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The dark side of interval temporal logic: marking the undecidability border

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Abstract Unlike the Moon, the dark side of interval temporal logics is the one we usually see: their ubiquitous undecidability. Identifying *minimal* undecidable interval logics is thus a natural and important issue in that research area. In this paper, we identify several new minimal undecidable logics amongst the fragments of Halpern and Shoham's logic HS, including the logic of the *overlaps* relation, over the classes of all finite linear orders and all linear orders, as well as the logic of the *meets* and *subinterval* relations, over the classes of all and dense linear orders. Together with previous undecidability results, this work contributes to bringing the identification of the dark side of interval temporal logics very close to the definitive picture.

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1 Introduction

Temporal reasoning plays a major role in computer science. In the most standard approach, the basic temporal entities are time points and temporal domains are represented as ordered structures of time points. The interval reasoning approach adopts another perspective on time, sometimes more natural, according to which the primitive ontological entities are time intervals instead of time points.

The tasks of representing and reasoning about time intervals arise naturally in various fields of computer science, artificial intelligence, and temporal databases, such as theories of action and change, natural language processing, and constraint satisfaction problems. Temporal logics with interval-based semantics have also been proposed as a useful formalism for the specification and verification of hardware [22] and of real-time systems [24].

Despite the relevance of interval-based temporal reasoning, however, interval temporal logics are far less studied and popular than point-based ones because of their higher conceptual and computational complexity (relations between intervals are more complex than those between points). Interval temporal logics typically feature modal operators that correspond to (binary) relations between intervals usually known as Allen's relations [1]. In [13], Halpern and Shoham introduce a modal logic for reasoning about interval structures (HS), with a modal operator for each Allen's relation. HS is undecidable under very weak assumptions on the class of interval structures [13]. In particular, undecidability holds for any class of interval structures over linear orders that contains at least one linear order with an infinite ascending (or descending) sequence of points, thus including the natural time flows $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$. For a long time, such sweeping undecidability results have discouraged attempts for practical applications and further research on interval logics. A renewed interest in the area has recently been stimulated by the discovery of some interesting decidable fragments of HS [7–10]. Gradually, the quest for more expressive yet still decidable fragments of HS has become one of the main focuses of the current research agenda for interval temporal logic.

In this paper, we contribute to delineating the border between decidable and undecidable fragments of HS by establishing new undecidability results. The initial undecidability results mentioned above have been strengthened further in a number of more recent publications, including [3–5, 19], where many fragments of HS have been shown to be undecidable. The present paper extends and partly subsumes some of these results. In particular, we exhibit the first known case of a single-modality HS fragment which is undecidable in the class of *all* linear orders, as well as in the class of all *finite* linear orders, strengthening previous results [4, 5]. Furthermore, most undecidability proofs for interval logics hinge on the existence of a linear ordering with an infinite sequence of points; here we show how to relax such an assumption. Although this paper is about undecidability, we believe that the presented results also give a better insight in the expressive power of interval-based temporal logics. While these results exclude the possibility of having correct and complete algorithmic decision procedures for some very natural fragments of HS, they may stimulate further research in this area and contribute to a deeper understanding of interval reasoning in artificial intelligence and the mathematics around it.

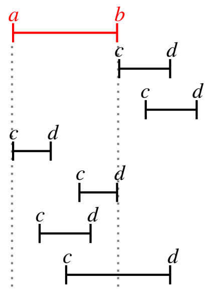
A complete picture of the state-of-the-art on the classification of HS fragments with respect to the decidability of satisfiability can be found in [11]. The web page [15] also provides a collection of online tools that enable one to check the status (decidable/undecidable/unknown) of any fragment of HS with respect to the satisfiability problem, over various classes of linear orders (all, dense, discrete, and finite). As a surprising outcome of the study of the family of fragments of HS with respect to the satisfiability problem, the borderline between decidable and undecidable such fragments turned out to be quite complicated, and they show an unexpected variety of behavior over different kinds of linear orders.

The rest of the paper is organized as follows. In Section 2, we introduce syntax and semantics of interval temporal logics. In Section 3, we give a short summary of undecidability results and proof techniques. The following two sections are devoted to the study of specific relevant fragments of HS: the logics O and $\overline{\mathsf{O}}$ of Allen's relation *overlaps* and its inverse (Section 4) and the logics AD, $\mathsf{A}\overline{\mathsf{D}}$, $\overline{\mathsf{A}}\mathsf{D}$, and $\overline{\mathsf{AD}}$ of Allen's relations *meets* and *during* and their inverses (Section 5). Section 6 provides an assessment of the work done and outlines future research directions.

2 Interval temporal logics: syntax and semantics

Let $\mathbb{D} = \langle D, < \rangle$ be a linearly ordered set. An *interval* over $\mathbb{D}$ is an ordered pair $[a, b]$, where $a, b \in D$ and $a \leq b$. Intervals of the form $[a, a]$ are called *point intervals*, while those where $a < b$ are *strict intervals*. There are 12 different non-trivial relations (excluding the identity) between two intervals in a linear order, often called *Allen's relations* [1]: the six relations depicted in Table 1 and their inverses, which are defined as follows: for each relation R_X, with $X \in \{A, L, B, E, D, O\}$, its inverse is the relation $R_{\overline{X}} = (R_X)^{-1}$. One can naturally associate a modal operator $\langle X \rangle$ with each Allen's relation R_X. For each operator $\langle X \rangle$, we denote by $\langle \overline{X} \rangle$ its *transpose*, corresponding to the inverse relation. The notion of sub-interval (the *contains* relation) can be defined in two variants, namely, *strict* sub-interval ($[a, b]$ is a strict sub-interval of $[c, d]$ if both $c < a$ and $b < d$) and *proper* sub-interval (when $c \leq a$,

Table 1 Allen's interval relations and the corresponding HS modalities

Relation	Operator	Formal definition	Pictorial example
Meets	$\langle A \rangle$	$[a, b] R_A [c, d] \Leftrightarrow b = c$	
Before	$\langle L \rangle$	$[a, b] R_L [c, d] \Leftrightarrow b < c$	
Started-by	$\langle B \rangle$	$[a, b] R_B [c, d] \Leftrightarrow a = c, d < b$	
Finished-by	$\langle E \rangle$	$[a, b] R_E [c, d] \Leftrightarrow b = d, a < c$	
Contains	$\langle D \rangle$	$[a, b] R_D [c, d] \Leftrightarrow a < c, d < b$	
Overlaps	$\langle O \rangle$	$[a, b] R_O [c, d] \Leftrightarrow a < c < b < d$	

$b \leq d$, and $[a, b] \neq [c, d]$). Both variants will be considered in this paper. Except when stated otherwise, we refer to proper ones.

Halpern and Shoham's logic HS is a multi-modal logic with formulae built over a set $\mathcal{AP}$ of proposition letters (aka atomic propositions), the propositional connectives $\vee$ and $\neg$, and unary modalities for Allen's relations. For each subset $\{R_{X_1}, \ldots, R_{X_k}\}$ of these relations, we define the HS fragment $\mathsf{X_1X_2 \ldots X_k}$, whose formulae are defined by the grammar:

$$\varphi ::= p \mid \pi \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle X_1 \rangle \varphi \mid \ldots \mid \langle X_k \rangle \varphi,$$

where π is a modal constant, true precisely at point intervals. We omit π when it is definable in the language or when point intervals are not allowed. The other propositional connectives, like $\wedge$ and $\rightarrow$, and universal modalities $[X]$ are defined as usual, e.g., $[X]\varphi \equiv \neg\langle X \rangle \neg\varphi$.

Let $\mathbb{I}(\mathbb{D})$ be the set of all intervals over $\mathbb{D}$. The *non-strict semantics* of an interval temporal logic is given in terms of *interval models* $M = \langle \mathbb{I}(\mathbb{D}), V \rangle$, where $V : \mathbb{I}(\mathbb{D}) \rightarrow 2^{\mathcal{AP}}$ is the *valuation function* that assigns to every interval $[a, b] \in \mathbb{I}(\mathbb{D})$ the set of proposition letters $V([a, b])$ that are true at it. The truth of a formula over an interval $[a, b]$ in a model M is defined by structural induction on formulae:

- $M, [a, b] \Vdash \pi$ iff $a = b$;
- $M, [a, b] \Vdash p$ iff $p \in V([a, b])$, for $p \in \mathcal{AP}$;
- $M, [a, b] \Vdash \neg\psi$ iff it is not the case that $M, [a, b] \Vdash \psi$;
- $M, [a, b] \Vdash \varphi \vee \psi$ iff $M, [a, b] \Vdash \varphi$ or $M, [a, b] \Vdash \psi$;
- $M, [a, b] \Vdash \langle X_i \rangle \psi$ iff there exists an interval $[c, d]$ such that $[a, b]\ R_{X_i}\ [c, d]$, and $M, [c, d] \Vdash \psi$.

Satisfiability is defined as usual: given a formula φ of HS, we say that φ is *satisfiable* if there exist a model M and an interval $[a, b]$ such that $M, [a, b] \Vdash \varphi$. Throughout the paper, for every proposition letter p and interval $[a, b]$, we say that $[a, b]$ is a *p-interval* if p holds at it (in the considered model).

Interval temporal logics can be given a *strict semantics* by excluding point intervals from the set of intervals over which formulae are interpreted (and by correspondingly excluding π from the language). In the following, we will restrict our attention to the strict semantics. However, our undecidability results hold for the non-strict semantics.

3 A short summary of undecidability results and proof techniques

In this section, we first summarize the main undecidability results for HS fragments, and then we state the main results of the present paper (Theorem 1), which extend known ones by providing new undecidability proofs for proper sub-fragments of logics that were already known to be undecidable.

3.1 Undecidability results

Undecidability of full HS was proved by Halpern and Shoham in [13]. Since then, several other undecidability results have been obtained. In [17], Lodaya proved undecidability of BE over dense linear orders, or, alternatively, over $\langle \omega, < \rangle$, provided that infinite intervals are allowed. In [3], Bresolin at al. showed undecidability of a number of HS fragments, namely, ADE, $\mathsf{AD\overline{E}}$, $\mathsf{A\overline{D}E}$, $\mathsf{A\overline{D}\overline{E}}$, $\mathsf{AD\overline{O}}$, $\mathsf{A\overline{D}\overline{O}}$, $\mathsf{\overline{A}DO}$, $\mathsf{\overline{A}\overline{D}O}$, $\mathsf{\overline{A}DB}$, $\mathsf{\overline{A}D\overline{B}}$, $\mathsf{\overline{A}\overline{D}B}$, $\mathsf{\overline{A}\overline{D}\overline{B}}$, $\mathsf{B\overline{E}}$, $\mathsf{\overline{B}E}$, and $\mathsf{\overline{B}\overline{E}}$. Undecidability of all (HS-)extensions of O (resp., $\mathsf{\overline{O}}$), except for those with modalities $\langle L\rangle$ and $\langle\overline{L}\rangle$, interpreted over any class of linear orders with at least one infinite sequence of points which, depending on the modalities of the fragment, may be required to be either ascending or descending, has been shown in [4]. In [5], the one-modality fragment O has been proved to be undecidable over the class of discrete linear orders. Finally, Marcinkowski et al. have shown the undecidability of BD, $\mathsf{B\overline{D}}$, $\mathsf{\overline{B}D}$, and $\mathsf{\overline{B}\overline{D}}$ on finite and discrete linear orders in [20], and later strengthened that result to the one-modality fragments D and $\mathsf{\overline{D}}$ in [19].

The present paper aims at contributing as much as possible to the completion of the undecidability picture. Its results can be summarized as follows.

Theorem 1 *The satisfiability problem for the* HS *fragments* O, $\mathsf{\overline{O}}$, AD, $\mathsf{A\overline{D}}$, $\mathsf{\overline{A}D}$, *and* $\mathsf{\overline{A}\overline{D}}$, *over any class of linear orders that contains, for each natural n, at least one linear order with cardinality greater than n, is undecidable.*

Theorem 1 states the undecidability of various HS fragments over (i) classes of linear orders that contain at least one linear order with an infinite ascending (fragments O, AD, $\mathsf{A\overline{D}}$) or descending (fragments $\mathsf{\overline{O}}$, $\mathsf{\overline{A}D}$, $\mathsf{\overline{A}\overline{D}}$) sequence of points, and (ii) those classes that contain arbitrarily large finite linear orders. The next sections are devoted to the proof of Theorem 1. The proof involve reductions of two different problems, depending on the considered class of linear orders: a reduction of the Octant Tiling Problem is used to deal with classes of linear orders containing an infinite sequence of points (we will call it "the infinite case"), while a reduction of the Finite Tiling Problem is used to deal with classes of unbounded finite linear orders ("the finite case"). It is important to point out that all natural classes of linear orders fit in one of these two cases, including the classes of all linear orders, dense linear orders, discrete linear orders, finite linear orders, as well as the linear orders of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ (singletons). The proof of Theorem 1 follows from Corollary 2 (infinite case for O and $\mathsf{\overline{O}}$), Corollary 3 (finite case for O and $\mathsf{\overline{O}}$), Corollary 5 (infinite case for AD), Corollary 9 (infinite case for $\mathsf{A\overline{D}}$), Corollary 10 (infinite case for $\mathsf{\overline{A}D}$ and $\mathsf{\overline{A}\overline{D}}$), and the undecidability results given in [19] (finite case for AD, $\mathsf{A\overline{D}}$, $\mathsf{\overline{A}D}$, and $\mathsf{\overline{A}\overline{D}}$).

Pairing known results with those given in this paper, we make a significant step toward the complete classification of HS fragments with respect to their decidable/undecidable status, as we can conclude that O, $\mathsf{\overline{O}}$, AD, $\mathsf{A\overline{D}}$, $\mathsf{\overline{A}D}$, $\mathsf{\overline{A}\overline{D}}$, BE, $\mathsf{B\overline{E}}$, $\mathsf{\overline{B}E}$, and $\mathsf{\overline{B}\overline{E}}$ are undecidable over all above referred classes of linear orders, that is, over the classes of all, dense, discrete, and finite linear orders. The undecidability proof for O and $\mathsf{\overline{O}}$ generalizes those given in [4, 5] as, unlike [5], it neither assumes discreteness nor, unlike [4], the existence of an infinite sequence. The undecidability proof for AD and $\mathsf{A\overline{D}}$ (resp., $\mathsf{\overline{A}D}$ and $\mathsf{\overline{A}\overline{D}}$), over any class of linear orders that contains at least a linear order with an infinite ascending (resp., descending) sequence of points, strengthens the undecidability results given in [3]. Since undecidability of these fragments over the class of finite linear orders immediately follows from that of D and $\mathsf{\overline{D}}$, undecidability spans all meaningful classes of linear orders. As a matter of fact, undecidability of finite satisfiability for D and $\mathsf{\overline{D}}$ can also be exploited to prove undecidability of AD and $\mathsf{A\overline{D}}$ (resp., $\mathsf{\overline{A}D}$ and $\mathsf{\overline{A}\overline{D}}$) over infinite, discrete, and right-bounded (resp., left-bounded) interval structures. Undecidability of BE, $\mathsf{B\overline{E}}$,

$\overline{\mathsf{B}}\mathsf{E}$, and $\overline{\mathsf{BE}}$ over all meaningful classes of linear orders follows from a collection of results. First, since $\langle D\rangle$ (resp., $\langle\overline{D}\rangle$) is definable in BE (resp., $\overline{\mathsf{BE}}$) by equation $\langle D\rangle\varphi = \langle B\rangle\langle E\rangle\varphi$ (resp., $\langle\overline{D}\rangle\varphi = \langle\overline{B}\rangle\langle\overline{E}\rangle\varphi$), undecidability of BE (resp., $\overline{\mathsf{BE}}$) over finite and discrete linear orders immediately follows from that of D (resp., $\overline{\mathsf{D}}$) [19]. Undecidability of BE over the class of all dense linear orders has been proved in [17] (since density is expressible in BE by a constant formula, undecidability over the class of all linear orders immediately follows [12]), while undecidability of $\overline{\mathsf{BE}}$ over the classes of dense and all linear orders has been shown in [3]. Finally, since O (resp., $\langle\overline{O}\rangle$) is definable in $\overline{\mathsf{B}}\mathsf{E}$ (resp., $\mathsf{B}\overline{\mathsf{E}}$) by equation $\langle O\rangle\varphi = \langle E\rangle\langle\overline{B}\rangle\varphi$ (resp., $\langle\overline{O}\rangle\varphi = \langle B\rangle\langle\overline{E}\rangle\varphi$), undecidability of $\mathsf{B}\overline{\mathsf{E}}$ (resp., $\overline{\mathsf{B}}\mathsf{E}$) over all meaningful classes of linear orders immediately follows from that of O (resp., $\overline{\mathsf{O}}$).

3.2 Proof techniques

The undecidability results given in this paper are proved by reduction of suitable instances of the Tiling Problem to the satisfiability problem for the considered HS fragments. Generally speaking, the Tiling (or Domino) Problem is the problem of deciding whether a set of tiles of a particular kind can tile a given portion of the plane. Starting from the seminal work by Wang [23], the Tiling Problem has been extensively used to prove undecidability and to give complexity bounds to the satisfiability problem for many different logical formalisms [2, 14]. As a matter of fact, a number of variants of the problem have been proposed in the literature, which differ from each other in the constraints they impose on the placement of the tiles and on the shape of the considered portion of the plane. In this paper, we will we make use of the Octant Tiling Problem (OTP) and of the Finite Tiling Problem (FTP).

OTP is the problem of establishing whether a given finite set of tile types $\mathcal{T} = \{t_1, \dots, t_k\}$ can *correctly* tile the second octant of the integer plane $\mathcal{O} = \{(i, j) : i, j \in \mathbb{N} \wedge 0 \le i \le j\}$. For every tile type $t_i \in \mathcal{T}$, let $\mathit{right}(t_i)$, $\mathit{left}(t_i)$, $\mathit{up}(t_i)$, and $\mathit{down}(t_i)$ be the colors of the corresponding sides of t_i. To obtain a *correct tiling* of the octant $\mathcal{O}$, one must find a function $f : \mathcal{O} \to \mathcal{T}$ such that $\mathit{right}(f(n, m)) = \mathit{left}(f(n+1, m))$, whenever $n < m$, and $\mathit{up}(f(n, m)) = \mathit{down}(f(n, m+1))$. Undecidability of OTP can be shown by means of an argument similar to the one used in [2] to prove undecidability of the Quadrant Tiling Problem.

FTP is a well-known undecidable problem, that has been studied in the literature in different, yet closely related, variants. Here, we refer to the one introduced and shown to be undecidable in [18]. Formally, we define FTP as the problem of establishing whether there exist two natural numbers k and l such that a finite set of tile types $\mathcal{T}$, containing two distinguished tile types t_0 and t_f, can correctly tile the $\{0, \dots, k\} \times \{0, \dots, l\}$ finite portion of the plane, under the additional restriction that $f(0, 0) = t_0$ and $f(k, l) = t_f$.

Given an HS fragment $\mathcal{F}$ and an instance OTP($\mathcal{T}$) of OTP, where $\mathcal{T}$ is a finite set of tile types, to reduce OTP to the satisfiability problem for $\mathcal{F}$, we build an $\mathcal{F}$-formula $\Phi_{\mathcal{T}}$ which is satisfiable if and only if $\mathcal{T}$ can correctly tile $\mathcal{O}$. The construction is similar to those used to prove undecidability of other HS fragments, but it is not readily derivable from them.

First, given an interval $[a, b]$, sometimes referred to as the *starting interval*, we identify a subset $\mathcal{G}_{[a,b]}$ of relevant intervals. Such a set contains all and only the intervals we need. Typically, $\mathcal{G}_{[a,b]}$ is the set of all intervals reachable from the starting interval $[a, b]$ using modalities of $\mathcal{F}$. However, this is not alway the case. For instance, in the proof for AD and $\mathsf{A\overline{D}}$ (Section 5), $\mathcal{G}_{[a,b]}$ is a subset of the set of intervals reachable from $[a, b]$. A characterization of the set $\mathcal{G}_{[a,b]}$ is given by means of a *global operator* $[G]$ (definable in $\mathcal{F}$) such that $[G]p$ is true if and only if p is true over all intervals in $\mathcal{G}_{[a,b]}$. Since all relevant formulae exclusively refer to intervals belonging to $\mathcal{G}_{[a,b]}$, hereafter, even if not explicitly said, we will only refer to intervals in $\mathcal{G}_{[a,b]}$, all other intervals being irrelevant.

Second, we set the tiling framework by forcing the existence of a unique infinite chain of u-intervals (u stands for *unit*), called u-*chain*, on the underlying linear order. The elements of the u-chain are used as cells to arrange the tiling. Furthermore, a new modality (definable in $\mathcal{F}$) is introduced to move from the current u-interval to the next one in the u-*chain*.

Third, we encode the octant by means of a unique infinite sequence of Id-intervals (Id stands for *identifier*), called Id-*chain*. Each Id-interval represents a row of the octant, and it consists of a sequence of u-intervals. Each u-interval is used either to represent a labeled position of the plane or to separate two consecutive rows. In the former case, it is labeled with tile; in the latter case, it is labeled with $*$.

Finally, by using suitable proposition letters, we encode the *above-* and *right-neighbor* relations, which connect each tile of the octant with, respectively, the one immediately above it and the one immediately at the right of it. Throughout the paper, if two tiles t_1 and t_2 are connected by the above-neighbor (resp., right-neighbor) relation, we say that t_1 is *above-connected* (resp., *right-connected*) to t_2, and similarly for tile-intervals (when they encode tiles of the octant that are above- or right-connected, respectively). The two neighbor relations must satisfy the following *commutativity property*.

Definition 1 An interval model has the *commutativity property* if for any pair of tile-intervals $[c, d]$ and $[e, f]$, if there exists a tile-interval $[d_1, e_1]$, such that $[c, d]$ is right-connected to $[d_1, e_1]$ and $[d_1, e_1]$ is above-connected to $[e, f]$, then there exists a tile-interval $[d_2, e_2]$ such that $[c, d]$ is above-connected to $[d_2, e_2]$ and $[d_2, e_2]$ is right-connected to $[e, f]$.

The rest of the paper is devoted to the proof of Theorem 1. As we already mentioned in Section 2, strict semantics is assumed. However, the proof can be easily adapted to non-strict semantics, because only proper intervals are used in the construction of models representing correct tilings.

4 The fragments O and $\overline{\mathsf{O}}$

The section is structured as follows. First, we prove undecidability of O over any class of linear orders containing at least one linear order with an infinite ascending sequence of points (infinite case) by building step-by step an O-formula that encodes OTP; then, to prove undecidability of O over any class of linear orders containing finite linear orders of unbounded cardinality (finite case), we show that an encoding of FTP can be obtained by a suitable adaptation of the construction for the infinite case. Since, by symmetry, analogous results hold for $\overline{\mathsf{O}}$, this suffices to prove Theorem 1 as far as O and $\overline{\mathsf{O}}$ are concerned. The proof for the infinite case is based on that given in [5] for the class of discrete linear orders, but dropping the discreteness assumption turned out to be far from being simple. The final achievement is a very general (and elegant) undecidability proof for O (and $\overline{\mathsf{O}}$).

4.1 Undecidability in the infinite case

Let $[a, b]$ be any interval of length at least 2, that is, such that there exists at least one point c in between a and b. Furthermore, let $\mathcal{G}_{[a,b]}$ be the set containing $[a, b]$ and all and only those intervals $[c, d]$ of length at least 2 such that $c > a$ and $d > b$. Finally, let modality $[G]$ be defined as follows: $[G]p \equiv p \wedge [O]p \wedge [O][O]p$ for $p \in \mathcal{AP}$. It can be easily checked that $[G]p$ holds over $[a, b]$ if and only if p holds over all intervals in $\mathcal{G}_{[a,b]}$.

Definition of the u*-chain* The definition of the u-chain is the most difficult step in the construction, due to the weakness of the language. It represents the most significant departure from the solution given in [5], where the definition of the u-chain hinges on the discreteness assumption. Here, a completely new approach is needed. It rests on three main ingredients: (a) existence of an infinite sequence of u-intervals $[b_0, b'_0], [b_1, b'_1], \ldots, [b_i, b'_i], \ldots$ such that $b \leq b_0$ and $b'_i = b_{i+1}$ for each $i \in \mathbb{N}$; (b) existence of an interleaved auxiliary chain $[c_0, c'_0], [c_1, c'_1], \ldots, [c_i, c'_i], \ldots$ such that $b_i < c_i < b'_i = b_{i+1} < c'_i < b'_{i+1}$ and $c'_i = c_{i+1}$ for each $i \in \mathbb{N}$; (c) uniqueness of chains. We call k-intervals the intervals of the auxiliary chain. Each k-interval overlaps exactly one u-interval, and it will be exploited to move from the current u-interval to next one in the u-chain. A graphical account of the relationships between the two chains is given in Fig. 1.

