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Constant-factor approximation of the domination number in sparse graphs

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ABSTRACT

The *k-domination number* of a graph is the minimum size of a set D such that every vertex of G is at distance at most k from D. We give a linear-time constant-factor algorithm for approximation of the k-domination number in classes of graphs with bounded expansion, which include e.g. proper minor-closed graph classes, proper classes closed on topological minors and classes of graphs that can be drawn on a fixed surface with bounded number of crossings on each edge.

The algorithm is based on the following approximate min–max characterization. A subset A of vertices of a graph G is *d-independent* if the distance between each two vertices in A is greater than d. Note that the size of the largest $2k$-independent set is a lower bound for the k-domination number. We show that every graph from a fixed class with bounded expansion contains a $2k$-independent set A and a k-dominating set D such that $|D| = O(|A|)$, and these sets can be found in linear time.

For a fixed value of k, the assumptions on the class can be formulated more precisely in terms of generalized coloring numbers. In particular, for the domination number ($k = 1$), the results hold for all graph classes with *arrangeability* bounded by a constant.

1. Introduction

For an undirected graph G, a set $D \subseteq V(G)$ is *dominating* if every vertex of G either belongs to D or has a neighbor in D. Determining the minimum size $\mathrm{dom}(G)$ of a dominating set (the *domination number*) in a graph G is a classical problem in algorithmic graph theory. It is known to be NP-complete in general (Karp [8]). Moreover, even approximating the domination number within a factor better

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than $O(\log |V(G)|)$ is NP-complete (Raz and Safra [16]). On the other hand, the problem becomes more manageable when restricted to some special classes of sparse graphs. For example, there exists a PTAS for the domination number of planar graphs (Baker [1]). Furthermore, there exists a linear-time algorithm approximating the domination number within a factor $O(c^2)$ for graphs with degeneracy at most c (Lenzen and Wattenhofer [12]).

In this paper, we follow the approach of Böhme and Mohar [2]. A subset A of vertices of a graph G is *d-independent* if the distance between each two vertices in A is greater than d. Denote by $\alpha_d(G)$ the maximum size of a d-independent set in G. Clearly, every vertex of G has at most one neighbor in a 2-independent set; hence, we have $\mathrm{dom}(G) \geq \alpha_2(G)$. In general, it is not possible to give an upper bound on $\mathrm{dom}(G)$ in terms of $\alpha_2(G)$, even for graphs with bounded degeneracy; see Section 4 for examples of 3-degenerate graphs with no 2-independent set of size greater than 2, but with arbitrarily large domination number. On the other hand, Böhme and Mohar [2] proved that for graphs in any proper minor-closed class, $\mathrm{dom}(G)$ is bounded by a linear function of $\alpha_2(G)$.

Theorem 1 (*Böhme and Mohar [2], Corollary 1.2*). *If G does not contain $K_{q,r}$ as a minor, then* $\mathrm{dom}(G) \leq (4r + (q-1)(r+1))\alpha_2(G) - 3r$.

The proof of the theorem is constructive, giving a polynomial-time algorithm that finds a dominating set D and a 2-independent set A such that $|D| \leq (4r + (q-1)(r+1))|A| - 3r$. Since $|A| \leq \mathrm{dom}(G)$, this approximates $\mathrm{dom}(G)$ within the constant factor $4r + (q-1)(r+1)$. Let us remark that all graphs in a proper minor-closed class have degeneracy bounded by a constant, and thus the algorithm of Lenzen and Wattenhofer [12] could also be applied in this case. However, Theorem 1 has the advantage of providing a simple certificate for the approximation factor (the 2-independent set A), and as we discuss below, it can be generalized for distance variants of domination.

