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r-Indecomposable and r-nearly decomposable matrices ☆

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Abstract

Let n, r be integers with $0 \leqslant r \leqslant n-1$. An $n \times n$ matrix A is called *r-partly decomposable* if it contains a $k \times l$ zero submatrix with $k+l=n-r+1$. A matrix which is not r-partly decomposable is called *r-indecomposable* (shortly, r-inde). Let E_{ij} be the $n \times n$ matrix with a 1 in the (i, j) position and 0's elsewhere. If A is r-indecomposable and, for each $a_{ij} \neq 0$, the matrix $A - a_{ij}E_{ij}$ is no longer r-indecomposable, then A is called *r-nearly decomposable* (shortly, r-nde). In this paper, we derive numerical and enumerative results concerning r-nde matrices of 0's and 1's. We also obtain some bounds on the index of convergence of r-inde matrices, especially for the adjacency matrices of primitive Cayley digraphs and circulant matrices. Finally, we propose an open problem for further research.

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1. Introduction

Let B_n denote the set of all matrices of order n over the Boolean algebra $\{0, 1\}$. Let $J_n \in B_n$ be the matrix whose entries are all equal to 1, and let $E_{ij} \in B_n$ be the matrix with a 1 in the (i, j) position and 0's elsewhere. There is a one to one correspondence between B_n and all digraphs $D = (V, E)$ with vertex set $V = \{1, 2, \dots, n\}$ and arc set $E = \{(i, j) \in V \times V : a_{ij} = 1\}$. The digraph D corresponding to $A \in B_n$ is called the *associated digraph* of A, denoted by $D(A)$; and the matrix A corresponding to D is called the *adjacency matrix* of D, denoted by $A(D)$. A digraph D is *strong* if, for any two vertices x and y, D contains a path from x to y and a path from y to x. A matrix $A \in B_n$ is *irreducible* if its associated digraph is strong.

Let n, r be integers with $0 \leqslant r \leqslant n - 1$. An $n \times n$ matrix A is called *r-partly decomposable* if it contains a $k \times l$ zero submatrix with $k + l = n - r + 1$. A matrix which is not r-partly decomposable is called *r-indecomposable* (shortly, r-inde). If a matrix A is r-inde and, for each $a_{ij} \neq 0$, the matrix $A - a_{ij}E_{ij}$ is no longer r-indecomposable, then A is called *r-nearly decomposable* (shortly, r-nde). In particular, a 0-inde matrix is called a Hall matrix, and a 1-inde (resp. 1-nde) matrix is called a fully indecomposable (resp. nearly decomposable) matrix. If A is r-inde (resp. r-nde), its associated digraph $D(A)$ is called r-inde (resp. r-nde).

A matrix A is said to have a positive diagonal if there exists a permutation φ of $\{1, 2, \dots, n\}$ such that all entries $a_{1\varphi(1)}, a_{2\varphi(2)}, \dots, a_{n\varphi(n)}$ are positive. If $a_{ii} > 0$ for all i, then A is said to have a positive main diagonal, denoted by $I \leqslant A$.

Let $A = (a_{ij}) \in B_n$, and let s and t be integers with $1 \leqslant s, t \leqslant n$. For $1 \leqslant i_1 < i_2 < \cdots < i_s \leqslant n$ and $1 \leqslant j_1 < j_2 < \cdots < j_t \leqslant n$, we write $\alpha = (i_1, i_2, \dots, i_s)$ and $\beta = (j_1, j_2, \dots, j_t)$. Let $A[i_1, i_2, \dots, i_s|j_1, j_2, \dots, j_t] = A[\alpha|\beta]$ denote the submatrix of A whose (p, q) entry is $a_{i_p j_q}$; that is, $A[\alpha|\beta]$ is obtained from A by deleting those rows not indexed in α and columns not indexed in β. Similarly, $A[\alpha|\beta)$ denotes the matrix obtained from A by deleting the rows not indexed in α and columns indexed in β. The matrices $A(\alpha|\beta]$ and $A(\alpha|\beta)$ are defined analogously. Let $\lfloor x \rfloor$ denote the maximum integer s such that $s \leqslant x$, and let $\lceil x \rceil$ denote the least integer s such that $s \geqslant x$.

Let $n \geqslant m \geqslant 1$ be integers and let $A = (a_{ij}) \in B_{m\times n}$. The permanent of A is

$$\mathrm{Per}(A) = \sum_{(i_1, i_2, \dots, i_m) \in P_m^n} a_{1i_1} a_{2i_2} \dots a_{mi_m},$$

where P_m^n is the set of all m-permutations of $\{1, 2, \dots, n\}$.

