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On Universal Fields of Fractions for Free Algebras

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Let L be an arbitrary Lie algebra. Then by Cohn's Theorem its universal enveloping algebra can be embedded in a skew field $D(L)$. We study this skew field in the case when L is a free Lie algebra and prove that in this case $D(L)$ is isomorphic to the universal field of fractions for the free associative algebra $U(L)$. We apply this theorem to obtain new results on free fields. © 2000 Academic Press

1. INTRODUCTION

The main goal of this article is to prove Theorem 1, which states that two known skew fields generated by free algebras are isomorphic: these skew fields are the universal field of fractions for free algebras and the skew fields obtained by Cohn's construction [4]. (See also Cohn [2, Section 2.6].) Theorem 1 provides also a new method for constructing universal fields of fractions for free algebras.

Let $K\langle x_i | i \in I\rangle$ be a free associative algebra with a free system of generators x_i $(i \in I)$ over a field K. The universal field of fractions for this algebra was constructed by Amitsur in [1] in connection with the study of rational identities in skew fields. For firs and semifirs the universal fields of fractions were constructed by Cohn (see Cohn [2; 3, Section 4.5]) by inverting all full matrices; this construction gives another method for obtaining the universal fields of fractions for free algebras. One more method for obtaining the universal field of fractions for a free group ring was given by Lewin in [8].

The universal fields of fractions for free algebras (and free group algebras) are called also "free field." A free field is the initial object in the

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category of skew fields where the morphisms are specializations from one skew field to another. One can compare also the role of free fields in the theory of skew fields to the role that free groups play in group theory.

We will denote the universal field of fractions for a free algebra $K\langle x_i | i \in I\rangle$ by $\Delta(x_i | i \in I)$, or simply by Δ. We give the definition of a free field in Section 2 and list a few facts and concepts which will be used in our proofs; the reader is referred to the books of Cohn [3] and Schofield [16] for a detailed exposition.

Now let L be an arbitrary Lie algebra over a field K and $U(L)$ be its universal enveloping algebra. Cohn proved in [4] that $U(L)$ can be imbedded in a skew field $\widetilde{D(L)}$. The skew subfield of $\widetilde{D(L)}$ generated by $U(L)$ will be denoted by $D(L)$; $D(L)$ is in fact a proper skew subfield of $\widetilde{D(L)}$ and $\widetilde{D(L)}$ is a topological completion of $D(L)$.

We will apply in our proofs another method for the construction of $D(L)$; it was developed by Lichtman [10] (see also Cohn [3, Section 2.6]).

We formulate now the main result of the article.

THEOREM 1. *Let H be a free Lie algebra. Then the skew field $D(H)$ is isomorphic to the universal field of fractions for the free associative algebra $U(H)$.*

In other words, we prove that if H is a free Lie algebra then $D(H)$ is a free field.

The following corollary follows immediately from Theorem 1 and Theorem 2 in Lichtman [10].

COROLLARY 1. *The free field $\Delta(x_i | i \in I)$ has a discrete valuation function $v(x)$ with the following properties: If S is the valuation ring of Δ, i.e., $S = \{x \in \Delta | v(x) \geq 0\}$, and $J(S)$ is the maximal ideal of S, then the quotient ring $S/J(S)$ is a field purely transcendental over K. Its transcendence basis is given by the images of the elements $e_j e_1^{-1}$ $(j > 1)$, where e_j $(j \in J)$ is an arbitrary ordered basis of L.*

Valuations in free fields with an abelian group of values and a commutative residue class field were constructed by Cohn (see Cohn [3, Section 9.2]), but these valuations are not discrete. On the other hand, there were examples of discrete valuations in free rings but their residue-class field was not commutative; sometimes it was more complicated than the initial free field itself (see Cohn [3, Section 9.1]). So the main content of Corollary 1 is that the valuation which is obtained is discrete and that the residue class-field has a special structure: it is purely transcendental over K with a transcendence basis $e_j e_1^{-1}$ $(j > 1)$ where e_j $(j \in J)$ is an arbitrary ordered basis of L.

The results from Lichtman [11] yield one more corollary of Theorem 1.

COROLLARY 2. *The multiplicative group* Δ^* *is residually nilpotent. Furthermore,*

$$\Delta^* \simeq K^* \times \Delta_1,$$

where Δ_1 *is residually torsion free nilpotent if* $\operatorname{char} K = 0$ *and residually a "nilpotent p-group of bounded exponent" if* $\operatorname{char} K = p > 0$.

Now let L be an arbitrary Lie algebra and L_1 be its subalgebra. We apply the method of Lichtman [10] and prove in Proposition 2.5 that the skew subfield of $D(L)$ generated by the subalgebra $U(L_1)$ is isomorphic to the skew field $D(L_1)$ and that every basis of $U(L)$ over $U(L_1)$ remains linearly independent over $D(L_1)$. We obtain from this Corollary 3 of Theorem 1.

COROLLARY 3. *Let* H_1 *be a subalgebra of a free Lie algebra* H *and* $\Delta(U(H)$ *be the universal field of fractions for the free associative algebra* $U(H)$. *Then the skew subfield of* $\Delta(U(H))$ *generated by* $U(H_1)$ *is isomorphic to the universal field of fractions* $\Delta(U(H_1))$ *for* $U(H_1)$. *Further, every basis of* $U(H)$ *over* $U(H_1)$ *remains linearly independent over* $\Delta(U(H_1)$.

We will obtain more corollaries of Theorem 1 in the paper [14]. We will prove in particular that if $\mathscr{A}$ is the set of all non-full matrices over $U(H)$ then $\mathscr{A} = \bigcap_{i=0}^{\infty} \mathscr{A}_i$ where $\mathscr{A}_i$ denotes the matrix ideal defined by the canonical embedding of $U(H/\gamma_i(H))$ into its skew field of fractions. Here $\gamma_i(H)$ $(i = 1, 2, \dots)$ are the terms of the lower central series.

Theorem 1 and Corollary 3 should be compared with Lewin's result [8]. Lewin proved that if F is a free group and $K((F))$ is the Malcev–Neumann power series skew field over a commutative field K then the skew subfield of it generated by the group ring KF is isomorphic to the universal field of fractions for KF. Further, he proved that if F_1 is a subgroup of F then KF_1 generates the universal field of fractions for KF_1 and the elements of any transversal for F_1 in F remain linearly independent over this skew subfield. He obtained these results as corollary of his theorem which states that if $K((F))$ is the Malcev–Neumann power series skew field over the free group F with coefficients in the field K then the skew subfield of it generated by the subring KF_1 is isomorphic to the universal field of fractions $\Delta(KF_1)$ for the free group ring KF. We give in Section 7 a sketch of a new proof of this theorem of Lewin; our proof makes essential use of Amitsur's Specialization Lemma (see Amitsur [1, Theorem 24B or Corollary 24D] or Cohn [3, Section 6.4.7]). We describe the difference between our proof and the original proof of Lewin in Section 7.

2. PRELIMINARIES

2.1. Let R be a domain. Assume that there exists a skew field $\operatorname{div}(R)$ which contains R and is generated by it. Assume now that $\phi: R \to \overline{R}$ is an epimorphism on a domain $\overline{R}$ and let $\operatorname{div}(\overline{R})$ be a skew field which contains $\overline{R}$ and is generated by it. If there exists a local subring $R \subseteq T \subseteq \operatorname{div}(R)$ with radical $J(T)$ such that $T/J(T) \cong \operatorname{div}(\overline{R})$ and $J(T) \cap R = \ker(\phi)$, then the homomorphism $\theta: T \to \operatorname{div}(\overline{R})$ is a specialization which extends the homomorphism ϕ and T is the domain of θ. If every such epimorphism on a ring $\overline{R}$, which is imbedded in a skew field $\operatorname{div}(\overline{R})$ and generates it, can be extended to a specialization, then $\operatorname{div}(R)$ is called the universal field of fractions for R. The universal field fractions is unique up to isomorphism.

