Algebra Universalis, 29 (1992) 151–171

0002–5240/92/020151–21$01.50 + 0.20/0

Problems and results in tame congruence theory. A survey of the '88 Budapest Workshop

EMIL W. KISS* AND PETER PRÖHLE*

Abstract. Tame congruence theory is a powerful new tool, developed by Ralph McKenzie, to investigate finite algebraic structures. In the summer of 1988, many prominent researchers in this field visited Budapest, Hungary. This paper is a survey of problems and ideas that came up during these visits. It is intended both for beginners and experts, who want to do research, or just want to see what is going on, in this new, active area. An Appendix, written in April, 1990, is attached to the paper to summarize new developments.

1. Introduction

First we list the names of those colleagues who were so kind as to accept our invitation to the Workshop. They are: Clifford Bergman, Joel Berman, Gábor Czédli, Keith Kearnes, George McNulty, Ralph McKenzie, James B. Nation, Peter Pálfy, Robert W. Quackenbush, Ágnes Szendrei, Matthew Valeriote. Most of them spent only one or two weeks in Budapest, but Joel Berman, Ágnes Szendrei, and Matthew Valeriote were here for almost a month. This meeting was most beneficial for the authors of this paper, who were the organizers; we learned a lot. The participants had the opportunity to discuss their ideas and problems informally, ideas and problems that one would not necessarily publish. With this paper, we want to serve the universal algebraic community by making this material available. We hope that many projects will originate from these questions. Therefore we mention some results and problems that have already been proved before the Workshop, or came up after it, but which we feel to be very important, and related to tame congruence theory.

There is no point in surveying a survey paper. The reader is requested to browse all the sections to find the questions and results he is interested in. We tried to

---

Presented by R. W. Quackenbusch.
Received May 3, 1990 and in final form October 2, 1990.
*Research supported by the Hungarian National Foundation for Scientific Research, grant number 1813.

motivate most of the problems by mentioning either known theorems, or ones which have been proved during the Workshop. Despite the short proofs, which are usually included, some of these new theorems and examples, in our opinion, are very elegant, nontrivial, and would deserve a paper in the Mailbox of Algebra Universalis.

In choosing our style, our main goal was to record all the information. Therefore, in most proofs, we assume a knowledge of the book [13] of Ralph McKenzie and David Hobby, which is referred to as *the book* throughout. However, it is sufficient to read Chapters 2, 3 and 4 from this book to understand our language. We included explicit references to the book in many proofs for those readers, who are beginners in this topic, and want to work on a particular problem. The harder to understand the environment of a problem, the less likely is that it can be solved without an appropriate knowledge of the theory.

Finally we would like to say thanks to Ervin Fried and Tamás Schmidt for helping us in overcoming many administrative difficulties that came up in organizing the Workshop.

2. Quackenbush's conjecture

CONJECTURE 2.1. (R. McKenzie, R. W. Quackenbush) *Let A be a finite algebra. If* $\mathrm{V}(A)$ *has infinitely many finite subdirectly irreducible algebras, then this variety has a subdirectly irreducible algebra of every infinite cardinality. The same conjecture holds for simple algebras.*

Ralph McKenzie investigated this problem for algebras $M(P)$ obtained from a finite, partially ordered set P by putting in all (finitary) monotone functions as basic operations. He proved a very important result in early 1988.

THEOREM 2.2. (R. McKenzie [11]) *If P is bounded (i.e., P has* 0 *and* 1*), then* $\mathrm{V}(M(P))$ *is residually small iff it is congruence distributive iff it is congruence modular.*

CONJECTURE 2.3. (R. McKenzie) *If P is bounded, then $M(P)$ has a near unanimity term iff it has Jónsson terms iff its clone of terms is finitely generated.*

Let us outline the classical proof of Ralph Freese and Ralph McKenzie [4] of Quackenbush's conjecture in modular varieties. Suppose that S is a finite subdirectly irreducible algebra in a modular variety generated by a finite algebra A, and let μ denote the monolith of S. We have to prove that if $\mathrm{V}(A)$ is residually small, then S is small. A certain construction shows that if S generates a residually small variety, then the annihilator of μ is Abelian, that is, it annihilates itself. On the other hand, $S \cong B/\theta$ for a subalgebra $B \leq A^n$, with μ corresponding to θ^*/θ. One of the projection kernels in this decomposition of B, say η, must annihilate θ^* modulo θ. Hence η annihilates itself modulo θ (as the annihilator of μ is Abelian). Therefore η must be small modulo θ. Since $B/\eta \in \mathrm{S}(A)$ is also small, we are done.

To generalize this argument for finite subdirectly irreducibles of type 5 monolith, we need a subdirectly irreducible construction, and a modified concept of the annihilator.

DEFINITION 2.4. (E. W. Kiss, R. McKenzie, P. Pröhle) *Let B be an algebra, e an idempotent unary polynomial of B and $1 \in e(B)$ a fixed element. We say that two binary relations P and R on B rectangulate each other if for very* $\mathbf{a}P\mathbf{b}$, $\mathbf{c}R\mathbf{d}$, *and* $f \in \mathrm{Pol}\,(B)$,

$$ef(\mathbf{a}, \mathbf{c}) = ef(\mathbf{b}, \mathbf{d}) = 1 \quad \Rightarrow \quad ef(\mathbf{a}, \mathbf{d}) = ef(\mathbf{b}, \mathbf{c}) = 1.$$

It is worth comparing this condition with TC and TC^*. The following lemma is a new construction of subdirectly irreducible algebras, analogous to the Freese–McKenzie result mentioned above.

LEMMA 2.5. (R. McKenzie) *Let B be a finite algebra and $\theta \prec \theta^*$ a type 5 prime quotient of B such that B/θ is subdirectly irreducible. Let $e(B)$ be a minimal set for this quotient with $e \in E(B)$, and $(0, 1) \in \theta - \theta^*$ elements of the unique trace of $e(B)$ such that $1 \cdot 1 = 1$ and $0 \cdot 1 = 1 \cdot 0 = 0 \cdot 0$, where $\cdot$ is the pseudo meet operation. Suppose that B generates a residually small variety. Then for every compatible tolerance T of B, if $\{0, 1\}^2$ rectangulates T, then T rectangulates itself.*

Proof. Suppose that T does not rectangulate itself. So we have $(u, v) \in T$, $(\mathbf{c}, \mathbf{d}) \in T$, and a polynomial h such that $eh(u, \mathbf{c}) = eh(v, \mathbf{d}) = 1$ but $eh(v, \mathbf{c}) \neq 1$. Let X be any set, $Q = \{p \in B^X \mid (\forall x, y \in X)(p(x), p(y)) \in T\}$ a subdirect subalgebra, and define the congruence ρ on Q by

$$(p, q) \in \rho \quad \Leftrightarrow \quad (\forall f \in \mathrm{Pol}_1\,(Q))(ef(p) = \bar{1} \leftrightarrow ef(q) = \bar{1}).$$

Here $\bar{1} \in Q$ is constant 1, and e on Q is defined componentwise. If $p, q \in \{u, v\}^X$, and $p \neq q$, then it is easy to see, using h, that $(p, q) \notin \rho$. So $|Q/\rho| \geq 2^{|X|}$. Therefore, to get a contradiction, it is sufficient to prove that Q/ρ is subdirectly irreducible.

