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Which finite monoids are syntactic monoids of rational ω-languages

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Abstract

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A notion of ω-rigid sets for a finite monoid is introduced. We prove that a finite monoid M is the Arnold's syntactic monoid of some rational ω-language (ω-syntactic for short) if and only if there exists an ω-rigid set for M. This property is shown to be decidable for the finite monoids. Relationship between the family of ω-syntactic monoids and that of $*$-syntactic monoids (i.e. the syntactic monoids of rational languages of finite words) is established.

Keywords: Formal languages, ω-languages, syntactic monoid

0. Introduction

Let M be a monoid and X be a subset of M. We denote by $\equiv_X$ the congruence on M defined by

$$m \equiv_X m' \quad \text{iff} \quad \forall x,y \in M\colon\ xmy \in M \Leftrightarrow xm'y \in M.$$

The subset X is called *rigid* [2] if

$$\forall m,m' \in M\colon \quad m \equiv_X m' \Rightarrow m = m',$$

or equivalently, if the natural morphism $M \to M/\equiv_X$ is an isomorphism. If $M = A^*$, the free monoid generated by an alphabet A, and $X \subseteq A^*$, then the congruence $\equiv_X$ and the quotient monoid $M_X = A^*/\equiv_X$ are called respectively the *syntactic congruence* and the *syntactic monoid* of X. A monoid M is said to be $*$*-syntactic* if there exist a finite alphabet A and a rational language $X \subseteq A^*$ such that M and M_X are isomorphic: $M \cong M_X$. The following result [2] allows to decide whether a given finite monoid is $*$-syntactic or not.

Proposition 0.1. *A finite monoid M is $*$-syntactic iff M has a rigid set.*

Let $h: A^* \to M$ be a morphism from a free monoid A^* into a monoid M. We say that the morphism h *saturates* an ω-language X in A^ω if, for any $(p, q) \in M \times M$,

$$h^{-1}(p)\left[h^{-1}(q)\right]^\omega \cap X \neq \emptyset \Rightarrow h^{-1}(p)\left[h^{-1}(q)\right]^\omega \subseteq X.$$

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We recall that for any subset X of A^*, $X^\omega = \{x_1 x_2 \dots \mid \varepsilon \neq x_i \in X\}$, where ε is the empty word. Let $\approx$ be a congruence on A^*. We denote by $[u]_\approx$, or simply by $[u]$ if no confusion can arise, the class of the word u w.r.t. $\approx$. We say that a congruence $\approx$ on A^* *saturates* X if the natural morphism $h: A^* \to A^*/\approx$ saturates X, or equivalently if

$$\forall u,v \in A^*:\quad [u][v]^\omega \cap X \neq \emptyset \Rightarrow [u][v]^\omega \subseteq X.$$

It is well known that an ω-language X in A^ω is rational (or recognizable) iff there exists a finite monoid M and a morphism $h: A^* \to M$ which saturates X. With every ω-language X in A^ω one associates a congruence $\approx_X$ on A^*, called *syntactic congruence* of X, defined as follows (see [1]): $u \approx_X v$ iff the following two conditions hold:

(a) $\forall x,y,z \in A^*$: $(xuy)z^\omega \in X \Leftrightarrow (xvy)z^\omega \in X$,

(b) $\forall x,y,z \in A^*$ with $xy \neq \varepsilon$: $z(xuy)^\omega \in X \Leftrightarrow z(xvy)^\omega \in X$.

If X is rational, then $\approx_X$ is the largest finite congruence saturating X.

Remark. The requirement "$xy \neq \varepsilon$" in (b) is necessary as shown in the following example.

Example 0.2. Let $A = \{a, b\}$, $X = \{a^\omega\}$ and let $U_1 = \{0, 1\}$ be the monoid with the ordinary multiplication of numbers. Consider the morphism $\phi: A^* \to U_1$ defined by: $\phi(a) = 1$, $\phi(b) = 0$. The congruence $\approx_\phi$ induced by ϕ is of index 2 with the corresponding classes $[a] = a^*$ and $[b] = A^*bA^*$. Since $X = [a][a]^\omega$, ϕ saturates X. On the other hand, it is easy to check that the words a, b and ε are not pairwise equivalent w.r.t. $\approx_X$ without the restriction "$xy \neq \varepsilon$".

