Realization of Continuous–Time Nonlinear Input–Output Equations: Polynomial Approach

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Abstract. The aim of the paper is to apply the polynomial methods to nonlinear realization problem. A new formula is presented which allows to compute the differentials of the state coordinates directly from the polynomial description of the nonlinear system, yielding a shorter and more compact program code in *Mathematica* implementation.

Keywords: nonlinear control system, continuous-time system, input-output models, polynomial methods, state space realization.

1 Introduction

Most results on nonlinear identification are achieved for systems, described by input-output (i/o) differential equations. At the same time the majority of techniques for nonlinear system analysis and control design are based on state-space description. The problem that we deal with in this paper is that of recovering the state-space model, whenever possible, starting from an arbitrary nonlinear higher order i/o differential equation.

In [1] the algebraic formalism, based on differential one-forms, has been applied for studying the realization problem. The coordinate-free necessary and sufficient realizability conditions were formulated in terms of the integrability of certain $\mathcal{H}_k$ subspaces of one-forms and the differentials of the state coordinates were defined as the basis elements of the last subspace. The algorithm to calculate the subspaces was given. Slightly different point of view in the studies of nonlinear control systems is provided by the polynomial approach in which the system is described by two polynomials from the non-commutative ring of left skew polynomials that act on input and output differentials. Polynomial approach has been used so far to study the reduction of the (set of nonlinear) i/o equations [2,3], the transfer equivalence [2] and used also in extending the concept of transfer function into the nonlinear domain [4,5].

The aim of the present paper is to apply the polynomial approach to the realization problem. This allows to simplify the step-by-step algorithm for computation of $\mathcal{H}_k$ subspaces given in [1]. A new formula is presented which allows to compute the $\mathcal{H}_k$ subspaces of one-forms directly from the polynomial description of the nonlinear system. The new method is more direct and therefore better

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suited for implementation in computer algebra packages like Mathematica or Maple. Note that the realization problem for discrete-time nonlinear systems was addressed within polynomial approach in [6], extending the linear case, discussed in [7].

The paper is organized as follows. Section 2 describes the realization problem studied in this paper and recalls its solution in terms of $\mathcal{H}_k$ subspaces. Section 3 introduces the polynomial framework and Section 4 presents the solution of the realization problem in terms of polynomials, describing the system. In Section 5 two examples and discussion are provided.

2 Problem Statement and the Algebraic Framework

Consider a nonlinear system Σ, described by a higher order i/o differential equation, relating the input u, the output y and a finite number of their time derivatives,

$$y^{(n)} = \phi(y, \ldots, y^{(n-1)}, u, \ldots, u^{(n-1)}). \tag{1}$$

In (1), $u \in U \subset I\!\!R$ is the scalar input variable, $y \in Y \subset I\!\!R$ is the scalar output variable, n is a nonnegative integer, and ϕ is a real analytic function, defined on $Y^n \times U^n$.

The realization problem is defined as follows. Given a nonlinear system, described by the i/o equation of the form (1), find, if possible, the state coordinates $x \in I\!\!R^n$, $x = \psi(y, \ldots, y^{(n-1)}, u, \ldots, u^{(n-1)})$ such that in these coordinates the system takes the classical state space form

$$\dot{x} = f(x, u), \quad y = h(x), \tag{2}$$

called the realization of (1).

Below we briefly recall the algebraic formalism, described in [1]. Let $\mathcal{K}$ denote the field of meromorphic functions in a finite number of the independent system variables $\{y, \ldots, y^{(n-1)}, u^{(k)},\ k \geq 0\}$ and $s : \mathcal{K} \to \mathcal{K}$ denote the time derivative operator $\mathrm{d}/\mathrm{d}t$. Then the pair $(\mathcal{K}, s)$ is differential field [8]. Over the field $\mathcal{K}$ one can define a differential vector space, $\mathcal{E} := \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\varphi \mid \varphi \in \mathcal{K}\}$ spanned by the differentials of the elements of $\mathcal{K}$. Consider a one-form $\omega \in \mathcal{E} : \omega = \sum_i \alpha_i \mathrm{d}\varphi_i$, $\alpha_i, \varphi_i \in \mathcal{K}$. Its derivative $\dot{\omega}$ is defined by $\dot{\omega} = \sum_i \dot{\alpha}_i \mathrm{d}\varphi_i + \alpha_i \mathrm{d}\dot{\varphi}_i$.

