Axiomatizing rectangular grids with no extra non-unary relations

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Abstract

We construct a formula ϕ which axiomatizes non-narrow rectangular grids without using any binary relations other than the grid neighborship relations. As a corollary, we prove that a set $A \subseteq \mathbb{N}$ is a spectrum of a formula which has only planar models if numbers $n \in A$ can be recognized by a non-deterministic Turing machine (or a one-dimensional cellular automaton) in time $t(n)$ and space $s(n)$, where $t(n)s(n) \leq n$ and $t(n), s(n) = \Omega(\log(n))$.

1 Introduction

The *spectrum* of a ϕ, denoted $\text{spec}(\phi)$ is the set of cardinalities of models of ϕ. Let SPEC be the set of $A \subseteq \mathbb{N}$ such that A is a spectrum of some formula ϕ is an interesting research area [Sch52, DJMM12]; it is known that SPEC=NE, i.e., A is a spectrum of a first order formula iff the set of binary representations of the elements of A is in the complexity class NE [Fag74, JS74]. However, the characterization of spectra remains open if we require our formula, or our models, to have additional properties. In [DK17] we study the complexity class FPSPEC (Forced Planar Spectra), which is the set of $A \subseteq \mathbb{N}$ such that there exists a formula ϕ such that $\text{spec}(\phi) = S$ and all models of ϕ are planar. It is shown there that FPSPEC $\supseteq$ NTISP$(n^{1-\epsilon}, \log(n))$, where NTISP$(t(n), s(n))$ the set fo $A \subseteq \mathbb{N}$ such that there exists a non-deterministic Turing machine which recognizes the binary representation of n in time $t(n)$ and space $s(n)$. However, this result is not satisfying, since space $\log(n)$ is very low; a construction of which allows more space is left as an open problem.

In this paper we construct a formula ϕ over a signature consisting of only binary relations U, D, L, R (neighbors in the grid in all directions) and unary relations, and which axiomatizes rectangular grids which are not *narrow*, i.e., grids of dimensions $x^* \times y^*$ where $x^* = \Omega(\log(y^*))$ and $y^* = \Omega(\log(x^*))$. We show that it is impossible to give a similar axiomatization of rectangular grids which includes the narrow ones. Non-narrow rectangular grids are planar graphs of bounded degree, and they can be used to simulate Turing machines, and thus we obtain the following corollary: FPSPEC $\supseteq$ NTISP$(t(n), s(n))$ for every pair

of functions $t(n), s(n)$ such that $t(n) \cdot s(n) \leq n$ and $t(n), s(n) = \Omega(\log(n))$. In fact, we get a bit more – we can actually simulate a non-deterministic one-dimensional cellular automaton (1DCA) working in the given time and memory. While 1DCAs are less commonly taught than Turing machines, they are simpler to define and more powerful, since they can perform computations on the whole tape at once [Kop17].

2 Axiomatizing a rectangular grid

We obtain our goal by showing a first-order formula whose all finite models are rectangular grids. A **rectangular grid** is a relational structure $G = (V(G), L, R, U, D)$ such that $V(G) = \{0..x^*\} \times \{0..y^*\}$, and the relations L, R, U, D hold only in the following situations: $L((x, y), (x-1, y))$, $R((x, y), (x+1), y)$, $U((x, y), (x, y-1))$, $D((x, y), (x, y+1))$, as long as these vertices exist.

Geometry We will use four binary relations L, R, U, D, which correspond to Left, Right, Up, Down, respectively. We will need axioms to specify that these four relations work according to the Euclidean square grid geometry.

- **Partial injectivity.** Our relations $X \in \{L, R, U, D\}$ are partial injective functions. That is, we have an axiom $\forall x \forall y X(x, y) \wedge X(x, z) \Rightarrow y = z$. For $X \in \{L, R, U, D\}$, we will write $X(x)$ for the element y such that $X(x, y)$ (if it exists).

