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FORCING AND REDUCIBILITIES. III. FORCING IN FRAGMENTS OF SET THEORY

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We conclude here the treatment of forcing in recursion theory begun in Part I and continued in Part II of [31]. The numbering of sections is the continuation of the numbering of the first two parts. The bibliography is independent.

§10. Forcing with trees on admissible sets. In Part I our language was a first-order language: the only set we considered was the (set constant for the) generic set. In Part II a second-order language was introduced, and we had to interpret the second-order variables in some way. What we did was to consider the ramified analytic hierarchy, defined by induction as:

$A_0 = \{X \subseteq \omega\colon X \text{ is arithmetic}\}$,
$A_{\alpha+1} = \{X \subseteq \omega\colon X \text{ is definable (in 2nd order arithmetic) over } A_\alpha\}$,
$A_\lambda = \bigcup_{\alpha<\lambda} A_\alpha$ (λ limit),
$RA = \bigcup_\alpha A_\alpha$.

We then used (a relativized version of) the fact that $A_{\omega_1} = \Delta^1_1$ (Kleene [27]). The definition of RA is obviously modeled on the definition of the constructible hierarchy introduced by Gödel [14]. For this we no longer work in a language for second-order arithmetic, but in a language for (first-order) set theory with membership as the only nonlogical relation:

$L_0 = \varnothing$,
$L_{\alpha+1} = \{X\colon X \text{ is (first-order) definable over } L_\alpha\}$,
$L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$ (λ limit),
$L = \bigcup_\alpha L_\alpha$.

There are some obvious differences between RA and L, the main one being that in the ramified analytic hierarchy we only consider subsets of ω and in the constructible hierarchy we do not: this will force us to restrict our attention to $L \cap \mathscr{P}(\omega)$. Actually, RA is only a small part of Δ^1_2, whereas $L \cap \mathscr{P}(\omega)$ might well be all of $\mathscr{P}(\omega)$ (in the sense that this assertion is consistent with ZFC). However, except for the minor difference given by the basis of the two definitions (in one case all the arithmetic sets, in the other the empty set), A_α and $L_\alpha \cap \mathscr{P}(\omega)$ are practically the same for all α's less than β_0 (the smallest ordinal β such that $A_{\beta+1} = A_\beta$). For details, see Boolos and Putnam [8]. In particular, $L_{\omega_1} \cap \mathscr{P}(\omega) = \Delta^1_1$, as was first proved by Spector [45]. A relativization of this gives the connection between hyperdegrees and constructible universe: $A \leq_h B$ iff $A \in L_{\omega_1^B}(B)$ (for $A, B \subseteq \omega$), where $L_\alpha(B)$ is defined as L_α, but letting $L_0(B) = B$. Hence if B (i.e. its hyperdegree)

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is an upper bound of a set of hyperdegrees, these hyperdegrees are contained in $L_{\omega_1^B}(B)$: for the problem of minimal upper bounds of sets of hyperdegrees (considered in §11) we must hence learn how to force (with trees) over such sets.

L_{ω_1} (and its relativization $L_{\omega_1^B}(B)$) are typical examples of (countable) *admissible sets*, i.e. they are transitive models of the fragment KP of set theory whose axioms are: extensionality, pair, union, foundation (scheme) and the following, where φ is a formula with no unbounded quantifiers:

$(\exists x)(\forall y)(y \in x \Leftrightarrow y \in a \wedge \varphi(x))$ (Δ_0-separation),

$(\forall x \in a)(\exists y)\varphi(x, y) \Rightarrow (\exists b)(\forall x \in a)(\exists y \in b)\varphi(x, y)$ (Δ_0-collection).

In other words KP is like ZF without the power axiom and the axiom of infinity, and with separation and collection only for the simplest formulas of the language. Actually all the axioms of KP except Δ_0-collection are trivial ones: e.g. they are satisfied by every L_α, α limit (the standard reference for admissible sets is Barwise [7]: all the facts we quote without proof can be found there). An ordinal α is called admissible if L_α is admissible.

Admissible sets are natural domains for recursion theory: e.g. it is possible to carry out in them Σ_1-recursion over Δ_1 well-founded relations (Σ_1 formulas are those whose only unbounded quantifiers are existential and occur positively; π_1 formulas are defined similarly; $\Delta_1 = \Sigma_1 \cap \pi_1$). This is particularly useful in the case of admissible sets of the form L_α, since then there is a Δ_1 well ordering of L_α itself (of ordinal α).

The basic property of Δ_1^1 used in Part II was that Δ_1^1 is the smallest model of Δ_1^1 comprehension. We could recast the whole treatment of forcing there in terms of forcing over L_{ω_1} and use the fact that L_{ω_1} is the smallest admissible set containing ω. We now do this in a generalized way.

Let α be a countable admissible ordinal greater than ω. The language $\mathscr{L}_\alpha$ is the (first-order) language of set theory with equality ($\in$ is the only nonlogical symbol), with individual constants for each integer, one set constant X and two types of (set) variables:

$x_\beta, y_\beta, z_\beta \ldots$ for any $\beta < \alpha$ (ranked variables of rank β),

$x, y, z \ldots$ (unranked variables).

The rank of a formula and the semantic interpretation of formulas are defined as in Part II. In the following A denotes a subset of ω.

DEFINITION. (a) $\mathscr{M}(A) = L_\alpha(A) \cap \mathscr{P}(\omega)$.

(b) If φ is a sentence of $\mathscr{L}_{\omega_1}$, we say $A \models \varphi$ if φ is true when the ranked variables of rank β are interpreted over $L_\beta(A)$ and the unranked variables are interpreted over $L_\alpha(A)$ (we also write $L_\alpha(A) \models \varphi$ for this).

Since the applications we have in mind are to minimal upper bounds of sets of hyperdegrees, it is natural to have as forcing conditions trees hyperarithmetically pointed, i.e. trees P such that $(\forall X \in P)(P \leq_h X)$. The definition of forcing is modelled after the one in §7:

DEFINITION. $A \Vdash \varphi$ means that for some $P \in L_\alpha$, $A \in P$ and $P \Vdash \varphi$ where $P \Vdash \varphi$ is defined by induction on φ as follows:

if φ is ranked and β is its rank, $P \Vdash \varphi$ iff $\beta < \omega_1^P$ and $(\forall X \in P)(L_\alpha(X) \models \varphi)$;

if $\varphi \equiv \exists X\psi(X)$, $P \Vdash \varphi$ iff for some $\beta < \alpha$, $P \Vdash \exists X_\beta \psi(x_\beta)$;

the other clauses are standard.

As we see, the conditions are no longer Δ_1^1 but are in L_α (the obvious connection being that $P \in \Delta_1^1$ iff $P \in L_{\omega_1}$). Similarly, P is supposed to control only sentences with sufficiently low rank (if $P \in \Delta_1^1$ then $\omega_1^P = \omega_1$ and hence in §7 there was no need for this: the ranks were all less than ω_1).

The basic definability result of §7 was that for Σ_1^1 sentences φ of $\mathscr{L}_{\omega_1}$, $P \Vdash \varphi$ was π_1^1. In the language of admissibility this says that if φ is a Σ_1 formula of $\mathscr{L}_{\omega_1}$ (for the new language of this section) then $P \Vdash \varphi$ is Σ_1 over L_{ω_1}. This holds in general:

PROPOSITION 10.1. *$\{(P, \varphi)$: φ is a Σ_1 sentence of $\mathscr{L}_\alpha$ and $P \Vdash \varphi\}$ is Σ_1 over L_α (Sacks* [37]).

PROOF. First we have to prove that the set of conditions is Σ_1 over L_α. This comes from the facts that it is a π_1^1 set ($P \leq_h X$ is π_1^1 in P, X and hence $(\forall X \in P)(P \leq_h X)$ is π_1^1 in P) and that every π_1^1 set of reals is Σ_1 over any admissible set containing ω (the shortest way to see this is perhaps to appeal to the representation of π_1^1 sets via well orderings, and to the fact that a set is well ordered iff there is an order-preserving map into an ordinal).

By the definition of forcing, it is then enough to consider ranked formulas. If φ is ranked, then $L_\alpha(X) \models \varphi$ is Σ_1 over $L_\alpha(X)$ because it can be defined by Σ_1-recursion, and $P \Vdash \varphi$ iff the rank of φ is recursive in P and $(\forall X \in P)(L_\alpha(X) \models \varphi)$. The result follows because $(\forall X \in P)$ is a bounded quantifier. $\square$

The crucial fact to prove now is the fusion lemma, since we know from §7 that the other facts we need follow from it. Let us try to do it intuitively in a simple but typical case. We are given $\varphi(n, x)$ with x as the only unranked variable and P such that $(\forall n)(P \Vdash^w \exists x \varphi(\bar{n}, x))$ (the definition of weak forcing is the standard one). We want to find $Q \subseteq P$ such that $(\forall n)(Q \Vdash \exists x \varphi(\bar{n}, x))$. Since the hypothesis is

$$(\forall n)(\forall Q \subseteq P)(\exists R \subseteq Q)(\exists \beta < \alpha)(R \Vdash \exists x_\beta \varphi(\bar{n}, x_\beta)),$$

by definability (10.1) we have a Σ_1 function over L_α that gives R and β in terms of n and Q. Forgetting about the problem of making our tree perfect (this can be obtained by the usual splittings), we can define

$P_0 = P$,

$P_{n+1} \subseteq P_n$ and $P_{n+1} \Vdash \exists x_{\beta_n} \varphi(\bar{n}, x_{\beta_n})$.

