

# A note on encoding infinity in ZFA with applications to register automata

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## Abstract

Working in Zermelo-Fraenkel Set Theory with Atoms over an  $\omega$ -categorical  $\omega$ -stable structure, we show how *infinite* constructions over definable sets can be encoded as *finite* constructions over the Stone-Ćech compactification of the sets. In particular, we show that for a definable set  $X$  with its Stone-Ćech compactification  $\overline{X}$  the following holds: a) the powerset  $\mathcal{P}(X)$  of  $X$  is isomorphic to the finite-powerset  $\mathcal{P}_{fin}(\overline{X})$  of  $\overline{X}$ , b) the vector space  $\mathcal{K}^X$  over a field  $\mathcal{K}$  is the free vector space  $F_{\mathcal{K}}(\overline{X})$  on  $\overline{X}$  over  $\mathcal{K}$ , c) every measure on  $X$  is tantamount to a *discrete* measure on  $\overline{X}$ . Moreover, we prove that the Stone-Ćech compactification of a definable set is still definable, which allows us to obtain some results about equivalence of certain formalizations of register machines.

## 1 Introduction

It is an old observation that goes back to Stanisław Ulam that one can separate “small sets” from “large sets” and “large sets” from “very large sets” by the existence of certain ultrafilters on the sets. For example, let us work in classical mathematics ZFC. Then a set is finite if and only if it is in a bijective correspondence with the set of ultrafilters on it, in which case, every ultrafilter is principal. Therefore, we may say that a set is *infinite* if there is a non-principal ultrafilter on it<sup>1</sup>. One may also ask about the existence of non-principal countably-additive ultrafilters on a set and it is well-known that the smallest set having such an ultrafilter<sup>2</sup> must be strongly inaccessible (therefore, it must be “very large”, as the sets below it form an inner model of ZFC).

A main theme of this paper is the structure of ultrafilters on definable sets in Fraenkel-Mostowski permutational models of Set Theory with Atoms (ZFA). In this setting the Axiom of Choice fails (unless the permutational model is trivial), and the Boolean Prime Ideal Theorem (BPIT) may hold or fail, but, counter-intuitively, it is mostly irrelevant for our results. In fact, our main results concern permutational models over  $\omega$ -categorical  $\omega$ -stable structures (although we will discuss other structures in the paper), in which case BPIT fails for general Boolean algebras, but holds for powerset algebras (see Theorem 2.5). Examples of such structures include Example 1.1 and Example 1.3, but not Example 1.2, Example 1.4 nor Example 1.5.

**Example 1.1** (Pure sets). *Let  $\mathcal{N} = \{0, 1, 2, \dots\}$  be a countably infinite set over empty signature  $\Xi$ . Then the first order theory of  $\mathcal{N}$  is  $\omega$ -categorical and  $\omega$ -stable, i.e. there is exactly one model of the theory up to an isomorphism for every infinite cardinal number. This theory is called the theory of “pure sets”.*

<sup>1</sup>Of course, we do not need the full power of the Axiom of Choice, Boolean Prime ideal Theorem is sufficient.

<sup>2</sup>If it exists, because its existence is not provable from ZFC alone.

**Example 1.2** (Pure sets with constants). *Let  $\mathcal{N} \sqcup N$  be the structure from Example 1.1 over an extended signature consisting of all constants  $n \in N$ . Then the first order theory of  $\mathcal{N} \sqcup N$  has countably many non-isomorphic countable models, therefore is not  $\omega$ -categorical. It is, however,  $\omega$ -stable, because adding countably many constants cannot change the stability of a structure.*

**Example 1.3** (Vector space over a finite field). *Let  $V_{\mathcal{F}}$  be the free  $\aleph_0$ -dimensional vector space over a finite field  $\mathcal{F}$ . We shall consider  $V_{\mathcal{F}}$  with its natural vector-space structure, i.e.  $\mathcal{V}_{\mathcal{F}} = \langle V_{\mathcal{F}}, +, (-)r \rangle$  for every  $r \in \mathcal{F}$ . This theory is both  $\omega$ -categorical and  $\omega$ -stable, because for every infinite cardinal  $\kappa$  it has exactly one model (up to isomorphism) of cardinality  $\kappa$  — the free vector space on  $\kappa$  base vectors.*

**Example 1.4** (Rational numbers with ordering). *Let  $\mathcal{Q} = \langle Q, \leq \rangle$  be the structure whose universe is interpreted as the set of rational numbers  $Q$  with a single binary relation  $\leq \subseteq Q \times Q$  interpreted as the natural ordering of rational numbers. Then the first order theory of  $\mathcal{Q}$  is  $\omega$ -categorical but not  $\omega$ -stable.*

**Example 1.5** (Random graph). *Let  $\mathcal{R}$  be a countable graph over signature consisting of a single binary relation  $E$  and satisfying the following two axioms: (Simplicity Axiom)  $R$  is symmetric and irreflexive; (Extension Axiom) if  $V_0, V_1 \subset R$  are finite disjoint subsets, then there is  $v \in R$  such that for every  $v_0 \in V_0$  the relation  $R(v, v_0)$  holds and for every  $v_1 \in V_1$  the relation  $R(v, v_1)$  does not hold. Structure  $\mathcal{R}$  is  $\omega$ -categorical, but not  $\omega$ -stable.*

Interestingly, definable sets in  $\omega$ -categorical  $\omega$ -stable structures behave like something intermediate between “small sets” and “large sets” — they enjoy many closure properties of finite sets, but the closure operators deviate significantly from the identity.

First of all, in classical ZFC, the distinction between “small sets” and “large sets” is not only a matter of a mere existence of non-principal ultrafilters, i.e. “large sets” have enormous number of non-principal ultrafilters, whereas small sets have none. That is, for a set  $X$  the number of non-principal ultrafilters is either 0 (in case  $X$  is finite) or doubly-exponential:  $2^{2^X}$  (in case  $X$  is infinite). In contrast (see Theorem 2.3), the number of non-principal ultrafilters on definable sets in our permutational models is always bounded by a polynomial. In fact, the set of ultrafilters on a definable set is always definable. For example, in the basic Fraenkel-Mostowski model, the set of atoms  $N$  has only one non-principal ultrafilter (consisting of all cofinite subsets of  $N$ ), and for the set of distinct pairs of atoms  $N^{[2]}$ , we have exactly  $2N + 1$  non-principal ultrafilters.

Secondly, in classical ZFC, a vector space is isomorphic to its dual if and only if it is finite dimensional. Let us assume for simplicity that our base field is 2. Then if  $V$  is an infinite-dimensional vector space with a base  $X$ , then the dimension of its dual space grows exponentially in  $X$ : i.e. the dimension of  $2^X$  is exactly  $2^X$ . Therefore, the base of  $2^X$  is isomorphic to the set of ultrafilters on  $X$  if and only if  $X$  is finite-dimensional. In contrast, for every definable set  $X$  in our permutational models, the set of ultrafilters on  $X$  is isomorphic to the base of  $2^X$ , which proves that dual spaces have basis and gives an explicit construction of the basis (see Theorem 3.3). Moreover, since the space  $2^X$  is just the power set  $\mathcal{P}(X)$  of  $X$  and the free vector space on a set is just the set  $\mathcal{P}_{fin}(X)$  of finite subsets of  $X$ , Theorem 3.3 implies that for every definable  $X$  we have that  $\mathcal{P}(X) \approx \mathcal{P}_{fin}(Y)$  for some definable  $Y$ , i.e.  $Y$  can be taken to be the set of ultrafilters on  $X$ . This means, that we can effectively, transfer theorems about *finite* subsets of definable sets to *all* subsets of definable sets. For some of the applications, see Subsection 1.2.1 below.

Finally, in classical ZFC, a set  $X$  is finite if and only if every measure  $\mu$  on the full algebra of all subsets of  $X$  is a *finite* combination of mass-measures, i.e.  $\mu = \sum_{i=1}^n r_i x_i$ , where  $\sum_{i=1}^n r_i = 1$ , each  $r_i$  is positive, and  $x_i$  is concentrated on a singleton. Of course, a mass measure on a set is

just a principal ultrafilter on the set. Moreover, every countably-additive ultrafilter is tantamount to a measure taking values in  $\{0, 1\}$ . But for definable sets  $X$  in our permutational models, being countably-additive is a vacuous condition, because every countable collection of subsets of  $X$  must be essentially finite. Therefore, every ultrafilter on a definable set is tantamount to a  $\{0, 1\}$ -measure. As it turns out, every measure on a definable set is a finite combination of ultrafilters on the set (see Theorem 4).

## 1.1 Set theory with atoms

Let  $\mathcal{A}$  be an algebraic structure (both operations and relations are allowed) with universum  $A$ . We shall think of elements of  $\mathcal{A}$  as “atoms”. A von Neumann-like hierarchy  $V_\alpha(\mathcal{A})$  of sets with atoms  $\mathcal{A}$  can be defined by transfinite induction [12], [5]:

- $V_0(\mathcal{A}) = A$
- $V_{\alpha+1}(\mathcal{A}) = \mathcal{P}(V_\alpha(\mathcal{A})) \cup V_\alpha(\mathcal{A})$
- $V_\lambda(\mathcal{A}) = \bigcup_{\alpha < \lambda} V_\alpha(\mathcal{A})$  if  $\lambda$  is a limit ordinal

Then the cumulative hierarchy of sets with atoms  $\mathcal{A}$  is just  $V(\mathcal{A}) = \bigcup_{\alpha: \text{Ord}} V_\alpha(\mathcal{A})$ . Observe, that the universe  $V(\mathcal{A})$  carries a natural action  $(\bullet): \text{Aut}(\mathcal{A}) \times V(\mathcal{A}) \rightarrow V(\mathcal{A})$  of the automorphism group  $\text{Aut}(\mathcal{A})$  of structure  $\mathcal{A}$  — it is just applied pointwise to the atoms of a set. If  $X \in V(\mathcal{A})$  is a set with atoms then by its set-wise stabiliser we shall mean the set:  $\text{Aut}(\mathcal{A})_X = \{\pi \in \text{Aut}(\mathcal{A}) : \pi \bullet X = X\}$ ; and by its point-wise stabiliser the set:  $\text{Aut}(\mathcal{A})_{(X)} = \{\pi \in \text{Aut}(\mathcal{A}) : \forall x \in X \pi \bullet x = x\}$ . Moreover, for every  $X$ , these sets inherit a group structure from  $\text{Aut}(\mathcal{A})$ .

