[9] Y. Cao and W. Ren, "Distributed coordinated tracking with reduced interaction via a variable structure approach," *IEEE Trans. Autom. Control*, vol. 56, no. 12, Dec. 2011, to be published.
[10] W. Ren, "Distributed leaderless consensus algorithms for networked Euler-Lagrange systems," *Int. J. Control*, vol. 82, no. 11, pp. 2137–2149, 2009.
[11] N. Chopra, D. M. Stipanovic, and M. W. Spong, "On synchronization and collision avoidance for mechanical systems," in *Proc. Amer. Control Conf.*, Seattle, WA, Jun. 2008, pp. 3713–3718.
[12] D. Sun, X. Zhao, and G. Feng, "A model-free cross-coupled control for position synchronization of multi-axis motions: Theory and experiments," *IEEE Trans. Control Syst. Technol.*, vol. 15, no. 2, pp. 306–314, Mar. 2007.
[13] M. W. Spong and N. Chopra, "Synchronization of networked Lagrangian systems," in *Lagrangian and Hamiltonian Methods for Nonlinear Control*, ser. Lecture Notes in Control and Information Sciences, F. Bullo and K. Fujimoto, Eds. New York: Springer-Verlag, 2007, pp. 47–59.
[14] S.-J. Chung and J.-J. E. Slotine, "Cooperative robot control and concurrent synchronization of Lagrangian systems," *IEEE Trans. Robotics*, vol. 25, no. 3, pp. 686–700, Jun. 2009.
[15] C. C. Cheah, S. P. Hou, and J. J. E. Slotine, "Region-based shape control for a warm of robots," *Automatica*, vol. 45, no. 10, pp. 2406–2411, Oct. 2009.
[16] P. F. Hokayem, D. M. Stipanovic, and M. W. Spong, "Semiautonomous control of multiple networked Lagrangian systems," *Int. J. Robust Nonlin. Control*, vol. 19, no. 18, pp. 2040–2055, 2009.
[17] J. Mei, W. Ren, and G. Ma, "Distributed coordinated tracking for multiple Euler-Lagrange systems," in *Proc. IEEE Conf. Decision Control*, Atlanta, GA, Dec. 2010, pp. 3208–3213.
[18] M. W. Spong, S. Hutchinson, and M. Vidyasagar, *Robot Modeling and Control*. New York: Wiley, 2006.
[19] E. Panteley and A. Loria, "Growth rate conditions for uniform asymptotic stability of cascaded time-varying systems," *Automatica*, vol. 37, no. 3, pp. 453–460, 2001.
[20] J.-J. E. Slotine and W. Li, *Applied Nonlinear Control*. Englewood Cliffs, NJ: Prentice-Hall, 1991.
[21] A. F. Filippov, *Differential Equations With Discontinuous Righthand Sides*. Amsterdam, The Netherlands: Kluwer, 1988.
[22] J. Cortes, "Discontinuous dynamical systems: A tutorial on solutions, nonsmooth analysis, and stability," *IEEE Control Syst. Mag.*, vol. 28, no. 3, pp. 36–73, Jun. 2008.
[23] B. Paden and S. S. Sastry, "A calculus for computing filippov's differential inclusion with application to the variable structure control of robot manipulators," *IEEE Trans. Circutis Syst.*, vol. CS-34, no. 1, pp. 73–82, Jan. 1987.

Linear Input-Output Equivalence and Row Reducedness of Discrete-Time Nonlinear Systems

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***Abstract*—The problem of linear input-output (i/o) equivalence of meromorphic nonlinear control systems, described by implicit higher order difference equations, is studied. It is proved that any system is linearly i/o equivalent to a row-reduced form. The constructive algorithm is given for finding the required transformation. The latter amounts to 1) multiply the set of i/o equations $\varphi = 0$ from left by a unimodular matrix $A(\delta)$, whose entries are non-commutative polynomials in the forward-shift operator δ, and 2) define certain multiplicative subset of the difference ring of analytic functions which introduces some inequations that should be satisfied.**

***Index Terms*—Discrete-time systems, input-output (i/o) models, linear input-output (i/o) equivalence transformations, meromorphic nonlinear control systems, polynomial approach.**

I. Introduction

One of the central themes in the system theory is the problem of representing a system in a form that is convenient for the particular purpose and of transforming one representation into another. Particularly, for linear systems, it is well-known that an arbitrary set of higher order input-output (i/o) difference equations can be always transformed into an i/o *equivalent* set of equations, having a row-reduced form [2], [20].

The main purpose of this technical note is to introduce and characterize the *linear* i/o equivalence for nonlinear control system, described by the set of meromorphic higher order i/o difference equations, and to transform the set of equations via linear i/o equivalence transformation into an equivalent set of equations in row-reduced form using the polynomial approach. Our equivalence transformation does not change the zeros (solutions) of the set of i/o equations. Note that for linear time-invariant systems, under the additional assumption that two sets have the same number of equations, linear i/o equivalence coincides with the i/o equivalence as defined in the paper [17].[1] Our interest in row-reduced form originates from the fact that this is a necessary step for realization of the i/o difference equations in the classical state space form. Note that the realization procedure in [16] as well as its extension to the MIMO case requires the system equations to be given in the explicit form, corresponding to the Popov canonical form in the linear case. This form defines explicitly the minimal set of independent system variables and allows to compute explicitly the dependent variables. Once the set of nonlinear higher order difference equations is in the row- and column-reduced form, it is extremely easy to transform these equations into the Popov form. So, the row- and column-reduced

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[1]This is not the same notion of i/o equivalence that is considered in [4]. The latter is called, according to [17], the transfer equivalence, and may change the solutions of the set of i/o equations.

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forms of the set of higher order i/o difference equations will be instrumental to all the further developments of multi-input multi-output (MIMO) realization problem.

The key for the success of polynomial systems theory in the linear case is its computational nature [2], [20]. Recently, the polynomial approach has gained popularity also in the study of nonlinear control systems, both in discrete and continuous time. The polynomial approach utilizes the algebraic properties of polynomials with coefficients from difference (or differential in the continuous-time case) field of meromorphic functions of system's variables and the strong interplay between the ring of noncommutative skew polynomials and the 'tangent linearized equations' of nonlinear higher order input-output difference (or differential) equations. Many different problems have been addressed up to now, among them controllability [21], irreducibility and system reduction [13], [15], realization [14], transfer equivalence [13] and model matching [11]. Besides that, the transfer function formalism has been recently introduced into nonlinear domain [9] and this formalism is also based on polynomial approach.

Note that the preliminary results on transformation of the set of nonlinear i/o difference equations into a row-reduced form have been reported in [12]. The results of [12] extend the analogous results for continuous-time system, see [18], [19]. The distinctive feature of our polynomial matrix approach is that it remarkably coincides with the corresponding result for linear systems. The basic difference is that, unlike in the linear case, the polynomials belong to a non-commutative twisted polynomial ring and that the polynomial matrices act on the differentials of inputs and outputs (not on inputs and outputs themselves). The algorithm is developed in [1] for transforming a polynomial matrix with entries in the ring of twisted polynomials into a row-reduced form. Row-reduced form of a matrix is achieved by multiplying it from the left by a unimodular matrix. However, this algorithm cannot be translated directly to systems since it operates over the field of fractions and not over the ring. Therefore, one has to define additionally a certain multiplicative subset of the difference ring of analytic functions which introduces the number of inequations[2] that should be satisfied. The notion of linear i/o equivalence is then defined that incorporates this multiplicative subset. As a main result we proved that any nonlinear control system, described by the set of meromorphic higher order i/o difference equations, is linearly i/o equivalent to a row-reduced system.

