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A NOTE ON THE COMPLETENESS OF KOZEN'S AXIOMATISATION OF THE PROPOSITIONAL μ-CALCULUS

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Abstract. The propositional μ-calculus is an extension of the modal system K with a least fixpoint operator. Kozen posed a question about completeness of the axiomatisation of the logic which is a small extension of the axiomatisation of the modal system K. It is shown that this axiomatisation is complete.

§1. Introduction. Propositional μ-calculus is an extension of the modal system K with a least fixpoint operator. In [8] Kozen proposed an elegant axiom system for the logic. It consists of an axiomatisation for the modal system K together with one axiom and one rule concerning the fixpoint operator. We show that this axiomatisation is complete, i.e., all valid formulas are provable. It is not possible to finitely axiomatise the semantic consequence relation because the logic is not compact.

The formulas of the logic are given by the following grammar:

$$\alpha ::= \top \mid \bot \mid p \mid X \mid \neg\alpha \mid \alpha \vee \beta \mid \alpha \wedge \beta \mid \langle a\rangle\alpha \mid [a]\alpha \mid \mu X.\alpha(X) \mid \nu X.\alpha(X)$$

where p ranges over the set *Prop* of *propositional constants,* X over the set *Var* of *propositional variables* and a over the set *Act* of *actions*. The last two constructs are the *least* and *greatest* fixpoint operators. In both constructions we require that the variable X appears only under an even number of negations in $\alpha(X)$. Both μ and ν *bind* the variable X. This means that when a substitution is performed we have to take care of possible clashes of variable names. We find it sometimes convenient to display a free variable in a formula, as for example in $\alpha(X)$. In this case a substitution of a formula φ for a variable X in $\alpha(X)$ is denoted just $\alpha(\varphi)$ instead of $\alpha(X)[\varphi/X]$. We use letters $\alpha, \beta, \gamma, \varphi, \psi$ to range over formulas.

The intended class of models for the logic is the class of all Kripke structures, i.e., triples of the form $\mathcal{M} = \langle S,\ R : Act \to \mathcal{P}(S \times S),\ \rho : Prop \to \mathcal{P}(S)\rangle$ where S is a nonempty set; we use $\mathcal{P}(S)$ to denote the set of all the subsets of S. The meaning of a formula α in a given model $\mathcal{M}$ and valuation

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$V : Var \to \mathcal{P}(S)$ is denoted by $\| \alpha \|_V^{\mathcal{M}}$. It is defined by induction with respect to the structure of the formula (usually we will omit the superscript $\mathcal{M}$):

$$\| \top \|_V = S \quad \| \bot \|_V = \emptyset$$
$$\| X \|_V = V(X) \quad \| p \|_V = \rho(p)$$
$$\| \neg\alpha \|_V = S - \| \alpha \|_V$$
$$\| \alpha \wedge \beta \|_V = \| \alpha \|_V \cap \| \beta \|_V$$
$$\| \alpha \vee \beta \|_V = \| \alpha \|_V \cup \| \beta \|_V$$
$$\| \langle a \rangle \alpha \|_V = \{s : \exists t.(s,t) \in R(a) \wedge t \in \| \alpha \|_V\}$$
$$\| [a] \alpha \|_V = \{s : \forall t.(s,t) \in R(a) \Rightarrow t \in \| \alpha \|_V\}$$
$$\| \mu X.\alpha(X) \|_V = \bigcap \{T \subseteq S : \| \alpha \|_{V[T/X]} \subseteq T\}$$
$$\| \nu X.\alpha(X) \|_V = \bigcup \{T \subseteq S : T \subseteq \| \alpha \|_{V[T/X]}\}.$$

Observe that $\| \mu X.\alpha(X) \|_V$ and $\| \nu X.\alpha(X) \|_V$ are respectively the least and the greatest fixpoints of the operator $\lambda T. \| \alpha(X) \|_{V[T/X]}$. Although the greatest fixpoint is definable from the least fixpoint by $\nu X.\alpha(X) \equiv \neg\mu X.\neg\alpha(\neg X)$ we prefer to have this construction in the syntax. We will write $\mathcal{M}, s, V \vDash \alpha$ for $s \in \| \alpha \|_V^{\mathcal{M}}$.

Let us recall Kozen's axiomatisation from [8]. A basic judgement of the system has the form $\alpha = \beta$ with the intended meaning that the two formulas are semantically equivalent. A judgement $\alpha \leq \beta$ is an abbreviation for $\alpha \wedge \beta = \alpha$. A formula α is *provable* if $\alpha = \top$ is provable. The axiomatisation consists of the axioms and rules of equational logic (including substitution of equals by equals, i.e., the cut rule) and the following axioms and rules:

$(K1)$	axioms for Boolean algebras
$(K2)$	$\langle a \rangle\varphi \vee \langle a \rangle\psi = \langle a \rangle(\varphi \vee \psi)$
$(K3)$	$\langle a \rangle\varphi \wedge [a]\psi \leq \langle a \rangle(\varphi \wedge \psi)$
$(K4)$	$\langle a \rangle\bot = \bot$
$(K5)$	$\alpha(\mu X.\alpha(X)) \leq \mu X.\alpha(X)$
$(K6)$	$\dfrac{\alpha(\varphi) \leq \varphi}{\mu X.\alpha(X) \leq \varphi}$
$(K7)$	$\nu X.\alpha(X) = \neg\mu X.\neg\alpha(\neg X)$.

The propositional μ-calculus is an interesting logic because it is both expressive and manageable. It subsumes most propositional temporal logics and propositional modal logics of programs [2, 10, 16]. Over Kripke structures which are binary trees the logic is as expressive as $S2S$ [12] (monadic second order logic of two successors [14]). On the other hand the validity problem for μ-calculus formulas is EXPTIME complete [3]. This makes it an interesting logic for applications in computer science.

We hope that the completeness theorem will help in finding an algebraic approach to the logic in the spirit of categorical logic. For example, it would be interesting to know a semantical characterisation of the quasi-variety defined by Kozen's axioms. Just to see that there are interesting models which are not Kripke structures consider a model consisting of all regular ω-languages. An algebraic understanding of the logic should have some influence on understanding automata on infinite trees and $S2S$. In this context let us mention that no finitary axiomatisation of $S2S$ is known (there is one for $S1S$ [15] but the proof of its completeness uses Ramsey's Theorem which is not so useful for trees).