The monoid $M_X = A^*/\approx_X$ and the natural morphism $h_X: A^* \to M_X$ are called respectively *syntactic monoid* and *syntactic morphism* of X. A monoid M is said to be *Arnold's syntactic* (resp. *ω-syntactic*) if there are an alphabet (resp. a finite alphabet) A and an ω-language (resp. a rational ω-language) X in A^ω such that $M \cong M_X$.

The aim of this note is to characterize the finite monoids which are ω-syntactic and to prove that the property of being ω-syntactic is decidable. The family of ω-syntactic monoids and that of $*$-syntactic monoids are shown to be uncomparable. This paper consists of two sections. In Section 1 we introduce a notion of ω-rigid sets for a monoid and prove that a finite monoid M is ω-syntactic iff there exists an ω-rigid set for M. This is an analogue of Proposition 0.1 for the infinitary case. In contrast with the notion of rigid sets introduced by Eilenberg, ω-rigid sets are subsets of $M \times M$ and not of M itself. In Section 2 we prove that the property of being an ω-rigid set, and therefore the property of being an ω-syntactic monoid, is decidable for the finite monoids. Different results, of the same nature, concerning however another notion of syntactic monoid for ω-languages can be found in [3,4]. Relationships between the two notions of syntactic congruence are studied in [7].

1. Syntactic monoids

Let M be a monoid. For every $(p, q) \in M \times M$, we denote by $C_{p,q}$ the set of all infinite sequences $(s_1, s_2, \dots)$ of elements of M such that there exists a strictly increasing sequence $(k_i)_{i \geq 1}$ of integers with $s_1 \dots s_{k_1} = p$ and $s_{k_i+1} \dots s_{k_{i+1}} = q$, for all $i \geq 1$.

By $\sim$ we denote the reflexive symmetric (but not transitive) relation on $M \times M$ defined by:

$$(p, q) \sim (p', q') \Leftrightarrow C_{p,q} \cap C_{p',q'} \neq \emptyset.$$

If $s \in C_{p,q} \cap C_{p',q'}$, we say that s *links* (p, q) and (p', q'). A subset $J \subseteq M \times M$ is said to be *closed by* $\sim$ if, for any $(p, q), (p', q') \in M \times M$,

$$(p, q) \sim (p', q') \;\&\; (p, q) \in J \Rightarrow (p', q') \in J.$$

The family of such subsets is denoted by $\mathscr{F}(M)$.

With every subset J of $M \times M$ we associate a congruence $\approx_J$ on M defined by: $m \approx_J m'$ iff for all $p,q,r \in M$ we have

(a′) $(pmq, r) \in J \Leftrightarrow (pm'q, r) \in J$,

(b′) $(r, pmq) \in J \Leftrightarrow (r, pm'q) \in J$.

We denote by $\mathscr{S}(M)$ the family of the subsets J of $M \times M$ such that

(S_1) $J \in \mathscr{F}(M)$,

(S_2) $\forall u,v \in M$: $u \approx_J v \Rightarrow u = v$, or equivalently, the natural morphism $M \to M/\approx_J$ is an isomorphism.

Let $h: A^* \to M$ be a monoid morphism and let $(p, q) \in M \times M$. From now on, for simplicity, $h^{-1}(p)[h^{-1}(q)]^\omega$ is denoted by $h^{-1}(p, q)$. With each subset X of A^ω we associate a subset of $M \times M$ defined by:

$$J_{X,h} = \{(p, q) \in M \times M \mid \emptyset \neq h^{-1}(p, q) \subseteq X\}.$$

Conversely, with each subset J of $M \times M$ we define a subset of A^ω which is

$$X_{J,h} = \bigcup_{(p,q) \in J} h^{-1}(p, q).$$

Instead of $J_{X,h}$ or $X_{J,h}$ we write simply J_X or X_J respectively if no confusion can arise. In the sequel, if otherwise not specified, 1 denotes the unit of a monoid.

