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A Characterization of Strictly Locally Testable Languages and Its Application to Subsemigroups of a Free Semigroup

ALDO DE LUCA AND ANTONIO RESTIVO

Laboratorio di Cibernetica del C.N.R., Arco Felice, Napoli, Italy

A syntactic characterization of strictly locally testable languages is given by means of the concept of *constant*. If S is a semigroup and X a subset of S, an element c of S is called constant for X if for all $p, q, r, s \in S^1$ $[pcq, rcs \in X \Rightarrow pcs \in X]$. The main result of the paper states that a recognizable subset X of a free semigroup A^+ is strictly locally testable if and only if all the idempotents of the syntactic semigroup $S(X)$ of X are constants for $X' = X\phi$, where $\phi : A^+ \to S(X)$ is the syntactic morphism. By this result some remarkable consequences are derived for recognizable subsemigroups of A^+. In particular we prove that if X is a recognizable free subsemigroup of A^+ and $Y = X/X^2$ its base then the following conditions are equivalent : (1) X is strictly locally testable. (2) X is locally testable. (3) X is locally parsable and Y is strictly locally testable. (4) X has a bounded synchronization delay and Y is strictly locally testable (5) A positive integer k exists such that all the elements of A^+ whose length is greater than or equal to k, are constants for X. (6) For all the idempotents e of the syntactic semigroup $S(X)$ of X, $eS(X)e \subseteq \{e, 0\}$.

0. INTRODUCTION

In this paper a characterization of *strictly locally testable languages* in the sense of McNaughton and Papert (1971) is given by means of the concept of *constant* introduced by Schützenberger (1975).

If S is a semigroup and X a subset of S, an element $c \in S$ is a constant for X if for all $p, q, r, s \in S^1$

$$pcq, rcs \in X \Rightarrow pcs \in X.$$

Let S be the free semigroup A^+ generated by a finite alphabet A. The characterization of strictly locally subsets X of A^+ is that X *is strictly locally testable if and only if all the sufficiently long words of A^+ are constants for X*. From this result one derives for recognizable languages a *syntactic characterization* of strictly local testability: All the idempotents of the *syntactic semigroup* $S(X)$ have to be constants for $X' = X\phi$, where $\phi: A^+ \to S(X)$ is the *syntactic morphism*.

This condition gives then an algebraic decision procedure for strictly local

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testability which seems to be more simple and manageable than that given by McNaughton (1974).

By this result some remarkable consequences are derived for recognizable subsemigroups of A^+. In fact under the hypothesis that a subsemigroups X of A^+ does not contain two-sided ideals (which is satisfied, for instance, if X is *nondense* or if X is a proper *free* subsemigroup of A^+) the condition that an element $c \in S(X)$ is a constant for $X\phi$ becomes simple $cS(X)c \subseteq \{c, 0\}$.

From this one derives that, under the previous hypotheses, X is strictly locally testable if and only if all the idempotents of $S(X)$ lie in the (unique) 0-minimal ideal J of $S(X)$ and the *group of structure* of J is trivial.

Another result is that if X is a strictly locally testable subsemigroup of A^+ then also its minimal set of generators $Y = X \setminus X^2$ will be so. The converse of this proposition, generally, is not true unless X (which is such that it does not contain two-sided ideals) does not possess strong synchronization properties (i.e. X is to have a bounded synchronization delay). This derives from the following proposition that generalizes a result of Schützenberger (1976) proved in a particular case: If X is a subsemigroup of A^+ that does not contain two-sided ideals and is generated by a strictly locally testable set then $X\phi$ meets all regular $\mathscr{D}$-classes of $S(X)\setminus 0$.

In the last section we consider some remarkable families of subsemigroups of A^+ which are defined by the properties of *local testability* (LT), *strictly local testability* (SLT), *local parsability* (LP), *bounded synchronization delay* (BSD) and of being *very pure* (VP).

When the subsemigroups are *free* these properties are of relevant interest in the theory of information transmission since they characterize the *codes* which are the bases of the corresponding subsemigroups. As it has been shown by Restivo (1974) and Hashiguchi and Honda (1976) all these properties coincide for *finitely generated free* subsemigroups of A^+. This is not, however, the case in general even if one concerns himself with *recognizable* free subsemigroups only. The main proposition of this section shows that *for recognizable free subsemigroups of A^+, $LT = SLT = LP = BSD$ if and only if the bases of subsemigroups are strictly locally testable.* Moreover the family of bounded synchronization delay free subsemigroups is *strictly* contained in that of very pure subsemigroups unless one does not make the further hypothesis that the base is such that any word of it does not contain as a factor an arbitrary large number of code-words.

1. Constants

For definitions and notations of semigroups and of automata theory used in this paper, unless otherwise mentioned, the reader is referred to Clifford and Preston (1961) and Eilenberg (1974) respectively.

Let S be a semigroup and X a subset of S. In the following $S(X)$ denotes the *syntactic semigroup* of X, i.e. $S(X) = S/\phi\phi^{-1}$ where ϕ is the *syntactic morphism* $\phi: S \to S(X)$ and $\phi\phi^{-1}$ is the *syntactic congruence* of X which is defined for all $s, t \in S$ as:

$$s \equiv t \ (\mathrm{mod}\, \phi\phi^{-1}) \Leftrightarrow \forall u, v \in S^1(usv \in X \Leftrightarrow utv \in X).$$

X is called a *recognizable subset* of S if $S(X)$ is a finite semigroup (cf. Eilenberg, 1974).

Following Schützenberger (1975) an element $c \in S$ is called a *constant* for X if for all $m_1, m_2, m_3, m_4 \in S^1$

$$m_1 c\, m_2,\ m_3 c\, m_4 \in X \Rightarrow m_1 c\, m_4 \in X.$$

If $c \in X$ is a constant for X then it follows that for all $m_1, m_2 \in S^1$

$$m_1 c\, m_2 \in X \Rightarrow m_1 c,\ c m_2 \in X.$$

For any $X \subseteq S$ we denote by $C(X)$ the set of all constants of S for X. One has that $C(X) \supseteq T(X)$ where

$$T(X) = \{s \in S \mid S^1 s S^1 \cap X = \varnothing\}.$$

Some elementary properties of $C(X)$, which are straightforward consequences of the previous definitions, are the following:

1. $C(X)$ is a two-sided ideal of S.
2. If $C(X\phi)$ is the set of the constants of $S(X)$ for $X\phi$ then:

$$C(X)\phi \subseteq C(X\phi), \qquad C(X\phi)\,\phi^{-1} \subseteq C(X).$$

From property 2 it follows that $C(X)\phi\phi^{-1} = C(X)$ so that if X is recognizable so will be $C(X)$.