The completeness proof presented here is rather syntactic in nature. We use an extension of the method of analytical tableaux which was introduced for modal logic by Kripke and Hintikka [1, 4]. The main reason that we use this method is that analytical tableaux are up till now essentially the only method of model construction for the μ-calculus [17] (see [9] for a slightly different approach). In particular this means that filtration based methods, so successful for proving completeness of various systems of modal logics and logics of programs [5, 10], do not adapt easily to the μ-calculus. This problem exhibits itself in the fact that μ-calculus does not have the *collapsed model property*. This property says that given a formula and its model, we can collapse the model by unifying states satisfying the same subformulas of the given formula, and the result will still be a model for the formula. For the μ-calculus this property does not hold no matter what notion of subformula we choose as long as there are only finitely many subformulas of a given formula.

Let us sketch the plan of the proof. In the next section we recall the method of model construction based on analytical tableaux. We define a notion of a *tableau* for a formula. It can be shown that a formula is satisfiable iff there is a subset of a tableau for the formula, called a *pre-model*. Using determinacy of Borel games (determinacy of $\Delta^0_3 = \Sigma^0_3 \cap \Pi^0_3$ games is enough) one can show that if there is no pre-model in a tableau then one can find a *refutation* in the tableau [13]. The method of model construction suggests the notion of *tableaux equivalence*. This is a syntactically defined relation which has the property that if two formulas have equivalent tableaux then they are semantically equivalent. Hence "to have equivalent tableaux" is a finer equivalence relation than semantical equivalence.

In Section 3 we will consider classes of formulas for which provability is easier than in the general case. We have mentioned already that every unsatisfiable formula has a refutation. Such a refutation looks almost like a proof, in a Gentzen-style system, of the negation of the initial formula.

The only problem is that some of the branches of the refutation may be infinite. Kozen [8] defined a notion of *aconjunctivity* and showed that one can convert a refutation into a proof of the formula if the formula we start with is aconjunctive. We define here two sets of formulas. One is the set of *weakly aconjunctive* formulas. A slight modification of Kozen's argument shows that the negation of an unsatisfiable weakly aconjunctive formula is provable. On the other hand, we show that every formula is semantically equivalent to a weakly aconjunctive formula, which is not true for aconjunctive formulas. We also define the set of *disjunctive formulas* which is a subset of the aconjunctive formulas. It turns out that for a disjunctive formula it is very easy to construct a model, in case the formula is satisfiable, or a proof of its negation, in case it is not satisfiable. Moreover we have:

(1) For every formula φ there is a disjunctive formula $\widehat{\varphi}$ with an equivalent tableau.

This is a stronger statement than just saying that φ and $\widehat{\varphi}$ are semantically equivalent. It has, moreover, the advantage that the tableau equivalence is a syntactically defined relation.

In the last section we prove our main theorem which is:

(2) For every formula φ there is a semantically equivalent disjunctive formula $\widehat{\varphi}$ such that $\varphi \leq \widehat{\varphi}$ is provable.

The theorem is proved by induction on the structure of the formula with nontrivial cases for conjunction, ν and μ (the most difficult).

Let us sketch the proof of (2) for the case that φ is $\nu X.\alpha(X)$. By induction assumption we have a disjunctive formula $\widehat{\alpha}(X)$ and a proof of $\alpha(X) \leq \widehat{\alpha}(X)$. Hence $\nu X.\alpha(X) \leq \nu X.\widehat{\alpha}(X)$ is provable. Because $\widehat{\alpha}(X)$ is a disjunctive formula, $\nu X.\widehat{\alpha}(X)$ is a weakly aconjunctive formula although it may not be a disjunctive formula. By (1) we have a disjunctive formula $\widehat{\varphi}$ with a tableau equivalent to a tableau for $\nu X.\widehat{\alpha}(X)$. We show that:

(3) If α is a weakly aconjunctive formula, δ is a disjunctive formula and the two formulas have equivalent tableaux then $\alpha \leq \delta$ is provable.

From this we can conclude that $\nu X.\widehat{\alpha}(X) \leq \widehat{\varphi}$ is provable, so $\varphi \leq \widehat{\varphi}$ is provable.

The case that φ is $\mu X.\alpha(X)$ is more difficult because weakly aconjunctive formulas are not closed under the least fixpoint operator. Although $\widehat{\alpha}(X)$ is a disjunctive formula $\mu X.\widehat{\alpha}(X)$ may not be an aconjunctive formula and we cannot use fact (3). Fortunately, by the rule $(K6)$, to prove $\mu X.\widehat{\alpha}(X) \leq \widehat{\varphi}$ it is enough to prove $\widehat{\alpha}(\widehat{\varphi}) \leq \widehat{\varphi}$. The formula $\widehat{\alpha}(\widehat{\varphi})$ is weakly aconjunctive but this time we meet another problem. There may be no tableau for $\widehat{\alpha}(\widehat{\varphi})$ that is equivalent to a tableau for $\widehat{\varphi}$. This should not come as a big surprise as the notion of tableau equivalence is very restrictive; it would be rather surprising if it worked all the time. We remedy this by introducing a weaker relation between tableaux which we call *tableau consequence*. This is a counterpart of a semantical consequence relation in a sense similar to the one in which tableau equivalence is a counterpart of semantical equivalence. We prove that there is a tableau for $\widehat{\alpha}(\widehat{\varphi})$ of which a tableau for $\widehat{\varphi}$ is a consequence. On the other hand, the notion of tableau consequence is still strong enough to show a statement similar to (3): if α is a weakly aconjunctive formula, δ is a disjunctive formula and a tableau for δ is a consequence of a tableau for α, then $\alpha \leq \delta$ is provable.

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§2. Tableaux and tableau equivalence. In this section we define a notion of a *tableau* for a formula and show how to construct a model from a tableau, if the initial formula is satisfiable, or a *refutation* otherwise. This suggests the definition of *tableau equivalence*, a notion we will use frequently in the following sections.

We start by stating some simple facts which allow us to restrict the set of formulas we have to consider. We also introduce some machinery to deal with fixpoint subformulas of a given formula.