For an $n \times n$ Boolean matrix A, the behavior of the sequence $A, A^2, A^3, \dots$ mainly depends on two parameters: the period of A and the index of convergence of A. The period of A, denoted by $p(A)$, is the least positive integer p such that $A^{l+p} = A^l$ for some l, and the index of convergence of A, denoted by $k(A)$, is the least value of l for which $A^{l+p} = A^l$ holds. If A is irreducible with $A \neq [0]$ and $p(A) = 1$, we call A primitive, and in which case the index of convergence of A is called the exponent of A, denoted by $\exp(A)$. It is well known that A is primitive if and only if there exists a positive integer k such that $A^k = J$.

Let $\sigma(A)$ be the number of entries of A equal to 1, and let $m(D)$ be the number of arcs in $D(A)$. Then $\sigma(A) = m(D)$. Let

$$f(n, r) = \max\{\sigma(A) : A \in B_n \text{ is } r\text{-nde}\}$$

and

$$g(n, r) = \min\{\sigma(A) : A \in B_n \text{ is } r\text{-nde}\}.$$

Clearly, a matrix $A \in B_n$ is $(n-1)$-inde or $(n-1)$-nde if and only if $A = J$. So throughout the paper we always assume $r < n - 1$ and only discuss this non-trivial case. In [1], Brualdi and Hedrick derived some structural, numerical, and enumerative results concerning nearly decomposable matrix of 0's and 1's. For $r = 1$ (and $n > r + 1 = 2$), they proved that

$$f(n, 1) = 3n - 3 \quad \text{and} \quad g(n, 1) = 2n.$$

In this paper, we continue the study of the two functions $f(n, r)$ and $g(n, r)$ for $r \geqslant 2$. The paper is organized as follows. In Section 2, we give some sufficient and necessary conditions on r-inde matrices by means of permanent and r-connected digraph (r-irreducible matrix), and give some examples for r-inde matrices. In Section 3, we prove that $g(n, r) = n(r + 1)$ and that

$$f(n, r) \geqslant f'(n, r) = \begin{cases} \left\lfloor \frac{(n+r+1)^2}{4} \right\rfloor & \text{if } n < 3r, \\ (n - r)(2r + 1) & \text{if } n \geqslant 3r. \end{cases}$$

Moreover, for each i with $g(n, r) \leqslant i \leqslant f'(n, r)$, we construct an r-nde matrix $A \in B_n$ with $\sigma(A) = i$. This extends a result [1, Theorem 3.4] for $r = 1$ by Brualdi and Hedrick. In Section 4, we discuss the exponent of r-inde and r-nde matrices and provide a new and simpler proof for a result [4, Theorem 2.1] by Huang. In Section 5, we propose an open problem on r-nde matrices as a suggestion for further research.

2. *r*-Indecomposable matrices

Theorem 2.A (Frobenius-Konig). *The permanent of an $n \times n$ non-negative matrix A is zero if and only if A contains an $s \times t$ zero submatrix with $s + t = n + 1$.*

Lemma 2.1. *Suppose $0 \leqslant r \leqslant n - 1$ and $A \in B_n$. Then the following are equivalent.*

(i) *The matrix A is r-inde.*
(ii) *For each k with $1 \leqslant k \leqslant n - r$, the matrix A does not have any $k \times l$ zero submatrix with $k + l = n - r + 1$.*
(iii) *For each k with $1 \leqslant k \leqslant n - r$, every $k \times n$ submatrix of A has at least $k + r$ non-zero columns.*

(iv) *For each subset* $S \subseteq V(D(A))$ *with* $1 \leqslant |S| = k \leqslant n - r$, *the inequality* $|N(S)| \geqslant k + r$ *holds, where* $N(S) = \{v \in V(D(A)) :$ *there is an arc* (u, v) *for some* $u \in S\}$ *is the set of out-neighbors of* S *in* $D(A)$.

Proof. (i) $\Longleftrightarrow$ (ii) follows from the definition of r-indecomposability. The matrix interpretation of (ii) is equivalent to (iii). The graph interpretation of (iii) is equivalent to (iv). □

It is evident that an r-inde matrix is a 1-inde matrix for any $r \geqslant 1$, and a 1-inde matrix is primitive. Let $B_{n,r}$ be the set of all r-inde matrices of order n.

Lemma 2.2. $\{J\} = B_{n,n-1} \subset B_{n,n-2} \subset \cdots \subset B_{n,2} \subset B_{n,1} \subset B_{n,0}$, *and every* r*-inde matrix with* $r \geqslant 1$ *is primitive.*

Theorem 2.1. *An* $n \times n$ *non-negative matrix* A *is* r*-inde if and only if* $\operatorname{Per}(A(\alpha|\beta)) > 0$ *for any* $\alpha, \beta \subseteq \{1, 2, \dots, n\}$ *with* $|\alpha| = |\beta| = r$.

