hindley and Pawel Urzyczyn (an editor of the TLCA List of Open Problems) that Problem #2 was posted in December 2006, and a claim was added in November 2010 that "A solution was announced by Vincent Padovani: the problem turns out to be decidable. A draft paper [Padovani, 2010] is now

[1] Kripke's abstract takes up no more than half a page, and contains none of the needed combinatorial ideas. Belnap and Wallace (1961) extends Kripke's result to the implication–negation fragments of R and E and in the process reveals the critical combinatorial components (including "Kripke's Lemma"). Kripke communicated his lemma to Belnap in September 1959 (Kripke would have then been just completing high school). We cite Belnap and Wallace (1961). See also Dunn (1986) and Riche and Meyer (1999) for further "deconstruction" of the Kripke argument.

[2] It is listed there because using the well-known translation of "formulas as types" it has an equivalent statement in terms of combinatory logic. As Hindley puts it: "The question is whether there is a decision-algorithm for the implicational fragment $T_{\rightarrow}$ of the propositional logic called ticket entailment. Equivalently, is there one for the simple type-theory of the restricted combinatory logic based on B, B', I, W?" The question is if there is an algorithm that for an arbitrary implicational formula viewed as a type will decide whether there is a combinator built of B, B', I and W, which inhabits that type.

[3] Some of the work on merge systems formed the basis of Belnap's (1959) Yale University doctoral dissertation *The Formalization of Entailment* and Belnap tells us this is (except for the preface) essentially identical to the Technical Report No. 7, Contract No. SAR/Nonr-609(16), Office of Naval Research. The work on the merge system for T, however, was not mentioned, but it certainly occurs in the discussion of the merge systems in Anderson and Belnap (1975). See the "Ticket restriction" on p. 59.

available."[4] We first discovered this announcement on 12/31/2010. Our proof was first announced in Bimbó and Dunn (2012a), which was submitted on 11/27/2010 before we knew of Padovani's proof. It is the abstract of a contributed paper presented on 3/25/2011 at the North American Annual Meeting of the Association for Symbolic Logic, in Berkeley CA. Padovani's proof and our proof are very different from each other.

We are presenting our own proof in two papers, of which this is the second.[5] We announced our result in Bimbó and Dunn (2012a) and followed up with our first paper, Bimbó and Dunn (2012b), where we developed a number of Gentzen-style calculi for the system $R^{\boldsymbol{t}}_{\to}$, the extension of implicational relevant logic by the sentential constant $\boldsymbol{t}$. We proved cut theorems and other important properties about these calculi—including that all agree on the pure implicational formulas. These provide the basis and a setting for the present paper. We refer the reader to Bimbó and Dunn (2012b) for further information on the calculi and on details of the proofs of the cut theorems. However, as a convenience to the reader, we list their rules in the *Appendix*. In labeling various calculi, we follow the notation of that paper, and also Dunn (1986), which occasionally differ from the labels used in Anderson and Belnap (1975). To set the context, we provide now a general description of their features.

We begin with the sequent calculus $LR_{\to}$, which is essentially the same as the implicational fragment of Gentzen's system LJ except that the rule weakening is dropped. But we also avoid the rule permutation, as suggested in effect in Dunn (1986), by using *multisets*—as in Meyer and McRobbie (1982)—to replace the finite sequences used by Kripke (1959) and Anderson and Belnap (1975). Multisets are kind of half-way between sequences and sets, and are determined only by the number of occurrences of each element and not by the order of the elements. From $LR_{\to}$ we go to $[LR_{\to}]$, by following Kripke, and dropping contraction but retaining its effect by building a limited number of contractions into an operational rule—thus bounding the number of contractions. We then (conservatively) add the sentential constant $\boldsymbol{t}$ and get the system $LR^{\boldsymbol{t}}_{\to}$ and then again a variant $[LR^{\boldsymbol{t}}_{\to}]$, which limits contractions. To obtain a consecution system $LT^{\boldsymbol{t}}_{\to}$ for $T^{\boldsymbol{t}}_{\to}$, one has to replace the multisets with structures, using ; as a *binary structural connective*, and adding appropriate structural rules corresponding to various combinators—together with $\boldsymbol{t}$. We use the terms 'sequent calculus' and 'consecution calculus' for "Gentzen-style" calculi. However, in the former, the structural connective is associative, whereas in the latter, is it not. Informally speaking, sequents and multisets are "flat," but consecutions have a "depth" to them.

We showed how one can obtain another consecution calculus for $R^{\boldsymbol{t}}_{\to}$ by adding to the calculus $LT^{\boldsymbol{t}}_{\to}$ a rule corresponding to the combinator C. To emphasize that this is built on binary grouping, we used the notation $LR^{\boldsymbol{t}}_{\to;}$, because the notation $LR_{\to}$ had been used already for a sequent calculus. Notice though that there is no semi-colon in the labels of our other consecution calculi.

$E_{\to}$ does not correspond to a set of proper combinators like $R_{\to}$ corresponds to $\{\mathsf{B}, \mathsf{C}, \mathsf{W}, \mathsf{I}\}$. Accordingly, full permutation cannot be included in a sequent calculus

[4] A version is "clickable" from Hindley's posting—and it is also available at the URL `http://arxiv.org/abs/1106.1875`, where it is stated that the paper is forthcoming.

[5] A talk based on parts of this paper was presented at the Winter meeting of the Association for Symbolic Logic in Boston, in January, 2012; Bimbó and Dunn (to appear) is the abstract for that talk.

formulation of $E_{\to}$, and if $\boldsymbol{t}$ and $\circ$ (fusion) are added, then $\boldsymbol{t}$ is not a full identity for $\circ$, because it is not a lower right identity. The latter observation is the insight behind a consecution calculus for $E^{\boldsymbol{t}}_{\to}$ in Bimbó (2007a), which contains a *special rule for* $\boldsymbol{t}$ to ensure that it is a partial identity of the right kind.

The idea about the *interaction* between $\boldsymbol{t}$ and $\to$ led to the consecution calculus $LT^{\textcircled{t}}_{\to}$ that we introduced in Bimbó and Dunn (2012b). $LT^{\textcircled{t}}_{\to}$ is equivalent to $R^{\boldsymbol{t}}_{\to}$, whereas $LT^{\boldsymbol{t}}_{\to}$ is equivalent to $T^{\boldsymbol{t}}_{\to}$. The difference between the two calculi is that in $LT^{\boldsymbol{t}}_{\to}$, $\boldsymbol{t}$ is left identity only, and in $LT^{\textcircled{t}}_{\to}$, it is full identity because of the two new rules $\mathsf{K}_{\boldsymbol{t}} \vdash$ and $\mathsf{T}_{\boldsymbol{t}} \vdash$, which are special versions of thinning and permutation.

Our decision procedure is not unlike Kripke's decision procedure for $R_{\to}$ in spirit. There are novel components (with respect to Kripke's proof) in our decision procedure such as turning an irredundant proof of a theorem into a finite set of proofs rather than into exactly one proof, and deciding theoremhood (for $T^{\boldsymbol{t}}_{\to}$) in $LT^{\textcircled{t}}_{\to}$ (which is equivalent to $R^{\boldsymbol{t}}_{\to}$). These complicate the proof, thereby, rendering it appropriate for $T^{\boldsymbol{t}}_{\to}$, which is a logic (unlike $R^{\boldsymbol{t}}_{\to}$) that differentiates between structures comprising the same formulas.

We define an algorithm (called π), which turns proofs in $LT^{\textcircled{t}}_{\to}$ into proofs in $LR^{\boldsymbol{t}}_{\to}$. The transformation π, so to speak, *simplifies* the $LT^{\textcircled{t}}_{\to}$ proofs. Every theorem of $T^{\boldsymbol{t}}_{\to}$ has an irredundant proof in $[LR^{\boldsymbol{t}}_{\to}]$, and it is straightforward to convert those proofs into proofs in $LR^{\boldsymbol{t}}_{\to}$ that do not contain the $\vdash \boldsymbol{t}$ axiom. Hence, our goal becomes then to undo the simplification. We define an algorithm (called τ), which turns proofs in $LR^{\boldsymbol{t}}_{\to}$ into proofs in $LT^{\textcircled{t}}_{\to}$, by restoring the *distinctions* inherent to $LT^{\textcircled{t}}_{\to}$.

In section 1, we prove that the addition of the constant $\boldsymbol{t}$ does not prevent the decidability of $R_{\to}$ to be extended to $R^{\boldsymbol{t}}_{\to}$. In section 2, we consider the relationship of the calculi $LT^{\textcircled{t}}_{\to}$ and $LR^{\boldsymbol{t}}_{\to}$, especially, from the point of view of transforming proofs in the former into proofs in the latter. Section 3 describes the transformations from $[LR^{\boldsymbol{t}}_{\to}]$ to $LR^{\boldsymbol{t}}_{\to}$, and further, to $LT^{\textcircled{t}}_{\to}$. Section 4 completes the proof of the decidability of $T_{\to}$. There we show that proofs that do not belong to $LT^{\boldsymbol{t}}_{\to}$ can be unequivocally distinguished from those that do.

§1. $R^{\boldsymbol{t}}_{\to}$ is decidable. The implicational fragment of the logic of relevant implication, $R_{\to}$ is known to be decidable.[6] Previously, in Bimbó and Dunn (2012b), we extended the sequent calculus $[LR_{\to}]$ by adding $\boldsymbol{t}$; this resulted in the sequent calculus $[LR^{\boldsymbol{t}}_{\to}]$, which is equivalent to $R^{\boldsymbol{t}}_{\to}$. It is well-known that $R^{\boldsymbol{t}}_{\to}$ is conservative over $R_{\to}$, which also follows from the cut theorem for $[LR^{\boldsymbol{t}}_{\to}]$. Our decidability result for $T_{\to}$ builds upon the decidability of $R_{\to}$, but via several intermediate steps. A consecution calculus for $T_{\to}$ has to include $\boldsymbol{t}$.[7] Therefore, we first prove that the conservative extension of $[LR_{\to}]$ with $\boldsymbol{t}$ inherits decidability from $[LR^{\boldsymbol{t}}_{\to}]$.

[6]See Kripke (1959) for the original statement of the result. A further application of the same technique is Belnap and Wallace (1961). We will rely on the exposition of the proof of the decidability of $R_{\to}$ in Dunn (1986, §3.6).

[7]There is a consecution calculus for $T_{\to}$ without $\boldsymbol{t}$ in Bimbó (2007b, §3.2). However, the absence of $\boldsymbol{t}$ makes the calculus intricate and difficult to use. There is a so-called merge calculus for $T_{\to}$ in Anderson and Belnap (1975), which is more similar to a sequent calculus than to a consecution calculus. We are not aware of the existence of a sequent calculus (of the usual sort) for $T_{\to}$.

To orient the reader, we very briefly outline the proof of the decidability of $R_{\rightarrow}$. The decision procedure relies on the calculus $[LR_{\rightarrow}]$. The notion of a proof in $[LR_{\rightarrow}]$, as well as in all the other sequent and consecution calculi is standard; therefore, each proof is a finite tree.

Curry's Lemma states that if a sequent $\alpha' \vdash \mathscr{A}$ is obtainable by contractions from a sequent $\alpha \vdash \mathscr{A}$ and the latter has a proof in $[LR_{\rightarrow}]$, then $\alpha' \vdash \mathscr{A}$ has a proof in $[LR_{\rightarrow}]$ of length less than or equal to the length of the proof of $\alpha \vdash \mathscr{A}$.[8]

Curry's lemma, obviously, goes some way to limiting proof searches in $[LR_{\rightarrow}]$—it says that contraction was successfully built into the rules and that a proof search tree does not need to be *redundant* by containing a branch with a sequent $\alpha' \vdash \mathscr{A}$ below another sequent $\alpha \vdash \mathscr{A}$ from which it can be obtained by contractions (including the degenerate case of the null contraction, i.e., where a sequent is simply repeated). This opens the door to Kripke's distinctive contribution, which shows that *every branch* in such an irredundant proof search tree is *finite*. First, two sequents $\alpha \vdash \mathscr{A}$ and $\alpha' \vdash \mathscr{A}$ are said to be *cognate*, when exactly the same formulas—not counting multiplicity—occur in them.[9] This partitions sequents into *cognation classes*.

Kripke's Lemma says that if a sequence of distinct cognate sequents $\Gamma_0, \Gamma_1, \ldots,$ is *irredundant* in the sense that for no Γ_i, Γ_j (where $i < j$), Γ_i is obtainable from Γ_j by contractions, then the sequence is *finite*.

The *cut rule* is admissible in $[LR_{\rightarrow}]$ (hence not listed), which means that the calculus has the *subformula property*. That is, if $\alpha \vdash \mathscr{A}$ is a sequent provable in $LR_{\rightarrow}$, then any formula occurring in any sequent in the proof is a *subformula* of some formula occurring in $\alpha \vdash \mathscr{A}$.