A subset $X \subseteq S$ is called *dense* if $T(X) = \varnothing$. If X is not dense then the syntactic semigroup $S(X)$ has a zero element $0 = T(X)\phi$. On the contrary $S(X)$ can have a zero element even if X is dense (this is the case for instance when X is a two-sided ideal of S).

In the following we simply denote by X' the homomorphic image $X\phi$ of X in $S(X)$ and by $S^1(X)$ the monoid $[S(X)]^1$.

PROPOSITION 1.1. *If c is a constant of $S(X)$ for X' then*

$$cS(X)c \subseteq \{c, 0\}.$$

Proof. Let c be a constant of $S(X)$ for X' and $s \in S(X)$. We have to show that either $csc = 0$ or $csc = c$. Let us then suppose $csc \neq 0$. This implies that there exist $u, v \in S^1(X)$ such that $ucscv \in X'$. Since c is a constant for X' it follows that $ucv \in X'$.

Conversely let $ucv \in X'$. Since $csc \neq 0$ there exist $w, w' \in S^1(X)$ such that $wcscw' \in X'$, that implies $ucscw' \in X'$ and then $ucscv \in X'$.

This shows that c is *syntactically equivalent to csc*. Since the syntactic congruence of X' is the identity, one has $c = csc$. Q.E.D.

Let X now be a *subsemigroup* of S, i.e. $X^2 \subseteq X$. In this case the concept of constant for X is related to that of *synchronizing pair* for X.

We recall that a pair $(u, v) \in S \times S$ is called a synchronizing pair for X if $uv \notin T(X)$ and for all $s, t \in S^1$

$$suvt \in X \rightarrow su, \qquad vt \in X.$$

X is called *synchronizing* if it admits at least a synchronizing pair.

From the definitions one easily derives that *if (u, v) is a synchronizing pair for X then uv is a constant for X. Conversely if $c \in X$ is a constant for X then the pair (c, c) is a synchronizing pair for X.* Hence one has that X is synchronizing if and only if there exists a constant $c \in X$.

Let us now consider a subsemigroup X of S which does not contain two-sided ideals of S. In terms of the syntactic semigroup this property is equivalent to the condition:

$$\text{if } 0 \in S(X) \text{ then } 0 \notin X'. \tag{1}$$

In fact if $S(X)$ has a zero element $0 \in X'$ then X contains the two-sided ideal $0\phi^{-1}$. Conversely if condition (1) is verified X cannot contain a two-sided ideal J. In fact, otherwise, $X \supseteq J = S^1JS^1$. This implies that all the elements of J are mapped by the syntactic morphism into a same element $J\phi$ which is a zero of $S(X)$ that belongs to X'.

The condition that X does not contain two-sided ideals of S is certainly verified when X is a nondense subsemigroup of S (i.e. $T(X) \neq \varnothing$) or when X is a proper subsemigroup of S *free in* S (in french *liberable*), i.e.

$$\forall s \in S,\, x, y \in X[sx, ys \in X \rightarrow s \in X].$$

In this last case, in fact, if one supposes that X contains a two-sided ideal J of S one has $X \supseteq J = S^1JS^1$. This implies that for all $s \in S$, $j \in J$ the elements sj, js, j belong to X and then $s \in X$. Thus $S = X$ whis is a contradiction.

When X is subsemigroup of S which does not contain two-sided ideals it holds the following converse of proposition 1.1.

PROPOSITION 1.2. *Let X be a subsemigroup of S which does not contain two-sided of S. If c is an element of $S(X)$ such that*

$$cS(X)c \subseteq \{c, 0\}$$

then c is a constant of $S(X)$ for X'.

Proof. Let $c \in S(X)$ be such that $cS(X)c \subseteq \{c, 0\}$. If $s, s', t, t' \in S^1(X)$ are such that

$$scs', tct' \in X'$$

then

$$scs'tct' \in X'.$$

Now $cs'tc$ has to be $\neq 0$, otherwise $0 \in X'$. This implies $cs'tc = c$ and then $sct' \in X'$. Q.E.D.

As we shall see in more details in the next section both the hypotheses made on X, i.e. that X is a subsemigroup of S and that X does not contain two-sided ideals, have to be, in general, required in order that proposition 1.2 holds.

At the end of this section we observe that when X is a recognizable subsemigroup of S which does not contain two-sided ideals then if X is nondense (resp. dense) $S(X)$ has a ***unique*** 0-minimal (resp. minimal) ideal J such that $X' \cap J \neq \varnothing$. Hence it follows that J is 0-simple (resp. simple) and $J\backslash 0$ (resp. J) coincide with a $\mathscr{D}$-class of $S(X)$. The $\mathscr{H}$-classes of $S(X)$ are isomorphic to a same abstract group called the ***structure group*** of J. Moreover one can prove that $S(X)$ admits a constant $c \neq 0$ for X'(i.e. X is synchronizing) if and only if the ***structure group of J is trivial.*** In this case $C(X') = J$ (see de Luca, Perrin, Restivo and Termini, 1979).

2. A Characterization of Strictly Locally Testable Languages

Let A be a *finite* set, or *alphabet*, and A^+ (resp. A^*) the *free semigroup* (resp. *free monoid*) generated by it. The elements w of A^+ are called *words* and the neutral element 1 of A the *empty word*. For any word $w \in A^+$, $|w|$ denotes its *length*. The length of the empty word is taken equal to 0.

A subset of A^* is called *language* over the alphabet A. Following McNaughton and Papert (1971) we give the following definition:

A subset X of A^+ is called *strictly locally testable* if a positive integer k and three subsets U, V, W of A^k exist such that:

$$X \cap A^kA^* = (UA^* \cap A^*V)\backslash A^*WA^*.$$

The least integer k for which the previous condition is satisfied is called the *order* of X and X will be also said *k-strictly locally testable.*

The intuitive containt of the definition is the following: a word f of length $\geqslant k$ belongs to X if and only if f has a left-factor (or prefix) in U, a right-factor (or suffix) in V and no one of its factors belongs to W.

