3.

THE SHORTEST AXIOM OF THE IMPLICATIONAL CALCULUS OF PROPOSITIONS.

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§ 1. Introductory remarks. § 2. History of the problem. § 3. Derivation of the Tarski-Bernays set of axioms from Axiom /1/. § 4. A certain theorem concerning the law of syllogism. § 5. Outline of a proof that Axiom /1/ is the shortest possible.

§ 1. *Introductory remarks.* The Implicational Calculus of Propositions constitutes that part of the Complete Propositional Calculus in which implication occurs as the only functor. I denote this functor by the letter "C" and put it before its arguments, thus dispensing with brackets. So the expression "$C\ p\ q$" means "if p, then q." Two propositional expressions belong to each "C" as its arguments and follow it immediately. By propositional expressions I understand propositional variables denoted by the small letters of the Latin alphabet or expressions of the form "$C\,\alpha\,\beta$" in which "α" and "β" are already propositional expressions. Propositional expressions which are either axioms or theorems derived from the axioms will be called theses. In derivations I will make use of the rule of substitution, according to which I can add to a set of theses a propositional expression derived from a thesis of the set by substituting any propositional expressions for the variables of the thesis, and the rule of detachment which enables me to add to a set of theses a propositional expression "β" provided expressions of the form "$C\,\alpha\,\beta$" and "α" are already members of the set.

In this article I intend to prove that all theses of the Implicational Calculus of Propositions can be derived from the following axiom

$$/1/ \quad C\,C\,C\,p\,q\,r\,C\,C\,r\,p\,C\,s\,p$$

by applying the rule of substitution and the rule of detachment.

Axiom /1/ consists of 13 letters, and is the shortest on the basis of which one can construct the Implicational Calculus of Propositions. I have mentioned this axiom twice in my previous articles, but on both occasions without proof.[1] The proof I am giving below will show

[1] For the first time in the article *W obronie logistyki*, Studia Gnesnensia XV, Poznan 1937, page 11 of the reprint; for the second time in the lecture *Die Logik und das Grundlagenproblem*, Les Entretiens de Zurich sur les fondements et la méth. des sciences mathém. 1938, Zurich 1941, page 95.

that the following three theses can be derived from axiom /1/:—

$$CpCqp$$
$$CCCpqpp$$
$$CCpqCCqrCpr.$$

These three theses are known as the "Tarski-Bernays" set of axioms, and —as A. *Tarski* has proved—form a sufficient basis for the Implicational Calculus of Propositions.[2] The first one is the so-called law of simplification; I have called the second one *Peirce's* law; the third thesis is the law of hypothetical syllogism. As the derivation of the law of syllogism is particularly difficult, it may prove useful to show how it can be done. The derivation of the law of syllogism will be followed by a certain theorem concerning this law. In the last paragraph I wish to outline a proof that there is no shorter thesis which could function as a sole axiom of the Implicational Calculus of Propositions.

§ 2. *History of the problem.* The problem of how to construct the Complete Propositional Calculus as well as the Implicational Calculus of Propositions on the basis of a single axiom was raised and solved in 1925 by *Tarski*, who gave a method of combining several axioms by applying the rule of substitution and the rule of detachment.[3] The first axioms arrived at, in accordance with this method, were very long. I tried to shorten them by modifying *Tarski's* method, and finally discovered the following axiom consisting of 25 letters:

$$/2/ \quad CCCpCqpCCCCCrstuCCsuCruvv.^{4}$$

This axiom is non-organic, as some constituents of it, namely:

$$CpCqp \quad \text{and} \quad CCCCrstuCCsuCru,$$

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[2] J. *Łukasiewicz* i A. *Tarski*, *Untersuchungen über den Aussagenkalkül*, Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie XXIII 1930, Cl. III, Satz 29.

[3] *Łukasiewicz-Tarski*, l.c. Satz 8 and 25.

[4] *Łukasiewicz-Tarski*, l.c. Satz 30. I am giving here the first derivational steps based on this axiom as they are not easy. To be brief I denote the thesis

$$CCCCrstuCCsuCru$$

contained in the axiom by the letter "α." This letter can denote any thesis in which the variables "p," "q" and "v" do not appear. It will be shown how thesis "α" can be derived from the axiom. As to the technique applied in derivational procedure see explanations given in § 3 of this article.