We strengthen Theorem 1, showing that instead of excluding a fixed minor, it suffices to assume that the arrangeability of the graph is bounded. The *arrangeability* of a graph is a parameter similar to degeneracy which takes into account neighborhoods up to distance two, and it appeared previously in works on graph coloring and Ramsey theory [3,9,17]. Let $v_1, v_2, \ldots, v_n$ be an ordering of the vertices of a graph G. A vertex v_a is *2-accessible* from v_b if $a < b$ and either v_a is adjacent to v_b, or there exists a path $v_a v_m v_b$ in G with $m > b$. For a fixed ordering of $V(G)$, let $r_2(v)$ denote the number of vertices that are 2-accessible from v. The *arrangeability* of the ordering is the maximum of $r_2(v)$ over $v \in V(G)$. The arrangeability of G is the minimum of the arrangeabilities of all orderings of $V(G)$. We can now state our main result for the domination number.

Theorem 2. *If the arrangeability of G is at most c, then* $\mathrm{dom}(G) \leq (c^2 + c + 1)^2 \alpha_2(G)$.

The proof gives a linear-time algorithm for finding the corresponding dominating and 2-independent sets, assuming that the ordering of the vertices of G with arrangeability at most c is given. To relate Theorem 2 with Theorem 1, we use the following bound on arrangeability. For an integer $t \geq 0$ and a graph G, let $\mathrm{sd}_t(G)$ denote the graph obtained from G by subdividing each edge exactly t times.

Theorem 3 (*Dvořák [5], Theorem 9*). *Let G be a graph and d an integer. If $\delta(H) < d$ for every H such that $H \subseteq G$ or $\mathrm{sd}_1(H) \subseteq G$, then the arrangeability of G is at most $4d^2(4d+5)$.*

Consider now a proper minor-closed graph class $\mathcal{G}$. There exists a constant d such that all graphs in $\mathcal{G}$ have minimum degree less than d (Kostochka [11]). Now, if $\mathrm{sd}_1(H) \subseteq G$ for a graph $G \in \mathcal{G}$, then H is a minor of G and belongs to $\mathcal{G}$ as well, and thus $\delta(H) < d$. Consequently, Theorem 3 implies that graphs in $\mathcal{G}$ have arrangeability $O(d^3)$ and we can apply Theorem 2 for them. Therefore, we do indeed generalize Theorem 1, although the multiplicative constant in our result may be greater. Note that a similar argument also shows that Theorem 2 applies to all proper graph classes closed on topological minors.

As we mentioned before, Böhme and Mohar [2] in fact proved a more general result concerning distance domination. A set $D \subseteq V(G)$ is *k-dominating* if the distance from every vertex of G to D is at most k; thus, 1-dominating sets are precisely dominating sets. Let $\mathrm{dom}_k(G)$ denote the size of the smallest k-dominating set in G. Clearly, $\mathrm{dom}_k(G) \geq \alpha_{2k}(G)$. Theorem 1.1 of [2] shows that in any proper minor-closed class of graphs, $\mathrm{dom}_k(G) = O(\alpha_m(G))$ for any $m < \frac{5}{4}(k+1)$. We strengthen this result by considering less restricted classes of graphs as well as increasing m to the natural bound.

In order to give the precise statement, we need to introduce generalized coloring numbers (first defined by Kierstead and Yang [10]). Let $v_1, v_2, \ldots, v_n$ be an ordering of the vertices of a graph G. A vertex v_a is *weakly k-accessible* from v_b if $a < b$ and there exists a path $v_a = v_{i_0}, v_{i_1}, \ldots, v_{i_\ell} = v_b$ of length $\ell \leq k$ in G such that $a \leq i_j$ for $0 \leq j \leq \ell$. For a fixed ordering of $V(G)$, let $Q_k(v)$ denote the set of vertices that are weakly k-accessible from v and let $q_k(v) = |Q_k(v)|$. The *weak k-coloring number* of the ordering is the maximum of $1 + q_k(v)$ over $v \in V(G)$. The weak k-coloring number $\mathrm{wcol}_k(G)$ of G is the minimum of the weak k-coloring numbers over all orderings of $V(G)$. Let us remark that $\mathrm{wcol}_1(G) - 1$ is the degeneracy of G. Furthermore, observe that if G has arrangeability c, then $c < \mathrm{wcol}_2(G) \leq c^2 + c + 1$—the definition of weak 2-accessibility allows paths $v_a v_m v_b$ with $a < m < b$; however, in an ordering with arrangeability c, v_b is adjacent to at most c vertices that appear before it in the ordering (candidates for v_m), and the same holds for v_m, and thus there are at most c^2 such paths from v_b.