There is an important sub-hierarchy of the cumulative hierarchy of sets with atoms  $\mathcal{A}$ , which consists of “symmetric sets” only. To define this hierarchy, we have to equip  $\text{Aut}(\mathcal{A})$  with the structure of a topological group. A set  $X \in V(\mathcal{A})$  is *symmetric* if the set-wise stabilisers of all of its descendants  $Y$  is an open set (an open subgroup of  $\text{Aut}(\mathcal{A})$ ), i.e. for every  $Y \in^* X$  we have that:  $\text{Aut}(\mathcal{A})_Y$  is open in  $\text{Aut}(\mathcal{A})$ , where  $\in^*$  is the reflexive-transitive closure of the membership relation  $\in$ . A function between symmetric sets is called symmetric if its graph is a symmetric set. Of a special interest is the topology on  $\text{Aut}(\mathcal{A})$  inherited from the product topology on  $\prod_A A = A^A$  (i.e. the Tychonoff topology). We shall call this topology the canonical topology on  $\text{Aut}(\mathcal{A})$ . In this topology, a subgroup  $\mathbb{H}$  of  $\text{Aut}(\mathcal{A})$  is open if there is a finite  $A_0 \subseteq A$  such that:  $\text{Aut}(\mathcal{A})_{(A_0)} \subseteq \mathbb{H}$ , i.e.: group  $\mathbb{H}$  contains a pointwise stabiliser of some finite set of atoms. The sub-hierarchy of  $V(\mathcal{A})$  that consists of symmetric sets according to the canonical topology on  $\text{Aut}(\mathcal{A})$  will be denoted by  $\mathbf{ZFA}(\mathcal{A})$  (it is a model of Zermelo-Fraenkel set theory with atoms).

**Remark 1.1.** *The above definition of hierarchy of symmetric sets is equivalent to another one used in model theory. By a normal filter of subgroups of a group  $\mathbb{G}$  we shall understand a filter  $\mathcal{F}$  on the poset of subgroups of  $\mathbb{G}$  closed under conjugation, i.e. if  $g \in \mathbb{G}$  and  $\mathbb{H} \in \mathcal{F}$  then  $g\mathbb{H}g^{-1} = \{g \bullet h \bullet g^{-1} : h \in \mathbb{H}\} \in \mathcal{F}$ . Let  $\mathcal{F}$  be a normal filter of subgroups of  $\text{Aut}(\mathcal{A})$ . We say that a set  $X \in V(\mathcal{A})$  is  $\mathcal{F}$ -symmetric if the set-wise stabilisers of all of its descendants  $Y$  belong to  $\mathcal{F}$  — i.e.  $Y \in^* \mathcal{F}$ . To see that the definitions of symmetric sets and  $\mathcal{F}$ -symmetric sets are equivalent, observe first that if  $\mathbb{G}$  is a topological group, then the set  $\mathcal{F}$  of all open subgroups of  $\mathbb{G}$  is a normal filter of subgroups. In the other direction, if  $\mathcal{F}$  is a normal filter of subgroups of a group  $\mathbb{G}$ , then we may define a topology on  $\mathbb{G}$  by declaring sets  $U \subseteq \mathbb{G}$  to be open if they satisfy the following property: for every  $g \in U$  there exists  $\mathbb{H} \in \mathcal{F}$  such that  $g\mathbb{H} \subseteq U$ . According to this topology a group  $\mathbb{U}$  is open iff  $\mathbb{U} \in \mathcal{F}$  — just observe that for every group  $\mathbb{U}$  and for every  $g \in \mathbb{U}$  we have that  $g\mathbb{U} = \mathbb{U}$ ; and if  $\mathbb{H} \in \mathcal{F}$  such that  $\mathbb{H} = 1\mathbb{H} \subseteq \mathbb{U}$  then by the property of the filter,  $\mathbb{U} \in \mathcal{F}$ .*

**Example 1.6** (The basic Fraenkel-Mostowski model). *Let  $\mathcal{N}$  be the structure from Example 1.1. We call  $\mathbf{ZFA}(\mathcal{N})$  the basic Fraenkel-Mostowski model of set theory with atoms. Observe that  $\text{Aut}(\mathcal{N})$  is the group of all bijections (permutations) on  $N$ . The following are examples of sets in  $\mathbf{ZFA}(\mathcal{N})$ :*

- all sets without atoms, e.g.  $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}, \dots\}, \dots$
- all finite subsets of  $N$ , e.g.  $\{0\}, \{0, 1, 2, 3\}, \dots$
- all cofinite subsets of  $N$ , e.g.  $\{1, 2, 3, \dots\}, \{4, 5, 6, \dots\}, \dots$
- $N \times N$
- $\{\langle a, b \rangle \in N^2 : a \neq b\}$
- $N^* = \bigcup_{k \in N} N^k$
- $\mathcal{P}_{\text{fin}}(N) = \{N_0 : N_0 \subseteq N, N_0 \text{ is finite}\}$
- $\mathcal{P}(N) = \{N_0 : N_0 \subseteq N, N_0 \text{ is symmetric}\}$

Here are examples of sets in  $V(\mathcal{N})$  which are not symmetric:

- $\{0, 2, 4, 6, \dots\}$
- $\{\langle n, m \rangle \in N^2 : n \leq m\}$
- the set of all functions from  $N$  to  $N$
- $\mathcal{P}_F(N) = \{N_0 : N_0 \subseteq N\}$

**Example 1.7** (The ordered Fraenkel-Mostowski model). *Let  $\mathcal{Q}$  be the structure from Example 1.4. We call  $\mathbf{ZFA}(\mathcal{Q})$  the ordered Fraenkel-Mostowski model of set theory with atoms. Observe that  $\text{Aut}(\mathcal{Q})$  is the group of all order-preserving bijections on  $Q$ . All symmetric sets from Example 1.6 are symmetric sets in  $\mathbf{ZFA}(\mathcal{Q})$  when  $N$  is replaced by  $Q$ . Here are some further symmetric sets:*

- $\{\langle p, q \rangle \in Q^2 : p \leq q\}$
- $\{\langle p, q \rangle \in Q^2 : 0 \leq p \leq q \leq 1\}$

Observe that the group  $\text{Aut}(\mathcal{A})_{(A_0)}$  is actually the group of automorphism of structure  $\mathcal{A}$  extended with constants  $A_0$ , i.e.:  $\text{Aut}(\mathcal{A})_{(A_0)} = \text{Aut}(\mathcal{A} \sqcup A_0)$ . Then a set  $X \in V(\mathcal{A})$  is symmetric if and only if there is a finite  $A_0 \in A$  such that  $\text{Aut}(\mathcal{A} \sqcup A_0) \subseteq \text{Aut}(\mathcal{A})_X$  and the canonical action of topological group  $\text{Aut}(\mathcal{A} \sqcup A_0)$  on discrete set  $X$  is continuous. A symmetric set is called  $A_0$ -equivariant (or equivariant in case  $A_0 = \emptyset$ ) if  $\text{Aut}(\mathcal{A} \sqcup A_0) \subseteq \text{Aut}(\mathcal{A})_X$ . Therefore, the (non-full) subcategory of  $\mathbf{ZFA}(\mathcal{A})$  on  $A_0$ -equivariant sets and  $A_0$ -equivariant functions (i.e. functions whose graphs are  $A_0$ -equivariant) is equivalent to the category  $\mathbf{Cont}(\text{Aut}(\mathcal{A} \sqcup A_0)) \subseteq \mathbf{Set}^{\text{Aut}(\mathcal{A} \sqcup A_0)}$  of continuous actions of the topological group  $\text{Aut}(\mathcal{A} \sqcup A_0)$  on discrete sets. We will heavily use the transfer principle developed in [13], which is based on the observation that adding finitely many constants to an  $\omega$ -categorical and  $\omega$ -stable structure and closing it under elimination of imaginaries, produces structure, which is  $\omega$ -categorical and  $\omega$ -stable.

**Definition 1.1** (Definable set). *We shall say that an  $A_0$ -equivariant set  $X \in \mathbf{ZFA}(\mathcal{A})$  is definable if its canonical action has only finitely many orbits, i.e. if the relation  $x \equiv y \Leftrightarrow \exists \pi \in \text{Aut}(\mathcal{A} \sqcup A_0) \ x = \pi \bullet y$  has finitely many equivalence classes.*

For an open subgroup  $\mathbb{H}$  of  $Aut(\mathcal{A})$  let us denote by  $Aut(\mathcal{A})/\mathbb{H}$  the quotient set  $\{\pi\mathbb{H} : \pi \in Aut(\mathcal{A})\}$ . This set carries a natural continuous action of  $Aut(\mathcal{A})$ , i.e. for  $\sigma, \pi \in Aut(\mathcal{A})$ , we have  $\sigma \bullet \pi\mathbb{H} = (\sigma \circ \pi)\mathbb{H}$ . All transitive (i.e. single orbit) actions of  $Aut(\mathcal{A})$  on discrete sets are essentially of this form (see for example Chapter III, Section 9 of [10]). Therefore, equivariant definable sets are essentially finite unions of sets of the form  $Aut(\mathcal{A})/\mathbb{H}$ . Moreover, if structure  $\mathcal{A}$  is  $\omega$ -categorical (Example 1.6, Example 1.7), then equivariant definable sets are the same as sets definable in the first order theory of  $\mathcal{A}$  extended with elimination of imaginaries [13].

## 1.2 Register machines

An important type of automata has been defined by Kaminski, Michael and Francez [8]. The authors called these type of automata “finite memory machines”, or “register machines”. A finite memory machine is a finite automaton augmented with a finite number of registers  $R_i$  that can store natural numbers. The movement of the machine can depend on the control state, on the letter and on the content of the registers. The dependency on the content of the registers is, however, limited — the machine can only test for equality (no formulas involving successor, addition, multiplication, etc. are allowed). Here is a suitable generalisation of this definition to a general structure  $\mathcal{A}$ .

A finite memory automata (over structure  $\mathcal{A}$ ) with  $k$  registers over alphabet  $\Sigma$  is a quadruple  $\langle S, \delta, I, F \rangle$  such that:

- $S$  is a finite set of states
- $I \subseteq S$  is a set of initial states, and  $\phi_I \subseteq A^k$  is a set of possible initial configurations of registers
- $F \subseteq S$  is a set of final states, and  $\phi_F \subseteq A^k$  is a set of possible final configurations of registers
- $\delta \subseteq (\Sigma \times S \times A^k) \times (S \times A^k)$  is a transition relation such that for every  $s, s' \in S$  the relation  $\delta(s, s') \subseteq (\Sigma \times A^k) \times A^k$  is  $\mathcal{A}$ -definable.

A finite memory automata is called deterministic if  $I$  is the singleton and the transition relation  $\sigma$  is functional.

