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The weak theory of monads

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Abstract

We construct a 'weak' version $\mathsf{EM}^w(\mathcal{K})$ of Lack and Street's 2-category of monads in a 2-category $\mathcal{K}$, by replacing their compatibility constraint of 1-cells with the units of monads by an additional condition on the 2-cells. A relation between monads in $\mathsf{EM}^w(\mathcal{K})$ and composite pre-monads in $\mathcal{K}$ is discussed. If $\mathcal{K}$ admits Eilenberg–Moore constructions for monads, we define two symmetrical notions of 'weak liftings' for monads in $\mathcal{K}$. If moreover idempotent 2-cells in $\mathcal{K}$ split, we describe both kinds of weak lifting via an appropriate pseudo-functor $\mathsf{EM}^w(\mathcal{K}) \to \mathcal{K}$. Weak entwining structures and partial entwining structures are shown to realize weak liftings of a comonad for a monad in these respective senses. Weak bialgebras are characterized as algebras and coalgebras, such that the corresponding monads weakly lift for the corresponding comonads and also the comonads weakly lift for the monads.

Keywords: 2-Category; Monad; Weak lifting

0. Introduction

Many constructions, developed independently in Hopf algebra theory, turn out to fit more general situations studied in category theory. For example, crossed products with a Hopf algebra in [21,2,12] are examples of a wreath product in [16]. As another example, the comonad induced by the underlying coalgebra in a Hopf algebra H, has a lifting to the category of modules over any H-comodule algebra. So-called Hopf modules are comodules (also called coalgebras) for the lifted comonad. Galois property of an algebra extension by a Hopf algebra turns out to correspond to comonadicity of an appropriate functor [14].

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Motivated by noncommutative differential geometry, these constructions were extended from Hopf algebras to coalgebras (over a commutative base ring) and to corings (over an arbitrary base ring), see e.g. the pioneering paper [8]. The resulting theory turns out to fit the same categorical framework, only the occurring (co)monads have slightly more complicated forms [14].

For a study of non-Tannakian monoidal categories (i.e. those that admit no strict monoidal fiber functor to the module category of some commutative ring), another direction of generalization was proposed in [6]. The essence of this approach is a weakening of the unitality of some maps and it leads to the replacing of a Hopf algebra by a 'weak' Hopf algebra. In the last decade many Hopf algebraic constructions were extended to the weak setting. Weak crossed products were studied e.g. in [9,13] and [18]. Weak Galois theory was developed, among other papers, in [9,8]. However, just because in these generalizations one deals with non-unital maps, they do not fit the categorical framework of (co)monads, their wreath products and liftings.

The aim of the current paper is to provide a categorical framework for 'weak' constructions. For this purpose, in Section 1 we construct, for any 2-category $\mathcal{K}$, a 2-category $\mathsf{EM}^w(\mathcal{K})$ which contains the 2-category $\mathsf{EM}(\mathcal{K})$ in [16] as a vertically full 2-subcategory. In $\mathsf{EM}^w(\mathcal{K})$ 0-cells are the same as in $\mathsf{EM}(\mathcal{K})$, i.e. monads in $\mathcal{K}$. Since we aim to describe constructions in terms of non-unital maps, in the definition of a 1-cell in $\mathsf{EM}^w(\mathcal{K})$ we impose the same compatibility condition with the multiplications of monads which is required in $\mathsf{EM}(\mathcal{K})$, but we relax the compatibility condition in $\mathsf{EM}(\mathcal{K})$ with the units of monads. Certainly, without compensating it with some other requirements, we would not obtain a 2-category. We show that imposing one further axiom on the 2-cells in addition to the axiom in $\mathsf{EM}(\mathcal{K})$, $\mathsf{EM}^w(\mathcal{K})$ becomes a 2-category, with the same horizontal and vertical composition laws used in $\mathsf{EM}(\mathcal{K})$.

It was observed in [9] that smash products (and more generally crossed products [18]) by weak bialgebras are not unital algebras. Motivated by the definition of a pre-unit in [9], we study pre-monads (defined in Section 2) in any 2-category $\mathcal{K}$. In Section 2 we interpret 'weak crossed products' in [13] as monads in $\mathsf{EM}^w(\mathcal{K})$. This leads to a bijection between monads $t \xrightarrow{s} t$ in $\mathsf{EM}^w(\mathcal{K})$ and pre-monad structures (in $\mathcal{K}$) on the composite 1-cell st with a 't-linear' multiplication.

Starting from Section 3, we restrict our studies to 2-categories $\mathcal{K}$ which admit Eilenberg–Moore constructions for monads (in the sense of [19]) and in which idempotent 2-cells split. These assumptions are motivated by applications to bimodules. The bicategory BIM of `[Algebras; Bimodules; Bimodule maps]`, over a commutative, associative and unital ring k, satisfies both assumptions. However, in order to avoid technical complications caused by non-strictness of the horizontal composition in a bicategory, we prefer to restrict to 2-categories. In the examples, instead of the bicategory BIM, we can work with its image in the 2-category CAT = `[Categories; Functors; Natural transformations]`, under the hom 2-functor $\mathsf{BIM}(k, -) : \mathsf{BIM} \to \mathsf{CAT}$, which image is a 2-category with the desired properties.

For a 2-category $\mathcal{K}$ which admits Eilenberg–Moore constructions for monads, the inclusion 2-functor $I : \mathcal{K} \to \mathsf{EM}(\mathcal{K})$ possesses a right 2-adjoint J, cf. [16]. In Section 3 we use the splitting property of idempotent 2-cells in $\mathcal{K}$ to construct a factorization of J through the inclusion 2-functor $\mathsf{EM}(\mathcal{K}) \hookrightarrow \mathsf{EM}^w(\mathcal{K})$ and an appropriate pseudo-functor $J^w : \mathsf{EM}^w(\mathcal{K}) \to \mathcal{K}$. For a monad $t \xrightarrow{s} t$ in $\mathsf{EM}^w(\mathcal{K})$ and any 0-cell k in $\mathcal{K}$, we prove that both monads $\mathcal{K}(k, J^w(s))$ and $\mathcal{K}(k, \widehat{st})$ in CAT possess isomorphic Eilenberg–Moore categories, where $\widehat{st}$ is a canonical retract monad of the pre-monad st.

In a 2-category $\mathcal{K}$ which admits Eilenberg–Moore constructions for monads, any monad $k \xrightarrow{t} k$ determines an adjunction $(k \xrightarrow{f} J(t), J(t) \xrightarrow{v} k)$ in $\mathcal{K}$, cf. [19]. A lifting of a 1-cell $k \xrightarrow{V} k'$

in $\mathcal{K}$ for monads $k \xrightarrow{t} k$ and $k' \xrightarrow{t'} k'$ is, by definition, a 1-cell $J(t) \xrightarrow{\overline{V}} J(t')$, such that $v'\overline{V} = Vv$, cf. [17]. In Section 4 we define a 'weak' lifting by replacing this equality with the existence of a 2-cell $v'\overline{V} \overset{\iota}{\Rightarrow} Vv$, possessing a retraction $Vv \overset{\pi}{\Rightarrow} v'\overline{V}$. This leads to two symmetrical notions of weak lifting of a 2-cell $V \overset{\omega}{\Rightarrow} W$ for the monads t and t'. A weak ι-lifting $\overline{V} \overset{\overrightarrow{\omega}}{\Rightarrow} \overline{W}$ is defined by the condition $\iota * v'\overrightarrow{\omega} = \omega v * \iota$ and a weak π-lifting $\overline{V} \overset{\overleftarrow{\omega}}{\Rightarrow} \overline{W}$ is defined by $v'\overleftarrow{\omega} * \pi = \pi * \omega v$. We show that any weak ι-lifting and any weak π-lifting of a 2-cell in $\mathcal{K}$, if it exists, is isomorphic to the image of an appropriate 2-cell in $\mathsf{EM}^w(\mathcal{K})$ under the pseudo-functor J^w. Both a weak ι-lifting and a weak π-lifting are proven to strictly preserve vertical composition and to preserve horizontal composition up to a coherent isomorphism. We also give sufficient and necessary conditions for the existence of weak ι- and π-liftings of a 2-cell in $\mathcal{K}$.

A powerful tool to treat algebra extensions by weak bialgebras is provided by 'weak entwining structures' in [9]. A weak entwining structure in a 2-category $\mathcal{K}$ consists of a monad t and a comonad c, together with a 2-cell $tc \Rightarrow ct$ relating both structures in a way which generalizes a mixed distributive law in [1]. It was observed in [8] that any weak entwining structure (in BIM) induces a comonad (called a 'coring' in the particular case of the bicategory BIM). In Section 5 we show that – in the same way as mixed distributive laws in a 2-category $\mathcal{K}$ provide examples of comonads in $\mathsf{EM}(\mathcal{K})$ – weak entwining structures provide examples of comonads in $\mathsf{EM}^w(\mathcal{K})$. Moreover, if the 2-category $\mathcal{K}$ satisfies the assumptions in Section 3, then the comonad in $\mathcal{K}$, induced by a weak entwining structure, is an example of a weak ι-lifting of a comonad for a monad.

Studying partial coactions of Hopf algebras, in [10] another generalization of a mixed distributive law, a so-called 'partial entwining structure' was introduced. Partial entwining structures (in BIM) were proven to induce comonads as well. We show that also partial entwining structures in a 2-category $\mathcal{K}$ provide examples of a comonad in $\mathsf{EM}^w(\mathcal{K})$. Moreover, if the 2-category $\mathcal{K}$ satisfies the assumptions in Section 3, then the comonad in $\mathcal{K}$, induced by a partial entwining structure, is an example of a weak π-lifting of a comonad for a monad.

As a final application, weak bialgebras are characterized via weak liftings. If a module H, over a commutative, associative and unital ring k, possesses both an algebra and a coalgebra structure, then it induces two monads $t_R = (-) \otimes_k H$ and $t_L = H \otimes_k (-)$, and two comonads $c_R = (-) \otimes_k H$ and $c_L = H \otimes_k (-)$ on the category of k-modules. We relate weak bialgebra structures of H to weak ι-liftings of c_R and c_L for t_R and t_L, respectively, and weak π-liftings t_R and t_L for c_R and c_L, respectively.

Notation. We assume that the reader is familiar with the theory of 2-categories. For a review of the occurring notions (such as a 2-category, a 2-functor and a 2-adjunction, monads, adjunctions and Eilenberg–Moore construction in a 2-category) we refer to the article [15].

In a 2-category $\mathcal{K}$, horizontal composition is denoted by juxtaposition and vertical composition is denoted by $*$, 1-cells are represented by an arrow $\rightarrow$ and 2-cells are represented by $\Rightarrow$.

For any 2-category $\mathcal{K}$, $\mathsf{Mnd}(\mathcal{K})$ denotes the 2-category of monads in $\mathcal{K}$ as in [19] and $\mathsf{Cmd}(\mathcal{K}) := \mathsf{Mnd}(\mathcal{K}_*)_*$ denotes the 2-category of comonads in $\mathcal{K}$, where $(-)_*$ refers to the vertical opposite of a 2-category. We denote by $\mathsf{EM}(\mathcal{K})$ the extended 2-category of monads in [16]. We use the reduced form of 2-cells in $\mathsf{EM}(\mathcal{K})$, see [16].

1. The 2-category $\mathsf{EM}^w(\mathcal{K})$

For any 2-category $\mathcal{K}$, the 2-category $\mathsf{EM}^w(\mathcal{K})$ introduced in this section extends the 2-category $\mathsf{EM}(\mathcal{K})$ in [16].

Theorem 1.1. *For any* 2*-category* $\mathcal{K}$*, the following data constitute a* 2*-category, to be denoted by* $\mathsf{EM}^w(\mathcal{K})$.

0-cells are monads $(k \xrightarrow{t} k, tt \overset{\mu}{\Rightarrow} t, k \overset{\eta}{\Rightarrow} t)$ *in* $\mathcal{K}$.

1-cells $(t,\mu,\eta) \to (t',\mu',\eta')$ *are pairs* (V,ψ)*, consisting of a* 1*-cell* $k \xrightarrow{V} k'$ *and a* 2*-cell* $t'V \overset{\psi}{\Rightarrow} Vt$ *in* $\mathcal{K}$*, such that*

$$V\mu * \psi t * t'\psi = \psi * \mu' V. \tag{1.1}$$

The identity 1*-cell is* $t \xrightarrow{(k,t)} t$.

2-cells $(V,\psi) \Rightarrow (W,\phi)$ *are* 2*-cells* $V \overset{\varrho}{\Rightarrow} Wt$ *in* $\mathcal{K}$*, such that*

$$W\mu * \varrho t * \psi = W\mu * \phi t * t'\varrho, \tag{1.2}$$

$$\varrho = W\mu * \phi t * \eta' W t * \varrho. \tag{1.3}$$

The identity 2*-cell is* $(W,\phi) \xrightarrow{\phi * \eta' W} (W,\phi)$.

Horizontal composition of 2*-cells* $(V,\psi) \overset{\varrho}{\Rightarrow} (W,\phi)$, $(V',\psi') \overset{\varrho'}{\Rightarrow} (W',\phi')$ *(for* 1*-cells* $(V,\psi),(W,\phi): t \to t'$ *and* $(V',\psi'),(W',\phi'): t' \to t''$*) is given by*

$$\varrho' \circ \varrho := W'W\mu * W'\varrho t * W'\psi * \varrho' V. \tag{1.4}$$

Vertical composition of 2*-cells* $(V,\psi) \overset{\varrho}{\Rightarrow} (W,\phi) \overset{\tau}{\Rightarrow} (U,\theta)$ *(for* 1*-cells* $(V,\psi),(W,\phi)$ *and* $(U,\theta): t \to t'$*) is given by*

$$\tau \bullet \varrho := U\mu * \tau t * \varrho. \tag{1.5}$$

Proof. We verify only those axioms whose proof is different from the proof of the respective axiom for $\mathsf{EM}(\mathcal{K})$.

The vertical composite of 2-cells $(V,\psi) \overset{\varrho}{\Rightarrow} (W,\phi) \overset{\tau}{\Rightarrow} (U,\theta)$ in $\mathsf{EM}^w(\mathcal{K})$ is checked to satisfy (1.2) by the same computation used to verify that the vertical composite of 2-cells in $\mathsf{EM}(\mathcal{K})$ is a 2-cell in $\mathsf{EM}(\mathcal{K})$. In order to see that $\tau \bullet \varrho$ satisfies (1.3), use the interchange law in $\mathcal{K}$, associativity of the multiplication μ of the monad t and the fact that τ satisfies (1.3):

$$\begin{aligned} U\mu * \theta t * \eta' U t * (\tau \bullet \varrho) &= U\mu * \theta t * \eta' U t * U\mu * \tau t * \varrho \\ &= U\mu * U\mu t * \theta t t * \eta' U t t * \tau t * \varrho = U\mu * \tau t * \varrho = \tau \bullet \varrho. \end{aligned}$$

Associativity of the vertical composition follows by the same reasoning as in the case of $\mathsf{EM}(\mathcal{K})$. Using the constraint (1.1), it follows for any 1-cell $t \xrightarrow{(W,\phi)} t'$ in $\mathsf{EM}^w(\mathcal{K})$ that

$$\begin{aligned} &W\mu * \phi t * t'\phi * t'\eta' W = \phi * \mu' W * t'\eta' W = \phi \quad \text{and} \\ &W\mu * \phi t * \eta' W t * \phi = W\mu * \phi t * t'\phi * \eta' t' W = \phi * \mu' W * \eta' t' W = \phi. \end{aligned}$$

Thus the 2-cell $(W,\phi) \xRightarrow{\phi * \eta' W} (W,\phi)$ in $\mathcal{K}$ satisfies (1.2) and (1.3), proving that it is a 2-cell in $\mathsf{EM}^w(\mathcal{K})$. It is immediate by condition (1.3) that for any 2-cell $(V,\psi) \overset{\varrho}{\Rightarrow} (W,\phi)$ in $\mathsf{EM}^w(\mathcal{K})$,

$$\left(\phi * \eta' W\right) \bullet \varrho = W\mu * \phi t * \eta' W t * \varrho = \varrho.$$

Using (1.2), the interchange law in $\mathcal{K}$ and then (1.3), one checks that also

$$\varrho \bullet \left(\psi * \eta' V\right) = W\mu * \varrho t * \psi * \eta' V = W\mu * \phi t * t'\varrho * \eta' V = W\mu * \phi t * \eta' W t * \varrho = \varrho.$$

Hence there are identity 2-cells of the stated form.