We can now state the general form of our main result.

Theorem 4. *If $k \geq 1$ and $1 \leq m \leq 2k+1$ are integers and G satisfies $\mathrm{wcol}_m(G) \leq c$, then $\mathrm{dom}_k(G) \leq c^2\alpha_m(G)$. Furthermore, if an ordering of $V(G)$ such that $q_m(v) < c$ for every $v \in V(G)$ is given, then a k-dominating set D and an m-independent set A such that $|D| \leq c^2|A|$ can be found in $O(c^2k|V(G)|)$ time.*

Let us remark that this result is tight in the sense that it would not be sufficient to bound $\mathrm{wcol}_{m-1}(G)$. An example showing this is given in Section 4. Furthermore, it is not possible to give a bound on $\mathrm{dom}_k(G)$ in terms of $\alpha_{2k+2}(G)$, even if G is a tree [2].

While it is natural to assume that $\mathrm{dom}_k(G)$ could be bounded in terms of $\alpha_{2k}(G)$, it may seem somewhat surprising that Theorem 4 gives a bound even in terms of $\alpha_{2k+1}(G)$. To see the intuition behind this, consider the following situation. Let T be a $2k$-independent set in G and suppose that there exists a vertex v such that for every pair of vertices $x, y \in T$, a shortest path between x and y passes through v. Since T is $2k$-independent, at most one vertex of T is at distance at most k from v. Therefore, T contains a $(2k+1)$-independent subset of size at least $|T| - 1$. On the basis of this observation, it is not difficult to prove that $\alpha_{2k}(G)$ is bounded by a function of $\alpha_{2k+1}(G)$, under the assumption that $\mathrm{wcol}_{2k+1}(G)$ is bounded.

In order to be able to apply the algorithm of Theorem 4 in general, we need an algorithm for finding an ordering of vertices of G with a bounded weak m-coloring number. We do not know of a polynomial-time algorithm for finding an ordering of vertices of G with weak m-coloring number exactly $\mathrm{wcol}_m(G)$. However, on the basis of Section 4.3 of the thesis of Dvořák [4], we can design a cubic-time algorithm that returns an ordering with weak m-coloring number bounded by a function of $\mathrm{wcol}_m(G)$. If $\mathrm{wcol}_m(G)$ is bounded by a constant, Theorem 4 applied with such an ordering returns a k-dominating and an m-independent set whose sizes differ only by a constant factor. More details are provided in Section 3.2.

For which graph classes does Theorem 4 provide a constant-factor approximation for every $k \geq 0$? That is, for which graph classes does there exist a function f such that $\mathrm{wcol}_m(G) \leq f(m)$ for every graph G in the class and every integer $m \geq 0$? By Zhu [19] and independently by Yang [18], these are precisely the graph classes with *bounded expansion* (see the survey [13] or the book [14] of Nešetřil and Ossona de Mendez for various equivalent definitions and properties of such graph classes). Let us note that most classes of "structurally sparse" graphs have bounded expansion, including graphs avoiding a fixed topological minor and graphs that can be drawn in a fixed surface with bounded number of crossings on each edge [15].

Furthermore, as discussed in Section 3.3, classes with bounded expansion admit efficient algorithms for first-order properties. This enables us to improve the time complexity of the algorithm for finding an ordering with bounded weak m-coloring number to linear. Consequently, for any class of graphs $\mathcal{G}$ with bounded expansion and for any integer $k \geq 0$, we obtain a linear-time algorithm approximating $\mathrm{dom}_k(G)$ and $\alpha_{2k+1}(G)$ within a constant factor for graphs $G \in \mathcal{G}$.

