Decidability of equality of elementary function solutions of constructible of rational differential equations

ALGEBRAIC PROPERTIES OF THE ELEMENTARY FUNCTIONS OF ANALYSIS.

By ROBERT H. RISCH.

Abstract. The elementary functions of a complex variable z are those functions built up from the rational functions of z by exponentiation, taking logarithms, and algebraic operations. The purpose of this paper is first, to prove a "structure theorem" which shows that if an algebraic relation holds among a set of elementary functions, then they must satisfy an algebraic relation of a special kind. Then we make four applications of this theorem, obtaining both new and old results which are described here briefly (and imprecisely).

(1) An algorithm is given for telling when two elementary expressions define the same function.
(2) A characterization is derived of those ordinary differential equations having elementary solutions.
(3) The four basic functions of elementary calculus—exp, log, tan, $\tan^{-1}$ —are shown to be "irredundant."
(4) A characterization is given of elementary functions possessing elementary inverses.

Introduction. The elementary functions of a complex variable z are those analytic functions that are built up from the rational functions of z by successively applying algebraic operations, exponentiating, and taking logarithms. As is well known, this class includes the trigonometric and basic inverse trigonometric functions.

In the 1830s and '40s Joseph Liouville wrote the first significant papers on the algebraic properties of these functions. A few subsequent writers discussed other properties (some of which will be mentioned shortly). The reader of these papers will notice a number of arguments and techniques they hold in common. This suggests that there might be a general theorem from which the other results would follow.

Here we have endeavored to give such a theorem: the Structure Theorem for Elementary Extensions of Part I. Roughly speaking, it says, in its most useful

Manuscript received August 8, 1975.

American Journal of Mathematics, Vol. 101, No. 4, pp. 743–759
0002-9327/79/1014-743 $01.50

case, that the only algebraic relations, holding among a given set of elementary functions, are of the type $\sum r_i y_i + \sum s_i \log z_i = c_1$ or $\prod (e^{y_i})^{r_i} \prod z_i^{s_i} = c_2$, where r_i, s_i are in $\mathbb{Z}$, c_1, c_2 are in $\mathbb{C}$, and $e^{y_i}, \log z_i$ are all the exponentials and logarithms used in building up the given set of elementary functions.

From this, we go on in Part II to obtain some consequences of the theorem. The four sections of Part II are completely independent of each other.

In Section 1 we outline an algorithm for determining the structure of finitely generated fields of elementary functions. In particular, it shows (again roughly speaking) how to tell if two expressions involving exponentials, logarithms, and algebraic operations define the same elementary function. We believe that the algorithm will work in most practical situations. However, its universal applicability depends on the solution of some extremely difficult problems in the theory of transcendental numbers.

Next, we turn to differential equations

$$\frac{dy_i}{dz} = f_i(z, y_1, \ldots, y_n), \qquad i = 1, \ldots, n,$$

How do we know that the system has a unique solution, and this solution is elementary?

with the f_i ~~algebraic~~ rational functions. Suppose such a system has a particular solution $y(z) = (y_1(z), \ldots, y_n(z))$ with all y_i elementary. Also suppose the smallest algebraic variety V in $\mathbb{C}^{n+1}$ that contains the graph of $y(z)$ is of dimension $k+1 \leqslant n+1$. Then a Zariski open subset of V is the union of elementary integral curves of the differential equation; i.e., we have a k parameter family of elementary solutions. Furthermore, such a solution is of a special type, being in a field of elementary functions that is obtained from the rational functions of z by constructing a tower of fields consisting of algebraic extensions alternating with k simple transcendental extensions $\mathfrak{D}(\theta)$, where θ is of the form $v_0 + \sum c_i \log v_i$ or $e^{v_0} \prod v_i^{c_i}$, $c_i \in \mathbb{C}$, $v_i \in \mathfrak{D}$. This result is due to Michael Singer [8, 9]. The case $k = 1$ was previously proved by D. D. Mordoukay-Boltovskoy [3, 4].

In Section 3 we give a real version of the structure theorem. As a consequence we show that the four basic transcendental functions of elementary calculus courses—logarithm, exponential, tangent, and inverse tangent—are irredundent; i.e., none can be expressed in terms of the others provided we restrict ourselves to algebraic operations which don't introduce nonreal complex numbers. Although this fact has undoubtedly been previously surmised, perhaps by millions, I've never seen it mentioned in print (except by myself in *Notices Amer. Math. Soc.*, 1970, p. 762).

In his most involved paper on the subject of elementary functions [6], J. F. Ritt proved that if $f(z)$ is elementary with an elementary inverse then $f(z) = \psi_n \circ \cdots \circ \psi_1(z)$, where each ψ_i is either an algebraic function of z or else e^z or

$\log z$. In Section 4 we derive this result, in a sharpened form, from the structure theorem.

The case of the structure theorem mentioned above was originally obtained as a corollary of our still unpublished general integration algorithm for elementary functions and appeared in an IBM research report in 1969. Shortly thereafter, Ax [1, and 2] generalized it under the name of the "Schanuel conjecture for differential fields." Ax showed how differential forms and the group $(\mathbb{C}\times\mathbb{C}^*)^n$ could be profitably introduced into the theory. These ideas are also used in the different generalization given here, with we hope greater simplicity and clarity, but without the attempt, as in [2], to relate the theory to the general theory of algebraic groups.

During the time when research leading to this paper was in progress, the author was partially supported by A.F.O.S.R. Contracts F44620-70C-0071 and F44620-73C-0018 with IBM and by A.F.O.S.R. Grant 2059-71 to the Institute for Advanced Study.

Part I. The Structure Theorem.

First some notation. We will work with fields $\mathfrak{D}$, $\mathfrak{E},\ldots$ that are always finitely generated over some algebraically closed field K of characteristic 0 (except, as will be noted, in Section 3 of Part II). If $x_1,\ldots,x_t$ is a transcendence basis of $\mathfrak{D}/K$, then all K derivations of $\mathfrak{D}$ are of the form $\sum_{i=1}^t g_i\,\partial/\partial x_i$, $g_i\in\mathfrak{D}$. $\Omega(\mathfrak{D})=$(the exterior differential algebra formed from $\mathfrak{D})=\sum_0^t\Omega^i(\mathfrak{D})$, with $\Omega^0(\mathfrak{D})=\mathfrak{D}$, $\Omega^1(\mathfrak{D})=\{\sum_{i=1}^t g_i\,dx_i : g_i\in\mathfrak{D}, dx_i(\partial/\partial x_j)=\delta_j^i\}=$dual space to the derivations of $\mathfrak{D}/K$. A pair $(\mathfrak{D},X)$, where X is a derivation of $\mathfrak{D}$, is called a differential field. We say that $(\mathfrak{F},Y)$ is a differential extension of $(\mathfrak{D},X)$ if $\mathfrak{F}\supseteq\mathfrak{D}$ and $Y|_{\mathfrak{D}}=X$. Alternatively we say here that $(\mathfrak{D},X)$ is a differential subfield of $(\mathfrak{F},Y)$. We also have a natural embedding $\Omega^1(\mathfrak{D})\rightarrow\Omega^1(\mathfrak{F})$ of $\mathfrak{D}$ vector spaces and hence a natural embedding of exterior differential algebras $\Omega(\mathfrak{D})\rightarrow\Omega(\mathfrak{F})$.