An example of such a machine is presented on Figure 1. The machine has a single register  $R$  and can test for equality and inequality only. It starts in state “SET PASSW”, where it awaits for the user to provide a password  $x$ . This password is then stored in register  $R$ , and the machine enters state “START”. Inside the top rectangle the machine can perform actions that do not require authentication, whereas the actions that require authentication are presented inside the bottom rectangle. The bottom rectangle can be entered by state “GRANT AUTH”, which can be accessed from one of three authentication states. In order to authorise, the machine moves to state “AUTH TRY 1”, where it gets input  $x$  from the user. If the input is the same as the value previously stored in register  $R$ , then the machine enters state “GRANT AUTH”. Otherwise, it moves to state “AUTH TRY 2” and repeats the procedure. Upon second unsuccessful authorisation, the machine moves to state “AUTH TRY 3”. But if the user provides a wrong password when the machine is in state “AUTH TRY 3”, the automaton moves back to “START” state — preventing the machine to reach any of the states from the bottom rectangle. Inside the bottom rectangle any action that requires authentication can be performed. For example, the user may request the change of the password. Observe, that our access control machine is not fully secure, because after the third unsuccessful attempt to authorise, the user can repeat the whole process trying new passwords. What if we wanted to erase the content of register  $R$  after the third unsuccessful attempt? We shall provide an answer in the following subsection. But before we do this, we have to make some general comments on finite memory automata.

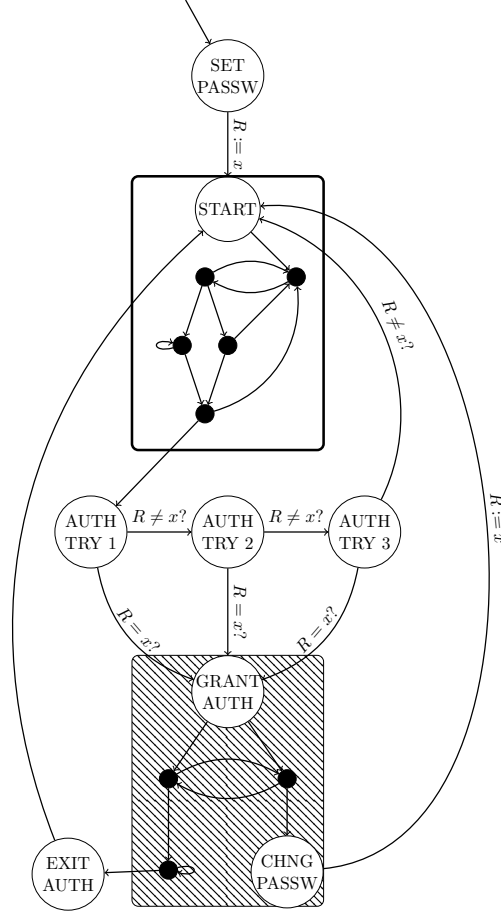


Figure 1: A register machine that models access control to the dashed part of the system.

It is well-known that finite memory automata in the above sense are equivalent to definable automata in ZFA, i.e. the set  $S \times A^k$  can be identified with a definable set, and then the transition relation becomes a definable relation between definable sets. Therefore, a definable deterministic automata is just a definable function  $\sigma: \Sigma \times S \rightarrow S$  between definable sets together with an initial state  $s_0 \in S$  and a set of final states  $F \subseteq S$ . To define the language  $L(A)$  recognised by such an automaton, we have to observe that functions  $\sigma: \Sigma \times S \rightarrow S$  are tantamount to functions  $\sigma^\dagger: \Sigma \rightarrow S^S$  and  $S^S$  carries a structure of a monoid under composition of functions  $S \rightarrow S$ , and so, one may extend  $\sigma^\dagger$  to the unique homomorphism  $h: \Sigma^* \rightarrow S^S$  from the free monoid on  $\Sigma$  generators. The language of  $A$  is just the set  $L(A) = \{w \in \Sigma^*: h(w)(s_0) \in F\}$ . Similarly, the crucial observation needed to define the language of a non-deterministic automaton is that the transition relation  $\sigma: \Sigma \times S \rightarrow \mathcal{P}(S)$  is tantamount to  $\sigma^\dagger: \Sigma \rightarrow \mathcal{P}(S)^S \approx \mathcal{P}(S \times S)$  and  $\mathcal{P}(S \times S)$  carries a monoidal structure induced by the composition of relations  $S \rightarrow S$ . One may wonder, if we can substitute the powerset operator with other operators on  $S$ . The answer is yes, provided that the operator is a *strong* monad (this is a sufficient, but not necessary condition) on the category  $\mathbf{ZFA}[A]$ , i.e. if  $T: \mathbf{ZFA}[A] \rightarrow \mathbf{ZFA}[A]$  is a strong monad, then  $T(S)^S$  is naturally a monoid under Kleisli composition of functions  $S \rightarrow T(S)$ .

### 1.2.1 On a machine that can erase information from its registers

Intuitively, erasing information from a register, should make all of the values in the register “equally likely” and each individual value “completely unlikely”. If  $R$  can hold a value from  $\mathcal{N}$ , then we can model this by assigning values to  $R$  in such a way that the probability for  $R$  to get values from any finite subset of  $\mathcal{N}$  is zero. This corresponds to the assignment of a value to  $R$  at “random” according to the only non-principal ultrafilter on  $N$ , i.e. the ultrafilter consisting of all cofinite subsets of  $N$ . This, in turn, suggests that we should model the operation of erasing information from registers via ultrafilter automata: that is, automata for the ultrafilter monad  $\overline{(-)}: \mathbf{ZFA}[A] \rightarrow \mathbf{ZFA}[A]$ . Notice that in the classical setting of finite automata, we do not speak about “finite ultrafilter automata”, because every ultrafilter on a finite set is principal. Here is the formal definition.

**Definition 1.2** (Ultra-automaton). *A deterministic ultra-automata (or erasing information automata) over a definable alphabet  $\Sigma$  consists of a definable set  $S$ , definable transition relation  $\sigma: \Sigma \times S \rightarrow \overline{S}$  an initial state  $s_0 \in S$  and a set of final states  $F$ .*

By Theorem 2.1, the set of ultrafilters carries a strong monad structure, therefore we can define the language of such an automaton in a natural way.

**Definition 1.3** (Language of an ultra-automaton). *The language  $L$  of an automaton  $\sigma: \Sigma \times S \rightarrow \overline{S}$  with initial state  $s_0$  and final states  $F$  is defined as  $L = \{w \in \Sigma^*: h(w)(s_0) \in F\}$ , where  $h: \Sigma^* \rightarrow \overline{S}^S$  is the unique homomorphism of monoids extending function  $\sigma^\dagger: \Sigma \rightarrow \overline{S}^S$ .*

One may extend the above definition to non-deterministic ultra-automaton by observing that the ultrafilter monad can be extended to internal relations. This is however unnecessary due to the next theorem and its proof.

**Theorem 1.1** (On the expressive power of ultra-automata). *Let  $\mathcal{A}$  be an  $\omega$ -categorical and  $\omega$ -stable structure. The languages in  $\mathbf{ZFA}[A]$  recognised by definable ultra-automata are exactly the same as the languages recognised by deterministic automata.*

*Proof.* According to Corollary 2.4, the ultrafilter monad restricts to the monad on definable sets. Thus,  $\overline{S}$  is definable. Moreover, because the monad is strong and the structure of the monad is equivariant, every definable function  $\sigma: \Sigma \times S \rightarrow \overline{S}$  extends to a definable function  $\sigma: \Sigma \times \overline{S} \rightarrow \overline{S}$ . Observe also that  $\overline{S}^S$  is a submonoid of  $\overline{S}^{\overline{S}}$  (actually, the full submonoid on continuous functions), therefore the languages recognised by  $\sigma: \Sigma \times S \rightarrow \overline{S}$  and  $\overline{\sigma}: \Sigma \times \overline{S} \rightarrow \overline{S}$  are the same.  $\square$

The above theorem effectively says that we can include the “erase information” operation to register machines without changing they properties.

While for general  $\omega$ -categorical structures the ultrafilter monad do not restrict to definable sets (see Example 2.1), we conjecture that Theorem 1.1 holds for every  $\omega$ -categorical structure.

**Conjecture 1.1.** *Let  $\mathcal{A}$  be an  $\omega$ -categorical structure. The languages in  $\mathbf{ZFA}[A]$  recognised by definable ultra-automata are exactly the same as the languages recognised by deterministic automata.*

### 1.2.2 Weighted register machine

In [2] M. Bojanczyk, B. Klin and M. Moerman introduced and studied weighted definable automata in  $\mathbf{ZFA}[A]$  for an  $\omega$ -categorical structure  $\mathcal{A}$ . Here is their definition.

**Definition 1.4** (Weighted automaton). *Let us fix a field  $\mathcal{K}$ . A weighted definable automaton consists of definable sets  $S$  and  $\Sigma$ , called the states and the alphabet, and symmetric functions:*

- $I: S \rightarrow \mathcal{K}$  for initial states
- $F: S \rightarrow \mathcal{K}$  for final states
- $\sigma: \Sigma \times S \times S \rightarrow \mathcal{K}$

subject to the following requirement: there are finitely many states with nonzero initial weight, and also for every state  $s \in S$  and input letter  $a \in \Sigma$ , there are finitely many states  $s' \in S$  such that the transition  $(a, s, s')$  has nonzero weight.