On identity 2-cells $(V,\psi) \xRightarrow{\psi * \eta' V} (V,\psi)$ and $(V',\psi') \xRightarrow{\psi' * \eta'' V'} (V',\psi')$, the horizontal composite comes out as

$$\left(\psi' * \eta'' V'\right) \circ \left(\psi * \eta' V\right) = V'\psi * \psi' V * \eta'' V' V,$$

where we applied (1.1) for ψ and unitality of the monad t'. That is to say, the horizontal composite of the 1-cells $t \xrightarrow{(V,\psi)} t' \xrightarrow{(V',\psi')} t''$ is the 1-cell $(V'V, V'\psi * \psi' V)$. By the same computations used in the case of $\mathsf{EM}(\mathcal{K})$, the horizontal composite of 2-cells $(V,\psi) \overset{\varrho}{\Rightarrow} (W,\phi)$, $(V',\psi') \overset{\varrho'}{\Rightarrow} (W',\phi')$ is checked to satisfy condition (1.2). It satisfies also (1.3), as

$$\begin{aligned}
W'W\mu \; * \; & W'\phi t * \phi' W t * \eta'' W' W t * \left(\varrho' \circ \varrho\right) \\
&= W'W\mu * W'\phi t * \phi' W t * \eta'' W' W t * W' W \mu * W' \varrho t * W' \psi * \varrho' V \\
&= W'W\mu * W'W\mu t * W'\phi t t * W' t' \varrho t * \phi' V t * \eta'' W' V t * W' \psi * \varrho' V \\
&= W'W\mu * W'W\mu t * W'\varrho t t * W' \psi t * \phi' V t * \eta'' W' V t * W' \psi * \varrho' V \\
&= W'W\mu * W'\varrho t * W' V \mu * W' \psi t * W' t' \psi * \phi' t' V * \eta'' W' t' V * \varrho' V \\
&= W'W\mu * W'\varrho t * W' \psi * W' \mu' V * \phi' t' V * \eta'' W' t' V * \varrho' V \\
&= W'W\mu * W'\varrho t * W' \psi * \varrho' V = \varrho' \circ \varrho.
\end{aligned}$$

The first and last equalities follow by (1.4). In the second and fourth equalities we used the interchange law in $\mathcal{K}$ and associativity of the multiplication μ of the monad t. In the third equality we used that ϱ satisfies (1.2). The fifth equality is derived by using that ψ satisfies (1.1). The penultimate equality follows by using that ϱ' satisfies (1.3). This proves that the horizontal composite of 2-cells is a 2-cell. Associativity of the horizontal composition in $\mathsf{EM}^w(\mathcal{K})$ is checked in the same way as it is done in $\mathsf{EM}(\mathcal{K})$. Obviously, for any 2-cell $(V,\psi) \overset{\varrho}{\Rightarrow} (W,\phi)$, $\varrho \circ \eta = W\mu * W\eta t * \varrho = \varrho$. By (1.2) and (1.3), also $\eta' \circ \varrho = W\mu * \varrho t * \psi * \eta' V = \varrho$. Hence the identity 2-cell $(k,t) \overset{\eta}{\Rightarrow} (k,t)$ is a unit for the horizontal composition, proving the stated form (k,t) of the identity 1-cell $t \to t$.

The interchange law in $\mathsf{EM}^w(\mathcal{K})$ is checked in the same way as it is done in $\mathsf{EM}(\mathcal{K})$. $\square$

Clearly, any 1-cell in $\mathsf{EM}(\mathcal{K})$ is a 1-cell also in $\mathsf{EM}^w(\mathcal{K})$. For a 1-cell $t \xrightarrow{(W,\phi)} t'$ in $\mathsf{EM}(\mathcal{K})$, any 2-cell ϱ in $\mathcal{K}$ of target Wt satisfies (1.3). Hence 2-cells in $\mathsf{EM}^w(\mathcal{K})$ between 1-cells of $\mathsf{EM}(\mathcal{K})$

are the same as the 2-cells in $\mathsf{EM}(\mathcal{K})$. Comparing the formulae of the horizontal and vertical compositions in $\mathsf{EM}(\mathcal{K})$ and $\mathsf{EM}^w(\mathcal{K})$, we conclude that $\mathsf{EM}(\mathcal{K})$ is a vertically full 2-subcategory of $\mathsf{EM}^w(\mathcal{K})$.

One may ask what 2-subcategory of $\mathsf{EM}^w(\mathcal{K})$ plays the role of the 2-subcategory, obtained as an image of $\mathsf{Mnd}(\mathcal{K})$ in $\mathsf{EM}(\mathcal{K})$. As the lemmata below show, there seems to be no unique answer to this question. This is because, for some 1-cells $t \xrightarrow{(V,\psi)} t'$ and $t \xrightarrow{(W,\phi)} t'$ in $\mathsf{EM}^w(\mathcal{K})$ and a 2-cell $V \overset{\omega}{\Rightarrow} W$ in $\mathcal{K}$, the 2-cells $\omega t * \psi * \eta' V$ and $\phi * \eta' W * \omega$ in $\mathcal{K}$ need not be equal. Still, there are two distinguished sets of 2-cells in $\mathsf{EM}^w(\mathcal{K})$ on equal footing, both closed under the horizontal and vertical compositions and both containing the identity 2-cells.

Lemma 1.2. *For any 2-category $\mathcal{K}$, let $t \xrightarrow{(V,\psi)} t'$ and $t \xrightarrow{(W,\phi)} t'$ be 1-cells in $\mathsf{EM}^w(\mathcal{K})$ and $V \overset{\omega}{\Rightarrow} W$ be a 2-cell in $\mathcal{K}$.*

(1) *The following assertions are equivalent:*
  (i) *$\omega t * \psi * \eta' V : (V,\psi) \Rightarrow (W,\phi)$ is a 2-cell in $\mathsf{EM}^w(\mathcal{K})$;*
  (ii) *$\omega t * \psi = W\mu * \phi t * t'\omega t * t'\psi * t'\eta' V$;*
  (iii) *$W\mu * \phi t * \eta' W t * \omega t * \psi * \eta' V = \omega t * \psi * \eta' V$ and $W\mu * \phi t * \eta' W t * \omega t * \psi = W\mu * \phi t * t'\omega t * t'\psi * t'\eta' V$.*
(2) *The following assertions are equivalent:*
  (i) *$\phi * \eta' W * \omega : (V,\psi) \Rightarrow (W,\phi)$ is a 2-cell in $\mathsf{EM}^w(\mathcal{K})$;*
  (ii) *$\phi * t'\omega = W\mu * \phi t * \eta' W t * \omega t * \psi$;*
  (iii) *$W\mu * \phi t * \eta' W t * \omega t * \psi * \eta' V = \phi * \eta' W * \omega$ and $W\mu * \phi t * \eta' W t * \omega t * \psi = W\mu * \phi t * t'\omega t * t'\psi * t'\eta' V$.*
(3) *The following assertions are equivalent:*
  (i) *$\phi * \eta' W * \omega$ and $\omega t * \psi * \eta' V$ are (necessarily equal) 2-cells $(V,\psi) \Rightarrow (W,\phi)$ in $\mathsf{EM}^w(\mathcal{K})$;*
  (ii) *$\phi * t'\omega = \omega t * \psi$.*

Proof. <u>(1)</u> <u>(i) $\Leftrightarrow$ (ii)</u> Using that ψ satisfies (1.1) together with the unitality of the monad t', condition (1.2) for $\varrho := \omega t * \psi * \eta' V$ comes out as the equality in part (ii). Hence in order to prove the equivalence of assertions (i) and (ii), we need to show that (ii) implies that ϱ satisfies (1.3). Indeed, applying the equality in part (ii) in the second step, we obtain

$$W\mu * \phi t * \eta' W t * \omega t * \psi * \eta' V = W\mu * \phi t * t'\omega t * t'\psi * t'\eta' V * \eta' V = \omega t * \psi * \eta' V. \quad (1.6)$$

<u>(ii) $\Leftrightarrow$ (iii)</u> We have seen in the proof of equivalence (i) $\Leftrightarrow$ (ii) above, that assertion (ii) implies (1.6), i.e. the first condition in part (iii). The second condition is checked as follows:

$$\begin{aligned} W\mu * \phi t * \eta' W t * \omega t * \psi &= W\mu * \phi t * \eta' W t * \omega t * V\mu * \psi t * t'\psi * \eta' t' V \\ &= W\mu * W\mu t * \phi t t * \eta' W t t * \omega t t * \psi t * \eta' V t * \psi \\ &= W\mu * \omega t t * \psi t * \eta' V t * \psi \\ &= \omega t * \psi = W\mu * \phi t * t'\omega t * t'\psi * t'\eta' V. \end{aligned} \quad (1.7)$$

The first equality follows by (1.1) and unitality of the monad t'. In the second equality we used associativity of μ and the interchange law. The third equality is obtained by using (1.6). The penultimate equality follows by (1.1) and unitality of the monad t' again. The last equality follows by assertion (ii).

Conversely, assume that the identities in part (iii) hold. Then

$$\begin{aligned}\omega t * \psi &= \omega t * V\mu * \psi t * t'\psi * \eta' t' V = W\mu * W\mu t * \phi tt * \eta' Wtt * \omega tt * \psi t * \eta' Vt * \psi \\ &= W\mu * \phi t * \eta' Wt * \omega t * \psi = W\mu * \phi t * t'\omega t * t'\psi * t'\eta' V.\end{aligned}$$

The first equality follows by (1.1) and unitality of the monad t'. In the second equality we applied the first condition in part (iii). The third equality is obtained by using the associativity of μ, the interchange law and (1.1) again. The last equality follows by the second condition in part (iii).

(2) (i) $\Leftrightarrow$ (ii) Using that ϕ satisfies (1.1) together with the unitality of the monad t', condition (1.2) for $\varrho := \phi * \eta' W * \omega$ comes out as the equality in part (ii). Condition (1.3) holds for ϱ automatically, i.e. it follows by applying (1.1) for ϕ.

(ii) $\Leftrightarrow$ (iii) If (iii) holds, then

$$\begin{aligned}\phi * t'\omega &= W\mu * \phi t * t'\phi * t'\eta' W * t'\omega \\ &= W\mu * \phi t * t' W\mu * t'\phi t * t'\eta' Wt * t'\omega t * t'\psi * t'\eta' V \\ &= W\mu * \phi t * t'\omega t * t'\psi * t'\eta' V = W\mu * \phi t * \eta' Wt * \omega t * \psi.\end{aligned}$$

The first equality follows by applying (1.1) for ϕ, together with the unitality of the monad t'. The second equality is obtained by the first identity in part (iii). In the penultimate equality we applied the interchange law, associativity of μ, (1.1) on ϕ and unitality of t'. The last equality follows by the second condition in part (iii).

Conversely, if assertion (ii) holds, then the first condition in part (iii) is proven by composing both sides of the equality in part (ii) by $\eta' V$ on the right. The second condition, i.e. equality (1.7), is proven by the following computation:

$$\begin{aligned}W\mu * \phi t * t'\omega t * t'\psi * t'\eta' V &= W\mu * W\mu t * \phi tt * t'\phi t * t'\eta' Wt * t'\omega t * t'\psi * t'\eta' V \\ &= W\mu * \phi t * t'\phi * t'\eta' W * t'\omega \\ &= \phi * t'\omega = W\mu * \phi t * \eta' Wt * \omega t * \psi.\end{aligned}$$

The first equality follows by applying (1.1) for ϕ, together with the unitality of the monad t'. The second equality is obtained by using the associativity of μ, the interchange law and the first condition in part (iii). The third equality follows by applying (1.1) for ϕ, together with the unitality of the monad t' again. The last equality follows by part (ii).

(3) Assume first that assertion (3)(i) holds. Then by parts (1) and (2), also (1)(ii) and (2)(ii) hold, implying (1.7). Hence

$$\omega t * \psi = W\mu * \phi t * t'\omega t * t'\psi * t'\eta' V' = W\mu * \phi t * \eta' Wt * \omega t * \psi = \phi * t'\omega.$$

Conversely, if assertion (3)(ii) holds, then

$$\begin{aligned}&W\mu * \phi t * t'\omega t * t'\psi * t'\eta' V = W\mu * \omega tt * \psi t * t'\psi * t'\eta' V = \omega t * \psi \quad \text{and} \\ &W\mu * \phi t * \eta' Wt * \omega t * \psi = W\mu * \phi t * \eta' Wt * \phi * t'\omega = \phi * t'\omega,\end{aligned}$$

where in both computations the first equality follows by (3)(ii) and the second equality follows by (1.1) and the unitality of the monad t'. We conclude by parts (1) and (2) that both $\omega t * \psi * \eta' V$ and $\phi * \eta' W * \omega$ are 2-cells in $\mathsf{EM}^w(\mathcal{K})$. It follows by comparing the first identities in (1)(iii) and (2)(iii) that $\phi * \eta' W * \omega = \omega t * \psi * \eta' V$, as stated. □

Next we investigate the behaviour of the correspondences in Lemma 1.2 with respect to the horizontal and vertical compositions in $\mathcal{K}$ and $\mathsf{EM}^w(\mathcal{K})$.

Lemma 1.3. *For any* 2*-category* $\mathcal{K}$*, let* (V, ψ)*,* (W, ϕ) *and* (U, θ) *be* 1*-cells* $t \to t'$ *and* (V', ψ') *and* (W', ϕ') *be* 1*-cells* $t' \to t''$ *in* $\mathsf{EM}^w(\mathcal{K})$*.*

(1) *If some* 2*-cells* $V \overset{\omega}{\Rightarrow} W$ *and* $V' \overset{\omega'}{\Rightarrow} W'$ *in* $\mathcal{K}$ *satisfy the equivalent conditions in Lemma* 1.2(1)*, then* $(\omega' t' * \psi' * \eta'' V') \circ (\omega t * \psi * \eta' V) = \omega' \omega t * V' \psi * \psi' V * \eta'' V' V$*. Hence in particular also* $\omega' \omega$ *satisfies the equivalent conditions in Lemma* 1.2(1)*.*
(2) *If some* 2*-cells* $V \overset{\omega}{\Rightarrow} W$ *and* $V' \overset{\omega'}{\Rightarrow} W'$ *in* $\mathcal{K}$ *satisfy the equivalent conditions in Lemma* 1.2(2)*, then* $(\phi' * \eta'' W' * \omega') \circ (\phi * \eta' W * \omega) = W' \phi * \phi' W * \eta'' W' W * \omega' \omega$*. Hence in particular also* $\omega' \omega$ *satisfies the equivalent conditions in Lemma* 1.2(2)*.*
(3) *If some* 2*-cells* $V \overset{\omega}{\Rightarrow} W \overset{\kappa}{\Rightarrow} U$ *in* $\mathcal{K}$ *satisfy the equivalent conditions in Lemma* 1.2(1)*, then* $(\kappa t * \phi * \eta' W) \bullet (\omega t * \psi * \eta' V) = \kappa t * \omega t * \psi * \eta' V$*. Hence in particular also* $\kappa * \omega$ *satisfies the equivalent conditions in Lemma* 1.2(1)*.*
(4) *If some* 2*-cells* $V \overset{\omega}{\Rightarrow} W \overset{\kappa}{\Rightarrow} U$ *in* $\mathcal{K}$ *satisfy the equivalent conditions in Lemma* 1.2(2)*, then* $(\theta * \eta' U * \kappa) \bullet (\phi * \eta' W * \omega) = \theta * \eta' U * \kappa * \omega$*. Hence in particular also* $\kappa * \omega$ *satisfies the equivalent conditions in Lemma* 1.2(2)*.*

Proof. <u>(1)</u> This compatibility with the horizontal composition follows by applying (1.1) for ψ, and unitality of the monad t'.

<u>(2)</u> This follows by using that ω obeys Lemma 1.2(2)(ii).

<u>(3)</u> This compatibility with the vertical composition follows using that, by Lemma 1.2(1), $\omega t * \psi * \eta' V$ satisfies (1.3).

<u>(4)</u> This assertion follows by using that κ satisfies Lemma 1.2(2)(ii). □

The message of Lemmas 1.2 and 1.3 can be summarized as follows.

Corollary 1.4. *Consider an arbitrary* 2*-category* $\mathcal{K}$*.*

(1) *There is a* 2*-category, to be denoted by* $\mathsf{Mnd}^\iota(\mathcal{K})$*, defined by the following data:*
 <u>*0-cells*</u> *are monads* t *in* $\mathcal{K}$*;*
 <u>*1-cells*</u> $t \xrightarrow{(V,\psi)} t'$ *are the same as* 1*-cells in* $\mathsf{EM}^w(\mathcal{K})$*, cf.* (1.1)*;*
 <u>*2-cells*</u> $(V, \psi) \overset{\omega}{\Rightarrow} (W, \phi)$ *are* 2*-cells* $V \overset{\omega}{\Rightarrow} W$ *in* $\mathcal{K}$*, satisfying the equivalent conditions in Lemma* 1.2(1)*;*
 <u>*horizontal and vertical compositions*</u> *are the same as in* $\mathcal{K}$*.*
 Furthermore, there is a 2*-functor* $G^\iota : \mathsf{Mnd}^\iota(\mathcal{K}) \to \mathsf{EM}^w(\mathcal{K})$*, acting on the* 0*- and* 1*-cells as the identity map and taking a* 2*-cell* $(V, \psi) \overset{\omega}{\Rightarrow} (W, \phi)$ *to* $\omega t * \psi * \eta' V$*.*
(2) *There is a* 2*-category, to be denoted by* $\mathsf{Mnd}^\pi(\mathcal{K})$*, defined by the following data:*
 <u>*0-cells*</u> *are monads* t *in* $\mathcal{K}$*;*

1-cells $t \xrightarrow{(V,\psi)} t'$ *are the same as* 1*-cells in* $\mathsf{EM}^w(\mathcal{K})$*, cf.* (1.1)*;*
2-cells $(V,\psi) \overset{\omega}{\Rightarrow} (W,\phi)$ *are* 2*-cells* $V \overset{\omega}{\Rightarrow} W$ *in* $\mathcal{K}$*, satisfying the equivalent conditions in Lemma* 1.2(2)*;*
horizontal and vertical compositions are the same as in $\mathcal{K}$*.*

Furthermore, there is a 2*-functor* $G^\pi : \mathsf{Mnd}^\pi(\mathcal{K}) \to \mathsf{EM}^w(\mathcal{K})$*, acting on the* 0*- and* 1*-cells as the identity map and taking a* 2*-cell* $(V,\psi) \overset{\omega}{\Rightarrow} (W,\phi)$ *to* $\phi * \eta' W * \omega$*.*

Clearly, both $\mathsf{Mnd}^\iota(\mathcal{K})$ and $\mathsf{Mnd}^\pi(\mathcal{K})$ contain $\mathsf{Mnd}(\mathcal{K})$ as a vertically full subcategory.

