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Abstract

We study the class of languages recognized by multi-counter finite state automata. These are finite automata reading letters from a finite alphabet $\mathbb{A}$, equipped with n counters of natural numbers, which can be incremented or decremented by transitions. The acceptance condition requires the last state to be from the final set of states. This is equivalent to the language acceptors associated with coverability problems for labelled Petri Nets or labelled Vector Addition Systems with States (VASS).

We show that the problem of whether the complement of the language has finitely many words (*i.e.*, whether it is 'almost' equal to $\mathbb{A}^*$) is decidable. We do this by a reduction to the universality problem (*i.e.*, whether it is equal to $\mathbb{A}^*$).

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We use the letter $\mathbb{A}$ to denote a finite alphabet, $\mathbb{Z}$ do denote the set of all integers, and $\mathbb{N}$ the set of non-negative integers. We use $A \subseteq_{\mathit{fin}} B$ to denote that A is a finite subset of B, and $\wp_{\mathit{fin}}(A)$ to denote the set of all finite subsets of A.

We consider a VASS of dimension $k \in \mathbb{N}$ as a tuple $\mathcal{A} = (\mathbb{A}, k, Q, I, \delta, F)$ with initial configurations, final states, with transitions that read letters from a finite alphabet $\mathbb{A}$. That is, $\mathbb{A}$ is a finite alphabet, Q is a finite statespace, $I \subseteq_{\mathit{fin}} Q \times \mathbb{N}^k$ is the set of initial configurations,[1] $F \subseteq Q$ is the set of final states, and $\delta \subseteq_{\mathit{fin}} Q \times \mathbb{A} \times \mathbb{Z}^k \times Q$ is the set of transitions. As usual we use $(q, \bar{x}) \xrightarrow{w} (p, \bar{y})$ to denote the run from $(q, \bar{x}) \in Q \times \mathbb{N}^k$ to $(p, \bar{y}) \in Q \times \mathbb{N}^k$ by reading the word $w \in \mathbb{A}^*$. More precisely, this relations is defined recursively: $(q, \bar{x}) \xrightarrow{a} (p, \bar{y})$ if $a \in \mathbb{A}$, $\bar{x}, \bar{y} \in \mathbb{N}^k$ and $(q, a, \bar{y} - \bar{x}, p) \in \delta$; and for non-empty words $u, v \in \mathbb{A}^*$, $(q, \bar{x}) \xrightarrow{u \cdot v} (p, \bar{y})$ if $(q, \bar{x}) \xrightarrow{u} (r, \bar{z})$ and $(r, \bar{z}) \xrightarrow{v} (p, \bar{y})$ for some $(r, \bar{z})$. Let us denote by $\mathcal{L}(\mathcal{A})$ the **coverability language** of $\mathcal{A}$, that is,

$$\mathcal{L}(\mathcal{A}) = \{w \in \mathbb{A}^* : (q_0, \bar{x}) \xrightarrow{w} (q_f, \bar{y}) \text{ for some } (q_0, \bar{x}) \in I \text{ and } (q_f, \bar{y}) \in F \times \mathbb{N}^k\}.$$

The **universality problem** is the problem of, given a VASS $\mathcal{A}$ over an alphabet $\mathbb{A}$, whether $\mathcal{L}(\mathcal{A}) = \mathbb{A}^*$. The **co-finiteness problem** is the problem of, given a VASS $\mathcal{A}$ over an alphabet $\mathbb{A}$, whether $\mathbb{A}^* \setminus \mathcal{L}(\mathcal{A})$ is finite. (One may alternatively call the universality problem the "co-emptiness problem".)

► **Theorem 1.** *The universality problem for VASS coverability languages is decidable.*

Proof. Consider finite sets of configurations $Q \times \mathbb{N}^k$, which we call **macro-states**. We say that a macro-state $X \subseteq_{\mathit{fin}} Q \times \mathbb{N}^k$ is **accepting** if $X \cap (F \times \mathbb{N}^k) \neq \emptyset$. Consider the ordering $\preceq$ over macro-states so that $X \preceq X'$ if for every $(q, \bar{x}) \in X$ there is $(q', \bar{y}) \in X'$ so that

[1] Note that we allow to have a *set* of initial configurations. Previous works [2] have studied the containment or universality problem assuming only one initial state. The humdrum extension to a set of initial states is albeit necessary for obtaining the decidability result of Theorem 2

$q = q'$ and $\bar{x} \leq \bar{y}$. It is easy to see, by Dickson's lemma, that it is a well-quasi-order. Notice that the set of accepting macro-states is $\preceq$-upward-closed or, in other words, that the set of non-accepting macro-states is $\preceq$-downward-closed.

Finally, consider the one-step relation between macro-states so that $X \xrightarrow{a} X'$ if $X' = \{(p, \bar{y}) \in Q \times \mathbb{N}^k :$ there is a run $(q, \bar{x}) \xrightarrow{a} (p, \bar{y})$ in $\mathcal{A}$ for some $(q, \bar{x}) \in X\}$, which we extend to words as usual. It is not hard to see that it is $\preceq$-downward-compatible: if $X \xrightarrow{a} Y$ and $X' \preceq X$ then $X' \xrightarrow{a} Y'$ for some $Y' \preceq Y$.