Lemma 1.1. *Let $h: A^* \to M$ be a surjective monoid morphism.*

(i) *If h saturates a subset X of A^ω, then $J_X \in \mathscr{F}(M)$. Moreover, if $h^{-1}(1) \neq \{\varepsilon\}$ or $h = h_X$, then $M/\approx_{J_X} \cong M_X$ in the sense that the morphism which maps $[u]_{\approx_X}$ into $[h(u)]_{\approx_{J_X}}$ is an isomorphism.*

(ii) *If $J \in \mathscr{F}(M)$, then h saturates X_J. Moreover, if $h^{-1}(1) \neq \{\varepsilon\}$, then $J_{X_J} = J$.*

Proof. (i) Firstly we prove that $J_X \in \mathscr{F}(M)$. Let us assume $(p, q) \sim (p', q')$ and $(p, q) \in J_X$. Then $\emptyset \neq h^{-1}(p, q) \subseteq X$. Let $(s_1, s_2, \ldots)$ be a sequence linking (p, q) and (p', q'). Because h is surjective, $h^{-1}(s_i) \neq \emptyset$, $\forall i \geqslant 1$. We can always choose $x_i \in h^{-1}(s_i)$ such that $x_i \neq \varepsilon$ for infinitely many i. Indeed, if it is not the case, we must have $h^{-1}(1) = \{\varepsilon\}$ and $s_i \neq 1$ only for finitely many i, this implies $q = 1$ and therefore $h^{-1}(p, q) = \emptyset$, a contradiction. Thus the infinite word $x_1 x_2 \ldots \in h^{-1}(p, q) \cap h^{-1}(p', q')$. This implies, since h saturates X, $\emptyset \neq h^{-1}(p', q') \subseteq X$. Hence $(p', q') \in J_X$. Thus $J_X \in \mathscr{F}(M)$. Now it is easy to verify that, for any $u,v \in A^*$, $h(u) \approx_{J_X} h(v)$ implies $u \approx_X v$. Moreover, if $h^{-1}(1) \neq \{\varepsilon\}$ or $h = h_X$, then the inverse implication holds too. Indeed, we first consider the case $h^{-1}(1) \neq \{\varepsilon\}$. Let us have $u \approx_X v$ and $(r, ph(u)q) \in J_X$ for some $p,q,r \in M$. Since $h^{-1}(1) \neq \{\varepsilon\}$, we can always choose $x,y,z \in A^*$ with $xy \neq \varepsilon$ such that $h(x) = p$, $h(y) = q$, $h(z) = r$. Then $(h(z), h(xuy)) \in J_X$. Hence $z(xuy)^\omega \in X$. Since $u \approx_X v$, this implies $z(xvy)^\omega \in X$ which in its turn gives $(h(z), h(xvy)) \in J_X$, or equivalently $(r, ph(v)q) \in J_X$. So we have $(r, ph(u)q) \in J_X \Rightarrow (r, ph(v)q) \in J_X$. The inverse is due to the symmetry, so we have (b′) for $h(u)$ and $h(v)$. In a similar way we have (a′) too, i.e. $h(u) \approx_{J_X} h(v)$. For the case $h = h_X$, obviously $u \approx_X v$ implies $h(u) = h(v)$. So $h(u) \approx_{J_X} h(v)$ is trivial. Thus in both cases the morphism which maps $[u] \approx_X$ into $[h(u)] \approx_{J_X}$ is an isomorphism.

