The Rabin Index of Parity Games
(Extended Abstract)

Michael Huth[1], Jim Huan-Pu Kuo[1], and Nir Piterman[2]

[1] Department of Computing
Imperial College London
London, SW7 2AZ, United Kingdom
{m.huth,jimhkuo}@imperial.ac.uk
[2] Department of Computer Science
University of Leicester
Leicester, LE1 7RH, United Kingdom
nir.piterman@leicester.ac.uk

Abstract. We study the descriptive complexity of parity games by taking into account the coloring of their game graphs whilst ignoring their ownership structure. Different colorings of the same graph are identified if they determine the same winning regions and strategies, for *all* ownership structures of nodes. The Rabin index of a parity game is the minimum of the maximal color taken over all equivalent coloring functions. We show that deciding whether the Rabin index is at least k is in P for $k = 1$ but NP-hard for all *fixed* $k \geq 2$. We present an EXPTIME algorithm that computes the Rabin index by simplifying its input coloring function. When replacing simple cycle with cycle detection in that algorithm, its output over-approximates the Rabin index in polynomial time. Experimental results show that this approximation yields good values in practice.

Parity games (see e.g. [1]) are infinite, 2-person, 0-sum, graph-based games that are hard to solve. Their nodes are colored with natural numbers, controlled by different players, and the winning condition of plays depends on the minimal color occurring in cycles. The condition for winning a node, therefore, is an alternation of existential and universal quantification. In practice, this means that the maximal color of its coloring function is the only exponential source for the worst-case complexity of most parity game solvers, e.g. for those in [1,2,3].

One approach taken in analyzing the complexity of such games is through the study of the descriptive complexity of their underlying game graph. This method therefore ignores the ownership structure on parity games.

An example of this approach is the notion of DAG-width in [4]. Every directed graph has a DAG-width, a natural number that specifies how well that graph can be decomposed into a directed acyclic graph (DAG). The decision problem for DAG-width, whether the DAG-width of a directed graph is at most k, is NP-complete [4] in k. But parity games whose DAG-width is below a given threshold have polynomial-time solutions [4]. The latter is a non-trivial result since DAG-width also ignores the colors of a parity game.

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In this abstract, we report a similar measure of the descriptive complexity of parity games, their *Rabin index*, a natural number that ignores the ownership of nodes, but does take into account the colors of a parity game.

The name for this measure is inspired by related work on the Wagner hierarchy for automata on infinite words [5]: Carton and Maceiras use similar ideas to compute and minimize the Rabin index of deterministic parity automata on infinite words [6]. To the best of our knowledge, our work is the first to study this notion in the realm of infinite, 2-person games.

The idea behind our Rabin index is that one may change the coloring function of a parity game to another one if that change neither affects the winning regions nor the choices of winning strategies. This yields an equivalence relation between coloring functions. For the coloring function of a parity game, we then seek an equivalent coloring function with the smallest possible maximal color, and call that minimal maximum the Rabin index of the respective parity game.

The results we report here about this Rabin index are similar in spirit to those developed for DAG-width in [4] but there are important differences:

- We propose a measure of descriptive complexity that is closer to the structure of the parity game as it only forgets ownership of nodes and not their colors.
- We prove that for every *fixed* $k \geq 2$, deciding whether the Rabin index of a parity game is at least k is NP-hard.
- We can characterize the above equivalence relation in terms of the parities of minimal colors on simple cycles in the game graph.
- We use that characterization to design an algorithm that computes the Rabin index and a witnessing coloring function in exponential time.
- We define an approximation of the Rabin index by replacing simple cycles in the definition of Rabin index by cycles.
- We show how to efficiently compute this approximation by replacing the search for simple cycles by search for cycles in the same algorithm.
- We conduct detailed experimental studies that corroborate the utility of that approximation, also as a preprocessor for solvers.

References

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