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THE LOGICAL WORK OF MORDCHAJ WAJSBERG

J. Srzednicki (ed.), *Initiatives in Logic*

i. Introductory Remarks

In this paper Mordchaj Wajsberg's life and research work in logic are described, and an attempt is made to situate the latter among the accomplishments of the rest of the Polish school of logic.

ii. Wajsberg and the Polish School of Logic

Wajsberg belonged to a research formation called in the course of time the Polish school of logic. The undisputed leaders of the school were S. Leśniewski, J. Łukasiewicz, and at a later stage, A. Tarski who since 1923 became responsible for many outstanding contributions and systematic studies in logic, metalogic and semantics.

Although the school members centered on modern logic and its applications where they promoted a number of new trends and opened many fresh fields of research, they also showed a lively interest in methodology of deductive and empirical sciences as they took up in modern logical form many of the traditional major philosophical questions at the same time putting outside the scope of philosophy some such philosophical problems which could be either clearly stated or investigated by the methods of science. In these efforts the school members were supported by the prominent philosophers T. Kotarbiński and K. Ajdukiewicz.

It is this school which emerged as the most dominant force in academic logic and philosophy of Poland as well as the Polish intellectual life between the two world wars. And it is this school which should be seen responsible for the spectacular rise to prominence of formal and philosophical logic.

The leaders of the school soon became surrounded by a large number of talented students, young assistants and followers. Among those who essentially contributed to the school's success there were A. Lindenbaum, B. Sobociński, S. Jaśkowski, J. Słupecki, and of course, M. Wajsberg.

A distinctive feature of the school and one of the secrets of its success was the spirit of teamwork. The mutual collaboration among the members was so close and intimate that it is often hard to decide who should be credited with which particular results. Another its feature is that its members seemed to care more about making research progress than about making the results actually published or otherwise documented. Consequently, many findings appeared in print only in the form of abstract with proofs and other essentials missing. Moreover, some important findings were never published during their authors' lifetime. They became more and more dependent on oral communication thus contributing to the growth of the school's 'oral tradition'. For a general school's background see KUZAWA 1968 and MOSTOWSKI 1957.

Many useful findings were summarized and systematized by Łukasiewicz and Tarski in their joint paper ŁUKASIEWICZ-TARSKI 1930. A great number of references to the school's results can be found in TARSKI 1956. See also JORDAN 1945, 1963, and 1967.

Wajsberg emerges as a prominent representative of the school. Many of his research results have profoundly influenced further studies in the field. Among other things, he became a pioneer in the axiomatization of many-valued logic. He was the first to provide an adequate semantics for one of Lewis's modal systems. He also worked out an original method for the separable axiomatization of intuitionistic propositional logic. Wajsberg made an impression on many things which he touched, perfected many results by others, particularly by Łukasiewicz, Leśniewski, Tarski, Lewis and Hilbert. His research work gave a new impetus to further studies. And, although, unlike his teachers, Wajsberg said directly nothing on philosophical subjects, his research work has borne unquestionable philosophical implications.

Wajsberg published twelve papers. For the availability of their English translations see SURMA 1977. See also McCALL 1967 which contains English translations of three of his papers, WAJSBERG 1931, 1937, and 1938b.

iii. Childhood

Mordchaj Wajsberg was born on May 10, 1902 at Łomża, Białystok district. The years of 1909 to 1912 he spent in a local primary school. Then he moved to an intermediary school but the school was closed two years later when the first world war had broken out. In 1920 a year of military service in the revived Polish army followed thus interrupting his preparations for final school certificate. After completion of the service he passed successfully his entrance examination to the last but one form of the local secondary school from which he graduated in June 1923.

iv. University Study

Wajsberg spent his formative years in Warsaw. In October, 1923 he enrolled as a mathematics student at the Philosophy Department of the Warsaw University. He specialized in mathematical logic which he studied under Łukasiewicz. Apart from those by Łukasiewicz he also attended lectures on logic given at that time by Leśniewski and Kotarbiński.