(ii) Assume that $h^{-1}(p, q) \cap X_J \neq \emptyset$ for some $(p, q) \in M \times M$. There must exist $(p', q') \in J$ such that $h^{-1}(p, q) \cap h^{-1}(p', q') \neq \emptyset$. Let $w = a_1 a_2 \ldots$, $a_i \in A$, be a word in this set. Put $s_i = h(a_i)$. Then the sequence $(s_1, s_2, \ldots)$ links (p, q) and (p', q'), that is $(p, q) \sim (p', q')$. Hence $(p, q) \in J$. Then $h^{-1}(p, q) \subseteq X_J$. Thus h saturates X_J.

Note that if $h^{-1}(1) \neq \{\varepsilon\}$, then $h^{-1}(p, q) \neq \emptyset$, $\forall (p, q) \in M \times M$. If $(p, q) \in J$, then $\emptyset \neq h^{-1}(p, q) \subseteq X_J$, hence $(p, q) \in J_{X_J}$. Thus $J \subseteq J_{X_J}$. Conversely, if $(p, q) \in J_{X_J}$, then $\emptyset \neq h^{-1}(p, q) \subseteq X_J$. There must exist $(p', q') \in J$ such that $h^{-1}(p, q) \cap h^{-1}(p', q') \neq \emptyset$. Like in the above part, this implies $(p, q) \sim (p', q')$. Then $(p, q) \in J$, that is $J_{X_J} \subseteq J$. Thus $J_{X_J} = J$. □

Lemma 1.2. *Let $f: M \to N$ be a surjective monoid morphism. Let $J \in \mathscr{F}(M)$ and $J' = \{(p, q) \in M \times M \mid (f(p), f(q)) \in J\}$. Then $J' \in \mathscr{F}(M)$ and $M/\approx_{J'} \cong N/\approx_J$.*

Proof. We first prove that $J' \in \mathscr{F}(M)$. Let us have $(p, q) \sim (p', q')$ and $(p, q) \in J'$. By the definition of J', $(f(p), f(q)) \in J$. Let $(s_1, s_2, \ldots)$ be a sequence linking (p, q) and (p', q'). Then evidently the sequence $(f(s_1), f(s_2), \ldots)$ links $(f(p), f(q))$ and $(f(p'), f(q'))$. So $(f(p), f(q)) \sim (f(p'), f(q'))$. Hence $(f(p'), f(q')) \in J$, and therefore $(p', q') \in J'$. Thus $J' \in \mathscr{F}(M)$. Now using the surjectivity of f it is easy to verify that

$\forall m,m' \in M : m \approx_{J'} m'$ iff $f(m) \approx_J f(m')$. Hence $M/\approx_{J'} \cong N/\approx_J$. $\square$

The following result provides a characterization of the monoids which are Arnold's syntactic.

Proposition 1.3. *A monoid M is Arnold's syntactic if and only if $\mathscr{S}(M) \neq \emptyset$.*

Proof. Let us have $\mathscr{S}(M) \neq \emptyset$ and $J \in \mathscr{S}(M)$. For a finite alphabet A large enough there exists a surjective morphism $h : A^* \to M$ such that $h^{-1}(1) \neq \{\varepsilon\}$. By Lemma 1.1(ii), h saturates X_J and $J_{X_J} = J$. Then, by Lemma 1.1(i), $M_{X_J} \cong M/\approx_{J_{X_J}} = M/\approx_J$. Since $J \in \mathscr{S}(M)$, $M/\approx_J \cong M$. Thus $M \cong M_{X_J}$, that is M is Arnold's syntactic. Conversely assume that M is Arnold's syntactic. There exist an alphabet A and an ω-language $X \subseteq A^\omega$ such that $M \cong M_X$. Consider the syntactic morphism $h_X : A^* \to M_X$. By Lemma 1.1(i), $J_{X,h_X} \in \mathscr{F}(M_X)$ and the natural morphism $M_X \to M_X/\approx_{J_X}$ is an isomorphism, that is $J_{X,h_X} \in \mathscr{S}(M_X)$. By Lemma 1.2, there exists J in $\mathscr{F}(M)$ such that the natural morphism $M \to M/\approx_J$ is an isomorphism, i.e. $J \in \mathscr{S}(M)$. Thus $\mathscr{S}(M) \neq \emptyset$. $\square$

