

Introducing Divergence for Infinite Probabilistic Models

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Abstract. Computing the reachability probability in infinite state probabilistic models has been the topic of numerous works. Here we introduce a new property called *divergence* that when satisfied allows to compute reachability probabilities up to an arbitrary precision. One of the main interest of divergence is that our algorithm does not require the reachability problem to be decidable. Then we study the decidability of divergence for probabilistic versions of pushdown automata and Petri nets where the weights associated with transitions may also depend on the current state. This should be contrasted with most of the existing works that assume weights independent of the state. Such an extended framework is motivated by the modeling of real case studies. Moreover, we exhibit some divergent subclasses of channel systems and pushdown automata, particularly suited for specifying open distributed systems and networks prone to performance collapsing in order to compute the probabilities related to service requirements.

Keywords: Reachability probability · Infinite state probabilistic models · Divergence.

1 Introduction

Probabilistic models. In the 1980’s, finite-state Markov chains have been considered for the modeling and analysis of probabilistic concurrent finite-state programs [22]. Since the 2000’s, many works have been done to verify the infinite-state Markov chains obtained from probabilistic versions of automata extended with unbounded data (like stacks, channels, counters and clocks)¹. The (qualitative and quantitative) model checking of *probabilistic pushdown automata* (pPDA) is studied in many papers, for example in [12,13,8,14,18] (see [7] for a survey). In 1997, Iyer and Narasimha [16] started the study of *probabilistic lossy channel systems* (pLCS) and later both some qualitative and quantitative

¹ Surprisingly, in 1972, to the best of our knowledge, Santos gave the first definition of *probabilistic pushdown automata* [21] that did not open up a new field of research at the time.

properties were shown decidable for pLCS [1]. *Probabilistic counter machines* (pCM) have also been studied [11,10,9].

Computing the reachability probability. In finite Markov chains, there is a well-known algorithm for computing exactly the reachability probabilities in polynomial time [3]. Here we focus on the problem of *Computing the Reachability Probability up to an arbitrary precision* (CRP) in *infinite* Markov chains. There are (at least) two possible research directions:

The first one is to consider the Markov chains associated with a particular class of probabilistic models (like pPDA or probabilistic Petri nets (pPN)) and some specific target sets and to exploit the properties of these models to design a CRP-algorithm. For instance in [7], the authors exhibit a PSPACE algorithm for pPDA and PTIME algorithms for single-state pPDA and for one-counter automata.

The second one consists in exhibiting a property of Markov chains that yields a generic algorithm for solving the CRP problem and then looking for models that generate Markov chains that fulfill this property. *Decisiveness* of Markov chains is such a property. Intuitively, decisiveness w.r.t. s_0 and A means that almost surely the random path σ starting from s_0 will reach A or some state s' from which A is unreachable. It has been shown that pPDA are not (in general) decisive but both pLCS and probabilistic Petri nets (pPN) are decisive (for pPN: when the target set is upward-closed [2]).

Two limits of the previous approaches. The generic approach based on the decisiveness property has numerous applications but suffers the restriction that the reachability problem must be decidable in the corresponding non deterministic model. To the best of our knowledge, all generic approaches rely on a *decidable reachability problem*.

In most of the works, the probabilistic models associate a *constant* weight for transitions and get transition probabilities by normalizing these weights among the enabled transitions in the current state. This *forbids to model phenomena* like congestion in networks (resp. performance collapsing in distributed systems) when the number of messages (resp. processes) exceeds some threshold leading to an increasing probability of message arrivals (resp. process creations) before message departures (resp. process terminations).

Our contributions.

- In order to handle realistic phenomena (like congestion in networks), we consider *dynamic* weights i.e., weights depending on the current state.
- We introduce the new *divergence* property of Markov chains w.r.t. s_0 and A : given some precision θ , one can discard a set of states with either a small probability to be reached from s_0 or a small probability to reach A such that the remaining subset of states is finite and thus allows for an approximate computation of the reachability probability up to θ . For divergent Markov chains, we provide a generic algorithm for the CRP-problem that *does not require* the decidability of the reachability problem. While decisiveness and divergence are not exclusive (both hold for finite Markov chains),

- they are complementary. In fact, divergence is somehow related to transience of Markov chains while decisiveness is somehow related to recurrence [15].
- In order to check divergence, we provide several simpler sufficient conditions based on existing and new results of martingale theory.
 - We study for different models the decidability of divergence. Our first undecidability result implies that whatever the infinite models, one must restrict the kind of dynamics weights. Here we limit to polynomial weights.
 - We prove, by a case study analysis, that divergence is decidable for a subclass of polynomial pPDA (i.e. pPDA with *polynomial* weights). We show that divergence is undecidable for polynomial pPNs w.r.t. an upward closed set.
 - We provide two classes of divergent polynomial models. The first one is a probabilistic version of channel systems particularly suited for the modeling of open queuing networks. The second one is the probabilistic version of pushdown automata restricted to some typical behaviors of dynamic systems.

Organisation. Section 2 recalls Markov chains, introduces divergent Markov chains, presents an algorithm for solving the CRP-problem. In Section 3, we study the decidability status of divergence for pPDA and pPN. Finally Section 4 presents two divergent subclasses of probabilistic channel systems and pPDA. All missing proofs and a second CRP-algorithm can be found in Appendix.

2 Divergence of Markov chains

2.1 Markov chains: definitions and properties

Notations. A set S is *countable* if there exists an injective function from S to the set of natural numbers: hence it could be finite or countably infinite. Let S be a countable set of elements called states. Then $Dist(S) = \{\Delta : S \rightarrow \mathbb{R}_{\geq 0} \mid \sum_{s \in S} \Delta(s) = 1\}$ is the set of *distributions* over S . Let $\Delta \in Dist(S)$, then $Supp(\Delta) = \Delta^{-1}(\mathbb{R}_{>0})$. Let $T \subseteq S$, then $S \setminus T$ will also be denoted \bar{T} .

Definition 1 (Effective Markov chain). A Markov chain $\mathcal{M} = (S, p)$ is a tuple where:

- S is a countable set of states,
- p is the transition function from S to $Dist(S)$;

When for all $s \in S$, $Supp(p(s))$ is finite and the function $s \mapsto p(s)$ is computable, one says that \mathcal{M} is effective.

Notations. Let $p^{(d)}$ denote the d^{th} power of the transition matrix p . When S is countably infinite, we say that \mathcal{M} is *infinite* and we sometimes identify S with \mathbb{N} . We also denote $p(s)(s')$ by $p(s, s')$ and $p(s, s') > 0$ by $s \xrightarrow{p(s, s')} s'$. A Markov chain is also viewed as a transition system whose transition relation \rightarrow is defined by $s \rightarrow s'$ if $p(s, s') > 0$. Let $A \subseteq S$, one denotes $Post_{\mathcal{M}}^*(A)$, the set of states that can be reached from some state of A and $Pre_{\mathcal{M}}^*(A)$, the set of states that can reach A . As usual, we denote \rightarrow^* , the transitive closure of \rightarrow and we say that s' is *reachable from* s if $s \rightarrow^* s'$. We say that a subset $A \subseteq S$ is *reachable*

from s if some $s' \in A$ is reachable from s . Note that every finite path of \mathcal{M} can be extended into (at least) one infinite path.

Example 1. Let \mathcal{M}_1 be the Markov chain of Figure 1. In any state $i > 0$, the probability for going to the “right”, $p(i, i + 1)$, is equal to $0 < p_i < 1$ and for going to the “left” $p(i, i - 1)$ is equal to $1 - p_i$. In state 0, one goes to 1 with probability 1. \mathcal{M}_1 is effective if the function $n \mapsto p_n$ is computable.

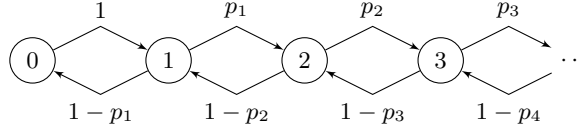


Fig. 1: A random walk \mathcal{M}_1

Given an initial state s_0 , the *sampling* of a Markov chain \mathcal{M} is an *infinite sequence of states* (i.e., a path) $\sigma = s_0 s_1 \dots$ such that for all $i \geq 0$, $s_i \rightarrow s_{i+1}$. As usual, the corresponding σ -algebra whose items are called events is generated by the finite prefixes of infinite paths and the probability of an event Ev given an initial state s_0 is denoted $\mathbf{Pr}_{\mathcal{M}, s_0}(Ev)$. In case of a finite path $s_0 \dots s_n$, $\mathbf{Pr}_{\mathcal{M}, s_0}(s_0 \dots s_n) = \prod_{0 \leq i < n} p(s_i, s_{i+1})$.

Notations. From now on, **G** (resp. **F**, **X**) denotes the always (resp. eventual, next) operator of LTL.

Let $A \subseteq S$. We say that σ *reaches* A if $\exists i \in \mathbb{N} s_i \in A$ and that σ *visits* A if $\exists i > 0 s_i \in A$. The probability that starting from s_0 , the path σ reaches (resp. visits) A will be denoted by $\mathbf{Pr}_{\mathcal{M}, s_0}(\mathbf{F}A)$ (resp. $\mathbf{Pr}_{\mathcal{M}, s_0}(\mathbf{XFA})$).

We now state qualitative and quantitative properties of a Markov chain.

Definition 2 (Irreducibility, recurrence, transience). Let $\mathcal{M} = (S, p)$ be a Markov chain and $s \in S$. Then \mathcal{M} is *irreducible* if for all $s, s' \in S$, $s \rightarrow^* s'$. s is *recurrent* if $\mathbf{Pr}_{\mathcal{M}, s}(\mathbf{XF}\{s\}) = 1$ otherwise s is *transient*.

In an irreducible Markov chain, all states are in the same category, either recurrent or transient [17]. Thus an irreducible Markov chain will be said transient or recurrent depending on the category of its states. In the remainder of this section, we will relate this category with techniques for computing reachability probabilities.

Example 2. Clearly \mathcal{M}_1 is irreducible. Moreover (see Appendix), \mathcal{M}_1 is recurrent if and only if $\sum_{n \in \mathbb{N}} \prod_{1 \leq m < n} \rho_m = \infty$ with $\rho_m = \frac{1-p_m}{p_m}$, and when transient, the probability that starting from i the random path visits 0 is equal to $\frac{\sum_{i \leq n} \prod_{1 \leq m < n} \rho_m}{\sum_{n \in \mathbb{N}} \prod_{1 \leq m < n} \rho_m}$.

One of our goals is to approximately compute reachability probabilities in infinite Markov chains. Let us formalize it. Given a finite representation of a subset $A \subseteq S$, one says that this representation is *effective* if one can decide the membership problem for A . With a slight abuse of language, we identify A with any effective representation of A .

The Computing of Reachability Probability (CRP) problem

- Input: an effective Markov chain \mathcal{M} , an (initial) state s_0 , an effective subset of states A , and a rational number $\theta > 0$.
- Output: an interval $[low, up]$ such that $up - low \leq \theta$ and $\mathbf{Pr}_{\mathcal{M}, s_0}(\mathbf{F}A) \in [low, up]$.

2.2 Divergent Markov chains

Let us first discuss two examples before introducing the notion of *divergent* Markov chains.

Example 3. Consider again the Markov chain \mathcal{M}_1 of Figure 1 with for all $n > 0$, $p_n = p > \frac{1}{2}$. In this case, for $m \geq 0$, $\mathbf{Pr}_{\mathcal{M}_1, m}(\mathbf{F}\{0\}) = \rho^m$ with $\rho = \frac{1-p}{p}$. Thus here the key point is that not only this reachability probability is less than 1 but it goes to 0 when m goes to ∞ . This means that given some precision θ , one could “prune” states $n \geq n_0$ and compute the reachability probabilities of A in a finite Markov chain.

Consider the Markov chain of Figure 2, where $\mathbf{Pr}_{\mathcal{M}, 0}(\mathbf{F}\{m, m+1, \dots\}) = \prod_{n < m} p_n$ goes to 0 when m goes to ∞ . As in the precedent example, computing the reachability probabilities of A can be also done in a finite Markov chain after pruning states $n \geq n_0$, given some precision θ .

