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On the number of labeled graphs of bounded treewidth ☆

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ABSTRACT

Let $T_{n,k}$ be the number of labeled graphs on n vertices and treewidth at most k (equivalently, the number of labeled partial k-trees). We show that

$$\left(c\,\frac{k\,2^k n}{\log k}\right)^n 2^{-\frac{k(k+3)}{2}} k^{-2k-2} \leqslant T_{n,k} \leqslant \left(k\,2^k n\right)^n 2^{-\frac{k(k+1)}{2}} k^{-k},$$

for $k > 1$ and some explicit absolute constant $c > 0$. Disregarding terms depending only on k, the gap between the lower and upper bound is of order $(\log k)^n$. The upper bound is a direct consequence of the well-known formula for the number of labeled k-trees, while the lower bound is obtained from an explicit construction. It follows from this construction that both bounds also apply to graphs of pathwidth and proper-pathwidth at most k.

1. Introduction

Given an integer $k > 0$, a *k-tree* is a graph that can be constructed starting from a $(k+1)$-clique and iteratively adding a vertex connected to k vertices that form a clique. They are natural extensions of trees, which correspond to 1-trees. A formula for the number of labeled k-trees on n vertices was first found by Beineke and Pippert [1], and alternative proofs were given by Moon [19] and Foata [9].

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1-trees: n^{n-2}

2-trees: $\frac{n(n-1)}{2} \cdot (2n-3)^{n-4}$

Theorem 1. *The number of n-vertex labeled k-trees is equal to*

$$\binom{n}{k}(kn-k^2+1)^{n-k-2}. \tag{1}$$

For completeness, we include in Appendix a short proof of Theorem 1 using the symbolic method [8] and avoiding recurrence relations.

A *partial k-tree* is a subgraph of a k-tree. For integers n, k with $0 < k \leqslant n-1$, let $T_{n,k}$ denote the number of n-vertex labeled partial k-trees. While the number of n-vertex labeled k-trees is given by Theorem 1, it appears that very little is known about $T_{n,k}$. Indeed, to the best of our knowledge, only the cases $k = 1$ (forests) and $k = 2$ (series-parallel graphs) have been studied. The number of n-vertex labeled forests is asymptotically $T_{n,1} \sim \sqrt{e}n^{n-2}$ [22], and the number of n-vertex labeled series-parallel graphs is asymptotically $T_{n,2} \sim g \cdot n^{-5/2}\gamma^n n!$ for some explicit constants g and $\gamma \approx 9.07$ [2].

Partial k-trees are exactly the graphs of *treewidth* at most k. Let us recall the definition of treewidth. A *tree-decomposition of width k* of a graph $G = (V, E)$ is a pair $(\mathsf{T}, \mathcal{B})$, where T is a tree and $\mathcal{B} = \{B_t \mid B_t \subseteq V, t \in V(\mathsf{T})\}$ such that:

1. $\bigcup_{t\in V(\mathsf{T})} B_t = V$.
2. For every edge $\{u, v\} \in E$ there is a $t \in V(\mathsf{T})$ such that $\{u, v\} \subseteq B_t$.
3. $B_i \cap B_\ell \subseteq B_j$ for all $\{i, j, \ell\} \subseteq V(\mathsf{T})$ such that j lies on the unique path from i to ℓ in T.
4. $\max_{t\in V(\mathsf{T})} |B_t| = k + 1$.

The sets of $\mathcal{B}$ are called *bags*. The *treewidth* of G, denoted by $\mathbf{tw}(G)$, is the smallest integer k such that there exists a tree-decomposition of G of width k. If T is a path, then $(\mathsf{T}, \mathcal{B})$ is also called a *path-decomposition*. The *pathwidth* of G, denoted by $\mathbf{pw}(G)$, is the smallest integer k such that there exists a path-decomposition of G of width k.

The following lemma is well-known and a proof can be found, for instance, in [17].

Lemma 1. *A graph has treewidth at most k if and only if it is a partial k-tree.*

In this article we are interested in counting n-vertex labeled graphs that have treewidth at most k. By Lemma 1, this number is equal to $T_{n,k}$, and actually our approach relies heavily on the definition of partial k-trees. In the following, when we consider asymptotic values, we assume that both k and n tend to infinity with $k = o(n)$.

