# Polynomial Identification of $\omega$ -Automata

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We study identification in the limit using polynomial time and data for models of  $\omega$ -automata. On the negative side we show that non-deterministic  $\omega$ -automata (of types Büchi, coBüchi, Parity, Rabin, Street, or Muller) cannot be polynomially learned in the limit. On the positive side we show that the  $\omega$ -language classes  $\mathbb{IB}$ ,  $\mathbb{IC}$ ,  $\mathbb{IP}$ ,  $\mathbb{IR}$ ,  $\mathbb{IS}$ , and  $\mathbb{IM}$ , which are defined by deterministic Büchi, coBüchi, Parity, Rabin, Streett, and Muller acceptors that are isomorphic to their right-congruence automata, are identifiable in the limit using polynomial time and data.

We give polynomial time inclusion and equivalence algorithms for deterministic Büchi, coBüchi, Parity, Rabin, Streett, and Muller acceptors, which are used to show that the characteristic samples for  $\mathbb{IB}$ ,  $\mathbb{IC}$ ,  $\mathbb{IP}$ ,  $\mathbb{IR}$ ,  $\mathbb{IS}$ , and  $\mathbb{IM}$  can be constructed in polynomial time. We also provide polynomial time algorithms to test whether a given deterministic automaton of type  $\mathbb{X}$  (for  $\mathbb{X} \in \{\mathbb{B}, \mathbb{C}, \mathbb{P}, \mathbb{R}, \mathbb{S}, \mathbb{M}\}$ ) is in the class  $\mathbb{IX}$  (i.e. recognizes a language that has a deterministic automaton that is isomorphic to its right congruence automaton). This is an extended version of a paper with the same name that appeared in TACAS'20 [10].

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### 1 INTRODUCTION

With the growing success of machine learning in efficiently solving a wide spectrum of problems, we are witnessing an increased use of machine learning techniques in formal methods for system design. One thread in recent literature uses general purpose machine learning techniques for obtaining more efficient verification/synthesis algorithms. Another thread, following the *automata theoretic approach to verification* [26, 41] works on developing grammatical inference algorithms for verification and synthesis purposes. *Grammatical inference* (aka *automata learning*) refers to the problem of automatically inferring from examples a finite representation (e.g. an automaton, a grammar, or a formula) for an unknown language. The term *model learning* [39] was coined for the task of learning an automaton model for an unknown system. A large body of works has developed learning techniques for different automata types (e.g. I/O automata [1], register automata [25], symbolic automata [20],  $\omega$ -automata [7], and program automata [30]) and has shown its usability in a diverse range of tasks.<sup>1</sup>

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<sup>&</sup>lt;sup>1</sup>E.g., tasks such as black-box checking [34], specification mining [2], assume-guarantee reasoning [18], regular model checking [24], learning verification fixed-points [40], learning interfaces [33], analyzing botnet protocols [16] or smart card readers [14], finding security bugs [14], error localization [15], and code refactoring [31, 36].

In grammatical inference, the learning algorithm does not learn a *language*, but rather a finite *representation* of it. Complexity of learning algorithms may vary greatly by switching representations. For instance, if one wishes to learn regular languages, she may consider representations using deterministic finite automata (DFAs), non-deterministic finite automata (NFAs), regular expressions, linear grammars etc. Since the translation results between two such formalisms are not necessarily polynomial, a polynomial learnability result for one representation does not necessarily imply a polynomial learnability result for another representation. Let  $\mathbb C$  be a class of representations C with a size measure size(C) (e.g. for DFAs the size measure can be the number of states in the minimal automaton). We extend  $size(\cdot)$  to the languages recognized by representations in  $\mathbb C$  by defining size(L) to be the minimum of size(C) over all C representing L. In this paper we restrict attention to automata representations, namely, acceptors.

There are various learning paradigms considered in the grammatical inference literature, roughly classified into passive and active. We mention here the two central ones. In passive learning the model of learning from finite data refers to the following problem: given a finite sample  $T \subseteq \Sigma^* \times \{0,1\}$  of labeled words, a learning algorithm A should return an acceptor C that agrees with the sample T. That is, for every  $(w,l) \in T$  the following holds:  $w \in \llbracket C \rrbracket$  iff l=1 (where  $\llbracket C \rrbracket$  is the language accepted by C). The class  $\mathbb C$  is identifiable in the limit using polynomial time and data if and only if there exists a polynomial time algorithm A that takes as input a labeled sample T and outputs an acceptor  $C \in \mathbb C$  that is consistent with T, and A also satisfies the following condition. If T is any language recognized by an automaton from class  $\mathbb C$ , then there exists a labeled sample T consistent with T of length bounded by a polynomial in size(T), and for any labeled sample T consistent with T such that T is the algorithm A produces an acceptor T that recognizes T in this case, T is termed a characteristic sample for the algorithm A. In some cases (e.g., DFAs) there is also a polynomial time algorithm to compute a characteristic sample for T, given an acceptor T is T.

In active learning the model of query learning [3] assumes the learner communicates with an oracle (sometimes called teacher) that can answer certain types of queries about the language. The most common type of queries are membership queries (is  $w \in L$  where L is the unknown language) and equivalence queries (is  $[\![\mathcal{A}]\!] = L$  where  $\mathcal{A}$  is the current hypothesis for an acceptor recognizing L). Equivalence queries are typically assumed to return a counterexample, i.e. a word in  $[\![\mathcal{A}]\!] \setminus L$  or in  $L \setminus [\![\mathcal{A}]\!]$ .

With regard to  $\omega$ -automata (automata on infinite words) most of the works consider *query learning* using *membership queries* and *equivalence queries*. The representations learned so far include:  $(L)_{\$}$  [21], a non-polynomial reduction to finite words; families of DFAs ( $\mathbb{FDFA}$ ) [6–8, 27]; strongly unambiguous Büchi automata ( $\mathbb{SUBA}$ ) [4]; mod-2-multiplicity automata ( $\mathbb{M2MA}$ ) [5]; and deterministic weak Parity automata ( $\mathbb{DWPA}$ ) [29]. Among these only the latter two are known to be learnable in polynomial time using membership queries and proper equivalence queries.<sup>2</sup>

One of the main obstacles in obtaining a polynomial learning algorithm for regular  $\omega$ -languages is that they do not in general have a Myhill-Nerode characterization; that is, there is no theorem correlating the states of a minimal automaton of some of the common automata types (Büchi, Parity, Muller, etc.) to the equivalence classes of the right congruence of the language. The *right congruence relation* for an  $\omega$ -language L relates two finite words x and y iff there is no infinite suffix z differentiating them, that is  $x \sim_L y$  (for  $x, y \in \Sigma^*$ ) iff  $\forall z \in \Sigma^\omega$ .  $xz \in L \iff yz \in L$ . In our quest for finding a polynomial query learning algorithm for a subclass of the regular  $\omega$ -languages, we have studied subclasses of languages for which such a relation holds [9], and termed them *fully informative*. We use  $\mathbb{IB}$ ,  $\mathbb{IC}$ ,  $\mathbb{IP}$ ,  $\mathbb{IR}$ ,  $\mathbb{IS}$ ,  $\mathbb{IM}$  to denote the classes of languages that are fully informative of type Büchi, coBüchi, Parity, Rabin, Streett, and Muller, respectively. A language L is said to be fully informative of type  $\mathbb{X}$  for  $\mathbb{X} \in \{\mathbb{B}, \mathbb{C}, \mathbb{P}, \mathbb{R}, \mathbb{S}, \mathbb{M}\}$  if there exists a deterministic automaton

<sup>&</sup>lt;sup>2</sup>Query learning with an additional type of query, *loop-index queries*, was studied for deterministic Büchi automata [32]. Manuscript submitted to ACM

of type  $\mathbb{X}$  which is isomorphic to the automaton derived from  $\sim_L$ . While many properties of these classes are now known, in particular that they span the entire hierarchy of  $\omega$ -regular properties [43], a polynomial learning algorithm for them is not known.

In this paper we show (in §4-8) that the classes  $\mathbb{IB}$ ,  $\mathbb{IC}$ ,  $\mathbb{IP}$ ,  $\mathbb{IR}$ ,  $\mathbb{IM}$  can be identified in the limit using polynomial time and data. We further show (in §9) that there is a polynomial time algorithm to compute a characteristic sample given an acceptor  $C \in \mathbb{IX}$ . A corollary of this result is that the class of languages accepted by  $\mathbb{DWPA}$ s (which as mentioned above is polynomially learnable in the query learning setting) also has a polynomial size characteristic sample. On the negative side, we show (in §3) that the classes  $\mathbb{NBA}$ ,  $\mathbb{NCA}$ ,  $\mathbb{NPA}$ ,  $\mathbb{NRA}$ ,  $\mathbb{NSA}$ ,  $\mathbb{NMA}$  of non-deterministic Büchi, coBüchi, Parity, Rabin, Streett, and Muller automata, respectively, cannot be identified in the limit using polynomial data.

To obtain the above results we needed a polynomial time algorithm for inclusion and equivalence of automata in these types. Such an algorithm is known to exist for the classes  $\mathbb{NBA}$ ,  $\mathbb{NCA}$ ,  $\mathbb{NPA}$ , since these classes have inclusion algorithms in NL. For the other classes, we are not aware of results in the literature showing a polynomial time algorithm. We provide such algorithms in §10-13.

The last part of this manuscript (§14-15) is devoted to the question of deciding whether a given automaton  $\mathcal{A}$  of type  $\mathbb{X}$  that is isomorphic to its right congruence, or if this is not the case whether there exists an automaton  $\mathcal{A}'$  of the same type that recognizes the same language and is isomorphic to its right congruence, namely whether the given automaton is in the class  $\mathbb{IX}$ .

#### 2 PRELIMINARIES

#### 2.1 Automata

An *automaton* is a tuple  $\mathcal{M} = \langle \Sigma, Q, q_i, \delta \rangle$  consisting of a finite alphabet  $\Sigma$  of symbols, a finite set Q of states, an initial state  $q_i \in Q$ , and a transition function  $\delta : Q \times \Sigma \to 2^Q$ . We extend  $\delta$  to domain  $Q \times \Sigma^*$  in the usual way:  $\delta(q, \varepsilon) = q$  and  $\delta(q, \sigma x) = \bigcup_{q' \in \delta(q, \sigma)} \delta(q', x)$  for all  $q \in Q$  and  $\sigma \in \Sigma$ .

We define the size of an automaton to be  $|\Sigma| \cdot |Q|$ . A state  $q \in Q$  is reachable iff there exists  $x \in \Sigma^*$  such that  $q = \delta(q_t, x)$ . For  $q \in Q$ ,  $\mathcal{M}^q$  is the automaton  $\mathcal{M}$  with its initial state replaced by q. We say that  $\mathcal{A}$  is deterministic if  $|\delta(q, \sigma)| \leq 1$  and complete if  $|\delta(q, \sigma)| \geq 1$ , for every  $q \in Q$  and  $\sigma \in \Sigma$ . For deterministic automata we abbreviate  $\delta(q, \sigma) = \{q'\}$  as  $\delta(q, \sigma) = q'$ . Two automata  $\mathcal{M}$  and  $\mathcal{M}'$  with the same alphabet  $\Sigma$  are isomorphic if there exists a bijection f from the states Q of  $\mathcal{M}$  to the states Q' of  $\mathcal{M}'$  such that  $f(q_t) = q'_t$  and for every  $q \in Q$  and  $\sigma \in \Sigma$ ,  $\{f(r) \mid r \in \delta(q, \sigma)\} = \delta'(f(q), \sigma)$ .

We assume a fixed total ordering on  $\Sigma$ , which induces the *shortlex* total ordering on  $\Sigma^*$ , defined as follows. For  $x, y \in \Sigma^*$ , x precedes y in the shortlex ordering if |x| < |y| or |x| = |y| and x precedes y in the lexicographic ordering induced by the ordering on  $\Sigma$ .

A run of an automaton on a finite word  $v=a_1a_2\ldots a_n$  is a sequence of states  $q_0,q_1,\ldots,q_n$  such that  $q_0=q_i$ , and for each  $i\geq 1, q_i\in \delta(q_{i-1},a_i)$ . A run on an infinite word is defined similarly and consists of an infinite sequence of states. For an infinite run  $\rho=q_0,q_1,\ldots$ , we define the set of states visited infinitely often, denoted  $inf_{\mathcal{M}}(\rho)$ , as the set of  $q\in Q$  such that  $q=q_i$  for infinitely many indices  $i\in \mathbb{N}$ . This is abbreviated to  $inf(\rho)$  if  $\mathcal{M}$  is understood.

