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The Instructor's Guide to Real Induction

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In this paper, we pursue inductive principles of ordered sets. To get a sense of what this means, consider the principle of mathematical induction. When applied, one thinks in terms of families of statements $P(n)$ indexed by the natural numbers $\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}$, but the cleanest enunciation is in terms of subsets. We call a subset $S \subseteq \mathbb{N}_0$ *inductive* if it satisfies both of the following properties:

(MI1) We have $0 \in S$.

(MI2) For all $n \in \mathbb{N}_0$, if $n \in S$, then also $n + 1 \in S$. The principle of mathematical

induction is that $\mathbb{N}_0$ has no proper inductive subset. (In this form, induction appears as the last and most important of the Peano Axioms.) To prove that $P(n)$ holds for all $n \in \mathbb{N}_0$ "by induction" one shows that the set $S = \{n \in \mathbb{N}_0 \mid P(n) \text{ holds}\}$ is inductive, and thus $S = \mathbb{N}_0$.

In the next section, we work in a closed, bounded interval $[a, b]$ on the real line. We define an inductive subset $S \subset [a, b]$ and state and prove the *principle of real induction* (Theorem 1): there are no proper inductive subsets of $[a, b]$. Just as mathematical induction is a powerful technique for proving families of statements indexed by the natural numbers, real induction can be used to prove families of statements indexed by intervals on the real line. This has applications in elementary analysis, especially to the basic interval theorems concerning a continuous function $f : [a, b] \to \mathbb{R}$. Students in a first real analysis course often find the standard proofs of these results hard to absorb, understand and remember. Proofs by mathematical induction have a common scaffolding that gives students a place to start, and so too do proofs by real induction: if one can "find the induction hypothesis," then the proof dissects into more manageable goals.

Comparing the real induction proofs of these results to the more standard proofs, one gets the sense that real induction functions as a sort of alternative to Dedekind's completeness axiom: every subset of $\mathbb{R}$ that is nonempty and bounded above has a supremum.

Recent years have seen the rise of a program that Propp has called real analysis in reverse [**24**] (see also [**7**, **30**] and the references cited therein)—given a result of real analysis that can be enunciated in any ordered field, one asks whether the truth of that theorem in an ordered field implies Dedekind completeness—in other words, forces that ordered field to be isomorphic to $\mathbb{R}$. (More ambitiously, one could seek to characterize the class of ordered fields in which that theorem holds.) Our definition of an inductive subset of $[a, b]$ makes sense for any elements $a < b$ in an ordered field, and it turns out that the absence of proper inductive subsets of $[a, b]$ is equivalent to Dedekind completeness. In other words, real induction characterizes $\mathbb{R}$ among ordered fields.

After the section on real induction, we go further, giving a definition of an inductive subset of any ordered set and showing the *principle of ordered induction*: the nonexistence of proper inductive subsets is equivalent to Dedekind completeness. Since well-

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ordered sets are Dedekind complete, real induction holds in any well-ordered set, and this recovers the *principle of transfinite induction*. Especially, $\mathbb{N}_0$ is well-ordered, and this extra special case recovers the principle of mathematical induction. An ordered space can be endowed with a canonical order topology, and ordered induction is a natural tool for exploring the interplay between certain topological properties of order topologies and completeness properties of the order. This material could be used in a general topology course.

And it has: in 2015 I taught such a course at the advanced undergraduate level in which I began with the topology of $\mathbb{R}$, including real induction, and spent some time on both order topologies and metric spaces—each of which generalizes and abstracts the real numbers, but in different ways—before moving on to topological spaces in general. This material also has a certain dialectic appeal, as it effects a synthesis of discrete and continuous induction, and in this regard could be of broad interest. The material on ordered induction is more abstract than that on real induction—necessarily so, since the unification afforded by abstraction is the major payoff. We have strived to make it accessible to the broadest possible audience. All that is assumed is the notion of a topological space; order theory and the connections between order and topology are developed from scratch.

Although it is natural to speak of these various forms of induction as axioms, our interest in them is not meta-mathematical. Rather, we seek to expose new proof techniques. The potential applicability of a good proof technique should be open-ended, not circumscribed in advance. In this regard, we have left some applications of real induction as challenges to the reader and also stated some problems for which I do not know definitive solutions. I hope thereby to entice the reader into the pleasure of independent or novel discovery, a pleasure this topic has afforded me several times over the years.

Real induction

Let $a < b$ be real numbers. We define a subset $S \subseteq [a, b]$ to be *inductive* if:
(RI1) We have $a \in S$.
(RI2) If $a \leq x < b$, then $x \in S \Rightarrow [x, y] \subseteq S$ for some $y > x$.
(RI3) If $a < x \leq b$ and $[a, x) \subseteq S$, then $x \in S$.

Theorem 1 (Principle of real induction)**.** *For a subset $S \subseteq [a, b]$, the following are equivalent:*
(i) S is inductive.
(ii) $S = [a, b]$.

Proof. (i) $\Rightarrow$ (ii). Let $S \subseteq [a, b]$ be inductive. Seeking a contradiction, suppose $S' = [a, b] \setminus S$ is nonempty, so $\inf S'$ exists and lies in $[a, b]$.

Case 1. Suppose $\inf S' = a$. By (RI1) we have $a \in S$, so by (RI2), there exists $y > a$ such that $[a, y] \subseteq S$. Thus y is a greater lower bound for S' than $a = \inf S'$, a contradiction.

Case 2. Suppose $a < \inf S' \in S$. If $\inf S' = b$, then $S = [a, b]$. Otherwise, by (RI2) there exists $y > \inf S'$ such that $[\inf S', y] \subseteq S$. Also, because $[a, \inf S') \subset S$, then $[a, y] \subset S$ and thus again y is a greater lower bound for S' than $\inf S'$, a contradiction.

Case 3. Suppose $a < \inf S' \in S'$. Then $[a, \inf S') \subseteq S$, so (RI3) gives $\inf S' \in S$, a contradiction.

The opposite direction, (ii) $\Rightarrow$ (i), is immediate. ∎

A little history Theorem 1 was first published by Hathaway [12]. I came up with it independently in 2010 [6] as a variation of Kalantari's induction over the continuum

[13, Section 3]. I was scheduled to give a seminar in my department that afternoon on a different topic, but instead I spoke on real induction. The audience shared my enthusiasm, which encouraged me to further develop and disseminate the material.

The enunciation of an inductive principle for subintervals of $\mathbb{R}$ is far from new. The earliest instance I know of is a 1923 work of Khinchin [15]. There is an earlier borderline case: a 1919 work of Chao [4] gives an inductive criterion for a subset of $[a, \infty)$ to be all of $[a, \infty)$. However, Chao's criterion involves a "discrete increment" $\Delta > 0$ and in fact can be proved by conventional mathematical induction.

