

Insignificant Choice Polynomial Time

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Abstract. In the late 1980s Gurevich conjectured that there is no logic capturing PTIME, where “logic” has to be understood in a very general way comprising computation models over isomorphism classes of structures. In this article we first show that Gurevich’s conjecture is false. For this we extend the seminal research of Blass, Gurevich and Shelah on *choiceless polynomial time* (CPT), which exploits deterministic Abstract State Machines (ASMs) supporting unbounded parallelism to capture the choiceless fragment of PTIME. CPT is strictly included in PTIME. We observe that choice is unavoidable, but that a restricted version suffices, which guarantees that the final result is independent from the choice. Such a version of polynomially bounded ASMs, which we call *insignificant choice polynomial time* (ICPT) will indeed capture PTIME. This can be expressed in the logic of non-deterministic ASMs plus inflationary fixed-point.

We use this result for our second contribution showing that PTIME differs from NP. For the proof we build again on the research on CPT first establishing a limitation on permutation classes of the sets that can be activated by an ICPT computation. We then prove an equivalence theorem, which characterises structures that cannot be distinguished by the logic. In particular, this implies that SAT cannot be decided by an ICPT computation.

Keywords. abstract state machine, non-determinism, insignificant choice, polynomial time, PTIME logic, descriptive complexity, Gurevich’s conjecture, inflationary fixed-point logic, ASM logic, NP, pebble game, SAT

1 Introduction

In 1982 Chandra and Harel raised the question whether there is a computation model over structures that captures PTIME rather than Turing machines that operate over strings [14]. Soon later Gurevich raised the related question how to define a computation model over structures that could serve as foundation for the notion of algorithm [22]. As there is typically a huge gap between the abstraction level of an algorithm and the one of Turing machines, Gurevich formulated a new thesis based on the observation that “if an abstraction level is fixed (disregarding low-level details and a possible higher-level picture) and the states of an algorithm

reflect all the relevant information, then a particular small instruction set suffices to model any algorithm, never mind how abstract, by a generalised machine very closely and faithfully”. This led to the definition of Abstract State Machines (ASMs), formerly known as evolving algebras [24].

Nonetheless, in 1988 Gurevich formulated the conjecture that there is no logic—understood in a very general way comprising computation models over isomorphism classes of structures—capturing PTIME [23]. If true an immediate implication would be that PTIME differs from NP. There is indeed a lot of evidence supporting this conjecture. Among the most important results in descriptive complexity theory (see Immerman’s monograph [28]) are Fagin’s theorem stating that the complexity class NP is captured by the existential fragment of second-order logic [17], and the theorem by Immerman, Livchak and Vardi stating that over ordered structures the complexity class PTIME is captured by first-order logic plus inflationary fixed-point [27,30,33]. Thus, if there is a logic capturing PTIME, it must be contained in \exists SO and extend IFP[FO]. As an extension by increase of order can be ruled out, the argumentation concentrates on the addition of generalised quantifiers, but there is very little evidence that all of PTIME can be captured by adding a set of quantifiers to IFP[FO] (see e.g. the rather detailed discussion in Libkin’s monograph [29, 204f.]).

Another strong argument supporting Gurevich’s conjecture comes from the work of Blass, Gurevich and Shelah on choiceless polynomial time (CPT), which exploits a polynomial time-bounded version of deterministic ASMs supporting unbounded parallelism, but no choice [9] (see also [5,10]). CPT extends IFP[FO], subsumes other models of computation on structures such relational machines [3], reflective relational machines [1] and generic machines [2], but still captures only a fragment of PTIME. As shown in [9, Thm.42,43] some PTIME problems such as Parity or Bipartite Matching cannot be expressed in CPT, and for extensions of CPT by adding quantifiers such as counting the perspective of capturing PTIME remains as dark as for IFP[FO], as all arguments given by Libkin in [29, 204f.] also apply to CPT.

If true, another consequence of Gurevich’s conjecture would be that complexity theory could not be based as a whole on more abstract models of computations on structures such as ASMs. In particular, it would not be possible to avoid dealing with string encodings using Turing Machines. However, this consequence appears to be less evident in view of the ASM success stories. Gurevich’s important sequential ASM thesis pro-

vides a purely logical definition of the notion of sequential algorithm and shows that these are captured by sequential ASMs [25], which provides solid mathematical support for the “new thesis” formulated by Gurevich in 1985 [22]. Generalisations of the theory have been developed for unbounded parallel algorithms [6,8,15,18] (differences concerning the axiomatic definition of the class of (synchronous) parallel algorithms are discussed in [18]), recursive algorithms [12], concurrent algorithms [11], and for reflective algorithms [31]. In addition, the usefulness of ASMs for high-level development of complex systems is stressed by Börger in [13]. Furthermore, logics that enable reasoning about ASMs have been developed for deterministic ASMs by Stärk and Nanchen [32] based on ideas from Glavan and Rosenzweig that had already been exploited for CPT [21], and extended to non-deterministic ASMs by Ferrarotti, Schewe, Tec and Wang [19,20], the latter work leading to a fragment of second-order logic with Henkin semantics [26].

Therefore, we dared to doubt that Gurevich’s conjecture is true. However, we abandoned looking for a solution by means of additional quantifiers. Instead, our research is based on a rather simple observation made during the study of CPT. While CPT captures only a fragment of PTIME, the counter-examples used in [9] show that non-deterministic choice cannot be avoided. In case of the Parity problem a PTIME-bounded computation is only possible, if we can select an arbitrary element from the (finite) input set, and similarly in case of bipartite matching it must be possible to select arbitrary unmatched boys and girls. In [18] the construction of an ASM rule capturing an abstractly given parallel algorithm draws on finite model theory (see [16] or [29]). A decisive argument is that logically indistinguishable tuples, i.e. with the same type, can only both give rise to updates in an update set or both not. In other words, it must be possible to break types, and this is only possible by using non-deterministic choice as used in non-deterministic ASMs [13].

However, simply adding arbitrary choice to the CPT computation model would be too strong. We observed that in all cases of PTIME problems not covered by CPT the required choice is insignificant in the sense that it does not matter, which choice is made: if one selection leads to a positive outcome for the decision problem, any other selection will lead to the same outcome. Hence the idea to formalise *insignificant choice polynomial time* (ICPT) as an extension of the CPT computation model and to show that in this way we are able to capture PTIME. This will be the first contribution of this article.

In view of the important theorem by Immerman, Livchak and Vardi it is not too hard to see that PTIME problems can be solved by polynomial time bounded ASMs with insignificant choice, as it suffices to create an order on the base set. To show that the extension will remain within PTIME it suffices to repeat the arguments for CPT. Actually, adding a choice does not have a big impact on the complexity of a run. We can then exploit that a choice being insignificant can be expressed in the logic of non-deterministic ASMs [19,20], so we can define a fragment capturing just a logic of insignificant choice ASMs. Then following again the work in [9] we can make insignificant choice ASMs time-explicit and obtain a fixed-point theorem characterising acceptance of a finite input structure by a computation using an insignificant choice ASM. Using a fairly standard embedding into an infinitary logic we define specific pebble games that can be used to characterise the expressiveness of the logic.

This leads to the second contribution of this article. We show the limitations of ICPT identifying problems that cannot be solved by an ICPT computation. First we establish a limitation on permutation classes of sets that can be activated by an ICPT computation. This gives a support theorem analogous to the one proven in [9]. It is quite plausible that such a result holds, as the added choice is insignificant. Then we prove an equivalence theorem characterising structures that cannot be distinguished by the logic. We apply this equivalence theorem to show that SAT cannot be decided by an ICPT computation, which implies that PTIME differs from NP.

Organisation of the Article. In Section 2 we introduce the background from ASMs and define a version of non-deterministic ASMs analogous to the one in [9], i.e. we assume a finite input structure, and consider only hereditarily finite sets as objects. This is then used to define PTIME-bounded ASMs. Section 3 is dedicated to the introduction of insignificant choice. We start with motivating examples, then formally restrict our computation model to PTIME-bounded insignificant choice ASMs, which we exploit to show the capture of PTIME. In Section 4 we continue with the logic of non-deterministic ASMs and PTIME logic analogous to the work on CPT. We then define the insignificant choice fragment of the logic and show a fixed-point theorem characterising acceptance of a finite input structure by a computation using an insignificant choice ASM. We show that such formulae can be expressed in an infinitary extension of the logic of ASMs restricted to insignificant choice. For this infinitary logic we

define a version of a pebble game, by means of which it becomes possible to characterise the expressiveness of the logic.

In Section 5 we address the power and limitations of ICPT. We first prove a generalisation of the support theorem for CPT to ICPT, by means of which we obtain a restriction on the size of permutation classes of sets that can be activated in an ICPT computation. The proof follows largely the argumentation in [9] for the extended version of coloured sets. The integer constant in the support theorem motivates the definition of structures over skew-symmetric orbits. Then we exploit this for a proof of an equivalence theorem which characterises structures over skew-symmetric objects that cannot be distinguished by the infinitary extension of the logic of ASMs restricted to insignificant choice, provided the structures are sufficiently large. As for the corresponding equivalence theorem for CPT the proof exploits that skew-symmetric objects can be defined by molecules and forms, which under certain conditions enables a winning strategy for the duplicator for the pebble game defined before. We apply this theorem to structures defined for the SAT problem and show that there are sufficiently large structures, one satisfiable, the other one not, which cannot be distinguished by our logic. This implies the second main result, the inequality of PTIME and NP. We conclude in Section 6 first summarising our results, then discussing the wider perspective of complexity theory on the grounds of ASMs, i.e. computations on isomorphism classes of structures rather than Turing machines.

2 Abstract State Machines

We assume familiarity with the basic concepts of ASMs. In general, ASMs including their foundations, semantics and usage in applications are the subject of the detailed monograph by Börger and Stärk [13]. In a nutshell, an ASM is defined by a signature, i.e. a finite set of function (and relation) symbols, a background, and a rule. The signature defines states as structures, out of which a set of initial states is defined. The sets of states and initial states are closed under isomorphisms. The background defines domains and fixed operations on them that appear in every state (see [7] for details), and the rule defines a relation between states and successor states and thus also runs.

Here we follow the development for CPT in [9] and adapt ASMs to the purpose of our study on complexity theory. In particular, we use hereditarily finite sets, which according to Barwise form a rather natural domain for computation [4].

2.1 Signature and Background

The *background* of an ASM, as we use them here, comprises logic names and set-theoretic names:

Logic names. These comprise the binary equality $=$, nullary function names **true** and **false** and the usual Boolean operations. All logic names are relational.

Set-theoretic names. These comprise the binary predicate \in , nullary function names \emptyset and *Atoms*, unary function names \bigcup and *The-Unique*, and the binary function name *Pair*.

We will use \emptyset also to denote undefinedness, for which usually another function name *undef* would be used. In this way we can concentrate on sets.

The signature \mathcal{Y} of an ASM, as we use them here, comprises input names and dynamic names:

Input names. These are given by a finite set of relation symbols, each with a fixed arity. Input names will be considered being static, i.e. locations defined by them will never be updated by the ASM.

Dynamic names. These are given by a finite set of function symbols, each with a fixed arity, including *Output* and a nullary function symbol *Halt*. Some of the dynamic names may be relational.

2.2 States

States S are defined as structures over the signature \mathcal{Y} , for which we assume specific base sets. A *base set* B comprises two disjoint parts: a finite set A of *atoms*, which are not sets, and the collection $HF(A)$ of hereditarily finite sets built over A .

If \mathcal{P} denotes the powerset operator and we define inductively $\mathcal{P}^0(A) = A$ and $\mathcal{P}^{i+1}(A) = \mathcal{P}(\bigcup_{j \leq i} \mathcal{P}^j(A))$, then we have

$$HF(A) = \bigcup_{i < \omega} \mathcal{P}^i(A) = A \cup \mathcal{P}(A) \cup \mathcal{P}(A \cup \mathcal{P}(A)) \cup \dots$$

We also use the following terminology. The atoms in A and the sets in $HF(A)$ are the *objects* of the base set B . A set X is called *transitive* iff $x \in X$ and $y \in x$ implies $y \in X$. If x is an object, then $TC(x)$ denotes the least transitive set X with $x \in X$. If $TC(x)$ is finite, the object x is called *hereditarily finite*.

The logic names are interpreted in the usual way, i.e. **true** and **false** are interpreted by 1 and 0, respectively (i.e. by $\{\emptyset\}$ and \emptyset). Boolean operations are undefined, i.e. give rise to the value 0, if at least one of the arguments is not Boolean.

The set-theoretic names \in and \emptyset are interpreted in the obvious way, and *Atoms* is interpreted by the set of atoms of the base set. If a_1, \dots, a_k are atoms and b_1, \dots, b_ℓ are sets, then $\bigcup\{a_1, \dots, a_k, b_1, \dots, b_\ell\} = b_1 \cup \dots \cup b_\ell$. For $b = \{a\}$ we have $TheUnique(b) = a$, otherwise it is undefined. Furthermore, we have $Pair(a, b) = \{a, b\}$.

An input name p is interpreted by a Boolean-valued function. If the arity is n and $p(a_1, \dots, a_n)$ holds, then each a_i must be an atom. Finally, a dynamic function symbol f of arity n is interpreted by a function $f_S : B^n \rightarrow B$ (or by $f_S : B^n \rightarrow \{0, 1\}$, if f is relational). The domain $\{(a_1, \dots, a_n) \mid f(a_1, \dots, a_n) \neq 0\}$ is required to be finite.

With such an interpretation we obtain the set of states over the signature \mathcal{T} and the given background. An *initial state* contains an *input structure*, which is a finite structure over the subsignature comprising only the input names¹. Equivalently we could require the domain of each dynamic function to be empty. If the finite set of atoms is A , then $|A|$ is referred to as the *size* of the input.

2.3 Terms and Rules

Terms and *Boolean terms* are defined in the usual way assuming a given set of variables V :

- Each variable $v \in V$ is a term.
- If f is a function name of arity n (in the signature \mathcal{T} or the background) and t_1, \dots, t_n are terms, then $f(t_1, \dots, t_n)$ is a term. If f is declared to be relational, the term is Boolean.
- If v is a variable, $t(v)$ is a term, s is a term without free occurrence of v , and $g(v)$ is a Boolean term, then $\{t(v) \mid v \in s : g(v)\}$ is a term.

The set $fr(t)$ of *free variables* in a term t is defined as usual, in particular $fr(\{t(v) \mid v \in s \wedge g(v)\}) = (fr(t(v)) \cup fr(s) \cup fr(g(v))) - \{v\}$. Also the interpretation of terms in a state S is standard.

ASM rules as we use them are defined as follows:

skip. **skip** is a rule.

¹ See the remark in [9, p.18] that without loss of generality it can be assumed that only atoms appear in the input.

assignment. If f is a dynamic function symbol in \mathcal{T} of arity n and t_0, \dots, t_n are terms, then $f(t_1, \dots, t_n) := t_0$ is a rule.

branching. If φ is a Boolean term and r_1, r_2 are rules, then also **if** φ **then** r_1 **else** r_2 **endif** is a rule.

parallelism. If v is a variable, t is a term with $v \notin fr(t)$, and $r(v)$ is a rule, then also **forall** $v \in t$ **do** $r(v)$ **enddo** is a rule.

choice. If v is a variable, t is a term with $v \notin fr(t)$, and $r(v)$ is a rule, then also **choose** $v \in t$ **do** $r(v)$ **enddo** is a rule.

Furthermore, we use the shortcut **par** $r_1 \dots r_k$ **endpar** for **forall** $i \in \{1, \dots, k\}$ **do** **if** $i = 1$ **then** r_1 **else** **if** $i = 2$ **then** r_2 **else** \dots **if** $i = k$ **then** r_k **endif** \dots **endif** **enddo**.

We further use the shortcut **let** $x = t$ **in** $r(x)$ for **choose** $x \in Pair(t, t)$ **do** $r(x)$ **enddo**.

The rule associated with an ASM must be closed. The semantics of ASM rules is defined via update sets that are built for the states of the machine. Applying an update set to a state defines a successor state.

If f is dynamic function symbol in \mathcal{T} of arity n , and a_1, \dots, a_n are objects of the base set B of a state S , then the pair $(f, (a_1, \dots, a_n))$ is a *location* of the state S . We use the abbreviation \bar{a} for tuples (a_1, \dots, a_n) , whenever the arity is known from the context. For a location $\ell = (f, \bar{a})$ we write $val_S(\ell) = b$ iff $f_S(a_1, \dots, a_n) = b$; we call b the value of the location ℓ in the state S .

An *update* is a pair (ℓ, a) consisting of a location ℓ and an object $a \in B$, and an *update set* (for a state S) is a set of updates with locations of S and objects a in the base set of S .

Now let S be a state with base set B , and let $\zeta : V \rightarrow B$ be a variable assignment. Let r be an ASM rule. We define a set of update sets $\Delta_{r,\zeta}(S)$ on state S for the rule r depending on ζ as follows:

- $\Delta_{\text{skip},\zeta}(S) = \{\emptyset\}$.
- For an assignment rule r of the form $f(t_1, \dots, t_n) := t_0$ taking the location $\ell = (f, (\text{val}_{S,\zeta}(t_1), \dots, \text{val}_{S,\zeta}(t_n)))$ and the object $a = \text{val}_{S,\zeta}(t_0)$ we have $\Delta_{r,\zeta}(S) = \{\{\ell, a\}\}$.
- For a branching rule r of the form **if** φ **then** r_1 **else** r_2 **endif** we have

$$\Delta_{r,\zeta}(S) = \begin{cases} \Delta_{r_1,\zeta}(S) & \text{if } \text{val}_{S,\zeta}(\varphi) = 1 \\ \Delta_{r_2,\zeta}(S) & \text{if } \text{val}_{S,\zeta}(\varphi) = 0 \end{cases}.$$

- For a parallel rule r of the form **forall** $v \in t$ **do** $r(v)$ **enddo** we have

$$\Delta_{r,\zeta}(S) = \left\{ \bigcup_{a \in \text{val}_{S,\zeta}(t)} \Delta_a \mid \Delta_a \in \Delta_{r(v),\zeta(v \mapsto a)}(S) \text{ for all } a \in \text{val}_{S,\zeta}(t) \right\}.$$

- For a choice rule r of the form **choose** $v \in t$ **do** $r(v)$ **enddo** we have

$$\Delta_{r,\zeta}(S) = \bigcup_{a \in \text{val}_{S,\zeta}(t)} \Delta_{r(v),\zeta(v \mapsto a)}(S).$$

2.4 Runs of ASMs

An update set Δ is *consistent* iff for any two updates $(\ell, a_1), (\ell, a_2) \in \Delta$ with the same location we have $a_1 = a_2$. This defines the notion of *successor state* $S' = S + \Delta$ of a state S . For a consistent update set $\Delta \in \Delta_{r,\zeta}(S)$ and a location ℓ we have

$$\text{val}_{S'}(\ell) = \begin{cases} a & \text{for } (\ell, a) \in \Delta \\ \text{val}_S(\ell) & \text{else} \end{cases}.$$

In addition, let $S + \Delta = S$ for inconsistent update sets Δ .

Then the (closed) rule r of an ASM defines a set of successor states for each state S . We write $\Delta_r(S, S')$ for an update set in $\Delta_r(S)$ with $S' = S + \Delta_r(S, S')$. This allows us to define the notion of run of an ASM.

A *run* of an ASM M with rule r is a finite or infinite sequence of states S_0, S_1, \dots such that S_0 is an initial state and $S_{i+1} = S_i + \Delta$ holds for some update set $\Delta \in \Delta_r(S_i)$. Furthermore, if k is the length of a run ($k = \omega$ for an infinite run), then *Halt* must fail on all states S_i with $i < k$.

Note that in a run all states have the same base set², which is in accordance with requirements from the behavioural theories of sequential and parallel algorithms (see [25] and [18], respectively, and [13, Sect.2.4.4]). However, not all atoms and sets are active in the sense that they appear as value or argument of a location with defined value. We therefore define active objects³ as follows.