The universal fields of fractions for free algebras (and for free group algebras) were constructed first by Amitsur in [1]. Let $K\langle h_i | i \in I\rangle$ be a free associative algebra over a commutative field K and Δ be the universal field of fractions for it; clearly K is a central subfield of Δ. We will need the following fact which is a special case of the results obtained in Section 5.3 of Lichtman [12]; it makes essential use of Amitsur's Specialization Lemma.

PROPOSITION 2.1. *Let*

$$x_j \qquad (j = 1, 2, \dots, m) \tag{2.1}$$

be given non-zero elements of Δ. Then there exists a homomorphism ϕ of $K\langle h_i | i \in I\rangle$ in a skew field D of prime index p such that

$$x_j, x_j^{-1} \in T \qquad (j = 1, 2, \dots, m),$$

where T is the domain of the specialization θ extending ϕ and

$$\theta(x_j) \neq 0 \qquad (j = 1, 2, \dots, m). \tag{2.2}$$

Remark. The assertion is true also for the case when Δ is the universal field of fractions for a free group algebra; the proof is obtained by the same argument.

Proof of Proposition 2.1. We will give only a sketch of the proof since it can be read off from Sections 5.2 and 5.3 in Lichtman [12].

Assume first that the field K is infinite and consider the group ring KF of a free metabelian group F. It is well known that KF is an Ore domain; we denote by S its skew field of fractions. The skew field S is infinite dimensional over its center, and in fact it is not difficult to prove that its center coincides with K. We apply now Amitsur's Specialization Lemma (see Amitsur [1, Theorem 24B or Corollary 24D] or Cohn [3, Section 6.4.7]) and find a non-zero homomorphism ϕ_1 of the algebra $K < h_i | i \in I \rangle$ in S and a specialization θ_1: $T_1 \to S$ that extends ϕ_1 such that the elements (2.1) belong to the domain T_1 of θ.

Let $\overline{T}_1 = T_1/J(T_1) = \theta(T_1)$ and $\theta(x_j) = \bar{x}_j$ $(j = 1, 2, \ldots, m)$. Then the elements

$$\bar{x}_j \qquad (j = 1, 2, \ldots, m) \tag{2.3}$$

are non-zero elements in the skew field $\overline{T}_1 \subseteq S$. We can now apply the results from [12, Section 5.3] and find a torsion-free group N, which is an extension of an abelian group by a cyclic group of prime order, and a homomorphism ϕ_2: $F \to N$ such that this homomorphism is extended to a specialization θ_2 from $\overline{S}$ to the skew field of fractions D of the group ring KN, and such that the elements (2.3) and their inverses belong to the domain T_2 of the specialization θ_2. The skew field D has dimension p^2 over its center because the group N is an extension of an abelian group by a cyclic group of order p.

Now let ϕ be the product of homomorphisms ϕ_1 and ϕ_2. Then ϕ is a homomorphism from $K\langle h_i | i \in I \rangle$ into D and it is extended to a specialization θ which is a product of the specializations θ_1 and θ_2, and (2.1) holds. This completes the proof for the case when the field K is infinite.

Now assume that the field K is finite and let $K_1 \supseteq K$ be an infinite extension; consider the free algebra $K_1\langle h_i | i \in I \rangle$ and its universal field of fractions Δ_1. Clearly $\Delta_1 \supseteq \Delta$. Let $\phi^{(1)}$ be a homomorphism of $K_1\langle h_i | i \in I \rangle$ into a K_1-skew subfield of $D^{(1)}$ of index p, and let $\theta^{(1)}$ be a specialization from Δ_1 in $D^{(1)}$ which extends $\phi^{(1)}$ and

$$x_j, x_j^{-1} \in T^{(1)} \qquad (j = 1, 2, \ldots, m),$$

where $T^{(1)}$ is the domain of $\theta^{(1)}$. Then we define ϕ as the restriction of $\phi^{(1)}$ on $K\langle h_i | i \in I \rangle$ and θ as the restriction of $\theta^{(1)}$ on Δ. We also see that ϕ and θ satisfy all the conclusions of the assertion. This completes the proof.

2.2. Now let L be an arbitrary Lie algebra over a commutative field K, $U(L)$ be its universal envelope, and $D(L)$ be the skew field constructed by Cohn for the embedding of $U(L)$. We will need here another method for constructing $D(L)$; this method was developed in Lichtman [10]. We will now outline this method.

Let e_j $(j \in J)$ be an ordered K-basis of the Lie algebra L and U^i $(i = 0, 1, \ldots)$ be the canonical filtration of the universal enveloping algebra (see Jacobson [5, V.3); it is known that this filtration defines a discrete valuation function $v(x)$ in $U(L)$ by the rule

$$v(0) = \infty,\ v(x) = -i \qquad \text{if } x \in U^i \setminus U^{i-1}. \tag{2.4}$$

(See, for instance, Cohn [3, Section 2.6].) Let $U(L)[t]$ be the polynomial ring over $U(L)$; the valuation function $v(x)$ can be extended to this polynomial ring in such a way that $v(t) = 1$ and it is extended then to the ring of fractions of $U(L)[t]$ with respect to the central subsemigroup generated by the element t. Let R be the valuation ring of this ring of fractions; i.e., R is the subset of elements with non-negative values. It is easy to see that the valuation $v(x)$ of R is now a t-adic valuation defined by the powers of the ideal t. The following fact is Proposition 3.3 in Lichtman [10].

PROPOSITION 2.2. *The quotient ring $R/(t)$ is isomorphic to the polynomial ring $K[\bar{t}_j]$ where $\bar{t}_j$ is the image of the element $t_j = e_j t$ $(j \in J)$.*

The following fact follows from the results in Section 2.1 of [10].

PROPOSITION 2.3. *The ring R is embedded in a skew field D and the valuation function $v(x)$ is extended to D. Let $S = \{s \in D | v(s) \geq 0\}$. Then S is a complete local ring with radical tS and S/tS is isomorphic to the field of rational functions $K(\bar{t}_j)$ $(j \in J)$ and D is the ring of fractions of S with respect to the central subsemigroup E generated by the element t. Further, if $\overline{X}$ is a system of coset representatives of S/tS then an arbitrary element $s \in S$ has a unique representation*

$$s = \sum_{i=0}^{\infty} \alpha_i t^i \qquad \left(\alpha_i \in \overline{X},\ i = 0, 1, \dots\right). \tag{2.5}$$

Now let L_1 be a Lie subalgebra of L. We denote by R_1 the subset of all the elements from R which have a representation of the form $x = ut^k$ with $u \in U(L_1)$. We select now an arbitrary ordered basis e_j $(j \in J_1)$ of L_1 and then an ordered system of elements e_j $(j \in J_2)$ which gives a basis of the vector space L/L_1. Then the system of elements e_j $(j \in J = J_1 \cup J_2)$ is an ordered basis of L (we assume that the elements of J_1 preceed the elements of J_2). We obtain now the following easy corollary of Propositions 2.2 and 2.3.

PROPOSITION 2.4. *Let S_1 be the complete subring of S generated by R_1. Then S_1 is a complete local subring of S with radical $tS_1 = S_1 \cap (tS)$ and S_1/tS_1 is isomorphic to the field of rational functions $K(\bar{t}_j | j \in J_1)$. The ring of fractions of S_1 with respect to the central subsemigroup E is a skew subfield D_1 of D. Further, an element s from S belongs to S_1 iff the coefficients α_i in its representation* (2.2) *are taken from the subsystem $X_1 \subseteq X$ of representatives of the commutative subfield $K(\bar{t}_j | j \in J_1)$.*

Now let D be the skew field obtained in Proposition 2.3. We recall that the skew subfield of D generated by the subalgebra $U(L)$ is isomorphic to the skew field $D(L)$ constructed by Cohn. We will need the following fact in the proof of our main result.

