The Reachability Problem For Vector Addition Systems

Jérôme Leroux

LaBRI (CNRS and University of Bordeaux), France.

Vector Addition Systems

Definition

Vector addition system (VAS) : finite set $\mathbf{A} \subseteq \mathbb{Z}^d$.
Actions : $\mathbf{a} \in \mathbf{A}$.

$\mathbf{A} = \{\mathbf{a}_1, \mathbf{a}_2\}$ with $\mathbf{a}_1 =$  $= (-1, 1)$

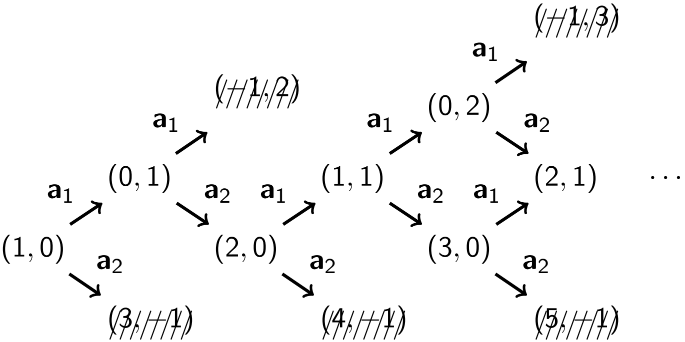
and $\mathbf{a}_2 =$  $= (2, -1)$

Semantics

Definition

Configurations : $\mathbf{x} \in \mathbb{N}^d$.
Transition relation : $\mathbf{x} \xrightarrow{\mathbf{a}} \mathbf{y}$ if $\mathbf{x}, \mathbf{y} \in \mathbb{N}^d$, $\mathbf{a} \in \mathbf{A}$ and $\mathbf{y} = \mathbf{x} + \mathbf{a}$.

$$\mathbf{A} = \{\mathbf{a}_1, \mathbf{a}_2\} \text{ with } \mathbf{a}_1 = (-1, 1) \text{ and } \mathbf{a}_2 = (2, -1).$$

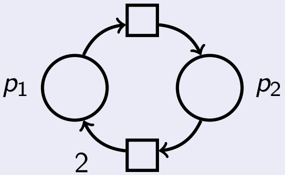


VAS

$\mathbf{A} = \{(-1,1),(2,-1)\}$

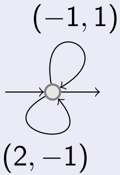
$\sim$

Petri nets



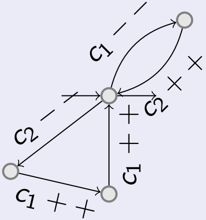
$\sim$

VAS with states



$\sim$

Minsky Machines without zero test



The Reachability Problem

Definition

$\overset{\mathbf{a}_1 \ldots \mathbf{a}_k}{\rightsquigarrow}$ is equal to $\xrightarrow{\mathbf{a}_1} \ldots \xrightarrow{\mathbf{a}_k}$

$$\overset{*}{\rightsquigarrow} \quad = \quad \bigcup_{w \in \mathbf{A}^*} \overset{w}{\rightsquigarrow}$$

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Reachability Problem

INPUT : $\mathbf{A}$, a VAS
$(\mathbf{c}_{\text{init}}, \mathbf{c}_{\text{final}})$, a pair of configurations.

OUTPUT : $\mathbf{c}_{\text{init}} \overset{*}{\rightsquigarrow} \mathbf{c}_{\text{final}}$?

Central Problem

- Many VAS Problems reduce to the VAS reachability:
  - Boundedness / Place boundedness.
  - Safety.
  - Reversibility.
  - Coverability.
  - ...
- Other problems reduce to the VAS reachability.
  - Satisifiability of some logics on data words [Bojanczyk & David & Muscholl & Schwentick & Segoufin '06 '11]
  - Software Model Checking [Heizmann & Hoenicke & Podelski '13]
  - ...

Story of the VAS Reachability Problem

- Story :

Story of the VAS Reachability Problem

- Story :
  - EXPSPACE-hard [Lipton '76]

"Clearly, lower bounds are a bit more interesting if the problem is decidable. [...] You do what you can."
[Lipton's blog, 2009]

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  - Partial proof of decidability [Sacerdote and Tenney '77]

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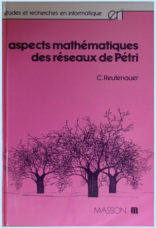

114 pages on the reachability problem, based on the Kosaraju paper.

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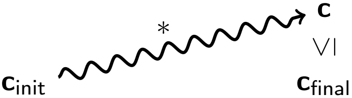
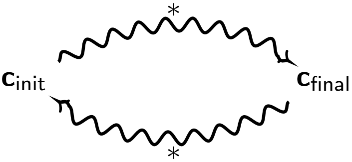
- Open problems:
  - Complexity gap.
  - Efficient algorithms.

Variants

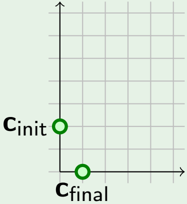
Two variants are EXPSPACE-complete:

Coverability / Rackoff '78	
Reversibility / Leroux '11	

Reachable Case

Example

$\mathbf{A} = \{$  ,  $\}$



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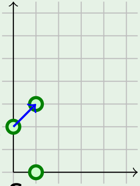

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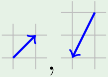
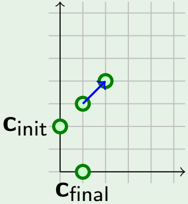
$\mathbf{A} = \{$  $\}$



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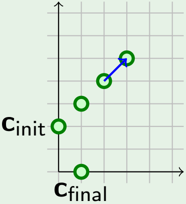

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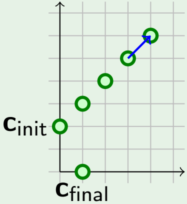

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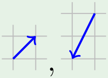
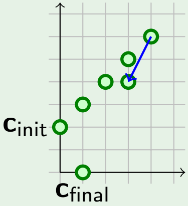
$\mathbf{A} = \{$  $\}$



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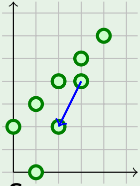

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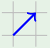
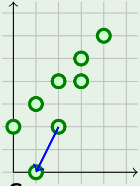
$\mathbf{A} = \{$  ,  $\}$



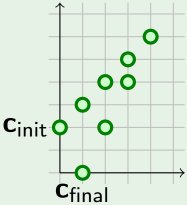
$\mathbf{c}_{\text{final}}$ is reachable from $\mathbf{c}_{\text{init}}$.

$$\rho = (0,2)\ (1,3)\ (2,4)\ (3,5)\ (4,6)\ (3,4)\ (2,2)\ (1,0)$$

Reachable Case

Example

$\mathbf{A} = \{$  ,  $\}$



$\mathbf{c}_{\text{final}}$ is reachable from $\mathbf{c}_{\text{init}}$.

$$\rho = (0,2)\ (1,3)\ (2,4)\ (3,5)\ (4,6)\ (3,4)\ (2,2)\ (1,0)$$

Definition

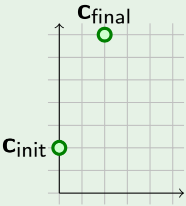
A run is a non-empty word $\rho = \mathbf{c}_0 \ldots \mathbf{c}_k$ over $\mathbb{N}^d$ such that:

$$\forall 1 \leq j \leq k \quad \mathbf{c}_j - \mathbf{c}_{j-1} \in \mathbf{A}$$

Unreachable Case

Example

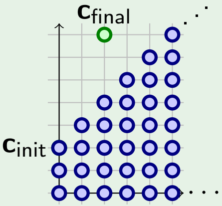
$\mathbf{A} = \{$  ,  $\}$



Unreachable Case

Example

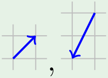
$\mathbf{A} = \{$  ,  $\}$

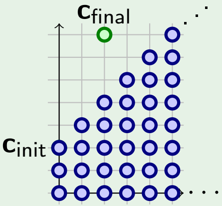


$\mathbf{c}_{\text{final}}$ is <u>not</u> in reachable from $\mathbf{c}_{\text{init}}$.

Unreachable Case

Example

$\mathbf{A} = \{$  $\}$



$\mathbf{c}_{\text{final}}$ is <u>not</u> in reachable from $\mathbf{c}_{\text{init}}$.

Definition

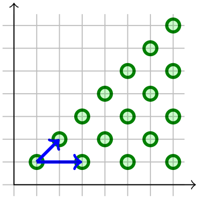
Inductive invariant : $\mathbf{X} \subseteq \mathbb{N}^d$ such that:

$$\mathbf{x} \in \mathbf{X} \quad \wedge \quad \mathbf{x} \xrightarrow{\mathbf{a}} \mathbf{y} \quad \Rightarrow \quad \mathbf{y} \in \mathbf{X}$$

Semilinear Sets

Definition (Ginsburg & Spanier '66)

Linear set : $\mathbf{b} + \mathbb{N}\mathbf{p}_1 + \cdots + \mathbb{N}\mathbf{p}_m$ with $\mathbf{b}, \mathbf{p}_1, \ldots, \mathbf{p}_m \in \mathbb{N}^d$.
Semilinear set : finite union of linear sets.



$$(1,1) + \mathbb{N}(1,1) + \mathbb{N}(2,0)$$

Witness of Unreachability

Theorem (Leroux '09 '10 '11 '12)

If $\mathbf{c}_{final}$ is not reachable from $\mathbf{c}_{init}$ there exists a semilinear inductive invariant $\mathbf{X}$ such that $\mathbf{c}_{init} \in \mathbf{X}$ and $\mathbf{c}_{final} \notin \mathbf{X}$.

Simple Algorithm

Reachability Algorithm

In $//$:

Enumerate reachable configurations $\mathbf{c}$ from $\mathbf{c}_{\text{init}}$

if $\mathbf{c} = \mathbf{c}_{\text{final}}$

return "reachable"

Enumerate semilinear sets $\mathbf{X}$

if $\mathbf{X}$ is an inductive invariant, $\mathbf{c}_{\text{init}} \in \mathbf{X}$, and $\mathbf{c}_{\text{final}} \notin \mathbf{X}$

return "unreachable"

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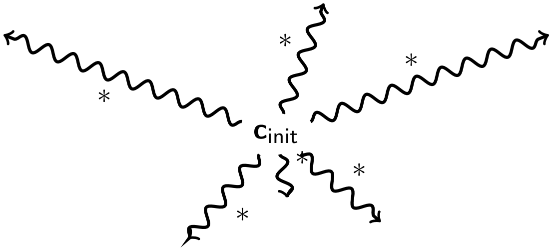
Reachability Sets

Definition

$$\text{Reachability set from } \mathbf{c}_{\mathrm{init}} \quad = \quad \left\{ \mathbf{c} \;\middle|\; \mathbf{c}_{\mathrm{init}} \overset{*}{\rightsquigarrow} \mathbf{c} \right\}$$



Reachability set from $\mathbf{c}_{\mathrm{init}}$

$=$

Most precise inductive invariant containing $\mathbf{c}_{\mathrm{init}}$.

