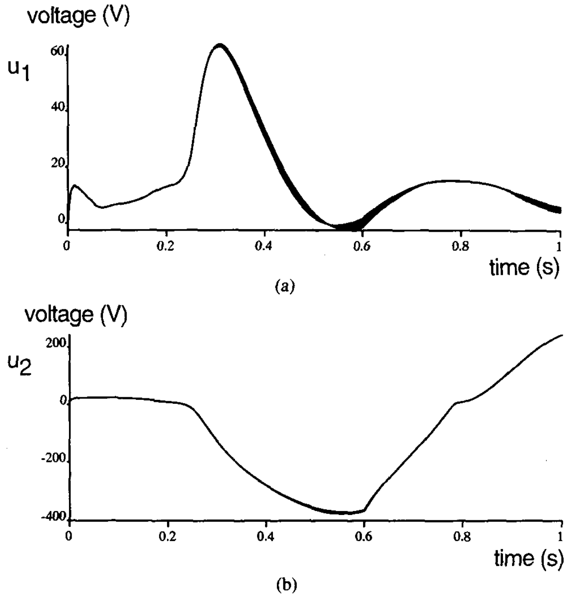

Fig. 5. The input voltages with respect to the $(d-q)$-axes frame: (a) $u_1 = V_{ds}$, (b) $u_2 = V_{qs}$.

4) The main open problem that remains to be solved is the case when we combine motor parameter estimation with state observers.

5) As pointed out in [8], the design technique used here applies directly to the fixed frame motor model of [4], [5], [7].

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On Invariant Polyhedra of Continuous-Time Linear Systems

E. B. Castelan* and J. C. Hennet

Abstract—**This note presents some conditions of existence of positively invariant polyhedra for linear continuous-time systems. These conditions are first described algebraically, then interpreted on the basis of the system eigenstructure. Then, a simple state-feedback placement method is described for solving some linear regulation problems under constraints.**

I. Preliminary Results on Positive Invariance and Polyhedral Sets

Any locally stable time invariant dynamical system admits some domains in its state-space from which any state-vector trajectory cannot escape. These domains are called positively invariant sets of the system. If a system is subject to constraints on its state vector and can be controlled, the purpose of a regulation law can be to stabilize it while maintaining its state-vector in a positively invariant set included in the admissible domain. Under a state feedback regulation law, this design technique can also be used to satisfy constraints on the control vector, possibly by transferring these constraints onto the state-space. The existence and characterization of positively invariant sets of dynamical systems is therefore a basic issue for many constrained regulation problems. To analyze the desired properties of a closed-loop time invariant linear system under a linear state feedback, it suffices to study the "autonomous" model:

$$\dot{x}(t) = Ax(t), \qquad x(t) \in \Re^n, \qquad A \in \Re^{n*n}, t \geq 0. \quad (1)$$

Definition 1: Positive Invariance. A nonempty set Ω is a positively invariant set of system (1) if and only if for any initial state $x_0 \in \Omega$, the complete trajectory of the state vector, $x(t)$, remains in Ω. Or, equivalently, Ω has the property $e^{At}\Omega \in \Omega \; \forall t$.

Definition 1 is general and the set Ω can for example be a bounded polyhedron, a cone or a vectorial subspace. In the last case, positive invariance is equivalent to the well-known property of A-invariance of subspaces [10].

Definition 2: Convex Polyhedron. Any nonempty convex polyhedron of $\Re^n$ can be characterized by a matrix $Q \in \Re^{r*n}$ and a vector $\rho \in \Re^r$, $r \in \mathcal{N} - \{0\}, n \in \mathcal{N} - \{0\}$. It is defined by: $R[Q, \rho] = \{x \in \Re^n; Qx \leq \rho\}$.

By convention, equalities and inequalities between vectors and between matrices are componentwise.

Without loss of generality, it can be assumed that the set of inequality constraints defining $R[Q, \rho]$ is nonredundant. Let Q_i be the ith row-vector of matrix Q, and ρ_i the ith component of vector ρ. Then (see [8]), there exists a one-to-one correspon-

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dence between the r facets of $R[Q, \rho]$ and the r systems (for $i \in (1,\cdots,r)$):

$$\begin{cases} Q_i x = \rho_i \\ Q_j x \le \rho_j, \quad \forall j \in (1,\cdots,r), j \ne i. \end{cases} \tag{2}$$

From Definitions 1 and 2, a convex polyhedron $R[Q, \rho]$ is a positively invariant set of system (1) if and only if:

$$Qx_0 \le \rho \Rightarrow Qe^{At}x_0 \le \rho, \forall x_0 \in R[Q, \rho], \forall t \ge 0. \tag{3}$$

The following Proposition is basic [2], [9]. It provides a necessary and sufficient algebraic condition for positive invariance of $R[Q, \rho]$. An original proof of this proposition is presented in this note. It requires the following definition, which, in particular, can be found in [7].

Definition 3: Essentially Nonnegative Matrices. A matrix $M \in \Re^{m*m}$ is essentially nonnegative if $M_{ij} \ge 0 \, \forall j \ne i, 1 \le i, j \le m$.

