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INTERSECTION OF A CORRESPONDENCE WITH A GRAPH OF FROBENIUS

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Abstract

The goal of this note is to give a short geometric proof of a theorem of Hrushovski asserting that an intersection of a correspondence with a graph of a sufficiently large power of Frobenius is non-empty.

Contents

Introduction

The goal of this work is to give a short geometric proof of the following theorem of Hrushovski [Hr, Cor 1.2], which has applications, for example, to algebraic dynamics [Fa], group theory [BS] and algebraic geometry [EM].

Let $\mathbb{F}_q$ be a finite field, $\mathbb{F}$ an algebraic closure of $\mathbb{F}_q$, and X^0 a scheme of finite type over $\mathbb{F}$, defined over $\mathbb{F}_q$. We denote by $\phi_q = \phi_{X^0,q} : X^0 \to X^0$ the geometric Frobenius morphism over $\mathbb{F}_q$, and by $\Gamma^0_{q^n} \subset X^0 \times X^0$ the graph of $\phi_{q^n} = (\phi_q)^n$, where the product here and later is taken over $\mathbb{F}$. Explicitly, $\Gamma^0_{q^n}$ is the image of the morphism $(\mathrm{Id}, \phi_{q^n}) : X^0 \to X^0 \times X^0$.

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Theorem 0.1. *Let $c^0=(c^0_1,c^0_2):C^0\to X^0\times X^0$ be a morphism of schemes of finite type over $\mathbb{F}$ such that X^0 and C^0 are irreducible, both c^0_1 and c^0_2 are dominant, and X^0 is defined over $\mathbb{F}_q$.*

Then for every sufficiently large n, the preimage $(c^0)^{-1}(\Gamma^0_{q^n})$ is non-empty.

Theorem 0.1 has the following corollary.

Corollary 0.2. *In the situation of Theorem 0.1, the union $\bigcup_n (c^0)^{-1}(\Gamma^0_{q^n})$ is Zariski dense in C^0.*

Proof of Corollary 0.2. Let $Z\subset C^0$ be the Zariski closure of $\bigcup_n (c^0)^{-1}(\Gamma^0_{q^n})$. Assume that $Z\neq C^0$l. Then $C':=C^0\smallsetminus Z$ is Zariski dense in C^0. Thus $c^0|_{C'}:C'\to X^0\times X^0$ satisfies all the assumptions of Theorem 0.1. Hence for every sufficiently large n, we have $(c^0)^{-1}(\Gamma^0_{q^n})\cap C'\neq\emptyset$, contradicting our choice of Z. □

Let $f:X^0\to X^0$ be a morphism. Following Borisov and Sapir [BS], we call a point $x\in X^0(\mathbb{F})$ *f-quasi-fixed* if $f(x)=(\phi_q)^n(x)$ for some $n\in\mathbb{N}$. The following result follows immediately from Corollary 0.2.

Corollary 0.3. *Let X^0 be an irreducible scheme of finite type over $\mathbb{F}$, defined over $\mathbb{F}_q$, and let $f:X^0\to X^0$ be a dominant morphism. Then the set of f-quasi-fixed points is Zariski dense.*

Let X^0 be a scheme of finite type over $\mathbb{F}$ and let $f:X^0\to X^0$ be a morphism. We say that $x\in X(\mathbb{F})$ is *f-periodic* if $f^m(x)=x$ for some $m\in\mathbb{N}$. Corollary 0.3 implies the following result (see [Fa, Prop. 5.5]).

Corollary 0.4. *Let X^0 be a scheme of finite type over $\mathbb{F}$, and let $f:X^0\to X^0$ be a dominant morphism. Then the set of f-periodic points is Zariski dense.*

Proof of Corollary 0.4. Since some power f^m stabilizes all irreducible components of X^0, we can assume that X^0 is irreducible. Replacing $\mathbb{F}_q$ by its finite extension, we can assume that both X^0 and f are defined over $\mathbb{F}_q$. Then every f-quasi-fixed point is f-periodic. Indeed, if $x\in X^0(\mathbb{F}_{q^m})$ satisfies $f(x)=(\phi_q)^n(x)$, then $f^m(x)=(\phi_q)^{mn}(x)=x$. Thus the assertion follows from Corollary 0.3. □

Our proof of Theorem 0.1 goes as follows. First we reduce to the case when X^0 is quasi-projective, $\dim C^0=\dim X^0$, and c^0 is a closed embedding.

Then we choose a compactification X of X^0 defined over $\mathbb{F}_q$, and a closed embedding $c=(c_1,c_2):C\to X\times X$, whose restriction to $X^0\times X^0$ is c^0. We say that $\partial X:=X\smallsetminus X^0$ is *locally c-invariant* if every point $x\in X$ has an open neighborhood $U\subset X$ such that $c_2^{-1}(\partial X\cap U)\cap c_1^{-1}(U)\subset c_1^{-1}(\partial X\cap U)$.

For every $m\in\mathbb{N}$, we denote by $c^{(m)}:C\to X\times X$ the map $((\phi_q)^m\circ c_1,c_2)$. The main step of our argument is to reduce to the situation where $\partial X:=X\smallsetminus X^0$ is locally $c^{(m)}$-invariant for all m. Namely, we show that this happens after we replace X^0 by its open subscheme and X by a certain blowup.

Next, using the de Jong theorem on alterations, we can further assume that X is smooth, and the boundary ∂X is a union of smooth divisors $X_i, i \in I$ with normal crossings, defined over $\mathbb{F}_q$.

Following Pink ([Pi]), we consider the blowup $\widetilde{Y} := \mathrm{Bl}_{\cup_{i\in I}(X_i\times X_i)}(X\times X)$, and denote by $\pi : \widetilde{Y} \to X\times X$ the projection map. Let $\Gamma_{q^n} \subset X\times X$ be the graph of ϕ_{X,q^n}, and denote by $\widetilde{C}\subset \widetilde{Y}$ and $\widetilde{\Gamma}_{q^n}\subset\widetilde{Y}$ the strict preimages of C and Γ_{q^n}, respectively.

Replacing c by $c^{(m)}$ for a sufficiently large m, we can get to the situation where $\widetilde{C}\cap\widetilde{\Gamma}_{q^n}\subset \pi^{-1}(X^0\times X^0)$. Then $\widetilde{C}\cap\widetilde{\Gamma}_{q^n} = (c^0)^{-1}(\Gamma^0_{q^n})$, so it remains to show that $\widetilde{C}\cap\widetilde{\Gamma}_{q^n}\neq\emptyset$. Note that $\widetilde{Y}$ is smooth, so it suffices to show that the intersection number $[\widetilde{C}]\cdot[\widetilde{\Gamma}_{q^n}]$ is non-zero.

For every subset $J\subset I$, we denote by X_J the intersection $\cap_{i\in J}X_i$. In particular, $X_\emptyset = X$. For every i, the correspondence c induces an endomorphism $H^i(c) = (c_2)_*\circ(c_1)^* : H^i(X,\mathbb{Q}_l)\to H^i(X,\mathbb{Q}_l)$. In particular, $H^{2d}(c)$ is the multiplication by $\deg(c_1)\neq 0$. More generally, for every $J\subset I$ and i, the correspondence c induces an endomorphism $H^i(c_J) : H^i(X_J,\mathbb{Q}_l)\to H^i(X_J,\mathbb{Q}_l)$ (compare [Laf, Ch. IV]).

Then for every $n\in\mathbb{N}$ we have the equality

$$[\widetilde{C}]\cdot[\widetilde{\Gamma}_{q^n}] = \sum_{J\subset I}(-1)^{|J|}\sum_{i=0}^{2(d-|J|)}(-1)^i\,\mathrm{Tr}((\phi_q^*)^n\circ H^i(c_J)). \tag{0.1}$$

Choose an embedding $\iota : \mathbb{Q}_l \hookrightarrow \mathbb{C}$. By a theorem of Deligne, every eigenvalue λ of $\phi_q^*\in \operatorname{End} H^i(X_J,\mathbb{Q}_l)$ satisfies $|\iota(\lambda)| = q^{i/2}$. Therefore the right hand side of (0.1) grows asymptoticically as $\deg(c_1)q^{dn}$, when n is large. In particular, $[\widetilde{C}]\cdot[\widetilde{\Gamma}_{q^n}]\neq 0$, when n is sufficiently large.

The paper is organized as follows. In the first section we introduce locally invariant subsets and show their simple properties. In the second section we show that every correspondence can be made locally invariant "near the boundary" after a blowup. In the third section we recall a beautiful geometric construction of Pink, and study its properties. In the fourth section we review basic facts about intersection theory and étale cohomology and prove a formula for the intersection number. Finally, in the last section we carry out the proof of Theorem 0.1.

Our proof is essentially self-contained and uses nothing beyond a theorem of de Jong on alterations, standard facts from intersection theory, the Grothendieck-Lefschetz trace formula and purity. Our argument was strongly motivated by the trace formula of Lafforgue [Laf, Prop. IV.6], which in its turn is based on the construction of Pink.

1. Locally invariant subsets

Notation 1.1. Let k be a field. In this work we will only be interested in the case when k is either algebraically closed or finite.

(a) By a *correspondence*, we mean a morphism $c = (c_1, c_2) : C \to X \times X$ between schemes of finite type over k.

(b) For a correspondence $c : C \to X \times X$ and open subsets $U \subset X$ and $W \subset C$, we denote by $c|_W : W \to X \times X$ and $c|_U : c_1^{-1}(U) \cap c_2^{-1}(U) \to U \times U$ the restrictions of c.

(c) Let $c : C \to X \times X$ and $\widetilde{c} : \widetilde{C} \to \widetilde{X} \times \widetilde{X}$ be two correspondences. By a *morphism* from $\widetilde{c}$ to c, we mean a pair of morphisms $[f] = (f, f_C)$, making the following diagram commutative:

$$\begin{array}{ccccc} \widetilde{X} & \xleftarrow{\widetilde{c}_1} & \widetilde{C} & \xrightarrow{\widetilde{c}_2} & \widetilde{X} \\ f\big\downarrow & & f_C\big\downarrow & & \big\downarrow f \\ X & \xleftarrow{c_1} & C & \xrightarrow{c_2} & X. \end{array} \tag{1.1}$$

(d) Suppose that we are given correspondences $\widetilde{c}$ and c as in (c) and a morphism $f : \widetilde{X} \to X$. We say that $\widetilde{c}$ *lifts* c if there exists a morphism $f_C : \widetilde{C} \to C$ such that $[f] = (f, f_C)$ is a morphism from $\widetilde{c}$ to c.

