

On the Parameterized Complexity of Computing Balanced Partitions in Graphs

René van Bevern · Andreas Emil Feldmann ·
Manuel Sorge · Ondřej Suchý

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Abstract A balanced partition is a clustering of a graph into a given number of equal-sized parts. For instance, the BISECTION problem asks to remove at most k edges in order to partition the vertices into two equal-sized parts. We prove that BISECTION is FPT for the distance to constant cliquewidth if we are given the deletion set. This implies FPT algorithms for some well-studied parameters such as cluster vertex deletion number and feedback vertex set. However, we show that BISECTION does not admit polynomial-size kernels for these parameters. For the VERTEX BISECTION problem, vertices need to be removed in order to obtain two equal-sized parts. We show that this problem is FPT for the number of removed vertices k if the solution cuts the graph into a constant number c of connected components. The latter condition is unavoidable, since we also prove that VERTEX BISECTION is W[1]-hard w.r.t. (k, c) . Our algorithms for finding bisections can easily be adapted to finding partitions into d equal-sized parts, which entails additional running time factors

An extended abstract of this article appeared at the *39th International Workshop on Graph-Theoretic Concepts in Computer Science (WG 2013)* [4]. The extended abstract contains results regarding BISECTION and VERTEX BISECTION, while this article additionally provides full proof details as well as parameterized complexity analyses of BALANCED PARTITIONING.

R. van Bevern · M. Sorge (✉)

Institut für Softwaretechnik und Theoretische Informatik, TU Berlin, Germany
e-mail: manuel.sorge@tu-berlin.de

R. van Bevern

e-mail: rene.vanbevern@tu-berlin.de

A. E. Feldmann

Combinatorics, Optimization, University of Waterloo, Waterloo, Ontario, Canada
e-mail: andreas.feldmann@uwaterloo.ca

O. Suchý

Faculty of Information Technology, Czech Technical University in Prague, Prague, Czech Republic
e-mail: Ondrej.suchy@fit.cvut.cz

of $n^{O(d)}$. We show that a substantial speed-up is unlikely since the corresponding task is W[1]-hard w.r.t. d , even on forests of maximum degree two. We can, however, show that it is FPT for the vertex cover number.

Keywords Bisection · NP-hard problems · Problem kernelization · Cliquewidth · Treewidth reduction

1 Introduction

In this article we consider partitioning problems on graphs. These are clustering-type problems in which the clusters need to be equal-sized, and the number of edges connecting the clusters needs to be minimized. At the same time the desired number of clusters is given. We begin with the setting in which only two clusters need to be found, and later generalize to more clusters. In particular we consider the BISECTION, VERTEX BISECTION, and BALANCED PARTITIONING problems, which are defined below. We study these problems from a parameterized complexity point of view and consider several parameters that naturally arise from the known results (see Table 1). That is, we consider a given parameter p of an input instance and ask whether an algorithm with running time $f(p) \cdot n^{O(1)}$ exists that optimally solves the problem. Here n is the instance size and $f(p)$ is a function that only depends on p . If there is such an algorithm, then the problem is called *fixed-parameter tractable* (or

Table 1 Overview of known and new parameterized results

Problem	Parameter	Results
BISECTION	cut size	FPT for planar graphs [10] FPT in general [15] No poly-size kernel (Theorem 6)
	treewidth	FPT [50, 52] No poly-size kernel (Theorem 6)
	union-oblivious (e.g. bandwidth)	No poly-size kernel (Theorem 6)
	cliquewidth- q deletion number	FPT (Theorem 7)
	cliquewidth	XP, W[1]-hard [28]
VERTEX BISECTION	cut size	FPT if nr of cut out components is constant (Theorem 5)
	cut size & nr of cut out components	W[1]-hard (Theorem 1)
BALANCED PARTITIONING	cut size & nr of cut out components	W[1]-hard (Theorem 9)
	treewidth	NP-hard for trees [24]
	cliquewidth	NP-hard for cluster graphs [1]
	vertex cover	FPT (Theorem 10)
	nr d of parts in the partition	W[1]-hard for forests (Theorem 8)

FPT for short) with respect to p . For in-depth introductions to parameterized complexity we refer to the literature [20, 27, 47]. Throughout this article we use standard terminology of graph theory [18].

1.1 The Bisection Problem

The first problem we consider is the NP-hard [33] BISECTION problem for which the n vertices of a graph $G = (V, E)$ need to be partitioned into two parts A and B of size at most $\lceil n/2 \rceil$ each, while minimizing the number of edges connecting A and B . The partition $\{A, B\}$ is called a *bisection* of G , and the number of edges connecting vertices in A with vertices in B is called the *cut size*. Throughout this article it will be convenient to consider BISECTION as a decision problem, which is defined as follows.

BISECTION

Input: A graph G and a positive integer k .

Output: Does G have a bisection with cut size at most k ?

The BISECTION problem is of importance both in theory and practice, and for instance has applications in divide-and-conquer algorithms [42], computer vision [41], and route planning [16]. As a consequence, the problem has been thoroughly studied in the past. It is known that it is NP-hard in general [33] and that the minimum cut size can be approximated within a factor of $O(\log n)$ [49]. Assuming the Unique Games Conjecture, no constant factor approximations exist [38]. For special graph classes such as trees [43] and solid grids [25] the optimum cut size can be computed in polynomial time. For planar graphs it is still open whether BISECTION is NP-hard, but it is known to be FPT with respect to the cut size [10].

It was recently shown by Cygan et al. [15] that BISECTION is FPT with respect to the cut size on general graphs. We complement this result by showing that BISECTION does not allow for polynomial-size problem kernels for this parameter unless $\text{coNP} \subseteq \text{NP/poly}$. Hence, presumably there is no polynomial-time algorithm that reduces an instance of BISECTION to an equivalent one that has size polynomial in the desired cut size. We prove this by giving a corresponding result for all parameters that are polynomial in the input size and that do not increase when taking the disjoint union of graphs. We call such parameters *union-oblivious*. This includes parameters such as treewidth, cliquewidth, bandwidth, and others.

Some of these parameters have been considered for the BISECTION problem before. For instance, we already mentioned the cut size, and it was shown that the problem is FPT with respect to treewidth [50, 52]. Even though treewidth is probably the most widely used graph parameter for sparse graphs, it is not suitable for dense graphs, although they can also have simple structure. For that purpose, Courcelle and Olariu [14] introduced the parameter *cliquewidth* [22]. Fomin et al. [28] showed that BISECTION is $\text{W}[1]$ -hard with respect to cliquewidth, that is, an FPT-algorithm is unlikely. On the positive side, they give an $n^{O(q)}$ -time algorithm if a cliquewidth- q expression is given. Generalizing the latter result we show that BISECTION is FPT with respect to the *cliquewidth- q vertex deletion number*: the number of

vertices that have to be deleted in order to obtain a graph of constant cliquewidth q .¹ To the best of our knowledge this parameter has not been considered in the past. The cliquewidth- q deletion number is a generalization of several well-studied graph parameters like vertex cover number ($q = 1$) [13], cluster vertex deletion number and cograph vertex deletion number ($q = 2$) [14], feedback vertex set number ($q = 3$) [40], and treewidth- t vertex deletion number ($q = 2^{t+1} + 1$) [14, 29].

1.2 The Vertex Bisection Problem

The next problem we consider is the VERTEX BISECTION problem, for which vertices instead of edges need to be removed in order to bisect the graph. More formally, let G be a graph and $S \subseteq V(G)$ be a subset of the vertices of G . We call S an A - B -separator for G if there are vertex sets $A, B \subseteq V(G)$ such that $\{S, A, B\}$ forms a partition of $V(G)$, and there are no edges between A and B in G . Moreover, we call S *balanced* if $||A| - |B|| \leq 1$. The problem then is the following.

VERTEX BISECTION

Input: A graph G and a positive integer k .

Question: Does G contain a balanced separator of size at most k ?

We show that this problem is more general than BISECTION in the sense that any solution to VERTEX BISECTION can be transformed into a solution to BISECTION having (almost) the same cut size and (almost) the same number of cut out connected components, in polynomial time. In contrast to BISECTION however, we prove that VERTEX BISECTION is W[1]-hard with respect to the cut size. In fact this still holds true when combining the cut size and the number of cut out connected components as a parameter. This means that to obtain a fixed-parameter algorithm it is unavoidable to impose some additional constraint.

We show that the VERTEX BISECTION problem is FPT with respect to the cut size, if an optimal solution cuts the graph into a given *constant* number of connected components. We chose this condition as a natural candidate: First, in practice optimal bisections often cut into very few connected components, typically only into two or three [2, 17, 37, 51]. Second, also for random regular graphs the sets A and B of the optimum bisection are connected with high probability [11]. And third, VERTEX BISECTION remains NP-hard even if all optimum bisections cut into exactly two connected components (this follows easily by combining the NP-hardness proof for BISECTION of Garey et al. [33] with our techniques from Section 2.2; see also [9] for a related problem). To achieve our FPT result for VERTEX BISECTION, we generalize the *treewidth reduction* technique for separation problems that has been recently introduced by Marx et al. [46]. By adapting it to the global balancedness constraint of our problem, we address an open question by Marx et al. [46] of whether this is possible.

¹To be precise, we need the vertex deletion set to be given to obtain an FPT algorithm for this parameter.

1.3 The Balanced Partitioning Problem

Apart from BISECTION and VERTEX BISECTION we also study the BALANCED PARTITIONING problem. This is a natural generalization of the BISECTION problem, in which the n vertices of a graph need to be partitioned into d equal-sized parts, for some arbitrary given number d (instead of only two). More formally the problem is defined as follows, where the *cut size* of a partition is the number of edges incident to vertices of different parts.

BALANCED PARTITIONING

Input: A graph G and two positive integers k and d .

Question: Is there a partition of the vertices of G into d sets of size at most $\lceil n/d \rceil$ each and with cut size at most k ?

Note that there is no explicit lower bound on the part sizes. This means that they can technically speaking be unbalanced. However the definition above is the most commonly used one in the literature. Also, for numerous applications such as parallel computing [3] or VLSI circuit design [5], only an upper bound is needed.

Our algorithms for the special case BISECTION can easily be extended to algorithms for BALANCED PARTITIONING, as we will describe in Section 5. However the algorithms have additional running time factors in the order of $n^{O(d)}$. We observe that BALANCED PARTITIONING is W[1]-hard for the number d of cut out parts even on forests of maximum degree two and hence it is unlikely that running time factors of $n^{f(d)}$ can be avoided. Furthermore, we show that the problem remains W[1]-hard on more general graphs for the larger number c of cut out connected components.

Regarding structural graph parameters, many of the known hardness results for BALANCED PARTITIONING already rule out FPT algorithms for parameters such as treewidth or cluster vertex deletion number (BALANCED PARTITIONING is NP-hard for trees [24] and graphs formed by a disjoint union of cliques [1]). On the positive side we can show that BALANCED PARTITIONING is FPT with respect to the vertex cover number τ . Recently Ganian and Obdržálek [31] developed an algorithmic framework with which they were able to show that BALANCED PARTITIONING is FPT with respect to the combined parameters τ and d . Hence we improve on this result by removing the dependence on d .

1.4 Organization of the Article

We begin with presenting our results for VERTEX BISECTION in Section 2. These include the hardness of VERTEX BISECTION, the FPT algorithm in case the number of cut out components is constant, and the reduction from BISECTION showing that VERTEX BISECTION is more general. Section 3 contains incompressibility results for BISECTION. In Section 4 we give our FPT algorithm for the cliquewidth- q deletion number. Hardness results for BALANCED PARTITIONING are given in Section 5, and Section 6 contains the FPT algorithm w.r.t. the vertex cover number.

2 Vertex Bisection and the Cut Size Parameter

In this section we show how to compute optimal bisections that cut into some constant number of connected components in FPT-time with respect to the cut size. As mentioned in the introduction, for VERTEX BISECTION one searches for a small set of vertices in order to bisect a given graph. We note below that VERTEX BISECTION generalizes BISECTION and hence, there is also a corresponding algorithm for BISECTION.

The outline of this section is as follows. First, we note that VERTEX BISECTION is $W[1]$ -hard with respect to k and the number c of cut out components (Section 2.1). Hence, an additional constraint like c being constant is unavoidable to get an FPT-algorithm. We then proceed to show that VERTEX BISECTION indeed generalizes BISECTION (Section 2.2). The FPT algorithm with respect to k and constant number of cut out components for VERTEX BISECTION is given in Section 2.3.

2.1 Hardness of Vertex Bisection

In this section we prove that VERTEX BISECTION is $W[1]$ -hard with respect to the combination of the desired separator size and the number of cut out components. It follows, that it is in particular $W[1]$ -hard for the parameter separator size.

Theorem 1 VERTEX BISECTION is $W[1]$ -hard with respect to the combined parameter (k, c) , where k is the desired separator size and c is the maximum number of components after removing any set of at most k vertices from the input graph.

We reduce from the $W[1]$ -hard CLIQUE problem [20]. The reduction is an adaption of the one Marx [44] used to show $W[1]$ -hardness for the CUTTING ℓ VERTICES problem. The construction we use is as follows.