In addition to the works [4, 12, 13, 15], each of the following papers introduces some form of continuous induction, in many cases without reference to past precedent: [2, 8–10, 16, 18, 19, 23, 25, 26].

Often for a mathematical principle, implicit use predates explicit formulation. The first explicit use of mathematical induction was in Pascal's 1665 *Traité du triangle arithmétique*, but most agree that Euclid's celebrated Proposition IX.20—There are infinitely many primes—of circa 300 BCE contains the crucial implicit use of an inductive principle. (Strictly speaking, Euclid assumes there are three primes and produces a fourth. Evidently some more general principle is intended.) Later we will encounter an important implicit use of real induction that predates the work of Khinchin and even of Chao.

Applications in analysis Let us see real induction in action.

Theorem 2 (Intermediate value theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function, and let $L \in \mathbb{R}$ be in between $f(a)$ and $f(b)$. Then there is $c \in [a, b]$ such that $f(c) = L$.*

Proof. Replacing f by $\pm(f - c)$, we reduce to the following special case: if $f : [a, b] \to \mathbb{R} \setminus \{0\}$ is continuous and $f(a) > 0$, then $f(b) > 0$. Let $S = \{x \in [a, b] \mid f(x) > 0\}$, so $f(b) > 0$ if and only if $b \in S$. We will use real induction to show that $S = [a, b]$. Thus $f(b) > 0$, completing the proof.

(RI1) Since $f(a) > 0$, we have $a \in S$.

(RI2) Let $x \in S$, $x < b$, so $f(x) > 0$. Since f is continuous at x, there exists $\delta > 0$ such that f is positive on $[x, x + \delta]$, and thus $[x, x + \delta] \subseteq S$.

(RI3) Let $x \in (a, b]$ be such that $[a, x) \subseteq S$, i.e., f is positive on $[a, x)$. We claim that $f(x) > 0$. Indeed, since $f(x) \neq 0$, the only other possibility is $f(x) < 0$, but if so, then by continuity there would exist $\delta > 0$ such that f is negative on $[x - \delta, x]$, i.e., f is both positive and negative at each point of $[x - \delta, x]$, a contradiction. ∎

Theorem 3. *A continuous function $f : [a, b] \to \mathbb{R}$ is bounded.*

Proof. Let $S = \{x \in [a, b] \mid f : [a, x] \to \mathbb{R}$ is bounded$\}$. We will use real induction to show that $S = [a, b]$.

(RI1): Evidently $a \in S$.

(RI2): Suppose $x \in S$, so that f is bounded on $[a, x]$. But then f is continuous at x, so is bounded near x: for instance, there exists $\delta > 0$ such that for all $y \in [x - \delta, x + \delta]$, $|f(y)| \leq |f(x)| + 1$. So f is bounded on $[a, x]$ and also on $[x, x + \delta]$ and thus on $[a, x + \delta]$.

(RI3): Suppose $x \in (a, b]$ and $[a, x) \subseteq S$. Since f is continuous at x, there exists $0 < \delta < x - a$ such that f is bounded on $[x - \delta, x]$. Since $a < x - \delta < x$, f is bounded on $[a, x - \delta]$, so f is bounded on $[a, x]$. ∎

When using real induction, one must beware the following pitfall. Often we have a family of statements $P(I)$ indexed by subintervals I of $[a, b]$. In the proof of Theorem 3, $P(I)$ is: f is bounded on I. In the proof of Theorem 2, $P(I)$ is: f is positive at all points of I. It can be tempting to construe (RI3) as: for all $a < x \le b$, assume $P([a, x))$ holds and prove $P([a, x])$. But this is not correct: we must assume $P([a, y])$ holds for all $a \le y < x$ and prove $P([a, x])$. Sometimes the distinction is immaterial: a function positive on $[a, y]$ for all $a \le y < x$ is positive on $[a, x)$. But sometimes it matters: a function bounded on $[a, y]$ for all $a \le y < x$ need not be bounded on $[a, x)$.

Here is an instance in which finding the right inductive hypothesis requires insight.

Theorem 4 (Integrability theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is Darboux integrable: for all $\epsilon > 0$, there is a partition $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$ of $[a, b]$ such that the difference between the associated upper sum*

$$U(f, \mathcal{P}) = \sum_{i=0}^{n-1} \sup(f, [x_i, x_{i+1}])(x_{i+1} - x_i)$$

and the associated lower sum

$$L(f, \mathcal{P}) = \sum_{i=0}^{n-1} \inf(f, [x_i, x_{i+1}])(x_{i+1} - x_i)$$

is less than ϵ.

Proof. For $\epsilon > 0$, let $S(\epsilon) = \{x \in [a, b] \mid$ there exists a partition $\mathcal{P}_x$ of $[a, x]$ where $U(f, \mathcal{P}_x) - L(f, \mathcal{P}_x) < (x - a)\epsilon\}$. We will use real induction to show that for all $\epsilon > 0$, we have $S(\epsilon) = [a, b]$. Then $b \in S(\frac{\epsilon}{b-a})$, completing the proof.

(RI1) As usual, this is clear.

(RI2) Suppose that for $x \in [a, b)$ we have $[a, x] \subseteq S(\epsilon)$, so that there is a partition $\mathcal{P}_x$ of $[a, x]$ such that $U(f, \mathcal{P}_x) - L(f, \mathcal{P}_x) < (x - a)\epsilon$. Since f is continuous at x, there is $\delta > 0$ such that $\sup(f, [x, x + \delta]) - \inf(f, [x, x + \delta]) < \epsilon$. Now let $y \in [x, x + \delta]$ and take the partition $\mathcal{P}_y = \mathcal{P}_x \cup \{y\}$ of $[a, y]$. Then

$$\begin{aligned} &U(f, \mathcal{P}_y) - L(f, \mathcal{P}_y) \\ &\quad = (U(f, \mathcal{P}_x) + (y - x)\sup(f, [a, y])) - (L(f, \mathcal{P}_x) + (y - x)\inf(f, [a, y])) \\ &\quad < (x - a)(\epsilon) + (y - x)(\epsilon) = (y - a)(\epsilon). \end{aligned}$$

(RI3) Suppose that for $x \in (a, b]$ we have $[a, x) \subseteq S(\epsilon)$. Since f is continuous at x, there is $\delta > 0$ such that $\sup(f, [x - \delta, x]) - \inf(f, [x - \delta, x]) < \epsilon$. Since $x - \delta < x$, $x - \delta \in S(\epsilon)$, there is a partition $\mathcal{P}_{x-\delta}$ of $[a, x - \delta]$ such that $U(f, \mathcal{P}_{x-\delta}) = L(f, \mathcal{P}_{x-\delta}) = (x - \delta - a)\epsilon$. Let $\mathcal{P}_x = \mathcal{P}_{x-\delta} \cup \{x\}$. Then as above we get

$$U(f, \mathcal{P}_x) - L(f, \mathcal{P}_x) < (x - \delta - a)\epsilon + \delta\epsilon = (x - a)\epsilon. \qquad \blacksquare$$

Applications in topology

Theorem 5 (Bolzano–Weierstrass)**.** *Each infinite subset $\mathcal{A}$ of $[a, b]$ has a limit point: there is $L \in [a, b]$ such that for all $\delta > 0$, the set $(L - \delta, L + \delta) \cap \mathcal{A}$ is infinite.*

Proof. Let S be the set of x in $[a, b]$ such that *if* $\mathcal{A} \cap [a, x]$ is infinite, it has a limit point. It suffices to show $S = [a, b]$, which we will do by real induction.

