- multivariate power series 1995
- what does the existence & uniqueness condition amount to over univariate power series?

Testing Identities of Series Defined by Algebraic Partial Differential Equations

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Abstract. In order to be able to manipulate solutions of systems of differential equations, one usually constructs differential extensions of differential rings, but the effectivity of the equality test in the extension is not trivial. In the ordinary differential case, the problem has been solved (see [13] and no power series [3]). We propose here a method in the case of extensions obtained by adjunction of formal power series defined as solutions of a system of non linear PDE's associated with a finite set of initial conditions.

1 Introduction

In the purely algebraic case, we know that an algebraic extension of an effective ring is effective, that is: one can actually compute in such a ring the usual arithmetical operation +,* and =. Of course, we are interested in knowing in which cases a differential extension of an effective differential ring R is effective (for example $R = \mathbb{Q}[x]$ is effective, and we would like to know if $\mathbb{Q}[x, sin(x)]$ is effective). More precisely: let k be an effective field of constants (notice that this means in particular that we can test equalities in k, which can be a difficult problem in number theory); we will focus on the equality test in extensions of $R = k[x]$ obtained by adjunction of formal power series $f_1, \ldots, f_m$. It is in fact the only operation that is not theoretically trivial. D. Zeilberger gave an algorithm for proving special function identities when the series are holonomic in one variable (for example, see [15]). The case when the f_i's are in $k[[x]]$ and defined by a system of (non linear) algebraic differential equations and a finite set of initial conditions, was (at least theoretically) solved by J. Denef and L. Lipshitz [3] (algebraic differential equations are equations of the form $P(x, f_1, f_1', \ldots, f_m, \ldots, f_m^{(n_m)}) = 0$ where P is a polynomial in all its variables with coefficients in k). In the special case of a triangular non singular system, J. Shackell [12] also gave an algorithm.

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However, if we wish to define the f_i's as solutions of PDE's, we must be much more cautious because of undecidability or uncomputability results. Indeed J. Denef and L. Lipshitz showed in [3], for example, that there does not exist an algorithm to check whether a linear PDE has a power series solution in $\mathbb{C}[[x_1, \ldots, x_n]]$ (for n large enough, say $n \geq 9$). They also proved that there does exist a system of linear PDE's having a power series solution over $\mathbb{Q}$ but no computable power series solutions (i.e.: the coefficients are not given by a primitive recursive function). H. Wilf and D. Zeilberger gave an algorithm to test multisum and/or integral identities involving hypergeometric series (with several variables) (see [14]). But up to my knowledge nothing has yet been done concerning series defined by non linear systems of PDE's. Hence, in this paper, we shall deal with this problem. As a first approach, we will reduce the frame of our work to very specific systems of PDE's associated with initial conditions, but we think that our methods could be extended. These systems have two important particularities: a finite number of initial conditions at the origin suffices to define the solution, and they are not singular at the origin in a sense that shall be specified later. In section 2, we formulate our question in terms of differential algebra. We thus characterize a class of systems for which there exists a unique and computable m-tuple solution $f_1, \ldots, f_m \in k[[x_1, \ldots, x_n]]$. In section 3, we give an algorithm to test equality in the extension of $R = k[x_1, \ldots, x_n]$ defined by the f_i's, i.e. if P is a given differential polynomial, the algorithm tests if $P(f_1, \ldots, f_m) = 0$ as a series.

2 Preliminaries

2.1 Some Notations and Terminology

To study differential equations in a computable way, we shall use differential algebra, a generalization of commutative algebra to differential equations. In this section we recall its outlines. For a complete exposition, the reader is referred to [8], [7], and [6].

Throughout what follows, k will denote a fixed effective field of constants of characteristic zero, and we will consider $R = k[x_1, \ldots, x_n]$, ring of polynomials, equipped with the derivations ∂_{x_i}, that commute one with each other; thus R is a differential effective ring. Θ will denote the free monoid generated by the ∂_{x_i}'s; an element $\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$ of Θ will be noted either θ, or ∂_α when more precision is needed. For example $\partial_{(1,2)} = \partial_{x_1}\partial_{x_2}^2$). Consider the ring of polynomials $\bar{\mathcal{R}} = R[\theta y_i, i = 1, \ldots, m,\ \theta \in \Theta]$; we define an action of Θ on $\bar{\mathcal{R}}$ by $\theta(\theta' y_i) = (\theta\theta')y_i$, hence $\bar{\mathcal{R}}$ is a differential ring, which will be noted $\bar{\mathcal{R}} = R\{y_1, \ldots, y_m\}$, and called *ring of partial differential polynomials.* If I is an ideal in $\bar{\mathcal{R}}$, I is a *differential ideal* if it is stable under $\partial_{x_i}, i = 1, \ldots, n$. If $P_1, \ldots, P_t$ are in $\bar{\mathcal{R}}$, we shall either consider the *algebraic ideal* generated by $P_1, \ldots, P_t$ which will be noted $(P_1, \ldots, P_t)$ or the *differential ideal* generated by the P_i's which will be noted $[P_1, \ldots, P_t]$, and which is in fact the algebraic ideal generated by all the θP_i's for $\theta \in \Theta$.

In the sequel, we will assume that we have defined an *admissible ordering* on the set of derivatives $\Gamma = \{\theta y_i, i = 1, \ldots, m, \theta \in \Theta\}$, i.e. an ordering such that

$(u < v \Rightarrow \theta u < \theta v)$, and $v \leq \theta v$ for all $v \in \Gamma$ and all $\theta \in \Theta$. Such an ordering is a well ordering, i.e.: there is no infinite decreasing sequence in Γ. For the examples below, we choose the following order: if $\partial_\alpha, \partial_{\alpha'}$ are in Θ, we will say that $\mathrm{ord}(\partial_\alpha) = |\alpha| = \sum_{i=1,\ldots,n} \alpha_i$, $\partial_\alpha < \partial_{\alpha'}$ if $|\alpha| < |\alpha'|$ or $|\alpha| = |\alpha'|$ and $\alpha < \alpha'$ for the lexicographical order. This order on Θ induces an order on the $\partial_\alpha y_i$'s: $\partial_\alpha y_i < \partial_{\alpha'} y_{i'}$ if $\partial_\alpha < \partial_{\alpha'}$ or $\partial_\alpha = \partial_{\alpha'}$ and $i < i'$.

Let P be a partial differential polynomial in $\bar{\mathcal{R}}$, (hereafter: "p.d.p."), its *leader* is the highest derivative $\partial_\alpha y_i$ involved in P and will be denoted v_P, its *order* is $\mathrm{ord}(v_P) = |\alpha|$, its *degree* is $\deg(P, v_P)$ (i.e.: the degree of P with respect to v_P), its *initial* is $I_P = \dfrac{\partial P}{\partial(v_P^{deg(P)})}$ (or the coefficient of $(v_P)^{deg(P)}$) and its *separant* is $S_P = \dfrac{\partial P}{\partial v_P}$ which is also $I_{\theta P}$ for every operator $\theta \neq Id$. We also define $\mathrm{rank}(P) = (v_P, \deg(P))$, and we order the rank lexicographically. Notice that there does not exist any infinite sequence of p.d.p. of decreasing rank.