Obviously, a function  $\sigma: \Sigma \times S \times S \rightarrow \mathcal{K}$  should be rewritten as  $\sigma: \Sigma \times S \rightarrow \mathcal{K}^S$  to exhibit more similarities with other types of automata. The problem with this definition is that  $\mathcal{K}^S$  is not the free vector space  $F(S)$  on  $S$ , and if try to define Kleisli composition, or equivalently, extend freely  $\sigma$  to the liner map in the second variable, we will obtain:  $F(\sigma): \Sigma \times F(S) \rightarrow \mathcal{K}^S$ . Generally, it seems that there is no natural way to induce the monoid structure, because functions  $F(S) \rightarrow \mathcal{K}^S$  do not compose. One way of dealing with this obstacle is to impose an extra condition on  $\sigma$  as in Definition 1.4. This condition is a convoluted way of saying that function  $\sigma: \Sigma \times S \rightarrow \mathcal{K}^S$  factors as  $\Sigma \times S \rightarrow F(S) \subseteq \mathcal{K}^S$ . However, when structure  $\mathcal{A}$  is  $\omega$ -stable, *there is a way* to define such a composition. By Theorem 3.3, vector space  $\mathcal{K}^S$  is isomorphic to the free vector space  $F(\overline{S})$  on the Stone–Čech compactification  $\overline{S}$  of  $S$ . Therefore, the transition relation  $\sigma: \Sigma \times S \rightarrow \mathcal{K}^S$  can be rewritten as  $\sigma: \Sigma \times S \rightarrow F(\overline{S})$  on the. Let us first assume that  $S$  consists of a single orbit. Then for every  $s \in \Sigma$ , function  $\sigma_a: S \rightarrow F(\overline{S})$  picks an orbit of a single vector, say  $\alpha = r_1p_1 + r_2p_2 + \dots + r_np_k$ , where  $p_i \in \overline{S}$ . Therefore, it factors as  $f_a: S \rightarrow \overline{S}^k$  followed by the assignment  $p_1, p_2, \dots, p_k \mapsto r_1p_1 + r_2p_2 + \dots + r_np_k$ . Function  $f_a$  may be extended to  $\overline{f_a}: \overline{S} \rightarrow \overline{S}^k$  coordinate-wise to induce a function  $\overline{\sigma_a}: \overline{S} \rightarrow F(\overline{S})$ . Because the ultrafilter monad is equivariant, if  $f_a$  is  $A_0$ -supported, then  $\overline{f_a}$  is equivariant, and so  $\overline{\sigma_a}$  is  $A_0$ -equivariant. But this means, that  $\overline{\sigma}: \Sigma \times \overline{S} \rightarrow F(\overline{S})$  is equivariant and can be naturally extended to an equivariant linear function in the second variable:  $\overline{\sigma}: \Sigma \times F(\overline{S}) \rightarrow F(\overline{S})$ . To get the general case, just observe that both the ultrafilter monad and Cartesian product functor preserve finite coproducts.

**Theorem 1.2** (On the extra requirement in the definition of weighted automaton). *Let  $\mathcal{A}$  be an  $\omega$ -categorical and  $\omega$ -stable structure. The languages in  $\mathbf{ZFA}[A]$  recognised by definable weighted-automata are exactly the same as the languages recognised by definable weighted-automata with the extra requirement dropped.*

### 1.2.3 Probabilistic register machine

In the classical setting of finite automata, probabilistic automata are a special kind of weighted automata — there is just an additional requirement that the weights of the transitions of any state must be non-negative and sum up to 1. This requirement does not translate directly to weighted automata in  $\mathbf{ZFA}[A]$  over definable sets for a single reason. If a set  $X$  is not finite then there are non-discrete probability measures on it. For example, if  $N$  is the set of atoms in the basic Fraenkel-Mostowski model  $\mathbf{ZFA}[N]$ , then there is a measure  $\mu$  that assigns to every finite set probability 0 and to every cofinite set, probability 1. Therefore, if the transition function assigns such a probability to a given state, then it violates the extra requirement in the original definition of a weighted automaton (Definition 1.4). Consequently, we have either restrict to the discrete measures on a set or drop the extra requirement from the definition as we did in Theorem ???. One can also think of the following definition as of a suitable generalisation of ultra-automaton from Definition 1.2.



**Definition 1.5** (Probabilistic automaton). *A probabilistic definable automaton consists of definable sets  $S$  and  $\Sigma$ , called the states and the alphabet, and the following data:*

- $p_0 \in \mathbf{m}(S)$  the initial probability on states  $S$
- $p_F \in \mathbf{m}(S)$  the final probability on states  $S$
- $\sigma: \Sigma \times S \rightarrow \mathbf{m}(S)$  the probabilistic transition relation

Such an automaton assigns to every word  $w \in \Sigma^*$  the probability that when starting in states  $s_0$  the automaton reach states  $s_F$  upon reading word  $w$ . By Theorem 4, it can be treated as a weighted automaton over the Stone–Čech compactification of  $S$ .

### 1.3 Organisation of the paper

The rest of the paper contains the proofs and some additional details of the abovementioned theorems. In the next section, we investigate the properties of ultrafilters in permutational models of set theory, with the main Theorem 2.3 saying that the Stone-Cech compactification of a definable set is definable. In Section 3 we shall study closure properties of vector spaces. The main technical theorem of the section is Theorem 3.3, which says that the free vector space on the Stone-Cech compactification of a definable set is the vector space of all functionals on the given set. In the last section, we study probabilistic measures on definable sets, also known as Keisler measures to model theorists. The main result of the section is Theorem 4, which says that all probabilistic measures on a definable set are discrete on a bit bigger definable set (i.e. on the Stone-Cech compactification of the set).

## 2 Ultrafilter monad

The aim of this section is to investigate ultrafilter monad on the category  $\mathbf{ZFA}[A]$  of symmetric sets over an  $\omega$ -categorical  $\omega$ -stable structure  $\mathcal{A}$ . First, let us observe that the ultrafilter monad exists on any Boolean topos, provided it satisfy some mild conditions about existence of free algebras. Moreover, such monad is always a *strong* monad. Explicitly, every topos can be regarded as a category enriched over itself, i.e. just put  $\text{hom}(A, B) = B^A$ , where  $B^A$  is the internal function space [9]. In particular, when working in  $\mathbf{ZFA}[A]$  it is natural to think that  $\text{hom}(A, B)$  carries the group action. The same is true for other algebraic structures studied here, especially: vector spaces (modules) and Boolean algebras. In fact, we have an enriched adjunction between the free vector space functor  $F: \mathbf{ZFA}[A] \rightarrow \mathbf{Vect}_{\mathbf{ZFA}[A]}$  and the forgetful functor  $S: \mathbf{Vect}_{\mathbf{ZFA}[A]} \rightarrow \mathbf{ZFA}[A]$ . Similarly, we have an enriched adjunction between the free Boolean algebra functor  $\mathbf{ZFA}[A] \rightarrow \mathbf{Bool}_{\mathbf{ZFA}[A]}$  and the underlying functor  $| - |: \mathbf{Bool}_{\mathbf{ZFA}[A]} \rightarrow \mathbf{ZFA}[A]$  (these follow from the transfer principle from [13] and the fact that both the theory of vector spaces and the theory of Boolean algebras are Lawvere theories.). Forgetful functors, being right adjoint, preserve all limits that exist, and free functors preserve all colimits that exists. Specifically, the enriched category of Boolean algebras have cotensors with all symmetric sets (these are just weighted limits), i.e. for every symmetric set  $A$  and an internal Boolean algebra  $B$  the cotensor  $B \pitchfork A$  exists and is preserved by the underlying functor  $|B \pitchfork A| = |B| \pitchfork A = |B|^A$ , where the last equality holds because cotensors coincide with exponents in the base of enrichment. Now, if we now consider the 2-element Boolean algebra  $2$ , we have a series of enriched natural isomorphisms:

$$\begin{array}{c}
\text{hom}_{\mathbf{Bool}^{\text{op}}_{\mathbf{ZFA}[A]}}(2 \pitchfork X, B) \\
\hline
\text{hom}_{\mathbf{Bool}_{\mathbf{ZFA}[A]}}(B, 2 \pitchfork X) \\
\hline
\text{hom}_{\mathbf{Bool}_{\mathbf{ZFA}[A]}}(B, 2)^X \\
\hline
\text{hom}_{\mathbf{ZFA}[A]}(X, \text{hom}_{\mathbf{Bool}_{\mathbf{ZFA}[A]}}(B, 2))
\end{array}$$

which means that  $2 \pitchfork (-): \mathbf{ZFA}[A] \rightarrow \mathbf{Bool}_{\mathbf{ZFA}[A]}$  is an internal left adjoint to enriched hom-functor  $\text{hom}_{\mathbf{Bool}_{\mathbf{ZFA}[A]}}(-, 2): \mathbf{Bool}_{\mathbf{ZFA}[A]} \rightarrow \mathbf{ZFA}[A]$ . By composing these two functors we obtain a strong (internal, enriched) monad on  $\mathbf{ZFA}[A]$ , i.e. the ultrafilter monad:  $\text{hom}(2 \pitchfork (-), 2): \mathbf{ZFA}[A] \rightarrow \mathbf{ZFA}[A]$ , where we still write  $2 \pitchfork (-)$  instead of  $2^{(-)}$  to indicate that this operation is *not* an exponent in Boolean algebras.

**Theorem 2.1** (Ultrafilter monad). *The ultrafilter monad on  $\mathbf{ZFA}[A]$  exists and is strong (equivalently, enriched over  $\mathbf{ZFA}[A]$ ).*

As usual, we shall call algebras of the ultrafilter monad *compact Hausdorff spaces*. We will also denote the monad by  $(-): \mathbf{ZFA}[A] \rightarrow \mathbf{ZFA}[A]$  to highlight the fact that the free algebra  $\overline{X}$  on a given set  $X$  is the “free compactification” of  $X$ , i.e. the internal Stone–Čech compactification of  $X$ .

**Lemma 2.2** (On preservation of finite coproducts). *Ultrafilter monad preserves binary coproducts.*

*Proof.* The proof is pretty standard. Let  $X$  and  $Y$  be two symmetric sets and consider an symmetric ultrafilter  $p$  on  $X$ . We shall define an ultrafilter  $p^*$  on  $X \sqcup Y$  as follows. For any  $S \subset X \sqcup Y$  put  $S \in p^*$  if and only if  $S \cap X \in p$ . Observe that  $X \sqcup Y \setminus S \in p^*$  if and only if  $(X \sqcup Y \setminus S) \cap X = X \setminus (S \cap X) \in p$ , thus  $(X \sqcup Y \setminus S) \notin p^*$ . Similarly, if  $S_1, S_2 \in p^*$  then  $S_1 \cap X, S_2 \cap X \in p$ , therefore  $(S_1 \cap S_2) \cap X \in p$ , so  $S_1 \cap S_2 \in p^*$ . Moreover,  $p^*$  is obviously upward-closed. In the other direction, given an ultrafilter  $q$  on  $X \sqcup Y$  either  $X \in q$  or  $Y \in q$ , but not both and we obtain an ultrafilter on  $X$  (resp.  $Y$ ) via restriction. It is also obvious that the operations are inverse of each other.  $\square$

Till the end of the section we shall assume that  $\mathcal{A}$  is  $\omega$ -categorical,  $\omega$ -stable and that  $\mathcal{A}$  eliminates imaginaries (extending a structure with elimination of imaginaries, as mentioned in the introduction, does not change the category  $\mathbf{ZFA}[A]$ ).