Definition 1.2. Let $c : Y \to X \times X$ be a correspondence, and let $Z \subset X$ be a closed subset.

(a) We say that Z is *c-invariant* if $c_1(c_2^{-1}(Z))$ is set-theoretically contained in Z.

(b) We say that Z is *locally c-invariant* if for every point $x \in Z$ there exists an open neighborhood $U \subset X$ of x such that $Z \cap U \subset U$ is $c|_U$-invariant.

Lemma 1.3. *Let $[f] = (f, f_C)$ be a morphism from a correspondence $\widetilde{c} : \widetilde{C} \to \widetilde{X} \times \widetilde{X}$ to $c : C \to X \times X$. If $Z \subset X$ is a locally c-invariant closed subset, then $f^{-1}(Z) \subset \widetilde{X}$ is locally $\widetilde{c}$-invariant.*

Proof. The assertion is local; therefore we can replace X by an open subset, thus assuming that Z is c-invariant. In this case, $\widetilde{c}_2^{-1}(f^{-1}(Z)) = f_C^{-1}(c_2^{-1}(Z))$ is set-theoretically contained in $f_C^{-1}(c_1^{-1}(Z)) = \widetilde{c}_1^{-1}(f^{-1}(Z))$; thus $f^{-1}(Z)$ is $\widetilde{c}$-invariant. $\square$

Notation 1.4. Let $c : Y \to X \times X$ be a correspondence, and $Z \subset X$ a closed subset. We set $F(c, Z) := c_2^{-1}(Z) \cap c_1^{-1}(X \smallsetminus Z)$, and let $G(c, Z)$ be the union $\bigcup_{S \in \mathrm{Irr}(F(c,Z))} [\overline{c_1(S)} \cap \overline{c_2(S)}] \subset X$, where $\mathrm{Irr}(F(c, Z))$ denotes the set of irreducible components of $F(c, Z)$, and $\overline{c_i(S)} \subset X$ is the closure of $c_i(S)$.

1.5. Remarks. (a) Note that Z is c-invariant if and only if $F(c, Z) = \emptyset$. More generally, if $U \subset X$ is an open subset, then $Z \cap U \subset U$ is $c|_U$-invariant if and only if $F(c, Z) \cap c_1^{-1}(U) \cap c_2^{-1}(U) = \emptyset$.

(b) For every $S \in \mathrm{Irr}(F(c,Z))$, we have $c_2(S) \subset Z$, hence $\overline{c_2(S)} \subset Z$. Therefore $G(c,Z)$ is contained in Z.

(c) Note that if $Z_1, Z_2 \subset X$ are two closed locally c-invariant subsets, then the union $Z_1 \cup Z_2$ is also locally c-invariant.

(d) Note that if $Z \subset X$ is locally c-invariant, then $Z \cap U \subset U$ is locally $c|_U$-invariant for every open $U \subset X$.

Lemma 1.6. *Let $c : Y \to X \times X$ be a correspondence, and $Z \subset X$ a closed subset. Then $X \smallsetminus G(c,Z) \subset X$ is the largest open subset $U \subset X$ such that $Z \cap U \subset U$ is locally $c|_U$-invariant.*

Proof. Let $U \subset X$ be an open subset. By Remark 1.5(a), $Z \cap U \subset U$ is $c|_U$-invariant if and only if $S \cap c_1^{-1}(U) \cap c_2^{-1}(U) = \emptyset$ for every $S \in \mathrm{Irr}(F(c,Z))$. Since S is irreducible, this happens if and only if either $S \cap c_1^{-1}(U) = \emptyset$ or $S \cap c_2^{-1}(U) = \emptyset$. But the condition $S \cap c_i^{-1}(U) = \emptyset$ is equivalent to $c_i(S) \cap U = \emptyset$ and hence to $U \subset X \smallsetminus \overline{c_i(S)}$.

By the proven above, a point $x \in X$ has an open neighborhood $U \subset X$ such that $Z \cap U \subset U$ is $c|_U$-invariant if and only if for every $S \in \mathrm{Irr}(F(c,Z))$, we have $x \notin \overline{c_1(S)}$ or $x \notin \overline{c_2(S)}$. Therefore this happens if and only if x does not belong to $\bigcup_{S \in \mathrm{Irr}(F(c,Z))} [\overline{c_1(S)} \cap \overline{c_2(S)}] = G(c,Z)$. $\square$

Corollary 1.7. *Let $c : Y \to X \times X$ be a correspondence.*

(a) A closed subset $Z \subset X$ is locally c-invariant if and only if $G(c,Z) = \emptyset$.

(b) For two closed subsets $Z_1, Z_2 \subset X$, we have

$$G(c, Z_1 \cup Z_2) \subset G(c, Z_1) \cup G(c, Z_2).$$

(c) If c_2 is quasi-finite, then $\dim G(c,Z) \leq \dim Z - 1$, where we set $\dim \emptyset$ to be $-\infty$.

Proof. (a) follows immediately from the lemma.

(b) Set $U_i := X \smallsetminus G(c, Z_i)$ and $U := U_1 \cap U_2$. Then every $Z_i \cap U_i$ is contained in U_i and is locally $c|_{U_i}$-invariant by the lemma; hence $(Z_1 \cup Z_2) \cap U \subset U$ is locally $c|_U$-invariant by Remarks 1.5(c), (d). Hence $G(c, Z_1 \cup Z_2) \subset X \smallsetminus U = G(c, Z_1) \cup G(c, Z_2)$ by the lemma.

(c) We may assume that Z is non-empty and irreducible (using (b)). We want to show that for every $S \in \mathrm{Irr}(F(c,Z))$, we have $\dim(\overline{c_1(S)} \cap \overline{c_2(S)}) < \dim Z$. Since $S \subset c_2^{-1}(Z)$ (see Remark 1.5(b)) and c_2 is quasi-finite, we have $\dim S \leq \dim Z$. Since $c_1(S) \cap \overline{c_2(S)} \subset (X \smallsetminus Z) \cap Z = \emptyset$ (see Remark 1.5(b)), we conclude that $\overline{c_1(S)} \cap \overline{c_2(S)} \subset \overline{c_1(S)} \smallsetminus c_1(S)$, thus $\dim(\overline{c_1(S)} \cap \overline{c_2(S)}) < \dim S \leq \dim Z$. $\square$

Definition 1.8. Let $c : C \to X \times X$ be a correspondence, and let $Z \subset X$ be a closed subset or, what is the same, a closed reduced subscheme.

(a) Denote by $\mathcal{I}_Z \subset \mathcal{O}_X$ the sheaf of ideals of Z, and let $c_i^{\cdot}(\mathcal{I}_Z) \subset \mathcal{O}_C$ be its inverse image (as a sheaf of sets).

(b) As in [Va], we say that c is *contracting near* Z if $c_1^{\cdot}(\mathcal{I}_Z) \subset c_2^{\cdot}(\mathcal{I}_Z) \cdot \mathcal{O}_C$ and there exists $n > 0$ such that $c_1^{\cdot}(\mathcal{I}_Z)^n \subset c_2^{\cdot}(\mathcal{I}_Z)^{n+1} \cdot \mathcal{O}_C$.

(c) We say that c is *locally contracting near* Z if for every $x \in X$ there exists an open neighborhood $U \subset X$ of x such that $c|_U$ is contracting near $Z \cap U$.

1.9. Correspondences over finite fields. Let $c : C \to X \times X$ be a correspondence over $\mathbb{F}$ such that X is defined over $\mathbb{F}_q$, and let $Z \subset X$ be a closed subset.

(a) For $n \in \mathbb{N}$, we denote by $c^{(n)} : C \to X \times X$ the correspondence $(\phi_{q^n} \circ c_1, c_2)$.

(b) We say that Z is *locally c-invariant over* $\mathbb{F}_q$ if the open neighborhood U of x from Definition 1.2(b) can be chosen to be defined over $\mathbb{F}_q$.

(c) We say that c is *locally contracting near Z over* $\mathbb{F}_q$ if the open neighborhood U from Definition 1.8(c) can be chosen to be defined over $\mathbb{F}_q$.

1.10. Remark. If Z is defined over $\mathbb{F}_q$ and locally c-invariant over $\mathbb{F}_q$ (see 1.9(b)), then Z is locally $c^{(n)}$-invariant over $\mathbb{F}_q$ for every $n \in \mathbb{N}$. Indeed, we immediately reduce to the case when Z is c-invariant. Then $\phi_{q^n}(Z) \subset Z$; thus Z is $c^{(n)}$-invariant.

1.11. The ramification degree. Let $f : Y \to X$ be a morphism of Noetherian schemes, and let Z be a closed subset of X. Let $f^{-1}(Z) \subset Y$ be the schematic preimage of Z, and let $\mathrm{ram}(f, Z)$ be the smallest positive integer m such that $\sqrt{\mathcal{I}_{f^{-1}(Z)}}^m \subset \mathcal{I}_{f^{-1}(Z)}$.

The following lemma and its proof are copied from [Va, Lem. 2.2.3].

Lemma 1.12. *In the situation of* 1.9, *assume that* $q^n > \mathrm{ram}(c_2, Z)$ *and* Z *is* $c^{(n)}$*-invariant. Then the correspondence* $c^{(n)}$ *is contracting near* Z.

Proof. Set $m := \mathrm{ram}(c_2, Z)$, and let φ_{q^n} be the inverse of the arithmetic Frobenius isomorphism $X \xrightarrow{\sim} X$ over $\mathbb{F}_{q^n}$. Then for every section f of $\mathcal{O}_X$, we have $\phi_{q^n}^{\cdot}(f) = (\varphi_{q^n})^{\cdot}(f)^{q^n}$. Therefore $(c_1^{(n)})^{\cdot}(\mathcal{I}_Z)$ equals $c_1^{\cdot}(\varphi_{q^n})^{\cdot}(\mathcal{I}_Z)^{q^n}$.