As a second year student he read two papers to the Philosophy of Mathematics Section of the Association of Philosophy Students, one on "Russell's Theory of Functions of Apparent Variable", the other one on "Invariants of Logistic Transformation".

v. Early Findings. Pure Implication

Already as a third-year student Wajsberg obtained some original results. He described a number of alternative axiomatic systems for various fragments of classical propositional logic. In particular, he found new axioms for the logic of pure implication and for that of pure equivalence. Among them there is his 25-letter single axiom for pure implication:

CCCpqCCrstCCuCCrstCCpuCst

(Explanation of the symbolism: the above formula is rendered using the so called Polish notation, due to Łukasiewicz (see ŁUKASIEWICZ 1929), where 'C' denotes the connective of implication, and where 'Cab' reads as 'If a, then b'). This axiom is organic in the sense that none of its proper subformulae is a tautology; the notion of organic formula was also introduced by Wajsberg (see ŁUKASIEWICZ-TARSKI 1930).

Unlike the above axiom, the 25-letter single axiom:

CCCpCqpCCCCCrstuCCsuCruvv

found by Łukasiewicz and also referred to in ŁUKASIEWICZ-TARSKI 1930, contains tautology CpCqp as a subformula, and so it is not organic.

It has been shown by Łukasiewicz later that the following 13-letter formula:

CCCpqrCCrpCsp

is the shortest single axiom for pure implication (see ŁUKASIEWICZ 1948). Still later Ivo Thomas, using the work of R. Tursman (see TURSMAN 1968), has finally shown that there are no more shortest single axioms for pure implication (see THOMAS 1970).

vi. Pure Equivalence

Investigations into the logic of pure equivalence were initiated in Poland by Leśniewski to whom we owe what is now called Leśniewski's decidability criterion to the effect that each purely equivalential formula is a tautology if and only if each propositional variable occurs in it an even number of times. Leśniewski was also the first to prove that all pure equivalential tautologies can be axiomatized with the help of substitution and ordinary detachment for equivalence:

Eab, a ⊢ b

together with the following axioms:

EEEprEqpErq, EEpEqrEEpqr.

For reference see LEŚNIEWSKI 1929. (Explanation of the symbolism: the above formulae are rendered using the Polish notation, where 'E' stands for the connective of equivalence, and where 'Eab' reads as 'a if and only if b').

The subject of pure equivalence attracted many members of the school. Among them was Wajsberg. To Wajsberg belongs the credit of showing that the logic of pure equivalence can be axiomatized with the help of single axiom. In 1925, still as a third-year student, he found the following two 15-letter single axioms (see WAJSBERG 1937, footnote 1):

EEEEpqrsEsEpEqr

and

EEEpEqrEErssEpq.

In 1930 five more 15-letter single axioms for pure equivalence were found by Łukasiewicz, Sobociński, and J. Bryman (see SOBOCIŃSKI 1932). Later all these results were sharpened by Łukasiewicz who in 1933 found the following three 11-letter single axioms for pure equivalence:

EEpqEErqEpr, EEpqEEprErq, and EEpqEErpEqr.

Łukasiewicz proved that each of these axioms is the shortest possible single axiom for pure equivalence, thus solving the problem of the length of single axioms for this logic (see ŁUKASIEWICZ 1939).

In 1963 C.A.Meredith found seven more shortest single axioms for pure equivalence (see MEREDITH 1963 and PETERSON 1976). One more such single axiom was added by J.A. Kalman in 1978 (see

KALMAN 1978). Continuing earlier efforts by Kalman and Peterson, L. Wos and S. Winker finally established that the number of all single axioms for pure equivalence is thirteen (WOS-WINKER 1980). More historical information concerning investigations into the logic of pure equivalence may be found in SURMA 1973b.

vii. Sheffer's Connective

As a fourth-year student Wajsberg made a contribution to the study of the Sheffer connective D (read as 'Not both'). He found the following axiom for D:

DDpDqrDDDsrDDpsDpsDpDpq

and he deduced from this axiom the following axiom:

DDpDqrDDtDttDDsqDDpsDpDps

which was found by J. Nicod as early as in 1917, and which became the first single axiom for propositional logic ever known (see NICOD 1917).