We are now in a position to prove the decidability of $[LR_{\rightarrow}]$. We start a *proof search tree* with the sequent to be proven, say $\alpha \vdash \mathscr{A}$. By Curry's lemma we can construct a complete proof search tree in $[LR_{\rightarrow}]$ so that every branch in it is irredundant. For each sequent that appears in the proof, there are only finitely many sequents that can appear above it among the premises. So the search tree we are constructing has the *finite fork property*. *König's Lemma* about trees tells us that the search tree will be finite if it has both the finite fork property and the *finite branch property*, where the latter simply means that every branch is finite. So if the search tree is infinite there must be at least one infinite branch. However, because of the subformula property, there will be only finitely many cognation classes represented in the proof, so at least one of those cognation classes would have to appear infinitely many times. Now Kripke's lemma comes to the rescue: this cannot happen, because the branch is irredundant!

Our proof of the decidability of $R^{t}_{\rightarrow}$ has a similar structure, however, it relies on the sequent calculus $[LR^{t}_{\rightarrow}]$.

LEMMA 1. *If a sequent $\alpha' \vdash \mathscr{A}$ is obtainable by contractions from a sequent $\alpha \vdash \mathscr{A}$ and the latter has a proof in $[LR^{t}_{\rightarrow}]$, then $\alpha' \vdash \mathscr{A}$ has* a proof of length less than or equal to the length of the proof *of* $\alpha \vdash \mathscr{A}$.

[8]See Curry (1963, 5E) for a similar lemma concerning other logics. Incidentally, Kripke's lemma is known to be equivalent to Dickson's lemma. See Riche and Meyer (1999).

[9]The idea of cognate sequents goes back to Kleene (1952, p. 480).

PROOF. The proof is by induction on the length of the proof of $\alpha \vdash \mathscr{A}$. We suppose the proof for $[LR_{\rightarrow}]$, and include here only the new cases. If $n = 1$ and the axiom is $\vdash \boldsymbol{t}$, then we note that this axiom does not permit any contractions, which means that the claim of Curry's lemma is trivially true.

If $n > 1$, then we have to consider, when the last rule applied is $\boldsymbol{t}\vdash$. If a sequent could be obtained from the lower sequent by one contraction on $\boldsymbol{t}$, then the upper sequent in the application of the $\boldsymbol{t} \vdash$ rule is identical to the sequent, which is the result of the contraction. This shows that the claim is true in this subcase. If several contractions are involved, which possibly, include contractions both on $\boldsymbol{t}$ and other formula(s), then by inductive hypothesis, there is a proof of the upper sequent (i.e., the lower sequent minus one occurrence of $\boldsymbol{t}$) of height $m \leq n - 1$. Then an application of $\boldsymbol{t} \vdash$ gives a proof of length $m + 1 \leq n$. ⊣

Having proved the appropriate version of Curry's lemma for $[LR^{\boldsymbol{t}}_{\rightarrow}]$, we can proceed to the following theorem.

THEOREM 2 (**$R^{\boldsymbol{t}}_{\rightarrow}$'s decidability**). *The implicational fragment of the logic of relevant implication with the truth constant is* decidable.

PROOF. The structure of this proof is similar to the structure of the proof demonstrating that $R_{\rightarrow}$ is decidable, which we outlined above.

We proved in Bimbó and Dunn (2012b) (see lemma 2.5) that $R^{\boldsymbol{t}}_{\rightarrow}$ and $[LR^{\boldsymbol{t}}_{\rightarrow}]$ have the same set of theorems. Kripke's lemma is independent of the concrete shape of formulas in a sequent, thus, it applies to $[LR^{\boldsymbol{t}}_{\rightarrow}]$. Similarly, König's lemma is true about trees in general. Lastly, we have just shown that Curry's lemma is true for $[LR^{\boldsymbol{t}}_{\rightarrow}]$.

These results together yield the decidability of $R^{\boldsymbol{t}}_{\rightarrow}$. ⊣

§2. The π transformation. The consecution calculus $LT^{\textcircled{t}}_{\rightarrow}$ was designed specifically to solve the decision problem of $T_{\rightarrow}$. It is important to underscore that $LT^{\textcircled{t}}_{\rightarrow}$ is equivalent to $R^{\boldsymbol{t}}_{\rightarrow}$ (not to $T^{\boldsymbol{t}}_{\rightarrow}$).[10] We proved in Bimbó and Dunn (2012b, §4) that the single cut rule is *admissible* in $LT^{\textcircled{t}}_{\rightarrow}$, and that $\mathscr{A}$ is a theorem of $R^{\boldsymbol{t}}_{\rightarrow}$ iff $\boldsymbol{t} \vdash \mathscr{A}$ is provable in $LT^{\textcircled{t}}_{\rightarrow}$. The cut theorem implies that $\boldsymbol{t} \vdash \mathscr{A}$ has a cut-free proof in $LT^{\textcircled{t}}_{\rightarrow}$—provided $\mathscr{A}$ is a theorem of $R^{\boldsymbol{t}}_{\rightarrow}$.

The differences between proofs in $LT^{\textcircled{t}}_{\rightarrow}$ and $LR^{\boldsymbol{t}}_{\rightarrow}$ are of a different sort than those in the case of $[LR^{\boldsymbol{t}}_{\rightarrow}]$ and $LR^{\boldsymbol{t}}_{\rightarrow}$. Therefore, we first prove that every proof in $LT^{\textcircled{t}}_{\rightarrow}$ can be transformed into a proof in $LR^{\boldsymbol{t}}_{\rightarrow}$. The steps in this transformation will not change the *order* of the application of those rules that closely resemble each other in the two calculi. Rather, the steps concern the differences between *structures* in $LT^{\textcircled{t}}_{\rightarrow}$ and *multisets* in $LR^{\boldsymbol{t}}_{\rightarrow}$. The observations we make about an $LT^{\textcircled{t}}_{\rightarrow}$ proof and its transform will become important, when we define the transformation of proofs in $LR^{\boldsymbol{t}}_{\rightarrow}$ into proofs in $LT^{\textcircled{t}}_{\rightarrow}$. There we will make sure that all the possible ways in which structures could be obtained from sequents are considered. The fact that the order of the $\rightarrow\vdash$, $\vdash\rightarrow$ and contraction rules does not need to be modified guarantees that no proofs in $LT^{\textcircled{t}}_{\rightarrow}$ are missed—including those that are proofs in $LT^{\boldsymbol{t}}_{\rightarrow}$ too.

[10]We explained in Bimbó and Dunn (2012b) how the notation is intended to remind the reader of certain features of the consecution calculus.

DEFINITION 3. Let Δ be a proof in $LT^{\textcircled{t}}_{\to}$. The *transformation* π of Δ is the consecutive application of the steps (1)–(4).

(1) Replace all ;'s by ,'s.
(2) Omit all the parentheses from complex structures.
(3) Omit the lower sequent, if the corresponding lower consecution was the result of an application of the $\mathsf{B}\vdash$, $\mathsf{B}'\vdash$ or $\mathsf{T}_t\vdash$ rule.
(4) Insert finitely many sequents between $\alpha, \mathscr{A}_1, \mathscr{A}_1, \dots, \mathscr{A}_n, \mathscr{A}_n \vdash \mathscr{B}$ and $\alpha, \mathscr{A}_1, \dots, \mathscr{A}_n \vdash \mathscr{B}$, if the corresponding lower consecution resulted by an application of $\mathsf{W}\vdash$ to two occurrences of a complex structure comprising n atomic structures that we denote by $\mathscr{A}_1, \dots, \mathscr{A}_n$.[11] The inserted $n-1$ sequents are consecutive contractions of $\mathscr{A}_1, \dots, \mathscr{A}_{n-1}$ starting from $\alpha, \mathscr{A}_1, \mathscr{A}_1, \dots, \mathscr{A}_n, \mathscr{A}_n \vdash \mathscr{B}$.

The transformation in effect "blurs" the proof Δ so that (1) semi-colons are seen as commas, (2) parentheses are not seen (unless they are in a formula), (3) identical sequents just above each other are squashed into one, and (4) contractions of complex structures are emulated by consecutive contractions on their atomic structures. For our purposes, it is even more important that certain features of the proofs are not destroyed.

Notice that if $n > 1$ in step (4), then different orderings of contractions (of different formulas) result in distinct proofs in $LR^{t}_{\to}$.

We give an example of the π transformation. The following tree is a proof of $\boldsymbol{t} \vdash (\mathscr{B} \to \mathscr{C}) \to (\mathscr{A} \to (\mathscr{B} \to \mathscr{C}) \to \mathscr{D}) \to \mathscr{A} \to \mathscr{D}$ in $LT^{\textcircled{t}}_{\to}$.[12] The formula $(\mathscr{B} \to \mathscr{C}) \to (\mathscr{A} \to (\mathscr{B} \to \mathscr{C}) \to \mathscr{D}) \to \mathscr{A} \to \mathscr{D}$ is the so-called restricted conditioned modus ponens, which is a theorem of $R^{t}_{\to}$ (indeed, of $E^{t}_{\to}$), but not of $T^{t}_{\to}$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathscr{A} \vdash \mathscr{A} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathscr{B} \to \mathscr{C} \vdash \mathscr{B} \to \mathscr{C}}{\mathscr{B} \to \mathscr{C}; \boldsymbol{t} \vdash \mathscr{B} \to \mathscr{C}}\,\mathsf{K}_t\vdash \qquad \mathscr{D} \vdash \mathscr{D}}{(\mathscr{B} \to \mathscr{C}) \to \mathscr{D}; (\mathscr{B} \to \mathscr{C}; \boldsymbol{t}) \vdash \mathscr{D}}\,{\to}\vdash}{\mathscr{B} \to \mathscr{C}; (\mathscr{B} \to \mathscr{C}) \to \mathscr{D}; \boldsymbol{t} \vdash \mathscr{D}}\,\mathsf{B}'\vdash}{\boldsymbol{t}; (\mathscr{B} \to \mathscr{C}; (\mathscr{B} \to \mathscr{C}) \to \mathscr{D}) \vdash \mathscr{D}}\,\mathsf{T}_t\vdash}{\boldsymbol{t}; \mathscr{B} \to \mathscr{C}; (\mathscr{B} \to \mathscr{C}) \to \mathscr{D} \vdash \mathscr{D}}\,\mathsf{B}\vdash}{}}{\boldsymbol{t}; \mathscr{B} \to \mathscr{C}; (\mathscr{A} \to (\mathscr{B} \to \mathscr{C}) \to \mathscr{D}; \mathscr{A}) \vdash \mathscr{D}}\,{\to}\vdash}{\boldsymbol{t}; \mathscr{B} \to \mathscr{C}; \mathscr{A} \to (\mathscr{B} \to \mathscr{C}) \to \mathscr{D}; \mathscr{A} \vdash \mathscr{D}}\,\mathsf{B}\vdash}{\boldsymbol{t}; \mathscr{B} \to \mathscr{C}; \mathscr{A} \to (\mathscr{B} \to \mathscr{C}) \to \mathscr{D} \vdash \mathscr{A} \to \mathscr{D}}\,\vdash{\to}}{\boldsymbol{t}; \mathscr{B} \to \mathscr{C} \vdash (\mathscr{A} \to (\mathscr{B} \to \mathscr{C}) \to \mathscr{D}) \to \mathscr{A} \to \mathscr{D}}\,\vdash{\to}}{\boldsymbol{t} \vdash (\mathscr{B} \to \mathscr{C}) \to (\mathscr{A} \to (\mathscr{B} \to \mathscr{C}) \to \mathscr{D}) \to \mathscr{A} \to \mathscr{D}}\,\vdash{\to}
$$

The result of the application of steps (1)–(2) is the following tree, which shows an intermediate stage in the transformation, and is not a proof in either $LT^{\textcircled{t}}_{\to}$ or $LR^{t}_{\to}$. We listed the formulas in the sequents so that the presence of identical sequents

[11]This list may contain multiple occurrences of a formula; however, for the purposes of describing this step it is easier to work with a listing of atomic structures whether they are identical formulas or not.