For example if $\bar{\mathcal{R}} = \mathbb{Q}[x_1, x_2]\{y_1, y_2\}$ and $P = x_1 x_2 y_1 (\partial_{(1,1)} y_2)^2 + y_2$, then $v_P = \partial_{(1,1)} y_2$, $\mathrm{ord}(P) = 2$, $\deg(P) = 2$, $I_P = x_1 x_2 y_1$, $S_P = 2 x_1 x_2 y_1 \partial_{(1,1)} y_2$, $\mathrm{rank}(P) = (\partial_{(1,1)} y_2, 2)$.

Let us now introduce the notion of *reduction*. If P, Q are two p.d.p.'s, then P is *reduced with respect to* Q if it contains no proper derivative of v_Q (i.e. no $\partial_\alpha v_Q$ with $|\alpha| > 0$) and $\deg(P, v_Q) < \deg(Q)$.

For example, the p.d.p. P written above is not reduced with respect to $Q = y_1^2 \partial_{(0,1)} y_2 + y_1$, because $v_P = \partial_{(1,0)} v_Q$.

We can pseudo-reduce one p.d.p. with respect to another: for all $P, Q \in \bar{\mathcal{R}}$ there exists a p.d.p. T reduced with respect to Q such that

$$(I_Q)^\nu (S_Q)^{\nu'} P = \sum M_i \theta_i Q + T,$$

where ν, ν' are positive integers and $M_1, \ldots, M_r$ are in $\bar{\mathcal{R}}$. We will note $P \xrightarrow{Q} T$. For the algorithm of reduction, see [8], page 165, or [7], page 77.

We will now show briefly, using the examples P and Q given above, how this algorithm works. The highest derivative of v_Q involved in P is $u = \partial_{x_1} v_Q$, and the degree of P with respect to u is 2. Let C be the coefficient of u^2 in P: $C = x_1 x_2 y_1$. Then $S_Q P - C u \partial_{x_1} Q = T_1$ is of degree at most 1 with respect to u, and no derivative of v_Q higher than u is involved in T_1. We shall say that we performed an elementary reduction of P with respect to Q. Proceeding repeatedly, we reduce until we get a remainder reduced with respect to Q. Here we would obtain $T = x_1 x_2 y_1 (\partial_{(1,0)} y_1)^2 + y_1^4 y_2$.

We will also use the notion of *partial reduction*: if P, Q are two p.d.p., then Q is *partially reduced with respect to* P if it contains no proper derivative of v_P. For example, the polynomial $P = x_1 x_2 y_1 (\partial_{(1,1)} y_2)^2 + y_2$ is partially reduced with respect to $Q' = \partial_{(1,1)} y_2$, but not reduced with respect to Q'. We can partially reduce one p.d.p. with respect to another, i.e. for all $P, Q \in \bar{\mathcal{R}}$ there exists a p.d.p. T partially reduced (but not necessarily reduced) with respect to Q such that

$$(S_Q)^\nu P = \sum_{M_i \in \mathcal{R}} M_i \theta_i Q + T, \text{ where } \nu \in \mathbb{N}.$$

The algorithm of partial reduction is very similar to that of reduction; the only difference is that you have less elementary reductions to perform.

Notice that we can also use this process in order to reduce a p.d.p. P with respect to a set $\mathcal{A} = \{A_1, \ldots, A_r\}$ of p.d.p.: we reduce P with respect to the A_i's until we get a remainder reduced with respect to all of them. But the result will depend on the way we proceeded; for example we could reduce P first with respect to A_1 and then with respect to A_2, or we could begin with A_2. We will note $P \xrightarrow{\mathcal{A}} T$, or $\mathrm{red}(P, \mathcal{A}) = T$.

2.2 Some Useful Properties of Auto-Reduced and Coherent Sets

Let us now define *auto-reduced and coherent sets* of polynomials, which are in some sense analogous to Groebner bases in commutative algebra (although the algebraic properties of auto-reduced coherent sets are much weaker).

Definition 1. A set $\mathcal{A} = \{A_1, \ldots, A_r\}$ of p.d.p. is *auto-reduced* (or a *chain* in [8], chap. 1) if each polynomial in $\mathcal{A}$ is reduced with respect to the others.
We adopt the standard notations: $H_{\mathcal{A}} = \prod_{i=1\ldots r} I_{A_i} S_{A_i}$, and $\mathcal{S}_{\mathcal{A}} = \prod_{i=1\ldots r} S_{A_i}$.

We now introduce an order on auto-reduced sets: let $\mathcal{Q} = \{Q_1, \ldots Q_r\}$ and $\mathcal{T} = \{T_1, \ldots T_s\}$ be two auto-reduced sets, with $\mathrm{rank}(Q_i) < \mathrm{rank}(Q_{i+1})$, and $\mathrm{rank}(T_j) < \mathrm{rank}(T_{j+1})$.

Then $\mathcal{Q} < \mathcal{T}$ if

• either there is $j \leq \min(r, s)$ such that $\mathrm{rank}(Q_i) = \mathrm{rank}(T_i)$ for $i < j$ and $\mathrm{rank}(Q_j) < \mathrm{rank}(T_j)$, or

• $r > s$ and $\mathrm{rank}(Q_i) = \mathrm{rank}(T_i)$ for $i \leq s$.

Example: If $\mathcal{Q} = \{Q_1 = (\partial_{x_2} y)^2 + y^2 - 1, Q_2 = (\partial_{x_1} 1y)^2 + y^2 - 1\}$, and $\mathcal{T} = \{T_1 = \partial^2_{x_2} y + y, T_2 = (\partial^2_{x_1} y) + y^2\}$, then $v_{Q_1} < v_{T_1}$, so $\mathrm{rank}(Q_1) < \mathrm{rank}(T_1)$, hence $\mathcal{Q} < \mathcal{T}$.

Further on, we shall use the following classic result:

Theorem 2. *There is no infinite decreasing sequence of auto-reduced sets in* $k\{y_1, \ldots y_m\}$.

Proof. See [8] or [7], chap. 1, page 81. □

This result is in fact the keystone of many methods in differential algebra: it can ensure termination of recursive algorithms (notice that rings of partial differential polynomials are not noetherian).