Theorem 5. *Let $\mathcal{G}$ be a class of graphs with bounded expansion and let $k \geq 1$ be an integer. There exists an algorithm that for each $G \in \mathcal{G}$ returns a k-dominating set D and a $(2k+1)$-independent set A such that $|D| = O(|A|)$. The algorithm runs in time $O(|V(G)|)$.*

2. Proof of the main result

Theorem 2 is implied by Theorem 4 with $k = 1$ and $m = 2$, since $\mathrm{wcol}_2(G) \le c^2 + c + 1$ if G has arrangeability at most c (and the ordering of the vertices of G with arrangeability at most c also has weak 2-coloring number at most $c^2 + c + 1$). Therefore, it suffices to prove the latter. We defer the discussion of the algorithmic aspects to Section 3.1, and prove here just the existence of the sets D and A with the required properties.

- initialize $D := \emptyset$, $A' := \emptyset$ and $R := V(G)$
- while R is nonempty, repeat:
  - let v be the least vertex of R in the ordering
  - set $A' := A' \cup \{v\}$
  - set $D := D \cup \{v\} \cup Q_m(v)$
  - remove from R all vertices whose distance from $\{v\} \cup Q_m(v)$ is at most k

Algorithm 1: Finding the dominating set.

Proof of Theorem 4. Let $v_1, \dots, v_n$ be an ordering of vertices of G such that $q_m(v) \le c - 1$ for every $v \in V(G)$. We construct sets D and A' using Algorithm 1. Clearly, D is a k-dominating set in G and $|D| \le c|A'|$. Therefore, it suffices to find an m-independent subset of A' of size at least $|A'|/c$.

Let $a_1, a_2, \dots, a_s$ be the vertices of A' in the order consistent with the ordering of $V(G)$. Let H be the graph with vertex set A' such that $uv \in E(H)$ iff the distance between u and v in G is at most m. Note that a subset of A' is m-independent in G if and only if it is independent in H.

Consider vertices $a_i, a_j \in A'$ with $j < i$ that are adjacent in H, and let P be a path in G of length at most m between a_i and a_j. Let z be the least vertex of P according to the ordering of $V(G)$. Since all other vertices of P are after z in the ordering, it follows that z is weakly m-accessible in G from every other vertex of P. In particular, we have $z \in Q_m(a_i) \cap (\{a_j\} \cup Q_m(a_j))$, and thus $z \in D$. Using Algorithm 1, the vertex a_j was added to A' before a_i. At that point, we also removed from R all vertices whose distance from $\{a_j\} \cup Q_m(a_j)$ is at most k. Since a_i was eventually added to A', it was not removed from R at that point, and thus the distance between a_i and $\{a_j\} \cup Q_m(a_j)$ is at least $k + 1$. Since z belongs to $\{a_j\} \cup Q_m(a_j)$, it follows that the length of the subpath of P between a_i and z is at least $k + 1$. Consequently, the length of the subpath of P between a_j and z is at most $m - (k + 1) \le k$. Therefore, either $z = a_j$ or z is weakly k-accessible from a_j, since z is the least vertex of P.

Let $k' = \min(k, m)$. For each $w \in D$, let T_w be the set of all vertices $a \in A'$ such that $w \in \{a\} \cup Q_{k'}(a)$. By the previous paragraph, if $a_i, a_j \in A'$ are adjacent in H and $j < i$, then $a_j \in \bigcup_{w \in Q_m(a_i)} T_w$.

Consider a vertex $w \in D$ and suppose that $T_w \neq \emptyset$. Let a be the least vertex of T_w according to the ordering of $V(G)$. Note that w belongs to $\{a\} \cup Q_m(a)$; hence, when Algorithm 1 added a to A', it also removed from R all vertices whose distance from w is at most k. Since all elements of T_w are at distance at most $k' \le k$ from w, we conclude that a is the only element of T_w. Therefore, we have $|T_w| \le 1$ for every $w \in D$.