If we let $Q = \bigcap_n P_n$ then $(\forall n)(Q \Vdash \exists x_\beta \varphi(\bar{n}, x_\beta))$ where $\beta = \sup \beta_n$, hence $(\forall n)(Q \Vdash \exists x \varphi(\bar{n}, x))$. But is Q hyperarithmetically pointed? Given $X \in Q$ we want $Q \leq_h X$, i.e. $Q \in L_{\omega_1^X}(X)$. Since $X \in P_n$ and P_n is hyperarithmetically pointed, $(\forall n)(P_n \in L_{\omega_1^X}(X))$: to show that $Q \in L_{\omega_1^X}(X)$ we should know that its definition (i.e. the definition of P_{n+1} from P_n) lives in $L_{\omega_1^X}(X)$. For the choice of P_{n+1} we may naturally make use of the well ordering of L_α, but there is no need for this to be compatible with the well ordering of $L_{\omega_1^X}(X)$ for any X as above. What we can do, however, is use, instead of hyperarithmetically pointed trees, trees which are absolutely so:

DEFINITION. P is *absolutely hyperarithmetically pointed* if $(\forall X \in P)(P \in L_{\omega_1^X})$.

The consideration of absolutely hyperarithmetically pointed trees does not affect 10.1. Moreover we have:

PROPOSITION 10.2. FUSION LEMMA (SACKS [37]). *If* $\{\varphi_n\}_{n\in\omega} \in L_\alpha$ *is a set of* Σ_1 *sentences of* $\mathscr{L}_\alpha$ *and* $(\forall n)(P \Vdash^w \varphi_n)$ *then* $(\exists Q \subseteq P)(\forall n)(Q \Vdash \varphi_n)$.

PROOF. Let $\varphi_n \equiv \exists x \psi_n(x)$ (ψ_n ranked) and define:

$P_0 = P$,

$P_{n+1} \subseteq P_n$ and $P_{n+1} \Vdash \exists x_{\beta_n} \psi_n(x_{\beta_n})$.

Note that by definition of forcing, $\beta_n < \omega_1^{P_n}$. The choice of P_n is the least in the natural well ordering of L_α. Let $Q = \bigcap_n P_n$: then $(\forall n)(P_n \in L_{\omega_1^X})$, hence $\beta_n < \omega_1^{P_n} \leq \omega_1^X$. Moreover the natural well orderings of L_α and $L_{\omega_1^X}$ are compatible ($L_{\omega_1^X}$ is admissible). Hence $Q \in L_{\omega_1^X}$. $\square$

We know from §7 that we have to use the fact that a generic set meets certain dense sets of conditions. We might as well request this by definition, to avoid some of the work:

DEFINITION. A is *Sacks α-generic* if $\{P: A \in P\}$ meets every dense set of conditions definable over L_α.

PROPOSITION 10.3. GENERIC EXISTENCE THEOREM. (a) *There is a perfect tree of Sacks α-generic sets (contained in any given condition).*

(b) *Sacks α-generic sets are not in L_α.*

PROOF. (a) The sets of conditions definable over L_α are countably many. We can meet all the dense ones in countably many stages.

(b) Suppose $A \in L_\alpha$: then $\{P: A \notin P\}$ is definable over L_α. It is also dense because every condition Q contains two incompatible strings: one of them is incompatible with A, and we let P be the full subtree of Q above it. By genericity $\{P: A \notin P\}$ is hence met by $\{P: A \in P\}$, contradiction. $\square$

It remains to prove that for generic sets $\mathscr{M}(A) = \Delta_1^{1,A}$, equivalently, that $\alpha = \omega_1^A$. Note that now we also have to prove the inequality $\alpha \leq \omega_1^A$; that was automatic in §7 because $\alpha = \omega_1$ there. We may expect, however, that the other inequality be proved similarly to the analogous part in §7: it turns out that it is so, but we need an additional assumption on α. Until further notice we assume that $L_\alpha \models$ "*every ordinal is countable*".

PROPOSITION 10.4. *Quasi-completeness*: $(\forall P)(\exists Q \subseteq P)(Q \Vdash \varphi$ *or* $Q \Vdash \sim\varphi)$.

PROOF. As usual, it is enough to consider ranked formulas. We do induction on φ as in §2. E.g. let $\varphi \equiv \exists x_\beta \psi(x_\beta)$. Since every ordinal is countable in L_α, we can enumerate over ω in L_α the formulas of rank $\leq \beta$ with one free variable, and hence the sets B_n they define (over L_α). By induction hypothesis, $(\forall n)(\forall P)(\exists Q \subseteq P)$ $(Q \Vdash \psi(B_n)$ or $Q \Vdash \sim\psi(B_n))$. Fix P: if for some n, $Q \subseteq P$ is $Q \Vdash \psi(B_n)$ then $Q \Vdash \varphi$. Otherwise, by the ind. hyp. $(\forall n)(\forall Q \subseteq P)(\exists R \subseteq Q)(R \Vdash \sim \psi(B_n))$, and by the fusion lemma $(\exists Q \subseteq P)(\forall n)(Q \Vdash \sim\psi(B_n))$, so $Q \Vdash \sim\varphi$. $\square$

Having quasi-completeness, we know that for every sentence φ of $\mathscr{L}_\alpha$ and every Sacks α-generic set A, $A \Vdash \varphi$ or $A \Vdash \sim\varphi$ (by 10.4 the set $\{P: P \Vdash \varphi$ or $P \Vdash \sim \varphi\}$ is dense, and by 10.1 it is definable, hence is met). Hence forcing = truth for generic sets. Note that the new assumption on L_α is essential: by definition we have $Q \Vdash \varphi$ or $Q \Vdash \sim\varphi$ only if the rank of φ is less than ω_1^Q, hence (since $Q \in L_\alpha$) only if rank(φ) is countable in L_α (being the ordinal of a well ordering of ω recursive in Q).

PROPOSITION 10.5. *If A is Sacks α-generic,* $\mathscr{M}(A) = \Delta_1^{1,A}$ (*Sacks* [37]).

PROOF. (a) $\Delta_1^{1,A} \subseteq \mathscr{M}(A)$. Since $L_{\omega_1^A}(A)$ is the smallest admissible containing A, it is enough to prove that $L_\alpha(A)$ is admissible. We only need to check Δ_0-collection. This is similar to the proofs of 7.4 and 7.6 using the fusion lemma. We only note that the assumption on L_α enters in two crucial ways: firstly it allows the step from truth to forcing; secondly it tells that the form of the fusion lemma is sufficiently general. The reason for the latter is that we have in fact a hypothesis of the kind: $(\forall x \in a)(\exists y)\varphi(x, y)$. Since $a \in L_\alpha$, a is countable in L_α and we are so reduced to a formula of the kind $(\forall n)(\exists y)\psi(n, y)$, which we can treat by the fusion lemma. We then get $\beta < \alpha$ such that $(\forall n)(\exists y_\beta)\psi(n, y_\beta)$ and we have the wanted bound.

(b) $\mathscr{M}(A) \subseteq \Delta_1^{1,A}$. Let $X \in \mathscr{M}(A)$: to prove $X \leq_h A$ it would be enough to find P such that $A \in P$ and $X \leq_h P$ (since by pointedness then $P \leq_h A$). Consider the set $\{Q : X \leq_h Q\}$: it is definable over L_α, and if we prove that it is dense then we get P by genericity. Given R we want to find $Q \subseteq R$, $X \leq_h Q$. Consider $R \oplus X$: it is an element of L_α and, by a result of Boolos and Putnam [8], we can find $B \in L_\alpha$ s.t. $R \oplus X \leq_T B$ and $B \in L_{\omega_1^B}$. Let $Q \subseteq R$ be pointed and $Q \equiv_h B$ (by the analogue of 2.6, since $R \leq_h B$): then $X \leq_h Q$. We have to prove that Q is absolutely pointed. Note that this is equivalent to have $Q \in L_{\omega_1^Q}$ (if $C \in Q$ then $Q \leq_h C$ by simple pointedness, so $\omega_1^Q \leq \omega_1^C$; moreover there is $C \in Q$ s.t. $Q \leq_h C$, say the leftmost branch, and $\omega_1^C = \omega_1^Q$ for it). But $\omega_1^Q = \omega_1^B$ (since $Q \equiv_h B$) and $L_{\omega_1^B}$ is closed under $\equiv_h$, so $Q \in L_{\omega_1^Q}$ since $B \in L_{\omega_1^B}$. $\square$

As a corollary of the above result we get: if $\alpha > \omega$ is a countable admissible ordinal and $L_\alpha \models$ "every ordinal is countable", there is A such that $\alpha = \omega_1^A$ and $\omega_1^B < \alpha$ for every $B <_h A$. The first assertion is immediate by 10.5. For the second, there are two cases:

if $\alpha = \omega_1^A$ for some $A \in L_\alpha$ we are finished: if $B \leq_h A$ then $B \in L_\alpha$, and if $\alpha = \omega_1^B$ then $L_\alpha = L_{\omega_1^B}(B)$ and $A \in L_{\omega_1^B}(B)$, so $A \leq_h B$.

otherwise, let A be Sacks α-generic: if $B \leq_h A$ and $\alpha = \omega_1^B$ then B is an upper bound for the hyperdegrees in L_α (since $L_\alpha \subseteq L_{\omega_1^B}(B)$) and is not in L_α (since the other case fails). In §11 we will prove that A is a minimal upper bound, hence $A \leq_h B$.

We discuss briefly the possibility of eliminating the hypothesis on L_α. Let $\alpha > \omega$ be any countable admissible ordinal: L_α may very well contain ordinals greater than ω that look like cardinals inside L_α. The obvious attack is to collapse every cardinal of L_α to ω, by using the Levy collapse. We run, however, into trouble when we try to reproduce the arguments of this section: we are working on a generic extension of L_α, and this is still an admissible set (see Zarach [50] for a treatment of unramified forcing on admissible sets) but it is not as nice as L_α was. In particular we cannot solve our problems by considering absolutely pointed trees as conditions. By using, however, uniformly hyperarithmetically pointed trees (see §9 for a similar notion) Sacks [37] was able to carry on the arguments of this section for countable admissible sets satisfying Σ_1 dependent choice (which is what is actually used in 10.2).

