WQO dichotomy for 3-graphs

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Abstract. We investigate data-enriched models, like Petri nets with data, where executability of a transition is conditioned by a relation between data values involved. Decidability status of various decision problems in such models may depend on the structure of data domain. According to the WQO Dichotomy Conjecture, if a data domain is homogeneous then it either exhibits a well quasi-order (in which case decidability follows by standard arguments), or essentially all the decision problems are undecidable for Petri nets over that data domain.
We confirm the conjecture for data domains being 3-graphs (graphs with 2-colored edges). On the technical level, this results is a significant step beyond known classification results for homogeneous structures.

1 Introduction

In Petri nets with data, tokens carry values from some data domain, and executability of transitions is conditioned by a relation between data values involved. One can consider *unordered data*, like in [24], i.e., an infinite data domain with the equality as the only relation; or *ordered data*, like in [20], i.e., an infinite densely totally ordered data domain; or timed data, like in timed Petri nets [1] and timed-arc Petri nets [15]. In [19] an abstract setting of Petri nets with an arbitrary fixed data domain $\mathbb{A}$ has been introduced, parametric in a relational structure $\mathbb{A}$. The setting uniformly subsumes unordered, ordered and timed data (represented by $\mathbb{A} = (\mathbb{N}, =)$, $\mathbb{A} = (\mathbb{Q}, \leq)$ and $\mathbb{A} = (\mathbb{Q}, \leq, +1)$, respectively).

Following [19], in order to enable finite presentation of Petri nets with data, and in particular to consider such models as input to algorithms, we restrict to relational structures $\mathbb{A}$ that are *homogeneous* [22] and *effective* (the formal definitions are given in Section 2). Certain standard decision problems (like the termination problem, the boundedness problem, or the coverability problem, jointly called from now on *standard problems*) are all decidable for Petri nets with ordered data [20] (and in consequence also for Petri nets with unordered data), as the model fits into the framework of well-structured transition systems of [11]. Most importantly, the structure $\mathbb{A} = (\mathbb{Q}, \leq)$ of ordered data *admits well quasi-order* (WQO) in the following sense: for any WQO X, the set of finite induced substructures of $(\mathbb{Q}, \leq)$ (i.e., finite total orders) labeled by elements of X, ordered

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naturally by embedding, is a WQO (this is exactly Higman's lemma). Moreover, essentially the same argument can be used for any other homogeneous effective data domain which admits WQO (see [19] for details). On the other hand, for certain homogeneous effective data domains $\mathbb{A}$ the standard problems become all undecidable. In the quest for understanding the decidability borderline, the following hypothesis has been formulated in [19]:

*Conjecture 1 (*WQO *Dichotomy Coinjecture [19]).* For an effective homogeneous structure $\mathbb{A}$, either $\mathbb{A}$ admits WQO (in which case the standard problems are decidable for Petri nets with data $\mathbb{A}$), or all the standard problems are undecidable for Petri nets with data $\mathbb{A}$.

According to [19], the conjecture could have been equivalently stated for another data-enriched models, e.g., for finite automata with one register [2]. In this paper we consider, for the sake of presentation, only Petri nets with data. WQO Dichotomy Conjecture holds in special cases when data domains $\mathbb{A}$ are undirected or directed graphs, due to the known classifications of homogeneous graphs [18,6].

Contributions. We confirm the WQO Dichotomy Conjecture for data domains $\mathbb{A}$ being *strongly*[1] homogeneous *3-graphs*. A 3-graph is a logical structure with three irreflexive symmetric binary relations such that every pair of elements of $\mathbb{A}$ belongs to exactly one of the relations (essentially, a clique with 3-colored edges).

Our main technical contribution is a complex analysis of possible shapes of strongly homogeneous 3-graphs, constituting the heart of the proof. We believe that this is a significant step towards full classification of homogeneous 3-graphs. The classification of homogeneous structures is a well-known challenge in model theory, and has been only solved in some cases by now: for undirected graphs [18], directed graphs (the proof of Cherlin spans a book [6]), multi-partite graphs [16], and few others (the survey [22] is an excellent overview of homogeneous structures). Although the full classification of homogeneous 3-graphs was not our primary objective, we believe that our analysis significantly improves our understanding of these structures and can be helpful for classification.

Our result does not fully settle the status of the WQO Dichotomy Conjecture. Dropping the (mild) strong homogeneity assumption, as well as extending the proof to arbitrarily many symmetric binary relations, is left for future work.

Related research. Net models similar to Petri nets with data have been continuously proposed since the 80s, including, among the others, high-level Petri nets [13], colored Petri nets [17], unordered and ordered data nets [20], ν-Petri nets [24], and constraint multiset rewriting [5,8,9]. Petri nets with data can be also considered as a reinterpretation of the classical definition of Petri nets in sets with atoms [3,4], where one allows for *orbit-finite* sets of places and transitions instead of just finite ones. The decidability and complexity of standard problems

[1] Strong homogeneity is a mild strengthening of homogeneity.

for Petri nets over various data domains has attracted a lot of attention recently, see for instance [14,20,21,23,24].

WQOs are important for their wide applicability in many areas. Studies of WQOs similar to ours, in case of graphs, have been conducted by Ding [10] and Cherlin [7]; their framework is different though, as they concentrate on subgraph ordering while we investigate *induced* subgraph (or substructure) ordering.

2 Petri nets with homogeneous data

In this section we provide all necessary preliminaries. Our setting follows [19] and is parametric in the underlying logical structure $\mathbb{A}$, which constitutes a *data domain*. Here are some example data domains:

- *Equality data domain*: natural numbers with equality $\mathbb{A}_= = (\mathbb{N}, =)$. Note that any other countably infinite set could be used instead of natural numbers, as the only available relation is equality.
- *Total order data domain*: rational numbers with the standard order $\mathbb{A}_\leq = (\mathbb{Q}, \leq)$. Again, any other countably infinite dense total order without extremal elements could be used instead.
- *Nested equality data domain*: $\mathbb{A}_1 = (\mathbb{N}^2, =_1, =)$ where $=_1$ is equality on the first component: $(n, m) =_1 (n', m')$ if $n = n'$ and $m \neq m'$. Essentially, $\mathbb{A}$ is an equivalence relation with infinitely many infinite equivalence classes.

Note that two latter structures essentially extend the first one: in each case the equality is either present explicitly, or is definable. From now on, we always assume a fixed countably infinite relational structure $\mathbb{A}$ with equality over a finite vocabulary (signature) Σ.

Petri nets with data. Petri nets with data are exactly like classical place/transition Petri nets, except that tokens carry data values and these data values must satisfy a prescribed constraint when a transition is executed. Formally, a *Petri net with data* $\mathbb{A}$ consists of two disjoint finite sets P (places) and T (transitions), the arcs $A \subseteq P \times T \cup T \times P$, and two labelings:

- arcs are labelled by pairwise disjoint finite nonempty sets of variables;
- transitions are labelled by first-order formulas over the vocabulary Σ of $\mathbb{A}$, such that free variables of the formula labeling a transition t belong to the union of labels of the arcs incident to t.

Example 1. For illustration consider a Petri net with equality data $\mathbb{A}_=$, with two places p_1, p_2 and two transitions t_1, t_2 depicted on Fig. 1. Transition t_1 outputs two tokens with arbitrary but distinct data values onto place p_1. Transition t_2 inputs two tokens with the same data value, say a, one from p_1 and one from p_2, and outputs 3 tokens: two tokens with arbitrary but equal data values, say b, one onto p_1 and the other onto p_2; and one token with a data value $c \neq a$ onto p_2. Note that the transition t_2 does not specify whether $b = a$, or $b = c$, or $b \neq a, c$, and therefore all three options are allowed. Variables y_1, y_2 can be considered as input variables of t_2, while variables z_1, z_2, z_3 can be considered as output ones; analogously, t_1 has no input variables, and two output ones x_1, x_2.

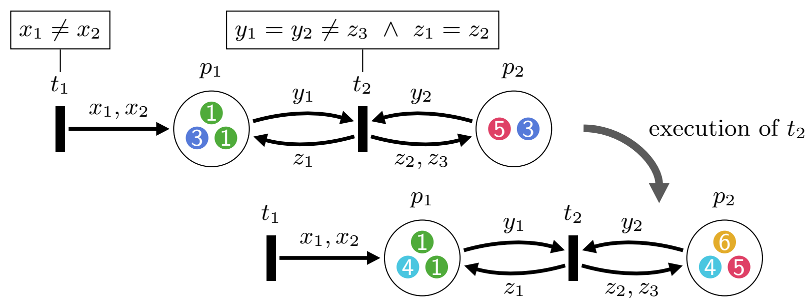


Fig. 1. A Petri net with equality data, with places $P = \{p_1, p_2\}$ and transitions $T = \{t_1, t_2\}$. In the shown configuration, t_2 can be fired: consume two tokens carrying 3, and put, e.g., token carrying 4 on p_1 and tokens carrying $4, 6$ on p_2.

The formal semantics of Petri nets with data is given by translation to multiset rewriting. Given a set X, finite or infinite, a finite multiset over X is a finite (possibly empty) partial function from X to positive integers. In the sequel let $\mathcal{M}(X)$ stand for the set of all finite multisets over X. A *multiset rewriting system* $(\mathcal{P}, \mathcal{T})$ consists of a set $\mathcal{P}$ together with a set of rewriting rules:

$$\mathcal{T} \quad \subseteq \quad \mathcal{M}(\mathcal{P}) \times \mathcal{M}(\mathcal{P}).$$

Configurations $C \in \mathcal{M}(\mathcal{P})$ are finite multisets over $\mathcal{P}$, and the step relation $\longrightarrow$ between configurations is defined as follows: for every $(I, O) \in \mathcal{T}$ and every $M \in \mathcal{M}(\mathcal{P})$, there is the step ($+$ stands for multiset union)

$$M + I \;\longrightarrow\; M + O.$$

For instance, a classical Petri net induces a multiset rewriting system where $\mathcal{P}$ is the set of places, and $\mathcal{T}$ is essentially the set of transitions, both $\mathcal{P}$ and $\mathcal{T}$ being finite. Configurations correspond to markings.

A Petri net with data $\mathbb{A}$ induces a multiset rewriting system $(\mathcal{P}, \mathcal{T})$, where $\mathcal{P} = P \times \mathbb{A}$ and thus is infinite. Configurations are finite multisets over $P \times \mathbb{A}$ (cf. a configuration depicted in Fig. 1). The rewriting rules $\mathcal{T}$ are defined as

$$\mathcal{T} \quad = \quad \bigcup_{t \in T} \mathcal{T}_t,$$

where the relation $\mathcal{T}_t \subseteq \mathcal{M}(\mathcal{P}) \times \mathcal{M}(\mathcal{P})$ is defined as follows: Let ϕ denote the formula labeling the transition t, and let X_i, X_o be the sets of input and output variables of t. Every valuation $v_i : X_i \to \mathbb{A}$ gives rise to a multiset M_{v_i} over $\mathcal{P}$, where $M_{v_i}(p, a)$ is the (positive) number of variables x labeling the arc (p, t) with $v_i(x) = a$. Likewise for valuations $v_o : X_o \to \mathbb{A}$. Then let

$$\mathcal{T}_t = \{\, (M_{v_i}, M_{v_o}) \,|\, v_i : X_i \to \mathbb{A},\; v_o : X_o \to \mathbb{A},\; v_i, v_o \vDash \phi \,\}\,.$$

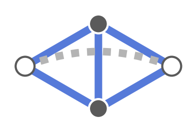
Like $\mathcal{P}$, the set of rewriting rules $\mathcal{T}$ is infinite in general.

As usual, for a net N and its configuration C, a run of (N, C) is a maximal, finite or infinite, sequence of steps starting in C.

Remark 1. As for classical Petri nets, an essentially equivalent definition can be given in terms of vector addition systems (such a variant has been used in [14] for equality data). Petri nets with equality data are equivalent to (even if defined differently than) unordered data Petri nets of [20], and Petri nets with total ordered data are equivalent to ordered data Petri nets of [20].

Effective homogeneous structures. For two relational Σ-structures $\mathcal{A}$ and $\mathcal{B}$ we say that $\mathcal{A}$ *embeds* in $\mathcal{B}$, written $\mathcal{A} \trianglelefteq \mathcal{B}$, if $\mathcal{A}$ is isomorphic to an induced substructure of $\mathcal{B}$, i.e., to a structure obtained by restricting $\mathcal{B}$ to a subset of its domain. This is witnessed by an injective function[2] $h : \mathcal{A} \to \mathcal{B}$, which we call *embedding*. We write $\textsc{Age}(\mathbb{A}) = \{ \mathcal{A} \text{ a finite structure} \mid \mathcal{A} \trianglelefteq \mathbb{A} \}$ for the class of all finite structures that embed into $\mathbb{A}$, and call it *the age of* $\mathbb{A}$.

Homogeneous structures are defined through their automorphisms: $\mathbb{A}$ is homogeneous if every isomorphism of two its finite induced substructures extends to an automorphism of $\mathbb{A}$. In the sequel we will also need an equivalent definition using amalgamation. An *amalgamation instance* consists of three structures $\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2 \in \textsc{Age}(\mathbb{A})$ and two embeddings $h_1 : \mathcal{A} \to \mathcal{B}_1$ and $h_2 : \mathcal{A} \to \mathcal{B}_2$. A solution of such instance is a structure $\mathcal{C} \in \textsc{Age}(\mathbb{A})$ and two embeddings $g_1 : \mathcal{B}_1 \to \mathcal{C}$ and $g_2 : \mathcal{B}_2 \to \mathcal{C}$ such that $g_1 \circ h_1 = g_2 \circ h_2$ (we refer the reader to [12] for further details). Intuitively, $\mathcal{C}$ represents 'gluing' of $\mathcal{B}_1$ and $\mathcal{B}_2$ along the partial bijection $h_2 \circ ({h_1}^{-1})$. In this paper we will restrict ourselves to *singleton* amalgamation instances, where only one element of $\mathcal{B}_1$ is outside of $h_1(\mathcal{A})$, and likewise for $\mathcal{B}_2$. An example singleton amalgamation instance is shown on the right, where the graph $\mathcal{A}$ consists of the single edge connecting two middle black nodes, $\mathcal{B}_1$ is the left triangle, and $\mathcal{B}_2$ the right one. The dashed line represents an edge that may (but does not have to) appear in a solution. $\mathbb{A}$ is homogeneous if, and only if every amalgamation instance has a solution; in such case we say that $\textsc{Age}(\mathbb{A})$ has the *amalgamation property*. See [22] for further details.



A solution $\mathcal{C}$ necessarily satisfies $g_1(h_1(\mathcal{A})) = g_2(h_2(\mathcal{A})) \subseteq g_1(\mathcal{B}_1) \cap g_2(\mathcal{B}_2)$; a solution is *strong* if $g_1(h_1(\mathcal{A})) = g_1(\mathcal{B}_1) \cap g_2(\mathcal{B}_2)$. Intuitively, this forbids additional gluing of $\mathcal{B}_1$ and $\mathcal{B}_2$ not specified by the partial bijection $h_2 \circ ({h_1}^{-1})$. If every amalgamation instance has a strong solution we call $\mathbb{A}$ *strongly homogeneous*. This is a mild restriction, as homogeneous structures are typically strongly homogeneous.

The equality, nested equality, and total order data domains are strongly homogeneous structures. For instance, in the latter case finite induced substructures are just finite total orders, which satisfy the strong amalgamation property. Many other natural classes of structures have the amalgamation property: finite graphs, finite directed graphs, finite partial orders, finite tournaments, etc. Each of these classes is the age of a strongly homogeneous relational structure, namely

[2] We deliberately do not distinguish a structure $\mathcal{A}$ from its domain set.

the *universal graph* (called also random graph), the universal directed graph, the universal partial order, the universal tournament, respectively. Examples of homogeneous structures abound [22].

Homogeneous structures admit quantifier elimination: every first-order formula is equivalent to (i.e., defines the same set as) a quantifier-free one [22]. Thus it is safe to assume that formulas labeling transitions are quantifier-free.