The Product of Two Automata. Let  $\mathcal{M}_1$  and  $\mathcal{M}_2$  be two deterministic complete automata with the same alphabet  $\Sigma$ , where for i=1,2,  $\mathcal{M}_i=\langle \Sigma,Q_i,(q_i)_i,\delta_i\rangle$ . Their product automaton, denoted  $\mathcal{M}_1\times\mathcal{M}_2$ , is the deterministic complete automaton  $\mathcal{M}=\langle \Sigma,Q,q_i,\delta\rangle$  such that  $Q=Q_1\times Q_2$ , the set of ordered pairs of states of  $\mathcal{M}_1$  and  $\mathcal{M}_2,q_i=((q_i)_1,(q_i)_2)$ ,

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the pair of initial states of the two automata, and for all  $(q_1, q_2) \in Q$  and  $\sigma \in \Sigma$ ,  $\delta((q_1, q_2), \sigma) = (\delta_1(q_1, \sigma), \delta_2(q_2, \sigma))$ . For i = 1, 2, let  $\pi_i$  be projection onto the ith coordinate, so that for a subset S of Q,  $\pi_1(S) = \{q_1 \mid \exists q_2(q_1, q_2) \in S\}$ , and analogously for  $\pi_2$ .

## 2.2 Acceptors

By augmenting an automaton  $\mathcal{M} = \langle \Sigma, Q, q_i, \delta \rangle$  with an acceptance condition  $\alpha$ , obtaining a tuple  $\mathcal{A} = \langle \Sigma, Q, q_i, \delta, \alpha \rangle$ , we get an *acceptor*, a machine that accepts some words and rejects others. We may also denote  $\mathcal{A}$  by  $(\mathcal{M}, \alpha)$ . An acceptor accepts a word if at least one of the runs on that word is accepting. If the automaton is not complete, a given word w may not have any run in the automaton, in which case w is rejected.

For finite words the acceptance condition is a set  $F \subseteq Q$  and a run on a word v is accepting if it ends in an accepting state, i.e., if  $\delta(v)$  contains an element of F. For infinite words, there are various acceptance conditions in the literature, and we consider six of them: Büchi, coBüchi, Parity, Rabin, Streett, and Muller, all based on the set of states visited infinitely often in a given run. For each model we define the related quantity of the size of the acceptor, taking into account the acceptance condition.

A Büchi or coBüchi acceptance condition is a set of states  $F \subseteq Q$ . A run  $\rho$  of a Büchi acceptor is accepting if it visits F infinitely often, that is,  $inf(\rho) \cap F \neq \emptyset$ . A run  $\rho$  of a coBüchi acceptor is accepting if it visits F only finitely many times, that is,  $inf(\rho) \cap F = \emptyset$ . The size of a Büchi or coBüchi acceptor is the size of its automaton.

A *Parity* acceptance condition is a map  $\kappa: Q \to \mathbb{N}$  assigning to each state a natural number termed a color (or priority). A run of a Parity acceptor is accepting if the **minimum** color visited infinitely often is **odd**. The *size* of a Parity acceptor is the size of its automaton.

A *Rabin* or *Streett* acceptance condition consists of a finite set of pairs of sets of states  $\alpha = \{(G_1, B_1), \dots, (G_k, B_k)\}$  for some  $k \in \mathbb{N}$  and  $G_i \subseteq Q$  and  $B_i \subseteq Q$  for  $i \in [1..k]$ . A run of a Rabin acceptor is accepting if there exists an  $i \in [1..k]$  such that  $G_i$  is visited infinitely often and  $B_i$  is visited finitely often. A run of a Streett acceptor is accepting if for all  $i \in [1..k]$ ,  $G_i$  is visited finitely often or  $B_i$  is visited infinitely often. The *size* of a Rabin or Streett acceptor is the sum of the size of its automaton and k-1.

A Muller acceptance condition is a set of sets of states  $\alpha = \{F_1, F_2, \dots, F_k\}$  for some  $k \in \mathbb{N}$  and  $F_i \subseteq Q$  for  $i \in [1..k]$ . A run of a Muller acceptor is accepting if the set S of states visited infinitely often in the run is a member of  $\alpha$ . The size of a Muller acceptor is the sum of the size of its automaton and k-1.

The set of words accepted by an acceptor  $\mathcal{A}$  is denoted by  $[\![\mathcal{A}]\!]$ . Two acceptors  $\mathcal{A}$  and  $\mathcal{B}$  are *equivalent* if they accept the same language, that is,  $[\![\mathcal{A}]\!] = [\![\mathcal{B}]\!]$ . For a state q, the acceptor  $\mathcal{A}^q$  is the acceptor  $\mathcal{A}$  with its automaton initial state replaced by q. We say that the  $\omega$ -word w is accepted from state q iff  $w \in [\![\mathcal{A}^q]\!]$ .

We use NBA, NCA, NPA, NRA, NSA, NMA (resp., DBA, DCA, DPA, DRA, DSA, DMA) for non-deterministic (resp. deterministic) Büchi, Parity, Rabin, Street, Muller and coBüchi, acceptors. We use NBA, NCA, NPA, NMA, NRA, and NSA (resp., DBA, DCA, DPA, DRA, DSA, and DMA) for the classes of languages they recognize. NCA is the same as DCA, and DCA and DBA are distinct proper subclasses of the regular  $\omega$ -languages. The other classes are the full class of regular  $\omega$ -languages.

# 2.3 Some relationships between the models

We observe the following known relationships.

CLAIM 2.1. Let  $\mathcal{A}$  be an acceptor of one of the types NBA, NCA, NPA, NRA, NSA, or NMA with n states over the alphabet  $\Sigma$ . There is an equivalent complete acceptor  $\mathcal{A}'$  of the same type whose size is at most  $|\Sigma|$  larger.  $\mathcal{A}'$  may be taken to be deterministic if  $\mathcal{A}$  is deterministic.

PROOF. If  $\mathcal A$  is incomplete, we modify  $\mathcal A$  to construct  $\mathcal A'$  as follows. A new *dead state*  $q_d$  is added to the state set, and the transition function is extended to define  $\delta'(q_d, \sigma) = q_d$  for all  $\sigma \in \Sigma$  and  $\delta'(q, \sigma) = q_d$  whenever  $\delta(q, \sigma) = \emptyset$ .

If necessary, the acceptance condition is modified to avoid classifying a run that ends in a consecutive sequence of visits to  $q_d$  as accepted. For Büchi, Rabin, and Muller acceptors, no modification is necessary. For Parity acceptors, we extend  $\kappa'(q_d) = 0$ . For coBüchi acceptors, we add  $q_d$  to F, and for Streett acceptors, we add  $q_d$  to  $G_i$  for all  $i \in [1..k]$ .  $\square$ 

CLAIM 2.2. Let  $\mathcal{B} = \langle \Sigma, Q, q_i, \delta, F \rangle$ , where  $\mathcal{B}$  is an NBA. Define the NPA  $\mathcal{P} = \langle \Sigma, Q, q_i, \delta, \kappa \rangle$  where  $\kappa(q) = 1$  if  $q \in F$  and  $\kappa(q) = 2$  otherwise. Then  $\mathcal{B}$  and  $\mathcal{P}$  are equivalent and have the same size.  $\mathcal{P}$  is deterministic if  $\mathcal{B}$  is.

PROOF. If  $\rho$  is any infinite run of the automaton then  $inf(\rho) \cap F \neq \emptyset$  iff the minimum value of  $\kappa(inf(\rho)) = 1$ .

CLAIM 2.3. Let  $\mathcal{B} = \langle \Sigma, Q, q_i, \delta, F \rangle$ , where  $\mathcal{B}$  is an NBA. Define the NRA  $\mathcal{R} = \langle \Sigma, Q, q_i, \delta, \{(F, \emptyset)\} \rangle$ . The  $\mathcal{B}$  and  $\mathcal{R}$  are equivalent and have the same size.  $\mathcal{R}$  is deterministic if  $\mathcal{B}$  is.

PROOF. If  $\rho$  is any infinite run of the automaton, then  $\inf(\rho) \cap F \neq \emptyset$  iff  $\inf(\rho) \cap F \neq \emptyset$  and  $\inf(\rho) \cap \emptyset = \emptyset$ .

CLAIM 2.4. Let  $C = \langle \Sigma, Q, q_i, \delta, F \rangle$ , where C is a NCA. Define the NSA  $S = \langle \Sigma, Q, q_i, \delta, \{(F, \emptyset)\} \rangle$ . Then S and C are equivalent and have the same size. S is deterministic if C is.

PROOF. If  $\rho$  is any infinite run of the automaton, then  $\inf(\rho) \cap F = \emptyset$  iff  $\inf(\rho) \cap F = \emptyset$  or  $\inf(\rho) \cap \emptyset \neq \emptyset$ .

CLAIM 2.5. Let  $\mathcal{B} = C = \langle \Sigma, Q, q_i, \delta, F \rangle$ , where  $\mathcal{B}$  is a complete DBA and C is a complete DCA. Then  $\mathcal{B}$  and C are the same size and the languages they recognize are complements of each other, that is,  $\|\mathcal{B}\| = \Sigma^{\omega} \setminus \|C\|$ .

PROOF. If  $w \in \Sigma^{\omega}$  then because the automaton is deterministic and complete, there is a unique run  $\rho$  on w. Then  $w \in \llbracket \mathcal{B} \rrbracket$  iff  $\inf(\rho) \cap F \neq \emptyset$  iff  $w \notin \llbracket \mathcal{C} \rrbracket$ .

CLAIM 2.6. Let  $\mathcal{R} = \mathcal{S} = \langle \Sigma, Q, q_i, \delta, \{(G_1, B_1), \dots, (G_k, B_k)\} \rangle$ , where  $\mathcal{R}$  is a complete DRA and  $\mathcal{S}$  is a complete DSA. Then  $\mathcal{R}$  and  $\mathcal{S}$  have the same size and the languages they recognize are complements of each other, that is,  $[\![\mathcal{R}]\!] = \Sigma^{\omega} \setminus [\![\mathcal{S}]\!]$ .

PROOF. If  $w \in \Sigma^{\omega}$  then because the automaton is deterministic and complete, there is a unique run  $\rho$  on w. Then  $w \in [\![\mathcal{R}]\!]$  iff there exists  $i \in [1..k]$  such that  $inf(\rho) \cap G_i \neq \emptyset$  and  $inf(\rho) \cap B_i = \emptyset$  iff it is not the case that for every  $i \in [1..k]$ ,  $inf(\rho) \cap G_i = \emptyset$  or  $inf(\rho) \cap B_i \neq \emptyset$  iff  $w \notin [\![S]\!]$ .

# 2.4 Right congruences

An equivalence relation  $\sim$  on  $\Sigma^*$  is a *right congruence* if  $x \sim y$  implies  $x\sigma \sim y\sigma$  for all  $x, y \in \Sigma^*$  and  $\sigma \in \Sigma$ . The *index* of  $\sim$ , denoted  $|\sim|$  is the number of equivalence classes of  $\sim$ . For a word  $x \in \Sigma^*$  the notation  $[x]_{\sim}$  denotes the equivalence class of  $\sim$  that contains x.

With a right congruence  $\sim$  of finite index one can naturally associate a complete deterministic automaton  $\mathcal{M}_{\sim} = \langle \Sigma, Q, q_i, \delta \rangle$  as follows: the set of states Q consists of the equivalence classes of  $\sim$ . The initial state  $q_i$  is the equivalence class  $[\varepsilon]_{\sim}$ . The transition function  $\delta$  is defined by  $\delta([u]_{\sim}, \sigma) = [u\sigma]_{\sim}$  for all  $\sigma \in \Sigma$ . Also, given a complete deterministic automaton  $\mathcal{M} = \langle \Sigma, Q, q_i, \delta \rangle$ , we can naturally associate with it a right congruence as follows:  $x \sim_{\mathcal{M}} y$  iff  $\mathcal{M}$  reaches the same state of  $\mathcal{M}$  when reading x or y, that is,  $\delta(q_i, x) = \delta(q_i, y)$ .