2. Monads in $\mathsf{EM}^w(\mathcal{K})$ and pre-monads in $\mathcal{K}$

It was observed in [16, p. 257] that monads in $\mathsf{EM}(\mathcal{K})$ induce monads in $\mathcal{K}$. The aim of this section is to give a similar interpretation of monads in $\mathsf{EM}^w(\mathcal{K})$.

A monad in $\mathsf{EM}^w(\mathcal{K})$ is given by a triple $((s,\psi),\nu,\vartheta)$, consisting of a 1-cell $(t,\mu,\eta) \xrightarrow{(s,\psi)} (t,\mu,\eta)$ and 2-cells $(s,\psi)\circ(s,\psi) \overset{\nu}{\Rightarrow} (s,\psi)$ and $(k,t) \overset{\vartheta}{\Rightarrow} (s,\psi)$ in $\mathsf{EM}^w(\mathcal{K})$, such that

$$\nu \bullet \big(\nu \circ (s,\psi)\big) = \nu \bullet \big((s,\psi)\circ \nu\big),$$
$$\nu \bullet \big(\vartheta \circ (s,\psi)\big) = (s,\psi) = \nu \bullet \big((s,\psi)\circ \vartheta\big).$$

In light of Theorem 1.1, this means a 1-cell $k \xrightarrow{s} k$, and 2-cells $ts \overset{\psi}{\Rightarrow} st$, $ss \overset{\nu}{\Rightarrow} st$ and $k \overset{\vartheta}{\Rightarrow} st$ in $\mathcal{K}$, subject to the following identities.

$$\psi * \mu s = s\mu * \psi t * t\psi, \tag{2.1}$$
$$s\mu * \psi t * t\nu = s\mu * \nu t * s\psi * \psi s, \tag{2.2}$$
$$s\mu * \psi t * \eta st * \nu = \nu, \tag{2.3}$$
$$s\mu * \psi t * t\vartheta = s\mu * \vartheta t, \tag{2.4}$$
$$s\mu * \nu t * s\nu = s\mu * \nu t * s\psi * \nu s, \tag{2.5}$$
$$s\mu * \nu t * s\psi * \vartheta s = \psi * \eta s, \tag{2.6}$$
$$s\mu * \nu t * s\vartheta = \psi * \eta s. \tag{2.7}$$

Condition (2.1) expresses the requirement that (s,ψ) is a 1-cell in $\mathsf{EM}^w(\mathcal{K})$, (2.2) and (2.3) together mean that ν is a 2-cell in $\mathsf{EM}^w(\mathcal{K})$, (2.4) means that ϑ is a 2-cell in $\mathsf{EM}^w(\mathcal{K})$ (condition (1.3) on ϑ follows by the interchange law in $\mathcal{K}$, (2.4) and unitality of the monad t). Conditions (2.5), (2.6) and (2.7) express associativity and unitality of the monad $((s,\psi),\nu,\vartheta)$, after being simplified using (2.1), (2.2), (2.3) and (2.4).

Note that a monad $(t \xrightarrow{(s,\psi)} t, \nu, \vartheta)$ in $\mathsf{EM}^w(\mathcal{K})$, for a monad $k \xrightarrow{t} k$ in $\mathcal{K}$, is identical to a 'crossed product system' (t,s,ψ,ν) in the monoidal category $\mathcal{K}(k,k)$, in the sense of [13, Definition 3.5], subject to the 'twisted' and 'cocycle' conditions in [13, Definitions 3.3 and 3.6], the normalization condition in [13, Proposition 3.7] and identities (11), (12) and (13) in [13, Theorem 3.11], for ϑ.

The following definition is inspired by [9, Section 3.1] (see also [13, Definition 2.3]).

Definition 2.1. A *pre-monad* in a 2-category $\mathcal{K}$ is a triple (t, μ, η), consisting of a 1-cell $k \xrightarrow{t} k$ and 2-cells $tt \overset{\mu}{\Rightarrow} t$ and $k \overset{\eta}{\Rightarrow} t$, such that the following conditions hold:

$$\mu * \mu t = \mu * t\mu, \tag{2.8}$$

$$\mu * \eta t = \mu * t\eta, \tag{2.9}$$

$$\mu * \eta\eta = \eta, \tag{2.10}$$

$$\mu * \mu t * \eta tt = \mu. \tag{2.11}$$

Note that if $k \xrightarrow{t} k$ is a 1-cell in a 2-category $\mathcal{K}$ and some 2-cells $tt \overset{\mu}{\Rightarrow} t$ and $k \overset{\eta}{\Rightarrow} t$ satisfy $\mu * \mu t = \mu * t\mu$ and $\mu * \eta t = \mu * t\eta = \mu * \mu t * \eta\eta t$ as in [9, Section 3.1], then $(t, \mu' := \mu * \mu t * \eta tt, \eta' := \mu * \eta\eta)$ is a pre-monad in the sense of Definition 2.1.

The motivation for a study of pre-monads stems from the following observation.

Lemma 2.2. *Consider a pre-monad (t, μ, η) in an arbitrary 2-category $\mathcal{K}$.*

(1) *The 2-cell $\mu * \eta t$ is idempotent.*
(2) *Assume that there exists a 1-cell $\hat{t}$ and 2-cells $t \overset{\pi}{\Rightarrow} \hat{t}$ and $\hat{t} \overset{\iota}{\Rightarrow} t$ in $\mathcal{K}$, such that $\mu * \eta t = \iota * \pi$ and $\hat{t} = \pi * \iota$. Then $(\hat{t}, \hat{\mu} := \pi * \mu * \iota\iota, \hat{\eta} := \pi * \eta)$ is a monad in $\mathcal{K}$.*

Proof. The proof is an easy computation using Definition 2.1 of a pre-monad and the properties ι and π obey, so it is left to the reader. □

Improving [13, Theorem 3.11], we obtain the following generalization of a correspondence between monads in $\mathsf{EM}(\mathcal{K})$ and in $\mathcal{K}$, observed by Lack and Street in [16].

Theorem 2.3. *For any monad $(k \xrightarrow{t} k, \mu, \eta)$ and any 1-cell $k \xrightarrow{s} k$ in an arbitrary 2-category $\mathcal{K}$, there is a bijective correspondence between the following structures:*

(i) *A monad $(t \xrightarrow{(s,\psi)} t, \nu, \vartheta)$ in $\mathsf{EM}^w(\mathcal{K})$;*
(ii) *A pre-monad (st, Θ, ϑ) in $\mathcal{K}$, such that*

$$\Theta * sts\mu = s\mu * \Theta t. \tag{2.12}$$

Proof. The proof is built on the same constructions as [13, Theorem 3.11].

Assume first that there exist 2-cells ψ and ν as in part (i). A multiplication Θ for the pre-monad in part (ii) is given by the same formula of a 'wreath product' in [16, p. 256]:

$$\Theta := s\mu * \nu t * ss\mu * s\psi t. \tag{2.13}$$

Its associativity is checked by the same computation as in the case of the wreath product in [16]. By (2.6) on one hand, and by (2.4) and (2.7) on the other,

$$\Theta * \vartheta st = s\mu * \psi t * \eta st = \Theta * st\vartheta, \tag{2.14}$$

proving (2.9). By applying (2.4) again, we conclude that $\Theta * \vartheta\vartheta = \vartheta$, i.e. also (2.10) holds true. Condition (2.11) is proven by the following computation:

$$\begin{aligned}
\Theta * \Theta st * \vartheta stst &= s\mu * \nu t * ss\mu * s\psi t * s\mu st * \psi tst * \eta stst \\
&= s\mu * \nu t * ss\mu * ss\mu t * s\psi tt * st\psi t * \psi tst * \eta stst \\
&= s\mu * s\mu t * s\mu tt * \nu ttt * s\psi tt * \psi stt * ts\psi t * \eta stst \\
&= s\mu * s\mu t * s\mu tt * \psi ttt * t\nu tt * ts\psi t * \eta stst \\
&= s\mu * s\mu t * \nu tt * s\psi t = \Theta.
\end{aligned}$$

The second equality follows by (2.1). The fourth and the fifth equalities follow by (2.2) and (2.3), respectively. Condition (2.12) follows by associativity of μ. This proves that the data in part (i) determine a pre-monad as in part (ii).

Conversely, assume that there is a 2-cell Θ as in part (ii). The 2-cells ψ and ν in part (i) are constructed as

$$\psi := \Theta * s\mu st * \vartheta tst * ts\eta, \qquad \nu := \Theta * s\eta s\eta. \tag{2.15}$$

By (2.12),

$$s\mu * \psi t = \Theta * s\mu st * \vartheta tst \quad \text{and} \quad s\mu * \nu t = \Theta * s\eta st. \tag{2.16}$$

Moreover, by (2.8), (2.12) and (2.9),

$$\Theta * st\psi = \Theta * \Theta st * sts\mu st * st\vartheta tst * stts\eta = \Theta * s\mu st * \Theta tst * \vartheta sttst * stts\eta. \tag{2.17}$$

With identities (2.16), (2.17), (2.12) and (2.10) at hand, (2.1) is verified as

$$\begin{aligned}
s\mu * \psi t * t\psi &= \Theta * st\psi * s\mu ts * \vartheta tts = \Theta * s\mu st * \Theta tst * sts\mu tst * \vartheta\vartheta ttst * tts\eta \\
&= \Theta * s\mu st * s\mu tst * \vartheta ttst * tts\eta = \psi * \mu s.
\end{aligned}$$

Use next (2.16), (2.17), (2.12) and (2.11) to compute

$$s\mu * \nu t * s\psi = \Theta * st\psi * s\eta ts = \Theta * \Theta st * \vartheta stst * sts\eta = \Theta * sts\eta. \tag{2.18}$$

In order to prove that (2.2) holds, apply (2.18), (2.8) and (2.16):

$$\begin{aligned}
s\mu * \nu t * s\psi * \psi s &= \Theta * \psi st * tss\eta = \Theta * \Theta st * s\mu stst * \vartheta tstst * ts\eta s\eta \\
&= s\mu * \psi t * t\nu.
\end{aligned} \tag{2.19}$$

Condition (2.3) is verified by comparing the last and third expressions in (2.19), and using (2.11):

$$s\mu * \psi t * \eta st * \nu = \Theta * \Theta st * \vartheta stst * s\eta s\eta = \Theta * s\eta s\eta = \nu.$$

Condition (2.4) is proven by using (2.16), (2.9), (2.12) and (2.10):

$$s\mu * \psi t * t\vartheta = \Theta * st\vartheta * s\mu * \vartheta t = \Theta * sts\mu * \vartheta\vartheta t = s\mu * \Theta t * \vartheta\vartheta t = s\mu * \vartheta t.$$

Condition (2.5) follows by (2.18), (2.8) and (2.16):

$$s\mu * \nu t * s\psi * \nu s = \Theta * \Theta st * s\eta s\eta s\eta = \Theta * s\eta st * s\Theta * ss\eta s\eta = s\mu * \nu t * s\nu.$$

Condition (2.6) is checked by applying (2.18):

$$s\mu * \nu t * s\psi * \vartheta s = \Theta * \vartheta st * s\eta = \Theta * s\mu st * \vartheta tst * \eta s\eta = \psi * \eta s. \tag{2.20}$$

Finally, (2.7) is proven by making use of (2.16), (2.9) and comparing the second and last expressions in (2.20):

$$s\mu * \nu t * s\vartheta = \Theta * st\vartheta * s\eta = \Theta * \vartheta st * s\eta = \psi * \eta s.$$

This proves that the data in part (ii) determine a monad as in part (i).

It remains to show that the above constructions are mutual inverses. Take 2-cells ν and ψ as in part (i). Use (2.13) to associate a 2-cell Θ as in part (ii) to them, and then use (2.15) to define 2-cells ν' and ψ' as in part (i) again. We obtain

$$\begin{aligned}\nu' &= s\mu * \nu t * s\psi * s\eta s = s\mu * s\mu t * \nu tt * s\nu t * ss\vartheta = s\mu * \nu t * ss\mu * s\psi t * st\vartheta * \nu \\ &= s\mu * s\mu t * \nu tt * s\vartheta t * \nu = s\mu * \psi t * \eta st * \nu = \nu.\end{aligned}$$

The first equality follows by unitality of μ and the second equality follows by (2.7) and associativity of μ. The third equality is obtained by (2.5) and the fourth equality follows by (2.4) and associativity of μ. The penultimate equality is a consequence of (2.7) and the last one follows by (2.3). Also,

$$\psi' = s\mu * \nu t * s\psi * s\mu s * \vartheta ts = s\mu * s\mu t * \nu tt * s\psi t * \vartheta st * \psi = s\mu * \psi t * t\psi * \eta ts = \psi.$$

The first equality follows by unitality of μ, the second equality follows by (2.1) and associativity of μ, and the third equality is obtained by (2.6). The last equality follows by (2.1) and by unitality of μ.

In the opposite order, start with a 2-cell Θ as in part (ii). Apply (2.15) to construct 2-cells ψ and ν as in part (i) and then apply (2.13) to obtain a 2-cell Θ' as in part (ii). It satisfies

$$\begin{aligned}\Theta' &= s\mu * \nu t * ss\mu * s\psi t = \Theta * s\eta st * ss\mu * s\psi t = s\mu * \Theta t * st\psi t * s\eta tst \\ &= \Theta * s\mu st * \Theta tst * \vartheta sttst * s\eta tst = \Theta * \Theta st * \vartheta stst = \Theta.\end{aligned}$$

The second equality follows by the second identity in (2.16). The third equality is obtained by (2.12). The fourth equality is a consequence of (2.17), (2.12) and unitality of μ. The penultimate equality follows by (2.12) and unitality of μ. The last equality is obtained by (2.11). □

Examples 2.4. Examples of a composite pre-monad as in Theorem 2.3(ii) are given, first of all, by wreath products in [16]. It is shown in [16, Example 3.3] that crossed products by Hopf algebras in [21,2,12] are examples of a wreath product. As it was observed by Ross Street [20] (see also [13, Example 3.18]), so are the crossed products with coalgebras in [7] and their generalizations to comonads in [22, Section 4.8].

Examples of a composite pre-monad, which are not monads, are provided by 'weak smash products' in [9, Section 3] (for a review see [13, Example 3.16]). This includes smash products with weak bialgebras [6]. Crossed products with weak bialgebras in [18] are also shown in [13, Section 4] to provide examples.

Note, however, that (weak) crossed products with bialgebroids in [5, Section 4 & Appendix] are not (pre-)monads of the kind in Theorem 2.3(ii). Let k be a commutative and associative unital ring. A k-algebra B, measured by a left bialgebroid H over a k-algebra L, determines two monads, $(-) \otimes_k B$ on the category of k-modules and $(-) \otimes_L B$ on the category of L-modules. In terms of the measuring $\cdot : H \otimes_k B \to B$ and the comultiplication $h \mapsto \sum h_1 \otimes_L h_2$ in H, consider the left L-module map

$$H \otimes_k B \to B \otimes_L H, \qquad h \otimes_k b \mapsto \sum h_1 \cdot b \otimes_L h_2.$$

It equips the left L-module (or L–k bimodule) H with the structure of a 1-cell $(-) \otimes_L B \xrightarrow{(-)\otimes_L H} (-) \otimes_k B$ in $\mathsf{EM}(\mathsf{CAT})$ (or in $\mathsf{EM}^w(\mathsf{CAT})$ if $\cdot$ is only a weak measuring). However, if L is a non-trivial k-algebra, this 1-cell has different source and target. Although in [5] the composite endofunctor $- \otimes_k B \otimes_L H$ on the category of k-modules is proven to carry a monad structure, it is not a composite of a monad with an endofunctor.

3. A pseudo-functor $\mathsf{EM}^w(\mathcal{K}) \to \mathcal{K}$

Throughout this section (and the next one), we make two basic assumptions on the 2-category $\mathcal{K}$ we deal with:

(i) Idempotent 2-cells in $\mathcal{K}$ split;
(ii) $\mathcal{K}$ admits Eilenberg–Moore constructions (EM constructions, for short) for monads.

In more details, assumption (i) means that for any 2-cell $V \overset{\varpi}{\Rightarrow} V$ in $\mathcal{K}$, such that $\varpi * \varpi = \varpi$, there exist a 1-cell $\hat{V}$ and 2-cells $\hat{V} \overset{\iota}{\Rightarrow} V$ and $V \overset{\pi}{\Rightarrow} \hat{V}$, such that $\pi * \iota = \hat{V}$ and $\iota * \pi = \varpi$. It is easy to see that the datum $(\hat{V}, \iota, \pi)$ is unique up to an isomorphism $\hat{V} \overset{\iota}{\Rightarrow} V \overset{\pi'}{\Rightarrow} \hat{V}'$.