Proof. "$\Longrightarrow$" Suppose otherwise $\operatorname{Per}(A(\alpha|\beta)) = 0$ for some α and β. Then by Theorem 2.A, the submatrix $A(\alpha|\beta)$ and thus A contain an $s \times t$ zero submatrix with $s + t = (n - r) + 1$, a contradiction to the r-indecomposability of A.

"$\Longleftarrow$" Suppose A has an $s \times t$ zero submatrix C such that $s + t = n - r + 1$. Then $s, t \leqslant n - r$. Thus C is a submatrix of $A(\alpha|\beta)$ for some $\alpha, \beta \subseteq \{1, 2, \dots, n\}$ with $|\alpha| = |\beta| = r$. By Theorem 2.A, we have $\operatorname{Per}(A(\alpha|\beta)) = 0$, a contradiction. Therefore A is r-inde. □

Let $r \geqslant 1$. A digraph D with at least $r + 1$ vertices is called *r-connected* if each digraph obtained from D by removing any $r - 1$ vertices is strong. Thus an r-connected digraph has the property that, for any $r + 1$ distinct vertices $i_1, i_2, \dots, i_r, i_{r+1}$, there exists a path from i_1 to i_{r+1} without containing $i_2, \dots, i_r$.

Theorem 2.2. *Suppose* $r \geqslant 1$ *and* $I_n \leqslant A \in B_n$. *Then* A *is* r*-inde if and only if* $D(A)$ *is* r*-connected.*

Proof. "$\Longrightarrow$" Suppose A is r-inde. By Theorem 2.1, $A(\alpha|\beta)$ is 1-inde for any $\alpha = \beta \subseteq \{1, 2, \dots, n\}$ with $|\alpha| = |\beta| = r - 1$. So $D(A)$ is r-connected.

"$\Longleftarrow$" Suppose A is r-partly decomposable. Then there exists a $p \times q$ zero submatrix C with $p + q = n - r + 1$. Since $I_n \leqslant A$, the zero submatrix C does not contain any entry on the main diagonal. Thus, there exist $i_1, i_2, \dots, i_p, j_1, j_2, \dots, j_{r-1} \in \{1, 2, \dots, n\}$ such that $A[i_1, i_2, \dots, i_p|i_1, i_2, \dots, i_p, j_1, j_2, \dots, j_{r-1}) = C$. Then the digraph obtained from $D(A)$ by removing the $r - 1$ vertices corresponding to the j_1th, j_2th, $\dots, j_{r-1}$th rows of A is not strong, a contradiction to the r-connectedness of $D(A)$. □

A matrix $A \in B_n$ is called *r-reducible* if by taking a simultaneous row permutation and column permutation, A is similar to the form $\begin{bmatrix} A_{11} & A_{12} & O \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$, where A_{11} and $\begin{bmatrix} A_{22} & A_{23} \end{bmatrix}$ are square matrices of order at least one and the size of O is $p \times (n-r+1-p)$. If A is not r-reducible, then A is called *r-irreducible*. The property of r-irreducibility has the following interpretation in terms of its associated digraph.

Lemma 2.3. *Suppose $r \geqslant 1$. Then a matrix A is r-irreducible if and only if its associated digraph D is r-connected.*

Proof. "$\Longleftarrow$" Suppose A is r-reducible. Then A contains a $p \times (n-r+1-p)$ zero submatrix. Without loss of generality, we suppose the rows of the zero submatrix correspond to vertices $1, 2, \ldots, p$ and the columns correspond to vertices $p+r, \ldots, n$. Let D_1 be obtained from D by deleting vertices $p+1, \ldots, p+r-1$. Then the adjacency matrix of D_1 is reducible. Thus D_1 is not strong, a contradiction.

"$\Longrightarrow$" Suppose that D is not r-connected. Then there exists a non-strong digraph D_1 obtained from D by removing $r-1$ vertices. Let A_1 be the adjacency matrix of D_1. Then there exists a permutation matrix P such that $PA_1P^{-1} = \begin{bmatrix} A_{11} & O \\ C & B \end{bmatrix}$, where A_{11} and B are square matrices of order at least one and O is a $p \times (n-r+1-p)$ zero matrix. Thus PAP^{-1} and so A contain a $p \times (n-r+1-p)$ zero matrix, a contradiction. $\square$

It is easy to see that the property of r-indecomposability is preserved under row permutations and column permutations. Suppose A is r-inde. Then there exist permutation matrices P and Q such that PAQ has a positive main diagonal and PAQ is r-inde.