Let $C_{\mathcal{A}} = \{w \in \mathbb{A}^* : I \xrightarrow{w} X$ with X non-accepting$\}$ and observe that $\mathbb{A}^* \setminus \mathcal{L}(\mathcal{A}) = C_{\mathcal{A}}$. If $C_{\mathcal{A}}$ is non-empty, consider any $w \in C_{\mathcal{A}}$ of minimal length. Then, $I = X_0 \xrightarrow{a_1} X_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} X_n$ for some non-accepting X_n and $w = a_1 \cdots a_n$. If there are two positions $0 \leq i < j \leq n$ so that $X_i \preceq X_j$, then by downward-closure of non-accepting macro states and downward-compatibility, it follows that the word $w' = (a_1 \cdots a_i) \cdot (a_{j+1} \cdots a_n)$ is neither in $\mathcal{L}(\mathcal{A})$ and thus that w was not length minimal. Therefore, there are no $i < j$ so that $X_i \preceq X_j$, which we usually define as $(X_i)_i$ being a "$\preceq$-bad sequence".

On the other hand, since $\preceq$ is decidable, and since $(\xrightarrow{a})_{a \in \mathbb{A}}$ is decidable, controlled (in the sense of [1]) and finite-branching (actually, each $\xrightarrow{a}$ is functional), there is a computable bound N on the maximal length of any sequence $X_0 \xrightarrow{a_1} X_1 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} X_{n-1} \xrightarrow{a_n} X_n$ so that $X_0 = I$ and $(X_i)_i$ is a $\preceq$-bad sequence.

Therefore, $\mathcal{L}(\mathcal{A}) \neq \mathbb{A}^*$ if, and only if, there is a word $w \in \mathbb{A}^*$ of length at most N such that $w \notin \mathcal{L}(\mathcal{A})$, whence we obtain decidability for universality. ◀

We show that the co-finiteness problem is also decidable, by reduction to the problem above.

▶ **Theorem 2.** *The co-finiteness problem for VASS coverability languages is decidable.*

Towards showing this, let us fix a VASS $\mathcal{A} = (\mathbb{A}, Q, I, \delta, F)$ of dimension k, and let us define $q \overset{w}{\dashrightarrow} p$ as the fact that there is a run on w from state q to state p in the underlying NFA of $\mathcal{A}$ (*i.e.*, we disregard the vectors), we call this a **pseudorun**. We will sometimes write $q \overset{w}{\dashrightarrow} F$ to denote $q \overset{w}{\dashrightarrow} q_f$ for some $q_f \in F$.

Let $\mathcal{P} = \wp_{fin}(Q \times \mathbb{N}^k) \times \wp(Q)$. We define the **profile** of $\mathcal{A}$ between two words $u, v \in \mathbb{A}^*$ as an element $(X, T) \in \mathcal{P}$, where X is the set of all configurations that sit between u and v in accepting pseudoruns on $u \cdot v$ which happen to be runs on u; and T is the set of the states leading to acceptance through pseudoruns. Formally, $\mathit{profile}(u|v) = (X, \mathit{right\text{-}type}(v)) \in \mathcal{P}$, where $\mathit{right\text{-}type}(v) = \{p \in Q : p \overset{v}{\dashrightarrow} F\}$, and $X = \{(q, \bar{y}) : (q_i, \bar{x}) \xrightarrow{u} (q, \bar{x}) \overset{v}{\dashrightarrow} F$ for some $(q_i, \bar{x}) \in I\}$.

We define the following ordering $\sqsubseteq$ over $\mathcal{P}$: $(X, \tau) \sqsubseteq (X', \tau')$ iff

- $\tau = \tau'$; and
- for all $(q, \bar{x}) \in X$ there is $(p, \bar{y}) \in X'$ such that $p = q$ and $\bar{x} \leq \bar{y}$.

▶ **Lemma 3.** *$(\sqsubseteq, \mathcal{P})$ is a well-quasi-order.*

Proof. By Dickson's lemma. ◀

▶ **Lemma 4.** *If $\mathit{profile}(u|v) \sqsubseteq \mathit{profile}(u'|v)$ and $uv \in \mathcal{L}(\mathcal{A})$, then $u'v \in \mathcal{L}(\mathcal{A})$.*

Proof. By monotonicity of VASS. ◀

▶ **Lemma 5.** *If $\mathit{profile}(uw|v) \sqsubseteq \mathit{profile}(u|wv)$ then $\mathit{profile}(uw^i|v) \sqsubseteq \mathit{profile}(uw|v)$ for all $i \geq 1$.*

Proof. Notice that it suffices to prove that $\mathit{profile}(uw^i|v) \sqsubseteq \mathit{profile}(uw^{i-1}|v)$ for any arbitrary $i \geq 1$; the statement then follows by transitivity of $\sqsubseteq$.