- **Inverses.** $\forall x \forall y R(x, y) \iff L(y, x) \wedge U(x, y) \iff D(y, x)$. This axiom formalizes our interpretation of directions (that Left is inverse to Right and Up is inverse to Down).

- **Commutativity.** Let $H \in \{L, R\}$ and $V \in \{U, D\}$. Then $\forall x \forall y \forall z H(x, y) \wedge V(x, z) \Rightarrow \exists t H(z, t) \wedge H(y, t)$. This axiom axiomatizes the Euclidean geometry of our grid: horizontal and vertical movements commute. Additionally, it enforces that whenever we can go horizontally and vertically from the given x, we can also combine these two movements and move diagonally.

Binary Counters We will require our grid to know its number of rows. To this end, we will introduce an extra relation B_V. Intuitively, replace every vertex v in the row $r = (x, R(x), R^2(x), \ldots)$, where $L(x)$ is not defined, with 1 if $B_V(v)$, and 0 otherwise. The axioms in this section will enforce that the obtained number (written in the little endian binary notation) is the index of our row.

- **Horizontal Zero.** $\forall x (\neg \exists y U(x, y)) \Rightarrow (\neg B_V(v))$. The binary number encoded in the first row is zero.

- **Horizontal Increment.** To increment a (little endian) binary number, we change every bit which is either the leftmost one, or such that its left neighbor changed from 1 to 0. This can be written as the following formula: $\forall x(\exists y U(x,y)) \Rightarrow ((B_v(x) \not\Longleftrightarrow B_v(U(x))) \Longleftrightarrow C(x)$ where $C(x) = ((\neg \exists y L(x,y)) \vee (\neg B_v(L(x)) \wedge B_v(U(L(x)))))$.

- **No Horizontal Overflow.** $\forall x(\neg \exists y R(x,y)) \Rightarrow (\neg B_V(v))$. This axiom makes sure that our binary counter does not overflow.

We also have analogous axioms for vertical binary counters, using an extra unary relation B_H, counting from right to left, with the least significant bit on the bottom. See Figures 1a and 1c, where the vertices of the grid satisfying respectively B_V and B_H are shown (ignore the small white circles and dark grey boxes for now – they will be essential for our further construction).

Let ϕ_1 be the conjunction of all axioms above.

Theorem 2.1 *If G is a connected finite model of ϕ_1 and there exists an $v \in V(G)$ and a relation $X \in \{L, R, U, D\}$ such that $X(v)$ is not defined, then G is a rectangular grid.*

Proof Take X and v such that $X(v)$ be not defined. Without loss of generality we can assume that $X \in \{L, R\}$ (horizontal and vertical axioms are symmetrical). Furthermore, we can also assume that $X = L$ (since R is the inverse of L, if R is not defined for some element, then so is L).

Let $v + (0, y) = D^y(v)$, where $x \geq 0$ and $y \geq 0$. From the commutativity axiom, $L(v + (0, y))$ is not defined for any y. Indeed, if $L(v + (0, y))$ was defined for $y > 0$, we have $L(v + (0, y))$ and $U(v + (0, y)) = v + (0, y - 1)$ defined, hence $L(v + (0, y - 1))$ is defined too.

Let $b_y = \sum 2^x [B_V(R^x(v + (0, y)))]$. From the Horizontal Increment and No Horizontal Overflow axioms, it is easy to show that $b_{y+1} = b_y + 1$. Furthermore, we have that $b_y < 2^{|V|}$. Therefore, there must exist y such that $D(v + (0, y))$ is not defined. Let $v' = v + (0, y)$. Let x^* be the greatest x such that $R^x(v')$ is defined, and y be the greatest y^* such that $U^y(v')$ is defined. Let $G = \{0, \ldots, x^*\} \times \{0, \ldots, y^*\}$, and for $(x, y) \in G$, let $m(x, y) = R^x(U^y(v'))$. It is straightforward that m gives an isomorphism between the rectangular grid G and V. ∎

3 Forbidding Tori

However, rectangular grids are not the only models of ϕ_1. Consider the torus $T = \{0, \ldots, x^*\} \times \{0, \ldots, y^*\}$, where (x^*, y) is additionally connected (with the R relation) to $(0, y)$, and (x, y^*) is additionally connected to $(x, 0)$ (with the U relation), and we add the respective inverses to L and D. If B_V and B_H are empty relations, the torus T satisifes all of our axioms. Additionally, if G is a model of ϕ_1, then the disjoint union $G \cup T$ is also a model of ϕ_1.