Property (ii) of a 2-category was introduced by Street in [19, p. 151]. It means that the inclusion 2-functor $\mathcal{K} \to \mathsf{Mnd}(\mathcal{K})$ (with underlying maps $k \mapsto (k \xrightarrow{k} k, k, k)$, $(k \xrightarrow{V} k') \mapsto (k \xrightarrow{V} k', V \overset{V}{\Rightarrow} V)$, $(V \overset{\varrho}{\Rightarrow} W) \mapsto (V \overset{\varrho}{\Rightarrow} W)$ on the 0-, 1-, and 2-cells, respectively) possesses a right 2-adjoint. By [16, Section 1], property (ii) can be formulated equivalently by saying that the inclusion 2-functor $I : \mathcal{K} \to \mathsf{EM}(\mathcal{K})$ possesses a right 2-adjoint J. Important properties of 2-categories admitting EM constructions for monads are formulated in the following theorem. It is recalled from [19, Theorem 2] and [16, Section 1].

Theorem 3.1. *In a 2-category $\mathcal{K}$ which admits EM constructions for monads, any monad $(k \xrightarrow{t} k, \mu, \eta)$ determines an adjunction $(k \xrightarrow{f} J(t), J(t) \xrightarrow{v} k, k \overset{\eta}{\Rightarrow} vf, fv \overset{\epsilon}{\Rightarrow} k)$ in $\mathcal{K}$, such that*

$(t,\mu,\eta)=(vf, v\epsilon f,\eta)$. One can choose $JI=\mathcal{K}$, and the isomorphism corresponding to the 2-adjunction (I,J) is given by the mutually inverse functors

$$\mathcal{K}\big(l,J(t)\big)\to \mathsf{EM}(\mathcal{K})\big(I(l),t\big),\qquad (V\overset{\omega}{\Rightarrow}W)\mapsto \big((vV,v\epsilon V)\overset{v\omega}{\Longrightarrow}(vW,v\epsilon W)\big),$$
$$\mathsf{EM}(\mathcal{K})\big(I(l),t\big)\to \mathcal{K}\big(l,J(t)\big),\qquad \big((A,\alpha)\overset{\varrho}{\Rightarrow}(B,\beta)\big)\mapsto \big(J(A,\alpha)\xrightarrow{J(\varrho)} J(B,\beta)\big), \tag{3.1}$$

for any 0-cell l and monad t in $\mathcal{K}$.

The notations in Theorem 3.1 are used throughout, without further explanation.

Lemma 3.2. *Consider a 2-category $\mathcal{K}$ which admits EM constructions for monads. For any 1-cell $(t,\mu,\eta)\xrightarrow{(V,\psi)}(t',\mu',\eta')$ in $\mathsf{EM}^w(\mathcal{K})$, the 2-cell $Vv\epsilon*\psi v*\eta' Vv : Vv\Rightarrow Vv$ in $\mathcal{K}$ is idempotent, and obeys the following identities:*

$$V\mu*\psi t*\eta' Vt*\psi=\psi; \tag{3.2}$$
$$Vv\epsilon*\psi v*t'Vv\epsilon*t'\psi v*t'\eta' Vv=Vv\epsilon*\psi v. \tag{3.3}$$

Proof. All statements follow easily by applying the interchange law, (1.1) and unitality of the monad t'. $\square$

The idempotent 2-cell in Lemma 3.2, corresponding to a 1-cell $(t,\mu,\eta)\xrightarrow{(V,\psi)}(t',\mu',\eta')$ in $\mathsf{EM}^w(\mathcal{K})$, is an identity 2-cell if and only if $\psi*\eta' V=V\eta$, i.e. (V,ψ) is a 1-cell in $\mathsf{EM}(\mathcal{K})$.

Our next aim is to extend the 2-functor J in Theorem 3.1 to $\mathsf{EM}^w(\mathcal{K})$. Our method is reminiscent to the way J is obtained from the right adjoint of the inclusion 2-functor $\mathcal{K}\to\mathsf{Mnd}(\mathcal{K})$.

Lemma 3.3. *Consider a 2-category $\mathcal{K}$ which admits EM constructions for monads and in which idempotent 2-cells split. For a 1-cell $(t,\mu,\eta)\xrightarrow{(V,\psi)}(t',\mu',\eta')$ in $\mathsf{EM}^w(\mathcal{K})$, denote a chosen splitting of the idempotent 2-cell in Lemma 3.2 by $Vv\overset{\pi}{\Rightarrow}\tilde V\overset{\iota}{\Rightarrow}Vv$. Then $(\tilde V,\tilde\psi:=\pi*Vv\epsilon*\psi v*t'\iota)$ is a 1-cell $IJ(t)\to t'$ in $\mathsf{EM}(\mathcal{K})$.*

Proof. By the interchange law, $\tilde\psi*\eta'\tilde V=\pi*\iota*\pi*\iota=\tilde V$. Furthermore, by (3.3) and (1.1),

$$\begin{aligned}\tilde\psi*t'\tilde\psi&=\pi*Vv\epsilon*\psi v*t'Vv\epsilon*t'\psi v*t't'\iota=\pi*Vv\epsilon*V\mu v*\psi tv*t'\psi v*t't'\iota\\&=\pi*Vv\epsilon*\psi v*\mu' Vv*t't'\iota=\tilde\psi*\mu'\tilde V.\qquad\square\end{aligned}$$

Lemma 3.4. *Consider a 2-category $\mathcal{K}$ which admits EM constructions for monads and in which idempotent 2-cells split. For any 2-cell $(V,\psi)\overset{\varrho}{\Rightarrow}(W,\phi)$ in $\mathsf{EM}^w(\mathcal{K})$, $\tilde\varrho:=\pi*Wv\epsilon*\varrho v*\iota$ is a 2-cell in $\mathsf{EM}(\mathcal{K})$, between the 1-cells $(\tilde V,\tilde\psi)$ and $(\tilde W,\tilde\phi)$ in Lemma 3.3 (where π and ι denote chosen splittings of both idempotent 2-cells in Lemma 3.2, corresponding to the 1-cells (V,ψ) and (W,ϕ)).*

Proof. Apply (3.2) (in the first equality), (1.2) (in the third equality) and (1.3) (in the penultimate equality) to conclude that

$$\begin{aligned}\tilde{\varrho} * \tilde{\psi} &= \pi * Wv\epsilon * \varrho v * Vv\epsilon * \psi v * t'\iota = \pi * Wv\epsilon * W\mu v * \varrho t v * \psi v * t'\iota \\ &= \pi * Wv\epsilon * W\mu v * \phi t v * t'\varrho v * t'\iota = \pi * Wv\epsilon * \phi v * t'\iota * t'\pi * t'Wv\epsilon * t'\varrho v * t'\iota \\ &= \tilde{\phi} * t'\tilde{\varrho}. \qquad \square\end{aligned}$$

Theorem 3.5. *Consider a 2-category $\mathcal{K}$ which admits EM constructions for monads and in which idempotent 2-cells split. The following maps determine a pseudo-functor $J^w : \mathsf{EM}^w(\mathcal{K}) \to \mathcal{K}$.*

For a 0-cell t, $J^w(t) := J(t)$.
For a 1-cell $t \xrightarrow{(V,\psi)} t'$, $J^w(V,\psi) := J(\tilde{V},\tilde{\psi})$, where the 1-cell $(\tilde{V},\tilde{\psi})$ in $\mathsf{EM}(\mathcal{K})$ is described in Lemma 3.3. That is, denoting by ι and π a chosen splitting of the idempotent 2-cell in Lemma 3.2, $J^w(V,\psi)$ is the unique 1-cell $J^w(t) \to J^w(t')$ in $\mathcal{K}$ for which $v'\epsilon' J^w(V,\psi) = \pi * Vv\epsilon * \psi v * t'\iota$.
For a 2-cell $(V,\psi) \overset{\varrho}{\Rightarrow} (W,\phi)$, $J^w(\varrho) := J(\tilde{\varrho})$, where the 2-cell $\tilde{\varrho}$ in $\mathsf{EM}(\mathcal{K})$ is described in Lemma 3.4. That is, $J^w(\varrho)$ is the unique 2-cell $J^w(V,\psi) \Rightarrow J^w(W,\phi)$ in $\mathcal{K}$ for which $v'J^w(\varrho) = \pi * Wv\epsilon * \varrho v * \iota$.

The pseudo-natural isomorphism class of J^w is independent of the choice of the 2-cells ι and π in its construction.

Proof. Let us fix splittings (π,ι) of the idempotent 2-cells in Lemma 3.2, for all 1-cells (V,ψ) in $\mathsf{EM}^w(\mathcal{K})$.

By construction, $v'J^w(\psi * \eta' V) = \pi * \iota * \pi * \iota = \tilde{V}$, for any 1-cell $t \xrightarrow{(V,\psi)} t'$ in $\mathsf{EM}^w(\mathcal{K})$. Hence J^w preserves identity 2-cells $(V,\psi) \xRightarrow{\psi * \eta' V} (V,\psi)$. For 2-cells $(V,\psi) \overset{\varrho}{\Rightarrow} (W,\phi) \overset{\tau}{\Rightarrow} (U,\theta)$ in $\mathsf{EM}^w(\mathcal{K})$, it follows by (1.3) (applied to ϱ) that

$$\begin{aligned}v'J^w(\tau) * v'J^w(\varrho) &= \pi * Uv\epsilon * \tau v * Wv\epsilon * \varrho v * \iota \\ &= \pi * Uv\epsilon * U\mu v * \tau t v * \varrho v * \iota = v'J^w(\tau \bullet \varrho).\end{aligned}$$

We conclude by the isomorphism (3.1) that J^w preserves the vertical composition.

For an identity 1-cell $t \xrightarrow{(k,t)} t$, the idempotent 2-cell in Lemma 3.2 is the identity 2-cell v by the adjunction relation $v\epsilon * \eta v = v$. Hence any splitting of it yields mutually inverse isomorphisms $v \xRightarrow{\pi_k} \tilde{k}$ and $\tilde{k} \xRightarrow{\iota_k} v$. They give rise to an isomorphism $J^w(t) = J(v, v\epsilon) \xRightarrow{J(\pi_k)} J^w(k,t) = J(\tilde{k}, \pi_k * v\epsilon * t\iota_k)$ with the inverse $J(\iota_k)$. Thus J^w preserves identity 1-cells up to isomorphism. (In particular, we can choose for the definition of J^w a trivial splitting $v \overset{v}{\Rightarrow} v \overset{v}{\Rightarrow} v$, in which case the 1-cell $(\tilde{k},\tilde{t})$ in Lemma 3.3 is equal to $(v, v\epsilon)$. Applying the isomorphism (3.1), we conclude that with this choice, J^w strictly preserves identity 1-cells, i.e. $J^w(k,t) = J(v,v\epsilon) = J^w(t)$.)

In order to investigate the preservation of the horizontal composition, consider different splittings (π,ι) and (π',ι') of the idempotent 2-cell in Lemma 3.2, for some 1-cell (V,ψ), and denote the corresponding 1-cells in Lemma 3.3 by $(\tilde{V},\tilde{\psi})$ and $(\tilde{V}',\tilde{\psi}')$, respectively. Applying (3.3) (for (π',ι')) and (3.2) (for (π,ι)),

$$\pi' * Vv\epsilon * \psi v * t'\iota' * t'\pi' * t'\iota = \pi' * Vv\epsilon * \psi v * t'\iota = \pi' * \iota * \pi * Vv\epsilon * \psi v * t'\iota.$$

Hence $(\tilde{V},\tilde{\psi}) \xrightarrow{\pi' * \iota} (\tilde{V}',\tilde{\psi}')$ is an iso 2-cell in $\mathsf{EM}(\mathcal{K})$, so $J(\tilde{V},\tilde{\psi}) \xrightarrow{J(\pi' * \iota)} J(\tilde{V}',\tilde{\psi}')$ is an iso 2-cell in $\mathcal{K}$.

For 1-cells $(V,\psi),(W,\phi): t \to t'$ and $(V',\psi'),(W',\phi'): t' \to t''$ and 2-cells $(V,\psi) \overset{\varrho}{\Rightarrow} (W,\phi)$ and $(V',\psi') \overset{\varrho'}{\Rightarrow} (W',\phi')$ in $\mathsf{EM}^w(\mathcal{K})$, the idempotent 2-cell in Lemma 3.2 corresponding to the 1-cell $(V',\psi') \circ (V,\psi)$ comes out as $V'Vv\epsilon * V'\psi v * \psi' Vv * \eta'' V'Vv$. We claim that it has a splitting given by the mono 2-cell $V'\iota * \iota' J^w(V,\psi)$ and the epi 2-cell $\pi' J^w(V,\psi) * V'\pi$, where (π,ι) and (π',ι') are the chosen splittings of the idempotent 2-cells in Lemma 3.2, corresponding to the 1-cells (V,ψ) and (V',ψ') in $\mathsf{EM}^w(\mathcal{K})$, respectively. Indeed, by construction of J^w (its action on a 1-cell (V,ψ)), (3.2) and (3.3),

$$\begin{aligned}
&V'\iota * \iota' J^w(V,\psi) * \pi' J^w(V,\psi) * V'\pi \\
&\quad = V'\iota * V'\pi * V'Vv\epsilon * V'\psi v * V't'\iota * V't'\pi * \psi' Vv * \eta'' V'Vv \\
&\quad = V'Vv\epsilon * V'\psi v * \psi' Vv * \eta'' V'Vv. \qquad (3.4)
\end{aligned}$$

Denote by $V'Vv \overset{\pi_2}{\Longrightarrow} \widetilde{V'V} \overset{\iota_2}{\Longrightarrow} V'Vv$ the canonical splitting of this idempotent which was chosen to define J^w on the 1-cell $(V',\psi') \circ (V,\psi) = (V'V, V'\psi * \psi' V)$. By considerations in the previous paragraph, there are mutually inverse iso 2-cells $J(\pi' J^w(V,\psi) * V'\pi * \iota_2) : J^w(V'V, V'\psi * \psi' V) \Rightarrow J^w(V',\psi')J^w(V,\psi)$ and $J(\pi_2 * V'\iota * \iota' J^w(V,\psi)) : J^w(V',\psi')J^w(V,\psi) \Rightarrow J^w(V'V, V'\psi * \psi' V)$. In order to see their naturality, observe that

$$v'' J^w(\varrho' \circ \varrho) = \pi_2 * W'Wv\epsilon * W'W\mu v * W'\varrho t v * W'\psi v * \varrho' Vv * \iota_2.$$

On the other hand,

$$\begin{aligned}
&v'' J^w(\varrho') J^w(\varrho) \\
&\quad = \pi' J^w(W,\phi) * W'\pi * W'Wv\epsilon * W'\phi v * \varrho' W v * V'\iota * V'\pi * V'Wv\epsilon * V'\varrho v \\
&\qquad * V'\iota * \iota' J^w(V,\psi) \\
&\quad = \pi' J^w(W,\phi) * W'\pi * W'Wv\epsilon * W'W\mu v * W'\phi t v * W't'\varrho v * \varrho' Vv * V'\iota * \iota' J^w(V,\psi) \\
&\quad = \pi' J^w(W,\phi) * W'\pi * W'Wv\epsilon * W'W\mu v * W'\varrho t v * W'\psi v * \varrho' Vv * V'\iota * \iota' J^w(V,\psi).
\end{aligned}$$

The second equality follows by applying (1.3), and the third one follows by applying (1.2), for ϱ. With this information in mind, we conclude that

$$\begin{aligned}
&v'' J^w(\varrho' \circ \varrho) * \pi_2 * V'\iota * \iota' J^w(V,\psi) \\
&\quad = \pi_2 * W'Wv\epsilon * W'W\mu v * W'\varrho t v * W'\psi v * \varrho' Vv * V'\iota * \iota' J^w(V,\psi) \\
&\quad = \pi_2 * W'\iota * \iota' J^w(W,\phi) * v'' J^w(\varrho') J^w(\varrho).
\end{aligned}$$

Thus naturality of $J(\pi_2 * V'\iota * \iota' J^w(V,\psi))$ follows by the isomorphism (3.1). It remains to check its associativity and unitality. For a further 1-cell $(V'',\psi''): t'' \to t'''$ in $\mathsf{EM}^w(\mathcal{K})$, use the notation $V''V'Vv \overset{\pi_3}{\Longrightarrow} \widetilde{V''V'V} \overset{\iota_3}{\Longrightarrow} V''V'Vv$ for the canonically split idempotent in the construction of J^w on $(V'', \psi'') \circ (V', \psi') \circ (V, \psi) = (V''V'V, V''V'\psi * V''\psi'V * \psi''V'V)$. By (3.4), the associativity condition

$$\begin{aligned}
&\pi_3 * V''\iota_2 * \iota'' J^w\big(V'V, V'\psi * \psi'V\big) * v''' J^w\big(V'', \psi''\big) J\big(\pi_2 * V'\iota * \iota' J^w(V, \psi)\big)\\
&\quad = \pi_3 * V''\iota_2 * V''\pi_2 * V''V'\iota * V''\iota' J^w(V, \psi) * \iota'' J^w\big(V', \psi'\big) J^w(V, \psi)\\
&\quad = \pi_3 * V''V'\iota * V''\iota' J^w(V, \psi) * \iota'' J^w\big(V', \psi'\big) J^w(V, \psi)\\
&\quad = \pi_3 * V''V'\iota * \iota_2' J^w(V, \psi) * \pi_2' J^w(V, \psi) * V''\iota' J^w(V, \psi) * \iota'' J^w\big(V', \psi'\big) J^w(V, \psi)
\end{aligned}$$

holds. The 2-cells ι_k and π_k, splitting the idempotent (identity) 2-cell in Lemma 3.2 corresponding to a unit 1-cell (k, t), are mutual inverses. Hence also the unitality conditions

$$\begin{aligned}
&v' J\big(\pi * V\iota_k * \iota J^w(k, t)\big) * v' J^w(V, \psi) J(\pi_k) = \pi * V\iota_k * V\pi_k * \iota = v' J^w(V, \psi),\\
&v' J\big(\iota_{k'} J^w(V, \psi)\big) * v' J(\pi_{k'}) J^w(V, \psi) = \iota_{k'} J^w(V, \psi) * \pi_{k'} J^w(V, \psi) = v' J^w(V, \psi)
\end{aligned}$$

hold. Thus we conclude by the isomorphism (3.1) that J^w preserves also the horizontal composition up to a coherent family of iso 2-cells, i.e. that it is a pseudo-functor.

