Semantical Analysis of Intuitionistic Logic I by Saul A. Kripke; J. N. Crossley; M. A. E. Dummett
Review by: G. Kreisel
The Journal of Symbolic Logic, Vol. 35, No. 2 (Jun., 1970), pp. 330-332
Published by: Association for Symbolic Logic
Stable URL: http://www.jstor.org/stable/2270547
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stages: (A) Transform Φ into the alternations of all its prime implicants and (B) delete from that alternation the largest possible combination of jointly superfluous clauses.

The author proves four theorems which justify a procedure of Samson and Mills for performing stage (A). He then quotes Ghazala, who has developed systematic methods for performing stage (B).

A normal formula Φ is called irredundant if no formula obtained from Φ by dropping either a clause or a literal of it is equivalent to Φ. The alternation of those clauses of Φ which appear in every irredundant equivalent of Φ is called the core of Φ. The significance of this concept for the problem of the shortest normal form is discussed and a simple test is given which decides whether a clause of an irredundant formula belongs to its core. S. RUDEANU

F. I. ANDON. *Ob odnom podhodé k minimizacii sistém bulévyh funkcij* (*On one approach to the minimization of systems of Boolean functions*). ***Kibérnétika*** (Kiev), no. 5 (1966), pp. 44–48.

F. I. ANDON. *Algoritm uproščéniá d.n.f. bulévyh funckij* (*A simplification alogrithm of a disjunctive normal form of the Boolean functions*). Ibid., no. 6 (1966), pp. 12–14.

The two reviewed papers generalize some results of an earlier article of Žuravlév (XXXV 162), and the technique is topological, like that of Žuravlév's paper.

Let us start with a finite set $\{\mathfrak{A}_1, \mathfrak{A}_2, \cdots, \mathfrak{A}_k\}$ of disjunctive normal forms (d.n.f.'s) $\mathfrak{A}_i$, and denote by f_i the function expressed by $\mathfrak{A}_i$ ($1 \leqq i \leqq k$). The papers define the complexity number of $\{\mathfrak{A}_1, \cdots, \mathfrak{A}_k\}$, the implicants and prime implicants of the system $\{f_1, \cdots, f_k\}$, and the irredundant and minimal d.n.f.'s of an f_i with respect to $\{f_1, \cdots, f_k\}$. The aim is to determine a system $\{\mathfrak{A}'_1, \cdots, \mathfrak{A}'_k\}$ of d.n.f.'s such that each $\mathfrak{A}'_i$ expresses f_i, and such that the system has the least possible complexity. This can be done in three phases: (1) For any i between 1 and k, all the irredundant d.n.f.'s of f_i with respect to $\{f_1, \cdots, f_k\}$ are determined; (2) the union of the sets of d.n.f.'s, obtained in the previous phase, is formed; and (3) the d.n.f.'s of minimal complexity are selected.

The main results of the papers are concerned with performing the three phases in a straightforward manner. They state sufficient conditions for an elementary conjunction to be contained in every minimal (or irredundant) d.n.f. of an f_i (with respect to $\{f_1, \cdots, f_k\}$), or in none of them. Á. ÁDÁM

PAUL LORENZEN. ***Einführung in die operative Logik und Mathematik.*** Second edition of XXII 289, with minor revisions. Die Grundlehren der mathematischen Wissenschaften, vol. 78. Springer-Verlag, Berlin-Heidelberg-New York 1969, VIII + 298 pp.

The author's purpose in preparing this new edition of XXII 289 was to bring its terminology and notation into agreement with that of his more recent publications. Several minor corrections and changes in wording have also been made, and the bibliography has been brought up to date. The content of the book is essentially unchanged. PERRY SMITH

SAUL A. KRIPKE. *Semantical analysis of intuitionistic logic I.* ***Formal systems and recursive functions, Proceedings of the Eighth Logic Colloquium, Oxford, July 1963,*** edited by J. N. Crossley and M. A. E. Dummett, Series in logic and the foundations of mathematics, North-Holland Publishing Company, Amsterdam 1965, pp. 92–130.