Definition. A differential extension $(\mathfrak{F},Y)$ of $(\mathfrak{D},X)$ is elementary over $(\mathfrak{D},X)$ iff $\mathfrak{F}=\mathfrak{D}(\theta_1,\ldots,\theta_n)$, where

(a) θ_i is algebraic over $\mathfrak{D}(\theta_1,\ldots,\theta_{i-1})$, or
(b) $Y\theta_i/\theta_i=Yb_i$ or $Yb_i/b_i=Y\theta_i$ for some $b_i\in\mathfrak{F}(\theta_1,\ldots,\theta_{i-1})$. We abbreviate these situations as $\theta_i=e^{b_i}$ and $\theta_i=\log b_i$ respectively .

We remark that a θ_i might satisfy both (a) and (b). When we use the expression "let $z_1=e^{y_1},\ldots,z_q=e^{y_q}$, $\log z_{q+1}=y_{q+1},\ldots,$ $\log z_r=y_r$ be all the exponentials and logs among $\theta_1,\ldots,\theta_n$," we mean that we have chosen one and only one of the three possibilities for each θ_i and fixed it for the discussion at hand.

Elementary functions of a single variable arise when our $\mathfrak{D}$ and $\mathfrak{F}$ are fields of meromorphic functions on some Riemann surface R, covering some region of the z plane with $X = Y = d/dz$ and $\mathfrak{D} \cong \mathbb{C}(z)$. If $y_1, \ldots, y_n$ are generators of $\mathfrak{F}$ over $\mathfrak{D}$, then $\mathfrak{F}$ is isomorphic to $\mathbb{C}(W)$, the field of rational functions on the Zariski closure W of the graph in $\mathbb{C}^{n+1}$ of the curve $y(z) = (y_1(z), \ldots, y_n(z))$. The isomorphism is given by associating $f \in \mathbb{C}(W)$ with $y^*f = f \circ y \in \mathfrak{F}$. We also have $dy_i/dz = y^*f_i$ for some $f_i \in \mathbb{C}(W)$. If $Z, Y_1, \ldots, Y_n$ are the coordinates on $\mathbb{C}^{n+1}$, then the rational vector field $Y = \partial/\partial Z + \sum_{i=1}^{n} \bar{f}_i \partial/\partial Y_i$ is tangent to W, where $\bar{f}_i \in \mathbb{C}(Z, Y_1, \ldots, Y_n)$ satisfies $f_i = \bar{f}_i | W$. $(z, y_1(z), \ldots, y_n(z))$ is an integral curve of the vector field. If we write Y for $Y|_W$ we have that under the field isomorphism y^*, Y and d/dz commute, i.e. $y^*(Yf) = (d/dz)(y^*(f))$. In particular, this means that Y has no nonconstant, rational first integrals; i.e., $Yf = 0$, $f \in \mathbb{C}(W)$ implies $f \in \mathbb{C}$.

Now back to the abstract situation. If $\omega_1, \ldots, \omega_k \in \Omega^1(V)$ $(= \Omega^1(\mathfrak{D}))$, V being any affine model of $\mathfrak{D}$) are a basis for the annihilator subspace of X, they generate an ideal I in $\Omega(V)$ that is closed under exterior differentiation (a "differential ideal"). The condition that kernel $X = K$ translates to: there is no nonzero, exact, rational one form df in I.

The structure theorem is an almost immediate consequence of the following lemma. Certain ideas in the proof go back to Abel and Liouville.

LEMMA. *Let $\mathfrak{F}$, $\mathfrak{D}$ be finitely generated extensions of K with $\mathfrak{F} = \mathfrak{D}(y_1, z_1, \ldots, y_n, z_n)$, I a differential ideal of $\Omega(\mathfrak{F})$ containing $dz_i/z_i - dy_i$, $i = 1, \ldots, n$, with an $\mathfrak{F}$ vector space basis of $\Omega^1(F) \cap I$ given by $\omega_1, \ldots, \omega_k$, $dz_1/z_1 - dy_1, \ldots, dz_r/z_r - dy_r$, $r < n$, where $\omega_1, \ldots, \omega_k$ generate $I \cap \Omega(\mathfrak{D})$. Furthermore we assume there is no $f \in \mathfrak{F}$, $f \notin K$ such that $df \in I$.*

Then for each $i > r$,

(1) *there are $c_{ij} \in K$, $h_i \in \tilde{\mathfrak{D}}$ $(=$ alg. closure of $\mathfrak{D}$ in $\mathfrak{F})$ such that*

$$y_i + \sum_{j=1}^{r} c_{ij} y_j = h_i.$$

If $1, c_{i,s+1}, \ldots, c_{i,r}$, $0 \leqslant s \leqslant r$, is a Q basis for the vector space generated by the c_{ij} (and this can always be arranged by a permutation), then

(2) *There are $n_i \neq 0$, $n_{ij} \in \mathbb{Z}$, $j_i \in \tilde{D}$ such that $z_i^{n_i} \prod_{j=1}^{s} z_j^{n_{ij}} = j_i$.*

Furthermore if $k = 0$, we can assert that h_i and j_i are in K, $s = r$ (i.e., the c_{ij} are in Q) and $c_{ij} = n_{ij}/n_i$, $j = 1, \ldots r$.

Proof. We can find affine models V, W of $\mathfrak{D}$ and $\mathfrak{F}$, respectively, such that $W \subset V \times (K \times K^*)^n$. The projection on the first factor π_1 sends W onto a Zariski dense subset of V. $\omega_1, \ldots, \omega_k$ are forms on W that are pullbacks, under π_1, of rational one forms on V. Fix $l > r$ and write y for y_l, z for z_l. Then

$$\left(\frac{dz}{z} - dy\right) + \sum_1^r f_i\left(\frac{dz_i}{z_i} - dy_i\right) = \sum_1^k g_i\omega_i.$$

on W, where $f_i, g_i \in \mathfrak{F}$.

Exterior differentiation yields

$$\sum_1^r df_i \wedge (dz_i/z_i - dy_i) = \sum_1^k \sigma_i \wedge \omega_i, \qquad \sigma_i \in \Omega^1(W).$$

Let $\lambda_i = \prod_{j \neq i}(dz_j/z_j - dy_j) \wedge \prod_1^k \omega_j$, and we find $df_i \wedge \lambda_i \wedge (dz_i/z_i - dy_i) = 0$. So $df_i \in I$ and thus $f_i \in K$.

We now write c_i for f_i and rewrite our equation as

$$\left(\frac{dz}{z} + \sum_1^r c_i\frac{dz_i}{z_i}\right) - \left(dy + \sum_i^r c_i dy_i\right) = \sum_1^k g_i\omega_i.$$

Let δ_1 be the form within the first pair of parentheses, and δ_2 that within the second. We claim that δ_2 (and consequently δ_1) is in $\mathfrak{F}\cdot\Omega^1(\mathfrak{D})$, the subspace of $\Omega^1(\mathfrak{F})$ generated by $\Omega^1(\mathfrak{D})$.