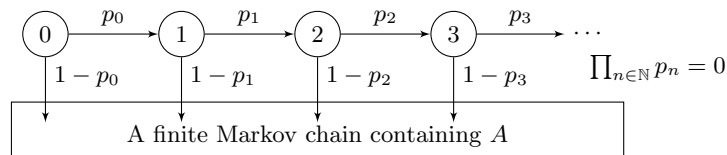


Fig. 2: An infinite (divergent) Markov chain

Intuitively, a divergent Markov chain w.r.t. s_0 and A generalizes these examples: given some precision θ , one can discard a set of states with either a small probability to be reached from s_0 or a small probability to reach A , such that the remaining subset of states is finite and thus allows for an approximate computation of the reachability probability up to θ .

Definition 3 (divergent Markov chain). Let \mathcal{M} be a Markov chain, $s_0 \in S$ and $A \subseteq S$. We say that \mathcal{M} is divergent w.r.t. s_0 and A if there exist two computable functions f_0 and f_1 from S to $\mathbb{R}_{\geq 0}$ such that:

- For all $0 < \theta < 1$, $\mathbf{Pr}_{\mathcal{M}, s_0}(\mathbf{F}f_0^{-1}([0, \theta])) \leq \theta$;
- For all $s \in S$, $\mathbf{Pr}_{\mathcal{M}, s}(\mathbf{F}A) \leq f_1(s)$;
- For all $0 < \theta < 1$, $\{s \mid f_0(s) \geq \theta \wedge f_1(s) \geq \theta\} \cap \text{Post}_{\mathcal{M}}^*(\{s_0\})$ is finite.

Observation and illustration. Let us remark that there cannot exist, for general Markov chains, an algorithm to decide the existence of such functions f_0, f_1 and if there exist, to find them. But there exist some simpler sufficient conditions for divergence.

A finite Markov chain is divergent (letting $f_0 = f_1 = 1$) w.r.t. any s_0 and any A . In the first Markov chain of Example 3, $f_0 = 1$ and $f_1(m) = \rho^m$ and in the second Markov chain, $f_1 = 1$, $f_0(m) = \prod_{0 \leq n < m} p_n$ and $f_0(s) = 1$ for all s in the finite Markov chain containing A . Generalizing these two examples, the next proposition introduces a sufficient condition for divergence. Its proof is immediate by choosing $(f = f_0 \text{ and } f_1 = 1)$ or $(f = f_1 \text{ and } f_0 = 1)$.

Proposition 1. Let \mathcal{M} be a Markov chain, $s_0 \in S$, $A \subseteq S$, and a computable function f from S to $\mathbb{R}_{\geq 0}$ such that:

- For all $0 < \theta < 1$, $\mathbf{Pr}_{\mathcal{M}, s_0}(\mathbf{F}f^{-1}([0, \theta])) \leq \theta$
or for all $s \in S$, $\mathbf{Pr}_{\mathcal{M}, s}(\mathbf{F}A) \leq f(s)$;
- For all $0 < \theta < 1$, $\{s \mid f(s) \geq \theta\} \cap \text{Post}_{\mathcal{M}}^*(\{s_0\})$ is finite.

Then \mathcal{M} is divergent w.r.t. s_0 and A .

2.3 An algorithm for divergent Markov chains

We now design an algorithm for accurately framing the reachability probability for a divergent (effective) Markov chain w.r.t. s_0 and an effective A .

Let us describe this algorithm. It performs an exploration of reachable states from s_0 maintaining S' , the set of visited states, and stopping an exploration when the current state s fulfills: either (1) for some $i \in \{0, 1\}$, $f_i(s) \leq \frac{\theta}{2}$ in which case s is inserted in the *AlmostLoose_i* set (initially empty), or (2) $s \in A$ in which case s is inserted in A' (initially empty). When the exploration is ended, if A' is empty, the algorithm returns the interval $[0, \theta]$. Otherwise it builds $\mathcal{M}' = (S', p')$ a finite Markov chain over S' whose transition probabilities are the ones of \mathcal{M} except for the states of *AlmostLoose₀* \cup *AlmostLoose₁* \cup A' , which are made absorbing. Finally it computes the vector of reachability probabilities starting from s_0 in \mathcal{M}' (function **CompFinProb**) and returns the interval $[\text{preach}(A'), \text{preach}(A') + \text{preach}(\text{AlmostLoose}_0) + \frac{\theta}{2} \cdot \text{preach}(\text{AlmostLoose}_1)]$. The next proposition establishes the correctness of the algorithm.

Proposition 2. Let \mathcal{M} be a divergent Markov chain with $s_0 \in S$, $A \subseteq S$ and $\theta > 0$. Then Algorithm 1 solves the CRP problem.

We also provide an algorithm for models with a decidable reachability problem that returns $[0, 0]$ when A is unreachable and $[\ell, u]$ with $\ell > 0$ otherwise. This algorithm and the proof of its correctness are both presented in Appendix.

Algorithm 1: Framing the reachability probability

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CompProb( $\mathcal{M}, s_0, A, \theta$ )
 $AlmostLoose_0 \leftarrow \emptyset$ ;  $AlmostLoose_1 \leftarrow \emptyset$ ;  $S' \leftarrow \emptyset$ 
 $A' \leftarrow \emptyset$ ;  $Front \leftarrow \emptyset$ ; Insert( $Front, s_0$ )
while  $Front \neq \emptyset$  do
     $s \leftarrow \text{Extract}(Front)$ ;  $S' \leftarrow S' \cup \{s\}$ 
    if  $f_0(s) \leq \frac{\theta}{2}$  then  $AlmostLoose_0 \leftarrow AlmostLoose_0 \cup \{s\}$ 
    else if  $f_1(s) \leq \frac{\theta}{2}$  then  $AlmostLoose_1 \leftarrow AlmostLoose_1 \cup \{s\}$ 
    else if  $s \in A$  then  $A' \leftarrow A' \cup \{s\}$ 
    else for  $s \rightarrow s' \wedge s' \notin S'$  do Insert( $Front, s'$ )
end
if  $A' = \emptyset$  then return  $(0, \theta)$ 
 $Abs \leftarrow AlmostLoose_0 \cup AlmostLoose_1 \cup A'$ 
for  $s \in Abs$  do  $p'(s, s) \leftarrow 1$ 
for  $s \in S' \setminus Abs \wedge s' \in S'$  do  $p'(s, s') \leftarrow p(s, s')$ 
 $preach \leftarrow \text{CompFinProb}(\mathcal{M}', s_0)$ 
return  $(preach(A'), preach(A') + preach(AlmostLoose_0) + \frac{\theta}{2} \cdot preach(AlmostLoose_1))$ 

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3 (Un)Decidability results

We now study probabilistic versions of well-known models like Pushdown Automaton (PDA) and Petri nets (PN), for which we analyse the decidability of the divergence property.

3.1 Probabilistic pushdown automata

Let Γ be a finite alphabet. $\Gamma^{\leq k}$ is the set of words over Γ with length at most k . Let $w \in \Gamma^*$, then $|w|$ denotes its length. ε denotes the empty word.

Definition 4 (pPDA). A (dynamic-)probabilistic pushdown automaton (*pPDA*) is a tuple $\mathcal{A} = (Q, \Gamma, \Delta, W)$ where:

- Q is a finite set of control states;
- Γ is a finite stack alphabet with $Q \cap \Gamma = \emptyset$;
- Δ is a subset of $Q \times \Gamma^{\leq 1} \times Q \times \Gamma^{\leq 2}$
such that for all $(q, \varepsilon, q', w) \in \Delta$, $|w| \leq 1$;
- W is a computable function from $\Delta \times \Sigma^*$ to $\mathbb{Q}_{>0}$.

In the version of pPDA presented in [14], the weight function W goes from Δ to $\mathbb{Q}_{>0}$. In order to emphasize this restriction here and later we say that, in this case, the weight function is *static* and the corresponding models will be called static pPDA. In what follows, pPDA denotes the dynamic version.

An item (q, a, q', w) of Δ is also denoted $q \xrightarrow{?a!w} q'$ and $?a!\varepsilon$ is also simply denoted by $?a$. A *configuration* of \mathcal{A} is a pair $(q, w) \in Q \times \Gamma^*$. We use the letters a, b, c, x, y for elements in Γ and w for a word in Γ^* .

Definition 5. Let \mathcal{A} be a pPDA. Then the Markov chain $\mathcal{M}_{\mathcal{A}} = (S_{\mathcal{A}}, p_{\mathcal{A}})$ is defined by:

- $S_{\mathcal{A}} = Q \times \Gamma^*$ is the set of configurations;
- For all $(q, \varepsilon) \in S_{\mathcal{A}}$ s.t. $\{t = q \xrightarrow{?\varepsilon!w_t} q'\}_{t \in \Delta} = \emptyset$, $p_{\mathcal{A}}((q, \varepsilon), (q, \varepsilon)) = 1$;
- For all $(q, \varepsilon) \in S_{\mathcal{A}}$ s.t. $\{t = q \xrightarrow{?\varepsilon!w_t} q'\}_{t \in \Delta} \neq \emptyset$,
let $W(q, \varepsilon) = \sum_{t=q \xrightarrow{?\varepsilon!w_t} q'} W(t, \varepsilon)$. Then:
for all $t = q \xrightarrow{?\varepsilon!w_t} q' \in \Delta$, $p_{\mathcal{A}}((q, \varepsilon), (q', w_t)) = \frac{W(t, \varepsilon)}{W(q, \varepsilon)}$
- For all $(q, wa) \in S_{\mathcal{A}}$ s.t. $\{t = q \xrightarrow{?a!w_t} q'\}_{t \in \Delta} = \emptyset$, $p_{\mathcal{A}}((q, wa), (q, wa)) = 1$;
- For all $(q, wa) \in S_{\mathcal{A}}$ s.t. $\{t = q \xrightarrow{?a!w_t} q'\}_{t \in \Delta} \neq \emptyset$,
let $W(q, wa) = \sum_{t=q \xrightarrow{?a!w_t} q'} W(t, wa)$. Then:
for all $t = q \xrightarrow{?a!w_t} q' \in \Delta$, $p_{\mathcal{A}}((q, wa), (q', ww_t)) = \frac{W(t, wa)}{W(q, wa)}$

We now show that even for pPDA with a single state and with a stack alphabet reduced to a singleton, divergence is undecidable.

Theorem 1. *The divergence problem for pPDA is undecidable even with a single state and stack alphabet $\{a\}$.*

Due to this negative result on such a basic model, it is clear that one must restrict the possible weight functions. A pPDA \mathcal{A} is said *polynomial* if for all $t \in \Delta$, $W(t, w)$ is a positive integer polynomial (i.e. whose coefficients are non negative and the constant one is positive) whose single variable is $|w|$.

Theorem 2. *The divergence problem w.r.t. s_0 and finite A for polynomial pPDA with a single state and stack alphabet $\{a\}$ is decidable (in linear time).*

3.2 Probabilistic Petri nets

A probabilistic Petri net (resp. a probabilistic VASS) is a Petri net (resp. a VASS) with a computable weight function W . In previous works [2,6], the weight function W is a *static* one: i.e., a function from a finite set of transitions T to $\mathbb{N}_{>0}$. As above, we call these models *static* probabilistic Petri nets. We introduce here a more powerful function where the weight of a transition depends on the current marking.

Definition 6. A (dynamic-)probabilistic Petri net (*pPN*)

$\mathcal{N} = (P, T, \mathbf{Pre}, \mathbf{Post}, W, \mathbf{m}_0)$ is defined by:

- P , a finite set of places;
- T , a finite set of transitions;
- $\mathbf{Pre}, \mathbf{Post} \in \mathbb{N}^{P \times T}$, resp. the pre and post condition matrices;
- W , a computable function from $T \times \mathbb{N}^P$ to $\mathbb{Q}_{>0}$ the weight function;
- $\mathbf{m}_0 \in \mathbb{N}^P$, the initial marking.

When for all $t \in T$, $W(t, -)$ is a positive polynomial whose variables are the place markings, we say that \mathcal{N} is a *polynomial* pPN.