The present paper follows the author's interpretation of the logical operations in modal logic (XXXI 120). He establishes the completeness of the usual intuitionistic predicate calculus HPC for his interpretation and relations of the latter to Beth's interpretation (XXII 363) and thus Mostowski's (XIV 137) and to P. J. Cohen's work on forcing. To avoid misunderstanding, some distinctions should be kept in mind which are basic for the correct assessment of (many) interpretations of HPC. First, "semantical" in the title means *set-theoretic semantical*; after all, Heyting himself gave a perfectly good semantic for HPC, that is, a meaning in terms of the basic notions of constructive function and proof. Secondly there are two kinds of relations between different interpretations. One, involved in completeness proofs, is elementary equivalence between possibly non-isomorphic structures; the other, which holds between the author's and Beth's interpretations (p. 124), consists in isomorphism results. In the present context, two isomorphic structures are by no means of equal value; philosophically, if one interpretation is intended, in which case the discovery of an isomorphism between it and a

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more familiar structure is a principal contribution; practically, if one of the structures is easier to handle than the other. It may fairly be said that the author's interpretation of HPC is as easy to handle as any in the literature (and as natural, in the sense that it has appealed to several people). Specifically, it has led to important new results on HPC; the author promises (p. 92) a proof of the recursive undecidability of the monadic fragment of HPC in Part II of the present paper.

Concerning its relation to other interpretations the author says on page 120 apropos of Cohen's notion of forcing that "the 'deeper' reasons ... may yet be unknown," but fails to mention the quite evident superficial reasons. Whenever mathematicians speak of "constructions" (even if only some kind of explicit definability is meant) their pseudo-constructions, for instance in model theory or set theory, may be expected to share some formal properties with constructions in the strict intuitionistic sense, for instance the properties expressed in HPC. In any case, the notion of forcing is nothing recondite, in the literal sense that it was the first thing that occurred to several people who played with "re-interpretations" of HPC. What merits analysis are the conditions when such elementary properties of this notion genuinely help one to define delicate models of set theory which solve specific independence problems in this subject. Put differently, it was remarkable how successfully Cohen used the (quite weak) analogy between constructions in the strict sense and invariance properties of definitions in the ramified hierarchy.

A good deal of the present paper is intended to connect Heyting's own interpretation with the author's, particularly with the version on pages 98–99 in terms of "stages of evidence" (corresponding to Heyting's proof), which he calls the "intuitive interpretation." In accordance with the superficial reasons given above, a formula valid in the author's sense is rather obviously valid for Heyting's, a fact which follows indirectly from completeness of HPC for the author's interpretation and its soundness for Heyting's. The converse is dubious because (some of) the author's counter models allowed on pages 98–99 picture an essentially more elementary process of treating "evidential situations" than allowed in intuitionistic mathematics. Specifically, the author considers ω-series (in time) of stages of evidence while, at least occasionally, Brouwer considered fully analyzed proofs with a transfinite number of steps. Also, and perhaps more important, the author's models do not bring in explicitly reflection on the process of reflection itself.

But, as the author realizes of course, a much more decisive difference between his interpretation and Heyting's would follow from incompleteness of HPC for Heyting's interpretation, since HPC is complete for the author's. He therefore considers results concerning this matter in the reviewer's XXXIV 119 but not very carefully. Thus since some considerations on the "intuitive interpretation" are formulated by use of the reviewer's system FC (XXXII 283) for lawless sequences the author asks on page 105, line 9 whether the incompleteness of HPC can be proved in FC. But, taken literally, the question cannot even be formulated in FC! Presumably he means completeness with respect to a class of domains and predicates definable in FC (containing parameters, as in Remark 1.1 on page 320 of the reviewer's XXXII 282). But then Theorem 3 of the reviewer's XXXIV 119, to which the author refers in somewhat different terms, does in fact give the following incompleteness result of HPC with respect to a non-derivable formula A expressing that all paths of a certain primitive recursive tree T_A are finite. Let A have the predicate symbols P_i $(1 \leqq i \leqq n)$ and let D^j and P_i^j $(1 \leqq i \leqq n; j = 1, 2, \ldots)$ be an enumeration of all domains and of all n-tuples of predicates (with the right number of arguments) definable in FC; parameters α are allowed. Let A^j be obtained from A by letting the individual variables range over D^j and replacing P_i by P_i^j. Then for no j is $[(\alpha)A^j \rightarrow \vdash(A)]$ provable in FC, where $\vdash$ means formal derivability in HPC. The steps are as follows. By Theorem 4 on page 377 of XXXII 283 there is a purely arithmetic B^j such that $B^j \leftrightarrow (\alpha)A^j$ is provable in FC. Each B^j is recursively realizable, essentially because all recursive paths of T_A are finite. Since FC is a conservative extension of Heyting's arithmetic, which is recursively realizable, and $[B^j \rightarrow \vdash(A)]$ is not, we have the result. (The conservative extension result was first published in the present paper; cf. page 118.) The author's *classical* completeness proof provides a j_0 for which $\neg B^{j_0}$ is classically provable, and so is $(B^{j_0} \rightarrow \vdash A)$; in other words the non-constructive character of the completeness result is not connected with delicate questions about choice