By using finite conditions for the collapse, Sacks [37] gets the following theorem *if $\alpha > \omega$ is a countable admissible ordinal, there is A such that $\alpha = \omega_1^A$*. Much simpler proofs have been found by Friedman and Jensen [13] using Barwise compactness, by Grilliott [16] using omitting types, and by Steel [46] using forcing with finite conditions.

By using perfect conditions for the collapse, Sacks [37] obtains this strong form of the theorem above: *if $\alpha > \omega$ is a countable admissible ordinal, there is A such that $\alpha = \omega_1^A$ and $\omega_1^B < \alpha$ for every $B <_h A$.* The proof is quite involved and takes care of the two remaining cases left open by the work done in this section: when there are cardinals of L_α of cofinality greater than ω (they can be bounded below α or not).

§11. Hyperdegrees again. By having at hand the results of §10, we can prove as in §2 that if $L_\alpha \models$ "every ordinal is countable", then the hyperdegrees in L_α have a continuum of minimal upper bounds, since every Sacks α-generic is a minimal upper bound. The hypothesis on L_α is a little disturbing, but we do not need much more work to get a series of nice results.

PROPOSITION 11.1. *If α is a countable ordinal then the hyperdegrees in L_α have a minimal upper bound* (*Sacks* [37]).

PROOF. Let γ be the least admissible $\geq \alpha$ and β the least ordinal such that $L_\beta \cap \mathscr{P}(\omega) = L_\gamma \cap \mathscr{P}(\omega)$. There are two cases:

(a) If $\beta \leq \alpha$ then $L_\beta \models$ "every ordinal is countable". In fact either $\beta < \gamma$ and then β looks like ω_1^L in L_γ, or $\beta = \gamma$ and hence β is an index ordinal or a limit of index ordinals (see Boolos and Putnam [8]). Then the hyperdegrees in L_β have a minimal upper bound and so do those in L_α because $\beta \leq \alpha \leq \gamma$ and $L_\beta \cap \mathscr{P}(\omega) = L_\alpha \cap \mathscr{P}(\omega)$.

(b) $\alpha < \beta$. Then $\beta = \gamma$ (since by part (a) β is admissible). By definition of β, there is $X \in L_\beta - L_\alpha$. X is certainly an upper bound for L_α ($\alpha < \omega_1^X$ and $L_\alpha \subseteq L_{\omega_1^X}(X)$). It is actually the least upper bound, since if Y is an upper bound for L_α then $L_\alpha \subseteq L_{\omega_1^Y}(Y)$ and $\alpha \leq \omega_1^Y$, and since $\gamma = \beta$ is the least admissible $\geq \alpha$, $\beta \leq \omega_1^Y$ and $X \leq_h Y$. □

PROPOSITION 11.2. *If $\alpha > \omega$ is a countable admissible ordinal and the hyperdegrees in L_α have least upper bound, then the least upper bound is contained in L_α* (*Sacks* [37]).

PROOF. In the proof of 11.1 the first case must apply (since α is admissible, hence $\alpha = \gamma$). Then the hyperdegrees of L_α have a continuum of minimal upper bounds. If the least upper bound exists, it must be below all of them and hence must be one of the hyperdegrees of L_α. □

The admissible ordinals may be thought of as a recursive analogue of the regular cardinals, since they can be characterized as those ordinals α such that $\alpha = \sigma 1\ \mathrm{cf}(\alpha)$, where $\sigma 1\ \mathrm{cf}(\alpha)$ is the recursive cofinality of α, i.e. the least ordinal β for which there is a cofinal map $f: \beta \to \alpha$ that is Σ_1 over L_α. We define the sequence of ordinals ω_α by induction on α: $\omega_0 = \omega$, $\omega_{\alpha+1} =$ the smallest admissible greater than ω_α, $\omega_\lambda =$ limit of $\{\omega_\alpha\}_{\alpha<\lambda}$ if λ is limit. This is consistent with our previous notation for ω_1. The step from ω_α to $\omega_{\alpha+1}$ corresponds to the step from a set to its hyperjump, but it is not true that if $\omega_\alpha = \omega_1^A$ then $\omega_{\alpha+1} = \omega_1^{\mathscr{O}A}$ in general: from 8.4(a) we know that there are A's such that $\omega_1 = \omega_1^A$ but $\omega_1^{\mathscr{O}A}$ is arbitrarily large. However Jensen (unpublished) has improved the theorem of Sacks discussed at the end of §10 in the following sense: given a countable ascending sequence $\{\beta_\alpha\}_{\alpha<\alpha_0}$ of admissible ordinals, there is a fixed A such that, for every $\alpha < \alpha_0$, β_α is the αth admissible relative to A.

By analogy with set theory, we call an ordinal α *recursively inaccessible* if it is admissible and a limit of admissibles. The first recursively inaccessible is denoted by $\omega_1^{E_1}$ (since it turns out to be the first ordinal not recursive in the type-2 object E_1 introduced by Tugué [49]).

PROPOSITION 11.3. *If α is countable and recursively inaccessible, the hyperdegrees in L_α do not have least upper bound* (*Sacks* [37]).

PROOF. By 11.2 the l.u.b. should be in L_α since α is admissible and greater than ω. But L_α is closed under hyperjump by recursive inaccessibility, so there is no l.u.b. □

These are the general results we can derive. Sacks [37] has more of them since he uses more powerful methods. He proves that:

(a) *The hyperdegrees in a countable admissible set closed under hyperjump have minimal upper bounds but no least upper bound.*

(b) *The hyperdegrees in a countable admissible set satisfying Σ_1 dependent choice have minimal upper bounds.*

At present it is not known if every countable set of hyperdegrees has minimal upper bounds. If the answer to this problem is affirmative, then new techniques must be discovered to prove it, since Abramson [1] has proved that there are admissible sets (not satisfying Σ_1 dependent choice and with ordinal ω_1) such that any upper bound for the hyperdegrees in it produced by any forcing, such that the forcing relation for Σ_1 formulas is Σ_1, is not a minimal upper bound.

We turn now to specific examples. The first one is implicit in 11.3:

PROPOSITION 11.4. *The hyperdegrees of Δ_1^1 sets have minimal upper bounds but no least upper bound.*

PROOF. By Shoenfield [38] $\Delta_2^1 = L_\alpha \cap \mathscr{P}(\omega)$, where α is the least ordinal which is not the ordinal of a Δ_2^1 well ordering of ω. By 11.1 there are minimal upper bounds. Since α is admissible and greater than ω, by 11.2 there is no least upper bound because (Addison and Kleene [5]) Δ_2^1 is closed under hyperjump. □

The same proof and an appeal to the closure properties of Δ_n^1 (Shoenfield [39]) give the same result for any Δ_n^1 ($n \geq 3$) under the hypothesis $\mathscr{P}(\omega) \subseteq L$. Actually this hypothesis is not necessary because Sacks [36] has proved that Δ_n^1 is the 1-section of a normal type-2 object, hence the real part of a countable admissible set (closed under hyperjump). The result for $n \geq 3$ is not, however, terribly important, since it is doubtful that we can obtain an interesting analysis of the Δ_n^1 ($n \geq 3$) sets in terms of hyperdegrees. For Δ_2^1 the situation is completely different, since such an analysis is available from Suzuki [47]. It is proved there that there is a well-ordered chain of hyperdegrees (consisting exactly of the hyperdegrees of π_1^1 singletons) containing $\mathbf{0}$, with successor given by the hyperjump, cofinal in Δ_2^1 and such that the Δ_2^1 sets are exactly the sets whose hyperdegrees are less than some element of the chain. Although this chain is not constructed from below, it nevertheless gives a nice stratification of the Δ_2^1 sets.

We turn now to the study of chains of hyperdegrees. Define $\mathbf{0}^{(\alpha)}$ by induction in the natural way: $\mathbf{0}^{(0)} = \mathbf{0}$, $\mathbf{0}^{(\alpha+1)} =$ the hyperjump of $\mathbf{0}^{(\alpha)}$, $\mathbf{0}^{(\lambda)} =$ the least upper bound of $\{\mathbf{0}^{(\alpha)}\}_{\alpha<\lambda}$ (when it exists) We know that the analogous chains of Turing degrees and arithmetical degrees are trivial, in the sense that $\mathbf{0}^{(\omega)}$ is not defined. For hyperdegrees the picture is completely different:

PROPOSITION 11.5. $\mathbf{0}^{(\alpha)}$ *is defined exactly for* $\alpha < \omega_1^{E_1}$ (*Richter* [33], *Sacks* [37]).

PROOF. We prove e.g. that $\mathbf{0}^{(\omega)}$ is defined. Let $X \in L_{\omega_{\omega+1}}$: we claim that the hyperdegree of X is the least upper bound of $\{\mathbf{0}^{(n)}\}_{n\in\omega}$. Certainly it is an upper bound. Suppose Y is another upper bound: then $L_{\omega_\omega} \subseteq L_{\omega_1^Y}(Y)$, hence $\omega_\omega \leq \omega_1^Y$. Since ω_1^Y is admissible and ω_ω is not, $\omega_\omega < \omega_1^Y$ and, by definition, $\omega_{\omega+1} \leq \omega_1^Y$. So $X \in L_{\omega_1^Y}$ and $X \leq_h Y$. The argument is perfectly general and works as long as we do not hit the first recursively inaccessible. Hence $\mathbf{0}^{(\alpha)}$ is defined for $\alpha < \omega_1^{E_1}$.