Admitting WQO. A *well quasi-order* (WQO) is a well-founded quasi-order with no infinite antichains. For instance, finite multisets $\mathcal{M}(P)$ over a finite set P, ordered by multiset inclusion $\sqsubseteq$, are a WQO. Another example is the embedding quasi-order $\trianglelefteq$ in $\mathrm{Age}(\mathbb{A}_{\leq})$ (= all finite total orders) isomorphic to the ordering of natural numbers. Finally, the embedding quasi-order in $\mathrm{Age}(\mathbb{A})$ can be lifted from finite structures to finite structures *labeled* by elements of some ordered set $(X, \leq)$: for two such labeled structures $a : \mathcal{A} \to X$ and $b : \mathcal{B} \to X$ we define $a \trianglelefteq_X b$ if some embedding $h : \mathcal{A} \to \mathcal{B}$ satisfies $a(x) \leq b(h(x))$ for every $x \in \mathcal{A}$. We say that $\mathbb{A}$ *admits* WQO when for every WQO $(X, \leq)$, the lifted embedding order $\trianglelefteq_X$ is a WQO too. For instance, $\mathbb{A}_{\leq}$ admits WQO by Higman's lemma. The WQO Dichotomy Conjecture for homogeneous undirected (and also directed) graphs is easily shown by inspection of the classifications thereof [18,6]:

Theorem 1. *A homogeneous graph $\mathbb{A}$ either admits* WQO*, or all standard problems are undecidable for Petri nets with data $\mathbb{A}$.*

Note the natural correspondence between configurations of a Petri net with data $\mathbb{A}$, and structures $\mathcal{A} \in \mathrm{Age}(\mathbb{A})$ labeled by finite multisets over the set P of places:

$$\mathcal{M}(P \times \mathbb{A}) \quad \equiv \quad \{\, m : \mathcal{A} \to \mathcal{M}(P) \mid \mathcal{A} \in \mathrm{Age}(\mathbb{A}) \,\}\,.$$

Thus the lifted embedding quasi-order $\trianglelefteq_{\mathcal{M}(P)}$ is an order on configurations.

Standard decision problems. A Petri net with data N can be finitely represented by finite sets P, T, A and appropriate labelings with variables and formulas. Due to the homogeneity of $\mathbb{A}$, a configuration C can be represented (up to automorphism of $\mathbb{A}$) by a structure $\mathcal{A} \in \mathrm{Age}(A)$ labeled by $\mathcal{M}(P)$. We can thus consider the classical decision problems that input Petri nets with data $\mathbb{A}$, like the *termination problem*: does a given (N, C) have only finite runs? The data domain is considered as a parameter, and hence itself does not constitute part of input. Another classical problem is the *place non-emptiness problem* (markability): given (N, C) and a place p of N, does (N, C) admit a run that puts at least one token on place p? One can also define the appropriate variants of the coverability problem (equivalent to the place non-emptiness problem), the boundedness problem, the evitability problem, etc. (see [19] for details). All the decision problems mentioned above we jointly call *standard problems*.

A Σ-structure $\mathbb{A}$ is called *effective* if the following *age problem* for $\mathbb{A}$ is decidable: given a finite Σ-structure $\mathcal{A}$, decide whether $\mathcal{A} \trianglelefteq \mathbb{A}$. If $\mathbb{A}$ admits WQO then application of the framework of well-structured transition systems [11] to the lifted embedding order $\trianglelefteq_{\mathcal{M}(P)}$ yields:

Theorem 2 ([19]). *If an effective homogeneous structure $\mathbb{A}$ admits* WQO *then all the standard problems are decidable for Petri nets with data $\mathbb{A}$.*

3 Results

A 3-graph $\mathbb{G} = (V, C_1, C_2, C_3)$ consists of a set V and three irreflexive symmetric binary relations $C_1, C_2, C_3 \subseteq V^2$ such that every pair of distinct elements of V belongs to exactly one of the three relations. In the sequel we treat a 3-graph as a clique with 3-colored edges. Any graph, including $\mathbb{A}_=$ and $\mathbb{A}_1$, can be seen as a 3-graph. Our main result confirms the Wqo Dichotomy Conjecture for strongly homogeneous 3-graphs:

Theorem 3. *An effective strongly homogeneous 3-graph $\mathbb{G}$ either admits* wqo*, or all standard problems are undecidable for Petri nets with data $\mathbb{G}$.*

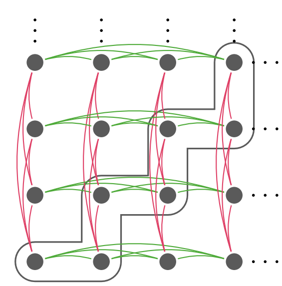
The core technical result of the paper is Theorem 4 below. A *path* is a finite graph with nodes $\{v_1, \ldots, v_n\}$ whose only edges are pairs $\{v_i, v_{i+1}\}$. The nodes v_1, v_n are *ends* of the path, and n is its length.

Theorem 4. *A strongly homogeneous 3-graph $\mathbb{G}$ either admits* wqo*, or for some $i, j \in \{1, 2, 3\}$ (not necessarily distinct) the graph $(V, C_i \cup C_j)$ contains arbitrarily long paths as induced subgraphs.*

We prove that Theorem 4 implies Theorem 3 in the next section. Then, in the rest of the paper we concentrate solely on the proof of Theorem 4.

Example 2. For a quasi-order $(X, \leq)$, the multiset inclusion is defined as follows for $m, m' \in \mathcal{M}(X)$: m' is included in m if m' is obtained from m by a sequence of operations, where each operation either removes some element, or replaces some element by a smaller one wrt. $\leq$. The structure $\mathbb{A}_= = (\mathbb{N}, =)$ admits wqo. Indeed, Age$(\mathbb{A}_=)$ contains just finite pure sets, thus $\trianglelefteq_X$ is quasi-order-isomorphic to the multiset inclusion on $\mathcal{M}(X)$, and is therefore a wqo whenever the underlying quasi-order $(X, \leq)$ is. Similarly, $\mathbb{A}_1 = (\mathbb{N}^2, =_1, =)$ also admits wqo, as $\trianglelefteq_X$ is quasi-order-isomorphic to the multiset inclusion on $\mathcal{M}(\mathcal{M}(X))$.

On the other hand, consider a 3-graph $(\mathbb{N}^2, =_1, =_2, \neq_{12})$ where $=_2$ is symmetric to $=_1$ and $(n, m) \neq_{12} (n', m')$ if $n \neq n'$ and $m \neq m'$. It refines $\mathbb{A}_1$ and does not admit wqo. Indeed, in agreement with Theorem 4, the graph $(\mathbb{N}^2, =_1 \cup =_2)$ contains arbitrarily long paths of the shape presented on the right, where the two colors depict $=_1$ and $=_2$, respectively, and lack of color corresponds to $\neq_{12}$. Note that $(\mathbb{N}^2, =_1, =_2, \neq_{12})$ is homogeneous but not strongly so.



4 Theorem 4 implies Theorem 3

Assume Theorem 4 holds. Towards proving Theorem 3 consider an effective strongly homogeneous 3-graph $\mathbb{A} = (V, C_1, C_2, C_3)$ that does not admit wqo

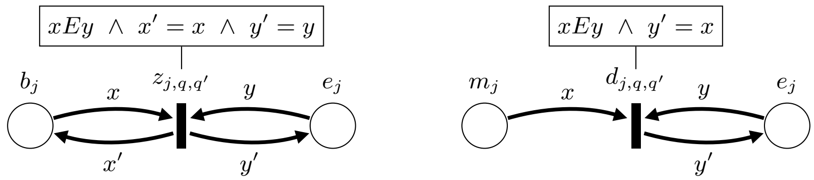


Fig. 2. Transition z_j and d_j simulating zero test and decrement of counter c_j, respectively. Places corresponding to control states of $\mathcal{M}$ are omitted for simplicity.

and let $E = C_i \cup C_j \subseteq V^2$ given by Theorem 4. Thus we know that the graph (V, E) contains arbitrarily long paths. We will demonstrate that Petri nets with data domain $\mathbb{A}$ can faithfully simulate computations of 2-counter machines. To this aim we fix an arbitrary deterministic counter machine $\mathcal{M}$ with two counters c_1, c_2, and states Q; and construct a Petri net $N_{\mathcal{M}}$ with data $\mathbb{A}$ that simulates the computation of $\mathcal{M}$ starting in the initial configuration: initial state q_{init} and the counter values $c_1 = c_2 = 0$. Places of the net will include

$$\{b_1, m_1, e_1, b_2, m_2, e_2, q, r\} \cup Q \ \subseteq \ P$$

plus some further auxiliary ones. In particular, every state of $\mathcal{M}$ will have a corresponding place in N. The idea is to represent a value $c_j = n$ by storing $n + 2$ tokens carrying, as its values, nodes of a path of length $n + 2$ in the graph (V, E). The two tokens carrying the ends of the path will be stored on places b_j and e_j, respectively, while the remaining n tokens will be stored on place m_j. Simulation of a zero test amounts then to checking if the ends are related by an edge. Simulation of a decrement amounts to replacing one end (say from place e_j) by its only neighbor from place m_j. And simulation of an increment amounts to moving the token from e_j to m_j, accompanied by production of a new token on place e_j carrying an arbitrary (guessed nondeterministically) value $v \in V$ not related by E to any of the other tokens on places b_j and m_j.

Zero test and decrement: If $\mathcal{M}$ does *zero test* for c_j in state q and goes to q', the net $N_{\mathcal{M}}$ has a transition $z_{j,q,q'}$ that inputs one token from b_j and one token from e_j, checks that data values they carry are related by E, and puts back the same tokens to the two places (cf. Fig. 2). In addition, the transition $z_{j,q,q'}$ moves one token from place q to q', irrespectively of the data values it carries. Similarly, *decrement* of c_j is performed by a transition $d_{j,q,q'}$ that inputs one token from m_j and one token from e_j, checks that data values they carry are related by E, and then puts back the former token to e_j while discarding the latter one.

Increment: Slightly more complicated is the simulation of increment of a counter c_j, as it involves creating a fresh value that must correctly extend, by one vertex, the path currently stored on places b_j, m_j, e_j. In the first step of the simulation, the net executes a transition i_j that guesses a data value $v \in V$ related by E

$$m\,E\,p \;\wedge\; m \neq r \;\wedge\; \neg m\,E\,e \;\wedge\; e' = e \;\wedge\; m' = m \;\wedge\; p' = m \;\wedge\; r' = p$$

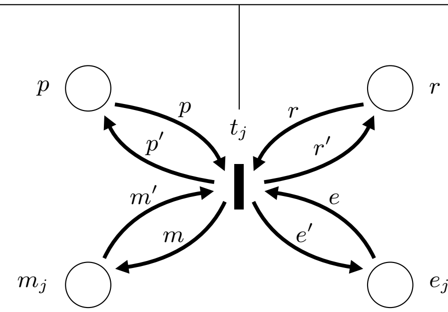


Fig. 3. Transition t_j used in the simulation of increment on counter c_j.

to the value v_e carried by the single token on place e_j but not to the value v_b carried by the single token on place b_j; the token from e_j is moved to m_j (and its copy is additionally put to an auxiliary place p for future use), and a new token carrying v is put on e_j (and its copy is additionally put to an auxiliary place r for future use). What remains to be checked in that v has been guessed correctly by i_j, namely that v is related by E to none of the data values carried by tokens on m_j except for v_e. To this end the net performs a traversal through the path, in the direction from v_e to v_b, in order to check the correctness of v. The traversal is done by iterative execution of the transition t_j, depicted on Fig. 3, which uses the places p, r to store the current edge of the path in the course of traversal. The condition $m\ E\ p\ \wedge\ m \neq r$ checks that the value of variable m is the other neighbour of p along the path; the condition $\neg m\ E\ e$ checks that the guessed value v, stored on place e_j, is indeed not related by E to the value of m; the condition $e'{=}e\ \wedge\ m'{=}m$ ensures that the same value returns to places m_j and e_j; and finally the condition $p'{=}m\ \wedge\ r'{=}p$ ensured that the current edge is moved along the path.

Finally, the simulation of increment of c_j finishes with a transition i'_j that is enabled when the value on place p is related by E to the value on place b_j; transition i'_j removes the tokens from places p and r.

Initial configuration $C_{\mathcal{M}}$ of $N_{\mathcal{M}}$ puts one token on places b_1, b_2, e_1, e_2 to encounter for $c_1 = c_2 = 0$; and one token on the place corresponding to the initial state q_{init}.

We have thus sketched a construction of a net $N_{\mathcal{M}}$ and the initial configuration $C_{\mathcal{M}}$. Observe that consecutive steps of $N_{\mathcal{M}}$ faithfully simulate consecutive steps of $\mathcal{M}$, using a path of sufficient length. N can however get stuck at some point of simulation, if the currently used path can not be extended to a longer one; a priori, this could happen if the fresh data values v used in the simulation of increments are not guessed appropriately. Nevertheless, since the net N stops when a token is put on p_{halt} (i.e., when no token is stored on places in $Q \setminus \{p_{\text{halt}}\}$), we have:

Claim. The place p_{halt} corresponding to the halting state of $\mathcal{M}$ is nonempty in some run of $(N_{\mathcal{M}}, C_{\mathcal{M}})$ if, and only if the machine $\mathcal{M}$ halts.

In one direction, a run of $(N_{\mathcal{M}}, C_{\mathcal{M}})$ putting a token on p_{halt} simulates the halting run of $\mathcal{M}$ from the initial configuration. In the other direction, if $\mathcal{M}$ halts then the net $N_{\mathcal{M}}$ can use a sufficiently long path in (V, E) for values v guessed in the simulation of increments to be able to simulate the whole computation of $\mathcal{M}$ and finally put a token on place p_{halt}. Thus the claim directly entails undecidability of the place non-emptiness, coverability and evitability problems. To treat other decision problems, we notice that (V, E) contains, in addition to arbitrarily long finite paths, also an infinite ω-path:

Claim. The graph (V, E) contains an ω-path.

Indeed, treat finite paths as finite words over a 2-letter alphabet, and arrange all finite paths into a tree. The tree contains arbitrarily long branches, thus it necessarily contains an infinite branch. Using homogeneity of $\mathbb{A}$ one argues that every infinite branch realizes as an ω-path in (V, E). With the last claim we obtain:

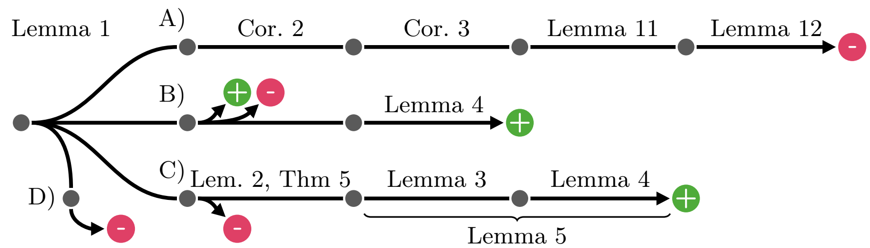
Claim. $(N_{\mathcal{M}}, C_{\mathcal{M}})$ terminates if and only if the machine $\mathcal{M}$ halts.

Indeed, when the computation of $\mathcal{M}$ from the initial configuration halts then $N_{\mathcal{M}}$ necessarily terminates. On the other hand, if the computation of $\mathcal{M}$ from the initial configuration is infinite, an infinite ω-path in (V, E) can be used for the simulation thus constituting an infinite run of $(N_{\mathcal{M}}, C_{\mathcal{M}})$. This entails undecidability of the termination problem, and hence of the boundedness problem too.

5 Proof of Theorem 4

From now on we consider a fixed 3-graph $\mathbb{G} = (V, C_1, C_2, C_3)$ as data domain, assuming $\mathbb{G}$ to be countably infinite and strongly homogeneous. We treat $\mathbb{G}$ as a clique with 3-colored edges: we call C_1, C_2 and C_3 *colors* and put *Colors* $= \{C_1, C_2, C_3\} \subset \mathcal{P}(V \times V)$. To denote individual colors from this set, we will use variables $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{x}, \mathbf{y}, \mathbf{z}$. A path in the graph $(V, \mathbf{a} \cup \mathbf{b})$ we call $\mathbf{ab}$-*path* ($\mathbf{ab} \in$ *Colors*); for simplicity, we will write $\mathbf{a}$-*path* instead of $\mathbf{aa}$-path. Likewise we speak of $\mathbf{ab}$-cliques, $\mathbf{a}$-cliques, $\mathbf{ab}$-cycles, etc. A *triangle* $\triangle\mathbf{abc}$ is a 3-clique with edges colored by $\mathbf{a}, \mathbf{b}, \mathbf{c}$. (Note that $\triangle\mathbf{abc} = \triangle\mathbf{bca} = \triangle\mathbf{cba}$).

