

# Causal Stream Inclusions

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**Abstract.** We study solutions to systems of stream inclusions  $f \in T(f)$ , where  $T$  is assumed to be *causal* in the sense that elements in output streams are determined by a finite history of inputs. For solving these inclusions we develop a correspondence of *causality* and *contraction* with respect to the prefix distance on streams. Now, based on this *causality-contraction correspondence*, we apply fixpoint principles for the spherically complete ultrametric space of streams to obtain solutions for causal stream inclusions. The underlying fixpoint iterations induce *fixpoint induction principles* for reasoning about solutions of causal stream inclusions. In addition, these fixpoint approximations induce *anytime algorithms* for computing finite stream prefixes of solutions. We illustrate the use of these developments for some central concepts of system design.

**Keywords:** Formal Methods · Fixpoint Approximation · Systems Engineering.

## 1 Introduction

We consider existence, uniqueness, approximations, and reasoning principles for solutions (fixpoints) of the *stream inclusions*

$$f \in T(f), \tag{1}$$

where  $f := (f_1, \dots, f_n)$ , for  $n \in \mathbb{N}$ , is a vector of infinite *streams*  $f_i$  of values, and the vector- and *multivalued stream transformer*  $T$  is *causal* in that every element in the output stream is completely determined by a finite history of inputs. These stream inclusions are ubiquitous in computer science and other fields of knowledge such as biology, economics, and artificial intelligence for modeling the evolution of systems under uncertainty. Typical sources of uncertainty hereby are lack of epistemic knowledge about the behavior of the system under consideration, as well as modeling artefacts such as nondeterminism and underspecification.

If the stream transformer  $T$  is deterministic then the stream inclusion (1) reduces to a system of stream equalities  $f = T(f)$ . Interacting systems of deterministic stream processors, for instance, are modeled as the least fixpoint of this

equality, where  $T$  is a Scott-continuous stream transformer in a complete partial order [21]. But this traditional denotational approach does not extend naïvely to unbounded nondeterminism [3,60,47,45,59,8,10].

We are pursuing a different route in that we are investigating conditions under which fixpoints for causal stream transformers with unbounded nondeterminism, which is naturally modeled by means of multivalued maps, exist. We hereby distinguish between *weakly* and *strongly causal* stream transformers, whereby the latter notion additionally implies a strict, bounded delay for which outputs are also determined. Our main results for solving stream inclusions (1) are as follows.

- (Theorem 1) For strongly causal stream transformers  $T$  with nonempty, compact codomains, solutions  $f \in T(f)$  are contained in the (unique) fixpoint  $F = \text{sp}_T(F)$  for the *strongest post* of  $T$ . This latter fixpoint is obtained as the limit of a Picard iteration with a bound on the approximation of each iteration.
- (Theorem 2)  $f \in T(f)$  has a solution if  $T$  is strongly causal, not identically empty, and all the codomains of  $T$  are closed (in the topology induced by the prefix metric on streams). In addition, we identify a slightly stronger condition than strong causality for establishing the uniqueness of solutions of the stream inclusion (1). A fixpoint induction principle is provided for reasoning about these solutions.
- (Theorem 3)  $f \in T(f)$  has a solution if  $T$  is strongly causal and all codomains of  $T$  are nonempty and compact (in the topology induced by the prefix metric on streams).
- (Theorem 4) If the stream transformer  $T$  is weakly causal then either  $f \in T(f)$  has a solution or there exists a ball of streams with positive radius on which the distance, as measured by the prefix metric on streams, between input stream and corresponding output sets with respect to  $T$  is, in a sense to be made more precise, invariant.

Our developments are heavily based on the correspondence of logic-based notions of causality with metric-based notions of contraction (and non-expansion). More precisely, we show that a vector- and multivalued stream transformer is **weakly causal if and only if it is *non-expansive***, and it is **strongly causal if and only if it is *contractive*** for the prefix metric. This approach is motivated by recent work of Broy [10] on stream-based system design calculus for strongly causal functionals, which is based on the existence of unique fixpoints for strongly causal, and therefore contracting functionals for the prefix metric on streams, by applying Banach’s fixpoint principle. Altogether, we proceed by:

1. Modeling nondeterministic and mutually dependent system components as conjunctions of causal stream inclusions;
2. Establishing the equivalence of vector- and multivalued causal stream transformers with contraction in a spherically complete ultrametric space of streams based on the prefix distance of streams;
3. Applying multivalued fixpoint principles in this ultrametric stream space for obtaining solutions of causal stream inclusions.

As a starting point, in Section 3, the prefix distance between two streams is measured in terms of the longest common prefix, thereby obtaining a spherically complete ultrametric space of streams. This notion of distance, which derives from Cantor sets, generalizes to arbitrary stream products and also to sets of streams via the induced Hausdorff distance. Next, Section 4 reviews the familiar notions of *stream transformers* together with some composition operators, refinement, and contractual specification as the basis of a more general framework for systematically constructing stream transformers and their realization on computational machinery. The focus in Section 5 is on *causal* stream transformers, whose outputs are determined by a finite history of inputs. For instance, causality is preserved under composition and refinement of stream transformers.

In Section 6 we develop a metric-based characterization of causal stream transformers. More specifically, we show that a nondeterministic stream transformer is causal if and only if it is contractive with respect to the given ultrametric on streams. This correspondence extends to a notion of Lipschitz contraction based on the Hausdorff distance between sets of streams, since stream transformers with non-empty compact images are Lipschitz contractive if and only if they are causal.

Since the induced Hausdorff metric is a complete metric space on the set of non-empty compact sets, we obtain in Section 7 unique fixpoints for causal set transformers such as the *weakest pre* and the *strongest post* set transformers. Moreover, there is a linearly, strictly decreasing upper bound of the prefix distance between this fixpoint and its approximation by the underlying Picard iteration, which suggests an *anytime* approximation algorithm, we formulate induction principles in the spirit of Park’s lemma for fixpoints of set transformers, and we show that the set of fixpoints of a multivalued stream transformer  $T$  is contained, and possibly strictly so, in the fixpoint of the strongest post transformer for  $T$ .

Additional fixpoint results in Section 7 construct solutions for the stream inclusion (1) by showing that every strongly causal vector- and multivalued stream transformer has a fixpoint, as long as it is not identically empty and its codomain is restricted to closed sets only. We also identify a slightly stronger condition for which this fixpoint is unique. The underlying Picard iteration enables us to derive an induction principle for reasoning about these fixpoints. In Section 7 we also state some immediate consequences of the correspondence of causality and Lipschitz contractivity for fixpoints of weakly causal maps. Section 8 discusses further consequences of the causality-contraction correspondence extensions, and Section 9 concludes with an outlook on the relevance of these developments for system design.

## 2 Preliminaries

We are summarizing a smorgasbord of concepts and notation for topological and metric spaces, as used in the remainder of this treatise. For a metric space  $(M, d)$  the sets  $B(x, r) := \{y \mid d(x, y) < r\}$  and  $B[x, r] := \{y \mid d(x, y) \leq r\}$  are called

the *open* and *closed balls* of *center*  $x$  and *radius*  $r$ , respectively. The family of open balls forms a base of neighborhoods for a uniquely determined Hausdorff topology on  $M$ , which is the *topology induced by  $d$*  (on  $M$ ). Open, closed, bounded, (dis)connected, convex, totally bounded (precompact), and compact sets are defined with respect to the metric-induced topology. The set of nonempty, closed, and bounded subsets of  $M$ , in particular, is denoted by  $\text{CB}(M)$ , and  $\text{Comp}(M)$  is the set of nonempty compact subsets of  $M$ .

The *distance*  $d(a, B)$  of an element  $a \in M$  to a non-empty set  $B \subseteq M$  is defined by  $d(a, B) = \inf_{b \in B} d(a, b)$ . Clearly,  $d(a, b) = d(a, \{b\})$  for all  $a, b \in M$ . For a bounded metric space  $(M, d)$  the *Hausdorff distance*  $\mathcal{H}_d(A, B)$  between two nonempty sets  $A, B \subseteq M$  measures the "longest path" to get from  $A$  to  $B$ , or vice versa, from  $B$  to  $A$ .

$$\mathcal{H}_d(A, B) := \max(\sup_{x \in A} (d(x, B)), \sup_{y \in B} (d(y, A))). \quad (2)$$

Clearly, all suprema exist for a bounded metric  $d$ , and  $d(a, b) = \mathcal{H}_d(\{a\}, \{b\})$ . Furthermore, if  $A, B$  are closed then their Hausdorff distance  $\mathcal{H}_d(A, B)$  is finite, and  $(\text{CB}(M), \mathcal{H}_d)$  is a metric space.

A sequence  $(x_k)_{k \in \mathbb{N}}$  in  $M$  is *Cauchy* if and only if for all  $\varepsilon > 0$  there exists  $N \in \mathbb{N}$  such that  $d(x_n, x_m) < \varepsilon$  for all  $n, m \geq N$ . Such a sequence  $x_k$  *converges* to  $x^* \in M$  if and only if for each neighborhood  $U$  of  $x$  there exists  $N \in \mathbb{N}$  such that  $x_n \in U$  for all  $n \geq N$ . Now, the space  $M$  is *Cauchy complete* if every Cauchy sequence converges to some  $x^* \in M$ . Equivalently,  $M$  is Cauchy complete if and only if the intersection of nested sequences of closed balls whose radius approaches to 0 are non-empty.

A map  $T : M \rightarrow M$  (in a metric space  $M$ ) is *contractive* if there exists a constant  $l$  with  $0 < l < 1$  such that  $d(T(x), T(y)) \leq l \cdot d(x, y)$  for all  $x, y \in M$ . Traditionally, *Banach's contraction principle* establishes that in a Cauchy complete metric space  $(M, d)$  there is a unique fixpoint for every contracting  $T : M \rightarrow M$  [29]. Starting with the *Picard iteration*  $x_{n+1} = T(x_n)$  for  $n \in \mathbb{N}$  with  $x_0$  arbitrary, one concludes from the contraction property with Lipschitz constant  $0 \leq l < 1$ , that  $d(x_{n+1}, x_n) \leq l \cdot d(x_n, x_{n-1})$ , and, therefore,  $d(x_{n+1}, x_n) \leq l^n / (1-l) \cdot d(x_0, x_1)$ . From this, one concludes that the sequence  $(x_n)_{n \in \mathbb{N}}$  is Cauchy, and, for completeness, that its limit  $x^* \in M$  is a fixpoint. This fixpoint is unique, since we get  $x^* = y^*$  for any two fixpoints from  $d(x^*, y^*) = d(T(x^*), T(y^*)) \leq l \cdot d(x^*, y^*)$ .

A map  $T : M \rightarrow M$  is said to be *shrinking* if  $d(T(x), T(y)) < d(x, y)$  for all  $x, y \in M$ . A shrinking map  $T$  need not have a fixed point in a complete metric space.

*Example 1.*  $T(x) := \ln(1 + e^x)$ , for  $x \in \mathbb{R}$ , is a shrinking map for the usual Euclidean distance on the reals, since  $T'(x) = e^x / (1 + e^x) < 1$  for all  $x \in \mathbb{R}$ . But  $T$  has no fixed point, since  $T(x) = x$  is equivalent to  $1 + e^x = e^x$ .

An *ultrametric space*  $(M, d)$  is a metric space with the *strong triangle inequality*, for all  $x, y, z \in M$

$$d(x, y) \leq \max(d(x, y), d(y, z)). \quad (3)$$

*Example 2 (Discrete Metric).* The discrete metric  $d(x, y)$ , which is 1 for  $x \neq y$  and 0 otherwise, is ultrametric.

As a consequence of (3) the *isosceles triangle principle*

$$d(x, z) = \max(d(x, y), d(y, z)) \quad (4)$$

holds whenever  $d(x, y) \neq d(y, z)$ . Further immediate consequences of the strong triangle inequality are: (1) every point inside a ball is its center, that is, if  $d(x, y) < r$  then  $B(x, r) = B(y, r)$ , (2) all balls of strictly positive radius  $r$  are *clopen*, that is both open and closed, (3) if two balls are not disjoint then one is included in the other, (4) the distance  $d(B_1, B_2) := \inf_{x \in B_1, y \in B_2} d(x, y)$  of two disjoint nonempty balls  $B_1, B_2$  is obtained as the distance of two arbitrarily chosen elements  $x \in B_1, y \in B_2$ . (4) ultrametric spaces are *totally disconnected*, that is, every superset of a singleton set is disconnected, and (5) a sequence  $(x_k)_{k \in \mathbb{N}}$  in an ultrametric space is Cauchy if and only if  $\lim_{k \rightarrow \infty} d(x_{k+1}, x_k) = 0$ . We will also make use of a generalized strong triangle inequality (see Appendix B) for ultrametric distances.

$$d(a, C) \leq \max(d(a, b), d(b, C)), \quad (5)$$

for all  $a, b \in M$  and  $\emptyset \neq C \subseteq M$ . For a proof of the following central fact see also Lemma 16 in the Appendix.