Monotonicity

Lemma (Monotonicity)

For any configuration $\mathbf{c}$*:*

$$\mathbf{c}_{init} \overset{w}{\rightsquigarrow} \mathbf{c}_{final}$$

$$\Rightarrow$$

$$\begin{matrix}\mathbf{c}_{init} \\ + \\ \mathbf{c}\end{matrix} \overset{w}{\rightsquigarrow} \begin{matrix}\mathbf{c}_{final} \\ + \\ \mathbf{c}\end{matrix}$$

Proof:

$\mathbf{x} \xrightarrow{\mathbf{a}} \mathbf{y}$

$\Rightarrow \mathbf{y} = \mathbf{x} + \mathbf{a}$

$\Rightarrow (\mathbf{y} + \mathbf{c}) = (\mathbf{x} + \mathbf{c}) + \mathbf{a}$

$\Rightarrow \begin{matrix}\mathbf{x} \\ + \\ \mathbf{c}\end{matrix} \xrightarrow{\mathbf{a}} \begin{matrix}\mathbf{y} \\ + \\ \mathbf{c}\end{matrix}$

Example of Computation

$\mathbf{A} = \{\mathbf{a}_1, \mathbf{a}_2\}$ with $\mathbf{a}_1 = (-1, 1)$ and $\mathbf{a}_2 = (2, -1)$.
$\mathbf{c}_{\text{init}} = (1, 0)$.

$(1, 0) \xrightarrow{\mathbf{a}_1} (0, 1) \xrightarrow{\mathbf{a}_2} (2, 0)$

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$(1,0) \xrightarrow{\mathbf{a}_1} (0,1) \xrightarrow{\mathbf{a}_2} (2,0)$
By monotonicity $\forall n \geq 0$:

$$(n+1,0) = \begin{matrix}(1,0)\\+\\(n,0)\end{matrix} \quad \overset{\mathbf{a}_1\mathbf{a}_2}{\rightsquigarrow} \quad \begin{matrix}(2,0)\\+\\(n,0)\end{matrix} = (n+2,0)$$

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By induction $\forall n \geq 0$:

$$(1, 0) \xrightarrow{(\mathbf{a}_1\mathbf{a}_2)^n} (n+1, 0).$$

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By induction $\forall n \geq 0$:

$$(1,0) \overset{(\mathbf{a}_1\mathbf{a}_2)^n}{\rightsquigarrow} (n+1, 0).$$

$$\mathbf{c}_{\text{init}} \overset{(\mathbf{a}_1\mathbf{a}_2)^*}{\rightsquigarrow} \mathbf{c} \quad \Longleftrightarrow \quad \mathbf{c} \in (1,0) + \mathbb{N}(1,0)$$

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By induction $\forall n \geq 0$:

$$(1, 0) \xrightarrow{(\mathbf{a}_1\mathbf{a}_2)^n} (n+1, 0).$$

$$\mathbf{c}_{\text{init}} \xrightarrow{(\mathbf{a}_1\mathbf{a}_2)^*} \mathbf{c} \iff \mathbf{c} \in (1, 0) + \mathbb{N}(1, 0)$$

$$\mathbf{c}_{\text{init}} \xrightarrow{(\mathbf{a}_1\mathbf{a}_2)^*\mathbf{a}_1^*} \mathbf{c} \iff \mathbf{c} \in \{(1, 0), (0, 1)\} + \mathbb{N}(1, 0) + \mathbb{N}(0, 1)$$

Flat Initialized VAS

Definition (Flat Initialized VAS)

A VAS $\mathbf{A}$ equipped with an initial configuration $\mathbf{c}_{\text{init}}$ such that:

$$\text{Reachability set from } \mathbf{c}_{\text{init}} = \left\{ \mathbf{c} \;\middle|\; \mathbf{c}_{\text{init}} \xrightarrow{\sigma_1^* \ldots \sigma_k^*} \mathbf{c} \right\}$$

for some $\sigma_1, \ldots, \sigma_k \in \mathbf{A}^*$.

Acceleration

Acceleration Algorithm:

$\mathbf{C} \leftarrow \{\mathbf{c}_{\mathsf{init}}\}$

while $\mathbf{C}$ is not inductive

select word σ

$$\mathbf{C} \leftarrow \left\{ \mathbf{c}' \;\middle|\; \exists \mathbf{c} \in \mathbf{C} \;\; \mathbf{c} \xrightarrow{\sigma^*} \mathbf{c}' \right\}$$

return $\mathbf{C}$

Remarks:

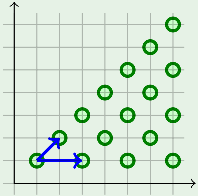
- Sets are effectively semilinear.
- Tools exist : FAST, LASH, TREX, ...
- When the algorithm terminates, it returns the reachability set from $\mathbf{c}_{\mathsf{init}}$.

Presburger Sets

Definition

A Presburger set is a set $\mathbf{X} \subseteq \mathbb{N}^d$ definable in $\mathrm{FO}(\mathbb{N}, +, \leq, 0, 1)$.

Example



$$(1,1) + \mathbb{N}(1,1) + \mathbb{N}(2,0)$$

Denoted by:

$$\phi(x, y) \quad := \quad \exists n_1 \exists n_2 \;\; x = 1 + n_1 + 2n_2 \wedge y = 1 + n_1$$

Presburger Sets = Semilinear Sets

Theorem (Ginsburg & Spanier '66)

Presburger sets $=$ *semilinear sets*

Corollary

Semilinear sets are closed under union, intersection, complement, projection of components, ...

Minimal Configuration for Executing Words

$\sigma = \mathbf{a}_1 \ldots \mathbf{a}_k$:

$$\exists \mathbf{y} \in \mathbb{N}^d \quad \mathbf{x} \xrightarrow{\sigma} \mathbf{y}$$

$$\iff$$

$$\bigwedge_{0 \leq p \leq k} \mathbf{x} + \sum_{j=1}^{p} \mathbf{a}_j \geq \mathbf{0}$$

$$\iff$$

$$\mathbf{x} \geq \mathbf{c}_\sigma$$

where $\mathbf{c}_\sigma(i) = \max_{0 \leq p \leq k} - \sum_{j=1}^{p} \mathbf{a}_j(i)$.

Transitive Closure with Presburger Arithmetic

Theorem (Fribourg '00)

$\xrightarrow{\sigma^*}$ *is effectively Presburger.*

Transitive Closure with Presburger Arithmetic

Theorem (Fribourg '00)

$\overset{\sigma^*}{\rightsquigarrow}$ *is effectively Presburger.*

$\sigma = \mathbf{a}_1 \ldots \mathbf{a}_k$:

$$\mathbf{x} \overset{\sigma^n}{\rightsquigarrow} \mathbf{y}$$

$$\Longleftrightarrow$$

$$\mathbf{x} + n\sum_{j=1}^{k} \mathbf{a}_j = \mathbf{y} \text{ and } \forall 0 \leq m < n \;\; \mathbf{x} + m(\sum_{j=1}^{k} \mathbf{a}_j) \geq \mathbf{c}_\sigma$$

Acceleration Algorithm

Acceleration Algorithm:

$\mathbf{C} \leftarrow \{\mathbf{c}_{\mathrm{init}}\}$
while $\mathbf{C}$ is not inductive
  select word σ
  $\mathbf{C} \leftarrow \left\{ \mathbf{c}' \;\middle|\; \exists \mathbf{c} \in \mathbf{C} \;\; \mathbf{c} \xrightarrow{\sigma^*} \mathbf{c}' \right\}$
return $\mathbf{C}$

- In theory : terminate on flat initialized VAS if all the finite sequences of words in $\mathbf{A}^*$ are subsequences of the infinite sequence $\sigma_1, \sigma_2, \ldots$ of selected words.
- In practice : find good heuristics.

Flat Counter Systems Almost Everywhere !

Theorem (Finkel & Leroux '02, Leroux & Sutre '05)

Reachability sets of flat Initialized VAS are effectively semilinear.

"Many known semilinear subclasses of counter automata are flat: reversal bounded counter machines, lossy vector addition systems with states, reversible Petri nets, persistent and conflict-free Petri nets, etc."
[Leroux & Sutre, ATVA 2005]

Theorem (Leroux '13)

An initialized VAS is flat if, and only if, its reachability set is semilinear.

Application:

- Completeness of acceleration techniques.
- Reachability semilinear $\Rightarrow$ effectively semilinear.

Application : Distance Of Reachability

Corollary

For any flat initialized VAS $<\mathbf{A}, \mathbf{c}_{init}>$ *there exists a constant m such that for every reachable configurations* $\mathbf{c}$ *from* $\mathbf{c}_{init}$*, there exists:*

$$\mathbf{c}_{init} \xrightarrow{\sigma} \mathbf{c}$$

with $|\sigma| \leq m.||\mathbf{c} - \mathbf{c}_{init}||_{\infty}$

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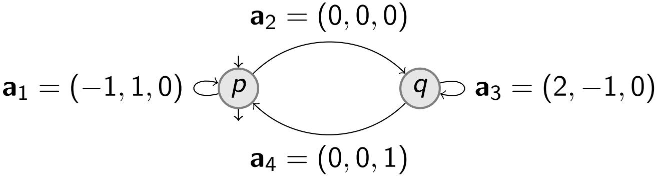
There exists $\sigma_1, \ldots, \sigma_k \in \mathbf{A}^*$ such that:

$$\text{Reachability set from } \mathbf{c}_{\text{init}} \quad = \quad \left\{ \mathbf{c} \;\middle|\; \mathbf{c}_{\text{init}} \xrightarrow{\sigma_1^* \ldots \sigma_k^*} \mathbf{c} \right\}$$

The Hopcroft-Pansiot 1979 Example



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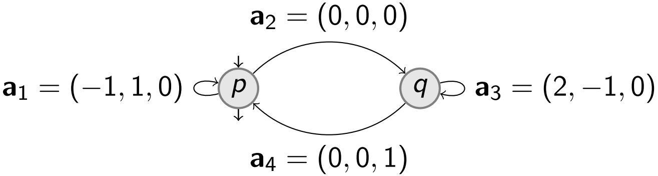

$$(1,0,0) \xrightarrow{\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4} (2,0,1) \xrightarrow{\mathbf{a}_1^2\mathbf{a}_2\mathbf{a}_3^2\mathbf{a}_4} (4,0,2)\cdots \xrightarrow{\mathbf{a}_1^{2^n}\mathbf{a}_2\mathbf{a}_3^{2^n}\mathbf{a}_4} (2^{n+1},0,n+1)$$

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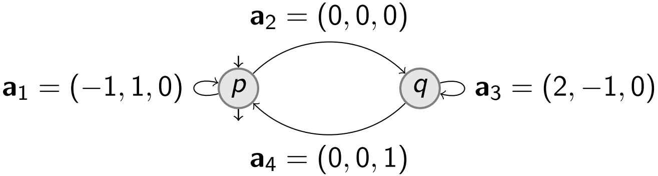

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Configurations reachable from $(1, 0, 0)$

$$\{(x, y, z) \in \mathbb{N}^3 \mid 1 \leq x + y \leq 2^z\}$$

There exist initialized VAS with non semilinear reachability sets:

- Semilinear inductive invariant proving that $\mathbf{c}_{\text{final}}$ is not reachable from $\mathbf{c}_{\text{init}}$ depends on $\mathbf{c}_{\text{init}}$ <u>and</u> $\mathbf{c}_{\text{final}}$.
- Semilinear inductive invariant cannot be as precise as reachability sets.