Let P, Q be two p.d.p. such that there exist θ, θ' with $v_{\theta P} = v_{\theta' Q}$ and let us choose the smallest possible θ, θ'. Then the *S-polynomial* of P and Q is defined by $\mathrm{SPol}(P, Q) = \frac{S_Q}{gcd(S_P, S_Q)} \theta P - \frac{S_P}{gcd(S_P, S_Q)} \theta' Q$. Notice that $v_{(\mathrm{SPol}(P,Q))} < v_{\theta P} = v_{\theta' Q}$.

Definition 3. If $\mathcal{A} = \{A_1, \ldots, A_r\}$ is such that $SPol(A_i, A_j) \xrightarrow{\mathcal{A}} 0$ then $\mathcal{A}$ is said to be *coherent.*

We will expose in section 2.3 the reason why we use coherent sets.

We now sketch very briefly an algorithm that will be used in section 3, but the reader can find more details in [1] or in [2].

Let us consider a set $\mathcal{L}$ of p.d.p.'s. In order to have more information about the solutions of the system defined by $\mathcal{L}$, we will compute an auto-reduced coherent set $\mathcal{M}$ of p.d.p.'s such that

- $[\mathcal{M}] \subset [\mathcal{L}]$
- $\ell \xrightarrow{\mathcal{M}} 0$, for all $\ell \in \mathcal{L}$.

(Recall that $[\mathcal{M}]$ denotes the differential ideal generated by $\mathcal{M}$.)

$\mathcal{M}$ will be called *an auto-reduced coherent set associated to* $\mathcal{L}$. Notice that such a set is not unique.

It is very easy to extract from $\mathcal{L}$ an autoreduced set of lowest possible rank (see [8] chapter 1, paragraph 5, page 5, characteristic set of a finite set), and we call such a set Extraction($\mathcal{L}$). The following is a rough version of the algorithm to compute $\mathcal{M}$, an autoreduced coherent set associated to a given set $\mathcal{L}$.

1. $L := \mathcal{L}$
2. $L_0 :=$ Extraction(L)
3. Compute $L_1 := \{\ell' = \text{red}(\ell, L_0)$ for $\ell \in L$ and $\ell' \neq 0\}$.
4. If $L_1 \neq \emptyset$, then $L := L \cup L_1$ and go back to step 2. Else go to step 5.
5. We have now an autoreduced set L_0 such that $[L_0] \subset [\mathcal{L}]$ and such that $\ell \xrightarrow{L_0} 0$, for all $\ell \in \mathcal{L}$.
   Compute $S(i,j) := \text{red}(\text{SPol}((\ell_i, \ell_j), L_0)$ for pairs of p.d.p. ℓ_i, ℓ_j in L_0 until $S(i,j) \neq 0$.
   If you find such a pair, then $L := \mathcal{L} \cup \{S(i,j)\} \cup L_0$ and go back to step 2.
   Otherwise, for all possible pairs ℓ_i, ℓ_j, we have $S(i,j) = 0$, hence L_0 is coherent and is the desired set. Return $\mathcal{M} = L_0$.

Theorem 2 garanties the termination of the algorithm because it computes a decreasing sequence of autoreduced sets (L_0). Little theoretical work has yet been done concerning the complexity of this algorithm, but it may well be comparable to that of Buchberger's algorithm. (See [11] for the complexity of a similar problem.)

2.3 Introduction of a Particular Class of Extension

Our aim is to be able to compute in particular differential extensions of $R = k[x_1, \ldots, x_n]$, and more precisely in extensions obtained by adjunction of formal power series $f_1, \ldots, f_m \in k[[x_1, \ldots x_n]]$. We consider only the case when the f_i's are uniquely defined by a system of partial differential equations, and by a finite set of initial conditions at the origin. We shall focus on a special class of systems. First, we give an example to enlighten the precise definitions we give hereafter.

$\partial_2((\partial_2 f)^2 + f^2 - 1) = 2(\partial_2 f)\partial_2^2 f + 2 f \partial_2 f = 0$

 $\Rightarrow \partial_2^2 f + f = 0$

Example 1. Consider the system:

$$\{A_1(f) = (\partial_{x_2} f)^2 + f^2 - 1 = 0, A_2(f) = (\partial_{x_1} f)^2 + f^2 - 1 = 0\},$$

associated with the initial conditions: $f(0) = 0, \partial_{x_1} f(0) = 1, \partial_{x_2} f(0) = 1$. This system has a unique solution in $\mathbb{Q}[[x_1, x_2]]$ (which is simply $f(x_1, x_2) = \sin(x_1 + x_2)$). Indeed, differentiating the partial differential equations, we obtain $\partial_{x_i}(f)(\partial^2_{x_i}(f) + f) = 0$. As $\partial_{x_i}(f)(0) \neq 0$, if f is a solution of the system, then of course $\partial_{x_i}(f) \neq 0$, and so we have $\partial^2_{x_i}(f) + f = 0$. This clearly gives us recurrence relations satisfied by the coefficients of an eventual solution f in $\mathbb{Q}[[x_1, x_2]]$. Notice that $\mathcal{A} = \{A_1(y), A_2(y)\}$ is an auto-reduced coherent set in $\bar{\mathcal{R}} = \mathbb{Q}[x_1, x_2]\{y\}$. The coherence of $\mathcal{A}$ ensures us that there is no contradiction between the different recurrence relations of the form $\theta(A_i)(f)(0) = 0$. E.g. to determine $\partial_{x_1}\partial_{x_2} f(0)$ we can either use $\partial_{x_1} A_2(f)$ or $\partial_{x_2} A_1(f)$. $\mathrm{SPol}(A_1, A_2) \xrightarrow{\mathcal{A}} 0$ simply expresses that the result will be the same. Hence the system associated with the initial conditions has a unique solution in $\mathbb{Q}[[x_1, x_2]]$, and it is possible to compute the coefficients of the series f up to any order.

This can be generalized as follows:

Definition 4. Let $\mathcal{A}$ be a subset of $\bar{\mathcal{R}}$. A derivative $\partial_\alpha y_i$ is said to be *under the stair of* $\mathcal{A}$ if it is not the leader of any θA where $\mathrm{ord}(\theta) > 0$, and $A \in \mathcal{A}$.

Definition 5. Let $\mathcal{A}$ be a subset of $\bar{\mathcal{R}}$. If the set of derivatives that are under the stair of $\mathcal{A}$ is finite, then $\mathcal{A}$ is said to be a *closed set.*

Remark. Closed sets provide us with "enough" recurrence relations on the coefficients of the series solution, so that we only need a finite set of initial conditions $\partial_\alpha(f_i)(0) = c_{(i,\alpha)}$, where $c_{(i,\alpha)} \in k$. Note that we had a closed set in example 1: only y is under the stair of $\mathcal{A}$.