PROPOSITION 2.5. *Let L be an arbitrary Lie algebra and L_1 be its subalgebra. Then*

(i) *the skew subfield generated by L_1 in $D(L)$ is isomorphic to $D(L_1)$;*

(ii) *the standard monomials π_i $(i \in I)$ on the set e_j $(j \in J_2)$ together with 1 are linearly independent over the skew subfield $D(L_1)$.*

Proof. We consider the ring R, which was obtained as the valuation ring of the ring of fractions of the algebra $U(L)[t]$, and then define a ring R_1 as the valuation ring in the ring of fractions of the algebra $U(L_1)$ with respect to the subsemigroup E. Let D be the skew field obtained from R in Proposition 2.3 and S be its valuation ring. We see now from Proposition 2.4 that the corresponding rings which are constructed for R_1 will be isomorphic to the skew subfield D_1 and its local subring S_1. Hence the skew subfield generated in D by $U(L_1)$ is isomorphic to $D(L_1)$. The first statement of the proposition now follows from the inclusion $U(L_1) \subseteq U(L) \subseteq D(L) \subseteq D$.

To prove the second statement we observe that $D(L_1) \subseteq D_1$, hence it is enough to prove the linear independence of these elements over D_1. Further, if π is a given monomial which has a form

$$\pi = e_{j_1}^{k_1} e_{j_2}^{k_2} \cdots e_{j_n}^{k_n},$$

then we can represent this monomial in the form

$$\pi = t_{j_1}^{k_1} t_{j_2}^{k_2} \cdots t_{j_n}^{k_n} t^{-(k_1+k_2+\cdots+k_n)},$$

where $t_j = e_j t$ $(j \in J_2)$. Since D_1 is the ring of fractions of S_1 with respect to the central subsemigroup generated by t it follows easily that it is enough to prove the linear independence over S_1 of monomials of the form

$$\pi' = t_{j_1}^{k_1} t_{j_2}^{k_2} \cdots t_{j_n}^{k_n}.$$

Assume that there exist non-zero elements $\lambda_0, \lambda_1, \ldots, \lambda_m$ in S_1 such that

$$\lambda_0 + \lambda_1 \pi_1' + \cdots + \lambda_m \pi_m' = 0 \tag{2.6}$$

or

$$\lambda_1 \pi_1' + \lambda_2 \pi_2' + \cdots + \lambda_m \pi_m' = 0. \tag{2.7}$$

We will consider the case of linear dependence of the form (2.6) since the proof in the second case is the same. We can assume that not all of the coefficients λ_i $(i = 0, 1, \dots, m)$ in the relation (2.3) belong to the ideal tS_1. We reduce now the relation (2.6) modulo the ideal (tS_1) and obtain that the images of the monomials π_i' $(i = 1, 2, \dots, m)$ together with 1 are linearly dependent over the subfield $K(\bar{t}_j | j \in J_1)$. Since these monomials contain only elements from J_2 we obtain a contradiction with the algebraic independence of elements $\bar{t}_j$ $(j \in J)$. This completes the proof.

2.3. Let R be an algebra without zero divisors. Assume that there exists a skew field $\operatorname{div}(R)$ which contains R and is generated by it. We define in $\operatorname{div}(R)$ the following system of subalgebras. We define $R_1 = R$. If for a natural number k the subalgebra R_k is already defined then R_{k+1} is defined as the subalgebra generated by all the non-zero elements of R_k and their inverses. We see that

$$R_1 \subseteq R_2 \subseteq \cdots$$

and

$$\operatorname{div}(R) = \bigcup_{k=1}^{\infty} R_k.$$

We will use this system of subalgebras in the proof of Lemma 2.1.

LEMMA 2.1. *Let L be a Lie algebra over a commutative field K. Let* $\operatorname{div}(L)$ *be a skew field which contains the universal enveloping algebra $U(L)$ and is generated by it, let L_1 be an ideal in L, and let* $\operatorname{div}(L_1)$ *be a skew subfield generated by the subalgebra $U(L_1)$. Then the subalgebra* $\operatorname{div}(L_1)$ *is L-invariant, i.e.,*

$$[x, u] \in \operatorname{div}(L_1) \tag{2.8}$$

for every $x \in \operatorname{div}(L_1)$, $u \in L$.

Proof. Clearly the relation (2.6) holds if $x \in R$, where R is the subalgebra generated by L_1. Consider the system of subalgebras R_k, $k = 1, 2, \dots,$ which was defined in this subsection. We have for every non-zero $x \in \operatorname{div}(L)$ and every $u \in \operatorname{div}(L)$ that

$$[x^{-1}, u] = -x^{-1}[x, u]x^{-1}, \tag{2.9}$$

and we obtain by induction that every R_k $(k = 1, 2, \dots)$ is L-invariant. Hence $\operatorname{div}(L_1)$ is L-invariant. This completes the proof.

COROLLARY 1. *Assume that the conditions of Lemma* 2.1 *hold. Let e_j $(j \in J)$ be a system of elements of L which gives a basis of the quotient* *algebra L/L_1, and let π_i $(i \in I)$ be the corresponding set of the standard monomials. Then the subring generated in* $\operatorname{div}(L)$ *by* $\operatorname{div}(L_1)$ *and L is isomorphic to a suitable smash product* $\operatorname{div}(L_1)\#U(L/L_1)$. *Further, if the ideal L_1 has a finite codimension then* $\operatorname{div}(L_1)\#U(L/L_1)$ *is a Noetherian ring and $D(L)$ is isomorphic to its skew field of fractions.*

Proof. The proof of the first statement is obtained from Lemma 2.1 and Proposition 2.5 in a routine way. The second statement is a known fact on smash products (see, for instance, McConnell and Robson [15, 1.7.14]).

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PROPOSITION 3.1. *Let R be an Ore domain and $R\#U(L)$ be a smash product of R with an arbitrary Lie algebra L. Then $R\#U(L)$ is imbedded in a skew field D.*

Proof. The ring $R\#U(L)$ has a filtration whose associated graded ring is a differential polynomial ring over R (see Lichtman [13, Lemma 3.3]). Hence this graded ring is a right Ore domain and we obtain from Cohn's Theorem that $R\#U(L)$ can be embedded in a skew field.

COROLLARY 1. *Let $R\#U(L)$ be a smash product of an Ore domain R with an arbitrary Lie algebra L. Then R is a right denominator set in $R\#U(L)$ and the ring of fractions of $R\#U(L)$ with respect to R is isomorphic to a suitable smash product $(Q(R))\#U(L)$, where $Q(R)$ is the skew field of fractions of R.*

Proof. Proposition 3.1 implies that $R\#U(L)$ can be embedded in a skew field D. Now the skew subfield generated by R is isomorphic to the ring of fractions of R; we denote it by $Q(R)$. Lemma 2.1 implies that $Q(R)$ is L-invariant. Let e_i $(i \in I)$ be a basis of L; then the standard monomials in these elements form a basis of $R\#U(L)$ over R and we obtain that these standard monomials are linearly independent over $Q(R)$. It follows easily from this that they generate over $Q(R)$ a smash product $(Q(R))\#U(L)$ and the assertion follows.

COROLLARY 2. *Let R be an Ore domain and L be a soluble Lie algebra. Then any smash product $R\#U(L)$ is a right Ore domain.*

Proof. An induction argument reduces the proof to the case when L is abelian. We can assume also that it is finitely generated and hence is finite dimensional. It is well known that in this case the smash product of $U(L)$ over a skew field $Q(R)$ is a Noetherian domain. Hence $Q(R)\#U(L)$ is a right Ore domain. This implies easily that $R\#U(L)$ is a right Ore domain.

LEMMA 3.1. *Let H be a free Lie algebra and N be an ideal in H such that the quotient algebra H/N is a finite-dimensional Lie algebra. Let $\Delta(U(H))$ be the universal field of fractions of $U(H)$ and* $\operatorname{div}(U(N))$ *be the skew subfield of $\Delta(U(H))$ generated by $U(N)$. Then the subalgebra* $\operatorname{div}\langle U(N), H\rangle$ *generated by* $\operatorname{div}(U(N))$ *and H is isomorphic to a suitable smash product* $\operatorname{div}(U(N))\#U(H/N)$.

Proof. Let h_i $(i \in I)$ be a system of elements of H which gives a basis of H/N. We have to prove that the standard monomials π_j $(j \in J)$ on the set of elements h_i $(i \in I)$ together with element 1 are linearly independent over $\operatorname{div}(U(N))$.