Wajsberg's axiom improves Nicod's one. First, it contains one less propositional variable. Besides, it is organic while Nicod's is not as it contains the tautology DtDtt as a subformula.

Wajsberg's own results on single axioms contributed to similar studies, already in their full swing, advanced considerably by Łukasiewicz, Tarski, and Sobociński, who found many single axioms for various fragments of propositional logic. For reference see ŁUKASIEWICZ-TARSKI 1930 and SOBOCIŃSKI 1932. All his early results were included into Wajsberg's master's thesis, entitled "Contribution to the Research on Mathematical Logic", which was written under Łukasiewicz's supervision. It is on the basis of this thesis that he was awarded his M.A. degree on October 2, 1928.

viii. Early Observation on Modal Logic

Still as a student Wajsberg became involved into the study of modal logic, the ancient subject which was revived by modern logicians, especially, by C.I. Lewis. He was the first to prove that none of Lewis's modal systems is equivalent to classical propositional logic. Following WAJSBERG 1937, footnote 7, his separating four-valued truth tables, used in the proof, were found by him as early as in 1926. He observed that formula 'La' (read as 'It is necessary that a') is already a theorem in Lewis's system S1, whenever 'a' itself is a classical tautology, an important fact pertaining to the so called Goedel-Lemmon-style formalization of modal logics. And for the first time in the history of modern modal logic he outlined an adequate semantic characterization of Lewis's system S5. A detailed description of the semantics was presented in his later paper WAJSBERG 1933a.

All these observations were communicated by Wajsberg to C.I. Lewis at least as early as in 1927, as it is acknowledged in Appendix ii of LEWIS-LANGFORD 1932. See also PARRY 1968.

ix. Wajsberg's Work on Many-Valued Logic

From August, 1929 to September, 1930 Wajsberg served in the army, first as a student in the cadet training unit, and then in the 4th Regiment of the Tatra Highland Gunners. In September, 1930 he qualified for Ph.D. studies at the Warsaw University. As a Ph.D. student he worked under Łukasiewicz's supervision. His research project centered on the three-valued logic of Łukasiewicz.

The three-valued logic of Łukasiewicz was discovered by Łukasiewicz in 1920, that is already a decade earlier (see ŁUKASIEWICZ 1919-1920 and 1921), and was described semantically with the help of his well-known three-valued truth tables, at that time referred to as the method of logical matrices (see ŁUKASIEWICZ 1930). In 1922 the three-valued logic was generalized by Łukasiewicz to n-valued logics, where n may be an arbitrary finite or even infinite number. Researches on Łukasiewicz's logics were carried out by a growing team of talented and devoted students and collaborators, which included not only Wajsberg but also Tarski, Lindenbaum, Sobociński, and, later, Słupecki and Jaśkowski.

Wajsberg accomplished his Ph.D. project in less than a year, entitled his manuscript "Axiomatization of the three-valued propositional logic", and submitted it oficially to the Warsaw University in fulfilment of the requirement for the degree of Doctor of Philosophy.

In his thesis Wajsberg found the following system of independent axioms for the three-valued logic of Łukasiewicz based on implication and negation as primitive connectives:

CpCqp, CCpqCCqrCpr, CCNpNqCqp, CCCpNppp

(Explanation of the symbolism: 'Cab' reads as 'if a, then b', as before; while 'Na' reads as 'it is not the case that a' so that 'N' stands for the connective of negation). He proved that each three-valued tautology and only such tautology can be deduced from the above axioms using the rules of detachment and substitution as the only rules of inference. This provided a solution to the completeness problem for the three-valued logic of Łukasiewicz, which was the first result of the kind in the history of many-valued logic.