[12]We omit parentheses from formulas by right association in order to enhance the readability of the formulas. Structures without parentheses are left-associated.

becomes obvious.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathscr{A}\vdash\mathscr{A}\qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathscr{B}\to\mathscr{C}\vdash\mathscr{B}\to\mathscr{C}}{\boldsymbol{t},\mathscr{B}\to\mathscr{C}\vdash\mathscr{B}\to\mathscr{C}}\qquad \mathscr{D}\vdash\mathscr{D}}{\boldsymbol{t},\mathscr{B}\to\mathscr{C},(\mathscr{B}\to\mathscr{C})\to\mathscr{D}\vdash\mathscr{D}}}{\boldsymbol{t},\mathscr{B}\to\mathscr{C},(\mathscr{B}\to\mathscr{C})\to\mathscr{D}\vdash\mathscr{D}}}{\boldsymbol{t},\mathscr{B}\to\mathscr{C},(\mathscr{B}\to\mathscr{C})\to\mathscr{D}\vdash\mathscr{D}}}{\boldsymbol{t},\mathscr{B}\to\mathscr{C},(\mathscr{B}\to\mathscr{C})\to\mathscr{D}\vdash\mathscr{D}}}{\boldsymbol{t},\mathscr{A},\mathscr{A}\to(\mathscr{B}\to\mathscr{C})\to\mathscr{D},\mathscr{B}\to\mathscr{C}\vdash\mathscr{D}}}{\boldsymbol{t},\mathscr{A},\mathscr{A}\to(\mathscr{B}\to\mathscr{C})\to\mathscr{D},\mathscr{B}\to\mathscr{C}\vdash\mathscr{D}}}{\boldsymbol{t},\mathscr{A}\to(\mathscr{B}\to\mathscr{C})\to\mathscr{D},\mathscr{B}\to\mathscr{C}\vdash\mathscr{A}\to\mathscr{D}}}{\boldsymbol{t},\mathscr{B}\to\mathscr{C}\vdash(\mathscr{A}\to(\mathscr{B}\to\mathscr{C})\to\mathscr{D})\to\mathscr{A}\to\mathscr{D}}}{\boldsymbol{t}\vdash(\mathscr{B}\to\mathscr{C})\to(\mathscr{A}\to(\mathscr{B}\to\mathscr{C})\to\mathscr{D})\to\mathscr{A}\to\mathscr{D}}$$

Lastly, the result of the application of (3) is shown below.[13] It is straightforward to verify that the tree is a proof in $LR^{\boldsymbol{t}}_{\to}$ (and we added some labels to make the verification easy).

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathscr{A}\vdash\mathscr{A}\qquad \dfrac{\dfrac{\mathscr{B}\to\mathscr{C}\vdash\mathscr{B}\to\mathscr{C}}{\boldsymbol{t},\mathscr{B}\to\mathscr{C}\vdash\mathscr{B}\to\mathscr{C}}\;{\scriptstyle \boldsymbol{t}\vdash}\qquad \mathscr{D}\vdash\mathscr{D}}{\boldsymbol{t},\mathscr{B}\to\mathscr{C},(\mathscr{B}\to\mathscr{C})\to\mathscr{D}\vdash\mathscr{D}}\;{\scriptstyle \to\vdash}}{\boldsymbol{t},\mathscr{A},\mathscr{A}\to(\mathscr{B}\to\mathscr{C})\to\mathscr{D},\mathscr{B}\to\mathscr{C}\vdash\mathscr{D}}\;{\scriptstyle \to\vdash}}{\boldsymbol{t},\mathscr{A}\to(\mathscr{B}\to\mathscr{C})\to\mathscr{D},\mathscr{B}\to\mathscr{C}\vdash\mathscr{A}\to\mathscr{D}}\;{\scriptstyle \vdash\to}}{\boldsymbol{t},\mathscr{B}\to\mathscr{C}\vdash(\mathscr{A}\to(\mathscr{B}\to\mathscr{C})\to\mathscr{D})\to\mathscr{A}\to\mathscr{D}}\;{\scriptstyle \vdash\to}}{\boldsymbol{t}\vdash(\mathscr{B}\to\mathscr{C})\to(\mathscr{A}\to(\mathscr{B}\to\mathscr{C})\to\mathscr{D})\to\mathscr{A}\to\mathscr{D}}\;{\scriptstyle \vdash\to}$$

If $\mathfrak{A}\vdash\mathscr{A}$ is a consecution in the sense of $LT^{\textcircled{t}}_{\to}$, then we denote by $\mathfrak{A}^{\flat}\vdash\mathscr{A}$ the result of an application of steps (1)–(2) from definition 3. It is obvious that $\mathfrak{A}^{\flat}\vdash\mathscr{A}$ is a sequent in the sense of $LR^{\boldsymbol{t}}_{\to}$.

Lemma 4 (Simplification). *If Δ is a proof in the sense of $LT^{\textcircled{t}}_{\to}$, then Δ', that is obtained by the π transformation is* a proof in $LR^{\boldsymbol{t}}_{\to}$.

Proof. We consider the structure of Δ.

1. If the consecution $\mathfrak{A}\vdash\mathscr{A}$ is an axiom of $LT^{\textcircled{t}}_{\to}$, then it is unchanged by the transformation, and $\mathfrak{A}\vdash\mathscr{A}$ is an axiom of $LR^{\boldsymbol{t}}_{\to}$. The rest of the proof is divided into nine cases—according to the last rule applied in Δ.

2.1 If $\mathfrak{A}[\boldsymbol{t};\mathfrak{B}]\vdash\mathscr{A}$ is by $\mathsf{KI}_{\boldsymbol{t}}\vdash$, then the premise's transformation yields $(\mathfrak{A}[\mathfrak{B}])^{\flat}\vdash\mathscr{A}$, and by the $\boldsymbol{t}\vdash$ rule in $LR^{\boldsymbol{t}}_{\to}$, we get $\boldsymbol{t},(\mathfrak{A}[\mathfrak{B}])^{\flat}\vdash\mathscr{A}$, which is $(\mathfrak{A}[\boldsymbol{t};\mathfrak{B}])^{\flat}\vdash\mathscr{A}$.

2.2 If $\mathfrak{A}[\boldsymbol{t}]\vdash\mathscr{A}$ is by $\mathsf{M}_{\boldsymbol{t}}\vdash$, then by an application of $W\vdash$ in $LR^{\boldsymbol{t}}_{\to}$, we obtain from $(\mathfrak{A}[\boldsymbol{t};\boldsymbol{t}])^{\flat}\vdash\mathscr{A}$ the sequent $(\mathfrak{A}[\boldsymbol{t}])^{\flat}\vdash\mathscr{A}$.

2.3 If $\mathfrak{A}\vdash\mathscr{A}\to\mathscr{B}$ is by $\vdash\to$, then the $\vdash\to$ rule in $LR^{\boldsymbol{t}}_{\to}$ yields $\mathfrak{A}^{\flat}\vdash\mathscr{A}\to\mathscr{B}$ from $(\mathfrak{A};\mathscr{A})^{\flat}\vdash\mathscr{A}$—as desired.

[13]The proof we started with did not contain an application of $\mathsf{W}\vdash$; therefore, (4) has not been applied in this example.

2.4 If $\mathfrak{A}[\mathscr{A} \to \mathscr{B}; \mathfrak{B}] \vdash \mathscr{C}$ is by $\to\vdash$, given the premises $\mathfrak{B} \vdash \mathscr{A}$ and $\mathfrak{A}[\mathscr{B}] \vdash \mathscr{C}$, then the premises for an application of $\to\vdash$ in $LR^{t}_{\to}$ are, respectively, $\mathfrak{B}^{\flat} \vdash \mathscr{A}$ and $(\mathfrak{A}[\mathscr{B}])^{\flat} \vdash \mathscr{C}$, from which $(\mathfrak{A}[\mathscr{A} \to \mathscr{B}; \mathfrak{B}])^{\flat} \vdash \mathscr{C}$ results.

2.5 If $\mathfrak{A}[(\mathfrak{B}; \mathfrak{C}); \mathfrak{D}] \vdash \mathscr{A}$ is by $\mathsf{B}\vdash$, then the premise is $\mathfrak{A}[\mathfrak{B}; (\mathfrak{C}; \mathfrak{D})] \vdash \mathscr{A}$. The two consecutions comprise the same formulas, each occurring as many times in one of the consecutions as in the other. That is, $(\mathfrak{A}[\mathfrak{B}; (\mathfrak{C}; \mathfrak{D})])^{\flat} \vdash \mathscr{A}$ is $(\mathfrak{A}[(\mathfrak{B}; \mathfrak{C}); \mathfrak{D}])^{\flat} \vdash \mathscr{A}$, and the latter of the two is omitted from Δ' because of (3).

2.6–2.7 These cases—with $\mathsf{B}'\vdash$ and $\mathsf{T}_{t}\vdash$—are similar to 2.5; hence, we omit the details.

2.8 If $\mathfrak{A}[\mathfrak{B}; \boldsymbol{t}] \vdash \mathscr{A}$ is by $\mathsf{K}_{t}\vdash$, then by the $\boldsymbol{t}\vdash$ rule in $LR^{t}_{\to}$ we get $(\mathfrak{A}[\mathfrak{B}; \boldsymbol{t}])^{\flat} \vdash \mathscr{A}$ from $(\mathfrak{A}[\mathfrak{B}])^{\flat} \vdash \mathscr{A}$.

2.9 If $\mathfrak{A}[\mathfrak{B}; \mathfrak{C}] \vdash \mathscr{A}$ is by $\mathsf{W}\vdash$, then $\mathfrak{C}$ is the structure an occurrence of which has been contracted. $\mathfrak{C}$ has finitely many formulas in it, let us say $\mathscr{A}_1, \ldots, \mathscr{A}_n$. (We do not insist in this listing to index different occurrences of a formula by the same number; rather, we assume that $\mathfrak{C}$ is built from n formula occurrences that we label by $\mathscr{A}_1, \ldots, \mathscr{A}_n$.) The left-hand side of $(\mathfrak{A}[\mathfrak{B}; \mathfrak{C}; \mathfrak{C}])^{\flat} \vdash \mathscr{A}$ will include $\mathscr{A}_1, \mathscr{A}_1, \ldots, \mathscr{A}_n, \mathscr{A}_n$. (4) stipulates the insertion of $n-1$ sequents between $(\mathfrak{A}[\mathfrak{B}; \mathfrak{C}; \mathfrak{C}])^{\flat} \vdash \mathscr{A}$ and $(\mathfrak{A}[\mathfrak{B}; \mathfrak{C}])^{\flat} \vdash \mathscr{A}$ that are justified by successive applications of the $W\vdash$ rule to the formulas denoted by $\mathscr{A}_1, \ldots, \mathscr{A}_{n-1}$. Then yet another application of $W\vdash$ yields the last sequent. The n applications of $W\vdash$ together give $(\mathfrak{A}[\mathfrak{B}; \mathfrak{C}])^{\flat} \vdash \mathscr{A}$ from $(\mathfrak{A}[\mathfrak{B}; \mathfrak{C}; \mathfrak{C}])^{\flat} \vdash \mathscr{A}$. ⊣

Our decision procedure relies on the sequent calculus $[LR^{t}_{\to}]$, to start with, and transforming proofs in that calculus into proofs in $LT^{\circledR}_{\to}$. We claim that our modifications of the proofs does not fail to produce a proof for a formula in $LT^{t}_{\to}$, if the formula is a theorem of $T^{t}_{\to}$, not only of $R^{t}_{\to}$.

LEMMA 5. *If $\mathscr{A}$ is a theorem of $T^{t}_{\to}$, then $\boldsymbol{t} \vdash \mathscr{A}$ is* provable *in* $[LR^{t}_{\to}]$.

PROOF. We take it that the claim is obviously true in view of what we established earlier about $[LR^{t}_{\to}]$. (See Bimbó and Dunn (2012b).) However, it may be useful to emphasize that $\mathscr{A}$ has an irredundant proof in $[LR^{t}_{\to}]$. ⊣

Of course, once a proof of $\mathscr{A}$ from $[LR^{t}_{\to}]$ is transformed into a proof in $LR^{t}_{\to}$, the proof may have become redundant—independently of whether the formula $\mathscr{A}$ is a theorem of $R^{t}_{\to}$ only, or a theorem of $T^{t}_{\to}$ too. However, if $\mathscr{A}$ is provable in $T^{t}_{\to}$, then it has a proof in $LR^{t}_{\to}$; moreover, $\mathscr{A}$ has a proof in $LR^{t}_{\to}$ in the set of proofs that are generated from proofs in $[LR^{t}_{\to}]$ by making contraction steps explicit.

Proofs in $LT^{\circledR}_{\to}$ may have an arbitrarily large number of $\boldsymbol{t}$'s added, and proofs of a formula $\mathscr{A}$ that is not a theorem of $T^{t}_{\to}$ typically will have additional occurrences of $\boldsymbol{t}$'s, because of the way full permutation is handled. The π transforms of these proofs may not be irredundant even if the contractions on the subformulas of $\mathscr{A}$ are built into the left-introduction rule for $\to$. All the $\boldsymbol{t}$'s can be contracted into one $\boldsymbol{t}$ in $LR^{t}_{\to}$, and they are reconstructed in our "inverse" transformation τ, when we allow the insertion of $\boldsymbol{t}$'s to simulate groupings and permutations that are unavailable in $LT^{t}_{\to}$.

Recall that the calculi $LT^{t}_{\to}$ and $LT^{\circledR}_{\to}$ both enjoy the cut property. Furthermore, $LT^{t}_{\to}$ is equivalent to $T^{t}_{\to}$, whereas $LT^{\circledR}_{\to}$ is equivalent to $R^{t}_{\to}$. We note that in an uninteresting way a theorem $\mathscr{A}$ of $R^{t}_{\to}$ (or of $T^{t}_{\to}$) has infinitely many proofs in $LT^{\circledR}_{\to}$: the rules $\mathsf{KI}_{t}\vdash$ and $\mathsf{M}_{t}\vdash$ can be applied in infinitely many ways. (This is not unlike, though not exactly the same as, how in LK or in LJ the permutation rule can be used as its own inverse.)