A wider family of languages is that of *locally testable languages* which is defined as the Boolean closure of strictly locally testable languages.

An equivalent definition of locally testable languages is the following (McNaughton, 1974). For any $f \in A^kA^*$ the *k-test vector* $t_k(f)$ of f is the triplet $t_k(f) = (p_k(f), I_k(f), s_k(f))$ where $p_k(f)$ (resp. $s_k(f)$) is the prefix (resp. suffix) of f length k and $I_k(f)$ is the set of all factors of f of length k. X is said *k-locally testable* if satisfies the following condition:

$$\forall f, g \in A^+ [(t_k(f) = t_k(g)) \Rightarrow (f \in X \Leftrightarrow g \in X)].$$

X is locally testable if it is k-locally testable for a suitable positive integer k.

From the definition a locally testable language X is recognizable, i.e. $S(X)$ is finite. We recall that a characterization of the syntactic semigroup of locally testable languages has been given independently by McNaughton (1974), Zalcstein (1973) and Brzozowski and Simon (1973).

PROPOSITION 2.1. *Let X be a recognizable subset of A^+. X is locally testable if and only if for all the idempotents of $S(X)$, $eS(X)e$ is a semilattice.*

The fact that $eS(X)e$ is a semilattice means that for all $s, t \in S(X)$

$$esese = ese \qquad \text{and} \qquad esete = etese.$$

Proposition 2.1 is of great interest since it gives, being $S(X)$ finite, an algebraic decision procedure for local testability. More complex, however, seems to be the algebraic characterization of strictly local testability given by McNaughton (1974) which makes use of six suitable conditions on the elements of $S(X)$.

Our aim is to present a simple characterization of strictly local testability in terms of the concept of constant introduced in the previous section.

PROPOSITION 2.2. *A language X is strictly locally testable if and only if a positive integer k exists such that all the words of A^kA^* are constants for X.*

Proof. ($\Rightarrow$) If X is a k-strictly locally testable language then

$$X \cap A^kA^* = (UA^* \cap A^*V) \setminus A^*WA^*$$

where U, V, W are subsets of A^k. We shall prove now that any word $g \in A^kA^*$ is a constant for X. In fact let $m_1, m_2, m_3, m_4 \in A^*$ be such that

$$m_1gm_2,\ m_3gm_4 \in X.$$

Since $|g| \geqslant k$ one has that $m_1g \in UA^*$, $gm_4 \in A^*V$, and $m_1gm_4 \notin A^*WA^*$. This implies $m_1gm_4 \in X$. Thus g is a constant for X.

($\Leftarrow$) The hypothesis is that a positive integer k exists such that all the words of A^kA^* are constants for X. Let us denote by U_k (resp. V_k) the set of prefixes (resp. suffixes) of X of length k. Moreover $T = T(X)$ is the set of the words of A^+ incompletable in X, i.e. $T = \{f \in A^+ \mid A^*fA^* \cap X = \varnothing\}$. We shall prove that:

1. $X \cap A^kA^* = (U_kA^* \cap A^*V_k)\setminus T$.
2. The ideal T is finitely generated, i.e. the *base* W of T is *finite*.

It is obvious that $X \cap A^kA^* \subseteq (U_kA^* \cap A^*V_k)\setminus T$. We have then prove the inverse inclusion.

Let $f \in (U_kA^* \cap A^*V_k)\setminus T$. We can write:

$$f = uf_1 = f_2v, \qquad \text{with} \quad f_1, f_2 \in A^*, \quad u \in U_k, \quad v \in V_k.$$

Since $f \notin T$ there exist $h_1, h_2 \in A^*$ *such that*

$$h_1fh_2 = h_1uf_1h_2 = h_1f_2vh_2 \in X.$$

Moreover, since $u \in U_k$ and $v \in V_k$ there exist $h', h'' \in A^*$ for which

$$uh', h''v \in X.$$

Since $|u| = |v| = k$, u and v are constants for X so that from the previous relations it follows

$$h_1f_2v = h_1uf_1 \in X$$

and then $uf_1 = f \in X$.

Let now $W = T\setminus(A^*TA^+ \cup A^+TA^*)$ be the base of the two-sided ideal $T = A^*WA^*$. We shall prove that the length of the words of W is less than or equal to $k + 1$.

Let $w \in W$ and suppose $|w| \geqslant k + 2$. We can then write $w = aub$ with $a, b \in A$ and $u \in A^kA^*$. Let us show that either $au \in T$ or $ub \in T$. In fact if $au \notin T$ and $ub \notin T$ there would exist $h_1, h_2, h_3, h_4 \in A^*$ such that

$$h_1auh_2, h_3ubh_4 \in X.$$

Since u is a constant for X it would follow

$$h_1aubh_4 = h_1wh_4 \in X,$$

which is absurd as $w \in T$. Thus either $W \cap AT \neq \varnothing$ or $W \cap TA \neq \varnothing$, which contradicts the fact that W is the base of T. Hence the length of the elements of W is $\leqslant k + 1$.

This proves that X is strictly locally testable of order $\leqslant k + 1$. Q.E.D.

A corollary of the previous proposition is the following characterization of the syntactic semigroup of strictly locally testable languages which gives also a decision procedure for strictly local testability.

COROLLARY 2.1. *Let X be a recognizable language. X is strictly locally testable if and only if all the idempotents of $S(X)$ are constants for $X' = X\phi$.*

Proof. ($\Rightarrow$) Since X is strictly locally testable from Proposition 2.2 there exists a positive integer k such that all the words of A^kA^* are constants for X. Let now e be an idempotent of $S(X)$ and f a word of A^+ such that $f\phi = e$ and $|f| \geqslant k$. Since f is a constant for X, e will be a constant for X'.

