Two logical hierarchies of optimization problems over the real numbers

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Dedicated to Professor Günter Asser on the occasion of his eightieth birthday

We introduce and study certain classes of optimization problems over the real numbers. The classes are defined by logical means, relying on metafinite model theory for so called $\mathbb{R}$-structures (see [12, 11]). More precisely, based on a real analogue of Fagin's theorem [12] we deal with two classes $\mathrm{MAX\text{-}NP}_{\mathbb{R}}$ and $\mathrm{MIN\text{-}NP}_{\mathbb{R}}$ of maximization and minimization problems, respectively, and figure out their intrinsic logical structure. It is proven that $\mathrm{MAX\text{-}NP}_{\mathbb{R}}$ decomposes into four natural subclasses, whereas $\mathrm{MIN\text{-}NP}_{\mathbb{R}}$ decomposes into two. This gives a real number analogue of a result by Kolaitis and Thakur [13] in the Turing model. Our proofs mainly use techniques from [17]. Finally, approximation issues are briefly discussed.

1 Introduction

Many important problems in mathematics and computer science appear as optimization problems. In the framework of complexity theory a huge number of such problems is studied in relation with the class NPO of combinatorial optimization problems with an exponential search space.

There are at least two directions with respect to studying problems in NPO. The first deals with the study of approximability properties of NP-hard problems in NPO, leading to the consideration of important subclasses of NPO such as APX, PTAS, FPTAS, see [1]. The other direction is descriptive complexity theory [14]. Here, optimization problems are studied from a logical definability standpoint. Concerning combinatorial optimization this line was started in a paper by Papadimitriou and Yannakakis [19]. They defined a subclass MAX-NP of NPO by logical means and studied (among other things) this class with respect to approximation algorithms.

A main aspect of all these problems dealt with in Turing (descriptive) complexity theory is that the space of feasible solutions for a given problem instance has at most exponential cardinality and thus is finite. However, many important optimization problems involving real numbers have an uncountable search space. A typical example is given by determining for a given set of real multivariate polynomials the maximal number of polynomials that have a common real zero. This problem, for example, plays a crucial role in fundamental algorithms for semi-algebraic problems like quantifier elimination, see [20, 2].

It is natural to ask whether a similar logical framework to the one mentioned above can be developed for such real number optimization problems. A real number model of computation together with a complexity theory was developed by Blum, Shub and Smale [4]. A descriptive complexity theory for the Blum-Shub-Smale (shortly: BSS) model was introduced in [12] and further studied in [7]. It is based on so-called meta-finite model theory introduced in [11]. Most of the fundamental real number complexity classes have been expressed logically using that approach. On the other side, a real number analogue of the class NPO has not been studied thoroughly so far (see some related remarks in [18]).

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In this paper we start such an investigation from a logical point of view. We define a class $\mathrm{NPO}_{\mathbb{R}}$ of certain real optimization problems. Then we focus on the definition and analysis of real number analogues of the hierarchies given in [13]. The techniques and problems we use to separate the different classes in these hierarchies are closely related to those used for the study of real number counting problems in [17]. Let us mention that the complexity of such counting problems in the BSS model recently was analyzed in some papers by Bürgisser and Cucker, see [5].

Our paper is organized as follows. Section 2 gives a short introduction of the BSS model and recalls basic concepts from meta-finite model theory for $\mathbb{R}$-structures. A real number version of NPO is defined. Using descriptive complexity theory over $\mathbb{R}$ we then introduce the central logical problem classes of this paper: $\mathrm{MAX}\text{-}\Sigma_{i,\mathbb{R}}$ and $\mathrm{MAX}\text{-}\Pi_{i,\mathbb{R}}$ for maximization problems as well as $\mathrm{MIN}\text{-}\Sigma_{i,\mathbb{R}}$ and $\mathrm{MIN}\text{-}\Pi_{i,\mathbb{R}}$ for minimization problems ($i = 0, 1, 2$). Some basic properties of these classes are listed. Sections 3 and 4 build the main part of the paper. We consider several natural real number optimization problems, express them in a logical manner and use them to separate the classes of our hierarchies. In Section 3 this is done for maximization problems, in Section 4 for minimization. Finally, we discuss briefly approximation issues for some of the problem classes.

2 Basic definitions and results

In this section we recall briefly the BSS model and descriptive complexity theory for real number problems, see [3, 12].

2.1 The BSS model; $\mathbb{R}$-structures and their logics

In the BSS model over $\mathbb{R}$ real numbers are considered as entities. The basic arithmetic operations $+, -, *, :$ can be performed at unit costs, and there is a test-operation "is $x \geq 0$?" reflecting the underlying ordering of the reals. Decision problems now are subsets $L \subseteq \mathbb{R}^\infty := \bigcup_{n=1}^{\infty} \mathbb{R}^n$. The (algebraic) size of a point $x \in \mathbb{R}^n$ is n. Having fixed these notions it is easy to define real analogues $\mathrm{P}_{\mathbb{R}}$ and $\mathrm{NP}_{\mathbb{R}}$ of the classes P and NP as well as the notion of $\mathrm{NP}_{\mathbb{R}}$-completeness. For more details on the BSS model we refer to [3].

Next, let us briefly recall the concept of meta-finite structures over the reals, called $\mathbb{R}$-structures, which were introduced in [12] and [11]. We assume the reader to be familiar with basic notions of finite model theory, see for example [8].

Definition 2.1 Let L_s, L_f be finite vocabularies, where L_s may contain relation and function symbols, and L_f contains function symbols only. An $\mathbb{R}$-*structure of signature* $\sigma = (L_s, L_f)$ is a pair $\mathfrak{D} = (\mathcal{A}, \mathcal{F})$ consisting of

(i) a finite structure $\mathcal{A}$ of vocabulary L_s, called *the skeleton of* $\mathfrak{D}$, whose universe A will also be said to be *the universe of* $\mathfrak{D}$, and

(ii) a finite set $\mathcal{F}$ of functions $X : A^k \longrightarrow \mathbb{R}$ interpreting the function symbols in L_f.

Fix a countable set $V = \{v_0, v_1, \ldots\}$ of variables. These variables range only over the skeleton; we do not use element variables taking values in $\mathbb{R}$.

Example 2.2 Consider a directed graph G with real-valued weights on edges. Such graphs can be encoded using the following signature, if A is a finite universe that contains the vertices in G: $L_s = \{E\}$; $E(u, v)$ if and only if there is a directed edge from vertex u to vertex v in G; $L_f = \{W\}$, $W : A^2 \longrightarrow \mathbb{R}$, $W(u, v)$ is the weight of the edge from vertex u to vertex v in G. Note that it is irrelevant whether W is defined for edges not present in G or not.

Definition 2.3 The language $\mathrm{FO}_{\mathbb{R}}$ contains, for each signature $\sigma = (L_s, L_f)$ a set of formulas and terms. Each term t takes, when interpreted in some $\mathbb{R}$-structure, values in either the skeleton, in which case we call it an *index term*, or in $\mathbb{R}$, in which case we call it a *number term*. Terms are defined inductively as follows:

(i) The set of index terms is the closure of the set V of variables under applications of function symbols of L_s.

(ii) Any real number is a number term.

(iii) If $h_1, \ldots, h_k$ are index terms and X is a k-ary function symbol of L_f, then $X(h_1, \ldots, h_k)$ is a number term.

(iv) If t, t' are number terms, then so are $t + t'$, $t - t'$, $t \times t'$, t/t' and $\mathrm{sign}(t)$.

Atomic formulas are equalities $h_1 = h_2$ of index terms, equalities $t_1 = t_2$ and inequalities $t_1 < t_2$ of number terms, and expressions $P(h_1, \ldots, h_k)$, where P is a k-ary predicate symbol in L_s and $h_1, \ldots, h_k$ are index terms.

