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SOME UNDECIDABLE PROBLEMS INVOLVING ELEMENTARY FUNCTIONS OF A REAL VARIABLE

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Introduction. Let E be a set of expressions representing real, single valued, partially defined functions of one real variable. E^* will be the set of functions represented by expressions in E.

If A is an expression in E, $A(x)$ is the function denoted by A.

It is assumed that E^* contains the identity function and the rational numbers as constant functions and that E^* is closed under addition, subtraction, multiplication and composition. In every case it is also supposed that given A and B in E there is an effective procedure for finding expressions in E to represent

$$\begin{aligned} &A(x) + B(x),\\ &A(x) - B(x),\\ &A(x)\cdot B(x),\\ &A(B(x)). \end{aligned}$$

$A(x) \equiv B(x)$ will mean that $A(x)$ and $B(x)$ are defined at the same points and equal wherever they are defined.

The identity problem for (E, E^*) is the problem of deciding, given A in E, whether $A(x) \equiv 0$.

The integration problem for (E, E^*) is the problem of deciding, given A in E, whether there is a function $f(x)$ in E^* so that $f'(x) \equiv A(x)$.

The following will be proved.

If E^* satisfies condition 1 below, the problem, *given an expression A in E, decide if there is a real number, x, such that $A(x)$ is less than zero* will be unsolvable.

If E^* satisfies conditions 1 and 2, the identity problem for (E, E^*) will be unsolvable.

If E^* satisfies conditions 1, 2, and 3, the integration problem for (E, E^*) will be unsolvable.

(1) E^* contains $\log 2$, π, e^x, $\sin x$.

(2) There is a function, $\mu(x)$, in E^* so that $\mu(x) = |x|$ for $x \neq 0$.

(3) There is a totally defined function, $\mathfrak{Z}(x)$, in E^* so that for no function, $f(x)$, in E^* and no interval I, is $f'(x) \equiv \mathfrak{Z}(x)$ on I.

There are simple pairs (E, E^*) satisfying all the above conditions. For example we could take E to be the smallest class of expressions obtained by iteration of addition, subtraction, multiplication and composition starting from x, e^x, $\sin x$, $\sqrt[+]{x}$, π, $\log 2$, and expressions for the rational numbers. E^* would be the class of functions of a real variable usually represented by these expressions. $\mathfrak{Z}(x)$ could be e^{x^2}. $\mu(x)$ could be $\sqrt[+]{x^2}$.

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§1 is devoted to preliminary theorems. The undecidability results mentioned above are obtained in §2.

§1. Let the subelementary expressions be those obtained by iteration of addition, subtraction, multiplication and composition starting with e^x, $\sin x$, variables $x_1, x_2, \cdots, \pi$, $\log 2$, and constants for the rational numbers.

The subelementary functions will be those functions of one or more real variables which are usually represented by these expressions. To be given a subelementary function will mean being given a subelementary expression standing for it.

We show the impossibility of recursively deciding whether there is a real number (hence, by the continuity of these functions, an interval of real numbers) where a given subelementary function of one variable is less than zero.

No method is known for deciding whether a subelementary expression with no variables in it represents a number less than zero. Whether or not such a method exists is an open question.

INTRODUCTION TO THEOREM ONE. It is known (see [1]) that there is a subelementary function $P(y, x_1, \cdots, x_n)$, a polynomial with integral coefficients in y, $x_1, \cdots, x_n$, $2^{x_1}, \cdots, 2^{x_n}$ for which the predicate *there exist natural numbers* $x_1, \cdots, x_n$ *such that* $P(y, x_1, \cdots, x_n) = 0$ is not recursive as y varies over the natural numbers. This implies that the predicate *there exist nonnegative real numbers* $x_1, \cdots, x_n$ *such that*

$$P^2(y, x_1, \cdots, x_n) + \sum_{i=1}^{n} \sin^2 \pi x_i = 0$$

is not recursive as y varies over the natural numbers.