A marking \mathbf{m} is an item of \mathbb{N}^P . Let t be a transition. Then t is *enabled* in \mathbf{m} if for all $p \in P$, $\mathbf{m}(p) \geq \mathbf{Pre}(p, t)$. When enabled, the *firing* of t leads to

marking \mathbf{m}' defined for all $p \in P$ by $\mathbf{m}'(p) = \mathbf{m}(p) + \mathbf{Post}(p, t) - \mathbf{Pre}(p, t)$ which is denoted by $\mathbf{m} \xrightarrow{t} \mathbf{m}'$. Let $\sigma = t_1 \dots t_n$ be a sequence of transitions. We define the enabling and the firing of σ by induction. The empty sequence is always enabled in \mathbf{m} and its firing leads to \mathbf{m} . When $n > 0$, σ is enabled if $\mathbf{m} \xrightarrow{t_1} \mathbf{m}_1$ and $t_2 \dots t_n$ is enabled in \mathbf{m}_1 . The firing of σ leads to the marking reached by $t_2 \dots t_n$ from \mathbf{m}_1 . A marking \mathbf{m} is reachable from \mathbf{m}_0 if there is a firing sequence σ that reaches \mathbf{m} from \mathbf{m}_0 .

Definition 7. Let \mathcal{N} be a pPN. Then the Markov chain $\mathcal{M}_{\mathcal{N}} = (S_{\mathcal{N}}, p_{\mathcal{N}})$ associated with \mathcal{N} is defined by:

- $S_{\mathcal{N}}$ is the set of reachable markings from \mathbf{m}_0 ;
- Let $\mathbf{m} \in S_{\mathcal{N}}$ and $T_{\mathbf{m}}$ be the set of transitions enabled in \mathbf{m} . If $T_{\mathbf{m}} = \emptyset$ then $p_{\mathcal{N}}(\mathbf{m}, \mathbf{m}) = 1$. Otherwise let $W(\mathbf{m}) = \sum_{\mathbf{m} \xrightarrow{t} \mathbf{m}_t} W(t, \mathbf{m})$. Then for all $\mathbf{m} \xrightarrow{t} \mathbf{m}_t$, $p_{\mathcal{N}}(\mathbf{m}, \mathbf{m}_t) = \frac{W(t, \mathbf{m})}{W(\mathbf{m})}$.

Contrary to the previous result, restricting the weight functions to be polynomials does not yield decidability for pPNs.

Theorem 3. The divergence problem of polynomial pPNs w.r.t. an upward closed set is undecidable.

4 Illustration of divergence

Due to the undecidability results, we propose syntactical restrictions for standard models like pushdown automata and channel systems that ensure divergence. Observing that function f_1 of Definition 3 is somewhat related to transience of Markov chains, we first establish a sufficient condition of transience from which we derive a sufficient condition of divergence for infinite Markov chains used for our two illustrations.

The next Theorem 4 extends the classical Theorem 5 (given in Appendix and whose proof is based on Martingale theory). Another interesting extension (Theorem 6), not used in this section, is also presented in the Appendix.

Theorem 4. Let \mathcal{M} be a Markov chain and f be a function from S to \mathbb{R} with $B = \{s \mid f(s) \leq 0\}$ fulfilling $\emptyset \subsetneq B \subsetneq S$, $\varepsilon, K \in \mathbb{R}_{>0}$ and $d \in \mathbb{N}^*$ such that:

$$\text{for all } s \in S \setminus B \quad \sum_{s' \in S} p^{(d)}(s, s') f(s') \geq f(s) + \varepsilon \text{ and } \sum_{|f(s') - f(s)| \leq K} p(s, s') = 1 \quad (1)$$

Then for all $s \in S$ such that $f(s) > dK$,

$$\Pr_{\mathcal{M}, s}(\mathbf{FB}) \leq c_1 e^{-c_2(f(s) - dK)}$$

where $c_1 = \sum_{n \geq 1} e^{-\frac{\varepsilon^2 n}{2(\varepsilon + K)^2}}$ and $c_2 = \frac{\varepsilon}{(\varepsilon + K)^2}$.
which implies transience of \mathcal{M} when it is irreducible.

Proposition 3. *Let \mathcal{M} be a Markov chain and f be a computable function from S to \mathbb{R} with $B = \{s \mid f(s) \leq 0\}$ fulfilling $\emptyset \subsetneq B \subsetneq S$, and for some $\varepsilon, K \in \mathbb{R}_{>0}$ and $d \in \mathbb{N}^*$, Equation (1). Assume in addition that for all $n \in \mathbb{N}$, $\{s \mid f(s) \leq n\}$ is finite. Then \mathcal{M} is divergent w.r.t. any s_0 and any finite A .*

4.1 Probabilistic channel systems

Now we introduce a probabilistic variant of channel systems particularly appropriate for the modelling of open queuing networks. Here a special input channel c_{in} (that works as a counter) only receives the arrivals of anonymous clients all denoted by $\$$ (item 1 of the next definition). Then the service of a client corresponds to a message circulating between the other channels with possibly change of message identity until the message disappears (items 2 and 3).

Definition 8. A probabilistic open channel system (pOCS) $\mathcal{S} = (Q, Ch, \Sigma, \Delta, W)$ is defined by:

- a finite set Q of states;
- a finite set Ch of channels, including c_{in} ;
- a finite alphabet Σ including $\$$;
- a transition relation $\Delta \subseteq Q \times Ch \times \Sigma_\varepsilon \times Ch \times \Sigma_\varepsilon \times Q$ that fulfills:
 1. For all $q \in Q$, $(q, c_{in}, \varepsilon, c_{in}, \$, q) \in \Delta$;
 2. For all $(q, c, a, c', a', q') \in \Delta$, $a = \varepsilon \Rightarrow a' = \$ \wedge c = c' = c_{in}$;
 3. For all $(q, c, a, c', a', q') \in \Delta$, $c \neq c_{in} \Rightarrow c' \neq c_{in}$;
- W is a function from $\Delta \times (\Sigma^*)^{Ch}$ to $\mathbb{Q}_{>0}$.

Illustration. We consider three types of transitions between configurations: sending messages to the input channel (i.e., $(q, c_{in}, \varepsilon, c_{in}, \$, q)$) representing client arrivals; transferring messages between different channels (i.e., (q, c, a, c', a', q') with $\varepsilon \notin \{a, a'\}$ and $c' \neq c_{in}$) describing client services; and terminating message processing (i.e., $(q, c, a, c', \varepsilon, q')$ with $a \neq \varepsilon$) meaning client departures. All messages entering c_{in} are anonymous (i.e., denoted by $\$$). The left part of Figure 3 is a schematic view of such systems. The left channel is c_{in} . All dashed lines represent message arrivals (to c_{in}) or departures. The solid lines model message transferrings.

The next definitions formalize the semantics of pOCS.

Definition 9. Let \mathcal{S} be a pOCS, $(q, \nu) \in Q \times (\Sigma^*)^{Ch}$ be a configuration and $t = (q, c, a, c', a', q') \in \Delta$. Then t is enabled in (q, ν) if $\nu(c) = aw$ for some w . The firing of t in (q, ν) leads to (q', ν') defined by:

- if $c = c'$ then $\nu'(c) = wa'$ and for all $c'' \neq c$, $\nu'(c'') = \nu(c'')$;
- if $c \neq c'$ then $\nu'(c) = w$, $\nu'(c') = \nu(c')a'$
and for all $c'' \notin \{c, c'\}$, $\nu'(c'') = \nu(c'')$.

As usual one denotes the firing by $(q, \nu) \xrightarrow{t} (q', \nu')$. Observe that from any configuration at least one transition (a client arrival) is enabled.

Definition 10. Let \mathcal{S} be a pOCS. Then the Markov chain $\mathcal{M}_{\mathcal{S}} = (S_{\mathcal{S}}, p_{\mathcal{S}})$ is defined by:

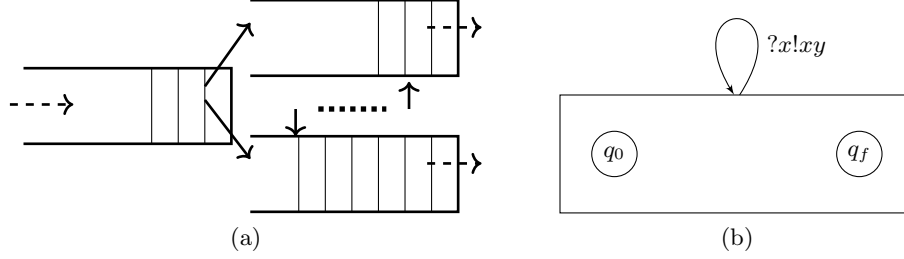


Fig. 3: A schematic view of pOCS (left) and a pPDA (right)

- $S_S = Q \times (\Sigma^*)^{Ch}$ is the set of configurations;
- For all $(q, \nu) \in S_S$ let $W(q, \nu) = \sum_{(q, \nu) \xrightarrow{t} (q', \nu')} W(t, \nu)$. Then:
for all $(q, \nu) \xrightarrow{t} (q', \nu')$, $p_S((q, \nu), (q', \nu')) = \frac{W(t, \nu)}{W(q, \nu)}$.

The restrictions on pOCS w.r.t. standard CS do not change the status of the reachability problem.

Proposition 4. *The reachability problem of pOCS is undecidable.*

As discussed in the introduction, when the number of clients exceeds some threshold, the performances of the system drastically decrease and thus the ratio of arrivals w.r.t. the achievement of a task increase. We formalize it by introducing *uncontrolled* pOCS where the weights of transitions are constant except the ones of client arrivals which are specified by positive non constant polynomials. Let $\nu \in (\Sigma^*)^{Ch}$. Then $|\nu|$ denotes $\sum_{c \in Ch} |\nu(c)|$.

Definition 11. *Let \mathcal{S} be a pOCS. Then \mathcal{S} is uncontrolled if:*

- For all $t = (q, c, a, c', a', q') \in \Delta$ with $a \neq \varepsilon$, $W(t, \nu)$ only depends on t and will be denoted $W(t)$;
- For all $t = (q, c_{in}, \varepsilon, c_{in}, \$, q)$, $W(t, \nu)$ is a positive non constant polynomial, whose single variable is $|\nu|$, and will be denoted $W_{in}(q, |\nu|)$.

The next proposition establishes that an uncontrolled pOCS generates a divergent Markov chain. This model illustrates the interest of divergence: while reachability of a pOCS is undecidable, we can apply Algorithm 1.

Proposition 5. *Let \mathcal{S} be a uncontrolled pOCS. Then $\mathcal{M}_{\mathcal{S}}$ is divergent.*

4.2 Probabilistic pushdown automata

Increasing pPDA We introduce the subset of *increasing pairs*, denoted as $Inc(\mathcal{A})$, which is a subset of $Q \times \Gamma$ that contains pairs (q, a) such that from state (q, wa) , the height of the stack can increase without decreasing before. When some conditions on $Inc(\mathcal{A})$ are satisfied, we obtain a syntactic *sufficient* condition for $\mathcal{M}_{\mathcal{A}}$ to be divergent.

Definition 12. Let $(q, a), (q', a') \in Q \times \Gamma$. Then (q', a') is reachable from (q, a) if either $(q, a) = (q', a')$ or there is a sequence of transitions of Δ , $(t_i)_{0 \leq i < d}$ such that: $t_i = q_i \xrightarrow{?a_i!a_{i+1}} q_{i+1}$, $(q_0, a_0) = (q, a)$, $(q_d, a_d) = (q', a')$ and for all i , $a_i \neq \varepsilon$. The set of increasing pairs $\text{Inc}(\mathcal{A}) \subseteq Q \times \Gamma$ is the set of pairs (q, a) that can reach a pair (q', a') with some $q' \xrightarrow{?a'!bc} q'' \in \Delta$.

$\text{Inc}(\mathcal{A})$ can be easily computed in polynomial time by a saturation algorithm.

Definition 13. A pPDA \mathcal{A} is increasing if:

- $\text{Inc}(\mathcal{A}) = Q \times \Gamma$;
- for all $t = q \xrightarrow{?a!w} q' \in \Delta$ such that $|w| \leq 1$, $W(t, -)$ is an integer constant denoted W_t ;
- for all $t = q \xrightarrow{?a!bc} q' \in \Delta$, $W(t, -)$ is a non constant integer polynomial where its single variable is the height of the stack denoted W_t ;
- for all $q \xrightarrow{?a} q' \in \Delta$, there exists $q \xrightarrow{?a!bc} q'' \in \Delta$.