Theorem 2.3. *A matrix A is r-inde if and only if there exist permutation matrices P and Q such that PAQ is r-irreducible and PAQ has a positive main diagonal.*

Now we show some examples of digraphs with high indecomposability. Let G be a multiplicative group with identity element e, and let $A = \{a_1, \ldots, a_k\}$ be a subset of G. The (*right*) *Cayley digraph* is the digraph $\mathrm{Cay}(G, A) = (V, E)$ where $V = G$ and $E = \{(x, y) : x^{-1}y \in A\}$. Thus $\mathrm{Cay}(G, A)$ is regular of outdegree $k = |A|$.

Lemma 2.4 (Shen and Gregory [3]). *Let $A = \{a_1, \ldots, a_k\}$ be a subset of an Abelian group G. Then* $\mathrm{Cay}(G, A)$ *is primitive if and only if* $\mathrm{Cay}(G, A_1)$ *is strong, where $A_1 = \{a_i a_1^{-1} : 1 \leqslant i \leqslant k\}$.*

Lemma 2.5 (Hamidoune [2]). *Any loopless strong vertex-transitive digraph with outdegree k is $(\lfloor k/2 \rfloor + 1)$-connected.*

Theorem 2.4. *Let $A = \{a_1, \ldots, a_k\}$ be a subset of an Abelian group G. Suppose* $\mathrm{Cay}(G, A)$ *is primitive. Then* $\mathrm{Cay}(G, A)$ *is* $\lceil k/2 \rceil$*-indecomposable.*

Proof. Let $A_1 = \{a_i a_1^{-1} : 1 \leqslant i \leqslant k\}$. Let M_1 and M_2 be the adjacency matrices of $\mathrm{Cay}(G, A)$ and $\mathrm{Cay}(G, A_1)$, respectively. Then $M_2 = M_1 P$, where P is the permutation matrix for $\sigma : g \to g a_1^{-1}$; that is, an entry of P is 1 if and only if this entry has row index g and column index $g a_1^{-1}$ for some $g \in G$. This implies that M_2 can be obtained from M_1 by permuting columns of M_1. Thus M_1 and M_2 have the same indecomposability, so do $\mathrm{Cay}(G, A)$ and $\mathrm{Cay}(G, A_1)$. Since $\mathrm{Cay}(G, A)$ is primitive, by Lemma 2.4, we know that $\mathrm{Cay}(G, A_1)$ is strong. Then, by Lemma 2.5, The digraph $\mathrm{Cay}(G, A_1)$ is $\lfloor (k-1)/2 \rfloor + 1 = \lceil k/2 \rceil$-connected. Since $\mathrm{Cay}(G, A_1)$ has a loop at each vertex, by Theorem 2.2, it is $\lceil k/2 \rceil$-inde. Therefore, the digraph $\mathrm{Cay}(G, A)$ is $\lceil k/2 \rceil$-inde. □

Note that Theorem 2.4 may not hold for non-Abelian Cayley digraphs since Lemma 2.4 does not hold for non-Abelian Cayley digraphs. Nevertheless, the techniques in Theorem 2.4 work for non-Abelian Cayley digraphs. That is, for any group G (Abelian or non-Abelian), the Cayley digraphs $\mathrm{Cay}(G, A)$ and $\mathrm{Cay}(G, A_1)$ have the same indecomposability. Furthermore, if $\mathrm{Cay}(G, A_1)$ is strong, then $\mathrm{Cay}(G, A)$ is $\lceil k/2 \rceil$-indecomposable.

3. Bounds on the number of 1's in r-nde matrices

Let $\sigma(A)$ be the number of entries of A equal to 1, and let $m(D)$ be the number of arcs in $D(A)$. Then $\sigma(A) = m(D)$. Let $f(n, r) = \max\{\sigma(A) : A \in B_n \text{ is } r\text{-nde}\}$ and $g(n, r) = \min\{\sigma(A) : A \in B_n \text{ is } r\text{-nde}\}$.

Example 3.1. Let N be the adjacency matrix of the Cayley digraph $\mathrm{Cay}(Z_n, \{1, 2, \ldots, r+1\})$. Then N is r-nde with $\sigma(N) = n(r+1)$.

