

Continuous Positional Payoffs

Alexander Kozachinskiy*

Department of Computer Science, University of Warwick, Coventry, UK

December 17, 2020

Abstract

What payoffs are positional for (deterministic) two-player antagonistic games on finite directed graphs? In this paper we study this question for payoffs that are *continuous*. The main reason why continuous positional payoffs are interesting is that they include *multi-discounted payoffs*.

First, we introduce so-called *contracting payoffs*. These payoffs are continuous and can be seen as a generalization of the multi-discounted payoffs. We establish positionality of contracting payoffs via a Shapley's fixed point argument. Our main result states that any continuous positional payoff is a composition of a non-decreasing continuous function and a contracting payoff (and, in turn, any such composition is a continuous positional payoff). This yields that for continuous payoffs positionality is equivalent to a simple property called *prefix-monotonicity*.

Finally, we study analogous questions for more general *stochastic* games. It turns out that all continuous stochastically positional payoffs are multi-discounted. As a byproduct, this gives a negative answer to a question of Gimbert [6], who conjectured that all deterministically positional payoffs are stochastically positional.

1 Introduction

We study games of the following kind. A game takes place on a finite directed graph. There is a token, initially located in one of the nodes. Before each turn there is exactly one node containing the token. In each turn one of the two antagonistic players called Max and Min chooses an edge starting in a node containing the token. As a result the token moves to the endpoint of this edge, and then the next turn starts. To determine who makes a move in a turn we are given in advance a partition of the nodes into two sets. If the token is in a node from the first set, then Max makes a move, otherwise Min.

*Alexander.Kozachinskiy@warwick.ac.uk. Supported by the EPSRC grant EP/P020992/1 (Solving Parity Games in Theory and Practice).

Players make infinitely many moves, and this yields an infinite trajectory of the token. Technically, we assume that each node of the graph has at least one out-going edge so that there is always at least one available move. To introduce competitiveness, we should somehow compare the trajectories of the token with each other. For that we first fix some finite set A and label the edges of the game graph by elements of A . We also fix a *payoff* φ which is a function from the set of infinite sequences of elements of A to \mathbb{R} . Each possible infinite trajectory of the token is then mapped to a real number called *the cost* of this trajectory as follows: we form an infinite sequence of elements of A by taking the labels of edges along the trajectory, and apply φ to this sequence. The larger the cost is the more Max is happy; on the contrary, Min wants to minimize the cost.

Assume that one of the players, say, Max, fixes a *strategy*, i.e., an instruction how to play in all possible developments of the game (these developments are formally possible finite trajectories of the token). How to measure a quality of such a strategy? One natural way is to take the infimum over all trajectories, consistent with the strategy, of the cost of the trajectory. This number can be interpreted as a *guarantee* of the strategy – no matter how Min plays against it, the cost cannot be smaller. Correspondingly, an *optimal* Max’s strategy would be a strategy with maximal guarantee.

One can define similar notions for the Min’s strategies, with minor modifications: we should take the supremum instead of the infimum in the definition of the guarantee, and an optimal Min’s strategy should be a strategy with the minimal guarantee.

The game is called *determined* if some Max’s strategy has the same guarantee as some Min’s strategy. Note that any two such strategies must be optimal: one proves the optimality of the other. Now, we say that a payoff is *determined* if any game on a finite directed graph with this payoff is determined.

For general determined payoffs an optimal strategy might be rather complicated (in fact, it might have no finite description). For what determined payoffs both players always have a “simple” optimal strategy? A word “simple” can be understood in different ways [1], and this leads to different classes of determined payoffs. Among these classes we study one for which “simple” is understood in, perhaps, the strongest sense possible. Namely, we study a class of *positional* payoffs.

For a positional payoff in any game graph both players must have a *positional strategy* which is optimal no matter in which node the game starts. Now, a positional strategy is a strategy which totally ignores the previous trajectory of the token and only looks at its current location. Formally, a positional strategy of Max maps each Max’s node to an edge which starts in this node (i.e., to a single edge which Max will use whenever this node contains the token). Min’s positional strategies are defined similarly.

A lot of works are devoted to concrete positional payoffs that are of particular interest in other areas of computer science. Classical examples of such payoffs are parity payoffs, mean payoffs and discounted payoffs [4, 12, 11, 14]. They constitute, respectively, Parity Games, Mean Payoff Games and Discounted Games. Parity Games have remarkable connections to logic, verification and finite automata theory [5, 10]. Mean Payoff Games

and Discounted Games are standard models of reactive systems and decision-making [13, 15]. All these three examples are also interesting from the viewpoint of algorithm design (see, for instance, [2]).

Along with this specialized research, in [7, 8] Gimbert and Zielonka undertook a thorough study of positional payoffs in general. In [7] they showed that all the so-called *fairly mixing* payoffs are positional. They also demonstrated that the parity, mean and discounted payoffs are fairly mixing. Next, in [8] they established a property of payoffs which is *equivalent* to positionality. Compared to the property of fairly mixing, this property has far more technical definition. Of course, fairly mixing property is only sufficient for positionality, so one might say that this is a price of being sufficient *and necessary*. This property from [8] also has a remarkable feature: any payoff which does not satisfy this property is not positional in some *one-player* game graph (a game graph where all the nodes belong to one of the players). As Gimbert and Zielonka indicate, this means that one-player positionality always implies two-player positionality. More precisely, to establish positionality of a payoff it is enough to do so only for one-player game graphs.

One could try to gain more understanding about positional payoffs that satisfy certain additional requirements. Of course, this is interesting only if there are practically important positional payoffs that satisfy these requirements. One such requirement studied in the literature is called *prefix independence* [3, 6]. A payoff is prefix-independent if it is invariant under throwing away any finite prefix from an infinite sequence of edge labels. For instance, the parity and mean payoffs are prefix-independent.

In [7] Gimbert and Zielonka briefly mention another interesting additional requirement, namely, *continuity*. They observe that discounted payoffs are continuous (this helps them to show that discounted payoffs are fairly mixing). In this paper we study continuous positional payoffs in more detail.

2 Our Results

First, when do we call a payoff continuous? Recall that a payoff is a function of the form $\varphi: A^\omega \rightarrow \mathbb{R}$, where A is a set of possible edge labels and A^ω denotes the set of infinite sequences of elements of A . For technical simplicity we assume that A is finite¹. Now, we call φ continuous if the following holds. Let $\alpha \in A^\omega$ and let $\{\beta_n\}_{n \in \mathbb{N}}$ be any infinite sequence of elements of A^ω with the following property: for all $n \in \mathbb{N}$ the first n elements of the sequences α and β_n coincide. Then the limit $\lim_{n \rightarrow \infty} \varphi(\beta_n)$ must exist and must be equal to $\varphi(\alpha)$.

Equivalently, one can define continuity through a product topology on A^ω (or, as Gimbert and Zielonka do in [7], using a metric).

¹This is not an essential restriction as any game graph contains only finitely many edge labels. So to establish positionality in the case when A is infinite it is enough to do so for all finite subsets of A .

Why it might be interesting to consider a class of continuous positional payoffs, besides the general importance of the concept of continuity? Our main motivation is that this class includes discounted payoffs, and even more general multi-discounted payoffs (see [9]). A payoff $\varphi: A^\omega \rightarrow \mathbb{R}$ is multi-discounted if there are functions $\lambda: A \rightarrow [0, 1)$ and $w: A \rightarrow \mathbb{R}$ such that

$$\varphi(a_1 a_2 a_3 \dots) = \sum_{n=1}^{\infty} \lambda(a_1) \cdot \dots \cdot \lambda(a_{n-1}) \cdot w(a_n) \quad (1)$$

for all $a_1 a_2 a_3 \dots \in A^\omega$. Observe that this series is convergent due to the condition $\lambda(a) \in [0, 1)$ for $a \in A$ (and because A is finite). This also immediately gives continuity. Indeed, a tail of this series converges to 0 uniformly over $a_1 a_2 a_3 \dots \in A^\omega$.

For the multi-discounted payoffs, the more shifts of the token have been made, the more definite becomes the cost of a play. Continuity captures and generalizes this feature of the multi-discounted payoffs. This contrasts with prefix-independent payoffs where any initial finite segment is irrelevant.

2.1 Contracting payoffs

Are there continuous positional payoffs that are not multi-discounted? To show that the answer is positive we introduce so-called *contracting payoffs*. Technically, we define these payoffs through *contracting representations*.

Definition 1. Let A be a finite set. A triple $\langle K, d, m \rangle$, where

- $K \subseteq \mathbb{R}$ is a compact set;
- $d: K \times K \rightarrow [0, +\infty)$ is a continuous monotone metric (a metric $d: K \times K \rightarrow [0, +\infty)$ is called monotone if for all $x, s, t, y \in K$ with $x \leq s \leq t \leq y$ it holds that $d(s, t) \leq d(x, y)$);
- $m: A \times K \rightarrow K$ is a mapping such that for every $a \in A$ a function $m(a, \cdot): K \rightarrow K$ is continuous, non-decreasing² and d -contracting (a function $f: K \rightarrow K$ is d -contracting if for all $x, y \in K$ with $x \neq y$ it holds that $d(f(x), f(y)) < d(x, y)$);

is called a **contracting representation**.

Lemma 1. Let A be a finite set and let $\langle K, d, m \rangle$ be a contracting representation. Then for all $a_1 a_2 a_3 \dots \in A^\omega$ the limit

$$\lim_{n \rightarrow \infty} m(a_1, \cdot) \circ m(a_2, \cdot) \circ \dots \circ m(a_n, \cdot)(x)$$

exists and is the same for all $x \in K$.

²Technically, throughout the paper we call a function $f: S \rightarrow \mathbb{R}$, $S \subseteq \mathbb{R}$ non-decreasing if for all $x, y \in S$ we have $x \leq y \implies f(x) \leq f(y)$. In turn, by continuity of a function $f: S \rightarrow \mathbb{R}$, where $S \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, we always mean continuity with respect to the standard topology of \mathbb{R}^n , restricted to S .

By this lemma every contracting representation $\langle K, d, m \rangle$ yields a payoff

$$\varphi: A^\omega \rightarrow \mathbb{R}, \quad \varphi(a_1 a_2 a_3 \dots) = \lim_{n \rightarrow \infty} m(a_1, \cdot) \circ m(a_2, \cdot) \circ \dots \circ m(a_n, \cdot)(x)$$

where $x \in K$ (by the lemma the choice of $x \in K$ is not important). Since K is compact (and hence is closed), φ will take values in K . We call payoffs that can be obtained in this way contracting payoffs.

Proposition 2. *Any contracting payoff is continuous and positional.*

We prove this proposition via a “fixed point argument” going back to Shapley [14] (Shapley’s argument is a standard way of establishing positionality of Discounted Games).

In a sense, a class of contracting payoffs is a non-linear generalization of the multi-discounted payoffs. To see why, first it is instructive to observe that any multi-discounted payoff is contracting. Indeed, we can rewrite (1) as follows:

$$\begin{aligned} \varphi(a_1 a_2 a_3 \dots) &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \lambda(a_1) \cdot \dots \cdot \lambda(a_{i-1}) \cdot w(a_i) \\ &= \lim_{n \rightarrow \infty} w(a_1) + \lambda(a_1) \left(w(a_2) + \lambda(a_2) \left(\dots \right) \right) \\ &= \lim_{n \rightarrow \infty} m(a_1, \cdot) \circ m(a_2, \cdot) \circ \dots \circ m(a_n, \cdot)(0), \end{aligned}$$

where $m(a, x) = \lambda(a)x + w(a)$ for $a \in A$. Since $\lambda(a) \in [0, 1)$ for all $a \in A$, the functions $m(a, \cdot)$ are non-decreasing and contracting with respect to the standard metric $d(x, y) = |x - y|$ (and of course they are continuous). Technically, the functions $m(a, \cdot)$ should be from K to K for some compact set K . This is not a problem because for some large enough $W > 0$ we have $m(a, \cdot)([-W, W]) \subseteq [-W, W]$ for all $a \in A$.