**Theorem 2.3** (Internal Stone–Čech compactification). *Let  $X$  be  $A_0$ -definable. The free Stone–Čech compactification  $\overline{X}$  of  $X$ , i.e. the set of ultrafilters on  $X$ , is  $A_0$ -definable.*

*Proof.* Let us first assume  $X = A^n$ . Let  $\mu: \mathcal{P}(X) \rightarrow 2$  be an  $A_1$ -supported ultrafilter on  $X$ . Consider any formula  $\phi(x, \overline{y})$ , where  $\overline{y}$  are treated as parameters. Because  $\mu$  is  $A_1$ -supported, the set:

$$D_\phi = \{\overline{q} \in A^{|\overline{y}|}: \mu(\phi(x, \overline{q})) = 1\}$$

is  $A_1$ -supported. Therefore, by  $\omega$ -categoricity of  $A$  set  $D_\phi$  may be thought of as a formula  $D_\phi(\overline{y}, \overline{a})$  with parameters  $\overline{a}$ . Therefore, the corresponding  $\phi$ -type is definable by  $D_\phi(\overline{y}, \overline{a})$ , so it is  $A_1$ -definable. Because, this is true for every formula  $\phi(x, \overline{y})$ , ultrafilter  $\mu: \mathcal{P}(X) \rightarrow 2$  corresponds to an  $A_1$ -definable type in  $S_n(A)$ . In the other direction, let us assume that a type  $p \in S_n(A)$  is  $A_1$ -definable for some finite  $A_1$ . Then for every  $\pi_{A_1}$  and every  $\phi(x, \overline{q})$  we have that:

$$\pi_{A_1}(\phi(x, \overline{q})) \in p \Leftrightarrow \phi(\pi_{A_1}(x), \overline{q}) \in p \Leftrightarrow \phi(\pi_{A_1}(\overline{q}), \overline{a}) \Leftrightarrow \phi(\overline{q}, \overline{a}) \Leftrightarrow \phi(x, \overline{q}) \in p$$

Thus,  $p$  is an  $A_1$ -supported function. By Theorem 2.6 space  $S_n(A)$  has finitely many orbits. Therefore,  $\overline{X}$  has finitely many orbits.

Now, moving to the general case, observe that arbitrary  $A_0$ -definable set  $X$  is an equivariant subset of  $A^n$  for some finite  $n$  in an expansion of structure  $A$  by finitely many constants  $A_0$ . Because such an expansion preserves both  $\omega$ -categoricity and  $\omega$ -stability of a structure, without loss of generality we may assume that  $X$  is an equivariant subset of  $A^n$  in  $\mathcal{A}$ . Let us denote  $X^c = A^n \setminus X$ . By Lemma 2.2 the ultrafilter monad preserves finite coproducts, therefore  $\overline{A^n} = \overline{X \sqcup (A^n \setminus X)} = \overline{X} \sqcup \overline{A^n \setminus X}$ . Because  $\overline{A^n}$  has finitely many orbits, both  $\overline{X}$  and  $\overline{A^n \setminus X}$  must have finitely many orbits.  $\square$

**Corollary 2.4.** *The ultrafilter monad restricts to the monad on the full subcategory of  $\mathbf{ZFA}[A]$  of definable sets.*

## 2.1 One theorem from the introduction

**Theorem 2.5** (Ultrafilters in ZFA over  $\omega$ -categorical  $\omega$ -stable structures). *Let  $A$  be a non-trivial  $\omega$ -categorical and  $\omega$ -stable structure. Then:*

1. *Boolean Prime Ideal Theorem does not hold in  $\mathbf{ZFA}[A]$*
2. *for every infinite set  $X$  in  $\mathbf{ZFA}[A]$  there is a non-principal ultrafilter on  $\mathcal{P}(X)$ .*

*Proof.* For (1) recall that BPIT is equivalent over  $\mathbf{ZF}(A)$  to the compactness theorem of propositional calculus (see, for example [1] or [7]). Let  $A$  be the set of atoms, and consider the following set of propositional variables  $Var = A^2$  with the following set of propositions:

- $\{\neg(\langle a, b \rangle \wedge \langle b, a \rangle) : \langle a, b \rangle, \langle b, a \rangle \in Var\}$
- $\{\langle a, b \rangle \vee \langle b, a \rangle : \langle a, b \rangle, \langle b, a \rangle \in Var \wedge a \neq b\}$
- $\{\langle a, b \rangle \wedge \langle b, c \rangle \rightarrow \langle a, c \rangle : \langle a, b \rangle, \langle b, c \rangle, \langle a, c \rangle \in Var\}$

Intuitively, the sets of propositions say that there exists a strict linear ordering on  $A$ . Because, every finite subset of the sets of propositions is satisfiable (i.e. there are definable orders in  $A$  of any finite length), by the compactness theorem for propositional calculus, the whole set is satisfiable, which means that there is a strict linear order on  $A$ . But this contradicts  $\omega$ -stability of structure  $A$ .

For (2), without loss of generality we assume that  $X$  is equivariant. Therefore, it is a disjoint union of  $I$  equivariant sets  $(X_i)_{i \in I}$  consisting of its orbits. If  $I$  is finite, then by  $\omega$ -categoricity of  $\mathcal{A}$  set  $X$  is definable and we have the structure theorems (Theorem 2.3, see also Remark 3.3) for the set of ultrafilters on  $X$ . So, let us assume that  $I$  is infinite. Then, assuming AC (or at least BPIT) in the external (meta-)mathematics, there is a non-principal ultrafilter  $\mu_I$  on  $I$ . Let us associate with every  $i \in I$  an equivariant ultrafilter (principal or not)  $\mu_i$  on  $X_i$ . Then, we may define an ultrafilter  $\mu$  on  $X$  as the Fubini-like product:

$$A \in \mu \Leftrightarrow \{i \in I : A \downarrow X_i \in \mu_i\} \in \mu_I$$

for every  $A \subseteq X$ , where  $A \downarrow X_i$  is the restriction of  $A$  to  $X_i$ . To see that  $\mu$  is symmetric, let us consider any permutation  $\pi$ . We have:

$$\pi(A) \in \mu \Leftrightarrow \pi(\{i \in I : A \downarrow X_i \in \mu_i\}) \in \mu_I \Leftrightarrow \{i \in I : \pi(A \downarrow X_i) \in \mu_i\} \in \mu_I$$

Obviously,  $A \downarrow X_i \subseteq X_i$  and  $X_i$  consists of a single orbit, thus  $\pi(A \downarrow X_i) \subseteq \pi(X_i) = X_i$ . Therefore, by equivariance of  $\mu_i$  we have that:  $\pi(A \downarrow X_i) \in \mu_i \Leftrightarrow A \downarrow X_i \in \mu_i$  and so:  $A \in \mu \Leftrightarrow \pi(A) \in \mu$ . It is a routine to check that such defined  $\mu$  is an ultrafilter:

- $\emptyset \in \mu \Leftrightarrow \{i \in I: \emptyset \in \mu_i\} \in \mu_I \Leftrightarrow \emptyset I$ , so  $\emptyset$  does not belong to  $\mu$  because it does not belong to  $I$
- if  $A, B \in \mu$  then both  $I_A = \{i \in I: A \downarrow X_i \in \mu_i\}$  and  $I_B = \{i \in I: B \downarrow X_i \in \mu_i\}$  and because  $I$  is an ultrafilter  $I_A \cap I_B \in \mu_I$ ; but  $I_A \cap I_B = \{i \in I: A \downarrow X_i \in \mu_i \wedge B \downarrow X_i \in \mu_i\}$  and because  $\mu_i$  is an ultrafilter  $(A \downarrow X_i) \cap (B \downarrow X_i) = (A \cap B) \downarrow X_i \in \mu_i$ , so  $A \cap B \in \mu$
- by definition  $(X \setminus A) \in \mu \Leftrightarrow \{i \in I: (X \setminus A) \cap X_i \in \mu_i\} \in \mu_I$ ; we have however,  $(X \setminus A) \cap X_i = X_i \setminus A = X_i \setminus (A \cap X_i)$ , so  $(X \setminus A) \cap X_i \in \mu_i \Leftrightarrow A \cap X_i \notin \mu_i$ ; but  $\{i \in I: A \cap X_i \notin \mu_i\} \in \mu_I \Leftrightarrow \{i \in I: A \cap X_i \in \mu_i\} \notin \mu_I \Leftrightarrow A \notin \mu$

□

## 2.2 An important mathematical lemma

**Lemma 2.6** (Types in  $\omega$ -categorical  $\omega$ -stable structures). *Let  $\omega$ -categorical  $\omega$ -stable structure and  $p$  a type in  $S_n(A)$ . Then  $p$  is supported by a finite tuple  $\bar{a} \in A^k$  and definable by an  $\bar{a}$ -supported formula  $\phi(x, \bar{a})$  of the same Morley Rank as  $p$  and Morley Degree 1. Moreover  $|\bar{a}| \leq 2|x|$ .*

The proof of the above lemma is strongly based on Theorem 6.3 from [3].

**Theorem 2.7** (Cherlin, Harrington, Lachlan [3]). *Let  $\mathcal{A}$  be  $\omega$ -categorical and  $\omega$ -stable. Then any type  $p \in S_1(A)$  is definable by a normalised formula with two parameters.*

This is a bit stronger than the original statement from [3], but it can be extracted from the proof. The authors show that the defining formula  $\phi(x, \bar{a})$  of type  $p$  can be chosen to be normalised: i.e. whenever  $\bar{a}$  and  $\bar{a}'$  are in the same orbit, and  $\phi(x, \bar{a})$  differs from  $\phi(x, \bar{a}')$  on a set of the Morley Rank strictly smaller than  $\phi(x, \bar{a})$  then in fact  $\phi(x, \bar{a}) = \phi(x, \bar{a}')$ . Moreover, the term “defining formula” refers to the following property: for every formula with parameters  $\psi$  we have that  $\psi(x, \bar{q}) \in p$  if and only if the Morley rank of  $\phi(x, \bar{a}) \cap \psi(x, \bar{q})$  has the same Morley rank as  $\phi(x, \bar{a})$ , therefore it has the minimal Morley rank possible in  $p$ . It is also a standard result in stability theory (see: [11], [6] or [14] for more details) that the above property is definable from parameters  $\bar{a}$ .