To prevent this, we use the following result of Berger [Ber66].

Theorem 3.1 *There exists a finite set of Wang tiles $K = \{k_1, \ldots, k_t\}$ and relations $T_R, T_D \subseteq K \times K$ such that there exists a tiling $C : \mathbb{Z} \times \mathbb{Z} \to K$ such that the following property holds:*

$$T_R(C(x,y), C(x+1,y)) \wedge T_D(C(x,y), C(x,y+1)) \text{ for each } x, y \in \mathbb{Z}. \quad (1)$$

However, no periodic tiling satisfying 1 holds. A tiling $C : \mathbb{Z} \times \mathbb{Z} \to K$ is **periodic** *iff there exists $(x_0, y_0) \neq (0,0)$ such that $C(x,y) = C(x + x_0, y + y_0)$ for each $x, y \in \mathbb{Z}$.*

The original coloring by Berger used 20426 tiles. It is sufficient to use 11 tiles [JR15].

We add a new relation C for every tile $C \in K$. We also add the following axioms:

- **Full tiling.** $\forall v \bigvee^!_{C \in K} C(v)$. Everything needs to have a color.

- **Correct tiling.** For every pair of tiles $C_1, C_2 \in K$ such that $\neg T_R(C_1, C_2)$, we have $\neg \exists v C_1(v) \wedge C_2(R(v))$. For every pair of tiles $C_1, C_2 \in K$ such that $\neg T_D(C_1, C_2)$, we have $\neg \exists v C_1(v) \wedge C_2(D(v))$.

Let ϕ_2 be the conjuction of ϕ_1 and the axioms above.

Theorem 3.2 *If G is a finite, connected model of ϕ_2 then G is a rectangular grid.*

Proof Take $v \in V$. If one of the relations L, R, U, D is not defined for some $v \in V$, then V is a rectangular grid by Theorem 2.1. Otherwise, let $C(x,y)$, for $x, y \geq 0$, be the relation $C \in K$ which is satisfied by $R^x(D^y(v))$. For $x < 0$ or $y < 0$, replace R^x by L^{-x} or D^y by U^{-y}. According to the correct tiling axiom, the property (1) holds.

Since V is finite, we must have $C(x_1, y_1)$ and $C(x_2, y_2)$ refer to the same element of our structure, even though $(x_1, y_1) \neq (x_2, y_2)$. It is easy to show that $(x_1 - x_2, y_1 - y_2)$ is then the period of the tiling C, which contradicts Theorem 3.1. ∎

Theorem 3.3 *There exists a formula ϕ_3 such that the models of ϕ, restricted to relations L, R, U, D, are precisely the rectangular grids $x^* \times y^*$ such that $y^* \leq 2^{x^*-1}$ and $x^* \leq 2^{y^*-1}$.*

Proof By adding an axiom that there exists exactly one element v^* such that $L(v^*)$ and $U(v^*)$ are not defined, we obtain a formula ϕ_3 whose all finite models are rectangular grids.