Let $K(y, x_1, \cdots, x_n)$ be a rapidly increasing subelementary function with $K(0, 0, \cdots, 0)$ very large. Then if $K(y, x_1, \cdots, x_n) \cdot [P^2(y, x_1, \cdots, x_n) + \sum_{i=1}^{n} \sin^2 \pi x_i]$ were less than 1 for a natural number value of y and nonnegative values of $x_1, \cdots, x_n$ this would imply that $x_1, \cdots, x_n$ were close to natural numbers and $|P(y, x_1, \cdots, x_n)|$ was small. Presumably, $|P(y, \langle x_1 \rangle, \cdots, \langle x_n \rangle)|$ would also be small. Here $\langle x \rangle$, defined for nonnegative real values of x, means the nearest natural number to x, or the nearest natural number below x if x is equidistant from two natural numbers.

If $|P(y, \langle x_1 \rangle, \cdots, \langle x_n \rangle)|$ is small (less than one) for a natural number value of y and nonnegative values of $x_1, \cdots, x_n$, it must be equal to zero since P maps natural numbers onto integers. This suggests that if the function K were chosen large enough we would have, for natural number values of y, *there are nonnegative real numbers* $x_1, \cdots, x_n$ *such that* $K(P^2 + \sum_{i=1}^{n} \sin^2 \pi x_i)$ *is less than one* equivalent with *there are natural numbers* $x_1, \cdots, x_n$ *such that* P *is equal to zero*. This is the basic idea of theorem one. The main difference will be that instead of multiplying $P^2 + \sum_{i=1}^{n} \sin^2 \pi x_i$ by a single function K, we will break it into parts and multiply each part by a rapidly increasing function.

LEMMA ONE. *For any subelementary function* $f(x_1, \cdots, x_n)$, *it is possible to find a subelementary function* $g(x_1, \cdots, x_n)$ *so that*

(1) $\forall x_1, \cdots, x_n g(x_1, \cdots, x_n) > 1$,

(2) $\forall x_1, \cdots, x_n, \Delta_1, \cdots, \Delta_n(|\Delta_1| \leq 1 \;\&\; |\Delta_2| \leq 1 \;\&\; \cdots \;\&\; |\Delta_n| \leq 1 \rightarrow$
$$g(x_1, \cdots, x_n) > |f(x_1 + \Delta_1, \cdots, x_n + \Delta_n)|.$$

PROOF. The proof is by induction on the number of steps which are needed to arrive at an expression for $f(x_1, \cdots, x_n)$. Expressions obtained in 0 steps are either constants or variables. Expressions obtained in $n + 1$ steps are of the form $A + B$, $A - B$, $A \cdot B$, e^A, or $\sin A$, where A and B are expressions obtained in $\leq n$ steps. All subelementary expressions are eventually obtained in this way.

First suppose that $f(x_1, \cdots, x_n)$ is obtained in 0 steps. If $f(x_1, \cdots, x_n) = c$, let $g(x_1, \cdots, x_n) = |c| + 2$. If $f(x_1, \cdots, x_n) = x_i$, let $g(x_1, \cdots, x_n) = x_i^2 + 2$. In either case the conditions of the lemma are met.

Next suppose that $A(x_1, \cdots, x_n)$ and $B(x_1, \cdots, x_n)$ are obtained in k or less steps. Suppose that

(i) $g_1(x_1, \cdots, x_n) > |A(x_1 + \Delta_1, \cdots, x_n + \Delta_n)|$; $g_1(x_1, \cdots, x_n) > 1$,

(ii) $g_2(x_1, \cdots, x_n) > |B(x_1 + \Delta_1, \cdots, x_n + \Delta_n)|$; $g_2(x_1, \cdots, x_n) > 1$ for any $x_1, \cdots, x_n$ and any $\Delta_1, \cdots, \Delta_n$ with $|\Delta_i| \leq 1$.

If $f(x_1, \cdots, x_n) = A(x_1, \cdots, x_n) \pm B(x_1, \cdots, x_n)$, let $g(x_1, \cdots, x_n) = g_1(x_1, \cdots, x_n) + g_2(x_1, \cdots, x_n)$.

If $f(x_1, \cdots x_n) = A(x_1, \cdots, x_n) \cdot B(x_1, \cdots, x_n)$, let $g(x_1, \cdots, x_n) = g_1(x_1, \cdots, x_n) \cdot g_2(x_1, \cdots, x_n)$.

If $f(x_1, \cdots, x_n) = e^{A(x_1, \cdots, x_n)}$, let $g(x_1, \cdots, x_n) = e^{g_1(x_1, \cdots, x_n)}$.