As usual, for any monoid M, we denote by $P(M)$ the set of all *bound* couples (see [5] for example) of elements of M:

$$P(M) = \{(f, e) \in M \times M \mid fe = f,\ e^2 = e\}.$$

Now let M be a finite monoid. We denote by $\varphi : M \times M \to P(M)$ the application associating with each couple (p, q) in $M \times M$ the couple (pq^k, q^k), where k is the smallest natural number such that q^k is an idempotent of M. For the sake of simplicity, instead of $\varphi((p, q))$ we write $\varphi(p, q)$.

For every subset I of $P(M)$ we define in the following way a congruence $\simeq_I$ on M: $m \simeq_I m'$ iff, for all $p,q,r \in M$, the following two conditions hold:

(a″) $\varphi(pmq, r) \in I \Leftrightarrow \varphi(pm'q, r) \in I$,

(b″) $\varphi(r, pmq) \in I \Leftrightarrow \varphi(r, pm'q) \in I$.

We denote by $\mathscr{G}(M)$ the family of the subsets I of $P(M)$ which are closed by $\sim$. Relationship between $\mathscr{F}(M)$ and $\mathscr{G}(M)$ is given by the following lemma.

Lemma 1.4. *The application ψ associating with each set $J \in \mathscr{F}(M)$ the set $J \cap P(M)$ is a bijection between $\mathscr{F}(M)$ and $\mathscr{G}(M)$. Moreover, for any $J \in \mathscr{F}(M)$, $\approx_J$ coincides with $\simeq_{\psi(J)}$.*

Proof. We first note that the following two facts are evident:

(φ_1) $\forall (p, q) \in M \times M$: $(p, q) \sim \varphi(p, q)$,

(φ_2) $\forall (p, q), (p', q') \in M \times M$: $(p, q) \sim (p', q') \Rightarrow \varphi(p, q) \sim \varphi(p'q')$.

Obviously if $J \in \mathscr{F}(M)$, then $\psi(J) \in \mathscr{G}(M)$. Let η be the application mapping every $I \in \mathscr{G}(M)$ into the set $J = \{(p, q) \in M \times M \mid \exists (f, e) \in I: (p, q) \sim (f, e)\}$. By using the properties (φ_1) and (φ_2) it is easy to verify that $\eta(I) \in \mathscr{F}(M)$ for any $I \in \mathscr{G}(M)$, and that $\eta\psi$ and $\psi\eta$ are the identity mappings on $\mathscr{F}(M)$ and on $\mathscr{G}(M)$ respectively. So ψ is a bijection between $\mathscr{F}(M)$ and $\mathscr{G}(M)$. Next, again by (φ_1) we have, $\forall J \in \mathscr{F}(M)$, $\forall (p, q) \in M \times M$, $(p, q) \in J \Leftrightarrow \varphi(p, q) \in \psi(J)$. This allows to verify that

$$\forall J \in \mathscr{F}(M), \forall m, m' \in M:$$
$$m \approx_J m' \text{ iff } m \simeq_{\psi(J)} m',$$

i.e. $\approx_J$ and $\simeq_{\psi(J)}$ are identical. $\square$

We call *ω-rigid* set for M any subset I of $P(M)$ such that

(R_1) $I \in \mathscr{G}(M)$,

(R_2) $\forall m,m' \in M : m \simeq_I m' \Rightarrow m = m'$, or equivalently, $M/\simeq_I \cong M$. The family of such sets is denoted by $\mathscr{R}(M)$. The main result of this section is the following.