The relative degree r of an one-form $\omega \in \mathcal{E}$ is defined to be the least integer such that $s^r\omega \notin \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \ldots, \mathrm{d}y^{(n-1)}, \mathrm{d}u, \ldots, \mathrm{d}u^{(n-1)}\}$. If such an integer does not exist, we set $r = \infty$. A sequence of subspaces $\{\mathcal{H}_k\}$ of $\mathcal{E}$ is defined by

$$\begin{aligned} \mathcal{H}_1 &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \ldots, \mathrm{d}y^{(n-1)}, \mathrm{d}u, \ldots, \mathrm{d}u^{(n-1)}\} \\ \mathcal{H}_{k+1} &= \{\omega \in \mathcal{H}_k \mid \dot{\omega} \in \mathcal{H}_k\},\ k \geq 1. \end{aligned} \tag{3}$$

Note that $\mathcal{H}_k$ contains the one-forms whose relative degree is equal to k or higher than k. It is clear that the sequence (3) is decreasing. Denote by k^* the least integer such that $\mathcal{H}_1 \supset \mathcal{H}_2 \supset \cdots \supset \mathcal{H}_{k^*} \supset \mathcal{H}_{k^*+1} = \mathcal{H}_{k^*+2} = \cdots =: \mathcal{H}_\infty$.

In what follows we assume that the i/o differential equation (1) is in the irreducible form, that is, $\mathcal{H}_\infty$ is trivial. A nth-order realization of equation (1) will be accessible if and only if system (1) is irreducible. If we find a nth-order realization for an i/o equation (1) which is in fact "reducible", the realization will be non-accessible.

System (2) is said to be single-experiment observable if the observability matrix has generically full rank $\operatorname{rank}_{\mathcal{K}}\left[\partial(h(x), sh(x), \ldots, s^{n-1}h(x))/\partial x\right] = n$.

Theorem 1. *The nonlinear system Σ, described by the irreducible i/o differential equation (1), has an observable and accessible state-space realization iff for $1 \leq k \leq n+1$ the subspaces $\mathcal{H}_k$, defined by (3), are completely integrable. Moreover, the state coordinates can be obtained by integrating the basis vectors of $\mathcal{H}_{n+1}$.*

We say that $\omega \in \mathcal{E}$ is exact, if there exists $\zeta \in \mathcal{K}$ such that $\mathrm{d}\zeta = \omega$. A subspace is integrable or closed, if it has a basis which consists only of closed one-forms. Note that closed one-forms are locally exact. Integrability of the subspace of one-forms can be checked by the Frobenius theorem.

Theorem 2. *[9] Let $\mathcal{V} = \operatorname{span}_{\mathcal{K}}\{\omega_1, \ldots, \omega_r\}$ be a subspace of $\mathcal{E}$. $\mathcal{V}$ is closed iff $\mathrm{d}\omega_k \wedge \omega_1 \wedge \ldots \wedge \omega_r = 0$, for all $k = 1, \ldots, r$.*

3 Polynomial Framework

Polynomial framework is built upon the linear algebraic framework. The differential field $(\mathcal{K}, s)$ induces a ring of left polynomials $\mathcal{K}[\partial, s]$. The elements of $\mathcal{K}[\partial, s]$ can be uniquely written in the form $a(\partial) = \sum_{i=0}^{n} a_i \partial^{n-i}$, $a_i \in \mathcal{K}$ where ∂ is a polynomial indeterminate and $a(\partial) \neq 0$ if and only if at least one of the functions a_i, $i = 0, \ldots, n$ is nonzero. If $a_0 \not\equiv 0$, then the positive integer n is called the degree of the left polynomial $a(\partial)$ and denoted by $\deg a(\partial)$. In addition, we set $\deg 0 = -\infty$. For $a \in \mathcal{K}$ let us define the multiplication