Construction 1 Suppose we want to construct an instance (G', k) of VERTEX BISECTION from an instance (G, k) of CLIQUE. Without loss of generality, assume that k is even. The graph G' is obtained by first copying G and then subdividing every edge, meaning to replace each $\{u, w\} \in E(G)$ by a new edge vertex $v_{u,w}$ and the edges $\{u, v_{u,w}\}, \{v_{u,w}, w\}$. Next, we make $V(G)$ into a clique in G' and we furthermore add to G' a disjoint clique D with $n + m - k - 2\binom{k}{2}$ vertices, where $n = |V(G)|$ and $m = |E(G)|$. Note that the overall number of vertices in G' is $2n + 2m - k - 2\binom{k}{2}$.

Let us prove that Construction 1 is a parameterized reduction.

Proof (Theorem 1) Assume that G contains a clique C of size k . We claim that C also induces a balanced separator in G' . Let V_C^E be the set of edge vertices in G' corresponding to the edges of $G[C]$. Consider the sets $A := V(G') \setminus (C \cup V_C^E \cup D)$ and $B := D \cup V_C^E$. Clearly, A, B, C form a partition of $V(G')$, and since C

is exactly the neighborhood of V_C^E in G' , there are no edges between A and B . Moreover,

$$\begin{aligned} |A| &= \left(2n + 2m - k - 2\binom{k}{2}\right) - k - \binom{k}{2} - \left(n + m - k - 2\binom{k}{2}\right) \\ &= n + m - k - \binom{k}{2} = n + m - k - 2\binom{k}{2} + \binom{k}{2} = |B|. \end{aligned}$$

Thus, indeed C is a balanced separator for G' .

For the reverse direction first consider any vertex set $S \subseteq V(G')$ with $|S| \leq k$. The graph G' consists of two cliques that, without loss of generality, contain more than k vertices, in addition to degree-two vertices attached to the clique on $V(G)$. Furthermore, any pair of vertices in $V(G)$ has at most one common degree-two neighbor. Hence, the number of connected components of $G' - S$ is at most $\binom{|S|}{2} + 2$, which means that $c \leq \binom{k}{2} + 2$.

Now assume additionally that S is a balanced A - B -separator for G' and, without loss of generality, assume that $(D \setminus S) \subseteq A$. We may furthermore assume that $S \cap D = \emptyset$. Otherwise we may successively replace all vertices in $S \cap D$ with arbitrary vertices from $(V(G') \setminus D) \cap A$, which is always non-empty since

$$\begin{aligned} |D \setminus S| &\leq |D| = n + m - k - 2\binom{k}{2} \\ &< \left(2n + 2m - k - 2\binom{k}{2} - k\right) / 2 \leq (|V(G')| - |S|) / 2. \end{aligned}$$

Note that, without loss of generality, we may assume further that $|S|$ is even. Otherwise, $|S| < k$ because k is even by assumption and we may simply add an arbitrary vertex from A or B to S . Hence, since $|V(G')|$ is also even and $||A| - |B|| \leq 1$, we have $|A| = |B|$. Thus, in addition to the vertices in D the set A needs to get at least

$$\begin{aligned} |V(G') \setminus S| / 2 - |D| \\ \geq \left(2n + 2m - 2k - 2\binom{k}{2}\right) / 2 - \left(n + m - k - 2\binom{k}{2}\right) = \binom{k}{2} \end{aligned}$$

more vertices. Hence, S needs to separate $\binom{k}{2}$ edge vertices from $V(G)$. This can only be achieved if $|S| \geq k$, and hence S induces a clique of size k in G . \square

2.2 Reducing Bisection to Vertex Bisection

Before turning to our algorithm for VERTEX BISECTION we show that it indeed transfers also to BISECTION. That is, VERTEX BISECTION is more general than BISECTION in our setting. We say that S is a c -component separator for G if there are exactly c connected components in $G - S$. A c -component bisection is defined analogously.

Theorem 2 *There is a polynomial-time many-one reduction from BISECTION to VERTEX BISECTION such that the desired separator size is one larger than the*

desired cut size. Furthermore, each c -component bisection for the BISECTION instance yields a $(c+2)$ -component balanced separator for the VERTEX BISECTION instance and vice versa.

The basic idea to prove Theorem 2 is to subdivide each edge and replace each vertex by a large clique in a given instance of BISECTION. As deleting edge vertices corresponding to cut edges in a bisection then yields imbalanced parts, we use a gadget consisting of two very large cliques connected by a path to rebalance the parts.

Construction 2 Let $(G = (V, E), k)$ be an instance of BISECTION and without loss of generality, assume that $k \leq m := |E|$ and denote $n := |V|$. Construct a graph $G' = (V', E')$ as follows. For each vertex $v \in V$, introduce a clique C_v with $3m + 2$ vertices. Next, for every edge $e = \{u, v\} \in E$ introduce a vertex v_e and make v_e adjacent to all vertices in both C_v and C_u . Call the set of such “edge vertices” V'_E . Finally, add two cliques D_1 and D_2 with $5nm$ vertices each and connect two arbitrary vertices of D_1 and D_2 by a path P with $m - 1$ inner vertices. Set the desired separator size to $k + 1$. This finishes the construction of a VERTEX BISECTION instance $(G', k + 1)$.

We claim that Construction 2 yields a proof for Theorem 2.

Proof (Theorem 2) First, Construction 2 can easily be seen to be doable in polynomial time. Let us prove that it is a many-one reduction.

Let $\{A, B\}$ be a bisection of G , that is $A, B \subseteq V$ such that $|A| = |B|$ and there are at most k edges between A and B in G ; call the set of these edges S . Let $A' := \bigcup_{v \in A} C_v \cup \{v_e \in V'_E \mid e \subseteq A\}$, $B' := \bigcup_{v \in B} C_v \cup \{v_e \in V'_E \mid e \subseteq B\}$, and $S' := \{v_e \mid e \in S\}$. Note that S' is an A' - B' -separator of $G'[V'_E \cup \bigcup_{v \in V} C_v]$. Furthermore $-m \leq |A'| - |B'| \leq m$. Observe also that for any $-m \leq i \leq m$ there is a vertex v on P such that $\{v\}$ is a A'' - B'' -separator of $G'[D_1 \cup D_2 \cup P]$ and $|A''| - |B''| \in \{i, i + 1\}$. Hence, choosing v on P appropriately yields a balanced separator $S \cup \{v\}$ for G' that has size $k + 1$. Further, if $\{A, B\}$ is a c -component bisection, then $S \cup \{v\}$ is a $(c + 2)$ -component balanced separator.

For the converse direction, let S be a balanced A - B -separator for G' . Observe that, for the cliques D_1, D_2 and for each of the cliques C_v , $v \in V$, removing S from their vertex set yields a set which is completely contained either in A or in B . Furthermore, not both $D_1 \setminus S$ and $D_2 \setminus S$ are contained in A or in B since, otherwise, this would contradict the balancedness of S . Let us assume without loss of generality, that $D_1 \setminus S \subseteq A$ and $D_2 \setminus S \subseteq B$. We claim that the number a of cliques C_v , $v \in V$, that intersect A is the same as the number b of cliques C_v , $v \in V$, that intersect B . By the above observation, all vertices of a clique C_v not contained in S are either completely contained in A or B . Hence, as A consists of $D_1 \setminus S$, its intersection with the cliques C_v , at most m edge-vertices $v_e \in V'_E$, and at most m vertices from P , we have

$$5nm + (3m + 2) \cdot a + 2m \geq |A| \geq 5nm + (3m + 2) \cdot a - k \geq 5nm + (3m + 2) \cdot a - m$$

and analogously for $|B|$. Without loss of generality, we may assume $|A| \geq |B|$. Using the above size bounds for $|A|$ and $|B|$, we thus obtain that $|A| - |B|$ is at least

$$5nm + (3m + 2) \cdot a - m - (5nm + (3m + 2) \cdot b + 2m) = (3m + 2) \cdot (a - b) - 3m,$$

and, thus, $a = b$ (recall that S is a balanced separator, and hence $||A| - |B|| \leq 1$). Since $D_1 \setminus S \subseteq A$ and $D_2 \setminus S \subseteq B$ we have at least one vertex in $S \cap P$, and thus the number of edge vertices v_e in the balanced separator S is at most k . We conclude that cutting the edges according to the edge vertices in S yields a bisection for G of cut size at most k . Here, too, the bound on the number of connected components is easy to see. \square

2.3 An FPT Algorithm for Cut Size and Constant Number of Cut Out Components

We now outline an FPT algorithm for VERTEX BISECTION. We say that S is an s - t -separator for vertices s, t if there are vertex sets $A, B \subseteq V(G)$ such that S is an A - B -separator and $s \in A$ and $t \in B$. We say that an s - t -separator S is *inclusion-wise minimal*, or just *minimal*, if there is no s - t -separator $S' \subsetneq S$. We first observe that a balanced separator consists of inclusion-wise minimal s - t -separators between a collection of “terminal” vertices s, t . The terminal vertices are chosen one from each of the connected components of the graph without the separator. Guessing the terminals, we can reduce VERTEX BISECTION to finding an “almost balanced” separator consisting of vertices contained in inclusion-wise minimal separators of pairs of terminals. To find such an almost balanced separator, we generalize the “treewidth reduction” technique introduced by Marx et al. [46]. We obtain an algorithm that constructs a graph G' that preserves all inclusion-wise minimal separators of size at most k between some given terminals and has treewidth bounded by some function $g(k, c)$, where c is the number of terminals. Moreover, the algorithm runs in time $f(k, c) \cdot (n + m)$ and also derives a mapping of the vertices between the input graph G and the constructed graph G' which allows to transfer balanced separators of G' to balanced separators of G . Using this algorithm it then only remains to show that weighted VERTEX BISECTION is fixed-parameter tractable with respect to the treewidth. Overall, the algorithm solving VERTEX BISECTION guesses the terminals, reduces the treewidth and then solves the bounded-treewidth problem.

The main ingredient in our FPT algorithm for VERTEX BISECTION is a generalization of the treewidth reduction technique of Marx et al. [46] to graphs with vertex weights. We aim to construct a graph of bounded treewidth that preserves all inclusion-wise minimal s - t -separators of a given size. To this end, we define trimmers.

Definition 1 Let $G = (V, E)$ be a graph, k an integer and $T \subseteq V$. A tuple (G^*, ϕ) of a graph $G^* = (V^*, E^*)$ and a total, surjective, but not necessarily injective mapping $\phi: V \rightarrow V^*$ is called a (k, T) -trimmer of G if the following holds. (Here, we let $\phi^{-1}(v) := \{v' \mid \phi(v') = v\}$, $\phi(V') := \bigcup_{v \in V'} \phi(v)$ for $V' \subseteq V(G)$, and define $\phi^{-1}(V')$ analogously.)

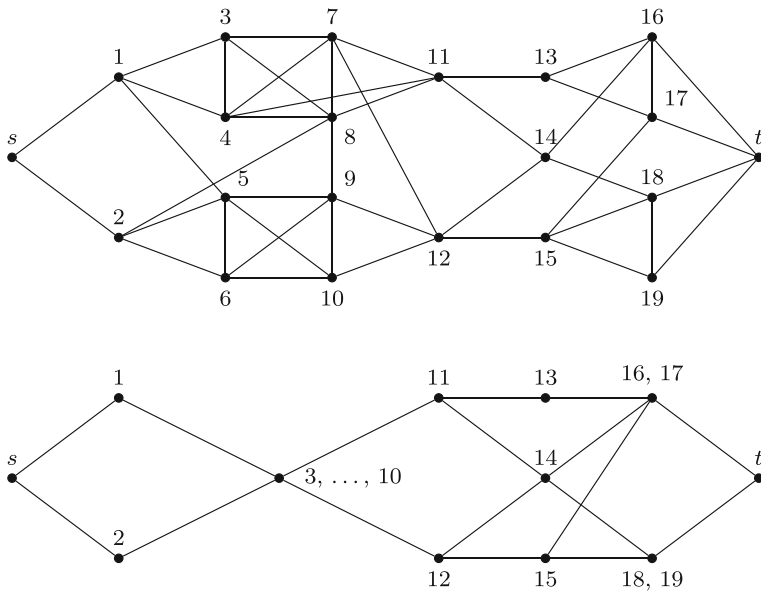


Fig. 1 A graph G (top) and the graph in a $(3, \{s, t\})$ -trimmer of G (bottom), computed as we show in Section 2.3.1. The associated mapping ϕ is indicated by the vertex labels

- (i) For any $S \subseteq V^*$, the mapping ϕ is a one-to-one mapping between the connected components of $G - \phi^{-1}(S)$ and $G^* - S$.
- (ii) If S is an inclusion-wise minimal s - t -separator for G with $|S| \leq k$ and $s, t \in T$, then $\phi(S) = S$ and S is an inclusion-wise minimal $\phi(s)$ - $\phi(t)$ -separator for G^* .

We refer to the above as *trimmer properties* (i) and (ii).

An example for a trimmer is given in Fig. 1: one can verify by inspecting the graph G depicted there, that only vertices 1, 2, 11, 12, 13, 14, and 15 are contained in any inclusion-wise minimal s - t -separator of size at most 3. For example, vertex 3 is not contained in any such separator because removing it from the graph leaves two vertex-disjoint s - t -paths and all pairs of such paths can only be destroyed by removing two further vertices if we delete either vertex 1 and 2, or vertex 11 and 12. However, both $\{1, 2\}$ and $\{11, 12\}$ are themselves s - t -separators and hence adding 3 does not yield an *inclusion-wise minimal* separator. Thus contracting every edge in $G[\{3, \dots, 10, 16, \dots, 19\}]$ we can derive a mapping ϕ that fulfills trimmer property (ii). Basically, trimmer property (i) is obtained by observing that contracting edges keeps intact all important paths.