II. Preliminaries

Consider a discrete-time MIMO nonlinear system Σ, described by the set of implicit higher order input-output difference equations

$$\varphi_i\left(y(k),\ldots,y(k+n),u(k),\ldots,u(k+n)\right)=0 \qquad (1)$$

$i=1,\ldots,p$, where $u\in\mathbb{R}^m$ is the input variable, $y\in\mathbb{R}^p$ is the output variable and φ_i is a real meromorphic function, defined on an open and dense subset of $\mathbb{R}^{(n+1)(p+m)}$. The sequence $\{u(k),y(k);k\geq r\}$, for $r\in\mathbb{Z}$, is called a solution of (1), if for any $k\geq r$, $u(k),y(k)$ satisfy (1). We assume that the system (1) has at least one solution.[3]

A. Forward-Shift Operator δ. Non-Commutative Ring of Polynomials

Consider the infinite sequences $Y=(\ldots,y(-1),y(0),y(1),y(2),\ldots,)$ and $U=(\ldots,u(-1),u(0),u(1),u(2),\ldots)$, where $y(k)\in\mathbb{R}^p$ and $u(k)\in\mathbb{R}^m$ for $k\in\mathbb{Z}$. We think of components of Y and U as independent variables. Let $\mathcal{A}$ be the set of all analytic functions with real values depending on finitely many elements of Y and U. $\mathcal{A}$ is a ring with addition and multiplication. Let $\delta:\mathcal{A}\to\mathcal{A}$ be defined as follows: $\delta y_i(k)=y_i(k+1)$, $\delta u_i(k)=u_i(k+1)$ and for $\varphi\in\mathcal{A}$, $(\delta\varphi)(Y,U):=\varphi(\delta Y,\delta U)$, where $\delta Y=\tilde{Y}:=(\ldots,\tilde{y}(0),\tilde{y}(1),\tilde{y}(2),\ldots)$, $\tilde{y}(k):=y(k+1)$, and $\delta U=\tilde{U}:=(\ldots,\tilde{u}(0),\tilde{u}(1),\tilde{u}(2),\ldots)$, $\tilde{u}(k):=u(k+1)$. Then $\mathcal{A}$ is a difference ring with a difference operator δ. Observe that δ is injective and onto, so it is an automorphism. Moreover, $\delta^{-1}\tilde{Y}=Y$ where $y(k)=\tilde{y}(k-1)$ and $\delta^{-1}\tilde{U}=U$ where $u(k)=\tilde{u}(k-1)$.

Let $\mathcal{K}$ denote the field of fractions of the ring $\mathcal{A}$. It consists of meromorphic functions of a finite number of variables from Y and U. As $\delta:\mathcal{A}\to\mathcal{A}$ is injective, it can be extended to $\mathcal{K}$ and the pair $(\mathcal{K},\delta)$ is a difference field [3]. Over the field $\mathcal{K}$ one can define a difference vector space, $\mathcal{E}:=\operatorname{span}_{\mathcal{K}}\{\mathrm{d}\varphi|\varphi\in\mathcal{K}\}$.

The field $\mathcal{K}$ and the shift operator δ induce the ring of polynomials in a variable Z over $\mathcal{K}$, denoted by $\mathcal{K}[Z;\delta]$. A polynomial $p(Z)\in\mathcal{K}[Z;\delta]$ is written as

$$p(Z)=a_mZ^m+a_{m-1}Z^{m-1}+\ldots+a_1Z+a_0 \qquad (2)$$

where $a_i\in\mathcal{K}$ for $0\leq i\leq m$. The addition of polynomials from $\mathcal{K}[Z;\delta]$ is standard. The multiplication is defined by the linear extension of the following rules:

$$Z\cdot a:=(\delta a)Z \quad \text{and} \quad a\cdot Z:=aZ \qquad (3)$$

where $a\in\mathcal{K}$ and δa means δ evaluated at a (so for example $(pZ^n)\cdot(qZ^m)=p(\delta^nq)Z^{n+m}$). Observe that an element $a\in\mathcal{K}\subset\mathcal{K}[Z;\delta]$ does not commute with Z, so the ring $\mathcal{K}[Z;\delta]$ is non-commutative. It is called the *twisted polynomial ring* and it satisfies both the left and right Øre conditions, i. e. it is an Øre ring [7]. Moreover, $\deg[p_1(Z)\cdot p_2(Z)]=\deg p_1(Z)+\deg p_2(Z)$. Let us define now the action of the ring $\mathcal{K}[Z;\delta]$ on the field $\mathcal{K}$ and the space $\mathcal{E}$ by the linear extension of the following formulas

$$Z^s\Rsh a:=\delta^sa \quad \text{and} \quad Z^s\Rsh(a\mathrm{d}\varphi):=(\delta^sa)\mathrm{d}(\delta^s\varphi) \qquad (4)$$

where a and φ are in $\mathcal{K}$. However we usually remove $\Rsh$ from the second formula, as this does not lead to confusion. Thus, in particular, we can write $Z^s\mathrm{d}y_j(k)=\mathrm{d}y_j(k+s)$. Observe that we have a natural property $p(Z)\Rsh[q(Z)\Rsh a]=[p(Z)q(Z)]\Rsh a$ for polynomials $p(Z)$ and $q(Z)$.

Instead of polynomials over $\mathcal{K}$ we can consider twisted polynomials over the difference ring $\mathcal{A}$. All the statements made above about $\mathcal{K}[Z;\delta]$ can be repeated for $\mathcal{A}[Z;\delta]$.

B. Polynomial Matrix Description of the Nonlinear System

We now represent the nonlinear system (1) in terms of two polynomial matrices, with the polynomials in $\mathcal{K}[Z;\delta]$. For that we apply the differential operation d to (1) to obtain

$$\sum_{j=1}^{p}\sum_{s=0}^{n}\frac{\partial\varphi_i}{\partial y_j(k+s)}\mathrm{d}y_j(k+s) = -\sum_{j=1}^{m}\sum_{s=0}^{n}\frac{\partial\varphi_i}{\partial u_j(k+s)}\mathrm{d}u_j(k+s), \quad i=1,\ldots,p. \qquad (5)$$

Since $\mathrm{d}y_j(k+s)=Z^s\mathrm{d}y_j(k)$, $\mathrm{d}u_j(k+s)=Z^s\mathrm{d}u_j(k)$, we can rewrite (5) as

$$P(Z)\mathrm{d}y(k)=Q(Z)\mathrm{d}u(k) \qquad (6)$$

where $P(Z)$ and $Q(Z)$ are $p\times p$ and $p\times m$-dimensional matrices, respectively, whose elements $p_{ij},q_{ij}\in\mathcal{K}[Z;\delta]$

$$p_{ij}(Z)=\sum_{s=0}^{n}\frac{\partial\varphi_i}{\partial y_j(k+s)}Z^s,\quad q_{ij}(Z)=-\sum_{s=0}^{n}\frac{\partial\varphi_i}{\partial u_j(k+s)}Z^s$$

[2]A word "inequations" should not be confused by a word "inequalities". An inequation is a statement that two expressions are not equal.

