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DIFFERENCE FIELDS AND DESCENT IN ALGEBRAIC DYNAMICS. I

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Abstract We draw a connection between the model-theoretic notions of modularity (or one-basedness), orthogonality and internality, as applied to difference fields, and questions of descent in in algebraic dynamics. In particular we prove in any dimension a strong dynamical version of Northcott's theorem for function fields, answering a question of Szpiro and Tucker and generalizing a theorem of Baker's for the projective line.

The paper comes in three parts. This first part contains an exposition some of the main results of the model theory of difference fields, and their immediate connection to questions of descent in algebraic dynamics. We present the model-theoretic notion of internality in a context that does not require a universal domain with quantifier-elimination. We also note a version of canonical heights that applies well beyond polarized algebraic dynamics. Part II sharpens the structure theory to arbitrary base fields and constructible maps where in part I we emphasize finite base change and correspondences. Part III will include precise structure theorems related to the Galois theory considered here, and will enable a sharpening of the descent results for non-modular dynamics.

Keywords: model theory; difference fields; algebraic dynamics

AMS 2000 *Mathematics subject classification:* Primary 03C60; 12H10; 14GXX

1. Introduction

Algebraic dynamics studies algebraic varieties V with an endomorphism ϕ. Most questions reduce to the case that ϕ is dominant, and we will assume this. The base field is taken to be a number field or a function field F; given an extension field K of F, one has the set $V(K)$, and a self-map $\phi : V(K) \to V(K)$.

The same objects $\boldsymbol{V} = (V, \phi)$ arise in model theory in a somewhat different way. Here we take not only K but also an endomorphism σ of K, and associate to $\boldsymbol{V}$ the fixed points of ϕ twisted by σ, i.e. the set $\boldsymbol{V}(K, \sigma) = \{x \in V(K) : \sigma(x) = \phi(x)\}$. The pair (K, σ) is called a difference field; the functor $(K, \sigma) \mapsto \boldsymbol{V}(K, \sigma)$ suffices to recover $\boldsymbol{V}$.

This point of view makes just as much sense when one takes ϕ to be any correspondence on V, and the theory is carried out in that generality; we will restrict attention in the introduction to the case that ϕ is a rational function.

Beyond the coincidence of the objects, the two subjects diverge. Much of the richness of algebraic dynamics comes from the arithmetic of the field, and in particular the interaction of the dynamics with valuations and absolute values on K. But the model theory of valued difference fields is still in its infancy, and the model theory of global fields is not yet conceived. Current model-theoretic results thus concern purely geometric aspects of algebraic dynamics. On the other hand, in a number of ways that will be detailed below, the treatment is more general; and in particular there is an effort to find precise dividing lines among quite differently behaved dynamics, in any dimension. It seems possible that the general properties that arise from this study can be useful in algebraic dynamics, and the purpose of this paper is to bring them out.

Our motivation lay in a question of Szpiro and Tucker concerning descent for algebraic dynamics, arising out of Northcott's theorem for dynamics over function fields. The question was originally formulated for polarized dynamical systems in terms of canonical heights; Szpiro translated it into the language of limited sets, where it admits a much more general form, and it is in this form that we solve the problem. A subset of the function field that can be parametrized by a constructible set over the base field will be called *limited** (see § 3). For algebraic dynamics ϕ on $\mathbb{P}^1$ with $\deg(\phi) > 1$, Baker showed that no infinite orbit can be contained in a limited set, unless the dynamics is isotrivial, i.e. comes from a dynamics defined over the base field. Generalizing this to higher dimensions requires first of all a definition: what is the right analogue, for higher dimensions, of the hypothesis that $\deg(\phi) > 1$? Of isotriviality? For Abelian varieties, the notion of a simple component is clear, and isotriviality needs to be refined to a notion of a *trace* of a family of Abelian varieties, roughly speaking the sum of simple components that stay constant in the family. What is the analogue when one generalizes to algebraic dynamics? The model-theoretic description of the category of algebraic dynamics gives clear answers; using them we sharpen and generalize Baker's theorem to arbitrary dimension. As Daniel Bertrand pointed out to us, in the case of Abelian varieties with a multiplication dynamics, our result is classical and due to Lang and Néron (see [**21**, Chapter 6, Theorem 5.4]).

Model theory usually begins by amalgamating all relevant structures into a homogeneous one, a universal domain $\mathcal{U} = (\mathcal{U}, \sigma)$. Then $\boldsymbol{V}(\mathcal{U})$ contains all the information in the functor $(K, \sigma) \mapsto \boldsymbol{V}(K, \sigma)$. Now difference fields admit amalgamation over algebraically closed subfields, but not over arbitrary subfields. This obliges us to consider more than one universal domain; $\mathcal{U}$ will be determined by its prime field F, and the conjugacy class of σ in $\mathrm{Aut}(F^{\mathrm{alg}}/F)$. Moreover, the family of sets $\boldsymbol{V}(\mathcal{U})$ (or their Boolean combinations) is not closed under projections. However, the fact that amalgamation holds over algebraicallly closed subfields means that it is not necessary to look further than projections under finite maps. By a *definable subset* of $V(\mathcal{U})$ we mean one defined by some formula

* The term 'bounded' may be more common. In [**22**] the authors speak of points 'belonging to a finite number of algebraic families'.

formed out of difference equations and inequations by means of the logical operators $(\exists x)$, $(\forall y)$, $\wedge$, $\neg$. The following proposition was proved for $V(F)$, the set of points of V in a pseudo-finite field F, by Ax. Van den Dries understood that Ax's theorem can be interpreted as applying to $\boldsymbol{V}(\mathcal{U})$ where V is a variety with the trivial dynamics, suggesting that the same form of quantifier elimination may be valid more generally. This was carried out in [**23**], as well as in [**7**]. Here we state it for algebraic dynamics, though a version valid for any difference variety is available.

Let K be a field, and AD_K the category of pairs $\boldsymbol{V} = (V, \phi)$ where V is an irreducible variety over K, and $\phi : V \to V$ a dominant rational map.

Proposition 1.1. *Every definable subset of $\boldsymbol{V}(\mathcal{U})$ is a finite Boolean combination of sets of the form $f(\boldsymbol{W}(\mathcal{U}))$, where $\boldsymbol{W} = (W, \phi') \in \mathrm{AD}_K$ and $f : W \to V$ is a quasi-finite morphism of varieties, with $\phi \circ f = f \circ \phi'$.*

We call $\boldsymbol{W}$ a *finite cover* of $\boldsymbol{V}$ (or rather of the closure of the image of f). Such finite covers cannot be avoided; indeed if $\deg(f) > 1$, possibly after passing from σ to σ^n, the set $f(\boldsymbol{W}(\mathcal{U}))$ is never a Boolean combination of quantifier-free definable sets; moreover if $f(\boldsymbol{W}(\mathcal{U}))$ is a Boolean combination of sets $f_i(\boldsymbol{W}_i(\mathcal{U}))$, then the finite cover $W \to V$ is itself a quotient of the fibre product of the covers W_i.

The theory of $\mathcal{U}$ is *simple*, and stable on a quantifier-free level. The methods and definitions of stability theory apply to $\boldsymbol{V}(\mathcal{U})$: stability, modularity, internality to fixed fields, analysis in terms of minimal types. We would like to view these as properties of $\boldsymbol{V}$; but the embedding into $\mathcal{U}$ is not canonically determined by the geometric data $\boldsymbol{V}$, and moreover, even if the variety V is absolutely irreducible, the embedding splits the generic point of $\boldsymbol{V}$ into a possibly infinite number of generic types. We show here that the above do not depend on the embedding, or the choice of generic type, but are really geometric properties of $\boldsymbol{V}$. Moreover, the decomposition theory can be carried out using rational maps. In particular the notion of descent depends little on whether one chooses rational, constructible or multivalued morphisms. We do this in two ways: in § 2, we develop the basic theory of internality from scratch for the category of difference fields, without a preliminary amalgamation into a universal domain. In this way the theory is more general, applying for instance to pairs of commuting automorphisms. In Part II of this paper, we use the usual model-theoretic language but show *a posteriori* the independence of the embedding; in this way we do not lose the intuitions associated with the model-theoretic viewpoint. In both approaches, a certain weakening of amalgamation plays an essential role. This weakening is associated to what one might loosely call 'Shelah's reflection principle'. Roughly speaking, stable interactions between a type P and external elements can be read off from inside P itself. Algebraically, the difficulty with difference algebra is that difference fields do not admit amalgamation; however if $K \leqslant L$ is an extension of difference fields, there is always a canonical amalgam of L with *itself* over K, identifying the algebraic part; using a version of the reflection principle, we show that this suffices for a definable Galois theory.

In this introduction, we will restrict attention to the category of algebraic dynamics. The results are valid in greater generality, for difference varieties that arise from correspondences rather than rational maps.

We are grateful to the referee for very helpful comments.

1.2. Modularity. In this paper, we will only consider 'birational' or 'generic' questions, i.e. we allow ourselves to ignore any given lower-dimensional subvariety of V. We correspondingly take a morphism $(V,\phi) \to (V',\phi')$ to be a dominant rational map $f : V \to V'$, such that $\phi' \circ f = f \circ \phi$. In particular (V,ϕ) is considered isomorphic to (V',ϕ') if there exists an isomorphism $V \setminus U \to V' \setminus U'$ for some lower-dimensional U, U', commuting (whenever defined) with ϕ, ϕ'. At this level we could dispense with varieties altogether, and speak of their function fields instead. However both the proofs of the statements, and intended future developments, require the geometric viewpoint.

If $\boldsymbol{U} = (U,\phi) \in \mathrm{AD}_K$, we let $\mathrm{AD}_{\boldsymbol{U}}$ be the category of AD_K-morphisms $\boldsymbol{V} \to \boldsymbol{U}$. Note that a fibre of $V \to U$ is a subvariety of V, which is not in general invariant under the dynamics. However if we generalize the notion of algebraic dynamics to difference fields, we can speak of the fibres of $\boldsymbol{V} \to \boldsymbol{U}$. In general, given a difference field $K = (K,\sigma)$, let AD_K be the category of triples (V, V^σ, ψ), with ψ a dominant rational map $V \to V^\sigma$. Here V^σ is obtained from V by applying σ to the coefficients. In our case, ϕ induces an endomorphism σ of the function field of U, making it into a difference field K. The generic fibre of $\boldsymbol{V} \to \boldsymbol{U}$ can be understood as an object of AD_K, and indeed $\mathrm{AD}_{\boldsymbol{U}}$ is isomorphic to AD_K. In this introduction we will stay with the geometric language.

For $\boldsymbol{V} \in \mathrm{AD}_K$, the irreducible components of $\boldsymbol{V} \times \boldsymbol{U}$, with the projection maps to $\boldsymbol{U}$, are elements of $\mathrm{AD}_{\boldsymbol{U}}$. If $\boldsymbol{V} \times \boldsymbol{U}$ is irreducible, we denote it $\boldsymbol{V}_{\boldsymbol{U}}$. More generally, if V remains irreducible over a difference field extension L of K, we write $\boldsymbol{V}_L$ for the change of basis.

If $\boldsymbol{W}$, $\boldsymbol{V}$ are objects of AD_K, we write $U \leqslant \boldsymbol{V} \times \boldsymbol{W}$ if U is an irreducible subvariety of $V \times W$ such that the projections $U \to V$, $U \to W$ are AD_K-morphisms with respect to some (unique) $\phi_{\boldsymbol{U}}$; in this case we write also $\boldsymbol{U} \leqslant \boldsymbol{V} \times \boldsymbol{W}$. We will sometimes think of U as *a family of difference subvarieties of* $\boldsymbol{V}$, indexed by $a \in W$. Assume that for generic $a \in W$, U_a is absolutely irreducible of dimension l, and if $b \neq a$ then $U_a \neq U_b$; then we say that it is a family of dimension $\dim(W)$ of l-dimensional difference subvarieties of $\boldsymbol{V}$. By a *difference subvariety* of $\boldsymbol{V}$ we mean a generic fibre of some such family.

In algebraic geometry there exist n-dimensional families of irreducible plane curves for arbitrarily large n. It is a fundamental attribute of algebraic dynamics that—with rare exceptions—the dimension of families is bounded. To state this precisely we need to demarcate off the exceptional sub-category, that of field-internal dynamics, that behaves like algebraic geometry.

Say $\boldsymbol{V} = (V,\phi)$ has constant dynamics if $\phi = \mathrm{Id}_V$; *periodic*, if for some n we have $\phi^n = \mathrm{Id}$; *twisted-periodic* if ϕ^n is a Frobenius morphism on V.*

* This requires V to be defined over a finite field; or in the more general situation of $\mathrm{AD}_{(L,\sigma)}$ considered below, that V descend to $\mathrm{Fix}(\sigma^n \,\mathrm{Frob}^m)$ for some $n \in \mathbb{N}$, $m \in \mathbb{Z}$.

An object of $\mathrm{AD}_{\boldsymbol{U}}$ is called constant (periodic, twisted-periodic) over $\boldsymbol{U}$ if it is isomorphic in $\mathrm{AD}_{\boldsymbol{U}}$ to some $\boldsymbol{V} \leqslant \boldsymbol{Q} \times \boldsymbol{U}$, with $\boldsymbol{Q}$ enjoying the corresponding property.

Call $\boldsymbol{V} \in \mathrm{AD}_{\boldsymbol{U}}$ *field-internal* if for some $\boldsymbol{W} \in \mathrm{AD}_{\boldsymbol{U}}$, some generic component $\boldsymbol{V}'$ of $\boldsymbol{V} \times_{\boldsymbol{U}} \boldsymbol{W}$ is twisted-periodic in $\mathrm{AD}_{\boldsymbol{W}}$.* The untwisted analogue will be referred to as *fixed-field-internal*.

The central example is given by *translation varieties*. Let $G \times X \to X$ be an algebraic group action defined over K. Let $g : U \to G$. Let $V = U \times X$, and define a dynamics by $(u, x) \mapsto (\phi_{\boldsymbol{U}}(u), g(u) \cdot x)$. In this case $\boldsymbol{V}$ is field-internal over $\boldsymbol{U}$. When $G = \mathrm{GL}_n$ and X is the natural module, this is the object of van der Put–Singer Picard–Vessiot theory.

Finally, $\boldsymbol{V}$ is *field-free* over $\boldsymbol{U}$ if it has no field-internal components, i.e if whenever $\boldsymbol{V} \to \boldsymbol{W} \to \boldsymbol{U}' \to \boldsymbol{U}$ factorizes the given morphism $\boldsymbol{V} \to \boldsymbol{U}$ and $\boldsymbol{W}$ is field-internal over $\boldsymbol{U}'$, then $\boldsymbol{W} \to \boldsymbol{U}$ has finite fibres (generically over $\boldsymbol{U}$). Similarly for fixed-field-free. For $\boldsymbol{U}$ a point, we say $\boldsymbol{V}$ is field-free.

Proposition 1.3.

(1) *If $\boldsymbol{V} \leqslant \boldsymbol{V}' \times \boldsymbol{V}''$ and $\boldsymbol{V}'$, $\boldsymbol{V}''$ are field-free, so is $\boldsymbol{V}$.*

(2) *Let $\boldsymbol{V}$ be field-free, and let $\boldsymbol{U}$ be field-internal. Let $R \leqslant V \times U$ be an irreducible difference subvariety, projecting dominantly to U, V. Then R is a component of $V \times U$.*

Part (1) of the proposition is just a closure property of the class of field-free dynamics. (2) is the quantifier-free part of *orthogonality to fixed fields*: there is no quantifier-free interaction between field-free and field-internal dynamics. We must speak of a component since $V \times U$ may not be irreducible, but if it is, the conclusion is that the only possible relation R is the trivial one. It suffices here to assume that $\boldsymbol{V}$ has no non-trivial field-internal quotients. We sketch a model-theoretic proof of a variant of this, illustrating how the solution set $\boldsymbol{V}(\mathcal{U})$, and especially the model-theoretic notion of induced structure on $\boldsymbol{V}(\mathcal{U})$, show up here.