For if not, we can find a fiber F of the map $\pi: W \to V$ and an algebraic curve $C \subseteq F$ such that $\delta_2|_C$ is a non-identically-vanishing exact element of $\Omega^1(C)$. Also $\delta_1|_C = \delta_2|_C$, a relation that extends to $\bar{C}$ the normalization of the projective completion of C. On $\bar{C}$, δ_1 has poles of order at most 1, while δ_2 must have at least one pole of order $\geqslant 2$, a contradiction. Therefore $\delta_2 = d(y + \Sigma_1^r c_i y_i)$ is in $\mathfrak{F}\cdot\Omega^1(\mathfrak{D})$, and so $y + \Sigma_1^r c_i y_i$ must be in $\tilde{\mathfrak{D}}$.

Next let $1, c_{s+1}, \ldots, c_r$, $0 \leqslant s \leqslant r$ be a maximal subset of $1, c_1, \ldots, c_r$ having the properties of containing 1 and being linearly independent over Q. Express $c_1, \ldots, c_s$ in terms of this basis and let n be a common denominator for all the coefficients. Then

$$n\omega_1 = \frac{dw}{w} + \sum_{s+1}^r c_i\frac{dw_i}{w_i},$$

where $w = z^n \prod_1^s z_i^{n_i}$, the n_i being in $\mathbb{Z}$, with similar expressions holding for the w_i. We claim that dw/w (indeed, each dw_i/w_i) is in $\mathfrak{F}\cdot\Omega^1(D)$.

For otherwise, as before, we can find a fibre F of $\pi: W \to V$ and a curve $C \subseteq F$ such that $dw/w \neq 0$ in $\Omega^1(C)$. On $\bar{C}$, the normalization of the projective completion of C, find a pole p of dw/w. There,

$$\operatorname{Res}_p\left(\frac{dw}{w}\right) + \sum_{s+1}^{r} c_i \operatorname{Res}_p\left(\frac{dw_i}{w_i}\right) = 0,$$

a contradiction. Thus dw/w and consequently dw is in $\mathcal{F} \cdot \Omega^1(\mathfrak{D})$. This means $w \in \tilde{\mathfrak{D}}$ as desired.

If $k=0$, the above computations yield successively that $\delta_2 = \delta_1 = 0$ and then $dw/w = 0$ so $w = z^n \prod_1^s z_i^{n_i} \in K$. From this we get

$$n\frac{dz}{z} + \sum_1^s n_i \frac{dz_i}{z_i} = 0,$$

so $d(ny + \sum_1^s n_i y_i) \in I$ and finally $y + \sum_1^s (n_i/n) y_i \in K$. Q.E.D.

The Structure Theorem for Elementary Functions. Let $(\mathcal{E}, Y)$ be elementary over $(\mathfrak{D}, X)$. kernel $Y = K$. $\mathfrak{D}_1 = \mathfrak{D}(\theta_1, \ldots, \theta_m)$, $\mathcal{E} = \mathfrak{D}_1(\theta)$. Let $z_1 = e^{y_1}, \ldots, z_q = e^{y_q}$, $\log z_{q+1} = y_{q+1}, \ldots, \log z_r = y_r$ be the exponentials and logarithms occurring among $\theta_1, \ldots, \theta_m$. (See remark following definition at beginning of this part.) Suppose either $\theta = z = e^y$ is algebraic over $\mathfrak{D}_1$ or $\theta = y = \log z$ is algebraic over $\mathfrak{D}_1$ (where y or respectively z is in $\mathfrak{D}_1$). Then

(1) there are $c_i \in K$, $f \in \tilde{\mathfrak{D}}$ (= alg. closure of $\mathfrak{D}$ in E) such that $y + \sum_1^r c_i y_i = f$, and (after a permutation of indices) $1, c_{s+1}, \ldots, c_r$, $0 \leqslant s \leqslant r$ form a maximal Q linearly independent set of the coefficients $1, c_1, \ldots, c_r$;

(2) there are $n \neq 0$, $n_i \in \mathbb{Z}$, $g \in \mathfrak{D}$ such that $z^n \prod_{i=1}^s z_i^{n_i} = g$.

Furthermore if $(\mathfrak{D}, X) = (K(z), d/dz)$, then f and g can be chosen to be in K, $r = s$, and $c_i = n_i/n$, $i = 1, \ldots, r$.

Proof. Let $\omega_1, \ldots, \omega_k$ be a basis for the annihilator of X in $\Omega^1(\mathfrak{D})$, $F = \mathfrak{D}(y_1, z_1, \ldots, y_r, z_r, y, z)$, J the annihilator of Y in $\Omega^1(\mathcal{E})$, and I be the ideal of $\Omega(\mathcal{F})$ generated by $J \cap \Omega^1(\mathcal{F})$. It is easy to check that I satisfies the conditions of the lemma with $n = r+1$. Note that we can make the slightly stronger assertion that f and g are in the algebraic closure of $\mathfrak{D}$ in $\mathcal{F}$ rather than that of $\mathfrak{D}$ in $\mathcal{E}$. Q.E.D.

Example. Let

$$\mathfrak{D} = \mathbb{C}\left(z, \int \left[\frac{1}{z} + \frac{\sqrt{2}}{z+1}\right]\right),$$
$$\mathcal{E} = \mathfrak{D}(\log z, \log(z+1))$$

be fields of meromorphic functions on a suitable complex domain. In a manner similar to one described above, one can associate abstract differential fields with them. It can be shown that $\mathfrak{D}$ and $\mathfrak{S}$ have respectively transcendence degree 2 and 3 over $\mathbb{C}$. In $\mathfrak{D}$ we have the algebraic relation

$$\log(z+1)+\frac{\sqrt{2}}{2}\log z=\frac{\sqrt{2}}{2}\int\left[\frac{1}{z}+\frac{\sqrt{2}}{z+1}\right].$$

From this it follows that in case (1) of the Structure Theorem, when $\mathfrak{D}\neq K(z)$, we cannot assert that the coefficients of the y's will be rational.

This example can be verified by proving the following old result of Leo Königsberger, which is a partial analogue of the structure theorem when elementary extensions are replaced by the more general "Liouville extensions" where logs and exponentials are replaced respectively by integrals and exponentials of integrals.

THEOREM. *Let* $(\mathfrak{D},X)$ *be a differential field. Let* $(\mathfrak{D}(y_1,\ldots,y_r,z_1,\ldots,z_s),Y)$ *be a differential extension where* $\operatorname{kernel} X=\operatorname{kernel} Y=K$ *and*

$$Yy_i=\alpha_i,\qquad \alpha_i\in\mathfrak{D},\quad i=1,\ldots,r,$$
$$Yz_j=\beta_j z_j,\qquad \beta_j\in\mathfrak{D},\quad j=1,\ldots,s.$$

Suppose $\operatorname{tr\,degree}\mathfrak{D}(y_1\ldots,y_r,z_1,\ldots,z_s)/\mathfrak{D}<r+s$. *Then either there are* $c_i\in K$, *not all* 0, *such that* $\sum c_iy_i=f$, $f\in\mathfrak{D}$, *or there are* $n_i\in\mathbb{Z}$, *not all* 0, *such that* $\prod z_i^{n_i}=g$, $g\in\mathfrak{D}$.

This can be proved by mimicking the proof of the structure theorem along with the trick of taking norms and traces as in e.g. [5, p. 5].