$$\partial \cdot a = a \cdot \partial + s(a). \tag{4}$$

If the multiplication is defined by (4), the ring $\mathcal{K}[\partial, s]$ is proved to satisfy left Ore condition [10], and $\partial^n \cdot a \in \mathcal{K}[\partial, s]$, for $n \geq 1$, and $\partial^n \cdot a = \sum_{i=0}^{n} C_n^i s^{n-i}(a) \partial^i$. A ring D is called an integral domain, if it does not contain any zero divisors. This means that if a and b are two elements of D such that $ab = 0$, then $a = 0$ or $b = 0$.

Lemma 1. *[10]*

(i) The ring $\mathcal{K}[\partial, s]$ is an integral domain.
(ii) If a and b are nonzero left polynomials, then $\deg(a\,b) = \deg a + \deg b$.

For $\Phi \in \mathcal{K}$ we define $\mathrm{d} : \mathcal{K} \to \mathcal{E}$ as follows: $\mathrm{d}\Phi := \sum_{i=0}^{n-1} \partial\Phi/\partial y^{(i)}\, \mathrm{d}y^{(i)} + \sum_{l=0}^{k} \partial\Phi/\partial u^{(l)}\, \mathrm{d}u^{(l)}$. $\mathrm{d}\Phi$ is said to be the total differential (or simply the differential) of the function Φ and it is a differential one-form. It is proved in [1] that $s(\mathrm{d}\Phi) = \mathrm{d}(s\Phi)$. Let us define $\partial^k \mathrm{d}y := \mathrm{d}(s^k y)$ and $\partial^l \mathrm{d}u := \mathrm{d}(s^l u)$, for $k, l \geq 0$ in the vector space $\mathcal{E}$. Since every one-form $\omega \in \mathcal{E}$ has the form $\omega = \sum_{i=0}^{n-1} a_i \mathrm{d}y^{(i)} + \sum_{j=0}^{k} b_j \mathrm{d}u^{(j)}$, where $a_i, b_j \in \mathcal{K}$, so ω can be expressed in terms of the left polynomials $\omega = \left(\sum_{i=0}^{n-1} a_i \partial^i\right) \mathrm{d}y + \left(\sum_{j=0}^{k} b_j \partial^j\right) \mathrm{d}u$. A left polynomial can be considered as an operator acting on the elements of $\mathcal{E}$:

$$\left(\sum_{i=0}^{k} a_i \partial^i\right)(\alpha \mathrm{d}\nu) := \sum_{i=0}^{k} a_i \left(\partial^i \cdot \alpha\right) \mathrm{d}\nu,$$

with $a_i, \alpha \in \mathcal{K}$ and $\mathrm{d}\nu \in \{\mathrm{d}y, \mathrm{d}u\}$. It is easy to notice that $\partial(\omega) = s(\omega)$, for $\omega \in \mathcal{E}$. Additionally, using the induction principle, one can show that $\partial^n(\mathrm{d}\Phi) = \mathrm{d}(s^n \Phi)$.

Instead of working with equation (1), describing the control system, we can work with its differential

$$\mathrm{d}y^{(n)} - \sum_{i=0}^{n-1} \frac{\partial \phi}{\partial y^{(i)}} \mathrm{d}y^{(i)} - \sum_{j=0}^{n-1} \frac{\partial \phi}{\partial u^{(j)}} \mathrm{d}u^{(j)} = 0 \tag{5}$$

that can be rewritten as

$$p(\partial)\mathrm{d}y = q(\partial)\mathrm{d}u, \tag{6}$$

with $p(\partial) = \partial^n - \sum_{i=0}^{n-1} p_i \partial^i$, $q(\partial) = \sum_{j=0}^{n-1} q_j \partial^j$ and $p_i = \partial\phi/\partial y^{(i)} \in \mathcal{K}$, $q_i = \partial\phi/\partial u^{(j)} \in \mathcal{K}$. Equation (6) describes the nonlinear system behavior in terms of two polynomials $\{p(\partial), q(\partial)\}$ in derivative operator $\partial := s$ over the differential field $\mathcal{K}$.