Proposition 1: The polyhedral set $R[Q, \rho]$ is a positively invariant set of system (1) if and only if there exists an essentially nonnegative matrix $\mathscr{H} \in \Re^{r*r}$ such that:

$$QA - \mathscr{H}Q = 0 \tag{4}$$

$$\mathscr{H}\rho \le 0. \tag{5}$$

Proof:

Necessity: Implication (3) should be valid for any $x_0 \in R[Q, \rho]$. In particular, it is true for any border point. Consider an infinitesimal move from any point of the facet (2). In order for this move to be feasible, we must have:

$$\begin{cases} Q_i x = \rho_i \\ Q_j x \le \rho_j, \forall j \ne i \end{cases} \Rightarrow Q_i \dot{x} = Q_i Ax \le 0.$$

By Farkas' Lemma [8], a necessary and sufficient condition for this implication to be true is:

$$\begin{cases} \exists h_{ii} \in \Re \\ \exists h_{ij} \in \Re, \\ h_{ij} \ge 0 \text{ for } j \ne i \end{cases} \text{such that} \begin{cases} \sum_{j=1}^{r} h_{ij} Q_j = Q_i A \\ \sum_{j=1}^{r} h_{ij} \rho_j \le 0. \end{cases} \tag{6}$$

The row vector $h_i = [h_{i1} h_{i2} \cdots h_{ir}]$ is the dual of x for the considered facet. The combination of the r similar dual conditions (6) for $i = 1,\cdots,r$ can be expressed as relations (4) and (5).

Sufficiency: Assume that relations (4) and (5) are satisfied and consider the power series representation of e^{At}:

$$e^{At} = I + A\frac{t}{1!} + A^2\frac{t^2}{2!} + \cdots + A^n\frac{t^n}{n!} + \cdots.$$

Relation (4) implies $QA^k = \mathscr{H}^k Q, \forall k \in \mathscr{N}$. Thus,

$$Qe^{At} = \left(I + \mathscr{H}\frac{t}{1!} + \cdots + \mathscr{H}^n\frac{t^n}{n!} + \cdots\right)Q = e^{\mathscr{H}t}Q. \tag{7}$$

It is a classical result [7] that:

$$\mathscr{H} \text{ essentially nonnegative} \Leftrightarrow e^{\mathscr{H}t} \text{ nonnegative: } e^{\mathscr{H}t} \ge 0, \forall t \ge 0. \tag{8}$$

From (5), $\mathscr{H}\rho = -\rho_1$ with $\rho_1 \in \Re^r, \rho_1 \ge 0$. Then, for any nonnegative value of t, $(e^{\mathscr{H}t} - I)\rho$ can be expanded as follows:

$$\begin{aligned}(e^{\mathscr{H}t} - I)\rho &= \left(t + \mathscr{H}\frac{t^2}{2} + \cdots + \frac{\mathscr{H}^n}{n!}\frac{t^{n+1}}{n+1} + \cdots\right)\mathscr{H}\rho \\ &= -\int_0^t e^{\mathscr{H}\tau}\rho_1 \, d\tau. \end{aligned} \tag{9}$$

But since vector ρ_1 is nonnegative and matrix $e^{\mathscr{H}\tau}$ is nonnegative for any value of τ such that $0 \le \tau \le t$, relation (9) implies, for all $t \ge 0$:

$$e^{\mathscr{H}t}\rho \le \rho. \tag{10}$$

Now, by the extended Farkas' Lemma [4], conditions (7), (8), and (10) are the equivalent dual formulation of the invariance property (3) for $R[Q, \rho]$. □

Remarks:

1) Note the equivalence between the two sets of conditions (4), (5) and (7), (10). For linear systems, the "local" invariance conditions are necessary and sufficient for the "global" invariance conditions to hold.

2) The invariance conditions (4) and (5) do not require any particular assumption on the rank of matrix Q and on the sign of the components of vector ρ. However, some additional properties can be obtained if it is assumed that $\operatorname{rank}(Q) = r \le n$ and that the zero-state belongs to the interior of the invariant domain $R[Q, \rho]$ (the components of vector ρ are then strictly positive). For $\rho > 0$, condition $\mathscr{H}\rho \le 0$ with $\mathscr{H}$ essentially nonnegative implies that $-\mathscr{H}$ is an M-matrix (see [7]). From a classical property of M-matrices, the real-parts of the eigenvalues of $\mathscr{H}$ are nonpositive. Under the assumption that $\operatorname{rank}(Q) = r \le n$ and $\mathscr{H}$ nonsingular, the restriction of A to $\Re^n/\mathscr{K}\text{er}\, Q$ is asymptotically stable.

Positive Invariance of Symmetrical Polyhedra: Consider the particular case of convex polyhedral domains which are symmetrical around the origin point. Let the symmetrical domain $S(G, \omega)$ be defined by:

$$\begin{aligned} S(G, \omega) = \{x \in \Re^n; -\omega \le Gx \le \omega\} &\text{ with } G \in \Re^{s*n}, \\ s \le n, \omega \in \Re^s, \quad &\omega_i > 0 \text{ for } i = 1,\cdots,s. \end{aligned} \tag{11}$$

The following proposition can be used for characterizing the positive invariance of $S(G, \omega)$. It is a particular consequence of the invariance conditions (4) and (5) for general polyhedral domains, presented in Section II.

