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CHARACTERIZATIONS OF LOCALLY TESTABLE EVENTS*

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Abstract. Let Σ be a finite alphabet, Σ^* the free monoid generated by Σ and $|x|$ the length of $x \in \Sigma^*$. For any integer $k \geq 0$, $f_k(x)$ $(t_k(x))$ is x if $|x| < k+1$, and it is the prefix (suffix) of x of length k, otherwise. Also let $m_{k+1}(x) = \{v | x = uvw \text{ and } |v| = k+1\}$. For $x, y \in \Sigma^*$ define $x \sim_{k+1} y$ iff $f_k(x) = f_k(y)$, $t_k(x) = t_k(y)$ and $m_{k+1}(x) = m_{k+1}(y)$. The relation $\sim_{k+1}$ is a congruence of finite index over Σ^*. An event $E \subseteq \Sigma^*$ is $(k+1)$-testable iff it is a union of congruence classes of $\sim_{k+1}$. E is locally testable (LT) if it is $k+1$-testable for some k. (This definition differs from that of [6] but is equivalent.)

We show that the family of LT events is a proper sub-family of star-free events of dot-depth 1. LT events and k-testable events are characterized in terms of (a) restricted star-free expressions based on finite and cofinite events; (b) finite automata accepting these events; (c) semigroups; and (d) structural decomposition of such automata. Algorithms are given for deciding whether a regular event is (a) LT and (b) $k+1$-testable. Generalized definite events are also characterized.

1. Notations and definitions

Our notation is based on that of [3]. Let Σ^* (Σ^+) denote the free monoid (semigroup) generated by the nonempty finite set Σ. Then $\Sigma^* = \Sigma^+ \cup \{\lambda\}$, where λ is the empty word. For $x, y \in \Sigma^*$, $xy \in \Sigma^*$ denotes the concatenation of x and y and $|x|$ denotes the length of x, defined by

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$|\lambda| = 0$, $|x\sigma| = |x| + 1$ for $\sigma \in \Sigma$. For $x, y, z \in \Sigma^*$, if $x = yz$ then y (z) is a prefix (suffix) of x. For a finite set Q, $\#Q$ denotes the cardinality of Q.

If f is a function f: $A \to B$, and $x \in A$, we denote the image of x by $f(x)$ or xf. If $X \subseteq A$, $Xf = \{xf | x \in X\}$. If g: $B \to C$, then the composition of f with g, fg, is fg: $A \to C$ where $x(fg) = (xf)g$. We also use xfg or $g(f(x))$ to denote $x(fg)$.

An *initialized semiautomaton* (ISA) is a quadruple $A = (Q, \Sigma, M, q_0)$, where Q and Σ are nonempty finite sets (of *states* and *inputs*), $q_0 \in Q$ is the *initial state* and M is a set of functions $\sigma^A : Q \to Q$, one for each $\sigma \in \Sigma$. For $q \in Q$, $q\sigma^A \in Q$ is the *next-state of* q *under input* $\sigma \in \Sigma$. For $x \in \Sigma^*$ the function $x^A : Q \to Q$ is defined inductively: λ^A is the identity on Q, and if $x = y\sigma$, $\sigma \in \Sigma$ then $x^A = y^A \sigma^A$. Clearly, for all $x, y \in \Sigma^*$, $(xy)^A = x^A y^A$. For $x \in \Sigma^*$, $x^0 = \lambda$, $(x^A)^0 = \lambda^A$, and for any integer $k \geq 0$, $x^{k+1} = x^k x$, $(x^A)^{k+1} = (x^A)^k x^A$. Clearly, $(x^k)^A = (x^A)^k$. The set of functions $\{x^A | x \in \Sigma^*\}$ $(\{x^A | x \in \Sigma^+\})$ is a finite monoid (semigroup) under composition of functions; it is denoted by G_A (S_A) and called the *monoid* (*semigroup*) *of* A. A is *connected* if for every $q \in Q$ there exists an $x \in \Sigma^*$ such that $q_0 x^A = q$. A is *permutation-free*, if for any $R \subseteq Q$, any $x \in \Sigma^*$, $Rx^A = R$ implies $rx^A = r$ for all $r \in R$.

An *automaton* is a quintuple $\hat{A} = (Q, \Sigma, M, q_0, F)$, where $A = (Q, \Sigma, M, q_0)$ is an ISA, called the ISA of $\hat{A}$, and $F \subseteq Q$. The event accepted by A is $E = \{x | x \in \Sigma^*, q_0 x^A \in F\}$. $\hat{A}$ is *reduced* if for every two distinct states $p, q \in Q$ there exists an $x \in \Sigma^*$ such that $px^A \in F$ iff $qx^A \notin F$, and A is connected.

Let $A = (Q, \Sigma, M, q_0)$ and $B = (R, \Sigma, N, r_0)$ be ISA's. The *direct product of* A *and* B is the ISA $A \times B = (Q \times R, \Sigma, P, (q_0, r_0))$, where for all $\sigma \in \Sigma$, $(q, r)\sigma^{A \times B} = (q\sigma^A, r\sigma^B)$. *$A$ is covered by B*, $A \leq B$, if there is an onto function η: $R_1 \to Q$, where $r_0 \in R_1 \subseteq R$, such that $r_0\eta = q_0$ and for all $r \in R_1$ and all $\sigma \in \Sigma$, $r\sigma^B \in R_1$ and $r\sigma^B\eta = r\eta\sigma^A$. Let π be a function π: $R \to 2^Q$; then *B is a π-factor of A*, $B = A/\pi$, if the following hold:

(i) $\bigcup_{r \in R} r\pi = Q$;

(ii) for every $r \in R$ and every $\sigma \in \Sigma$, $r\pi\sigma^A \subseteq r\sigma^B\pi$;

(iii) $q_0 \in r_0\pi$.

Let $A = (Q, \Sigma, M, q_0)$ and $B = (R, Q \times \Sigma, N, r_0)$ be ISA's. The *cascade product of* A *and* B is the ISA

$$A \circ B = (Q \times R, \Sigma, P, (q_0, r_0)), \text{ where } (q,r)\sigma^{A \circ B} = (q\sigma^A, r(q,\sigma)^B).$$

A semigroup S is *group-free* if all of its subgroups are trivial, i.e., contain one element only.

2. Locally testable and star-free events

The material up to Definition 2.1 is based on previous work [2]. An event is star-free (SF) iff it can be denoted by a regular expression using only concatenations and Boolean operations. We shall use the symbol I for Σ^*; I is SF since $I = \overline{\emptyset}$. If $\mathcal{K}$ is any family of events, let $\mathrm{B}(\mathcal{K})$ ($\mathrm{M}(\mathcal{K})$) denote the smallest family of events containing $\mathcal{K}$ and closed under Boolean operations (concatenation). In forming the family of SF events one can begin with the *basic* family $\mathcal{E}_0 = \{\{\sigma_1\}, ..., \{\sigma_m\}, \{\lambda\}, \emptyset\}$ of events requiring no operations. Then any SF event can be obtained from $\mathcal{E}_0$ by applying a finite number of Boolean operations and concatenations. Let $\mathcal{B}_1 = \mathrm{B}(\mathcal{E}_0)$, $\mathcal{M}_n = \mathrm{M}(\mathcal{B}_n)$ for $n \geq 1$, and $\mathcal{B}_n = \mathrm{B}(\mathcal{M}_{n-1})$ for $n > 1$. If Boolean operations are applied first we can define the sequence (2.1) of families:

$$(2.1) \qquad \mathcal{E}_0 \subseteq \mathcal{B}_1 \subseteq \mathcal{M}_1 \subseteq \mathcal{B}_2 \subseteq \mathcal{M}_2 \subseteq \dots .$$

Considering only the Boolean algebras leads to sequence (2.2) which defines the *"dot-depth" hierarchy*:

$$(2.2) \qquad \mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \mathcal{B}_3 \subseteq \dots ,$$

where for any SF event E, the *dot-depth* $d(E)$ is $d(E) = 0$ if $E \in \mathcal{B}_1$ and, for $n > 0$, $d(E) = n$ if $E \in \mathcal{B}_{n+1} \setminus \mathcal{B}_n$.

If concatenation is applied to $\mathcal{E}_0$ first, we obtain the sequence (2.3):

$$(2.3) \qquad \mathcal{E}_0 \subseteq \hat{\mathcal{M}}_1 \subseteq \hat{\mathcal{B}}_1 \subseteq \hat{\mathcal{M}}_2 \subsetneq \hat{\mathcal{B}}_2 \subseteq \dots ,$$

where $\hat{\mathcal{M}}_1 = \mathrm{M}(\mathcal{E}_0)$, $\hat{\mathcal{B}}_n = \mathrm{B}(\hat{\mathcal{M}}_n)$ for $n \geq 1$, and $\hat{\mathcal{M}}_n = \mathrm{M}(\hat{\mathcal{B}}_{n-1})$ for $n > 1$. It has been shown that for $n \geq 2$, $\mathcal{B}_n = \hat{\mathcal{B}}_n$, $\hat{\mathcal{M}}_{n+1} = \mathcal{M}_n$, i.e., the two sequences (2.1) and (2.3) are identical. Thus it is necessary to consider only the initial segment of sequence (2.3); in particular, $\hat{\mathcal{B}}_1$ and $\hat{\mathcal{B}}_2 = \mathcal{B}_2$ will be of interest.