Note that in the polynomial description, the systems $\phi(\cdot) = 0$ and $\phi(\cdot) + \text{constant} = 0$ are not distinguished for arbitrary constant value. In order to overcome such situations one has to fix the constant and assume it to be defined by the equilibrium point of the system, around which the one forms will be integrated to get the state coordinates. Note that in many papers the assumption $\phi(0, \ldots, 0) = 0$ was made which however is sometimes restrictive if the i/o equation does not admit a zero equilibrium point. Though we make the same assumption for simplicity, note that this assumption can be relaxed.

4 Problem Solution: Polynomial Approach

We introduce certain one-forms in terms of which our main result (Theorem 3) will be formulated. The one-forms

$$\omega_{k,l} := \bar{p}_{k,l}(\partial)\mathrm{d}y + \bar{q}_{k,l}(\partial)\mathrm{d}u, \tag{7}$$

for $k = 1, \ldots, n+1$, $l = 1, \ldots, n$, where $\bar{p}_{k,l}(\partial)$ and $\bar{q}_{k,l}(\partial)$ are Ore polynomials, which can be recursively calculated as left quotients from equalities

$$\begin{aligned} \bar{p}_{k,l-1}(\partial) &= s\bar{p}_{k,l}(\partial) + r_{k,l}, \deg r_{k,l} = 0, \\ \bar{q}_{k,l-1}(\partial) &= s\bar{q}_{k,l}(\partial) + \rho_{k,l}, \deg \rho_{k,l} = 0, \end{aligned} \tag{8}$$

with initial polynomials $\bar{p}_{10} = -s$, $\bar{q}_{10} = 0$, for $k = 1$ and

$$\bar{p}_{k,0}(\partial) = \sum_{i=n-k+1}^{n-1} p_i \partial^i - \partial^n, \quad \bar{q}_{k,0}(\partial) = \sum_{i=n-k+1}^{n-1} q_i \partial^i \tag{9}$$

for $2 \leq k \leq n+1$. The following lemmas are necessary to prove the main result. The proofs are omitted due to lack of space.

Lemma 2. *The one-forms $\omega_{k,l}$, for $k = 1, \ldots, n$, defined by (7), satisfy the relationships*

(i) for $l = 1, \ldots, n-k$

$$\omega_{k+1,l} = \omega_{k,l} + \sum_{i=0}^{n-k-l} \binom{-l}{n-k-l-i} \left[p_{n-k}^{(n-k-l-i)} \mathrm{d}y^{(i)} + q_{n-k}^{(n-k-l-i)} \mathrm{d}u^{(i)} \right],$$

(ii) for $l = n-k+1, \ldots, n$ $\omega_{k+1,l} = \omega_{k,l}$.

Lemma 3. *For the one-forms $\omega_{k,l}$, defined by (7), the property $\dot{\omega}_{k,l} = \omega_{k,l-1} - r_{k,l}\mathrm{d}y - \rho_{k,l}\mathrm{d}u$, holds, where $\deg r_{k,l} = \deg \rho_{k,l} = 0$ for $l = 2, \ldots, n$ and for $k = 1, \ldots, n+1$.*

Theorem 3. *For the i/o model (1), the subspaces $\mathcal{H}_k$ can be calculated as*

$$\mathcal{H}_k = \mathrm{span}_{\mathcal{K}}\{\omega_{k,l}, \mathrm{d}u, \ldots, \mathrm{d}u^{(n-k)}\}, \quad k = 1, \ldots, n \tag{10}$$

and

$$\mathcal{H}_{n+1} = \mathrm{span}_{\mathcal{K}}\{\omega_{n+1,l}\}, \tag{11}$$

where $\omega_{k,l}$ for $l = 1, \ldots, n$ are defined by (7).

