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THE AXIOMATIZATION OF ARITHMETIC

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1. Introduction. I once asked myself the question: How were the famous axiom systems, such as Euclid's for geometry, Zermelo's for set theory, Peano's for arithmetic, originally obtained? This was to me more than merely a historical question, as I wished to know how the basic concepts and axioms were to be singled out, and, once they were singled out, how one could establish their adequacy. One possible approach which suggests itself is to take typical theorems, proofs, definitions, and examine case by case what assumptions and concepts are involved. The obstacle in such an empirical study is, apart from the obvious demand of excessive time and energy, the lack of conclusiveness in both result and justification.

The attempt to find an answer to this question led me to some interesting fragments of history. For example, in 1899 Cantor distinguished consistent collections (the "sets") from inconsistent collections ([1], p. 443), anticipating partly the distinction between the two kinds of classes stressed by von Neumann and Quine. Cantor had already proposed a form of the axiom of substitution ([1], p. 444, line 3), although Fraenkel and Skolem, more than twenty years later, had to adjoin it to Zermelo's list of axioms as a supplement. In another direction, the history of the development of axioms of geometry makes clear how natural it was for Hilbert to raise in 1900 the consistency question of analysis ([2], p. 299) quite independently of the emphasis on set-theoretical paradoxes.

By far the best piece of good fortune I had in these historical researches was, however, my findings with regard to Peano's axioms for arithmetic. It is rather well-known, through Peano's own acknowledgement ([3], p. 273), that Peano borrowed his axioms from Dedekind and made extensive use of Grassmann's work in his development of the axioms. It is not so well-known that Grassmann had essentially the characterization of the set of all integers, now customary in texts of modern algebra, that it forms an ordered integral domain in which each set of positive elements has a least member. Very few people know (cf. [4], p. 490) that Dedekind wrote a very interesting letter (dated 27 February, 1890, addressed to Headmaster Dr. H. Keferstein of Hamburg) to explain how he arrived at the Peano axioms. In what follows I intend to quote at length (by written permission of the Niedersächsische Staats- und Universitätsbibliothek at Göttingen given in the autumn of 1954) and to comment on this letter. The notion of non-standard models (unintended interpretations) of axioms for positive integers is, for instance, brought out quite clearly in Dedekind's letter. To clarify

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the letter, I shall also dwell at length on the contents of Dedekind's famous essay on the nature and meaning of positive integers ([4], pp. 335–391).

2. Grassmann's calculus. A more elementary question related to the axiomatization of arithmetic would seem to be an explicit statement of some adequate group of natural rules and conventions which enables us to justify all the true numerical formulae containing 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, $+$, $\times$, $=$, (,) such as $7+5=12$. Since we must not use "infinitely long expressions" or indispensable and-so-on's, such a task is not as simple as one might think at first. For example, we would need the multiplication table, the "addition table," the convention about the positional system of notation, the ordinary commutative, associative, distributive laws, the substitution of equals for equals. It would be much too tiresome to include the result of any such attempt here, although I am inclined to think that such an exercise could be of pedagogic interest if one wished to explain in an elementary textbook the process of formalizing implicit assumptions. The advantages are that addition and multiplication of positive integers are familiar to everybody and that they are of fundamental importance everywhere. Of course I do not claim that I can predict whether such an example will induce respect for formalization or destroy the pupil's interest in logic. In any case, our interest here is not so much with numerical formulae in the Arabic notation but primarily with general statements in arithmetic.

Many mathematicians have studied natural numbers in complete isolation from their applications. In this way, a large body of concepts and theorems concerning natural numbers has been accumulated. This more or less determined area constitutes the mathematical theory of natural numbers. A transparent characterization of the domain should be interesting by itself and of assistance toward understanding the concept of number. We are familiar with the two usual methods of developing mathematics: the genetic or constructive approach customary in the extension of numbers to integers, fractions, real numbers, complex numbers; and the axiomatic method usually adopted for the teaching of elementary geometry. There are many differences between elementary textbooks of geometry and the sharper formulations of axioms of geometry, so that we might wish to call the former intuitive axiomatics, the latter formal axiomatics. The application of the axiomatic method in the development of numbers is not natural. Its rather late appearance is evidence. If, however, our problem is to have a systematic understanding of the domain of number theory, the axiomatic method suggests itself. Indeed, we now realize that the method can be applied to numbers, and that for certain purposes the application is suitable. It remains a rare thing to teach arithmetic from an axiomatic approach.