Let $\mathcal{F}$ and $\mathcal{C}$ denote the families of finite and cofinite events (E is cofinite iff its complement is finite), respectively. Let β_1 be the family $\mathcal{F} \cup \mathcal{C}$. It is easily seen that $\beta_1 = \hat{\mathcal{B}}_1$. For $n \geq 1$ let $(\mathcal{F} \cup \mathcal{C})^n$ be the family of events expressible as concatenations of n factors, each of which is either finite or cofinite, and let $\beta_n = \mathrm{B}((\mathcal{F} \cup \mathcal{C})^n)$. Since $\mathcal{F} \cup \mathcal{C}$ is a Boolean algebra, we have $\beta_1 = \mathrm{B}((\mathcal{F} \cup \mathcal{C})^1) = \mathcal{F} \cup \mathcal{C}$, so that the notation is consistent for β_1. Note that $\beta_n \subseteq \beta_{n+1}$ for all $n \geq 1$.

Definition 2.1. The $\mathcal{B}_2$ *hierarchy* is the sequence (2.4) of Boolean algebras:

(2.4) $\quad \beta_1 \subseteq \beta_2 \subseteq \beta_3 \subseteq \beta_4 \subseteq \ldots$

Clearly $\mathcal{B}_2 = \bigcup_{n=1}^{\infty} \beta_n$.

We will show that β_1, β_2 and β_3 are all distinct and contain several well-known families of events. For all $n \geq 1$, we now claim that $\beta_{2n+1} = \beta_{2n+2}$. Note that β_n can be defined equivalently as the smallest Boolean algebra containing all events in the family $[w, I]^n$, which we define as the set of all concatenations of n factors, each of which is either a word w in Σ^* or is I. This follows from the fact that each finite event is a finite union of words, and each cofinite event is expressible as a finite union of words and of events of the form wI, $w \in \Sigma^*$. Thus $\beta_n = \mathrm{B}((\mathcal{F} \cup \mathcal{C})^n) = \mathrm{B}([w, I]^n)$. Now, the only products in $[w, I]^{2n+2} \setminus [w, I]^{2n+1}$ are those of the form $E = w(Iw_1Iw_2 \ldots Iw_nI)$ or $E' = (Iw_1Iw_2 \ldots Iw_nI)w$. However,

$$E = wI \cap \Sigma^{|w|}(Iw_1I \ldots Iw_nI) = wI \cap I(\Sigma^{|w|}w_1)I \ldots Iw_nI.$$

Thus E can be expressed as a Boolean function of products in $[w, I]^{2n+1}$, and the same is true for E'. Hence, for $n \geq 1$,

$$[w, I]^{2n+2} \subseteq \mathrm{B}([w, I]^{2n+1}) = \beta_{2n+1}\ .$$

Therefore, $\beta_{2n+2} \subseteq \beta_{2n+1}$ and the claim follows.

Definition 2.2. An event is *definite* [1; 5; 8] (*reverse definite* [1,4], *generalized definite* [4], iff it can be expressed in the form (2.5)–(2.7):

(2.5) $\qquad$ Definite $\quad E = F \cup GI$, (2.6) Reverse Definite $E = F \cup IG$,

(2.7) Generalized Definite $E = F \cup (\bigcup_{i=1}^{j} H_i I G_i)$, for some j,

where F, G and H_i, G_i for $i = 1, ..., j$ are finite events.

The material up to Theorem 2.1 is based on [6]. For reasons which become apparent later, we modify some of the definitions from [6].

For any $x \in \Sigma^*$ and any integer $k \geq 0$, $f_k(x)$ ($t_k(x)$) is x if $|x| < k+1$, and it is the prefix (suffix) of x of length k, otherwise. Let $m_{k+1}(x) = \{v | x = uvw \text{ and } |v| = k+1\}$. For $x, y \in \Sigma^*$ define $x \sim_{k+1} y$ iff $f_k(x) = f_k(y)$, $t_k(x) = t_k(y)$ and $m_{k+1}(x) = m_{k+1}(y)$. It is easily seen that the relation $\sim_{k+1}$ is a congruence of finite index over Σ^*, and that $x \sim_{k+2} y$ implies $x \sim_{k+1} y$. Let $[x]_{k+1}$ denote the congruence class containing x.

Definition 2.3. An event $E \subseteq \Sigma^*$ is $(k+1)$-*testable* iff it is a union of congruence classes of $\sim_{k+1}$. E is *locally testable* (LT) if it is $(k+1)$-testable for some k. (The verification that the family of LT events defined here coincides with that of [6] is straightforward.)

One verifies that if $|x| \geq k+1$ and $u = f_k(x)$, $v = t_k(x)$ and $m_{k+1}(x) = \{w_1, w_2, ..., w_t\}$ then

$$(2.8) \qquad [x]_{k+1} = uI \cap Iv \cap (Iw_1 I \cap ... \cap Iw_t I) \cap \overline{I(\Sigma^{k+1} \setminus m_{k+1}(x))I}.$$

Theorem 2.1. *Let* $\beta_{2\text{L}} = \text{B}(\mathcal{F}^2 \cup \mathcal{C}\mathcal{F} \cup \mathcal{C}^2)$ *and* $\beta_{2\text{R}} = \text{B}(\mathcal{F}^2 \cup \mathcal{F}\mathcal{C} \cup \mathcal{C}^2)$. *Then*:

(i) *E is definite iff $E \in \beta_{2\text{L}}$;*
(ii) *E is reverse definite iff $E \in \beta_{2\text{R}}$;*
(iii) *E is generalized definite iff $E \in \beta_2$;*
(iv) *E is locally testable iff $E \in \beta_3$.*

Proof. Let $\mathcal{D}$, $\mathcal{RD}$, $\mathcal{GD}$ and $\mathcal{LT}$ denote the families of definite, reverse definite, generalized definite and locally testable events, respectively.

(i) From (2.5), if E is definite then $E \in \beta_{2\text{L}}$. Conversely, one verifies that any event in $(\mathcal{F}^2 \cup \mathcal{C}\mathcal{F} \cup \mathcal{C}^2)$ is definite. (Note that if E and E' are finite (cofinite) then EE' is finite (cofinite)). Since $\mathcal{D}$ is a Boolean algebra [8], we have $\mathcal{D} \supseteq B(\mathcal{F}^2 \cup \mathcal{CF} \cup \mathcal{C}^2) = \beta_{2L}$, and (i) follows.

(ii) The proof is similar to that of (i).

(iii) From (2.7) and the fact that for any $u, v \in \Sigma^*$, $uIv = uI\Sigma^{|v|} \cap \Sigma^{|u|}Iv = (u\Sigma^{|v|})I \cap I(\Sigma^{|u|}v)$, it follows that $E \in \mathcal{GD}$ implies $E \in \beta_2$. Next, one verifies that any event in $(\mathcal{F} \cup \mathcal{C})^2$ is in $\mathcal{GD}$. From the definition of $\mathcal{GD}$ events, $\mathcal{GD}$ is closed under union, and from [4] it is closed under complementation. Thus $\mathcal{GD}$ is a Boolean algebra and $\mathcal{GD} \supseteq B((\mathcal{F} \cup \mathcal{C})^2) = \beta_2$.

(iv) From (2.8), each equivalence class $[x]_{k+1}$ is in β_3. Hence any finite union of such equivalence classes, i.e., any LT event, is in β_3. Now, β_3 can be defined alternately as $B([w, I]^3)$. One easily checks that each product in $[w, I]^3$ is in $\mathcal{LT}$. Since $\mathcal{LT}$ is a Boolean algebra [6], we have $\mathcal{LT} \supseteq B([w, I]^3) = \beta_3$.

Let $\Sigma = \{0, 1\}$. The event $I0$ is neither finite nor cofinite; thus $\beta_{2L} \neq \beta_1$. Similarly $0I \in \beta_{2R} \setminus \beta_1$ and $\beta_{2R} \supsetneqq \beta_1$. In fact $\beta_1 = \beta_{2L} \cap \beta_{2R}$. For suppose $K \in \beta_{2L} \cap \beta_{2R}$. Then K can be expressed as [1; 8], $K = E \cup IF = G \cup HI$, where E and G are sets of words of length $< m$ and $F, H \subseteq \Sigma^m$, for some m. We must have $E = G$ and $IF = HI$. If either F or H is empty then K is finite. Otherwise, for any $f \in F$ and $x \in \Sigma^m$, $xf \in IF = HI$. Hence $x \in H$ and therefore $H = \Sigma^m$. Thus $K = G \cup \Sigma^m I$ is cofinite. One verifies also that the event $0I0$ is generalized definite, but it is neither definite nor reverse definite. Thus $\beta_2 \supsetneqq \beta_{2R}$ and $\beta_2 \supsetneqq \beta_{2L}$. Similarly, the event $I0I$ is locally testable but is not generalized definite, so $\mathcal{LT} \supsetneqq \mathcal{GD}$. It will be shown later that $\mathcal{LT}$ is a proper subfamily of $\mathcal{B}_2$.

The position of LT events in the family $\mathcal{SF}$ of star-free events is illustrated in Fig. 1.

3. A necessary condition for local testability

We now prove some properties of the congruence $\sim_{k+1}$ which will motivate the next definitions and will finally lead to the characterization of LT events.

Lemma 3.1. *For $x, y, z \in \Sigma^*$,*

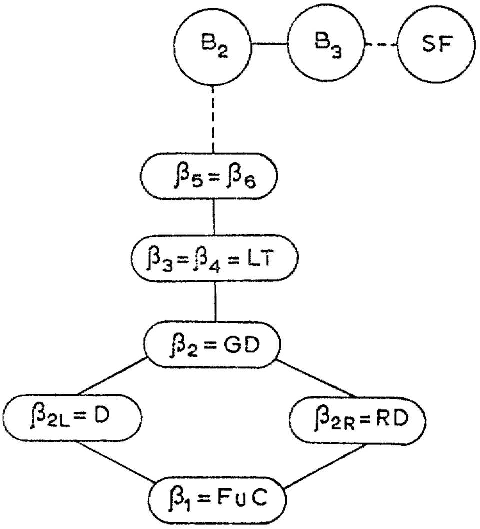


Fig. 1. The $\dot{B}_2$ hierarchy and the dot-depth hierarchy.