LEMMA 6. *If* Δ *is a proof in* $LT^{\textcircled{t}}_{\to}$, *then* $\pi(\Delta)$ *contains the applications of the connective rules and of the contraction rule* in the original order *in* Δ.

PROOF. The proof is by scrutinizing the algorithm π. The omission of identical sequents cannot change the order of the rules that remain in the proof $\pi(\Delta)$. Step (4) yields either one contraction step or a block of contraction steps (in which the order of contractions depends on our choice of the order in which the atomic structures are listed). This potentially expands one step into several similar steps (i.e., all $W\vdash$ steps), but it does not permute an application of the contraction rule with another rule. $\dashv$

If $\mathscr{A}$ is a theorem of $R^{\mathbf{t}}_{\to}$, then it is provable in $LT^{\textcircled{t}}_{\to}$ (without the cut rule). In $LT^{\textcircled{t}}_{\to}$ (like in $LR^{\mathbf{t}}_{\to}$ for ,), there are two ways to add a new occurrence of the structural connective ; —by the left introduction rule for $\to$ or by thinning in a $\mathbf{t}$. If $\mathbf{t}$ is not a subformula of the theorem $\mathscr{A}$, then the roles that the $\mathbf{t}$'s can play in $LT^{\textcircled{t}}_{\to}$ is to provide the non-emptiness of the antecedent and to allow to create suitable groupings and permutations of formulas. Although, in principle, an arbitrarily large number of $\mathbf{t}$'s can be thinned in (and then contracted out), permutations of $\mathbf{t}$ with $\mathbf{t}$ yield the same sequent, thus there is no real reason to have more $\mathbf{t}$'s than atomic structures that are not $\mathbf{t}$ in any consecution.

The applications of $\to\vdash$ and $\vdash\to$ in Δ and $\pi(\Delta)$ are not only in the same order, but they have the same principal and side formulas. All the calculi here have the subformula property and only $\mathbf{t}$'s can be thinned in. Therefore, the complex subformulas of a theorem are created by the connective rules.

Given these observations, our strategy, when we get to defining τ (on p. 227), an "inverse" of the π transformation will be to preserve the exact multiplicities and formulas to which the connective rules and $W\vdash$ are applied in $LR^{\mathbf{t}}_{\to}$, but to allow the introduction of $\mathbf{t}$'s whenever needed to produce a particular permutation or grouping.

§3. **Decision procedure for $T_{\to}$.** Our main goal is to prove that $T_{\to}$ is decidable. Every theorem of $T_{\to}$ is of the form $\mathscr{A}_1 \to \cdots (\mathscr{A}_n \to p)$, where $n \in \mathbb{N}^+$ and p is a propositional variable. The whole decision procedure involves working with various sequent and consecution calculi. Thus—practically speaking—it might be useful first to exclude formulas that are either obviously theorems or obviously not theorems of $T_{\to}$. Then only the formulas that could not be categorized will be subjected to the proof search procedure using the calculi $[LR^{\mathbf{t}}_{\to}]$, $LR^{\mathbf{t}}_{\to}$ and $LT^{\textcircled{t}}_{\to}$.

We will assume that a given formula $\mathscr{A}$ is tested for provability using $[LR^{\mathbf{t}}_{\to}]$. This can mean—equivalently—searching for a proof of either of the sequents $\mathbf{t} \vdash \mathscr{A}$ or $\vdash \mathscr{A}$. Just as in the case of $[LR_{\to}]$, we can generate a proof search tree. If we find a branch in the tree that is a proof, then $\mathscr{A}$ is a theorem of $R^{\mathbf{t}}_{\to}$. To answer the question if $\mathscr{A}$ is provable in $R^{\mathbf{t}}_{\to}$, the existence of one proof is sufficient. However, we want to find *all the irredundant proofs of* $\mathscr{A}$ in $[LR^{\mathbf{t}}_{\to}]$. Therefore, we assume that the proof search does not stop when a proof has been found; rather, it proceeds as in the case when no proof has been found yet (except that the proofs are recorded and collected). This stipulation is innocuous, because the proof search tree is finite.

To summarize, if a formula $\mathscr{A}$ is given, then the (modified) decision procedure for $[LR^t_\to]$ yields either an answer "no," or a finite set of proofs of $\mathscr{A}$ in $[LR^t_\to]$. "No" for $R^t_\to$ implies "no" for $T^t_\to$. Therefore, from now on, we assume that we are dealing with a formula $\mathscr{A}$ that has turned out to be a theorem of $R^t_\to$.

3.1. From $[LR^t_\to]$ to $LR^t_\to$. Let us assume that $\mathscr{A}$ is a $\boldsymbol{t}$-free implicational formula, which is potentially provable in $T_\to$. The search for a proof of the sequent $\boldsymbol{t} \vdash \mathscr{A}$ in the calculus $[LR^t_\to]$ generates a finite set of *cut-free proofs*. These are all the irredundant cut-free proofs of $\boldsymbol{t} \vdash \mathscr{A}$ in $[LR^t_\to]$. An "irredundant" proof is one that satisfies the condition from section 1, namely that no sequent in the proof has an ancestor in the proof tree from which it can be obtained by contractions.

Let Δ be one of the proofs of $\boldsymbol{t} \vdash \mathscr{A}$. We stipulate that for each proof the following transformations are repeated, which turn the proof Δ into a proof in $LR^t_\to$. The aim of the transformation is to make explicit the contractions that are built into the $[\to\vdash]$ rule in $[LR^t_\to]$.

Starting with the leaves of the proof tree, and moving from the top to the bottom, at each node we consider how that node is justified in the proof. Without loss of generality, we may assume that the tree is scrutinized level by level from left to right.[14] For our purposes, the only important feature of this ordering is that it guarantees that at each node we know that the node or nodes that are immediately above the one being examined have already been processed at some earlier step.

Notice that the two axioms can be distinguished from each other due to their form. Each application of one of the three rules is uniquely identifiable when all their sequents are given. In other words, Δ *uniquely* determines which axioms occur in Δ and which rules have been applied in Δ.

(1) Each axiom in $[LR^t_\to]$ is an axiom of $LR^t_\to$; that is, the leaves of Δ remain unchanged.

(2) If a sequent is the result of an application of $\vdash\to$ or $\boldsymbol{t}\vdash$, then the proof tree is not modified.

(3) If a sequent is the result of an application of the rule $[\to\vdash]$, then that might include some contraction on the left (i.e., in $[\alpha, \beta, \mathscr{A} \to \mathscr{B}]$). To preclude a potential misunderstanding, we stress that the bracket notation does not lead to any ambiguity in a concrete proof. The rule permits some contractions—though it does not prescribe any. But once the rule has been applied to a pair of sequents (which contain finitely many formulas), no uncertainty remains with respect to the contractions that have been performed as part of the application of the $[\to\vdash]$ rule. Since we are dealing with three concrete sequents, the discernment of the contractions amounts to counting formula occurrences. The sequents are *finite*, which implies that only finitely many contractions could have been applied. At each application of $[\to\vdash]$, we can create a finite list $\langle \mathscr{A}_1, \ldots, \mathscr{A}_n \rangle$ comprising the formulas to which contraction has been applied. (This list may contain some formulas more than once in some cases.) The $W\vdash$ rule reduces the number of occurrences of a formula by one; thus, if the length of the list is n, then n applications of $W\vdash$—following an application of $\to\vdash$—yield the lower sequent in Δ.

[14]The levels of the tree are counted from the root—as usual. Thus, replacement of subtrees, as in (3) below, will not affect the levels to which yet unprocessed nodes belong.

In other words, if $\alpha \vdash \mathscr{A}$ and $\beta, \mathscr{B} \vdash \mathscr{C}$ were the premises of an application of the $[\to\vdash]$ rule, then we add—as the node immediately below both of them—the sequent $\alpha, \beta, \mathscr{A} \to \mathscr{B} \vdash \mathscr{C}$, which is justified by $\to\vdash$ in $LR^{t}_{\to}$. If the list of contracted formulas is empty, then this sequent is the original lower sequent in Δ; that is, no modifications of Δ are required at this point. If the list of contracted formulas is not empty, then for each $\mathscr{A}_i$ (where $1 \leq i \leq n$) in $\langle \mathscr{A}_1, \ldots, \mathscr{A}_n \rangle$, we add a new node supported by an application of $W\vdash$ on that formula. After the addition of n new nodes, the lower sequent coincides in its shape with $[\alpha, \beta, \mathscr{A} \to \mathscr{B}] \vdash \mathscr{C}$. Now we replace the subtree rooted in this sequent in the so-far modified Δ with the new subtree that contains contractions.

If there is a modification to the proof, then each new sequent added is justified by an application of a rule of $LR^{t}_{\to}$. The old sequents are justified either as axioms or by the rules that bear the same label (save the brackets in $[\to\vdash]$). This means that (1), (2) and (3) produce a proof in $LR^{t}_{\to}$.

Each proof is finite; hence, after scrutinizing finitely many nodes, and possibly modifying the proof Δ, we have a proof, let us say Δ', in $LR^{t}_{\to}$ with the root of Δ' identical to the root of Δ.

If $\Delta_1, \ldots, \Delta_m$ are proofs of $\boldsymbol{t} \vdash \mathscr{A}$ in $[LR^{t}_{\to}]$, then $\Delta'_1, \ldots, \Delta'_m$ are the proofs obtained according to (1)–(3) in $LR^{t}_{\to}$.

3.2. Within $LR^{t}_{\to}$. A difference between $LT^{\textcircled{t}}_{\to}$ and $LR^{t}_{\to}$ is that $LR^{t}_{\to}$ has an axiom that is absent from $LT^{\textcircled{t}}_{\to}$, namely, $\vdash \boldsymbol{t}$. The next lemma ensures that we do not need to worry about proofs in $LR^{t}_{\to}$ that contain the $\vdash \boldsymbol{t}$ axiom.

LEMMA 7. *If Δ is a proof of $\boldsymbol{t} \vdash \mathscr{A}$ in $LR^{t}_{\to}$, then there is a proof Δ' of the same end sequent that* does not contain *occurrences of the axiom* $\vdash \boldsymbol{t}$.

PROOF. We consider the proof Δ. First we show, by structural induction, that if there are n occurrences of $\vdash \boldsymbol{t}$ in the subproof ending in $\alpha \vdash \mathscr{A}$, then there is a proof of $\boldsymbol{t}^n, \alpha \vdash \mathscr{A}$, where all the occurrences of $\vdash \boldsymbol{t}$ have been replaced by $\boldsymbol{t} \vdash \boldsymbol{t}$. ($\boldsymbol{t}^n$ stands for n occurrences of $\boldsymbol{t}$.)

1.1 If an axiom is $\mathscr{A} \vdash \mathscr{A}$, then Δ is unchanged. If an axiom is $\vdash \boldsymbol{t}$, then it is replaced by $\boldsymbol{t} \vdash \boldsymbol{t}$, which is an instance of id.

1.2 There are four subcases to consider, each of which amounts to observing that the same rule is applicable in a context with one or more occurrences of $\boldsymbol{t}$ added, and in the case of $\to\vdash$, adding up two numbers. For example, if there are n occurrences of $\vdash \boldsymbol{t}$ above an application of $W\vdash$, then instead of

$$\frac{\alpha, \mathscr{A}, \mathscr{A} \vdash \mathscr{B}}{\alpha, \mathscr{A}, \vdash \mathscr{B}} \quad \text{we have} \quad \frac{\boldsymbol{t}^n, \alpha, \mathscr{A}, \mathscr{A} \vdash \mathscr{B}}{\boldsymbol{t}^n, \alpha, \mathscr{A} \vdash \mathscr{B}}.$$

If the rule is $\to\vdash$, then by inductive hypothesis, $\boldsymbol{t}^n, \alpha \vdash \mathscr{A}$ and $\boldsymbol{t}^m, \mathscr{B}, \beta \vdash \mathscr{C}$ are provable. The application of $\to\vdash$ yields $\boldsymbol{t}^{n+m}, \alpha, \beta, \mathscr{A} \to \mathscr{B} \vdash \mathscr{C}$, while $n + m$ occurrences of $\vdash \boldsymbol{t}$ have been replaced above this sequent.

2. By the assumption of the lemma, Δ ended in the sequent $\boldsymbol{t} \vdash \mathscr{A}$. Now we have a proof of $\boldsymbol{t}^{n+1} \vdash \mathscr{A}$, where n is the number of $\vdash \boldsymbol{t}$ axioms in Δ. n contractions on $\boldsymbol{t}$ result in the original end sequent $\boldsymbol{t} \vdash \mathscr{A}$. ⊣

The proof shows that any proof of $\boldsymbol{t} \vdash \mathscr{A}$ containing occurrences of the $\vdash \boldsymbol{t}$ axiom can be effectively turned into a proof without occurrences of the $\vdash \boldsymbol{t}$ axiom. This, of course, is also true for those proofs in $LR^{t}_{\to}$, which resulted from proofs in $[LR^{t}_{\to}]$.