In his thesis Wajsberg also proved that no subsystems of classical propositional logic can be axiomatized with the help of axioms built up of at most two propositional variables. An algebraic proof of this fact was given by A.H. Diamond and J.C.C. McKinsey in 1947 (see DIAMOND-McKINSEY 1947).

A paper based on the results contained in Wajsberg's thesis was presented by Łukasiewicz to the Warsaw Scientific Society for publication as early as January 19, 1931. It appeared in the Proceedings of the Society in the same year (see WAJSBERG 1931).

Formal defence of Wajsberg's Ph.D. thesis followed, with Łukasiewicz and S. Mazurkiewicz as referees, and the degree of Doctor of Philosophy was conferred upon him at the promotion ceremony on May 29, 1931.

Wajsberg's Ph.D. thesis did not contain all of his findings concerning Łukasiewicz's many-valued logics. At about the same time he proved axiomatizability of all those n-valued Łukasiewicz's logics, for which (n-1) is a prime number. This result was later extended by Lindenbaum to all natural n (see ŁUKASIEWICZ-TARSKI 1930).

Wajsberg also confirmed Łukasiewicz's conjecture on the axiomatizability of the infinite-valued Łukasiewicz's logics, namely, that the logic can be axiomatized by the detachments and substitution rules together with the following axioms:

CpCqp, CCpqCCqrCpr, CCCpqqCCqpp, CCCpqCqpCqp, CCNpNqCqp.

He announced in WAJSBERG 1936, p.240, that he had found proof for the conjecture but his proof has never been published (see ŁUKASIEWICZ-TARSKI 1930). The proof that the above axioms suffice for Łukasiewicz's infinite-valued logic was shown in print by A. Rose and J.B. Rosser only in 1958 (see ROSE-ROSSER 1958). C.A. Meredith and C.C. Chang then showed, independently, that axiom CCCpqCqpCqp is redundant and so can be omitted from above list (see MEREDITH 1958 and CHANG 1958).

Wajsberg also found a relatively simple axiomatization of the so called extended three-valued logic of Łukasiewicz. An extended propositional logic was defined in the school as a propositional logic admitting quantification over propositional variables (see ŁUKASIEWICZ-TARSKI 1930). As such it can be viewed as a particular case of Leśniewski's protothetic (see LEŚNIEWSKI 1929) in which also quantification over variable connectives is admissible.

x. Axiomatizability of Negation

In the year of 1931, apart from the paper containing his Ph.D. thesis, Wajsberg also prepared for publication his papers WAJSBERG 1932a and 1932b, which were published in the next year. In WAJSBERG 1932a he presented an axiomatizability criterion for the classical propositional logic based on implication and negation. According to this criterion, a set X of formulae built up of implication and negation in such a way that negation may only be followed by propositional variables, when added to the axioms for pure implication, axiomatizes the logic based on implication and negation if and only if each unary connective different from negation does not satisfy at least one formula from X. For reference see also ŻARNECKA-BIAŁY 1973.

In the paper WAJSBERG 1932b we find Wajsberg's organic axiom for the Sheffer connective along with his findings involving pure implication and pure equivalence which he found already as an undergraduate student.

xi. Wajsberg's Semantics for Lewis's Modal System S5

The year of 1932 Wajsberg also spent in Warsaw. In February and March he presented two papers to the Section of Logic of the Warsaw Philosophical Society, entitled "From the Research on the Theory of Deduction", and "Axiomatization of Predicate Logic", respectively. The precise contents of the papers is unknown. One may only guess that they were related to his papers WAJSBERG 1933a and 1933b which he prepared for publication at around that time.

In the paper WAJSBERG 1933a the author constructed an adequate semantic characterization of Lewis's system S5, the first example of an adequate semantics in the history of modal logic. As we mentioned in Section viii, this semantics was known to Wajsberg long before 1933 (see LEWIS-LANGFORD 1932, Appendix ii).