The third ingredient is definitely the most difficult one to deal with. To guarantee uniqueness, we will show that, under certain conditions, O can impose suitable conditions on proper sub-intervals of a given interval (which is quite surprising for a very weak HS fragment like O). In particular, we will show that, under appropriate constraints on the valuation of $p \in \mathcal{AP}$ (see Definition 2 below), it is possible to express properties like: "for each interval $[c, d]$, if $[c, d]$ satisfies p, then no proper sub-interval of $[c, d]$ satisfies p".

Definition 2 Let M be a model, $[a, b]$ be an interval over M, and $p, q \in \mathcal{AP}$. We say that p and q are *disjoint in* $\langle M, [a, b]\rangle$ if, for every pair of intervals $[c, d], [e, f] \in \mathcal{G}_{[a,b]}$ such that $M, [c, d] \Vdash p$ and $M, [e, f] \Vdash q$, either $d \leq e$ or $f \leq c$. Furthermore, we say that q is a *disjoint consequent* of p in $\langle M, [a, b]\rangle$ if p and q are disjoint in $\langle M, [a, b]\rangle$ and any p-interval is followed by a q-interval, that is, for each p-interval $[c, d] \in \mathcal{G}_{[a,b]}$, there exists a q-interval $[e, f] \in \mathcal{G}_{[a,b]}$ such that $e \geq d$. Finally, we say that p is *disjointly-bounded in* $\langle M, [a, b]\rangle$ *with disjoint consequent* q if (i) $[a, b]$ neither is a p-interval nor overlaps a p-interval, that is, if $[c, d]$ is a p-interval, then $c \geq b$; (ii) there

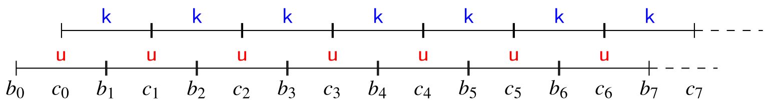


Fig. 1 Encoding of the u-chain in O

are no two p-intervals $[c, d]$ and $[e, f]$ with $c < e < d < f$, that is, p-intervals do not overlap; (iii) q is a disjoint consequent of p in $\langle M, [a, b]\rangle$.

The definition of u-chain makes use of the auxiliary proposition letters u_1, u_2, k_1, and k_2. The following formulae constrain u_1, u_2, k_1, and k_2 to be disjointly-bounded in $\langle M, [a, b]\rangle$ with disjoint consequents u_2, u_1, k_2, k_1, respectively.

$$\langle O\rangle\top \wedge \neg\mathsf{u} \wedge \neg\mathsf{k} \wedge [O](\neg\mathsf{u} \wedge \neg\mathsf{k}) \tag{1}$$

$$[G]((\mathsf{u} \leftrightarrow \mathsf{u}_1 \vee \mathsf{u}_2) \wedge (\mathsf{k} \leftrightarrow \mathsf{k}_1 \vee \mathsf{k}_2) \wedge (\mathsf{u}_1 \rightarrow \neg\mathsf{u}_2) \wedge (\mathsf{k}_1 \rightarrow \neg\mathsf{k}_2)) \tag{2}$$

$$[G]((\mathsf{u}_1 \rightarrow [O](\neg\mathsf{u} \wedge \neg\mathsf{k}_2)) \wedge (\mathsf{u}_2 \rightarrow [O](\neg\mathsf{u} \wedge \neg\mathsf{k}_1))) \tag{3}$$

$$[G]((\mathsf{k}_1 \rightarrow [O](\neg\mathsf{k} \wedge \neg\mathsf{u}_1)) \wedge (\mathsf{k}_2 \rightarrow [O](\neg\mathsf{k} \wedge \neg\mathsf{u}_2))) \tag{4}$$

$$[G]((\langle O\rangle\mathsf{u}_1 \rightarrow \neg\langle O\rangle\mathsf{u}_2) \wedge (\langle O\rangle\mathsf{k}_1 \rightarrow \neg\langle O\rangle\mathsf{k}_2)) \tag{5}$$

$$[G]((\mathsf{u}_1 \rightarrow \langle O\rangle\mathsf{k}_1) \wedge (\mathsf{k}_1 \rightarrow \langle O\rangle\mathsf{u}_2) \wedge (\mathsf{u}_2 \rightarrow \langle O\rangle\mathsf{k}_2) \wedge (\mathsf{k}_2 \rightarrow \langle O\rangle\mathsf{u}_1)) \tag{6}$$

$$(1) \wedge \ldots \wedge (6) \tag{disj-bnd$^{\mathsf{k}}_{\mathsf{u}}$}$$

Lemma 1 *Let M be a model and $[a, b]$ be interval over M such that $M, [a, b] \Vdash$ (disj-bndu$^{\mathsf{k}}_{\mathsf{u}}$). Then u_1, u_2, k_1, and k_2 are disjointly-bounded in $\langle M, [a, b]\rangle$ with disjoint consequents u_2, u_1, k_2, k_1, respectively.*

Proof We prove the statement for u_1. The proof for u_2, k_1, and k_2 is analogous. We show that u_1 satisfies the three conditions for disjointly-bounded proposition letters of Definition 2. By (1) and (2), $[a, b]$ neither satisfies u_1 nor overlaps an u_1-interval (condition (i)). By (2) and (3), u_1-intervals do not overlap each other (condition (ii)). We show now that u_2 is a disjoint consequent of u_1 (condition (iii)). First, we prove that u_1 and u_2 are disjoint. To this end, suppose, for the sake of contradiction, that there are an u_1-interval $[c, d]$ and an u_2-interval $[e, f]$ such that $d > e$ and $f > c$. We distinguish three cases:

1. $e < c$. If $f < d$, (3) is violated, while if $f \geq d$, (5) is violated. The former case is straightforward; as for the latter case, consider an interval starting strictly inside $[a, b]$ and ending strictly inside $[c, d]$. Such an interval overlaps both the u_1-interval $[c, d]$ and the u_2-interval $[e, f]$ (by the first conjunct of (1), the length of $[a, b]$ is greater than or equal to 2, and, by definition of $[G]$, the same holds for each interval reachable from $[a, b]$);
2. $e = c$. Equation (5) is violated (same argument of point 1);
3. $e > c$. If $f \leq d$, (5) is violated (same argument of point 1), while if $f > d$, (3) is violated.

To complete the proof, we need to show that, for each u_1-interval $[c, d]$, there exists an u_2-interval $[e, f]$ such that $e \geq d$. This follows from (6) and disjointedness of u_1 and u_2. □

We now show that whenever a proposition letter p is disjointly-bounded in $\langle M, [a, b]\rangle$ with disjoint consequent q, we can introduce an auxiliary proposition letter inside_p and force it to be true over all proper sub-intervals (in $\mathcal{G}_{[a,b]}$) of p-intervals. Then, by simply stating that inside_p-intervals and p-intervals cannot overlap, we prove that no p-interval can be a proper sub-interval of another one. To properly constrain the behavior of inside_p, we introduce an auxiliary proposition letter $\overrightarrow{p}$ and we force it to be true over intervals that start inside a p-interval and end inside a q-interval (and thus outside the p-interval they start from). The following three formulae express the above conditions for a generic proposition letter p disjointly-bounded in $\langle M, [a, b]\rangle$ with disjoint consequent q.

$$[G](p \rightarrow [O](\langle O\rangle q \rightarrow \overrightarrow{p})) \tag{7}$$

$$[G](\neg p \wedge [O](\langle O\rangle q \rightarrow \overrightarrow{p}) \rightarrow \mathsf{inside}_p) \tag{8}$$

$$[G]((\mathsf{inside}_p \rightarrow \neg\langle O\rangle p) \wedge (p \rightarrow \neg\langle O\rangle \mathsf{inside}_p)) \tag{9}$$

Lemma 2 *Let M be a model, $[a, b]$ be an interval over M, and $p, q \in \mathcal{AP}$ be two proposition letters such that p is disjointly-bounded in $\langle M, [a, b]\rangle$ with disjoint consequent q. If $M, [a, b] \Vdash$ (7) $\wedge$ (8) $\wedge$ (9), then no p-interval can be a proper sub-interval of another one in $\mathcal{G}_{[a,b]}$.*

Proof Suppose, for the sake of contradiction, that there exist two p-intervals $[c, d]$ and $[e, f]$ in $\mathcal{G}_{[a,b]}$ such that $[e, f]$ is a proper sub-interval of $[c, d]$. By definition of proper sub-interval, we have that $c < e$ or $f < d$. Without loss of generality, let us suppose that $c < e$ (the other case is analogous). Since $[e, f] \in \mathcal{G}_{[a,b]}$, then there exists a point in between e and f, call it e'. The interval $[c, e']$ cannot satisfy p, since it overlaps the p-interval $[e, f]$ and p is disjointly-bounded in $\langle M, [a, b]\rangle$. Furthermore, $[c, e']$ is a sub-interval of $[c, d]$, and, by (7), each interval starting at a point between c and d and ending inside a q-interval (q being a disjoint consequent of p) satisfies $\overrightarrow{p}$. Hence, $[c, e']$ satisfies $\neg p$ and $[O](\langle O\rangle q \rightarrow \overrightarrow{p})$. By (8), it immediately follows that $[c, e']$ satisfies inside_p as well, and then, by (9), that $[c, e']$ overlaps no p-interval (contradiction). □

Hereafter, we will denote by *non-sub* (x, y), with x, y proposition letters in $\mathcal{AP}$, the formula obtained from (7) $\wedge$ (8) $\wedge$ (9) by replacing p by x, $\overrightarrow{p}$ by $\overrightarrow{x}$, inside_p by inside_x, and q by y, stating that no x-interval is a sub-interval of another one.

To complete the construction of the u-chain, the following formulae are needed, where first is used to identify the first interval of the u-chain.

$$\langle O\rangle\langle O\rangle(\mathsf{u}_1 \wedge \mathsf{first}) \tag{10}$$

$$[G](\mathsf{u} \vee \mathsf{k} \rightarrow [O]\neg\mathsf{first} \wedge [O][O]\neg\mathsf{first}) \tag{11}$$

$$[G]((\mathsf{first} \rightarrow \mathsf{u}_1) \wedge (\mathsf{first} \rightarrow [O][O]\neg\mathsf{first})) \tag{12}$$

$$\textit{non-sub}\,(\mathsf{u}_1, \mathsf{u}_2) \wedge \textit{non-sub}\,(\mathsf{u}_2, \mathsf{u}_1) \wedge \textit{non-sub}\,(\mathsf{k}_1, \mathsf{k}_2) \wedge \textit{non-sub}\,(\mathsf{k}_2, \mathsf{k}_1) \tag{13}$$

$$[G](\mathsf{u} \vee \mathsf{k} \rightarrow [O]\langle O\rangle(\mathsf{u} \vee \mathsf{k})) \tag{14}$$

$$(10) \wedge \ldots \wedge (14) \tag{u-chain}$$

Lemma 3 *Let M be a model and $[a, b]$ be an interval such that $M, [a, b] \Vdash$ (disj-bndu$^{\mathsf{k}}_{\mathsf{u}}$) $\wedge$ (u-chain). Then,*

(a) *there exists an infinite sequence of u-intervals $[b_0, b'_0], [b_1, b'_1], \ldots, [b_i, b'_i], \ldots$, with $b \le b_0$ and $b'_i = b_{i+1}$ for every $i \in \mathbb{N}$, such that $M, [b_0, b'_0] \Vdash \mathsf{first}$,*

(b) *there exists an infinite sequence of* k*-intervals* $[c_0, c_0'], [c_1, c_1'], \ldots, [c_i, c_i'], \ldots$ *such that* $b_i < c_i < b_i'$, $b_{i+1} < c_i' < b_{i+1}'$, *and* $c_i' = c_{i+1}$ *for every* $i \in \mathbb{N}$, *and*

(c) *any other interval* $[c, d] \in \mathcal{G}_{[a,b]}$ *satisfies no one of* u, k, *and* first, *unless* $c > b_i$ *for every* $i \in \mathbb{N}$.

Proof For the sake of simplicity, we first prove a weaker version of (a) and (b), namely,

(a') there exists an infinite sequence of u-intervals $[b_0, b_0'], [b_1, b_1'], \ldots, [b_i, b_i'], \ldots$, with $b \leq b_0$ and $b_i' \leq b_{i+1}$ (instead of $b_i' = b_{i+1}$) for every $i \in \mathbb{N}$, such that $M, [b_0, b_0'] \Vdash \mathsf{first}$, and

(b') there exists an infinite sequence of k-intervals $[c_0, c_0'], [c_1, c_1'], \ldots, [c_i, c_i'], \ldots$ such that $b_i < c_i < b_i'$, $b_{i+1} < c_i' < b_{i+1}'$, and $c_i' \leq c_{i+1}$ (instead of $c_i' = c_{i+1}$) for every $i \in \mathbb{N}$.

Then, we will prove (c), and, finally, we will force $b_i' = b_{i+1}$ and $c_i' = c_{i+1}$ for every $i \in \mathbb{N}$.

As a matter of fact, the existence of an u-chain and a k-chain that respectively satisfy (a') and (b') easily follows from formulae (1)–(4), (6), and (10).

To prove (c) , we show that any other interval (in $\mathcal{G}_{[a,b]}$) satisfies neither u nor k, unless its starting point is greater than all points belonging to the intervals of the u-chain and the k-chain. As a preliminary step, we show that no u-interval (resp., k-interval) belonging to $\mathcal{G}_{[a,b]}$ can be a proper sub-interval of an u-interval or a k-interval. Formula (5) guarantees that no u_1-interval (resp., k_1-interval) can be a sub-interval of an u_2-interval (resp., k_2-interval) and vice versa, that is, no u_2-interval (resp., k_2-interval) can be a sub-interval of an u_1-interval (resp., k_1-interval). Furthermore, by Lemma 1, u_1, u_2, k_1, and k_2 are disjointly-bounded, and thus from (13) it follows that no u_1-interval (resp., u_2-interval, k_1-interval, k_2-interval) can be a sub-interval of another u_1-interval (resp., u_2-interval, k_1-interval, k_2-interval). It remains to show that no u-interval can be a sub-interval of any k-interval, and vice versa. Suppose, for the sake of contradiction, that there exist an u-interval $[c', d']$ which is a sub-interval of a k-interval $[c'', d'']$. By (6), it follows that there exists a k-interval which starts in between c' and d'. Let that be $[c''', d''']$. Then, either $d''' \leq d''$ and the k-interval $[c''', d''']$ is a sub-interval of the k-interval $[c'', d'']$ (contradiction, as we just proved that this cannot be the case) or $d''' > d''$ and the k-interval $[c'', d'']$ overlaps the k-interval $[c''', d''']$ (contradiction, as (4) constrains k-intervals to not overlap). With a similar argument, one can show that no k-interval can be a sub-interval of an u-interval.

To conclude the proof of (c), suppose, for the sake of contradiction, that there exists an u-interval $[c, d]$ in $\mathcal{G}_{[a,b]}$ such that $[c, d] \neq [b_j, b_j']$, for every $j \in \mathbb{N}$, and $c \leq b_k$, for some $k \in \mathbb{N}$. By (1), $c \geq b$. We show that all possible choices for c lead to contradiction.

– If $b \leq c < b_0$, then one of the following two cases applies: (i) if $d < b_0'$, then (11) is violated; (ii) if $d \geq b_0'$, then the u-interval $[b_0, b_0']$ is a sub-interval of the u-interval $[c, d]$ (contradiction).
– If $c = b_i$ for some $i \leq k$, then one of the following cases applies: (i) if $d < b_i'$, then the u-interval $[c, d]$ is a sub-interval of the u-interval $[b_i, b_i']$ (contradiction); (ii) if $d = b_i'$, then $[c, d] = [b_i, b_i']$, against the assumption that $[c, d] \neq [b_i, b_i']$ for any

$i \in \mathbb{N}$; (iii) if $d > b'_i$, then the u-interval $[b_i, b'_i]$ is a sub-interval of the u-interval $[c, d]$ (contradiction).
- If $b_i < c < b'_i$ for some $i \leq k$, then one of the following two cases applies: (i) if $d \leq b'_i$, then the u-interval $[c, d]$ is a sub-interval of the u-interval $[b_i, b'_i]$ (contradiction); (ii) if $d > b'_i$, then the u-interval $[b_i, b'_i]$ overlaps the u-interval $[c, d]$, thus violating (3).
- If $b'_i \leq c < b_{i+1}$ for some $i < k$, then one of the following cases applies: (i) if $d \leq b_{i+1}$, then the u-interval $[c, d]$ is a sub-interval of the k-interval $[c_i, c'_i]$ (contradiction); (ii) if $b_{i+1} < d < b'_{i+1}$, then the u-interval $[c, d]$ overlaps the u-interval $[b_{i+1}, b'_{i+1}]$, thus violating (3); (iii) if $d \geq b'_{i+1}$, then the u-interval $[b_{i+1}, b'_{i+1}]$ is a sub-interval of the u-interval $[c, d]$ (contradiction).

A similar argument can be used to prove that there exists no k-interval $[c, d] \in \mathcal{G}_{[a,b]}$ except $[c_i, c'_i]$, $i \in \mathbb{N}$, unless $c > b_i$ for every $i \in \mathbb{N}$. Finally, assume that there exists a first-interval $[c, d]$ in $\mathcal{G}_{[a,b]}$ such that $[c, d] \neq [b_0, b'_0]$ and $c \leq b_k$ for some $k \in \mathbb{N}$ for the sake of contradiction. By the first conjunct of (12), $[c, d] = [b_i, b'_i]$ for some $i \in \mathbb{N}$, with $i \neq 0$ (we just proved that there are no other u-intervals), but this immediately leads to a violation of the second conjunct of (12).

We conclude the proof by showing that $b'_i = b_{i+1}$ and $c'_i = c_{i+1}$ for every $i \in \mathbb{N}$. Suppose, for the sake of contradiction, that $b'_i < b_{i+1}$ for some $i \in \mathbb{N}$. By (a') and (b'), there exist b_i, c_i, c'_i, b'_{i+1} such that $b_i < c_i < b'_i$, $b_{i+1} < c'_i < b'_{i+1}$, and $[c_i, c'_i]$ satisfies k. Furthermore, by (c), there exists no u- or k-interval starting in between c_i and b_{i+1}. It can be easily checked that the u-interval $[b_i, b'_i]$ overlaps the interval $[c_i, b_{i+1}]$ that, in turn, overlaps no u- or k-interval, thus violating (14). A similar argument can be used to prove that $c'_i = c_{i+1}$ for every $i \in \mathbb{N}$. $\square$

In the following, we will make use of a derived modality $\langle X_{\mathsf{u}} \rangle$ to access the first u-interval of the u-chain and to step from any given u-interval to the next one in the u-chain. $\langle X_{\mathsf{u}} \rangle$ is defined as follows:

$$\langle X_{\mathsf{u}} \rangle \varphi \equiv \langle O \rangle \langle O \rangle (\mathsf{first} \wedge \varphi) \vee (\mathsf{u} \wedge \langle O \rangle (\mathsf{k} \wedge \langle O \rangle (\mathsf{u} \wedge \varphi)))$$

Definition of the ld*-chain* The ld-chain is defined by the following set of formulae:

$$\neg \mathsf{ld} \wedge \neg \langle O \rangle \mathsf{ld} \wedge [G](\mathsf{ld} \rightarrow \neg \langle O \rangle \mathsf{ld}) \tag{15}$$

$$\langle X_{\mathsf{u}} \rangle (* \wedge \langle X_{\mathsf{u}} \rangle (\mathsf{tile} \wedge \mathsf{ld} \wedge \langle X_{\mathsf{u}} \rangle * \wedge [G](* \rightarrow \langle X_{\mathsf{u}} \rangle (\mathsf{tile} \wedge \langle X_{\mathsf{u}} \rangle \mathsf{tile})))) \tag{16}$$

$$[G]((\mathsf{u} \leftrightarrow * \vee \mathsf{tile}) \wedge (* \rightarrow \neg \mathsf{tile})) \tag{17}$$

$$[G](* \rightarrow \langle O \rangle (\mathsf{k} \wedge \langle O \rangle \mathsf{ld})) \tag{18}$$

$$[G](\mathsf{ld} \rightarrow \langle O \rangle (\mathsf{k} \wedge \langle O \rangle *)) \tag{19}$$

$$[G]((\mathsf{u} \rightarrow \neg \langle O \rangle \mathsf{ld}) \wedge (\mathsf{ld} \rightarrow \neg \langle O \rangle \mathsf{u})) \tag{20}$$

$$[G](\langle O \rangle * \rightarrow \neg \langle O \rangle \mathsf{ld}) \tag{21}$$

$$\textit{non-sub}\,(\mathsf{ld}, *) \tag{22}$$

$$(15) \wedge \ldots \wedge (22) \tag{ld-chain}$$

Lemma 4 *Let M be a model and $[a, b]$ be an interval such that $M, [a, b] \Vdash$ (disj-bndu$^{\mathsf{k}}_{\mathsf{u}}$) $\wedge$ (u-chain) $\wedge$ (ld-chain). Furthermore, let $b \leq b^0_1 < c^0_1 < b^1_1 < \ldots < b^{k_1-1}_1 < c^{k_1-1}_1 < b^{k_1}_1 = b^0_2 < c^{k_1}_1 = c^0_2 < b^1_2 < \ldots < b^{k_2}_2 = b^0_3 < \ldots$ be a sequence of points such that $[b^i_j, b^{i+1}_j]$ is an* u*-interval and $[c^i_j, c^{i+1}_j]$ is a* k*-interval, for every $j \geq 1, 0 \leq i < k_j$, whose existence is guaranteed by Lemma 3. Then for every $j \geq 1$:*

(a) $M, [b^0_j, b^1_j] \Vdash *$;
(b) $M, [b^i_j, b^{i+1}_j] \Vdash$ tile *for each* $0 < i < k_j$;
(c) $M, [b^1_j, b^0_{j+1}] \Vdash$ ld;
(d) $k_1 = 2$, *and* $k_\ell > 2$ *for* $\ell > 1$.

Furthermore, any other interval $[c, d] \in \mathcal{G}_{[a,b]}$ satisfies no one of $*$, tile*, and* ld*, unless $c > b^i_j$ for every $i, j > 0$.*

Proof As a first step, we show that ld is a disjointly-bounded proposition in $\langle M, [a, b]\rangle$ with disjoint consequent $*$. By (15), it can be easily checked that ld meets the first two conditions of Definition 2. By (17), (20) and (21), $*$ and ld are disjoint, and, by (19), $*$ is a disjoint consequent of ld. Now, we prove the statements of the lemma one by one.

(a) From (16), (18), and (19), it immediately follows that there exists an infinite sequence of $*$-intervals. Without loss of generality, we can assume it to be $[b^0_1, b^1_1], [b^0_2, b^1_2], \ldots, [b^0_j, b^1_j], \ldots$. Furthermore, by the first conjunct of (17), we can assume that, for every $j > 0$, there exists no a $*$-interval between $[b^0_j, b^1_j]$ and $[b^0_{j+1}, b^1_{j+1}]$.
(b) By (17), every interval satisfying $*$ or tile is an u-interval and every u-interval satisfies either $*$ or tile. Then all u-intervals between two consecutive $*$-intervals (if any) must be tile-intervals.
(c) By (18), for every k-interval $[c^0_j, c^1_j]$ overlapped by a $*$-interval, there exists an ld-interval $[c, d]$ such that $c^0_j < c < c^1_j < d$. We show that $c = b^1_j$ and $d = b^0_{j+1}$. For the sake of contradiction, suppose that $c \neq b^1_j$. If $c < b^1_j$, then the u-interval $[b^0_j, b^1_j]$ overlaps the ld-interval $[c, d]$, thus violating (20). In case $c > b^1_j$, two alternatives must be considered, both leading to contradiction.