An easy upper bound on $T_{n,k}$ is obtained as follows. Since every partial k-tree is a subgraph of a k-tree, and a k-tree has exactly $kn - \frac{k(k+1)}{2}$ edges, Theorem 1 gives

$$T_{n,k} \leqslant 2^{kn-\frac{k(k+1)}{2}} \binom{n}{k}(kn-k^2+1)^{n-k-2}. \tag{2}$$

Simple calculations yield, disregarding lower-order terms, that

$$T_{n,k} \leqslant (k2^k n)^n 2^{-\frac{k(k+1)}{2}} k^{-k} \leqslant (k2^k n)^n. \tag{3}$$

We can provide a lower bound with the following construction. Starting from an $(n-k+1)$-vertex forest, we add $k-1$ *apices*, that is, $k-1$ vertices with an arbitrary neighborhood in the forest. Every graph created in this way has exactly n vertices and is of treewidth at most k, since adding an apex increases treewidth by at most one. The number of labeled forests on $n-k+1$ vertices is at least the number of trees on $n-k+1$ vertices, which is well-known to be $(n-k+1)^{n-k-1}$. Since each apex can be connected to the ground forest in 2^{n-k+1} different ways, we obtain

$$T_{n,k} \geqslant (n-k+1)^{n-k-1} 2^{(k-1)(n-k+1)}. \tag{4}$$

If we assume that n/k tends to infinity then asymptotically

$$T_{n,k} \geqslant \left(2^{k-1} n\right)^{(1-o(1))n}. \tag{5}$$

We conclude that $T_{n,k}$ is essentially between $(2^k n)^n$ and $(k2^k n)^n$. These bounds differ by a factor k^n. For constant k this does not matter much since (except when $k = 1, 2$) we do not have a precise estimate on $T_{n,k}$. However, when k goes to infinity, the gap k^n is quite significant. Our main result considerably reduces the gap. Throughout the paper, logarithms are natural.

Theorem 2. *For integers n, k with $k \geqslant 2$ and $k + 1 \leqslant n$, the number $T_{n,k}$ of n-vertex labeled graphs with treewidth at most k satisfies*

$$T_{n,k} \geqslant \left(\frac{1}{64e^2} \cdot \frac{k2^k n}{\log k} \right)^n 2^{-\frac{k(k+3)}{2}} k^{-2k-2}. \tag{6}$$

It follows that $T_{n,k}$ is asymptotically between $\left(\frac{k}{\log k} 2^k n\right)^n$ and $(k2^k n)^n$ when n and k grow. Thus the gap is now of order $(\log k)^n$ instead of k^n.

In order to prove Theorem 2, we present in Section 2 an algorithmic construction of a family of n-vertex labeled partial k-trees, which is inspired by the definition of k-trees. When exhibiting such a construction toward a lower bound, one has to play with the trade-off of, on the one hand, constructing as many graphs as possible and, on the other hand, being able to bound the number of duplicates; we perform this analysis in Section 3. Namely, we first count the number of elements created by the construction, and then we bound the number of times that the same element may have been created. We conclude in Section 4 with some remarks and a discussion of further research.

2. The construction

Let n and k be fixed positive integers with $0 < k \leqslant n - 1$. In this section we construct a set $\mathcal{R}_{n,k}$ of n-vertex labeled partial k-trees. We let $R_{n,k} = |\mathcal{R}_{n,k}|$. In Section 2.1 we introduce some notation and definitions used in the construction, in Section 2.2 we describe the construction, and in Section 2.3 we prove that the treewidth of the graphs generated this way is indeed at most k. In fact, we prove a stronger property, namely that the graphs we construct have *proper-pathwidth* at most k, where the proper-pathwidth, defined later, is a graph invariant that is at least the pathwidth, which is at least the treewidth.

2.1. Notation and definitions

For the construction, we use a *labeling function* σ defined by a permutation of $\{1, \ldots, n\}$ with the constraint that $\sigma(1) = 1$. Inspired by the definition of k-trees, we will introduce vertices $\{v_1, v_2, \ldots, v_n\}$ one by one following the order $\sigma(1), \sigma(2), \ldots, \sigma(n)$ given by σ. If $i, j \in \{1, \ldots, n\}$, then i is called the *index* of $v_{\sigma(i)}$, the vertex $v_{\sigma(i)}$ is the ith introduced vertex and, if $j < i$, the vertex $v_{\sigma(j)}$ is said to be *to the left* of $v_{\sigma(i)}$.