A more precise description of computing a trimmer, a formal proof of the properties, and an upper bound on the treewidth of the trimmer is given in Section 2.3.1: as we show in Theorem 3, if k and $|T|$ are small, then there are trimmers of small treewidth that can be computed efficiently.

Theorem 3 *Let G be a graph. For every constant $k \in \mathbb{N}$ and constant-size $T \subseteq V$, we can compute a (k, T) -trimmer (G^*, ϕ) for G in $O(n + m)$ time such that the treewidth of G^* is at most $g(k, |T|)$ for some function g depending only on k and $|T|$.*

The final ingredient for our FPT algorithm for VERTEX BISECTION is an efficient algorithm for small treewidth and vertex weights.

Theorem 4 *Let G be a graph with treewidth ω and integer vertex-weights λ . Let Λ be the sum of all vertex weights and let $c \geq 2$ be an integer. We can find in $\omega^{O(\omega)} \cdot c^2 \cdot \Lambda^2 \cdot n$ time, for all integers $1 \leq s \leq \Lambda$, a partition $\{A, B, S\}$ of $V(G)$ such that $\lambda(A) = s$ and S is a minimum-weight c -component A - B -separator, or reveal that no such partition exists.*

We defer also this proof until later in Section 2.3.2. If we suppose for the moment that the above Theorems 3 and 4 hold, then we arrive at the main theorem of this section.

Theorem 5 *Let G be a graph. Given non-negative integers c and k , in $h(c, k) \cdot n^{c+3}$ time we can find a c -component balanced separator for G of size at most k if it exists. Here, $h(c, k)$ is a function depending only on c and k .*

Proof The algorithm proceeds as follows. For each $T \subseteq V(G)$ of size exactly c we compute a (k, T) -trimmer (G^*, ϕ) using Theorem 3. We create a vertex weight function λ for G^* by letting $\lambda(v) = |\phi^{-1}(v)|$. Then, for each s , $|V(G)|/2 - 1 - k \leq s \leq |V(G)|/2 + k$, we compute a minimum-weight c -component A' - B' -separator for G^* and the corresponding sets A' , B' with $\lambda(A') = s$ using Theorem 4. If among the separators there is an A' - B' -separator S' with $|\lambda(A') - \lambda(B')| \leq k - \lambda(S') + 1$, then we compute $S := \phi^{-1}(S')$, $A := \phi^{-1}(A')$, and $B := \phi^{-1}(B')$. Note that, by trimmer property (i), S is a c -component A - B -separator for G . Moreover, since ϕ is a total mapping, $||A| - |B|| \leq k - |S| + 1$. We move $k - |S|$ vertices from A or B to S in such a way that S is a c -component balanced separator for G and we output S . If no suitable separator is found, we output that there is no c -component balanced separator of size at most k for G . Note that, unless $|V(G)|$ is bounded by a function of k (i.e. the problem is trivially FPT), moving the vertices from A or B to S without changing the number of components of $G - S$ is always possible. This is because not every vertex of a connected component can separate it into multiple ones and there is always a component of size at least two.

Let S be a c -component balanced separator of size at most k for G and pick vertices v_1, \dots, v_c , one from each connected component of $G - S$. Let us observe that the above algorithm finds a c -component balanced separator of size at most k . Note that S is a v_i - v_j -separator for each $1 \leq i < j \leq c$. Hence, S contains inclusion-wise minimal v_i - v_j -separators $S_{i,j}$ of size at most k . Let $\hat{S} = \bigcup_{1 \leq i < j \leq c} S_{i,j}$, call a connected component in $G - \hat{S}$ *odd* if it does not contain any v_i , and let \tilde{S} be the union of \hat{S} and all odd components. Note that odd components are contained in S . Hence, \tilde{S} is a c -component \tilde{A} - \tilde{B} -separator for G with $||\tilde{A}| - |\tilde{B}|| \leq k - |\tilde{S}| + 1$ and $|V(G)|/2 - 1 - k \leq |\tilde{A}| \leq |V(G)|/2 + k$. Clearly, at some point in the algorithm

$T = \{v_1, \dots, v_c\}$ and the (k, T) -trimmer (G^*, ϕ) of G is computed. It remains to show that separator \tilde{S} induces a separator in G^* that is found by the algorithm. By trimmer property (ii) we have that $\phi(\hat{S}) = \hat{S}$ is contained in G^* . Trimmer property (i) gives that ϕ is a one-to-one mapping of connected components C in $G - \hat{S}$ and their counterparts $\phi(C)$ in $G^* - \hat{S}$. In particular, there is a corresponding mapping for all odd connected components. Thus, $\phi(\tilde{S})$ is a c -component $\phi(\tilde{A})$ - $\phi(\tilde{B})$ -separator for G^* and we have $\lambda(\phi(\tilde{S})) = |\tilde{S}|$, $\lambda(\phi(\tilde{A})) = |\tilde{A}|$, and $\lambda(\phi(\tilde{B})) = |\tilde{B}|$. Hence, an A' - B' -separator S' for G^* with $\lambda(S') \leq \lambda(\phi(\tilde{S}))$ and $\lambda(A') = \lambda(\phi(\tilde{A}))$ is enumerated by the algorithm of Theorem 4. Applying the size bounds of $\tilde{A}, \tilde{B}, \tilde{S}$ we have $|\lambda(A') - \lambda(B')| \leq k - \lambda(S') + 1$ and $|V(G)|/2 - 1 - k \leq \lambda(A') \leq |V(G)|/2 + k$. Thus, the algorithm described above finds a c -component balanced separator of size at most k for G .

Concerning the running time, there are at most n^c computations of the trimmer, each of which can be done in $f(k, c) \cdot (n + m)$ time (Theorem 3), where $m = |E(G)|$. Then we compute the separators for G^* and since the treewidth of G^* is bounded by some function $g(k, c)$ (Theorem 3), this can be done in time $g(k, c)^{O(g(k, c))} \cdot n^3$ using Theorem 4. Next, the parts A', B' can be computed in linear time from the parts A, B and we can modify the algorithm for Theorem 4 to also output A, B without increasing the running time bound (see Section 2.3.2). Finally, moving the vertices from the parts A or B to S can be done in $O(k(n + m))$ time because we move at most k vertices and a vertex that does not change the number of components can be found in $O(n + m)$ time by taking a leaf of a BFS tree of a component that contains at least two vertices. Hence, the overall running time is bounded by $n^c \cdot (f(k, c) \cdot (n + m) + g(k, c)^{O(g(k, c))} \cdot n^3 + O(n + m) + O(k(n + m)))$ which in turn is bounded by $h(c, k) \cdot n^{c+3}$ for a suitable function h . \square

We remark that an upper bound on the treewidth of the trimmer is $2^{O(k^2)}$ (see Remark 2 below), yielding a $2^{2^{O(k^2)} \cdot c^5} \cdot n^{c+3}$ -time algorithm. This bound can most certainly be improved, and it would be interesting to know what kinds of lower bounds exist. Applying Theorem 2 we can derive the following.

Corollary 1 *Let G be a graph. Given non-negative integers c and k , in $h(c, k) \cdot n^{c+9}$ time we can find a c -component bisection for G of size at most k if it exists. Here, $h(c, k)$ is a function depending only on c and k .*

Note that a direct application of Theorem 2 yields only a factor of $n^{3(c+5)}$ in the running time. The trick is, however, that two of the terminals in instances created by the corresponding Construction 2 do not need to be guessed, while for the others we only have to guess in which vertex clique they appear. They can be assumed to be arbitrary vertices in the appropriate cliques. We omit the straightforward details. We remark that the FPT algorithm given by Cygan et al. [15] has a much better singly exponential runtime, even if c is constant. However, as said above, we believe that the runtime of our algorithm can be significantly improved. Hence our observations might still yield faster FPT algorithms for constant c in instances from practice. As described in the introduction, this would have implications for applications, as there c often is a small constant.

Let us now deliver the proofs of Theorems 3 and 4.

2.3.1 Treewidth Reduction

In order to prove Theorem 3, we need to generalize the results of Marx et al. [46], which we do using the following definition.

Definition 2 Let $G = (V, E)$ be a graph. The *annotated torso* $\text{atorso}(G, W)$, for a set $W \subseteq V$, is a tuple (G', ϕ) of a graph $G' = (V', E')$ and a total, surjective, but not necessarily injective mapping $\phi: V \rightarrow V'$ defined as follows. The graph G' is obtained from G by contracting all edges that have empty intersection with W and removing all loops and parallel edges created in the process. Hence, each connected component of $G - W$ has a corresponding vertex in G' which we call *component vertex*. The mapping ϕ is defined as the identity when restricted to W , that is, for all $v \in W$ we have $\phi(v) = v$. For all remaining vertices $v \in V \setminus W$, ϕ maps v to the component vertex of the connected component of $G - W$ which contains v .

The *torso* $\text{torso}(G, W)$ is obtained from G' in $(G', \phi) = \text{atorso}(G, W)$ as follows. First, make the neighborhood of each component vertex into a clique, creating *shortcut edges*. Then remove all component vertices.²

Some graphs and their (annotated) torso graphs are depicted in Fig. 2. We begin with an observation about annotated torsos and their connected components if some set of vertices is removed.

Lemma 1 Let G be a graph, $W \subseteq V(G)$, and $(G', \phi) = \text{atorso}(G, W)$. For any $S \subseteq V(G')$, the mapping ϕ is a one-to-one mapping between the connected components of $G - \phi^{-1}(S)$ and $G' - S$.

Proof We prove that ϕ maps connected components of $G - \phi^{-1}(S)$ to connected components of $G' - S$ and then prove that it is indeed a one-to-one mapping. Before proving the first part, we note that $s'-t'$ -paths in $G' - S$ translate to s - t -paths in $G - \phi^{-1}(S)$ for all $s \in \phi^{-1}(s')$, $t \in \phi^{-1}(t')$ and vice versa.

- (i) If v_1, \dots, v_ℓ is a path P in $G - \phi^{-1}(S)$, then $\phi(v_1), \dots, \phi(v_\ell)$ is a walk P' in $G' - S$ where multiple consecutive occurrences of a vertex in the second sequence are omitted. Hence there is a $\phi(v_1)$ - $\phi(v_\ell)$ -path in $G' - S$.

It is enough to show that ϕ maps each pair x, y of adjacent vertices either to the same vertex or to adjacent vertices. This is clear if both x, y are in W . If both x and y are in $V(G) \setminus W$ then they are mapped to the same component vertex of G' , as they are in the same connected component of $G - W$. Finally, if x in W and $y \in V(G) \setminus W$ or vice versa, then $\phi(x)$ and $\phi(y)$ are adjacent by the definition of annotated torso.

- (ii) If v_1, \dots, v_ℓ is a path P' in $G' - S$, then there is an s - t -path P in $G - \phi^{-1}(S)$ for every $s \in \phi^{-1}(v_1)$, $t \in \phi^{-1}(v_\ell)$.

²This definition of torso is equivalent to the one used by Marx et al. [46].

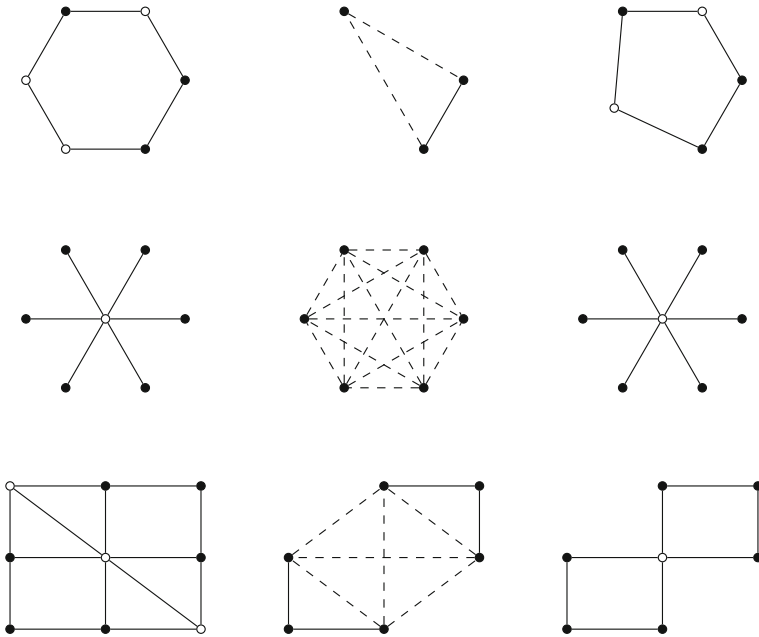


Fig. 2 Graphs G and vertex subsets W (black vertices) on the left, $\text{torso}(G, W)$ in the middle (new shortcut edges are dashed), and the graph in $\text{atorso}(G, W)$ on the right (component vertices are white)

Construct P' as follows. First, consider a component vertex $v_i \in V(P')$ with $1 < i < \ell$, if there is any. Note that v_{i-1} and v_{i+1} are not component vertices, because component vertices are not adjacent with each other. We know that $\phi^{-1}(v_i)$ is a connected component in $G - W$ and, since $v_i \notin S$, also $\phi^{-1}(S) \cap \phi^{-1}(v_i) = \emptyset$. Since v_i is a component vertex adjacent to v_{i-1} and v_{i+1} , there are v'_{i-1} and v'_{i+1} in $\phi^{-1}(v_i)$ adjacent to v_{i-1} and v_{i+1} , respectively. Modify P' by replacing v_i with a v_{i-1} - v_{i+1} -path in $G' - \phi^{-1}(S)$ formed by the edges $v_{i-1}v'_{i-1}$, $v'_{i+1}v_{i+1}$ and a v'_{i-1} - v'_{i+1} -path inside the component $\phi^{-1}(v_i)$.

