Ranking based Techniques for Disambiguating Büchi Automata

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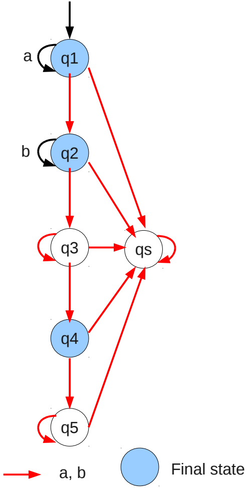
Non-deterministic Büchi automata over words (NBW)

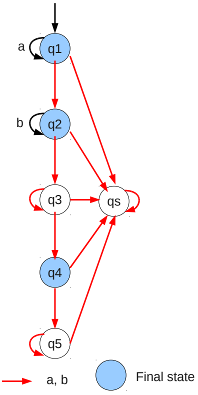
A 5−tuple $(\Sigma, Q, Q_0, \delta, F)$, where

- Σ : Input alphabet
- Q : Finite set of states
- $Q_0 \subseteq Q$: Initial states
- $\delta \subseteq Q \times \Sigma \times Q$: State transition relation
- F : Set of final/accepting states



Runs and acceptance

- A *run* of $\mathcal{A}$ on $\alpha \in \Sigma^\omega$ is a sequence $\rho : \mathbb{N} \to Q$ such that
  - $\rho(0) \in Q_0$
  - $\rho(i+1) \in \delta(\rho(i), \alpha(i))$
- An automaton may have several runs on α.
- ρ is accepting iff $\inf(\rho) \cap F \neq \emptyset$
- α is accepted by $\mathcal{A}$ ($\alpha \in L(\mathcal{A})$) iff there is an accepting run of $\mathcal{A}$ on α.

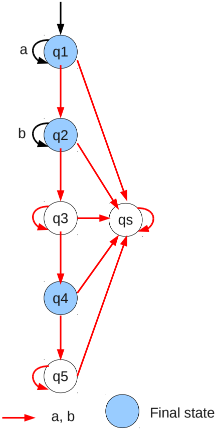


- $\alpha = abbbbb\cdots$, $\rho_1 = q_1q_2q_2q_2q_2q_2q_2\cdots$
- $\rho_2 = q_1q_1q_2q_2q_2q_2q_2\cdots$

Ambiguous automata

$\mathcal{A}$ is *ambiguous* if there exists $\alpha \in L(A)$ such that there are ≥ 2 accepting runs of $\mathcal{A}$ on α Otherwise, $\mathcal{A}$ is *unambiguous*.

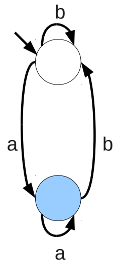
An ambiguous NBW



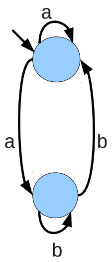
- $\alpha = ab^\omega$
- $\rho_1 = q_1 q_2^\omega$
- $\rho_2 = q_1 q_1 q_2^\omega$

Strongly Unambiguous automata

- *Final run* of $\mathcal{A}$ on α: A run ρ starting from *any* state in Q such that $\inf(\rho) \cap F \neq \emptyset$.
  - A word $\notin L(\mathcal{A})$ may have 0 or more final runs
  - A word $\in L(A)$ has ≥ 1 final runs
- NBW $\mathcal{A}$ is *strongly unambiguous* if for every $\alpha \in \Sigma^\omega$, there is exactly one final run.
- Not all unambiguous automata are strongly unambiguous.



Deterministic (hence unambiguous) but not strongly unambiguous



Strongly unambiguous

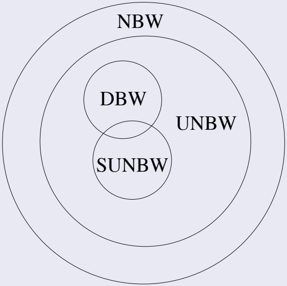
Containment relations

UNBW: Unambiguous NBW, SUNBW: Strongly unambiguous NBW, DBW: Deterministic Büchi automata over words

Expressive power-wise

$$\text{DBW} \subsetneq \text{NBW} \equiv \text{UNBW} \equiv \text{SUNBW}$$

Automata structure-wise



What this talk is about

Given an NBW, construct UNBW accepting the same language and using as few states as possible.