[3]For polynomial systems this property may be checked by the method in [8].

and $\mathrm{d}y(k) = [\mathrm{d}y_1(k), \ldots, \mathrm{d}y_p(k)]^T$, $\mathrm{d}u(k) = [\mathrm{d}u_1(k), \ldots, \mathrm{d}u_m(k)]^T$. We write $\mathcal{K}^{p\times q}[Z;\delta]$ for the set of $p \times q$ matrices with entries in $\mathcal{K}[Z;\delta]$.

C. System of Implicit Input-Output Equations

Let S be a multiplicative subset of the ring $\mathcal{A}$. This means that $1 \in S$ and if a and b belong to S, so does ab. We shall assume that S is invariant with respect to both δ and δ^{-1}. Then $S^{-1}\mathcal{A}$ denotes the localization of the ring $\mathcal{A}$ with respect to S. It consists of elements of $\mathcal{K}$ whose denominators belong to S. Observe that $S^{-1}\mathcal{A}$ is an inversive difference ring with the difference operator δ and, via the natural injection $\alpha \mapsto (\alpha/1)$, S may be interpreted as a subset of $S^{-1}\mathcal{A}$.

Let us recall that an *ideal* I of a commutative ring A is a specific subring of A with the property that if $a \in A$ and $i \in I$, then $ai \in I$. If A is a difference ring, then an ideal I of A is a *difference ideal* of A if it is closed with respect to the difference operator.

Let $\Phi = \{\varphi_1, \ldots, \varphi_p\}$ be a finite subset of $S^{-1}\mathcal{A}$. Φ may be interpreted as a system of implicit input-output equations. Let $\langle\langle\Phi\rangle\rangle_S$ denote the smallest ideal of $S^{-1}\mathcal{A}$ that contains all the shifts $\delta^k(\varphi_i)$ for $i = 1, \ldots, p$ and $k \in \mathbb{Z}$, i.e. the forward and backward shifts of φ_i. Observe that $\langle\langle\Phi\rangle\rangle_S$ is a difference ideal of $S^{-1}\mathcal{A}$ and

$$\delta\left(\langle\langle\Phi\rangle\rangle_S\right) = \langle\langle\Phi\rangle\rangle_S = \delta^{-1}\left(\langle\langle\Phi\rangle\rangle_S\right). \tag{7}$$

Observe that Φ may be considered as a subset of $\tilde{S}^{-1}\mathcal{A}$ for some other multiplicative set $\tilde{S}$. For that reason we put S in the notation of the ideal $\langle\langle\Phi\rangle\rangle_S$.

We make the following

Assumption: The ideal $\langle\langle\Phi\rangle\rangle_S$ is prime, i.e. if $\alpha, \beta \in S^{-1}\mathcal{A}$ and $\alpha\beta \in \langle\langle\Phi\rangle\rangle_S$, then $\alpha \in \langle\langle\Phi\rangle\rangle_S$ or $\beta \in \langle\langle\Phi\rangle\rangle_S$, and is proper, i.e. different from the entire ring.[4]

Properness of the ideal $\langle\langle\Phi\rangle\rangle_S$ is equivalent to the condition

$$S \cap \langle\langle\Phi\rangle\rangle_S = \emptyset. \tag{8}$$

In particular, numerators of φ_i's do not belong to S.

Let $S^{-1}\mathcal{A}/\langle\langle\Phi\rangle\rangle_S$ be the quotient ring. It consists of cosets $\bar{\varphi} = \varphi + \langle\langle\Phi\rangle\rangle_S$ for $\varphi \in S^{-1}\mathcal{A}$. We define "+" and "$\cdot$" in this new ring by $\bar{\varphi} + \bar{\psi} := \overline{\varphi + \psi}$ and $\bar{\varphi} \cdot \bar{\psi} := \overline{\varphi \cdot \psi}$. These definitions do not depend on the choice of a representative in a coset. In particular $\bar{\varphi}_i = 0$, for $i = 1, \ldots, p$. Since, by Assumption, $\langle\langle\Phi\rangle\rangle_S$ is a prime ideal, $S^{-1}\mathcal{A}/\langle\langle\Phi\rangle\rangle_S$ is an integral ring. Now we can redefine δ on $S^{-1}\mathcal{A}/\langle\langle\Phi\rangle\rangle_S$ (denoted now by δ_Φ to indicate its dependence on Φ) as follows: $\delta_\Phi\bar{\varphi} = \overline{\delta\varphi}$. This again is well defined, for if $\bar{\varphi} = \bar{\psi}$, then $\varphi + \langle\langle\Phi\rangle\rangle_S = \psi + \langle\langle\Phi\rangle\rangle_S$. Since $\delta(\langle\langle\Phi\rangle\rangle_S) \subset \langle\langle\Phi\rangle\rangle_S$ and $\delta(\langle\langle\Phi\rangle\rangle_S) + \langle\langle\Phi\rangle\rangle_S = \langle\langle\Phi\rangle\rangle_S$, then $\delta\varphi + \delta(\langle\langle\Phi\rangle\rangle_S) = \delta\psi + \delta(\langle\langle\Phi\rangle\rangle_S)$. Moreover, the operator δ_Φ is bijective, so δ_Φ^{-1} is well defined on $S^{-1}\mathcal{A}/\langle\langle\Phi\rangle\rangle_S$. Let $\mathcal{Q}_S^\Phi$ denote the field of fractions of the ring $S^{-1}\mathcal{A}/\langle\langle\Phi\rangle\rangle_S$. As δ_Φ can be naturally extended to the field of fractions, $\mathcal{Q}_S^\Phi$ is now an inversive difference field with the difference operator δ_Φ.

Proposition 1: Assume that S and $\tilde{S}$ are multiplicative subsets of $\mathcal{A}$ and $S \subset \tilde{S}$, invariant with respect to δ and δ^{-1}. Let $\Phi \subset S^{-1}\mathcal{A}$ and the ideal $\langle\langle\Phi\rangle\rangle_S$ be prime and proper. Let $\tilde{S} \cap \langle\langle\Phi\rangle\rangle_{\tilde{S}} = \emptyset$. Then:

a) $S^{-1}\mathcal{A} \subset \tilde{S}^{-1}\mathcal{A} \subset \mathcal{K}$;
b) the ideal $\langle\langle\Phi\rangle\rangle_{\tilde{S}}$ of $\tilde{S}^{-1}\mathcal{A}$ is prime and proper;
c) there is a natural monomorphism of difference rings

$$\tau : S^{-1}\mathcal{A}/\langle\langle\Phi\rangle\rangle_S \to \tilde{S}^{-1}\mathcal{A}/\langle\langle\Phi\rangle\rangle_{\tilde{S}}$$

d) τ may be extended to a monomorphism of difference fields $\mathcal{Q}_S^\Phi \to \mathcal{Q}_{\tilde{S}}^\Phi$.