General contracting payoffs evaluate sequences of edge labels quite similarly to the multi-discounted payoffs. Indeed, observe that for a contracting payoff appending a label a to a sequence with the cost x results in a sequence with the cost $m(a, x)$. Thus, after reading a finite segment $a_1 a_2 \dots a_n$ from a sequence of edge labels we know that the cost of this sequence is in the image of a composed function:

$$m(a_1, \cdot) \circ m(a_2, \cdot) \circ \dots \circ m(a_n, \cdot).$$

Due to contractivity, this image shrinks to a single point as $n \rightarrow \infty$. Multi-discounted payoffs form a special case when the functions $m(a, \cdot)$ are linear. Now, for general contracting payoffs these functions may not be linear. Still, they should be continuous, non-decreasing and contracting (but not necessarily with respect to the standard metric).

It is not hard to obtain a contracting payoff which is not multi-discounted. We also observe that there are contracting payoffs that differ from all multi-discounted payoffs even on some fixed finite subset of A^ω .

Proposition 3. *There exist a finite set A , a contracting payoff $\varphi: A^\omega \rightarrow \mathbb{R}$ and three pairs $(\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3) \in A^\omega \times A^\omega$ such that*

- for all $i \in \{1, 2, 3\}$ we have $\varphi(\alpha_i) > \varphi(\beta_i)$
- for every multi-discounted payoff $\psi: A^\omega \rightarrow \mathbb{R}$ there exists $i \in \{1, 2, 3\}$ such that $\psi(\alpha_i) \leq \psi(\beta_i)$.

2.2 Characterizing continuous positional payoffs

One can generate more examples of continuous positional payoffs using the following observation: *a composition of a non-decreasing function and a positional payoff is always³ a positional payoff*. So for a contracting payoff $\varphi: A^\omega \rightarrow \mathbb{R}$ and for an arbitrary non-decreasing function $g: \varphi(A^\omega) \rightarrow \mathbb{R}$ a composition $g \circ \varphi$ will be a positional payoff. If g is additionally continuous, then $g \circ \varphi$ is a continuous positional payoff. Our main result states that in this way one can obtain *all* continuous positional payoffs.

Theorem 4. *Let A be a finite set. Then for every continuous positional payoff $\varphi: A^\omega \rightarrow \mathbb{R}$ there exists a contracting payoff $\psi: A^\omega \rightarrow \mathbb{R}$ and a continuous non-decreasing function $g: \psi(A^\omega) \rightarrow \mathbb{R}$ such that $\varphi = g \circ \psi$.*

In turn, if $\psi: A^\omega \rightarrow \mathbb{R}$ is a contracting payoff and $g: \psi(A^\omega) \rightarrow \mathbb{R}$ is a continuous non-decreasing function, then $g \circ \psi$ is a continuous positional payoff.

Given Proposition 2, the second part of this theorem is immediate. To show the first part of Theorem 4 we introduce an auxiliary property of payoffs which we call *prefix-monotonicity*. In the definition below we use a notation uv to denote the concatenation of $u \in A^*$ and $v \in A^* \cup A^\omega$.

Definition 2. *Let A be a finite set. A payoff $\varphi: A^\omega \rightarrow \mathbb{R}$ is called **prefix-monotone** if there are no $u, v \in A^*$, $\alpha, \beta \in A^\omega$ such that*

$$\varphi(u\alpha) > \varphi(u\beta) \text{ and } \varphi(v\alpha) < \varphi(v\beta).$$

First, it is easy to see prefix-monotonicity is necessary for positionality (for continuous payoffs).

Proposition 5. *Let A be a finite set. If a continuous payoff $\varphi: A^\omega \rightarrow \mathbb{R}$ is not prefix-monotone, then φ is not positional.*

So a continuous positional payoff must be prefix-monotone. To show Theorem 4 it is now enough to show the following proposition (which is the main technical contribution of this paper).

Proposition 6. *Let A be a finite set. Then for every continuous **prefix-monotone** payoff $\varphi: A^\omega \rightarrow \mathbb{R}$ there exists a contracting payoff $\psi: A^\omega \rightarrow \mathbb{R}$ and a continuous non-decreasing function $g: \psi(A^\omega) \rightarrow \mathbb{R}$ such that $\varphi = g \circ \psi$.*

As a byproduct, we obtain the following simple characterization of the class of continuous positional payoffs.

³See Proposition 9 below.

Theorem 7. *Let A be a finite set and $\varphi: A^\omega \rightarrow \mathbb{R}$ be a continuous payoff. Then φ is positional if and only if φ is prefix-monotone.*

Indeed, the “only if” part is proved in Proposition 5. For the “if” part observe that any continuous prefix-monotone payoff by Proposition 6 is a composition of a non-decreasing function and a contracting payoff. It remains to refer to the positionality of contracting payoffs (Proposition 2).

2.3 Comparing our results with the results about continuous payoffs from [7]

In [7] Gimbert and Zielonka gave the following definition.

Definition 3. *Let A be a finite set. A payoff $\varphi: A^\omega \rightarrow \mathbb{R}$ is called **fairly mixing** if the following holds:*

- **(a)** *for all $u \in A^*$ and $\alpha, \beta \in A^\omega$ we have that*

$$\varphi(\alpha) \leq \varphi(\beta) \implies \varphi(u\alpha) \leq \varphi(u\beta).$$

- **(b)** *for all non-empty $u \in A^*$ and for all $\alpha \in A^\omega$ we have that*

$$\min\{\varphi(u^\omega), \varphi(\alpha)\} \leq \varphi(u\alpha) \leq \max\{\varphi(u^\omega), \varphi(\alpha)\}$$

(here u^ω denotes an infinite sequence obtained by concatenating infinitely many copies of $u \in A^$).*

- **(c)** *for any infinite sequence $\{u_i\}_{i \in \mathbb{N}}$ of non-empty strings from A^* we have that:*

$$\begin{aligned} \min\{\varphi(u_0 u_2 u_4 \dots), \varphi(u_1 u_3 u_5 \dots), \inf_{i \in \mathbb{N}} \varphi(u_i^\omega)\} &\leq \varphi(u_0 u_1 u_2 u_3 \dots) \\ &\leq \max\{\varphi(u_0 u_2 u_4 \dots), \varphi(u_1 u_3 u_5 \dots), \sup_{i \in \mathbb{N}} \varphi(u_i^\omega)\}. \end{aligned}$$

They prove [7, Theorem 1] that any fairly mixing payoff is positional. They also show [7, Lemma 2] that for continuous payoffs the conditions **(a–b)** of Definition 3 imply the condition **(c)** of this definition (they use this to simplify a proof that multi-discounted payoffs are fairly mixing). In fact, it is easy to see that we only need **(a)**.

Observation 1. *For continuous payoffs the condition **(a)** of Definition 3 implies the condition **(b)** of Definition 3.*

Proof. We only show that $\varphi(u\alpha) \leq \max\{\varphi(u^\omega), \varphi(\alpha)\}$, the other inequality can be proved similarly. If $\varphi(u\alpha) \leq \varphi(\alpha)$, then we are done. Assume now that $\varphi(u\alpha) > \varphi(\alpha)$. By repeatedly applying **(a)** we obtain $\varphi(u^{i+1}\alpha) \geq \varphi(u^i\alpha)$ for every $i \in \mathbb{N}$. In particular, for every $i \geq 1$ we get that $\varphi(u^i\alpha) \geq \varphi(u\alpha)$. By continuity of φ the limit of $\varphi(u^i\alpha)$ as $i \rightarrow \infty$ exists and equals $\varphi(u^\omega)$. Hence $\varphi(u^\omega) \geq \varphi(u\alpha)$. \square

It is immediate that contracting payoffs satisfy the condition **(a)** of Definition 3. So alternatively one can establish the positionality of contracting payoffs (Proposition 2) via [7, Theorem 1].

The condition **(a)** of Definition 3 looks quite similar to prefix-monotonicity (which, in turn, is equivalent to positionality for continuous payoffs by Theorem 7). A close examination shows that prefix-monotonicity is *weaker*, i.e., the condition **(a)** of Definition 3 implies prefix-monotonicity but not vice versa.

In fact, one can give an example of a continuous positional payoff which does not satisfy the condition **(a)** of Definition 3. This shows that [7, Theorem 1] is not by itself sufficient to obtain our characterization of continuous positional payoffs from Theorem 7.

Indeed, let $A = \{0, 1\}$ and let $\varphi: \{0, 1\}^\omega \rightarrow \mathbb{R}$ be a discounted payoff with $\lambda(0) = \lambda(1) = 1/2$ and $w(0) = 0, w(1) = 1$. Next, define a function

$$g: [0, 2] \rightarrow \mathbb{R}, \quad g(x) = \begin{cases} 0 & 0 \leq x \leq 1, \\ x - 1 & 1 \leq x \leq 2. \end{cases}$$

A composition $g \circ \varphi$ is obviously a continuous positional payoff. But $0 = g \circ \varphi(1(0)^\omega) \leq g \circ \varphi(01(0)^\omega) = 0$ while $1/2 = g \circ \varphi(11(0)^\omega) > g \circ \varphi(101(0)^\omega) = 1/4$, so $g \circ \varphi$ does not satisfy the condition **(a)** of Definition 3.

2.4 MDP-positionality for continuous payoffs

Finally, we investigate continuous payoffs that are positional for a more general class of games, namely *stochastic* games. Let us first briefly discuss how these games are defined (we follow a formalization of Gimbert from [6]). In fact, we will need only one-player stochastic games, also known as Markov Decision Processes (MDPs). For definiteness, we assume player Max. As before there is some finite set of possible locations of the token (for MDPs they are usually called *states* rather than nodes). Each state is equipped with a set of *actions* available in this state. An action is just a probability distribution over the set of nodes. In each turn Max chooses an action available at the state with the token. The next location of the token is then sampled according to the corresponding distribution. In a special case when all actions have support of size 1 we get deterministic one-player games.

Fixing a Max's strategy induces a probability distribution over the infinite trajectories of the token. An optimal strategy is one which maximizes the *expected cost* of the trajectory (with respect to its induced distribution). To compute the cost of a trajectory we first fix a *labeling* of our MDP so that each trajectory gets an infinite sequence of labels. As before, the cost is then the value of a payoff on this sequence of labels. We postpone technical details to Preliminaries.

A payoff for which in every MDP Max has an optimal positional strategy is called *MDP-positional*. Similarly to deterministic games, a positional strategy is a strategy which never chooses two different actions in the same state (in this context positional strategies are called sometimes *pure stationary* strategies).

We show the following result.

Theorem 8. *Any continuous MDP-positional payoff is a multi-discounted payoff.*

In turn, it is classical that multi-discounted payoffs are not only MDP-positional, but also positional for two-player stochastic games.

As a byproduct, this disproves a conjecture of Gimbert [6]. Namely, Gimbert conjectured the following: “Any payoff function which is positional for the class of non-stochastic one-player games is positional for the class of Markov decision processes”. To show that this is not the case, one can just take any payoff from Theorem 4 which is not multi-discounted. It will be positional even for two-player deterministic games; but it will be not MDP-positional by Theorem 8.

2.5 Organization of the paper

- In Section 3 we give preliminaries.
- In Section 4 we establish Lemma 1, Proposition 2, Proposition 3 and Proposition 5.
- In Section 5 we prove our main technical result, Proposition 6. Together with the previous section this will prove Theorems 4 and 7.
- In Section 6 we prove Theorem 8.

3 Preliminaries

We denote the function composition by \circ .

Sets and sequences. For two sets A and B by A^B we denote the set of all functions from B to A . We write $C = A \sqcup B$ for three sets A, B, C if A and B are disjoint and $C = A \cup B$.