(i) *If* $xy = zx$, *then* $xy \sim_{k+1} xy^2$, *where* $k = |x|$;
(ii) $xyx \sim_{k+1} xyxyx$, *where* $k = |x|$;
(iii) $xyxzx \sim_{k+1} xzxyx$, *where* $k = |x|$;
(iv) $x^n yx^n \sim_{k+1} x^n yx^n yx^n$, *where* $|x| > 0$, *for any* k, $0 \le k \le n$;
(v) $x^n yx^n zx^n \sim_{k+1} x^n zx^n yx^n$, *where* $|x| > 0$, *for any* k, $0 \le k \le n$.

Proof. (i) One verifies that for $u, v, w \in \Sigma^*$ and $|v| = k$, the relation $m_{k+1}(uvw) = m_{k+1}(uv) \cup m_{k+1}(vw)$ holds. Now $xy = zx$ implies that $xy^2 = z^2x$. Thus, the length k prefixes and suffixes of both xy and xy^2 are equal to x. Furthermore

$$m_{k+1}(xy^2) = m_{k+1}(zxy) = m_{k+1}(zx) \cup m_{k+1}(xy) = m_{k+1}(xy).$$

(ii) Replace y by yx and z by xy in (i).

(iii) The length k prefixes and suffixes of $xyxzx$ and $xzxyx$ are equal to x, and

$$m_{k+1}(xyxzx) = m_{k+1}(xyx) \cup m_{k+1}(xzx) = m_{k+1}(xzxyx).$$

(iv) In (ii) replace x by x^n and let $l = |x^n|$. Then $x^n yx^n \sim_{l+1} x^n yx^n yx^n$.

If $|x| > 0$, then $l = |x^n| \geq n$. Hence $x^n y x^n \sim_{k+1} x^n y x^n y x^n$ for any $k \leq n$.

(v) Follows analogously to (iv).

Definition 3.1. Let $A = (Q, \Sigma, M, q_0)$ be an ISA and let $k \geq 0$ be an integer. Then A is *$(k+1)$-testable,* if for all $x, y, z \in \Sigma^*$ such that $|x| = k$,

$$(3.1) \qquad xy = zx \text{ implies } (xy)^A = (xy^2)^A ,$$

and

$$(3.2) \qquad (xyxzx)^A = (xzxyx)^A .$$

Notice that the condition $xy = zx$ is required only in (3.1).

Let $n = \#Q$. Then A is *locally testable* (LT) if for all $x \in \Sigma^+$ and for all $y, z \in \Sigma^*$

$$(3.3) \qquad (x^n y x^n)^A = (x^n y x^n y x^n)^A$$

and

$$(3.4) \qquad (x^n y x^n z x^n)^A = (x^n z x^n y x^n)^A .$$

Note that for a given ISA A one can effectively decide whether A is $(k+1)$-testable and whether it is LT. This can be done because the semigroup S_A of functions $x^A : Q \to Q$ is finite.

Definition 3.2. A finite semigroup S is *locally testable* (LT) if for every idempotent $e \in S$, the subsemigroup eSe of S is an idempotent commutative monoid.

Lemma 3.2. *Let $A = (Q, \Sigma, M, q_0)$ be an ISA with semigroup S_A. If S_A is LT then A is permutation-free.*

Proof. Let G be any subgroup of S_A ; then $G = eGe$, where e is the identity of G. Hence G is a subset of $eS_A e$. Since e is idempotent and S_A is LT, every element of G is idempotent. This implies that G consists solely of the element e and that S_A is group-free. It follows by a theorem of [6] that A is permutation free.

Lemma 3.3. *Let $A = (Q, \Sigma, M, q_0)$ be an ISA. If A is permutation free, and $\#Q = n$ then $(x^n)^A = (x^{n+1})^A$, for every $x \in \Sigma^*$.*

Proof. It is clear that $Q \supseteq Qx^A \supseteq Q(x^2)^A \supseteq \ldots \supseteq Q(x^n)^A \supseteq Q(x^{n+1})^A$. Each inclusion above is either proper or can be replaced by an equality. Furthermore, $Q(x^{p+1})^A \supsetneq Q(x^{p+2})^A$, $p \geq 0$, implies $Q(x^p)^A \supsetneq Q(x^{p+1})^A$. Thus $Q(x^n)^A \supsetneq Q(x^{n+1})^A$ implies that all $n + 1$ inclusions in the chain are proper; this is a contradiction since $\#Q = n$. Therefore $Q(x^n)^A = Q(x^{n+1})^A$. Since A is permutation-free it follows that for all $q \in Q$, $q(x^n)^A = q(x^{n+1})^A$. Hence $(x^n)^A = (x^{n+1})^A$.

Theorem 3.1. *Let $A = (Q, \Sigma, M, q_0)$ be an ISA and let S_A be the semigroup of A. Then A is LT iff S_A is LT.*

Proof. Let A be LT and e an idempotent in S_A. Clearly, eS_Ae is a subsemigroup of S_A, in fact a monoid with unit e. Let $a \in S_A$; then there exist $x, y \in \Sigma^+$ such that $e = x^A$ and $a = y^A$. Thus if $\#Q = n$, $eae = (xyx)^A = (x^n y x^n)^A$, since $(x^2)^A = x^A$. By (3.3), $(x^n y x^n)^A = e^n a e^n a e^n = eaeeae$. Thus $eae = (eae)^2$, for all $a \in S_A$. One shows in a similar way, using (3.4) that eS_Ae is commutative.

Conversely suppose S_A is LT. By Lemma 3.2, A is permutation-free and by Lemma 3.3, for any $x \in \Sigma^+$, $(x^n)^A = (x^{n+1})^A$, where $n = \#Q$. Thus $(x^n)^A$ is idempotent. Since S_A is LT, $(x^n)^A \; S_A \; (x^n)^A$ is an idempotent and commutative monoid. Thus for all $y \in \Sigma^+$, $(x^n y x^n)^A = (x^n y x^n)^A \; (x^n y x^n)^A = (x^n y x^n y x^n)^A$. This is also true for $y = \lambda$, and (3.3) follows. By a similar argument, (3.4) follows and A is LT.

The last result yields another decision procedure for testing whether a given ISA A is LT. It is sufficient to test whether the semigroup S_A is LT; this can always be done since S_A is finite.

The next result gives a necessary condition for $(k+1)$-testability.

Theorem 3.2. *Let $\hat{A} = (Q, \Sigma, M, q_0, F)$ be the reduced automaton accepting the $(k+1)$-testable event E. Then A is $(k+1)$-testable.*

Proof. Assume that (3.1) does not hold, i.e., there exist $x, y, z \in \Sigma^+$ such that $|x| = k$, $xy = zx$, and $(xy)^A \neq (xyy)^A$. Then, for some $q \in Q$,

$q(xy)^A \neq q(xyy)^A$. Since A is reduced, there exist $u, v \in \Sigma^*$ such that $uxyv \in E$ iff $uxyyv \notin E$. By Lemma 3.1 and the fact that $\sim_{k+1}$ is a congruence relation, $uxyv \sim_{k+1} uxyyv$. Thus E is not $(k+1)$-testable, which is a contradiction. A similar argument holds if we assume that (3.2) does not hold; hence A is $(k+1)$-testable.

Theorem 3.3. *Let $A = (Q, \Sigma, M, q_0)$ be an ISA, Then A is LT iff A is $(k+1)$-testable for some integer $k \geq 0$.*

Proof. Let A be LT and let $k = (\#S_A) + 1$. We will prove that A is $(k+1)$-testable. Let $x \in \Sigma^*$ be such that $|x| = k$. It follows from the choice of k that there exist $x_1, x_3 \in \Sigma^*$ and $x_2 \in \Sigma^+$ such that

$$(3.5) \qquad x = x_1 x_2 x_3 ,$$

and

$$(3.6) \qquad x_1^A = (x_1 x_2)^A .$$

Now, (3.6) implies that for all $m \geq 0$

$$(3.7) \qquad x_1^A = (x_1 x_2^m)^A .$$

To see that (3.1) holds, let $y, z \in \Sigma^*$ be such that $xy = zx$. If $y = \lambda$ we have nothing to prove, so assume that $y \in \Sigma^+$. From (3.5) it follows that there is a shortest $v \in \Sigma^*$, such that for some $u \in \Sigma^*$

$$(3.8) \qquad x = ux_1 x_2 v .$$

We claim that $|v| < |y|$. In fact, from $xy = zx$ and (3.8) we have

$$(3.9) \qquad ux_1 x_2 vy = zux_1 x_2 v .$$

Now, if $|v| \geq |y|$, then $v = v'y$ for some $v' \in \Sigma^*$ and $ux_1 x_2 v = zux_1 x_2 v'$. By (3.8), $x = zux_1 x_2 v'$. Since $|y| > 0$ it follows that $|v'| < |v|$, i.e., there is a $v' \in \Sigma^*$ shorter than v which satisfies (3.8). This is a contradiction; hence $|v| < |y|$. Thus, from (3.9) we have that, for some $y_1 \in \Sigma^*$, $y =$