Part II. Applications of the Structure Theorem.

1. Algorithm for Determining the Structure of Fields of Elementary Functions. Suppose $\mathbb{C}(z,\theta_1,\ldots,\theta_m)=\mathfrak{D}_m$ is the abstract field, isomorphic to a field of meromorphic functions on some region R of the complex plane, $(\mathfrak{D}_m, X)$ being elementary over $(\mathbb{C}(z),d/dz)$, where X is the appropriate derivation. Suppose we are told the minimal algebraic equation satisfied by θ_i over $\mathfrak{D}_{i-1}$, or else that $\theta_i=e_i^y$ or $\theta_i=\log z_i$ for y_i, z_i in $\mathfrak{D}_{i-1}$. The problem is then to determine which θ_i of the latter two kinds are algebraic over $\mathfrak{D}_{i-1}$ and give the minimal equations for those that are. In other words, we wish to know the precise algebraic structure of $\mathfrak{D}_m$. This would enable us to determine, in particular, when two elements of $\mathfrak{D}_m$ are equal.

The structure theorem gives a solution, provided the algebraic structure of the field generated over Q by the complex numbers appearing in the following process, is known.

By induction, we have the problem solved for $\mathbb{C}(z,\theta_1,\ldots,\theta_m)=\mathfrak{D}_m$, where $z_1=e^{y_1},\ldots,z_q=e^{y_q}$, $y_{q+1}=\log z_{q+1},\ldots,y_r=\log z_r$ is a maximal set of algebraically independent exponentials and logarithms occurring among $\theta_1,\ldots,\theta_m$.

Let $z=e^y$, $y\in\mathfrak{D}_m$. z is algebraic over $\mathfrak{D}_m$ iff there are integers $n\neq 0$, n_i such that $n\,dy=\sum_{i=1}^r n_i dy_i$. To determine whether this is so reduces to being able to solve linear equations with coefficients in the subfield of $\mathbb{C}$ that is generated over Q by the constants used to define $\theta_1,\ldots,\theta_m$. Determine the minimal positive n (provided it exists). Then $e^{ny}=c\prod_{i=1}^q e^{n_i y_i}\prod_{i=q+1}^r z_i^{n_i}$ where $c\in\mathbb{C}$. We determine c by evaluating both sides of this equation at some point in R. Then e^y satisfies $x^n-c\prod_{i=1}^r z_i^{n_i}=0$. This equation is irreducible over $\mathfrak{D}_m$, since that field contains the roots of unity.

Let $y=\log z$, $z\in\mathfrak{D}_m$. Then y is algebraic over $\mathfrak{D}_m$ iff there are rational r_i such that $dz/z=\sum_{i=1}^n r_i\,dy_i$. Then, as before, we determine by evaluation a $c\in\mathbb{C}$ so that $\log z=\sum_{i=1}^q r_i y_i+\sum_{i=q+1}^r r_i\log z_i+c$.

Thus, our problem is reduced to that of performing arithmetic operations of the c's that come up in this process.

2. Intermediate Extension Fields of Elementary Extensions. Suppose the system of algebraic differential equations $dy_i/dz=f_i(z,y_1,\ldots,y_n)$, $i=1,\ldots,n$, with the f_i rational functions, has an elementary particular solution; i.e., there are n elementary functions $y_1(z),\ldots,y_n(z)$ on some Riemann surface covering a region of the z plane satisfying the above system. If $\mathfrak{E}=\mathbb{C}(z,y_1,\ldots,y_n)$, then $(\mathfrak{E},d/dz)$ is an intermediate differential extension between $(\mathbb{C}(z),d/dz)$ and some elementary extension $(\mathfrak{F},d/dz)$ of $(\mathbb{C}(z),d/dz)$. $(\mathfrak{E},d/dz)$ itself will not, in general, be elementary over $(\mathbb{C}(z),d/dz)$. For example, $dy/dz=1/z+\sqrt{2}\,/(z+1)$ gives rise to $(\mathfrak{D},d/dz)$ at the end of Part I. It follows by the Königsberger theorem mentioned there, along with the trick of taking residues, that $(\mathfrak{D},d/dz)$ is not an elementary extension of $(\mathbb{C}(z),d/dz)$. It is, however, a differential subfield of the elementary extension $(\mathbb{C}(z,\log z,\log(z+1)),d/dz)$.

The theorem of Mordoukhay-Boltovskoy and Singer gives a characterization of such intermediate differential extensions.

Definition. A differential extension $(\mathfrak{E},Y)$ of $(\mathfrak{D},X)$ is *generalized elementary* over $(\mathfrak{D},X)$ iff $\mathfrak{E}=\mathfrak{D}(\theta_1,\ldots,\theta_m)$, where

(a) θ_i is algebraic over $\mathfrak{D}(\theta_1,\ldots,\theta_m)$, or

(b) $Y\theta_i/\theta_i=Xv_0+\sum_{j=1}^s c_jXv_j$ or $Y\theta_i=Xv_0+\sum_{j=1}^s c_jXv_j/v_j$ with $c_j\in K$ and $v_j\in\mathfrak{D}(\theta_1,\ldots,\theta_{i-1})$.

It is clear that $(\mathcal{E}, Y)$ is an intermediate differential extension of an elementary extension of $(\mathcal{D}, X)$. We can establish the converse direction for $\mathcal{E}$ algebraically closed in an elementary extension of $\mathcal{D}$.

THEOREM. *Let $(\mathcal{F}, X)$ be elementary over $(\mathcal{D}, X)$ (we will abuse notation here by letting X be the symbol for the derivation, no matter what the field is). Assume kernel $X = K$. Let $(\mathcal{E}, X)$ be an intermediate differential extension, $(\mathcal{D}, X) \subseteq (\mathcal{E}, X) \subseteq (\mathcal{F}, X)$. Assume that $\mathcal{E}$ is algebraically closed in $\mathcal{F}$ and of transcendence degree k over $\mathcal{D}$.*

Then $\mathcal{E}$ is generalized elementary over $\mathcal{D}$, there being a tower of simple extensions from $\mathcal{D}$ to $\mathcal{E}$ with precisely k transcendental extensions satisfying (b).

Thus an intermediate differential extension of an elementary extension (indeed, a generalized elementary extension) has a finite algebraic extension that is generalized elementary.

Proof. The result is obvious for $k = 0$.

Let $k \geqslant 1$. We assume the theorem is true for $\mathcal{E}$ of transcendence degree $k-1$ over $\mathcal{D}$ and prove it true for $\mathcal{E}$ of transcendence degree k over $\mathcal{D}$. If $\mathcal{F} = \mathcal{D}(\theta_1, \ldots, \theta_n)$ exhibits $\mathcal{F}$ as an elementary extension of $\mathcal{D}$, then find the first m so that $\mathcal{E}(\theta_1, \ldots, \theta_m, \theta)$ satisfies: $\theta = \theta_{m+1}$ is transcendental over $\mathcal{D}(\theta_1, \ldots, \theta_m)$ but algebraic over $\mathcal{E}(\theta_1, \ldots, \theta_m)$. Thus $\mathcal{E} \cap \mathcal{D}(\theta_1, \ldots, \theta_m) \subseteq \tilde{\mathcal{D}}$, the algebraic closure of $\mathcal{D}$ in $\mathcal{E}$. $(\tilde{\mathcal{D}}, X)$ is easily shown to be a differential extension of $(\mathcal{D}, X)$.