Let $p \in e(Q)$. We show that if $p \neq \bar{1}$, then $(p, p \cdot \bar{0}) \in \rho$. Suppose that $f \in \mathrm{Pol}_1\,(Q)$ and one of $ef(p)$ and $ef(p \cdot \bar{0})$ equals $\bar{1}$. Let $x \in X$ be such that $p(x) \neq 1$. Then $(p(x), p(x) \cdot 0) \in \theta$ by the properties of a pseudo meet operation.

Since 1 is contained in a one element block of θ, $ef(p)(x) = ef(p \cdot \bar{0})(x) = 1$. Now the facts that $\{0, 1\}^2$ rectangulates T and $p \cdot \bar{1} = p$ imply that $ef(p)(y) = ef(p \cdot \bar{0})(y) = 1$ for every $y \in X$. Hence $(p, p \cdot \bar{0}) \in \rho$ indeed.

To show that ρ is meet irreducible, let $(p, q) \notin \rho$, that is, say, $ef(p) = \bar{1}$ but $ef(q) \neq \bar{1}$. Then $(\bar{1}, ef(q))$ and $(ef(q) \cdot \bar{0}, \bar{1} \cdot \bar{0})$ are in $\mathrm{Cg}(p, q)$, and $(ef(q), ef(q) \cdot \bar{0}) \in \rho$ by the result of the previous paragraph. Therefore $(\bar{0}, \bar{1}) \in \mathrm{Cg}(p, q) \vee \rho$ by transitivity. □

Now we start investigating a subdirect decomposition of B.

LEMMA 2.6. (R. McKenzie) *Let B be a finite algebra and $\theta \prec \theta^*$ a type 5 prime quotient of B such that B/θ is subdirectly irreducible. Suppose that $B \le A^n$ for an algebra A generating a residually small variety. Choose a congruence λ of B that is minimal with respect to $\lambda \vee \theta = \theta^*$ and let η be a projection kernel not containing λ. Then* $\mathrm{typ}(\eta \wedge \theta, \eta) \subseteq \{1, 5\}$.

Proof. Let $e(B)$ be a $\langle\theta, \theta^*\rangle$-minimal set with $e \in E(B)$. As $\lambda \vee \theta = \theta^*$, there must exist elements 0 and 1 satisfying the conditions of Lemma 2.5 such that $\lambda = \mathrm{Cg}(0, 1)$. We show that $\{0, 1\}^2$, moreover λ, rectangulates η. Indeed, by the minimality of λ we have $\eta \wedge \lambda \le \theta$. Thus if $\mathbf{a}\lambda\mathbf{b}$, $\mathbf{c}\eta\mathbf{d}$, and $f \in \mathrm{Pol}(B)$, then $ef(\mathbf{a}, \mathbf{c}) = ef(\mathbf{b}, \mathbf{d}) = 1$ implies that $ef(\mathbf{a}, \mathbf{c})(\lambda \wedge \eta)ef(\mathbf{a}, \mathbf{d})$. Hence they are in θ, and since $\{1\}$ is a block of θ, $ef(\mathbf{a}, \mathbf{d}) = 1$. Similarly, $ef(\mathbf{b}, \mathbf{c}) = 1$.

By Lemma 2.5, η rectangulates itself. Suppose that $\eta \wedge \theta \le \alpha \prec \beta \le \eta$ has type 2, 3 or 4, let N be a corresponding trace and $a, b \in N$ such that $(a, b) \in \beta - \alpha$. Then $(a, b) \in \eta - \theta$, hence $\mathrm{Cg}(a, b) \vee \theta \ge \theta^*$. Therefore there exists a unary polynomial f with $ef(a) = 1$ and $ef(b) \neq 1$. Let p be a 1-snag on (a, b). Then $ef(p(a, b)) = ef(p(b, a)) = ef(a) = 1$ but $ef(p(b, b)) = ef(b) \neq 1$. This is a contradiction, since η rectangulates itself. □

PROBLEM 2.7. *Describe the structure of those congruences that rectangulate themselves. Continue the above investigation to show that B/θ must have a small cardinality.*

3. Abelian and strongly Abelian algebras

Every congruence modular, Abelian variety is polynomially equivalent to a veriety of modules. It would be crucial to prove a similar structure theorem for locally finite, Abelian (but not necessarily modular) varieites. At the moment, we do not even have a conjecture. There is more hope for strongly Abelian varieties.

M. Valeriote has outlined a very reasonable way, based on the ideas of Ralph McKenzie's paper [10], to prove the following assertion.

CONJECTURE 3.1. (E. W. Kiss, M. Valeriote) *Every locally finite, strongly Abelian variety is polynomially equivalent to a subreduct of a matrix power of a unary variety. (A subreduct of an algebra is a subalgebra of a reduct of it.)*

The first step towards a structure theorem would be to find properties, which must be satisfied by all locally finite, Abelian varieties.

PROBLEM 3.2. (Ralph McKenzie) *Is every locally finite, Abelian variety Hamiltonian? Does it always have enough injectives?*

Hamiltonian means: every subalgebra is a congruence class. There are two partial results concerning this problem.

THEOREM 3.3. (E. W. Kiss, M. Valeriote [8]) *Every locally finite, strongly Abelian variety is Hamiltonian. Moreover, if $\mathscr{V}$ is strongly Abelian, and there is a finite bound on the essential arities of the term functions of $\mathscr{V}$, then $\mathscr{V}$ is Hamiltonian.*

THEOREM 3.4. (M. Valeriote [17]) *If A is a finite, simple, Abelian algebra, then A has no subalgebras other than singletons and A itself.*

The following example shows that an Abelian variety can be quite nasty.

EXAMPLE 3.5. (M. Valeriote) *The following three element algebra A satisfies: $\mathbf{V}(A)$ is Abelian, Hamiltonian, residually large, and not quasi affine. (An algebra is called quasi affine if it is a subreduct of a module.)*

$+$	0	$0'$	1
0	0	0	1
$0'$	0	0	1
1	1	1	0

f	
0	$0'$
$0'$	$0'$
1	1

$\underline{0}$	
0	0
$0'$	0
1	0

We conclude this section by a list of related problems. They are due to E. W. Kiss, M. Valeriote, and partly to Ralph McKenzie.

By the results of M. Valeriote (see [14]), if $\mathbf{HS}(A^2)$ is Abelian, then every strongly solvable congruence quotient of A is strongly Abelian.

PROBLEM 3.6. *Let A be a finite algebra such that $\mathbf{HS}(A^2)$ is Abelian. Does it follow that $\mathbf{V}(A)$ is Abelian? Is A necessarily Hamiltonian?*

J. Shapiro [16] proved that every locally finite, strongly Abelian variety is residually small.

PROBLEM 3.7. *Characterize those locally finite Abelian varieties that are residually small. Are they always quasi affine? If A is finite, Abelian, and $\mathbf{V}(A)$ is residually small, is $\mathbf{V}(A)$ Abelian?*

PROBLEM 3.8. *Characterize those finite algebras that are homomorphic images of finite, Abelian algebras. (They must be solvable.)*

PROBLEM 3.9. *Investigate the transfer principles (see [14]) in locally finite Abelian varieties. Do they always hold in quasi affine or residually small (locally finite, Abelian) varieties? If they hold in the generator algebras, do they hold in the whole variety?*

4. The type set of a finitely generated variety

It is usually a difficult question to determine typ $\{\mathbf{V}(A)\}$ for a given finite algebra A. If A has only type 1 and 2, that is, if A is solvable, then so is $\mathbf{V}(A)$, and if A has only type 1, then $\mathbf{V}(A)$ is also strongly solvable. Can we state similar results for other sets of types? The first negative result in this direction is the following surprising example, due to Ralph McKenzie.