Finally, we investigate the ambiguity of the pseudo-functor J^w, caused by a free choice of the splittings of the idempotent 2-cells in Lemma 3.2. Take two collections $\{(\pi, \iota)\}$ and $\{(\pi', \iota')\}$ of splittings (indexed by the 1-cells in $\mathsf{EM}^w(\mathcal{K})$). The pseudo-functors J^w and J'^w, associated to both families of splittings, are pseudo-naturally isomorphic via $J^w(t) = J'^w(t)$ and $J^w(V, \psi) \xRightarrow{J(\pi' * \iota)} J'^w(V, \psi)$, for any 0-cell t and 1-cell (V, ψ) in $\mathsf{EM}^w(\mathcal{K})$. $\square$

Consider a 2-category $\mathcal{K}$ which admits EM constructions for monads and in which idempotent 2-cells split. We can regard a 1-cell $t \xrightarrow{(V,\psi)} t'$ in $\mathsf{EM}(\mathcal{K})$ as a 1-cell in $\mathsf{EM}^w(\mathcal{K})$. Choosing a trivial splitting $Vv \xRightarrow{Vv} Vv \xRightarrow{Vv} Vv$ of the identity 2-cell, the corresponding 1-cell $IJ(t) \xrightarrow{(\tilde{V}, \tilde{\psi})} t'$ in Lemma 3.3 comes out as the 1-cell $(Vv, Vv\epsilon * \psi v)$ in $\mathsf{EM}(\mathcal{K})$. By 1-naturality of the counit of the 2-adjunction (I, J), we have $(v' J(V, \psi), v'\epsilon' J(V, \psi)) = (Vv, Vv\epsilon * \psi v)$. From this and from the isomorphism (3.1) it follows that

$$J^w(V, \psi) = J(Vv, Vv\epsilon * \psi v) = J(V, \psi).$$

Similarly, we can regard a 2-cell $(V, \psi) \overset{\varrho}{\Rightarrow} (W, \phi)$ in $\mathsf{EM}(\mathcal{K})$ as a 2-cell in $\mathsf{EM}^w(\mathcal{K})$. The corresponding 2-cell $(\tilde{V}, \tilde{\psi}) \overset{\tilde{\varrho}}{\Rightarrow} (\tilde{W}, \tilde{\phi})$ in Lemma 3.4 is equal to the 2-cell $Wv\epsilon * \varrho v : (Vv, Vv\epsilon * \psi v) \Rightarrow (Wv, Wv\epsilon * \phi v)$ in $\mathsf{EM}(\mathcal{K})$. By the 2-naturality condition $v' J(\varrho) = Wv\epsilon * \varrho v$ and the isomorphism (3.1) we obtain that

$$J^w(\varrho) = J(Wv\epsilon * \varrho v) = J(\varrho).$$

Summarizing, we proved that the pseudo-functor $J^w : \mathsf{EM}^w(\mathcal{K}) \to \mathcal{K}$ in Theorem 3.5 can be chosen such that the 2-functor $J : \mathsf{EM}(\mathcal{K}) \to \mathcal{K}$ in Theorem 3.1 factorizes through the obvious inclusion $\mathsf{EM}(\mathcal{K}) \hookrightarrow \mathsf{EM}^w(\mathcal{K})$ and J^w.

The pseudo-functor J^w in Theorem 3.5 takes a monad $((s, \psi), \nu, \vartheta)$ in $\mathsf{EM}^w(\mathcal{K})$ to a monad $J^w(s, \psi)$ in $\mathcal{K}$, with multiplication $J^w(s, \psi) J^w(s, \psi) \overset{\cong}{\Rightarrow} J^w((s, \psi) \circ\ (s, \psi)) \xRightarrow{J^w(\nu)} J^w(s, \psi)$

and unit $J^w(t) \overset{\cong}{\Rrightarrow} J^w(k,t) \xRightarrow{J^w(\vartheta)} J^w(s,\psi)$. Applying to this monad in $\mathcal{K}$ a hom 2-functor $\mathcal{K}(l,-):\mathcal{K}\to \mathsf{CAT}$ (for any 0-cell l in $\mathcal{K}$), we obtain a monad in CAT. Our next aim is to describe its Eilenberg–Moore category.

Lemma 3.6. *Consider a 2-category $\mathcal{K}$ which admits EM constructions for monads and in which idempotent 2-cells split. Let l be a 0-cell and $(k \xrightarrow{t} k, \mu, \eta)$ be a monad in $\mathcal{K}$ and let $t \xrightarrow{(s,\psi)} t$ be a 1-cell in $\mathsf{EM}^w(\mathcal{K})$. There is a bijective correspondence between the following structures:*

(i) *Pairs $(l \xrightarrow{V} J^w(t), J^w(s,\psi)V \overset{\zeta}{\Rightarrow} V)$, consisting of a 1-cell V and a 2-cell ζ in $\mathcal{K}$;*
(ii) *Pairs $((l \xrightarrow{W} k, tW \overset{\varrho}{\Rightarrow} W), sW \overset{\lambda}{\Rightarrow} W)$, consisting of a 1-cell $I(l) \xrightarrow{(W,\varrho)} t$ in $\mathsf{EM}(\mathcal{K})$ and (regarding (W,ϱ) as a 1-cell in $\mathsf{EM}^w(\mathcal{K})$), a 2-cell $(s,\psi)\circ(W,\varrho) \overset{\lambda}{\Rightarrow} (W,\varrho)$ in $\mathsf{EM}^w(\mathcal{K})$.*

Proof. Denote by ι and π the splitting of the idempotent 2-cell in Lemma 3.2, corresponding to the 1-cell (s,ψ) in $\mathsf{EM}^w(\mathcal{K})$, that was chosen to construct $J^w(s,\psi)$. For the 1-cell (W,ϱ) in $\mathsf{EM}^w(\mathcal{K})$, choose the trivial splitting of the identity 2-cell $W \overset{W}{\Rightarrow} W$, so that $J^w(W,\varrho) = J(W,\varrho)$.

By (3.1), there is a bijection between the 1-cells $I(l) \xrightarrow{(W,\varrho)} t$ in $\mathsf{EM}(\mathcal{K})$ as in part (ii), and the 1-cells $V := J(W,\varrho) = J^w(W,\varrho) : l \to J(t) = J^w(t)$ in $\mathcal{K}$ as in part (i). In order to see that it extends to the stated bijection, take first a 2-cell λ in $\mathsf{EM}^w(\mathcal{K})$ as in part (ii). Then there is a 2-cell $\zeta := (J^w(s,\psi)V \overset{\cong}{\Rrightarrow} J^w((s,\psi)\circ (W,\varrho)) \xRightarrow{J^w(\lambda)} V)$ in $\mathcal{K}$ as in part (i). Conversely, for a 2-cell ζ in $\mathcal{K}$ as in part (i), put $\lambda := v\zeta * \pi V$. It satisfies

$$\begin{aligned} \varrho * t\lambda &= v\epsilon V * tv\zeta * t\pi V = v\zeta * v\epsilon J^w(s,\psi)V * t\pi V \\ &= v\zeta * \pi V * sv\epsilon V * \psi v V * t\iota V * t\pi V = \lambda * s\varrho * \psi W. \end{aligned} \tag{3.5}$$

The second equality follows by the interchange law and the third one follows by construction of the pseudo-functor J^w, cf. Theorem 3.5. The last equality follows by (3.3). Together with the unitality of ϱ, this proves that λ is a 2-cell in $\mathsf{EM}^w(\mathcal{K})$, as needed.

The above two constructions can be seen to be mutual inverses. Take first a pair (V,ζ) as in part (i) and iterate both constructions. The result is $(V, J^w(s,\psi)V \overset{\cong}{\Rrightarrow} J^w((s,\psi)\circ (W,\varrho)) \xRightarrow{J^w(v\zeta*\pi v)} V) = (V,\zeta)$. In the opposite order, a datum $((W,\varrho),\lambda)$ is taken to $((W,\varrho), sW \xRightarrow{\pi V} vJ^w(s,\psi)V \overset{\cong}{\Rrightarrow} vJ^w((s,\psi)\circ(W,\varrho)) \xRightarrow{vJ^w(\lambda)} W) = ((W,\varrho), \lambda * \iota V * \pi V)$. The resulting 2-cell $\lambda * \iota V * \pi V$ in $\mathcal{K}$ is equal to λ since by (3.5) and unitality of ϱ,

$$\lambda * \iota V * \pi V = \lambda * s\varrho * \psi W * \eta s W = \varrho * t\lambda * \eta s W = \lambda. \qquad \square \tag{3.6}$$

The following extends [16, Proposition 3.1] and also [9, Theorem 3.4].

Proposition 3.7. *Consider a 2-category $\mathcal{K}$ which admits EM constructions for monads and in which idempotent 2-cells split. Let l be a 0-cell and $(k \xrightarrow{t} k, \mu, \eta)$ be a monad in $\mathcal{K}$ and let $(t \xrightarrow{(s,\psi)} t, \nu, \vartheta)$ be a monad in $\mathsf{EM}^w(\mathcal{K})$. The following categories are isomorphic:*

(i) *The Eilenberg–Moore category $\mathsf{EM}(\mathcal{K})(I(l), J^w(s,\psi))$ of the monad $\mathcal{K}(l, J^w(s,\psi)) : \mathcal{K}(l, J^w(t)) \to \mathcal{K}(l, J^w(t))$;*

(ii) *The Eilenberg–Moore category* $\mathsf{EM}(\mathcal{K})(I(l), \widehat{st})$ *of the monad* $\mathcal{K}(l, \widehat{st}) : \mathcal{K}(l,k) \to \mathcal{K}(l,k)$, *where the monad* $\widehat{st}$ *is obtained from the pre-monad* st *in Theorem* 2.3 *in the way described in Lemma* 2.2*;*

(iii) *The category* $\mathcal{C}$*, with*

objects that are pairs $((W,\varrho),\lambda)$*, consisting of a* 1*-cell* $I(l) \xrightarrow{(W,\varrho)} t$ *in* $\mathsf{EM}(\mathcal{K})$ *and a* 2*-cell* $(s,\psi)\circ(W,\varrho) \overset{\lambda}{\Rightarrow} (W,\varrho)$ *in* $\mathsf{EM}^w(\mathcal{K})$*, satisfying*

$$\lambda \bullet \big((s,\psi)\circ\lambda\big) = \lambda \bullet \big(\nu\circ(W,\varrho)\big); \tag{3.7}$$

$$(W,\varrho) = \lambda \bullet \big(\vartheta\circ(W,\varrho)\big); \tag{3.8}$$

morphisms $((W,\varrho),\lambda) \to ((W',\varrho'),\lambda')$ *that are* 2*-cells* $(W,\varrho) \overset{\alpha}{\Rightarrow} (W',\varrho')$ *in* $\mathsf{EM}(\mathcal{K})$ *such that*

$$\lambda' \bullet \big((s,\psi)\circ\alpha\big) = \alpha \bullet \lambda. \tag{3.9}$$

Proof. Denote by ι and π the splitting of the idempotent 2-cell in Lemma 3.2, corresponding to the 1-cell (s,ψ) in $\mathsf{EM}^w(\mathcal{K})$, that was chosen to construct $J^w(s,\psi)$. For the 1-cell (W,ϱ) in $\mathsf{EM}^w(\mathcal{K})$, choose the trivial splitting of the identity 2-cell $W \overset{W}{\Rightarrow} W$, so that $J^w(W,\varrho) = J(W,\varrho)$. Introduce shorthand notations $\bar{s} := J^w(s,\psi)$ and $V := J^w(W,\varrho)$.

Isomorphism of $\mathsf{EM}(\mathcal{K})(I(l), J^w(s,\psi))$ and $\mathcal{C}$. In light of Lemma 3.6, any object in $\mathsf{EM}(\mathcal{K})(I(l), J^w(s,\psi))$ is of the form $(J^w(W,\varrho), \bar{s}V \overset{\cong}{\Rightarrow} J^w((s,\psi)\circ\ (W,\varrho)) \xRightarrow{J^w(\lambda)} V)$, for a unique 1-cell $I(l) \xrightarrow{(W,\varrho)} t$ in $\mathsf{EM}(\mathcal{K})$ and a unique 2-cell $(s,\psi)\circ(W,\varrho) \overset{\lambda}{\Rightarrow} (W,\varrho)$ in $\mathsf{EM}^w(\mathcal{K})$. So we only need to show that λ satisfies (3.7), i.e. the equality

$$\lambda * s\lambda = \lambda * s\varrho * \nu W \tag{3.10}$$

of 2-cells in $\mathcal{K}$, if and only if $\bar{s}V \overset{\cong}{\Rightarrow} J^w((s,\psi)\circ(W,\varrho)) \xRightarrow{J^w(\lambda)} V$ is an associative action, and λ satisfies (3.8), i.e.

$$W = \lambda * s\varrho * \vartheta W \tag{3.11}$$

if and only if this $\bar{s}$-action on V is unital. Compose the associativity condition $J^w(\lambda) * J^w((s,\psi)\circ\lambda) = J^w(\lambda) * J^w(\nu\circ(W,\varrho))$ with v horizontally on the left, and compose it with the chosen split epimorphism $ssW \to vJ^w(ssW, ss\varrho * s\psi W * \psi sW)$ on the right (i.e. on the 'top'). It yields the equality

$$\lambda * s\lambda * (ss\varrho * s\psi W * \psi sW * \eta ssW) = \lambda * s\varrho * \nu W * (ss\varrho * s\psi W * \psi sW * \eta ssW).$$

Making use of (3.5), the left-hand side is easily shown to be equal to $\lambda * s\lambda$. As far as the right-hand side is concerned, use associativity of ϱ (in the first equality), (2.2) and (2.3) (in the second and third equalities, respectively) to see that

$$\lambda * s\varrho * \nu W * ss\varrho * s\psi W * \psi s W * \eta ss W = \lambda * s\varrho * s\mu W * \nu t W * s\psi W * \psi s W * \eta ss W$$
$$= \lambda * s\varrho * s\mu W * \psi t W * t\nu W * \eta ss W = \lambda * s\varrho * \nu W.$$

This proves that the $\bar{s}$-action on V is associative if and only if (3.10) holds. Similarly, the unitality condition $J^w(\lambda) * J^w(\vartheta \circ (W, \varrho)) = V$ is equivalent to $\lambda * \iota V * \pi V * s\varrho * \vartheta W = W$, hence by (3.6) it is equivalent to (3.11). Thus the bijection in Lemma 3.6 restricts to a bijection between the objects in $\mathsf{EM}(\mathcal{K})(I(l), J^w(s, \psi))$ and the objects in $\mathcal{C}$.

For a 2-cell $(W, \varrho) \overset{\alpha}{\Rightarrow} (W', \varrho')$ in $\mathsf{EM}(\mathcal{K})$, the condition $J^w(\lambda') * J^w((s, \psi) \circ \alpha) = J^w(\alpha) * J^w(\lambda)$ (expressing that $J^w(\alpha) = J(\alpha)$ is a morphism in $\mathsf{EM}(\mathcal{K})(I(l), J^w(s, \psi))$) is equivalent to $\lambda' * s\alpha * \iota V * \pi V = \alpha * \lambda * \iota V * \pi V$. The right-hand side is equal to $\alpha * \lambda$ by (3.6) and the left-hand side is equal to

$$\lambda' * s\alpha * s\varrho * \psi W * \eta s W = \lambda' * s\varrho' * \psi W' * \eta s W' * s\alpha = \varrho' * t\lambda' * \eta s W' * s\alpha = \lambda' * s\alpha,$$

using that α is a 2-cell in $\mathsf{EM}(\mathcal{K})$, (3.5) and unitality of ϱ'. Hence $J^w(\alpha) = J(\alpha)$ is a morphism in $\mathsf{EM}(\mathcal{K})(I(l), J^w(s, \psi))$ if and only if (3.9) holds. Thus we conclude by the isomorphism (3.1) that the 2-functor J induces an (obviously functorial) bijection between the morphisms in $\mathsf{EM}(\mathcal{K})(I(l), J^w(s, \psi))$ and the morphisms in $\mathcal{C}$.