Illustration. The right part of Figure 3 is an abstract view of a pPDA modelling of a server simultaneously handling multiple requests. The requests may occur at any time and are stored in the stack. The loop labelled by $?x!xy$ is a symbolic representation of several loops: one per triple (q, x, y) with $q \in Q$, $x \in \Gamma$ and $y \in \Gamma$. Due to the symbolic loop, the set of increasing pairs of the $pPDA_{\text{server}}$ is equal to $Q \times \Gamma$ and there is always a transition increasing the height of the stack outgoing from any (q, a) . Assume now that for any other transition, its weight does not depend on the size of the stack and that a transition $t = q \xrightarrow{?a!ab} q$ has weight $W_t(n) = c_t \times n$. Then \mathcal{A} is increasing. The dependance on n means that due to congestion, the time to execute tasks of the server increases with the number of requests in the system and thus increase the probability of a new request that occurs at a constant rate. One is interested in computing the probability to reach (q_f, ε) from (q_0, ε) representing the probability that the server reaches an idle state having served all the incoming requests.

The next proposition establishes that an increasing pPDA generates a divergent Markov chain.

Proposition 6. Let \mathcal{A} be an increasing pPDA. Then the Markov chain $\mathcal{M}_{\mathcal{A}}$ is divergent w.r.t. any s_0 and finite A .

5 Conclusion and perspectives

We have introduced the divergence property of Markov chains and designed two generic CRP-algorithms depending on the status of the reachability problem. Then we have studied the decidability of divergence for pPDA and for pPN for different kinds of weights and target sets. Finally, we have provided two useful classes of divergent models within pCS and pPDA.

In the future, we plan to study the model checking of polynomial pPDA (as a possible extension of [14]) and some heuristics to find functions f_0 and f_1 .

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6 Appendix

6.1 Proofs for divergent Markov chains (Section 2)

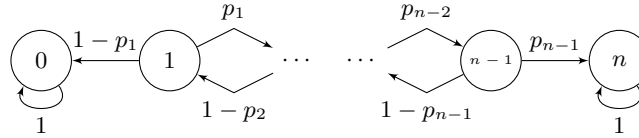


Fig. 4: The (generalized) gambler's ruin

Proposition 7. For all $0 \leq m \leq n$, let $preach_m$ be the probability to reach state 0 from state m in the Markov chain of Figure 4. Then:

$$preach_m = \frac{\sum_{m \leq j < n} \prod_{0 < k \leq j} \rho_k}{\sum_{0 \leq j < n} \prod_{0 < k \leq j} \rho_k}$$

with for all m , $\rho_m = \frac{1-p_m}{p_m}$.

Proof. $preach_0 = 1$, $preach_n = 0$ and for $0 < m < n$,

$$preach_m = p_m preach_{m+1} + (1 - p_m) preach_{m-1}$$

Thus for $0 < m < n$,

$$preach_m - preach_{m+1} = \rho_m (preach_{m-1} - preach_m)$$

Thus for $0 < m < n$,

$$preach_m - preach_{m+1} = \prod_{0 < k \leq m} \rho_k (preach_0 - preach_1)$$

So for $0 < m < n$,

$$preach_0 - preach_{m+1} = \sum_{0 \leq j \leq m} \prod_{0 < k \leq j} \rho_k (preach_0 - preach_1)$$

Let $m = n - 1$. Then:

$$1 = \sum_{0 \leq j < n} \prod_{0 < k \leq j} \rho_k (1 - preach_1)$$

implying

$$1 - preach_1 = \frac{1}{\sum_{0 \leq j \leq n} \prod_{0 < k \leq j} \rho_k}$$

and consequently for $0 \leq m \leq n$,

$$1 - preach_m = \frac{\sum_{0 \leq j < m} \prod_{0 < k \leq j} \rho_k}{\sum_{0 \leq j < n} \prod_{0 < k \leq j} \rho_k}$$

which can be rewritten for $0 \leq m \leq n$,

$$preach_m = \frac{\sum_{m \leq j < n} \prod_{0 < k \leq j} \rho_k}{\sum_{0 \leq j < n} \prod_{0 < k \leq j} \rho_k}$$

■

Proposition 8. *Let \mathcal{M}_1 be the Markov chain of Figure 1. Then \mathcal{M}_1 is recurrent if and only if:*

$$\sum_{n \in \mathbb{N}} \prod_{1 \leq m < n} \rho_m = \infty \text{ with } \rho_m = \frac{1 - p_m}{p_m},$$

and when transient, the probability that starting from m the random path visits 0 is equal to:

$$\frac{\sum_{m \leq n} \prod_{1 \leq m < n} \rho_m}{\sum_{n \in \mathbb{N}} \prod_{1 \leq m < n} \rho_m}.$$

Proof. Let $0 < m$ be some state of \mathcal{M}_1 and $n > m$. Consider the set of paths Ω_n that, starting from m visit state 0 without having previously visited state n and Ω_∞ the set of paths that, starting from m visit state 0. Obviously $(\Omega_n)_{n > m}$ is a non decreasing sequence of sets and $\bigcup_{n > m} \Omega_n = \Omega_\infty$. Thus $\mathbf{Pr}_{\mathcal{M}_1, m}(\Omega_\infty) = \lim_{n \rightarrow \infty} \mathbf{Pr}_{\mathcal{M}_1, m}(\Omega_n)$.

Observe now that $\mathbf{Pr}_{\mathcal{M}_1, m}(\Omega_n)$ is the probability in the Markov chain of Figure 4 that, starting from m the random path reaches 0, i.e., $\frac{\sum_{m \leq j < n} \prod_{0 < k \leq j} \rho_k}{\sum_{0 \leq j < n} \prod_{0 < k \leq j} \rho_k}$. Thus

$$\mathbf{Pr}_{\mathcal{M}_1, m}(\Omega_\infty) = \lim_{n \rightarrow \infty} \frac{\sum_{m \leq j < n} \prod_{0 < k \leq j} \rho_k}{\sum_{0 \leq j < n} \prod_{0 < k \leq j} \rho_k}.$$

This probability is equal to 1 if and only if $\sum_{n \in \mathbb{N}} \prod_{1 \leq m < n} \rho_m = \infty$ which is equivalent to the recurrence of \mathcal{M}_1 .

If \mathcal{M}_1 is transient then $\Pr_{\mathcal{M}_1, m}(\Omega_\infty) = \frac{\sum_{m \leq j} \prod_{0 \leq k \leq j} \rho_k}{\sum_{j \in \mathbb{N}} \prod_{0 \leq k \leq j} \rho_k}$ which concludes the proof. ■

Proposition 2. *Let \mathcal{M} be a divergent Markov chain with $s_0 \in S$, $A \subseteq S$ and $\theta > 0$. Then Algorithm 1 solves the CRP problem.*

Proof. Let us establish the termination of Algorithm 1. Any state s inserted in S' fulfills $\min(f_0(s), f_1(s)) > \frac{\theta}{2}$ or is reachable by a transition from such a state. Due to the third condition of divergence, there is a finite number of such states, the termination of Algorithm 1 is ensured.

Every state s inserted in $AlmostLoose_0$ fulfills $f_0(s) \leq \frac{\theta}{2}$. Due to the first item of divergence, $\Pr_{\mathcal{M}, s_0}(\mathbf{F}AlmostLoose_0) \leq \frac{\theta}{2}$. Since a path in \mathcal{M}' from s_0 that reaches $AlmostLoose_0$ is also a path in \mathcal{M} , $\Pr_{\mathcal{M}', s_0}(\mathbf{F}AlmostLoose_0) \leq \Pr_{\mathcal{M}, s_0}(\mathbf{F}AlmostLoose_0)$. Thus the interval returned by the algorithm has length at most θ . It remains to prove that $\Pr_{\mathcal{M}, s_0}(\mathbf{F}A)$ belongs to this interval. Since a path in \mathcal{M}' from s_0 that reaches $A' \subseteq A$ is also a path in \mathcal{M} , $pr_{\mathcal{M}}(A') \leq \Pr_{\mathcal{M}, s_0}(\mathbf{F}A)$. Consider in \mathcal{M} a path starting from s_0 that reaches A . There are three possible (exclusive) cases:

- either it is a path in \mathcal{M}' ;
- or the last state that it reaches in \mathcal{M}' is a state s fulfilling $f_0(s) \leq \frac{\theta}{2}$. The probability of such paths is thus bounded by $\Pr_{\mathcal{M}', s_0}(\mathbf{F}AlmostLoose_0)$;
- or the last state that it reaches in \mathcal{M}' is a state s fulfilling $f_0(s) > \frac{\theta}{2}$ and $f_1(s) \leq \frac{\theta}{2}$. For any such state the probability to reach A is at most $\frac{\theta}{2}$. Thus the cumulated probability of such paths is bounded by $\Pr_{\mathcal{M}', s_0}(\mathbf{F}AlmostLoose_1) \cdot \frac{\theta}{2}$.

Thus the upper bound of the interval returned by the algorithm is correct. Observe that the interval $[0, \theta]$ returned when $A' = \emptyset$ may be larger than the one that could have been returned but avoids the computation of the reachability probabilities in \mathcal{M}' . ■

When the reachability problem is decidable, one can modify the previous algorithm such that when A is reachable from s_0 , the algorithm returns an interval whose lower bound is strictly greater than 0. It first decides whether A is reachable from s_0 . In the positive case, by a breadth first exploration, it discovers a path from s_0 to A and decreases θ according to the values of f_0 and f_1 along this path. This ensures that the states of this path will belong to S' to be built later on. The remaining part of the algorithm is similar to Algorithm 1 except that using decidability of the reachability problem, we insert in $Loose$ the encountered states from which one cannot reach A . These states will also be made absorbing in \mathcal{M}' .

Proposition 9. *Algorithm 2 terminates and computes the interval $[0, 0]$ if A is unreachable from s_0 and otherwise an interval of length at most θ , not containing 0 and containing $\Pr_{\mathcal{M}, s_0}(\mathbf{F}A)$.*

Proof. The proof of correctness of Algorithm 2 is very similar to the one of Algorithm 1. There are only two observations to be done. First a path in \mathcal{M}

Algorithm 2: Framing the reachability probability when reachability is decidable

```

CompProb( $\mathcal{M}, s_0, A, \theta$ )
if not  $s_0 \rightarrow^* A$  then return (0, 0)
Insert( $Queue, s_0$ );  $S' \leftarrow \emptyset$ 
 $cont \leftarrow \mathbf{true}$ ;  $pere[s_0] \leftarrow s_0$ 
while  $cont$  do
     $s \leftarrow \text{Extract}(Queue)$ ;  $S' \leftarrow S' \cup \{s\}$ 
    if  $s \in A$  then  $cont \leftarrow \mathbf{false}$ 
    else if  $s \rightarrow^* A$  then
        for  $s \rightarrow s' \wedge s' \notin S'$  do Insert( $Queue, s'$ );  $pere[s'] \leftarrow s$ 
    end
end
 $\theta \leftarrow \min(\theta, f_0(s), f_1(s))$ 
while  $s \neq pere[s]$  do  $s \leftarrow pere[s]$ ;  $\theta \leftarrow \min(\theta, f_0(s), f_1(s))$ 
 $S' \leftarrow \emptyset$ ;  $AlmostLoose_0 \leftarrow \emptyset$ ;  $AlmostLoose_1 \leftarrow \emptyset$ 
 $Loose \leftarrow \emptyset$ ;  $Front \leftarrow \emptyset$ ; Insert( $Front, s_0$ )
while  $Front \neq \emptyset$  do
     $s \leftarrow \text{Extract}(Front)$ ;  $S' \leftarrow S' \cup \{s\}$ 
    if not  $s \rightarrow^* A$  then
         $Loose \leftarrow Loose \cup \{s\}$ 
    else if  $f_0(s) \leq \frac{\theta}{2}$  then
         $AlmostLoose_0 \leftarrow AlmostLoose_0 \cup \{s\}$ 
    else if  $f_1(s) \leq \frac{\theta}{2}$  then
         $AlmostLoose_1 \leftarrow AlmostLoose_1 \cup \{s\}$ 
    else if  $s \in A$  then
         $A' \leftarrow A' \cup \{s\}$ 
    else
        for  $s \rightarrow s' \wedge s' \notin S'$  do Insert( $Front, s'$ )
    end
end
 $Abs \leftarrow Loose \cup AlmostLoose_0 \cup AlmostLoose_1 \cup A'$ 
for  $s \in Abs$  do  $p'(s, s) \leftarrow 1$ 
for  $s \in S' \setminus Abs \wedge s' \in S'$  do  $p'(s, s') \leftarrow p(s, s')$ 
 $preach \leftarrow \text{CompFinProb}(\mathcal{M}', s_0)$ 
return
 $(preach(A'), preach(A') + preach(AlmostLoose_0) + \frac{\theta}{2} \cdot preach(AlmostLoose_1))$ 

```

that reaches a state of *Loose* without visiting before *A* cannot reach afterwards *A* which justifies the fact that they are absorbing in \mathcal{M}' and do not occur in the computation of the interval.