$y_1 v$ and

$$(3.10) \qquad ux_1x_2vy_1 = zux_1x_2 \; .$$

Now, from (3.8), (3.7) and (3.10) it follows that for all $m \geq 0$, $(xy_1)^A = (xy_1x_2^m)^A$. Thus, if $n = \#Q$, then $(xy)^A = (xy_1v)^A = (xy_1x_2^nv)^A$. From (3.8) and (3.7), we also have that $(xy_1x_2^nv)^A = (ux_1x_2^nvy_1x_2^nv)^A$. Hence,

$$(3.11) \qquad (xy)^A = (ux_1x_2^nvy_1x_2^nv)^A \; .$$

On the other hand, $(xyy)^A = (zxy)^A$ and from (3.11) and (3.7), $(xyy)^A = (zux_1x_2{}^{n+1}vy_1x_2^nv)^A$. Using (3.10) and (3.7), $(xyy)^A = (ux_1x_2vy_1x_2^nvy_1x_2^nv)^A$. Since A is LT, (3.3) holds, and from (3.11) and the last equality, $(xy)^A = (xyy)^A$, i.e., (3.1) also holds. To see that (3.2) holds, we have from (3.5) and (3.7), for any $y, z \in \Sigma^*$,

$$(xyxzx)^A = (x_1x_2^nx_3yx_1x_2^nx_3zx_1x_2^nx_3)^A$$

and

$$(xzxyx)^A = (x_1x_2^nx_3zx_1x_2^nx_3yx_1x_2^nx_3)^A \; .$$

From (3.4) it follows that $(xyxzx)^A = (xzxyx)^A$. Hence A is $(k+1)$-testable.

Conversely, let A be $(k+1)$-testable for some $k \geq 0$. We first prove that A is permutation-free. In fact let $x \in \Sigma^*$ and $R \subseteq Q$ be such that $Rx^A = R$; we have to prove that for all $r \in R$, $rx^A = r$. If $R = \emptyset$, or $x = \lambda$, this holds, so assume that $x \in \Sigma^+$ and $R \neq \emptyset$. $Rx^A = R$ implies that the restriction of x^A to R is a permutation on the finite set R. Therefore there exists an $m \geq 1$ such that for all $r \in R$

$$(3.12) \qquad r(x^m)^A = r \; .$$

On the other hand, $|x^{(k+1)m-1}| \geq k$. Therefore, there exist x_1, $w \in \Sigma^*$ such that $|w| = k$ and $x_1w = x^{(k+1)m-1}$. We also have $x^{(k+1)m} = xx_1w = x_1wx$ which implies that $wx = x_2w$ for some $x_2 \in \Sigma^*$. Thus, by (3.1) $(wx)^A = (wxx)^A$ and hence $(x_1wx)^A = (x_1wxx)^A$. From this and (3.12), it follows that for all $r \in R$, $r = r(x^{(k+1)m})^A = r(x^{(k+1)m+1})^A = rx^A$; hence A is permutation-free. Thus by Lemma 3.3, for all $x \in \Sigma^*$

$$(3.13) \qquad (x^n)^A = (x^{n+1})^A \ ,$$

where $n = \#Q$. Now, let $x \in \Sigma^+$ and $y \in \Sigma^*$. Since $|x^{n+k}| > k$ there exist $x_1, w \in \Sigma^*$ such that $|w| = k$ and $x_1 w = x^{n+k}$. On the other hand, replacing in (3.1) x by w, y by $yx_1 w$ and z by wyx_1 we have $(wyx_1 w)^A = (wyx_1 wyx_1 w)^A$ and thus

$$(x_1 wyx_1 w)^A = (x_1 wyx_1 wyx_1 w)^A \ .$$

Since $x_1 w = x^{n+k}$ and (3.13) implies that $(x^{n+k})^A = (x^n)^A$, we have $(x^n yx^n)^A = (x^n yx^n yx^n)^A$, i.e., (3.3.) holds for A. Similarly (3.4) also holds for A and hence A is LT.

Corollary 3.1.[1] *Let $\hat{A} = (Q, \Sigma, M, q_0, F)$ be the reduced automaton accepting the LT event E. Then A is LT.*

Proof. If E is an LT event then it is $(k+1)$-testable for some $k \geq 0$. By Theorem 3.1, A is $(k+1)$-testable and by Theorem 3.4, A is LT.

Note that this corollary can also be proved, much in the same way as Theorem 3.2, by using (iv) and (v) of Lemma 3.1 and Lemma 3.3.

4. Idempotent and commutative automata

In this section we study a restricted class of LT automata which will play an important role in the proof of the converse of Theorem 3.2 and Corollary 3.1, as well as in the decomposition of LT automata.

Definition 4.1. An ISA $A = (Q, \Sigma, M, q_0)$ is *idempotent* if, for all $x \in \Sigma^*$, $x^A = (x^2)^A$; it is *commutative* if, for all $x, y \in \Sigma^*$, $(xy)^A = (yx)^A$. If A is both idempotent and commutative, we will write A is IC.

[1] An equivalent result has also been independently obtained in [9].

Note that an ISA A is IC iff it is 1-testable iff the monoid G_A of A is idempotent and commutative. Thus it is effectively decidable whether A is IC. However, the next result establishes that it is sufficient to verify the idempotent and commutative properties for $\{\sigma^A \mid \sigma \in \Sigma\}$.

Theorem 4.1. *Let $A = (Q, \Sigma, M, q_0)$ be an ISA. Then the following are equivalent*:

(i) *A is IC.*

(ii) *For all $x, y \in \Sigma^*$, $m_1(x) = m_1(y)$ implies $x^A = y^A$.*

(iii) *For all $\sigma_1, \sigma_2 \in \Sigma$, $\sigma_1^A = (\sigma_1^2)^A$ and $(\sigma_1\sigma_2)^A = (\sigma_2\sigma_1)^A$.*

Proof. (i) *implies* (ii). Let $x \in \Sigma^*$ and $\sigma \in m_1(x)$. Then $x = u\sigma v$ for some $u, v \in \Sigma^*$ and $x^A = (u\sigma v)^A = (u\sigma\sigma v)^A = (u\sigma v\sigma)^A = (x\sigma)^A$, since A is IC. This clearly implies that, if $y \in \Sigma^*$ and $m_1(y) \subseteq m_1(x)$, then $x^A = (xy)^A$. Since by hypothesis $m_1(x) = m_1(y)$, we also have $y^A = (yx)^A$. Since A is IC, $(xy)^A = (yx)^A$ and so $x^A = y^A$.

(ii) *implies* (iii). $m_1(\sigma_1) = m_1(\sigma_1^2) = \{\sigma_1\}$ and $m_1(\sigma_1\sigma_2) = m_1(\sigma_1\sigma_2) = \{\sigma_1, \sigma_2\}$. Hence (iii) follows from (ii).

(iii) *implies* (i). We first prove that

$$(4.1) \qquad (u\sigma)^A = (\sigma u)^A$$

for all $u \in \Sigma^*$ and $\sigma \in \Sigma$. This is done by induction on $|u|$.

Obviously, (4.1) is true when $|u| = 0$, i.e. $u = \lambda$. Assume (4.1) holds for u and let $\sigma_1 \in \Sigma$. Then $(u\sigma_1\sigma)^A = (u\sigma\sigma_1)^A = (\sigma u\sigma_1)^A$, by (iii) and by the induction hypothesis. Thus (4.1) holds for $u\sigma_1$. We now show by induction on $|x|$ that $x^A = (x^2)^A$. The claim is obviously true for $x = \lambda$. Assume it is true for x and let $\sigma \in \Sigma$. Then $(x\sigma)^A = (xx\sigma\sigma)^A = (x\sigma x\sigma)^A$, by the induction hypothesis, by (iii) and by (4.1). Next, for a fixed $y \in \Sigma^*$ we show by induction on $|x|$ that $(xy)^A = (yx)^A$. For $x = \lambda$ this is obvious. Assume it is true for x and let $\sigma \in \Sigma$. Then $(x\sigma y)^A = (xy\sigma)^A = (yx\sigma)^A$, by (4.1) and by the induction hypothesis. Thus (iii) implies that A is IC.

We now proceed to give a structural characterization of IC ISA's.

Definition 4.2. A *half-reset* is an ISA $D = (\{q_0, q_1\}, \Sigma, M, q_0)$, where for every $\sigma \in \Sigma$, σ^D is either an identity or a reset to q_1 (σ^D is a reset to q_1 iff $q_0\sigma^D = q_1\sigma^D = q_1$).

Note that a half-reset is one of the units in the Krohn-Rhodes decomposition theory [3].

Definition 4.3. Let $\Delta = (Q, \Sigma, M, q_0)$ be an ISA, where $Q = 2^\Sigma$, $q_0 = \emptyset$ and for $\theta \subseteq \Sigma$ and $\sigma \in \Sigma$, $\theta\sigma^\Delta = \theta \cup \{\sigma\}$. One verifies that Δ is an IC ISA; it will be called the *free IC ISA over* Σ.

The following result which holds for any ISA's A and B will be required in the proof of Theorem 4.2.

Lemma 4.1. *Let* $A = (Q, \Sigma, M, q_0)$ *and* $B = (R, \Sigma, N, r_0)$ *be ISA's and assume that* A *is connected. Then* $A \leq B$ *iff, for all* $x, y \in \Sigma^*$, $r_0x^B = r_0y^B$ *implies* $q_0x^A = q_0y^A$.