Relevant earlier work:

- Arnold 1983: UNBW expressively equivalent to NBW
- Carton & Michel 2003: Effective construction of SUNBW, size bound $O((12n)^n)$
- Kähler and Wilke 2008: Effective construction of UNBW, size bound $O((3n)^n)$.
- Bousquet and Löding 2010: Equivalence and inclusion problems for SUNBW are poly-time

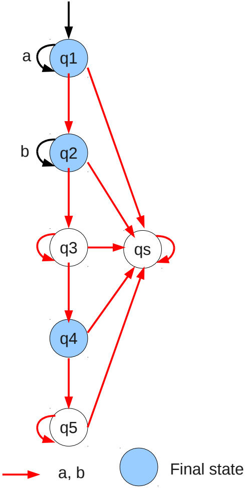
Our contribution

- Effective construction of UNBW, size bound $O(n^2.(0.76n)^n)$
  - Same as best known bound for NBW complementation!

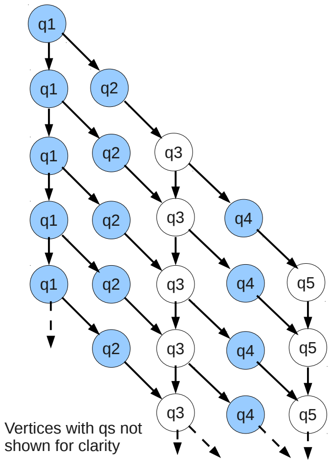
Why care about disambiguation?

- Of course, a theoretically interesting problem
- Can it lead to a better understanding of what kinds of NBW admit easy determinization?
- Practical application? Seek inputs from the audience.

Run DAGs



Run DAG for a^ω



Ranking run DAGs

- Intuitively, assign a metric to each vertex in run DAG such that the metric changes in a desirable way only along "good" runs.
- Early work by Michel (1984?), Klarlund (1991): Ranking functions/progress measures for Büchi complementation
- Recent spurt of work triggered by similar metrics defined by Kupferman & Vardi (2001 onwards)
    - Schewe (2009) used this approach to match upper bound of NBW complementation within $O(n^2)$ of lower bound
    - We use Kupferman-Vardi style rankings

Kupferman-Vardi style ranking

n: Number of states in NBW
V: Set of run DAG vertices
$r : V \to \{1, 2, \ldots 2n + 1\}$: Ranking function

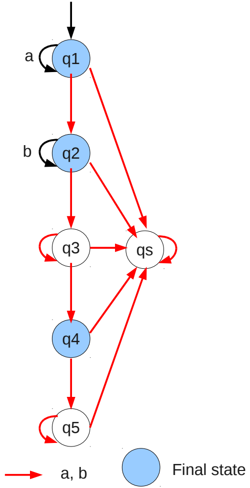
Constraints on ranks

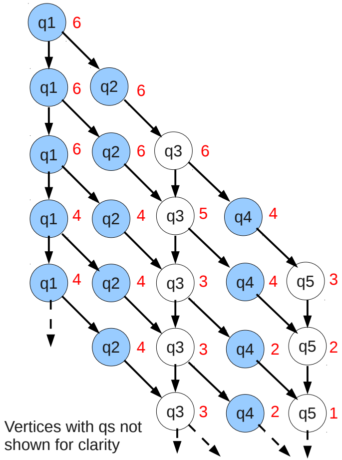
- Vertices corresponding to final states must not get odd ranks
- Ranking cannot increase along any path in run DAG

- *Odd ranking*: Every path eventually trapped in an odd rank
- *Even ranking* otherwise