In 1861, Hermann Grassmann published his ***Lehrbuch der Arithmetik*** ([5]). This was probably the first serious and rather successful attempt to put numbers on a more or less axiomatic basis. Instead of just the positive integers, Grassmann dealt with the totality of all integers, positive, negative, and 0. Much of his method can be used to handle the smaller totality of all positive integers, too. He was probably the first to introduce recursive definitions for addition and multiplication, and prove on such a basis ordinary laws of arithmetic by mathematical induction. The work is of interest not only in Grassmann's success in making implicit things explicit, but also in his failure or choice not to characterize formally certain concepts which are neutral or belong to logic.

I shall compare Grassmann's system with the postulational characterization of integers which is customary in present day abstract algebra. According to the latter, integers form an ordered integral domain in which each set of positive integers has a least element. (Compare, for example, [6], p. 36, p. 3, p. 7.)

With non-essential changes we can formulate this into a calculus L_1:

I. Atoms: 0, 1; $+$, $-$, $.$; Pos: a, b, c, etc. (variables).

II. Terms: 0 is a term; 1 is a term; a variable is a term; if s and t are terms, $(-s)$, $(s+t)$, $(s.t)$ are terms.

III. Formulae: if s and t are terms, $s=t$ is a formula, $s \epsilon \text{Pos}$ is a formula.

IV. Axioms:

(2.1) $a+(b+c) = (a+b)+c$.
(2.2) $a+b = b+a$.
(2.3) $a.(b.c) = (a.b).c$.
(2.4) $a.b = b.a$.
(2.5) $(a.b)+(a.c) = a.(b+c)$.
(2.6) $a+0 = a$.
(2.7) $a.1 = a$.
(2.8) $a+(-a) = 0$.
(2.9) $c \neq 0,\ c.a=c.b \rightarrow a=b$.
(2.10) $a \epsilon \text{Pos},\ b \epsilon \text{Pos} \rightarrow a+b \epsilon \text{Pos}$.
(2.11) $a \epsilon \text{Pos},\ b \epsilon \text{Pos} \rightarrow a.b \epsilon \text{Pos}$.
(2.12) Either $a \epsilon \text{Pos}$, or $a=0$, or $-a \epsilon \text{Pos}$.
(2.13) If $1 \epsilon A$ and for all b, $b \epsilon A \rightarrow b+1 \epsilon A$, then for all a, $a \epsilon \text{Pos} \rightarrow a \epsilon A$.

It may be noted that in the above formulation, logical particles such as "or", "$\rightarrow$", "for all", "$=$", "ϵ", and the set variable "A" are used though not listed in advance. The rules governing the use of these are also taken for granted.

Grassmann did not present his development in an axiomatic form, although such a recasting is not difficult. To avoid tedious details, I shall reformulate it without indicating at every place the exact reference to his book. By consulting the book, the interested reader will undoubtedly be able to verify the general historical accuracy and discover minor deviations in the following presentation.

Grassmann's calculus L_2:

A. Atoms: $=$, (,); a, b, c, d, etc. (letters); 1, $+$, $-$, $.$; Pos.

B. Terms: 1 is a term; -1 is a term; a letter is a term; if s and t are terms, then $(s+t)$ and $(s.t)$ are terms.

C. Definitions (observe that Grassmann takes only "negative of" applied to 1 as primitive and defines the general case together with "difference" therefrom; a purist might wish to use two distinct symbols for the two uses of "$-$"):

(2.20) $0 = 1+-1$.
(2.21) For any a and b, $a-b$ is the number such that $b+(a-b) = a$.
(2.22) $-a = 0-a$.
(2.23) $a>b \leftrightarrow a-b\epsilon\mathrm{Pos}$.