1 $CCCpCqpC\alpha vv$

1 $p/C\alpha Cq\alpha,\ q/\alpha,\ v/Cq\alpha \ + \ C1\,p/\alpha,\ v/C\alpha Cq\alpha \ - \ 2$

2 $Cq\alpha$

2 $q/1 \ + \ C1 \ - \ 3$

3 α

I think that further steps should not be hindered by any difficulty.

are theses of the Calculus making the whole expression a conglomeration of two theses. Later I abandoned the idea of constructing shorter axioms in the way just mentioned, as in 1926 M. *Wajsberg* has shown that one could base the Implicational Calculus of Propositions on the following organic axiom, that is, on an axiom, no constituent of which, was a thesis of the Calculus. *Wajsberg's* axiom consisted too of 25 letters:

$$/3/ \quad C\,C\,C\,p\,q\,C\,C\,r\,s\,t\,C\,C\,u\,C\,C\,r\,s\,t\,C\,C\,p\,u\,C\,s\,t.$$ [5]

This discovery made me hope that there might exist shorter organic axioms, while, at the same time, I realised that the shortest axiom must be organic, as conglomerations of several axioms are naturally bound to be longer. In 1930 I found an organic axiom which was shorter than *Wajsberg's* thesis and consisted of 17 letters:

$$/4/ \quad C\,C\,C\,p\,q\,C\,r\,s\,C\,t\,C\,C\,s\,p\,C\,r\,p.$$

In 1932 I found another such axiom:

$$/5/ \quad C\,C\,C\,p\,q\,C\,r\,s\,C\,C\,s\,p\,C\,t\,C\,r\,p.$$ [6]

Then in 1936 I discovered the shortest axiom /1/, cited above, and thus terminated the examination of the problem.

§ 3. *Derivation of Tarski-Bernays set of axioms from axiom /1/.* The proof which follows is fully formalised in accordance with the method adopted by me in my previous publications.[7] Every thesis which is not the axiom—all theses have their numbers and thus are distinguished as theses—is preceded by a line without its number. I call this line a derivational line. Every derivational line consists of two parts separated from each other by the cross "+." The cross is preceded by the substitution which has to be performed on a previously given thesis, and followed by

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[5] *Łukasiewicz-Tarski,* l.c. Satz 30. As to the terms "organic" and "non-organic" see l.c. Satz 9. See also J. *Łukasiewicz, Uwagi o aksjomacie Nicod'a i o "dedukcji uogólniającej,"* Księga Pamiątkowa Polskiego Towarzystwa Filozoficznego we Lwowie *Lwów* 1931, page 15 of the reprint. The proof that thesis /3/ can be a sole axiom of the Implicational Calculus of Propositions, was given by M. *Wajsberg* in his article: *Ein Neues Axiom des Aussagenkalküls in der Symbolik von Sheffer,* Monatshefte f. Math. u. Phys. XXXIX, 1932.

[6] See e.g. J. *Łukasiewicz, Uwagi o aksjomacie Nicod'a,* page 17 of the reprint, and B. *Sobocinski, Z badan nad teorią dedukcji,* Przegląd, Filozoficzny, XXXV, Warszawa 1932, page 7 and 8.

[7] See e.g. J. *Łukasiewicz, Philosophische Bemerkungen zu mehrwertigen Systemen des Aussagenkalküls,* Comptes Rendus des séances de la Société des Sciences et des Lettres de Varsovie XXIII 1930, Cl. III, page 56.

detachment, which has to be performed on the thesis arrived at by the substitution. An example will clarify the methods: In the derivational line belonging to thesis *2* the expression "*1 p/C p q, q/r, r/C C r p C s p, s/r*" means that in thesis *1* "*C p q*" has to be substituted for "*p*," "*r*" for "*q*," "*C C r p C s p*" for "*r*," and "*r*" for "*s*." The thesis generated by this substitution is omitted in the actual proof to save space. It would be of the following form:

1′ C C C C p q r C C r p C s p C C C C r p C s p C p q C r C p q.

The expression following the cross "+," i.e. "*C 1 – 2*," shows how thesis *1′* is constructed, making obvious that the rule of detachment can be applied to thesis *1′*. Thesis *1′* begins with "*C*," then follows axiom /1/ as the antecedent and thesis *2* as the consequent. Thus thesis *1′* is an expression of the form "$C\,\alpha\,\beta$," and both "$C\,\alpha\,\beta$" and "α" are theses. One can therefore detach "β," i.e.: *2* as a new thesis. In some derivational lines detachment is performed twice; as, for instance, in the derivational line preceding thesis *4*. In the same line, instead of substituting the whole first thesis I substitute only its number. The stroke "/" is the sign of substitution and the hyphen "–" is the sign of detachment. I think that after these explanatory remarks the reader will be able to understand and check the proof without any difficulty.