Proposition 2: The convex symmetrical polyhedron $S(G, \omega)$ is positively invariant for system (1) if and only if there exists a matrix $K \in \Re^{s*s}$ and a scalar $s_0 > 0$ such that:

$$(-s_0 I_s + K)G = GA \tag{12}$$

$$(-s_0 I_s + |K|)\omega \le 0. \tag{13}$$

By definition, the components of matrix $|K|$ are the absolute values of the components of matrix K. The proof of this proposition is similar to the one presented in [1] in the discrete-time case.

Relation (12) can be equivalently replaced by:

$$HG = GA \text{ with, by definition } H = -s_0 I_q + K. \tag{14}$$

If we now assume $\operatorname{rank}(G) = s$, conditions (12) and (13) can be extended to the case of nonsymmetrical polyhedra of the following form: $-\omega_m \le Gx \le \omega_M$ with $\omega_m \in \Re^s_+$, $\omega_M \in \Re^s_+$. Positive invariance of the domain defined by these constraints is equivalent to the existence of two nonnegative matrices K^+ and K^-, both in $\Re^{s*s}$, and a scalar $s_0 > 0$ such that:

$$[-s_0 I_s + (K^+ - K^-)]G = GA \tag{15}$$

$$\left[-s_0 I_{2s} + \begin{bmatrix} K^+ & K^- \\ K^- & K^+ \end{bmatrix}\right]\begin{bmatrix} \omega_M \\ \omega_m \end{bmatrix} \le 0. \tag{16}$$

II. Geometric and Spectral Properties for the Existence of Positively Invariant Symmetrical Polyhedra

Assuming that matrix $G \in \Re^{s*n}$, with $s \le n$, is full rank, relation (14) can be interpreted as a canonical projection relation of the map represented by A on the subspace $(\Re^n/\mathscr{K}\text{er } G)$. And, from the following Lemma, the map represented by matrix H describes the restriction of A to $(\Re^n/\mathscr{K}\text{er } G)$.

Lemma 1: A necessary and sufficient condition for the existence of a matrix $H \in \Re^{s*s}$ satisfying $HG = GA$ is that $\mathscr{K}\text{er } G$ is an A-invariant subspace.

Note that from any matrix $H \in \Re^{s*s}$ and any positive scalar s_0, it is always possible to construct a matrix $K = H + s_0 I_s$. Relation (14) also corresponds to a particular eigenstructure, described in the following corollary.

Corollary 1: If $\mathscr{K}\text{er } G$ is A-invariant, to any set of generalized real eigenvectors of the restriction of A to $(\Re^n/\mathscr{K}\text{er } G)$, represented by the columns of a nonsingular matrix $E \in \Re^{s*s}$, there corresponds a set of generalized real eigenvectors of matrix A, represented by the columns of a matrix $[V1 \mid V_2]$, with $V_1 \in \Re^{n*(n-s)}$ and $V_2 \in \Re^{n*s}$, such that:

$$G[V1 \mid V_2] = [0_{n*(n-s)} \mid E]. \tag{17}$$

Proof: If $\mathscr{K}\text{er } G$ is A-invariant, by Lemma 1, there exists a matrix $H \in \Re^{n*s}$ such that $HG = GA$. And consider a matrix $E \in \Re^{s*s}$ whose columns are eigenvectors of $H: HE = E\Lambda_2$. Let

$$\Lambda = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}$$

be the *real Jordan representation* of A:

$$A[V1 \mid \bar{V}_2] = [V1 \mid \bar{V}_2]\Lambda. \tag{18}$$

$\Lambda_1 \in \Re^{(n-s)*(n-s)}$ is the real Jordan representation of the restriction of A to $\mathscr{K}\text{er } G$;

$\Lambda_2 \in \Re^{s*s}$ the real Jordan representation of the restriction of A to $(\Re^n/\mathscr{K}\text{er } G)$;

$V_1 \in \Re^{n*(n-s)}$ a matrix whose columns span $\mathscr{K}\text{er } G$;

$\bar{V}_2 \in \Re^{n*s}$ a matrix whose columns span a subspace $R \subset \Re^n$ such that $R \oplus \mathscr{K}\text{er } G = \Re^n$.