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sequences, but is reduced to a purely arithmetic problem. Obviously the negative result is not wholly satisfactory since it only asserts formal underivability; also because of the restriction to domains and predicates definable in FC. Of course for a (positive) completeness result such a restriction would be an advantage, as emphasized bottom of page 318 of XXXII 282. In this connection it should, perhaps, be mentioned that people are often tempted to define validity of A (in FC) by interpreting the predicate variables of A as lawless predicates and letting individual variables range over the natural numbers. This is meaningful (but has little to do with validity in Heyting's sense): HPC is evidently incomplete since $(x)(Px \vee \neg Px)$ is valid here.

A minor, but possibly disturbing, weakness of the present paper is its imprecise use of "Markov's principle," e.g. on pages 93, 104 or 130; the author applies it to

$$(\alpha)[\neg\neg(Ex)(\alpha x = 0) \rightarrow (Ex)(\alpha x = 0)]$$

for lawless α while Markov is concerned with recursive functions, not lawless sequences. Incidentally, what the author calls "Markov's principle" is already refuted in Remark 7.4 on page 382 of XXXII 283. G. KREISEL

LUITZEN EGBERTUS JAN BROUWER. *On the significance of the principle of excluded middle in mathematics, especially in function theory.* English translation of 155*16* by Stefan Bauer-Mengelberg and Jean van Heijenoort. ***From Frege to Gödel, A source book in mathematical logic, 1879–1931,*** edited by Jean van Heijenoort, Harvard University Press, Cambridge, Massachusetts, 1967, pp. 334–341. *Addenda and corrigenda.* English translation of XXIV 189(12) by Stefan Bauer-Mengelberg, Claske M. Berndes Franck, Dirk van Dalen, and Jean van Heijenoort. Ibid., pp. 341–342. *Further addenda and corrigenda.* English translation of XXIV 189(13) by Stefan Bauer-Mengelberg, Dirk van Dalen, and Jean van Heijenoort. Ibid., pp. 342–345.

LUITZEN EGBERTUS JAN BROUWER. *On the domains of definition of functions.* Ibid., pp. 446–463. English translation of §§1–3 of *Über Definitionsbereiche von Funktionen* (***Mathematische Annalen,*** vol. 97 (1926–27), pp. 60–75) by Stefan Bauer-Mengelberg.

LUITZEN EGBERTUS JAN BROUWER. *Intuitionistic reflections on formalism.* English translation of §1 of 155*20* by Stefan Bauer-Mengelberg. Ibid., pp. 490–492.

These selections from Brouwer's writings, together with Charles Parsons' very scholarly introduction to *On the domains of definition of functions*, are welcome additions to English-language intuitionistic literature. The choice of papers (based partly on inaccessibility of the originals) is excellent, and all translations read smoothly.

In the first paper Brouwer speculates about the historical reasons for the uncritical acceptance of the law of the excluded middle for infinite systems, and traces some of the consequences, for function theory, of its rejection. Among the latter are the loss of the linear ordering of the continuum, the failure of the Bolzano-Weierstrass theorem and (a form of) the Heine-Borel covering theorem, the existence of a real-valued continuous function defined everywhere on the closed unit interval but possessing no maximum, and an intuitionistic splitting of the notion "convergent" (for sequences or series) into three classically equivalent notions ("non-oscillating," "negatively convergent," "positively convergent"). The examples are based on the familiar partial recursive function $k(n)$ = [the number of places after the decimal point at which the nth sequence 0123456789 in the decimal expansion of π begins], whose domain is not known to be either empty or non-empty.

The first addendum (published thirty years later) remarks that in intuitionistic mathematics one obtains contradictions by applying the law of the excluded middle simultaneously to each element of an infinite species, and that the Heine-Borel theorem holds intuitionistically when "closed" is defined as "containing all accumulation points" (but not if "accumulation" is replaced by "limit"). The second gives a new example of a continuous, monotonic, nowhere-differentiable function on the closed unit interval, and by a clever time-lag argument exposes the inadequacy of the example given in the main paper.

In his introduction to the next paper, Parsons gives many essential definitions culled from

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