Approximating the Existential Theory of the Reals

Argyrios Deligkas[1,2*], John Fearnley[1†], Themistoklis Melissourgos[1], and Paul G. Spirakis[1,3‡]

[1] Department of Computer Science, University of Liverpool, UK
{John.Fearnley,T.Melissourgos,P.Spirakis}@liverpool.ac.uk
[2] Leverhulme Research Centre for Functional Materials Design, Liverpool, UK
{Argyrios.Deligkas}@liverpool.ac.uk
[3] Computer Engineering and Informatics Department, University of Patras, Greece

Abstract. The existential theory of the reals (ETR) consists of existentially quantified boolean formulas over equalities and inequalities of real-valued polynomials. We propose the approximate existential theory of the reals (ϵ-ETR), in which the constraints only need to be satisfied approximately. We first show that unconstrained ϵ-ETR = ETR, and then study the ϵ-ETR problem when the solution is constrained to lie in a given convex set. Our main theorem is a sampling theorem, similar to those that have been proved for approximate equilibria in normal form games. It states that if an ETR problem has an exact solution, then it has a k-uniform approximate solution, where k depends on various properties of the formula. A consequence of our theorem is that we obtain a quasi-polynomial time approximation scheme (QPTAS) for a fragment of constrained ϵ-ETR. We use our theorem to create several new PTAS and QPTAS algorithms for problems from a variety of fields.

1 Introduction

Sampling techniques. The Lipton-Markakis-Mehta algorithm (LMM) is a well known method for computing approximate Nash equilibria in normal form games [28]. The key idea behind their technique is to prove that there exist approximate Nash equilibria where both players use *simple* strategies.

Suppose that we have a convex set $C = \text{conv}(c_1, c_2, \dots, c_l)$ defined by vectors c_1 through c_l. A vector $x \in C$ is *k-uniform* if it can be written as a sum of the form $(\beta_1/k)\cdot c_1 + (\beta_2/k)\cdot c_2 + \cdots + (\beta_l/k)\cdot c_l$, where each β_i is a non-negative integer and $\sum_{i=1}^{l} \beta_i = k$.

Since there are at most $l^{O(k)}$ k-uniform vectors, one can enumerate all k-uniform vectors in $l^{O(k)}$ time. For approximate equilibria in $n \times n$ bimatrix games, Lipton, Markakis, and Mehta showed that for every $\epsilon > 0$ there exists an ϵ-Nash equilibrium where both players use k-uniform strategies where $k \in O(\log n/\epsilon^2)$, and so they obtained a quasi-polynomial approximation scheme (QPTAS) for finding an ϵ-Nash equilibrium.

Their proof of this fact uses a sampling argument. Every bimatrix game has an exact Nash equilibrium (NE), and each player's strategy in this NE is a probability distribution. If we sample from each of these distributions k times, and then construct new k-uniform strategies using these samples, then when $k \in O(\log n/\epsilon^2)$ there is a positive probability the new strategies form an ϵ-NE. So by the probabilistic method, there must exist a k-uniform ϵ-NE.

The sampling technique has been widely applied. It was initially used by Althöfer [1] in zero-sum games, before being applied to non-zero sum games by Lipton, Markakis, and Mehta [28]. Subsequently, it was used to produce algorithms for finding approximate equilibria in normal form games with many players [3], sparse bimatrix games [4], tree polymatrix [5], and Lipschitz games [20]. It has also been used to find constrained approximate equilibria in polymatrix games with bounded treewidth [18].

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At their core, each of these results uses the sampling technique in the same way as the LMM algorithm: first take an exact solution to the problem, then sample from this solution k times, and finally prove that with positive probability the sampled vector is an approximate solution to the problem. The details of the proofs, and the value of k, are often tailored to the specific application, but the underlying technique is the same.

The existential theory of the reals. In this paper we ask the following question: *is there a broader class of problems to which the sampling technique can be applied?* We answer this by providing a sampling theorem for the existential theory of the reals. The existential theory of the reals consists of existentially quantified formulae using the connectives $\{\wedge, \vee, \neg\}$ over polynomials compared with the operators $\{=, \leq, <, \geq, >\}$. For example, each of the following is a formula in the existential theory of the reals.

$$\exists x \exists y \exists z \cdot (x = y) \wedge (x > z) \qquad\qquad \exists x \cdot (x^2 = 2)$$
$$\exists x \exists y \cdot \neg(x^{10} = y^{100}) \vee (y \geq 4) \qquad\qquad \exists x \exists y \exists z \cdot (x^2 + y^2 = z^2)$$

Given a formula in the existential theory of the reals, we must decide whether the formula is *true*, that is, whether there do indeed exist values for the variables that satisfy the formula.

The complexity class ETR is defined to be all problems that can be reduced in polynomial time to the existential theory of the reals. It is known that ETR $\subseteq$ PSPACE [12], and NP $\subseteq$ ETR since the problem can easily encode Boolean satisfiability. However, the class is not known to be equal to either PSPACE or NP, and it seems to be a distinct class of problems between the two. Many problems are now known to be ETR-complete, including various problems involving constrained equilibria in normal form games with at least three players [6,7,8,9,22].

Our contribution. In this paper we propose the *approximate* existential theory of the reals (ϵ-ETR), where we seek a solution that approximately satisfies the constraints of the formula. We show a subsampling theorem for a large fragment of ϵ-ETR, which can be used to obtain PTASs and QPTASs for the problems that lie within it. We believe that this will be useful for future research: instead of laboriously reproving subsampling results for specific games, it now suffices to simply write a formula in ϵ-ETR and then apply our theorem to immediately get the desired result. To exemplify this, we prove several new QPTAS and PTAS results using our theorem.

Our first result is actually that ϵ-ETR = ETR, meaning that the problem of finding an approximate solution to an ETR formula is as hard as finding an exact solution. However, this result crucially relies on the fact that ETR formulas can have solutions that are doubly-exponentially large. This motivates the study of *constrained* ϵ-ETR, where the solutions are required to lie within a given convex set.

Our main theorem (Theorem 2) gives a subsampling result for constrained ϵ-ETR. It states that if the formula has an exact solution, then it also has a k-uniform approximate solution, where the value of k depends on various parameters of the formula, such as the number of constraints and the number of variables. The theorem allows for the formula to be written using *tensor* constraints, which are a type of constraint that is useful in formulating game-theoretic problems.

The consequence of the main theorem is that, when various parameters of the formula are constant (see Corollary 1), we are able to obtain a QPTAS for approximating the existential theory of the reals. Specifically, this algorithm either finds an approximate solution of the constraints, or verifies that no exact solution exists. In many game theoretic applications an exact solution always exists, and so this algorithm will always find an approximate solution.

It should be noted that we are not just applying the well-known subsampling techniques in order to derive our main theorem. Our main theorem incorporates a new method for dealing with polynomials of degree d, which prior subsampling techniques were not able to deal with.

Our theorem can be applied to a wide variety of problems. In the game theoretic setting, we prove new results for constrained approximate equilibria in normal form games, and approximating the value vector of a Shapley game. We also show optimization results. Specifically, we give approximation algorithms for optimizing polynomial functions over a convex set, subject to polynomial constraints. We also give algorithms for approximating eigenvalues and eigenvectors of tensors. Finally, we apply the theorem to some problems from computational geometry.

2 The Existential Theory of the Reals

Let $x_1, x_2, \ldots, x_q \in \mathbb{R}$ be distinct variables, which we will treat as a vector $x \in \mathbb{R}^q$. A *term* of a multivariate polynomial is a function $T(x) := a \cdot x_1^{d_1} \cdot x_2^{d_2} \cdot \cdots \cdot x_q^{d_q}$, where $d_1, d_2, \ldots, d_q$ are non negative integers and $a \in \mathbb{R}$. A multivariate polynomial is a function $p(x) := T_1(x) + T_2(x) + \cdots + T_t(x) + c$, where each T_i is a term as defined above, and $c \in \mathbb{R}$ is a constant.

We now define *Boolean formulae* over multivariate polynomials. The atoms of the formula are polynomials compared with $\{<, \leq, =, \geq, >\}$, and the formula itself can use the connectives $\{\wedge, \vee, \neg\}$.

Definition 1. *The* existential theory of the reals *consists of every true sentence of the form* $\exists x_1 \exists x_2 \ldots \exists x_q \cdot F(x)$*, where F is a Boolean formula over multivariate polynomials of x_1 through x_q.*

Given a Boolean formula F, the ETR problem is to decide whether F is a true sentence in the existential theory of the reals. We will say that F has m constraints if it uses m operators from the set $\{<, \leq, =, \geq, >\}$ in its definition.

The approximate ETR. In the *approximate* existential theory of the reals, we replace the operators $\{<, \leq, \geq, >\}$ with their approximate counterparts. We define the operators $<_\epsilon$ and $>_\epsilon$ with the interpretation that $x <_\epsilon y$ holds if and only if $x < y + \epsilon$ and $x >_\epsilon y$ if and only if $x > y - \epsilon$. The operators $\leq_\epsilon$ and $\geq_\epsilon$ are defined analogously.

We do not allow equality tests in the approximate ETR. Instead, we require that every constraint of the form $x = y$ should be translated to $(x \leq y) \wedge (y \leq x)$ before being weakened to $(x \leq_\epsilon y) \wedge (y \leq_\epsilon x)$.

We also do not allow negation in Boolean formulas. Instead, we require that all negations are first pushed to atoms, using De Morgan's laws, and then further pushed into the atoms by changing the inequalities. So the formula $\neg((x \leq y) \wedge (a \geq b))$ would first be translated to $(x \geq y) \vee (a \leq b)$ before then being weakened to $(x \geq_\epsilon y) \vee (a \leq_\epsilon y)$.