Consequently, $|\bigcup_{w \in Q_m(a)} T_w| \le |Q_m(a)| \le c - 1$ for every $a \in A'$, and thus a is adjacent to at most $c - 1$ vertices of H that appear before a in the ordering of $V(H)$. This means that H is $(c-1)$-degenerate; hence, H is c-colorable. Therefore, H has an independent set A of size at least $|A'|/c$. By the definition of H, the set A is m-independent in G, and we have $|D| \le c|A'| \le c^2|A|$ as required. $\square$

3. Algorithmic aspects

3.1. Implementation of Algorithm 1

Let G be a graph on n vertices such that $\mathrm{wcol}_m(G) \le c$, and assume that we are given an ordering of $V(G)$ such that $q_m(v) < c$ for every $v \in V(G)$. Since $m \ge 1$, this implies that G is c-degenerate, and thus it has at most cn edges.

For each $i \leq m$ and $v \in V(G)$, we determine the set $Q_i(v)$ (whose size is bounded by c) using the following algorithm: For $i = 1$, $Q_1(v)$ is the set of neighbors of v that appear before it in the ordering, which can be determined by enumerating all the edges incident with v. For $i > 1$, $Q_i(v)$ is the subset of $Q_1(v) \cup \bigcup_{uv \in E(G)} Q_{i-1}(u)$ consisting of the vertices before v in the ordering. Note that $Q_i(v)$ can be determined in time $O(c(\deg(v) + 1))$, assuming that Q_{i-1} was already computed before. Therefore, each Q_i can be computed for all vertices of G in time $O(c^2n)$, and in total we take time $O(c^2mn)$ to determine $Q_m(v)$ for every vertex of G.

With this information, we can implement Algorithm 1 in time $O(ckn)$. The only nontrivial part is the removal of the vertices from R. For each vertex v of $V(G)$ we maintain the value $p(v) = \min(k+1, d(v))$, where $d(v)$ is the distance of v from D. In each step of the algorithm, we have $v \in R$ iff $p(v) = k + 1$ and $v \in D$ iff $p(v) = 0$. When a vertex v is added to D, we decrease $p(v)$ to 0. For each vertex w, whenever the value of $p(w)$ decreases, we recursively propagate this change to the neighbors of w: if $uw \in E(G)$ and $p(u) > p(w) + 1$, then we decrease $p(u)$ to $p(w) + 1$. Clearly, the value of $p(w)$ decreases at most $(k + 1)$ times during the run of the algorithm, and we spend time $O(k \deg(v))$ in updating it and propagating the decrease to the neighbors. Therefore, the total time for maintaining the set R is bounded by $O(ckn)$.

Next, we compute the set T_w for each vertex $w \in D$: we initialize these sets to $\emptyset$, and then for each $a \in A'$, we add a to T_w for each $w \in \{a\} \cup Q_{k'}(a)$. For the final part of the algorithm, we do not determine the graph H exactly. Instead, a $(c - 1)$-degenerate supergraph H' of H is obtained by joining each $a \in A'$ with all the elements of $\bigcup_{w \in Q_m(a)} T_w$ that precede a in the ordering. We find a proper coloring of H' with at most c colors using the standard greedy algorithm, and choose A as the largest color class in this coloring. The time for this phase is $O(cn)$.

Since $m \leq 2k+1$, the total complexity of the algorithm is $O(c^2kn)$. The space complexity is bounded by the space needed to represent $Q_{k'}$ and Q_m, and thus it is $O(cn)$.

3.2. *The weak coloring number*

Let us now turn our attention to the problem of finding a suitable ordering of vertices. We were not able to find a polynomial-time algorithm for determining $\mathrm{wcol}_m(G)$ for $m \geq 2$, and we conjecture that the problem is NP-complete. However, we can at least find an ordering whose weak m-coloring number is bounded by a function of $\mathrm{wcol}_m(G)$, by optimizing a related parameter.

Consider an ordering $v_1, v_2, \ldots, v_n$ of the vertices of a graph G. The *m-backconnectivity* $b_m(v)$ of v with respect to this ordering is the maximum number of paths of length at most m in G that start in v, share no other vertices and end before v in the ordering (clearly, we can assume that the internal vertices of the paths appear after v in the ordering). The *m-admissibility* of the ordering is the maximum of $b_m(v)$ over $v \in V(G)$. The m-admissibility $\mathrm{adm}_m(G)$ of G is the minimum of m-admissibilities of all orderings of $V(G)$.