**Proposition 1.** *If  $(M, d)$  is ultrametric then  $(\text{CB}(M), \mathcal{H}_d)$  is also ultrametric.*

An ultrametric space  $(M, d)$  is *spherically complete* if  $\bigcap_{B \in \mathcal{C}} B \neq \emptyset$  for every chain  $\mathcal{C}$  of balls  $B_0 \supseteq B_1 \supseteq B_2 \supseteq \dots$

*Example 3.* Let  $M = \{\alpha, \beta, \gamma, \delta\}$  with  $d(x, x) = 0$  for all  $x \in M$ ,  $d(\alpha, \beta) = d(\gamma, \delta) = 1/2$ ,  $d(\alpha, \gamma) = d(\alpha, \delta) = d(\beta, \gamma) = d(\beta, \delta) = 1$ , and  $d(y, x) = d(x, y)$  for all  $x, y \in M$ . Then  $(M, d)$  is a spherically complete ultrametric space.

Clearly, spherical completeness implies Cauchy completeness.

**Proposition 2** ([17], p. 59). *If  $(M, d)$  is ultrametric then  $(\text{Comp}(M), \mathcal{H}_d)$  is Cauchy complete.*

For a spherically complete ultrametric space  $M$ , every strictly contracting  $T : M \rightarrow M$  has a unique fixpoint in  $M$  ([43], Theorem 1). Moreover, if  $T : M \rightarrow M$  is *nonexpansive*, that is  $d(T(x), T(y)) \leq d(x, y)$  for all  $x, y \in M$ , then either  $T$  has at least one fixpoint or there exists a ball  $B$  of radius  $r > 0$  such that  $T : B \rightarrow B$  and for which  $d(u, T(u)) = r$  for each  $u \in B$  ([43], Theorem 2). Such a ball  $B$  is sometimes also said to be *minimal  $T$ -invariant*.

*Example 4.* For the spherically complete ultrametric space  $(M, d)$  from Example 3 define  $T : M \rightarrow M$  such that  $\alpha \mapsto \gamma$ ,  $\beta \mapsto \delta$ ,  $\gamma \mapsto \alpha$ , and  $\delta \mapsto \beta$ . Then  $T$  has no fixpoint and for each  $x \in X$  there exists  $r > 0$  such that  $d(u, T(u)) = r$  for all  $u \in B[x, r]$ .

### 3 Streams

An  $A$ -valued *stream* in  $A^\omega := \mathbb{N} \rightarrow A$ , for a given nonempty set  $A$  of *values*, is an infinite sequence  $(a_k)_{k \in \mathbb{N}}$  with  $a_k \in A$ . Depending on the application context, streams are also referred to as discrete streams or signals,  $\omega$ -streams,  $\omega$ -sequences, or  $\omega$ -words. The *generating function* [12] of a stream  $(a_k)_{k \in \mathbb{N}}$  is a *formal power series*

$$\sum_{k \in \mathbb{N}} a_k X^k$$

in the *indefinite*  $X$ . These power series are *formal* as, in the algebraic view, the symbol  $X$  is not being instantiated and there is no notion of convergence. We call  $a_k$  the *coefficient* of  $X^k$ , and the set of formal power series with coefficients in  $A$  is denoted by  $A[[X]]$ . We also write  $[X^k]f$  for the coefficient of  $X^k$  in the formal power series  $f$ . Now,  $\text{hd}(f) := [X^0]f$  and  $\text{tl}(f)$  is the unique stream such that  $f = \text{hd}(f) + X \cdot \text{tl}(f)$ . A *polynomial* in  $A[X]$  of degree  $d \in \mathbb{N}$  is a formal power series  $f$  which is *dull*, that is  $[X^d]f \neq 0$  and  $[X^n]f = 0$  for all  $n > d$ . For the one-to-one relationship between streams  $A^\omega$  and formal power series  $A[[X]]$  we use these notions interchangeably. Streams are added componentwise and they are multiplied by *discrete convolution*.

$$\left(\sum_{k \in \mathbb{N}} a_k X^k\right) + \left(\sum_{k \in \mathbb{N}} b_k X^k\right) := \sum_{k \in \mathbb{N}} (a_k + b_k) X^k \quad (6)$$

$$\left(\sum_{k \in \mathbb{N}} a_k X^k\right) \cdot \left(\sum_{k \in \mathbb{N}} b_k X^k\right) := \sum_{k \in \mathbb{N}} \left(\sum_{i=0}^k a_i b_{k-i}\right) X^k \quad (7)$$

With these operations and  $A$  an (integrity) ring,  $A^\omega$  becomes a commutative (integrity) ring with zero element  $0 := \bar{0}$  and multiplicative identity  $1 := \bar{1}$ . Hereby  $\bar{a} := (a + \sum_{k \geq 1} 0X^k)$ , for  $a \in A$ , is the injective and homomorphic *embedding* of the ring  $A$  into the ring  $A[[X]]$  of formal power series. Similarly, the ring of polynomials in  $A[X]$  is injectively and homomorphically embedded in  $A[[X]]$  as dull formal power series.

If  $A$  is a field, then  $A^\omega \simeq A[[X]]$  is a *principal ideal domain* with the ideal  $(X)$ , which consists of all streams  $f \in X \cdot A^\omega$ , the only non-zero maximal ideal. Moreover, for  $A$  a field or a division ring,  $(A^\omega, +, (a \cdot)_{a \in K})$  is a *linear space*, whereby the *dot product*  $(a \cdot f)$ , for  $f \in A^\omega$ , is defined by  $\bar{a} \cdot f$ . Unless stated otherwise, we assume the values  $A$  in  $A^\omega$  to be a field.

The multiplicative inverse  $f^{-1}$  for  $f \in A^\omega$  exists (in which case it is unique) if and only if  $[X^0]f \neq 0$ . We also write  $f/g$  instead of  $f \cdot g^{-1}$ . In particular, the multiplicative inverse of  $X$  does not exist in  $A^\omega$ . As a consequence,  $A^\omega$  is not a field even when  $A$  is a field.

*Example 5 (Stream Calculus).* A stream is said to be *rational* [55] if it can be expressed as a quotient  $p/q$  of polynomial streams  $p, q \in A[X]$  such that  $[X^0]q \neq 0$ .

0.

$$\begin{aligned}
1/(1-X) &= (1, 1, 1, 1, \dots) \\
1/(1-X)^2 &= (1, 2, 3, 4, \dots) \\
1/(1-aX) &= (1, a, a^2, a^3, \dots) && \text{for } a \in A \\
X/(1-X-X^2) &= (0, 1, 1, 2, 3, 5, 8, \dots) && \text{(Fibonacci)} \\
2/(1+\sqrt{1-4X}) &= (1, 1, 2, 5, 14, \dots) && \text{(Catalan)}
\end{aligned}$$

The square root of a stream is defined, for instance, in [52]. Ultimately periodic streams such as  $\omega$ -words are a special case of rational streams [55]. On the other hand, by Theorem 5.4 in [55], there is no such quotient of polynomials for characterizing the stream  $(1, 1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, \dots)$ .

*Remark 1 (Rational Streams vs. Rational Functions).* The set of rational streams as defined in Example 5 is a subset of the real-valued formal power series. The ring of rational streams should not be confused, however, with the field  $\mathbb{R}(X)$  of so-called *rational functions* with real-valued coefficients, which is the quotient field of the polynomial ring  $\mathbb{R}[X]$ . For example,  $1/X \in \mathbb{R}(X)$  but  $1/X$  is, by definition, not a rational stream.

The *index*  $v(f)$  of a stream  $f \in A^\omega$  is the minimal  $k \in \mathbb{N}$  such that  $[X^k]f \neq 0$ , if any exists; otherwise  $v(f) := \infty$ .

**Proposition 3 (Valuation).** *For all  $f, g \in A^\omega$ :*

1.  $v(f) = \infty$  iff  $f = 0$
2.  $v(f \cdot g) = v(f) + v(g)$
3.  $v(f + g) \geq \min(v(f), v(g))$

Immediate consequences include: (1)  $v(1) = 0$ , (2)  $v(f^{-1}) = -v(f)$  for all streams  $f$  for which  $f^{-1}$  exists, and, (3)  $v(f + g) = \min(v(f), v(g))$  whenever  $v(f) \neq v(g)$ . By construction,

$$|f| := 2^{-v(f)}, \tag{8}$$

with  $2^{-\infty} := 0$ , is the *non-Archimedean absolute value* on  $A^\omega$  induced by the index  $v(\cdot)$  [39].

**Proposition 4 (Non-Archimedean Absolute Value).** *For all  $f, g \in A^\omega$ :*

1.  $0 \leq |f| \leq 1$
2.  $|f| = 0$  if and only if  $f = 0$
3.  $|f \cdot g| = |f| \cdot |g|$
4.  $|f + g| \leq \max(|f|, |g|)$

Some immediate consequences of Proposition 4 include: (1)  $|1| = 1$ , (2)  $|r| = 1$  for all roots  $r$  of unity, (3)  $|f^{-1}| = |f|^{-1}$  for streams  $f$  with an inverse, (4)  $||f| - |g|| \leq |f - g|$ , and (5)  $|f + g| = \max(|f|, |g|)$  whenever  $|f| \neq |g|$ . Notice also

that stream valuation trivially has the *non-Archimedean* property  $|1 + \dots + 1| \leq |1| = 1$ .

The *prefix* distance between streams  $f$  and  $g$  is measured in terms of the longest common prefix: the longer the common prefix, the closer a pair of streams. Via the distance function

$$d(f, g) := |f - g| \quad (9)$$

the set  $A^\omega$  is a metric with a discrete *set of values*  $\Delta_d := \{2^{-n} \mid n \in \mathbb{N} \cup \{\infty\}\}$ . In fact,  $d$  is an *ultrametric*, since the *strong triangle inequality*

$$d(f, h) \leq \max(d(f, g), d(g, h)). \quad (10)$$

holds for all streams  $f, g, h$  (see Appendix A). As a consequence, the *isosceles triangle principle* holds for  $d$ .

*Remark 2.*  $\mathbb{B}^\omega$  with the ultrametric  $d$  is known as the *Cantor space*, and  $\mathbb{N}^\omega$  is the *Baire space*.

**Proposition 5.** *Both addition and multiplication of streams are continuous with respect to the topology induced by  $d$ .*

Since this metric-induced topology is identical to the product topology  $A^\mathbb{N}$ , where each copy of  $A$  is the discrete topology, Tychonoff's theorem applies.

**Proposition 6.**  *$A^\omega$  is compact if and only if  $A$  is finite.*

A sequence  $(f_k)_{k \in \mathbb{N}}$  is *convergent in  $A[[X]]$*  provided that for every  $n \in \mathbb{N}$  there exists  $K_n \in \mathbb{N}$  and  $A_n \in A$  such that if  $k \geq K_n$  then  $[X^n]f_k = A_n$ . This condition says that if we focus attention on only the  $n$ -th power of  $X$ , and consider the sequence  $[X^n]f_k$  of coefficients of  $X^n$  in  $f_k$  as  $k \rightarrow \infty$ , then this sequence (of elements of  $A$ ) is eventually constant, with the ultimate value  $A_n$ . In this case,

$$f := \sum_{n=0}^{\infty} A_n X^n \quad (11)$$

is a well-defined formal power series, called the *limit* of the convergent sequence  $(f_k)_{k \in \mathbb{N}}$ . We use the notation  $\lim_{k \rightarrow \infty} f_k = f$  to denote this relationship.

*Example 6.*

1.  $\lim_{k \rightarrow \infty} X^k = 0$ ,
2.  $\lim_{k \rightarrow \infty} \sum_{i=0}^k X^i = 1/(1-X)$ .