For a set A by A^* we denote the set of all finite sequences of elements of A and by A^ω we denote the set of all infinite sequences of elements of A . For $w \in A^*$ we let $|w|$ be the length of w . For $\alpha \in A^\omega$ we let $|\alpha| = \infty$.

For $u \in A^*$ and $v \in A^* \cup A^\omega$ we let uv denote the concatenation of u and v . We call $u \in A^*$ a prefix of $v \in A^* \cup A^\omega$ if for some $w \in A^* \cup A^\omega$ we have $u = vw$. For $w \in A^*$ by wA^ω we denote the set $\{w\alpha \mid \alpha \in A^\omega\}$. Alternatively, wA^ω is the set of all $\beta \in A^\omega$ such that w is a prefix of β .

For $u \in A^*$ and $k \in \mathbb{N}$ we use a notation

$$u^k = \underbrace{uu \dots u}_{k \text{ times}}.$$

In turn, we let $u^\omega \in A^\omega$ be a unique element of A^ω such that u^k is a prefix of u^ω for every $k \in \mathbb{N}$. We call $\alpha \in A^\omega$ ultimately periodic if α is a concatenation of u and v^ω for some $u, v \in A^*$.

Graphs notation. By a finite directed graph G we mean a pair $G = (V, E)$ of two finite sets V and E equipped with two functions $\text{source}, \text{target}: E \rightarrow V$. Elements of V are called nodes of G and elements of E are called edges of G .

The out-degree of a node $a \in V$ is $|\{e \in E \mid \text{source}(e) = a\}|$. A node $a \in V$ is called a sink if its out-degree is 0. We call a graph G sinkless if there are no sinks in G .

A path in G is a non-empty (finite or infinite) sequence of edges of G with the property that $\text{target}(e) = \text{source}(e')$ for any two subsequent edges e and e' from the sequence. For a path p we define $\text{source}(p) = \text{source}(e)$, where e is the first edge of p . For a finite path p we define $\text{target}(p) = \text{target}(e')$, where e' is the last edge of p .

For technical convenience we also consider 0-length paths. Each 0-length path is associated with some node of G (so that there are $|V|$ different 0-length paths). For a 0-length path p , associated with $a \in V$, we define $\text{source}(p) = \text{target}(p) = a$.

When we write pq for two paths p and q , we mean the concatenation of p and q (recall that p and q are sequences of edges). Of course, this is well-defined only if p is finite. Note that pq is not necessarily a path. Namely, pq is a path if and only if $\text{target}(p) = \text{source}(q)$.

3.1 (Deterministic) infinite duration games on finite directed graphs

Mechanics of the game. A *game graph* is a sinkless finite directed graph $G = (V, E)$, equipped with two sets V_{Max} and V_{Min} such that $V = V_{\text{Max}} \sqcup V_{\text{Min}}$.

A game graph $G = (V = V_{\text{Max}} \sqcup V_{\text{Min}}, E)$ induces a so-called *infinite duration game* (IDG for short) on G . The game is always between two players called Max and Min. Positions of the game are finite paths in G (informally, these are possible finite trajectories of the token). We call a finite path p a Max's (a Min's) position if $\text{target}(p) \in V_{\text{Max}}$ (if $\text{target}(p) \in V_{\text{Min}}$). Max makes moves in Max's positions and Min makes moves in Min's positions.

The set of moves available at a position p is the set $\{e \in E \mid \text{source}(e) = \text{target}(p)\}$. A move e from a position p leads to a position pe .

A Max's strategy σ in a game graph G is a mapping assigning to every Max's position p a move available at p . Similarly, a Min's strategy τ in a game graph G is a mapping assigning to every Min's position p a move available at p .

Given a Max's strategy σ , a Min's strategy τ , and a finite path (position) p , we let $\mathcal{P}_p^{\sigma, \tau}$ be the (infinite) *play* between σ and τ with the initial position p . Formally,

$$\mathcal{P}_p^{\sigma, \tau} = pe_1e_2e_3 \dots,$$

where $e_1, e_2, e_3, \dots \in E$ is an infinite sequences of edges defined by

- $e_{i+1} = \sigma(pe_1e_2 \dots e_i)$ if $\text{target}(pe_1e_2 \dots e_i) \in V_{\text{Max}}$;
- $e_{i+1} = \tau(pe_1e_2 \dots e_i)$ if $\text{target}(pe_1e_2 \dots e_i) \in V_{\text{Min}}$.

We only use the notation $\mathcal{P}_p^{\sigma,\tau}$ in case when p is a 0-length path, i.e., when p is identified with some node $s \in V$. In this case we simply write $\mathcal{P}_s^{\sigma,\tau}$.

Positional strategies. A Max's strategy σ in a game graph $G = (V = V_{\text{Max}} \sqcup V_{\text{Min}}, E)$ is called *positional* if $\sigma(p) = \sigma(q)$ for all finite paths p and q in G with $\text{target}(p) = \text{target}(q) \in V_{\text{Max}}$. Clearly, a Max's positional strategy σ can be alternatively defined as a mapping $\sigma: V_{\text{Max}} \rightarrow E$ satisfying $\text{source}(\sigma(u)) = u$ for all $u \in V_{\text{Max}}$. We define Min's positional strategies analogously.

Labels. Let A be a finite set. A game graph $G = (V = V_{\text{Max}} \sqcup V_{\text{Min}}, E)$ equipped with a function $\text{lab}: E \rightarrow A$ is called an *A-labeled game graph*. If $p = e_1 e_2 e_3 \dots$ is a (finite or infinite) path in an A -labeled game graph $G = (V = V_{\text{Max}} \sqcup V_{\text{Min}}, E)$, $\text{lab}: E \rightarrow A$, we define $\text{lab}(p) = \text{lab}(e_1)\text{lab}(e_2)\text{lab}(e_3)\dots \in A^* \cup A^\omega$.

Payoffs and optimal strategies. Let A be a finite set. A *payoff* is a function of the form $\varphi: A^\omega \rightarrow \mathbb{R}$.

Let $\varphi: A^\omega \rightarrow \mathbb{R}$ be a payoff and let $G = (V = V_{\text{Max}} \sqcup V_{\text{Min}}, E)$, $\text{lab}: E \rightarrow A$ be an A -labeled game graph. Next, let σ be a Max's strategy in G and let τ be a Min's strategy in G . We say that τ is an *optimal response* to σ (with respect to φ) if for every Min's strategy τ' in G and for every $s \in V$ we have

$$\varphi \circ \text{lab}(\mathcal{P}_s^{\sigma,\tau}) \leq \varphi \circ \text{lab}(\mathcal{P}_s^{\sigma,\tau'}). \quad (2)$$

Similarly, we call σ an *optimal response* to τ (with respect to φ) if for every Max's strategy σ' in G and for every $s \in V$ we have

$$\varphi \circ \text{lab}(\mathcal{P}_s^{\sigma,\tau}) \geq \varphi \circ \text{lab}(\mathcal{P}_s^{\sigma',\tau}). \quad (3)$$

A pair (σ, τ) of a Max's strategy σ in G and a Min's strategy τ in G is called *φ -optimal* if σ is an optimal response to τ and τ is an optimal response to σ . If a Max's strategy σ (a Min's positional strategy τ) appears in at least one optimal pair, then we also say that the strategy σ (the strategy τ) is *φ -optimal*. It is classical that a set of φ -optimal pairs is always a Cartesian product. Hence, if σ is a Max's φ -optimal strategy and τ is a Min's φ -optimal strategy, then (σ, τ) is a φ -optimal pair.

Positional payoffs. Let A be a finite set and $\varphi: A^\omega \rightarrow \mathbb{R}$ be a payoff. We call φ *positional in an A-labeled game graph G* if in G there exists a φ -optimal pair of positional strategies of the players. We call φ *positional* if φ is positional in all A -labeled game graphs.

Proposition 9. *If A is a finite set, $\varphi: A^\omega \rightarrow \mathbb{R}$ is a positional payoff and $g: \varphi(A^\omega) \rightarrow \mathbb{R}$ is a non-decreasing function, then $g \circ \varphi$ is a positional payoff.*

Proof. Any φ -optimal pair of positional strategies is also $(g \circ \varphi)$ -optimal (g preserves (2-3)). \square

3.2 Continuous payoffs

For a finite set A , we consider the set A^ω as a topological space. Namely, we take the discrete topology on A and the corresponding product topology on A^ω . In this product topology open sets are sets of the form

$$\mathcal{S} = \bigcup_{u \in S} uA^\omega,$$

where $S \subseteq A^*$. In particular, when we say that a function of the form $\varphi: A^\omega \rightarrow \mathbb{R}$ is continuous we always mean continuity with respect to this product topology (and with respect to the standard topology on \mathbb{R}).

Proposition 10. *Let A be a finite set. A payoff $\varphi: A^\omega \rightarrow \mathbb{R}$ is continuous if and only if the following holds. For any $\alpha \in A^\omega$ and for any infinite sequence $\{\beta_n\}_{n=1}^\infty$ of elements of A^ω , such that for every $n \geq 1$ the sequences α and β_n coincide in the first n elements, the limit $\lim_{n \rightarrow \infty} \varphi(\beta_n)$ exists and equals $\varphi(\alpha)$.*

Proof. First, assume that φ is continuous. Take any $\varepsilon > 0$. We have to show that for some n_0 it holds that $\varphi(\beta_n) \in (\varphi(\alpha) - \varepsilon, \varphi(\alpha) + \varepsilon)$ for all $n \geq n_0$. The set $\varphi^{-1}((\varphi(\alpha) - \varepsilon, \varphi(\alpha) + \varepsilon))$ must be open. So for some $S \subseteq A^*$ we have:

$$\varphi^{-1}((\varphi(\alpha) - \varepsilon, \varphi(\alpha) + \varepsilon)) = \bigcup_{u \in S} uA^\omega.$$

Since obviously $\alpha \in \varphi^{-1}((\varphi(\alpha) - \varepsilon, \varphi(\alpha) + \varepsilon))$, there exists $u \in S$ such that $\alpha \in uA^\omega$. Hence for $n \geq |u|$ we have $\beta_n \in uA^\omega \subseteq \varphi^{-1}((\varphi(\alpha) - \varepsilon, \varphi(\alpha) + \varepsilon))$, as required.

Let us now show the other direction of the proposition. It is enough to show that for any $x, y \in \mathbb{R}$, $x < y$ the set $\varphi^{-1}((x, y))$ is open. Take any $\alpha \in \varphi^{-1}((x, y))$. Let us show that there exists $n(\alpha)$ such that all $\beta \in A^\omega$ that coincide with α in the first $n(\alpha)$ elements belong to $\varphi^{-1}((x, y))$. Indeed, otherwise for any n there exists β_n coinciding with α in the first n elements such that $\beta_n \notin \varphi^{-1}((x, y))$. Now, the limit $\lim_{n \rightarrow \infty} \varphi(\beta_n)$ must exist and must be equal to $\varphi(\alpha)$. But $\varphi(\alpha) \in (x, y)$ and all $\varphi(\beta_n)$ are not in this interval, contradiction.

Now, for $\alpha \in \varphi^{-1}((x, y))$ let $u_\alpha \in A^{n(\alpha)}$ be the $n(\alpha)$ -length prefix of α . Observe that

$$\varphi^{-1}((x, y)) = \bigcup_{\alpha \in \varphi^{-1}((x, y))} u_\alpha A^\omega.$$

So the set $\varphi^{-1}((x, y))$ is open, as required. \square

For a finite A by Tychonoff's theorem the space A^ω is compact (because any finite set A with the discrete topology is compact).

3.3 MDPs and MDP-positionality

Let A be a finite set. By \mathfrak{S}_A^{bor} we mean a σ -algebra of Borel subsets of A^ω (with respect to the product topology from the previous subsection). By $\Delta(S)$ we denote the set of all probability distributions over a finite set S .

