# Deciding semantic finiteness of pushdown processes and first-order grammars w.r.t. bisimulation equivalence \*

Petr Jančar Dept Comp. Sci., FEI, Techn. Univ. Ostrava, Czech Rep.

#### Abstract

The problem if a given configuration of a pushdown automaton (PDA) is bisimilar with some (unspecified) finite-state process is shown to be decidable. The decidability is proven in the framework of first-order grammars, which are given by finite sets of labelled rules that rewrite roots of first-order terms. The framework is equivalent to PDA where also deterministic (i.e. alternative-free) epsilon-steps are allowed, i.e. to the model for which Sénizergues showed an involved procedure deciding bisimilarity (1998, 2005). Such a procedure is here used as a black-box part of the algorithm.

The result extends the decidability of the regularity problem for deterministic PDA that was shown by Stearns (1967), and later improved by Valiant (1975) regarding the complexity. The decidability question for nondeterministic PDA, answered positively here, had been open (as indicated, e.g., by Broadbent and Göller, 2012).

It might be worth to note that the first-order grammars here can be viewed as describing machine behaviours in a more "syntax-independent" way than in the classical works like by Courcelle (1995), and the regularity question is thus more abstract than there.

# 1 Introduction

The question of deciding semantic equivalences of systems, like language equivalence, has been a frequent topic in computer science. A closely related question asks if a given system in a class  $C_1$  has an equivalent in a subclass  $C_2$ . Pushdown automata (PDA) constitute a well-known example; language equivalence and regularity are undecidable for PDA. In the case of deterministic PDA (DPDA), the decidability and complexity results for regularity [16, 18] preceded the famous decidability result for equivalence by Sénizergues [13].

In concurrency theory, logic, verification, and other areas, a finer equivalence, called *bisimulation equivalence* or *bisimilarity*, has emerged as another fundamental behavioural equivalence; on deterministic systems it essentially coincides with language equivalence. An on-line survey of the results which study this equivalence in a specific area of process rewrite systems is maintained by Srba [15].

One of the most involved results in this area is the decidability of bisimilarity for pushdown processes generated by (nondeterministic) PDA in which  $\varepsilon$ -steps are restricted so that each  $\varepsilon$ -step has no alternative (and can be restricted to be popping); this result was shown by Sénizergues [14] who thus generalized his above mentioned result for DPDA. There is no known upper bound on the complexity of this decidable problem. The nonelementary lower

<sup>\*</sup>This paper contains a full and simplified version of the proof sketched in Proc. of MFCS'16.

bound established in [1] is, in fact, TOWER-hardness in the terminology of [12], and it holds even for real-time PDA, i.e. PDA with no  $\varepsilon$ -steps. For the above mentioned PDA with restricted  $\varepsilon$ -steps the bisimilarity problem is even not primitive recursive; its Ackermann-hardness is shown in [6]. In the deterministic case, the equivalence problem is known to be PTIME-hard, and has a primitive recursive upper bound shown by Stirling [17] (where a finer analysis places the problem in TOWER [6]).

Extrapolating the deterministic case, we might expect that for PDA the "regularity" problem w.r.t. bisimilarity (asking if a given PDA-configuration is bisimilar with a state in a finite-state system) is decidable as well, and that this problem might be easier than the equivalence problem solved in [14]; only EXPTIME-hardness is known here (see [10], and [15] for detailed references). Nevertheless, this decidability question has been open so far, as also indicated in [2] (besides [15]).

Contribution of this paper. We show that semantic finiteness of pushdown configurations w.r.t. bisimilarity is decidable. The decidability is proven in the framework of first-order grammars, i.e. of finite sets of labelled rules that rewrite roots of first-order terms. Though we do not use (explicit)  $\varepsilon$ -steps, the framework is equivalent to the model of PDA with restricted  $\varepsilon$ -steps for which Sénizergues's general decidability proof [14] applies. (A simplified proof directly in the first-order grammar framework, hence an alternative to the proof in [14], is given in [5].)

The presented algorithm, answering if a given configuration, i.e. a first-order term  $E_0$  in the labelled transition system generated by a first-order grammar, has a bisimilar finite-state system, uses the result of [14] (or of [5]) as a black-box procedure. By [6] we cannot get a primitive recursive upper bound via a black-box use of the decision procedure for bisimilarity.

Semidecidability of the semantic finiteness problem has been long clear, hence it is the existence of finite effectively verifiable witnesses of the negative case that is the crucial point here. It turns out that a witness of semantic infiniteness of a term (i.e., of a configuration)  $E_0$ is a specific path  $E_0 \xrightarrow{u} \xrightarrow{w}$  in the respective labelled transition system where the sequence w of actions can be repeated forever. The idea how to verify if the respective infinite path  $E_0 \xrightarrow{u} \xrightarrow{w} \xrightarrow{w} \xrightarrow{w} \cdots$ , denoted  $E_0 \xrightarrow{u} \xrightarrow{w^{\omega}}$ , visits terms (configurations) from infinitely many equivalence classes is to consider the "limit term" LIM that is "reached" by  $E_0 \xrightarrow{u} \xrightarrow{w^{\omega}}$ ; the term LIM is generally infinite but regular (i.e., it has only finitely many subterms). The (black-box) procedure deciding equivalence is used for computing a finite number e such that we are guaranteed that if  $E_0 \xrightarrow{u} \xrightarrow{w^e}$  does not reach a term equivalent to LIM then  $E_0 \xrightarrow{u} \xrightarrow{w^k}$ does not reach such a term for any  $k \ge e$ . In this case the path  $E_0 \xrightarrow{u} \xrightarrow{w^{\omega}}$  indeed visits terms in infinitely many equivalence classes since the visited terms approach LIM syntactically and thus also semantically (by increasing the "equivalence-level" with Lim) but never belong to the equivalence class of Lim. To show the existence of a respective witness  $E_0 \xrightarrow{u} \xrightarrow{w}$  for each semantically infinite  $E_0$  is not trivial but it can done by a detailed study of the paths  $E_0 \xrightarrow{a_1} E_1 \xrightarrow{a_2} E_2 \xrightarrow{a_3} \cdots$  where  $E_i$  are from pairwise different equivalence classes.

Remark on the relation to other uses of first-order grammars. In this paper the first-order grammars are used for slightly different aims than in the works on higher-order grammars (or higher-order recursion schemes) and higher-order pushdown automata, where the first order is a particular case; we can exemplify such works by [4, 9], while many other references can be found, e.g., in the survey papers [11, 19]. There a grammar is used to describe an infinite labelled tree (the syntax tree of an infinite applicative term produced by a unique outermost derivation from an initial nonterminal), and the questions like, e.g., the decidability

of monadic second-order (MSO) properties for such trees are studied. In this paper, a first-order grammar can be also seen as a tool describing an infinite tree, namely the tree-unfolding of a nondeterministic labelled transition system with an initial state. The question if this tree is regular (i.e., if it has only finitely many subtrees) would correspond to the regularity question studied in [16, 18]; but here we ask a different question, namely if identifying bisimilar subtrees results in a regular tree.

We can also note that the question if a given first-order grammar generates a regular tree refers to a particular formalism (namely to the respective infinite applicative term) while the regularity question studied here is more "syntax-independent."

Some further remarks are given at the end of Section 2 and in Section 4.

# 2 Basic Notions, and Result

In this section we define the basic notions and state the result in the form of a theorem. Some standard definitions are restricted when we do not need the full generality. We finish the section by a note about a transformation of pushdown automata to first-order grammars.

By N and N<sub>+</sub> we denote the sets of nonnegative integers and of positive integers, respectively. By [i,j], for  $i,j\in\mathbb{N}$ , we denote the set  $\{i,i+1,\ldots,j\}$ . For a set  $\mathcal{A}$ , by  $\mathcal{A}^*$  we denote the set of finite sequences of elements of  $\mathcal{A}$ , which are also called words (over  $\mathcal{A}$ ). By |w| we denote the length of  $w\in\mathcal{A}^*$ , and by  $\varepsilon$  the empty sequence; hence  $|\varepsilon|=0$ . We put  $\mathcal{A}^+=\mathcal{A}^*\setminus\{\varepsilon\}$ ,  $w^0=\varepsilon$ , and  $w^{j+1}=ww^j$  for  $j\in\mathbb{N}$ ;  $w^\omega$  denotes the infinite sequence  $www\cdots$ .

**Labelled transition systems.** A labelled transition system, an LTS for short, is a tuple  $\mathcal{L} = (\mathcal{S}, \Sigma, (\stackrel{a}{\longrightarrow})_{a \in \Sigma})$  where  $\mathcal{S}$  is a finite or countable set of states,  $\Sigma$  is a finite set of actions (or letters), and  $\stackrel{a}{\longrightarrow} \subseteq \mathcal{S} \times \mathcal{S}$  is a set of a-transitions (for each  $a \in \Sigma$ ). We say that  $\mathcal{L}$  is a deterministic LTS if for each pair  $s \in \mathcal{S}$ ,  $a \in \Sigma$  there is at most one s' such that  $s \stackrel{a}{\longrightarrow} s'$  (which stands for  $(s, s') \in \stackrel{a}{\longrightarrow}$ ). By  $s \stackrel{w}{\longrightarrow} s'$ , where  $w = a_1 a_2 \dots a_n \in \Sigma^*$ , we denote that there is a path  $s = s_0 \stackrel{a_1}{\longrightarrow} s_1 \stackrel{a_2}{\longrightarrow} s_2 \cdots \stackrel{a_n}{\longrightarrow} s_n = s'$ ; if  $s \stackrel{w}{\longrightarrow} s'$ , then s' is reachable from s. By  $s \stackrel{w}{\longrightarrow}$  we denote that w is enabled in s, i.e.,  $s \stackrel{w}{\longrightarrow} s'$  for some s'. If  $\mathcal{L}$  is deterministic, then  $s \stackrel{w}{\longrightarrow} s'$  and  $s \stackrel{w}{\longrightarrow}$  also denote a unique path.

**Bisimilarity.** Given  $\mathcal{L} = (\mathcal{S}, \Sigma, (\stackrel{a}{\longrightarrow})_{a \in \Sigma})$ , a set  $\mathcal{B} \subseteq \mathcal{S} \times \mathcal{S}$  covers  $(s, t) \in \mathcal{S} \times \mathcal{S}$  if for any  $s \stackrel{a}{\longrightarrow} s'$  there is  $t \stackrel{a}{\longrightarrow} t'$  such that  $(s', t') \in \mathcal{B}$ , and for any  $t \stackrel{a}{\longrightarrow} t'$  there is  $s \stackrel{a}{\longrightarrow} s'$  such that  $(s', t') \in \mathcal{B}$ . For  $\mathcal{B}, \mathcal{B}' \subseteq \mathcal{S} \times \mathcal{S}$  we say that  $\mathcal{B}'$  covers  $\mathcal{B}$  if  $\mathcal{B}'$  covers each  $(s, t) \in \mathcal{B}$ . A set  $\mathcal{B} \subseteq \mathcal{S} \times \mathcal{S}$  is a bisimulation if  $\mathcal{B}$  covers  $\mathcal{B}$ . States  $s, t \in \mathcal{S}$  are bisimilar, written  $s \sim t$ , if there is a bisimulation  $\mathcal{B}$  containing (s, t). A standard fact is that  $\sim \subseteq \mathcal{S} \times \mathcal{S}$  is an equivalence relation, and it is the largest bisimulation, namely the union of all bisimulations.

E.g., in the LTS in Fig. 1 we have  $s_3 \sim s_4$  and  $s_1 \not\sim s_2$  (though  $s_1, s_2$  are trace-equivalent, i.e., the sets  $\{w \mid s_1 \xrightarrow{w}\}$  and  $\{w \mid s_2 \xrightarrow{w}\}$  are the same).

**Semantic finiteness.** Given  $\mathcal{L} = (\mathcal{S}, \Sigma, (\stackrel{a}{\longrightarrow})_{a \in \Sigma})$ , we say that  $s_0 \in \mathcal{S}$  is finite up to bisimilarity, or bisim-finite for short, if there is some state f in some finite LTS such that  $s_0 \sim f$ ; otherwise  $s_0$  is infinite up to bisimilarity, or bisim-infinite. We should add that when comparing states from different LTSs, we implicitly refer to the disjoint union of these LTSs.

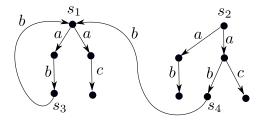


Figure 1: Example of a finite (nondeterministic) labelled transition system

First-order terms, regular terms, finite graph presentations. We will consider LTSs with countable sets of states in which the states are first-order regular terms. (In Fig. 1 the states are depicted as unstructured black dots; Fig. 6 depicts three states of an LTS with terms "inside the black dots".)

The terms are built from variables taken from a fixed countable set

$$VAR = \{x_1, x_2, x_3, \dots\}$$

and from function symbols, also called (ranked) nonterminals, from some specified finite set  $\mathcal{N}$ ; each  $A \in \mathcal{N}$  has  $arity(A) \in \mathbb{N}$ . We reserve symbols A, B, C, D to range over nonterminals, and E, F, G, H to range over terms.

On the left in Fig. 2 we can see the syntactic tree of a term  $E_1$ , namely of  $E_1 = A(D(x_5, C(x_2, B)), x_5, B)$ , where the arities of nonterminals A, B, C, D are 3, 0, 2, 2, respectively. The numbers at the arcs just highlight the fact that the outgoing arcs of each node are ordered.

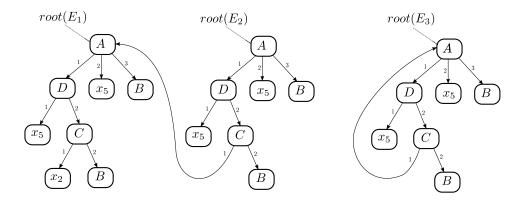


Figure 2: Finite terms  $E_1$ ,  $E_2$ , and a graph presenting a regular infinite term  $E_3$ 

We identify terms with their syntactic trees. Thus a term over  $\mathcal{N}$  is (viewed as) a rooted, ordered, finite or infinite tree where each node has a label from  $\mathcal{N} \cup VAR$ ; if the label of a node is  $x_i \in VAR$ , then the node has no successors, and if the label is  $A \in \mathcal{N}$ , then it has m (immediate) successor-nodes where m = arity(A). A subtree of a term E is also called a subterm of E. We make no difference between isomorphic (sub)trees, and thus a subterm can have more (maybe infinitely many) occurrences in E. Each subterm-occurrence has its (nesting) depth in E, which is its (naturally defined) distance from the root of E.

E.g., the term  $C(x_2, B)$  is a subterm of the term  $E_1$  in Fig. 2, with one depth-2 occurrence; it is also a subterm of  $E_2$ , with one depth-5 occurrence. The term B has two occurrences in  $E_1$ , one in depth 1 and another in depth 3, and four occurrences in  $E_2$ , in depths 1, 3, 4, 6.

We also use the standard notation for terms: we write  $E = x_i$  or  $E = A(G_1, \ldots, G_m)$  with the obvious meaning; in the latter case ROOT $(E) = A \in \mathcal{N}$ , m = arity(A), and  $G_1, \ldots, G_m$  are the ordered depth-1 occurrences of subterms of E, which are also called the *root-successors* in E.

A term is finite if the respective tree is finite. A (possibly infinite) term is regular if it has only finitely many subterms (though the subterms may be infinite and can have infinitely many occurrences). We note that any regular term has at least one graph presentation, i.e. a finite directed graph, with a designated root, where each node has a label from  $\mathcal{N} \cup \text{VAR}$ ; if the label of a node is  $x_i \in \text{VAR}$ , then the node has no outgoing arcs, if the label is  $A \in \mathcal{N}$ , then it has m ordered outgoing arcs where m = arity(A).

We can see an example of such a graph presenting a term  $E_3$  on the right in Fig. 2.

The standard tree-unfolding of the graph is the respective term, which is infinite if there are cycles in the graph. There is a bijection between the nodes in the *least* graph presentation of E and (the roots of) the subterms of E.

To get the least presentation of  $E_3$  in Fig. 2, we should unify the roots of the same subterms, in our case the nodes labelled with B and the nodes labelled with  $x_5$ . We can also note that  $E_3$  contains infinitely many occurrences of itself, in depths  $0, 3, 6, \ldots$ .

**Convention.** In what follows, by a "term" we mean a "regular term" unless the context makes clear that the term is finite. (We do not consider non-regular terms.) By Terms<sub>N</sub> we denote the set of all (regular) terms over a set N of (ranked) nonterminals (and over the set Var of variables). As already said, we reserve symbols A, B, C, D to range over nonterminals, and E, F, G, H to range over (regular) terms.

#### Substitutions, associative composition, iterated (eligible) substitutions.

A substitution  $\sigma$  is a mapping  $\sigma: VAR \to TERMS_{\mathcal{N}}$  whose support

$$SUPP(\sigma) = \{x_i \mid \sigma(x_i) \neq x_i\}$$

is finite; we reserve the symbol  $\sigma$  for substitutions. By applying a substitution  $\sigma$  to a term E we get the term  $E\sigma$  that arises from E by replacing each occurrence of  $x_i$  with  $\sigma(x_i)$ ; given graph presentations, in the graph of E we just redirect each arc leading to  $x_i$  towards the root of  $\sigma(x_i)$  (which includes the special "root-designating arc" when  $E = x_i$ ). Hence  $E = x_i$  implies  $E\sigma = x_i \sigma = \sigma(x_i)$ .

E.g., for the terms in Fig. 2, by applying the substitution  $\sigma = \{(x_2, E_1)\}$  (by which we denote that  $\sigma$  satisfies  $x_2\sigma = E_1$  and  $x_i\sigma = x_i$  for all  $i \neq 2$ ) to  $E_1$  we get the term  $E_1\sigma$ , which is  $E_2$ .

The natural composition of substitutions, where  $\sigma = \sigma_1 \sigma_2$  is defined by  $x_i \sigma = (x_i \sigma_1) \sigma_2$ , can be easily verified to be associative. We thus write simply  $E \sigma_1 \sigma_2$  when meaning  $(E \sigma_1) \sigma_2$  or  $E(\sigma_1 \sigma_2)$ . We let  $\sigma^0$  be the empty-support substitution, and we put  $\sigma^{i+1} = \sigma \sigma^i$ . We will also use the limit substitution

$$\sigma^{\omega} = \sigma \sigma \sigma \cdots$$

when this is well defined, i.e., when there is no "unguarded cycle"  $x_{i_1}\sigma = x_{i_2}, x_{i_2}\sigma = x_{i_3}, \ldots, x_{i_{k-1}}\sigma = x_{i_k}, x_{i_k}\sigma = x_{i_1}$  where  $x_{i_1} \neq x_{i_2}$ . In fact, we will use  $\sigma^{\omega}$  only for eligible substitutions  $\sigma$ ; we say that  $\sigma$  is *eligible* if  $x_j\sigma = x_i$  implies  $x_i\sigma = x_i$ , for all  $j, i \in \mathbb{N}_+$ . (The reason for this definition will be clarified later.)

Operationally, to get some graph presentations of terms  $x_i\sigma^{\omega}$  from some graph presentations of  $x_i\sigma$  (for all  $x_i \in \text{SUPP}(\sigma)$  for an eligible substitution  $\sigma$ ), we redirect any arc leading to  $x_i$ , where  $x_i \in \text{SUPP}(\sigma)$ , towards the root of (the presentation of)  $x_i\sigma$ .

In Fig. 2, for  $\sigma = \{(x_2, E_1)\}$  we have  $E_1 \sigma = E_2$  and  $E_1 \sigma \sigma \sigma \cdots = E_1 \sigma^{\omega} = E_3$ ; we also note that  $\sigma^{\omega} = \{(x_2, E_3)\}$  and that  $x_2$  does not occur in  $E_3$ .

We note that no variable  $x_i \in \text{SUPP}(\sigma)$  occurs in any term  $E\sigma^{\omega}$ ; such variables "disappear" by applying  $\sigma^{\omega}$ ; hence  $E\sigma^{\omega}$  can only contain variables  $x_i$  for which  $x_i\sigma = x_i$ .