*Proof of Lemma 2.6.* Let us recall that a type  $p$  being definable from parameters  $\bar{a}$  means just that for every formula  $\psi(x, y)$  without parameters, there is a formula  $D_\psi(y, \bar{a})$  with parameters  $\bar{a}$  such that  $\psi(x, \bar{q}) \in p$  if and only if  $D_\psi(\bar{q}, \bar{a})$ . Therefore, definability by a single formula in the sense of [3] is a stronger property. Fortunately, it is not stronger for  $\omega$ -categorical  $\omega$ -stable structures. We must show that parameters  $\bar{a}$  are necessary, i.e. that  $p$  is not definable with a smaller number of parameters. But it follows from our choice of  $\phi(x, \bar{a})$  to be normalised. The fact that  $\phi(x, \bar{a})$  is normalised means that every permutation  $\pi$  that fixes  $p$  must also fix  $\phi(x, \bar{a})$ , otherwise we would have  $\phi(x, \pi^{-1}(\bar{a})) \in p$  and therefore  $\phi(x, \pi^{-1}(\bar{a})) \cap \phi(x, \bar{a}) \in p$ . But because  $\phi(x, \bar{a})$  was of the smallest Morley Rank, say  $r$ , then the Morley rank of  $\phi(x, \pi^{-1}(\bar{a})) \cap \phi(x, \bar{a})$  must be equal to  $r$  too. Therefore,  $\phi(x, \bar{a}) \setminus (\phi(x, \pi^{-1}(\bar{a})) \cap \phi(x, \bar{a}))$  must be of Morley Rank strictly smaller than  $r$  – otherwise,  $\phi(x, \bar{a})$  would be a disjoint sum of two sets of Morley Rank  $r$ , what would contradict the choice of  $\phi(x, \bar{a})$  with Morley Degree 1. The same argument works for  $\phi(x, \pi^{-1}(\bar{a})) \setminus (\phi(x, \pi^{-1}(\bar{a})) \cap \phi(x, \bar{a}))$ , what means that  $\phi(x, \bar{a})$  and  $\phi(x, \pi(\bar{a}))$  differs on a set of the Morley Rank strictly smaller than  $r$ . Therefore, must be equal.

It remains to prove the bound on the number of parameters for types in  $S_n(A)$ . Note however that the general case for  $n$ -types reduces to the case of 1-types. It suffices to observe that the reduct of an  $\omega$ -categorical  $\omega$ -stable structure is itself  $\omega$ -categorical and  $\omega$ -stable and one may easily construct a reduct of  $A$  on  $n$ -element tuples of  $A$  whose 1-types encode  $n$ -types of  $A$ . □

**Example 2.1** (Ultrafilters in random graphs). *As mentioned in the introduction the structure  $\mathcal{R}$  of the Random Graph from Example 1.5 is  $\omega$ -categorical, but not  $\omega$ -stable. Here we will show, that the set of ultrafilters on  $R$  has infinitely many orbits. First, observe that by the extension property of the Random Graph, every set of formulas:*

$$S = \{E(x, a_1), E(x, a_2), \dots, \neg E(x, b_1), \neg E(x, b_2), \dots\}$$

*for pairwise distinct elements  $a_i, b_j$  is finitely satisfiable. Therefore, by the compactness of the First-Order Logic, it is satisfiable. Because, Random Graphs admit elimination of quantifiers, if  $a_i, b_j$  enumerate the whole  $R$ , then  $S$  generates an ultrafilter  $\mu$  on  $\mathcal{P}(R)$ . Moreover,  $\mu$  is symmetric if and only if the set  $\{a_i: E(x, a_i) \in S\}$  is definable. Therefore, symmetric ultrafilters on  $R$  are tantamount to definable subsets of  $R$ .*

### 3 Closure properties of vector spaces on definable sets

The aim of this section is to prove that the category of vector spaces having definable bases enjoy many closure properties somehow similar and somehow different from the closure properties of the category of vector spaces on finite bases. Let  $V, W$  be vector spaces on  $A_0$ -definable bases  $\Lambda$  and  $\Gamma$  respectively, in  $\mathbf{ZFA}[A]$  for an  $\omega$ -categorical and  $\omega$ -stable structure  $\mathcal{A}$ . Then:

- the finite coproduct space  $V + W$  is the same as the finite product space  $V \times W$  and has basis  $\lambda \sqcup \Gamma$
- the tensor product space  $V \otimes W$  has basis  $\lambda \times \Gamma$
- the dual space  $V^* = \mathcal{K}^V$  has basis  $\overline{\Lambda}$
- the space of linear exponent  $V \multimap W$  has a basis that is an  $A_0$ -equivariant subset of  $\overline{\Lambda \times \Gamma}$ .

The last closure property is quite remarkable, because it means that the category of vector spaces on definable sets is monoidally closed. This is in contrast to the category of definable sets, where the exponents are not definable. It also bears saying, that this last property follows from the previous ones. The space  $V \multimap W$  is a complemented subspace of  $(V \otimes W)^*$  and as such must have a basis. Therefore, the main difficulty is in proving the characterisation of the basis of the dual space in terms of the basis of the space, to which is devoted this section.

**Theorem 3.1** (On the existence of dual basis). *For every  $\omega$ -categorical  $\omega$ -stable structure  $A$ , the set theory with atoms over  $A$  satisfies the following: For every  $X$  of bounded support the vector space  $\mathcal{K}^X$  has a basis of a bounded support for every classical field  $\mathcal{K}$ . Moreover, if  $X$  is  $A_0$ -equivariant (resp.  $A_0$ -definable), then we may choose the basis to be  $A_0$ -equivariant (resp.  $A_0$ -definable).*

*Proof.* Without loss of generality, we shall assume that  $X$  is equivariant and  $A$  eliminates imaginaries. Let  $b$  be the bound on the size of the support of each element  $x \in X$ . Equivariant set  $X$  can be written as a disjoint union of its equivariant orbits  $(X_i)_{i \in I}$ , where  $I$  is a cardinal number. By elimination of imaginaries of  $A$ , every orbit  $X_i$  is isomorphic to an equivariant orbit of  $A^b$  and by  $\omega$ -categoricity of  $A$  there are only finitely many of them. Therefore, there are some  $X_{i_1}, X_{i_2}, \dots, X_{i_n}$  and cardinals  $\kappa_1, \kappa_2, \dots, \kappa_n$  such that:

$$X \approx \coprod_{1 \leq j \leq n} \kappa_j \times X_{i_j}$$

Because, the free vector space functor  $F$  preserves colimits, and exponents map colimits to limits:

$$\mathcal{K}^X \approx \text{Lin}(F(X), \mathcal{K}) \approx \text{Lin}(F(\coprod_{1 \leq j \leq n} \kappa_j \times X_{i_j}), \mathcal{K}) \approx \prod_{1 \leq j \leq n} \text{Lin}(F(\kappa_j \times X_{i_j}), \mathcal{K}) = \prod_{1 \leq j \leq n} \text{Lin}(F(\kappa_j \times X_{i_j}), \mathcal{K})$$

where the last equality follows from the fact that finite coproducts coincide with finite products for vector spaces. By Lemma 3.2 we have that:  $\mathcal{K}^{\kappa_j \times X_{i_j}} \approx F(\mathcal{B}(\kappa_j) \overline{X_{i_j}})$  and so:

$$\mathcal{K}^X \approx \prod_{1 \leq i \leq n} F(\mathcal{B}(\kappa_j) \times \overline{X_{i_j}}) \approx F(\prod_{1 \leq i \leq n} \mathcal{B}(\kappa_j) \times \overline{X_{i_j}})$$

Therefore,  $\prod_{1 \leq i \leq n} \mathcal{B}(\kappa_j) \times \overline{X_{i_j}}$  is (isomorphic to) a basis of  $\mathcal{K}^X$ . Observe that because each  $\mathcal{B}(\kappa_j)$  and  $\overline{X_{i_j}}$  are equivariant, the constructed basis is equivariant. By Theorem 2.3 each  $\overline{X_{i_j}}$  is definable, therefore the support of its elements is bounded by some finite  $b_j$ . So  $\prod_{1 \leq i \leq n} \mathcal{B}(\kappa_j) \times \overline{X_{i_j}}$  is of a bounded support  $\max_{1 \leq i \leq n} b_j$ . Moreover, if  $X$  is definable, then the cardinals  $\kappa_j$  must be finite, and by Theorem 2.3 the basis consists of finitely many orbits, thus is definable.  $\square$

**Lemma 3.2** (On dual basis of  $\kappa \times X$ ). *Let  $X$  be an equivariant set consisting of a single orbit and  $\kappa$  a cardinal. Then the vector space  $\mathcal{K}^{\kappa \times X}$  has an equivariant basis  $\mathcal{B}(\kappa) \times \overline{X}$ , where  $\mathcal{B}(\kappa)$  is a basis of  $\mathcal{K}^\kappa$ .*

*Proof.* We claim that for  $\lambda \in \mathcal{B}(\kappa)$  and  $p \in \overline{X}$  vectors  $(\lambda, p): \kappa \times X \rightarrow \mathcal{K}$  defined as  $(\lambda, p)(i, x) = \lambda(i)p(x)$  form the basis of  $\mathcal{K}^{\kappa \times X}$ . Linear independence of the vectors is obvious, therefore let us show that every definable function  $f: \kappa \times X \rightarrow \mathcal{K}$  is a finite combination of these vectors. Because  $f$  is definable it is  $A_0$ -supported for some finite  $A_0$ . The crucial observation is that for every  $i \in \kappa$  the function  $f(i, -): X \rightarrow \mathcal{K}$  must be  $A_0$ -supported, so there are only finitely many  $X_j$  such that  $X = \coprod_{1 \leq j \leq n} X_j$  and the restrictions  $f_j(i, -): X_j \rightarrow \mathcal{K}$  are constant. Let us denote the constant associated to the pair  $i, j$  by  $r_{i,j}$ . Then  $r_{(-),j}$  is a function  $\kappa \rightarrow \mathcal{K}$  and as such has a unique decomposition in the basis  $\mathcal{B}(\kappa)$ , say:  $r_{(-),j} = \sum_{1 \leq s_j \leq N} c_{s_j} \lambda_{s_j}$ . Moreover, by Theorem 3.3 each  $X_j$  has its own decomposition in  $\overline{X}$  as  $X_j = \sum_{1 \leq t_j \leq M} b_{t_j} p_{t_j}$ , where  $M, N$  can be chosen to not depend on  $j$ . So:

$$f(i, x) = \sum_{1 \leq j \leq n} f(i, x) X_j = \sum_{1 \leq j \leq n} r_{i,j} X_j = \sum_{1 \leq j \leq n} \sum_{1 \leq s_j \leq N} \sum_{1 \leq t_j \leq n} c_{s_j} b_{t_j} \lambda_{s_j} p_{t_j}$$

$\square$

**Theorem 3.3** (Free space on ultrafilters). *Let  $X$  be definable. The free  $\mathcal{K}$ -vector space  $F(\overline{X})$  over the set  $\overline{X}$  of ultrafilters on  $X$  is isomorphic to the space of functions  $\mathcal{K}^X$ .*