Definition 6. A *complete system* is the given of a closed auto-reduced coherent set $\mathcal{A} = \{A_1, \ldots, A_r\} \subset \bar{\mathcal{R}} = R\{y_1, \ldots, y_m\}$ associated with a finite set of initial conditions of the form

$$I.C. = \{\partial_\alpha f_i(0) = c_{(i,\alpha)}, \text{ where } c_{(i,\alpha)} \in k \text{ and } \partial_\alpha y_i \text{ is under the stair of } \mathcal{A}\}.$$

And the set $I.C.$ does satisfy the following additional condition:

$$A(f)(0) = 0, S_A(f)(0) \neq 0 \text{ for all } A \in \mathcal{A}.$$

Example 1 was a complete system. Let us consider another simple example.

Example 2.

$$\begin{aligned} \mathcal{A} := \{\ & A_1 := \partial_{(1,1)}y - 2\partial_{(0,1)}y - 5\partial_{(0,2)}y, \\ & A_2 := \partial_{(2,0)}y - 2\partial_{(1,0)}y + 5y, \\ & A_3 := 5\partial_{(0,3)}y + 2\partial_{(0,2)}y + \partial_{(0,1)}y \quad \} \end{aligned}$$

is a closed auto-reduced coherent set. To have a complete system, we choose a set of initial conditions. For example

$$I.C. = \{f(0) = 1, \partial_{(1,0)}f(0) = 0, \partial_{(2,0)}f(0) = -5, \partial_{(0,1)}f(0) = 5,$$
$$\partial_{(1,1)}f(0) = 0, \partial_{(0,2)}f(0) = 2, \partial_{(0,3)}f(0) = -9/5\}$$

As all separants are in $\mathbb{Q}$, they don't vanish at zero under the substitutions $x_i = 0$, $\partial_\alpha y = c_\alpha$. To ensure that it is a complete system, we should also check that these initial conditions are compatible with the equations $A_i(f)(0) = 0$.

In fact, in this example, $\partial_{(1,1)}f(0), \partial_{(2,0)}f(0), \partial_{(0,3)}f(0)$ do not need to be assigned: their values are given by $A_i(f)(0) = 0, i = 1, 2, 3$, because the A_i's are of degree 1. If $\deg(A) > 1$ for some $A \in \mathcal{A}$, and $v_A = \partial_\alpha y$, then $\partial_\alpha(f)(0)$ needs to be assigned (as in example 1).

Theorem 7. *Let $(\mathcal{A}, I.C.)$ be a given complete system. There exists a unique and computable* m*-tuple of power series $f = (f_1, \ldots, f_m)$ where $f_i \in k[[x_1, \ldots, x_n]]$ such that: $A(f_1, \ldots, f_m) = 0$ for all $A \in \mathcal{A}$, and f satisfies the equations in I.C..*

Proof. This is in fact a well known result, see [5] , [9]. We shall give here only a sketch of the proof.

The problem is to derive, from the differential equations, recurrence relations on the coefficients of the series $f_i = \sum_{\alpha \in \mathbb{N}^n} \frac{u_{(i,\alpha)}}{\alpha!} x^\alpha$ (with the standard multi-index notation). We notice that the the p.d.p.'s θA (where $A \in \mathcal{A}$, and $\theta \in \Theta$ is such that $\mathrm{ord}(\theta) > 0$) are all quasi-linear, i.e.: linear with respect to their leader. The relations of the form $\theta A(f)(0) = (S_A v_{\theta A} + T)(f)(0) = 0$ give us recurrence relations upon the coefficients, if the separants do not vanish under the substitutions $x_i = 0$, $\partial_\alpha y_i = c_{(i,\alpha)}$. These recurrence relations give us the uniqueness of the solution, provided it does exist.

The problem is that we may have several ways of expressing a given coefficient $u_{(i,\alpha)}$ as a function of lower coefficients. Indeed, if $\partial_\alpha y_i = v_{\theta_j A_j} = v_{\theta_t A_t}$ where $j \neq t$ and θ_j and θ_t are two operators in Θ, then the equations $\theta_j A_j(f)(0) = 0$, and $\theta_t A_t(f)(0) = 0$ provide us with us two ways of expressing $u_{i,\alpha}$ as a function of lower coefficients. However, as $\mathcal{A}$ is a coherent set, these possible different expressions will not be contradictory. And this gives us the existence of a solution. □

We now give a short algorithm to compute a fixed coefficient $u_{(i,\alpha)}$ of a series given by a complete system.

1. If $\partial_\alpha y_i$ is under the stair of $\mathcal{A}$, then $u_{(i,\alpha)} := c_{(i,\alpha)}$.
2. If $\partial_\alpha y_i$ is not under the stair of $\mathcal{A}$, then find θA (where $\theta \in \Theta, A \in \mathcal{A}$) such that $\partial_\alpha y_i = v_{\theta A}$.
3. Using the equation $\theta A(f)(0) = 0$, write $u_{(i,\alpha)}$ as a rational function of lower coefficients.
4. Compute the coefficients involved in the expression obtained in 3.

Remark. This is a recursive algorithm (see step 4), and the reason why it does terminate is that there is no infinite decreasing sequence of derivatives.

Let us show how this works on example 2. We compute $u_{(0,4)}$, which is the coefficient of $\frac{x_2^4}{2!3!4!}$ in the series f. As $\partial_{(0,4)}y$ is not under the stair of $\mathcal{A}$, we go to step 2 of the algorithm. To compute $u_{(0,4)}$, we use the equation $\partial_{(0,1)}(A_3)(f)(0) = 0$, i.e. $5 * u_{(0,4)} + 2 * u_{(0,2)} + u_{(0,1)} = 0$, hence $u_{(0,4)} = -1/5(2 * u_{(0,2)} + u_{(0,1)}) = -9/5$.

3 The Algorithm

3.1 Preliminaries

In the sequel, $\bar{\mathcal{R}} = R\{y_1, \dots y_m\}$ and $f = (f_1, \dots, f_m)$ is a m-tuple of series defined by a given complete system $(\mathcal{A}, I.C.)$. Let $P_1, \dots, P_t$ be a given set of p.d.p. in $\bar{\mathcal{R}}$. We want to test whether $P_1(f) = \dots = P_t(f) = 0$.

If one of the P_j's is in R, by hypothesis (R is an effective ring) we know how to test $P_j(f) = 0$. So let us suppose that all the P_j's do indeed involve at least one of the indeterminates y_i. We first test whether $P_j(f)(0) = 0$ for $j = 1, \dots, t$. (That is, we test whether the constant coefficient of the series $P_j(f)$ is zero). If one of them is not zero, it is finished. Otherwise, we wish in fact to test whether f is a solution of the system $(\mathcal{A}, P_1, \dots, P_t)$. First of all, we compute another system, having 'more or less' the same solutions, but which will enable us to compute. More precisely, we shall compute an auto-reduced coherent set $\mathcal{B}$ associated to $(\mathcal{A}, P_1, \dots, P_t)$, (see section 2.2). Recall that it means:

1. $[\mathcal{B}] \subset [\mathcal{A}, P_1, \dots, P_t]$,
2. for all $A \in \mathcal{A}$,$A \xrightarrow{\mathcal{B}} 0$,
3. for all P_j, $P_j \xrightarrow{\mathcal{B}} 0$.