EXAMPLE 4.1. *Consider the eight element Tardos poset P shown on Figure 1, and let $A = M(P)$ be the algebra obtained from P by putting in all monotone operations. Then A is simple of type 4 and has no subalgebras. However,* $3 \in \text{typ}\,\{\mathbf{V}(A)\}$.

Proof. To exhibit a type 3 quotient in $\mathbf{V}(A)$ we define a subdirect power of A. Let Q be the poset shown above and $T = Q - \{r, s\}$ with the ordering inherited from Q. Let B be the set of all monotone functions $T \to P$ that can be extended to Q. Clearly, B is a subalgebra of A^6. For a subset $X \subseteq T$ denote by η_X the projection kernel corresponding to X. That is, $g, h \in B$ are congruent modulo η_X iff they agree on X. Set $\beta = \eta_V$ and $\alpha = \eta_{V_0} \vee \eta_{V_1}$, where $V = \{u_0, u_1, b_0, b_1\}$, $V_0 = V \cup \{m_0\}$, and $V_1 = V \cup \{m_1\}$. We shall prove that β covers α, and this quotient has type 3.

Let u be the automorphism of P switching u_0 and u_1 and fixing all the other elements, and define b analogously. These mappings are obviously polynomials of A. Set

$$f_0 = (u_0, u_1, m_0, m_0, b_0, b_1) \qquad f_1 = (u_0, u_1, m_1, m_1, b_0, b_1).$$

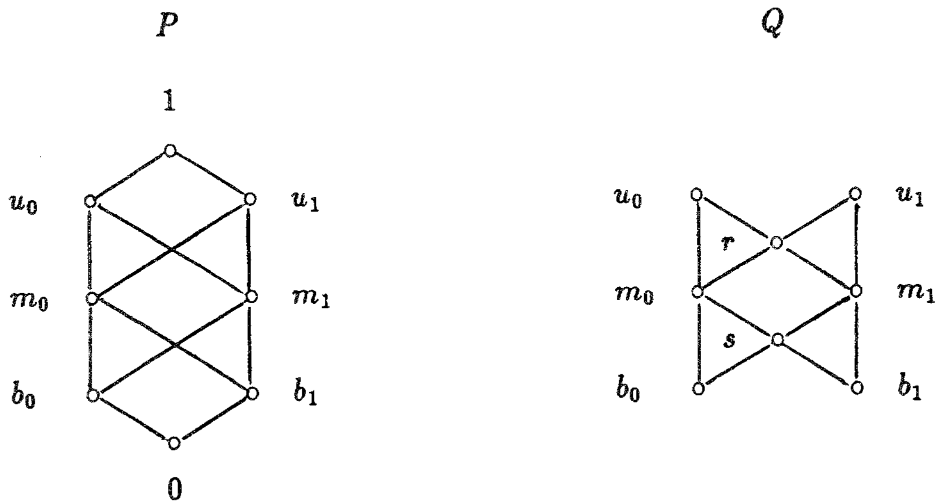


Figure 1

By this notation, we mean a mapping $f_0 : T \to P$, which maps u_0 to u_1, u_1 to u_1, m_0 to m_0, m_1 to m_0, b_0 to b_0, and b_1 to b_1. Clearly, f_0 and f_1 are elements of B. It is easy to check that

$$\{f_0, f_1\}, \qquad \{u \circ f_0, u \circ f_1\}, \qquad \{b \circ f_0, b \circ f_1\}, \qquad \{b \circ u \circ f_0, b \circ u \circ f_1\}$$

are β blocks, that every other block of β is also a block of α, and that these eight elements all form a one element α block. Since these four blocks are polynomially isomorphic (since u and b are polynomials), β indeed covers α, and each of these blocks must be an $\langle \alpha, \beta \rangle$ trace. It is easy to design a unary and two binary polynomials of A that yield the Boolean operations on $\{f_0, f_1\}$. □

There are several problems arising from this example. Ralph McKenzie has proved that typ $\{\mathbf{V}(A)\} \subseteq \{2, 3, 4\}$ by exhibiting terms satisfying his Mal'cev condition for omitting 1 and 5. Nobody knows at the moment whether this variety has type 2. On the other hand, it is easy to take a reduct of A such that the above construction yields a type 2 quotient (with A still being simple of type 4, with no subalgebras). These investigations led to the following result showing that the only restriction on the type set of a variety is the one mentioned at the beginning of this section, namely solvability and strong solvability.

THEOREM 4.2. (J. Berman) *For every $i \in \{3, 4, 5\}$ and $j \in \{1, 2, 3, 4, 5\}$ there exists a seven element algebra C_{ij} which is simple, has no proper subalgebras, has type i, and C_{ij}^2 has a subalgebra having a type j prime quotient.*

The proof of this result will appear in the paper by Joel Berman, Emil W. Kiss, Peter Pröhle, Ágnes Szendrei [3]. It is worth mentioning that such anomalies cannot occur in modular varieties.

CLAIM 4.3. (E. W. Kiss) *Let A be a finite algebra. If $\mathbf{V}(A)$ is congruence modular, then* $\operatorname{typ}\{\mathbf{V}(A)\} = \operatorname{typ}\{\mathbf{S}(A)\}$.

Proof. Suppose that a given type i occurs in $\operatorname{typ}\{\mathbf{V}(A)\}$, say in a finite $B \in \mathbf{P}_s\mathbf{S}(A)$. Choose $\alpha \prec \beta \in \operatorname{Con}(B)$ of type i, where β is as small as possible. Let η be a projection kernel not containing β. Then $\eta \wedge \alpha = \eta \wedge \beta$, since otherwise this would be a covering pair by modularity, of type i, contradicting the minimality of β. Hence, by modularity, $\eta \vee \alpha \prec \eta \vee \beta$, yielding a covering pair of congruences of type i in $B/\eta \in \mathbf{S}(A)$. □

PROBLEM 4.4. *Let A be a finite algebra. How can one determine the type set of $\mathbf{V}(A)$? Does there exist a natural number n (not depending on the size of A) such that every type occurring in $\mathbf{V}(A)$ occurs already in $F_{\mathbf{V}(A)}(n)$?*

For type one the answer is affirmative:

CLAIM 4.5. *If type 1 occurs in $\mathbf{V}(A)$, then it occurs in $F_{\mathbf{V}(A)}(2)$.*

Proof. Let $\alpha \prec \beta \in \operatorname{Con}(B)$ be a type one pair for a finite $B \in \mathbf{V}(A)$. Pick $(a, b) \in \beta - \alpha$ and let C be the subalgebra generated by a and b. Then $\alpha \restriction C < \beta \restriction C$ satisfies the strong term condition, hence every covering pair between these two congruences has type one. Since C is a homomorphic image of $F_{\mathbf{V}(A)}(2)$, type one occurs in this free algebra also. □

This means that type one already occurs in a subalgebra of A^{A^2}. On the other hand, it may be necessary to go up as far as A^A:

EXAMPLE 4.6. (cf. Exercise 6.23.8 in the book) *On any finite set define an algebra A by putting in all unary operations, and all operations that are not surjective (the so called Slupecki clone). Then $A^{|A|}$ admits type 1 (in fact, it has a two element unary factor), but $\mathbf{S}(A^n)$ has only Boolean type for every $n < |A|$.*