(RI1) is clear.

(RI2) Suppose $x \in [a, b) \cap S$. If $\mathcal{A} \cap [a, x]$ is infinite, then it has a limit point and hence so does $\mathcal{A} \cap [a, b]$: thus $S = [a, b]$. If for some $\delta > 0$, $\mathcal{A} \cap [a, x + \delta]$ is finite, then $[x, x + \delta] \subseteq S$. Otherwise $\mathcal{A} \cap [a, x]$ is finite but $\mathcal{A} \cap [a, x + \delta]$ is infinite for all $\delta > 0$, and then x is a limit point for $\mathcal{A}$ and $S = [a, b]$ as above.

(RI3) If $[a, x) \subseteq S$, then either $\mathcal{A} \cap [a, y]$ is infinite for some $y < x$, so $x \in S$; or $\mathcal{A} \cap [a, x]$ is finite, so $x \in S$; or $\mathcal{A} \cap [a, y]$ is finite for all $y < x$ and $\mathcal{A} \cap [a, x]$ is infinite, so x is a limit point of $\mathcal{A} \cap [a, x]$ and $x \in S$. ■

Recall that a subset U of $[a, b]$ is *open* if for all $x \in U$, there is $\delta > 0$ such that

$$\begin{cases} (x - \delta, x + \delta) \subset U & x \notin \{a, b\} \\ [a, x + \delta) \subset U & x = a \\ (x - \delta, b] \subset U & x = b \end{cases}.$$

A subset is closed if its complement is open.

Theorem 6. *The interval $[a, b]$ is connected: if U and V are disjoint open subsets of $[a, b]$ such that $U \cup V = [a, b]$, then $U = [a, b]$ or $V = [a, b]$.*

Proof. Suppose $[a, b] = U \cup V$, with U and V open and $U \cap V = \emptyset$. We assume $a \in U$ and prove by real induction that $U = [a, b]$: (RI1) is immediate, (RI2) holds because U is open, and (RI3) holds because U is closed. We're done! ■

Theorem 7 (Heine–Borel)**.** *The interval $[a, b]$ is compact: if $\{U_i\}_{i \in I}$ is a family of open subsets of $[a, b]$ such that $\bigcup_{i \in I} U_i = [a, b]$, then there is a finite subset $J \subset I$ such that $\bigcup_{i \in J} U_i = [a, b]$.*

Proof. For an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of $[a, b]$, let

$$S = \{x \in [a, b] \mid \mathcal{U} \cap [a, x] \text{ has a finite subcovering}\}.$$

We prove $S = [a, b]$ by real induction. (RI1) is clear. (RI2): If $U_1, \ldots, U_n$ covers $[a, x]$, then some U_i contains $[x, x + \delta]$ for some $\delta > 0$. (RI3): If $[a, x) \subseteq S$, then $x \in U_i$ for some $i \in I$; let $y < x$ be such that $(y, x] \in U_i$. There is a finite $J \subseteq I$ with $\bigcup_{i \in J} U_i \supset [a, y]$, so $\{U_i\}_{i \in J} \cup U_i$ covers $[a, x]$. We're done! ■

Some real induction proofs for the reader Here are more results amenable to real induction. The proofs are left to you.

Theorem 8 (Mean value inequality)**.** *Let $f : [a, b] \to \mathbb{R}$ be differentiable. Suppose that there exists $M \in \mathbb{R}$ such that for all $x \in [a, b]$ we have $f'(x) \geq M$. Then for all $x < y \in \mathbb{R}$, we have $f(y) - f(x) \geq M(y - x)$.*

Theorem 9 (Uniform continuity theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is uniformly continuous on $[a, b]$.*

Theorem 10 (Cantor intersection theorem)**.** *Let $\{F_n\}_{n=1}^{\infty}$ be a decreasing sequence of closed subsets of $[a, b]$. Put $F = \bigcap_n F_n$. Then* either *$F \neq \varnothing$ or there exists $n \in \mathbb{Z}^+$ such that $F_n = \varnothing$.*

Theorem 11 (Lebesgue number lemma)**.** *If $\{U_i\}_{i \in I}$ is an open covering of $[a, b]$, then there is $\delta > 0$ such that if $A \subseteq [a, b]$ has diameter at most δ, then $A \subseteq U_i$ for some $i \in I$.*

Theorem 12 (Dini's lemma)**.** *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous real-valued functions on the interval $[a, b]$ that is pointwise decreasing: for all $x \in [a, b]$ and all $n \in \mathbb{Z}^+$, $f_{n+1}(x) \leq f_n(x)$. If $f : [a, b] \to \mathbb{R}$ is continuous and $f_n \to f$ pointwise, then $f_n \to f$ uniformly.*

Theorem 13 (Arzelà–Ascoli). *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions on $[a, b]$ such that:*
(i) There is $M \in \mathbb{R}$ such that for all $n \in \mathbb{Z}^+$ and all $x \in [a, b]$, $|f_n(x)| \leq M$, and
(ii) For all $x \in [a, b]$ and all $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - y| < \delta$, then for all $n \in \mathbb{Z}^+$, $|f_n(x) - f_n(y)| < \epsilon$.
Then there is a subsequence $\{f_{n_k}\}$ that is uniformly convergent on $[a, b]$.

Theorem 8 is a consequence of the mean value theorem. It is one of several results that have been advocated (by some; we will not weigh in on this issue) as being pedagogically preferable to the mean value theorem. The other variants can also be proved by real induction. But what about the mean value theorem itself?

Problem 1. The standard proof of the mean value theorem is a deduction from the extreme value theorem. Either prove the mean value theorem directly by real induction or explain why it is not possible to do so.

Problem 2. Find other theorems that can be proved via real induction.