Condition 1. implies that if $P_j(f) = 0, i = 1, \dots, t$ then $\mathcal{B}(f) = 0$. Under which condition is the converse true? Conditions 2. and 3. imply that there exist positive integers ν_j, ν_j' such that

$$(H_{\mathcal{B}})^{\nu_j'} P_j = \sum M_\ell \theta_\ell B_\ell, \tag{1}$$

$$(H_{\mathcal{B}})^{\nu_j} A_j = \sum M_\ell \theta_\ell B_\ell, \tag{2}$$

where the M_ℓ's are in $R\{y_1, \dots, y_m\}$, and the θ_ℓ's are in Θ. So if $H_{\mathcal{B}}(f) \neq 0$ and $\mathcal{B}(f) = 0$, then $P_j(f) = 0, j = 1 \dots t$.

Example 3. We give here a trivial example. Consider f defined as in example 1. Suppose one wants to test whether $P(f) = \partial_{x_1}^2 f + f = 0$. Using $\partial_{x_1} A_1(f)(0) = 0$, $\partial_{x_1}^2 f(0)$ is easily computed $\partial_{x_1}^2 f(0)$ and one sees that $P(f)(0) = 0$. So, one computes an autoreduced coherent set $\mathcal{B}$ associated to $\{A_1, A_2, P\}$. In this case, as $P \xrightarrow{\mathcal{A}} 0$, it is clear that $\mathcal{B} = \mathcal{A}$. Now, recall that $S_{A_i}(f)(0) \neq 0, i = 1, 2$ and notice that the initials of the A_i's are in $\mathbb{Q}$, and hence don't vanish at zero. So $H_{\mathcal{B}}(f) \neq 0$ and $P(f) = 0$ if and only if $\mathcal{B}(f) = 0$, which is the true (because $\mathcal{B}(f) = \mathcal{A}(f) = 0$). So $P(f) = 0$.

However, it is not always so simple. We show in the subsections 3.2 and 3.3 that when $H_{\mathcal{B}}(f) \neq 0$ we are able to test whether $\mathcal{B}(f) = 0$, which is then equivalent to $P_1(f) = \cdots = P_t(f) = 0$.

And in section 3.4 , we shall explain how to use this result recursively to test $P_j(f) = 0, j = 1, \ldots, t$ in the general case. In fact, we shall apply the algorithm to the set of p.d.p. $\{P_1, \ldots, P_t, H_{\mathcal{B}}\}$, and this will eventually lead to the computation of an autoreduced coherent set of lower rank than $\mathcal{B}$. And as there is no infinite sequence of such sets of decreasing rank (see theorem 2), the procedure will stop.

3.2 Effective Test in a Regular Case

In this section, we suppose $H_{\mathcal{B}}(f)(0) \neq 0$, (as in example 3). (This is clearly very easy to test.) As we noticed in the previous section, the relation $P_1(f) = \ldots = P_t(f) = 0$ is then equivalent to $\mathcal{B}(f) = 0$.

If $\mathcal{B}$ contains a p.d.p. not involving any derivative $\partial_\alpha y_i$, then clearly $\mathcal{B}(f) \neq 0$. So let us suppose that all p.d.p. in $\mathcal{B}$ involve at least one of the differential indeterminates y_i. In fact $\mathcal{B}(f) = 0$ is equivalent to

$$\theta B(f)(0) = 0, \forall \theta \in \Theta, \forall B \in \mathcal{B}. \tag{3}$$

Indeed, $B(f) \in k[[x_1, \ldots, x_n]]$, so $B(f) = \sum_{\alpha \in \mathbb{N}^n} \frac{b_\alpha}{\alpha!} x^\alpha$, where $b_\alpha = \partial_\alpha B(f)(0) \in k$. As $B(f) = 0$ if and only if ($b_\alpha = 0, \forall \alpha \in \mathbb{N}^n$), it is equivalent to equation (3).

But as we know that $\mathcal{A}(f) = 0$, we will only have to check equation (3) for a finite number of $\theta \in \Theta$. To prove this, we need the following lemma:

Lemma 8. *For all θ in Θ, and every B in $\mathcal{B}$ there exist positive integers ν, ν' such that:*

$$(H_{\mathcal{B}})^\nu (S_{\mathcal{A}})^{\nu'} \theta B = \sum_{\theta_l \in \Theta, A_l \in \mathcal{A}} M_l \theta_l A_l + \sum_{\theta_l B_l \in \overline{\mathcal{B}}} M_l \theta_l B_l, \tag{4}$$

where $\overline{\mathcal{B}} = \{\theta B$ such that $v_{\theta B}$ is under the stair of $\mathcal{A}\}$ and the M_l's are in $\bar{\mathcal{R}}$.

Proof. Let us consider a fixed $\theta_0 B_{i_0}$, where $\theta_0 \in \Theta$, and $B_{i_0} \in \mathcal{B}$. We reduce partially $\theta_0 B_{i_0}$ with respect to $\mathcal{A}$ and we obtain

$$(S_{\mathcal{A}})^{\nu_1} \theta_0 B_{i_0} = \sum_{M \in \bar{\mathcal{R}}, A \in \mathcal{A}} M \theta A + T. \tag{5}$$

As $\mathcal{A} \xrightarrow{\mathcal{B}} 0$, $[\mathcal{A}] \subset [\mathcal{B}] : (H_{\mathcal{B}})^\infty$, where $[\mathcal{B}] : (H_{\mathcal{B}})^\infty$ denotes the differential ideal of all p.d.p. P such that there exists $\nu \in \mathbb{N}$ satisfying $(H_{\mathcal{B}})^\nu P \in [\mathcal{B}]$. (Indeed,

$\mathcal{A} \xrightarrow{\mathcal{B}} 0$ implies that $(H_{\mathcal{B}})^{\nu} A = \sum M_{\ell}\theta_{\ell}B_{\ell}$ where $\nu \in \mathbb{N}$ and the M_{ℓ}'s are in $\bar{\mathcal{R}}$.) And as $[\mathcal{B}] \subset [\mathcal{B}] : (H_{\mathcal{B}})^{\infty}$, it is clear that $T \in [\mathcal{B}] : (H_{\mathcal{B}})^{\infty}$.