Proof. Since there exists a permutation matrix P such that $M = NP$ with $D(M) = \mathrm{Cay}(Z_n, \{0, 1, 2, \ldots, r\})$, it suffices to prove M is r-nde. By Theorem 2.2, it suffices to prove $\mathrm{Cay}(Z_n, \{0, 1, 2, \ldots, r\})$ is r-connected. We use induction to prove the latter statement. For $r = 1$, certainly $\mathrm{Cay}(Z_n, \{0, 1\})$ is 1-connected. Now suppose $\mathrm{Cay}(Z_{n-1}, \{0, 1, 2, \ldots, r-1\})$ is $(r-1)$-connected. We observe that the removal of any vertex from $\mathrm{Cay}(Z_n, \{0, 1, 2, \ldots, r\})$ results a superdigraph of $\mathrm{Cay}(Z_{n-1}, \{0, 1, 2, \ldots, r-1\})$ on the same vertex set. Thus the resultant superdigraph is $(r-1)$-connected. This implies that $\mathrm{Cay}(Z_n, \{0, 1, 2, \ldots, r\})$ is r-connected. □

The following lemma follows from the above example and the fact that an r-inde matrix has at least $r+1$ 1's in each row and each column.

Lemma 3.1. $g(n, r) = n(r + 1)$.

Definition 3.1. Suppose $M \in B_{n-t}$ is an $(r - t)$-nde matrix with $\sigma(M) = (r - t + 1)(n - t)$. We define

$$A_t = \left[\begin{array}{c|c} O_{t\times t} & J_{t\times(n-t)} \\ \hline J_{(n-t)\times t} & M_{(n-t)\times(n-t)} \end{array}\right].$$

Clearly, $\sigma(A_t) = (n - t)(r + 1 + t)$.

Lemma 3.2. *Suppose $n \geqslant r + 2p$, $1 \leqslant p \leqslant r$. Then $A_1, A_2, \ldots, A_p$ are r-nde.*

Proof. First, we want to prove that A_t is r-inde for all t, $1 \leqslant t \leqslant p$. Let $Y = V(D(M))$, $X = V(D(A_t)) - V(D(M))$. Then $|Y| = n - t$, $|X| = t$. Let $S \subseteq V(D(A_t))$ with $1 \leqslant |S| = k \leqslant n - r$.

Case 1: $S \subseteq X$. Then $k \leqslant t$ and

$$|N(S)| = |Y| = n - t \geqslant (r + 2p) - t = r + p + (p - t) \geqslant r + p \geqslant r + k.$$

Case 2: $S \subseteq Y$. Since M is $(r - t)$-nde, by Lemma 2.1, we have $|N(S) \cap Y| \geqslant |S| + r - t$. Thus

$$|N(S)| \geqslant |S| + r - t + |X| = |S| + r - t + t = k + r.$$

Case 3: $S = S_1 \cup S_2$, where $S_1 \subseteq X$, $S_2 \subseteq Y$. Then

$$|N(S)| = |N(S_1) \cup N(S_2)| = |Y| + |X| = n \geqslant k + r.$$

Thus, by Lemma 2.1, A_t is r-inde. Furthermore, A_t is r-nde since, for each nonzero (i, j) entry of A_t, either row i or column j contains exactly $(r + 1)$ 1's. □

$$\text{Let } f'(n, r) = \begin{cases} \left\lfloor \frac{(n+r+1)^2}{4} \right\rfloor & \text{if } n < 3r, \\ (n - r)(2r + 1) & \text{if } n \geqslant 3r. \end{cases}$$

Theorem 3.1. *Let $f(n, r)$ and $f'(n, r)$ be defined as above, where $n > r + 1$, then $f(n, r) \geqslant f'(n, r)$.*

Proof. Let $p = \min\{r, \lfloor (n - r)/2 \rfloor\}$. Then $n \geqslant r + 2p$ and $p \leqslant r$. By Lemma 3.2, the matrix A_p is r-nde. Thus $f(n, r) \geqslant \sigma(A_p) = (n - p)(r + p + 1) = f'(n, r)$. □

Now we want to show that, for any integer i with $g(n, r) \leqslant i \leqslant f'(n, r)$, there is always an r-nde matrix A with order n and $\sigma(A) = i$. We make use of a particular class of matrices from Definition 3.1. Let N be the adjacency matrix of

$\mathrm{Cay}(Z_{n-t}, \{1, 2, \ldots, r-t+1\})$. Then Example 3.1 shows that N is $(r-t)$-nde with $\sigma(N) = (n-t)(r-t+1)$. Let

$$C_t = \left[\begin{array}{c|c} O_{t\times t} & J_{t\times(n-t)} \\ \hline J_{(n-t)\times t} & N \end{array}\right].$$

By Lemma 3.2, C_t is r-nde if $n \geqslant r+2t$ and $1 \leqslant t \leqslant r$. For a fixed integer l with $t+1 \leqslant l \leqslant n$, we define a matrix $D = (d_{ij}) \in B_n$ as follows:

$$d_{ij} = \begin{cases} 0 & \text{if either } i = l,\ j = t \text{ or } i = t,\ j = l, \\ 1 & \text{if } i = j = l, \\ c_{ij} & \text{otherwise.} \end{cases}$$

We say that D is obtained from C_t by the $\langle l, t\rangle$-*interchanges*.

