AN APPLICATION OF THE GENERATING FUNCTION TO DIFFERENTIAL EQUATIONS

T. A. NEWTON, Washington State University

The calculus discussed here is based upon a power series transform for sequences which was introduced by Laplace in 1812 [4], a year before he introduced his now well-known integral transform. This series transform has been extensively studied as a *generating function* of a given sequence (see Erdélyi *et al.* [1], pp. 228–282). A discussion of its applications in the theory of finite differences is found in Jordan ([3], pp. 20–44). Goldberg ([2], pp. 189–207) discusses the application of this transform to the solution of linear difference equations with constant coefficients. First, we will briefly outline some of the basic theory of this transform and its inverse. We will then show how the inverse transform can be applied to formalize the finding of the power series expansion of the solution of a linear differential equation in a manner that seems to be new.

Let $\mathfrak{F}$ denote the set of complex valued functions, defined and holomorphic on some open connected subset (depending upon the individual function) of the complex plane containing the origin. Let $\mathfrak{R}$ denote the set of equivalence classes of $\mathfrak{F}$ so defined that two elements of $\mathfrak{F}$ are equivalent if their restrictions to some neighborhood of zero are identical. For convenience sake, we will not distinguish between an element of $\mathfrak{F}$ and the equivalence class to which it belongs. Now let $\mathcal{S}$ denote the set of all sequences $a=\{a_n\}$ such that $\overline{\lim}\sqrt[n]{|a_n|}$ is finite. We then consider the mapping $G: \mathcal{S}\rightarrow\mathfrak{R}$ for which $u=G(a)$ if

$$u(t) = \sum_{n=0}^{\infty} a_n t^n$$

for $|t|<(\lim \sqrt[n]{|a_n|})^{-1}$. The function $u=G(a)$ is called the *generating function* of the sequence a. It follows from the theory of the Taylor series that this mapping $G: \mathcal{S}\rightarrow\mathfrak{R}$ is one-to-one; and for the inverse mapping, G^{-1}, $a=G^{-1}(u)$ if and only if

$$a_n = u^{(n)}(0)/n! \tag{1}$$

for $n=0, 1, \cdots$. It is immediate that for $a\in\mathcal{S}$, $a=G^{-1}(G(a))$; while for $u\in\mathfrak{R}$, $u=G(G^{-1}(u))$.

It follows immediately from their definition that the mappings G and G^{-1} are linear. That is, for α and β constants: if $a\in\mathcal{S}$, $b\in\mathcal{S}$, then $(\alpha a+\beta b)\in\mathcal{S}$ and

$$G(\alpha a + \beta b) = \alpha G(a) + \beta G(b); \tag{2}$$

if $u\in\mathfrak{R}$, $v\in\mathfrak{R}$, then $(\alpha u+\beta v)\in\mathfrak{R}$ and

$$G^{-1}(\alpha u + \beta v) = \alpha G^{-1}(u) + \beta G^{-1}(v). \tag{3}$$

Define the mapping $E^m: \mathcal{S}\rightarrow\mathcal{S}$, $m=0, 1, \cdots$, by $(E^m a)_n = a_{n+m}$ for $n=0, 1, \cdots$, where $E^1=E$. It can be established by induction that if $U_m=G(E^m a)$, then

$$U_m(t) = t^{-m}[u(t) - a_0 - ta_1 - \cdots - t^{m-1}a_{m-1}] \tag{4}$$

for $t \neq 0$, and $m = 1, 2, \cdots$.

Upon checking the above reference to Goldberg ([2], pp. 189–207), one finds that (2), (3), and (4) are the basis for the operational calculus which makes the generating function applicable for the solution of difference equations. As with the Laplace transform, a table of generating functions is a valuable tool for such applications. Any table of power series expansions can be used as a starting point for such a table, and further entries with their derivations can be found in the references cited above.

We shall now show how the inverse transform, $G^{-1}: \mathcal{R} \to \mathcal{S}$, is applicable to the solution of differential equations.

It follows from the Cauchy product theorem for power series that if $u \in \mathcal{R}$, $v \in \mathcal{R}$, then $uv \in \mathcal{R}$ and if $a = G^{-1}(u)$, $b = G^{-1}(v)$, then

$$(G^{-1}(uv))_n = \sum_{j=0}^{n} a_j b_{n-j} \tag{5}$$

for $n = 0, 1, \cdots$. We might well call the sequence defined by (5) the *convolution* of the sequences a and b, and if we were to denote it by $a * b$, we would then have $G(a)G(b) = G(a*b)$.