Now we reduce T with respect to $\mathcal{B}$. So

$$(H_{\mathcal{B}})^{\nu_2} T = \sum_{B_{\ell} \in \mathcal{B}} M_{\ell}\theta_{\ell}B_{\ell} + T' = \Sigma_1 + T'. \tag{6}$$

Notice that T is partially reduced with respect to $\mathcal{A}$ and so it does only involve derivatives that are *under* the stair of $\mathcal{A}$. So, the sum Σ_1 only involves partial differential polynomials θB's where $v_{\theta B}$ is under the stair of $\mathcal{A}$. In other words:

$$(H_{\mathcal{B}})^{\nu_2} T = \sum_{\theta_{\ell} B_{\ell} \in \overline{\mathcal{B}}} M_{\ell}\theta_{\ell}B_{\ell} + T'. \tag{7}$$

Besides it is clear that $T' \in [\mathcal{B}] : (H_{\mathcal{B}})^{\infty}$. We shall note $(\mathcal{B}) : (H_{\mathcal{B}})^{\infty}$ the algebraic ideal of all p.d.p. P such that there exists a positive integer ν satisfying $(H_{\mathcal{B}})^{\nu} P \in (\mathcal{B})$ (recall that $(\mathcal{B})$ denotes the *algebraic* ideal generated by $\mathcal{B}$). Now we can use Rosenfeld's lemma (see [10], section I.2): as $T' \in [\mathcal{B}] : (H_{\mathcal{B}})^{\infty}$, T' is reduced with respect to $\mathcal{B}$ and $\mathcal{B}$ is autoreduced and coherent, Rosenfeld's lemma implies that $T' \in (\mathcal{B}) : (H_{\mathcal{B}})^{\infty}$. In other words:

$$(H_{\mathcal{B}})^{\nu_3} T' = \sum M_i B_i. \tag{8}$$

So using (5), (7), (8), we obtain the formula of the lemma. □

Remark. The notion of coherent set we defined in section 2.2 is not exactly the same as in [10], but if a set is coherent with respect to our definition, it is also coherent with respect to the definition given in [10], and so we can use Rosenfeld's lemma.

Now let us show that in order to check $\mathcal{B}(f) = 0, \forall B \in \mathcal{B}$, we only have to check $\theta B(f)(0) = 0$ for a finite number of $\theta \in \Theta$.

As we supposed $H_{\mathcal{B}}(f)(0) \neq 0$, and as $\theta A(f)(0) = 0$ for all $\theta \in \Theta$, and for all $A \in \mathcal{A}$, lemma 8 implies that the values of all the $\theta B(f)(0)$ depend only on

$$C_{\mathcal{B}} = \{\theta B(f)(0), \text{ where } \theta B \in \overline{\mathcal{B}}\}.$$

More precisely, $\mathcal{B}(f) = 0$ if and only if all elements in $C_{\mathcal{B}}$ are zero. So we compute $C_{\mathcal{B}}$. If we get a set containing only zero then $P_1(f) = \ldots = P_t(f) = 0$. If one element in $C_{\mathcal{B}}$ is not zero, then $\mathcal{B}(f) \neq 0$, which means that f is not a solution of $(\mathcal{A}, P_1, \ldots, P_t)$. In this case, there is at least one i_0 such that $P_{i_0}(f) \neq 0$. In order to find it, we can compute successively the sets $\mathrm{Coeff}_{\theta} = \{\theta P_1(f)(0), \ldots, \theta P_t(f)(0)\}$, for increasing θ until we find $P_{i_0}(f)(0) \neq 0$, for some $i_0 \in 1 \ldots k$.

Example 4. Let us consider again $f(x_1,x_2) := sin(x_1+x_2)$, but this time defining it as the solution of the following system: $A_1(f) = \partial^2_{x_2} f + f = 0, A_2(f) = \partial^2_{x_1} f + f = 0$, associated with the initial conditions $f(0) = 0, \partial_{x_1} f(0) = 1, \partial_{x_2} f(0) = 1, \partial_{x_2}\partial_{x_1} f(0) = 0, \partial^2_{x_1} f(0) = 0, \partial^2_{x_2} f(0) = 0$. We shall check that $P(f) = (\partial_{x_2} f)^2 + f^2 - 1 = 0$. We have $P(f)(0) = 0$ and so we compute an autoreduced coherent set $\mathcal{B}$ associated to $\{A_1, A_2, P\}$, in $\mathbb{Q}[x_1,x_2]\{y\}$. We find $\mathcal{B} := \{B_1 = P, B_2 = \partial_{x_1} y^2 + y^2 - 1\}$. Fortunately, $H_{\mathcal{B}}(f) = 4\partial_{x_1} f \partial_{x_2} f$, therefore $H_{\mathcal{B}}(f)(0) \neq 0$. And so we can apply the result of this section: we only have to test $\bar{\mathcal{B}}(f)(0) = 0$ where $\bar{\mathcal{B}} = \{B_1, B_2, \partial_{x_1} B1, \partial_{x_2} B1, \partial_{x_1} B2, \partial_{x_2} B2\}$.

3.3 Effective Test in a Semi-Singular Case

In this section, we suppose that $H_{\mathcal{B}}(f) \neq 0$ but $H_{\mathcal{B}}(f)(0) = 0$. So, again in this case, $P_1(f) = \ldots = P_t(f) = 0$ is equivalent to $\mathcal{B}(f) = 0$ (see section 3.1).

We will show that the initial conditions, considered as a vector, belong to the closure of a semi-algebraic set, to be described more precisely in the next paragraph. (This semi-algebraic set is made of vectors of initial conditions corresponding to the case dealt with in the previous section.) We therefore only have to treat a problem of commutative algebra.

We now introduce a few more notations. Let N be the number of initial conditions. Note that N is equal to the number of derivatives $\partial_\alpha y_i$ that are under the stair of $\mathcal{A}$. An element $c = (0_{k^n}, c_{(i,\alpha)})$ of k^{n+N} will be called *vector of initial conditions* if the corresponding set $I.C.$ of initial conditions is such that $(\mathcal{A}, I.C.)$ is a complete system ($I.C. = \{\partial_\alpha f_i(0) = c_{(i,\alpha)}$, where $\partial_\alpha y_i$ is under the stair of $\mathcal{A}\}$, see section 2.3, definition 4). We call $\bar{\mathcal{R}}^\star$ the ring of polynomials in $n+N$ indeterminates $\bar{\mathcal{R}}^\star = k[x_1, \ldots, x_n, Y_{(i,\alpha)}$, where $\partial_\alpha y_i$ is under the stair of $\mathcal{A}]$. If P is a p.d.p. of $\bar{\mathcal{R}}$ involving only derivatives that are under the stair of $\mathcal{A}$, we shall note $P^\star$ the element of $\bar{\mathcal{R}}^\star$ obtained by substituting $Y_{(i,\alpha)}$ to $\partial_\alpha y_i$.