Equality of Reachability Sets

Definition (Equivalence Problem)

INPUT : Two initialized VAS $< \mathbf{A}_1, \mathbf{c}_1 >$ and $< \mathbf{A}_2, \mathbf{c}_2 >$.

OUTPUT : Decide the equality of the reachability sets.

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OUTPUT : Decide the equality of the reachability sets.

Theorem (Hack 1976)

The equivalence problem is undecidable.

$\Longrightarrow$

- No decidable logic for denoting reachability sets.
- Inductive invariant in decidable logics cannot be as precise as reachability sets.

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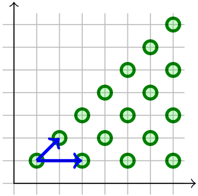
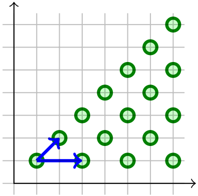
From Linear Sets To Periodic Sets



The linear set $(1,1)+\underbrace{\mathbb{N}(1,1)+\mathbb{N}(2,0)}_{\text{Periodic Set}}$

From Linear Sets To Periodic Sets



The linear set $(1,1) + \underbrace{\mathbb{N}(1,1) + \mathbb{N}(2,0)}_{\text{Periodic Set}}$

Definition (Periodic Sets)

Sets $\mathbf{P} \subseteq \mathbb{N}^d$ such that:

- $\mathbf{0} \in \mathbf{P}$
- $\mathbf{P} + \mathbf{P} \subseteq \mathbf{P}$

where $\mathbf{X} + \mathbf{Y} = \{\mathbf{x} + \mathbf{y} \mid (\mathbf{x}, \mathbf{y}) \in \mathbf{X} \times \mathbf{Y}\}$.

Finitely Generated Periodic Sets

Definition

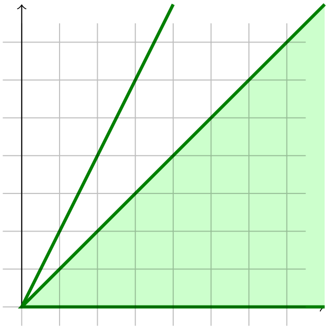
A Periodic set $\mathbf{P}$ is said finitely-generated if $\mathbf{P} = \mathbb{N}\mathbf{p}_1 + \cdots + \mathbb{N}\mathbf{p}_k$ for some $\mathbf{p}_1, \ldots, \mathbf{p}_k$.

Example

Linear sets are decomposed as $\mathbf{b} + \mathbf{P}$ where $\mathbf{b} \in \mathbb{N}^d$ and $\mathbf{P} \subseteq \mathbb{N}^d$ is a finitely generated periodic set.

Additive Logic over the non Negative Rational Numbers

Set definable in $\text{FO}(\mathbb{Q}_{\geq 0}, +, \leq, 0)$:



$$\{(x, y) \in \mathbb{Q}^2_{\geq 0} \mid \quad x \leq y \quad \vee \quad y = x + x\}$$

Quantifier Elimination

Theorem

The logic $\mathsf{FO}(\mathbb{Q}_{\geq 0}, +, \leq, 0)$ *admits a quantifier elimination algorithm.*

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Theorem

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$\Rightarrow$

Sets definable in $\mathrm{FO}(\mathbb{Q}_{\geq 0}, +, \leq, 0)$ are Boolean combinations of:

$$\{(x_1, \ldots, x_d) \in \mathbb{Q}_{\geq 0}^d \mid h_1 x_1 + \cdots + h_d x_d \leq 0\}$$

where $h_1, \ldots, h_d \in \mathbb{Z}$.

Asymptotically Definable Periodic Sets

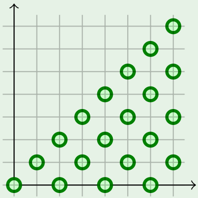
Definition

A periodic set $\mathbf{P}$ is said to be asymptotically definable if

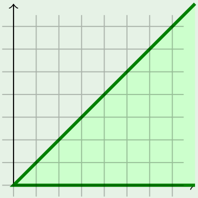
$$\mathbb{Q}_{\geq 0}\mathbf{P} = \{\lambda \mathbf{p} \mid (\lambda, \mathbf{p}) \in \mathbb{Q}_{\geq 0} \times \mathbf{P}\}$$

is definable in $\mathsf{FO}(\mathbb{Q}_{\geq 0}, +, \leq, 0)$.

Example



$\mathbf{P} = \mathbb{N}(1,1) + \mathbb{N}(2,0)$



$\mathbb{Q}_{\geq 0}\mathbf{P} = \mathbb{Q}_{\geq 0}(1,1) + \mathbb{Q}_{\geq 0}(2,0)$

Examples

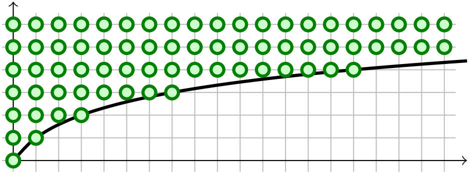
Lemma

$\mathbf{P} = \mathbb{N}\mathbf{p}_1 + \cdots + \mathbb{N}\mathbf{p}_k$ *is an asymptotically definable periodic set for any* $\mathbf{p}_1, \ldots, \mathbf{p}_k \in \mathbb{N}^d$*.*

$\mathbb{Q}_{\geq 0}\mathbf{P} = \mathbb{Q}_{\geq 0}\mathbf{p}_1 + \cdots + \mathbb{Q}_{\geq 0}\mathbf{p}_k$ is denoted by:

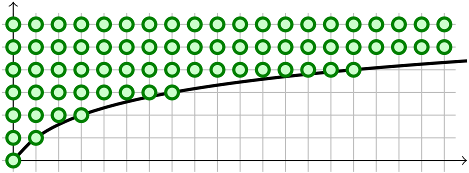
$$\phi(x_1, \ldots, x_d) \quad := \quad \exists \lambda_1 \ldots \exists \lambda_k \quad \bigwedge_{i=1}^{d} x_i = \lambda_1 \mathbf{p}_1(i) + \cdots + \lambda_k \mathbf{p}_k(i)$$

Another Example



$$\mathbf{P} = \{(p_1, p_2) \in \mathbb{N}^2 \mid \quad p_1 \leq 2^{p_2} - 1\}$$

Another Example



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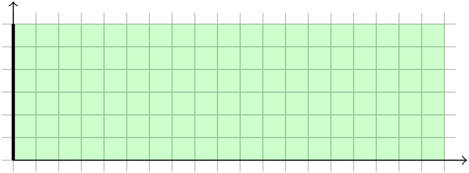
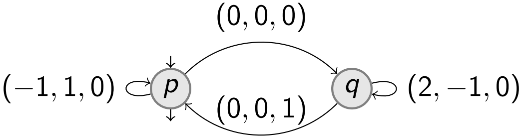


$$\mathbb{Q}_{\geq 0}\mathbf{P} = \{(p_1, p_2) \in \mathbb{Q}^2_{\geq 0} \mid \quad p_1 = 0 \quad \vee \quad p_2 > 0\}$$

Back to the Hopcroft-Pansiot 1979 Example



Configurations reachable from $(1, 0, 0)$

$$\{(x, y, z) \in \mathbb{N}^3 \mid 1 \leq x + y \leq 2^z\}$$

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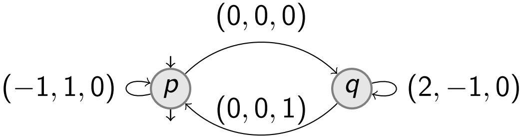


Configurations reachable from $(1, 0, 0)$

$$\{(x, y, z) \in \mathbb{N}^3 \mid 1 \leq x + y \leq 2^z\}$$
$$= \{(1, 0, 0), (0, 1, 0)\} \quad + \quad \{(x, y, z) \in \mathbb{N}^3 \mid x + y \leq 2^z - 1\}$$

Almost Semilinear Sets

Definition

An almost linear set is a set of the form $\mathbf{b} + \mathbf{P}$ where:

- $\mathbf{b} \in \mathbb{N}^d$, and
- $\mathbf{P} \subseteq \mathbb{N}^d$ is an asymptotically definable periodic set.

An almost semilinear set is a finite union of almost linear sets.

Example

Linear sets $\mathbf{b} + \mathbb{N}\mathbf{p}_1 + \cdots + \mathbb{N}\mathbf{p}_k$ are almost linear.
Semilinear sets are almost semilinear.

The reachability set of the Hopcroft-Pansiot example is almost semilinear.

In fact, it is not specific to the Hopcroft-Pansiot example !