Part (2) of the proposition reduces to the case that $\boldsymbol{U}$ is twisted-periodic, so that each point of $\boldsymbol{U}(\mathcal{U})$ is contained in a difference field generated over K by $\mathrm{Fix}(\tau)$, $\tau = \sigma^m \mathrm{Frob}^l$. Take $\tau = \sigma$ for simplicity. Then (2) is actually true under the weaker assumption $(*)$: $\boldsymbol{V}$ has no positive-dimensional quotients with constant dynamics. We sketch the proof.

By $(*)$, for a generic point b of $\boldsymbol{V}(\mathcal{U})$, we have $K(b) \cap \mathrm{Fix}(\sigma) \subseteq K^{\mathrm{alg}}$. It follows that $K(b)^{\mathrm{alg}} \cap \mathrm{Fix}(\sigma) \subseteq K^{\mathrm{alg}}$ (see Lemma 2.9).

On the other hand, using quantifier elimination to the level of images of finite maps (see Proposition 1.1) one sees that definable closure is contained in the field-theoretic algebraic closure. It follows that every K-definable map $\boldsymbol{V}(\mathcal{U}) \to \mathrm{Fix}(\sigma)$ is generically constant. Using elimination of imaginaries and stable embeddedness of $\mathrm{Fix}(\sigma)$, it follows that if c is a generic point of $\boldsymbol{V}(\mathcal{U})$, then every $K(c)$-definable subset of $\mathrm{Fix}(\sigma)^n$ is

* We quote a sentence from the referee report regarding the terminology: the 'notions of "field-internal", "fixed-field-internal", and "field-free" are applied to algebraic dynamics but they make essential reference to the difference field point of view ... these terms were selected to facilitate the difference field interpretation'.

K^{alg}-definable: the code for such a set being a definable function of c. It follows that every definable relation on $\boldsymbol{V}(\mathcal{U}) \times \mathrm{Fix}(\sigma)^n$ is a finite Boolean combination of rectangles $X \times Y$, with $X \subseteq \boldsymbol{V}(\mathcal{U})$ and $Y \subseteq \mathrm{Fix}(\sigma)^n$, and of relations whose projection to $\boldsymbol{V}(\mathcal{U})$ is not generic. From this (2) is immediate.

In fact (2) does not exhaust the strength of the model-theoretic orthogonality just proved; it is stated only for quantifier-free definable sets, whereas we showed that there is no interaction using subsets of $\boldsymbol{V}(\mathcal{U}) \times \boldsymbol{U}(\mathcal{U})$ that are defined using quantifiers, either: any $\boldsymbol{U}(\mathcal{U})$-parametrized family of definable subsets of $\boldsymbol{V}(\mathcal{U})$ must be finite. Using the remark following Proposition 1.1, we can translate this to a statement about finite covers. If $\mathcal{F}$ is a $\boldsymbol{U}(\mathcal{U})$-parametrized family of finite covers $f : \boldsymbol{W} \to \boldsymbol{V}$, then the family of sets $f(\boldsymbol{W}(\mathcal{U}))$ must be finite. From the converse to Proposition 1.1 it follows that the elements of $\mathcal{F}$ themselves arise from a finite family of finite covers. More precisely, we have:

(2^+) let $L = K(\boldsymbol{U})$, and let $\boldsymbol{V}''$ be a finite cover of $\boldsymbol{V}_L$. Then there exists a finite cover of $\boldsymbol{V}''$ of the form $\boldsymbol{V}'_L$, where $\boldsymbol{V}'$ is a finite cover of $\boldsymbol{V}$.

This is Lemma 4.2 of [**9**]. The same proof applies in the twisted-periodic case.

So far we only noted closure and orthogonality properties. Here is the essential statement.

Theorem 1.4. *Assume (V, ϕ) is field-free. Let $R \leqslant \boldsymbol{V} \times \boldsymbol{Q}$ be an irreducible k-dimensional family of l-dimensional irreducible difference subvarieties of $\boldsymbol{V}$. Then $k + l \leqslant \dim(V)$.*

This property is characteristic of theories of modules. We say $\boldsymbol{V}$ is *modular* if (3) holds for all powers $\boldsymbol{V}^n$ of $\boldsymbol{V}$. Thus for algebraic dynamics, modularity is equivalent to being field-free.

Modularity is a fundamental dividing line in model theory, and has many equivalent formulations. Here we will just note an algebraic equivalent that will be used in Part II. Call $\boldsymbol{V}$ *one-based* if for any difference field L extending K, any tuple a from $\boldsymbol{V}(L)$, and any tuple b from L, the fields $K(a)^{\mathrm{alg}}$, $K(b)^{\mathrm{alg}}$ are free over their intersection. To see that modularity implies one-basedness and modularity, say $\boldsymbol{V}, \boldsymbol{U}, \boldsymbol{R} \in \mathrm{AD}_K$ have function fields isomorphic to $K(a)_\sigma$, $K(b)_\sigma$, $K(a, b)_\sigma$ respectively. Let $R_b = \{v \in V : (v, b) \in R\}$, and let U be the irreducible component of R_b containing a; let of $K(b')$ be the field of definition of U. Then $b' \in K(b)^{\mathrm{alg}}$. By the modular rank inequality we have $\mathrm{tr\,deg}_K(K(b')) + \mathrm{tr\,deg}_{K(b')} K(a, b') \leqslant \mathrm{tr\,deg}_K K(a)$; so $b' \in K(a)^{\mathrm{alg}}$. Thus $K(a)$, $K(b)$ are free over the intersection of their algebraic closures. The converse is proved similarly. We will reserve the term *one-based* to the solution set $\boldsymbol{V}(\mathcal{U})$, and *modular* to the geometric data (V, ϕ) itself.

Theorem 1.4 expresses a dichotomy between modular and field-like behaviour in algebraic dynamics. This strengthens [**8**], where in effect modularity is proved assuming every finite cover of (V, ϕ) is field-free.

The dichotomy is an expression of a general philosophy of Zilber's, and indeed it is through this general principle that the theorem is proved in [**8**]. The proof is easy to explain in the case of algebraic dynamics over a field. We say $\boldsymbol{V} = (V, \phi)$ is *primitive* if there is no $f : \boldsymbol{V} \to \boldsymbol{U}$, $0 < \dim(U) < \dim(V)$. To avoid technicalities, assume ϕ

is an endomorphism of a smooth variety V, and that for any $m \geqslant 1$ there are no ϕ^m-invariant proper, infinite subvarieties, nor any dominant, equivariant $f : (V, \phi^m) \to (U, \psi)$ with $0 < \dim(U) < \dim(V)$. Consider V as a structure, with n-ary relations given by subvarieties of V^n left invariant by a power of ϕ. Then it is very easy to check that we have a Zariski geometry, i.e. a structure with a topology satisfying the basic properties of Zariski closed sets.* Even if $\dim(V) > 1$, this derived structure will be intrinsically one-dimensional: it has no infinite, co-infinite definable sets. From this it follows by a general theory that either modularity holds, or else a field can be found essentially as a quotient of a difference subvariety of V^n; in the latter case one shows that $\boldsymbol{V}$ is not field-free. We refer to [**8**] for details.

Note that taking all invariant subvarieties would not have worked: the intersection of two invariant two-dimensional subvarieties of a smooth three-dimensional one may be a union of two one-dimensional parts, whose intersection in turn is a point. If we allowed only invariant subvarieties, we would see two codimension 1 sets intersecting in a codimension 3 set, contradicting a basic property of dimension. The device of finding more regular behaviour by replacing ϕ by ϕ^n will be used repeatedly in the present paper; we will also check that field-internality and modularity are insensitive to this.

Let us relate modularity to some degree-based notions used commonly in dynamics.

Proposition 1.5. *Let $\boldsymbol{V} = (V, \phi)$ be primitive. If $\deg(\phi) > 1$ then $\boldsymbol{V}$ is fixed-field free. If ϕ has separable degree greater than 1 then $\boldsymbol{V}$ is modular.*

If V is a curve, then $\boldsymbol{V}$ is primitive, and the converse of both statements is true. But this is a purely one-dimensional phenomenon. In fact, if (V, ϕ^n) is primitive for all n, and remains primitive after base change, then $\boldsymbol{V}$ is *necessarily* modular when $\dim(V) > 1$. Thus for such strongly primitive $\boldsymbol{V}$, modularity is equivalent to having either dimension or separable degree greater than 1.

If V is not primitive, neither implication is true; the condition $\deg(\phi) > 1$ becomes rather weak, implying only the existence of one fixed-field-free subquotient in a decomposition. The very strong condition of 'polarization' (see [**6**]) implies that every subquotient has positive degree and hence fixed-field-freeness, thus modularity in characteristic 0.

A natural question is to what extent the results described here can be formulated at the level of varieties, rather than birationally. If a primitive difference variety is field-internal in the generic sense used here, then in fact it contains a difference subvariety that is field-internal. We do not know if the same holds for modularity.

1.6. Isotriviality. Let $K \leqslant L$ be difference fields, and let $V \in \mathrm{AD}_L$. Roughly speaking, we say that V is isotrivial if it derives from an algebraic dynamics over K. This splits into a number of technically distinct notions.

* The main point is that if U_1, U_2 are periodic then so is each component of $U_1 \cap U_2$. It is in this that the notion of 'periodic' behaves better than 'invariant'. The Boolean combinations of periodic subvarieties are also closed under projections, since both the closure and the boundary of the projection are invariant for the same power of ϕ. We define a new dimension for a subvariety U of V^n by $\dim_{\mathrm{Zar}}(U) = \dim(V)^{-1} \dim(U)$, and show by induction on n that it is integral. The dimension theorem follows from the above property of intersections.

(1) $\boldsymbol{V}$ *descends to* K if it is AD_L-isomorphic to $\boldsymbol{W}_L$ for some $\boldsymbol{W} \in \mathrm{AD}_K$. This is the finest notion we will use; as above our isomorphisms are birational.

(2) $\boldsymbol{V}$ is *constructibly isotrivial* if it is isomorphic to $\boldsymbol{W}_L$ using constructible maps, i.e. compositions of rational maps with inverses of purely inseparable rational maps.

(3) $\boldsymbol{V}$, $\boldsymbol{V}'$ are *isogenous* if they admit a common finite cover. $\boldsymbol{V}$ is *isogeny isotrivial* if there exists $\boldsymbol{W} \in \mathrm{AD}_K$ such that $\boldsymbol{V}$, $\boldsymbol{W}_L$ are isogenous in AD_L.

In dimension 1 the notions (1) and (2) coincide, since a constructible bijection can always be made birational by a Frobenius twist. Beyond this remark we will consider only (2) and (3) here. Condition (3) is of course too coarse to be of interest in pure algebraic geometry, but is the most basic equivalence relation to consider for varieties with a dynamics.

Proposition 1.7. *Let $K \leqslant L$ be algebraically closed fields, $\boldsymbol{V} \in \mathrm{AD}_L$. Assume no positive-dimensional quotient of $\boldsymbol{V}$ has constant dynamics. Then isogeny isotriviality is equivalent to constructible isotriviality for $\boldsymbol{V}$.*

We sketch the proof. Assume condition (3) holds. Then there exists a common finite cover $\boldsymbol{Y}$ of $\boldsymbol{V}$, $\boldsymbol{W}_L$. It follows that $\boldsymbol{W}_L$ has no positive-dimensional quotients with constant dynamics (see the discussion following Proposition 1.3), hence the same is true for $\boldsymbol{W}$. By (2^+), there exists a finite cover $\boldsymbol{W}'$ of $\boldsymbol{W}$, with $\boldsymbol{W}'_L$ a finite cover of $\boldsymbol{Y}$. Let $f : W' \to V$ be the L-definable map coming from the composition $\boldsymbol{W}'_L \to \boldsymbol{Y} \to \boldsymbol{V}$. Define E on W' by wEw' if and only if $f(w) = f(w')$. This is a constructible equivalence relation on W' defined over L. Using the proposition on orthogonality (Proposition 1.3 (2)) we see that E must be K-definable. At the constructible level this gives immediately a dynamics $\boldsymbol{W}'/E \in \mathrm{AD}_K$, isomorphic over L to $\boldsymbol{V}$, hence constructible isotriviality (2).

Given $\boldsymbol{U} \in \mathrm{AD}_K$ and $\boldsymbol{V} \in \mathrm{AD}_{\boldsymbol{U}}$, we say $\boldsymbol{V}$ is isotrivial (in any of these senses) if it is isotrivial in AD_L, with L the function field of $\boldsymbol{U}$. The proposition applies in particular when $\boldsymbol{V}/\boldsymbol{U}$ is modular, and $\boldsymbol{U}$ has constant dynamics.

1.8. Modularity and descent. Let $\boldsymbol{U}, \boldsymbol{V}'' \in \mathrm{AD}_K$, $\boldsymbol{V}, \boldsymbol{V}' \in \mathrm{AD}_{\boldsymbol{U}}$. Say $\boldsymbol{V}$ is *dominated by* $\boldsymbol{V}' \in \mathrm{AD}_{\boldsymbol{U}}$ if there exists a (dominant) morphism $\boldsymbol{V}' \to \boldsymbol{V}$ in $\mathrm{AD}_{\boldsymbol{U}}$; and by $\boldsymbol{V}'' \in \mathrm{AD}_K$ if it is dominated by $\boldsymbol{V}''_{\boldsymbol{U}}$.

Proposition 1.9. *Let $K = K^{\mathrm{alg}}$, $\boldsymbol{U} \in \mathrm{AD}_K$ and let $\boldsymbol{W} \in \mathrm{AD}_{\boldsymbol{U}}$ be field-free. Assume $\boldsymbol{W}$ is dominated by an object of AD_K. Then $\boldsymbol{W}$ is isogeny isotrivial.*

Proof. Say $\boldsymbol{W}$ is dominated by $f : (\boldsymbol{V} \times \boldsymbol{U}) \to \boldsymbol{W}$. Consider the dominant rational map $f : V \times U \to W$, ignoring the dynamics. The following two statements are basic in Shelah's stability theory, and easy to establish directly in the case of algebraic varieties.

(i) There exists a dominant $g : V \to V'$ such that $g(v) = g(v')$ if and only if $f(v, u) = f(v', u)$ for generic $u \in U$ (i.e. whenever defined). The map f factors through g and some $f' : V' \times U \to W$. For $v, v' \in V'$ we have $f'(v, u) = f'(v', u)$ for generic $u \in U$ if and only if $v = v'$.

(ii) Define $f_m : V' \times U^n \to W^n$ by $f_m(v, u_1, \dots, u_n) = (f(v, u_1), \dots, f(v, u_m))$. Then for some m, (f_m, Id) embeds $\boldsymbol{V}' \times \boldsymbol{U}^m$ into $\boldsymbol{W}^m$ (generically). (This follows from the last statement of (i).)