Definition 2. *The approximate existential theory of the reals consists of every true sentence of the form $\exists x_1 \exists x_2 \ldots \exists x_q \cdot F(x)$, where F is a negation-free Boolean formula using the operators $\{<_\epsilon, \leq_\epsilon, \geq_\epsilon, >_\epsilon\}$ over multivariate polynomials of x_1 through x_q.*

Given a Boolean formula F, the ϵ-ETR problem asks us to decide whether F is a true sentence in the approximate existential theory of the reals, where the operators $\{<_\epsilon, \leq_\epsilon, \geq_\epsilon, >_\epsilon\}$ are used.

Unconstrained ϵ-ETR. Our first result is that if no constraints are placed on the value of the variables, that is if each x_i can be arbitrarily large, then ϵ-ETR = ETR for *all* values of $\epsilon \in \mathbb{R}$. We show this via a two way reduction between ϵ-ETR and ETR. The reduction from ϵ-ETR to ETR is trivial, since we can just rewrite each constraint $x <_\epsilon y$ as $x < y + \epsilon$, and likewise for the other operators.

For the other direction, we show that the ETR-complete problem Feas, which asks us to decide whether a system of multivariate polynomials $(p_i)_{i=1,\ldots,k}$ has a shared root, can be formulated in ϵ-ETR. Here we rely on a result of Schaefer and Stefankovic [30], which showed that Feas has a solution if and only if there is a point x such that $|p_i(x)| < 2^{-2^{L+5}}$ for all i, where L is the number of bits used to represent the polynomials. To formulate the problem in ϵ-ETR, we blow-up the instance by multiplying each polynomial by a doubly-exponentially large number t that is bigger than $\epsilon \cdot 2^{2^{L+5}}$. The number t can be constructed by a polynomially-sized formula that uses repeated squaring. So if we write down the constraint $t \cdot p_i(x) \leq_\epsilon 0$ in ϵ-ETR, then this implies that $t \cdot p_i(x) \leq \epsilon$ and therefore $p_i(x) < 2^{-2^{L+5}}$. Thus, via the lemma of Schaefer and Stefankovic, we can formulate Feas in the ϵ-ETR. The full details of this reduction are given in Appendix A.

Theorem 1. *ϵ-ETR = ETR for all $\epsilon \in \mathbb{R}$.*

Constrained ϵ-ETR. In our negative result for unconstrained ϵ-ETR, we abused the fact that variables could be arbitrarily large to construct the doubly-exponentially large number t. So, it makes sense to ask whether ϵ-ETR gets easier if we *constrain* the problem so that variables cannot be arbitrarily large.

In this paper, we consider ϵ-ETR problems that are constrained by a convex set in $\mathbb{R}^q$. For vectors $c_1, c_2, \ldots, c_l \in \mathbb{R}^q$ we use $\text{conv}(c_1, c_2, \ldots, c_l)$ to denote the set containing every vector that lies in the convex hull of c_1 through c_l. In the constrained ϵ-ETR, we require that the solution of the ϵ-ETR problem should also lie in the convex hull of c_1 through c_l.

Definition 3. *Given a Boolean formula F and vectors $c_1, c_2, \ldots, c_l \in \mathbb{R}^q$, the constrained ϵ-ETR problem asks us to decide whether*

$$\exists x_1 \exists x_2 \ldots \exists x_q \cdot \big(x \in \text{conv}(c_1, c_2, \ldots, c_l) \wedge F(x)\big).$$

Note that, unlike the constraints used in F, the convex hull constraints are not weakened. So the resulting solution $x_1, x_2, \ldots, x_q$, must actually lie in the convex set.

3 Approximating Constrained ϵ-ETR

Polynomial classes. To state our main theorem, we will use a certain class of polynomials where the coefficients are given as a tensor. This will be particularly useful when we apply our theorem to certain problems, such as normal form games. To be clear though, this is not a further restriction on the constrained ϵ-ETR problem, since all polynomials can be written down in this form.

The variables of the polynomials we will study will be p-dimensional vectors denoted as $x_1, x_2, \ldots, x_n$, where $x_j(i)$ will denote the i-th element of vector x_j. The coefficients of the polynomials will be a tensor denoted by A. Given a $\times_{j=1}^{n} p$ tensor A, we denote by $a(i_1, \ldots, i_n)$ its element with coordinates $(i_1, \ldots, i_n)$ on the tensor's dimensions $1, \ldots, n$, respectively, and by α we denote the maximum absolute value of these elements. We define the following two classes of polynomials.

– **Simple tensor multivariate.**
We will use $\text{STM}(A, x_1^{d_1}, \ldots, x_n^{d_n})$ denote an STM polynomial with n variables where each variable x_j, $j \in [n]$ is applied d_j times on tensor A that defines the coefficients. Tensor A has $\sum_{j=1}^{n} d_j$ dimensions with p indices each. We will say that an STM polynomial is of maximum degree d, if $d = \max_j d_j$. Here is an example of a degree 2 simple tensor polynomial with two variables:

$$\text{STM}(A, x^2, y) = \sum_{i=1}^{p} \sum_{j=1}^{p} \sum_{k=1}^{p} x(i) \cdot x(j) \cdot y(k) \cdot a(i, j, k) + 10.$$

This polynomial itself is written as follows.

$$\begin{aligned}&\text{STM}(A, x_1^{d_1}, \ldots, x_n^{d_n}) = \\ &\sum_{i_{1,1} \in [p]} \cdots \sum_{i_{n,d_n} \in [p]} (x_1(i_{1,1})) \cdot \ldots \cdot (x_1(i_{1,d_1})) \cdot \ldots \cdot (x_n(i_{n,1})) \cdot \ldots \cdot (x_n(i_{n,d_n})) \cdot \\ &\qquad\qquad \cdot a(i_{1,1}, \ldots, i_{1,d_1} \ldots, i_{n,1}, \ldots, i_{n,d_n}) + a_0.\end{aligned}$$

– **Tensor multivariate.** A tensor multivariate (TMV) polynomial is the sum over a number of simple tensor multivariate polynomials. We will use $\text{TMV}(x_1, \ldots, x_n)$ to denote a tensor multivariate polynomial with n vector variables, which is formally defined as

$$\text{TMV}(x_1, \ldots, x_n) = \sum_{i \in [t]} \text{STM}(A_i, x_1^{d_{i1}}, \ldots, x_n^{d_{in}}),$$

where the exponents $d_{i1}, \ldots, d_{in}$ depend on i, and t is the number of simple multivariate polynomials. We will say that $\text{TMV}(x_1, \ldots, x_n)$ has length t if it is the sum of t STM polynomials, and that it is of degree d if $d = \max_{i \in [t], j \in [n]} d_{ij}$.

ϵ-ETR with tensor constraints. We focus on ϵ-ETR instances F where all constraints are of the form $\mathrm{TMV}(x_1, \ldots, x_n) \bowtie 0$, where $\bowtie$ is an operator from the set $\{<_\epsilon, \leq_\epsilon, >_\epsilon, \geq_\epsilon\}$. Recall that each TMV constraint considers vector variables. We consider the number of variables used in F (denoted as n) to be the number of vector variables used in the TMV constraints. So the value of n used in our main theorem may be constant if only a constant number of vectors are used, even if the underlying ϵ-ETR instance actually has a non-constant number of variables. For example, if x and y and w are p-dimensional probability distributions and A_1 and A_2 are $p \times p$ tensors, the TMV constraint $x^T A_1 y + w^T A_2 x > 0$ has three variables, degree 1, length two; though the underlying problem has $3 \cdot p$ variables.

Note that every ϵ-ETR constraint can be written as a TMV constraint, because all multivariate polynomials can be written down as a TMV polynomial. Every term of a TMV can be written as a STM polynomial where the tensor entry is non zero for exactly the combination of variables used in the term, and 0 otherwise. Then a TMV polynomial can be constructed by summing over the STM polynomial for each individual term.

The main theorem. Given an ϵ-ETR formula F, we define $\mathrm{exact}(F)$ to be a Boolean formula in which every approximate constraint is replaced with its exact variant, meaning that every instance of $x \leq_\epsilon y$ is replaced with $x \leq y$, and likewise for the other operators.

Our main theorem is as follows.

Theorem 2. *Let F be an ϵ-ETR instance with n vector variables and m multivariate-polynomial constraints each one of maximum length t and maximum degree d, constrained by a convex set defined by $c_1, c_2, \ldots, c_l \in \mathbb{R}^{np}$. Let α be the maximum absolute value of the coefficients of constraints of F, and let $\gamma = \max_i \|c_i\|_\infty$. If $\mathrm{exact}(F)$ has a solution in $\mathrm{conv}(c_1, c_2, \ldots, c_l)$, then F has a k-uniform solution in $\mathrm{conv}(c_1, c_2, \ldots, c_l)$ where*

$$k = \frac{48 \cdot \alpha^6 \cdot \gamma^{2d+2} \cdot d^5 \cdot t^4 \cdot n^6 \cdot \ln(2 \cdot \alpha \cdot \gamma \cdot d \cdot t \cdot n \cdot m)}{\epsilon^4}.$$

Consequences of the main theorem. Our main theorem gives a QPTAS for approximating a fragment of ϵ-ETR. The total number of k-uniform vectors in a convex set $C = \mathrm{conv}(c_1, c_2, \ldots, c_l)$ is $l^{O(k)}$. So, if the parameters α, γ, d, t, and n are all constant, then our main theorem tells us that the total number of k-uniform vectors is $l^{O(\log m)}$, where m is the number of constraints. So if we enumerate each k-uniform vector x, we can check whether F holds, and if it does, we can output x. If no k-uniform vector exists that satisfies F, then we can determine that $\mathrm{exact}(F)$ has no solution. This gives us the following result.

Corollary 1. *Let F be an ϵ-ETR instance constrained by the convex set defined by $c_1, c_2, \ldots, c_l$. If α, γ, d, t, and n are constant, and l is polynomial, then we have an algorithm that runs in time $l^{O(\log m)}$ that either finds a solution to F, or determines that $\mathrm{exact}(F)$ has no solution.*

If m is constant and l is polynomial then this gives a PTAS, while if m and l are polynomial, then this gives a QPTAS.