3.3. From $LR^{t}_{\to}$ to $LT^{\textcircled{t}}_{\to}$. We are interested in turning proofs in $LR^{t}_{\to}$ into proofs in $LT^{\textcircled{t}}_{\to}$. The notion of sequents has "built-in permutation," so to speak, that we want to simulate in $LT^{\textcircled{t}}_{\to}$.

One of the roles of $\boldsymbol{t}$ in $LT^{\textcircled{t}}_{\to}$ (together with the rules $\mathsf{B}\vdash$, $\mathsf{B}'\vdash$ and $\mathsf{T}_{\boldsymbol{t}}\vdash$) is to guarantee that all the permutations and groupings of the formulas that constitute the antecedent of a sequent can be emulated in $LT^{\textcircled{t}}_{\to}$. The following three lemmas show that $\boldsymbol{t}$ is instrumental in achieving this effect, and at the same time, extra occurrences of $\boldsymbol{t}$ can be positioned so as to be eventually contracted by applications of $\mathsf{M}_{\boldsymbol{t}}\vdash$.

LEMMA 8. *If* $\mathfrak{C}[\mathfrak{A};\mathfrak{B}]\vdash\mathscr{A}$ *is provable in* $LT^{\textcircled{t}}_{\to}$, *then* $\mathfrak{C}[\boldsymbol{t};\mathfrak{B};\mathfrak{A}]\vdash\mathscr{A}$ *and* $\mathfrak{C}[\boldsymbol{t};(\mathfrak{B};\mathfrak{A})]\vdash\mathscr{A}$ *are* provable *too.*

PROOF. The following sequence of steps demonstrates both claims.

$$\begin{array}{c l}
\mathfrak{C}[\mathfrak{A};\mathfrak{B}]\vdash\mathscr{A} & \\ \hline
\mathfrak{C}[\mathfrak{A};(\mathfrak{B};\boldsymbol{t})]\vdash\mathscr{A} & \mathsf{K}_{\boldsymbol{t}}\vdash \\ \hline
\mathfrak{C}[\mathfrak{B};\mathfrak{A};\boldsymbol{t}]\vdash\mathscr{A} & \mathsf{B}'\vdash \\ \hline
\mathfrak{C}[\boldsymbol{t};(\mathfrak{B};\mathfrak{A})]\vdash\mathscr{A} & \mathsf{T}_{\boldsymbol{t}}\vdash \\ \hline
\mathfrak{C}[\boldsymbol{t};\mathfrak{B};\mathfrak{A}]\vdash\mathscr{A} & \mathsf{B}\vdash
\end{array}$$

⊣

LEMMA 9. *If* $\mathfrak{D}[\mathfrak{A};\mathfrak{B};\mathfrak{C}]\vdash\mathscr{A}$ *is provable in* $LT^{\textcircled{t}}_{\to}$, *then* $\mathfrak{D}[\mathfrak{G}]\vdash\mathscr{A}$ *is provable too, where* $\mathfrak{G}$ *is any* permutation *and* grouping *of the three structures* $\mathfrak{A},\mathfrak{B}$ *and* $\mathfrak{C}$ with a $\boldsymbol{t}$ *on the left and associated with the leftmost of those three structures.*

PROOF. There are 3! permutations of 3 elements, and each has 2 groupings. In total this gives 12 possibilities. Since $\boldsymbol{t}$ is always grouped with the leftmost structure, the total number of the cases does not change.

1. The simplest case is when the consecution to be proven has the form $\mathfrak{D}[\boldsymbol{t};\mathfrak{A};\mathfrak{B};\mathfrak{C}]\vdash\mathscr{A}$. Obviously, an application of $\mathsf{KI}_{\boldsymbol{t}}\vdash$ yields this consecution in one step.

2. The next consecution we wish to prove is $\mathfrak{D}[\boldsymbol{t};\mathfrak{C};(\mathfrak{B};\mathfrak{A})]\vdash\mathscr{A}$. The following segment can be appended to the proof of $\mathfrak{D}[\mathfrak{A};\mathfrak{B};\mathfrak{C}]\vdash\mathscr{A}$ to obtain the desired proof.

$$\begin{array}{c l}
\mathfrak{D}[\mathfrak{A};\mathfrak{B};\mathfrak{C}]\vdash\mathscr{A} & \\ \hline
\mathfrak{D}[\mathfrak{A};\mathfrak{B};(\mathfrak{C};\boldsymbol{t})]\vdash\mathscr{A} & \mathsf{K}_{\boldsymbol{t}}\vdash \\ \hline
\mathfrak{D}[\mathfrak{C};(\mathfrak{A};\mathfrak{B});\boldsymbol{t}]\vdash\mathscr{A} & \mathsf{B}'\vdash \\ \hline
\mathfrak{D}[\mathfrak{C};(\mathfrak{A};(\mathfrak{B};\boldsymbol{t}));\boldsymbol{t}]\vdash\mathscr{A} & \mathsf{K}_{\boldsymbol{t}}\vdash \\ \hline
\mathfrak{D}[\mathfrak{C};(\mathfrak{B};\mathfrak{A};\boldsymbol{t});\boldsymbol{t}]\vdash\mathscr{A} & \mathsf{B}'\vdash \\ \hline
\mathfrak{D}[\boldsymbol{t};(\mathfrak{C};(\mathfrak{B};\mathfrak{A};\boldsymbol{t}))]\vdash\mathscr{A} & \mathsf{T}_{\boldsymbol{t}}\vdash \\ \hline
\mathfrak{D}[\boldsymbol{t};(\mathfrak{C};(\boldsymbol{t};(\mathfrak{B};\mathfrak{A})))]\vdash\mathscr{A} & \mathsf{T}_{\boldsymbol{t}}\vdash \\ \hline
\mathfrak{D}[\boldsymbol{t};(\boldsymbol{t};\mathfrak{C};(\mathfrak{B};\mathfrak{A}))]\vdash\mathscr{A} & \mathsf{B}'\vdash \\ \hline
\mathfrak{D}[\boldsymbol{t};(\boldsymbol{t};\mathfrak{C});(\mathfrak{B};\mathfrak{A})]\vdash\mathscr{A} & \mathsf{B}\vdash \\ \hline
\mathfrak{D}[\boldsymbol{t};\boldsymbol{t};\mathfrak{C};(\mathfrak{B};\mathfrak{A})]\vdash\mathscr{A} & \mathsf{B}\vdash \\ \hline
\mathfrak{D}[\boldsymbol{t};\mathfrak{C};(\mathfrak{B};\mathfrak{A})]\vdash\mathscr{A} & \mathsf{M}_{\boldsymbol{t}}\vdash
\end{array}$$

3. We consider the pair of the previous case which differs only with respect to the grouping of the three structures $\mathfrak{A},\mathfrak{B}$ and $\mathfrak{C}$; that is, our goal is to prove $\mathfrak{D}[\boldsymbol{t};\mathfrak{C};\mathfrak{B};\mathfrak{A}]\vdash\mathscr{A}$. Given that we have already shown that $\mathfrak{D}[\boldsymbol{t};\mathfrak{C};(\mathfrak{B};\mathfrak{A})]\vdash\mathscr{A}$ is provable, one application of the $\mathsf{B}\vdash$ rule yields the desired consecution.

The consecutions that are to be proven in the remaining cases are listed in the following table. (We omit the details of those cases.)

4. $\mathfrak{D}[\boldsymbol{t};\mathfrak{A};(\mathfrak{B};\mathfrak{C})]\vdash\mathscr{A}$.	7. $\mathfrak{D}[\boldsymbol{t};\mathfrak{B};\mathfrak{A};\mathfrak{C}]\vdash\mathscr{A}$.	10. $\mathfrak{D}[\boldsymbol{t};\mathfrak{B};(\mathfrak{C};\mathfrak{A})]\vdash\mathscr{A}$.
5. $\mathfrak{D}[\boldsymbol{t};\mathfrak{A};\mathfrak{C};\mathfrak{B}]\vdash\mathscr{A}$.	8. $\mathfrak{D}[\boldsymbol{t};\mathfrak{B};(\mathfrak{A};\mathfrak{C})]\vdash\mathscr{A}$.	11. $\mathfrak{D}[\boldsymbol{t};\mathfrak{C};\mathfrak{A};\mathfrak{B}]\vdash\mathscr{A}$.
6. $\mathfrak{D}[\boldsymbol{t};\mathfrak{A};(\mathfrak{C};\mathfrak{B})]\vdash\mathscr{A}$.	9. $\mathfrak{D}[\boldsymbol{t};\mathfrak{B};\mathfrak{C};\mathfrak{A}]\vdash\mathscr{A}$.	12. $\mathfrak{D}[\boldsymbol{t};\mathfrak{C};(\mathfrak{A};\mathfrak{B})]\vdash\mathscr{A}$. ⊣

The next lemma parallels the previous one.

Lemma 10. *If* $\mathfrak{D}[\mathfrak{A};\mathfrak{B};\mathfrak{C}]\vdash\mathscr{A}$ *is provable in* $LT^{\textcircled{t}}_{\to}$, *then* $\mathfrak{D}[\mathfrak{G}]\vdash\mathscr{A}$ *is provable too, where* $\mathfrak{G}$ *is any* permutation *and* grouping *of the three structures* $\mathfrak{A},\mathfrak{B}$ *and* $\mathfrak{C}$ *associated together* with a $\boldsymbol{t}$ *as the leftmost structure.*

Proof. The proof is similar to the proof of the previous lemma. There are 12 cases again. We only give the details of one case, which is nearly obvious.

1. From $\mathfrak{D}[\mathfrak{A};\mathfrak{B};\mathfrak{C}]\vdash\mathscr{A}$, by a single application of the $\mathsf{KI}_{\boldsymbol{t}}\vdash$ rule, we get $\mathfrak{D}[\boldsymbol{t};(\mathfrak{A};\mathfrak{B};\mathfrak{C})]\vdash\mathscr{A}$. ⊣

In all the cases covered by the last three lemmas, there are minimal sequences of steps that produce the desired lower consecution. Given that the lower consecutions are provable from the left-associated structure $\mathfrak{A};\mathfrak{B};\mathfrak{C}$, it follows that the same lower consecutions are provable from $\mathfrak{A};(\mathfrak{B};\mathfrak{C})$ too, because an application of $\mathsf{B}\vdash$ yields $\mathfrak{A};\mathfrak{B};\mathfrak{C}$.

The introduction of $\boldsymbol{t}$ in many of the cases is by $\mathsf{K}_{\boldsymbol{t}}\vdash$, which immediately shows that the proof does not belong to $LT^{\boldsymbol{t}}_{\to}$.

Now we define the transformation τ of $LR^{\boldsymbol{t}}_{\to}$ proofs (without $\vdash\boldsymbol{t}$ axioms) into $LT^{\textcircled{t}}_{\to}$ proofs. This is in effect a "deblurring" transformation, and undoes the "blurring" done by the transformation π by reconstructing the distinctions inherent to $LT^{\textcircled{t}}_{\to}$. This reconstruction can often be done in various ways. Hence an application of the transformation τ to an $LR^{\boldsymbol{t}}_{\to}$ proof typically does not result in a singleton set of $LT^{\textcircled{t}}_{\to}$ proofs, rather it yields a finite set of proofs with more that one element.

An application of τ proceeds from the top to the bottom: at each step we construct lower consecutions according to the rule that was last applied in the proof in $LR^{\boldsymbol{t}}_{\to}$. We assume that we process the nodes in the proof tree in $LR^{\boldsymbol{t}}_{\to}$ level by level, from left to right, and the lower consecutions in the proofs constructed in $LT^{\textcircled{t}}_{\to}$ correspond to a particular sequent in the $LR^{\boldsymbol{t}}_{\to}$ proof. As a rule, a transformation step results in multiple lower consecutions, let us say $\mathfrak{A}_1\vdash\mathscr{A},\ldots,\mathfrak{A}_n\vdash\mathscr{A}$, each corresponding to $\alpha\vdash\mathscr{A}$. We may assume, without loss of generality, that when in a later step $\alpha\vdash\mathscr{A}$ is a premise of a rule, then each of the $\mathfrak{A}_i\vdash\mathscr{A}$'s are considered in that transformation step. Since we have finitely many proofs at each step, repeated applications of a step terminate.

1. If a node is an instance of the id axiom, then the same node is added to the proof in $LT^{\textcircled{t}}_{\to}$, and it is justified by the id axiom in that calculus.

There are four rules in $LR^{\boldsymbol{t}}_{\to}$. 2–5 deal with them.