Using modern terminology and notation Wajsberg's semantics may be described as follows. Let A be a non-empty set, and let P(A) denote the set of all subsets of A. Let $-_A$ denote the set-complementation operation within A, i.e., if X is a subset of A, then $-_A(X)$ denotes the set of all those elements in A which are not members of X. Let $\cap$ and $\cup$ denote, as usual, the set-intersection and the set-union, respectively. Let I_A be a unary operation in P(A) defined, for every $X \subseteq A$, as shown below:

$$I_A(X) = \begin{cases} A & \text{if } X = A \\ \emptyset & \text{if } X \neq A \end{cases}$$

where, of course, $\emptyset$ denotes the empty set. Let $\underline{P(A)}$ denote the sequence:

$$(P(A), -_A, \cap, \cup, I_A)$$

Thus $\underline{P(A)}$ is a Boolean algebra of subsets of A with the additional unary operation I. A formula is defined as true in $\underline{P(A)}$ if and only if it takes on value A under every assignment of members of $\underline{P(A)}$ to its propositional variables, where the valuation function is defined in such a way that propositional connectives: $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), and L (necessity) correspond to the operations: $-_A$, $\cap$, $\cup$, and I_A, respectively. Now, the main result of Wajsberg may be expressed as follows:

> An arbitrary formula is provable as a theorem of Lewis's system S5 if and only if the formula is true in the system $\underline{P(A)}$, for every non-empty set A.

For reference see also ZACHOROWSKI 1973.

To prove this theorem Wajsberg introduced a kind of normal form procedure. More specifically, he showed that every propositional formula of the form 'La' (read as 'It is necessary that a') is reducible in S5 to a kind of conjunctive normal form where each disjunct consist of 'L' or '¬L' followed by a disjunction of variables (negated or un-negated). It should be noted, however, that this form cannot be used as a general normalization procedure for S5 because only formulae of the form 'La' and not all formulae are so reducible. G.F. Schumm has observed (see SCHUMM 1975) that a slight modification of Wajsberg's original form could do the normalization job. Namely, each formula is reducible in S5 to another conjunctive normal form where each disjunct consist of either 'L' or '¬L' followed by a disjunction of variable or a negated variable.

Notice that Wajsberg's sequence $\underline{P(A)}$, as constructed above, appears to be a kind of the so called nowadays McKinsey-Tarski topological Boolean algebras (see McKINSEY-TARSKI 1944) which are widely used to construct algebraic-type semantics for modal logics.

An inspection of Wajsberg's proof also reveals that a formula containing precisely n propositional variables is a theorem of Lewis's system S5 if and only if it becomes true in Wajsberg's sequence $\underline{P(A)}$, for every set A consisting of 2^n elements. It follows here from that implicit in Wajsberg's proof is a decidability procedure for S5.

At the end of the paper WAJSBERG 1933a the author observed that the replacement of propositional variables: $p_1, p_2, p_3, \ldots$ by monadic formulae of predicate logic of one and the same variable x: $P_1x, P_2x, P_3x, \ldots$ and the replacement of the connective 'L' by the universal quantifier '$\forall$' binding 'x' we can get the (non-modal) monadic predicate logic in one individual variable 'x'. It should be added that similar relation between a modal system and a system of predicate logic has since been found also in respect to some other modal systems (see, for instance THOMAS 1962).

xii. Papers on Predicate Logic

Unlike previous papers, the paper WAJSBERG 1933b concerns the first order predicate logic. Let us call formula of predicate logic k-true if and only if it is true in any of its k-element models, and let us define a k-true formula, which is not (k+1)-true, as exactly k-true. In WAJSBERG 1933b the author constructed an exactly k-true formula from which every k-true formula is deducible,

$$(Ax_k) \qquad \bigvee^{i}_{i\leq k} F_i x_i \supset \bigvee^{m}_{i<m<k+1} F_i x_m$$

where the expression '$\bigvee^{i}_{i\leq k} B(x_i)$' abbreviates the disjunction '$B(x_1) \vee B(x_2) \vee \ldots \vee B(x_k)$', and where '$\supset$' stands for the connective of implication. To see better the syntactic structure of (Ax_k) we give below three particular cases, for k = 1, k = 2, and k = 3, respectively

(Ax_1) $F_1x_1 \supset F_1x_2$,

(Ax_2) $(F_1x_1 \supset F_1x_2 \vee F_1x_3) \vee (F_2x_2 \supset F_2x_3)$,

(Ax_3) $(F_1x_1 \supset F_1x_2 \vee F_1x_3 \vee F_1x_4) \vee (F_2x_2 \supset F_2x_3 \vee F_2x_4) \vee (F_3x_3 \supset F_3x_4)$

For reference see also WOLEŃSKI 1973.