   (i) $j = 1$. By (16), $[b^1_1, b^2_1]$ is the ld-interval corresponding to the first row of the octant. Now, if $d > b^2_1$, then the u-interval $[b^1_1, b^2_1]$ overlaps the ld-interval $[c, d]$, violating (20); otherwise, if $d \leq b^2_1$, then the ld-interval $[c, d]$ is a sub-interval of the ld-interval $[b^1_1, b^2_1]$, violating (22).
   (ii) $j > 1$. By (16), $[b^1_j, b^2_j]$ is not the last tile-interval of the jth row, and thus the k-interval $[c^1_j, c^2_j]$ overlaps no $*$-interval ($[b^2_j, b^3_j]$ is a tile-interval). By (19), it must be $d \geq c^2_j$, from which it follows that the u-interval $[b^1_j, b^2_j]$ overlaps the ld-interval $[c, d]$, thus violating (20).

   We show now that $d = b^0_{j+1}$, that is, the ld-interval starting at the right endpoint of the $*$-interval $[b^0_j, b^1_j]$ ends at the left endpoint of the next $*$-interval $[b^0_{j+1}, b^1_{j+1}]$. Suppose, for the sake of contradiction, that $d \neq b^0_{j+1}$. Two cases must be considered: $j = 1$ and $j > 1$. If $j = 1$, then from $d < b^0_2$ (resp., $d > b^0_2$), it follows that the ld-interval $[c, d]$ (resp., $[b^1_1, b^2_1]$) is a sub-interval of the

ld-interval $[b_1^1, b_1^2]$ (resp., $[c, d]$), violating (22). In case $j > 1$, several alternatives must be considered, all leading to contradiction:

(i) if $d \le c_j^{k_j-1}$, then (19) is violated, since either $[c, d]$ overlaps no k-interval or it overlaps a k-interval that overlaps no $*$-interval;

(ii) if $c_j^{k_j-1} < d < b_{j+1}^0$, then the ld-interval $[c, d]$ overlaps the u-interval $[b_j^{k_j-1}, b_j^{k_j}]$, thus violating (20);

(iii) if $b_{j+1}^0 < d < b_{j+1}^1$, then the ld-interval $[c, d]$ overlaps the u-interval $[b_{j+1}^0, b_{j+1}^1]$, again violating (20);

(iv) if $d \ge b_{j+1}^1$, then (21) is violated, since any interval $[a', c_{j+1}^0]$ such that $a < a' < b$ (by (1), there exists at least one such interval) overlaps both the $*$-interval $[b_{j+1}^0, b_{j+1}^1]$ and the ld-interval $[c, d]$.

(d) It immediately follows from (16).

Finally, for the sake of contradiction, suppose that there exists an ld-interval $[c, d] \in \mathcal{G}_{[a,b]}$ such that $[c, d] \ne [b_j^1, b_{j+1}^0]$, for every $j > 0$, and $c \le b_j^i$, for some $i, j > 0$. By (15), $[a, b]$ is not an ld-interval and it overlaps no ld-interval. Hence, $c \ge b$. We show now that all possible choices for c ($\ge b$) lead to contradiction:

(i) if $b \le c < b_1^0$, then, by (19), it must be $d > c_1^0$, which causes a violation of (21);

(ii) if $b_j^0 \le c < c_j^0$ for some $j > 0$, then (21) is violated;

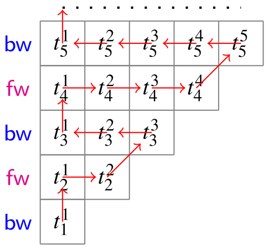
(iii) if $c_j^0 \le c < b_j^1$ for some $j > 0$, then, by (19), it must be $d > c_j^1$ and thus the u-interval $[b_j^0, b_j^1]$ overlaps the ld-interval $[c, d]$, violating (20);

(iv) if $c = b_j^1$ for some $j > 0$, then it must be $d = b_{j+1}^0$ (see proof of statement (c));

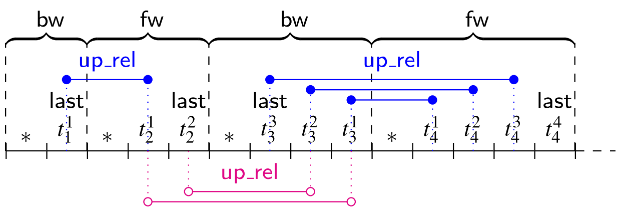
(v) if $b_j^1 < c < b_{j+1}^0$ for some $j > 0$ then either $d \le b_{j+1}^0$ and the ld-interval $[c, d]$ is a sub-interval of the ld-interval $[b_j^1, b_{j+1}^0]$, violating (22), or $d > b_{j+1}^0$ and the ld-interval $[b_j^1, b_{j+1}^0]$ overlaps the ld-interval $[c, d]$, violating (15).

A similar argument can be used to prove that no other interval $[c, d] \in \mathcal{G}_{[a,b]}$ satisfies $*$ or tile, unless $c > b_j^i$ for every $i, j > 0$ □

Above-neighbor relation The next (difficult) step is the encoding of the above-neighbor relation (the idea is depicted in Fig. 2). The main role here is played by proposition letter $\mathsf{up\_rel}$, used to connect each tile-interval with its upper level



(a) Cartesian representation.



(b) Interval representation.

Fig. 2 Encoding of the above-neighbor relation in O

neighbor in the octant, e.g., t_2^2 with t_3^2 in Fig. 2). For technical reasons, we need to partition rows on $\mathcal{O}$ into *backward* and *forward* rows making use of proposition letters bw and fw. Intuitively, tiles belonging to forward rows on $\mathcal{O}$ are encoded in ascending order, while those belonging to backward rows are encoded in descending order (the tiling is encoded in a zig-zag manner by suitably connecting forward and backward rows). In particular, this means that the leftmost tile-interval of a backward row encodes the last tile of that row of $\mathcal{O}$ (not the first one). Let α, β range over $\{\mathsf{bw}, \mathsf{fw}\}$ such that $\alpha \neq \beta$. We label each u-interval with bw (resp., fw) if it belongs to a backward (resp., forward) row:

$$\langle X_{\mathsf{u}}\rangle \mathsf{bw} \wedge [G]((\mathsf{u} \leftrightarrow \mathsf{bw} \vee \mathsf{fw}) \wedge (\mathsf{bw} \rightarrow \neg \mathsf{fw})) \tag{23}$$

$$[G]((\alpha \wedge \neg \langle X_{\mathsf{u}}\rangle * \rightarrow \langle X_{\mathsf{u}}\rangle \alpha) \wedge (\alpha \wedge \langle X_{\mathsf{u}}\rangle * \rightarrow \langle X_{\mathsf{u}}\rangle \beta)) \tag{24}$$

$$(23) \wedge (24) \tag{bw/fw}$$

Lemma 5 *Let M be a model, $[a, b]$ be an interval over M, and $b \le b_1^0 < c_1^0 < b_1^1 < \ldots < b_1^{k_1-1} < c_1^{k_1-1} < b_1^{k_1} = b_2^0 < c_1^{k_1} = c_2^0 < b_2^1 < \ldots < b_2^{k_2} = b_3^0 < \ldots$ be the sequence of points defined by Lemma 4. If $M, [a, b] \Vdash$ (disj-bndu$_{\mathsf{u}}^{\mathsf{k}}$) $\wedge$ (u-chain) $\wedge$ (ld-chain) $\wedge$ (bw/fw), then $M, [b_j^i, b_j^{i+1}] \Vdash \mathsf{bw}$ (resp., $M, [b_j^i, b_j^{i+1}] \Vdash \mathsf{fw}$) if and only if j is an odd (resp., even) number, for every $j \ge 1$. Furthermore, no other interval $[c, d] \in \mathcal{G}_{[a,b]}$ satisfies bw or fw, unless $c > b_j^i$ for every $i, j > 0$.*

The alternation between backward and forward rows makes it possible to correctly encode the above-neighbor relation by constraining every $\mathsf{up\_rel}$-interval starting from a backward (resp., forward) row not to overlap any other $\mathsf{up\_rel}$-interval starting from a backward (resp., forward) row. The structure of the encoding is shown in Fig. 2b, where $\mathsf{up\_rel}$-intervals starting from a given forward (resp., backward) row are placed one inside the other. Consider, for instance, how the 3rd and 4th rows on the octant are represented in Fig. 2b. The 1st tile-interval of the 3rd row (t_3^3) is connected to the second-last tile-interval of the 4th row (t_4^3), the 2nd tile-interval of the 3rd row (t_3^2) is connected to the third-last tile-interval of the 4th row (t_4^2), and so on. Notice that, in forward (resp., backward) rows, the last (resp., first) tile-interval has no tile-intervals above-connected to it, in order to constrain each row to have exactly one tile-interval more than the previous one (these tile-intervals are labeled with last).

Formally, the above-neighbor relation is defined as follows. We constrain every tile-interval $[b_j^i, b_j^{i+1}]$ belonging to a forward (resp., backward) row to be above-connected to the tile-interval $[b_{j+1}^{j+2-i}, b_{j+1}^{j+2-i+1}]$ (resp., $[b_{j+1}^{j+2-i-1}, b_{j+1}^{j+2-i}]$) by labeling the interval $[c_j^i, c_{j+1}^{j+2-i}]$ (resp., $[c_j^i, c_{j+1}^{j+2-i-1}]$) with $\mathsf{up\_rel}$. We distinguish between $\mathsf{up\_rel}$-intervals starting from forward and backward rows and, within each case, between those starting from odd and even tile-intervals. To this end, we use a new proposition letter $\mathsf{up\_rel}_{\mathsf{o}}^{\mathsf{bw}}$ (resp., $\mathsf{up\_rel}_{\mathsf{e}}^{\mathsf{bw}}$, $\mathsf{up\_rel}_{\mathsf{o}}^{\mathsf{fw}}$, $\mathsf{up\_rel}_{\mathsf{e}}^{\mathsf{fw}}$) to label $\mathsf{up\_rel}$-intervals starting from an odd tile-interval of a backward row (resp., even tile-interval/backward row, odd/forward, even/forward). To ease the reading of formulae, we group $\mathsf{up\_rel}_{\mathsf{o}}^{\mathsf{bw}}$ and $\mathsf{up\_rel}_{\mathsf{e}}^{\mathsf{bw}}$ in $\mathsf{up\_rel}^{\mathsf{bw}}$ ($\mathsf{up\_rel}^{\mathsf{bw}} \leftrightarrow \mathsf{up\_rel}_{\mathsf{o}}^{\mathsf{bw}} \oplus \mathsf{up\_rel}_{\mathsf{e}}^{\mathsf{bw}}$, where $\oplus$ denotes "exclusive or"), and similarly for $\mathsf{up\_rel}^{\mathsf{fw}}$. Finally, $\mathsf{up\_rel}$ is one of $\mathsf{up\_rel}^{\mathsf{bw}}$ and $\mathsf{up\_rel}^{\mathsf{fw}}$ ($\mathsf{up\_rel} \leftrightarrow \mathsf{up\_rel}^{\mathsf{bw}} \oplus \mathsf{up\_rel}^{\mathsf{fw}}$). Let α, β range over $\{\mathsf{bw}, \mathsf{fw}\}$ and let γ, δ range over $\{\mathsf{o}, \mathsf{e}\}$, with $\alpha \neq \beta$ and $\gamma \neq \delta$. We encode the correspondence between tiles of consecutive rows on the plane induced by the above-neighbor relation as follows:

$$\neg \mathsf{up\_rel} \wedge \neg\langle O\rangle \mathsf{up\_rel} \tag{25}$$

$$[G]((\mathsf{up\_rel} \leftrightarrow \mathsf{up\_rel}^{\mathsf{bw}} \vee \mathsf{up\_rel}^{\mathsf{fw}}) \wedge (\mathsf{up\_rel}^{\alpha} \leftrightarrow \mathsf{up\_rel}^{\alpha}_{\mathsf{o}} \vee \mathsf{up\_rel}^{\alpha}_{\mathsf{e}})) \tag{26}$$

$$[G]((\mathsf{k} \vee * \rightarrow \neg\langle O\rangle \mathsf{up\_rel}) \wedge (\mathsf{up\_rel} \rightarrow \neg\langle O\rangle \mathsf{k})) \tag{27}$$

$$[G](\mathsf{u} \wedge \langle O\rangle \mathsf{up\_rel}^{\alpha}_{\gamma} \rightarrow \neg\langle O\rangle \mathsf{up\_rel}^{\alpha}_{\delta} \wedge \neg\langle O\rangle \mathsf{up\_rel}^{\beta}) \tag{28}$$

$$[G](\mathsf{up\_rel}^{\alpha} \rightarrow \neg\langle O\rangle \mathsf{up\_rel}^{\alpha}) \tag{29}$$

$$[G](\mathsf{up\_rel} \rightarrow \langle O\rangle \mathsf{ld}) \tag{30}$$

$$[G](\langle O\rangle \mathsf{up\_rel} \rightarrow \neg\langle O\rangle \mathsf{first}) \tag{31}$$

$$[G](\mathsf{up\_rel}^{\alpha}_{\gamma} \rightarrow \langle O\rangle(\mathsf{tile} \wedge \langle O\rangle \mathsf{up\_rel}^{\beta}_{\gamma})) \tag{32}$$

$$(25) \wedge \ldots \wedge (32) \tag{$\mathsf{up\_rel\text{-}def}_1$}$$

Lemma 6 *Let M be a model, $[a, b]$ be an interval over M, and $b \leq b^0_1 < c^0_1 < b^1_1 < \ldots < b^{k_1-1}_1 < c^{k_1-1}_1 < b^{k_1}_1 = b^0_2 < c^{k_1}_1 = c^0_2 < b^1_2 < \ldots < b^{k_2}_2 = b^0_3 < \ldots$ be the sequence of points defined by Lemma 4. If $M, [a, b] \Vdash$ ($\mathsf{disj\text{-}bndu}^{\mathsf{k}}_{\mathsf{u}}$) $\wedge$ ($\mathsf{u\text{-}chain}$) $\wedge$ ($\mathsf{ld\text{-}chain}$) $\wedge$ ($\mathsf{bw/fw}$) $\wedge$ ($\mathsf{up\_rel\text{-}def}_1$), then, it holds that:*

(a) *if $[c, d]$ is an $\mathsf{up\_rel}$-interval, then $c = c^i_j$ and $d = c^{i'}_{j'}$ for some $i, i', j, j' > 0$;*
(b) *for all $i, i', j, j' > 0$, $[c^i_j, c^{i'}_{j'}]$ is an $\mathsf{up\_rel}$-interval iff it is either an $\mathsf{up\_rel}^{\mathsf{bw}}$-interval or an $\mathsf{up\_rel}^{\mathsf{fw}}$-interval, and $[c^i_j, c^{i'}_{j'}]$ is an $\mathsf{up\_rel}^{\mathsf{bw}}$-interval (resp., $\mathsf{up\_rel}^{\mathsf{fw}}$-interval) iff it is either an $\mathsf{up\_rel}^{\mathsf{bw}}_{\mathsf{o}}$-interval or an $\mathsf{up\_rel}^{\mathsf{bw}}_{\mathsf{e}}$-interval (resp., $\mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{o}}$-interval or $\mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{e}}$-interval);*
(c) *for all $i, i', j, j' > 0$, $\alpha, \beta \in \{\mathsf{bw}, \mathsf{fw}\}$, and $\gamma, \delta \in \{\mathsf{o}, \mathsf{e}\}$, if $[c^i_j, c^{i'}_{j'}]$ is an $\mathsf{up\_rel}^{\alpha}_{\gamma}$-interval, then there exists no $\mathsf{up\_rel}^{\beta}_{\delta}$-interval starting at c^i_j with $\mathsf{up\_rel}^{\alpha}_{\gamma} \neq \mathsf{up\_rel}^{\beta}_{\delta}$;*
(d) *no $\mathsf{up\_rel}^{\mathsf{bw}}$-interval (resp., $\mathsf{up\_rel}^{\mathsf{fw}}$-interval) overlaps another $\mathsf{up\_rel}^{\mathsf{bw}}$-interval (resp., $\mathsf{up\_rel}^{\mathsf{fw}}$-interval);*
(e) *for all $i, i', j, j' > 0$, if $[c^i_j, c^{i'}_{j'}]$ is an $\mathsf{up\_rel}^{\mathsf{bw}}_{\mathsf{o}}$-interval (resp., $\mathsf{up\_rel}^{\mathsf{bw}}_{\mathsf{e}}$-interval, $\mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{o}}$-interval, $\mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{e}}$-interval), then there exists an $\mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{o}}$-interval (resp., $\mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{e}}$-interval, $\mathsf{up\_rel}^{\mathsf{bw}}_{\mathsf{o}}$-interval, $\mathsf{up\_rel}^{\mathsf{bw}}_{\mathsf{e}}$-interval) starting at $c^{i'}_{j'}$.*

Proof We only prove (a), which can be rephrased as follows: each $\mathsf{up\_rel}$-interval starts (resp., ends) at the same point in which a k-interval starts (resp., ends). The other statements of the lemma are rather straightforward. Let $[c, d]$ be an $\mathsf{up\_rel}$-interval. As a preliminary step, we observe that it cannot be the case that $c = c^0_j$ (resp., $d = c^0_{j'}$) for some $j \geq 1$ (resp., $j' \geq 1$), since this would imply the existence of an $\mathsf{up\_rel}$-interval overlapped by (resp., overlapping) a $*$-interval, thus violating (27).

We first show that $c = c^i_j$, for some $i, j > 0$. By (25), $c \geq b$. Since, by (1), the length of $[a, b]$ is greater than or equal to 2, it immediately follows that there exists at least one point e, with $a < e < b$, such that no $\mathsf{up\_rel}$-interval starts at e. Together with (31) and (32), this allows us to conclude that $c \geq c^0_1$. Furthermore, by (27) and (32), it cannot be the case that $b^i_j \leq c < c^i_j$ for any $i \geq 0$, $j > 0$. It only remains to exclude that $c^i_j < c < b^{i+1}_j$ for some $i \geq 0$, $j > 0$. For the sake of contradiction, suppose that $c^i_j <$

$c < b_j^{i+1}$ for some $i \geq 0$, $j > 0$. If $d > c_j^{i+1}$, then (27) is violated. Thus, let us assume $d \leq c_j^{i+1}$. By (30), $[c, d]$ overlaps an ld-interval, which necessarily starts at b_j^{i+1}, and thus $[b_j^i, b_j^{i+1}]$ is a $*$-interval. Since $[b_j^i, b_j^{i+1}]$ overlaps the $\mathsf{up\_rel}$-interval $[c, d]$, (27) is violated.

We now show that $d = c_{j'}^{i'}$ for some $i', j' > 0$. For the sake of contradiction, we assume that $d \neq c_{j'}^{i'}$ for every $i', j' > 0$. Two cases are possible, both leading to contradiction.

(i) $c = c_j^i$ and $d < c_j^{i+1}$. By (30), $d > b_j^{i+1}$ and an ld-interval starts at b_j^{i+1}. $[b_j^i, b_j^{i+1}]$ is thus a $*$-interval. Since $[b_j^i, b_j^{i+1}]$ overlaps the $\mathsf{up\_rel}$-interval $[c, d]$, (27) is violated.

(ii) $c = c_j^i$ and $d > c_j^{i+1}$. Hence, the $\mathsf{up\_rel}$-interval $[c, d]$ overlaps the k-interval starting at c_j^{i+1}, thus violating (27). □

To complete the encoding of the above-neighbor relation, we constrain each tile-interval, apart from those encoding the last tile of a row, to have a tile-interval above-connected to it. To this end, we first label any tile interval representing the last tile of a row with the new proposition letter last (formulae (38)–(40)). Then, we force all tile-intervals which are not labeled with last to have a tile-interval above-connected to them (formulae (41)–(44)):

$$[G](\mathsf{tile} \rightarrow \langle O\rangle \mathsf{up\_rel}) \tag{33}$$

$$[G](\alpha \rightarrow [O](\mathsf{up\_rel} \rightarrow \mathsf{up\_rel}^{\alpha})) \tag{34}$$

$$[G](\mathsf{up\_rel}^{\alpha} \rightarrow \langle O\rangle \beta) \tag{35}$$

$$[G](\langle O\rangle * \rightarrow \neg(\langle O\rangle \mathsf{up\_rel}^{\mathsf{bw}} \wedge \langle O\rangle \mathsf{up\_rel}^{\mathsf{fw}})) \tag{36}$$

$$[G](\mathsf{tile} \wedge \langle O\rangle \mathsf{up\_rel}^{\alpha}_{\gamma} \wedge \langle X_{\mathsf{u}}\rangle \mathsf{tile} \rightarrow \langle X_{\mathsf{u}}\rangle(\mathsf{tile} \wedge \langle O\rangle \mathsf{up\_rel}^{\alpha}_{\delta})) \tag{37}$$

$$[G](\mathsf{last} \rightarrow \mathsf{tile}) \tag{38}$$

$$[G]((* \wedge \mathsf{bw} \rightarrow \langle X_{\mathsf{u}}\rangle \mathsf{last}) \wedge (\mathsf{fw} \wedge \langle X_{\mathsf{u}}\rangle * \rightarrow \mathsf{last})) \tag{39}$$

$$[G]((\mathsf{last} \wedge \mathsf{fw} \rightarrow \langle X_{\mathsf{u}}\rangle *) \wedge (\mathsf{bw} \wedge \langle X_{\mathsf{u}}\rangle \mathsf{last} \rightarrow *)) \tag{40}$$

$$[G](* \wedge \mathsf{fw} \rightarrow \langle X_{\mathsf{u}}\rangle(\mathsf{tile} \wedge \langle O\rangle(\mathsf{up\_rel} \wedge \langle O\rangle(\mathsf{tile} \wedge \langle X_{\mathsf{u}}\rangle *)))) \tag{41}$$

$$[G](\mathsf{last} \wedge \mathsf{bw} \rightarrow \langle O\rangle(\mathsf{up\_rel} \wedge \langle O\rangle(\mathsf{tile} \wedge \langle X_{\mathsf{u}}\rangle(\mathsf{tile} \wedge \langle X_{\mathsf{u}}\rangle *)))) \tag{42}$$

$$\begin{aligned} &[G](\mathsf{k} \wedge \langle O\rangle(\mathsf{tile} \wedge \langle O\rangle \mathsf{up\_rel}^{\alpha}_{\gamma}) \\ &\quad \rightarrow [O](\langle O\rangle \mathsf{up\_rel}^{\alpha}_{\gamma} \wedge \langle O\rangle(\mathsf{k} \wedge \langle O\rangle(\mathsf{tile} \wedge \langle O\rangle \mathsf{up\_rel}^{\beta}_{\delta} \wedge \neg \mathsf{last})) \\ &\quad \rightarrow \langle O\rangle \mathsf{up\_rel}^{\alpha}_{\delta})) \end{aligned} \tag{43}$$

$$[G](\mathsf{up\_rel} \rightarrow \neg\langle O\rangle \mathsf{last}) \tag{44}$$

$$(33) \wedge \ldots \wedge (44) \tag{$\mathsf{up\_rel\text{-}def}_2$}$$

Lemma 7 *Let M be a model, $[a, b]$ be an interval over M, and $b \leq b_1^0 < c_1^0 < b_1^1 < \ldots < b_1^{k_1-1} < c_1^{k_1-1} < b_1^{k_1} = b_2^0 < c_1^{k_1} = c_2^0 < b_2^1 < \ldots < b_2^{k_2} = b_3^0 < \ldots$ be the* *sequence of points defined by Lemma 4. If* $M, [a, b] \Vdash$ (disj-bndu$_\mathsf{u}^\mathsf{k}$) $\wedge$ (u-chain) $\wedge$ (ld-chain) $\wedge$ (bw/fw) $\wedge$ (up_rel-def$_1$) $\wedge$ (up_rel-def$_2$), *then it holds that:*

(a) *for every* up_rel-*interval* $[c^i_j, c^{i'}_{j'}]$ *connecting the* tile-*interval* $[b^i_j, b^{i+1}_j]$ *to the* tile-*interval* $[b^{i'}_{j'}, b^{i'+1}_{j'}]$, *if* $[c^i_j, c^{i'}_{j'}]$ *is an* up_rel$^\mathsf{bw}$-*interval (resp.,* up_rel$^\mathsf{fw}$-*interval), then* $[b^i_j, b^{i+1}_j]$ *is a* bw-*interval (resp.,* fw-*interval) and* $[b^{i'}_{j'}, b^{i'+1}_{j'}]$ *is a* fw-*interval (resp.,* bw-*interval);*

(b) *for every* tile-*interval* $[b^i_j, b^{i+1}_j]$, *with* $i < k_j - 1$, *such that there exists an* up_rel$_\mathsf{o}^\mathsf{bw}$-*interval (resp.,* up_rel$_\mathsf{e}^\mathsf{bw}$-*interval,* up_rel$_\mathsf{o}^\mathsf{fw}$-*interval,* up_rel$_\mathsf{e}^\mathsf{fw}$-*interval) starting at* c^i_j, *there exists an* up_rel$_\mathsf{e}^\mathsf{bw}$-*interval (resp.,* up_rel$_\mathsf{o}^\mathsf{bw}$-*interval,* up_rel$_\mathsf{e}^\mathsf{fw}$-*interval,* up_rel$_\mathsf{o}^\mathsf{fw}$-*interval) starting at* c^{i+1}_j *(strict alternation property);*

(c) *for all i, j, if* $[b^i_j, b^{i+1}_j]$ *satisfies* tile *and* last, *then intervals ending ată*c^i_j *are not* up_rel*;*

(d) *for each* up_rel-*interval* $[c^i_j, c^{i'}_{j'}]$ *such that* $0 < i < k_j$, $j' = j + 1$.