Sketch of the proof. Lemma 1 bellow states that any 3-graph $\mathbb{G}$ has to meet one of the four listed cases. It splits the proof into four separate paths:

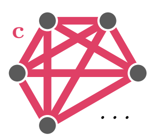


(-) – $\mathbb{G}$ embeds arbitrarily long paths (+) – $\mathbb{G}$ admits WQO

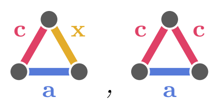
After stating and proving Lemma 1 we proceed with the proofs of Cases A), B) and C). Case A) constitutes the most difficult part of the proof and involves a complex and delicate analysis of consequences of the amalgamation property. It consists of four step that deduce extension of the assumed induced substructures by individual vertices (cf. Cor. 2), individual edges (cf. Cor. 3), paths of length 2 (cf. Lemma 11), resp., culminating in derivation of arbitrarily long paths (cf. Lemma 12). Thus in case A) only the second condition of Theorem 4 is possible, while in the other two cases both conditions of Theorem 4 may hold true.

Lemma 1. *Every homogeneous 3-graph $\mathbb{G} = (V, C_1, C_2, C_3)$ satisfies one of the following conditions:*

A) for some color $\mathbf{c} \in Colors$, $\mathbb{G}$ contains the following induced substructures:



a) arbitrarily large $\mathbf{c}$-cliques



b) two triangles: $\triangle\mathbf{axc}$ and $\triangle\mathbf{acc}$ for some colors $\mathbf{a}, \mathbf{x}$ different than $\mathbf{c}$

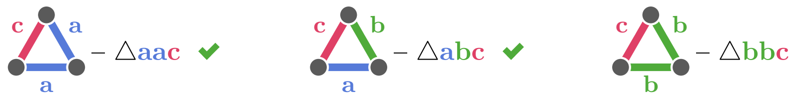
B) for some colors $\mathbf{x} \neq \mathbf{y}$, $(V, \mathbf{x} \cup \mathbf{y})$ is a union of disjoint cliques,
C) for some color $\mathbf{x}$, $(V, \mathbf{x})$ is a union of finitely many disjoint infinite cliques,
D) for some colors $\mathbf{x} \neq \mathbf{y}$, $(V, \mathbf{x} \cup \mathbf{y})$ contains arbitrarily long paths.

Proof. By Ramsey theorem, $\mathbb{G}$ contains an arbitrarily large monochromatic cliques. Let us state a bit stronger requirement:

Condition ♠ For some $\mathbf{a}, \mathbf{c} \in Colors$, $\mathbb{G}$ contains arbitrarily large $\mathbf{c}$-cliques and a triangle $\triangle\mathbf{acc}$ with exactly two $\mathbf{c}$-edges ($\mathbf{a} \neq \mathbf{c}$).
Consider two cases, depending on whether the condition ♠ is satisfied or not.

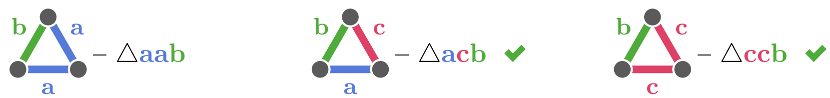
Case 1° Assume that $\mathbb{G}$ contains both arbitrarily large $\mathbf{c}$-cliques and a triangle $\triangle\mathbf{acc}$ for some $\mathbf{a}, \mathbf{c} \in Colors$. Let $\mathbf{b}$ be the third, remaining color. Our goal will be to show that either A) or B) holds.

If the graph $(V, \mathbf{a} \cup \mathbf{b})$ is a disjoint sum of cliques, we immediately obtain B). Suppose the contrary. We get that $\mathbb{G}$ has to contain one of the three possible counterexamples for transitivity of relation $\mathbf{a} \cup \mathbf{b}$:



If it contains the triangle $\triangle$aac or $\triangle$abc, case A) holds.

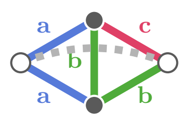
Suppose we got $\triangle$bbc. Let us check this time whether colors a and c form a union of disjoint cliques. Again, if it is so, we easily get B), so we assume the contrary. Similarly, we necessarily obtain one of the following triangles:



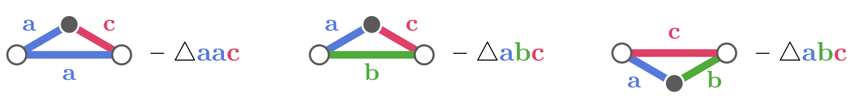
This time case A) also holds for two out of the three triangles above:

- for $\triangle$acb, because together with subgraphs resulting from assumption ♠ (i.e. with triangle $\triangle$acc and the c-cliques) we get all graphs required by A).
- for $\triangle$ccb paired with the triangle $\triangle$bbc we just obtained, using color b appearing in those triangles in place of a in condition A).

It only remains to consider the situation when we got $\triangle$aab. We use it together with previously obtained triangle $\triangle$bbc to build the following instance of singleton amalgamation:



Depending on the color of the dashed edge, in the solution we get one of the following triangles:



and each one alone completes the requirements of A). This closes case 1°.

Case 2° Suppose condition ♠ is false. Remind that $\mathbb{G}$ contains arbitrarily large c-cliques for some c $\in \mathbb{G}$. Since ♠ does not hold, the graph does not contain a triangle $\triangle$cca – in other words, the color c appears only within cliques. We conclude that (V, c) is a union of disjoint cliques. Clearly at least one of such cliques has to be infinite. By homogeneity we get that all the cliques in (V, c) have to be infinite. Now our target is to show that either C) or D) holds.

The case C) is fulfilled when there are only finitely many c-cliques. Let us assume the contrary. In each of the c-cliques we chose one vertex. Edges between the chosen vertices form an infinite ab-clique K. Using Ramsey theorem again, we conclude that in K one of the colors a, b forms arbitrarily large monochromatic cliques. W.l.o.g. suppose that this is color b.

If the graph $\mathbb{G}$ contained $\triangle$ybb for some y $\neq$ b, then the assumptions of ♠ would be met, leading to a contradiction. Therefore we conclude that (V, b) is a union of disjoint infinite b-cliques.

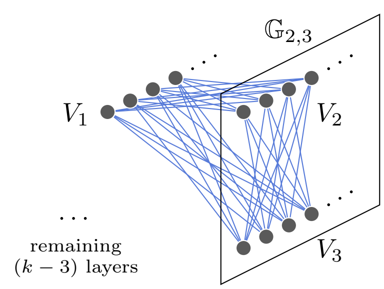
When there are only finitely many b-cliques, condition C) is fulfilled. Otherwise we know that $\mathbb{G}$ is a union of infinitely many x-cliques for both $\mathbf{x} = \mathbf{c}$ and $\mathbf{x} = \mathbf{b}$. Using homogenity, it is easy to show that then every pair of differently colored cliques has *exactly one* common vertex, so the graph $\mathbb{G}$ takes the form as depicted in Example 2. A graph of such form contains arbitrarily long bc-path, so the requirements of D) are met. ☐

6 Case C) in the proof of Theorem 4

Let c be the color that satisfies condition C), and a, b — the remaining two colors. In this section we often treat $\mathbb{G}$ as the k-partite graph $(V, \mathbf{a} \cup \mathbf{b})$ (for some $k \in \mathbb{N}$): k cliques of color c allow to distinguish k groups of vertices $V_1 \cup V_2 \cup \cdots \cup V_k = V$ (from now on we will refer to them as layers). The remaining two colors can be interpreted as existence (a) and nonexistence (b) of edges between these groups.

Remark ★ We observe that the special color c between vertices within each layer V_i ensures that the automorphisms of $\mathbb{G}$ will not 'mix' those layers: when two vertices u, v belong to a common layer V_i, then their images $f(u), f(v)$ will also belong to some common layer V_j, no matter what automorphism $f \in \mathrm{Aut}(\mathbb{G})$ we choose. Obviously, the automorphisms can switch positions of whole layers, e.g. move all vertices from V_i to some V_j and vice versa — in this respect the layers are undistinguishable.

Lemma 2. *For every $i, j \in \{1, 2, \ldots, k\}$ and $\mathbf{a} \in$ Colors $(\mathbf{a} \neq \mathbf{c})$ the bipartite graph $\mathbb{G}_{i,j} = (V_i \cup V_j, \mathbf{a} \cap (V_i \cup V_j)^2, V_i, V_j)$ (with two distinguishable sides V_i, V_j) is homogeneous.*



The vertex sets V_i and V_j are used here as unary relations that allow to tell the two layers of $\mathbb{G}_{i,j}$ (sides of $\mathbb{G}_{i,j}$) apart. An example is shown on the right, with three layers V_1, V_2 and V_3, and three bipartite graphs $\mathbb{G}_{1,2}$, $\mathbb{G}_{2,3}$ and $\mathbb{G}_{1,3}$.

Proof. Fix $\mathbb{G}_{i,j}$ a bipartite graph. To prove its homogeneity we have to show that each isomorphism of two of its finite induced subgraphs may be extended to some automorphism of $\mathbb{G}_{i,j}$. Let us then take some given automorphism $f : G_1 \to G_2$ for some finite induced subgraphs G_1, G_2 of $\mathbb{G}_{i,j}$. It is easy to extend it to a full automorphism when it 'touches' both layers of $G_{i,j}$, i.e.:

$$V(G_1) \cap V_i \neq \emptyset \quad \wedge \quad V(G_1) \cap V_j \neq \emptyset$$

where $V(G_1)$ is the set of vertices of G_1. In this case, by homogeneity of $\mathbb{G}$, we construct a full automorphism $f' : \mathbb{G} \to \mathbb{G}$, which extends f. It is easy to see that in this case f' has to fix the layers V_i and V_j, and hence f' restricted to the graph $\mathbb{G}_{i,j}$ is a correct automorphism of this graph.

Things get more complicated when f operates only on some single layer of $\mathbb{G}_{i,j}$. W.l.o.g. suppose that it 'touches' only V_i, so $V(G_1) \cap V_j = \emptyset$. Now the above construction will not work out of the box — if we were unlucky, the automorphism of $\mathbb{G}$ we get by homogeneity moves the whole layer V_j to some V_n located 'outside' the graph $\mathbb{G}_{i,j}$ ($n \notin \{i, j\}$).

It will be handy to make the following observation: when f 'touches' only V_i we may assume that $V(G_1) \cap V(G_2) = \emptyset$. Indeed, every function $g : G_1 \to G_2$ that violates this condition may be decomposed as $g = f_2 \circ f_1$ for some f_1, f_2:

$$G_1 \xrightarrow{f_1} H \xrightarrow{f_2} G_2$$

such that H is disjoint both with G_1 and with G_2.

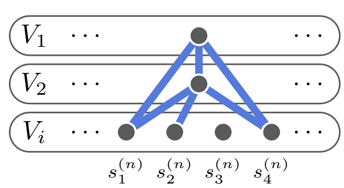
Now, let $N = |V(G_1)| = |V(G_2)|$ be the size of the domain of isomorphism f. Let us take an arbitrary infinite family $(S_n)_{n\in\mathbb{N}}$ of subgraphs of $\mathbb{G}$ with disjoint vertex sets, such that the following conditions are met:

– $|V(S_n) \cap V_m| = 1$ for $m \neq i$ (and this single vertex will be denoted as $v_m^{(n)}$),
– $|V(S_n) \cap V_i| = N$ (denote these vertices as $s_1^{(n)}, s_2^{(n)}, s_3^{(n)}, \ldots, s_N^{(n)}$).

We define a *connection type* of a layer V_i with V_m in the graph S_n as the N-element sequence of colors of edges from the list bellow:

$$(\{s_1^{(n)}, v_m^{(n)}\}, \{s_2^{(n)}, v_m^{(n)}\}, \ldots, \{s_N^{(n)}, v_m^{(n)}\})$$

E.g. in the graph bellow, the connection type of layer $V_i = V_3$ with V_1 is **abba**, and with V_2 — **aaba** (remembering that **b** is treated as lack of an edge):



Furthermore, we define the type of graph S_n to be the sequence of types arising between V_i and other layers plus the list of edge-colors between all pairs of vertices $v_\bullet^{(n)}$ (enumerated in some consistent way). As there are only finitely many such types, by pigeonhole principle there exists a pair of graphs S_a and S_b with the same type.

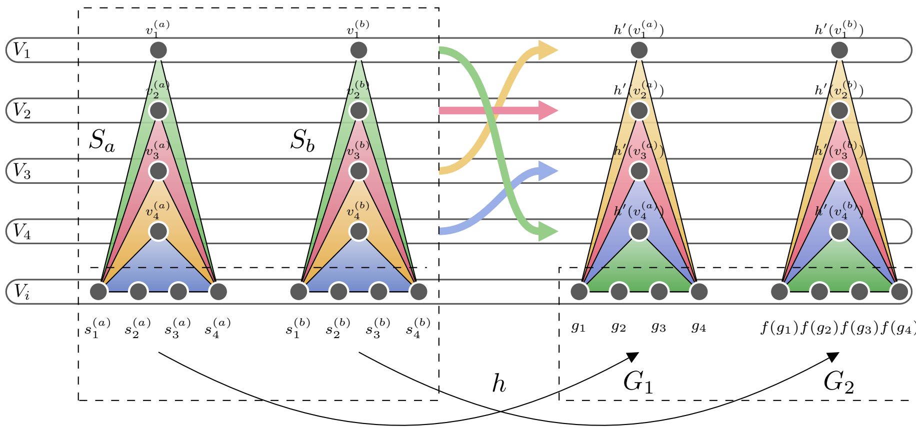
Let us fix some order on vertices of G_1: $V(G_1) = \{g_1, g_2, \ldots, g_N\}$. Let h be the partial isomorphism that moves the vertices as follows:

$$\begin{array}{ccc} s_1^{(a)} \to g_1 & \qquad & s_1^{(b)} \to f(g_1) \\ \cdots & & \cdots \\ s_N^{(a)} \to g_N & & s_N^{(b)} \to f(g_N) \end{array}$$

By homogeneity, it has to extend to a full automorphism $h' \in \mathrm{Aut}(\mathbb{G})$. In particular, in the neighbourhood of G_1 and G_2 there will be images of all vertices $v_\bullet^{(\alpha)}$ of graphs S_a and S_b:

$$h'\left(v_1^{(\alpha)}\right), h'\left(v_2^{(\alpha)}\right), \ \ldots \ , h'\left(v_{i-1}^{(\alpha)}\right), h'\left(v_{i+1}^{(\alpha)}\right), \ \ldots \ , h'\left(v_k^{(\alpha)}\right)$$

(for α in $\{a, b\}$). What follows is that G_1 with added vertices $h'(v_\bullet^{(a)})$ has the same type as G_2 with $h'(v_\bullet^{(b)})$ respectively (that type may differ from the type of S_a and S_b though!). It is best illustrated on a picture:



Above, the colored triangles represent the types of connections. The order of those types may get permuted when applying h', but still — in line with the remark ★ — for each $\beta \in \{1, 2, \ldots, k\} \setminus \{i\}$ the vertex $h'\left(v_\beta^{(a)}\right)$ must stay in the same layer as $h'\left(v_\beta^{(b)}\right)$, furthermore their type of connection with layer V_i is preserved.

Extending the isomorphism f in a natural way (thanks to the compatibility of types) on those newly obtained vertices:

$$h'\left(v_\bullet^{(a)}\right) \xrightarrow{\ f\ } h'\left(v_\bullet^{(b)}\right)$$

we get an isomorphism that this time 'operates' on all layers $V_\bullet$. If we now extend it to an automorphism of the whole $\mathbb{G}$, we will get a function that fixes all layers $V_\bullet$. This function may be safely restricted to $V_i \cup V_j$, staying a correct automorphism of our initial bipartite graph $\mathbb{G}_{i,j}$, which completes the proof. □

We are going to apply to graphs $\mathbb{G}_{i,j}$ the following classification result:

Theorem 5 ([16]). *A countably infinite homogeneous bipartite graph (with distinguishable sides) is either empty, or full, or a perfect matching, or the complement of a perfect matching, or a* universal *graph.*

From our point of view, all we need to know about the universal graph is that it contains arbitrarily long paths which – translated to our notation – would mean that $\mathbb{G}_{i,j}$ contains arbitrarily long $\mathbf{a}$-paths. Therefore in our further considerations we assume that $\mathbb{G}_{i,j}$ is not universal which, in our notation, leaves two types of $\mathbb{G}_{i,j}$:

1. all edges of $\mathbb{G}_{i,j}$ have the same color $\mathbf{x} \in \{\mathbf{a}, \mathbf{b}\}$, i.e. $\mathbb{G}_{i,j}$ is a full or empty bipartite graph,
2. one of the colors $\mathbf{x} \in \{\mathbf{a}, \mathbf{b}\}$ forms a perfect matching in $\mathbb{G}_{i,j}$, the second one ($\mathbf{y} \neq \mathbf{x}$) is then the complement of this matching.