Proof. Define the congruence ψ of A^A to have just two blocks, one consisting of all permutations of A. Then A/ψ is clearly essentially unary. It is also clear that A is simple, has no subalgebras (and automorphisms), and $\operatorname{typ}\{A\} = \{3\}$. For any proper subset H of A construct a "local discriminator" term function d_H such that $d_H(x, x, z) = z$ if $z \in H$, and $d_H(x, y, z) = x$ whenever $x \in H$ and $x \neq y$. These functions show that the patchwork property along equalizers and coequalizers holds in every A^n for $n < |A|$. Indeed, if $f \in A^n$ is the function we want to patch together, then we can use d_H for H being the range of f. Thus every congruence on A^n is a projection kernel, and a standard, easy argument shows that every subalgebra of A^n is isomorphic to A^k for some $k \leq n$. Now the proof of Claim 4.3 shows that $\operatorname{typ}\{A^n\} = \{3\}$. □

Suppose that we can solve the Quackenbush problem in the following, stronger form: there exists a function $f: \omega \to \omega$ such that for every finite algebra A, if $\mathbf{V}(A)$ has a subdirectly irreducible algebra of size bigger than $f(|A|)$, then $\mathbf{V}(A)$ is residually large. In that case, to look for the type set of a residually small variety $\mathbf{V}(A)$, it is sufficient to check the algebras of size not bigger than $f(|A|)$ (since every type occurs as a monolith of a subdirectly irreducible quotient). Example 4.6 shows that the following question might have a positive answer.

PROBLEM 4.7. *Let A be a finite algebra and suppose that type* 1 *occurs in* $\mathbf{V}(A)$. *Does it occur in a member B of* $\mathbf{V}(A)$ *which has a "small" cardinality (say, $|B| \leq |A|^4$)?*

For type 2 we have a negative answer to Problem 4.4:

THEOREM 4.8. (J. Berman, October 1988) *For each integer $m > 1$ there exists a five element algebra A such that $2 \in \operatorname{typ}\{\mathbf{V}(A)\}$ but $2 \notin \operatorname{typ}\{\mathbf{S}(A^r)\}$ for $r < m$.*

For direct powers only, the answer is positive for every type.

CLAIM 4.9. (J. Berman) *If a type occurs in A^k, then it occurs in A^n for some $n \leq |A|^2$.*

To prove this statement, we need a new characterization of the type set of a finite algebra.

DEFINITION 4.10. *Let $\alpha \prec \beta \in \operatorname{Con}(A)$. An $\langle \alpha, \beta \rangle$ subtrace of A is defined to be any subset N of any β trace of any $\langle \alpha, \beta \rangle$ minimal set of A such that N is not contained in a single α class.*

LEMMA 4.11. (J. Berman, E. W. Kiss) *A subset N of a finite algebra A is an $\langle \alpha, \beta \rangle$ subtrace for a suitable covering pair of congruences of A iff*

$$N^2 \not\subseteq \mathrm{LCg}_A(N) \stackrel{\mathrm{def}}{=} \mathrm{Cg}_A\,\{f(N)^2 \mid f \in \mathrm{Pol}_1(A), f(N) \not\simeq N\}.$$

Here $\not\simeq$ means: not polynomially isomorphic.

Proof. If N is an $\langle\alpha, \beta\rangle$ subtrace, then $f(N)^2 \nsubseteq \alpha$ implies that $f(N) \simeq N$, thus $\mathrm{LCg}_A(N) \subseteq \alpha$. Conversely, suppose that $N^2 \nsubseteq \mathrm{LCg}_A(N)$. Set $\beta = \mathrm{Cg}_A(N^2)$ and let α be any lower cover of β containing $\mathrm{LCg}_A(N)$. Then N can be mapped into an $\langle\alpha, \beta\rangle$ trace by a unary polynomial f such that $f(N)^2 \nsubseteq \alpha$. So by the definition of $\mathrm{LCg}_A(N)$ we have $N \simeq f(N)$. Thus N is a subtrace. □

This lemma enables us to determine the subtraces without computing the minimal sets. On the other hand, we can determine the type set of an algebra by considering the induced *partial* algebras on the two element subtraces.

DEFINITION 4.12. *Let N be a two element $\langle\alpha, \beta\rangle$ subtrace of A. We define the type of N in the following way.*

(1) *If $A|_N$ is polynomially equivalent to a Boolean algebra, then N has type* 3.
(2) *If $A|_N$ is polynomially equivalent to a lattice, then N has type* 4.
(3) *If $A|_N$ is polynomially equivalent to a semilattice, then N has type* 5.
(4) *If none of the above cases hold, but for $N = \{a, b\}$, either (a, b) or (b, a) is a* 1*-snag, then N has type* 2.
(5) *In all other cases, N has type* 1.

OBSERVATION 4.13. (J. Berman, E. W. Kiss) *Let A be a finite algebra and $\alpha \prec \beta$ in* $\mathrm{Con}(A)$. *Then every two element $\langle\alpha, \beta\rangle$ subtrace has type* $\mathrm{typ}\langle\alpha, \beta\rangle$. *Hence A has type i iff it has a two element subtrace of type i (for $i \in \{1, 2, 3, 4, 5\}$).* □

To determine the type set of A using Lemma 4.11 and Observation 4.13 is computationally feasible. These results also prove Claim 4.9. Indeed, it is easy to check that if B is a retract of C (that is, B is the range of an idempotent endomorphism of C), and N is a subset of B which is a subtrace in C, then it is a subtrace in B (use Lemma 4.11). Now suppose that $\{a, b\}$ is a type i two element subtrace of $C = A^k$. Let B be the subalgebra of A^k consisting of those elements which agree at every pair of indices, where both a and b agree. Then B is a retract of C, it is isomorphic to A^n for some $n \leq |A|^2$, and has type i by our remark above.

If a finite algebra A has type 3 or 4, then it has a trace also (and not just a subtrace), on which the induced algebra is polynomially equivalent to a Boolean algebra or a lattice, respectively (see Lemma 4.17 in the book). A similar statement holds for type 2 as well.

CLAIM 4.14. (E. W. Kiss) *Let A be a finite algebra having type* 2. *Choose $\alpha \prec \beta$ in* $\mathrm{Con}(A)$ *of type* 2, *where β is as small as possible. (Notice that β is join irreducible.) Let B be the body of any $\langle\alpha, \beta\rangle$ minimal set U. Then $\alpha \restriction B = 0$. Hence the induced algebra on every trace is a vector space.*

Proof. To get a contradiction choose $(a, b) \in \alpha$, $a \neq b$, $a, b \in B$, and let $\gamma \prec \delta \leq \alpha$ be congruences of A with $(a, b) \in \delta - \gamma$. By the minimality of β, the type of $\langle \gamma, \delta \rangle$ is either 1 or it is not Abelian. The first case is impossible, since the induced algebra on B is Mal'cev, hence it cannot have a strongly Abelian quotient. In the second case connect a and b with a chain of $\langle \gamma, \delta \rangle$ traces. Let $U = e(A)$ for an idempotent polynomial e of A and apply e to this chain. One of these $\langle \gamma, \delta \rangle$ traces, say N, satisfies $e(N)^2 \nsubseteq \gamma$. Hence $e(N)$ is also a $\langle \gamma, \delta \rangle$ trace, sitting in B. This is impossible, since $A|_B$ is nilpotent and $e(N)$ is a 2-snag. □

It was asked by Keith Kearnes whether every equationally complete variety admits only one possible type. This is certainly the case with modular varieties by Claim 4.3. The answer is negative in general. Our example is based on a construction by Clifford Bergman and Ralph McKenzie.