Definition 4. A MDP is a pair $\mathcal{M} = (S, B)$ of two finite sets S and B equipped with two function **source**: $B \rightarrow S$, **Dist**: $B \rightarrow \Delta(S)$ such that for any $s \in S$ there exists $b \in B$ with $s = \text{source}(b)$. Elements of S are called states of \mathcal{M} and elements of B are called actions of \mathcal{M} .

Let $\mathcal{M} = (S, B)$ be an MDP. We call pairs $(b, s) \in B \times S$ *transitions* of \mathcal{M} . A non-empty (finite or infinite) sequence of transitions $h = (b_1, s_1)(b_2, s_2)(b_3, s_3) \dots$ is called a *history* if for every $1 \leq i < |h|$ we have $s_i = \text{source}(b_{i+1})$. For a history $h = (b_1, s_1)(b_2, s_2)(b_3, s_3) \dots$ we define $\text{source}(h) = \text{source}(b_1)$. For finite h we define $\text{target}(h) = s_{|h|}$.

For consistency, we also consider $|S|$ histories of length 0, each identified with some state. For a 0-length history h identified with $s \in S$ we write $\text{source}(h) = \text{target}(h) = s$.

Definition 5. A strategy σ in an MDP $\mathcal{M} = (S, B)$ is a mapping which to every finite history h assigns an action $b \in B$ such that $\text{target}(h) = \text{source}(b)$.

Let $\mathcal{M} = (S, B)$ be an MDP, $s \in S$ be a state of \mathcal{M} and σ be a strategy in \mathcal{M} . Denote by $T = B \times S$ the set of transitions of \mathcal{M} . Let us define a probability measure \mathcal{P}_s^σ on $\mathfrak{S}_T^{\text{bor}}$. For that consider the following random process generating a history (a sequence of transitions) in \mathcal{M} :

Algorithm 1: Generating \mathcal{P}_s^σ .

initialization:

h is a history in $\mathcal{M}, h \leftarrow s$;

$b \in B$;

$u \in S$;

while *true* **do**

$b \leftarrow \sigma(h)$;

 sample $u \in S$ at random according to $\text{Dist}(b)$ (and independently of the previous samples);

 append (b, u) to h ;

end

We let \mathcal{P}_s^σ be a probability measure on $\mathfrak{S}_T^{\text{bor}}$ such that for any $h \in T^*$ we have that $\mathcal{P}_s^\sigma(hT^\omega)$ equals the probability of the appearance of h in Algorithm 1. Existence of such \mathcal{P}_s^σ needs some justification. A standard consequence of the Caratheodory's extension theorem is that for any $p: T^* \rightarrow [0, 1]$ with

- $p(\text{empty string}) = 1$;
- $p(h) = \sum_{t \in T} p(ht)$ for all $h \in T^*$

there exists a unique probability measure P on $\mathfrak{S}_T^{\text{bor}}$ with $P(hT^\omega) = p(h)$ for all $h \in T^*$. We apply this to p which to $h \in T^*$ assigns the probability of the appearance of h in Algorithm 1.

Let A be a finite set. An A -labeled MDP is an MDP $\mathcal{M} = (S, B)$ equipped with a function lab from the set T of transitions of \mathcal{M} to A . Similarly to deterministic games we set $\text{lab}(t_1 t_2 t_3 \dots) = \text{lab}(t_1) \text{lab}(t_2) \text{lab}(t_3) \dots$ for $t_1 t_2 t_3 \dots \in T^* \cup T^\omega$.

Let $\varphi: A^\omega \rightarrow \mathbb{R}$ be a bounded measurable payoff (with respect to $\mathfrak{S}_A^{\text{bor}}$). It is easy to see that a composition $\varphi \circ \text{lab}: T^\omega \rightarrow \mathbb{R}$ is also measurable (now with respect to $\mathfrak{S}_T^{\text{bor}}$) and bounded. In particular, $\varphi \circ \text{lab}$ is integrable with respect to a probability measure \mathcal{P}_s^σ , for any $s \in S$ and for any strategy σ in \mathcal{M} . Thus we can consider the following quantity:

$$\mathbb{E}_{x \sim \mathcal{P}_s^\sigma} \varphi \circ \text{lab}(x).$$

For the sake of readability we will use the following notation for this quantity:

$$\mathbb{E} \varphi \circ \text{lab}(\mathcal{P}_s^\sigma).$$

Definition 6. Let A be a finite set, $\varphi: A^\omega \rightarrow \mathbb{R}$ be a bounded measurable payoff and $\mathcal{M} = (S, B)$, $\text{lab}: B \times S \rightarrow A$ be an A -labeled MDP. Next, let σ be a strategy in \mathcal{M} . We say that σ is φ -optimal if for every strategy σ' in \mathcal{M} and for every $s \in S$ we have:

$$\mathbb{E} \varphi \circ \text{lab}(\mathcal{P}_s^\sigma) \geq \mathbb{E} \varphi \circ \text{lab}(\mathcal{P}_s^{\sigma'}).$$

Definition 7. Let A be a finite set and $\varphi: A^\omega \rightarrow \mathbb{R}$ be a continuous payoff. We call φ MDP-positional if for every A -colored MPD $\mathcal{M} = (S, B)$, $\text{lab}: B \times S \rightarrow A$ there exists a positional strategy σ in \mathcal{M} which is φ -optimal.

Clearly, any continuous payoff is bounded (because A^ω is compact) and measurable, so we can apply these definitions to continuous payoffs without any clauses.

4 Establishing Preliminary Results

In this section we prove Lemma 1, Proposition 2, Proposition 3 and Proposition 5.

Let A be a finite set and $\langle K, d, m \rangle$ be a contracting representation. For the sake of readability, we will use a notation $m[a] = m(a, \cdot): K \rightarrow K$ and, more generally,

$$m[a_1 a_2 \dots a_n] = m[a_1] \circ m[a_2] \circ \dots \circ m[a_n]$$

for $n \in \mathbb{N}$, $a_1 \dots a_n \in A^n$ (in particular, $m[\text{empty string}]$ is just an identity function).

4.1 Proof of Lemma 1

Lemma 1 follows from the following

Lemma 11. Let A be a finite set and $\langle K, d, m \rangle$ be a contracting representation. Then for every $\varepsilon > 0$ there exists n such that for all $a_1 a_2 \dots a_n \in A^n$ we have:

$$m[a_1 a_2 \dots a_n](\max K) - m[a_1 a_2 \dots a_n](\min K) \leq \varepsilon.$$

Indeed, consider for any $a_1 a_2 a_3 \dots \in A^\omega$ and $x, y \in K$ the sequences

$$\{m[a_1 a_2 \dots a_n](x)\}_{n \in \mathbb{N}}, \quad \{m[a_1 a_2 \dots a_n](y)\}_{n \in \mathbb{N}}.$$

All their elements (except the first n of the first sequence and the first n of the second sequence) are in $m[a_1 a_2 \dots a_n](K)$. In turn, by Lemma 11 for every $\varepsilon > 0$ there exists n such that the set $m[a_1 a_2 \dots a_n](K)$ is a subset of an interval of length ε . Hence both sequences are Cauchy sequences. Clearly, they also must have the same limit.

(We will also need Lemma 11 for Proposition 2).

Proof of Lemma 11. Define $T = \{(x, y) \in K \times K \mid |x - y| \geq \varepsilon\}$. Obviously, T is a compact set. A function $d(x, y)/|x - y|$ is continuous on T (on T we never have 0 in the denominator). Hence there exists

$$z = \min_{(x, y) \in T} d(x, y)/|x - y|.$$

Observe that $z > 0$. Indeed, for some $(x, y) \in T$ we have $z = d(x, y)/|x - y|$. By definition of T we have $|x - y| \geq \varepsilon$. Hence $x \neq y$ and $d(x, y)$ is positive, as well as z .

Now, define $S = \{(x, y) \in K \times K \mid d(x, y) \geq z \cdot \varepsilon\}$. Again, S is a compact set. Consider a function:

$$f(x, y) = \max_{a \in A} \frac{d(m[a](x), m[a](y))}{d(x, y)}.$$

The function f is continuous on S (again, we never have 0 in its denominator on S). Hence there exists

$$\lambda = \max_{(x, y) \in S} f(x, y).$$

The function f is non-negative, so $\lambda \geq 0$. Let us show that $\lambda < 1$. Indeed, for some $(x, y) \in S$ we have $\lambda = f(x, y)$. By definition of f for some $a \in A$ we have:

$$\lambda = \frac{d(m[a](x), m[a](y))}{d(x, y)}.$$

Since $(x, y) \in S$, we have $d(x, y) \geq z \cdot \varepsilon > 0$. Hence $x \neq y$. Now, $m[a]$ is d -contracting. Therefore $d(m[a](x), m[a](y)) < d(x, y)$ and $\lambda < 1$.

Take any $n \in \mathbb{N}$ such that

$$\lambda^n < \frac{z\varepsilon}{d(\min K, \max K)}$$

(if $d(\min K, \max K) = 0$, then $\min K = \max K$ and there is nothing to prove). We claim that for any $w = a_1 a_2 \dots a_n \in A^n$ we have

$$m[w](\max K) - m[w](\min K) \leq \varepsilon.$$

First, it is easy to see that $d(m[w](\min K), m[w](\max K)) \leq z\varepsilon$. Indeed, define $w_{\geq i} = a_i a_{i+1} \dots a_n$ for $i = 1, \dots, n$, and let $w_{\geq n+1}$ be the empty string. Notice that for every $i = 1, \dots, n$ we have

$$d(m[w_{\geq i}](\min K), m[w_{\geq i}](\max K)) \leq d(m[w_{\geq i+1}](\min K), m[w_{\geq i+1}](\max K))$$

(because $m[w_{\geq i}] = m[a_i] \circ m[w_{\geq i+1}]$ and because $m[a_i]$ is d -contracting). In fact, if the right-hand side is at least $z\varepsilon$, then by definition of λ the left-hand side is at least $1/\lambda$ times smaller than the right-hand side. So if for contradiction $d(m[w](\min K), m[w](\max K)) > z\varepsilon$, then $d(m[w](\min K), m[w](\max K))$ is at least $(1/\lambda)^n$ times smaller than $d(\min K, \max K)$. On the other hand by definition of n we have $\lambda^n d(\min K, \max K) < z\varepsilon$, contradiction.

It remains to establish that $d(m[w](\min K), m[w](\max K)) \leq z\varepsilon \implies m[w](\max K) - m[w](\min K) \leq \varepsilon$. We show the contraposition, i.e., if $m[w](\max K) - m[w](\min K) > \varepsilon \implies d(m[w](\min K), m[w](\max K)) > z\varepsilon$. Indeed, $m[w](\max K) - m[w](\min K) > \varepsilon$, then $(m[w](\min K), m[w](\max K)) \in T$. Hence

$$\frac{d(m[w](\min K), m[w](\max K))}{m[w](\max K) - m[w](\min K)} \geq \min_{(x,y) \in T} \frac{d(x,y)}{|x-y|} = z,$$

and $d(m[w](\min K), m[w](\max K)) \geq (m[w](\max K) - m[w](\min K)) \cdot z > z\varepsilon$. \square

4.2 Proof of Proposition 2

Let A be a finite set, $\langle K, d, m \rangle$ be a contracting representation and $\varphi: A^\omega \rightarrow \mathbb{R}$ be the corresponding contracting payoff. The fact that φ is continuous follows from Lemma 11. Indeed, note that by continuity of $m[a]$, $a \in A$, for any n we can write:

$$\begin{aligned} \varphi(a_1 a_2 a_3 \dots) &= \lim_{k \rightarrow \infty} m[a_1 \dots a_k](x) = \lim_{k \rightarrow \infty} m[a_1 \dots a_n] \circ m[a_{n+1} \dots a_k](x) \\ &= m[a_1 \dots a_n] \left(\lim_{k \rightarrow \infty} m[a_{n+1} \dots a_k](x) \right) \\ &= m[a_1 \dots a_n] \left(\varphi(a_{n+1} a_{n+2} a_{n+3} \dots) \right). \end{aligned}$$

Hence by Lemma 11 we have that for all $\varepsilon > 0$ there exists n such that for all $\alpha, \beta \in A^\omega$ coinciding in the first n elements it holds that $|\varphi(\alpha) - \varphi(\beta)| \leq \varepsilon$ (both $\varphi(\alpha)$ and $\varphi(\beta)$ will be inside $m[w](K)$ for some $w \in A^n$).