In order to build explicitly a class of partial k-trees, for every $i \geqslant k + 1$ we define:

1. A set $A_i \subseteq \{j \mid j < i\}$ of indices of *active* vertices, corresponding to the clique to which a new vertex can be connected in the definition of k-trees, such that $|A_i| = k$.
2. An element $a_i \in A_i$, called the index of the *anchor*, whose role will be described in the next paragraph.
3. An element $f(i) \in A_i$, called the index of the *frozen* vertex, which corresponds to a vertex that will not be active anymore.
4. A set $N(i) \subseteq A_i$, which corresponds to the indices of the neighbors of $v_{\sigma(i)}$ to the left.

The construction works with *blocks* of size s, for some integer s depending of n and k, to be specified later. Namely, we insert the vertices by consecutive blocks of size s, with the property that all vertices of the same block share the same anchor and are adjacent to it.

In the description of the construction, we use the term *choose* for the elements for which there are several choices, which will allow us to provide a lower bound on the number of elements in $\mathcal{R}_{n,k}$. This will be the case for the functions σ, f, and N. As will become clear later (see Section 3), once σ, f, and N are fixed, all the other elements of the construction are uniquely defined.

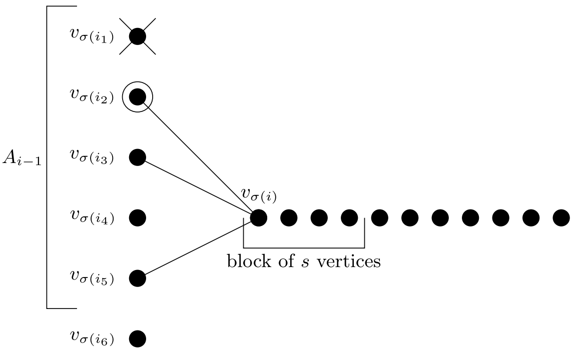

Fig. 1. Introduction of $v_{\sigma(i)}$ with $k+2 \leqslant i \leqslant n$ and $i \equiv k+2 \pmod{s}$, $s = 4$, and $k = 5$. We assume that $i_1 < i_2 < i_3 < i_4 < i_5 < i_6 < i$, and note that $i_5 = i-2$ and $i_6 = i-1$. We have defined $f(i) = i_1$ and $a_i = i_2$. The frozen vertex, with index $f(i)$, is marked with a cross, and the anchor, with index a_i, is marked with a circle. We choose $N(i) = \{i_2, i_3, i_5\}$.

For every index $i \geqslant k+2$, we impose that

$$|N(i)| \geqslant \frac{k+1}{2},$$

in order to have simultaneously enough choices for $N(i)$ and enough choices for the frozen vertex $f(i)$, which will be chosen among the vertices in $N(i-1)$. On the other hand, as will become clear later, the role of the anchor vertices is to determine uniquely the vertices belonging to "its" block. To this end, when a new block starts, its anchor is defined as the smallest currently active vertex.

2.2. Description of the construction

We say that a triple (σ, f, N), with σ a permutation of $\{1, \ldots, n\}$, $f : \{k+2, \ldots, n\} \to \{1, \ldots, n\}$, and $N : \{2, \ldots, n\} \to 2^{\{1,\ldots,n\}}$, is *constructible* if it is one of the possible outputs of the following algorithm:

Choose σ, a permutation of $\{1, \ldots, n\}$ such that $\sigma(1) = 1$.
for *i=2* **to** k **do**
  Choose $N(i) \subseteq \{j \mid j < i\}$, such that $1 \in N(i)$.
for *i=k+1* **do**
  Define $A_{k+1} = \{j \mid j < k+1\}$.
  Define $a_{k+1} = 1$.
  Choose $N(k+1) \subseteq \{j \mid j < i\}$, such that $1 \in N(k+1)$.
for *i=k+2* **to** n **do**
  if $i \equiv k+2 \pmod{s}$ **then**
    Define $f(i) = a_{i-1}$.
    Define $A_i = (A_{i-1} \setminus \{f(i)\}) \cup \{i-1\}$.
    Define $a_i = \min A_i$.
    Choose $N(i) \subseteq A_i$ such that $a_i \in N(i)$ and $|N(i)| \geqslant \frac{k+1}{2}$; cf. Fig. 1.
  else
    Choose $f(i) \in (A_{i-1} \setminus \{a_{i-1}\}) \cap N(i-1)$.
    Define $A_i = (A_{i-1} \setminus \{f(i)\}) \cup \{i-1\}$.
    Define $a_i = a_{i-1}$.
    Choose $N(i) \subseteq A_i$ such that $a_i \in N(i)$ and $|N(i)| \geqslant \frac{k+1}{2}$; cf. Fig. 2.