Given a language  $L\subseteq \Sigma^*$  its canonical right congruence  $\sim_L$  is defined as follows:  $x\sim_L y$  iff  $\forall z\in \Sigma^*$  we have  $xz\in L\iff yz\in L$ . The Myhill-Nerode Theorem states that a language  $L\subseteq \Sigma^*$  is regular iff  $\sim_L$  is of finite index. Moreover, if L is accepted by a complete DFA  $\mathcal{A}$ , then  $\sim_{\mathcal{M}}$  refines  $\sim_L$ , where  $\mathcal{M}$  is the automaton of  $\mathcal{A}$ . Finally, any complete DFA of minimum size that accepts L has an automaton that is isomorphic to  $\mathcal{M}_{\sim_L}$ .

For an  $\omega$ -language  $L \subseteq \Sigma^{\omega}$ , its canonical right congruence  $\sim_L$  is defined similarly, by quantifying over  $\omega$ -words. That is,  $x \sim_L y$  iff  $\forall z \in \Sigma^{\omega}$  we have  $xz \in L \iff yz \in L$ . If L is a regular  $\omega$ -language then  $\sim_L$  is of finite index, and for any complete DBA (resp., DCA, DPA, DRA, DSA, DMA)  $\mathcal A$  that accepts L,  $\sim_{\mathcal M}$  refines  $\sim_L$ , where  $\mathcal M$  is the automaton of the acceptor.

However, for regular  $\omega$ -languages, the relation  $\sim_L$  does not suffice to obtain a "Myhill-Nerode" characterization. In particular, for a regular  $\omega$ -language L there may be no way to define an acceptance condition for  $\mathcal{M}_{\sim_L}$  that yields a DBA (resp., DCA, DPA, DRA, DSA, DMA) that accepts L. As an example consider the language  $L = (a+b)^*(bba)^\omega$ . Then  $\sim_L$  consists of just one equivalence class, because for any  $x \in \Sigma^*$  and  $w \in \Sigma^\omega$  we have that  $xw \in L$  iff w has  $(bba)^\omega$  as a suffix. But a DBA (resp., DCA, DPA, DRA, DSA, DMA) that accepts L clearly needs more than a single state.

# 2.5 The classes IB, IC, IP, IR, IS and IM

In light of the lack of a Myhill-Nerode result for regular  $\omega$ -languages, we define a restricted type of deterministic Büchi (resp., coBüchi, Parity, Rabin, Streett, Muller) acceptor. A DBA (resp., DCA, DPA, DRA, DSA, DMA)  $\mathcal A$  that accepts the language  $L = [\![\mathcal A]\!]$  is an IBA (resp., ICA, IPA, IRA, ISA, IMA) if it is complete and its automaton is isomorphic to  $\mathcal M_{\sim L}$ . A language L is in  $\mathbb IB$  (resp.,  $\mathbb IC$ ,  $\mathbb IP$ ,  $\mathbb IR$ ,  $\mathbb IS$ ,  $\mathbb IM$ ) if there exists an IBA (resp., ICA, IPA, IRA, ISA, IMA)  $\mathcal A$  such that  $L = [\![\mathcal A]\!]$ . We note that every state of an IBA (resp., ICA, IPA, IRA, ISA, IMA) is reachable because every state of  $\mathcal M_{\sim L}$  is reachable.

Despite the fact that each of these classes is a proper subset of its corresponding deterministic class (e.g.,  $\mathbb{IB}$  is a proper subset of  $\mathbb{DBA}$ ), these classes are more expressive than one might first conjecture. It was shown in [9] that in every class of the infinite Wagner hierarchy [43] there are languages in  $\mathbb{IM}$  and  $\mathbb{IP}$ . Moreover, in a small experiment reported in [9], among randomly generated Muller automata, the vast majority turned out to be in  $\mathbb{IM}$ .

# 2.6 Examples and samples

Because we require finite representations of examples,  $\omega$ -words in our case, we work with ultimately periodic words, that is, words of the form  $u(v)^{\omega}$  where  $u \in \Sigma^*$  and  $v \in \Sigma^+$ . It is known that two regular  $\omega$ -languages are equivalent iff they agree on the set of ultimately periodic words, so this choice is not limiting.

The example  $u(v)^{\omega}$  is concretely represented by the pair (u,v) of finite strings, and its length is |u|+|v|. A labeled example is a pair  $(u(v)^{\omega},l)$ , where the label l is either 0 or 1. A sample is a finite set of labeled examples such that no example is assigned two different labels. The length of a sample is the sum of the lengths of the examples that appear in it. A sample T and a language T are consistent with each other if and only if for every labeled example T and an acceptor T are consistent with each other if and only if T is consistent with T and T are consistent with each other if and only if T is consistent with T and T are consistent with each other if and only if T is consistent with T and T are consistent with each other if and only if T is consistent with T and T are consistent with each other if and only if T is consistent with T and T are consistent with each other if and only if T is consistent with T and T are consistent with each other if and only if T is consistent with T and T are consistent with each other if and only if T is consistent with T and T are consistent with each other if and only if T is consistent with T and T are consistent with each other if and only if T is consistent with T and T are consistent with each other if T and T are consistent with each other if T and T are consistent with each other if T and T are consistent with each other if T and T are consistent with each other if T are consistent with each other if T and T are consistent with each other if T and T are consistent with each other if T and T are consistent with each other if T are consistent with each other if T and T are consistent with each other if T and T are consistent with each other if T are consistent with each other if T are consistent with each other if T and T are consistent with each other in T and T are consistent with each other in T and T are consistent with

PROPOSITION 2.7. Let  $u_1, u_2 \in \Sigma^*$  and  $v_1, v_2 \in \Sigma^+$ . If  $u_1(v_1)^\omega \neq u_2(v_2)^\omega$  then they differ in at least one of the first  $\ell$  symbols, for  $\ell = \max(|u_1|, |u_2|) + |v_1| \cdot |v_2|$ .

Let  $suffixes(u(v)^{\omega})$  denote the set of all  $\omega$ -words that are suffixes of  $u(v)^{\omega}$ . Manuscript submitted to ACM PROPOSITION 2.8. The set suffixes  $(u(v)^{\omega})$  consists of at most |u| + |v| different examples: one of the form  $u'(v)^{\omega}$  for every nonempty suffix u' of u, and one of the form  $(v_2v_1)^{\omega}$  for every division of v into a non-empty prefix and suffix as  $v = v_1v_2$ .

# 2.7 Identification in the limit using polynomial time and data

We consider the notion of identification in the limit using polynomial time and data. This criterion of learning was introduced by Gold [22], who showed that regular languages of finite strings represented by DFAs are learnable in this sense. We follow a more general definition given by de la Higuera [19]. The definition has two requirements: (1) a learning algorithm A that runs in polynomial time on a set of labeled examples and produces a hypothesis consistent with the examples, and (2) that for every language L in the class, there exists a set  $T_L$  of labeled examples of size polynomial in a measure of size of L such that on any set of labeled examples containing  $T_L$ , the algorithm A outputs a hypothesis correct for L. Condition (1) ensures polynomial time, while condition (2) ensures polynomial data. The latter is not a worst-case measure; there could be arbitrarily large finite samples for which A outputs an incorrect hypothesis. However, de la Higuera shows that identifiability in the limit with polynomial time and data is closely related to a model of a learner and a helpful teacher introduced by Goldman and Mathias [23].

### 3 NEGATIVE RESULTS

We start with negative results. We show that when the representation at hand is non-deterministic, polynomial identification is not feasible.

THEOREM 3.1. NBA cannot be identified in the limit using polynomial data.

PROOF. The proof follows the idea given in de la Higuera's negative result for learning NFAs in the limit from polynomial data [19]. For any integer  $M \ge 2$ , let  $p_1, \ldots, p_m$  be the set of all primes less than or equal to M. For each such M, consider the NBA  $\mathcal{B}_M$  over a two letter alphabet  $\Sigma = \{a, b\}$  with  $p_1 + p_2 + \ldots + p_m + 2$  states, where state 0 has a-transitions to state  $(\mathbf{p}, 1)$  for each  $\mathbf{p} \in \{p_1, p_2, \ldots, p_m\}$ . State  $(\mathbf{p}, i)$  has an a-transition to state  $(\mathbf{p}, i \oplus_p 1)$  where  $\oplus_p$  is addition modulo p. All states except the states  $(\mathbf{p}, 0)$  have a b-transition to state b. The state b has a self-loop on b. The only accepting state is b. The NBA  $\mathcal{B}_M$  for M = 5 is given in Fig. 1.

The NBA  $\mathcal{B}_M$  accepts the set of all words of the form  $a^kb^\omega$  such that k is not a positive multiple of  $\ell=p_1\cdot p_2\cdots p_m$ . Note that the size of the shortest ultimately periodic word in  $a^*b^\omega\setminus \llbracket \mathcal{B}_M\rrbracket$  is  $\ell+1$ , and thus, to distinguish the language  $\llbracket \mathcal{B}_M\rrbracket$  from the language  $a^*b^\omega$ , a word of at least this size must be provided. Since the number of primes not greater than M is  $\Theta(M/\log M)$  and since each prime is of size at least 2 the data must be of size at least  $2^{\Theta(M/\log M)}$  while the number of states of  $\mathcal{B}_M$  is  $O(M^2)$ .

COROLLARY 3.2. NPA, NRA, and NMA cannot be identified in the limit using polynomial data.

PROOF. The NBA in the proof of Theorem 3.1 can be converted to an equivalent NPA of the same size using Claim 2.2, and to an equivalent NRA of the same size using Claim 2.3. It can be converted to an equivalent NMA of the same size with the same automaton by specifying the acceptance condition  $\{\{b\}\}$ .

While NBAs are not a special case of non-deterministic coBüchi automata (NCA) it can be shown that  $\mathbb{NCA}$  also cannot be identified in the limit from polynomial data, which is in some sense surprising, since NCAs are not more expressive than DCAs, their deterministic counterpart, and accept a very small subclass of the regular  $\omega$ -languages.

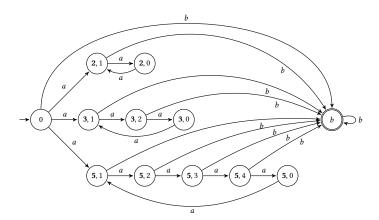


Fig. 1. The NBA  $\mathcal{B}_M$  for M=5.

THEOREM 3.3. NCA and NSA cannot be identified in the limit using polynomial data.

PROOF. Define the NCA acceptor  $C_M$  to have the same automaton as  $\mathcal{B}_M$  from the proof of Theorem 3.1, with all states except b accepting. Then  $C_M$  accepts the same language as  $\mathcal{B}_M$  and has the same size. This NCA can be converted to an equivalent NSA of the same size using Claim 2.4.

### 4 TOWARDS THE POSITIVE RESULTS

The rest of the paper is devoted to the positive results. This section covers some preliminary issues and gives an overview of the positive results.

# 4.1 Duality

There are reductions of the problem of identification in the limit using polynomial time and data between  $\mathbb{IB}$  and  $\mathbb{IC}$  and between  $\mathbb{IR}$  and  $\mathbb{IS}$ , using the duality between these types of acceptors. Consequently we focus on the classes  $\mathbb{IB}$ ,  $\mathbb{IP}$ ,  $\mathbb{IR}$ , and  $\mathbb{IM}$  in what follows.

PROPOSITION 4.1. If IB (resp., IR) is identifiable in the limit using polynomial time and data, then so is IC (resp., IS).

PROOF. Let  $\mathcal{A}$  be an ICA. Because  $\mathcal{A}$  is deterministic and complete, if we let  $\mathcal{A}'$  denote the IBA with the same components as  $\mathcal{A}$ , then  $\mathcal{A}'$  accepts the complement of the language  $\mathcal{A}$ , by Claim 2.5.

We modify the characteristic sample for  $\mathcal{A}'$  by complementing all its labels to get a characteristic sample for  $\mathcal{A}$ . The algorithm to learn an ICA from a sample T is obtained by complementing all the labels in the sample T and calling the algorithm to learn an IBA from a sample. The resulting IBA, now considered to be an ICA, is returned as the answer.

Exactly the same conversion may be done with acceptors of types IRA and ISA, by Claim 2.6.