Using the equality $HG = GA$, we obtain:

$$GA[V1 \mid \bar{V}_2] = HG[V1 \mid \bar{V}_2] = G[V1 \mid \bar{V}_2]\begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}.$$

In particular, we have: $HG\bar{V}_2 = G\bar{V}_2\Lambda_2$. Then, the columns of the nonsingular matrix $G\bar{V}_2 = \bar{E}$ are, necessarily, generalized real eigenvectors of matrix H. They span $\Re^{s*s}$ and satisfy: $H\bar{E} = \bar{E}\Lambda_2$. Now, replace $\bar{E}$ by EQ, with Q a nonsingular matrix in $\Re^{s*s}$, to obtain: $Q^{-1}\Lambda_2 = \Lambda_2 Q^{-1}$. Right-hand multiplication of relation (18) by matrix

$$\begin{bmatrix} I_{n-s} & 0 \\ 0 & Q^{-1} \end{bmatrix}$$

yields:

$$A[V1 \mid \bar{V}_2 Q^{-1}] = [V1 \mid \bar{V}_2 Q^{-1}]\begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}.$$

The matrix of eigenvectors $V = [V_1 \mid V_2]$, with $V_2 = \bar{V}_2 Q^{-1}$, satisfies condition (17). □

Let us now investigate the possible use of the real Jordan canonical representation of matrix A, $\Lambda = V^{-1}AV$, for constructing invariant sets. The rows of V^{-1} are left generalized real eigenvectors of A. Blocks $L_1, \cdots, L_{p_1}$ are associated with the p_1 (not necessarily distinct) real eigenvalues of A. Each block L_i concerns the eigenvalue λ_i with the order of multiplicity q_i in this block. Blocks D_m, $(m = 1, \cdots, p_2)$ are associated with the p_2 (not necessarily distinct) couples of complex eigenvalues $\mu_m \pm j\sigma_m$, with order of multiplicity s_m in this block:

$$D_m = \begin{bmatrix} \mu_m & -\sigma_m & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ \sigma_m & \mu_m & 0 & 1 & \cdot & & & \cdot \\ 0 & 0 & \ddots & \ddots & \ddots & & & \cdot \\ \cdot & \cdot & & \ddots & \cdot & \cdot & 0 & 1 \\ \cdot & \cdot & & & \cdot & 0 & \mu_m & -\sigma_m \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & \sigma_m & \mu_m \end{bmatrix}.$$

To each such block D_m corresponds a matrix $G_m^c \in \Re^{2s_m*n}$ which satisfies (19). Its rows constitute an associated chain of left generalized real eigenvectors:

$$G_m^c A = D_m G_m^c. \tag{19}$$

Lemma 2: Let $\mu_m \pm j\sigma_m$ be a couple of complex conjugate eigenvalues of A. A necessary and sufficient condition for the existence of a strictly positive vector ω_m^c such that $S(G_m^c, \omega_m^c)$ is a positively invariant set of system (1), is:

$$\mu_m \le -|\sigma_m| \tag{20}$$

under the additional condition that if $\mu_m = -|\sigma_m|$, $s_m = 1$.

Proof:

Necessity: Suppose that $S(G_m^c, \omega_m^c)$ is a positively invariant symmetrical set of system (1). Then, from Proposition 2, there exists a matrix $K_m \in \Re^{2s_m*2s_m}$ and a scalar $s_0 > 0$ such that:

$$(-s_0 I_{2s_m} + K_m)G_m^c = G_m^c A \tag{21}$$

$$(-s_0 I_{2s_m} + |K_m|)\omega_m^c \le 0. \tag{22}$$

Under the assumption of independence of the set of generalized eigenvectors, the left kernel of G_m^c is {0} and from (19) and (21), we obtain: $D_m = -s_0 I_{2s_m} + K_m$. The existence of a vector $\omega_m^c \in \Re^{2s_m}$ with positive components such that $J_m \omega_m^c \ge 0$ where $J_m = s_0 I_{2s_m} - |K_m|$, implies that J_m is an M-matrix. Consider any matrix $K \in \Re^{n*n}$. From the Perron–Frobenius theorem, the spectral radius of $|K|, r(|K|)$, is an eigenvalue of $|K|$. Then, $s_0 - r(|K|)$ is an eigenvalue of $J = s_0 I_n - |K|$. If J is an M-matrix we must have $s_0 - r(|K|) \ge 0$. The converse property is obviously true since $s_0 - r(|K|)$ is the smallest real part of the eigenvalues of J. Therefore, $J = s_0 I_n - |K|$ is an M-matrix if and only if $s_0 \ge r(|K|)$. By construction, $|K_m|$ has the same structure as D_m. The eigenvalues corresponding to block $|K_m|$ are real:

$$\begin{cases} \lambda_m^1 = |s_0 + \mu_m| + |\sigma_m| \\ \lambda_m^2 = |s_0 + \mu_m| - |\sigma_m| \end{cases}.$$

In order for J_m to be an M-matrix, we should have: $s_0 \ge r(|D_m|) = |\lambda_m^1|$. But, from the definition of the spectral radius, we also have: $r(|D_m|) \ge r(D_m) \ge |\mu_m|$. Therefore, $s_0 \ge -\mu_m$ and $|s_0 + \mu_m| = (s_0 + \mu_m)$. Condition $\mu_m \le -|\sigma_m|$ is thus required for the eigenvalues of J_m to be nonnegative.