Comments and complements Our proof of Theorem 2 is not so different from the usual proof using suprema. That proof is probably even cleaner: it suffices to assume that $f(a) > 0$ and $f(b) < 0$ and show that there is $c \in (a, b)$ with $f(c) = 0$. For this, let $c = \sup\{x \in [a, b] \mid f(x) \leq 0\}$. Then—as follows from the definition of continuity—we must have $f(c) = 0$.

But this proof has within it a germ of the idea for real induction. In fact, one can motivate real induction in a classroom setting by asking for a proof of Theorem 3 along the lines of the above proof of Theorem 2: i.e., we start by defining $c = \sup\{x \in [a, b] \mid f \text{ is bounded on } [a, x]\}$. Then it emerges naturally that we want to show that $c = b$ and that we can establish this by showing (RI2) and (RI3). (Here, as in every application I know of, (RI1) is obvious.) It is then an interesting exercise to see how to modify the standard proof of Theorem 2 to get the proof by real induction.

We have used real induction to prove the interval theorems of elementary real analysis (cf. [28, Chapter 7]) with one exception: we are missing the extreme value theorem, which asserts that every continuous $f : [a, b] \to \mathbb{R}$ assumes its maximum and minimum values. This result may be deduced from Theorem 3 by an easy argument using suprema: by Theorem 3, $M = \sup(f, [a, b])$ is finite. If M were not attained on the interval $[a, b]$, then the function $g : x \mapsto \frac{1}{M - f(x)}$ would be continuous and unbounded on $[a, b]$, contradicting Theorem 3. To reiterate: we do not advocate using real induction *in place of* Dedekind's completeness axiom but rather—when helpful!—as a proof technique.

Theorem 4 is usually proved using the uniform continuity of continuous functions $f : [a, b] \to \mathbb{R}$. In [28, pp. 292–293], Spivak gives a different proof, establishing equality of the upper and lower integrals by differentiation. This method goes back at least to M.J. Norris [22]. Our proof seems different from both of these.

Standard proofs of Theorem 5 use monotone subsequences, dissection/nested intervals or the compactness of $[a, b]$. Our proof appears to be new.

Perhaps the best argument for real induction in the classroom is the proofs it affords for Theorems 6 and 7: not only are they short and simple, but initiates in real induction will find them easily.

On the one hand this suggests that the concepts of connectedness and compactness may be inherently inductive in some sense. There seems to be something to this: see, e.g., induction on connectedness and induction on compactness in [31]. We will give a different kind of generalization in the next section when we explore connectedness and compactness in order topologies.

On the other hand, it raises the question of why this proof technique—which, recall, has appeared in many variations in more than a dozen prior works—is not more popular. The situation becomes even more curious once one learns that our proof of Theorem 7 is essentially the same as one given in 1904 by Henri Lebesgue. In [**17**], Lebesgue proves the result as follows: he says that $x \in [a, b]$ is "reached" if there is a finite subcovering of the interval $[a, x]$, and proceeds by considering the supremum of the set of all points x that are reached. This is the last of the early proofs of Theorem 7 surveyed in [**1**]; they also discuss proofs by Borel, Cousin, Schoenflies, and Young. The authors are quite enthusiastic about Lebesgue's proof, writing "This is the one! The proof is thoroughly modern and simple to follow. In comparison, all previous arguments are cumbersome and overly complicated."

Of course Theorem 2 and the extreme value theorem are quick consequences of Theorems 6 and 7, via the following basic result.

Proposition 1. *Let $f : X \to Y$ be a continuous surjection of topological spaces.*
*(a) [**20**, Theorem 23.5] If X is connected, then so is Y.*
*(b) [**20**, Theorem 26.5] If X is compact, then so is Y.*

Ordered induction

In this section we pursue induction in ordered sets, obtaining a common generalization of mathematical induction and real induction.

Ordered sets An *ordered set* (sometimes called a linearly ordered or totally ordered set) is a set X endowed with a binary relation $\leq$ that satisfies:
- reflexivity: for all $x \in X$, $x \leq x$;
- anti-symmetry: for all $x, y \in X$, if $x \leq y$ and $y \leq x$, then $x = y$;
- transitivity: for all $x, y, z \in X$, if $x \leq y$ and $y \leq z$, then $x \leq z$; and
- totality: for all $x, y \in X$, at least one of $x \leq y$ and $y \leq x$ holds.

Our distinguished example is the interval $[a, b] \subseteq \mathbb{R}$.

A *top element* (resp. a *bottom element*) of an ordered set $(X, \leq)$ is an element $\top$ (resp. $\perp$) such that $x \leq \top$ (resp. $\perp \leq x$) for all $x \in X$. Clearly X can have at most one top (resp. bottom) element. If X lacks a top element, then we can simply adjoin such an element, denoted $\top$—that is, $\top$ is not an element of X and decreed to satisfy $x < \top$ for all $x \in X$. Similarly, if X lacks a bottom element we can adjoin one, denoted $\perp$. We denote by $\tilde{X}$ the set X extended by a top element if it lacks one and extended by a bottom element if it lacks one. Applying this construction to the real numbers, we get the extended real numbers, in which *every* subset has a supremum and an infimum.

If X and Y are ordered sets, a map $f : X \to Y$ is *isotone* (also called order-preserving), increasing or monotone, though the latter is used in analysis also for antitone (order-reversing) maps if for all $x_1 \leq x_2$ in X, we have $f(x_1) \leq f(x_2)$ in Y. A map $f : X \to Y$ is an *order-isomorphism* if it is isotone and admits an isotone inverse $g : Y \to X$. (An isotone bijection is an order-isomorphism, but defining an isomorphism as a bijective morphism is certainly wrong in other contexts, e.g., for topological spaces or partially ordered sets.)

Next we define intervals in an ordered set. The empty set is decreed to be an open interval in X. A closed, bounded interval in X is either the empty set or a subset of the form $[a, b] = \{x \in X \mid a \leq x \leq b\}$ for elements $a \leq b$ in X. A nonempty subset $I \subseteq X$ is an interval if $\inf I$ and $\sup I$ both exist in $\tilde{X}$ and $I \cup \{\inf I, \sup I\}$ is a closed, bounded interval in $\tilde{X}$. A nonempty interval I is open if the following hold:

(i) if $\inf I \in I$ then $\inf I$ is the bottom element of X and (ii) if $\sup I \in I$ then $\sup I$ is the top element of X. For $x \in X$, we explicitly define the following intervals:

$$^{<}x = \{y \in X \mid y < x\}, \ ^{>}x = \{y \in X \mid y > x\},$$
$$^{\leq}x = \{y \in X \mid y \leq x\}, \ ^{\geq}x = \{y \in X \mid y \geq x\}.$$

The open intervals form a base for a topology on X, called the *order topology*. The bounded open intervals—those of the form (a, b) for elements $a < b$ of X, $[\bot, b)$ for $b \in X$ if X has a bottom element, $(a, \top]$ for $a \in X$ if X has a top element and $X = [\bot, \top]$ if X has both top and bottom elements—are a base for the same topology. If $X = \mathbb{R}$ this is the usual Euclidean topology.