**Lemma 1.** *The ultrametric space  $(A^\omega, d)$  of streams is spherically complete.*

*Proof.* Let  $B_0 \supseteq B_1 \supseteq \dots$  be any non-increasing sequence of non-empty balls in  $A^\omega$ . Then, the sequence  $(r_k)_{k \in \mathbb{N}}$  of radii  $r_k$  of the balls  $B_k$  is a non-increasing sequence of numbers in  $\mathbb{R}$ , which, by the discreteness of  $d$ , either becomes constant or converges to zero. If it becomes constant then  $\bigcap_{k \in \mathbb{N}} B_k$  even contains a ball, and if it converges to zero then this intersection of balls also is nonempty, since, for the ultrametricity of  $d$ , every point inside a ball is its center.



Lemma 1 implies Cauchy completeness of the metric space of streams (see also Appendix A, Lemma 15 for a direct proof).

**Corollary 1.**  $(A^\omega, d)$  is Cauchy complete.

Indeed,  $A^\omega \simeq A[[X]]$  is the Cauchy completion of the polynomials  $A[X]$  for the prefix metric  $d$ . Since any metric space is compact if and only if it is Cauchy complete and totally bounded, Corollary 1 and Proposition 6 together imply that

**Corollary 2.**  $(A^\omega, d)$  is totally bounded if and only if  $A$  is finite.

*Remark 3.* The metric subspace  $\mathbb{R}(X)$  of  $\mathbb{R}^\omega$  is not Cauchy complete. Indeed the sequence  $(\sum_{k=0}^n \frac{1}{k!} X^k)_{n \in \mathbb{N}}$  of polynomials is Cauchy but it does not converge in  $\mathbb{R}(X)$ .

*Remark 4.* The set  $A((X))$  of formal Laurent series of the form  $\sum_{k \geq k_0} a_k X^k$ , for  $k \in \mathbb{Z}$ , is the quotient field of the formal power series  $A[[X]]$ . Now,  $A((X))$  is spherically complete and therefore also Cauchy complete [16]. Indeed,  $A((X))$  is the Cauchy completion of  $A(X)$  for the prefix metric  $d$ .

*Stream Products.* Let  $\mathcal{I}$  be a nonempty index set,  $A_\iota$  a set of values for each index  $\iota \in \mathcal{I}$ , and  $(\prod_{\iota \in \mathcal{I}} A_\iota)^\omega$  the set of  $\mathcal{I}$ -indexed product of formal power series. The valuation

$$|\bar{f}|_{\mathcal{I}} := \sup_{\iota \in \mathcal{I}} |f_\iota| \quad (12)$$

for products  $\bar{f}$  of the form  $(f_\iota)_{\iota \in \mathcal{I}}$  is just the supremum of the valuation of its components.

*Remark 5.* In particular, the valuation of finite dimensional products of the form  $(f_1, \dots, f_n)$ , for  $n \in \mathbb{N}$ , is  $2^{-k}$ , where  $k \in \mathbb{N} \cup \{\infty\}$  is the maximal position such that all prefixes  $f_i|_k$ , for  $i = 1, \dots, n$ , only contain zeros.

*Remark 6.* Likewise, when interpreting  $\bar{f} \in (\prod_{\iota \in \mathcal{I}} A_\iota)^\omega$  as a (dependent) function with domain  $\mathcal{I}$  and codomains  $A_\iota^\omega$  for  $\iota \in \mathcal{I}$ , then the valuation  $|\bar{f}|_{\mathcal{I}}$  is obtained as  $\sup_{g \in \bar{f}(\mathcal{I})} |g|$ .

A metric on the  $\mathcal{I}$ -indexed product space  $(\prod_{\iota \in \mathcal{I}} A_\iota)^\omega$  is induced by the valuation (12).

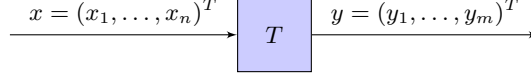
$$d_{\mathcal{I}}(\bar{f}, \bar{g}) := |\bar{f} - \bar{g}|_{\mathcal{I}} \quad (13)$$

**Lemma 2.** For nonempty  $\mathcal{I}$  and  $A_\iota$  for each  $\iota \in \mathcal{I}$ , the space  $(\prod_{\iota \in \mathcal{I}} A_\iota)^\omega, d_{\mathcal{I}}$  is ultrametric and spherically complete.

Since  $d_{\mathcal{I}}$  specializes for a singleton index set to the distance  $d$  on streams, we usually drop the subindex  $\mathcal{I}$ .

## 4 Transformers

Stream transformers are the basic building blocks for modeling (discrete) dynamical systems, say  $T$ , with a vector  $x$  of  $n$  input streams and a vector  $y$  of  $m$  output streams (see Figure 1). We write  $A^{\omega,n}$  for the set  $(A^\omega)^n$  of  $n$ -dimensional vectors of  $A$ -valued streams.



**Fig. 1.** Stream Transformer.

**Definition 1 (Stream Transformers).** A stream transformer is a vector- and multi-valued map  $T : A^{\omega,n} \rightarrow \mathcal{P}(B^{\omega,m})$ , for  $n, m \geq 1$ . If  $A$  ( $B$ ) is a singleton set, then  $T$  is a source (sink). If  $|T(f)| = 1$  (componentwise) for all  $f \in (A^{\omega,n})$  then  $T$  is a deterministic stream transformer; otherwise the stream transformer is said to be non-deterministic. For a deterministic stream transformer we also write  $T : A^{\omega,n} \rightarrow B^{\omega,m}$ . A stream transformer is said to be identically empty if  $T(A^{\omega,n}) = \emptyset$ .

Hereby, the direct image of a stream transformer  $T$  with respect to a set  $F$  of streams is denoted by  $T(F)$ .

*Remark 7.* The restriction to vector-valued stream transformer is mainly motivated by notational convenience, as most of the developments here generalize to transformers with heterogenous coefficient sets of the form

$$(A_1^\omega \times \dots \times A_n^\omega) \rightarrow \mathcal{P}(B_1^\omega \times \dots \times B_m^\omega),$$

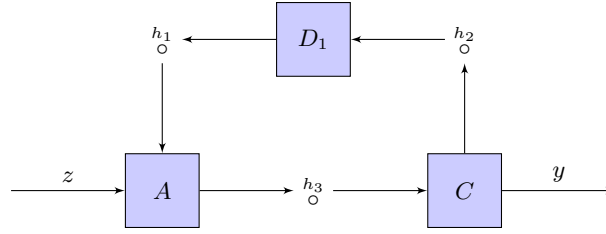
and also to infinite products.

*Remark 8.* Vector-valued stream transformers  $A^{\omega,n} \rightarrow \mathcal{P}(B^{\omega,m})$  may be identified with stream transformers in  $(A^n)^\omega \rightarrow \mathcal{P}((B^m)^\omega)$ , because of the one-to-one relationship of  $(A^n)^\omega$  with  $A^{\omega,n}$ , and  $(B^m)^\omega$  with  $B^{\omega,m}$ . If  $A$  ( $B$ ) is a field then  $A^n$  ( $B^m$ ) can be made into a field in the usual way.

**Definition 2.** Rational stream transformers  $T$  map a vector  $z$  of  $n$  rational input streams to a vector of  $m$  rational output streams by means of the linear mapping

$$z \mapsto R \cdot z, \tag{14}$$

where  $R$  is an  $(n \times m)$ -dimensional matrix  $R$  of rational streams.



**Fig. 2.** Finite Stream Circuit.

*Example 7.* *Stream circuits* [55] are clocked, hardware-like finite structures, which are obtained by (finite) compositions of the rational stream transformers (for  $r \in \mathbb{R}$  and  $\tau$  the vector transpose)

$$\begin{aligned} M_r(z) &:= r \cdot z \\ A(z_1, z_2)^\tau &:= z_1 + z_2 \\ C(z) &:= (z, z)^\tau \\ D_1(z) &:= X \cdot z, \end{aligned}$$

for multiplying a stream by a constant, adding two streams, copying, and delaying a stream. Such a stream circuit is visualized graphically in Figure 2.

*Remark 9.* We can systematically analyze stream circuits, such as the one in Figure 2, by solving the underlying system of stream equations

$$h_1 = X \cdot h_2 \quad h_3 = z + h_1 \quad h_2 = h_3 \quad y = h_3,$$

for the output stream  $y$  to obtain  $y = (1/1-X) \cdot z$ . Thus, the rational stream transformer  $\mathbb{R}^\omega \rightarrow \mathbb{R}^\omega$  that is implemented by the feedback circuit in Figure 2 is given by  $z \mapsto (1/1-X) \cdot z$ . Since  $1/1-X = (1, 1, 1, \dots)$ , the output stream  $y$  is, by stream multiplication (7), of the form  $(\sum_{k=0}^n [X^k]z)_{n \in \mathbb{N}}$ .

Every rational stream transformers can be realized by a stream circuit [55]. In addition, rational stream transformers and weighted automata are equally expressive [55]. Not every feedback loop however gives rise to a stream transformer.

*Remark 10.* Consider for instance the circuit obtained by replacing the one unit delay  $D_1$  in the feedback loop in Figure 2 by a 1-multiplier  $M_1$ . From the definitions of the basic stream transformers we read off the corresponding system of stream equations  $h_1 = 1 \cdot h_2$ ,  $h_3 = z + h_1$ ,  $h_2 = h_3$ , and  $y = h_3$ . This implies  $h_3 = z \cdot h_3$  and therefore  $z = 0$ . But  $z$  is supposed to be an arbitrary input stream. Circuits in which every feedback loop passes through at least one register, however, can be shown to represent a rational stream transformer [55].

*Example 8 (Functional Composition).* A formal power series may also be viewed as a stream transformer  $f(X)$ , where  $X$  is now instantiated with an input stream, say,  $g(X)$  by the "composition"  $f(g(X))$ . Formally, the *functional composition*  $g \circ f$  of two power series  $f := \sum_{k \geq 0} a_k X^k$  and  $g := \sum_{k \geq 1} b_k X^k$ , that is,  $b_0 = 0$ , is a power series with coefficients

$$c_k := \sum_{n \geq 0} \sum_{j_1 + \dots + j_n = k} b_n a_{j_1} \dots a_{j_n}. \quad (15)$$

In case  $A$  is a field of characteristic 0, *Faà di Bruno's formula* provides a more explicit description of these coefficients.

*Example 9 (Compositional Inverse).* Inverses for the functional composition (15) of two power series exist under certain conditions, and may be used for *reverse* computation. Let  $f \in A^\omega$  with  $[X^0]f = 0$  and  $[X^1]f$  is an invertible element of  $A$ . Then there exists the *composition inverse*  $g \in A^\omega$  with  $f \circ g = 1$ . In the case when the coefficient ring is a field of characteristic 0, the  $k$ -th coefficient of  $g^n$ , for  $n \in \mathbb{N}^+$ , is obtained from *Lagrange inversion*  $k[X^k]g^n = n[X^{-n}]f^{-k}$ .

*Composing Stream Transformers.* System design is based on gradually composing or decomposing stream transformers.

**Definition 3.** For suitable stream transformers  $S, T$ :

$$\begin{aligned} \text{magic} &:= (\lambda_-)A^{\omega, n} \\ \text{abort} &:= (\lambda_-)\emptyset \\ S;T &:= (\lambda f)T(S(f)) \\ S \otimes T &:= (\lambda f)(T(f), S(f)) \\ S \sqcup T &:= (\lambda f)S(f) \cup T(f) \\ S \sqcap T &:= (\lambda f)S(f) \cap T(f). \end{aligned}$$

$S;T$  is sequential composition,  $S \otimes T$  is parallel composition,  $S \sqcup T$  realizes angelical and  $S \sqcap T$  demonic non-determinism, which is possibly unbounded.

Notice that these definitions work for arbitrary vector- and multi-valued maps.

*Example 10.* Parallel composition  $S \otimes T$  is visualized in Figure 4, where  $C : A^\omega \rightarrow A^{\omega, 2}$  provides two copies of the input stream, and  $M : A^{\omega, 2} \rightarrow (A^2)^\omega$  merges its two input stream  $(a_k)_{k \in \mathbb{N}}$  and  $(b_k)_{k \in \mathbb{N}}$  into the single output stream  $((a_k, b_k))_{k \in \mathbb{N}}$ .

*Example 11.* Let  $b \in \mathbb{B}^\omega$ , and  $f, g \in A^\omega$ ; then the *conditional stream*  $(b ? f : g)$  is the unique stream satisfying the identities

$$\begin{aligned} \text{hd}(b ? f : g) &= \begin{cases} \text{hd}(f) & \text{if } \text{hd}(b) = 1 \\ \text{hd}(g) & \text{if } \text{hd}(b) = 0 \end{cases} \\ \text{tl}(b ? f : g) &= (\text{tl}(b) ? \text{tl}(f) : \text{tl}(g)). \end{aligned}$$

As a coinductive definition [54], the conditional in Definition 11 is well-defined.