Graphs of type 2. may be seen as bijections between their sets of vertices (layers). Lemma 3 states that those bijections have to agree with each other.

Lemma 3. *Let V_i, V_j, V_k be some arbitrary pairwise different layers, such that $\mathbb{G}_{i,j}$ is of type 2 and $\psi : V_i \to V_j$ is the bijection it determines. Then ψ takes $\mathbf{a} \cap (V_i \cup V_k)$ to $\mathbf{a} \cap (V_j \cup V_k)$, or to its complement. Formally:*

$$\left(\underset{u \in V_i}{\forall}\ \underset{v \in V_k}{\forall}\ \underbrace{u\ \mathbf{a}\ v}_{\clubsuit} \Leftrightarrow \underbrace{\psi(u)\ \mathbf{a}\ v}_{\spadesuit}\right) \vee \left(\underset{u \in V_i}{\forall}\ \underset{v \in V_k}{\forall}\ \underbrace{\neg u\ \mathbf{a}\ v}_{\heartsuit} \Leftrightarrow \underbrace{\psi(u)\ \mathbf{a}\ v}_{\diamondsuit}\right)$$

Proof. We head towards a contradiction. Negating the claim we get:

$$\left(\underset{u \in V_i}{\exists}\ \underset{v \in V_k}{\exists} \neg\clubsuit \wedge \spadesuit \quad \vee \quad \clubsuit \wedge \neg\spadesuit\right) \wedge \left(\underset{u \in V_i}{\exists}\ \underset{v \in V_k}{\exists} \neg\heartsuit \wedge \diamondsuit \qquad \vee \quad \heartsuit \wedge \neg\diamondsuit\right)$$

which leads to four cases with similar proofs. We will consider one of them (corresponding to $\neg\heartsuit \wedge \diamondsuit$ and $\clubsuit \wedge \neg\spadesuit$) and omit the other. Let us then assume that there exist $x, x' \in V_i$ and $y, y' \in V_k$ such that:

$$x\ \mathbf{a}\ y \quad \wedge \quad x'\ \mathbf{a}\ y' \quad \wedge \quad \psi(x)\ \mathbf{a}\ y \quad \wedge \quad \neg\psi(x')\ \mathbf{a}\ y'.$$

Let g be a partial isomorphism of the form $g = \{x \to x', y \to y'\}$. By homogeneity of $\mathbb{G}$, there is some full automorphism $g' \in \mathrm{Aut}(\mathbb{G})$ extending g. If additionally we were able to force g to fix the layer V_j, we would be almost done. Let us try to achieve that property.

For that purpose, in V_j we choose a vertex v such that:

I. $v \notin \psi(\{x, x'\})$,
II. if $\mathbb{G}_{j,k}$ is a graph of type 2. defining a bijection $\phi : V_k \to V_j$, then also $v \notin \phi(\{y, y'\})$.

Clearly such vertex must exist – two above conditions exclude at most 4 different vertices from the infinite set of candidates. The function g extended with $v \xrightarrow{g} v$ stays a correct isomorphism, because:

- in $\mathbb{G}_{i,j}$ by definition of isomorphism we need the edges $\{x, v\}$ and $\{g(x), g(v)\}$ to be equally colored, and, in fact, they are. We get this thanks to the condition I.: x is connected with all vertices from $V_j \setminus \{\psi(x)\}$ by $\mathbf{x}$-edges, $\mathbf{x} \in \{\mathbf{a}, \mathbf{b}\}$. We similarly handle x'.

– in turn in $\mathbb{G}_{j,k}$ — if it is a graph of type 1., the needed equality of colors of edges $\{y, v\}$ and $\{g(y), g(v)\}$ trivially holds. If it is a graph of type 2., the equality of colors is derived similarly as in $\mathbb{G}_{i,j}$, using the condition II.

Presence of the vertex v ensures that layer V_j is preserved by the full automorphism $g' \in \mathrm{Aut}(\mathbb{G})$ we get by homogeneity.

Since $\mathbb{G}_{i,j}$ is of type 2., the vertex $\psi(x')$ is the only possible choice for the image of $\psi(x)$ under g' — this is the only vertex x' is connected to by an appropriately colored edge. Because g' is an automorphism, we get that $\psi(x')$ a y', which leads us to the contradiction. $\square$

From the lemma we have just proved one easily derives the following corollary:

Corollary 1. *The following relation $\equiv$ on layers is transitive:*

$$V_i \equiv V_j \Leftrightarrow \text{ the graph } \mathbb{G}_{i,j} \text{ is of type 2.}$$

Furthermore, if $V_i \equiv V_j$ and $V_j \equiv V_k$ then $f_{j,k} \circ f_{i,j} = f_{i,k}$, where $f_{i,j}, f_{i,k}, f_{j,k}$ are the bijections determined by graphs $\mathbb{G}_{i,j}, \mathbb{G}_{i,k}$ and $\mathbb{G}_{j,k}$.

In Lemma 5 below, which is the last step of the proof of case C), we will apply the following fact:

Lemma 4. *Consider a homogeneous 3-graph $\mathbb{G}$ and a partition of its vertex set $V = \bigcup_{n\in\mathbb{N}} U_n$ into sets $U_\bullet$ of equal finite cardinality. Suppose further that for every $n \in \mathbb{N}$, there is an automorphism π_n of $\mathbb{G}$ that swaps U_0 with U_n and is identity elsewhere. Then $\mathbb{G}$ admits* WQO.

Proof. Let $\mathbb{G} = (V, \mathbf{a}, \mathbf{b}, \mathbf{c})$ be a 3-graph. Define for $u \in U_0$ the sets $V_u \subseteq V$, which we call *layers*:

$$V_u = \{\, \pi_n(u) \mid n \in \mathbb{N} \,\}\,.$$

We will prove that the structure $\mathbb{G}' = (V, \mathbf{a}, \mathbf{b}, \mathbf{c}, (V_u)_{u\in U_0})$ admits WQO. This will imply that $\mathbb{G}$ admits WQO as well; indeed, compared to $\mathbb{G}$, structure $\mathbb{G}'$ is equipped with additional unary relations $V_\bullet$, which only makes the order $\trianglelefteq$ in $\mathrm{Age}(\mathbb{G}')$ finer than the analogous order in $\mathrm{Age}(\mathbb{G})$.

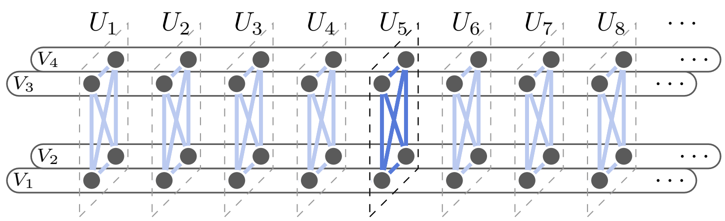
Let G_n denote the induced substructure of $\mathbb{G}'$ on vertex set U_n. By the assumptions, for every $n, m \in \mathbb{N}$ there is a swap of U_n and U_m that, extended with identity elsewhere, is an automorphism of $\mathbb{G}'$. In consequence, all structures $G_\bullet$ are isomorphic, and the embedding order $\trianglelefteq$ of induced substructures of $\mathbb{G}'$ is isomorphic to finite multisets over $\mathrm{Age}(G_0)$, ordered by multiset inclusion. Thus $(\mathrm{Age}(\mathbb{G}'), \trianglelefteq)$ is isomorphic to the multiset inclusion in $\mathcal{M}(\mathrm{Age}(G_0))$, which is a WQO as U_0 is finite. For any WQO $(X, \leq)$, analogous isomorphism holds between the lifted embedding order $(\mathrm{Age}(\mathbb{G}'), \trianglelefteq_X)$ and the multiset inclusion in multisets over induced substructures of G_0 labeled by elements of X, and again the latter order is a WQO. Thus $\mathbb{G}'$ admits WQO. $\square$

Lemma 5. *The 3-graph $\mathbb{G}$ admits* WQO.

Proof. We are going to prepare the ground for the use of Lemma 4. By Corollary 1. the vertex set V partitions into $V = \bigcup_{n\in\mathbb{N}} U_n$ so that

a) every layer V_i shares with every set U_n exactly one vertex: $U_n \cap V_i = \{v_i^{(n)}\}$,
b) if $f_{i,j}$ is the bijection determined by $\mathbb{G}_{i,j}$ (a graph of type 2.), then $f_{i,j}(v_i^{(n)}) \in U_n$, so all the bijections preserve every set $U_\bullet$.

Intuitively, $\mathbb{G}$ can by cut into thin 'slices' perpendicular to the layers $V_\bullet$. By thin we mean that the slices have exactly one vertex in each layer. The cut is made along the bijections dictated by the graphs of type 2. as in the picture bellow:



We observe that for every n, the bijection $h_n : V \to V$ that swaps U_1 and U_n along the only bijection $U_1 \to U_n$ that preserves layers, and is identity elsewhere, is an automorphism of $\mathbb{G}$. Indeed, for any three slices U_a, U_b, U_c we have that:

$$v_i^{(a)} \ \mathbf{a} \ v_j^{(c)} \Leftrightarrow v_i^{(b)} \ \mathbf{a} \ v_j^{(c)}$$

so the edges $\left\{v_i^{(a)}, v_j^{(c)}\right\}$ and $\left\{v_i^{(b)}, v_j^{(c)}\right\}$ are colored the same way. The above equivalence is obvious in case when $\mathbb{G}_{i,j}$ is a graph of type 1. In the case of graph of type 2., the vertex $v_i^{(c)}$ is connected with all vertices from V_j but one by $\mathbf{x}$-edges for some $\mathbf{x} \in \{\mathbf{a}, \mathbf{b}\}$. However, the special vertex $f_{i,j}(v_i^{(c)})$ that is not connected by a $\mathbf{x}$-edge, by the condition b), also belongs to U_c, so it does not interfere with above equivalence.

By Lemma 4 we deduce that $\mathbb{G}$ admits WQO, which completes the proof. □

7 Case B) in the proof of Theorem 4

Let $\mathbf{a}$, $\mathbf{b}$ be the two colors such that the graph $H = (V, \mathbf{a} \cup \mathbf{b})$ is a sum of disjoint cliques. The color appearing between the cliques we mark as $\mathbf{c}$. Since the set of vertices V is infinite, the graph H cannot be a finite sum of finite cliques. Furthermore, by homogeneity we have that all $\mathbf{ab}$-cliques in $\mathbb{G}$ are isomorphic, so their sizes are equal. We then have three cases to investigate:

1. H is a sum of *infinite* number of *infinite* $\mathbf{ab}$-cliques,
2. H is a sum of *finite* number of *infinite* $\mathbf{ab}$-cliques,
3. H is a sum of *infinite* number of *finite* $\mathbf{ab}$-cliques.

Let us concentrate on the first case. Because each ab-clique $K \trianglelefteq \mathbb{G}$ maximal in terms of relation '$\trianglelefteq$' is homogeneous, we can apply Theorem 1 to deduce that either K admits WQO, or it contains arbitrarily long x-paths for some $x \in \{a, b\}$. We only need to consider the former case.

The crucial observation is that the embedding order on induced substructures of $\mathbb{G}$ is isomorphic to the multiset inclusion in $\mathcal{M}(\textsc{Age}(K))$. Indeed, any induced substructure $\mathcal{X} \trianglelefteq \mathbb{G}$ splits into the ab-cliques, and as there are only c-edges between the cliques, this split of $\mathcal{X}$ determines $\mathcal{X}$ uniquely. Finally, the choice of particular ab-cliques is irrelevant, as they are all isomorphic.

As the multiset inclusion in $\mathcal{M}(\textsc{Age}(K))$ admits WQO by assumption, being itself a WQO in particular, we deduce that $(\textsc{Age}(\mathbb{G}), \trianglelefteq)$ is a WQO too. Similarly one observes that the lifted order $\trianglelefteq_X$ is a WQO, for any underlying WQO $(X, \leq)$.

The second case, when H is a sum of k infinite ab-cliques, is dealt analogously with the only difference that multisets over $\textsc{Age}(K)$ of size at most k are considered instead of multisets of unbounded size.

Finally the third case, when H is a sum of finite ab-cliques, follows immediately by Lemma 4.

8 Case A) in the proof of Theorem 4

This is the most extensive part of the proof. Now we assume that case A) of Lemma 1 holds and analyze the consequences. We are going to present a chain of lemmas that eventually gives us the existence of arbitrarily long paths in $\mathbb{G}$.

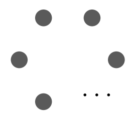
From now on we fix the color c appearing in case A) of lemma 1 and consider it as the no-edge relation. Consequently, we will treat $\mathbb{G}$ as a 2-edge-colored graph. For that reason we define $\mathit{Colors}' = \mathit{Colors} \setminus \{c\}$. In all pictures in this section, the lack of an edge between some two vertices of graph will mean that they are connected by a c-edge.

Let us introduce a few new notations:

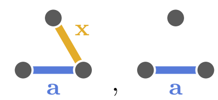
- $\underline{xyz\ldots}$ will denote an ab-path with consecutive edges colored by x, y, z, etc. ($a, b \in \mathit{Colors}'$). E.g., $\underline{aba}$ corresponds to the following path: $\bullet\overset{a}{—}\bullet\overset{b}{—}\bullet\overset{a}{—}\bullet$. The single-vertex path will be written as $\bullet$.
- For cycles we will use similar notation: $\circ xy\ldots z$ stands for a ab-cycle with consecutive edges painted $x, y, \ldots, z$.
- For two given graphs G_1 and G_2, a graph $G_1 + G_2$ is built as follows: We take disjoint copies of G_1 and G_2 and connect the two parts with c-edges. E.g., $\underline{aa} + \bullet$ denotes the graph: " $\bullet\overset{a}{—}\bullet\overset{a}{—}\bullet \quad \bullet$ ".
- For a given graph G_1, a sum of its k copies (in the above sense) is written as $k \cdot G_1$, e.g. $3 \cdot \bullet = \bullet + \bullet + \bullet$.
- Discrete graph D_k is a graph $k \cdot \bullet$.

Now we can reformulate the case A) of Lemma 1 using the new convention:

$\mathbb{G}$ contains the following induced subgraphs



a) arbitrarily large discrete graphs D_k for $k \in \mathbb{N}$



b) above graphs: ax and $\mathrm{a} + \bullet$ for some colors $\mathrm{a}, \mathrm{x} \in \mathit{Colors}'$

8.1 Adding isolated vertices

Our first goal is to show that $\mathbb{G}$ embeds a graph $\underline{\mathrm{ax}} + k \cdot \bullet$ for each $k \in \mathbb{N}$. The proof will be inductive. The induction base follows easily by the assumed condition A). Two coming lemmas, when combined, will form the inductive step. From now on, the expression $k \cdot \bullet$ will appear many times, so for readability we will emphasize it as $\boxed{k \cdot \bullet}$.

Lemma 6. *Let $\mathbb{G}$ be the strongly homogeneous graph that embeds arbitrarily large discrete graphs and also the subgraphs $\underline{\mathrm{ax}} + \boxed{k \cdot \bullet}$ and $\underline{\mathrm{a}} + \bullet + \boxed{k \cdot \bullet}$ for some $\mathrm{a}, \mathrm{x} \in \mathit{Colors}'$ and $k \in \mathbb{N}$. Then $\mathbb{G}$ for some $\mathrm{a}_2, \mathrm{y} \in \mathit{Colors}'$ embeds the graphs:*

1. $\underline{\mathrm{a}_2\mathrm{y}} + \boxed{k \cdot \bullet}$,
2. $\underline{\mathrm{a}_2} + \bullet + \bullet + \boxed{k \cdot \bullet}$.

It is important to note that a does not have to be equal to a_2.