D. Axioms:

(2.26) $a = (a+1)+-1$.
(2.27) $a = (a+-1)+1$.
(2.28) $a+(b+1) = (a+b)+1$.
(2.29) $a.0 = 0$.
(2.30) $1\epsilon\mathrm{Pos}$.
(2.31) $a\epsilon\mathrm{Pos} \rightarrow a+1\epsilon\mathrm{Pos}$.
(2.32) $b=0$ or $b\epsilon\mathrm{Pos} \rightarrow a.(b+1) = (a.b)+a$.
(2.33) $b\epsilon\mathrm{Pos} \rightarrow a.(-b) = -(a.b)$.
(2.34) If $1\epsilon A$, for all b, $b\epsilon A \rightarrow b+1\epsilon A$, and $b\epsilon A \rightarrow b+-1\epsilon A$; then for all a, $a\epsilon A$.
(2.35) If $1\epsilon A$, and for all b, $b\epsilon A \rightarrow b+1\epsilon A$, then for all a, $a\epsilon\mathrm{Pos} \rightarrow a\epsilon A$.

As a matter of fact, assuming rules which govern the notions of elementary logic, it is possible to derive all the axioms (2.1)–(2.13) of L_1 from the definitions and axioms of L_2. I shall not present these proofs but merely indicate, by referring to Grassmann's book, how this could be done. (2.1) is proved in Nr. 22 of his book; (2.2) in Nr. 23; (2.3) in Nr. 70; (2.4) in Nr. 72; (2.5) in Nr. 66; (2.6) in Nr. 18; (2.7) can be proved from (2.6), (2.2), (2.32), (2.29); (2.8) can be proved by (2.21), (2.22), and Nr. 2.6; (2.9) is proved in Nr. 96 of Grassmann's book; (2.10) in Nr. 88; (2.11) in Nr. 93; (2.13) is (2.35). The derivation of (2.12) uses primarily (2.21), (2.22), (2.30), (2.31), (2.34). This may serve as an exercise for the reader.

It may be observed that the definition (2.21) makes use of a description. In order to formalize Grassmann's work thoroughly, one would have to consider explicit rules for the truth functional connectives "or", "and",

"$\rightarrow$", the quantifiers "for all", equality "$=$", membership "ϵ", the set variable "A", and descriptions. Carrying out such a program would yield almost a formalization of logic. We can, for example, regard more modern works of Frege, Whitehead-Russell, Quine as realizations of such a program, except that their works are entangled with their particular theories of sets. An alternative approach would be to use properties instead of classes and formalize number theory in complete isolation from set theory. Such an approach would enjoy a purer underlying logic unaffected by the difficulties of formalizing set theory.

Grassmann's calculus is defective in at least one important respect. There is no explicit mention of the fact that different numbers have different successors, or the fact that 1 is not the successor of any positive integer. As a result, Grassmann's axioms would be satisfied if all integers are taken as identical with one another. The omission is easily understandable since we more or less take for granted that different numerals represent different objects. The surprising thing is not so much that Grassmann neglected to mention them but rather that shortly afterwards others did state these facts explicitly.

Peano, it is often believed, showed that the entire theory of natural numbers could be derived from three primitive concepts and five axioms in addition to those of pure logic. The basic concepts are: 1, number, successor. The axioms are:

(P1) 1 is a number.

(P2) The successor of any number is a number.

(P3) No two numbers have the same successor.

(P4) 1 is not the successor of any number.

(P5) Any property which belongs to 1, and also to the successor of every number which has the property, belongs to all numbers.

Of these concepts and axioms, it is possible to absorb the concept "number" and the first two axioms into an explicit specification of notation, and use just two primitive concepts and three axioms ([7], p. 219). Leaving, however, these and related simplifications aside, we may consider the informal question as to how the selection of these basic concepts and axioms could be made and justified.

Historically, Peano borrowed his axioms from Dedekind who in his remarkable booklet ***Was sind und was sollen die Zahlen***?(1888, reprinted in [4]) defined natural numbers as any set of objects which satisfy these axioms, for a suitably chosen successor relation. It is very fortunate that a letter by Dedekind has been preserved in which he explains in great detail how he first arrived at what are now known as the Peano axioms. Since this letter is highly illuminating, I quote it at length. (Notice that Dedekind uses the word "system" as we would now use the word "set" or "class.")