*Remark 11.* Coinductive stream specifications are also said to be *behavioral differential equations* by identifying the tail operation  $\text{tl}(\cdot)$  with differentiation [53]. These behavioural differential equations have an operational reading, from which algorithms for producing the elements of this infinite stream one-by-one can be easily and progressively derived.

There are unique operators on streams satisfying co-inductive identities for addition, multiplication, and inverse ([53], Theorem 3.1).

*Example 12.*

$$\begin{aligned} \text{tl}(f + g) &= \text{tl}(f) + \text{tl}(g) & \text{hd}(f + g) &= \text{hd}(f) + \text{hd}(g) \\ \text{tl}(f \cdot g) &= \text{tl}(f) \cdot g + \text{hd}(f) \cdot \text{tl}(g) & \text{hd}(f \cdot g) &= \text{hd}(f) \cdot \text{hd}(g) \\ \text{tl}(f^{-1}) &= -(\text{hd}(f^{-1}) \cdot \text{tl}(f)) \cdot f^{-1} & \text{hd}(f^{-1}) &= \text{hd}(f)^{-1} \end{aligned}$$

*Example 13 (Map Transformer).* For  $\phi : A \rightarrow B$ , the stream transformer  $M_\phi : A^\omega \rightarrow B^\omega$  is co-inductively defined as the unique solution of the identities ( $f \in A^\omega$ )

$$\begin{aligned} \text{hd}(M_\phi(f)) &= \phi(\text{hd}(f)) \\ \text{tl}(M_\phi(f)) &= M_\phi(\text{tl}(f)). \end{aligned}$$

*Example 14 (Boolean Transformers).* Let  $b, c \in \mathbb{B}^\omega$  and  $Neg$  such that  $Neg(1) = 0$  and  $Neg(0) = 1$ ; then:

$$\begin{aligned} -b &:= M_{Neg}(b) \\ b \&c &:= b ? c : 0 \\ b | c &:= b ? 1 : c. \end{aligned}$$

#### 4.1 Refinement

A nondeterministic stream transformer  $S$  is a *refinement* of another nondeterministic stream transformer  $T$ , written  $S \preceq T$  if  $S$  is *more deterministic* than  $T$ ; that is:

$$S \preceq T := (\forall f \in A^\omega) S(f) \subseteq T(f), \quad (16)$$

where  $\subseteq$  is interpreted component-wise. Clearly,  $\preceq$  is a partial order, and refinement is compatible with composition operators.

**Lemma 3 (Compatibility).** *Let  $S_1 \preceq T_1$ ,  $S_2 \preceq T_2$ ; then:*

1.  $S_1; S_2 \preceq T_1; T_2$
2.  $S_1 \otimes S_2 \preceq T_1 \otimes T_2$

3.  $S_1 \sqcup S_2 \preceq T_1 \sqcup T_2$
4.  $S_1 \sqcap S_2 \preceq T_1 \sqcap T_2$

**Definition 4 (Stream Predicates).** A set  $P \in \mathcal{P}(A^{\omega,n})$  is also said to be a stream predicate. We simply write  $A^{\omega,n} \upharpoonright_P$  instead of the set comprehension  $\{f \in A^{\omega,n} \mid f \in P\}$ .

**Definition 5 (Contracts).**

Let  $P \in A^\omega$  be a stream predicate, and  $Q := (Q_f)_{f \in A^\omega}$  be an  $A^\omega$ -indexed family of stream predicate. Then, a contractual stream transformer  $T$  has the (dependent) signature

$$T : (\Pi f \in A^{\omega,n} \upharpoonright_P) B^{\omega,m} \upharpoonright_{Q_f} \quad (17)$$

Now,  $P$  is said to be the precondition of  $T$  and  $Q$  is the postcondition of  $T$ . The interface  $[P, Q]$  denotes the set of stream transformers with precondition  $P$  and postcondition  $Q$ .

Whenever we write  $T(f)$  we implicitly assume that this application is well-defined in that  $f \in P$  and also that  $T(f) \in Q_f$ . Moreover,  $[P', Q'] \subseteq [P, Q]$  whenever  $P \subseteq P'$  and  $Q' \subseteq Q$  (pointwise). Notice also that contracts usually include *fairness conditions* for constraining nondeterminism [41].

## 5 Causality

We restrict our investigations to stream transformers whose outputs are determined by the history of inputs. The *prefix*  $f \upharpoonright_n$  of length  $n \in \mathbb{N}$  for a stream  $f$  is the finite word  $([X^0]f) \dots ([X^{n-1}]f)$  of the first  $n$  coefficients of  $f$ . Now, for a set  $F$  of streams,  $F \upharpoonright_n$  denotes the set of prefixes  $f \upharpoonright_n$  for  $f \in F$ . Similarly,  $[X^k]F$  denotes the set  $\{[X^k]f \mid f \in F\}$ . Causal stream transformers, at least in the single-valued case, are discussed in [11].

**Definition 6 (Causal Stream Transformers).** The stream transformer  $T : A^{\omega,n} \rightarrow \mathcal{P}(B^{\omega,m})$  is  $\delta$ -causal, for  $\delta \in \mathbb{N}$ , if for all  $k \in \mathbb{N}$  and  $f, g \in A^{\omega,n}$

$$f \upharpoonright_k = g \upharpoonright_k \text{ implies } [X^{k+\delta}]T(f) = [X^{k+\delta}]T(g). \quad (18)$$

In case  $T$  is 0-causal then  $T$  is also said to be (weakly) causal, and if  $T$  is  $\delta$ -causal for  $\delta > 0$  then  $T$  is strongly causal.

Every  $\delta$ -causal transformer is also  $\delta'$ -causal for  $\delta' < \delta$ . In particular, every strongly causal transformer is also weakly causal. This notion of strong causality is also closely related to the *guardedness* condition of de Bakker [4].

An alternative characterization of  $\delta$ -causality to the one in Definition 6 is easily established by natural induction on  $n$ .

**Proposition 7.** A stream transformer  $T$  is  $\delta$ -causal, for  $\delta \geq 0$ , if and only if for all  $f, g \in A^\omega$  and  $k \in \mathbb{N}$ ,

$$f \upharpoonright_k = g \upharpoonright_k \text{ implies } T(f) \upharpoonright_{k+\delta} = T(g) \upharpoonright_{k+\delta}. \quad (19)$$

*Remark 12.* Strongly causal stream transformer are often used for specifying clocked, hardware-like finite structures. In system design, however, working with the larger set of causal stream transformers is often preferred [6].

*Example 15.* A clocked  $N$ -bit register holding a value belonging to the alphabet  $A := \mathbb{B}^N$  may be viewed as a causal stream transformer  $(A \times \mathbb{B})^\omega \rightarrow A^\omega$ , where the second input  $\mathbb{B} := \{0, 1\}$  corresponds to an enabling stream stating whether the incoming value on the first component ought to be loaded into the register or be ignored at a given time (or clock tick).

*Example 16.* The stream transformers from Example 16 are all (0-) causal, and  $D_1$  is also 1-causal.

*Example 17 (Consing).* Consider, for  $a \in A$ , the family  $Cons_a : A^\omega \rightarrow A^\omega$  of stream transformers, which map  $z \mapsto a + X \cdot z$ . These are 1-causal transformers.  $Cons_0$ , in particular, yields the unit delay  $D_1$  from Example 16.

If one is to implement a causal map by means of a machine that consumes successive values in an input stream, and produces a stream of successive values, it seems necessary that the map is continuous. Otherwise, the whole input stream would be needed at once, and an output would be forthcoming only *at the end of time*. Continuity means that finite information concerning the output of the function is determined by finite information concerning its input.

*Example 18.* The stream transformer  $\text{pos}(f)$  which returns the stream 1 if all elements of  $f$  are positive, and the stream 0 otherwise, is not causal.

### 5.1 Type 2 Computability

Type 2 Turing machines are extensions of Turing machine with a read-only input tape, a write-once output tape, and a working tape. Programs on a Type 2 machine run forever, generating increasingly more digits of the output [61]. The map  $T$  is considered to be computable on a Type 2 machine if there is a *total* program, which takes finite time to write any number of symbols on the output tape regardless of the input. The working of a Type 2 machine up to any point of computation clearly only depends on a finite prefix of the input tape, and, therefore, every input tape that agrees with the given one up to that point would have caused the machine to write the same answer to the same output cells. Therefore all Type 2 computable functions are causal. As we will see below, causal functions are continuous. This implies the well-known fact that all Type 2 computable functions are continuous (see also Appendix E).

### 5.2 Mealy Realizability

A Mealy machine is an intensional description of a causal stream transformer  $T$ , where the state holds enough information to determine  $[X^k]T(f)$  from  $[X^k]f$ . More precisely, the set of weakly causal stream transformers  $A^\omega \rightarrow B^\omega$  is isomorphic to the functions  $A^+ \rightarrow B$  from finite words over  $A$  in  $A^+$  to  $B$ , and a Mealy machine can be constructed for realizing such a word level function.

*Example 19.* The  $N$ -bit register from Example 15 can be realized by a Mealy machine with state-space  $A := 2^N$ , initial state  $a_0 \in A$ , where  $a_0$  is chosen arbitrarily, and the transition function  $\delta : (A \times 2) \times A \rightarrow A \times A$  defined as

$$\delta((a, b), q) := \begin{cases} (q, q) & \text{if } b = 0 \\ (a, a) & \text{if } b = 1. \end{cases}$$

Not every causal stream transformer, however, can be realized by a finite-state Mealy machine.

*Example 20.* Let  $\text{Prime} : \mathbb{U}^\omega \rightarrow \mathbb{B}^\omega$ , where  $\mathbb{U}$  is a singular set, be the Boolean-valued stream such that  $[X^k]\text{Prime}$  holds if and only if  $k$  is a prime number. Now, the minimal Mealy machine realizing  $\text{Prime}$  has a countably infinite number of states.

### 5.3 Composition

**Proposition 8.** *Let  $S$  be  $\delta_S$ -causal and  $T$  be  $\delta_T$ -causal; then:*

1. *magic, abort are weakly causal;*
2.  *$S;T$  is  $(\delta_S + \delta_T)$ -causal;*
3.  *$S \otimes T$  is  $\min(\delta_S, \delta_T)$ -causal;*
4.  *$S \sqcup T$  and  $S \sqcap T$  are  $\max(\delta_S, \delta_T)$ -causal.*

As a consequence,  $\delta$ -causal stream transformers, for  $\delta > 0$ , are not preserved under composition, which results in an unfortunate inflexibility in system design. For instance, if we want to represent a stream transformer  $T$  as a sequential composition  $T_1; T_2$  of two 1-causal stream transformers then we always have to accept a delay by at least two. Both weakly and strongly causal stream transformers, however, are preserved, as a consequence of Proposition 8 under sequential and parallel composition.

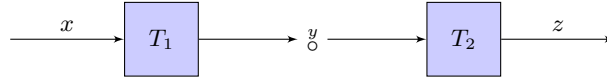
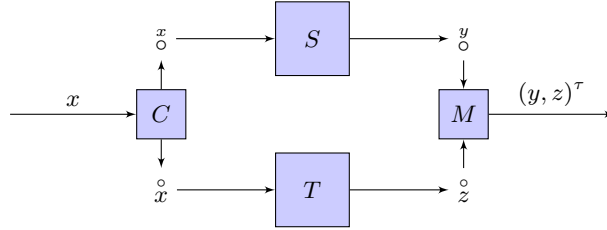
**Lemma 4.** *For weakly (strongly) causal stream transformers  $S, T$  (with suitable domains and codomains)  $S;T$ ,  $S \otimes T$ ,  $S \sqcap T$ , and  $S \sqcup T$  are weakly (strongly) causal.*

### 5.4 Fixpoints

Strongly causal stream transformers have a unique fixpoint.

*Example 21.* For a 1-causal stream transformer  $T : A^\omega \rightarrow A^\omega$  one obtains a unique fixpoint  $f^* \in A^\omega$  with  $f^* = T(f^*)$  by, recursively constructing Mealy machines whose transitions may also depend on outputs [46].



**Fig. 3.** Sequential Composition.**Fig. 4.** Parallel Composition with Copying and Merging.

In contrast, fixpoints for weakly causal transformers may not exist and they may not be unique.

*Example 22.*

1.  $Succ(z) := z + 1$  has no fixpoint in  $\mathbb{R}^\omega$ ;
2. Every stream in  $A^\omega$  is a fixpoint of the identity transformer  $Id(z) := z$ .