Proof. The proof is by induction. We first show that formula (3) holds for $k = 1$. From (7), the quotient polynomials $\bar{p}_{1,l}(\partial) = -\partial^{n-l}$ and $\bar{q}_{1,l}(\partial) = 0$ for $l = 1, \ldots, n$. Consequently, $\omega_l = -s^{n-l}\mathrm{d}y = -\mathrm{d}y^{(n-l)}$ and $\mathcal{H}_1 = \mathrm{span}_{\mathcal{K}}\{-\mathrm{d}y^{(n-1)}, \ldots, -\mathrm{d}y, \mathrm{d}u, \ldots, \mathrm{d}u^{(n-1)}\}$ that agrees with the definition of $\mathcal{H}_1$.

Assume next that formula (10) holds for k and we prove it to be valid for $k+1$. The proof is based on definition of the subspaces $\mathcal{H}_k$. We have to prove that $\mathcal{H}_{k+1} = \mathrm{span}_{\mathcal{K}}\{\omega_{k+1,l}, \mathrm{d}u, \ldots, \mathrm{d}u^{(n-k-1)}\}$, calculated according to formula (10), satisfies the condition (3).

First, we show that basis one-forms $\omega_{k+1,l}, \mathrm{d}u, \ldots, \mathrm{d}u^{(n-k-1)}$ are in $\mathcal{H}_k$. It is obvious that $\mathrm{d}u, \ldots, \mathrm{d}u^{(n-k-1)} \in \mathcal{H}_k$. Lemma 2 represents the one-forms $\omega_{k+1,l}$ as a linear combination of vectors $\omega_{k,l}$, $\mathrm{d}y$, ..., $\mathrm{d}y^{(n-k)}$, $\mathrm{d}u$, ..., $\mathrm{d}u^{(n-k)}$. Though $\mathrm{d}y$, ..., $\mathrm{d}y^{(n-k)}$ are not listed explicitly among the basis vectors of $\mathcal{H}_k$ in (10), they can be expressed as a linear combination of the other basis vectors. From (8) and (9) follows that the coefficients of the higher order terms of polynomials $\bar{p}_{k,l}(\partial)$ are always 1 as well as $\deg \bar{p}_{k,l}(\partial) = n-l$ and $\deg \bar{q}_{k,l}(\partial) = n-l-1$ for $l = 1, \ldots, n-1$. It means that $\bar{p}_{k,l}(\partial)$ and $\bar{q}_{k,l}(\partial)$ for $l = 1, \ldots, n-1$ have the form $\bar{p}_{k,l}(\partial) = \sum_{j=0}^{n-l-1} p_{k,l,j}\partial^j - \partial^{n-l}$, $\bar{q}_{k,l}(\partial) = \sum_{j=0}^{n-l-1} q_{k,l,j}\partial^j$. For $l = n$

we get $\bar{p}_{k,n}(\partial) = 1$ and $\bar{q}_{k,n}(\partial) = 0$. Consequently, $\omega_{k,n} = \mathrm{d}y$. The rest of the differentials $\mathrm{d}\dot{y}, \ldots, \mathrm{d}y^{(n-k)}$ can be recursively computed from (7) as follows: $\mathrm{d}y^{(l)} = \omega_{k,n-l} - \sum_{j=0}^{l-1} \bar{p}_{k,n-l,j}\mathrm{d}y^{(j)} - \sum_{j=0}^{l-1} \bar{q}_{k,n-l,j}\mathrm{d}u^{(j)}$ for $l = 1, \ldots, n-k$.