Lemma 7. *If $\mathbb{G}$ embeds graphs $\underline{\mathrm{ay}} + \boxed{k \cdot \bullet}$ and $\underline{\mathrm{a}} + \bullet + \bullet + \boxed{k \cdot \bullet}$ for some $\mathrm{a}, \mathrm{y} \in \mathit{Colors}'$ and $k \in \mathbb{N}$, it also embeds graph $\underline{\mathrm{az}} + \bullet + \boxed{k \cdot \bullet}$ for some $\mathrm{z} \in \mathit{Colors}'$.*

Juxtaposition of those lemmas allows us to 'add' arbitrarily many isolated vertices:

$$\left\{ \begin{array}{l} \mathrm{ax} + \boxed{k \cdot \bullet} \\ \mathrm{a} + \bullet + \boxed{k \cdot \bullet} \end{array} \right\} \xrightarrow{\text{Lem. 6.}} \left\{ \begin{array}{l} \mathrm{a}_2\mathrm{y} + \boxed{k \cdot \bullet} \\ \mathrm{a}_2 + \bullet + \bullet + \boxed{k \cdot \bullet} \end{array} \right\} \xrightarrow{\text{Lem. 7.}} \left\{ \begin{array}{l} \mathrm{a}_2\mathrm{z} + \boxed{\bullet + \boxed{k \cdot \bullet}} \\ \mathrm{a}_2 + \bullet + \boxed{\bullet + \boxed{k \cdot \bullet}} \end{array} \right\}$$

Similar scheme will emerge also in subsequent parts of the proof: in analogous way we will later be adding isolated edges and two-edge paths.

Now, let us move on to the proof of Lemmas 6. and 7. They will be the first from a group of lemmas making a heavy use of the amalgamation property.

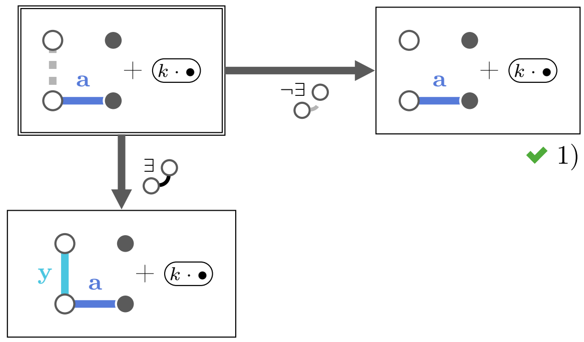
Proof (of Lemma 6). By assumptions we know that $G_1 = \underline{\mathrm{ax}} + \boxed{k \cdot \bullet} \trianglelefteq \mathbb{G}$ as well as $G_2 = \underline{\mathrm{a}} + \bullet + \boxed{k \cdot \bullet} \trianglelefteq \mathbb{G}$ for some given colors $\mathrm{a}, \mathrm{x} \in \mathit{Colors}'$. The set Colors' has two elements — let b be the second of its elements, different from a.

Current target. To prove the lemma, it suffices to show one of the following statements:

✔ 1) $\mathbb{G}$ embeds a graph $\underline{\mathrm{a}} + \bullet + \bullet + \boxed{k \cdot \bullet}$
(paired with G_1, it will give us the thesis of lemma),

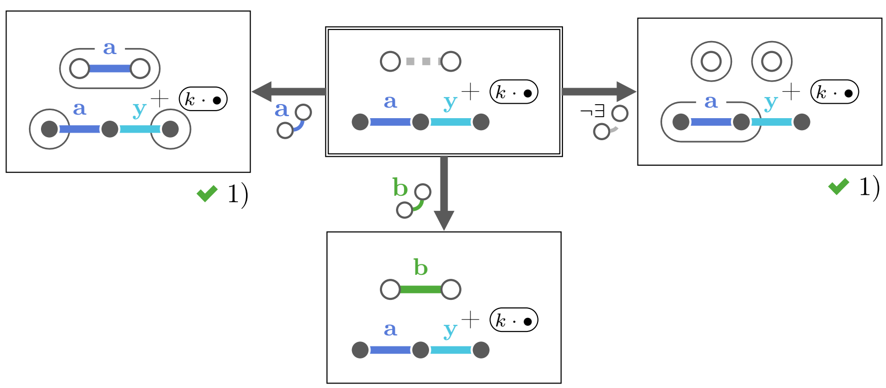
✔ 2) $\mathbb{G}$ embeds graphs $\underline{\mathrm{b}} + \bullet + \bullet + \boxed{k \cdot \bullet}$ and $\underline{\mathrm{by}} + \boxed{k \cdot \bullet}$
(here G_1 would not help, since lemma requires compatibility of edge colors, yet G_1 may not contain b-edge if $\mathrm{x} = \mathrm{a}$).

Instance 6.1. We begin by considering the following amalgamation instance:



If in its solution the edge is not present, we get graph $\underline{\mathrm{a}} + \bullet + \bullet + \boxed{k \cdot \bullet}$, so ✔ 1) is obtained immediately. Assume the contrary — that some y-edge appeared $\mathrm{y} \in \mathit{Colors}'$.

Instance 6.2. Using the obtained graph, we build a new instance:



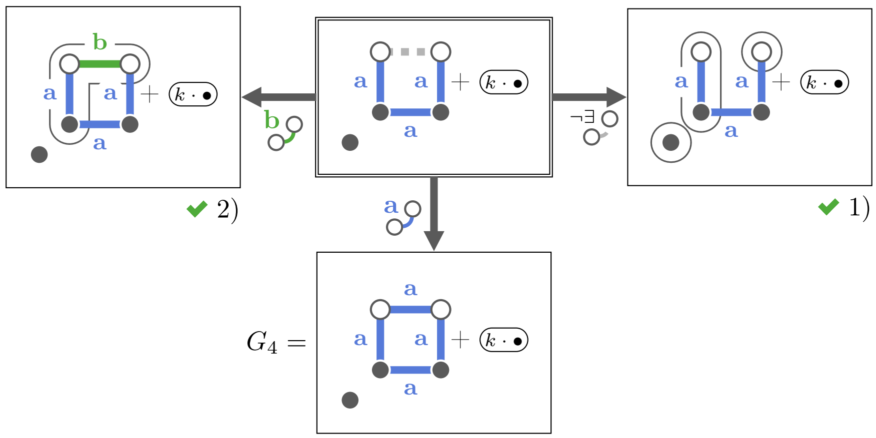
The above instance is one of the few that actually use the strong amalgamation property. As shown on the picture, in cases when we get an a-edge or we do not get an edge at all, condition ✔ 1). is easily met. Let us assume we obtained a b-edge.

At this point we have to notice that $\underline{\mathrm{b}} + \underline{\mathrm{ay}} + \boxed{k \cdot \bullet}$ embeds a graph $\underline{\mathrm{b}} + \bullet + \bullet + \boxed{k \cdot \bullet}$, so from now on to prove ✔ 2), it suffices to obtain $\underline{\mathrm{ab}} + \boxed{k \cdot \bullet}$. Hence, if $\mathrm{y} = \mathrm{b}$, we would have the missing graph $\underline{\mathrm{ab}} + \boxed{k \cdot \bullet}$ as a subgraph of $\underline{\mathrm{b}} + \underline{\mathrm{ay}} + \boxed{k \cdot \bullet}$. It then only remains to consider the case $\mathrm{y} = \mathrm{a}$.

For later use, from $\underline{\mathrm{b}} + \underline{\mathrm{aa}} + \boxed{k \cdot \bullet}$ we take the following subgraph G_3:

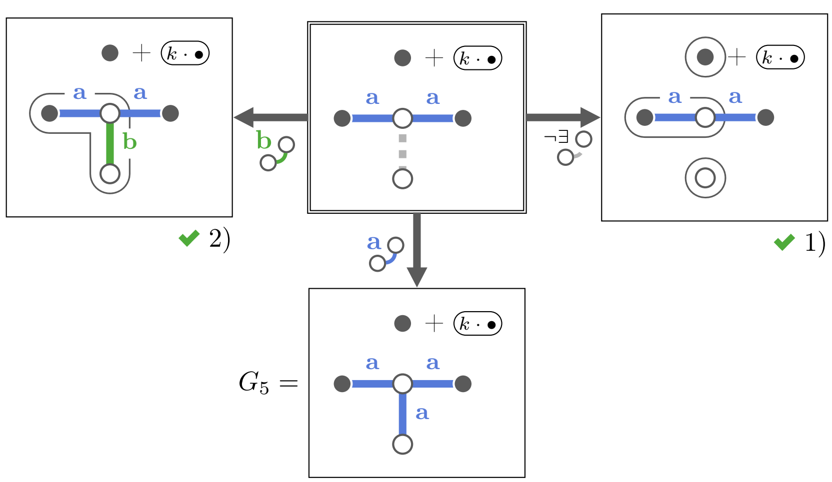
$G_3 =$ 

Instance 6.3. We use it to construct a new instance of amalgamation:



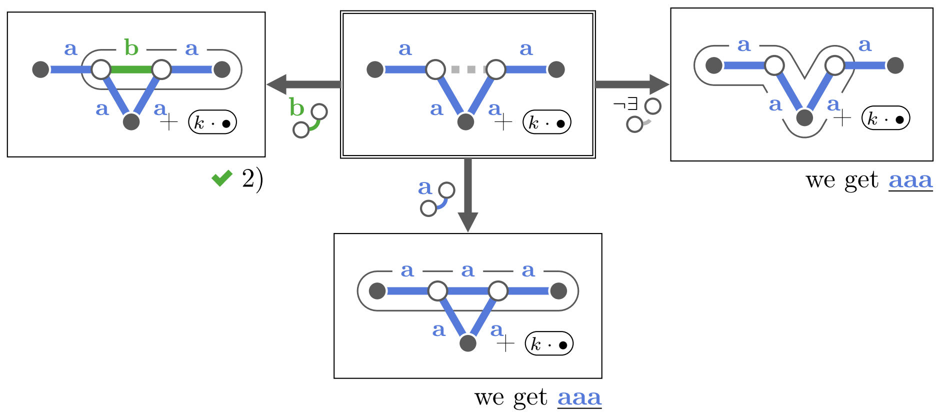
Again, two cases immediately lead us to the end of the proof (see the picture), so only one needs further examination: If an a-edge is present in the solution, we have the graph $G_4 = \circ$aaaa $+ \bullet + (k \cdot \bullet)$. It will come useful in a moment (at the end of the proof), but first we have to 'construct' yet another one. The construction will take three upcoming amalgamations, then we will return to G_4.

Instance 6.4. To build the instance we again use graph G_3, this time paired with the discrete graph D_{k+4} — we can afford to do that, since in $\mathbb{G}$ embeds arbitrarily large discrete graphs.



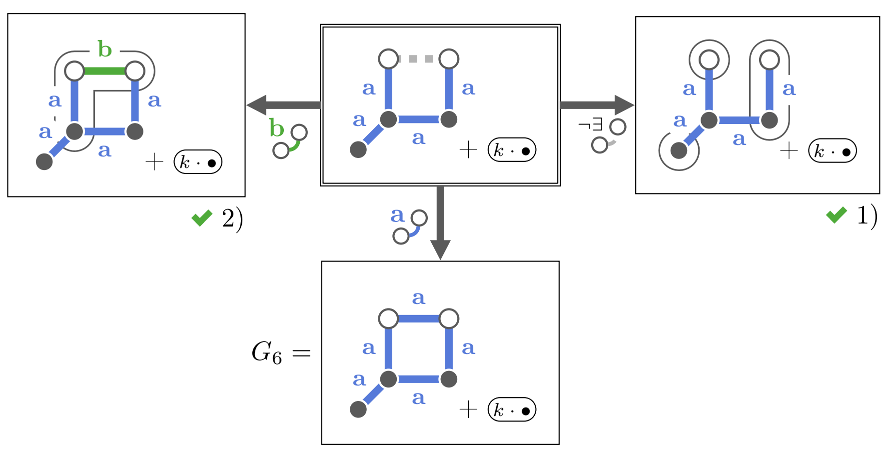
The acquired graph G_5 will be used in Instance 6.6. To complete the proof of lemma, we still need one more graph — namely aaa$+(k \cdot \bullet)$. We will get it quickly in the following instance of amalgamation:

Instance 6.5. This time we put together two copies of G_3:



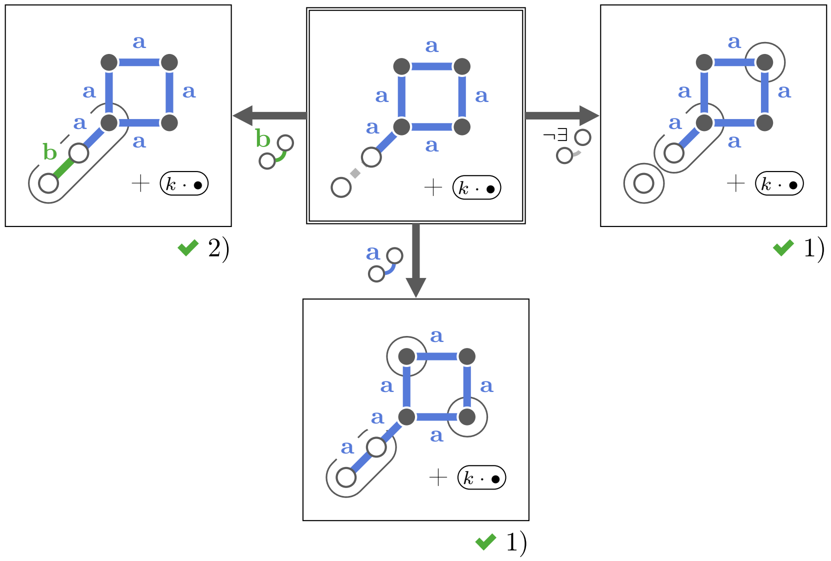
If we obtained a b-edge, we luckily end, having met the condition ✓ 2). In both remaining cases from the resulting graph we derive a path aaa.

Instance 6.6. Using that path together with G_5 (from Instance 6.4), we construct another instance of amalgamation. Fortunately, it is the penultimate instance in the proof of the current lemma.



Similarly as in all previous instances, only one case does not end immediately by satisfying one of the conditions ✓ 1) or ✓ 2). Let G_6 be the graph we get in the a-edge–case.

Instance 6.7. We have nearly made it through to the end of the proof of Lemma 6. For construction of the last amalgamation instance we need graphs G_4 (from Instance 6.3) and G_6 (just created).



Each of three possible outcomes of this instance allows to fulfill the conditions ✔ 1) or ✔ 2), thus we finally completed the proof of Lemma 6.

□

There is nothing left to do but to proceed with proving the next lemma. This proof will be a bit shorter, as it consists only of four amalgamation instances.

Proof (of Lemma 7). The assumptions of the lemma require $\mathbb{G}$ to embed the following graphs:

– graph $G_1 = \underline{\mathrm{ay}} + (k \cdot \bullet)$,
– graph $G_2 = \underline{\mathrm{a}} + \bullet + \bullet + (k \cdot \bullet)$ obtained as the result of previous lemma.

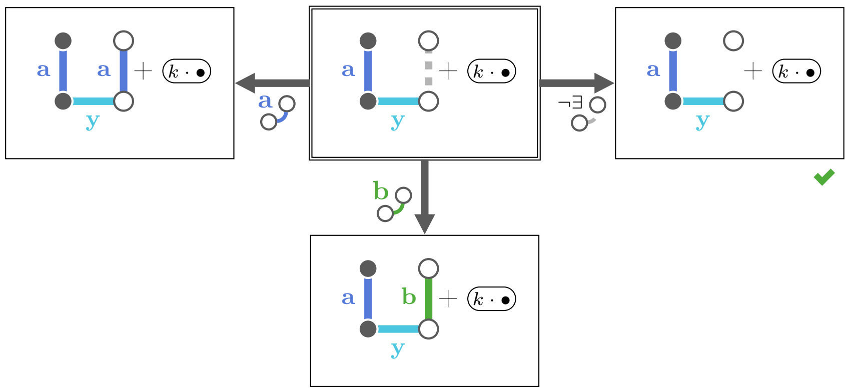
for some colors $\mathrm{a}, \mathrm{y} \in Colors'$. As before, let b denote the second (i.e. different than a) color from two-element set $Colors'$.

Proof structure. Present lemma aims at showing that $\mathbb{G}$ embeds a graph of the form $\underline{\mathrm{az}} + \bullet + (k \cdot \bullet)$. The structure of the proof has a slight subtlety: depending on color y two different cases may occur:

1. if $\mathrm{y} = \mathrm{b}$, then we are bound to succeed with finding the required graph $\underline{\mathrm{az}} + \bullet + (k \cdot \bullet)$,
2. however, if $\mathrm{y} = \mathrm{a}$, in some case we may not immediately find such graph. Instead of it, first we will find graph $G_1' = \underline{\mathrm{ab}} + (k \cdot \bullet)$ — a graph that looks like G_1 we have in our assumptions, but with one edge recolored from y to b. This graph allows us to repeat the whole reasoning, but now with the guarantee that we will end in the first case ($\mathrm{y} = \mathrm{b}$).