*Remark 13.* Synchronous languages such as Lustre [25] or Esterel [7] therefore use syntactic restrictions for avoiding causal loops. The approach taken in the symbolic analysis laboratory SAL [5] relies on *verification conditions* asserting the absence of causal loops. The approach of SAL of semantically characterizing causality errors is more general. But it is also undecidable in general, since it can depend on arbitrary data properties.

*Example 23.* Let  $c, d, e$  be Boolean-valued streams. Then a solution  $(f, g) \in (\mathbb{B} \times \mathbb{B})^\omega$  of the stream equalities

$$\begin{aligned} f &= c ? \neg g : d \\ g &= c ? e : f \end{aligned}$$

is a fixpoint of

$$T(f, g) := c ? (\neg g, e) : (d, f)$$

There is no causal loop, because  $f$  is causally dependent on  $g$  only when  $c$  is true, and vice-versa only when it is false. Therefore, these stream equations are acceptable in SAL.

## 6 Contraction

We develop metric-based characterizations both of weakly and of strongly causal stream transformers.

**Definition 7 (Contractions for Multivalued Maps).** *Let  $(M, d_M), (N, d_N)$  be metric spaces. A multivalued map  $T : M \rightarrow \mathcal{P}(N)$  is said to be an  $l$ -contraction, for  $0 \leq l \leq 1$ , if for all  $x, y \in M$  and for all  $u \in T(x)$  there exists  $v \in T(y)$  such that*

$$d_M(u, v) \leq l \cdot d_N(x, y). \quad (20)$$

*If  $l < 1$  then the multivalued  $T$  is said to be contractive, and if  $l = 1$  then  $T$  is nonexpansive.*

*Remark 14.* Contraction for multivalued maps  $x \mapsto \{x\}$  coincides with the usual notion of contraction on singlevalued maps as stated, for example, in Section 2.

**Lemma 5.** *A stream transformer  $T : A^{\omega, n} \rightarrow \mathcal{P}(A^{\omega, m}) \setminus \emptyset$  is  $\delta$ -causal, for  $\delta \geq 0$ , if and only if it is  $2^{-\delta}$ -contractive.*

*Proof.* Let  $f, g \in A^{\omega}$  and let  $k \in \mathbb{N} \cup \{+\infty\}$  such that  $d(f, g) = 2^{-k}$ , and therefore  $f \upharpoonright_k = g \upharpoonright_k$  ( $h \upharpoonright_{\infty} := h$ ).

1. ( $\Rightarrow$ ) By assumption,  $T$  is  $\delta$ -causal, and therefore

$$T(f) \upharpoonright_{k+\delta} = T(g) \upharpoonright_{k+\delta}.$$

Consequently, for all  $u \in T(f)$  there exists  $v \in T(g)$  with  $u \upharpoonright_{k+\delta} = v \upharpoonright_{k+\delta}$ , that is  $d(u, v) \leq 2^{-(k+\delta)} = 2^{-\delta} \cdot d(f, g)$ .

2. ( $\Leftarrow$ ) By assumption, for all  $u \in T(f)$  there exists  $v \in T(g)$  such that

$$d(u, v) \leq 2^{-\delta} \cdot d(f, g) = 2^{-(k+\delta)},$$

and therefore  $T(f) \upharpoonright_{k+\delta} \subseteq T(g) \upharpoonright_{k+\delta}$ . Similarly, also  $T(g) \upharpoonright_{k+\delta} \subseteq T(f) \upharpoonright_{k+\delta}$  holds.

From Lemma 5 one obtains metric-based characterizations for weak and also for strong causality.

**Corollary 3.** *A stream transformer  $T : A^{\omega, n} \rightarrow \mathcal{P}(B^{\omega, m}) \setminus \emptyset$  is*

1. *weakly causal if and only if it is nonexpansive, and*
2. *strongly causal if and only if it is contractive.*

### 6.1 Fixpoints of Stream Transformers

Recall that a *strictly contracting* or *shrinking* map  $T$  satisfies  $d(T(f), T(g)) < d(f, g)$  for all  $f, g$ . Now, any shrinking map in a spherically complete ultrametric space has a unique fixpoint ([43], Theorem 1).

**Definition 8.** For  $T : A^{\omega, n} \rightarrow A^{\omega, n}$  a strictly contractive stream transformer,  $T^+$  is the unique fixpoint of  $\mathcal{T} := (\lambda S) S; T$ .

$T^+$  is well-defined by ([43], Theorem 1), since, using Lemma 2, maps of (deterministic) stream transformers form an ultrametric and spherically complete space, and  $\mathcal{T}$  is strictly contractive in this space.

$$\begin{aligned} d(\mathcal{T}(S_1), \mathcal{T}(S_2)) &= d((S_1; T), (S_2; T)) \\ &= \sup_f d(T(S_1(f)), T(S_2(f))) \\ &< \sup_f d(S_1(f), S_2(f)) \\ &= d(S_1, S_2). \end{aligned}$$

Using fixpoint results for strictly contractive multivalued maps (Theorem 2 in [35]) Definition 8 is generalized to obtain a unique fixpoint of transformers of strictly contracting multivalued stream transformers.

### 6.2 Lipschitz Contraction

We show that contraction on multivalued maps with nonempty compact codomains coincides with contraction with respect to the Hausdorff metric as defined in [38].

**Definition 9 (Lipschitz Contraction).** Let  $(M, d_M)$  and  $(N, d_N)$  be metric spaces. A multivalued map  $T : M \rightarrow \text{CB}(N)$  is a Lipschitz mapping if and only if

$$\mathcal{H}_{d_N}(T(x), T(y)) \leq l \cdot d_M(x, y), \quad (21)$$

for all  $x, y \in M$ , where  $l > 0$  is the Lipschitz constant for  $T$ . In these cases we also say that  $T$  is  $l$ -Lipschitz. Furthermore, if  $l = 1$  then  $T$  is Lipschitz nonexpansive, and if  $0 < l < 1$  then  $T$  is Lipschitz contracting.

*Remark 15.* A multivalued mapping  $T$  with Lipschitz constant  $l$  is uniformly continuous, since for arbitrary  $\varepsilon > 0$  set  $\delta := \varepsilon/l$  to obtain

$$d(T(x), T(y)) \leq l \cdot d(x, y) < l \cdot \delta = \varepsilon$$

from  $d(x, y) < \delta$ .

The continuous image of a compact set is compact. Assume an open cover of  $T(K)$ . As  $T$  is continuous, the inverse image of those open sets form an open cover for  $K$ . Since  $K$  is compact, there is a finite subcover of  $T(K)$ , and, by construction, the images of the finite subcover give a finite subcover of  $T(K)$ , and therefore  $T(K)$  is compact.

**Lemma 6.** *If  $T : M \rightarrow \text{Comp}(N)$  is continuous and  $K \in \text{Comp}(M)$  then  $T(K) \in \text{Comp}(N)$ .*

In particular, the image  $T(K)$  of a compact set  $K$  with respect to a Lipschitz map  $T$  is compact.

We present some elementary results which will be used in later sections. The proofs of many of these facts are straightforward (see also Proposition 8).

**Proposition 9.** *If  $T : L \rightarrow \text{Comp}(M)$  is  $l_T$ -Lipschitz and  $S : M \rightarrow \text{Comp}(N)$  is  $l_S$ -Lipschitz, then  $S \circ T : L \rightarrow \text{Comp}(N)$  is  $(l_T \cdot l_S)$ -Lipschitz.*

**Proposition 10.** *If  $S, T : M \rightarrow \text{Comp}(N)$  are  $l_S$ - and  $l_T$ -Lipschitz, respectively, then  $(S \sqcup T) : M \rightarrow \text{Comp}(N)$  is  $\max(l_S, l_T)$ -Lipschitz.*

Proposition 10 gives a technique for constructing a Lipschitz mapping from a finite number of single-valued Lipschitz mapping by "unioning their graphs at each point". The closure condition (10) of Lipschitz maps under *angelic non-determinism*  $S \sqcup T$  can be generalized to an arbitrary  $\mathcal{I}$ -index family  $(T_i)_{i \in \mathcal{I}}$ .

*Remark 16.* The set of Lipschitz maps, however, is not closed under *demonic nondeterminism*  $(S \sqcap T)$ , which maps  $x \mapsto S(x) \cap T(x)$  (assuming all these intersections are non-empty). Indeed, the demonic combination of Lipschitz maps might not even be continuous [38].

For stream transformers with non-empty compact images, Lipschitz contraction (Definition 9) is equivalent to the notion of contraction mappings in Definition 7.

**Lemma 7.** *Let  $(M, d_M)$ ,  $(N, d_N)$  be metric spaces,  $T : M \rightarrow \text{Comp}(N)$  a multivalued map, and constant  $l$  with  $0 < l \leq 1$ ; then:  $T$  is an  $l$ -contraction if and only if  $T$  is  $l$ -Lipschitz.*

*Proof.* For given  $x, y \in M$  we may, without loss of generality, assume

$$\mathcal{H}_{d_N}(T(x), T(y)) = \sup_{u \in T(x)} (d(u, T(y))).$$

1. ( $\Rightarrow$ ) As a consequence of this assumption,  $\mathcal{H}_{d_N}(T(x), T(y)) \leq d(u, T(y))$  for all  $u \in T(x)$ . Since,  $T$  is  $l$ -contractive, there is, by definition,  $v_0 \in T(y)$  such that  $d_N(u, v_0) \leq l \cdot d(x, y)$ , and consequently:

$$d(u, T(y)) = \inf_{v \in T(y)} d_N(u, v) \leq d_N(u, v_0) \leq l \cdot d(x, y).$$

Altogether,  $\mathcal{H}_{d_N}(T(x), T(y)) = \sup_{u \in T(x)} d(u, T(y)) \leq l \cdot d(x, y)$ .

2. ( $\Leftarrow$ ) For all  $u \in F$ ,

$$\inf_{v \in T(y)} d(u, v) = d(u, T(y)) \leq \mathcal{H}_{d_N}(T(x), T(y)).$$

By compactness of  $T(y)$ ,<sup>1</sup> there exists  $v_0 \in T(y)$  such that

$$d(u, v_0) = d(u, T(y)) \leq \mathcal{H}_{d_N}(T(x), T(y)).$$

*Remark 17.* The proof of the left-to-right statement of Lemma 7 only requires the non-empty images of  $T$  to be closed (and bounded), and not necessarily compact.

Lemma 7 and Corollary 3 together yield the correspondence between causal and Lipschitz contractive maps.

**Corollary 4.** *The stream transformer  $T : A^{\omega, n} \rightarrow \text{Comp}(B^{\omega, m})$  is*

1. *weakly causal if and only if it is Lipschitz nonexpansive, and*
2. *strongly causal if and only if it is Lipschitz contractive.*

## 7 Fixpoints

We have already seen how fixpoints for strictly causal functions can be constructed using recursive Mealy machines (Example 21). We pursue, however, a less syntax- and machine-oriented path, for solving stream equations, and we compute fixpoints for vector- and multivalued maps.

We are interested in fixpoints of stream transformers  $T : A^{\omega, n} \rightarrow \mathcal{P}(A^{\omega, n})$ , that is, vectors of streams  $f^*$  with

$$f^* \in T(f^*). \quad (22)$$

For deterministic maps  $T$  this inequality reduces to the fixpoint equality  $f^* = T(f^*)$ . Since the mapping  $\iota : A^{\omega, n} \rightarrow \text{Comp}(A^{\omega, n})$ , given by  $f \mapsto \{f\}$  for each  $f \in A^{\omega, n}$ , is an isometry, the fixpoint theorems in this paper for multivalued mappings are generalizations of their single-valued analogues.

### 7.1 Set Transformers

We show that the *weakest pre* and *strongest post* of Lipschitz contraction (equivalently, strongly causal) maps have a unique fixpoint, and we formulate a corresponding fixpoint induction principle. Moreover, the unique fixpoint of  $\text{sp}_T$  includes all fixpoints of the multivalued map  $T$ .

**Definition 10.** *Let  $T : M \rightarrow \text{Comp}(N)$  be a Lipschitz map; then:*

1. *The strongest post  $\text{sp}_T : \text{Comp}(M) \rightarrow \text{Comp}(N)$  is the set transformer*

$$\text{sp}_T(P) := T(P); \quad (23)$$

---

<sup>1</sup> For every compact topological space  $M$  and  $f : M \rightarrow \mathbb{R}$  continuous: the image  $f(M)$  is bounded and there exists  $a, b \in M$  such that  $f(a) = \inf_{x \in M} f(x)$  and  $f(b) = \sup_{x \in M} f(x)$ .