Isomorphism of $\underline{\mathsf{EM}(\mathcal{K})(I(l), \widehat{st}) \text{ and } \mathcal{C}}$. In view of (2.14), we can choose $\widehat{st} = v\bar{s}f$ as 1-cells in $\mathcal{K}$. Moreover, taking axioms (2.8), (2.9) and (2.11) of a pre-monad into account,

$$\iota f * \pi f * \Theta = \Theta * \iota f st * \pi f st = \Theta * st\iota f * st\pi f = \Theta. \tag{3.12}$$

For an object $(l \xrightarrow{W} k, v\bar{s}fW \overset{\gamma}{\Rightarrow} W)$ in $\mathsf{EM}(\mathcal{K})(I(l), \widehat{st})$, put

$$\varrho := \gamma * \pi f W * s\mu W * \vartheta t W \quad \text{and} \quad \lambda := \gamma * \pi f W * s\eta W. \tag{3.13}$$

We show that $((W, \varrho), \lambda)$ is an object in $\mathcal{C}$. Recall that associativity and unitality of γ read as

$$\gamma * v\bar{s}f\gamma = \gamma * \pi f W * \Theta W * \iota f \iota f W \quad \text{and} \quad W = \gamma * \pi f W * \vartheta W,$$

respectively. Hence using associativity of γ and (3.12) (in the second equality) and applying the first identity in (2.16) (in the last equality),

$$\varrho * t\gamma * t\pi f W = \gamma * v\bar{s}f\gamma * \pi f \pi f W * s\mu st W * \vartheta tst W$$
$$= \gamma * \pi f W * \Theta W * s\mu st W * \vartheta tst W = \gamma * \pi f W * s\mu W * \psi t W. \tag{3.14}$$

Moreover, apply associativity of γ and (3.12) (in the second equality) and use (2.12) (in the third equality) to obtain

$$\lambda * s\varrho = \gamma * v\bar{s}f\gamma * \pi f \pi f W * s\eta st W * ss\mu W * s\vartheta t W$$
$$= \gamma * \pi f W * \Theta W * sts\mu W * st\vartheta t W * s\eta t W$$
$$= \gamma * \pi f W * s\mu W * \Theta t W * st\vartheta t W * s\eta t W = \gamma * \pi f W. \tag{3.15}$$

In the last equality we used (2.14) and that (since $\mu = v\epsilon f$) the interchange law yields $s\mu * \iota f t * \pi f t = \iota f * \pi f * s\mu$. With these identities at hand, associativity of ϱ is checked as

$$\begin{aligned}\varrho * t\varrho &= \varrho * t\gamma * t\pi f W * ts\mu W * t\vartheta t W = \gamma * \pi f W * s\mu W * s\mu t W * \psi tt W * t\vartheta t W\\ &= \gamma * \pi f W * s\mu W * s\mu t W * \vartheta tt W = \varrho * \mu W.\end{aligned}$$

The second equality follows by (3.14) and by associativity of μ. In the third equality we applied (2.4). The last equality follows by associativity of μ and the form of ϱ in (3.13). The unitality condition $\varrho * \eta W = W$ follows by unitality of μ and unitality of γ. Conditions (3.5), (3.10) and (3.11) are proven by

$$\begin{aligned}\varrho * t\lambda &= \varrho * t\gamma * t\pi f W * ts\eta W = \gamma * \pi f W * s\mu W * \psi t W * ts\eta W = \lambda * s\varrho * \psi W;\\ \lambda * s\lambda &= \gamma * v\bar{s} f\gamma * \pi f \pi f W * s\eta s\eta W = \gamma * \pi f W * \nu W = \lambda * s\varrho * \nu W;\\ W &= \gamma * \pi f W * \vartheta W = \lambda * s\varrho * \vartheta W.\end{aligned}$$

In each case, the last equality follows by (3.15). In the first computation, the second equality follows by (3.14). In the second equality of the second computation we used associativity of γ together with (3.12) and we applied the expression of ν in (2.15). In the first equality of the last computation we used unitality of γ. This proves that $((W, \varrho), \lambda)$ is an object in $\mathcal{C}$.

Conversely, for an object $((W, \varrho), \lambda)$ in $\mathcal{C}$, put

$$\gamma := \lambda * s\varrho * \iota f W. \tag{3.16}$$

It is associative as

$$\begin{aligned}\gamma * \pi f W * \Theta W * \iota f \iota f W &= \lambda * s\varrho * \Theta W * \iota f \iota f W\\ &= \lambda * s\varrho * \nu W * ss\varrho * s\psi W * sts\varrho * \iota f \iota f W\\ &= \lambda * s\lambda * ss\varrho * s\psi W * sts\varrho * \iota f \iota f W\\ &= \lambda * s\varrho * st\lambda * sts\varrho * \iota f \iota f W = \gamma * v\bar{s} f\gamma.\end{aligned}$$

The first equality follows by (3.16) and (3.12). In the second equality we substituted Θ by its expression in (2.13) and we used associativity of ϱ twice. In the third equality we applied (3.10) and in the fourth one we used (3.5). By (2.4) and unitality of μ, $\iota f * \pi f * \vartheta = s\mu * \psi t * \eta st * \vartheta = \vartheta$. Hence the unitality condition $\gamma * \pi f W * \vartheta W = W$ follows by (3.11). This proves that (W, γ) is an object in $\mathsf{EM}(\mathcal{K})(I(l), \widehat{st})$.

Let us see that the above constructions are mutual inverses. Starting with an object (W, γ) of $\mathsf{EM}(\mathcal{K})(I(l), \widehat{st})$ and iterating the above constructions, we re-obtain (W, γ) by (3.15). In the opposite order, applying both constructions to an object $((W, \varrho), \lambda)$ of $\mathcal{C}$, we obtain $((W, \lambda * s\varrho * \iota f W * \pi f W * s\mu W * \vartheta t W), \lambda * s\varrho * \iota f W * \pi f W * s\eta W)$. Since $\iota f * \pi f * s\mu = s\mu * \iota f t * \pi f t = s\mu * \Theta t * \vartheta stt$, axiom (2.10) of a pre-monad, associativity of ϱ and (3.11) imply that

$$\lambda * s\varrho * \iota f W * \pi f W * s\mu W * \vartheta t W = \lambda * s\varrho * s\mu W * \vartheta t W = \lambda * s\varrho * \vartheta W * \varrho = \varrho.$$

Also, by (2.14), unitality of μ, (3.5) and unitality of ϱ,

$$\lambda * s\varrho * \iota f W * \pi f W * s\eta W = \lambda * s\varrho * s\mu W * \psi t W * \eta s\eta W = \lambda * s\varrho * \psi W * \eta s W = \lambda.$$

Hence we constructed a bijection between the objects of $\mathsf{EM}(\mathcal{K})(I(l), \widehat{st})$ and $\mathcal{C}$. It is immediate by the form of the bijection between the objects that a 2-cell $W \overset{\alpha}{\Rightarrow} W'$ in $\mathcal{K}$ is a morphism in $\mathsf{EM}(\mathcal{K})(I(l), \widehat{st})$ if and only if it is a morphism in $\mathcal{C}$. □

4. Weak liftings

If $\mathcal{K}$ is a 2-category which admits EM constructions for monads, then 'liftings' of 1- and 2-cells for monads in $\mathcal{K}$ arise as images under the right 2-adjoint J of the inclusion 2-functor $\mathcal{K} \to \mathsf{EM}(\mathcal{K})$, see [17]. In this section we discuss 'weak' liftings and the role what the pseudo-functor J^w plays in their description.

Definition 4.1. Consider a 2-category $\mathcal{K}$ which admits EM constructions for monads. We say that a 1-cell $k \xrightarrow{V} k'$ in $\mathcal{K}$ possesses a *weak lifting* for some monads $(k \xrightarrow{t} k, \mu, \eta)$ and $(k' \xrightarrow{t'} k', \mu', \eta')$ in $\mathcal{K}$ if there exist a 1-cell $J(t) \xrightarrow{\overline{V}} J(t')$ and a split mono 2-cell $v'\overline{V} \overset{\iota}{\Rightarrow} Vv$ (with a retraction denoted by $Vv \overset{\pi}{\Rightarrow} v'\overline{V}$).

If in a 2-category $\mathcal{K}$ which admits EM constructions for monads also idempotent 2-cells split, then we know by Theorem 3.5 that, for every 1-cell $t \xrightarrow{(V,\psi)} t'$ in $\mathsf{EM}^w(\mathcal{K})$, the underlying 1-cell $k \xrightarrow{V} k'$ in $\mathcal{K}$ possesses a weak lifting $J^w(V, \psi)$ for t and t'. As we will see later in this section, in fact in such a 2-category $\mathcal{K}$, up-to an isomorphism, every weak lifting arises in this way. This extends assertions about 1-cells in [17, Lemma 3.9 and Theorem 3.10].

Definition 4.2. Consider a 2-category $\mathcal{K}$ which admits EM constructions for monads. Let $(k \xrightarrow{t} k, \mu, \eta)$ and $(k' \xrightarrow{t'} k', \mu', \eta')$ be monads, and $k \xrightarrow{V} k'$ and $k \xrightarrow{W} k'$ be 1-cells in $\mathcal{K}$, such that there exist their weak liftings $(J(t) \xrightarrow{\overline{V}} J(t'), \iota_V, \pi_V)$ and $(J(t) \xrightarrow{\overline{W}} J(t'), \iota_W, \pi_W)$ for t and t'. For a 2-cell $V \overset{\omega}{\Rightarrow} W$ in $\mathcal{K}$, we say that

a 2-cell $\overline{V} \overset{\overrightarrow{\omega}}{\Rightarrow} \overline{W}$ is a *weak ι-lifting* of ω if $\iota_W * v'\overrightarrow{\omega} = \omega v * \iota_V$;
a 2-cell $\overline{V} \overset{\overleftarrow{\omega}}{\Rightarrow} \overline{W}$ is a *weak π-lifting* of ω if $v'\overleftarrow{\omega} * \pi_V = \pi_W * \omega v$.

Throughout, indices of ι and π are omitted, as they can be reconstructed without ambiguity from the context.

By the isomorphism (3.1), the weak ι-lifting or weak π-lifting of a 2-cell is unique, provided that it exists. Moreover, if a 2-cell ω in Definition 4.2 possesses both a weak ι-lifting $\overrightarrow{\omega}$ and a weak π-lifting $\overleftarrow{\omega}$, then

$$v'\overrightarrow{\omega} = \pi * \iota * v'\overrightarrow{\omega} = \pi * \omega v * \iota = v'\overleftarrow{\omega} * \pi * \iota = v'\overleftarrow{\omega}.$$

Hence in view of the isomorphism (3.1), $\overrightarrow{\omega} = \overleftarrow{\omega}$.

Proposition 4.3 below, about weak liftings of 2-cells in a (nice enough) 2-category, extends statements about 2-cells in [17, Lemma 3.9 and Theorem 10]. Therein, notions and notations introduced in Corollary 1.4 are used.

Proposition 4.3. *Consider a 2-category $\mathcal{K}$ which admits EM constructions for monads and in which idempotent 2-cells split. Let $t \xrightarrow{(V,\psi)} t'$ and $t \xrightarrow{(W,\phi)} t'$ be 1-cells in $\mathsf{EM}^w(\mathcal{K})$ and $V \overset{\omega}{\Rightarrow} W$ be a 2-cell in $\mathcal{K}$. Denote by ι and π the splittings of both idempotent 2-cells in Lemma 3.2, corresponding to the 1-cells (V,ψ) and (W,ϕ), that were chosen to construct $J^w(V,\psi)$ and $J^w(W,\phi)$, respectively.*

(1) *The following assertions are equivalent:*
  (i) *ω is a 2-cell $(V,\psi) \Rightarrow (W,\phi)$ in $\mathsf{Mnd}^\iota(\mathcal{K})$;*
  (ii) *$\omega t * \psi * \eta' V : (V,\psi) \Rightarrow (W,\phi)$ is a 2-cell in $\mathsf{EM}^w(\mathcal{K})$;*
  (iii) *$\pi * \omega v * \iota : (v' J^w(V,\psi), v'\epsilon' J^w(V,\psi)) \Rightarrow (v' J^w(W,\phi), v'\epsilon' J^w(W,\phi))$ is a 2-cell in $\mathsf{EM}(\mathcal{K})$ such that $\iota * \pi * \omega v * \iota = \omega v * \iota$;*
  (iv) *ω possesses a weak ι-lifting $\vec{\omega} : (J^w(V,\psi),\iota,\pi) \to (J^w(W,\phi),\iota,\pi)$.*
  If these equivalent statements hold, then $v' J^w G^\iota(\omega) = \pi * \omega v * \iota$, that is, $\vec{\omega} = J^w G^\iota(\omega)$.

(2) *The following assertions are equivalent:*
  (i) *ω is a 2-cell $(V,\psi) \Rightarrow (W,\phi)$ in $\mathsf{Mnd}^\pi(\mathcal{K})$;*
  (ii) *$\phi * \eta' W * \omega : (V,\psi) \Rightarrow (W,\phi)$ is a 2-cell in $\mathsf{EM}^w(\mathcal{K})$;*
  (iii) *$\pi * \omega v * \iota : (v' J^w(V,\psi), v'\epsilon' J^w(V,\psi)) \Rightarrow (v' J^w(W,\phi), v'\epsilon' J^w(W,\phi))$ is a 2-cell in $\mathsf{EM}(\mathcal{K})$ such that $\pi * \omega v * \iota * \pi = \pi * \omega v$;*
  (iv) *ω possesses a weak π-lifting $\overleftarrow{\omega} : (J^w(V,\psi),\iota,\pi) \to (J^w(W,\phi),\iota,\pi)$.*
  If these equivalent statements hold, then $v' J^w G^\pi(\omega) = \pi * \omega v * \iota$, that is, $\overleftarrow{\omega} = J^w G^\pi(\omega)$.

(3) *The following assertions are equivalent:*
  (i) *$\phi * t'\omega = \omega t * \psi$;*
  (ii) *$\phi * \eta' W * \omega$ and $\omega t * \psi * \eta' V$ are (necessarily equal) 2-cells $(V,\psi) \Rightarrow (W,\phi)$ in $\mathsf{EM}^w(\mathcal{K})$;*
  (iii) *$\pi * \omega v * \iota : (v' J^w(V,\psi), v'\epsilon' J^w(V,\psi)) \Rightarrow (v' J^w(W,\phi), v'\epsilon' J^w(W,\phi))$ is a 2-cell in $\mathsf{EM}(\mathcal{K})$ such that $\iota * \pi * \omega v = \omega v * \iota * \pi$;*
  (iv) *ω possesses both a weak ι-lifting and a weak π-lifting 2-cell $(J^w(V,\psi),\iota,\pi) \to (J^w(W,\phi),\iota,\pi)$ (which are necessarily equal).*

Proof. <u>(1) (i) ⇔ (ii)</u> This equivalence follows by Lemma 1.2(1).

<u>(ii) ⇔ (iii)</u> The 2-cell $\pi * \omega v * \iota$ in $\mathcal{K}$ is a 2-cell in $\mathsf{EM}(\mathcal{K})$ if and only if $v'\epsilon' J^w(W,\phi) * t'\pi * t'\omega v * t'\iota = \pi * \omega v * \iota * v'\epsilon' J^w(V,\psi)$. Compose this equality by f horizontally on the right, and compose it vertically by ιf on the left and by $t'\pi f * t'V\eta$ on the right. By virtue of (3.2), (3.3) and the adjunction relation $\epsilon f * f\eta = f$, the resulting equivalent condition is identical to (1.7). Property $\iota * \pi * \omega v * \iota = \omega v * \iota$ is equivalent to $\iota * \pi * \omega v * \iota * \pi = \omega v * \iota * \pi$, what is easily seen to be equivalent to (1.6). Thus we conclude by Lemma 1.2(1)(i) ⇔ (iii).

<u>(iii) ⇔ (iv)</u> By the isomorphism (3.1), $\pi * \omega v * \iota$ is a 2-cell in $\mathsf{EM}(\mathcal{K})$ if and only if there is a 2-cell $J^w(V,\psi) \overset{\vec{\omega}}{\Rightarrow} J^w(W,\phi)$ in $\mathcal{K}$ such that $v'\vec{\omega} = \pi * \omega v * \iota$. Clearly, $\iota * \pi * \omega v * \iota = \omega v * \iota$ if and only if $\vec{\omega}$ is a weak ι-lifting of ω.

If the equivalent statements (i)–(iv) hold, then

$$v' J^w\big(\omega t * \psi * \eta' V\big) = \pi * W v\epsilon * \omega t v * \psi v * \eta' V v * \iota = \pi * \omega v * \iota * \pi * \iota = \pi * \omega v * \iota.$$

(2) (i) $\Leftrightarrow$ (ii) This equivalence follows by Lemma 1.2(2).

(ii) $\Leftrightarrow$ (iii) As we have seen in the proof of part (1), $\pi * \omega v * \iota$ is a 2-cell in $\mathsf{EM}(\mathcal{K})$ if and only if (1.7) holds. Property $\pi * \omega v * \iota * \pi = \pi * \omega v$ is equivalent to $\iota * \pi * \omega v * \iota * \pi = \iota * \pi * \omega v$ hence to the first condition in Lemma 1.2(2)(iii). Thus we conclude by Lemma 1.2(2)(i) $\Leftrightarrow$ (iii).

(iii) $\Leftrightarrow$ (iv) is proven by the same reasoning as in part (1).

If the equivalent statements (i)–(iv) hold, then

$$v' J^w\big(\phi * \eta' W * \omega\big) = \pi * W v\epsilon * \phi v * \eta' W v * \omega v * \iota = \pi * \iota * \pi * \omega v * \iota = \pi * \omega v * \iota.$$

(3) These equivalences follow immediately by Lemma 1.2(3) and parts (1) and (2) in the current theorem. $\square$

For suggesting the following theorem, the author is grateful to the referee.