Proof

(a) Let $[c^i_j, c^{i'}_{j'}]$ be an up_rel-interval connecting the tile-interval $[b^i_j, b^{i+1}_j]$ to the tile-interval $[b^{i'}_{j'}, b^{i'+1}_{j'}]$. Suppose, for the sake of contradiction, that $[c^i_j, c^{i'}_{j'}]$ is an up_rel$^\mathsf{bw}$-interval (the other case is symmetric) and $[b^i_j, b^{i+1}_j]$ is an fw-interval. Then, (34) is violated. Similarly, if $[b^{i'}_{j'}, b^{i'+1}_{j'}]$ is a bw-interval, then (35) is violated.

(b) Straightforward (by (37)).

(c) Straightforward (by (44)).

(d) Let $[c^i_j, c^{i'}_{j'}]$ be an up_rel-interval such that $0 < i < k_j$. We assume $[c^i_j, c^{i'}_{j'}]$ to be an up_rel$^\mathsf{bw}$-interval (the other case is symmetric). For the sake of contradiction, suppose that $j' \neq j + 1$. We just proved (item (7)) that $[b^i_j, b^{i+1}_j]$ is a bw-interval and $[b^{i'}_{j'}, b^{i'+1}_{j'}]$ is a fw-interval. Two cases must be considered.

   (i) Let $j' = j$. Then $[b^i_j, b^{i+1}_j]$ and $[b^{i'}_{j'}, b^{i'+1}_{j'}]$ belong to the same ld-interval. By Lemma 5, both of them must be labeled either by bw or by fw (contradiction).

   (ii) Let $j' > j + 1$. Consider a tile-interval $[b^h_{j+1}, b^{h+1}_{j+1}]$ belonging to the $(j + 1)$th row. Since $[b^i_j, b^{i+1}_j]$ is a bw-interval, by Lemma 5, $[b^h_{j+1}, b^{h+1}_{j+1}]$ is a fw-interval. By (33) and (34), there exists an up_rel$^\mathsf{fw}$-interval starting at c^h_{j+1} and ending at some $c^{h'}_{j''}$, with $j'' > j + 1$ (by (i), j'' cannot be equal to $j + 1$). Now consider the $*$-interval $[b^0_{j+2}, b^1_{j+2}]$. Since, by (1), the length of $[a, b]$ is at least 2, there exists an e such that $a < e < b$ and $[e, c^0_{j+2}]$ overlaps the $*$-interval $[b^0_{j+2}, b^1_{j+2}]$, the up_rel$^\mathsf{bw}$-interval $[c^i_j, c^{i'}_{j'}]$, and the up_rel$^\mathsf{fw}$-interval $[c^h_{j+1}, c^{h'}_{j''}]$, thus violating (36). $\square$

Lemma 8 *Let M be a model,* $[a, b]$ *be an interval over M, and* $b \leq b^0_1 < c^0_1 < b^1_1 < \ldots < b^{k_1-1}_1 < c^{k_1-1}_1 < b^{k_1}_1 = b^0_2 < c^{k_1}_1 = c^0_2 < b^1_2 < \ldots < b^{k_2}_2 = b^0_3 < \ldots$ *be the sequence of points defined by Lemma 4. Then, each* tile-*interval is above-connected* *to exactly one* tile*-interval and, if it does not satisfy* last*, then there exists exactly one* tile*-interval which is above-connected to it.*

Proof

Step 1. From (33) and Lemma 6(a), it immediately follows that each tile-interval is above-connected with at least one tile-interval.

Step 2. We prove now that if a tile-interval is not a last-interval, then there exists a tile-interval which is above-connected to it. Assume the contrary for the sake of contradiction. The proof is by induction on the position of the tile-interval within the ld-interval it belongs to.

Base case Let $[b^i_j, b^{i+1}_j]$ be the rightmost tile-interval, belonging to the jth ld-interval, which is not a last-interval. If $[b^i_j, b^{i+1}_j]$ is a fw-interval (resp., bw-interval), then $i = k_j - 2$ (resp., $i = k_j - 1$). Formula (42) (resp., (41)) guarantees the existence of an up_rel-interval ending at c^i_j (contradiction).

Inductive step Let $[b^i_j, b^{i+1}_j]$ be tile-interval, belonging to the jth ld-interval, which is not a last-interval, but not the rightmost one. By inductive hypothesis, there exists an up_rel-interval ending at c^{i+1}_j and starting at some point $c^{i'}_{j-1}$. We prove that there exists also an up_rel-interval ending at c^i_j. Without loss of generality, suppose that $[c^{i'}_{j-1}, c^{i+1}_j]$ satisfies $\mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{o}}$. By item (e) of Lemma 6, an $\mathsf{up\_rel}^{\mathsf{bw}}_{\mathsf{o}}$-interval starts at c^{i+1}_j, and then, by Lemma 7(b) (strict alternation property), an $\mathsf{up\_rel}^{\mathsf{bw}}_{\mathsf{e}}$-interval starts at c^i_j.

We now focus our attention on the k-interval $[c^{i'-1}_{j-1}, c^{i'}_{j-1}]$, showing that we get contradiction with the condition expressed by formula (43). First, we observe that $[c^{i'-1}_{j-1}, c^{i'}_{j-1}]$ satisfies the formula $\mathsf{k} \wedge \langle O\rangle(\mathsf{tile} \wedge \langle O\rangle \mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{o}})$ and it overlaps the u-interval $[b^{i'}_{j-1}, b^i_j]$, which satisfies the formula $\langle O\rangle \mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{o}}$, as $[c^{i'}_{j-1}, c^{i+1}_j]$ is a $\mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{o}}$-interval, and the formula $\langle O\rangle(\mathsf{k} \wedge \langle O\rangle(\mathsf{tile} \wedge \langle O\rangle \mathsf{up\_rel}^{\mathsf{bw}}_{\mathsf{e}} \wedge \neg \mathsf{last}))$, as $[c^{i-1}_j, c^i_j]$ is a k-interval that overlaps the tile-interval $[b^i_j, b^{i+1}_j]$, which is not a last-interval (by hypothesis) and overlaps an $\mathsf{up\_rel}^{\mathsf{bw}}_{\mathsf{e}}$-interval (the one starting at c^i_j).

To realize that (43) is violated it suffices to establish that $[b^{i'}_{j-1}, b^i_j]$ does not satisfy $\langle O\rangle \mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{e}}$. For the sake of contradiction, suppose that there exists an $\mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{e}}$-interval $[e, f]$ such that $b^{i'}_{j-1} < e < b^i_j < f$. We show that all possible choices for e and f lead to contradiction.

– If $f > c^{i+1}_j$ and $e > c^{i'}_{j-1}$, then the $\mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{o}}$-interval $[c^{i'}_{j-1}, c^{i+1}_j]$ overlaps the $\mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{e}}$-interval $[e, f]$ (contradiction with Lemma 6(d)).
– If $f > c^{i+1}_j$ and $e = c^{i'}_{j-1}$, then both the $\mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{o}}$-interval $[c^{i'}_{j-1}, c^{i+1}_j]$ and the $\mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{e}}$-interval $[e, f]$ start at $c^{i'}_{j-1}$ (contradiction with item (c) of Lemma 6).
– If $f = c^{i+1}_j$, then both the $\mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{o}}$-interval $[c^{i'}_{j-1}, c^{i+1}_j]$ and the $\mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{e}}$-interval $[e, f]$ end at c^{i+1}_j and thus, by Lemma 6(e), there are an $\mathsf{up\_rel}^{\mathsf{bw}}_{\mathsf{o}}$-interval and an $\mathsf{up\_rel}^{\mathsf{bw}}_{\mathsf{e}}$-interval that both start at c^{i+1}_j (contradiction with item (c) of Lemma 6).

– If $f = c^i_j$, then there exists a tile-interval above-connected to $[b^i_j, b^{i+1}_j]$ via $[e, f]$ against the assumption.

Step 3. To complete the proof, we need to guarantee uniqueness. For the sake of contradiction, suppose that for some c^i_j, $c^{i'}_{j+1}$, and $c^{i''}_{j+1}$ such that $c^{i'}_{j+1} < c^{i''}_{j+1}$ (the case $c^{i'}_{j+1} > c^{i''}_{j+1}$ is symmetric), both $[c^i_j, c^{i'}_{j+1}]$ and $[c^i_j, c^{i''}_{j+1}]$ are $\mathsf{up\_rel}$-intervals. By Lemma 6(c), they both satisfy one of $\mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{o}}$, $\mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{e}}$, $\mathsf{up\_rel}^{\mathsf{bw}}_{\mathsf{o}}$, and $\mathsf{up\_rel}^{\mathsf{bw}}_{\mathsf{e}}$. Let that be $\mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{o}}$ (the other cases are analogous). Then, by Lemma 6(e), both $c^{i'}_{j+1}$ and $c^{i''}_{j+1}$ start an $\mathsf{up\_rel}^{\mathsf{bw}}_{\mathsf{o}}$-interval. By Lemma 7(b) (strict alternation property), an $\mathsf{up\_rel}^{\mathsf{bw}}_{\mathsf{e}}$-interval starts at $c^{i'+1}_{j+1}$. Since $[b^{i'+1}_{j+1}, b^{i'+2}_{j+1}]$ is not a last-interval (it is neither the rightmost nor the leftmost tile-interval of the $(j+1)$th ld-interval), there exists a c such that $[c, c^{i'+1}_{j+1}]$ is an $\mathsf{up\_rel}$-interval (step 2 above). By Lemma 6(c) and (e), $[c, c^{i'+1}_{j+1}]$ is an $\mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{e}}$-interval. We show now that all possible choices for c lead to contradiction: (i) if $c < c^i_j$, then $[c, c^{i'+1}_{j+1}]$ overlaps the $\mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{o}}$-interval $[c^i_j, c^{i''}_{j+1}]$ (contradiction with Lemma 6(d)); (ii) if $c = c^i_j$, then c^i_j starts both an $\mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{o}}$-interval and an $\mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{e}}$-interval (contradiction with Lemma 6(c)); and (iii) if $c > c^i_j$, then the $\mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{o}}$-interval $[c^i_j, c^{i'}_{j+1}]$ overlaps $[c, c^{i'+1}_{j+1}]$ (contradiction with Lemma 6(d)). In a similar way, one can prove that no two distinct $\mathsf{up\_rel}$-intervals can end at the same point. □

Right-neighbor relation The right-neighbor relation connects each tile with its horizontal (right) neighbor in the octant, if any (e.g., t^2_3 with t^3_3 in Fig. 2). Again, in order to encode the right-neighbor relation, we must distinguish between forward and backward rows: a tile-interval belonging to a forward row is right-connected to the tile-interval immediately to the right, if any, while a tile-interval belonging to a backward row is right-connected to the tile-interval immediately to the left, if any. For example, in Fig. 2b, the 2nd tile-interval of the 4th row (t^2_4) is right-connected to the tile-interval immediately to the right (t^3_4), since the 4th row is a forward one, while the 2nd tile-interval of the 3rd row (t^2_3) is right-connected to the tile-interval immediately to the left (t^3_3), since the 3rd row is a backward one.

We define the right-neighbor relation as follows: if $[b^i_j, b^{i+1}_j]$ is a tile-interval belonging to a forward (resp., backward) ld-interval such that $i \neq k_j - 1$ (resp., $i \neq 1$), then we say that it is right-connected to the tile-interval $[b^{i+1}_j, b^{i+2}_j]$ (resp., $[b^{i-1}_j, b^i_j]$). As a matter of fact, no additional proposition letter is needed to encode right-connectedness.

The following lemma proves that the commutativity property holds.

Lemma 9 *Let M be a model, $[a, b]$ be an interval over M, and $b \leq b^0_1 < c^0_1 < b^1_1 < \ldots < b^{k_1-1}_1 < c^{k_1-1}_1 < b^{k_1}_1 = b^0_2 < c^{k_1}_1 = c^0_2 < b^1_2 < \ldots < b^{k_2}_2 = b^0_3 < \ldots$ be the sequence of points defined by Lemma 4 If $M, [a, b] \Vdash$ ($\mathsf{disj\text{-}bndu}^{\mathsf{k}}_{\mathsf{u}}$) $\wedge$ ($\mathsf{u\text{-}chain}$) $\wedge$ ($\mathsf{ld\text{-}chain}$) $\wedge$ ($\mathsf{bw/fw}$) $\wedge$ ($\mathsf{up\_rel\text{-}def}_1$) $\wedge$ ($\mathsf{up\_rel\text{-}def}_2$), then M satisfies the commutativity property.*

Proof Given two tile-intervals $[b^i_j, b^{i+1}_j]$ and $[b^{i'}_{j'}, b^{i'+1}_{j'}]$, let $[c, d]$ be a tile-interval such that $[b^i_j, b^{i+1}_j]$ is right-connected to $[c, d]$ and $[c, d]$ is above-connected to $[b^{i'}_{j'}, b^{i'+1}_{j'}]$. We assume $[b^i_j, b^{i+1}_j]$ to be a fw-interval, that is, to belong to a forward ld-interval. The case in which $[b^i_j, b^{i+1}_j]$ is a bw-interval can be dealt with in a similar way, and thus it is omitted.

By the definitions of the right-neighbor and above-neighbor relations, $[c, d] = [b^{i+1}_j, b^{i+2}_j]$ and $[c^{i+1}_j, c^{i'}_{j'}]$ is an up_rel-interval. It follows that $j' = j+1$. Since $[b^{i+1}_j, b^{i+2}_j]$ is a fw-interval, by Lemma 7(a), $[c^{i+1}_j, c^{i'}_{j+1}]$ is either an $\mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{o}}$-interval or an $\mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{e}}$-interval. Without loss of generality, we assume $[c^{i+1}_j, c^{i'}_{j+1}]$ to be an $\mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{o}}$-interval (the case in which $[c^{i+1}_j, c^{i'}_{j+1}]$ is an $\mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{e}}$-interval is similar and thus omitted).

Since $[b^i_j, b^{i+1}_j]$ is a tile-interval, by Lemma 8, it is above-connected to exactly one tile-interval, say $[b^{i''}_{j+1}, b^{i''+1}_{j+1}]$. Thus, $[c^i_j, c^{i''}_{j+1}]$ is an up_rel-interval. Since $[c^{i+1}_j, c^{i'}_{j+1}]$ is an $\mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{o}}$-interval, by Lemma 7(b), it is an $\mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{e}}$-interval.

To complete the proof, we must show that $[b^{i''}_{j+1}, b^{i''+1}_{j+1}]$ is right-connected to $[b^{i'}_{j+1}, b^{i'+1}_{j+1}]$. Since $[b^i_j, b^{i+1}_j]$ is a fw-interval, by Lemma 5, it follows that $[b^{i'}_{j+1}, b^{i'+1}_{j+1}]$ is a bw-interval. Hence, the only interval that is right-connected to $[b^{i'}_{j+1}, b^{i'+1}_{j+1}]$, if any, is the interval $[b^{i'+1}_{j+1}, b^{i'+2}_{j+1}]$. Then it suffices to show that $[b^{i''}_{j+1}, b^{i''+1}_{j+1}] = [b^{i'+1}_{j+1}, b^{i'+2}_{j+1}]$, which amounts to proving that $c^{i''}_{j+1} = c^{i'+1}_{j+1}$. For the sake of contradiction, suppose that this is not the case. Two cases must be considered, both leading to contradiction.

(i) $c^{i''}_{j+1} < c^{i'+1}_{j+1}$. If $c^{i''}_{j+1} = c^{i'}_{j+1}$, then the up_rel-intervals $[c^i_j, c^{i'}_{j+1}]$ and $[c^{i+1}_j, c^{i'}_{j+1}]$ end at the same point (contradiction with Lemma 8); otherwise, if $c^{i''}_{j+1} < c^{i'}_{j+1}$, then the $\mathsf{up\_rel}^{\mathsf{fw}}$-interval $[c^i_j, c^{i''}_{j+1}]$ overlaps the $\mathsf{up\_rel}^{\mathsf{fw}}$-interval $[c^{i+1}_j, c^{i'}_{j+1}]$ (contradiction with item (d) of Lemma 6).

(ii) $c^{i''}_{j+1} > c^{i'+1}_{j+1}$. By Lemma 8, there exists a point $c^{i'''}_j$ such that $[c^{i'''}_j, c^{i'+1}_{j+1}]$ is an up_rel-interval. By Lemmas 6(e) and 7(b), $[c^{i'''}_j, c^{i'+1}_{j+1}]$ is an $\mathsf{up\_rel}^{\mathsf{fw}}_{\mathsf{e}}$-interval. We show now that all possible choices for $c^{i'''}_j$ lead to contradiction:

 – if $c^{i'''}_j > c^{i+1}_j$, then the $\mathsf{up\_rel}^{\mathsf{fw}}$-interval $[c^{i+1}_j, c^{i'}_{j+1}]$ overlaps the $\mathsf{up\_rel}^{\mathsf{fw}}$-interval $[c^{i'''}_j, c^{i'+1}_{j+1}]$ (contradiction with item (d) of Lemma 6);
 – if $c^{i'''}_j = c^{i+1}_j$, then the up_rel-intervals $[c^{i+1}_j, c^{i'}_{j+1}]$ and $[c^{i+1}_j, c^{i'+1}_{j+1}]$ begin at the same point (contradiction with Lemma 8);
 – if $c^{i'''}_j = c^i_j$, then the up_rel-intervals $[c^i_j, c^{i''}_{j+1}]$ and $[c^i_j, c^{i'+1}_{j+1}]$ begin at the same point (contradiction with Lemma 8);
 – if $c^{i'''}_j < c^i_j$, then the $\mathsf{up\_rel}^{\mathsf{fw}}$-interval $[c^{i'''}_j, c^{i'+1}_{j+1}]$ overlaps the $\mathsf{up\_rel}^{\mathsf{fw}}$-interval $[c^i_j, c^{i''}_{j+1}]$ (contradiction with item (d) of Lemma 6)

Hence, $c^{i''}_{j+1} = c^{i'+1}_{j+1}$, and thus $[b^{i''}_{j+1}, b^{i''+1}_{j+1}] = [b^{i'+1}_{j+1}, b^{i'+2}_{j+1}]$. This allows us to conclude that $[b^i_j, b^{i+1}_j]$ is above-connected to $[b^{i'+1}_{j+1}, b^{i'+2}_{j+1}]$, which is right-connected to $[b^{i'}_{j+1}, b^{i'+1}_{j+1}]$, thus proving the thesis. □

Corollary 1 *The ith* tile-*interval of the jth row (*ld-*interval) is above-connected to the ith* tile-*interval of the $(j+1)$th row.*

Tiling the plane The following formulae constrain each tile-interval (and no other one) to be tiled by exactly one tile (formula (45)) and tiles that are right- or above-connected to respect color constraints (formulae (46)–(48)):

$$[G]\big(\big(\bigvee_{i=1}^{k} \mathtt{t}_i \leftrightarrow \mathsf{tile}\big) \wedge \big(\bigwedge_{i,j=1,i\neq j}^{k} \neg(\mathtt{t}_i \wedge \mathtt{t}_j)\big)\big) \tag{45}$$

$$[G]\big(\mathsf{tile} \rightarrow \bigvee_{\mathrm{up}(t_i)=\mathrm{down}(t_j)} (\mathtt{t}_i \wedge \langle O\rangle(\mathsf{up\_rel} \wedge \langle O\rangle \mathtt{t}_j))\big) \tag{46}$$

$$[G]\big(\mathsf{tile} \wedge \mathsf{fw} \wedge \langle X_{\mathsf{u}}\rangle \mathsf{tile} \rightarrow \bigvee_{\mathrm{right}(t_i)=\mathrm{left}(t_j)} (\mathtt{t}_i \wedge \langle X_{\mathsf{u}}\rangle \mathtt{t}_j)\big) \tag{47}$$

$$[G]\big(\mathsf{tile} \wedge \mathsf{bw} \wedge \langle X_{\mathsf{u}}\rangle \mathsf{tile} \rightarrow \bigvee_{\mathrm{left}(t_i)=\mathrm{right}(t_j)} (\mathtt{t}_i \wedge \langle X_{\mathsf{u}}\rangle \mathtt{t}_j)\big) \tag{48}$$

$$(45) \wedge \ldots \wedge (48) \tag{tiles}$$

Let $\mathcal{T}$ be the set of tile types $\{t_1, t_2, \ldots, t_k\}$ and $\Phi_{\mathcal{T}}$ be the formula (disj-bndu$_{\mathsf{u}}^{\mathsf{k}}$) $\wedge$ (u-chain) $\wedge$ (ld-chain) $\wedge$ (bw/fw) $\wedge$ (up_rel-def$_1$) $\wedge$ (up_rel-def$_2$) $\wedge$ (tiles) over $\mathcal{T}$.

Lemma 10 *For any linear order $\mathbb{D}$ with an infinite ascending sequence of points, the formula $\Phi_{\mathcal{T}}$ is satisfiable in $\mathbb{D}$ if and only if $\mathcal{T}$ can tile the second octant $\mathcal{O}$.*

Proof ("only if" direction) Let $\mathbb{D}$ be a linear order with an infinite ascending sequence of points such that $M, [a, b] \Vdash \Phi_{\mathcal{T}}$ for some model $M = \langle \mathbb{I}(\mathbb{D}), V\rangle$ and interval $[a, b] \in \mathbb{I}(\mathbb{D})$. Let $b \leq b_1^0 < b_1^1 < b_1^2 = b_2^0 < \ldots < b_2^{k_2} = b_3^0 < \ldots < b_j^0 < b_j^1 < \ \ldots < b_j^{k_j} = b_{j+1}^0 < \ldots$ be the sequence of points defined by Lemma 4. For each $j > 0$ and $0 < i < k_j$, $[b_j^i, b_j^{i+1}]$ is a tile-interval, and thus $M, [b_j^i, b_j^{i+1}] \Vdash \mathtt{t}_v$ for a unique v. Then, for each i, j such that $0 \leq i \leq j$, we put $f(i, j) = t_v$, where $\mathtt{t}_v$ is the unique proposition letter in the set $\mathtt{T} = \{\mathtt{t}_1, \mathtt{t}_2, \ldots, \mathtt{t}_k\}$ such that $M, [b_{j+1}^{i+1}, b_{j+1}^{i+2}] \Vdash \mathtt{t}_v$. By Lemmas 8 and 9, Corollary 1, and formula (tiles), the function $f : \mathcal{O} \mapsto \mathcal{T}$ defines a correct tiling of $\mathcal{O}$.