The paper WAJSBERG 1933-1934 also deals with predicate logic. Applying Tarski's notion of the degree of completeness of a deductive system (see TARSKI 1930) Wajsberg provided in the paper a detailed proof that the degree of completeness, i.e., the number of all maximal consistent extensions of the first order logic is equal to the number of the continuum.

xiii. Wajsberg's Criterion of Axiomatizability of Finite Matrices

From Warsaw Wajsberg moved to Kowl in Volhynia where he worked as a teacher to the end of June, 1933. Then he returned to his native Łomża where he continued his teaching career and where the rest of his works were written.

The paper WAJSBERG 1935 included author's well-known result concerning the conditions of axiomatizability of finite logical matrices, including Łukasiewicz's matrices and the so called finite intermediate logics among others. According to his theorem if the formulae below:

CCpqCCqrCpr, CCqrCCpqCpr, CCpqCNqNp, CNqCCpqNp, CCqqCpp

are all satisfied in a finite logical matrix, then the matrix must be axiomatizable. The theorem, with formula CCqrCpp replacing formula CCqqCpp, was stated as Wajsberg's theorem without proof in ŁUKASIEWICZ-TARSKI 1930, i.e., as early as in 1930. Wajsberg's own proof of this theorem, included in his paper, is lengthy and rather difficult to comprehend. A detailed exposition of his proof, with only small changes in notation, can be found in ACKERMANN 1971. See also SZCZĘCH 1973.

xiv. General Approach to Logical Matrices

The general notion of logical matrix was introduced by Tarski (see ŁUKASIEWICZ-TARSKI 1930). The paper WAJSBERG 1936, written in 1934 and published two years later, was conceived as a contribution to the study of logical matrices. In this rather technical paper Wajsberg made an effort to classify logical matrices into types (distinguished in the paper are various special types of matrices such as congruence matrices; linear congruence matrices and sum-matrices as their special case; infinite linear matrices; and conditional matrices along with interval matrices as a special case of the latter). He also described some systematic methods for deciding which formulae built up of implication and negation are satisfied in which matrices of a given type. For reference see also SUCHOŃ 1973. It is rather striking that the discussed Wajsberg's paper has attracted almost no attention from the subsequent researches in the field. In particular, in Łoś's monograph ŁOŚ 1948 Wajsberg's paper is not even mentioned. Neither is it referred to in

J. Kalicki's works on logical matrices (see, for instance, ZYGMUNT 1981).

xv. Separability Property of Intuitionistic Connectives

In WAJSBERG 1938a the separability theorem for a system of intuitionistic propositional logic of axiomatic type was established to the effect that no intuitionistic theorem, from which any one of the four connectives:

(★) ¬ (negation), ⊃ (implication), ∧ (conjunction), ∨ (disjunction)

is absent, requires for its proof any axiom in which the connective is present.

Wajsberg also added a number of interesting results concerning definability of propositional connectives to the effect that none of the mentioned in (★) can be expressed in intuitionistic logic in terms of the remaining three, the result which was also arrived at, independently, in a paper by J.C.C. McKinsey published one year later (see McKINSEY 1939).