The additional preliminary stage when *A* is reachable from s_0 consists in (1) finding a reachability path from s_0 to *A* and then decreasing θ in such a way that no state along this path will be discarded during the main exploration. Since the path discovered during the first stage belong to \mathcal{M}' , the (non null) probability

of this path will lower bound the lower bound of the interval returned by the algorithm. ■

6.2 (Un)Decidability proofs (Section 3)

Theorem 1. *The divergence problem for pPDA is undecidable even with a single state and stack alphabet $\{a\}$.*

Proof. We will reduce the Hilbert's tenth problem to the divergence problem. Let $P \in \mathbb{Z}[X_1, \dots, X_k]$ be an integer polynomial with k variables. This problem asks whether there exist $n_1, \dots, n_k \in \mathbb{N}$ such that $P(n_1, \dots, n_k) = 0$.

We define \mathcal{A}_P as follows. There is a single state q . There are two transitions:

- $pop : q \xrightarrow{?a} q;$
- $push : q \xrightarrow{?a!aa} q;$
- $restart : q \xrightarrow{?\epsilon!a} q.$

The weights of pop and $restart$ are the constant function 1 while the weight of $push$ is defined by the following (non polynomial) function:

$$W(push, a^n) = \min(P^2(n_1, \dots, n_k) + 1 \mid n_1 + \dots + n_k \leq n)$$

This function is obviously computable. One studies the divergence w.r.t. $s_0 = (q, a)$ and $A = \{(q, \epsilon)\}$. Now the new Markov chain $\mathcal{M}_{\mathcal{A}_P}$ is isomorphic to the Markov chain \mathcal{M}_1 of Figure 1.

- Assume there exist $n_1, \dots, n_k \in \mathbb{N}$ s.t. $P(n_1, \dots, n_k) = 0$. Then $\mathcal{M}_{\mathcal{A}_P}$ is recurrent and if there are functions f_0, f_1 fulfilling the first two items of Definition 3, then these functions are necessarily the constant function 1, thus falsifying the last item of the definition for any $\theta < 1$. Thus $\mathcal{M}_{\mathcal{A}_P}$ is not divergent.
- Assume there do not exist $n_1, \dots, n_k \in \mathbb{N}$ such that $P(n_1, \dots, n_k) = 0$. Then $\mathcal{M}_{\mathcal{A}_P}$ fulfills the hypotheses of Proposition 1 with function f_1 defined by $f_1(q, a^n) = \frac{1}{2^n}$. ■

Theorem 2. *The divergence problem w.r.t. s_0 and finite A for polynomial pPDA with a single state and stack alphabet $\{a\}$ is decidable (in linear time).*

Proof. Let q be the single state of the pPDA. W.l.o.g. we assume that $s_0 = (q, a^{n_i})$ and $A = \{(q, a^{n_f})\}$ with $n_i > n_f$. The other cases either reduce to this one or do not present difficulties.

Such pPDA \mathcal{A} can be described as follows since self-loops over states do not change the behaviour of pPDA and the other cases are straightforward (no pop or no push). There are three transitions:

- $pop : q \xrightarrow{?a} q;$
- $push : q \xrightarrow{?a!aa} q;$
- $restart : q \xrightarrow{?\epsilon!a} q.$

The weight of transitions *pop* and *push* are polynomial depending on the number of a in the stack while the weight of transition *restart* is irrelevant since it is the single transition outgoing from (q, ε) . These will be denoted as W_{pop} and W_{push} . As for the previous theorem, we remark $\mathcal{M}_{\mathcal{A}}$ is isomorphic to \mathcal{M}_1 . Thus we introduce $\rho_n = \frac{W_{pop}(n)}{W_{push}(n)}$ for $n > 0$ and we will show how to decide whether $\mathcal{M}_{\mathcal{A}}$ is recurrent.

Let us write $W_{pop}(n) = \sum_{i \leq d} a_i n^i$ and $W_{push}(n) = \sum_{i \leq d'} a'_i n^i$ with coefficients in \mathbb{N} . We perform a case study analysis.

• When:

- $d' < d$
- or $d' = d$, $i_0 = \max(i \mid a_i \neq a'_i)$ exists and $a_{i_0} > a'_{i_0}$
- or $W_{push} = W_{pop}$

Then there exists n_0 such that for all $n \geq n_0$,

$W_{push}(n) \leq W_{pop}(n)$ implying $\rho_n \geq 1$.

Thus for all $n \geq n_0$, $\prod_{1 \leq m \leq n} \rho_m \geq \prod_{1 \leq m \leq n_0} \rho_m$ implying $\sum_{n \in \mathbb{N}} \prod_{1 \leq m \leq n} \rho_m = \infty$ yielding recurrence.

• **Case** $d' = d$ and $i_0 = \max(i \mid a_i \neq a'_i)$ exists and $a_{i_0} < a'_{i_0}$ and $i_0 \leq d - 2$. Then there exists n_0 and $\alpha > 0$ such that for all $n > n_0$, $\rho_n \geq 1 - \frac{\alpha}{n^2}$. Observe that:

$$\begin{aligned} \prod_{n_0 < m \leq n} \rho_m &\geq \prod_{n_0 < k \leq n} 1 - \frac{\alpha}{m^2} \geq \prod_{n_0 < m \leq n} e^{-\frac{2\alpha}{m^2}} \\ &= e^{-\sum_{n_0 < m \leq n} \frac{2\alpha}{m^2}} \geq e^{-\sum_{n_0 < m} \frac{2\alpha}{m^2}} > 0 \end{aligned}$$

Thus $\sum_{n \in \mathbb{N}} \prod_{1 \leq m \leq n} \rho_m = \infty$ yielding recurrence.

• **Case** $d' = d$ and $i_0 = \max(i \mid a_i \neq a'_i)$ exists, $i_0 = d - 1$ and $0 < \frac{a'_{d-1} - a_{d-1}}{a_d} \leq 1$.

Let $\alpha = \frac{a'_{d-1} - a_{d-1}}{a_d}$.

Then there exists n_0 and $\beta > 0$ such that for all $n > n_0$,

$\rho_n \geq 1 - \frac{\alpha}{n} - \frac{\beta}{n^2}$. Observe that:

$$\begin{aligned} \prod_{n_0 < m \leq n} \rho_m &\geq \prod_{n_0 < m \leq n} 1 - \frac{\alpha}{m} - \frac{\beta}{m^2} \\ &\geq \prod_{n_0 < m \leq n} e^{-\frac{\alpha}{m} - \frac{\beta}{m^2} - (\frac{\alpha}{m} - \frac{\beta}{m^2})^2} \\ &\geq \prod_{n_0 < m \leq n} e^{-\frac{\alpha}{m} - \frac{\beta'}{m^2}} \\ &\text{for some } \beta' \\ &= e^{-\sum_{n_0 < m \leq n} \frac{\alpha}{m} + \frac{\beta'}{m^2}} \\ &\geq e^{-\sum_{n_0 < m} \frac{\beta'}{m^2}} e^{-\alpha \log(n)} = \frac{e^{-\sum_{n_0 < m} \frac{\beta'}{m^2}}}{n^\alpha} \end{aligned}$$

Thus $\sum_{n \in \mathbb{N}} \prod_{1 \leq m \leq n} \rho_m = \infty$ yielding recurrence.

- **Case** $d' = d$ and $i_0 = \max(i \mid a_i \neq a'_i)$ exists, $i_0 = d - 1$ and $\frac{a'_{d-1} - a_{d-1}}{a_d} > 1$.

Let $\alpha = \frac{a'_{d-1} - a_{d-1}}{a_d}$.

Then there exists n_0 and $1 < \alpha' < \alpha$ such that for all $n \geq n_0$, $\rho_n \leq 1 - \frac{\alpha'}{n}$. Observe that:

$$\begin{aligned} \prod_{n_0 < m \leq n} \rho_m &\leq \prod_{n_0 < m \leq n} 1 - \frac{\alpha'}{m} \\ &\leq \prod_{n_0 < m \leq n} e^{-\frac{\alpha'}{m}} \\ &= e^{-\sum_{n_0 < m \leq n} \frac{\alpha'}{m}} \\ &\leq e^{-\alpha'(\log(n+1) - \log(n_0+1))} \\ &= \frac{e^{\alpha' \log(n_0+1)}}{(n+1)^{\alpha'}} \end{aligned}$$

Thus $\sum_{n \in \mathbb{N}} \prod_{1 \leq m \leq n} \rho_m < \infty$ implying non recurrence.

- When:
 - $d = d'$ and $a'_d > a_d$ or
 - $d < d'$.

Then there exists n_0 and $\alpha < 1$ such that for all $n \geq n_0$, $\rho_n \leq \alpha$. Thus $\sum_{n \in \mathbb{N}} \prod_{1 \leq m \leq n} \rho_m < \infty$ implying non recurrence.

This concludes the proof that the recurrence is decidable for this model. Let us focus on divergence.

- Assume that $\mathcal{M}_{\mathcal{A}}$ is recurrent. Then the probability to reach any state from (q, n_i) is 1 and the probability to reach from any state (q, n_f) is 1. if there are functions f_0, f_1 fulfilling the first two items of Definition 3, then these functions are necessarily the constant function 1, thus falsifying the last item of the definition for any $\theta < 1$. Thus $\mathcal{M}_{\mathcal{A}}$ is not divergent.
- Assume that $\mathcal{M}_{\mathcal{A}}$ is transient. The probability $preach(n)$ starting from (q, n) with $n > n_f$ to reach (q, n_f) is given by this formula.

$$preach(n) = \frac{\sum_{m \geq n} \prod_{n_f < k \leq m} \rho_k}{\sum_{m \geq n_f} \prod_{n_f < k \leq m} \rho_k}$$

Then an upper bound of $preach(n)$ which corresponds to function f of Proposition 1 is defined by :

- when $(d = d'$ and $a'_d > a_d)$ or $d < d'$, $\sum_{m \geq n} \alpha^{m-n_f} = \frac{\alpha^{n-n_f}}{1-\alpha}$;
- when $d' = d$ and $i_0 = \max(i \mid a_i \neq a'_i)$ exists,
 $i_0 = d - 1$ and $\frac{a'_{d-1} - a_{d-1}}{a_d} > 1$, $e^{\alpha' \log(n_0+1)} \sum_{m \geq n} \frac{1}{(m+1)^{\alpha'}} \leq \frac{e^{\alpha' \log(n_0+1)}}{(1-\alpha')n^{\alpha'-1}}$.

So $\mathcal{M}_{\mathcal{A}}$ is divergent.

Since recurrence of \mathcal{A} is decidable, divergence is also decidable. ■

We establish below that for polynomial pPNs, divergence is undecidable. We will proceed using a reduction of the following undecidable problem.

Let us recall that a two-counter (Minsky) machine \mathcal{C} is defined by a set of two counters $\{c_1, c_2\}$ and a set of $n+1$ instructions labelled by $\{0, \dots, n\}$, where for all $i < n$, the instruction i is of type

- either (1) $c_j \leftarrow c_j + 1$; **goto** i' with $j \in \{1, 2\}$ and $0 \leq i' \leq n$
- or (2) **if** $c_j > 0$ **then** $c_j \leftarrow c_j - 1$; **goto** i' **else goto** i'' with $j \in \{1, 2\}$ and $0 \leq i', i'' \leq n$

and the instruction n is **halt**. The program machine starts at instruction 0 and halts if it reaches instruction n .

The halting problem for two-counter machines asks, given a two-counter machine \mathcal{C} and initial values of counters, whether \mathcal{C} eventually halts. It is undecidable [20].

We introduce a subclass of two-counter machines that we call *normalized*. A normalized two-counter machine \mathcal{C} starts by resetting its counters and on termination resets its counters before halting.

In a normalized two-counter machine \mathcal{C} , given any initial values v_1, v_2 , \mathcal{C} halts with v_1, v_2 if and only if \mathcal{C} halts with initial values 0, 0. Moreover when \mathcal{C} halts, the values of the counters are null.