#### 4.2 The default acceptor

One condition of the definition of identification in the limit using polynomial time and data is that the learning algorithm must run in polynomial time and return an acceptor of the required type that is consistent with the input Manuscript submitted to ACM

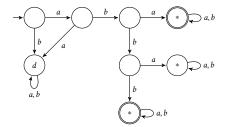


Fig. 2. Default acceptor of type DBA for  $T = \{(a(b)^{\omega}, 1), ((ab)^{\omega}, 1), (ab(baa)^{\omega}, 0)\}$ . Leaf states are marked with \* and the dead state with d.

sample *T*, even if the sample *T* does not subsume a characteristic sample. To meet this condition, we use the strategy of Gold's construction, that is, the learning algorithm optimistically assumes that the sample includes a characteristic sample, and if that assumption fails to produce an acceptor consistent with the sample, the algorithm instead produces a *default acceptor* to ensure that its hypothesis is consistent with the sample. In practical applications much more effort would be expended to find good heuristic choices to avoid defaulting too easily.

PROPOSITION 4.2. There is a polynomial time algorithm that takes a sample T and returns a DBA (resp., DCA, DPA, DRA, DSA, DMA) consistent with T.

PROOF. Given a sample T, we find the shortest prefix of each example  $u(v)^{\omega}$  in T that distinguishes it from all other examples in T and place these prefixes in a deterministic prefix-tree automaton. By Prop. 2.7, this prefix-tree automaton can be constructed in time polynomial in the length of the sample T. For each leaf state we define self-transitions on each symbol in  $\Sigma$ , and if the automaton is incomplete, we add a new dead state with self-transitions on each symbol in  $\Sigma$ , and define all undefined transitions to go to the dead state.

For a DBA, the acceptance condition F consists of all the leaf states that are prefixes of positive examples in T. For a DMA, the acceptance condition consists of  $\{\{q\} \mid q \in F\}$ . For a DCA, the acceptance condition consists of the dead state (if one was added) and all the leaf states that are not prefixes of positive examples in T. The DBA thus constructed may be transformed to a DPA or a DRA using Claim 2.2 or Claim 2.3, and the DCA may be transformed to a DSA using Claim 2.4.

As an example of this construction, let the sample be

$$T = \{(a(b)^{\omega}, 1), ((ab)^{\omega}, 1), (ab(baa)^{\omega}, 0)\}.$$

The corresponding prefixes are abbb, aba, and abba, and the default acceptor of type DBA for T is shown in Figure 2.

# 4.3 Strongly connected components

The acceptance conditions that we consider are all based on the set of states visited infinitely often in a run of the automaton on an input  $w \in \Sigma^{\omega}$ . We consider only acceptors whose automata are deterministic and complete, so for any  $w \in \Sigma^{\omega}$  there is exactly one run, which we denote  $\rho(w)$ , of the automaton on input w. Thus we may define  $\inf(w) = \inf(\rho(w))$ , the set of states visited infinitely often in this unique run. In the run  $\rho(w)$ , there is some point after which none of the states visited finitely often is visited. Because each state in  $\inf(w)$  is visited infinitely often, for Manuscript submitted to ACM

any states  $q_1, q_2 \in inf(w)$ , there exists a non-empty word  $x \in \Sigma^*$  such that  $\delta(q_1, x) = q_2$  and for each prefix x' of x,  $\delta(q_1, x') \in inf(w)$ , that is, the path from  $q_1$  to  $q_2$  on x does not visit any state outside the set inf(w).

These properties characterize the following definition. Given an automaton  $\mathcal{M}$ , a strongly connected component (SCC) of  $\mathcal{M}$  is a nonempty set of states C such that for every  $q_1, q_2 \in C$ , there exists a nonempty string  $x \in \Sigma^*$  such that  $\delta(q_1, x) = q_2$  and for any prefix x' of x,  $\delta(q_1, x') \in C$ .

Note that an SCC need not be maximal, and that a singleton state set  $\{q\}$  is an SCC if and only if the state q has a self-loop, that is,  $\delta(q, \sigma) = q$  for some  $\sigma \in \Sigma$ . There is a close relationship between SCCs and the set of states visited infinitely often in a run.

PROPOSITION 4.3. Let  $\mathcal{M}$  be a complete deterministic automaton and  $w \in \Sigma^{\omega}$ . Then  $\inf(w)$  is an SCC of  $\mathcal{M}$ . If w is the ultimately periodic word  $u(v)^{\omega}$ , then  $\inf(w)$  may be computed in time polynomial in the size of  $\mathcal{M}$  and the length of  $u(v)^{\omega}$ .

Proposition 4.4. For any deterministic automaton  $\mathcal{M} = \langle \Sigma, Q, q_t, \delta \rangle$  and any reachable SCC C of  $\mathcal{M}$ , there exists an ultimately periodic word  $u(v)^{\omega}$  of length at most  $|Q| + |C|^2$  such that  $C = \inf(w)$ . Such a word may be found in time polynomial in |Q| and  $|\Sigma|$ .

PROOF. Because C is reachable, a word  $u \in \Sigma^*$  of minimum length such that  $\delta(q_i, u) \in C$  may be found by breadth first search. The length of u is at most |Q|. If  $C = \{q\}$ , then there is at least one symbol  $\sigma \in \Sigma$  such that  $\delta(q, \sigma) = q$ . Then the  $\omega$ -word  $w = u(\sigma)^{\omega}$  is such that  $C = \inf(w)$ . The length of this ultimately periodic word is at most |Q| + 1.

If C contains at least two states, let  $q_1, \ldots, q_k$  be the states in C that are not q. Then for each i, there exist two nonempty finite words  $x_i$  and  $y_i$  each of length at most n such that  $\delta(q, x_i) = q_i$  and  $\delta(q_i, y_i) = q$ , and the path on  $x_i$  from q to  $q_i$  and the path on  $y_i$  from  $q_i$  to q do not visit any states outside of C. The words  $x_i$  and  $y_i$  may be found in polynomial time by breadth-first search. Then the word  $w = u(x_1y_1 \cdots x_ky_k)^{\omega}$  is such that inf(w) = C. The length of this ultimately periodic word is at most  $|Q| + |C|^2$ .

We let **Witness**  $(C, \mathcal{M})$  denote the ultimately periodic word  $u(v)^{\omega}$  returned by the algorithm described in the proof above for the reachable SCC C of automaton  $\mathcal{M}$ .

PROPOSITION 4.5. If  $C_1$  and  $C_2$  are SCCs of automaton  $\mathcal{M}$  and  $C_1 \cap C_2 \neq \emptyset$ , then  $C_1 \cup C_2$  is also an SCC of  $\mathcal{M}$ .

If  $\mathcal{M}$  is an automaton and S is any set of its states, define SCCs(S) to be the set of all C such that  $C \subseteq S$  and C is an SCC of  $\mathcal{M}$ . Also define maxSCCs(S) to be the maximal elements of SCCs(S) with respect to the subset ordering. The following is a consequence of Prop. 4.5.

Proposition 4.6. If  $\mathcal{M}$  is an automaton and S is any set of its states, then the elements of maxSCCs(S) are pairwise disjoint, and every set  $C \in SCCs(S)$  is a subset of exactly one element of maxSCCs(S).

There are some differences in the terminology related to strong connectivity between graph theory and omega automata, which we resolve as follows. In graph theory, a *path* of length k from u to v in a directed graph (V, E) is a finite sequence of vertices  $v_0, v_1, \ldots, v_k$  such that  $u = v_0, v = v_k$  and for each i with  $i \in [1..k], (v_{i-1}, v_i) \in E$ . Thus, for every vertex v, there is a path of length 0 from v to v. A set of vertices S is *strongly connected* if and only if for all  $u, v \in S$ , there is a path of some nonnegative length from u to v and all the vertices in the path are elements of S. Thus, for every vertex v, the singleton set  $\{v\}$  is a strongly connected set of vertices. A *strongly connected component* of a Manuscript submitted to ACM

directed graph is a maximal strongly connected set of vertices. There is a linear time algorithm to find the set of strong components of a directed graph [38].

In this paper, we use the terminology SCC and maximal SCC to refer to the definitions from the theory of omega automata, and the terminology graph theoretic strongly connected components to refer to the definitions from graph theory. We use the term trivial strong component to refer to a graph theoretic strongly connected component that is a singleton vertex  $\{v\}$  such that there is no edge (v, v).

If  $\mathcal{M}$  is an automaton, we may define a related directed graph  $G(\mathcal{M})$  whose vertices are the states of  $\mathcal{M}$  and whose edges  $(q_1,q_2)$  are the pairs of states such that  $q_2 \in \delta(q_1,\sigma)$  for some  $\sigma \in \Sigma$ . Then for any set S of states of  $\mathcal{M}$ , the maximal SCCs in S, maxSCCs(S), are the graph theoretic strongly connected components of the subgraph of  $G(\mathcal{M})$  induced by S, with any trivial strong components removed.

Proposition 4.7. For automaton M and any subset S of its states, maxSCCs(S) can be computed in time linear in the size of M.

## 4.4 A decreasing forest of SCCs of an automaton

Let  $\mathcal{M}$  be a deterministic automaton, and let S be a subset of its states. A decreasing forest of SCCs of  $\mathcal{M}$  rooted in S is a finite rooted forest  $\mathcal{F}$  in which every node C is an SCC of  $\mathcal{M}$  that is contained in S, and the following properties are satisfied.

- (1) The roots of  $\mathcal{F}$  are the elements of maxSCCs(S).
- (2) Whenever  $D_1, \ldots, D_k$  are the children of node C, we have  $D_1 \cup \ldots \cup D_k \subsetneq C$ . Also, letting  $\Delta(C) = C \setminus (D_1 \cup \ldots \cup D_k)$ , the children  $D_1, \ldots, D_k$  are exactly the elements of  $maxSCCs(C \setminus \Delta(C))$ .

PROPOSITION 4.8. Let  $\mathcal{M}$  be a deterministic automaton, S a subset of its states, and  $\mathcal{F}$  a decreasing forest of SCCs of  $\mathcal{M}$  rooted in S. Then the following are true.

- (1) The roots of  $\mathcal{F}$  are pairwise disjoint.
- (2) The children of any node are pairwise disjoint.
- (3)  $\mathcal{F}$  has at most |S| nodes.
- (4) For any  $D \subseteq S$  that is an SCC of M, there is a unique node C in F such that  $D \subseteq C$  and D is not a subset of any of the children of C.

PROOF. The roots of  $\mathcal{F}$  are the elements of maxSCCs(S), which are pairwise disjoint. The children of a node C are the elements of  $maxSCCs(C \setminus \Delta(C))$ , which are pairwise disjoint. The sets  $\Delta(C)$  for nodes C in  $\mathcal{F}$  are contained in S, nonempty, and pairwise disjoint, so the number of nodes is at most |S|. If  $D \subseteq S$  is an SCC of  $\mathcal{M}$ , then D is a subset of exactly one of the roots of  $\mathcal{F}$ , say  $C_1$ . If  $D \cap \Delta(C_1) \neq \emptyset$ , then D is not a subset of any of the children of  $C_1$ . Otherwise, D must be a subset of exactly one of the children of  $C_1$ , say  $C_2$ . If  $D \cap \Delta(C_2) \neq \emptyset$ , then D is not a subset of any of the children of  $C_2$ . Continuing in this way, we eventually arrive at the required node C.

Given a decreasing forest  $\mathcal{F}$  of SCCs of automaton  $\mathcal{M}$  rooted in S, and an SCC  $D \subseteq S$ , we denote by  $\mathbf{Node}(D, \mathcal{F})$  the unique node C of  $\mathcal{F}$  such that  $D \subseteq C$  and D is not a subset of any of the children of C. We note that if  $C = \mathbf{Node}(D, \mathcal{F})$  then  $D \cap \Delta(C) \neq \emptyset$ . If D is a child of some C in  $\mathcal{F}$ , we define *merging* D *into* C as the operation of removing D from  $\mathcal{F}$  and making the children of D (if any) direct children of C.

PROPOSITION 4.9. Let  $\mathcal{M}$  be a deterministic automaton and S a subset of its states. Let  $\mathcal{F}$  be a decreasing forest of SCCs of  $\mathcal{M}$  rooted in S. Let D be a child of C in  $\mathcal{F}$  and let  $\mathcal{F}$ ' be obtained from  $\mathcal{F}$  by merging D into C. Then  $\mathcal{F}$ ' is also a decreasing forest of SCCS of  $\mathcal{M}$  rooted in S.