The set of formulas of $\mathrm{FO}_{\mathbb{R}}$ is the smallest set containing all atomic formulas and which is closed under Boolean connectives and quantification $\exists v\,\psi$ and $\forall v\,\psi$. Note that we do *not* consider formulas $\exists x\,\psi$ where x ranges over $\mathbb{R}$. Neither do we consider functions from $\mathbb{R}$ to $\mathbb{R}$.

Example 2.4 Consider the following first-order formula over the signature given in Example 2.2:

$$\forall x \forall y\,(E(x,y) \Rightarrow W(x,y) \geq 0).$$

Clearly a graph G is a model of this formula if and only if G has no edges with negative weight.

In order to describe the class $\mathrm{NP}_{\mathbb{R}}$ using logic over $\mathbb{R}$-structures it turns out to be fruitful also considering an extension of first-order logic.

Definition 2.5 We say that ψ *is an existential second-order sentence* (*of signature* $\sigma = (L_s, L_f)$) if

$$\psi = \exists Y_1 \cdots \exists Y_r\,\varphi,$$

where φ is a first-order sentence in $\mathrm{FO}_{\mathbb{R}}$ of signature $(L_s, L_f \cup \{Y_1, \ldots, Y_r\})$. The symbols $Y_1, \ldots, Y_r$ will be called *function variables*. The sentence ψ is *true* in an $\mathbb{R}$-structure $\mathfrak{D}$ of signature σ when there exist interpretations of $Y_1, \ldots, Y_r$ such that φ holds true on $\mathfrak{D}$. The set of existential second-order sentences will be denoted by $\exists\mathrm{SO}_{\mathbb{R}}$. Together with the interpretation above it constitutes *existential second-order logic*.

It is an easy exercise to establish a natural correspondence between elements in $\mathbb{R}^{\infty}$ and $\mathbb{R}$-structures. We use this correspondence when talking about decision problems of $\mathbb{R}$-structures that belong to certain real number complexity classes. For full details see [12].

For the characterization of our class $\mathrm{MAX}\text{-}\mathrm{NP}_{\mathbb{R}}$ introduced below we also need the following extension of Fagin's theorem to $\mathbb{R}$-structures.

Theorem 2.6 [12] *Let (F, F^{+}) be a decision problem of $\mathbb{R}$-structures. Then $(F, F^{+}) \in \mathrm{NP}_{\mathbb{R}}$ if and only if there exists an $\exists\mathrm{SO}_{\mathbb{R}}$-formula ψ such that $F^{+} = \{\mathfrak{D} \in F \mid \mathfrak{D} \models \psi\}$.*

2.2 The classes $\mathbf{NPO}_{\mathbb{R},\mathbf{max}}$ and $\mathbf{NPO}_{\mathbb{R},\mathbf{min}}$

The problems we deal with in this paper are certain optimization problems over $\mathbb{R}$-structures. As a starting point we consider the well known definition of combinatorial optimization problems in class NPO. We extend this definition in order to introduce a similar class in the BSS model. The main additional aspect of this new class $\mathrm{NPO}_{\mathbb{R}}$ is a potentially uncountable set of feasible solutions among which the optimum is searched for. This makes some changes in the definitions necessary, one of which is the lacking requirement of explicitly computing a feasible solution. Moreover, we consider functions with integer values in order to avoid (numerical) approximation issues. Though interesting in its own, the latter is expected to result in completely different issues, see [6, 15].

Let us start with introducing a real analogue of the class NPO of combinatorial optimization problems. Note that generalizations of NPO to arithmetical structures (i. e. metafinite structures with the natural numbers as infinite part) were studied in [16].

The definition below is tailored for the logical framework we study thereafter. It can easily be re-translated to a purely complexity theoretic definition, see [18] for such a definition.

Definition 2.7

a) An optimization problem $\mathcal{P} := (\mathcal{I}, \{\mathrm{Sol}(\mathfrak{D})\}_{\mathfrak{D}\in\mathcal{I}}, f)$ in class $\mathrm{NPO}_{\mathbb{R},\max}$ is a maximization problem over $\mathbb{R}$-structures that consists of three parts:

(i) a set $\mathcal{I}$ of $\mathbb{R}$-structures (over a fixed vocabulary) as instances for the problem;

(ii) for every instance $\mathfrak{D} \in \mathcal{I}$ a set $\mathrm{Sol}(\mathfrak{D})$ of feasible solutions. Elements in $\mathrm{Sol}(\mathfrak{D})$ are tuples

$$\mathfrak{S} = (S_1, \ldots, S_r)$$

of functions from the universe A of $\mathfrak{D}$ to the reals. Each S_i has a fixed arity $d_i \in \mathbb{N}$, so $S_i : A^{d_i} \longrightarrow \mathbb{R}$. The number r of components of $\mathfrak{S}$ depends on the problem only, but not on $\mathfrak{D}$.

For the purposes of this paper it is reasonable to require that $\mathrm{Sol}(\mathfrak{D})$ is non-empty.

(iii) A function $f : \{(\mathfrak{D}, \mathfrak{S}) \mid \mathfrak{D} \in \mathcal{I}, \mathfrak{S} \in \mathrm{Sol}(\mathfrak{D})\} \longrightarrow \mathbb{N}$. The value $f(\mathfrak{D}, \mathfrak{S})$ is called *the value of the feasible solution* $\mathfrak{S}$.

In addition, we require the following conditions to hold:

(iv) For any input $\mathfrak{D} \in \mathcal{I}$ and for any arbitrary $\mathfrak{S} \in \mathbb{R}^\infty$ of size at most $p(|\mathfrak{D}|)$ (for a fixed polynomial p) it is decidable in polynomial time in the BSS model whether $\mathfrak{S} \in \mathrm{Sol}(\mathfrak{D})$.

(v) The measure function f is computable in polynomial time with respect to $|\mathfrak{D}|$ in the BSS model.

We are looking for the maximal solution value $\max_{\mathfrak{S}\in\mathrm{Sol}(\mathfrak{D})} f(\mathfrak{D}, \mathfrak{S})$. Finally, we require that the maximum always exists.

b) The class $\mathrm{NPO}_{\mathbb{R},\min}$ of *minimization problems* is defined similarly.

Some remarks are appropriate here.

Remark 2.8

a) An important $\mathrm{NP}_\mathbb{R}$-hard problem is the minimization of a polynomial p, over all possible variable assignments $x \in \mathbb{R}$. Notice that given the remarks before the previous definition this problem is not in class $\mathrm{NPO}_{\mathbb{R},\min}$ as the value f is not integral.

Due to the structure of classes we consider in this paper it is natural to require the measure function f to yield values in $\mathbb{N}$. This is done because below we define classes via the number of satisfying assignments in some A^u for certain formulas. For a general theory of a class $\mathrm{NPO}_\mathbb{R}$ of optimization problems over $\mathbb{R}$ not inspired by a logical framework it certainly makes sense to remove this condition.

b) The logical framework we use to define $\mathrm{NPO}_\mathbb{R}$ directly guarantees that a feasible solution has a polynomial (algebraic) size. However, the set of feasible solutions can be uncountable. In many examples of such problems it might be impossible to compute a feasible solution; therefore, we do not require it.

c) Later on for sake of simplicity in many of our proofs we consider $\mathrm{NPO}_\mathbb{R}$ problems where $\mathfrak{S}$ consists of a single function only. For all proofs it can easily be seen that this is no serious restriction. An extension to the general case can always be done (almost) word by word and just increases the notational complexity.

Definition 2.9 An optimization problem $\mathcal{P}$ in class $\mathrm{NPO}_{\mathbb{R},\max}$ or class $\mathrm{NPO}_{\mathbb{R},\min}$ is *polynomially bounded* if there exists a polynomial p such that for all $\mathfrak{D}$ the value $f(\mathfrak{D}, \mathfrak{S})$ is bounded by $p(|\mathfrak{D}|)$ for all feasible solutions.