Proof:

a) is obvious.
b) Since $\tilde{S} \cap \langle\langle\Phi\rangle\rangle_{\tilde{S}} = \emptyset$, $\langle\langle\Phi\rangle\rangle_{\tilde{S}}$ is proper. To show that it is also prime, observe that $\langle\langle\Phi\rangle\rangle_S = S^{-1}I$, where I is a prime ideal of $\mathcal{A}$ and $I \cap S = \emptyset$. Then $\langle\langle\Phi\rangle\rangle_{\tilde{S}} = \tilde{S}^{-1}I$ is prime as well.
c) Define $\tau((\alpha/s) + \langle\langle\Phi\rangle\rangle_S) := (\alpha/s) + \langle\langle\Phi\rangle\rangle_{\tilde{S}}$, where α/s is treated as an element of $\tilde{S}^{-1}\mathcal{A}$. Then τ is a homomorphism of rings. If $\tau((\alpha/s) + \langle\langle\Phi\rangle\rangle_S) = 0$, then $\alpha/s \in \langle\langle\Phi\rangle\rangle_{\tilde{S}}$. This implies that $\alpha/s \in \langle\langle\Phi\rangle\rangle_S$, so τ is a monomorphism.
d) Define $\tau(a/b) := \tau(a)/\tau(b)$ for a/b from the field $\mathcal{Q}_S^\Phi$. ■

We shall later need the rings of twisted polynomials with coefficients in the field $\mathcal{Q}_S^\Phi$. They will be denoted by $\mathcal{Q}_S^\Phi[Z, \delta_\Phi]$.

A system may be described by different sets of equations. This leads to the problem of equivalence.

Definition 1: Consider two discrete-time control systems $\varphi_i = 0$, $i = 1, \ldots, p$, and $\tilde{\varphi}_j = 0$, $j = 1, \ldots, \tilde{p}$, of the form (1). The systems are called *linearly input-output equivalent* if there exists a multiplicative subset S of the ring $\mathcal{A}$, such that φ_i and $\tilde{\varphi}_j$ belong to $S^{-1}\mathcal{A}$ for $i = 1, \ldots, p$ and $j = 1, \ldots, \tilde{p}$ and Assumption is satisfied for both systems, and there exist two matrices $A(Z)$ and $B(Z)$ with coefficients in $S^{-1}\mathcal{A}[Z;\delta]$ such that $\tilde{\varphi} = A(Z) \Rsh \varphi$ and $\varphi = B(Z) \Rsh \tilde{\varphi}$, where $\varphi = (\varphi_1, \ldots, \varphi_p)^T$ and $\tilde{\varphi} = (\tilde{\varphi}_1, \ldots, \tilde{\varphi}_{\tilde{p}})^T$ are treated as column vectors with elements in $S^{-1}\mathcal{A}$.

Note that linear input-output equivalence is an equivalence relation in the set of all systems of the form (1). For linear time-invariant systems linear input-output equivalence coincides with the standard input-output equivalence concept. We will explain a difference between these two notions in Section VI.

Proposition 2: The systems $\varphi_i = 0$, $i = 1, \ldots, p$, and $\tilde{\varphi}_j = 0$, $j = 1, \ldots, \tilde{p}$, are linearly input-output equivalent if and only if there exists a multiplicative set S such that the systems satisfy Assumption and difference ideals of $S^{-1}\mathcal{A}$, $\langle\langle\Phi\rangle\rangle_S$ and $\langle\langle\tilde{\Phi}\rangle\rangle_S$, generated by $\Phi = \{\varphi_i : i = 1, \ldots, p\}$ and $\tilde{\Phi} = \{\tilde{\varphi}_j : j = 1, \ldots, \tilde{p}\}$, are equal.

Proof: Since $\tilde{\varphi} = A(Z) \Rsh \varphi$ and $\varphi = B(Z) \Rsh \tilde{\varphi}$, the generators of the first ideal can be expressed by the elements of the second ideal and vice versa. Thus the ideals are equal. Similar reasoning shows the reverse implication. ■

III. Polynomial Matrices With Elements in $\mathcal{Q}_S^\Phi[Z, \delta_\Phi]$

The purpose of this section is to show that as in the linear case, where the polynomials have real coefficients, the polynomial matrix in $\mathcal{Q}_S^{\Phi^{p\times q}}[Z;\delta_\Phi]$ can be transformed by a sequence of elementary row operations into the row-reduced form. This result allows us later to transform the set of i/o difference equations into an equivalent system in the row-reduced form.

Definition 2: By an *elementary row operation* we mean any of the following row operations on polynomial matrices from $\mathcal{Q}_S^{\Phi^{p\times q}}[Z;\delta_\Phi]$:

1) Interchange of rows i and j.
2) Multiplication of row i by nonzero scalar in $\mathcal{Q}_S^\Phi$.
3) Replacement of row i by itself plus any polynomial multiplied by any other row j.

Observe that all elementary row operations are invertible and any elementary row operation may be interpreted as premultiplication (left multiplication) by an invertible matrix $E(Z)$ from $\mathcal{Q}_S^{\Phi^{p\times p}}[Z;\delta]$.

Definition 3: A matrix $U(Z)$ in a ring of polynomial matrices $\mathcal{Q}_S^{\Phi^{p\times p}}[Z;\delta_\Phi]$ is called *unimodular* if it has an inverse matrix $U(Z)^{-1}$ in $\mathcal{Q}_S^{\Phi^{p\times p}}[Z;\delta]$ such that $U(Z)U(Z)^{-1} = U(Z)^{-1}U(Z) = I$.

A unimodular matrix admits a product decomposition in terms of elementary matrices $E_i(Z) : U(Z) = E_N(Z) \ldots E_2(Z)E_1(Z)$. The inverse matrix of the unimodular matrix is also a unimodular matrix as $U(Z)^{-1} = E_1(Z)^{-1}E_2(Z)^{-1} \ldots E_N(Z)^{-1}$.

[4]Note that the assumption is an analogue of the submersivity assumption for the explicitly defined systems. In [10] it has been proved that the explicitly defined discrete-time nonlinear control system is submersive iff the associated ideal is prime, proper and reflexive. Note that $\langle\langle\Phi\rangle\rangle_S$ is reflexive by (7).

For a nonzero polynomial row $P(Z) \in \mathcal{Q}_S^{\Phi^{1\times q}}[Z;\delta]$, we define its degree as the exponent of the highest power in Z present in $P(Z)$. We denote this degree by $\deg P(Z)$ and it follows that $\deg P(Z) \geq 0$. If $P(Z) \equiv 0$ we define $\deg P(Z) = -\infty$. Let $P(Z)$ be a polynomial matrix in $\mathcal{Q}_S^{\Phi^{p\times q}}[Z;\delta]$ with nonzero rows $P_1(Z),\ldots,P_p(Z) \in \mathcal{Q}_S^{\Phi^{1\times q}}[Z;\delta]$. We denote $\deg P_i(Z) = \sigma_i$ for $i = 1,\ldots,p$ and $\deg P(Z) = \max\{\sigma_1,\ldots,\sigma_p\}$. Then we write $P_i(Z) = P_{i0}Z^{\sigma_i} + P_{i1}Z^{\sigma_i - 1} + \ldots + P_{i\sigma_i}$ with P_{ij} being a row vector of functions in $\mathcal{Q}_S^{\Phi}$ for $j = 0,\ldots,\sigma_i$. Let us introduce the vector of row degrees $\sigma = [\sigma_1,\ldots,\sigma_p]$. Let $N = \deg P(Z)$, $e = (1,\ldots,1)$ and $M = (m_1,\ldots,m_p) = N\cdot e - \sigma$. By Z^M we will denote the diagonal $p\times p$ matrix with the diagonal elements $Z^{m_1},\ldots,Z^{m_p}$.

Definition 4: The matrix $L = L(P(Z))$ such that

$$Z^M P(Z) = LZ^N + \text{lower degree terms} \tag{9}$$

is called the *leading row coefficient matrix*[5] of $P(Z)$.