Lemma 3.3. *Suppose $n \geqslant r+2t$, $1 \leqslant t \leqslant r$ and $n \geqslant r+t+2$. Let $D \in B_n$ be obtained from C_t by the $\langle i, t\rangle$-interchanges where $t+1 \leqslant i \leqslant n$. Then D is r-nde.*

Proof. Suppose on the contrary that D is r-partly decomposable. Then D contains an $s \times (n-r+1-s)$ zero submatrix. Since $n-t-1 \geqslant r+1$, every row and column of D has at least $r+1$ positive entries. Thus $s \geqslant 2$ and $n-r+1-s \geqslant 2$. Since C_t is r-inde and it differs from D only in the (i, t), (t, i) and (i, i) positions, the zero submatrix contains $d_{i,t}$ or $d_{t,i}$. Without loss of generality, we suppose the zero submatrix contains $d_{i,t}$. But since $d_{i,1} = d_{i,2} = \cdots = d_{i,t-1} = 1$, $d_{i,i} = d_{i,i+1} = \cdots = d_{i,i+(r-t)+1} = 1$, the $s \times (n-r+1-s)$ zero submatrix is contained in $D[1, 2, \ldots, n|1, 2, \ldots, t-1, i, i+1, \ldots, i+(r-t)+1)$, where addition is taken modulo $n-t$. However, this implies that the submatrix F obtained from C_t by deleting columns $1, 2, \ldots, t-1, t, i, i+1, \ldots, i+(r-t)+1$ would contain an $(s+1) \times (n-r-s)$ zero submatrix, since all entries in row $i-1$ of F are zero. This shows that C_t contains the zero submatrix F, a contradiction to the r-indecomposability of C_t. Thus D is r-inde. We observe that, for any non-zero (i, j) entry of D, either row i or column j of D has exactly $(r+1)$ 1's. Therefore D is r-nde. □

We construct a series of matrices from C_t as follows:

1. The matrix $L_{1,t}$ is obtained from C_t by the $\langle n, t\rangle$-interchanges.
2. For each $i \in \{1, 2, \ldots, i_t - 1\}$ with $i_t = \min\{\lfloor \frac{n-t}{2} \rfloor, n-t-r-1\}$, the matrix $L_{2i+1,t}$ is obtained from $L_{2i-1,t}$ by the $\langle n-2i, t\rangle$-interchanges.

The following lemma can be proved similarly by applying the same proof techniques shown in Lemma 3.3.

Lemma 3.4. *For any $i \in \{1, 2, \ldots, i_t\}$ with $i_t = \min\{\lfloor \frac{n-t}{2} \rfloor, n-t-r-1\}$, the matrix $L_{2i-1,t}$ is r-nde and $\sigma(C_t) - \sigma(L_{2i-1,t}) = i$.*

We define $i_t = \min\{\lfloor\frac{n-t}{2}\rfloor, n-t-r-1\}$ and $i'_t = \begin{cases} i_t - 1 & \text{if } i_t = \lfloor\frac{n-t}{2}\rfloor, \\ i_t & \text{otherwise.} \end{cases}$

Lemma 3.5. *Suppose $n \geqslant r+2t$, $2 \leqslant t \leqslant r$ and $1 \leqslant i \leqslant t$. Let D be obtained from $L_{2i_t-1,t}$ by the $\langle n-2i+1, t-1\rangle$-interchanges. Then D is r-nde.*