The construction above also proves that $\{\mathbf{0}^{(\alpha)}\}_{\alpha<\omega_1^{E_1}}$ is contained in $L_{\omega_1^{E_1}}$ and cofinal in it. By 11.3 it does not have least upper bound. □

Of course the same argument of 11.6 proves that $\{\mathbf{0}^{(\alpha)}\}_{\alpha<\omega_1^{E_1}}$ has indeed a 1-least upper bound. We can generalize the definition at limit ordinals and let $\mathbf{0}^{(\lambda)} = \gamma$-least upper bound of $\{\mathbf{0}^{(\alpha)}\}_{\alpha<\lambda}$ for the least γ for which this exists. As can be thought, the places in which 1-least upper bounds are considered are the recursively inaccessible; those in which 2-least upper bounds are considered are the recursively inaccessible limits of recursively inaccessible (i.e. the recursively hyperinaccessible); those in which $\mathbf{0}^{(\alpha)}$ is defined as the α-least upper bound are the fixed points of iterations of recursive inaccessibility (i.e. the recursivly mahlo), etc. It is possible to see that γ-least upper bounds are considered for any $\gamma < \omega_1^L$, that ω_1^L is the stopping point of the process, and that the chain $\{\mathbf{0}^{(\alpha)}\}_{\alpha<\omega_1^L}$ is a natural hierarchy of hyperdegrees for the constructible sets of integers. It is also a well ordering of the set $\mathscr{C}_1 = \{X\colon X \in L_{\omega_1^X}\}$, the largest thin π_1^1 set ($\mathscr{C}_1$ plays an important role in descriptive set theory). This chain of hyperdegrees coincides with the Suzuki chain up to the first π_1^1-reflecting ordinal. The reference for all this is Kechris [23]. □

The problem of characterizing the sets (or even the chains) of hyperdegrees with least upper bound is open. We know from above that there are chains of hyperdegrees without least upper bound. Here is another result:

PROPOSITION 11.6. *For any countable ordinal* α *there is a hyperdegree* $\boldsymbol{a}$ *such that the chain of hyperdegrees defined starting with* $\boldsymbol{a}$ *and iterating hyperjumps at successor stages and least upper bounds at limit stages has length greater than* α.

PROOF. Let $\beta \geq \alpha$ be a countable admissible and let A be such that $\omega_1^A = \beta$. If $\boldsymbol{a}$ is the hyperdegree of A, the chain above has length at least equal to the first recursively inaccessible in A, in particular at least α. □

§12. Δ_2^1-degrees and Σ_2^1 sets. The connections between constructibility and the second level of the analytical hierarchy are even more striking than those between constructibility and the first level. Some of them were noted by Gödel [14] already and proofs were published by Addison [4]: they basically say that $\mathscr{P}(\omega) \cap L$ is a Σ_2^1 set (this is nontrivial only in the case $\mathscr{P}(\omega) \not\subseteq L$) and that the natural well ordering $<_L$ of the constructible sets is Δ_2^1 on $\mathscr{P}(\omega)$. Both these results are proved by arithmetization (see Devlin [9] for a standard reference on constructibility). An immediate consequence of the last one is that if $\mathscr{P}(\omega) \subseteq L$ then the Δ_2^1-degrees (i.e. the degrees obtained from the relation $A \leq_2 B$ iff A is Δ_2^1 in B) are well ordered and hence their structure is very well behaved.

To go beyond this a finer analysis of L is needed, and the basic tool is the use of trees. We will be sketchy in our treatment of the basic facts we need. A comprehensive treatment is in Moschovakis [28]. The basic result for the study of Δ_2^1-degrees is in Shoenfield [38]: $A \leq_2 B$ *iff* $A \in L_{\delta_2^1(B)}(B)$ (for $A, B \subseteq \omega$) where $\delta_2^1(B)$ is the least ordinal which is not the ordinal of a $\Delta_2^{1,B}$ well ordering of ω. Thus the ordinals $\delta_2^1(B)$ now play the role played by ω_1^B for hyperdegrees. The result is an immediate consequence of the basis theorem for Σ_2^1 sets [28, p. 236] and the following: $\Sigma_2^1 \subseteq L$. This is again due to Shoenfield and is obtained by analyzing Σ_2^1 sets in terms of (non) well-founded trees (recall that the notion of well foundedness is absolute for models of ZFC).

Another useful result is: *every Σ_2^1 set of reals containing a nonconstructible element contains a perfect tree* (Mansfield, see [44]). Again the proof uses the representation of Σ_2^1 sets via trees and consists in the analysis of trees by iterating the Cantor derivative (i.e. chopping out of the tree the isolated nodes). Since the process is absolute for models of ZFC, either we get a perfect kernel or every branch of the tree is constructible from the tree (because it is uniquely determined by a finite initial segment). An immediate consequence is: *if $\mathscr{P}(\omega) \cap L$ is countable then there is a largest countable Σ_2^1 set of reals* (Solovay [44]), since $\mathscr{P}(\omega) \cap L$ is Σ_2^1. A similar argument gives the following: *every Σ_1^1 set of reals containing a nonhyperarithmetical element contains a perfect tree* (Harrison [20]), the reason now being that every Σ_1^1 implicitly definable set is Δ_1^1.

We introduce now the machinery of forcing over $L_{\delta_2^1}$. Since δ_2^1 is admissible, we have from §10 the language $\mathscr{L}_{\delta_2^1}$ and the structure $\mathscr{M}(A) = L_{\delta_2^1}(A) \cap \mathscr{P}(\omega)$. For the moment we consider, e.g., forcing with Δ_2^1 trees. Let $P, Q, \ldots$ denote forcing conditions.

DEFINITION. A is *Sacks δ_2^1-generic* if $\{P: A \in P\}$ meets every dense set of conditions definable over $L_{\delta_2^1}$.

Of course our main goal is to prove that for A Sacks δ_2^1-generic, $\mathscr{M}(A) = \Delta_2^{1,A}$. The proof of the corresponding result in §10 made use of a very peculiar property, namely the fact that L_{ω_1} is the smallest admissible set containing ω. But no system of axioms with the property that $L_{\delta_2^1}$ is the smallest model of it exists (by the Kondo-Addison theorem). The approach of §7 fails similarly: there we used the characterization of Δ_1^1 as the smallest model of Δ_1^1-comprehension, but Δ_2^1 is not the smallest model of Δ_2^1-comprehension (this is $L_{\omega_1^{E_1}} \cap \mathscr{P}(\omega)$). Moreover, there is another trouble: the result we look for is simply false if $A \in L$, since a generic set is certainly not Δ_2^1 and if A is constructible and not Δ_2^1, $\delta_2^1 < \delta_2^1(A)$ (the result we seek is, as usual, equivalent to $\delta_2^1 = \delta_2^1(A)$). In particular the result is false if $\mathscr{P}(\omega) \subseteq L$.

Although the situation looks quite unpromising, it turns out that the solution is not very difficult. The reason is that it is easy to control the ordinal $\delta_2^1(A)$ directly. The method we use is due to Moschovakis (see [28]). Let $R(x, A)$ be a Σ_2^1 set such that, for each A, $\{x: R(x, A)\}$ is $\Sigma_2^{1,A}$-complete. Let $R(x, A) \Leftrightarrow (\exists B)S(x, A, B)$, where $S \in \pi_1^1$. We may also suppose, by the Kondo-Addison theorem, that S is uniformized over B (i.e. if there is a B such that $S(x, A, B)$ then B is unique). We may also suppose that S is π_1^1-complete. There is a π_1^1-norm $\tau(x, A, B)$ on S (i.e. τ assigns ordinals to x, A, B in such a way that both the relations

$$R(x, A, B) \wedge \tau(x, A, B) \leq \tau(x', A', B'),$$
$$R(x, A, B) \wedge \tau(x, A, B) < \tau(x', A', B'),$$

are uniformly Δ^1_1). Then the function $\sigma(x, A) = \tau(x, A, B)$, defined when $R(x, A)$ (note that B is uniquely determined) is a Σ^1_2-norm on R (in a similar sense). By the completeness property of S, $\delta^1_2(A) = \sup\{\sigma(x, A)\colon R(x, A)\}$. The basic result we are seeking comes from the following lemma:

PROPOSITION 12.1. *If* $A \in P$, $A \notin L$ *and* $R(x, A)$ *then* $(\exists Q \subseteq P)(\forall X \in Q)(\sigma(x, X) < \delta^1_2)$ (*Friedman* [12]).

PROOF. For simplicity of notation we write $\tau(X)$ for $\tau(x, X, X')$ where X' is the unique Y such that $S(x, X, Y)$ (x can be skipped because it is held constant). We claim that

$$(\exists X)(\tau(A) = \tau(X) \wedge A \not\leq_h X, P).$$

Suppose otherwise: the set of Y's such that

$$(\forall X)(\tau(Y) = \tau(X) \Rightarrow Y \leq_h X, P)$$

is hence a π^1_1 (in P) set containing A. By the relativization of Mansfield's theorem and the facts that $A \notin L$ and $P \in L$ (since $P \in \Delta^1_2$), it contains a perfect subset. But for each possible value of $\tau(Y)$ there are only countably many such Y's (since $Y \leq_h X$ for some X) and the class of π^1_1 thin (i.e. without perfect subsets) sets is additive (Kechris [23]), contradiction. Alternatively, we can directly build X as wanted by considering the first ordinal $\beta > \tau(A)$ which is admissible in P, taking X constructible, and collapsing $\tau(A)$ to ω. Then $L_\beta(X, P)$ is admissible and $\beta = \omega_1^{X,P}$. Moreover, $A \notin L_\beta(X, P)$ because $A \notin L$ and X, P are constructible. So $A \not\leq_h X, P$.