Normalized two counters machine. The two first instructions of a normalized two counters machine are:

- 0 : **if** $c_1 > 0$ **then** $c_1 \leftarrow c_1 - 1$; **goto** 0
 else goto 1
- 1 : **if** $c_2 > 0$ **then** $c_2 \leftarrow c_2 - 1$; **goto** 1
 else goto 2

The three last instructions of a normalized CM are:

- $n-2$: **if** $c_1 > 0$ **then** $c_1 \leftarrow c_1 - 1$; **goto** $n-2$
 else goto $n-1$
- $n-1$: **if** $c_2 > 0$ **then** $c_2 \leftarrow c_2 - 1$; **goto** $n-1$
 else goto n
- n : **halt**

For $2 < i < n-2$, the labels occurring in instruction i belong to $\{0, \dots, n-2\}$.

The halting problem for two-counter machines can be reduced to the halting problem for normalized two-counter machines, which is thus undecidable.

Lemma 1. *The halting problem of normalized counter machine is undecidable.*

Proof. Let \mathcal{C} be a two-counter machine with initial values v_1, v_2 , one builds the normalized two-counter machine \mathcal{C}_{v_1, v_2} by adding after the two first instructions of a normalized two-counter machine, v_1 incrementations of c_1 followed by v_2 incrementations of c_2 followed by the instructions \mathcal{C} where the halting instruction has been replaced by the last three instructions of a normalized two-counter machine. The normalized two-counter machine \mathcal{C}_{v_1, v_2} halts if and only if \mathcal{C} with initial values v_1, v_2 halts. ■

The following fact will be used in the next proofs. Let X be a random variable with range in \mathbb{N} and h be a strictly decreasing function from $\mathbb{R}_{\geq 0}$ to $\mathbb{R}_{\geq 0}$. Assume that for all $n \geq n_0$, $\Pr(X \geq n) \leq h(n)$. Then for all θ such that $\theta \leq h(n_0)$,

$$\begin{aligned} \Pr(h(X) \leq \theta) &= \Pr(X \geq h^{-1}(\theta)) \\ &= \Pr(X \geq \lceil h^{-1}(\theta) \rceil) \\ &\leq h(\lceil h^{-1}(\theta) \rceil) \leq \theta \end{aligned}$$

Theorem 3. *The divergence problem of polynomial pPNs w.r.t. an upward closed set is undecidable.*

Proof. We reduce the reachability problem of normalized two-counter machines to the divergence problem of polynomial pPN. Let \mathcal{C} be a normalized two-counter machine. The pPN $\mathcal{N}_{\mathcal{C}}$ is built as follows. Its set of places is $\{p_i \mid 0 \leq i \leq n\} \cup \{q_i \mid i \text{ is a test instruction}\} \cup \{c_j, c'_j \mid 1 \leq j \leq 2\} \cup \{init, sim, stop\}$. The initial marking is $\mathbf{m}_0 = init$ and the set A for divergence is defined by $A = \{\mathbf{m} \mid \mathbf{m}(p_0) \geq 1\}$.

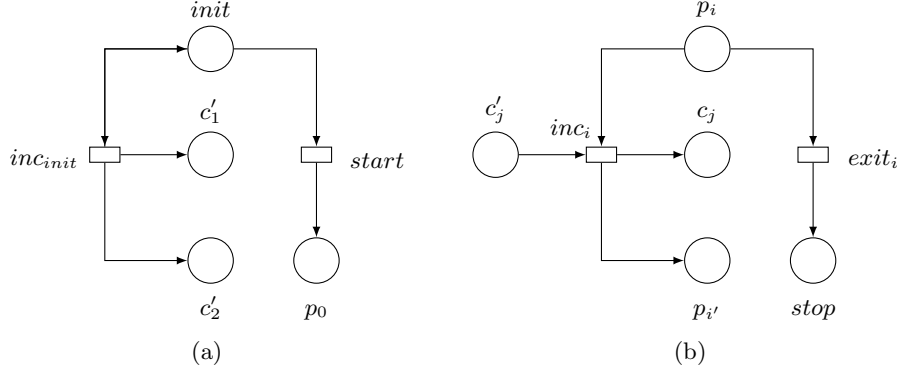


Fig. 5: Initialization stage (a); $i : c_j \leftarrow c_j + 1$; **goto** i' (b)

Places c'_1 and c'_2 are “complementary places” for the counters. During the initialization stage, transition inc_{init} repeatedly increments these places until transition $start$ is fired unmarking $init$. The weight of inc_{init} and $start$ is 1.

Afterwards the simulation proceeds as previously described with one main difference. For $j \in \{1, 2\}$, the sum of markings of c_j and c'_j remains constant fixed by the number of firings inc_{init} during the initialization stage. Thus the simulation of the i^{th} instruction when i increments c_j cannot proceed if c_j has reached this value letting only fireable transition $exit_i$.

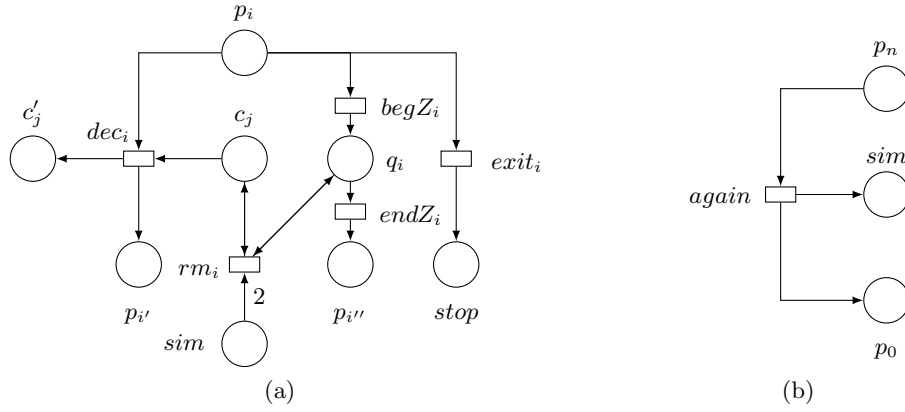


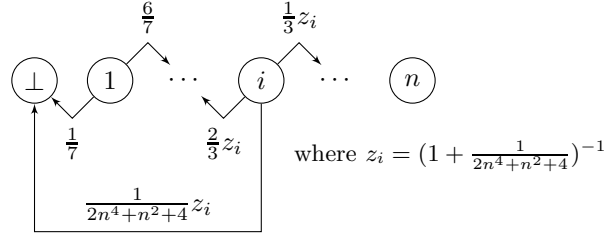
Fig. 6: i : if $c_j > 0$ then $c_j \leftarrow c_j - 1$; goto i' else goto i'' (a); halt instruction (b)

We will establish that $\mathcal{N}_{\mathcal{C}}$ is divergent w.r.t. to \mathbf{m}_0 and A if and only if \mathcal{C} does not halt.

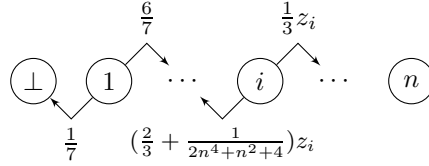
- Assume that \mathcal{C} halts and consider its execution ρ with initial values $(0, 0)$. Let $\ell = |\rho|$ the length of this execution. Consider now σ the infinite sequence of $\mathcal{N}_{\mathcal{C}}$ that first fires ℓ times inc_{init} , then fires $start$ and infinitely performs the correct simulation of this execution (possible due to the initialization stage). We have already established that the probability of σ is strictly greater than 0, say θ_0 . Observe that the set of markings visited during the simulation (say B) is infinite since after every simulation, sim is incremented. Moreover par definition of σ , for all $\mathbf{m} \in B$, $\Pr_{\mathcal{N}_{\mathcal{C}}, \mathbf{m}_0}(\mathbf{Fm}) \geq \theta_0$ and $\Pr_{\mathcal{N}_{\mathcal{C}}, \mathbf{m}}(\mathbf{FA}) \geq \theta_0$.

Assume by contradiction that $\mathcal{N}_{\mathcal{C}}$ is divergent w.r.t. to \mathbf{m}_0 and A and consider the corresponding functions f_0 and f_1 . Then $B \subseteq \{\mathbf{m} \mid f_0(\mathbf{m}) \geq \theta_0 \wedge f_1(\mathbf{m}) \geq \theta_0\} \cap Reach(\mathcal{N}_{\mathcal{C}}, \mathbf{m}_0)$ falsifying the third condition of divergence.

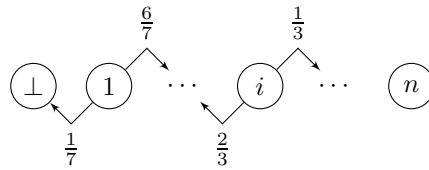
- Assume that \mathcal{C} does not halt. Due to the initialization stage for all $j \in \{1, 2\}$ the probability that a reachable marking \mathbf{m} fulfills $\mathbf{m}(c_j) + \mathbf{m}(c'_j) \geq n$ is exactly 2^{-n} . Let us bound the probability that a reachable marking \mathbf{m} fulfills $\mathbf{m}(sim) \geq n$. Recalling that to mark again p_0 after a necessarily faulty simulation, when sim is marked implies that rm_i can fire, the probabilistic behaviour of the marking $\mathbf{m}(sim)$ after every firing of $again$ or some $exit_i$ until it possibly reaches n or remains unchanged is depicted as follows.



Here we have made the most pessimistic assumptions over this behaviour i.e., (1) we have not considered the possibility of multiple firings of rm_i and (2) we have only considered the possibility of a single firing of $exit_i$ during a simulation. The state \perp of this random walk represents the fact that *stop* is marked (and so the marking of *sim* remains unchanged in the future). This random walk can be stochastically bounded by this one:



and then by this one:



Now applying the standard results of gambler's ruin, one gets that the probability that the marking of *sim* reaches n (for $n > 2$) is bounded by: $\frac{6}{2^{n-2}-1}$.

We now define function f_0 and show that it fulfills the hypotheses of Proposition 1. $f_0(\mathbf{m})$ is defined by:

- if $\mathbf{m}(sim) \leq 2$ then $f_0(\mathbf{m}) = 1$
- else $f_0(\mathbf{m}) = 2^{-(\mathbf{m}(c_1) + \mathbf{m}(c'_1))} + \frac{6}{2^{\mathbf{m}(sim)-2}-1}$.

Let us verify the conditions of divergence. Let $0 < \theta < 1$.

$$\begin{aligned}
& \Pr_{\mathcal{N}_C, \mathbf{m}_0}(\mathbf{F}(f_0(\mathbf{m}) \leq \theta)) \\
&= \Pr_{\mathcal{N}_C, \mathbf{m}_0}(\mathbf{F}(\mathbf{m}(\text{sim}) > 2 \\
&\quad \wedge 2^{-(\mathbf{m}(c_1) + \mathbf{m}(c'_1))} + \frac{6}{2^{\mathbf{m}(\text{sim})-2} - 1} \leq \theta)) \\
&\leq \Pr_{\mathcal{N}_C, \mathbf{m}_0}(\mathbf{F}((\mathbf{m}(\text{sim}) > 2 \wedge \frac{6}{2^{\mathbf{m}(\text{sim})-2} - 1} \leq \frac{\theta}{2}) \\
&\quad \vee (2^{-(\mathbf{m}(c_1) + \mathbf{m}(c'_1))} \leq \frac{\theta}{2}))) \\
&\leq \Pr_{\mathcal{N}_C, \mathbf{m}_0}(\mathbf{F}(\mathbf{m}(\text{sim}) > 2 \wedge \frac{6}{2^{\mathbf{m}(\text{sim})-2} - 1} \leq \frac{\theta}{2})) \\
&\quad + \Pr_{\mathcal{N}_C, \mathbf{m}_0}(\mathbf{F}(2^{-(\mathbf{m}(c_1) + \mathbf{m}(c'_1))} \leq \frac{\theta}{2})) \\
&\leq \frac{\theta}{2} + \frac{\theta}{2} = \theta
\end{aligned}$$

On the other hand $f_0(\mathbf{m}) \geq \theta$ implies that $\mathbf{m}(c_1)$, $\mathbf{m}(c'_1)$ and $\mathbf{m}(\text{sim})$ are bounded by some constant. Since for all reachable \mathbf{m} , $\mathbf{m}(c_2) + \mathbf{m}(c'_2) = \mathbf{m}(c_1) + \mathbf{m}(c'_1)$ and $\sum \mathbf{m}(p_i) + \sum \mathbf{m}(q_i) + \mathbf{m}(\text{stop}) = 1$, all places are bounded and thus the set of reachable markings \mathbf{m} such that $f_0(\mathbf{m}) \geq \theta$ is finite. ■

6.3 Proofs for the illustration of divergence (Section 4)

Theorem 5 ([19]). *Let \mathcal{M} be a Markov chain and f be a function from S to \mathbb{R} with $B = \{s \mid f(s) \leq 0\}$ fulfilling $\emptyset \subsetneq B \subsetneq S$ and $\varepsilon, K \in \mathbb{R}_{>0}$ such that:*

$$\begin{aligned}
& \text{for all } s \in S \setminus B \quad \sum_{s' \in S} p(s, s') f(s') \geq f(s) + \varepsilon \text{ and} \\
& \sum_{|f(s') - f(s)| \leq K} p(s, s') = 1
\end{aligned} \tag{2}$$

Then:

$$\text{for all } s \in S \setminus B \quad \Pr_{\mathcal{M}, s}(\mathbf{F}B) \leq c_1 e^{-c_2 f(s)}$$

where $c_1 = \sum_{n \geq 1} e^{-\frac{\varepsilon^2 n}{2(\varepsilon + K)^2}}$ and $c_2 = \frac{\varepsilon}{(\varepsilon + K)^2}$.