At the top left in Fig. 3 we can see a depiction of the substitution  $\sigma$  defined by

$$x_1\sigma = B(C(x_4, x_2), x_3), \ x_3\sigma = C(x_1, A(C(x_4, x_2), B(x_3, x_5), x_3, B(x_3, x_5), x_4))),$$
  
 $x_4\sigma = x_2, x_6\sigma = B(x_3, x_5), x_7\sigma = C(x_5, x_6).$ 

Fig. 3 also depicts explicitly that  $x_2\sigma = x_2$  and  $x_5\sigma = x_5$ ; each dashed line connects a variable  $x_i$  (above the bar) with the root of the term  $x_i\sigma$ . The substitution  $\sigma$  is eligible ( $x_4\sigma = x_2$  requires  $x_2\sigma = x_2$ , which indeed holds), and at the bottom right we can see a graph presentation of the substitution  $\sigma^{\omega}$ .

At the bottom left we can see a more transparent depiction of  $\sigma^{\omega}$ . Here we also use an auxiliary depiction device, namely some "fictitious" nodes that are not labelled with nonterminals or variables. Such a node in our figures can be called a *collector node*: it might "collect" several incoming arcs that are in reality deemed to proceed to the target specified by the (precisely one) outgoing arc of the collector node. (E.g., the arc from the rightmost node C in Fig. 3 to B is depicted as going through two collector nodes in the bottom left part.)

We have noted that no  $x_i \in \text{SUPP}(\sigma)$  (in our case  $\text{SUPP}(\sigma) = \{x_1, x_3, x_4, x_6, x_7\}$ ) can occur in  $x_j \sigma^{\omega}$  (for any j). In our example,  $x_1, x_3, x_4$  occur in some of the terms from the set  $\{x_i \sigma^k \mid x_i \in \text{SUPP}(\sigma)\}$  for any  $k \in \mathbb{N}$ , "drowning" to increasing depths with increasing k, but they disappear in the limit (since not occurring in any term from the set  $\{x_i \sigma^{\omega} \mid x_i \in \text{SUPP}(\sigma)\}$ ). The variables  $x_7$  and  $x_6$  disappear "earlier":  $x_7$  does not occur in any  $x_j \sigma$ , and  $x_6$  does not occur in any  $x_j \sigma \sigma$ .

**First-order grammars.** The set  $TERMS_{\mathcal{N}}$  (of regular terms over a finite set  $\mathcal{N}$  of nonterminals) will serve us as the set of states of an LTS. The transitions will be determined by a finite set of (schematic) *root-rewriting* rules, illustrated in Fig. 4. This is now defined formally.

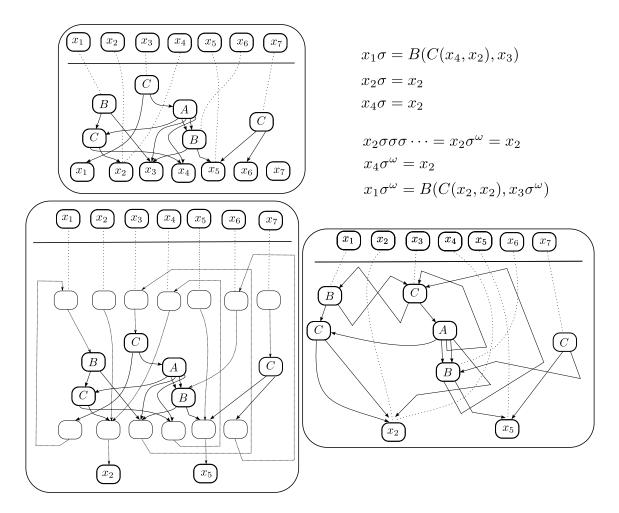


Figure 3: Depictions of a substitution  $\sigma$  and of the substitution  $\sigma^{\omega}$ 

Remark. As mentioned in the introduction (at the end Section 1), we use first-order grammars for describing labelled transition systems (LTSs); given an initial state (i.e. a term in our case), a grammar can be viewed as describing the tree-unfolding of the respective LTS from the initial state.

A first-order grammar, or just a grammar for short, is a tuple  $\mathcal{G} = (\mathcal{N}, \Sigma, \mathcal{R})$  where  $\mathcal{N} = \{A_1, A_2, \dots, A_{|\mathcal{N}|}\}$  is a finite set of ranked nonterminals, viewed as function symbols with arities,  $\Sigma = \{a_1, a_2, \dots, a_{|\Sigma|}\}$  is a finite set of actions (or letters), and  $\mathcal{R} = \{r_1, r_2, \dots, r_{|\mathcal{R}|}\}$  is a finite set of rules of the form

$$A(x_1, x_2, \dots, x_m) \stackrel{a}{\longrightarrow} E$$
 (1)

where  $A \in \mathcal{N}$ , arity(A) = m,  $a \in \Sigma$ , and E is a finite term over  $\mathcal{N}$  in which each occurring variable is from the set  $\{x_1, x_2, \ldots, x_m\}$ . A rule of the form (1) is called a *sink rule* if  $E = x_i$  (for some  $i \in [1, m]$ ); otherwise it is a *non-sink rule*.

Fig. 4 shows a non-sink rule,  $A(x_1, x_2, x_3) \xrightarrow{b} C(A(x_2, x_1, B), x_2)$ , and a sink rule,  $A(x_1, x_2, x_3) \xrightarrow{b} x_3$ . The depiction stresses that the variables  $x_1, x_2, x_3$  serve as the "place-holders" for the "root-successors" (RS), i.e. the depth-1 occurrences of

subterms of a term with the root A; the (root of the) term might be rewritten by performing action b (as defined below).

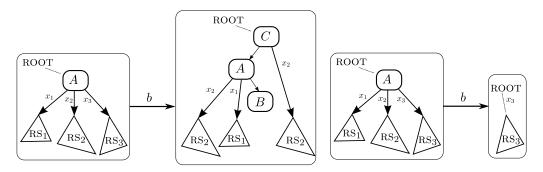


Figure 4: Depiction of rules  $A(x_1, x_2, x_3) \xrightarrow{b} C(A(x_2, x_1, B), x_2)$  and  $A(x_1, x_2, x_3) \xrightarrow{b} x_3$ 

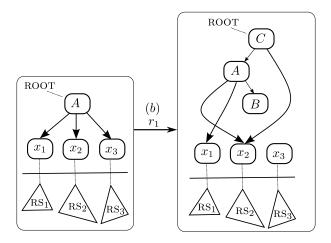


Figure 5: Another presentation of the rule  $r_1: A(x_1, x_2, x_3) \xrightarrow{b} C(A(x_2, x_1, B), x_2)$ 

LTSs generated by grammars. Given  $\mathcal{G} = (\mathcal{N}, \Sigma, \mathcal{R})$ , by  $\mathcal{L}_{\mathcal{G}}^{\mathbb{R}}$  we denote the (rule-based) LTS  $\mathcal{L}_{\mathcal{G}}^{\mathbb{R}} = (\text{Terms}_{\mathcal{N}}, \mathcal{R}, (\stackrel{r}{\longrightarrow})_{r \in \mathcal{R}})$  where each rule r of the form  $A(x_1, x_2, \dots, x_m) \stackrel{a}{\longrightarrow} E$  induces transitions  $A(x_1, \dots, x_m) \sigma \stackrel{r}{\longrightarrow} E \sigma$  for any substitution  $\sigma$ . The transition induced by  $\sigma$  with  $\text{SUPP}(\sigma) = \emptyset$  is  $A(x_1, \dots, x_m) \stackrel{r}{\longrightarrow} E$ .

Fig. 5 shows another presentation of the rule  $A(x_1, x_2, x_3) \xrightarrow{b} C(A(x_2, x_1, B), x_2)$  (from the left of Fig. 4), denoted  $r_1$ . It makes more explicit that the application of the same substitution to both sides yields a transition; it also highlights the fact that RS<sub>3</sub> "disappears" by applying the rule since it loses the connection with the root.

Fig. 6 shows two examples of transitions in  $\mathcal{L}_{\mathcal{G}}^{\mathbb{R}}$ . One is generated by the non-sink rule  $r_1$  of the form  $A(x_1, x_2, x_3) \xrightarrow{b} C(A(x_2, x_1, B), x_2)$  (depicted in Figures 4

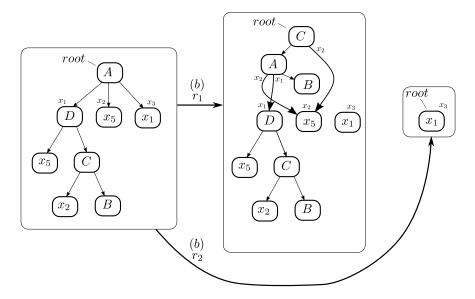


Figure 6: One  $r_1$ -transition and one  $r_2$ -transition in  $\mathcal{L}_{\mathcal{G}}^{\mathbb{R}}$  (both labelled with b)

and 5), and the other by the sink rule  $r_2$  of the form  $A(x_1, x_2, x_3) \xrightarrow{b} x_3$ ; in both cases we apply the substitution  $\sigma = \{(x_1, D(x_5, C(x_2, B))), (x_2, x_5), (x_3, x_1)\}$  to (the both sides of) the respective rule. The small symbols  $x_1, x_2, x_3$  in Fig. 6 are only auxiliary, highlighting the use of our rules, and they are no part of the respective terms. The middle term is here given by an (acyclic) graph presentation of its syntactic tree (and the node with label  $x_1$  is no part of it).

Fig. 7 shows the transitions resulting by the applications of two rules to a graph presenting an infinite regular term. (The small symbols  $x_1, x_2, x_3$  are again just auxiliary.)

By definition the LTS  $\mathcal{L}_{\mathcal{G}}^{\mathbb{R}}$  is deterministic (for each F and r there is at most one H such that  $F \xrightarrow{r} H$ ). We note that variables are dead (have no outgoing transitions). We also note that  $F \xrightarrow{w} H$  implies that each variable occurring in H also occurs in F (but not necessarily vice versa).

Since the rhs (right-hand sides) E in the rules (1) are finite, all terms reachable from a finite term are finite. It is convenient to have the rhs finite while including regular terms into our LTSs; the other options are in principle equivalent.

The deterministic rule-based LTS  $\mathcal{L}_{\mathcal{G}}^{\mathbb{R}}$  is helpful technically, but we are primarily interested in the (generally nondeterministic) action-based LTS  $\mathcal{L}_{\mathcal{G}}^{\mathbb{A}} = (\text{Terms}_{\mathcal{N}}, \Sigma, (\stackrel{a}{\longrightarrow})_{a \in \Sigma})$  where each rule  $A(x_1, \ldots, x_m) \stackrel{a}{\longrightarrow} E$  induces the transitions  $A(x_1, \ldots, x_m) \sigma \stackrel{a}{\longrightarrow} E \sigma$  for all substitutions  $\sigma$ .

Figures 6 and 7 also show transitions in  $\mathcal{L}_{\mathcal{G}}^{A}$ , when we ignore the symbols  $r_1, r_2, r_3$  and consider the "labels" b, a instead. Figure 6 also exemplifies nondeterminism in  $\mathcal{L}_{\mathcal{G}}^{A}$ , since there are two different outgoing b-transitions from a state.

Given a grammar  $\mathcal{G} = (\mathcal{N}, \Sigma, \mathcal{R})$ , two terms from Terms, are bisimilar if they are bisimilar as states in the action-based LTS  $\mathcal{L}_{\mathcal{G}}^{\Lambda}$ . By our definitions all variables are bisimilar,

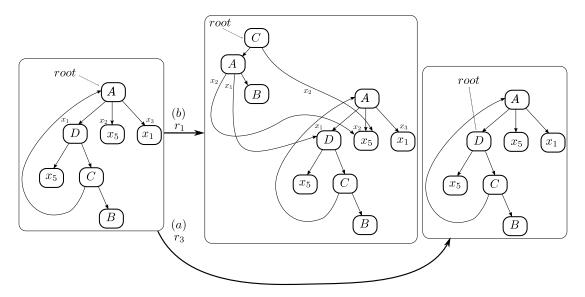


Figure 7: Applying  $r_1: A(x_1, x_2, x_3) \xrightarrow{b} C(A(x_2, x_1, B), x_2)$  and  $r_3: A(x_1, x_2, x_3) \xrightarrow{a} x_1$  to a graph of an (infinite regular) term

since they are dead terms. The variables serve us primarily as "place-holders for subtermoccurrences" in terms (which might themselves be variable-free); such a use of variables as place-holders has been already exemplified in the rules (1).

Main result, and its relation to pushdown automata. We now state the theorem, to be proven in the next section, and we mention why the result also applies to pushdown automata (PDA) with deterministic popping  $\varepsilon$ -steps.

**Theorem 1.** There is an algorithm that, given a grammar  $\mathcal{G} = (\mathcal{N}, \Sigma, \mathcal{R})$  and (a finite presentation of) a term  $E_0 \in \text{Terms}(\mathcal{N})$ , decides if  $E_0$  is bisim-finite (i.e., if  $E_0 \sim f$  for a state f in some finite LTS).

A transformation of (nondeterministic) PDA in which deterministic popping  $\varepsilon$ -steps are allowed to first-order grammars (with no  $\varepsilon$ -steps) is recalled in the appendix (at the end of this paper).

This makes clear that the semantic finiteness of PDA with deterministic popping  $\varepsilon$ -steps (w.r.t. bisimilarity) is also decidable. In fact, the problems are interreducible; the close relationship between (D)PDA and first-order schemes has been long known (see, e.g., [3]). The proof of Theorem 1 presented here uses the fact that bisimilarity of first-order grammars is decidable; this was shown for the above mentioned PDA model by Sénizergues [14], and a direct proof in the first-order-term framework was presented in [5].

We note that for PDA where popping  $\varepsilon$ -steps can be in conflict with "visible" steps bisimilarity is already undecidable [7]; hence the proof presented here does not yield the decidability of semantic finiteness in this more general model. The decidability status of semantic finiteness is also unclear for second-order PDA (that operate on a stack of stacks; besides the standard work on the topmost stack, they can also push a copy of the topmost stack or to pop the topmost stack in one move). Bisimilarity is undecidable for second-order PDA even without any use of  $\varepsilon$ -steps [2] (some remarks are also added in [8]).

# 3 Proof of Theorem 1

## 3.1 Computability of eq-levels, and semidecidability of bisim-finiteness

We will soon note that the semidecidability of bisim-finiteness is clear, but we first recall the computability of eq-levels, which is one crucial ingredient in our proof of semidecidability of bisim-infiniteness.

Stratified equivalence, and eq-levels. Assuming an LTS  $\mathcal{L} = (\mathcal{S}, \Sigma, (\stackrel{a}{\longrightarrow})_{a \in \Sigma})$ , we put  $\sim_0 = \mathcal{S} \times \mathcal{S}$ , and define  $\sim_{k+1} \subseteq \mathcal{S} \times \mathcal{S}$  (for  $k \in \mathbb{N}$ ) as the set of pairs covered by  $\sim_k$ . (Hence  $s \sim_{k+1} t$  iff for any  $s \stackrel{a}{\longrightarrow} s'$  there is  $t \stackrel{a}{\longrightarrow} t'$  such that  $s' \sim_k t'$  and for any  $t \stackrel{a}{\longrightarrow} t'$  there is  $s \stackrel{a}{\longrightarrow} s'$  such that  $s' \sim_k t'$ .)

We easily verify that  $\sim_k$  are equivalence relations, and that  $\sim_0 \supseteq \sim_1 \supseteq \sim_2 \supseteq \cdots \supseteq \sim_k$ . For the (first infinite) ordinal  $\omega$  we put  $s \sim_\omega t$  if  $s \sim_k t$  for all  $k \in \mathbb{N}$ ; hence  $\sim_\omega = \cap_{k \in \mathbb{N}} \sim_k$ . It is standard (and can be easily checked) that  $\cap_{k \in \mathbb{N}} \sim_k$  is a bisimulation in image-finite LTSs, and thus  $\sim = \cap_{k \in \mathbb{N}} \sim_k = \sim_\omega$  for them. We recall that  $\mathcal{L}$  is image-finite if the set  $\{s' \mid s \xrightarrow{a} s'\}$  is finite for each pair  $s \in \mathcal{S}$ ,  $a \in \Sigma$ . Our grammar-generated LTSs  $\mathcal{L}_{\mathcal{G}}^{\mathbb{A}}$  are obviously image-finite (while  $\mathcal{L}_{\mathcal{G}}^{\mathbb{R}}$  are even deterministic); we thus further assume image-finiteness.

To each pair of states s, t we attach their equivalence level (eq-level):

$$EQLV(s,t) = \max\{k \in \mathbb{N} \cup \{\omega\} \mid s \sim_k t\}.$$

In Fig. 1 we have, e.g.,  $EQLV(s_1, s_3) = 0$ ,  $EQLV(s_1, s_2) = 1$ ,  $EQLV(s_3, s_4) = \omega$ .

Eq-levels are computable for first-order grammars. We now state an important lemma that follows easily from the involved decidability proof in [14] (and a transformation to first-order grammars); as already mentioned, a proof given directly for the first-order grammars was presented in [5]. (The lemma is surely a fundamental theorem in general, here the name lemma has been chosen to reflect that it is a prerequisite for Theorem 1 proven in this paper.)

**Lemma 2.** There is an algorithm that, given  $\mathcal{G} = (\mathcal{N}, \Sigma, \mathcal{R})$  and  $E_0, F_0 \in \text{Terms}(\mathcal{N})$ , computes  $\text{EqLv}(E_0, F_0)$  in  $\mathcal{L}_{\mathcal{G}}^{\Lambda}$  (and thus also decides if  $E_0 \sim F_0$ ).

*Proof.* The question  $E_0 \stackrel{?}{\sim} F_0$ , i.e.  $\text{EQLV}(E_0, F_0) \stackrel{?}{=} \omega$ , can be decided by [14] (and [5]). In the case  $\text{EQLV}(E_0, F_0) \neq \omega$ , a straightforward brute-force algorithm finds the least  $k+1 \in \mathbb{N}$  such that  $E_0 \not\sim_{k+1} F_0$ , thus finding that  $\text{EQLV}(E_0, F_0) = k$ .

**Semidecidability of bisim-finiteness.** Given  $\mathcal{G}$  and  $E_0$ , we can systematically generate all finite LTSs, presenting them by first-order grammars with nullary nonterminals (which then coincide with states); for each state f of each generated system we can check if  $E_0 \sim f$  by Lemma 2. In fact, Lemma 2 is not crucial here, since the decidability of  $E_0 \sim f$  can be shown in a much simpler way (see, e.g., [10]).

## 3.2 Semidecidability of bisim-infiniteness

In Section 3.2.1 we note a few simple general facts on bisim-infiniteness, and also note the obvious compositionality (congruence properties) of bisimulation equivalence in our framework of first-order terms.

In Section 3.2.2 we describe some finite structures that are candidates for witnessing bisim-infiniteness of a given term  $E_0$ ; such a candidate is, in fact, a rule sequence uw such that the infinite (ultimately periodic) word  $uw^{\omega}$  is performable from  $E_0$  in  $\mathcal{L}_{\mathcal{G}}^{\mathbb{R}}$ . Then we show an algorithm checking if a candidate is indeed a witness, i.e., if the respective infinite path  $E_0 \xrightarrow{u} \xrightarrow{w} \xrightarrow{w} \xrightarrow{w} \cdots$  visits terms from infinitely many equivalence classes. The crucial idea is that we can naturally define a (regular) term, called the limit LIM =  $E_{\omega}$ , that could be viewed as "reached" from  $E_0$  by performing the infinite word  $uw^{\omega}$ . The terms  $E_j$  such that  $E_0 \xrightarrow{u} \xrightarrow{w^j} E_j$  will approach  $E_{\omega}$  syntactically with increasing j ( $E_j$  coincides with  $E_{\omega}$  up to depth j at least), which also entails that EqLv( $E_j, E_{\omega}$ ) will grow above any bound. If we can verify that EqLv( $E_j, E_{\omega}$ ) are finite for infinitely many j, in particular if EqLv( $E_j, E_{\omega}$ ) never reaches  $\omega$  (hence  $E_j \not\sim E_{\omega}$  for all j), then uw is indeed a witness of bisim-infiniteness of  $E_0$ ; Lemma 2 will play an important role in such a verification.

In Section 3.2.3 we show that each bisim-infinite term has a witness of the above form. By this a proof of Theorem 1 will be finished.