From the claim we have

$$(\exists X)(\exists Y)(Y \in P \wedge \tau(Y) = \tau(X) \wedge Y \not\leq_h X, P).$$

By the basis theorem, there is $X \in \Delta^1_2$ (actually Δ^1_2 in P, but P is Δ^1_2 itself). So $\tau(X) < \delta^1_2$ since $\sup \sigma(x, X) < \delta^1_2$. Fix such an X, and consider the Y's such that

$$Y \in P \wedge \tau(Y) = \tau(X) \wedge Y \not\leq_h X, P.$$

They form a set Σ^1_1 in P, X which is nonempty (because A is in it), hence with a member not Δ^1_1 in P, X. By the Harrison theorem, there is a perfect subset Q of it. $\square$

PROPOSITION 12.2. *If A is Sacks δ^1_2-generic and A is not constructible,* $\mathscr{M}(A) = \Delta^{1, A}_2$.

PROOF. By 12.1.

Note that the simple hypothesis $\mathscr{P}(\omega) \not\subseteq L$ already insures the existence of nonconstructible Sacks δ^1_2-generic: it is possible to build directly (with no hypothesis) a constructible perfect tree of Sacks δ^1_2-generics, and this induces a constructible homeomorphism between the tree and the reals. If $\mathscr{P}(\omega) \not\subseteq L$ then there is a nonconstructible real, hence a nonconstructible branch of the tree.

We do not know if the analogue of 12.2 holds for nonconstructible δ^1_2-generic reals (i.e. obtained by forcing with finite conditions). It is, however, possible to use 12.1 to build a δ_2^1-generic real A such that $\mathscr{M}(A) = \Delta_2^{1,A}$ (under the hypothesis $\mathscr{P}(\omega) \not\subseteq L$).

We are now ready for the study of the structure of Δ_2^1-degrees. Our notations are the usual ones: $\boldsymbol{a}$ is a Δ_2^1-degree, $\mathbf{0}$ is the smallest Δ_2^1-degree containing exactly the Δ_2^1 sets, $\boldsymbol{a}'$ is the degree of the complete Σ_2^1 set in A for any $A \in \boldsymbol{a}$. Almost all the structural results will be dependent on some set-theoretical assumptions. The surprising fact is that $\mathscr{P}(\omega) \not\subseteq L$ or its relativizations are usually characterizing the behaviour of Δ_2^1-degrees.

PROPOSITION 12.3. (a) $\mathscr{P}(\omega) \subseteq L$ *iff the Δ_2^1-degrees are well ordered* (*Guaspari* [15]).

(b) $\mathscr{P}(\omega) \not\subseteq L$ *iff every countable partial ordering is embeddable in the Δ_2^1-degrees.*

PROOF. If $\mathscr{P}(\omega) \subseteq L$ then the Δ_2^1-degrees are well ordered. If $\mathscr{P}(\omega) \not\subseteq L$ then there is a (nonconstructible) δ_2^1-generic real such that $\mathscr{M}(A) = \Delta_2^{1,A}$: as usual its components are independent. □

PROPOSITION 12.4. $\mathbf{0}'$ *is a minimal Δ_2^1-degree* (*Shoenfield* [40]).

PROOF. It follows from the basis theorem and the fact that $\mathscr{P}(\omega) \cap L_{\delta_2^1} = \Delta_2^1$, that the Σ_2^1 sets are exactly the sets Σ_1 over Δ_2^1 and they are all in $L_{\delta_2^1+1}$. Let $B \in \Sigma_2^1 - \Delta_2^1$ and $A \leq_2 B$: since Δ_2^1 operations are constructible and $B \in L$, $A \in L$. Either $A \in L_{\delta_2^1}$ and then $A \in \Delta_2^1$ or $B \in L_{\delta_2^1(A)}(A)$ because then $\delta_2^1(A) > \delta_2^1$, so $B \leq_2 A$. □

A relativization of the above proposition shows that (without any assumptions beyond ZFC) the chain of the Δ_2^1 degrees of the constructible sets is well ordered, and the successor operation is given by the Δ_2^1-jump.

PROPOSITION 12.5. *The completeness theorem is false for the Δ_2^1-jump, i.e. there is $\boldsymbol{a}$ such that for no $\boldsymbol{b}$, $\boldsymbol{b}' = \boldsymbol{a}$.*

PROOF. Let $\boldsymbol{a}$ be any limit point in the chain of Δ_2^1-degrees of the constructible sets. If $\boldsymbol{b}' = \boldsymbol{a}$ then $\boldsymbol{b}$ must contain (only) constructible sets, because $\boldsymbol{b} < \boldsymbol{a}$ and Δ_2^1 operations are constructible. Contradiction. □

PROPOSITION 12.6. (a) $\mathscr{P}(\omega) \subseteq L$ *iff there is only one minimal Δ_2^1-degree.*

(b) $\mathscr{P}(\omega) \not\subseteq L$ *iff there are uncountably many minimal Δ_2^1-degrees* (*Friedman* [12]).

PROOF. If $\mathscr{P}(\omega) \subseteq L$ the result follows from 12.3 and 12.4. If $\mathscr{P}(\omega) \not\subseteq L$ then we can build a tree of nonconstructible Sacks δ_2^1-generic sets: by 12.2 and usual arguments, they have minimal Δ_2^1-degrees. □

Note that the situation here is similar to that of §4 (arithmetical degrees): using 12.2 we can construct directly minimal degrees without any appeal to forcing. Genericity is used above only to get the result automatically. This observation is crucial because then the treatment of minimal upper bounds does not cause the kind of problems we faced in §11.

PROPOSITION 12.7. *Every countable ideal of Δ_2^1-degrees has a minimal upper bound* (*Friedman* [12]).

PROOF. Let $\{\boldsymbol{a}_n\}_{n\in\omega}$ be an enumeration of the given ideal and let $\{\boldsymbol{b}_n\}_{n\in\omega}$ be the associated chain: $\boldsymbol{b}_0 = \boldsymbol{a}_0$, $\boldsymbol{b}_{n+1} = b_n \cup \boldsymbol{a}_{n+1}$. Choose $B_n \in \boldsymbol{b}_n$. There are two cases.

If $(\forall n)(\mathscr{P}(\omega) \not\subseteq L(B_n))$ then we can easily relativize the proof of 12.6(b) (using Δ_2^1-pointed trees whose Δ_2^1-degree is in the given ideal) and get a continuum of minimal upper bounds.

If for some n, $\mathscr{P}(\omega) \subseteq L(B_n)$ then there is a well ordering of $\mathscr{P}(\omega)$ which is Δ^1_2 in B_n. Let C be the least (in this well ordering) among the upper bounds (in the sense of Δ^1_2-degrees) of our ideal. Then C is the least upper bound because if D is any other upper bound then C is Δ^1_2 in B_n and D (by the choice of C) and $B_n \leq_2 D$ (since D is an upper bound). So $C \leq_2 D$. $\square$

The situation for minimal degrees and minimal upper bounds is similar: we can prove in ZFC that they exist, but their number depends on further set-theoretical assumptions.

PROPOSITION 12.8. (a) *$\mathscr{P}(\omega) \subseteq L$ iff every countable set of Δ^1_2-degrees has least upper bound.*

(b) *$(\forall X)(\mathscr{P}(\omega) \not\subseteq L(X))$ iff no ascending sequence of Δ^1_2-degrees has least upper bound.*

(c) *$(\forall X)(\mathscr{P}(\omega) \not\subseteq L(X)$ iff every countable set of Δ^1_2-degrees has a continuum of minimal upper bounds* (*Friedman* [12].)

PROOF. We first dispose of the left-to-right implications. Part (a) follows from the well orderings of Δ^1_2-degrees (under the hypothesis $\mathscr{P}(\omega) \subseteq L$); parts (b) and (c) follow from the proof of 12.7 since then every countable set of Δ^1_2-degrees has a continuum of minimal upper bounds, hence no ascending sequence has least upper bound.

Now for the right-to-left implications. We prove their contrapositives. Let $\mathscr{P}(\omega) \not\subseteq L$: then the proof of 12.7 gives a continuum of minimal upper bounds for $\{\mathbf{0}^{(n)}\}_{n\in\omega}$ (which consists only of constructible sets), hence there is no least upper bound for it. This proves part (a). Let now $\mathscr{P}(\omega) \subseteq L(X)$ for some X and consider the chain $\{\boldsymbol{b}_n\}_{n\in\omega}$ defined as: $\boldsymbol{b}_0 = \Delta^1_2$-degree of X, $\boldsymbol{b}_{n+1} = \boldsymbol{b}_n'$. The proof of 12.7 gives a least upper bound for it (this proves part (b)), hence it cannot have a continuum of minimal upper bounds because is ascending (this proves part (c)). $\square$

We may define the natural chain of Δ^1_2-degrees as: $\mathbf{0}^{(0)} = \mathbf{0}$, $\mathbf{0}^{(\alpha+1)}$ = the Δ^1_2-jump of $\mathbf{0}^{(\alpha)}$, $\mathbf{0}^{(\lambda)}$ = the least upper bound of $\{\mathbf{0}^{(\alpha)}\}_{\alpha<\lambda}$ (when it exists). Let α_0 be the least ordinal α for which $\mathbf{0}^{(\alpha)}$ is not defined.

PROPOSITION 12.9. (a) *If $\mathscr{P}(\omega) \subseteq L$ then $\alpha_0 =$ (real) ω_1.*

(b) *If $\mathscr{P}(\omega) \not\subseteq L$ then $\alpha_0 = \omega$.*

PROOF. (a) The Δ^1_2-degrees are well ordered with ordinal $\omega_1^L = \omega_1$.