Which implies transience of \mathcal{M} when it is irreducible.

Theorem 4. *Let \mathcal{M} be a Markov chain and f be a function from S to \mathbb{R} with $B = \{s \mid f(s) \leq 0\}$ fulfilling $\emptyset \subsetneq B \subsetneq S$, $\varepsilon, K \in \mathbb{R}_{>0}$ and $d \in \mathbb{N}^*$ such that:*

$$\begin{aligned}
& \text{for all } s \in S \setminus B \quad \sum_{s' \in S} p^{(d)}(s, s') f(s') \geq f(s) + \varepsilon \text{ and} \quad \sum_{|f(s') - f(s)| \leq K} p(s, s') = 1
\end{aligned} \tag{1}$$

Then for all $s \in S$ such that $f(s) > dK$,

$$\Pr_{\mathcal{M}, s}(\mathbf{F}B) \leq c_1 e^{-c_2(f(s) - dK)}$$

where $c_1 = \sum_{n \geq 1} e^{-\frac{\varepsilon^2 n}{2(\varepsilon + K)^2}}$ and $c_2 = \frac{\varepsilon}{(\varepsilon + K)^2}$.

which implies transience of \mathcal{M} when it is irreducible.

Proof. Consider $\mathcal{M}^{(d)} = (S, p^{(d)})$.

Since $\sum_{|f(s')-f(s)| \leq dK} p^{(d)}(s, s') = 1$, $\mathcal{M}^{(d)}$ fulfills the conditions of Theorem 5 and so:

$$\text{for all } s \in S \setminus B \quad \Pr_{\mathcal{M}^{(d)},s}(\mathbf{F}B) \leq c_1 e^{-c_2 f(s)}$$

From this result one gets for all s such that $f(s) > \alpha$:

$$\Pr_{\mathcal{M}^{(d)},s}(\mathbf{F}\{s' \mid f(s') \leq \alpha\}) \leq c_1 e^{-c_2(f(s)-\alpha)}$$

Consider $d' \leq d$ transitions in \mathcal{M} , leading from $s \in S \setminus B$ to s' . Then $f(s) - f(s') \leq dK$ as long as B is not reached. So for all s , $\Pr_{\mathcal{M},s}(\mathbf{F}B) \leq \Pr_{\mathcal{M}^{(d)},s}(\mathbf{F}\{s' \mid f(s') \leq Kd\})$. ■

Proposition 3. *Let \mathcal{M} be a Markov chain and f be a computable function from S to \mathbb{R} with $B = \{s \mid f(s) \leq 0\}$ fulfilling $\emptyset \subsetneq B \subsetneq S$, and for some $\varepsilon, K \in \mathbb{R}_{>0}$ and $d \in \mathbb{N}^*$, Equation (1). Assume in addition that for all $n \in \mathbb{N}$, $\{s \mid f(s) \leq n\}$ is finite. Then \mathcal{M} is divergent w.r.t. any s_0 and any finite A .*

Proof. First observe that one can compute $c'_1 \geq c_1$ and $0 < c'_2 \leq c_2$ so that for all $s \in S$ such that $f(s) > dK$, $\Pr_{\mathcal{M},s}(\mathbf{F}B) \leq c_1 e^{-c_2(f(s)-dK)}$. Fix some s_0 and some finite A . Let $a = \max(0, \max_{s \in A}(f(s)))$.

Let $f_a(s) = f(s) - a$. Then $A \cup B \subseteq \{s \in S \mid f_a(s) \leq 0\}$. Since f_a is a translation of f , Equation (1) is satisfied for all $s \in S \setminus B$ and thus for all $s \in S \setminus \{s \mid f_a(s) \leq 0\}$. So one can apply Theorem 4 for f_a . Let us define $g(s) = c'_1 e^{-c'_2(f(s)-dK)}$ or more precisely some computable function such that $g(s) \leq 2c'_1 e^{-c'_2(f(s)-dK)}$. Then since $A \subseteq \{s \mid f_a(s) \leq 0\}$, $\Pr_{\mathcal{M},s}(\mathbf{F}A) \leq g(s)$. Furthermore since for all $n \in \mathbb{N}$, $\{s \mid f(s) \leq n\}$ is finite, we deduce that for all $\theta > 0$, $\{s \mid g(s) \geq \theta\}$ is finite. So g fulfills the condition of Proposition 1 and \mathcal{M} is divergent w.r.t. s_0 and A . ■

We recall some definitions related to martingale theory. Let (Ω, \mathcal{F}) and (S, \mathcal{E}) be sets equipped with a σ -algebra. A (discrete-time) stochastic process is a sequence $(X_n)_{n \in \mathbb{N}}$ of random variables from (Ω, \mathcal{F}) to (S, \mathcal{E}) . A filtration of \mathcal{F} is a non decreasing sequence $(\mathcal{F}_n)_{n \in \mathbb{N}}$ of σ -algebras included in \mathcal{F} . A stochastic process $(X_n)_{n \in \mathbb{N}}$ is adapted to a filtration $(\mathcal{F}_n)_{n \in \mathbb{N}}$ if for all $n \in \mathbb{N}$, X_n is \mathcal{F}_n -measurable. Let τ be a random variable from (Ω, \mathcal{F}) to $\mathbb{N} \cup \{\infty\}$. τ is a stopping time if for all $n \in \mathbb{N}$, the event $\tau = n$ belongs to \mathcal{F}_n . Let $n \in \mathbb{N}$, we denote $n \wedge \tau$, the random variable $\min(n, \tau)$.

Definition 14. *Let $(X_n)_{n \in \mathbb{N}}$ be a real-valued stochastic process adapted to a filtration $(\mathcal{F}_n)_{n \in \mathbb{N}}$. $(X_n)_{n \in \mathbb{N}}$ is a submartingale if for all $n \in \mathbb{N}$, $\mathbf{E}(X_{n+1} - X_n \mid \mathcal{F}_n) \geq 0$.*

Below we recall a result about transience of Markov chains based on martingale theory. We will use this classic lemma in order to establish Theorem 6.

Lemma 2 (Azuma-Hoeffding inequality[19]). *Let $(X_k)_{k \in \mathbb{N}}$ be a submartingale and $\{c_k\}_{k \geq 1}$ be positive reals such that for all $k \geq 1$, $|X_k - X_{k-1}| \leq c_k$. Let $a > 0$. Then for all $n \geq 0$:*

$$\Pr(X_n - X_0 \leq -a \mid \mathcal{F}_0) \leq e^{\frac{-a^2}{2 \sum_{k=1}^n c_k^2}}$$

The interest of the next theorem lies in the fact that f can now decrease by a non constant factor i.e., some $O(f(s)^\alpha)$ for $0 < \alpha < \frac{1}{2}$. Since we do not use it in the illustration of Section 4, we omit the corresponding sufficient condition for divergence. Observe that $\sum_{n \geq 1} \gamma^{n^{1-2\alpha}} < \infty$ since $0 < \gamma < 1$ and $1 - 2\alpha > 0$.

Theorem 6. *Let \mathcal{M} be a Markov chain and f be a function from S to \mathbb{R} with $B = \{s \mid f(s) \leq 0\}$ fulfilling $\emptyset \subsetneq B \subsetneq S$, $0 < \varepsilon \leq 1$, $K \geq 2$, $K' \geq 0$ and $0 \leq \alpha < \frac{1}{2}$ such that:*

$$\begin{aligned} \text{for all } s \in S \setminus B \quad f(s) + \varepsilon &\leq \sum_{s' \in S} p(s, s') f(s') \quad \text{and} \\ \sum_{-K - K' f(s)^\alpha \leq f(s') - f(s) \leq K} p(s, s') &= 1 \end{aligned} \tag{3}$$

Then for all s such that $f(s) \geq 2$:

$$\Pr_{\mathcal{M}, s}(\mathbf{FB}) \leq \beta^{f(s)^{1-2\alpha}} \sum_{n \geq 1} \gamma^{n^{1-2\alpha}}$$

where $0 < \beta, \gamma < 1$ are defined by:

$$\beta = e^{-\left(\frac{2(\varepsilon+K)K^\alpha}{\varepsilon} + \frac{4K'(\varepsilon+K)K^\alpha}{\varepsilon^{1+\alpha}(1+\alpha)} + \frac{2K'^2 K^{2\alpha}}{\varepsilon^{1+2\alpha}}(1+2\alpha)\right)^{-1}} \quad \text{and} \quad \gamma = \beta^{\varepsilon^{1-2\alpha}}$$

which implies transience of \mathcal{M} when it is irreducible.

Proof. Pick some $s_0 \in S \setminus B$. Due to Equation (3), there exists s_1 with $f(s_1) \geq f(s_0) + \varepsilon$ and by induction for all $n \in \mathbb{N}$, there exists s_n with $f(s_n) \geq f(s_0) + n\varepsilon$. Thus $\lim_{n \rightarrow \infty} f(s_n) = \infty$. Assume that the inequation of the conclusion holds, there exists some n_0 with $\Pr_{\mathcal{M}, s_{n_0}}(\mathbf{FB}) < 1$. If \mathcal{M} is irreducible, this establishes transience of \mathcal{M} .

Let us establish this inequation. Let $a = f(X_0) \geq 2$. Let τ be the stopping time associated with the entry in B . Let $Y_n = f(X_n) - n\varepsilon$. Equation (3) implies that $Y_{n \wedge \tau}$ is a submartingale. Furthermore Equation (3) implies that:

$$Y_{n \wedge \tau} \leq a + nK$$

and so $|Y_{n+1 \wedge \tau} - Y_{n \wedge \tau}| \leq \varepsilon + K + K'(a + nK)^\alpha$.