Next, if v_1 is a component vertex, choose an arbitrary s - v_2 -path \hat{P}' in $G' - \phi^{-1}(S)$. Such a path exists by a similar argument as above. Replace v_1 with \hat{P}' in P' . Proceed analogously if v_ℓ is a component vertex. Since, in this way, we replaced all vertices in $V(P') \setminus W$ with paths that exist in $G - \phi^{-1}(S)$, we have obtained an s - t -walk in $G - \phi^{-1}(S)$. Hence also (ii) is proved.

For a graph G and a set $T \subseteq V(G)$, let us call a set $X \subseteq V(G) \setminus T$ of vertices T -unbroken if there is a path in $G - T$ between any two vertices in X . Note that a connected component of $G - T$ is an inclusion-wise maximal T -unbroken set. Claims (i) and (ii) show that if C is $(\phi^{-1}(S))$ -unbroken in G , then $\phi(C)$ is S -unbroken in G' and if C' is S -unbroken in G' , then $\phi^{-1}(C')$ is $(\phi^{-1}(S))$ -unbroken in G .

Now let us prove that ϕ indeed provides a mapping between the connected components.

(iii) If a set $C \subseteq V(G)$ is a connected component in $G - \phi^{-1}(S)$ then $\phi(C)$ is a connected component in $G' - S$.

If C is a connected component in $G - \phi^{-1}(S)$ then $\phi(C)$ is S -unbroken in G' . Now for the sake of contradiction assume that there is a connected component $K' \supsetneq \phi(C)$ in $G' - S$. Then $\phi^{-1}(K')$ is $(\phi^{-1}(S))$ -unbroken in G and $\phi^{-1}(K') \supsetneq C$ as ϕ is surjective. This contradicts C being a connected component. Hence $\phi(C)$ is a connected component of $G' - S$.

Finally, let us prove that the mapping is one-to-one.

(iv) If a set $C' \subseteq V(G')$ is a connected component in $G' - S$ then $\phi^{-1}(C')$ is a connected component in $G - \phi^{-1}(S)$.

If C' is a connected component in $G' - S$ then $\phi^{-1}(C')$ is $(\phi^{-1}(S))$ -unbroken in G . Now for the sake of contradiction assume that there is a connected component $K \supsetneq \phi^{-1}(C')$ in $G - \phi^{-1}(S)$. Then $\phi(K)$ is S -unbroken in G' and $\phi(K) \supsetneq C'$ by the definition of ϕ^{-1} as $K \supsetneq \phi^{-1}(C')$. This contradicts C' being a connected component. Hence $\phi^{-1}(C')$ is a connected component of $G - \phi^{-1}(S)$. \square

We now show that the treewidth of an annotated torso is at most one larger than the treewidth of the corresponding torso. For this we need to formally introduce the treewidth first. A *tree decomposition* of a graph $G = (V, E)$ is a pair (T, τ) , where T is a rooted tree and τ is a mapping $V(T) \rightarrow 2^V$ such that

- for every $e \in E(G)$, there is an $x \in V(T)$ with $e \subseteq \tau(x)$, and
- for every $v \in V$ the set $T(v) = \{x \in V(T) \mid v \in \tau(x)\}$ induces a non-empty subtree of T .

The sets $\tau(x)$ are sometimes called *bags*. The *width* of a tree decomposition (T, τ) is $\max\{|\tau(x)| \mid x \in V(T)\} - 1$ and the *treewidth* $\omega(G)$ of a graph G is the minimum width of a tree decomposition for G .

Lemma 2 *Let $G = (V, E)$ be a graph, let $W \subseteq V(G)$ and let $\text{torso}(G, W)$ be of treewidth ω . Then, the graph in $\text{atorso}(G, W)$ has treewidth at most $\omega + 1$.*

Proof Let (T, τ) be a tree-decomposition for $\text{torso}(G, W)$. Note that, to obtain a tree-decomposition for $\text{atorso}(G, W)$ we only need to incorporate the component vertices and their incident edges. For this, successively consider each component vertex v and let N be its neighborhood in $\text{atorso}(G, W)$. Due to the shortcut edges between neighbors of v , N induces a clique in $\text{torso}(G, W)$. Hence there is a bag $\tau(t)$, $t \in V(T)$, that contains N [38, Lemma 2.2.2.]. Add a new vertex t' to T adjacent only to t and define $\tau(t') = \tau(t) \cup \{v\}$. Hence, the bags containing v or N , respectively, (still) induce a subtree in $V(T)$. Furthermore, each edge incident to v is contained in $\tau(t')$. Note that we never need to introduce a copy of a bag that does not occur in the tree-decomposition of $\text{torso}(G, W)$ because component vertices are not adjacent with each other. Thus, introducing a new bag for all component vertices in the above-described way yields a tree-decomposition for $\text{atorso}(G, W)$ and increases the maximum bag size by at most one. \square

Remark 1 We mention in passing that the notion of annotated torso is more robust with respect to the treewidth than the notion of torso introduced by Marx et al. [46] in the following sense. Adding vertices to the set W increases the treewidth of both $\text{torso}(G, W)$ and the graph in $\text{atorso}(G, W)$ by at most one, as Corollary 2 (below) shows. However, removing a vertex from W may increase the treewidth of $\text{torso}(G, W)$ arbitrarily. This is witnessed by a star where W contains all vertices; removing the center of the star from W yields a clique in the torso (see Fig. 2). In contrast, the treewidth of annotated torsos cannot increase when removing a vertex v from W . This is because, either the graph in the annotated torso stays the same, or removing v is equivalent to contracting an edge between v and a component vertex. That is, the resulting graph is a minor of the original annotated torso graph.

As we observe next, annotated torsos can be computed in linear time.

Lemma 3 *Let $G = (V, E)$ be a graph and let $W \subseteq V$. Then, $(G', \phi) = \text{atorso}(G, W)$ and ϕ^{-1} can be computed in $O(n + m)$ time.*

Proof To compute the annotated torso, color the connected components of $G - W$ with distinct colors $1, \dots, c$. Next, create a copy G' of G and modify G' by contracting all edges that have empty intersection with W and removing all self-loops and parallel edges. In order to achieve a linear time bound for this, we proceed as follows (we assume G' to be represented as an adjacency list structure).

We start with vertices in W having their adjacency lists as in G and introduce new vertices v_1, \dots, v_c (the component vertices) with empty lists. For each w in W we process its adjacency list and for each vertex x on the list we distinguish the following cases:

- Case 1: the vertex x is in W . We make no changes to the adjacency lists and proceed with the next vertex in the adjacency list.
- Case 2: the vertex x is in $V \setminus W$, in the component of color i and w is the last entry in the adjacency list of vertex v_i . Then we delete x from the adjacency list of vertex w .
- Case 3: the vertex x is in $V \setminus W$, in the component of color i and w is not the last entry in the adjacency list of vertex v_i . Then we add w to the end of adjacency list of v_i and replace x by v_i in the adjacency list of vertex w .

Note that if w appears on the list of v_i , it must be at the end, since we only process another vertex in W after we finish processing w . Hence, this procedure indeed computes G' according to the definition of $\text{atorso}(G, W)$. This proves the running time bound of computing the graph G' .

Using the colors of the vertices in G we can create an array to compute ϕ efficiently. To compute ϕ^{-1} we use an array indexed by the colors which at index i contains a list of all vertices with color i in G . Filling the array can be done in linear time. \square

To prove Theorem 3 we reuse some results by Marx et al. [46]. To state them we need to define the excess of a cut.

Definition 3 Let G be a graph and $s, t \in V(G)$. Let ℓ be the minimum size of an s - t -separator in G . The *excess* of an s - t -separator S is $|S| - \ell$.

The following three statements have been proved by Marx et al. [46]; they correspond to Lemma 2.11, Lemma 2.8, and Corollary 2.10, respectively.

Lemma 4 (Marx et al. [46]) *Let s, t be two vertices of a graph G and let ℓ be the minimum size of an s - t -separator. For some $e > 0$ let C be the union of all inclusion-wise minimal s - t -separators having excess at most e (that is, having size at most $k = \ell + e$). Then, there is an $f(\ell, e) \cdot (n+m)$ -time algorithm that returns a set $C' \supseteq C$ disjoint from $\{s, t\}$ such that $\omega(\text{torso}(G, C')) \leq g(\ell, e)$, for some functions f and g depending only on ℓ and e .*

Lemma 5 (Marx et al. [46]) *Let $G = (V, E)$ be a graph and $W_1, \dots, W_r \subseteq V$ and let $W := \bigcup_{i=1}^r W_i$. We have $\omega(\text{torso}(G, W)) \leq 1 + \sum_{i=1}^r \omega(\text{torso}(G, W_i))$.*

Corollary 2 (Marx et al. [46]) *For every graph G and sets $C, X \subseteq V(G)$, we have $\omega(\text{torso}(G, C \cup X)) \leq \omega(\text{torso}(G, C)) + |X|$.*

Finally we put all the above results together in order to prove Theorem 3. We have to only slightly adapt the proof of the treewidth-reduction theorem by Marx et al. [46].

Proof (Theorem 3) We first use Lemma 4 to compute a set $C_{s,t}$ for each pair $s, t \in T$ such that $C_{s,t}$ contains each inclusion-wise minimal s - t -separator of size at most k . Let C' be the union of all the $O(|T|^2)$ sets $C_{s,t}$. Using the upper bound on the treewidth of Lemma 4 and combining it with the bound of Lemma 5, we obtain that $\text{torso}(G, C')$ has treewidth bounded by some function $g(k, |T|)$ depending only on k and $|T|$. Hence, by Corollary 2 also $\text{torso}(G, C' \cup T)$ has treewidth bounded by such a function. Using Lemma 2 we have a similar bound on the graph G^* in $(G^*, \phi) = \text{atorso}(G, C' \cup T)$. We claim that (G^*, ϕ) is a (k, T) -trimmer of G ; by the above, the treewidth of G^* is bounded and (G^*, ϕ) can be computed within a time bound as claimed by the theorem and it only remains to prove that the two trimmer properties hold. For trimmer property (i), observe that it follows directly from Lemma 1. Trimmer property (ii) is also not hard to obtain. Consider an inclusion-wise minimal s - t -separator S for G with $s, t \in T$. Clearly, since $S \subseteq C' \cup T$ and ϕ is one-to-one on $C' \cup T$ we have $\phi(S) = S$ and by Lemma 1 $\phi(S)$ is a $\phi(s)$ - $\phi(t)$ -separator for G^* . Since there is an s - t -path P in $G - S'$ for every $S' \subsetneq S$ and P is preserved by contracting edges contained in $V \setminus (C' \cup T)$ (which is disjoint to S), there is also a $\phi(s)$ - $\phi(t)$ -path in $G^* - \phi(S')$. Hence, since ϕ is one-to-one on $C' \cup T$ there is a $\phi(s)$ - $\phi(t)$ -path in $G^* - S''$ for any $S'' \subsetneq \phi(S)$. Thus, trimmer property (ii) holds. \square

Remark 2 The concrete function of g' such that the treewidth of G^* is at most $g'(k, |T|)$ depends mainly on the function $g(\ell, e)$ from Lemma 4. As Marx et al. [46] note, $g = 2^{O(e\ell)}$ and they also show that hypercubes yield a lower bound

on g which is exponential in ℓ and in \sqrt{e} (see [46, Remark 2.14]): they note that there are choices of vertices s and t in an n -dimensional hypercube graph such that each remaining vertex is contained in an inclusion-wise minimal s - t -separator of size at most $n(n-1)$. Hence, taking the torso with respect to the vertex set containing s , t , and their inclusion-wise minimal separators of size at most $n(n-1)$ yields an unchanged hypercube graph. Since hypercube graphs have treewidth $\Omega(2^n/\sqrt{n})$ [12], this yields the lower bound on g . The same is true for taking the corresponding annotated torso and, hence, there is also a corresponding lower bound on g' .

To prove Theorem 5 it now only remains to show Theorem 4 which is done in the next section.

2.3.2 An FPT Algorithm for Vertex Bisection w.r.t. Treewidth

To prove Theorem 4 we define a table that can be filled by dynamic programming over a tree-decomposition. Let us fix some more notation.

Let G be a graph with non-negative integer vertex weights and weight function λ . Furthermore, let (T, τ) be a tree decomposition for G . For $t \in V(T)$ denote by G_t the graph induced by the subtree of T rooted at t ; that is, G_t is the graph induced by the vertex set $\bigcup_{t'} \tau(t')$, where the union is taken over all successors t' of t in T . For a partition P of some set, a pair of partitions P_1, P_2 of the same set is called a *splitting* of P if P is the finest common coarsening of P_1 and P_2 . In other words, the transitive closure of the union of the equivalence relations corresponding to P_1 and P_2 yield the equivalence relation corresponding to P . By $P - v$ we denote the partition derived from P by removing v from the part it is contained in.