In Section 5 we will show that the problem of approximating the best social welfare achievable by an approximate Nash equilibrium in a two-player normal form game can be written down as a constrained ϵ-ETR formula where α, γ, d, and m are constant. It has been shown that, assuming the exponential time hypothesis, this problem cannot be solved faster than quasi-polynomial time [11,19], so this also implies that constrained ϵ-ETR where α, γ, d, and m are constant cannot be solved faster than quasi-polynomial time unless the exponential time hypothesis is false.

Many ϵ-ETR problems are naturally constrained by sets that are defined by the convex hull of exponentially many vectors. The cube $[0,1]^n$ is a natural example of one such set. Brute force enumeration does not give an efficient algorithm for these problems, since we need to enumerate $l^{O(k)}$ vectors, and l is already exponential. However, our main theorem is able to provide non-deterministic polynomial time algorithms for these problems.

This is because each k-uniform vector is, by definition, the convex combination of at most k of the vectors in the convex set, and this holds even if l is exponential. So, provided that k is polynomial, we can guess the subset of vectors that are used, and then verify that the formula holds. This is particularly useful for problems where $\text{exact}(F)$ always has a solution, which is often the case in game theory applications, since it places the approximation problem in NP, whereas computing the exact solution may be ETR-complete.

Corollary 2. *Let F be an ϵ-ETR instance constrained by the convex set defined by $c_1, c_2, \ldots, c_l$. If α, γ, d, t, n, are polynomial, then there is a non-deterministic polynomial time algorithm that either finds a solution to F, or determines that $\text{exact}(F)$ has no solution. Moreover, if $\text{exact}(F)$ is guaranteed to have a solution, then the problem of finding an approximate solution for F is in NP.*

A theorem for non-tensor formulas. One downside of Theorem 2 is that it requires that the formula is written down using tensor constraints. We have argued that every ETR formula can be written down in this way, but the translation introduces a new vector-variable for each variable in the ETR formula. When we apply Theorem 2 to obtain PTASs or QPTASs we require that the number of vector variables is at most polylogarithmic, and so this limits the application of the theorem to ETR formulas that have at most polylogarithmically many variables.

The following theorem is a sampling result for ϵ-ETR with non-tensor constraints, which is proved in Appendix B.

Theorem 3. *Let F be an ϵ-ETR instance constrained over the convex set defined by $c_1, c_2, \ldots, c_l \in \mathbb{R}^q$. Let m be the number of constraints used in F, Let $\gamma = \max_i \|c_i\|_\infty$, let α be the largest constant coefficient used in F, let t be the number of terms used in F, and let d be the maximum degree of the polynomials in F. If $\text{exact}(F)$ has a solution in $\text{conv}(c_1, c_2, \ldots, c_l)$, then F has a k-uniform solution in $\text{conv}(c_1, c_2, \ldots, c_l)$ where*

$$k = \alpha^2 \cdot \gamma^{2d-2} \cdot (2^d - 1)^2 \cdot t^2 \cdot \log l/\epsilon^2.$$

The key feature here is that the number of variables does not appear in the formula for k, which allows the theorem to be applied to some formulas for which Theorem 2 cannot. However, since the theorem does not allow tensor constraints, its applicability is more limited because the number of terms t will be much larger in non-tensor formulas. For example, as we will see in Section 5, we can formulate bimatrix games using tensor constraints over constantly many vector variables, and this gives a result using Theorem 2. No such result can be obtained via Theorem 3, because when we formulate problem without tensor constraints, the number of terms t used in the inequalities becomes polynomial.

4 The Proof of the Main Theorem

In this section we prove Theorem 2. Before we proceed with the technical results let us provide a roadmap. We begin by considering two special cases, which when combined will be the backbone of the proof of the main theorem.

Firstly, we will show how to deal with problems where every constraint of the Boolean formula is a *multilinear polynomial*, which we will define formally later. We deal with this kind of problems using Hoeffding's inequality and the union bound, which is similar to how such constraints have been handled in prior work.

Then, we study problems where the Boolean formula consists of a *single* degree d polynomial constraint. We reduce this kind of problems to a constrained $\epsilon/2$-ETR problem with multilinear constraints, so we can use our previous result to handle the reduced problem. Degree d polynomials have not been considered in previous work, and so this reduction is a novel extension of sampling based techniques to a broader class of ϵ-ETR formulas.

Finally, we deal with the main theorem: we reduce the original ETR problem with multivariate constraints to a set of $\epsilon' -$ ETR problems with a single standard degree d constraint, and then we use the last result to derive a bound on k.

Problems with multilinear constraints. We begin by considering constrained ϵ-ETR problems where the Boolean formula F consists of tensor-multilinear polynomial constraints. We will use $\mathrm{TML}(A, x_1, \ldots, x_n)$ to denote a tensor-multilinear polynomial with n variables and coefficients defined by tensor A of size $\times_{j=1}^{n} p$. Formally,

$$\mathrm{TML}(A, x_1, \ldots, x_n) = \sum_{i_1 \in [p]} \cdots \sum_{i_n \in [p]} x_1(i_1) \cdot \ldots \cdot x_n(i_n) \cdot a(i_1, \ldots, i_n) + c.$$

We will use α to denote the maximum entry of tensor A in the absolute value sense and γ to denote the infinite norm of the convex set that constrains the variables.

The following lemma is proved in Appendix C. The proof uses Hoeffding's inequality and the union bound, and is similar to previous applications of the sampling technique.

Lemma 1. *Let F be a Boolean formula with n variables and m tensor-multilinear polynomial constraints and let $\mathcal{Y}$ be a convex set in the variables space. If the constrained ETR problem defined by* $\mathrm{exact}(F)$ *and $\mathcal{Y}$ has a solution, then the constrained ϵ-ETR problem defined by F and $\mathcal{Y}$ has a k uniform solution where*

$$k = \frac{2 \cdot \alpha^2 \cdot \gamma^2 \cdot n^2 \cdot \ln(2 \cdot n \cdot m)}{\epsilon^2}.$$

Problems with a standard degree d constraints. We now consider constrained ϵ-ETR problems with *exactly one* tensor polynomial constraint of standard degree d. We will use $\mathrm{TSD}(A, x, d)$ to denote a standard degree d tensor-polynomial with coefficients defined by the $\times_{j=1}^{d} l$ tensor A. Here, d identical vectors x are applied on A. Formally,

$$\mathrm{TSD}(A, x, d) = \sum_{i_1 \in [p]} \cdots \sum_{i_d \in [p]} x(i_1) \cdot \ldots \cdot x(i_d) \cdot a(i_1, \ldots, i_d) + c.$$

The following lemma is proved in Appendix D. To prove the lemma we consider the variable x to be defined as the average of $r = O(\frac{\alpha^2 \cdot \gamma^d \cdot d^2}{\epsilon})$ variables. This allows us to "break" the standard degree d tensor polynomial to a sum of multilinear tensor polynomials and to a sum of not-too-many multivariate polynomials. Then, the choice of r allows us to upper bound the error occurred by the multivariate polynomials by $\frac{\epsilon}{2}$. Then, we observe that in order to prove the lemma we can write the sum of multilinear tensor polynomials as an $\frac{\epsilon}{2}$-ETR problem with r variables and roughly r^d multilinear constraints. This allows us to use Lemma 1 to complete the proof.

Lemma 2. *Let F be a Boolean formula with variable x and one tensor-polynomial constraint of standard degree d, and let $\mathcal{Y}$ be a convex set. If the constrained ETR problem defined by* $\mathrm{exact}(F)$ *and $\mathcal{Y}$ has a solution, then the constrained ϵ-ETR problem defined by F and $\mathcal{Y}$ has a k-uniform solution where*

$$k = \frac{24 \cdot \alpha^6 \cdot \gamma^{2d+2} \cdot d^5 \cdot \ln(2 \cdot \alpha \cdot \gamma \cdot d)}{\epsilon^4}.$$

Problems with simple multivariate constraints. We now assume that we are given a constraint-ϵ-ETR problem defined by a Boolean formula F of tensor simple multilinear polynomial constraints and a convex set $\mathcal{Y}$. As before $\gamma = \|\mathcal{Y}\|_\infty$ and let α be the maximum absolute value of the coefficients of the constraints. We will say that the constraints are of maximum degree d if d is the maximum degree among all variables. The following lemma is proved in Appendix E. The idea is to rewrite the problem as an equivalent problem with standard degree d constraints and then apply Lemmas 2 and 1 to derive the bound for k.

Lemma 3. *Let F be a Boolean formula with n variables and m simple tensor-multivariate polynomial constraints of maximum degree d and let $\mathcal{Y}$ be a convex set in the variables space. If the constrained ETR problem defined by* $\mathrm{exact}(F)$ *and $\mathcal{Y}$ has a solution, then the constrained ϵ-ETR problem defined by F and $\mathcal{Y}$ has a k uniform solution where*

$$k = \frac{48 \cdot \alpha^6 \cdot \gamma^{2d+2} \cdot d^5 \cdot n^6 \cdot \ln(2 \cdot \alpha \cdot \gamma \cdot d \cdot n \cdot m)}{\epsilon^4}.$$

The proof of Theorem 2.

Proof. Assume that $x_1^*, \dots, x_n^* \in \mathcal{Y}$ is a solution for exact(F). Consider now a multivariate constraint $i \in [m]$ of F defined by $TMV_i(x_1, \dots, x_n)$. Firstly, we replace this constraint by

$$|TMV_i(x_1, \dots, x_n) - TMV_i(x_1^*, \dots, x_n^*)| \leq \epsilon. \tag{1}$$

Then, replace Constraint (1) by t constraints of the form

$$|STM_{i,j}(x_1, \dots, x_n) - STM_{i,j}(x_1^*, \dots, x_n^*)| \leq \frac{\epsilon}{t} \tag{2}$$

where $STM_{i,1}(x_1, \dots, x_n), \dots, STM_{i,t}(x_1, \dots, x_n)$ are the simple tensor multivariate polynomials $TMV_i(x_1, \dots, x_n)$ consists of. By the triangle inequality we get that if all t constraints given by (2) hold, then Constraint (1) holds as well. Hence, we can reduce the problem to an equivalent problem with the same n variables and $m \cdot t$ constraints that all of them are simple tensor multivariate polynomials. So, we can apply Lemma 3 where we replace m with $m \cdot t$ and ϵ with $\frac{\epsilon}{t}$. This completes the proof of the theorem. $\square$

5 Applications

We now show how our theorems can be applied to derive new approximation algorithms for a variety of problems.