2.3 The classes $\mathbf{MAX\text{-}NP}_\mathbb{R}$ and $\mathbf{MIN\text{-}NP}_\mathbb{R}$

We next introduce a class $\mathrm{MAX\text{-}NP}_\mathbb{R}$ as a subclass of problems in $\mathrm{NPO}_{\mathbb{R},\max}$ that is defined by certain logical conditions. These classes are extensions of the corresponding discrete ones introduced in [19].

Definition 2.10

a) Let $\mathcal{P}$ be an optimization problem in $\mathrm{NPO}_{\mathbb{R},\max}$ whose instances are given as $\mathbb{R}$-structures over a fixed vocabulary V. Then $\mathcal{P}$ belongs to $\mathrm{MAX\text{-}NP}_\mathbb{R}$ if and only if for every instance $\mathfrak{D}$ of $\mathcal{P}$ having universe A the maximum can be expressed as

$$\max_{\mathfrak{S}\in\mathrm{Sol}(\mathfrak{D})} |\{x \in A^u \mid \mathfrak{D} \models \varphi(x, \mathfrak{S})\}|,$$

where $\mathfrak{S}$ is as defined in Definition 2.7 and φ is a first-order formula over $V \cup \{\mathfrak{S}\}$.

b) We obtain the subclasses $\mathrm{MAX\text{-}\Sigma}_{0,\mathbb{R}}$, $\mathrm{MAX\text{-}\Sigma}_{1,\mathbb{R}}$, $\mathrm{MAX\text{-}\Pi}_{1,\mathbb{R}}$, $\mathrm{MAX\text{-}\Sigma}_{2,\mathbb{R}}$ and $\mathrm{MAX\text{-}\Pi}_{2,\mathbb{R}}$, respectively, by restricting φ above to be of the corresponding format. For example, $\mathrm{MAX\text{-}\Sigma}_{1,\mathbb{R}}$ is the class of maximization problems whose maximum is expressible via

$$\max_{\mathfrak{S}\in\mathrm{Sol}(\mathfrak{D})} |\{x \in A^u \mid \mathfrak{D} \models (\exists y \in A^s)\, \psi(x, y, \mathfrak{S})\}|,$$

where ψ is first-order quantifier free.[1)]

c) In the same way the classes $\mathrm{MIN\text{-}NP}_\mathbb{R}$, $\mathrm{MIN\text{-}\Pi}_{0,\mathbb{R}}$, $\mathrm{MIN\text{-}\Sigma}_{1,\mathbb{R}}$, $\mathrm{MIN\text{-}\Pi}_{1,\mathbb{R}}$, $\mathrm{MIN\text{-}\Sigma}_{2,\mathbb{R}}$ are defined.

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1) In order to avoid confusion we point out that $\Sigma_{1,\mathbb{R}}$ in this paper is defined by means of existential first-order formulas in the framework of decsriptive complexity theory over $\mathbb{R}$. It should not be mixed up with the existential part of the first level of the polynomial hierarchy over $\mathbb{R}$, i. e. with $\mathrm{NP}_\mathbb{R}$.

Example 2.11 Let us consider a typical optimization problem in our framework and express it logically. The input are natural numbers n, m together with m polynomials $p_1, \ldots, p_m \in \mathbb{R}[x_1, \ldots, x_n]$. Each p_i has degree 2 and depends on precisely three variables among $\{x_1, \ldots, x_n\}$. The task is to compute the maximal number of p_i's that have a common zero in $\mathbb{R}^n$. This task is $\mathrm{NP}_{\mathbb{R}}$-hard. We give a logical description that places the problem (or better: its used logical version) in $\mathrm{MAX}\text{-}\Sigma_{0,\mathbb{R}}$.

As vocabulary we choose a nullary function $\mathbf{0}$, a unary relation Pol, a unary function $\varrho : A \longrightarrow \mathbb{R}$ and a function $C : A^3 \longrightarrow \mathbb{R}$ (where A is the universe). For an $\mathbb{R}$-structure representing a polynomial system as above the interpretations of these symbols are as follows: The universe A splits into two parts. The first part $A_1 = \{0, \ldots, n\}$ represents the variables $x_1, \ldots, x_n$ plus an additional one x_0 used for homogenization; the second part $A_2 := \{n+1, \ldots, n+m\}$ represents the indices for the polynomials p_i (i. e. $n+i$ stands for p_i). If $\ell \in A$ satisfies $\mathrm{Pol}(\ell)$, then ℓ is the index of a polynomial $p_{\ell-n}$. The nullary function $\mathbf{0}$ represents $0 \in A$. The unary function $\varrho : A_1 \longrightarrow \{0, \ldots, n\} \subset \mathbb{R}$ is interpreted in the natural way as a total linear order for A_1. Finally, $C : A_1^2 \times A_2 \subset A^3 \longrightarrow \mathbb{R}$ stands for the coefficients of the p_i; for $(i,j) \in A_1^2$, $\ell \in A_2$ the coefficient of $x_i \cdot x_j$ in $p_{\ell-n}$ is $C(i,j,\ell)$. Here, we use $x_0 = 1$ as homogenization variable. Note that the way we define it C is symmetric in the first two components. For other arguments not mentioned above we define C to give the result 0.

The maximal number of polynomials having a zero in common can be described as follows: A common root is coded via a function $X : A_1 \longrightarrow \mathbb{R}$. This function corresponds to $\mathfrak{S}$ from Definition 2.7 (with $r = 1$).

Then we look for

$$\max_{X : A_1 \longrightarrow \mathbb{R}} |\{(i,j,k,\ell) \mid \mathrm{Pol}(\ell) \wedge X(0) = 1 \wedge \varrho(i) < \varrho(j) < \varrho(k) \\ \wedge \varphi_1(i,j,k,\ell) \wedge \varphi_2(i,j,k,\ell)\}|.$$

Here, $\varphi_1(i,j,k,\ell)$ is a formula guaranteeing that x_i, x_j, x_k are the three variables $p_{\ell-n}$ depends on, i. e.

$$\begin{aligned}\varphi_1(i,j,k,\ell) \equiv \quad & (C(i,0,\ell) \neq 0 \vee C(i,i,\ell) \neq 0 \vee C(j,i,\ell) \neq 0 \vee C(i,k,\ell) \neq 0) \\ \wedge\ & (C(j,0,\ell) \neq 0 \vee C(j,j,\ell) \neq 0 \vee C(j,i,\ell) \neq 0 \vee C(j,k,\ell) \neq 0) \\ \wedge\ & (C(k,0,\ell) \neq 0 \vee C(k,k,\ell) \neq 0 \vee C(k,i,\ell) \neq 0 \vee C(k,j,\ell) \neq 0).\end{aligned}$$

Similarly, $\varphi_2(i,j,k,\ell)$ expresses that the evaluation of $p_{\ell-n}$ in the point represented by X will give result 0. Note that the knowledge coded in φ_1 can be used to design φ_2 as a quantifier free formula:

$$\begin{aligned}\varphi_2(i,j,k,\ell) \equiv C(0,0,\ell) &+ C(i,0,\ell) \cdot X(i) + C(j,0,\ell) \cdot X(j) \\ &+ C(k,0,\ell) \cdot X(k) + C(i,i,\ell) \cdot X(i)^2 + \cdots = 0.\end{aligned}$$

Altogether, we see that the problem lies in $\mathrm{MAX}\text{-}\Sigma_{0,\mathbb{R}}$.

We remark that since finding the maximum is $\mathrm{NP}_{\mathbb{R}}$-hard our results have more a model-theoretic than a complexity theoretic flavor.

2.4 Establishing maximization and minimization hierarchies

Next, we list some basic properties of problems in $\mathrm{MAX}\text{-}\mathrm{NP}_{\mathbb{R}}$. They mirror the corresponding properties over finite structures [13] and can basically be shown by similar techniques using additional results for $\mathbb{R}$-structures given in [12].