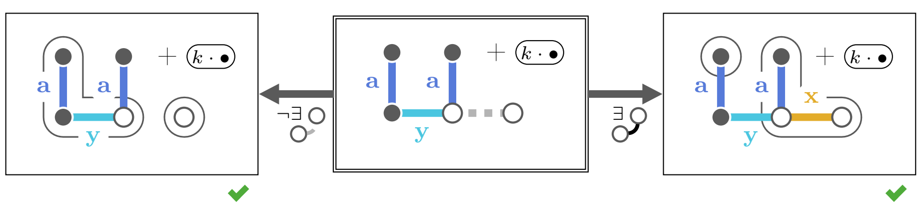
Let us now move on to the proof — even if the subtlety is not entirely clear now, everything should get more evident, when we will get to the problematic point.

Instance 7.1. The first amalgamation instance is built using the graphs G_1 and G_2 following from the assumptions:



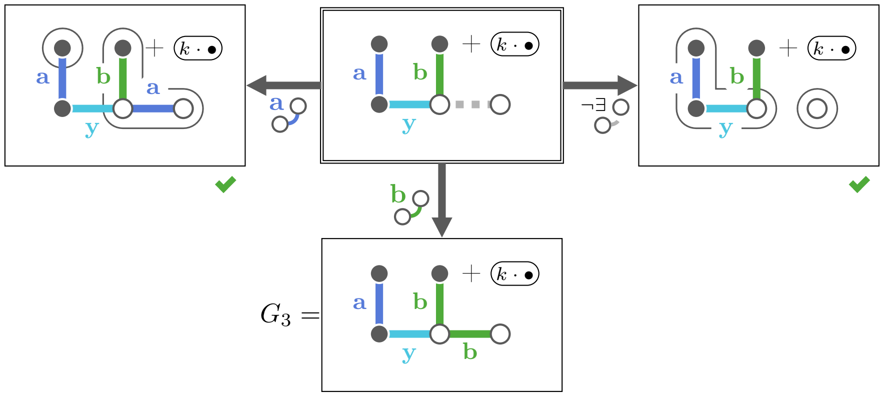
In case where the solution does not contain a new edge, we directly get the graph we are looking for. The case of and a-edge is not much difficult – to successfully deal with it, we only need one additional amalgamation. It turns out, that the appearance of a b-edge is the most cumbersome case. We will return to it in instance 7.3.

Instance 7.2. Here we use the graph axa we just obtained (in case of a-edge) together with G_2.



In each of possible cases we get a graph that matches the pattern we look for — a graph az $+ \bullet +$ $(k \cdot \bullet)$ for some z $\in \mathit{Colors}'$. Let us return to the omitted b-edge case of Instance 7.1:

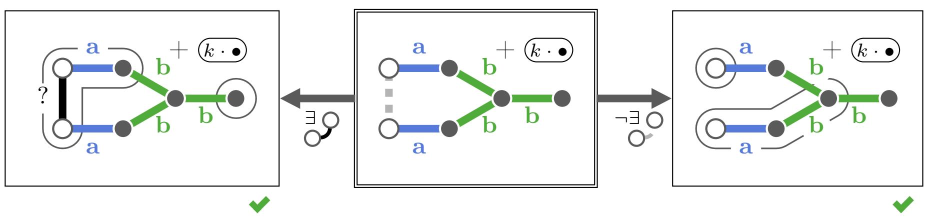
Instance 7.3. Present instance differs from the previous one only with the color of one edge, but it has substantial consequences for our proof.



Let us now consider two possible values of edge color y in the resulting graph.

Case 1° (y = b). Here, to get the graph we look for, it suffices to build one additional amalgamation instance. As the ingredients we take two copies of graph G_3, having in mind the assumed color substitution y = b:

Instance 7.4.



It is easy to see that in each case we get an appropriate subgraph required by the lemma. We may thus move on to the second case.

Case 2° (y = a). Color y has originally appeared in our considerations, because we started with the assumed graph $G_1 = \underline{\text{ay}} + (k \cdot \bullet)$. If y is equal to a, we cannot directly use the technique from the case 1°., however – happily – not everything is lost. After the instance 7.3. we obtained (as a subgraph of G_3) the following graph: $G_1' = \underline{\text{ab}} + (k \cdot \bullet)$. It enables us to repeat the whole proof of the lemma 7. with a new value of variable y, now being certain, that we will succeed: even if none of the previous instances yields the graph we want, we will necessarily fall to the case 1°.

Above observation completes the proof of Lemma 7. □

Lemmas 6 and 7 — in accordance to the previous remarks — form an inductive step that allows to easily prove the following corollary:

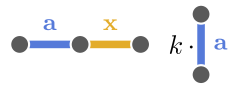
Corollary 2. *If $\mathbb{G}$ satisfies the condition A) of Lemma 1 then for every $k \in \mathbb{N}$ there exist such* a, x $\in$ *Colors′, that $\mathbb{G}$ embeds graph:*

$$\underline{\text{ax}} + (k \cdot \bullet)$$

We omit the simple proof.

8.2 Adding isolated edges

In this part of proof we will be showing a fact similar to the one stated in Corollary 2, but respecting the existence of graphs $\underline{\mathrm{ax}} + k \cdot \underline{\mathrm{a}} \trianglelefteq \mathbb{G}$ for some $\mathrm{a}, \mathrm{x} \in \mathit{Colors}'$:



This time the whole reasoning is divided into three lemmas. Their proofs will be a bit simpler, but the way we should connect them to form a valid inductive step will be less obvious.

Notational remark. Some parts of the statements of the three lemmas were circled. Those expressions are required from the formal point of view, but in fact they make the idea behind the lemmas harder to grasp. It should be noted that the graph S present in those fragments never changes — the lemma 'gets' it from the assumptions and yields it in its thesis in an unchanged form. Similarly, the discrete graphs $n \cdot \bullet$ contribute to the proof in a very simple way: each lemma 'uses' a few their isolated vertices (constants $M_\bullet$) and returns the remaining $(n - M_\bullet)$ vertices. Due to that fact, when reading the lemmas, one should not pay a great attention to the circled fragments. All we have to know is that they exist, then we may safely ignore them.

Lemma 8. *Let $\mathbb{G}$ be a homogeneous, 2-edge-colored graph which embeds $\underline{\mathrm{aa}} + n \cdot \bullet + S$ for some given $n \in \mathbb{N}$ and colors $\mathrm{a}, \mathrm{b} \in \mathit{Colors}'$ ($\mathrm{a} \neq \mathrm{b}$). Then, if $n \geq M_8$, $\mathbb{G}$ embeds also one of the following graphs:*

1. $\underline{\mathrm{a}} + \underline{\mathrm{a}} + (n - M_8) \cdot \bullet + S$,
2. $\underline{\mathrm{ab}} \quad + (n - M_8) \cdot \bullet + S$

for some constant $M_8 \in \mathbb{N}$ (its precise value is not important).

Lemma 9. *Let $\mathbb{G}$ be a homogeneous, 2-edge-colored graph that embeds a graph $\underline{\mathrm{ab}} + n \cdot \bullet + S$ for some given $n \geq M_9$ and colors $\mathrm{a}, \mathrm{b} \in \mathit{Colors}'$ ($\mathrm{a} \neq \mathrm{b}$). Then $\mathbb{G}$ also embeds the graph:*

$$\underline{\mathrm{x}} + \underline{\mathrm{y}} + (n - M_9) \cdot \bullet + S$$

for some constant $M_9 \in \mathbb{N}$ and colors $\mathrm{x}, \mathrm{y} \in \mathit{Colors}'$.

Lemma 10. *Let $\mathbb{G}$ be the* strongly *homogeneous, 2-edge-colored graph that embeds the following graphs:*

1. $\underline{\mathrm{ax}} \quad + n \cdot \bullet + S$,
2. $\underline{\mathrm{a}} + \underline{\mathrm{y}} + n \cdot \bullet + S$,

for some $n \geq M_{10}$ and colors $\mathrm{a}, \mathrm{b}, \mathrm{x} \in$ Colors$'$ ($\mathrm{a} \neq \mathrm{b}$) and $\mathrm{y} \in \{\mathrm{a}, \mathrm{x}\}$. Then one of the following cases holds:

1. *$\mathbb{G}$ embeds a graph $\underline{\mathrm{a}_1\mathrm{x}_1} + \underline{\mathrm{a}_1} + Ⓡ$ for some $\mathrm{a}_1, \mathrm{x}_1 \in$ Colors$'$,*
2. *$\mathbb{G}$ embeds a graph $\underline{\mathrm{ab}}+Ⓡ$ and also embeds either $\underline{\mathrm{aa}}+\underline{\mathrm{b}}+Ⓡ$ or $\underline{\mathrm{bb}}+\underline{\mathrm{a}}+Ⓡ$.*

Above $Ⓡ = ((n - M_{10}) \cdot \bullet + S)$, and $M_{10} \in \mathbb{N}$ is — as in previous lemmas — some constant resulting from the structure of the proof.

Proofs of the above lemmas will help us to show the following corollary:

Corollary 3. *If a strongly homogeneous, 2-edge-colored graph $\mathbb{G}$ embeds a graph $\underline{\mathrm{a}_0\mathrm{x}_0} + k \cdot \bullet$ ($\mathrm{a}_0, \mathrm{x}_0 \in$ Colors$'$) for each $k \in \mathbb{N}$, then also for each $k \in \mathbb{N}$ there exist (potentially new) colors $\mathrm{a}', \mathrm{x}' \in$ Colors$'$ such that $\mathbb{G}$ embeds the following graph:*

$$\underline{\mathrm{a}'\mathrm{x}'} + k \cdot \underline{\mathrm{a}'}$$

We first show how we derive the above corollary from the lemmas, and only later will we focus on proving the three lemmas.

Proof. The procedure of 'producing' the desired graph $\underline{\mathrm{a}'\mathrm{x}'} + k \cdot \underline{\mathrm{a}'}$ will be inductive. Using it, we will be successively getting the following graphs:

$$\begin{array}{lclcl}
\underline{\mathrm{a}_0\mathrm{x}_0} & + & (3k) \cdot M \cdot \bullet & & \\
\underline{\mathrm{a}_1\mathrm{x}_1} & + & (3k-1) \cdot M \cdot \bullet & + & \underline{\mathrm{a}_1} \\
\underline{\mathrm{a}_2\mathrm{x}_2} & + & (3k-2) \cdot M \cdot \bullet & + & \underline{\mathrm{a}_1} + \underline{\mathrm{a}_2} \\
\underline{\mathrm{a}_3\mathrm{x}_3} & + & (3k-3) \cdot M \cdot \bullet & + & \underline{\mathrm{a}_1} + \underline{\mathrm{a}_2} + \underline{\mathrm{a}_3} \\
\dots & & & & \\
\underline{\mathrm{a}_{2k}\mathrm{x}_{2k}} & + & (\mathrm{k}) \cdot M \cdot \bullet & + & \underline{\mathrm{a}_1} + \underline{\mathrm{a}_2} + \underline{\mathrm{a}_3} + \dots + \underline{\mathrm{a}_{2k}} \\
\dots & & & & \\
\underline{\mathrm{a}_{3k}\mathrm{x}_{3k}} & + & (0) \cdot M \cdot \bullet & + & \underline{\mathrm{a}_1} + \underline{\mathrm{a}_2} + \underline{\mathrm{a}_3} + \dots + \underline{\mathrm{a}_{2k}} + \dots + \underline{\mathrm{a}_{3k}}
\end{array}$$

After repeating the inductive step $2k$ times, we will get the graph that – apart from the path $\underline{\mathrm{a}_{2k}\mathrm{x}_{2k}}$ – will contain $2k$ isolated edges colored by $\mathrm{a}_1, \mathrm{a}_2, \dots, \mathrm{a}_{2k} \in$ *Colors$'$* respectively. It is clear there exists a group of at least k edges painted with a common color $\mathbf{w}$. If $\mathrm{a}_{2k} = \mathbf{w}$ we get the thesis of the corollary — we just found a graph:

$$\underline{\mathrm{a}_{2k}\mathrm{x}_{2k}} + k \cdot \underline{\mathrm{a}_{2k}}$$

Similarly, if in one of the next k steps we will get $\mathrm{a}_{2k+i} = \mathbf{w}$ ($i \in \{1, 2, \dots, k\}$), the requirements of the corollary are met. Otherwise, we have $\mathrm{a}_{2k} = \mathrm{a}_{2k+1} = \mathrm{a}_{2k+2} = \dots = \mathrm{a}_{3k} = \overline{\mathbf{w}}$ (where $\overline{\mathbf{w}} \in$ *Colors$'$*, $\overline{\mathbf{w}} \neq \mathbf{w}$), so we have just obtained k isolated edges in a color $\overline{\mathbf{w}}$ together with a path $\overline{\mathbf{w}}\mathrm{x}_{3k}$. It would complete the proof of the corollary.

It remains to show how to use the three lemmas to build the inductive step.

Inductive step. At this point it is easy to guess, what was the purpose of the circled fragments of the form $n \cdot \bullet + S$ appearing in the lemmas:

$$\underline{\mathrm{a}_{2k}\mathrm{x}_{2k}} \quad + \quad \underbrace{(\mathrm{k} \qquad\quad) \cdot M \cdot \bullet}_{k \cdot M \cdot \bullet} \quad + \quad \underbrace{\underline{\mathrm{a}_1} + \underline{\mathrm{a}_2} + \underline{\mathrm{a}_3} + \cdots + \underline{\mathrm{a}_{2k}}}_{S_{2k}}$$

The first part $n \cdot \bullet$ corresponds to a 'resource' of vertices that is used by the lemmas to 'produce' the new edges $\underline{\mathrm{a}_i}$ that appear in the induction scheme we presented earlier. In turn S is a common notation for the edges that are already produced: we begin with empty S_0 and after each inductive step we add one edge to it. After i steps we get $S_i = \underline{\mathrm{a}_1} + \underline{\mathrm{a}_2} + \cdots + \underline{\mathrm{a}_i}$. For the sake of simplicity, we **will omit** both kinds of graphs in the further considerations, only indicating their presence with symbol $\clubsuit$.

Let us assume we have already shown that $\mathbb{G}$ embeds:

$$\underline{\mathrm{ax}} + \clubsuit$$

Our current goal is to show, that $\mathbb{G}$ also embeds:

$$\underline{\mathrm{a}'\mathrm{x}'} + \underline{\mathrm{a}'} + \clubsuit$$

If $\mathrm{x} = \mathrm{a}$ (so we have $\underline{\mathrm{aa}} + \clubsuit$), we use lemma 8., trying to show that $\mathbb{G}$ embeds $\underline{\mathrm{a}} + \underline{\mathrm{a}} + \clubsuit$. If we fail because the second option from lemma takes place, we get the graph $\underline{\mathrm{ab}} + \clubsuit$ (for $\mathrm{b} \neq \mathrm{a}$). It allows us to move on to the case $\mathrm{x} = \mathrm{b}$. If $\mathrm{x} = \mathrm{b}$ (and then we have $\underline{\mathrm{ab}} + \clubsuit$), we can now use lemma 9. In this case we will certainly get the graph $\underline{\mathbf{v}} + \underline{\mathbf{w}} + \clubsuit$ (where $\mathbf{v}, \mathbf{w} \in \mathit{Colors}'$).

Summing the two above cases up, we may end getting one of the three graphs:

$$\underline{\mathrm{a}} + \underline{\mathrm{a}} + \clubsuit, \qquad\qquad \underline{\mathrm{a}} + \underline{\mathrm{b}} + \clubsuit, \qquad\qquad \underline{\mathrm{b}} + \underline{\mathrm{b}} + \clubsuit,$$

wherein the latter two are obtained only when $\mathrm{x} = \mathrm{b}$. In other words, we now have:

1. $\underline{\mathrm{ax}} + \clubsuit$, (from assumptions)
2. $\underline{\mathrm{a}} + \underline{\mathrm{y}} + \clubsuit$ for $\mathrm{y} \in \{\mathrm{a}, \mathrm{x}\}$. (just obtained)

It turns out that those are exactly the assumptions of Lemma 10. Let us use it then.