Sufficiency: Now, assume that condition (20) is satisfied with a strict inequality and consider the block D_m. Select $s_0 > -\mu_m$ and define $K_m = s_0 I_{2s_m} + D_m$ and $J_m = s_0 I_{2s_m} - |K_m|$. Then, J_m has the same structure as D_m, with nonpositive off-diagonal terms, and its eigenvalues are real and positive: $s_0 - |s_0 + \mu_m| \pm |\sigma_m| = -\mu_m \pm |\sigma_m| > 0$. This implies that J_m is an M-matrix and there exists $\omega_m^c > 0$ such that:

$$J_m \omega_m^c \ge 0 \tag{23}$$

ω_m^c being a positive vector. The equality in (23) is true only when $\mu_m = -|\sigma_m|$, and the system of equations $J_m \omega_m^c = 0$ has a solution only if $s_m = 1$. The positive two-component vector ω_m^c should then satisfy: $\omega_m^c = [\tau \ \tau]^T$, for any positive τ.

From Proposition 2, relations (19) and (23) imply the positive invariance of the symmetrical domain $S(G_m^c, \omega_m^c)$. □

The strictly positive vector ω_m^c can be constructed as follows:

- If $s_m = 1$, $\omega_m^c = [\tau_1 \ \tau_2]^T$ should satisfy:
$$\begin{cases} -\mu_m\tau_1 - |\sigma_m|\tau_2 \geq 0 \\ -|\sigma_m|\tau_2 - \mu_m\tau_1 \geq 0 \end{cases}.$$
- If $s_m > 1$, set: $\omega_m^c = [\omega_{m1} \ \omega_{m2} \cdots \omega_{ms_m}]^T$, with $\omega_{mj} = [\tau_{j1} \ \tau_{j2}]^T$, for $j = 1, \cdots, s_m$; and ω_{mj} should satisfy, for $j = 1, \cdots, s_m$:
$$\begin{cases} -\mu_m\tau_{j1} - |\sigma_m|\tau_{j2} > \tau_{(j+1)1} \\ -|\sigma_m|\tau_{j2} - \mu_m\tau_{j1} > \tau_{(j+1)2} \end{cases},$$
with by convention $\tau_{(s_m+1)1} = \tau_{(s_m+1)2} = 0$.

Similarly, to each block L_i corresponds a matrix $G_i^r \in \Re^{q_i * n}$ such that: $G_i^r A = L_i G_i^r$. By a similar proof as above, the following lemma can be obtained.

Lemma 3: Let λ_i be a real eigenvalue of A. A necessary and sufficient condition for the existence of a strictly positive vector ω_i^r such that $S(G_i^r, \omega_i^r)$ is a positively invariant set of system (1), is that λ_i is a stable eigenvalue of A.

Any candidate vector ω_i^r should satisfy the following conditions:

- If the order of multiplicity of λ_i in the associated block of H is $q_i = 1$, ω_i^r can be any positive number.
- If $q_i > 1$, define the strictly positive number $\epsilon_i = |\lambda_i|$ and set $\omega_i^r = [\omega_{i1} \ \omega_{i2} \cdots \omega_{iq_i}]^t$. The following relations should be satisfied: $\omega_{i1} > (\omega_{i2}/\epsilon_i), \cdots, \omega_{iq_i - 1} > (\omega_{iq_i}/\epsilon_i)$. Now, it is easy to derive the following Proposition.

Proposition 3: A necessary and sufficient condition for positive invariance of the symmetrical n-polytope $S(G, \omega) = \{x \in \Re^n; -\omega \leq Gx \leq \omega\}$ with $G = V^{-1}$ any matrix of left generalized real eigenvectors, and ω a suitable vector in $\Re_+^n$, is that all the eigenvalues of A (real and complex), denoted $\mu_i + j\sigma_i$, satisfy:

$$\mu_i \leq -|\sigma_i|, \text{ for } i = 1, \cdots, p_1 + p_2 \tag{24}$$

provided that the eigenvalues for which relation (24) becomes an equality correspond to simple blocks in the real Jordan representation of A.

III. Imposing the Positive Invariance of a Given Polyhedron by Eigenstructure Assignment

Consider a controllable linear continuous-time dynamical system under a closed-loop linear regulation law $u(t) = Fx(t)$:

$$\dot{x}(t) = (A + BF)x(t) \text{ with } B \in \Re^{n*m}, \quad m \leq n \text{ and } \operatorname{rank}(B) = m. \tag{25}$$

Without loss of generality, any given symmetrical polyhedron having the zero state as an interior point can be put under the form $S(G, 1_s)$, with $G \in \Re^{s*n}$, $1_s = [1 \cdots 1]^T$, by appropriately scaling the norms of the row-vectors of G. An efficient way of satisfying the state constraints $x(t) \in S(G, 1_s) \forall t \geq 0$, from any feasible initial state, is to impose, if possible, the positive invariance of the polyhedron $S(G, 1_s)$ for system (25).