We define the table $Sep(t, S_t, P_A, P_B, c, \ell)$, where $t \in V(T)$, $S_t \subseteq \tau(t)$, $P_A \cup P_B$ is a partition of $\tau(t) \setminus S_t$, and c, ℓ are non-negative integers. The entry $Sep(t, S_t, P_A, P_B, c, \ell)$ contains the minimum weight $\lambda(S)$ of an A - B -separator S of G_t such that

- i) $\lambda(A) = \ell$,
- ii) $S \cap \tau(t) = S_t$,
- iii) S is a $(|P_A \cup P_B| + c)$ -component separator for G_t ,
- iv) taking the set of intersections of each connected component of $G_t[A]$ with $\tau(t)$ yields exactly P_A (after removing the empty set if present) and analogously for $G_t[B]$ and P_B .

If no such separator exists, we let $Sep(t, S_t, P_A, P_B, c, \ell) = \perp$. Note that, for non- \perp values of Sep , the above conditions imply that no two vertices from different parts of $P_A \cup P_B$ are adjacent in G_t . In the following we will tacitly assume that this is always the case. This is no restriction, since it is easily checkable in $O(\omega^2)$ time.

Lemma 6 *Let G be a graph with non-negative integer weights λ on the vertices and let Λ be the sum of all weights. Furthermore, let (T, τ) be a tree decomposition for G of width ω with root t . Then, we can compute all the values $Sep(t, S_t, P_A, P_B, c, \ell)$ in $O(\omega^{O(\omega)} \cdot c^2 \cdot \Lambda^2 \cdot n)$ time.*

Proof By [39, Lemma 13.1.3] we may assume that (T, τ) is a nice tree decomposition with at most $4n$ bags, that is

- T is a binary tree,
- if a node x in T has two children y, z , then $\tau(x) = \tau(y) = \tau(z)$ (in this case x is called a *join node*), and
- if a node x in T has one child y , then one of the following situations must hold
 - $\tau(x) = \tau(y) \cup \{v\}$ for some $v \in V(G) \setminus \tau(y)$ (in this case x is called an *introduce node*), or
 - $\tau(x) \setminus \{v\} = \tau(y)$ for some $v \in V(G) \setminus \tau(x)$ (in this case x is called a *forget node*).

We prove that for any node t of T the corresponding values for Sep can be computed in $\omega^{O(\omega)} \cdot c^2 \cdot \Lambda^2$ time if the values for the children of t in T are already known or if t is a leaf. The result then follows by computing the values Sep in a bottom-up fashion on the at most $4n$ nodes of T .

First if t is a leaf in T , then $G_t = G[\tau(t)]$ and it is trivial to obtain the values of Sep : Simply check whether S_t is a separator adhering to (i) through (iv) for all sets $S_t \subseteq \tau(t)$, all bipartitions of $\tau(t) \setminus S_t$ into W_A and W_B , all pairs of partitions P_A, P_B of W_A and W_B , and all values for ℓ . The check takes $O(\omega^2)$ time and there are at most 2^ω sets S_t , at most 2^ω bipartitions of $\tau(t) \setminus S_t$ into W_A and W_B , at most $\omega^{O(\omega)}$ pairs of partitions P_A, P_B of W_A and W_B , and at most Λ values for ℓ . Filling the entries of Sep for leaves t can thus be done in $2^\omega \cdot 2^\omega \cdot \omega^{O(\omega)} \cdot \Lambda \cdot \omega^2 = \omega^{O(\omega)} \cdot \Lambda$ time.

If t is a join node, consider its children t_1, t_2 in T . We claim that

$$Sep(t, S_t, P_A, P_B, c, \ell) = \min(Sep(t_1, S_t, P_{A_1}, P_{B_1}, c_1, \ell_1) + Sep(t_2, S_t, P_{A_2}, P_{B_2}, c_2, \ell_2) - \lambda(S_t)), \quad (1)$$

where the minimum is taken over all $0 \leq \ell_1, \ell_2 \leq \ell$ such that $\ell_1 + \ell_2 = \ell - \lambda(\bigcup P_A)$, over all $0 \leq c_1, c_2 \leq c$ such that $c_1 + c_2 = c$, and over all pairs of splittings P_{A_1}, P_{A_2} and P_{B_1}, P_{B_2} of the partitions P_A and P_B , respectively. Let us prove the claim.

“ \geq ”: Let S be an A - B -separator for G_t corresponding to the left-hand side $Sep(t, S_t, P_A, P_B, c, \ell)$ and let S_1, S_2 be the intersections of S with the vertex sets of G_{t_1}, G_{t_2} , respectively. Let us show that S_1, S_2 both adhere to (i) through (iv) for different but tightly related values of P_A, P_B, c , and ℓ . Clearly, for $\phi \in \{1, 2\}$ we have that S_ϕ is an A_ϕ - B_ϕ -separator in G_{t_ϕ} , where A_ϕ, B_ϕ are the intersections of A and B with the vertex set of G_{t_ϕ} . Furthermore, $S_\phi \cap \tau(t_\phi) = S_t$ that is (ii) holds for S_ϕ . It is also clear that the set S_ϕ is an A_ϕ - B_ϕ -separator for G_{t_ϕ} for some pair of partitions P_{A_ϕ}, P_{B_ϕ} such that (iv) holds. Further, for $\Gamma \in \{A, B\}$ consider the intersection graph F_Γ of the sets in $P_{\Gamma_1} \cup P_{\Gamma_2}$. The sets in P_Γ correspond exactly to the connected components of F_Γ . Hence, the pair $P_{\Gamma_1}, P_{\Gamma_2}$ is a splitting of P_Γ . Next, for $\phi \in \{1, 2\}$ the set S_ϕ is a $(|P_{A_\phi} \cup P_{B_\phi}| + c_\phi)$ -component separator in G_{t_ϕ} for some c_ϕ , i.e. (iii) holds for this value. Since all connected components left by removing S from G_t that do not intersect $\tau(t)$ are either contained in G_{t_1} or G_{t_2} , we have $c_1 + c_2 = c$. Finally, since $A_1 \cap A_2$ is exactly the vertex set partitioned by each

of P_{A_1} , P_{A_2} , and P_A , if we set $\ell_\phi = |A_\phi|$ for $\phi \in \{1, 2\}$, we have that (i) holds for S_ϕ and ℓ_ϕ . Furthermore, $\ell_1 + \ell_2 = \ell - \lambda(P_A)$. This proves the “ \geq -inequality part” of (1).

It is not hard to check that also two separators corresponding to the values on the right hand side of (1) give a separator corresponding to the left hand side. Hence (1) holds. As to computing $\text{Sep}(t, S_t, P_A, P_B, c, \ell)$, observe that c_2 is fixed once we fix c_1 and analogously for ℓ_1 and ℓ_2 . Thus, $\text{Sep}(t, S_t, P_A, P_B, c, \ell)$ can be computed by considering $c \cdot \Lambda$ times all pairs of splittings of P_A and P_B . To iterate over all splittings, one can simply iterate over all partitions of the corresponding vertex sets and check whether the partitions induce splittings in $\omega^{O(1)}$ time. Hence the overall running time for computing $\text{Sep}(t, S_t, P_A, P_B, c, \ell)$ is $c \cdot \Lambda \cdot \omega^{O(\omega)}$ and computing all values Sep for a join node t can be done in $c^2 \cdot \Lambda^2 \cdot \omega^{O(\omega)}$ time.

If t is an introduce node, consider the child t' of t and $\{v\} = \tau(t) \setminus \tau(t')$. Considering an A - B -separator for G_t corresponding to $\text{Sep}(t, S_t, P_A, P_B, c, \ell)$ and the separator it induces in $G_{t'}$, the following is easy to observe.

$$\text{Sep}(t, S_t, P_A, P_B, c, \ell) = \begin{cases} \text{Sep}(t', S_t \setminus \{v\}, P_A, P_B, c, \ell) + \lambda(v), & \text{if } v \in S_t \\ \min \text{Sep}(t', S_t, P'_A, P_B, c, \ell - \lambda(v)), & \text{if } v \in \bigcup P_A \\ \min \text{Sep}(t', S_t, P_A, P'_B, c, \ell), & \text{if } v \in \bigcup P_B. \end{cases} \quad (2)$$

Here the minimum in the second case is taken over all sets P'_A such that P'_A and $\{N[v] \cap (\tau(t) \setminus S_t)\} \cup \{\{u\} \mid u \in \bigcup P_A \setminus N[v]\}$ form a splitting of P_A . Similarly, the minimum in the third case is taken over all sets P'_B such that P'_B and $\{N[v] \cap (\tau(t) \setminus S_t)\} \cup \{\{u\} \mid u \in \bigcup P_B \setminus N[v]\}$ form a splitting of P_B . Note that in the second case v has no neighbors in any set of P_B and in the third case v has no neighbors in any set of P_A . If this is not true then, technically (2) may not hold. However, as explained above, we may ignore this situation since then, clearly, $\text{Sep}(t, S_t, P_A, P_B, c, \ell) = \perp$ and it is easily checkable in $O(\omega^2)$ time. Hence, using (2) we can compute $\text{Sep}(t, S_t, P_A, P_B, c, \ell)$ in $c \cdot \Lambda \cdot \omega^{O(\omega)}$ time for all table entries at an introduce node t .

If t is a forget node, consider its child t' and $\tau(t') = \tau(t) \cup \{v\}$. Consider an A - B -separator S corresponding to $\text{Sep}(t, S_t, P_A, P_B, c, \ell)$. There are again three cases to consider. First, v can be part of the desired minimum separator $S \supseteq S_t$, then $\text{Sep}(t, S_t, P_A, P_B, c, \ell) = \text{Sep}(t, S_t \cup \{v\}, P_A, P_B, c, \ell)$. Second, v can be in A , then either v is not connected to any vertex in $\tau(t') \setminus S$ and $\text{Sep}(t, S_t, P_A, P_B, c, \ell) = \text{Sep}(t, S_t, P_A \cup \{v\}, P_B, c - 1, \ell)$. Otherwise, $\text{Sep}(t, S_t, P_A, P_B, c, \ell) = \min \text{Sep}(t, S_t, P'_A, P_B, c, \ell)$, where the minimum is taken over all P'_A derived from P_A by adding v to a part. The third case $v \in B$ is analogous to the second case. Hence, to compute $\text{Sep}(t, S_t, P_A, P_B, c, \ell)$ we have to simply keep the minimum weight assumed in one of the cases. It is easy to check that, using the above case-distinction, we can also compute all the table entries for forget nodes in $c \cdot \Lambda \cdot \omega^{O(\omega)}$ time. This concludes the proof. \square

Theorem 4 now follows as an easy corollary.

Proof (Theorem 4, Sketch) We first compute a nice tree-decomposition (T, τ) of width $O(\omega)$ for the input graph G , which is possible in $2^{O(\omega)} \cdot n$ time [7]. We

then compute all the values $Sep(t, S_t, P_A, P_B, c, \ell)$ where t is the root of T using Lemma 6. Using standard techniques, we retrace the minima in the corresponding dynamic program to find an A - B -separator S and the corresponding sets A, B for all the values $Sep(t, S_t, P_A, P_B, c, \ell)$. For every $1 \leq \ell \leq s$ and for every of the $O(n)$ bags of (T, τ) only a constant number of disjoint set union operations is needed to obtain the sets A, B and S . This amounts to $O(n^3)$ time spent. Hence, deriving the sets does not increase the running time bound of Lemma 6. \square

3 Incompressibility of Bisection

Problem kernelization is a powerful preprocessing tool in attacking NP-hard problems [6, 34]. A *reduction to a problem kernel* is an algorithm that, given an instance I with parameter p of a parameterized problem, in time polynomial in $(|I| + p)$ outputs an instance I' of the same problem and a parameter p' such that

- i) I is a yes-instance if and only if I' is a yes-instance,
- ii) $|I'| + p' \leq f(p)$, where f is a function only depending on p .

The function f is called the *size* of the problem kernel. It is desirable to find problem kernels of size polynomial in the parameter p .

In this section, we show that, unless a reasonable complexity-theoretic assumption fails, BISECTION has no polynomial-size kernel with respect to the cut size (the “standard parameter”) and any parameter that is polynomial in the input size and does not increase when taking disjoint unions of graphs. Let us call such parameters *union-oblivious*. Our result excludes polynomial-size problem kernels for the parameters treewidth, cliquewidth, or bandwidth, for example.

Theorem 6 *Unless $coNP \subseteq NP/poly$, BISECTION does not admit polynomial-size kernels with respect to the desired cut size and any union-oblivious parameter.*

To prove Theorem 6, we first show that a version of BISECTION with integer edge weights does not have a polynomial-size kernel, and then show how to remove the weights. To obtain that EDGE-WEIGHTED BISECTION does not have a polynomial-size kernel, it is sufficient to show a cross composition (cf. Bodlaender et al. [8]) from the NP-hard [32] MAXIMUM CUT problem to EDGE-WEIGHTED BISECTION. MAXIMUM CUT is defined as follows.

MAXIMUM CUT

Input: A graph $G = (V, E)$ and an integer k .

Question: Is there a partition of V into sets A and B such that at least k edges have one endpoint in A and one in B ?

Showing the cross composition amounts to the following. We give a polynomial-time algorithm that transforms input instances $(G_1, k_1), \dots, (G_t, k_t)$ of MAXIMUM CUT into one instance (G^*, k^*) of EDGE-WEIGHTED BISECTION such that (G^*, k^*) is a yes-instance if and only if one of the MAXIMUM CUT instances is, and such that k^* is polynomial in the size of the largest input instance.