($\Leftarrow$) Since X is recognizable $S(X)$ is finite. Let $g \in A^{k+1}A^*$ where $k =$ Card $S(X)$. We can write $g = a_1a_2 \cdots a_h$, where the a_i $(i = 1,..., h)$ belong to A and $h \geqslant k + 1$. There must exist then two integers r, s such that $1 \leqslant r < s \leqslant k + 1$ and

$$a_1 \cdots a_r \equiv a_1 \cdots a_r(a_{r+1} \cdots a_s)^p \pmod{\phi\phi^{-1}}$$

for all $p \geqslant 1$. For a sufficiently large p, $(a_{r+1} \cdots a_s)^p\phi$ is an idempotent of $S(X)$ so that $(a_{r+1} \cdots a_s)^p$ is a constant for X. Since $g = a_1 \cdots a_r(a_{r+1} \cdots a_s)^{p+1}a_{s+1} \cdots a_{k+1}$ it follows that g is a constant for X. By proposition 2.2, X is strictly locally testable. Q.E.D.

We remark that from propositions 1.1 and 2.1 one has that *if X is a strictly locally testable language then for all the idempotents e of $S(X)$, $eS(X)e \subseteq \{e, 0\}$.* However for a recognizable language X of A^+ the converse of this proposition is not generally true unless one does not make the hypothesis that *X is a subsemigroup of A^+ which does not contain two-sided ideals* (see, proposition 3.1). Both these two conditions on X have to be, in general, required as it is shown by the following examples.

We recall that according to the characterization given by Brzozowski and Simon (1973), *a language X is generally definite if and only if for all the idempotents $e \in S(X)$, $eS(X)e = \{e\}$.*

Now the language $X = F \cup aA^*b \cup bA^*a$ where $a, b \in A$ and F is a finite set *is generally definite but not strictly locally testable.* This implies that for all the idempotents e of $S(X)$, $eS(X)e = \{e\}$. However for all $k > \max\{|w| \mid w \in F\}$ the word a^k is not a constant for X. Otherwise since $a^kb, ba^k \in X$ it would follow $a^k \in X$ which is absurd.

Let us now consider over the alphabet $A = \{a, b\}$ the two-sided ideal $X = A^*aA^*$. The complement $\bar{X}$ of X is given by $\bar{X} = b^*$. Since $\bar{X}$ is strictly locally testable and $S(\bar{X}) = S(X)$ for all idempotents $e \in S(X)$, $eS(X)e \subseteq \{e, 0\}$. However X is not strictly locally testable (cf. corollary 2.2).

COROLLARY 2.2. *Let X be a strictly locally testable language such that for all positive integers k, $X \cap A^kA^* \subset A^kA^*$. One has that X cannot contain two-sided ideals of A^+. In particular if X is a finitely generated two-sided ideal of A^+ then X is locally testable but not strictly locally testable.*

Proof. Suppose that $X \subseteq A^+$ contains a two-sided ideal J. If X is k-strictly locally testable then by proposition 2.2 all the words $f \in A^kA^*$ are constants for X. Since J is a two-sided ideal of A^+, $J = A^*JA^*$ so that words $f_1, f_2 \in A^*$ exist such that $ff_1, f_2f \in J \subseteq X$. Since f is a constant for X then $f \in X$. This implies $A^kA^* \subseteq X \cap A^kA^*$, i.e. $X \cap A^kA^* = A^kA^*$ that contradicts the hypothesis made on X.

When X is a two-sided ideal of A^+ having a finite base W, i.e. $X = A^*WA^*$ then X is trivially locally testable since its complement $A^+\backslash A^*WA^*$ is strictly locally testable. However X cannot be strictly locally testable. Q.E.D.

3. The Case of Subsemigroups of A^+

In this section we shall consider languages $X \subseteq A^+$ which are subsemigroups of A^+, i.e. $X^2 \subseteq X$. The first proposition shows that for a recognizable subsemigroup X of A^+ which does not contain two-sided ideals the condition that for all the idempotents e of $S(X)$, $eS(X)e \subseteq \{e, 0\}$ is sufficient to assure that X is strictly locally testable.

Proposition 3.1. *Let X be a recognizable subsemigroup of A^+ which does not contain two-sided ideals. X is strictly locally testable if and only if for all the idempotents e of $S(X)$, $eS(X)e \subseteq \{e, 0\}$.*

Proof. ($\Rightarrow$) If X is strictly locally testable then by corollary 2.1 all the idempotents $e \in S(X)$ are constants for X' so that from proposition 1.1 $eS(X)e \subseteq \{e, 0\}$.

($\Leftarrow$) Since X is a subsemigroup of A^+ which does not contain two-sided ideals by proposition 1.2 if e is an idempotent of $S(X)$ such that $eS(X)e = \{e, 0\}$ then e is a constant for X'. Thus all the idempotents of $S(X)$ are constants for X' so that, since X is recognizable, from corollary 2.1 it follows that X is strictly locally testable. Q.E.D.

Let us observe that an equivalent formulation (cf. Section 1) of the previous proposition is the following: under the same hypotheses of proposition 3.1 *X is strictly locally testable if and only if the structure group of the 0-minimal ideal J of $S(X)$ is trivial and all the idempotents of $S(X)$ are in J.*

Let X be a subsemigroup of A^+ and $Y = X/X^2$ its (unique) *minimal set of generators*, i.e. $Y^+ = X$ and any word of Y cannot be expressed as the product of two words of X.

Proposition 3.2. *If X is strictly locally testable so will be $Y = X\backslash X^2$.*

Proof. By proposition 2.2 it is sufficient to show that all sufficiently long words of A^* are constants for Y. If X is k-strictly locally testable all the words of A^kA^* are constants for X. We shall prove that all the words of $A^{2k}A^*$ are constants for Y.

Let $u \in A^{2k}A^*$ and $m_1, m_2, m_3, m_4 \in A^*$ be such that

$$m_1um_2, \quad m_3um_4 \in Y.$$

Since u is a constant for X it follows that $w = m_1um_4 \in X$. If $m_1um_4 \in Y$ the proof is achieved. Let us then suppose that $w \in Y^2Y^*$ and consider a factorization of w in terms of the elements of Y.

We have to distinguish two cases.

Case 1. There exist $u_1, u_2 \in A^*$ such that

$$u_1u_2 = u; \quad m_1u_1, u_2m_4 \in X.$$

Since $|u| \geqslant 2k$ either $|u_1| \geqslant k$ or $|u_2| \geqslant k$. Let us first suppose $|u_1| \geqslant k$ so that u_1 is a constant for X. From the relations:

$$m_1u_1 \in X \qquad \text{and} \qquad m_3u_1u_2m_4 \in Y$$

it follows $m_3u_1 \in X$ and then $m_3um_4 = (m_3u_1)(u_2m_4) \in Y \cap X^2$ which is a contradiction. In a symmetric way if $|u_2| \geqslant k$ then from the relations $u_2m_4 \in X$ and $m_1u_1u_2m_2 \in Y$ one has $u_2m_2 \in X$ and then $m_1um_2 = (m_1u_1)(u_2m_2) \in Y \cap X^2$ that is a contradiction.