2. If the rule applied is $\boldsymbol{t}\vdash$, then we take as many copies of the consecution that corresponds to the upper sequent $\alpha\vdash\mathscr{A}$ as the number of substructures in the consecution. In each copy, we insert one $\boldsymbol{t}$ on the left-hand side of one of the substructures. We note that using $\mathsf{KI}_{\boldsymbol{t}}\vdash$ guarantees that we do not accidentally generate a proof that does not belong to $LT^{\boldsymbol{t}}_{\to}$. If there was already an occurrence of $\boldsymbol{t}$ in the consecution corresponding to $\alpha\vdash\mathscr{A}$, then we also keep a copy of that

consecution without adding another $\boldsymbol{t}$. All these lower consecutions correspond to the lower sequent $\boldsymbol{t}, \alpha \vdash \mathscr{A}$.

3. If the rule is $\vdash\rightarrow$, then the lower sequent determines the antecedent and the consequent of the $\rightarrow$ formula. If $\mathfrak{A} \vdash \mathscr{B}$ is the consecution corresponding to $\alpha, \mathscr{A} \vdash \mathscr{B}$, then we generate all the permutations and groupings of $\mathfrak{A}$ of the form $\mathfrak{B}; \mathscr{A}$, where $\mathfrak{B}$ is $\mathfrak{A}$ save one occurrence of $\mathscr{A}$ with possibly some additional occurrences of $\boldsymbol{t}$, if there are applications of the rule $\mathsf{T}_t \vdash$ preceded by an application of $\mathsf{K}_t \vdash$. If $\mathfrak{A}$ is $\mathscr{A}$, then we replace $\mathfrak{A} \vdash \mathscr{B}$ with one consecution $\boldsymbol{t}; \mathscr{A} \vdash \mathscr{B}$ (which is justified by $\mathsf{KI}_t \vdash$). We note that in generating the permutations and groupings, only $\mathsf{K}_t \vdash$ can be used from among the two rules that introduce $\boldsymbol{t}$, because $\mathsf{KI}_t \vdash$ is never required. The other rules involved, $\mathsf{B} \vdash$, $\mathsf{B}' \vdash$ and $\mathsf{T}_t \vdash$, do not add new formulas. Furthermore, at most as many $\boldsymbol{t}$'s can be introduced as the number of atomic structures in $\mathfrak{A}$. If in any of the $\mathfrak{B}$'s there is an occurrence of $\boldsymbol{t}; \boldsymbol{t}$, then we replace that by $\boldsymbol{t}$ by an application of $\mathsf{M}_t \vdash$. There are finitely many $\mathfrak{B}$'s that result, and the lower consecutions that correspond to $\alpha \vdash \mathscr{A} \rightarrow \mathscr{B}$ are the consecutions $\mathfrak{B} \vdash \mathscr{A} \rightarrow \mathscr{B}$. The proof segment that yields the permutations and groupings is justified by the five rules mentioned, whereas the last consecution is obtained by an application of the $\vdash\rightarrow$ rule.

4. If the rule applied is $\rightarrow\vdash$, then we take the consecutions that correspond to $\alpha \vdash \mathscr{A}$ and $\beta, \mathscr{B} \vdash \mathscr{C}$. Let us say, these are $\mathfrak{A} \vdash \mathscr{A}$ and $\mathfrak{B} \vdash \mathscr{C}$. We generate all the permutations and groupings of the atomic structures, possibly, with extra occurrences of $\boldsymbol{t}$ (as in the previous step) from $\mathfrak{A}$ and $\mathfrak{B}$. If there are adjacent occurrences of $\boldsymbol{t}$, then we add $\mathsf{M}_t \vdash$ steps. Each resulting $\mathfrak{B}'$ must contain an atomic structure $\mathscr{B}$, which we replace by $\mathscr{A} \rightarrow \mathscr{B}; \mathfrak{A}'$, where $\mathfrak{A}'$ is the antecedent of the consecution obtained from $\mathfrak{A} \vdash \mathscr{A}$ by the permutation and grouping steps. The lower consecutions $\mathfrak{B}'[\mathscr{A} \rightarrow \mathscr{B}; \mathfrak{A}'] \vdash \mathscr{C}$ correspond to the sequent $\alpha, \beta, \mathscr{A} \rightarrow \mathscr{B} \vdash \mathscr{C}$. The derivation of permutations and groupings is justified by some of the $\mathsf{B} \vdash$, $\mathsf{B}' \vdash$, $\mathsf{T}_t \vdash$, $\mathsf{K}_t \vdash$ and $\mathsf{M}_t \vdash$ rules, and the last step is justified by an application of the $\rightarrow\vdash$ rule.

5. If the rule is $W \vdash$, then, first of all, we determine if there are further $W \vdash$ steps immediately below the current one. If there is no other $W \vdash$, then from the antecedent of the consecution corresponding to $\alpha, \mathscr{A}, \mathscr{A} \vdash \mathscr{B}$, we generate all the permutations and groupings of atomic structures that contain a substructure of the form $\mathfrak{A}; \mathscr{A}; \mathscr{A}$. (Note that if the whole antecedent of the starting consecution is $\mathscr{A}; \mathscr{A}$, then a suitable $\mathfrak{A}$ can be supplied by $\mathsf{KI}_t \vdash$.) There are finitely many such consecutions and they are derivable as above. If there is a substructure of the form $\boldsymbol{t}; \boldsymbol{t}$, then we apply $\mathsf{M}_t \vdash$. The lower consecutions corresponding to $\alpha, \mathscr{A} \vdash \mathscr{B}$ are obtained by replacing $\mathfrak{A}; \mathscr{A}; \mathscr{A}$ with $\mathfrak{A}; \mathscr{A}$. The last step is an instance of the $\mathsf{W} \vdash$ rule.

If there are $n > 1$ consecutive applications of the $W \vdash$ rule in Δ at this point, then we consider them together. First, we divide n into all the possible ordered sums (including n itself). (For example, if $n = 3$ then we get $1+1+1$, $1+2$, $2+1$ and 3.) It is trivial that there are finitely many possibilities for each n. The total number of ordered sums is 2^{n-1}.[15]

[15]We give a proof of this in the *Appendix*, because we could not find a reference, though we are sure that this simple claim has been proved before.

In the case a summand is 1, we proceed as described in the first paragraph (in step 5).

If a summand is > 1, then we consider all the structures that can be formed from the contracted formulas. The number and the shape of the contracted formulas is always determined by the proof in $LR^{t}_{\to}$. There are finitely many possible structures. Let us assume that $\mathfrak{B}$ is one of the possible structures. Then from the consecution that corresponds to the top sequent in the series of $W\vdash$ steps, we generate all the permutations and groupings of atomic structures that contain an occurrence of a substructure of the form $\mathfrak{A};\mathfrak{B};\mathfrak{B}$. The lower consecution corresponding to the bottom sequent in the series of $W\vdash$ steps (in the segment delineated by the summand) is obtained by replacing $\mathfrak{A};\mathfrak{B};\mathfrak{B}$ with $\mathfrak{A};\mathfrak{B}$. The generation of permutations and groupings is justified by some of the rules $\mathsf{B}\vdash$, $\mathsf{B}'\vdash$, $\mathsf{T}_{t}\vdash$, $\mathsf{K}_{t}\vdash$ and $\mathsf{M}_{t}\vdash$; the last steps is an application of the $\mathsf{W}\vdash$ rule.

We give an *example*, or more precisely, a proof in $[LR^{t}_{\to}]$ and some of its transforms. To provide an example in full details (beyond a proof of something nearly trivial such as $\boldsymbol{t}\vdash\mathscr{A}\to\mathscr{A}$) would require an excessive amount of space. The formula was suggested to us by a referee, and it seems to us to provide an interesting example to illustrate the differences between an $[LR^{t}_{\to}]$ proof and an $LT^{\circledcirc}_{\to}$ proof of a theorem of $T_{\to}$.

$$
\begin{array}{rl}
\mathscr{B}\vdash\mathscr{B}\quad \mathscr{B}\vdash\mathscr{B} & \\ \hline
\mathscr{A}\vdash\mathscr{A}\quad \mathscr{B}\to\mathscr{B},\mathscr{B}\vdash\mathscr{B} & [\to\vdash] \\ \hline
\mathscr{C}\vdash\mathscr{C}\quad \mathscr{B}\to\mathscr{B},\mathscr{A}\to\mathscr{B},\mathscr{A}\vdash\mathscr{B} & [\to\vdash] \\ \hline
\mathscr{B}\to\mathscr{B},\mathscr{A}\to\mathscr{B},\mathscr{C}\to\mathscr{A},\mathscr{C}\vdash\mathscr{B} & [\to\vdash] \\ \hline
\mathscr{A}\vdash\mathscr{A}\quad \mathscr{B}\to\mathscr{B},\mathscr{A}\to\mathscr{B},\mathscr{C}\to\mathscr{A}\vdash\mathscr{C}\to\mathscr{B} & \vdash\to \\ \hline
\mathscr{C}\vdash\mathscr{C}\quad \mathscr{A}\to\mathscr{B},\mathscr{C}\to\mathscr{A},\mathscr{A}\to\mathscr{B}\to\mathscr{B},\mathscr{A}\vdash\mathscr{C}\to\mathscr{B} & [\to\vdash] \\ \hline
\mathscr{C}\vdash\mathscr{C}\quad \mathscr{A}\to\mathscr{B},\mathscr{C}\to\mathscr{A}\to\mathscr{B}\to\mathscr{B},\mathscr{C}\to\mathscr{A},\mathscr{A},\mathscr{C}\vdash\mathscr{C}\to\mathscr{B} & [\to\vdash] \\ \hline
\mathscr{B}\vdash\mathscr{B}\quad \mathscr{A}\to\mathscr{B},\mathscr{C}\to\mathscr{A}\to\mathscr{B}\to\mathscr{B},\mathscr{C}\to\mathscr{A},\mathscr{C}\vdash\mathscr{C}\to\mathscr{B} & [\to\vdash]^{\dagger} \\ \hline
(\mathscr{C}\to\mathscr{B})\to\mathscr{B},\mathscr{A}\to\mathscr{B},\mathscr{C}\to\mathscr{A}\to\mathscr{B}\to\mathscr{B},\mathscr{C}\to\mathscr{A},\mathscr{C}\vdash\mathscr{B} & [\to\vdash] \\ \hline
\boldsymbol{t},(\mathscr{C}\to\mathscr{B})\to\mathscr{B},\mathscr{A}\to\mathscr{B},\mathscr{C}\to\mathscr{A}\to\mathscr{B}\to\mathscr{B},\mathscr{C}\to\mathscr{A},\mathscr{C}\vdash\mathscr{B} & \boldsymbol{t}\vdash \\ \hline
\boldsymbol{t}\vdash((\mathscr{C}\to\mathscr{B})\to\mathscr{B})\to(\mathscr{A}\to\mathscr{B})\to(\mathscr{C}\to\mathscr{A}\to\mathscr{B}\to\mathscr{B})\to(\mathscr{C}\to\mathscr{A})\to\mathscr{C}\to\mathscr{B} & \vdash\to\text{'s}
\end{array}
$$

The proof shows that the contractions are due solely to applications of the $[\to\vdash]$ rule, that is, the proof is irredundant. (A careful look reveals that there are *no* cognate sequents in the proof except pairs of axioms, which are not above each other.) The next proof is the result of the transformation from $[LR^{t}_{\to}]$ into $LR^{t}_{\to}$. Here the contraction steps are made explicit, and we excerpt the section of the proof where this modification occurs. (We †'d the step in the above proof that contains contraction.)

$$
\begin{array}{cl}
\vdots & \\
\mathscr{C}\vdash\mathscr{C}\quad \mathscr{A}\to\mathscr{B},\mathscr{C}\to\mathscr{A}\to\mathscr{B}\to\mathscr{B},\mathscr{C}\to\mathscr{A},\mathscr{A},\mathscr{C}\vdash\mathscr{C}\to\mathscr{B} & \\ \hline
\mathscr{A}\to\mathscr{B},\mathscr{C}\to\mathscr{A}\to\mathscr{B}\to\mathscr{B},\mathscr{C}\to\mathscr{A},\mathscr{C}\to\mathscr{A},\mathscr{C},\mathscr{C}\vdash\mathscr{C}\to\mathscr{B} & \to\vdash \\ \hline
\mathscr{A}\to\mathscr{B},\mathscr{C}\to\mathscr{A}\to\mathscr{B}\to\mathscr{B},\mathscr{C}\to\mathscr{A},\mathscr{C}\to\mathscr{A},\mathscr{C}\vdash\mathscr{C}\to\mathscr{B} & W\vdash \\ \hline
\mathscr{A}\to\mathscr{B},\mathscr{C}\to\mathscr{A}\to\mathscr{B}\to\mathscr{B},\mathscr{C}\to\mathscr{A},\mathscr{C}\vdash\mathscr{C}\to\mathscr{B} & W\vdash
\end{array}
$$

The proof does not have any $\vdash\boldsymbol{t}$ axioms, hence, we can proceed immediately to transform the proof from $LR^{t}_{\to}$ into $LT^{\circledcirc}_{\to}$. Multiplicity of all permutations and groupings gets quickly in the way of presenting all of them. However, from the proof that we give here, it should be clear how the generation of all possible consecutions blends smoothly into the structure of the whole proof that is determined by the structure of the proof in $LR^{t}_{\to}$.