First, it is clear that the objective function f of a problem $\mathcal{P}$ in $\mathrm{MAX}\text{-}\mathrm{NP}_{\mathbb{R}}$ is polynomially bounded. This is true because the definition of these problems asks for the maximal cardinality of certain subsets of A^u; this value is bounded by $|A|^u$. Vice versa, the following proposition shows that the converse holds as well.

Proposition 2.12

a) *If $\mathcal{P}$ is a polynomially bounded problem in* $\mathrm{NPO}_{\mathbb{R},\max}$, *then* $\mathcal{P} \in \mathrm{MAX}\text{-}\mathrm{NP}_{\mathbb{R}}$.

b) $\mathrm{MAX}\text{-}\mathrm{NP}_{\mathbb{R}} = \mathrm{MAX}\text{-}\Pi_{2,\mathbb{R}}$.

c) $\mathrm{MAX}\text{-}\Sigma_{2,\mathbb{R}} = \mathrm{MAX}\text{-}\Pi_{1,\mathbb{R}}$.

Proof.

a) The proof idea is given in [13]; we just sketch the differences when dealing with $\mathbb{R}$-structures. Let $\mathfrak{D}$ be an instance of an $\mathbb{R}$-structure with universe A, $S : A^k \longrightarrow \mathbb{R}$ a feasible solution and $f : (\mathfrak{D}, S) \longrightarrow \mathbb{N}$ the objective function which has to be maximized with respect to S (for considering just a single function S confer the earlier Remark 2.8).

Polynomial boundedness gives $\max_S f(\mathfrak{D}, S) \leq n^m$ for some fixed m, $n := |A|$. Define a decision problem in $\mathrm{NP}_{\mathbb{R}}$ as follows: Given an $\mathbb{R}$-structure $\mathfrak{D}$ and a function $T : A^m \longrightarrow \{0,1\} \subset \mathbb{R}$, is there an $S : A^k \longrightarrow \mathbb{R}$ such that

$$f(\mathfrak{D}, S) \geq |\{x \in A^m \mid T(x) = 1\}|? \tag{1}$$

According to the analogue of Fagin's theorem for $\mathbb{R}$-structures [12] there is a first-order formula ψ over $\mathbb{R}$-structures such that (1) holds if and only if there exists a second-order variable Y with $(\mathfrak{D}, T) \vDash \psi(Y, T)$. Then the following problem belongs to $\mathrm{MAX}\text{-}\mathrm{NP}_{\mathbb{R}}$ and gives the maximal value for f we are looking for:

$$\max_{\substack{T:A^m \longrightarrow \{0,1\} \\ Y:A^k \longrightarrow \mathbb{R}}} |\{x \in A^m \mid \psi(Y, T) \wedge T(x) = 1\}|.$$

b) The proof of Fagin's theorem for $\mathbb{R}$-structures in [12] (or, more elementary, a result from [10]) shows that in the proof of part a) we can choose ψ to be of the form

$$\psi \equiv (\forall v \in A^{m_1})\,(\exists w \in A^{m_2})\,\varphi,$$

where φ is first-order quantifier free and thus $\psi \in \Pi_{2,\mathbb{R}}$.

c) Here, the corresponding proof from [13] can be adapted almost word by word. The idea is to remove the first existential quantifier $\exists x$ in a formula $\exists x \forall y\, \varphi$ by introducing an additional $\mathbb{R}$-valued function R which forces x to be unique. For sake of completeness consider a $\mathrm{MAX}\text{-}\Sigma_{2,\mathbb{R}}$ problem

$$\max_{S:A^k \longrightarrow \mathbb{R}} |\{w \in A^u \mid \mathfrak{D} \vDash (\exists x \in A^t)\,(\forall y \in A^r)\,\varphi(x, y, w, S)\}|. \tag{2}$$

Now, for each w^* that satisfies the above formula a witness $x^* \in A^t$ is forced to be unique by adding a new function $R : A^u \times A^t \longrightarrow \{0,1\} \subset \mathbb{R}$ together with the $\Pi_{1,\mathbb{R}}$-formula

$$\psi \equiv R(w^*, x^*) = 1 \wedge \forall x_1, x_2\, ([R(w^*, x_1) = 1 \wedge R(w^*, x_2) = 1] \Rightarrow x_1 = x_2).$$

Then the maximum of (2) is given as well by the $\mathrm{MAX}\text{-}\Pi_{1,\mathbb{R}}$ description

$$\max_{\substack{S:A^k \longrightarrow \mathbb{R} \\ R:A^u \times A^t \longrightarrow \{0,1\}}} |\{(w^*, x^*) \mid (\forall y \in A^r)\,\varphi(x^*, y, w^*, S) \wedge \psi(w^*, x^*)\}|. \qquad \square$$

Likewise we get:

Proposition 2.13

a) *If $\mathcal{P}$ is a problem in* $\mathrm{NPO}_{\mathbb{R},\min}$ *which is polynomially bounded, then* $\mathcal{P} \in \mathrm{MIN}\text{-}\mathrm{NP}_{\mathbb{R}}$.

b) $\mathrm{MIN}\text{-}\mathrm{NP}_{\mathbb{R}} = \mathrm{MIN}\text{-}\Sigma_{2,\mathbb{R}}$.

c) $\mathrm{MIN}\text{-}\Pi_{1,\mathbb{R}} = \mathrm{MIN}\text{-}\Sigma_{2,\mathbb{R}}$.

d) $\mathrm{MIN}\text{-}\Pi_{0,\mathbb{R}} = \mathrm{MIN}\text{-}\Sigma_{1,\mathbb{R}}$.

Proof.

a) The proof is very similar to that of Proposition 2.12. Let $\mathfrak{D}$ be the instance encoded as an $\mathbb{R}$-structure with universe A, $S : A^k \longrightarrow \mathbb{R}$ a feasible solution and $f : (\mathfrak{D}, S) \longrightarrow \mathbb{N}$ the objective function which has to be minimized with respect to S. Since the problem is polynomially bounded we have $\min_S f(\mathfrak{D}, S) \leq n^m$ for some fixed m, and $n = |A|$. As before we define a decision problem in $\mathrm{NP}_{\mathbb{R}}$: Given an $\mathbb{R}$-structure $\mathfrak{D}$ and a function $T : A^m \longrightarrow \{0,1\} \subset \mathbb{R}$, is there an existential second-order variable $S : A^k \longrightarrow \mathbb{R}$ such that

$$f(\mathfrak{D}, S) \leq |\{x \in A^m \mid T(x) = 1\}|? \tag{3}$$

As in the proof of Proposition 2.12 we use the analogue of Fagin's theorem and get a first-order formula ψ and the following problem in $\text{MIN-NP}_{\mathbb{R}}$ which gives the minimal value for f:

$$\begin{aligned}\min_{\substack{T:A^m\longrightarrow\{0,1\}\\ Y:A^k\longrightarrow\mathbb{R}}} &|\{x\in A^m \mid \psi(Y,T)\Rightarrow T(x)=1\}|\\ &= \min_{\substack{T:A^m\longrightarrow\{0,1\}\\ Y:A^k\longrightarrow\mathbb{R}}} |\{x\in A^m \mid \neg\psi(Y,T)\vee T(x)=1\}|.\end{aligned}$$

b) Similar to part b) of Proposition 2.12. However, as in the proof for the classical setting [13], note that when dealing with minimization problems we consider the negation of a $\Pi_{2,\mathbb{R}}$ formula, i. e. a $\Sigma_{2,\mathbb{R}}$ formula.

c) See proof of part d) (except in the present case the φ formula contains universal quantifiers).

d) The corresponding proof in [13] can be transferred to $\mathbb{R}$-structures very easily. We need to remove an existential quantifier from our formula, ending up with a quantifier-free formula which produces the same optimal value. The proof how to do this, and that the optimal value remains the same, is equivalent to that in the classical setting and will not be repeated here. □

3 The 4-level maximization hierarchy

We turn to the main results of this paper. In this section we prove that $\text{MAX-NP}_{\mathbb{R}}$ can be decomposed into a hierarchy of four distinct levels. More precisely, we show:

Theorem 3.1 $\text{MAX-}\Sigma_{0,\mathbb{R}} \subsetneq \text{MAX-}\Sigma_{1,\mathbb{R}} \subsetneq \text{MAX-}\Pi_{1,\mathbb{R}} \subsetneq \text{MAX-}\Pi_{2,\mathbb{R}} = \text{MAX-NP}_{\mathbb{R}}$.