Now, take an $x^* \times y^*$ rectangular grid G. From Theorem 3.1 there exists a tiling $C : \mathbb{Z} \times \mathbb{Z} \to K$ satisfying 1. Assign the relation $C(x,y)$ to each $(x,y) \in G$. If $y^* \leq 2^{x^*-1}$ and $x^* \leq 2^{y^*-1}$, we can also set $B_H(x,y)$ iff x-th bit of y is 1,

and $B_V(x, y)$ iff y-th bit of x is 1. Such a model will satisfy ϕ_3. Note that if $x^* > 2^{y^*-1}$ or $y^* > 2^{x^*-1}$, the respective overflow axiom will not be satisfied. ■

The number 2 in the theorem above can be changed to an integer $b \geq 2$ by using b-ary counters instead of the binary ones. However:

Theorem 3.4 There is no formula ϕ over a signature consisting of L, R, U, D, and possibly extra unary relations whose all models restricted to relations L, R, U, D are precisely all rectangular grids. Furthermore, there is no such ϕ such that all models of ϕ are rectangular grids, and there exists $y*$ such that for every $x*$ a rectangular model $x^* \times y^*$ of ϕ exists.

Proof We will be using Hanf's locality lemma [Han65]. Let a r-neighborhood of the vertex $v \in V$, $N_r(v)$ be the set of all vertices whose distance from v is at most r. Let a r-**type** of the vertex v, $\tau(v)$, be the isomorphism type of $N_r(v)$. When we restrict to models of degree bounded by d, there are only finitely many such types. Let T_r be the set of all types. Let $f_{r,M}(G) : T \rightarrow \{0..M\}$ be the function that assigns to each type $\tau \in T$ the minimum of M and the number of vertices of type τ in G.

Theorem 3.5 (Hanf's locality lemma[Han65]) *Let ϕ be a FO formula. Then there exist numbers r and M such that, for each graph $G = (V, E)$, $G \models \phi$ depends only on $f_{r,M}$.*

Let ϕ be a FO formula such that all models of ϕ are rectangular grids. Take r and M from Theorem 3.5. Let the rectangular grid G be a model of phi, where $V(G) = \{0, \dots, x^*\} \times \{0, \dots, y^*\}$. Let $\tau(x)$ be the type of column x, i.e., $\tau(x) = (\tau(x, 0), \dots, \tau(x, y*))$. For sufficiently large x^* there will be x_1 and x_2 such that $\tau(x_1+i) = \tau(x_2+i)$ for $i = -r, \dots, r$ and such that $f_{r,M}(G)(\tau(x, y)) \geq M$ for every $x \in \{x_1, \dots, x_2\}$. Construct a new structure G' by adding a cylinder of dimensions $(x_2 - x_1) \times y*$ to G, i.e., $V(G') = V(G) \cup \{(1, x, y) : x \in \{x_1, \dots, x_2-1\}, y \in \{0, \dots, y*\}, U(1, x, y) = (1, x, y-1), D(1, x, y) = (1, x, y+1), R(1, x, y) = (1, x+1, y), L(1, x, y) = (1, x-1, y), R(1, x_2-1, y) = (1, x_1, y), L(1, x_1, y) = (1, x_2-1, y)$, whenever the point on the right hand side exists, and undefined otherwise. For every unary relation U we have $U(1, x, y)$ iff $U(x, y)$. It is easy to verify that $\tau(1, x, y) = \tau(x, y)$, and every of these types already appeared at least M times, and thus from Theorem 3.5, $G' \models \phi$. ■

4 Forced Planar Spectra

Corollary 4.1 *Let $S \subseteq \mathbb{N}$ be a set such that there exists a non-deterministic Turing machine (or 1DCA) recognizing the set of binary representations of elements of S in time $t(n)$ and memory $s(n)$, where $t(n) \cdot s(n) \leq n$ and $t(n), s(n) \geq \Omega(log(n))$. Then there exists a first-order formula ϕ such that all models of ϕ are planar graphs, and the set of cardinalities of models of ϕ is S.*

Proof Let $A \subseteq \mathbb{N}$, and let M be a non-deterministic Turing machine or a non-deterministic 1DCA recognizing A in time $t(n)$ and space $s(n)$ such that $t(n) \cdot s(n) \leq n$. A non-deterministic 1DCA is $M = (\Sigma, R, F)$ where $R \subseteq \Sigma^4$, and the final symbol $F \in \Sigma$. It is defined similar to a Turing machine, but where computations are performed in parallel on all the tape cells: if $t(x, y)$ is the content of the tape at position x and time y, then the relation $R(t(x - 1, y), t(x, y), t(x+1, y), t(x, y+1))$ must hold. The 1DCA accepts when it writes the symbol F.