Now recall ϕ_V, ϕ_W. If $g(v) = g(v')$ then $g(\phi_V(v)) = g(\phi_V(v'))$: indeed if $f(v, u) = f(v', u)$ for generic u, then

$$f(\phi_V(v), \phi_U(u)) = \phi_W(f(v, u)) = \phi_W(f(v', u)) = f(\phi_V(v'), \phi_W(u));$$

and ϕ_W was assumed dominant. So we can define a rational $\phi' : V' \to V'$ with $\phi'(g(v)) = g(\phi_V(v))$. Let $\boldsymbol{V}' = (V', \phi')$. By (ii) we have $f(V) \leqslant \boldsymbol{W} \times_{\boldsymbol{U}} \cdots \times_{\boldsymbol{U}} \boldsymbol{W}$. By Proposition 1.3 (1) over $\boldsymbol{U}$, $\boldsymbol{V}'$ is field-free, hence modular.

Now view $\boldsymbol{V}'$ as parametrizing a family of functions $\boldsymbol{U} \to \boldsymbol{W}$; their graphs are irreducible subvarieties of $\boldsymbol{U} \times \boldsymbol{W}$ of dimension equal to $\boldsymbol{U}$; so by modularity, $\boldsymbol{V}'$ may again be replaced by a quotient $\boldsymbol{V}''$ of dimension $\dim(U \times W) - \dim(U) = \dim(W)$. Hence the graph of f is a finite cover of $\boldsymbol{V}''_{\boldsymbol{U}}$ and also of $\boldsymbol{W}$, showing they are isogenous, and isotriviality (3) holds. □

Each of the steps in this proof is valid in great generality, in categories of definable sets of a wide class of theories, including the theory ACFA of existentially closed difference fields. Given the basic properties of such definable sets, the proof generalizes to dynamics given by correspondences. This will be done in Part II.

Assume now that the dynamics on $\boldsymbol{U}$ is trivial. Then in Proposition 1.9 one can conclude that $\boldsymbol{W}$ descends constructibly to K. We have a morphism $f : \boldsymbol{U} \times \boldsymbol{V} \to \boldsymbol{W}$ in $\mathrm{AD}_{\boldsymbol{U}}$, and as above we may take $\boldsymbol{V}$ to be fixed-field-free. Given $u \in U$ we have a constructible equivalence relation E_u on $\boldsymbol{V}^2$, namely xE_uy if $f(u, x) = f(u, y)$. If $u \in \boldsymbol{U}(\mathcal{U})$ then E_u is compatible with ϕ_V, as we saw in the proof above. By the orthogonality principle of Proposition 1.3, E_u cannot really depend on $u \in \boldsymbol{U}(\mathcal{U})$; so $E_u = E_{u'}$ for generic $u, u' \in \boldsymbol{U}(\mathcal{U})$, and hence $E_u = E$ generically for some E. Now $\boldsymbol{W}$ descends constructibly to $\boldsymbol{V}' = \boldsymbol{V}/E$.

The same argument shows that $\boldsymbol{W}$ descends together with any additional structure it may carry; notably if $\boldsymbol{W} = (W, \phi)$ and $\phi' : W \to W$ commutes with ϕ, then (W, ϕ, ϕ') descends. For the image of the graph of ϕ' under the isomorphism $\boldsymbol{W} \to \boldsymbol{V}'_{\boldsymbol{U}}$ is a difference subvariety of $\boldsymbol{V}^2$, definable over the generic point of $\boldsymbol{U}$; as noted above for E_u, any such variety is K-definable. Thus it suffices to assume that (W, ϕ^l) is dominated by an object of AD_K, for some l (for then (W, ϕ^l) descends, and ϕ commutes with ϕ^l).

Part II contains more general statements. We show by example (see Example 3.5 in Part II) that Proposition 1.9 (or Corollary 1.10 below) fails without a modularity or orthogonality assumption, even when the dynamics on $\boldsymbol{U}$ is trivial.

1.10. Dynamical Northcott for function fields. The assumption of domination by a difference variety over K can be rephrased in the language of limited subsets of the function field $L = K(U)$. If $g : V \times U \to W$ is the dominating map, and a a generic point of U, then the image of $V(K)$ under $x \mapsto g(a, x)$ is a typical *limited subset of* $K(U)$. When $V(K)$ has an Zariski dense ϕ-orbit, $W(L)$ will have a Zariski dense orbit contained in a limited set. Conversely, if $W(L)$ will have a Zariski dense orbit contained in a limited set, then W is dominated by a difference variety over K. We thus have the following corollary.

Corollary. *Let (W, ϕ) be a fixed-field-free algebraic dynamics over L, L a finitely generated extension field of $K = K^{\mathrm{alg}}$. Assume (W, ϕ) does not constructibly descend to K. Then no limited subset of $W(L)$ can contain a Zariski dense ϕ-orbit.*

The case $V = (\mathbb{P}^1)_L$, is the result of [**1**]. In fact the hypothesis $K = K^{\mathrm{alg}}$ is not needed when L/K is regular, answering a question posed there for $\mathbb{P}^1$.

Theorem 1.11. *Let (V, ϕ) be a primitive algebraic dynamics over L, L a finitely generated regular extension field of a field K. Assume (V, ϕ) does not constructibly descend to K. Then no limited subset Y of $V(L)$ contains a Zariski dense ϕ orbit; in fact for some $n = n(Y)$ and a finite number of proper subvarieties $U_1, \ldots, U_j$ of V, defined over L, there is no $a \in V(L) \setminus \bigcup U_i(L)$ with $a, \phi(a), \ldots, \phi^n(a) \in Y$.*

The finite bounds n, j follow by compactness. In some cases, there exists canonical height, i.e. a function $h : V(L) \to \mathbb{R}^{\geqslant 0}$ such that the inverse image of a compact set is limited, and $h(\phi(a)) = \kappa h(a)$ for some $\kappa > 1$. In this case, taking $Y = h^{-1}([0, 1])$, there will be no $a \in V(L) \setminus \bigcup U_i(L)$ with $h(a) < \kappa^{-n(Y)}$.

We first prove this theorem under the additional assumption that (V, ϕ) is fixed-field-free. The methods presented in the present part suffice to prove this for isogeny-isotriviality. Using Theorem 3.3 of Part II, we improve to constructible isotriviality; see § 3.* Finally, in Part III, the primitive, fixed-field internal dynamics are explicitly described; they are associated with translations in one-dimensional algebraic groups; using the classification of these, it turns out that the fixed-field-free assumption can be removed.

1.12. Difference varieties. If (V, ϕ) is an algebraic dynamics over K, the function field L of V is a difference field that is finitely generated over K *as a field*. This can be made geometric in one of two ways.

(1) One can consider pairs (V, ϕ) with V a pro-algebraic variety.

(2) One can stay with varieties but replace the rational map ϕ by a *correspondence* on V, i.e. a subvariety S of $V \times V$ with generically finite projections. For any difference field (L, σ) extending K it is possible, using the Ritt–Raudenbusch basis theorem, to find (V, S) such that $L = K(a)_\sigma \stackrel{\mathrm{def}}{=} K(a, \sigma(a), \sigma^2(a), \ldots)$ with (a_0, a_1) a generic point of S, and such that moreover any other difference field of this description is isomorphic over K to L. The extension to such 'nondeterministic dynamics' should allow for greater flexibility.

* Theorem 3.3 of Part II assumes $\deg(\phi) > 1$, but in fact uses this only via the conclusion of Proposition 3.1, that $\boldsymbol{V}$ is fixed-field-free. It is in this form that we use it here.

All our structural results remain valid in the more general context, when L is finitely generated over K as a difference field. Theorem 1.11 applies to difference varieties presented via a correspondence (V, S), but in the following form: for some n, and some finite union Z of proper subvarieties, there is no n-cycle a, $\phi(a), \ldots, \phi^n(a)$ each of which is in $Y \cap V(L) \setminus Z$. In other words, for some Z' and any $a \in V(L) \setminus Z'$, no choice of ϕ-preimages can be iterated n times.

1.13. Definable Galois theory. Some of our results have been relegated to a future paper. This includes more precise descriptions of non-disintegrated modular sets, and of of field-internal dynamics. We briefly discuss the latter here. The issue is the elaboration in the present context of definable Galois theory, introduced into model theory, in the differential setting, by Poizat.

Let K be a field, and let $\boldsymbol{V} \in \mathrm{AD}_K$ be field-internal. Assume V is absolutely irreducible. Then $\boldsymbol{V}$ can be shown to be a translation variety, or fibred over a periodic dynamics by translation varieties. The automorphism group of $\boldsymbol{V}(\mathcal{U})$ over the fixed field of $\mathcal{U}$ can be seen to be a part of an algebraic group G, and ϕ will be an element of (another part of) G.

In the relative case, when K is a difference field and V is not necessarily defined over the fixed field, more definable Galois theory is needed, including the theory of the opposite group (a group over the constants, isomorphic after base change to the automorphism group) and the existence of a bi-torsor for the two groups. We obtain essentially the same result, reducing field-internal dynamics to translation varieties, when the fixed field of K is pseudo-finite. The pseudo-finiteness condition is analogous to the requirement in differential Picard–Vessiot theory that the field of constants be algebraically closed. Over more general fields, delicate questions of Galois cohomology arise.

2. Internality without quantifiers

In the context of a universal domain $\mathcal{U}$, one says that a type P is *internal* to a definable set (or union of type-definable sets) π if any realization of P is definable over a fixed finite set, and elements of π. The theory began in the stable setting [**15**, **16**], with three components.

(a) Existence of a canonical maximal quotient P_π of P, internal to π.

(b) Domination: any interaction between a realization b of P and elements of π must involve the image of b in P_π.

(c) Definable Galois theory: the interaction of P_π and π is controlled by ∞-definable automorphism groups, or in a more precise version, groupoids.

Parts (a) and (b) had their origins in Shelah's semi-regular types (based on algebraic rather than definable closure). The theory was generalized to arbitrary first order theories in [**17**, Appendix B], and part (c) was transformed to a quantifier-free setting in [**19**]; but a general treatment of (a), (b) without quantifiers is missing.

In the simple context, all parts of the theory were generalized, at least 'up to algebraic (or bounded) closure' (see [**3**, **12**, **24**]). Roughly speaking, (a), (b) are carried out for a certain notion of internality, (c) for a finer one, and the internal parts P_π in the two senses are shown to be related by multivalued maps. This corresponds precisely to what we do in Parts II (quantifier-free internality, later abbreviated as qf-internality) and III (definable Galois theory), except that in our context we are able to identify the finer version as quantifier-free internal; for our purposes it is vital to identify the maps as rational maps.

Simple theories are often obtained from expansions of stable theories by an amalgamation process; the additional structure results from the failure of amalgamation over non-algebraically closed sets. We place ourselves in such an enriched stable context here. In place of amalgamating, defining internality using quantified formulae, and then show ing that it depends only on the quantifier-free part, we shortcut and define internality before amalgamation. Examples include any number of derivations and automorphisms, commuting or not.

Simplicity corresponds essentially to uniqueness of amalgamation of the expansions, relative to uniqueness for the stable layer. This is valid in the examples mentioned above. We do not assume this, except for Remark 2.7. Our framework thus includes non-simple structures, but we have not investigated what it means for them.

We consider below a class $\mathcal{C}$ of $\mathcal{L}$-substructures, with a stable reduct to a language $\mathcal{L}_0$. We assume $\mathcal{C}$ is closed under substructures. The stable reduct is used to obtain a notion of free amalgamation on $\mathcal{C}$, enjoying the usual properties, and in particular the existence of canonical bases: for any structure C in the class and any $A, B \leqslant C$, there exists a unique minimal substructure B' of B such that A, B are independent over B'. It would be possible to replace the 'stable sublanguage' hypothesis by assuming abstractly that $\mathcal{C}$ is given with an amalgamation notion, having the usual properties, and with canonical bases.

2.1. Internality and orthogonality. Let $\mathcal{T}_0$ be a stable theory in a language $\mathcal{L}_0$. We assume $\mathcal{T}_0$ eliminates quantifiers and imaginaries, and that substructures of models of $\mathcal{T}_0$ are definably closed.

We recall some facts and notation: let K, L, L' be substructures of some model of $\mathcal{T}_0$. The substructure generated by $K \cup L$ is denoted KL. We write L^{alg} for the algebraic closure of L. We say K_1, K_2 are strongly free over L if they are independent over L, and whenever K_1', K_2' are independent over L and $\alpha_i : K_i \to K_i'$ is an L-isomorphism, then $\alpha_1 \cup \alpha_2$ is an L-isomorphism. We say L/K is *stationary* if L is strongly free from K^{alg} over K; equivalently, $\mathrm{dcl}(L) \cap \mathrm{acl}_\sigma(K) = \mathrm{dcl}(K)$.

$\mathrm{Cb}(K/L)$ denotes the smallest substructure L' of L such that K, L are strongly free over L'; equivalently they are free over L', and $\mathrm{dcl}(L', K) \cap \mathrm{dcl}(L) \subseteq \mathrm{dcl}(L')$. If $e \in \mathrm{Cb}(K/(L')^{\mathrm{alg}})$ then $e \in KL$; let f code the the finite set of realizations of $\mathrm{tp}(e/L')$; then $f \in \mathrm{Cb}(K/L)$; and such elements f generate $\mathrm{Cb}(K/L)$. Equivalent forms of these notions appear in Shelah and in Bouscaren [**4**].

Let $\mathcal{L}$ be a language containing $\mathcal{L}_0$. Let $\mathcal{C}$ be a class of $\mathcal{L}$-structures, closed under isomorphism and substructures.

The reduct of an $\mathcal{L}$-structure L to $\mathcal{L}_0$ is denoted L_0. Let $\mathcal{C}_0 = \{L_0 : L \in \mathcal{C}\}$. We assume the following for the rest of this section.

(I0) If $A \leqslant B, C \in \mathcal{C}$, and B_0, C_0 are strongly free over A_0, then their free amalgam D_0 can be expanded to some $D \in \mathcal{C}$, with $B, C \leqslant D$. Conversely, for any $B, C \leqslant D \in \mathcal{C}$ we have $(BC)_0 = B_0 C_0$.

Let us immediately explain how (I0) will be used. Assume $B^1, B^2, B^3, \dots \in \mathcal{C}$ and $B_0^1, B_0^2, \dots$ are free over A_0. Let $A_0^i = A_0^{\mathrm{alg}} \cap B_0^i$, and let A^i be the substructure of B^i with reduct A_0^i. Assume there exists isomorphisms $f^i : B_0^1 \to B_0^j$ such that $f^i | A^1$ is an isomorphism $A^1 \to A^i$. So all A^i are copies of a single structure A'. Now $B_0^1, B_0^2, \dots$ are strongly free over A_0'. By (I0), their free amalgam D_0 can be expanded to some $D \in \mathcal{C}$, with each $B^i \leqslant D$. In particular, the B^i can be amalgamated over A within $\mathcal{C}$. If only B^1 is given, it is always possible to find $B_0^2, \dots$ free over A_0, and expand them to $B^2, \dots$ in such a way that even B^1, B^i are isomorphic; so $B^1, B^2, \dots$ are jointly freely embeddable into an element of $\mathcal{C}$.

We will write K, L, K', J, etc., for elements of $\mathcal{C}$. We let qf-Cb(K/L) be the $\mathcal{L}$-structure generated by $\mathrm{Cb}(K_0/L_0)$. $\mathrm{tp}_0(a/L)$ denotes the $\mathcal{L}_0$-type.

Let π be a set of $\mathcal{C}$-isomorphism types. We write $L \in \pi$ to mean that the isomorphism type of L is in π. We assume the following.

(I1) Let $L \in \mathcal{C}$ be generated by some collection $\mathcal{A}$ of substructures of L. Then $L \in \pi$ if and only if $\mathcal{A} \subseteq \pi$.

If $L \in \mathcal{C}$, let L_π be the join of all 1-generated substructures of L in π; this is the largest substructure of L in π.

Consider the following properties.