In connection with the separability problem it may be worthwhile recalling that on A. Church's suspicion (see his errata to CHURCH 1956, footnote 211) Wajsberg's proof were to contain an error difficult to correct. Without further discussion of the nature of the alleged error Church seemed to suggest that the result should be, therefore, credited to H.B.Curry whose paper CURRY 1939, solving independently, the separability problem by a Gentzen's sequents' technique, appeared one year later. In his monograph CURRY 1963 the author confessed that though he had never examined Wajsberg's proof, he trusted others in considering it erroneous. The suspicion of error has since been repeated by many, among others, by A. Horn who provided the first modern-style algebraic proof of separability property of intuitionistic logic (see HORN 1962). Of the papers which have attempted a detailed reconstruction of Wajsberg's argument two are in order, KABZIŃSKI-PORĘBSKA 1975, and BEZHANISHVILI 1981. In the first paper it is shown that some of Wajsberg's preparatory lemmas admit, in fact slight strengthening which then implies the separability property without complications. In the second paper Wajsberg's Definition 2, #8 of an n-order thesis, claimed to be the source of the alleged error, was changed and so was the proof of Wajsberg's Theorem 14, #8. For reference see also KABZIŃSKI 1973b.

The formulation and the solution of the separability problem for intuitionistic logic did not come as a surprise. It was well motivated by the parallel investigations into the axiomatization of various fragments of the expressively complete ordinary, two-valued propositional logic. The latter can be viewed an investigations into the separation of properties of various classical connectives.

xvi. Miscellany on Propositional Logic

In WAJSBERG 1937 and 1938b the author included a rich crop of various 'incidental' results and remarks on different axiom systems of classical propositional logic and its fragments. Some of them come from his unpublished master's thesis. Various axiom systems for pure implication are listed in WAJSBERG 1937, #1 and #6, and in WAJSBERG 1938b, #1; axiom systems for implication and falsum are discussed in WAJSBERG 1937, #9, and in WAJSBERG 1938b, #2; paper WAJSBERG 1938, #2 also contains various axioms for implication and negation; paper WAJSBERG 1937, #7 is devoted to axiom systems for equivalence. The completeness property of each of the systems referred to above was established by syntactic means, i.e., by deducing from each of them another axiom system, already known to be complete. Paragraph 2 of WAJSBERG 1937 discussed the independence property of various axiom systems. For reference see also STĘPIEŃ 1973.

Paragraph 4 of paper WAJSBERG 1937, entitled "General Scheme of a Completeness Proof for the C-Pure", contains a schematic description of Wajsberg's method of proof of the completeness property for the propositional logic based on implication as the only primitive connective. To solve the completeness problem for the three-valued logic of Łukasiewicz as well as for some other many-valued logics, Wajsberg had to work out an original completeness argument. Later he adjusted the argument to provide a new proof of the completeness theorem for the two-valued logic of pure implication (see WAJSBERG 1937). It should be mentioned that the first proof of the completeness theorem for pure implication was found by Tarski but it was not published by the author (see ŁUKASIEWICZ-TARSKI 1930, and TARSKI 1934-1935).

Wajsberg's method can be characterized briefly as follows:

i. first one must prove that all tautologies built up of one propositional variable are formally deductible by the axioms and rules of an axiomatic system under consideration (a most laborious part of the completeness proof);oof);

ii. then, assuming that all tautologies built up of n different variables each are formally deducible, one must prove that all tautologies built up of (n+1) different variables each are also formally deducible in the axiomatic system (a comparatively easy part of the completeness proof).

Wajsberg's completeness argument has since been in frequent use. In 1938 W.V. Quine followed the plan sketched in WAJSBERG 1937 to solve the completeness problem for the logic based on implication and falsum as the only primitive connectives (see QUINE 1938). In 1943 K. Schroeter provided a Wajsberg-type completeness argument for the full classical logic based on all usual connectives as primitives (see SCHROETER 1943). Wajsberg's method has often been used in Poland (see, for instance, SADOWSKI 1961, and SURMA 1973a). A detailed discussion of Wajsberg's method is also contained in the monograph ASSER 1959.

xvi. Closing Remark

Since the outbreak of the second world war there has been no reliable information concerning Wajsberg's fate. The only fact known for certain is that he has perished prematurely and so all his unpublished manuscripts have been lost.