1 *C C C p q r C C r p C s p.*
1 p/C p q, q/r, r/C C r p C s p, s/r + C 1 – 2.
2 *C C C C r p C s p C p q C r C p q.*
1 p/C C r p C s p, q/C p q, r/C r C p q, s/t + C 2 – 3.
3 *C C C r C p q C C r p C s p C t C C r p C s p.*
3 r/C p q, t/1 + C 1 r/C p q – C 1 – 4.
4 *C C C p q p C s p.*
1 p/C p q, q/p, r/C s p, s/r + C 4 – 5.
5 *C C C s p C p q C r C p q.*
1 p/C s p, q/C p q, r/C r C p q, s/t + C 5 – 6.
6 *C C C r C p q C s p C t C s p.*
1 p/C r C p q, q/C s p, r/C t C s p, s/u + C 6 – 7.
7 *C C C t C s p C r C p q C u C r C p q.*
7 t/C p q, p/q, r/C C s q p, q/p, u/1 + C 1 r/C s q, s/q – C 1 – 8.
8 *C C C s q p C q p.*
8 s/C p q, q/r, p/C C r p C s p + C 1 – 9.
9 *C r C C r p C s p.*
1 p/r, r/C C C r q p C s p, s/t + C 9 r/C r q – 10.
10 *C C C C C r q p C s p r C t r.*
1 p/C C C r q p C s p, q/r, r/C t r, s/u + C 10 – 11.

11 $C\,C\,C\,t\,r\,C\,C\,C\,r\,q\,p\,C\,s\,p\,C\,u\,C\,C\,C\,r\,q\,p\,C\,s\,p.$
$1\ p/C\,t\,r,\ \ q/C\,C\,C\,r\,q\,p\,C\,s\,p,\ \ r/C\,u\,C\,C\,C\,r\,q\,p\,C\,s\,p,$
$s/v + C\ 11 - 12.$

12 $C\,C\,C\,u\,C\,C\,C\,r\,q\,p\,C\,s\,p\,C\,t\,r\,C\,v\,C\,t\,r.$
$1\ p/C\,u\,C\,C\,C\,r\,q\,p\,C\,s\,p,\ q/C\,t\,r,\ r/C\,v\,C\,t\,r,\ s/w + C\ 12 - 13.$

13 $C\,C\,C\,v\,C\,t\,r\,C\,u\,C\,C\,C\,r\,q\,p\,C\,s\,p\,C\,w\,C\,u\,C\,C\,C\,r\,q\,p\,C\,s\,p.$
$13\ v/C\,C\,s\,p\,q,\ \ u/C\,C\,t\,r\,C\,s\,p,\ \ w/1 + C\ 1\ p/C\,s\,p,\ \ r/C\,t\,r,$
$s/C\,C\,r\,q\,p - C\ 1 - 14.$

14 $C\,C\,C\,t\,r\,C\,s\,p\,C\,C\,C\,r\,q\,p\,C\,s\,p.$
$14\ t/C\,p\,q,\ \ s/C\,r\,p,\ \ p/C\,s\,p + C\ 1 - 15.$

15 $C\,C\,C\,r\,q\,C\,s\,p\,C\,C\,r\,p\,C\,s\,p.$
$15\ s/C\,C\,r\,q\,p,\ \ p/C\,s\,p + C\ 9\ r/C\,r\,q - 16.$

16 $C\,C\,r\,C\,s\,p\,C\,C\,C\,r\,q\,p\,C\,s\,p.$
$16\ r/C\,C\,p\,q\,r,\ \ s/C\,r\,p,\ \ p/C\,s\,p,\ \ q/t + C\ 1 - 17.$

17 $C\,C\,C\,C\,C\,p\,q\,r\,t\,C\,s\,p\,C\,C\,r\,p\,C\,s\,p.$
$1\ p/C\,C\,C\,p\,q\,r\,t,\ \ q/C\,s\,p,\ \ r/C\,C\,r\,p\,C\,s\,p,\ \ s/u + C\ 17 - 18.$

18 $C\,C\,C\,C\,r\,p\,C\,s\,p\,C\,C\,C\,p\,q\,r\,t\,C\,u\,C\,C\,C\,p\,q\,r\,t.$
$18\ r/C\,r\,p,\ \ p/C\,s\,p,\ \ s/C\,C\,p\,q\,r,\ \ t/C\,C\,C\,p\,q\,r\,C\,s\,p,$
$u/18 + C\ 18\ t/C\,s\,p,\ \ u/C\,C\,C\,s\,p\,q\,C\,r\,p - C\ 18 - 19.$