Let $z_1 = e^{y_1}, \ldots, z_q = e^{y_q}$ and $y_{q+1} = \log z_{q+1}, \ldots, y_r = \log z_r$ be, respectively, the exponentials and logarithms among $\theta_1, \ldots, \theta_m$.

Case 1. $\theta = \log z$, $z \in \mathcal{D}(\theta_1, \ldots, \theta_m)$. Then $\theta + \sum_1^r c_i y_i = f$, $f \in \mathcal{E}, c_i \in K$. f must be transcendental over $\mathcal{D}$, since θ is transcendental over $\mathcal{D}(\theta_1, \ldots, \theta_m)$. Then

$$Xf = \frac{Xz}{z} + \sum_1^r c_i X y_i \in \mathcal{E} \cap \mathcal{D}(\theta_1, \ldots, \theta_m).$$

$Xf \in \tilde{\mathcal{D}}$. The Liouville theorem on integration in finite terms (see [5] for a proof in the spirit of the present paper) then asserts that $Xf = Xv_0 + \sum_{j=1}^s c_j X v_j / v_j$ with $c_j \in K$, $v_j \in \tilde{\mathcal{D}}$.

Case 2. $\theta = e^y$, $y \in \mathcal{D}(\theta_1, \ldots, \theta_m)$. Then there are integers $n \neq 0$, n_i such that $\theta^n \prod_1^r z_i^{n_i} = f$, $f \in \mathcal{E}$ and is transcendental over $\mathcal{D}$. Then $Xf/f = X(ny + \sum_1^r n_i y_i) \in \mathcal{E} \cap \mathcal{D}(\theta_1, \ldots, \theta_m)$. So as before, we get by Liouville's theorem $Xf/f = Xv_0 + \sum_{j=1}^s c_j X v_j / v_j$ with $c_j \in K, v_j \in \tilde{\mathcal{D}}$.

In both cases $\tilde{\mathfrak{D}}(f)$ is a generalized elementary extension of transcendence degree 1 over $\mathfrak{D}$.

We now apply the induction hypothesis to the situation $(\tilde{\mathfrak{D}}(f), X) \subseteq (\mathfrak{E}, X) \subseteq (\mathfrak{F}, X)$ and obtain the theorem. Q.E.D.

We remark that this result answers, positively, a conjecture of Ritt [7, pp. 57–58]. Apparently Ritt did not relate his question to the work of Chapter VII of [7].

The following example, taken from [8], shows that the condition that $\mathfrak{E}$ is algebraically closed in $\mathfrak{F}$ cannot be omitted:

Let $(\mathfrak{D}, X)$ be $(\mathbb{C}(z), d/dz)$. Let $\mathfrak{F} = \mathfrak{D}(\sqrt{1-z^2}, \sin^{-1} z)$ and $\mathfrak{E} = \mathbb{C}(z, y)$, where $y = \sqrt{1-z^2}\ \sin^{-1} z$, so $Xy = 1 - (zy/(1-z^2))$. Although $\mathfrak{E}$ clearly has an algebraic extension of degree 2 that is elementary over $\mathfrak{D}$, it itself is not generalized elementary over $\mathfrak{D}$.

We see this from the Königsberger theorem, which yields a relation $dz/\sqrt{1-z^2} + c\omega = df$, where $c \in \mathbb{C}$, $\omega \in \Omega^1(\mathbb{C}(z))$, and $f \in \mathbb{C}(z, \sqrt{1-z^2})$. By looking at the two places above $z = \infty$ we get a contradiction.

Returning to the system $dy_i/dz = f_i(z, y_1, \ldots, y_n)$ with elementary particular solution $y(z) = (y_1(z), \ldots, y_n(z))$, let V^{k+1} be the Zariski closure in $\mathbb{C} \times \mathbb{C}^n$ of the graph of the curve $y(z)$. Let $Z, Y_1, \ldots, Y_n$ be coordinates in $\mathbb{C} \times \mathbb{C}^n$. Then $X = \partial/\partial Z + \sum_{i=1}^n f_i \partial/\partial Y_i$ is a rational vector field tangent to V (cf. the beginning of Part I). The theorem implies that there is a variety W^{k+1}, rational vector field Y on W and branched covering map π from W onto a Zariski open set of V with $\pi_* Y = X$ and $(\mathbb{C}(W), Y)$ generalized elementary over $(\mathbb{C}(Z), d/dZ)$. The differential equation may be said to have a k parameter family of generalized elementary particular integrals.

We don't go into how the Zariski closure V of the graph of $y(z)$ varies as we vary the initial conditions. To do this we would have to generalize the work here to the case of a non-algebraically-closed constant field K. K is then the field of "first integrals" of X, and also the function field on the reduced base scheme (in the sense of Grothendieck) of a family whose fibres are the different V's.

3. The Basic Real Elementary Operations. For our next application we show that the four basic transcendental functions of elementary calculus—the exponential. logarithm, tangent and inverse tangent—are independent in the sense that none can be expressed in terms of the other three if we are restricted to using auxilliary algebraic operations ~~that don't introduce nonreal complex numbers~~. We do this by first establishing an analogue of the structure theorem in order to handle such a situation. using only real numbers

We consider $(R(x), d/dx)$, where $R(x)$ is the field of rational functions with real coefficients and d/dx is the usual derivation.

Definition. We say that $(\mathfrak{D}, X)$ is real elementary over $(R(x), d/dx)$ if $\mathfrak{D} = R(x, \psi_1, \ldots, \psi_n)$, where the kernel of the derivation X is R and either ψ_j is algebraic over $R(x, \psi_1, \ldots, \psi_{j-1})$ or for some $f_j \in R(x, \psi_1, \ldots, \psi_{j-1})$,

(1) $X\psi_j/\psi_j = Xf_j$, i.e. $\psi_j = e^{f_j}$, or
(2) $X\psi_j = Xf_j/f_j$, i.e. $\psi_j = \log f_j$, or
(3) $X\psi_i/(1+\psi_j^2) = \frac{1}{2}Xf_j$, i.e., $\psi_j = \tan\frac{1}{2}f_j$, or
(4) $X\psi_j = Xf_j/(1+f_j^2)$, i.e., $\psi_j = \tan^{-1} f_j$.

With such a tower of fields leading to $\mathfrak{D}$ (and assignment of exactly one of the above five conditions to each ψ_j) we associate an elementary tower from $\mathbb{C}(x)$ to $\mathfrak{D}(i)$, where $i = \sqrt{-1}$, in the following way:

$$\mathfrak{D}(i) = \mathbb{C}(x, \theta_1, \ldots, \theta_n),$$

where $\theta_j = \psi_j$ if ψ_j is algebraic or of type (1) or (2). X, of course, extends to be a derivation of $\mathfrak{D}(i)$.

If $\psi_j = \tan\frac{1}{2}f_j$, then let $\theta_j = (-\psi_j + i)/(\psi_j + i)$. θ_j satisfies $X\theta_j/\theta_j = X(if_j)$, so $\theta_j = e_j^{if}$.