The theorem is a real number version of [13, Theorem 2]. However, the problems we use (as well as our proofs) to establish it are different since they have to involve meta-finite structures.

In order to show the above separations we consider the following real number optimization problems.

Definition 3.2

a) For fixed $d\in\mathbb{N}$ the $\text{MAX-HNS}_{\mathbb{R}}(d)$ *problem* (*maximal Hilbert-Nullstellensatz*) is given as:

INPUT: $n, m\in\mathbb{N}$ and polynomials $p_1,\dots,p_m$ of degree at most d in variables $x_1,\dots,x_n$.

QUESTION: What is the maximal number of polynomials p_i, $1\le i\le m$, that have a common zero $x\in\mathbb{R}^n$?

b) The *sign-changes problem* is given as:

INPUT: $n\in\mathbb{N}$ together with a sequence of n ordered reals $(x_1,\dots,x_n)$.

QUESTION: Find the number of components i with $x_i\neq 0$ for which there exists j such that $x_j\neq 1$ and $x_i\cdot x_j<0$.

The above problems in our framework first become interesting after we formalize them as problems for meta-finite structures. There are several ways to do so depending on which information we include in the structure. This will in particular have impact on the question to which $\text{MAX}_{\mathbb{R}}$-classes the problems belong. Moreover, it will be crucial for our separation results.

3.1 The non-ordered version of the $\text{MAX-HNS}_{\mathbb{R}}(d)$ problem

Let us start with $\text{MAX-HNS}_{\mathbb{R}}(d)$, $d\in\mathbb{N}$. In the first formalization we take, similarly as in Example 2.11, an $\mathbb{R}$-structure with universe $A:=\{0,\dots,n\}\cup\{n+1,\dots,n+m\}$. The vocabulary includes one unary relation $\text{Pol}\subseteq A$ indicating whether an $x\in A$ is a polynomial or a variable. It as well includes a function $C: A^{d+1}\longrightarrow\mathbb{R}$ that is interpreted as representing the coefficients of the monomials in the corresponding polynomials.

Thus, we consider a polynomial system given as $\mathbb{R}$-structure $\mathfrak{D}=(A,\text{Pol},C)$, where $A=\{0,\dots,n+m\}$, $\text{Pol}(i)$ if and only if $i\in\{n+1,\dots,n+m\}$ and

$$C(i_1,\dots,i_d,k)=\begin{cases}0 & \text{if } \neg\text{Pol}(k),\\ \text{coefficient of } x_{i_1}\cdot x_{i_2}\cdot\dots\cdot x_{i_d} \text{ in equation } k & \text{if } \text{Pol}(k).\end{cases}$$

In order to represent as well monomials of degree strictly less than d we again guarantee in all our formulas $x_0:=1$.

This time we do not include a linear ordering on A in the vocabulary. We therefore denote this formalization of the Hilbert-Nullstellensatz problem by $\mathrm{NORD\text{-}MAX\text{-}HNS}_{\mathbb{R}}(d)$.

It is almost straightforward to see that $\mathrm{NORD\text{-}MAX\text{-}HNS}_{\mathbb{R}}(d)$ belongs to $\mathrm{MAX\text{-}NP}_{\mathbb{R}}$, and thus by Proposition 2.12 to $\mathrm{MAX\text{-}}\Pi_{2,\mathbb{R}}$. The interested reader might try to find directly (without use of Proposition 2.12) a $\Pi_{2,\mathbb{R}}$-formula which establishes that membership.

Theorem 3.3 $\mathrm{NORD\text{-}MAX\text{-}HNS}_{\mathbb{R}}(4) \notin \mathrm{MAX\text{-}}\Pi_{1,\mathbb{R}}$.

P r o o f. Suppose the claim to be false. For a corresponding input structure $\mathfrak{D} = (A, \mathrm{Pol}, C)$ of the problem $\mathrm{NORD\text{-}MAX\text{-}HNS}_{\mathbb{R}}(4)$ the maximum can be expressed as

$$\max_{S:A^t \longrightarrow \mathbb{R}} |\{x \in A^u \mid \mathfrak{D} \models (\forall y \in A^s)\, \varphi(x, y, C, \mathrm{Pol}, S)\}|, \tag{4}$$

where φ is first-order quantifier free, $s, t, u \in \mathbb{N}$ fixed. For an even $n \in \mathbb{N}$ consider the following polynomials f_i, $1 \le i \le \frac{n}{2}$, in variables $x_1, \ldots, x_n$ and of degree 4 which will be important for constructing an appropriate input-structure:

$$\begin{aligned}
f_1(x_1, x_2) &= (x_1 \cdot x_2 - 1)^2 + x_1^2,\\
f_2(x_3, x_4) &= (x_3 \cdot x_4 - 1)^2 + x_3^2,\\
&\ \ \vdots\\
f_i(x_{2i-1}, x_{2i}) &= (x_{2i-1} \cdot x_{2i} - 1)^2 + x_{2i-1}^2,\\
&\ \ \vdots\\
f_{\frac{n}{2}}(x_{n-1}, x_n) &= (x_{n-1} \cdot x_n - 1)^2 + x_{n-1}^2.
\end{aligned}$$

Define a new polynomial $p(x_1, \ldots, x_n)$ of degree 4 by

$$p(x_1, \ldots, x_n) = \textstyle\sum_{i=1}^{\frac{n}{2}} f_i(x_{2i-1}, x_{2i}) - \varepsilon,$$

where ε is an arbitrary, fixed real number in $(0, 1)$. It is the (single) polynomial equation $p = 0$ that we now consider as input for $\mathrm{NORD\text{-}MAX\text{-}HNS}_{\mathbb{R}}(4)$. As input $\mathbb{R}$-structure it is represented as $\mathfrak{D} = (A, \mathrm{Pol}, C)$ with $A := \{0, 1, \ldots, n\} \cup \{n+1\}$, $\mathrm{Pol}(i)$ if and only if $i = n+1$ and $C(i, j, k, \ell, n+1)$ gives the coefficient of $x_i \cdot x_j \cdot x_k \cdot x_\ell$ in p, where $x_0 := 1$ once again is used to represent monomials of degree less than 4. For example, the constant part $\frac{n}{2} - \varepsilon$ is given as $C(0, 0, 0, 0, n+1)$.

We claim that p has a real zero for any choice $\varepsilon > 0$. In order to see this note that each of the polynomials f_i satisfies $f_i(x_{2i-1}, x_{2i}) > 0$. Non-negativity is obvious by definition, whereas strict positivity follows from the fact that $x_{2i-1} := 0$ results in the function value 1. Moreover, $\inf_{x_{2i-1}, x_{2i}} f_i(x_{2i-1}, x_{2i}) = 0$ by choosing $x_{2i-1} := x_{2i}^{-1}$ for $x_{2i} > 0$ and now considering the limit $\lim_{x_{2i} \to \infty} f_i(\frac{1}{x_{2i}}, x_{2i})$. For any $\varepsilon > 0$ if we choose the x_{2i}'s large enough such that $f_i(\frac{1}{x_{2i}}, x_{2i}) < \frac{2\varepsilon}{n}$, we get a negative function value for p. Since p clearly has positive values continuity implies the claim.

Thus, the above formula (4) has to give the result 1. Let $X^* : A^t \longrightarrow \mathbb{R}$, $x^* \in A^u$ be an assignment such that $\mathfrak{D} \models (\forall y \in A^s)\, \varphi(x^*, y, X^*, \mathrm{Pol}, C)$. We construct a substructure $\mathfrak{D}'$ of $\mathfrak{D}$ that still gives a result of at least 1 for (4) but codes a polynomial without real zeros.