In some ways, order topologies are closer relatives to $\mathbb{R}$ than an arbitrary metric space, while in other ways they are more exotic. For example, they need not be metrizable or even first countable. Order topologies are always Hausdorff, so a compact subset must be closed. Moreover a compact subset C must be bounded—that is, contained in a closed, bounded interval: for each $x \in C$, let I_x be a bounded open interval containing x. Then there is a finite subset $Y \subseteq X$ such that $C \subseteq \bigcup_{x\in Y} I_x$ and $C \subseteq [\min_{x\in Y} \inf I_x, \max_{x\in Y} \sup I_x]$.

An ordered set is *Dedekind complete* if every nonempty subset that is bounded above has a supremum. This holds if and only if every nonempty subset that is bounded below has an infimum. An ordered set is complete if *every* subset has a supremum (if and only if every subset has an infimum).

Proposition 2. *Let X be an ordered set. Then,*
(a) X is Dedekind complete if and only if $\tilde{X}$ is complete.
(b) X is complete if and only if it is Dedekind complete and has top and bottom elements.

Proof. We observe that $\inf \varnothing$ exists if and only if X has a top element—in which case $\inf \varnothing = \top$—and $\sup \varnothing$ exists if and only if X has a bottom element—in which case $\inf \varnothing = \bot$. The rest is straightforward and left to the reader. ■

Ordered induction A subset S of an ordered set $(X, \leq)$ is *inductive* if it satisfies all of the following:

(IS1) There is $a \in X$ such that $^{\leq}a \subseteq S$.
(IS2) For all $x \in S$, either $x = \top$ or there is $y > x$ such that $[x, y] \subseteq S$.
(IS3) For all $x \in X$, if $^{<}x \subseteq S$, then $x \in S$.

Theorem 14 (Principle of ordered induction)**.** *For a nonempty ordered set X, the following are equivalent:*
(i) X is Dedekind complete.
(ii) The only inductive subset of X is X itself.

Proof. (i) $\Rightarrow$ (ii). Let $S \subseteq X$ be inductive. Seeking a contradiction, we suppose $S' = X \setminus S$ is nonempty. Fix $a \in X$ satisfying (IS1). Then a is a lower bound for S', so by hypothesis S' has an infimum, say y. Any element less than y is strictly less than every element of S', so $^{<}y \subset S$. By (IS3), $y \in S$. If $y = \top$, then $S' = \{\top\}$ or $S' = \varnothing$: both are contradictions. So $y < \top$, and then by (IS2) there exists $z > y$ such that $[y, z] \subseteq S$ and thus $^{\leq}z \subseteq S$. Thus z is a lower bound for S' that is strictly larger than y, a contradiction.

(ii) $\Rightarrow$ (i). Let $T \subseteq X$ be nonempty and bounded below by a. Let S be the set of lower bounds for T. Then $^{\leq}a \subseteq S$, so S satisfies (IS1).

Case 1. Suppose S does not satisfy (IS2): there is $x \in S$ with no $y \in X$ such that $[x, y] \subseteq S$. Since S is downward closed, x is the top element of S and $x = \inf T$.

Case 2. Suppose S does not satisfy (IS3): there is $x \in X$ such that ${}^{<}x \in S$ but $x \notin S$, i.e., there exists $t \in T$ such that $t < x$. Then also $t \in S$, so t is the least element of T: in particular $t = \inf T$.

Case 3. If S satisfies (IS2) and (IS3), then $S = X$, $T = \{\top\}$ and $\inf T = \top$. ■

Transfinite induction An ordered set X is well-ordered if every nonempty subset has a bottom element. If X is well-ordered and $\varnothing \subsetneq Y \subset X$ is bounded above, then (as usual) if Y has a top element $\top_Y$ then $\sup Y = \top_Y$; otherwise there is an element $x \in X$ such that $x > y$ for all $y \in Y$ and thus, by well ordering, a least such element, which is $\sup Y$. That is, well-ordered subsets are Dedekind complete, and thus, in view of Theorem 14, the only inductive subset of a well-ordered set X is X itself.

Let $x < y$ be elements of an ordered set X. If $[x, y] = \{x, y\}$ then we say that y is the successor of x and that x is the predecessor of y. If X is a nonempty well-ordered set, then every $y \neq \top$ has a successor. Clearly $\bot$ has no predecessor. The natural numbers form an infinite well-ordered set in which every $x \neq \bot$ has a predecessor, and this characterizes $\mathbb{N}_0$ up to order-isomorphism.

In a well-ordered set, (IS2) is equivalent to

(IS2′) For all every $x \in S$, either $x = \top$ or the successor of x also lies in S.

Thus we recover the following important result.

Theorem 15 (Principle of transfinite induction)**.** *Let X be a nonempty well-ordered set. Let S be a subset of X such that:*
(T1) We have $\bot \in S$.
(T2) If $x \in X$, either $x = \top$ or the successor of x also lies in S.
(T3) For all $y \in X$, if ${}^{<}y \subseteq S$, then $y \in S$.
Then $S = X$.

This statement is in fact rather redundant: applying (T3) with $y = \bot$ we get (T1); applying (T3) with y the successor of a non-top element x, we get (T2). The redundancy could be eliminated by requiring (T3) only for non-bottom elements that have no predecessors. But it is harmless, and moreover it is often natural to treat the three cases separately. (For instance, the three cases correspond to the three types of ordinal numbers.) As mentioned above, in $\mathbb{N}_0$ there is no non-bottom element without a predecessor, and thus applied therein Theorem 15 becomes the principle of mathematical induction.

Transfinite induction does not seem to have the ubiquitous presence in mathematics students' toolkits that it once did. If true, that is both unfortunate and beyond the scope of this article to remedy. However, we can recommend [**14**] which gives this topic the elegant presentation, context and range of applications that it deserves.

Completeness of subsets Let X be a Dedekind complete ordered set, and let $\varnothing \neq Y \subseteq X$. Then Y is an ordered set in its own right—when is it Dedekind complete? The analogy with metric spaces (and Cauchy completeness: that is, in which every Cauchy sequence converges) suggests that this holds if and only if Y is closed, but a little thought shows that this cannot be quite right. For example, the arctangent function gives an order isomorphism from $\mathbb{R}$ to $(\frac{-\pi}{2}, \frac{\pi}{2})$, so $(\frac{-\pi}{2}, \frac{\pi}{2})$ is Dedekind complete but not closed in $\mathbb{R}$. The precise answer is as follows: as usual, let $\tilde{X}$ be X augmented with a bottom element and/or a top element if and only if X lacks them. Then $\tilde{X}$ is complete by Proposition 2, so $\inf Y$ and $\sup Y$ exist in $\tilde{X}$. This allows us to view $\tilde{Y} = Y \cup \{\inf Y, \sup Y\}$ as a subset of the complete ordered set $\tilde{X}$.