(I2) Let $K, J \leqslant L \in \mathcal{C}$, with $J \in \pi$ and $J_0 \leqslant K_0^{\mathrm{alg}}$. Then there exists $N \in \mathcal{C}$, $J \leqslant N$, with $N_0 \leqslant K_0^{\mathrm{alg}}$, and such that N_π is $\mathrm{Aut}(K_0^{\mathrm{alg}}/K_0)$-invariant.

(I3) Let $L \in \pi$ and let $L' \in \mathcal{C}$. Then qf-Cb$(L/L') \in \pi$.

Claim. If π has (I1), (I2), it also enjoys (I3).

Indeed let $L_0, L_0^1, \dots$ be an independent sequence over L_0' in some model of $\mathcal{T}_0$, such that all L_0^j satisfy the same type over $(L_0')^{\mathrm{alg}}$. Expand each L_0^i to L^i so that $L^i \cong L$. Then each $L^i \in \pi$, and by the remark following (I0) they jointly embed into some L_0^*, $L^* \in \mathcal{C}$. Let $E = \mathrm{Cb}(L_0/(L_0')^{\mathrm{alg}}) \subseteq L_0^1 L_0^2 \dots$. By (I1) (the 'if' part) we have $L^1 L^2 \dots \in \pi$. Since π is closed under substructures (the 'only if' part of (I1)), we have $E \in \pi$. For $e \in E$ we have $e \in (L_0')^{\mathrm{alg}}$; let $e' \in L_0'$ code the $\mathrm{Aut}((L_0')^{\mathrm{alg}}/L_0')$-orbit of e; by (I2) we have $e' \in L_\pi^*$. By the discussion of canonical bases in stable theories above, the set of such e' generates $\mathrm{Cb}(L_0/L_0') = $ qf-Cb(L/L').

We will say that A, B are (strongly) free over C if A_0, B_0 are so.

Definition 2.2. Let $J \in \mathcal{C}$.

- We say that J/K is *qf-internal to* π if there exists $\hat{L}$ with $\hat{L}_0/J_0$ stationary, and $K \leqslant L \leqslant \hat{L}$, such that J, L are free over K, and $\hat{L} = L\hat{L}_\pi = LJ$.
- Let $K \leqslant K_1 \leqslant K_2$. We say that K_2 is π-dominated by K_1/K if for any $\hat{L} \in \mathcal{C}$ with $K_2 \leqslant \hat{L}$ and $K \leqslant L \leqslant \hat{L}$, if K_1 is free from L over K then K_2 is strongly free from $K_1L\hat{L}_\pi$ over K_1L.

Theorem 2.3. *Let $K \leqslant J \in \mathcal{C}$. Then there exists a unique sub-extension K_1 with with K_1/K qf-internal to π, and J π-dominated by K_1/K.*

Proof. Let K_1 be a sub-extension of J/K. Call K_1/K *weakly qf-internal to* π if there exist $K \leqslant L$, $K_1 \leqslant \hat{L}_1$ with $\hat{L}_1/K_1$ stationary, K_1, L free over K, and $K_1 \subseteq L(\hat{L}_1)_\pi$.

Claim 1. If K_1/K is weakly qf-internal, and J is π-dominated by K_2/K, then $K_1 \subseteq K_2$.

To see this, let $K \leqslant L$, $K_1 \leqslant \hat{L}_1$ be such that $\hat{L}_1/K_1$ is stationary, K_1, L are free over K, and and $\hat{L}_1 = L(\hat{L}_1)_\pi$ contains K_1. We may jointly embed $\hat{L}_1$, J into some $\hat{L} \in \mathcal{C}$ with $K_1 \leqslant \hat{L}$ and $\hat{L}_1$, J free over K_1; hence L, J are free over K; we may take $\hat{L} = J\hat{L}_1$, so $\hat{L} = JL(\hat{L}_1)_\pi$ and in particular $K_1 \subseteq L\hat{L}_\pi$.

By definition of domination, J is strongly free over K_2L from $K_2L\hat{L}_\pi$. As $K_1 \subseteq L\hat{L}_\pi$ we have $K_1 \subseteq K_2L$. But $K_2 \leqslant J$ and (by stationarity) J, L are strongly free over K_1 so $K_1 \subseteq K_2$.

Uniqueness follows immediately from Claim 1, since if K_1, K_2 are two candidates, then K_i/K is weakly qf-internal to π and J is π-dominated by K_{2-i}/K, so $K_i \subseteq K_{2-i}$.

Existence: let K_1 be the join of all sub-extensions of J/K that are qf-internal to π. By amalgamating the relevant witnesses L over K,we see that K_1/K is qf-internal to π. We have to show that J is π-dominated by K_1/K. In other words, consider $J, L \leqslant \hat{L}$, with L free from K_1 over K. We have to show that $M = \hat{L}_\pi$ is strongly free from J over LK_1.

Suppose otherwise. Then there exist $J, L \leqslant \hat{L}$, with L free from K_1 over K, and $M = \hat{L}_\pi$ not strongly free from J over LK_1.

Claim 2. We may choose L, $\hat{L}$ so that LJ/J is stationary.

Proof. For $i = 1, 2, \dots$ let $J^i, M^i \in \mathcal{C}$, contained in some $L^* \in \mathcal{C}$ with $L \leqslant L^*$ and $J^iM^i \cong_{L^{\mathrm{alg}}} JM$ by an isomorphism taking J, M to J^i, M^i, and with J^iM^i an independent sequence of extensions of L. Take $J^0 = J$, $M^0 = M$. Then the J^i are independent over L, and each one is independent from L over K, so $L, J, J^1, \dots$ are independent over K. Let $L' = J^1J^2 \dots$. Since the J^i have the same type over K^{alg}, it follows that J^i/J is stationary; hence L'/J is stationary. Let $E = \text{qf-Cb}(JM/L)$. So JM, L are strongly free over E. Thus J, ME are not strongly free over K_1, hence (since J, L' are strongly free over K_1) J, ME are not strongly free over $L'K_1$. We have $E \subseteq J^1M^1J^2M^2 \dots = L'M^1M^2 \dots$, so $ME \subseteq L'M^*_\pi$. Thus J, $L'M^*_\pi$ are not strongly free over $L'K_1$. $\square$

Assume now that L, $\hat{L}$ are as in Claim 2. Let $M' = \mathrm{Cb}(M/LJ)$ Then by Claim 2, M' is not contained in LK_1. On the other hand, $M' \subseteq LJ$ and by (I3), $M'/K \in \pi$.

Finally, let $K_1' = \mathrm{Cb}(M'L/J)$. Then K_1'/K is qf-internal to π. For L/K is stationary, and if L_i, M_i' are indiscernible independent copies of $M'L/J$, $L_\infty = L_1L_2\dots$, $M_\infty' = M_1'M_2'\dots$ then $L_\infty M_\infty' = L_\infty K_1'$. By maximality of K_1 we have $K_1' = K_1$. Since $M' \subseteq LJ$ we have $M' \subseteq LK_1$. This contradiction proves the proposition. □

Here is a corollary to the proof. Part (2) of the corollary is a version of the 'Shelah reflection principle' referred to in the introduction.

Corollary 2.4. *Assume* (I1), (I2).

(1) *Weakly qf-internal is the same as qf-internal.*

(2) *Let J/K be finitely generated. In the definition of qf-internal (2.2), one can take L to be the amalgam of finitely many freely joined indiscernible copies of J/K.*

It will be worthwhile to review the statement of (2) in terms of definable maps. Since J_0/K_0 may not be finitely generated, we need to work in the pro-definable category (see [**18**]); this is really a side issue here, and the reader may prefer at first to assume finite generation or ignore it. By an $\mathcal{L}_0$-morphism (or an $\mathcal{L}$-morphism) we mean a term of $\mathcal{L}_0$ (respectively $\mathcal{L}$). Assume the language $\mathcal{L}_0$ has constants for the elements of K_0. Let $\mathcal{C}_0^*$ be the category whose objects are complete quantifier-free $*$-types of $\mathcal{L}_0$. We can think of an object of $\mathcal{C}_0^*$ as a pair (L, a) with $L \in \mathcal{C}$ and a a sequence of elements of L_0, generating L_0, up to isomorphisms $(L, a) \to (L', a')$ with $a \mapsto a'$. A *morphism* from $(L, (a_i)_{i\in I}) \to (L, (b_\upsilon)_{\upsilon\in\Upsilon})$ consists of a choice of functions f_υ given by terms in $\mathcal{L}_0$, such that $f_\upsilon = f_\upsilon((x_i)_{i\in I(\upsilon)})$ is defined on $(a_i)_{i\in I(\upsilon)}$ and $f_\upsilon((a_i)_{i\in I(\upsilon)}) = b_\upsilon$. Given $p_0 \in \mathcal{C}_0^*$ we have the usual notion of the type of an n-tuple from an indiscernible, independent sequence in p_0; it is denoted $p_0^{(n)}$, and is canonically determined even if p is not stationary. Here is the restatement of Corollary 2.4 (2). Below, as usual, $K(c)$ denotes the substructure of J generated by c over K.

Corollary 2.5. *J/K is qf-internal to π if and only if there exists $p = (J, a) \in \mathcal{C}_0^*$, and an invertible morphism h of $\mathcal{C}_0^*$ with domain $p_0^{(n)}$, such that $h(a_1, \dots, a_n) = (a_1, \dots, a_{n-1}, c)$ with $K(c) \in \pi$.*

This gives the assumption used in [**19**] to obtain the quantifier-free liaison group.

2.6. Internality in a universal domain. Let $\mathcal{U}$ be a universal domain for $\mathcal{C}$, in the sense that if $A \leqslant \mathcal{U}$ is small, $A \leqslant B \in \mathcal{C}$, and B_0/A_0 is stationary then B embeds into $\mathcal{U}$ over A. We say $\mathrm{tp}(a/K)$ is qf-internal to π if $K(a)/K$ is. Let $D = D_\pi(\mathcal{U}) = \{a : K(a) \in \pi\}$. We also speak of qf-internality to D.

Say (I0!) holds if the expansion in (I0) is unique. For instance this is the case if $\mathcal{C}$ is the class of fields with some distinguished derivations and automorphisms, see below.

Remark 2.7. Assume (I0!), (I1). Then qf-internality to π implies that $\mathrm{tp}(a/K)$ is D-internal in $\mathcal{U}$, i.e. for some fixed finite set F of parameters, every realization of $\mathrm{tp}(a/K)$ is definable over F together with finitely many elements of D.

The theory of liaison groups for qf-internal types will be described in the sequel. By a $\mathcal{C}$-group we will mean an $\mathcal{L}$-quantifier-free definable subgroup G of an $\mathcal{L}_0$-definable group H. We will show the existence of an $\mathcal{C}$-group G such that for any $L \in \mathcal{C}$, $G(L)$ is canonically isomorphic to the image of $\mathrm{Aut}(L/\pi(L))$ in $\mathrm{Sym}(p)$. Moreover, there exists a π-groupoid H with torsor Q such (fibred over an object set Y) such that H_y and G act regularly on Q_y, by commuting actions. The orbits of G on p do not always coincide with the quantifier-free types over π, but if T_0 is ω-stable they cut them in finitely may pieces.

2.8. We now check that (I1), (I2) hold for difference-differential fields.

Let $\mathcal{T}_0 = \mathrm{ACF}_F$ be the theory of algebraically closed fields over some base field F. Let $\mathcal{L}$ be the language of fields with $l + l'$ operators and let $\mathcal{C}$ be a class of $\mathcal{L}$-structures where the first l operators are automorphisms and the rest are derivations. We may ask that certain pairs commute. A central case is that π is the subfield fixed by some of the l difference operators and annihilated by some of the l' derivations. If $l > 1$ we will assume that we are in this case.

For $k \in \mathbb{N}$, $L \in \mathcal{C}$, let $L[k]$ denote L with σ replaced by σ^k.

Recall that L/K is called primary if L_0/K_0 is primary, i.e. if and only if L_0/K_0 is stationary.

Lemma 2.9. *Let (L, σ) be any difference field, K a difference subfield, $k = K \cap \mathrm{Fix}(\sigma)$, and $a \in \mathrm{Fix}(\sigma) \cap K^{\mathrm{alg}}$. Then $a \in k^{\mathrm{alg}}$.*

Proof. Let $F \in K[X]$ be the monic minimal polynomial of a over K. Then $F^\sigma(a) = F^\sigma(\sigma(a)) = \sigma(F(a)) = 0$ and F^σ is monic, so $F^\sigma = F$. Hence $F \in k[X]$. □

Lemma 2.10. *Assume* (I1) *holds for π. Then* (I2) *holds too if we make the following assumption.*

(I2′) *Assume $L[k] \cong_{K[k]} L'[k]$. Then $L \in \pi$ if and only if $L' \in \pi$.*

Proof. Let L, K, J be as in (I2); we may take $K = L(a)$, $J = L(a')$. Let N be the normal closure of $L(a)$ over L, $\tau \in \mathrm{Aut}(N/L)$ with $\tau(a) = a'$. By [**8**, (1.11)], for some ℓ, τ extends to an automorphism of the σ^ℓ-difference field generated by N. So $L(a)_\sigma[\ell] \cong_{L[\ell]} L(a')_\sigma[\ell]$. In particular $K(a)_\sigma[\ell] \cong_{K[\ell]} K(a')_\sigma[\ell]$. Since $K(a) \in \pi$, by (I2′) we have $K(a') \in \pi$. □

For several difference operators, one may consider an analogous condition, considering subgroups of finite index in place of σ^k. Does the lemma remain valid? At all events, (I2) certainly holds if π is the fixed field of some of the operators; this follows directly from Lemma 2.9.

An element of a difference field is (twisted) *periodic* if $\sigma^m(a) = a^{p^r}$ for some $m \geqslant 1$ and $r \in \mathbb{Z}$, where $p = 1$ in characteristic 0 and $p = \mathrm{char}(K)$ otherwise. Note that a is periodic in L if and only if a is periodic in $L[k]$.

2.11. Modular and fixed-field internality. We return to difference fields. If we take also derivations the results below regarding the fixed field generate to the constant fixed field of any subset of the derivations; but we do not know if (I2) holds for modularity in these cases.

Let π_{fix} (over K) be the set of difference field extensions L/K generated by periodic elements. Obviously, (I1) holds. In fact (I2′) holds too; indeed L/K is generated by periodic elements if and only if $L[k]/K[k]$ is. Let $\pi_{\text{fix-int}}$ be the set of L/K that are qf-internal to π_{fix}. It follows easily from the definition that $\pi_{\text{fix-int}}$ also satisfies (I1), (I2′).

Say L/K is qf-π-analysable if there exists a sequence $K \leqslant K_1 \leqslant \cdots \leqslant K_n = L$ with K_{i+1}/K_i qf-internal to π.

Corollary 2.12.