Let i_0 be such that no component of the particular $x^* \in A^u$ chosen above equals x_{2i_0}. For n large enough (for example $n > 2u + 1$) such an i_0 exists since k is fixed (independently of n). Define a new input structure $\mathfrak{D}'$ by deleting $2i_0$ from the universe (and identifying the other elements of A with those in the new universe A' correspondingly). Furthermore, Pol' and C' are defined as for $\mathfrak{D}$ on the remaining arguments. The related polynomial p' then is given by

$$p'(x_1, \ldots, x_{2i_0-1}, x_{2i_0+1}, \ldots, x_n) = \sum_{\substack{i=1,\\ i \ne 2i_0}}^{\frac{n}{2}} f_i - \varepsilon + (-1)^2 + x_{2i_0-1}^2.$$

As a substructure $\mathfrak{D}'$ still satisfies the universal formula $(\forall y \in (A')^s)\, \varphi(x^*, y, X'^*, \mathrm{Pol}', C')$. Therefore,

$$\max_{X':(A')^t \longrightarrow \mathbb{R}} |\{x \in (A')^u \mid (\forall y \in (A')^s)\, \varphi(x, y, X', \mathrm{Pol}', C')\}| \ge 1.$$

But, since $f_i \geq 0$,

$$p'(x_1, \dots, x_{2i_0-1}, x_{2i_0+1}, \dots, x_n) \geq x_{2i_0-1}^2 + 1 - \varepsilon \geq 1 - \varepsilon > 0.$$

Thus, p' has no real zero but our formula φ counts at least one. Contradiction. $\square$

3.2 The ordered version of the $\mathrm{MAX\text{-}HNS}_{\mathbb{R}}(d)$ problem

In order to separate $\mathrm{MAX\text{-}}\Sigma_{1,\mathbb{R}}$ from $\mathrm{MAX\text{-}}\Pi_{1,\mathbb{R}}$ we represent instances from $\mathrm{MAX\text{-}HNS}_{\mathbb{R}}(d)$ in a different manner as $\mathbb{R}$-structures. This will push the problem into class $\mathrm{MAX\text{-}}\Pi_{1,\mathbb{R}}$. For a system $p_1, \dots, p_m$ over $x_1, \dots, x_n$ the universe again is $A_1 \cup A_2$, where $A_1 = \{0, \dots, n\}$ and $A_2 = \{n+1, \dots, n+m\}$. A unary relation Pol again identifies the polynomials: $\mathrm{Pol}(i)$ if and only if $i \in A_2$. As before, $C : A^{d+1} \longrightarrow \mathbb{R}$ denotes the coefficients of the p_i's. In addition, the vocabulary will contain a linear ordering $\varrho : A_1 \longrightarrow \{0, \dots, n\} \subset \mathbb{R}$ as well as nullary relations $\mathbf{0}$ and $\mathbf{n}$ giving the first and the last element in A_1 (w. r. t. ϱ). Note that $\mathbf{0}$ and $\mathbf{n}$ as well as extensions ϱ^d, $\mathbf{0}^d$, $\mathbf{n}^d$ of ϱ, $\mathbf{0}$ and $\mathbf{n}$ to A_1^d can be defined by a universally quantified first-order formula, see [7].

The presence of this linear ordering is the reason why the proof of Theorem 3.3 cannot be applied in this setting: If we remove an element from the universe, the ordering will be invalid.

Representing the $\mathrm{MAX\text{-}HNS}_{\mathbb{R}}(d)$ problem that way we obtain its ordered version which we denote by $\mathrm{ORD\text{-}MAX\text{-}HNS}_{\mathbb{R}}(d)$. In what follows we fix $d = 4$.

Theorem 3.4 $\mathrm{ORD\text{-}MAX\text{-}HNS}_{\mathbb{R}}(4) \in \mathrm{MAX\text{-}}\Pi_{1,\mathbb{R}} \setminus \mathrm{MAX\text{-}}\Sigma_{1,\mathbb{R}}$.

P r o o f. Concerning the membership in $\mathrm{MAX\text{-}}\Pi_{1,\mathbb{R}}$ let ϱ^4, $\mathbf{0}^4$, $\mathbf{n}^4$ denote the above mentioned extensions of the ranking ϱ and $\mathbf{0}$ and $\mathbf{n}$ to A_1^4 (see [7] for how to express that in $\Pi_{1,\mathbb{R}}$). Then for an input $\mathbb{R}$-structure $\mathfrak{D}(A, \mathrm{Pol}, C, \varrho, \mathbf{0}, \mathbf{n})$ of $\mathrm{ORD\text{-}MAX\text{-}HNS}_{\mathbb{R}}(4)$ the maximal number of polynomials having a common real zero is given as

$$\max_{\substack{X: A_1 \longrightarrow \mathbb{R}, \\ Y: A_1^4 \times A_2 \longrightarrow \mathbb{R}}} |\{i \in A_2 \mid \mathfrak{D} \models \mathrm{Pol}(i) \wedge \varphi_1 \wedge \varphi_2 \wedge \varphi_3\}|,$$

where

$$\begin{aligned} &\varphi_1 \equiv Y(\mathbf{0}^4, i) = C(\mathbf{0}^4, i), \quad \varphi_2 \equiv Y(\mathbf{n}^4, i) = 0, \\ &\varphi_3 \equiv (\forall u, v \in A_1^4)\, (\varrho^4(u) = \varrho^4(v) + 1 \\ &\qquad\qquad \Rightarrow Y(u, i) = Y(v, i) + C(u, i) \cdot X(u_1) \cdot X(u_2) \cdot X(u_3) \cdot X(u_4)). \end{aligned}$$

Here, X is interpreted as a zero giving the maximum, $Y(\cdot, i)$ describes the intermediate sum when evaluating polynomial i in X by cycling through all monomials given by $u \in A_1^4$ (expressed by φ_3). Finally, φ_1 and φ_2 guarantee to start and finish the evaluation process with the correct values.

In order to establish non-membership in class $\mathrm{MAX\text{-}}\Sigma_{1,\mathbb{R}}$ assume that the maximum is computed by

$$\max_{S: A^t \longrightarrow \mathbb{R}} |\{x \in A^u \mid \mathfrak{D} \models (\exists y \in A^s)\, \varphi(x, y, S)\}|, \tag{5}$$

where φ is first-order quantifier free. Consider once more the polynomial system (consisting of a single polynomial) used to prove Theorem 3.3. This time we represent the system by an ordered $\mathbb{R}$-structure

$$\mathfrak{D} = (A, \mathrm{Pol}, C, \varrho, \mathbf{0}, \mathbf{n}).$$

Let $x^* \in A^u$, $y^* \in A^s$, $S^* : A^t \longrightarrow \mathbb{R}$ satisfy $\mathfrak{D} \models \varphi(x^*, y^*, S^*)$ according to our assumption that formula (5) works correctly and the earlier proven fact that polynomial p has a real zero. φ is quantifier-free, so $\varphi(x^*, y^*, S^*)$ contains at most r elements from A, where r is a constant independent from $\mathfrak{D}$. Choose the size n of universe A such that n is even and $r < \frac{n}{2}$. Then there is a variable among $\{x_2, x_4, \dots, x_n\}$ which does not occur in $\varphi(x^*, y^*, S^*)$. Without loss of generality let x_n be that variable. Now define a new structure $\mathfrak{D}'$representing a polynomial p' which is generated as before by the polynomials f_i, $1 \le i \le \frac{n}{2}$. The only difference between p and p' is the polynomial $f_{\frac{n}{2}}$ which now has the form

$$f_{\frac{n}{2}}(x_{n-1}, x_n) := (x_{n-1} \cdot x_n - 1)^2 + x_{n-1}^2 + x_n^2.$$

Therefore, in contrast to p the polynomial p' contains the monomial x_n^2 with coefficient 1, whereas in p the coefficient was 0. The other f_i's remain unchanged.