### 3.2.1 Some general facts on bisim-infiniteness, and compositionality of terms

**Bisimilarity quotient.** Given an LTS  $\mathcal{L} = (\mathcal{S}, \Sigma, (\stackrel{a}{\longrightarrow})_{a \in \Sigma})$ , the quotient-LTS  $\mathcal{L}_{\sim}$  is the tuple  $(\{[s]_{\sim} \mid s \in \mathcal{S}\}, \Sigma, (\stackrel{a}{\longrightarrow})_{a \in \Sigma})$  where  $[s]_{\sim} = \{s' \mid s' \sim s\}$ , and  $[s]_{\sim} \stackrel{a}{\longrightarrow} [t]_{\sim}$  if  $s' \stackrel{a}{\longrightarrow} t'$  for some  $s' \in [s]_{\sim}$  and  $t' \in [t]_{\sim}$ ; in fact,  $[s]_{\sim} \stackrel{a}{\longrightarrow} [t]_{\sim}$  implies that for each  $s' \in [s]_{\sim}$  there is  $t' \in [t]_{\sim}$  such that  $s' \stackrel{a}{\longrightarrow} t'$ . We have  $s \sim [s]_{\sim}$ , since  $\{(s, [s]_{\sim}) \mid s \in \mathcal{S}\}$  is a bisimulation (in the union of  $\mathcal{L}$  and  $\mathcal{L}_{\sim}$ ). We refer to the states of  $\mathcal{L}_{\sim}$  as to the bisim-classes (of  $\mathcal{L}$ ).

A sufficient condition for bisim-infiniteness. We recall that  $s_0 \in \mathcal{S}$  is bisim-finite if there is some state f in a finite LTS such that  $s_0 \sim f$ ; otherwise  $s_0$  is bisim-infinite. We observe that  $s_0$  is bisim-infinite in  $\mathcal{L}$  iff the reachability set of  $[s_0]_{\sim}$  in  $\mathcal{L}_{\sim}$ , i.e. the set of states reachable from  $[s_0]_{\sim}$  in  $\mathcal{L}_{\sim}$ , is infinite.

reachable from  $[s_0]_{\sim}$  in  $\mathcal{L}_{\sim}$ , is infinite. An LTS  $\mathcal{L} = (\mathcal{S}, \Sigma, (\stackrel{a}{\longrightarrow})_{a \in \Sigma})$  is *finitely branching* if the set  $\{s' \mid s \stackrel{a}{\longrightarrow} s' \text{ for some } a\}$  is finite for each  $s \in \mathcal{S}$ . The LTSs generated by first-order grammars are finitely branching, and thus the following fact applies to them:

**Proposition 3.** A state  $s_0$  of a finitely branching LTS is bisim-infinite iff there is an infinite path  $s_0 \xrightarrow{a_1} s_1 \xrightarrow{a_2} s_2 \xrightarrow{a_3} \cdots$  where  $s_i \not\sim s_j$  for all  $i \neq j$ .

Proof. The "if" direction is trivial; our goal is thus to show the "only if" direction. For a finitely branching LTS  $\mathcal{L}$  and its state  $s_0$  we consider the reachability tree in  $\mathcal{L}_{\sim}$  where  $[s_0]_{\sim}$  is the root (and the successors of a node  $[s]_{\sim}$  constitute the set  $\{[s']_{\sim} \mid [s]_{\sim} \xrightarrow{a} [s']_{\sim}$  for some  $a \in \Sigma\}$ ); we "trim" the tree so that we finish each branch when it visits a state (i.e., a bisimclass) for the second time (if it happens). If  $s_0$  is bisim-infinite in  $\mathcal{L}$ , then the reachability set of  $[s_0]_{\sim}$  is infinite, and by König's Lemma there is an infinite branch  $[s_0]_{\sim} \xrightarrow{a_1} [s_1]_{\sim} \xrightarrow{a_2} [s_2]_{\sim} \xrightarrow{a_3} \cdots$  in our trimmed tree. We thus have a path  $s_0 \xrightarrow{a_1} s'_1 \xrightarrow{a_2} s'_2 \xrightarrow{a_3} \cdots$  in  $\mathcal{L}$  where  $s'_i \in [s_i]_{\sim}$ ; this path proves the claim.

To demonstrate that  $s_0$  is bisim-infinite, it suffices to show that its reachability set contains states with arbitrarily large *finite* eq-levels w.r.t. a "test state" t; we now formalize this observation.

**Proposition 4.** Given  $\mathcal{L} = (\mathcal{S}, \Sigma, (\stackrel{a}{\longrightarrow})_{a \in \Sigma})$  and states  $s_0, t$ , if for every  $e \in \mathbb{N}$  there is s' that is reachable from  $s_0$  and satisfies  $e < \text{EQLV}(s', t) < \omega$ , then  $s_0$  is bisim-infinite.

*Proof.* We note that  $s_1 \sim_k s_2$ ,  $s_2 \sim_k t$  implies  $s_1 \sim_k t$ , and  $s_1 \sim_k s_2$ ,  $s_2 \not\sim_k t$  implies  $s_1 \not\sim_k t$  (since  $\sim_k$  are equivalence relations). Hence  $s_1 \sim s_2$  implies that  $\mathrm{EQLv}(s_1,t) = \mathrm{EQLv}(s_2,t)$  for any t. Thus the assumption of the claim implies that from  $s_0$  we can reach states from infinitely many bisim-classes.

Eq-levels yielded by states in a bounded region and test states. Our final general observation (tailored to a later use) is also straightforward: if two states are bisimilar, then the states in their equally bounded reachability regions must yield the same eq-levels when compared with states from a fixed (test) set. This observation is informally depicted in Fig. 8, and formalized in what follows. (Despite the depiction in Fig. 8, the test states can be also inside the regions.)

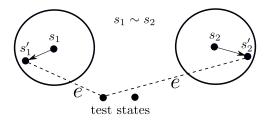


Figure 8: Bounded regions of bisimilar states yield the same eq-levels w.r.t. test states

For any  $s \in \mathcal{S}$  and  $d \in \mathbb{N}$  (a distance, or a "radius") we put

REGION
$$(s, d) = \{s' \mid s \xrightarrow{w} s' \text{ for some } w \in \Sigma^* \text{ where } |w| \le d\}.$$

For any  $s \in \mathcal{S}$ ,  $d \in \mathbb{N}$ , and  $\mathcal{T} \subseteq \mathcal{S}$  (a test set), we define the following subset of  $\mathbb{N}$  (finite TestEqLevels):

$$\text{TEL}(s, d, \mathcal{T}) = \{e \in \mathbb{N} \mid e = \text{EqLv}(s', t) \text{ for some } s' \in \text{Region}(s, d) \text{ and some } t \in \mathcal{T}\}.$$

For  $X \subseteq \mathbb{N}$ , by the supremum  $\sup(X)$  we mean -1 if  $X = \emptyset$ ,  $\max(X)$  if X is finite and nonempty, and  $\omega$  if X is infinite. (The next proposition will be later applied to the LTSs  $\mathcal{L}_{\mathcal{G}}^{\mathbb{A}}$  with finite test sets, hence the sets  $\operatorname{REGION}(s,d)$  and  $\operatorname{TEL}(s,d,\mathcal{T})$  will be finite.)

**Proposition 5.** If  $s_1 \sim s_2$ , then  $\text{TEL}(s_1, d, \mathcal{T}) = \text{TEL}(s_2, d, \mathcal{T})$  for all  $d \in \mathbb{N}$  and  $\mathcal{T} \subseteq \mathcal{S}$ , which also entails that  $\sup(\text{TEL}(s_1, d, \mathcal{T})) = \sup(\text{TEL}(s_2, d, \mathcal{T}))$ .

Proof. Suppose  $s_1 \sim s_2$  and  $s_1' \in \text{REGION}(s_1, d)$ ; let  $s_1 \xrightarrow{w} s_1'$  where  $|w| \leq d$ . From the definition of bisimilarity we deduce that  $s_2 \xrightarrow{w} s_2'$  for some  $s_2'$  such that  $s_1' \sim s_2'$ ; we have  $s_2' \in \text{REGION}(s_2, d)$ . Since  $s_1' \sim s_2'$  implies  $\text{EQLV}(s_1', t) = \text{EQLV}(s_2', t)$  for any t (as noted in the proof of Prop. 4), the claim is clear.

Remark. The fact that  $s_1 \sim s_2$  and  $s_1 \stackrel{a}{\longrightarrow} s_1'$  implies that there is  $s_2'$  such that  $s_2 \stackrel{a}{\longrightarrow} s_2'$  and  $s_1' \sim s_2'$  is a crucial property of bisimilarity that we use for our decision procedure. Hence our approach does not apply to trace equivalence or simulation equivalence. (E.g., the states  $s_1, s_2$  in Fig. 1 are trace equivalent but their a-successors are from pairwise different trace equivalence classes.) On the other hand, the below mentioned compositionality of bisimilarity holds for other equivalences as well.

Compositionality of the states of the grammar-generated LTSs. Regarding the congruence properties, in principle it suffices for us to observe the fact depicted in Fig. 9: if in a term E we replace a subterm F with F' such that  $F' \sim F$  then the resulting term E' satisfies  $E' \sim E$ .

Figure 9: Replacing a subterm with an equivalent term does not change the bisim-class

Hence we also have that  $A(G_1, \ldots, G_m) \not\sim A(G'_1, \ldots, G'_m)$  implies  $G_i \not\sim G'_i$  for some  $i \in [1, m]$ . Formally, we put  $\sigma \sim \sigma'$  if  $x_i \sigma \sim x_i \sigma'$  for each  $x_i$ , and we note:

**Proposition 6.** If  $\sigma \sim \sigma'$ , then  $E\sigma \sim E\sigma'$ .

(Hence  $E\sigma \nsim E\sigma'$  implies that  $x_i\sigma \nsim x_i\sigma'$  for some  $x_i$  occurring in E.)

*Proof.* By induction on k it is obvious that  $\sigma \sim_k \sigma'$  (meaning  $x_i \sigma \sim_k x_i \sigma'$  for each  $x_i$ ) implies  $E \sigma \sim_k E \sigma'$ . Since  $\sigma \sim \sigma'$  iff  $\sigma \sim_k \sigma'$  for all  $k \in \mathbb{N}$ , we are done.

#### Conventions.

• To make some later discussions easier, we further consider only the normalized grammars  $\mathcal{G} = (\mathcal{N}, \Sigma, \mathcal{R})$ , i.e. those satisfying the following condition: for any  $A(x_1, \ldots, x_m)$  and any  $i \in [1, m]$  there is a word  $w_{(A,i)}$  such that  $A(x_1, \ldots, x_m) \xrightarrow{w_{(A,i)}} x_i$ ; hence for any E it is possible to "sink" to any of its subterm-occurrences by applying a sequence of the grammar-rules. (E.g., in the middle term in Fig. 7 we can "sink" to the root-successor term with the root A by applying some rule sequence  $u_1$ , then "to D" by applying some  $u_2$ , then to C by some  $u_3$ , then to A by some  $u_4$ , ...) This does not exclude that by applying another rule sequence the respective subterm "disappears", by losing the connection to the root. (E.g., in Fig. 6 by applying  $r_1$  the third root-successor disappears, while by applying  $r_2$  the first two root-successors disappear.)

Such a normalization can be efficiently achieved by harmless modifications of the non-terminal arities and of the rules in  $\mathcal{R}$ , while the bisimilarity quotient of the LTS  $\mathcal{L}_{\mathcal{G}}^{A}$  remains the same (up to isomorphism). Now we simply assume this, the details are given in Appendix.

- In our notation we use m as the arity of all nonterminals in the considered grammar, though m is deemed to denote the maximum arity, in fact. Formally we could replace our expressions of the form  $A(G_1, \ldots, G_m)$  with  $A(G_1, \ldots, G_{m_A})$  where  $m_A = arity(A)$ , and adjust the respective discussions accordingly, but it would be unnecessarily cumbersome. In fact, such uniformity of arities can be even achieved by a construction while keeping the previously discussed normalization condition, when a slight problem with arity 0 is handled. The details are also given in Appendix.
- For technical convenience we further view the expressions like  $G \xrightarrow{w} H$  as referring to the deterministic LTS  $\mathcal{L}_{\mathcal{G}}^{\mathbb{R}}$  (hence  $w \in \mathcal{R}^*$  and any expression  $G \xrightarrow{w}$  refers to a unique path in  $\mathcal{L}_{\mathcal{G}}^{\mathbb{R}}$ ), while  $\sim_k$ ,  $\sim$ , and the eq-levels are always considered w.r.t. (the action-based LTS)  $\mathcal{L}_{\mathcal{G}}^{\mathbb{A}}$ .

#### 3.2.2 Witnesses of bisim-infiniteness

Assuming a grammar  $\mathcal{G} = (\mathcal{N}, \Sigma, \mathcal{R})$ , we now describe candidates for witnesses of bisiminfiniteness of terms; we start with defining a technical notion of (eligible) stairs.

Stairs, eligible stairs. A nonempty sequence of rules  $w = r_1 r_2 \dots r_\ell \in \mathcal{R}^+$  is a stair if we have  $A(x_1, \dots, x_m) \xrightarrow{w} F$  where F has a nonterminal root (note that  $A(x_1, \dots, x_m)$  is the left-hand side of the rule  $r_1$ ); in this case the path  $A(x_1, \dots, x_m) \xrightarrow{w} F$  does not "sink" to a root-successor  $x_i$ , and we can write  $A(x_1, \dots, x_m) \xrightarrow{w} B(x_1, \dots, x_m) \sigma$  for some  $B \in \mathcal{N}$  and some substitution  $\sigma$ .

E.g., the sequence  $r_1r_2r_3$  of rules used in the path

$$A(G_1, G_2, G_3) \xrightarrow{r_1} C_1(D(G_3, B), G_2) \xrightarrow{r_2} C_2(G_2, D(G_3, B)) \xrightarrow{r_3} D(G_3, B)$$

(an instance of 
$$A(x_1, x_2, x_3) \xrightarrow{r_1} C_1(D(x_3, B), x_2) \xrightarrow{r_2} C_2(x_2, D(x_3, B)) \xrightarrow{r_3} D(x_3, B)$$
)

is a stair;  $r_2$  is also a stair, but  $r_2r_3$  and  $r_3$  are no stairs.

We have defined a substitution  $\sigma$  as eligible if  $x_j\sigma = x_i$  implies  $x_i\sigma = x_i$  (for all  $x_j \in VAR$ ); in this case  $\sigma^{\omega}$  is well defined (recall Fig. 3).

We say that a stair  $w \in \mathbb{R}^+$  is eligible (for "pumping") if  $A(x_1, \ldots, x_m) \xrightarrow{w} A(x_1, \ldots, x_m) \sigma$  for some  $A \in \mathcal{N}$  and some eligible substitution  $\sigma$ . In such a case we have  $A(x_1, \ldots, x_m) \xrightarrow{w^k} A(x_1, \ldots, x_m) \sigma^k$  for all  $k \in \mathbb{N}$  and, moreover, the term  $A(x_1, \ldots, x_m) \sigma^\omega$  is well defined. We note that the variables occurring in  $A(x_1, \ldots, x_m) \sigma^\omega$  are precisely those  $x_i \in \{x_1, \ldots, x_m\}$  for which  $x_i \sigma = x_i$ .

Remark. We can note that each stair w of the type  $A(x_1, \ldots, x_m) \xrightarrow{w} A(x_1, \ldots, x_m) \sigma$  can be "pumped"; we have  $A(x_1, \ldots, x_m) \xrightarrow{w^k} A(x_1, \ldots, x_m) \sigma^k$  in this case, but  $A(x_1, \ldots, x_m) \sigma^\omega$  might be undefined (e.g., when  $x_1 \sigma = x_2$  and  $x_2 \sigma = x_1$ ). Our restriction to eligible substitutions (where  $x_j \sigma = x_i$  implies  $x_i \sigma = x_i$ ) in the "pumped stairs" w in the candidates  $E_0 \xrightarrow{w} \xrightarrow{w}$  for witnesses (defined below) is technically convenient, and still allows us to prove the existence of such witnesses (in Section 3.2.3).

For instance, the sequence w in Figures 10 and 11 is an eligible stair, where  $A(x_1, \ldots, x_5) \xrightarrow{w} A(x_1, \ldots, x_5)\sigma$  and for the only case  $x_j\sigma = x_i$  with  $j \neq i$ , namely  $x_4\sigma = x_2$ , we indeed have  $x_i\sigma = x_i$  (i.e.,  $x_2\sigma = x_2$ ).

Here we can note that  $x_1, \ldots, x_5$  occur in  $A(x_1, \ldots, x_5)\sigma^j$  for all  $j \in \mathbb{N}$ , but only  $x_2, x_5$  occur in  $A(x_1, \ldots, x_5)\sigma^\omega$  (as is also implicitly depicted in Fig. 12).

Another example of an eligible stair (for another grammar) could be of the form  $A(x_1, \ldots, x_7) \xrightarrow{w} A(x_1, \ldots, x_7) \sigma$  where  $\sigma$  is depicted in Fig. 3. In this case  $x_7$  does not occur in  $A(x_1, \ldots, x_7) \sigma^j$  for  $j \geq 1$ ,  $x_6$  does not occur in  $A(x_1, \ldots, x_7) \sigma^j$  for  $j \geq 2$ ,  $x_1, x_2, \ldots, x_5$  occur in  $A(x_1, \ldots, x_7) \sigma^j$  for all  $j \in \mathbb{N}$ , and only  $x_2, x_5$  occur in  $A(x_1, \ldots, x_7) \sigma^\omega$ .

For an eligible stair w, of the form  $A(x_1, \ldots, x_m) \xrightarrow{w} A(x_1, \ldots, x_m) \sigma$ , we define the terms  $G_{(w,z)}$  for all  $z \in \mathbb{N} \cup \{\omega\}$  by putting

$$G_{(w,z)} = A(x_1,\ldots,x_m)\sigma^z.$$

We have noted that  $A(x_1, \ldots, x_m) \xrightarrow{w^j} G_{(w,j)}$  for all  $j \in \mathbb{N}$ ; this entails  $A(x_1, \ldots, x_m)\sigma_0 \xrightarrow{w^j} G_{(w,j)}\sigma_0$  for any substitution  $\sigma_0$ .

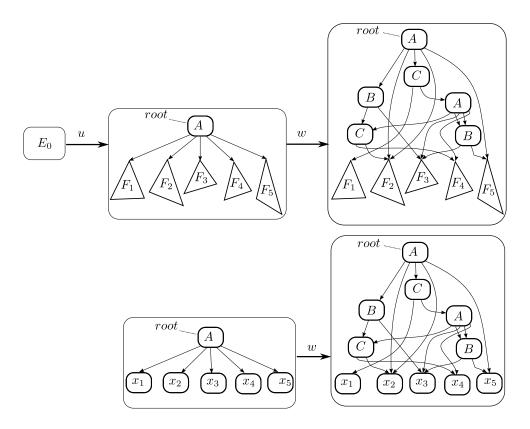


Figure 10: Candidate for a witness of bisim-infiniteness of  $E_0$ 

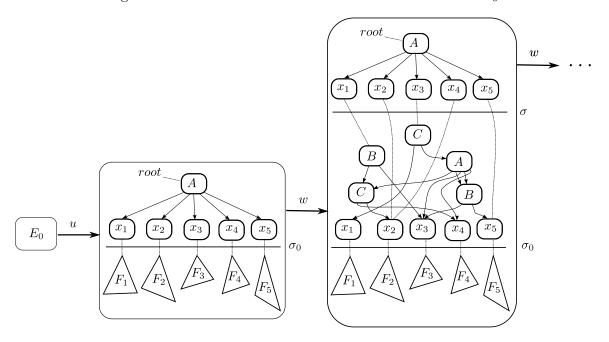


Figure 11: Candidate induces the infinite path  $E_0 \xrightarrow{u} \xrightarrow{w} \xrightarrow{w} \cdots$ 

Candidates for witnesses of bisim-infiniteness. Given a grammar  $\mathcal{G} = (\mathcal{N}, \Sigma, \mathcal{R})$ , by a candidate for a witness of bisim-infiniteness of a term  $E_0$ , or by a candidate for  $E_0$  for

short, we mean a pair (u, w) where  $u \in \mathcal{R}^*$ ,  $w \in \mathcal{R}^+$ ,  $E_0 \xrightarrow{uw}$ , and w is an eligible stair. (Figures 10 and 11 show an example.)