Definition 2.1. Let S be a state with base set B . An object $a \in B$ is called *critical* iff a is an atom or $a \in \{0, 1\}$ or a is the value of a location ℓ of S or there is a location $\ell = (f, \bar{a})$ with $\text{val}_S(\ell) \neq \emptyset$ and a appears in \bar{a} . An object $a \in B$ is called *active* in S iff there exists a critical object a' with $a \in TC(a')$.

In addition, if $R = S_0, S_1, \dots$ is a run of an ASM, then we call an object $a \in B$ *active* in R iff a is active in at least one state S_i of R .

² See the discussion in [25, Sect.4.5].

³ We adopt here the notion of critical and active object as defined in the work on CPT [9]. It should be noted that there is a close relationship to critical elements in a state defined via a minimal bounded exploration witness in the sequential, recursive and parallel ASM theses (see [25,12] and [18], respectively), and the active objects then correspond to the closure of a bounded exploration witness under subterms.

2.5 Polynomial-Time-Bounded ASMs

In order to define a polynomial time bound on an ASM we have to count steps of a run. If we only take the length of a run, each step would be a macrostep that involves many elementary updates, e.g. the use of unbounded parallelism does not impose any restriction on the number of updates in an update set employed in a transition from one state to a successor state. So we better take the size of update sets into account as well. Furthermore, as objects are sets, their size also matters in estimating what an appropriate microstep is. We therefore adopt the notion of PTIME bound from CPT [9].

Definition 2.2. A *PTIME (bounded) ASM* is a triple $\tilde{M} = (M, p(n), q(n))$ comprising an ASM M and two integer polynomials $p(n)$ and $q(n)$. A *run* of \tilde{M} is a maximal initial segment of a run of M of length at most $p(n)$ and a total number of at most $q(n)$ active objects, where n is the size of the input in the initial state of the run.

We say that a PTIME ASM \tilde{M} *accepts* the input structure I iff there is a run of \tilde{M} with initial state defined by I and ending in a state, in which *Halt* holds and the value of *Output* is 1. Analogously, a PTIME ASM \tilde{M} *rejects* the input structure I iff there is a run of \tilde{M} with initial state defined by I and ending in a state, in which *Halt* holds and the value of *Output* is 0.

3 Insignificant Choice

PTIME bounded ASMs as defined in the previous section are the non-deterministic analog of the PTIME bounded machines used to define CPT [9]. As such they also allow the usual set-theoretic expressions to be used freely (see [9, Sect.6.1]). In this section we will motivate and then define a restriction of this non-deterministic computation model.

3.1 Examples

We look at two rather simple problems in PTIME and their solution using PTIME ASMs, the Parity problem and the bipartite matching problem. For both problems Blass, Gurevich and Shelah showed that they are not in CPT [9], thus adding choice to the computation model adds strength.

Example 3.1. Let us consider ASMs without input names. So the input structure is just the naked set of atoms. In addition to *Output* and *Halt*

use nullary function symbols $mode$, set and $parity$, and assume that in an initial state $mode = \text{init}$ holds.

Consider the following ASM rule

```

par  if       $mode = \text{init}$ 
      then par   $mode := \text{progress}$ 
                 $set := \text{Atoms}$ 
                 $parity := \text{false}$ 
      endpar
endif
if     $mode = \text{progress}$ 
then if  $set \neq \emptyset$ 
      then choose  $x \in set$  do
        par   $set := set - \text{Pair}(x, x)$ 
               $parity := \neg parity$ 
        endpar
      enddo
      else par   $Output := parity$ 
                 $Halt := \text{true}$ 
      endpar
endif
endif
endpar

```

Clearly, we obtain a PTIME bounded ASM, and $Output$ will become true iff the size of the input structure is odd.

Without choice the solution to Parity in Example 3.1 would not be possible within a polynomial time bound. We could replace the choice by unbounded parallelism, but then the computation would explore all possible orderings of the set of atoms, whereas with the choice only a single ordering is considered. Note that this is sufficient for the Parity problem. Further note that the Subset Parity problem could be handled in an analogous way.

Example 3.2. For the bipartite matching problem we are given a finite bipartite graph (V, E) , where the set V of vertices is partitioned into two sets *Boys* and *Girls* of equal size. Thus, the set E of edges contains sets $\{x, y\}$ with $x \in \text{Boys}$ and $y \in \text{Girls}$. A *perfect matching* is a subset $F \subseteq E$ such that every vertex is incident to exactly one edge in F . A *partial matching* is a subset $F \subseteq E$ such that every vertex is incident to at most one edge in F . So the algorithm will create larger and larger partial

matchings until no more unmatched boys and girls are left, otherwise no perfect matching exists.

We use functions `girls_to_boys` and `boys_to_girls` turning sets of unordered edges into sets of ordered pairs:

$$\begin{aligned}\text{girls_to_boys}(X) &= \{(g, b) \mid b \in \text{Boys} \wedge g \in \text{Girls} \wedge \{b, g\} \in X\} \\ \text{boys_to_girls}(X) &= \{(b, g) \mid b \in \text{Boys} \wedge g \in \text{Girls} \wedge \{b, g\} \in X\}\end{aligned}$$

Conversely, the function `unordered` turns a set of ordered pairs (b, g) or (g, b) into a set of two-element sets:

$$\text{unordered}(X) = \{\{x, y\} \mid (x, y) \in X\}$$

We further use a predicate `reachable` and a function `path`. For the former one we have `reachable(b, X, g)` iff there is a path from b to g using the directed edges in X . For the latter one `path(b, X, g)` is a set of ordered pairs representing a path from b to g using the directed edges in X . Both functions are defined elsewhere.

Then an algorithm for bipartite matching is realised by an ASM with the following rule:

```

par if      mode = init
  then par   mode := examine
             partial_match :=  $\emptyset$ 
          endpar
  endif
  if        mode = examine
  then if     $\exists b \in \text{Boys} . \forall g \in \text{Girls} . \{b, g\} \notin \text{partial\_match}$ 
  then      mode := build-digraph
  else par   Output := true
             Halt := true
             mode := final
          endpar
  endif
  endif
  if        mode = build-digraph
  then par   di_graph := girls_to_boys(partial_match)
              $\cup$  boys_to_girls(E - partial_match)
             mode := build-path
          endpar
  endif
  if        mode = build-path

```

```

then choose  $b \in \{x \mid x \in Boys : \forall g \in Girls. \{b, g\} \notin partial\_match\}$ 
  do    if  $\exists g' \in Girls. \forall b' \in Boys. \{b', g'\} \notin partial\_match$ 
         $\wedge reachable(b, di\_graph, g')$ 
      then choose  $g \in \{y \mid y \in Girls. \forall x \in Boys. \{x, y\}$ 
         $\notin partial\_match \wedge reachable(b, di\_graph, y)\}$ 
      do par  $path := path(b, di\_graph, g)$ 
         $mode := modify$ 
      endpar
      enddo
    else par  $Output := false$ 
       $Halt := true$ 
       $mode := final$ 
    endpar
  endif
enddo
endif
if     $mode = modify$ 
then par     $partial\_match := (partial\_match - unordered(path))$ 
               $\cup (unordered(path) - partial\_match)$ 
           $mode := examine$ 
endpar
endif
endpar

```

Clearly, we obtain a PTIME bounded ASM, and *Output* will become true iff there exists a perfect matching.

Again the use of choice-rules in Example 3.2 cannot be dispensed with. However, we observe that in both cases, i.e. for the Parity problem and the bipartite matching problem, that the choices used are insignificant in the sense that if the final output is true for one choice made, then it is also true for any other possible choice. For the case of Parity this corresponds to the implicit creation of different orderings, while for bipartite matching different perfect matchings are constructed. We will formalise this observation in the sequel.

3.2 Insignificant Choice Polynomial Time

We now formalise the observation above concerning insignificant choice. We further restrict PTIME ASMs, which defines the ICPT logic, which in Section 4 will be linked to the logic of non-deterministic ASMs [19,20].

Definition 3.1. An *insignificant choice ASM* (for short: icASM) is an ASM M such that for every run S_0, \dots, S_k of length k such that *Halt* holds in S_k , every $i \in \{0, \dots, k-1\}$ and every update set $\Delta \in \mathbf{\Delta}(S_i)$ there exists a run $S_0, \dots, S_i, S'_{i+1}, \dots, S'_m$ such that $S'_{i+1} = S_i + \Delta$, *Halt* holds in S'_m , and *Output* = **true** (or **false**, respectively) holds in S_k iff *Output* = **true** (or **false**, respectively) holds in S'_m .

A *PTIME (bounded) insignificant choice ASM* (for short: PTIME icASM) is a triple $\tilde{M} = (M, p(n), q(n))$ comprising an icASM M and two integer polynomials $p(n)$ and $q(n)$ with runs defined as in Definition 2.2 such that whenever an input structure I is accepted by \tilde{M} (or rejected, respectively) then every run on input structure I is accepting (or rejecting, respectively).

According to this definition choices are insignificant in two respects. First, whenever there exists an accepting or rejecting run, then all other runs on the same input structure, i.e. runs that result making different choices, are also accepting or rejecting, respectively. Second, when an accepting (or rejecting) run remains within the polynomial time bounds, then all other accepting (or rejecting, respectively) runs on the same input also remain within these time bounds.

Note that the insignificant choice restriction is a semantic one expressed by means of runs. We will see in the next section how insignificant choice can be characterised in a logical way.

Definition 3.2. The complexity class *insignificant choice polynomial time* (ICPT) is the collection of pairs (K_1, K_2) , where K_1 and K_2 are disjoint classes of finite structures of the same signature, such that there exists a PTIME icASM that accepts all structures in K_1 and rejects all structures in K_2 .

We also that a pair $(K_1, K_2) \in \text{ICPT}$ is *ICPT separable*. As for the analogous definition of CPT a PTIME icASM may accept structures not in K_1 and reject structures not in K_2 . Therefore, we also say that a class K of finite structures is in ICPT, if $(K, K') \in \text{ICPT}$ holds for the complement K' of structures over the same signature.

Let us link the definition of ICPT to PTIME logics as defined in [9]. In general, a logic \mathcal{L} can be defined by a pair (Sen, Sat) of functions satisfying the following conditions:

- Sen assigns to every signature \mathcal{Y} a recursive set $\text{Sen}(\mathcal{Y})$, the set of \mathcal{L} -sentences of signature \mathcal{Y} .

- *Sat* assigns to every signature \mathcal{T} a recursive binary relation $Sat_{\mathcal{T}}$ over structures S over \mathcal{T} and sentences $\varphi \in Sen(\mathcal{T})$. We assume that $Sat_{\mathcal{T}}(S, \varphi) \Leftrightarrow Sat_{\mathcal{T}}(S', \varphi)$ holds, whenever S and S' are isomorphic.

We say that a structure S over \mathcal{T} *satisfies* $\varphi \in Sen(\mathcal{T})$ (notation: $S \models \varphi$) iff $Sat_{\mathcal{T}}(S, \varphi)$ holds.

If \mathcal{L} is a logic in this general sense, then for each signature \mathcal{T} and each sentence $\varphi \in Sen(\mathcal{T})$ let $K(\mathcal{T}, \varphi)$ be the class of structures S with $S \models \varphi$. We then say that \mathcal{L} is a *PTIME logic*, if every class $K(\mathcal{T}, \varphi)$ is PTIME in the sense that it is closed under isomorphisms and there exists a PTIME Turing machine that accepts exactly the standard encodings of ordered versions of the structures in the class.

We further say that a logic \mathcal{L} *captures PTIME* iff it is a PTIME logic and for every signature \mathcal{T} every PTIME class of \mathcal{T} -structures coincides with some class $K(\mathcal{T}, \varphi)$.

These definitions of PTIME logics can be generalised to three-valued logics, in which case $Sat_{\mathcal{T}}(S, \varphi)$ may be true, false or unknown. For these possibilities we say that φ *accepts* S or φ *rejects* S or neither, respectively. Then two disjoint classes K_1 and K_2 of structures over \mathcal{T} are called *\mathcal{L} -separable* iff there exists a sentence φ accepting all structures in K_1 and rejecting all those in K_2 .

In this sense, ICPT defines a three-valued PTIME logic that separates pairs of structures in ICPT. The sentences of this logic are PTIME icASMs, for which \mathcal{T} is the signature of the input structure. By abuse of terminology we also denote this logic as ICPT.

3.3 The Capture of PTIME

We now present our first main result, the capture of PTIME by ICPT. For the proof that ICPT is subsumed by PTIME we basically use the same arguments as in the proof of the upper bound theorem for CPT [9, Thm.3]. For the proof that PTIME is subsumed by ICPT we exploit the fundamental theorem by Immerman, Livchak and Vardi that over ordered structures PTIME is captured by first-order logic plus inflationary or least fixed-point, i.e. by IFP[FO] [27,30,33]. As CPT and hence also ICPT covers IFP[FO] we only have to show that with a PTIME icASM we can create an order.

Theorem 3.1. *The logic of ICPT captures PTIME on arbitrary finite structures, i.e. $ICPT = PTIME$.*

Proof. **PTIME** \subseteq **ICPT**. Consider a PTIME problem represented by a Boolean query ϕ and an input structure I for ϕ , for which there exists a PTIME Turing machine T accepting I iff I satisfies ϕ . We combine three steps. First we show that with a PTIME icASM we can construct an arbitrary order on the set of atoms of I , so we obtain an ordered structure $(I, <)$. Then using an ASM rule **CREATE_ENCODING** we build the binary encoding of $(I, <)$ (see [29, p.88]), which can be done by a PTIME ASM without choice in polynomial time. In the third step we use an ASM rule **RUN_SIMULATION** to simulate T (see [13, p.289]), which defines another PTIME ASM without choice. The second and the third steps are standard. The rule of the combined PTIME icASM looks as follows:

```

par if       $mode = \text{init}$ 
  then par    $mode := \text{create-order}$ 
               $A := \text{Atoms}$ 
               $A^c := \emptyset$ 
            endpar
  endif
  if       $mode = \text{create-order}$ 
  then if     $A \neq \emptyset$ 
    then choose  $a \in A$ 
      do   par   forall  $a' \in A^c$ 
        do  $<(a', a) := \text{true}$ 
        enddo
         $A := A - \{a\}$ 
         $A^c := A^c \cup \{a\}$ 
      endpar
    enddo
  else  $mode := \text{build-tm}$ 
  endif
endif
if       $mode = \text{build-tm}$ 
then par   CREATE_ENCODING
             $mode := \text{simulate-tm}$ 
  endpar
endif
if       $mode = \text{simulate-tm}$ 
then RUN_SIMULATION
endif
endpar

```


ICPT \subseteq PTIME. Assume a PTIME icASM $\tilde{M} = (M, p(n), q(n))$. Analogously to the proof of [9, Thm.3] we create a simulating PTIME Turing machine, which takes strings encoding ordered versions of input structures I of \tilde{M} as input. The bounds in set terms appearing in **forall** and **choose** rules ensures that the number of immediate subcomputations in the former case as well as the number of possible choices is bounded by the number of active elements and thus by $q(n)$. This defines a polynomial bound on the number of steps of the simulating Turing machine. Furthermore, as the simulating Turing machine operates on encodings of ordered versions of structures, any choice is simulated by selecting the smallest element in the order. As all choices are insignificant, this has no effect on the final result. Hence the simulating Turing machine accepts (or rejects) its input iff \tilde{M} accepts the corresponding input structure or rejects it, respectively. \square

Note that the first part of the proof shows that PTIME is included in a fragment of ICPT, where every single choice leads to isomorphic update sets, which together with the second part of the proof gives us a handier characterisation of ICPT. We will exploit this characterisation in the section on fixed-point definability in the logic of non-deterministic ASMs.

Still the theorem is not yet fully satisfactory, because so far we defined ICPT by means of a semantic restriction of PTIME ASMs. We will address this problem in the next section linking ICPT to the logic of non-deterministic ASMs [19,20].

4 The Logic of ASMs and Insignificant Choice

We now look further into the capture of PTIME by ICPT, for which we exploit the logic of non-deterministic ASMs, which we tailor to our purposes here.

4.1 The Logic of Non-Deterministic ASMs

The logic of ASMs introduced by Stärk and Nanchen is a definitional extension of first-order logic that allows us to express statements about a step of a computation of a deterministic ASM, i.e. without choice-rules. Decisive for the logic are predicates $upd(r, f, \bar{x}, y)$ with a rule r and a (dynamic) function symbol f in the signature \mathcal{T} of an ASM. Informally, it expresses that after application of the rule r in the given state the location (f, \bar{x}) (so the length of the tuple \bar{x} is the arity of f) will have the

value y . In [21] and also in [9] this predicate was written as $Update_{r,f}(\bar{x}, y)$ to emphasise the extra-logical character of r and f .

The extension of the logic to the case of non-deterministic ASMs was left open due to problems concerning the expression of consistency (see the detailed discussion in [13, p.326f.]). This gap was closed by Ferrarotti et al. in [19] first emphasising database transformations, then in general⁴ in [20]. The key idea underlying this work is to replace $upd(r, f, \bar{x}, y)$ by a new predicate $upd(r, X)$ with the informal meaning that the rule r in the given state yields the update set X . Then $[X]\varphi$ (instead of $[r]\varphi$ in the Stärk/Nanchen logic) is used to express that after the application of the update set X , i.e. in the successor state, the formula φ will hold.

This makes the logic a fragment of second-order logic. Nonetheless, a completeness result could be achieved on the grounds of Henkin semantics for the logic.

The logic addresses many subtleties considered necessary for reasoning about non-deterministic ASMs, but several of these are not needed for our purposes here:

- The ASMs supported by the logic are based on meta-finite states. Therefore, three types of function symbols are distinguished: database functions defining finite structures, algorithmic functions defining arbitrary structures, and bridge functions linking them. In Section 2 for the ASMs in this article we required that the domain of each dynamic function shall be finite in every state. With this requirement the subtle distinction obtained by meta-finite states becomes irrelevant.
- The logic is grounded on ASMs that permit multiset functions (defined in the background) as synchronisation operators for synchronous parallelism as expressed by **forall** rules. The version of ASMs in this article, however, does not foresee such operators. Synchronisation can be handled in a different way, which will require more microsteps, but otherwise has no effect on the complexity as defined here. Therefore, we can dispense with the so-called ϱ -terms of the logic.
- In order to feed the synchronisation operators the logic handles not only update sets, but also update multisets, and a corresponding predicate $upm(r, \ddot{X})$ expressing that rule r yields the update multiset \ddot{X} in the given state. Clearly, for our purposes here we can omit update multisets.

⁴ This includes the possibility to simulate concurrent ASMs by non-deterministic ones, which permits to exploit the logic also for reasoning about concurrent systems.

With these simplifications in mind we define the logic \mathcal{L}^{nd} of non-deterministic ASMs. As the logic has to deal with update sets, we extend the signature of our ASMs by fresh constant symbols c_f for each dynamic function symbol $f \in \mathcal{T}$, i.e. c_f is not dynamic and has arity 0. We also extend the base set by fresh elements that interpret c_f in every state. By abuse of notation we write $(c_f)_S = c_f$. Now let X be a second-order variable of order 3. For a variable assignment ζ we say that $\zeta(X)$ *represents an update set* Δ iff for each $((f, \bar{a}), b) \in \Delta$ we have $(c_f, \bar{a}, b) \in \zeta(X)$ and vice versa.

As for the syntax, with this extension the terms of \mathcal{L}^{nd} are ASM terms as defined in Section 2. The formulae of the logic are defined inductively as follows:

- If t and t' are terms, then $t = t'$ is a formula.
- If X is an n -ary second-order variable and t_1, \dots, t_n are terms, then $X(t_1, \dots, t_n)$ is a formula.
- If r is an ASM rule and X is a second-order variable of arity 3, then $\text{upd}_r(X)$ is a formula.
- If φ and ψ are formulae, then also $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$ and $\varphi \rightarrow \psi$ are formulae.
- If φ is a formula, x is a first-order variable and X is a second-order variable, then also $\forall x.\varphi$, $\exists x.\varphi$, $\forall X.\varphi$, $\exists X.\varphi$ are formulae.
- If φ is a formula and X is a second-order variable of order 3, then $[X]\varphi$ is formula.