The gain matrix to be constructed should also drive the state vector to zero with the desired dynamics and enough robustness. But the main purpose of this section is essentially to analyze the possibility of making a given polyhedron invariant by state-feedback. The previous results on the existence of invariant polyhedra can be applied to the closed-loop system (25). In particular the condition of A-invariance of $\mathscr{K}\text{er}\, G$ in Lemma 1 is now replaced by the condition of (A, B)-invariance defined in [10]: a subspace $\mathscr{S}$ is (A, B)-invariant if and only if there exists a gain matrix F (called a "friend" of $\mathscr{S}$) such that $\mathscr{S}$ is $(A + BF)$-invariant.

Let $P(\lambda)$ be the *system matrix* defined by:

$$P(\lambda) = \begin{bmatrix} \lambda I - A & -B \\ G & 0_{s*m} \end{bmatrix}.$$

$(A + BF)$-invariance of $\mathscr{K}\text{er}\, G$ can be obtained if and only if one can find $(n - s)$ finite frequencies λ_i and $(n - s)$ independent associated state directions (v_i), with their input directions $w_i = Fv_i$, satisfying:

$$P(\lambda_i)\begin{bmatrix} v_i \\ w_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{26}$$

The set of finite complex frequencies λ_i which make $P(\lambda_i)$ rank deficient, define the *invariant zeros* of system (A, B, G). To each invariant zero with algebraic multiplicity γ_i, there corresponds an invariant subspace with dimension γ_i included in $\mathscr{K}\text{er}\, G$ and spanned by γ_i independent generalized (or pseudo) zero-directions [6].

When applied to the problem of positive invariance of $S(G, 1_s)$, these results on zeros and zero-directions lead to the following proposition.

Proposition 4: If system (A, B) is controllable and if $s \leq m$, with $G \in \Re^{s*n}$, $\operatorname{rank}(GB) = s$, it is always possible to construct a gain matrix $F \in \Re^{m*n}$ for which the polyhedron $S(G, 1_s)$ is positively invariant for the closed-loop system (25). Stability of the closed-loop system (25) can also be obtained if system (A, B, G) has no unstable invariant zeros.

Proof: The proof of this result is constructive.

If $s = m$, the solutions of (26) are the invariant zeros of (A, B, G). Under condition $\operatorname{rank}(GB) = s$, (26) has exactly $n - s$ finite invariant zeros with $n - s$ associated state zero directions spanning $\mathscr{K}\text{er}\, G$.

In the case $s < m$ and $\operatorname{rank}(GB) = s$, (26) may have two types of solutions ([3]):

- finite invariant zeros, which are the frequencies λ_i for which $P(\lambda_i)$ is rank-deficient. The associated zero directions span a p dimensional subspace of $\mathscr{K}\text{er}\, G$, with $p \leq s$,
- any stable complex frequency and its associated directions satisfying (26); $s - p$ independent state directions associated with such "controllable zeros" can be generated, to span a complementary subspace of $\mathscr{K}\text{er}\, G$ with dimension $s - p$.

The $(A + BF)$-invariance of $\mathscr{K}\text{er}\, G$ can then be obtained by locating $n - s$ closed-loop eigenvalues and associated (generalized) eigenvectors at $n - s$ independent solutions of (26). But positive invariance of $S(G, 1_s)$ and stability of the closed-loop system are then simultaneously obtained only if all the invariant zeros are stable (located in the left complex half-plane).

Thus, if $s \leq m$, $\operatorname{rank}(GB) = s$ and if all the invariant zeros of (A, B, G) are stable, the requirement of closed-loop positive invariance of $S(G, 1_s)$ can then be met by a two-stage eigenstructure assignment technique:

1) Assignment of $n - s$ generalized eigenvectors in $\mathscr{K}\text{er}\, G$. These vectors are the columns of a matrix $V_1 \in \Re^{n*(n-s)}$ such that $GV_1 = 0$ and $\Lambda_1 V_1 = AV_1 + BW_1$, with $\Lambda_1 \in \Re^{(n-s)*(n-s)}$ under the real Jordan form.

2) Assignment of s generalized eigenvectors in a complementary space of $\mathscr{K}er\,G$. They constitute the columns of a matrix $V_2 \in \Re^{n*s}$ such that $GV_2 = I_{n-s}$ and $\Lambda_2 V_2 = AV_2 + BW_2$, with $\Lambda_2 \in \Re^{s*s}$ under the real Jordan form. The eigenvalues of Λ_2 are selected different from the eigenvalues of Λ_1. Furthermore, each of them is simple and satisfies conditions (24).

The feedback gain matrix that produces this assignment is $F = [W_1\ W_2][V_1\ V_2]^{-1}$. This placement scheme possesses the property:

$$G[V_1\ V_2] = [0\ \ I_{n-s}]. \tag{27}$$

The row vectors of G constitute a set of left generalized real eigenvector of $A + BF$ associated with Λ_2. The real Jordan matrix Λ_2 can be selected as a candidate matrix H such that: $HG = G(A + BF)$. And from the proofs of Lemmas 2 and 3 and Proposition 3, there exists a positive scalar s_0 and a matrix K such that $\Lambda_2 = -s_0 I_s + K$ and $(-s_0 I_s + |K_2|)1_s \le 0$. The two conditions of Proposition 2 are satisfied. $S(G, 1_s)$ is a positively invariant symmetrical polyhedron of system (25). □

If the triplet (A, B, G) has unstable invariant zeros, the scheme based on the closed-loop positive invariance of $S(G, 1_s)$ cannot be directly applied. But it is possible to construct a positively invariant symmetrical polytope that perfectly sticks to the constraints, as described in the following proposition.