Second, we show that the derivatives of the one-forms, computed according to (10) and (7), also belong to $\mathcal{H}_k$. Again, it is obvious that $\mathrm{d}\dot{u}, \ldots, \mathrm{d}u^{(n-k)} \in \mathcal{H}_k$. We have to prove that $\dot{\omega}_{k+1,l} \in \mathcal{H}_k$. According to Lemma 3, $\dot{\omega}_{k+1,l} = \omega_{k+1,l-1} - r_{k+1,l}\mathrm{d}y - \rho_{k+1,l}\mathrm{d}u$, where $\deg r_{k+1,l} = \deg \rho_{k+1,l} = 0$ for $l = 2, \ldots, n$. It was proved in the previous step that $\omega_{k+1,l}$ for $l = 2, \ldots, n$ and $\mathrm{d}y$ are in $\mathcal{H}_k$. For $l = 1$ we have to show separately that $\dot{\omega}_{k+1,1} \in \mathcal{H}_k$. From (7) we have: $\dot{\omega}_{k+1,1} = s\bar{p}_{k+1,1}(\partial)\mathrm{d}y + s\bar{q}_{k+1,1}(\partial)\mathrm{d}u$. Increasing k by 1 and taking $l = 1$ in (8) allows us to express $s\bar{p}_{k+1,1}$ and $s\bar{q}_{k+1,1}$ and substitute them into the previous equality. $\dot{\omega}_{k+1,1} = (\bar{p}_{k+1,0}(\partial) - r_{k+1,1})\mathrm{d}y + (\bar{q}_{k+1,0}(\partial) - \rho_{k+1,1})\mathrm{d}u$. Replacing in the above equality the initial polynomials $\bar{p}_{k+1,0}(\partial)$ and $\bar{q}_{k+1,0}(\partial)$ by their expressions (9) and using the relations $\partial^i \mathrm{d}y = \mathrm{d}y^{(i)}$ for $i = n-k, \ldots, n$ and $\partial^j \mathrm{d}u = \mathrm{d}u^{(j)}$, for $j = n-k, \ldots, n-1$, we obtain $\dot{\omega}_{k+1,1} = \sum_{i=n-k}^{n-1} p_i \mathrm{d}y^{(i)} - \mathrm{d}y^{(n)} + \sum_{j=n-k}^{n-1} q_j \mathrm{d}u^{(j)} - r_{k+1,1}\mathrm{d}y - \rho_{k+1,1}\mathrm{d}u$. Finally, replacing $\mathrm{d}y^{(n)}$ in the above equality by the right-hand side of (5), we get: $\dot{\omega}_{k+1,1} = -\sum_{i=0}^{n-k-1} p_i \mathrm{d}y^{(i)} - \sum_{j=0}^{n-k-1} q_j \mathrm{d}u^{(j)} - r_{k+1,1}\mathrm{d}y - \rho_{k+1,1}\mathrm{d}u$. The latter means that the one-forms $\dot{\omega}_{k+1,1}$ can be expressed as a linear combination of the basis vectors of $\mathcal{H}_k$. This completes the proof.

The differentials of the state coordinates can be found from the subspace $\mathcal{H}_{n+1}$, see Theorem 1. Though in case of realizable i/o equation, $\mathcal{H}_{n+1}$, defined by (11), is completely integrable, the one-forms $\omega_{n+1,l}$ for $l = 1, \ldots, n$, are not necessarily always exact. Therefore, one has to find for $\mathcal{H}_{n+1}$ a new integrable bases, using linear transformations. From Theorem 3 the next corollary can be concluded[1].

Corollary 1. *For realizable i/o equation (1), the differentials of the state coordinates can be calculated as the integrable linear combinations of the one-forms*

$$\omega_l = \bar{p}_l(\partial)\mathrm{d}y + \bar{q}_l(\partial)\mathrm{d}u, \quad l = 1, \ldots, n \tag{12}$$

where $\bar{p}_l(\partial)$ and $\bar{q}_l(\partial)$ can be computed recursively from

$$\bar{p}_{l-1}(\partial) = s\bar{p}_l(\partial) + r_l, \quad \bar{q}_{l-1}(\partial) = s\bar{q}_l(\partial) + \rho_l, \tag{13}$$

with the initial polynomials

$$\bar{p}_0(\partial) = p(\partial), \quad \bar{q}_0(\partial) = q(\partial). \tag{14}$$

Remark 1. Note that in the linear case integrability aspect does not come into the play since all the one-forms $\omega_1, \ldots, \omega_n$ are integrable. This is so because all the polynomial coefficients p_i, q_i, $i = 0, \ldots, n-1$ are real numbers and not the functions, depending on control system variables.