We use the usual shortcuts and parentheses when needed. We also write simply $f(t_1, \dots, t_n)$ for the formula $f(t_1, \dots, t_n) = \mathbf{true}$ in case f is relational.

For the semantics of \mathcal{L}^{nd} we use Henkin structures [26]. We first define Henkin prestructures.

Definition 4.1. A *Henkin prestructure* \tilde{S} over signature \mathcal{Y} is a structure S over \mathcal{T} as defined in Section 2 with base set B together with sets of relations $D_n \subseteq \mathcal{P}(B^n)$ for all $n \geq 1$.

With this definition we inherit all our specific assumptions on structures from Section 2, in particular the finiteness of the domain of all dynamic function symbols. For a consistent update set Δ we assume that $S + \Delta$ preserves the sets of relations D_n of \tilde{S} , i.e. $\tilde{S} + \Delta$ is the Henkin prestructure $(S + \Delta, \{D_n\}_{n \geq 1})$.

As the logic uses second-order variables we need extended variable assignments ζ into a Henkin prestructure. For first-order variables x we

have $\zeta(x) \in B$ as usual, but for second-order variables X of arity n we request $\zeta(X) \in D_n$. Then with respect to a Henkin prestructure \tilde{S} and such a variable assignment terms are interpreted as usual. The interpretation $\llbracket \varphi \rrbracket_{\tilde{S}, \zeta}$ for formulae φ is defined as follows (we omit the interpretation for the shortcuts):

– If φ has the form $t = t'$, then $\llbracket \varphi \rrbracket_{\tilde{S}, \zeta} = \begin{cases} 1 & \text{if } \text{val}_{S, \zeta}(t) = \text{val}_{S, \zeta}(t') \\ 0 & \text{else} \end{cases}$.

– If φ has the form $X(t_1, \dots, t_n)$, then

$$\llbracket \varphi \rrbracket_{\tilde{S}, \zeta} = \begin{cases} 1 & \text{if } (\text{val}_{S, \zeta}(t_1), \dots, \text{val}_{S, \zeta}(t_n)) \in \text{val}_{S, \zeta}(X) \\ 0 & \text{else} \end{cases}.$$

– If φ has the form $\text{upd}_r(X)$, then

$$\llbracket \varphi \rrbracket_{\tilde{S}, \zeta} = \begin{cases} 1 & \text{if } \text{val}_{S, \zeta}(X) \text{ represents an update set } \Delta \in \mathbf{\Delta}_{r, \zeta}(S) \\ 0 & \text{else} \end{cases}.$$

– If φ has the form $\neg\psi$, then $\llbracket \varphi \rrbracket_{\tilde{S}, \zeta} = \begin{cases} 1 & \text{if } \llbracket \psi \rrbracket_{\tilde{S}, \zeta} = 0 \\ 0 & \text{else} \end{cases}$.

– If φ has the form $\psi_1 \vee \psi_2$, then $\llbracket \varphi \rrbracket_{\tilde{S}, \zeta} = \begin{cases} 0 & \text{if } \llbracket \psi_1 \rrbracket_{\tilde{S}, \zeta} = \llbracket \psi_2 \rrbracket_{\tilde{S}, \zeta} = 0 \\ 1 & \text{else} \end{cases}$.

– If φ has the form $\forall x.\psi$, then

$$\llbracket \varphi \rrbracket_{\tilde{S}, \zeta} = \begin{cases} 1 & \text{if } \llbracket \psi \rrbracket_{\tilde{S}, \zeta[x \mapsto a]} = 1 \text{ for all } a \in B \\ 0 & \text{else} \end{cases}.$$

– If φ has the form $\forall X.\psi$ with a second-order variable X of order n , then

$$\llbracket \varphi \rrbracket_{\tilde{S}, \zeta} = \begin{cases} 1 & \text{if } \llbracket \psi \rrbracket_{\tilde{S}, \zeta[X \mapsto A]} = 1 \text{ for all } A \in D_n \\ 0 & \text{else} \end{cases}.$$

– If φ has the form $[X]\psi$, then

$$\llbracket \varphi \rrbracket_{\tilde{S}, \zeta} = \begin{cases} 0 & \text{if } \text{val}_{S, \zeta}(X) \text{ represents a consistent update set } \Delta \\ & \text{with } \llbracket \psi \rrbracket_{\tilde{S} + \Delta, \zeta} = 0 \\ 1 & \text{else} \end{cases}.$$

While this interpretation is defined for arbitrary Henkin prestructures, it makes sense to restrict the collections D_n of n -ary relations to those that are closed under definability, which defines the notion of Henkin structure. We then say that a sentence is *valid* iff it is interpreted as 1 (i.e., true) in all Henkin structures.

Definition 4.2. A *Henkin structure* over signature \mathcal{T} is a Henkin pre-structure $\tilde{S} = (S, \{D_n\}_{n \geq 1})$ that is closed under definability, i.e. for every formula φ , every variable assignment ζ and every $n \geq 1$ we have

$$\{(a_1, \dots, a_n) \in B^n \mid \llbracket \varphi \rrbracket_{\tilde{S}, \zeta[x_1 \mapsto a_1, \dots, x_n \mapsto a_n]} = 1\} \in D_n.$$

Note that we merely need second-order variables to capture update sets and isomorphisms defined by choice rules. Therefore, the use of Henkin structures is not really a restriction.

4.2 ICPT Restriction

We now approach a characterisation of the semantic insignificant choice restriction in the logic \mathcal{L}^{nd} defined above. First we show how to express that X represents an update set, or is consistent. For these we have

$$\text{isUSet}(X) \equiv \forall x_1, \bar{x}_2, x_3. X(x_1, \bar{x}_2, x_3) \rightarrow \bigvee_{f \in \mathcal{T}_{dyn}} (x_1 = c_f \wedge f(\bar{x}_2) = f(\bar{x}_3))$$

using \mathcal{T}_{dyn} to denote the set of dynamic function symbols and

$$\begin{aligned} \text{conUSet}(X) &\equiv \text{isUSet}(X) \wedge \forall x_1, \bar{x}_2, x_3, x_4. \\ &\quad (X(x_1, \bar{x}_2, x_3) \wedge X(x_1, \bar{x}_2, x_4) \rightarrow x_3 = x_4). \end{aligned}$$

Next we exploit that a choice in state S will be insignificant iff all update sets in $\Delta_r(S)$ are isomorphic, so we express that X is an isomorphism by

$$\begin{aligned} \text{iso}(X) &\equiv \forall x, y_1, y_2. (X(x, y_1) \wedge X(x, y_2) \rightarrow y_1 = y_2) \wedge \\ &\quad \forall x_1, x_2, y. (X(x_1, y) \wedge X(x_2, y) \rightarrow x_1 = x_2) \wedge \\ &\quad \forall x \exists y. X(x, y) \wedge \forall y \exists x. X(x, y) \end{aligned}$$

This leads to the following *insignificance constraint*:

$$\begin{aligned} &\forall X_1, X_2. \text{upd}_r(X_1) \wedge \text{upd}_r(X_2) \rightarrow \exists X. (\text{iso}(X) \\ &\quad \wedge \forall x_0, \bar{x}_1, x_2, \bar{y}_1, y_2. (X_1(x_0, \bar{x}_1, x_2) \wedge \bigwedge_{1 \leq i \leq n} X(x_{1i}, y_{1i}) \wedge X(x_2, y_2) \\ &\quad \rightarrow X_2(x_0, \bar{y}_1, y_2)) \\ &\quad \wedge \forall x_0, \bar{x}_1, x_2, \bar{y}_1, y_2. (X_2(x_0, \bar{x}_1, x_2) \wedge \bigwedge_{1 \leq i \leq n} X(x_{1i}, y_{1i}) \wedge X(x_2, y_2) \\ &\quad \rightarrow X_1(x_0, \bar{y}_1, y_2)) \end{aligned}$$

We can use this characterisation of insignificant choice to modify the logic in such a way that a choice rule will either become an insignificant choice or interpreted as **skip**. First we recall the axiomatic definition of $\text{upd}_r(X)$ from [19, Sect.7.2] (adapted to our simplifications of \mathcal{L}^{nd}).

U0. For $r = \mathbf{skip}$ we have

$$\text{upd}_r(X) \leftrightarrow \text{isUSet}(X) \wedge \forall x_1, \bar{x}_2, x_3. X(x_1, \bar{x}_2, x_3) = \mathbf{false}$$

U1. For an assignment r of the form $f(t_1, \dots, t_n) := t_0$ we have

$$\begin{aligned} \text{upd}_r(X) \leftrightarrow & \text{isUSet}(X) \wedge \forall x_1, \bar{x}_2, x_3. (X(x_1, \bar{x}_2, x_3) \leftrightarrow \\ & x_1 = c_f \wedge \bigwedge_{1 \leq i \leq n} x_{2i} = t_i \wedge x_3 = t_0) \end{aligned}$$

U2. For a branching rule r of the form **if** φ **then** r_1 **else** r_2 **endif** we have

$$\text{upd}_r(X) \leftrightarrow (\varphi \wedge \text{upd}_{r_1}(X)) \vee (\neg\varphi \wedge \text{upd}_{r_2}(X))$$

U3. For a parallel rule r of the form **forall** $v \in t$ **do** $r'(v)$ **enddo** we have

$$\begin{aligned} \text{upd}_r(X) \leftrightarrow & \text{isUSet}(X) \wedge \forall x_1, \bar{x}_2, x_3. (X(x_1, \bar{x}_2, x_3) \leftrightarrow \\ & \exists v. v \in t \wedge \exists Y. \text{upd}_{r'(v)}(Y) \wedge Y(x_1, \bar{x}_2, x_3)) \end{aligned}$$

U4. For a choice rule of the form **choose** $v \in t$ **do** $r'(v)$ **enddo** we have

$$\text{upd}_r(X) \leftrightarrow \exists v. v \in t \wedge \text{upd}_{r'(v)}(X)$$

In order to express insignificant choice introduce new formulae of the form $\text{upd}_r^{ic}(X)$. If r is not a choice rule, we simply keep the definitions above in **U0–U3** replacing upd by upd^{ic} on the right hand side of the equivalence.

U4'. For a choice rule r of the form **choose** $v \in t$ **do** $r'(v)$ **enddo** we define

$$\begin{aligned} \text{upd}_r^{ic}(X) \leftrightarrow & \exists v. v \in t \wedge \text{upd}_{r'(v)}^{ic}(X) \wedge \\ & \forall Y. (\exists x. x \in t \wedge \text{upd}_{r'(x)}^{ic}(Y)) \rightarrow \exists Z. (\text{iso}(Z) \\ & \wedge \forall x_0, \bar{x}_1, x_2, \bar{y}_1, y_2. (X(x_0, \bar{x}_1, x_2) \wedge \bigwedge_{1 \leq i \leq n} Z(x_{1i}, y_{1i}) \wedge Z(x_2, y_2) \\ & \rightarrow Y(x_0, \bar{y}_1, y_2)) \\ & \wedge \forall x_0, \bar{x}_1, x_2, \bar{y}_1, y_2. (Y(x_0, \bar{x}_1, x_2) \wedge \bigwedge_{1 \leq i \leq n} Z(x_{1i}, y_{1i}) \wedge Z(x_2, y_2) \\ & \rightarrow X(x_0, \bar{y}_1, y_2))) \end{aligned}$$

The axiom **U4'** redefines the semantics of choice rules in the desired way. This also addresses the concern expressed at the end of the previous section. Analogous to [19, Lemma 7.3] the axioms above for $\text{upd}_r^{ic}(X)$ show that every formulae in our logic can be written in an equivalent form not containing upd^{ic} at all.

4.3 Fixed-Point Definability

We now address a generalisation of the fixed point theorems for CPT (see [9, Thm.18&19]) to ICPT. Basically, we replace first-order logic by a fragment of \mathcal{L}^{nd} .

First, analogous to CPT we make our ASMs *time-explicit*, i.e. we replace the rule r by a new rule of the form

par r **if** $Halt = \text{false}$ **then** $ct := ct \cup \{ct\}$ **endif endpar**

That is, we use a counter ct taking non-negative integer values, which is initially set to 0 and incremented in every step of the machine. Then for a PTIME bounded ASM \tilde{M} the polynomial bounds have to be slightly adjusted, but still the time-explicit ASM will be PTIME bounded, and apart from the counter does exactly the same as the originally given ASM.

Now let $\tilde{M} = (M, p(n), q(n))$ be a PTIME icASM with time-explicit M . For an input structure I we use the notation $HF(I)$ for the hereditarily finite sets over its base set and $\text{State}(I)$ for the initial state generated by I . Recall that the base set of every state contains all hereditarily finite sets built over the atoms. Then let $\text{Active}(I)$ denote the set of active objects in a run starting in an initial state defined by I . Clearly, due to the definition of active objects $\text{Active}(I)$ is a transitive set. By abuse of notation let $\text{Active}(I)$ also denote the structure $(\text{Active}(I), \in, \emptyset, \bar{R})$, where \bar{R} stands for the relations in the input structure I .

Let S_0, S_1, \dots, S_ℓ be a run of M of length ℓ . For each $f \in \mathcal{Y}_{dyn}$ we introduce a new relation symbol D_f of arity $n+2$, where n is the arity of f with the intended semantics that $D_f(i, \bar{x}, y)$ should hold iff $S_i \models f(\bar{x}) = y \neq 0$. Analogous to [9, Thm.18] we obtain the following Theorem 4.1.

Before formulating this theorem and its proof, let us observe that without loss of generality we can write formulae of our logic in an equivalent *simple* way, where all atomic subformulae take the form $g(\bar{u}) = t$ with t being either a variable or a constant **true** or **false**.

Theorem 4.1. *The relations $D_f(i, \bar{x}, y)$ for $f \in \mathcal{Y}_{dyn}$ can be defined by an inflationary fixed-point formula $[\text{ifp}_{D_f} \phi(D_f, i, \bar{x}, y)]$ independent of the input structure I over the structure $\text{Active}(I)$.*

Proof. Let r be the rule of the ASM M . We define D_f by simultaneous recursion using

$$\begin{aligned} D_f(i, \bar{x}, y) &\leftrightarrow \text{PosInteger}(i) \wedge y \neq 0 \wedge \\ &(((D_f(i-1, \bar{x}, y) \wedge \neg \exists X.(U_r(i-1, X) \wedge \exists z.(z \neq y \wedge X(c_f, \bar{x}, z)))) \vee \\ &\quad \exists X.(U_r(i-1, X) \wedge X(c_f, \bar{x}, y))), \end{aligned}$$

where $U_r(j, X)$ is the formula that results from $\text{upd}_r^{ic}(X)$ by replacing every atomic subformula of the form $g(\bar{u}) = t$ by

$$D_g(j, \bar{u}, t) \vee (t = 0 \wedge \neg \exists z.D_g(j, \bar{u}, z)).$$

This defines the formula ϕ as desired. \square

The theorem actually shows that there is a sentence φ defined in our logic plus inflationary fixed-point which asserts that a (time-explicit) PTIME ic-ASM \tilde{M} accepts the input structure I , i.e. there exists some i such such $D_f(i, \mathbf{true})$ holds in $\text{Active}(I)$, where f is *Output* and *Halt*. Actually, we only require a fragment of our logic:

- predicates $\text{upd}_r(X)$ are replaced by $\text{upd}_r^{ic}(X)$;
- there is no occurrence of formulae of the form $[X]\varphi$;
- all second-order variables have either arity 2 and are bound to isomorphisms using $\text{iso}(X)$, or they have arity 3 and are bound to consistent update sets using $\text{conUSet}(X)$.

Furthermore, all universally quantified second-order variables are only needed to express the insignificant choice constraint. Let us use the notation \mathcal{L}^{ic} for this fragment of the logic \mathcal{L}^{nd} .

Furthermore, in analogy to [9, Thm.19] we can extend fixed-point definability to transitive structures T that subsume $\text{Active}(I)$. For this we just have to show fixed-point definability of $\text{Active}(I)$ over $\text{HF}(I)$.

Theorem 4.2. *Consider input structures I such that \tilde{M} halts on runs starting in an initial state defined by I . Then $\text{Active}(I)$ can be uniformly (i.e. independent of I) defined by an inflationary fixed-point formula over $\text{HF}(I)$.*

Proof. Consider the formula (where n_f is the arity of $f \in \mathcal{T}_{\text{dyn}}$)

$$\begin{aligned} \text{critical}(x) &\equiv x \in \text{Atoms} \vee x \in \{0, 1\} \vee \\ &\quad \bigvee_{f \in \mathcal{T}_{\text{dyn}}} \exists i, x_0, \dots, x_{n_f}. (D_f(i, x_0, \dots, x_{n_f}) \wedge \\ &\quad (x = x_0 \vee \dots \vee x = x_{n_f}) \wedge \forall j. (j \in i \rightarrow \neg D_{\text{Halt}}(j, \mathbf{true}))), \end{aligned}$$

which expresses that x is critical. Furthermore,

$$transitive(x) \equiv \forall y, z. (y \in x \wedge z \in y \rightarrow z \in x)$$

expresses that x is transitive. Hence, we can define $Active(I)$ by the formula

$$x \in Active(I) \leftrightarrow \forall y. (transitive(y) \wedge \forall z. (critical(z) \rightarrow z \in y) \rightarrow x \in y).$$

That is x is active if and only if it is a member of every transitive set containing all critical elements. If x is in each such transitive sets, then in particular $x \in TC(y)$ for some critical y , so x is active. Conversely, for active x there exists a critical y with $x \in TC(y)$ and further x must be an element of all transitive sets containing all critical elements, as these contain $TC(y)$ as subset.

As D_f has been defined by an inflationary fixed-point formula, so is $Active(I)$. \square

As a consequence, first-order variables in our logic are always bound to active objects.

Isomorphisms are defined as permutations of the set of atoms that are extended to sets using $\theta(x) = \{\theta(y) \mid y \in x\}$. Let $Sym(I)$ denote the permutation group of $State(I)$ and let $Aut(I)$ denote the automorphism group. Different to CPT there is more than one run with an initial state defined by I , and we have to take all these runs into account. For this we exploit *orbits* (y, G) comprising an object y and a subgroup G of $Sym(I)$ such that G permutes with each $\theta \in Aut(I)$.

Definition 4.3. A *critical orbit* of a PTIME ic-ASM $\tilde{M} = (M, p(n), q(n))$ for an input structure I is an orbit (y, G) such that y is a critical object in some state S_i in a run S_0, S_1, \dots of M on input structure I with $\Delta_r(S_i) = \{\sigma \Delta_1 \mid \sigma \in G\}$.

If (y, G) is a critical orbit, then every orbit (z, G) with $z \in TC(y)$ is called an *active orbit*.

By abuse of terminology we also refer to the set $G(y)$ as the orbit (y, G) . Note that an object may appear in different active orbits. Let $\mathcal{A}[I]$ denote the set of active orbits of \tilde{M} for the input structure I . With this we call a substructure J of $HF(I)$ *semi-transitive* iff for all orbits (y, G) with $Gy \subseteq J$ and any $z \in TC(y)$ we also have $Gz \subseteq J$ for the orbit (z, G) .