Example of KV-ranking

Example KV-ranking of run DAG for a^ω

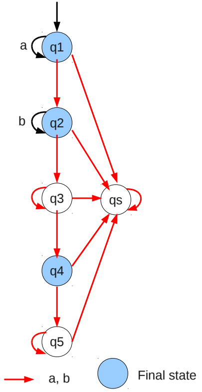




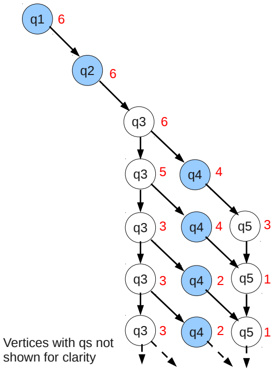
Ranking based complementation

Theorem (Kupferman-Vardi 2001)

An ω-word $\alpha \in \overline{L(\mathcal{A})}$ iff there is an odd ranking of the run DAG of $\mathcal{A}$ on α



Example ranking for ba^ω



Applications of Kupferman-Vardi's theorem

- Series of followup work on NBW complementation using KV-ranking
- Schewe (2009) finally gave a construction yielding a complement NBW of size $O(n^2.(0.76n)^n)$
  - Lower bound $\Omega((0.76n)^n)$.
- Several optimizations possible on basic construction
- One such set of optimizations leads to an **unambiguous complementation** construction, and a **disambiguation** construction too!
  - Achieves same bound of $O(n^2, (0.76n)^n)$.

Extending KV-ranks

Recall KV-ranking

n: Number of states in NBW

V: Set of run DAG vertices

$r : V \to \{1, 2, \ldots 2n+1\} \boxed{\cup\{\infty\}}$: Ranking function

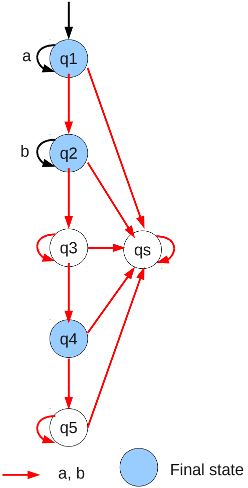
Constraints on ranks

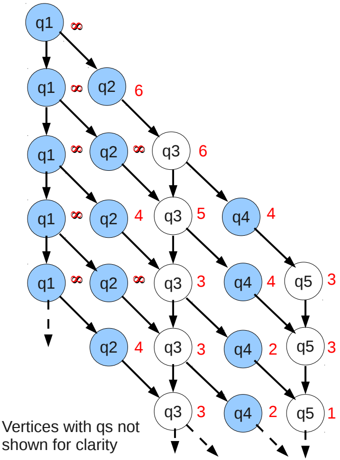
- Vertices corresponding to final states must not get odd ranks
- Ranking cannot increase along any path in run DAG
- Every path eventually trapped in odd rank or in ∞

We call this a **full ranking** of the run DAG.

Example of full ranking

Example full ranking for a^ω

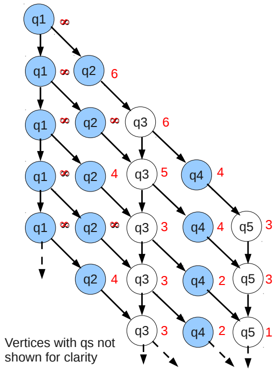




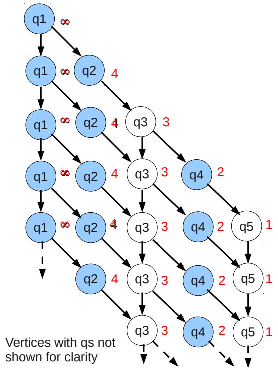
Minimal full rankings

Given run DAG G, full ranking r^* of G is **minimal** iff for all full rankings r of G, $r^*(v) \leq r(v)$ for all vertices v in G.

Non-minimal full ranking



Minimal full ranking



Properties of minimal full rankings

Theorem

For every run DAG, there exists a unique minimal full ranking. A word α is accepted by $\mathcal{A}$ iff the minimal full ranking of the run DAG assigns ∞ as the rank of the root vertex.