Theorem 1.5. *A finite monoid M is ω-syntactic if and only if $\mathscr{R}(M) \neq \emptyset$.*

Proof. Let us have $\mathscr{R}(M) \neq \emptyset$ and $I \in \mathscr{R}(M)$. Then, by Lemma 1.4, $J = \psi^{-1}(I) \in \mathscr{F}(M)$ and $\approx_J$ coincides with $\simeq_I$. For a finite alphabet A large enough there exists a surjective morphism $h: A^* \to M$ such that $h^{-1}(1) \neq \{\varepsilon\}$. By Lemma 1.1(ii), h saturates X_J and $J_{X_J} = J$. So X_J is a rational ω-language. Next, by Lemma 1.1(i), M_{X_J}

$\cong M/\approx_{J_{X_I}} = M/\approx_J = M/\simeq_I$. Since I is ω-rigid, $M/\simeq_I \cong M$. Thus $M \cong M_{X_I}$, which means that M is ω-syntactic. Conversely assume that M is ω-syntactic. Then M is Arnold's syntactic too, and therefore, by Proposition 1.3, $\mathscr{S}(M) \neq \emptyset$, i.e. there exists $J \in \mathscr{F}(M)$ such that $M/\approx_J \cong M$. Let $I = \psi(J)$. By Lemma 1.4, $I \in \mathscr{G}(M)$ and $\simeq_I$ coincides with $\approx_J$. Hence $M/\simeq_I = M/\approx_J \cong M$ which implies $I \in \mathscr{R}(M)$, that is $\mathscr{R}(M) \neq \emptyset$. □

As an immediate consequence of the first part in the proof of Proposition 1.3 and of Theorem 1.5, we have the following corollary.

Corollary 1.6. *For any monoid M (resp. for any finite monoid M), for any $J \in \mathscr{F}(M)$ (resp. for any $I \in \mathscr{G}(M)$), the monoid $M/\approx_J$ (resp. the monoid $M/\simeq_I$) is Arnold's syntactic (resp. ω-syntactic).*

2. Decidability of being ω-syntactic

Two couples (f, e) and (f', e') in $P(M)$ are *conjugate* [5] if there exist $x, y \in M$ such that

$$f = f'x, \quad e' = xy, \quad e = yx$$

(which imply $f' = fy$).

Lemma 2.1. *Let M be a finite monoid. For any (f,e), $(f', e') \in P(M)$, $(f, e) \sim (f', e')$ if and only if (f, e) and (f', e') are conjugate.*

Proof. Let $(s_1, s_2, \ldots)$ be a sequence linking (f, e) and (f', e'). Then there exist $(\alpha_i, \beta_i) \in M \times M$, $i \geqslant 1$, such that

$$f = f'\alpha_i, \quad \alpha_i e \beta_i = e', \quad \beta_i \alpha_{i+1} = e, \qquad i \geqslant 1.$$

Because M is finite, there are k and m, $1 \leqslant k < m \leqslant |M \times M| + 1$ such that $(\alpha_k, \beta_k) = (\alpha_m, \beta_m)$. Put $x = \alpha_k$ $(= \alpha_m)$, $y = e\beta_k \alpha_{k+1} e \beta_{k+1} \cdot \alpha_{k+2} e \ldots e\beta_{m-1}$. We have $f = f'z$, $xy = e'$, $yx = e$, i.e. (f, e) and (f', e') are conjugate. Conversely, if (f, e) and (f', e') are conjugate, then the sequence $(f', x, y, x, y, \ldots)$ links them. □

Theorem 2.2. *One can decide, for any finite monoid M, whether M is ω-syntactic or not.*

Proof. It suffices to show that one can decide, for every subset I of $P(M) = \phi(M \times M)$, whether I is ω-rigid or not, i.e. whether the conditions (R_1) and (R_2) hold. The decidability of (R_2) is evident. That of (R_1) is guaranteed by Lemma 2.1. □

We denoted by Syn^* and Syn^ω the families of $*$-syntactic and ω-syntactic monoids respectively. The following examples make clear the relationship between these families and also detail the algorithm proposed in the proof of Theorem 2.2.