Since x_n was not occuring in $\varphi(x^*, y^*, S^*)$ the new structure $\mathfrak{D}'$ as well satisfies $\mathfrak{D}' \models \varphi(x^*, y^*, S^*)$. This implies

$$\max_{S:A^t \longrightarrow \mathbb{R}} |\{x \in A^u \mid \mathfrak{D}' \models (\exists y \in A^s)\, \varphi(x, y, S)\}| \ge 1.$$

But

$$\begin{aligned} p' &= \textstyle\sum_{i=1}^{\frac{n}{2}-1} f_i(x_{2i-1}, x_{2i}) + (x_{n-1} \cdot x_n - 1)^2 + x_{n-1}^2 + x_n^2 - \varepsilon \\ &> x_{n-1}^2 \cdot x_n^2 - 2x_{n-1} \cdot x_n + 1 + x_{n-1}^2 + x_n^2 - \varepsilon \\ &\ge (x_{n-1} - x_n)^2 + 1 - \varepsilon \\ &\ge 1 - \varepsilon \\ &> 0. \end{aligned}$$

Thus, p' has no real zero, and (5) does not give the correct result. The theorem is proven. $\square$

3.3 Separation between $\mathbf{MAX}$-$\boldsymbol{\Sigma}_{\mathbf{0},\mathbb{R}}$ and $\mathbf{MAX}$-$\boldsymbol{\Sigma}_{\mathbf{1},\mathbb{R}}$

The separation will be established using the sign-changes problem introduced in part b) of Definition 3.2. We represent its instances as $\mathbb{R}$-structures $\mathfrak{D} = (A, C)$, where $A = \{1, \dots, n\}$, $C : A \longrightarrow \mathbb{R}$. The number we are looking for is given as

(6) $$|\{i \in A \mid \exists \ell\, (C(\ell) \ne 1 \wedge C(i) \cdot C(\ell) < 0)\}|.$$

We express this problem a bit artificially as a problem in MAX-$\Sigma_{1,\mathbb{R}}$ by noting that the value given in (6) equals

$$\max_{S:A^t \longrightarrow \mathbb{R}} |\{i \in A \mid \exists \ell\, (C(\ell) \ne 1 \wedge C(i) \cdot C(\ell) < 0)\}|$$

(since the $\Sigma_{1,\mathbb{R}}$-formula does not at all depend on S). However, we shall now see how the particular form of the problem is useful in order to separate the two lowest classes of our hierarchy.

Theorem 3.5 MAX-$\Sigma_{0,\mathbb{R}} \subsetneq$ MAX-$\Sigma_{1,\mathbb{R}}$.

P r o o f. Assume the sign-changes problem to belong to MAX-$\Sigma_{0,\mathbb{R}}$. Let

(7) $$\max_{S:A^t \longrightarrow \mathbb{R}} |\{x \in A^u \mid \mathfrak{D} \models \varphi(x, C, S)\}|$$

be a corresponding formula, φ first-order quantifier free. Without loss of generality assume $t = 1$ (the following arguments hold as well for arbitrary t and more than one function S). For $n \in \mathbb{N}$ and each $1 \le i \le n$ consider the structures $\mathfrak{D}^{(i)}$ representing the sequence $(1, 0, \dots, 0, \underset{\substack{\uparrow \\ i}}{-1}, 0, \dots, 0)$. For these structures we get the result 1 as maximum (note that the condition $\exists \ell\, (C(\ell) \ne 1)$ in the problem's definition excludes position i as solution). Let $x^{(i)} \in A^u$, $S^{(i)} : A \longrightarrow \mathbb{R}$ be such that $\mathfrak{D}^{(i)} \models \varphi(x^{(i)}, C, S^{(i)})$. If n is large enough, we can find $j \in A$ such that $\varphi(x^{(i)}, C, S^{(i)})$ does not depend on j. Therefore, we can assume that $S^{(i)}(j) := S^{(i)}(i) \in \mathbb{R}$, i.e. we just require the value $S^{(i)}(j)$ to be the same as $S^{(i)}(i)$. This does not affect validity of $\varphi(x^{(i)}, C, S^{(i)})$. On the other hand, $x^{(i)}$ has to depend on i because otherwise the formula would not realize a change in the input structure from $C(i) = -1$ to $C(i) = 0$.

The following observations now are crucial:

(i) Each atomic subformula of the finite-structural part $\mathcal{A} := (A, =)$ in φ either has the form $i_1 = i_2$ or $i_1 \neq i_2$, for $i_1, i_2 \in A$. Since $x^{(i)}$ does not depend on j we can define another tuple $x^{(j)} \in A^u$ by replacing each occurence of i as component in $x^{(i)}$ by j. The above fact implies that $x^{(j)}$ as well satisfies those subformulas of φ which just refer to the finite structure $\mathcal{A}$. This is true because for all $k \in A$, $k \neq \{i, j\}$, it is $k = i$ if and only if $k = j$ (recall that the only relation occuring in $\mathcal{A}$ is equality).

(ii) Next, consider the meta-finite parts of φ, i.e. those subformulas including real constants or constructs depending on C or S. The atomic subformulas of this type are polynomial (in-)equalities with real constants originating from the constants occuring in φ (a finite set), real numbers $C(\ell)$ for some $\ell \in A$ and real variables $S(\ell)$ for some $\ell \in A$. More precisely, searching for an $S : A \longrightarrow \mathbb{R}$ that realizes the maximum value in (7) corresponds to searching adequate reals $S(\ell)$.

Consider the structure $\mathfrak{D}^{(i,j)}$ representing the sequence $(1, 0, \ldots, 0, \underset{\substack{\uparrow \\ i}}{-1}, 0, \ldots, 0, \underset{\substack{\uparrow \\ j}}{-1}, 0, \ldots, 0)$. Then

$$\mathfrak{D}^{(i,j)} \models \varphi(x^{(i)}, C, S^{(i)})$$

because the latter does not depend on j and the two structures $\mathfrak{D}^{(i,j)}$ and $\mathfrak{D}^{(i)}$ only differ in the value $C(j)$.

Next, (i) and (ii) show that $\mathfrak{D}^{(i,j)} \models \varphi(x^{(j)}, C, S^{(i)})$ because replacing all occurences of i in $\varphi(x^{(i)}, C, S^{(i)})$ by j will result in the tuple $x^{(j)}$ that satisfies those parts of φ which only involve the underlying finite structure, whereas for all meta-finite parts of φ the values we obtain from C and $S^{(i)}$ do not change under the above replacement (due to our requirement $S^{(i)}(j) = S^{(i)}(i)$). Since obviously $x^{(i)} \neq x^{(j)}$ it follows

$$\max_{S : A \longrightarrow \mathbb{R}} |\{x \in A^u \mid \mathfrak{D}^{(i,j)} \models \varphi(x, C, S)\}| \geq 2,$$

whereas the correct result for $\mathfrak{D}^{(i,j)}$ is 1 (here it is important that we required the value we are looking for to differ from 1). We obtain a contradiction. □

4 The 2-level minimization hierarchy

In this section we show that the hierarchy of polynomially bounded minimization problems consists of the two classes remaining from Proposition 2.13:

Theorem 4.1 MIN-$\Pi_{0,\mathbb{R}} \subsetneq$ MIN-$\Pi_{1,\mathbb{R}}$.

The theorem is a real number version of [13, Theorem 4].

Definition 4.2 We define the problem MIN-QPS-VALUES as follows:

INPUT: $n, m \in \mathbb{N}$ and polynomials $p_1, \ldots, p_m$ each of degree at most 2, and each depends on at most 3 of the variables $x_1, \ldots, x_n$.

