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On Combining Probability and Nondeterminism [1]

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Abstract

The problem of combining nondeterminism and probability within a denotational model has been the subject of much research. Early work used schedulers to model probabilistic choice, interleaving their execution with that of nondeterministic choice, a theme that continues in some operational models today. More recent work has focused on providing a principled account of the interactions of these operators, with the aim of devising models that support both operators so that neither is related with the other. In this paper we recount the results along this line, and point out some places where further research is warranted.

Keywords: Probabilistic choice, valuations, domain theory, distributive law, CCS

1 Introduction

Nondeterminism has been a staple of process algebra since its inception, but more recently probabilistic choice has been included. While early work substituted probabilistic choice for nondeterminism, the more recent trend has to include both nondeterminism and probabilistic choice within the same algebra. One rationale for this is that nondeterminism represents a user's approach to electing which action to take, while probabilistic choice could capture the vagaries of the environment as random events occur during the running of a process. The problem in including both operators within the same algebra has been to find models that capture both types of choice, but in which neither has an influence on the other. For example, a model would be unsatisfactory if probabilistic choice depended on the nondeterministic choices that preceded it, or if a nondeterministic choice were determined by how probabilistic choices were resolved. The easiest way to assure this doesn't happen is to find a model in which the laws characterizing each operator are obeyed,

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and where there is no relation between a probabilistic choice of two processes and their nondeterministic choice. It has proved very difficult to find such a model. In fact, there are results that indicate such a model may not exist.

To begin, we recall that the laws for nondeterministic choice are those of a semilattice – nondeterministic choice should be commutative, associative and idempotent. Over a finite, unordered state space, an appropriate model is the power set, but if the underlying model is a domain, [2] then now-familiar results of Hennessy and Plotkin [5] show that there are three models to choose from: the *lower power domain*, the *upper power domain*, and the *convex power domain.* [3] Each of these forms the object level of a monad on various categories of domains, and in each case, the algebras of the monad are domain semilattices of the appropriate type.

The *probabilistic power domain* is the family of *valuations* $\phi\colon (\sigma(D), \subseteq) \to ([0,1], \leq)$ from the family of Scott-open sets of D to the unit interval which are Scott continuous, take the empty set to 0, and satisfy the inclusion—exclusion principle: $\phi(U \cup V) + \phi(U \cap V) = \phi(U) + \phi(V)$. They generate a model that satisfies the laws for probabilistic choice first elaborated by Graham [4]: If for $\lambda \in [0,1]$ we let $p \oplus_\lambda q$ denote the process with probability λ of acting like p and probability $1 - \lambda$ of acting like q, then for $\lambda, \xi \in [0,1]$ and processes p, q, r:

- $p \oplus_\lambda p = p$; • $p \oplus_\lambda q = q \oplus_{1-\lambda} p$; • $p \oplus_1 q = p$;
- $(p \oplus_\lambda q) \oplus_\xi r = \begin{cases} p \oplus_{\lambda\xi} (q \oplus_{\frac{1-\lambda}{1-\lambda\xi}} r), \text{ provided } \lambda < 1, \\ p \oplus_\xi r \qquad\qquad \text{otherwise.} \end{cases}$

The probabilistic power domain is a monad over domains, but beyond preserving continuity and coherence, [4] it is not known whether it is an endofunctor on any of the cartesian closed categories of domains.

An obvious way to create a model which would support both nondeterminism and probabilistic choice would be simply to apply one monad after the other. For example, Morgan, et al [12] take this approach with CSP by applying the probabilistic power domain to the failures–divergences model. The result is a model where probabilistic choice obeys the expected laws (because its monad was applied last), but nondeterminism is no longer idempotent. To see why, let $\sqcap$ denote nondeterministic choice and note that $\sqcap$, being lifted from the failures-divergence model to its probabilistic power domain in a pointwise fashion, distributes through $\oplus_\lambda$ for all λ; using the laws of probabilistic choice above, it follows that

$$\begin{aligned}(P \oplus_{\frac12} Q) \sqcap (P \oplus_{\frac12} Q) &= (P \oplus_{\frac12} (P \sqcap Q)) \oplus_{\frac12} ((Q \sqcap P) \oplus_{\frac12} Q) \\ &= P \oplus_{\frac14} ((P \sqcap Q) \oplus_{\frac23} Q),\end{aligned}$$

so we see nondeterminism and probabilistic choice have become intermingled.

[2] By a *domain* we mean a directed complete partial order in which every element is the directed sup of those elements that are way-below it; cf. [1] for details.

[3] These are the initial sup-semilattice domain, the initial inf-semilattice domain and the initial ordered semilattice domain over the underlying domain.