If $f(x_1, \cdots, x_n) = \sin A(x_1, \cdots, x_n)$, let $g(x_1, \cdots, x_n) = 2$.

In every case, $g(x_1, \cdots, x_n) > 1$ and $g(x_1, \cdots, x_n) > |f(x_1 + \Delta_1, \cdots, x_n + \Delta_n)|$ for any $x_1, \cdots, x_n$ and any $\Delta_1, \cdots, \Delta_n$ with $|\Delta_i| \leq 1$.

The derivative or partial derivative of a subelementary function is again a subelementary function. To these derivatives the lemma can be applied. In particular, let P be a subelementary function such that

(i) For natural number values of y, there is no algorithm for deciding whether there are natural numbers $x_1, \cdots, x_n$ such that $P(y, x_1, \cdots, x_n) = 0$.

(ii) P maps natural numbers into integers.

According to the lemma, there are functions $k_i(y, x_1, \cdots, x_n)$ so that $|x_1 - \bar{x}_1| \leq 1$ and $|x_2 - \bar{x}_2| \leq 1$ and ... and $|x_n - \bar{x}_n| \leq 1 \rightarrow k_i(y, x_1, \cdots, x_n) > |D_{x_i}(P^2(y, \bar{x}_1, \cdots, \bar{x}_n))|$. We now let $f(y, x_1, \cdots, x_n) = (n + 1)^4[P^2(y, x_1, \cdots, x_n) + \sum_{i=1}^n (\sin^2 \pi x_i) k_i^4(y, x_1, \cdots, x_n)]$.

THEOREM ONE. *For every natural number y, the following conditions are equivalent.*

(1) *There are natural numbers $x_1, \cdots, x_n$ such that $P(y, x_1, \cdots, x_n) = 0$.*

(2) *There are nonegative real numbers $x_1, \cdots, x_n$ such that $f(y, x_1, \cdots, x_n) = 0$.*

(3) *There are nonnegative real numbers $x_1, \cdots, x_n$ such that $f(y, x_1, \cdots, x_n) \leq 1$.*

PROOF. We have immediately all the implications from top to bottom since $P(y, x_1, \cdots, x_n) = 0$ and $x_1, \cdots, x_n$ are natural numbers imply $f(y, x_1, \cdots, x_n) = 0$; and, of course, $f = 0$ implies that $f \leq 1$. To prove the theorem it is only necessary to establish

$$\exists \text{ nonnegative real } x_1, \cdots, x_n f(y, x_1, \cdots, x_n) \leq 1$$
$$\rightarrow \exists \text{ natural numbers } x_1, \cdots, x_n P(y, x_1, \cdots, x_n) = 0$$

under the assumption that y is a natural number.

$$f(y, x_1, \cdots, x_n) \leq 1$$
$$\rightarrow P^2(y, x_1, \cdots, x_n) + \sum_{i=1}^n (\sin^2 \pi x_i) k_i^4(y, x_1, \cdots, x_n) \leq 1/(n + 1)^4.$$

Then

(1) $P^2(y, x_1, \cdots, x_n) \leq 1/(n+1)^4$.

(2) For all i, $(\sin^2 \pi x_i)^{1/4} \cdot k_i(y, x_1, \cdots, x_n) \leq 1/(n+1)$.

Let $\langle x \rangle$ be the nearest natural number to x. We will need to know that, for all nonnegative x, $|\langle x \rangle - x| \leq (\sin^2 \pi x)^{1/4}$. It is sufficient to prove that $x^2 \leq \sin \pi x$ for x between 0 and $\frac{1}{2}$. This is true when $x = 0$ and when $x = \frac{1}{2}$. Letting $g(x) = \sin \pi x - x^2$, $g''(x) < 0$ in the interval, $g'(0) > 0$, and $g'(\frac{1}{2}) < 0$. Therefore the minimum of $g(x)$ in the interval is the left endpoint.

From (2) above:

$$(\sin^2 \pi x_i)^{1/4} \cdot k_i(y, x_1, \cdots, x_n) \leq 1/(n+1).$$

Then $|\langle x_i \rangle - x_i| k_i(y, x_1, \cdots, x_n) \leq 1/(n+1)$.

$$\sum_{i=1}^{n} |\langle x_i \rangle - x_i| k_i(y, x_1, \cdots, x_n) \leq n/(n+1).$$

(3) By (1): $P^2(y, x_1, \cdots, x_n) + \sum_{i=1}^{n} |\langle x_i \rangle - x_i| k_i(y, x_1, \cdots, x_n) < 1$.