(b) See the proof of 12.8. $\square$

PROPOSITION 12.10. *$\mathscr{P}(\omega) \subseteq L$ iff the Δ^1_2-degrees are a lattice.*

PROOF. We only have to worry about greatest lower bounds of pairs of degrees. If $\mathscr{P}(\omega) \subseteq L$ they always exist. If $\mathscr{P}(\omega) \not\subseteq L$ then the chain $\{\mathbf{0}^{(n)}\}_{n\in\omega}$ has a continuum of minimal upper bounds: any two of them of distinct degrees are without g.l.b. $\square$

PROPOSITION 12.11. (a) *$\mathscr{P}(\omega) \subseteq L$ iff the theory of Δ^1_2-degrees is decidable.*

(b) *If $(\forall X)(\mathscr{P}(\omega) \not\subseteq L(X))$ then the theory of Δ^1_2-degrees is recursively isomorphic to second order arithmetic.*

PROOF. (a) One direction is trivial. For the other, note that $\mathscr{P}(\omega) \not\subseteq L$ is enough to embed all the countable (distributive) lattices as initial segments, hence the theory of degrees is undecidable (see Epstein [10]).

(b) Nerode and Shore [29] prove that the result follows from the following assumptions: every countable distributive lattice is isomorphic to some segment of the degrees (this follows from $\mathscr{P}(\omega) \not\subseteq L$) and every countable ideal has an exact pair (this follows from the whole assumption, as in 12.8). $\square$

We discuss briefly the analogue of Sacks' theorem on countable admissible ordinals. The basic fact in that result was that ω_1 is the first ordinal of a certain kind (namely, the first admissible ordinal greater than ω). The result then said that every other countable ordinal of the same kind was of the form ω_1^A for some A. It is known (see Barwise [7]) that δ_2^1 is the first ordinal of a certain kind, namely the first stable ordinal (α is stable if L_α is an elementary substructure of L for Σ_1 formulas). By relativization, $\delta_2^1(A)$ is a countable stable ordinal. $\mathscr{P}(\omega) \not\subseteq L$ *iff for every countable stable ordinal* α *there is* A *such that* $\alpha = \delta_2^1(A)$. One direction is immediate: if $\mathscr{P}(\omega) \subseteq L$ then the limit of the first ω stable ordinals is stable but not of the form $\delta_2^1(A)$. For the other direction, Friedman [12] proved it under the hypothesis that ω_1 is inaccessible in L by using the Levy collapse with finite conditions. By collapsing with perfect conditions in the style of the strong form of Sacks' theorem, Thomas John (unpublished) was able to handle the remaining cases.

We conclude this section with some words about Σ_2^1 sets. It is possible to rewrite the work done in the first part of §9 by substituting Σ_2^1 and δ_2^1 for π_1^1 and ω_1. We get a notion of δ_2^1-calculability, and the basic connection with the work of this section is: for $A, B \subseteq \omega$,

$$A \leq_2 B \text{ iff } A \leq_{c\delta_2^1} B \text{ iff } A \leq_{\delta_2^1(B)} B.$$

The basic results are:

PROPOSITION 12.12. (a) *Every* Σ_2^1-Δ_2^1 *set is in the the* Δ_2^1*-degree* $\mathbf{0}'$.

(b) *There are two* Σ_2^1 *sets* δ_2^1*-incomparable* (*Sacks* [35]).

We do not know of any interesting characterization of the Turing degree of a Σ_2^1-complete set similar to the one obtained for $\mathcal{O}$ in §9.

§13. L-degrees and Δ_3^1-sets. We have seen in previous sections that the notion of constructibility was a useful tool in the study of hyperdegrees and Δ_2^1-degrees, the basic connections being (for subsets of ω): $A \leq_h B \Leftrightarrow A \in L_{\omega_1^B}(B)$ and $A \leq_2 B \Leftrightarrow A \in L_{\delta_2^1(B)}$. In this section we study the degrees of nonconstructibility (L-degrees) obtained from the relation $A \leq_L B$ iff A is constructible from B. From Gödel [14] it follows that the relevant ordinals here are ω_1^L (the first ordinal uncountable in L) and its relativization $\omega_1^{L(B)}$, in the sense that $A \leq_L B \Leftrightarrow A \in L_{\omega_1^{L(B)}}(B)$. In the following *we assume the consistency of* ZFC *and work in* ZFC (until further notice). We begin with some consistency results. We follow the usual convention and call Cohen-generic a set generic with respect to the notion of forcing whose conditions are strings.

PROPOSITION 13.1. *The following are consistent with* ZFC:

(a) *There is only one* L*-degree* (*Gödel* [14]).

(b) *Every countable partial ordering is embeddable in the* L*-degrees* (*Feferman* [11]).

PROOF. (a) By the consistency of $\mathscr{P}(\omega) \subseteq L$.

(b) By the Levy collapse it is consistent that ω_1^L is countable. Under this assumption, we can add a Cohen-generic real over $L_{\omega_1^L}$: by the usual arguments, this is not in $L_{\omega_1^L}$ (hence not in L) and its components are L-independent. This implies the result. □

PROPOSITION 13.2. *The following are consistent with* ZFC:

(a) *There are uncountably many L-degrees but no minimal one* (*Solovay*).

(b) *There are uncountably many minimal L-degrees* (*Sacks* [34]).

PROOF. (a) Under the assumption that ω_1^L is countable, we can add a tree of Cohen-generic reals over $L_{\omega_1^L}$. Solovay has noted that if A is a Cohen-generic real over $L_{\omega_1^L}$ and $B \in L(A) - L$ then for some Cohen-generic C is $L(B) = L(C)$. Hence any L-degree below the L-degree of a Cohen-generic real is itself the L-degree of a Cohen-generic real, so it is not minimal (see 13.1(b)).

(b) By the Levy collapse, it is consistent that ω_2^L is countable. Under this assumption, we can add a tree of Sacks-generic reals over $L_{\omega_2^L}$: by the usual argument they have minimal L-degree. The countability of ω_2^L is needed because the forcing conditions are basically subsets of ω, and we need to know that there are only countably many sets of (dense) conditions to meet. □

Note that the proof of (b) is now a type of forcing argument with a set of conditions, as opposed to the proof of the same result for hyperdegrees of Δ_2^1-degrees, where we forced with classes of conditions.

As usual, the join operation gives the least upper bound of a pair of L-degrees.

PROPOSITION 13.3. *The following are consistent with* ZFC:

(a) *The L-degrees are a* (*nontrivial*) *lattice.*

(b) *The L-degrees are not a lattice* (*Balcar and Hajek* [6]).

PROOF. (a) Start with ω_2^L countable and add a pair of Sacks-generic reals by product forcing: the L-degrees form a diamond.

(b) As usual this follows from the existence of a Cohen-generic real, hence from ω_1^L countable. □

PROPOSITION 13.4. *The following are consistent with* ZFC:

(a) *Every ascending sequence of L-degrees has least upper bound.*

(b) *No ascending sequence of L-degrees has minimal upper bounds.*

PROOF. (a) By using the usual techniques from the theory of Turing degrees (see e.g. Epstein [10]) and Sacks forcing (from ω_2^L countable), embed $\omega + 1$ as initial segment of the L-degrees.

(b) Similar, by embedding $\omega + \omega^*$ (where ω^* is the reverse ordering of ω). □

We quote a general result on initial segments: it is consistent with ZFC that the L-degrees are isomorphic to any bottomed, topped upper semilattice which is countable and well founded in L or which is finite (Adamowicz [2], [3]). As usual the proofs are mixtures of Sacks forcing and the techniques used to get the analogous results for Turing degrees.

Since the picture of L-degrees can vary widely from one model to another, it is perhaps more interesting—instead of simply giving consistency results—to analyze the structure of L-degrees under some fixed set-theoretical assumptions. We briefly discuss here the most straightforward one: ω_1^L *is countable.* Some of the results above were directly derived from it. Jensen [21] has shown how to use iterated forcing to obtain minimal L-degrees from this hypothesis, and his method works fine for other simple initial segments. Hence from ω_1^L countable we have the following picture of L-degrees: every countable partial ordering is embeddable; there are minimal L-degrees and L-degrees with no minimal predecessors: the L-degrees are not a lattice.

In connection with ω_1^L and its relativizations, we quote the analogue of Sacks' theorem of §10. The analogy to keep in mind is that the admissible ordinals are the recursively regular ordinals. The result is: *if $\alpha > \omega$ is a countable regular L-cardinal, then $\alpha = \omega_1^{L(A)}$ for some $A \subseteq \omega$* (Jensen and Solovay [22]). The nontrivial case occurs when α is inaccessible in L, and the Levy collapse allows the coding of it by a subset of (real) ω_1. The new step introduced in [22] is a technique (of independent great interest) to code, under certain conditions, subsets of ω_1 by subsets of ω. This is referred to as almost-disjoint forcing.

Sacks [37] has obtained a sharper version of this theorem, similar to the strong version of his theorem for countable admissible ordinals, by using iterated forcing with uniformly constructibly pointed trees. The result is obviously: *if $\alpha > \omega$* is a *countable regular L-cardinal, there is $A \subseteq \omega$ such that $\alpha = \omega_1^{L(A)}$ and $\omega_1^{L(B)} < \alpha$ for every $B <_L A$.*

We turn now to the jump operator for L-degrees. Here the task turns out to be more delicate than for the other kind of degrees we have been studying, the reason being that any reasonable notion of jump would take a constructible set into a nonconstructible one: the existence of such operations is hence not absolute, since it implies $\mathscr{P}(\omega) \subseteq L$. Moreover we are in a situation different from that for Turing degrees, hyperdegrees and Δ_2^1-degrees, where we had a natural class of sets (respectively Σ_1^0, π_1^1 and Σ_2^1) whose self-dual class (Δ_1^0, Δ_1^1, Δ_2^1) was the basis for that notion of degree: it was natural there to take as jump the relativization of a complete set in that class. We do not have anything similar here, but we already overrode once a difficulty of this kind in dealing with arithmetical degrees: the basic idea to get a natural nonarithmetical set went actually back to Tarski and consisted of taking $0^{(\omega)}$, the set of Gödel numbers of true sentences of arithmetic (briefly, the truth set for arithmetic). By Tarski [48] this was not arithmetic, and it was easy to prove that every arithmetical set was recursive in it (by using its arithmetical definition). We can try the same path here, and define $0^\#$ as the truth set for L. If we knew it exists, then we immediately would have that it is not in L and every constructible set is recursive in it.