Note that the m-admissibility of an ordering is smaller than its weak m-coloring number, as the endvertices of the paths in the definition of m-backconnectivity of v are weakly m-accessible from v. Conversely, the weak m-coloring number is bounded by a function of m-admissibility (this generalizes a well-known relationship between arrangeability and 2-admissibility [9]).

Lemma 6. *Let $m \geq 1$ and $c \geq 2$ be integers. Let G be a graph and $v_1, v_2, \ldots, v_n$ an ordering of its vertices. If the m-admissibility of the ordering is at most c, then its weak m-coloring number is at most $\frac{c^{m+1}-1}{c-1}$.*

Proof. For a vertex $v_b \in V(G)$ and an integer $k \geq 1$, let $R_k^0(v_b)$ denote the set of vertices $v_a \in V(G)$ such that $a < b$ and there exists a path $v_a = v_{i_0}, v_{i_1}, \ldots, v_{i_\ell} = v_b$ of length $\ell \leq k$ in G such that $b \leq i_j$ for $1 \leq j \leq \ell$. That is, v_a is weakly k-accessible from v_b and all the vertices of a path that certifies this lie after v_b in the ordering. Let $R_1(v_b) = R_1^0(v_b)$ and let $R_k(v_b) = R_k^0(v_b) \setminus R_{k-1}^0(v_b)$ for $k \geq 2$.

Consider a vertex $v \in V(G)$ and an integer $i \leq m$. The set $R_i^0(v)$ can be found by breadth-first search from v, stopping when a vertex that appears before v in the ordering is encountered. This gives a tree $T \subseteq G$ containing v such that

- $R_i^0(v)$ is the set of leaves of T distinct from v,

- for $1 \le k \le i$, every path from v to $R_k(v)$ in T has length exactly k, and
- all vertices of $V(T) \setminus \left(\{v\} \cup R_i^0(v)\right)$ appear after v in the ordering.

Since the m-admissibility of the ordering is at most c, every vertex $u \in V(T) \setminus R_i^0(v)$ has degree at most c in T; otherwise, the m-backconnectivity of u would be at least $c+1$ due to the paths from u to $\{v\} \cup R_i^0(v)$ in T sharing no vertex other than u. We conclude that

$$\sum_{k=1}^{i} |R_k(v)|(c-1)^{i-k} \le c(c-1)^{i-1}.$$

Observe that

$$Q_i(v) = R_i^0(v) \cup \bigcup_{k=1}^{i-1} \bigcup_{u \in R_k(v)} Q_{i-k}(u)$$

for every integer $i \ge 1$: consider the path from v to $x \in Q_i(v)$ certifying that x is weakly i-accessible from v, and let u be the first vertex of this path that is smaller than v in the ordering. If $u = x$, then x belongs to $R_i^0(v)$. Otherwise, u belongs to $R_k(v)$ for some $k \le i-1$, and the rest of the path shows that x belongs to $Q_{i-k}(u)$.

We show that $|Q_i(u)| \le \frac{c^{i+1}-1}{c-1} - 1$ for $1 \le i \le m$ and for all $u \in V(G)$ by induction. For $i = 1$ the claim holds, as $Q_1(v) = R_1(v)$. Assume now that $i > 1$ and that the claim holds for all smaller values of i. We have

$$\begin{aligned}
|Q_i(v)| &\le |R_i^0(v)| + \sum_{k=1}^{i-1} \sum_{u \in R_k(v)} |Q_{i-k}(u)| \\
&\le |R_i^0(v)| + \sum_{k=1}^{i-1} |R_k(v)| \left(\frac{c^{i-k+1}-1}{c-1} - 1 \right) \\
&= \sum_{k=1}^{i} |R_k(v)| + \sum_{k=1}^{i-1} |R_k(v)| \left(\frac{c^{i-k+1}-1}{c-1} - 1 \right) \\
&= |R_i(v)| + \sum_{k=1}^{i-1} |R_k(v)| \frac{c^{i-k+1}-1}{c-1} \\
&= \sum_{k=1}^{i} |R_k(v)| \frac{c^{i-k+1}-1}{c-1} = \sum_{k=1}^{i} |R_k(v)|(c-1)^{i-k} \frac{c^{i-k+1}-1}{(c-1)^{i-k+1}} \\
&\le \frac{c^i-1}{(c-1)^i} \sum_{k=1}^{i} |R_k(v)|(c-1)^{i-k} \\
&\le \frac{c^i-1}{(c-1)^i} c(c-1)^{i-1} = c\frac{c^i-1}{c-1} = \frac{c^{i+1}-1}{c-1} - 1.
\end{aligned}$$