Proof. If $A \leq B$, there is an onto function $\eta: R_1 \to Q$; $r_0 \in R_1 \subseteq R$, $r_0\eta = q_0$ and for all $r \in R_1$ and $\sigma \in \Sigma$, $r\sigma^B \in R_1$ and $r\sigma^B\eta = r\eta\sigma^A$. This implies that for all $r \in R_1$ and for all $x \in \Sigma^*$, $rx^B \in R_1$ and $rx^B\eta = r\eta x^A$. Thus, if $r_0x^B = r_0y^B$, then $r_0x^B\eta = r_0y^B\eta$ and also $r_0x^B\eta = r_0\eta x^A = q_0x^A$ and $r_0y^B\eta = r_0\eta y^A = q_0y^A$. Hence $q_0x^A = q_0y^A$. Conversely, if for all $x, y \in \Sigma^*$, $r_0x^B = r_0y^B$ implies $q_0x^A = q_0y^A$, then we can define a function η by $r_0x^B\eta = q_0x^A$. Since A is connected, η is onto Q and clearly η satisfies the conditions in the definition of $A \leq B$.

The following result about IC ISA's will also be required in the proof of Theorem 4.2.

Lemma 4.2. *Let* $A = (Q, \Sigma, M, q_0)$ *and* $B = (R, \Sigma, N, r_0)$ *be ISA's. If* $A \leq B$ *and* B *is IC, then* A *is also IC.*

Proof. Let η be the function relating B to A as in the definition of $A \leq B$, $\eta: R_1 \to Q$, where $r_0 \in R_1 \subseteq R$. Let $q \in Q$; then, since η is onto, there exists $r \in R_1$ such that $r\eta = q$. Thus

$$q\sigma^A = r\eta\sigma^A = r\sigma^B\eta = r\sigma^B\sigma^B\eta = (r\sigma^B)\eta\sigma^A$$
$$= r\eta\sigma^A\sigma^A = q(\sigma^2)^A \ ,$$

where the third equality above follows from the fact that B is idempotent. By similar reasoning, using the fact that B is commutative, we verify that for $\sigma_1 \in \Sigma$, $q(\sigma\sigma_1)^A = q(\sigma_1\sigma)^A$, and (by Theorem 4.1) A is IC.

Theorem 4.2. *Let $A = (Q, \Sigma, M, q_0)$ be a connected ISA and let Δ be the free IC ISA over Σ. Then the following are equivalent:*

(i) *A is IC.*

(ii) *$A \leq \Delta$.*

(iii) *There exists an integer $l \geq 1$ and l half-resets $D_1, D_2, ..., D_l$, such that $A \leq D_1 \times D_2 \times ... \times D_l$.*

Proof. (i) *implies* (ii). In view of Lemma 4.1, it is sufficient to prove that for $x, y \in \Sigma^*$, $\emptyset x^\Delta = \emptyset y^\Delta$ implies $q_0x^A = q_0y^A$. From the definition of Δ we have $\emptyset x^\Delta = \{\sigma | \sigma \in \Sigma$ and $x = u\sigma v$ for some $u, v \in \Sigma^*\} = m_1(x)$. Thus $\emptyset x^\Delta = \emptyset y^\Delta$ implies $m_1(x) = m_1(y)$. Since A is IC, it follows from Theorem 4.1 that $x^A = y^A$. In particular, $q_0x^A = q_0y^A$, and $A \leq \Delta$.

(ii) *implies* (iii). We first show that, if $l = \#\Sigma$, there exist l half-resets $D_1, D_2, ..., D_l$ such that $\Delta \leq D_1 \times D_2 \times ... \times D_l$. Let $\Sigma = \{\sigma_1, \sigma_2, ..., \sigma_l\}$ and let $D_i = (\{q_{i,0}, q_{i,1}\}, \Sigma, M_i, q_{i,0})$ for $1 \leq i \leq l$, where $\sigma_j^{D_i}$ is a reset to $q_{i,1}$ if $i = j$ and is an identity otherwise. Let $B = D_1 \times D_2 \times ... \times D_l$, and $x \in \Sigma^*$; then

$$(q_{1,0}, q_{2,0}, ..., q_{l,0})\, x^B = (q_{1,0}x^{D_1}, q_{2,0}x^{D_2}, ..., q_{l,0}x^{D_l}).$$

Now $q_{i,0}x^{D_i}$ is $q_{i,1}$ if $\sigma_i \in m_1(x)$ and it is $q_{i,0}$ otherwise. Thus $(q_{1,0}, q_{2,0}, ..., q_{l,0})x^B = (q_{1,0}, q_{2,0}, ..., q_{l,0})y^B$ implies $m_1(x) = m_1(y)$, and therefore $\emptyset x^\Delta = \emptyset y^\Delta$. Since Δ is connected, Lemma 4.1 applies, and $\Delta \leq B$. Since covering of ISA's is a transitive relation [3], $A \leq \Delta$ and $\Delta \leq B$ implies $A \leq B$. (In fact one can verify that also $B \leq \Delta$ and thus Δ is isomorphic to B.)

(iii) *implies* (i). One can verify that a half-reset is an IC ISA. Also, if C and D are IC ISA's, then so is $C \times D$. It follows that $B = D_1 \times D_2 \times ... \times D_l$ is IC. Since $A \leq B$, Lemma 4.2 implies that A is an IC ISA.

Finally, we have the following definitions and the main theorem of this section.

Definition 4.4. Let $A = (Q, \Sigma, M, q_0)$ be an ISA and let $B = (R, \Sigma, N, r_0)$ be a π-factor of A. We say that B is an *idempotent π-factor* of A, if, for all $r \in R$, for all $q \in r\pi$ and for all $x \in \Sigma^*$, $rx^B = r$ implies $qx^A = q(x^2)^A$.. B is a *commutative π-factor of A* if, for all $r \in R$, for all $q \in r\pi$ and for all $x, y \in \Sigma^*$, $rx^B = ry^B = r$ implies $q(xy)^A = q(yx)^A$. We say that B is an *IC π-factor of A* if it satisfies both of the above conditions.

Note that, given a π-factor $B = A/\pi$, one can verify whether it is an IC π-factor. In fact, for $r \in R$ and for $x \in \Sigma^*$ such that $rx^B = r$, the restriction of x^A to $r\pi$ is a function from $r\pi$ into $r\pi$. The set of all such functions, say S_r, is a finite monoid under composition of functions, which can be effectively found. B is an IC π-factor of A iff for all $r \in R$, S_r is an idempotent and commutative monoid.

Definition 4.5. Let $A = (Q, \Sigma, M, q_0)$ be a connected ISA and let $B = (R, \Sigma, N, r_0)$ be a π-factor of A. We say that the π-factor B is a *minimal π-factor of A* if, for all $r \in R$ and for all $q \in r\pi$, there exists an $x \in \Sigma^*$ such that $r_0x^B = r$ and $q_0x^A = q$.

Theorem 4.3. *Let $A = (Q, \Sigma, M, q_0)$ be a connected ISA, let $B = (R, \Sigma, N, r_0)$ be an IC π-factor of A and let Δ be the free IC ISA over $R \times \Sigma$. Then $A \leq B \circ \Delta$. Conversely, if B is a minimal π-factor of A, then $A \leq B \circ \Delta$ implies that B is an IC π-factor of A.*

Proof. First we prove the last statement. Thus, let $r \in R$ and $x, y \in \Sigma^*$ be such that, $rx^B = ry^B = r$, and let $q \in r\pi$. Since B is a minimal π-factor of A, there exists $z \in \Sigma^*$ such that $q_0z^A = q$ and $r_0z^B = r$. Now, for some $\theta \subseteq R \times \Sigma$, $(r_0, \emptyset)z^{B\circ\Delta} = (r, \theta)$ and, if η is the function relating $B\circ\Delta$ to A $(A \leq B\circ\Delta)$, then

$$(r, \theta)\eta = (r_0, \emptyset)z^{B\circ\Delta}\eta = (r_0, \emptyset)\eta z^A = q_0z^A = q\ ,$$

i.e., $(r, \theta)\eta = q$. Since $rx^B = r$ and Δ is IC, it follows that $(r, \theta)x^{B\circ\Delta} = (r, \theta)(x^2)^{B\circ\Delta}$. Hence, $(r, \theta)x^{B\circ\Delta}\eta = (r, \theta)(x^2)^{B\circ\Delta}\eta$ and $qx^A = q(x^2)^A$.

Similarly, $q(xy)^A = q(yx)^A$ and therefore B is an IC π-factor of A.

Now, we proceed to prove the first part. We will refer, without explicitly stating it, to ISA's A, B and Δ as in the statement of the theorem and we assume that B is an IC π-factor of A. We begin with two definitions.

For $x \in \Sigma^*$ and $r \in R$ we define the function θ as: $\theta(r, x) = \{(s, \sigma) | s \in R, \sigma \in \Sigma$, and $x = y\sigma z$ for some $y, z \in \Sigma^*$ such that $ry^B = s\}$. Intuitively, "$\theta(r, x)$ is the subset of the set $R \times \Sigma$ of transitions of B traveled when spelling x in B, starting from r".

For $x, y \in \Sigma^*$ and $r \in R$ we define the relation $\leftrightarrow_r$ over Σ^* as: $x \leftrightarrow_r y$ iff $rx^B = ry^B$ and, for all $q \in r\pi$, $qx^A = qy^A$.

We state, without proof, the following properties of the relation $\leftrightarrow_r$:

(4.2) $\leftrightarrow_r$ is an equivalence relation on Σ^*.

(4.3) If $x \leftrightarrow_r y$ then, for all $z \in \Sigma^*$, $xz \leftrightarrow_r yz$.

(4.4) If $z \in \Sigma^*$ and $s \in R$ are such that $sz^B = r$, and if $x \leftrightarrow_r y$, then $zx \leftrightarrow_s zy$.