Consider a 2-category $\mathcal{K}$ which admits EM constructions for monads. To any monads $(k \xrightarrow{t} k, \mu, \eta)$ and $(k' \xrightarrow{t'} k', \mu', \eta')$ in $\mathcal{K}$, we can associate categories $\mathsf{Lift}^\iota(t, t')$ and $\mathsf{Lift}^\pi(t, t')$, as follows. In both categories objects are quadruples $(V, \overline{V}, \iota, \pi)$ such that the 1-cell $J(t) \xrightarrow{\overline{V}} J(t')$ in $\mathcal{K}$ is a weak lifting of the 1-cell $k \xrightarrow{V} k'$, corresponding to the split monic 2-cell $v'\overline{V} \overset{\iota}{\Rightarrow} Vv$, with a retraction $Vv \overset{\pi}{\Rightarrow} v'\overline{V}$. Morphisms $(V, \overline{V}, \iota, \pi) \to (W, \overline{W}, \iota, \pi)$ are pairs of 2-cells $(\omega, \overline{\omega})$ in $\mathcal{K}$ such that $\overline{V} \overset{\overline{\omega}}{\Rightarrow} \overline{W}$ is a weak ι-lifting, respectively, a weak π-lifting, of $V \overset{\omega}{\Rightarrow} W$. Composition of morphisms is defined via component-wise composition of 2-cells in $\mathcal{K}$.

Theorem 4.4. *For any* 2*-category* $\mathcal{K}$ *which admits EM constructions for monads and in which idempotent* 2*-cells split, and for any monads* $(k \xrightarrow{t} k, \mu, \eta)$ *and* $(k' \xrightarrow{t'} k', \mu', \eta')$ *in* $\mathcal{K}$*, the following assertions hold.*

(1) $\mathsf{Lift}^\iota(t, t')$ *is equivalent to the category* $\mathsf{Mnd}^\iota(\mathcal{K})(t, t')$.
(2) $\mathsf{Lift}^\pi(t, t')$ *is equivalent to the category* $\mathsf{Mnd}^\pi(\mathcal{K})(t, t')$.

For $t = t'$*, these equivalences are also strong monoidal, with respect to the monoidal structure of* $\mathsf{Lift}^{\iota/\pi}(t, t)$ *induced by the horizontal composition in* $\mathcal{K}$.

Proof. (1) For any 1-cell $t \xrightarrow{(V,\psi)} t'$ in $\mathsf{EM}^w(\mathcal{K})$, denote by $Vv \overset{\pi_c}{\Longrightarrow} v' J^w(V, \psi) \overset{\iota_c}{\Longrightarrow} Vv$ the chosen splitting of the idempotent 2-cell in Lemma 3.2, used to construct $J^w(V, \psi)$.

By Corollary 1.4 and Theorem 3.5, there is a pseudo-functor $J^w G^\iota : \mathsf{Mnd}^\iota(\mathcal{K}) \to \mathcal{K}$. By Proposition 4.3(1)(i) $\Rightarrow$ (iv), it induces a functor $G : \mathsf{Mnd}^\iota(\mathcal{K})(t, t') \to \mathsf{Lift}^\iota(t, t')$, with object map $(V, \psi) \mapsto (V, J^w(V, \psi), \iota_c, \pi_c)$ and morphism map $\omega \mapsto (\omega, J^w G^\iota(\omega))$.

In the opposite direction, consider a functor $F : \mathsf{Lift}^\iota(t, t') \to \mathsf{Mnd}^\iota(\mathcal{K})(t, t')$ with the object map

$$(V, \overline{V}, \iota, \pi) \mapsto \big(V, \psi := \iota f * v'\epsilon'\overline{V} f * t'\pi f * t'V\eta : t'V \Rightarrow Vt\big) \tag{4.1}$$

and morphism map $(\omega, \overline{\omega}) \mapsto \omega$. By the form of ψ in (4.1) and the adjunction relation $v\epsilon * \eta v = v$, it follows that

$$Vv\epsilon * \psi v = \iota * v'\epsilon'\overline{V} * t'\pi. \tag{4.2}$$

Using (4.2) together with the form of ψ in (4.1) and naturality, it is easily checked that ψ satisfies (1.1), i.e. (V,ψ) is an object in $\mathsf{Mnd}^\iota(\mathcal{K})(t,t')$. By (4.2), the associated idempotent in Lemma 3.2 obeys $\iota_c * \pi_c = Vv\epsilon * \psi v * \eta' Vv = \iota * \pi$. Applying (4.2) together with (3.2) and (3.3), respectively, we conclude that

$$v'\epsilon' J^w(V,\psi) * t'\pi_c * t'\iota = \pi_c * \iota * v'\epsilon'\overline{V} \quad \text{and} \quad \pi * \iota_c * v'\epsilon' J^w(V,\psi) = v'\epsilon'\overline{V} * t'\pi * t'\iota_c.$$

That is, there are 2-cells $(v'J^w(V,\psi),\ v'\epsilon' J^w(V,\psi)) \xRightarrow{\pi*\iota_c} (v'\overline{V}, v'\epsilon'\overline{V})$ and $(v'\overline{V}, v'\epsilon'\overline{V}) \xRightarrow{\pi_c*\iota} (v'J^w(V,\psi), v'\epsilon' J^w(V,\psi))$ in $\mathsf{EM}(\mathcal{K})$. By (3.1) they induce mutually inverse isomorphisms $J^w(V,\psi) \xRightarrow{J(\pi*\iota_c)} \overline{V}$ and $\overline{V} \xRightarrow{J(\pi_c*\iota)} J^w(V,\psi)$ in $\mathcal{K}$. Both of these 2-cells are weak ι-liftings of the identity 2-cell $V \overset{V}{\Rightarrow} V$. Hence, for any morphism $(V,\overline{V},\iota,\pi) \xrightarrow{(\omega,\overline{\omega})} (W,\overline{W},\iota,\pi)$ in $\mathsf{Lift}^\iota(t,t')$, the composite $J(\pi_c * \iota) * \overline{\omega} * J(\pi * \iota_c) : J^w(V,\psi) \Rightarrow J^w(W,\phi)$ is a weak ι_c-lifting of ω (where both ψ and ϕ are defined via (4.1)). Thus it follows by Proposition 4.3(1)(iv) $\Rightarrow$ (i) that ω is a morphism $(V,\psi) \to (W,\phi)$ in $\mathsf{Mnd}^\iota(\mathcal{K})(t,t')$. This proves that F is a well-defined functor.

For any object (V,ψ) in $\mathsf{Mnd}^\iota(\mathcal{K})(t,t')$, we obtain $FG(V,\psi) = (V,\psi)$ by (3.2), (3.3) and unitality of μ. Evidently, also $FG(\omega) = \omega$. For any object $(V,\overline{V},\iota,\pi)$ of $\mathsf{Lift}^\iota(t,t')$, we obtain $GF(V,\overline{V},\iota,\pi) = (V, J^w(V,\psi),\iota_c,\pi_c)$. The mutually inverse isomorphisms $(V,\overline{V},\iota,\pi) \xrightarrow{(V,J(\pi_c*\iota))} (V,J^w(V,\psi),\iota_c,\pi_c)$ and $(V,J^w(V,\psi),\iota_c,\pi_c) \xrightarrow{(V,J(\pi*\iota_c))} (V,\overline{V},\iota,\pi)$ in $\mathsf{Lift}^\iota(t,t')$ define, in turn, mutually inverse natural isomorphisms between the identity functor and GF. Indeed, for any morphism $(V,\overline{V},\iota,\pi) \xrightarrow{(\omega,\overline{\omega})} (W,\overline{W},\iota,\pi)$ in $\mathsf{Lift}^\iota(t,t')$, we conclude by Corollary 1.4 and Theorem 3.5 that

$$v'J^w G^\iota(\omega) * \pi_c * \iota = \pi_c * Wv\epsilon * \omega t v * \psi v * \eta' Vv * \iota = \pi_c * \omega v * \iota = \pi_c * \iota * v'\overline{\omega}.$$

Hence naturality follows by the isomorphism (3.1).

It remains to prove strong monoidality of G in the $t = t'$ case. Recall from the proof of Theorem 3.5 that the coherent natural isomorphisms $J^w(V',\psi')J^w(V,\psi) \xRightarrow{j_{V',V}} J^w((V',\psi') \circ (V,\psi))$ and $J^w(t) \xRightarrow{j_0} J^w(k,t)$, rendering J^w (hence $J^w G^\iota$) a pseudo-functor, arise as weak ι-liftings of identity 2-cells, for any 1-cells $(V',\psi'),(V,\psi) : t \to t$ in $\mathsf{EM}^w(\mathcal{K})$. Hence they induce a strong monoidal structure $G(V',\psi')G(V,\psi) \xRightarrow{(V'V,j_{V',V})} G((V',\psi') \circ (V,\psi))$ and $(k, J^w(t)) \xRightarrow{(k,j_0)} G(k,t)$ of G.

Part (2) is proven symmetrically. $\square$

5. Applications

In this section we collect from the literature several situations where weak liftings occur. The following corollary is a consequence of Proposition 4.3 and Theorem 4.4.

Corollary 5.1. *Consider a 2-category $\mathcal{K}$ which admits EM constructions for monads and in which idempotent 2-cells split. Let $(k \xrightarrow{t} k, \mu, \eta)$ be a monad and $(k \xrightarrow{c} k, \delta, \varepsilon)$ be a comonad in $\mathcal{K}$.*

(1) *The following assertions are equivalent:*

(i) *There exists a comonad* $((c,\psi),\delta,\varepsilon)$ *in* $\mathsf{Mnd}^\iota(\mathcal{K})$. *That is, there exists a* 1*-cell* $t \xrightarrow{(c,\psi)} t$ *in* $\mathsf{EM}^w(\mathcal{K})$*, satisfying*

$$\delta t * \psi = cc\mu * c\psi t * \psi ct * t\delta t * t\psi * t\eta c; \tag{5.1}$$

$$\varepsilon t * \psi = \mu * t\varepsilon t * t\psi * t\eta c. \tag{5.2}$$

(ii) *There is a comonad* $(t \xrightarrow{(c,\psi)} t, \delta t * \psi * \eta c, \varepsilon t * \psi * \eta c)$ *in* $\mathsf{EM}^w(\mathcal{K})$.

(iii) *There are a comonad* $(J^w(t) \xrightarrow{\bar{c}} J^w(t), \overrightarrow{\delta}, \overrightarrow{\varepsilon})$ *and a split monic* 2*-cell* $v\bar{c} \overset{\iota}{\Rightarrow} cv$ *in* $\mathcal{K}$ *such that* $\overrightarrow{\delta}$ *is a weak* ι*-lifting of* δ *and* $\overrightarrow{\varepsilon}$ *is a weak* ι*-lifting of* ε.

If these equivalent statements hold, then we say shortly that the comonad $(\bar{c}, \overrightarrow{\delta}, \overrightarrow{\varepsilon})$ *is a weak* ι*-lifting of the comonad* (c,δ,ε) *for the monad* (t,μ,η).

(2) *The following assertions are equivalent:*

(i) *There exists a comonad* $((c,\psi),\delta,\varepsilon)$ *in* $\mathsf{Mnd}^\pi(\mathcal{K})$. *That is, there exists a* 1*-cell* $t \xrightarrow{(c,\psi)} t$ *in* $\mathsf{EM}^w(\mathcal{K})$*, satisfying*

$$c\psi * \psi c * t\delta = cc\mu * c\psi t * \psi ct * \eta cct * \delta t * \psi; \tag{5.3}$$

$$t\varepsilon = \varepsilon t * \psi. \tag{5.4}$$

(ii) *There is a comonad* $(t \xrightarrow{(c,\psi)} t, c\psi * \psi c * \eta cc * \delta, \eta * \varepsilon)$ *in* $\mathsf{EM}^w(\mathcal{K})$.

(iii) *There are a comonad* $(J^w(t) \xrightarrow{\bar{c}} J^w(t), \overleftarrow{\delta}, \overleftarrow{\varepsilon})$ *and a split epi* 2*-cell* $cv \overset{\pi}{\Rightarrow} v\bar{c}$ *in* $\mathcal{K}$ *such that* $\overleftarrow{\delta}$ *is a weak* π*-lifting of* δ *and* $\overleftarrow{\varepsilon}$ *is a weak* π*-lifting of* ε.

If these equivalent statements hold, then we say shortly that the comonad $(\bar{c}, \overleftarrow{\delta}, \overleftarrow{\varepsilon})$ *is a weak* π*-lifting of the comonad* (c,δ,ε) *for the monad* (t,μ,η).

(3) *The following assertions are equivalent:*

(i) *There exists a* 1*-cell* $t \xrightarrow{(c,\psi)} t$ *in* $\mathsf{EM}^w(\mathcal{K})$*, satisfying*

$$c\psi * \psi c * t\delta = \delta t * \psi; \tag{5.5}$$

$$t\varepsilon = \varepsilon t * \psi. \tag{5.6}$$

(ii) *There are a comonad* $(J^w(t) \xrightarrow{\bar{c}} J^w(t), \bar{\delta}, \bar{\varepsilon})$ *and a split epi-mono pair of* 2*-cells* $cv \overset{\pi}{\Rightarrow} v\bar{c} \overset{\iota}{\Rightarrow} cv$ *in* $\mathcal{K}$ *such that* $\bar{\delta}$ *is both a weak* ι*-lifting and a weak* π*-lifting of* δ *and* $\bar{\varepsilon}$ *is both a weak* ι*-lifting and a weak* π*-lifting of* ε.

Note that by Lemmas 1.2 and 1.3, in parts (1) and (2) of Corollary 5.1, assertions (i) and (ii) are equivalent in case of an arbitrary 2-category $\mathcal{K}$.

Let us stress the (tiny) difference between a 2-cell $tc \overset{\psi}{\Rightarrow} ct$ in $\mathcal{K}$ occurring in Corollary 5.1(3)(i), and a mixed distributive law. A 2-cell ψ in Corollary 5.1(3)(i) satisfies three of the identities defining a mixed distributive law: compatibility with the multiplication of the monad (as (c,ψ) is a 1-cell in $\mathsf{EM}^w(\mathcal{K})$), compatibility with the comultiplication of the comonad (by (5.5)) and compatibility with the counit of the comonad (by (5.6)). However, the fourth condition on a mixed distributive law, compatibility $\psi * \eta c = c\eta$ with the unit of the monad, does not appear in Corollary 5.1(3)(i) – it plays no role in a weak lifting.

Example 5.2. Generalizing a mixed distributive law of a monad and a comonad (in particular in the bicategory BIM), weak entwining structures were introduced by Caenepeel and De Groot in [9]. The axioms are obtained by weakening the compatibility conditions of a mixed distributive law with the unit of the monad and the counit of the comonad. Precisely, a *weak entwining structure* in an arbitrary 2-category $\mathcal{K}$ consists of a monad $(k \xrightarrow{t} k, \mu, \eta)$, a comonad $(k \xrightarrow{c} k, \delta, \varepsilon)$ and a 2-cell $tc \overset{\psi}{\Rightarrow} ct$ subject to the following conditions:

$$\psi * \mu c = c\mu * \psi t * t\psi; \tag{5.7}$$

$$\delta t * \psi = c\psi * \psi c * t\delta; \tag{5.8}$$

$$\psi * \eta c = c\varepsilon t * c\psi * c\eta c * \delta; \tag{5.9}$$

$$\varepsilon t * \psi = \mu * t\varepsilon t * t\psi * t\eta c. \tag{5.10}$$

We claim that under these assumptions $((c, \psi), \delta t * \psi * \eta c, \varepsilon t * \psi * \eta c)$ is a comonad in $\mathsf{EM}^w(\mathcal{K})$. For that, we need to show that axioms (5.7)–(5.10) imply (5.1). Indeed,

$$cc\mu * c\psi t * \psi ct * t\delta t * t\psi * t\eta c = cc\mu * \delta tt * \psi t * t\psi * t\eta c = \delta t * \psi.$$

The first equality follows by (5.8) and the second one follows by (5.7) and unitality of the monad t.

Hence if moreover $\mathcal{K}$ admits EM constructions for monads and idempotent 2-cells in $\mathcal{K}$ split (hence there exists the pseudo-functor J^w) then, by Corollary 5.1(1), the comonad c has a weak ι-lifting for the monad t.

For a commutative, associative and unital ring k, consider a k-algebra A and a k-coalgebra C. Let $\Psi : C \otimes_k A \to A \otimes_k C$ be a k-module map such that the triple $((-) \otimes_k A, (-) \otimes_k C, (-) \otimes_k \Psi)$ is a weak entwining structure in CAT. (If we are ready to cope with the more involved situation of a bicategory, we can say simply that (A, C, Ψ) is a weak entwining structure in BIM.) The corresponding weak ι-lifting of the comonad $(-) \otimes_k C$ for the monad $(-) \otimes_k A$ is studied in [9, Section 2]. Brzeziński showed in [8, Proposition 2.3] that it can be described as a comonad $(-) \otimes_A \overline{C}$ on the category of right A-modules, where the A-coring (i.e. comonad $A \to A$ in BIM) $\overline{C}$ is constructed as a k-module retract of $A \otimes_k C$.

Examples of weak entwining structures, thus examples of weak ι-liftings of comonads for monads, are provided by weak Doi–Koppinen data in [3] (see [9]), i.e. by comodule algebras and module coalgebras of weak bialgebras. Further examples are weak comodule algebras of bialgebras in [11, Proposition 2.3].