For such a candidate (u, w) we thus have  $E_0 \xrightarrow{u} A(x_1, \dots, x_m)\sigma_0$  and  $A(x_1, \dots, x_m) \xrightarrow{w} A(x_1, \dots, x_m)\sigma$  for some nonterminal A and some substitutions  $\sigma_0, \sigma$ , where  $\sigma$  is eligible; moreover, there is the infinite path

$$E_0 \xrightarrow{u} G_{(w,0)} \sigma_0 \xrightarrow{w} G_{(w,1)} \sigma_0 \xrightarrow{w} G_{(w,2)} \sigma_0 \xrightarrow{w} G_{(w,3)} \sigma_0 \xrightarrow{w} \cdots$$

We will now observe that for increasing  $k \in \mathbb{N}$  the terms  $G_{(w,k)}$  converge (syntactically and semantically) to the term  $G_{(w,\omega)}$ ; thus also  $G_{(w,k)}\sigma_0$  converge to the "limit" term  $G_{(w,\omega)}\sigma_0$ . This is illustrated in Fig. 12, where some of the auxiliary "collector nodes" have special labels to ease a future discussion.

Tops of terms  $G_{(w,k)}\sigma_0$  converge to  $G_{(w,\omega)}\sigma_0$ . For a term H and  $d \in \mathbb{N}$ , by  $\operatorname{Top}_d(H)$  (the d-top of H) we refer to the tree corresponding to H up to depth d. Hence  $\operatorname{Top}_0(H)$  is the tree consisting solely of the root labelled with  $\operatorname{ROOT}(H)$ . For d > 0, we have  $\operatorname{Top}_d(x_i) = \operatorname{Top}_0(x_i)$ , and  $\operatorname{Top}_d(A(G_1, \ldots, G_m)) = A(\operatorname{Top}_{d-1}(G_1), \ldots, \operatorname{Top}_{d-1}(G_m))$ , which denotes the (ordered labelled) tree with the A-labelled root and with the (ordered) depth-1 subtrees  $\operatorname{Top}_{d-1}(G_1), \ldots, \operatorname{Top}_{d-1}(G_m)$ . (Hence  $\operatorname{Top}_d(H)$  is not a term in general, since it arises by "cutting-off" the depth-(d+1) subterm-occurrences.) We also define  $\operatorname{Top}_{-1}(H)$  as the "empty tree", and use the consequence that  $\operatorname{Top}_{-1}(H_1) = \operatorname{Top}_{-1}(H_2)$  for all  $H_1, H_2$ .

The next observation is trivial, due to the root-rewriting form of the transitions in the grammar-generated labelled transition systems.

**Proposition 7.** For any 
$$k \in \mathbb{N}$$
, if  $ToP_{k-1}(H_1) = ToP_{k-1}(H_2)$ , then  $H_1 \sim_k H_2$ .

Proof. We proceed by induction on k; for k = 0 the claim is trivial. If  $\operatorname{TOP}_k(H_1) = \operatorname{TOP}_k(H_2)$  for  $k \geq 0$  (hence  $\operatorname{ROOT}(H_1) = \operatorname{ROOT}(H_2)$  in particular), and  $H_1 \stackrel{r}{\longrightarrow} H_1'$  (in the LTS  $\mathcal{L}_{\mathcal{G}}^{\mathrm{R}}$ ), then we trivially have  $H_2 \stackrel{r}{\longrightarrow} H_2'$  where  $\operatorname{TOP}_{k-1}(H_1') = \operatorname{TOP}_{k-1}(H_2')$ , and  $H_1' \sim_k H_2'$  by the induction hypothesis. Hence  $\operatorname{TOP}_k(H_1) = \operatorname{TOP}_k(H_2)$  implies  $H_1 \sim_{k+1} H_2$ .

The next proposition captures another obvious fact: for increasing  $j \in \mathbb{N}$ , the term  $G_{(w,j)}\sigma_0$  (illustrated on the left of Fig. 12) converges to the limit term LIM =  $G_{(w,\omega)}\sigma_0$  (on the right of Fig. 12). This also entails that for "large"  $j \in \mathbb{N}$  the term determined by "the first-row collector node"  $p_{1i}$  coincides with its counterpart determined by  $q_i$  in the limit term up to a large depth. In Fig. 12 we can also note that some occurrences of  $F_2$ ,  $F_5$  keep being present in  $G_{(w,j)}\sigma_0$  in a bounded distance from the root (being always among root-successors, in fact), and they also occur in LIM. On the other hand, though  $F_1$ ,  $F_3$ ,  $F_4$  also keep present in  $G_{(w,j)}\sigma_0$ , they are stepwise "drowning" (the depths of their shallowest occurrences are stepwise increasing with increasing j); these "drowning" terms  $F_1$ ,  $F_3$ ,  $F_4$  are "completely drowned" in LIM, i.e., they do not occur there.

**Proposition 8.** Let w be an eligible stair, where  $A(x_1, \ldots, x_m) \xrightarrow{w} A(x_1, \ldots, x_m) \sigma$ , and let  $\sigma_0$  be a substitution. Using the notation  $G_{(w,z)} = A(x_1, \ldots, x_m) \sigma^z$  (for  $z \in \mathbb{N} \cup \{\omega\}$ ) and LIM  $= G_{(w,\omega)}\sigma_0$ , the following conditions hold for all  $k \in \mathbb{N}$  and  $i \in [1, m]$ :

1. 
$$\operatorname{ToP}_{k-1}(x_i\sigma^k\sigma_0) = \operatorname{ToP}_{k-1}(x_i\sigma^\omega\sigma_0)$$
; hence  $\operatorname{ToP}_k(G_{(w,k)}\sigma_0) = \operatorname{ToP}_k(\operatorname{Lim})$ .

2. EQLV
$$(x_i \sigma^k \sigma_0, x_i \sigma^\omega \sigma_0) \ge k$$
 and EQLV $(G_{(w,k)} \sigma_0, \text{LIM}) \ge k+1$ .

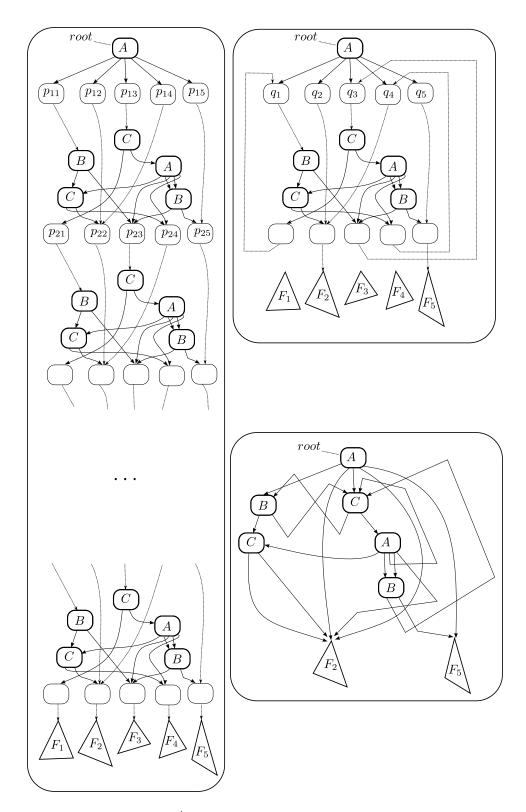


Figure 12:  $A(x_1,x_2,x_3,x_4,x_5)\sigma^j\sigma_0$  (left) and two presentations of  $A(x_1,x_2,x_3,x_4,x_5)\sigma^\omega\sigma_0$ 

*Proof.* 1. For k=0 the claim is trivial, so we assume  $k \geq 1$ . If  $x_i\sigma = x_j$ , then  $x_j\sigma = x_j$  (by the eligibility property), and  $x_i\sigma^k = x_i\sigma^\omega = x_j$ , which implies that the terms  $x_i\sigma^k\sigma_0$  and  $x_i\sigma^\omega\sigma_0$  are both equal to  $x_j\sigma_0$ . If  $x_i\sigma$  has a nonterminal root, then clearly

$$\operatorname{Top}_0((x_i\sigma)\sigma^{k-1}\sigma_0) = \operatorname{Top}_0((x_i\sigma)\sigma^{\omega}\sigma_0).$$

Using now the induction hypothesis

$$\operatorname{Top}_{k-2}(x_j\sigma^{k-1}\sigma_0) = \operatorname{Top}_{k-2}(x_j\sigma^{\omega}\sigma_0) \text{ for all } j \in [1, m],$$

we deduce for any  $i \in [1, m]$  that  $\text{ToP}_{k-1}((x_i \sigma) \sigma^{k-1} \sigma_0) = \text{ToP}_{k-1}((x_i \sigma) \sigma^{\omega} \sigma_0)$ ; hence

$$\operatorname{Top}_{k-1}(x_i\sigma^k\sigma_0) = \operatorname{Top}_{k-1}(x_i\sigma^\omega\sigma_0) \text{ for all } i \in [1, m].$$

Now we easily deduce that  $\operatorname{Top}_k(A(x_1,\ldots,x_m)\sigma^k\sigma_0) = \operatorname{Top}_k(A(x_1,\ldots,x_m)\sigma^\omega\sigma_0)$ , which can be written as  $\operatorname{Top}_k(G_{(w,k)}\sigma_0) = \operatorname{Top}_k(\operatorname{Lim})$  in our notation.

2. Using Prop. 7, the claim follows from the point 1.

Checking if a candidate is a witness. A candidate (u, w) for  $E_0$ , yielding the path  $E_0 \xrightarrow{u} G_{(w,0)}\sigma_0 \xrightarrow{w} G_{(w,1)}\sigma_0 \xrightarrow{w} G_{(w,2)}\sigma_0 \xrightarrow{w} G_{(w,3)}\sigma_0 \xrightarrow{w} \cdots$ , is a witness (of bisiminfiniteness) for  $E_0$  if  $G_{(w,k)}\sigma_0 \not\sim \text{LIM}$  for infinitely many  $k \in \mathbb{N}$ , where

$$Lim = G_{(w,\omega)}\sigma_0.$$

Since  $\text{EqLv}(G_{(w,k)}\sigma_0, \text{Lim}) > k$  (Prop. 8(2)), we then have

$$e < \text{EQLV}(G_{(w,e)} \sigma_0, \text{Lim}) < \omega \text{ for infinitely many } e \in \mathbb{N},$$

and Prop. 4 thus confirms that  $E_0$  is indeed bisim-infinite if it has a witness.

The existence of an algorithm checking if a candidate is a witness follows from the next lemma (which we prove by using the fundamental fact captured by Lemma 2).

**Lemma 9.** Given an eligible stair  $A(x_1, ..., x_m) \xrightarrow{w} A(x_1, ..., x_m)\sigma$ , and a term  $F_i$  for each  $i \in I = \{\ell \in [1, m]; x_{\ell}\sigma = x_{\ell}\}$ , there is a computable number  $e \in \mathbb{N}$  such that for any  $\sigma_0$  satisfying  $x_i\sigma_0 = F_i$  for all  $i \in I$  one of the following conditions holds, using the notation  $G_{(w,z)} = A(x_1, ..., x_m)\sigma^z$  (for  $z \in \mathbb{N} \cup \{\omega\}$ ) and  $\text{Lim} = G_{(w,\omega)}\sigma_0$ :

- 1.  $G_{(w,e)}\sigma_0 \not\sim \text{Lim}$ , in which case  $G_{(w,k)}\sigma_0 \not\sim \text{Lim}$  for all integer  $k \geq e$ , or
- 2.  $G_{(w,e)}\sigma_0 \sim \text{Lim}$ , in which case  $G_{(w,k)}\sigma_0 \sim \text{Lim}$  for all  $k \geq e$ .

We note that the term LIM is independent of the terms  $x_i\sigma_0$  where  $i \notin I$  (since the variables  $x_i, i \notin I$ , do not occur in  $G_{(w,\omega)} = A(x_1,\ldots,x_m)\sigma^{\omega}$ ). If we have a candidate (u,w), where  $E_0 \xrightarrow{u} A(x_1,\ldots,x_m)\sigma_0 \xrightarrow{w} A(x_1,\ldots,x_m)\sigma\sigma_0$ , and the above condition 1  $(G_{(w,e)}\sigma_0 \not\sim \text{LIM})$  holds (for w, and  $F_i = x_i\sigma_0$  for  $i \in I$ ), then (u,w) is clearly a witness; in the case 2 it is no witness. Now we prove the lemma.

*Proof.* Assume an eligible stair  $A(x_1, \ldots, x_m) \xrightarrow{w} A(x_1, \ldots, x_m) \sigma$ , and a substitution  $\sigma_0$ , putting  $F_i = x_i \sigma_0$  when  $x_i \sigma = x_i$  (for  $i \in [1, m]$ ). Now for  $\ell \in \mathbb{N}_+$  we put

$$\mathcal{V}_{\ell} = \{x_i \mid x_i \text{ occurs in } A(x_1, \dots, x_m) \sigma^{\ell-1}\}.$$

Hence  $\mathcal{V}_1 = \{x_1, \dots, x_m\} \supseteq \mathcal{V}_2 \supseteq \mathcal{V}_3 \supseteq \dots$ . Let  $\ell_0$  be the least number such that  $\mathcal{V}_{\ell_0} = \mathcal{V}_{\ell_0+1}$  (hence  $\ell_0 \leq m+1$ ); we then have  $\mathcal{V}_{\ell_0} = \mathcal{V}_{\ell_0+1} = \mathcal{V}_{\ell_0+2} = \dots$ .

We can surely compute  $\ell_0$ , and also a number  $d \in \mathbb{N}$  so that any  $x_i \in \mathcal{V}_{\ell_0}$  belongs to both of the (reachability) regions REGION $(A(x_1,\ldots,x_m)\sigma^{\ell_0-1},d)$  and REGION $(A(x_1,\ldots,x_m)\sigma^{\ell_0},d)$ .

For  $\sigma$  in Figures 11 and 12 we have  $\ell_0 = 1$ ; for the case depicted in Fig. 3 we would get  $\ell_0 = 3$ . In both examples we have  $\mathcal{V}_{\ell_0} = \{x_1, x_2, x_3, x_4, x_5\}$ . The radius d is chosen so that each (term determined by any) collector node in the  $\ell_0$ -th row and the  $(\ell_0+1)$ -th row is reachable from  $G_{(w,j)}\sigma_0$  within d steps (when  $j \geq \ell_0$ ).

We define the test set  $\mathcal{T} = \{x_i \sigma^{\omega} \sigma_0 \mid x_i \in \mathcal{V}_{\ell_0}\}$ , and compute the maximum test-eq-level  $\text{MAX}_{\text{TEL}} = \sup(\text{TEL}(\text{Lim}, d, \mathcal{T})).$ 

We can recall Fig. 8 and imagine that the state  $s_1$  is, in fact, the term LIM in the LTS  $\mathcal{L}_{\mathcal{G}}^{\Lambda}$ . The test states (corresponding to the terms determined by the collector nodes  $q_i$  in Fig. 12) are inside REGION(LIM, d), unlike the test states depicted in Fig. 8.

The set REGION(LIM, d) and its subset  $\mathcal{T}$  are finite (easily constructible) sets, and the set  $TEL(LIM, d, \mathcal{T})$ , and thus also the number  $MAX_{TEL} \in \{-1\} \cup \mathbb{N}$ , are computable by Lemma 2. We now put

$$e = \text{Max}_{\text{TEL}} + \ell_0$$

and show that this e satisfies the claim. The fact that the term LIM =  $A(x_1, \ldots, x_m)\sigma^{\omega}\sigma_0$ , the numbers d,  $\ell_0$ , the elements  $x_i\sigma^{\omega}\sigma_0$  of the test set  $\mathcal{T}$ , and thus also the number e, are determined by the eligible stair w and the terms  $x_i\sigma_0$  where  $x_i\sigma = x_i$  is obvious. Hence it remains to consider the following two cases.

1. Suppose that  $G_{(w,e)}\sigma_0 \not\sim \text{Lim}$ , i.e.,

$$(A(x_1,\ldots,x_m)\sigma^{\ell_0-1})\sigma^{e-\ell_0+1}\sigma_0 \nsim (A(x_1,\ldots,x_m)\sigma^{\ell_0-1})\sigma^{\omega}\sigma_0.$$

By compositionality (Prop. 6) we then have  $x_i \sigma^{e-\ell_0+1} \sigma_0 \not\sim x_i \sigma^{\omega} \sigma_0$  for some  $x_i$  occurring in  $A(x_1, \ldots, x_m) \sigma^{\ell_0-1}$ , i.e., for some  $x_i \in \mathcal{V}_{\ell_0}$ ; we fix such  $x_i$  and recall that it also occurs in  $A(x_1, \ldots, x_m) \sigma^{\ell_0}$  (since  $\mathcal{V}_{\ell_0} = \mathcal{V}_{\ell_0+1}$ ).

Since EqLv $(x_i \sigma^{e-\ell_0+1} \sigma_0, x_i \sigma^{\omega} \sigma_0) \ge e-\ell_0+1$  (by Prop. 8(2)), we have

$$\text{Max}_{\text{TEL}} < \text{EqLv}(x_i \sigma^{e-\ell_0+1} \sigma_0, x_i \sigma^{\omega} \sigma_0) < \omega,$$

where  $x_i \sigma^{\omega} \sigma_0$  belongs to the test set  $\mathcal{T}$ . Our choice of d guarantees that  $x_i$  belongs to REGION $(A(x_1,\ldots,x_m)\sigma^{\ell_0},d)$ , and thus

$$x_i \sigma^{e-\ell_0+1} \sigma_0$$
 belongs to Region $(A(x_1, \dots, x_m) \sigma^{\ell_0} \sigma^{e-\ell_0+1} \sigma_0, d)$ ,

hence  $x_i \sigma^{e-\ell_0+1} \sigma_0$  belongs to REGION $(G_{(w,e+1)} \sigma_0, d)$ ; this implies that  $G_{(w,e+1)} \sigma_0 \not\sim \text{Lim}$  (by Prop. 5). Repeating the above reasoning for  $G_{(w,e+1)} \sigma_0 \not\sim \text{Lim}$ , we deduce that  $G_{(w,e+2)} \sigma_0 \not\sim \text{Lim}$ , etc. Hence indeed  $G_{(w,k)} \sigma_0 \not\sim \text{Lim}$  for all integer  $k \geq e$ .

2. Now suppose  $G_{(w,e)} \sigma_0 \sim \text{Lim}$ . Then  $x_i \sigma^{e-\ell_0+1} \sigma_0 \sim x_i \sigma^{\omega} \sigma_0$  for all  $x_i \in \mathcal{V}_{\ell_0}$ , since otherwise the sets  $\text{TEL}(G_{(w,e)} \sigma_0, d, \mathcal{T})$  and  $\text{TEL}(\text{Lim}, d, \mathcal{T})$  would differ (we would have  $\text{Max}_{\text{TEL}} < \text{EqLv}(x_i \sigma^{e-\ell_0+1} \sigma_0, x_i \sigma^{\omega} \sigma_0) < \omega$  for some  $x_i \in \mathcal{V}_{\ell_0}$ ).

For  $k \geq e$  this entails that  $G_{(w,k)} \sigma_0$ , presented as  $A(x_1, \ldots, x_m) \sigma^{k-e+\ell_0-1} \sigma^{e-\ell_0+1} \sigma_0$ , is bisimilar with  $A(x_1, \ldots, x_m) \sigma^{k-e+\ell_0-1} \sigma^{\omega} \sigma_0$ , i.e. with  $A(x_1, \ldots, x_m) \sigma^{\omega} \sigma_0 = \text{Lim}$ .

Remark. We note that we do not need to invoke Lemma 2 for a demonstration of the fact  $G_{(w,e)}\sigma_0 \not\sim \text{Lim}$ , if it is the case. But we have relied on Lemma 2 when computing e.

## 3.2.3 Each bisim-infinite term has a witness

Once we show that there is a witness for any bisim-infinite term  $E_0$ , the proof of Theorem 1 will be finished. We show this witness existence by Lemma 14, which is preceded by introducing some useful notions and facts.