PROOF. After the merge, the roots of  $\mathcal{F}$  remain the elements of maxSCCs(S). Let  $D_1, \ldots, D_k$  be the children of C in F, where  $D = D_k$ , and let  $E_1, \ldots, E_\ell$  be the children of D in F. Because the union of  $E_1, \ldots, E_\ell$  is a proper subset of  $D = D_k$  and the union of  $D_1, \ldots, D_k$  is a proper subset of C, the union of  $D_1, \ldots, D_{k-1}$  with the union of  $E_1, \ldots, E_\ell$  is a proper subset of C, therefore the union of the children of C in  $\mathcal{F}$  is a proper subset of C. Also, the children of C in  $\mathcal{F}$  are the maximum SCCs of  $C \setminus \Delta_{\mathcal{F}}(C)$ , and no other nodes are affected, so  $\mathcal{F}$  is a decreasing forest of SCCs of  $\mathcal{M}$  rooted in S.

### 4.5 Overview of the positive results

To show that a class is identified in the limit using polynomial time and data there are two parts: (i) constructing a sample of words  $T_L$  of size polynomial in the size of the given acceptor  $\mathcal{A}$  for the language L at hand, called the *characteristic sample*, and (ii) providing a polynomial time learning algorithm  $\mathbf{A}$  that for every given sample T returns an acceptor consistent with T, and, moreover, for any sample T that subsumes  $T_L$ , returns an acceptor that accepts T.

The definition of an acceptor has two parts: (a) the definition of the automaton and (b) the definition of the acceptance condition. Correspondingly, we view the characteristic sample as a union of two parts:  $T_{Aut}$  (to specify the automaton) and  $T_{Acc}$  (to specify the acceptance condition). In Section 5 we discuss the construction of  $T_{Aut}$ , which is common to all the classes we consider, as they all are isomorphic to  $\mathcal{M}_{\sim L}$  for the target language L. We also describe a polynomial time algorithm to construct the automaton using the labeled words in  $T_{Aut}$ .

Because the acceptance conditions differ,  $T_{Acc}$  is different for each type of acceptor we consider. In Section 6 we describe the construction of  $T_{Acc}$  for acceptors of types IMA and IBA and learning algorithms for acceptors of these types, showing that IM, IB, and IC are identifiable in the limit using polynomial time and data. In Section 7 we describe the construction of  $T_{Acc}$  for acceptors of type IPA and a learning algorithm for acceptors of this type, showing that IPA is identifiable in the limit using polynomial time and data. In Section 8 we describe the construction of  $T_{Acc}$  for acceptors of type IRA and a learning algorithm for acceptors of this type, showing that IR and IS are identifiable in the limit using polynomial time and data.

In Section 9 we show that the characteristic samples we have defined can be computed in polynomial time in the size of the acceptor. These results rely on polynomial time algorithms for the inclusion and equivalence problems for the acceptors. These are described in Sections 10, 11, 12, and 13. In Section 14 we show the right congruence automaton  $\mathcal{M}_{\sim L}$  can be computed in polynomial time, given a DBA, DCA, DPA, DRA, DSA, or DMA accepting L, which yields polynomial time algorithms to test whether a DBA is an IBA, and similarly for the other acceptor types. In Section 15, we consider the harder problem of whether a DBA accepts a language in  $\mathbb{IB}$ , and give polynomial time algorithms for DBAs, DCAs, DPAs, and DMAs, leaving open the problem for DRAs and DSAs. In Section 16 we summarize our results and describe some additional open problems.

# 5 THE SAMPLE $T_{Aut}$ FOR THE AUTOMATON

In this section we describe construction of the  $T_{Aut}$  part of the sample. We first show that if two states of the automaton are distinguishable, they are distinguishable by words of length polynomial in the number of states of the automaton. Manuscript submitted to ACM

#### 5.1 Existence of short distinguishing words

Let A be an acceptor of one of the types DBA, DCA, DPA, DRA, DSA, or DMA over alphabet  $\Sigma$ . We say that states  $q_1$  and  $q_2$  of  $\mathcal{M}$  are distinguishable if there exists a word  $w \in \Sigma^{\omega}$  that is accepted from one state but not the other, that is,  $w \in [\![\mathcal{A}^{q_1}]\!] \setminus [\![\mathcal{A}^{q_2}]\!]$  or  $w \in [\![\mathcal{A}^{q_2}]\!] \setminus [\![\mathcal{A}^{q_1}]\!]$ . In this case we say that w is a distinguishing word.

PROPOSITION 5.1. If two states of a complete DBA, DCA, DPA, DRA, DSA, or DMA of n states are distinguishable, then they are distinguishable by an ultimately periodic  $\omega$ -word of length bounded by  $O(n^4)$ .

PROOF. We prove the result for a DMA. Because any DBA, DCA, DPA, DRA, or DSA is equivalent to a DMA with the same automaton, this result holds for these types of acceptors as well.

Let  $\mathcal{A}$  be a complete DMA of n states such that the states  $q_1$  and  $q_2$  are distinguishable. Then there exists an  $\omega$ -word w that is accepted from exactly one of the two states, that is, w is accepted by exactly one of  $\mathcal{A}^{q_1}$  and  $\mathcal{A}^{q_2}$ .

Let  $\mathcal{M}_i$  denote the automaton of  $\mathcal{A}$  with its initial state replaced by  $q_i$  for i=1,2. Let  $\mathcal{M}$  denote the product automaton  $\mathcal{M}_1 \times \mathcal{M}_2$ . The number of states of  $\mathcal{M}$  is  $n^2$ . By Prop. 4.3,  $\inf_{\mathcal{M}}(w)$  is a reachable SCC C of  $\mathcal{M}$ , and by Prop. 4.4 there exists an ultimately periodic word  $u(v)^{\omega}$  of length bounded by  $O(n^4)$  such that  $\inf_{\mathcal{M}}(u(v)^{\omega}) = C$ .

Then for i = 1, 2,  $\inf_{\mathcal{M}_i}(u(v)^{\omega}) = \pi_i(C) = \inf_{\mathcal{M}_i}(w)$ , so  $u(v)^{\omega}$  is also accepted by exactly one of  $\mathcal{A}^{q_1}$  and  $\mathcal{A}^{q_2}$ , and  $u(v)^{\omega}$  distinguishes  $q_1$  and  $q_2$ .

# 5.2 Defining the sample $T_{Aut}$ for the automaton

We now define the  $T_{Aut}$  part of the characteristic sample, given an acceptor  $\mathcal{A} = \langle \Sigma, Q, q_t, \delta, \alpha \rangle$  that is an IBA, ICA, IPA, IRA, ISA, or IMA. This construction is analogous to that of the corresponding part of a characteristic sample for a DFA, with distinguishing experiments that are ultimately periodic  $\omega$ -words instead of finite strings. Let  $\mathcal{M}$  be the automaton of  $\mathcal{A}$  and let n be the number of states of  $\mathcal{M}$ .

Because  $\mathcal{A}$  is an IBA, ICA, IPA, IRA, ISA, or IMA, every state is reachable and every pair of states is distinguishable. We define a distinguished set of n access strings for the states of  $\mathcal{M}$  as follows. For each state q, access(q) is the least string x in the shortlex ordering such that  $\delta(q_t, x) = q$ . Given  $\mathcal{A}$ , the access strings may be computed in polynomial time by breadth first search.

Because every pair of states is distinguishable, by Proposition 5.1, there exists a set E of at most n distinguishing experiments, each of length at most  $n^2 + n^4$ , that distinguish every pair of states. The issue of computing E is addressed in Section 9.

The sample  $T_{Aut}$  consists of all the examples in  $(S \cdot E) \cup (S \cdot \Sigma \cdot E)$ , labeled to be consistent with  $\mathcal{A}$ . There are at most  $(1 + |\Sigma|)n^2$  labeled examples in  $T_{Aut}$ , each of length bounded by a polynomial in n. A learning algorithm using  $T_{Aut}$  is described next.

# 5.3 Learning the automaton from $T_{Aut}$

We now describe a learning algorithm  $\mathbf{L}_{Aut}$  and prove the following.

THEOREM 5.2. The algorithm  $L_{Aut}$  with a sample T as input runs in polynomial time and returns a deterministic complete automaton M. Let  $\mathcal{A}$  be an an acceptor of type IBA, ICA, IPA, IRA, ISA, or IMA. If T is consistent with  $\mathcal{A}$  and subsumes  $T_{Aut}$  then the returned automaton M is isomorphic to the automaton of  $\mathcal{A}$ .

Algorithm  $\mathbf{L}_{Aut}$  on input T constructs a set E of words that serve as experiments used to distinguish candidate states. For each  $(u(v)^{\omega}, l)$  in T, all of the elements of  $suffixes(u(v)^{\omega})$  are placed in E. Two strings  $x, y \in \Sigma^*$  are consistent with Manuscript submitted to ACM

respect to T if and only if there does not exist any  $u(v)^{\omega} \in E$  such that the examples  $xu(v)^{\omega}$  and  $yu(v)^{\omega}$  are oppositely labeled in T.

Starting with the empty string  $\varepsilon$ , the algorithm builds up a prefix-closed set S of finite strings as follows. Initially,  $S_1 = \{\varepsilon\}$ . After  $S_k$  has been constructed, the algorithm considers each  $s \in S_k$  in shortlex order, and each symbol  $\sigma \in \Sigma$  in the ordering defined on  $\Sigma$ . If there exists no  $s' \in S_k$  such that  $s\sigma$  is consistent with s' with respect to T, then  $S_{k+1}$  is set to  $S_k \cup \{s\sigma\}$  and k is set to k+1. If no such pair s and  $\sigma$  is found, then the final set S is  $S_k$ .

In the second phase, the algorithm uses the strings in S as names for states and constructs a transition function  $\delta$  using S and E. For each  $s \in S$  and  $\sigma \in \Sigma$ , there is at least one  $s' \in S$  such that  $s\sigma$  and s' are consistent with respect to T. The algorithm selects any such s' and defines  $\delta(s,\sigma)=s'$ . Once S and  $\delta$  are defined, the algorithm returns the automaton  $\mathcal{M}=\langle \Sigma, S, \varepsilon, \delta \rangle$ .

PROOF OF THEOREM 5.2. E may be computed in time polynomial in the length of T, by Prop. 2.8. Because the default acceptor for T has a polynomial number of states and is consistent with T, the number of distinguishable states, and the number of strings added to S, is bounded by a polynomial in the length of T. The returned automaton  $\mathcal{M}$  is deterministic and complete by construction.

Assume the sample T is consistent with  $\mathcal{A}$  and subsumes  $T_{Aut}$ . For any pair of states of  $\mathcal{A}$ , the set E includes an experiment to distinguish them. Also, if x and y reach the same state of  $\mathcal{A}$ , there is no experiment in E that distinguishes them. Then the set S is precisely the access strings of  $\mathcal{A}$ . The choice of s' for  $\delta(s, \sigma)$  is unique in each case, and the returned automaton  $\mathcal{M}$  is isomorphic to the automaton of  $\mathcal{A}$ .

Although the processes of constructing  $T_{Aut}$  and learning an automaton from it are the same for acceptors of types IBA, ICA, IPA, IRA, ISA, or IMA, different types of acceptance condition require different kinds of characteristic samples and learning algorithms.

In the following sections we describe for each type of acceptor the corresponding sample  $T_{Acc}$  and learning algorithm. Each learning algorithm takes as input an automaton  $\mathcal{M}$  and a sample T and returns in polynomial time an acceptor of the appropriate type consistent with T. We show that for each type of acceptor  $\mathcal{A}$ , if the input automaton  $\mathcal{M}$  is isomorphic to the automaton of  $\mathcal{A}$  and the sample T is consistent with  $\mathcal{A}$  and subsumes the  $T_{Acc}$  for  $\mathcal{A}$ , then the learning algorithm returns an acceptor that is equivalent to  $\mathcal{A}$ . This learning algorithm is then combined with  $\mathbf{L}_{Aut}$  to prove identification in the limit using polynomial time and data for the relevant class of languages.

# 6 THE SAMPLES $T_{Acc}$ AND LEARNING ALGORITHMS FOR IMA AND IBA

The straightforward cases of Muller, Büchi, and coBüchi acceptance conditions are covered in this section. Subsequent sections cover the cases of Parity, Rabin, and Street acceptance conditions, which are somewhat more involved.