Example 5.3. Another generalization of a mixed distributive law, motivated by partial coactions of Hopf algebras, is due to Caenepeel and Janssen. Following [10, Proposition 2.6], a *partial entwining structure* in a 2-category $\mathcal{K}$ consists of a monad $(k \xrightarrow{t} k, \mu, \eta)$, a comonad $(k \xrightarrow{c} k, \delta, \varepsilon)$ and a 2-cell $tc \overset{\psi}{\Rightarrow} ct$ in $\mathcal{K}$, such that identities (5.4) and (5.7) hold, together with

$$cc\mu * c\psi t * c\eta ct * \delta t * \psi = c\psi * \psi c * t\delta. \tag{5.11}$$

Observe that axiom (5.11) implies (5.3):

$$cc\mu * c\psi t * \psi ct * \eta cct * \delta t * \psi = cc\mu * cc\mu t * c\psi tt * c\eta ctt * \delta tt * \psi t * \eta ct * \psi$$
$$= cc\mu * c\psi t * c\eta ct * \delta t * \psi = c\psi * \psi c * t\delta.$$

The first and last equalities follow by (5.11) and the second equality is obtained using associativity of μ and (3.2). This implies that $((c, \psi), c\psi * \psi c * \eta cc * \delta, \eta * \varepsilon)$ is a comonad in $\mathsf{EM}^w(\mathcal{K})$. Thus if moreover $\mathcal{K}$ is a 2-category which admits EM constructions for monads and in which idempotent 2-cells split, then we conclude by Corollary 5.1(2) that a partial entwining structure (t, c, ψ) in $\mathcal{K}$ induces a weak π-lifting of the comonad c for the monad t.

Consider the particular case when a monad $t := (-) \otimes_k A$ in CAT is induced by an algebra A over a commutative, associative and unital ring k, a comonad $c := (-) \otimes_k C$ is induced by a k-coalgebra C and a natural transformation $tc \overset{\psi}{\Rightarrow} ct$ is induced by a k-module map $C \otimes_k A \to A \otimes_k C$. Then the weak π-lifting of the comonad c for the monad t, induced by a partial entwining ψ, is a comonad $(-) \otimes_A \overline{C}$ on the category of right A-modules. The A-coring $\overline{C}$ was constructed in [10, Proposition 2.6] as a k-module retract of $A \otimes_k C$.

Examples of partial entwining structures (hence of weak π-liftings of a comonad for a monad) are provided by partial comodule algebras of bialgebras in [11, Proposition 2.6].

Example 5.4. Yet another way to generalize a mixed distributive law was proposed in [10]. Following [10, Proposition 2.5], a *lax entwining structure* in a 2-category $\mathcal{K}$ consists of a monad $(k \xrightarrow{t} k, \mu, \eta)$, a comonad $(k \xrightarrow{c} k, \delta, \varepsilon)$ and a 2-cell $tc \overset{\psi}{\Rightarrow} ct$ in $\mathcal{K}$, such that identities (5.7), (5.10) and (5.11) hold, together with

$$c\mu * ct\varepsilon t * ct\psi * ct\eta c * \psi c * t\delta * \eta c = \psi * \eta c.$$

As we observed in Example 5.3, (5.11) implies (5.3), and (5.10) is identical to (5.2). However, none of (5.1) and (5.4) seems to hold for an arbitrary lax entwining structure. Still, the axioms of a lax entwining structure allow us to prove that there is a comonad $((c, \psi), c\psi * \psi c * \eta cc * \delta, \varepsilon t * \psi * \eta c)$ in $\mathsf{EM}^w(\mathcal{K})$. Therefore, if $\mathcal{K}$ admits EM constructions for monads and idempotent 2-cells in $\mathcal{K}$ split, then J^w takes it to a comonad $(J^w(t) \xrightarrow{\bar{c}} J^w(t), \overleftarrow{\delta}, \overrightarrow{\varepsilon})$ in $\mathcal{K}$. However, it is neither a weak ι-lifting nor a weak π-lifting of the comonad c, it is of a mixed nature.

In the particular case when a lax entwining structure in CAT is induced by modules over a commutative associative and unital ring, the comonad $(\bar{c}, \overleftarrow{\delta}, \overrightarrow{\varepsilon})$ is induced by a coring, which was computed in [10, Proposition 2.5]. Examples of lax entwining structures are provided by lax comodule algebras of bialgebras in [11, Proposition 2.5].

A fourth logical possibility, to obtain a comonad structure on a weak lifting for a monad t of a 1-cell c underlying a comonad (c, δ, ε), is to allow the comultiplication to be a weak ι-lifting of δ and the counit to be a weak π-lifting of ε. That is, to require a 1-cell $t \xrightarrow{(c,\psi)} t$ in $\mathsf{EM}^w(\mathcal{K})$ to satisfy (5.1) and (5.4). By (the proof of) Lemma 1.3, it yields a coassociative and counital comonad $(\bar{c}, \overrightarrow{\delta}, \overleftarrow{\varepsilon})$ in $\mathcal{K}$.

For any 2-category $\mathcal{K}$, one may consider the vertically-opposite 2-category $\mathcal{K}_*$. The 2-category $\mathcal{K}_*$ has the same 0-, 1-, and 2-cells as $\mathcal{K}$, the same horizontal composition and the opposite vertical composition. Obviously, 2-cells in $\mathcal{K}$ split if and only if 2-cells in $\mathcal{K}_*$ split. Since monads in $\mathcal{K}_*$ are the same as the comonads in $\mathcal{K}$, the 2-category $\mathcal{K}_*$ admits EM constructions for monads if and only if $\mathcal{K}$ admits EM constructions for comonads, cf. [17]. In this case we denote by $J^w_* : \mathsf{EM}^w(\mathcal{K}_*)_* \to \mathcal{K}$ the pseudo-functor in Theorem 3.5.

Definition 5.5. Consider a 2-category $\mathcal{K}$ which admits EM constructions for comonads. We say that a 1-cell V in $\mathcal{K}$ possesses a *weak lifting* $\overline{V}$ *for some comonads* c and c', provided that, regarded as 1-cells in $\mathcal{K}_*$, $\overline{V}$ is a weak lifting of V for the monads c and c' in $\mathcal{K}_*$.

For a 2-cell ω in $\mathcal{K}$, a *weak ι-lifting* (resp. *weak π-lifting*) for some comonads c and c' in $\mathcal{K}$ is defined as a weak π-lifting (resp. weak ι-lifting) of ω, regarded as a 2-cell in $\mathcal{K}_*$, for the monads c and c' in $\mathcal{K}_*$.

The following corollary is obtained by applying Corollary 5.1 to the vertically-opposite of a 2-category. Therein, the symbol $*$ denotes the vertical composition in $\mathcal{K}$ (not its opposite).

Corollary 5.6. *Consider a 2-category $\mathcal{K}$ which admits EM constructions for comonads and in which idempotent 2-cells split. Let $(k \xrightarrow{t} k, \mu, \eta)$ be a monad and $(k \xrightarrow{c} k, \delta, \varepsilon)$ be a comonad in $\mathcal{K}$.*

(1) *The following assertions are equivalent:*
   (i) *There is a monad $((t, \psi), \mu, \eta)$ in $\mathsf{Mnd}^\iota(\mathcal{K}_*)_*$. That is, there exists a 1-cell $c \xrightarrow{(t,\psi)} c$ in $\mathsf{EM}^w(\mathcal{K}_*)_*$ (i.e. a 2-cell $tc \overset{\psi}{\Rightarrow} ct$ in $\mathcal{K}$ such that $\delta t * \psi = c\psi * \psi c * t\delta$), satisfying*

$$\psi * \mu c = c\varepsilon t * c\psi * c\mu c * \psi tc * t\psi c * tt\delta; \tag{5.12}$$

$$\psi * \eta c = c\varepsilon t * c\psi * c\eta c * \delta. \tag{5.13}$$

   (ii) *There is a monad $(c \xrightarrow{(t,\psi)} c, \varepsilon t * \psi * \mu c, \varepsilon t * \psi * \eta c)$ in $\mathsf{EM}^w(\mathcal{K}_*)_*$.*
   (iii) *There are a monad $(J_*^w(c) \xrightarrow{\bar{t}} J_*^w(c), \overleftarrow{\mu}, \overleftarrow{\eta})$ and a split epi 2-cell π in $\mathcal{K}$ such that $\overleftarrow{\mu}$ is a weak π-lifting of μ and $\overleftarrow{\eta}$ is a weak π-lifting of η.*

   If these equivalent statements hold, then we say shortly that the monad $(\bar{t}, \overleftarrow{\mu}, \overleftarrow{\eta})$ is a weak π-lifting of the monad (t, μ, η) for the comonad (c, δ, ε).
(2) *The following assertions are equivalent:*
   (i) *There is a monad $((t, \psi), \mu, \eta)$ in $\mathsf{Mnd}^\pi(\mathcal{K}_*)_*$. That is, there exists a 1-cell $c \xrightarrow{(t,\psi)} c$ in $\mathsf{EM}^w(\mathcal{K}_*)_*$, satisfying*

$$c\mu * \psi t * t\psi = \psi * \mu c * \varepsilon ttc * \psi tc * t\psi c * tt\delta; \tag{5.14}$$

$$c\eta = \psi * \eta c. \tag{5.15}$$

   (ii) *There is a monad $(c \xrightarrow{(t,\psi)} c, \mu * \varepsilon tt * \psi t * t\psi, \eta * \varepsilon)$ in $\mathsf{EM}^w(\mathcal{K}_*)_*$.*
   (iii) *There are a monad $(J_*^w(c) \xrightarrow{\bar{t}} J_*^w(c), \overrightarrow{\mu}, \overrightarrow{\eta})$ and a split monic 2-cell ι in $\mathcal{K}$ such that $\overrightarrow{\mu}$ is a weak ι-lifting of μ and $\overrightarrow{\eta}$ is a weak ι-lifting of η.*

   If these equivalent statements hold, then we say shortly that the monad $(\bar{t}, \overrightarrow{\mu}, \overrightarrow{\eta})$ is a weak ι-lifting of the monad (t, μ, η) for the comonad (c, δ, ε).
(3) *The following assertions are equivalent:*
   (i) *There exists a 1-cell $t \xrightarrow{(c,\psi)} t$ in $\mathsf{EM}^w(\mathcal{K}_*)_*$, satisfying*

$$c\mu * \psi t * t\psi = \psi * \mu c; \tag{5.16}$$

$$c\eta = \psi * \eta c. \tag{5.17}$$

(ii) *There are a monad* $(J_*^w(c) \xrightarrow{\bar{t}} J_*^w(c), \overline{\mu}, \overline{\eta})$ *and a split epi-mono pair* (π, ι) *of* 2*-cells in* $\mathcal{K}$ *such that* $\overline{\mu}$ *is both a weak* ι*-lifting and a weak* π*-lifting of* μ *and* $\overline{\eta}$ *is both a weak* ι*-lifting and a weak* π*-lifting of* η.

A 2-cell ψ in Corollary 5.6(3)(i) differs from a mixed distributive law by the compatibility condition with the counit of the comonad.

In a 2-category $\mathcal{K}$ which admits EM constructions for both monads and comonads and in which idempotent 2-cells split, one can say more about weak entwining structures than it was said in Example 5.2.

Proposition 5.7. *Consider a* 2*-category* $\mathcal{K}$ *which admits EM constructions for both monads and comonads and in which idempotent* 2*-cells split. For a monad* $(k \xrightarrow{t} k, \mu, \eta)$*, a comonad* $(k \xrightarrow{c} k, \delta, \varepsilon)$*, and a* 2*-cell* $tc \overset{\psi}{\Rightarrow} ct$ *in* $\mathcal{K}$ *(with a chosen splitting* (π, ι) *of the associated idempotent in Lemma* 3.2*), the following assertions are equivalent:*

(i) *The triple* (t, c, ψ) *is a weak entwining structure, that is, axioms* (5.7)–(5.10) *are satisfied.*
(ii) *The* 2*-cell* ψ *induces both a weak* ι*-lifting of the comonad c for the monad t and a weak* π*-lifting of the monad t for the comonad c. That is to say, the assertions in Corollary* 5.1(1) *and Corollary* 5.6(1) *hold.*

Proof. We have seen in Example 5.2 that axioms (5.7)–(5.10) imply (5.1). Similarly, they can be seen to imply (5.12) as well, applying first (5.7) and next (5.8). $\square$

Proposition 5.7 is the basis of a construction in [4] of a 2-category of weak entwining structures in any 2-category. In that paper, for a weak entwining structure in a 2-category $\mathcal{K}$ which admits EM constructions for both monads and comonads and in which idempotent 2-cells split, it is proven that the weakly lifted monad, and the weakly lifted comonad, occurring in part (ii) of Proposition 5.7, possess equivalent Eilenberg–Moore objects.

The characterization of weak entwining structures in Proposition 5.7 can be used, in particular, to describe weak bialgebras [6] in terms of weak liftings. Recall that a *weak bialgebra* H over a commutative, associative and unital ring k, is a k-module which possesses both a k-algebra and a k-coalgebra structure, subject to the following compatibility conditions. Denote the multiplication $H \otimes_k H \to H$ in H by juxtaposition of elements. Write 1 for the unit element of the algebra H and write $\varepsilon : H \to k$ for the counit. For the comultiplication $H \to H \otimes_k H$, use a Sweedler type index notation $h \mapsto \sum h_1 \otimes_k h_2$. With these notations, the axioms

$$\sum (hh')_1 \otimes_k (hh')_2 = \sum h_1 h'_1 \otimes_k h_2 h'_2; \tag{5.18}$$

$$\sum 1_1 \otimes 1_2 1_{1'} \otimes 1_{2'} = \sum 1_1 \otimes_k 1_2 \otimes_k 1_3 = \sum 1_1 \otimes_k 1_{1'} 1_2 \otimes_k 1_{2'}; \tag{5.19}$$

$$\sum \varepsilon(h 1_1)\varepsilon(1_2 h') = \varepsilon(hh') = \sum \varepsilon(h 1_2)\varepsilon(1_1 h') \tag{5.20}$$

are required to hold, for all elements h and h' of H. However, the comultiplication is not required to preserve the unit, i.e. $\sum 1_1 \otimes_k 1_2$ is not required to be equal to $1 \otimes 1$ and the counit is not required to be multiplicative, i.e. $\varepsilon(hh')$ is not required to be equal to $\varepsilon(h)\varepsilon(h')$, for elements $h, h' \in H$.

In the following proposition we deal with the (co)monads $H \otimes_k (-)$ and $(-) \otimes_k H$, induced by a k-(co)algebra H on the category of modules over a commutative, associative and unital ring k.

Proposition 5.8. *For a commutative, associative and unital ring k, consider a k-module H which possesses both a k-algebra and a k-coalgebra structure. Using the notations introduced above the proposition, the following assertions are equivalent:*

(i) *The algebra and coalgebra structures of H constitute a weak bialgebra;*
(ii) *The k-module map*

$$\Psi_R : H \otimes_k H \to H \otimes_k H, \qquad h \otimes_k h' \mapsto \sum h'_1 \otimes_k hh'_2 \tag{5.21}$$

induces a weak ι-lifting of the comonad $(-) \otimes_k H$ for the monad $(-) \otimes_k H$ and a weak π-lifting of the monad $(-) \otimes_k H$ for the comonad $(-) \otimes_k H$, and the k-module map

$$\Psi_L : H \otimes_k H \to H \otimes_k H, \qquad h \otimes_k h' \mapsto \sum h_1 h' \otimes_k h_2 \tag{5.22}$$

induces a weak ι-lifting of the comonad $H \otimes_k (-)$ for the monad $H \otimes_k (-)$ and a weak π-lifting of the monad $H \otimes_k (-)$ for the comonad $H \otimes_k (-)$. That is to say,

$$\big((-) \otimes_k H, (-) \otimes_k \Psi_R\big) \quad \textit{and} \quad \big(H \otimes_k (-), \Psi_L \otimes_k (-)\big)$$

are comonads in $\mathsf{Mnd}^\iota(\mathsf{CAT})$, via the comultiplication and counit induced by the coalgebra H, and they are monads in $\mathsf{Mnd}^\iota(\mathsf{CAT}_*)_*$, via the multiplication and unit induced by the algebra H.

Proof. Note first that assertion (ii) implies (5.18). Indeed, (5.21) determines a 1-cell $((-) \otimes_k H, (-) \otimes_k \Psi_R)$ in $\mathsf{EM}^w(\mathsf{CAT})$ if and only if

$$\sum \big(h'h''\big)_1 \otimes_k h\big(h'h''\big)_2 = \sum h'_1 h''_1 \otimes_k hh'_2 h''_2,$$

for any elements h, h' and h'' of H. Putting $h = 1$ we obtain (5.18).

By Proposition 5.7, assertion (ii) is equivalent to saying that $((-) \otimes_k H, (-) \otimes_k H, (-) \otimes_k \Psi_R)$ and $(H \otimes_k (-), H \otimes_k (-), \Psi_L \otimes_k (-))$ are weak entwining structures in CAT (or, in the terminology of [9], (H, H, Ψ_R) is a right–right weak entwining structure and (H, H, Ψ_L) is a left–left weak entwining structure in BIM). This statement was proven to be equivalent to (i) in [9, Theorem 4.7]. □

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