Theorem 4.3. *Consider input structures I such that \tilde{M} halts on runs starting in an initial state defined by I . Then $\mathcal{A}[I]$ can be uniformly defined by an inflationary fixed-point formula over $HF(I)$.*

Proof. Analogous to Theorem 4.2 define the formula

$$\begin{aligned}
critical-orbit(x) \equiv & critical(x) \vee \\
& ((\forall y \in x. critical(y) \wedge (\bigvee_{f \in \mathcal{T}_{dyn}} \exists i, x_0, \dots, x_{n_f}. PosInteger(i) \wedge \\
& (y = x_0 \vee \dots \vee y = x_{n_f}) \wedge \exists X. U_r(i-1, X) \wedge X(c_f, x_0, \dots, x_{n_f})) \rightarrow \\
& \forall Y. (U_r(i-1, Y) \rightarrow \exists \Phi. (\text{iso}(\Phi) \wedge \forall z. (\Phi(y, z) \rightarrow z \in x)))) \wedge \\
& (\forall y_1, y_2 \in x. (\bigvee_{f \in \mathcal{T}_{dyn}} \exists i. PosInteger(i) \wedge \exists X_1, X_2. U_r(i-1, X_1) \wedge \\
& U_r(i-1, X_2) \wedge \exists x_0, \dots, x_{n_f}, x'_0, \dots, x'_{n_f}. (y_1 = x_0 \vee \dots \vee y_1 = x_{n_f}) \wedge \\
& (y_2 = x'_0 \vee \dots \vee y_2 = x'_{n_f}) \rightarrow \exists \Phi. (\text{iso}(\Phi) \wedge \Phi(y_1, y_2))))))
\end{aligned}$$

Then we can define

$$x' \in TC(z) \leftrightarrow \forall z'. (\text{transitive}(z') \wedge \forall u. (u \in z \rightarrow u \in z') \rightarrow x' \in z') ,$$

and with this we finally get

$$active-orbit(x) \equiv \exists y, z. critical-orbit(y) \wedge z \in y \wedge \forall x' \in x. x' \in TC(z) . \quad \square$$

Then the following corollary summarises the discussion in this subsection.

Corollary 4.1. *Let I be an input structure for a (time-explicit) PTIME ic-ASM \tilde{M} . There exists a formula φ in the fixed-point logic $IFP[\mathcal{L}^{ic}]$ that asserts that \tilde{M} accepts I , which is the case iff φ holds in some or equivalently all semi-transitive substructures of $HF(I)$ containing all objects of all active orbits in $\mathcal{A}[I]$.*

Proof. By the first fixed-point Theorem 4.1 there exists an $IFP[\mathcal{L}^{ic}]$ sentence φ that asserts that \tilde{M} accepts I . The sentence asserts that there is some positive integer i such that $D_f(i, \mathbf{true})$ holds in $\mathcal{A}[I]_2$ where f is *Output* or *Halt*. Using the second fixed-point Theorem 4.3 \tilde{M} accepts I iff φ is true in some or equivalently all semi-transitive substructures of $HF(I)$ containing all objects of active orbits in $\mathcal{A}[I]$. \square

4.4 Finite Variable Infinitary Logic

In the remainder of this section we investigate the expressiveness of the fixed-point logic $\text{IFP}[\mathcal{L}^{ic}]$. Following [29, Ch.11] we embed the logic into an infinitary extension of \mathcal{L}^{ic} , then in the next subsection we develop a pebble game by means of which we can characterise structures that cannot be distinguished by formulae of the logic. We will use this later for the investigation of the limitations of ICPT.

As usual let $\mathcal{L}_{\infty\omega}^{ic}$ be the logic that extends \mathcal{L}^{ic} by infinitary conjunctions \bigwedge and disjunctions \bigvee . Let $\mathcal{L}_{\infty\omega}^{ic,m}$ denote the fragment of $\mathcal{L}_{\infty\omega}^{ic}$ with formulae that contain at most m variables, and let $\mathcal{L}_{\infty\omega}^{ic,\omega}$ be the fragment of formulae that contain only finitely many variables, i.e.

$$\mathcal{L}_{\infty\omega}^{ic,\omega} = \bigcup_{m \in \mathbb{N}} \mathcal{L}_{\infty\omega}^{ic,m}.$$

Now let $[\text{ifp}_R \varphi(R, \bar{X}, \bar{x})](\bar{X}, \bar{x})$ be a formula in $\text{IFP}[\mathcal{L}^{ic}]$. Assume that φ in addition to the second-order variables \bar{X} and the first-order variables $\bar{x} = (x_1, \dots, x_m)$ uses also first-order variables $\bar{z} = (z_1, \dots, z_\ell)$. We use additional first-order variables $\bar{y} = (y_1, \dots, y_m)$ to inductively define formulae $\varphi^\beta(\bar{X}, \bar{x})$ for each ordinal β . For finite structures it suffices to consider $\beta \in \mathbb{N}$. For $\beta = 0$ let $\varphi^0(\bar{X}, \bar{x}) \equiv \neg(x_1 = x_1)$, which is equivalent to **false**. Then define $\varphi^{\beta+1}(\bar{X}, \bar{x})$ replacing in $\varphi(R, \bar{X}, \bar{x})$ each occurrence $R(u_1, \dots, u_m)$, where u_1, \dots, u_m are variables among \bar{x} and \bar{z} by

$$\exists \bar{y}. \bar{y} = \bar{u} \wedge \exists \bar{x}. (\bar{x} = \bar{y} \wedge \varphi^\beta(\bar{X}, \bar{x})).$$

For a limit ordinal γ we simply have $\varphi^\gamma(\bar{X}, \bar{x}) \equiv \bigvee_{\beta < \gamma} \varphi^\beta(\bar{X}, \bar{x})$.

Then for each ordinal β the formula $\varphi^\beta(\bar{X}, \bar{x})$ represents R in the β 'th stage of the fixed-point iteration. Hence $[\text{ifp}_R \varphi(R, \bar{X}, \bar{x})](\bar{X}, \bar{x})$ is equivalent to $\bigvee_\beta \varphi^\beta(\bar{X}, \bar{x})$. Note that at most the number of first-order variables is doubled, whereas the number of second-order variables remains unchanged.

For complex nested fixed-point formulae we proceed by induction. We can use the same replacement as above to define $\varphi^{\beta+1}(\bar{X}, \bar{x})$; the only difference is that $\varphi^\beta(\bar{X}, \bar{x})$ is already a formula that includes infinitary connectives. The same holds for limit ordinals.

With this embedding of $\text{IFP}[\mathcal{L}^{ic}]$ into $\mathcal{L}_{\infty\omega}^{ic,\omega}$ we can rephrase Corollary 4.1 as follows:

Theorem 4.4. *Let I be an input structure for a (time-explicit) PTIME ic-ASM \tilde{M} . There exists a formula φ in $\mathcal{L}_{\infty\omega}^{ic,m}$ for some $m \in \mathbb{N}$ that asserts*

that \tilde{M} accepts I , which is the case iff φ holds in some or equivalently all semi-transitive substructures of $HF(I)$ containing all objects of all active orbits in $\mathcal{A}[I]$.

For later use we recall the notion of *quantifier rank* of a formula. For a formula φ of $\mathcal{L}_{\infty\omega}^{ic,m}$ we define $qr(\varphi)$ as follows:

- for an atomic formula φ we have $qr(\varphi) = 0$;
- for Boolean connectives we have $qr(\neg\varphi) = qr(\varphi)$ and $qr(\varphi \wedge \psi) = qr(\varphi \vee \psi) = qr(\varphi \rightarrow \psi) = \max\{qr(\varphi), qr(\psi)\}$;
- for infinitary connectives we have $qr(\bigwedge_{i \in I} \varphi_i) = qr(\bigvee_{i \in I} \varphi_i) = \sup_{i \in I} qr(\varphi_i)$;
- for quantified formulae we have $qr(\exists x.\varphi) = qr(\forall x.\varphi) = qr(\exists X.\varphi) = qr(\forall X.\varphi) = qr(\varphi) + 1$.

4.5 IC Pebble Games

Infinitary logics with finitely many variables can be characterised by pebble games (see [29, Sect.11.2]). We will now define an analogous pebble game for $\mathcal{L}_{\infty\omega}^{ic,m}$. We concentrate on structures, where the signature \mathcal{Y} contains a constant \emptyset and set membership \in in addition to finitely many relation symbols. Denote such structures as $\mathcal{A} = (T, \in, \emptyset, \bar{R})$, where \bar{R} stands for the finite set of relations over T . Assume that T contains a finite set of special constants c_f corresponding to dynamic functions symbols f , each with an arity n . The presence of such constants allows us to talk about *update sets* in \mathcal{A} , i.e. sets of updates in the form of triples (c_f, \bar{a}, b) , where the length of the tuple \bar{a} equals the arity of f .

Definition 4.4. Let \mathcal{A} and \mathcal{B} be structures over \mathcal{Y} . Let $\bar{\Delta} = (\Delta_1, \dots, \Delta_k)$ and $\bar{\Delta}' = (\Delta'_1, \dots, \Delta'_k)$ be finite sequences of update sets of equal length over \mathcal{A} and \mathcal{B} , respectively, and let $\bar{a} = (a_1, \dots, a_\ell)$ and $\bar{b} = (b_1, \dots, b_\ell)$ be finite, equally long sequences of elements of \mathcal{A} and \mathcal{B} , respectively, not including the special constants c_f . Then $(\bar{\Delta}, \bar{a})$ and $(\bar{\Delta}', \bar{b})$ define a *partial isomorphism* between structures \mathcal{A} and \mathcal{B} iff the following conditions hold:

- (i) for all $i, j \leq k$ we have $\Delta_i = \Delta_j$ iff $\Delta'_i = \Delta'_j$, and $\Delta_i \subseteq \Delta_j$ iff $\Delta'_i \subseteq \Delta'_j$;
- (ii) for all $i, j \leq \ell$ we have $a_i = a_j$ iff $b_i = b_j$, $a_i \in a_j$ iff $b_i \in b_j$, and $a_i \subseteq a_j$ iff $b_i \subseteq b_j$;
- (iii) for $f \in \mathcal{Y}_{dyn}$, $i \leq k$, $i_0 \leq \ell$ and all sequences (i_1, \dots, i_n) of indices $i_j \leq \ell$ we have $(c_f, (a_{i_1}, \dots, a_{i_n}), a_{i_0}) \in \Delta_i$ iff $(c_f, (b_{i_1}, \dots, b_{i_n}), b_{i_0}) \in \Delta'_i$;
- (iv) for all sequences (i_1, \dots, i_n) of indices $i_j \leq \ell$ and all relation symbols $R \in \mathcal{Y}$ of arity n we have $R^{\mathcal{A}}(a_{i_1}, \dots, a_{i_n})$ iff $R^{\mathcal{B}}(b_{i_1}, \dots, b_{i_n})$.

The partial isomorphism is called *ic-compatible* iff for every automorphism σ of \mathcal{A} and every automorphism σ' of \mathcal{B} whenever all Δ_i and a_j are replaced by $\sigma\Delta_i$ and $\sigma(a_j)$, respectively, in $\bar{\Delta}$ and \bar{a} and likewise Δ'_i and b_j by $\sigma'\Delta'_i$ and $\sigma'(b_j)$, respectively, in $\bar{\Delta}'$ and \bar{b} we obtain another partial isomorphism between \mathcal{A} and \mathcal{B} .

Then an *IC pebble game* over structures \mathcal{A} and \mathcal{B} is played by two players, called *spoiler* and *duplicator* both having m pebbles marked by numbers $1, \dots, m$. In every round first the spoiler makes a move, which is either an *(update) set move* or an *object move*, and the duplicator reacts with a set move or an object move, respectively:

- (update) set move.** The spoiler chooses a structure \mathcal{A} or \mathcal{B} and places one of his pebbles (regardless, if used or not) on an update set over the chosen structure. The duplicator reacts by placing his pebble with the same number on an update set of the other structure.
- object move.** The spoiler chooses a structure \mathcal{A} or \mathcal{B} and places one of his pebbles (regardless, if used or not) on an object of the chosen structure. The duplicator reacts by placing his pebble with the same number on an object of the other structure.

After each round, if the k update sets and ℓ objects marked by pebbles (so $k + \ell \leq m$) define a partial isomorphism between \mathcal{A} and \mathcal{B} that is ic-compatible, the game continues with another round. If there is no such isomorphism, the game is terminated and the spoiler has won. If the game continues forever, the duplicator wins.

Thus, we say that the duplicator has a *winning strategy* for an IC pebble game iff he can ensure that after each move of the spoiler he is able to react with a move that defines a partial ic-compatible isomorphism between the two structures. If this is the case, we write $\mathcal{A} \equiv_{\infty\omega}^{ic,m} \mathcal{B}$.

Note that for our purposes here we included the case of infinite structures⁵ \mathcal{A} and \mathcal{B} . This is, because Theorem 4.4 shows that a formula expressing acceptance of an input structure I is to hold in semi-transitive substructures of $HF(I)$ containing all active orbits in $\mathcal{A}[I]$, and these structures are not necessarily finite.

We can now approach the main result of this subsection. Let A and B denote the base sets of the structures \mathcal{A} and \mathcal{B} , respectively. We have to consider partial maps $f : A \rightarrow B$ on objects as well as partial maps

$$g : \mathcal{P}(\{c_f \mid f \in \mathcal{I}_{dyn}\} \times \bigcup_{n \in \mathbb{N}} A^n \times A) \rightarrow \mathcal{P}(\{c_f \mid f \in \mathcal{I}_{dyn}\} \times \bigcup_{n \in \mathbb{N}} B^n \times B)$$

⁵ A similar pebble game has been exploited for CPT in [9], where also infinite structures had to be taken into account.

on update sets, and their union $h = f \oplus g$. Let $\text{dom}(h)$ and $\text{rng}(h)$ denote the domain and the range of such partial functions, respectively.

As the IC pebble games may require infinitely many rounds, we consider ordinals denoted as α, β, \dots . For an ordinal β let \mathcal{I}_β denote a set of partial isomorphisms between \mathcal{A} and \mathcal{B} that are ic-compatible. Then let $\mathcal{J}_\alpha = \{\mathcal{I}_\beta \mid \beta < \alpha\}$.

Definition 4.5. The set \mathcal{J}_α of sets of partial isomorphisms has the *m-back-and-forth property* iff the following are satisfied:

- (i) Every set \mathcal{I}_β with $\beta < \alpha$ is non-empty.
- (ii) $\mathcal{I}_\gamma \subseteq \mathcal{I}_\beta$ holds for $\beta < \gamma$.
- (iii) Each set \mathcal{I}_β is downward-closed, i.e. for $h \in \mathcal{I}_\beta$ and $h' \subseteq h$ (i.e. $\text{dom}(h') \subseteq \text{dom}(h)$ and h, h' coincide on $\text{dom}(h')$) also $h' \in \mathcal{I}_\beta$.
- (iv) For every $h = f \oplus g \in \mathcal{I}_{\beta+1}$ with $|\text{dom}(h)| < m$ the following two conditions hold:

forth: for every object $a \in A$ and every update set Δ over A , there is a $h' \in \mathcal{I}_\beta$ with $h \subseteq h'$ and $a \in \text{dom}(h')$ (then $h' = f' \oplus g$) or $\Delta \in \text{dom}(h')$ (then $h' = f \oplus g'$), respectively;

back: for every object $b \in B$ and every update set Δ' over B , there is a $h' \in \mathcal{I}_\beta$ with $h \subseteq h'$ and $b \in \text{rng}(h')$ (then $h' = f' \oplus g$) or $\Delta' \in \text{rng}(h')$ (then $h' = f \oplus g'$), respectively.

In the Ehrenfeucht-Fraïssé method games are just a reformulation of the back-and-forth property and vice versa. Indeed, the partial, ic-compatible isomorphisms in \mathcal{I}_β represent configurations of the game, when there are β rounds left to play. For this properties (i)-(iii) in Definition 4.5 are obvious, and property (iv) expresses that whatever move the spoiler may make, the duplicator finds a suitable reaction that establishes a partial, ic-compatible isomorphism.

Theorem 4.5. Two structures \mathcal{A} and \mathcal{B} over \mathcal{U} are $\mathcal{L}_{\infty\omega}^{ic,m}$ -equivalent for $m \in \mathbb{N}$ iff $\mathcal{A} \equiv_{\infty\omega}^{ic,m} \mathcal{B}$ holds.

Proof. Due to our considerations above it suffices to show that \mathcal{A} and \mathcal{B} agree on all sentences of $\mathcal{L}_{\infty\omega}^{ic,m}$ of quantifier rank $< \alpha$ iff there is a family $\mathcal{J}_\alpha = \{\mathcal{I}_\beta \mid \beta < \alpha\}$ of partial isomorphisms between \mathcal{A} and \mathcal{B} that are ic-compatible with the *m-back-and-forth property*.

Part I. First assume that \mathcal{A} and \mathcal{B} agree on all sentences of $\mathcal{L}_{\infty\omega}^{ic,m}$ of quantifier rank $< \alpha$. For $\beta < \alpha$ we define \mathcal{I}_β as the set of partial isomorphisms h between \mathcal{A} and \mathcal{B} that are ic-compatible with $|\text{dom}(h)| < m$,

such that for every formula $\varphi(\bar{X}, \bar{x})$ of $\mathcal{L}_{\infty\omega}^{ic,m}$ with $qr(\varphi) \leq \beta$ and every $(\bar{\Delta}, \bar{a})$ contained in $dom(h)$ (i.e. $\bar{\Delta} \subseteq dom(f)$ and $\bar{a} \subseteq dom(g)$ for $h = f \oplus g$) we have

$$\mathcal{A} \models \varphi(\bar{\Delta}, \bar{a}) \quad \Leftrightarrow \quad \mathcal{B} \models \varphi(f(\bar{\Delta}), g(\bar{a})).$$

We show that $\mathcal{I}_\alpha = \{\mathcal{I}_\beta \mid \beta < \alpha\}$ has the m -back-and-forth property.

The empty partial isomorphism is in \mathcal{I}_β , which gives property (i) of Definition 4.5. The containment property (ii) and the downward closure property (iii) are immediate consequences of the definition of \mathcal{I}_β . Therefore, it remains to show property (iv).

For this first assume that there exists some $h = f \oplus g \in \mathcal{I}_{\beta+1}$ with $\beta + 1 < \alpha$ and $|dom(h)| = \ell < m$ violating the **forth** condition. Consider two cases:

Case a. There is an object $a \in A$ but no object $b \in B$ such that h can be extended to $h' \in \mathcal{I}_\beta$ with $h'(a) = b$. By the definition of \mathcal{I}_β for every object $b \in B$ we find a formula $\varphi_b(\bar{X}, x_0, \bar{x})$ of $\mathcal{L}_{\infty\omega}^{ic,m}$ with $qr(\varphi_b) \leq \beta$ such that for some $\bar{\Delta}$ and \bar{a} contained in $dom(h)$ we have

$$\mathcal{A} \models \varphi_b(\bar{\Delta}, a, \bar{a}) \quad \text{and} \quad \mathcal{B} \models \neg \varphi_b(f(\bar{\Delta}), b, g(\bar{a})).$$

Define the formula

$$\varphi(\bar{X}, \bar{x}) \equiv \exists x_0. \bigwedge_{b \in B} \varphi_b((\bar{X}, x_0, \bar{x})).$$

We have $qr(\varphi) \leq \beta + 1$ and $\mathcal{A} \models \varphi(\bar{\Delta}, \bar{a})$, but $\mathcal{B} \models \neg \varphi(f(\bar{\Delta}), g(\bar{a}))$, which contradicts $h = f \oplus g \in \mathcal{I}_{\beta+1}$.