Now, take an arbitrary A -labeled game graph $G = (V = V_{\text{Max}} \sqcup V_{\text{Min}}, E)$, $\text{lab}: E \rightarrow A$. We shall show that φ is positional in G .

Consider the following system of equations in $\mathbf{x} \in K^V$:

$$\mathbf{x}_u = \max_{e \in E, \text{source}(e)=u} m[\text{lab}(e)](\mathbf{x}_{\text{target}(e)}), \quad \text{for } u \in V_{\text{Max}}, \quad (4)$$

$$\mathbf{x}_u = \min_{e \in E, \text{source}(e)=u} m[\text{lab}(e)](\mathbf{x}_{\text{target}(e)}), \quad \text{for } u \in V_{\text{Min}}. \quad (5)$$

Lemma 12. *There exists a solution $\mathbf{x}^* \in K^V$ to (4–5).*

Proof. Define a mapping $T: K^V \rightarrow K^V$ as follows:

$$\begin{aligned} T(\mathbf{x})_u &= \max_{e \in E, \text{source}(e)=u} m[\text{lab}(e)](\mathbf{x}_{\text{target}(e)}), & \text{for } u \in V_{\text{Max}}, \\ T(\mathbf{x})_u &= \min_{e \in E, \text{source}(e)=u} m[\text{lab}(e)](\mathbf{x}_{\text{target}(e)}), & \text{for } u \in V_{\text{Min}}. \end{aligned}$$

It is enough to show that T has a fixed point. Define a metric $D: K^V \times K^V \rightarrow [0, +\infty)$ as follows:

$$D(\mathbf{x}, \mathbf{y}) = \max_{u \in V} d(\mathbf{x}_u, \mathbf{y}_u).$$

It is enough to show $D(T(\mathbf{x}), T(\mathbf{y})) < D(\mathbf{x}, \mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in K^V, \mathbf{x} \neq \mathbf{y}$. Indeed, assume it is proved and consider a point $\mathbf{x}^* \in K^V$ minimizing $D(\mathbf{x}, T(\mathbf{x}))$ (such \mathbf{x}^* exists because $D(\mathbf{x}, T(\mathbf{x}))$ is continuous and $K^V \times K^V$ is a compact set). If $\mathbf{x}^* \neq T(\mathbf{x}^*)$, then $D(T(\mathbf{x}^*), T \circ T(\mathbf{x}^*)) < D(\mathbf{x}^*, T(\mathbf{x}^*))$, contradiction.

Now, take any $\mathbf{x}, \mathbf{y} \in K^V, \mathbf{x} \neq \mathbf{y}$. Let $u \in V$ be such that $D(T(\mathbf{x}), T(\mathbf{y})) = d(T(\mathbf{x})_u, T(\mathbf{y})_u)$. Assume without loss of generality that $u \in V_{\text{Max}}$. Also, up to swapping x and y , we may assume that $T(\mathbf{x})_u \leq T(\mathbf{y})_u$. Take $e \in E$ such that $\text{source}(e) = u$ and

$$T(\mathbf{y})_u = m[\text{lab}(e)](\mathbf{x}_{\text{target}(e)}).$$

Denote $\text{target}(e) = w$. Then we have:

$$m[\text{lab}(e)](\mathbf{x}_w) \leq T(\mathbf{x})_u \leq T(\mathbf{y})_u = m[\text{lab}(e)](\mathbf{y}_w).$$

Since d is a monotone metric, we obtain:

$$d(T(\mathbf{x})_u, T(\mathbf{y})_u) \leq d(m[\text{lab}(e)](\mathbf{x}_w), m[\text{lab}(e)](\mathbf{y}_w)).$$

If $\mathbf{x}_w = \mathbf{y}_w$, then $0 = d(T(\mathbf{x})_u, T(\mathbf{y})_u) = D(T(\mathbf{x}), T(\mathbf{y})) < D(\mathbf{x}, \mathbf{y})$, because $\mathbf{x} \neq \mathbf{y}$. Now, if $\mathbf{x}_w \neq \mathbf{y}_w$, then, since the function $m[\text{lab}(e)]$ is d -contracting, we have:

$$d(m[\text{lab}(e)](\mathbf{x}_w), m[\text{lab}(e)](\mathbf{y}_w)) < d(\mathbf{x}_w, \mathbf{y}_w) \leq D(\mathbf{x}, \mathbf{y}),$$

as required. \square

Let \mathbf{x}^* be a solution to (4–5). For $u \in V_{\text{Max}}$, let $\sigma^*(u)$ be an edge on which the maximum in (4) is attained for \mathbf{x}^* . Similarly, for $u \in V_{\text{Min}}$ let $\tau^*(u)$ be an edge on which the minimum in (5) is attained for \mathbf{x}^* . We claim that (σ^*, τ^*) is a φ -optimal pair of positional strategies. For that it is enough to show that

- (a) for any Min's strategy τ and for any $s \in V$ we have

$$\varphi \circ \text{lab}(\mathcal{P}_s^{\sigma^*, \tau}) \geq \mathbf{x}_s^*.$$

- **(b)** for any Max's strategy σ and for any $s \in V$ we have

$$\varphi \circ \text{lab}(\mathcal{P}_s^{\sigma, \tau^*}) \leq \mathbf{x}_s^*.$$

(Indeed, from these two inequalities we also obtain $\varphi \circ \text{lab}(\mathcal{P}_s^{\sigma^*, \tau^*}) = \mathbf{x}_s^*$ for every $s \in V$, so σ^* is an optimal response to τ^* and vice versa). We only show **(a)**, a proof of **(b)** is similar. Let $e_1, e_2, e_3, \dots \in E$ be such that $\mathcal{P}_s^{\sigma^*, \tau^*} = e_1 e_2 e_3 \dots$. For $n \geq 1$ define $s_n = \text{target}(e_1 e_2 \dots e_n)$ and

$$V_n = m[\text{lab}(e_1 \dots e_n)](\mathbf{x}_{s_n}^*)$$

(by definition we set $s_0 = s$ and $V_0 = \mathbf{x}_s^*$). By Lemma 1 the limit $\lim_{n \rightarrow \infty} V_n$ exists and equals $\varphi \circ \text{lab}(\mathcal{P}_s^{\sigma^*, \tau^*})$ (note that $\mathbf{x}_{s_n}^*$ takes only finitely many different values). Thus, it is enough to show that $V_{n+1} \geq V_n$ for every $n \geq 0$. Indeed, assume first that $s_n \in V_{\text{Max}}$. Then $e_{n+1} = \sigma^*(s_n)$ and $\text{target}(e_{n+1}) = s_{n+1}$. By definition of σ^* , we have $m[\text{lab}(e_{n+1})](\mathbf{x}_{s_{n+1}}^*) = \mathbf{x}_{s_n}^*$. After applying a function $m[\text{lab}(e_1 e_2 \dots e_n)]$ to this equality, we obtain $V_{n+1} = V_n$.

Now, if $s_n \in V_{\text{Min}}$, then $m[\text{lab}(e_{n+1})](\mathbf{x}_{s_{n+1}}^*) \geq \mathbf{x}_{s_n}^*$ by (5). The function $m[\text{lab}(e_1 e_2 \dots e_n)]$ is composed from non-decreasing functions. Hence after applying this function to $m[\text{lab}(e_{n+1})](\mathbf{x}_{s_{n+1}}^*) \geq \mathbf{x}_{s_n}^*$ we obtain an inequality $V_{n+1} \geq V_n$.

4.3 Proof of Proposition 3

Lemma 13. *Let A be a finite set and $\psi: A^\omega \rightarrow \mathbb{R}$ be a multi-discounted payoff. Then there are no $a, b \in A, \gamma \in A^\omega$ such that*

$$\begin{aligned} \psi(a\gamma) &> \psi(b\gamma), \\ \psi(aa\gamma) &< \psi(bb\gamma), \\ \psi(aaa\gamma) &> \psi(bbb\gamma). \end{aligned}$$

Proof. Assume for contradiction that such a, b, γ exist. Set $\lambda = \lambda(a)$, $\mu = \lambda(b)$, $u = w(a)$, $v = w(b)$ and $x = \psi(\gamma)$. Then $\lambda, \mu \in [0, 1)$ and

$$\lambda x + u > \mu x + v, \tag{6}$$

$$\lambda^2 x + (1 + \lambda)u < \mu^2 x + (1 + \mu)v, \tag{7}$$

$$\lambda^3 x + (1 + \lambda + \lambda^2)u > \mu^3 x + (1 + \mu + \mu^2)v. \tag{8}$$

Multiply (6) by $\lambda + \mu + \lambda\mu$, multiply (7) by $-(1 + \lambda + \mu)$, multiply (8) by 1 and take the sum. It can be checked that this will give us $0 > 0$. \square

To finish a proof of Proposition 3 we construct a contracting payoff $\varphi: \{0, 1, 2\}^\omega \rightarrow \mathbb{R}$ such that:

$$\begin{aligned} \varphi(02^\omega) &> \varphi(12^\omega), \\ \varphi(002^\omega) &< \varphi(112^\omega), \\ \varphi(0002^\omega) &> \varphi(1112^\omega). \end{aligned}$$

Namely, φ is defined through the following contracting representation $\langle K, d, m \rangle$. First we set $K = [0, 1]$ and $d(x, y) = |x - y|$. Next, we let $m(0, x) = \frac{x}{2}$, $m(2, x) = \frac{x}{2} + \frac{1}{2}$. These two functions are clearly d -contracting. Finally, we let $m(1, x)$ be a piece-wise linear function whose graph has the following break-points:

$$(0, 0), (0.26, 0.11), (0.49, 0.26), (1, 0.49).$$

Observe that its slope is always from $[0, 1)$, so $m(1, \cdot)$ is also d -contracting. It is easy to see that $\varphi(2^\omega) = 1$ and

$$\begin{aligned}\varphi(02^\omega) &= 0.5 > \varphi(12^\omega) = 0.49, \\ \varphi(002^\omega) &= 0.25 < \varphi(112^\omega) = 0.26, \\ \varphi(0002^\omega) &= 0.125 > \varphi(1112^\omega) = 0.11.\end{aligned}$$

4.4 Proof of Proposition 5

Assume that φ is not prefix-monotone. Then for some $u, v \in A^*$ and $\alpha, \beta \in A^\omega$ we have

$$\varphi(u\alpha) > \varphi(u\beta) \text{ and } \varphi(v\alpha) < \varphi(v\beta). \quad (9)$$

First, notice that by continuity of φ we may assume that α and β are ultimately periodic. Indeed, consider any two sequences $\{\alpha_n\}_{n \in \mathbb{N}}$ and $\{\beta_n\}_{n \in \mathbb{N}}$ of ultimately periodic sequences from A^ω such that α_n and α (respectively, β_n and β) have the same prefix of length n . Then from continuity of φ we have:

$$\begin{aligned}\lim_{n \rightarrow \infty} \varphi(u\alpha_n) &= \varphi(u\alpha), & \lim_{n \rightarrow \infty} \varphi(v\alpha_n) &= \varphi(v\alpha), \\ \lim_{n \rightarrow \infty} \varphi(u\beta_n) &= \varphi(u\beta), & \lim_{n \rightarrow \infty} \varphi(v\beta_n) &= \varphi(v\beta).\end{aligned}$$

So if u, v, α, β violate prefix-monotonicity, then so do u, v, α_n, β_n for some $n \in \mathbb{N}$.

Now, if α, β are ultimately periodic, then $\alpha = pq^\omega$ and $\beta = wr^\omega$ for some $p, q, w, r \in A^*$. Consider the following A -labeled game graph (where all nodes belong to Max).