3. Dedekind's letter.

"I should like to ask you to lend your attention to the following train of thought which constitutes the genesis of my essay. How did my essay come to be written? Certainly not in one day, but rather it is the result of a synthesis which has been constructed after protracted labour. The synthesis is preceded by and based upon an analysis of the sequence of natural numbers, just as it presents itself, in practice so to speak, to the mind. Which are the mutually independent fundamental properties of this sequence N, i.e. those properties which are not deducible from one another and from which all others follow? How should we divest these properties of their specifically arithmetical character so that they are subsumed under more general concepts and such activities of the understanding, which are *necessary* for all thinking, but at the same time *sufficient*, to secure reliability and completeness of the proofs, and to permit the construction of consistent concepts and definitions?

When the problem is put in this manner, one is, I believe, forced to accept the following facts:

(1) The number-sequence N is a *system* of individuals or elements which are called numbers. This leads to the general study of systems as such (§ 1 of my essay).

(2) The elements of the system N stand in a certain relation to one another, they are in a certain order determined, in the first place, by the fact that to each definite number n, *belongs* again a definite number n', the number which succeeds or is next after n. This leads to the consideration of the general concept of a *mapping* ϕ of a system (§ 2). Since the image $\phi(n)$ of each number n is again a *number* n' and therefore $\phi(\mathrm{N})$ is a part of N, we are here concerned with the mapping ϕ of a system N *into itself*. And so this must be studied in its full generality (§ 4).

(3) Given distinct numbers a, b, their successors a', b' are also distinct; the mapping ϕ has therefore the character of distinctness or *similarity* (§ 3).

(4) Not every number is a successor n', i.e. $\phi(\mathrm{N})$ is a proper part of N; this (together with the preceding paragraph) constitutes the infinitude of the number-sequence N (§ 5).

(5) More precisely, 1 is the *only* number which does not lie in $\phi(\mathrm{N})$. Thus we have listed those facts which you regard as the complete characterization of an ordered simply infinite system N.

(6) But I have shown in my reply that these facts are still far from being adequate for a complete characterization of the nature of the number-sequence. Indeed, all these facts also apply to every system S which, in addition to the number-sequence N, contains also a system T of arbitrary other elements t. One can always define the mapping ϕ so as to preserve the character of similarity and so as to make $\phi(T) = T$. But such a system S is obviously something quite different from our number-sequence N, and I could so choose the system that scarcely a single arithmetic theorem holds for it. What must we now add to the facts above in order to cleanse our system S from such alien intruders t which disturb every vestige of order, and to restrict ourselves to the system N? This was one of the most difficult points of my analysis and its mastery required much thought. If one assumes knowledge of the sequence N of natural numbers to begin with and accordingly permits himself an arithmetic terminology, then he has of course an easy time of it. He needs only to say: an element n belongs to the sequence N if and only if by starting with the element 1, and going on counting, i.e. by a finite number of iterations of the mapping ϕ (compare the conclusion of 131 of my essay) I eventually reach the element n; on the other hand, I never reach an element t outside the sequence N by means of this process. But it is quite useless for our purpose to adopt this manner of distinguishing between those elements t which are to be ejected from S, and those elements n which alone are to remain in S. Such a procedure would surely involve the most pernicious and obvious

kind of circulus vitiosus. The mere words "finally get there" of course will not do either. They would be of no more use than, say, the words "karam sipso tatura", which I invent at this instant, without giving them any clearly defined meaning. Thus: how can I, without assuming any arithmetical knowledge, determine formally and without exception the distinction between the elements n and t? Merely by the consideration of the *chains* (37 and 44 of my essay), and yet completely! When I wish to avoid my expression "chain," I shall say: an element n of S belongs to the sequence N if and only if n is an element of *every such* part K of S which possesses the two properties (i) that the element 1 belongs to K and (ii) that the image $\phi(K)$ is part of K. In my technical language: N is the intersection 1_0 or $\phi_0(1)$ of all those chains K (in S) to which the element 1 belongs. Only after this addition is the complete character of the sequence N determined. — In this connection I remark in passing the following. For a brief period last Summer (1889) Frege's "Begriffschrift" and "Grundlagen der Arithmetik" came, for the first time, into my possession. I noted with pleasure that this method of defining a relation between an element and another which it follows, not necessarily immediately, in a sequence, agrees in *essence* with my concept of chains (37,44). Only one must not be put off by his somewhat inconvenient terminology.