If $\psi_j = \tan^{-1} f_j$, then let $\theta_j = 2i\psi_j$. θ_j then satisfies $X\theta_j = 2iXf_j/(1+f_j^2) = Xw_j/w_j$ where $w_j = (f_j - i)/(f_j + i)$, so $\theta_j = \log[(f_j - i)/(f_j + i)]$.

Real Version of the Structure Theorem. *Let $(\mathfrak{D}(\psi), X)$ be real elementary over $(R(x), d/dx)$. $\mathfrak{D} = R(x, \psi_1, \ldots, \psi_n)$. Let $z_1 = e^{y_1}, \ldots, z_p = e^{y_p}$; $y_{p+1} = \log z_{p+1} = \log z_r$; $v_1 = \tan\frac{1}{2}u_1, \ldots, v_s = \tan\frac{1}{2}u_s$; $u_{s+1} = \tan^{-1} v_{s+1}, \ldots, u_t = \tan^{-1} u_t$ be the quantities among the ψ_j, $j = 1, \ldots, n$, which satisfy* (1)–(4). *Suppose ψ is algebraic over $\mathfrak{D}$. Then if:*

(1) *$\psi = y = \log z$, $z \in \mathfrak{D}$ or $\psi = z = e^y$, $y \in \mathfrak{D}$, then there are $m_i \in Q$, $a \in R$, such that $y + \sum_{i=1}^r m_i y_i = a$.*
(2) *$\psi = v = \tan\frac{1}{2}u$, $u \in \mathfrak{D}$ or $\psi = u = \tan^{-1} v$, $v \in \mathfrak{D}$, then there are $q_i \in Q$, $b \in R$, such that $u + \sum_{i=1}^t q_i u_i = b$.*

Proof. As indicated previously, we form the elementary tower from $\mathbb{C}(x)$ to $\mathfrak{D}(i) = \mathbb{C}(x)(\theta_1, \ldots, \theta_n)$. The exponentials and logs occurring among the θ_j are $z_1, \ldots, z_p, y_{p+1}, \ldots, y_r$; $e^{iu_1}, \ldots, e^{iu_s}$; $\log w_{s+1}, \ldots, \log w_t$, where $w_k = (v_k - 1)/(v_k + 1)$ $k = s+1, \ldots, t$.

By applying the structure theorem to the extension $\mathfrak{D}(i)(\theta)/\mathbb{C}(x)$ we obtain that there are $m_j, q_k \in \mathbb{Z}$ such that, corresponding to (1) or (2) of the hypotheses, we obtain:

$$y + \sum_1^r m_j y_j + i\sum_1^s q_k u_k + \sum_{s+1}^t q_k \log w_k = c, \tag{1}$$

or

$$\left\{\begin{matrix} iu \\ \text{or} \\ \log w \end{matrix}\right\} + \sum_1^r m_j y_j + i\sum_1^s u_k + u_k + \sum_{s+1}^t q_k \log w_k = c, \tag{2}$$

where $c \in \mathbb{C}$. On applying X, we have:

$$Xy + \sum_1^r m_j X y_j + i\sum_1^s q_k X u_k + \sum_{s+1}^t \frac{q_k X w_k}{w_k} = 0, \tag{1}$$

or

$$\left\{\begin{matrix} iXu \\ \text{or} \\ Xw/w \end{matrix}\right\} + \sum_1^r m_j X y_j + i\sum_1^s q_k X u_k + \sum_{s+1}^t \frac{q_k X w_k}{w_k} = 0 \tag{2}$$

These lead to

$$Xy + \sum_1^r m_j X y_j + i\sum_1^s q_k X u_k + 2i\sum_{s+1}^t \frac{q_k X v_k}{(1+v_k)^2} = 0 \tag{1}$$

or

$$\left\{\begin{matrix} iXu \\ \text{or} \\ \dfrac{2iXv}{1+v^2} \end{matrix}\right\} + \sum_1^r m_j X y_j + i\Sigma q_k X u_k + 2i\sum_{s+1}^t \frac{q_k X v_k}{(1+v_k)^2} = 0. \tag{2}$$

By equating real and imaginary parts we get the desired result. Q.E.D.

Corresponding to the four types of element ψ_j in the definition above of real elementary extension, we consider the following four differential equations:

(1) $X\psi/\psi = 1$;
(2) $X\psi = 1/x$;
(3) $X\psi/(1+\psi^2) = 1/2$;
(4) $X\psi = 1/(1+x^2)$.

Then we have the following result:

IRREDUNDANCY THEOREM. *Let $\mathfrak{D}$ be real elementary over $R(x)$, and let h denote one of the numbers* 1, 2, 3, 4. *Suppose that no sequence $\psi_1,\dots,\psi_n$ as in the definition of real elementary extension contains a ψ_j of type (h). Then no element of $\mathfrak{D}$ is a solution of the equation (h).*

Proof. We write out cases (1) and (4) only. We stick to the notation used in the real structure theorem.

(1) Suppose case (1) is false and we have such a $\mathfrak{D}=R(x,\psi_1,\dots,\psi_n)$ containing a solution of $X\psi/\psi=1=Xx$. We may assume, without loss of generality, that the logarithms among the ψ_j are algebraically independent over $R(x)$. Then by the real structure theorem $x=\sum_{j=1}^{r}q_j\log z_j+a$, $q_j\in Q$, $a\in R$, a contradiction to the algebraic independence of the logarithms over $R(x)$.

(4) Suppose case (4) is false. Assume we have such a tower from $R(x)$ to $\mathfrak{D}$ where the number t of $\tan u_j$ is minimal. The real structure theorem yields here

$$\tan^{-1}x=\sum_{k=1}^{t}m_k u_k+b,\qquad m_k\in Q,\quad b\in R.$$

But this is an expression for $\tan^{-1}(x)$ using at least one less $\tan u_j$, a contradiction.

4. Invertible Elementary Functions. We determine the structure of the groupoid (a category in which every morphism has an inverse), of elementary functions with elementary inverses. We will work in the subcategory $\mathcal{G}$ of the category of groupoids, whose objects are groupoids G such that $\mathrm{Obj}(G)=\{$all open, connected subsets of $\mathbb{C}$ in the usual topology$\}$ and $\mathrm{Morph}_{\mathcal{G}}(G,H)=\{$all functors from G to H that leave Obj (G) fixed$\}$.

We mention three particular groupoids in $\mathcal{G}$ that are of interest to us:

A, in which $\mathrm{Morph}_A(R,S)$, $R,S\in\mathrm{Obj}(A)$, consists of all analytic homeomorphisms from R onto S that are given by restricting a single valued branch of an algebraic function of a single variable z to R.

B, in which $\mathrm{Morph}_B(R,S)$ consists of all analytic homeomorphisms from R onto S that are given by restricting a single valued branch of some iterate of e^z or $\log z$ to R.

C, in which $\mathrm{Morph}_C(R,S)$ consists of all analytic homeomorphisms of R onto S given by restricting a single valued branch of an elementary function f, such that f^{-1} is also elementary, to R. For example $f=z-a\sin z$, $a\in\mathbb{C}^*$, is *not* such an elementary function. See [7, pp. 56–57], or prove it from the structure theorem.