(1) *Let $L \in \mathcal{C}$, $\operatorname{tr} \deg_K L \leqslant n$. Let $\mathcal{U}$ be a universal domain containing L. Then L/K is not modular in $\mathcal{U}$ if and only if there exist $K \leqslant K' < L' \leqslant L$ with L'/K' non-algebraic and in $\pi_{\text{fix-int}}$.*

(2) *Let $\pi_{\mathrm{mod};K;n}$ be the set of difference field extensions L/K that are modular in some $\mathcal{U}$, of transcendence degree less than or equal to n. Then any $L \in \pi_{\mathrm{mod};K;n}$ is modular in any $\mathcal{U}$ containing L; and* (I1), (I2′) *hold.*

Proof. Note that (1) implies (2) immediately. One direction of (1) is clear: if L/K is modular in some universal domain $\mathcal{U}$, then certainly K', L' cannot exist. We prove the converse part of (1) by induction on n. By (2), properties (I1), (I2′) hold for $\pi_{\mathrm{mod};K;m}$ for $m < n$. Now suppose L/K is not modular in $\mathcal{U}$. Then by [**7**], L/K is not almost-orthogonal to some $\mathrm{Fix}(\tau)$, or else to a modular type q over K of SU-rank one. In the first case by Theorem 2.3 there exists K_1 with K_1/K non-algebraic and qf-internal to π_{fix}. Assume the second possibility. If q has transcendence degree n then L/K is itself of SU-rank one, and modular, a contradiction. Otherwise, let L' be the difference field generated by a realization of q_{qf}. Then $L' \in \pi_{\mathrm{mod};K';<n}$, and L/K is not orthogonal to L'. Since (I1), (I2) hold for $\pi_{\mathrm{mod};K';<n}$ there exists $K_0' \leqslant L$, $K \neq K_0'$, with K_0'/K qf-internal to $\pi_{\mathrm{mod};K';<n}$. So L/K_0' is not modular. Again using induction we may find $K_0' \leqslant K' < L' \leqslant L$ as required. □

Corollary 2.13. *Let $L \in \mathcal{C}$. Then there exists a finite sequence $K = L_0 \leqslant L_1 \leqslant \cdots \leqslant L_n = L$ such that for each i, L_{i+1}/L_i is L_i-primitive and either finite, or qf-internal to π_{fix}, or else it is modular and internal to $\pi_{\mathrm{mod};K} = \bigcup_n \pi_{\mathrm{mod};K;n}$.*

Proof. Immediate from the fact that π_{fix}, $\pi_{\mathrm{mod};K;n}$ satisfy (I1), (I2). □

3. Limited sets

Let k be a field, L a finitely generated extension field, and V a variety over L. The original formulation of the problem we solve requires the notion of a *limited* subset of $V(L)$. Geometers are familiar with constructions interpreting varieties over fields such as $k((t))$, with inductive and projective systems of varieties over k. See for instance [**2**, **11**].

In this language, viewing $V(L)$ as an Ind-variety over k, a subset of $V(L)$ is *limited* if it is contained in a finite-dimensional k-subvariety of $V(L)$. We now give an account of this for model theorists; see [**18**] for a detailed explanation of these ideas.

Let k be a structure. We will say *constructible* for quantifier-free definable over k.

A structure N for a finite relational language L is *piecewise-definable* over another structure k if there exist constructible L-structures N_i over k and definable L-embeddings $N_i \to N_{i+1}$ such that $\lim_i N_i$ is isomorphic to N. A *k-definable subset* of $\lim_i N_i$ is just a definable subset of some N_i. Similarly for *piecewise-constructible*.

Lemma 3.1. *Let $N = \lim_i N_i$ be piecewise-constructible over k, and let S be a quantifier-free definable subset of N^k. Let $\alpha_i : N_i \to N$ be the canonical map, and write α_i also for the induced map $N_i^k \to N^k$. Then $\alpha_i^{-1}(S)$ is a constructible subset of N_i^k.*

Let N' be an L-structure, quantifier-free definable over N. If N is piecewise-constructible over k, then so is N'.

Proof. Immediate from the definitions. □

We now consider fields and k-algebras. We take L_k to be the relational language with a relation for any k-constructible set.

Lemma 3.2.

(1) *If N is quantifier-free constructible over L, and L is piecewise-constructible over k, then N is piecewise-definable over k.*

(2) *Let k be a field, and let $L = k(b_1, \ldots, b_n)$ be a finitely generated field extension of k. Then $(L, +, \cdot, b_1, \ldots, b_n, k)$ is piecewise-constructible over k (with parameters in k). More precisely there exists a piecewise-constructible k-algebra L' and an isomorphism $\psi : L \to L'$ of k-algebras.*

(3) *For any variety V over L, $V(L)$ can be viewed as piecewise-constructible over k. (That is, $\psi(V(L)) = V^{\psi}(L')$ is piecewise-constructible over k.)*

Proof. Part (1) is clear. For (2), L is a finite extension of a purely transcendental extension $k(t) = k(t_1, \ldots, t_n)$ of k. Clearly, L is quantifier-free definable over $k(t)$. Hence by (1) it suffices to show that $k(t)$ is piecewise-constructible over k. Indeed let S_n be the set of rational functions $f(t)/g(t)$ with $\deg(f), \deg(g) \leqslant n$, and let $+$, $\cdot$ be the graphs of addition and multiplication restricted to S_n^3. Then $\lim_n S_n = k(t)$.

Note that the k-algebra isomorphism ψ induces a map $V(L) \to V^{\psi}(L')$, also denoted ψ. Part (3) follows from (1) and (2). □

Let L' be constructed as in Lemma 3.2 (1); we can view L' as the union of an increasing system of k-constructible L_k-structures L'_i. Let b be a finite tuple of generators of L' over k. For any n, $Y_n(b) = \{f(b)/g(b); \deg f, \deg g \leqslant n,\ g(b) \neq 0\}$ is contained in some L'_i. Conversely, any L'_i is contained in some $Y_n(b)$. It follows that any k-algebra automorphism of L' preserves the family of sets contained in some L'_i.

Definition. Let L be a finitely generated extension field of k, and let V be a variety over L. A subset Y of $V(L)$ is called *limited* if for some isomorphism $\psi : L \to L'$ to a piecewise-constructible k-algebra as in Lemma 3.2, $\psi(Y)$ is contained in an k-constructible subset of the piecewise-definable set $V^{\psi}(L')$.

If L is a piecewise-constructible k-algebra, V a variety over L, the k-constructible subsets of $V(L)$ can be characterized as the images under piecewise rational constructible functions defined over L, of a constructible set over k. This class is preserved under automorphisms of k-algebras. In the above definition, one can therefore replace 'some' by 'every' isomorphism $\psi : L \to L'$.

Lemma 3.3. *$Y \subseteq V(L)$ is limited if and only if there exists a constructible set U over k, and a constructible map $g : U_L \to V$, such that $Y \subseteq g(U(k))$. If Y is limited, one can choose g to be injective on $U(k)$.*

Proof. We have an isomorphism $\psi^{-1} : \cup L_i \to L$, where $\bigcup_i L_i$ is a piecewise-definable field over k. This induces $\psi^{-1} : \bigcup V(L_i) \to V$. Now $V(L_i)$ is a constructible set over k, and $h_i\psi^{-1}|V(L_i)$ is a constructible map. By definition, if Y is limited then Y is contained in the image of one of these maps. This gives g, U with $Y \subseteq g(U(k))$. By Lemma 3.1, $g^{-1}(=)$ is a constructible equivalence relation on $U(k)$. Factoring it out, we may take g to be injective on $U(k)$. □

Let us mention two further equivalent formulations, one geometric and one model-theoretic.

(1) Assume V comes with a projective embedding and a notion of height applies, see [**21**]. Then a limited subset of $V(L)$ is a set of bounded height. This equivalence is standard. For instance over $k(x)$, a point of $\mathbb{P}^n(k(x))$ can be written in projective coordinates as $(f_0(x) : \cdots : f_n(x))$ with f_i polynomials without common factors, and then the height is the maximal degree of f_i. More generally see Proposition 3.2 in Chapter 3 of [**21**], or the box below it; and recall that the set of rational functions on W whose polar divisor is bounded by some fixed divisor forms a limited set, in fact a finite-dimensional k-space. Compare also Lemma 4.7.

(2) Let T be the ω-stable theory of pairs (k, K) of algebraically closed fields, with $k < K$ (see [**20**]). The completions of T are obtained by specifying the characteristic. If $D \subset k^N$ is definable (with parameters) in the $\mathcal{L}_Q$-structure (K, k), then it is k-constructible.

Assume $(k, K) \models T$. Then a subset Y of $V(K)$ is *limited* if it is k-internal, i.e. $Y \subseteq \mathrm{dcl}(b, k)$ for some finite b. In this case $Y \subseteq V(L)$ for some subfield L of K, finitely generated over k, and Y is limited in the sense defined above for finitely generated extensions.

So far, the discussion involved only algebraic varieties over fields k, K, and no dynamics. Now assume given also a subvariety S of $V \times V$. We continue to assume: k is algebraically closed, K a finitely generated extension field of k, S, V are defined over K.

Lemma 3.4. *The following are equivalent.*

(1) *There exists a possible reducible variety V' over k, $S' \leqslant (V')^2$ with quasi-finite, dominant projections to V', and a dominant rational map $V' \to V$ carrying S' to S.*

(2) *For some limited subset Y of V, for any n and any non-empty open subvariety W of V, there exist $a_0, \ldots, a_n \in W \cap Y$, with $(a_i, a_{i+1}) \in S$.*

Proof. Assume (1). Let $\phi : V' \to V$ be a dominant rational map taking S' to S. Let W be as in (2). Let W' be the set of points $v \in V'$ such that $\phi(v)$ is defined and $\phi(v) \in W$. Choose $b_0, \ldots, b_n \in W'(k)$, $(b_i, b_{i+1}) \in S'$; and let $a_i = \phi(b_i)$.

Now assume (2). We may assume here that k is saturated (i.e. has infinite transcendence degree over the prime field). In this case by compactness there exist $a_i \in Y$ for $i \in \mathbb{Z}$, with a_i avoiding any K-definable proper subvariety of V, and with $(a_i, a_{i+1}) \in S$. Let U, g be as in Lemma 3.3, with g to be injective on $U(k)$. We will use the following general principle.

($*$) For any constructible $W \subseteq V^2$, the pullback $g^{-1}(W)$ is a constructible subset of U. By Lemma 3.1, we obtain a constructible $S'' = g^{-1}(S) \leqslant U^2$, such that g carries S' to S. The projections $S'' \to U$ have finite fibres since this is true for the projections $S \to V$. Let $a'_i = g^{-1}(a_i)$. We may take U to be a finite union of varieties, and g piecewise rational (pre-compose with a Frobenius power if necessary). Let V' be the Zariski closure of $\{a'_i : i \in \mathbb{Z}\}$. Let $S' = S'' \cap (V')^2$. Then the projections $S' \to V'$ are dominant, since their image contains all the a'_i.

□

Remark 3.5. Suppose this weakening of (2) holds: $S \leqslant V^2$ has quasi-finite projections; and

($2'$) for some limited subset Y of V, for any n, there exist distinct $a_0, \ldots, a_n \in Y$, with $(a_i, a_{i+1}) \in S$.

Equivalently, after saturating we obtain

($2''$) there exist distinct $a_i \in Y (i \in \mathbb{Z})$, with $(a_i, a_{i+1}) \in S$.

Let V' be the Zariski closure of $\{a_i : i \in \mathbb{Z}\}$, and $S' = S \cap (V')^2$. Note that S' projects dominantly and quasi-finitely to V'; moreover now (2) holds.

We will thus investigate the consequences of (2); assuming ($2'$) instead, they will automatically apply to some infinite $(V', S') \leqslant (V, S)$.

This can also be reformulated using canonical heights, when they are available. Assume S is the graph of a morphism s. Assume h is a function on $V(K)$ such that

(i) a subset Z of $V(K)$ is limited if and only if $h(Z)$ is bounded above in $\mathbb{R}$;

(ii) for some $\kappa > 1$, if $a \in V(K)$ and $b = s(a)$ then $h(b) = \kappa h(a)$.

Lemma 3.5 ($2'$) is equivalent to the statement: for some $\varepsilon > 0$, for any n, there exist distinct $a_0, \ldots, a_n \in V(K)$, with $(a_i, a_{i+1}) \in S$, and $h(a_i) < \varepsilon$. If this holds, then $h(a_0) < \varepsilon\kappa^{-n}$. Conversely, if just a_0 can be found with $h(a_0) < \kappa^{-n}$, letting $a_i = s^i(a_0)$, we have $h(a_i) = \kappa^{-i} \leqslant 1$. In this situation, ($2''$) is equivalent to the following.

(2h) For any $\varepsilon > 0$ and n, there exist $a_0 \in V(K)$ with $s^n > (a_0) \neq 0$ and $h(a_0) < \varepsilon$.

If one knows that for each n there are only finitely many fixed points of s^n, the condition $h^n(a_0) \neq 0$ can be replaced by the following.

($2\mathrm{h}'$) There exist infinitely many $a_0 \in V(K)$ with $h(a_0) < \varepsilon$.

This is the formulation used in [**1**].

Proof of Theorem 1.11. The finite version follows by compactness from the qualitative one. So assume some ϕ-orbit $a_1, a_2, \ldots$ is contained in a limited subset of $V(L)$. View $V(L)$ as a direct limit of constructible sets V_i over K. ϕ induces a function $F : V'(L) \to V(L)$ (V' being the domain of definition of ϕ); this is a morphism of Ind-constructible sets; i.e. for each i, for some j, $F|V_i$ is a constructible function $V_i \to V_j$. By assumption $\{a_1, a_2, \ldots\} \subseteq V_i(K)$ for some i. Let U' be the Zariski closure in V_i of this set. Then $F(U') \subseteq U'$. For some m, and some component U of U' of maximal dimension, we have $F^m(U) = U$. We can view $\boldsymbol{U} = (U, F^m) \in \mathrm{AD}_K$. Then $\boldsymbol{U}_L$ dominates (V, ϕ^m). By Theorem 3.3 of Part II of this paper, (V, ϕ) descends to K. □

Remark 3.6. The proof goes through for difference varieties, not necessarily algebraic dynamics. If the difference variety is generated by a relation $(a, b) \in S$, with S a correspondence, we conclude that there for any limited subset Y of L, for some n are no $a_1, \ldots, a_n \in L$ with $(a_i, a_{i+1}) \in S$.

Question 3.7. If (V, ϕ) is a field-free algebraic dynamics over $\mathbb{Q}$, or over $K(t)$, do there exist $a, b \in \mathbb{Q}^{\mathrm{alg}}$ or $K(t)^{\mathrm{alg}}$ in the same orbit of Galois, and with $\phi(a) = b$?

It would follow that a is periodic. When (V, ϕ) is polarizable, Fakhruddin has shown that at least periodic points exist (and are Zariski dense).

4. Canonical heights

Perhaps the most characteristic feature of algebraic dynamics is the *canonical height* associated to an algebraic dynamics (X, ϕ) over a field K (see [**5, 14, 21**]). The descent questions treated in the present paper are usually stated in terms of canonical height. We review this concept here in the case of a a function field $K = k(C)$, with C a curve over k. Our presentation is more general than the usual one in three ways, compatible with the more general setting of the paper. (1) We allow k to be a difference field, so that $\phi : X \to X^\sigma$; this generalization requires no effort, and the reader interested in the basic case may take $\sigma = \mathrm{Id}$. (2) We allow 'probabilistic' dynamics given by correspondences, rather than morphisms or rational maps. (3) In the classical case $\sigma = \mathrm{Id}$, ϕ a morphism, we give a construction that does not depend on a polarization assumption; and show that it is non-trivial under much weaker conditions than polarizability.

The dynamics will take place on complete normal varieties V over K. Such a variety can be viewed as the generic fibre of a complete normal variety $\boldsymbol{V}$ over C, i.e. a variety over k with a dominant morphism $j : \boldsymbol{V} \to C$; we fix $\boldsymbol{V}$ too. In this section we use boldface for varieties over C; difference varieties will be considered only in the explicit geometric form (V, ϕ) or (V, S).

The basic properties of 1-cycles and divisors are reviewed below. For any variety V let $\mathrm{NS}(V) = \mathrm{Pic}(V)/\mathrm{Pic}^0(V)$ denotes the group of Cartier divisors on V, up to algebraic equivalence (see [**22**]).