Construction 3 The construction resembles the reduction given for the NP-hardness of BISECTION by Garey et al. [33]. To ease the presentation of the construction we assume the following without loss of generality.

- i) Each of the G_i , $1 \leq i \leq t$, has exactly n vertices and $k_1 = \dots = k_t =: k$. We may assume this because it implies a polynomial-time computable equivalence relation on the instances of MAXIMUM CUT, see Bodlaender et al. [8].
- ii) It holds that $1 \leq k \leq n^2$. Indeed, if $k = 0$ then all instances are yes-instances, and if $k > n^2$ then all instances are no-instances. Hence, if not $1 \leq k \leq n^2$, we can return a trivial yes-instance or no-instance of EDGE-WEIGHTED BISECTION.
- iii) The number t of input instances is odd. Otherwise, we can add a no-instance to the list of input instances that consists of the edgeless graph on n vertices.

We create G^* as follows. For each input graph $G_i = (V_i, E_i)$, $1 \leq i \leq t$, add to G^* the vertices in V_i and a clique V'_i with $|V_i|$ vertices and edges of weight $W := n^2$ each. We make all vertices in V'_i adjacent to all vertices in V_i in G^* via an edge of weight W . Now, for each pair $v, w \in V_i$, we add an edge $\{v, w\}$ to G^* with weight W if $\{v, w\} \notin E_i$ and with weight $W - 1$ if $\{v, w\} \in E_i$. We set $k^* := Wn^2 - k$.

Let us prove that Construction 3 is the promised cross composition.

Lemma 7 *Construction 3 is a cross composition from MAXIMUM CUT to EDGE-WEIGHTED BISECTION with respect to the desired cut weight and any union-oblivious parameter.*

Proof First, it is clear that the desired cut weight in an instance created by Construction 3 is bounded by a polynomial in n because $W = n^2$. Furthermore, let us note that any union-oblivious parameter as above is polynomial in n . This is true because the constructed output graph G^* consists of connected components, each having at most $2n$ vertices. (Recall that union-oblivious parameters are polynomial in the input size and do not increase when taking the disjoint union of graphs.)

It is also clear that Construction 3 can be carried out in polynomial time. It thus remains to show that the instance (G^*, k^*) output by Construction 3 is a yes-instance for EDGE-WEIGHTED BISECTION if and only if there is an $i \in \{1, \dots, t\}$ such that the input instance (G_i, k) is a yes-instance for MAXIMUM CUT.

(\Leftarrow) Without loss of generality, let (G_1, k) be a yes-instance for MAXIMUM CUT. Then, $V_1 = A \uplus B$ such that there are at least k edges with one vertex in A and the other in B .³ We show how to construct a solution for EDGE-WEIGHTED BISECTION in G^* by partitioning G^* into two vertex sets A' and B' , where we choose A' and B' as follows.

- i) A' contains $V_1 \cap A$, arbitrary $|B|$ vertices of V'_1 , and $\bigcup_{i=2}^{\lceil t/2 \rceil} V_i \cup V'_i$;
- ii) B' contains $V_1 \cap B$, the $|A|$ vertices of $V'_1 \setminus A'$, and $\bigcup_{i=\lceil t/2 \rceil+1}^t V_i \cup V'_i$.

³Here, \uplus denotes the disjoint union of sets.

Obviously, $|A'| = |B'|$ since t is odd. We analyze the total weight of edges cut by the partition into A' and B' . Only edges between vertices in $V_1 \uplus V'_1$ are cut. The graph induced by $V_1 \uplus V'_1$ is a clique and each of A' and B' contains exactly n vertices of this clique. Since for k out of the n^2 cut edges we pay the cheaper weight of $W - 1$ instead of W , the total weight of the cut edges is $Wn^2 - k = k^*$. It follows that (G^*, k^*) is a yes-instance for EDGE-WEIGHTED BISECTION.

(\Rightarrow) Assume that, for all $i \in \{1, \dots, t\}$, (G_i, k) is a no-instance for MAXIMUM CUT. We analyze the number of edges cut by a bisection of G^* into $A \uplus B$. Consider a connected component induced by $V_i \uplus V'_i$ of G^* . Let $a_i = |(V_i \uplus V'_i) \cap A|$ be the number of vertices cut from this component by the bisection. Since (G_i, k) is a no-instance, the maximum cut in each G_i cuts at most $k - 1$ edges. Hence the weight of the edges between $(V_i \uplus V'_i) \cap A$ and $(V_i \uplus V'_i) \cap B$ is at least $Wa_i(2n - a_i) - (k - 1)$: for at most $k - 1$ edges we pay the cheaper cost of $W - 1$ instead of W .

Since t is odd, at least one component of G^* has to be cut. First, assume that only one of the components of G^* is cut by the bisection. In this case $a_i = n$ for the corresponding value i since the partition into A and B has to be balanced, and the cut size of the bisection is at least $Wn^2 - k + 1 > k^*$. Hence (G^*, k^*) is a no-instance.

We now show that there is indeed only one component of G^* that is cut. Assume that this is not the case. Then, there are non-zero values a_i, a_j for some $i \neq j$. We may assume without loss of generality that $a_i + a_j \leq 2n$. Otherwise, we can define the a_i values as the number of vertices cut out by B instead of A . By the argument above, the total weight of edges cut in both components is at least

$$Wa_i(2n - a_i) + Wa_j(2n - a_j) - 2(k - 1) = 2nW(a_i + a_j) - W(a_i^2 + a_j^2) - 2(k - 1).$$

Now, consider cutting all $a_i + a_j$ vertices from only one component instead, which is possible since $a_i + a_j \leq 2n$. The total weight of edges cut in the considered components would be at most

$$W(a_i + a_j)(2n - (a_i + a_j)) = 2nW(a_i + a_j) - W(a_i^2 + a_j^2) - 2Wa_ia_j.$$

We assumed however that $k \leq n^2 = W$, and $a_i, a_j \neq 0$. Hence the cut size would drop after changing the bisection, which contradicts its minimality. \square

We are now ready to prove Theorem 6.

Proof (Theorem 6) It remains to show that the instance (G^*, k^*) of EDGE-WEIGHTED BISECTION resulting from Construction 3 can be converted to a BISECTION instance such that the considered parameters remain polynomial in n .

Let (G^*, k^*) be the instance of EDGE-WEIGHTED BISECTION resulting from Construction 3. We create an equivalent instance (G', k^*) of BISECTION as follows.

- i) For each vertex v of G^* , introduce a clique C_v with $W + k^* + 2$ vertices to G' .
- ii) For an edge $\{v, w\}$ of G^* with weight ω , add ω pairwise disjoint edges from C_v to C_w .

Now it is easy to see that (G^*, k^*) is a yes-instance if and only if (G', k^*) is, since no bisection with cut size at most k^* can cut a clique C_v that was introduced for a vertex v .

It is clear that the desired cut size is polynomial in n . It remains to show $\phi(G') \in \text{poly}(n)$ for any union-oblivious parameter ϕ . To this end, observe that G' is the disjoint union of its connected components. Hence, $\phi(G') \leq \phi(C')$ for some connected component C' of G' . By construction of G' from G^* , have $|C'| \in \text{poly}(n)$, since each connected component of G^* has $\text{poly}(n)$ vertices. Hence, $\phi(G') \leq \phi(C') \in \text{poly}(|C'|) \subseteq \text{poly}(n)$. \square

4 Bisection and the Cliquewidth- q Vertex Deletion Number

In this section we show that BISECTION is fixed-parameter tractable with respect to the number of vertices that have to be removed from a graph to reduce its cliquewidth to some constant q . Thus, we generalize many well-studied graph parameters like vertex cover number ($q = 1$) [13], cluster vertex deletion number and cograph vertex deletion number ($q = 2$) [14], or feedback vertex set number ($q = 3$) [40] and treewidth- t vertex deletion number [29]. Let us formally define cliquewidth. The definition is inspired by Hliněný et al. [35].

Let q be a positive integer. We call (G, λ) a q -labeled graph if G is a graph and $\lambda : V(G) \rightarrow \{1, 2, \dots, q\}$ is a mapping. The number $\lambda(v)$ is called *label* of a vertex v . We introduce the following operations on labeled graphs.

- i) For every i in $\{1, \dots, q\}$, we let \bullet_i denote the graph with only one vertex that is labeled by i (a constant operation).
- ii) For every pair of distinct $i, j \in \{1, 2, \dots, q\}$, we define a unary operator $\eta_{i,j}$ such that $\eta_{i,j}(G, \lambda) = (G', \lambda)$, where $V(G') = V(G)$, and $E(G') = E(G) \cup \{(v, w) \mid v, w \in V, \lambda(v) = i, \lambda(w) = j\}$. In other words, the operator adds all edges between label- i vertices and label- j vertices.
- iii) For every pair of distinct $i, j \in \{1, 2, \dots, q\}$, we let $\rho_{i \rightarrow j}$ be the unary operator such that $\rho_{i \rightarrow j}(G, \lambda) = (G, \lambda')$, where $\lambda'(v) = j$ if $\lambda(v) = i$, and $\lambda'(v) = \lambda(v)$ otherwise. The operator only changes the labels of vertices labeled i to j .
- iv) Finally, \oplus is a binary operation that makes the disjoint union, while keeping the labels of the vertices unchanged. Note explicitly that the union is disjoint in the sense that $(G, \lambda) \oplus (G, \lambda)$ has twice the number of vertices of G .

A q -expression is a well-formed expression φ written with these symbols. The q -labeled graph produced by performing these operations therefore has a vertex for each occurrence of the constant symbol in φ ; and this q -labeled graph (and any q -labeled graph isomorphic to it) is called the *value* $\text{val}(\varphi)$ of φ . If a q -expression φ has value (G, λ) , we say that φ is a q -expression of G . The *cliquewidth* of a graph G , denoted by $\text{cwd}(G)$, is the minimum q such that there is a q -expression of G . We say that a join $\eta_{i,j}$ is *full* if there is no edge between vertices of label i and j in the labeled graph on which the join is applied.

Proposition 1 *For any q -expression for an n -vertex graph there is an equivalent one which is at most as long as φ , contains $O(q^2n)$ symbols, and for which every join is full.*

Proof (Sketch) For the first statement, observe that there are n symbols \bullet_i and $n - 1$ unions. Between any two unions, obviously the number of ρ 's and η 's can be reduced to $O(q^2)$.

The second statement follows from inspecting the proof of Corollary 2.17 by Courcelle and Olariu [14], which states that for every q -expression, there is an equivalent “irredundant” one, meaning that every join is full. \square

In the following, we show how to compute an optimal bisection using the q -expression of a given graph G . This will naturally also solve the decision problem BISECTION. Let $D \subseteq V(G)$ and φ be a q -expression for $G - D$, i.e. $\text{val}(\varphi) = (G - D, \lambda)$. Let A_0, B_0 be a partition of D . For now, we assume that there are no edges between A_0 and B_0 . Let $n_i(\varphi)$ for $i \in \{1, \dots, q\}$ be the number of vertices of $G - D$ with label i . For every pair of vectors $\mathbf{a} = (a_1, \dots, a_q), \mathbf{b} = (b_1, \dots, b_q) \in \mathbb{N}^q$ with $a_i + b_i = n_i(\varphi)$, let us denote by $\text{Cut}_{A_0, B_0}(\varphi, \mathbf{a}, \mathbf{b})$ the minimum number of edges between different parts of a partition (A, B) of $V(G)$ which satisfies the following conditions.

- i) $A_0 \subseteq A, B_0 \subseteq B$, and
- ii) the number of vertices in $A \setminus D$ and $B \setminus D$ of label i are a_i and b_i , respectively.

In the following we use x_i to denote the i 'th entry of a vector \mathbf{x} .

Lemma 8 *For given G, A_0, B_0 and φ in time $O(n^{2q} \cdot q \cdot |\varphi|)$ we can compute all the numbers $\text{Cut}_{A_0, B_0}(\varphi, \mathbf{a}, \mathbf{b})$.*

Proof We prove the lemma by induction on the length of the q -expression. By Proposition 1 we can assume that every join in φ is full. If $\varphi = \bullet_i$, then we have $n_i(\varphi) = 1$ and $n_j(\varphi) = 0$ for every $j \neq i$. Hence, in each pair of q -dimensional vectors \mathbf{a}, \mathbf{b} of $\text{Cut}_{A_0, B_0}(\varphi, \mathbf{a}, \mathbf{b})$ there is either $a_i = 1$ or $b_i = 1$ and the other numbers are zero. In this case, there is exactly one partition fulfilling the conditions (i) and (ii), namely the one which puts the only vertex of $G - D$ to set A or B as required. It is easy to compute the number of edges between the parts in this partition.

Now, suppose $\varphi = \eta_{i,j}(\varphi')$. Since φ' is shorter than φ , by the induction hypothesis we can compute all the numbers $\text{Cut}_{A_0, B_0}(\varphi', \mathbf{a}, \mathbf{b})$ and store them in a table. Note that $\text{val}(\varphi')$ differs from $G - D$ only in that $G - D$ has an edge between every vertex of label i and every vertex of label j , while $\text{val}(\varphi')$ has no such edges (as the join is full). Therefore, every partition (A, B) of $G - D$ fulfilling the conditions (i) and (ii), is also a partition for $\text{val}(\varphi')$ fulfilling these conditions, but in $G - D$ there are exactly $a_i \cdot b_j + a_j \cdot b_i$ more edges between the parts. Hence, we can output $\text{Cut}_{A_0, B_0}(\varphi, \mathbf{a}, \mathbf{b}) = \text{Cut}_{A_0, B_0}(\varphi', \mathbf{a}, \mathbf{b}) + a_i \cdot b_j + a_j \cdot b_i$.