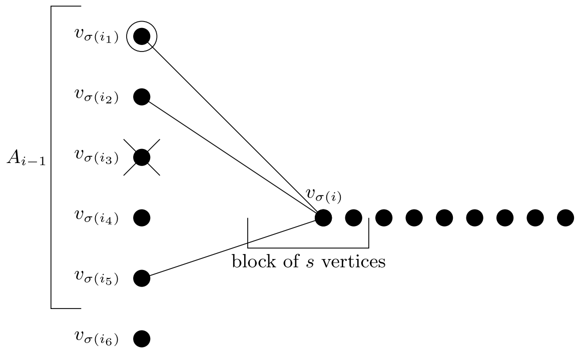

Fig. 2. Introduction of $v_{\sigma(i)}$ with $k+2 \leqslant i \leqslant n$ and $i \not\equiv k+2 \pmod{s}$, $s=4$, and $k=5$. We assume that $i_1 < i_2 < i_3 < i_4 < i_5 < i_6 < i$, and note that $i_5 = i-2$ and $i_6 = i-1$. We have defined $a_i = a_{i-1} = i_1$. The frozen vertex, with index $f(i)$, is marked with a cross, and the anchor, with index a_i, is marked with a circle. We choose $f(i) = i_3, f(i) = v_{\sigma(i_3)}$, assuming $v_{\sigma(i_3)}$ is a neighbor of $v_{\sigma(i_5)}$, and $N(i) = \{i_1, i_2, i_5\}$.

Let (σ, f, N) be a constructible triple. We define the graph $G(\sigma, f, N) = (V, E)$ such that $V = \{v_i \mid i \in \{1, \ldots, n\}\}$, and $E = \{\{v_{\sigma(i)}, v_{\sigma(j)}\} \mid j \in N(i)\}$. Note that, given (σ, f, N), the graph $G(\sigma, f, N)$ is well-defined. We denote by $\mathcal{R}_{n,k}$ the set of all graphs $G(\sigma, f, N)$ such that (σ, f, N) is constructible.

2.3. Bounding the treewidth

We start by defining the notion of proper-pathwidth of a graph. This parameter was introduced by Takahashi et al. [23] and its relation with search games has been studied in [24].

Let G be a graph and let $\mathcal{X} = \{X_1, X_2, \ldots, X_r\}$ be a sequence of subsets of $V(G)$. The *width* of $\mathcal{X}$ is $\max_{1 \leqslant i \leqslant r} |X_i| - 1$. $\mathcal{X}$ is called a *proper-path decomposition* of G if the following conditions are satisfied:

1. For any distinct i and j, $X_i \not\subseteq X_j$.
2. $\bigcup_{i=1}^{r} X_i = V(G)$.
3. For every edge $\{u, v\} \in E(G)$, there exists an i such that $u, v \in X_i$.
4. For all a, b, and c with $1 \leqslant a \leqslant b \leqslant c \leqslant r$, $X_a \cap X_c \subseteq X_b$.
5. For all a, b, and c with $1 \leqslant a < b < c \leqslant r$, $|X_a \cap X_c| \leqslant |X_b| - 2$.

The *proper-pathwidth* of G, denoted by $\mathbf{ppw}(G)$, is the minimum width over all proper-path decompositions of G. Note that if $\mathcal{X}$ satisfies only conditions 1-4 above, then $\mathcal{X}$ is a path-decomposition as defined in Section 1.