One way to insure the existence of $0^\#$ (actually, in the literature the phrase "$0^\#$ exists" means that the two following conditions are satisfied) is the following. Suppose that:

(1) if $\alpha < \beta$ are (real) uncountable cardinals then $L_\alpha \prec L_\beta$ or, equivalently, that:

(1′) if α is a real uncountable cardinal, then $L_\alpha \prec L$.

Then to decide if a formula φ of n free variables is true in L for the parameters $x_1, \ldots, x_n$ it is enought to look for the smallest real uncountable cardinal α such that all the parameters are in L_α and check if $\varphi(x_1, \ldots, x_n)$ holds in L_α. To avoid the consideration of arbitrary parameters it is enough to assume that:

(2) there is a closed unbounded class (of ordinals) C of indiscernibles for L, containing all real uncountable cardinals and finitely generating L (i.e. every element of L is definable in L from a finite number of elements of C).

Then it is enough to consider ω many indiscernibles, say the real ω_n's, and $0^\#$ becomes the truth set of L_{ω_ω} (when the free variables are interpreted as the ω_n's), in particular, it is definable.

Many set-theoretical assumptions imply the existence of $0^\#$, among them the existence of a Ramsey cardinal (Silver [41]) and the determinacy of Σ_1^1 games (Harrington [18]). It is not possible however to prove in ZFC that if ZFC is consistent then so is ZFC + "$0^\#$ exists". The reason is that from assumption (1′) it follows that, in particular, $L_{\omega_1} \prec L$, hence everything definable in L (without parameters) is actually in L_{ω_1} and is then countable. In particular so are ω_1^L, ω_2^L, ... and the first inaccessible in L (and the existence of an inaccessible in L is not provably consistent with ZFC).

The definition of $0^\#$ can be relativized (by substituting L with $L(x)$) and gives $x^\#$. Until the end of the section we assume that *for every x, $x^\#$ exists.* We stress the fact that this is a very strong set-theoretical hypothesis. We use the usual notations: $\boldsymbol{a}$ is an L-degree, $\mathbf{0}$ is the smallest L-degree (containing exactly the constructible sets), $\boldsymbol{a}'$ is the L-degree of $x^\#$ for any $x \in \boldsymbol{a}$. Hence the jump operation for L-degrees is induced by the sharp operation on sets. The work done in §4 for arithmetical degrees readily extends to L-degrees.

PROPOSITION 13.5. *Any countable partial ordering is embeddable in the L-degrees below* $\mathbf{0}'$.

PROOF. Build a Cohen-generic real over $L_{\omega_1^L}$ (note: ω_1^L is countable since $0^\#$ exists) recursively in $0^\#$. □

PROPOSITION 13.6. *The L-degrees below* $\mathbf{0}'$ *are not a lattice.*

PROPOSITION 13.7. *There is a minimal L-degree below* $\mathbf{0}'$.

PROOF. Build a Sacks-generic real over $L_{\omega_2^L}$ (ω_2^L is countable) recursively in $0^\#$. □

PROPOSITION 13.8. *The L-jump operation has range* $\{\boldsymbol{a}: \boldsymbol{a} \geq \mathbf{0}'\}$ *and is never one-one on its range.*

PROOF. The basic fact to note is that if A is Cohen-generic over $L_{\omega_1^L}$, then $A^\# \equiv_T A \oplus 0^\#$. □

By building a tree of Cohen-generic reals over $L_{\omega_1^L}$ recursively in $0^\#$ we can get the following strong form of completeness for the L-jump: if $0^\# \leq_T A$ then for some B, $B^\# \equiv_T A$. The completeness follows from this because if $0^\# \leq_L A$ then $0^\# \leq_T A^\#$ (by relativization of the fact that any constructible set is recursive in $0^\#$).

To get results such as: if $\boldsymbol{a} < \mathbf{0}'$ then $\boldsymbol{a}' = \mathbf{0}'$; every minimal L-degree realizes the minimal L-jump; the L-jump operation restricted to minimal L-degrees has range $\{\boldsymbol{a}: \boldsymbol{a} \geq \mathbf{0}'\}$ (Simpson [42]), we need (to reproduce the proofs of the analogous results for hyperdegrees) an ordinal assignment $x \to \lambda^x$ with the following properties:

(a) $x \leq_L y \Rightarrow \lambda^x \leq \lambda^y$,

(b) $x \leq_L y \Rightarrow (x^\# \leq_L y \Leftrightarrow \lambda^x < \lambda^y)$.

At least two ordinal assignments like this are known to exist (see Harrington and Kechris [19]). Since, however, no natural minimal assignment is known, we do not pursue the matter here. We only note that the assignment $x \to \omega_1^{L(x)}$ does not work: in fact (b) fails even in unrelativized form, since ω_2^L is countable and hence (by the result of Jensen and Solovay [22] quoted above) $\omega_2^L = \omega_1^{L(x)}$ for some x. Since $\omega_1^L < \omega_2^L$, if (b) held it would yield $0^\# \leq_L x$ and hence $\omega_1^{L(0^\#)} \leq \omega_2^L$, against the properties of $0^\#$.

We may define the natural chain of L-degrees as usual: $\mathbf{0}^{(0)} = \mathbf{0}, \mathbf{0}^{(\alpha+1)} =$ the L-jump of $\mathbf{0}^{(\alpha)}$, $\mathbf{0}^{(\lambda)} =$ the least upper bound of $\{\mathbf{0}^{(\alpha)}\}_{\alpha<\lambda}$ (when it exists). Friedman [12] notes that $\mathbf{0}^{(\omega)}$ is defined. It is, however, easy to see the following: if (for λ limit) $M_\lambda = L[\langle 0^{(\beta)} : \beta < \lambda \rangle]$, let α_0 be the first λ such that $\lambda = \omega_1^{M_\lambda}$. Then:

PROPOSITION 13.9. *$\mathbf{0}^{(\alpha)}$ is defined exactly for $\alpha < \alpha_0$.*

PROOF. We only need to consider limit ordinals. If $\lambda < \omega_1^{M_\lambda}$ then there is f: $\omega \to \lambda$ onto, one-one in M_λ. We then define $\langle x, n \rangle \in 0^{(\lambda)} \Leftrightarrow x \in 0^{(f(n))}$: $\mathbf{0}^{(\lambda)}$ is obviously an upper bound for $\{\mathbf{0}^{(\alpha)}\}_{\alpha<\lambda}$. Suppose x is an upper bound: then $(\forall \alpha < \lambda)$ $(0^{(\alpha)} \in L(x))$ and $M_\lambda \subseteq L[x]$. Since $\mathbf{0}^{(\lambda)}$ is constructible from $\{0^{(\alpha)}\}_{\alpha<\lambda}$ and f, $0^{(\lambda)} \in L[x]$ and $0^{(\lambda)} \leq_L x$.

If α_0 is the real ω_1 then we are finished. Otherwise α_0 is countable and we can get (by product forcing) an exact pair for the chain: this forces the least upper bound (if it exists) to be in M_{α_0}. But then there is no upper bound because, by general facts about relative constructibility, anything in $M_\lambda = L[\langle 0^{(\beta)} : \beta < \omega_1^{M_\lambda} \rangle]$ is already in $L[\langle 0^{(\beta)} : \beta < \beta_0 \rangle]$ for some β_0 countable in M_λ. $\square$

It would be interesting to characterize α_0 in some other way. Certainly $\alpha_0 \leq$ (real)ω_1 because we assume $(\forall x)(x^\#$ exists), hence $(\forall x)(\omega_1^{L(x)}$ is countable) and every L-degree has only countably many predecessors.

We turn now to the structure of Δ_3^1 sets under the hypothesis $(\forall x)(x^\#$ exists). The basic result is due to Solovay [43]: $0^\#$ is a Δ_3^1 set and $x^\#$ is Δ_3^1 in x.

PROPOSITION 13.10. *The L-degrees of Δ_3^1 sets are an initial segment of the L-degrees closed under L-jump.*

PROOF. The closure under L-jump comes from Solovay's result and the closure properties of Δ_3^1 (Shoenfield [39]): if x is Δ_3^1 then so is $x^\#$, which is Δ_3^1 in x.

Let now $x \leq_L y$ and $y \in \Delta_3^1$. Then $y^\# \in \Delta_3^1$ and x is recursive in $y^\#$, so $x \in \Delta_3^1$. $\square$

In particular the L-degrees below $\mathbf{0}'$ are all L-degrees of Δ_3^1 sets, and the structure of L-degrees of Δ_3^1 sets is very rich. Hajek [17] has noted that the existence of $0^\#$ is enough to insure the embeddability of every finite lattice as an initial segment below $0^\#$, hence *the theory of L-degrees of Δ_3^1 sets is undecidable.*

We may ask which $\mathbf{0}^{(\alpha)}$ are degrees of Δ_3^1 sets. This depends on further set-theoretical assumptions, but it is certainly true that if α is the ordinal of a Δ_2^1 well ordering of ω (in particular if $\alpha < \delta_2^1$) then $0^{(\alpha)} \in \Delta_3^1$ (by the definition and the fact that Σ_2^1 formulas are absolute). We do not know if it is possible to prove (only from our assumptions) that there is $\alpha < \alpha_0$ such that $\mathbf{0}^{(\alpha)}$ is not the degree of a Δ_3^1 set. This is certainly true inside M_{α_0} (because $\alpha_0 = \omega_1^{M_{\alpha_0}}$ and M_{α_0} thinks that only countably many sets in his sense are Δ_3^1), but Δ_3^1 formulas are not absolute.