OBSERVATION 4.15. (Ágnes Szendrei) *There exists an equationally complete variety* $\mathscr{V}$ *such that* $1, 5 \in \operatorname{typ}\{\mathscr{V}\}$. *Every subquasivariety* $\mathscr{Q}$ *of* $\mathscr{V}$ *also admits* 1 *and* 5 (*since* $\mathscr{V} = \mathbf{H}(\mathscr{Q})$), *including the unique minimal subquasivariety.*

Proof. Consider the algebra $A = \langle \{0, 1, 2\}, \cdot, f, g, \underline{0}, \underline{1}, \underline{2} \rangle$ constructed in [2], where the operations are given by the following tables:

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	1	2

f	
0	0
1	0
2	2

g	
0	0
1	1
2	0

It is proved in [2] that $\mathscr{V} \stackrel{\text{def}}{=} \mathbf{V}(A)$ is equationally complete. As 0 is an absorbing element, A, which is simple, has type 5. Define a congruence θ on A^2 by collapsing all pairs that contain 0. Then A^2/θ is subdirectly irreducible of type 1 monolith. □

5. Miscellaneous results and problems

An algebra is called *affine complete* if every congruence preserving finitary function is a polynomial. In an arithmetical variety, every finite algebra is affine complete (K. Baker, A. Pixley, see H. Werner [18]). On the other hand, Ralph McKenzie proved, using tame congruence theory, that if every algebra is affine complete in a locally finite variety $\mathscr{V}$, then $\mathscr{V}$ is congruence distributive.

EXAMPLE 5.1. (H. Werner) *In the variety of sets, every nonsimple finite algebra is affine complete. The same property holds in the variety of vector spaces over any finite field.* □

DEFINITION 5.2. *A variety is called affine complete (in Werner's sense) if every finite, nonsimple member is affine complete.*

PROBLEM 5.3. (E. W. Kiss, R. McKenzie) *Characterize locally finite affine complete varieties using tame congruence theory. If every algebra is affine complete in a (not necessarily locally finite) variety $\mathscr{V}$, is $\mathscr{V}$ congruence distributive?*

A possible starting point would be to investigate the direct product of two finite, simple algebras A and B, namely, its type set, the existence of skew congruences, and affine completeness. The following nice result, answering a question of E. W. Kiss and M. Valeriote, also grew out from the problem of affine completeness.

THEOREM 5.4. (K. Kearnes) *Let $\mathscr{V}$ be a locally finite variety. Then $\mathscr{V}$ is congruence modular iff the minimal sets for the prime quotients of the finite algebras of $\mathscr{V}$ have empty tail.*

To prove this assertion we introduce a very useful concept.

DEFINITION 5.5. (E. W. Kiss) *Let A be an algebra. A subset of A is called an E-trace of A, if it is the intersection of a congruence class and of the range of an idempotent polynomial.*

Of course, traces of $\langle \alpha, \beta \rangle$-minimal sets are E-traces. On the other hand, by Theorem 6.17 and Lemma 6.18 in the book, if 1 or 5 is contained in typ $\{\mathscr{V}\}$, then every finite set or finite semilattice, respectively, is polynomially equivalent to the induced algebra *on an E-trace* of a suitable finite algebra of $\mathscr{V}$. Both in the variety of sets and in the variety of semilattices, the tails are not always empty (take three element algebras). Hence it is important to relate the minimal sets of the E-traces to the minimal sets of A.

LEMMA 5.6. (E. W. Kiss) *Let A be a finite algebra, E an E-trace of A and $\alpha \prec \beta$ congruences of the induced algebra $A|_E$. Then there exist congruences $\alpha' \prec \beta'$ of A such that $\alpha' \restriction E = \alpha$, $\beta' \restriction E = \beta$. Moreover, if U is an $\langle \alpha, \beta \rangle$-minimal set of the induced algebra $A|_E$, then there exists an $\langle \alpha', \beta' \rangle$-minimal set U' of A such that $U = U' \cap E$, and U is a union of $\beta' \restriction U'$ classes.*

First let us see how Theorem 5.4 follows from this lemma. Assume the conditions of the lemma and notice that the tail of U is contained in the tail of U' (since U is a union of $\beta' \restriction U'$ classes). Hence, by our remark above, if a variety $\mathscr{V}$ admits 1 or 5, then the tails are not empty (moreover, we can see that the minimal sets of $\mathscr{V}$ must contain big sets and semilattices, respectively). So by Theorem 8.5 in the book, we are done.

Now we prove Lemma 5.6. Let $E = e(A) \cap (a/\gamma)$, where e is an idempotent polynomial and $\gamma \in \mathrm{Con}(A)$. By Lemma 2.4 in the book, both α and β can be extended to A, moreover, β has a smallest extension β' to A, and $\beta' \le \gamma$ (actually, β' is the congruence of A generated by β). Now α has an extension α' that is maximal among all congruences $\delta \le \beta'$ of A, for which $\delta \restriction E = \alpha$. Then β' clearly covers α'.

Next we construct U'. Let $f_0 \in \mathrm{Pol}_1(A)$ be such that $f_0(E) \subseteq E$, $f_0 \restriction E$ is idempotent, and has range U. Let f be an idempotent power of ef_0. Then $ef = f$, hence the range V of f is contained in $e(A)$, and $f(E) = U$. Pick a pair $(a, b) \in \beta - \alpha$ with $a, b \in U$. Then $(a, b) \in \beta' - \alpha'$, so we can connect a and b with a series of $\langle \alpha, \beta \rangle$ traces, and pairs in α'. Apply f to this chain. The resulting chain goes already in $V = f(A) \subseteq e(A)$, and in a $\beta' \le \gamma$-class, too, hence in E. It still connects a and b, and therefore it has a pair $(a', b') \in \beta - \alpha$. Let U_0 be the $\langle \alpha', \beta' \rangle$ minimal set of A from which (a', b') originated. Then $f(U_0)^2$ is not contained in β', hence $U' = f(U_0)$ is also an $\langle \alpha', \beta' \rangle$-minimal set of A, containing a' and b'. Let $g(A) = U'$, where g is an idempotent polynomial of A.

We show that $g(E) \subseteq E$. Indeed, the range of g is contained in $f(A) \subseteq e(A)$, and if $x \in E$ then $x\gamma a'$, hence $g(x)\gamma g(a') = a' \in E$. Hence $g(x) \in E$. This means that $g \restriction E$ is a unary polynomial of $A|_E$, and (a', b') witnesses that $g(\beta) \nsubseteq \alpha$. On the other hand, if $x \in E$, then $g(x) = fg(x) \in f(E) = U$, hence $g(E) \subseteq U$. By the minimality of U, $g(E) = U$, hence $U' = g(A) \supseteq U$. Conversely, let $x \in U' \cap E$, then $x = g(x) \in g(E) = U$, thus $U = U' \cap E$ as desired. As $\beta' \le \gamma$ and $U' \subseteq e(A)$, U indeed is a union of $\beta' \restriction U'$-classes. □

Other applications (and a comprehensive study) of E-traces are found in the paper [7]. Those results show that it may be worth splitting type one into two subclasses.