Before giving technical details, we give an informal overview of the idea. We fix a bisiminfinite  $E_0$  and assume that it has no witness, for the sake of contradiction. We know that there is an infinite path  $E_0 \xrightarrow{r_1} E_1 \xrightarrow{r_2} E_2 \xrightarrow{r_3} \cdots$  where  $E_i \not\sim E_j$  for all  $i \neq j$  (by Prop. 3). Then we easily deduce the existence of a path

$$E_0 \xrightarrow{u} H_0 \xrightarrow{w_1} H_1 \xrightarrow{w_2} H_2 \xrightarrow{w_3} \cdots$$

where  $H_i \not\sim H_j$  for all  $i \neq j$  and  $w_i$  are stairs from a bounded set. Hence each segment  $H_j \xrightarrow{w_{j+1}} H_{j+1}$  never sinks to a root-successor of  $H_j$ , and the path  $H_0 \xrightarrow{w_1} H_1 \xrightarrow{w_2} H_2 \xrightarrow{w_3} \cdots$  can be presented as

$$A_0(x_1,\ldots,x_m)\sigma_0 \xrightarrow{w_1} A_1(x_1,\ldots,x_m)\sigma_1\sigma_0 \xrightarrow{w_2} A_2(x_1,\ldots,x_m)\sigma_2\sigma_1\sigma_0 \xrightarrow{w_3} \cdots$$

(where also  $\sigma_i$  are from a bounded set). By straightforward applications of the pigeonhole principle we have frequent occurrences of eligible stairs w of the form  $w_iw_{i+1}\dots w_j$  for  $i\leq j$ . Since none of them corresponds to a witness for  $E_0$  (of the form  $(uw_1\dots w_{i-1},w_i\dots w_j)$ ) by our assumption, we always have that by repeating the stair w a certain number of times we reach the equivalence class of the respective limit (as formalized in Lemma 9 where the case 2 is now relevant).

If an eligible stair  $A(x_1, \ldots, x_m) \xrightarrow{w} A(x_1, \ldots, x_m) \sigma$  has no "root-sticks", i.e., the terms  $x_i \sigma$  have nonterminal roots for all  $i \in [1, m]$  (hence all variables, i.e. the root-successors at the start, have "sunk", none of them has an occurrence as a root-successor at the end), then we have the case  $I = \emptyset$  in Lemma 9, and thus the term LIM and the respective number e are fully determined by w. This entails that for every  $A(x_1, \ldots, x_m) \sigma'$  reachable from  $E_0$  we have  $A(x_1, \ldots, x_m) \sigma' \xrightarrow{w^e} A(x_1, \ldots, x_m) \sigma^e \sigma' \sim \text{LIM}$ ; this further entails that for all  $x_i$  occurring in  $A(x_1, \ldots, x_m) \sigma^e$  we have that  $x_i \sigma'$  must belong to the bisim-class of a term in a bounded reachability-distance of LIM, hence  $x_i \sigma'$  must belong to one of boundedly many bisim-classes.

Such observations lead to a desired contradiction in a straightforward way in the case of grammars with no "root-stick problem" (in particular in the case corresponding to pushdown automata with no  $\varepsilon$ -steps). In the general case we consider a suffix of the path  $H_0 \xrightarrow{w_1} H_1 \xrightarrow{w_2} H_2 \xrightarrow{w_3} \cdots$  with the maximal number of "permanent root-sticks", and partition it into finite blocks (of possibly unbounded lengths) that have no other root-sticks than the permanent

ones. This allows us to mimic the above situation where a stair w determines LIM and the number e; now the set I in Lemma 9 might be nonempty but the parameters  $F_i$ ,  $i \in I$ , are fixed since we consider a fixed path from  $E_0$  (and  $F_i$  correspond to the permanent root-sticks).

We now formalize these ideas, again assuming a fixed grammar  $\mathcal{G} = (\mathcal{N}, \Sigma, \mathcal{R})$ .

An infinite direct-rhs-stair path demonstrating bisim-infiniteness. A stair  $w \in \mathbb{R}^+$ , where  $A(x_1, \ldots, x_m) \stackrel{w}{\longrightarrow} F$ , is called a *direct stair* if there is no v such that |v| < |w| and  $A(x_1, \ldots, x_m) \stackrel{v}{\longrightarrow} F$ . If  $w \in \mathbb{R}^+$  is a direct stair, where  $A(x_1, \ldots, x_m) \stackrel{w}{\longrightarrow} F$ , and F is a subterm of the right-hand side E of some rule from  $\mathbb{R}$  in the grammar  $\mathcal{G}$ , then w is called a *direct-rhs stair*. (For technical reasons we do not require F to be a subterm of the right-hand side of the first rule in w.) We note easily:

**Proposition 10.** Each grammar  $\mathcal{G}$  has a finite computable set of direct-rhs stairs.

Some direct-rhs stairs can be of the form  $A(x_1, \ldots, x_m) \xrightarrow{w} A'(x_{i_1}, \ldots, x_{i_m})$ , where  $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, m\}$ ; we might even have A = A' but in this case  $(x_1, \ldots, x_m) \neq (x_{i_1}, \ldots, x_{i_m})$  since  $A(x_1, \ldots, x_m) \xrightarrow{w} A(x_1, \ldots, x_m)$  is no direct stair.

**Proposition 11.** If a term  $E_0$  is bisim-infinite, then there is an infinite path

$$E_0 \xrightarrow{u} H_0 \xrightarrow{w_1} H_1 \xrightarrow{w_2} H_2 \xrightarrow{w_3} \cdots$$
 (2)

where all  $w_i$  are direct-rhs stairs and  $H_i \not\sim H_j$  for all  $i \neq j$ .

*Proof.* We assume a bisim-infinite term  $E_0$  and we fix an infinite path

$$E_0 \xrightarrow{r_1} E_1 \xrightarrow{r_2} E_2 \xrightarrow{r_3} \cdots$$
 (3)

in the LTS  $\mathcal{L}_{\mathcal{G}}^{\mathbb{R}}$  such that  $E_i \not\sim E_j$  (in  $\mathcal{L}_{\mathcal{G}}^{\mathbb{A}}$ ) for all  $i \neq j$ ; the existence of such a path follows from Prop. 3. In (3) we thus have no repeat, i.e., we have  $E_i \neq E_j$  for all  $i \neq j$ . Hence there is the least  $i_0 \in \mathbb{N}$  such that  $r_{i_0+1}r_{i_0+2}\dots r_{i_0+\ell}$  is a stair for each  $\ell \in \mathbb{N}_+$ ; if there was no such  $i_0$ , there would be a repeat: we would have an infinite sequence  $0 = j_0 < j_1 < j_2 < \cdots$  where  $E_{j_{k+1}}$  is a depth-1 subterm of  $E_{j_k}$  for each  $k \in \mathbb{N}$  (since  $E_{j_k} \xrightarrow{r_{j_k+1}r_{j_k+2}\dots r_{j_{k+1}}} E_{j_{k+1}}$  sinks to a root-successor in  $E_{j_k}$ ), and we recall that  $E_0$  has only finitely many subterms (since it is a regular term).

Having defined  $i_0, i_1, \ldots, i_j$ , we define  $i_{j+1}$  as the least number i such that  $i_j < i$  and  $r_{i+1}r_{i+2}\ldots r_{i+\ell}$  is a stair for each  $\ell \in \mathbb{N}_+$ . (The existence of  $i_{j+1}$  is deduced similarly as the existence of  $i_0$ ; below we discuss this in more detail.) For each  $j \in \mathbb{N}$  we put  $H_j = E_{i_j}$  and  $w_{j+1} = r_{i_j+1}r_{i_j+2}\cdots r_{i_{j+1}}$ ; hence the (infinite) suffix of the path (3) that starts with  $E_{i_0}$  can be presented as

$$H_0 \xrightarrow{w_1} H_1 \xrightarrow{w_2} H_2 \xrightarrow{w_3} \cdots$$

which is illustrated in Fig. 13 in two forms. By the choice of (3) we have  $H_i \not\sim H_j$  for  $i \neq j$ . For each  $j \in \mathbb{N}$  we present  $H_j$  as  $A_j(x_1, \ldots, x_m) \sigma'_j$  (where  $\text{SUPP}(\sigma'_j) \subseteq \{x_1, \ldots, x_m\}$ ) and the segment  $H_j \xrightarrow{w_{j+1}} H_{j+1}$  as  $A_j(x_1, \ldots, x_m) \sigma'_j \xrightarrow{w_{j+1}} A_{j+1}(x_1, \ldots, x_m) \sigma_{j+1} \sigma'_j$  where  $A_j(x_1, \ldots, x_m) \xrightarrow{w_{j+1}} A_{j+1}(x_1, \ldots, x_m) \sigma_{j+1}$ ; we thus have  $\sigma'_j = \sigma_j \sigma_{j-1} \cdots \sigma_0$ , when putting  $\sigma_0 = \sigma'_0$ .

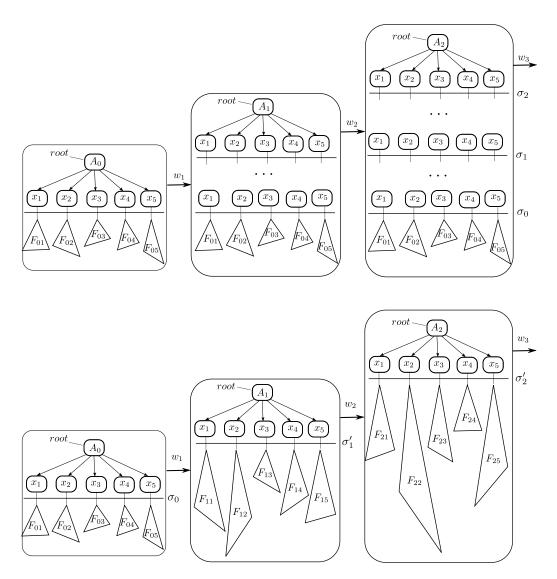


Figure 13:  $H_0 \xrightarrow{w_1} H_1 \xrightarrow{w_2} H_2 \xrightarrow{w_3} \cdots$ , depicted in two forms

By our choice of  $i_{j+1}$  (recall that  $E_{i_{j+1}} = H_{j+1} = A_{j+1}(x_1, \ldots, x_m)\sigma_{j+1}\sigma'_j)$  we have that  $A_{j+1}(x_1, \ldots, x_m)\sigma_{j+1}$  is a subterm of the right-hand side E of the first rule r in  $w_{j+1}$ , where r is  $A_j(x_1, \ldots, x_m) \stackrel{a}{\longrightarrow} E$ , say (and E is thus a finite term). Indeed, either  $w_{j+1} = r$  and  $A_{j+1}(x_1, \ldots, x_m)\sigma_{j+1} = E$ , or a prefix rv of  $w_{j+1}$  "sinks" to a depth-1 subterm F of E (we have  $A_j(x_1, \ldots, x_m) \stackrel{r}{\longrightarrow} E \stackrel{v}{\longrightarrow} F$ ); in the latter case, if  $A_{j+1}(x_1, \ldots, x_m)\sigma_{j+1} \neq F$ , then a longer prefix of  $w_{j+1}$  sinks to a depth-1 subterm F' of F (hence to a depth-2 subterm of E), etc. Since  $w_{j+1}$  is a stair, no prefix can sink to a variable in E.

Hence if  $w_{j+1}$  is a direct stair, then it is a direct-rhs stair; otherwise it can be replaced with a shorter word  $w'_{j+1}$  that is a direct stair and satisfies  $A_j(x_1,\ldots,x_m) \stackrel{w'_{j+1}}{\longrightarrow} A_{j+1}(x_1,\ldots,x_m)\sigma_{j+1}$ , hence  $H_j \stackrel{w'_{j+1}}{\longrightarrow} H_{j+1}$ . By our definitions,  $w'_{j+1}$  is a direct-rhs stair (though  $A_{j+1}(x_1,\ldots,x_m)\sigma_{j+1}$  might be not a subterm of the right-hand side of the first rule in  $w'_{j+1}$ ).

Hence the existence of the path (2) is clear.

Partitioning a suffix of the path (2) into blocks with the same root-sticks. Given a stair w, where  $A(x_1, \ldots, x_m) \xrightarrow{w} B(x_1, \ldots, x_m) \sigma$ , we say that the *i-th root-successor* is root-sticking for w, or shortly that  $i \in [1, m]$  is a root-stick for w, if  $x_j \sigma = x_i$  for some  $j \in [1, m]$  (which implies that in  $F \xrightarrow{w} F'$  the *i-th* root successor before performing w is a depth-1 subterm also after w, though maybe at different position(s)).

E.g., 2 is the only root-stick for the stair  $r_1$  in Fig. 7. In Fig. 11, the stair w has two root-sticks, namely 2 and 5; one reason for 2 being the root-stick is that  $x_4\sigma=x_2$ . In fact, 2,5 are here the root-sticks for each stair  $w^\ell$ ,  $\ell\in\mathbb{N}_+$ .

If i is a root-stick for a stair w, then i is called a root-stick also for any path  $F \xrightarrow{w} F'$ . We observe that if  $F_0 \xrightarrow{w_1} F_1 \xrightarrow{w_2} F_2$  where  $w_1, w_2$  are stairs and i is no root-stick for  $w_1$ , then i is no root-stick for  $w_1w_2$  (since  $F_1 \xrightarrow{w_2} F_2$  does not sink to any root-successor).

Given a fixed presentation  $F \xrightarrow{w_1} \xrightarrow{w_2} \xrightarrow{w_3} \cdots$  of an infinite path where  $w_j$  are stairs, we say that  $i \in [1, m]$  is a permanent root-stick if i is a root-stick for the stair  $w_1 w_2 \cdots w_\ell$  for each  $\ell \in \mathbb{N}_+$ . In this definition we could equivalently replace "for each  $\ell \in \mathbb{N}_+$ " with "for infinitely many  $\ell \in \mathbb{N}_+$ ", since if i is a root-stick for  $w_1 w_2 \cdots w_{\ell+1}$  then i is a root-stick for  $w_1 w_2 \cdots w_\ell$  as well (as we observed above).

By a direct-rhs-stair path we further mean an infinite path

$$A_0(x_1, \dots, x_m)\sigma_0 \xrightarrow{w_1} A_1(x_1, \dots, x_m)\sigma_1\sigma_0 \xrightarrow{w_2} A_2(x_1, \dots, x_m)\sigma_2\sigma_1\sigma_0 \xrightarrow{w_3} \dots$$
 (4)

where  $w_j$  are direct-rhs stairs. (The suffix  $H_0 \xrightarrow{w_1} H_1 \xrightarrow{w_2} H_2 \xrightarrow{w_3} \cdots$  of (2) is an example of a direct-rhs-stair path.) For  $i, j \in \mathbb{N}$  we put

$$w_{(i,j)} = w_i w_{i+1} \cdots w_j$$
 and  $\sigma_{(j,i)} = \sigma_j \sigma_{j-1} \cdots \sigma_i$ 

(where  $w_{(i,j)} = \varepsilon$  and  $\sigma_{(j,i)}$  is the empty-support substitution if i > j); we also put  $\sigma'_j = \sigma_{(j,0)}$ . For any i < j, we define the (i,j)-segment of (4) as the path

$$A_i(x_1,\ldots,x_m)\sigma_i' \xrightarrow{w_{(i+1,j)}} A_j(x_1,\ldots,x_m)\sigma_j' = A_j(x_1,\ldots,x_m)\sigma_{(j,i+1)}\sigma_i'.$$

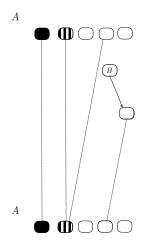
In the path  $A(x_1, \ldots, x_5)\sigma_0 \xrightarrow{w} \xrightarrow{w} \xrightarrow{w} \cdots$  in Fig. 11, there are two permanent root-sticks, namely 2 and 5. On the left in Fig. 14 we can see a depiction of the "stair-terms"  $A_i(x_1, \ldots, x_m)\sigma'_i$  of a direct-rhs-stair path (4); each stair-term is depicted by a row of auxiliary fictitious nodes representing its root-successors while its root  $A_i$  is written on the left. The substitutions  $\sigma_{i+1}$  ("filling the space" between the rows i and i+1) are depicted only partly. The depicted direct-rhs-stair path has a suffix with two permanent root-sticks: one is depicted by the black fictitious nodes, the other by the striped fictitious nodes. The whole Fig. 14 illustrates a partition of a direct-rhs-stair path into certain finite blocks, as stated in Prop. 12 below.

**Proposition 12.** (Block partition.) For a fixed direct-rhs-stair path in the presentation (4) let  $s \in [0, m]$  be the maximum such that some suffix of (4), starting with  $A_i(x_1, \ldots, x_m)\sigma'_i$  for some i, has s permanent root-sticks. There are some  $A \in \mathcal{N}$  and pairwise different  $k_1, k_2, \ldots, k_s \in [1, m]$  such that there is an infinite sequence  $0 \le i_0 < i_1 < i_2 < \cdots$  satisfying the following conditions:

 $A_{6}$   $A_{5}$   $A_{4}$   $A_{2}$   $A_{3}$   $A_{4}$   $A_{4}$   $A_{5}$   $A_{6}$   $A_{7}$   $A_{8}$   $A_{9}$   $A_{1}$   $A_{1}$   $A_{2}$   $A_{3}$   $A_{4}$   $A_{5}$   $A_{5}$   $A_{6}$   $A_{7}$   $A_{8}$   $A_{9}$   $A_{1}$   $A_{1}$   $A_{2}$   $A_{3}$   $A_{4}$   $A_{5}$   $A_{5}$   $A_{5}$   $A_{5}$   $A_{6}$   $A_{7}$   $A_{8}$   $A_{9}$   $A_{1}$   $A_{1}$   $A_{2}$   $A_{3}$   $A_{4}$   $A_{5}$   $A_{5$ 

Consider the suffix with the maximum number of permanent root-sticks (2 in our case).

Partition the suffix into finite blocks (using the pigeonhole principle)



only permanent r-sticks are block r-sticks

Figure 14: Partition of an (infinite) direct-rhs-stair path into finite blocks

- 1. The suffix of the fixed direct-rhs-stair path that starts at  $A_{i_0}(x_1, \ldots, x_m)\sigma'_{i_0}$  has s permanent root-sticks.
- 2. For each  $j \in \mathbb{N}$ , the  $(i_j, i_{j+1})$  segment, i.e. the path  $A_{i_j}(x_1, \ldots, x_m)\sigma'_{i_j} \xrightarrow{w_{(i_j+1, i_{j+1})}} A_{i_{j+1}}(x_1, \ldots, x_m)\sigma_{(i_{j+1}, i_j+1)}\sigma'_{i_j}$  has precisely s root-sticks. (I.e., the cardinality of the set  $\{k \mid \exists \ell \in [1, m] : x_{\ell}\sigma_{(i_{j+1}, i_{j+1})} = x_k\}$  is s.)
- 3.  $A = A_{i_0} = A_{i_1} = A_{i_2} = \cdots$
- 4. For all  $j \in \mathbb{N}$  and  $\ell \in [1, s]$  we have  $x_{k_{\ell}}\sigma_{(i_{j+1}, i_j + 1)} = x_{k_{\ell}}$ . (Hence  $\sigma_{(i_{j+1}, i_j + 1)}$  is eligible.)

Proof. Let us fix a direct-rhs-stair path in the presentation (4); there is certainly the respective maximum  $s \in [0, m]$ , and we can choose  $i_0$  so that  $A_{i_0}(x_1, \ldots, x_m)\sigma'_{i_0}$  is the starting term of a suffix with s permanent root-sticks (we might have s = 0 and/or  $i_0 = 0$ ). Hence for each  $i \geq i_0$  the suffix starting at  $A_i(x_1, \ldots, x_m)\sigma'_i$  has also s permanent root-sticks. Moreover, if  $i_0 \leq i < i'$  and  $k \in [1, m]$  is a permanent root-stick in the suffix starting at  $A_i(x_1, \ldots, x_m)\sigma'_i$  then there is precisely one  $k' \in [1, m]$  that is a permanent root-stick in the suffix starting at  $A_{i'}(x_1, \ldots, x_m)\sigma'_{i'}$  and that satisfies  $x_{k'}\sigma_{(i',i+1)} = x_k$ .

In Fig. 14 we have s = 2,  $i_0 = 4$ , and the last above fact is illustrated by the sequence of the black fictitious nodes for one permanent root-stick, and by the

sequence of the striped fictitious nodes for the other permanent root-stick. Though the striped also occurs as the first root-successor at level 5 and as the second rootsuccessor at level 6, the respective "root-successor sequence" must be finite since we assume that no suffix has more than 2 permanent root-sticks.

By the maximality of s, for each  $i \geq i_0$  there is i' > i such that the (i, i')-segment, i.e.  $A_i(x_1, \ldots, x_m)\sigma_i' \xrightarrow{w_{(i+1,i')}} A_{i'}(x_1, \ldots, x_m)\sigma_{i'}'$ , has precisely s root-sticks (and not more).

The existence of a sequence  $0 \le i_0 < i_1 < i_2 < \cdots$  satisfying the conditions 1 and 2 is thus clear; moreover, any infinite subsequence  $i_{j_0}, i_{j_1}, i_{j_2}, \ldots$  of  $i_0, i_1, i_2, \ldots$  satisfies 1 and 2 as well (after the notational change which renames  $i_{j_0}, i_{j_1}, i_{j_2}, \ldots$  to  $i_0, i_1, i_2, \ldots$ ).