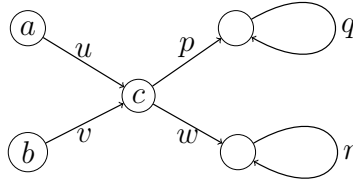


Figure 1: A game graph where φ is not positional

In this game graph there are two positional strategies of Max, one which from c goes by p and the other which goes from c by w . The first one is not optimal when the game starts in b , and the second one is not optimal when the game starts in a (because of (9)). So φ is not positional in this game graph.

5 Proof of Proposition 6

Define a payoff $\psi: A^\omega \rightarrow \mathbb{R}$ as follows:

$$\psi(\gamma) = \sum_{w \in A^*} \left(\frac{1}{|A| + 1} \right)^{|w|} \varphi(w\gamma), \quad \gamma \in A^\omega. \quad (10)$$

First, why is ψ well-defined, i.e., why does this series converge? Since A^ω is compact, so is $\varphi(A^\omega) \subseteq \mathbb{R}$, because φ is continuous. Hence $\varphi(A^\omega) \subseteq [-W, W]$ for some $W > 0$. So (10) is bounded by the following absolutely converging series:

$$\sum_{w \in A^*} W \cdot \left(\frac{1}{|A| + 1} \right)^{|w|}.$$

The proof has two parts

- we show that there exists a continuous non-decreasing $g: \psi(A^\omega) \rightarrow \mathbb{R}$ such that $\varphi = g \circ \psi$ (Subsection 5.1).
- we show that ψ is a contracting payoff (Subsection 5.2).

Before that it is convenient to show that ψ is continuous and prefix-monotone.

To show continuity of ψ , consider any $\alpha \in A^\omega$ and any infinite sequence $\{\beta_n\}_{n \in \mathbb{N}}$ of elements of A^ω such that for all n the sequences α and β_n coincide in the first n elements. We have to show that $\psi(\beta_n)$ converges to $\psi(\alpha)$ as $n \rightarrow \infty$. By definition:

$$\psi(\beta_n) = \sum_{w \in A^*} \left(\frac{1}{|A| + 1} \right)^{|w|} \varphi(w\beta_n), \quad \psi(\alpha) = \sum_{w \in A^*} \left(\frac{1}{|A| + 1} \right)^{|w|} \varphi(w\alpha).$$

The first series, as we have seen, is bounded uniformly (in n) by an absolutely converging series. So it remains to note that the first series converges to the second one term-wise, by continuity of φ .

Now, let us establish prefix-monotonicity of ψ . Let $\alpha, \beta \in A^\omega$. We have to show that either $\psi(u\alpha) \geq \psi(u\beta)$ for all $u \in A^*$ or $\psi(u\alpha) \leq \psi(u\beta)$ for all $u \in A^*$.

Since φ is prefix-monotone, then either $\varphi(w\alpha) \geq \varphi(w\beta)$ for all $w \in A^*$ or $\varphi(w\alpha) \leq \varphi(w\beta)$ for all $w \in A^*$. Up to swapping α and β we may assume that $\varphi(w\alpha) \geq \varphi(w\beta)$ for all $w \in A^*$. Then for any $u \in A^*$ the difference

$$\psi(u\alpha) - \psi(u\beta) = \sum_{w \in A^*} \left(\frac{1}{|A| + 1} \right)^{|w|} [\varphi(wu\alpha) - \varphi(wu\beta)]$$

consists of non-negative terms. Hence $\psi(u\alpha) \geq \psi(u\beta)$ for all $u \in A^*$, as required.

5.1 Proving that $\varphi = g \circ \psi$ for a continuous non-decreasing g

Let us first show that

$$\varphi(\alpha) > \varphi(\beta) \implies \psi(\alpha) > \psi(\beta) \text{ for all } \alpha, \beta \in A^\omega. \quad (11)$$

Indeed, if $\varphi(\alpha) > \varphi(\beta)$, then we also have $\varphi(w\alpha) \geq \varphi(w\beta)$ for every $w \in A^*$, by prefix-monotonicity of φ . Now, by definition,

$$\psi(\alpha) - \psi(\beta) = \sum_{w \in A^*} \left(\frac{1}{|A| + 1} \right)^{|w|} [\varphi(w\alpha) - \varphi(w\beta)].$$

All the terms in this series are non-negative, and the term corresponding to the empty w is strictly positive. So we have $\psi(\alpha) > \psi(\beta)$, as required.

Now, let us demonstrate that (11) implies that $\varphi = g \circ \psi$ for some non-decreasing $g: \psi(A^\omega) \rightarrow \mathbb{R}$. Namely, define g as follows. For $x \in \psi(A^\omega)$ take an arbitrary $\gamma \in \psi^{-1}(x)$ and set $g(x) = \varphi(\gamma)$. First of all, why do we have $\varphi = g \circ \psi$? By definition, $g(\psi(\alpha)) = \varphi(\gamma)$ for some $\gamma \in A^\omega$ with $\psi(\alpha) = \psi(\gamma)$. By (11) we also have $\varphi(\alpha) = \varphi(\beta)$, so $g(\psi(\alpha)) = \varphi(\gamma) = \varphi(\alpha)$, as required. Now, why is g non-decreasing? I.e., why for all $x, y \in \psi(A^\omega)$ we have $x \leq y \implies g(x) \leq g(y)$? Indeed, $g(x) = \varphi(\gamma_x)$, $g(y) = \varphi(\gamma_y)$ for some for some $\gamma_x \in \psi^{-1}(x)$ and $\gamma_y \in \psi^{-1}(y)$. Now, since $x \leq y$, we have $x = \psi(\gamma_x) \leq \psi(\gamma_y) = y$. By taking the contraposition of (11) we get that $g(x) = \varphi(\gamma_x) \leq \varphi(\gamma_y) = g(y)$, as required.

Finally, we show that any $g: \psi(A^\omega) \rightarrow \mathbb{R}$ with $\varphi = g \circ \psi$ must be continuous. For that we show that $|g(x) - g(y)| \leq |x - y|$ for all $x, y \in \psi(A^\omega)$. Take any $\alpha, \beta \in A^\omega$ with $x = \psi(\alpha)$ and $y = \psi(\beta)$. By prefix-monotonicity of φ we have that either $\varphi(w\alpha) \geq \varphi(w\beta)$ for all $w \in A^*$ or $\varphi(w\alpha) \leq \varphi(w\beta)$ for all $w \in A^*$. Up to swapping x and y we may assume that the first option holds. Then

$$\psi(\alpha) - \psi(\beta) = \sum_{w \in A^*} \left(\frac{1}{|A| + 1} \right)^{|w|} [\varphi(w\alpha) - \varphi(w\beta)] \geq \varphi(\alpha) - \varphi(\beta) \geq 0.$$

On the left here we have $x - y$, and on the right we have $\varphi(\alpha) - \varphi(\beta) = g \circ \psi(\alpha) - g \circ \psi(\beta) = g(x) - g(y)$.

5.2 Proving that ψ is a contracting payoff

We shall establish a contracting representation $\langle K, d, m \rangle$ for ψ . First, we set $K = \psi(A^\omega)$ (this will be a compact set because A^ω is compact and ψ is continuous).

To define m , we first have to establish the following claim.

$$\psi(\beta) = \psi(\gamma) \implies \psi(a\beta) = \psi(a\gamma) \quad \text{for all } a \in A, \beta, \gamma \in A^*. \quad (12)$$

Indeed, assume that

$$0 = \psi(\beta) - \psi(\gamma) = \sum_{w \in A^*} \left(\frac{1}{|A| + 1} \right)^{|w|} [\varphi(w\beta) - \varphi(w\gamma)].$$

If this series contains a non-zero term, then it must contain a positive term and a negative term. But this contradicts prefix-monotonicity of φ . So all the terms in this series must be 0. The same then must hold for a series:

$$\psi(a\beta) - \psi(a\gamma) = \sum_{w \in A^*} \left(\frac{1}{|A| + 1} \right)^{|w|} [\varphi(wa\beta) - \varphi(wa\gamma)]$$

(all the terms in this series also appear in the series for $\psi(\beta) - \psi(\gamma)$). So we must have $\psi(a\beta) = \psi(a\gamma)$, and thus (12) is proved.

By (12) for any $a \in A$ there exists a function $m[a]: \psi(A^\omega) \rightarrow \psi(A^\omega)$ such that

$$m[a](\psi(\beta)) = \psi(a\beta) \text{ for all } \beta \in A^\omega.$$

As in the previous section, we will use a notation

$$m[a_1 a_2 \dots a_n] = m[a_1] \circ m[a_2] \circ \dots \circ m[a_n]$$

for $n \in \mathbb{N}$, $a_1 \dots a_n \in A^n$. Formally, we define a mapping $m: A \times K \rightarrow K$ for our contracting representation $\langle K, d, m \rangle$ as follows $m(a, x) = m[a](x)$.

It remains to show that

- **(a)** for all $a \in A$ the function $m[a]$ is non-decreasing and continuous.
- **(b)** there exists a continuous monotone metric $d: K \times K \rightarrow [0, +\infty)$ such that $m[a]$ is d -contracting for every $a \in A$;
- **(c)** a contracting payoff, defined by $\langle K, d, m \rangle$, coincides with ψ .

First, let us explain why **(c)** holds assuming **(a)** and **(b)** are proved. Indeed, by Lemma 1 we only have to show that for any $a_1 a_2 a_3 \dots \in A^\omega$ the limit

$$\lim_{n \rightarrow \infty} m[a_1 a_2 \dots a_n](x)$$

exists and equals $\psi(a_1 a_2 a_3 \dots)$ for some $x \in K$. Indeed, take an arbitrary $x \in K = \psi(A^\omega)$. Let $\beta \in A^\omega$ be such that $x = \psi(\beta)$. Observe that $m[a_1 a_2 \dots a_n](x) = \psi(a_1 a_2 \dots a_n \beta)$. By continuity of ψ , the quantity $\psi(a_1 a_2 \dots a_n \beta)$ must converge to $\psi(a_1 a_2 a_3 \dots)$ as $n \rightarrow \infty$, as required.

Now, let us show **(a)**. The fact that $m[a]$ must be non-decreasing follows from prefix-monotonicity of ψ . Indeed, if $m[a]$ is not non-decreasing, then for some $\beta, \gamma \in A^\omega$ with $\psi(\beta) < \psi(\gamma)$ we have $\psi(a\beta) = m[a](\psi(\beta)) > m[a](\psi(\gamma)) = \psi(a\gamma)$, and this contradicts prefix-monotonicity of ψ .

Now, to show that $m[a]$ is continuous we show that $|m[a](x) - m[a](y)| \leq (|A| + 1) \cdot |x - y|$ for all $x, y \in \psi(A^\omega)$. Indeed, let $\beta, \gamma \in A^\omega$ be such that $x = \psi(\beta), y = \psi(\gamma)$. Up to swapping x and y , from prefix-monotonicity of φ we may assume that $\varphi(w\beta) \geq \varphi(w\gamma)$ for all $w \in A^*$. Then we have that:

$$m[a](x) - m[a](y) = \psi(a\beta) - \psi(a\gamma) = \sum_{w \in A^*} \left(\frac{1}{|A| + 1} \right)^{|w|} \cdot [\varphi(wa\beta) - \varphi(wa\gamma)] \geq 0.$$

On the other hand:

$$\begin{aligned}
m[a](x) - m[a](y) &= \sum_{w \in A^*, |w|=a} \left(\frac{1}{|A|+1} \right)^{|w|-1} \cdot [\varphi(w\beta) - \varphi(w\gamma)] \\
&= (|A|+1) \cdot \sum_{w \in A^*, |w|=a} \left(\frac{1}{|A|+1} \right)^{|w|} \cdot [\varphi(w\beta) - \varphi(w\gamma)] \\
&\leq (|A|+1) \cdot \sum_{w \in A^*} \left(\frac{1}{|A|+1} \right)^{|w|-1} \cdot [\varphi(w\beta) - \varphi(w\gamma)] \\
&= (|A|+1) \cdot (\psi(\beta) - \psi(\gamma)) = (|A|+1) \cdot (x - y).
\end{aligned}$$

Thus, we get $0 \leq m[a](x) - m[a](y) \leq (|A|+1) \cdot (x - y)$, as required.