Assume that there exist non-zero elements

$$\lambda_\alpha \in \operatorname{div}(U(N)) \qquad (\alpha = 1, 2, \ldots, m)$$

such that

$$\sum_{\alpha=1}^{m} \lambda_{\alpha\alpha} = 0. \tag{3.1}$$

Proposition 2.1 implies that there exists an epimorphism φ: $U(H) \to \overline{U(H)}$ such that $\overline{U(H)}$ is a PI-domain with a skew field of fractions $\operatorname{div}(\overline{U(H)})$ which either has a prime index p or is commutative. The epimorphism φ is extended to a specialization θ: $T \to \operatorname{div}(\overline{U(H)})$, where $U(H) \subseteq T \subseteq \operatorname{div}(U(H))$,

$$\lambda_\alpha \in T \qquad (\alpha = 1, 2, \ldots, m),$$

and $\theta(\lambda_\alpha) \neq 0$. Let $S = T \cap \operatorname{div}(U(N))$. Then $S \supseteq U(N)$ because $T \supseteq U(H) \supseteq U(N)$ and $\operatorname{div}(U(N)) \supseteq U(N)$. Since $U(N)$ is H-invariant, Lemma 2.1 implies that $\operatorname{div}(U(N))$ is H-invariant; also T is H-invariant because $T \supseteq U(H)$. Further, S is H-invariant because $\operatorname{div}(U(N))$ and T are H-invariant, and the ring $\overline{U(N)} = \phi(U(N))$ is an Ore domain because it is a PI-domain. The corollary of Proposition 3.1 implies that the ring $\overline{U(N)}\#U(H/N)$ has a skew field of fractions, say $\operatorname{div}(\overline{U(N)}\#U(H/N))$.

Let $A = \ker(\varphi) \cap U(N)$. Then the ideal $A \subseteq U(N)$ generates in $U(H)$ an ideal $A\#U(H/N)$ with the quotient ring

$$\begin{aligned} U(H)/(A)U(H) &\simeq (U(N)\#(U(H/N)))/(A\#U(H/N)) \\ &\simeq (U(N)/A)\#U(H/N) \simeq \overline{U(N)}\#U(H/N). \end{aligned}$$

The epimorphism ψ: $U(H) \to U(H)/A$ can be extended to a specialization π_1: $\operatorname{div}(U(H)) \to \operatorname{div}(\overline{U(N)}\#U(H/N))$. Let Q be the domain of π_1. Clearly, $Q \supseteq U(H)$ and $Q \cap \operatorname{div}(U(N)) = S$, $J(Q) \cap U(N) = A$, i.e., the restriction of π_1 on $\operatorname{div}(U(N))$ coincides with the specialization π and the restriction of φ on $U(N)$ is the homomorphism $U(N) \to U(N)/A \simeq \overline{U(N)}$.

Now consider the relation (3.1) in the subring $Q \subseteq \operatorname{div}(U(H))$. We obtain from it the relation in $Q/J(Q) \simeq \operatorname{div}(\overline{U(N)} \# U(H/N))$,

$$\sum_{\alpha=1}^{m} \bar{\lambda}_\alpha \overline{\pi_\alpha} = 0, \tag{3.2}$$

where $\bar{x}$ denotes the image of an element $x \in Q$ under the homomorphism $\pi_1\colon Q \to Q/J(Q)$. We recall also that $\bar{\lambda}_\alpha = \varphi(\lambda_\alpha) \neq 0$ $(\alpha = 1, 2, \dots, m)$.

We see now from (3.2) that the elements $\overline{\pi_\alpha}$ $(\alpha = 1, 2, \dots, m)$ are linearly dependent in $\operatorname{div}(\overline{U(N)} \# U(H/N))$ over $\operatorname{div}(\overline{U(N)})$; since $\operatorname{div}(\overline{U(N)})$ is the field of fractions of $\overline{U(N)}$ they must be linearly dependent over $\overline{U(N)}$ in the ring $\overline{U(N)} \# U(H/N)$. This contradiction shows that the relation (3.1) is impossible and the proof is complete.

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Let D be an arbitrary skew field and $D[u, v]$ be a differential polynomial ring in variables u, v such that $[D, u] \subseteq D$, $[D, v] \subseteq D$, and $[u, v] \in D$. Let R be a subring of D such that $[R, u] \subseteq R$, $[R, v] \subseteq R$ and $[u, v] \in R$. We see that $D[u, v]$ is a smash product of D with an abelian algebra of dimension 2 and it contains the smash product $R[u, v]$. Let $D(u)$ be the skew field of fractions of $D[u]$. It is well known (see Cohn [3, Section 2.1.2]) that the derivation d of $D[u]$ defined by the map $x \to [x, v]$ can be uniquely extended to a derivation of $D(u)$ and in particular $d(u^{-1}) = -u^{-1}d(u)u^{-1}$; we will denote the extended derivation also by d. Let $z = u^{-1}$; one can form now the differential Taylor power series ring $D[[z]]$ and the Laurent power series skew field $D((z))$; since $[R, u] \subseteq R$ we obtain that R and z topologically generate in $D((z))$ a subring $R((z))$.

LEMMA 4.1. *Let $z = u^{-1}$ and r be an arbitrary element of D. Then for an arbitrary natural n the element $z^n r$ has the representation in $D[[z]]$,*

$$z^n r = \left(\sum_{i=0}^{\infty} c_i z^i \right) z^n \qquad (c_i \in D;\ i = 0, 1, \dots). \tag{4.1}$$

If $r \in R$ then all the coefficients c_i $(i = 0, 1, \dots)$ in (4.1) belong to R.

Proof. We have

$$zr = (zr - rz) + rz = (u^{-1}r - ru^{-1}) + rz = u^{-1}r'u^{-1} + rz = zr'z + rz \tag{4.2}$$

where

$$r' = [r, u] = (ru - ur) \in D.$$

It is not difficult to obtain from (4.2) (see [3, Section 2.3]) that

$$zr = rz + r'z^2 + r''z^3 + \cdots = (r + r'z + r''z^2 + \cdots)z, \tag{4.3}$$

where

$$r^{(n)} = [r, \underbrace{u, u, \dots, u}_{n\text{-times}}].$$

Now let $n > 1$. Assume that it has already been proved that

$$z^{n-1}r = \left(\sum_{i=0}^{\infty} \alpha_i z^i\right) z^{n-1} \qquad (\alpha_i \in D;\ i = 0, 1, \dots). \tag{4.4}$$

Then

$$z^n r = z(z^{n-1}r) = z\left(\sum_{i=0}^{\infty} \alpha_i z^i\right) z^{n-1} \tag{4.5}$$

and by (4.3)

$$z(\alpha_i z^i) = \left(\sum_{k=0}^{\infty} \alpha_i^{(k)} z^k\right) z^{i+1} \qquad (\alpha_i^{(k)} \in D;\ k = 0, 1, \dots;\ i = 0, 1, \dots), \tag{4.6}$$

where

$$\alpha_i^{(k)} = [\alpha_i, \underbrace{u, u, \dots, u}_{k\text{-times}}]$$

It is important that if we substitute all of the series (4.6) instead of the corresponding terms of the series $z(\sum_{i=0}^{\infty} \alpha_i z^i)z^{n-1}$ in the right side of (4.5) we would obtain a well-defined element of $D[\![z]\!]$; this follows from the fact that the coefficient of every power of z is a sum of a finite number of elements of D. We obtain now (4.1) from (4.4), (4.5), and (4.6) and it is easy to see that if $r \in R$ then all the coefficients $\alpha_i, \alpha_i^{(k)}$ $(i = 0, 1, \dots;$ $k = 0, 1, \dots)$ in (4.4)–(4.6) belong to R.