("if" direction) Let $\mathbb{D}$ be a linear order with an infinite ascending sequence of points, and let $f : \mathcal{O} \mapsto \mathcal{T}$ be a correct tiling of $\mathcal{O}$. We provide a model $M = \langle \mathbb{I}(\mathbb{D}), V\rangle$ and an interval $[a, b] \in M$ such that $M, [a, b] \Vdash \Phi_{\mathcal{T}}$ (see Fig. 3). Let $\sigma = b_0, b_1, \ldots$ be the infinite ascending sequence of points in $\mathbb{D}$ whose existence is guaranteed by

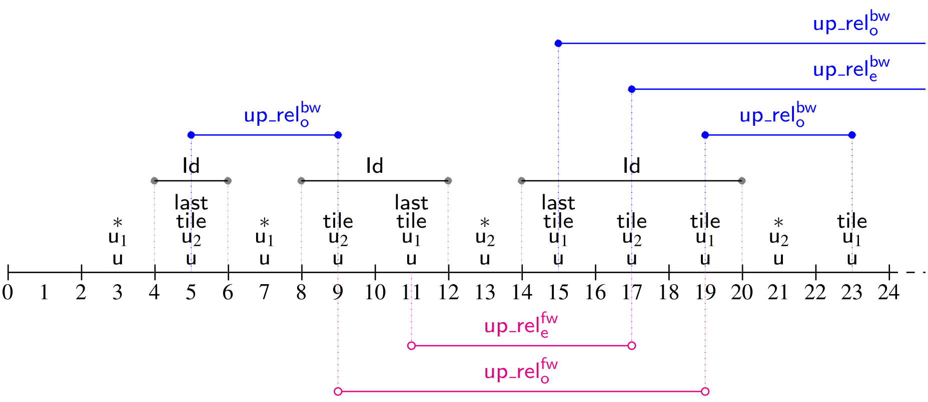


Fig. 3 A model satisfying the formula $\Phi_{\mathcal{T}}$ (for the sake of readability, we write i for b_i, $i = 0, \ldots, 24$)

hypothesis. The valuation function V is defined as follows, where $g(n)$ abbreviates $(n+1)(n+2)$:

- $\mathsf{u} \in V([c,d])$ iff $[c,d] = [b_i, b_j]$, $i = 2n$, and $j = i+2$ for some $n(>0) \in \mathbb{N}$;
- $\mathsf{k} \in V([c,d])$ iff $[c,d] = [b_i, b_j]$ and $\mathsf{u} \in V([b_{i-1}, b_{j-1}])$ for some $i, j \in \mathbb{N}$;
- $\mathsf{u}_1 \in V([c,d])$ iff $[c,d] = [b_i, b_j]$, $\mathsf{u} \in V([b_i, b_j])$, and $i = 2n$ for some odd $n \in \mathbb{N}$;
- $\mathsf{u}_2 \in V([c,d])$ iff $[c,d] = [b_i, b_j]$, $\mathsf{u} \in V([b_i, b_j])$, and $\mathsf{u}_1 \notin V([b_i, b_j])$ for some $i, j \in \mathbb{N}$;
- $\mathsf{k}_1 \in V([c,d])$ iff $[c,d] = [b_i, b_j]$, $\mathsf{u}_1 \in V(b_{i-1}, b_{j-1}])$ for some $i, j \in \mathbb{N}$;
- $\mathsf{k}_2 \in V([c,d])$ iff $[c,d] = [b_i, b_j]$, $\mathsf{u}_2 \in V([b_{i-1}, b_{j-1}])$ for some $i, j \in \mathbb{N}$;
- $\mathsf{inside}_{\mathsf{u}_1} \in V([c,d])$ iff $\mathsf{u}_1 \in V([b_i, b_j])$ for some $i, j \in \mathbb{N}$ such that $b_i \leq c, d \leq b_j$ and $[c,d] \neq [b_i, b_j]$;
- $\overrightarrow{\mathsf{u}_1} \in V([c,d])$ iff $\mathsf{u}_1 \in V([b_i, b_j])$ and $\mathsf{u}_2 \in V([b_k, b_l])$ for some $i, j, k, l \in \mathbb{N}$, with $b_i < c < b_j < d, c < b_k < d < b_l$;
- $\mathsf{inside}_{\mathsf{u}_2} \in V([c,d])$ iff $\mathsf{u}_2 \in V([b_i, b_j])$ for some $i, j \in \mathbb{N}$ such that $b_i \leq c, d \leq b_j$ and $[c,d] \neq [b_i, b_j]$;
- $\overrightarrow{\mathsf{u}_2} \in V([c,d])$ iff $\mathsf{u}_2 \in V([b_i, b_j])$ and $\mathsf{u}_1 \in V([b_k, b_l])$ for some $i, j, k, l \in \mathbb{N}$, with $b_i < c < b_j < d, c < b_k < d < b_l$;
- $\mathsf{inside}_{\mathsf{k}_1} \in V([c,d])$ iff $\mathsf{k}_1 \in V([b_i, b_j])$ for some $i, j \in \mathbb{N}$ such that $b_i \leq c, d \leq b_j$ and $[c,d] \neq [b_i, b_j]$;
- $\overrightarrow{\mathsf{k}_1} \in V([c,d])$ iff $\mathsf{k}_1 \in V([b_i, b_j])$ and $\mathsf{k}_2 \in V([b_k, b_l])$ for some $i, j, k, l \in \mathbb{N}$, with $b_i < c < b_j < d, c < b_k < d < b_l$;
- $\mathsf{inside}_{\mathsf{k}_2} \in V([c,d])$ iff $\mathsf{k}_2 \in V([b_i, b_j])$ for some $i, j \in \mathbb{N}$ such that $b_i \leq c, d \leq b_j$ and $[c,d] \neq [b_i, b_j]$;
- $\overrightarrow{\mathsf{k}_2} \in V([c,d])$ iff $\mathsf{k}_2 \in V([b_i, b_j])$ and $\mathsf{k}_1 \in V([b_k, b_l])$ for some $i, j, k, l \in \mathbb{N}$, with $b_i < c < b_j < d, c < b_k < d < b_l$;
- $\mathsf{first} \in V([c,d])$ iff $[c,d] = [b_2, b_4]$;
- $* \in V([c,d])$ iff $[c,d] = [b_i, b_j]$, $\mathsf{u} \in V([b_i, b_j])$, and $i = g(n)$ for some $n \geq 0$;
- $\mathsf{Id} \in V([c,d])$ iff $[c,d] = [b_i, b_j]$, $* \in V([b_{i-2}, b_i])$, $* \in V([b_j, b_{j+2}])$, $i = g(n) + 2$, and $j = g(n+1)$ for some $n \geq 0$;
- $\mathsf{inside}_{\mathsf{Id}} \in V([c,d])$ iff $\mathsf{Id} \in V([b_i, b_j])$ for some $i, j \in \mathbb{N}$, with $b_i \leq c, d \leq b_j$, and $[c,d] \neq [b_i, b_j]$;
- $\overrightarrow{\mathsf{Id}} \in V([c,d])$ iff $\mathsf{Id} \in V([b_i, b_j])$ and $* \in V([b_k, b_l])$ for some $i, j, k, l \in \mathbb{N}$, with $b_i < c < b_j < d, c < b_k < d < b_l$;
- $\mathsf{tile} \in V([c,d])$ iff $[c,d] = [b_i, b_j]$, $\mathsf{u} \in V([b_i, b_j])$, and $* \notin V([b_i, b_j])$ for some $i, j \in \mathbb{N}$;
- $\mathsf{fw} \in V([c,d])$ iff $[c,d] = [b_i, b_j]$, $\mathsf{u} \in V([b_i, b_j])$, $\mathsf{Id} \in V([b_k, b_l])$ for some $i, j, k, l \in \mathbb{N}$ such that $k \leq i$ and $j \leq l$, and $k = g(n)$ for some odd n;
- $\mathsf{bw} \in V([c,d])$ iff $[c,d] = [b_i, b_j]$, $\mathsf{u} \in V([b_i, b_j])$, and $\mathsf{fw} \notin V([b_i, b_j])$ for some $i, j \in \mathbb{N}$;
- for each $h \in \{1, \ldots, k\}$, $\mathtt{t}_h \in V([c,d])$ iff $[c,d] = [b_i, b_j]$, $\mathsf{tile} \in V([b_i, b_j])$, $f(l,m) = t_h$ for some l, m such that $0 \leq l \leq m$, and either (i) $\mathsf{fw} \in V([b_{g(m)}, b_{g(m)+2}])$ and $i = g(m) + 2l + 2$, or (ii) $\mathsf{bw} \in V([b_{g(m)}, b_{g(m)+2}])$ and $i = g(m+1) - 2l - 2$;
- $\mathsf{up\_rel}_{\mathsf{o}}^{\mathsf{fw}} \in V([c,d])$ iff $[c,d] = [b_i, b_j]$, $\mathsf{tile} \in V([b_{i-1}, b_{i+1}])$, $\mathsf{tile} \in V([b_{j-1}, b_{j+1}])$, $\mathsf{fw} \in V([b_{i-1}, b_{i+1}])$, $i - 1 = g(m) + 2l + 2$, and $j - 1 = g(m+2) - 2l - 2$ for some l, m such that $0 \leq l \leq m$ and $l = 2n$ for some $n \geq 0$;

- $\mathsf{up\_rel}_{\mathsf{e}}^{\mathsf{fw}} \in V([c,d])$ iff $[c,d] = [b_i, b_j]$, $\mathsf{tile} \in V([b_{i-1}, b_{i+1}])$, $\mathsf{tile} \in V([b_{j-1}, b_{j+1}])$, $\mathsf{fw} \in V([b_{i-1}, b_{i+1}])$, $i-1 = g(m)+2l+2$, and $j-1 = g(m+2)-2l-2$ for some l, m such that $0 \leq l \leq m$ and $l = 2n+1$ for some $n \geq 0$;
- $\mathsf{up\_rel}_{\mathsf{o}}^{\mathsf{bw}} \in V([c,d])$ iff $[c,d] = [b_i, b_j]$, $\mathsf{tile} \in V([b_{i-1}, b_{i+1}])$, $\mathsf{tile} \in V([b_{j-1}, b_{j+1}])$, $\mathsf{bw} \in V([b_{i-1}, b_{i+1}])$, $i-1 = g(m)+2l+2$, and $j-1 = g(m+2)-2l-4$ for some l, m such that $0 \leq l \leq m$ and $l = 2n$ for some $n \geq 0$;
- $\mathsf{up\_rel}_{\mathsf{e}}^{\mathsf{bw}} \in V([c,d])$ iff $[c,d] = [b_i, b_j]$, $\mathsf{tile} \in V([b_{i-1}, b_{i+1}])$, $\mathsf{tile} \in V([b_{j-1}, b_{j+1}])$, $\mathsf{bw} \in V([b_{i-1}, b_{i+1}])$, $i-1 = g(m)+2l+2$, and $j-1 = g(m+2)-2l-4$ for some l, m such that $0 \leq l \leq m$ and $l = 2n+1$ for some $n \geq 0$;
- $\mathsf{up\_rel}^{\mathsf{fw}} \in V([c,d])$ iff $[c,d] = [b_i, b_j]$, $\mathsf{up\_rel}_{\mathsf{o}}^{\mathsf{fw}} \in V([b_i, b_j])$ or $\mathsf{up\_rel}_{\mathsf{e}}^{\mathsf{fw}} \in V([b_i, b_j])$ for some $i, j \in \mathbb{N}$;
- $\mathsf{up\_rel}^{\mathsf{bw}} \in V([c,d])$ iff $[c,d] = [b_i, b_j]$, $\mathsf{up\_rel}_{\mathsf{o}}^{\mathsf{bw}} \in V([b_i, b_j])$ or $\mathsf{up\_rel}_{\mathsf{e}}^{\mathsf{bw}} \in V([b_i, b_j])$ for some $i, j \in \mathbb{N}$;
- $\mathsf{up\_rel} \in V([c,d])$ iff $[c,d] = [b_i, b_j]$, $\mathsf{up\_rel}^{\mathsf{fw}} \in V([b_i, b_j])$ or $\mathsf{up\_rel}^{\mathsf{bw}} \in V([b_i, b_j])$ for some $i, j \in \mathbb{N}$;
- $\mathsf{last} \in V([c,d])$ iff $[c,d] = [b_i, b_j]$, $\mathsf{tile} \in V([b_i, b_j])$ and either (i) $\mathsf{bw} \in V([b_i, b_j])$ and $* \in V([b_{i-2}, b_i])$ or (ii) $\mathsf{fw} \in V([b_i, b_j])$ and $* \in V([b_j, b_{j+2}])$, for some $i, j \in \mathbb{N}$;

It can be easily checked that $M, [b_0, b_2] \Vdash \Phi_{\mathcal{T}}$. □

Corollary 2 *The satisfiability problem for* O *(resp.,* $\overline{\mathsf{O}}$*) is undecidable over any class of linear orders that contains at least one linear order with an infinite ascending (resp., descending) sequence of points.*

4.2 Undecidability in the finite case

The encoding of OTP we provided makes essential use of an infinite sequence of points, and thus it cannot be exploited to prove undecidability of O over classes of finite linear orders. In the following, we show how to adapt the given construction in order to encode FTP. The resulting reduction proves the undecidability of O over any class of linear orders that contains finite linear orders of unbounded cardinality. In particular, it proves undecidability of O when interpreted over the class of all finite linear orders.

Definition of the u*-chain* The main difference between the reduction of FTP and that of OTP is the finiteness of the rectangular area. Such a condition requires the existence of an arbitrarily long, but finite, u-chain. To deal with it, we introduce an auxiliary proposition letter $\mathsf{last_u}$ to denote the last u-interval of the (finite) u-chain. Properties of $\mathsf{last_u}$ are expressed by the following formulae:

$$\langle O\rangle\langle O\rangle \mathsf{last_u} \tag{49}$$

$$[G](\mathsf{last_u} \rightarrow * \wedge [O](\neg \mathsf{u} \wedge \neg \mathsf{k}) \wedge [O][O](\neg \mathsf{u} \wedge \neg \mathsf{k})) \tag{50}$$

Most formulae introduced in the previous section can still be used for the present reduction. In the following, we simply identify which ones must be replaced, and we provide their replacements. Formula (6) must be replaced by (51) to guarantee the existence of the u- and k-chains.

$$\begin{aligned}&[G]((\mathsf{u}_1 \wedge \neg \mathsf{last}_\mathsf{u} \rightarrow \langle O\rangle \mathsf{k}_1) \wedge (\mathsf{k}_1 \rightarrow \langle O\rangle \mathsf{u}_2)\\ &\quad \wedge (\mathsf{u}_2 \wedge \neg \mathsf{last}_\mathsf{u} \rightarrow \langle O\rangle \mathsf{k}_2) \wedge (\mathsf{k}_2 \rightarrow \langle O\rangle \mathsf{u}_1))\end{aligned} \tag{51}$$

Since u_1- and u_2-intervals (resp., k_1- and k_2-intervals) do not infinitely alternate with each other, to force u_1, u_2, k_1, and k_2 to be disjointly-bounded, we introduce a new proposition cons and we constrain it to be a disjoint consequent of u and k.

$$\neg \mathsf{cons} \wedge [O]\neg \mathsf{cons} \wedge [G](\mathsf{u} \wedge \mathsf{k} \rightarrow \langle O\rangle\langle O\rangle \mathsf{cons}) \tag{52}$$

$$[G](\langle O\rangle \mathsf{u} \vee \langle O\rangle \mathsf{k} \rightarrow \neg\langle O\rangle \mathsf{cons}) \tag{53}$$

$$[G]((\mathsf{u} \vee \mathsf{k} \rightarrow \neg\langle O\rangle \mathsf{cons}) \wedge (\mathsf{cons} \rightarrow [O](\neg \mathsf{u} \wedge \neg \mathsf{k}))) \tag{54}$$

Finally, we replace formulae (13) and (14) by formulae (55) and (56), respectively:

$$\begin{aligned}&\textit{non-sub}\,(\mathsf{u}_1, \mathsf{cons}) \wedge \textit{non-sub}\,(\mathsf{u}_2, \mathsf{cons})\\ &\quad \wedge \textit{non-sub}\,(\mathsf{k}_1, \mathsf{cons}) \wedge \textit{non-sub}\,(\mathsf{k}_2, \mathsf{cons})\end{aligned} \tag{55}$$

$$[G](\mathsf{u} \vee \mathsf{k} \rightarrow [O](\langle O\rangle\langle O\rangle \mathsf{last}_\mathsf{u} \rightarrow \langle O\rangle(\mathsf{u} \vee \mathsf{k}))) \tag{56}$$

It is worth pointing out that formulae (49)–(56) guarantee the existence of the u-chain also when interpreted over arbitrary linear orders. The finiteness assumption guarantees the finiteness of the chain. As a counterexample, consider the $\mathbb{Q}$-model depicted in Fig. 4, where u_1 holds over every interval $[2-\frac{1}{2^{2n}}, 2-\frac{1}{2^{2n+1}}]$ and u_2 holds over every interval $[2-\frac{1}{2^{2n+1}}, 2-\frac{1}{2^{2n+2}}]$, the sequences of k_1- and k_2-intervals are defined consistently, and last_u holds over the interval $[2, 2+\frac{1}{2}]$. This model satisfies formulae (49)–(56), but it contains an infinite u-chain.

Definition of the ld-chain Like the u-chain, the ld-chain must be finite. To deal with such a condition, we introduce a proposition letter $\mathsf{last}_\mathsf{ld}$ that denotes the last ld-interval of the (finite) ld-chain.

$$[G]((\mathsf{last}_\mathsf{ld} \rightarrow \mathsf{ld}) \wedge (\mathsf{ld} \wedge \langle O\rangle(\mathsf{k} \wedge \langle O\rangle \mathsf{last}_\mathsf{u}) \rightarrow \mathsf{last}_\mathsf{ld})) \tag{57}$$

Furthermore, we replace formulae (16) and (18) by formulae (58) and (59), respectively:

$$\langle X_\mathsf{u}\rangle * \wedge [G](* \wedge \neg \mathsf{last}_\mathsf{u} \rightarrow \langle X_\mathsf{u}\rangle \mathsf{tile}) \tag{58}$$

$$[G](* \wedge \neg \mathsf{last}_\mathsf{u} \rightarrow \langle O\rangle(\mathsf{k} \wedge \langle O\rangle \mathsf{ld})) \tag{59}$$

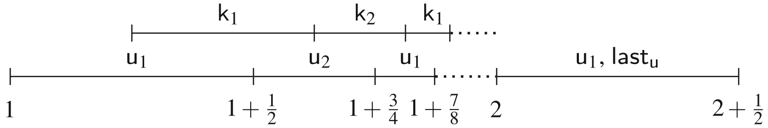


Fig. 4 An infinite bounded-above u-chain

Above-neighbor relation In the finite case, each row has exactly the same number of tiles, and thus formulae (38), (39), (40), (42), and (44) can be dismissed. Formulae (32), (33), (41), and (43) are respectively replaced by the following ones.

$$[G](\mathsf{up\_rel}^{\alpha}_{\gamma} \rightarrow (\langle O\rangle \mathsf{tile} \wedge (\langle O\rangle\langle O\rangle(* \wedge \neg \mathsf{last_u}) \rightarrow \langle O\rangle(\mathsf{tile} \wedge \langle O\rangle \mathsf{up\_rel}^{\beta}_{\gamma})))) \tag{60}$$

$$[G](\mathsf{tile} \wedge \langle O\rangle\langle O\rangle(* \wedge \neg \mathsf{last_u}) \rightarrow \langle O\rangle \mathsf{up\_rel}) \tag{61}$$

$$[G](* \wedge \langle O\rangle\langle O\rangle(* \wedge \neg \mathsf{last_u}) \\ \rightarrow \langle X_{\mathsf{u}}\rangle(\mathsf{tile} \wedge \langle O\rangle(\mathsf{up\_rel} \wedge \langle O\rangle(\mathsf{tile} \wedge \langle X_{\mathsf{u}}\rangle *)))) \tag{62}$$

$$[G](\mathsf{k} \wedge \langle O\rangle(\mathsf{tile} \wedge \langle O\rangle \mathsf{up\_rel}^{\alpha}_{\gamma}) \\ \rightarrow [O](\langle O\rangle \mathsf{up\_rel}^{\alpha}_{\gamma} \wedge \langle O\rangle(\mathsf{k} \wedge \langle O\rangle(\mathsf{tile} \wedge \langle O\rangle \mathsf{up\_rel}^{\beta}_{\delta})) \rightarrow \langle O\rangle \mathsf{up\_rel}^{\alpha}_{\delta})) \tag{63}$$

To complete the construction, it suffices to add the constraints on the first and last tile of the plane. Therefore, O (resp., $\overline{\mathsf{O}}$) turns out be undecidable over finite linear orders as well.

Corollary 3 *The satisfiability problem for* O *(resp.,* $\overline{\mathsf{O}}$*) is undecidable over any class of finite linear orders that contains, for every* $n > 0$*, at least one linear order with cardinality greater than* n*.*

5 The fragments AD, A$\overline{\mathsf{D}}$, $\overline{\mathsf{A}}$D, and $\overline{\mathsf{AD}}$

In this section, we focus our attention on the interval logics of Allen's relations *meets/met by* and *during/contains* AD, A$\overline{\mathsf{D}}$, $\overline{\mathsf{A}}$D, and $\overline{\mathsf{AD}}$. We assume the sub-interval (resp., super-interval) relation to be strict. First, we prove undecidability of the satisfiability problem for AD over any class of linear orders containing at least one linear order with an infinite ascending sequence of points by reducing OTP to it. Then, we show how the proof can be adapted to A$\overline{\mathsf{D}}$ over the same classes of linear orders. Undecidability of $\overline{\mathsf{A}}$D and $\overline{\mathsf{AD}}$ over any class of linear orders containing at least one linear order with an infinite descending sequence of points follows by symmetry from undecidability of AD and A$\overline{\mathsf{D}}$, respectively. Finally, undecidability of AD, A$\overline{\mathsf{D}}$, $\overline{\mathsf{A}}$D, and $\overline{\mathsf{AD}}$ over any class of finite linear orders immediately follows from undecidability of D and $\overline{\mathsf{D}}$ over finite linear orders [19].

5.1 The fragment AD

Let $[a, b]$ be an interval. We define the set of relevant intervals $\mathcal{G}_{[a,b]}$ as the set containing $[a, b]$ and all intervals $[c, d]$ such that $c \geq b$. Accordingly, the global modality $[G]$ is defined as:

$$[G]p \equiv p \wedge [A]p \wedge [A][A]p.$$

It is worth noticing that the definition of $[G]$ makes no reference to $[D]$, and thus $\mathcal{G}_{[a,b]}$ is only a subset of all the intervals reachable from $[a, b]$ using modalities in AD.

Definition of the u*-chain* In order to build a chain of u-intervals, we need to chop each u-interval into a pair (u_1-interval, u_2-interval). Proposition letters u_1 and u_2 are instrumental to the construction of the u-chain. The following formulae define the u-chain:

$$\neg \mathsf{u} \wedge \neg \mathsf{u}_1 \wedge \neg \mathsf{u}_2 \wedge \langle A\rangle \mathsf{u} \wedge [G](\mathsf{u} \rightarrow \langle A\rangle \mathsf{u}) \tag{64}$$

$$[G](\langle A\rangle \mathsf{u} \leftrightarrow \langle A\rangle \mathsf{u}_1) \tag{65}$$

$$[G](\mathsf{u} \rightarrow \neg \mathsf{u}_1) \tag{66}$$

$$[G](\mathsf{u}_1 \rightarrow \langle A\rangle \mathsf{u}_2) \tag{67}$$

$$[G](\mathsf{u}_2 \rightarrow \langle A\rangle \mathsf{u}) \tag{68}$$

$$[G](\langle A\rangle \mathsf{u}_1 \rightarrow \neg\langle A\rangle \mathsf{u}_2) \tag{69}$$

$$[G](\mathsf{u} \rightarrow \neg\langle D\rangle\langle A\rangle \mathsf{u} \wedge \neg\langle D\rangle \mathsf{u}_1 \wedge \neg\langle D\rangle \mathsf{u}_2) \tag{70}$$

$$[G](\mathsf{u} \rightarrow \langle D\rangle \top) \tag{71}$$

$$[G](\mathsf{u}_1 \rightarrow \neg\langle D\rangle\langle A\rangle \mathsf{u}) \tag{72}$$

$$[G](\mathsf{u}_2 \rightarrow \neg\langle D\rangle\langle A\rangle \mathsf{u} \wedge \neg\langle D\rangle \mathsf{u}_1) \tag{73}$$

$$[G](\mathsf{u}_2 \rightarrow \langle D\rangle \top) \tag{74}$$

$$(64) \wedge \ldots \wedge (74) \tag{u-chain$^{\mathsf{AD}}$}$$

From now on, for each formula of the form $[G](\varphi \rightarrow \varphi_1 \wedge \ldots \wedge \varphi_n)$, where, for each i, φ_i is conjunction-free, labeled by (k), we use k followed by a Roman numeral to restrict the consequent of the implication to the conjunct pointed by the Roman numeral, that is, (k-I) denotes the formula $[G](\varphi \rightarrow \varphi_1)$, ($k$-II) denotes the formula $[G](\varphi \rightarrow \varphi_2)$, and so on. As an example, (70-III) denotes the formula $[G](\mathsf{u} \rightarrow \neg\langle D\rangle \mathsf{u}_2)$.