Observe that:

$$\begin{aligned} \Pr(\tau < \infty) &= \Pr\left(\bigcup_{n \geq 1} f(X_{n \wedge \tau}) - f(X_0) \leq -a\right) \\ &= \Pr\left(\bigcup_{n \geq 1} Y_{n \wedge \tau} - Y_0 \leq -a - n\varepsilon\right) \\ &\leq \sum_{n \geq 1} \Pr(Y_{n \wedge \tau} - Y_0 \leq -a - n\varepsilon) \end{aligned}$$

Applying Lemma 2, one gets:

$$\begin{aligned}
& \Pr(\tau < \infty) \\
& \leq \sum_{n \geq 1} e^{-\frac{(a+n\varepsilon)^2}{2 \sum_{k=1}^n (\varepsilon+K+K'(a+kK)^\alpha)^2}} \\
& = \sum_{n \geq 1} e^{-\frac{(a+n\varepsilon)^2}{2n(\varepsilon+K)^2 + \sum_{k=1}^n 4(\varepsilon+K)K'(a+kK)^\alpha + 2K'^2 \sum_{k=1}^n (a+kK)^{2\alpha}}}
\end{aligned}$$

Observe that for $u \geq 0$ and $\theta, v > 0$,

$$\sum_{k=1}^n (u + vk)^\theta \leq \int_1^{n+1} (u + vx)^\theta dx \leq \frac{(u+v(n+1))^{1+\theta}}{v(1+\theta)}.$$

Thus:

$$\begin{aligned}
& \Pr(\tau < \infty) \\
& \leq \sum_{n \geq 1} e^{-\frac{(a+n\varepsilon)^2}{2n(\varepsilon+K)^2 + \frac{4K'(\varepsilon+K)}{K(1+\alpha)}(a+(n+1)K)^{1+\alpha} + \frac{2K'^2}{K(1+2\alpha)}(a+(n+1)K)^{1+2\alpha}}}
\end{aligned}$$

Since $n \leq \frac{a+n\varepsilon}{\varepsilon}$ and $a + (n+1)K \leq \frac{K}{\varepsilon}(a+n\varepsilon)$,

$$\begin{aligned}
& \Pr(\tau < \infty) \\
& \leq \sum_{n \geq 1} e^{-\frac{(a+n\varepsilon)^2}{\frac{2(\varepsilon+K)^2(a+n\varepsilon)}{\varepsilon} + \frac{4K'(\varepsilon+K)K^\alpha(a+n\varepsilon)^{1+\alpha}}{\varepsilon^{1+\alpha}(1+\alpha)} + \frac{2K'^2K^{2\alpha}(1+2\alpha)(a+n\varepsilon)^{1+2\alpha}}{\varepsilon^{1+2\alpha}}}}
\end{aligned}$$

After dividing numerator and denominator by $(a+n\varepsilon)^2$,

$$\begin{aligned}
& \Pr(\tau < \infty) \\
& \leq \sum_{n \geq 1} e^{-\frac{1}{\frac{2(\varepsilon+K)}{\varepsilon(a+n\varepsilon)} + \frac{4K'(\varepsilon+K)K^\alpha}{\varepsilon^{1+\alpha}(1+\alpha)(a+n\varepsilon)^{1-\alpha}} + \frac{2K'^2K^{2\alpha}(1+2\alpha)}{\varepsilon^{1+2\alpha}(a+n\varepsilon)^{1-2\alpha}}}}
\end{aligned}$$

Since $a + n\varepsilon \geq 1$,

$$\begin{aligned}
& \Pr(\tau < \infty) \\
& \leq \sum_{n \geq 1} e^{-\frac{1}{\left(\frac{2(\varepsilon+K)}{\varepsilon} + \frac{4K'(\varepsilon+K)K^\alpha}{\varepsilon^{1+\alpha}(1+\alpha)} + \frac{2K'^2K^{2\alpha}}{\varepsilon^{1+2\alpha}}(1+2\alpha)\right)(a+n\varepsilon)^{2\alpha-1}}} \\
& = \sum_{n \geq 1} e^{-\frac{(a+n\varepsilon)^{1-2\alpha}}{\left(\frac{2(\varepsilon+K)K^\alpha}{\varepsilon} + \frac{4K'(\varepsilon+K)K^\alpha}{\varepsilon^{1+\alpha}(1+\alpha)} + \frac{2K'^2K^{2\alpha}}{\varepsilon^{1+2\alpha}}(1+2\alpha)\right)}}
\end{aligned}$$

So:

$$\begin{aligned}
\Pr(\tau < \infty) &\leq \sum_{n \geq 1} \beta^{(a+n\varepsilon)^{1-2\alpha}} \\
&\leq \sum_{n \geq 1} \beta^{a^{1-2\alpha} + (n\varepsilon)^{1-2\alpha}} \\
&= \left(\sum_{n \geq 1} \beta^{(n\varepsilon)^{1-2\alpha}} \right) \beta^{a^{1-2\alpha}} \\
&= \left(\sum_{n \geq 1} \gamma^{n^{1-2\alpha}} \right) \beta^{a^{1-2\alpha}}
\end{aligned}$$

■

Channel systems are a classical model for protocols where components communicate asynchronously via FIFO channels [5,4]. In the following, let Σ be an alphabet. One denotes $\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$.

Definition 15. A channel system (CS) $\mathcal{S} = (Q, Ch, \Sigma, \Delta)$ is defined by:

- a finite set Q of states;
- a finite set Ch of channels;
- a finite alphabet Σ ;
- a transition relation $\Delta \subseteq Q \times Ch \times \{!, ?\} \times \Sigma_\varepsilon \times Q$;

Semantic. A configuration of \mathcal{S} is a pair $(q, \nu) \in Q \times (\Sigma^*)^{Ch}$. Let $t = (q, c, ?, a, q')$ and (q, ν) where $\nu(c) = aw$. Then $(q, \nu) \xrightarrow{t} (q', \nu')$ with $\nu'(c) = w$ and for all $c' \neq c$, $\nu'(c) = \nu(c)$. Let $t = (q, c, !, a, q')$ and (q, ν) . Then $(q, \nu) \xrightarrow{t} (q', \nu')$ with $\nu'(c) = \nu(c)a$ and for all $c' \neq c$, $\nu'(c) = \nu(c)$. The reachability relation is defined as usual and the reachability problem of CS is undecidable.

Notation. To precise that a transition occurs in a (pO)CS \mathcal{C} , we add the subscript \mathcal{C} to \rightarrow (and to \rightarrow^* , its transitive closure).

Proposition 4. The reachability problem of pOCS is undecidable.

Proof. We reduce the reachability problem of CS to the one of pOCS. Let \mathcal{C} be a CS where w.l.o.g. $c_{in} \notin Ch$ and $\$ \notin \Sigma$. Then the pOCS \mathcal{C}' is defined as follows:

- $Q' = Q$; $Ch' = Ch \cup \{c_{in}\}$; $\Sigma' = \Sigma \cup \{\$\}$;
- For all $(q, c, ?, a, q') \in \Delta$, $(q, c, a, c, \varepsilon, q') \in \Delta'$;
- For all $(q, c, !, a, q') \in \Delta$, $(q, c_{in}, \$, c, a, q') \in \Delta'$;
- For all $q \in Q$, $(q, c_{in}, \varepsilon, c_{in}, \$, q) \in \Delta'$.

Consider the relations \sim and \sim_0 between a configuration (q, ν) of \mathcal{C} and a configuration (q', ν') of \mathcal{C}' defined by (1) $(q, \nu) \sim (q', \nu')$ if $q = q'$ and for all $c \in Ch$, $\nu(c) = \nu'(c)$, and (2) $(q, \nu) \sim_0 (q', \nu')$ if $(q, \nu) \sim (q', \nu')$ and $\nu'(c_{in}) = \varepsilon$.

Due to the definition of \mathcal{C}' , for all $(q, \nu) \sim_0 (q', \nu')$, $(q, \nu) \xrightarrow{t}_{\mathcal{C}} (q_1, \nu_1)$ there exists (q'_1, ν'_1) such that $(q_1, \nu_1) \sim_0 (q'_1, \nu'_1)$ and $(q', \nu') \xrightarrow{*}_{\mathcal{C}'} (q'_1, \nu'_1)$.

Similarly, for all $(q, \nu) \sim (q', \nu')$, $(q', \nu') \xrightarrow{t}_{\mathcal{C}'} (q'_1, \nu'_1)$ there exists (q_1, ν_1) such that $(q_1, \nu_1) \sim (q'_1, \nu'_1)$ and $(q, \nu) \rightarrow_{\mathcal{C}}^* (q_1, \nu_1)$.
 Now consider (q_0, ν_0) (resp. (q_f, ν_f)) a configuration of \mathcal{C} and (q'_0, ν'_0) (resp. (q'_f, ν'_f)) the single configuration of \mathcal{C}' such that $(q_0, \nu_0) \sim_0 (q'_0, \nu'_0)$ (resp. $(q_f, \nu_f) \sim_0 (q'_f, \nu'_f)$). Then due to the simulation properties above, (q_f, ν_f) is reachable from (q_0, ν_0) in \mathcal{C} iff (q'_f, ν'_f) is reachable from (q'_0, ν'_0) in \mathcal{C}' . ■

Proposition 5. *Let \mathcal{S} be an uncontrolled pOCS. Then $\mathcal{M}_{\mathcal{S}}$ is divergent.*

Proof. We show that the conditions of Proposition 3 are satisfied with $d = 1$. Let n_0 be an integer such that for all $n \geq n_0$,

$$\min_{q \in Q} (W_{in}(q, n)) \geq \sum_{t=(q,c,a,c',a',q') \in \Delta \wedge a' \neq \$} W(t) + 1$$

Let $f(q, \nu) = |\nu|$. Then Equation 1 is satisfied with $\varepsilon = 1$ and $K = 1$. Furthermore let $n \in \mathbb{N}$, $\{(q, \nu) \mid |\nu| \leq n\}$ is finite. Thus $\mathcal{M}_{\mathcal{S}}$ is divergent. ■

Notation. $B_{\mathcal{A}} = \max(W_t \mid t = q \xrightarrow{?a!w} q' \in \Delta \wedge |w| \leq 1)$.

Proposition 6. *Let \mathcal{A} be an increasing pPDA. Then the Markov chain $\mathcal{M}_{\mathcal{A}}$ is divergent w.r.t. any s_0 and finite A .*

Proof. Due to the condition on W for increasing \mathcal{A} , there exists some n_0 such that for all $n \geq n_0$,

$$2|\Delta|B_{\mathcal{A}} \leq \min(W_t(n) \mid t = q \xrightarrow{?a!bc} q' \in \Delta)$$

Let $f(q, w) = |w|$ and $d = |Q||\Gamma|$. Let us show by induction on $d' \leq d$ that for all (q, w) such that $|w| \geq n_0 + d'$,

$$\sum_{(q', w')} p^{(d')}((q, w), (q', w')) f(q', w') \geq f(q, w) \quad (4)$$

and if a pair (q', a') with a transition $q' \xrightarrow{?a'!bc} q''$ is reachable from pair $(q, w[|w|])$ in less than d' steps then:

$$\sum_{(q', w')} p^{(d')}((q, w), (q', w')) f(q', w') \geq f(q, w) + \frac{1}{3B_{\mathcal{A}}^{d'}} \quad (5)$$

The basis case $d' = 0$ is immediate. Assume the property is satisfied for some d' . There are two cases to be considered. Let $a = w[|w|]$. Either (1) there is no $q \xrightarrow{?a} q' \in \Delta$ or (2) there is some $q \xrightarrow{?a} q' \in \Delta$.

Case (1) Since the size of the stack cannot decrease after a step and all successors of (q, w) satisfy Equation (4) w.r.t. d' , Equation (4) is satisfied for (q, w) w.r.t. $d' + 1$. Assume there exists a pair (q', a') with a transition $q' \xrightarrow{?a'!bc} q''$ reachable from (q, a) in less than $d' + 1$ steps. Then there is a successor (q_1, w_1)

of (q, w) such that (q', a') is reachable from $(q_1, w_1[|w_1|])$ in less than d' steps. Thus Equation (5) is satisfied for (q_1, w_1) w.r.t. d' .

Since $p((q, w), (q_1, w_1)) \geq \frac{1}{B_{\mathcal{A}}}$,

Equation (5) is satisfied for (q, w) w.r.t. $d' + 1$.

Case (2) There is some $q \xrightarrow{?a} q'' \in \Delta$. So there is some $q \xrightarrow{?a!bc} q' \in \Delta$ implying that there is a successor (q', w') of (q, w) with $|w'| = |w| + 1$. Every other successor (q_1, w_1) fulfill $|w_1| \geq |w| - 1$. Whatever the successor, they all satisfy Equation (4) w.r.t. d' . Since $n \geq n_0$, $p((q, w), (q', w')) \geq \frac{2}{3}$,

$$\sum_{(q'', w'')} p^{(d'+1)}((q, w), (q', w')) f(q'', w'') \geq f(q, w) + \frac{1}{3}$$

which achieves the proof of this case.

By definition of increasing pPDA, from all (q, a) there exists a pair (q', a') with a transition $q' \xrightarrow{?a'!bc} q''$ reachable from pair (q, a) in less than d steps. So for all (q, w) such that $|w| \geq n_0 + d$,

$$\sum_{(q', w')} p^{(d)}((q, w), (q', w')) f(q', w') \geq f(q, w) + \varepsilon$$

with $\varepsilon = \frac{1}{3B_{\mathcal{A}}^d}$.

Observe that in d steps, the size of the stack can change by at most d and that the number of configurations with size of stacks bounded by some n is finite. So the hypotheses of Proposition 3 are satisfied and $\mathcal{M}_{\mathcal{A}}$ is divergent. ■