DEFINITION 1 (Positive, guarded formulas). We call a formula *positive* iff all negations in the formula appear only before propositional constants and free variables.

The variable X in $\mu X.\alpha(X)$ is called *guarded* iff every occurrence of X in α is in the scope of some modality operator $\langle a \rangle$ or $[a]$. We say that a *formula is guarded* iff every bound variable in the formula is guarded.

PROPOSITION 2 (Kozen). *Every formula is provably equivalent to a positive guarded formula.*

DEFINITION 3 (Binding). We call a formula *well-named* iff every variable is bound at most once in the formula and free variables are distinct from bound variables. For a variable X bound in a well-named formula α there exists a unique subformula of α of the form $\sigma X.\beta(X)$, from now on called the *binding definition of X in α* and denoted $\mathcal{D}_\alpha(X)$. We will omit the subscript α when it causes no ambiguity. We call X a *ν-variable* if $\sigma = \nu$, otherwise we call X a *μ-variable*.

The function $\mathcal{D}_\alpha$ assigning to every bound variable its *binding definition* in α will be called the *binding function* associated with α.

Remark. Every formula is equivalent to a well-named formula which can be obtained by some consistent renaming of bound variables. The substitution of a formula β for all free occurrences of a variable X in α, denoted $\alpha[\beta/X]$, can be made modulo some consistent renaming of bound variables of β, so that the resulting formula is still well-named.

Definition 4 (Dependency order). Given a well-named formula α, we define the *dependency order*, $\leq_\alpha$, on the bound variables of α as the least partial order relation such that if X occurs free in $\mathcal{D}_\alpha(Y)$ then $X \leq_\alpha Y$. We will say that a bound variable Y depends on a bound variable X in α when $X \leq_\alpha Y$.

Example. In case $\alpha = \mu X.\langle a\rangle X \vee \nu Y.\langle b\rangle Y$, variables X and Y are incomparable in the $\leq_\alpha$ ordering. On the other hand, if α is $\mu X.\nu Y.\langle a\rangle X \vee \mu Z.\langle a\rangle(Z \vee Y)$ then $X \leq_\alpha Z$. ⊣

Definition 5. We extend the syntax of the μ-calculus by allowing a new construct of the form $(a \to \Psi)$, where a is an action and Ψ is a finite set of formulas. As for its semantics, we will consider such a formula to be an abbreviation of the formula $\bigwedge\{\langle a\rangle\psi : \psi \in \Psi\} \wedge [a]\bigvee\Psi$.

Remark. By itself the $(a \to \Psi)$ construction is nothing but a way to hide some conjunctions. This construct arises when one tries to find a notion of automata corresponding to the μ-calculus that is able to cope with potentially unbounded branching. In our case, we use this construction to provide a more symmetric rule for reducing modalities. It also makes the definition of a special conjunction (Definition 16) more natural. With this one construct it is possible to express both $[a]$ and $\langle a\rangle$ modalities. A formula $[a]\psi$ is equivalent to $(a \to \emptyset) \vee (a \to \{\psi\})$ and a formula $\langle a\rangle\psi$ is equivalent to $(a \to \{\psi, \top\})$. All the notions from this section such as guarded formula, binding function etc. extend to formulas with this new construct.

Definition 6 (Terminal formulas, literals). A formula of the form $(a \to \emptyset)$ will be called a *terminal formula* because its meaning is that there are no states reachable by action a from a given state.

Propositional constants (including $\top, \bot$), variables and the negations of each of these will be called *literals*.

Proviso. If not otherwise stated all formulas are assumed to be well-named, positive and guarded and to contain $(a \to \Psi)$ construct instead of the modalities $\langle a\rangle$ and $[a]$. By the Remark stated above, this is not a restriction as far as semantics is concerned. It is also easy to check that every formula is provably equivalent to a formula of this kind.

Definition 7 (Tableau rules). For a formula φ and its binding function $\mathcal{D}_\varphi$ we define the system of tableau rules $\mathcal{S}^\varphi$ parametrised by φ, or rather its binding function. The system is presented in Figure 1 (we use $\{\alpha, \Gamma\}$ as a shorthand for $\{\alpha\} \cup \Gamma$).

$$\text{(and)}\ \frac{\{\alpha, \beta, \Gamma\}}{\{\alpha \wedge \beta, \Gamma\}} \qquad \text{(or)}\ \frac{\{\alpha, \Gamma\} \quad \{\beta, \Gamma\}}{\{\alpha \vee \beta, \Gamma\}}$$

$$(\mu)\ \frac{\{\alpha(X), \Gamma\}}{\{\mu X.\alpha(X), \Gamma\}} \qquad (\nu)\ \frac{\{\alpha(X), \Gamma\}}{\{\nu X.\alpha(X), \Gamma\}}$$

$$\text{(reg)}\ \frac{\{\alpha(X), \Gamma\}}{\{X, \Gamma\}} \quad \text{whenever } X \text{ is a bound variable of } \varphi \text{ and } \mathcal{D}_\varphi(X) = \sigma X.\alpha(X)$$

$$\text{(mod)}\ \frac{\{\psi\} \cup \{\bigvee \theta : (a \to \theta) \in \Gamma, \theta \neq \Psi\} \text{ for every } (a \to \Psi) \in \Gamma, \psi \in \Psi}{\Gamma}$$

FIGURE 1. The system $\mathcal{S}^\varphi$

REMARK. 1. We see applications of the rules as a process of reduction. Given a finite set of formulas Γ that we want to derive, we look for a rule the conclusion of which matches our set. Then we apply the rule and obtain the assumptions of the instance of the rule in which Γ is the conclusion.

2. There is no rule for reducing formulas of the form $\langle a \rangle\varphi$ or $[a]\varphi$ because we assume that these formulas are replaced by equivalent formulas using the $(a \to \Psi)$ notation.

3. The rule (mod) has as many assumptions as there are formulas in the sets Ψ for which $(a \to \Psi) \in \Gamma$. For example

$$\frac{\{\varphi_1, \varphi_3\} \quad \{\varphi_2, \varphi_3\} \quad \{\varphi_1 \vee \varphi_2, \varphi_3\} \quad \{\psi_1\} \quad \{\psi_2\}}{\{(a \to \{\varphi_1, \varphi_2\}), (a \to \{\varphi_3\}), (b \to \{\psi_1, \psi_2\})\}}$$

is an instance of the rule. We will call a son labeled by an assumption obtained by *reducing* an action a an $\langle a \rangle$*-son*. In our example if a node n of a tableau is labeled by the conclusion of the rule then its son labeled by $\{\varphi_1, \varphi_3\}$ is an $\langle a \rangle$-son of n and a son labeled by $\{\psi_1\}$ is a $\langle b \rangle$-son of n.