It follows from (1) that if $u \in \mathcal{R}$, then $u^{(1)} \in \mathcal{R}$, and

$$(G^{-1}(u^{(1)}))_n = \frac{u^{(n+1)}(0)}{n!} = (n+1)\frac{u^{(n+1)}(0)}{(n+1)!} = (n+1)(EG^{-1}(u))_n$$

for $n = 0, 1, \cdots$. We can now establish by induction that $u^{(m)} \in \mathcal{R}$ and

$$(G^{-1}(u^{(m)}))_n = (n+1)(n+2) \cdots (n+m)(E^m G^{-1}(u))_n \tag{6}$$

for $m = 1, 2, \cdots, n = 0, 1, \cdots$. Finally, we combine (5) and (6) to conclude that if $u \in \mathcal{R}$, $v \in \mathcal{R}$, then $uv^{(m)} \in \mathcal{R}$, and if $a = G^{-1}(u)$, $b = G^{-1}(v)$, then

$$(G^{-1}(uv^{(m)}))_n = \sum_{j=0}^{n} a_j(n-j+1)(n-j+2) \cdots (n-j+m) b_{n-j+m} \tag{7}$$

for $n = 0, 1, \cdots$ and $m = 1, 2, \cdots$.

Consider now the differential equation $(1+t^2)v'' + 2tv' - 2v = 0$. If we define $u_0(t) = (1+t^2)$, $u_1(t) = 2t$, $u_2(t) = -2$, and apply (3), we get

$$G^{-1}(u_0 v'') + G^{-1}(u_1 v') + G^{-1}(u_2 v) = 0. \tag{8}$$

Substituting into (7) with $m = 2$, $a_0 = 1$, $a_1 = 0$, $a_2 = 1$, and $a_j = 0$ for $j > 2$, it follows that if $G^{-1}(v) = b$, then

$$(G^{-1}(u_0 v''))_0 = 2b_2, \quad (G^{-1}(u_0 v''))_1 = 2 \cdot 3 b_3,$$

$$(G^{-1}(u_0 v''))_n = (n+1)(n+2) b_{n+2} + (n-1)n b_n \qquad \text{for } n \geqq 2.$$

Likewise, we find

$$(G^{-1}(u_1v'))_0 = 0,$$
$$(G^{-1}(u_1v'))_n = 2nb_n \qquad \text{for } n \geqq 1,$$
$$(G^{-1}(u_2v))_n = -2b_n \qquad \text{for } n \geqq 0.$$

Substituting into (8) and simplifying we get the difference system

$$b_2 = b_0, \; b_3 = 0, \qquad (n+1)b_{n+2} + (n-1)b_n = 0, \qquad \text{for } n \geqq 2.$$

It follows that b_0 and b_1 are arbitrary, $b_{2k+1}=0$, and $b_{2k}=[(-1)^{k+1}/(2k-1)]b_0$ for $k=1, 2, \cdots$. That is,

$$v(t) = b_0 + b_1t + b_0\left[\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)} t^{2k}\right] = b_1t + b_0(1 + t\cdot\tan^{-1} t).$$

The above steps are precisely those met in the usual text-book solution of the differential equation (8) when one assumes a solution of the form

$$v(t) = \sum_{n=0}^{\infty} b_nt^n.$$

References

1. A. Erdelyi *et al.*, Higher Transcendental Functions, vol. 3, New York, 1955.
2. S. Goldberg, Introduction to Difference Equations, New York, 1958.
3. C. Jordan, Calculus of Finite Differences (2nd ed.), New York, 1947.
4. Pierre Simon de Laplace, Théorie Analytique des Probabilités, Paris, 1812.

AN ALGEBRAIC PROOF OF KIRCHHOFF'S NETWORK THEOREM*

A. NERODE AND H. SHANK, The University of Chicago

1. Introduction. Weyl [10] gave probably the first complete proof of the existence and uniqueness of currents in a resistive network subject to Ohm's law and Kirchhoff's voltage and current laws (Eckmann [2, 3], Roth [7, 8]). Much earlier Kirchhoff [5] gave an explicit expression for these currents in terms of maximal trees of the correlated graph. There are a number of papers on this theorem scattered through the literature, notably those of Ahrens [1] and Franklin [4]; and there has been considerable interest recently in its possible use in connection with computation of solutions to network problems (MacWilliams [6]). We give a simple linear algebraic proof of this ancient theorem, and incidentally obtain for graphs an expression for the projection of a real 1-chain on the space of 1-cycles in terms of maximal trees.

2. Preliminaries. A (finite oriented) graph G consists of a finite set V (of *vertices*) and a subset G of $V\times V$ (of *branches*) such that if $(v_1, v_2)\in G$ then

* This research was supported in part by the U. S. Air Force under Contract No. AF33(616)-2797, monitored by the Aeronautical Research Laboratory, Wright Air Development Center.