Case b. There is an update set Δ_0 over A but no update set Δ'_0 over B such that h can be extended to $h' \in \mathcal{I}_\beta$ with $h'(\Delta_0) = \Delta'_0$. In this case the definition of \mathcal{I}_β permits two possibilities:

- As in Case a for every update set Δ'_0 over B we find a formula $\varphi_{\Delta'_0}(X_0, \bar{X}, \bar{x})$ of $\mathcal{L}_{\infty\omega}^{ic,m}$ with $qr(\varphi_{\Delta'_0}) \leq \beta$ such that for some $\bar{\Delta}$ and \bar{a} contained in $dom(h)$ we have

$$\mathcal{A} \models \varphi_{\Delta'_0}(\Delta_0, \bar{\Delta}, \bar{a}) \quad \text{and} \quad \mathcal{B} \models \neg \varphi_{\Delta'_0}(\Delta'_0, f(\bar{\Delta}), g(\bar{a})).$$

Then define the formula

$$\phi(\bar{X}, \bar{x}) \equiv \exists X_0. \bigwedge_{\Delta'_0 \text{ update set over } B} \varphi_{\Delta'_0}(X_0, (\bar{X}, \bar{x})).$$

We have $qr(\phi) \leq \beta + 1$ and $\mathcal{A} \models \phi(\bar{\Delta}, \bar{a})$, but $\mathcal{B} \models \neg \phi(f(\bar{\Delta}), g(\bar{a}))$, which contradicts $h = f \oplus g \in \mathcal{I}_{\beta+1}$.

- Alternatively, for all update sets Δ'_0 over B we cannot find such a formula $\varphi_{\Delta'_0}(X_0, \bar{X}, \bar{x})$, but the extension cannot be ic-compatible. Then for every update set Δ'_0 over B we find a formula

$$\begin{aligned} \phi_{\Delta'_0}(X_0, \dots, X_k, x_1, \dots, x_\ell) \equiv \\ \forall X. \text{iso}(X) \rightarrow \forall Y_1, \dots, Y_k. \forall y_1, \dots, y_\ell. \text{map}(X, X_0, Y_0) \wedge \dots \\ \wedge \text{map}(X, X_k, Y_k) \wedge X(x_1, y_1) \wedge \dots \wedge X(x_\ell, y_\ell) \rightarrow \\ \varphi_{\Delta'_0}(Y_0, \dots, Y_k, y_1, \dots, y_\ell), \end{aligned}$$

where we use $\text{map}(X, X', Y')$ as a shortcut for

$$\begin{aligned} \forall x_0, \bar{x}_1, x_2. X'(x_0, \bar{x}_1, x_2) \rightarrow \\ \exists \bar{y}_1, y_2. \bigwedge_{1 \leq i \leq n_{x_0}} X(x_{1i}, y_{1i}) \wedge X(x_2, y_2) \wedge Y'(x_0, \bar{y}_1, y_2) \\ \wedge \forall x_0, \bar{y}_1, y_2. Y'(x_0, \bar{y}_1, y_2) \rightarrow \\ \exists \bar{x}_1, x_2. \bigwedge_{1 \leq i \leq n_{x_0}} X(x_{1i}, y_{1i}) \wedge X(x_2, y_2) \wedge X'(x_0, \bar{x}_1, x_2), \end{aligned}$$

such that

$$\begin{aligned} \mathcal{A} \models \phi_{\Delta'_0}(\Delta_0, \Delta_1, \dots, \Delta_k, a_1, \dots, a_\ell) \text{ and} \\ \mathcal{B} \models \neg \phi_{\Delta'_0}(\Delta'_0, f(\Delta_1), \dots, f(\Delta_k), g(a_1), \dots, g(a_\ell)) \end{aligned}$$

hold. Then we define

$$\begin{aligned} \phi(X_1, \dots, X_k, x_1, \dots, x_\ell) \equiv \\ \exists X_0. \bigwedge_{\Delta'_0 \text{ update set over } B} \phi_{\Delta'_0}(X_0, X_1, \dots, X_k, x_1, \dots, x_\ell), \end{aligned}$$

which gives us

$$\begin{aligned} \mathcal{A} \models \phi(\Delta_1, \dots, \Delta_k, a_1, \dots, a_\ell) \text{ and} \\ \mathcal{B} \models \neg \phi(f(\Delta_1), \dots, f(\Delta_k), g(a_1), \dots, g(a_\ell)). \end{aligned}$$

As we have $qr(\phi) \leq \beta + 1$, this contradicts again $h = f \oplus g \in \mathcal{I}_{\beta+1}$.

Next assume that there exists some $h = f \oplus g \in \mathcal{I}_{\beta+1}$ with $|dom(h)| < m$ violating the **back** condition. We proceed analogously to the case of the **forth** condition. Again we have two cases:

Case a. There is an object $b \in B$ but no object $a \in A$ such that h can be extended to $h' \in \mathcal{I}_\beta$ with $h'(a) = b$. By the definition of \mathcal{I}_β for every object $a \in A$ we find a formula $\varphi_a(\bar{X}, x_0, \bar{x})$ of $\mathcal{L}_{\infty\omega}^{ic,m}$ with $qr(\varphi_a) \leq \beta$ such that for some $\bar{\Delta}$ and \bar{a} contained in $dom(h)$ we have

$$\mathcal{A} \models \neg \varphi_a(\bar{\Delta}, a, \bar{a}) \quad \text{and} \quad \mathcal{B} \models \varphi_a(f(\bar{\Delta}), b, g(\bar{a})).$$

Define the formula

$$\varphi(\bar{X}, \bar{x}) \equiv \exists x_0. \bigwedge_{a \in A} \varphi_a((\bar{X}, x_0, \bar{x}).$$

We have $qr(\varphi) \leq \beta + 1$ and $\mathcal{A} \models \neg \varphi(\bar{\Delta}, \bar{a})$, but $\mathcal{B} \models \varphi(f(\bar{\Delta}), g(\bar{a}))$, which contradicts $h = f \oplus g \in \mathcal{I}_{\beta+1}$.

Case b. There is an update set Δ'_0 over B but no update set Δ_0 over A such that h can be extended to $h' \in \mathcal{I}_\beta$ with $h'(\Delta_0) = \Delta'_0$. In this case the definition of \mathcal{I}_β permits two possibilities as in Case b of the **forth** case. The construction of a formula $\phi(X_1, \dots, X_k, x_1, \dots, x_\ell)$ with $qr(\phi) \leq \beta + 1$ such that

$$\begin{aligned} \mathcal{A} &\models \neg \phi(\Delta_1, \dots, \Delta_k, a_1, \dots, a_\ell) \text{ and} \\ \mathcal{B} &\models \phi(f(\Delta_1), \dots, f(\Delta_k), g(a_1), \dots, g(a_\ell)) \end{aligned}$$

hold is done in exactly the same way as for the **forth** case, which gives a contradiction to $h = f \oplus g \in \mathcal{I}_{\beta+1}$.

Part II. Now assume that there is a family $\mathcal{J}_\alpha = \{\mathcal{I}_\beta \mid \beta < \alpha\}$ of partial isomorphisms between \mathcal{A} and \mathcal{B} that are ic-compatible with the m -back-and-forth property. We have to show that \mathcal{A} and \mathcal{B} agree on all sentences of $\mathcal{L}_{\infty\omega}^{ic,m}$ of quantifier rank $< \alpha$. We use transfinite induction over β to show that for every formula $\varphi(\bar{X}, \bar{x})$ of $\mathcal{L}_{\infty\omega}^{ic,m}$ with $qr(\varphi) \leq \beta < \alpha$, every $h = f \oplus g \in \mathcal{I}_\beta$ and all sequences $\bar{\Delta}$ and \bar{a} contained in $dom(h)$ we have

$$\mathcal{A} \models \varphi(\bar{\Delta}, \bar{a}) \quad \Leftrightarrow \quad \mathcal{B} \models \varphi(f(\bar{\Delta}), g(\bar{a})). \quad (*)$$

For the induction base let $\beta = 0$. Then φ is a Boolean or infinitary combination of atomic formulae. We may exclude atomic formulae of the form $\text{upd}_r^{ic}(X)$, as these represent complex second-order formula (see the axioms **U1-U4** and **U4'** in Section 4.2) with quantifier rank > 0 . For a Boolean combination (*) follows immediately from h being a partial, ic-compatible isomorphism between \mathcal{A} and \mathcal{B} . For a formula $\varphi \equiv \bigvee_{i \in I} \varphi_i$ we must have $qr(\varphi_i) = 0$ for all $i \in I$. As up to equivalence there are only

finitely many different \mathcal{L}^{ic} formulae with quantifier rank 0, φ is equivalent to $\bigvee_{i \in I'} \varphi_i$ for some finite subset $I' \subseteq I$, i.e. φ is equivalent to a Boolean combination of atomic formulae.

For the induction step let $\beta > 0$ and assume that $(*)$ holds for all $\gamma < \beta$. We proceed by structural induction over the formulae φ . The cases of Boolean combinations follows immediately from the induction hypothesis. Consider an infinitary combination $\varphi \equiv \bigvee_{i \in I} \varphi_i$. Then we can apply the induction hypothesis, which assures $(*)$ for all φ_i , hence also for φ .

For quantified formulae it suffices to consider the cases $\varphi(\bar{X}, \bar{x}) \equiv \exists x_0. \psi(\bar{X}, x_0, \bar{x})$ and $\varphi(\bar{X}, \bar{x}) \equiv \exists X_0. \psi(X_0, \bar{X}, \bar{x})$.

(1.) First consider the case of first-order quantification. Let $\varphi(\bar{X}, \bar{x}) \equiv \exists x_0. \psi(\bar{X}, x_0, \bar{x})$ with $qr(\varphi) = \beta + 1 < \alpha$ and $qr(\psi) = \beta$. Without loss of generality we can assume that x_0 does not occur in \bar{x} and thus $k + \ell < m$, where k is the length of \bar{X} and ℓ the length of \bar{x} .

Let $h \in \mathcal{I}_{\beta+1}$, and take a sequence $\bar{\Delta} = \Delta_1, \dots, \Delta_k$ of update sets over \mathcal{A} as well as a sequence $\bar{a} = (a_1, \dots, a_\ell)$ of objects of \mathcal{A} . As $\mathcal{I}_{\beta+1}$ is downward-closed, we can assume $dom(h) = \{\Delta_1, \dots, \Delta_k, a_1, \dots, a_\ell\}$.

Further assume that $\mathcal{A} \models \varphi(\bar{\Delta}, \bar{a})$ holds, so we get $\mathcal{A} \models \psi(\bar{\Delta}, a_0, \bar{a})$ for some object $a_0 \in A$. Using the **forth** property for h we find an extension $h' \in \mathcal{I}_\beta$ with $a_0 \in dom(h')$. Applying the induction hypothesis for ψ we derive $\mathcal{B} \models \psi(f(\bar{\Delta}), h'(a_0), g(\bar{a}))$ and further $\mathcal{B} \models \varphi(f(\bar{\Delta}), g(\bar{a}))$.

Conversely, assume that $\mathcal{B} \models \varphi(f(\bar{\Delta}), g(\bar{a}))$ holds, so we get $\mathcal{B} \models \psi(f(\bar{\Delta}), b_0, g(\bar{a}))$ for some object $b_0 \in B$. Using the **back** property for h we find an extension $h' \in \mathcal{I}_\beta$ with $b_0 \in rng(h')$, say $b_0 = h'(a_0)$. Applying the induction hypothesis for ψ we derive $\mathcal{A} \models \psi(\bar{\Delta}, a_0, \bar{a})$ and further $\mathcal{A} \models \varphi(\bar{\Delta}, \bar{a})$.

(2.) Next let $\varphi(\bar{X}, \bar{x}) \equiv \exists X_0. \psi(X_0, \bar{X}, \bar{x})$ with $qr(\varphi) = \beta + 1 < \alpha$ and $qr(\psi) = \beta$. Without loss of generality we can assume that X_0 does not occur in \bar{X} and thus $k + \ell < m$. We can further assume that X has arity 3 and ranges over update sets, as isomorphisms only appear within the scope of a quantification over update sets (see the remark following the proof of Theorem 4.1).

Let $h \in \mathcal{I}_{\beta+1}$, and take a sequence $\bar{\Delta} = \Delta_1, \dots, \Delta_k$ of update sets over \mathcal{A} as well as a sequence $\bar{a} = (a_1, \dots, a_\ell)$ of objects of \mathcal{A} . As $\mathcal{I}_{\beta+1}$ is downward-closed, we can assume $dom(h) = \{\Delta_1, \dots, \Delta_k, a_1, \dots, a_\ell\}$.

First assume that $\mathcal{A} \models \varphi(\bar{\Delta}, \bar{a})$ holds, so we get $\mathcal{A} \models \psi(\Delta_0, \bar{\Delta}, \bar{a})$ for some update set Δ_0 over \mathcal{A} . Using the **forth** property for h we find an extension $h' \in \mathcal{I}_\beta$ with $\Delta_0 \in dom(h')$. Applying the induction hypothesis for ψ we derive $\mathcal{B} \models \psi(h'(\Delta_0), f(\bar{\Delta}), g(\bar{a}))$ and further $\mathcal{B} \models \varphi(f(\bar{\Delta}), g(\bar{a}))$.

The other direction that $\mathcal{B} \models \varphi(f(\bar{\Delta}), g(\bar{a}))$ implies $\mathcal{A} \models \varphi(\bar{\Delta}, \bar{a})$ is completely analogous using the **back** property.

This leaves the case, where $\varphi \equiv \text{upd}_r^{ic}(X)$ holds, as this represents a complex second-order formula (see the axioms **U1-U4** and **U4'** in Section 4.2). For these we proceed by structural induction over r . The only non-trivial case arises for a choice rule r of the form **choose** $v \in t$ **do** $r'(v)$ **enddo**; for all other cases an immediate inductive argument applies.

Let $h \in \mathcal{I}_{\beta+1}$ with $\text{dom}(h) = \{\Delta\}$ and assume that $\mathcal{A} \models \text{upd}_r(\Delta)$. According to axiom **U4'** we have

$$\begin{aligned} \text{upd}_r^{ic}(X) &\leftrightarrow \exists v. v \in t \wedge \text{upd}_{r'(v)}^{ic}(X) \wedge \\ &\quad \forall Y. (\exists x. x \in t \wedge \text{upd}_{r'(x)}^{ic}(Y)) \rightarrow \exists Z. (\text{iso}(Z) \wedge \text{map}(Z, X, Y)) \end{aligned}$$

Consider the subformulae

$$\begin{aligned} \psi(X, v) &\equiv v \in t \wedge \text{upd}_{r'(v)}^{ic}(X) \wedge \\ &\quad \forall Y. (\exists x. x \in t \wedge \text{upd}_{r'(x)}^{ic}(Y)) \rightarrow \exists Z. (\text{iso}(Z) \wedge \text{map}(Z, X, Y)) \end{aligned}$$

and

$$\chi(X) \equiv \forall Y. (\exists x. x \in t \wedge \text{upd}_{r'(x)}^{ic}(Y)) \rightarrow \exists Z. (\text{iso}(Z) \wedge \text{map}(Z, X, Y)).$$

Then we have $\mathcal{A} \models \psi(\Delta, a)$ for some object $a \in A$, so we have $\mathcal{A} \models a \in t$, $\mathcal{A} \models \text{upd}_{r'(a)}^{ic}(\Delta)$ and $\mathcal{A} \models \chi(\Delta)$. We apply the **forth** property to extend h to $h_1 \in \mathcal{I}_\beta$ with $h_1(a) = b$. Then by induction we obtain $\mathcal{B} \models b \in t$ and $\mathcal{B} \models \text{upd}_{r'(b)}^{ic}(h(\Delta))$. We have to show that $\mathcal{B} \models \chi(h(\Delta))$ holds.

For this take an arbitrary update set Δ' over \mathcal{B} and assume that $\mathcal{B} \models \exists x. x \in t \wedge \text{upd}_{r'(x)}^{ic}(\Delta')$ holds. Then we have $\mathcal{B} \models b' \in t \wedge \text{upd}_{r'(b')}^{ic}(\Delta')$ for some object $b' \in B$.

If neither β nor $\beta - 1$ is a limit ordinal, i.e. $\beta = \gamma + 2$, we can apply the **back** property twice (for b' and Δ') to extend h_1 to $h_2 \in \mathcal{I}_\gamma$ with $h_2(\Delta'') = \Delta'$ and $h_2(a') = b'$ for some update set Δ'' over \mathcal{A} and some object $a' \in A$. Then by induction we get $\mathcal{A} \models a' \in t \wedge \text{upd}_{r'(a')}^{ic}(\Delta'')$ and further $\mathcal{A} \models \exists x. x \in t \wedge \text{upd}_{r'(x)}^{ic}(\Delta'')$. As $\mathcal{A} \models \chi(\Delta)$ holds, we derive $\mathcal{A} \models \exists Z. (\text{iso}(Z) \wedge \text{map}(Z, \Delta, \Delta''))$. By induction we then get $\mathcal{B} \models \exists Z. (\text{iso}(Z) \wedge \text{map}(Z, h(\Delta), h_2(\Delta'')))$, which implies the desired $\mathcal{B} \models \chi(h(\Delta))$.

Now assume that β is a limit ordinal. We have (for the partial isomorphism h_1)

$$\mathcal{A} \models a \in t \wedge \text{upd}_{r'(a)}^{ic}(\Delta) \quad \leftrightarrow \quad \mathcal{B} \models b \in t \wedge \text{upd}_{r'(b)}^{ic}(h(\Delta)).$$

Furthermore, we assumed $\mathcal{B} \models b' \in t \wedge \text{upd}_{r'(b')}^{ic}(\Delta')$. Take an automorphism σ of \mathcal{A} with $\sigma(a) = a'$ and $\sigma(\Delta) = \Delta'$, and an automorphism τ of \mathcal{B} with $\tau(b) = b'$ and $\tau(h(\Delta)) = \Delta'$. As h_1 is ic-compatible, we derive $\mathcal{A} \models a' \in t \wedge \text{upd}_{r'(a')}^{ic}(\Delta'')$ as in the case, where β is not a limit ordinal. Applying the same arguments gives again $\mathcal{B} \models \chi(h(\Delta))$.

The proof of the other direction that $\mathcal{B} \models \text{upd}_r(h(\Delta))$ implies $\mathcal{A} \models \text{upd}_r(\Delta)$ is completely analogous. This completes the proof of the theorem. \square

Theorem 4.5 provides the decisive means to show limitations of ICPT, which we will exploit in the next section.

5 Power and Limitations of ICPT

The primary goal of this section is to show that SAT, i.e. the satisfiability problem for a set of propositional clauses, is not in ICPT, but our method will allow us to establish further restrictions to ICPT. We will follow the argumentation path of [9, Sect.8/9] with a few generalisations that are necessary to capture structures over non-empty signatures and the extension by insignificant choice.

5.1 A Support Theorem for ICPT

Consider a PTIME ic-ASM $\tilde{M} = (M, p(n), q(n))$. Let I be an input structure for \tilde{M} and $\text{State}(I)$ be the initial state generated by I . Let \bar{R} denote the set of input relations of I . Recall that the base set of every state contains all hereditarily finite sets built over the atoms. Isomorphisms are defined as permutations of the set of atoms that are extended to sets using $\theta(x) = \{\theta(y) \mid y \in x\}$. By means of the Kuratowski representation of ordered pairs as sets, i.e. $(x, y) = \{\{x\}, \{x, y\}\}$, extended to n -tuples using $(x_1, \dots, x_n) = (x_1, (x_2, \dots, x_n))$, we extend permutations also to tuples and thus to the input relations and update sets. If θ maps the input relations R_i of I onto themselves, we obtain an *automorphism* of $\text{State}(I)$. Let $\text{Aut}(I)$ denote the automorphism group of $\text{State}(I)$. Note that for every automorphism $\theta \in \text{Aut}(I)$ we have $\theta(c_f) = c_f$ for the special constants c_f with $f \in \mathcal{T}_{dyn}$.

Definition 5.1. A set X of atoms of I is called a *support* of an object y iff every automorphism of $\text{State}(I)$ that pointwise fixes X also fixes y . A *support* of an orbit (y, G) is a support of $G(y) = \{\sigma(y) \mid \sigma \in G\}$.