**Remark 3.1** (On necessity of stability). *Let us consider the set of atoms  $Q$  in the ordered Fraenkel-Mostowski model  $\mathbf{ZFA}[Q]$ . We claim that both  $\overline{Q}$  and the basis  $\Lambda$  of  $\mathcal{R}^Q$  exist, but are not isomorphic. In fact,  $\Lambda$  is a proper subset of  $\overline{Q}$ . To see this, consider a symmetric function  $f: Q \rightarrow \mathbb{R}$ . Then there is a finite decomposition  $Q = I_1 \sqcup I_2 \sqcup \dots \sqcup I_n$  on intervals  $I_k$ , such that  $f$  is constant on each  $I_k$ . Therefore, the set of vectors:*

$$\{1, p_1^*, p_2^*, \dots, p_1^<, p_2^<, \dots\} \approx Q \sqcup Q \sqcup 1$$

where:

- 1 is the constant function, i.e.  $1(a) = 1$

- $p^*$  is the characteristic function, i.e.  $p^*(q) = [p = q]$
- $p^<$  is the open down set of  $p$ , i.e.  $p^<(q) = [p > q]$

generates  $\mathbb{R}^Q$ . Moreover, these vectors are linearly independent: if  $S$  is any non-empty finite set of the above vectors, then there exists vector  $m \in S$  and an atom  $p$  with  $m(p) = 1$  such that for every  $s \in S \setminus \{m\}$  we have that  $s(p) = 0$ , so  $m$  cannot be a linear combination of  $S \setminus \{m\}$ . On the other hand it is easy to compute:

$$\overline{Q} = \{-\infty, +\infty, (q)_{q \in Q}, (q^-)_{q \in Q}, (q^+)_{q \in Q}\} \approx Q \sqcup Q \sqcup Q \sqcup 2$$

where:

- $-\infty$  is the ultrafilter generated by  $\{x : x < q\}_{q \in Q}$
- $+\infty$  is the ultrafilter generated by  $\{x : x > q\}_{q \in Q}$
- $(q)_{q \in Q}$  are all principal ultrafilters
- $(q^-)_{q \in Q}$  are ultrafilters of generated by the left neighbourhoods of  $q$ , i.e. all sets  $\{x : p < x < q\}_{p < q}$
- $(q^+)_{q \in Q}$  are ultrafilters of generated by the right neighbourhoods of  $q$ , i.e. all sets  $\{x : q < x < p\}_{p > q}$

**Remark 3.2** (A few words about dual modules). *Although this section is devoted to vector spaces, that is: modules over a field, an inspection of Lemma 3.6 and Lemma 3.7 shows that Theorem 3.3 holds for modules over arbitrary ring.*

The rest of this section is devoted to the proof of Theorem 3.3. For a set  $X$  let us denote by  $U$  the set of non-principal ultrafilters on  $X$ , i.e.  $U(X) = \overline{X} \setminus X$ , where  $X$  is identified with the image of  $X$  under  $\eta$ . With every subset  $Y \subseteq \overline{X}$  we may associate the set  $Y'$  of limit points of  $Y$ , i.e. the image of  $U(Y) \subseteq U(\overline{X}) \subseteq \overline{\overline{X}} \rightarrow^\mu \overline{X}$ . It is clear that if  $Y$  is closed under limits, then  $Y' \subseteq Y$  and  $Y'$  is also closed under limits (i.e. it is defined as the subspace of limit points). Moreover, if  $Y$  is  $A_0$ -supported then  $Y'$  is  $A_0$ -supported, because both  $\eta$  and  $\mu$  are equivariant. Therefore, the operation of taking limit points produces a descending sequence of  $A_0$ -equivariant subsets of  $Y$ :

$$\dots \subseteq Y^{(n+1)} \subseteq Y^{(n)} \subseteq \dots \subseteq Y'' \subseteq Y' \subseteq Y$$

Because  $A$  is  $\omega$ -categorical, there are only finitely many  $A_0$ -supported subsets of  $Y$ , so there must be  $n$  such that  $(Y^{(n)})' = Y^{(n)}$ . Therefore, the sequence gives a decomposition of  $Y$  on  $n$ -disjoint  $A_0$ -equivariant subsets  $Y_{(i)} = Y^{(i-1)} \setminus Y^{(i)}$ . Observe also, that by the construction, points in  $Y_{(i)}$  are isolated in  $Y^{(i-1)}$ . We have the following lemma.

**Lemma 3.4** (Existence of a binary tree). *If the sequence  $Y^{(n)} \subseteq \dots \subseteq Y'' \subseteq Y' \subseteq Y$  ends in a non-empty set  $Y^{(n)}$  then structure  $A$  is not  $\omega$ -stable.*

*Proof.* Let us write  $P = Y^{(n)}$ . Observe that if  $P$  is finite then  $P' = \emptyset$  so  $P = \emptyset$ . If  $P$  is infinite, then we may choose any two distinct points  $a \neq b \in P$  and two non-principal ultrafilters  $p, q \in U(P)$  such that  $p \rightarrow a$  and  $q \rightarrow b$ . Obviously,  $p \neq q$ , so there must be a set  $P_0$  such that  $P_0 \in p$  and  $P_0 \notin q$ . This means that  $P_1 = (P \setminus P_0) \in q$ . Because,  $p$  and  $q$  are non-principal, both  $P_1$  and  $P_2$  are infinite. Therefore,  $P_1$  and  $P_2$  give a decomposition of  $P$  on two disjoint infinite subsets and because  $U(P) = U(P_1 \sqcup P_2) = U(P_1) \sqcup U(P_2)$ , we may construct by induction an infinite binary tree  $(P_w)_{w \in \{0,1\}^*}$  what contradicts  $\omega$ -stability of  $A$ .  $\square$

**Remark 3.3** (Non-principal ultrafilters on infinite sets). *The proof of Lemma 3.4 shows that if  $U(X)$  is empty, i.e. there are no non-principal ultrafilters on  $U(X)$ , then  $X$  must be finite. Therefore, for every infinite definable set  $X$  the set of non-principal ultrafilters on  $X$  is non-empty.*

For the rest of the proof, we shall assume that  $A$  is  $\omega$ -stable (therefore,  $Y^{(i-1)} = \emptyset$ ) and  $Y = \overline{X}$  is equivariant.

**Remark 3.4.** *Because  $A$  is  $\omega$ -stable, every type in  $S_n(A)$  is definable with a finite set of parameters. Therefore, by the considerations in the proof of Theorem 2.3,  $S_n(A)$  is isomorphic to  $\overline{A^n}$ . Moreover, a formula  $\phi$  (with parameters from  $A$ ) has Morley rank  $r$  and Morley degree  $d$  if and only if it belongs to exactly  $d$  types in  $\overline{A^n}_{(r)}$ . According to this setting, Lemma 3.4 gives an internal proof of the fact that formulas in an  $\omega$ -categorical  $\omega$ -stable theory must have finite Morley rank.*

**Lemma 3.5** (On nice isolated sets). *For every  $y \in Y_{(i)}$  for  $1 \leq i \leq n$  there is a set  $\lambda_y$  that isolates  $y$  in  $Y^{(i-1)}$  and such that if  $A_0$  supports  $y$  then  $A_0$  supports  $\lambda_y$ . Moreover, we may choose  $\lambda_y$  uniformly for the orbit of  $y$ , i.e.  $\pi(\lambda_y) = \lambda_{\pi(y)}$ .*

*Proof.* Set  $\lambda_y$  may be chosen to be the normalised defining formula for  $y$  from Lemma 2.6. Consider any permutation  $\pi$ . If  $\lambda_y$  is normalised, then  $\pi(\lambda_y)$  must be normalised (by the definition of normality) and belongs to  $\pi(y)$ . Because both the rank and the degree are preserved by automorphisms,  $\pi(\lambda_y)$  can be chosen as a defining formula for  $\pi(y)$ .  $\square$

The above lemma implies the existence of an equivariant injection  $\overline{X} \rightarrow \mathcal{P}(X)$  sending an ultrafilter  $p$  to a nice set contained in  $p$ .

**Lemma 3.6** (Nice isolated sets are linearly independent in  $\mathcal{K}^X$ ). *The proof is by induction on sets  $\overline{X}_{(i)}$ . For  $i = 1$  sets  $\lambda_x$  are singletons  $\{x\}$ , so they are linearly independent as functions  $X \rightarrow \{0, 1\} \rightarrow \mathcal{K}$  for any ring  $\mathcal{K}$ . Let us assume that the set  $\{\lambda_p : \exists_{k < i} p \in \overline{X}_{(k)}\}$  is linearly independent. Consider any linear combination that equals 0:*

$$\alpha + a_1\lambda_{p_1} + a_2\lambda_{p_2} + \cdots a_n\lambda_{p_n} = 0$$

where  $p_1, p_2, \dots, p_n$  belong to  $\overline{X}_{(i)}$  and  $\alpha$  is a linear combination of some  $\lambda_p$  for  $p \in \overline{X}_{(k)}$  for  $k < i$ . Because  $\alpha$  is a finite combination of functions that are zero at every  $p \in \overline{X}_{(i)}$ , function  $\alpha$  must be itself zero at every  $p \in \overline{X}_{(i)}$ , therefore  $a_1\lambda_{p_1} + a_2\lambda_{p_2} + \cdots a_n\lambda_{p_n}$  must be zero at every  $p \in \overline{X}_{(i)}$ . But  $(a_1\lambda_{p_1} + a_2\lambda_{p_2} + \cdots a_n\lambda_{p_n})(p) = a_j$  for  $p = p_j$  and so  $a_1 = a_2 = \cdots = a_n = 0$ . And then,  $\alpha = 0$ .