By the pigeonhole principle there must be an infinite subsequence  $i_{j_0}, i_{j_1}, i_{j_2}, \ldots$  of the sequence  $i_0, i_1, i_2, \ldots$  that, moreover, satisfies the conditions 3 and 4 for some  $A \in \mathcal{N}$  and some  $k_1, k_2, \ldots, k_s \in [1, m]$ , after the respective notational change. (Fig. 14 illustrates this on the right, assuming the set  $\{k_1, k_2\}$  happens to be  $\{1, 2\}$ .)

To show the last claim in more detail, assume that we have fixed some  $i_0, i_1, i_2, \ldots$  satisfying 1 and 2; let  $k_{01}, k_{02}, \ldots, k_{0s}$  be the permanent root-sticks of the suffix starting at  $A_{i_0}(x_1, \ldots, x_m)\sigma'_{i_0}$ . (In Fig. 14 we have  $i_0 = 4$ ,  $k_{01} = 2$ ,  $k_{02} = 5$ .) To each  $j \in \mathbb{N}_+$  we attach the stamp

$$S(j) = (A_{i_1}, k_{j1}, k_{j2}, \dots, k_{js})$$

where  $\{k_{j1}, k_{j2}, \ldots, k_{js}\}$  is the set of permanent root-sticks in the suffix starting with the term  $A_{i_j}(x_1, \ldots, x_m)\sigma'_{i_j}$ , and the order in S(j) is such that  $x_{k_{j\ell}}\sigma_{(i_j,i_0+1)} = x_{k_{0\ell}}$  for each  $\ell \in [1,s]$ . (As discussed above, for each  $j \in \mathbb{N}_+$  and each  $\ell \in [1,s]$  there is precisely one  $k_{j\ell}$  that satisfies the required conditions. In Fig. 14, if  $i_1 = 6$  then  $S(1) = (A_6, 1, 3)$ .)

This also entails that for any 0 < j < j' and any  $\ell \in [1, s]$  we have  $x_{k_{j'\ell}}\sigma_{(i_{j'},i_j+1)} = x_{k_{j\ell}}$ . Indeed, by our choices the set  $\{k \mid x_{k'}\sigma_{(i_{j'},i_j+1)} = x_k \text{ for some } k' \in [1, m]\}$  must be equal to  $\{k_{j1}, k_{j2}, \dots, k_{js}\}$ . Since  $x_{k_{j'\ell}}\sigma_{(i_{j'},i_0+1)} = x_{k_{0\ell}}$  and  $\sigma_{(i_{j'},i_0+1)} = \sigma_{(i_{j'},i_j+1)}\sigma_{(i_j,i_0+1)}$ , we must have  $x_{k_{j'\ell}}\sigma_{(i_{j'},i_j+1)} = x_n$  and  $x_n\sigma_{(i_j,i_0+1)} = x_{k_{0\ell}}$  for some  $n \in [1, m]$ . This entails  $n = k_{j\ell}$ .

There must be an infinite sequence  $0 < j_1 < j_2 < j_3 < \cdots$  where  $S(j_1) = S(j_2) = S(j_3) = \cdots$ ; hence all elements of  $i_{j_1}, i_{j_2}, i_{j_3}, \ldots$  yield the same stamp  $(A, k_1, k_2, \ldots, k_s)$  for some  $A \in \mathcal{N}$  and some pairwise different  $k_1, k_2, \ldots, k_s$ . The above discussion also entails that for each  $\ell \in [1, s]$  we have

$$x_{k_{\ell}}\sigma_{(i_{j_2},i_{j_1}+1)} = x_{k_{\ell}}, x_{k_{\ell}}\sigma_{(i_{j_3},i_{j_2}+1)} = x_{k_{\ell}}, x_{k_{\ell}}\sigma_{(i_{j_4},i_{j_3}+1)} = x_{k_{\ell}}, \ldots$$

Hence the sequence  $i_{j_1}, i_{j_2}, i_{j_3}...$  satisfies the conditions 1, 2, 3, 4 (after being renamed to  $i_0, i_1, i_2...$ ).

Terms keeping root-connected for a few blocks are in finitely many classes. We fix a direct-rhs-stair path

$$A_0(x_1,\ldots,x_m)\sigma_0 \xrightarrow{w_1} A_1(x_1,\ldots,x_m)\sigma_1\sigma_0 \xrightarrow{w_2} \cdots$$

in the presentation (4), and we also fix  $s \in [0, m]$ ,  $\{k_1, k_2, \ldots, k_s\} \subseteq [1, m]$ ,  $A \in \mathcal{N}$ , and a sequence  $i_0, i_1, i_2, \ldots$  for which the conditions 1–4 in Prop. 12 are satisfied. We have thus partitioned (a suffix of) the fixed path into blocks, i.e. segments  $(i_0, i_1), (i_1, i_2), \ldots$  where the  $(i_j, i_{j+1})$ -block is the segment

$$A(x_1, \dots, x_m)\sigma'_{i_j} \xrightarrow{w_{(i_j+1, i_{j+1})}} A(x_1, \dots, x_m)\sigma'_{i_{j+1}} = A(x_1, \dots, x_m)\sigma_{(i_{j+1}, i_j+1)}\sigma'_{i_j}.$$

For  $k \in [1, m]$  and  $\ell \in \mathbb{N}_+$  we say that k survives  $\ell$  blocks (meaning some consecutive  $\ell$  blocks) if there is  $j \in \mathbb{N}$  such that  $x_k$  occurs in the term  $A(x_1, \ldots, x_m)\sigma_{(i_j+\ell, i_j+1)}$  (i.e., the k-th root-successor in the term starting the  $(i_j, i_{j+1})$ -block has not lost the root-connection till the end of the  $(i_{j+\ell-1}, i_{j+\ell})$ -block).

We note that if k is a permanent root-stick at level  $i_j$ , i.e.,  $k \in \{k_1, k_2, ..., k_s\}$  by our choice of blocks, then k survives forever, i.e. any number of blocks; there can be also other k that survive forever, though not as root-successors but in stepwise increasing depths. (The definition of surviving k is restricted to the levels  $i_j$  that start a block since this technically suffices for our aims.)

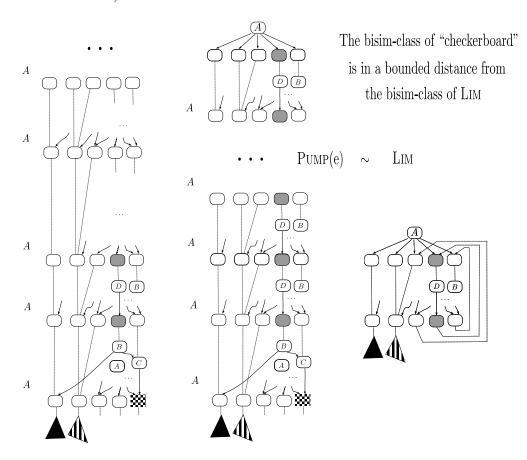


Figure 15: Witness nonexistence implies that long-surviving terms are in a few bisim-classes

The next proposition states that if  $A_0(x_1, \ldots, x_m)\sigma_0$  (the starting term of our fixed directrhs-stair path) has no witness (of bisim-infiniteness) and k survives m blocks (where m is the arity of nonterminals) then the k-th root successors at all levels  $i_j$  are from a finite set of bisim-classes.

**Proposition 13.** Suppose the term  $A_0(x_1, \ldots, x_m)\sigma_0$ , in our fixed path partitioned into blocks, has no witness. Then there is a finite set  $\mathcal{F}$  of bisim-classes (in the LTS  $\mathcal{L}_{\mathcal{G}}^{\mathbf{A}}$ ) such that for any  $k \in [1, m]$  that survives (at least) m blocks we have  $[x_k \sigma'_{i_j}]_{\sim} \in \mathcal{F}$  for all  $j \in \mathbb{N}$ .

*Proof.* If  $A_0(x_1, \ldots, x_m)\sigma_0$  is bisim-finite, then the claim is obvious (in this case the terms  $A(x_1, \ldots, x_m)\sigma'_{i_j}$ ,  $j \in \mathbb{N}$ , belong to finitely many bisim-classes, which entails that  $x_k\sigma'_{i_j}$  belong

to finitely many bisim-classes). But we have not yet excluded that  $A_0(x_1, \ldots, x_m)\sigma_0$  is bisiminfinite and has no witness.

It clearly suffices to show for each fixed  $k \in [1, m]$  that there is a finite set  $\mathcal{F}_k$  satisfying the following condition: if k survives m blocks then  $[x_k \sigma'_{i_j}]_{\sim} \in \mathcal{F}_k$  for all  $j \in \mathbb{N}$ . The crux of the proof is depicted in Fig. 15.

So suppose k survives m blocks from the level  $i_j$  for some j. If k is a permanent root-stick, hence k belongs to the set  $\{k_1, k_2, \ldots, k_s\}$  yielded by the block-partition according to Prop. 12, then the claim is trivial (since  $x_k \sigma'_{i_j}$  is the same term for all  $j \in \mathbb{N}$ ); so suppose  $k \notin \{k_1, k_2, \ldots, k_s\}$ . (On the left in Fig. 15,  $i_j$  is the bottom-row, and k is the "checkerboard" 5, which is different from the permanent root-sticks, the black 1 and the striped 2.)

We thus have a sequence  $\ell_0, \ell_1, \ell_2, \ldots, \ell_m$  of elements of [1, m] such that  $\ell_0 = k$  and  $x_{\ell_p}$  occurs in  $x_{\ell_{p+1}}\sigma_{(i_{j+p+1},i_{j+p+1})}$  for all  $p=0,1,2,\ldots,m-1$ . By the pigeonhole principle there must be  $d,d', 0 \leq d < d' \leq m$ , and some  $k' \in [1,m]$  such that  $x_{k'}$  occurs in  $x_{k'}\sigma_{(i_{j+d'},i_{j+d}+1)}$  and  $x_k$  occurs in  $x_{k'}\sigma_{(i_{j+d},i_{j+1})}$ . (On the left in Fig. 15, we can see the case with d=1, d'=2, and k'=4, denoted as grey.) We can thus consider the candidate

$$A_0(x_1, \dots, x_m)\sigma_0 \xrightarrow{w_{(1, i_{j+d})}} A(x_1, \dots, x_m)\sigma'_{i_{j+d}} \xrightarrow{w} A(x_1, \dots, x_m)\sigma\sigma'_{i_{j+d}}$$
 (5)

where  $w = w_{(i_{j+d}+1,i_{j+d'})}$  and  $\sigma = \sigma_{(i_{j+d'},i_{j+d}+1)}$ ; by our choice of blocks, with only s root-sticks,  $\sigma$  is eligible (since  $x_{\ell}\sigma_{(i_{j+1},i_{j}+1)} = x_{\ell'}$  implies  $x_{\ell'}\sigma_{(i_{j+1},i_{j}+1)} = x_{\ell'}$  and thus  $\ell' \in \{k_1,k_2,\ldots,k_s\}$ , for any j).

Since we assume that  $A_0(x_1,\ldots,x_m)\sigma_0$  has no witness, by Lemma 9 we get

$$A(x_1,\ldots,x_m)\sigma^e\sigma'_{i_{j+d}}\sim A(x_1,\ldots,x_m)\sigma^\omega\sigma'_{i_{j+d}}$$

for some  $e \in \mathbb{N}$ , determined by w (which entails A and  $\sigma$ ) and the terms  $x_{k_1}\sigma'_{i_{j+d}}$ ,  $x_{k_2}\sigma'_{i_{j+d}}$ , ...,  $x_{k_s}\sigma'_{i_{j+d}}$  (since only the permanent root-sticks, the black and the striped in Fig. 15, matter for creating LIM). In Fig 15, PUMP(e) denotes the term  $A(x_1,\ldots,x_m)\sigma^e\sigma'_{i_{j+d}}$  and LIM denotes  $A(x_1,\ldots,x_m)\sigma^\omega\sigma'_{i_{j+d}}$ . By our choice of blocks we have  $x_{k_\ell}\sigma'_{i_{j+d}}=x_{k_\ell}\sigma'_{i_0}$  for all  $\ell \in [1,s]$ , and

LIM = 
$$A(x_1, \ldots, x_m) \sigma^{\omega} \sigma'_{i_{i+1}} = A(x_1, \ldots, x_m) \sigma^{\omega} \sigma'_{i_0}$$
.

Since the variable  $x_k$  (the "checkerboard" in Fig. 15) occurs in  $A(x_1, \ldots, x_m)\sigma^e \sigma_{(i_{j+d}, i_j+1)}$ , and  $A(x_1, \ldots, x_m)\sigma^e \sigma'_{i_{j+d}} = A(x_1, \ldots, x_m)\sigma^e \sigma_{(i_{j+d}, i_j+1)}\sigma'_{i_j}$ , there is some rule-sequence v such that  $A(x_1, \ldots, x_m)\sigma^e \sigma'_{i_{j+d}} \stackrel{v}{\longrightarrow} x_k \sigma'_{i_j}$ ; hence  $x_k \sigma'_{i_j}$  belongs to a bisim-class whose representant is reachable in |v| steps from LIM (since  $A(x_1, \ldots, x_m)\sigma^e \sigma'_{i_{j+d}} \sim \text{LIM}$ , we must have LIM  $\stackrel{v}{\longrightarrow} F$  where  $x_k \sigma'_{i_j} \sim F$ ).

Now for any  $\ell \in \mathbb{N}$  we can consider a candidate

$$A_0(x_1, \dots, x_m)\sigma_0 \xrightarrow{w_{(1,i_\ell)} w'} A(x_1, \dots, x_m)\sigma'\sigma'_{i_\ell} \xrightarrow{w} A(x_1, \dots, x_m)\sigma\sigma'\sigma'_{i_\ell}$$
 (6)

where  $w' = w_{(i_j+1,i_{j+d})}$ ,  $\sigma' = \sigma_{(i_{j+d},i_{j}+1)}$ , and  $j, d, w, \sigma$  are the same as in (5); we do not assume that  $A(x_1,\ldots,x_m)\sigma'_{i_\ell} \xrightarrow{w'w}$  is a segment of our fixed direct-rhs-stair path but (6) surely is a path in the LTS  $\mathcal{L}^g_{\mathcal{G}}$ . The candidate (6) is no witness either, by our assumption on  $A_0(x_1,\ldots,x_m)\sigma_0$ . The respective LIM and the number e are the same for all  $\ell \in \mathbb{N}$ , and the facts  $A(x_1,\ldots,x_m)\sigma^e\sigma'\sigma'_{i_\ell} \sim \text{LIM}$  and  $A(x_1,\ldots,x_m)\sigma^e\sigma'\sigma'_{i_\ell} \xrightarrow{v} x_k\sigma'_{i_\ell}$  entail that the bisimclasses of  $x_k\sigma'_{i_\ell}$  are reachable by |v| steps from LIM, for all  $\ell \in \mathbb{N}$ ; hence the classes whose representants are reachable by |v| steps from LIM constitute the required finite set  $\mathcal{F}_k$ .  $\square$ 

Bisim-infinite  $E_0$  indeed has a witness. By showing the next lemma we finish the proof of Theorem 1. In the proof of the lemma we use two applications of Prop. 13.

**Lemma 14.** For any grammar  $\mathcal{G}$  and any bisim-infinite  $E_0$  there is a witness (satisfying the condition 1, namely  $G_{(w,e)}\sigma_0 \not\sim \text{Lim}$ , in Lemma 9).

*Proof.* Let  $E_0$  be a bisim-infinite term; for the sake of contradiction we assume that  $E_0$  has no witness. Let us fix a direct-rhs-stair path  $E_0 \xrightarrow{u} H_0 \xrightarrow{w_1} H_1 \xrightarrow{w_2} H_2 \xrightarrow{w_3} \cdots$  guaranteed by Prop. 11 ( $w_i$  are direct-rhs stairs and  $H_i \not\sim H_j$  for  $i \neq j$ ), and consider its suffix

$$H_0 \xrightarrow{w_1} H_1 \xrightarrow{w_2} H_2 \xrightarrow{w_3} \cdots$$
 (7)

in the form (4), also depicted in Fig. 13; hence

$$H_j = A_j(x_1, \dots, x_m)\sigma_j\sigma_{j-1}\cdots\sigma_0 = A_j(x_1, \dots, x_m)\sigma_{(j,0)} = A_j(x_1, \dots, x_m)\sigma'_j.$$

We fix a partition of the path (7) into blocks as in Prop. 12 (which is illustrated in Fig. 14); let A be the respective nonterminal and  $i_0, i_1, i_2, \ldots$  the sequence partitioning a suffix of (7) into  $(i_j, i_{j+1})$ -blocks

$$A(x_1, ..., x_m)\sigma'_{i_j} \xrightarrow{w_{(i_j+1, i_{j+1})}} A(x_1, ..., x_m)\sigma_{(i_{j+1}, i_j+1)}\sigma'_{i_j}.$$

We also fix a finite set  $\mathcal{F}$  of bisim-classes guaranteed by Prop. 13; hence if the k-th root-successor keeps connected to the root in some consecutive m blocks (i.e.,  $x_k$  occurs in  $A(x_1,\ldots,x_m)\sigma_{(i_{j+m},i_j+1)}$  for some j), then  $[x_k\sigma'_{i_j}]_{\sim} \in \mathcal{F}$  for all  $j \in \mathbb{N}$ .

Let us now fix an arbitrary  $j \in \{i_0, i_1, i_2, \dots\}$  (a level at a block border). There must be the least  $j' \geq j$  (a level in (7), not necessarily at a block border) such that for each variable  $x_k$  occurring in the term  $A_{j'}(x_1, \dots, x_m)\sigma_{(j',j+1)}$  we have that  $[x_k\sigma'_j]_{\sim} \in \mathcal{F}$ . Hence during the (j, j')-segment of the path (7), i.e. during

$$A(x_1, \dots, x_m)\sigma'_j \xrightarrow{w_{(j+1,j')}} A_{j'}(x_1, \dots, x_m)\sigma_{(j',j+1)}\sigma'_j \tag{8}$$

all root successors  $x_k \sigma'_j$  in  $A(x_1, \ldots, x_m) \sigma'_j$  such that  $[x_k \sigma'_j]_{\sim} \notin \mathcal{F}$  (if there are any) lose their root-connections (and thus have no impact on the bisim-class of  $A_{j'}(x_1, \ldots, x_m) \sigma_{(j',j+1)} \sigma'_j$ ).

For  $\ell \in [j,j']$  we say that the k-th root-successor at the level  $\ell$  is end-surviving (in the (j,j')-segment (8)), or that  $k@\ell$  is end-surviving for short, if  $x_k$  occurs in  $A_{j'}(x_1,\ldots,x_m)\sigma_{(j',\ell+1)}$  (i.e., the k-th root successor at level  $\ell$  is connected to the root also in the end-term of the segment (8)). For  $\ell = j'$  we trivially have that  $k@\ell$  is end-surviving for all  $k \in [1,m]$  (recall that  $\sigma_{(j',j'+1)}$  is the identity, i.e. the empty-support substitution). For  $\ell \in [j,j'-1]$  we have that  $k@\ell$  is end-surviving iff  $x_k$  occurs in the term  $x_{k'}\sigma_{\ell+1}$  for some k' such that  $k'@\ell+1$  is end-surviving. By our choice of j', if  $[x_k\sigma_j']_{\sim} \notin \mathcal{F}$  then k@j is not end-surviving.

Fig. 16 sketches a (j, j')-segment, as a sequence of direct-rhs stairs. In the middle we present each direct-rhs stair by the respective row of auxiliary fictitious nodes, while ignoring the bounded number of real nonterminal nodes between two neighbouring rows. In each row the black nodes correspond to the end-surviving root-successors, and the other nodes to those that have no connection to the root in the end-term  $A_{j'}(x_1, \ldots, x_m)\sigma_{(j',j+1)}\sigma'_{j}$ .