Let c be a vector of initial conditions, and $f = (f_1, \ldots, f_m)$ the m-tuple of series solution of $(\mathcal{A}, I.C.)$: $f_i = \sum_{\alpha \in \mathbb{N}^n} \frac{u_{(i,\alpha)}}{\alpha!} x^\alpha$. Using the algorithm given at the end of section 2.3, we can easily prove that $u_{(i,\alpha)} = \left(\frac{U_{(i,\alpha)}}{(\mathcal{S}^\star_{\mathcal{A}})^\nu}\right)(c)$ where $U_{(i,\alpha)} \in \bar{\mathcal{R}}^\star$ and ν is a positive integer. So for all $P \in \bar{\mathcal{R}}$, $P(f) \in k[[x_1, \ldots, x_n]]$ and $P(f) = \sum_{\alpha \in \mathbb{N}^n} p_\alpha(c) \frac{x^\alpha}{\alpha!}$, where $p_\alpha \in \bar{\mathcal{R}}^\star : (\mathcal{S}^\star_{\mathcal{A}})^\infty$. From this we deduce that $\partial_\alpha P(f)(0) = p_\alpha(c)$ continuously depends on c, for all $\alpha \in \mathbb{N}^n$.

Let W be the variety of k^{n+N} defined by the ideal $I^\star$ of $\bar{\mathcal{R}}^\star$ generated by the set of polynomials $\{(\theta B)^\star, \theta B \in \bar{\mathcal{B}}\}$ and let W' be the variety defined by $H^\star_{\mathcal{B}} = 0$. In the previous section, we showed that for each vector of initial conditions c we have the implication

$$c \in W - W' \Rightarrow \mathcal{B}(f) = 0. \tag{9}$$

In other words:

$$c \in W - W' \Rightarrow \forall \theta \in \Theta, \theta B(f)(0) = 0. \tag{10}$$

As $\theta B(f)(0)$ continuously depends on c for all θ, from (10) we deduce that

$$c \in \overline{W - W'} \Rightarrow \forall \theta \in \Theta, \theta B(f)(0) = 0. \tag{11}$$

Here the Zarisky and the metric closure coincide. We shall show a partial converse, namely:

Lemma 9. *Let c be a vector of initial conditions, such that $H_{\mathcal{B}}(f) \neq 0$ then*

$$\mathcal{B}(f) = 0 \Rightarrow c \in \overline{W - W'}. \tag{12}$$

Proof. Let us prove it by abstract nonsense. Consider a vector of initial conditions c such that $\mathcal{B}(f) = 0, H_{\mathcal{B}}(f) \neq 0$ and assume $c \notin \overline{W - W'}$. This means that there is a polynomial $Q^\star$ in $\sqrt{I^\star : (H^\star_{\mathcal{B}})^\infty}$ such that $Q^\star(c) \neq 0$. In other words, there exist $\nu_1, \nu_2 \in \mathbb{N}$ such that $(H^\star_{\mathcal{B}})^{\nu_1}(Q^\star)^{\nu_2} \in I^\star$. So

$$(H^\star_{\mathcal{B}})^{\nu_1}(Q^\star)^{\nu_2} = \sum_{\theta B \in \overline{\mathcal{B}}, M^\star \in \bar{\mathcal{R}}^\star} M^\star(\theta B)^\star. \tag{13}$$

To each polynomial $P^\star$ of $\bar{\mathcal{R}}^\star$ we can associate $P \in \bar{\mathcal{R}}$ by substituting $\partial_\alpha y_i$ to $Y_{(i,\alpha)}$. The equation (13) then becomes:

$$(H_{\mathcal{B}})^{\nu_1} Q^{\nu_2} = \sum_{\theta B \in \overline{\mathcal{B}}} M(\theta B). \tag{14}$$

We assumed that $Q(f)(0) = Q^\star(c) \neq 0$ (hence $Q(f) \neq 0$), $H_{\mathcal{B}}(f) \neq 0$ and $\mathcal{B}(f) = 0$ (hence $\theta B(f) = 0$ for all $\theta B \in \overline{\mathcal{B}}$), which is not compatible with equation (14), and this proves the lemma. □

Now let us come back to our problem: assume we know that $H_{\mathcal{B}}(f) \neq 0$, but we cannot apply the previous section because $H_{\mathcal{B}}(f)(0) = 0$. Testing $P_1(f) = \ldots = P_t(f) = 0$ is equivalent to testing whether $c \in \overline{W - W'}$, which is equivalent to $c \in V(I^\star : (H^\star_{\mathcal{B}})^\infty)$, or in other words: $P^\star(c) = 0, \forall P^\star \in I^\star : (H^\star_{\mathcal{B}})^\infty$. In order to do this we compute a Groebner basis of $I^\star : (H^\star_{\mathcal{B}})^\infty$. Here, we explain briefly how to proceed, but for more detail the reader can refer to [1], theorem 3, p. 57. We compute the set of polynomials generating $I^\star$, i.e. $\chi = \{(\theta B)^\star$ where $\theta B \in \overline{\mathcal{B}}\}$. Now compute in $\bar{\mathcal{R}}^\star[z]$ a Groebner basis of the ideal generated by $\chi \cup \{zH^\star_{\mathcal{B}} - 1\}$ with respect to a lexicographical order where $z > Y_{(i,\alpha)}$. Take in this Groebner basis the polynomials which do not involve the indeterminate z, and you have a basis G of $I^\star : (H^\star_{\mathcal{B}})^\infty$ (in $\bar{\mathcal{R}}^\star$). Now $c \in \overline{W - W'}$ if and only if $g(c) = 0$ for all $g \in G$. Now either we know that $P_i(f) = 0$ for all $i \in 1, \ldots, t$, or we know that $P_{i_0}(f) \neq 0$ for at least one $i_0 \in 1, \ldots, t$. In this case, to find i_0 we proceed as we did at the end of section 3.2.

3.4 Algorithm in the General Case

We shall give a recursive algorithm.

Let $P_1, \ldots, P_t$ be a set of p.d.p. in $k\{y_1, \ldots y_m\}$. If $P_{i_0}(f)(0) \neq 0$ for some $i_0 \in 1, \ldots, t$, there is nothing to do. Otherwise, we compute an auto-reduced coherent set $\mathcal{B} = \{B_1, \ldots, B_s\}$ associated to $(\mathcal{A}, P_1, \ldots, P_t)$. As we noticed in section 3.1, if $H_{\mathcal{B}}(f) \neq 0$, testing $P_1(f) = \cdots = P_t(f) = 0$ is equivalent to testing $\mathcal{B}(f) = 0$. And we showed how to proceed in this case in sections 3.2 and 3.3. So we first compute $H_{\mathcal{B}}(f)(0)$ (we only have to substitute the initial conditions to the indeterminates in $H_{\mathcal{B}}$). If it is not zero, we apply the result of section 3.2. Otherwise, we shall proceed recursively, testing successively $\Sigma := \{P_1(f) = \cdots = P_t(f) = Q(f) = 0\}$ where Q is the initial or the separant of some $B \in \mathcal{B}$. Indeed, if $Q(f)(0) = 0$, we will compute an auto-reduced coherent set $\mathcal{C}$ associated to $(\mathcal{A}, P_1, \ldots, P_t, Q)$. And as Q is of lower rank than all the B's in $\mathcal{B}$, it is clear that $\mathcal{C}$ will be lower than $\mathcal{B}$. Hence theorem 2 will ensure that this recursive procedure will stop. We now describe the algorithm more precisely.