Let $Active(I)$ denote the set of active objects in a run S_0, S_1, \dots , where the initial state S_0 is defined by I . Clearly, Definition 2.1 implies that $Active(I)$ is transitive and closed under $Aut(I)$. Clearly, if y is active, then there exists an active orbit (y, G) with $y \in G(y)$. The set of active orbits $\mathcal{A}[I]$ is also closed under $Aut(I)$.

Our aim is to generalise the support theorem for CPT (see [9, Thm.24 & Cor.33]) to ICPT, i.e. to show that every active orbit in $\mathcal{A}[I]$ has a support bounded by a fixed positive integer k .

Let us first address the problem arising from the presence of input relations, say $\bar{R} = \{R_1, \dots, R_m\}$, where the arity of R_i is $n_i \leq \ell$. In addition let $R_0 = I$. For indices $i_1, \dots, i_q \in \{1, \dots, m\}$ such that $n_{i_x} = n_{i_y}$ we build

$$R_{i_1, \dots, i_q} = \bigcap_{1 \leq j \leq q} R_{i_j} - \bigcup_{i \notin \{i_1, \dots, i_q\}} R_i.$$

We may omit those R_{i_1, \dots, i_q} that by definition must be empty. In this way we partition the set of tuples over I including I itself. We refer to all sets in the partition as *colours*. In the following let C denote the set of colours.

In addition, let us introduce finitely many new atoms representing tuples over I of length between 2 and ℓ . Use \tilde{I} to denote the extension of I by these new atoms. Then the colours refer to pairwise disjoint sets of atoms of \tilde{I} . For a real number $\varepsilon > 0$, we call C ε -level iff $|R_{i_1, \dots, i_q}| > \varepsilon \cdot n$ holds for every colour in C and $n = |\tilde{I}|$. As each $\theta \in Aut(I)$ preserves the colours defined this way, it defines an automorphism of $State(\tilde{I})$. We will therefore consider (\tilde{I}, C) instead of (I, \bar{R}) , which brings us to the case of a structure of coloured sets handled in [9]. Note that M can be considered as an ASM operating on the input structure (\tilde{I}, C) instead of (I, \bar{R}) , but whenever an atom in \tilde{I} representing a tuple over I is needed, it is converted to the actual tuples. The conversion function can be assumed to be part of the background structure. This only occurs when the input relations are accessed. In particular, $Active(\tilde{I})$ extends $Active(I)$ only by the new atoms, and also $\mathcal{A}[\tilde{I}]$ only extends $\mathcal{A}[I]$ by the new atoms.

In doing so we can as well define a support with respect to atoms in \tilde{I} rather than I . In the following, if not clear from the context we will therefore distinguish between an I -support and an \tilde{I} -support.

We will next exploit a sequence of lemmata similar to those that have been proven in [9, Sect.8] for the case of CPT with input structures over an empty signature, i.e. naked sets, and for an extension to coloured sets. We assume that our set C of colours is ε -level for some $\varepsilon > 0$. The first lemma is a purely combinatorial lemma on so-called Δ -systems. A Δ -

system is a collection K of sets such that all pairs of sets $X, Y \in K$ with $X \neq Y$ have the same intersection $X \cap Y$. The second lemma allows us to minimize supports (see also [9, Lemma 26]).

Lemma 5.1 ([9, Lemma 25]). *Every family $\{X_i\}_{i \in I}$ of sets with $|X_i| \leq \ell$ for all $i \in I$ and $|I| \geq \ell!p^{\ell+1}$ contains a Δ -system K with $|K| = p$.*

Lemma 5.2. *Let X_1 and X_2 support $(y, G) \in \mathcal{A}[\tilde{I}]$ with $(\tilde{I} - (X_1 \cup X_2)) \cap c \neq \emptyset$ for all colours $c \in C$. Then also $X_1 \cap X_2$ is a support of (y, G) .*

Proof. For each $c \in C$ fix an atom $a_c \in \tilde{I} - (X_1 \cup X_2)$, and let b_c range over atoms in $(\tilde{I} - (X_1 \cap X_2)) \cap c$. Let π_{b_c} be a transposition swapping a_c and b_c , so it is an automorphism of \tilde{I} . We have $b_c \notin X_1$ or $b_c \notin X_2$. In the former case π_{b_c} pointwise fixes X_1 , in the latter case it pointwise fixes X_2 . As X_1 and X_2 both support $G(y)$, we derive $\pi_{b_c}(G(y)) = G(y)$ in both cases. Transpositions generate all permutations, and consequently all these transpositions π_{b_c} generate all automorphisms of \tilde{I} that pointwise fix $X_1 \cap X_2$. Therefore, all these automorphisms fix $G(y)$, and thus $X_1 \cap X_2$ supports (y, G) . \square

For an active orbit $(y, G) \in \mathcal{A}[\tilde{I}]$ with a support X such that $|X \cap c| < |c|/2$ holds for all colours $c \in C$ Lemma 5.2 justifies to define

$$\text{Supp}(y, G) = \bigcap \{X \mid X \text{ supports } (y, G) \text{ and } |X \cap c| < |c|/2 \text{ for all } c \in C\},$$

which gives the smallest support of (y, G) .

Let $n = |\tilde{I}|$. As \tilde{M} is a PTIME ic-ASM, we have $|\mathcal{A}[\tilde{I}]| \leq n^{k'}$ for some positive integer k' . Consider only input structures I , for which n is so large that $\binom{\lceil \varepsilon n \rceil}{k' + 1} > n^{k'}$ holds. In the following let $k = k' \cdot |C|$. This leads to the next lemma (see also [9, Lemma 27]).

Lemma 5.3. *If $(y, G) \in \mathcal{A}[\tilde{I}]$ has a support X with $|X \cap c| < |c|/2$ for all colours $c \in C$, then $|\text{Supp}(y, G) \cap c| \leq k'$ and $|\text{Supp}(y, G)| \leq k$.*

Proof. Let $Q = G(y)$. For an automorphism θ with $\theta(Q) = Q'$ we also have $\theta(\text{Supp}(Q)) = \text{Supp}(Q')$ and $\theta(\text{Supp}(Q) \cap c) = \text{Supp}(Q') \cap c$ for every colour $c \in C$. Let $s_c = |\text{Supp}(Q) \cap c|$ and assume $s_c > k'$. This implies the contradiction

$$\begin{aligned} n^{k'} &\geq |\mathcal{A}[\tilde{I}]| \geq |\{\theta(Q) \cap c \mid \theta \in \text{Aut}(\tilde{I})\}| \\ &\geq |\{\theta(\text{Supp}(Q) \cap c) \mid \theta \in \text{Aut}(\tilde{I})\}| = \binom{\lceil \varepsilon n \rceil}{s_c} \geq \binom{\lceil \varepsilon n \rceil}{k' + 1} > n^{k'}. \end{aligned}$$

Hence $s_c \leq k'$ and further $|\text{Supp}(Q)| \leq |C| \cdot k' = k$ holds. \square

To reach our claim it only remains to show that the prerequisite in Lemma 5.3 is always satisfied. This will be done by the next lemma (see also [9, Lemma 28]).

Lemma 5.4. *If $n = |\tilde{I}|$ is sufficiently large, then every active orbit $(y, G) \in \mathcal{A}[\tilde{I}]$ has a support X with $|X \cap c| < |c|/2$ for all colours $c \in C$.*

Note that in the following proof the arguments are the same as in the proof of [9, Lemma 28] except for claim I.

Proof. Let k and k' be as above, let $n' = \min\{|c| \mid c \in C\}$ and take $m = \lfloor n'/4k' \rfloor$. Assume that there is an orbit (x, G) with no support X with $|X \cap c| < |c|/2$ for all colours $c \in C$. We choose such an orbit such that x has minimal rank. Then x must be a set. Otherwise if x were an atom, $G(x)$ would be a set of atoms with support $\{x\}$.

Claim (I). There exists a sequence of quintuples $(\theta_j, y_j, z_j, Y_j, Z_j)$ ($1 \leq j \leq m$) satisfying the following conditions:

- $\theta_j \in \text{Aut}(\tilde{I})$ is an automorphism, $Y_j = \text{Supp}(y_j, G)$, $Z_j = \text{Supp}(z_j, G)$;
- $y_j \in x$ and $z_j \notin x$;
- $\theta_j(y_j) = z_j$, and $\theta_j(Y_j) = Z_j$;
- θ_j pointwise fixes $Y_i \cup Z_i$ for all $i < j$.

Proof (of the claim). We construct the sequence by induction on j . For $j = 1$ there exists an automorphism θ_1 with $\theta_1(x) \neq x$ —otherwise \emptyset would be a support of $G(x)$, as every automorphism θ would satisfy $\theta(x) = x$ and hence also $\theta(\sigma(x)) = \sigma(\theta(x)) = \sigma(x) \in G(x)$ for all $\sigma \in G$. Then there exists a $y_1 \in x$ such that $z_1 = \theta_1(y_1) \notin x$. With $Y_1 = \text{Supp}(y_1, G)$ and $Z_1 = \text{Supp}(z_1, G)$ we get $\theta_1(Y_1) = \theta_1(\text{Supp}(G(y_1))) = \text{Supp}(\theta_1(G(y_1))) = \text{Supp}(G(z_1)) = Z_1$ and all required conditions are satisfied.

Next suppose that $(\theta_i, y_i, z_i, Y_i, Z_i)$ ($1 \leq i < j$) satisfying all the conditions of the claim has been constructed. As x has minimal rank and $y_i \in x$ holds, the support Y_i of (y_i, G) satisfies $|Y_i \cap c| < |c|/2$ for all colours $c \in C$. As z_i and Z_i are automorphic images, the same holds for Z_i . In Lemma 5.3 we showed $|Y_i \cap c| \leq k'$, hence also $|Z_i \cap c| \leq k'$.

Define $X_j = \bigcup_{i < j} (Y_i \cup Z_i)$. Then

$$|X_j \cap c| \leq (j-1) \cdot 2 \cdot k' < m \cdot 2 \cdot k' \leq \frac{n'}{4k'} \cdot 2 \cdot k' = \frac{n'}{2} \leq \frac{|c|}{2}$$

for all $c \in C$.

Hence there must exist an automorphism θ_j that pointwise fixes X_j with $\theta_j(x) \neq x$ —otherwise X_j would be a support for $G(x)$ with $|X_j \cap c| < |c|/2$ for all colours $c \in C$ contradicting our assumption above. Then there exists an $y_j \in x$ such that $z_j = \theta_j(y_j) \notin x$. With $Y_j = \text{Supp}(y_j, G)$ and $Z_j = \text{Supp}(z_j, G)$ we get $\theta(Y_j) = Z_j$ as above. Hence $(\theta_j, y_j, z_j, Y_j, Z_j)$ satisfies the conditions above, which proves the claim.

For a given value of n let p be the largest integer with $(2k)! \cdot p^{2k+1} \leq m$. For increasing values of n both m and p increase, but k remains fixed. Therefore, for sufficiently large n we have

$$n^k < ((m+1) \cdot 4k')^k \leq ((2k)! \cdot p^{2k+1} \cdot 4k')^k < 2^{p-1},$$

so we can also ensure $2^{p-1} > n^k$.

Claim (II). There exists a sequence of quintuples $(\theta_j, y_j, z_j, Y_j, Z_j)$ ($1 \leq j \leq p$) satisfying the conditions from the previous claim such that the sets $Y_i \cup Z_i$ form a Δ -system.

Proof (of the claim). Let $(\theta_j, y_j, z_j, Y_j, Z_j)$ ($1 \leq j \leq m$) be the sequence guaranteed by claim I. According to Lemma 5.3 we have $|Y_i| \leq k$. As Z_i is an automorphic image, we also get $|Z_i| \leq k$ and thus $|Y_i \cup Z_i| \leq 2k$. As we have $(2k)! \cdot p^{2k+1} \leq m$, we can apply Lemma 5.1 with $\ell = 2k$ to obtain a subsequence of length p , in which the sets $Y_i \cup Z_i$ form a Δ -system. This proves the claim.

Now fix a sequence $(\theta_j, y_j, z_j, Y_j, Z_j)$ ($1 \leq j \leq p$) as in claim II, and define $X_0 = (Y_i \cup Z_i) \cap (Y_j \cup Z_j)$ with $1 \leq i \neq j \leq p$. As we have a Δ -system, X_0 is uniquely determined independent of the choice of $i \neq j$. Let $U = \{2, \dots, p\}$. Then for any $i \in U$ θ_i pointwise fixes $Y_1 \cup Z_1$ and hence pointwise fixes also X_0 .

Claim (III). For each $V \subseteq U$ there exists an automorphism θ_V such that $z_i = \theta_V(z_i)$ for $i \in V$ and $z_i = \theta_V(y_i)$ for $i \in U - V$.

Proof (of the claim). We construct a permutation π of atoms as follows. For $a \in X_0$ or $a \in Y_i \cup Z_i$ for some $i \in V$ let $\pi(a) = a$. For $a \in Y_i - X_0$ let $\pi(a) = \theta_i(a)$. Then π extends to an automorphism, which we denote as θ_V . Due to this construction π pointwise fixes $Z_i = \text{Supp}(z_i, G)$ for $i \in V$, which implies $\pi(z_i) = z_i$ and hence $\theta_V(z_i) = z_i$ for $i \in V$.

Furthermore, θ_V and θ_i coincide on $Y_i = \text{Supp}(y_i, G)$ for $i \notin V$, which implies $\theta_V^{-1}(\theta_i(y_i)) = y_i$. Hence $\theta_V(y_i) = \theta_i(y_i) = z_i$, which proves the claim.

Finally, fix the automorphisms θ_V in claim III for all $V \subseteq U$.

Claim (IV). Let $V, W \subseteq U$ with $V \neq W$. Then $\theta_V(x) \neq \theta_W(x)$ and $\theta_V(G(x)) \neq \theta_W(G(x))$.

Proof (of the claim). Without loss of generality let $V - W \neq \emptyset$, then take any $i \in V - W$. Claim III implies $\theta_V(z_i) = z_i$. As $z_i \notin x$, we have $z_i = \theta_V(z_i) \notin \theta_V(x)$. Claim III further implies $\theta_W(y_i) = z_i$. As $y_i \in x$, we have $z_i = \theta_W(y_i) \in \theta_W(x)$, which completes the proof of the claim.

Due to claim IV there are 2^{p-1} different automorphic images of $G(x)$. This implies the contradiction

$$|\mathcal{A}[\tilde{I}]| \geq 2^{p-1} > n^k \geq |\mathcal{A}[\tilde{I}]|,$$

which completes the proof of the lemma. \square

With Lemmata 5.3 and 5.4 we get a positive integer k such that every active orbit $(y, G) \in \mathcal{A}[\tilde{I}]$ has an \tilde{I} -support $\text{Supp}(y, G)$ of size $|\text{Supp}(y, G)| \leq k$. This completes the proof of the support theorem.

Theorem 5.1 (Support Theorem I). *There exists a positive integer k such that for sufficiently large I and ϵ -level colours R_{i_1, \dots, i_q} every active orbit $(y, G) \in \mathcal{A}[\tilde{I}]$ has a support (in \tilde{I}) of cardinality at most k .*

Such a support contains atoms of \tilde{I} , so we get $\text{Supp}(y, G) = X \cup X_{ext}$, where $X \subseteq I$ and X_{ext} is a set of atoms representing tuples. As every automorphism $\theta \in \text{Aut}(I)$ defines an automorphism of $\text{State}(\tilde{I})$ and θ fixes a tuple (x_1, \dots, x_n) iff θ fixes each x_i , each automorphism $\theta \in \text{Aut}(I)$ pointwise fixing all atoms of I in $X \cup X_{ext}$ also fixes $G(y)$. This gives rise for a support of (y, G) with atoms in I of size at most $\bar{k} = k \cdot \ell$, where ℓ is the maximum arity of tuples. This gives rise to the following corollary.

Corollary 5.1 (Support Theorem II). *There exists a positive integer \bar{k} such that for sufficiently large I and ϵ -level colours R_{i_1, \dots, i_q} every active orbit $(y, G) \in \mathcal{A}[\tilde{I}]$ has a support (in I) of cardinality at most \bar{k} .*

5.2 Skew-Symmetric Objects

Fix an input signature $\mathcal{Y}_0 \subseteq \mathcal{Y}$, and let I denote an input structure over \mathcal{Y}_0 . As in the previous subsection let \tilde{I} denote the extension of I by new atoms representing tuples over I of length between 2 and ℓ , and let C be

the set of colours defined by the input relations $\bar{R} = \{R_1, \dots, R_m\}$, where the arity of R_i is $n_i \leq \ell$. We assume that C is ε -level for some $\varepsilon > 0$.

Given a positive integer k we call an object $x \in HF(\tilde{I})$ *k-skew-symmetric* iff there exists an orbit (y, G) with $x = Gy$ such that for every object $z \in TC(y)$ the orbit (z, G) has a support $\leq k$.

In the following let \bar{I}_k denote the set of *k-skew-symmetric* objects in $HF(\tilde{I})$. As the colours $c \in C$ in \tilde{I} subsume the input relations \bar{R} , we ignore these relations in this subsection.

According to [9, Sect.9] an object is called *k-symmetric* iff every $z \in TC(x)$ has a support $\leq k$. Clearly, if x is *k-symmetric*, it is also *k-skew-symmetric*. Thus, our definition extends the corresponding one in the work on CPT.

The main result of this subsection will be that every $x \in \bar{I}_k$ can be constructed out of its support, which will be represented by an ordered sequence of k distinct atoms called *k-molecule*, by means of a so-called *k-form*. For this we adapt again the work in [9]. This will be the basis for an equivalence theorem in the next subsection, which characterises, when structures \bar{I}_k and \bar{J}_k are $\mathcal{L}_{\infty}^{ic,m}$ -equivalent.

Definition 5.2. A *k-molecule* is an injective mapping $\sigma : k \rightarrow \tilde{I}$, i.e. a sequence of k distinct atoms.

Now let $\bar{\sigma} = (\sigma_0, \dots, \sigma_{\ell-1})$ be a finite sequence of *k-molecules* of length ℓ . The *configuration* $conf(\bar{\sigma})$ of $\bar{\sigma}$ is a pair $(\sim_{\bar{\sigma}}, col_{\bar{\sigma}})$, where

- $\sim_{\bar{\sigma}}$ is an equivalence relation on $\ell \times k$ defined by

$$(i, p) \sim_{\bar{\sigma}} (j, q) \Leftrightarrow \sigma_i(p) = \sigma_j(q) ,$$

- $col_{\bar{\sigma}}$ is a function $\ell \times k \rightarrow C$ with

$$col_{\bar{\sigma}}(i, p) = c \Leftrightarrow \sigma_i(p) \in c .$$

A configuration describes how the *k-molecules* in the sequence $\bar{\sigma}$ overlap. We see that $conf(\bar{\sigma})$ is uniquely determined by the configurations $conf(\sigma_i, \sigma_j)$ for $i \neq j$. We now define a more abstract notion of configuration.

Definition 5.3. For $\ell \in \mathbb{N}$, $\ell \neq 0$ an *abstract ℓ -configuration* is a pair (\sim, col) , where \sim is an equivalence relation on $\ell \times k$ satisfying $(i, p) \sim (i, q) \Leftrightarrow p = q$, and $col : \ell \times k \rightarrow C$ is a function such that $(i, p) \sim (j, q) \Rightarrow col(i, p) = col(j, q)$.

Clearly, every configuration $\text{conf}(\bar{\sigma})$ is an abstract ℓ -configuration. Conversely, given an abstract ℓ -configuration (\sim, col) , choose a different atom $x_{(i,p)} \in c$ for the equivalence class $[(i,p)]_\sim$ with $c = \text{col}(i,p)$. Then using $\sigma_i(p) = x_{(i,p)}$ we define a configuration $\bar{\sigma} = (\sigma_0, \dots, \sigma_{\ell-1})$ that realises the abstract ℓ -configuration.