Definition 5: Let $P(Z)$ be a polynomial matrix in $\mathcal{Q}_S^{\Phi^{p\times q}}[Z;\delta_\Phi]$ with nonzero rows. Then $P(Z)$ is called *row-reduced* if and only if its leading row coefficient matrix $L(P(Z))$ has full row rank over $\mathcal{Q}_S^{\Phi}$. If $P(Z)$ contains zero rows, then $P(Z)$ is called *row-reduced* if and only if its submatrix consisting of nonzero rows is row-reduced.

Theorem 1: For any $p\times q$ polynomial matrix $P(Z)$ there exists a unimodular matrix $U(Z)$ over $\mathcal{Q}_S^{\Phi}[Z;\delta_\Phi]$ such that for $T(Z) = U(Z)P(Z)$ the following holds:

i) $\deg(T(Z)) \leq \deg(P(Z))$;

ii) $T(Z)$ is row-reduced.

The proof of the theorem follows from Theorem 2.2 in [1], where we specify the ring automorphism σ to be the forward shift operator δ and the σ-derivation to be identically zero. The proof yields the following algorithm of constructing $U(Z)$ that transforms $P(Z)$ into the required form:

Step 1) Set $k := 0$ and $\tilde{P}(Z) := P(Z)$.

Step 2) Compute the row degrees $\sigma_1,\ldots,\sigma_p$ of the matrix $\tilde{P}(Z)$. Reorder the nonzero rows of $\tilde{P}(Z)$ with respect to the row degrees starting from the lowest and put all zero rows at the end. This operation can be expressed in the form of left multiplication of $\tilde{P}(Z)$ by a certain permutation matrix R_k, $\tilde{P}(Z) := R_k\tilde{P}(Z)$. Let $N = \max\{\sigma_1,\ldots,\sigma_p\}$. Compute the *leading coefficient matrix* L of the submatrix $\tilde{P}(Z)$ that consists of all nonzero rows. If the rows of the matrix L are linearly independent over $\mathcal{Q}_s^{\Phi}$, then the matrix $\tilde{P}(Z)$ is in the required form. Define $E_k(Z) = I$, and go to STEP 3.
If the rows of the matrix L are linearly dependent, then there exists an integer i such that the first i rows of L, that is, $L_1,\ldots,L_i$, are independent, and the $(i+1)$st row L_{i+1} is a linear combination of the first i rows; that is there exist $\alpha_1,\ldots,\alpha_i \in \mathcal{Q}_s^{\Phi}$ such that

$$\alpha_1 L_1 + \alpha_2 L_2 + \ldots + \alpha_i L_i + L_{i+1} = 0. \tag{10}$$

Set $\gamma := N - \sigma_{i+1}$, $\nu_j := \sigma_{j+1} - \sigma_j$, $j = 1,\ldots,i$, and replace the matrix $\tilde{P}(Z)$ by $E_k(Z)\tilde{P}(Z)$, where $E_k(Z)$ equals to

$$\begin{pmatrix} 1 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ Z^{-\gamma}(\alpha_1)Z^{\nu_1} & \ldots & Z^{-\gamma}(\alpha_i)Z^{\nu_i} & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & 0 & \ldots & 1 \end{pmatrix} \tag{11}$$

i.e., E_k is obtained from the identity $p\times p$ matrix by changing its $(i+1)$st row. Then the degree of $(i+1)$st row of $E_k(Z)\tilde{P}(Z)$ (new $\tilde{P}(Z)$) is reduced. Repeat STEP 2 with $k := k+1$.

Step 3) Denote $U(Z) = E_s(Z)\cdot R_s\cdot\ldots\cdot E_0(Z)R_0$ where s is the last k value. It is the required unimodular matrix and $T(Z) = U(Z)P(Z)$.

[5] Construction of the leading row coefficient matrix for discrete-time nonlinear systems is different from that for continuous-time nonlinear systems (see [18]) or for linear systems, as $P(Z)$ (or system equations) has (have) to be multiplied by a certain diagonal matrix Z^M first.

IV. Row-Reducedness of the Input-Output System

In this section we develop a method for transforming the set of nonlinear input-output difference equations into a row-reduced form. Consider the set of i/o difference equations (1). Assume that there is a multiplicative set S such that for all $i = 1,\ldots,p$, $\varphi_i \in S^{-1}\mathcal{A}$ and Assumption is satisfied. Then the matrices $P(Z)$ and $Q(Z)$ in the associated polynomial description (6) have elements that are polynomials in Z with coefficients in $S^{-1}\mathcal{A}$. Let e_S^{Φ} denote the map: $S^{-1}\mathcal{A} \to \mathcal{Q}_S^{\Phi}$: $\varphi \mapsto (\varphi + \langle\langle\Phi\rangle\rangle_S)/1$ (later we will usually skip the denominator). If $p(Z) = \sum_i p_i Z^i$ with $p_i \in S^{-1}\mathcal{A}$, then we define $e_S^{\Phi}(p(Z)) = \sum_i e_S^{\Phi}(p_i)Z^i$. This is a polynomial in $\mathcal{Q}_S^{\Phi}[Z;\delta_\Phi]$. For a matrix $P(Z)$ with elements in $S^{-1}\mathcal{A}[Z;\delta]$, by $e_S^{\Phi}(P(Z))$ we mean the matrix defined by $e_S^{\Phi}(P(Z))_{ij} = e_S^{\Phi}(P(Z)_{ij})$. We will also often use notation $e_S^{\Phi}(p(Z)) = p(Z) + \langle\langle\Phi\rangle\rangle_S$ for polynomials and $e_S^{\Phi}(P(Z)) = P(Z) + \langle\langle\Phi\rangle\rangle_S$ for polynomial matrices.

Definition 6: The set of i/o difference equations (1) is said to be *row-reduced* if there is a multiplicative subset S of $\mathcal{A}$ such that the matrix $\bar{P}(Z) = e_S^{\Phi}(P(Z))$ is row-reduced over $\mathcal{Q}_S^{\Phi}[Z;\delta_\Phi]$.

Lemma 1: Let T be a multiplicative subset of $S^{-1}\mathcal{A}$, invariant with respect to δ and δ^{-1}, such that $T\cap\langle\langle\Phi\rangle\rangle_S = \emptyset$. Then $T/\langle\langle\Phi\rangle\rangle_S$ is a multiplicative subset of the ring $S^{-1}\mathcal{A}/\langle\langle\Phi\rangle\rangle_S$, $T^{-1}\langle\langle\Phi\rangle\rangle_S$ is an ideal of the difference ring $T^{-1}(S^{-1}\mathcal{A})$, and the map

$$\left(T/\langle\langle\Phi\rangle\rangle_S\right)^{-1}\left(S^{-1}\mathcal{A}/\langle\langle\Phi\rangle\rangle_S\right) \to T^{-1}(S^{-1}\mathcal{A})/T^{-1}\langle\langle\Phi\rangle\rangle_S$$

given by

$$\frac{\gamma + \langle\langle\Phi\rangle\rangle_S}{t + \langle\langle\Phi\rangle\rangle_S} \mapsto \frac{\gamma}{t} + T^{-1}\langle\langle\Phi\rangle\rangle_S$$

is an isomorphism of difference rings.