(7) After the essential nature of the simple infinite system, whose abstract type is the sequence N of numbers, had been recognized from my analysis (71, 73), the question arose; does there *exist* at all such a system in our realm of ideas? Without a logical proof of existence there would always remain a doubt, whether the concept of such a system contains internal contradictions. Hence the need for such proofs (66 and 72 of my essay).

(8) After this had also been settled, there was the question: does the analysis thus far contain also a general *method of proof* sufficient to establish theorems which are intended to hold for *all* numbers n? Indeed! The famous proof by induction rests on the secure foundation of the concept of chains (59, 60, 80 of my essay).

(9) Finally: is it also possible to set up consistently *definitions* of numbers and operations for *all* numbers n? Indeed! This is in fact accomplished by Theorem 126 of my essay.

Thus the analysis was complete and the synthesis could begin. Yet this has still caused me enough trouble! Also the reader does not have an easy task; in order to work through everything completely, he needs, apart from sound common sense, also a very strong determination."

4. Dedekind's essay. The letter is of much historical interest and can also serve to guide a study of Dedekind's essay on the nature and meaning of number. A few explanatory remarks on his essay will supplement the letter.

The first section of Dedekind's essay deals with general principles of set theory. Thus, for any two things (objects of thought) a and b, $a=b$ if there is no property belonging to the one that does not belong to the other. A set M is completely determined when with respect to everything it is determined whether it is an element of S or not; it is explicitly stated, mentioning Kronecker's opposite view, that the determination need not be effective. Two sets are the same, $M=N$, when every element of M is one of N and vice versa (the axiom of extensionality). Set inclusion is defined in the usual manner in terms of the membership relation; M is a part of

N if every element of M is an element of N. Yet the same symbol is used for inclusion and membership. Given a set M of sets there is a set which contains every element which belongs to some set in M (axiom of union or sum set). Given a set M of sets there is a set (intersection) which contains every element which belongs to all sets in M. Given anything a, there is a unit set which contains only a as element. While acknowledging the possibility of an empty set, Dedekind excludes it in his development "for certain reasons." The historical interest of this section lies in the fact that it is probably the first partial attempt to state explicitly intuitive principles in the formation of sets. Later on, Zermelo, in his construction of an axiom system, makes use of this and other sections of Dedekind's essay.

The second section deals with the general concept of function (mapping, transformation), more or less following Dirichlet. One-to-one correspondence (similar transformation) is treated in § 3. Mapping of a set into itself is treated in § 4, where the important concept of a chain is also introduced: a set M forms a chain relative to a mapping ϕ if the set of all images of elements of M, $\phi(M)$, is included in M. The intersection of all chains (relative to a mapping ϕ) which contains a as an element is *the* chain of a: $\phi_0(a)$.

In § 5, Dedekind introduces his definition of infinity: a set is infinite if there is a one-to-one correspondence between it and a proper subset of it. It is proved that a set containing an infinite set as part is infinite, and that an infinite set remains infinite after deletion of a single element. A proof that infinite sets exist is given in the manner of Parmenides and Bolzano. Afterwards, Zermelo, in setting up his axiom system disregards the proof but states as an axiom (the axiom of infinity) the existence of some set similar to the one constructed in Dedekind's proof.