Since Z is $c^{(n)}$-invariant, we get an inclusion $\mathcal{I}_{(c_1^{(n)})^{-1}(Z)} \subset \sqrt{\mathcal{I}_{c_2^{-1}(Z)}}$. Hence $c_1^{\cdot}(\varphi_{q^n})^{\cdot}(\mathcal{I}_Z) \subset \sqrt{\mathcal{I}_{c_2^{-1}(Z)}}$; therefore $(c_1^{(n)})^{\cdot}(\mathcal{I}_Z) \subset \sqrt{\mathcal{I}_{c_2^{-1}(Z)}}^{q^n}$. As $q^n \geq m+1$, we conclude that $(c_1^{(n)})^{\cdot}(\mathcal{I}_Z) \subset \sqrt{\mathcal{I}_{c_2^{-1}(Z)}}^{m+1} \subset \mathcal{I}_{c_2^{-1}(Z)}$. Furthermore, $(c_1^{(n)})^{\cdot}(\mathcal{I}_Z)^m$ is contained in $\sqrt{\mathcal{I}_{c_2^{-1}(Z)}}^{m(m+1)} \subset (\mathcal{I}_{c_2^{-1}(Z)})^{m+1}$. Hence $c^{(n)}$ is contracting near Z, as claimed. □

Corollary 1.13. *In the situation of* 1.9, *assume that* $q^n > \mathrm{ram}(c_2, Z)$ *and* Z *is locally* $c^{(n)}$*-invariant over* $\mathbb{F}_q$. *Then the correspondence* $c^{(n)}$ *is locally contracting near* Z *over* $\mathbb{F}_q$.

Proof. For every open subset $U \subset X$ we have $\mathrm{ram}((c|_U)_2, Z \cap U) \leq \mathrm{ram}(c_2, Z)$. Thus the assertion follows from Lemma 1.12. □

2. Main technical result

2.1. Set up. Let $c : C \to X \times X$ be a correspondence over k, and let $X^0 \subset X$ be a non-empty open subset such that X and C are irreducible, c_2 is dominant, and $\dim C = \dim X$.

Lemma 2.2. *In the situation of* 2.1, *there exist non-empty open subsets* $V \subset U \subset X^0$ *such that*

(i) $c_1^{-1}(V) \subset c_2^{-1}(U)$;

(ii) the closed subset $U \smallsetminus V \subset U$ *is locally* $c|_U$*-invariant.*

Proof. Since $\dim C = \dim X$ and c_2 is dominant, there exists a non-empty open subset $U_0 \subset X^0$ such that $c_2|_{c_2^{-1}(U_0)}$ is quasi-finite. By induction, we define for every $j \geq 0$ open subsets $V_j \subset U_j \subset X^0$ by the rules

$$V_j := U_j \smallsetminus \overline{c_1(c_2^{-1}(X \smallsetminus U_j))}, \tag{2.1}$$

$$Z_j := G(c|_{U_j}, U_j \smallsetminus V_j), \text{ and } U_{j+1} := U_j \smallsetminus Z_j \subset U_j. \tag{2.2}$$

First we claim that U_j and V_j are non-empty. Indeed, $U_0 \neq \emptyset$ by construction, and if $U_j \neq \emptyset$, then $c_2^{-1}(X \smallsetminus U_j) \neq C$ since c_2 is dominant; hence $\dim c_2^{-1}(X \smallsetminus U_j) < \dim C$, since C is irreducible. Thus

$$\dim \overline{c_1(c_2^{-1}(X \smallsetminus U_j))} \leq \dim c_2^{-1}(X \smallsetminus U_j) < \dim C = \dim X = \dim U_j,$$

hence $V_j \neq \emptyset$. Finally, $Z_j \subset U_j \smallsetminus V_j$ (see Remark 1.5(b)), thus $U_{j+1} \supset V_j$, hence $U_{j+1} \neq \emptyset$.

We claim that for every $j \geq 0$, we have

(i)$'$ $c_1^{-1}(V_j) \subset c_2^{-1}(U_j)$, and

(ii)$'$ $U_{j+1} \smallsetminus V_j = U_{j+1} \cap (U_j \smallsetminus V_j)$ is locally $c|_{U_{j+1}}$-invariant.

Indeed, (i)$'$ is equivalent to the equality $c_1^{-1}(V_j) \cap c_2^{-1}(X \smallsetminus U_j) = \emptyset$, hence to the equality $V_j \cap c_1(c_2^{-1}(X \smallsetminus U_j)) = \emptyset$, so (i)$'$ follows from (2.1). Next, (ii)$'$ follows from Lemma 1.6.

We will show that $Z_j = \emptyset$ for some j. In this case, $U := U_j = U_{j+1}$ and $V := V_j$ satisfy the properties of the lemma. Indeed, properties (i) and (ii) would follow from (i)$'$ and (ii)$'$, respectively.

Since $U_{j+1} \subset U_j$, formula (2.1) implies that $V_{j+1} \subset V_j$. It suffices to show that for every j we have inequalities

$$\dim Z_{j+1} + 1 \leq \dim(V_j \smallsetminus V_{j+1}) \leq \dim Z_j. \tag{2.3}$$

Let $\overline{V_j \smallsetminus V_{j+1}} \subset U_{j+1}$ be the closure of $V_j \smallsetminus V_{j+1}$. Then $U_{j+1} \smallsetminus V_{j+1}$ equals $(U_{j+1} \smallsetminus V_j) \cup (V_j \smallsetminus V_{j+1}) = (U_{j+1} \smallsetminus V_j) \cup \overline{V_j \smallsetminus V_{j+1}}$; hence we conclude from Corollary 1.7(b) that $Z_{j+1} = G(c|_{U_{j+1}}, U_{j+1} \smallsetminus V_{j+1})$ is contained in

$$G(c|_{U_{j+1}}, \overline{V_j \smallsetminus V_{j+1}}) \cup G(c|_{U_{j+1}}, U_{j+1} \smallsetminus V_j).$$

Using (ii)′ and Corollary 1.7(a), we conclude that $G(c|_{U_{j+1}}, U_{j+1} \smallsetminus V_j)$ is empty, thus $Z_{j+1} \subset G(c|_{U_{j+1}}, \overline{V_j \smallsetminus V_{j+1}})$. Therefore, by Corollary 1.7(c), we get inequalities $\dim Z_{j+1}+1 \leq \dim G(c|_{U_{j+1}}, \overline{V_j \smallsetminus V_{j+1}})+1 \leq \dim \overline{V_j \smallsetminus V_{j+1}} = \dim(V_j \smallsetminus V_{j+1})$.

Next, since $X \smallsetminus U_{j+1} = (X \smallsetminus U_j) \cup Z_j$, it follows from (2.1) and (2.2) that $V_{j+1} = V_j \smallsetminus [Z_j \cup \overline{c_1(c_2^{-1}(Z_j))}]$, thus $V_{j+1} \smallsetminus V_j \subset Z_j \cup \overline{c_1(c_2^{-1}(Z_j))}$.

Finally, since $Z_j \subset U_j \subset U_0$, we conclude that $c_2|_{c_2^{-1}(Z_j)}$ is quasi-finite, hence $\dim \overline{c_1(c_2^{-1}(Z_j))} \leq \dim c_2^{-1}(Z_j) \leq \dim Z_j$. Therefore $\dim(V_j \smallsetminus V_{j+1}) \leq \dim Z_j$, and the proof of (2.3) is complete. □

Proposition 2.3. *In the situation of* 2.1*, there exists a non-empty open subset $V \subset X^0$ and a blowup $\pi : \widetilde{X} \to X$, which is an isomorphism over V, such that for every correspondence $\widetilde{c} : \widetilde{C} \to \widetilde{X} \times \widetilde{X}$ lifting c, the closed subset $\widetilde{X} \smallsetminus \pi^{-1}(V) \subset \widetilde{X}$ is locally $\widetilde{c}$-invariant.*

Proof. The argument goes similarly to that of [Va, Lem. 1.5.4]. Let $V \subset U \subset X^0$ be as in Lemma 2.2. Set $F := F(c, X \smallsetminus V) = c_2^{-1}(X \smallsetminus V) \cap c_1^{-1}(V)$. For every $S \in \mathrm{Irr}(F)$, we denote by $\mathcal{K}_S$ the sheaf of ideals $\mathcal{I}_{\overline{c_1(Z)}} + \mathcal{I}_{\overline{c_2(Z)}} \subset \mathcal{O}_X$ of the schematic intersection $\overline{c_1(S)} \cap \overline{c_2(S)}$, and set $\mathcal{K} := \prod_{S \in \mathrm{Irr}(F)} \mathcal{K}_S \subset \mathcal{O}_X$. Let $\widetilde{X}$ be the blowup $\mathrm{Bl}_{\mathcal{K}}(X)$, and denote by $\pi : \widetilde{X} \to X$ the canonical projection.

We claim that V and π satisfy the required properties. Notice that the support of $\mathcal{O}_X/\mathcal{K}$ equals $\cup_S(\overline{c_1(S)} \cap \overline{c_2(S)}) = G(c, X \smallsetminus V)$. Since $U \smallsetminus V$ is locally $c|_U$-invariant, $G(c, X \smallsetminus V)$ is therefore contained in $X \smallsetminus U$ (by Lemma 1.6). In particular, π is an isomorphism over U, hence over $V \subset U$.

Next, we show that for all $S \in \mathrm{Irr}(F)$ we have $\overline{\pi^{-1}(c_1(S))} \cap \overline{\pi^{-1}(c_2(S))} = \emptyset$. By the definition of $\pi : \widetilde{X} \to X$, the strict preimages of $\overline{c_1(S)}$ and $\overline{c_2(S)}$ in $\widetilde{X}$ do not intersect. Thus it suffices to show that every $\overline{\pi^{-1}(c_i(S))}$ is a strict preimage of $\overline{c_i(S)}$. Since π is an isomorphism over U, it suffices to show that both $c_1(S)$ and $c_2(S)$ are contained in U. But S is contained in $F \subset c_1^{-1}(V) \subset c_2^{-1}(U)$ (by Lemma 2.2(i)), hence $c_1(S) \subset V \subset U$ and $c_2(S) \subset U$.