Proof: Let $t_1, t_2 \in T$. Then $(t_1 + \langle\langle\Phi\rangle\rangle_S)(t_2 + \langle\langle\Phi\rangle\rangle_S) = t_1t_2 + \langle\langle\Phi\rangle\rangle_S \in T/\langle\langle\Phi\rangle\rangle_S$. If $(a_1/s_1), (a_2/s_2) \in T^{-1}\langle\langle\Phi\rangle\rangle_S$, then $(a_1/s_1) + (a_2/s_2) = (a_1s_2 + a_2s_1)/s_1s_2 \in T^{-1}\langle\langle\Phi\rangle\rangle_S$. If $(b/s_3) \in T^{-1}(S^{-1}\mathcal{A})$, then $(a_1/s_1)(b/s_3) = a_1b/s_1s_3 \in T^{-1}\langle\langle\Phi\rangle\rangle_S$. Thus $T^{-1}\langle\langle\Phi\rangle\rangle_S$ is an ideal of $T^{-1}(S^{-1}\mathcal{A})$. Let $\iota(\gamma + \langle\langle\Phi\rangle\rangle_S/t + \langle\langle\Phi\rangle\rangle_S) = (\gamma/t) + T^{-1}\langle\langle\Phi\rangle\rangle_S$ for $\gamma + \langle\langle\Phi\rangle\rangle_S/t + \langle\langle\Phi\rangle\rangle_S \in (T/\langle\langle\Phi\rangle\rangle_S)^{-1}(S^{-1}\mathcal{A}/\langle\langle\Phi\rangle\rangle_S)$. It is easy to see that ι is a homomorphism of difference rings. If $\iota(\gamma + \langle\langle\Phi\rangle\rangle_S/t + \langle\langle\Phi\rangle\rangle_S) = 0$, then $(\gamma/t) \in T^{-1}\langle\langle\Phi\rangle\rangle_S$, so $\gamma \in \langle\langle\Phi\rangle\rangle_S$. This means that ι is injective. It is also surjective as any element of $T^{-1}(S^{-1}\mathcal{A})/T^{-1}\langle\langle\Phi\rangle\rangle_S$ has the form $(\gamma/t) + T^{-1}\langle\langle\Phi\rangle\rangle_S$ for some $\gamma \in S^{-1}\mathcal{A}$ and some $t \in T$. Thus it belongs to the image of ι. ■

Theorem 2: Any nonlinear system of form (1) is linearly input-output equivalent to a row-reduced system.

Proof: Let $\bar{P}(Z) := e_S^{\Phi}(P(Z)) = P(Z) + \langle\langle\Phi\rangle\rangle_S$ and let $U(Z)$ be a unimodular matrix over $\mathcal{Q}_S^{\Phi}[Z;\delta_\Phi]$ such that $\breve{P}(Z) = U(Z)\bar{P}(Z)$ is row-reduced. Then $U(Z) = \sum_k U_kZ^k$, where U_k is a matrix over $\mathcal{Q}_S^{\Phi}$, $U_k = (U_{kij})$ and

$$U_{kij} = \frac{\frac{\alpha_{kij}}{s_{kij}} + \langle\langle\Phi\rangle\rangle_S}{t_{kij} + \langle\langle\Phi\rangle\rangle_S}$$

with $\alpha_{kij} \in \mathcal{A}$, $s_{kij} \in S$ and $t_{kij} \in S^{-1}\mathcal{A}$, $t_{kij} \notin \langle\langle\Phi\rangle\rangle_S$. Similarly, $V(Z) = U(Z)^{-1}$ can be expressed as $V(Z) = \sum_k V_kZ^k$, where $V_k = (V_{kij})$ and

$$V_{kij} = \frac{\frac{\beta_{kij}}{\tilde{s}_{kij}} + \langle\langle\Phi\rangle\rangle_S}{\tilde{t}_{kij} + \langle\langle\Phi\rangle\rangle_S}$$

with $\beta_{kij} \in \mathcal{A}$, $\tilde{s}_{kij} \in S$ and $\tilde{t}_{kij} \in S^{-1}\mathcal{A}$, $\tilde{t}_{kij} \notin \langle\langle\Phi\rangle\rangle_S$. Let T be the multiplicative set generated by all t_{kij}, $\tilde{t}_{kij}$ and all their forward and backward shifts. Observe that $T \cap \langle\langle\Phi\rangle\rangle_S = \emptyset$. Then, by Lemma 1, U_{kij} may be identified with $(\alpha_{kij}/s_{kij}t_{kij}) + T^{-1}\langle\langle\Phi\rangle\rangle_S$ and V_{kij} with $(\beta_{kij}/\tilde{s}_{kij}\tilde{t}_{kij}) + T^{-1}\langle\langle\Phi\rangle\rangle_S$, which are elements of $T^{-1}(S^{-1}\mathcal{A})/T^{-1}\langle\langle\Phi\rangle\rangle_S$. Let $A(Z) = \sum_k A_k Z^k$, where $A_k = (A_{kij})$ with elements $A_{kij} = \alpha_{kij}/s_{kij}t_{kij}$ belonging to $T^{-1}(S^{-1}\mathcal{A})$. Similarly, let $B(Z) = \sum_k B_k Z^k$, where $B_k = (B_{kij})$ with elements $B_{kij} = \beta_{kij}/\tilde{s}_{kij}\tilde{t}_{kij}$. Then $U(Z) = A(Z) + T^{-1}\langle\langle\Phi\rangle\rangle_S$ and $V(Z) = B(Z) + T^{-1}\langle\langle\Phi\rangle\rangle_S$. This means that $B(Z)A(Z) + T^{-1}\langle\langle\Phi\rangle\rangle_S = I$. In other words, $B(Z)A(Z) = C$ is a matrix over the ring $T^{-1}(S^{-1}\mathcal{A})$ whose determinant does not belong to $T^{-1}\langle\langle\Phi\rangle\rangle_S$. Let us notice that the ring $T^{-1}(S^{-1}\mathcal{A})$ may be written in the form $\tilde{S}^{-1}\mathcal{A}$ for some multiplicative set $\tilde{S}$ containing S. Thus, in particular, $S^{-1}\mathcal{A} \subset \tilde{S}^{-1}\mathcal{A} \subset \mathcal{K}$ and $T^{-1}\langle\langle\Phi\rangle\rangle_S = \langle\langle\Phi\rangle\rangle_{\tilde{S}}$. If $\det C \in \tilde{S}$, then C is invertible over $\tilde{S}^{-1}\mathcal{A}$. Otherwise, we can extend $\tilde{S}$ so that this is satisfied and the condition $\tilde{S} \cap \langle\langle\Phi\rangle\rangle_{\tilde{S}}$ still holds.

Let $\tilde{\varphi} := A(Z) \upharpoonright \varphi$. We treat now the components of φ and $\tilde{\varphi}$ as elements of $\tilde{S}^{-1}\mathcal{A}$. First observe that $B(Z) \upharpoonright \tilde{\varphi} = [B(Z)A(Z)] \upharpoonright \varphi = C \upharpoonright \varphi$ and $\varphi = [C^{-1}B(Z)] \upharpoonright \tilde{\varphi}$. Moreover, $C^{-1}B(Z)$ is a matrix over $\tilde{S}^{-1}\mathcal{A}[Z,\delta]$. This means that the systems given by Φ and $\tilde{\Phi}$ are linearly input-output equivalent.