Constrained approximate Nash equilibria. A *constrained* Nash equilibrium is a Nash equilibrium that satisfies some extra constraints, like specific bounds on the payoffs of the players. Constrained Nash equilibria attracted the attention of many authors, who proved NP-completeness for two-player games [23,14,6] and ETR-completeness for three-player games [6,7,8,9,22] for constrained *exact* Nash equilibria.

Constrained approximate equilibria have been studied, but so far only lower bounds have been derived [2,25,11,19,18]. It has been observed that sampling methods can give QPTASs for finding constrained approximate Nash equilibria for certain constraints in two player games [19].

By applying Theorem 2, we get the following result for games with a constant number of players: *Any property of an approximate equilibrium that can be formulated in ϵ-ETR where α, γ, d, t and n are constant has a QPTAS.* This generalises past results to a much broader class of constraints, and provides results for games with more than two players, which had not previously been studied in this setting. The details of this result are given in Appendix F.

Shapley games. Shapley's stochastic games [31] describe a two-player infinite-duration zero-sum game. The game consists of N *states*. Each state specifies a two-player $M \times M$ matrix game where the players compete over: (1) a reward (which may be negative) that is paid by player two to player one, and (2) a probability distribution over the next state of the game. So each round consists of the players playing a bimatrix game at some state s, which generates a reward, and the next state s' of the game. The reward in round i is discounted by λ^{i-1}, where $0 < \lambda < 1$ is a *discount factor*. The overall payoff to player 1 is the discounted sum of the infinite sequence of rewards generated during the course of the game.

Shapley showed that these games are determined, meaning that there exists a value vector v, where v_s is the value of the game starting at state s. A polynomial-time algorithm has been devised for computing the value vector of a Shapley game when the number of states N is constant [24]. However, since the values may be irrational, this algorithm needs to deal with algebraic numbers, and the *degree* of the polynomial is $O(N)^{N^2}$, so if N is even mildly super-constant, then the algorithm is not polynomial.

Shapley showed that the value vector is the unique solution of a system of polynomial optimality equations, which can be formulated in ETR. Any approximate solution of these equations gives an approximation of the value vector, and applying Theorem 2 gives us a QPTAS. This algorithm works when $N \in O(\sqrt[6]{\log M})$, which is a value of N that prior work cannot handle. The downside of our algorithm is that, since we require the solution to be bounded by a convex set, the algorithm only works when the value vector is reasonably small. Specifically, the algorithm takes a constant bound $B \in \mathbb{R}$, and either finds the approximate value of the game, or verifies that the value is strictly greater than B. The details of the algorithm are given in Appendix G.

Optimization problems. Our framework can provide approximation schemes for optimization problems with one vector variable $x \in \mathbb{R}^p$ with polynomial constraints over bounded convex sets. Formally,

$$\begin{aligned} \max \quad & h(x) \\ s.t. \quad & h_1(x) > 0, \dots, h_m(x) > 0 \\ & x \in \text{conv}(c_1, \dots, c_l) \end{aligned}$$

where $h(x), h_1(x), \dots, h_m(x)$ are polynomials with respect to vector x; for example $h(x) = x^T Ax$, where A is an $p \times p$ matrix, subject to $h_1(x) = x^T x > \frac{1}{10}$ and $x \in \Delta^p$. We will call the polynomials h_i *solution-constraints*. Optimization problems of this kind received a lot of attention over the years [15,16,17,21].

For optimization problems, we sample from the solution that achieves the maximum when we apply Theorem 2, in order to prove that there is a k-uniform solution that is close to the maximum. Our algorithm enumerates all k-uniform profiles, and outputs the one that maximizes the objective function. Using this technique, Theorem 2 gives the following results.

1. There is a PTAS if $h(x)$ is a STM polynomial of maximum degree independent of p, the number of solution-constraints is independent of p, and $l = \text{poly}(p)$.
2. There is QPTAS if $h(x)$ is an STM polynomial of maximum degree up to $\text{poly} \log p$, the number of solution-constraints is $\text{poly}(p)$, and $l = \text{poly}(p)$.

To the best of our knowledge, the second result is new. The first result was already known, however it was proven using completely different techniques: in [10] it was proven for the special case of degree two, in [21] it was extended to any fixed degree, and alternative proofs of the fixed degree case were also given in [16,17]. We highlight that in all of the aforementioned results solution constraints were not allowed. Note that unless NP=ZPP there is no FPTAS for quadratic programming even when the variables are constrained in the simplex [15]. Hence, our results can be seen as a partial answer to the important question posed in [15]: "*What is a complete classification of functions that allow a PTAS?*"

Tensor problems. Our framework provides quasi-polynomial time algorithms for deciding the existence of approximate eigenvalues and approximate eigenvectors of tensors in $\mathbb{R}^{p \times p \times p}$, where the elements are bounded by a constant, where the solutions are required to be in a convex set. In [26] it is proven that there is no PTAS for these problems when the domain is unrestricted. To the best of our knowledge this is the first positive result for the problem even in this, restricted, setting. The details of the algorithm are given in Appendix H.

Computational geometry. Finally, we note that our theorem can be applied to problems in computational geometry, although the results are not as general as one may hope. Many problems in this field are known to be ETR-complete, including, for example, the Steinitz problem for 4-polytopes, inscribed polytopes and Delaunay triangulations, polyhedral complexes, segment intersection graphs, disk intersection graphs, dot product graphs, linkages, unit distance graphs, point visibility graphs, rectilinear crossing number, and simultaneous graph embeddings. We refer the reader to the survey of Cardinal [13] for further details.

All of these problems can be formulated in ϵ-ETR, and indeed our theorem does give results for these problems. However, our requirement that the bounding convex set be given explicitly limits their applicability. Most computational geometry problems are naturally constrained by a cube, so while Corollary 2 does give NP algorithms, we do not get QPTASs unless we further restrict the convex set. In Appendix I we formulate QPTASs for the segment intersection graph and the unit disk intersection graph problems when the solutions are restricted to lie in a simplex. While it is not clear that either problem has natural applications that are restricted in this way, we do think that future work may be able to derive sampling theorems that are more tailored towards the computational geometry setting.

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P. Spirakis wishes to dedicate this paper to the memory of his late father in law Mathematician and Professor Dimitrios Chrysofakis, who was among the first in Greece to work on tensor analysis.

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A Proof of Theorem 1

In this section we will show that unconstrained ϵ-ETR = ETR for all $\epsilon \in \mathbb{R}$. Every ϵ-ETR instance can be trivially reduced in polynomial time to an ETR instance by replacing each constraint of the form $x =_\epsilon y$ with the constraint $x = y + \epsilon$, and likewise translating $<_\epsilon$ and $\leq_\epsilon$ to their exact counterparts.

It is less obvious that every ETR formula can be reformulated as an ϵ-ETR formula. We will prove this by modifying a technique of Schaefer and Stefankovic [30]. They considered the following problem, which asks us to find a shared root of a system of polynomials.

Definition 4 (Feas). *Given a system of k multi-variate polynomials $p_i : \mathbb{R}^n \to \mathbb{R}^n$, decide whether there exists an $x \in \mathbb{R}^n$ such that $p_i(x) = 0$ for all i.*

Schaefer and Stefankovic showed that this problem is ETR-complete.

Theorem 4 ([30]). Feas *is ETR-complete.*

We will reduce Feas to ϵ-ETR. Let $P = (p_i)_{i=1,\dots,k}$ be an instance of Feas, and let L be the number of bits needed to represent this instance. We define $\operatorname{gap}(P) = 2^{-2^{L+5}}$. The following lemma was shown by Schaefer and Stefankovic.

Lemma 4 ([30]). *Let $P = (p_i)_{i=1,\dots,k}$ be an instance of* Feas. *If there does not exist an $x \in \mathbb{R}^n$ such that $p_i(x) = 0$ for all i, then for every $x \in \mathbb{R}^n$ there exists an i such that $|p_i(x)| > \operatorname{gap}(P)$.*

In other words, if the instance of Feas is not solvable, then one of the polynomials will always be bounded away from 0 by at least $\operatorname{gap}(P)$.

The reduction. The first task is to build an ϵ-ETR formula that ensures that a variable $t \in \mathbb{R}$ satisfies $t \geq \epsilon/\operatorname{gap}(P)$. This can be done by the standard trick of repeated squaring, but we must ensure that the ϵ-inequalities do not interfere with the process. We define the following formula over the variables $t, g_1, g_2, \dots, g_{L+5} \in \mathbb{R}^n$, where all of the following constraints are required to hold.

$$\begin{aligned} g_1 &\geq_\epsilon 2 + \epsilon, \\ g_i &\geq_\epsilon g_{i-1}^2 + \epsilon && \text{for all } i > 1. \\ t &\geq_\epsilon \epsilon \cdot g_{L+5} + \epsilon \end{aligned}$$

In other words, this requires that $g_1 \geq 2$, and $g_i \geq g_{i-1}^2$. So we have $g_{L+5} \geq 2^{2^{L+5}}$, and hence $t \geq \epsilon/\operatorname{gap}(P)$. Note that the size of this formula is polynomial in the size of P.