Proposition 3. *Let Y be a subset of a Dedekind complete ordered set $(X, \leq)$ Then Y is Dedekind complete if and only if $\tilde{Y}$ is closed in $\tilde{X}$. It follows that every interval in a Dedekind complete ordered set is Dedekind complete.*

Proof. Suppose that Y is a Dedekind complete subset of the ordered set X, and let $x \in \tilde{X}$ be a point such that every neighborhood U of x in $\tilde{X}$ meets Y. Seeking a contradiction, we suppose that $x \notin \tilde{Y}$. Then at least one of the following holds: (i) for all elements $x' < x$ of X, we have $(x', x) \cap Y \neq \varnothing$, in which case $x = \sup\{y \in Y \mid y \leq x\} \in \tilde{Y}$, or (ii) for all elements $x'' > x$ of X we have $(x, x'') \cap Y \neq \varnothing$, in which case $x = \inf\{y \in Y \mid x \leq y\} \in \tilde{Y}$. This contradiction shows that $\tilde{Y}$ is closed in $\tilde{X}$.

Now suppose that X is Dedekind complete and that $Y \subset X$ is such that $\tilde{Y}$ is closed in $\tilde{X}$. Let A be a nonempty subset of Y that is bounded above by $M \in Y$, and let $a \in A$. Since X is Dedekind complete, $\sup A$ exists in X, and clearly $\sup A \in [a, M]$. Since $A \subset Y$, every open interval in X centered at $\sup A$ contains points of Y, so $\sup A \in \tilde{Y} \cap [a, M] \subset Y$, and Y is Dedekind complete.

The empty interval is Dedekind complete, and for every nonempty interval $I \subseteq X$, the set $\tilde{I}$ is a closed interval in $\tilde{X}$, so I is Dedekind complete. ■

Ordered fields An *ordered field* F is a field that is endowed with an ordering $\leq$ that satisfies the following compatibilities:

$$\text{for all } x, y, z \in F,\ x \leq y \text{ implies } x + z \leq y + z \text{ and}$$

$$\text{for all } x, y \in F,\ x \geq 0,\ y \geq 0 \text{ implies } xy \geq 0.$$

The real numbers $\mathbb{R}$ form an ordered field. Every subfield of an ordered field is again an ordered field, so one gets many examples by taking subfields of $\mathbb{R}$. The standard orderings on $\mathbb{R}$ and $\mathbb{Q}$ are in fact the unique orderings on these fields [**5**, Section 15.2]. Every ordered field has characteristic 0, so admits $\mathbb{Q}$ as an ordered subfield [**5**, *loc. cit.*].

Here is another example. Let $\mathbb{R}(t)$ be the field of rational functions $\frac{p(t)}{q(t)}$, that is, $p(t)$ and $q(t)$ are real polynomial functions and $q(t)$ is not identically zero. If $r, s \in \mathbb{R}(t)$ are distinct rational functions, then either $r(x) > s(x)$ for all sufficiently large x, which we denote $r > s$, or $r(x) < s(x)$ for all sufficiently large x, which we denote $r < s$. This makes $\mathbb{R}(t)$ into an ordered field. An element x of an ordered field is called infinitely large if $x > n$ for all $n \in \mathbb{Z}^+$. In the field $\mathbb{R}(t)$, the element t is infinitely large.

An ordered field that admits an infinitely large element is called non-Archimedean. The subfields of $\mathbb{R}$ are Archimedean, and conversely every Archimedean ordered field admits a unique isotone field embedding into $\mathbb{R}$ and thus may be identified with a subfield of $\mathbb{R}$ [**5**, Corollary 15.48]. If F is a proper subfield of $\mathbb{R}$, then F contains $\mathbb{Q}$ so is dense in $\mathbb{R}$. It follows that $\tilde{F}$ is not closed in the extended real numbers $\tilde{\mathbb{R}}$, so F is not Dedekind complete by Proposition 3. On the other hand, if F is non-Archimedean, then the upper bounds for $\mathbb{N}_0$ are precisely the infinitely large elements, which exist by definition, but if x is infinitely large then so is $x - 1$, so $\mathbb{N}_0$ has no supremum in F. We conclude that any Dedekind complete ordered field is isomorphic to $\mathbb{R}$ [**5**, Theorem 15.56].

Ordered induction is also a characteristic property of $\mathbb{R}$ among ordered fields, as the following corollary indicates.

Corollary 1. *In an ordered field $(F, +, \cdot, \leq)$, the following are equivalent:*
(i) F is Dedekind complete (and thus isomorphic to $\mathbb{R}$).

(ii) For all $a < b$ in F, the interval $[a, b]$ is complete.
(ii′) For all $a < b$ in F, the only inductive subset of $[a, b]$ is $[a, b]$.
(iii) There are $a < b$ in F such that the interval $[a, b]$ is complete.
(iii′) There are $a < b$ in F such that the only inductive subset of $[a, b]$ is $[a, b]$.

Proof. (i) $\Rightarrow$ (ii) by Proposition 3.

(ii) $\Leftrightarrow$ (ii′) and (iii) $\Leftrightarrow$ (iii′) by Theorem 14.

(ii) $\Rightarrow$ (iii) is clear.

(iii) $\Rightarrow$ (i). Suppose $[a, b]$ is complete. Let $S \subset F$ be any subset that is nonempty and bounded above, say by B. Let $A \in S$. Then S has a supremum in F if and only if $T = \{x \in S \mid A \le x\} \subseteq [A, B]$ does. The map

$$\ell : [a, b] \to [A, B],\ x \mapsto \frac{B - A}{b - a}(x - a) + A$$

is an order-isomorphism, so $[A, B]$ is complete and T has a supremum in F. ■

Problem 3. Characterize the inductive subsets of $[a, b] \cap \mathbb{Q}$.

Completeness and connectedness A subset Y of an ordered set $(X, \le)$ is convex if for all $x, z, y \in X$ with $x < z < y$, if $x, y \in Y$ then also $z \in Y$. In any ordered set $(X, \le)$, both intervals and connected sets are convex. The former is clear; as for the latter, if $Y \subseteq X$ is not convex, there are $x < z < y \in X$ with $x, y \in Y$ and $z \notin Y$, and then $Y_1 = {}^{<}z \cap Y,\ Y_2 = {}^{>}z \cap Y$ is a separation of Y. The converse implications depend on completeness, as indicated in the following proposition.