Proposition 5: If the triplet (A, B, G) has unstable invariant zeros, the constrained regulation problem can be solved by restricting the initial states of the system to a positively invariant symmetrical polytope $S(R, 1_n)$, where R is obtained by adding $n - s$ independent row vectors to G.

Proof: In this nonminimum phase case, a state-feedback eigenstructure assignment technique can still be used. A set of stable closed-loop eigenvalues is selected. This set may include the stable zeros of (A, B, G). The associated generalized real eigenvectors will have to satisfy some relations (33) which differ from (27). So far, let us only impose that the matrix of generalized real eigenvectors can be decomposed as $V = [V_1\ V_2]$. The column vectors of V_1 and V_2 span two independent complementary subspaces with dimensions s and $n - s$. The matrix of left generalized real eigenvectors is accordingly decomposed. It satisfies:

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}(A + BF) = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}. \tag{28}$$

Moreover, the eigenvalues of $\Lambda_2 \in \Re^{s*s}$ and of $\Lambda_1 \in \Re^{(n-s)*(n-s)}$ are all selected simple. They satisfy the spectral conditions $\mu_i \le -|\sigma_i|$, and this inequality is strict for the eigenvalues of Λ_2. Then, there exists a positive scalar s_0 and two matrices K_1, K_2 such that $\Lambda_1 = -s_0 I_s + K_1$, $\Lambda_2 = -s_0 I_s + K_2$,

$$(-s_0 I_s + |K_1|)1_s \le 0 \tag{29}$$

and it is possible to construct a non null matrix, $M \in \Re^{s*(n-s)}$ such that:

$$|M|1_{n-s} + (s_0 I_s + |K_2|)1_s \le 0. \tag{30}$$

Under such an assignment, $S(Q_2, 1_s)$ is a positively invariant symmetrical polyhedron of the closed-loop system. From a matrix M satisfying (30), compute a matrix $E \in \Re^{s*(n-s)}$ satisfying the equation:

$$E\Lambda_1 - \Lambda_2 E = M. \tag{31}$$

This equation can be solved column by column by as assignment type equation: $e_j = (\lambda_{1j} I_s - \Lambda_2)^{-1} m_j$. Relation (31) implies:

$$\begin{bmatrix} I_{n-s} & 0 \\ E & I_s \end{bmatrix}\begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} = \begin{bmatrix} \Lambda_1 & 0 \\ M & \Lambda_2 \end{bmatrix}\begin{bmatrix} I_{n-s} & 0 \\ E & I_s \end{bmatrix}. \tag{32}$$

The matrix of generalized real eigenvectors, $V = [V_1\ V_2]$, can be selected so as to satisfy:

$$G[V1\ V_2] = [E\ \ I_s]. \tag{33}$$

Then, the matrix of left generalized real eigenvectors satisfies:

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} G = [E\ I_s]\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}.$$

Relations (28) and (32) yield:

$$\begin{bmatrix} Q_1 \\ G \end{bmatrix}(A + BF) = \begin{bmatrix} \Lambda_1 & 0 \\ M & \Lambda_2 \end{bmatrix}\begin{bmatrix} Q_1 \\ G \end{bmatrix}.$$

This relation, combined with relations (29) and (30) shows the positive invariance of the symmetrical polytope

$$S\left(\begin{bmatrix} Q_1 \\ G \end{bmatrix}, 1_n\right)$$

for the closed-loop system. □

Note that if the spectral conditions (24) are satisfied by all the closed-loop eigenvalues, the assignment technique described in the proof of Proposition 4 also defines some positively invariant symmetrical polytopes $S(V^{-1}, \rho)$. And for the robustness concern, the choice of a compact set of admissible initial states is usually preferable ([5]). Conversely, the positively invariant domain constructed in the proof of Proposition 5 is compact. But if we want to enlarge the set of admissible initial states, we can generally set to zero some columns of matrix M. Some unbounded symmetrical polyhedral domains included in $S(G, 1_s)$ are then also constructed by the assignment technique presented in the proof of Proposition 5.

In any practical case, the choice of the closed-loop eigenstructure can result from a trade-off between maximizing the robustness of the control scheme and maximizing the size of the set of admissible initial states.