The set of natural numbers is defined in § 6. A set N is simply infinite if there exists a one-to-one mapping of N into itself such that there is an element in N which we shall denote by 1 and call the base-element of N, such that 1 does not belong to $\phi(N)$ and the chain $\phi_0(1)$ coincides with N. Any simply infinite set can be taken as the set of natural numbers except that we seem no longer to have a unique set of natural numbers. Dedekind proposes to neglect entirely the special character of the elements and seems to identify the set of natural numbers with what is common to (the essence of) all simply infinite sets. This is very much in the spirit of taking postulates as constituting an implicit definition. Dedekind comes back to his favourite philosophy of number: "With reference to this freeing the elements from every other content (abstraction) we are justified in calling numbers a free creation of the human mind." Indeed, Dedekind proves in § 10 that any two simply infinite systems N and M are isomorphic with regard to their mappings ϕ and ψ and their base elements 1 and 1^*: there is a one-to-one correspondence between N and M such that 1 corresponds to 1^* and if a corresponds to a^*, then $\phi(a)$ corresponds to $\psi(a^*)$. It follows that the abstract structure of a simply infinite set is entirely determined by its definition. Identifying the set of natural numbers with a simply infinite system N, Dedekind supplies in § 10 the Peano axioms: 1, the basic element, belongs to N; $\phi(N)$ is included in N, or, in other words, the successor of a number is again a number; ϕ is one-to-one, so that no two numbers have the same successor; 1 does not belong to $\phi(N)$, or, in other words, 1 is not a successor; $N=\phi_0(1)$, so that induction holds since any set M satisfying the induction hypothesis $(1\epsilon M,\ a\epsilon M \rightarrow \phi(a)\epsilon M)$ is a chain containing 1 and therefore includes $\phi_0(1)$ which is, by definition, the intersection of all chains containing 1.

In § 7, a number n is defined as less than m, $n<m$, if $\phi_0(n)$ contains $\phi_0(m)$ as a proper part. A "counter" set is introduced for each natural number n: Z_n is the set consisting of the numbers from 1 to n. Properties of $<$ and Z_n are proved in § 7 and § 8. It is also proved in § 8 that every unbounded set of natural numbers is simply infinite.

§ 9 contains the earliest set-theoretical treatment of recursive definitions (definitions by induction), such as those of addition and multiplication. As we know, these definitions are not entirely explicit since the operations defined occur in the defining conditions too; as a result, we can only eliminate the defined symbols when no variables occur in the context. To admit outright such definitions in a theory amounts to taking them as axioms, and the operations defined as primitive operations. Since these operations do not deal directly with sets, anybody who wishes to develop arithmetic on the basis of set theory is obliged to supply a method of eliminating them. Dedekind proves here a general theorem that given any mapping θ of an arbitrary set M into itself and an arbitrary element b of M, we can always find a function ψ such that $\psi(1) = b$, and $\psi(n+1) = \psi(\theta(n))$. The general theorem is then applied in § 11, § 12, § 13 to define addition, multiplication, exponentiation, and derive the usual properties.

Finally, in § 14, Dedekind defines the *Anzahl* of a finite set M as the number n such that the counter set Z_n is similar to M, and also gives an elaborate proof of the equivalence of his own definition of infinity and the usual definition according to which a nonempty set is infinite if it is similar to no Z_n. This proof is interesting for two reasons: that the apparently obvious conclusion should require such a complex proof, and that the proof contains an unacknowledged appeal to the axiom of choice, the method of proof first made famous by Zermelo sixteen years later.

5. Adequacy of Dedekind's characterization. It is remarkable that Dedekind obtained the Peano axioms entirely by analyzing the sequence of natural numbers. What is more remarkable is, once he had completed his analysis, he believed that properties of and theorems about natural numbers can all be derived from his characterization. This belief has to a large extent been confirmed by later developments. Clearly Dedekind did not look at a great number of theorems and proofs about natural numbers to see that no other characteristics are needed. Rather, he verified to his own satisfaction that the sequence of natural numbers is completely determined by his axioms, and then concluded that the axioms are adequate to the derivation of theorems as well. The mystery is how he made his verification.

His letter supplies a useful clue, when he discusses under (6) the question of excluding undesirable interpretations of the set N for which some ordinary arithmetic theorems would fail to hold. This suggests the following line of argument which may have been followed by Dedekind. The definition of natural numbers in terms of the chain of 1 enables us to determine the abstract character of the set of natural numbers entirely: witness his proof that any two sets satisfying the definition are isomorphic. If a theorem is independent of his definition, then there are two possible interpretations of the definition according to one of which the theorem is true and according to another the theorem is false. If the definition determines a unique interpretation of the theory, such situations cannot arise. Therefore, by the uniqueness of interpretation, all theorems about natural numbers must be derivable. This argument is plausible but not entirely rigorous because, among other things, the notion of interpretation has not been made sufficiently explicit to assure that any undecidable theorem will necessarily yield two different interpretations of the definition.