Let $u(n) = n - t(n)s(n)$; without loss of generality we can assume $u(n) < t(n)$. It is well known that a first order formula on a grid can be used to simulate a Turing machine (or 1DCA): the bottom row is the initial tape, and our formula ensures that each other row above it is a correct successor of the row below it.

Let $n \in A$. We will construct a formula ϕ which will have a model consisting of:

- A rectangular grid $G' = \{0, \ldots, s\} \times \{0, \ldots, t\}$, where $t = t(n) - 1$ and $s = s(n) - 1$. The structure of the grid is given by relations L, R, U and D just as in Section 2; we also have all the auxiliary relations required by Theorem 3.3.

- $u(n)$ elements which are not in the grid. The relation P will hold for all the extra elements and only for them. The relation Q will hold only for the elements $(0, t - i) \in G'$ where $i \leq u(n) - 1$. The relation B gives a bijection between elements x such that $P(x)$, and the elements x such that $Q(x)$.

- Encoding of the number n. We encode the number n in the leftmost cells in the initial tape using two relations D_n and E_n in the following way: $D_a(x, t)$ is the x-th digit of n, and the relation E_a signifies the end of the encoding: $E_a(x, y) \Rightarrow E_a(x + 1, t) \wedge \neg D_a(x, t)$ (if $(x + 1, t)$ exists).

- Similarly we encode the numbers s, t and u.

- An encoded run of M which accepts the encoded value n as the input.

- An encoded run of an one-dimensional cellular automaton M_2 which verifies that the relation $n = (s + 1) \times (t + 1) + u$ holds for the encoded numbers. A one-dimensional cellular automaton can add and multiply k-digit numbers in time $O(k)$ [Atr65], hence our space s will be sufficient.

- Our grid already has the binary representations of s and t computed as the relations B_H and B_V. In the case of B_V the computed t is already where we need it (we only need to define the relation E_t in the straightforward way). In the case of B_H the computed s is in the rightmost column, so we add extra wiring relations W to move it to the beginning of the initial tape. In the case of u, we need to compute the binary representation of the number of rows i such that $Q(0, i)$; this can be computed in the same way as we have computed the number of all rows (using the relation B_U similar to B_V).
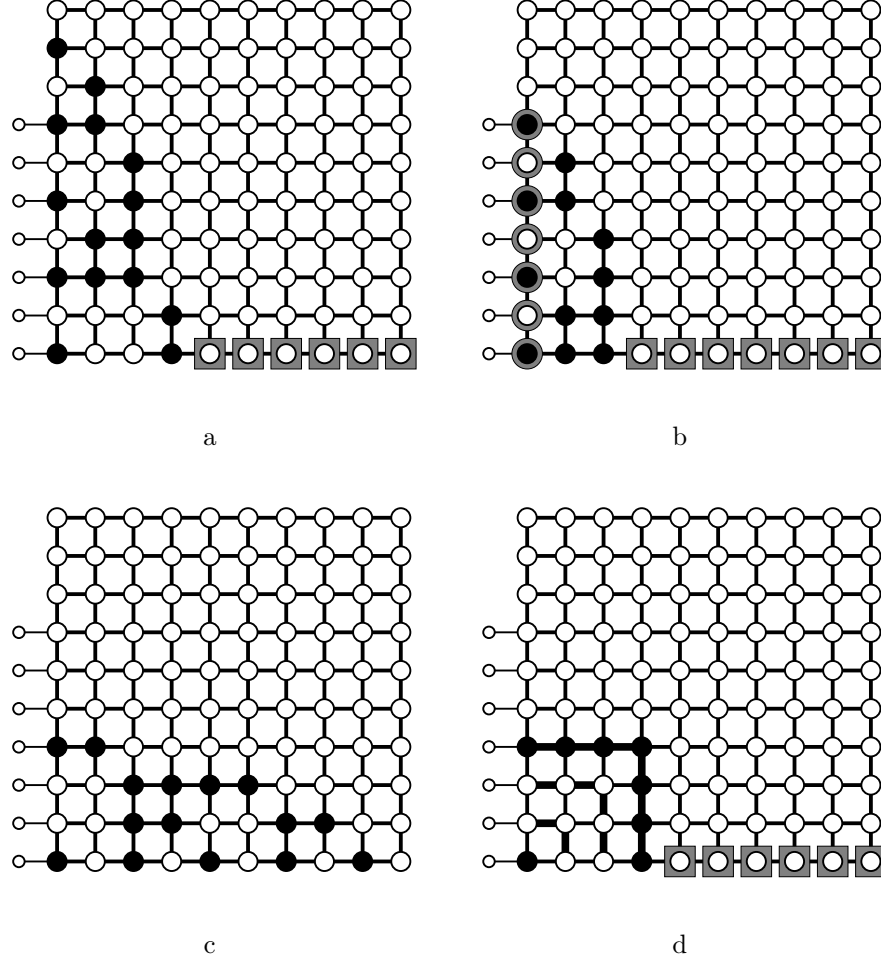
Figure 1: Computing the size of our model.