Given an instance $P = (p_i)_{i=1,\dots,k}$ of Feas we create the following ϵ-ETR instance ψ, where all of the following are required to hold.

$$t \cdot p_i(x) \leq_\epsilon 0 \qquad \text{for all } i, \tag{3}$$
$$t \cdot p_i(x) \geq_\epsilon 0 \qquad \text{for all } i, \tag{4}$$
$$t \geq \epsilon/\operatorname{gap}(P), \tag{5}$$

where the final inequality is implemented using the construction given above.

Lemma 5. *ψ is satisfiable if and only if P has a solution.*

Proof. First, let us assume that P has a solution. This means that there exists an $x \in \mathbb{R}^n$ such that $p_i(x) = 0$ for all i. Note that x clearly satisfies Inequalities 3 and 4, while Inequality 5 can be satisfied by fixing t to be any number greater than $\epsilon/\operatorname{gap}(P)$. So we have proved that ψ is satisfiable.

On the other hand, now we will assume that $x \in \mathbb{R}^n$ satisfies ψ. Note that we must have

$$p_i(x) \leq \epsilon/t \leq \operatorname{gap}(P)$$

and likewise

$$p_i(x) \geq -\epsilon/t \geq -\operatorname{gap}(P),$$

and hence $|p_i(x)| \leq \operatorname{gap}(P)$ for all i. But Lemma 4 states that this is only possible in the case where P has a solution. ◻

This completes the proof of Theorem 1.

B Proof of Theorem 3

We will use the following theorem of Barman.

Theorem 5 ([4]). *Let $c_1, c_2, \ldots, c_l \in \mathbb{R}^q$ with $\max_i \|c_i\|_\infty \leq 1$. For every $x \in \mathrm{conv}(c_1, c_2, \ldots, c_l)$ and every $\epsilon > 0$ there exists a $O(\log l/\epsilon^2)$-uniform vector $x' \in \mathrm{conv}(c_1, c_2, \ldots, c_l)$ such that $\|x - x'\|_\infty \leq \epsilon$.*

The following lemma shows that if we take two vectors x and x' that are close in the L_∞ norm, then for all polynomials p the value of $|p(x) - p(x')|$ cannot be too large.

Lemma 6. *Let $p(x)$ be a multivariate polynomial over $x \in \mathbb{R}^q$ with degree d and let $\epsilon \in (0, \gamma]$ for some constant $\gamma > 0$. For every pair of vectors $x, x' \in [0, \gamma]^q$ with $\|x - x'\|_\infty \leq \epsilon$ we have:*

$$|p(x) - p(x')| \leq \gamma^{d-1} \cdot (2^d - 1) \cdot consts(p) \cdot terms(p) \cdot \epsilon.$$

Proof. Consider a term of $p(x)$, which can without loss of generality be written as $t(x) = c \cdot \prod_{\substack{i \in [q] \\ \sum_i = d}} x_i$, where it could be the case that any number of x_i's are the same. We have

$$\begin{aligned}
|t(x) - t(x')| &= \left| c \cdot \prod_{\substack{i \in [q] \\ \sum_i = d}} x_i - c \cdot \prod_{\substack{i \in [q] \\ \sum_i = d}} x'_i \right| \\
&= c \cdot \left| \prod_{\substack{i \in [q] \\ \sum_i = d}} x_i - \prod_{\substack{i \in [q] \\ \sum_i = d}} x'_i \right| \\
&\leq c \cdot \left| \prod_{\substack{i \in [q] \\ \sum_i = d}} x_i - \prod_{\substack{i \in [q] \\ \sum_i = d}} (x_i + \epsilon) \right| \\
&\leq c \cdot \left| \prod_{\substack{i \in [q] \\ \sum_i = d}} x_i - \left[\prod_{\substack{i \in [q] \\ \sum_i = d}} x_i + \binom{d}{1} \gamma^{d-1} \epsilon + \binom{d}{2} \gamma^{d-2} \epsilon^2 + \cdots + \binom{d}{d} \gamma^0 \epsilon^d \right] \right| \\
&\leq c \cdot \left| \epsilon \cdot \sum_{k=1}^{d} \binom{d}{k} \gamma^{d-1} \right| \\
&\leq c \cdot \epsilon \cdot \gamma^{d-1} \cdot \sum_{k=1}^{d} \binom{d}{k} \\
&= \epsilon \cdot c \cdot \gamma^{d-1} \cdot (2^d - 1),
\end{aligned}$$

where the second to last four lines use the fact that x_i's, and ϵ are all less than or equal to γ.

Next consider a term $t(x)$ of $p(x)$ of degree $d' \leq d$. This can be written similarly to the aforementioned term. Then $|t(x) - t(x')| \leq c \cdot \epsilon \cdot \gamma^{d-1} \cdot (2^{d'} - 1) \leq c \cdot \epsilon \cdot \gamma^{d-1} \cdot (2^d - 1)$. Since there are $terms(p)$ many terms in p, we therefore have that

$$|p(x) - p(x')| \leq \gamma^{d-1} \cdot (2^d - 1) \cdot consts(p) \cdot terms(p) \cdot \epsilon.$$

□

We now apply this to prove Theorem 3.

Proof (of Theorem 3). Let x be the solution to exact(F). First we apply Theorem 5 to find a point y that is k-uniform, where $k = \alpha^2 \cdot \gamma^{2d-2} \cdot (2^d - 1)^2 \cdot t^2 \cdot \log l / \epsilon^2$, such that

$$\|x - y\|_\infty \leq \epsilon / (\alpha \cdot \gamma^{d-1} \cdot (2^d - 1) \cdot t).$$

Next we can apply Lemma 6 to argue that, for each polynomial p used in F, we have

$$\begin{aligned} |p(x) - p(y)| &\leq \alpha \cdot \gamma^{d-1} \cdot (2^d - 1) \cdot t \cdot \left(\frac{\epsilon}{\alpha \cdot \gamma^{d-1} \cdot (2^d - 1) \cdot t} \right) \\ &= \epsilon. \end{aligned}$$

Since all constraints of F have a tolerance of ϵ, and since x satisfies exact(F), we can conclude that $F(y)$ is satisfied. □

C Proof of Lemma 1

Proof. Let $(x_1^*, x_2^*, \ldots, x_n^*) \in \mathcal{Y}$ be a solution for exact(F). Since we assume the $\mathcal{Y}$ is a convex set of $c_1, \ldots, c_l$ any $x \in \mathcal{Y}$ can be written as a convex combination of the c_i's, i.e., $x = \sum_{i \in [l]} a_i \cdot c_i$, where $a_i \geq 0$ for every $i \in [l]$, and $\sum_{i \in [l]} a_i = 1$. Observe, $a = (a_1, \ldots, a_l)$ corresponds to a probability distribution over $c_1, \ldots, c_l$, where vector c_i is drawn with probability a_i, and x can be thought of as the mean of a. So, we can "sample" a point by sampling over c_is according to the probability that define this point.

For every $i \in [n]$, let x'_i be a k-uniform vector sampled independently from x_i^*. To prove the lemma, we will show that, because of the choice of k, with positive probability the sampled vectors satisfy every constraint of the ϵ-ETR problem. Then, by the probabilistic method the lemma will follow.

Let $\text{TML}_j(A_j, x_1, \ldots, x_n)$ be a multilinear polynomial that defines a constraint of F. For every $j \in [m]$ we define the following constraint

$$|\,\text{TML}_j(A_j, x'_1, \ldots, x'_n) - \text{TML}_j(A_j, x_1^*, \ldots, x_n^*)| \leq \epsilon. \tag{6}$$

Observe that if $x'_1, \ldots, x'_n$, satisfy inequality (6) for every $j \in [m]$, then the lemma follows.

For every $j \in [m]$, we replace the corresponding Constraint (6) with n *linear* constraints. For notation simplicity, let us denote ML_j^i the multivariate polynomial $ML_j(A_j, x_1, \ldots, x_n)$ where we set $x_1 = x'_1, x_2 = x'_2, \ldots, x_i = x'_i$ and $x_{i+1} = x_{i+1}^*, x_{i+2} = x_{i+2}^*, \ldots, x_n = x_n^*$. Furthermore, let $ML_j^0 = ML_j(A_j, x_1^*, \ldots, x_n^*)$. Then, for every $i \in [n]$ we create the constraint

$$|ML_j^i - ML_j^{i-1}| \leq \frac{\epsilon}{n}. \tag{7}$$

Observe that, if all n constraints defined in (7) are satisfied, then by the triangle inequality, the corresponding Constraint (6) is satisfied as well.

Consider now ML_j^i. This can be seen as a random variable that depends on the choice of x'_i and takes values in $[-\gamma \cdot \alpha, \gamma \cdot \alpha]$. But recall that the x'_i's are sampled from x_i^* using k samples, and that they are mutually independent, so $\mathbb{E}\big[ML_j^i\big] = ML_j^{i-1}$. Thus, we can bound the probability that a constraint (7) is not satisfied, i.e. bound the probability that $|ML_j^i - ML_j^{i-1}| > \frac{\epsilon}{n}$, using Hoeffding's inequality [27]. So,

$$\begin{aligned} \Pr\left(\big|ML_j^i - ML_j^{i-1}\big| > \frac{\epsilon}{n}\right) &= \Pr\left(\big|ML_j^i - \mathbb{E}\big[ML_j^i\big]\big| > \frac{\epsilon}{n}\right) \\ &\leq 2 \cdot exp\left(-\frac{2 \cdot k^2 \cdot \left(\frac{\epsilon}{n}\right)^2}{4 \cdot k \cdot \gamma^2 \cdot \alpha^2}\right) \\ &= 2 \cdot exp\left(-\frac{k \cdot \epsilon^2}{2 \cdot n^2 \cdot \gamma^2 \cdot \alpha^2}\right). \end{aligned} \tag{8}$$

Recall, that we have $n \cdot m$ constraints of the form (7). We can bound the probability that any of those constraints is violated, via the union bound. So, using (8) and the union bound, the probability that any of these constraints is violated is upper bounded by

$$2 \cdot m \cdot n \cdot exp\left(-\frac{k \cdot \epsilon^2}{2 \cdot n^2 \cdot \gamma^2 \cdot \alpha^2}\right). \tag{9}$$

Hence, if the value of (9) is strictly less than 1, then $x'_1, \dots, x'_m$ satisfy all of the $n \cdot m$ linear constraints, and thus the lemma follows. If we solve (9) for k, such that it is strictly less than 1, we get

$$k > \frac{\alpha^2 \cdot \gamma^2 \cdot n^2 \cdot \ln(2 \cdot n \cdot m)}{\epsilon^2}$$

which holds, by our choice of k.