Now we are ready to show the assertion. Let $\widetilde{c} : \widetilde{C} \to \widetilde{X} \times \widetilde{X}$ be any correspondence lifting c, and let π_C be the corresponding morphism $\widetilde{C} \to C$. We claim that $\widetilde{Z} := \widetilde{X} \smallsetminus \pi^{-1}(V) = \pi^{-1}(X \smallsetminus V)$ is locally $\widetilde{c}$-invariant. Set $\widetilde{F} := F(\widetilde{c}, \widetilde{Z})$ and fix $\widetilde{S} \in \mathrm{Irr}(\widetilde{F})$. We want to show that $\overline{\widetilde{c}_1(\widetilde{S})} \cap \overline{\widetilde{c}_1(\widetilde{S})} = \emptyset$

(use Corollary 1.7(a)). Observe that $\widetilde{F}$ equals

$$\widetilde{c}_2^{-1}(\pi^{-1}(X \smallsetminus V)) \cap \widetilde{c}_1^{-1}(\pi^{-1}(V)) = \pi_C^{-1}(c_2^{-1}(X \smallsetminus V) \cap c_1^{-1}(V)) = \pi_C^{-1}(F).$$

Therefore $\pi_C(\widetilde{F})$ is contained in F. Hence there exists $S \in \mathrm{Irr}(F)$ such that $\pi_C(\widetilde{S}) \subset S$. Then for every $i = 1, 2$, we have $\pi(\overline{\widetilde{c}_i(\widetilde{S})}) = \overline{c_i(\pi_C(\widetilde{S}))} \subset c_i(S)$. Thus $\widetilde{c}_i(\widetilde{S}) \subset \pi^{-1}(c_i(S))$. Hence the intersection $\overline{\widetilde{c}_1(\widetilde{S})} \cap \overline{\widetilde{c}_1(\widetilde{S})}$ is contained in $\overline{\pi^{-1}(c_1(S))} \cap \overline{\pi^{-1}(c_2(S))} = \emptyset$. Therefore $\overline{\widetilde{c}_1(\widetilde{S})} \cap \overline{\widetilde{c}_1(\widetilde{S})} = \emptyset$, as claimed. □

For the applications, we will need the following version of Proposition 2.3.

Corollary 2.4. *In the situation of* 2.1, *assume that* $k = \mathbb{F}$ *and that* X *and* X^0 *are defined over* $\mathbb{F}_q$.

Then there exists an open subset $V \subset X^0$ *and a blowup* $\pi : \widetilde{X} \to X$ *which is an isomorphism over* V *such that both* V *and* π *are defined over* $\mathbb{F}_q$, *and for every map* $\widetilde{c} : \widetilde{C} \to \widetilde{X} \times \widetilde{X}$ *lifting* c *the closed subset* $\widetilde{X} \smallsetminus \pi^{-1}(V) \subset \widetilde{X}$ *is locally* $\widetilde{c}$*-invariant over* $\mathbb{F}_q$.

Proof. By assumption, there exists a scheme of finite type $\underline{X}$ over $\mathbb{F}_q$ and an open subscheme $\underline{X}^0$ of $\underline{X}$, whose base change to $\mathbb{F}$ are X and X^0, respectively. Let $\omega : X \to \underline{X}$ be the canonical morphism.

Choose $r \in \mathbb{N}$ such that C and $c : C \to X \times X$ are defined over $\mathbb{F}_{q^r}$. Then there exists $r \in \mathbb{N}$ and a scheme of finite type $\underline{C}$ over $\mathbb{F}_{q^r}$, whose base change to $\mathbb{F}$ is C, such that the composition $(\omega \times \omega) \circ c : C \to \underline{X} \times \underline{X}$ factors through $\underline{c} : \underline{C} \to \underline{X} \times \underline{X}$. Then $\underline{c}$ satisfies all the assumptions of 2.1 for $k = \mathbb{F}_q$.

Let $\underline{V} \subset \underline{X}^0$ and $\underline{\pi} : \underline{\widetilde{X}} \to \underline{X}$ be an open subset and a blowup from Proposition 2.3, respectively, and let $V \subset X^0$ and $\pi : \widetilde{X} \to X$ be their base changes to $\mathbb{F}$. Then V and π satisfy the required properties.

Indeed, since $\widetilde{c}$ lifts c, the composition $\widetilde{d} := (\omega \times \omega) \circ \widetilde{c} : \widetilde{C} \to \underline{\widetilde{X}} \times \underline{\widetilde{X}}$ lifts $\underline{c}$. Hence, by the assumption on $\underline{V}$ and $\underline{\pi}$, the closed subset $\underline{\widetilde{X}} \smallsetminus \underline{p}^{-1}(\underline{V}) \subset \underline{\widetilde{X}}$ is locally $\widetilde{d}$-invariant. Finally, since $\widetilde{c}$ is a lift of $\widetilde{d}$, we conclude as in Lemma 1.3 that $\widetilde{X} \smallsetminus \pi^{-1}(V) \subset \widetilde{X}$ is locally $\widetilde{c}$-invariant over $\mathbb{F}_q$. □

3. Geometric construction of Pink [Pi]

3.1. The construction.

(a) Let X be a smooth scheme of relative dimension d over a field k, and let $X_i \subset X$, $i \in I$ be a finite collection of smooth divisors with normal crossings. We set $\partial X := \bigcup_{i \in I} X_i \subset X$ and $X^0 := X \smallsetminus \partial X$.

(b) For every $J \subset I$, we set $X_J := \bigcap_{i \in J} X_i$. In particular, $X_\emptyset = X$. Then every X_J is either empty or smooth over k of relative dimension $d - |J|$. We also set $\partial X_J := \cup_{i \in I \smallsetminus J} X_{J \cup \{i\}}$, and $X_J^0 := X_J \smallsetminus \partial X_J$.

(c) Set $Y := X \times X$, $Y_i := X_i \times X_i$, $i \in I$, $\partial Y := \cup_{i\in I} Y_i$ and $Y^0 := Y \smallsetminus \partial Y$. We denote by $\mathcal{K}_i := \mathcal{I}_{Y_i} \subset \mathcal{O}_Y$ the sheaf of ideals of Y_i, set $\mathcal{K} := \prod_{i\in I} \mathcal{K}_i$, let $\widetilde{Y}$ be the blowup $\mathrm{Bl}_{\mathcal{K}}(Y)$, and let $\pi : \widetilde{Y} \to Y$ be the projection map. Then π is an isomorphism over Y^0.

(d) For every $J \subset I$, we set $Y_J := X_J \times X_J \subset Y$ and $E_J := \pi^{-1}(Y_J) \subset \widetilde{Y}$, denote by i_J the inclusion $E_J \hookrightarrow \widetilde{Y}$, and by π_J the projection $E_J \to Y_J$. We also set $\partial Y_J := \cup_{i\in I\smallsetminus J} Y_{J\cup\{i\}}$ and $Y_J^0 := Y_J \smallsetminus \partial Y_J$. Explicitly, a point $y \in Y$ belongs to Y_J^0 if and only if $y \in Y_j$ for every $j \in J$, and $y \notin Y_j$ for every $j \in I \smallsetminus J$.

3.2. Basic case. (a) Assume that $X = \mathbb{A}^I$ with coordinates $\{x_i\}_{i\in I}$, and let $X_i = Z(x_i) \subset X$ (the zero scheme of x_i) for all $i \in I$.

(b) The product $Y = X \times X$ is the affine space $(\mathbb{A}^2)^I$ with coordinates $x_i, y_i, i \in I$, and $\widetilde{Y} = (\widetilde{\mathbb{A}^2})^m$, where $\widetilde{\mathbb{A}^2} := \mathrm{Bl}_{(0,0)}(\mathbb{A}^2)$. Explicitly, $\widetilde{Y}$ is a closed subscheme of the product $(\mathbb{A}^2 \times \mathbb{P}^1)^I$ with coordinates $(x_i, y_i, (a_i : b_i))_{i\in I}$ given by equations $x_i b_i = y_i a_i$.

(c) For every $J \subset I$, the subschemes $Y_J \subset Y$ and $E_J \subset \widetilde{Y}$ are given by equations $x_j = y_j = 0$ for all $j \in J$. Thus we have natural isomorphisms $Y_J \cong (\mathbb{A}^2)^{I\smallsetminus J}$, $E_J \cong (\widetilde{\mathbb{A}^2})^{I-J} \times (\mathbb{P}^1)^J$, and $\pi_J : E_J \to Y_J$ is the projection $(\widetilde{\mathbb{A}^2})^{I\smallsetminus J} \times (\mathbb{P}^1)^J \to (\widetilde{\mathbb{A}^2})^{I\smallsetminus J} \to (\mathbb{A}^2)^{I\smallsetminus J}$.

3.3. Local coordinates. Suppose that we are in the situation of 3.1.

(a) For every $J \subset I$, we set $\mathcal{K}_J := \prod_{j\in J} \mathcal{K}_j$, and denote by $\widetilde{Y}_J$ the blowup $\mathrm{Bl}_{\mathcal{K}_J}(Y)$. Then we have a natural projection $\widetilde{Y} \to \widetilde{Y}_J$, which is an isomorphism over $Y \smallsetminus (\cup_{i\in I\smallsetminus J} Y_i)$.

(b) Let $a \in X_I \subset X$ be a closed point. Then there exists an open neighborhood $U \subset X$ of a and regular functions $\{\psi_i\}_{i\in I}$ on U such that $\psi = (\psi_i)_{i\in I}$ is a smooth morphism $U \to \mathbb{A}^I$, and $X_i \cap U$ is the scheme of zeros $Z(\psi_i)$ of ψ_i for every $i \in I$. Then $X_i \cap U$ is the schematic preimage $\psi^{-1}(Z(x_i))$ for all $i \in I$.