The lemma lists two possible cases. When the first one holds, we directly get what we wanted — the graph:

$$\underline{\mathrm{a}'\mathrm{x}'} + \underline{\mathrm{a}'} + \clubsuit$$

for some $\mathrm{a}', \mathrm{x}' \in \mathit{Colors}'$.

The second case makes the situation a bit more complicated. Although just as we wanted, we get two separate paths — w.l.o.g. $\underline{\mathrm{aa}} + \underline{\mathrm{b}} + \clubsuit$ — but they do not share a color of some edge, this being needed to complete the proof. We have to repeat all the steps we made so far, adding the obtained edge $\underline{b}$ to $\textcircled{\clubsuit}$:

$$\textcircled{\clubsuit}' = \underline{b} + \textcircled{\clubsuit}$$

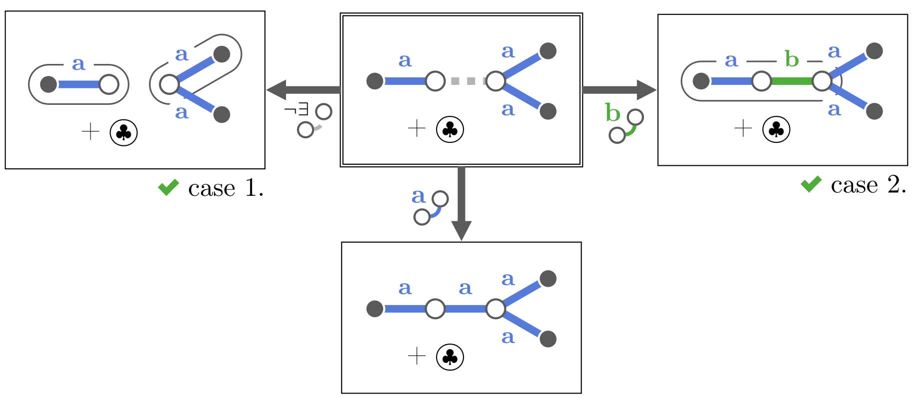
If we again end up in this 'unfortunate' second case of lemma 10, this time we will finally succeed closing the proof. Indeed, in that situation we will get the graph:

$$\underline{ab} + \textcircled{\clubsuit}' = \underline{ab} + \underline{b} + \textcircled{\clubsuit}$$

which corresponds to the graph $\underline{a'x'}$ + $\underline{b'} + \textcircled{\clubsuit}$ appearing in the induction scheme (for $a' = b$ and $x' = a$). $\square$

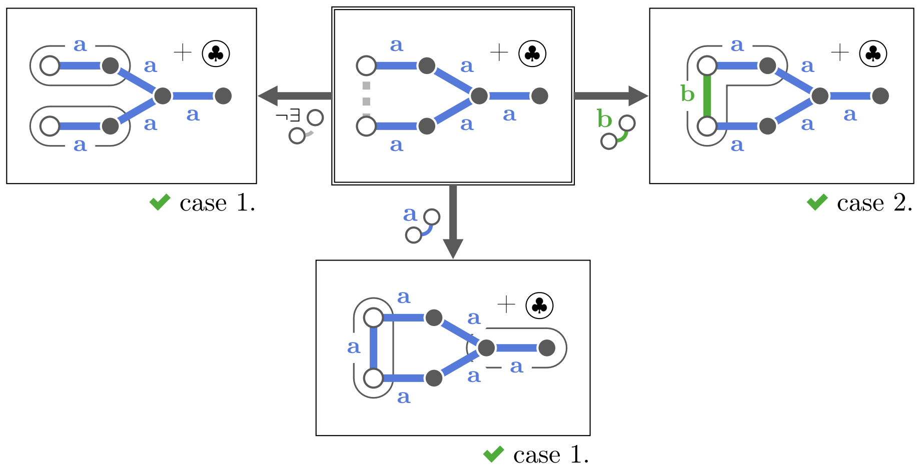
Proof (of Lemma 8).

Instance 8.1. A simple proof of this lemma consists of two amalgamations only. To construct the first one, we use two subgraphs of graph $\underline{aa} + \textcircled{\clubsuit}$ that is present in the assumptions:



If an a-edge does not appear, either case 1. or 2. of the lemma holds, so we are done. If it does, we use two copies of the resulting graph to form the next amalgamation:

Instance 8.2.

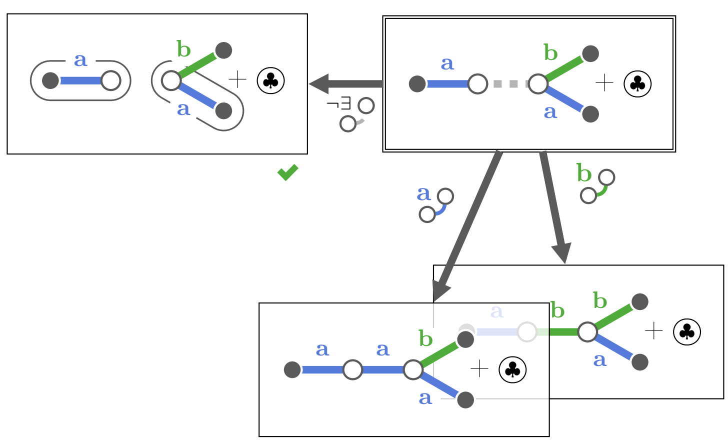


Here, no matter what the result is, we get one of the cases stated in the lemma, what ends the proof. (We may notice here, that for this lemma the constant M_8 is equal to 2, both isolated vertices were consumed in the first instance of amalgamation.) ☐

Proof of the next lemma is equally simple — it is built from three amalgamations, wherein two of them are very similar, so we omit one of them.

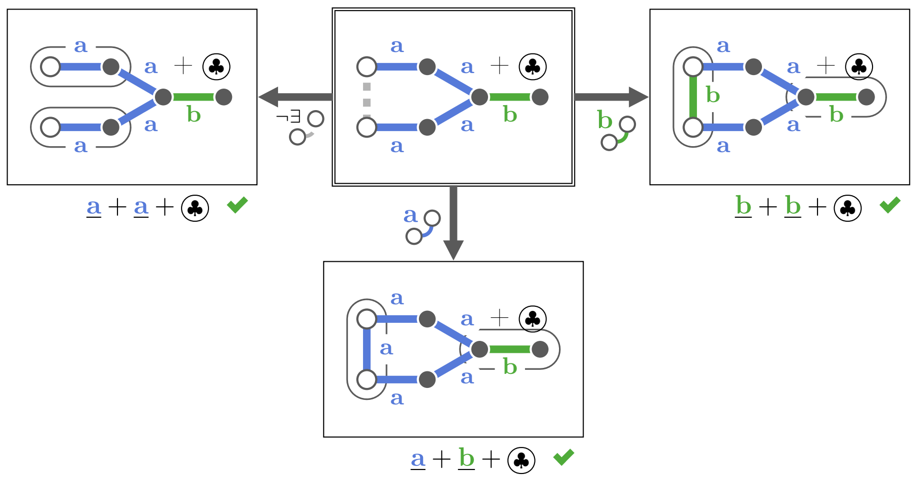
Proof (of Lemma 9).

Instance 9.1. Now in the assumptions we have the graph $\underline{ab} + \clubsuit \trianglelefteq \mathbb{G}$. We build the first instance as in the previous proof:



In the case of nonexistent edge we get what we were looking for — two disjoint edges ($+\clubsuit$). However, if the edge exists, we have to use two further instances — for a and for b. Again, they are similar, so we omit the second one.

Instance 9.2. (The third one is analogous.)



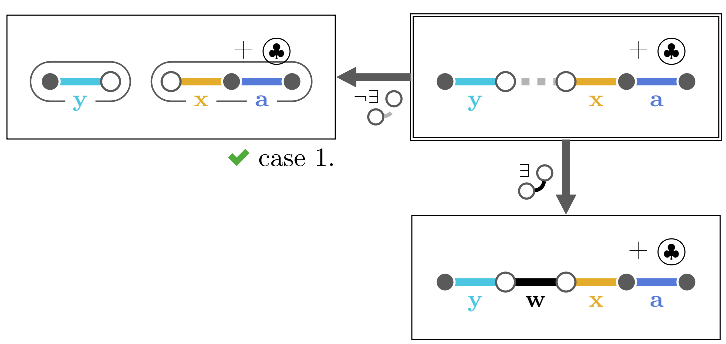
This ends the proof of the lemma, since in all above cases we get the subgraphs we need. ◻

Now the only remaining part is the proof of Lemma 10.

Proof (of Lemma 10). From the assumptions we get the subgraphs $\underline{\mathrm{ax}} + \clubsuit \trianglelefteq \mathbb{G}$ and $\underline{\mathrm{a}} + \underline{\mathrm{y}} + \clubsuit \trianglelefteq \mathbb{G}$ for some colors $\mathrm{a}, \mathrm{b}, \mathrm{x} \in \mathit{Colors}'$ ($\mathrm{a} \neq \mathrm{b}$) and $\mathrm{y} \in \{\mathrm{a}, \mathrm{x}\}$.

We will start by considering the following instance:

Instance 10.1.



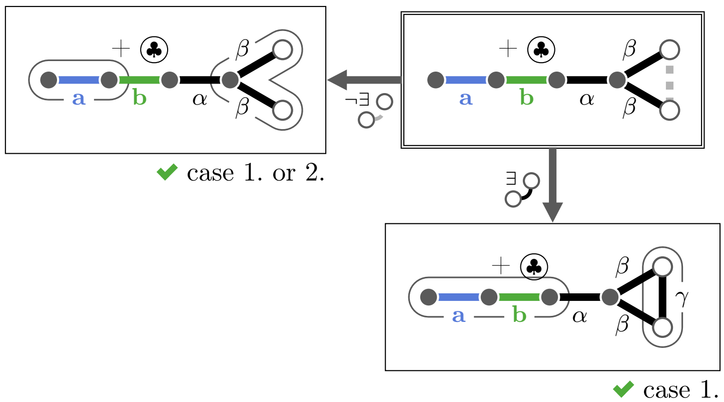
If in its solution the edge will not emerge, we get appropriate graph: since $\mathrm{y} \in \{\mathrm{a}, \mathrm{x}\}$, we know that y will appear somewhere on the path $\underline{\mathrm{ax}}$, and this suffices to fulfill the case 1. of the lemma we are proving.

Otherwise we get a path $P = \underline{\mathrm{y}\mathbf{w}\mathrm{xa}} + \clubsuit$. We will consider two cases, depending on whether it has the form $W = \underline{\mathrm{ab}??} + \clubsuit$ or not.

We should first observe, that there is only one case when P does not match W. Indeed: When $\mathrm{x} = \mathrm{b}$, P is bound to have the form W. In the other case P takes the shape $\underline{\mathrm{a}\mathbf{w}\mathrm{aa}} + \clubsuit$, since $\mathrm{y} \in \{\mathrm{a}, \mathrm{x}\}$, and yet now $\mathrm{x} = \mathrm{a}$. It follow immediately that the only case when P is not of the form W is $P = \underline{\mathrm{aaaa}} + \clubsuit$.

Case 1°. ($P = \underline{ab\alpha\beta} + ♣$, where $\alpha, \beta \in \mathit{Colors}'$) Here, the only amalgamation instance is built as follows:

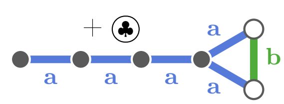
Instance 10.2.



If as a result of amalgamation we get an edge, we may easily fulfill case 1. of our lemma — the only thing we need is that color γ appears on the path $\underline{ab}$, and this of course is happening, since $\gamma \in \{a, b\} = \mathit{Colors}'$.

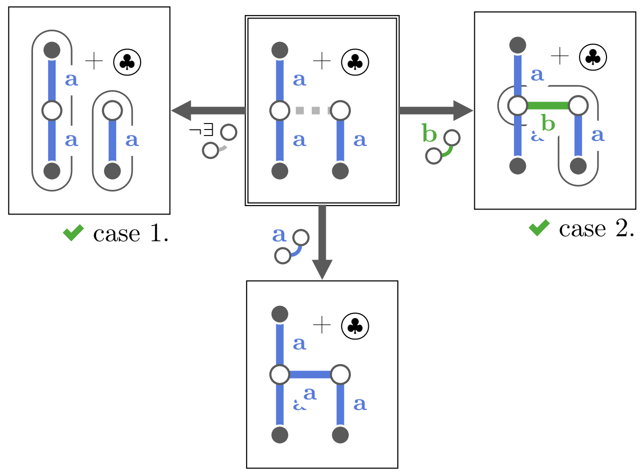
If in turn the edge was not produced, we get (as a aubgraph): $\underline{\beta\beta} + \underline{a} + ♣$. Now, depending on the value of β, either case 1. or 2. is fulfilled. Indeed, when $\beta = a$ we obtain the subgraph $\underline{aa} + \underline{a} + ♣$ and case 1. of the lemma holds. When we get $\beta = b$, then (together with graph $\underline{ab} + ♣ \trianglelefteq P$) we have all what is needed for case 2. of the lemma.

Case 2°. ($P = \underline{aaaa} + ♣$) Here, the simple amalgamation instance similar to the one from case 1° (picture omitted) completes the proof only in cases ¬∃ and a. If instead we got the following result



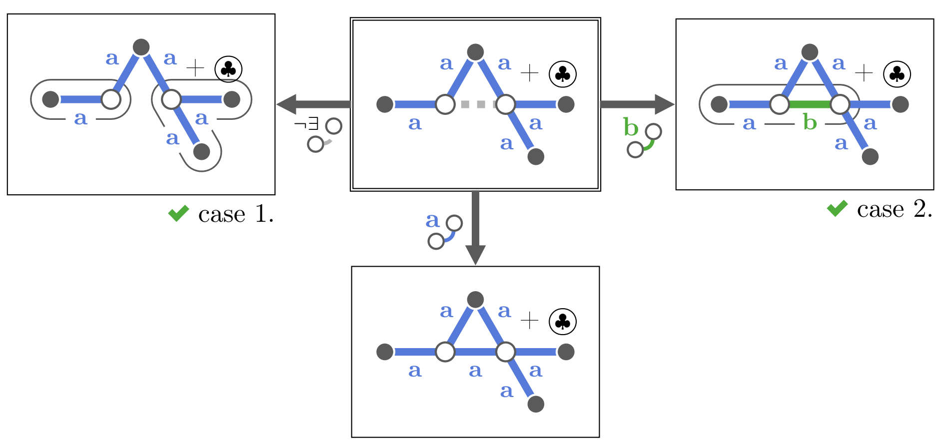
we cannot use it for case 1. of the lemma, and to satisfy the case 2. an additional graph $\heartsuit = \underline{ab} + ♣$ is required. Another sequence of amalgamations awaits — four extra instances will be needed.

Instance 10.3.



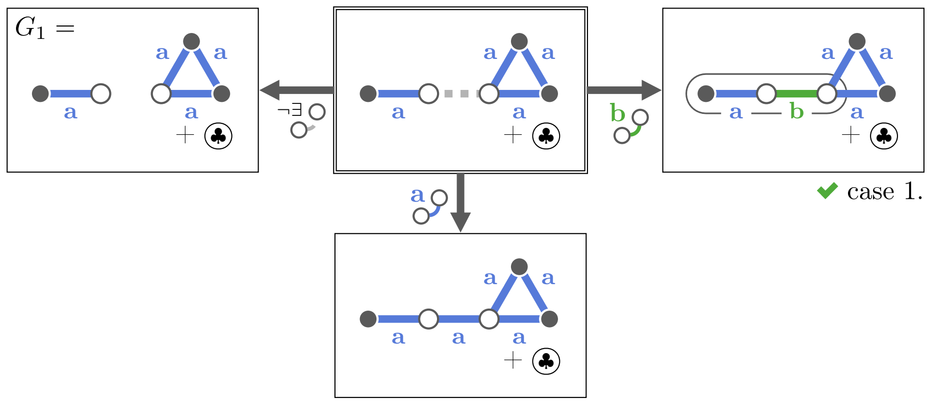
Above, when the edge does not exist case 1. of lemma easily follows. If in turn we get a b-edge, there appears the graph ♡ we are searching for, allowing to meet the requirements of case 2. Let us assume then, that we got an a-edge.