Another way to analyze the structure of Δ_3^1 sets is suggested by the work of Suzuki (see §11), namely, studying the structure of π_2^1 singletons (note that Σ_2^1 singletons are Δ_2^1). It is easy to see that not every Δ_3^1 set is a π_2^1 singleton (let A be a Cohen-generic real recursive in $0^\#$: A is Δ_3^1 but is not implicitly definable in L by the usual argument, in particular it is not a π_2^1 singleton by the absoluteness of π_2^1 formulas). Solovay [43] has proved that $0^\#$ is a π_2^1 singleton and by relativization it follows that $\mathbf{0}^{(n)}$ is the degree of a π_2^1 singleton for every n. By definition so is $\mathbf{0}^{(\omega)}$. However it is now possible to prove that not all the $\mathbf{0}^{(\alpha)}$ are degrees of π_2^1

singletons, since M_{α_0} thinks that only countably many $\mathbf{0}^{(\alpha)}$ are degrees of π_2^1 singletons, and π_2^1 formulas are absolute (note that for this same reason the definition of $\mathbf{0}^{(\alpha)}$ is absolute for any model of ZFC containing all the ordinals and every $\mathbf{0}^{(\beta)}$, $\beta \prec \alpha$).

In analogy with Suzuki's result, Solovay has conjectured that the L-degrees of π_2^1 singletons are well ordered with successors given by the L-jump. It is not known if there are π_2^1 singletons of degree strictly between $\mathbf{0}$ and $\mathbf{0}'$. For some related work see Harrington and Kechris [19]. We also do not know if our assumptions imply that every Δ_3^1 set is recursive in (or even constructible from) a π_2^1 singleton (this follows from determinacy hypothesis, see Moschovakis [28]).

We conclude this section by noting that, although the existence of $0^\#$ is not provably consistent with ZFC, some of its consequences are, in particular, so is the existence of a nonconstructible Δ_3^1 set (actually a π_2^1 singleton); see Jensen and Solovay [22].

§14. **Δ_3^1-degrees and beyond.** We discuss in this final section some notions of degrees obtained from the various levels of the analytical hierarchy. We begin with the structure of Δ_3^1-degrees obtained from the relation $A \leq_3 B$ iff A is Δ_3^1 in B. The Δ_3^1-jump is induced by the operation consisting of taking, given a set A, any set which is $\Sigma_3^{1,A}$-complete (or $\pi_3^{1,A}$-complete). We adopt the usual notations: $\boldsymbol{a}$ for a Δ_3^1-degree, $\boldsymbol{a}'$ for its Δ_3^1-jump, $\mathbf{0}$ for the degree of the Δ_3^1 sets. A natural assignment of ordinals is: $\delta_3^1(A)$ = the first ordinal which is not the ordinal of a $\Delta_3^{1,A}$ well ordering of ω.

PROPOSITION 14.1. *If $\mathscr{P}(\omega) \subseteq L$ then the Δ_3^1-degrees are well ordered with successor given by the Δ_3^1-jump.*

PROOF. The assertion about well order is immediate, since there is a Δ_2^1 well ordering of $\mathscr{P}(\omega)$. The assumption $\mathscr{P}(\omega) \subseteq L$ implies that $\mathscr{P}(\omega) \cap L_{\delta_3^1} = \Delta_3^1$ and its relativizations, and the rest of the proposition follows as in 12.4. □

Although $\mathbf{0}'$ is a minimal Δ_3^1-degree under the assumption $\mathscr{P}(\omega) \subseteq L$, we are unable to prove in ZFC that it is a minimal Δ_3^1-degree. The missing link is the fact that Δ_3^1 operations are not provably constructible. Harrington has actually shown (to us) that there is a model of ZFC with no minimal Δ_3^1-degrees.

Since there are no connections between constructibility and the third analytical level, we cannot simply use the hypothesis $\mathscr{P}(\omega) \not\subseteq L$ to get a consistent picture of the Δ_3^1-degrees contrasting the one given by 14.1, as was possible for Δ_2^1-degrees. However stronger hypotheses do the job, and we consider here one which has been studied extensively: *Determinacy* (Δ_2^1). A treatment of the elementary consequences of it is in Moschovakis [28]. By mixing game-theoretical techniques with the machinery of forcing, Kechris [25], [26] has been able to extend to Δ_3^1-degrees a good part of the theory of Δ_2^1-degrees. We review here some of the results. The first one is proved, as can be expected, by forcing with Δ_3^1 perfect sets.

PROPOSITION 14.2. *There is a continuum of minimal Δ_3^1-degrees* (*Kechris* [26]).

In §11 we discussed a natural well-ordered chain of hyperdegrees (with successor given by the hyperjump) consisting of the hyperdegrees of the sets in $\mathscr{C}_1$ = the largest thin π_1^1 set. In §12 we saw that the chain of Δ_2^1-degrees of the constructible sets had similar properties (being well ordered with successor given by the Δ^1_2-jump). Moreover the set $\mathscr{C}_2 = \mathscr{P}(\omega) \cap L$ was the largest countable Σ^1_2 set (under the hypothesis of it being countable) and the Δ^1_2-degrees of sets in $\mathscr{C}_2$ were actually an initial segment of the Δ^1_2-degrees. If we let $\mathscr{C}_3 =$ the largest countable π^1_3 set, then the Δ^1_3-degrees of sets in $\mathscr{C}_3$ are well ordered with successor given by Δ^1_3-jump and there is a long initial segment of them which is actually an initial segment of the Δ^1_3-degrees (Kechris [23]). From this it is easy to get:

PROPOSITION 14.3. $\mathbf{0}'$ *is a minimal Δ^1_3-degree* (*Kechris* [23]).

PROPOSITION 14.4. *The completeness criterion fails for the Δ^1_3-jump.*

The general situation for minimal upper bounds is not known. However Kechris [26] proves that short sequences (in a precise sense) have a continuum of minimal upper bounds. From this follows:

PROPOSITION 14.5. *The natural chain of Δ^1_3-degrees, obtained by starting with* $\mathbf{0}$, *iterating Δ^1_3-jumps at successor stages, and taking least upper bounds at limit stages, stops at ω* (*Kechris* [26]).

PROPOSITION 14.6. *The Δ^1_3-degrees are not a lattice.*

The treatment of Δ^1_3-degrees is actually more general and gives the same picture for the Δ^1_n-degrees ($n \geq 2$) under determinacy hypotheses. For each $n \geq 2$, $\mathscr{C}_n$ is defined as the largest countable Σ^1_n set (if n is even) or π^1_n set (if n is odd). Note that Determinacy (Δ^2_1) implies that ω^L_1 is countable, hence $\mathscr{C}_2 = \mathscr{P}(\omega) \cap L$.

The structure of Δ^1_n-degrees for n even is actually very similar to the one of Δ^1_2-degrees. In fact the Δ^1_n-degrees of sets in $\mathscr{C}_n$ are actually an initial segment of the Δ^1_n-degrees and every sequence of Δ^1_n-degrees is short (in the sense referred to above). It follows that:

PROPOSITION 14.7. *If n is even then*:

(a) *Every countable set of Δ-degrees has a minimal upper bound.*

(b) $(\forall X)(\mathscr{P}(\omega) \not\subseteq \mathscr{C}_n(X))$ *iff every countable set of Δ^1_n-degrees has a continuum of minimal upper bounds* (*Kechris* [26]).

Thus there is a pleasant analogy among the structural theories of Δ^1_n-degrees for n even. There is a similar analogy among the theories of Δ^1_n-degrees for n odd and $n > 1$, but they differ markedly from the theory of hyperdegrees. This situation is regarded as unsatisfactory, and Kechris [23] has argued that the Δ^1_3-degrees are not the correct analogue of the hyperdegrees. He has introduced, for n odd and $n > 1$, a notion of Q_n-degree. This basically regards as the correct analogue of Δ^1_1 not Δ^1_3 but the larger set Q_3, defined as the largest initial segment of $\mathscr{C}_3$ closed under $\leq_3$. Q_3-degrees are defined by relativization. The natural object which is not in Q_3 is the first nontrivial π^1_3 singleton (first in the sense of $\leq_3$, among the sets in $\mathscr{C}_3$): this naturally induces a Q_3-jump operation, and agrees with the situation at level one, since $\mathscr{O}$ is the first nontrivial π^1_1 singleton. Kechris conjectures that the theory of Q_3-degrees is the correct analogue of the theory of hyperdegrees. In particular, he conjectures that the completeness criterion holds.

Conclusion. We have surveyed the applications of the forcing technique to the study of the structure of many different notions of degrees. These are, however, by no means the only applications of forcing to recursion theory. We content ourselves by quoting some of the most important ones we could not go over in this work. We will treat some of them in our book [32].

Sacks [36] proves the plus-one theorem for recursion in higher-types: the 1-section of a normal object is the 1-section of a normal type-2 object. After §10 the reader will not be surprised to notice that in the general treatment given by Sacks the basic assumptions he needs are local countability and Σ_1 dependent choice.

We noted that the technique of almost-disjoint forcing introduced by Jensen and Solovay [22] was used by them to prove the analogue of Sacks' theorem on admissible ordinals for countable regular L-cardinals. They also used it to prove the consistency with ZFC of the existence of a nonconstructible Δ^1_3 set. Harrington has subsequently elaborated on this technique and proved the following consistency results with ZFC: prewellordering for π^1_3; failure of prewellordering for both π^1_3 and Σ^1_3; $\mathscr{P}(\omega) \cap L = \Delta^1_3$; existence of two incomparable Δ^1_3-degrees containing π^1_3 sets (all these results are still unpublished).

Recently Maass [30] has introduced a notion of forcing on L_ω (the hereditarily finite sets) that admits r.e. sets as generic sets. As usual every such set is low and not recursive, and its components are recursively independent.

Finally, Rubin (unpublished) has used forcing techniques to get a great deal of uncountable initial segments of Turing degrees, in particular a chain of length (real) ω_1.

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