LEMMA 4.2. *The derivation d of $D[u]$ defined by the rule $d(x) = [x, v]$, $x \in D[u]$, can be extended to a derivation of $D((z))$ and the subring $R((z))$ is invariant with respect to this derivation.*

Proof. We have already observed that the derivation d can be extended to a derivation of the skew field $D(u)$, which is isomorphic to $D(z)$. Now pick an arbitrary positive power z^m and consider the element $d(z^m)$. We will now prove that this element has the representation in $R((z))$,

$$d(z^m) = \left(\sum_{i=0}^{\infty} c_i z^i\right) z^m \qquad (c_i \in R;\ i = 0, 1, \dots), \tag{4.7}$$

for a suitable element $(\sum_{i=0}^{\infty} c_i z^i) \in R[\![z]\!]$.

If $m = 1$ then

$$d(z) = [z, v] = [u^{-1}, v] = -u^{-1}[u, v]u^{-1} = -zrz$$

where $r = [u, v] \in R$, and (4.7) follows now from Lemma 4.1. If $m > 1$ then

$$d(z^m) = d(z^{m-1})z + z^{m-1}d(z)$$

and (4.7) now follows from the last relation via Lemma 4.1 by an induction argument.

We define now for an arbitrary element $\sum_{i=0}^{\infty} a_i z^i$ $(a_i \in D;\ i = 0, 1, \dots)$

$$d\left(\sum_{i=0}^{\infty} a_i z^i\right) = \sum_{i=0}^{\infty} d\left(a_i z^i\right) = \sum_{i=0}^{\infty} \left[\left(d(a_i)z^i + a_i d(z^i)\right)\right]. \tag{4.8}$$

Since the second series in the right side of (4.8) converges via (4.7) we see that the map is well defined; clearly, it is linear. To prove for arbitrary $x, y \in D[\![z]\!]$ the relation

$$d(xy) = d(x)y + xd(y) \tag{4.9}$$

we observe that (4.9) holds if x and y are elements from the polynomial ring $D[z]$. This follows from the fact that $D[z] \subseteq D(u)$ and d has already been extended to $D(u)$.

If now x and y are arbitrary elements of $D[\![z]\!]$ then to prove (4.9) it is enough to verify that for arbitrary m the coefficients of z^m on the left side and on the right side of (4.9) are equal. Let

$$x = \sum_{i=1}^{\infty} \alpha_i z^i \tag{4.10}$$

$$y = \sum_{i=1}^{\infty} \beta_i z^i \tag{4.11}$$

$$xy = \sum_{i=1}^{\infty} \gamma_i z^i . \tag{4.12}$$

The relations (4.7) and (4.8) together with Lemma 4.1 imply that z^m can occur in the left side of (4.9) only from the terms $d(\gamma_k z^k)$ with $k \le m$; once again, we obtain from (4.7), (4.8), and (4.1) that the term z^m on the right side of (4.9) can come only from the products $d(a_r z^r)(\beta_s z^s)$ or $(a_r z^r) d(\beta_s z^s)$ with $(r+s) \le m$. We take now arbitrary partial sums for x and y which include the term x^m. Let

$$x_1 = \sum_{i=0}^{m_1} \alpha_i z^i ,\ y_1 = \sum_{i=0}^{m_1} \beta_i z^i \qquad (m_1 \ge m).$$

Since the elements x_1 and y_1 are polynomials in z we obtain

$$d(x_1 y_1) = d(x_1) y_1 + x_1 d(y_1) \tag{4.13}$$

and we see that the coefficients of z^m on the left side of (4.13) and on the right side are equal. This implies that z^m occurs on the left and right sides of (4.9) with the same coefficients and we proved that d is extended to $D[\![z]\!]$. Further, d can now be uniquely extended to $D((z))$ because $D((z))$ is the Ore ring of fractions of $D[\![z]\!]$ with respect to the subset of elements $\{1, z, z^2, \dots\}$ (see [3, Section 2.3]).

We see also from (4.8) that $d(R[\![z]\!]) \subseteq R[\![z]\!]$ because $d(R) \subseteq R$ and $d(z) \subseteq R[\![z]\!]$ and this implies easily that $R((z))$ is also d-invariant. This completes the proof.

We will summarize now the results of this section. Let $D[u, v]$ be a differential polynomial ring, where $D[u] \subseteq D$, $D[v] \subseteq D$, and $[u, v] \in D$. Let R be a subring of D such that $[R, u] \subseteq R$, $[R, v] \subseteq R$, and $[u, v] \in R$. We obtain a natural imbedding of the differential polynomial ring $R[u, v]$ in $D[u, v]$. Let $z = u^{-1}$. Lemma 4.2 implies that v defines a derivation in $D((z))$ and $R((z))$ is d-invariant. We can now consider a differential Laurent power series ring in the variable $w = v^{-1}$ with coefficients from $D((z))$. We will denote this ring by $D((z, w))$; it is important to remember that this is an iterated Laurent power series ring, i.e., we take first the ring $D((z))$ and then the Laurent power series ring in w over $D((z))$. The ring

$D((z, w))$ contains a subring $R((z, w))$. Further since D is a skew field we obtain from Theorem 2.3.1. in Cohn [3] that the ring $D((z))$ is a skew field, and hence $D((z, w))$ is also a skew field.

5

We will now apply the same method for constructing differential Laurent power series rings in a more general situation. Let $R\#U(L)$ be a smash product of a domain R with nilpotent Lie algebra L. We assume that this ring is a subring of a smash product $D\#U(L)$ where D is a skew field and the embedding $(R\#U(L)) \subseteq (D\#U(L))$ is a natural extension of the embedding $R \subseteq D$. Let $L = L_1 \supseteq L_2 \supseteq \cdots \supseteq L_{k-1} \supseteq L_k = 0$ be a central series in L. For every given $1 \leq i \leq (k-1)$ we pick in L_i ($i = 1, 2, \ldots, k-1$) a system of elements E_i which gives a basis of E_i modulo E_{i+1}. Then $E = \bigcup_{i=1}^{k-1} E_i$ is a basis of L. We order every basis E_i ($i = 1, 2, \ldots, k-1$) in an arbitrary way and extend this order to E assuming that the elements of E_{i+1} are greater than the elements of E_i.
If $u \in E_{k-1}$ and v is an arbitrary element of E then $[v, u] \in D$. This implies via Lemma 4.2 that the inner derivation d_v of $D\#U(L)$ defined by the element v can be extended to a derivation of the skew Laurent power series field $D((z))$ where $z = u^{-1}$ and the subring $R((z)) \subseteq D((z))$ is d_v-invariant. If now $u_1, u_2, \ldots, u_n$ are arbitrary elements of E_{k-1}, then they generate a differential polynomial ring $D[u_1, u_2, \ldots, u_n]$ and we can define the iterated Laurent power series field $D((z_1, z_2, \ldots, z_n))$ where $z_j = u_j^{-1}$ ($j = 1, 2, \ldots, n$); clearly it contains $D[u_1, u_2, \ldots, u_n]$ and Lemma 4.2 implies once again that d_v can be extended to $D((z_1, z_2, \ldots, z_n))$.
Now let u_j ($j \in J$) be all the elements of E_{k-1}. We use a transfinite induction and obtain an embedding of the smash product $D\#U(L_{k-1})$ into the skew field $D((z_j \mid j \in J))$ where $z_j = u_j^{-1}$ ($j \in J$). Clearly we obtain simultaneously an imbedding of $R\#U(L_{k-1})$ in the ring $R((z_j \mid j \in J))$, the derivation d_v can be extended to $D((z_j \mid j \in J))$ and the subring $R((z_j \mid j \in J))$ is d_v-invariant. We denote $D((z_j \mid j \in J)) = D((L_{k-1}))$ and $R((z_j \mid j \in J)) = R((L_{k-1}))$.
The smash product $D\#U(L)$ is isomorphic to a suitable smash product of the ring $D\#U(L_{k-1})$ with the Lie algebra L/L_{k-1}. Then this smash product is naturally embedded into a suitable smash product of the skew field $D_1 = D((L_{k-1}))$ with the Lie algebra L/L_{k-1} which has nilpotency class $k-1$ and the induction by k now yields that the ring $D\#U(L)$ can be embedded in the ring $D_1((L/L_{k-1}))$ and $R\#U(L)$ is simultaneously embedded in the ring $R_1((L/L_{k-1}))$ where $R_1 = R((L_{k-1}))$; we denote these rings by $D((L))$ and $R((L))$ correspondingly. It is worth remarking

that in fact the skew field $D((L))$ was constructed in k steps: we define $D_0 = D$, then $D_1 = D((L_{k-1}))$, then $D_2 = D_1((L_{k-2}/L_{k-1})), \dots, D((L)) = D_{k-1}((L_1/L_2))$. Further, if an ordered basis E_i is chosen then every element of D_i has a unique representation as an iterated Laurent power series in the variables z_j $(j \in J)$ over the skew subfield D_{i-1} and hence if the ordered basis E is chosen then every element $x \in D(L)$ has a unique representation as an iterated Laurent power series over D. The properties of the rings $D((L))$ and $R((L))$ depend on the choice of the central series and the basis in L but we always have a natural isomorphic embedding of $R\#U(L)$ into $R((L))$.