(c) Let a and b be two closed points of X_I, and let $\psi_a : U_a \to \mathbb{A}^I$ and $\psi_b : U_b \to \mathbb{A}^I$ be two smooth morphisms as in (b). Then $\psi := (\psi_a, \psi_b)$ is a smooth morphism $U := U_a \times U_b \to (\mathbb{A}^2)^I$, which induces an isomorphism $\widetilde{Y} \times_Y U \to (\widetilde{\mathbb{A}^2})^I \times_{(\mathbb{A}^2)^I} U$.

Lemma 3.4. *In the situation of* 3.1, *for every $J \subset I$ the closed subscheme $E_J \subset \widetilde{Y}$ is smooth of dimension $2d - |J|$.*

Proof. In the basic case 3.2, the assertion follows from the explicit description in 3.2(c) and the observation that $\widetilde{\mathbb{A}}^2$ is smooth of dimension two. Since the assertion is local on Y, the general case follows from this and 3.3. Namely, it suffices to show that for every closed point $a \in Y$ there exists an open neighborhood U such that $\pi^{-1}(U) \cap E_J$ is smooth of dimension $2d - |J|$.

Choose $J \subset I$ such that $a \in Y^0_J$. Then $a \in Y \smallsetminus (\bigcup_{i \in I \smallsetminus J} Y_i)$; thus it follows from 3.3(a) that we can replace I by J, thus assuming that $a \in Y_I = X_I \times X_I$. In this case, by 3.3(c), there exists an open neighborhood $U \subset Y$ of a and a smooth morphism $U \to (\mathbb{A}^2)^I$, which induces an isomorphism $\widetilde{Y} \times_Y U \to (\widetilde{\mathbb{A}^2})^I \times_{(\mathbb{A}^2)^I} U$. Thus the assertion in general follows from the basic case. $\square$

3.5. Notation. For every morphism $c : C \to Y$, we denote by $\widetilde{c} : \widetilde{C} \to \widetilde{Y}$ the *strict preimage* of c. Explicitly, $\widetilde{C}$ is the schematic closure of $c^{-1}(Y^0) \times_Y \widetilde{Y}$ in $C \times_Y \widetilde{Y}$. Notice that $\widetilde{c}$ is a closed embedding (resp. finite) if c is such.

3.6. Set-up. In the situation of 3.1, assume that $k = \mathbb{F}$ and that X and all the X_i's are defined over $\mathbb{F}_q$.

Lemma 3.7. *In the situation of* 3.6*, assume that the correspondence $c : C \to X \times X$ is locally contracting near $Z := \partial X$ over $\mathbb{F}_q$. Let $\Gamma \subset X \times X$ be the graph of ϕ_q, and let $\widetilde{\Gamma} \subset \widetilde{Y}$ be the strict preimage of Γ. Then*

$$\widetilde{c}(\widetilde{C}) \cap \widetilde{\Gamma} \subset \pi^{-1}(X^0 \times X^0).$$

Proof. Choose a closed point $\widetilde{y} \in \widetilde{c}(\widetilde{C}) \cap \widetilde{\Gamma}$, and set $y = (y_1, y_2) := \pi(\widetilde{y}) \in \Gamma$. Then $y_2 = \phi_{q^n}(y_1)$, and it remains to show that $y_1 \in X^0$.

By assumption (see 1.9(c)), there exists an open neighborhood U of y_1 defined over $\mathbb{F}_q$ such that $c|_U$ is contracting near $Z \cap U$. Then $y_2 = \phi_{q^n}(y_1)$ belongs to U, so we can replace c by $c|_U$, thus assuming that the correspondence c is contracting near Z.

Assume that $y_1 \notin X^0$. Arguing as in the proof of Lemma 3.4 and using 3.3(a), we can assume that $y_1 \in X_I$, thus also $y_2 = \phi_{q^n}(y_1) \in X_I$. By 3.3(b), there exists an open neighborhood U of y_1 and a smooth morphism $\psi : U \to \mathbb{A}^I$ such that $\psi^{-1}(Z(x_i)) = X_i$ for every $i \in I$. Moreover, we can assume that U and ψ are defined over $\mathbb{F}_q$. Again $y_2 = \phi_{q^n}(y_1) \in U$; thus replacing c by $c|_U$ we can assume that $U = X$.

Consider correspondence $d := (\psi, \psi) \circ c : C \to \mathbb{A}^I \times \mathbb{A}^I$. Then equalities $\psi^{-1}(Z(x_i)) = X_i$ imply that d is contracting near $\partial \mathbb{A}^I := \cup_{i \in I} Z(x_i)$. Then the assertion for c, X and X_i follows from the corresponding assertion for $d, \mathbb{A}^I$ and $Z(x_i)$. In other words, we can assume that we are in the basic case 3.2.

We denote by $\widetilde{Y}' \subset \widetilde{Y}$ the open subscheme, given by inequalities $a_i \neq 0$ for all i. The assertion now follows from the part (a) of the following claim. $\square$

Claim 3.8. (a) We have inclusions $\widetilde{\Gamma} \subset \widetilde{Y}'$, and $\widetilde{c}(\widetilde{C}) \cap \widetilde{Y}' \subset \pi^{-1}(X^0 \times X^0)$.

(b) The projection $\widetilde{\Gamma} \to \Gamma$, induced by π, is an isomorphism.

Proof. Notice that every $\overline{b}_i := b_i/a_i$ is a regular function on $\widetilde{Y}'$ and that $\widetilde{Y}'$ is a closed subscheme of the affine space $(\mathbb{A}^3)^I$ with coordinates $x_i, y_i, \overline{b}_i$, given by equations $y_i = x_i \overline{b}_i$.

Consider the closed subscheme $\widetilde{\Gamma}' \subset \widetilde{Y}'$ given by equations $y_i = x_i^q$ and $\overline{b}_i = x_i^{q-1}$. Then $\pi : \widetilde{Y} \to Y$ induces an isomorphism $\widetilde{\Gamma}' \to \Gamma$. In particular, the projection $\widetilde{\Gamma}' \to \Gamma$ is proper. Since Γ is closed in Y and π is proper, this implies that $\widetilde{\Gamma}'$ is closed in $\widetilde{Y}$ and therefore $\widetilde{\Gamma}'$ has to coincide with $\widetilde{\Gamma}$. This implies assertion (b) and proves the inclusion $\widetilde{\Gamma} \subset \widetilde{Y}'$.

Set $\widetilde{C}' := \widetilde{c}^{-1}(\widetilde{Y}') \subset \widetilde{C}$. We have to show that $\widetilde{C}' \subset \widetilde{c}^{-1}(\pi^{-1}(X^0 \times X^0))$. Consider regular functions $x := \prod_{i\in I} x_i, y := \prod_{i\in I} y_i$ and $\overline{b} := \prod_{i\in I} \overline{b}_i$ on $\widetilde{Y}'$. It remains to show that both pullbacks $\widetilde{c}^{\cdot}(x)$ and $\widetilde{c}^{\cdot}(y)$ to $\widetilde{C}'$ are invertible.

Since c is contracting near ∂X, there exist $n > 0$ and a regular function f on C such that $c^{\cdot}(x)^n = c^{\cdot}(y)^{n+1} \cdot f$. Taking pullbacks to $\widetilde{C}'$, we get the equality

$$\widetilde{c}^{\cdot}(x)^n = \widetilde{c}^{\cdot}(y)^{n+1} \cdot \widetilde{f}, \tag{3.1}$$

so it remains to show that $\widetilde{c}^{\cdot}(x)$ is invertible on $\widetilde{C}'$. Since on $\widetilde{Y}'$ we have the equality $y = x \cdot \overline{b}$, equation (3.1) can be rewritten as

$$\widetilde{c}^{\cdot}(x)^n = \widetilde{c}^{\cdot}(x)^{n+1} \cdot \widetilde{c}^{\cdot}(\overline{b})^{n+1} \cdot \widetilde{f}. \tag{3.2}$$

We claim that $\widetilde{c}^{\cdot}(x) \cdot \widetilde{c}^{\cdot}(\overline{b})^{n+1} \cdot \widetilde{f} = 1$ on $\widetilde{C}'$, which obviously implies that $\widetilde{c}^{\cdot}(x)$ is invertible.

Since $\widetilde{C}$ is the closure of $c^{-1}(Y^0) \times_Y \widetilde{Y} = C \times_Y (\widetilde{Y} \times_Y Y^0)$, we conclude that $\widetilde{C}'$ is the closure of $C \times_Y (\widetilde{Y}' \times_Y Y^0) \subset C \times_Y \widetilde{Y}'$. Thus it suffices to show that $\widetilde{c}^{\cdot}(x) \cdot \widetilde{c}^{\cdot}(\overline{b}^{n+1}) \cdot \widetilde{f} = 1$ on $C \times_Y (\widetilde{Y}' \times_Y Y^0)$. By (3.2), it is enough to show that x is invertible on $\widetilde{Y}' \times_Y Y^0$, that is, $x_i(a) \neq 0$ for every closed point $a \in \widetilde{Y}' \times_Y Y^0$ and every $i \in I$. Since $y_i = x_i \overline{b}_i$, it suffices to show that for every $i \in I$ we have either $x_i(a) \neq 0$ or $y_i(a) \neq 0$, but this follows from the definition of Y^0. □

Lemma 3.9. *In the situation of* 3.6, *let $J \subset I$, and let $\Gamma_J \subset X_J \times X_J$ (resp. $\Gamma^0_J \subset X^0_J \times X^0_J$) be the graph of ϕ_q.*

Then the schematic closure $\overline{\pi_J^{-1}(\Gamma^0_J)} \subset E_J$ is smooth of dimension d, and the schematic preimage $\pi_J^{-1}(\Gamma_J) \subset E_J$ is the schematic union of the $\overline{\pi_{J'}^{-1}(\Gamma_{J'})}$'s with $J' \supset J$.

Proof. Assume first that we are in the basic case. By 3.2(c), we immediately reduce to the case $J = \emptyset$. In this case, $\overline{\pi_J^{-1}(\Gamma^0)} = \widetilde{\Gamma}$ is isomorphic to Γ (by Claim 3.8(b)), thus it is smooth of dimension d. This shows the first assertion.