From the definitions, for any graph G, it clearly holds that

$$\mathbf{ppw}(G) \geqslant \mathbf{pw}(G) \geqslant \mathbf{tw}(G). \tag{7}$$

Let us show that any element of $\mathcal{R}_{n,k}$ has proper-pathwidth at most k. Let (σ, f, N) be constructible such that $G(\sigma, f, N) \in \mathcal{R}_{n,k}$ and let A_i be defined as in Section 2.2. We define for every $i \in \{k+1, \ldots, n\}$ the bag $X_i = \{v_{\sigma(j)} \mid j \in A_i \cup \{i\}\}$. The sequence $\mathcal{X} = \{X_{k+1}, X_{k+2}, \ldots, X_n\}$ is a path-decomposition satisfying the five conditions of the above definition, and for every $i \in \{k+1, \ldots, n\}$, $|X_i| = k+1$. It follows that $G(\sigma, f, N)$ has proper-pathwidth at most k, so it also has treewidth at most k, and therefore $G(\sigma, f, N)$ is a partial k-tree by Lemma 1.

3. Proof of the main result

In this section we analyze our construction and give a lower bound on $R_{n,k}$. We first start by counting the number of constructible triples (σ, f, N) generated by the algorithm, and then we provide an upper bound on the number of duplicates. Finally, we determine the best choice for the parameter s defined in the construction.

3.1. Number of constructible triples (σ, f, N)

We proceed to count the number of constructible triples (σ, f, N) created by the algorithm given in Section 2.2. As σ is a permutation of $\{1, \ldots, n\}$ with the constraint that $\sigma(1) = 1$, there are $(n-1)!$ distinct possibilities for the choice of σ. The function f can take more than one value only for $k+2 \leqslant i \leqslant n$ and $i \not\equiv k+2 \pmod s$. This represents $n-(k+1)-\lceil \frac{n-(k+1)}{s} \rceil$ cases. In each of these cases, there are at least $\frac{k-1}{2}$ distinct possible values for $f(i)$. Thus, we have at least $(\frac{k-1}{2})^{(n-(k+1)-\lceil \frac{n-(k+1)}{s} \rceil)}$ distinct possibilities for the choice of f. For every $i \in \{2, \ldots, k+1\}$, $N(i)$ can be chosen as any subset of $i-1$ vertices containing the fixed vertex $v_{\sigma(1)}$. This yields $\prod_{i=2}^{k+1} 2^{i-2} = 2^{\frac{k(k-1)}{2}}$ ways to define N over $\{2, \ldots, k+1\}$. For $i \geqslant k+2$, $N(i)$ can be chosen as any subset of size at least $\frac{k+1}{2}$ of a set of k elements with one element that is imposed. This results in $\sum_{i=\lceil \frac{k+1}{2} \rceil}^{k} \binom{k-1}{i-1} \geqslant 2^{k-2}$ possible choices for $N(i)$. Thus, we have at least $2^{\frac{k(k+1)}{2}} 2^{(n-(k+1))(k-2)}$ distinct possibilities to construct N.

By combining everything, we obtain at least

$$(n-1)!\left(\frac{k-1}{2}\right)^{n-(k+1)-\lceil \frac{n-(k+1)}{s} \rceil} 2^{\frac{k(k-1)}{2}} 2^{(n-(k+1))(k-2)} \tag{8}$$

distinct possible constructible triples (σ, f, N).

3.2. Bounding the number of duplicates

Let H be an element of $\mathcal{R}_{n,k}$. Our objective is to obtain an upper bound on the number of constructible triples (σ, f, N) such that $H = G(\sigma, f, N)$.

Given H, we start by reconstructing σ. Firstly, we know by construction that $\sigma(1) = 1$. Secondly, we know that $f(k+2) = 1$ and so, for every $i > k+1$, $1 \notin A_i$, implying that $1 \notin N(i)$. It follows that the only neighbors of $v_{\sigma(1)}$ are the vertices $\{v_{\sigma(i)} \mid 1 < i \leqslant k+1\}$. So the set of images under σ of $\{2, \ldots, k+1\}$ is uniquely determined. Then we guess the function σ over this set $\{2, \ldots, k+1\}$. Overall, this results in $k!$ possible guesses for σ.