19 $C\,C\,C\,C\,s\,p\,q\,C\,r\,p\,C\,C\,C\,p\,q\,r\,C\,s\,p.$
$14\ t/C\,C\,s\,p\,q,\ \ r/C\,r\,p,\ \ s/C\,C\,p\,q\,r,\ \ p/C\,s\,p,$
$q/p + C\ 19 - 20.$

20 $C\,C\,C\,C\,r\,p\,p\,C\,s\,p\,C\,C\,C\,p\,q\,r\,C\,s\,p.$
$20\ r/q,\ p/C\,p\,r,\ s/C\,q\,r + C\ 15\ r/q,\ q/C\,p\,r,\ s/p;\ p/r - 21.$

21 $C\,C\,C\,C\,p\,r\,q\,q\,C\,C\,q\,r\,C\,p\,r.$
$5\ s/C\,p\,q,\ \ q/p,\ \ r/4 + C\ 4\ s/p - C\ 4 - 22.$

22 $C\,p\,p.$
$20\ s/C\,r\,p + C\ 22\ p/C\,C\,r\,p\,p - 23.$

23 $C\,C\,C\,p\,q\,r\,C\,C\,r\,p\,p.$
$8\ s/C\,p\,q,\ \ q/r,\ \ p/C\,C\,r\,p\,p + C\ 23 - 24.$

24 $C\,r\,C\,C\,r\,p\,p.$
$15\ r/p,\ \ q/r,\ \ s/C\,C\,p\,r\,q,\ \ p/q + C\ 24\ r/C\,p\,r,\ p/q - 25.$

25 $C\,C\,p\,q\,C\,C\,C\,p\,r\,q\,q.$
$25\ p/C\,p\,q,\ \ q/C\,C\,C\,p\,r\,q\,q,\ \ r/C\,C\,q\,r\,C\,p\,r + C\ 25 - 26.$

26 $C\,C\,C\,C\,p\,q\,C\,C\,q\,r\,C\,p\,r\,C\,C\,C\,p\,r\,q\,q\,C\,C\,C\,p\,r\,q\,q.$
$8\ s/C\,s\,q,\ \ q/p,\ \ p/C\,q\,p + C\ 8 - 27.$

27 $C\,p\,C\,q\,p.$
$25\ q/p,\ \ r/q + C\ 22 - 28.$

28 $C\,C\,C\,p\,q\,p\,p.$
$21\ p/C\,p\,q,\ \ r/C\,C\,q\,r\,C\,p\,r,\ \ q/C\,C\,C\,p\,r\,q\,q + C\ 26 -$
$C\ 21 - 29.$

29 $C\,C\,p\,q\,C\,C\,q\,r\,C\,p\,r.$

§ 4. *A certain theorem concerning the law of syllogism.* A formalised proof can be checked mechanically but cannot be mechanically discovered. I do not know of any other method of finding proofs in the Propositional Calculus than the method of "trial and error." In the above proof the most difficult step was to find the hypothetical syllogism. The task was made easier, thanks to a certain theorem discovered by myself in 1933. This theorem is not without more general importance.

I have derived the law of syllogism from two theses:

25 $C\,C\,p\,q\,C\,C\,C\,p\,r\,q\,q$, and

21 $C\,C\,C\,C\,p\,r\,q\,q\,C\,C\,q\,r\,C\,p\,r$.

From thesis *25* follows thesis *26*, and from theses *21* and *26* follows thesis *29*, i.e. the law of syllogism. It had been known to me before that if we have two expressions of the form:

$$C\,C\,p\,q\,\alpha \quad \text{and} \quad C\,\alpha\,C\,C\,q\,r\,C\,p\,r,$$

where "α" is so constructed that the two expressions are theses, we can always derive the law of syllogism by applying the rules of substitution and detachment to these theses.

I write this theorem in the following symbols:

$$/\mathrm{A}/ \quad C\,C\,p\,q\,\alpha, \quad C\,\alpha\,C\,C\,q\,r\,C\,p\,r \rightarrow C\,C\,p\,q\,C\,C\,q\,r\,C\,p\,r.$$

It is easy to see that "α" must include both "p" and "q." Because if, for instance, "p" does not appear in "α," then by substituting "q" for "p" we can derive "$C\,C\,q\,q\,\alpha$," and as "$C\,q\,q$" is a thesis, "α" must be a thesis also, and so must "$C\,C\,q\,r\,C\,p\,r$," which is not possible. The same reasoning applies to the variable "q." The variable "r" and other variables, e.g.: "s," can be constituents of "α" or not, as it does not affect the proof. While performing substitution for "p" and "q" we change "α," but if the substitution is of the same kind "S," "α" changes always into the same expression, which I denote by "$\alpha[\mathrm{S}]$." The proof of the theorem /A/ is based on this observation.