Instance 10.4. The graph we just obtained allows to build the following instance:



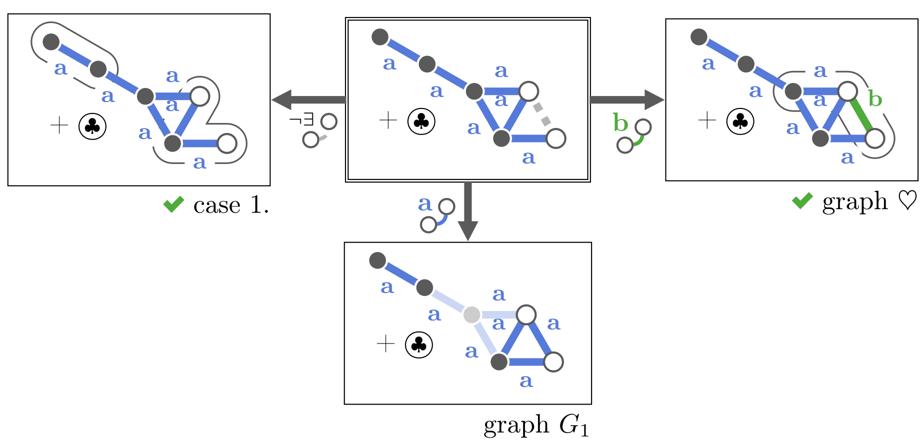
As before, the lack of an edge of the appearance of a b-edge lead us straight to the cases 1. or 2. Again, we assume we unluckily got an a-edge.

Instance 10.5. From the result of previous instance we take the subgraph ∘aaa+ ♣, and, pairing it with the graph $\underline{\text{a}}$+$\underline{\text{a}}$+♣, we build an amalgamation as follows:



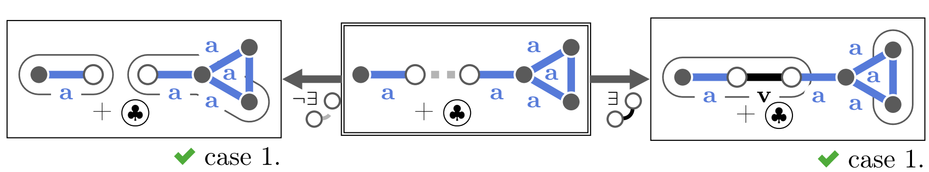
Here, if we got a b-edge, we finish with case 2., having found the graph ♡. If the edge was not present, we immediately get a graph G_1 that later will help us to finish the proof. If in turn an a-edge appeared, we need to perform one additional amalgamation in order to get the same G_1.

Instance 10.6. (building G_1) Now, we pair the previous result with the path P:



In two out of three possible cases we finish immediately, while in the third one the expected graph G_1 appears as a subgraph.

Instance 10.7. Using the graph G_1 and the result of instance 10.4., we perform the last amalgamation in the proof of this lemma, thus providing the final missing link needed to finalize the proof of corollary 3.



No matter if the edge appeared or not, case 1. of the lemma gets fulfilled, what finishes the proof. □

8.3 Adding paths of length 2

In the previous part of the proof we had a 'resource' ♣ of isolated vertices and we could use them as needed to construct successive instances of amalgamation. From now on — thanks to Corollary 3. — we may afford to maintain an arbitrarily large collection of edges $\underline{a}$ ($a \in Colors'$).

The aim of the next four amalgamations will be to show, that we actually can afford even more — a collection of 2-edge paths of the form $\underline{ax}$ (for some $x \in Colors'$). It is the last step we need to make before showing the ultimate goal of this branch of the proof — deriving the existence of arbitrarily long ab-paths in $\mathbb{G}$.

Let us formalize the lemma we intend to prove:

Lemma 11. *If a strongly homogeneous, 2-edge-colored graph $\mathbb{G}$ satisfies corollary 3, i.e., for every $k \in \mathbb{N}$ we may find colors $a, x \in Colors'$ such that $\mathbb{G}$ embeds the graph $\underline{ax} + k \cdot \underline{a}$, then the following condition also holds:*

$$\forall_{n \in \mathbb{N}} \ \exists_{y \in Colors'} \ n \cdot \underline{ay} \trianglelefteq \mathbb{G}$$

Proof. (*lemat 11.*) As in the previous part, the proof will be inductive. This time, aiming to find $n \cdot \underline{ay} \trianglelefteq \mathbb{G}$ (for some $y \in Colors'$), we will produce successively all the graphs bellow:

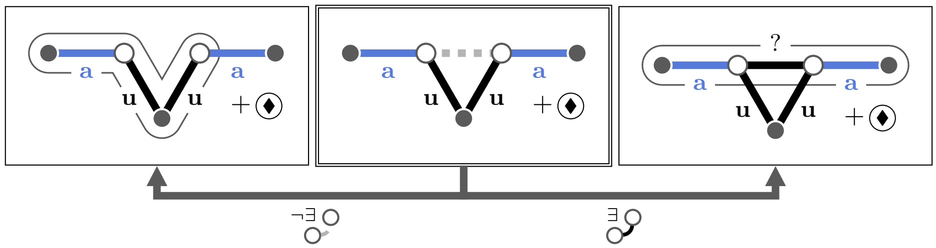
$$\begin{array}{lllll}
\underline{ax_0} & + & (2n\quad\ ) \cdot M \cdot \underline{a} & & \\
\underline{ax_1} & + & (2n-1) \cdot M \cdot \underline{a} & + & \underline{ay_1} \\
\underline{ax_2} & + & (2n-2) \cdot M \cdot \underline{a} & + & \underline{ay_1} + \underline{ay_2} \\
\underline{ax_3} & + & (2n-3) \cdot M \cdot \underline{a} & + & \underline{ay_1} + \underline{ay_2} + \underline{ay_3} \\
\dots & & & & \\
\underline{ax_{2n}} & + & (0\qquad\ ) \cdot M \cdot \underline{a} & + & \underline{ay_1} + \underline{ay_2} + \underline{ay_3} + \dots + \underline{ay_{2n}}
\end{array}$$

At each point, to produce one isolated path $\underline{ay_i}$ we will have to get some constant number $M \in \mathbb{N}$ of isolated edges $\underline{a}$ from our 'resource'. After completing $2n$ steps, among the resulting paths $\underline{ay_\bullet}$, by pigeonhole principle, there exists a subset of n paths all colored the same way. This will finish the proof.

Similarly as before, to hide the unnecessary details, we will use the symbol ♦ for the frequently appearing graphs of the form $\alpha \cdot \underline{a} \ + \ \underline{ay_1} + \dots + \underline{ay_i}$ — they are almost passive in the steps of the coming proof. It is enough to remember, that each time we need a new isolated edge $\underline{a}$, we take it from ♦. Moreover, after each inductive step we add to ♦ a new isolated path $\underline{ay_i}$.

Inductive step From the assumptions we have the graph $\underline{a\mathbf{u}}$ + ♦ (for some $a, \mathbf{u} \in Colors'$), and this time our goal is to prove that $\mathbb{G}$ embeds a graph $\underline{a\mathbf{v}} + \underline{a\mathbf{w}}$ + ♦. As we have already mentioned, we only have to consider four instances of amalgamation.

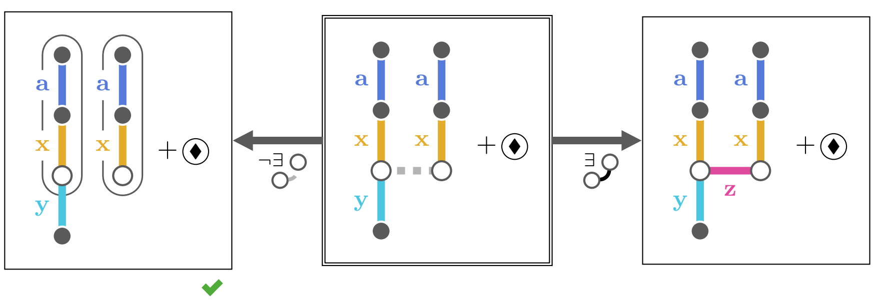
Instance 11.1.



No matter what the result will be, we will get the following path:

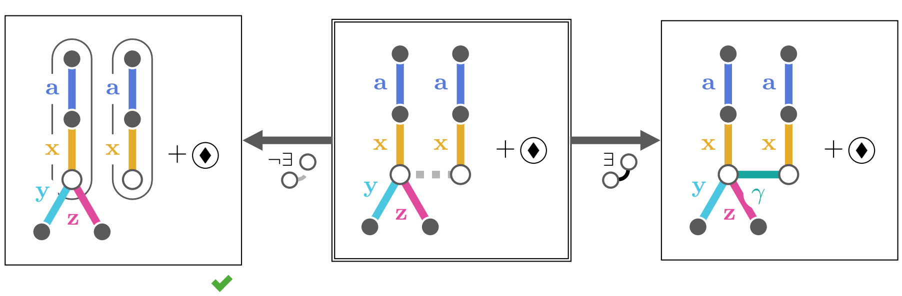
for some $x, y \in \mathit{Colors}'$.

Instance 11.2. Using it (with an additional edge $\underline{a}$ taken from $\blacklozenge$), we build the following instance:



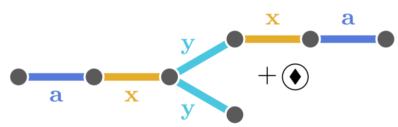
If the edge is not present in the solution, we readily get two disjoint paths of length 2. Suppose then that some z-edge appeared ($z \in \mathit{Colors}'$). If $z = y$, we move on straight to the instance 4. If in turn $z \neq y$, an additional step in necessary:

Instance 11.3. Once more we get one edge from $\blacklozenge$ and build an instance similar to the previous one — the only difference is the new edge colored with $z \neq y$.



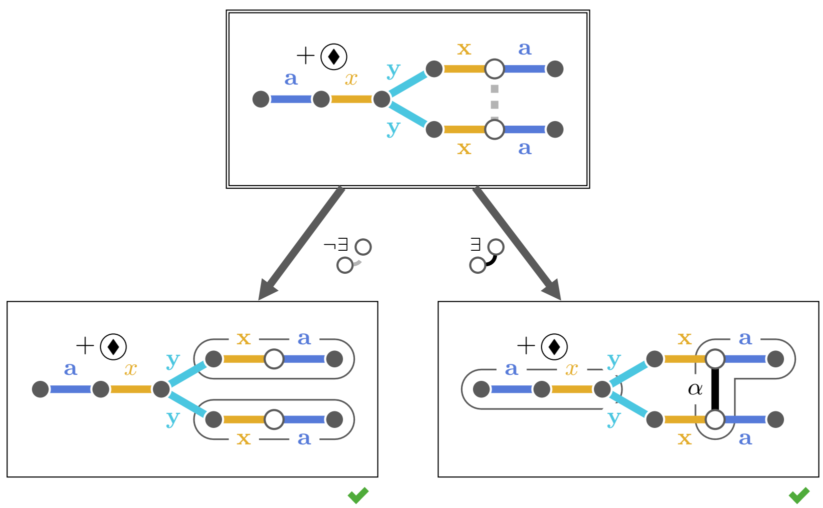
If we do not obtain an edge, we end having – as before – two disjoint paths. When some edge exists, we are sure that its color $\gamma \in \mathit{Colors}'$ is either equal $\mathbf{y}$ or $\mathbf{z}$, since $\mathbf{y} \neq \mathbf{z}$ and Colors' has only two elements. W.l.o.g. let us assume, that $\gamma = \mathbf{y}$.

Then we have, as a result of Instance 11.2 or 11.3, a graph of the form:



Using it we may create the last amalgamation instance and finalize the proof.

Instance 11.4.



Independently form the existence of an edge, in the result we may fine a subgraph of the following form:

$$\underline{\mathbf{av}} + \underline{\mathbf{aw}} + \blacklozenge$$

Its presence ends the proof of the lemma. □

8.4 Producing arbitrarily long paths

There is the last thing to do in case A) — showing that in $\mathbb{G}$ arbitrarily long paths exist. It is formalized by the following lemma:

Lemma 12. *If a homogeneous and 2-edge-colored graph $\mathbb{G}$ embeds a graph $n \cdot \mathbf{ax}$ (for some $\mathbf{a}, \mathbf{x} \in \mathit{Colors}'$) for each $n \in \mathbb{N}$, then $\mathbb{G}$ also embeds an arbitrarily long $\mathbf{ab}$-path.*

Proof. Once more we conduct an induction.

Inductive step Here we will be showing how from shorter paths we may produce longer ones: Assuming that we have a graph that is a sum of paths of length $d \in \mathbb{N}, d \geq 2.$, we will build (using amalgamation) a graph that is a sum of (fewer) paths of length $2d - 1$.

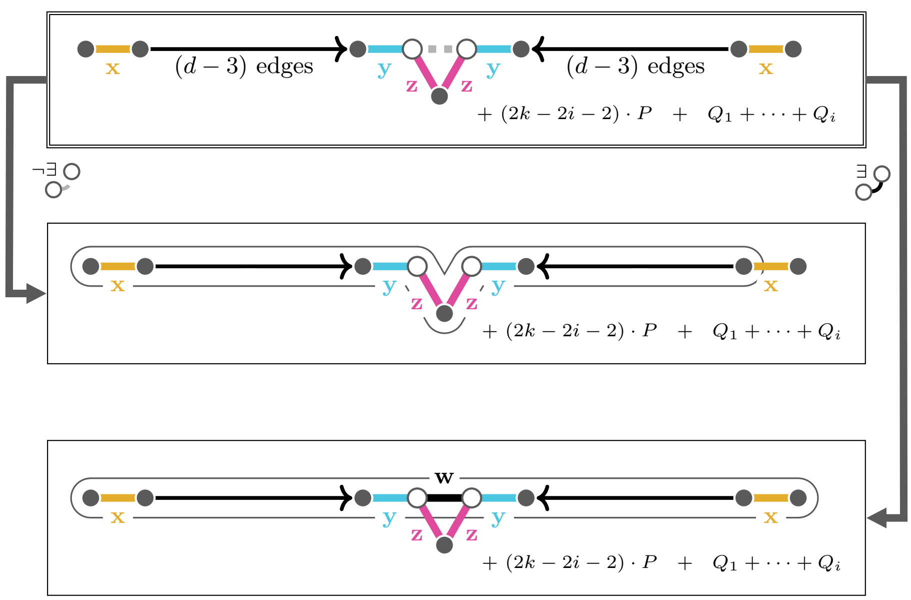
More precisely, we will show how from the graph $2k \cdot P$ (where $k \in \mathbb{N}$ and P is some ab-path of length d) we can derive in k steps a graph $Q_1 + Q_2 + \cdots + Q_k$, where $Q_\bullet$ are ab-paths of length $2d-1$. Taking sufficiently large k we will ensure, that among those k paths (by pigeonhole principle) there will be a group of size n of equally colored ones.

The outline of the procedure is as follows:

$$\begin{array}{lll}
(k \qquad) \cdot P & & \\
(k-1) \cdot P & + & Q_1 \\
(k-2) \cdot P & + & Q_1 + Q_2 \\
(k-3) \cdot P & + & Q_1 + Q_2 + Q_3 \\
\ldots & & \\
(0 \qquad) \cdot P & + & Q_1 + Q_2 + Q_3 + \cdots + Q_k
\end{array}$$

To show a single step, one amalgamation will be enough:

Instance 12.1.



In both cases we get the path of length $2d - 1$ we wanted. It should be noted here, that the construction required a pair of equally colored paths.

Repeating the above amalgamation k times (according to the previously mentioned outline) we get a collection of k disjoint paths, each of length $2d-1$. They are not necessarily painted the same way, but fixing some sufficiently large k (e.g. $k = 2^{d-1} \cdot n$ surely would do), we may choose a subset of n same-looking paths. We are allowed to do that, since by assumption for each $k \in \mathbb{N}$ we can produce a graph $k \cdot P$ (for some ab-path P of length d). This ends the proof of the inductive step.

Because at the very beginning we can choose an arbitrarily large sum of equal paths ax, then using the inductive step repeatedly we will be proving the possibility of producing collections of paths of increasing lengths:

$$2 \longrightarrow (2 \cdot 2) - 1 = 3 \longrightarrow 5 \longrightarrow 9 \longrightarrow 17 \longrightarrow \ldots$$

This observation completes our proof. $\square$

Summary We made our way to the end of the Section 8. The chain of lemmas that were stated has its beginning at the case A) of Lemma 1. As we move along this chain, we show the possibility of adding to the initial graphs respectively:

– first, an arbitrary number of isolated vertices,
– then, edges,
– next, 2-edge paths,
– finally, arbitrarily long paths.

At the end of the chain, we have obtained the second case of Theorem 4, so we may at last consider the case A) as resolved.

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