Thirdly, assume that we have correctly guessed σ on $\{1, \ldots, k+1+ps\}$ for some non-negative integer p with $k+1+ps < n$. Then $a_{k+1+ps+1}$ is the smallest active vertex that is adjacent to at least one element that is still not introduced after step $k+1+ps$. Then the neighbors of $a_{k+1+ps+1}$ over the elements that are not introduced yet after step $k+1+ps$ are the elements whose indices are between $k+1+ps+1$ and $k+1+(p+1)s$, and these vertices constitute the next block of the construction; see Fig. 3 for an illustration. As before, the set of images by σ of $\{k+1+ps+1, \ldots, k+1+(p+1)s\}$ is uniquely determined, and we guess σ over this set. We have at most $s!$ possible such guesses. Fourthly, if $k+1+(p+1)s > n$ (that is, for the last block, which may have size smaller than s), we have $t!$ possible guesses with $t = n-(k+1)-s\lfloor \frac{n-(k+1)}{s} \rfloor$.

We know that the first, the second, and the fourth cases can occur only once in the construction, and the third case can occur at most $\lfloor \frac{n-(k+1)}{s} \rfloor$ times. Therefore, an upper bound on the number of distinct possible guesses for σ is $k!(s!)^{\lfloor \frac{n-(k+1)}{s} \rfloor} t!$, where $t = n-(k+1)-s\lfloor \frac{n-(k+1)}{s} \rfloor$.

Let us now fix σ. Then the function N is uniquely determined. Indeed, for every $i \in \{1, \ldots, n\}$, $N(i)$ corresponds to the neighbors of $v_{\sigma(i)}$ to the left. It remains to bound the number of possible functions f. In order to do this, we define for every $i > 1$, $D_i = \{j \in N(i) \mid \forall j' > i, \{v_{\sigma(j)}, v_{\sigma(j')}\} \notin E(H)\}$. Then, for every $i \geqslant k+2$, by definition of $f(i)$, $f(i) \in D_{i-1}$. Moreover, for $i, j > k+1$ with $i \neq j$, it holds that, by definition of D_i and D_j, $D_i \cap D_j = \emptyset$. Indeed, assume w.l.o.g. that $i < j$, and suppose for contradiction that there exists $a \in D_i \cap D_j$. As $a \in D_j$, it holds that $a \in N(j)$, but as $a \in D_i$, for every $j' > i$, $a \notin N(j')$, hence $a \notin N(j)$, a contradiction.

We obtain that the number of distinct functions f is bounded by $\prod_{i=k+1}^{n} |D_i|$. As $D_i \cap D_j = \emptyset$ for every $i, j \geqslant k+1$ with $i \neq j$ and $D_i \subseteq \{1, \ldots, n\}$ for every $i \geqslant k+1$, we have that $\sum_{i=k+1}^{n} |D_i| \leqslant n$. Let $I = \{i \in \{k+1, \ldots, n\} \mid |D_i| \geqslant 2\}$, and note that $|I| \leqslant k$. By the previous discussion, it holds that $\sum_{i \in I} |D_i| \leqslant 2k$. So it follows that, by using Cauchy–Schwarz inequality,

$$\prod_{i=k+1}^{n} |D_i| = \prod_{i \in I} |D_i| \leqslant \left(\frac{\sum_{i \in I} |D_i|}{k}\right)^k \leqslant \left(\frac{2k}{k}\right)^k = 2^k. \tag{9}$$

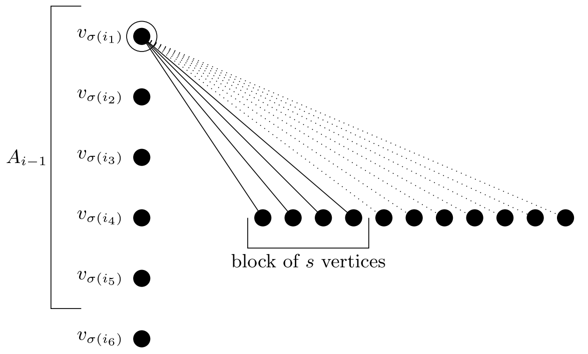

Fig. 3. The current anchor $v_{\sigma(i_1)}$ is connected to all the s vertices of the current block but will not be connected to any of the remaining non-introduced vertices.