Next, the second assertion immediately reduces to the case $m = 1$, in which it is easy. Indeed, $\Gamma \subset \mathbb{A}^2$ is given by equation $y = x^q$; thus $\pi^{-1}(\Gamma) \subset \mathbb{A}^2 \times \mathbb{P}^1$ is given by equations $y = x^q$ and $xb = x^q a$. Thus $\pi^{-1}(\Gamma)$ equals the schematic union of $\widetilde{\Gamma}$, given by $y = x^q$ and $b = x^{q-1}a$, and the exceptional divisor $x = y = 0$.

Finally, arguing as in Lemmas 3.4 and 3.7, we reduce the general case to the basic case. Namely, replacing X by its open subset, we may assume that there exists a smooth map $\psi : X \to \mathbb{A}^I$ defined over $\mathbb{F}_q$ such that $\psi^{-1}(Z(x_i)) = X_i$ for every $i \in I$. Then ψ induces an isomorphism $\widetilde{Y} \xrightarrow{\sim} Y \times_{(\mathbb{A}^2)^I} (\widetilde{\mathbb{A}^2})^I$. Moreover, if we denote by $(\cdot)_{\mathbb{A}^I}$ the objects corresponding to $\mathbb{A}^I$ instead of X, then ψ induces a smooth morphism $\Gamma_J \to (\Gamma_J)_{\mathbb{A}^I}$, hence smooth morphisms $\pi_J^{-1}(\Gamma_J) \to \pi_J^{-1}(\Gamma_J)_{\mathbb{A}^I}$ and $\pi_J^{-1}(\Gamma_J^0) \to \pi_J^{-1}(\Gamma_J^0)_{\mathbb{A}^I}$. Thus both assertions follow from the basic case. $\square$

4. Formula for the intersection number

4.1. Intersection theory. (a) Let X be a scheme of finite type over a field k. For $i \in \mathbb{N}$, we denote by $A_i(X)$ the group of i-cycles modulo rational equivalence (see [Fu, 1.3]). For a closed subscheme $Z \subset X$ of pure dimension i, we denote by $[Z]$ its class in $A_i(X)$ (see [Fu, 1.5]). For a proper morphism of schemes $f : X \to Y$, we denote by $f_* : A_i(X) \to A_i(Y)$ the induced morphism (see [Fu, 1.4]). In particular, if X is proper over k, then the projection $p_X : X \to \operatorname{Spec} k$ gives rise to the degree map $\deg := (p_X)_* : A_0(X) \to A_0(\operatorname{Spec} k) = \mathbb{Z}$.

(b) Let X be a smooth connected scheme over k of dimension d. For every i, we set $A^i(X) := A_{d-i}(X)$. For every i, j, we have the intersection product $\cap : A^i(X) \times A^j(X) \to A^{i+j}(X)$ (see [Fu, 8.3]). In particular, if X is also proper, we have the intersection product $\cdot := \deg \circ \cap : A_i(X) \times A^i(X) \to \mathbb{Z}$.

(c) For every morphism $f : X \to Y$ between smooth connected schemes, we have a pullback map $f^* : A^i(Y) \to A^i(X)$ (see [Fu, 8.1]). Moreover, if f is proper, then we have the equality $f^*(x) \cdot y = x \cdot f_*(y)$ for every $x \in A^i(Y)$ and $y \in A_i(X)$, called the projection formula (see [Fu, Prop. 8.3(c)]).

4.2. Example. Let $f : X \to Y$ be a morphism between smooth connected schemes, and let $C \subset Y$ be a closed subscheme such that both inclusions $i : C \hookrightarrow Y$ and $i' : f^{-1}(C) := C \times_Y X \hookrightarrow X$ are regular embeddings of codimension d. Then $f^*([C]) = [f^{-1}(C)]$.

Indeed, f is a composition $X \xrightarrow{(\mathrm{Id},f)} X \times Y \xrightarrow{\mathrm{pr}_2} Y$. Since the assertion for pr_2 is clear and (Id, f) is a regular embedding, we may assume that f is a regular embedding. In this case, the induced morphism $f^{-1}(C) \to C$ is a regular embedding as well, so the assertion follows, for example, from [Fu, Thm. 6.2 (a) and Rem. 6.2.1].

From now on we assume that k is an algebraically closed field and that l is a prime, different from the characteristic of k.

4.3. The cycle map. Let X be a d-dimensional smooth connected scheme over k.

(a) Recall (see [Gr]) that for every closed integral subscheme $C \subset X$ of codimension i one can associate its class $\mathrm{cl}(C) \in H^{2i}(X, \mathbb{Q}_l(i))$. Namely, by the Poincaré duality, $\mathrm{cl}(C)$ corresponds to the composition

$$H_c^{2(d-i)}(X, \mathbb{Q}_l(d-i)) \xrightarrow{\mathrm{res}_C} H_c^{2(d-i)}(C, \mathbb{Q}_l(d-i)) \xrightarrow{\mathrm{Tr}_{C/k}} \mathbb{Q}_l$$

of the restriction map and the trace map. In other words, $\mathrm{cl}(C) \in H^{2i}(X, \mathbb{Q}_l(i))$ is characterized by the condition that $\mathrm{Tr}_{X/k}(x \cup \mathrm{cl}(C)) = \mathrm{Tr}_{C/k}(\mathrm{res}_C^*(x))$ for every $x \in H_c^{2(d-i)}(X, \mathbb{Q}_l(d-i))$.

(b) Note that $\mathrm{cl}(C)$ only depends on the class $[C] \in A^i(X)$ (see, for example, [Lau, Thm. 6.3]); thus cl induces a map $A^i(X) \to H^{2i}(X, \mathbb{Q}_l(i))$. Furthermore, for every $x \in A^i(X)$ and $y \in A^j(X)$, we have an equality $\mathrm{cl}(x \cap y) = \mathrm{cl}(x) \cup \mathrm{cl}(y)$ (see [Lau, Cor. 7.2.1] when X is quasi-projective, which suffices for the purpose of this note, or use [KS, Lem. 2.1.2] in the general case).

(c) If X is proper, then it follows from the description of (a) that $\mathrm{Tr}_{X/k}(\mathrm{cl}(x)) = \deg(x)$ for every $x \in A_0(X)$.

4.4. Endomorphism of the cohomology. Let X_1 and X_2 be smooth connected proper schemes over k of dimensions d_1 and d_2, respectively, and set $Y := X_1 \times X_2$.

(a) Fix an element $u \in H^{2d_1}(Y, \mathbb{Q}_l(d_1))$. Then u induces a morphism $H^i(u) : H^i(X_1, \mathbb{Q}_l) \to H^i(X_2, \mathbb{Q}_l)$ for every i. Namely, by the Poincaré duality, $H^i(u)$ corresponds to the map $H^i(X_1, \mathbb{Q}_l) \times H^{2d_2-i}(X_2, \mathbb{Q}_l(d_2)) \to \mathbb{Q}_l$, which sends (x, y) to $\mathrm{Tr}_{Y/k}(u \cup (x \boxtimes y))$. Here we set $x \boxtimes y := p_1^* x \cup p_2^* y \in H^{2d_2}(X_2, \mathbb{Q}_l(d_2))$.

(b) As in (a), an element $v \in H^{2d_2}(Y, \mathbb{Q}_l(d_2))$ induces a morphism $H^i(v) : H^i(X_2, \mathbb{Q}_l) \to H^i(X_1, \mathbb{Q}_l)$. Moreover, we have

$$u \cup v \in H^{2(d_1+d_2)}(Y, \mathbb{Q}_l(d_1 + d_2)),$$

and it was shown in [Gr, Prop. 3.3] that $\mathrm{Tr}_{Y/k}(u \cup v)$ equals the alternate trace

$$\mathrm{Tr}(H^*(v) \circ H^*(u)) := \sum_i (-1)^i \, \mathrm{Tr}(H^i(v) \circ H^i(u)).$$

4.5. Connection with the cycle map. In the situation of 4.4, let C be a closed integral subscheme of $X_1 \times X_2$ of dimension d_2, hence of codimension d_1.

(a) Then C gives rise to an element $\mathrm{cl}(C) \in H^{2d_1}(Y, \mathbb{Q}_l(d_1))$; hence it induces an endomorphism $H^i([C]) := H^i(\mathrm{cl}(C)) : H^i(X_1, \mathbb{Q}_l) \to H^i(X_2, \mathbb{Q}_l)$. Let $(p_1, p_2) : C \hookrightarrow X_1 \times X_2$ be the inclusion. Denote by $(p_2)_* : H^i(C, \mathbb{Q}_l) \to H^i(X_2, \mathbb{Q}_l)$ the push-forward map, corresponding by duality to the map $H^i(C, \mathbb{Q}_l) \times H^{2d_2-i}(X_2, \mathbb{Q}_l(d_2)) \to \mathbb{Q}_l$, defined by $(x, y) \mapsto \mathrm{Tr}_{C/k}(x \cup p_2^*(y))$.

(b) We claim that the endomorphism $H^i([C]) : H^i(X_1, \mathbb{Q}_l) \to H^i(X_2, \mathbb{Q}_l)$ decomposes as $H^i(X_1, \mathbb{Q}_l) \xrightarrow{(p_1)^*} H^i(C, \mathbb{Q}_l) \xrightarrow{(p_2)_*} H^i(X_2, \mathbb{Q}_l)$. Indeed, by 4.3(a) and 4.4(a), $H^i([C])$ corresponds by duality to the map

$$H^i(X_1, \mathbb{Q}_l) \times H^{2d_2-i}(X_2, \mathbb{Q}_l(d_2)) \to \mathbb{Q}_l,$$

defined by the rule $(x, y) \mapsto \mathrm{Tr}_{Y/k}((x \boxtimes y) \cup \mathrm{cl}(C)) = \mathrm{Tr}_{C/k}(p_1^*(x) \cup p_2^*(y))$. Thus $H^i([C])$ decomposes as $(p_2)_* \circ (p_1)^*$.

(c) It follows from (b) that if $\Gamma_f \subset X_1 \times X_2$ is the graph of the map $f : X_2 \to X_1$, then $H^i([\Gamma_f])$ is simply the pullback map $f^* = H^i(f)$.