(4.5) It follows from (4.3) and (4.4) that, if $x \leftrightarrow_r y$ and $su^B = r$, then $uxv \leftrightarrow_s uyv$.

Since B is an IC π-factor of A, it follows that for $x, y \in \Sigma^*$ and $r \in R$ such that $rx^B = ry^B = r$,

(4.6) $x \leftrightarrow_r x^2$

and

(4.7) $xy \leftrightarrow_r yx$.

Now we relate these two definitions by the following lemmas:

Lemma 4.3. *Let $x, y \in \Sigma^*$ and $r \in R$ be such that $\theta(r, x) = \theta(r, xy)$ and $rx^B = r(xy)^B$. Then $x \leftrightarrow_r xy$.*

Proof. Let $s = rx^B$. It is clear that $\theta(r, xy) = \theta(r, x) \cup \theta(s, y)$. Since $\theta(r, x) = \theta(r, xy)$ it follows that

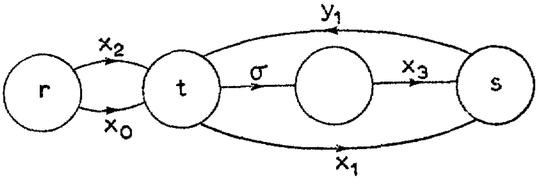
$$(4.8) \qquad \theta(s, y) \subseteq \theta(r, x) .$$

Now, we prove that for any prefix y_1 of y there exist $x_0, x_1 \in \Sigma^*$, such that

$$(4.9) \qquad x = x_0x_1,\ rx_0^B = sy_1^B \text{ and } x \leftrightarrow_r xy_1x_1 .$$

We proceed by induction on $|y_1|$. For $|y_1| = 0$, i.e. $y_1 = \lambda$, we take $x_0 = x$ and $x_1 = \lambda$ which clearly satisfy (4.9). Assume that the assertion holds for y_1, i.e., there exist $x_0, x_1 \in \Sigma^*$ such that $x = x_0x_1$, $rx_0^B = sy_1^B$ and

$$(4.10) \qquad x \leftrightarrow_r xy_1x_1 .$$



$x = x_0x_1 = x_2\sigma x_3$

Fig. 2.

If $y = y_1$, we are finished. Otherwise for some $\sigma \in \Sigma$ and $y_2 \in \Sigma^*$, $y = y_1\sigma y_2$. Let $t = sy_1^B$; then it follows that $tx_1^B = s$. Thus $(t, \sigma) \in \theta(s, y)$ and, by (4.8), $(t, \sigma) \in \theta(r, x)$. Hence, there exist x_2, $x_3 \in \Sigma^*$ such that $x = x_2\sigma x_3$ and $rx_2^B = t$. Clearly $t(\sigma x_3)^B = s$ (see Fig. 2). Thus, $t(\sigma x_3y_1)^B = t$ and it follows from (4.6) that $\sigma x_3y_1 \leftrightarrow_t \sigma x_3y_1\sigma x_3y_1$. Thus letting $u = x_2$ and $v = x_1$, since $rx_2^B = t$, we have from (4.5) that

$$x_2\sigma x_3y_1x_1 \leftrightarrow_r x_2\sigma x_3y_1\sigma x_3y_1x_1 .$$

Since $x_2\sigma x_3 = x$, we have

(4.11) $\quad xy_1x_1 \leftrightarrow_r xy_1\sigma x_3y_1x_1 .$

On the other hand, since $s(y_1\sigma x_3)^B = s(y_1x_1)^B = s$, it follows from (4.7) that $y_1\sigma x_3y_1x_1 \leftrightarrow_s y_1x_1y_1\sigma x_3$. Thus letting $z = x$, since $rx^B = s$, we have from (4.4) that

(4.12) $\quad xy_1\sigma x_3y_1x_1 \leftrightarrow_r xy_1x_1y_1\sigma x_3 .$

Now, from (4.10) and letting $z = y_1\sigma x_3$, we have from (4.3) that

(4.13) $\quad xy_1\sigma x_3 \leftrightarrow_r xy_1x_1y_1\sigma x_3 .$

Finally, by transitivity, from (4.10)–(4.13) it follows that $x \leftrightarrow_r xy_1\sigma x_3$ which proves the induction step, since $x = x_2\sigma x_3$ and $r(x_2\sigma)^B = s(y_1\sigma)^B = t\sigma^B$.

Now, since y itself is a prefix of y, it follows from (4.10) that there are $x_0, x_1 \in \Sigma^*$ such that $x = x_0x_1$, $rx_0^B = sy^B$ and

(4.14) $\quad x \leftrightarrow_r xyx_1 .$

Since $sy^B = s$ it follows that $rx_0^B = s$ and $sx_1^B = s$. Thus we have from (4.6) and (4.7) that $x_1y \leftrightarrow_s x_1yx_1$ and letting $z = x_0$, since $rx_0^B = s$, it follows from (4.4) that $x_0x_1y \leftrightarrow_r x_0x_1yx_1$, i.e. $xy \leftrightarrow_r xyx_1$, since $x_0x_1 = x$. Now, from (4.14) by transitivity, $x \leftrightarrow_r xy$.

Lemma 4.4. *Let $x, y \in \Sigma^*$ and $r \in R$ be such that $\theta(r, x) = \theta(r, y)$ and $rx^B = ry^B$. Then $x \leftrightarrow_r y$.*

Proof. We proceed by induction on $\#\theta(r, x)$. If $\#\theta(r, x) = 0$ then $x = \lambda$ and, since $\theta(r, x) = \theta(r, y)$, also $y = \lambda$. Hence $x \leftrightarrow_r y$. Assume now that $\#\theta(r, x) > 0$ and that for all $r' \in R$ and for all $x', y' \in \Sigma^*$ such that $r'(x')^B = r'(y')^B$, $\theta(r', x') = \theta(r', y')$ and $\#\theta(r', x') < \#\theta(r, x)$ we have $x' \leftrightarrow_{r'} y'$. Let $s = rx^B = ry^B$ and let $P = \{sz^B \,|\, z \in \Sigma^* \text{ and } \theta(s, z) \subseteq \theta(r, x)\}$

[Intuitively, "P is the subset of Q, reachable from s, using only transitions in $\theta(r, x)$. In other words, if we delete from the state graph of B the transitions not in $\theta(r, x)$, P will be the set of states in the strongly connected component which contains s."]

We have:

Case 1. If $r \in P$ then there exists $z \in \Sigma^*$ such that $sz^B = r$ and $\theta(s, z) \subseteq \theta(r, x)$. Clearly $r(xz)^B = r(yz)^B = r$; therefore by (4.3) and (4.7), $xzyzx \leftrightarrow_r yzxzx$. On the other hand,

$$\theta(r, x) = \theta(r, y) = \theta(r, xzyzx) = \theta(r, yzxzx) .$$

Hence, by Lemma 4.3

$$x \leftrightarrow_r xzyzx \text{ and } y \leftrightarrow_r yzxzx .$$

Thus by transitivity, $x \leftrightarrow_r y$.

Case 2. If $r \notin P$, then $x = x_1 \sigma x_2$ for some $x_1, x_2 \in \Sigma^*$ and $\sigma \in \Sigma$, such that $p = rx_1^B \notin P$ and $q = r(x_1\sigma)^B \in P$. This is so, since $rx^B = s \in P$ and $r \notin P$. Now, since $\theta(r, x) = \theta(r, y)$ and $(p, \sigma) \in \theta(r, x)$ it follows that $(p, \sigma) \in \theta(r, y)$, i.e., there are $y_1, y_2 \in \Sigma^*$, such that $y = y_1 \sigma y_2$ and $ry_1^B = p$. Clearly, $qy_2^B = qx_2^B = s$. (See Fig. 3.) Now we claim that

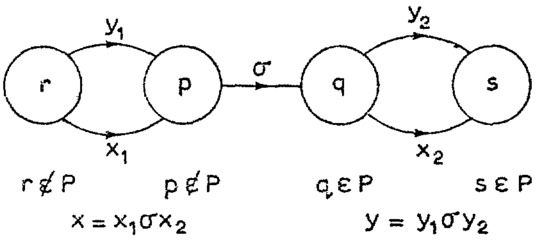


Fig. 3.

(4.15) $\quad (p, \sigma) \notin \theta(r, x_1) \cup \theta(q, x_2)$

To see this, suppose $(p, \sigma) \in \theta(r, x_1)$, i.e., $x_1 = x_3 \sigma x_4$ for some $x_3, x_4 \in \Sigma^*$ such that $rx_3^B = p$. It follows that $qx_4^B = rx_1^B = p$. Since $q \in P$ and clearly $\theta(q, x_4) \subseteq \theta(r, x)$, we also have $p \in P$; this is a contradiction. If we suppose that $(p, \sigma) \in \theta(q, x_2)$, then $x_2 = x_3 \sigma x_4$, where $qx_3^B = p$, which is again a contradiction of $p \notin P$. A similar argument shows that

(4.16) $\quad (p, \sigma) \notin \theta(r, y_1) \cup \theta(q, y_2)$.