(4) The system is really two systems in one and will be used for two different purposes. First, one can read it as a system for showing that a set of formulas is not satisfiable. In this reading it would be enough if the rule (mod) had only one assumption. The other way is to look at it as a system for showing that a set of formulas is satisfiable. In this case, all the assumptions of the rule (mod) are needed but the (or) rule needs only one assumption. These are precisely the intuitions behind Definition 12.

DEFINITION 8 (Tableaux). A *tableau* for a formula φ is a pair $\langle T, L\rangle$, where T is a tree and L is a labelling function such that:

1. The root of T is labeled by $\{\varphi\}$.
2. The sons of any internal node n are created and labeled according to the rules of the system $\mathcal{S}_\varphi$. Additionally, we require that the rule (mod) is applied only when no other rule is applicable.

Without rules for fixpoints and variables these tableaux are essentially the same as the analytical tableaux of Kripke and Hintikka. As our tableaux may be infinite we will be interested not only in the form of the leaves but also in the internal structure of tableaux. We are now going to distinguish some nodes of tableaux and define a notion of *trace* which captures the idea of a history of a regeneration of a formula.

DEFINITION 9 (Modal and choice nodes, neighbourhoods). Leaves and nodes where reduction of modalities is performed, i.e., the rule (mod) is used, will be called *modal nodes*. The root of the tableau and sons of modal nodes will be called *choice nodes*.

If φ is a guarded formula then the sequence of all the choice nodes on a path of a tableau for φ induces a partition of the path into finite intervals beginning in choice nodes and ending in modal nodes. We will say that a modal node m is *near* a choice node n iff they are both in the same interval, i.e., in the tableau there is a path from n to m without an application of the rule (mod). Observe that in some cases a choice node may be also a modal node.

DEFINITION 10 (Trace). Given a path $\mathcal{P}$ of a tableau $\mathcal{T} = \langle T, L\rangle$, a *trace* on $\mathcal{P}$ will be a function $\mathcal{T}r$ assigning a formula to every node in some initial segment of $\mathcal{P}$ (possibly to all of $\mathcal{P}$) and satisfying the following conditions:

- If $\mathcal{T}r(m)$ is defined then $\mathcal{T}r(m) \in L(m)$.
- Let m be a node with $\mathcal{T}r(m)$ defined and let $n \in \mathcal{P}$ be a son of m. If the rule applied in m does not reduce the formula $\mathcal{T}r(m)$ then $\mathcal{T}r(n) = \mathcal{T}r(m)$. If $\mathcal{T}r(m)$ is reduced in m then $\mathcal{T}r(n)$ is one of the results of the reduction. This should be clear for all the rules, except possibly for (mod). If m is a modal node and n is labeled by $\{\psi\} \cup \{\bigvee\theta : (a \to \theta) \in \Gamma, \theta \neq \Psi\}$ for some $(a \to \Psi) \in L(m)$ and $\psi \in \Psi$, then $\mathcal{T}r(n) = \psi$ if $\mathcal{T}r(m) = (a \to \Psi)$ and $\mathcal{T}r(n) = \bigvee\theta$ if $\mathcal{T}r(m) = (a \to \theta)$ for some $(a \to \theta) \in \Gamma$, $\theta \neq \Psi$. Traces from all the other formulas end in the node m.

DEFINITION 11 (μ-trace). We say that there is a *regeneration* of a variable X on a trace $\mathcal{T}r$ on some path of a tableau for γ iff for some node m and its son n on the path $\mathcal{T}r(m) = X$ and $\mathcal{T}r(n) = \alpha(X)$, where $\mathcal{D}_\gamma(X) = \sigma X.\alpha(X)$.

We call a trace a *μ-trace* iff it is an infinite trace (defined for the whole path) on which the smallest variable (with respect to the $\leq_\gamma$ ordering) regenerated infinitely often is a μ-variable. Similarly, a trace will be called a ν-*trace* iff it is an infinite trace where the smallest variable regenerated infinitely often is a ν-variable.

REMARK. Every infinite trace is either a μ-trace or a ν-trace because all the rules except (reg) decrease the size of formulas and formulas are guarded so every formula is eventually reduced.

DEFINITION 12 (Pre-models, refutations). A *pre-model* $\mathcal{PM}$ of a formula γ is a set of nodes of a tableau $\mathcal{T}$ for γ which contains the root and such that for every n belonging to $\mathcal{PM}$: if a rule other than (mod) was applied in n then exactly one of the sons of n belongs to $\mathcal{PM}$; if the (mod) rule was applied in n then all the sons of n belong to $\mathcal{PM}$. A pre-model must satisfy two consistency conditions: (i) there cannot be $\perp$ or a literal and its negation in any of the labels of the nodes of $\mathcal{PM}$; (ii) there cannot be a μ-trace on any infinite path of $\mathcal{PM}$.

Dually, we define a notion of a *refutation*, $\mathcal{R}$. It is a subset of the set of nodes of $\mathcal{T}$ which contains the root and such that for every n belonging to $\mathcal{R}$: if a rule other than (mod) was applied in n then all of the sons of n belong to $\mathcal{R}$; if the (mod) rule was applied in n then exactly one of the sons of n must belong to $\mathcal{R}$. A refutation must also satisfy two other conditions: (i) there must be $\perp$ or a literal and its negation in the the label of every leaf of $\mathcal{R}$; (ii) there must be a μ-trace on every infinite path of $\mathcal{R}$.

The first two statements of the theorem below were proved by Streett and Emerson [17]; for the last statement see [13].

THEOREM 13. *Every pre-model for a formula γ can be converted into a model for γ. Given a model for γ one can construct a pre-model for γ. A formula γ is not satisfiable iff there is a refutation for γ.*

This theorem suggests a notion of *tableau equivalence*. This will be an equivalence relation which guarantees that for every pre-model in a tableau we can find a "very similar" pre-model in any other equivalent tableau.