To finish the proof of Lemma 6 it remains to establish **(b)**.

Proof of (b). First we need the following lemma.

Lemma 14. *For every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that for all $w \in A^*$ with $|w| \geq n$ we have*

$$m[w](\max K) - m[w](\min K) \leq \varepsilon.$$

Proof. Take any $\varepsilon > 0$. Call $w \in A^*$ *bad* if

$$m[w](\max K) - m[w](\min K) > \varepsilon.$$

Assume for contradiction that the set of bad $w \in A^*$ is infinite. We will show that there exists $\alpha = a_1 a_2 a_3 \dots \in A^\omega$ such that any finite prefix of α is bad. By Kőnig's Lemma it is enough to show that any prefix of a bad $w \in A^*$ is also bad. Indeed, assume that $w \in A^*$ is bad and $w = uv$ for some $u, v \in A^*$. Then

$$m[u](\max K) - m[u](\min K) \geq m[u](m[v](\max K)) - m[u](m[v](\min K)).$$

This is because $m[u]$ is non-decreasing and because $m[v](\max K), m[v](\min K) \in K$. On the other hand:

$$m[u](m[v](\max K)) - m[u](m[v](\min K)) = m[w](\max K) - m[w](\min K) > \varepsilon.$$

So we get that u is bad, as required.

Thus, there exists $\alpha = a_1 a_2 a_3 \dots \in A^\omega$ such that any finite prefix of α is bad. Take any $\beta, \gamma \in A^\omega$ such that

$$\min K = \min \psi(A^\omega) = \psi(\beta), \quad \max K = \max \psi(A^\omega) = \psi(\gamma)$$

(such β and γ exist because A^ω is compact). For every $n \in \mathbb{N}$ define

$$\beta_n = a_1 a_2 \dots a_n \beta \in A^\omega, \quad \gamma_n = a_1 a_2 \dots a_n \gamma \in A^\omega.$$

Both β_n, γ_n coincide in the first n elements with α , so since ψ is continuous, we have $\psi(\alpha) = \lim_{n \rightarrow \infty} \psi(\beta_n) = \lim_{n \rightarrow \infty} \psi(\gamma_n)$. On the other hand, for every $n \in \mathbb{N}$ we have:

$$\begin{aligned} \psi(\gamma_n) - \psi(\beta_n) &= m[a_1 a_2 \dots a_n](\psi(\gamma)) - m[a_1 a_2 \dots a_n](\psi(\beta)) \\ &= m[a_1 a_2 \dots a_n](\max K) - m[a_1 a_2 \dots a_n](\min K) > \varepsilon. \end{aligned}$$

The last inequality is because $a_1 a_2 \dots a_n$ is a prefix of α and hence is bad. We obtained a contradiction, so the set of bad $w \in A^*$ could not be infinite. \square

Define

$$d: K \times K \rightarrow [0, +\infty), \quad d(x, y) = \sup_{w \in A^*} (2 - 2^{-|w|}) \cdot |m[w](x) - m[w](y)|. \quad (13)$$

First, we obviously have $d(x, x) = 0$ and $d(x, y) = d(y, x)$. Notice also that $d(x, y) \geq |x - y|$, so $d(x, y) > 0$ for $x \neq y$. In turn, $d(x, y) \leq d(x, z) + d(z, y)$ because first we can write a similar inequality for every $w \in A^*$ in (13), and then it remains to notice that the supremum of the sums is at most the sum of the supremums. These considerations show that d is a metric.

To show that d is a monotone metric take any $x, s, t, y \in K$ with $x \leq s \leq t \leq y$. Since $m[w]$ is composed of non-decreasing functions for all $w \in A^*$, we also have $m[w](x) \leq m[w](s) \leq m[w](t) \leq m[w](y)$. Thus, $|m[w](x) - m[w](y)| \geq |m[w](s) - m[w](t)|$ for all $w \in A^*$ and hence $d(x, y) \geq d(s, t)$.

Let us show that the supremum in (13) is always attained on some $w \in A^*$. Indeed, if $d(x, y) = 0$, then it is attained already for the empty string. Assume now that $d(x, y) > 0$. By Lemma 14 there exists n such that for all $w \in A^*$ with $|w| \geq n$ we have:

$$m[w](\max K) - m[w](\min K) \leq d(x, y)/3.$$

In particular, all the terms in (13) with $|w| \geq n$ are smaller than $d(x, y)$. Hence the supremum there must be attained on some w with $|w| < n$.

This already implies that $m[a]$ is d -contracting for every $a \in A$. Indeed, take any $x, y \in K$. Then for some $w \in A^*$ we have:

$$d(m[a](x), m[a](y)) = (2 - 2^{-|w|}) \cdot |m[w](m[a](x)) - m[w](m[a](y))|.$$

We have to show that if $x \neq y$, then $d(m[a](x), m[a](y)) < d(x, y)$. If $d(m[a](x), m[a](y)) = 0$, there is nothing to prove. Otherwise the quantity

$$|m[w](m[a](x)) - m[w](m[a](y))|$$

is positive. Therefore we can write:

$$\begin{aligned} d(m[a](x), m[a](y)) &< (2 - 2^{-|w|-1}) \cdot |m[w](m[a](x)) - m[w](m[a](y))| \\ &= (2 - 2^{-|wa|}) \cdot |m[wa](x) - m[wa](y)| \leq d(x, y). \end{aligned}$$

It remains to show that d is continuous. Consider any $(x_0, y_0) \in K \times K$. We have to show that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $(x, y) \in K \times K$ with $|x - x_0| + |y - y_0| \leq \delta$ we have $|d(x, y) - d(x_0, y_0)| \leq \varepsilon$.

By Lemma 14 there exists $n \in \mathbb{N}$ such that for all $w \in A^*$ with $|w| \geq n$ we have:

$$m[w](\max K) - m[w](\min K) \leq \varepsilon/6.$$

In particular, this means that in (13) all the terms corresponding to $w \in A^*$ with $|w| \geq n$ are at most $\varepsilon/3$. Hence for every $(x, y) \in K \times K$ we have that $d(x, y)$ is $(\varepsilon/3)$ -close to $d_n(x, y)$, where

$$d_n(x, y) = \max_{w \in A^*, |w| < n} \left(2 - 2^{-|w|} \right) \cdot |m[w](x) - m[w](y)|.$$

Now, notice that the function d_n is continuous (as a composition of finitely many continuous functions). Hence there exist $\delta > 0$ such that for all $(x, y) \in K \times K$ with $|x - x_0| + |y - y_0| \leq \delta$ we have $|d_n(x, y) - d_n(x_0, y_0)| \leq \varepsilon/3$. Obviously, for all such (x, y) we also have $|d(x, y) - d(x_0, y_0)| \leq \varepsilon$.

6 Proof of Theorem 8

First we establish two conditions that are necessary for continuous MDP-positional payoffs.

Proposition 15. *Let A be a finite set and $\varphi: A^\omega \rightarrow \mathbb{R}$ be a continuous MDP-positional payoff. Then there are no $a \in A$, $\beta, \gamma, \delta \in A^\omega$, $(p_1, p_2, p_3), (q_1, q_2, q_3) \in [0, +\infty)^3$ such that $p_1 + p_2 + p_3 = q_1 + q_2 + q_3 = 1$ and*

$$\begin{aligned} p_1\varphi(\beta) + p_2\varphi(\gamma) + p_3\varphi(\delta) &> q_1\varphi(\beta) + q_2\varphi(\gamma) + q_3\varphi(\delta), \\ p_1\varphi(a\beta) + p_2\varphi(a\gamma) + p_3\varphi(a\delta) &< q_1\varphi(a\beta) + q_2\varphi(a\gamma) + q_3\varphi(a\delta). \end{aligned}$$

Proposition 16. *Any continuous MDP-positional payoff is prefix-monotone.*

In fact, Proposition 16 is already proved. Indeed, in the proof of Proposition 5 we have shown that for any continuous payoff which is not prefix-monotone there exists a game graph *with all the nodes belonging to Max* where φ is not positional. This game graph is a deterministic MDP, so any continuous payoff which is not prefix monotone is not MDP-positional.

To finish a proof of Theorem 8 we show the following technical claim.

Proposition 17. *Let A be a finite set and $\varphi: A^\omega \rightarrow \mathbb{R}$ be a continuous prefix-monotone payoff. Assume that there are no $a \in A$, $\beta, \gamma, \delta \in A^\omega$, $(p_1, p_2, p_3), (q_1, q_2, q_3) \in [0, +\infty)^3$ such that $p_1 + p_2 + p_3 = q_1 + q_2 + q_3 = 1$ and*

$$\begin{aligned} p_1\varphi(\beta) + p_2\varphi(\gamma) + p_3\varphi(\delta) &> q_1\varphi(\beta) + q_2\varphi(\gamma) + q_3\varphi(\delta), \\ p_1\varphi(a\beta) + p_2\varphi(a\gamma) + p_3\varphi(a\delta) &< q_1\varphi(a\beta) + q_2\varphi(a\gamma) + q_3\varphi(a\delta). \end{aligned}$$

Then φ is a multi-discounted payoff.

6.1 Proof of Proposition 15

Assume for contradiction that such $a, \beta, \gamma, \delta, (p_1, p_2, p_3), (q_1, q_2, q_3)$ exist. Similarly to the proof of Proposition 5, due to continuity of φ we may assume that β, γ and δ are ultimately periodic. We construct an A -labeled MDP \mathcal{M} where φ is not MDP-positional. To define \mathcal{M} first consider the following A -labeled game graph:

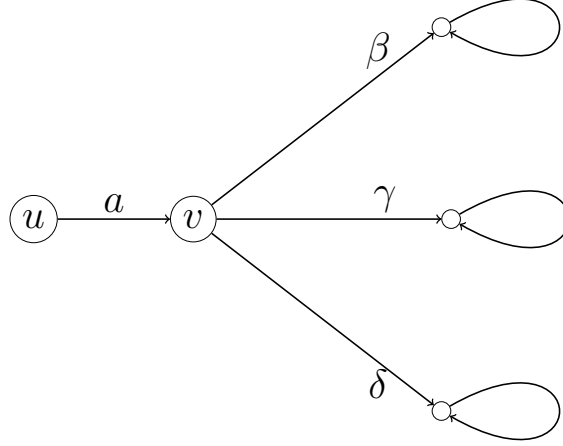


Figure 2: A graph for an MDP where φ is not MDP-positional

In this graph there are exactly 3 infinite paths (“lassos”) $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ that start at v . We label edges in such a way that $\text{lab}(\mathcal{P}_1) = \beta, \text{lab}(\mathcal{P}_2) = \gamma, \text{lab}(\mathcal{P}_3) = \delta$. Clearly, this is possible due to ultimate periodicity of β, γ and δ .

Next, we turn this graph into an MDP (formally, nodes of the graph will be states of the MDP). There will be two actions available at the node v . Both will be distributed on the three successors of v , one with probabilities p_1, p_2, p_3 , and the other with probabilities q_1, q_2, q_3 (in the descending order if one looks at Figure 2). For each node different from v there will be only one action with the source in this node, leading with probability 1 to its unique successor.

Note that each possible transition is along some edge of the graph from Figure 2, and the label of this edge will also serve as the label of the transition. This concludes a description of \mathcal{M} .

To show that φ is not MDP-positional in \mathcal{M} , note that in \mathcal{M} there are exactly 2 positional strategies, σ_p and σ_q , corresponding to two actions available at v . We show that none of these two strategies is optimal.