Lemma 11 *Let M be a model and $[a, b]$ be an interval over M. If $M, [a, b] \Vdash$ (u-chain$^{\mathsf{AD}}$), then there exists an infinite sequence of points $b = b_0 < b_1 < \ldots$ in M such that:*

- (a) *for each $i \geq 0$, $M, [b_i, b_{i+1}] \Vdash \mathsf{u}$;*
- (b) *for each $i \geq 0$, there exists c_i such that $b_i < c_i < b_{i+1}$, $M, [b_i, c_i] \Vdash \mathsf{u}_1$, and $M, [c_i, b_{i+1}] \Vdash \mathsf{u}_2$;*
- (c) *no other interval $[c, d] \in \mathcal{G}_{[a,b]}$ satisfies u, unless $c > b_i$ for each $i \geq 0$.*

Proof

- (a) The existence of an infinite sequence of u-intervals follows immediately from (64).
- (b) Consider any interval $[b_i, b_{i+1}]$ of the sequence. We first prove that there exists a point c_i such that $b_i < c_i < b_{i+1}$ and $[b_i, c_i]$ satisfies u_1. By (71) there exists at least one point d_i such that $b_i < d_i < b_{i+1}$. By (65), there exists a point $c_i > b_i$ such that $[b_i, c_i]$ satisfies u_1, and, by (66), $c_i \neq b_{i+1}$. For the sake of contradiction, suppose that $c_i > b_{i+1}$. Then, $[d_i, b_{i+1}]$ is strictly contained in the u_1-interval $[b_i, c_i]$ and it meets the u-interval $[b_{i+1}, b_{i+2}]$, violating (72). Hence $b_i < c_i < b_{i+1}$. We prove now that $[c_i, b_{i+1}]$ satisfies u_2. By (67), there exists an $f_i > c_i$ such that $[c_i, f_i]$ satisfies u_2. We show that $f_i = b_{i+1}$. For the sake of contradiction, suppose that $f_i \neq b_{i+1}$. If $f_i < b_{i+1}$, then (70-III) is violated.

Then, let $f_i > b_{i+1}$. By (74), the u_2-interval $[c_i, f_i]$ must contain some interval $[g_i, h_i]$. We show that there exists no way to properly locate such an interval. If $g_i < b_{i+1}$, then $[g_i, b_{i+1}]$ is strictly contained in the u_2-interval $[c_i, f_i]$ and it meets the u-interval $[b_{i+1}, b_{i+2}]$, thus violating (73-I). Then, let $g_i \geq b_{i+1}$. It immediately follows that $b_{i+1} < h_i < f_i$. To show that such an alternative is inconsistent, we compare the relative position of f_i and c_{i+1} (we just proved that, for each $j \geq 0$, $[b_j, c_j]$ is a u_1-interval): (i) if $f_i < c_{i+1}$, then the interval $[h_i, f_i]$ is strictly contained in the u_1-interval $[b_{i+1}, c_{i+1}]$, and since, by (68), f_i starts a u-interval, (72) is violated; (ii) if $f_i = c_{i+1}$, then, by (68) and (65), f_i starts a u_1-interval ($[c_i, f_i]$ is a u_2-interval), and, by (67), it also starts a u_2-interval ($[b_{i+1}, c_{i+1}]$ is a u_1-interval), thus violating (69); (iii) if $f_i > c_{i+1}$, then the u_1-interval $[b_{i+1}, c_{i+1}]$ is contained in the u_2-interval $[c_i. f_i]$, violating (73-II).

(c) To show that there exists no other u-interval $[c, d] \in \mathcal{G}_{[a,b]}$, unless $c > b_i$ for each i, suppose, for the sake of contradiction, that there exists one such interval $[c, d]$. By the last conjunct of (64), $[c, d]$ starts an infinite u-chain. Then, by applying the argument we use to prove (b), we can show that there exists a point e such that $c < e < d$, $[c, e]$ is a u_1-interval and $[e, d]$ is a u_2-interval. We show that all possible choices for c lead to contradiction.

If $b_i < c < b_{i+1}$ for some i, then there exists no way to properly locate d. If $d \leq b_{i+1}$, then the u-interval $[b_i, b_{i+1}]$ contains the u_1-interval $[c, e]$, violating (70-II). If $d > b_{i+1}$, then two options are given. If $d \geq b_{i+2}$, then the u-interval $[c, d]$ contains the u_1-interval $[b_{i+1}, c_{i+1}]$, violating (70-II). If $d < b_{i+2}$, then there exists no way to properly locate e: (i) if $e < b_{i+1}$, then the u-interval $[b_i, b_{i+1}]$ contains the u_1-interval $[c, e]$, violating (70-II); (ii) if $e = b_{i+1}$, then both a u_1-interval and a u_2-interval start at e, violating (69); (iii) if $e > b_{i+1}$, then the u-interval $[b_{i+1}, b_{i+2}]$ contains the u_2-interval $[e, d]$, violating (70-III).

If $c = b_i$ for some i, then $d \neq b_{i+1}$. If $(b_i <)\ d < b_{i+1}$, the above argument about $b_i < c < b_{i+1}$ can be reused. The same applies to $d > b_{i+1}$, as the u-interval $[b_{i+1}, b_{i+2}]$ is such that $c < b_{i+1} < d$. □

Definition of the ld-chain The following set of formulae defines the ld-chain:

$$[G]((\mathsf{u} \leftrightarrow (* \vee \mathsf{tile})) \wedge (* \rightarrow \neg\mathsf{tile})) \tag{75}$$

$$\neg\mathsf{ld} \wedge \langle A\rangle(* \wedge \langle A\rangle(\mathsf{tile} \wedge \langle A\rangle(* \wedge [G](* \rightarrow \langle A\rangle(\mathsf{tile} \wedge \langle A\rangle\mathsf{tile}))))) \tag{76}$$

$$[G](\langle A\rangle\mathsf{ld} \leftrightarrow \langle A\rangle *) \tag{77}$$

$$[G](\mathsf{ld} \rightarrow \langle A\rangle *) \tag{78}$$

$$[G](\mathsf{ld} \rightarrow \neg\langle D\rangle *) \tag{79}$$

$$(75) \wedge \ldots \wedge (79) \tag{ld-chain$^{\mathsf{AD}}$}$$

Lemma 12 *Let M be a model, $[a, b]$ be an interval over M, and $b = b_1^0 < b_1^1 < \ldots < b_1^{k_1} = b_2^0 < b_2^1 < \ldots < b_2^{k_2} = b_3^0 < \ldots$ be the sequence of points defined by Lemma 11. If $M, [a, b] \Vdash$ (u-chain$^{\mathsf{AD}}$) $\wedge$ (ld-chain$^{\mathsf{AD}}$), then:*

(a) *for each $j \geq 1$, $M, [b_j^0, b_j^{k_j}] \Vdash \mathsf{ld}$;*
(b) *for each $j \geq 1$, $M, [b_j^0, b_j^1] \Vdash *$;*

(c) *for each* $j \geq 1$, $M, [b^i_j, b^{i+1}_j] \Vdash \mathsf{tile}$ *for each* $0 < i < k_j$;
(d) $k_1 = 2$, *and* $k_\ell > 2$ *for each* $\ell > 1$,

and no other interval $[c, d] \in \mathcal{G}_{[a,b]}$ *satisfies* ld (*resp.*, $*$, tile), *unless* $c > b^i_j$ *for each* $i, j > 0$.

Proof

(a) By (75), (77) (left-to-right direction), and (78), each ld-interval begins and ends with u-intervals. The existence of the ld-chain, beginning at $b_0 = b$, is guaranteed by (76), (77) (right-to-left direction), and (78).
(b) By (77) (left-to-right direction), the first u-interval of every ld-interval ($[b^0_j, b^1_j]$) is a $*$-interval.
(c) Suppose, for the sake of contradiction, that there is a u-interval $[b^i_j, b^{i+1}_j]$ such that $1 \leq i \leq k_j - 1$ (that is, $[b^i_j, b^{i+1}_j]$ is not the first u-interval of the jth ld-interval $[b^0_j, b^{k_j}_j]$), which is a $*$-interval. There are two possibilities, both leading to contradiction: (i) if $i < k_j - 1$, then $[b^i_j, b^{i+1}_j]$ is not the last u-interval of $[b^0_j, b^{k_j}_j]$, and thus there exists a $*$-interval strictly contained in an ld-interval, thus violating (79). If $i = k_j - 1$, then the $*$-interval $[b^i_j, b^{i+1}_j]$ meets the $*$-interval $[b^0_{j+1}, b^1_{j+1}]$, violating (76).
(d) Direct consequence of (76).

To conclude the proof we have to show that no other interval $[c, d] \in \mathcal{G}_{[a,b]}$ satisfies ld, $*$, or tile, unless $c > b^i_j$ for every $i, j > 0$. For the sake of contradiction, suppose that there exists an ld-interval $[c, d] \in \mathcal{G}_{[a,b]}$ with $[c, d] \neq [b^0_j, b^{k_j}_j]$ for every $j > 0$. By Lemma 11, (75), and (78), $d = b^{i'}_{j'}$ for some $j' > 0$ and $i' \geq 0$, and $[b^{i'}_{j'}, b^{i'+1}_{j'}]$ is a $*$-interval. We have two options for c, both leading to contradiction.

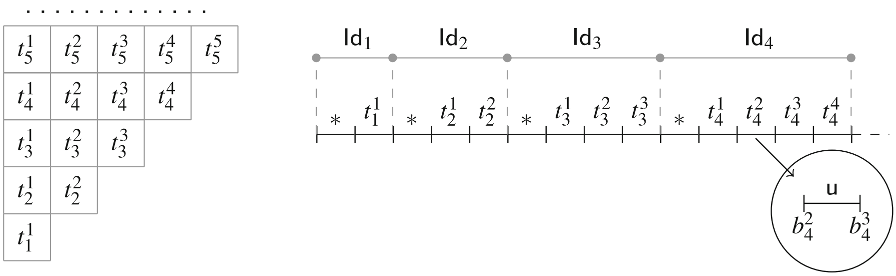
If $c = b^0_j$ for some $j > 0$, then $d \neq b^{k_j}_j$. If $d < b^{k_j}_j$, then $d = b^i_j$, for some $i > 0$. Hence $[b^i_j, b^{i+1}_j]$ satisfies both $*$ (by (78)) and tile (by item (c)), violating (75). If $d > b^{k_j}_j$, then two cases are possible: (i) if $d = b^1_{j+1}$, then, by (78), both $[b^1_{j+1}, b^2_{j+1}]$ and $[b^0_{j+1}, b^1_{j+1}]$ satisfy $*$, violating (76); (ii) if $d > b^1_{j+1}$, then the ld-interval $[c, d]$ contains the $*$-interval $[b^0_{j+1}, b^1_{j+1}]$, violating (79).

If for every $j > 0$ $c \neq b^0_j$, then, by Lemma 11, (75), and (77) (left-to-right direction), $c = b^i_j$ for some $i, j > 0$ and $[b^i_j, b^{i+1}_j]$ satisfies $*$. By item (c), $[b^i_j, b^{i+1}_j]$ also satisfies tile, thus violating (75).

For the sake of contradiction, suppose now that there exists a $*$-interval $[c, d] \in \mathcal{G}_{[a,b]}$ such that $[c, d] \neq [b^0_j, b^1_j]$ for every $j > 0$. By Lemma 11 and (75), $[c, d] = [b^i_j, b^{i+1}_j]$ for some $i, j > 0$. By item (c), $[b^i_j, b^{i+1}_j]$ is a tile-interval as well, thus violating (75).

A similar argument can be use to show that no other interval $[c, d] \in \mathcal{G}_{[a,b]}$ satisfies tile, unless $c > b^i_j$ for every $i, j > 0$. □

Figure 5 shows how to exploit the u-chain and ld-chain to encode the octant plane. Notice that there exists no need to distinguish between forward and backward rows: tiles are always encoded in ascending order. As a matter of fact, so far we have only encoded the rows of the octant by means of ld-intervals, the first one featuring exactly

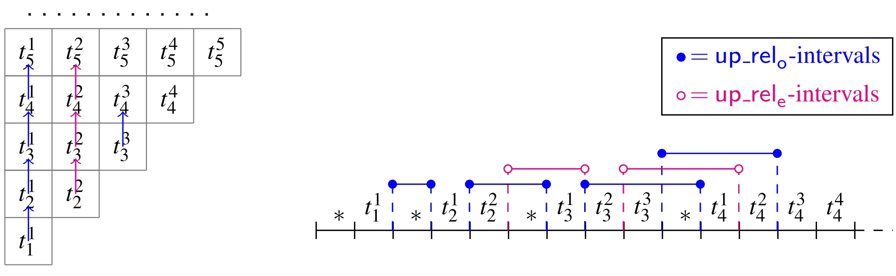


(a) Cartesian representation. (b) Interval representation

Fig. 5 Encoding of the octant plane in AD

one tile, the other ones at least two tiles. We show now how to encode some neighbor relations that connect each tile with its above neighbor and its right neighbor, if any, in the octant. This will allow us to force the jth Id-interval to contain exactly j tile-intervals.

Above-neighbor relation The above-neighbor relation connects each tile with its above neighbor in the octant. If $[b^i_j, b^{i+1}_j]$ and $[b^{i'}_{j'}, b^{i'+1}_{j'}]$ are, respectively, the ith tile-interval of the jth Id-interval and the i'th tile-interval of the j'th Id-interval, then we say that $[b^i_j, b^{i+1}_j]$ is *above-connected* to $[b^{i'}_{j'}, b^{i'+1}_{j'}]$ if and only if $j' = j + 1$ and $i = i'$. To encode the above-neighbor relation, we make use of a proposition letter up_rel: the up_rel-interval $[b^{i+1}_j, b^i_{j+1}]$ connects the tile-interval $[b^i_j, b^{i+1}_j]$ with the tile-interval $[b^i_{j+1}, b^{i+1}_{j+1}]$. Let $[b^i_j, b^{i+1}_j]$ be a tile-interval. We say that it is an *odd* (resp., *even*) tile-interval if i is odd (resp., even). The relation up_rel is encoded by means of the additional proposition letters $\mathsf{up\_rel_o}$ (connecting odd tile-intervals) and $\mathsf{up\_rel_e}$ (connecting even tile-intervals) such that $\mathsf{up\_rel} \leftrightarrow \mathsf{up\_rel_o} \oplus \mathsf{up\_rel_e}$. As shown on Fig. 6, intervals $\mathsf{up\_rel_o}$ and $\mathsf{up\_rel_e}$ alternate (*strict interleaving property*), namely, if $[b^i_j, b^{i+1}_j]$ is a tile-interval such that b^{i+1}_j is the starting point of an $\mathsf{up\_rel_o}$-interval (resp., $\mathsf{up\_rel_e}$-interval), then the next tile-interval $[b^{i+1}_j, b^{i+2}_j]$, if any, is connected to its above-neighbor by means of an $\mathsf{up\_rel_e}$-interval (resp., $\mathsf{up\_rel_o}$-interval). Furthermore, we prevent any two up_rel-intervals from starting or ending at the same point and from being contained in each other. Finally, for any row, each



(a) Cartesian representation. (b) Interval representation.

Fig. 6 Encoding of the above-neighbor relation in AD: $\mathsf{up\_rel_o}$- and $\mathsf{up\_rel_e}$-intervals alternate tile-interval must be above-connected to some tile-interval of the next row and for each tile-interval, except for the last one of the row, there must be some tile-interval of the previous row, if any, which is above-connected to it (formula (93) below). This guarantees that each row has exactly one tile-interval more than the previous one. Let α, β range over $\{\mathsf{o}, \mathsf{e}\}$, with $\alpha \neq \beta$. The following formulae encode the properties of the above-neighbor relation:

$$\neg \mathsf{up\_rel} \wedge \neg\langle A\rangle \mathsf{up\_rel} \wedge \langle A\rangle(* \wedge \langle A\rangle(\mathsf{tile} \wedge \langle A\rangle(* \wedge \mathsf{up\_rel}_{\mathsf{o}}))) \tag{80}$$

$$[G](\mathsf{up\_rel} \leftrightarrow (\mathsf{up\_rel}_{\mathsf{o}} \vee \mathsf{up\_rel}_{\mathsf{e}})) \tag{81}$$

$$[G](\langle A\rangle \mathsf{up\_rel}_{\mathsf{o}} \rightarrow \neg\langle A\rangle \mathsf{up\_rel}_{\mathsf{e}}) \tag{82}$$

$$[G](\mathsf{tile} \rightarrow \langle A\rangle \mathsf{up\_rel}) \tag{83}$$

$$[G](\mathsf{up\_rel}_{\alpha} \rightarrow \langle A\rangle(\mathsf{tile} \wedge \langle A\rangle \mathsf{up\_rel}_{\alpha})) \tag{84}$$

$$[G](\langle A\rangle \mathsf{up\_rel} \rightarrow \langle A\rangle \mathsf{u}) \tag{85}$$

$$[G](\mathsf{u} \wedge \langle A\rangle \mathsf{up\_rel} \rightarrow \mathsf{tile}) \tag{86}$$

$$[G](\mathsf{up\_rel} \rightarrow \langle A\rangle(\mathsf{tile} \wedge \langle A\rangle \mathsf{tile})) \tag{87}$$

$$[G](\langle A\rangle \mathsf{up\_rel}_{\alpha} \wedge \langle A\rangle \mathsf{tile} \rightarrow \langle A\rangle(\mathsf{tile} \wedge \langle A\rangle \mathsf{up\_rel}_{\beta})) \tag{88}$$

$$[G](\langle A\rangle * \rightarrow [A](\mathsf{up\_rel} \rightarrow \neg\langle D\rangle\langle A\rangle *)) \tag{89}$$

$$[G]((\mathsf{up\_rel} \rightarrow \neg\langle D\rangle \mathsf{Id}) \wedge (\mathsf{Id} \rightarrow \neg\langle D\rangle \mathsf{up\_rel})) \tag{90}$$

$$[G](* \rightarrow \langle A\rangle(\mathsf{tile} \wedge [A](\mathsf{up\_rel} \rightarrow \neg\langle D\rangle *))) \tag{91}$$

$$[G](\mathsf{up\_rel} \rightarrow \neg\langle D\rangle \mathsf{up\_rel}) \tag{92}$$

$$[G](\langle D\rangle \mathsf{up\_rel}_{\alpha} \wedge \langle A\rangle(\mathsf{u}_2 \wedge \langle A\rangle \mathsf{tile} \wedge \langle A\rangle \mathsf{up\_rel}_{\beta}) \rightarrow \langle D\rangle \mathsf{up\_rel}_{\beta}) \tag{93}$$

$$(80) \wedge \ldots \wedge (93) \tag{$\mathsf{up\_rel\text{-}def}^{\mathsf{AD}}$}$$

Lemma 13 *Let M be a model, $[a, b]$ be an interval over M, and $b = b_1^0 < b_1^1 < \ldots < b_1^{k_1} = b_2^0 < b_2^1 < \ldots < b_2^{k_2} = b_3^0 < \ldots$ be the sequence of points defined by Lemma 12. If $M, [a, b] \Vdash$ (u-chain$^{\mathsf{AD}}$) $\wedge$ (Id-chain$^{\mathsf{AD}}$) $\wedge$ (80) $\wedge$ (85) $\wedge \ldots \wedge$ (90), then:*

(a) *for every* up_rel*-interval $[c, d]$, there are c' and d' such that $[c', c]$ and $[d, d']$ are* tile*-intervals;*
(b) *for every* tile*-interval $[b_j^i, b_j^{i+1}]$ such that $i < k_j - 1$, if there exists an* $\mathsf{up\_rel}_{\mathsf{o}}$*-interval (resp.,* $\mathsf{up\_rel}_{\mathsf{e}}$*-interval) starting at b_j^{i+1}, then there is an* $\mathsf{up\_rel}_{\mathsf{e}}$*-interval (resp.,* $\mathsf{up\_rel}_{\mathsf{o}}$*-interval) starting at b_j^{i+2} (strict interleaving property);*
(c) *for every $j > 0$, $b_j^{k_j - 1}$ is the right endpoint of no* up_rel*-interval;*
(d) *for every* up_rel*-interval $[b_j^i, b_{j'}^{i'}]$ such that $1 < i \leq k_j$, $j' = j + 1$.*

Proof

(a) Let $[c, d]$ be an up_rel-interval. By (87), there exists d' such that $[d, d']$ is a tile-interval, and, by (80) (second conjunct), (85), (86), and Lemma 11, there exists c' such that $[c', c]$ is a tile-interval.
(b) Straightforward by (88).

(c) Straightforward by (87).

(d) Let $[b^{i}_{j}, b^{i'}_{j'}]$ be an up_rel-interval such that $1 < i \leq k_j$. For the sake of contradiction, suppose that $j' \neq j+1$. Two cases are possible, both leading to contradiction.

- Let $j' > j+1$. If $i = k_j$ ($[b^{i-1}_{j}, b^{i}_{j}]$ is the last tile-interval of the jth ld-interval), then $[b^{i-1}_{j}, b^{i}_{j}]$ satisfies $\langle A \rangle *$ and the last tile-interval of the $(j+1)$-th ld-interval $[b^{k_{j+1}-1}_{j+1}, b^{k_{j+1}}_{j+1}]$ meets the $*$-interval $[b^{0}_{j+2}, b^{1}_{j+2}]$ and is contained in the up_rel-interval $[b^{i}_{j}, b^{i'}_{j'}]$, thus violating (89).If $i < k_j$, the up_rel-interval $[b^{i}_{j}, b^{i'}_{j'}]$ contains the ld-interval $[b^{0}_{j+1}, b^{k_{j+1}}_{j+1}]$, violating (90-I);
- Let $j' = j$. Then, it necessarily holds that $i < i'$ and thus the up_rel-interval $[b^{i}_{j}, b^{i'}_{j}]$ is contained in the jth ld-interval, violating (90-II). □

Lemma 13 states some basic properties of the above-neighbor relation. However, it does not guarantee the existence of the above neighbor of a tile-interval: it may be the case that a tile-interval is not above-connected to any other tile-interval. Existence and uniqueness of the above-neighbor is enforced by the following lemma.

Lemma 14 *Let M be a model, $[a, b]$ be an interval over M, and $b = b^{0}_{1} < b^{1}_{1} < \ldots < b^{k_1}_{1} = b^{0}_{2} < b^{1}_{2} < \ldots < b^{k_2}_{2} = b^{0}_{3} < \ldots$ be the sequence of points defined by Lemma 12. If $M, [a, b] \Vdash$ (u-chain^{AD}) $\wedge$ (ld-chain^{AD}) $\wedge$ (up_rel-def^{AD}), then each tile-interval $[b^{i}_{j}, b^{i+1}_{j}]$ is above-connected to exactly one tile-interval and, if $i < k_j - 1$, then there exists exactly one tile-interval which is above-connected to it.*

Proof

Step 1 By (83) and (87), every tile-interval is above-connected to at least one tile-interval.

Step 2 Consider a tile-interval $[b^{i}_{j}, b^{i+1}_{j}]$, with $i < k_j - 1$. Since the first row consists of only one tile-interval, it holds that $j > 1$. By Lemma 13, there exists $b^{i'}_{j+1}$ such that $[b^{i+1}_{j}, b^{i'}_{j+1}]$ is an up_rel-interval. By (81) and (82), $[b^{i+1}_{j}, b^{i'}_{j+1}]$ is either an up_rel$_o$-interval or an up_rel$_e$-interval. Let us assume it to be an up_rel$_o$-interval (the other case is analogous). We prove that there exists a point c such that $[c, b^{i}_{j}]$ is an up_rel-interval. For the sake of contradiction, suppose that there exists no such a point. The proof is by induction on i.