2. The weakest pre  $\text{wp}_T : \text{Comp}(M) \rightarrow \text{Comp}(N)$  is the set transformer

$$\text{wp}_T(Q) := \{x \in \text{Comp}(M) \mid T(x) \subseteq Q\}. \quad (24)$$

It follows that the  $\text{sp}_T$  and  $\text{wp}_T$  are *adjoint*.

$$\text{sp}_T(P) \subseteq Q \iff P \subseteq \text{wp}_T(Q) \quad (25)$$

for all  $P \in \text{Comp}(M)$ ,  $Q \in \text{Comp}(N)$ . As a consequence,  $P \subseteq (\text{wp}_T \circ \text{sp}_T)(P)$ ,  $(\text{sp}_T \circ \text{wp}_T)(Q) \subseteq Q$ , and  $\text{sp}_T$  and  $\text{wp}_T$  are both *monotonic* with respect to set inclusion.

**Lemma 8 (WP Stream Calculus).**

Let  $Q \in \text{Comp}(A^{\omega,n})$  and  $S, T : A^{\omega,n} \rightarrow \text{Comp}(B^{\omega,m})$  strongly causal; then:

$$\begin{aligned} \text{wp}_{\text{magic}}(Q) &= A^{\omega,n} \\ \text{wp}_{\text{abort}}(Q) &= \emptyset \\ \text{wp}_{S;T}(Q) &= \text{wp}_S(\text{wp}_T(Q)) \\ \text{wp}_{S \sqcap T}(Q) &= \text{wp}_S(Q) \cap \text{wp}_T(Q) \\ \text{wp}_{S \sqcup T}(Q) &= \text{wp}_S(Q) \cup \text{wp}_T(Q) \\ \text{wp}_{T+}(Q) &= \text{wp}_{T+}(\text{wp}_T(Q)) \end{aligned}$$

*Remark 18.* A Hoare-like stream contract  $\{P\} T \{Q\}$  for a nondeterministic stream transformer  $T$  with *precondition*  $P$  and *postcondition*  $Q$  holds if and only if  $P \subseteq \text{wp}_T(Q)$ , or, equivalently,  $\text{sp}_T(P) \subseteq Q$ . Corresponding rules of a Hoare-like calculus are derived from the weakest precondition calculus (Lemma 8).

**Proposition 11.** If  $T : M \rightarrow \text{Comp}(N)$  is  $l$ -Lipschitz then both  $\text{sp}_T$  and  $\text{wp}_T$  are  $l$ -Lipschitz.

Consequently, if  $T$  is Lipschitz contracting, then both  $\text{sp}_T$  and  $\text{wp}_T$  are Lipschitz contracting, and we obtain unique fixpoints for  $\text{sp}_T, \text{wp}_T : \text{Comp}(M) \rightarrow \text{Comp}(M)$  from Banach's contraction principle, since  $(\text{Comp}(M), \mathcal{H}_{d_M})$  is a Cauchy complete metric space.

**Lemma 9.** Let  $(M, d_M)$  be a metric space. If  $T : M \rightarrow \text{Comp}(M)$  is Lipschitz contractive then

1.  $\text{sp}_T, \text{wp}_T : \text{Comp}(M) \rightarrow \text{Comp}(M)$  have unique fixpoints, say,  $\text{fix}(\text{sp}_T)$  and  $\text{fix}(\text{wp}_T)$ , respectively.
2. For  $F_{k+1} = \text{sp}_T(F_k)$ ,  $G_{k+1} = \text{wp}_T(G_k)$  and arbitrary  $F_0, G_0 \in \text{Comp}(M)$ ,

$$\begin{aligned} \text{fix}(\text{sp}_T) &= \lim_{k \rightarrow \infty} F_k \\ \text{fix}(\text{wp}_T) &= \lim_{k \rightarrow \infty} G_k. \end{aligned}$$

3. For  $k \in \mathbb{N}$ :

$$\begin{aligned} \mathcal{H}_{d_M}(F_k, \text{fix}(\text{sp}_T)) &\leq l^k / (1-l) \cdot \mathcal{H}_{d_M}(F_0, F_1) \\ \mathcal{H}_{d_M}(G_k, \text{fix}(\text{wp}_T)) &\leq l^k / (1-l) \cdot \mathcal{H}_{d_M}(G_0, G_1). \end{aligned}$$

*Remark 19.* Compared with the usual Knaster-Tarski fixpoint iteration of Scott-continuous transformers, the main advantage of the iteration in Lemma 9 is that, at any iteration, there is a quantitative measure of the distance to the fixpoint and also of the progress towards this fixpoint. On the other hand, the iterations in Lemma 9 generally neither are under- nor overapproximations of fixpoints.

**Lemma 10.** *Every fixpoint of a contractive  $T : M \rightarrow \text{Comp}(M)$  is included in  $\text{fix}(\text{sp}_T)$ .*

*Proof.* For a fixpoint  $f^*$  of  $T$ , that is,  $f^* \in T(f^*)$  define the iteration  $F_0 := \{f^*\}$   $F_{k+1} := \text{sp}_T(F_k) = \{g \in T(f) \mid f \in F_k\}$ . By induction on  $k$ ,  $f^* \in F_k$  for each  $k \in \mathbb{N}$ , and therefore  $f^* \in \lim_{k \rightarrow \infty} F_k = \text{fix}(\text{sp}_T)$ .

The subset relation in Lemma 10 may be strict [38].

*Example 24 ([38]).* For simplicity, we work in the unit interval  $[0, 1]$  with the usual Euclidean metric. Let

$$t(x) := \begin{cases} 1/2(x+1) & \text{if } 0 \leq x \leq 1/2 \\ (-1/2)x + 1 & \text{if } 1/2 < x \leq 1 \end{cases}$$

and the multivalued  $T : I \rightarrow \text{Comp}(I)$  maps  $x \mapsto \{0\} \cup \{t(x)\}$ . Evidently,  $T$  is a multivalued contraction, the fixpoints of the multivalued  $T$  are  $\{0, 2/3\}$ , and the fixpoint of the set transformer  $\text{sp}_T$  induced by  $T$  is  $\{0, 2/3, t(0), t^2(0), t^3(0), \dots\}$ .

When applying Lemma 9 to streams  $A^{\omega, n}$  and its ultrametric  $d$ , then the convergence bounds can be improved to  $l^k$  with  $l = (1/2)^\delta$  for some  $\delta > 0$ , since  $d$  is bounded and the application of the strengthened triangle inequality in the proof of Banach's contraction principle yields the improved bound  $l^k$ .

**Theorem 1.** *Let  $T : A^{\omega, n} \rightarrow \text{Comp}(B^{\omega, m})$  be  $\delta$ -causal, for  $\delta > 0$ . With the notation and the iterations  $F_k, G_k$  as in Lemma 9, the unique fixpoints  $\text{fix}(\text{sp}_T)$  and  $\text{fix}(\text{wp}_T)$  are obtained as the limits of  $F_k$  and  $G_k$ , respectively, as  $k \rightarrow \infty$ . Furthermore, for  $k \in \mathbb{N}$ :*

$$\begin{aligned} \mathcal{H}_d(F_k, \text{fix}(\text{sp}_T)) &\leq (1/2)^{k\delta} \\ \mathcal{H}_d(G_k, \text{fix}(\text{wp}_T)) &\leq (1/2)^{k\delta}. \end{aligned}$$

**Definition 11 (Inclusive Predicates).** *A stream predicate  $P \subseteq A^{\omega, n}$  is said to be inclusive if for any convergent sequence  $(f_k)_{k \in \mathbb{N}}$  of streams in  $A^{\omega, n}$  the following holds:*

$$((\forall k \in \mathbb{N}) f_k \in P) \Rightarrow (\lim_{k \rightarrow \infty} f_k) \in P$$

**Corollary 5 (Induction Principle for  $\text{sp}_T$ ).**

*Let  $P \subseteq A^{\omega, n}$  be an inclusive predicate and  $T : A^{\omega, n} \rightarrow \text{Comp}(A^{\omega, n})$  contracting; then*

$$(\forall Q \subseteq P) \text{sp}_T(Q) \subseteq P \text{ implies } \text{fix}(\text{sp}_T) \subseteq P. \quad (26)$$

*Proof.* For  $F_0 := \emptyset$  and  $F_{k+1} = \text{sp}_T F_k$  we get, by induction on  $k$  and the assumption above, that  $F_k \subseteq P$  for all  $k \in \mathbb{N}$ . Since  $P$  is inclusive,  $\text{fix}(\text{sp}_T) = \lim_{k \rightarrow \infty} F_k \subseteq P$ .

*Remark 20.* For the adjointness of  $\text{sp}_T$  and  $\text{wp}_T$  we might also use the equivalent assumption  $(\forall Q \subseteq P) Q \subseteq \text{wp}_T(P)$  in the induction principle for  $\text{sp}_T$  (see Corollary 5).

## 7.2 Contractive Maps

An identically empty mapping  $T$  can not have fixed points, since  $f \notin \emptyset = T(f)$ . Also, we restrict the codomain of multivalued mappings to closed sets only, since any closed subset of a Cauchy complete space is Cauchy complete.

**Lemma 11.** *A nondeterministic contractive stream transformer  $T$  has a fixpoint if (1)  $T$  is not identically empty and (2)  $T(f)$  is closed for all streams  $f$ .*

*Proof.* Since  $T$  is not identically empty, there exist  $f_0, f_1 \in A^\omega$  such that  $f_1 \in T(f_0)$ . Now, since  $T$  is contractive there is  $0 \leq l < 1$  and a  $f_2 \in T(f_1)$  with  $d(f_1, f_2) \leq l \cdot d(f_0, f_1)$ . In this way, we recursively construct a sequence  $(f_k)$  such that  $f_{k+1} \in T(f_k)$  and

$$d(f_{k+1}, f_{k+2}) \leq l \cdot d(f_k, f_{k+1}) \leq \dots \leq l^k \cdot d(f_0, f_1) \leq l^k,$$

for every  $k \in \mathbb{N}$ . The strengthened triangle inequality for the ultrametric  $d$  yields

$$d(f_k, f_m) \leq \max(d(f_k, f_{k+1}), \dots, d(f_{m-1}, f_m)) \leq l^k, \quad (27)$$

which implies that the sequence  $(f_k)$  is Cauchy in  $(A^{\omega, n}, d)$ . For Cauchy completeness,  $(f_k)$  therefore converges to some  $f^* \in A^{\omega, n}$ . From the inequality (27) we conclude in the limit  $m \rightarrow \infty$  that

$$d(f_k, f^*) \leq l^k. \quad (28)$$

Since  $T$  is a contraction, there is a sequence  $(g_k)_{k \in \mathbb{N}}$  with  $d(f_{k+1}, g_k) \leq l \cdot d(f_k, f^*)$ . Therefore, using inequality (28):

$$\begin{aligned} d(f^*, g_k) &\leq \max(d(f^*, f_{k+1}), d(f_{k+1}, g_k)) \\ &\leq \max(d(f^*, f_{k+1}), l \cdot d(f_k, f^*)) \\ &= l^{k+1}. \end{aligned}$$

Now, since  $\lim_{k \rightarrow \infty} d(f^*, g_k) = 0$ ,

$$\lim_{k \rightarrow \infty} g_k = f^*.$$

But  $g_k \in T(f^*)$  and all images of  $T$  are closed, and therefore the limit  $f^*$  of the sequence  $(g_k)$  is an element of  $T(f^*)$ . Altogether,  $f^* \in T(f^*)$ .



In general, fixpoints of contractive multivalued maps in the sense of Definition 7 are not unique, but a slightly stronger requirement on contractiveness implies uniqueness of fixpoints.

**Definition 12 (Strong Contractions).**

Let  $T$  be a nondeterministic stream transformer and  $0 \leq l < 1$ . If for all  $f, g \in A^{\omega, n}$  and for all  $u \in T(f)$  and  $v \in T(g)$  such that  $d(u, v) \leq l \cdot d(T(f), T(g))$ , then  $T$  is a strong contraction with Lipschitz constant  $l$ .

*Remark 21.* If  $\emptyset \notin T(A^{\omega, n})$  then every strong  $l$ -contraction also is a (weak)  $l$ -contraction in the sense of Definition 7. Moreover, a deterministic stream transformer  $T$  is  $l$ -contractive if and only if it is strongly  $l$ -contractive.