We assume V and $\boldsymbol{V}$ are smooth. It follows that the notions of Weil and Cartier divisors are the same on V or $\boldsymbol{V}$; we will refer to them as divisors. The coincidence of Weil and Cartier is used to show that the natural map $\mathrm{NS}(\boldsymbol{V}) \to \mathrm{NS}(V)$ is surjective, and to define a pushforward $\mathrm{NS}(S) \to \mathrm{NS}(V)$ when $S \to V$ is finite. Together with a certain statement on Galois covers in case the dynamics is not rational, this will be our only use of the smoothness assumption.

We assume k and K come with compatible endomorphisms σ, and let $V' = V^\sigma$; the reader may take $\sigma = \mathrm{Id}$ if desired. Let $S \leqslant V \times V'$ be a subvariety, with finite projection $p : S \to V$ of degree d, and generically finite projection $q : S \to V'$ of degree d'. We make no smoothness assumption on the dynamics S. We will explain below how an action $S_* = \sigma^{-1} q_* p^*$ is induced on 0-cycles over V. We can view S_* as a non-deterministic dynamics on $V(K^{\mathrm{alg}})$, going from a to b_i with probability m_i/d if $S_*(a) = \sum_{i=1}^d m_i b_i$. In case S is the graph of a morphism ϕ, this is compatible with the action of ϕ on $V(K^{\mathrm{alg}})$. Similarly, if h is a function on $V(K^{\mathrm{alg}})$, and $c = \sum m_i a_i$ is a 0-cycle, we let $h(c) = \sum m_i h(a_i)$.

Definition 4.1. A *canonical height* is a a finite-dimensional $\mathbb{R}$-vector space Λ, an expanding linear transformation $\lambda : \Lambda \to \Lambda$ and a function $h : V(K^{\mathrm{alg}}) \to \Lambda$ such that for any 0-cycle a on V,

$$h(S_*(a)) = \lambda(h(a)).$$

By *expanding* we mean that every complex eigenvalue lies outside the unit circle. Classically (see [**5**], discussed below), one only takes the case $\dim(\Lambda) = 1$.

By complexifying and taking eigenvectors v_ν, with eigenvalues λ_ν we could obtain maps h_ν into $\mathbb{C}$ with $h_\nu(S_*(a)) = \lambda_\nu(h_{\boldsymbol{u}}(a))$ and take absolute values to get $|\nu|$ into $\mathbb{R}$, with $|\lambda_\nu| > 1$. This may look closer to the classical case. However the construction of Λ will be more canonical, and gives more information in the non-semisimple case, so we do not choose eigenvectors. At all events we do not assume the existence of real eigenvalues.

Theorem 4.2. *There exists a canonical height $h : V(K^{\mathrm{alg}}) \to \Lambda$, canonically associated to (V, S).*

We think Λ is rarely trivial. We formulate two statements in this direction when S is the graph of a rational map ϕ. One is simply that ϕ is not birational. Another, applying even in the birational case, links with the Northcott results of this paper (see Corollary 1.10, Theorem 1.11 and Remark 3.5). Assume $\phi : V \to V^\sigma$ is a morphism; then ϕ induces a homomorphism $\mathrm{NS}(V^\sigma) \to \mathrm{NS}(V)$, and by composing with σ^{-1} we obtain an endomorphism ϕ^* of $\mathrm{NS}(V)$.

Theorem 4.3. *Let V be a smooth projective variety over K, and S is the graph of a rational map $\phi : V \to V^\sigma$.*

(a) *Assume ϕ is not birational. Then the canonical height h of Theorem 4.2 is non-trivial.*

(b) *Let $\phi : V \to V^\sigma$ be a morphism, and assume no non-torsion element of $\mathrm{NS}(V)$ is fixed by a power of ϕ^*. Then $h(a) = 0$ if and only if the ϕ-orbit of a is contained in a limited set.*

In fact if in (b) the morphism $\phi : V \to V'$ is finite, then any ample height is bounded uniformly on the set of all elements of canonical height 0.

Note that if (V, ϕ) is primitive, non-isotrivial, and has no periodic subvarieties of positive dimension, then by Theorem 1.11 the condition in (b) is equivalent to pre-periodicity of a. Conversely, if a is pre-periodic then any canonical height must vanish on it. Hence in this case at least h is the universal canonical height, i.e. any canonical height $h' : V \to \Lambda'$ is the composition of h with a linear map $\Lambda \to \Lambda'$.

This construction of canonical heights would formally extend to number fields K given some Arakelov theoretic information, of which we are uncertain. We would need an Arakelov variety $\boldsymbol{V}$ with $\boldsymbol{V}_K = V$, a definition of $\mathrm{NS}(\boldsymbol{V})$ (with real coefficients) and a pairing $A_0(\boldsymbol{V}) \times \mathrm{NS}(\boldsymbol{V}) \to \mathbb{R}$, and a surjective $\mathbb{R}$-linear ('Gysin') map $\mathrm{NS}(\boldsymbol{V}) \to \mathrm{NS}(V) \otimes \mathbb{R}$. Moreover, given a correspondence $S \leqslant V \times V^\sigma$, we require pullback maps $\mathrm{NS}(\boldsymbol{V}) \to \mathrm{NS}(\boldsymbol{S})$, and pushforward maps for 1-cycles, related by a projection formula. No finiteness statement on $\mathrm{NS}(\boldsymbol{V})$ is needed.

We begin with a review of (non-dynamical) heights over function fields.

4.4. Cycles. Fix a complete variety W of dimension n over an algebraically closed field. We outline the most basic concepts of intersection theory on W; for this purpose we take W to be smooth and discuss all dimensions, but we will really use only cycles of dimensions 0, 1 and $n-1$. See a similar summary in [**13**], or [**10**].

Let U, V be subvarieties of W of complementary dimension. If the intersection $U \cap V$ is transversal, we write $U \cdot V$ for the number of intersection points. From this basic geometric data one forms an algebraic structure as follows.

Let $C_l(W)$ be the Abelian group freely generated by the l-dimensional irreducible subvarieties of W. If $U = \sum_{i=1}^m \alpha_i U_i \in C_l(W)$, $V = \sum_{i=1}^m \beta_i V_i \in C_{n-l}(W)$ and each intersection $U_i \cap V_j$ is transversal, we say that U, V are transversal and define $U \cdot V = \sum_{i,j} \alpha_i \beta_j U_i \cdot V_j$. This symbol is then extended to the non-transversal case, as follows. Two cycles $U, U' \in C_l(W)$ are said to be *numerically equivalent* if for any $V \in C_{n-l}(W)$ transversal to U and to U', we have $U \cdot V = U' \cdot V$. Write $[U]$ for the class of U up to numerical equivalence, and let $A_l(W) = \{[U] : U \in C_l(W)\}$ be the quotient of $C_l(W)$ by the cycles numerically equivalent to 0. Write $A^l(W)$ for $A_{n-l}(W)$. For reducible subvarieties U of dimension l, we let $[U] = \sum_V [V]$, where V ranges over the irreducible components of U of dimension l. There exists a unique bilinear pairing $A_l(W) \times A^l(W) \to \mathbb{Z}$ such that $[U] \cdot [V] = U \cdot V$ when the right-hand side is transversal.

If $l_1 + l_2 + l_3 = n$, it is similarly possible to define a trilinear pairing $A_{l_1} \times A_{l_2} \times A_{l_3} \to \mathbb{Z}$, $(U_1, U_2, U_3) \mapsto U_1 \cdot U_2 \cdot U_3$, agreeing with the number of intersection points in the transversal case. In fact there exists a bilinear map $A_{l_1} \times A_{l_2} \to A_{n-l_1-l_2}$ such that $U_1 \cdot U_2 \cdot U_3 = (U_1 \cdot U_2) \cdot U_3$.

Let W' be another smooth complete variety, and $f : W \to W'$ a morphism. For an irreducible subvariety U of W, define $f_*([U]) = \deg(f|U)[U'] \in C_l(W')$, where $U' = f(U)$ the image in W', and $\deg(f|U)$ is defined to be field extension degree $[k(U) : k(U')]$ if this is finite, 0 otherwise. Extend by linearity to a linear map $f_* : C_l(W) \to C_l(W')$. Then f_* induces a homomorphism $f_* : A_l(W) \to A_l(W')$.

$A_0(W)$ can be identified with $\mathbb{Z}$ via the degree map $\sum \alpha_i[u_i] \mapsto \sum \alpha_i$. The pushforward f_* preserves degree on A_0.

When $W = \mathbb{P}^n$ and V is a hyperplane, $U \cdot V$ is the projective degree of U. For any integer m, the family of all curves U on W with $U \cdot V \leqslant m$ is therefore a limited family.

For our purposes a *divisor* is an element of $A^1(W)$. By Néron–Severi (see [**22**]), this is a finitely generated Abelian group. We will also write $\mathrm{NS}(W)$ for $A^1(W)$, and $\mathrm{NS}_{\mathbb{R}}(W)$ for $\mathbb{R} \otimes \mathrm{NS}(W)$. NS is a contravariant functor for surjective morphisms. We have the projection formula [**10**, 2.3(c)]: given $f : W \to W'$, for $U \in A_l(W)$, $D \in \mathrm{NS}(W')$ we have $f_*(U \cdot f^*(D)) = f_*(U) \cdot D$. In case $l = 1$, using our identification of A_0 with $\mathbb{Z}$, and the fact that f_* preserves degrees, this also reads: $U \cdot f^*(D) = f_*(U) \cdot D$.

A divisor D on W is called *very ample* if it is the pullback of a hyperplane, under some projective embedding of W; *ample* if mD is very ample for some $m > 0$. Such a divisor D inherits the property noted for the hyperplane divisor on $\mathbb{P}^n$: the family of curves U on W with $U \cdot D \leqslant m$ is a limited family.

We can define heights using *either* intersections of subvarieties S on $W = V \times V'$ with certain divisors and curves from V and V', *or* using intersections of divisors and curves on S itself. The latter is more efficient since it can be defined for Cartier divisors without assuming V is smooth. However, we need to push forward Cartier divisors under generically finite morphisms, and cannot do it unless Cartier divisors coincide with Weil divisors, at least up to algebraic equivalence. So at all events some smoothness assumption is needed. At all events, smoothness of the ambient algebraic variety is not an overly restrictive assumption; in characteristic 0 in particular, it can be achieved with at worst a birational change to the variety, and then $\boldsymbol{V}$ can be chosen smooth once V is. We impose no smoothness condition on the dynamics S.

4.5. Weil height. We work with the data k, $K = k(C)$, $j : \boldsymbol{V} \to C$ with generic fibre V, as above. Cycles on $\boldsymbol{V}$ will always be assumed to be defined over k. If $U \leqslant \boldsymbol{V}$ is an l-dimensional irreducible variety defined over k, then $U \cap V$ is either empty or an $(l-1)$-dimensional K-irreducible subvariety of V. Any K-irreducible subvariety of V can be written uniquely in this way. This gives a homomorphism

$$\rho : C_l(\boldsymbol{V}) \to C_{l-1}(V)$$

whose kernel is generated by the classes of subvarieties that project onto a non-Zariski dense subset of C (i.e. a finite subset of C).

In the opposite direction, given $A \in C_{l-1}(V)$ there exists a unique $\boldsymbol{A} \in C_l(\boldsymbol{V})$ whose support has no component projecting to a point of C, and with $\rho(\boldsymbol{A}) = A$. We denote $\beta(A) = \boldsymbol{A}$.

Up to Galois conjugacy, a point of $V(K^{\mathrm{alg}})$ can be identified with an irreducible element a of $C_0(V)$, i.e. a cycle with non-negative coefficients, not all zero, which is not the sum of two other such. In this case $\boldsymbol{a} = \beta(a)$ is a curve on $\boldsymbol{V}$, the morphism $j|\boldsymbol{a} : \boldsymbol{a} \to C$ is finite, and has degree equal to the degree $d(a)$ of a as a 0-cycle.

Let D be a divisor on V, and let $\boldsymbol{D}$ be a divisor on $\boldsymbol{V}$, defined over k, restricting to D on V.

Let $a \in C_0(V)$. We define the *Weil height* of a by $h_{\boldsymbol{D}}(a) = \beta(a) \cdot \boldsymbol{D}/d(a)$. Here $\beta(a) \cdot \boldsymbol{D}$ is the intersection number. Note that $h_{\boldsymbol{D}}(na) = h_{\boldsymbol{D}}(a)$, so h factors through the projectivization $\mathbb{Q} \otimes C_0(V)/\mathbb{Q}^*$.

If $\boldsymbol{V} = C \times \mathbb{P}^n$ and D is the hyperplane divisor on $\mathbb{P}^n$ pulled back to $\boldsymbol{V}$, then we have the usual Weil height over $k(C)$ on $\mathbb{P}^n$.

Lemma 4.6. *If $\boldsymbol{D}'$ is another divisor on $\boldsymbol{V}$, restricting to D on V, then $h_{\boldsymbol{D}} - h_{\boldsymbol{D}'}$ is a bounded function on $V(K^{\mathrm{alg}})$.*

Proof. Let $L = \boldsymbol{D} - \boldsymbol{D}'$; write $L = \sum m_i L_i$ where L_i is an irreducible hypersurface of $\boldsymbol{V}$; since $\boldsymbol{D}$, $\boldsymbol{D}'$ agree on a generic fibre of j, we have $j(L_i)$ non-Zariski-dense for each i, i.e. $j(L_i)$ is finite. So L_i is contained in a fibre of j. Let $a \in V(K^{\mathrm{alg}})$, $\boldsymbol{a} = \beta(a)$. For a generic fibre, hence for each fibre L' of j we have $\boldsymbol{a} \cdot L' = d(a)$. It follows that $|\boldsymbol{a} \cdot L| \leqslant \sum |m_i| d(a)$. So $|h_{\boldsymbol{D}}(a) - h_{\boldsymbol{D}'}(a)| \leqslant \sum |m_i|$. □

Let $\mathcal{F}(V)$ be the space of functions $V(K^{\mathrm{alg}}) \to \mathbb{R}$, modulo the bounded functions. The class $h_{\boldsymbol{D}}$ in $\mathcal{F}(V)$ depends only on D; we denote it h_D.

4.7. Bounded height and limited families. We recall the following lemma (see [**22**, Property 1F]).

Lemma. *Assume D is very ample. Fix $d_0 \in \mathbb{N}$, and $\alpha \in \mathbb{R}$. Then*

$$\{a \in V(K^{\mathrm{alg}}) : d(a) \leqslant d_0,\ h_{\boldsymbol{D}}(a) \leqslant \alpha\}$$

is a limited set.