We have obtained the following lemma.

LEMMA 5.1. *Let L be a nilpotent Lie algebra over a field K, $R\#U(L)$ be smash product of L with a domain R. Assume that R is embedded in a skew field D and this embedding is extended to an embedding $(R\#U(L)) \subseteq (D\#U(L))$. Then the smash product $R\#(U(L))$ can be embedded into a differential Laurent power series ring $R((L))$.*

Now let x be an element of $R((L))$. If L is one dimensional then $x \in R((z))$ and we denote by $\alpha(x)$ the coefficient of the term with the lowest degree of z. We will call it the first coefficient of x. If now L has a finite dimension greater than 1 then $R((L)) = R((z_1, z_2, \dots, z_n))$ and $\alpha_1(x)$ is the first coefficient of x as a power series in z_n with coefficients from $R((z_1, z_2, \dots, z_{n-1}))$, then we define $\alpha_2(x)$ as the first coefficient of $\alpha_1(x)$ in its representation as a power series in z_n with coefficient over $R((z_1, z_2, \dots, z_{n-1}))$ and after n steps we obtain an element $\alpha_n(x) \in R$ which we will call the first coefficient of x and denote by $\alpha(x)$. If L is abelian and has infinite dimension and we have an ordered basis E in it then we can find elements $z_{j_\alpha} = e_{j_\alpha}^{-1}$ $(\alpha = 1, 2, \dots, n)$ such that $x \in R((z_{j_1}, z_{j_2}, \dots, z_{j_n}))$ and x is represented uniquely as a Laurent power series in these elements. The first coefficient $\alpha(x) \in R$ of the element $x \in R(L)$ is now defined as the first coefficient of x in $R((z_{j_1}, z_{j_2}, \dots, z_{j_n}))$. It follows immediately that $\alpha(x)$ does not depend on the finitely generated subset $\{z_{j_1}, z_{j_2}, \dots, z_{j_n}\}$ of the set z_j $(j \in J)$ and we see that the function $\alpha(x)$ is uniquely defined for every element $x \in R((L))$. If now L is nilpotent of class greater than 1 then we define first $\alpha(x)$ for elements of $R((L_{(k-1)}))$; after this we define it in the ring $R((L_{(k-1)}))((L_{(k-2)}/L_{(k-1)}))$ and after $k-1$ steps we obtain the first coefficient $\alpha(x)$ of the element $x \in R((L))$.

LEMMA 5.2. *Let L be a nilpotent Lie algebra and $x \in R((L))$. The element x is a unit in $R((L))$ if its first coefficient $\alpha(x)$ is a unit in R.*

Proof. Assume first that L is one-dimensional and hence $x \in R((z))$. (This case is considered in Cohn [3, Section 2.3].) Let

$$x = \sum_{i=k}^{\infty} \alpha_i z^i \qquad (\alpha_i \in R;\ i = k, k+1, \dots),$$

with $\alpha(x) = \alpha_i$ a unit in R. Then the element $y = \alpha^{-1} x z^{-k}$ has a form $y = 1 + y_1$, where

$$y_1 = \sum_{j=0}^{\infty} b_j z^j \qquad \big(b_j \in R;\ j = 0, 1, \dots\big).$$

The element y is invertible and its inverse is

$$y^{-1} = 1 - y_1 + y_1^2 + \cdots \tag{5.1}$$

because the series in the right side of (5.1) represents an element of $R((z))$; this follows easily from Section 4. (See [3, Section 2.3] for details.) We see also that if $\alpha(x)$ is invertible then the element x is invertible; its inverse is $z^k y^{-1} \alpha^{-1}$. Now let L be an abelian algebra. We can assume once again as above that $x \in R((z_{i_1}, z_{i_2}, \dots, z_{i_n}))$. We recall that $\alpha(x) = \alpha_n(x)$ and see that the truth of the assertion in the one-dimensional case implies that $\alpha_{n-1}(x)$ is invertible in $R((z_{i_1}))$, and after n steps we obtain that x is invertible in $R((z_{i_1}, z_{i_2}, \dots, z_{i_n}))$. This completes the proof in the case when L is abelian. A similar argument can be used now to obtain the proof in the general case when L is nilpotent.

COROLLARY. *Let L be a nilpotent Lie algebra and R be a skew field. Then $R((L))$ is a skew field.*

Remark. It is not difficult to verify that the sufficient condition on $\alpha(x)$ in Lemma 4.2 is also necessary for the invertibility of x; we will not need this fact in our proofs.

6

Throughout this section let H be a graded Lie algebra: H is a direct sum of subspaces H_i $(i = 1, 2, \dots)$, and if $x \in H_{i_1}$, $y \in H_{i_2}$ then $[x, y] \in H_{i_1 + i_2}$.

Let E_i be an arbitrary well ordered basis of H_i and let $H^{(k)} = \sum_{i=k}^{\infty} H_i$. Then $H^{(k)}$ is an ideal in H and $E^{(k)} = \bigcup_{i=1}^{k} E_i$ is a basis of the quotient algebra $H/H^{(k)} \simeq H_1 + H_2 + \cdots + H_k$. The set $E = \bigcup_{i=1}^{\infty} E_i$ is a basis of H. We extend first the orders of E_i $(i = 1, 2, \dots)$ on $E^{(i)}$, assuming that the elements of E_{i_1} are greater than the elements of E_{i_2} if $i_1 < i_2$. We recall that the standard monomials on the set $E^{(k)}$ form a basis of $U(H)$ over $U(H^{(k)})$ (see Jacobson [5]) and hence $U(H) \simeq U(H^{(k)})\#U(H/H^{(k)})$. Further, Lemma 5.1 implies that $U(H)$ can be imbedded in $U(H^{(k)})((H/H^{(k)}))$ for every natural number k. We observe also that

$$U(H^{(k)})\big((H/(H^{(k)})\big) \subseteq U(H^l)\big((H/H^l)\big) \tag{6.1}$$

if $k < l$ because $E^{(k)} \subseteq E^{(l)}$.

THEOREM 2. *Let H be a graded Lie algebra. Then the ring*

$$R = \bigcup_{j=1}^{\infty} U(H^{(j)})\big((H/H^{(j)})\big) \tag{6.2}$$

is a skew field which contains $U(H)$.

We need first the following lemma.

LEMMA 6.1. *Let $0 \neq x \in U(H)$. Then there exists an m such that the embedding $\psi: U(H) \to U(H^{(m)})((H/H^{(m)}))$ maps x on an invertible element $\psi(x) \in U(H^{(m)})((H/H^{(m)}))$.*

Proof. Let $\pi_1, \pi_2, \ldots, \pi_r$ be all of the standard monomials which occur in the representation of x and let $e_1, e_2, \ldots, e_s$ be all of the basic elements from E which occur in the monomials π_α $(\alpha = 1, 2, \ldots, r)$. Find m such that $e_\beta \in E^{(m)} = E_1 \cup E_2 \cup \cdots \cup E_m$ $(\beta = 1, 2, \ldots, s)$. We consider x as an element of $U(H^{(m)})((H/H^{(m)}))$, and it is easy to see that its first element belongs to the field K and is non-zero. Lemma 5.2 now implies that x is invertible in $U(H^{(m)})((H/H^{(m)}))$.