DEFINITION 5.7. (E. W. Kiss) *A prime quotient of a finite algebra is defined to have type zero ("constant type"), if the traces of the corresponding minimal algebras have no unary polynomials that are permutations other than the identity map.*

PROBLEM 5.8. (E. W. Kiss) *Give a characterization of type zero quotients that is independent of minimal sets. Describe type zero simple algebras and varieties. When are they residually small?*

EXAMPLE 5.9. (P. P. Pálfy) *Let G be a finite directed graph that is weakly connected and contains no directed cycles. Add all mappings $f: G \to G$ as unary operations that satisfy: if $u \to v$, then either $f(u) \to f(v)$ or $f(u) = f(v)$. Then the resulting unary algebra is simple of type zero.*

Proof. For vertices $u \ne v$, $u \to v$ and a of G define a function f such that $f(x) = v$ if there is a directed path from a to x, and $f(x) = u$ otherwise. Then f is a unary operation showing that the algebra is simple, and every arrow is a minimal set of type zero. □

Now we turn to the problem of investigating locally finite varieties having enough injectives. The first step is to deal with the congruence extension property. In the modular case, as shown by Emil W. Kiss [6], such varieties have to satisfy the following two commutator conditions:

$$x, y \in \mathrm{Con}\,(A) \quad \Rightarrow \quad [x, y] = x \cdot y \cdot [1_A, 1_A] \tag{C2}$$

$$x, y \in \mathrm{Con}\,(A), B \leq A \quad \Rightarrow \quad [x, y] \restriction B = [x \restriction B, y \restriction B]. \tag{R}$$

OBSERVATION 5.10. (K. Kearnes, E. W. Kiss) *Let A be a finite algebra in a CEP variety and $\alpha, \beta \in \mathrm{Con}\,(A)$ such that $[\alpha, \beta] = 0$. Let $a\alpha b$ and $f \in \mathrm{Pol}\,(A)$. Then*

$$\begin{array}{ccc} f(a, \mathbf{c}) & = & f(a, \mathbf{d}) \\ |\alpha & & |\alpha \\ f(b, \mathbf{c}) & \beta & f(b, \mathbf{d}) \end{array} \quad \Rightarrow \quad f(b, \mathbf{c}) = f(b, \mathbf{d}).$$

In particular, if $[\alpha, \alpha] = 0$, then $[\alpha, 1] = 0$.

Proof. Let Δ be the congruence on the subalgebra β of $A \times A$ generated by the set $\{((a, a), (b, b)) \mid a\alpha b\}$. Then $[\alpha, \beta] = 0$ implies that $(z, z)\,\Delta\,(x, y) \Rightarrow x = y$. Extend Δ to a congruence Δ' of $A \times A$. Set $z = f(a, \mathbf{c}) = f(a, \mathbf{d})$, $x = f(b, \mathbf{c})$ and $y = f(b, \mathbf{d})$. Then $(z, z)\,\Delta'\,(x, y)$, and both pairs are in the algebra β. Therefore $(z, z)\,\Delta\,(x, y)$, implying $x = y$. □

PROBLEM 5.11. (K. Kearnes, E. W. Kiss) *Let A be a finite, solvable algebra in a CEP variety. Is A Abelian?*

The answer is yes for modular varieties by (C2). More generally, we show that if S is a finite, solvable, but not Abelian algebra in a CEP variety of minimal possible cardinality, then S is subdirectly irreducible of type 1 monolith. Indeed, as subdirect products of Abelian algebras are Abelian, S must be subdirectly irreducible. Let μ be its monolith. Then $[\mu, \mu] = 0$, hence $[\mu, 1] = 0$ by the previous Observation. By the minimality of S, S/μ is Abelian. As S is not Abelian, we have $f(a, \mathbf{c}) = f(a, \mathbf{d})$ but $f(b, \mathbf{c}) \neq f(b, \mathbf{d})$ for suitable elements and polynomial. By TC in S/μ, we have $f(b, \mathbf{c})\mu f(b, \mathbf{d})$. Therefore, by the Observation above, it is sufficient to prove that $[1, \mu] = 0$. This is indeed the case if μ has type 2, since in this case the $\langle 0, \mu \rangle$-minimal algebras are full bodied by Lemma 4.27 (4/ii) in the book, and therefore Mal'cev (see the proof of Lemma 4.36 in the book for the commutativity of the commutator in this case).

To generalize (R), let A be a finite algebra in a CEP variety and B a subalgebra of A. Let $\alpha \prec \beta$ be a type i quotient of A. Using CEP we can show that if $\alpha \restriction B \neq \beta \restriction B$, then $\alpha \restriction B \prec \beta \restriction B$. Assume this and let $j = \mathrm{typ}(\alpha \restriction B, \beta \restriction B)$.

THEOREM 5.12. (K. Kearnes) *The relationship between i and j is given by the following table. Here 2_p means a type two quotient, where the traces (which are vector spaces) have characteristic p. The (i, j) entry in the table is Y if that situation can happen, N if it cannot happen, and ? if we do not know the answer.*

$i \backslash j$	1	2_p	2_q	3	4	5
1	Y	N	N	N	N	N
2_p	N	Y	N	N	N	N
2_q	N	N	Y	N	N	N
3	N	N	N	Y	Y	N
4	N	N	N	Y	Y	N
5	N	N	N	?	?	Y

In CEP varieties, Quackenbush's conjecture holds in the following, stronger form.

OBSERVATION 5.13. (E. W. Kiss) *There exists a function $f: \omega \to \omega$ such that for every finite algebra A generating a CEP variety, if $\mathbf{V}(A)$ has a subdirectly irreducible algebra of size bigger than $f(|A|)$, then $\mathbf{V}(A)$ is residually large.*

Proof. By the results of J. Baldwin and J. Berman [1], every locally finite CEP variety has definable principal congruences, and therefore satisfies Quackenbush's conjecture. For a finite algebra A generating a CEP variety construct a new algebra $\hat{A}$, which has the same underlying set as that of A, and the basic operations of $\hat{A}$ are exactly the four variable terms of A. For every cardinal κ, the four generated subalgebras of A^κ and $\hat{A}^\kappa$ are the same, hence, by CEP in A^κ, we have $\mathrm{Con}(A^\kappa) = \mathrm{Con}(\hat{A}^\kappa)$, and $\hat{A}^\kappa$ also has CEP. Thus, if B is a homomorphic image if $\hat{A}^\kappa$, then B is a reduct of an algebra from $\mathbf{H}(A^\kappa)$, and these two algebras have the same four element subalgebras. Therefore, B has CEP, too. On the other hand, $\mathbf{V}(\hat{A}) = \mathbf{SHP}(\hat{A})$, hence this variety has CEP.

Let $S = B/\psi$ be subdirectly irreducible, where $B \leq A^\kappa$. Let θ be a maximal extension of ψ to A^κ. Then A^κ/θ is also subdirectly irreducible, and contains B as a subalgebra. The same argument holds for $\mathbf{V}(\hat{A})$ as well. However, $\mathrm{Con}(A^\kappa) = \mathrm{Con}(\hat{A}^\kappa)$, hence the subdirectly irreducible factors of these two powers have the same cardinality. Thus $\mathbf{V}(A)$ and $\mathbf{V}(\hat{A})$ are residually small at the same time, and if they are residually finite, then the maximal size of their subdirectly irreducibles is the same. Therefore, to prove the Observation, it is sufficient to consider algebras that have at most 4-ary basic operations.

On every finite set, there are only finitely many such algebras. Let $f(n)$ be the maximal possible size of a subdirectly irreducible algebra in a residually small CEP variety generated by an n element algebra that has at most 4-ary basic operations. Then f clearly satisfies the conditions. □

The following observation generalizes H. P. Gumm's permutability results for modular varieties.

