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NOTE

THE EXPONENTIAL GENERATING FUNCTION OF LABELLED BLOCKS

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An *(n, q) graph* is a graph on n points and q edges (no loops, no parallel lines); except where we state otherwise, the n points are labelled. A *network* is a graph in which two points are distinguished as a *positive pole* and a *negative pole* respectively. A *block* is a 2-connected graph (i.e. a graph from which at least 2 points and their adjacent edges have to be removed to disconnect the graph) or a maximal 2-connected sub-graph of a graph which is not itself 2-connected; conventionally the $(2, 1)$ graph is a block and the $(1, 0)$ graph is not. We write $N = \frac{1}{2}n(n-1)$ and $b(n, q)$ is the number of (n, q) blocks. If

$$F(X, Y) = \sum_n \sum_q f(n, q) X^n Y^q / n!,$$

we say that F is the *exponential generating function* (e.g.f.) of f and write $F = E(f)$. If $f(n, q)$ is the number of graphs of a particular family on n points and q edges, we say that F is the e.g.f. of that family of graphs. We write $B = E(b)$, i.e.

$$B(X, Y) = \tfrac{1}{2}X^2 Y + \sum_{n=3}^{\infty} \sum_{q=n}^{N} b(n, q) X^n Y^q / n!,$$

so that B is the e.g.f. of the family of blocks. We use suffixes to denote partial differentiation.

It is well known that

$$\log C_X = \partial B(Z, Y)/\partial Z, \tag{1}$$

where $C = C(X, Y)$ is the e.g.f. of connected graphs and $Z = XC_X$. (See [1, pp. 10, 11] for a proof and references). Temperley [2] used the calculus to deduce

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from (1) that

$$X^2\{1+B_{XX}(1-XB_{XX})^{-1}=2(1+Y)B_Y. \tag{2}$$

(2) has obvious advantages over (1) for computing $b(n, q)$ and has been used by one of us (E.M.W.) to find exact formulae [5] for $b(n, n+k)$ for successive k and general n and asymptotic formulae [6] for large n. Our object here is to produce a direct combinatorial proof of (2).

If we form an $(n, q+1)$ block in every possible way by adding a line to an (n, q) graph, we have a collection $\mathscr{C}$ of $(n, q+1)$ blocks. In $\mathscr{C}$, every $(n, q+1)$ block occurs just $q+1$ times, since each of its edges occurs once as the added edge. Hence $|\mathscr{C}|=(q+1)b(n, q+1)$ and so $B_Y=E(|\mathscr{C}|)$, if the added edge of a block in $\mathscr{C}$ is excluded from the edge count. We separate $\mathscr{C}$ into the three collections $\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3$. Of these, $\mathscr{C}_1$ consists of those $(n, q+1)$ blocks formed by adding an edge to an (n, q) block. There are $b(n, q)$ of the latter and to each of them an edge can be added in $N-q$ different ways. Hence $|\mathscr{C}_1|=(N-q)b(n, q)$ and

$$E(|\mathscr{C}_1|)=\tfrac{1}{2}X^2B_{XX}-YB_Y.$$

$\mathscr{C}_2$ is empty except when $n=2$, when it contains the (2, 1) graph, the only block formed by adding an edge to a disconnected graph; thus $E(|\mathscr{C}_2|)=\frac{1}{2}X^2$. $\mathscr{C}_3$ consists of the $(n, q+1)$ blocks formed by the addition of an edge to a connected graph, not itself a block. We have then

$$\begin{aligned} E(|\mathscr{C}_3|) &= E(|\mathscr{C}|)-E(|\mathscr{C}_1|)-E(|\mathscr{C}_2|) \\ &= (1+Y)B_Y-\tfrac{1}{2}X^2(1+B_{XX}). \end{aligned} \tag{3}$$

It remains to find another expression for $E(|\mathscr{C}_3|)$, which we can equate to this.

We take each member of $\mathscr{C}_3$, distinguish the ends of the added edge as positive and negative poles and remove the edge. We can do this in just two ways and so we have a collection $\mathscr{C}_4$ of (n, q) networks, all different, and

$$|\mathscr{C}_4|=2|\mathscr{C}_3|. \tag{4}$$

Each network M in $\mathscr{C}_4$ is connected but not a block. It must therefore contain s cut-points, where $s\geq 1$. Neither pole can be a cut-point, for, if it were, it would have been a cut-point in the original $(n, q+1)$ block and a block has no cut-points. If we remove a cut-point and its adjacent edges from M, the resulting disconnected graph can have only two components, for the subsequent addition of the line joining the two poles must produce a connected graph. It follows that each cut-point of the network M lies on just two blocks of M and that every path in M joining the two poles must pass through every cut-point. Hence M consists of a chain of $s+1$ blocks, each having a single cut-point in common with each of its neighbours. The two end blocks each contain a pole and a cut-point; every other block contains two cut-points.

Let F be the e.g.f. of a family of networks $\mathscr{F}$, all of whose points are labelled, and let G be the e.g.f. of a family of networks $\mathscr{G}$, in each of which the negative pole is unlabelled. Then the e.g.f. of the number of ordered pairs $(\mathscr{F}, \mathscr{G})$ is FG. This is unaltered if in each pair we now fasten $\mathscr{F}$ and $\mathscr{G}$ together by identifying the unlabelled negative pole of $\mathscr{G}$ with the labelled positive pole of $\mathscr{F}$ and regard the new point as labelled but not a pole. The resulting graph is, of course, a network.

The number of different networks which can be formed from an (n, q) graph by the selection of a positive and a negative pole is $n(n-1)$ and so the e.g.f. of the family of networks formed from blocks in this way is $X^2 B_{XX}$. If, however, the negative pole is to be unlabelled and excluded from the point count, the e.g.f. is XB_{XX}. Hence, if D_s is the e.g.f. of the number of members of $\mathscr{C}_4$ which have s cut-points, we have

$$D_1 = X^3 B_{XX}^2, \qquad D_{s+1} = XB_{XX} D_s. \tag{5}$$

It follows that

$$E(|\mathscr{C}_4|) = \sum_{s=1}^{\infty} D_s = \sum_{s=1}^{\infty} X^{s+2} B_{XX}^{s+1} = X^3 B_{XX}^2 (1 - XB_{XX})^{-1}.$$

From this and (3) and (4), we have (2).

A minor variant on the above is to consider what we obtain if we attach a single block with two poles in the way described above to our network M. The result is a new network (with different n and q) of the same kind, but with more than one cut-point, i.e. with $s \geqslant 2$. Hence the e.g.f. of the family of all M for which $s \geqslant 2$ is $XB_{XX} E(|\mathscr{C}_4|)$ and so

$$E(|\mathscr{C}_4|) = D_1 + XB_{XX} E(|\mathscr{C}_4|) = X^3 B_{XX}^2 + XB_{XX} E(|\mathscr{C}_4|).$$

By (3) and (4) this gives us

$$\{2(1+Y)B_Y - X^2(1 + B_{XX})\}(1 - XB_{XX}) = X^3 B_{XX}^2.$$

which is (2), multiplied through by $(1 - XB_{XX})$. We have thus a combinatorial interpretation of this form of (2).

One of us (N.W.) used this latter method [4] to find the partial differential equation satisfied by the exponential generating function of 3-connected labelled graphs. The problem is a much more difficult one than that of the present paper. Walsh [3] has found the equation corresponding to (1) for the 3-connected case by a development of the method used by Mayer and Riddell to prove (1) (for which see [1]).

References

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