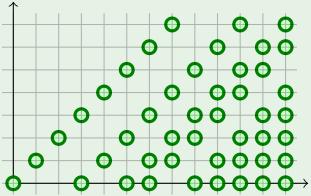
Limits

Definition

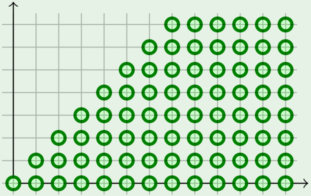
The limit of a periodic set $\mathbf{P} \subseteq \mathbb{N}^d$:

$$\lim(\mathbf{P}) = \{\mathbf{v} \in \mathbb{N}^d \mid \exists n \in \mathbb{N} \ \ (n + \mathbb{N})\mathbf{v} \subseteq \mathbf{P}\}$$

Example

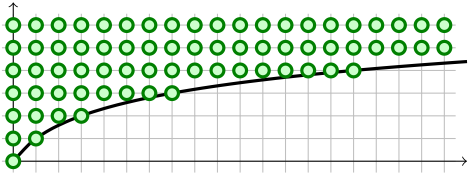


$\mathbf{P} = \mathbb{N}(1,1) + \mathbb{N}(3,0) + \mathbb{N}(5,0)$



$\lim(\mathbf{P})$

Another Example



$$\mathbf{P} = \{(p_1, p_2) \in \mathbb{N}^2 \mid \quad p_1 \leq 2^{p_2} - 1\}$$

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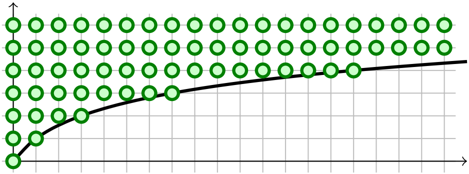

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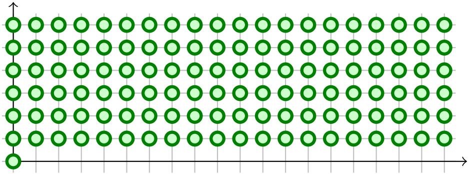

$\lim(\mathbf{P})$

Over-approximation

Lemma

For every periodic set $\mathbf{P}$*:*

$$\mathbf{P} \subseteq \lim(\mathbf{P})$$

Proof:

$\mathbb{N}\mathbf{p} \subseteq \mathbf{P}$ for every $\mathbf{p} \in \mathbf{P}$.

Semilinear Characterization

Theorem

P *is asymptotically definable if, and only if,* $\lim(\mathbf{P})$ *is semilinear.*

Proof by induction over the dimension.

$\Rightarrow$

Simple way for approximating asymptotically definable periodic sets by semilinear sets.

Semilinear Approximation

Definition

A linearization of an almost semilinear set $\bigcup_{j=1}^{k} \mathbf{L}_j$ where $\mathbf{L}_j = \mathbf{b}_j + \mathbf{P}_j$ and $\mathbf{P}_j$ is an asymptotically definable periodic set is the semilinear set:

$$\bigcup_{j=1}^{k} \mathbf{b}_j + \lim(\mathbf{P}_j)$$

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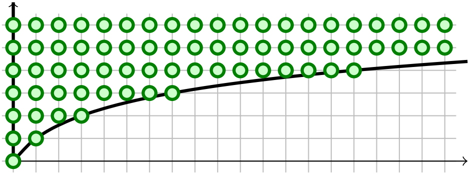
Multiple linearizations.



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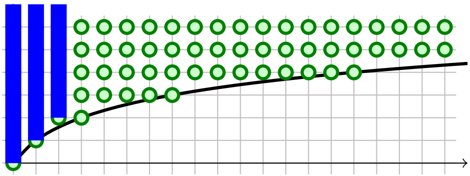

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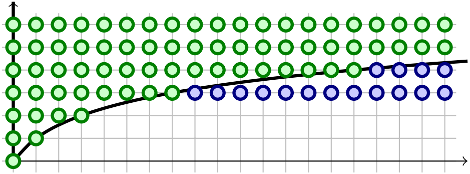

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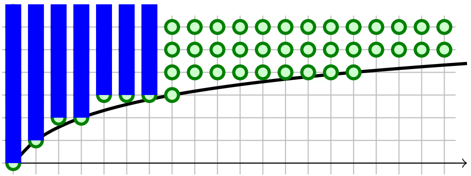

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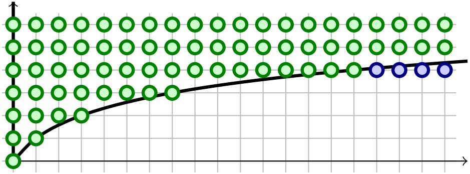


$$\mathbf{P} = \{(p_1, p_2) \in \mathbb{N}^2 \mid \quad p_1 \leq 2^{p_2} - 1\}$$

A Simple Observation

Let $\mathbf{S}, \mathbf{T}$ be linearizations of almost semilinear sets $\mathbf{X}, \mathbf{Y}$ such that:

$$\mathbf{X} \cap \mathbf{Y} = \emptyset$$

In general:

$$\mathbf{S} \cap \mathbf{T} \neq \emptyset$$

Dimension

Definition

The dimension $\dim(\mathbf{X})$ of a set $\mathbf{X} \subseteq \mathbb{N}^d$ is the minimal integer $r \in \{-1, \ldots, d\}$ such that:

$$\sup_{k \in \mathbb{N}} \frac{|\mathbf{X} \cap \{0, \ldots, k\}^d|}{(k+1)^r} < \infty$$

Example

$\dim(\emptyset) = -1$

$\dim(\mathbb{N}) = 1$

$\dim(\{(0,1),(1,0)\}) = 0$

$\dim(\{(x,y) \in \mathbb{N}^2 \mid x \leq y\}) = 2$

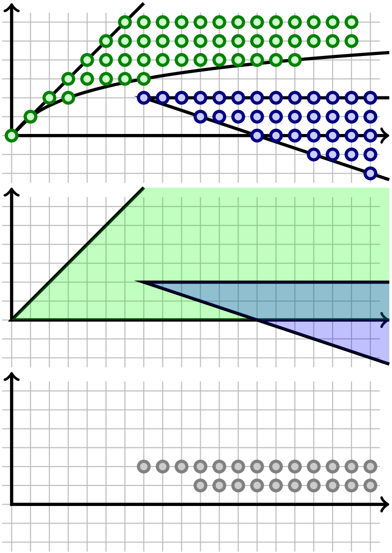
Preciseness

Lemma

Let $\mathbf{S}, \mathbf{T}$ *be linearizations of non-empty almost semilinear sets* $\mathbf{X}, \mathbf{Y}$ *with an empty intersection. We have:*

$$\dim(\mathbf{S} \cap \mathbf{T}) < \dim(\mathbf{X} \cup \mathbf{Y})$$

Example



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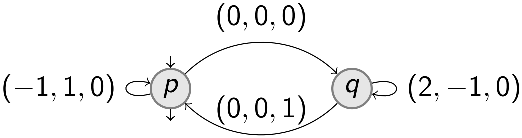

Configurations reachable from $(1,0,0)$

$$\begin{aligned}&\{(x,y,z)\in\mathbb{N}^3 \mid 1 \leq x+y \leq 2^z\}\\ &=\{(1,0,0),(0,1,0)\} \quad + \quad \{(x,y,z)\in\mathbb{N}^3 \mid x+y \leq 2^z-1\}\end{aligned}$$

Well Partial Orders

Definition

A partial order $\sqsubseteq$ over a set S is said to be well if for every sequence $(s_n)_{n \in \mathbb{N}}$ of elements $s_n \in S$ there exists $i < j$ such that $s_i \sqsubseteq s_j$.

Example

$(\mathbb{Z}, \leq)$ is not well, e.g., $0, -1, -2, \ldots$.

Example (Pigeon Hole Principle)

$(S, =)$ is well if and only if S is finite.

Example

$(\mathbb{N}, \leq)$ is well.

Minimal Elements

Definition

$(S, \sqsubseteq)$: a partially ordered set.

$$\min_{\sqsubseteq}(S) \quad = \quad \{e \in S \mid \forall s \in S \quad s \sqsubseteq e \quad \Rightarrow \quad s = e\}$$

Example

$\mathbf{S} = \{(1,2),(1,3),(2,1)\}$.
$\min_{\leq}(\mathbf{S}) = \{(1,2),(2,1)\}$.

Lemma

For every well partially ordered set $(S, \sqsubseteq)$:

- $E = \min_{\sqsubseteq}(S)$ *is finite, and*
- *for any $s \in S$, there exists $e \in E$ such that $e \sqsubseteq s$.*

Dickson's Lemma

Definition

$(S_1, \sqsubseteq_1)$ and $(S_2, \sqsubseteq_2)$: partially ordered sets.
$S_1 \times S_2$ is partially ordered by $\sqsubseteq_1 \times \sqsubseteq_2$ defined by:

$$(s_1, s_2) \ (\sqsubseteq_1 \times \sqsubseteq_2) \ (t_1, t_2) \ \text{ if } \ s_1 \sqsubseteq_1 t_1 \ \text{ and } \ s_2 \sqsubseteq_2 t_2$$

Lemma (Dickson's Lemma)

$(S_1 \times S_2, \sqsubseteq_1 \times \sqsubseteq_2)$ is well if $(S_1, \sqsubseteq_1)$ and $(S_2, \sqsubseteq_2)$ are well.

Example

$(\mathbb{N}^d, \leq)$ is well.

An Example

Definition

$\mathbf{P}$: Periodic set.

$$\mathbf{p} \leq_{\mathbf{P}} \mathbf{q} \quad \text{if} \quad \mathbf{q} \in \mathbf{p} + \mathbf{P}$$

Lemma

$(\mathbf{P}, \leq_{\mathbf{P}})$ *is well if, and only if,* $\mathbf{P}$ *is finitely-generated.*

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$(\Rightarrow)$:
$(\mathbf{P}, \leq_{\mathbf{P}})$ is well
$\Rightarrow$ $\min_{\leq_{\mathbf{P}}}(\mathbf{P} \setminus \{\mathbf{0}\})$ is a finite set $\{\mathbf{p}_1, \ldots, \mathbf{p}_k\}$.
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$(\Leftarrow)$:
$(\mathbb{N}^k, \leq)$ well.
$\Rightarrow$ $(\mathbf{P}, \leq_{\mathbf{P}})$ well.

An application: The Boundedness Problem

Boundedness problem
INPUT : An initialized VAS $(\mathbf{A}, \mathbf{c}_{\text{init}})$.
OUTPUT : The reachability set from $\mathbf{c}_{\text{init}}$ is finite.

Theorem (Karp & Miller '69)

The boundedness problem is decidable.

Proof.

Unbounded iff there exists $\mathbf{x} < \mathbf{y}$ and $u, v \in \mathbf{A}^*$ such that:

$$\mathbf{c}_{\text{init}} \overset{u}{\rightsquigarrow} \mathbf{x} \overset{v}{\rightsquigarrow} \mathbf{y}$$

$\square$

Higman's Lemma

Definition

$(S, \sqsubseteq)$: a partially ordered set.
$(S^*, \sqsubseteq^*)$ is defined by $u \sqsubseteq^* v$ if:

$$\begin{array}{ccccccccc} v & = & w_0 & t_1 & w_1 & \dots & t_d & w_d \\ & & & \sqcup\!\!| & & & \sqcup\!\!| & \\ u & = & & s_1 & & \dots & s_d & \end{array}$$

where:

- $s_1, t_1, \dots, s_d, t_d \in S$, and
- $w_0, \dots, w_d \in S^*$.

Lemma (Higman's Lemma)

$(S^*, \sqsubseteq^*)$ is well if $(S, \sqsubseteq)$ is well.