Proof. Using Lemma 4.6, we may replace $\boldsymbol{D}$ by any divisor on $\boldsymbol{V}$ restricting to D at the generic fibre. So we can add to $\boldsymbol{D}$ any divisor whose projection to C is not Zariski dense. The linear system $L(D)$ of rational functions on V with poles at most at D contains elements $f_0, \dots, f_l$, such that $v \mapsto (f_0(v) : \cdots : f_l(v))$ is an embedding $V \to \mathbb{P}^l$. These f_i can be taken to be defined over K; they are restrictions of functions F_i on $\boldsymbol{V}$. By adding to $\boldsymbol{D}$ some divisors whose projection to C is finite, we may assume $F_i \in L(\boldsymbol{D})$. By further adding to $\boldsymbol{D}$ a divisor $j^*(D_C)$, where D_C is a very ample divisor on C, we may assume there exist functions $G_i = g_i \circ j$ in $L(\boldsymbol{D})$, such that $c \mapsto (g_0(c) : \cdots : g_{l'}(c))$ is a projective embedding of C. In particular $G_0, \dots, G_{l'}$ do not simultaneously vanish on $\boldsymbol{V}$. Let $J(v) = (F_0(v) : \cdots : F_l(v) : G_0(v) : \cdots : G_{l'}(v))$. Then J is a morphism

$\boldsymbol{V} \to \mathbb{P}^{l+l'+1}$. If $J(v) = J(v')$ then $j(v) = j(v')$; and for some proper subvariety W of C, if $j(v) \notin W$, then $J(v) = J(v')$ implies $v = v'$. Now given $a \in V(K^{\mathrm{alg}})$, $\boldsymbol{a} = \beta(a)$, $J(\boldsymbol{a})$ is a curve in $\mathbb{P}^{l+l'+1}$ of projective degree $d(a)h_{\boldsymbol{D}}(a) \leqslant d_0\alpha$. Thus the $J(\boldsymbol{a})$ span a limited family. It is clear that $\boldsymbol{a}$ is determined by $J(\boldsymbol{a})$ in a uniformly definable fashion (away from W we have $\boldsymbol{a} = J^{-1}(J(\boldsymbol{a}))$). So the family in question is also limited. $\square$

4.8. Effect of Aut(K^{alg}) on heights. Let $\gamma : C' \to C$ be a finite morphism of curves, $K' = k(C')$. and assume there exists a smooth $\boldsymbol{V}'$, $j' : \boldsymbol{V}' \to C'$, and a morphism $e : \boldsymbol{V}' \to \boldsymbol{V}$ with $je = \gamma j'$. Let V' be the generic fibre of $\boldsymbol{V}'$, and $D' = (e|V')^*(D)$. Then $k(C')^{\mathrm{alg}} = k(C)^{\mathrm{alg}}$, so we can compare the height of $a \in V(K^{\mathrm{alg}})$ from the point of view of C and of C'. The point a corresponds to a curve F'_a on $\boldsymbol{V}'$, with $e(F'_a) = \beta(a)$. Thus both $h_D(a) := h_D(\beta(a))$ and $h_{D'}(a) := h_{D'}(F'_a)$ are defined (up to bounded functions).

Lemma 4.9. *Let $\gamma : C' \to C$ be a finite morphism of curves, and let h_D, $h_{D'}$ be as above. Then in the space $\mathcal{F}(V)$ we have*

$$\deg(e)h_D = \deg(\gamma)h_{D'}.$$

Proof. Choose a divisor $\boldsymbol{D}'$ restricting to D' at V'. So $h_{\boldsymbol{D}'} = h_{D'}$ up to bounded functions. Let $d'(a)$ be the degree of $j'|F'$. Fix a and let $F' = F'_a$, $F = \beta(a)$. Then $e_*[F'] = \deg(e|F')F$ By the projection formula [**10**, 2.3(c)], and since e_* preserves the degree of 0-cycles, we have

$$[F'] \cdot \boldsymbol{D}' = e_*([F'] \cdot \boldsymbol{D}') = \deg(e|F')F \cdot D = \deg(e|F')[F] \cdot [\boldsymbol{D}].$$

So $d'(a)h_{\boldsymbol{D}'}(a) = \deg(e|F')d(a)h_{\boldsymbol{D}}(a)$. Now since $je = \gamma j'$ we have $\deg(\gamma)\deg(j'|F') = \deg(e|F')\deg(j|F)$ and the claim follows. $\square$

For any finitely generated extension field K of k we have the modular function $\delta : \mathrm{Aut}(K^{\mathrm{alg}}/k) \to \mathbb{Q}$. It can be defined by the ratio of field degree extensions:

$$\delta(\sigma) = \frac{[KK^\sigma : K^\sigma]}{[KK^\sigma : K]},$$

where $K^\sigma = \sigma(K)$ and KK^σ is the field compositum.

Assume now that V is defined over k, i.e. $\boldsymbol{V} = V \times C$. Any $\sigma \in \mathrm{Aut}(K^{\mathrm{alg}}/k)$ induces a function $\sigma_V : V(K^{\mathrm{alg}}) \to V(K^{\mathrm{alg}})$, and composition induces an action of σ on $\mathcal{F}(V)$. Lemma 4.9 implies the following corollary.

Corollary 4.10. *Let $\sigma \in \mathrm{Aut}(K^{\mathrm{alg}}/k)$. Then in $\mathcal{F}(V)$ we have*

$$h_D \circ \sigma_V = \delta(\sigma)h_D.$$

The same relation thus holds for the canonical height h. Corollary 4.10 implies in a variety of cases that any algebraic solution of a difference equation $(x, \sigma(x)) \in S$ has canonical height 0. We have $h(\sigma(a)) = \delta(\sigma)h(a)$ while $h(s(a)) = \lambda h(a)$, so if $\sigma(a) = s(a)$ and $h(a) \neq 0$, then $h(a)$ must be an eigenvector for λ with eigenvalue $\delta(\sigma)$. Thus we have the following corollary.

Corollary 4.11. *Assume V is defined over k, $a \in V(K^{\mathrm{alg}})$ and $k(a) \cong_k k(s(a))$ by an isomorphism σ taking a to $s(a)$. Then any of the conditions below implies that a has canonical height* 0*:*

(1) *S is the graph of a rational function, and λ has no rational eigenvalues;*

(2) *S is the graph of a rational function, and $\delta(\sigma)$ is smaller than any real eigenvalue of λ;*

(3) *$\delta(\sigma)$ is greater in absolute value than any eigenvalue of λ.*

4.12. Dynamics of a correspondence. Let V be a variety defined over K, as above. If $K = (K, \sigma)$ carries a difference field structure, leaving k invariant, let $V' = V^\sigma$. (If one wishes to think of K as a field, let $\sigma = \mathrm{Id}$.) Let S be a complete variety of dimension $n = \dim(V)$, defined over K, and let $p : S \to V$ and $q : S \to V'$ be morphisms. We assume p is finite, of degree d, and that q is generically finite, of degree d'. For simplicity, we assume p is separable.

We have $p^* : \mathrm{NS}(V) \to \mathrm{NS}(S)$ and $q^* : \mathrm{NS}(V') \to \mathrm{NS}(S)$. Since p is finite, we also have $p_* : \mathrm{NS}(S) \subset A^1(S) \to A^1(V) = \mathrm{NS}(V)$. We obtain an endomorphism S^t of $\mathrm{NS}(V)$, namely $S^t(D) = p_* q^* D^\sigma$. Let $S^* = d^{-1} S^t$.

Similarly we have $p^* : C_0(V) \to C_0(S)$, using finiteness of p. And we have $q_* : C_0(S) \to C_0(V')$. We obtain an endomorphism S_* of $C_0(V)$, namely $S_*(a) = q_* p^*(a)^{\sigma^{-1}}$.

We will consider two cases: subvarieties of $V \times V'$ with the projection maps to V and V'; or normalizations S of such varieties. We note that an arbitrary correspondence S gives equivalent dynamics to one of this form. Let S, p, q be as above. Let S' be the image of S under $(p, q) : S \to V \times V'$. We can define a dynamics using S' with the projection maps to V, V'. Now the map $\pi : S \to S'$ is finite, and we have $\pi_* \pi^* = \deg(\pi)$. It follows that the dynamics given by S and S' on $\mathrm{NS}(V)$ and on $PA_0(V)$ are the same (on $A_0(V)$ they differ by a constant multiple $\deg(\pi)$). Hence we can work with subvarieties of $V \times V'$.

However, it is convenient to have S' normal and Galois over V. Hence we show how to replace a given $S' \leqslant V \times V'$ by (S, p, q) with these properties, and with $(p, q)(S) = S'$. Let $\pi : S \to S'$ be the normalization of S' in the Galois hull L of $K(S')/K(V)$. Then $H = \mathrm{Gal}(L/K(V))$ acts on S, over V. For $D \in \mathrm{NS}(S)$, at least if D is represented by a divisor with support not contained in the ramification divisor $\mathrm{Ram}(p)$ of p, we have $p^* p_* D = \sum_{h \in H} h^* D$.

When $S' \leqslant V \times V'$, let $\boldsymbol{S}' = \beta(S')$ be the unique irreducible subvariety of $\boldsymbol{V} \times_C \boldsymbol{V}$ with $\boldsymbol{S}' \cap (V \times V) = S'$. So $\dim(\boldsymbol{S}') = n + 1$. In general, let $(\boldsymbol{S}, p : \boldsymbol{S} \to \boldsymbol{V}, q : \boldsymbol{S} \to \boldsymbol{V}^\sigma)$ be a triple with generic fibre (S, p, q). If S is the normalization of S' as above, we can let $\boldsymbol{S}$ be a similar normalization of $\boldsymbol{S}'$, so that H acts on $\boldsymbol{S}$, extending the action on S.

Let $\boldsymbol{D}$ be a divisor on $\boldsymbol{S}$.

We have an intersection pairing $C_1(\boldsymbol{V}) \times \mathrm{NS}(\boldsymbol{V}) \to \mathbb{Z}$, and also $C_1(\boldsymbol{S}) \times \mathrm{NS}(\boldsymbol{S}) \to \mathbb{Z}$. They are compatible via

$$p_* x \cdot y = x \cdot p^* y \tag{4.1}$$

for $x \in C_1(\boldsymbol{S})$, $y \in \mathrm{NS}(\boldsymbol{V})$; and similarly for q.

Recall $\beta_V : C_0(V) \to C_1(\boldsymbol{V})$, $\beta_S : C_0(S) \to C_1(\boldsymbol{S})$. Let $\beta_{V'}$ be the σ-conjugate.

The 'Gysin' homomorphism $\rho : \mathrm{NS}_{\mathbb{Q}}(\boldsymbol{V}) \to \mathrm{NS}_{\mathbb{Q}}(V)$ is surjective, using the fact that every Weil divisor on $\boldsymbol{V}$ is Cartier (or, has a multiple numerically equivalent to one). Let $\gamma_V : \mathrm{NS}_{\mathbb{Q}}(V) \to \mathrm{NS}_{\mathbb{Q}}(\boldsymbol{V})$ be a section of this linear map, and similarly $\gamma_{V'}$. We omit the subscripts when possible.

We have

$$\beta_{V'} q_* = q_* \beta_S : C_0(S) \to C_1(\boldsymbol{V}'), \qquad \beta_V p_* = p_* \beta_S : C_0(S) \to C_1(\boldsymbol{V}). \tag{4.2}$$

Note that on $\boldsymbol{S}$ the morphism p may not be finite, and so p^*, β need not commute.

Given two functions f, g of a variable x in $C_0(V)$ and possibly other variables y, we write $f \sim g$ if for any y, $f(x,y) - g(x,y)$ is bounded on $C_0(V)$. For elements $b, b' \in \mathrm{NS}(\boldsymbol{V})$ we write $b \sim b'$ if $h_b \sim h_{b'}$.

Let $\xi \in C_0(S)$, $y \in \mathrm{NS}(V')$. From (4.1) and (4.2) we obtain

$$\beta(\xi) \cdot q^* \gamma(y) \sim \beta(q_* \xi) \cdot \gamma(y). \tag{4.3}$$

Lemma 4.13. *Let $x \in C_0(V)$, $y \in \mathrm{NS}(V')$. Then $\beta(q_* p^* x) \cdot \gamma(y) \sim \beta(x) \cdot \gamma(p_* q^*(y))$. Hence $h_y \circ S_* \sim h_{S^* y}$.*

Proof. The 'hence' follows immediately from the main statement, using the definition of h_y and the fact that q_* preserves degrees, while p^* multiplies them by d. To begin with, we have the following statement.

Claim. Let $D' \in \mathrm{NS}(V')$, $D = q^* D'$. Then $p^* p_* D = \sum_{h \in H} h^* D$.

Proof. Let Ram_p be the ramification divisor of $p : S \to V$, viewed as a divisor on S. Then $q(\mathrm{Ram}_p)$ is a proper subvariety of V', and we may choose a representative of the linear equivalence class of D' whose support has no component contained in $q(\mathrm{Ram}_p)$. We show equality of the Cartier divisors $p^* p_* D$ and $\sum_{h \in H} h^* D$. Since S is normal, equality as Weil divisors suffices. Since p is finite, neither of these has a component of the support contained in Ram_p; and to show equality we may work away from Ram_p. There, p is étale, and S is smooth. Since p is Galois, the equality is clear. $\square$

We explain below, in sequence, the following chain of equalities and '$\sim$':

$$\begin{aligned}
\beta(q_* p^* x) \cdot \gamma(y) &= \beta(p^* x) \cdot q^* \gamma(y) \\
&\sim d^{-1} \beta(p^* x) \cdot \sum_{h \in H} h q^* \gamma(y) \\
&= d^{-1} \beta(p^* x) \cdot p^* p_* q^* \gamma(y) \\
&= d^{-1} p_* \beta(p^* x) \cdot p_* q^* \gamma(y) \\
&= d^{-1} \beta(p_* p^* x) \cdot p_* q^* \gamma(y) \\
&\sim \beta(x) \cdot \gamma(p_* q^*(y)).
\end{aligned}$$

- By (4.3) applied to $\xi = p^* x$.
- Here we use the fact that H acts by automorphisms on $\boldsymbol{S}$, and so respects the intersection product; thus $u \cdot v = d^{-1} \sum_{h \in H} (hu) \cdot (hv)$; and $h \beta p^* x = \beta h p^* x = \beta p^* x$.

- By the claim.
- By the projection formula (4.1).
- By (4.2).
- While γ is not defined on $\mathrm{NS}(S)$, ρ is defined on V, S, S' and commutes with p_* and with q^*. Hence ρ commutes with p_*q^*, so $\rho(p_*q^*\gamma(y)) = p_*q^*(y)$, and hence by Lemma 4.6 we have $\beta(x) \cdot (p_*q^*\gamma(y)) \sim \beta(x) \cdot \gamma(p_*q^*(y))$. We also use $p_*p^*x = dx$ on $C_0(V)$.

$\square$

For the sake of a later observation (4.21), we record an easier case.

Lemma 4.14. *Assume p is finite above a Zariski open neighbourhood of the support of $\boldsymbol{a} \in C_1(\boldsymbol{V})$. Then*

$$h_{\boldsymbol{D}} \circ S_*(a) = h_{\boldsymbol{S}^*(\boldsymbol{D})}(a).$$

Proof. In this case we have $\beta p^*(a) = p^*\beta(a)$, since using the finiteness assumption on $\boldsymbol{S} \to \boldsymbol{V}$ near a, there will be no components of $p^*\beta(a)$ projecting to a point of C. The projection formula then immediately gives the lemma. $\square$

4.15. Canonical heights. Let $E = \mathrm{NS}_{\mathbb{R}}(V) = \mathbb{R} \otimes \mathrm{NS}(V)$. Then S^* induces a linear endomorphism $S^*|E$ of E. We can (uniquely) express E as a direct sum of two $S^*|E$-invariant subspaces E_-, E_+, such that every complex eigenvalue of S^* on E_- (respectively E_+) has absolute value less than or equal to 1 (respectively greater than 1). In particular, $S^*|E_+$ is invertible; let s denote the inverse. When $d = 1$, the eigenvalues on E_- are 0 and roots of 1 (see Lemma 4.18).

For any $e \in E_+$, the sequence $s^n(e)$ approaches 0 exponentially fast.

Let κ^{-1} be spectral radius of s, i.e. $\kappa = \min\{|\alpha| : \alpha \in \mathrm{spec}(S^*|E),\ |\alpha| > 1\}$, where $\mathrm{spec}(\cdot)$ is the set of eigenvalues. So $\kappa > 1$.

Let $\Lambda = E_+^*$ be the dual space to E_+. Let λ be the dual linear transformation to S^*, i.e. $\lambda(F) = F \circ S^*$.