Case 2. There exist $h, k \in A^*$, $h_1, k_1 \in X \cup \{1\}$ such that

$$h_1h = m_1, \qquad kk_1 = m_4 \qquad \text{and} \quad huk \in Y.$$

Since u is a constant for X it follows that hum_2, $m_3uk \in X$ and

$$m_1um_2 = (h_1)(hum_2) \in Y$$
$$m_3um_4 = (m_3uk)\, k_1 \in Y.$$

If $h_1 = k = 1$ then $w \in Y$ which has been excluded. Thus either h_1 or k_1 have to be $\neq 1$. Thus implies from the above relations $Y \cap X^2 \neq \varnothing$, which is absurd.

Hence the only possibility is that $m_1um_4 \in Y$. Thus u is a constant for Y and Y is strictly locally testable. Q.E.D.

Let us observe that one can prove proposition 3.2 also by showing (cf. Appendix) that if

$$X \cap A^kA^* = (UA^* \cap A^*V)\backslash A^*WA^*$$

with $U, V, W \subseteq A^k$ then

$$Y \cap A^{2k}A^* = (U'A^* \cap A^*V')\backslash A^*W'A^*$$

where

$$W' = W \cup VU$$
$$U' = UA^k \backslash X_k U$$
$$V' = A^k V \backslash VX_k$$
$$X_k = \bigcup_{i<k} X \cap A^i.$$

Thus if X is k-strictly locally testable then $Y = X \backslash X^2$ is strictly locally testable of order $\leqslant 2k$.

In terms of the syntactic semigroups $S(X)$ and $S(Y)$ the proposition 3.2 states that for a recognizable subsemigroup X if all the idempotents of $S(X)$ are constants for $X' = X\phi$ then also the idempotents of $S(Y)$ are constants for $Y' = Y\psi$, where ψ is the syntactic morphism $\psi: A^+ \to S(Y)$.

The converse of proposition 3.2 is not generally true. However the following proposition that generalizes a result proven by Schützenberger (1976) in a particular case, will allow us to find a remarkable condition under which a subsemigroup which does not contain two-sided ideals and is generated by a strictly locally testable set, is strictly locally testable.

PROPOSITION 3.3. *Let X be a subsemigroup of A^+ which does not contain two-sided ideals. If X is generated by a set Y which is strictly locally testable then X' meets all regular $\mathscr{D}$-classes of $S(X) \backslash 0$.*

Proof. It is well known (cf. de Luca, Perrin, Restivo and Termini, 1979) that under the made hypotheses on X, $S(X)$ is finite and has a unique 0-minimal ideal J such that $X' \cap J \neq \varnothing$. Moreover $J \backslash 0$ is a $\mathscr{D}$-class of $S(X)$. Let $e \in S(X)$ be an idempotent. If $e \in J \backslash 0$ the result is trivial since $X' \cap J \neq \varnothing$ and $0 \notin X'$.

Let us then suppose $e \notin J$. We shall prove than an idempotent $f \in X'$ exists such that $f \mathscr{D} e$. By hypothesis Y is strictly locally testable so that a positive integer k exists such that all the words of $A^k A^*$ are constants for Y (cf. proposition 2.2). Let $g \in A^k A^*$ be such that $g\phi = e$ and let us take $h > |g|$. Since $g^h \phi = e \neq 0$ then $A^* g^h A^* \cap X \neq \varnothing$. We have to distinguish two cases:

Case 1. There exists a context $(u, v) \in A^* \times A^*$ such that $w = ug^h v \in X$ and, moreover, there exists a factorization of w in terms of the elements of Y such that each g in w contains a factorization line.

In this case since $h > |g|$ two factorization lines will lie in the same position. This implies that words $g_1, g_2 \in A^*$ and nonnegative integers p, q, r exist such that:

$$ug^p g_1, (g_2 g_1)^{r+1}, g_2 g^q v \in X, \qquad g_1 g_2 = g$$
$$p + q + r + 2 = h.$$

From these relations one derives:

$$(g_2g_1)^{r+1}\phi = (g_2g_1)^2\phi \in X',$$

and $(g_1g_1)^2\phi = f = f^2$.

Moreover $f\mathscr{L}g_1\phi\mathscr{R}e$, so that f and e lie in the same $\mathscr{D}$-class.

Case 2. For each context $(u, v) \in A^* \times A^*$ such that $w = ug^hv \in X$ any factorization of w in terms of the elements of Y is such that a g, at least, in w is factor of a word of Y.

We shall prove that in such a case g^h is a constant for X.

Let $m_1, m_2, m_3, m_4 \in A^*$ be such that

$$m_1g^hm_2, \qquad m_3g^hm_4 \in X.$$

By the made hypothesis integers $r, s \geq 1$, $p, q, m, n \geq 0$ and words $g_1, g_2, g_3, g_4, g_1', g_2', g_3', g_4' \in A^*$ have to exist such that:

$$m_1g^pg_1, \quad g_4g^qm_2 \in X, \quad g_2g^rg_3 \in Y$$
$$m_3g^ng_1', \quad g_4'g^mm_4 \in X, \quad g_2'g^sg_3' \in Y$$
$$g_1g_2 = g_3g_4 = g_1'g_2' = g_3'g_4' = g$$
$$p + q + r + 2 = n + m + s + 2 = h.$$

Since g is a constant for Y, from the relations $g_2g^rg_3, g_2'g^sg_3' \in Y$ it follows $g_2g^tg_3' \in Y$, where $t = \min\{r, s\}$. From this and the previous relations one derives:

$$m_1g^pg_1(g_2g^tg_3')\,g_4'g^mm_4 = m_1g^im_4 \in X$$

with $i = p + t + m + 2$. Since $g^h\phi = g^i\phi$ it follows, syntactically, that $m_1g^hm_4 \in X$. Thus g^h is a constant for X and $e = g^h\phi$ is a constant for X'. This would imply $e \in J$ that is a contradiction.