Figure 1 shows the elements of our construction. In all the pictures, the small circles are the extra elements (where P holds), and the other elements are the grid; the thin lines represent the relation B, the thick lines represent the relations U, D, L and R. In 1a the black circles represent $B_H \equiv D_t$ and gray boxes represent E_t. In 1b the gray circles represent Q, black circles represent B_U and gray circles represent E_U. In 1c the black circles represent B_V, while in 1d the extra thick lines represent W, black circles represent E_s, and gray boxes represent E_s.

The formula ϕ will be the conjuction of the following axioms:

- (1) ϕ_3, restricted to elements for which P does not hold. This requires

that we indeed have a rectangular grid.

- (2) Axiomatiziations of the Turing machine M.

- (3) B is a bijection.

- (4) The set of elements satisfying Q has the correct shape: $Q(v) \Rightarrow \neg P(v) \wedge (\neg \exists y L(x,y)) \wedge (\exists y D(x,y) \Rightarrow Q(y))$,

- (5) Axiomatiziation of B_U, similar to the axiomatization of B_V, but where we add 1 only in the rows y where $Q(0,y)$ holds.

- (6) Axiomatiziation of the wiring W moving s. The axioms are as follows: $W(v,w) \wedge D_t(v) \Rightarrow D_t(w)$; $W(v,w) \Rightarrow W(w,v)$; every v is connected to either (a) only $R(v)$ and $L(v)$ is undefined, (b) only $L(v)$ and $R(v)$; (c) only $L(v)$ and $D(v)$; (d) only $D(v)$ and $U(v)$; (e) only $U(v)$ and $D(v)$ is not defined; (f) nothing. Furthermore, in case (c), $L(D(v))$ must either be also case (c) or the bottom left corner; $D_t(v) \iff B_H(v)$ whenever $L(v)$ is undefined; and the case (a) holds whenever $L(v)$ undefined, $D_t(v)$, and $D(v)$ defined.

- (7) $\forall v D_t(v) \iff V_H(y)$.

- (8) For every encoded number a, $E_a(v) \to (\neg D_a(v) \wedge \exists w R(v,w) \to E_a(w)$.

- (9) Axiomatiziations of the automaton M_2.

Our model satisfies all these axioms.

On the other hand, suppose that ϕ has a model G of size n. By (1) this model constists of a rectangular grid and a number of u extra elements. By (3) and (4) the relation Q is satisfied only for u bottommost elements in the leftmost column. By (5) the encoded number u equals the number of these elements. By (6) and (7) the encoded numbers s and t equal the dimensions of the grid. By (8) and (9) we know that the encoded number n indeed equals the size of G. By (2) we know that M accepts n, therefore $n \in A$.

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