THEOREM 5.14. (R. McKenzie) *Let A be a finite algebra and $\alpha, \beta \in \mathrm{Con}(A)$. If $\mathbf{V}(A)$ omits type 1, then*

$$\alpha \circ \beta \subseteq [\alpha, \alpha] \circ \beta \circ \alpha \circ [\beta, \beta].$$

Proof. By Theorem 7.12 in the book, $\mathbf{V}(A)$ has a ternary term operation d such that d is Mal'cev on every block of every solvable congruence. Thus if $x\alpha y\beta z$, then

$$x[\alpha, \alpha]d(x, y, y)\beta d(x, y, z)\alpha d(y, y, z)[\beta, \beta]z.$$ □

In varieties admitting type 3 and 4 only, the condition that replaces distributivity is semi-distributivity. Therefore it is natural to try to extend the modular commutator to varieties omitting 1 and 5, and require semi-distributivity in both variables, instead of distributivity.

PROBLEM 5.15. (M. Valeriote) *Let $\mathcal{V}$ be a locally finite variety omitting 1 and 5. Does there exist a semi-distributive commutator in $\mathcal{V}$? That is, a binary operation $[x, y]$ on the congruence lattice of every algebra from $\mathcal{V}$ satisfying the following conditions.*

(1) $[x, y] \leq x \wedge y$;
(2) $[x, z] = [y, z] \quad \Rightarrow \quad [x, z] = [x \vee y, z]$;
(3) $[z, x] = [z, y] \quad \Rightarrow \quad [z, x] = [z, x \vee y]$;
(4) $[x, x] = 0$ *iff x is an Abelian congruence.*

Can one require a good behaviour with respect to homomorphisms? Does the ordinary TC commutator satisfy these conditions?

OBSERVATION 5.16. *The ordinary TC commutator is left semi-distributive, that is, satisfies* (2) *of the above problem.* □

This Observation follows trivially from the (elementary) facts in Chapter 3 of the book. *It does not follow*, however, that the TC commutator is left distributive. The following example shows that we cannot expect commutativity, either.

EXAMPLE 5.17. (E. W. Kiss) *The algebra A given below generates a variety omitting* 1 *and* 5, *but the TC commutator in A is not commutative. The algebra A is minimal of type* 2 *with respect to its unique minimal congruence.*

$+$	0	1	∞
0	0	1	∞
1	1	0	∞
∞	∞	∞	∞

$\cdot$	0	1	∞
0	0	0	0
1	0	0	0
∞	0	0	∞

Proof. Let α be the congruence of A collapsing 0 and 1. It is easy to see by induction that for every unary polynomial f, if f is not constant, then $f(\infty) = \infty$, and if $f(0)$ or $f(1)$ is ∞, then f is constant ∞. Therefore A has only seven unary polynomials, which are easy to compute, and they show that A is indeed $\langle 0, \alpha \rangle$-minimal. Next we show that $[\alpha, 1] = 0 \neq [1, \alpha]$. Suppose that $a\alpha b$, $f(a, \mathbf{c}) = f(a, \mathbf{d})$, but $f(b, \mathbf{c}) \neq f(b, \mathbf{d})$. Since these four elements are α related, they must be contained in the trace $\{0, 1\}$. We have seen above that if a unary polynomial satisfies $g(\infty, \mathbf{a}) \neq \infty$, then g is constant in its first variable. In other words, we can replace the ∞ by, say, 0, and g will still take the same value. Therefore we can change all occurrences of ∞ to 0 in $\mathbf{c}$ and $\mathbf{d}$ without changing our assumptions. But $\{0, 1\}$ is a subalgebra of A which is term equivalent to an Abelian group, hence it satisfies TC. Therefore $[\alpha, 1] = 0$ indeed. On the other hand, $[1, \alpha] = 0$ does not hold, since $\infty + 0 = \infty + 1$ but $0 + 0 \neq 0 + 1$.

Finally, we show that typ $\{\mathbf{V}(A)\} \subseteq \{2, 3, 4\}$ by exhibiting terms satisfying the conditions of Theorem 9.8 (4) in the book for $n = 2$. These are:

$$d_0(x, y, z) = d_1(x, y, z) = x, \qquad e_1(x, y, z) = e_2(x, y, z) = z,$$
$$d_2(x, y, z) = x + yz, \qquad p(x, y, z) = x + y + z, \qquad e_0(x, y, z) = xy + z. \qquad \square$$

It would be interesting to find the congruence variety of $\mathbf{V}(A)$, in connection with Problem 13 in the book.

Next we mention another problem that is related to the modular commutator. The following concepts are due to R. W. Quackenbush. Let A be any algebra, and consider the set of all formal linear combinations formed from the elements of A using integer coefficients. Extend all operations of A linearly, and add the operations $+$ and $-$ to obtain a new algebra $\mathbf{Z}(A)$. It is easy to see that every congruence α of A has a unique minimal extension α' to $\mathbf{Z}(A)$. For $\alpha, \beta \in \operatorname{Con}(A)$ we define the *linear commutator* of these congruences by $L(\alpha, \beta) = [\alpha', \beta'] \restriction A$. Here we used the ordinary commutator in the Mal'cev algebra $\mathbf{Z}(A)$.

PROBLEM 5.18. (R. W. Quackenbush) *For congruence modular varieties, does the linear commutator coincide with the ordinary TC commutator?*

The two commutators do coincide for comparable pairs of congruences (E. W. Kiss), and the above problem has a positive answer for congruence permutable varieites. See R. W. Quackenbush's paper [15] for details on the linear commutator.

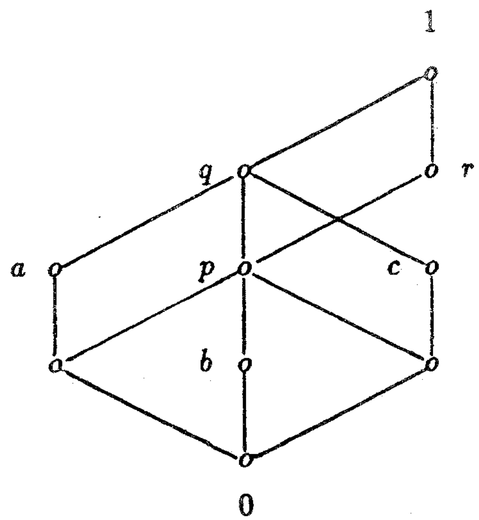
PROBLEM 5.19. (R. McKenzie, I. G. Rosenberg) *On a finite set, find the number of clones containing a Mal'cev operation. Can it be infinite?*

In the paper [9] W. A. Lampe found a famous condition that has to be satisfied by all lattices of subvarieties. It goes as follows. Let L be the dual of the lattice of subvarieties of any variety and 1 the biggest element of L. Then

$$\left.\begin{array}{l} a_1 \vee \cdots \vee a_n = 1 \\ a_i \vee u = v \quad (1 \le i \le n) \end{array}\right\} \quad \Rightarrow \quad u = v. \tag{$*$}$$

This condition covers R. McKenzie's results on subvariety lattices of locally finite varieties found in [10].

EXAMPLE 5.20. (E. W. Kiss, P. P. Pröhle) *The lattice drawn below satisfies W. Lampe's condition* ($*$), *but it is not isomorphic to the dual of the lattice of subvarieties of any locally finite variety.*



The "*nightmare*".