(d) Assume now that $X_1 = X_2 = X$ is of dimension d and that both $p_1, p_2 : C \to X$ are dominant. Then p_1 is generically finite, and it follows from the description of (b) and basic properties of the trace map that $H^{2d}([C]) = \deg(p_1)\,\mathrm{Id}$.

4.6. Example. In the situation of 4.4, let $C \subset X_1 \times X_2$ be a closed integral subscheme of dimension d_1, and let $f : X_2 \to X_1$ be a morphism. Then C defines a class $[C] \in A_{d_1}(Y) = A^{d_2}(Y)$ and a morphism $H^i([C]) : H^i(X_2, \mathbb{Q}_l) \to H^i(X_1, \mathbb{Q}_l)$ (see 4.5(a)).

On the other hand, morphism f defines a class $[\Gamma_f] \in A^{d_1}(Y)$, and a morphism $f^* = H^i(f) : H^i(X_1, \mathbb{Q}_l) \to H^i(X_2, \mathbb{Q}_l)$. Then we have the equality

$$[C] \cdot [\Gamma_f] = \mathrm{Tr}(f^* \circ H^*([C])). \tag{4.1}$$

Indeed, since $f^* = H^*([\Gamma_f])$ by 4.5(c), the right hand side of (4.1) equals $\mathrm{Tr}_{Y/k}(\mathrm{cl}(C) \cup \mathrm{cl}(\Gamma_f))$ by 4.4(b), hence to $\mathrm{Tr}_{Y/k}(\mathrm{cl}([C] \cap [\Gamma_f])) = [C] \cdot [\Gamma_f]$ by 4.3(b),(c).

4.7. Purity. Let X be a smooth proper variety of dimension d over $\mathbb{F}$, defined over $\mathbb{F}_q$. It is well known that $H^i(X, \mathbb{Q}_l) = 0$ if $i > 2d$. Moreover, by a theorem of Deligne [De], for every i and every embedding $\iota : \overline{\mathbb{Q}_l} \hookrightarrow \mathbb{C}$, all eigenvalues λ of $\phi_q^* : H^i(X, \mathbb{Q}_l) \to H^i(X, \mathbb{Q}_l)$ satisfy $|\iota(\lambda)| = q^{i/2}$.

4.8. Notation. Assume that we are in the situation of 3.1. Let $C \subset Y$ be a closed integral subscheme of dimension d such that $C \cap Y^0 \neq \emptyset$, and let $\widetilde{C} \subset \widetilde{Y}$ be the strict transform of C.

(a) Consider cycle classes $[C] \in A_d(Y)$ and $[\widetilde{C}] \in A_d(\widetilde{Y})$. For every $J \subset I$, schemes Y_J and E_J are smooth and proper over k; therefore we can form a cycle class $[\widetilde{C}]_J := (\pi_J)_* i_J^* [\widetilde{C}] \in A_{d-|J|}(Y_J)$. In particular, we have $[\widetilde{C}]_J = 0$ if $X_J = \emptyset$.

(b) Assume in addition that $k = \mathbb{F}$ and that X and all the X_i's are defined over $\mathbb{F}_q$. Fix $n \geq 0$. Let $\Gamma = \Gamma_{q^n} \subset Y = X \times X$, $\Gamma_J = \Gamma_{J,q^n} \subset Y_J = X_J \times X_J$ and $\Gamma_J^0 \subset X_J^0 \times X_J^0$ be the graphs of ϕ_{q^n}, and let $\widetilde{\Gamma} = \widetilde{\Gamma}_{q^n} \subset \widetilde{Y}$ be the strict preimage of Γ. We denote by $[\widetilde{\Gamma}] \in A_d(\widetilde{Y})$, $[\Gamma] \in A_d(Y)$ and $[\Gamma_J] \in A_{d-|J|}(Y_J)$ the corresponding classes.

The following result is an analog of [Laf, Prop. IV.6].

Lemma 4.9. *In the situation of* 4.8*, we have the equality*

$$[\widetilde{C}]\cdot[\widetilde{\Gamma}] = [C]\cdot[\Gamma] + \sum_{J\neq\emptyset}(-1)^{|J|}[\widetilde{C}]_J\cdot[\Gamma_J]. \tag{4.2}$$

Proof. Since $[C] = \pi_*[\widetilde{C}]$ and $[\widetilde{C}]_J = (\pi_J)_* i_J^*[\widetilde{C}]$, we conclude from the projection formula that $[C]\cdot[\Gamma] = [\widetilde{C}]\cdot\pi^*[\Gamma]$ and $[\widetilde{C}]_J\cdot[\Gamma_J] = [\widetilde{C}]\cdot(i_J)_*\pi_J^*[\Gamma_J]$. Hence it suffices to show that

$$[\widetilde{\Gamma}] = \pi^*[\Gamma] + \sum_{J\neq\emptyset}(-1)^{|J|}(i_J)_*\pi_J^*[\Gamma_J]. \tag{4.3}$$

Note that for every J, scheme X_J is smooth; thus the inclusion $\Gamma_J \hookrightarrow Y_J$ is regular of codimension $\dim X_J = d - |J|$. On the other hand, by Lemma 3.9, every schematic irreducible component of $\pi_J^{-1}(\Gamma_J)$ is of dimension $d - |J|$; thus the inclusion $\pi_J^{-1}(\Gamma_J) \hookrightarrow E_J$ is also regular of codimension $d-|J|$. It therefore follows from Example 4.2 that $\pi_J^*[\Gamma_J] = [\pi_J^{-1}(\Gamma_J)]$.

Using Lemma 3.9 again, we get that $[\pi_J^{-1}(\Gamma_J)] = \sum_{J'\supset J}[\overline{\pi_{J'}^{-1}(\Gamma_{J'}^0)}]$, thus

$$(i_J)_*\pi_J^*[\Gamma_J] = \sum_{J'\supset J}[\overline{\pi_{J'}^{-1}(\Gamma_{J'}^0)}]. \tag{4.4}$$

Applying this in the case $J=\emptyset$, we get the equality

$$\pi^*[\Gamma] = [\widetilde{\Gamma}] + \sum_{J'\neq\emptyset}[\overline{\pi_{J'}^{-1}(\Gamma_{J'}^0)}]. \tag{4.5}$$

Finally, equation (4.3) follows from (4.4), (4.5) and the observation that for every $J'\neq 0$, we have the equality $\sum_{J\subset J'}(-1)^{|J|} = 0$. □

Corollary 4.10. *In the situation of Lemma* 4.9*, assume that both projections $p_1, p_2 : C \to X$ are dominant. Then for every sufficiently large n, we have $[\widetilde{C}]\cdot[\widetilde{\Gamma}_{q^n}] \neq 0$, thus $\widetilde{C}\cap\widetilde{\Gamma}_{q^n}\neq\emptyset$.*

Proof. By equation (4.1) from Example 4.6, we have

$$[C]\cdot[\Gamma_{q^n}] = \mathrm{Tr}((\phi_q^*)^n \circ H^*([C]), H^*(X,\mathbb{Q}_l))$$

and

$$[\widetilde{C}]_J\cdot[\Gamma_{J,q^n}] = \mathrm{Tr}((\phi_q^*)^n \circ H^*([\widetilde{C}]_J), H^*(X_J,\mathbb{Q}_l))$$

for all n and J.

By 4.5(d), we have $H^{2d}([C]) = \deg(p_1)\,\mathrm{Id}$. Therefore we conclude from a combination of Deligne's theorem (see 4.7) and Lemma 4.11 below that for large n we have $[C]\cdot[\Gamma_{q^n}] \sim \deg(p_1)q^{dn}$, and $[\widetilde{C}]_J\cdot[\Gamma_{J,q^n}] = O(q^{2(d-|J|)n})$ for $J\neq\emptyset$. Hence, by (4.2), for large n we have $[\widetilde{C}]\cdot[\widetilde{\Gamma}_{q^n}] \sim \deg(p_1)q^{dn}$, thus $[\widetilde{C}]\cdot[\widetilde{\Gamma}_{q^n}]\neq 0$. □

Lemma 4.11. *Let $a > 1$, and let $A, B \in \mathrm{Mat}_d(\mathbb{C})$ be such that every eigenvalue λ of A satisfies $|\lambda| \leq a$. Then $\mathrm{Tr}(A^n B)$ is of magnitude $O(n^{d-1}a^n)$.*

Proof. We can assume that A has a Jordan form. Then all entries of A^n are of magnitude $O(n^{d-1}a^n)$. This implies the assertion. □

5. Proof of the main theorem

5.1. Reduction steps. (a) Assume that we are given a commutative diagram of schemes of finite type over $\mathbb{F}$:

$$\begin{array}{ccc} \widetilde{C}^0 & \xrightarrow{\widetilde{c}^0} & \widetilde{X}^0 \times \widetilde{X}^0 \\ {\scriptstyle f_C}\big\downarrow & & {\scriptstyle f\times f}\big\downarrow \\ C^0 & \xrightarrow{c^0} & X^0 \times X^0, \end{array} \tag{5.1}$$

such that c^0 and $\widetilde{c}^0$ satisfy the assumptions of Theorem 0.1, and $f : \widetilde{X} \to X$ is defined over $\mathbb{F}_q$. Then Theorem 0.1 for $\widetilde{c}^0$ implies that for c^0.

Indeed, for every $n \geq 0$, we have an inclusion $f_C((\widetilde{c}^0)^{-1}(\Gamma^0_{q^n})) \subset (c^0)^{-1}(\Gamma^0_{q^n})$. Thus we have $(c^0)^{-1}(\Gamma^0_{q^n}) \neq \emptyset$ if $(\widetilde{c}^0)^{-1}(\Gamma^0_{q^n}) \neq \emptyset$.

(b) By (a), for every open subset $U \subset X^0$ over $\mathbb{F}_q$, Theorem 0.1 for c^0 follows from that for $c^0|_U$. Also for every open $W \subset C^0$, Theorem 0.1 for c^0 follows from that for $c^0|_W$.

(c) Assume that the commutative diagram (5.1) satisfies $f = \mathrm{Id}_{X^0}$ and f_C is surjective. Then $f_C((\widetilde{c}^0)^{-1}(\Gamma^0_{q^n})) = (c^0)^{-1}(\Gamma^0_{q^n})$; thus Theorem 0.1 for c^0 implies that for $\widetilde{c}^0$.