Example

With $(\{a, b, c, d, r\}, =)$:

$$baba =^* abracadabra$$

Example

With $(\mathbb{N}, \leq)$:

$$1\ 3\ 7 \leq^* 0\ 2\ 2\ 2\ 3\ 5\ 8\ 2\ 4$$

Relation On Runs

Remember:

Definition

A run is a non-empty word $\rho = \mathbf{c}_0 \ldots \mathbf{c}_k$ over $\mathbb{N}^d$ such that:

$$\forall 1 \leq j \leq k \;\; \mathbf{a}_j = \mathbf{c}_j - \mathbf{c}_{j-1} \in \mathbf{A}$$

$\mathrm{src}(\rho) = \mathbf{c}_0$ and $\mathrm{tgt}(\rho) = \mathbf{c}_k$.

$$\rho \;=\; \mathbf{c}_0 \xrightarrow{\mathbf{a}_1} \mathbf{c}_1 \xrightarrow{\mathbf{a}_2} \qquad \xrightarrow{\mathbf{a}_k} \mathbf{c}_k$$

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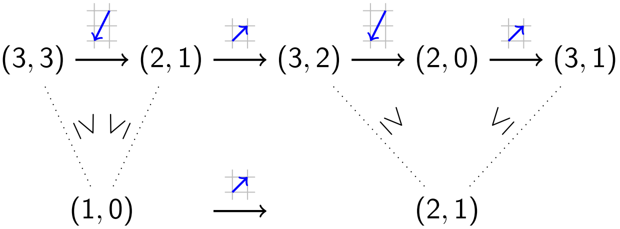
$\mathrm{src}(\rho) = \mathbf{c}_0$ and $\mathrm{tgt}(\rho) = \mathbf{c}_k$.

$$\begin{array}{ccccccccc}
\rho' & = & \mathbf{c}'_0 \cdots \mathbf{c}''_0 & \xrightarrow{\mathbf{a}_1} & \mathbf{c}'_1 \cdots \mathbf{c}''_1 & \xrightarrow{\mathbf{a}_2} & \cdots & \xrightarrow{\mathbf{a}_k} & \mathbf{c}'_k \cdots \mathbf{c}''_k \\
\rotatebox{90}{$\trianglelefteq$} & & \rotatebox{-45}{$\leq$} \quad \rotatebox{45}{$\leq$} & & \rotatebox{-45}{$\leq$} \quad \rotatebox{45}{$\leq$} & & & & \rotatebox{-45}{$\leq$} \quad \rotatebox{45}{$\leq$} \\
\rho & = & \mathbf{c}_0 & \xrightarrow{\mathbf{a}_1} & \mathbf{c}_1 & \xrightarrow{\mathbf{a}_2} & & \xrightarrow{\mathbf{a}_k} & \mathbf{c}_k
\end{array}$$

An Example



$$(1,0)(2,1) \quad \trianglelefteq \quad (3,3)(2,1)(3,2)(2,0)(3,1)$$

Definition (Transformer Relations for Configurations $\mathbf{c}$)

$$\mathbf{x} \overset{\mathbf{c}}{\curvearrowright} \mathbf{y} \quad \text{if} \quad \begin{matrix}\mathbf{x}\\ + \\ \mathbf{c}\end{matrix} \overset{*}{\rightsquigarrow} \begin{matrix}\mathbf{y}\\ + \\ \mathbf{c}\end{matrix}$$

$$\begin{tikzcd}
\begin{matrix}(2,3)\\ + \\ (1,0)\end{matrix} \arrow[d, equal] & & \begin{matrix}(1,1)\\ + \\ (1,0)\end{matrix} \arrow[d, equal] & & \begin{matrix}(1,1)\\ + \\ (2,1)\end{matrix} \arrow[d, equal] & & & & \begin{matrix}(1,0)\\ + \\ (2,1)\end{matrix} \arrow[d, equal] \\
(3,3) \arrow[rr] & & (2,1) \arrow[rr] & & (3,2) \arrow[rr] & & (2,0) \arrow[rr] & & (3,1) \\
& (1,0) \arrow[ul, phantom, sloped, "\leq"] \arrow[ur, phantom, sloped, "\leq"] \arrow[rrrrr] & & & & & (2,1) \arrow[ull, phantom, sloped, "\leq"] \arrow[urr, phantom, sloped, "\leq"] & &
\end{tikzcd}$$

Important Observation

Definition

$\rho = \mathbf{c}_0 \ldots \mathbf{c}_k$

$$\overset{\rho}{\curvearrowright} \quad = \quad \overset{\mathbf{c}_0}{\curvearrowright} \circ \cdots \circ \overset{\mathbf{c}_k}{\curvearrowright}$$

Lemma

$$\rho \trianglelefteq \rho' \quad \Rightarrow \quad (\mathrm{src}(\rho') - \mathrm{src}(\rho)) \overset{\rho}{\curvearrowright} (\mathrm{tgt}(\rho') - \mathrm{tgt}(\rho))$$

$$
\begin{array}{cccccccccccc}
 & & \mathbf{v}_0 & \mathbf{v}_1 & & \mathbf{v}_1 & \mathbf{v}_2 & & & & \mathbf{v}_k & \mathbf{v}_{k+1} \\
 & & + & + & & + & + & & & & + & + \\
 & & \mathbf{c}_0 & \mathbf{c}_0 & & \mathbf{c}_1 & \mathbf{c}_1 & & & & \mathbf{c}_k & \mathbf{c}_k \\
 & & \| & \| & & \| & \| & & & & \| & \| \\
\rho' & = & \mathbf{c}'_0 \cdots & \mathbf{c}''_0 & \xrightarrow{\mathbf{a}_1} & \mathbf{c}'_1 \cdots & \mathbf{c}''_1 & \xrightarrow{\mathbf{a}_2} & \cdots & \xrightarrow{\mathbf{a}_k} & \mathbf{c}'_k \cdots & \mathbf{c}''_k \\
\trianglerighteq & & \rotatebox{90}{$\leq$} & \rotatebox{90}{$\geq$} & & \rotatebox{90}{$\leq$} & \rotatebox{90}{$\geq$} & & & & \rotatebox{90}{$\leq$} & \rotatebox{90}{$\geq$} \\
\rho & = & \multicolumn{2}{c}{\mathbf{c}_0} & \xrightarrow{\mathbf{a}_1} & \multicolumn{2}{c}{\mathbf{c}_1} & \xrightarrow{\mathbf{a}_2} & & \xrightarrow{\mathbf{a}_k} & \multicolumn{2}{c}{\mathbf{c}_k}
\end{array}
$$

Well Partial Order On Runs

Theorem (Jančar '90)

$\trianglelefteq$ *is a well partial order.*

Proof (Leroux '11 '12).

$\mathbf{A} \times \mathbb{N}^d$ is partially ordered by $(\mathbf{a}, \mathbf{x}) \sqsubseteq (\mathbf{b}, \mathbf{y})$ if $\mathbf{a} = \mathbf{b}$ and $\mathbf{x} \leq \mathbf{y}$.

$$\alpha(\mathbf{c}_0 \xrightarrow{\mathbf{a}_1} \mathbf{c}_1 \ldots \xrightarrow{\mathbf{a}_k} \mathbf{c}_k) \quad = \quad (\mathbf{a}_1, \mathbf{c}_1) \ldots (\mathbf{a}_k, \mathbf{c}_k)$$

We observe:

$$\rho \trianglelefteq \rho' \quad \Longleftrightarrow \quad \mathrm{src}(\rho) \leq \mathrm{src}(\rho') \ \wedge \ \mathrm{tgt}(\rho) \leq \mathrm{tgt}(\rho') \ \wedge \ \alpha(\rho) \sqsubseteq^* \alpha(\rho')$$

$\square$

Decomposing Reachability Sets

$\mathbf{X}, \mathbf{Y} \subseteq \mathbb{N}^d$ semilinear sets.

$$\bigcup_{\mathbf{x} \in \mathbf{X}} \left\{ \mathbf{y} \in \mathbf{Y} \;\middle|\; \mathbf{x} \overset{*}{\rightsquigarrow} \mathbf{y} \right\}$$

Decomposing Reachability Sets

$\mathbf{X}, \mathbf{Y} \subseteq \mathbb{N}^d$ ~~semilinear sets~~ linear sets $\mathbf{X} = \mathbf{x} + \mathbf{P}$ and $\mathbf{Y} = \mathbf{y} + \mathbf{Q}$.

$$\bigcup_{\mathbf{x} \in \mathbf{X}} \left\{ \mathbf{y} \in \mathbf{Y} \;\middle|\; \mathbf{x} \xrightarrow{*} \mathbf{y} \right\}$$

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$$\bigcup_{\mathbf{x} \in \mathbf{X}} \left\{ \mathbf{y} \in \mathbf{Y} \;\middle|\; \mathbf{x} \xrightarrow{*} \mathbf{y} \right\}$$
$$= \bigcup_{\rho \in \Omega} \mathrm{tgt}(\rho)$$

where:

- Ω set of runs from $\mathbf{X}$ to $\mathbf{Y}$

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$$= \bigcup_{\rho \in \min_{\trianglelefteq_{\mathbf{P} \times \mathbf{Q}}}(\Omega)} \{\mathrm{tgt}(\rho') \mid \rho' \in \Omega \ \wedge \ \rho \trianglelefteq_{\mathbf{P} \times \mathbf{Q}} \rho'\}$$

where:

- Ω set of runs from $\mathbf{X}$ to $\mathbf{Y}$
- $\rho \trianglelefteq_{\mathbf{P} \times \mathbf{Q}} \rho'$ if $\rho \trianglelefteq \rho' \ \wedge \ \mathrm{src}(\rho') \in \mathrm{src}(\rho) + \mathbf{P} \ \wedge \ \mathrm{tgt}(\rho') \in \mathrm{tgt}(\rho) + \mathbf{Q}$

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$$\begin{aligned}
&\bigcup_{\mathbf{x} \in \mathbf{X}} \left\{ \mathbf{y} \in \mathbf{Y} \;\middle|\; \mathbf{x} \overset{*}{\rightsquigarrow} \mathbf{y} \right\} \\
&= \bigcup_{\rho \in \Omega} \mathrm{tgt}(\rho) \\
&= \bigcup_{\rho \in \min_{\trianglelefteq_{\mathbf{P} \times \mathbf{Q}}}(\Omega)} \{ \mathrm{tgt}(\rho') \mid \rho' \in \Omega \ \wedge \ \rho \trianglelefteq_{\mathbf{P} \times \mathbf{Q}} \rho' \} \\
&= \bigcup_{\rho \in \min_{\trianglelefteq_{\mathbf{P} \times \mathbf{Q}}}(\Omega)} \mathrm{tgt}(\rho) + \{ \mathrm{tgt}(\rho') - \mathrm{tgt}(\rho) \mid \rho' \in \Omega \ \wedge \ \rho \trianglelefteq_{\mathbf{P} \times \mathbf{Q}} \rho' \}
\end{aligned}$$

where:

- Ω set of runs from $\mathbf{X}$ to $\mathbf{Y}$
- $\rho \trianglelefteq_{\mathbf{P} \times \mathbf{Q}} \rho'$ if $\rho \trianglelefteq \rho' \ \wedge \ \mathrm{src}(\rho') \in \mathrm{src}(\rho) + \mathbf{P} \ \wedge \ \mathrm{tgt}(\rho') \in \mathrm{tgt}(\rho) + \mathbf{Q}$

Decomposing Reachability Sets

$\mathbf{X}, \mathbf{Y} \subseteq \mathbb{N}^d$ ~~semilinear sets~~ linear sets $\mathbf{X} = \mathbf{x} + \mathbf{P}$ and $\mathbf{Y} = \mathbf{y} + \mathbf{Q}$.

$$\bigcup_{\mathbf{x} \in \mathbf{X}} \left\{ \mathbf{y} \in \mathbf{Y} \;\middle|\; \mathbf{x} \overset{*}{\rightsquigarrow} \mathbf{y} \right\}$$

$$= \bigcup_{\rho \in \Omega} \operatorname{tgt}(\rho)$$

$$= \bigcup_{\rho \in \min_{\trianglelefteq_{\mathbf{P} \times \mathbf{Q}}}(\Omega)} \{ \operatorname{tgt}(\rho') \mid \rho' \in \Omega \ \wedge \ \rho \trianglelefteq_{\mathbf{P} \times \mathbf{Q}} \rho' \}$$

$$= \bigcup_{\rho \in \min_{\trianglelefteq_{\mathbf{P} \times \mathbf{Q}}}(\Omega)} \operatorname{tgt}(\rho) + \{ \operatorname{tgt}(\rho') - \operatorname{tgt}(\rho) \mid \rho' \in \Omega \ \wedge \ \rho \trianglelefteq_{\mathbf{P} \times \mathbf{Q}} \rho' \}$$

$$= \bigcup_{\rho \in \min_{\trianglelefteq_{\mathbf{P} \times \mathbf{Q}}}(\Omega)} \operatorname{tgt}(\rho) + \left\{ \mathbf{q} \in \mathbf{Q} \;\middle|\; \exists \mathbf{p} \in \mathbf{P} \ \ \mathbf{p} \overset{\rho}{\curvearrowright} \mathbf{q} \right\}$$

where:

- Ω set of runs from $\mathbf{X}$ to $\mathbf{Y}$
- $\rho \trianglelefteq_{\mathbf{P} \times \mathbf{Q}} \rho'$ if $\rho \trianglelefteq \rho' \ \wedge \ \operatorname{src}(\rho') \in \operatorname{src}(\rho) + \mathbf{P} \ \wedge \ \operatorname{tgt}(\rho') \in \operatorname{tgt}(\rho) + \mathbf{Q}$

Theorem

If the sets

$$\left\{\mathbf{q} \in \mathbf{Q} \;\middle|\; \exists \mathbf{p} \in \mathbf{P} \;\; \mathbf{p} \overset{\rho}{\curvearrowright} \mathbf{q}\right\}$$

$$\left\{\mathbf{p} \in \mathbf{P} \;\middle|\; \exists \mathbf{q} \in \mathbf{Q} \;\; \mathbf{p} \overset{\rho}{\curvearrowright} \mathbf{q}\right\}$$

are asymptotically definable periodic sets for every finitely-generated periodic sets $\mathbf{P}, \mathbf{Q} \subseteq \mathbb{N}^d$ *and for every run* ρ
then for every semilinear sets $\mathbf{X}, \mathbf{Y} \subseteq \mathbb{N}^d$ *the sets:*

$$\bigcup_{\mathbf{x} \in \mathbf{X}} \left\{\mathbf{y} \in \mathbf{Y} \;\middle|\; \mathbf{x} \overset{*}{\rightsquigarrow} \mathbf{y}\right\}$$

$$\bigcup_{\mathbf{y} \in \mathbf{Y}} \left\{\mathbf{x} \in \mathbf{X} \;\middle|\; \mathbf{x} \overset{*}{\rightsquigarrow} \mathbf{y}\right\}$$

are almost semilinear.

Stability Properties

Lemma

For every periodic relations $R, R_1, R_2 \subseteq \mathbb{N}^d \times \mathbb{N}^d$, we have:

$$\mathbb{Q}_{\geq 0}(R_1 \cap R_2) = (\mathbb{Q}_{\geq 0} R_1) \cap (\mathbb{Q}_{\geq 0} R_2)$$

$$\mathbb{Q}_{\geq 0}(R_1 \circ R_2) = (\mathbb{Q}_{\geq 0} R_1) \circ (\mathbb{Q}_{\geq 0} R_2)$$

$$\mathbb{Q}_{\geq 0}\{\mathbf{y} \in \mathbb{N}^d \mid (\mathbf{x}, \mathbf{y}) \in R\} = \{\mathbf{y} \in \mathbb{Q}_{\geq 0}^d \mid (\mathbf{x}, \mathbf{y}) \in \mathbb{Q}_{\geq 0} R\}$$

Corollary

With $\rho = \mathbf{c}_0 \ldots \mathbf{c}_k$:

$$\mathbb{Q}_{\geq 0}\left\{\mathbf{q} \in \mathbf{Q} \;\middle|\; \exists \mathbf{p} \in \mathbf{P} \;\; \mathbf{p} \overset{\rho}{\curvearrowright} \mathbf{q}\right\}$$
$$= \left\{\mathbf{q} \in \mathbb{Q}_{\geq 0}\mathbf{Q} \;\middle|\; \exists \mathbf{p} \in \mathbb{Q}_{\geq 0}\mathbf{P} \;\; (\mathbf{p}, \mathbf{q}) \in (\mathbb{Q}_{\geq 0} \overset{\mathbf{c}_0}{\curvearrowright}) \circ \cdots \circ (\mathbb{Q}_{\geq 0} \overset{\mathbf{c}_k}{\curvearrowright})\right\}$$

$$\mathbb{Q}_{\geq 0}\left\{\mathbf{p} \in \mathbf{P} \;\middle|\; \exists \mathbf{q} \in \mathbf{Q} \;\; \mathbf{p} \overset{\rho}{\curvearrowright} \mathbf{q}\right\}$$
$$= \left\{\mathbf{p} \in \mathbb{Q}_{\geq 0}\mathbf{P} \;\middle|\; \exists \mathbf{q} \in \mathbb{Q}_{\geq 0}\mathbf{Q} \;\; (\mathbf{p}, \mathbf{q}) \in (\mathbb{Q}_{\geq 0} \overset{\mathbf{c}_0}{\curvearrowright}) \circ \cdots \circ (\mathbb{Q}_{\geq 0} \overset{\mathbf{c}_k}{\curvearrowright})\right\}$$

Theorem

If transformer relations are asymptotically definable, then:

$$\bigcup_{\mathbf{x}\in\mathbf{X}} \left\{\mathbf{y} \in \mathbf{Y} \;\middle|\; \mathbf{x} \overset{*}{\rightsquigarrow} \mathbf{y}\right\}$$

$$\bigcup_{\mathbf{y}\in\mathbf{Y}} \left\{\mathbf{x} \in \mathbf{X} \;\middle|\; \mathbf{x} \overset{*}{\rightsquigarrow} \mathbf{y}\right\}$$

are almost semilinear for every semilinear sets $\mathbf{X}, \mathbf{Y} \subseteq \mathbb{N}^d$.

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Recall

Definition (Transformer Relation)

$$\mathbf{x} \overset{\mathbf{c}}{\curvearrowright} \mathbf{y} \quad \text{if} \quad \begin{array}{c}\mathbf{x}\\ +\\ \mathbf{c}\end{array} \overset{*}{\rightsquigarrow} \begin{array}{c}\mathbf{y}\\ +\\ \mathbf{c}\end{array}$$

Example

$$\overset{\mathbf{0}}{\curvearrowright} \quad = \quad \overset{*}{\rightsquigarrow}$$

$\overset{\mathbf{c}}{\curvearrowright}$ is an asymptotically definable periodic relation iff:

- $\mathbf{0} \overset{\mathbf{c}}{\curvearrowright} \mathbf{0}$, and
- $\mathbf{x}_1 \overset{\mathbf{c}}{\curvearrowright} \mathbf{y}_1 \ \wedge \ \mathbf{x}_2 \overset{\mathbf{c}}{\curvearrowright} \mathbf{y}_2 \ \Rightarrow \ (\mathbf{x}_1 + \mathbf{x}_2) \overset{\mathbf{c}}{\curvearrowright} (\mathbf{y}_1 + \mathbf{y}_2)$, and
- $\mathbb{Q}_{\geq 0} \overset{\mathbf{c}}{\curvearrowright}$ is definable in $\mathrm{FO}(\mathbb{Q}_{\geq 0}, +, \leq, 0)$.

Transformer Relations are Periodic

$$\mathbf{x}_1 \overset{\mathbf{c}}{\curvearrowright} \mathbf{y}_1 \implies \begin{matrix}\mathbf{x}_1\\+\\\mathbf{c}\end{matrix} \overset{*}{\rightsquigarrow} \begin{matrix}\mathbf{y}_1\\+\\\mathbf{c}\end{matrix}$$

$$\implies \mathbf{x}_2 \begin{matrix}\mathbf{x}_1\\+\\\mathbf{c}\end{matrix} \overset{*}{\rightsquigarrow} \begin{matrix}\mathbf{y}_1\\+\\\mathbf{c}\end{matrix} \mathbf{x}_2$$

$$\mathbf{x}_2 \overset{\mathbf{c}}{\curvearrowright} \mathbf{y}_2 \implies \begin{matrix}\mathbf{x}_2\\+\\\mathbf{c}\end{matrix} \overset{*}{\rightsquigarrow} \begin{matrix}\mathbf{y}_2\\+\\\mathbf{c}\end{matrix}$$

$$\implies \mathbf{y}_1 \begin{matrix}\mathbf{x}_2\\+\\\mathbf{c}\end{matrix} \overset{*}{\rightsquigarrow} \begin{matrix}\mathbf{y}_2\\+\\\mathbf{c}\end{matrix} \mathbf{y}_1$$

Main Problem

Show that

$$\mathbb{Q}_{\geq 0}\overset{\mathbf{c}}{\curvearrowright} \quad = \quad \{\lambda(\mathbf{x},\mathbf{y}) \mid \lambda \in \mathbb{Q}_{\geq 0} \ \wedge \ \mathbf{x} \overset{\mathbf{c}}{\curvearrowright} \mathbf{y}\}$$

is definable in $\mathrm{FO}(\mathbb{Q}_{\geq 0}, +, \leq, 0)$.