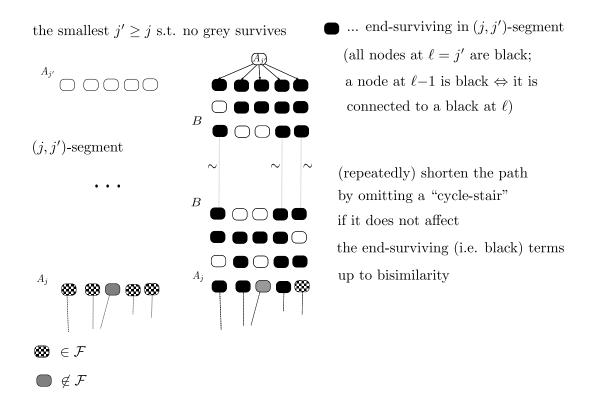


Figure 16: End-surviving terms in (j, j')'-segments

In Fig. 16 we suppose just one k such that  $[x_k \sigma'_j]_{\sim} \notin \mathcal{F}$ , coloured grey. It can be connected only to the above white nodes that have no connection to the root of the end-term either. The (checkerboard) nodes, corresponding to terms belonging to the bisim-classes in  $\mathcal{F}$ , might or might not have connections to the root in the end-term.

We now make a simple observation, potentially enabling to shorten the path of the considered (j, j') segment (8) without changing the start-term and without changing the bisim-class of the end-term (since some of its subterms might be just replaced with equivalent ones).

Suppose there are  $j_1, j_2$  such that  $j \leq j_1 < j_2 \leq j'$ , and the following conditions hold:

- 1.  $A_{j_1} = A_{j_2}$  (denoted *B* in Fig. 16),
- 2. for each  $k \in [1, m]$ ,  $k@j_1$  is end-surviving iff  $k@j_2$  is end-surviving,
- 3.  $x_k \sigma'_{j_1} \sim x_k \sigma'_{j_2}$  for each k such that  $k@j_1$  (and thus also  $k@j_2$ ) is end-surviving.

Let us consider the shorter path arising from (8) by omitting  $w_{(j_1+1,j_2)}$  (the "B-B cycle-stair" in Fig. 16); we get

$$A(x_1, \dots, x_m)\sigma'_i \xrightarrow{w_{(j+1,j_1)}w_{(j_2+1,j')}} A_{j'}(x_1, \dots, x_m)\sigma_{(j',j_2+1)}\sigma'_{j_1}.$$

The end-term  $A_{j'}(x_1,\ldots,x_m)\sigma_{(j',j_2+1)}\sigma'_{j_1}$  of the shorter path is bisimilar with the end-term  $A_{j'}(x_1,\ldots,x_m)\sigma_{(j',j_2+1)}\sigma'_{j_2}$  of (8), since for any  $x_k$  occurring in  $A_{j'}(x_1,\ldots,x_m)\sigma_{(j',j_2+1)}$  we

have that  $x_k \sigma'_{j_1} \sim x_k \sigma'_{j_2}$  by the above assumptions. (We invoke the compositionality captured by Prop. 6.)

It might be possible to repeatedly shorten the shorter path in the same way (by stepwise omitting "cycle-stairs" that do not affect the distribution and the bisim-classes of the respective end-surviving root-successors), keeping the same bisim-class of the end-term, until a basic (j, j')-path arises that cannot be further shortened in the above described way. A formal description of this process would require a straightforward adjustment of the notation. Such a basic (j, j')-path is generally not a segment of (7) but its end-term is still bisimilar with  $H_{j'}$ .

We can thus fix one basic (j, j')-path for the above fixed  $j \in \{i_0, i_1, i_2, ...\}$ . Now we assume that we have fixed one basic (j, j')-path for each  $j \in \{i_0, i_1, i_2, ...\}$ . For each  $j \in \{i_0, i_1, i_2, ...\}$  we present the fixed basic (j, j')-path as

$$A(x_1, \ldots, x_m) \sigma'_j \xrightarrow{v_j} H'_j$$
 (i.e.  $H_j \xrightarrow{v_j} H'_j$ ) where  $v_j = u_{j1} u_{j2} \ldots u_{jn_j}$ 

is a sequence of direct-rhs stairs  $u_{ji}$ ; we have  $H'_j \sim H_{j'}$  (where  $H_j$  and  $H_{j'}$  are from (7)).

For any  $j \in \{i_0, i_1, i_2, ...\}$ , we can partition the set [1, m] into the sets  $\{k \mid [x_k \sigma'_j]_{\sim} \in \mathcal{F}\}$  and  $\{k \mid [x_k \sigma'_j]_{\sim} \notin \mathcal{F}\}$ . By the pigeonhole principle this partition is the same for infinitely many j. Moreover, since  $\mathcal{F}$  is finite, the pigeonhole principle also yields the following: there is a set  $\mathcal{C} \subseteq [1, m]$  and a mapping  $g : \mathcal{C} \to \mathcal{F}$  such that there is an infinite subset  $\mathcal{D} \subseteq \{i_0, i_1, i_2, ...\}$  where for each  $j \in \mathcal{D}$  we have  $(A_j = A \text{ and})$ 

$$\{k \mid [x_k \sigma_i']_{\sim} \in \mathcal{F}\} = \mathcal{C}, \text{ and } [x_k \sigma_i']_{\sim} = g(k) \text{ for each } k \in \mathcal{C}.$$

We now restrict our attention to the basic (j, j')-paths  $A(x_1, \ldots, x_m)\sigma'_j \xrightarrow{v_j} H'_j$  for  $j \in \mathcal{D}$ .

(The starting "checkerboard-grey" row depicted in Fig. 16 is the same for all  $j \in \mathcal{D}$  regarding the checkerboard-grey distribution and the bisim-classes of the "checkerboard terms"; the "grey terms" can be wildly different for different  $j \in \mathcal{D}$  but they are not end-surviving in the fixed basic (j, j')-paths.)

For technical convenience we even assume that

$$\mathcal{D} = \{j_0, j_1, j_2, \dots\}$$

where  $j'_0 < j_1, \ j'_1 < j_2, \ldots$  (i.e.,  $j'_{\ell} < j_{\ell+1}$  for all  $\ell \in \mathbb{N}$ ); we might simply remove some elements of the original  $\mathcal{D}$  to guarantee this. We thus get  $H'_{j_k} \nsim H'_{j_\ell}$  for  $k \neq \ell$  (since  $H'_{j_k} \sim H_{j'_k}, \ H'_{j_\ell} \sim H_{j'_\ell}$ , and  $j'_k \neq j'_\ell$  implies  $H_{j'_k} \nsim H_{j'_\ell}$  due to our choice of (7)). For  $k \neq \ell$  we thus also have that  $v_{j_k} \neq v_{j_\ell}$ , since otherwise we would have  $H'_{j_k} \sim H'_{j_\ell}$  by compositionality. For  $j_p \in \mathcal{D}$   $(p \in \mathbb{N})$ , by the enriched prefix of length  $\ell$  of the sequence  $v_{j_p} = u_{j_p 1} u_{j_p 2} \cdots u_{j_p n_{j_p}}$ 

For  $j_p \in \mathcal{D}$   $(p \in \mathbb{N})$ , by the enriched prefix of length  $\ell$  of the sequence  $v_{j_p} = u_{j_p 1} u_{j_p 2} \cdots u_{j_p n_{j_p}}$  of direct-rhs stairs (where  $\ell \leq n_{j_p}$ ) we mean the sequence  $\mathcal{E}_0, u_{j_p 1}, \mathcal{E}_1, u_{j_p 2}, \mathcal{E}_2, \dots, u_{j_p \ell}, \mathcal{E}_{\ell}$  where for  $i \in [0, \ell]$  we have

$$\mathcal{E}_i = \{k \in [1, m] \mid k@(j_p + i) \text{ is end-surviving in the fixed basic } (j_p, j_p') - \text{path}\}.$$

(Hence  $\mathcal{E}_i$  correspond to the distribution of black nodes in the rows in Fig. 16, after the path has been shortened by omitting the respective cycle-stairs.)

For each  $\ell \in \mathbb{N}$  there are clearly only finitely many enriched prefixes of length  $\ell$  of all  $v_{j_p}$ ,  $p \in \mathbb{N}$ . By an (implicit) use of König's Lemma, there must be an infinite sequence

$$SEQ = \mathcal{E}_0, u_1, \mathcal{E}_1, u_2, \mathcal{E}_2, \dots,$$

such that for any  $\ell \in \mathbb{N}$  the prefix  $\mathcal{E}_0, u_1, \mathcal{E}_1, u_2, \mathcal{E}_2, \dots, u_\ell, \mathcal{E}_\ell$  of SEQ is an enriched prefix of  $v_{j_p}$  for infinitely many  $p \in \mathbb{N}$ .

We note that  $u_1u_2u_3...$  is obviously performable from  $A(x_1,...,x_m)$  (in the LTS  $\mathcal{L}_{\mathcal{G}}^{\mathbb{R}}$ ), and we use the notation

$$A(x_1,\ldots,x_m)=G_0\xrightarrow{u_1}G_1\xrightarrow{u_2}G_2\xrightarrow{u_3}G_3\cdots;$$

in more detail, we write  $G_i \xrightarrow{u_{i+1}} G_{i+1}$  as

$$B_i(x_1,\ldots,x_m)\sigma_i'' \xrightarrow{u_{i+1}} B_{i+1}(x_1,\ldots,x_m)\overline{\sigma}_{i+1}\sigma_i''.$$

By the above definition of SEQ, for each  $k \in [1, m]$  we have

$$k \in \mathcal{E}_i \text{ iff } x_k \text{ occurs in } x_{k'} \overline{\sigma}_{i+1} \text{ for some } k' \in \mathcal{E}_{i+1}.$$
 (9)

For an arbitrary fixed  $p \in \mathbb{N}$  (referring to  $j_p \in \mathcal{D}$ ) let us now consider the infinite path

$$E_0 \xrightarrow{u w_{(1,j_p)}} A(x_1, \dots, x_m) \sigma'_{j_p} \xrightarrow{u_1} G_1 \sigma'_{j_p} \xrightarrow{u_2} G_2 \sigma'_{j_p} \xrightarrow{u_3} \cdots$$

By Prop. 12 there is a sequence  $0 \le i'_0 < i'_1 < i'_2 < \cdots$  corresponding to the block partition of the direct-rhs-stair path in the presentation  $G_0\sigma'_{j_p} \xrightarrow{u_1} G_1\sigma'_{j_p} \xrightarrow{u_2} G_2\sigma'_{j_p} \xrightarrow{u_3} \cdots$ . We now apply Prop. 13 to this direct-rhs-stair path partitioned to blocks.

Since we assume that  $E_0$  has no witness,  $G_0\sigma'_{j_p}$  has no witness either. There is thus some finite set  $\mathcal{F}'$  of bisim-classes such that for each  $k \in [1, m]$  that survives some (consecutive) m blocks we have  $[x_k\sigma''_i\sigma'_{j_p}]_{\sim} \in \mathcal{F}'$  for all  $i \in \{i'_0, i'_1, i'_2, \ldots\}$ . Since any  $k \in \mathcal{E}_i$  survives forever (due to the above noted fact (9)), we have  $[x_k\sigma''_i\sigma'_{j_p}]_{\sim} \in \mathcal{F}'$  for all  $i \in \{i'_0, i'_1, i'_2, \ldots\}$  and  $k \in \mathcal{E}_i$ .

By the pigeonhole principle, there are i < i' in  $\{i'_0, i'_1, i'_2, \dots\}$  such that  $\mathcal{E}_i = \mathcal{E}_{i'}$  and  $x_k \sigma''_i \sigma'_{j_p} \sim x_k \sigma''_{i'} \sigma'_{j_p}$  for each  $k \in \mathcal{E}_i = \mathcal{E}_{i'}$  (and we have  $B_i = B_{i'}$  by our block-partition).

By our assumptions on SEQ, there must be some  $j_r \in \mathcal{D}$   $(r \in \mathbb{N})$  with a corresponding enriched prefix

$$\mathcal{E}_0, u_1, \mathcal{E}_1, u_2, \mathcal{E}_2, \dots, u_i, \mathcal{E}_i, \dots, u_{i'}, \mathcal{E}_{i'}.$$

We now note that  $x_k \sigma'_{j_r} \sim x_k \sigma'_{j_p}$  for each  $k \in \mathcal{E}_0$ : by definition such k is end-surviving in the basic  $(j_r, j'_r)$ -path, hence  $[x_k \sigma'_{j_r}]_{\sim} \in \mathcal{F}$ , and thus  $[x_k \sigma'_{j_r}]_{\sim} = [x_k \sigma'_{j_p}]_{\sim}$  by our choice of  $\mathcal{D}$ . This entails that  $x_k \sigma''_i \sigma'_{j_r} \sim x_k \sigma''_i \sigma'_{j_p}$  for all  $i \in \mathbb{N}$  and  $k \in \mathcal{E}_i$  (since  $k \in \mathcal{E}_i$  and  $x_\ell \in x_k \sigma''_i$  imply  $\ell \in \mathcal{E}_0$ ). But this implies that the (supposedly) basic  $(j_r, j'_r)$ -path could have been shortened  $(x_k \sigma''_i \sigma'_{j_r} \sim x_k \sigma''_{i'} \sigma'_{j_r})$  for each  $k \in \mathcal{E}_i = \mathcal{E}_{i'}$ ; this is a contradiction. Hence  $E_0$  must have a witness.

# 4 Additional Remarks

The mentioned deterministic case studied in [16, 18] could be roughly explained in our framework as follows: for a deterministic grammar (with at most one rule  $A(x_1, \ldots, x_m) \stackrel{a}{\longrightarrow} \ldots$  for each nonterminal A and each action a), if an eligible stair is reachable from  $E_0$  where the start and the end of the stair are non-equivalent, then  $E_0$  is bisim-infinite. Hence by compositionality a bound on the size of the potential equivalent finite system can be derived, and thus decidability of the full equivalence is not needed here.

In the case equivalent to *normed* pushdown processes, the regularity problem essentially coincides with the boundedness problem, and is thus much simpler. (See, e.g., [15] for a further discussion.)

# **Appendix**

At the ends of Sections 2 and 3.2.1 we mentioned the issues of transforming pushdown automata to first-order grammars, of normalizing the grammars, and of unifying the nonterminal arities. We now deal with these issues in more detail.

## Transforming pushdown automata to first-order grammars

A pushdown automaton (PDA) is a tuple  $M=(Q,\Sigma,\Gamma,\Delta)$  of finite sets where the elements of  $Q,\Sigma,\Gamma$  are called control states, actions (or terminal letters), and stack symbols, respectively;  $\Delta$  contains transition rules of the form  $pY \stackrel{a}{\longrightarrow} q\alpha$  where  $p,q \in Q, Y \in \Gamma, a \in \Sigma \cup \{\varepsilon\}$ , and  $\alpha \in \Gamma^*$ . (We assume  $\varepsilon \notin \Sigma$ .) A PDA  $M=(Q,\Sigma,\Gamma,\Delta)$  generates the labelled transition system

$$\mathcal{L}_M = (Q \times \Gamma^*, \Sigma \cup \{\varepsilon\}, (\stackrel{a}{\longrightarrow})_{a \in \Sigma \cup \{\varepsilon\}})$$

where each rule  $pY \xrightarrow{a} q\alpha$  induces transitions  $pY\beta \xrightarrow{a} q\alpha\beta$  for all  $\beta \in \Gamma^*$ .

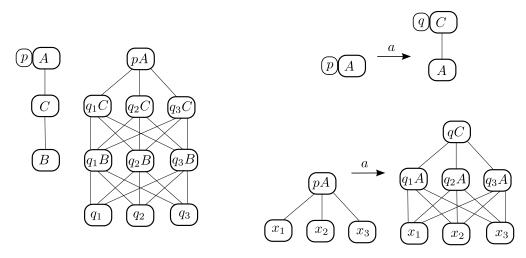


Figure 17: PDA configuration as a term (left), and transforming a rule (right)

Fig. 17 (left) presents a PDA-configuration (i.e. a state in  $\mathcal{L}_M$ ) pACB as a term; here we assume that  $Q = \{q_1, q_2, q_3\}$ . (The string pACB, depicted on the left in a convenient vertical form, is transformed into a term presented by an acyclic graph in the figure.) On the right in Fig. 17 we can see a transformation of a PDA-rule  $pA \stackrel{a}{\longrightarrow} qCA$  into a grammar-rule.

Formally, for a PDA  $M = (Q, \Sigma, \Gamma, \Delta)$ , where  $Q = \{q_1, q_2, \dots, q_m\}$ , we can define the first-order grammar  $\mathcal{G}_M = (\mathcal{N}, \Sigma \cup \{\varepsilon\}, \mathcal{R})$  where  $\mathcal{N} = Q \cup (Q \times \Gamma)$ , with arity(q) = 0 and arity((q, X)) = m; the set  $\mathcal{R}$  is defined below. We write [q] and [qY] for nonterminals q and (q, Y), respectively, and we map each configuration  $p\alpha$  to the term  $\mathcal{T}(p\alpha)$  by structural induction:  $\mathcal{T}(p\varepsilon) = [p]$ , and  $\mathcal{T}(pY\alpha) = [pY](\mathcal{T}(q_1\alpha), \mathcal{T}(q_2\alpha), \dots, \mathcal{T}(q_m\alpha))$ .

For a smooth transformation of rules we introduce a special "stack" symbol x, and we put  $\mathcal{T}(q_ix) = x_i$  (for all  $i \in [1, m]$ ). A PDA-rule  $pY \xrightarrow{a} q\alpha$  in  $\Delta$  is transformed to the grammar rule  $\mathcal{T}(pYx) \xrightarrow{a} \mathcal{T}(q\alpha x)$  in  $\mathcal{R}$ . (Hence  $pY \xrightarrow{a} q_i$  is transformed to  $[pY](x_1, \ldots, x_m) \xrightarrow{a} x_i$  and  $pY \xrightarrow{a} qZ\alpha$  is transformed to  $[pY](x_1, \ldots, x_m) \xrightarrow{a} [qZ](\mathcal{T}(q_1\alpha x), \ldots, \mathcal{T}(q_m\alpha x))$ .)

It is obvious that the LTS  $\mathcal{L}_M$  is isomorphic with the restriction of the LTS  $\mathcal{L}_{\mathcal{G}_M}^{\Lambda}$  to the states  $\mathcal{T}(p\alpha)$  where  $p\alpha$  are configurations of M; moreover, the set  $\{\mathcal{T}(p\alpha) \mid p \in Q, \alpha \in \Gamma^*\}$  is

closed w.r.t. reachability in  $\mathcal{L}_{\mathcal{G}_M}^{A}$  (if  $\mathcal{T}(p\alpha) \stackrel{a}{\longrightarrow} F$  in  $\mathcal{L}_{\mathcal{G}_M}^{A}$ , then  $F = \mathcal{T}(q\beta)$  where  $p\alpha \stackrel{a}{\longrightarrow} q\beta$  in  $\mathcal{L}_M$ ).

In fact, we have not allowed  $\varepsilon$ -rules  $A(x_1, \ldots, x_m) \xrightarrow{\varepsilon} E$  in our definition of first-order grammars. We would consider a variant of so called *weak bisimilarity* in such a case, which is undecidable in general (see, e.g., [7] for a further discussion).

In our discussion at the end of Section 2 we mention restricted PDAs where  $\varepsilon$ -rules  $pY \stackrel{\varepsilon}{\longrightarrow} q\alpha$  can be only popping, i.e.  $\alpha = \varepsilon$  in such rules, and deterministic (or having no alternative), which means that if there is a rule  $pY \stackrel{\varepsilon}{\longrightarrow} q$  in  $\Delta$  then there is no other rule with the left-hand side pY (of the form  $pY \stackrel{a}{\longrightarrow} q'\alpha$  where  $a \in \Sigma \cup \{\varepsilon\}$ ). We define the stable configurations as  $p\varepsilon$  and  $pY\alpha$  where there is no rule  $pY \stackrel{\varepsilon}{\longrightarrow} ...$  in  $\Delta$ ; for the restricted PDAs we have that any unstable configuration  $p\alpha$  only allows to perform a finite sequence of  $\varepsilon$ -transitions that reaches a stable configuration. Hence it is natural to restrict the attention to the "visible" transitions  $p\alpha \stackrel{a}{\longrightarrow} q\beta$  ( $a \in \Sigma$ ) between stable configurations; such transitions might encompass sequences of  $\varepsilon$ -steps. In defining the grammar  $\mathcal{G}_M$  we can naturally avoid the explicit use of deterministic popping  $\varepsilon$ -transitions, by "preprocessing" them: in our inductive definition of  $\mathcal{T}(p\alpha)$  (and  $\mathcal{T}(p\alpha x)$ ) we add the following item: if pY is unstable, since there is a rule  $pY \stackrel{\varepsilon}{\longrightarrow} q$ , then  $\mathcal{T}(pY\alpha) = \mathcal{T}(q\alpha)$ . Fig. 18 (right) shows the grammar-rule  $\mathcal{T}(pAx) \stackrel{a}{\longrightarrow} \mathcal{T}(qCAx)$  (arising

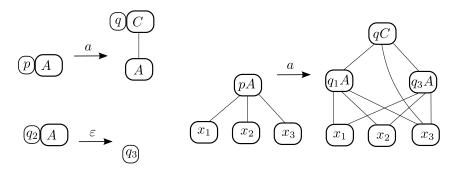


Figure 18: Deterministic popping  $\varepsilon$ -transitions are "preprocessed"

from the PDA-rule  $pA \xrightarrow{a} qCA$ ), when  $Q = \{q_1, q_2, q_3\}$  and there is a PDA-rule  $q_2A \xrightarrow{\varepsilon} q_3$ , while  $q_1A$ ,  $q_3A$  are stable. Such preprocessing causes that the term  $\mathcal{T}(p\alpha)$  can have branches of varying lengths.