Example: Consider the following data:

$$A = \begin{bmatrix} 9.10 & 0.47 & -6.33 \\ 7.62 & 0.00 & 7.56 \\ 2.62 & -3.28 & 9.91 \end{bmatrix} \quad B = \begin{bmatrix} 1.82 & 3.61 \\ 1.24 & -3.77 \\ -4.91 & 0.00 \end{bmatrix}.$$

The open-loop system has eigenvalues:

$$\begin{cases} 9.50 \\ 4.75 \pm j3.88 \end{cases}.$$

Constraints on the state vector are defined by:

$$G = \begin{bmatrix} 5.69 & 1.97 & -1.68 \\ 2.24 & -1.68 & 5.59 \end{bmatrix} \text{ and } g = \begin{bmatrix} 8.75 \\ 10.50 \end{bmatrix}.$$

System $\mathscr{S}(A, B, G)$ has a stable zero $\lambda_1 = -3.00$. This value is selected as closed-loop eigenvalue, for which the associated eigenvector spans $\mathscr{K}er\,G$. The two other poles have been chosen as follows:

$$\begin{cases} \lambda_1' = -3.00 + j2.00 \\ \lambda_2' = -3.00 - j2.00 \end{cases}.$$

Then, we obtain

$$A + BF = \begin{bmatrix} -8.06 & -1.26 & 0.26 \\ 22.81 & -0.61 & 7.22 \\ 6.87 & 0.52 & -0.32 \end{bmatrix},$$

$$F = \begin{bmatrix} -0.86 & -0.77 & 2.08 \\ -4.32 & -0.09 & 0.77 \end{bmatrix}.$$

Fig. 1 shows the projection of the positively invariant polyhedron and of a trajectory respecting the constraints. x_1' and x_2' are the

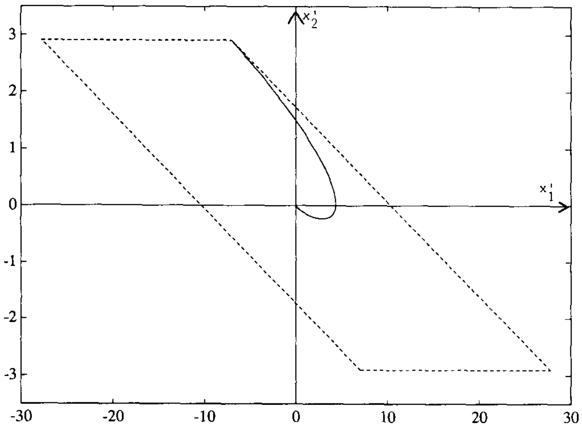

Fig. 1. A trajectory in projection.

coordinates of the state vector in an orthonormal basis of the plan spanned by the last two columns of V_2 for the initial condition: $x_0 = [1.88\ \ -0.01\ \ 1.12]^T$.

IV. Conclusion

Positive invariance of some domains (polyhedral or not) is a generic property of any dynamical system having a stable restriction in some subspace. For continuous-time linear systems, it has been shown that under a condition slightly stronger than the stability of the restriction mapping, the system also admits some easily constructed positively invariant symmetrical polyhedra. For this property to hold, the eigenvalues of the restriction have to belong to a particular domain which has been characterized. This condition has also been specialized in the case where the right terms of the inequalities defining the polyhedron are given by the magnitude of constraints on the state-vector. Moreover, the problem of a controllable continuous-time linear system subject to symmetrical linear constraints on its state vector has been solved in this note. The proposed eigenstructure assignment algorithm provides the closed-loop system with the structural and spectral conditions for positive invariance of the domain of constraints or of a domain included in it.

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On Some Asymptotic Uncertainty Bounds in Recursive Least Squares Identification

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Abstract—**This note deals with the performance of the recursive least squares algorithm when it is applied to problems where the measured signal is corrupted by bounded noise. Using ideas from bounding ellipsoid algorithms we derive an asymptotic expression for the bound on the uncertainty of the parameter estimate for a simple choice of design variables. This bound is also transformed to a bound on the uncertainty of the transfer function estimate.**

I. Introduction

The recursive least squares (RLS) algorithm is a well known tool in automatic control and signal processing, and its properties have been thoroughly investigated. See, for example, [1]. When analyzing the properties of the RLS algorithm this is typically done in a probabilistic framework, where the disturbance acting on the system is described as a stochastic process. Measures of the model quality are then obtained in terms of, for example, the variance of the estimated parameters or the estimated transfer function.

In this note we shall however investigate the RLS algorithm in a different framework by assuming that the disturbance is bounded. This will be done by applying ideas from so called bounding ellipsoid algorithms. These algorithms are however only one group of methods dealing with bounded disturbances. For more general presentations of the topic we refer to, for example, [2] and [3]. In the analysis of the RLS method presented below we shall concentrate on the case when the design variables are chosen as $\lambda(t) \equiv 1$ and $\alpha(t) \equiv \alpha$, which corresponds to equal weighting of all measurements.

II. System and Signal Description

We shall consider time invariant systems that can be described by the linear regression

$$y(t) = \varphi^T(t)\theta + v(t), \tag{1}$$

where $y(t)$ and $v(t)$ denote output and disturbance signal, respectively. The vector θ represents the unknown parameters, and the regression vector $\varphi(t)$ contains delayed input and output signals, i.e.,

$$\varphi(t) = (-y(t-1), \cdots, -y(t-n), u(t-1), \cdots, u(t-n))^T. \tag{2}$$

Both the input signal $u(t)$ and the disturbance signal are assumed to be quasi stationary, see [4], which implies that the

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