Main idea:

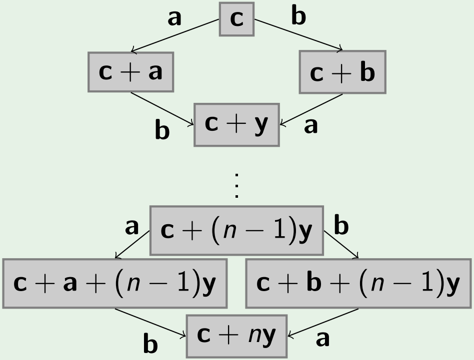
$$\boxed{\begin{array}{c}\text{Pick}\\ \gamma = (\mathbf{x},\mathbf{c},\mathbf{y})\\ \mathbf{x}\overset{\mathbf{c}}{\curvearrowright}\mathbf{y}\end{array}} \longrightarrow \boxed{\begin{array}{c}\text{Build}\\ \text{a graph } G_\gamma\end{array}} \longrightarrow \boxed{\begin{array}{c}\text{Build}\\ \text{a relation } R_\gamma\\ (\mathbf{x},\mathbf{y}) \in R_\gamma \subseteq \mathbb{Q}_{\geq 0}\overset{\mathbf{c}}{\curvearrowright}\end{array}}$$

$$\mathbb{Q}_{\geq 0}\overset{\mathbf{c}}{\curvearrowright} \quad = \quad \bigcup_{\substack{\gamma = (\mathbf{x},\mathbf{c},\mathbf{y}) \\ \mathbf{x}\overset{\mathbf{c}}{\curvearrowright}\mathbf{y}}} R_\gamma$$

$$\Omega_\gamma = \bigcup_{n\in\mathbb{N}} \{\text{runs from } \mathbf{c} + n\mathbf{x} \text{ to } \mathbf{c} + n\mathbf{y}\}$$

Example

$\mathbf{A} = \{\mathbf{a}, \mathbf{b}\}$ where $\mathbf{a} = (1, 1, -1)$ and $\mathbf{b} = (-1, 0, 1)$
$\gamma = (\mathbf{x}, \mathbf{c}, \mathbf{y})$ where $\mathbf{x} = (0, 0, 0)$, $\mathbf{c} = (1, 0, 1)$ and $\mathbf{y} = (0, 1, 0)$



Bounded Components

$$I_\gamma = \{\text{bounded components of configurations occurring in runs of } \Omega_\gamma\}$$

Lemma

$\mathbf{x}(i) = 0 \wedge \mathbf{y}(i) = 0$ *for all* $i \in I_\gamma$.

Proof.

$$\mathbf{x} \overset{\mathbf{c}}{\curvearrowright} \mathbf{y} \Rightarrow \begin{array}{c}\mathbf{x}\\+\\\mathbf{c}\end{array} \xrightarrow{w} \begin{array}{c}\mathbf{y}\\+\\\mathbf{c}\end{array}$$

$$\Rightarrow \begin{array}{c}n\mathbf{x}\\+\\\mathbf{c}\end{array} \xrightarrow{w^n} \begin{array}{c}n\mathbf{y}\\+\\\mathbf{c}\end{array}$$

$\square$

Unbounded Component Projection

$$\begin{array}{rccc} \pi_\gamma : & \mathbb{N}^d & \to & \mathbb{N}^{I_\gamma} \\ & \mathbf{x} & \to & (\mathbf{x}(i))_{i \in I_\gamma} \end{array}$$

Corollary

For every run ρ in Ω_γ:

$$\pi_\gamma(\mathrm{src}(\rho)) = \pi_\gamma(\mathbf{c}) = \pi_\gamma(\mathrm{tgt}(\rho))$$

Example

$\mathbf{A} = \{\mathbf{a}, \mathbf{b}\}$ where $\mathbf{a} = (1, 1, -1)$ and $\mathbf{b} = (-1, 0, 1)$
$\gamma = (\mathbf{x}, \mathbf{c}, \mathbf{y})$ where $\mathbf{x} = (0, 0, 0)$, $\mathbf{c} = (1, 0, 1)$ and $\mathbf{y} = (0, 1, 0)$

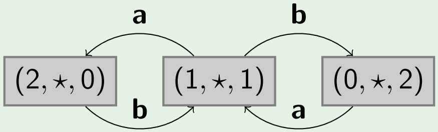
$$\begin{tikzcd} \boxed{(2, \star, 0)} \arrow[r, bend right, "\mathbf{b}"'] & \boxed{(1, \star, 1)} \arrow[l, bend right, "\mathbf{a}"'] \arrow[r, bend left, "\mathbf{b}"] & \boxed{(0, \star, 2)} \arrow[l, bend left, "\mathbf{a}"] \end{tikzcd}$$

Relation R_γ

$$R_\gamma = \left\{ (\mathbf{e}, \mathbf{f}) \in \mathbb{Q}_{\geq 0}^d \times \mathbb{Q}_{\geq 0}^d \;\middle|\; \begin{array}{c} \bigwedge_{i \mid \mathbf{x}(i) > 0} \mathbf{e}(i) > 0 \wedge \\ \bigwedge_{i \mid \mathbf{y}(i) > 0} \mathbf{f}(i) > 0 \wedge \\ \mathbf{f} - \mathbf{e} \in \mathbb{Q}_{>0}(\mathbf{a}_1 + \cdots + \mathbf{a}_k) \\ \text{where } \mathbf{a}_1 \ldots \mathbf{a}_k \text{ label of a total cycle in } G_\gamma \end{array} \right\}$$

Example

$\mathbf{A} = \{\mathbf{a}, \mathbf{b}\}$ where $\mathbf{a} = (1, 1, -1)$ and $\mathbf{b} = (-1, 0, 1)$
$\gamma = (\mathbf{x}, \mathbf{c}, \mathbf{y})$ where $\mathbf{x} = (0, 0, 0)$, $\mathbf{c} = (1, 0, 1)$ and $\mathbf{y} = (0, 1, 0)$



$$R_\gamma = \left\{ (\mathbf{e}, \mathbf{e}) + \mathbb{Q}_{>0}(\mathbf{0}, \mathbf{y}) \;\middle|\; \mathbf{e} \in \mathbb{Q}_{\geq 0}^3 \right\}$$

Facts:

- $\{G_{(\mathbf{x},\mathbf{c},\mathbf{y})} \mid \mathbf{x} \overset{\mathbf{c}}{\curvearrowright} \mathbf{y}\}$ is finite for all $\mathbf{c}$.
- $\{R_{(\mathbf{x},\mathbf{c},\mathbf{y})} \mid \mathbf{x} \overset{\mathbf{c}}{\curvearrowright} \mathbf{y}\}$ is finite for all $\mathbf{c}$.
- R_γ is definable in $\mathrm{FO}(\mathbb{Q}_{\geq 0}, +, \leq, 0)$.
- $(\mathbf{x}, \mathbf{y}) \in R_\gamma$.
- $R_\gamma \subseteq \mathbb{Q}_{\geq 0} \overset{\mathbf{c}}{\curvearrowright}$

Thus

$$\mathbb{Q}_{\geq 0} \overset{\mathbf{c}}{\curvearrowright} = \bigcup_{\substack{\gamma = (\mathbf{x},\mathbf{c},\mathbf{y}) \\ \mathbf{x} \overset{\mathbf{c}}{\curvearrowright} \mathbf{y}}} R_\gamma$$

is definable in $\mathrm{FO}(\mathbb{Q}_{\geq 0}, +, \leq, 0)$.

Theorem

$\overset{\mathbf{c}}{\curvearrowright}$ *is an asymptotically definable periodic relation.*

Main Application

Theorem

If transformer relations are asymptotically definable, then:

$$\bigcup_{\mathbf{x}\in\mathbf{X}} \left\{\mathbf{y} \in \mathbf{Y} \;\middle|\; \mathbf{x} \overset{*}{\rightsquigarrow} \mathbf{y}\right\}$$

$$\bigcup_{\mathbf{y}\in\mathbf{Y}} \left\{\mathbf{x} \in \mathbf{X} \;\middle|\; \mathbf{x} \overset{*}{\rightsquigarrow} \mathbf{y}\right\}$$

are almost semilinear for every semilinear sets $\mathbf{X}, \mathbf{Y} \subseteq \mathbb{N}^d$.

Main Application

Theorem

~~*If transformer relations are asymptotically definable, then:*~~

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are almost semilinear for every semilinear sets $\mathbf{X}, \mathbf{Y} \subseteq \mathbb{N}^d$.

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Main Objective

Show the following:

Theorem (Leroux '09 '10 '11 '12)

If $\mathbf{c}_{final}$ is not reachable from $\mathbf{c}_{init}$ there exists a semilinear inductive invariant $\mathbf{X}$ such that $\mathbf{c}_{init} \in \mathbf{X}$ and $\mathbf{c}_{final} \notin \mathbf{X}$.

Forward and Backward Images

$$\text{Forward}(\mathbf{X}) = \bigcup_{\mathbf{x} \in \mathbf{X}} \{\mathbf{y} \in \mathbb{N}^d \mid \mathbf{x} \xrightarrow{*} \mathbf{y}\}$$

$$\text{Backward}(\mathbf{Y}) = \bigcup_{\mathbf{y} \in \mathbf{Y}} \{\mathbf{x} \in \mathbb{N}^d \mid \mathbf{x} \xrightarrow{*} \mathbf{y}\}$$

Corollary

For every semilinear sets $\mathbf{X}, \mathbf{Y}$*:*

$$\text{Forward}(\mathbf{X}) \backslash \mathbf{X}$$

$$\text{Backward}(\mathbf{Y}) \backslash \mathbf{Y}$$

are almost semilinear.