1. Compute $P_i(f)(0)$ for $i = 1, \ldots, t$. If one of them is not zero, then return $P_{i_0}(f) \neq 0$.
2. Else compute an auto-reduced coherent set $\mathcal{B}$ associated to $(\mathcal{A}, P_1, \ldots, P_t)$. Compute $H_{\mathcal{B}}(f)(0)$. If it is not zero, just apply section 3.2.
3. Else call $Q_1, \ldots, Q_{2s}$ the initial and separants of the B's in $\mathcal{B}$. Start with $i = 1$ and go to step 4.
4. Apply the algorithm to the set of polynomials $\{P_1, \ldots, P_t, Q_i\}$. We can get now three possible different answers: either $P_1(f) = \cdots = P_t(f) = Q_i(f) = 0$, or $\exists i_0$ such that $P_{i_0}(f) \neq 0$, or $Q_i(f) \neq 0$. In the two first cases, return the result. If $Q_i(f) \neq 0$, and $i < 2s$, then start step 4 again with $i + 1$. If $i = 2s$, then go to step 5.
5. At this stage, we have $H_{\mathcal{B}}(f) \neq 0$, but $H_{\mathcal{B}}(f)(0) = 0$ and we just apply section 3.3

Let us now study a very simple example.

Example 5. Let us consider $f(x_1, x_2) := cos(x_1 + x_2)$: it is the solution of the system $A_1(f) = \partial^2_{x_2} f + f = 0, A_2(f) = \partial^2_{x_1} f + f = 0$, associated with the initial conditions $f(0) = 1, \partial_{x_1} f(0) = 0, \partial_{x_2} f(0) = 0, \partial_{x_2}\partial_{x_1} f(0) = -1, \partial^2_{x_1} f(0) = -1, \partial^2_{x_2} f(0) = -1$. (It is clear that $\{A_1(F), A_2(F)\}$ is an auto-reduced coherent set in $Q\{F\}$) We shall check that $P(f) = (\partial_{x_2} f)^2 + f^2 - 1 = 0$. We have $P(f)(0) = 0$ and so we go to step 2 of the algorithm.

Then, computing in $\mathbb{Q}[x_1, x_2]\{y\}$ an auto-reduced coherent set associated to (A_1, A_2, P), we find $\mathcal{B} := \{B_1 = P, B_2 = \partial_{x_1} y^2 + y^2 - 1\}$. Unfortunately, $H_{\mathcal{B}}(f) = 4\partial_{x_1} f \partial_{x_2} f$ hence $H_{\mathcal{B}}(f)(0) = 0$. So we have to proceed recursively to test if $H_{\mathcal{B}}(f) = 0$. We therefore go to step 3.

We call $Q_1 = S_{B_1}$, $Q_2 = S_{B_2}$, $Q_3 = I_{B_1}$, $Q_4 = I_{B_2}$. We start with $i = 1$ and we go to step 4.

$Q_1(f)(0) = S_{B_1}(f)(0) = 0$ so we compute an auto-reduced coherent set associated to (A_1, A_2, P, Q_1), and we obtain only a constant non zero polynomial.

So either $P(f) \neq 0$ or $Q_1(f) = S_{B_1}(f) \neq 0$. Computing successively the coefficients of the series $P(f)$ and $S_{B_1}(f)$, we find that $\partial_{x_2} S_{B_1}(f)(0) = -1$, hence $S_{B_1}(f) \neq 0$. Now we take $i = 2$ and we start step 4 again.

$Q_2(f)(0) = S_{B_2}(f)(0) = 0$ so we compute an auto-reduced coherent set associated to (A_1, A_2, P, S_{B_2}), and again we obtain only a constant non zero polynomial, and $\partial_{x_1} S_{B_2}(f)(0) = -1$. Now we take $i = 3$ and start step 4 again.

As $Q_3(f)(0) = I_{B_1}(f)(0) = 1$, so take $i = 4$ and start step 4 again.

As $Q_3(f)(0) = I_{B_2}(f)(0) = 1$, so $Q_4(f) \neq 0$ and we go to step 5.

We now know that $H_{\mathcal{B}}(f) \neq 0$, and we can apply the result of section 3.3. We shall check whether the vector of initial conditions belongs to $\overline{W - W'}$ (see section 3.3). This means that we have to check that

$$P^\star(c) = 0 \text{ for all } P^\star \in I^\star : (H_{\mathcal{B}}^\star)^\infty.$$

In order to do this, we compute the set $\chi = \{(\theta B)^\star \text{ for } \theta B \in \overline{\mathcal{B}}\}$ in $\bar{\mathcal{R}}^\star = \mathbb{Q}[x_1, x_2, Y_{(0,0)}, Y_{(1,0)}, Y_{(0,1)}, Y_{(1,1)}, Y_{(2,0)}, Y_{(0,2)}]$. For example $(B1)^\star = Y_{(0,1)}^2 + Y_{(0,0)}^2 - 1$. Here $\chi = \{(B1)^\star, (\partial_{x_1} B1)^\star, (\partial_{x_2} B1)^\star, (B2)^\star, (\partial_{x_1} B2)^\star, (\partial_{x_2} B2)^\star\}$. We compute in $\bar{\mathcal{R}}^\star[Z]$ a Groebner basis G_1 of the ideal generated by $\chi \cup \{ZH_{\mathcal{B}}^\star - 1\}$, (where $ZH_{\mathcal{B}}^\star - 1 = 4ZY_{(0,1)}Y_{(1,0)} - 1$). (This computation performed with Maple lasted 2 seconds.) Then we take in G_1 only the equations not involving Z, and we obtain a Groebner basis G_2 of $I^\star : (H_{\mathcal{B}}^\star)^\infty$ (in $\bar{\mathcal{R}}^\star$). Now to check that $P^\star(c) = 0$ for all $P^\star \in I^\star : (H_{\mathcal{B}}^\star)^\infty$, we only have to plug the initial conditions in the polynomials of G_2.

Remark. Notice that this algorithm also applies to the ordinary differential case. It will even be much simpler because, as there are no S-polynomials in this case, we will only have to compute auto-reduced sets. It does not reduce to any of the known algorithms mentioned in the introduction. The method exposed in [4] is more general: J. Denef and L. Lipshitz consider a wider type of extension, but their algorithm may suffer from too high a complexity. However, the ordinary differential case will probably be studied in a forthcoming paper.

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