DEFINITION 14 (Tableau equivalence). We will say that two tableaux $\mathcal{T}_1$ and $\mathcal{T}_2$ are *equivalent* iff there is a bijection $\mathcal{E}$ between the choice and modal nodes of $\mathcal{T}_1$ and $\mathcal{T}_2$ such that:

1. $\mathcal{E}$ maps the root of $\mathcal{T}_1$ onto the root of $\mathcal{T}_2$, it maps choice nodes to choice nodes and modal nodes to modal nodes.
2. If n is a descendant of m then $\mathcal{E}(n)$ is a descendant of $\mathcal{E}(m)$. Moreover, if for some action a, node n is a $\langle a\rangle$-son of a modal node m then $\mathcal{E}(n)$ is a $\langle a\rangle$-son of $\mathcal{E}(m)$.

3. For every modal node m, the sets of literals and terminal formulas (i.e., formulas of the form $(a \rightarrow \emptyset)$) occurring in $L(m)$ and in $L(\mathcal{E}(m))$ are equal.
4. There is a μ-trace on a path $\mathcal{P}$ of $\mathcal{T}_1$ iff there is a μ-trace on the path of $\mathcal{T}_2$ determined by the image of $\mathcal{P}$ under $\mathcal{E}$.

THEOREM 15. *If two formulas have equivalent tableaux then they are semantically equivalent.*

We omit the proof which is quite straightforward once one has the notions needed to prove Theorem 13.

§3. Aconjunctive and disjunctive formulas. In this section we consider a class of formulas for which provability is easier than in the general case. We recall the notion of *aconjunctive formulas* from [8] and propose a slight generalisation called *weakly aconjunctive formulas*. We obtain a slight generalisation of the result from [8] which states that the negation of every unsatisfiable weakly aconjunctive formula is provable. Finally we define a notion of *disjunctive* formula and show that for every formula and every tableau for this formula which is a regular tree, there is a disjunctive formula with an equivalent tableau.

DEFINITION 16 (Special conjunctions, weakly aconjunctive formulas). A conjunction $\alpha_1 \wedge \cdots \wedge \alpha_n$ is called *special* iff every α_i is either a literal or a formula of a form $(a \rightarrow \Psi)$ and for every action a there is at most one Ψ such that $(a \rightarrow \Psi)$ is one of the conjuncts.

Let φ be a formula, let $\mathcal{D}_\varphi$ be its binding function and let $\leq_\varphi$ be the dependency ordering (see Definitions 3 and 4).

— We say that a variable X is *active* in ψ, a subformula of φ, iff there is a variable Y appearing in ψ and $X \leq_\varphi Y$.

— Let X be a variable with its binding definition $\mathcal{D}_\varphi(X) = \mu X.\gamma(X)$. The variable X is called *aconjunctive* iff for all subformulas of γ of the form $\alpha \wedge \beta$ it is not the case that X is active in α as well as in β.

— A variable X as above is called *weakly aconjunctive* iff for all subformulas of γ of the form $\alpha \wedge \beta$: if X is active in both α and β then $\alpha \wedge \beta$ is a special conjunction as defined above.

— A formula φ is called (*weakly*) *aconjunctive* iff all μ-variables in φ are (weakly) aconjunctive.

In the following we will be interested only in weakly aconjunctive formulas. The definition of aconjunctive formulas was restated just to compare the two notions.

The next proposition states some closure properties of the class of weakly aconjunctive formulas. Observe that weakly aconjunctive formulas are not closed under negation nor under the least fixpoint operation.

PROPOSITION 17 (Composition). *If $\gamma(X)$ and δ are weakly aconjunctive formulas then $\gamma[\delta/X]$, $\nu X.\gamma(X)$ and $\delta \wedge \gamma(X)$ are also weakly aconjunctive formulas.*

DEFINITION 18 (Thin refutations). We call a refutation *thin* iff whenever a formula of the form $\alpha \wedge \beta$ is reduced in some node of the refutation and some variable is active in α as well as in β then either $\alpha \wedge \beta$ is a special conjunction or one of the conjuncts is immediately discarded by the use of the weakening rule.

REMARK. Hence for thin refutations we allow the use of the weakening rule which was not the case for refutations. We did not allow weakening in the last section because we wanted to show a correspondence between pre-models and refutations, and the weakening rule is not a sound rule for pre-models. The rule is sound for refutations in the sense that if there is a refutation constructed using the weakening rule then there is also one where the rule is not used.

FACT 19. Every refutation for a weakly aconjunctive formula is a thin refutation.

By Theorem 13 every unsatisfiable formula has a refutation. Hence the next theorem implies that one can prove the negation of an unsatisfiable weakly aconjunctive formula. The theorem is stated in greater generality because in Lemma 25 we deal with thin refutations for formulas which may not be weakly aconjunctive.

THEOREM 20. *If a formula has a thin refutation then its negation is provable.*

The proof of this theorem is rather involved and will not be given here. Let us instead comment why it is difficult to extend this theorem to all formulas. This has to do with the fact that the conditions imposed on the infinite paths of refutations are of the form: on every path there exists a sequence satisfying some Δ^0_3 condition. In other words this condition is a Σ^1_1 condition. A refutation may be seen as a strategy in the game on a tableau. Hence at first glance we have to deal with Σ^1_1 games. It is possible to replace a tableau for a formula with a different arena for the game so that the game becomes Δ^0_3. Nevertheless it seems to be impossible to simulate this translation in the proof system (it is possible to do this using a stronger, but still finitary, axiom system [18]). For thin refutations the game is easier. In the general situation traces can split (because of the (and) rule) and merge (because of the implicit contraction). In thin refutations the traces (almost) never merge and splitting is in some sense deterministic because conjuncts have no active variable in common.

Next we define the notion of disjunctive formula.

DEFINITION 21 (Disjunctive formulas). The set of *disjunctive formulas*, $\mathcal{F}_d$ is the smallest set defined by the following clauses:

1. Every literal is a disjunctive formula.
2. If $\alpha, \beta \in \mathcal{F}_d$ then $\alpha \vee \beta \in \mathcal{F}_d$; moreover, if X occurs only positively in α and not in the context $X \wedge \gamma$ for some γ, then $\mu X.\alpha, \nu X.\alpha \in \mathcal{F}_d$.
3. $(a \to \Psi) \in \mathcal{F}_d$ if $\Psi \subseteq \mathcal{F}_d$.
4. A special conjunction (see Definition 16) of disjunctive formulas is a disjunctive formula.