Proof. Suppose on the contrary that D is r-partly decomposable. Then D contains an $s \times (n-r+1-s)$ zero submatrix. Since $n-t-i_t \geqslant r+1$, every row and column of D has at least $r+1$ positive entries. Also one can easily verify that every $2 \times n$ submatrix of D has at least $r+2$ non-zero columns and every $n \times 2$ submatrix of D has at least $r+2$ non-zero rows. Thus $s \geqslant 3$ and $n-r+1-s \geqslant 3$. Since $L_{2i_t-1,t}$ is r-inde and it differs from D only in the $(n-2i+1, t-1)$, $(t-1, n-2i+1)$ and $(n-2i+1, n-2i+1)$ positions, the zero submatrix contains $d_{n-2i+1,t-1}$ or $d_{t-1,n-2i+1}$. Without loss of generality, we suppose the zero submatrix contains $d_{n-2i+1,t-1}$. But since $d_{n-2i+1,1} = d_{n-2i+1,2} = \cdots = d_{n-2i+1,t-2} = d_{n-2i+1,t} = 1$, $d_{n-2i+1,n-2i+1} = d_{n-2i+1,n-2i+1+1} = \cdots = d_{n-2i+1,n-2i+1+(r-t)+1} = 1$, the $s \times (n-r+1-s)$ zero submatrix is contained in $D[1, 2, \ldots, n | 1, 2, \ldots, t-2, t, n-2i+1, n-2i+1+1, \ldots, n-2i+1+(r-t)+1)$. However, this implies that the submatrix F obtained from $L_{2i_t-1,t}$ by deleting columns $1, 2, \ldots, t-2, t-1, t, n-2i, n-2i+1, n-2i+1+1, \ldots, n-2i+1+(r-t)+1$ would contain an $(s+2) \times (n-r-s-1)$ zero submatrix, since all entries in rows $n-2i$ and $n-2i-1$ of F are zero. This is a contradiction to the r-indecomposability of $L_{2i_t-1,t}$. Thus D is r-inde. Furthermore, D is r-nde since, for each non-zero (i, j) entry of D, either row i or column j has exactly $(r+1)$ 1's. $\square$

We construct a series of r-nde matrices from $L_{2i_t-1,t}$ as follows:

1. The matrix $L_{2,t}$ is obtained from $L_{2i_t-1,t}$ by the $\langle n-1, t-1\rangle$-interchanges.
2. For each $i \in \{1, 2, \ldots, i'_t - 1\}$, the matrix $L_{2i+2,t}$ is obtained from $L_{2i,t}$ by the $\langle n-2i-1, t-1\rangle$-interchanges.

Lemma 3.6. *Suppose $n \geqslant r+2t$, $2 \leqslant t \leqslant r$ and $1 \leqslant i \leqslant t$. Then $L_{2i,t}$ is r-nde and*

$$\sigma(C_t) - \sigma(L_{2i'_t,t}) = i_t + i'_t = \begin{cases} 2\lfloor\frac{n-t}{2}\rfloor - 1 & \textit{if } \lfloor\frac{n-t}{2}\rfloor \leqslant n-t-r-1, \\ 2(n-t-r-1) & \textit{if } \lfloor\frac{n-t}{2}\rfloor > n-t-r-1. \end{cases}$$

For $n \geqslant r+2t$ with $t = 1$, we have the following construction:

1. The $L_{1,1}$ is obtained from C_1 by the $\langle n, 1\rangle$-interchanges.
2. For each $i \in \{1, 2, \ldots, n-r-3\}$, the matrix $L_{i+1,1}$ is obtained from $L_{i,1}$ by the $\langle n-i, 1\rangle$-interchanges.

Lemma 3.7. *Suppose $n \geqslant r + 2t$ with $t = 1$. Then, for each $i \in \{1, 2, \dots, n - r - 2\}$, $L_{i,1}$ is r-nde and $\sigma(C_1) - \sigma(L_{i,1}) = i$.*

Brualdi and Hedrick [1, Theorem 3.4] showed that for each i with $g(n, 1) \leqslant i \leqslant f'(n, 1)$ and $n \geqslant 2$, there is always a 1-nde matrix A with order n and $\sigma(A) = i$. Our next theorem extends their result.

Theorem 3.2. *For any $r \in \{1, 2, \dots, n - 2\}$ and any $i \in [g(n, r), f'(n, r)]$, there exists an r-nde matrix A with order n and $\sigma(A) = i$.*

Proof. *Case 1: $r = 1$.* Then, by Lemma 3.7, $C_1, L_{1,1}, L_{2,1}, \dots, L_{n-3,1}$ are all 1-nde matrices and the number of positive entries in these matrices cover the entire interval from $2n$ to $3n - 3$.