### 6.1 Muller acceptors

Let  $\mathcal{A}$  be an IMA with acceptance condition  $\alpha = \{F_1, \dots, F_k\}$ . By Prop. 4.3, we may assume that each  $F_i$  is a reachable SCC of  $\mathcal{A}$ . The sample  $T_{Acc}^{\text{IMA}}$  consists of k positive examples, one for each set  $F_i$ . The example for  $F_i$  is  $(u(v)^{\omega}, 1)$  where  $inf(u(v)^{\omega}) = F_i$ . These examples may be found in polynomial time in the size of  $\mathcal{A}$  by Prop. 4.4.

The learning algorithm  $L_{Acc}^{IMA}$  takes as input a deterministic complete automaton  $\mathcal{M}$  and a sample T. It constructs an acceptance condition  $\alpha'$  as follows. For each positive labeled example  $(u(v)^{\omega}, 1) \in T$ , it computes the set  $C = \inf_{\mathcal{M}}(u(v)^{\omega})$  and makes C a member of  $\alpha'$ . Once the set  $\alpha'$  is complete, the algorithm checks whether the DMA  $(\mathcal{M}, \alpha')$  is consistent with T. If so, it returns  $(\mathcal{M}, \alpha')$ ; if not, it returns the default acceptor of type DMA for T.

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Theorem 6.1. Algorithm  $\mathbf{L}_{Acc}^{IMA}$  runs in time polynomial in the sizes of the inputs  $\mathcal{M}$  and T. Let  $\mathcal{A}$  be an IMA. If the input automaton  $\mathcal{M}$  is isomorphic to the automaton of  $\mathcal{A}$ , and the sample T is consistent with  $\mathcal{A}$  and subsumes  $T_{Acc}^{IMA}$ , then algorithm  $\mathbf{L}_{Acc}^{IMA}$  returns an IMA  $(\mathcal{M}, \alpha')$  equivalent to  $\mathcal{A}$ .

PROOF. The construction of  $\alpha'$  can be done in time polynomial in the sizes of  $\mathcal{M}$  and T by Prop. 4.3. The returned acceptor is consistent with T by construction.

Assume  $\mathcal{M}$  is isomorphic to the automaton of  $\mathcal{A}$  and that T is consistent with  $\mathcal{A}$ . For ease of notation, assume the isomorphism is the identity. Then for each positive example  $(u(v)^{\omega}, 1)$  in T, the set  $F = \inf(u(v)^{\omega})$  must be in  $\alpha$ , so  $\alpha'$  is a subset of  $\alpha$ .

If T subsumes  $T_{Acc}^{\text{IMA}}$ , then for every set  $F \in \alpha$  there is a positive example  $(u(v)^{\omega}, 1)$  in T with  $F = \inf(u(v)^{\omega})$ . Thus the set F is added to  $\alpha'$ , and  $\alpha$  is a subset of  $\alpha'$ . Thus,  $(\mathcal{M}, \alpha')$  is equivalent to  $\mathcal{A}$ , and because T is consistent with  $\mathcal{A}$ . the IMA  $(\mathcal{M}, \alpha')$  is returned by  $\mathbf{L}_{Acc}^{\text{IMA}}$ .

THEOREM 6.2. The class IM is identifiable in the limit using polynomial time and data.

PROOF. Let  $\mathcal{A}$  be an IMA accepting a language L. The characteristic sample  $T_L = T_{Aut} \cup T_{Acc}^{\text{IMA}}$  is of size polynomial in the size of  $\mathcal{A}$ .

The combined learning algorithm  $\mathbf{L}^{\text{IMA}}$  takes a sample T as input and runs  $\mathbf{L}_{Aut}$  on T to produce an automaton  $\mathcal{M}$  and then runs  $\mathbf{L}_{Acc}^{\text{IMA}}$  on  $\mathcal{M}$  and T and returns the resulting acceptor. It runs in polynomial time in the size of T because it is the composition of two polynomial time algorithms, and the acceptor it returns is guaranteed to be consistent with T.

If the sample T is consistent with  $\mathcal{A}$  and subsumes  $T_L$ , then by Theorem 5.2 the automaton  $\mathcal{M}$  returned by  $\mathbf{L}_{Aut}$  is isomorphic to the automaton of  $\mathcal{A}$ . Then by Theorem 6.1 the acceptor returned by  $\mathbf{L}_{Acc}^{\mathrm{IMA}}$  with inputs  $\mathcal{M}$  and T is an IMA equivalent to  $\mathcal{A}$ .

# 6.2 Büchi acceptors

The case of Büchi acceptors is nearly as straightforward as that of Muller acceptors. Let  $\mathcal A$  be an IBA with n states and acceptance condition F. For every state q of  $\mathcal A$ , if there is an  $\omega$ -word w such that  $\mathcal A$  rejects w and  $q \in inf(w)$ , then there is an example  $u(v)^\omega$  of length  $O(n^2)$  such that  $\mathcal A$  rejects  $u(v)^\omega$  and  $q \in inf(u(v)^\omega)$ , by Prop. 4.4. The negative labeled example  $(u(v)^\omega, 0)$  is included in  $T_{Acc}^{\text{IBA}}$ .

The learning algorithm  $L_{Acc}^{IBA}$  takes as input a deterministic complete automaton  $\mathcal{M}$  and a sample T. The acceptance condition F' consists of all the states q of  $\mathcal{M}$  such that for no negative example  $(u(v)^{\omega}, 0)$  in T do we have  $q \in inf_{\mathcal{M}}(u(v)^{\omega})$ . Once F' has been computed, the algorithm checks whether the DBA  $(\mathcal{M}, F')$  is consistent with the sample T. If so, it returns  $(\mathcal{M}, F')$ ; if not, it returns the default acceptor of type DBA for T.

THEOREM 6.3. Algorithm  $\mathbf{L}_{Acc}^{IBA}$  runs in time polynomial in the sizes of the inputs  $\mathcal{M}$  and T. Let  $\mathcal{A}$  be an IBA. If the input automaton  $\mathcal{M}$  is isomorphic to the automaton of  $\mathcal{A}$ , and the sample T is consistent with  $\mathcal{A}$  and subsumes  $T_{Acc}^{IBA}$ , then algorithm  $\mathbf{L}_{Acc}^{IBA}$  returns an IBA  $(\mathcal{M}, F')$  equivalent to  $\mathcal{A}$ .

PROOF. The construction of F' can be done in time polynomial in the sizes of  $\mathcal{M}$  and T by Prop. 4.3. The returned acceptor is consistent with T by construction.

Assume the input  $\mathcal{M}$  is isomorphic to the automaton of  $\mathcal{A}$ , and that T is consistent with  $\mathcal{A}$  and subsumes  $T_{Acc}^{IBA}$ . For ease of notation, assume the isomorphism is the identity. We show that the DBA  $(\mathcal{M}, F')$  is equivalent to  $\mathcal{A}$ .

If  $\mathcal A$  rejects the word  $u(v)^\omega$  then let  $C=\inf_{\mathcal M}(u(v)^\omega)$ . Because T subsumes  $T_{Acc}^{\mathrm{IBA}}$ , for each  $q\in C$ , there is a negative example  $(u'(v')^\omega,0)$  in T such that  $q\in\inf_{\mathcal M}(u'(v')^\omega)$ . Thus no  $q\in C$  is in F' and  $(\mathcal M,F')$  also rejects  $u(v)^\omega$ .

Conversely, if  $\mathcal{A}$  accepts the word  $u(v)^{\omega}$ , then there is at least one state  $q \in F$  such that  $q \in \inf_{\mathcal{M}}(u(v)^{\omega})$ . Because T is consistent with  $\mathcal{A}$ , there is no negative example  $(u(v)^{\omega}, 0)$  in T such that  $q \in \inf_{\mathcal{M}}(u(v)^{\omega})$ , so  $q \in F'$  and  $(\mathcal{M}, F')$  also accepts  $u(v)^{\omega}$ . Thus  $(\mathcal{M}, F')$  is equivalent to  $\mathcal{A}$ . Because T is consistent with  $\mathcal{A}$ , the IBA  $(\mathcal{M}, F')$  is returned by  $L_{Acc}^{IBA}$ .

THEOREM 6.4. The classes  $\mathbb{IB}$  and  $\mathbb{IC}$  are identifiable in the limit using polynomial time and data.

PROOF. The result for  $\mathbb{IC}$  follows from that for  $\mathbb{IB}$  by Prop. 4.1. Let  $\mathcal{A}$  be an IBA accepting language L. The characteristic sample  $T_L = T_{Aut} \cup T_{Acc}^{\text{IBA}}$  is of size polynomial in the size of  $\mathcal{A}$ .

The combined learning algorithm  $\mathbf{L}^{\mathrm{IBA}}$  takes a sample T as input and runs  $\mathbf{L}_{Aut}$  to get a deterministic complete automaton  $\mathcal{M}$ . It then runs  $\mathbf{L}_{Acc}^{\mathrm{IBA}}$  on inputs  $\mathcal{M}$  and T, and returns the resulting acceptor.  $\mathbf{L}^{\mathrm{IBA}}$  runs in polynomial time in the length of T and returns a DBA consistent with T.

If the sample T is consistent with  $\mathcal{A}$  and subsumes  $T_L$  then  $\mathbf{L}_{Aut}$  returns an automaton  $\mathcal{M}$  isomorphic to the automaton of  $\mathcal{A}$  by Theorem 5.2. Then the acceptor returned by  $\mathbf{L}_{Acc}^{\mathrm{IBA}}$  on inputs  $\mathcal{M}$  and T is an IBA equivalent to  $\mathcal{A}$  by Theorem 6.3.

# 7 THE SAMPLE $T_{Acc}$ AND LEARNING FOR IPA

The construction of  $T_{Acc}^{\text{IPA}}$  for an IPA  $\mathcal{P}$  builds on the construction of the canonical forest of SCCs for  $\mathcal{P}$ , whose construction and properties are described next.

# 7.1 Constructing the Canonical Forest and Coloring of a DPA

Let  $\mathcal{P} = \langle \Sigma, Q, q_t, \delta, \kappa \rangle$  be a complete DPA. We extend the coloring function  $\kappa$  to nonempty sets of states by  $\kappa(S) = \min\{k(q) \mid q \in S\}$ , the minimum color of any state in S. We define the  $\kappa$ -parity of S to be 1 if  $\kappa(S)$  is odd, and 0 if  $\kappa(S)$  is even. A word  $w \in \Sigma^{\omega}$  is accepted by  $\mathcal{P}$  iff the  $\kappa$ -parity of  $\inf(w)$  is 1. Note that the union of two sets of  $\kappa$ -parity b is also of  $\kappa$ -parity b. For any nonempty  $S \subseteq Q$ , we define  $\min States(S) = \{q \in S \mid \kappa(q) = \kappa(S)\}$ , the states of S that are assigned the minimum color among all states of S.

7.1.1 The minStates-Forest. We describe an algorithm to construct the minStates-forest of  $\mathcal{P}$ . The roots of the minStates-forest are the elements of maxSCCs(Q), each marked as unprocessed. If C is unprocessed, then  $\mathcal{D} = maxSCCs(C \setminus minStates(C))$  is computed. If  $\mathcal{D}$  is empty, C becomes a leaf in the forest, and is marked as processed. Otherwise, C is marked as processed and the elements of  $\mathcal{D}$  are made the children of C and are marked as unprocessed.

PROPOSITION 7.1. Let  $\mathcal{P} = \langle \Sigma, Q, q_t, \delta, \kappa \rangle$  be a complete DPA with automaton  $\mathcal{M}$ . Let  $\mathcal{F}$  be the minStates-forest of  $\mathcal{P}$ . Then  $\mathcal{F}$  is a decreasing forest of SCCs of  $\mathcal{M}$  rooted in Q, and can be computed in polynomial time. For any SCC D,  $\kappa(D) = \kappa(\mathbf{Node}(D, \mathcal{F}))$ .

PROOF. Referring to the construction of the *minStates*-forest  $\mathcal{F}$ , its roots are the elements of maxSCCs(Q). When a node C is processed, the nonempty set minStates(C) is removed and the maximum SCCs (if any) of the result become the children of C, so the union of the children of C is a proper subset of C, and the children are the maximum SCCs of  $C \setminus \Delta(C)$ . Let D be an SCC. Then  $D \subseteq Q$  and for the node  $C = \mathbf{Node}(D, \mathcal{F})$ , we have that  $D \subseteq C$  and D is not a subset of any child of C. Thus  $D \cap minStates(C) \neq \emptyset$ , because otherwise D would be a subset of some child of C. This implies Manuscript submitted to ACM

that  $\kappa(D) = \kappa(C)$ . The *minStates*-forest of  $\mathcal{P}$  can be computed in polynomial time because it has at most |Q| nodes, and each set maxSCCs(S) can be computed in polynomial time by Prop. 4.7.