Proposition 4.16. *There exists a unique function $h : V(K^{\mathrm{alg}}) \to \Lambda$ (extending to a linear $h : C_0(V) \to \Lambda$) such that*

(1) *for any $e \in E$, the function $a \mapsto h(a)(e)$ represents h_e in $\mathcal{F}(V)$;*

(2) *for all $a \in V(K^{\mathrm{alg}})$, $h(S_*(a)) = \lambda(h(a))$.*

When E_+ contains an ample divisor, h is 'proper' in the sense that the inverse image of a bounded subset of E^* is a limited subset of $V(K^{\mathrm{alg}})$.

Proof. Recall the lemma behind Tate's canonical heights construction (see, for example, [**25**, Theorem 3.20]). let X be a set, $T : X \to X$ a function, $h : X \to \mathbb{R}$, $\kappa \in \mathbb{R}$, $\kappa > 1$. Assume $h \circ T - \kappa h$ is a bounded function on X. Then there exists a unique $h' : X \to \mathbb{R}$ with $h - h'$ bounded, and such that $h' \circ T = \kappa h'$. (Namely $h' = \lim \kappa^{-n} h \circ T^n$.)

We will apply this to an appropriate function on $X = V(K^{\mathrm{alg}}) \times Y$ for various compact subsets Y of E_+. We write V for $V(K^{\mathrm{alg}})$ for the rest of this proof.

Let $\mathrm{Bdd}(X)$ be the space of functions $\phi : V \times E_+ \to \mathbb{R}$, such that for any compact $Y \subseteq E_+$, $\phi|(V \times Y)$ is a bounded function.

We have a surjective homomorphism $\rho : \mathrm{NS}_{\mathbb{R}}(\boldsymbol{V}) \to \mathrm{NS}_{\mathbb{R}}(V)$. Choose a linear map $\gamma : \mathrm{NS}_{\mathbb{R}}(V) \to \mathrm{NS}_{\mathbb{R}}(\boldsymbol{V})$, with $\rho \circ \gamma = \mathrm{Id}_V$. We use the fact that $\mathrm{NS}(V)$ is a finite-dimensional space; $E = \mathrm{NS}_{\mathbb{R}}(V)$ has a uniquely determined topology of a real topological vector space.

Define $h_0 : C_0(V) \times \mathrm{NS}(V) \to \mathbb{R}$ by $h_0(a,e) = h_{\gamma(e)}(a)$, and extend to $C_0(V) \times \mathrm{NS}_{\mathbb{R}}(V)$ by linearity in the second variable. For any fixed $e \in \mathrm{NS}(V)$, $\rho \boldsymbol{S}^t(\gamma(e)) = \rho\gamma(S^t(e))$. Hence by Lemma 4.6, $h_{\boldsymbol{S}^t(\gamma(e))} - h_{\gamma(S^t(e))}$ is bounded on $C_0(V)$. Now $h_0(S_*(a), e) - d^{-1} h_{\boldsymbol{S}^t(\boldsymbol{D})}(a)$ is bounded as a function of a, while by definition $h_0(a, S^t(D)) = h_{\gamma(S^t(e))}(a)$. Hence for fixed e, $\delta(a,e) = h_0(S_*(a), e) - d^{-1} h_0(a, S^t(e))$ is bounded on $C_0(V)$. Let $e_1, \ldots, e_r \in \mathrm{NS}(V)$ be a basis for $\mathrm{NS}_{\mathbb{R}}(V)$. If $Y \subset E$ is compact then for some $B \in \mathbb{R}$, any $y \in Y$ can be written $y = \sum \alpha_i e_i$ with $|\alpha_i| \leqslant B$; so $\delta(a,y) = \sum_i \alpha_i \delta(a, e_i)$ is bounded on $C_0(V) \times Y$.

In particular, restricting to $X = C_0(V) \times E_+$, we see that $h_0(S_*(a), e) - d^{-1} h_0(a, S^t(e))$ lies in $\mathrm{Bdd}(X)$.

Choose κ_1 with $1 < \kappa_1 < \kappa$. Define $T : X \to X$ by $T(a,e) = (S_*(a), \kappa_1 s(e))$. The spectral radius of $\kappa_1 s(e)$ is $\kappa_1 \kappa^{-1} < 1$, so that any orbit of $\kappa_1 s$ is bounded (since it approaches 0), and more generally any compact subset of E_+ is contained in a compact, $\kappa_1 s$-invariant subset. On the other hand, modulo $\mathrm{Bdd}(X)$ we have

$$h_0(T(a,e)) = \kappa_1 h_0(S_*(a), s(e)) = \kappa_1 d^{-1} h_0(a, S^t(s(e))) = \kappa_1 h_0(a,e).$$

Since $\kappa_1 > 1$, the Tate lemma applies on $C_0(V) \times Y$ for any compact, $\kappa_1 s$-invariant subset Y of E_+. So for each such Y there exists a unique $h_1^Y : C_0(V) \times Y \to \mathbb{R}$ with $h_1^Y - h_0$ bounded and $h_1^Y(T(a,e)) = \kappa_1 h_1^Y(a,e)$. Hence there exists a unique $h_1 : X \to \mathbb{R}$ with $h_1 - h_0 \in \mathrm{Bdd}(X)$, and $h_1(T(a,e)) = \kappa_1 h_1(a,e)$. For any fixed a, the function $h_1(a,e)$ is defined as a limit of linear functions in e, so $h_1(a,e)$ is linear in e. We have

$$h_1(S_*(a), e) = h_1(T(a, \kappa_1^{-1} S^*(e))) = \kappa_1 h_1(a, \kappa_1^{-1} S^*(e)) = h_1(a, S^*(e)).$$

Let $h(a)$ be the linear map: $e \mapsto h_1(a,e)$. Then (1), (2) are clear. The remark on the ample divisor is also clear. □

Consider now the case $d = 1$. When is $E_+ \neq 0$? Recall $d' = \deg(q)$, the forward degree of S.

Lemma 4.17. *If $d' > 1$ then $E_+ \neq 0$.*

Proof. We have on E an n-multilinear form, the intersection product. Write e^n for $e \cdot \cdots \cdot e$. If $e \in E$ is the class of an ample divisor, then $e^n > 0$, and using the projection formula

$$((S^*)^m e)^n = (d')^m e^n.$$

As this grows exponentially in m, the lemma follows from the following fact from linear algebra.

Claim 1. Let V be a finite-dimensional complex vector space, and let $m : V^k \to \mathbb{C}$ be a multilinear map. Let $v \in V$. Let $T \in \mathrm{End}(V)$, and suppose every eigenvalue of T has absolute value less than or equal to 1. Then $|m(T^a v, \ldots, T^a v)|$ is bounded by a polynomial in a.

Proof. Let $v_1, \ldots, v_n$ be a basis with respect to which T has Jordan normal form. Let $v = \sum \alpha_i v_i$. Then $T^a v = \sum q_i(a) v_i$ where $|q_i(a)|$ is polynomially bounded in a. It follows from multilinearity that $m(T^a v, \ldots, T^a v)$ is polynomially bounded too. □

□

We thus obtain a non-trivial function into a finite-dimensional real dynamical system whenever the algebraic dynamics is non-birational. But even in the birational case, it would seem to be rare that E_+ is trivial, and it would be nice to obtain more geometric information. We have the following from global linear algebra, showing that E_+ is trivial only when S^* is essentially uni-by-nilpotent on E. Here we use the fact that the action of S^* on E_+ arises from the action on a finitely generated group, the Néron–Severi group N of V; the lemma would not be true if N were allowed to be an arbitrary $\mathbb{Q}$-space.

Lemma 4.18. *Let N be a finitely generated Abelian group, and $T \in \mathrm{End}(N)$. Let $V = N \otimes \mathbb{C} \cong \mathbb{C}^n$, and write T for $T \otimes \mathbb{C}$. Assume every eigenvalue of T has complex absolute value less than or equal to* 1. *Then every such eigenvalue is* 0 *or a root of* 1.

Proof. These eigenvalues lie in some number field L, and form a set closed under conjugation. For each non-Archimedean absolute value p of L, each eigenvalue has absolute value less than or equal to 1, since the compact open set $L_p \otimes N$ is left invariant by T. Thus each eigenvalue is an algebraic integer, all of whose conjugates have complex norm at most 1; by the product formula, it is 0 or else every conjugate has complex norm 1; in the latter case it is a root of unity. □

Hence if $d = 1$; then E_+ is the image of $(S^t)^n((S^t)^n - 1)^m$, for large enough m, n.

Proof of Theorem 4.3. Part (a) has been proved above. As for (b), by assumption, and by Lemma 4.18, every eigenvalue of ϕ^* is either 0 or of absolute value greater than 1. Let D be an ample divisor. Then we may write $D = D_1 + D_2$ where $(S^*)^m D_1 = 0$ for some m, and $D_2 \in E_+$. Assume $\lambda(a) = 0$. Then $\lambda(\phi^k(a)) = 0$ for all k, so $h_{D_2}(\phi^k(a))$ is bounded uniformly in k. On the other hand, $h_{D_1}(\phi^m(b)) \sim 0$ since $h_{D_1} \circ \phi^m \sim h_{(S^*)^m D_1}$ but $(S^*)^k D_1 = 0$. This holds uniformly for all b, in particular for $b = \phi^k(a)$, so $h_{D_1}(\phi^{m+k}(a)) \sim 0$ uniformly in k. Thus $h_D(\phi^k(a)) \sim 0$. □

Examples.

(1) If V is a curve, then $\mathrm{NS}(V)$ is one dimensional, so S acts by multiplication by a scalar. In the case of a curve the scalar is the degree of π_2 on S. Thus the condition $(*)$ in this case is that the degree of π_2 is greater than that of π_1. The classical case has $d = 1$, and the condition is the π_2 has degree greater than 1.

(2) If $V = \mathbb{P}^n$ then $\mathrm{NS}(V)$ is one dimensional, and S acts by multiplication by a scalar, which is positive unless S is linear.

(3) Suppose S is the graph of a morphism s. Call and Silverman [**5**] assume the existence of $e \in E$ with $S^*(e) = \kappa e$, $\kappa > 1$. In this case we have $e \in E_+$, and the Call–Silverman canonical height $h_{V,e,s}$ is given by $h_{V,e,s}(a) = h(a)(e)$.

If all the eigenvalues of $S^*|E_+$ are real, then h is captured by $\dim(E_+)$ canonical heights into $\mathbb{R}$ as in (3). But using SL_n actions on compactifications of $(G_m)^n$, it is easy to find examples with no real eigenvalues.

4.19. Generic algebraicity. For simplicity, assume S is the graph of a morphism ϕ, and that $S^*(D) = \lambda D$ for some ample divisor D and some $\lambda > 1$, so that we have a canonical height function $H_D : V(K^{\mathrm{alg}}) \to \mathbb{R}$. For Abelian varieties over number fields, Silverman suggested that the canonical height may be transcendental for sufficiently general algebraic points. We note here that for many (if not all) dynamics over function fields, the canonical height of sufficiently irrational algebraic points is an algebraic number.

Let $\boldsymbol{E}$ be the blowup locus of $\boldsymbol{S} \to \boldsymbol{V}$. $\boldsymbol{E}$ projects to a finite subset $\{c_1, \dots, c_k\}$ of C, so $\boldsymbol{E} = \bigcup_i E_i$, with $E_i = E_{c_i}$. Even for non-isotrivial dynamics, $\boldsymbol{E}$ may be empty. In the case of dynamics on curves it is at most finite.

Lemma 4.20. *Assume ϕ is a rational map $V \to V$, defined outside a finite set E, over a field k. Let l be a prime bigger than $|E|$ and $\deg(\phi)$. Let $d \in V(k^{\mathrm{alg}})$ with $l|[k(d) : k]$. Then $\phi^m(d)$ is defined for any $m \in \mathbb{N}$.*

Proof. If $e \in E$ then $[k(e) : k] \leqslant |E| < l$; hence $d \notin E$, so $\phi(d)$ is defined. We have $[k(d, \phi(d)) : k(\phi(d))] \leqslant \deg(\phi)$, so if $[k(\phi(d)) : k]$ is prime to l then so is $[k(d, \phi(d)) : k]$ and hence $[k(d) : k]$, a contradiction. Thus $[k(\phi(d)) : k]$ is also divisible by l. We continue inductively. □

Let $\tilde{V}$ be the set of algebraic points a of V, such that curve $\rho(\phi^n(a))$ corresponding to $\phi^n(a)$ does not meet $\boldsymbol{E}$, for any n. Let $c_1, \dots, c_n$ be the support of the projection of $\boldsymbol{E}$ to C, $V_i = V_{c_i}$, ϕ_i the restriction of S to V_i. Assume $\boldsymbol{E}$ is finite, so that ϕ_i is a rational map (defined away from E_i). Assume the base field k is not real closed or algebraically closed. (The heights can be evaluated with respect to $k^{\mathrm{alg}}(C)$; the points in question will be in $k(C) \setminus k^{\mathrm{alg}}$.) Let $d_i \in V_i(k^{\mathrm{alg}})$ be such that $[k(d_i, c_i) : k(c_i)]$ is divisible by a large prime l. By the lemma, $\phi_i^m(d_i)$ is defined for all m. By the approximation lemma, choose $a \in \boldsymbol{V}(K^{\mathrm{alg}})$ such that under the map $\mathrm{res} : \boldsymbol{V}(K^{\mathrm{alg}}) \to V(k^{\mathrm{alg}})$ associated with the valuation at c_i, we have $\mathrm{res}(a) = d_i$. Then $a \in \tilde{V}$. In this sense, 'sufficiently general' algebraic points lie in $\tilde{V}$; meaning that the residue at a finite number of places is to be highly irrational, i.e. of high degree.

Presumably this is true in general, and not only when $\boldsymbol{E}$ is finite.

Lemma 4.21. *Let $a \in \tilde{V}$. Then $H_D(a)$ is an algebraic number.*

Proof. Let $f = \deg(a)$. Let $\boldsymbol{S}$, $\boldsymbol{D}$ be subvarieties of $\boldsymbol{V} \times_C \boldsymbol{V}$, $\boldsymbol{V}$ defined over k and restricting to S, D, as above. Let $d_m = (\boldsymbol{S}_*^m(\boldsymbol{a})) \cdot \boldsymbol{D}$. By Lemma 4.14 we have $d_m = \boldsymbol{a} \cdot (\boldsymbol{S}^t)^m(\boldsymbol{D})$, and $h_{\boldsymbol{D}}(\phi^m(a)) = f^{-1}d_m$. So $H_D(a) = \lim_{m\to\infty} f^{-1}\kappa^{-m}d_m$. We will prove the stronger claim, that the generating series $\sum d_m t^m$ is rational with algebraic coefficients.

We use the finite dimensionality of $W = \mathrm{NS}(\boldsymbol{V})$. $\boldsymbol{S}^t$ acts on $\mathrm{NS}(\boldsymbol{V})$ linearly. We have a linear map $l : \mathrm{End}(W) \to \mathbb{R}$, namely $w \mapsto \beta(a) \cdot w$. It follows by using Jordan form over $\mathbb{Q}^{\mathrm{alg}}$ that $d_m = \sum p_i(m)\kappa_i^m$ for some polynomials p_i with algebraic coefficients, and some $\kappa_i \in \mathbb{Q}^{\mathrm{alg}}$ and $p_i \in \mathbb{Q}^{\mathrm{alg}}[T]$. It follows that $\lim_{m\to\infty} f^{-1}d_m\kappa^{-m}$ (being finite) is algebraic. $\square$

If for example V is the generic point of an Abelian scheme over C, with dynamics given by multiplication, there will be no blowup locus, and the canonical height will be algebraic everywhere.

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