□

D Proof of Lemma 2

To prove the lemma we will first prove the following auxiliary Lemma.

Lemma 7. *Let F be a Boolean formula with one variable and one tensor-polynomial constraint of standard degree d, let $\mathcal{Y}$ be a convex set, and let $r = \frac{2 \cdot \alpha^2 \cdot \gamma^d \cdot (d-1)^2}{\epsilon}$. If the constrained ETR problem* exact(F) *has a solution in $\mathcal{Y}$, then there exists a satisfiable constrained $\frac{\epsilon}{2}$ − ETR problem Π_{ML} with r variables and $\prod_{i=0}^{d-1}(r-i)$ tensor multilinear constraints, such that every solution of Π_{ML} in $\mathcal{Y}$ can be transformed to a solution for the constrained ϵ-ETR problem defined by F and $\mathcal{Y}$.*

Proof. Assume that $x^* \in \mathcal{Y}$ is a solution for F. Let $\mathrm{TSD}(A, x, d)$ denote the tensor polynomial of standard degree d used in F. For notation simplicity, let $\mathrm{TSD}(A, x, d) = A(x^d)$. Create r new variables $x_1, \dots, x_r \in \mathcal{Y}$ and set $x = \frac{1}{r}(x_1 + \ldots + x_r)$. Let $\mathcal{X} = \bigcup_{i=1}^{r}\{x_i\}$ be a *multiset* of $\mathcal{Y}$ with cardinality r, meaning that multiple copies of an element of $\mathcal{Y}$ are allowed in $\mathcal{X}$. In the sequel we will treat the elements of $\mathcal{X}$ as distinct, even though some might correspond to the same element of $\mathcal{Y}$. Then, note that $A(x^d)$ can be written as a sum of simple tensor-multivariate polynomials where some of them are multilinear and have as variables $x_1, \dots, x_r$. Now, let $\mathcal{S}$ be the set of all ordered d-tuples that can be made by drawing d elements from $\mathcal{X}$ with replacement. Formally, $\mathcal{S} = \{(\hat{x}_1, \dots, \hat{x}_d) : \hat{x}_1, \dots, \hat{x}_d \in \mathcal{X}\}$. Let us also define $\mathcal{S}_d$ to be the set of all ordered d-tuples that can be made by drawing d elements from $\mathcal{X}$ without replacement. Formally, $\mathcal{S}_d = \{(\hat{x}_1, \dots, \hat{x}_d) : \hat{x}_1, \dots, \hat{x}_d \in \mathcal{X}, \quad \hat{x}_1, \dots, \hat{x}_d \text{ are pairwise different}\}$, and observe that $|S_d| = \prod_{i=0}^{d-1}(r-i)$. So, any element of $\mathcal{S}_d$, combined with tensor A, produces a multilinear polynomial. Hence, using the notation introduced, we get that $|A(x^d) - A({x^*}^d)|$ is less than or equal to sum of

$$\frac{1}{r^d} \sum_{(\hat{x}_1, \dots, \hat{x}_d) \in \mathcal{S}_d} \left| A(\hat{x}_1, \dots, \hat{x}_d) - A({x^*}^d) \right| \qquad \text{and} \tag{10}$$

$$\frac{1}{r^d} \sum_{(\hat{x}_1, \dots, \hat{x}_d) \in \mathcal{S} - \mathcal{S}_d} \left| A(\hat{x}_1, \dots, \hat{x}_d) - A({x^*}^d) \right|. \tag{11}$$

Observe, $|\mathcal{S} - \mathcal{S}_d| = r^d - |\mathcal{S}_d|$ and that $\left|A(\hat{x}_1, \dots, \hat{x}_d) - A(x^{*^d})\right| \leq \gamma^d \cdot \alpha$ for every $A(\hat{x}_1, \dots, \hat{x}_d)$. Then, for the sum given in (11) we get

$$\begin{aligned}
\frac{1}{r^d} \sum_{(\hat{x}_1,\dots,\hat{x}_d)\in\mathcal{S}-\mathcal{S}_d} & \left|A(\hat{x}_1, \dots, \hat{x}_d) - A(x^{*^d})\right| \\
&\leq \left(1 - \frac{r\cdot(r-1)\cdots(r-d+1)}{r^d}\right)\cdot\gamma^d\cdot\alpha \\
&\leq \left(1 - \left(1-\frac{1}{r}\right)\left(1-\frac{2}{r}\right)\cdot\left(1-\frac{d-1}{r}\right)\right)\cdot\gamma^d\cdot\alpha \\
&\leq \left(1 - \left(1-\frac{d-1}{r}\right)^{d-1}\right)\cdot\gamma^d\cdot\alpha \\
&\leq \left(1 - \left(1-\frac{(d-1)^2}{r}\right)\right)\cdot\gamma^d\cdot\alpha \\
&= \frac{(d-1)^2}{r}\cdot\gamma^d\cdot\alpha \\
&\leq \frac{\epsilon}{2}.
\end{aligned}$$

Hence, in order for the original constraint to be satisfied, it suffices to satisfy the constraint

$$\frac{1}{r^d} \sum_{(\hat{x}_1,\dots,\hat{x}_d)\in\mathcal{S}_d} \left|A(\hat{x}_1, \dots, \hat{x}_d) - A(x^{*^d})\right| \leq \frac{\epsilon}{2}. \tag{12}$$

Observe that $|\mathcal{S}_d| = \prod_{i=0}^{d-1}(r-i) < r^d$, therefore, instead of the constraint (12), it suffices to deal with the following $|S_d|$ constraints (we introduce one constraint for every $(\hat{x}_1, \dots, \hat{x}_d) \in \mathcal{S}_d$)

$$\left|A(\hat{x}_1, \dots, \hat{x}_d) - A(x^{*^d})\right| \leq \frac{\epsilon}{2}. \tag{13}$$

The proof is completed by setting the set of tensor-multilinear constraints to be the union of $\prod_{i=0}^{d-1}(r-i)$ constraints defined by (13).

□

We can combine Lemma 1 with Lemma 7 and prove Lemma 2.

Proof (Proof of Lemma 2). Firstly, use Lemma 7 to construct the constrained $\frac{\epsilon}{2}$ − ETR problem Π_{ML} with tensor-multilinear constraints. Recall that Π_{ML} has $r = \frac{2\cdot\alpha^2\cdot\gamma^d\cdot(d-1)^2}{\epsilon}$ variables and $\prod_{i=0}^{d-1}(r-i) < r^d$ constraints. Then, from Lemma 1 we know that if Π_{ML} is satisfiable, then there exist $\frac{k}{r}$-uniform vectors $x'_1 \in Y, \dots, x'_r \in \mathcal{Y}$ that $\epsilon/2$-satisfy Π_{ML}. Finally, observe that $x' = \frac{1}{r}\cdot(x'_1+\ldots+x'_r)$ is k-uniform. □

E Proof of Lemma 3

Proof. Let $x_1^*, \dots, x_m^*$ be a solution for exact(F). Let $r = \frac{2\cdot\alpha^2\cdot\gamma^d\cdot(d-1)^2\cdot n}{\epsilon}$ and let $\hat{x}_1^i, \dots, \hat{x}_r^i$ be $\frac{k}{r}$-uniform vectors sampled from x_i^*. Define $x'_i = \frac{1}{r}(\hat{x}_1^i + \ldots + \hat{x}_r^i)$.

Consider a constraint in $j \in [m]$ defined by the simple tensor-multivariate polynomial $\mathrm{STM}_j(A_j, x_1^{d_1}, \dots, x_m^{d_m})$. We will use the same technique we used in Lemma 1 to create n constraints, where constraint $i \in [n]$ is defined via a simple degree d_i polynomial. Again, for notation simplicity for every $i \in [m]$ we use STM_j^i to denote the polynomial $\mathrm{STM}_j(A_j, x_1^{d_1}, \dots, x_m^{d_m})$ where we set $x_1 = x'_1, \dots, x_i = x'_i$ and $x_{i+1} = x_{i+1}^*, \dots, x_n = x_n^*$. Let $\mathrm{STM}_j^0 := \mathrm{STM}_j(A_j, (x_1^*)^{d_1}, \dots, (x_n^*)^{d_n})$. Then, we define the following n constraints

$$|\,\mathrm{STM}_j^i - \mathrm{STM}_j^{i-1}\,| \leq \frac{\epsilon}{2n}. \tag{14}$$

Observe that every constraint i of the form (14) defines simple degree d_i polynomial with respect to variable x'_i. Furthermore, observe that if every such constraint is satisfied, then the initial constraint defined by $\mathrm{STM}_j(A_j, x_1^{d_1}, \dots, x_n^{d_n})$ is satisfied too. Then, we convert each such constraint to a set of $\prod_{i=0}^{d-1}(r-i)$ multilinear constraints with r variables, using Lemma 7 where we demand that every multilinear constraint is $\frac{\epsilon}{2\cdot n}$-satisfied. The proof is then completed by using Lemma 1 where we observe that we have $r \cdot n < \frac{2\cdot\alpha^2\cdot\gamma^d\cdot d^2\cdot n^2}{\epsilon}$ variables and $\prod_{i=0}^{d-1}(r-i)\cdot n\cdot m < r^d \cdot n \cdot m$ constraints and we set ϵ to $\frac{\epsilon}{2\cdot n}$. □

F Constrained Approximate Nash equilibria application

A game is defined by the set of players, the set of actions for every player, and the payoff function of every player. In normal form games, the payoff function is given by a multilinear function on a tensor of appropriate size. Consider an n-player game where every player has l-actions, and let A_j denote the payoff tensor of player j with elements in $[0,1]$; A_j has size $\times_{i=1}^{n} l$. The interpretation of the tensor A_j is the following: the element $A_j(i_1, \dots, i_n)$ of the tensor corresponds to the payoff of player j when Player 1 chooses action i_1, Player 1 chooses action i_2, and so on. To play the game, every player j chooses a probability distribution $x_j \in \Delta^l$, a.k.a. a *strategy*, over their actions. A collection of strategies is called *strategy profile*. The expected payoff of player j under the strategy profile $(x_1, \dots, x_n)$ is given by $ML(A_j, x_1, \dots, x_n)$. For notation simplicity, let $u_j(x_j, x_{-j}) := ML(A_j, x_1, \dots, x_n)$, where x_{-j} is the strategy profile of all players except player j. A strategy profile $(x_1^*, \dots, x_n^*)$ is a Nash equilibrium if for every player j it holds that $u_j(x_j^*, x_{-j}^*) \geq u_j(x_j, x_{-j}^*)$ for every $x_j \in \Delta^l$, or equivalently $u_j(x_j^*, x_{-j}^*) \geq u_j(s_p, x_{-j}^*)$ for every possible s_p, where s_p denotes the case where player j chooses their action p with probability 1.