- Base case ($i = 1$). If $j = 2$, then, by (80), $[b^{2}_{1}, b^{1}_{2}]$ is an up_rel-interval (contradiction). If $j > 2$, let us consider the interval $[b^{0}_{j-1}, b^{1}_{j-1}]$. By (91), (83), and Lemma 13, $[b^{2}_{j-1}, b^{1}_{j}]$ is an up_rel-interval (contradiction).
- Inductive step ($i > 1$). By Lemma 13(b) (strict interleaving property), there exists a point $b^{i''}_{j+1}$ such that $[b^{i}_{j}, b^{i''}_{j+1}]$ is an up_rel$_e$-interval. Furthermore, by the inductive hypothesis, there exists a point $b^{i'''}_{j-1}$ such that $[b^{i'''}_{j-1}, b^{i-1}_{j}]$ is an up_rel-interval. Such an interval $[b^{i'''}_{j-1}, b^{i-1}_{j}]$ is in fact an up_rel$_e$-interval (if it was an up_rel$_o$-interval, by (84), both an up_rel$_o$-interval and an up_rel$_e$-interval would begin at b^{i}_{j}, violating (82)).

  Now, let $[\overline{c}, \overline{d}]$ be an interval such that its left endpoint $\overline{c}$ lies strictly in between $b^{i'''-1}_{j-1}$ and $b^{i'''}_{j-1}$, and its right endpoint $\overline{d}$ lies strictly in between

b^i_j and b^{i+1}_j and $[\overline{d}, b^{i+1}_j]$ is a u_2-interval (by Lemma 11, such a point $\overline{d}$ always exists). We focus our attention on $[\overline{c}, \overline{d}]$, showing that we get contradiction with the condition expressed by formula (93).

First, it can be easily checked that $[\overline{c}, \overline{d}]$ satisfies the formulae (i) $\langle D\rangle \mathsf{up\_rel_e}$, as $[b^{i'''}_{j-1}, b^{i-1}_j]$ is an $\mathsf{up\_rel_e}$-interval and $\overline{c} < b^{i'''}_{j-1} < b^{i-1}_j < \overline{d}$, and (ii) $\langle A\rangle(\mathsf{u}_2 \wedge \langle A\rangle \mathsf{tile} \wedge \langle A\rangle \mathsf{up\_rel_o})$, as $[b^i_j, b^{i+1}_j]$ is not the last tile of the jth ld-interval and $[b^{i+1}_j, b^{i'}_{j+1}]$ is an $\mathsf{up\_rel_o}$-interval.

To violate (93), it suffices to show that $[\overline{c}, \overline{d}]$ does not satisfy $\langle D\rangle \mathsf{up\_rel_o}$. For the sake contradiction, suppose that there exists an $\mathsf{up\_rel_o}$-interval $[h, h']$ such that $\overline{c} < h < h' < \overline{d}$. We show that all possible choices for $[h, h']$ lead to contradiction.

- If $h = b^{i'''}_{j-1}$, then both an $\mathsf{up\_rel_o}$- and an $\mathsf{up\_rel_e}$-interval begin at h, violating (82).
- If $h > b^{i'''}_{j-1}$ and $h' < b^{i-1}_j$, then the $\mathsf{up\_rel}$-interval $[h, h']$ is contained in the $\mathsf{up\_rel}$-interval $[b^{i'''}_{j-1}, b^{i-1}_j]$, violating (92).
- If $h > b^{i'''}_{j-1}$ and $h' = b^{i-1}_j$, then both the $\mathsf{up\_rel_e}$-interval $[b^{i'''}_{j-1}, b^{i-1}_j]$ and the $\mathsf{up\_rel_o}$-interval $[h, b^{i-1}_j]$ end at b^{i-1}_j, and thus, by (84), b^i_j begins both an $\mathsf{up\_rel_e}$-interval and an $\mathsf{up\_rel_o}$-interval, violating (82).
- If $h > b^{i'''}_{j-1}$ and $h' = b^i_j$, then there exists an $\mathsf{up\_rel}$-interval ending at b^i_j, against the initial hypothesis.

Step 3 We just proved that for every tile-interval $[b^i_j, b^{i+1}_j]$ which is not the last tile-interval of the jth ld-interval there exists a point c such that $[c, b^i_j]$ is an $\mathsf{up\_rel}$-interval. To complete the proof, we must show that every tile-interval is above-connected to at most one tile-interval and there exists at most one tile-interval above-connected to it. For the sake of contradiction, suppose that there exist $b^i_j, b^{i'}_{j+1}$, and $b^{i''}_{j+1}$, with $b^{i'}_{j+1} < b^{i''}_{j+1}$ (the case $b^{i'}_{j+1} > b^{i''}_{j+1}$ is symmetric) such that both $[b^i_j, b^{i'}_{j+1}]$ and $[b^i_j, b^{i''}_{j+1}]$ are $\mathsf{up\_rel}$-intervals. If $[b^i_j, b^{i'}_{j+1}]$ is an $\mathsf{up\_rel_o}$-interval and $[b^i_j, b^{i''}_{j+1}]$ is an $\mathsf{up\_rel_e}$-interval, or vice versa, then (82) is violated. Let us assume that both $[b^i_j, b^{i'}_{j+1}]$ and $[b^i_j, b^{i''}_{j+1}]$ are $\mathsf{up\_rel_o}$-intervals (the case in which both are $\mathsf{up\_rel_e}$-intervals is symmetric). By (84), both $b^{i'+1}_{j+1}$ and $b^{i''+1}_{j+1}$ start an $\mathsf{up\_rel_o}$-interval. By Lemma 13(b) (strict interleaving property), an $\mathsf{up\_rel_e}$-interval starts at $b^{i'+2}_{j+1}$. Since $[b^{i'+1}_{j+1}, b^{i'+2}_{j+1}]$ is not the last tile of the $(j+1)$th ld-interval, there exists a point c such that $[c, b^{i'+1}_{j+1}]$ is an $\mathsf{up\_rel}$-interval (step 2 above). By (84) and (82), $[c, b^{i'+1}_{j+1}]$ is a $\mathsf{up\_rel_e}$-interval. We show that all possible choices for c lead to contradiction: (i) if $c < b^i_j$, then the $\mathsf{up\_rel}$-interval $[c, b^{i'+1}_{j+1}]$ contains the $\mathsf{up\_rel}$-interval $[b^i_j, b^{i'}_{j+1}]$, violating (92); (ii) if $c = b^i_j$, then b^i_j starts both an $\mathsf{up\_rel_o}$- and an $\mathsf{up\_rel_e}$-interval, violating (82); (iii) if $c > b^i_j$, then the $\mathsf{up\_rel}$-interval $[b^i_j, b^{i''}_{j+1}]$ contains the $\mathsf{up\_rel}$-interval $[c, b^{i'+1}_{j+1}]$, violating (92). In a similar way, we can prove that two $\mathsf{up\_rel}$-intervals cannot end at the same point. $\square$

Right-neighbor relation The right-neighbor relation connects two consecutive tiles belonging to the same row. We say that two tile-intervals $[b^i_j, b^{i+1}_j]$ and $[b^{i'}_{j'}, b^{i'+1}_{j'}]$ are *right-connected* if and only if $j' = j$ and $i' = i + 1$. The encoding of the right-neighbor relation is trivial, as it exploits the adjacency of consecutive pairs of tiles belonging to the same row, that is, from a tile-interval $[b^i_j, b^{i+1}_j]$ it is possible to access the tile-interval $[b^{i+1}_j, b^{i+2}_j]$, if any, with which it is right-connected, simply by applying modality $\langle A \rangle$.

The following lemma shows that the commutativity property holds.

Lemma 15 *Let M be a model, $[a, b]$ be an interval over M, and $b = b^0_1 < b^1_1 < \ldots < b^{k_1}_1 = b^0_2 < b^1_2 < \ldots < b^{k_2}_2 = b^0_3 < \ldots$ be the sequence of points defined by Lemma 12. If $M, [a, b] \Vdash$ (u-chain^{AD}) $\wedge$ (ld-chain^{AD}) $\wedge$ (up_rel-def^{AD}), then M satisfies the commutativity property.*

Proof Let $[b^i_j, b^{i+1}_j]$ and $[b^{i'}_{j'}, b^{i'+1}_{j'}]$ be two tile-intervals and suppose that there exists a tile-interval $[c, d]$ such that $[b^i_j, b^{i+1}_j]$ is right-connected to $[c, d]$ and $[c, d]$ is above-connected to $[b^{i'}_{j'}, b^{i'+1}_{j'}]$. It holds that $[c, d] = [b^{i+1}_j, b^{i+2}_j]$ and $[b^{i+2}_j, b^{i'}_{j'}]$ is an up_rel-interval. It immediately follows that $j' = j + 1$. Since $[b^i_j, b^{i+1}_j]$ is a tile-interval, by Lemma 14, it is above-connected to exactly one tile-interval. Let that be $[b^{i''}_{j+1}, b^{i''+1}_{j+1}]$. Thus, $[b^{i+1}_j, b^{i''}_{j+1}]$ is an up_rel-interval. We show that $[b^{i''}_{j+1}, b^{i''+1}_{j+1}]$ is right connected to $[b^{i'}_{j+1}, b^{i'+1}_{j+1}]$. Since the only interval that is right-connected to $[b^{i'}_{j+1}, b^{i'+1}_{j+1}]$, if any, is the interval $[b^{i'-1}_{j+1}, b^{i'}_{j+1}]$, then it suffices to show that $[b^{i''}_{j+1}, b^{i''+1}_{j+1}] = [b^{i'-1}_{j+1}, b^{i'}_{j+1}]$, that is, $b^{i''}_{j+1} = b^{i'-1}_{j+1}$. For the sake of contradiction, suppose that this is not the case. Two cases must be considered, both leading to contradiction.

(i) $b^{i''}_{j+1} > b^{i'-1}_{j+1}$. If $b^{i''}_{j+1} = b^{i'}_{j+1}$, then the up_rel-intervals $[b^{i+1}_j, b^{i'}_{j+1}]$ and $[b^{i+2}_j, b^{i'}_{j+1}]$ end at the same point (contradiction with Lemma 14); otherwise, if $b^{i''}_{j+1} > b^{i'}_{j+1}$, then the up_rel-interval $[b^{i+1}_j, b^{i''}_{j+1}]$ contains the up_rel-interval $[b^{i+2}_j, b^{i'}_{j+1}]$, violating (92);

(ii) $b^{i''}_{j+1} < b^{i'-1}_{j+1}$. By Lemma 14, there exists a point $b^{i'''}_j$ such that $[b^{i'''}_j, b^{i'-1}_{j+1}]$ is an up_rel-interval. We show that all possible choices for $b^{i'''}_j$ lead to contradiction:

- if $b^{i'''}_j > b^{i+2}_j$, then the up_rel-interval $[b^{i+2}_j, b^{i'}_{j+1}]$ contains the up_rel-interval $[b^{i'''}_j, b^{i'-1}_{j+1}]$, violating (92);
- if $b^{i'''}_j = b^{i+2}_j$, then the two up_rel-intervals $[b^{i+2}_j, b^{i'}_{j+1}]$ and $[b^{i+2}_j, b^{i'-1}_{j+1}]$ begin at the same point (contradiction with Lemma 14);
- if $b^{i'''}_j = b^{i+1}_j$, then the two up_rel-intervals $[b^{i+1}_j, b^{i''}_{j+1}]$ and $[b^{i+1}_j, b^{i'-1}_{j+1}]$ begin at the same point (contradiction with Lemma 14);
- if $b^{i'''}_j < b^{i+1}_j$, then the up_rel-interval $[b^{i'''}_j, b^{i'-1}_{j+1}]$ contains the up_rel-interval $[b^{i+1}_j, b^{i''}_{j+1}]$, violating (92).

Hence $b^{i''}_{j+1} = b^{i'-1}_{j+1}$, which implies that $[b^i_j, b^{i+1}_j]$ is above-connected to $[b^{i'-1}_{j+1}, b^{i'}_{j+1}]$. □

Corollary 4 *The ith tile-interval of the jth row (ld-interval) is above-connected to the ith tile-interval of the $(j+1)$th row (ld-interval).*

Tiling the plane To complete the encoding of OTP, we must constrain each tile-interval (and no other one) to be tiled by exactly one tile, and force tiles that are right- or above-connected to respect the color constraints. We do it as follows:

$$[G]\left(\left(\bigvee_{i=1}^{k} \mathtt{t}_i \leftrightarrow \mathsf{tile}\right) \wedge \left(\bigwedge_{i,j=1,i\neq j}^{k} \neg(\mathtt{t}_i \wedge \mathtt{t}_j)\right)\right) \tag{94}$$

$$[G](\mathsf{tile} \rightarrow \bigvee_{\mathrm{up}(t_i)=\mathrm{down}(t_j)} (\mathtt{t}_i \wedge \langle A\rangle(\mathsf{up\_rel} \wedge \langle A\rangle \mathtt{t}_j))) \tag{95}$$

$$[G](\mathsf{tile} \wedge \langle A\rangle \mathsf{tile} \rightarrow \bigvee_{\mathrm{right}(t_i)=\mathrm{left}(t_j)} (\mathtt{t}_i \wedge \langle A\rangle \mathtt{t}_j)) \tag{96}$$

$$(94) \wedge (95) \wedge (96) \tag{tiles$^{\mathsf{AD}}$}$$

Let $\mathcal{T}$ be the set of tile types $\{t_1, t_2, \ldots, t_k\}$ and $\Phi_{\mathcal{T}}$ be the formula (u-chain$^{\mathsf{AD}}$) $\wedge$ (Id-chain$^{\mathsf{AD}}$) $\wedge$ ($\mathsf{up\_rel}$-def$^{\mathsf{AD}}$) $\wedge$ (tiles$^{\mathsf{AD}}$). The following lemma holds.

Lemma 16 *For any linear order $\mathbb{D}$ with an infinite ascending sequence of points, the formula $\Phi_{\mathcal{T}}$ is satisfiable in $\mathbb{D}$ if and only if $\mathcal{T}$ can tile the second octant $\mathcal{O}$.*

Proof ("only if" direction) Let $\mathbb{D}$ be a linear order with an infinite ascending sequence of points such that $M, [a, b] \Vdash \Phi_{\mathcal{T}}$ for some model $M = \langle \mathbb{I}(\mathbb{D}), V\rangle$ and interval $[a, b] \in \mathbb{I}(\mathbb{D})$. Let $b = b_1^0 < b_1^1 < b_1^2 = b_2^0 < \ldots < b_2^{k_2} = b_3^0 < \ldots < b_j^0 < b_j^1 < \ldots < b_j^{k_j} = b_{j+1}^0 < \ldots$ be the sequence of points defined by Lemma 12. For each $j > 0$ and $0 < i < k_j$, $[b_j^i, b_j^{i+1}]$ is a tile-interval, and thus $M, [b_j^i, b_j^{i+1}] \Vdash \mathtt{t}_v$ for an unique v. Then, for each i, j such that $0 \leq i \leq j$, we put $f(i, j) = t_v$, where $\mathtt{t}_v$ is the unique proposition letter in the set $\mathtt{T} = \{\mathtt{t}_1, \mathtt{t}_2, \ldots, \mathtt{t}_k\}$ such that $M, [b_{j+1}^{i+1}, b_{j+1}^{i+2}] \Vdash \mathtt{t}_v$. By Lemmas 14 and 15, Corollary 4, and formula (tiles$^{\mathsf{AD}}$), the function $f : \mathcal{O} \mapsto \mathcal{T}$ defines a correct tiling of $\mathcal{O}$.

("if" direction) Let $\mathbb{D}$ be a linear order with an infinite ascending sequence of points, and let $f : \mathcal{O} \mapsto \mathcal{T}$ be a correct tiling of $\mathcal{O}$. We provide a model $M = \langle \mathbb{I}(\mathbb{D}), V\rangle$ and an interval $[a, b] \in M$ such that $M, [a, b] \Vdash \Phi_{\mathcal{T}}$ (see Fig. 7). Let $\sigma = b_0, b_1, \ldots$ be the infinite ascending sequence of points in $\mathbb{D}$ whose existence is guaranteed by

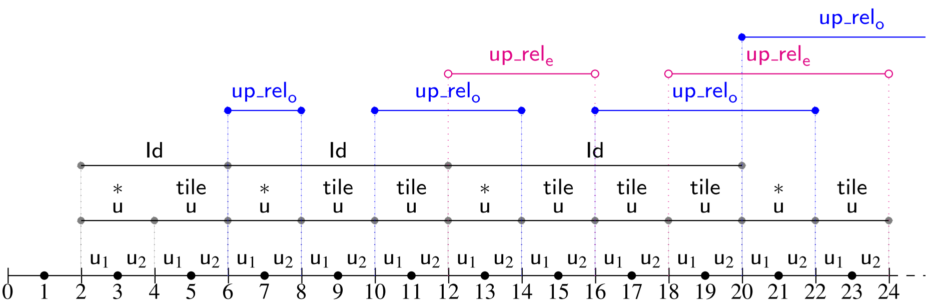


Fig. 7 A model satisfying the formula $\Phi_{\mathcal{T}}$ (for the sake of readability, we write i for b_i, $i = 0, \ldots, 24$)

hypothesis. The valuation function V is defined as follows, where $g(n)$ abbreviates $(n+1)(n+2)$:

- $\mathsf{u} \in V([c,d])$ iff $[c,d] = [b_i, b_j]$, $i = 2n$ and $j = i+2$ for some $n(>0) \in \mathbb{N}$;
- $\mathsf{u}_1 \in V([c,d])$ iff $c = b_i$, $d = b_{i+1}$, and $\mathsf{u} \in V([b_i, b_{i+2}])$ for some $i \in \mathbb{N}$;
- $\mathsf{u}_2 \in V([c,d])$ iff $c = b_i$, $d = b_{i+1}$, and $\mathsf{u} \in V([b_{i-1}, b_{i+1}])$ for some $i \in \mathbb{N}$;
- $* \in V([c,d])$ iff $[c,d] = [b_i, b_j]$, $\mathsf{u} \in V([b_i, b_j])$, and $i = g(n)$ for some $n \geq 0$;
- $\mathsf{Id} \in V([c,d])$ iff $[c,d] = [b_i, b_j]$, $* \in V([b_i, b_{i+2}])$, $* \in V([b_j, b_{j+2}])$, $i = g(n)$, and $j = g(n+1)$ for some $n \geq 0$;
- $\mathsf{tile} \in V([c,d])$ iff $[c,d] = [b_i, b_j]$, $\mathsf{u} \in V([b_i, b_j])$, and $* \notin V([b_i, b_j])$ for some $i, j \in \mathbb{N}$;
- for each $h \in \{1, \dots, k\}$, $\mathtt{t}_h \in V([c,d])$ iff $[c,d] = [b_i, b_j]$, $\mathsf{tile} \in V([b_i, b_j])$, $f(l,m) = t_h$, and $i = g(m) + 2l + 2$ for some l, m such that $0 \leq l \leq m$;
- $\mathsf{up\_rel_o} \in V([c,d])$ iff $[c,d] = [b_i, b_j]$, $\mathsf{tile} \in V([b_{i-2}, b_i])$, $\mathsf{tile} \in V([b_j, b_{j+2}])$, $i - 2 = g(m) + 2l + 2$, and $j = g(m+1) + 2l + 2$ for some l, m such that $0 \leq l \leq m$ and $l = 2n$ for some $n \geq 0$;
- $\mathsf{up\_rel_e} \in V([c,d])$ iff $[c,d] = [b_i, b_j]$, $\mathsf{tile} \in V([b_{i-2}, b_i])$, $\mathsf{tile} \in V([b_j, b_{j+2}])$, $i - 2 = g(m) + 2l + 2$, and $j = g(m+1) + 2l + 2$ for some l, m such that $0 \leq l \leq m$ and $l = 2n+1$ for some $n \geq 0$;
- $\mathsf{up\_rel} \in V([c,d])$ iff $[c,d] = [b_i, b_j]$, $\mathsf{up\_rel_o} \in V([b_i, b_j])$ or $\mathsf{up\_rel_e} \in V([b_i, b_j])$ for some $i, j \in \mathbb{N}$.

It can be easily checked that $M, [b_0, b_2] \models \Phi_{\mathcal{T}}$. $\square$

Corollary 5 *The satisfiability problem for* AD *is undecidable over any class of linear orders that contains at least one linear order with an infinite ascending sequence of points.*

5.2 The fragments $\mathsf{A\overline{D}}$, $\mathsf{\overline{A}D}$, and $\mathsf{\overline{A}\overline{D}}$

To adapt the construction we devised for AD to $\mathsf{A\overline{D}}$, we will replace each formula containing the operator D with a formula (or a set of formulae) belonging to the language of $\mathsf{A\overline{D}}$. Thus, such a replacement involves formulae (70)–(74), (79) and (89)–(93). Most of them can be rewritten with minimum effort, but some of them need to be completely reformulated.

As a preliminary step, let us consider the AD formula $[G](p \to [D]q)$ and the $\mathsf{A\overline{D}}$ formula $[G](\langle\overline{D}\rangle p \to q)$. Apparently, both formulae force every interval in $\mathcal{G}_{[a,b]}$ contained in a p-interval to be a q-interval. However, due to the way in which we defined modality $[G]$, the two formulae are not equivalent, as they behave differently on intervals that do not belong to $\mathcal{G}_{[a,b]}$. The former imposes no constraint on these intervals, while the latter does: starting from an interval in $\mathcal{G}_{[a,b]}$, the $\mathsf{A\overline{D}}$ formula allows one to reach a p-interval outside $\mathcal{G}_{[a,b]}$ (via modality $\langle\overline{D}\rangle$) and then it forces such a p-interval to contain a q-interval. Hence, the $\mathsf{A\overline{D}}$-formula is stronger than the AD one, since it can constrain the behavior of a larger set of intervals. This is formally stated by the following lemma.

Lemma 17 *Let $p, q \in \mathcal{AP}$ and $M, [a,b] \Vdash \neg p$. Then it holds that:*

(i) *if $M, [a,b] \Vdash [G](\langle\overline{D}\rangle p \to \neg q)$, then $M, [a,b] \Vdash [G](p \to \neg\langle D\rangle q)$, and*
(ii) *if $M, [a,b] \Vdash [G](\langle\overline{D}\rangle p \to \neg\langle A\rangle q)$, then $M, [a,b] \Vdash [G](p \to \neg\langle D\rangle\langle A\rangle q)$.*

Proof We only prove item (ii). The proof of item (i) is simpler and thus omitted. Let $M, [a,b] \Vdash [G](\langle\overline{D}\rangle p \rightarrow \neg\langle A\rangle q)$ and $[c,d] \in \mathcal{G}_{[a,b]}$. Then if there exists a p-interval $[e, f]$ such that $e < c < d < f$, then for every $d' > d$ $[d, d']$ is not a q-interval. For the sake of contradiction, suppose that $M, [a,b] \nVdash [G](p \rightarrow \neg\langle D\rangle\langle A\rangle q)$. Then there exists a p-interval $[e, f]$ in $\mathcal{G}_{[a,b]}$ such that there exist a sub-interval $[c,d]$ of $[e, f]$ and a q-interval $[d, d']$, for some $d' > d$. Since, by hypothesis, $[a,b]$ is not a p-interval, $[e, f] \neq [a,b]$. It can be easily checked that $[c,d]$ belongs to $\mathcal{G}_{[a,b]}$ ($[e, f]$ belongs to $\mathcal{G}_{[a,b]} \setminus \{[a,b]\}$, and $[c,d]$ is a sub-interval of $[e, f]$). From $M, [a,b] \Vdash [G](\langle\overline{D}\rangle p \rightarrow \neg\langle A\rangle q)$, it follows that $M, [c,d] \Vdash \neg\langle A\rangle q$ (contradiction). □

By Lemma 17, formulae (70), (72), (73), (79), (90), and (92) can be rewritten as follows:

$$[G](\langle\overline{D}\rangle \mathsf{u} \rightarrow \neg\langle A\rangle \mathsf{u} \wedge \neg \mathsf{u}_1 \wedge \neg \mathsf{u}_2) \tag{97}$$

$$[G](\langle\overline{D}\rangle \mathsf{u}_1 \rightarrow \neg\langle A\rangle \mathsf{u}) \tag{98}$$

$$[G](\langle\overline{D}\rangle \mathsf{u}_2 \rightarrow \neg\langle A\rangle \mathsf{u} \wedge \neg \mathsf{u}_1) \tag{99}$$

$$[G](\langle\overline{D}\rangle \mathsf{Id} \rightarrow \neg *) \tag{100}$$

$$[G]((\langle\overline{D}\rangle \mathsf{up\_rel} \rightarrow \neg \mathsf{Id}) \wedge (\langle\overline{D}\rangle \mathsf{Id} \rightarrow \neg \mathsf{up\_rel})) \tag{101}$$

$$[G](\langle\overline{D}\rangle \mathsf{up\_rel} \rightarrow \neg \mathsf{up\_rel}) \tag{102}$$

Since all proposition letters occurring in the above formulae are not satisfied by the initial interval $[a,b]$, from Lemma 17 it immediately follows that:

$$\begin{array}{ll} M,[a,b] \Vdash (97) \Rightarrow M,[a,b] \Vdash (70) & M,[a,b] \Vdash (98) \Rightarrow M,[a,b] \Vdash (72) \\ M,[a,b] \Vdash (99) \Rightarrow M,[a,b] \Vdash (73) & M,[a,b] \Vdash (100) \Rightarrow M,[a,b] \Vdash (79) \\ M,[a,b] \Vdash (101) \Rightarrow M,[a,b] \Vdash (90) & M,[a,b] \Vdash (102) \Rightarrow M,[a,b] \Vdash (92) \end{array} \tag{$\mathsf{A}\overline{\mathsf{D}}1$}$$

The replacement of the remaining formulae is more complex. Let us start with formulae (71) and (74). First, we expand the set $\mathcal{AP}$ of proposition letters with two additional letters k_1 and k_2, whose meaning is expressed by the following set of formulae:

$$[G](\langle A\rangle \mathsf{u} \rightarrow \langle A\rangle\langle\overline{D}\rangle \mathsf{k}_1) \tag{103}$$

$$[G](\mathsf{u} \rightarrow \neg\langle\overline{D}\rangle \mathsf{k}_1) \tag{104}$$

$$[G][\overline{D}](\mathsf{k}_1 \rightarrow \neg\langle A\rangle \mathsf{u}) \tag{105}$$

$$[G](\langle A\rangle \mathsf{u}_2 \rightarrow \langle A\rangle\langle\overline{D}\rangle \mathsf{k}_2) \tag{106}$$

$$[G](\mathsf{u}_2 \rightarrow \neg\langle\overline{D}\rangle \mathsf{k}_2) \tag{107}$$

$$[G][\overline{D}](\mathsf{k}_2 \rightarrow \neg\langle A\rangle \mathsf{u}) \tag{108}$$

Formulae (103), (104), and (105) replace formula (71), while formulae (106), (107), and (108) replace formula (74), as formally stated by the next lemma.