Definition 5.4. The set of k -forms is the smallest set \mathcal{F} with

- $\{c_0, \dots, c_{k-1}\} \subseteq \mathcal{F}$, where the c_p are new symbols,
- whenever $\varphi_1, \dots, \varphi_n \in \mathcal{F}$ and E_1, \dots, E_n are abstract 2-configurations, then the set of pairs $\varphi = \{(\varphi_i, E_i) \mid 1 \leq i \leq n\}$ is a form in \mathcal{F} .

Each k -form $\varphi \in \mathcal{F}$ has a *rank* $rk(\varphi)$. We have $rk(c_p) = 0$ and

$$rk(\{(\varphi_i, E_i) \mid 1 \leq i \leq n\}) = 1 + \max\{rk(\varphi_i) \mid 1 \leq i \leq n\}.$$

We extend the definition of k -forms to *orbit-forms* (φ, G) , where $\varphi \in \mathcal{F}$ and $G \subseteq \text{Sym}(I)$ is a group of isomorphisms commuting with $\text{Aut}(I)$.

Following the idea in [9, Sect.9.2] a k -molecule σ together with a k -form $\varphi \in \mathcal{F}$ define a unique object $\varphi * \sigma \in HF(\tilde{I})$:

- For $\varphi = c_p$ we have $\varphi * \sigma = \sigma(p)$;
- For $\varphi = \{(\varphi_i, E_i) \mid 1 \leq i \leq n\}$ we have

$$\varphi * \sigma = \{\varphi_i * \tau \mid E_i = \text{conf}(\tau, \sigma)\}.$$

This extends to orbit-forms by $(\varphi, G) * \sigma = G(\varphi * \sigma)$.

Lemma 5.5. For any automorphism $\pi \in \text{Aut}(I)$ we have $\pi((\varphi, G) * \sigma) = (\varphi, G) * \pi\sigma$.

Proof. As we have $\pi((\varphi, G) * \sigma) = \pi(G(\varphi * \sigma)) = G(\pi(\varphi * \sigma))$ and $(\varphi, G) * \pi\sigma = G(\varphi * \pi\sigma)$, it suffices to consider $G = 1$, for which we proceed by induction over φ .

We have $\pi(c_p * \sigma) = \pi\sigma(p) = c_p * \pi\sigma$ covering the base case $\varphi = c_p$. If $\varphi = \{(\varphi_i, E_i) \mid 1 \leq i \leq n\}$ is a set, we get

$$\begin{aligned} \pi(\varphi * \sigma) &= \pi\{\varphi_i * \tau \mid E_i = \text{conf}(\tau, \sigma)\} \\ &= \{\pi(\varphi_i * \tau) \mid E_i = \text{conf}(\tau, \sigma)\} \\ &= \{\varphi_i * \pi\tau \mid E_i = \text{conf}(\tau, \sigma)\} && \text{(by induction hypothesis)} \\ &= \{\varphi_i * \rho \mid E_i = \text{conf}(\pi^{-1}\rho, \sigma)\} && (\rho = \pi\tau) \\ &= \{\varphi_i * \rho \mid E_i = \text{conf}(\rho, \pi\sigma)\} && \text{(to be shown)} \\ &= \varphi * \pi\sigma \end{aligned}$$

It remains to show that $\text{conf}(\pi^{-1}\rho, \sigma) = \text{conf}(\rho, \pi\sigma)$. For the equivalence relations we have

$$\begin{aligned} (0, p) \sim_{(\pi^{-1}\rho, \sigma)} (1, q) &\Leftrightarrow \pi^{-1}\rho(p) = \sigma(q) \Leftrightarrow \\ \rho(p) = \pi\sigma(q) &\Leftrightarrow (0, p) \sim_{(\rho, \pi\sigma)} (1, q) , \end{aligned}$$

i.e. $\sim_{(\pi^{-1}\rho, \sigma)} = \sim_{(\rho, \pi\sigma)}$. As automorphisms preserve colours, we further have

$$\text{col}_{(\pi^{-1}\rho, \sigma)}(0, p) = c \Leftrightarrow \pi^{-1}\rho(p) \in c \Leftrightarrow \rho(p) \in c \Leftrightarrow \text{col}_{(\rho, \pi\sigma)}(0, p) = c$$

and

$$\text{col}_{(\pi^{-1}\rho, \sigma)}(1, q) = c \Leftrightarrow \sigma(q) \in c \Leftrightarrow \pi\sigma(q) \in c \Leftrightarrow \text{col}_{(\rho, \pi\sigma)}(1, q) = c ,$$

which completes the proof. \square

Corollary 5.2. *Every object $(\varphi, G) * \sigma$ is k -skew-symmetric and thus an object in \bar{I}_k .*

Proof. If $\pi \in \text{Aut}(I)$ pointwise fixes $\text{range}(\sigma)$, Lemma 5.5 implies

$$\pi((\varphi, G) * \sigma) = (\varphi, G) * \pi\sigma = (\varphi, G) * \sigma ,$$

i.e. $\text{range}(\sigma)$ is a support for $(\varphi, G) * \sigma = G(\varphi * \sigma)$.

As every element $z \in TC(\varphi * \sigma)$ has the form $z = \psi * \tau$ and $Gz = (\psi, G) * \tau$, also Gz has a support of size $\leq k$. \square

Now we are ready to prove the main result of this subsection.

Theorem 5.2. *Every k -skew-symmetric object $x = Gy \in \bar{I}_k$ can be written in the form $x = (\varphi, G) * \sigma$ with a k -form φ and a k -molecule σ .*

Proof. As $x = Gy$ is k -skew-symmetric, every orbit (z, G) with $z \in TC(y)$ has a support $\leq k$ in \tilde{I} .

We proceed by induction over y . If $y \in \tilde{I}$ is an atom, then $y = c_0 * \sigma$ with $\sigma(0) = y$, which implies

$$x = Gy = G(c_0 * \sigma) = (c_0, G) * \sigma .$$

Now let σ be a k -molecule such that $\text{range}(\sigma)$ is a support for $x = Gy$. Assume that y is a set. Then Gz is k -skew-symmetric for $z \in y$, and by the induction hypothesis we have

$$Gz = (\psi_z, G) * \tau_z = G(\psi_z * \tau_z) .$$

Then we define the k -form $\varphi = \{(\psi_z, \text{conf}(\tau_z, \sigma)) \mid z \in y\}$. We claim that $x = (\varphi, G) * \sigma = G(\varphi * \sigma)$ holds.

First consider $\rho z \in x$ with $z \in y$ and $\rho \in G$; we have $x = \{\rho z \mid z \in y, \rho \in G\}$. We can write $\rho z = \rho(\psi_z * \tau_z)$. This implies $\varphi * \sigma = \{\psi_z * \tau_z \mid z \in y\}$ and thus $G(\varphi * \sigma) = \{\rho(\psi_z * \tau_z) \mid z \in y\}$, i.e. $\rho z \in G(\varphi * \sigma) = (\varphi, G) * \sigma$.

Conversely, let $u \in \varphi * \sigma$ and $\rho u \in (\varphi, G) * \sigma$ for some $\rho \in G$. We have $\varphi * \sigma = \{\psi * \tau \mid (\psi, \text{conf}(\tau, \sigma)) \in \varphi\}$, thus $u = \psi_z * \tau$ for some $z \in y$ and $\text{conf}(\tau, \sigma) = \text{conf}(\tau_z, \sigma)$, but not necessarily $z = u$ nor $\tau_z = \tau$.

We now construct an automorphism $\pi \in \text{Aut}(I)$ with $\pi(z) = u$, which pointwise fixes $\text{range}(\sigma)$. Then π fixes x , as σ is a support for x . Hence $\rho u = \rho\pi(z) = \pi\rho(z) \in x$, as $z \in y$ and thus also $\rho z \in Gy = x$.

It remains to construct the automorphism π with the desired properties. For this define first $\pi_0 : \text{range}(\tau_z) \cup \text{range}(\sigma) \rightarrow \tilde{I}$ by

$$\pi_0(a) = \begin{cases} \tau(p) & \text{if } a = \tau_z(p) \\ \sigma(q) & \text{if } a = \sigma(q) \end{cases}.$$

As $\text{conf}(\tau, \sigma) = \text{conf}(\tau_z, \sigma)$ holds, we get $\sim_{(\tau, \sigma)} = \sim_{(\tau_z, \sigma)}$. This implies that π_0 is *well defined*, as we have

$$\tau_z(p) = \sigma(q) \Rightarrow (0, p) \sim_{(\tau_z, \sigma)} (1, q) \Rightarrow (0, p) \sim_{(\tau, \sigma)} (1, q) \Rightarrow \tau(p) = \sigma(q).$$

Furthermore, we must have $\text{col}_{(\tau, \sigma)} = \text{col}_{(\tau_z, \sigma)}$, which implies that π_0 *preserves colours*. This is trivial for $a = \sigma(q)$. For $a = \tau_z(p)$ we have $\pi_0(a) = \tau(p)$ and thus

$$\tau_z(p) \in c \Rightarrow \text{col}_{(\tau_z, \sigma)}(0, p) = c \Rightarrow \text{col}_{(\tau, \sigma)}(0, p) = c \Rightarrow \tau(p) \in c.$$

We can further show that π_0 is *injective*. For $\pi_0(a) = \tau(p_1) = \tau(p_2) = \pi_0(b)$ we get $p_1 = p_2$, as τ is injective and hence $a = \tau_z(p_1) = \tau_z(p_2) = b$. For $\pi_0(a) = \sigma(q_1) = \sigma(q_2) = \pi_0(b)$ we trivially have $a = \sigma(q_1) = \sigma(q_2) = b$. For $\pi_0(a) = \tau(p) = \sigma(q) = \pi_0(b)$ we have $(0, p) \sim_{(\tau, \sigma)} (1, q)$ and thus $(0, p) \sim_{(\tau_z, \sigma)} (1, q)$, which implies $a = \tau_z(p) = \sigma(q) = b$.

As π_0 is injective and colour-preserving, it extends to an automorphism π of \tilde{I} , which is defined by an automorphism $\pi \in \text{Aut}(I)$. By the definition of π_0 above π pointwise fixes $\text{range}(\sigma)$ and $\pi\tau_z = \tau$. As $\text{range}(\sigma)$ is a support for x , we have $\pi(x) = x$ and hence $\pi((\varphi, G) * \sigma) = (\varphi, G) * \pi\sigma$ due to Lemma 5.5. This finally implies

$$\pi(z) = \pi(\psi_z * \tau_z) = \psi_z * \pi\tau_z = \psi_z * \tau = u$$

which shows our claim above and completes the proof. \square

5.3 Relations over Forms and Configurations

Now let $m \geq 3$ and assume $|\tilde{I}| \geq km$. The following lemma shows that if the configurations of two sequences of k -molecules of length $\ell < m$ over different structures I and J coincide, then the sequences can be uniformly extended.

Lemma 5.6. *Let $\bar{\sigma} = \sigma_1, \dots, \sigma_\ell$ and $\bar{\tau} = \tau_1, \dots, \tau_\ell$ be sequences of k -molecules over \tilde{I} and \tilde{J} , respectively, with $\ell < m$. If $\text{conf}(\bar{\sigma}) = \text{conf}(\bar{\tau})$ holds and σ_0 is another k -molecule over \tilde{I} , then there exists another k -molecule τ_0 over \tilde{J} with $\text{conf}(\sigma_0, \bar{\sigma}) = \text{conf}(\tau_0, \bar{\tau})$.*

Proof. For $1 \leq p \leq k$ and $1 \leq i \leq \ell$ we clearly must have $\tau_0(p) = \tau_i(q_i)$, if $(0, p) \sim_{\sigma_0, \bar{\sigma}} (i, q_i)$ holds. Therefore, we define a partial molecule $\tau'_0(p) = \tau_i(q_i)$ for such p and q_i . This partial molecule τ'_0 is well defined: if $(0, p) \sim_{\sigma_0, \bar{\sigma}} (i, q_i)$ and $(0, p) \sim_{\sigma_0, \bar{\sigma}} (j, q_j)$ hold, we must have $(i, q_i) \sim_{\bar{\sigma}} (j, q_j)$ and hence $\tau_i(q_i) = \tau_j(q_j)$.

For $(0, p) \sim_{\sigma_0, \bar{\sigma}} (i, q_i)$ we also have $\sigma_0(p) = \sigma_i(q_i)$. As $\text{col}_{\bar{\sigma}}(i, q_i) = \text{col}_{\bar{\tau}}(i, q_i)$ holds, we also get that $\tau'_0(p)$ and $\sigma_0(p)$ have the same colour.

Furthermore, τ'_0 is injective. If we have $\tau'_0(p_1) = \tau'_0(p_2)$, then we get $\tau_i(q_i) = \tau_j(q_j)$ for $(0, p_1) \sim_{\sigma_0, \bar{\sigma}} (i, q_i)$ and $(0, p_2) \sim_{\sigma_0, \bar{\sigma}} (j, q_j)$. Hence $(i, q_i) \sim_{\bar{\sigma}} (j, q_j)$ and consequently $(0, p_1) \sim_{\sigma_0, \bar{\sigma}} (0, p_2)$ and $p_1 = p_2$.

So we can extend τ'_0 to a k -molecule τ_0 over \tilde{J} such that $\tau'_0(p)$ and $\sigma_0(p)$ have the same colour for all $1 \leq p \leq k$. This is possible, because there are enough elements for each colour. \square

With this lemma we can now express relationships between elements of k -skew-symmetric objects using relations over forms and abstract configurations that do not depend on the input structure. The decisive point is that the k -molecules needed to construct the k -skew-symmetric objects only enter via their configurations. The following theorem was already proven in [9, Sect.9.3], but due to its importance for our generalised equivalence theorem in the next subsection we repeat the proof here.

Theorem 5.3. *There exist ternary relations In , Eq and Sub such that for every input structure I we have*

$$\psi * \tau \in \varphi * \sigma \quad \Leftrightarrow \quad In(\psi, \varphi, \text{conf}(\tau, \sigma)) \quad (1)$$

$$\psi * \tau = \varphi * \sigma \quad \Leftrightarrow \quad Eq(\psi, \varphi, \text{conf}(\tau, \sigma)) \quad (2)$$

$$\psi * \tau \subseteq \varphi * \sigma \quad \Leftrightarrow \quad Sub(\psi, \varphi, \text{conf}(\tau, \sigma)) \quad (3)$$

for all k -forms φ, ψ and all k -molecules σ, τ .

Proof. We explicitly define the relations In , Eq and Sub recursively. For an n -ary configuration Q and $0 \leq i, j \leq n - 1$ we use the notation $Q_{i,j}$ for the binary configuration resulting from projection of Q to $\{i, j\} \times k$ and re-indexing, i.e. $(0, p) \sim_{Q_{i,j}} (1, q)$ holds iff $(i, p) \sim_Q (j, q)$ holds, and $col_{Q_{i,j}}(0, p) = col_Q(i, p)$ and $col_{Q_{i,j}}(1, q) = col_Q(j, q)$. Now define

- $In(\psi, \varphi, E)$ holds iff φ is a set and there exists a k -form χ and a ternary configuration Q with $Q_{1,2} = E$ such that $(\chi, Q_{0,2}) \in \varphi$ and $Eq(\chi, \psi, Q_{0,1})$ hold.
- $Eq(\psi, \varphi, E)$ holds iff either $\psi = c_p$ and $\varphi = c_q$ hold for some $p, q \in k$ with $(0, p) \sim_E (1, q)$ or φ, ψ are sets and for all k -forms χ and all ternary configurations Q with $Q_{1,2} = E$ we have

$$\begin{aligned} (\chi, Q_{0,2}) \in \varphi &\Rightarrow In(\chi, \psi, Q_{0,1}) \text{ and} \\ (\chi, Q_{0,1}) \in \psi &\Rightarrow In(\chi, \varphi, Q_{0,2}) . \end{aligned}$$

- $Sub(\psi, \varphi, E)$ holds iff φ, ψ are sets and for all k -forms χ and all ternary configurations Q with $Q_{1,2} = E$ we have

$$(\chi, Q_{0,1}) \in \psi \Rightarrow In(\chi, \varphi, Q_{0,2}) .$$

We now proceed by simultaneous induction over the sum of ranks of ψ and φ .

(1) For $\varphi = c_p$ both the left and right hand sides of (1) are false. Therefore, let φ be a set.

Assume that $\psi * \tau \in \varphi * \sigma$ holds. Then there exists a pair $(\chi, conf(\rho, \sigma)) \in \varphi$ with a k -form χ and a k -molecule ρ such that $\psi * \tau = \chi * \rho$ holds. By the induction hypothesis we have $Eq(\psi, \chi, conf(\tau, \rho))$. Use the ternary configuration $Q = conf(\rho, \tau, \sigma)$, for which obviously $Q_{1,2} = conf(\tau, \sigma)$ holds. Furthermore, $(\chi, Q_{0,2}) = (\chi, conf(\rho, \sigma)) \in \varphi$, and $Eq(\chi, \psi, Q_{0,1})$ holds, because $Q_{0,1} = conf(\rho, \tau)$. So by definition we infer $In(\psi, \varphi, conf(\tau, \sigma))$.

Conversely, assume that $In(\psi, \varphi, conf(\tau, \sigma))$ holds. Then according to our definition above there exists a k -form χ and a ternary configuration Q with $Q_{1,2} = conf(\tau, \sigma)$ such that $(\chi, Q_{0,2}) \in \varphi$ and $Eq(\chi, \psi, Q_{0,1})$ hold. By Lemma 5.6 there exists a k -molecule ρ such that $Q = conf(\rho, \tau, \sigma)$ holds. Then we have $(\chi, conf(\rho, \sigma)) = (\chi, Q_{0,2}) \in \varphi$, and $Eq(\chi, \psi, conf(\rho, \tau))$ because of $Q_{0,1} = conf(\rho, \tau)$. By the induction hypothesis we get $\chi * \rho = \psi * \tau$ and hence $\psi * \tau \in \varphi * \sigma$.

(2) Both sides of (2) are false, if one of φ, ψ is symbol c_p , while the other one is a set. For the case $\varphi = c_q$ and $\psi = c_p$ we have

$$\psi * \tau = \varphi * \sigma \Leftrightarrow \tau(p) = \sigma(q) \Leftrightarrow (0, p) \sim_{\tau, \sigma} (1, q) \Leftrightarrow Eq(\psi, \varphi, conf(\tau, \sigma)) .$$

Thus, it remains to consider the case, where both φ and ψ are sets. First assume $\psi * \tau = \varphi * \sigma$. Let χ be an arbitrary k -form and Q an arbitrary ternary configuration Q with $Q_{1,2} = conf(\tau, \sigma)$. Assume that $(\chi, Q_{0,2}) \in \varphi$ holds. By Lemma 5.6 there exists a k -molecule ρ such that $Q = conf(\rho, \tau, \sigma)$ holds. Then we get $(\chi, conf(\rho, \sigma)) = (\chi, Q_{0,2}) \in \varphi$, and hence $\chi * \rho \in \varphi * \sigma = \psi * \tau$. By the induction hypothesis $In(\chi, \psi, conf(\rho, \tau))$ holds, i.e. $In(\chi, \psi, Q_{0,1})$. The second implication required in our definition above follows analogously, hence $Eq(\psi, \varphi, conf(\tau, \sigma))$ holds.

Conversely, assume that $Eq(\psi, \varphi, conf(\tau, \sigma))$ holds. We only show $\psi * \tau \subseteq \varphi * \sigma$; the proof of the other subset-relationship is completely analogous. Let $\chi * \rho \in \psi * \tau$ for $(\chi, conf(\rho, \tau)) \in \psi$. Use the ternary configuration $Q = conf(\rho, \tau, \sigma)$ with $Q_{1,2} = conf(\tau, \sigma)$. By our definition of Eq above we must have $(\chi, Q_{0,1}) \in \psi \Rightarrow In(\chi, \varphi, Q_{0,2})$. As $(\chi, Q_{0,1}) = (\chi, conf(\rho, \tau)) \in \psi$, we get $In(\chi, \varphi, Q_{0,2})$. As $Q_{0,2} = conf(\rho, \sigma)$ holds, we have $In(\chi, \varphi, conf(\rho, \sigma))$. By the induction hypothesis this implies $\chi * \rho \in \varphi * \sigma$ as required.