[1] The first index $n+1$ is omitted in Corollary.

Remark 2. The results of [3] allow to check in terms of polynomials $p(\partial)$ and $q(\partial)$ if the i/o differential equation is irreducible or not, and in case if it is reducible, to find a reduced system description. Namely, equation (1) is irreducible if and only if $p(\partial)$ and $q(\partial)$ have no common left divisors. So, our results complement those of [3], allowing to find the minimal state space realization, no matter whether one starts from irreducible or reducible system description.

5 Examples and Discussion

Example 1. Consider the control system $\ddot{y} = 3uy + 2u\dot{u} + \dot{y}^2$ that can be described by two polynomials $p(\partial) = -\partial^2 + 2\dot{y}\partial + 3u$, $q(\partial) = 2u\partial + (3y + 2\dot{u})$.

To find the state coordinates, one has to, according to Corollary 1, compute the one-forms ω_1, ω_2, defined by (12). From equalities $\bar{p}_0(\partial) := p(\partial) = s\bar{p}_1(\partial) + r_1$, $\bar{q}_0(\partial) := q(\partial) = s\bar{q}_1(\partial) + \rho_1$ one can find the left quotients $\bar{p}_1(\partial) = -\partial + 2\dot{y}$, $\bar{q}_1(\partial) = 2u$. Furthermore, from equalities $\bar{p}_1(\partial) = s\bar{p}_2(\partial) + r_2$, $\bar{q}_1(\partial) = s\bar{q}_2(\partial) + \rho_2$ one can find the left quotients $\bar{p}_2(\partial) = -1$, $\bar{q}_2(\partial) = 0$. Finally, the one-forms that define the differentials of the state coordinates, can be computed according to (12) as follows

$$
\begin{aligned}
\omega_1 &= \bar{p}_1(\partial)\mathrm{d}y + \bar{q}_1(\partial)\mathrm{d}u = -\mathrm{d}\dot{y} + 2\dot{y}\mathrm{d}y + 2u\mathrm{d}u \\
\omega_2 &= \bar{p}_2(\partial)\mathrm{d}y + \bar{q}_2(\partial)\mathrm{d}u = -\mathrm{d}y.
\end{aligned}
$$

Though the subspace $\mathrm{span}_{\mathcal{K}}\{\omega_1, \omega_2\}$ is completely integrable, ω_1 is not exact and we have to replace ω_1 by an integrable linear combination of ω_1 and ω_2 to obtain the differentials of the state coordinates $\mathrm{d}x_1 = -\omega_2 = \mathrm{d}y$, $\mathrm{d}x_2 = -\omega_1 - 2\dot{y}\,\omega_2 = \mathrm{d}(\dot{y} - u^2)$ yielding the state equations $\dot{x}_1 = u^2 + x_2$, $\dot{x}_2 = 3ux_1 + (u + x_2)^2$.

Example 2. Consider the control system $y^{(3)} = \sqrt{\dot{y}\ddot{u}}$. We use again Corollary 1. Initial polynomials in (14) are for this example

$$
\bar{p}_0(\partial) := p(\partial) = -\partial^3 + \frac{\ddot{u}}{2\sqrt{\dot{y}\ddot{u}}}\partial, \quad \bar{q}_0(\partial) := q(\partial) = \frac{\dot{y}}{2\sqrt{\dot{y}\ddot{u}}}\partial^2.
$$

Recursive computation of the left quotients, according to (13), yields

$$
\begin{aligned}
&\bar{p}_1(\partial) = -\partial^2 + \frac{\sqrt{\ddot{u}}}{2\sqrt{\dot{y}}}, && \bar{p}_2(\partial) = -\partial, && \bar{p}_3(\partial) = -1, \\
&\bar{q}_1(\partial) = \frac{\sqrt{\dot{y}}}{\sqrt{\ddot{u}}} - \frac{\ddot{u}\ddot{y} - \dot{y}u^{(3)}}{4\sqrt{\dot{y}\ddot{u}^3}}, && \bar{q}_2(\partial) = \frac{\sqrt{\dot{y}}}{\sqrt{2\ddot{u}}}, && \bar{q}_3(\partial) = 0.
\end{aligned}
$$