The order of the $\to\vdash$, $\vdash\to$, $W\vdash$ and $\boldsymbol{t}\vdash$ rules as well as the principal formulas in the applications of the rules are retained. In this particular example, the successive contractions are not combined into one contraction on a complex structure. The new consecutions are inserted because of applications of the $\mathsf{B}\vdash$ and $\mathsf{B}'\vdash$ rules. As the middle segment of the proof shows, these rules can produce quite sophisticated rearrangements of a structure. The formula is a theorem of $T_{\to}$, and the particular product of the τ algorithm that we give here does not use either of the two rules $\mathsf{K}_{\boldsymbol{t}}\vdash$ or $\mathsf{T}_{\boldsymbol{t}}\vdash$.

$$
\begin{array}{rl}
\mathscr{B}\vdash\mathscr{B} \quad \mathscr{B}\vdash\mathscr{B} & \\ \hline
\mathscr{A}\vdash\mathscr{A} \quad \mathscr{B}\to\mathscr{B};\mathscr{B}\vdash\mathscr{B} & \to\vdash \\ \hline
\mathscr{C}\vdash\mathscr{C} \quad \mathscr{B}\to\mathscr{B};(\mathscr{A}\to\mathscr{B};\mathscr{A})\vdash\mathscr{B} & \to\vdash \\ \hline
\mathscr{B}\to\mathscr{B};(\mathscr{A}\to\mathscr{B};(\mathscr{C}\to\mathscr{A};\mathscr{C}))\vdash\mathscr{B} & \to\vdash \\ \hline
\mathscr{B}\to\mathscr{B};(\mathscr{A}\to\mathscr{B};\mathscr{C}\to\mathscr{A};\mathscr{C})\vdash\mathscr{B} & \mathsf{B}\vdash \\ \hline
\mathscr{A}\to\mathscr{B};\mathscr{C}\to\mathscr{A};\mathscr{B}\to\mathscr{B};\mathscr{C}\vdash\mathscr{B} & \mathsf{B}'\vdash \\ \hline
\mathscr{A}\vdash\mathscr{A} \quad \mathscr{A}\to\mathscr{B};\mathscr{C}\to\mathscr{A};\mathscr{B}\to\mathscr{B}\vdash\mathscr{C}\to\mathscr{B} & \vdash\to \\ \hline
\mathscr{C}\vdash\mathscr{C} \quad \mathscr{A}\to\mathscr{B};\mathscr{C}\to\mathscr{A};(\mathscr{A}\to\mathscr{B}\to\mathscr{B};\mathscr{A})\vdash\mathscr{C}\to\mathscr{B} & \to\vdash \\ \hline
\mathscr{C}\vdash\mathscr{C} \quad \mathscr{A}\to\mathscr{B};\mathscr{C}\to\mathscr{A};(\mathscr{C}\to\mathscr{A}\to\mathscr{B}\to\mathscr{B};\mathscr{C};\mathscr{A})\vdash\mathscr{C}\to\mathscr{B} & \to\vdash \\ \hline
\mathscr{A}\to\mathscr{B};\mathscr{C}\to\mathscr{A};(\mathscr{C}\to\mathscr{A}\to\mathscr{B}\to\mathscr{B};\mathscr{C};(\mathscr{C}\to\mathscr{A};\mathscr{C}))\vdash\mathscr{C}\to\mathscr{B} & \to\vdash \\ \hline
\mathscr{A}\to\mathscr{B};\mathscr{C}\to\mathscr{A};(\mathscr{C}\to\mathscr{A};(\mathscr{C}\to\mathscr{A}\to\mathscr{B}\to\mathscr{B};\mathscr{C});\mathscr{C})\vdash\mathscr{C}\to\mathscr{B} & \mathsf{B}'\vdash \\ \hline
\mathscr{A}\to\mathscr{B};\mathscr{C}\to\mathscr{A};(\mathscr{C}\to\mathscr{A}\to\mathscr{B}\to\mathscr{B};\mathscr{C}\to\mathscr{A};\mathscr{C};\mathscr{C})\vdash\mathscr{C}\to\mathscr{B} & \mathsf{B}'\vdash \\ \hline
\mathscr{A}\to\mathscr{B};\mathscr{C}\to\mathscr{A};(\mathscr{C}\to\mathscr{A}\to\mathscr{B}\to\mathscr{B};\mathscr{C}\to\mathscr{A};\mathscr{C});\mathscr{C}\vdash\mathscr{C}\to\mathscr{B} & \mathsf{B}\vdash \\ \hline
\mathscr{A}\to\mathscr{B};\mathscr{C}\to\mathscr{A};(\mathscr{C}\to\mathscr{A}\to\mathscr{B}\to\mathscr{B};\mathscr{C}\to\mathscr{A});\mathscr{C};\mathscr{C}\vdash\mathscr{C}\to\mathscr{B} & \mathsf{B}\vdash \\ \hline
\mathscr{C}\to\mathscr{A}\to\mathscr{B}\to\mathscr{B};(\mathscr{A}\to\mathscr{B};\mathscr{C}\to\mathscr{A});\mathscr{C}\to\mathscr{A};\mathscr{C};\mathscr{C}\vdash\mathscr{C}\to\mathscr{B} & \mathsf{B}'\vdash \\ \hline
\mathscr{A}\to\mathscr{B};\mathscr{C}\to\mathscr{A}\to\mathscr{B}\to\mathscr{B};\mathscr{C}\to\mathscr{A};\mathscr{C}\to\mathscr{A};\mathscr{C};\mathscr{C}\vdash\mathscr{C}\to\mathscr{B} & \mathsf{B}'\vdash \\ \hline
\mathscr{A}\to\mathscr{B};\mathscr{C}\to\mathscr{A}\to\mathscr{B}\to\mathscr{B};\mathscr{C}\to\mathscr{A};\mathscr{C}\to\mathscr{A};\mathscr{C}\vdash\mathscr{C}\to\mathscr{B} & \mathsf{W}\vdash \\ \hline
\mathscr{B}\vdash\mathscr{B} \quad \mathscr{A}\to\mathscr{B};\mathscr{C}\to\mathscr{A}\to\mathscr{B}\to\mathscr{B};\mathscr{C}\to\mathscr{A};\mathscr{C}\vdash\mathscr{C}\to\mathscr{B} & \mathsf{W}\vdash \\ \hline
(\mathscr{C}\to\mathscr{B})\to\mathscr{B};(\mathscr{A}\to\mathscr{B};\mathscr{C}\to\mathscr{A}\to\mathscr{B}\to\mathscr{B};\mathscr{C}\to\mathscr{A};\mathscr{C})\vdash\mathscr{B} & \to\vdash \\ \hline
(\mathscr{C}\to\mathscr{B})\to\mathscr{B};\mathscr{A}\to\mathscr{B};\mathscr{C}\to\mathscr{A}\to\mathscr{B}\to\mathscr{B};\mathscr{C}\to\mathscr{A};\mathscr{C}\vdash\mathscr{B} & \mathsf{B}\vdash\text{'s} \\ \hline
\boldsymbol{t};(\mathscr{C}\to\mathscr{B})\to\mathscr{B};\mathscr{A}\to\mathscr{B};\mathscr{C}\to\mathscr{A}\to\mathscr{B}\to\mathscr{B};\mathscr{C}\to\mathscr{A};\mathscr{C}\vdash\mathscr{B} & \boldsymbol{t}\vdash \\ \hline
\boldsymbol{t}\vdash((\mathscr{C}\to\mathscr{B})\to\mathscr{B})\to(\mathscr{A}\to\mathscr{B})\to(\mathscr{C}\to\mathscr{A}\to\mathscr{B}\to\mathscr{B})\to(\mathscr{C}\to\mathscr{A})\to\mathscr{C}\to\mathscr{B} & \vdash\to\text{'s}
\end{array}
$$

Scrutinizing this proof, we see that it is within $LT^{t}_{\to}$, hence the formula

$$((\mathscr{C}\to\mathscr{B})\to\mathscr{B})\to(\mathscr{A}\to\mathscr{B})\to(\mathscr{C}\to\mathscr{A}\to\mathscr{B}\to\mathscr{B})\to(\mathscr{C}\to\mathscr{A})\to\mathscr{C}\to\mathscr{B}$$

is a theorem of $T_{\to}$. Of course, another proof that on line 3 has a consecution $\mathscr{B}\to\mathscr{B};(\mathscr{B};\boldsymbol{t})\vdash\mathscr{B}$, and then uses some other permutation steps to yield a proof of the same formula is—obviously—not a proof in $LT^{t}_{\to}$, because $\mathsf{K}_{\boldsymbol{t}}\vdash$ is not a rule in $LT^{t}_{\to}$.

As a preparation for the next lemma, first we note that $\mathsf{W}\vdash$ is a rule in $LT^{t}_{\to}$ which can be applied to atomic structures (i.e., formulas). Thus, the step that makes explicit the hidden contractions in $[LR^{t}_{\to}]$ proofs as $W\vdash$ steps in the corresponding

$LR^t_\to$ proof is unproblematic with respect to preserving the provability of $T^t_\to$ theorems.

Second, lemma 7 guarantees that an irredundant proof in $[LR^t_\to]$, which contains occurrences of the $\vdash \boldsymbol{t}$ axiom can be modified into an $LR^t_\to$ proof with axioms that have exact counterparts in $LT^t_\to$, hence, in $LT^{\textcircled{t}}_\to$ too. Although non-contractibility is—obviously—not preserved by these two modifications, provability is ensured.

LEMMA 11 (Distinction). *If $\mathscr{A}$ is a theorem of $T^t_\to$ with $\vdash$ $\boldsymbol{t}$-free proofs $\Delta_1, \ldots, \Delta_n$ in $LR^t_\to$ obtained from the irredundant proofs of $\boldsymbol{t} \vdash \mathscr{A}$ in $[LR^t_\to]$, then $\bigcup_{i=1}^n \tau(\Delta_i)$ contains a Δ' such that Δ' is a* proof *of $\boldsymbol{t} \vdash \mathscr{A}$ in $LT^t_\to$.*

PROOF. We consider the τ transformation.

1. Steps 1 and 2 clearly preserve provability, because $LT^{\textcircled{t}}_\to$ matches the axiom and emulates the $\boldsymbol{t} \vdash$ rule by $\mathsf{KI}_t \vdash$. The use of $\mathsf{KI}_t \vdash$ guarantees that if the formula in question is provable in $T^t_\to$, then at this point, no problematic step is introduced into the proof.

2. Steps 3 and 4 of τ are guaranteed to produce a valid continuation of a proof, because $LT^{\textcircled{t}}_\to$ can generate all the permutations and groupings of formulas in a sequent. Since all the possible permutations and grouping are considered in both cases, those that do not use $\mathsf{K}_t \vdash$ or $\mathsf{T}_t \vdash$ are also included, thereby generating an $LT^t_\to$ continuation.

3. Step 5 allows the same as well as more powerful contractions that $LR^t_\to$ does, which guarantees that a contraction step that is legitimate in $LT^t_\to$ (and is a series of $W \vdash$ steps in $LR^t_\to$), does not need to be rendered as a series of contractions on atomic formulas, which would require a regrouping of formulas in a way that is potentially not allowed for a $T^t_\to$ theorem.

The essential difference between $LT^{\textcircled{t}}_\to$ and $LR^t_\to$ proofs is in the way how permutation and grouping is handled, because of the difference between structures and sequences of formulas. In $LT^t_\to$, all the permutation that there is, results from $\mathsf{B} \vdash$ and $\mathsf{B}' \vdash$ steps (as it is illustrated by the previous example). It is easy to see that no interplay between $\mathsf{KI}_t \vdash$ and $\mathsf{B}' \vdash$ (let alone, $\mathsf{B} \vdash$) can yield new permutations, effectively excluding $\mathsf{KI}_t \vdash$'s usefuleness for creating new permutations in a consecution. Thus, the π *transformation* applied to a proof in $LT^t_\to$ is an analogue of *building in contractions* into the $[\to\vdash]$ rule in $[LR^t_\to]$. ⊣

The τ transformation applied to all the proofs of a formula generated from the proof search in $[LR^t_\to]$ produces at least one proof. However, typically, the procedure will produce more than one proof.

Thus it is useful to recall here that the permutations and groupings have *upper bounds* on the number of generated objects, and all the additional $\boldsymbol{t}$'s introduced (if any) are occurrences of one formula, which means that adjacent occurrences can be contracted away and unless the target formula contains an occurrence of $\boldsymbol{t}$, the $\boldsymbol{t}$'s do not even need to be counted as atomic structures, because they must be collapsed into one $\boldsymbol{t}$ in the final consecution.

Each sequent in a proof is finite, in particular, the antecedent comprises finitely many formula occurrences. Therefore, there are finitely many *permutations* of each antecedent. If the cardinality of the multiset, which is the antecedent, is n, and each element occurs once, then there are $n!$ permutations. If there are multiple occurrences of some formula, then the number of distinct sequences is less. If there are n formula occurrences of i distinct formulas, each with $m_1, \ldots, m_i$ occurrences, then the number of distinct sequences is $\dfrac{n!}{m_1! \cdot \cdots \cdot m_i!}$.