Dedekind's conclusion that these determine adequately the sequence of natural numbers is often expressed equivalently by saying that the Peano axioms are *categorical* or have no essentially different interpretations. As we know, the axioms do admit different interpretations such as taking 100 as 1 or taking the square of a number as the successor of a number. But they are all essentially the same in a technical sense of being isomorphic.

The proof of this is very easy once we concede that the axiom of induction (P5) does assure that the number sequence contains no "alien intruders" beyond the true natural numbers each of which is either the base-element or can be reached from the base-element by a finite number of steps. Granting this for the moment, we can assume given two interpretations of the Peano axioms: say 1^{**}, ϕ, N stand for 1, successor, number in one interpretation, 1^*, ψ, M in the other. Correlate 1^{**} with 1^*, and for every successor $\phi(a)$, if a is correlated with a^*, correlate $\phi(a)$ with $\psi(a^*)$. Since we are granting that no "alien intruders" can occur in any interpretation the correlation ensures a one-one correspondence between all objects in N and M which preserve the transit from each element to its successor. And this is the sense in which Peano's axioms are said to be categorical: any two models are isomorphic.

The proof will break down if the axiom of induction cannot exclude entirely the "alien intruders." For then we shall have two models for the axioms, one of which fits our intentions while the other contains additional alien elements which cannot be reached from the base-element by finite numbers of steps.

The question of an adequate specification of the class of natural numbers independently of arithmetic notions may also be viewed as a desire to analyse one use of the phrase "and so on" without explicit appeal to the general concept of finite numbers. Dedekind's discussion in his letter, especially under (6), is very articulate and instructive except that the last step in his argument, which leads to the conclusion that N is completely determined, can be elaborated further. This same conclusion comes out as a theorem in his essay (item 79). His proof comes to this: the class of natural numbers satisfies the conditions that 1 belongs to it and the successor of a member of it again belongs to it. If we consider all classes which satisfy the two conditions, their common part or intersection must be exactly the desired class: it cannot contain less members because every number must be in every one of the original classes; it cannot contain more because if it did there would be a smaller class which again satisfies the conditions.

Now there is, however, the question of specifying the arbitrary classes. How do we know whether a given specification of classes will include all classes or at least enough classes to yield as an intersection the class of numbers as desired? Or, what explicitly are the properties to which (P5) is applicable? In recent years a good deal of research in mathematical logic has been devoted to the question of unintended interpretations (nonstandard models) of theories of positive integers. It is therefore of interest to find this question raised in Dedekind's letter. I give a simple example.

A somewhat trivial interpretation of the principle of induction (P5) can be constructed if we restrict properties to those only which are expressible in a rather weak language. For example, we can imagine a notation in which every expressible property holds either for only a finite (possibly empty) set of numbers, or for all except a finite set of members. One such notation is obtained if $a_1, a_2, \ldots$ are constant names and every property $F(n)$ can only be a truth-function of finitely many equalities $n=a_i$. For example, no expression in it can represent the set of all odd positive integers.

If we agree to use such a language, we can easily find an unintended model for the whole set of axioms (P1)–(P5). Thus, we can take the domain as consisting of not only the positive integers but in addition the positive and negative fractions of the form $(2b+1)/2$ (b an integer), 1 as 1, and $a+1$ as the successor of a. It can be verified that (P1)—(P4) are satisfied. Moreover, (P5) is satisfied because any property for which "$F(1)$" and "$F(m) \rightarrow F(m+1)$" hold must hold for all numbers. This is evident with the "true" positive integers. If there were a "positive integer" $a/2$ such that "F(a/2)" is not true, "F(($a-2$)/2)" would also be false, a/2 being the "successor" of ($a-2$)/2 and "F(m)" implying "F($m+1$)"; similarly, "F(($a-4$)/2)", "F(($a-6$)/2)", etc. would all be false. Hence, we would have a property expressible in the given language, such that there is an infinity of numbers which possess it and there is also an infinity of numbers which do not possess it, contrary to our assumption. Hence, we get two non-isomorphic models for the weakened Peano axioms. The assumption is only true for very restricted languages. For any reasonably rich language, the problem of unintended models becomes more complex. It follows, however, from certain fundamental results of Gödel and Skolem that whenever a language can be effectively set up and proofs can be effectively checked, there are always unintended models of positive integers which satisfy all of the Dedekind-Peano axioms, provided that the properties in the axiom of induction are restricted to ones expressible in the given language. Usually the models directly obtained from these general theorems are quite complex or artificial, and to find intuitively transparent unintended models is in most cases very difficult. Indeed, it is known that for reasonably adequate languages, such non-standard models will have to consist of rather non-constructive sets, e.g., not recursively enumerable.