Finally, by (12),

$$
\omega_1 = \frac{\sqrt{\ddot{u}}}{2\sqrt{\dot{y}}}\mathrm{d}y - \mathrm{d}\ddot{y} - \frac{\ddot{u}\ddot{y} - \dot{y}u^{(3)}}{4\sqrt{\dot{y}\ddot{u}^3}}\mathrm{d}u + \frac{\sqrt{\dot{y}}}{\sqrt{2\ddot{u}}}, \quad \omega_2 = -\mathrm{d}\dot{y} + \frac{\sqrt{\dot{y}}}{\sqrt{2\ddot{u}}}, \quad \omega_3 = -\mathrm{d}y.
$$

Unfortunately, the subspace $\mathrm{span}_{\mathcal{K}}\{\omega_1, \omega_2, \omega_3\}$ is not integrable and therefore, the i/o equation does not admit a state space form.

Theorem 3 provides an alternative, polynomial method for computing the bases vectors for the subspaces $\mathcal{H}_k$. Note that the polynomial method has some advantages in computer implementation. First, it is direct, meaning that there is no need to compute step-by-step all the $\mathcal{H}_k$ subspaces in order to find $\mathcal{H}_{n+1}$. Second, its program code is shorter and more compact. Algebraic method requires to solve a pseudolinear system of equations, which is linear with respect to unknowns, but those coefficients are nonlinear functions, and not real numbers. If the expressions, found on previous steps, have been not enough simplified, there is a chance that *Mathematica* may be unable to solve the pseudolinear system of equations and the computation fails. Polynomial method does not require solving any system of equations. In case of the second example from Section 5, the algebraic method requires to insert the additional simplification commands into the program code, while the polynomial method was able to produce the result without any intermediate simplification. The main disadvantage of the polynomial method is computation time. However, for discrete-time systems, polynomial method is also less time-consuming, if compared to the algebraic method. Unfortunately, this is not the case for continuous-time systems. The reason lies in the complex multiplication rule (4), compared to that one needs in the discrete-time case: $\partial \cdot a = \delta(a) \cdot \partial$, where δ is the shift operator.

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References

1. Conte, G., Moog, C., Perdon, A.: Algebraic Methods for Nonlinear Control Systems. Springer, London (2007)
2. Kotta, Ü., Kotta, P., Nõmm, S., Tõnso, M.: Irreducibility conditions for continuous-time multi-input multi-output nonlinear systems. In: 9th International Conference on Control, Automation, Robotics and Vision, Singapore, pp. 807–811 (2006)
3. Zheng, Y., Willems, J., Zhang, C.: A polynomial approach to nonlinear system controllability. IEEE Transaction on Automatic Control 46, 1782–1788 (2001)
4. Halás, M.: An algebraic framework generalizing the concept of transfer functions to nonlinear systems. Automatica 44(5), 1181–1190 (2008)
5. Zheng, Y., Cao, L.: Transfer function description for nonlinear systems. J. East China Normal Univ. (Nat. Sci.) 2, 15–26 (1995)
6. Kotta, Ü., Tõnso, M.: Realization of discrete-time nonlinear input-output equations: polynomial approach. In: Proc. of the 7th World Congress on Intelligent Control and Automation, Chongqing, China, pp. 529–534 (2008)
7. Rapisarda, P., Willems, J.: State maps for linear systems. SIAM J. Control. Optim. 35(3), 1053–1091 (1997)
8. Kolchin, E.: Differential algebra and algebraic groups. Academic Press, London (1973)
9. Choquet-Bruhat, Y., DeWitt-Morette, C., Dillard-Bleichi, M.: Analysis, manifolds and physics: Part I: Basics. North-Holland, Amsterdam (1982)
10. McConnel, J., Robson, J.: Noncommutative Noetherian Rings. Birkhäuser, Basel (1987)