Example 2.3. Consider the two-element monoid $M_1 = \{0, 1\}$ whose multiplication table is given by:

$$1.0 = 0.1 = 0, \qquad 1.1 = 0.0 = 1.$$

The subset $J = \{0\}$ is a rigid set of M_1. Indeed, we have $1.0.1 = 0 \in J$ whereas $1.1.1 = 1 \notin J$. This means that $0 \not\approx_J 1$, i.e. J is rigid. By Proposition 0.1, $M_1 \in Syn^*$. Checking whether M_1 is ω-syntactic can be done in the following steps:

(1) Computing the function $\varphi : M_1 \times M_1 \to P(M_1)$:

(x, y)	$\varphi(x, y)$
$(0, 0)$	$(0, 1)$
$(0, 1)$	$(0, 1)$
$(1, 0)$	$(1, 1)$
$(1, 1)$	$(1, 1)$

(2) Computing the set $P(M_1)$ of the bound couples:

$$P(M_1) = \varphi(M_1 \times M_1) = \{(0, 1), (1, 1)\}.$$

(3) Finding out in $P(M_1)$ all the couples which are in relation $\sim$ by verifying whether they are conjugate (Lemma 2.1). Note that the conjugacy relation is an equivalence one.

$$(0, 1) \sim (1, 1) \quad \text{because } 0 = 1.0,\ 0.0 = 1.$$

(4) Checking, for each subset I of $P(M_1)$, whether I is closed by conjugacy relation to discover all the members of $\mathscr{G}(M_1)$:

$$\mathscr{G}(M_1) = \{\emptyset, M_1 \times M_1\}.$$

(5) Checking, for each $I \in \mathscr{G}(M_1)$, whether I satisfies (R_2). Evidently for any $I \in \mathscr{G}(M_1)$, $0 \simeq_I 1$, i.e. (R_2) does not hold. So $\mathscr{R}(M_1) = \emptyset$, hence $M_1 \notin Syn^\omega$. Thus we have shown that $M_1 \in Syn^* - Syn^\omega$.

Example 2.4. Let $M_2 = \{1, p, q, s\}$ be the monoid having 1 as the unit and whose multiplication law is given by:

$$p^2 = pq = ps = p,$$

$$q^2 = qp = qs = q,$$

$$s^2 = sp = sq = s.$$

On the one hand, it is not difficult to show that

$$\forall J \subseteq M_2,\ \exists m_1, m_2 \in M_2:$$
$$m_1 \neq m_2 \text{ and } m_1 \equiv_J m_2,$$

i.e. J is not rigid. So $M_2 \notin Syn^*$. On the other hand, by applying the algorithm described in Example 2.3 one can verify that the set

$$I = \{(p, p), (p, q), (p, s), (s, 1)\}$$

is an ω-rigid set for M_2. Thus we have $M_2 \in Syn^\omega - Syn^*$. Let $A = \{a, b, c, d\}$. A rational ω-language whose syntactic monoid is M_2 can be chosen as follows

$$X = (a^*bA^*)^\omega + (a^*bA^*)(a^*cA^*)^\omega + (a^*bA^*)(a^*dA^*)^\omega + (a^*dA^*)a^\omega.$$

Example 2.5. The monoid $U_1 = \{0, 1\}$ considered in Example 0.2 is obviously in $Syn^* \cap Syn^\omega$.

Example 2.6. Let us consider the monoid $M_3 = \{1, p, q, s\}$, where 1 is the unit and the multiplication law is the following:

$$p^2 = qp = sp = p,$$

$$q^2 = pq = sq = q,$$

$$s^2 = ps = qs = s.$$

It is not difficult to show that

- $\forall J \subseteq M_3$, $\exists m_1, m_2 \in M_3$: $m_1 \neq m_2$ and $m_1 \equiv_J m_2$. So $M_3 \notin Syn^*$.
- $\forall I \in \mathscr{G}(M_3)$, $\exists m_1, m_2 \in M_3$: $m_1 \neq m_2$ and $m_1 \simeq_I m_2$. So $M_3 \notin Syn^\omega$.

Thus $M_3 \notin Syn^* \cup Syn^\omega$.

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