It would be interesting to relate these groupoids to their associated monoids of correspondences on $\mathbb{C}\times\mathbb{C}$. We use groupoids here instead of (as might be expected) pseudogroups because we wish to work in a category where one can form free products and take quotients. We are grateful to C. Elgot and R. Solovay for their suggestions concerning these and related points.

The free product $A*B$ has $\mathrm{Morph}(R,S)$ consisting of finite sequences of morphisms from A and B, $\{\gamma_i\}_1^e$, with $\mathrm{Domain}\,\gamma_1=R$, $\mathrm{Range}\,\gamma_i=\mathrm{Domain}\,\gamma_{i+1}=R_i$, $\mathrm{Range}\,\gamma_e=S$. Sequences in $A*B$ are written from right to left. When writing the associated function from R to S we also compose from right to left.

A subgroupoid $\hat{C}$ of C (in $\mathcal{G}$) is said to be abundant in C if for every $f\in\mathrm{Morph}_C(R,S)$ there are $\hat{R}\subseteq R$, $\hat{S}\subseteq S$, $\hat{R}$ and $\hat{S}$ in $\mathrm{Obj}\,C$ $(=\mathrm{Obj}\,\hat{C})$ and an $\hat{f}\in\mathrm{Morph}_{\hat{C}}(\hat{R},\hat{S})$ such that $f|_{\hat{R}}=\hat{f}$.

THEOREM. *There is an abundant subgroupoid $\hat{C}$ of C such that $\hat{C}$ is isomorphic to $A*B/N$ where $N\in\mathrm{Obj}\,\mathcal{G}$, $\mathrm{Morph}_N(R,S)=\varnothing$ except if $R=S$. Then $\mathrm{Morph}_N(R,R)$ consists of all morphisms $\prod_{i=1}^q w_i\sigma_i w_i^{-1}$ of $A*B$, where there are R_i, $i=1,\dots,q$, such that $w_i\in\mathrm{Morph}_{A*B}(R_i,R)$, and $\sigma_i\in\mathrm{Morph}_{A*B}(R_i,R_i)$ is of form $\{(1/q)(z-\log a),\log z,az^q,e^z\}$. Here, $a\in C^*$, $q\in Q^*$, and the branch of $\log z$ are chosen arbitrarily, but $\log a$ is determined by these choices.*

Intuitively, the invertible elementary functions of z are those for which a suitable restriction can be written as a finite composition of e^z, $\log z$, and algebraic functions of z. The only way such a sequence can represent the identity function is if it is of the form of a morphism of N.

We first describe the guiding idea of the proof. Let f be an elementary function. f is in a field $\mathfrak{D}$ of elementary meromorphic functions, $\mathfrak{D}=\mathbb{C}(z,\theta_1(z),\dots,\theta_m(z))$. Let $\mathfrak{E}$ be another field of elementary meromorphic functions $\mathfrak{E}=\mathbb{C}(w,\psi_1(w),\dots,\psi_n(w))$. If all compositions $\psi_i(f)$ are defined, $i=1,\dots,n$, the field

$$\mathfrak{F}=\mathbb{C}(z,\theta_1(z),\dots,\theta_m(z),\psi_1(f),\dots,\psi_n(f))$$

is also an elementary extension of $\mathbb{C}(z)$. Suppose there is a $g\in\mathfrak{E}$ such that $g(f)=z$. This implies that the exponentials and logarithms used to build up $\mathfrak{F}$ are algebraically dependent. The structure theorem tells us what this algebraic relation is, and this, in turn, gives us information on f.

To begin the proof, we first describe composition of functions in our algebro-geometric setup.

Let $(\mathfrak{D},X)$ and $(\mathfrak{E},Y)$ be elementary extensions of $(K(z),d/dz)$ and $(K(w),d/dw)$ respectively. $\mathfrak{D}=K(z,\theta_1,\dots,\theta_m)$, where $z_1=e^{y_1},\dots,z_q=e^{y_q}$,

$\log z_{q+1},\ldots,\log z_r = y_r$ are the exponentials and logs occurring among the θ's. $\mathcal{E} = K(w,\psi_1,\ldots,\psi_n)$, where $v_1 = e^{u_1},\ldots,\ v_s = e^{u_s}$, $\log v_{s+1} = u_{s+1},\ldots,\log v_t = u_t$ are chosen from the ψ's in the same way. Assume kernel $X =$ kernel $Y = K$. Let Z and W be affine models of $\mathfrak{D}$ and $\mathcal{E}$, respectively. Suppose $f \in \mathfrak{D}$.

In $Z \times W$, consider the hypersurface U which is the Zariski closure of $w - f = 0$. Suppose all the components of the vector field $T = X + (Xf)Y$ are in the local ring of U in $Z \times W$. [If we are working with meromorphic functions, this corresponds to asking that all compositions $\psi_1(f),\ldots,\psi_n(f)$ be defined.] Then T is tangent to U, i.e., T sends $I(U)$, the maximal ideal of the local ring of U in $K[Z \times W]$, into itself and therefore can be considered as an operator on $K(U)$. Also an induction shows that $(K(U),T)$ is elementary over $(\mathfrak{D},X)$ [and therefore over $(K(z),d/dz)$]. We may write $K(U) = \mathfrak{D}(\bar{\psi}_1,\ldots,\bar{\psi}_n)$, where $\bar{\psi}_i = (\pi_2^*\psi_i)|_U$, π_2 being the projection of $Z \times W$ onto the second factor.

However, here the kernel of $T|_{K(U)}$ may not equal K. For example, let

$\mathfrak{D} = K(z,z_1), z, z_1$ algebraically independent, $X = \partial/\partial z + z_1\,\partial/\partial z_1$,
$\mathcal{E} = K(w,w_1), w, w_1$ algebraically independent, $Y = \partial/\partial w + (1/w)\,\partial/\partial w_1$.

Let $f = z_1$. Then $K(U) = K(z,z_1,w_1)$ with z, z_1, w_1 algebraically independent and

$$T|_{K(U)} = \frac{\partial}{\partial z} + z_1\frac{\partial}{\partial z_1} + \frac{\partial}{\partial \bar{w}_1} \quad \text{and} \quad T(\bar{w}_1 - z) = 0.$$

We can always find subvarieties S of U such that $K(S) \supseteq \mathfrak{D}$, T is tangent to S, and kernel T, when T is considered an operator on $K(S)$, equals K. When working with meromorphic functions, we would take the Zariski closure of $(z, \theta_1(z),\ldots,\theta_m(z),\psi_1(f),\ldots,\psi_n(f))$. We still write $K(S) = K(z,\theta_1,\ldots,\theta_m,\bar{\psi}_1,\ldots,\bar{\psi}_n)$ and T for $T|_s$. Still we have that $(K(S),T)$ is elementary over $(\mathfrak{D},X)$.

Lemma. *In the situation of the last paragraph, suppose we have either*

(Case 1) that $\psi_n = v_s = e^{u_s} = e^u = v$, with $u = u_s$ in $K(w,\psi_1,\ldots,\psi_{n-1})$ and $\bar{\psi}_n$ algebraic over $K(z,\theta_1,\ldots,\theta_m,\bar{\psi}_1,\ldots,\bar{\psi}_{n-1})$, or

(Case 2) that $\psi_n = u_t = \log v_t = \log v = u$, with $v = v_t$ in $K(w,\psi_1,\ldots,\psi_{n-1})$ and $\bar{\psi}_n$ algebraic over $K(z,\theta_1,\ldots,\theta_m,\bar{\psi}_1,\ldots,\bar{\psi}_{n-1})$.