Thus if $e \notin J$ only the case 1 can occur, that concludes the proof. Q.E.D.

Under the same hypothesis of the previous proposition if X is such that all the idempotents of X' are in J (cf. proposition 4.3) then X is strictly locally testable. In fact in this case all the idempotents of $S(X)$ have to lie in J. This implies that all the idempotents of $S(X)$ are constants and then by corollary 2.1 that X is strictly locally testable.

4. On Some Families of Subsemigroups of a Free Semigroup

In this section we shall concern with some remarkable families of subsemigroups of A^+ which have been studied by many authors mainly when the subsemigroups are *free* and *finitely generated.*

We recall that from a classical result of Schützenberger (1956) a subsemigroup X of A^+ is *free* if and only if it is *free* in A^+. The base $Y = X\backslash X^2$ of a free subsemigroup X of A^+ is usually called *code*. Codes play an essential role in the theory of information transmission (cf. Perrot, 1977).

First of all we can consider subsemigroups of A^+ which are strictly locally testable or locally testable. We shall denote these two families by SLT and LT respectively. When the subsemigroups are *free* and *finitely generated* a result of Restivo (1974) and of Hashiguchi and Honda (1976) shows that the property SLT is equivalent to LT and moreover equivalent to other properties called, respectively, "local parsability" (LP), "bounded synchronization delay" (BSD) and of being "very pure" (VP).

To have the previous equivalence two hypotheses are necessary namely that the subsemigroups are *recognizable* (since so are locally testable languages) and *free* (since so are very pure subsemigroups). However these two hypotheses are not, in general, sufficient in order to have the previous equivalence. A consequence of the main result of this section will be that for recognizable free subsemigroups of A^+ *the coincidence of the classes SLT, LT, BSD and LP holds if and only if the bases of the subsemigroups are strictly locally testable.*

In order to give this result we need, first of all, some precise definitions of the classes VP, BSD and LP.

DEFINITION 4.1. A subsemigroup X of A^+ is called "very pure" if

$$\forall u, v \in A^+[uv, vu \in X \Rightarrow u, v \in X].$$

From the definition one has that a very pure subsemigroup X of A^+ is free in A^+ and then *free*. Let us recall that the notion of "very pure" plaies an important role in some problems *of algebra* (Schützenberger, 1959, 1965), *information theory* (Perrin and Schützenberger, 1977) and *language theory* (Pin, 1979).

DEFINITION 4.2. A subsemigroup X of A^+ has a *bounded synchronization delay* if a positive integer k exists such that any pair in $X^k \times X^k$ is synchronizing. The least integer k for which this condition is verified is called the *synchronization delay* of X.

Let us observe that a pair $(x, y) \in X^k \times X^k$ is synchronizing for X if and only if $xy \in X^{2k}$ is a constant for X. Thus the property BSD is equivalent to say that a positive integer k exists such that all the elements of X^k are constants for X.

This family of subsemigroups has been studied in the case in which X is free and finitely generated by Golomb and Gordon (1965).

DEFINITION 4.3. Let X be a subsemigroup of A^+ and denote, for all $k \geqslant 1$, by U_k (resp. V_k) the set of *prefixes* (resp. *suffixes*) of *length k* of the words of X.

X is *k-locally parsable* if all the elements of $V_k \times U_k$ are synchronizing pairs for X.

Let us explicitly note that this definition of local parsability, given in terms of the notion of synchronizing pairs, is equivalent to that of McNaughton and Papert (1971) and Hashiguchi and Honda (1976).

PROPOSITION 4.1. *Let X be a finitely generated free subsemigroup of A^+. The following conditions are equivalent*:

1. $X \in VP$
2. $X \in SLT$
3. $X \in LT$
4. $X \in LP$
5. $X \in BSD$
6. *There exists a positive integer k such that all the words of A^kA^* are constants for X.*
7. *For all idempotents $e \in S(X)$, $eS(X)e \subseteq \{e, 0\}$.*

The proof of the equivalence of 1, 2, 5 is due to Restivo (1974). The equivalence of 2, 3, 4 has been proven by Hashiguchi and Honda (1976). At last the equivalence of 2, 7, 6 is a consequence of propositions 2.2 and 3.1. One has only to observe that if X is free and $X \subset A^+$ then X does not contain two-sided ideals of A^+. When $X = A^+$ then the result is trivial.

Let us now leave out the hypothesis that X is free and finitely generated. In the general case one has that $SLT \subset LT$. In fact, SLT by the definition is obviously contained in LT. However the subsemigroup $X \subsetneq A^+$, $A = \{a, b\}$, generated by the finite set

$$Y = \{a^2, b^2, ab, ba, a^3, b^3, b^2a, a^2b, ba^2, ab^2\},$$

is an example of *finitely* generated (nonfree) subsemigroup of A^+ which is locally testable but not strictly locally testable (Hashiguchi and Honda, 1976). In fact $X = Y^+$ is locally testable since its complement $\bar{X} = b(ab)^* \cup a(ba)^*$ is the union of two strictly locally testable languages. However for all $n \geqslant 1$, $(ab)^n$, $b(ab)^na \in X$ so that if $X \in SLT$ then for a sufficiently large n, $(ab)^n$ should be a constant that would imply $b(ab)^n \in X$ which is absurd. Furthermore, as we have seen in the previous section except for trivial case a two-sided ideal J having a finite base (this does not mean that $J \setminus J^2$ is finite) is locally testable but not strictly locally testable.

The following proposition shows the inclusion $SLT \subsetneq LP \subsetneq BSD$.

PROPOSITION 4.2. *Let X be a subsemigroup of A^+. If X is k-strictly locally testable then it is k-locally parsable. If X is k-locally parsable then it has a bounded synchronization delay $s \leqslant k$.*

Proof. If X is k-strictly locally testable then all the words of A^kA^* are constants for X. Thus all the elements of the set U_k (resp. V_k) of the prefixes (resp. suffixes) of length k of the words of X are constants for X. We prove that all the pairs in $V_k \times U_k$ are synchronizing pairs for X. Let $(v, u) \in V_k \times U_k$ and $f, g \in A^*$ be such that:

$$fvug \in X.$$

Since $u \in U_k$, $v \in V_k$ there exist $h_1, h_2 \in A^*$ for which uh_1, $h_2v \in X$. As u and v are constants for X it follows that $fv, ug \in X$.