Next, let us assume that $\varphi = \rho_{i \rightarrow j}(\varphi')$, and the values of $\text{Cut}_{A_0, B_0}(\varphi', \mathbf{a}', \mathbf{b}')$ are already computed and stored in a table. Note that in $G - D$ there are no vertices of label i , so we have $0 = n_i(\varphi) = a_i = b_i$. On the other hand, some of the vertices which have label j in $G - D$ had label i in $\text{val}(\varphi')$. A minimal partition for $G - D$, \mathbf{a} , and \mathbf{b} which satisfies the conditions (i) and (ii) is also a partition for $\text{val}(\varphi')$ which satisfies the conditions (i) and (ii) for some \mathbf{a}', \mathbf{b}' and a corresponding distribution

of a_j to a'_j and a'_i and of b_j to b'_j and b'_i . Therefore $Cut_{A_0, B_0}(\varphi, \mathbf{a}, \mathbf{b})$ can be computed as $\min\{Cut_{A_0, B_0}(\varphi', \mathbf{a}', \mathbf{b}')\}$, where the minimum is taken over all pairs \mathbf{a}', \mathbf{b}' with $a'_t = a_t$ and $b'_t = b_t$ for every $t \in \{1, \dots, q\} \setminus \{i, j\}$, $a_j = a'_j + a'_i$, $b_j = b'_j + b'_i$, and $a'_i + b'_i = n_i(\varphi')$ for $t \in \{i, j\}$. As every pair \mathbf{a}', \mathbf{b}' gives rise to exactly one \mathbf{a}, \mathbf{b} , all the minima can be computed in one pass over all \mathbf{a}', \mathbf{b}' .

Finally, let $\varphi = \varphi^1 \oplus \varphi^2$ and let the values of $Cut_{A_0, B_0}(\varphi^1, \mathbf{a}^1, \mathbf{b}^1)$ and $Cut_{A_0, B_0}(\varphi^2, \mathbf{a}^2, \mathbf{b}^2)$ be already computed and stored in a table. A minimal partition for $G - D$ and \mathbf{a}, \mathbf{b} satisfying the conditions (i) and (ii) also induces partitions for $\text{val}(\varphi^1)$ and $\text{val}(\varphi^2)$, which satisfy the conditions (i) and (ii) for some $\mathbf{a}^1, \mathbf{b}^1$ and $\mathbf{a}^2, \mathbf{b}^2$ and corresponding distributions of a_i to a_i^1 and a_i^2 and of b_i to b_i^1 and b_i^2 . Moreover, there are no edges between $\text{val}(\varphi^1)$ and $\text{val}(\varphi^2)$. Thus

$$Cut_{A_0, B_0}(\varphi, \mathbf{a}, \mathbf{b}) = \min\{Cut_{A_0, B_0}(\varphi^1, \mathbf{a}^1, \mathbf{b}^1) + Cut_{A_0, B_0}(\varphi^2, \mathbf{a}^2, \mathbf{b}^2)\},$$

where the minimum is taken over all $\mathbf{a}^1, \mathbf{b}^1$ and $\mathbf{a}^2, \mathbf{b}^2$ where for every $i \in \{1, \dots, q\}$, $a_i = a_i^1 + a_i^2$, $b_i = b_i^1 + b_i^2$, and $a_i^l + b_i^l = n_i(\varphi^l)$ for $l \in \{1, 2\}$. As every pair of pairs $\mathbf{a}^1, \mathbf{b}^1$ and $\mathbf{a}^2, \mathbf{b}^2$ gives rise to exactly one pair \mathbf{a}, \mathbf{b} , all the minima can be computed in one pass over all combinations of $\mathbf{a}^1, \mathbf{b}^1$ and $\mathbf{a}^2, \mathbf{b}^2$.

Concerning the running time, we again argue by induction to show that the overall time is $O(n^{2q} \cdot q \cdot |\varphi|)$. If $\varphi = \bullet_i$, then $|\varphi| = 1$ and the computation of Cut_{A_0, B_0} for the only two possible pairs of q -dimensional vectors takes $O(m+n) \subseteq O(n^{2q} \cdot q)$ time. This constitutes the induction basis. Otherwise, for any sub-expression φ' of a given expression φ , the computation of the table for φ' takes $O(n^{2q} \cdot q \cdot |\varphi'|)$ time by the induction hypothesis. Observe that there are $O(n^q)$ different pairs of q -dimensional vectors \mathbf{a}, \mathbf{b} with $a_i + b_i = n_i(\varphi)$. If $\varphi = \eta_{i,j}(\varphi')$, then the computation for each pair of vectors takes $O(q)$ time. For $\varphi = \rho_{i \rightarrow j}(\varphi')$, one pass through the table of φ' is obviously accomplished in $O(n^q)$ time, spending $O(1)$ time per entry. Since in both cases $|\varphi| = |\varphi'| + 1$, this proves the time bound for φ for these expressions. Finally, if $\varphi = \varphi^1 \oplus \varphi^2$ then the tables for φ^1 and φ^2 can be computed in $O(n^{2q} \cdot q \cdot (|\varphi^1| + |\varphi^2|))$ time. Then we cycle over the entries of both tables and for each combination we spend $O(q)$ time, so this can be accomplished in $O(n^{2q} \cdot q)$ time. Since $|\varphi| = |\varphi^1| + |\varphi^2| + 1$, also in this case the algorithm runs in $O(n^{2q} \cdot q \cdot |\varphi|)$ time. \square

Note that Lemma 8 yields an XP-algorithm for BISECTION with respect to q by simply setting $A_0 = B_0 = \emptyset$. Furthermore, we can derive the following.

Theorem 7 *Let G be a graph, $D \subseteq V(G)$ a vertex subset, and φ a q -expression for $G - D$. There is an $O(2^{|D|} \cdot n^{2q+1} q^3)$ time algorithm which computes the optimal bisection of G .*

Proof It is enough to find the minimum of $Cut_{A_0, B_0}(\varphi, \mathbf{a}, \mathbf{b})$ over all partitions A_0, B_0 and pairs of q -dimensional vectors \mathbf{a}, \mathbf{b} with $|A_0| + \sum_{i=1}^q a_i$ equal to $|B_0| + \sum_{i=1}^q b_i$. Since Lemma 8 only applies when there are no edges between A_0 and B_0 , we delete them and add the number of them to the sum. As the size of φ is $O(q^2 \cdot n)$ by Proposition 1, the running time follows from Lemma 8. \square

Given D , a $(2^{2+3q} - 1)$ -expression for $G - D$ can be computed in $O(n^4)$ time, where q is the cliquewidth of $G - D$ [48]. Thus, BISECTION is FPT with respect to the size of any constant-cliquewidth vertex-deletion set that is obtainable in FPT time.

Corollary 3 BISECTION is fixed-parameter tractable with respect to the size of a feedback vertex set, the size of a cluster vertex deletion set, and the size of a treewidth- t vertex deletion set.

5 The Hardness of Balanced Partitioning

In this section we consider the BALANCED PARTITIONING problem, for which the vertices of a graph need to be partitioned into d parts of equal size. As before, the cut size, i.e. the number of edges connecting vertices of different parts, needs to be minimized. The formal definition is stated in Section 1. It is easy to generalize Theorems 5 and 7 to BALANCED PARTITIONING as follows. At the heart of each of these algorithms is a dynamic program which recurses on the structure of the given graph. It fills a table with an entry for each subgraph on which the algorithm recurses, and all integers a and b such that $a + b$ is the size of the subgraph. Every entry contains the optimal way to partition the vertices of a subgraph into two parts of sizes a and b . By expanding the table to store the best way to partition the vertices of a subgraph into d parts of sizes a_1, \dots, a_d , where now $\sum_{i=1}^d a_i$ is the size of the subgraph, the optimum solution to BALANCED PARTITIONING can be found. This results in algorithms with additional running time factors in the order of $n^{O(d)}$. In particular, the running time achieved by adapting Theorem 7 to BALANCED PARTITIONING is $O(d^{|D|+1} \cdot n^{2(d-1)q+1} q^3)$. Therefore it is a natural question to ask whether corresponding FPT algorithms can be found. We note however, that even for forests (that is, for graphs of cliquewidth at most 3 [14]) any algorithm optimally solving BALANCED PARTITIONING has to include a running time factor of $n^{f(d)}$ unless $\text{FPT} = \text{W}[1]$.

Theorem 8 The BALANCED PARTITIONING problem is $\text{W}[1]$ -hard with respect to the number d of parts in the partition, even on forests with maximum degree two.

We give a reduction from UNARY BIN PACKING which is defined as follows.

UNARY BIN PACKING

Input: Positive integers w_1, \dots, w_ℓ, b, C each encoded in unary.

Question: Is there an assignment of ℓ items with weights w_1, \dots, w_ℓ to at most b bins such that none of the bins exceeds weight C ?

Jansen et al. [36] showed that UNARY BIN PACKING is $\text{W}[1]$ -hard with respect to the number b of bins.

Proof (Theorem 8) Let us construct an instance of BALANCED PARTITIONING from a UNARY BIN PACKING instance $(w_1, \dots, w_\ell, b, C)$. It is clear that $W := \sum_{i=1}^{\ell}$

$w_i \leq b \cdot C$ since otherwise we may output a trivial no-instance. We may furthermore assume that $W = b \cdot C$ since otherwise we may add $b \cdot C - W$ items of weight 1 each. Now the task given by the UNARY BIN PACKING instance is to find a partition of the items into b sets such that each set has weight at most $C = W/b = \lceil W/b \rceil$. Hence, an equivalent instance (G, k, d) of BALANCED PARTITIONING is created by taking G to be the disjoint union of ℓ paths with w_1, \dots, w_ℓ vertices respectively, setting $d = b$ and $k = 0$. \square

As mentioned above, also Theorem 5 can be generalized to BALANCED PARTITIONING, yielding a running time of $h(c, k) \cdot n^{O(c)}$ to find a balanced d -partition with cut size at most k that cuts into c connected components. (Note that $c \geq d$.) We already showed that for VERTEX BISECTION an $h(c, k) \cdot n^{O(1)}$ -time algorithm is out of reach (Theorem 1). We next show that this is also true for BALANCED PARTITIONING.

Theorem 9 BALANCED PARTITIONING is $W[1]$ -hard with respect to the combined parameter (k, c) , where k is the desired cut size and c is the maximum number of components after removing any set of at most k edges from the input graph.

We use a slight modification of the construction used by Enciso et al. [21] to show hardness for EQUITABLE CONNECTED PARTITION. This problem is defined as follows.

EQUITABLE CONNECTED PARTITION

Input: A graph G and a positive integer d .

Question: Is there a partition of the vertices of G into d parts C_1, \dots, C_d such that for all i, j , $1 \leq i < j \leq d$, we have $||C_i| - |C_j|| \leq 1$ and $G[C_i]$ is connected?

We show that for graphs constructed by the corresponding hardness reduction with some small tweaks, each balanced d -partition with cut size at most $3d/2$ consists of parts that are connected and, thus, we obtain a hardness reduction for BALANCED PARTITIONING. The difference of the construction of Enciso et al. [21] and Construction 4 below lies in making the part sizes precise and giving the upper bound on the cut size. Therefore, in the correctness proof, we may rely on the correctness of the reduction to EQUITABLE CONNECTED PARTITION and use it to prove correctness also for BALANCED PARTITIONING.

The reduction for hardness of EQUITABLE CONNECTED PARTITION is from the $W[1]$ -hard MULTICOLORED CLIQUE problem [26]. In MULTICOLORED CLIQUE one is given a graph $G = (V, E)$, where each vertex $v \in V$ is colored by a color $c(v) \in \{1, \dots, s\}$ and the question is whether G contains a clique with s vertices such that each vertex it contains has a distinct color.

Construction 4 Let $(G = (V, E), c, s)$ be an instance of MULTICOLORED CLIQUE and denote $V_i = \{v \in V \mid c(v) = i\}$. We construct an instance of BALANCED PARTITIONING in four steps. First, we construct a “skeleton” graph and then we successively replace vertices and edges by more complicated gadgets. In each step,

we will refer to the skeleton graph as the graph with all previous substitutions. The number of parts we are looking for in the instance of BALANCED PARTITIONING will be set to $d := 2s(s - 1)$ and the cut size to $k := 3s(s - 1)$ yielding an average of three cut edges incident with each part.

Step 1: The skeleton graph contains s cycles, each with $2(s - 1)$ vertices. For the i 'th cycle, let us denote its vertices "clock-wise" by $N_1^i, P_1^i, N_2^i, P_2^i, \dots, N_s^i, P_s^i$, where we omit N_i^i, P_i^i in the sequence. We call these vertices *anchors*. We sometimes need to refer to the next index in the sequence; for this we define

$$\text{succ}_i(j) := \begin{cases} (i + 1) \bmod s, & j = i - 1 \\ 1, & j = s \\ j + 1, & \text{otherwise.} \end{cases}$$

The i 'th cycle corresponds to the i th color in the MULTICOLORED CLIQUE and will serve as a "vertex-chooser", choosing a vertex with color i to be in the clique. For all i, j with $1 \leq i < j \leq s$ we connect the i th and the j th cycle by the edges $\{N_j^i, P_i^j\}$ and $\{N_i^j, P_j^i\}$. These connections will "transmit" the choices of the clique-vertices to the neighboring cycles and ensure that the chosen vertices are adjacent. This concludes the description of the skeleton graph. We will call the s cycles *vertex choosers* and the inter-cycle edges *transmitters*.