Using the mean value theorem, $P^2(y, \langle x_1 \rangle, \cdots, \langle x_n \rangle) \leq P^2(y, x_1, \cdots, x_n) + \sum_{i=1}^{n} |\langle x_i \rangle - x_i| \cdot |D_{x_i} P^2(y, \bar{x}_1, \cdots, \bar{x}_n)|$ where $\bar{x}_i$ is between x_i and $\langle x_i \rangle$. But for $\bar{x}_i$ in this interval, $|D_{x_i} P^2(y, \bar{x}_1, \cdots, \bar{x}_n)| < k_i(y, x_1, \cdots, x_n)$. By (3), $P^2(y, \langle x_1 \rangle, \cdots, \langle x_n \rangle) < 1$. We suppose y is a natural number. P^2 maps natural numbers into integers. Therefore, $P^2(y, \langle x_1 \rangle, \cdots, \langle x_n \rangle) = 0$ and $P(y, \langle x_1 \rangle, \cdots, \langle x_n \rangle) = 0$.

COROLLARY. *There is a subelementary function of $n+1$ variables—which will be written $F(y, x_1, \cdots, x_n)$ from now on—for which, as y varies over the natural numbers,*

(1) *There are real numbers $x_1, \cdots, x_n$ such that $F(y, x_1, \cdots, x_n) \leq 1$ if and only if there are real numbers $x_1, \cdots, x_n$ such that $F(y, x_1, \cdots, x_n) = 0$.*

(2) *There is no algorithm for deciding whether or not there are real numbers $x_1, \cdots, x_n$ such that $F(y, x_1, \cdots, x_n) = 0$.*

PROOF. Let $F(y, x_1, \cdots, x_n) = f(y, x_1^2, \cdots, x_n^2)$ where f is the function which appeared in Theorem One.

INTRODUCTION TO THEOREM TWO. The purpose of Theorem Two is to replace the variables $x_1, \cdots, x_n$ in $F(y, x_1, \cdots, x_n)$ by different functions of x. It is shown that if $h(x) = x \sin x$, and if $g(x) = x \sin x^3$, and if $f(y, x_1, \cdots, x_n)$ is any subelementary function, *there are real numbers, $x_1, \cdots, x_n$, such that $f(y, x_1, \cdots, x_n)$ is less than one* is equivalent with *there is a real number x greater than zero such that $f(y, h(x), h(g(x)), \cdots, h(g(g \cdots g(x)) \cdots), g(g(g(\cdots g(x)) \cdots)$ is less than one.* Since the subelementary functions are all continuous, it will be sufficient to prove that for any δ between zero and one, and for any values $x_1, \cdots, x_n$ we can find a number x greater than zero so that

$$\begin{aligned}
&|h(x) - x_1| < \delta \\
&|h(g(x)) - x_2| < \delta \\
&\vdots \\
&|h(g(g(\cdots g(x)) \cdots) - x_{n-1}| < \delta \\
&g(g(g(\cdots g(x)) \cdots) - x_n = 0.
\end{aligned}$$

The nth function is the same as the $(n-1)$th except that the initial h has been changed to a g. For example, for four variables, the replacing functions would be $h(x), h(g(x)), h(g(g(x))), g(g(g(x)))$.

Consider any particular values x_1, x_2 and δ. There are an infinite number of positive intervals in which $|h(x) - x_1| < \delta$. Let D_1 be the first of these, D_2 the second and so on. D_{n+1} is within 2π of D_n. On D_1, $g(x)$ takes all values in a certain range, R_1. Similarly, on D_n, $g(x)$ takes all values in a certain range, R_n. Since $g(x)$ varies more precipitously than $h(x)$, the size of the R's is increasing. Eventually, the R's become so large that they include x_2. In Theorem Two this argument—in a different form— is extended by induction.

LEMMA TWO. *For any x_1 and x_2 and any number $\delta > 0$, there are numbers w_1 and w_2 so that*

(1) $w_2 > w_1 > |x_2|$,
(2) $w_2 \sin w_2 = x_1$,
(3) $w_2^3 - w_1^3 > 2\pi$,
(4) $(w_2 - w_1)(w_2 + 1) < \delta$.