(3) Both sides of (3) are false, if one of φ, ψ is a symbol c_p , so we can assume that both φ, ψ are sets. Then the proof is completely analogous to (2). \square

5.4 An Equivalence Theorem for ICPT

We now investigate the expressiveness of the logic $\mathcal{L}_{\infty\omega}^{ic,m}$ for some m . We consider input structures I over the input signature \mathcal{T}_0 , and let \tilde{I} denote its extension by new atoms defined by relations \bar{R} . Colours defined this way are assumed to be ε -level. For fixed k we consider the structure \bar{I}_k defined by all objects in the k -skew-symmetric objects in $HF(\tilde{I})$. We also use the notation \bar{I}_k to denote the semi-transitive structure with this base set and signature $\{\in, \emptyset\}$. As in the previous subsection we assume that input structures are sufficiently large.

The following lemma shows that $\mathcal{L}_{\infty\omega}^{ic,m}$ is quite powerful, as it is always possible to distinguish structures with base sets of different cardinality, or more general with colours of different cardinalities.

Lemma 5.7. *Let I, J be sufficiently large input structures with $|\bar{I}_k| \neq |\bar{J}_k|$, and let $m \geq 3$. Then \bar{I}_k and \bar{J}_k are not $\mathcal{L}_{\infty\omega}^{ic,m}$ -equivalent.*

Proof. Without loss of generality we can assume $I \subseteq J$, so there must exist an k -skew-symmetric orbit $(x, G) \in \mathcal{A}[\tilde{J}] - \mathcal{A}[\tilde{I}]$. We choose such an orbit, for which x is maximal. Let the spoiler start with an update set move on structure \bar{J}_k , such that x appears as critical element in the update set. The requirements of Definition 4.4 can only be satisfied, if the duplicator can choose an update set over the structure \bar{J}_k with a critical element y , which is a set with the same cardinality as x . However, this is not possible. \square

Note that the argument in the proof could not be applied to CPT, because the support theorem for CPT [9, Sect.8] does not permit the creation of large sets, and this is independent of the base set. Using orbits, however, we can create such sets as elements of an orbit, even though the orbit has a bounded support. For instance, for the Parity example orbits have empty support.

Clearly, the argument in the proof generalises to structures I, J with $|\bar{I}_k| = |\bar{J}_k|$, if we have colours C (i.e. unary, disjoint relations) and $|I \cap c| \neq |J \cap c|$ holds for at least one colour c . Just play the same game only using objects of one colour, for which $|I \cap c| \neq |J \cap c|$ holds.

So in the following we can concentrate on structures I, J with $|\bar{I}_k| = |\bar{J}_k|$ and $|\tilde{I} \cap c| = |\tilde{J} \cap c|$. In order for the duplicator to win an IC pebble game on structures \bar{I}_k, \bar{J}_k in every move the conditions of a partial isomorphism from Definition 4.4 must be satisfied. In particular, condition (iv) requires that $R^{\bar{I}_k}(a_{i_1}, \dots, a_{i_n})$ holds in \bar{I}_k if and only if $R^{\bar{J}_k}(b_{i_1}, \dots, b_{i_n})$ holds in \bar{J}_k , where (a_i, b_i) is a pair of objects (in this case atoms from I, J , respectively) corresponding to a pair of pebbles of the spoiler and the duplicator.

This leads us to another necessary condition for the duplicator to win: the duplicator must have a winning strategy for the *associated static game*, where only object moves are allowed, but no (update) set moves [9]. We formulate this condition in another lemma.

Lemma 5.8. *Let I, J be sufficiently large input structures with $|\bar{I}_k| = |\bar{J}_k|$, and let $m \geq 3$. If the duplicator has a winning strategy for the IC pebble game on \bar{I}_k, \bar{J}_k , then he also has a winning strategy for the associated static game characterising L_{∞}^m -equivalence.*

Proof. If the duplicator has a winning strategy, then it can be applied also for the case, where the spoiler only makes object moves. This defines the winning strategy for the associated static game. \square

Now we can formulate and prove the main result of this subsection, the generalised equivalence theorem, which basically states that the two

necessary conditions for $\mathcal{L}_{\infty\omega}^{ic,m}$ -equivalence given by Lemmata 5.7 and 5.8 are also sufficient. The proof will exploit the relations over forms and configurations from Theorem 5.3 and the representation of skew-symmetric objects from Theorem 5.2.

Theorem 5.4 (Equivalence Theorem). *Let I, J be sufficiently large input structures and let $k \geq 1, m \geq 3$. Then structures \bar{I}_k, \bar{J}_k are $\mathcal{L}_{\infty\omega}^{ic,m}$ -equivalent if and only if $|\bar{I}_k| = |\bar{J}_k|$ and the duplicator has a winning strategy for the associated static game.*

Proof. By Lemma 5.7 the structures \bar{I}_k, \bar{J}_k are not $\mathcal{L}_{\infty\omega}^{ic,m}$ -equivalent if $|\bar{I}_k| \neq |\bar{J}_k|$ holds. By Lemma 5.8 the structures are also not equivalent, if the duplicator has no winning strategy for the associated static game.

We have to show the converse, i.e. according to Theorem 4.5 we have to show that the duplicator has a winning strategy for the m IC pebble game on structures \bar{I}_k, \bar{J}_k . After each move let x_i, y_i ($0 \leq i \leq m-1$) be the objects or update sets covered by the i 'th pair of pebbles. Without loss of generality we can assume that the game starts with all $2m$ pebbles used such that $x_i = y_i = \emptyset$. We define a winning strategy for the duplicator which ensures that after each step there exist k -form φ and k -molecules σ_i, τ_i over \tilde{I} and \tilde{J} , respectively, such that

$$x_i = \varphi_i * \sigma_i, \quad y_i = \varphi_i * \tau_i \quad \text{and} \quad \text{conf}(\bar{\sigma}) = \text{conf}(\bar{\tau}) \quad (*)$$

hold for all $0 \leq i \leq m-1$.

First consider the move the duplicator has to perform in response to a move by the spoiler to maintain condition (*). The condition is obviously satisfied at the beginning of the game with $\varphi_i = \emptyset, \sigma_i = \sigma_j$ and $\tau_i = \tau_j$ for all i, j . Now consider a move by the spoiler. By symmetry we can assume without loss of generality that the spoiler uses the pebble 0 for \bar{I}_k replacing x_0 by x'_0 . As $x'_0 \in \bar{I}_k$, Theorem 5.2 implies that there is a k -form φ'_0 , a group $G \subseteq \text{Sym}(\tilde{I})$ and a k -molecule ρ over \tilde{I} such that $x'_0 \in (\varphi'_0, G) * \rho$.

Thus, $x'_0 = g(\varphi'_0 * \rho)$ for some $g \in G$. As spoiler and duplicator always choose elements in $HF(I)$ or atoms in \tilde{I} (and not arbitrary elements of $HF(\tilde{I})$), we have two cases: either $x'_0 \in \tilde{I}$ or $x'_0 \in HF(I)$. In the former case $\varphi'_0 * \rho$ is an atom in \tilde{I} , which is only possible for $\varphi'_0 = c_p$ for some p . In this case $x'_0 = g\rho(p) = \varphi'_0 * g\rho$ follows immediately. In the latter case ρ must be a k -molecule over I , which implies that colours other than I itself are irrelevant for the configuration. So we can use Lemma 5.5 to obtain $x'_0 = g(\varphi'_0 * \rho) = \varphi'_0 * g\rho$ again. Let $\sigma'_0 = g\rho$.

According to Lemma 5.6 there exists a k -molecule τ'_0 such that

$$\text{conf}(\sigma'_0, \sigma_1, \dots, \sigma_{m-1}) = \text{conf}(\tau'_0, \tau_1, \dots, \tau_{m-1})$$

holds. Then choosing $y'_0 = \varphi'_0 * \tau'_0$ preserves the condition (*).

We now have to show that $x_i \mapsto y_i$ defines a partial isomorphism between \bar{I}_k and \bar{J}_k . For this first consider any pair i, j , so we have $\text{conf}(\sigma_i, \sigma_j) = \text{conf}(\tau_i, \tau_j)$ due to (*). With Theorem 5.3 we obtain:

$$\begin{aligned} x_i \in x_j &\Leftrightarrow \varphi_i * \sigma_i \in \varphi_j * \sigma_j \Leftrightarrow \text{In}(\varphi_i, \varphi_j, \text{conf}(\sigma_i, \sigma_j)) \Leftrightarrow \\ &\quad \text{In}(\varphi_i, \varphi_j, \text{conf}(\tau_i, \tau_j)) \Leftrightarrow \varphi_i * \tau_i \in \varphi_j * \tau_j \Leftrightarrow y_i \in y_j \\ x_i = x_j &\Leftrightarrow \varphi_i * \sigma_i = \varphi_j * \sigma_j \Leftrightarrow \text{Eq}(\varphi_i, \varphi_j, \text{conf}(\sigma_i, \sigma_j)) \Leftrightarrow \\ &\quad \text{Eq}(\varphi_i, \varphi_j, \text{conf}(\tau_i, \tau_j)) \Leftrightarrow \varphi_i * \tau_i = \varphi_j * \tau_j \Leftrightarrow y_i = y_j \\ x_i \subseteq x_j &\Leftrightarrow \varphi_i * \sigma_i \subseteq \varphi_j * \sigma_j \Leftrightarrow \text{Sub}(\varphi_i, \varphi_j, \text{conf}(\sigma_i, \sigma_j)) \Leftrightarrow \\ &\quad \text{Sub}(\varphi_i, \varphi_j, \text{conf}(\tau_i, \tau_j)) \Leftrightarrow \varphi_i * \tau_i \subseteq \varphi_j * \tau_j \Leftrightarrow y_i \subseteq y_j \end{aligned}$$

Further consider an input relation symbol $R \in \mathcal{Y}_0$. If we have $\bar{I}_k \models R(a_1, \dots, a_n)$ such that $a_i = x_{j(i)}$ or $a_i = x'_0$ holds, then this is only possible, if $\varphi'_0 = c_p$ for some $1 \leq p \leq k$, i.e. $x'_0 = \sigma'_0(p)$. Our choice above gives $y'_0 = \tau'_0(p)$. If we have $(0, p) \sim_{\sigma'_0, \sigma_1, \dots, \sigma_{m-1}} (j, q)$, then we have $x'_0 = \sigma'_0(p) = \sigma_j(q)$ and also $y'_0 = \tau'_0(p) = \tau_j(q)$. That is, all a_i are elements in $\{x_0, \dots, x_{m-1}\}$ and all corresponding objects b_i in the structure \bar{J}_k are elements of $\{y_0, \dots, y_{m-1}\}$. Before the move of the spoiler we already had a partial isomorphism, which shows that $\bar{I}_k \models R(b_1, \dots, b_n)$ holds. The inverse implication follows analogously.

So it remains to consider the case, where there is no (j, q) with $(0, p) \sim_{\sigma'_0, \sigma_1, \dots, \sigma_{m-1}} (j, q)$. In this case the winning strategy for the static game allows us to choose some y'_0 which guarantees property (iv) of Definition 4.4. Then there exists a k -molecule τ'_0 with $\tau'_0(p) = y'_0$, i.e. $y'_0 = \varphi'_0 * \tau'_0$.

Finally, we have to show ic-compatibility. For this let $\pi_1, \pi_2 \in \text{Aut}(\bar{I}_k)$. Then we have $\pi_1 x_i = \varphi_i * \pi_1 x_i$, $\pi_2 y_i = \varphi_i * \pi_2 y_i$ and $\text{conf}(\pi_1 \bar{\sigma}) = \text{conf}(\bar{\sigma}) = \text{conf}(\bar{\tau}) = \text{conf}(\pi_2 \bar{\tau})$. Thus, if the spoiler chooses a new object $x'_0 = \varphi'_0 * \sigma'_0$, we get $\pi_1 x'_0 = \varphi'_0 * \pi_1 \sigma'_0$. If the duplicator chooses $y'_0 = \varphi'_0 * \tau'_0$, we get $\pi_2 y'_0 = \varphi'_0 * \pi_2 \tau'_0$. Then we apply the same arguments as above to $\pi_1 \sigma'_0$ and $\pi_2 \tau'_0$ to see that $\pi_1 x_i \mapsto \pi_2 y_i$ defines a partial isomorphism.

This completes the proof of the equivalence theorem. \square

5.5 The SAT Problem

We now look at SAT, the satisfiability problem for a set of clauses. It is commonly known (see e.g. [28]) that the problem can be represented by a

signature \mathcal{T} with two binary relation symbols P and N . We have $P(i, j)$, if the atom a_j appears positively in the clause c_i , and $N(i, j)$ if the atom a_j appears negatively in the clause c_i . Without loss of generality we can assume that we never have an atom $P(i, j)$ and $N(i, j)$ together.

We now show that for arbitrary $k, m > 0$ (with $m \geq 3$) we always find different structures for SAT of arbitrary size that are $\mathcal{L}_{\infty\omega}^{ic,m}$ -equivalent. According to Theorem 5.4 the structures have to be built over \bar{I}_k and we have to show that the duplicator has a winning strategy for the associated static game over the defined structures.

Lemma 5.9. *Let K_1 be the class of satisfiable SAT structures and K_2 the class of unsatisfiable SAT structures. For $k > 0$, $m \geq 3$ there are infinitely many pairs of structures $(\bar{I}_k, \bar{R}_{(1)}) \in K_1$, $(\bar{I}_k, \bar{R}_{(2)}) \in K_2$ that are $\mathcal{L}_{\infty\omega}^{ic,m}$ -equivalent.*

Proof. Choose atoms a_1, \dots, a_m . For $I \subseteq \{1, \dots, m\}$ let c_I be the clause, in which a_i appears positively for $i \in I$ and negatively for $i \notin I$. Let $R_{(1)}$ contain all these clauses except one with (almost) equal number of positive and negative atoms, and let $R_{(2)}$ contain all these clauses.

Furthermore, let b_0, \dots, b_m be different atoms. Define clauses c'_j containing all b_i positively except b_j . Add all these clauses to $R_{(1)}$, and add all these clauses with one exception to $R_{(2)}$. So the number of clauses and atoms in $R_{(1)}$ and $R_{(2)}$ is the same. We can further extend $R_{(1)}$ and $R_{(2)}$ arbitrarily adding the same clauses with different atoms to both structures.

Then clearly $(\bar{I}_k, \bar{R}_{(1)})$ is satisfiable, i.e. in K_1 , and $(\bar{I}_k, \bar{R}_{(2)})$ is not satisfiable, i.e. in K_2 .

Now consider the bipartite graphs $G_{(1)}$ and $G_{(2)}$ with clauses and atoms as vertices and edges labelled by P and N , such that $G_{(i)}$ is defined by the structure $(\bar{I}_k, \bar{R}_{(i)})$ in the obvious way. For both graphs consider subgraphs with m vertices. The sets of such subgraphs are partitioned into isomorphism classes, and the sets of isomorphisms classes for $G_{(1)}$ and $G_{(2)}$ are isomorphic.

Therefore, in the winning strategy for the associated static game for $\varphi'_0 = c_p$ when τ'_0 is selected by the duplicator we can ensure that the selection remains in an isomorphic subgraph. \square

Theorem 5.5 (SAT Theorem). *The pair (K_1, K_2) of satisfiable/non-satisfiable SAT structures cannot be separated by ICPT.*

Proof. Let \tilde{M} be a PTIME ic-ASM with input signature $\mathcal{T}_0 = \{P, N\}$. According to Theorem 4.4 there exists a formula φ in $\mathcal{L}_{\infty\omega}^{ic,m}$ for some

$m > 0$ that asserts that \tilde{M} accepts an input structure I . This is the case, if φ holds in some semi-transitive substructures of $HF(I)$ containing all elements of all active orbits in $\mathcal{A}[I]$. In particular, \tilde{M} accepts I if and only if \tilde{I}_k satisfies φ .

However, if I is sufficiently large, then Theorem 5.5 states that $\mathcal{L}_{\infty\omega}^{ic,m}$ cannot distinguish structures in K_1 and K_2 , so \tilde{M} accepts both satisfiable and non-satisfiable structures. This shows that (K_1, K_2) cannot be separated by ICPT. \square

As SAT is in NP and ICPT captures PTIME Theorem 3.1 implies the following corollary.

Corollary 5.3. *PTIME differs from NP.*

6 Conclusions

Originally, we only wanted to study how complexity theory could be conducted in connection with ASMs, and the work on CPT [9] was one of the few studies in this direction. We saw that CPT exploits (synchronous, parallel) ASMs to characterise the choiceless fragment of PTIME. In view of the behavioural theory of parallel algorithms, in particular using the more convincing set of postulates in [18], it is indeed justified to say that CPT captures THE choiceless fragment of PTIME. This brought us to Gurevich's conjecture that there is no logic capturing PTIME [23], in other words that the gap between PTIME and its choiceless fragment could not be covered by a logical extension of CPT. This would also imply a negative answer to Chandra and Harel's question if there exists a computation model on structures rather than strings that can capture PTIME [14].

This study brought us to the idea of insignificant choice. Retrospectively, after showing that ICPT captures PTIME and Gurevich's conjecture does not hold despite all the supporting evidence, it appears very natural to investigate ICPT. A computation not covered by CPT must either be outside PTIME or it must contain some form of non-deterministic choice. There cannot be any PTIME elimination of the choice, as this would bring the computation back into CPT. There can also be no polynomial bound on the number of choices, because otherwise the choice could be replaced by a parallel execution of all choices, which again would lead back to CPT. However, if nonetheless a well-defined result comes out, the choice must have been insignificant.

Our first major result gives an answer to the question raised by Chandra and Harel: the appropriate model of computation on structure is the model of ASMs (formerly known as evolving algebras and occasionally also called Gurevich machines), and the particular form of ASMs needed for the capture of PTIME are the PTIME ic-ASMs. The corresponding Theorem 3.1 is simple, but only becomes convincing by the logical fixed-point characterisation in Theorems 4.1 and 4.4, which depend on the logic of non-deterministic ASMs [20].

With the logical capture of PTIME it becomes quite natural to investigate also the relationship between PTIME and NP. Here we adopted first the Ehrenfeucht-Fraïssé method [29] and defined a pebble game that allows us to characterise equivalence of structures in the logic $\mathcal{L}_{\infty\omega}^{ic,m}$. Then we exploited the proof that CPT is strictly included in PTIME, which led us to a generalised equivalence Theorem 5.4 showing under which conditions structures cannot be distinguished by $\mathcal{L}_{\infty\omega}^{ic,m}$. Naturally, we chose the NP-complete SAT problem to create an example of such structures. This brought us to the SAT Theorem 5.5 showing that for sufficiently large input structures satisfiability of a set of clauses cannot be decided using ICPT. The consequence is that PTIME differs from NP.

Besides the value of the two major technical results, the refutation of Gurevich’s conjecture and the proof that PTIME and NP differ, we see a third, even more important contribution made by this article. There exist already several behavioural theories showing how particular classes of algorithms are captured by particular classes of ASMs. This justifies to qualify ASMs as THE general computation model on structures. This article shows that a presumed limitation with respect to the ability to deal with complexity theory on the level of computations on structures does not exist. To the contrary, fundamental complexity theoretical results only result when we adopt computations on structures.

Last but not least we like to remark that in this article analogous to Blass, Gurevich and Shelah we adopted a rather specific view on ASMs, which brought them closer to the usual treatment in complexity theory. We see, however, no reason to stick to such a specific version of ASMs. We could as well treat ASMs on arbitrary structures, and a logical characterisation by fixed-point formulae over the logic of ASMs would still work. This offers countless opportunities for the study of descriptive complexity theory without having to refer all the time to the level of Turing machines.

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