Next, we claim that

$$(4.17) \qquad \theta(r, x_1) \cap \theta(q, y_2) = \emptyset \, .$$

In fact, suppose that there are $t \in R$ and $\sigma_1 \in \Sigma$, such that $(t, \sigma_1) \in \theta(r, x_1) \cap \theta(q, y_2)$. Then, (see Fig. 4) $x_1 = x_3 \sigma_1 x_4$ and $y_2 = y_3 \sigma_1 y_4$ for some $x_3, x_4, y_3, y_4 \in \Sigma^*$, such that $rx_3^B = qy_3^B = t$. It follows that $q(y_3 \sigma_1 x_4)^B = p$ and thus since $\theta(q, y_3 \sigma_1 x_4) \subseteq \theta(r, x)$, $p \in P$; this is a contradiction. On the other hand,

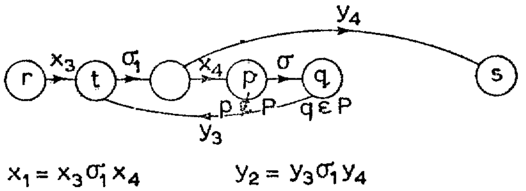


Fig. 4.

$$(4.18) \qquad \theta(r, x_1) \cup \{(p, \sigma)\} \cup \theta(q, x_2) = \theta(r, x) = \theta(r, y)$$
$$= \theta(r, y_1) \cup \{(p, \sigma)\} \cup \theta(q, y_2).$$

It follows from (4.15), (4.17) and (4.18) that $\theta(r, x_1) \subseteq \theta(r, y_1)$ and from (4.16)–(4.18) that $\theta(q, y_2) \subseteq \theta(q, x_2)$. Similarly , $\theta(r, y_1) \cap \theta(q, x_2) = \emptyset$ and then from (4.15), (4.16) and (4.18), $\theta(r, y_1) \subseteq \theta(r, x_1)$ and $\theta(q, x_2) \subseteq \theta(q, y_2)$. Altogether,

$$\theta(r, x_1) = \theta(r, y_1) \subsetneqq \theta(r, x)$$

and

$$\theta(q, x_2) = \theta(q, y_2) \subsetneqq \theta(r, x) \, .$$

Since $rx_1^B = ry_1^B = p$ and $qx_2^B = qy_2^B = s$, it follows from the induction hypothesis that $x_1 \leftrightarrow_r y_1$ and $x_2 \leftrightarrow_q y_2$. Finally, we have from (4.5) that $x = x_1 \sigma x_2 \leftrightarrow_r y_1 \sigma x_2$ and $y_1 \sigma x_2 \leftrightarrow_r y_1 \sigma y_2 = y$. Thus $x \leftrightarrow_r y$.

Now, it is easy to prove that $A \leq B \circ \Delta$. In fact, in view of Lemma 4.1, it is sufficient to prove that, if $x, y \in \Sigma^*$ are such that $(r_0, \emptyset) x^{B \circ \Delta} = (r_0, \emptyset) y^{B \circ \Delta}$, then $q_0 x^A = q_0 y^A$. But $(r_0, \emptyset) x^{B \circ \Delta} = (r_0, \emptyset) y^{B \circ \Delta}$ implies that $r_0x^B = r_0y^B$ and $\theta(r_0, x) = \theta(r_0, y)$. Hence, by Lemma 4.3, $x \leftrightarrow_{r_0} y$ and since $q_0 \in r_0\pi$ it follows that $q_0x^A = q_0y^A$.

Note that the minimality of B in the second statement of Theorem 4.3 is necessary. In fact, if B is isomorphic to A and $r\pi = Q$ for all $r \in R$, then clearly $A \leq B \circ \Delta$. However, in general, B is not an IC π-factor of A. We also note that the equivalence of (i) and (ii) of Theorem 4.2 also follows from Theorem 4.3 if we take $B = (\{r_0\}, \Sigma, N, r_0)$ and $r_0\pi = Q$. We also have:

Corollary 4.1. *Let $A = (Q, \Sigma, M, q_0)$ be a connected ISA and $B = (R, \Sigma, N, r_0)$ be a minimal π-factor of A. Then there is an IC ISA C such that $A \leq B \circ C$ iff B is an IC π-factor of A.*

Proof. For the *if* part, take $C = \Delta$. For the *only if*, $A \leq B \circ C$ and A connected imply that $A \leq B \circ \Delta$.

5. Definite π-factors

In this section, we show that if A is a $(k+1)$-testable ISA then there is an IC π-factor B of A which is a k-definite ISA. We have:

Definition 5.1. Let $A = (Q, \Sigma, M, q_0)$ be an ISA and let $k \geq 0$ be an integer. We say that A is *k-definite* if for all $x \in \Sigma^*$ such that $|x| = k$, $\#(Qx^A) = 1$. A is *definite* if it is k-definite for some k.

Definition 5.2. For $k \geq 0$, the ISA $B = (R, \Sigma, N, r_0)$, where $R = \{\langle x\rangle | x \in \Sigma^* \text{ and } |x| \leq k\}$, $r_0 = \langle\lambda\rangle$,

$$\langle x\rangle\sigma^A = \begin{cases} \langle x\sigma\rangle, & \text{if } |x| < k \\ \langle y\rangle, & \text{if } |x| = k, \end{cases}$$

where $x\sigma = \sigma' y$ for some $\sigma' \in \Sigma$, $y \in \Sigma^*$, is called the *free k-definite* ISA over Σ [8].

Definition 5.3. Let $A = (Q, \Sigma, M, q_0)$ be a connected ISA and let $B = (R, \Sigma, N, r_0)$ be the free k-definite ISA over Σ. Let π: $R \to 2^Q$ be the function defined by

$$\langle x\rangle\pi = \begin{cases} \{q_0 x^A\}\,, & \text{if } |x| < k\,, \\ Qx^A\,, & \text{if } |x| = k. \end{cases}$$

One can verify that B is a π-factor of A and we call B the *free k-definite π-factor of A.*

Theorem 5.1. *Let $k \geq 0$ and let $A = (Q, \Sigma, M, q_0)$ be a connected $(k{+}1)$-testable ISA. Then the free k-definite π-factor of A is an IC π-factor.*

Proof. Let $B = (R, \Sigma, N, r_0)$ be the free k-definite π-factor of A. For $x \in \Sigma^*$ such that $|x| < k$, $\langle x\rangle\pi = \{q_0 x^A\}$. Hence, $\#(\langle x\rangle\pi) = 1$ and the conditions of Definition 4.4 are trivially satisfied. Thus, let $x, y, z \in \Sigma^*$ be such that $|x| = k$ and $\langle x\rangle y^B = \langle x\rangle z^B = \langle x\rangle$. We have to prove that for all $q \in \langle x\rangle\pi$, $qy^A = q(y^2)^A$ and $q(yz)^A = q(zy)^A$. Since $\langle x\rangle\pi = Qx^A$ this holds iff for all $q \in Q$, $q(xy)^A = q(xy^2)^A$ and $q(xyz)^A = q(xzy)^A$, i.e.,

$$(5.1) \qquad (xy)^A = (xyy)^A \ ,$$

and

$$(5.2) \qquad (xyz)^A = (xzy)^A \ .$$

If $y = \lambda$ or $z = \lambda$ this is trivial, so let's assume $y, z \in \Sigma^+$. Let $y_1, x' \in \Sigma^*$ be such that $|x'| = k$ and $xy = y_1x'$. Since B is the free k-definite ISA over Σ, and $|x| = |x'| = k$, it is easily seen that $Rx^B = \{\langle x\rangle\}$ and $R(y_1x')^B = \{\langle x'\rangle\}$. Thus

$$\{\langle x'\rangle\} = R(y_1x')^B = R(xy)^B = \{\langle x\rangle\}y^B \ .$$

Since $\langle x\rangle y^B = \langle x\rangle$, it follows that $x = x'$ and hence $xy = y_1x$. Since A is $(k{+}1)$-testable, (5.1) holds. Similarly $(xz)^A = (xzz)^A$. Therefore,

$$(xyz)^A = (y_1xz)^A = (y_1xz^{k+1})^A = (xyz^{k+1})^A = (xy^{k+1}z^{k+1})^A,$$

i.e.,

$$(5.3)\qquad (xyz)^A = (xy^{k+1}z^{k+1})^A \ .$$

Similarly,

$$(5.4)\qquad (xzy)^A = (xz^{k+1}y^{k+1})^A \ .$$

Now, $xy = y_1x$ implies that $xy^{k+1} = y_1^{k+1}x$ and since $y \in \Sigma^+$, $|y^{k+1}| > k$. Hence, $y^{k+1} = y_2x$ for some $y_2 \in \Sigma^*$. Similarly, $z^{k+1} = z_2x$ for some $z_2 \in \Sigma^*$. Thus

$$(xy^{k+1}z^{k+1})^A = (xy_2xz_2x)^A \ .$$

Since A is $(k+1)$-testable, $(xy_2xz_2x)^A = (xz_2xy_2x)^A$ and it follows that $(xy^{k+1}z^{k+1})^A = (xz^{k+1}y^{k+1})^A$. By (5.3) and (5.4), $(xyz)^A = (xzy)^A$. Hence (5.2) also holds. Thus B is an IC π-factor of A.

6. Characterization of (k+1)-testable and LT events

Before proceeding we need the following:

Notation. Let $A = (Q, \Sigma, M, q_0)$ be an ISA and let $q \in Q$. We denote by E_q^A the event accepted by the automaton $(Q, \Sigma, M, q_0, \{q\})$.

