Game bisimulations between basic positions

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- proceed in rounds
- moving from one special, basic position to another.

Parity Games

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- parity game: $\langle V_0, V_1, E, v_I, \Omega \rangle$
- V_0 and V_1 positions for players 0 and 1, respectively
- arbitrary player: π, with opponent $\overline{\pi}$
- $V := V_0 \cup V_1$
- $E \subseteq V \times V$
- $\Omega : V \to \mathbb{N}$, finite range, highest priority counts
- v_I initial position

Basic positions

Let $\mathbb{G} = \langle V_0, V_1, E, v_I, \Omega \rangle$ be a parity game.

A set $B \subseteq V_0 \cup V_1$ is basic if

- $v_I \in B$;
- any full match starting at some $b \in B$
  (i) ends in a terminal position, or
  (ii) passes through another position in B;
- $\Omega(v) > 0$ iff $v \in B$.

Elements of B will be called basic positions.

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- may compare player π in $\mathbb{G}$ with $\overline{\pi}$ in $\mathbb{G}'$
- may relate $v \in V_\pi$ to $v' \in V'_\pi$ or $v' \in V'_{\overline{\pi}}$

Local Game Tree

Fix parity game $\mathbb{G} = \langle V_0, V_1, E, v_I, \Omega \rangle$ and basic set $B \subseteq V$

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- $V^b := \left\{ \beta \text{ finite path } \mid \text{first}(\beta) = p \text{ and } \beta \cap B \subseteq \{\text{first}(\beta), \text{last}(\beta)\} \right\}$
- $V_\pi^b := \{\beta \in V^b \mid \text{last}(\beta) \in V_\pi\}$
- $E^b := \{(\beta, \beta v) \mid v \in E(\text{last}(\beta))\}$
- $v_I^b := \langle b \rangle$

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$N(b)$ is the set of possible next basic positions, after b:

- $\mathrm{Leaves}(T^b) := \{\beta \in V^b \mid |\beta| > 1 \text{ and } \mathrm{last}(\beta) \in B\}$
- $N(b) := \{\mathrm{last}(\beta) \mid \beta \in \mathrm{Leaves}(T^b)\}$.

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Formally first define $P_\pi(\beta) \subseteq \wp(N(b))$ for $\beta \in V^b$:

- If $\beta \in \text{Leaves}(T^b)$, we put $P_\pi(\beta) := \Big\{\{\text{last}(\beta)\}\Big\}$.
- If $\beta \notin \text{Leaves}(T^b)$, we put

$$P_\pi(\beta) := \begin{cases} \bigcup\{P_\pi(\gamma) \mid \gamma \in E^b(\beta)\} & \text{if } \beta \in V^b_\pi, \\ \Big\{\bigcup_{\gamma \in E^b(\beta)} Y_\gamma \mid Y_\gamma \in P_\pi(\gamma), \text{ all } \gamma\Big\} & \text{if } \beta \in V^b_{\overline{\pi}}. \end{cases}$$

Finally, $P_\pi(b) := P_\pi(\langle b \rangle)$.

Basic Game Bisimulation

$\mathbb{G}, \mathbb{G}'$ parity games, B, B' basic sets, π, σ players.

$Z \subseteq B \times B'$ is a π, σ-bisimulation if

(1) Z satisfies, for all $v \in B, v' \in B'$ with vZv', the structural conditions

$$
\begin{array}{llllll}
(\pi,\text{forth}) & \forall U \in P^{\mathbb{G}}_{\pi}(v) & \exists U' \in P^{\mathbb{G}'}_{\sigma}(v') & \forall u' \in U' & \exists u \in U & (u,u') \in Z, \\
(\overline{\pi},\text{forth}) & \forall U \in P^{\mathbb{G}}_{\overline{\pi}}(v) & \exists U' \in P^{\mathbb{G}'}_{\overline{\sigma}}(v') & \forall u' \in U' & \exists u \in U & (u,u') \in Z, \\
(\sigma,\text{back}) & \forall U' \in P^{\mathbb{G}'}_{\sigma}(v') & \exists U \in P^{\mathbb{G}}_{\pi}(v) & \forall u \in U & \exists u' \in U & (u,u') \in Z, \\
(\overline{\sigma},\text{back}) & \forall U' \in P^{\mathbb{G}'}_{\overline{\sigma}}(v') & \exists U \in P^{\mathbb{G}}_{\overline{\pi}}(v) & \forall u \in U & \exists u' \in U & (u,u') \in Z,
\end{array}
$$

(2) Z satisfies the priority conditions

(parity) for all $v \in B, v' \in B'$ with vZv':

$$\Omega(v)\ mod\ 2 = \pi \text{ iff } \Omega'(v')\ mod\ 2 = \sigma,$$

(contraction) for all $v, w \in B$ and $v', w' \in B'$ with vZv' and wZw':

$$\Omega(v) \leq \Omega(w) \text{ iff } \Omega(v') \leq \Omega(w').$$

Variations

May change priority condition into:

for all paths $\alpha = (v_i)_{i\in\omega}$, $\alpha' = (v'_i)_{i\in\omega}$ such that $v_i Z v'_i$ for all i:

$$\pi \text{ wins } \alpha \text{ in } \mathbb{G} \text{ iff } \sigma \text{ wins } \alpha' \text{ in } \mathbb{G}'.$$

Main Theorem

Theorem $\mathbb{G}, \mathbb{G}'$ parity games, B, B' basic sets, π, σ players.
If $\mathbb{G}, \pi, v \leftrightarroweq \mathbb{G}', \sigma, v'$ (i.e. vZv' for some π, σ-bisimulation $Z \subseteq B \times B'$), then

$$v \in Win_\pi(\mathbb{G}) \text{ iff } v' \in Win_\sigma(\mathbb{G}').$$

Application

Let L be some language of 'one-step' formulas, and let C be an alphabet/set of colors.
An L-automaton is a triple $\mathbb{A} = \langle A, \delta, \Omega \rangle$ with

$$\delta : A \times C \to L(A)$$

Corollary If L is closed under taking Boolean duals, then $\mathrm{Aut}(L)$ is closed under complementation.

Reference

C. Kissig & Y. Venema, Complementation of Coalgebra Automata, CALCO 2009 (LNCS 5728), pp 81–96.