7.1.2 The Canonical Forest and Coloring. The canonical forest of  $\mathcal{P}$  is constructed as follows, starting with the minStates-forest of  $\mathcal{P}$ . While there exist in the forest a node D and its parent C of the same  $\kappa$ -parity, one such pair D and C is selected, and the child node D is merged into the parent node C. When no such pair remains, the result is the canonical forest of  $\mathcal{P}$ , denoted  $\mathcal{F}^*(\mathcal{P})$ . The canonical forest  $\mathcal{F}^*(\mathcal{P})$  can be computed from  $\mathcal{P}$  in polynomial time.

From the canonical forest  $\mathcal{F}^*(\mathcal{P})$ , we define the *canonical coloring*  $\kappa^*$ . The states in  $(Q \setminus \bigcup maxSCCs(Q))$  are not contained in any SCC of  $\mathcal{P}$  and do not affect the acceptance or rejection of any  $\omega$ -word. For definiteness, we assign them  $\kappa^*(q) = 0$ . For a root node C of  $\kappa$ -parity b, we define  $\kappa^*(q) = b$  for all  $q \in \Delta(C)$ . Let C be an arbitrary node of  $\mathcal{F}^*(\mathcal{P})$ . If the states of  $\Delta(C)$  have been assigned color k by  $\kappa^*$  and D is a child of C, then the states of  $\Delta(D)$  are assigned color k + 1 by  $\kappa^*$ . Clearly  $\kappa^*$  can be computed from  $\mathcal{P}$  in polynomial time.

An Example. Figure 3(a) shows the graph  $G(\mathcal{P})$  of a DPA  $\mathcal{P}$  with states a through m, labeled by the colors assigned by  $\kappa$ . Figure 3(b) shows the minStates-forest of  $\mathcal{P}$ , with the nodes labeled by their  $\kappa$ -parities. Figure 3(c) shows the canonical forest  $\mathcal{F}^*(\mathcal{P})$  of  $\mathcal{P}$ , with the nodes labeled by their  $\kappa$ -parities. Figure 3(d) shows the graph  $G(\mathcal{P})$  re-colored using the canonical coloring  $\kappa^*$ .

THEOREM 7.2. Let  $\mathcal{P} = \langle \Sigma, Q, q_i, \delta, \kappa \rangle$  be a complete DPA with automaton  $\mathcal{M}$ . The canonical forest  $\mathcal{F}^*(\mathcal{P})$  is a decreasing forest of SCCs of  $\mathcal{M}$  rooted in Q and has the following properties.

- (1) For any SCC D, D and Node(D,  $\mathcal{F}^*(\mathcal{P})$ ) have the same  $\kappa$ -parity.
- (2) For every node C of  $\mathcal{F}^*(\mathcal{P})$ , the  $\kappa$ -parity of C is the same as the  $\kappa^*$ -parity of C.
- (3) The children in  $\mathcal{F}^*(\mathcal{P})$  of a node C of  $\kappa$ -parity b are the maximal SCCs  $D \subseteq C$  of  $\kappa$ -parity 1-b.

PROOF. Because  $\mathcal{F}^*(\mathcal{P})$  is obtained from the *minStates*-forest of  $\mathcal{P}$  by a sequence of merges,  $\mathcal{F}^*(\mathcal{P})$  is a decreasing forest of SCCs of  $\mathcal{M}$  rooted in  $\mathcal{Q}$  by Prop. 4.9. Let  $\mathcal{F}_0$  denote the *minStates*-forest of  $\mathcal{P}$ , and let  $\mathcal{F}_i$  denote the forest after i merges have been performed in the computation to produce the canonical acceptor  $\mathcal{F}^*(\mathcal{P})$ .

By Prop. 7.1, for any SCC D,  $\kappa(D) = \kappa(\operatorname{Node}(D, \mathcal{F}_0))$ , so property (1) holds for  $\mathcal{F}_0$ . We show by induction that it holds for each  $\mathcal{F}_i$  and therefore for  $\mathcal{F}^*(\mathcal{P})$ . Assume that property (1) holds of  $\mathcal{F}_i$ , and  $\mathcal{F}_{i+1}$  is obtained from  $\mathcal{F}_i$  by merging child node D into parent node C. Let D' be any SCC. If  $\operatorname{Node}(D', \mathcal{F}_i) = \operatorname{Node}(D', \mathcal{F}_{i+1})$ , then because property (1) holds in  $\mathcal{F}_i$ , we have that the  $\kappa$ -parity of D' is the same as the  $\kappa$ -parity of  $\operatorname{Node}(D', \mathcal{F}_{i+1})$ . Otherwise, it must be that  $D = \operatorname{Node}(D', \mathcal{F}_i)$  and  $C = \operatorname{Node}(D', \mathcal{F}_{i+1})$ . Because D is only merged to C if they are of the same  $\kappa$ -parity, this implies that the  $\kappa$ -parity of D' is the same as the  $\kappa$ -parity of C. Thus, property (1) holds also in  $\mathcal{F}_{i+1}$ .

For property (2), we note that the  $\kappa$ -parity and the  $\kappa^*$ -parity of each root node of  $\mathcal{F}^*(\mathcal{P})$  is the same. Suppose C is a node of  $\mathcal{F}^*(\mathcal{P})$  whose  $\kappa$ -parity and  $\kappa^*$ -parity are equal to b, and D is a child of C. Then by the construction of  $\mathcal{F}^*(\mathcal{P})$ , the  $\kappa$ -parity of D is 1-b. And by the definition of  $\kappa^*$ , the  $\kappa^*$  parity of the elements of  $\Delta(D)$  is the opposite of the  $\kappa^*$ -parity of the elements of  $\Delta(C)$ . Because the  $\kappa^*$ -parity of C is b, the  $\kappa^*$ -parity of D is also 1-b.

For property (3), let D be any maximal SCC of  $\mathcal{P}$  that is contained in C and has  $\kappa$ -parity 1-b. Then  $\mathbf{Node}(D, \mathcal{F}_0)$  is contained in C and has the same  $\kappa$ -parity as D. Because D is maximal, we must have  $D = \mathbf{Node}(D, \mathcal{F}_0)$ , and any nodes on the path between C and D must have  $\kappa$ -parity b and must be merged into C to form  $\mathcal{F}^*(\mathcal{P})$ . Thus, D is a child of C in  $\mathcal{F}^*(\mathcal{P})$ .

Conversely, if D is a child of C in  $\mathcal{F}^*(\mathcal{P})$  then  $D \subseteq C$  and the  $\kappa$ -parity of D is 1-b. Assume D' is an SCC such that  $D' \subseteq C$ , the  $\kappa$ -parity of D' is 1-b and  $D \subseteq D'$ . Then  $D'' = \mathbf{Node}(D', \mathcal{F}_0)$  is a descendant of C in  $\mathcal{F}_0$  that has  $\kappa$ -parity Manuscript submitted to ACM

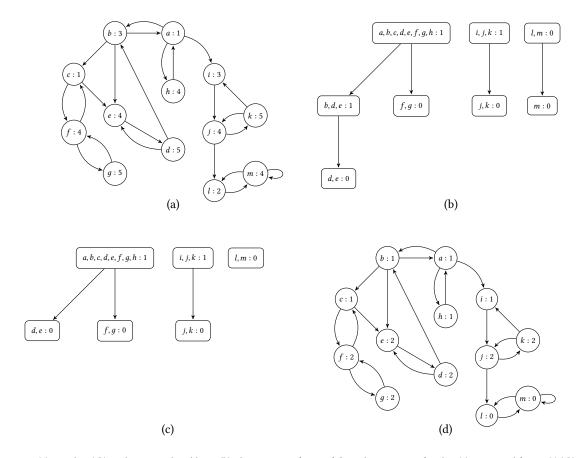


Fig. 3. (a) Graph  $G(\mathcal{P})$  with states colored by  $\kappa$ . (b) The *minStates*-forest of  $\mathcal{P}$ , with  $\kappa$ -parities of nodes. (c) Canonical forest  $\mathcal{F}^*(\mathcal{P})$ , with  $\kappa$ -parities of nodes. (d) Graph  $G(\mathcal{P})$  with the canonical coloring  $\kappa^*$ .

1-b, and  $D \subsetneq D''$ , so because there is another node of parity 1-b on the path between D and C in  $\mathcal{F}_0$ , D cannot be a child of C in  $\mathcal{F}^*(\mathcal{P})$ , a contradiction.

Replacing the coloring function of  $\mathcal{P}$  by the canonical coloring does not change the  $\omega$ -language accepted.

THEOREM 7.3. Let  $\mathcal{P} = \langle \Sigma, Q, q_i, \delta, \kappa \rangle$  be a complete DPA, and  $\mathcal{P}^*$  be  $\mathcal{P}$  with the canonical coloring  $\kappa^*$  for  $\mathcal{P}$  in place of  $\kappa$ . Then  $\mathcal{P}$  and  $\mathcal{P}^*$  recognize the same  $\omega$ -language.

PROOF. Let w be an  $\omega$ -word and let  $D = \inf(w)$ . This is an SCC of the (common) automaton of  $\mathcal{P}$  and  $\mathcal{P}^*$ . Let  $C = \mathbf{Node}(D, \mathcal{F}^*(\mathcal{P}))$ . Then  $D \cap \Delta(C) \neq \emptyset$ , and  $\kappa^*(D) = \kappa^*(C)$ , by the definition of  $\kappa^*$ , The  $\kappa^*$  parity of C is the same as the  $\kappa$ -parity of C, by property (2) of Theorem 7.2. The  $\kappa$ -parity of C is the same as the  $\kappa$ -parity of C, by property (1) of Theorem 7.2. Thus, the  $\kappa^*$ -parity of D is the same as the  $\kappa$ -parity of D, and  $C = \mathbb{P}$  iff  $C = \mathbb{P}$  if  $C = \mathbb{P}$  if C =

# 7.2 Constructing $T_{Acc}^{IPA}$

We now describe the construction of  $T_{Acc}^{\text{IPA}}$ , the second part of the characteristic sample for an IPA  $\mathcal P$  with the automaton  $\mathcal M$  of n states. The sample  $T_{Acc}^{\text{IPA}}$  consists of one example  $u(v)^\omega = \text{Witness}(C,\mathcal M)$  of length  $O(n^2)$  for each reachable Manuscript submitted to ACM

SCC C in the canonical forest  $\mathcal{F}^*(\mathcal{P})$ . The example  $u(v)^{\omega}$  is labeled 1 if it is accepted by  $\mathcal{P}$  and otherwise is labeled 0. Thus  $T_{Acc}^{IPA}$  contains at most n labeled examples, each of length  $O(n^2)$ .

# 7.3 The learning algorithm $L_{Acc}^{IPA}$

Given a complete deterministic automaton  $\mathcal{M} = \langle \Sigma, Q, q_i, \delta \rangle$  and a sample T as input, the learning algorithm  $\mathbf{L}_{Acc}^{IPA}$  attempts to construct a coloring of the states of  $\mathcal{M}$  consistent with T.

The algorithm first constructs the set Z of all  $C \subseteq Q$  such that for some labeled example  $(u(v)^{\omega}, l)$  in T we have  $C = \inf_{\mathcal{M}} (u(v)^{\omega})$ . If two examples with different labels are found to yield the same set C, this is evidence that the automaton  $\mathcal{M}$  is not correct, and the learning algorithm returns the default acceptor of type DPA for T.

Otherwise, each set C in Z is associated with the label of the one or more examples that yield C. The set Z is partially ordered by the subset relation. The learning algorithm then attempts to construct a rooted forest  $\mathcal{F}'$  with nodes that are elements of Z, corresponding to the canonical forest of the target acceptor. Initially,  $\mathcal{F}'$  contains as roots all the maximal elements of Z. If these are not pairwise disjoint, it returns the default acceptor of type DPA for T. Otherwise, the root nodes are all marked as unprocessed.