We shall show now that the system $\tilde{\varphi} = 0$ is row-reduced. Let $A(Z) = \sum_i A_i Z^i$. Then

$$\begin{aligned}
\mathrm{d}\tilde{\varphi} &= \mathrm{d}\left(\sum_i A_i(\delta^i\varphi)\right) = \sum_i \mathrm{d}A_i(\delta^i\varphi) + \sum_i A_i \mathrm{d}(\delta^i\varphi)\\
&= \sum_i (T_i(Z)\mathrm{d}y - S_i(Z)\mathrm{d}u)(\delta^i\varphi)\\
&\quad + \sum_i A_i Z^i (P(Z)\mathrm{d}y - Q(Z)\mathrm{d}u)\\
&= \widetilde{P}(Z)\mathrm{d}y - \widetilde{Q}(Z)\mathrm{d}u,
\end{aligned}$$

where $\tilde{P}(Z) = \sum_i \sum_j T_{ij}(\delta^i\varphi)Z^j + A(Z)P(Z)$ is a matrix over $\tilde{S}^{-1}\mathcal{A}[Z;\delta]$ and $T_i(Z) = \sum_j T_{ij}Z^j$. We have to show that $e_{\tilde{S}}^{\tilde{\Phi}}(\tilde{P}(Z))$ is row-reduced.

Since Φ and $\tilde{\Phi}$ are linearly input-output equivalent, from Proposition 2 it follows that the ideals $\langle\langle\Phi\rangle\rangle_{\tilde{S}}$ and $\langle\langle\tilde{\Phi}\rangle\rangle_{\tilde{S}}$ are equal, so $Q_{\tilde{S}}^{\Phi} = Q_{\tilde{S}}^{\tilde{\Phi}}$ and $e_{\tilde{S}}^{\tilde{\Phi}} = e_{\tilde{S}}^{\Phi}$.

Thus $\check{P}(Z) := e_{\tilde{S}}^{\tilde{\Phi}}(\tilde{P}(Z)) = e_{\tilde{S}}^{\Phi}(\tilde{P}(Z)) =(A(Z) + \langle\langle\Phi\rangle\rangle_{\tilde{S}})(P(Z) + \langle\langle\Phi\rangle\rangle_{\tilde{S}})$ is a matrix of polynomials in Z with coefficients in $\tilde{S}^{-1}\mathcal{A}/\langle\langle\Phi\rangle\rangle_{\tilde{S}}$, which is contained in its field of quotients $Q_{\tilde{S}}^{\Phi}$. On the other hand, $A(Z) + \langle\langle\Phi\rangle\rangle_{\tilde{S}} = U(Z)$ is a matrix with elements that are polynomials with coefficients in $Q_{\tilde{S}}^{\Phi}$. Since coefficients of $P(Z) + \langle\langle\Phi\rangle\rangle_S$ belong to $S^{-1}\mathcal{A}/\langle\langle\Phi\rangle\rangle_S$, they are also members of the field of fractions Q_S^{Φ}. Thus coefficients of $\check{P}(Z) = U(Z)(P(Z) + \langle\langle\Phi\rangle\rangle_S)$ belong Q_S^{Φ}. They are mapped onto respective coefficients of $\check{P}(Z)$ under the monomorphism of fields $\tau : Q_S^{\Phi} \to Q_{\tilde{S}}^{\Phi} : (\alpha + \langle\langle\Phi\rangle\rangle_S)/(\beta + \langle\langle\Phi\rangle\rangle_S) \mapsto (\alpha + \langle\langle\Phi\rangle\rangle_{\tilde{S}})/(\beta + \langle\langle\Phi\rangle\rangle_{\tilde{S}})$. Since $\check{P}(Z)$ is row-reduced over Q_S^{Φ}, $\check{P}(Z)$ is row-reduced over $Q_{\tilde{S}}^{\Phi}$. ■

Remark 1: In some sense the linear i/o equivalence is more restrictive (stronger) concept than the i/o equivalence since we only use the linear transformations whereas in the i/o equivalence also certain nonlinear transformations are allowed. Note that nonlinear transformations are unnecessary in order to transform the system equations into the row-reduced form. From the other side, while transforming the system equations, we may introduce additional restrictions, requiring that certain expressions are not zero. The latter means increasing the multiplicative set S. Emergence of inequations is not uncommon in nonlinear control. For example, they may emerge in passing from state space description to i/o description [6]. The latter includes, in general, in addition to system equations also a number of inequations. Since in the linear case it may never happen that additional restrictions show up and obviously only linear transformations are allowed to transform the system equations, the new equivalence concept introduced by us, the linear i/o equivalence, coincides in the linear case with the concept of i/o equivalence.

Example: Consider the system

$$\begin{aligned}
\varphi_1 &= y_1(k+2) + \frac{y_2(k+2)y_3(k+1)}{y_1(k)}\\
&\quad - \frac{y_2(k+2)u_3(k)}{y_1(k)} - u_1(k) = 0\\
\varphi_2 &= y_1(k+3)u_1(k) + y_2(k+1)\\
&\quad - u_1(k+1)u_1(k) - u_2(k) = 0\\
\varphi_3 &= y_3(k+1) - u_3(k) = 0.
\end{aligned} \tag{12}$$

Since φ_1 contains a denominator $y_1(k)$, we set $S := \{y_1(k); k \in \mathbb{Z}\}$. Then $S^{-1}\mathcal{A}$ is a localization of the ring $\mathcal{A}$ with respect to the multiplicative subset S. Note that the required condition $S \cap \langle\langle\Phi\rangle\rangle_S = \emptyset$ holds. Compute for system (12) the matrix $P(Z)$ in the associated polynomial description (6) with the elements being polynomials in Z with coefficients in $S^{-1}\mathcal{A}$

$$\begin{bmatrix} Z^2 - \frac{y_2(k+2)D}{y_1^2(k)} & \frac{D}{y_1(k)}Z^2 & \frac{y_2(k+2)}{y_1(k)}Z\\ u_1(k)Z^3 & Z & 0\\ 0 & 0 & Z \end{bmatrix}$$

where $D := y_3(k+1) - u_3(k)$. The representative of $\bar{P}(Z) := e_S^{\Phi}(P(Z))$ is

$$\begin{bmatrix} Z^2 & 0 & \frac{y_2(k+2)}{y_1(k)}Z\\ u_1(k)Z^3 & Z & 0\\ 0 & 0 & Z \end{bmatrix}. \tag{13}$$

It is easy to check that the latter matrix is not in the row-reduced form. For that, compute the row degrees $\sigma = (2,3,1)$ and reorder the rows of $\bar{P}(Z)$ with respect to the row degrees starting from the lowest. Now, $N = \max\{1,2,3\} = 3$ and $M = (m_1, m_2, m_3) = N \cdot (1,1,1) - \sigma = (2,1,0)$. Next, in order to find the leading coefficient matrix L, we have to multiply (13) first by a diagonal matrix $\mathrm{diag}\{Z^{m_1}, Z^{m_2}, Z^{m_3}\} = \mathrm{diag}\{Z^2, Z, 1\}$. So

$$L = \begin{bmatrix} 0 & 0 & 1\\ 1 & 0 & 0\\ u_1(k) & 0 & 0 \end{bmatrix}.$$

Obviously, L is not of full rank meaning that the (12) are not yet in the row-reduced form. In order to transform the matrix (13) into the row reduced form we have, according to the algorithm in Section III, multiply it by the unimodular matrix

$$U(Z) = \begin{bmatrix} 0 & 0 & 1\\ -u_1(k)Z & 1 & \frac{u_1(k)y_2(k+3)}{y_1(k+1)}Z\\ 1 & 0 & 0 \end{bmatrix}$$

being a product of intermediate unimodular matrices, constructed at the consecutive steps of the algorithm. Since in $U(Z)$ a denominator $y_1(k+1)$ appears, that is already in our S and since $U^{-1}(Z)$ does not introduce additional denominators, there is no need to extend the multiplicative set S. Then, by the proof of Theorem 2, $U(Z) = A(Z) + \langle\langle\Phi\rangle\rangle_S$ and the row-reduced equations are given by $\tilde{\varphi} := A(Z) \upharpoonright \varphi$ yielding, after simplification, the linear i/o equivalent system description in row-reduced form $\tilde{\varphi}_1 = y_3(k+1) - u_3(k) = 0$, $\tilde{\varphi}_2 = y_2(k+1) - u_2(k) = 0$, $\tilde{\varphi}_3 = y_1(k+2) - u_1(k) = 0$ with $y_1(k) \neq 0$, $k \in \mathbb{Z}$.