#### Normalization of grammars

We call a grammar  $\mathcal{G} = (\mathcal{N}, \Sigma, \mathcal{R})$  normalized if for each  $A \in \mathcal{N}$  and each  $i \in [1, arity(A)]$  there is a ("sink") word  $w_{(A,i)} \in \mathcal{R}^+$  such that  $A(x_1, \dots, x_{arity(A)}) \xrightarrow{w_{(A,i)}} x_i$ .

For any grammar  $\mathcal{G} = (\mathcal{N}, \Sigma, \mathcal{R})$  we can find some words  $w_{(A,i)}$  or find out their non-existence, for all  $A \in \mathcal{N}$  and  $i \in [1, arity(A)]$ , as shown below. For technical convenience we will also find some words  $w_{(E',x_i)} \in \mathcal{R}^*$  satisfying  $E' \xrightarrow{w_{(E',x_i)}} x_i$  for subterms E' of the rhs (right-hand sides) E of the rules  $A(x_1, \ldots, x_m) \xrightarrow{a} E$  in  $\mathcal{R}$ .

We put  $w_{(x_i,x_i)} = \varepsilon$  (for subterms  $x_i$  of the rhs), while all other  $w_{(E',x_i)}$  and all  $w_{(A,i)}$  are undefined in the beginning. Then we repeatedly define so far undefined  $w_{(A,i)}$  or  $w_{(E',x_i)}$  by applying the following constructions, as long as possible:

• put  $w_{(A,i)} = r w_{(E,x_i)}$  if there is a rule  $r : A(x_1,\ldots,x_m) \stackrel{a}{\longrightarrow} E$  and  $w_{(E,x_i)}$  is defined;

• put  $w_{(E',x_i)} = w_{(A,j)} w_{(E'',x_i)}$  if ROOT(E') = A, E'' is the j-th root-successor in E', and  $w_{(A,j)}, w_{(E'',x_i)}$  are defined.

The correctness is obvious. The process could be modified to find some shortest  $w_{(A,i)}$  (and  $w_{(E',x_i)}$ ) that exist but this is not important here. For any pair (A,i) for which  $w_{(A,i)}$  has remained undefined such word obviously does not exist, hence the *i*-th root-successor  $G_i$  of any term  $A(G_1,\ldots,G_m)$  is "non-exposable" and thus plays "no role" (not affecting the bisim-class of  $A(G_1,\ldots,G_m)$ ). We will now show a safe removal of such non-exposable root-successors.

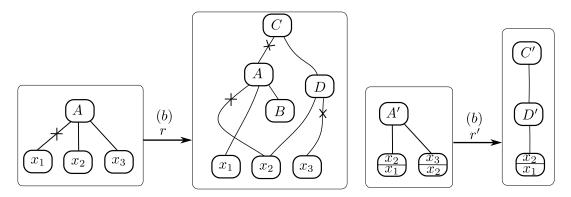


Figure 19: Modifying (cutting) a rule r, when  $w_{(A,1)}$ ,  $w_{(C,1)}$ , and  $w_{(D,2)}$  do not exist

For  $\mathcal{G} = (\mathcal{N}, \Sigma, \mathcal{R})$  we put  $\mathcal{G}' = (\mathcal{N}', \Sigma, \mathcal{R}')$  where the sets  $\mathcal{N}' = \{A' \mid A \in \mathcal{N}\}$  and  $\mathcal{R}' = \{r' \mid r \in \mathcal{R}\}$  are defined below. For  $A \in \mathcal{N}$ , we put

$$SINK(A) = \{i \in [1, arity(A)] \mid \text{ there is some } w_{(A,i)}\}, \text{ and } arity(A') = |SINK(A)|.$$

We define the mapping  $Cut : Terms_{\mathcal{N}} \to Terms_{\mathcal{N}'}$  by the following structural induction:

- 1.  $Cut(x_i) = x_i$ ;
- 2.  $Cut(A(G_1, G_2, ..., G_m)) = A'(Cut(G_{i_1}), Cut(G_{i_2}), ..., Cut(G_{i_{m'}})),$ where  $1 \le i_1 < i_2 < \cdots < i_{m'} \le m$  and  $\{i_1, i_2, ..., i_{m'}\} = Sink(A).$

The set of rules  $\mathcal{R}' = \{r' \mid r \in \mathcal{R}\}$  is defined as follows:

for 
$$r: A(x_1, \ldots, x_m) \xrightarrow{a} E$$
 we put  $r': Cut(A(x_1, \ldots, x_m)) \sigma \xrightarrow{a} Cut(E) \sigma$ 

where  $\sigma = \{(x_{i_1}, x_1), (x_{i_2}, x_2), \dots, (x_{i_{m'}}, x_{m'})\}$  for  $\{i_1, i_2, \dots, i_{m'}\} = \operatorname{SINK}(A)$ . (Fig. 19 depicts the transformation of  $r: A(x_1, x_2, x_3) \xrightarrow{b} C(A(x_2, x_1, B), D(x_2, x_3))$  to  $r': A'(x_2, x_3)\sigma \xrightarrow{b} C'(D'(x_2))\sigma$  where  $\sigma = \{(x_2, x_1), (x_3, x_2)\}$ .) We note that for each variable  $x_i$  occurring in  $\operatorname{CUT}(E)$  we must have  $i \in \operatorname{SINK}(A) = \{i_1, i_2, \dots, i_{m'}\}$ ; hence  $\sigma$  yields a one-to-one renaming of "place-holders" in  $\operatorname{CUT}(A(x_1, \dots, x_m)) \xrightarrow{a} \operatorname{CUT}(E)$ .

It is easy to check that Cut maps Terms<sub>N</sub> onto Terms<sub>N'</sub>, and that  $G \xrightarrow{r} H$  in  $\mathcal{L}_{\mathcal{G}}^{\mathbb{R}}$  implies  $\text{Cut}(G) \xrightarrow{r'} \text{Cut}(H)$  in  $\mathcal{L}_{\mathcal{G}'}^{\mathbb{R}}$ ; moreover, if  $G' \xrightarrow{r'} H'$  in  $\mathcal{L}_{\mathcal{G}'}^{\mathbb{R}}$  and  $G \in \text{Cut}^{-1}(G')$ , then there is  $H \in \text{Cut}^{-1}(H')$  such that  $G \xrightarrow{r} H$  in  $\mathcal{L}_{\mathcal{G}}^{\mathbb{R}}$ .

Grammar  $\mathcal{G}'$  is normalized: if  $\operatorname{SINK}(A) = \{i_1, i_2, \dots, i_{m'}\}$  then for each  $j \in [1, m']$  we have  $A(x_1, \dots, x_m) \xrightarrow{w_{(A,i_j)}} x_{i_j}$ , and thus  $\operatorname{CUT}(A(x_1, \dots, x_m)) \xrightarrow{(w_{(A,i_j)})'} \operatorname{CUT}(x_{i_j})$ , i.e.  $A'(x_{i_1}, \dots, x_{i_{m'}}) \xrightarrow{(w_{(A,i_j)})'} x_{i_j}$ , where w' arises from w by replacing each element r with r'.

We also have that the set  $\{(F, \text{Cut}(F)) \mid F \in \text{Terms}_{\mathcal{N}}\}$  is a bisimulation in the union of  $\mathcal{L}_{\mathcal{G}}^{\text{A}}$  and  $\mathcal{L}_{\mathcal{G}'}^{\text{A}}$ ; hence  $F \sim \text{Cut}(F)$ , and  $E_0$  is bisim-finite in  $\mathcal{L}_{\mathcal{G}}^{\text{A}}$  iff  $\text{Cut}(E_0)$  is bisim-finite in  $\mathcal{L}_{\mathcal{G}'}^{\text{A}}$ . (We can also note that the quotient-LTS  $(\mathcal{L}_{\mathcal{G}}^{\text{A}})_{\equiv_{\text{Cut}}}$  is isomorphic with  $\mathcal{L}_{\mathcal{G}'}^{\text{A}}$ , and that the bisimilarity quotients of  $\mathcal{L}_{\mathcal{G}}^{\text{A}}$  and  $\mathcal{L}_{\mathcal{G}'}^{\text{A}}$  are the same, up to isomorphism.)

## Unification of nonterminal arities

In the proof of Theorem 1 we used m for denoting the arity of each nonterminal in the considered normalized grammar, instead of using  $m_A$  for arity(A). This was not crucial, since it would be straightforward to modify the relevant arguments in the proof, but we also mentioned that we could "harmlessly" achieve the uniformity of nonterminal arities by a construction, while keeping the adjusted grammar normalized. We now sketch such a construction.

Suppose  $\mathcal{G} = (\mathcal{N}, \Sigma, \mathcal{R})$  is normalized and the arities of nonterminals are not all the same; let m be the maximum arity. If there are no nullary nonterminals, then the arities can be unified to m by a straightforward "padding with superfluous copies of root-successors". But we will pad with a special (infinite regular) term, which handles the case of nullary nonterminals as well. (This is illustrated in Figures 20 and 21.)

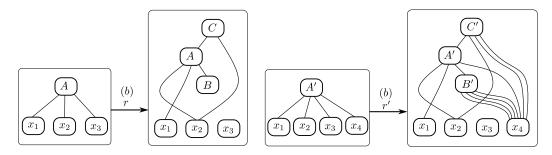


Figure 20: Padding a rule in  $\mathcal{R}$  (left) with  $x_4$ , when m=3

We define the grammar  $\mathcal{G}' = (\mathcal{N}', \Sigma', \mathcal{R}')$  where  $\mathcal{N}' = \{A' \mid A \in \mathcal{N}\} \cup \{A_{SP}\}, \Sigma' = \Sigma \cup \{a_{SP}\},$  and  $\mathcal{R}' = \{r' \mid r \in \mathcal{R}\} \cup \mathcal{R}_{SP}$  as defined below. Each nonterminal in  $\mathcal{N}'$ , including the special nonterminal  $A_{SP}$ , has arity m+1. The set  $\mathcal{R}_{SP}$  (of the rules with the special added action  $a_{SP}$ ) contains the following rules:

- $A_{\text{SP}}(x_1,\ldots,x_{m+1}) \xrightarrow{a_{\text{SP}}} x_i$ , for all  $i \in [1,m+1]$ ;
- $A'(x_1, \ldots, x_{m+1}) \xrightarrow{a_{SP}} x_i$ , for all  $A \in \mathcal{N}$  and  $i \in [arity(A)+1, m+1]$ .

The definition of  $\mathcal{R}' = \mathcal{R}_{SP} \cup \{r' \mid r \in \mathcal{R}\}$  is finished by the following point (see Fig. 20):

• for 
$$r: A(x_1, \ldots, x_{arity(A)}) \stackrel{a}{\longrightarrow} E$$
 we put  $r': A'(x_1, \ldots, x_{m+1}) \stackrel{a}{\longrightarrow} PAD(E, x_{m+1})$ .

The expression  $PAD(E, x_{m+1})$  is clarified by the following inductive definition of PAD(F, H) (padding  $F \in TERMS_N$  with certain H):

- 1. PAD $(x_i, H) = x_i$ ;
- 2.  $PAD(A(G_1, ..., G_{m'}), H) = A'(PAD(G_1, H), ..., PAD(G_{m'}, H), H, ..., H),$ where m' = arity(A), and m+1-m' copies of H are used to "fill" the arity m+1 of A'.

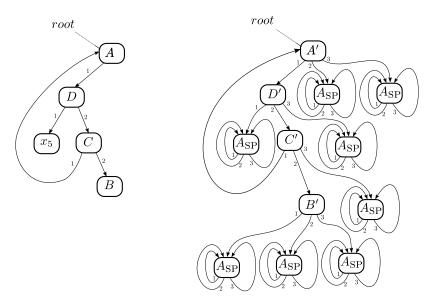


Figure 21: A term  $F \in \text{Terms}_{\mathcal{N}}$  (left) and  $\text{Pad}_{SP}(F)$  (right), when m=2

Besides the above case  $PAD(E, x_{m+1})$  we use the definition of PAD(F, H) also for  $H = E_{SP}$ , i.e., for the special (infinite regular) term

$$E_{\text{SP}} = A_{\text{SP}}(x_1, \dots, x_{m+1}) \sigma^{\omega}$$
 where  $x_i \sigma = A_{\text{SP}}(x_1, \dots, x_{m+1})$  for all  $i \in [1, m+1]$ .

(In Fig. 21 there are several copies of the least presentation of  $E_{\rm SP}$  when m=2.) The behaviour of  $E_{\rm SP}$  is trivial: its only outgoing transition in  $\mathcal{L}_{\mathcal{G}'}^{\rm A}$  is the loop  $E_{\rm SP} \xrightarrow{a_{\rm SP}} E_{\rm SP}$ .

We define the mapping  $PAD_{SP}$ :  $TERMS_{\mathcal{N}} \to TERMS_{\mathcal{N}'}$  by  $PAD_{SP}(F) = PAD(F, E_{SP})\sigma_{SP}$ where  $x_i\sigma_{SP} = E_{SP}$  for each variable  $x_i$ . (The support of  $\sigma_{SP}$  is infinite but this causes no problem.) Hence there are no variables in  $PAD_{SP}(F)$ . (Fig. 21 shows an example. We note that if the nullary nonterminal B happens to be dead in  $\mathcal{L}_{\mathcal{G}}^{A}$ , then  $B' \sim E_{SP}$  in  $\mathcal{L}_{\mathcal{G}'}^{A}$ ; this is the reason for replacing the variables with  $E_{SP}$ .)

The mapping  $Pad_{SP}$  is injective (but not onto  $Terms_{\mathcal{N}'}$ ) and the following conditions obviously hold:

- if  $G \xrightarrow{r} H$  (in  $\mathcal{L}_{\mathcal{G}}^{\mathbb{R}}$ ) then  $PAD_{SP}(G) \xrightarrow{r'} PAD_{SP}(H)$  (in  $\mathcal{L}_{\mathcal{G}'}^{\mathbb{R}}$ );
- if  $PAD_{SP}(G) \xrightarrow{r'} H'$  then there is H such that  $PAD_{SP}(H) = H'$  and  $G \xrightarrow{r} H$ .

The rules in  $\mathcal{R}_{SP}$  guarantee that  $\mathcal{G}'$  is normalized (if  $\mathcal{G}$  is normalized), and they also induce that the special action  $a_{SP}$  is enabled in any term  $PAD_{SP}(G)$  (in  $\mathcal{L}_{\mathcal{G}'}^{A}$ ); moreover,  $PAD_{SP}(G) \xrightarrow{a_{SP}} H'$  entails that  $H' = E_{SP}$ .

We now note that any set  $\mathcal{B} \subseteq \text{Terms}_{\mathcal{N}} \times \text{Terms}_{\mathcal{N}}$  is a bisimulation in  $\mathcal{L}_{\mathcal{G}}^{\text{A}}$  iff  $\mathcal{B}' = \{(E_{\text{SP}}, E_{\text{SP}})\} \cup \{(\text{Pad}_{\text{SP}}(F), \text{Pad}_{\text{SP}}(G)) \mid (F, G) \in \mathcal{B})\}$  is a bisimulation in  $\mathcal{L}_{\mathcal{G}'}^{\text{A}}$ . We deduce that  $E \sim F$  in  $\mathcal{L}_{\mathcal{G}}^{\text{A}}$  iff  $\text{Pad}_{\text{SP}}(E) \sim \text{Pad}_{\text{SP}}(F)$  in  $\mathcal{L}_{\mathcal{G}'}^{\text{A}}$ , and  $E_0$  is bisim-finite in  $\mathcal{L}_{\mathcal{G}}^{\text{A}}$  iff  $\text{Pad}_{\text{SP}}(E_0)$  is bisim-finite in  $\mathcal{L}_{\mathcal{G}'}^{\text{A}}$ .

**Author's acknowledgements.** This work has been supported by the Grant Agency of the Czech Rep., project GAČR:15-13784S. I also thank Stefan Göller for drawing my attention to the decidability question for regularity of pushdown processes, for discussions about some related works (like [18]), and for detailed comments on a previous version of this paper.

# References

- [1] Michael Benedikt, Stefan Göller, Stefan Kiefer, and Andrzej S. Murawski. Bisimilarity of pushdown automata is nonelementary. In *Proc. LICS 2013*, pages 488–498. IEEE Computer Society, 2013.
- [2] Christopher H. Broadbent and Stefan Göller. On bisimilarity of higher-order pushdown automata: Undecidability at order two. In *FSTTCS 2012*, volume 18 of *LIPIcs*, pages 160–172. Schloss Dagstuhl Leibniz-Zentrum für Informatik, 2012.
- [3] Bruno Courcelle. Recursive applicative program schemes. In Jan van Leeuwen, editor, Handbook of Theoretical Computer Science, vol. B, pages 459–492. Elsevier, MIT Press, 1990.
- [4] Bruno Courcelle. The monadic second-order logic of graphs IX: machines and their behaviours. *Theor. Comput. Sci.*, 151(1):125–162, 1995.
- [5] Petr Jančar. Bisimulation equivalence of first-order grammars. In *Proc. ICALP'14 (II)*, volume 8573 of *LNCS*, pages 232–243. Springer, 2014.
- [6] Petr Jančar. Equivalences of pushdown systems are hard. In *Proc. FOSSACS 2014*, volume 8412 of *LNCS*, pages 1–28. Springer, 2014.
- [7] Petr Jančar and Jiri Srba. Undecidability of bisimilarity by defender's forcing. *J. ACM*, 55(1), 2008.
- [8] Petr Jančar and Jiri Srba. Note on undecidability of bisimilarity for second-order push-down processes. CoRR, abs/1303.0780, 2013. URL: http://arxiv.org/abs/1303.0780.
- [9] Teodor Knapik, Damian Niwinski, and Pawel Urzyczyn. Higher-order pushdown trees are easy. In *Proc. FOSSACS 2002*, volume 2303 of *LNCS*, pages 205–222. Springer, 2002.
- [10] Antonín Kučera and Richard Mayr. On the complexity of checking semantic equivalences between pushdown processes and finite-state processes. *Inf. Comput.*, 208(7):772–796, 2010.
- [11] Luke Ong. Higher-order model checking: An overview. In *Proc. LICS 2015*, pages 1–15. IEEE Computer Society, 2015.
- [12] Sylvain Schmitz. Complexity hierarchies beyond elementary. TOCT, 8(1):3, 2016.
- [13] Géraud Sénizergues. L(A)=L(B)? Decidability results from complete formal systems. Theoretical Computer Science, 251(1-2):1-166, 2001.
- [14] Géraud Sénizergues. The bisimulation problem for equational graphs of finite out-degree. SIAM J. Comput., 34(5):1025–1106, 2005.
- [15] Jiri Srba. Roadmap of infinite results. In Current Trends In Theoretical Computer Science, The Challenge of the New Century, volume 2, pages 337–350. World Scientific Publishing Co., 2004. Updated version at http://users-cs.au.dk/srba/roadmap/.
- [16] Richard Edwin Stearns. A regularity test for pushdown machines. Information and Control, 11(3):323–340, 1967.

- [17] Colin Stirling. Deciding DPDA equivalence is primitive recursive. In *Proc. ICALP'02*, volume 2380 of *LNCS*, pages 821–832. Springer, 2002.
- [18] Leslie G. Valiant. Regularity and related problems for deterministic pushdown automata.  $J.\ ACM,\ 22(1):1-10,\ 1975.$
- [19] Igor Walukiewicz. Automata theory and higher-order model-checking. ACM SIGLOG News, 3(4):13-31, 2016.