PROOF. Let w_1 range over values $> |x_2| + 1$. $(w_2 - w_1)(w_2 + 1) < (w_2 - w_1)(w_2 + w_1)$ if $w_2 > w_1$. For each w_1, choose w_2 as $(w_1^2 + \delta/2)^{1/2}$. (1) and (4) are satisfied by all such pairs (w_1, w_2). $w_2^3 - w_1^3 = w_2(w_1^2 + \delta/2) - w_1^3 > \delta w_2/2$. Then if $w_1 > 4\pi/\delta$ (w_1, w_2) satisfies (1), (3) and (4). Pick w_1 in $(4\pi/\delta, \infty)$ so that $w_2 \sin w_2 = x_1$.

LEMMA THREE. *Let $h(w) = w \sin w$, $g(w) = w \sin w^3$. For any x_1 and x_2 and any $\delta > 0$, there is a $w > 0$ so that*

$$|h(w) - x_1| < \delta, \qquad g(w) = x_2.$$

PROOF. Let w_2 and w_1 be two numbers such that

$$\begin{aligned} &w_2 > w_1 > |x_2|, \\ &h(w_2) = x_1, \\ &w_2^3 - w_1^3 > 2\pi, \\ &(w_2 - w_1)(w_2 + 1) < \delta. \end{aligned}$$

Since w_2 and w_1 are greater than $|x_2|$ and since $w_2^3 - w_1^3 > 2\pi$ there is a number w between w_1 and w_2 so that $g(w) = x_2$.

We want to show that $|h(w) - x_1| < \delta$.

$$\begin{aligned} |h(w) - x_1| &= |h(w) - h(w_2)| \\ &\le (w_2 - w_1)|(\sin x + x \cos x)| \quad \text{for some } x \text{ between } w_2 \text{ and } w_1 \\ &< (w_2 - w_1)(w_2 + 1) \\ &< \delta. \end{aligned}$$

THEOREM TWO. *For any real numbers $x_1, \cdots, x_n$ and any $\delta > 0$, there is a number w so that*

$$\begin{aligned} &|h(w) - x_1| < \delta \\ &|h(g(w)) - x_2| < \delta \\ &\vdots \\ &|h(g(g(\cdots g(w)) \cdots) - x_{n-1}| < \delta \\ &g(g(g(\cdots g(w)) \cdots) = x_n. \end{aligned}$$

PROOF. By the preceding lemma, the result is known when n is two. Suppose we have the result when $n = k$. There is a number w^* so that

$$\begin{aligned}&|h(w^*) - x_2| < \delta\\ &|h(g(w^*)) - x_3| < \delta\\ &\vdots\\ &|h(g(g(\cdots g(w^*))\cdots) - x_k| < \delta\\ &g(g(g(\cdots g(w^*))\cdots) = x_{k+1}.\end{aligned}$$

By Lemma Three, there is a number w so that

$$|h(w) - x_1| < \delta, \qquad g(w) = w^*.$$

This is the number that we were required to find.

COROLLARY TO THEOREM TWO. *There is a subelementary function of two variables, $G(y, x)$, such that, as y varies over the natural numbers,*

(1) *There is no algorithm for deciding whether there is a real number x such that $G(y, x) \leq 1$.*

(2) *There is a real number x such that $G(y, x) \leq 1$ if and only if, for every $z > 0$, there is a real number x such that $G(y, x) < z$.*

PROOF. Let

$$G(y, x) = F(y, h(x), h(g(x)), \cdots, h(g(g(\cdots g(x))\cdots), g(g(g(\cdots g(x))\cdots)).$$

Here F is the subelementary function obtained in the corollary to Theorem One. Suppose y is a natural number.

$$\begin{aligned}\exists \text{ real number } xG(y, x) \leq 1 &\rightarrow \exists \text{ real numbers } x_1, \cdots, x_nF(y, x_1, \cdots, x_n) = 0\\ &\rightarrow \forall z > 0 \exists \text{ real number } xG(y, x) < z.\end{aligned}$$

$\exists$ *real number* $xG(y, x) \leq 1$ is not decidable because it is equivalent to $\exists$ *real numbers* $x_1, \cdots, x_nF(y, x_1, \cdots, x_n) \leq 1$.