Our framework formally captures the set of constrains that we can get a QPTAS for computing approximate Nash equilibria in games with constant number of players.

Theorem 6. *Let Γ be an n-player l-action normal form game Γ where n is constant. Furthermore, let F be a Boolean formula with $c = poly(l)$ TSM constraints of constant degree and of constant length with variables the strategies of the players. Then in a quasi-polynomial time we can compute a approximate NE of Γ constrained by F, or decide that no such constrained approximate NE exists.*

Proof. Observe that we can write the problem of the existence of a constrained Nash equilibrium as an ETR problem. The constraints of the problem will be the constraints of F plus the constraint

$$u_j(s_l, x_{-j}) - u_j(x_j, x_{-j}) \leq 0$$

for every player $i \in [m]$ and every action s_l of player j.

Thus, we can use Theorem 2 and complete the proof since we produced an ϵ-ETR problem with $m = c + n \cdot l = \mathrm{poly}(l)$ constraints, which is polynomial in the input size; d and t are constants by assumption; $\gamma = 1$ since every variable is a probability distribution; $\alpha = 1$ by the definition of normal form games. □

G Shapley's stochastic games application

To formally define a Shapley game, we use N to denote the number of states, and M to denote the number of actions. The game is defined by the following two functions.

- For each $s \leq N$ and $j, k \leq M$ the function $r(s, j, k)$ gives the reward at state s when player one chooses action j and player two chooses action k.
- For each $s, s' \leq N$ and $j, k \leq M$ the function $p(s, s', j, k)$ gives the probability of moving from state s to state s' when player one chooses action j and player two chooses action k. It is required that $\sum_{s'=1}^{N} p(s, s', j, k) = 1$ for all s, j, and k.

The game begins at a given starting state. In each round of the game the players are at a state s, and play the matrix game at that state by picking an action from the set $\{1, 2, \dots, M\}$. The players are allowed to use randomization to make this choice. Supposing that the first player chose action j and the second player chose the action k, the first player receives the reward $r(s, j, k)$, and then a new state s' is chosen according to the probability distribution given by $p(s, \cdot, j, k)$.

The reward in future rounds is *discounted* by a factor of λ where $0 < \lambda < 1$ in each round. So if r_1, r_2, $\ldots$ is the infinite sequence of rewards, the total reward paid by player two to player one is $\sum_{i=1}^{\infty} \lambda^{i-1} \cdot r_i$, which, due to the choice of λ, is always a finite value.

The two players play the game by specifying a probability distribution at each state, which represents their strategy for playing at that state. Let Δ^M denote the M-dimensional simplex, which represents the strategy space for both players at a single state. For each $x, y \in \Delta^M$, we overload notation by defining the expected reward and next state functions.

$$r(s, x, y) = \sum_{j=1}^{M} \sum_{k=1}^{M} x(j) \cdot y(k) \cdot r(s, i, j),$$

$$p(s, s', x, y) = \sum_{j=1}^{M} \sum_{k=1}^{M} x(j) \cdot y(k) \cdot p(s, s', i, j).$$

Shapley showed that these games are *determined* [31], meaning that there is a unique vector $v \in \mathbb{R}^N$ such that v_s is the *value* of the game starting at state s: player one has a strategy to ensure that the expected reward is at least $v(s)$, while player two has a strategy to ensure that the expected reward is at most $v(s)$. Furthermore, Shapley showed that this value vector is the unique solution of the following *optimality equations* [31]. For each state s we have the equation

$$v(s) = \min_{x \in \Delta^M} \max_{y \in \Delta^M} \left(r(s, x, y) + \lambda \cdot \sum_{s'=1}^{N} p(s, s', x, y) \cdot v_{s'} \right). \tag{15}$$

In other words, v_s must be the value of the one-shot zero-sum game at s, where the payoffs of this zero-sum game are determined by the values of the other states given by $v_{s'}$.

Theorem 7. *Let Γ be a Shapley game with $N \in O(\sqrt[6]{\log m})$, unbounded number of actions per state, and rewards in $[-c, c]$ for every state-action combination, where c is a constant. Furthermore, let s be the starting state of the game. Let $B \in \mathbb{R}$ be a constant. In a quasi-polynomial time we can approximately compute the value of Γ starting from s, if the value of every state less than or equal to B, or decide that at least one of these values is greater than or equal to B.*

Proof. Let $v = (v(1), v(2), \ldots, v(N))$, and for every state s let x_s and y_s denote the strategy player one and player two choose at state s respectively. Observe that $r(s, x_s, y_s)$ is an STM polynomial with variables x and y of the form

$$\mathrm{STM}(A_{s1}, x_s, y_s) = \sum_{j=1}^{M} \sum_{k=1}^{M} x_s(j) \cdot y_s(k) \cdot a_{s1}(j, k)$$

where $a_{s1}(i, j, k) = r(s, j, k)$.

Observe furthermore that $\lambda \cdot \sum_{s'=1}^{N} p(s, s', x_s, y_s) \cdot v_{s'}$ can be written as an STM polynomial with variables x, y and v of the form

$$\mathrm{STM}(A_{s2}, x, y, v) = \sum_{j=1}^{M} \sum_{k=1}^{M} \sum_{l=1}^{N} x_s(j) \cdot y_s(k) \cdot v(l) \cdot a_{s2}(j, k, l)$$

where $a_{s2}(i, j, k) = \lambda \cdot p(s, l, j, k)$.

Let us define $\mathrm{TMV}_s(x_s, y_s, v) = \mathrm{STM}(A_{s1}, x_s, y_s) + \mathrm{STM}(A_{s2}, x_s, y_s, v)$; $\mathrm{TMV}_s(x_s, y_s, v)$ has length 2 and degree 1.

Observe, we can replace Equation (15) with the following $2 \cdot M$ TMV polynomial constraints

$$\mathrm{TMV}(x_s, y_s, v) - \mathrm{TMV}(j, y_s, v) \leq 0 \qquad \text{for every action } j \leq M \text{ of player one}$$

$$\mathrm{TMV}(x_s, k, v) - \mathrm{TMV}(x_s, y_s, v) \geq 0 \qquad \text{for every action } k \leq M \text{ of player two.}$$

So, to approximate $v(s)$ it suffices to solve the ϵ-ETR problem defined by the $2 \cdot M \cdot N$ constraints defined as above for every state $s \leq N$. Observe, the ϵ-ETR problem has: $2N + 1$ variables (x_1 through x_N, y_1 through y_N, and v); $2 \cdot M \cdot N$ TMV constraints; $\gamma = \max\{1, \max_s v(s)\}$; $\alpha = \max\{c, \lambda \cdot \max_{s,s',j,k} p(s, s', j, k)\} = \max\{c, 1\}$, since $\lambda < 1$ and $\max_{s,s',j,k} p(s, s', j, k) < 1$. So, if $N \in \sqrt[6]{\log m}$, $\max_s v(s)$ is constant, and c is a constant, we can use Theorem 2 and derive a QPTAS for (15).

Finally, we note that an approximate solution to (15) gives an approximation of the value vector itself. This is because Shapley has shown that, when v is treated as a variable, the optimality equation given in (15) is a *contraction map*. The value vector is a fixed point of this contraction map, and the uniqueness of the value vector is guaranteed by Banach's fixed point theorem. Our algorithm produces an approximate fixed point of the optimality equations. It is easy to show, using the contraction map property, that an approximate fixed point must be close to an exact fixed point. $\square$

H Tensor Problems

Definition 5. *The nonzero vector $x \in \mathbb{R}^p$ is an* eigenvector *of tensor $A \in \mathbb{R}^{p\times p\times p}$ if there exists an* eigenvalue $\lambda \in \mathbb{R}$ *such that for every $k \in [p]$ it holds that*

$$\sum_i^n \sum_j^n a(i, j, k) \cdot x(i) \cdot x(j) = \lambda \cdot x(k). \tag{16}$$

Theorem 8. *Let A be an $\mathbb{R}^{p\times p\times p}$ tensor with entries in $[-c, c]$, where c is a constant. Furthermore, let $B \in \mathbb{R}$ be a constant and let $\mathcal{Y}$ be a convex set where $\|\mathcal{Y}\|_\infty$ is a constant. In a quasi-polynomial time we can compute an eigenvalue-eigenvector pair (λ, x) that approximately satisfy* (16) *such that $\lambda \leq B$ and $x \in \mathcal{Y}$, or decide that no such pair exists.*

Proof. Observe that $\sum_i^n \sum_j^n a(i, j, k) \cdot x(i) \cdot x(j)$ can be written as an STM polynomial $\mathrm{STM}(A_1, x^2)$ where $a_1(i, j) = a(i, j, k)$. Furthermore, let ℓ be a p-dimensional vector. Then, $\lambda \cdot x(k)$ can be written as an STM polynomial $\mathrm{STM}(A_2, x, \ell)$, where $a_2(k, 1) = 1$ and zero otherwise.

