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A REGULARITY CONDITION FOR PARALLEL REWRITING SYSTEMS

Jan van Leeuwen

1. In recent years a considerable amount of work in formal language theory has focused on the study of parallel rewriting systems.

Once upon a time the study was almost exclusively biologically motivated, in view of the fact that one may consider linear arrays of cells in varying internal states as strings of symbols over some finite vocabulary and that in many cases the uniform growth of such a filament can be modelled by a type of symbol-replacement performed in parallel throughout the corresponding string during a series of discrete time-steps.

Nowadays the underlying structure is recognized in a growing variety of problems in computer science ranging from pure formal language theory to programming, while at the same time the analysis has led to interesting mathematical problems.

A class of parallel rewriting systems which received much attention in particular is described in the following definition.

Definition. An ETOL-grammar is a 4-tuple $G = \langle V,\Sigma,P,S\rangle$, where V, Σ, and $S \in V - \Sigma$ are as usual, and $P = \{\tau_1,\ldots,\tau_k\}$ is a finite set of finite substitutions over V. The language generated by G is the set $(\tau_1,\ldots,\tau_k)^*(S) \cap \Sigma^*$. Any such language is called an ETOL-language.

The generative mechanism underlying ETOL-languages is of interest not so much because it was once biologically motivated, but largely because it appeared to be in many ways mathematically attractive and fundamental in the wider context of language theory.

Deriving strings with an ETOL-grammar quite often shows an exponentially expanding or otherwise non-regular pattern. In this note we show how a theorem from the theory of well-partially-ordered sets may be used to characterize a large class of ETOL-grammars which, nonetheless, can generate regular languages only.

We show

<u>Theorem 1.1.</u> Let L be generated by an ETOL-grammar $\langle V,\Sigma, \{\tau_1,\ldots,\tau_k\},S\rangle$ for which there is an $m > 0$ such that for all $\tau \in (\tau_1,\ldots,\tau_k)^m$ and $a \in \Sigma$: $\lambda \in \tau(a)$. L is regular.

Thus, if in an ETOL-grammar any symbol can "die" within m steps when it chooses to no matter what sequence of tables is applied then the generated language must be regular.

2. To prove 1.1, we need some preliminaries first. An introduction to ETOL-grammars can be found in Herman & Rozenberg [2].

For strings v and w, let $v \le w$ if and only if there are strings $v_1,\ldots,v_n$ and $w_1,\ldots,w_{n+1}$ (some n) such that $v = v_1\ldots v_n$ and $w = w_1v_1\ldots w_nv_nw_{n+1}$.

<u>Definition.</u> For languages L, let $\underset{\sim}{L} = \{v \mid \exists_{w\in L}\ v \le w\}$.

It is known from the theory of well-partially-ordered sets that $\underset{\sim}{L}$ is always regular no matter how L is choosen. (It is part of "Haines' theorem". See Haines [1], Kruskal [3], van Leeuwen [4]).

Let K be a family of languages.

Definition. A K-iteration grammar is a 4-tuple $G = \langle V, \Sigma, P, M\rangle$ where V and Σ are as usual, $M \varepsilon K$ (and $M \subseteq V^*$), and $P = \{\tau_1, \ldots, \tau_k\}$ is a finite set of K-substitutions. The language generated by G is the set $(\tau_1, \ldots, \tau_k)^*(M) \cap \Sigma^*$.

We call K natural if and only if it is closed under $\cap\Sigma^*$ for all Σ. Let $Sub(R,K)$ be the family obtained from substituting K into the regular languages.

Theorem 2.1. Let K be natural. Let L generated by a K-iteration grammar $\langle V, \Sigma, \{\tau_1, \ldots, \tau_k\}, M\rangle$ such that for all i $(1 \le i \le k)$ and $a \varepsilon V: \lambda \varepsilon \tau_i(a)$. L belongs to $Sub(R,K)$.

Proof. Define a substitution σ by: $\sigma(a) = \{\lambda, a\}$ for $a \varepsilon V$.

Let $L' = (\tau_1, \ldots, \tau_k)^*(M)$, $M' = M \cap \Sigma^* (\varepsilon K)$, and consider $L = L' \cap \Sigma^*$.

Using that for all i $(1 \le i \le k)$ and $a \varepsilon \Sigma$: $\tau_i(a) = \tau_i(\sigma(a))$, we can derive

$$
\begin{aligned}
L &= (M \cup (\tau_1, \ldots, \tau_k)(L')) \cap \Sigma^* = \\
&= ((\tau_1, \ldots, \tau_k)(\sigma(L')) \cap \Sigma^*) \cup M' = \\
&= ((\tau_1, \ldots, \tau_k)(\underset{\sim}{L}') \cap \Sigma^*) \cup M' = \\
&= \bigcup_{1 \le i \le k} (\tau_i(\underset{\sim}{L}') \cap \Sigma^*) \cup M'.
\end{aligned}
$$

Defining K-substitutions τ_i' by : $\tau_i'(a) = \tau_i(a) \cap \Sigma^*$, it follows that

$$
L = \bigcup_{1 \le i \le k} \tau_i'(\underset{\sim}{L}') \cup M'
$$

and thus $L \varepsilon Sub(R,K)$, using that the latter family is closed under union and that $K \subseteq Sub(R,K)$. $\square$

3. The statement of 1.1. immediately follows from 2.1. in case $m = 1$. For $m > 1$ the following argument applies.

Let FIN be the family of finite languages. Let $\sigma_1, \ldots, \sigma_t$ $(t = k^m)$ be an enumeration of $(\tau_1, \ldots, \tau_k)^m$, and let for each i, $0 \leq i \leq m-1$, $L_i \in \mathrm{FIN}$ be the set of words over V derivable by G in i steps.

L is seen to be the (finite) union of languages generated by the FIN-grammars $\langle V, \Sigma, \{\sigma_1, \ldots, \sigma_t\}, L_i \rangle$, which all satisfy the condition of 2.1. It follows that L is regular.

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