1 $C\,C\,p\,q\,\alpha$.

2 $C\,\alpha\,C\,C\,q\,r\,C\,p\,r$.

  1 $p/C\,p\,q, \quad q/\alpha, \quad r/C\,C\,q\,r\,C\,p\,r \times C\,1 - 3$.

3 $\alpha[p/C\,p\,q, \quad q/\alpha, \quad r/C\,C\,q\,r\,C\,p\,r]$.

  2 $p/C\,p\,q, \quad q/\alpha, \quad r/C\,C\,q\,r\,C\,p\,r \times C\,3 - C\,2 - 4$.

4 $C\,C\,p\,q\,C\,C\,q\,r\,C\,p\,r$.

Let the premises *1* and *2* be theses. I perform the same substitution on both of them: "*1* $p/C\,p\,q, \quad q/\alpha, \quad r/C\,C\,q\,r\,C\,p\,r$." So in both cases "$\alpha$"

changes into the following expression: "$\alpha[p/C\,p\,q,\ q/\alpha,\ r/C\,C\,q\,r\,C\,p\,r]$," which is a thesis. In the proof given in § 3 "α" is of the form "$C\,C\,C\,p\,r\,q\,q$." Here are other expressions which can function as "α" the premises *1* and *2* remaining theses: "$C\,p\,C\,p\,q$," "$C\,C\,C\,q\,s\,p\,q$," "$C\,C\,C\,p\,q\,r\,r$," "$C\,C\,r\,s\,C\,p\,q$."

§ 5. *Outline of a proof that Axiom /1/ is the shortest possible.* I proved that there exists no thesis, shorter than axiom /1/, on which one could construct the Implicational Calculus of Propositions by examining all shorter theses and not finding one among them sufficient to be the sole axiom of the Calculus. I cannot give here the full proof as it would take too much space, but I wish to outline the way of reasoning which led me to arrive at the above result.[8]

All propositional expressions of the Implicational Calculus of Propositions, therefore all its theses, consist of an odd number of letters, as in every propositional expression the number of variables is greater by one than the number of functors. The shortest implicational thesis is the law of identity "$C\,p\,p$," which consists of 3 letters. The theses shorter than axiom /1/, consisting itself of 13 letters, are theses consisting of 3, 5, 7, 9 or 11 letters. For our purpose it is enough to examine the theses consisting of 11 letters as if a shorter thesis "α" were a sole axiom, then it is easy to prove that the thesis "$C\,z\,\alpha$," longer by two letters than "α" and in which "z" is a variable not appearing in "α," would also have been a sole axiom. By substituting "$C\,z\,\alpha$" for "z" we derive "$C\,C\,z\,\alpha\,\alpha$" and then "α" by detachment. As "α" was supposed to be a sole axiom, "$C\,z\,\alpha$" must be a sole axiom also.

After careful scrutiny I have come to the conclusion that there are 92 theses consisting of 11 letters if one disregards theses derived from shorter theses by applying the rule of substitution. For example, the thesis "$C\,C\,C\,p\,C\,q\,C\,r\,s\,p\,p$" is derived from the thesis "$C\,C\,C\,p\,q\,p\,p$" by substituting the expression "$C\,q\,C\,r\,s$" for "q." One can also disregard theses derived from theses consisting of 11 letters by identifying some of their variables. The thesis "$C\,C\,C\,p\,q\,q\,C\,C\,q\,p\,p$" is derived, for instance, from the thesis "$C\,C\,C\,p\,q\,r\,C\,C\,r\,p\,p$" by identifying the variables "q" and "r." The set of 92 theses can be divided into three groups. The first group, which is most numerous, contains theses belonging to the so-called Positive Logic in the meaning introduced by Bernays. These theses are generated by the following three axioms:

$$
/\mathrm{B}/\qquad
\begin{array}{l}
C\,p\,C\,q\,p.\\
C\,C\,p\,C\,p\,q\,C\,p\,q.\\
C\,C\,p\,q\,C\,C\,q\,r\,C\,p\,r.
\end{array}
$$

[8] J. *Słupecki* has constructed a shorter proof based on certain general theorems. The proof as far as I know has not yet been published.