Since $\mathit{right\text{-}type}(v) = \mathit{right\text{-}type}(wv)$ by the hypothesis $\mathit{profile}(uw|v) \sqsubseteq \mathit{profile}(u|wv)$, we obtain

$$\mathit{right\text{-}type}(v) = \mathit{right\text{-}type}(w^j v) \quad \text{for every } j. \tag{1}$$

Given an arbitrary $(r, \bar{n})$ inside the first component of $\mathit{profile}(uw^i|v)$, we show that there is some $(r, \bar{n}^+)$ in $\mathit{profile}(uw^{i-1}|v)$ for some $\bar{n}^+ \geq \bar{n}$. Suppose that $(r, \bar{n})$ originates from some $(q, \bar{m})$ in $\mathit{profile}(uw|w^{i-1}v)$, that is,

$$I \ni (q_0, \bar{x}) \xrightarrow{uw} (q, \bar{m}) \xrightarrow{w^{i-1}} (r, n) \overset{v}{\dashrightarrow} F. \tag{2}$$

Since $q \overset{w^{i-1}v}{\dashrightarrow} F$ by (2), we have $q \in \mathit{right\text{-}type}(w^{i-1}v) = \mathit{right\text{-}type}(v)$ by (1). In other words, $q \overset{v}{\dashrightarrow} F$.

Therefore, $(q, \bar{m})$ is in $\mathit{profile}(uw|v) \sqsubseteq \mathit{profile}(u|wv)$, and thus some $(q, \bar{m}^+)$ is in $\mathit{profile}(u|wv)$ for some $\bar{m}^+ \geq \bar{m}$, that is,

$$I \ni (q_0, \bar{x}) \xrightarrow{u} (q, \bar{m}^+) \overset{wv}{\dashrightarrow} F. \tag{3}$$

Since $(q, \bar{m}) \xrightarrow{w^{i-1}} (r, \bar{n})$ by (2), then $(q, \bar{m}^+) \xrightarrow{w^{i-1}} (r, \bar{n}^+)$ for some $\bar{n}^+ \geq \bar{n}$, by monotonicity. Further, since $r \overset{v}{\dashrightarrow} F$ by (2), we obtain

$$I \ni (q_0, \bar{x}) \xrightarrow{u} (q, \bar{m}^+) \xrightarrow{w^{i-1}} (r, \bar{n}^+) \overset{v}{\dashrightarrow} F, \tag{4}$$

meaning that $(r, \bar{n}^+)$ is in $\mathit{profile}(uw^{i-1}|v)$. This, together with (1), means that $\mathit{profile}(uw^i|v) \sqsubseteq \mathit{profile}(uw^{i-1}|v)$. ◀

Using the above three lemmas, we can now prove Theorem 2 by reduction to the universality problem.

Proof of Theorem 2. Let $\mathcal{A} = (Q, I, \delta, F)$ be a k-dimension VASS on the alphabet $\mathbb{A}$. Notice that there is a computable function f so that for every letter $a \in \mathbb{A}$ and words $u, v \in \mathbb{A}^*$, $|\mathit{profile}(u|av)| \leq f(|\mathit{profile}(ua|v)|)$, where the size $|\cdot|$ consists of the number of bits needed to encode the profile. Further, for every word $w \notin \mathcal{L}(\mathcal{A})$ we have $\mathit{profile}(w|\varepsilon) = (\emptyset, \emptyset)$. Therefore, since $\sqsubseteq$ is a wqo (Lemma 3), there is a computable bound N so that every $z \notin \mathcal{L}(\mathcal{A})$ of size larger than N can be decomposed $z = u \cdot w \cdot v$ so that $\mathit{profile}(uw|v) \sqsubseteq \mathit{profile}(u|wv)$.

Suppose there is a word in the complement of $\mathcal{L}(\mathcal{A})$ of size larger than N. Then, by Lemma 5 and the discussion before we have, for all j, that $\mathit{profile}(uw^j|v) \sqsubseteq \mathit{profile}(uw|v)$ for some $uwv \notin \mathcal{L}(\mathcal{A})$, $w \neq \varepsilon$. By the counter-positive statement of Lemma 4, there are infinitely many distinct words in the complement of $\mathcal{L}(\mathcal{A})$. Summing up, the complement of $\mathcal{L}(\mathcal{A})$ is finite if, and only if, it contains words of size at most N.

Therefore, we have to test if there is a word of size $> N$ missing from $\mathcal{L}(\mathcal{A})$. For doing this, we first guess a prefix $w \in \mathbb{A}^*$ of size $N+1$, and we compute $I' = \{(q, \bar{y}) : (q_0, \bar{x}) \xrightarrow{w} (q, \bar{y}) \text{ for some } (q_0, \bar{x}) \in I\}$. We now need to check that for some $v \in \mathbb{A}^*$ we have $wv \notin \mathcal{L}(\mathcal{A})$, in other words, that the universality problem for $\mathcal{A}' = (Q, I', \delta, F)$ is negative. By Theorem 1 this is decidable and thus so is the co-finiteness problem. ◀

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