[4] A domain is *coherent* if its Lawson topology is compact. These domains arise often in applications; for example, both retracts of bfinite domains and FS-domains are coherent. However, coherent domains don't form a ccc.

The explanation for the intermingling of choice operators just witnessed is that the composition of monads is not always another monad. Beck [2] explored this question, and proved that monads compose if and only if there is a distributive law [5] of one over the other. The unfortunate fact is that Plotkin and Varacca [14,15] have shown that there is no distributive law of any of the nondeterminism monads over the probabilistic power domain, or vice versa, so composing any of the monads for nondeterminism with the probabilistic power domain won't result in another monad.

One approach to resolving this was described by Tix [13] (later revised and elaborated in [7]) and independently by the author [9]. It involves first applying the probabilistic power domain and then one of the power domains for nondeterminism, but then refining the nondeterminism monad to take account of the geometrically convex structure of the domain of probability measures. This results in analogs to the three power domains, each of which is realized as a retract of the usual power domain onto its subfamily of *geometrically-convex* [6] elements; for example, in the case of the upper power domain, the result is the power domain of geometrically convex, Scott-compact upper sets of the underlying domain. The resulting domains model both nondeterministic choice and probabilistic choice so that the laws of each are obeyed, but there is a relation between the resulting operators. For example, in the analog of the upper power domain, which is an inf-semilattice, the inequation $p \sqcap q \sqsubseteq p \oplus_\lambda q$ holds for every p, q and every $\lambda \in [0, 1]$. These models show that the standard power domain monads can be adjusted to account for geometrically convex structure, and in each case, the subdomain of geometrically convex elements is a retract of the original power domain. Moreover, it can be shown that in the case of the lower and upper power domains, these constructions applied to a coherent domain D yield a bounded complete domain.

An operational justification of one of the models devised by Tix / Mislove ([13,9]) was presented in [11], where using the theory of labeled Markov processes, it was shown that the construction gives a denotational model for a probabilistic extension of a simple sublanguage of CCS that is fully abstract with respect to a notion of *partial probabilistic bisimulation:* processes P and Q satisfy $[\![P]\!] \sqsubseteq [\![Q]\!]$ in the model iff whenever P satisfies a formula from a particular domain logic $\mathcal{L}$, [7] then Q also satisfies the formula. Moreover, this probabilistic extension is conservative over CCS, meaning that purely CCS processes are identified in the model iff they are identified as CCS processes. It remains to expand this line of research to include a more representative subalgebra of CCS with probabilistic choice appended.

Another resolution of the search for a model for nondeterminism and probabilistic choice was devised by Varacca [14], who realized that altering the laws defining the monads would allow such a distributive law. Varraca took his cue from a result of Gautem [3] that asserts that an algebraic theory modeled on a set lifts pointwise to the power set iff each equation in the theory mentions each variable at most once

[5] A *distributive law* of a monad S over a monad T is a natural transformation $d: S \circ T \stackrel{\cdot}{\longrightarrow} T \circ S$ satisfying additional laws. These generalize the usual notion of one algebraic operation distributing over another.

[6] A set X is *geometrically convex* if $x, y \in X$ and $\lambda \in [0, 1]$ imply $x \oplus_\lambda y \in X$.

[7] By a *domain logic*, we mean one that characterizes the order on the domain of interest.

on each side of the equation. The problem in the case of probabilistic choice is the law $p \oplus_\lambda p = p$, and so he eliminated this law. The result was a theory in which this equality is replaced in one of three ways – as an inequality in one direction or the other, or with no relation between the components. Varacca devised models called *indexed valuations*—one for each of the three possible relations between $p \oplus_\lambda p$ and p—that define monads each of which enjoys a distributive law with respect to at least one of the nondeterminism monads. Varacca also provides an operational justification of his construction (at least in the case that the state space is a set) by proving adequacy theorems for his construction as denotational models. The operational model makes much finer distinctions than usual, however, since it records how each probabilistic choice is resolved.

Further work using Varacca's ideas can be found in [8] where it is shown that one of the constructions can be viewed as the family of discrete random variables over a domain, and a slight modification of this construction leaves the ccc's RB and FS of (continuous) domains[8] both invariant. This provides the first model of probabilistic computation that has this property. The work in [8] relies on some interesting results about the structure of bag domains over a domain [10]. For example, one construction shows how the partial order on an initial domain monoid can be refined so that a given embedding–surjection pair becomes an embedding–projection pair.

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[8] RB is the category of retracts of bifinite domains, and FS is the category of domains for which the identity is the supremum of maps finitely separated from the identity; each is a ccc, the latter being maximal.

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