It is easy to see that:

$$\begin{aligned}\mathbb{E}\varphi \circ \text{lab}(\mathcal{P}_u^{\sigma_p}) &= p_1 \cdot \varphi(a\beta) + p_2 \cdot \varphi(a\gamma) + p_3 \cdot \varphi(a\delta), \\ \mathbb{E}\varphi \circ \text{lab}(\mathcal{P}_v^{\sigma_p}) &= p_1 \cdot \varphi(\beta) + p_2 \cdot \varphi(\gamma) + p_3 \cdot \varphi(\delta), \\ \mathbb{E}\varphi \circ \text{lab}(\mathcal{P}_u^{\sigma_q}) &= q_1 \cdot \varphi(a\beta) + q_2 \cdot \varphi(a\gamma) + q_3 \cdot \varphi(a\delta), \\ \mathbb{E}\varphi \circ \text{lab}(\mathcal{P}_v^{\sigma_q}) &= q_1 \cdot \varphi(\beta) + q_2 \cdot \varphi(\gamma) + q_3 \cdot \varphi(\delta).\end{aligned}$$

Due to our assumptions about $(p_1, p_2, p_3), (q_1, q_2, q_3)$ we obtain:

$$\mathbb{E}\varphi \circ \text{lab}(\mathcal{P}_u^{\sigma_p}) < \mathbb{E}\varphi \circ \text{lab}(\mathcal{P}_u^{\sigma_q}), \quad \mathbb{E}\varphi \circ \text{lab}(\mathcal{P}_v^{\sigma_p}) > \mathbb{E}\varphi \circ \text{lab}(\mathcal{P}_v^{\sigma_q}).$$

Therefore, neither σ_p nor σ_q is optimal for φ .

6.2 Proof of Proposition 17

If $\varphi(\gamma) = \varphi(\delta)$ for all $\beta, \gamma \in A^\omega$, then clearly φ is multi-discounted (one can define $\lambda(a) = 0, w(a) = \varphi(\gamma)$ for all $a \in A$ and for an arbitrary $\gamma \in A^\omega$). So in what follows we fix any $\gamma, \delta \in A^\omega$ with $\varphi(\gamma) \neq \varphi(\delta)$. First we derive from the conditions of Proposition 17 the following:

Lemma 18. *For any $a \in A$ there exist $\lambda(a), w(a) \in \mathbb{R}$ such that for any $\beta \in A^\omega$ we have:*

$$\varphi(a\beta) = \lambda(a)\varphi(\beta) + w(a).$$

Proof. The following system in (λ, w) has a unique solution:

$$\begin{pmatrix} \varphi(a\gamma) \\ \varphi(a\delta) \end{pmatrix} = \begin{pmatrix} \varphi(\gamma) & 1 \\ \varphi(\delta) & 1 \end{pmatrix} \cdot \begin{pmatrix} \lambda \\ w \end{pmatrix}, \quad (14)$$

(because $\varphi(\gamma) \neq \varphi(\delta)$). Let its solution be $(\lambda(a), w(a))$. We show that $\varphi(a\beta) = \lambda(a)\varphi(\beta) + w(a)$ for all $\beta \in A^\omega$. Let us first show that

$$\det \begin{pmatrix} 1 & 1 & 1 \\ \varphi(\beta) & \varphi(\gamma) & \varphi(\delta) \\ \varphi(a\beta) & \varphi(a\gamma) & \varphi(a\delta) \end{pmatrix} = 0. \quad (15)$$

Indeed, otherwise there exists a vector $(x, y, z) \in \mathbb{R}^3$ such that

$$\begin{pmatrix} 1 & 1 & 1 \\ \varphi(\beta) & \varphi(\gamma) & \varphi(\delta) \\ \varphi(a\beta) & \varphi(a\gamma) & \varphi(a\delta) \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}. \quad (16)$$

Let $P_1, P_2, P_3, Q_1, Q_2, Q_3$ be any positive real numbers such that $x = P_1 - Q_1, y = P_2 - Q_2, z = P_3 - Q_3$. From the first equality in (16) it follows that there exists $S > 0$ such that

$$S = P_1 + P_2 + P_3 = Q_1 + Q_2 + Q_3.$$

Define $p_i = P_i/S, q_i = Q_i/S$ for $i \in \{1, 2, 3\}$. Observe that $a, \beta, \gamma, \rho, (p_1, p_2, p_3), (q_1, q_2, q_3)$ violate the conditions of Proposition 17 (this can be seen from the second and the third equality in (16)), contradiction. Therefore (15) is proved.

The first two rows of the matrix from (15) are linearly independent because $\varphi(\gamma) \neq \varphi(\delta)$. Hence the third one must be a linear combination of the first two. I.e., there must exist $\lambda, w \in \mathbb{R}$ such that

$$(\varphi(a\beta), \varphi(a\gamma), \varphi(a\delta)) = \lambda(\varphi(\beta), \varphi(\gamma), \varphi(\delta)) + w(1, 1, 1).$$

From the second and the third coordinate we conclude that (λ, w) must be a solution to (14), so $\lambda = \lambda(a), w = w(a)$. Now, by looking at the first coordinate we obtain that $\varphi(a\beta) = \lambda(a)\varphi(\beta) + w(a)$, as required. \square

From now on let $\lambda(a), w(a)$ for $a \in A$ be as in Lemma 18. Let us show that $\lambda(a) \in [0, 1)$ for all $a \in A$.

Assume first that for some $a \in A$ we have $\lambda(a) < 0$. Without loss of generality we may also assume that $\varphi(\gamma) < \varphi(\delta)$. Then $\varphi(a\gamma) = \lambda(a)\varphi(\gamma) + w(a) > \lambda(a)\varphi(\delta) + w(a) = \varphi(a\delta)$. Therefore φ is not prefix-monotone, contradiction.

Next, assume that $\lambda(a) \geq 1$ for some $a \in A$. Consider the following two sequences $\{x_n\}_{n \in \mathbb{N}}$ and $\{y_n\}_{n \in \mathbb{N}}$ of real numbers:

$$x_n = \varphi(\underbrace{aa \dots a}_n \gamma), \quad y_n = \varphi(\underbrace{aa \dots a}_n \delta).$$

By definition we set $x_0 = \varphi(\gamma)$ and $y_0 = \varphi(\delta)$. Note that by our choice of γ and δ we have $x_0 \neq y_0$. Next, since φ is continuous, we have:

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n = \varphi(aaa \dots). \quad (17)$$

On the other hand, we can directly compute x_n and y_n by repeatedly applying Lemma 18:

$$x_n = \lambda(a)^n x_0 + w(a)(1 + \lambda(a) + \dots + \lambda(a)^{n-1}), \quad (18)$$

$$y_n = \lambda(a)^n y_0 + w(a)(1 + \lambda(a) + \dots + \lambda(a)^{n-1}). \quad (19)$$

We will show that $\lambda(a) \geq 1$ contradicts (17–19).

First consider the case $\lambda(a) = 1$. Then x_n and y_n look as follows:

$$x_n = x_0 + nw(a), \quad y_n = y_0 + nw(a).$$

If $w(a) \neq 0$, then the sequences $\{x_n\}_{n \in \mathbb{N}}$ and $\{y_n\}_{n \in \mathbb{N}}$ are not convergent, contradiction with (17). If $w(a) = 0$, then one sequence converges to x_0 , and the other to y_0 . But $x_0 \neq y_0$, so this again gives a contradiction with (17).

Now consider the case $\lambda(a) > 1$. Then we can rewrite (18–19) as follows:

$$x_n = \lambda(a)^n \left(x_0 + \frac{w(a)}{\lambda(a) - 1} \right) - \frac{w(a)}{\lambda(a) - 1}, \quad y_n = \lambda(a)^n \left(y_0 + \frac{w(a)}{\lambda(a) - 1} \right) - \frac{w(a)}{\lambda(a) - 1}.$$

Since $x_0 \neq y_0$, for at least one of these two expressions the coefficient before $\lambda(a)^n$ is non-zero. Hence either $\{x_n\}_{n \in \mathbb{N}}$ or $\{y_n\}_{n \in \mathbb{N}}$ diverges. This contradicts (17).

We have established that $\lambda(a) \in [0, 1)$ for every $a \in A$. All that remains to do is to show that φ satisfies (1). For that we again employ continuity of φ . Take any $\beta \in A^\omega$. Note that by Lemma 18 we have:

$$\varphi(a_1 a_2 \dots a_n \beta) = \lambda(a_1) \cdot \dots \cdot \lambda(a_n) \varphi(\beta) + \sum_{i=1}^n \lambda(a_1) \cdot \dots \cdot \lambda(a_{i-1}) \cdot w(a_i). \quad (20)$$

We know that $\lambda(a_i)$ are all from $[0, 1)$. Since the set A is finite, all $\lambda(a_i)$ are uniformly bounded away from 1. Hence the first term in the right-hand side of (20) converges to 0 as $n \rightarrow \infty$. On the other hand, the second term in the right-hand side of (20) converges to the series from the right-hand side of (1). Finally, due to continuity of φ , the left-hand side of (20) converges to $\varphi(a_1 a_2 a_3 \dots)$. Thus φ is multi-discounted.

References

- [1] BOUYER, P., LE ROUX, S., OUALHADJ, Y., RANDOUR, M., AND VANDENHOVE, P. Games where you can play optimally with arena-independent finite memory. In *31st International Conference on Concurrency Theory (CONCUR 2020)* (2020), Schloss Dagstuhl-Leibniz-Zentrum für Informatik.
- [2] CALUDE, C. S., JAIN, S., KHOUSSAINOV, B., LI, W., AND STEPHAN, F. Deciding parity games in quasi-polynomial time. *SIAM Journal on Computing*, 0 (2020), STOC17–152.
- [3] COLCOMBET, T., AND NIWIŃSKI, D. On the positional determinacy of edge-labeled games. *Theoretical Computer Science* 352, 1-3 (2006), 190–196.
- [4] EHRENFUCHT, A., AND MYCIELSKI, J. Positional strategies for mean payoff games. *International Journal of Game Theory* 8, 2 (1979), 109–113.
- [5] EMERSON, E. A., AND JUTLA, C. S. Tree automata, mu-calculus and determinacy. In *FoCS* (1991), vol. 91, Citeseer, pp. 368–377.
- [6] GIMBERT, H. Pure stationary optimal strategies in markov decision processes. In *Annual Symposium on Theoretical Aspects of Computer Science* (2007), Springer, pp. 200–211.
- [7] GIMBERT, H., AND ZIELONKA, W. When can you play positionally? In *International Symposium on Mathematical Foundations of Computer Science* (2004), Springer, pp. 686–697.
- [8] GIMBERT, H., AND ZIELONKA, W. Games where you can play optimally without any memory. In *International Conference on Concurrency Theory* (2005), Springer, pp. 428–442.

- [9] GIMBERT, H., AND ZIELONKA, W. Applying blackwell optimality: priority mean-payoff games as limits of multi-discounted games. In *Logic and automata* (2008), pp. 331–356.
- [10] GRADEL, E., AND THOMAS, W. *Automata, logics, and infinite games: a guide to current research*, vol. 2500. Springer Science & Business Media, 2002.
- [11] MCNAUGHTON, R. Infinite games played on finite graphs. *Annals of Pure and Applied Logic* 65, 2 (1993), 149–184.
- [12] MOSTOWSKI, A. W. Games with forbidden positions. Tech. Rep. 78, Uniwersytet Gdański, Instytut Matematyki, 1991.
- [13] PUTERMAN, M. L. *Markov decision processes: discrete stochastic dynamic programming*. John Wiley & Sons, 2014.
- [14] SHAPLEY, L. S. Stochastic games. *Proceedings of the national academy of sciences* 39, 10 (1953), 1095–1100.
- [15] SUTTON, R. S., BARTO, A. G., ET AL. *Introduction to reinforcement learning*, vol. 135. MIT press Cambridge, 1998.