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CONCATENATION AS BASIS FOR A COMPLETE SYSTEM OF ARITHMETIC

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In [3] Myhill has constructed a complete system K which allows in it the development of a large and important section of classical mathematics. Completeness is achieved essentially by sacrificing universal quantification and introducing instead the proper ancestral as a primitive idea.

In the following we are presenting a system K_0 which will be shown to be equivalent to K (i.e. the primitive operators of both systems are mutually definable in terms of one another). K_0 is also complete and covers the same ground as K. K_0, however, differs from K by the introduction of the limited universal quantifier instead of the proper ancestral and of concatenation instead of the ordered-pair function as primitive operators. By a further reduction K_0 will be shown to be equivalent to the system K_1 not containing the abstraction-operator and the class-membership relation.

I. Primitive constants of K_0 are: 'w', 'L', 'R', '$=$', '$\frown$', 'ϵ', '$\hat{}$', '$.$', '$\vee$', '$(x)y$' (limited universal quantifier with operator variable x and parameter y), '(Ex)' (existential quantifier with operator variable x). 'w', 'L', 'R' may be thought of as denoting '0', '(', ')', respectively, but for the formal development of K_0 these interpretations are not strictly relevant. It will rather be seen that 'w' itself will play the part of '0' in K_0. '$\frown$' denotes the concatenation-function which produces a string from two strings one being placed to the right of the other.

We now state the formation-rules of K_0, which will be found to be in part modifications of Myhill's formation-rules for K.

Definition of Chain Matrix.

(1) 'w', 'L', 'R' are chain-matrices.
(2) If x is a variable, x is a chain-matrix.
(3) If x and y are chain-matrices, so is $(x\frown y)$.

Definition of Formula, Chain, Class-name.

(1) If x and y are chain-matrices, $(x = y)$ is a statement-matrix.
(2) If p and q are statement-matrices, so are $(p \,.\, q)$ and $(p \vee q)$.
(3) If p is a statement-matrix, x and y variables, $(Ex)p$ and $(x)yp$ are statement-matrices, and $\hat{x}p$ is a class-matrix.
(4) If x is a chain-matrix, y a class-matrix, $x \epsilon y$ is a statement-matrix.

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(5) A given occurrence of a variable x is bound to a given occurrence of (Ex), (x), or $\hat{x}$, if it stands in the statement-matrix or class-matrix beginning with the occurrence of (Ex), (x), or $\hat{x}$.
(6) A given occurrence is bound in an expression, if that expression contains an occurrence of (Ex), (x), or $\hat{x}$ to which the given occurrence is bound.
(7) An occurrence of a variable in an expression is free in that expression, if it is not bound in that expression.
(8) An expression is closed, if no variable is free in that expression.
(9) A closed statement-matrix is a formula. A closed chain-matrix is a chain. A closed class-matrix is a class-name.

Axiom Schema for Theorems.

A. If a and b become identical when all parentheses and concatenation signs are dropped, then $(a = b)$ is a theorem.

Rules of Inference.

For the sake of brevity we state the rules of inference of K_0 by giving inference schemata in which we shall make use of the inference-line and the function $\begin{bmatrix} x \\ y \end{bmatrix} p$, where $\begin{bmatrix} x \\ y \end{bmatrix} p$ is like p except in containing y in place of all free occurrences of x.

I1. $$\frac{\ldots a -,\ a = b}{\ldots b -.}$$

I2. $$\frac{p,\ q}{p \cdot q.}$$

I3a. $$\frac{p}{p \vee q.}$$

I3b. $$\frac{q}{p \vee q.}$$

I4. $$\frac{\begin{bmatrix} x \\ a \end{bmatrix} p}{(Ex)p.}$$

I5. $$\frac{\begin{bmatrix} x \\ a \end{bmatrix} p}{a \in \hat{x}p.}$$

I6. $$\frac{\left(\begin{bmatrix} x \\ w \end{bmatrix} p \cdot \begin{bmatrix} x \\ L \end{bmatrix} p \cdot \begin{bmatrix} x \\ R \end{bmatrix} p\right)}{(x)\xi\, p} \qquad (\xi = w,\ L \text{ or } R).$$

I7. $$\frac{(x)a\{p \cdot [x^\frown w]p \cdot [x^\frown L]p \cdot [x^\frown R]p\}}{(x)a^\frown \xi\, p} \qquad (\xi = w,\ L \text{ or } R).$$

Interpretation of K_0.

Int. 1. 'w', 'L', and 'R' designate '0', '(' and ')', respectively.

Int. 2. If a designates x and b designates y, then $(a^\frown b)$ designates the string in which the string y stands to right of the string x.

Int. 3. $(a = b)$ is true, if and only if a and b designate the same string.