Lemma 6.1. *Let $k \geq 0$ be an integer and let $A = (Q, \Sigma, M, q_0)$, $B = (R, \Sigma, N, r_0)$ and $C = (S, R \times \Sigma, P, s_0)$ be ISA's such that B is k-definite, C is IC and $A \leq B \circ C$. Then for all $q \in Q$, E_q^A is a $(k+1)$-testable event.*

Proof. Without loss of generality we assume that all ISA's are connected. It is clear that if $A_i = (Q_i, \Sigma, M_i, q_{0,i})$ $(i = 1, 2)$ are ISA's such that $A_1 \leq A_2$, then for all $q_1 \in Q_1$,

$$E_{q_1}^{A_1} = \bigcup_{Q_3} E_{q_2}^{A_2} \ ,$$

where $Q_3 = \{q_2 \in Q_2 | q_2\eta = q_1\}$. On the other hand, by Theorem 4.2, there is an integer $l \geq 1$ and half-resets $D_1, D_2, ..., D_l$ such that $C \leq D$, where $D = D_1 \times D_2 ... \times D_l$. By a theorem of [3], $B \circ C \leq B \circ D$ and hence $A \leq B \circ D$, since $\leq$ is transitive. Thus, in view of the earlier remark and the fact that $(k+1)$-testable events form a Boolean algebra, it is sufficient to prove that, if C is the direct product of half-resets, then each $E^{B \circ C}_{(r,s)}$ is a $(k+1)$-testable event. Furthermore, if $C = C_1 \times C_2$, then clearly

$$E^{B \circ C}_{(r,s)} = E^{B \circ C_1}_{(r,q_1)} \cap E^{B \circ C_2}_{(r,q_2)} ,$$

where $s = (q_1, q_2)$, and q_1 and q_2 are states of C_1 and C_2, respectively. Since $(k+1)$-testable events are closed under intersection, it is sufficient to prove that, if C is a half-reset, then $E^{B \circ C}_{(r,s)}$ is $(k+1)$-testable. Thus, let

$$C = (\{s_0, s_1\}, R \times \Sigma, P, s_0),$$

$$\theta = \{(p, \sigma) | p \in R,\ \sigma \in \Sigma, \text{ and } (p, \sigma)^C \text{ is a reset}\}.$$

Then,

$$E^{B \circ C}_{(r,s_1)} = E^B_r \cap (\bigcup_{(p,\sigma)\in\theta} E^B_p \sigma I) ,$$

where $I = \Sigma^*$. Now, since B is k-definite it follows that for all $p \in R$ there are finite sets F_p and G_p such that $E^B_p = F_p \cup IG_p$ and F_p and G_p contain words of length less than k and k, respectively [8]. Thus,

$$E^{B \circ C}_{(r,s_1)} = E^B_r \cap (\bigcup_{(p,\sigma)\in\theta} (F_p \sigma I \cup IG_p \sigma I)).$$

Clearly E^B_r, $F_p \sigma I$ and $IG_p \sigma I$ are $(k+1)$-testable events; hence so is $E^{B \circ C}_{(r,s_1)}$. Finally, we have

$$E^{B \circ C}_{(r,s_0)} = E^B_r \cap \overline{E^{B \circ C}_{(r,s_1)}} .$$

Hence $E^{B \circ C}_{(r,s_0)}$ is also $(k+1)$-testable.

Now we combine our previous results in the following two theorems.

Theorem 6.1. *Let $k \geq 0$, let $\hat{A} = (Q, \Sigma, M, q_0, F)$ be the reduced automaton accepting the event E, and let A be the ISA of $\hat{A}$. Then the following are equivalent:*

(i) *E is $(k+1)$-testable.*

(ii) *A is $(k+1)$-testable.*

(iii) *There is an IC π-factor of A which is a k-definite ISA.*

(iv) *There are ISA's B and C, B k-definite and C IC, such that $A \leq B \circ C$.*

Proof. (i) implies (ii) by Theorem 3.2. (ii) implies (iii) by Theorem 5.1.. (iii) implies (iv) by Theorem 4.3. (iv) implies (i) by Lemma 6.1.

Theorem 6.2.[2] *Let $\hat{A}$ be the reduced automaton accepting the event E, let A be the ISA of $\hat{A}$ and S_A the semigroup of A. Then the following are equivalent:*

(i) *E is LT.*

(ii) *A is LT.*

(iii) *S_A is LT.*

(iv) *There is an IC π-factor of A which is a definite ISA.*

(v) *There are ISA's B and C, B definite and C IC, such that $A \leq B \circ C$.*

(vi) *$E \in \beta_3$.*

Proof. (i) implies (ii) by Corollary 3.1. (ii) implies (iv) by Theorems 3.3 and 5.1. (iv) implies (v) by Theorem 4.3. (v) implies (i) by Lemma 6.1. (ii) iff (iii) by Theorem 3.1. (i) iff (vi) by Theorem 2.1.

Let $\Sigma = \{0, 1\}$. One verifies that the event $I0I0I$ is in β_5 but is not LT. Hence LT is a proper subset of $\mathcal{B}_2$.

Finally, we mention the following open problems regarding LT events. Theorem 3.3 implies that the event E, accepted by the reduced automaton $\hat{A}$, is LT iff A is $(k+1)$-testable, where $k = \#S_A + 1$. Can this bound on k be improved? A related problem is to find "efficiently" the smallest k such that a given LT ISA is $(k+1)$-testable. With the present methods, one would test if it is $(k+1)$-testable for $k = 0, 1, ..., \#S_A + 1$. Finally, find a step by step method to decompose a LT ISA, e.g. like that of Zeiger [3]. Our approach uses the free k-definite and the free IC ISA's

[2] The equivalence of (i) and (iii) as well as another characterization of LT events has also been independently obtained in [7].

to do this, and it succeeds since these are finite for a fixed alphabet. However, in general, there are smaller ISA's, the cascade product of which covers A. This problem could be of interest since in general the free automata are infinite and their use would be impossible.

7. Generalized definite events

For the sake of completeness we include a characterization of generalized definite events which is obtained by methods similar to those used for LT events. We leave it to the reader to verify the following:

Lemma 7.1. *An event E is generalized definite iff there exists an integer $k \geq 0$ such that for all $u, v \in \Sigma^*$, if $f_k(u) = f_k(v)$ and $t_k(u) = t_k(v)$ then $u \in E$ iff $v \in E$.*

Now we have:

Theorem 7.1.[3] *Let $\hat{A} = (Q, \Sigma, M, q_0, F)$ be the reduced automaton accepting the event E. Then E is generalized definite iff for every idempotent $e \in S_A$ the monoid $eS_A e$ consists solely of the element e.*

Proof. Let us suppose E is generalized definite. Then, by Lemma 7.1, there is a $k \geq 0$ such that for all $u, v \in \Sigma^*$, if $f_k(u) = f_k(v)$ and $t_k(u) = t_k(v)$, then $u \in E$ iff $v \in E$. Let e be an idempotent of S_A and let $a \in S_A$. Then there exist $x, y \in \Sigma^+$, such that $e = x^A$ and $a = y^A$. Let us assume that $(xyx)^A \neq x^A$. Since $x^A = (x^2)^A$, it follows that $(x^{k+1}yx^{k+1})^A \neq (x^{k+1})^A$. Since A is reduced, there are $z_1, z_2 \in \Sigma^*$ such that $w_1 = z_1 x^{k+1} y x^{k+1} z_2 \in E$ iff $w_2 = z_1 x^{k+1} z_2 \notin E$. Now, $x \in \Sigma^+$ implies that $f_k(w_1) = f_k(w_2) = f_k(z_1 x^k)$ and $t_k(w_1) = t_k(w_2) = t_k(x^k z_2)$. Thus, $w_1 \in E$ iff $w_2 \in E$ which is a contradiction. Hence $(xyx)^A = x^A$ and therefore $eS_A e$ consists of e only.

Conversely, let us assume that for every idempotent $e \in S_A$, $eS_A e$ consists of e only. Let $l = \#S_A + 1$ and let $k = 2l$. Furthermore, let $u, v \in \Sigma^*$ be such that $f_k(u) = f_k(v)$ and $t_k(u) = t_k(v)$. We will prove

[3] This theorem has also been independently obtained in [9].

that $u^A = v^A$. In fact, if $|u| < k$ then $f_k(u) = f_k(v)$ implies that $u = v$, and hence $u^A = v^A$. if $|u| \geq k$ then also $|v| \geq k$. Since $f_k(u) = f_k(v)$ and $k = 2l$ also $f_l(u) = f_l(v)$. Similarly, $t_l(u) = t_l(v)$. Let $x = f_l(u)$ and $y = t_l(u)$. Since $k = 2l$, it follows that there exist $u_1, v_1 \in \Sigma^*$ such that $u = xu_1y$ and $v = xv_1y$. Now, since $l = \#S_A + 1$ there exist $x_1, x_3, y_1, y_3 \in \Sigma^*$ and $x_2, y_2 \in \Sigma^+$ such that $x = x_1x_2x_3$, $y = y_1y_2y_3$, $x_1^A = (x_1x_2)^A$ and $y_1^A = (y_1y_2)^A$. On the other hand, similarly to Lemma 3.2, we have that A is permutation-free and thus, by Lemma 3.3 if $n = \#Q$ then $(x_2^n)^A$ and $(y_2^n)^A$ are idempotents of S_A. Now, clearly

$$u^A = (x_1x_2^nx_3u_1y_1y_2^ny_3)^A \ .$$

On the other hand, since $(y_2^n)^A$ is an idempotent, $(y_2^n)^A = (y_2^nx_2^ny_2^n)^A$. Thus,

$$u^A = (x_1x_2^nx_3u_1y_1y_2^nx_2^ny_2^ny_3^n)^A \ .$$

Since $(x_2^n)^A$ is also idempotent

$$(x_2^nx_3u_1y_1y_2^nx_2^n)^A = (x_2^n)^A \ .$$

Hence $u^A = (x_1x_2^ny_2^ny_3)^A$. Similarly, $v^A = (x_1x_2^ny_2^ny_3)^A$ and therefore $u^A = v^A$. Thus $u \in E$ iff $v \in E$ and, by Lemma 7.1, E is generalized definite.

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