Case 2: $r \geqslant 2$. Let positive integer $p = \min\{r, \lfloor (n - r)/2 \rfloor\}$. Then $n \geqslant r + 2p$ and $1 \leqslant p \leqslant r$. By Lemmas 3.4 and 3.6, the following matrices are all r-nde:

$$
\begin{aligned}
&C_p, L_{1,p}, L_{3,p}, \dots, L_{2i_p-1,p}, L_{2,p}, L_{4,p}, \dots, L_{2i'_p,p};\\
&C_{p-1}, L_{1,p-1}, L_{3,p-1}, \dots, L_{2i_{p-1}-1,p-1}, L_{2,p-1}, L_{4,p-1}, \dots, L_{2i'_{p-1},p-1};\\
&\dots;\\
&C_2, L_{1,2}, L_{3,2}, \dots, L_{2i_2-1,2}, L_{2,2}, L_{4,2}, \dots, L_{2i'_2,2};\\
&C_1, L_{1,1}, L_{2,1}, \dots, L_{n-r-2,1}.
\end{aligned}
$$

Since

$$
\begin{aligned}
\sigma(C_t) - \sigma(C_{t-1}) &= n - 2t - r\\
&\leqslant \begin{cases} 2\lfloor \frac{n-t}{2} \rfloor - 1 & \text{if } \lfloor \frac{n-t}{2} \rfloor \leqslant n - t - r - 1,\\ 2(n - t - r - 1) & \text{if } \lfloor \frac{n-t}{2} \rfloor > n - t - r - 1, \end{cases}
\end{aligned}
$$

by Lemma 3.6, $\sigma(L_{2i'_t,t}) \leqslant \sigma(C_{t-1})$ for any $2 \leqslant t \leqslant p$. So the number of positive entries in the above sequence of matrices cover the entire interval $[g(n, r), f'(n, r)]$. $\square$

4. The exponents of r-inde and r-nde matrices

The product of r-inde matrices behaves nicely with respect to the indecomposability. This quickly leads to some upper bounds on the index of convergence of r-inde matrices.

Lemma 4.1. *Suppose $A_1, A_2 \in B_n$ are r_1-inde and r_2-inde, respectively. Then the product A_1A_2 is r-inde, where $r = \min\{n - 1, r_1 + r_2\}$.*

Proof. Let v be a row vector of length n with all entries non-negative. Let $|v|$ be the number of positive entries in v. Then, by Lemma 2.1(iv), it is easy to see that a matrix A is r-inde if and only if $|vA| \geqslant \min\{n, |v| + r\}$ for all such v with $|v| > 0$. Since A_1, A_2 are r_1-inde and r_2-inde, respectively, we have $|vA_1A_2| \geqslant \min\{n, |vA_1| + r_2\} \geqslant \min\{n, |v| + r_1 + r_2\}$. Therefore A_1A_2 is r-inde, where $r = \min\{n - 1, r_1 + r_2\}$. □

Corollary 4.1. *Suppose $r \geqslant 1$ and A is r-inde. Then* $\exp(A) \leqslant \lceil (n-1)/r \rceil$.

Proof. Let $k = \lceil (n-1)/r \rceil$. Then $rk \geqslant n - 1$. Since A is r-inde, by Lemma 4.1, the matrix A^k is $\min\{n - 1, rk\} = (n - 1)$-inde. Thus $A^k = J$. □

The following corollary follows from Theorem 2.4 and Corollary 4.1.

Corollary 4.2. *Let $A = \{a_1, \ldots, a_k\}$ be a subset of an Abelian group G. Suppose* $\mathrm{Cay}(G, A)$ *is primitive. Then* $\exp(\mathrm{Cay}(G, A)) \leqslant \lceil \frac{n-1}{\lceil k/2 \rceil} \rceil$.

Let

$$P = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \end{bmatrix}_{n\times n}.$$

A circulant Boolean matrix is a matrix of the form $C = P^{a_1} + P^{a_2} + \cdots + P^{a_k}$ $(0 \leqslant a_1 < a_2 < \cdots < a_k < n)$. We denote it by $C\langle a_1, a_2, \ldots, a_k; n\rangle$ for convenience. The set of all circulants of order n forms a multiplicative semigroup C_n with $|C_n| = 2^n$. The following corollary was originally proved by Huang by using several lemmas. Now we can see that it follows immediately from Corollary 4.2 since the associated digraph of $C\langle a_1, a_2, \ldots, a_k; n\rangle$ is $\mathrm{Cay}(Z_n, \{a_1, a_2, \ldots, a_k\})$.

Corollary 4.3. (Huang [4, Theorem 2.1]) *Suppose $C = C\langle a_1, a_2, \ldots, a_k; n\rangle$ is primitive. Then either* $\exp(C) = n - 1$ *or* $\exp(C) \leqslant \lfloor \frac{n}{2} \rfloor$.

5. Further research

We proved $f(n, r) \geqslant f'(n, r)$ in Theorem 3.1. For $r = 1$ and $n \geqslant 2$, Brualdi and Hedrick [1, Theorems 3.3, 3.4] confirmed that $f(n, r) = f'(n, r)$. Now it is natural to ask the following question:

Question. Does $f(n, r) = f'(n, r)$ hold for all $n > r + 1$, $r \geqslant 2$?

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