REMARK. Modulo the order of application of (and) rules, disjunctive formulas have unique tableaux. Moreover, on every infinite path there is one and only one infinite trace.

The following theorem from [6] shows that every formula is equivalent to a disjunctive formula. This is unfortunately not a normal form because there may be many equivalent disjunctive formulas.

THEOREM 22. *For every formula φ and every regular tableau $\mathcal{T}$ for φ (i.e., a tableau which can be presented as a finite graph) there is a disjunctive formula $\widehat{\varphi}$ with a tableau equivalent to $\mathcal{T}$.*

PROOF. We give an outline of the proof. A *tree with back edges* is a tree with added edges leading from some of the leaves to their ancestors. First, using some automata theory and, in particular, automata with parity conditions [11], we prove:

LEMMA 22.1. *It is possible to construct a finite tree with back edges $\mathcal{T}_l$, satisfying the following conditions:*

1. *$\mathcal{T}_l$ unwinds to $\mathcal{T}$ (i.e., $\langle T_l^*, L_l^* \rangle$ is isomorphic to $\mathcal{T}$ where T_l^* is the set of paths in T_l starting from the root and $L_l^*(wn) = L_l(n)$ for a path wn ending at a node n).*

2. *Every node to which a back edge points can be assigned the color magenta or navy in such a way that there is a μ-trace on a path of the unwinding of $\mathcal{T}_l$ iff the node of $\mathcal{T}_l$ closest to the root through which the path goes infinitely often is colored magenta.*

Having such a tree one constructs from it a disjunctive formula $\widehat{\varphi}$ which has a tableau equivalent to $\mathcal{T}$. The construction starts at the leaves of the tree and proceeds to the root. To every edge leading to a leaf n we assign a formula $\widehat{n}$ which is a conjunction of all the literals in the label of n. To every back edge leading to a node n we assign the variable X_n. For every internal (or) node we take a disjunction of the two formulas assigned to the edges going from it. For a (mod) node we construct an appropriate special conjunction. When coming to a node n to which some back edge points, we know that all the back edges pointing to this node are assigned the same variable X_n. The color of the node is used to decide which fixpoint operator should be used to bind this variable; we use μ if n is colored magenta. We take for $\widehat{\varphi}$ the formula assigned to the root of $\mathcal{T}_l$. ⊣

§4. Completeness.

Our main goal is:

THEOREM 23 (Completeness). *For every unsatisfiable formula φ, the formula $\neg\varphi$ is provable.*

From Fact 19 and Theorems 13 and 20 we know that the negation of every unsatisfiable disjunctive formula is provable. Hence, to show completeness it is enough to show that for every unsatisfiable formula φ there is an unsatisfiable disjunctive formula $\widehat{\varphi}$ such that $\varphi \leq \widehat{\varphi}$ is provable. This follows from the theorem below. Its proof will occupy the rest of this section.

THEOREM 24. *For every formula φ there is a semantically equivalent disjunctive formula $\widehat{\varphi}$ such that $\varphi \leq \widehat{\varphi}$ is provable. Moreover, if a free variable occurs only positively in φ then it occurs only positively in $\widehat{\varphi}$.*

PROOF. The proof is by induction on the structure of the formula φ.

Case: φ is a literal. In this case $\widehat{\varphi}$ is just φ.

Case: φ is $\alpha \vee \beta$. By induction assumption there are disjunctive formulas $\widehat{\alpha}, \widehat{\beta}$ equivalent to α and β respectively. We let $\widehat{\alpha \vee \beta}$ be $\widehat{\alpha} \vee \widehat{\beta}$. Because $\alpha \leq \widehat{\alpha}$ and $\beta \leq \widehat{\beta}$ are provable, $\alpha \vee \beta \leq (\widehat{\alpha} \vee \widehat{\beta})$ is also provable.

Case: φ is $(a \to \Phi)$. This case is very similar to the previous one.

Case: φ is $\nu X.\alpha(X)$. By the induction assumption, there is a disjunctive formula $\widehat{\alpha}(X)$ equivalent to $\alpha(X)$. Of course $\nu X.\alpha(X)$ is semantically equivalent to $\nu X.\widehat{\alpha}(X)$ and $\nu X.\alpha(X) \leq \nu X.\widehat{\alpha}(X)$ is provable. Unfortunately $\nu X.\widehat{\alpha}(X)$ may not be a disjunctive formula. This is because X may occur in a context $X \wedge \gamma$ for some γ. Therefore we have to construct $\widehat{\varphi}$ from scratch.

By Theorem 22 there is a disjunctive formula $\widehat{\varphi}$ which has a tableau equivalent to some regular tableau $\mathcal{T}$ for $\nu X.\widehat{\alpha}(X)$. By Theorem 15 the two formulas are equivalent. We are left to show that $\nu X.\widehat{\alpha}(X) \leq \widehat{\varphi}$ is provable in Kozen's system. As every disjunctive formula is a weakly aconjunctive formula, by Proposition 17 we have that $\nu X.\widehat{\alpha}(X)$ is a weakly aconjunctive formula.

If $\neg\widehat{\varphi}$ were a weakly aconjunctive formula then by Theorem 15 and Fact 19 we would have a thin refutation for $(\nu X.\widehat{\alpha}(X)) \wedge \neg\widehat{\varphi}$. So by Theorem 13 we would have a proof of $\nu X.\widehat{\alpha}(X) \leq \widehat{\varphi}$. Unfortunately $\neg\widehat{\varphi}$ may not be weakly aconjunctive so this approach does not work. Nevertheless, we know that the two formulas have equivalent tableaux and we can use this information.

LEMMA 25. *Suppose that we have a weakly aconjunctive formula α and a disjunctive formula δ. If the two formulas have equivalent tableaux then $\alpha \leq \delta$ is provable.*

PROOF. We would like to sketch how tableau equivalence is used to construct a proof.