Lemma 18 *Let* $M, [a, b] \Vdash (64) \wedge (68)$. *Then it holds that:*

(i) *if* $M, [a, b] \Vdash (103) \wedge (104) \wedge (105)$, *then* $M, [a, b] \Vdash (71)$, *and*
(ii) *if* $M, [a, b] \Vdash (106) \wedge (107) \wedge (108)$, *then* $M, [a, b] \Vdash (74)$.

Proof We prove item (i). Item (ii) can be proved in the very same way, and thus its proof is omitted. Let $M, [a, b] \Vdash (64) \wedge (103) \wedge (104) \wedge (105)$. For the sake of contradiction, suppose that $[a, b]$ does not satisfy (71), that is, $M, [a, b] \nVdash [G](\mathsf{u} \to \langle D\rangle\top)$. Then there exists an interval $[c, d]$ in $\mathcal{G}_{[a,b]}$ such that $M, [c, d] \Vdash \mathsf{u} \wedge \neg\langle D\rangle\top$. By (64), $[a, b]$ does not satisfy u, and thus there exists an interval $[e, c]$ in $\mathcal{G}_{[a,b]}$ such that $M, [e, c] \Vdash \langle A\rangle\mathsf{u}$. By (103), it follows that there exist an interval $[c, f]$ such that $M, [c, f] \Vdash \langle\overline{D}\rangle\mathsf{k}_1$, and an interval $[g, h]$, which is a super-interval of $[c, f]$, such that $M, [g, h] \Vdash \mathsf{k}_1$. We show that all choices for the relative positions of h and d lead to contradiction: (i) if $h > d$, then the k_1-interval $[g, h]$ contains the u-interval $[c, d]$, thus violating (104); (ii) if $h = d$, then, by (64), the k_1-interval $[g, h]$ meets some u-interval $[d, i]$, violating (105); (iii) if $h < d$, then $[f, h]$ is strictly contained in $[c, d]$, against the hypothesis that $M, [c, d] \Vdash \neg\langle D\rangle\top$. □

Lemma 18 allows us to rewrite formula (u-chain$^{\mathsf{AD}}$), that defines the u-chain, as follows:

$$(64) \wedge \ldots \wedge (69) \wedge (97) \wedge (98) \wedge (99) \wedge (103) \wedge \ldots \wedge (108) \qquad (\mathsf{u}\text{-chain}^{\mathsf{A}\overline{\mathsf{D}}})$$

Corollary 6 *If* $M, [a, b] \Vdash (\mathsf{u}\text{-chain}^{\mathsf{A}\overline{\mathsf{D}}})$, *then* $M, [a, b] \Vdash (\mathsf{u}\text{-chain}^{\mathsf{AD}})$.

As for formula (ld-chain$^{\mathsf{AD}}$), it suffices to replace formula (79) (the only one making use of modality $\langle D\rangle$) by formula (100):

$$(75) \wedge \ldots \wedge (78) \wedge (100) \qquad (\mathsf{ld}\text{-chain}^{\mathsf{A}\overline{\mathsf{D}}})$$

Corollary 7 *If* $M, [a, b] \Vdash (\mathsf{ld}\text{-chain}^{\mathsf{A}\overline{\mathsf{D}}})$, *then* $M, [a, b] \Vdash (\mathsf{ld}\text{-chain}^{\mathsf{AD}})$.

The case of formula ($\mathsf{up\_rel\text{-}def}^{\mathsf{AD}}$) is more involved. Formulae (90) and (92) can be replaced by formulae (101) and (102), but there exists no such a direct replacement for formulae (89), (91), and (93). These formulae can be replaced by the following set of formulae, which makes use of a new proposition letter first:

$$[G](* \to [A](\langle A\rangle * \to \neg\langle\overline{D}\rangle\mathsf{up\_rel})) \tag{109}$$

$$[G](* \to \langle A\rangle(\mathsf{tile} \wedge [A](\mathsf{up\_rel} \to \mathsf{first}))) \tag{110}$$

$$[G](* \to \neg\langle\overline{D}\rangle\mathsf{first}) \tag{111}$$

$$[G](\langle A\rangle * \to [A](\mathsf{up\_rel} \to \langle A\rangle(\mathsf{tile} \wedge \langle A\rangle(\mathsf{tile} \wedge \langle A\rangle *)))) \tag{112}$$

$$\begin{aligned}&[G](\langle A\rangle(\mathsf{u}_2 \wedge \langle A\rangle(\mathsf{tile} \wedge \langle A\rangle\mathsf{up\_rel}_\alpha \wedge \langle A\rangle\mathsf{tile}))\\ &\quad \to \langle\overline{D}\rangle\mathsf{up\_rel}_\alpha \vee \neg\langle\overline{D}\rangle\mathsf{up\_rel}_\beta)\end{aligned} \tag{113}$$

First, we show that (89) and (91) can be replaced by (109) and by (110) and (111), respectively.

Lemma 19 *It holds that:*

(i) *if* $M, [a, b] \Vdash$ (u-chain$^{\mathsf{A\overline{D}}}$) $\wedge$ (ld-chain$^{\mathsf{A\overline{D}}}$) $\wedge$ (109)*, then* $M, [a, b] \Vdash$ (89)*;*
(ii) *if* $M, [a, b] \Vdash (110) \wedge (111)$*, then* $M, [a, b] \Vdash (91)$*.*

Proof We first prove item (i). For the sake of contradiction, suppose that $M, [a, b] \Vdash (76) \wedge (109)$, but $M, [a, b] \not\Vdash [G](\langle A\rangle * \rightarrow [A](\mathsf{up\_rel} \rightarrow \neg\langle D\rangle\langle A\rangle *))$. Then there exists a $[c, d] \in \mathcal{G}_{[a,b]}$ such that $M, [c, d] \Vdash \langle A\rangle * \wedge \langle A\rangle(\mathsf{up\_rel} \wedge \langle D\rangle\langle A\rangle *)$. Hence, there exist $[d, e], [d, f] \in \mathcal{G}_{[a,b]}$ such that $M, [d, e] \Vdash *$ and $M, [d, f] \Vdash \mathsf{up\_rel} \wedge \langle D\rangle\langle A\rangle *$. By definition of $\langle D\rangle$, it follows that there exists a $*$-interval $[g, h] \in \mathcal{G}_{[a,b]}$ such that $d < g < f$. By Lemmas 11 and 12, $e < g$. Since both $[d, e]$ and $[g, h]$ are $*$-intervals, from (109), it follows that $M, [e, g] \Vdash \neg\langle\overline{D}\rangle\mathsf{up\_rel}$ (contradiction with $M, [d, f] \Vdash \mathsf{up\_rel}$).

We now prove item (ii). For the sake of contradiction, suppose that $M, [a, b] \Vdash (110) \wedge (111)$ but $M, [a, b] \not\Vdash [G](* \rightarrow \langle A\rangle(\mathsf{tile} \wedge [A](\mathsf{up\_rel} \rightarrow \neg\langle D\rangle *)))$. Then there exists a $[c, d] \in \mathcal{G}_{[a,b]}$ such that $M, [c, d] \Vdash * \wedge [A](\neg\mathsf{tile} \vee \langle A\rangle(\mathsf{up\_rel} \wedge \langle D\rangle *))$. From (110), it follows that there exists a $[d, e] \in \mathcal{G}_{[a,b]}$ such that $M, [d, e] \Vdash \mathsf{tile} \wedge [A](\mathsf{up\_rel} \rightarrow \mathsf{first}) \wedge \langle A\rangle(\mathsf{up\_rel} \wedge \langle D\rangle *)$. Then there exists an $[e, f] \in \mathcal{G}_{[a,b]}$ such that $M, [e, f] \Vdash \mathsf{up\_rel} \wedge \mathsf{first} \wedge \langle D\rangle *$. Finally, by the definition of $\langle D\rangle$, it follows that $[e, f]$ contains a $*$-interval $[g, h] \in \mathcal{G}_{[a,b]}$ (contradiction with (111)). □

Formula (up_rel-def$^{\mathsf{AD}}$) can be replaced by the following one:

$$(80) \wedge \ldots \wedge (88) \wedge (101) \wedge (102) \wedge (109) \wedge \ldots \wedge (113) \qquad (\mathsf{up\_rel\text{-}def}^{\mathsf{A\overline{D}}})$$

To prove that formula (up_rel-def$^{\mathsf{A\overline{D}}}$) correctly defines the above-neighbor relation in $\mathsf{A\overline{D}}$, we need to show that formula (93) can be replaced by formulae (112) and (113).

Lemma 20 *If* $M, [a, b] \Vdash$ (u-chain$^{\mathsf{A\overline{D}}}$) $\wedge$ (ld-chain$^{\mathsf{A\overline{D}}}$) $\wedge$ (up_rel-def$^{\mathsf{A\overline{D}}}$)*, then* $M, [a, b] \Vdash (93)$*.*

Proof As a preliminary remark, we observe that from Corollaries 6 and 7, and Lemma 19, it follows that Lemmas 11, 12, and 13 still hold.

To prove the statement of the lemma, suppose, for the sake of contradiction, that $M, [a, b] \Vdash$ (u-chain$^{\mathsf{A\overline{D}}}$) $\wedge$ (ld-chain$^{\mathsf{A\overline{D}}}$) $\wedge$ (up_rel-def$^{\mathsf{A\overline{D}}}$), but $M, [a, b] \not\Vdash [G](\langle D\rangle\mathsf{up\_rel}_\alpha \wedge \langle A\rangle(\mathsf{u}_2 \wedge \langle A\rangle\mathsf{tile} \wedge \langle A\rangle\mathsf{up\_rel}_\beta) \rightarrow \langle D\rangle\mathsf{up\_rel}_\beta)$. This implies the existence of $[c, d]$ such that:

$$M, [c, d] \Vdash \langle D\rangle\mathsf{up\_rel}_\alpha \wedge \langle A\rangle(\mathsf{u}_2 \wedge \langle A\rangle\mathsf{tile} \wedge \langle A\rangle\mathsf{up\_rel}_\beta) \wedge \neg\langle D\rangle\mathsf{up\_rel}_\beta, \qquad (\mathrm{H1})$$

for some $\alpha \neq \beta \in \{\mathsf{o}, \mathsf{e}\}$.

From the first conjunct, it follows that there exists an $\mathsf{up\_rel}_\alpha$-interval $[e, f]$ strictly contained in $[c, d]$. Without loss of generality, we can assume $[e, f]$ to be initiated by some tile-interval $[e, g]$. Suppose that this is not the case. By (85) and (75), $[e, g]$ is a $*$-interval. Hence, by (112), there exists a tile-interval $[f, f']$, followed by a tile-interval $[f', g']$, followed by $*$-interval $[g', g'']$. We show now that $g'' < d$. For the sake of contradiction, suppose that $g'' \geq d$. Let $\overline{f}$ (resp., $\overline{f'}, \overline{g'}$) be such that $f < \overline{f} < f'$ (resp., $f' < \overline{f'} < g', g' < \overline{g'} < g''$) and $M, [f, \overline{f}] \Vdash u_1$ and $M, [\overline{f}, f'] \Vdash u_2$ (resp., $M, [f', \overline{f'}] \Vdash u_1$ and $M, [\overline{f'}, g'] \Vdash u_2$, $M, [g', \overline{g'}] \Vdash u_1$ and $M, [\overline{g'}, g''] \Vdash u_2$).

Since $M, [c, d] \Vdash \langle A \rangle \mathsf{u}_2$, either $d = \overline{f}$ or $d = \overline{f'}$ or $d = \overline{g'}$. If $d = \overline{f}$, then, by (H1) (second conjunct), there exists an $\mathsf{up\_rel}_\beta$-interval starting at f' and, by (84), there exists an $\mathsf{up\_rel}_\alpha$-interval which starts at f' and violates (82). If $d = \overline{f'}$, then by (H1) (second conjunct), $[g', g'']$ is both a tile-interval and $*$-interval, violating (75). Finally, if $d = \overline{g'}$, then, by (H1) (second conjunct), the $*$-interval $[g', g'']$ meets an $\mathsf{up\_rel}$-interval, violating (86). Hence, $g'' < d$, and thus $[g', g'']$ is contained in $[c, d]$. Now, by (76), the $*$-interval $[e, g]$ meets some tile-interval $[g, t]$ and, by (83) and (91), there exists a $\mathsf{up\_rel}$-interval starting at t and ending at some point t', with $t'(\leq g'') < d$. Thus $[t, t']$ is strictly contained in $[c, d]$ and it is started by a tile-interval. Hence, whenever $[e, f]$ is started by a $*$-interval, we can substitute $[t, t']$ for it in our argument.

From the assumption that $[e, g]$ is a tile-interval and (88), it follows that there exists an $\mathsf{up\_rel}_\beta$-interval $[g, h]$ for some h. Since, by (H1) (third conjunct), $M, [c, d] \Vdash \neg\langle D \rangle \mathsf{up\_rel}_\beta$, it holds that $h \geq d$. Now, by (H1) (second conjunct), there exists a u_2-interval $[d, d']$, for some d', and an $\mathsf{up\_rel}_\beta$-interval starting at d'. By Lemma 11 and (86), there exist i, j such that $j < i < d$ and $[i, d']$ is a tile-interval and $[j, i]$ is a u_2-interval. Now, the interval $[g, j]$ satisfies $\langle A \rangle(\mathsf{u}_2 \wedge \langle A \rangle(\mathsf{tile} \wedge \langle A \rangle \mathsf{up\_rel}_\beta \wedge \langle A \rangle \mathsf{tile}))$. Hence, by (113), it also satisfies $\langle \overline{D} \rangle \mathsf{up\_rel}_\beta \vee \neg\langle \overline{D} \rangle \mathsf{up\_rel}_\alpha$. Suppose $M, [g, j] \Vdash \langle \overline{D} \rangle \mathsf{up\_rel}_\beta$. Then there exists an $\mathsf{up\_rel}_\beta$-interval $[\hat{g}, \hat{j}]$ such that $\hat{g} < g$ and $\hat{j} > j$. In fact, $\hat{g} < e$, since $\hat{g} = e$ would violate (82). This implies that the $\mathsf{up\_rel}_\alpha$-interval $[e, f]$ is contained in the $\mathsf{up\_rel}_\beta$-interval $[\hat{g}, \hat{j}]$, thus violating (102). Hence, we can conclude that $M, [g, j] \Vdash \neg\langle \overline{D} \rangle \mathsf{up\_rel}_\alpha$.

Now, let $[k, h]$ be the u-interval that ends the $\mathsf{up\_rel}_\beta$-interval $[g, h]$, whose existence is guaranteed by Lemmas 11 and 13. Since both $[e, f]$ and $[g, h]$ are $\mathsf{up\_rel}$-intervals starting from the same Id-interval (row), say l, by Lemma 13, they both must end on the next Id-interval (row) $l + 1$. It immediately follows that $[k, h]$ cannot be a $*$-interval, as, otherwise, f and h would belong to different Id-intervals (rows), and thus it is a tile-interval. By the strict interleaving property, h starts a $\mathsf{up\_rel}_\alpha$-interval. Now, let consider the interval $[m, n]$, where m and n are such that $[g, m]$ is a u_1-interval and $[n, k]$ is a u_2-interval (the existence of these points is guaranteed by Lemma 11). $[m, n]$ satisfies $\langle A \rangle(\mathsf{u}_2 \wedge \langle A \rangle(\mathsf{tile} \wedge \langle A \rangle \mathsf{up\_rel}_\alpha \wedge \langle A \rangle \mathsf{tile}))$, and thus, by (113), it satisfies $\langle \overline{D} \rangle \mathsf{up\_rel}_\alpha \vee \neg\langle \overline{D} \rangle \mathsf{up\_rel}_\beta$. We show that in both cases we get a contradiction.

- If $M, [m, n] \Vdash \langle \overline{D} \rangle \mathsf{up\_rel}_\alpha$, then there exists a $\mathsf{up\_rel}_\alpha$-interval $[m', n']$ such that $m' < m < n < n'$. By Lemma 13, $m' < g$, and thus $[g, j]$ is contained in the $\mathsf{up\_rel}_\alpha$-interval $[m', n']$, against the hypothesis that $M, [g, j] \Vdash \neg\langle \overline{D} \rangle \mathsf{up\_rel}_\alpha$.
- If $M, [m, n] \Vdash \neg\langle \overline{D} \rangle \mathsf{up\_rel}_\beta$, we immediately get a contradiction, since $[g, h]$ is an $\mathsf{up\_rel}_\beta$-interval. □

Corollary 8 *If* $M, [a, b] \Vdash$ (u-chain$^{\mathsf{A\overline{D}}}$) $\wedge$ (Id-chain$^{\mathsf{A\overline{D}}}$) $\wedge$ (up_rel-def$^{\mathsf{A\overline{D}}}$), *then* $M, [a, b] \Vdash$ (up_rel-def$^{\mathsf{AD}}$).

Since modality $\langle D \rangle$ does not occur in formula (tiles$^{\mathsf{AD}}$), no replacement is necessary. The encoding of OTP is thus complete. Let $\mathcal{T}$ be the set of tile types $\{t_1, t_2, \ldots, t_k\}$ and $\Psi_{\mathcal{T}}$ be the formula (u-chain$^{\mathsf{A\overline{D}}}$) $\wedge$ (Id-chain$^{\mathsf{A\overline{D}}}$) $\wedge$ (up_rel-def$^{\mathsf{A\overline{D}}}$) $\wedge$ (tiles$^{\mathsf{AD}}$). The following lemma holds.

Lemma 21 *For any linear order $\mathbb{D}$ with an infinite ascending sequence of points, the formula $\Psi_{\mathcal{T}}$ is satisfiable in $\mathbb{D}$ if and only if $\mathcal{T}$ can tile the second octant $\mathcal{O}$.*

Proof ("only if" direction) Let $\mathbb{D}$ be a linear order with an infinite ascending sequence of points such that $M, [a, b] \Vdash \Psi_{\mathcal{T}}$ for some model $M = \langle \mathbb{I}(\mathbb{D}), V \rangle$ and interval $[a, b] \in \mathbb{I}(\mathbb{D})$. By Corollary 6, Corollary 7, and Corollary 8, $M, [a, b] \Vdash \Phi_{\mathcal{T}}$, where $\Phi_{\mathcal{T}}$ is the formula (u-chain) $\wedge$ (ld-chain$^{\mathsf{AD}}$) $\wedge$ (up_rel-def$^{\mathsf{AD}}$) $\wedge$ (tiles$^{\mathsf{AD}}$). By Lemma 16, it immediately follows that $\mathcal{T}$ can tile the second octant $\mathcal{O}$.

("if" direction) Let $f : \mathcal{O} \mapsto \mathcal{T}$ be a correct tiling of $\mathcal{O}$. It can be easily shown that a model for $\Psi_{\mathcal{T}}$ can be obtained from the model $M = \langle \mathbb{I}(\mathbb{D}), V \rangle$ for $\Phi_{\mathcal{T}}$ given in the proof of Lemma 16, by extending the valuation function to the new proposition letters k_1, k_2, and first. This completes the proof. □

Corollary 9 *The satisfiability problem for $\mathsf{A}\overline{\mathsf{D}}$ is undecidable over any class of linear orders that contains at least one linear order with an infinite ascending sequence of points.*

The previous reductions for AD and $\mathsf{A}\overline{\mathsf{D}}$ can be easily adapted, by symmetry, to $\overline{\mathsf{A}}\mathsf{D}$ and $\overline{\mathsf{A}}\overline{\mathsf{D}}$, provided that there exists an infinite descending sequence of points.

Corollary 10 *The satisfiability problem for $\overline{\mathsf{A}}\mathsf{D}$ and $\overline{\mathsf{A}}\overline{\mathsf{D}}$ is undecidable over any class of linear orders that contains at least one linear order with an infinite descending sequence of points.*

6 Conclusions and future work

In this paper, we solved various open problems in the characterization of HS fragments with respect to decidability/undecidability. First, we showed undecidability of the satisfiability problem for O and $\overline{\mathsf{O}}$ over all meaningful classes of linear orders, including the classes of all, discrete, dense, and finite linear orders. As a direct consequence of this result, we got undecidability of $\mathsf{B}\overline{\mathsf{E}}$ and $\overline{\mathsf{B}}\mathsf{E}$, whose decidability status over the class of finite linear orders was still unknown. Then, we proved undecidability of the satisfiability problem for AD, $\mathsf{A}\overline{\mathsf{D}}$, $\overline{\mathsf{A}}\mathsf{D}$, and $\overline{\mathsf{A}}\overline{\mathsf{D}}$ over all classes of linear orders containing at least a linear order with an infinite sequence of points. Since undecidability of AD, $\mathsf{A}\overline{\mathsf{D}}$, $\overline{\mathsf{A}}\mathsf{D}$, and $\overline{\mathsf{A}}\overline{\mathsf{D}}$ over the class of finite linear orders was already known, we can conclude that also for these fragments undecidability spans all important classes of linear orders.

Even though this paper solved a number of open problems about (un)decidability of HS fragments, it does not allow us to get the complete picture: the status of some interesting fragments is still unknown. In particular, the satisfiability problem is still open for the following fragments:

(i) $\overline{\mathsf{A}}\mathsf{B}$, $\overline{\mathsf{A}}\overline{\mathsf{B}}$, AE, and $\mathsf{A}\overline{\mathsf{E}}$ over all meaningful classes of linear orders, except for the class of finite linear orders, over which the problem is known to be decidable [21]. We believe it possible to adapt the undecidability results for $\mathsf{A}\overline{\mathsf{A}}\mathsf{B}\overline{\mathsf{B}}$ and $\mathsf{A}\overline{\mathsf{A}}\mathsf{E}\overline{\mathsf{E}}$ over (Dedekind-complete) infinite linear orders given in [21].

(ii) D and $\overline{\mathsf{D}}$ over the class of all linear orders. This is the most challenging problem, since neither the decidability proof for the dense case [7] nor undecidability proofs for the finite and discrete cases [19] can be transferred directly to the case of all linear orders.

We believe it's possible to exploit the proof technique developed in the paper to prove undecidability of other logical formalisms, not necessarily interval-based. As an example, we have explored their applicability in the setting of first-order and monadic second-order logics with matching relations [16], which are binary relations whose semantics resembles the one of Allen's relations over intervals. We have already obtained some preliminary results showing that the proofs given in this paper can be adapted to prove undecidability of first-order logic with two matching relations over linear orders.

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