To conclude, the number of constructible triples that can give rise to H is at most $2^k(s!)^{\lfloor\frac{n-(k+1)}{s}\rfloor}t!$ where $t = n-(k+1)-s\lfloor\frac{n-(k+1)}{s}\rfloor$. Thus, we obtain that

$$R_{n,k} \geqslant \frac{(n-1)!\left(\frac{k-1}{2}\right)^{n-(k+1)-\lceil\frac{n-(k+1)}{s}\rceil} 2^{\frac{k(k-1)}{2}} 2^{(n-(k+1))(k-2)}}{2^k k!(s!)^{\lfloor\frac{n-(k+1)}{s}\rfloor}(n-(k+1)-s\lfloor\frac{n-(k+1)}{s}\rfloor)!}. \tag{10}$$

For better readability, we bound separately each of the terms on the right-hand side:

- $(n-1)! \geqslant (\frac{n}{e})^n 2^{-n}$, $2^{\frac{k(k-1)}{2}} 2^{(n-(k+1))(k-2)} \geqslant 2^{kn-\frac{k(k+3)}{2}} 2^{-2n}$.
- $(k-1)^{(n-(k+1)-\lceil\frac{n-(k+1)}{s}\rceil)} \geqslant 2^{-n} k^{(n-\frac{n}{s}-k-2)}$, since $k \geqslant 2$.
- $2^k k! \leqslant 2^n k^k$, $(s!)^{\lfloor\frac{n-(k+1)}{s}\rfloor}(n-(k+1)-s\lfloor\frac{n-(k+1)}{s}\rfloor)! \leqslant s^n$.

Applying these relations to (10) gives

$$R_{n,k} \geqslant \left(\frac{1}{64e} \cdot \frac{k2^k n}{k^{1/s} s}\right)^n 2^{-\frac{k(k+3)}{2}} k^{-2k-2}. \tag{11}$$

3.3. *Choosing the parameter s*

We now discuss how to choose the size s of the blocks in the construction. In order to obtain the largest possible lower bound for $R_{n,k}$, we would like to choose s minimizing the factor $k^{1/s}s$ in the denominator of (11).

Let $t = s/\log k$. Then

$$\log\left(k^{1/s}s\right) = \frac{\log k}{s} + \log s = \frac{1}{t} + \log t + \log\log k. \tag{12}$$

The minimum of $\frac{1}{t} + \log t$ is achieved for $t = 1$, thus $s = \log(k)$ (in fact $s = \lceil \log k \rceil$ since s must be an integer, but we omit this precision). Then $k^{1/s}s = k^{1/\log(k)}s = e\log(k)$ maximizes the lower bound given by Eq. (11). Therefore, we obtain that

$$R_{n,k} \geqslant \left(\frac{1}{64e^2} \cdot \frac{k2^k n}{\log k}\right)^n 2^{-\frac{k(k+3)}{2}} k^{-2k-2}, \tag{13}$$

concluding the proof of Theorem 2, where we assume that $k \geqslant 2$.

4. Concluding remarks and further research

Comparing Eqs. (3) and (6), there is still a gap of $(64e^2 \cdot \log k)^n$ in the dominant term of $T_{n,k}$, and closing this gap remains an open problem. The factor $(\log k)^n$ appears because, in our construction, when a new block starts, we force the frozen vertex to be the previous anchor. Therefore, this factor is somehow artificial, and we believe that it could be avoided.

One way to improve the upper bound would be to show that *every* partial k-tree with n vertices and m edges can be extended to at least a large number $\alpha(n, m)$ of k-trees, and then use *double counting*. This is the approach taken in [20] for bounding the number of planar graphs, but so far we have not been able to obtain a significant improvement using this technique.

As mentioned before, our results also apply to other relevant graph parameters such as pathwidth and proper-pathwidth. For both parameters, besides improving the lower bound given by our construction, it may be also possible to improve the upper bound given by Eq. (3). For proper-pathwidth, a modest improvement can be obtained as follows. It follows easily from the definition of proper-pathwidth that the edge-maximal graphs of proper-pathwidth k, which we call *proper linear k-trees*, can be constructed starting from a $(k+1)$-clique and iteratively adding a vertex v_i connected to a clique K_{v_i} of size k, with the constraints that $v_{i-1} \in K_{v_i}$ and $K_{v_i} \setminus \{v_{i-1}\} \subseteq K_{v_{i-1}}$. From this observation, and taking into account that the order of the first k vertices is not relevant and that there are $2k$ initial cliques giving rise to the same graph, it follows that the number of n-vertex labeled proper linear k-trees is equal to

$$n!k^{n-k-1}\frac{1}{(2k)k!}. \tag{14}$$

From this and the fact that a k-tree has $kn - \frac{k(k+1)}{2}$ edges, an easy calculation yields that the number of n-vertex labeled graphs of proper-pathwidth at most k is at most $\left(\frac{k2^k n}{c}\right)^n$, for some absolute constant $c \geqslant 1.88$.