Separators

Definition (Separators)

A pair $(\mathbf{X}, \mathbf{Y})$ of subsets of $\mathbb{N}^d$ is called a separator if:

$$\text{Forward}(\mathbf{X}) \cap \text{Backward}(\mathbf{Y}) = \emptyset$$

$\mathbf{D} = \mathbb{N}^d \backslash (\mathbf{X} \cup \mathbf{Y})$: the domain.

$$(\mathbf{X}, \mathbf{Y}) \text{ not a separator}$$

$$\Longleftrightarrow$$

$$\exists (\mathbf{x}, \mathbf{y}) \in \mathbf{X} \times \mathbf{Y} \quad \mathbf{x} \overset{*}{\rightsquigarrow} \mathbf{y}$$

Separators with Empty Domains

$$(\mathbf{X},\mathbf{Y}) \text{ separator with empty domain}$$
$$\Longleftrightarrow$$
$$(\mathbf{X},\mathbf{Y}) \text{ partition of } \mathbb{N}^d \text{ with } \mathrm{Forward}(\mathbf{X}) = \mathbf{X} \text{ and } \mathrm{Backward}(\mathbf{Y}) = \mathbf{Y}.$$

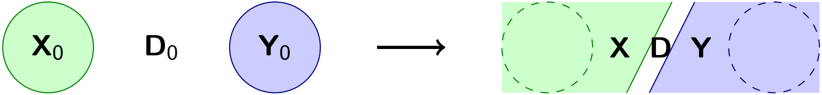
Example

Separators $(\mathbf{X},\mathbf{Y})$ are included in separators with empty domains:

$$(\mathrm{Forward}(\mathbf{X})\ ,\ \mathbb{N}^d \setminus \mathrm{Forward}(\mathbf{X}))$$

$$(\mathbb{N}^d \setminus \mathrm{Backward}(\mathbf{Y})\ ,\ \mathrm{Backward}(\mathbf{Y}))$$

Reduce The Domain



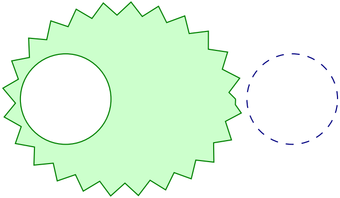
$(\mathbf{X}_0, \mathbf{Y}_0)$ semilinear separator $\longrightarrow$ $(\mathbf{X}, \mathbf{Y})$ semilinear separator

with $\dim(\mathbf{D}_0) > \dim(\mathbf{D})$

Facts:

- $(\mathbf{X}_0, \mathbf{Y}_0)$ semilinear separator

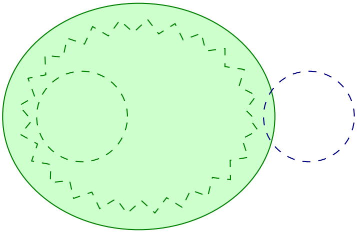
Forward($\mathbf{X}_0$)$\setminus\mathbf{X}_0$ is an almost semilinear set



Let $\mathbf{S}$ be a linearization.

Facts:

- $(\mathbf{X}_0, \mathbf{Y}_0)$ semilinear separator
- $\mathbf{S}$ linearization of Forward($\mathbf{X}_0$)$\setminus\mathbf{X}_0$

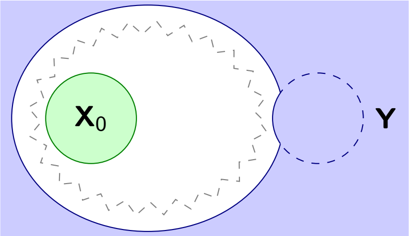


$\mathbf{X}_0 \cup \mathbf{S}$ is an over-approximation of Forward($\mathbf{X}_0$).

Facts:

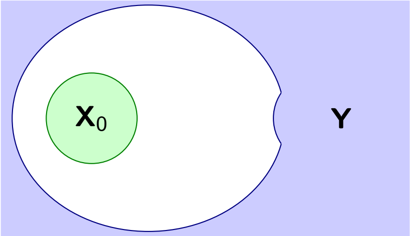
- $(\mathbf{X}_0, \mathbf{Y}_0)$ semilinear separator
- $\mathbf{S}$ linearization of Forward$(\mathbf{X}_0) \setminus \mathbf{X}_0$

$$\mathbf{Y} := \mathbf{Y}_0 \cup (\mathbb{N}^d \backslash (\mathbf{X}_0 \cup \mathbf{S}))$$



Facts:

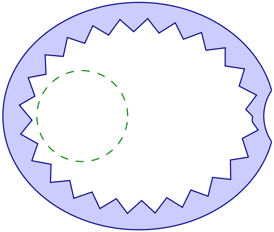
- $(\mathbf{X}_0, \mathbf{Y}_0)$ semilinear separator
- $\mathbf{S}$ linearization of Forward$(\mathbf{X}_0) \backslash \mathbf{X}_0$
- $\mathbf{Y} = \mathbf{Y}_0 \cup (\mathbb{N}^d \backslash (\mathbf{X}_0 \cup \mathbf{S}))$
- $(\mathbf{X}_0, \mathbf{Y})$ semilinear separator.



Facts:

- $(\mathbf{X}_0, \mathbf{Y}_0)$ semilinear separator
- $\mathbf{S}$ linearization of Forward$(\mathbf{X}_0)\backslash\mathbf{X}_0$
- $\mathbf{Y} = \mathbf{Y}_0 \cup (\mathbb{N}^d \backslash (\mathbf{X}_0 \cup \mathbf{S}))$
- $(\mathbf{X}_0, \mathbf{Y})$ semilinear separator.

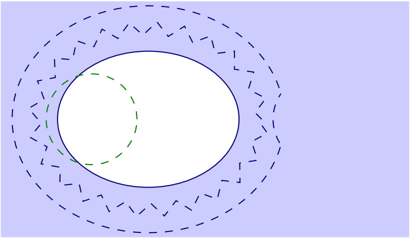
Backward($\mathbf{Y}$)$\backslash\mathbf{Y}$ is an almost semilinear set



Let $\mathbf{T}$ be a linearization.

Facts:

- $(\mathbf{X}_0, \mathbf{Y}_0)$ semilinear separator
- $\mathbf{S}$ linearization of Forward$(\mathbf{X}_0)\backslash\mathbf{X}_0$
- $\mathbf{Y} = \mathbf{Y}_0 \cup (\mathbb{N}^d \backslash (\mathbf{X}_0 \cup \mathbf{S}))$
- $(\mathbf{X}_0, \mathbf{Y})$ semilinear separator.
- $\mathbf{T}$ linearization of Backward$(\mathbf{Y})\backslash\mathbf{Y}$

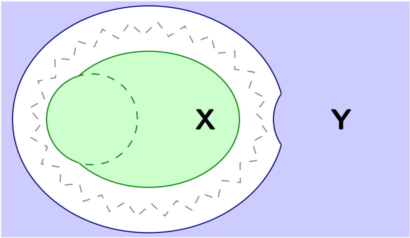


$\mathbf{Y} \cup \mathbf{T}$ is an over-approximation of Backward($\mathbf{Y}$).

Facts:

- $(\mathbf{X}_0, \mathbf{Y}_0)$ semilinear separator
- $\mathbf{S}$ linearization of Forward$(\mathbf{X}_0)\backslash\mathbf{X}_0$
- $\mathbf{Y} = \mathbf{Y}_0 \cup (\mathbb{N}^d \backslash (\mathbf{X}_0 \cup \mathbf{S}))$
- $(\mathbf{X}_0, \mathbf{Y})$ semilinear separator.
- $\mathbf{T}$ linearization of Backward$(\mathbf{Y})\backslash\mathbf{Y}$

$$\mathbf{X} := \mathbf{X}_0 \cup (\mathbb{N}^d \backslash (\mathbf{Y} \cup \mathbf{T}))$$



$(\mathbf{X}, \mathbf{Y})$ is a semilinear separator such that $\mathbf{X}_0 \subseteq \mathbf{X}$

Facts:

- $(\mathbf{X}_0, \mathbf{Y}_0)$ semilinear separator
- $\mathbf{S}$ linearization of Forward$(\mathbf{X}_0)\backslash\mathbf{X}_0$
- $\mathbf{Y} = \mathbf{Y}_0 \cup (\mathbb{N}^d \backslash (\mathbf{X}_0 \cup \mathbf{S}))$
- $(\mathbf{X}_0, \mathbf{Y})$ semilinear separator.
- $\mathbf{T}$ linearization of Backward$(\mathbf{Y})\backslash\mathbf{Y}$
- $\mathbf{X} = \mathbf{X}_0 \cup (\mathbb{N}^d \backslash (\mathbf{Y} \cup \mathbf{T}))$
- $(\mathbf{X}, \mathbf{Y})$ semilinear separator.

Induction

The domain $\mathbf{D}$ of $(\mathbf{X}, \mathbf{Y})$ satisfies $\mathbf{D} = \mathbf{D}_0 \cap \mathbf{S} \cap \mathbf{T}$ where:

- $\mathbf{S}$ is a linearization of $\text{Forward}(\mathbf{X}_0) \backslash \mathbf{X}_0 \subseteq \mathbf{D}_0$.
- $\mathbf{T}$ is a linearization of $\text{Backward}(\mathbf{Y}) \backslash \mathbf{Y} \subseteq \mathbf{D}_0$.

Since $(\mathbf{X}, \mathbf{Y})$ is a separator we deduce that $(\mathbf{X}_0, \mathbf{Y})$ is also a separator. Hence the sets $\text{Forward}(\mathbf{X}_0)$ and $\text{Backward}(\mathbf{Y})$ have an empty intersection.

We get $\dim(\mathbf{S} \cap \mathbf{T}) < \dim(\mathbf{D}_0)$. Hence $\dim(\mathbf{D}) < \dim(\mathbf{D}_0)$.

Theorem

$$\neg(\mathbf{c}_{init} \overset{*}{\rightsquigarrow} \mathbf{c}_{final})$$

$$\Longrightarrow$$

There exists a semilinear separator $(\mathbf{X}, \mathbf{Y})$ *with an empty domain such that:*

$$\mathbf{c}_{init} \in \mathbf{X} \qquad \textit{and} \qquad \mathbf{c}_{final} \in \mathbf{Y}$$

Proof.

$(\{\mathbf{c}_{\text{init}}\}, \{\mathbf{c}_{\text{final}}\})$ is a semilinear separator. □

Main result

Theorem (Leroux '09 '10 '11 '12)

If $\mathbf{c}_{final}$ is not reachable from $\mathbf{c}_{init}$ there exists a semilinear inductive invariant $\mathbf{X}$ such that $\mathbf{c}_{init} \in \mathbf{X}$ and $\mathbf{c}_{final} \notin \mathbf{X}$.

Corollary

The reachability problem for VAS is decidable.

Table of Contents

Sum Up

Recent advances presented:

- Geometrical properties satisfied by VAS reachability sets.
- Presburger arithmetic is sufficient for denoting certificates of non-reachability.
- Reachability problem with a simple algorithm.
- Reachability computation of semilinear Petri nets based on acceleration.
- The well order $\trianglelefteq$ over the runs.

Not presented:

- Complexity results for Dickson's/ Higman's lemma (and others...)
- Rackoff's techniques : Regularity, boundedness, place-bounded, context-freeness,...
- Simulation / Bisimulation / Games for Petri nets.
- Extensions : Branching VAS, VAS + 1 zero test, VAS + 1 stack.

Conclusion : Open Problems

- Simple criterion for detecting the initialized VAS not semilinear.
- Improve acceleration techniques with on-demand over-approximations.
- Close the complexity gap.
- At least, provide a clear upper bound (in the fast growing hierarchy).