**Lemma 12.** *A nondeterministic strongly contractive stream transformer has at most one fixpoint.*

*Proof.* We assume fixpoints  $f^*, g^*$  of  $T$ . Since  $T$  is strongly contractive, there is an  $l$  with  $0 < l < 1$  such that

$$d(f^*, g^*) \leq \max\{d(u, v) \mid u \in T(f^*), v \in T(g^*)\} \leq l \cdot d(f^*, g^*).$$

Consequently,  $d(f^*, g^*) = 0$ , and any two fixpoints of  $T$  are equal.

*Remark 22.* For deterministic maps, the fixpoint iteration with  $f_{k+1} \in T(f_k)$ , as constructed in the proof of Lemma 11 reduces to the *Picard iteration*  $f_{k+1} = T(f_k)$ . Moreover, uniqueness of fixpoints for deterministic,  $\delta > 0$ -causal maps can also be shown directly, since the equality of two arbitrary fixpoints  $f^*, g^*$  directly follows from  $d(f^*, g^*) = d(T(f^*), T(g^*)) \leq 2^{-\delta} \cdot d(f^*, g^*)$ .

The following result directly follows from Lemmata 5, 11, and 12.

**Theorem 2.** *A strongly causal stream transformer  $T$  has a fixpoint if it is (1) not identically empty, and (2)  $T(f)$  is closed for each stream  $f$ . If, in addition,  $T$  is strongly contracting then this fixpoint is unique.*

As an immediate consequence of Theorem 2 we get an automata-free construction of unique fixpoints for strongly causal deterministic stream transformers over streams with infinite value sets.

**Corollary 6.** *Strongly causal deterministic stream transformers have a unique fixpoint.*

### 7.3 Fixpoint Induction

The fixpoint iteration for strongly causal stream transformers in Theorem 2 is used to derive a *fixpoint induction principle*. The overall approach is analogous to deriving fixpoint induction for Scott-continuous functions on complete partial orders [62]. for strongly causal stream transformers.

**Lemma 13 (Fixpoint Induction).** *Let  $T$  be a stream transformer as in Theorem 2 with a unique fixpoint  $\text{fix}(T)$  and  $P$  an inclusive stream predicate; then:*

$$((\forall f \in P) T(f) \in P) \Rightarrow \text{fix}(T) \in P. \quad (29)$$

*Proof.* Let  $(f_k)$  be the sequence with  $f_{k+1} \in T(f_k)$  for  $k \in \mathbb{N}$  and  $\text{fix}(T) = \lim_{k \rightarrow \infty} f_k$  as constructed in the proof of Lemma 11. From the assumption and natural induction on  $k$  we obtain that  $P(f_k)$  for all  $k \in \mathbb{N}$ . But  $P$  is inclusive, and therefore  $\text{fix}(T) \in P$ .

#### 7.4 Shrinking Maps

If  $(M, d)$  is a spherically complete ultrametric space, then every shrinking map  $T : M \rightarrow M$  has a unique fixpoint ([43], Theorem 1). The proof of this result relies on Zorn's lemma for showing the existence of a maximal, with respect to set inclusion, ball  $B_z$  in the set of balls of the form  $B_x := B[x, d(x, T(x))]$  for  $x \in X$ ; this  $z \in X$  is the unique fixpoint of  $T$ . A "more constructive proof", not relying on Zorn's lemma, also shows that there is fixpoint of  $T$  in every ball of the form  $B[x, d(x, T(x))]$  for  $x \in X$  (Corollary 5 in [33], see also [34], Chapter 5.5).

The following statement follows directly from Theorem 2.1 in [35] and the fact that  $A^{\omega, n}$  is a spherically complete ultrametric space 2. This extension of the results in [43] for multivalued functions also relies on the application of Zorn's lemma. Notice that these results are stated for complete non-Archimedean normed spaces, but they evidently hold also in case of spherically complete ultrametric spaces.

**Lemma 14.**  *$T : A^{\omega, n} \rightarrow \text{Comp}(A^{\omega, n})$  has a fixpoint if*

$$\mathcal{H}_d(T(f), T(g)) < d(f, g) \quad (30)$$

*for any distinct  $f, g \in A^{\omega, n}$ .*

Notice that every strongly causal map  $T$  as above satisfies the shrinking condition (30).

**Theorem 3.** *Every strongly causal stream transformer  $T : A^{\omega, n} \rightarrow \text{Comp}(A^{\omega, n})$  has a fixpoint.*

Since every singleton set is compact, we obtain the existence of a fixpoint, in particular, for deterministic maps. Uniqueness of this fixpoint follows, for example, from ([33]; see also [34], Theorem 5.4).

**Corollary 7.** *Every strongly causal stream transformer  $T : A^{\omega, n} \rightarrow A^{\omega, n}$  has a unique fixpoint.*

*Remark 23.* The fixpoint for shrinking deterministic stream transformers in Corollary 7 is obtained as the limit of a transfinite iteration ([50]; see also [34], Remark 5.5). Now, one obtains an induction principle for shrinking (and therefore also strongly causal) deterministic stream transformers analogously to Lemma 13.

### 7.5 Non-expansive Maps

In the light of Lemma 7 we can use causality instead of the equivalent non-expansion property  $(\mathcal{H}_d(T(f), T(g)) \leq d(f, g)$ , for all  $f, g$ ) in Theorem 2.2 of [35].

**Theorem 4.** *If  $T : A^{\omega, n} \rightarrow \text{Comp}(A^{\omega, n})$  is causal then either  $T$  has a fixpoint or there exists a ball  $B$  with radius  $r > 0$  that that  $d(u, T(u)) = r$  for all  $u \in B$ .*

*Example 25.*  $\text{Succ}(x) := x + 1$  is causal, for all  $u \in \mathbb{R}^\omega = B[0, 1]$  we have  $d(u, \text{Succ}(u)) = 1$ , and  $\text{Succ}$  does not have a fixpoint in  $\mathbb{R}^\omega$ .

*Remark 24.* Theorem 4 provides a concise verification condition for establishing the existence of fixpoint for causal stream transformers in SAL such as the one discussed in Example 23.

## 8 Remarks

Stream transformers may be specified using a rich set of interrelated formalisms, including

1. contractual stream transformers,
2. rational stream transformers,
3. formal power series as transfer functions,
4. co-inductive definitions,
5. behavioral differential equations,
6. Mealy machines, and
7. finite stream circuits.

For instance, for every causal stream transformer one may construct a corresponding (not necessarily finite-state) Mealy machine, and vice versa. There is a similar correspondence between rational stream transformers and finite stream circuits. In this way, there is a *duality* between the functional and a more machine-oriented view of causal stream transformers. This duality may advantageously be used in system design for switching between these intertwined view points as needed.

Based on the equivalence of causality and contraction for vector- and multivalued maps we have been formulating fixpoint results for nondeterministic stream transformers. Hereby, it is key that the underlying space of streams is equipped with the prefix ultrametric, which is spherically complete. In this way we obtain fixpoints for strongly causal, and therefore contractive, stream transformers with compact codomains (Theorem 2). If such a multivalued stream transformer is, in addition, also strongly contractive then the fixpoint is unique. Also notice that the definitions of weak and strong contraction for multivalued maps, respectively, resemble powerdomain constructions as commonly used for modeling nondeterminism in the denotational semantics of programming languages [1].

The statement and proof of Theorem 2 is based on similar fixpoint result for multivalued maps by Nadler ([38], Theorem 5). Generalizations are described, for example, in [49,20,24,51]. In particular, the papers [49,48] are concerned with ultrametric spaces whose distance functions take their values in an arbitrary partially ordered set, not just in the real numbers, and Theorem 2 may also have been obtained from ([49], 3.1), but here we prefer to stay in the framework of real-valued metrics.

Khamsi [31] uses a generalization of the Banach fixpoint principle to multivalued maps on a Cauchy complete metric space for developing a fixpoint semantics of stratified disjunctive logic programs (see also [26]). We are not aware, however, of previous attempts for developing fundamental concepts of system design based on multivalued fixpoints principles in the ultrametric space of streams.

The derivation of the fixpoint induction principle (Lemma 13) relies on fixpoint iteration. This induction principle is reminiscent, of course, of the fixpoint induction principle for Scott-continuous functions and Kleene's fixpoint theorem [62]. Moreover, the induction principle for multi-valued stream transformers is analogous to Park's lemma, which is an immediate consequence of the Knaster-Tarski fixpoint theorem for monotone functions on complete lattices.

We may also develop metric-based fixpoint approximations of stream transformers over a concrete domain by corresponding fixpoints in an abstracted domain as follows. A *stream abstraction*  $\alpha : C^\omega \rightarrow A^\omega$  relates concrete  $C$ -valued streams with abstract  $A$ -valued streams, and a corresponding *stream concretization*  $\gamma : A^\omega \rightarrow \mathcal{P}(C^\omega) \setminus \{\emptyset\}$  maps abstract streams to a corresponding set of concrete streams. Such a pair  $(\alpha, \gamma)$  forms, for  $c$  a constant, what we may call a *c-bounded Galois connection* between  $C^\omega$  and  $A^\omega$  if and only if

$$d(\alpha(f), g) < c \quad \text{iff} \quad d(f, \gamma(g)) < c \quad (31)$$

for all  $f \in C^\omega$  and  $g \in A^\omega$ . Now, for strongly causal (deterministic) stream transformers  $T_C : C^\omega \rightarrow C^\omega$  and  $T_A : A^\omega \rightarrow A^\omega$  such that  $d(T_C, \gamma \circ T_A \circ \alpha) < c$  one obtains  $d(\text{fix}(T_C), \gamma(\text{fix}(T_A))) < c$  for approximating the concrete fixpoint by an abstract fixpoint. These kinds of fixpoint approximation are useful for stream verification procedures based on *model checking* [14,56,13] and symbolic execution [32,28] for the WP (or SP) stream calculus.

Vector-valued metrics with codomain  $\mathbb{R}^n$ , for  $n \geq 1$ , are a viable alternative to our use of the supremum valuation for products of streams. The classical Banach contraction principle was extended by Perov [42] for contraction mappings on spaces endowed with (finite dimensional) vector-valued metrics with codomain  $\mathbb{R}^n$ . Generalizations to multivalued contraction mappings have been developed, among others, by Filip and Perusel [19,2]. These fixpoint theorems rely on contractive mappings. Therefore, they are applicable to strongly causal but not to weakly causal stream transformers. The advantage of taking the supremum valuation is that fixpoint results for multivalued mappings apply readily, whereas the use of vector-valued metrics requires explicit extensions of these results to this modified setting. On the other hand, contractions of endomorphisms with respect to vector-valued metric can be defined more generally by requiring a

square matrix  $L$  with nonnegative entries such that  $L^k \rightarrow 0$  as  $k \rightarrow \infty$ . Here we restricted ourselves to the cases where  $L$  is of the specific form  $l \cdot I$  for  $l < 1$  a Lipschitz constants and  $I$  the identity matrix.

Additional restrictions are needed for the existence of fixpoints for causal transformers. Consider, for example, the *Browder-Göhde-Kirk* fixpoint theorem: for  $C$  a closed, convex, bounded subset of a uniformly convex Banach space, every nonexpansive  $T : C \rightarrow C$  has at least one fixpoint. In fact, if  $x_0$  is any point in  $C$ , and a sequence  $(x_k)_{k \in \mathbb{N}}$  is defined by  $x_{k+1} = T(x_k)$ , then the *asymptotic center*<sup>2</sup> of the sequence  $(x_k)$  with respect to  $C$  is a fixpoint of  $T$  [57]. Since  $(A^\omega, d)$  is uniformly convex (see Appendix A), the *Browder-Göhde-Kirk* fixpoint principle applies to non-expansive, and therefore weakly causal, stream transformers. For an overview on fixpoints for nonexpansive maps see [30,34].

The causality condition on stream maps is rather restrictive as it includes continuity. This condition may be relaxed, however, since the space  $(A^\omega, d)$  is  $\varepsilon$ -chainable (for given  $\varepsilon > 0$ ) (Appendix D), which intuitively states that there is a finite path between any two streams with intermediate "jumps" less than  $\varepsilon$ . For  $\varepsilon$ -chainable spaces a local contraction condition suffices to obtain fixpoints for single- [18] and multivalued maps [38].