Step 2: We now replace some edges in the vertex choosers by a "choice" gadget. Let (A, b) be a tuple of an integer $b > 0$ and a set $A = \{a_1, \dots, a_t\}$ of integers such that $0 \leq a_i < a_{i+1} \leq b$, $1 \leq i \leq t$. An (A, b) -choice is a path v_1, \dots, v_{t+1} with $t + 1$ vertices, each possibly having additional vertices pending on it (that is, each vertex can have additional degree-one neighbors). The number of pending vertices on v_i , $1 \leq i \leq t$ is determined by

$$\begin{cases} a_1, & \text{if } i = 1, \\ a_i - a_{i-1} - 1, & \text{if } 2 \leq i \leq t, \text{ and} \\ b - a_t, & \text{if } i = t + 1. \end{cases}$$

The choices will be cut exactly once between vertex v_p and vertex v_{p+1} by any feasible partition and if they are cut in this way, observe that, except for the first and the last vertex, they contribute $a_1 + \sum_{i=1}^p ((a_i - a_{i-1} - 1) + 1) = a_p$ vertices to one of the parts and $b - a_t + \sum_{i=p+1}^t ((a_i - a_{i-1} - 1) + 1) = b - a_p$ to the other part.

Using the definition of choice, for every $1 \leq i \leq s$, we replace each edge $\{P_j^i, N_{\text{succ}_i(j)}^i\}$ in the skeleton graph by an (A_i, \max, A_i) -choice: we identify $N_{\text{succ}_i(j)}^i$ and the first vertex of the choice (the vertex with the lowest index in the path), and we identify P_j^i and the last vertex of the choice. For the definition of A_i , first let $z_0 = 2|E| + 10$. Then, $A_i = \{p \cdot z_0 \mid 1 \leq p \leq |V_i|\}$.

Intuitively, the vertex choosers in the skeleton graph choose vertices as follows. Assume that also each edge $\{N_j^i, P_j^i\}$ is replaced by an (A_i, \max, A_i) -choice (in order to represent adjacency between the chosen vertices, the actual choices

used will be more complicated). Furthermore, assume that each part of any solution partition is connected and contains exactly one of any of the anchors (vertices N_j^i, P_j^i). Let us ignore the transmitters and consider the i th vertex chooser. Then, in order to have equal-size parts⁴ the cut between the parts of two “neighboring” anchors N_j^i, P_j^i has to be at the same position for every choice in the vertex chooser, that is, every choice is cut exactly once between the vertex v_p and v_{p+1} for some p . The position of the cut in the choices of vertex chooser i corresponds to the chosen clique-vertex for color i in the MULTICOLORED CLIQUE instance (G, c, s) .

Step 3: In order to represent the adjacency of two chosen vertices, we now substitute choices for both the transmitter edges and some further edges in the vertex choosers. Fix arbitrary one-to-one mappings $\phi_i : V_i \rightarrow \{1, \dots, |V_i|\}$ and $\psi : E \rightarrow \{1, \dots, |E|\}$. We replace the edge between the anchors N_j^i, P_j^i in each vertex chooser i , $1 \leq i \leq s$, and for each j , $1 \leq j \leq s, i \neq j$, by the $(A_j^i, |V_i| \cdot z_0 + |E|)$ -choice by identifying N_j^i and the first vertex of the choice and identifying P_j^i and the last vertex of the choice. The choice is supposed to choose vertices as the choice used in Step 2 but shall also choose an edge incident with the chosen vertex and a vertex with color j . Thus, we define $A_j^i = \{p \cdot z_0 + \psi(\{u, v\}) \mid \{u, v\} \in E \wedge \phi_i(v) = p \wedge c(u) = j\}$. To ensure that vertex choosers for different colors agree on the chosen edge, we replace each transmitter edge $\{N_j^i, P_i^j\}$ by a $(\{1, \dots, |E|\}, |E|)$ -choice by identifying P_i^j with the first vertex of the choice and identifying N_j^i with the last vertex of the choice.

Step 4: We now add further pending vertices to anchors (vertices N_j^i, P_j^i) in order to ensure that the parts of the desired partition are connected and have equal size, and to ensure that the choices work as intended. We add $10 \cdot z_0 \cdot (|V| + |E|) - z_0 \cdot |V_i| - |E|$ pending vertices to each of N_j^i, P_j^i for i, j , $1 \leq i, j \leq s, i \neq j$. This concludes the construction of the BALANCED PARTITIONING instance $(G', k = 3s(s-1), d = 2s(s-1))$.

Let us prove that Construction 4 is the promised parameterized reduction.

Proof (Theorem 9) First, we derive the total number of vertices in G' . The number of pending vertices added to anchors in Step 4 is

$$\begin{aligned} & \sum_{i=1}^s 2(s-1) \cdot (10 \cdot z_0 \cdot (|V| + |E|) - z_0 \cdot |V_i| - |E|) \\ &= 2(s-1) \cdot (s \cdot 10 \cdot z_0 \cdot (|V| + |E|) - z_0 \cdot |V| - s \cdot |E|). \end{aligned}$$

Note that any (A, b) -choice contains exactly $b + 2$ vertices. Let us count the number of vertices contained in choices in vertex chooser i , $1 \leq i \leq s$, without the anchors:

$$(s-1) \cdot z_0 \cdot |V_i| + (s-1) \cdot (|V_i| \cdot z_0 + |E|).$$

⁴Parts in a feasible partition for BALANCED PARTITIONING may not be of equal size; we will consider this issue more closely below.

The number of vertices in choices of vertex choosers thus totals at

$$2(s-1) \cdot z_0 \cdot |V| + s \cdot |E|.$$

Counting the number of vertices in transmitters without the anchors yields $s(s-1) \cdot |E|$. Adding all the vertices and the anchors we thus obtain $2s(s-1) \cdot (10 \cdot z_0 \cdot (|V| + |E|) + |E|/2 + 1)$. Without loss of generality, we may assume that $|E|$ is even because otherwise we may simply add an isolated edge to G . Then, note that the number of vertices in G' is a multiple of $2s(s-1)$. Thus, any partition of the vertices into $d = 2s(s-1)$ parts such that each part has size at most $\lceil |V(G')|/d \rceil$ contains only parts of size exactly

$$n_0 := 10 \cdot z_0 \cdot (|V| + |E|) + |E|/2 + 1.$$

This implies that the required upper bound on the part sizes of BALANCED PARTITIONING and the equal-size requirement of EQUITABLE CONNECTED PARTITION coincide on the instances created by Construction 4.

Enciso et al. [21] proved that Construction 4 is correct if every part in the desired partition is connected and the cut size can be arbitrary. We simply prove that setting the cut size to $3(s-1)s$ ensures connectivity of each part. For this, we consider one part and the number of cut edges it contributes. We prove the following.

Claim: Each part in a balanced partition has at least three incident cut edges and if it has exactly three, then it is connected.

This claim implies that each of the $2(s-1)s$ parts is connected in a yes-instance, since otherwise the cut size would be at least $3(s-1)s+1$. Let us now consider a connected component in one of the parts. If it contains exactly one vertex, then we call it *small* and it has at least one incident cut edge. If it contains at least two vertices but none of the anchors (vertices N_j^i, P_j^i), then we call it *medium*. Observe that medium connected components have at least two incident cut edges. Note also that medium connected components have size at most $\max\{z_0|V_i| + |E| \mid 1 \leq i \leq s\}$ because this number is the maximum number of vertices in a choice. If the connected component contains at least one of the anchors, then we call it *large*. It is not hard to see that large connected components have at least three incident cut edges. This is clear if the component contains only one anchor. If it contains at least two anchors, then, since each part of the partition has size exactly n_0 and each anchor has at least $9z_0(|V| + |E|)$ pending vertices, there are at least three pending vertices cut off. Note also that every connected component of size greater than $\max\{z_0|V_i| + |E| \mid 1 \leq i \leq s\}$ is large.

For the first part of the claim, assume that there is a part with at most two incident cut edges. Thus, it can either contain two small connected components or one medium one. In both cases, the part is smaller than n_0 for every large-enough MULTICOLORED CLIQUE instance which contradicts the balancedness of the partition.

For the second part of the claim, assume that there is a part with exactly three incident cut edges and at least two connected components. Since the part size is exactly n_0 we conclude that it contains at least one large connected component. This contradicts the fact that this part has at most three incident cut edges.

This finishes the proof of the above claim and, thus, Construction 4 is a reduction from MULTICOLORED CLIQUE to BALANCED PARTITIONING. It is easy to verify that it is computable in polynomial time. It is also clear that k is bounded by some function of s and also c is because the graph obtained by Construction 4 is connected. Hence, Construction 4 is also a parameterized reduction with respect to these parameters. \square

We conjecture that this hardness result can be extended to planar graphs using a similar technique as Enciso et al. [21] and even to two-dimensional grid graphs using a specific kind of planar embedding.

Interestingly, it seems pivotal that the treewidth is $\Omega(s^2)$ for the above construction. Hence, we can not trivially infer that BALANCED PARTITIONING is W[1]-hard with respect to k and d on trees, for example. This is left as an interesting open question.

6 Balanced Partitioning and the Vertex Cover Number

In the following we present an FPT algorithm for BALANCED PARTITIONING and parameter τ , which is the size of a minimum vertex cover of the graph. Recently Ganian and Obdržálek [31] gave an FPT algorithm for the combined parameters τ and d with a running time of $2^{2^{O(d+\tau)}} + 2^\tau n$. We improve on this by removing the dependence on d : our algorithm has a running time of $O(\tau^\tau n^3)$. Some of the ideas of our algorithm are inspired by Doucha and Kratochvíl [19].

Theorem 10 BALANCED PARTITIONING is fixed-parameter tractable with respect to the size of a minimum vertex cover of the input graph.

Proof Let a vertex cover C of size τ be given. We consider all partitions of C into d sets. Up to isomorphism, there are at most τ^τ possible such partitions. This can be seen as follows. If $d \geq \tau$ we can pick τ representatives of the d sets, since the maximal number of sets into which the τ vertices can be partitioned is τ . Otherwise d is upper bounded by τ . In both cases each of the τ vertices can be put into one of at most τ sets, which gives the upper bound on the number of partitions.

For each partition of C we now check whether some set has more than $\lceil n/d \rceil$ vertices. If this is the case, the partition is discarded. Otherwise we need to partition the vertex set $I = V \setminus C$ not belonging to the vertex cover. For $v \in I$ let $c_v(j)$ denote the number of edges that are cut when putting v into set j . Since I is the complement of a vertex cover, it induces an independent set. Hence the cost $c_v(j)$ is solely determined by the partition of C which at this point is fixed. Also depending on this partition, each set j can still hold at most some s_j vertices until it has reached its full capacity of $\lceil n/d \rceil$.

We need to compute an assignment of the vertices in I to sets such that the capacities of the sets are not exceeded and the total introduced cost is minimal. This can be done using a minimum cost maximum matching in an auxiliary graph as follows. Introduce a vertex w_v for each vertex $v \in I$, and s_j vertices $w_j^1, \dots, w_j^{s_j}$ for each

set $j \in \{1, \dots, d\}$. Now for each $v \in I$, $j \in \{1, \dots, d\}$, and $l \in \{1, \dots, s_j\}$, connect vertex w_v with vertex w_j^l using an edge of cost $c_v(j)$.

Now a min-cost maximum matching in the auxiliary graph corresponds to an assignment of each vertex v to a set j . The resulting partition of the graph does not have sets containing more than $\lceil n/d \rceil$ vertices. Moreover this partition has minimum cut size for the fixed partition of the vertex cover, since the costs of the edges in the auxiliary graph reflect the incurred cut edges due to the partition of I .

By going through the above steps and picking the best solution among all partitions of C that are not discarded, the minimum cut size can be computed. Note that the auxiliary graph is bipartite with at most $3n$ vertices and $O(n^2)$ edges. Hence the algorithm runs in time $O(\tau^\tau \cdot n^3)$, using Dijkstra's algorithm in combination with Fibonacci heaps to solve the matching problem [30]. \square

7 Open Problems

We presented a $h(k, c) \cdot n^{c+9}$ -time algorithm for finding a c -component bisection of size at most k . However the function $h(k, c)$ we gave is doubly exponential in k , whereas Cygan et al. [15] give a $2^{O(k^3)} \cdot n^3 \log^3 n$ -time algorithm for BISECTION. Hence even for constant c the latter algorithm improves over ours. However it is quite conceivable that better running times should be achievable when combining the parameters k and c . Since c is a small constant in most practical applications, it would be of real practical value to find improved FPT algorithms for the BISECTION problem with combined parameter (k, c) .

Concerning the BALANCED PARTITIONING problem, it was already known that it is considerably harder than BISECTION. Even for simple graph classes such as trees or grids, the problem is NP-hard to approximate [23]. It was therefore asked in [23] whether practical algorithms beyond the standard deterministic worst-case scenario exist. In this article we ruled out FPT algorithms for several parameters. However the general question remains only partially answered by our negative results. One possible direction for further research is finding fixed parameter approximation algorithms [45] for the problem.

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