Int. 4. $(p \centerdot q)$ is true, if and only if p is true and q is true.

Int. 5. $(p \vee q)$ is true, if and only if p and q are closed and at least one of them is true.

Int. 6. $(x)ap$ is true, if and only if the result of writing any chain denoting a string of length equal to or less than that denoted by a for all free occurrences of x in p is true.

Int. 7. $(Ex)p$ is true, if and only if there is a chain a the result of writing which for all free occurrences of x in p is true.

Int. 8. $(a \in \hat{x}p)$ is true, if and only if the result of writing a for all free occurrences of x is true.

II. Completeness of K_0. We define the order of a formula as the number of occurrences of operators, where an operator is any of the signs '$\centerdot$', '$\vee$', 'E', 'ϵ', and '( )' where the parentheses of the last sign enclose an arbitrary variable. Formulae of order 0 which do not contain any operators are of the form $(x = y)$. Such formulae, if true, are obviously theorems in virtue of A.

Suppose that all formulae of order n are provable if true. Consider the true formulae of order $n+1$. They fall into the following five classes of different forms:

I. $(p \centerdot q)$. II. $(p \vee q)$. III. $(Ex)p$. IV. $x \in \hat{y}p$. V. $(x)yp$.

In I and II p and q are formulae of order n. In I both p and q must be true (Int. 4), they are thus provable by the inductive hypothesis. Therefore I is also provable (I2).

In II both p and q must be closed and at least one of them true (Int. 5). One of them is then provable by the inductive hypothesis. Hence II is also provable (I3a or I3b).

In III there must be a chain y such that if y is written for all occurrences of x in p the resulting formula which is of order n is true (Int. 7), thus also provable by the inductive hypothesis. Hence III is provable (I4).

In IV the result of writing x for all occurrences of y in p which is of order n is true (Int. 8), and thus also provable by the inductive hypothesis. Hence IV is provable (I5).

In V all formulae resulting from p by writing any chain denoting a string of length equal to or less than the length of the string denoted by y for all occurrences of x in p is true (Int. 6). These are of order n, and thus provable by the inductive hypothesis. Hence by successive application of I1, I2, I6, and I7, V may be deduced from them.

Hence all true formulae of K_0 are provable, which completes the completeness-proof for K_0.

III. K_0 is equivalent to K. It is clear that K_0 is a subsystem of K, for all primitive operators of K_0 except '$\frown$' and '$(x)y$' are primitive operators of K. Further $x\frown y = z$ and $(x)yp$ are primitive recursive relations and hence definiable in K.

We shall now show that K is also a subsystem of K_0. For this purpose we have to state definitions of '$;$' and '$*$' in terms of the primitive operators of K_0. In framing these definitions we make use of the sequence-technique first employed by Quine in [4] with certain modifications.

We begin by quoting three auxiliary definitions from [4]. Parentheses will be dropped frequently where there is no danger of ambiguity.

Def. 1. $x \mathrm{B}\, y$ for $((x = y) \vee (Ez)\, (x\frown z = y))$.
Interpretation: x is the beginning of y.

Def. 2. $x \mathrm{E}\, y$ for $((x = y) \vee (Ez)\, (z\frown x = y))$.
I: x is the end of y.

Def. 3. $x \mathrm{P}\, y$ for $(Ez)\, (z \mathrm{B}\, y \,.\, x\, E\, z)$.
I: x is a part of y.

The further definitions of [4] needed here contain negation which is not available in K_0. We can, however, render these definitions in K_0 by realizing that whenever an existential quantifier occurs in them within the scope of a negation-symbol this can without loss of meaning be replaced by a limited existential quantifier. To this, however, we have a negative in K_0, namely '$(x)y$' provided '$\sim$' operates in turn on existential quantifiers which are in essence limited. We have yet to account for the negation of formulae of the form $a = b$.

Def. 4. $a \neq b$ for $(Ex)(Ey)(Ez)(Et)\, (a\frown x = b \vee b\frown x = a \vee$
$\{(u)y\, (u = y \vee u = z \vee u = t) \,.\, (x\frown y \,\mathrm{B}\, z\frown a \,.\, x\frown t \,\mathrm{B}\, z\frown b)\})$.

Def. 5. $\simeq x \mathrm{B}\, y$ for $(x \neq y \,.\, (z)y\, (x\frown z \neq y))$.
I: x is not the beginning of y.

Def. 6. $\simeq x \mathrm{E}\, y$ for $(x \neq y \,.\, (z)y(z\frown x \neq y))$.
I: x is not the end of y.

Def. 7. $\simeq x \mathrm{P}\, y$ for $(z)y(\simeq z \mathrm{B}\, y) \vee (\simeq x \mathrm{E}\, z)$.
I: x is not a part of y.