There are 64 such theses. None of them can function as the sole axiom, the Positive Logic being a fragment of the Implicational Calculus of Propositions. Neither *Peirce's* law nor axiom /1/ can be derived from it. The strict proof is based on matrix I shown below:

$$I \qquad \begin{array}{c|ccc} C & 1 & 2 & 3 \\ \hline {}^{*}1 & 1 & 2 & 3 \\ 2 & 1 & 1 & 3 \\ 3 & 1 & 1 & 1 \end{array} .$$

In this matrix values of the implication "$C\,\alpha\,\beta$" are given with respect to "α" and "β" assuming values: *1, 2* and *3*. The first argument is in the left column, the second one in the top line of the matrix. Thus in accordance with the matrix "$C\,2\,3$" has the value "*3*." For every combination made by substituting the figures *1, 2* and *3* for the variables in the axioms of set /B/, reduction having been done according to the matrix, we obtain "*1*," i.e. the selected value marked with the asterisk. For example, if on the third axiom we perform the following substitution: "$p/1$," "$q/2$," "$r/3$" we obtain:

$$C\,C\,1\,2\,C\,C\,2\,3\,C\,1\,3 = C\,2\,C\,3\,3 = C\,2\,1 = 1.$$

A thesis is verified by a matrix if for every combination of substitutions of figures for variables it generates the selected value provided the reduction has been done according to the matrix. The axioms of set /B/ are verified by matrix I, which is hereditary with regard to the rule of substitution and the rule of detachment, i.e. all consequences of theses verified by it are also verified. All of the 64 theses that consist of 11 letters and belong to the Positive Logic are verified by matrix I, whereas axiom /1/ is not verified by this matrix; if we perform the following substitution: "$p/2$," "$q/3$," "$r/3$," "$s/1$," we obtain:

$$C\,C\,C\,2\,3\,3\,C\,C\,3\,2\,C\,1\,2 = C\,C\,3\,3\,C\,1\,2 = C\,1\,2 = 2.$$

Thus axiom /1/ cannot be a consequence of any of those 64 theses, neither can any of them be a sole axiom of the Implicational Calculus of Propositions, as it is unable to generate all implicational theses.

The same method is applied to the remaining 28 theses which depend in one way or other on *Peirce's* law. They can be divided into two groups: the first one contains 24 theses, which can be deduced from the following set of axioms:

$$/C/ \qquad \begin{array}{l} C\,p\,C\,q\,p. \\ C\,p\,C\,C\,p\,q\,q. \\ C\,C\,p\,q\,C\,C\,C\,p\,r\,q\,q. \\ C\,C\,p\,C\,q\,r\,C\,C\,C\,q\,r\,s\,C\,p\,s. \end{array}$$

All the theses of this set are verified by the four-valued matrix given below, "*1*" being again the selected value.

II

C	1	2	3	4
$*1$	1	2	3	3
2	1	1	3	3
3	1	2	1	1
4	1	1	1	1

Axiom /1/ cannot be a consequence either of set /C/ or of any of the 24 theses verified by matrix II, because if we perform the following substitution, "*p/2,*" "*q/1,*" "*r/4,*" "*s/3,*" we obtain:

$$CCC214CC42C32 = CC14C12 = C32 = 2.$$

The remaining 4 theses consisting of 11 letters, but not verified by either matrix I or II, are the following:

$$CCCpqpCCprr.$$
$$CCCpqrCCrpp.$$
$$CCpqCCCprpq.$$
$$CCpqCCCqrpq.$$

These theses are verified by matrix III, four-valued with "*1*" as the selected value.

III

C	1	2	3	4
$*1$	1	2	3	3
2	1	1	3	3
3	1	2	1	1
4	1	2	2	1

Axiom /1/ is not verified by matrix III because, if we perform the following substitution, "*p/3,*" "*q/2,*" "*r/1,*" "*s/4,*" we obtain:

$$CCC321CC13C43 = CC21C32 = C12 = 2.$$

It follows from the above considerations that every thesis shorter than axiom /1/ is verified by at least one of the matrices I, II, or III, whereas axiom /1/ is not verified by any of them. Therefore, there does not exist a thesis shorter than one consisting of 13 letters, which could be used as the sole axiom of the Implicational Calculus of Propositions. Whether besides axiom /1/ there are any other theses consisting of 13 letters which could function as sole axioms of the Calculus is not known.