V. Conclusion

The technical note introduced a notion of linear i/o equivalence transformation for the set of meromorphic nonlinear higher order i/o difference equations. Then, it was proved that using the linear i/o equivalence transformations, the set of nonlinear equations can be transformed into the row-reduced form. Finally, the constructive algorithm is given for finding the equivalence transformation which extends the corresponding transformation for linear systems. The future task is to find out under which additional assumptions the concept of linear i/o equivalence coincides with the conventional i/o equivalence definition based on the i/o pairs.

The problem of transforming the set of i/o difference equations into a doubly-reduced (i.e., both row- and column-reduced) form is the topic of the future paper. For that purpose the paper [5] addressing the transformation the matrix over a skew polynomial ring into a doubly-reduced form, may be helpful. Note that the doubly-reduced form is instrumental in the solution of the realization problem of the i/o difference equations into the state space form.

References

[1] B. Beckermann, H. Cheng, and G. Labahn, "Fraction-free row reduction of matrices of ore polynomials," *J. Symbolic Comput.*, vol. 41, pp. 513–543, 2006.

[2] H. Blomberg and R. Ylinen, *Algebraic Theory for Multivariable Linear Systems*. London, U.K.: Academic Press, 1983.

[3] R. Cohn, *Difference Algebra*. New York: Wiley-Interscience, 1965.

[4] G. Conte, C. H. Moog, and A. M. Perdon, *Algebraic Methods for Nonlinear Control Systems. Theory and Applications*. London, U.K.: Springer-Verlag, 2007.

[5] P. Davies, H. Cheng, and G. Labahn, "Computing popov form of general ore polynomial matrices," *Milestones Comp. Algebra (MICA)*, pp. 149–156, 2008.

[6] S. Diop, "Elimination in control theory," *Math. Control, Signals, Syst.*, vol. 4, pp. 17–33, 1991.

[7] B. Farb and R. K. Dennis, *Noncommutative Algebra*. New York: Springer-Verlag, 1993.

[8] X.-S. Gao, Y. Luo, and C. Yuan, "A characteristic set method for ordinary difference polynomial systems," *J. Symbolic Comput.*, vol. 44, pp. 242–260, 2009.

[9] M. Halás, "An algebraic framework generalizing the concept of transfer functions to nonlinear systems," *Automatica*, vol. 44(5), pp. 1181–1190, 2008.

[10] M. Halás, Ü. Kotta, Z. Li, H. Wang, and C. Yuan, "Submersive rational difference systems and their accessibility," in *Proc. Int. Symp. Symbolic Algebraic Comput.*, Seoul, Korea, 2009, pp. 175–182.

[11] M. Halás, Ü. Kotta, and C. H. Moog, "Transfer function approach to the model matching problem of nonlinear systems," in *Proc. 17th IFAC World Congress*, Seoul, Korea, 2008, pp. 15197–15202.

[12] Ü. Kotta, "Towards a solution of the state-space realization problem of a set of multi-input multi-output nonlinear difference equations," in *Proc. Eur. Control Conf.*, Karlsruhe, Germany, 1999, [CD ROM].

[13] Ü. Kotta, Z. Bartosiewicz, E. Pawłuszewicz, and M. Wyrwas, "Irreducibility, reduction and transfer equivalence of nonlinear input-output equations on homogeneous time scales," *Syst. Control Lett.*, vol. 59, no. 9, pp. 646–651, 2009.

[14] Ü. Kotta and M. Tõnso, "Realization of discrete-time nonlinear input-output equations: Polynomial approach," in *Proc. 6th World Congress Intell. Control Autom.*, Chongqing, China, 2008, pp. 529–534.

[15] Ü. Kotta and M. Tõnso, "Irreducibility conditions for discrete-time nonlinear multi-input multi-output systems," in *Proc. NOLCOS'04: 6th IFAC Symp. Nonlin. Control Syst.*, 2004, pp. 269–274.

[16] Ü. Kotta, A. S. I. Zinober, and P. Liu, "Transfer equivalence and realization of nonlinear higher order input-output difference equations," *Automatica*, vol. 37, pp. 1771–1778, 2001.

[17] J. Rudolph, "Viewing input-output system equivalence from differential algebra," *J. Math. Syst., Estim. Control*, vol. 4, no. 3, pp. 353–383, 1994.

[18] A. J. van der Schaft, "Transformations of nonlinear systems under external equivalence," in *Lecture Notes in Control and Information Science*. New York: Springer-Verlag, 1988, pp. 33–43.

[19] A. J. van der Schaft, "On realization of nonlinear systems described by higher-order differential equations," *Math. Syst. Theory*, vol. 19, pp. 239–275, 1987.

[20] W. A. Wolovich, *Linear Multivariable Systems*. New York: Springer-Verlag, 1974.

[21] Y. Zheng, J. C. Willems, and C. Zhang, "A polynomial approach to nonlinear systems controllability," *IEEE Trans. Autom. Copntrol*, vol. 46, no. 1, pp. 1782–1788, Jan. 2001.

Local Agent Requirements for Stable Emergent Group Distributions

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***Abstract*—This note introduces a model of a generic team formation problem. We derive general conditions under which a group of scattered decision-making agents converge to a particular distribution. The desired distribution is achieved when the agents divide themselves into a fixed number of sub-groups while consenting on "gains" which are associated to every subgroup. The model allows us to quantify the impact of limited sensing, motion, and communication capabilities on the rate at which the distribution is achieved. Finally, we show how this theory is useful in solving a cooperative surveillance problem.**

***Index Terms*—Cooperative systems, distributed decision-making, team formation.**

I. Introduction

There is growing interest in designing and understanding multi-agent systems composed of independent decision-makers. Some well-known examples include cooperative groups of agents trying to accomplish a common *global* objective like: (*i*) agreeing upon a particular variable of interest (e.g., consensus problems [1], [2]); (*ii*) achieving collective group motion and formation patterns (e.g., [3], [4]); and (*iii*) allocating a group across spatially distributed tasks (e.g., [5]–[7]). While each of these problems has unique features, common constraints include (*i*) motion dynamics; (*ii*) range-limited and inaccurate sensing; or (*iii*) agent-to-agent limited and delayed communications.

In all of these multi-agent to multi-task studies there is a strong agent-to-task assignment coupling that occurs through the shared set of spatially distributed tasks that are to be accomplished by the group. For instance, suppose that tasks lie in distinct spatially separated areas. When a particular agent gets assigned to an area to perform tasks, the benefit of assigning all other agents to this area decreases since the same agent can usually perform several tasks in the same vicinity with

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