QUESTION: Minimizing over $x \in \mathbb{R}^n$ what is the minimal number of different function values we can obtain when we evaluate the polynomials p_i, $1 \leq i \leq m$, in x?

We present polynomial systems by $\mathbb{R}$-structures as done in Example 2.11.

The problem is different from maximizing the amount of polynomials that have a common zero. Clearly, should they have a common zero the optimal value as MIN-QPS-VALUES instance is 1. The problem is $\mathrm{NP}_{\mathbb{R}}$-hard: Given such a system together with an additional polynomial that is constantly 0 the new system yields result 1 for MIN-QPS-VALUES if and only if the original system has a zero.

The problem trivially is polynomially bounded and thus belongs to class MIN-$\Pi_{1,\mathbb{R}}$ using Proposition 2.13. The interested reader might try to design a corresponding formula directly without using that proposition.

Theorem 4.3 MIN-QPS-VALUES $\notin$ MIN-$\Pi_{0,\mathbb{R}}$.

Proof. Let $\mathcal{H}_1$ be an instance with optimal value $\mathrm{opt}(\mathcal{H}_1) = k$, where $k \geq 2$. We denote the corresponding interpretations of relation and function symbols by $\mathbf{0}_1$, Pol_1, C_1, compare Example 2.11. Let $\mathcal{H}_2$ with $\mathbf{0}_2$, Pol_2, C_2 be an isomorphic copy of $\mathcal{H}_1$. Clearly $\mathrm{opt}(\mathcal{H}_2) = k$ as well. We define a new structure $\mathcal{H}$ representing a polynomial system consisting of the original set of polynomials and its copy (in new variables). The $\mathbb{R}$-structure $\mathcal{H}$ is obtained as follows. The universe A of $\mathcal{H}$ is the disjoint union of the universes of $\mathcal{H}_1$ and $\mathcal{H}_2$ except that we identify the elements $\mathbf{0}_1$ and $\mathbf{0}_2$ in the new structure (we use the same homogenization variable).

The interpretation of the relation Pol as well as of the function C in $\mathcal{H}$ is by using the obvious extensions of the corresponding interpretations in $\mathcal{H}_1, \mathcal{H}_2$. For those arguments where C_1, C_2 are undefined we define C to have the value 0. Clearly $\mathrm{opt}(\mathcal{H}) = k$ since the two sets of polynomials in the union have no variables in common and each set has optimal value k.

Assume that there is a quantifier free formula ψ over $\mathbb{R}$-structures which gives the optimal value of MIN-QPS-VALUES by minimizing over a real valued function S. For the structure $\mathcal{H}$ let X denote an assignment to this function that realizes the minimal value k:

$$k = |\{w \in A^t \mid (\mathcal{H}, X) \models \psi(w, X)\}|.$$

Let X_1, X_2 be the restrictions of X to the structures $\mathcal{H}_1, \mathcal{H}_2$, respectively. For both structures the minimal value is k and we obtain

$$k \le |\{w \in A^t_{\mathcal{H}_1} \mid (\mathcal{H}_1, X_1) \models \psi(w, X_1)\}|$$

and

$$k = |\{w \in A^t_{\mathcal{H}_2} \mid (\mathcal{H}_2, X_2) \models \psi(w, X_2)\}|.$$

Since $A_{\mathcal{H}_1}$ and $A_{\mathcal{H}_2}$ contain only the element 0 in common, at most one element w (namely the one with all components 0) can occur as satisfying assignment in both above formulas. All the others are different when considered as elements in the disjoint union A^t.

Now ψ is quantifier free; therefore, all w satisfying the formula in one of the two structures $\mathcal{H}_1, \mathcal{H}_2$ yield (with respect to X) a satisfying assignment in $\mathcal{H}$ as well, and at most a single w can occur twice. Because $k \ge 2$ it follows

$$k < 2k - 1 \le |\{w \mid (\mathcal{H}, X) \models \psi(w, X)\}|.$$

We arrive at a contradiction. □

5 Conclusions

In this paper we have introduced and studied from a logical point of view classes of optimization problems over the real numbers. Using tools from descriptive complexity theory over the real numbers two logical hierarchies of such problems were obtained. One for maximization problems consisting of four distinct levels and one for minimization problems containing two levels. Our results provide a real number analogue of corresponding results for the Turing model given in [13].

A most interesting direction for future research in our opinion is to study the relation between the logical description of real number maximization problems and approximation issues. For the Turing setting this line of research was started in [19] by showing that all problems in the discrete version of $\mathrm{MAX}\text{-}\Sigma_{1,\mathbb{R}}$ have approximation algorithms in class APX. Further related results were given in [13].

However, in the real number model no serious investigation of approximation classes has been performed so far. Some initial ideas can be found in [18] in relation with probabilistically checkable proofs, but a concise theory is waiting to be developed here. At a first sight, it seems at least unclear in how far the techniques used in [19] to combine descriptive complexity with approximation could be applied to the real number setting. The same seems to be the case for the techniques used in [16]. The reason might be that on the real number side fixing certain values for variables in a formula results in much stronger backtracking problems than on finite structures. We consider it to be an interesting future research area to develop approximation concepts in the BSS model as well as its relation to descriptive complexity theory for real number maximization problems.

For minimization problems we can show a result which, however, is a negative one (compare with the discrete analogue in [13]).

Theorem 5.1 *There exist problems in* $\mathrm{MIN}\text{-}\Pi_{0,\mathbb{R}}$ *that cannot be approximated in polynomial time by any constant factor unless* $\mathrm{P}_{\mathbb{R}} = \mathrm{NP}_{\mathbb{R}}$.

P r o o f. We consider once again Example 2.11. However, instead of finding the maximal number of polynomials that have a common zero, we want to find the minimal number of polynomials that have a common non-zero.

It is an easy exercise using the reasoning in Example 2.11 to show this problem is in MIN-$\Pi_{0,\mathbb{R}}$. Next suppose there is a constant $c > 0$ and a polynomial time BSS algorithm $\mathcal{A}$ computing for each polynomial system $\mathcal{P}$ represented that way as an $\mathbb{R}$-structure a natural number $\mathcal{A}(\mathcal{P})$ such that[2)]

$$\frac{\mathcal{A}(\mathcal{P})}{\mathrm{OPT}(\mathcal{P})} \leq 1 + c.$$

We show how this implies $\mathrm{P}_{\mathbb{R}} = \mathrm{NP}_{\mathbb{R}}$. Let $\mathcal{P} = (p_0, p_1, \ldots, p_m)$ be a polynomial system, where p_0 does not have a real root and all p_i, $1 \leq i \leq m$, are of degree 2 depending on precisely 3 variables among $\{x_1, \ldots, x_n\}$. Clearly, $\mathrm{OPT}(\mathcal{P}) = 1$ if and only if the system $(p_1, \ldots, p_m)$ has a common real zero. Therefore, computing $\mathrm{OPT}(\mathcal{P})$ is $\mathrm{NP}_{\mathbb{R}}$-hard. Let $K \in \mathbb{N}$ be larger than c. Construct a new system $\mathcal{P}'$ as follows: For each $1 \leq j \leq K$ include p_0 together with the m many polynomials p_i, $1 \leq i \leq m$, in $\mathcal{P}'$. Thus

$$\mathrm{OPT}(\mathcal{P}') = \begin{cases} 1 & \text{if and only if } p_1, \ldots, p_m \text{ have a common zero,} \\ \alpha \geq 1 + K & \text{otherwise.} \end{cases}$$

Performing $\mathcal{A}$ on $\mathcal{P}'$ we have to check whether $\mathcal{A}(\mathcal{P}') \leq 1 + c$ in order to decide in polynomial time whether a system $p_1, \ldots, p_m$ has a common zero. This would imply $\mathrm{P}_{\mathbb{R}} = \mathrm{NP}_{\mathbb{R}}$. □

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[2)] The definition of a real version of the class APX requires some care, see [18], but for our purposes it is not necessary to go into detail here.

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