Also suppose that $t = \operatorname{tr deg}.\mathcal{E}/K$ is at least one, but minimal for a field $\mathcal{E}$ satisfying case 1 or 2 with respect to the given $\mathfrak{D}$ and f.

Then $t = 1$, so u in case 1 and v in case 2 are algebraic over $K(w)$, and in either case there are $c_1 \in K$, $c_2 \in K^*$ and k, p_i in $\mathbb{Z}$ such that $k\bar{u} = c_1 + \sum_{i=1}^r p_i y_i$ and $\bar{v}^k = c_2\prod_1^r z_1^{p_i}$.

Proof. We have, by the structure theorem that there are a $c_1 \in K$, $c_2 \in K^*$, $k_i (k_t \neq 0)$ and $p_i \in \mathbb{Z}$ such that

$$\sum_{i=1}^{t} k_i \bar{u}_i = c_1 + \sum_{i=1}^{r} p_i y_i$$

and

$$\prod_{i=1}^{t} \bar{v}^{k_i} = c_2 \prod_{i=1}^{r} z_i^{p_i}$$

both hold on S.

In case (1) we can conclude from the minimality of t that first, $\sum_{i=1}^{t} k_i \bar{u}_i$ is algebraic over $K(w)$. Then we observe that $t=1$. Case (2) is similar. Q.E.D.

Proof of the theorem using the lemma. Let f be an invertible nonalgebraic elementary function, restricted to a connected open set R, where it is an analytic homeomorphism onto another open set S. $f \in \mathfrak{D}_m \cong \mathbb{C}(z, \theta_1, \ldots, \theta_m)$, with m minimal for f. The existence of an elementary f^{-1} on S implies, by the lemma and minimality of m, that f is algebraic over $\mathbb{C}(e^{h(z)})$ or $\mathbb{C}(\log j(z))$ with $h(z)$ and $j(z)$ in $\mathfrak{D}_{m-1} \cong \mathbb{C}(z, \theta_1, \ldots, \theta_{m-1})$ and analytic homeomorphisms on some open connected subset $\hat{R} \subseteq R$. $h(z)$ and $j(z)$ also have elementary inverses, so the fact that A and B generate some abundant subcategory $\hat{C}$ of C follows by induction on m.

We now have that there is a homomorphism of $A * B$ onto $\hat{C}$ whose kernel includes N. We conclude the proof by showing that anything in the kernel is equivalent, modulo N, to an appropriate identity function.

Suppose we have a sequence $\{\gamma_i\}_1^e$ of elements of Morph_{A*B}, not all of them being Morph_A or Morph_B, such that $\gamma_e \circ \cdots \circ \gamma_1$ is the identity function on the domain of γ_1. We may assume that each γ_i from B is of the form e^z or $\log z$ and that no consecutive pair γ_{i+1}, γ_i are both in A.

Let $\mathbb{C}(z, \theta_1, \ldots, \theta_m, \bar{\psi}_1, \ldots, \bar{\psi}_n)$ be the elementary extension of $\mathbb{C}(z)$ that is associated to this sequence in the obvious manner: $m+n=e$, $\bar{\psi}_1$, corresponding to γ_{m+1}, is the first exponential or logarithm such that $\operatorname{tr deg} \mathbb{C}(z, \theta_1, \ldots, \theta_m, \bar{\psi}_1)/\mathbb{C}(z) = \operatorname{tr deg} \mathbb{C}(z, \theta_1, \ldots, \theta_m)/\mathbb{C}(z)$. Thus all the exponentials and logarithms occurring among $\theta_1, \ldots, \theta_m$, viz. $z_1 = e^{y_1}, \ldots, z_q = e^{y_q}$; $y_{q+1} = \log z_{q+1}, \ldots,$ $y_r = \log z_r$, are algebraically independent over $\mathbb{C}(z)$. We assume here that $\{z_i\}_1^q$ and $\{y_i\}_{q+1}$ are subsequences (same order but different indexing) of $\{\theta_i\}_1^m$.

Suppose $\bar{\psi}_1 = \log\theta_m$. The lemma yields that $\theta_m^k = c\prod_1^q (e^{y_i})^{p_i} \prod_{q+1}^r z_i^{p_i}$ with $c \in \mathbb{C}^*$, and $k \neq 0$, $p_i \in \mathbb{Z}$. But from the nature of $\{y_i\}_1^e$ we also know that θ_m is algebraic over $\mathbb{C}(e^{y_q})$ or $\mathbb{C}(\log z_r)$ where e^{y_q} or $\log z_r$ is the last exp or log occurring in the sequence of θ_i's. Thus it must be that $\theta_m^k = c(e^{y_q})^{p_q}$. Thus the subsequence $\{\gamma_{m+1}, \gamma_m, \gamma_{m-1}\}$ had to be of the form $\{\log z, cz^{p_q/k}, e^z\}$. $\{\gamma_i\}_1^e$ is then equivalent, modulo N, to a sequence of length $e-2$ with this triple replaced by $\{\log c + (p_q/k)z\}$. It is clear how to complete the argument by induction.

If $\bar{\psi}_1 = e^{\theta_m}$, then the argument is very similar. We show that there is a subsequence $\{\gamma_{m+1}, \gamma_m, \gamma_{m-1}\}$ of the form $\{e^z, qz + c, \log z\}$ with $q \in Q^*$, $c \in \mathbb{C}$. We replace this by $\{e^c z^q\}$ and again apply induction.

IBM Thomas J. Watson Research Center.

REFERENCES.

[1] J Ax, On Schanuel's conjectures, Ann. Math. **93** (1971), pp. 252–268.

[2] ———, Some topics in differential algebraic geometry I: Analytic subgroups of algebraic groups, *Amer. J. Math.* **94** (1972), pp. 1195–1204.

[3] D. D. Mordoukhay-Boltovskoy, General investigations on integration in finite terms of first order differential equations, Paper II (in Russian), *Bull. Kharkov Math. Soc.* (*Ser.* 2) **X** (1909), pp. 231–269.

[4] ———, Sur la résolution des equations différentielles de premier ordre en forme finie, *Rendiconti Circ. Mat. Palermo* **LXI** (1937), pp. 49–72.

[5] R. H. Risch, Implicitly elementary integrals, *Proc. Amer. Math. Soc.* **57** (No. 1, May 1976), pp. 1–7.

[6] J. F. Ritt, Elementary functions and their inverses, *Trans. Amer. Math. Soc.* **27** (1925), pp. 68–90.

[7] ———, *Integration in Finite Terms*, Columbia U. P., New York, 1948.

[8] M. Rosenlicht, and M. Singer, On elementary, generalized elementary, and Liouvillian extension fields, in *Contributions to Algebra* (Hyman Bass, Phyllis J. Cassidy, and Jerald Kovacic, Eds.), Academic, New York, 1977, pp. 329–342.

[9] M. Singer, Elementary solutions of differential equations, *Pacific J. Math.* **59** (No. 2, 1975), pp. 535–547.