Combinatory terms (and λ-terms) are not simply strings, but they are formed by the binary function application operation, which is not assumed to be associative or commutative. It is a well-known result in combinatorics (and it is often mentioned in the literature on λ-calculi) that the number of possible (binary) groupings of a string of length n is given by the $n - 1$th *Catalan number*.[16] The Catalan numbers can be calculated by the formula

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

If the length of the antecedent is n and the order of the formulas is already fixed, then for each such "string" or sequence of formulas, there are C_{n-1} structures.

The number of structures might get large quickly, but each step produces finitely many further potential continuations for the transformed proof Δ'. In each step, we deal with at least one node of the original proof Δ. This means that the transformation *terminates* after finitely many steps, and it yields a finite set of proofs in $LT^{\circledcirc{t}}_{\rightarrow}$ for each Δ in $LR^{t}_{\rightarrow}$. By the decidability of $R^{t}_{\rightarrow}$, there are finitely many proofs for a provable formula in $[LR^{t}_{\rightarrow}]$, yielding finitely many ($\vdash \boldsymbol{t}$-less) proofs in $LR^{t}_{\rightarrow}$; thus, in total we have *finitely many proofs* in $LT^{\circledcirc{t}}_{\rightarrow}$ obtained via the transformation procedure, for any theorem of $R^{t}_{\rightarrow}$.

§4. Inspection of proofs in $LT^{\circledcirc{t}}_{\rightarrow}$. The proofs in $LT^{\circledcirc{t}}_{\rightarrow}$ may be inspected. There are more rules in $LT^{\circledcirc{t}}_{\rightarrow}$ than in $[LR^{t}_{\rightarrow}]$, and in fact, it is no longer true that every concrete proof uniquely determines which rule has been applied at each step. For example, the following step may be an instance of the $\mathsf{B} \vdash$ or of the $\mathsf{B}' \vdash$ rule.

$$\frac{\mathfrak{A}[\mathfrak{B}; (\mathfrak{B}; \mathfrak{C})] \vdash \mathscr{A}}{\mathfrak{A}[\mathfrak{B}; \mathfrak{B}; \mathfrak{C}] \vdash \mathscr{A}}$$

However, the ambiguity in this example does not cause a difficulty for our purposes, since $\mathsf{B} \vdash$ and $\mathsf{B}' \vdash$ are both rules of $LT^{t}_{\rightarrow}$, and we may choose in each case, where $\mathsf{B} \vdash$ and $\mathsf{B}' \vdash$ are both applicable, to go with $\mathsf{B}' \vdash$.

A similar ambiguity may occur in some cases between $\mathsf{M}_{\boldsymbol{t}} \vdash$ and $\mathsf{W} \vdash$, and we may stipulate that in each case $\mathsf{M}_{\boldsymbol{t}} \vdash$ is the rule that has been applied.

The rules $\mathsf{KI}_{\boldsymbol{t}} \vdash$ and $\mathsf{K}_{\boldsymbol{t}} \vdash$ may also be applicable at the same time, as the following shows.

$$\frac{\mathfrak{A}[\boldsymbol{t}] \vdash \mathscr{A}}{\mathfrak{A}[\boldsymbol{t}; \boldsymbol{t}] \vdash \mathscr{A}}$$

In this case, we always assume that the rule that was applied is $\mathsf{KI}_{\boldsymbol{t}} \vdash$. This is a rule of $LT^{t}_{\rightarrow}$, whereas $\mathsf{K}_{\boldsymbol{t}} \vdash$ is not.

Except in these cases, the rule that has been applied in a concrete proof Δ' can be uniquely determined from the consecutions involved.

The last step in the decision procedure is to scrutinize all the proofs of a formula for applications of the rules $\mathsf{K}_{\boldsymbol{t}} \vdash$ and $\mathsf{T}_{\boldsymbol{t}} \vdash$. If the formula has at least *one proof*

[16]See, for example, Bimbó (2012, A.1).

that is free of applications of these rules, then the formula is a theorem of $T^{t}_{\rightarrow}$. Otherwise, the formula is not a theorem of $T^{t}_{\rightarrow}$ (though it is a theorem of $R^{t}_{\rightarrow}$).

At the beginning of section 3.1, we assumed that $\mathscr{A}$, a potential theorem of $T_{\rightarrow}$, is t-free. (Obviously, t is not in the language of $T_{\rightarrow}$.) However, neither the transformation from $[LR^{t}_{\rightarrow}]$ to $LR^{t}_{\rightarrow}$, nor the further transformation into $LT^{\textcircled{t}}_{\rightarrow}$ depends on t occurring or not occurring in $\mathscr{A}$ itself. Thus, both $T_{\rightarrow}$ and $T^{t}_{\rightarrow}$ are *decidable*, and the same decision procedure may be used to determine theoremhood in them.

§5. **Conclusions.** The logic of ticket entailment can be easily formulated and it is a well-motivated logic. Fusion ($\circ$, or intensional conjunction) may be added conservatively to $T^{t}_{\rightarrow}$ and it has been known for a long time that t is a *left identity* for fusion.

In the absence of t, it is not clear how to "Gentzenize" $T_{\rightarrow}$ into a calculus that is easy to handle. In the presence of t, it was not clear until some of our earlier work how to prove the *admissibility of the cut rule*.

Once a well-behaved consecution calculus for $T^{t}_{\rightarrow}$ has been defined, we needed another insight, namely, that by making t the full identity we get $R^{t}_{\rightarrow}$. The interaction between $\circ$ (or ;) and t (as identity) produces the commutativity of fusion (or the provability of the principal type of C). Using t with special structural rules goes against the common-sense way to extend $LT^{t}_{\rightarrow}$ to a calculus equivalent to $R^{t}_{\rightarrow}$, which would simply add the missing permutation by a rule that corresponds to C.

Our proof of the decidability is unusual in the sense that instead of getting by with as few connectives as possible, we extended the logic for the decidability proof. Adding t proved useful in other relevance logics for other purposes, such as the algebraization of a logic. However, it was far from clear at first that t would turn out to be useful for the proof of the decidability of $T_{\rightarrow}$.[17]

With the technical obstacles arising in consecution calculi now overcome, our proof seems to us quite transparent. We conjecture that the complexity of our decision procedure for $T^{t}_{\rightarrow}$ is the same as the complexity of Kripke's decision procedure for $R_{\rightarrow}$. We will address this question in a subsequent paper.

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[17]Of course, it is well possible that in another proof t is not needed.

Appendix.

Axioms and rules of $LR_{\to}$ and $LR^{t}_{\to}$.

$$\mathscr{A} \vdash \mathscr{A} \ \text{(id)} \qquad\qquad \vdash \boldsymbol{t} \ (\vdash \boldsymbol{t})$$

$$\frac{\alpha \vdash \mathscr{A} \qquad \mathscr{B}, \beta \vdash \mathscr{C}}{\alpha, \mathscr{A} \to \mathscr{B}, \beta \vdash \mathscr{C}} \ (\to\vdash) \qquad\qquad \frac{\alpha, \mathscr{A} \vdash \mathscr{B}}{\alpha \vdash \mathscr{A} \to \mathscr{B}} \ (\vdash\to)$$

$$\frac{\alpha \vdash \mathscr{A}}{\boldsymbol{t}, \alpha \vdash \mathscr{A}} \ (\boldsymbol{t}\vdash) \qquad\qquad \frac{\alpha, \mathscr{A}, \mathscr{A} \vdash \mathscr{B}}{\alpha, \mathscr{A} \vdash \mathscr{B}} \ (W\vdash)$$

Axioms and rules of $[LR_{\to}]$ and $[LR^{t}_{\to}]$. The rules $\to\vdash$ and $W\vdash$ are omitted, and the following rule is added.

$$\frac{\alpha \vdash \mathscr{A} \qquad \beta, \mathscr{B} \vdash \mathscr{C}}{[\alpha, \beta, \mathscr{A} \to \mathscr{B}] \vdash \mathscr{C}} \ ([\to\vdash]),$$

where $[\alpha, \beta, \mathscr{A} \to \mathscr{B}]$ is $\alpha, \beta, \mathscr{A} \to \mathscr{B}$, but possibly, with fewer occurrences of formulas (without omitting any formula). The number of occurrences of $\mathscr{A} \to \mathscr{B}$ may be reduced by 1 or 2, whereas the number of occurrences of other formulas may be reduced by 1.

Axiom and rules of $LT^{t}_{\to}$, $LR^{t}_{\to;}$ and $LT^{\textcircled{t}}_{\to}$.

$$\mathscr{A} \vdash \mathscr{A} \ \text{(id)}$$

$$\frac{\mathfrak{A} \vdash \mathscr{B} \quad \mathfrak{C}[\mathscr{D}] \vdash \mathscr{E}}{\mathfrak{C}[\mathscr{B} \to \mathscr{D}; \mathfrak{A}] \vdash \mathscr{E}} \ (\to\vdash) \qquad\qquad \frac{\mathfrak{A}; \mathscr{B} \vdash \mathscr{C}}{\mathfrak{A} \vdash \mathscr{B} \to \mathscr{C}} \ (\vdash\to)$$

$$\frac{\mathfrak{A}[\mathfrak{B}; (\mathfrak{C}; \mathfrak{D})] \vdash \mathscr{E}}{\mathfrak{A}[\mathfrak{B}; \mathfrak{C}; \mathfrak{D}] \vdash \mathscr{E}} \ (\mathsf{B}\vdash) \qquad \frac{\mathfrak{A}[\mathfrak{B}; (\mathfrak{C}; \mathfrak{D})] \vdash \mathscr{E}}{\mathfrak{A}[\mathfrak{C}; \mathfrak{B}; \mathfrak{D}] \vdash \mathscr{E}} \ (\mathsf{B}'\vdash) \qquad \frac{\mathfrak{A}[\mathfrak{B}; \mathfrak{C}; \mathfrak{C}] \vdash \mathscr{D}}{\mathfrak{A}[\mathfrak{B}; \mathfrak{C}] \vdash \mathscr{D}} \ (\mathsf{W}\vdash)$$

$$\frac{\mathfrak{A}[\mathfrak{B}] \vdash \mathscr{C}}{\mathfrak{A}[\boldsymbol{t}; \mathfrak{B}] \vdash \mathscr{C}} \ (\mathsf{KI}_t\vdash) \qquad\qquad \frac{\mathfrak{A}[\boldsymbol{t}; \boldsymbol{t}] \vdash \mathscr{B}}{\mathfrak{A}[\boldsymbol{t}] \vdash \mathscr{B}} \ (\mathsf{M}_t\vdash)$$

$$\frac{\mathfrak{A}[\mathfrak{B}; \mathfrak{C}; \mathfrak{D}] \vdash \mathscr{E}}{\mathfrak{A}[\mathfrak{B}; \mathfrak{D}; \mathfrak{C}] \vdash \mathscr{E}} \ (\mathsf{C}\vdash)$$

$$\frac{\mathfrak{A}[\mathfrak{B}] \vdash \mathscr{C}}{\mathfrak{A}[\mathfrak{B}; \boldsymbol{t}] \vdash \mathscr{C}} \ (\mathsf{K}_t\vdash) \qquad\qquad \frac{\mathfrak{A}[\mathfrak{B}; \boldsymbol{t}] \vdash \mathscr{C}}{\mathfrak{A}[\boldsymbol{t}; \mathfrak{B}] \vdash \mathscr{C}} \ (\mathsf{T}_t\vdash)$$

$LT^{t}_{\to}$ includes the axiom and the first *seven* rules. $LR^{t}_{\to;}$ adds $\mathsf{C}\vdash$, whereas $LT^{\textcircled{t}}_{\to}$ adds $\mathsf{K}_t\vdash$ and $\mathsf{T}_t\vdash$ to $LT^{t}_{\to}$. (Incidentally, the brackets [ ] in these calculi single out an occurrence of a structure in a structure and have no connection to implicit contractions.)

The number of ordered sums. A positive natural number n may be represented in unary notation as a sequence of n 1's. The number of ordered sums yielding n can then easily be seen to be the number of ways to divide the sequence representing the number n into subsequences. The natural numbers can be inductively generated by adding one 1 on the right. If $n = 1$, then there is one ordered sum, 1 itself. $n + 1$ is obtained from n by adding a 1 on the right. If there are m ordered sums for n, then each of these with one more 1 concatenated on the right (as a subsequence itself) is an ordered sum of $n + 1$. Also, if the m ordered sums of n are of the form $o + p$ (with o possibly ε), then $o + (p + 1)$ is an ordered sum of $n + 1$. There is no other way to generate an ordered sum of $n + 1$; hence, there are $2 \cdot m$ ordered sums of $n + 1$.

$1 = 2^0$, and the successor doubles the number of ordered sums, which means that for n, there are 2^{n-1} ordered sums.

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