Therefore, we have $q_m(v) \le \frac{c^{m+1}-1}{c-1} - 1$ for every $v \in V(G)$, and the claim of the lemma follows. □

For a set $S \subseteq V(G)$ and a vertex $v \in S$, we let $b_m(S, v)$ denote the maximum number of paths from v to S of length at most m that intersect only in v and whose internal vertices belong to $V(G) \setminus S$. The ordering of $V(G)$ with the smallest m-admissibility can be found greedily, using Algorithm 2.

Clearly, the resulting ordering has m-admissibility $\max(p_1, \dots, p_n)$, and it is easy to see that this is equal to $\mathrm{adm}_m(G)$: Let X be an ordering of $V(G)$ with m-admissibility $p = \mathrm{adm}_m(G)$, and consider an arbitrary set $S \subseteq V(G)$. Let v be the last vertex of S according to the ordering X. Then $b_m(S, v) \le p$, since all vertices of S are before v in the ordering X and the m-backconnectivity of v in X is at most p. Therefore, we have $p_i \le p$ for $1 \le i \le n$ in the algorithm.

- initialize $S := V(G)$
- for $i = n, n-1, \ldots, 1$:
  - choose $v_i \in S$ minimizing $p_i = b_m(S, v_i)$
  - set $S := S \setminus \{v_i\}$

Algorithm 2: Determining m-admissibility.

A slightly problematic step in the algorithm is finding the vertex $v \in S$ minimizing $b_m(S, v)$, since for $m \geq 5$, determining $b_m(S, v)$ is NP-complete in general (Itai, Perl and Shiloach [7]). Nevertheless, $b_m(S, v)$ can be approximated within a factor of m, by repeatedly taking any path from v to S of length at most m with all internal vertices in $V(G) \setminus S$ and removing its vertices distinct from v (each such removal can interrupt at most m paths in the optimal solution, and hence we will be able to pick at least $b_m(S, v)/m$ paths this way). A straightforward implementation gives an $O(mn^3)$ algorithm for approximating $\mathrm{adm}_m(G)$ within a factor of m.

When the weak m-coloring number of G is bounded by a constant $c \geq 2$, then $\mathrm{adm}_m(G) \leq c$, and thus the algorithm described in the previous paragraph returns an ordering with m-admissibility at most cm. By Lemma 6, this ordering has weak m-coloring number at most $\frac{(cm)^{m+1}-1}{cm-1}$, which is a constant. Together with the algorithm of Theorem 4, this gives a cubic-time algorithm for approximating $\mathrm{dom}_k(G)$ within a constant factor, for any $k \leq m/2$.

3.3. Classes with bounded expansion

Let us remark that when the m-admissibility of G is bounded by a constant p, we can design a polynomial-time algorithm to determine $\mathrm{adm}_m(G)$ exactly. To test whether $b_m(S, v) \leq p$, we simply enumerate all sets of at most $p+1$ paths of length at most m starting in v. A straightforward implementation gives an algorithm with time complexity $O(n^{mp+m+2})$.

This time complexity can be improved significantly if the class of graphs $\mathcal{G}$ considered has bounded expansion. Dvořák et al. [6] described a data structure for representing a graph in such a class and answer first-order queries for it in a constant time. In particular, suppose that $\varphi(x)$ is a first-order formula with one free variable x using a binary predicate e and a unary predicate s. This data structure can be used to represent a graph in $\mathcal{G}$ and a subset S of its vertices and the following holds

- the data structure can be initialized in linear time,
- we can add a vertex to S or remove it from S in constant time, and
- we can find in constant time a vertex $v \in V(G)$ such that $\varphi(v)$ holds, with e interpreted as the adjacency in G and s as the membership in S, or decide that no such vertex exists.