F-vertex: Vertex in run DAG for which the state is final.

Local properties (successors)

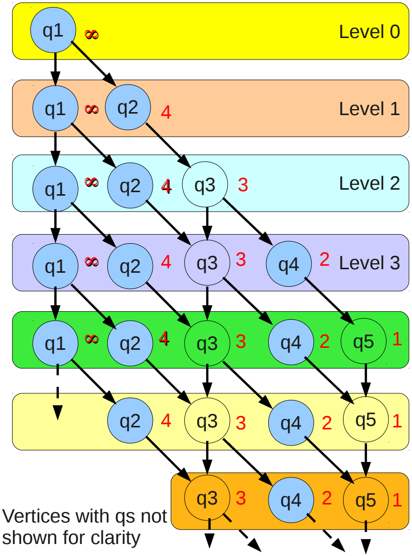
- Every vertex that is not a F-vertex has a successor with the same rank
- Every even ranked vertex either has a successor with the same rank or one with the next lower odd rank

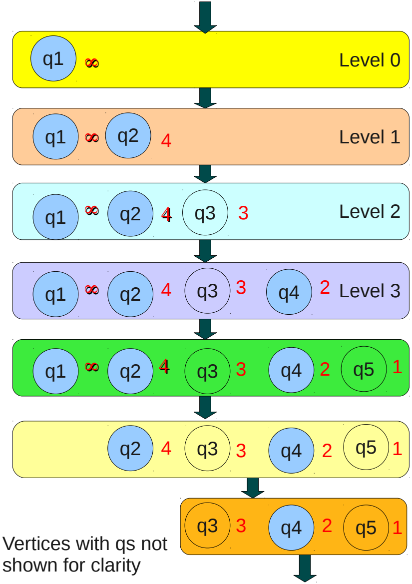
Properties of minimal full rankings

Global properties (descendants)

- Every even ranked vertex has at least one descendant with the next lower odd rank
- Every odd ranked (> 1) vertex has at least one F-vertex descendant with the next lower even rank
- Every path from every even ranked vertex eventually encounters a vertex with a lower rank
- Every ∞ ranked vertex has at least one ∞ ranked F-vertex descendant
- Every ∞ ranked vertex has at least one descendant with the largest non-infinity rank in range of the ranking function.

Intuition of disambiguation construction





Disambiguation construction

- Construct an automaton whose states are full-ranked levels of run DAG
  - Goal: Only minimally full-ranked levels must be accepted
- Local properties of minimal full-ranking easy to enforce in transition relation
- Enforcing global properties requires maintaining additional book-keeping information
  - Global properties checked one vertex (and also one rank) at a time
  - Decompose every global property of an infinite run into properties of finite segments of the run, which can then be concatenated.
  - Ensure that each finite segment satisfies relevant property checkable over finite steps
- Acceptance condition simply ensures that every finite segment of an infinite run satisfies relevant properties and root vertex is ranked ∞

State representation

State of resulting automaton:

$$(S, O, X, f, i), \text{ where}$$

- S : subset of states of NBW in current level
- f : ranking function at current level
- i : rank of vertices for which (decomposed) global properties are currently being checked
- $O \subseteq S$: subset of states with rank i for which global properties yet to be checked
- $X \subseteq S$: subset of states being used to check global property of one state with rank i

Total count of states is $O(n^2.(0.76n)^n)$

- Uses a modification of a counting argument used by Schewe (2009) for NBW complementation

Why is it unambiguous?

- Recall minimal full-ranking for every run DAG is unique.
- Our construction accepts only those runs that enforce both local and global properties of minimal full-ranking
  - Accepted full-ranking is minimal
- Any two accepting runs must differ in the ranking of at least one level
- Since minimal ranking is unique, only one accepting run possible

Conclusion

- Using a variant of KV-ranking (similar to that used by Carton and Michel), we obtain a UNBW (not SUNBW) with better bound than reported in the literature
- We conjecture that this matches the lower bound for disambiguation
- Shows potential close connection between disambiguation and complementation