So, Equation 16 can be written as an TMV polynomial constraint of degree 2 and length 2, with two vector variables, x and ℓ. So, the problem of computing an eigenvalue-eigenvector pair that approximately satisfy (16) can be written as an ϵ-ETR problem with p TMV polynomial constraints of degree 2 and length 2 and two vector variables. Hence, we can use Theorem 2 with $\gamma = \|\mathcal{Y}\|_\infty$ which is constant, $\alpha = c$, $n = 2$, $t = 2$, $d = 2$, and $m = p$ to find a solution if exists, or decide that no such solution exists. $\square$

I Computational Geometry Applications

I.1 Segment intersection graphs

Definitions. Let G be an undirected graph with vertex set $\{v_1, v_2, \ldots, v_n\}$. We say that G is a *segment graph* if there are straight segments $s_1, s_2, \ldots, s_n$ in the plane such that, for every $i, j, 1 \leq i < j \leq n$, the segments s_i and s_j have a common point if and only if $\{v_i, v_j\} \in E(G)$.

By a suitable rotation of the co-ordinate system we can achieve that none of the segments is vertical. Then the segment s_i representing vertex v_i can be algebraically described as the set $\{(x, y) \in \mathbb{R}^2 : y = a_i x + b_i, c_i \leq x \leq d_i\}$ for some real numbers a_i, b_i, c_i, d_i. We say that G is a *simplex K segment graph* if the real numbers a_i, b_i, c_i, d_i, $i = 1, 2, \ldots n$ are under the constraints

$$\begin{aligned} &a_i, b_i, c_i, d_i \geq 0, \text{ for every } i = 1, 2, \ldots n, \text{ and} \\ &\sum_{i=1}^n (a_i + b_i + c_i + d_i) = K, \quad \text{where } K > 0 \text{ is a given constant.} \end{aligned}$$

We let SIM-K-SEG denote the class of all simplex K segment graphs with parameter $K > 0$.

The problem ϵ-RECOG(SIM-K-SEG) is defined as follows. Given an abstract undirected graph G, does it belong with tolerance ϵ to SIM-K-SEG?

Formulation of ϵ-RECOG(SIM-K-SEG). We first give a description for the problem with $\epsilon = 0$ and then we generalize for arbitrary $\epsilon \geq 0$. The formulation is taken from [29].

Letting l_i be the line containing s_i, we note that $s_i \cap s_j \neq \emptyset$ if l_i and l_j intersect in a single point whose x-coordinate lies in both the intervals $[c_i, d_i]$ and $[c_j, d_j]$. As is easy to calculate, that x-coordinate equals $\frac{b_j - b_i}{a_i - a_j}$.

Now we turn to the general case where $\epsilon \geq 0$. Let us introduce variables A_i, B_i, C_i, D_i representing the unknown quantities a_i, b_i, c_i, d_i, $i = 1, 2, \ldots, n$. By the problem's definition we require the vector $(A_1, B_1, C_1, D_1, \ldots, A_n, B_n, C_n, D_n)$ to be in the $(4n-1)$-simplex with parameter K. Then $s_i \cap s_j \neq \emptyset$ can be expressed by the following predicate:

$$\begin{aligned}\mathrm{INTS}(A_i, B_i, C_i, D_i, &A_j, B_j, C_j, D_j) = \\ &(A_i >_\epsilon A_j \wedge C_i(A_i - A_j) \leq_\epsilon B_j - B_i \leq_\epsilon D_i(A_i - A_j) \\ &\wedge C_j(A_i - A_j) \leq_\epsilon B_j - B_i \leq_\epsilon D_j(A_i - A_j)) \\ &\vee (A_i <_\epsilon A_j \wedge C_i(A_i - A_j) \geq_\epsilon B_j - B_i \geq_\epsilon D_i(A_i - A_j) \\ &\wedge C_j(A_i - A_j) \geq_\epsilon B_j - B_i \geq_\epsilon D_j(A_i - A_j))\end{aligned}$$

(this is only correct if we "globally" assume that $C_i \leq_\epsilon D_i$ for all i). The existence of a SEG-representation of G can then be expressed by the formula

$$\begin{aligned}(\exists A_1 B_1 C_1 D_1 \ldots A_n B_n C_n D_n K) &\left(\bigwedge_{i=1}^{n} C_i \leq_\epsilon D_i \right) \\ \wedge &\left(\bigwedge_{\{i,j\} \in E} \mathrm{INTS}(A_i, B_i, C_i, D_i, A_j, B_j, C_j, D_j) \right) \\ \wedge &\left(\bigwedge_{\{i,j\} \notin E} \neg \mathrm{INTS}(A_i, B_i, C_i, D_i, A_j, B_j, C_j, D_j) \right)\end{aligned}$$

Theorem 9. *There is an algorithm that runs in time $n^{O(K^2 \cdot \log n / \epsilon^2)}$ and either finds a vector $(A_1, B_1, C_1, D_1, \ldots, A_n, B_n$ that is a solution to ϵ-RECOG(SIM-K-SEG), or determines that there is no solution to 0-RECOG(SIM-K-SEG).*

Proof. We set $x = (A_1, B_1, C_1, D_1, \ldots, A_n, B_n, C_n, D_n)$ and $F(x)$ to be the above formula that we constructed. Their combination makes an ϵ-ETR instance. Vector x is constrained over the convex set defined by the vertices of the $(4n-1)$-simplex, i.e. vectors $v_i \in \mathbb{R}^{4n}$, $i \in \{1, 2, \ldots 4n\}$ with their i-th element equal to K and the rest equal to 0. Therefore the cardinality of our convex set is $m = 4n$, and $\gamma = K$. By looking at the formula we can conclude that $a = 1$, $t = 4$, and $d = 2$. By Theorem 3 the result follows. □

I.2 Unit disk intersection graphs

Definitions. Let G be an undirected graph with vertex set $\{v_1, v_2, \ldots, v_n\}$. We say that G is a *unit disk intersection graph* or *unit disk graph* if there are disks $d_1, d_2, \ldots, d_n$ (in the plane) with radius 1 such that, for every $i, j, 1 \leq i < j \leq n$, the disks d_i and d_j have more than one points common (i.e. an area) if and only if $\{v_i, v_j\} \in E(G)$.

The disk d_i representing vertex v_i can be algebraically described as the set $\{(x, y) \in \mathbb{R}^2 : (x - x_i)^2 + (y - y_i)^2 \leq 1\}$ for some real numbers x_i, y_i that determine the centre of the disk. We say that G is a *simplex K unit disk graph* if the real numbers x_i, y_i, $i = 1, 2, \ldots n$ are under the constraints

$$\begin{aligned}&x_i, y_i \geq 0, \text{ for every } i = 1, 2, \ldots n, \text{ and} \\ &\sum_{i=1}^{n} (x_i + y_i) = K, \quad \text{where } K > 0 \text{ is a given constant.}\end{aligned}$$

We let SIM-K-UDG denote the class of all simplex K unit disk graphs with parameter $K > 0$.

The problem ϵ-RECOG(SIM-K-UDG) is defined as follows. Given an abstract undirected graph G, does it belong with tolerance ϵ to SIM-K-UDG?

Formulation of ϵ-RECOG(SIM-K-UDG). Let us introduce variables X_i, Y_i representing the unknown quantities x_i, y_i, $i = 1, 2, \ldots, n$. We require the vector $(X_1, Y_1, \ldots, X_n, Y_n)$ to be in the $(2n-1)$-simplex with parameter K. Then we consider an ϵ-intersection $d_i \cap_\epsilon d_j \neq \emptyset$ to happen if:

$$\sqrt{(X_i - X_j)^2 + (Y_i - Y_j)^2} < 2 + \epsilon$$

and an ϵ-non-intersection $d_i \cap_\epsilon d_j = \emptyset$ to happen if:

$$\sqrt{(X_i - X_j)^2 + (Y_i - Y_j)^2} \geq 2 - \epsilon$$

The existence of a UDG-representation of G can then be expressed by the formula

$$\begin{aligned}
&(\exists X_1 Y_1 \ldots X_n Y_n) \\
&\qquad \left(\bigwedge_{\{i,j\} \in E} (X_i - X_j) \cdot (X_i - X_j) + (Y_i - Y_j) \cdot (Y_i - Y_j) < 4 + 2\epsilon + \epsilon^2 \right) \\
&\qquad \wedge \left(\bigwedge_{\{i,j\} \notin E} (X_i - X_j) \cdot (X_i - X_j) + (Y_i - Y_j) \cdot (Y_i - Y_j) \geq 4 - 2\epsilon + \epsilon^2 \right)
\end{aligned}$$

Theorem 10. *There is an algorithm that runs in time $n^{O(K^2 \cdot \log n/\epsilon^2)}$ and either finds a vector $(X_1, Y_1, \ldots, X_n, Y_n)$ that is a solution to ϵ-RECOG(SIM-K-UDG), or determines that there is no solution to 0-RECOG(SIM-K-UDG).*

Proof. We set $x = (X_1, Y_1, \ldots, X_n, Y_n)$ and $F(x)$ to be the above formula that we constructed. Their combination makes an ϵ-ETR instance. Vector x is constrained over the convex set defined by the vertices of the $(2n-1)$-simplex, i.e. vectors $v_i \in \mathbb{R}^{2n}$, $i \in \{1, 2, \ldots 2n\}$ with their i-th element equal to K and the rest equal to 0. Therefore the cardinality of our convex set is $m = 2n$, and $\gamma = K$. By looking at the formula we can conclude that $a = 2$, $t = 7$, and $d = 2$. By Theorem 3 the result follows. $\square$