For the purpose of the algorithm for m-admissibility, to test whether $b_m(S, v_i) \leq p$, we apply the data structure for the property

$$\varphi(x) = s(x) \wedge \neg \left[(\exists y_1) \ldots (\exists y_{m(p+1)}) \psi(x, y_1, \ldots, y_{m(p+1)})\right],$$

where ψ is the formula describing that the subgraph induced by $\{x, y_1, \ldots, y_{m(p+1)}\}$ contains $p+1$ paths from x of length at most m, intersecting only in x, and with endvertices satisfying s and internal vertices not satisfying s.

Using this data structure, we repeatedly find $x \in S$ such that $b_m(S, x) \leq p$ and remove it from S, thus obtaining an ordering of $V(G)$ with admissibility at most p or determining that $\mathrm{adm}_m(G) > p$ in linear time. By Zhu [19] or Yang [18], for each class $\mathcal{G}$ with bounded expansion, there exists a function f such that $\mathrm{adm}_m(G) \leq f(m)$ for each $G \in \mathcal{G}$ and each $m \geq 1$. Therefore, we can determine the exact value of the m-admissibility by applying this test for $p = 1, \ldots, f(m)$.

Theorem 7. *Let $\mathcal{G}$ be a class of graphs with bounded expansion and $m \geq 1$ an integer. There exists a linear-time algorithm that for each $G \in \mathcal{G}$ determines $\mathrm{adm}_m(G)$ and outputs the corresponding ordering of $V(G)$.*

This implies Theorem 5.

4. The lower bound

Let us now explore the limits for the possible extensions of Theorem 4. Let us fix an integer $k \geq 1$. For $n \geq 4$, let $G'_n = \mathrm{sd}_{2k-1}(K_n)$, let X be the set of the middle vertices of the paths corresponding to the edges of K_n in G'_n and let Y be the set of vertices of G'_n of degree $n-1$. Let G_n be the graph obtained from G'_n by adding a new vertex v adjacent to all the vertices of X.

The distance between any two vertices of $V(G_n) \setminus Y$ is at most $2k$, since all these vertices are at distance at most k from v. Furthermore, the distance between any two vertices of Y is most $2k$, since they are joined by a path of length $2k$ corresponding to an edge of K_n. Therefore, every $2k$-independent set in G_n contains at most one vertex in Y and at most one vertex in $V(G_n) \setminus Y$, and consequently $\alpha_{2k}(G_n) \leq 2$. On the other hand, for any $w \in V(G_n) \setminus X$, there is at most one vertex of Y whose distance from w is at most k, and each vertex of X has distance at most k from exactly two vertices of Y. We conclude that $\mathrm{dom}_k(G_n) \geq n/2$, since at least $n/2$ vertices of G are needed to dominate Y. Therefore, k-domination number cannot be bounded by a function of $2k$-independence number on any class of graphs that contains $\{G_i : i \geq 4\}$. Let us remark that G_n is 3-degenerate if $k = 1$ and 2-degenerate if $k \geq 2$.

Let us consider the following ordering of the vertices of G_n: the first vertex is v, followed by Y in an arbitrary order, followed by the rest of vertices of G_n in an arbitrary order. Since the distance between any two vertices of Y is $2k$, we have $q_{2k-1}(w) \leq 1$ for $w \in Y$, and similarly $q_{2k-1}(w) \leq 2k+1$ for $w \in V(G_n) \setminus Y$. Therefore, $\mathrm{wcol}_{2k-1}(G_n) \leq 2k+2$ for every $n \geq 4$. It follows that at least in the case where $m = 2k$, it is not sufficient to restrict $\mathrm{wcol}_{2k-1}(G)$ in Theorem 4.

Another possible extension, bounding $\mathrm{dom}_k(G)$ by a function of $\alpha_{2k+2}(G)$, is impossible even for trees [2], as the graph $\mathrm{sd}_k(K_{1,n})$ demonstrates.

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