Let us now suppose that all the elements of $V_k \times U_k$ are synchronizing pairs for X; we shall prove that all the elements of X^{2k} are constants for X. Let $w = xy \in X^{2k}$ with $x, y \in X^k$. Since $|x|, |y| \geqslant k$ we can write $x = h_1v$, $y = uh_2$ with $u \in U_k$, $v \in V_k$ and $h_1, h_2 \in A^*$, and $w = h_1vuh_2$. As vu is a constant for X so will be w. Q.E.D.

The following two examples show that $SLT \subset LP \subset BSD$ even if one takes into account only recognizable free subsemigroups of A^+.

EXAMPLE 1. Let $A = \{a, b, c, d, e\}$ and $X = \{ab^+c \cup db^+e\}^+$. X is 1-locally parsable since the elements of $V_1 \times U_1$, where $V_1 = \{c, e\}$, $U_1 = \{a, d\}$, are synchronizing pairs for X. However $X \notin SLT$ since $Y = X\backslash X^2$ is not strictly locally testable (cf. proposition 3.2). A more direct proof is obtained by observing that for all $k \geqslant 1$ $w = ab^kcab^kcdb^ke \in X$ and $w' = ab^kcdb^kcab^ke \notin X$ even though w and w' have the same k-test vector.

EXAMPLE 2. Let $A = \{a, b, c\}$ and $X = \{ab^+c, cb^+, c\}^+$. X has a bounded synchronization delay equal to 1. However X is not locally parsable. In fact for each $k \geqslant 1$ the pair $(b^k, c^k) \in V_k \times U_k$ is not synchronizing for X. In fact let us consider the word $ab^kc^k \in X$. If X is k-locally parsable then $ab^k \in X$ which is absurd.

The following proposition gives a syntactic characterization of BSD property in the case in which the subsemigroups are recognizable and do not contain two-sided ideals.

PROPOSITION 4.3. *Let X be a recognizable subsemigroup of A^+ which does not contain two-sided ideals of A^+. $X \in BSD$ if and only if for all idempotents $e \in X'$, $eS(X)e \subseteq \{e, 0\}$.*

Proof. ($\Rightarrow$) If $X \in BSD$ then there exists a positive integer k such that all the words of X^k are constants for X. Let e be an idempotent of X' and $f \in X$ be such that $f\phi = e$. Since $f^k\phi = f\phi = e$ it follows that e is a constant for X' and then $eS(X)e \subseteq \{e, 0\}$.

($\Leftarrow$) Let us recall that if the idempotents of a semigroup S are contained in a two-sided ideal I then $S^k \subseteq I$ where $k = \mathrm{Card}(S \setminus I) + 1$ (cf. Eilenberg, 1976, p. 81). Let us now observe that if $C(X')$ is the set of constants of $S(X)$ for X' then $D(X') = C(X') \cap X'$ is a two-sided ideal of X'. By hypothesis the set of the idempotents of X' is contained in $D(X')$. Thus $(X')^k \subseteq D(X')$ where $k = \mathrm{Card}(X' \setminus D(X')) + 1$. This implies $X^k\phi \subseteq (X')^k \subseteq C(X')$. Hence all the elements of X^k are constants for X, i.e. $X \in BSD$. Q.E.D.

PROPOSITION 4.4. *Let X be a subsemigroup of A^+ which does not contain two-sided ideals of A^+. If $X \in BSD$ and X is generated by a set Y which is strictly locally testable then $X \in SLT$.*

Proof. From proposition 4.3 for all the idempotents $e \in X'$, $eS(X)e \subseteq \{e, 0\}$. This implies that all the idempotents of X' lie in the 0-minimal ideal J of $S(X)$. By proposition 3.3 it follows that all the idempotents of $S(X)$ lie in J. Since $C(X') = J$ one derives that for all the idempotents $e \in S(X)$, $eS(X)e \subseteq \{e, 0\}$. By proposition 3.1 it follows that $X \in SLT$. Q.E.D.

We remark that a result analogous to that of Proposition 4.4 has been given by Schützenberger (1975) for *aperiodic languages*. He, in fact, proves that *if $X \in BSD$ and it is generated by a set Y which is aperiodic, then X is aperiodic.* The techniques of Schützenberger's proof cannot be, however, extended to our case: indeed he uses in an essential way the fact that the family of aperiodic languages is closed under product which is not the case for strictly locally testable languages.

We say that a subsemigroup X of A^+ satisfies the condition $F(p)$ if

$$A^*Y^pA^* \cap Y = \varnothing,$$

where p is a nonnegative integer and $Y = X \setminus X^2$. This condition means that no word of Y contains as factor a product of a number of words of Y greater than or equal to p. When Y is finite then condition $F(p)$ is verified for $p > \max\{|y| \mid y \in Y\}$.

The following proposition relates for free subsemigroups of A^+ the notions of "very pure" and "bounded synchronization delay".

PROPOSITION 4.5. *Let X be a free subsemigroup of A^+. One has that:*

$X \in BSD \Rightarrow X \in VP$ and satisfies the condition $F(p)$ for a suitable nonnegative integer p. Moreover if X is recognizable then the following conditions are equivalent:

1. *$X \in BSD$*
2. *$X \in VP$ and satisfies the condition $F(p)$ for a suitable nonnegative integer p.*
3. *For all the idempotents $e \in X'$, $eS(X)e \subseteq \{e, 0\}$.*

The proof that $BSD \subsetneq VP$ and that 1 and 2 are equivalent is due to Restivo (1975). The equivalence of 1 and 3 is a straightforward consequence of proposition 4.3 (see, also de Luca, 1980, de Luca, Perrin, Restivo and Termini, 1979). Let us now prove our main result.

PROPOSITION 4.5. *Let X be a recognizable and free subsemigroup of A^+. The following conditions are equivalent*:

1. $X \in LT$
2. $X \in SLT$
3. $X \in LP$ *and its base* $Y = X \backslash X^2$ *is strictly locally testable.*
4. $X \in BSD$ *and its base* $Y = X \backslash X^2$ *is strictly locally testable.*
5. *for all the idempotents e of $S(X)$, $eS(X)e \subsetneq \{e, 0\}$.*
6. *there exists a positive integer k such that all the words of A^kA^* are constants for X.*

Proof. If $X = A^+$ the result is trivial. Let us then suppose that $X \subset A^+$. We observe that since X is free it does not contain two-sided ideals.