Proof. It is routine to check that $(*)$ is satisfied. In the paper [10] R. McKenzie proves that if the lattice of subvarieties of a locally finite variety is finite, then its dual is isomorphic to the congruence lattice of a finite algebra having a reduct that is a groupoid with left zero and left one. Such algebras obviously do not have nontrivial TC factors. Suppose that the lattice above is the congruence lattice of such an algebra. Then by Lemma 6.6 in the book, applied to the M_3 spanned by a, b, and c, we see that the quotient $\langle p, q \rangle$ has type 1 or 2. By Lemma 6.2 in the book, projective quotients have the same type. Hence $\langle r, 1 \rangle$ also has type 1 or 2. This is a contradiction, since we cannot have an Abelian quotient at the top. □

PROBLEM 5.21. (W. A. Lampe, J. B. Nation) *Is this lattice isomorphic to the dual of the lattice of subvarieties of any variety? Is it isomorphic to the congruence lattice of an algebra having a reduct that is a groupoid with left zero and left one?*

It was an open question (Problem 16 in the book) whether the congruence variety of all groups, that is, the lattice variety generated by the congruence lattices of all groups, is the same as the congruence variety of all Abelian groups. The following result shows that this is not the case.

THEOREM 5.22. (C. Herrmann, E. W. Kiss, P. P. Pálfy) *There exists a lattice identity that holds in the subgroup lattices of all Abelian groups, but it does not hold in the normal subgroup lattice of the free group on three generators in the variety generated by the eight element quaternion group.*

PROBLEM 5.23. (P. P. Pálfy) *Present explicitly such a lattice identity.*

The following related question arrived to us from Professor Norman R. Reilly.

PROBLEM 5.24. *Let $\mathscr{V}_1$ be the congruence variety of all groups and $\mathscr{V}_2$ the lattice variety generated by the lattice of fully invariant congruences of the absolutely free group on countably many generators, that is, the lattice variety generated by the subvariety lattices of all group varieties. Is $\mathscr{V}_1 = \mathscr{V}_2$? Is a basis of identities known for either of these varieties?*

Appendix

Since the Workshop, many problems mentioned in this paper have been solved, at least partially. In this Appendix, written in April 1990, we summarize some of these developments.

Conjecture 3.1 is false. A counterexample is found in Kiss, Valeriote [8]. Ralph McKenzie [12] improved Theorem 3.4 by showing that for a finite algebra in an Abelian variety, every maximal subalgebra is a congruence block. This result implies that a locally finite, Abelian variety with the CEP is Hamiltonian, which is a new partial answer to Problem 3.2. Added in proof: E. W. Kiss and M. Valeriote proved in 1991 that locally finite Abelian varieties are Hamiltonian.

Concerning Example 4.1, Ralph McKenzie has shown that typ $\{\mathbf{V}(A)\} = \{3, 4\}$, by constructing the appropriate monotone operations to satisfy Theorem 9.11 in the book. He claims that these operations are not hard to find. This result is not published.

Theorem 4.8 has been strengthened by Joel Berman. For every type k other than 1 and for every integer $m > 1$ there exists an at most five element algebra A having at most three basic operations, all $m + 2$-ary, such that the type k appears in the variety generated by A, but it does not appear in the subalgebras of A^{m-1}. This same variety has no subdirectly irreducibles of size less than $m + 2$ with type k monolith. For all such k and m an algebra B can also be constructed such that k occurs in B but the $m - 1$-generated free algebra in $\mathbf{V}(B)$ is essentially unary. So the answer to the second question of Problem 4.4 is strongly negative. These results appear in [3].

Progress has also been made to understand the congruence extension property. Ralph McKenzie [12] has answered Problem 5.11 affirmatively, by showing that every locally solvable congruence of an algebra in a CEP variety is Abelian. He also constructed CEP varieties, which are generated by simple algebras, their type set can be anything, and which satisfy no transfer principles. Thus the answer to the last question of Problem 3.9 is negative. This result is unpublished.

Keith Kearnes [5] has also many new results on CEP. He proved that if $\mathscr{V}$ is a locally finite variety with CEP, then typ $\{\mathscr{V}\} \subseteq$ typ $\{F_{\mathscr{V}}(2)\} \cup \{3\}$, and if $4 \notin$ typ $\{F_{\mathscr{V}}(2)\}$, then typ $\{\mathscr{V}\} =$ typ $\{F_{\mathscr{V}}(2)\}$. Conversely, for any m there is a locally finite congruence distributive variety $\mathscr{V}_m$ with CEP such that 3 occurs in $\mathscr{V}_m$, but 3 does not occur in the m-generated free algebra of $\mathscr{V}_m$. Thus the second statement of Problem 4.4 holds in CEP varieties, except for type 3, which is hard to distinguish from 4. Theorem 5.12 is also improved in the same paper (see Theorem 2.13 of [5]).

Finally, Theorem 5.22 has to be modified. It is true that the congruence lattice of the three generated free group in the variety generated by the quaternion group cannot be embedded into the congruence lattice of any Abelian group. However, C. Herrmann's proof of the existence of a lattice identity, as stated in the theorem, is not yet complete. Added in proof: P. P. Pálfy and Cs. Szabo found such an identity in 1990.

REFERENCES

[1] BALDWIN, J. T. and BERMAN, J., *The number of subdirectly irreducible algebras in a variety.* Algebra Universalis 5 (1975), 379–389.

[2] BERGMAN, C. and MCKENZIE, R., *Minimal varieties and quasivarieties*. J. Austral. Math. Soc. (Series A) *48* (1990), 133–147.

[3] BERMAN, J., KISS, E. W., PRÖHLE, P. and SZENDREI, Á., *On the set of types of a finitely generated variety*. (Manuscript.)

[4] FREESE, R. and MCKENZIE, R., *Residually small varieties with modular congruence lattices*. Trans. Amer. Math. Soc. *264* (1981), 419–430.

[5] KEARNES, K., *Type preservation in locally finite varieties with the CEP*. (Manuscript.)

[6] KISS, E. W., *Injectivity and related concepts in modular varieties, I–II*. Bull. Austral. Math. Soc. *32* (1985), 33–53.

[7] KISS, E. W., *An interpretation of permutation groups into varieties*. (Manuscript.)

[8] KISS, E. W. and VALERIOTE, M., *Strongly Abelian varieties and the Hamiltonian property*. (Manuscript.)

[9] LAMPE, W. A., *A property of the lattice of equational theories*. Algebra Universalis *23* (1986), 61–69.

[10] MCKENZIE, R., *Finite forbidden lattices*, Springer Lecture Notes No. *1004* (1983), 176–205.

[11] MCKENZIE, R., *Monotone clones, residual smallness and congruence distributivity*. (Manuscript.)

[12] MCKENZIE, R., *Congruence extension, Hamiltonian and Abelian properties in locally finite varieties*. Algebra Universalis *28* (1991), 589–603.

[13] MCKENZIE, R. and HOBBY, D., "The structure of finite algebras (Tame congruence theory)," American Mathematical Society, Contemporary Mathematics series, vol. 76, 1988.

[14] MCKENZIE, R. and VALERIOTE, M., "The structure of locally finite decidable varieties," Birkhäuser Verlag, Boston, 1989.

[15] QUACKENBUSH, R. W., *General commutator theory*. (Manuscript.)

[16] SHAPIRO, J., *Finite algebras with Abelian properties*, Algebra Universalis *25* (1988), 334–364.

[17] VALERIOTE, M., *Finite simple Abelian algebras are strictly simple*. (Manuscript.)

[18] WERNER, H., "Discriminator Algebras," Studien zur Algebra und ihre Anwendungen, Akademie-Verlag, Berlin, 1978.

Mathematical Institute of the
Hungarian Academy of Sciences
1364. Budapest, P.O. Box 127
Hungary