6. Dedekind and Frege. To turn once again to Dedekind's letter. There is a reference to Frege's works of 1879 and 1884 both of which appeared before Dedekind's essay (1888). As Dedekind says, Frege's definition of the set of possible integers agrees essentially with his own, in terms of what he calls chains. Both of them were interested in characterizing arithmetic properties by general notions independent of arithmetic. Frege's view that arithmetic is a part of logic is well-known; "arithmetic thus becomes simply a development of logic, and every proposition of arithmetic a law of logic, albeit a derivative one". ([8], § 87.) In the preface to his essay, Dedekind, also says,

"In speaking of arithmetic (algebra, analysis) as a part of logic I mean to imply that I consider the number-concept entirely independent of notions or intuitions of space and time, that I consider it an immediate result from the laws of thought."

They do not agree on what logic is, though. Frege uses concept and relation as the foundation stones upon which to erect his structure; Dedekind uses classes and the relation of an element belonging to a class. Nowadays we would think that they employ essentially the same thing. But not for Frege: in his ***Grundgesetze*** he says,

"Dedekind also is of the opinion that the theory of number is a part of logic; but his work hardly goes to strengthen this opinion, because the expressions 'system' ['class'] and 'a thing belongs to a thing' used by him are not usual in logic and are not reduced to accepted logical notions." ([9], p. 139).

In addition, Frege thought that his reduction refuted Kant's contention that arithmetic truths are synthetic. The reduction, however, cuts both ways. It is not easy to see how Frege can avoid the seemingly frivolous argument that if his reduction is really successful, one who believes firmly in the synthetic character of arithmetic can conclude that Frege's logic is thus proved to be synthetic rather than that arithmetic is proved to be analytic. Indeed, Russell at one time came close to drawing such a conclusion,

> "In the first place, Kant never doubted for a moment that the propositions of logic are analytic, whereas he rightly perceived that those of mathematics are synthetic. It has since appeared that logic is just as synthetic as all other kinds of truth." ([10], p. 457).

In the same vein, if one believes firmly in the irreducibility of arithmetic to logic, he will conclude from Frege's or Dedekind's successful reduction that what they take to be logic contains a good deal that lies outside the domain of logic. On account of the basic ambiguities of the words "logic," "arithmetic," "analytic," these arguments embody more than strikes the eye.

In contrast with Dedekind, Frege tries to tie his definition of number more directly to application and defines individual numbers. Frege begins with 0 instead of 1, defining 0 as the extension of the concept "equal to the concept 'not identical with itself'." If we disregard the distinction between concept and class, the definition amounts to identifying 0 with the class of all empty classes or since there is only one empty class by the axiom of extensionality, with the unit class of the empty class. The distinction is, however, important for Frege who apparently finds an empty class absurd, although an empty concept is entirely in order ([9], p. 150). Frege also defines the successor function separately which amounts to the function mapping a class n of similar classes to a new class $\phi(n)$ of classes each of which is obtained from some class in n by adding a new element. The class of natural numbers is then defined as the chain (in Dedekind's sense) of the number 0 thus defined relative to the successor function thus defined.

Already in his ***Begriffschrift*** Frege has presented the basic ideas of his derivation of arithmetic and also an exact formulation of the underlying logic of deduction, containing both the propositional calculus and the laws of quantification (the restricted predicate calculus). Dedekind has never attempted to formulate explicitly the logic of deduction. Frege's ***Grundlagen*** contains a good deal of stimulating philosophical discussion on the nature of number.

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