(2 $\Rightarrow$ 1) It is trivial by the definition.

(2 $\Leftrightarrow$ 6) By proposition 2.2

(2 $\Rightarrow$ 3) By proposition 4.2 and 3.2

(3 $\Rightarrow$ 4) By proposition 4.2

(4 $\Rightarrow$ 2) By proposition 4.4

(2 $\Leftrightarrow$ 5) By proposition 3.1.

It remains to show that $1 \Rightarrow 2$. We shall prove equivalently that $1 \Rightarrow 6$. Suppose that X is k-locally testable and let $u \in A^kA^*$ and $m_1, m_2, m_3, m_4 \in A^*$ be such that:

$$m_1um_2, \qquad m_3um_4 \in X.$$

One has then

$$w = m_1um_2m_3um_4m_1um_2m_3um_4 \in X.$$

Moreover

$$\begin{aligned} w_1 &= m_1um_4m_1um_2m_3um_4 \in X \\ w_2 &= m_1um_2m_3um_4m_1um_4 \in X. \end{aligned}$$

In fact, since X is k-locally testable, w_1 and w_2 have the same k-test vector as w. Since X is a free subsemigroup of A^+ and $(m_1um_4)X \cap X \neq \varnothing$, $X \cap X(m_1um_4) \neq \varnothing$ from the theorem of Schützenberger (1956) it follows that $m_1um_4 \in X$. Thus u is a constant. This proves that all the words of A^kA^* are constants for X.

Q.E.D.

From proposition 4.5 one derives that *for a recognizable free subsemigroup X of A^+ the properties SLT, LT, LP, BSD are equivalent if and only if the base of X is strictly locally testable.* However they are not equivalent, in general, to VP as shown by the following example.

EXAMPLE 3. Let us consider on the alphabet $A = \{a, b, c\}$ the free subsemigroup $X = Y^+$ where $Y = ba^*c + a$. One has that Y is strictly locally testable, X is very pure but it has not a bounded synchronization delay (cf. Restivo, 1975).

Of course all the above properties coincide with VP if and only if Y is strictly locally testable and satisfies the condition $F(p)$ for a suitable nonnegative integer p.

A remarkable corollary of proposition 4.5 is the following:

COROLLARY 4.1. *Let X_i^+, $i \in \{1,..., m\}$, be a finite number of free subsemigroups having a bounded synchronization delay and such that X_i, $i \in \{1,..., m\}$, are strictly locally testable. Then $\bigcap_{i=1}^m X_i^+$ is a free subsemigroup having a bounded synchronization delay and it is generated by a code which is strictly locally testable.*

Proof. From proposition 4.5 any X_i^+, $i \in \{1,..., m\}$, is strictly locally testable and then $\bigcap_{i=1}^m X_i^+$ is a free strictly locally testable subsemigroup of A^+ since the meet of free subsemigroups of A^+ is still a free subsemigroup of A^+ and the meet of a finite number of strictly locally testable languages is still a strictly locally testable language. Hence by using again proposition 4.5 one derives that $\bigcap_{i=1}^m X_i^+$ has a bounded synchronization delay and its base is strictly locally testable.

Q.E.D.

However, as shown in (de Luca and Restivo, 1979) by an example, the intersection of an *infinite* number of free subsemigroups of A^+ having a bounded synchronization delay and generated by strictly locally testable codes, has not, in general, a bounded synchronization delay.

APPENDIX

PROPOSITION. *Let X be a k-strictly locally testable subsemigroup of A^+:*

$$X \cap A^kA^* = (UA^* \cap A^*V)\backslash A^*WA^*,$$

with U, V, $W \subseteq A^k$. One has then:

$$Y \cap A^{2k}A^* = (U'A^* \cap A^*V')\backslash A^*W'A^*$$

where $Y = X\backslash X^2$, $W' = W \cup VU$, $U' = UA^k\backslash X_kU$, $V' = A^kV\backslash VX_k$, $X_k = \bigcup_{i<k} X \cap A^i$.

Proof. Let $f \in Y \cap A^{2k}A^*$ and suppose that $f \in A^*W'A^*$. Since $f \notin A^*WA^*$ then $f \in A^*VUA^*$. Thus we can write:

$$f = f_1vuf_2 \qquad \text{with} \quad (v, u) \in V \times U, \qquad f_1, f_2 \in A^*.$$

Since $f \in X$ and $|f_1v|, |uf_2| \geqslant k$ it follows $f_1v \in UA^*$, $uf_2 \in A^*V$. This implies $f_1v, uf_2 \in X$ and then $Y \cap X^2 \neq \varnothing$ which is a contradiction. We show now that $f \in (U'A^* \cap A^*V')$. Since $f \in UA^kA^*$ we have to prove that $f \notin U_kUA^*$. Let $f = xuf_1$ with $x \in X_k$, $u \in U$ and $f_1 \in A^*$. Since $|f| \geqslant 2k$ and $|x| < k$ it follows $|uf_1| > k$ so that $uf_1 \in A^*V$. This implies $uf_1 \in X$ and then $Y \cap X^2 \neq \varnothing$ which is a contradiction. In a symmetric way one can show that $f \notin A^*VX_k$. Hence $f \in (U'A^* \cap A^*V')\backslash A^*W'A^*$.

Conversely let $f \in (U'A^* \cap A^*V')\backslash A^*W'A^*$. This implies that $f \in X$. Let us suppose that $f \in X^2$. We can then write $f = x_1x_2$ with $|x_1| + |x_2| \geqslant 2k$. Let us first assume $|x_1|, |x_2| \geqslant k$. One has then $x_1 = f_1v$, $x_2 = uf_2$, with $u \in U$, $v \in V$, $f_1, f_2 \in A^*$, and $f = f_1vuf_2$ which is a contradiction.

If $|x_1| < k$ and $|x_2| \geqslant k$ then $f = x_1uf_2$ with $x_1 \in X_k$, $u \in U$, $f_2 \in A^*$, i.e. $f \in X_kUA^*$ that is a contradiction. Similarly if $|x_1| \geqslant k$ and $|x_2| < k$ one obtains $f \in A^*VX_k$ that is a contradiction. Q.E.D.

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