§2. E is a set of expressions and E^* is a set of functions as described in the introduction.

THEOREM ONE. *If E^* satisfies condition one of the list of conditions given in the introduction, there is no algorithm for deciding, given an expression A in E, whether there is a real number x such that $A(x) < 0$.*

PROOF. The rational numbers, x, $\log 2$, π, e^x, $\sin x$ are in E^*.

Then there are expressions in E to represent each of these. Every subelementary function of one variable is built up from these functions by iteration of composition, addition, subtraction and multiplication.

We are assuming that given any expressions A and B in E, there is an effective procedure for finding expressions in E to represent

$$\begin{aligned}&A(x) + B(x),\\ &A(x) - B(x),\\ &A(x)\cdot B(x),\\ &A(B(x)).\end{aligned}$$

Thus, given any subelementary function of one variable, $f(x)$, there is an effective procedure for finding an expression A in E so that $A(x) \equiv f(x)$.

Let $G(n, x)$ be the subelementary function produced in the corollary to Theorem Two, §1. For each natural number, n_0, we can find an expression A_{n_0} in E so that $A_{n_0}(x) \equiv G(n_0, x) - 1$.

If we could decide $\exists xA(x) < 0$ for each expression A in E we could also decide $\exists xG(n, x) < 1$ for each natural number, n. But this is not possible.

Theorem Two. *If E^* satisfies conditions one and two the identity problems for (E, E^*) is unsolvable.*

Proof. We have the function $\mu(x)$ in E^*.

By the same argument as was used in Theorem One, if we could decide whether $A(x) \equiv 0$ for each A in E, we could also decide whether

$$\mu(G(n,x) - 1) - (G(n,x) - 1) \equiv 0$$

for each natural number n.

Call this function $B_n(x)$.

$$\forall x G(n,x) > 1 \rightarrow B_n(x) \equiv 0.$$

$$\exists x G(n,x) \leq 1 \rightarrow \exists x G(n,x) < 1 \rightarrow B_n(x) \not\equiv 0.$$

So,

$$\exists x G(n,x) < 1 \leftrightarrow B_n(x) \not\equiv 0.$$

Lemma Three. *Define*

$$x \dot{-} y = \frac{\mu(x - y) + (x - y)}{2}.$$

Define $m(x, y) = x - (x \dot{-} y)$.
Then

(1) $\neg\exists x G(n,x) < 1 \leftrightarrow m(1, 2 \dot{-} 2G(n,x)) \equiv 0,$

(2) $\exists x G(n,x) < 1 \leftrightarrow$ *on some interval, I,*

$$m(1, 2 \dot{-} 2G(n,x)) \equiv 1.$$

Proof.

$$x > y \rightarrow x \dot{-} y = x - y$$
$$x < y \rightarrow x \dot{-} y = 0$$
$$x > y \rightarrow m(x,y) = y$$
$$x < y \rightarrow m(x,y) = x$$

$$\begin{aligned}
&\neg\exists x G(n,x) < 1\\
\leftrightarrow\ &\neg\exists x G(n,x) \leq 1\\
\leftrightarrow\ &2 \dot{-} 2G(n,x) \equiv 0\\
\leftrightarrow\ &m(1, 2 \dot{-} 2G(n,x)) \equiv 0\\
&\exists x G(n,x) < 1\\
\leftrightarrow\ &\exists x G(n,x) < \tfrac{1}{2}\\
\leftrightarrow\ &\text{there is an interval, } I, \text{ on which } 2 \dot{-} 2G(n,x) > 1\\
\leftrightarrow\ &\text{there is an interval, } I, \text{ on which } m(1, 2 \dot{-} 2G(n,x)) \equiv 1.
\end{aligned}$$

Theorem Three. *If E^* satisfies conditions* 1, 2, *and* 3 *the integration problem for (E, E^*) is unsolvable.*

Proof. If the integration problem for (E, E^*) were solvable, we would be able to decide, for each natural number n, whether there were a function $f(x)$ in E^* so that

$$f'(x) \equiv \mathfrak{Z}(x) \cdot m(1, 2 \dot{-} 2G(n,x))$$

where $\mathfrak{Z}(x)$ is a function which has no integral in E^*.

But by the previous lemma, this is impossible.

REFERENCES

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