It would be interesting to count graphs of bounded "width" in other cases. For instance, branchwidth seems to be a good candidate, as it is known that, if we denote by $\mathbf{bw}(G)$ the branchwidth of a graph G and $|E(G)| \geqslant 3$, then $\mathbf{bw}(G) \leqslant \mathbf{tw}(G) + 1 \leqslant \frac{3}{2}\mathbf{bw}(G)$ [21]. Other relevant graph parameters are cliquewidth, rankwidth, tree-cutwidth, or booleanwidth. For any of these parameters, a first natural step would be to find a "canonical" way to build such graphs, as in the case of partial k-trees.

Our results find algorithmic applications, specially in the area of Parameterized Complexity [6]. When designing a parameterized algorithm, usually a crucial step is to solve the problem at hand restricted to graphs decomposable along small separators by performing dynamic programming (see [15] for a recent example). For instance, precise bounds on $T_{n,k}$ are useful when dealing with the Treewidth-k Vertex Deletion problem, which has recently attracted significant attention in the area [10,13,16]. In this problem, given a graph G and a fixed integer $k > 0$, the objective is to remove as few vertices from G as possible in order to obtain a graph of treewidth at most k. When solving Treewidth-k Vertex Deletion by dynamic programming, the natural approach is to enumerate, for any partial solution at a given separator of the decomposition, all possible graphs of treewidth at most k that are "rooted" at the separator. In this setting, the value of $T_{n,k}$, as well as an explicit construction to generate such graphs, may be crucial in order to speed-up the running time of the algorithm. Other recent algorithmic applications of knowing the precise number of graphs of bounded treewidth are finding path- or tree-decompositions with minimum number of bags [4] and subgraph embedding problems on sparse graphs [5].

Finally, a challenging open problem is to count the number of *unlabeled* partial k-trees, for which nothing is known except for some results concerning random models [3,14,18]. Note that the number of unlabeled k-trees was an open problem for long time, until it was recently solved by Gainer-Dewar [11] (see also [7,12]).

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Appendix. A short proof of Theorem 1

We notice from the recursive definition of k-trees that the number of k-cliques is $kn - k^2 + 1$. Let a_n be the number of labeled k-trees with n vertices and a distinguished labeled k-clique C, and let $A(z) = \sum a_n z^n/n!$ be the associated exponential generating function. The clique C belongs to a set (an unordered collection) of $(k+1)$-cliques $C_1, \ldots, C_s$. Each of the C_i's gives rise to k new k-trees rooted at a k-clique. This combinatorial construction corresponds to the algebraic expression $e^{A(z)^k}$: the power $A(z)^k$ corresponds to an ordered collection of objects counted by $A(z)$, and the exponential operator corresponds to sets of these structures.

Correcting the exponent of z so that each vertex is labeled exactly once, a direct application of the symbolic method [8] gives the equation

$$A(z) = z^k \exp\left(\frac{A(z)^k}{z^{k^2-1}}\right).$$

$$\partial A = k z^{k-1} \cdot e'' + A \cdot \frac{k A^{k-1} \partial A \cdot (z^{k^2}-1) + A^k}{(z^{k^2}-1)^2}$$

The change of variable $B(z) = A(z^k)/z^{k^2-1}$ gives a simpler equation

$$B(z) = z\exp(B(z)^k),$$

$$\partial B = \frac{B}{z} + B \cdot k B^{k-1} \partial B \Rightarrow \partial B = \frac{B}{z(1 - kB^k)} \quad (p\partial_j(z))$$

which can be solved directly using the Lagrange's inversion theorem [8], namely

$$[z^n]B(z) = \frac{1}{n}[u^{n-1}]\exp(u^k n).$$

A routine calculation gives then

$$a_n = \frac{n!}{(n-k)!}(kn - k^2 + 1)^{n-k-1}.$$

Finally, we have to divide between the number $k!(kn - k^2 + 1)$ of choices for the distinguished clique C, giving the formula of Theorem 1.

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