(d) For every $m \in \mathbb{N}$, the Frobenius twist $(c^0)^{(m)} : C^0 \to X^0 \times X^0$ (see 1.9(a)) satisfies $((c^0)^{(m)})^{-1}(\Gamma^0_{q^n}) = (c^0)^{-1}(\Gamma^0_{q^{n+m}})$. Thus Theorem 0.1 for c^0 follows from that for $(c^0)^{(m)}$.

(e) Fix $r \in \mathbb{N}$. Then every $n \in \mathbb{N}$ has a form $n = rn' + m$ with $m \in \{0, \ldots, r-1\}$. Thus the assertion Theorem 0.1 for c^0 over $\mathbb{F}_q$ is equivalent to the corresponding assertion for $(c^0)^{(m)}$ over $\mathbb{F}_{q^r}$ for $m = 0, \ldots, r-1$.

Claim 5.2. It is enough to prove Theorem 0.1 under the assumption that there exists a Cartesian diagram

$$\begin{array}{ccc} C^0 & \xrightarrow{c^0} & X^0 \times X^0 \\ {\scriptstyle j_C}\big\downarrow & & {\scriptstyle j\times j}\big\downarrow \\ C & \xrightarrow{c} & X \times X \end{array} \tag{5.2}$$

of schemes of finite type over $\mathbb{F}$ such that:

(i) X is irreducible projective, and j is an open embedding, both defined over $\mathbb{F}_q$.

(ii) C is irreducible of dimension $\dim X$, and c is finite.

(iii) X is smooth, and the complement $\partial X := X \smallsetminus j(X^0)$ is a union of smooth divisors X_i with normal crossings, defined over $\mathbb{F}_q$.

(iv) The correspondence c is locally contracting near ∂X over $\mathbb{F}_q$.

Proof. We carry out the proof in six steps.

Step 1. We may assume that X^0 is quasi-projective and $\dim C^0 = \dim X^0$.

Indeed, by 5.1(b), we may replace X^0 and C^0 by their open neighborhoods, thus assuming that X^0 and C^0 are affine. Next, since $c_1^0, c_2^0 : C^0 \to X^0$ are dominant, the difference $r := \dim C^0 - \dim X^0$ is non-negative, and there exist maps $d_1^0, d_2^0 : C^0 \to \mathbb{A}^r$ such that $e_i^0 := c_i^0 \times d_i^0 : C^0 \to X^0 \times \mathbb{A}^r$ is dominant for $i = 1, 2$. Thus by 5.1(a), it suffices to prove the theorem for $e^0 := (e_1^0, e_2^0)$ instead of c^0; thus we may assume that $\dim C^0 = \dim X^0$.

Step 2. In addition to the assumptions of Step 1, we may assume that c^0 is a closed embedding.

Indeed, assume that we are in the situation of Step 1, and let C' be the closure $\overline{c^0(C^0)} \subset X^0 \times X^0$. Then the image $c^0(C^0)$ contains a non-empty open subset $U \subset C'$. Then by 5.1(b), (c), we may replace c^0 by the inclusion $U \hookrightarrow X^0 \times X^0$, thus assuming that c^0 is a locally closed embedding. Moreover, since $\dim(C' \smallsetminus U) < \dim C^0 = \dim X^0$, we can replace X^0 by its open subscheme $X_0' := X^0 \smallsetminus (\overline{c_1^0(C' \smallsetminus U)} \cup \overline{c_2^0(C' \smallsetminus U)})$ and c^0 by $c^0|_{X_0'}$, thus assuming that c^0 is a closed embedding.

Step 3. We may assume that there exists a Cartesian diagram (5.2) satisfying (i) and (ii).

Indeed, assume that we are in the situation of Step 2. Since X^0 is quasi-projective, there exists an open embedding with dense image $j : X^0 \hookrightarrow X$ over $\mathbb{F}_q$ with X projective. Let $C \subset X \times X$ be the closure of $c^0(C^0)$, and let $c : C \hookrightarrow X \times X$ be the inclusion map. Then c defines a Cartesian diagram (5.2) satisfying (i) and (ii).

Step 4. In addition to the assumptions of Step 3, we may assume that $\partial X := X \smallsetminus j(X_0)$ is locally c-invariant over $\mathbb{F}_q$.

Indeed, assume that we are in the situation of Step 3. Then, by Corollary 2.4, there exist an open subset $V \subset X^0$ and a blowup $\pi : \widetilde{X} \to X$, which is an isomorphism over V, such that both V and π are defined over $\mathbb{F}_q$ and for every map $\widetilde{c} : \widetilde{C} \to \widetilde{X} \times \widetilde{X}$ lifting c, the closed subset $\widetilde{X} \smallsetminus \pi^{-1}(V) \subset \widetilde{X}$ is locally $\widetilde{c}$-invariant over $\mathbb{F}_q$.

Replacing X^0 by V and c^0 by $c^0|_V$, we may assume that $V = X^0$ (use 5.1(b)). Let $\widetilde{j}$ be the inclusion $X^0 \cong \pi^{-1}(X^0) \hookrightarrow \widetilde{X}$, let $\widetilde{C} \subset \widetilde{X} \times \widetilde{X}$ be the closure of $C^0 \subset C \times_{(X\times X)} (\widetilde{X} \times \widetilde{X})$, and let $\widetilde{c} : \widetilde{C} \to \widetilde{X} \times \widetilde{X}$ be the projection. Then, replacing c by $\widetilde{c}$, we get the Cartesian diagram we are looking for.

Step 5. In addition to the assumptions of Step 4, we may assume that property (iii) is satisfied.

Indeed, assume that we are in the situation of Step 4. By a theorem of de Jong on alterations (see [dJ, Thm. 4.1 and Rem. 4.2]), there exists a proper generically finite map $\pi : \widetilde{X} \to X$ such that $\widetilde{X}$ is smooth and geometrically connected over $\mathbb{F}$, and $\pi^{-1}(\partial X) \subset \widetilde{X}$ is a union of smooth divisors with strict normal crossings $\widetilde{X}_i$. Choose r such that $\widetilde{X}$, $\widetilde{X}_i$ and π are defined over $\mathbb{F}_{q^r}$.

By 5.1(e), in order to prove Theorem 0.1 for c^0 over $\mathbb{F}_q$ it suffices to prove Theorem 0.1 over $\mathbb{F}_{q^r}$ for $(c^0)^{(m)}$ for $m = 0, \ldots, r-1$. Note that every $(c^0)^{(m)}$ satisfies all the assumptions of Step 4. Indeed, the twist $c^{(m)} : C \to X \times X$ is finite, because c and $\phi_q : X \to X$ are finite, and $c^{(m)}$ gives rise to the Cartesian diagram we are looking for. Thus replacing $\mathbb{F}_q$ by its finite extension $\mathbb{F}_{q^r}$ and c^0 by $(c^0)^{(m)}$, we may assume that $\widetilde{X}$, $\widetilde{X}_i$ and π are defined over $\mathbb{F}_q$.

Since $c_1, c_2 : C \to X$ are dominant, there exists a unique irreducible component $\widetilde{C}$ of $C \times_{(X\times X)} (\widetilde{X} \times \widetilde{X})$ such that both projections $\widetilde{c}_1, \widetilde{c}_2 : \widetilde{C} \to \widetilde{X}$ are dominant. Replacing X^0 by $\pi^{-1}(X^0)$ and c by $\widetilde{c} = (\widetilde{c}_1, \widetilde{c}_2)$, we get the required Cartesian diagram. Indeed, properties (i)-(iii) are satisfied by construction, while the locally invariance property of Step 4 is preserved by Lemma 1.3.

Step 6. We may assume that all the assumptions of Claim 5.2 are satisfied.

Assume that we are in the situation of Step 5. Choose $m \in \mathbb{N}$ such that $q^m > \mathrm{ram}(c_2, \partial X)$. Then ∂X is locally c-invariant over $\mathbb{F}_q$ by Step 4; then it is locally $c^{(m)}$-invariant over $\mathbb{F}_q$ by Remark 1.10. Thus $c^{(m)}$ is locally contracting near ∂X over $\mathbb{F}_q$ by Lemma 1.12. By 5.1(d), we can replace c^0 by $(c^0)^{(m)}$ (and c by $c^{(m)}$); thus we can assume that all the assumptions of Claim 5.2 are satisfied. □

5.3. *Proof of Theorem* 0.1. By Claim 5.2, we can assume that there exists a Cartesian diagram (5.2) satisfying properties (i)-(iv). To simplify the notation, we will identify X^0 with $j(X^0) \subset X$. By property (iv) and Lemma 3.7, we have an inclusion $\pi(\widetilde{c}(\widetilde{C}) \cap \widetilde{\Gamma}_{q^n}) \subset X^0 \times X^0$ for every $n \in \mathbb{N}$. Since $\pi_C(\widetilde{C}) \subset C$ and $\pi(\widetilde{\Gamma}_{q^n}) \subset \Gamma_{q^n}$, we conclude that

(5.3)
$$\pi(\widetilde{c}(\widetilde{C}) \cap \widetilde{\Gamma}_{q^n}) \subset (c(C) \cap \Gamma_{q^n}) \cap (X^0 \times X^0) = c^0(C^0) \cap \Gamma^0_{q^n} = c^0((c^0)^{-1}(\Gamma^0_{q^n})).$$

On the other hand, since c is finite its image $C' := c(C) \subset Y$ is a closed integral subscheme of dimension d, and its strict preimage $\widetilde{C'} \subset \widetilde{Y}$ equals $\widetilde{c}(\widetilde{C})$. Thus, it follows from Corollary 4.10, applied to C', that for every sufficiently large $n \in \mathbb{N}$ we have $\widetilde{c}(\widetilde{C}) \cap \widetilde{\Gamma}_{q^n} \neq \emptyset$. Thus, by (5.3), we get $(c^0)^{-1}(\Gamma^0_{q^n}) \neq \emptyset$, and the proof of Theorem 0.1 is complete. □

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