Proposition 4. *In an ordered set $(X, \le)$, the following are equivalent:*
(i) X is Dedekind complete.
(ii) Every convex subset $Y \subseteq X$ is an interval.

Proof. (i) $\Rightarrow$ (ii). We may assume that Y is nonempty. Consider $\tilde{Y} \subseteq \tilde{X}$. We have $\tilde{Y} \subseteq [\inf Y, \sup Y]$. Conversely, if $\inf Y < z < \sup Y$ then there are $x, y \in Y$ with $x < z < y$, so $z \in Y$. Thus $\tilde{Y} = [\inf Y, \sup Y]$, so Y is an interval.

(ii) $\Rightarrow$ (i). We proceed by contraposition. Suppose X is not Dedekind complete, and let $Y \subseteq X$ be nonempty, bounded above and without a supremum in X. Let

$$D(Y) = \{x \in X \mid x \le y \text{ for some } y \in Y\}.$$

Then $D(Y)$ is convex, bounded above and has no supremum, so not an interval. ■

The next question is when intervals are connected. For this, even completeness is not sufficient. For example, a finite ordered set with more than one element is complete but not connected: the order topology is discrete. The extra condition we need is as follows: an ordered set $(X, \le)$ is densely ordered if for all $x < y$ in X there is $z \in (x, y)$. A convex subset of a densely ordered set is again densely ordered.

Theorem 16. *For an ordered set X, the following are equivalent:*
(i) X is densely ordered and Dedekind complete.
(ii) X is connected in the order topology.

Proof. (i) $\Rightarrow$ (ii). Step 1. We suppose $\perp \in X$. Since X is densely ordered, a subset $S \subseteq X$ which contains $\perp$ and is both open and closed in the order topology is inductive. Since X is Dedekind complete, by Theorem 14, $S = X$. This shows X is connected.

Step 2. We may assume $X \neq \varnothing$ and choose $a \in X$. By Proposition 3, Step 1 applies to show ${}^{\geq}a$ is connected. A similar downward induction argument shows ${}^{\leq}a$ is connected. Since $X = {}^{\leq} a \cup {}^{\geq}a$ and ${}^{\leq}a \cap {}^{\geq}a \neq \varnothing$, X is connected.

(ii) $\Rightarrow$ (i). We proceed by contraposition. First, if X is not densely ordered, there are $a < b$ in X with $[a, b] = \{a, b\}$, so $A = {}^{\leq}a$, $B = {}^{\geq}b$ is a separation of X.

Next, suppose that X is densely ordered but there is a subset $S \subseteq X$ that is nonempty, bounded below by a and with no infimum. Let L be the set of lower bounds for X. Since S has no infimum, for all $\ell \in L$ there is $\ell' \in L$ with $\ell' > \ell$, and thus

$$\ell \in {}^{<}\ell' \subset L.$$

This shows that L is open. Now let $x \in X \setminus L$. Then x is not a lower bound for S, so there is $s \in S$ with $s < x$. Since X is densely ordered, there is $y \in X$ with $s < y < x$, and then

$$x \in {}^{>}y \subset X \setminus L.$$

This shows that L is closed. Since $a \in L$ and X is connected, we must have $L = X$. Thus $L \cap S = S$ is nonempty, but any element of $L \cap S$ is an infimum for S, a contradiction. ∎

Corollary 2. *Let $(X, \leq)$ be densely ordered and Dedekind complete. For a subset $Y \subseteq X$, the following are equivalent:*
(i) Y is connected in the order topology.
(ii) Y is convex.
(iii) Y is an interval.

Proof. (i) $\Rightarrow$ (ii) was shown above for any order topology.

(ii) $\Rightarrow$ (iii) by Proposition 4.

(iii) $\Rightarrow$ (i): Being an interval, Y is a convex subset of a densely ordered set, so Y is densely ordered. By Proposition 3, Y is Dedekind complete, so by Theorem 16, Y is connected in the order topology. ∎

Above we saw that an ordered field F is Dedekind complete if and only if there are $a < b$ in F such that the interval $[a, b]$ is complete. This has the following consequence.

Corollary 3. *Let $(F, +, \cdot, <)$ be an ordered field. The following are equivalent:*
(i) F is Dedekind complete (and thus isomorphic to $\mathbb{R}$).
(ii) Every closed interval $[a, b]$ of F is connected in the order topology.
(iiii) For some $a < b$ in F, the interval $[a, b]$ is connected in the order topology.

It follows that if $I \subseteq \mathbb{R}$ is an interval and $f : I \to \mathbb{R}$ is continuous, then $f(I)$ is again an interval. If I is closed and bounded, then it is compact, so $f(I)$ is again closed and bounded. Conversely, it is a nice exercise to show that if $I, J \subseteq \mathbb{R}$ are intervals, each consisting of more than one point, and J is closed and bounded if I is, then there is a continuous function $f : I \to \mathbb{R}$ with $f(I) = J$.

Completeness and compactness

Theorem 17. *For an ordered set X, the following are equivalent:*
(i) X is complete.
(ii) X is compact in the order topology.

Proof. (i) $\Rightarrow$ (ii). Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of X. Let S be the set of $x \in X$ such that the covering $\mathcal{U} \cap [\perp, x]$ of $[\perp, x]$ admits a finite subcovering. We have $\perp \in S$, so S satisfies (IS1). Suppose $U_1 \cap [\perp, x], \ldots, U_n \cap [\perp, x]$ covers $[\perp, x]$. If there exists $y \in X$ such that $[x, y] = \{x, y\}$, then adding to the covering any element U_y containing y gives a finite covering of $[\perp, y]$. Otherwise some U_i contains x and hence also $[x, y]$ for some $y > x$. So S satisfies (IS2). Now suppose that $x \neq \perp$ and $[\perp, x) \subseteq S$. Let $i_x \in I$ be such that $x \in U_{i_x}$, and let $y < x$ be such that $(y, x] \subseteq U_{i_x}$. Since $y \in S$, there is a finite $J \subseteq I$ with $\bigcup_{i \in J} U_i \supset [a, y]$, so $\bigcup_{i \in J} U_i \cup U_{i_x} \supset [a, x]$. Thus $x \in S$ and S satisfies (IS3). Thus S is inductive; since X is Dedekind complete, we have $S = X$. In particular $\top \in S$, hence the covering has a finite subcovering.