Let $\mathcal{T}_\alpha = \langle T_\alpha, L_\alpha \rangle$ and $\mathcal{T}_\delta = \langle T_\delta, L_\delta \rangle$ be tableaux for α and δ respectively. Let $\mathcal{E} : \mathcal{T}_\alpha \to \mathcal{T}_\delta$ be an equivalence function. We will construct a thin refutation $\mathcal{R}$ for $\alpha \wedge \neg\delta$.

To facilitate the construction we will define a correspondence function $\mathcal{C}_\alpha$ which assigns to every considered node n of $\mathcal{R}$ (that is not to every node) a node $\mathcal{C}_\alpha(n)$ of $\mathcal{T}_\alpha$ such that:

$$(*) \qquad L(n) = L_\alpha(\mathcal{C}_\alpha(n)) \cup \{\neg \bigwedge L_\delta(\mathcal{E}(\mathcal{C}_\alpha(n)))\}.$$

Of course, the root of $\mathcal{R}$ will be labeled by $\{\alpha, \neg\delta\}$. Setting $\mathcal{C}_\alpha(r)$ to be the root of $\mathcal{T}_\alpha$ establishes condition $(*)$.

Now suppose that n is a node of $\mathcal{R}$ under construction and $\mathcal{C}_\alpha(n)$ is a choice node such that $(*)$ holds. As δ is a disjunctive formula, $L_\delta(\mathcal{E}(\mathcal{C}_\alpha(n)))$ is a one element set, say $\{\gamma\}$. Hence $L(n) = L_\alpha(\mathcal{C}_\alpha(n)) \cup \{\neg\gamma\}$.

We show how to extend $\mathcal{R}$ from n. Let us apply, as long as possible, rules other than (mod) and weakening to all the formulas in $L(n)$ except $\neg\gamma$ in the same order as they were applied from $\mathcal{C}_\alpha(n)$. In this way, we obtain a finite part of a tree rooted in n with leaves $n_1, \dots, n_k$. For every $j = 1, \dots, k$ the label $L(n_j)$ contains $\neg\gamma$ and some set of formulas Γ_j to which only (mod) may be applicable. It is easy to see that every n_j corresponds to some modal node near $\mathcal{C}_\alpha(n)$, call it $\mathcal{C}_\alpha(n_j)$, with the property that $L_\alpha(\mathcal{C}_\alpha(n_j)) = \Gamma_j$.

Let us look at the path from $\mathcal{E}(\mathcal{C}_\alpha(n))$ to $\mathcal{E}(\mathcal{C}_\alpha(n_j))$ in $\mathcal{T}_\delta$. Because δ is a disjunctive formula, on this path first only (μ), (ν), (reg) and (or) rules may be applied and then we have zero or more applications of the (and) rule. Let us apply dual rules to $\neg\gamma$: dual to (μ) is (ν), (reg) is self-dual, dual to (or) is (and) but when we apply this rule we immediately use weakening to leave only the conjunct which appears on the path to $\mathcal{E}(\mathcal{C}_\alpha(n_j))$. We do not apply (or) rules.

In this way, we arrive at a node m_j. If we define $\mathcal{C}_\alpha(m_j) = \mathcal{C}_\alpha(n_j)$ then its label is $L_\alpha(\mathcal{C}_\alpha(m_j)) \cup \{\neg \bigwedge L_\delta(\mathcal{E}(\mathcal{C}_\alpha(m_j)))\}$. Hence $(*)$ is satisfied. We can now apply similar reasoning to extend $\mathcal{R}$ from this node.

Let $\mathcal{P}$ be an infinite path of $\mathcal{R}$ constructed in this way. We have two possibilities. There may be a μ-trace on the path of $\mathcal{T}_\alpha$ designated by the image of $\mathcal{P}$ under $\mathcal{C}_\alpha$. If so, then, by construction, the same trace appears on $\mathcal{P}$. If there is no μ-trace on $\mathcal{C}_\alpha(\mathcal{P})$ then, by the definition of tableau equivalence, there cannot be a μ-trace on $\mathcal{E}(\mathcal{C}_\alpha(\mathcal{P}))$. Hence there is a ν-trace on $\mathcal{E}(\mathcal{C}_\alpha(\mathcal{P}))$ which when negated in $\mathcal{R}$ becomes a μ-trace.

This means that every finite path of $\mathcal{R}$ ends in a set containing $\bot$ or a literal and its negation and there is a μ-trace on every infinite path. Hence $\mathcal{R}$ is a refutation. It is also a thin refutation because α is a weakly aconjunctive formula and whenever we reduce a conjunction coming from $\neg\delta$ we leave only one of the conjuncts. Hence by Theorem 20, $\alpha \leq \delta$ is provable. ⊣

Case: $\varphi = \alpha \wedge \beta$. The argument is very similar to that of the previous case. Lemma 25 is also used here.

Case: $\varphi = \mu X.\alpha(X)$. This case is more complicated than the case for the greatest fixpoint. As in that case, we have, by the induction assumption a disjunctive formula $\widehat{\alpha}(X)$ equivalent to $\alpha(X)$. The difference is that $\mu X.\widehat{\alpha}(X)$ may not be a weakly aconjunctive formula so we cannot use the same argument. Let us nevertheless try to carry on and see where modifications are needed.

By Theorem 22 there is a disjunctive formula $\widehat{\varphi}$ which has a tableau equivalent to some regular tableau $\mathcal{T}$ for $\mu X.\widehat{\alpha}(X)$. By Theorem 15 the two formulas are equivalent. We are left to show that $\mu X.\widehat{\alpha}(X) \leq \widehat{\varphi}$ is provable in Kozen's system. Because $\mu X.\widehat{\alpha}(X)$ may not be an aconjunctive formula we cannot just apply Lemma 25 to $\mu X.\widehat{\alpha}(X)$ and $\widehat{\varphi}$. What we could do is to use the fixpoint rule $(K6)$ if we knew that $\widehat{\alpha}(\widehat{\varphi}) \leq \widehat{\varphi}$ is provable.

By Proposition 17 we know that $\widehat{\alpha}(\widehat{\varphi})$ is a weakly aconjunctive formula but we meet another obstacle preventing us from using Lemma 25. We don't know whether $\widehat{\alpha}(\widehat{\varphi})$ and $\widehat{\varphi}$ have equivalent tableaux. Actually, it may happen that the two formulas do not have equivalent tableaux.