Def. 8. x Ing y for $(L\frown R\frown x\frown L\frown R \,\mathrm{P}\, y) \,.\, (\simeq L\frown R \,\mathrm{P}\, x)$.
I: x is an ingredient of y.

Def. 9. $\simeq x \,\mathrm{Ing}\, y$ for $(\simeq L\frown R\frown x\frown L\frown R \,\mathrm{P}\, y) \vee (L\frown R \,\mathrm{P}\, x)$.
I: x is not an ingredient of y.

Def. 10. $x \,\mathrm{Pr}_y\, z$ for $(Eu)\, (u\mathrm{B}y) \,.\, (x \,\mathrm{Ing}\, u) \,.\, (z \,\mathrm{Ing}\, y) \,.\, (\simeq z \,\mathrm{Ing}\, u)$.
I: x and z are ingredients of y, and x is prior to z.

Def. 11. $\mathrm{Br}(x)$ for $L\frown x\frown R$.
I: x in parentheses.

To define the ordered-pair function we first define the class of terms.

Def. 12. Term for $\hat{x}(Ey)\{x \text{ Ing } y \,.\, (z)y\, [\simeq z \text{ Ing } y \vee z = w$
$\vee (Eu)\,(Ev)\,(u \text{ Pr}_y\, z \,.\, v \text{ Pr}_y\, z) \,.\, (z = \text{Br}(v^\frown u) \vee z = \text{Br}(u^\frown v))]\}$.

I: x is a term, if it is an ingredient of a string y, any ingredient z of which is either equal to (i) w, or to (ii) $\text{Br}(u^\frown v)$ or $\text{Br}(v^\frown u)$, where u and v are ingredients of y prior to z.

Def. 13. $x\,;\,y = z$ for $x \in \text{Term} \,.\, y \in \text{Term} \,.\, \text{Br}\,(x^\frown y) = z$.
This is the definition of the ordered-pair function of K.

Def. 14. $*c$ for $\hat{x}\,(Eu)\,(x \text{ Ing } u \,.\, (t)u\, (\simeq t \text{ Ing } u \vee (Es)\,(Er)\,(t = (s\,;\,r) \,.$
$((s\,;\,r) \in c \vee (Em)\,(En)\,(Eq)\,\, (m = (s\,;\,n) \,.\, q = (n\,;\,r) \,.$
$m \text{ Pr}_u\, t \,.\, q \text{ Pr}_u\, t)))))$.

I: x is a member of the ancestral of a class c, if x is an ingredient of a string u any ingredient t of which is equal to a pair $(s\,;\,r)$, and is either a member of c or two pairs $(s\,;\,n)$ and $(n\,;\,r)$ are ingredients of u prior to t.

We have thus shown that the primitive operators of K and K_0 are mutually definable and hence that K and K_0 are equivalent.

IV. K_0 may be somewhat reduced. Firstly, the presence of the operators 'ϵ' and '$\hat{}$' are not strictly essential since we can eliminate them by the definition:

$$a \in \hat{x}p \text{ for } \begin{bmatrix} a \\ x \end{bmatrix} p \text{ where } a \text{ is a chain-matrix.}$$

Further, we may reduce the number of primitive individual symbols from 3 to 2 by using paraphrases for them, e.g.

'*aba*' for 'w',
'*abba*' for 'L', and
'*abbba*' for 'R',

and suitably adjusting the inferential schemata for '$(x)a$'. By these reductions we are led to a system K_1 with 'a', 'b', '$=$', '$^\frown$', '$.$', '$\vee$', '$(x)\ldots$', '(Ex)' as the only primitive constants. K_1 is equivalent to K_0 and hence also to K.

In [5] Quine showed that classical arithmetic can be built up in a system whose primitive constants are 'a', 'b', '$^\frown$', '$=$', '$.$', '$\vee$', '(x)', '$\sim$'. But '$\sim$' is definable in terms of the other primitives of the system together with (Ex), since we can define B as in Def. 1 and then define

$(x \neq y)$ for $(Ez)\,(x^\frown z = y \vee y^\frown z = x \vee (z^\frown a\; B\; a^\frown x \,.\, z^\frown b\; B\; a^\frown y)$
$\vee (z^\frown b\; B\; a^\frown x \,.\, z^\frown a\; B\; a^\frown y))$,
$\sim (p \,.\, q)$ for $\sim p \vee \sim q$,
$\sim (p \vee q)$ for $\sim p \,.\, \sim q$,
$\sim (x)\, p$ for $(Ex) \sim p$,
$\sim (Ex)\, p$ for $(x) \sim p$.

Quine's system differs from K_1 only by containing an unlimited universal quantifier in place of the limited universal quantifier of K_1. It other words, the step from classical arithmetic as presented by Quine to constructive arithmetic in the sense of K_1 lies in limiting universal quantification.

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