(ii) $\Rightarrow$ (i). For each $x \in X$ there is a bounded open interval I_x containing x. If X is compact, $\{I_x\}_{x \in X}$ has a finite subcovering, so X is bounded, i.e., has $\perp$ and $\top$. Let $S \subseteq X$. Since $\inf \varnothing = \top$, we may assume $S \neq \varnothing$. Let L be the set of lower bounds for S. For each $(b, s) \in L \times S$, consider the closed interval $C_{b,s} := [b, s]$. For any finite subset $\{(b_1, s_1), \ldots, (b_n, s_n)\}$ of $L \times S$, we have

$$\bigcap_{i=1}^{N} [b_i, s_i] \supset [\max b_i, \min s_i] \neq \varnothing.$$

Since X is compact, there is $y \in \bigcap_{L \times S} [b, s]$ and then $y = \inf S$. ■

Corollary 4 (Generalized Heine–Borel)**.** *(a) For an ordered set X, the following are equivalent:*
(i) X is Dedekind complete.
(ii) A subset S of X is compact in the order topology if and only if it is closed and bounded.
(b) For an ordered field F, the following are equivalent:
(i) F is Dedekind complete (and thus isomorphic to $\mathbb{R}$).
(ii) Every closed bounded interval $[a, b] \subseteq F$ is compact.
(iii) For some $a < b$ in F, the interval $[a, b]$ is compact.

Proof. (a) (i) $\Rightarrow$ (ii). A compact subset of any ordered space is closed and bounded. Conversely, if X is Dedekind complete and $S \subseteq X$ is closed and bounded, then by Proposition 3, S is complete and then by Theorem 17, S is compact.

(ii) $\Rightarrow$ (i). If $S \subseteq X$ is nonempty and bounded above, let $a \in S$. Then $S' = S \cap {}^{\geq}a$ is bounded, so $\overline{S'}$ is compact and thus $\overline{S'}$ is complete by Theorem 17. The least upper bound of $\overline{S'}$ is also the least upper bound of S.

(b) This follows immediately from part (a) and Corollary 1. ■

Comments and complements The notion of Dedekind completeness goes back to Dedekind's construction of $\mathbb{R}$ using Dedekind cuts. In an ordered set X, a *Dedekind cut* is a pair (L, R) of subsets of X such that R is the set of upper bounds for L and L is the set of lower bounds for R. Then $L \cup R = X$. Indeed, let $x \in X$; if $x \notin R$ then there is $\ell \in L$ with $x < \ell$, and if $x \notin L$ then there is $r \in R$ with $r < x$, but then $r < x < \ell$, so $r \in R$ is not an upper bound for L. Similarly, $L \cap R$ is either empty or consists of a single point $x = \top_L = \perp_R$. In the latter case we say the cut is principal.

Example 1. If the ordered set X has a bottom element $\perp$, then $(\{\perp\}, X)$ is a principal cut in X; otherwise $(\varnothing, X)$ is a nonprincipal cut in X. Similarly, if X has a top element $\top$, then $(X, \{\top\})$ is a principal cut in X; otherwise $(X, \varnothing)$ is a nonprincipal cut in X.

Example 2. In $\mathbb{Q}$,

$$L = \{x \in \mathbb{Q} \mid x < \sqrt{2}\}, \ R = \{x \in \mathbb{Q} \mid \sqrt{2} < x\}$$

is a nonprincipal cut. To define a similar cut in $\mathbb{R}$ we must place $\sqrt{2}$ in both L and R, giving a principal cut. However $(\varnothing, \mathbb{R})$ and $(\mathbb{R}, \varnothing)$ are nonprincipal cuts in $\mathbb{R}$.

An ordered set X is complete (resp. Dedekind complete) if and only if every cut (resp. every cut different from $(\varnothing, X)$ and $(X, \varnothing)$) is principal. (We leave this for the interested reader to prove—by the principle of ordered induction or otherwise!) Let $\mathcal{C}(X)$ be the set of cuts in X. For (L_1, R_1), (L_2, R_2) in X, we put $(L_1, R_1) \le (L_2, R_2)$ if and only if $L_1 \subseteq L_2$. This makes $\mathcal{C}(X)$ into a complete ordered set. For $x \in X$, we define

$$\iota(x) = (\{y \in X \mid y \le x\}, \{z \in X \mid x \le z\}).$$

Then $\iota : X \hookrightarrow \mathcal{C}(X)$ is an isotone injection. Thus every ordered set can be canonically embedded in a complete ordered set, which may expand the range of applicability of the Principle of Ordered Induction.

Problem 4. Use the embedding $\iota : X \hookrightarrow \mathcal{C}(X)$ to give applications of the principle of ordered induction to ordered sets X that are *not* Dedekind complete.

The principality of Dedekind cuts may not be the most initially appealing completeness axiom, but it can be an elegant proof technique. Propp puts it to good use in [**24**].

Most of the above results can be found piecemeal in various places. The implication (i) $\Rightarrow$ (ii) in Theorem 17 is due to Frink [**11**]. This is probably the more interesting direction. A different proof of (ii) $\Rightarrow$ (i) goes by contraposition: if X is not complete, then there is a nonprincipal Dedekind cut (L, R), which one can use to construct an open cover without a finite subcover: cf. [**29**, p. 67]. The implication (i) $\Rightarrow$ (ii) of Theorem 16 is treated by Munkres [**20**, Theorem 24.1]. Similarly, [**20**, Theorem 27.1] gives a portion of Corollary 4.

A subtlety arises when considering the topology on a subset Y of an ordered set $(X, \le)$. On the one hand, restricting $\le$ to Y makes Y an ordered set in its own right, and thus it gets an order topology. On the other hand, we can endow Y with the topology it inherits as a subspace of the order topology on X. In general the order topology is coarser than the subspace topology, and they need not coincide. For example, consider $Y = \{0\} \cup (1/2, 1] \subseteq \mathbb{R}$. Then $\{0\}$ is open in the subspace topology on Y but not in the order topology—with the order topology, Y is homeomorphic to $[\frac{1}{2}, 1]$. Moreover there is no ordering on Y that induces the subspace topology. Thus in the above results we were careful to specify "in the order topology."

For a convex subset $Y \subseteq X$ the two topologies coincide. We give an example in which the distinction matters. An ordered field F that is *not* Dedekind complete is totally disconnected in the order topology. That is, if $Y \subseteq F$ consists of more than a single point, then Y is not connected *in the subspace topology*. However, if $F \supset \mathbb{R}$—e.g., $F = \mathbb{R}(t)$—then the subset $\mathbb{R}$ is connected in the order topology.

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Summary. We introduce real induction, a proof technique analogous to Mathematical Induction but applicable to statements indexed by an interval on the real line. We apply these principles to give streamlined, conceptual proofs of basic results in elementary real analysis and topology. Then we pursue inductive principles in arbitrary ordered sets. Applications are given, e.g., to "real analysis in reverse."

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