**Lemma 3.7** (Nice isolated sets spans  $\mathcal{K}^X$ ). *Let  $f : X \rightarrow \mathcal{K}$  be a finitely supported function. Because  $\mathcal{K}$  is classical and  $X$  has finitely many orbits, such  $f$  must take only finitely many values in  $\mathcal{K}$ , say  $r_1, r_2, \dots, r_n$ . These values induce decomposition of  $X$  into  $n$  disjoint subsets  $\phi_i = f^{-1}[r_i] \subseteq X$ , such that  $f = \sum_{i=1}^n r_i\phi_i$ , where  $\phi_i$  are treated as characteristic functions  $\phi_i : X \rightarrow \{0, 1\} \rightarrow \mathcal{K}$ . Moreover, we may drop  $i$  such that  $r_i = 0$  from the sum. In the below we shall restrict to  $\phi_j$  such that  $r_j \neq 0$ .*

*The proof is by induction on  $k$  such that  $k = \max_{0 \leq i < n} p \in \overline{X}_{(i)} \wedge \phi_j \in p$ , i.e. the biggest  $i$  such that  $\phi_j$  belongs to an ultrafilter in  $\overline{X}_{(i)}$ . For  $k = 0$  subset  $\phi_j$  must be finite, therefore it is the sum of singletons  $\{x\}$  such that  $x \in \phi_j$  and the sum is disjoint, thus interpreted the same way in any ring  $\mathcal{K}$ . Let us now assume that the theorem is true for all  $k' < k$ . By the assumption  $\phi_j$  does not belong to an ultrafilter from  $\overline{X}_{(k'')}$  for  $k'' > k$ , so the number of ultrafilters  $p$  from  $\overline{X}_{(k)}$  that  $\phi_j$  belongs to  $p$  must be finite, say the ultrafilters are  $p_1, p_2, \dots, p_m$ . Then  $g = \phi_j - (\lambda_{p_1} + \lambda_{p_2} + \cdots + \lambda_{p_m})$  is a finitely supported function such that none of  $g^{-1}[r]$  for  $r \neq 0$  is contained in an ultrafilter from  $\overline{X}_{(k'')}$  for  $k'' > k - 1$ . By inductive hypothesis  $g$  is a linear combination of nice isolated sets, say  $a_1\lambda_{p'_1} + a_2\lambda_{p'_2} + \cdots + a_t\lambda_{p'_t}$ , therefore  $\phi_j = a_1\lambda_{p'_1} + a_2\lambda_{p'_2} + \cdots + a_t\lambda_{p'_t} + \lambda_{p_1} + \lambda_{p_2} + \cdots + \lambda_{p_m}$ .*



## 4 Probability measures

By a measurable space we shall mean a tuple  $\langle X, \sigma \rangle$ , where  $X$  is a set, and  $\sigma \subseteq \mathcal{P}(X)$  is a subset of the power-set of  $X$  closed under Boolean operations and countable unions/intersections:

- $\emptyset \in \sigma$
- if  $X_0 \in \sigma$  then  $X \setminus X_0 \in \sigma$
- if  $(X_i)_{i \in \mathcal{N}}$  is a family of  $X_i \in \sigma$  then  $(\bigcup_{i \in \mathcal{N}} X_i) \in \sigma$

Set  $\sigma$  is usually called a  $\sigma$ -algebra on  $X$ , or a Borel space on  $X$ . A measurable function from a measurable space  $\langle X, \sigma_X \rangle$  to a measurable space  $\langle Y, \sigma_Y \rangle$  is a function  $f: X \rightarrow Y$  such that if  $Y_0 \in \sigma_Y$  then  $f^{-1}[Y_0] \in \sigma_X$ . A sub-probability measure  $\mu$  on a measurable space  $\sigma$ , is a countably additive function  $\mu: X \rightarrow [0, 1]$ , i.e. for every countable family  $(X_i)_{i \in \mathcal{N}}$  in  $\sigma$  of pairwise disjoint sets  $X_i \neq X_j$  whenever  $i \neq j$ , we have that:

$$\mu\left(\bigcup_{i \in \mathcal{N}} X_i\right) = \sum_{i \in \mathcal{N}} \mu(X_i)$$

We call a sub-probability measure  $\mu$  a probability measure if  $\mu(X) = 1$ . Of a special interest are measurable spaces  $\langle X, \sigma \rangle$  whose  $\sigma$ -algebra  $\sigma$  is the full powerset on  $X$ . The reason is that every function from  $X$  is measurable according to such  $\langle X, \sigma \rangle$ . Let us denote by  $\mathbf{m}(X)$  the set of all probability measures on the powerset  $\mathcal{P}(X)$  of  $X$ . Note, that the structure of  $\mathbf{m}(X)$  for an arbitrary  $X$  may be difficult to describe (i.e. this structure highly depends on the foundations of the ambient set theory, in particular on the existence of large cardinals). Nonetheless, if  $X$  is at most countable then one may easily describe the structure of  $\mathbf{m}(X)$ , i.e. every measure  $\mu$  on  $\mathcal{P}(X)$  is discrete in the sense that  $\mu$  is fully determined by its values on atoms  $\{x\} \in \mathcal{P}(X)$ . Therefore, every such a measure is tantamount to a function  $\mu \downarrow X: X \rightarrow [0, 1]$  such that  $\sum_{x \in X} (\mu \downarrow X)(x) = 1$ . The next theorem extends this characterisation to all definable sets in  $\mathbf{ZFA}[A]$  for an  $\omega$ -categorical  $\omega$ -stable structure  $\mathcal{A}$ .

**Theorem 4.1** (Characterisation of measures on a definable set). *. Let  $\mathcal{A}$  be an  $\omega$ -categorical and  $\omega$ -stable structure. For every definable set  $X$  in  $\mathbf{ZFA}[A]$  every  $A_0$ -supported probability measure on  $X$  is a finite combination of  $A_0$ -supported ultrafilters on  $X$ , i.e. every  $A_0$ -equivariant measure  $\mu: \mathcal{P}(X) \rightarrow [0, 1]$  is of the form  $\mu = r_1 p_1 + r_2 p_2 + \dots + r_n p_n$  for some real numbers  $0 < r_i \leq 1$ , and ultrafilters  $p_i \in \overline{X}$  for  $1 \leq i \leq n$ .*

We claim that  $(\lambda_p)_{p \in \overline{X}}$  generate measures on  $X$  in the following sense: every assignment  $f: \overline{X} \rightarrow [0, 1]$  extends to at most one measure  $\mu$  with  $\mu(\lambda_p) = f(p)$ . Moreover, if the measure is  $A_0$ -definable then  $f$  must be also  $A_0$ -definable. This will give us an upper-bound on the size of structure  $\mathbf{m}(X)$ .

We prove the claim by induction on  $k$  such that  $\lambda_p$  for  $p \in \overline{X}_{(k)}$  generate measures on  $Y$  that are not contained in ultrafilters outside of  $\overline{X}_{(k)}$ . Let us take any subset  $Y \subseteq X$ . If  $Y$  does not belong to any non-principal ultrafilter, then  $Y$  is finite and  $\mu(Y) = \sum_{y \in Y} \{y\}$ , where the singletons  $\{y\}$  are  $\lambda_y$  for  $y$  treated as principal ultrafilter. Assume that the theorem is true for all  $k' < k$  and the maximal  $p$  such that  $Y \in p$  belongs to  $\overline{X}_{(k)}$ . Then by Theorem 3.3 for the 2-element field  $Y = \lambda_{p_1} \oplus \lambda_{p_2} \oplus \dots \oplus \lambda_{p_n} \oplus Y_0$  for  $p_i \in \overline{X}_{(k)}$  and  $Y_0$  such that if  $Y_0 \in p$  then  $p \in \overline{X}_{(k)}$  for  $k' < k$ . Observe, that by our choice of  $\lambda_p$  we have that  $\lambda_{p_i} \cap \lambda_{p_j}$  does not belong to a type  $p \in \overline{X}_{(k')}$  for  $k' \geq k$ . Therefore, we can write the measure on each pair as  $\mu(\lambda_{p_i} \oplus \lambda_{p_{i+1}}) = \mu(\lambda_{p_i}) + \mu(\lambda_{p_{i+1}}) - 2\mu(\lambda_{p_i} \cap \lambda_{p_j})$ . Thus, we can restrict to the case with at most

one  $p_1$ , i.e.  $Y = \lambda_{p_1} \oplus Y_0$ . But then:  $\mu(Y) = \mu(\lambda_{p_1}) - 2\mu(\lambda_{p_1} \cap Y) + \mu(Y)$ , where  $\mu(\lambda_{p_1} \cap Y)$  and  $\mu(Y)$  are given by the inductive hypothesis.

It is possible to impose some restrictions on  $f: \overline{X} \rightarrow [0, 1]$  to induce at least one measure on  $\mathcal{P}(X)$  and give a direct proof of Theorem along this line. Instead, we use a characterisation of measures in  $\omega$ -stable structures from [4] (Remark 2.2), which is originally due to H.J Keisler.

**Theorem 4.2** (Keisler on Keisler measures in  $\omega$ -stable structures). *Let  $T$  be  $\omega$ -stable and  $\mu$  a probability measure over the Monster model  $\mathcal{U}$  of  $T$ . Then  $\mu = \sum_{i=0}^{\infty} r_i p_i$  for  $p_i \in S_n(U)$  and  $r_i \in [0, 1]$  such that  $\sum_{i=0}^{\infty} r_i = 1$ .*

Because we are working with  $\omega$ -categorical structure  $\mathcal{A}$ , we can replace the Monster model  $\mathcal{U}$  with  $\mathcal{A}$ . So the theorem says that every probability measure on  $A^n$  is an infinite positive convex combination of countably many ultrafilters on  $A^n$ . But in case of  $\omega$ -categorical structure this result can be improved to *finitely* many ultrafilters.

*Proof of Theorem 4.* Let an  $A_0$ -supported measure  $\mu: \mathcal{P}(A^n) \rightarrow [0, 1]$  be given. By Theorem 4.2 we have that:  $\mu = \sum_{i=0}^{\infty} r_i p_i$ . Because  $\overline{A}^n$  has finitely many orbits (by Lemma 3.5), if the number of non-zero  $r_i$  is infinite, then there must be an orbit (using the notation from the proof of Theorem 3.3) in  $\overline{A}^n_{(k)}$  that contains an infinite sequence of  $p_{i_j}$  such that  $r_{i_j}$  is strictly decreasing to zero. Let us assume that  $k$  is the smallest number with this property. This means, that there are only finitely many ultrafilters  $p_{k_1}, p_{k_2}, \dots, p_{k_m}$  belonging to  $\overline{A}^n_{(k')}$  for  $k' > k$ . Consider the measures of  $\lambda_{p_{i_j}}$ . If  $p \in \overline{A}^n_{(k)}$  then  $\lambda_{p_{i_j}} \in p$  only if  $p = p_{i_j}$ , because  $\lambda_{p_{i_j}}$  isolates  $p_{i_j}$  in  $\overline{A}^n_{(k)}$ . Therefore:

$$\mu(\lambda_{p_{i_j}}) = r_{i_j} + r_{k_1}[\lambda_{p_{i_j}} \in p_{k_1}] + r_{k_2}[\lambda_{p_{i_j}} \in p_{k_2}] + \dots + r_{k_m}[\lambda_{p_{i_j}} \in p_{k_m}]$$

But there are only  $2^m$  distinct subsets of  $r_{k_1}, r_{k_2}, \dots, r_{k_m}$ , thus they can produce at most  $2^m$  distinct values. Therefore,  $\mu(\lambda_{p_{i_j}})$  must take infinitely many distinct values, what contradicts the fact  $\mu$  is finitely supported. For general equivariant set  $X$ , observe that by elimination of imaginaries  $X \subseteq A^n$  for some  $n$  and  $A^n \setminus X$  is also equivariant. Therefore, by additivity, every measure on  $X$  is just a restriction of a measure on  $A^n$ .  $\square$

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