Proof of Theorem 2. Let $0 \neq x \in R$. We will now prove that x is invertible. Find n such that $x \in U(H^{(n)})((H/H^{(n)}))$ and let $x_1 \in U(H^{(n)})$ be the first coefficient of x. Now apply Lemma 6.1 and find m such that x_1 is invertible in $U(H^{(m)})((H/H^{(m)}))$; clearly, we can assume that $m \geq n$. Lemma 5.2 now implies that x is invertible in $U(H^{(m)})((H/H^{(m)}))$ and hence in R. This proves that R is a skew field. Lemma 5.1 implies that $U(H)$ is naturally imbedded in R. This completes the proof.

LEMMA 6.2. *Let H be an arbitrary Lie algebra and let* $\operatorname{div}(U(H))$ *be a skew field which contains $U(H)$ and is generated by it. Assume that for each k the standard monomials on the set $E^{(k)}$ remain linearly independent over the skew subfield* $\operatorname{div}(U(H^{(k)})$ *generated by $U(H^{(k)})$. Then the natural embedding $\phi: U(H) \cong U(H^{(k)})\#U(H/H^{(k)}) \to U(H^k)((H/H^{(k)}))$ can be extended to an embedding ψ:* $\operatorname{div}(U(H)) \to \operatorname{div}(U(H^{(k)}))((H/H^{(k)}))$.

Proof. The corollary of Lemma 2.1 implies that $\operatorname{div}(U(H))$ is isomorphic to a suitable smash product $\operatorname{div}(U(H^{(k)}))\#U(H/H^{(k)})$. The assertion now follows from Lemma 5.1.

We observe now that if the conditions of Lemma 6.2 hold, then similarly to (6.1),

$$\operatorname{div}\big(U(H^{(k)})\big)\big((H/H^{(k)})\big) \subseteq \operatorname{div}\big(U(H^{(l)})\big)\big((H/H^{(l)})\big) \qquad (6.3)$$

if $k < l$.

THEOREM 3. *Let H be a graded Lie algebra and let* $\operatorname{div}(U(H)$ *be a skew field which contains $U(H)$ and is generated by it. Assume that for each k the standard monomials on the set $E^{(k)}$ remain linearly independent over the skew subfield* $\operatorname{div}(U(H^{(k)})$ *generated by $U(H^{(k)})$. Let $S = \bigcup_{k=1}^{\infty} \operatorname{div}(U(H^{(k)}((H/H^{(k)}))$. Then S is a skew field and $U(H) \subseteq R \subseteq S$, where R is the skew field* (6.2).

Proof. The inclusion $R \subseteq S$ follows from the definition of R and S together with Lemma 5.1. The proof of the fact that S is a skew field is obtained by an easy modification of the argument that was applied in the proof of Theorem 2 and thus we omit it.

COROLLARY. *Let H be a graded Lie algebra, R be the skew field obtained in Theorem* 2, *and* $\operatorname{div}_R(U(H)$ *be the skew subfield of R generated by $U(H)$, and let* $\operatorname{div}(U(H))$ *be an arbitrary skew field which contains $U(H)$ and is generated by it and has the property that for every number k the standard monomials on the set $E^{(k)}$ remain linearly independent over the skew subfield* $\operatorname{div}(U(H^{(k)})$. *Then the skew field* $\operatorname{div}(U(H)$ *is isomorphic to* $\operatorname{div}_R(U(H))$.

Proof. The first statement follows immediately from Theorems 2 and 3. The second statement follows now from the fact that the skew field $D(H)$ satisfies the conditions of Theorem 2 (see Proposition 2.5) and hence $D(H)$ is also isomorphic to $\operatorname{div}_R(U(H))$.

Proof of Theorem 1. Let H be a free Lie algebra over a field K. We consider the natural grading of H: the ith homogeneous component H_i is the subspace generated by the Lie monomials of degree i and $\sum_{i=j}^{\infty} H_i$ is the jth term $\gamma_j(H)$ of the lower central series. Lemma 3.1 implies that the skew field $\Delta(U(H))$ satisfies the conditions of the corollary of Theorem 3 and Theorem 1 follows now from the corollary of Theorem 3.

7

Throughout this section let H be a residually torsion free nilpotent group, KH be the group ring of H over a field K. Let

$$H = H_1 \supseteq H_2 \supseteq \cdots \tag{7.1}$$

be a central series of normal subgroups in H with torsion free abelian factors H_i/H_{i+1} $(i = 1, 2, \dots)$ and $\bigcap_{i=1}^{\infty} H_i = 1$. It is known (see, for instance, Eizenbud and Lichtman [6], Section 4) that the groups H_i $(i = 1, 2, \dots)$ and H can be ordered in such a way that all the homomorphisms $H \to H/H_i$ and $H/H_{i+1} \to H/H_i$ are homomorphisms of ordered groups. We pick for an arbitrary term H_i a transversal X_i; we can assume that this transversal is ordered, that $X_{i+1} \supseteq X_i$ $(i = 1, 2, \dots)$ and the orderings in all the transversals X_i are coherent (see [6]). Let $K((H))$ be the Malcev–Neumann power series skew field; we recall that the elements of $K((H))$ are series over K with well-ordered support in H (see [3], Section 2.4 for the main properties of these skew fields).

For a given i let R_i be the subset of all the elements from $K((H))$ which have a representation of the form

$$x = \sum_{j \in J} \lambda_j x_j, \tag{7.2}$$

where λ_j $(j \in J)$ are elements from the group ring KH_i and the set x_j $(j \in J)$ is a well-ordered subset of X_i. It is clear that the group ring KH is naturally embedded in every ring R_i $(i \in I)$ and

$$R_1 \subseteq R_2 \subseteq \cdots.$$

PROPOSITION 7.1. *Let ρ be an ordering in H which is coherent with orderings in the quotient groups H/H_i $(i = 1, 2, \dots)$. Let*

$$R = \bigcup_{i=0}^{\infty} R_i.$$

Then R is a skew subfield of $K((H))$ and it contains $K(H)$.

We need first the following analogue of Lemma 6.1.

LEMMA 7.1. *Let u be a nonzero element of KH. Then u is invertible in R.*

Proof. The argument is similar to the one which was used in the proof of Lemma 5.2 and we will give only a sketch of the proof. Let $u = \sum_{\alpha=1}^{k} \mu_\alpha h_a$. We can find i such that the elements h_α $(\alpha = 1, 2, \dots)$ in the support of u belong to different cosets modulo H_i and hence the element

u has a representation

$$u = \sum_{\alpha=1}^{k} \lambda_\alpha x_{j_\alpha}, \tag{7.3}$$

where $\lambda_\alpha \in KH_i$ $(\alpha = 1, 2, \ldots, k)$ and x_{j_α} $(\alpha = 1, 2, \ldots, k)$ are the elements of the transversal X_i of H_i in H. We can assume that the smallest element x_{j_1} in (7.3) is equal to 1 and hence all the other elements x_{j_α} are greater than 1; this implies easily that we can represent u in the form $u = 1 + u_1$ and obtain that $u^{-1} = 1 - u_1 + u_1^2 - u_1^3 + \cdots$. This completes the proof of Lemma 7.1.

Proof of Proposition 7.1. We give only a sketch of the proof, since the argument is similar to the one which was used in the proof of Theorem 2. Let $0 \neq x \in R$. Then there exists $i \in I$ such that $x \in R_i$. Consider the representation (7.2) for the element x; let λ be the coefficient of the minimal element x_{j_0} in the support of x. It is enough to show that λ is invertible in R. Since $0 \neq \lambda \in KH$ this follows from Lemma 7.1.

The second statement is obvious.