Our developments may also be extended to *dense*,  $A$ -valued streams  $(a_t)_{t \in \mathbb{R}}$  by defining valuations on dense streams along the line of valuations for discrete streams. In this way, one may also mix dense and discrete streams in heterogeneous products of streams for modeling *hybrid systems*. Moreover, *probabilistic systems* are modeled by Menger's probabilistic metric space (or any variant thereof), in which the distance between any two points is a probability distribution function [58,36]. Such an endeavor yields a comprehensive mathematical foundation for the formal construction of cyber-physical systems [37,9] and their realization on conventional (von Neumann, dataflow, quantum) computers, but eventually also on dedicated, still to be developed, hypercomputing [40] machinery for capturing unbounded nondeterminism.

## 9 Conclusions

Fundamental concepts of system design such as interfaces, composition, refinement, and abstraction, but also mutual recursion and corresponding induction principles are defined and derived almost seamlessly from the causality-contraction correspondence together with established results in fixpoint and approximation theory. It is well beyond the scope and the ambition of this treatise, however, to develop a comprehensive and readily applicable framework for system design. For such an endeavor the stream-based calculus for distributed, concurrent, real-time systems in [10] may serve as a starting point.

In contrast to traditional denotational approaches for complete partial orders we do not need to be constantly concerned with modeling and reasoning

<sup>2</sup> For the mapping  $f : C \rightarrow [0, \infty)$  such that  $u \mapsto \limsup_{k \rightarrow \infty} d(u, x_k)$ , the unique point  $u_0 \in C$  satisfying  $f(u_0) = \min(f(C))$  is the asymptotic center of the sequence  $(x_k)$  with respect to the set  $C$ .

about undefinedness. Reasoning about undefinedness seems to be an appropriate concept for modeling concrete computations with their multitude of sources of erroneous behavior. Simultaneously chasing these all potential sources of errors and their consequences, however, tends to be overwhelming in practice. Here, principled system design offers a suitable alternative by gradually introducing "real-world" issues and features to an abstract system design or algorithm in a compositional manner [23]. In general, this kind of incremental design can be accomplished even when features are mutually dependent [44].

Quantifiable fixpoint approximation is another advantage of the proposed metric-based approach to system design. These bounds provide useful *anytime information* to human system designers and also to mechanized verification and design engines, such as "the 1024 element prefix of the stream solution has already been established in the current Picard iteration". On the other hand, these approximations are neither under- nor over-approximations according to some partial order as is the case, for example, with Kleene-style fixpoint iteration.

**Disclaimer.** The author thanks Manfred Broy who pointed out to him the close correspondence between the logical notion of causality and metric-based contraction. Indeed these notes grew out of curiosity in investigating at least some of the immediate consequences of this profound correspondence.

## A Stream Ultrametric

For the prefix metric  $d$  on streams the *strong triangle inequality* holds.

**Proposition 12.**

$$d(f, h) \leq \max(d(f, g), d(g, h)). \quad (32)$$

for all streams  $f, g, h$ .

*Proof.* It is the case that  $f(j) = g(j)$  for  $j < v(f - g)$  and  $g(j) = h(j)$  for  $j < v(g - h)$ . By transitivity,  $f(j) = h(j)$  for each  $j < \min(v(f - g), v(g - h))$ . Therefore,  $v(f - h) \geq \min(v(f - g), v(g - h))$ , which is equivalent to  $2^{-v(f-h)} \leq \max(2^{-v(f-g)}, 2^{-v(g-h)})$ . Thus, inequality (32) holds.

As a consequence, the *isosceles triangle principle*

**Corollary 8.**

$$d(f, h) = \max(d(f, g), d(g, h)) \quad \text{when } d(f, g) \neq d(g, h) \quad (33)$$

holds for streams  $f, g, h$ .

*Proof.* For establishing (33) suppose  $d(f, g) < d(g, h)$ . Then,

$$d(f, h) \leq \max(d(f, g), d(g, h)) = d(g, h),$$

but also, since  $d(g, h)$  is not  $\leq d(f, g)$ ,  $d(g, h) \leq \max(d(f, g), d(f, h)) = d(f, h)$ , and we are done.

Among the numbers  $d(f, h)$ ,  $d(f, g)$ ,  $d(g, h)$  the largest and second largest are equal, and therefore (33) states that each three points are vertices of an isosceles triangle.

A consequence of (33) is: if  $B(f, r_1)$  and  $B(g, r_2)$  are two closed balls in an ultrametric space, with  $r_1 < r_2$ , then either  $B(f, r_1) \cap B(g, r_2) = \emptyset$  or  $B(f, r_1) \subseteq B(g, r_2)$ .

**Lemma 15.**  $(A^\omega, d)$  is Cauchy complete.

*Proof.* Let  $(f_n)_{n \in \mathbb{N}}$  be a Cauchy sequence with  $f_n \in A^\omega$ . Then, for all  $k \in \mathbb{N}$  there is  $N_k \in \mathbb{N}$  such that  $|f_n - f_m| < |X^k|$  for all  $n \geq m \geq N_k$ . This means that  $f_n - f_m \in X^k A^\omega$ . Writing  $f_n = \sum_{j \in \mathbb{N}} a_{n,j} X^j$  we get that  $(a_{n,j})_{n \in \mathbb{N}}$  is constant for  $n$  large enough, and

$$\lim_{n \rightarrow \infty} f_n = \sum_{j \in \mathbb{N}} \left( \lim_{n \rightarrow \infty} a_{n,j} \right) X^j \in A^\omega.$$

## B Hausdorff Distance

**Proposition 13.**

$$d(f, H) = \max(d(f, g), d(g, H))$$

for all  $f, g \in A^\omega$  and nonempty  $H \subseteq A^\omega$ .

*Proof.* 1. In case  $d(g, H) < d(f, g)$ , then there is  $h \in H$  such that  $d(g, h) \leq d(f, g)$ , and therefore

$$\begin{aligned} d(f, H) &\leq d(f, h) \leq \max(d(f, g), d(g, h)) \\ &= d(f, g) = \max(d(f, g), d(g, H)) \end{aligned}$$

2. In case  $d(f, g) \leq d(g, H)$ , then for all  $h \in H$ ,  $d(f, g) \leq d(g, h)$ , and hence

$$d(f, H) \leq d(f, h) \leq \max(d(f, g), d(g, h)) = d(g, h).$$

Therefore,  $d(f, H) \leq \inf_{h \in H} d(g, h) = d(g, H) = \max(d(f, g), d(g, H))$ .

**Lemma 16.**  $(\text{CB}(A^\omega), \mathcal{H}_d)$  is an ultrametric space.

*Proof.* Let  $F, G, H \in \text{CB}(A^\omega)$ , and let  $f \in F$  be arbitrary.

1. In case there is a  $g \in G$  with  $d(f, g) \leq d(g, H)$ , then

$$\begin{aligned} d(f, H) &\leq \max(d(f, g), d(g, H)) = d(g, H) \\ &\leq \mathcal{H}_d(G, H) \leq \max(\mathcal{H}_d(F, G), \mathcal{H}_d(G, H)). \end{aligned}$$

2. Otherwise, in case  $d(g, H) < d(f, g)$  for all  $g \in G$ , then

$$d(f, H) \leq \max(d(f, g), d(g, H)) = d(f, g),$$

and therefore

$$d(f, H) \leq \inf_{g \in G} d(f, g) \leq \mathcal{H}_d(F, G) \leq \max(\mathcal{H}_d(F, G), \mathcal{H}_d(G, H)).$$

Similarly,  $d(h, F) \leq \max(\mathcal{H}_d(F, G), \mathcal{H}_d(G, H))$  for each  $h \in H$  and therefore

$$\mathcal{H}_d(F, H) \leq \max(\mathcal{H}_d(F, G), \mathcal{H}_d(G, H)).$$

## C Uniform Convexity

Streams equipped with  $|\cdot|$  are *uniformly convex*. Expressed in geometrical terms this property is simple: "it is that the mid-point of a variable chord of the unit sphere of the space cannot approach the surface of the sphere unless the length of the chord goes to zero" [15].

**Proposition 14.**  $(A^\omega, |\cdot|)$ , for  $A$  a field, is *uniformly convex*.

*Proof.* We need to show that for every  $0 < \varepsilon \leq 2$  there is some  $\delta > 0$  such that for any two streams  $f, g$  on the unit ball, that is  $|f| = |g| = 1$ , the condition  $|f - g| \geq \varepsilon$  implies that

$$\left| \frac{f+g}{2} \right| \leq 1 - \delta.$$

The condition  $|f - g| \geq \varepsilon$  implies that  $[X^0](f - g) = 0$ , and therefore  $[X^0]f = [X^0]g$ . Moreover,  $[X^0]f, [X^0]g \neq 0$ , since, by assumption, both  $f$  and  $g$  lie on the unit ball. Now,

$$[X^0]\left(\frac{f+g}{2}\right) = [X^0]f \neq 0$$

and the norm  $\left|\frac{f+g}{2}\right|$  is strictly less than 1.

As a consequence of uniform convexity, the triangle inequality for linear independent vectors is strict with a uniform bound [22].

## D $\varepsilon$ -Chainability

For given  $\varepsilon > 0$ , a metric space  $(M, d)$  is said to be  $\varepsilon$ -chainable if and only if for any given  $a, b \in M$  there is an  $\varepsilon$ -chain from  $a$  to  $b$ , that is a finite set of points  $x_0, \dots, x_n \in M$  with  $a = x_0, b = x_n$ , and  $d(x_{i-1}, x_i) < \varepsilon$  for all  $i = 1, 2, \dots, n$ .

**Proposition 15.**  $(A^\omega, d)$  is  $\varepsilon$ -chainable.

*Proof.* Let  $\varepsilon > 0$  and  $f, g \in A^\omega$ . Let  $k$  be minimum with  $2^{-k} \leq \varepsilon$ . In case  $f, g \neq 0$  there are  $n, m \in \mathbb{N}$  with  $f \upharpoonright_n = 0$  and  $g \upharpoonright_m = 0$ . We may assume  $n < m$ : now patch  $f$  with trailing zeros from  $n$  to  $n+k$ . If the distance of the patched stream to  $g$  is less than  $\varepsilon$  we are done, and if not we repeat patching (a finite number of times) until we get close enough to  $g$ . In case,  $f \neq 0, g = 0$  a similar procedure shows  $\varepsilon$ -chainability. The case  $f = g = 0$  trivially holds.

**Definition 13.** ([38], p. 480) Let  $(M, d)$  be a metric space. A multivalued function  $T : M \rightarrow \text{CB}(M)$  is said to be  $(\varepsilon, \lambda)$ -uniformly locally contractive mapping (where  $\varepsilon > 0$  and  $0 \leq \lambda < 1$  provided that, if  $x, y \in M$  then  $\mathcal{H}_d(T(x), T(y)) \leq \lambda d(x, y)$ ).

**Theorem 5.** ([38], Theorem 6, p. 481) Let  $(M, d)$  be a complete  $\varepsilon$ -chainable metric space. If  $T : M \rightarrow \text{Comp}(M)$  is an  $(\varepsilon, \lambda)$ -uniformly locally contractive mapping, then  $T$  has a fixpoint.



## E Type 2 Turing Machines

Recall that a basic open set on  $A^\omega$  the product topology, and therefore also of the topology induced by  $d$ , is of the form  $U(\alpha) := \{f \in A^\omega \mid |\alpha| = n, f|_n = \alpha\}$  for  $\alpha \in A^*$  a finite word.

**Proposition 16.** *Stream transformers  $T : A^\omega \rightarrow B^\omega$  as computed by Type 2 Turing machines are continuous (with respect to the topology induced by  $d$ ).*

*Proof.* (Sketch) For a basic open set  $V := U(\alpha)$ , for arbitrary finite word  $\alpha \in B^\omega$ , we need to verify that  $W := T^{-1}(V)$  is open. Consider  $f \in W$ , then  $T(f) \in V$ . Thus on input  $f$  the machine produces an output tape starting with the finite word  $\alpha$ . By the time it writes out these cells it has inspected at most a finite prefix of the input tape.<sup>3</sup> We verify that  $f \in U(\alpha) \subseteq W$ . It is obvious that  $f \in U(\alpha)$ . To prove  $U(\alpha) \subseteq W$  take any  $g \in U(\alpha)$  and observe that  $T(f)$  and  $T(g)$  agree on the first  $|\alpha|$  values of the output. This implies that  $T(g) \in V$  and hence  $g \in W$ , as required.

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Within a particular type of space-time (which our universe may or may not possess), however, there can be situations in which a form of observer-relative acceleration occurs and an observer can witness the evolution of an entire infinite world-line by some finite time point [27].

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