We need some weaker notion of correspondence between tableaux but it should be strong enough to allow us to construct a proof of $\widehat{\alpha}(\widehat{\varphi}) \leq \widehat{\varphi}$. Below, we propose such a notion which we call tableau consequence. It is defined in terms of games.

DEFINITION 26 (Tableau consequence). For a given pair of tableaux $\widetilde{\mathcal{T}} = \langle \widetilde{T}, \widetilde{L} \rangle$ and $\mathcal{T} = \langle T, L \rangle$, we define a two player game $\mathcal{G}(\widetilde{\mathcal{T}}, \mathcal{T})$ with the following rules:

1. The starting position is the pair of the roots of each tableaux.

2. Suppose a position of a play is $(\widetilde{n}, n)$, both nodes being choice nodes of $\widetilde{\mathcal{T}}$ and $\mathcal{T}$ respectively. Player I must choose a modal node $\widetilde{m}$ near $\widetilde{n}$ and player II must respond by choosing a modal node m near n. Node m must have the property that every literal and terminal formula from $L(m)$ appears in $\widetilde{L}(\widetilde{m})$.

3. Suppose a position of a play is $(\widetilde{N}, N)$ with $\widetilde{N}, N$ being sets of choice nodes of $\widetilde{\mathcal{T}}$ and $\mathcal{T}$ respectively. Player I must choose a modal node $\widetilde{m}$ near some $\widetilde{n} \in \widetilde{N}$ and player II must respond with a modal node m near some

$n \in N$, such that, every literal and terminal formula from $L(m)$ appears in $\widetilde{L}(\widetilde{m})$.

4. Suppose a position consists of a pair of modal nodes $(\widetilde{m}, m)$ from $\widetilde{\mathcal{T}}$ and $\mathcal{T}$, respectively. Player I chooses some action a and has two possibilities afterwards. He can choose a $\langle a \rangle$-son n of m and player II then has to respond with a $\langle a \rangle$-son $\widetilde{n}$ of $\widetilde{m}$. Otherwise, player I can choose all $\langle a \rangle$-sons of m and player II must respond with the set of all $\langle a \rangle$-sons of $\widetilde{m}$.

The game may end in a finite number of steps because one of the players cannot make a move. In this case, the opposite player wins. When the game has infinitely many steps we get as the result two infinite paths: $\widetilde{\mathcal{P}}$ from $\widetilde{\mathcal{T}}$ and $\mathcal{P}$ from $\mathcal{T}$. Player I wins if there is no μ-trace on $\widetilde{\mathcal{P}}$ but there is a μ-trace on $\mathcal{P}$; otherwise, player II is the winner.

We will say that a tableau $\mathcal{T}$ is a *consequence* of a tableau $\widetilde{\mathcal{T}}$ iff player II has a winning strategy in $\mathcal{G}(\widetilde{\mathcal{T}}, \mathcal{T})$.

The definition of the game is based on the following intuition about tableaux. A tableau for a formula describes "operationally" the semantics of a formula. In order to satisfy formulas in a choice node n, we must provide a state which satisfies the label of one of the modal nodes near n. The sons of a modal node describe the transitions from a hypothetical state satisfying its label. Every $\langle a \rangle$-son describes an a-successor which is required. The set of all $\langle a \rangle$-sons puts a restriction on all possible a-successors of the node. In this way, a tableau of a formula describes all possible models of that formula.

The game is defined so that whenever player II has a winning strategy from a position $(\widetilde{n}, n)$ then every model of the label of $\widetilde{n}$, $\widetilde{L}(\widetilde{n})$, is also a model of the label of n, $L(n)$. If $\widetilde{n}$ and n are both choice nodes then a model of $\widetilde{L}(\widetilde{n})$ must satisfy the label of one of the modal nodes near $\widetilde{n}$. Hence, for every modal node near $\widetilde{n}$, we must find a modal node near n whose label is implied by it. If $\widetilde{n}$, n are modal nodes then every $\langle a \rangle$-son of n describes a state the existence of which is required in order to satisfy $L(n)$. We must show that existence of such a state is also required by $\widetilde{L}(\widetilde{n})$. The set of all the $\langle a \rangle$-sons represents general requirements, imposed by $L(n)$, on states reachable by action a. We must show that they are implied by the general requirements in $\widetilde{L}(\widetilde{n})$.

The following lemma can be proved using exactly the same method as in Lemma 25.

LEMMA 27. *Suppose that we have a weakly aconjunctive formula α and a disjunctive formula δ. If there is a tableau for δ which is a consequence of a tableau for α then $\alpha \leq \delta$ is provable.*

It is easy to show that if two tableaux are equivalent then one is a consequence of the other. Hence, we could use tableau consequence also for the case of the ν operator. To prove Theorem 24 it remains to show:

LEMMA 28. *Let $\mathcal{T}$ be a tableau for $\mu X.\widehat{\alpha}(X)$ that can be presented as a finite graph. Let $\widehat{\varphi}$ be a disjunctive formula with an equivalent tableau. The tableau $\mathcal{T}$ is a consequence of a tableau $\widetilde{\mathcal{T}}$ for $\widehat{\alpha}(\widehat{\varphi})$.*

We will not give the proof of this lemma here. The proof is quite technical and uses an argument similar to that in the proof of Lemma 25.

We can now complete the case when φ is $\mu X.\widehat{\alpha}(X)$. Tableau $\mathcal{T}$ is a consequence of $\widetilde{\mathcal{T}}$ and the unique tableau $\widehat{\mathcal{T}}$ for $\widehat{\varphi}$ is equivalent to $\mathcal{T}$. Hence, $\widehat{\mathcal{T}}$ is a consequence of $\widetilde{\mathcal{T}}$. Using Lemma 27, we get a proof of $\widehat{\alpha}(\widehat{\varphi}) \leq \widehat{\varphi}$. By rule $(K6)$, we get a proof of $\mu X.\widehat{\alpha}(X) \leq \widehat{\varphi}$. By induction assumption, we have a proof of $\mu X.\alpha(X) \leq \mu X.\widehat{\alpha}(X)$. Hence, by transitivity $\mu X.\alpha(X) \leq \widehat{\varphi}$ is provable. ⊣

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