Now let $\operatorname{div}(KH)$ be an arbitrary skew field generated by the group ring $K(H)$. Assume that for each given i the elements of the transversal X_i remain linearly independent over the skew subfield $\operatorname{div}(KH_i)$ generated by KH_i. It is easy to see that in this case the subring generated in $\operatorname{div}(KH)$ by $\operatorname{div}(KH_i)$ and KH is isomorphic to a suitable crossed product $\operatorname{div}(KH_i) * (H/H_i)$ and $\operatorname{div}(KH)$ is isomorphic to the skew field of fractions of this crossed product. We assume once again that the group H and all the quotient groups H/H_i $(i = 1, 2, \ldots)$ are ordered in a coherent way. We can consider now the skew Malcev–Neumann field $\operatorname{div}(KH_i) * ((H/H_i))$ and its subring $R_i = (KH_i) * ((H/H_i))$. The elements of this skew field are series with coefficients from $\operatorname{div}(KH_i)$ and support a well-ordered subset of the transversal X_i.

Clearly, for every $i \in I$ we have a natural embedding.

$$KH \subseteq (KH_i) * (H/H_i) \subseteq \operatorname{div}(KH_i) * (H/H_i) \subseteq \operatorname{div}(KH_i) * ((H/H_i)). \tag{7.4}$$

We have now analogs of Lemma 6.1 and of Theorem 3.

LEMMA 7.2. *Assume that for each given i the elements of the transversal* X_i *remain linearly independent over the skew subfield* $\operatorname{div}(KH_i)$. *Then the* *embedding* (7.4) *can be extended to an embedding*

$$\operatorname{div}(KH) \subseteq \operatorname{div}(KH_i) * ((H/H_i)). \tag{7.5}$$

Proof. We have already pointed out that $\operatorname{div}(KH)$ is isomorphic to the ring of fractions of $\operatorname{div}(KH_i) * (H/H_i)$. But the crossed product $\operatorname{div}(KH_i) * (H/H_i)$ is a subring of the skew field $\operatorname{div}(KH_i) * ((H/H_i))$; hence the skew field of fractions of this crossed product is a subring of $\operatorname{div}(KH_i) * ((H/H_i))$. This completes the proof.

Once again, as in (6.2), we have

$$\operatorname{div}(KH_{i_1}) * \left((H/H_{i_1})\right) \subseteq \operatorname{div}(KH_{i_2}) * \left((H/H_{i_2})\right), \tag{7.6}$$

if $i_1 < i_2$.

THEOREM 4. *Let R be as in Proposition* 7.1 *and* $\operatorname{div}(KH)$ *be an arbitrary skew field which contains KH and is generated by it. Assume that for each i the elements of the transversal X_i remain linearly independent over the skew subfield* $\operatorname{div}(KH_i)$. *Then* $\operatorname{div}(KH)$ *is isomorphic to the skew subfield* $\operatorname{div}_R(KH)$ *generated by the subring* $(KH) \subseteq R$.

Proof. We define in H the same ordering ρ as in Proposition 7.1. We obtain now from (7.4) and (7.5) the following embeddings:

$$\operatorname{div}(KH) \subseteq \left(\bigcup_{i=1}^{\infty} \operatorname{div}(KH_i) * ((H/H_i)) \right) \tag{7.7}$$

and

$$\begin{aligned} (KH) &\subseteq \left(\bigcup_{i=1}^{\infty} (KH_i) * ((H/H_i)) \right) \\ &\subseteq \left(\bigcup_{i=1}^{\infty} \operatorname{div}(KH_i) * ((H/H_i)) \right). \end{aligned} \tag{7.8}$$

We recall that $R_i = (KH_i) * ((H/H_i))$ and obtain from Proposition 7.1

$$(KH) \subseteq R \subseteq \left(\bigcup_{i=1}^{\infty} \operatorname{div}(KH_i) * ((H/H_i)) \right). \tag{7.9}$$

The relations (7.6) and (7.8) imply that $\operatorname{div}(KH)$ is the skew subfield of R generated by KH and the proof is complete.

THEOREM 5. *Let H be a residually torsion free nilpotent group and* (7.1) *be a given central series in H with torsion free abelian factors and unit* *intersection. Let* $H = N_1 \supseteq N_2 \supseteq \cdots$ *be an arbitrary central series in* H *with torsion free abelian factors and unit intersection,* τ *be an ordering in* H *which is coherent with orderings in the quotient groups* H/N_i $(i = 1, 2, \dots)$, *and* $K((H))_\tau$ *be the corresponding Malcev–Neumann skew field. Then the skew subfield* $\mathrm{div}(KH) \subseteq K((H))_\tau$ *generated by* KH *is isomorphic to the skew subfield* $\mathrm{div}_R(KH)$ *obtained in Theorem* 4 *and Proposition* 7.1. *Hence the skew subfield* $\mathrm{div}(KH)$ *does not depend on the choice of the central series in* H *and the ordering defined by this series.*

Proof. Let H_i be an arbitrary term of the series (7.1), X be its transversal in H. We consider the skew subfield $K((H_i))$ of $K((H))$ whose elements are series with support in H_i. Clearly the skew subfield $\mathrm{div}(KH_i)$ generated by KH_i is contained in $K((H_i))$. It is easy to see that the elements of X are left linearly independent over $K((H_i))$; hence they are linearly independent over $\mathrm{div}(KH_i)$ and the assertion now follows from Theorem 4.

Remark. The last statement of Theorem 5 coincides in fact with Theorem 6.2 in Eizenbud and Lichtman [6] and hence Theorem 5 yields a new proof of Theorem 6.2 in [6]. We point out that Proposition 3.1 in [6] is incorrect.

Now let F be a free group, N_i $(i = 1, 2, \dots)$ be a central series in F with torsion-free abelian factors and unit intersection, $K((F))$ be the corresponding Malcev–Neumann power series skew field.

We can now give a new proof of the following theorem of Lewin (see [8]).

THEOREM 6. *The skew subfield generated in* $K((F))$ *by the free group ring* KF *is isomorphic to the universal field of fractions* $\Delta(KF)$.

Proof. The theorem follows from Theorem 5 and the following analogue of Lemma 3.1.

LEMMA 7.4. *Let* $\Delta(KF)$ *be the universal field of fractions for the free group ring* KF, N *be a normal subgroup in* F *such that the quotient group is torsion free nilpotent,* X *be a transversal for* N *in* F. *Then the elements of* X *are left linearly independent over the skew subfield* $\mathrm{div}(KN)$ *and hence the subring generated by* $\mathrm{div}(KN)$ *and* KF *is isomorphic to a suitable crossed product* $\mathrm{div}(KN) * (F/N)$.

The proof of Lemma 7.4 is obtained by essentially the same argument based on the Specialization Lemma as the proof of Lemma 3.1 and we omit it.

8. CONCLUDING REMARKS

The proof of our main result, Theorem 1, needs Lemma 3.1; similarly, Lemma 7.3 is needed in the proof of Theorem 5. (It is worth remarking that eventually these theorems imply corollaries that are stronger than the statements of the lemmas—such as Corollary 3 of our Theorem 1 and Theorem 2 in Lewin's paper [8].) These facts are related to the so-called free embedding of group rings and enveloping algebras in skew fields. This embedding was first studied by Hughes in [7]; he proved that every two free embeddings of a group ring of a locally indicable group are isomorphic. Lewin's original proof of his Theorem 2 in [8] makes essential use of Hughes' theorem and the properties of free embeddings of group rings. (See also Lewin and Lewin [9] for more applications of these properties.)

Our proof of Lewin's theorem is based on a different idea, and in particular it makes essential use of Amitsur's Specialization Lemma. It is reasonable to conjecture that an analogue of Hughes' theorem will hold for enveloping algebras. This analogue should have the following form: If L is a Lie algebra such that every finitely generated subalgebra of it does not coincide with its commutator ideal then every two free embeddings of $U(L)$ are isomorphic and hence every free embedding of $U(L)$ gives the skew field $D(L)$ constructed by Cohn. (Here an embedding of $U(L)$ in a skew field is free if for every subalgebra $L_1 \subseteq L$ a basis of $U(L)$ over $U(L_1)$ remains linearly independent over the skew subfield $\operatorname{div}(U(L_1))$ generated by $U(L_1)$).

Such an analogue of Hughes' theorem would make it possible to simplify the proof of Theorem 1.

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