For each unprocessed node C in  $\mathcal{F}'$ , it computes the set of all  $D \in Z$  such that  $D \subseteq C$ , D has the opposite label to C, and D is maximal with these properties, and makes D a child of C and marks D as unprocessed. When all the children of a node C have been determined, the algorithm checks two conditions: (1) that the children of C are pairwise disjoint, and (2) there is at least one  $Q \in C$  that is not in any child of C. If either of these conditions fail, then it returns the default acceptor of type DPA for C. If both conditions are satisfied, then the node C is marked as processed. When there are no more unprocessed nodes, the construction of C is complete. Note that C has at most C nodes.

When the construction of  $\mathcal{F}'$  is complete, for each node C in  $\mathcal{F}'$  let  $\Delta(C)$  denote the elements of C that do not appear in any of its children. Then the learning algorithm assigns colors to the elements of Q starting from the roots of  $\mathcal{F}'$ , as follows. If C is a root with label l, then  $\kappa'(q) = l$  for all  $q \in \Delta(C)$ . If the elements of  $\Delta(C)$  have been assigned color k and D is a child of C, then  $\kappa'(q) = k + 1$  for all  $s \in \Delta(D)$ . When this process is complete, any uncolored states q are assigned  $\kappa'(q) = 0$ .

If the resulting DPA  $(\mathcal{M}, \kappa')$  is consistent with the sample T, the algorithm  $\mathbf{L}_{Acc}^{\text{IPA}}$  returns  $(\mathcal{M}, \kappa')$ . If not, it returns the default acceptor of type DPA for T.

Theorem 7.4. Algorithm  $\mathbf{L}_{Acc}^{IPA}$  runs in time polynomial in the sizes of the inputs  $\mathcal{M}$  and T. Let  $\mathcal{P}$  be an IPA. If the input automaton  $\mathcal{M}$  is isomorphic to the automaton of  $\mathcal{P}$ , and the sample T is consistent with  $\mathcal{P}$  and subsumes  $T_{Acc}^{IPA}$ , then algorithm  $\mathbf{L}_{Acc}^{IPA}$  returns an IPA  $(\mathcal{M}, \kappa')$  equivalent to  $\mathcal{P}$ .

PROOF. The construction of  $\kappa'$  can be done in time polynomial in the sizes of  $\mathcal{M}$  and T. The returned acceptor is consistent with T by construction.

Assume the input  $\mathcal{M} = \langle \Sigma, Q, q_l, \delta \rangle$  is isomorphic to the automaton of  $\mathcal{P}$ , and that T is consistent with  $\mathcal{P}$  and subsumes  $T_{Acc}^{IPA}$ . For ease of notation, assume the isomorphism is the identity. We show that the forest  $\mathcal{F}'$  constructed by the learning algorithm is equal to the canonical forest of  $\mathcal{F}^*(\mathcal{P})$ , the coloring  $\kappa'$  is equal to the canonical coloring  $\kappa^*$ , and therefore the acceptor  $(\mathcal{M}, \kappa')$  is equivalent to  $\mathcal{P}$ .

The roots of  $\mathcal{F}^*(\mathcal{P})$  are the maximal SCCs contained in Q, and for each such root C,  $T_{Acc}^{IPA}$  contains an example  $(u(v)^{\omega}, l)$  such that  $C = \inf(u(v)^{\omega})$ . Thus, the set of maximal elements of Z is equal to the set of roots of  $\mathcal{F}^*(\mathcal{P})$ .

Let C be any node of  $\mathcal{F}^*(\mathcal{P})$ , and let D be a child of C in  $\mathcal{F}^*(\mathcal{P})$ . Then D is an SCC,  $D \subseteq C$ , the parity of D is opposite to the parity of C, and D is maximal in the subset ordering with these properties, by property (3) of Theorem 7.2. In

the sample  $T_{Acc}^{IPA}$  there is an example  $(u(v)^{\omega}, l)$  with  $D = \inf(u(v)^{\omega})$ , so D is an element of Z, and will be made a child of C in  $\mathcal{F}'$  because  $D \subseteq C$ , the label l is the opposite of the label of C, and D is maximal in Z with these properties. Conversely, if D is made a child of C in  $\mathcal{F}'$ , then  $D \subseteq C$ , the label of D is opposite to the label of C (that is, they are of opposite  $\kappa$ -parity), and D is maximal in D with these properties. This implies D is a child of D in D is property (3) of Theorem 7.2.

By induction,  $\mathcal{F}'$  is equal to  $\mathcal{F}^*(\mathcal{P})$ , and therefore  $\kappa'$  is equal to the canonical coloring  $\kappa^*$ . Then the IPA  $(\mathcal{M}, \kappa')$  is equivalent to  $\mathcal{P}$ , by Theorem 7.3. Because T is consistent with  $\mathcal{P}$ , the IPA  $(\mathcal{M}, \kappa')$  is returned by  $L_{Acc}^{IBA}$ .

THEOREM 7.5. The class  $\mathbb{P}$  is identifiable in the limit using polynomial time and data.

PROOF. Let  $\mathcal{P}$  be an IPA accepting the language L. The characteristic sample  $T_L = T_{Aut} \cup T_{Acc}$  for  $\mathcal{P}$  is of size polynomial in the size of  $\mathcal{P}$ .

The combined learning algorithm  $\mathbf{L}^{\text{IPA}}$  with a sample T as input first runs  $\mathbf{L}_{Aut}$  on T to get a complete deterministic automaton  $\mathcal{M}$  and then runs  $\mathbf{L}_{Acc}^{\text{IPA}}$  on inputs  $\mathcal{M}$  and T and returns the resulting acceptor. The running time of  $\mathbf{L}^{\text{IPA}}$  is polynomial in the length of T and the returned acceptor is consistent with T.

Assume that the sample T is consistent with  $\mathcal{P}$  and subsumes  $T_L$ . Then by Theorem 5.2, the automaton  $\mathcal{M}$  is isomorphic to the automaton of  $\mathcal{P}$ . By Theorem 7.4, the acceptor  $(\mathcal{M}, \kappa')$  is equivalent to  $\mathcal{P}$ . Because it is consistent with T, the IPA  $(\mathcal{M}, \kappa')$  is the acceptor returned by  $\mathbf{L}^{\text{IPA}}$ .

A corollary of Theorem 7.5 is that the class of languages recognized by deterministic weak Parity acceptors ( $\mathbb{DWPA}$ ), which was shown to be polynomially learnable using membership and equivalence queries by Maler and Pnueli [28], is identifiable in the limit using polynomial time and data. This class (which is equivalent to the intersection of classes  $\mathbb{DBA} \cap \mathbb{DCA}$ ) was shown to be a subset of  $\mathbb{IM}$  in [37], and to be a subset of  $\mathbb{IM}$  in [9].

COROLLARY 7.6. The class DWPA is identifiable in the limit using polynomial time and data.

# 8 THE SAMPLE $T_{Acc}$ AND LEARNING FOR IRA

In this section we introduce some terminology, establish a normal form for Rabin acceptors, define an ordering on sets of states of an automaton, and then describe the learning algorithm  $\mathbf{L}_{Acc}^{\mathrm{IRA}}$  and sample  $T_{Acc}^{\mathrm{IRA}}$  for IRAs. We then prove that the classes  $\mathbb{IR}$  and  $\mathbb{IS}$  are identifiable in the limit using polynomial time and data.

Let  $\mathcal{R} = \langle \Sigma, Q, q_i, \delta, \alpha \rangle$  be a Rabin acceptor, where the acceptance condition  $\alpha = \{(G_1, B_1), \dots, (G_k, B_k)\}$  is a set of ordered pairs of states. We say that an  $\omega$ -word w satisfies a pair of state sets (G, B) iff  $\inf(w) \cap G \neq \emptyset$  and  $\inf(w) \cap B = \emptyset$ . Also, w satisfies the acceptance condition  $\alpha$  iff there exists  $i \in [1..k]$  such that w satisfies  $(G_i, B_i)$ . Then an  $\omega$ -word w is accepted by  $\mathcal{R}$  iff w satisfies the acceptance condition  $\alpha$  of  $\mathcal{R}$ .

# 8.1 Singleton normal form for a Rabin acceptor

We say that a Rabin acceptor  $\mathcal{R}$  is in *singleton normal form* iff for every pair  $(G_i, B_i)$  in its acceptance condition we have  $|G_i| = 1$ , that is, every  $G_i$  is a singleton set. To avoid extra braces, we abbreviate the pair  $(\{q\}, B)$  by (q, B). Every Rabin acceptor may be put into singleton normal form by a polynomial time algorithm.

PROPOSITION 8.1. Let  $\mathcal{R} = \langle \Sigma, Q, q_i, \delta, \alpha \rangle$  be a Rabin acceptor with acceptance condition  $\alpha = \{(G_1, B_1), \ldots, (G_k, B_k)\}$ . Define the acceptance condition  $\alpha'$  to contain  $(q, B_i)$  for every  $i \in [1..k]$  and  $q \in G_i$ , and let  $\mathcal{R}'$  be  $\mathcal{R}$  with  $\alpha$  replaced by  $\alpha'$ . Then  $\mathcal{R}'$  is in singleton normal form and accepts the same language as  $\mathcal{R}$ . Also,  $\mathcal{R}'$  is of size at most |Q| times the size of  $\mathcal{R}$ .

PROOF. If an  $\omega$ -word w satisfies a pair (q, B) of  $\alpha'$ , then there exists a pair  $(G_i, B_i)$  of  $\alpha$  with  $q \in G_i$  and  $B_i = B$ , so w also satisfies the pair  $(G_i, B_i)$  in  $\alpha$ . Conversely, if w satisfies a pair  $(G_i, B_i)$  in  $\alpha$ , then there exists  $q \in G_i$  such that  $q \in inf(w)$ , so w also satisfies  $(q, B_i)$  in  $\alpha'$ . Each pair  $(G_i, B_i)$  in  $\alpha$  is replaced by at most |Q| pairs in  $\alpha'$ .

### 8.2 An ordering on sets of states

Given an automaton, we define an ordering  $\leq$  on sets of its states that is used to coordinate between the characteristic sample and the learning algorithm for an IRA.

Let  $\mathcal{M} = \langle \Sigma, Q, q_i, \delta \rangle$  be a deterministic complete automaton in which every state is reachable from the start state. Recall from Section 5.2 that access(q) is the shortlex least string  $s \in \Sigma^*$  such that  $\delta(q_i, s) = q$ .

For a set of states S, we define access(S) to be the sequence of values access(q) for  $q \in S$ , sorted into increasing shortlex order. We define the total ordering  $\leq$  on sets of states of  $\mathcal{M}$  as follows. If  $S_1, S_2 \subseteq Q$ , then  $S_1 \leq S_2$  iff either  $|S_1| < |S_2|$  or  $|S_1| = |S_2|$  and the sequence  $access(S_1)$  is less than or equal to the sequence  $access(S_2)$  in the lexicographic ordering, using the shortlex ordering on  $\Sigma^*$  to compare the component entries. For example, if  $access(S_1) = \langle \epsilon, a, baa \rangle$ ,  $access(S_2) = \langle a, ba \rangle$ , and  $access(S_3) = \langle \epsilon, ab, ba \rangle$  then  $S_2 \leq S_1 \leq S_3$ .

# 8.3 The learning algorithm $L_{Acc}^{IRA}$

We begin with the description of the learning algorithm  $\mathbf{L}_{Acc}^{\text{IRA}}$ , which is used in the definition of the sample  $T_{Acc}^{\text{IRA}}$  in the next section. The inputs to  $\mathbf{L}_{Acc}^{\text{IRA}}$  are a deterministic complete automaton  $\mathcal{M}$  and a sample T. The algorithm attempts to construct a singleton normal form Rabin acceptance condition  $\beta$  to produce an acceptor  $(\mathcal{M}, \beta)$  consistent with T.

The processing of an example w from T depends only on  $\inf_{\mathcal{M}}(w)$ . The algorithm first computes the set  $\{\inf_{\mathcal{M}}(w) \mid (w,1) \in T\}$ , sorts its elements into decreasing order  $C_1, C_2, \ldots, C_\ell$  using  $\leq$ , and for each  $i \in [1..\ell]$  chooses a positive example  $(z_i, 1) \in T$  with  $C_i = \inf_{\mathcal{M}}(z_i)$ .

At each stage k of the learning algorithm,  $\beta_k$  is a singleton normal form acceptance condition. Initially,  $\beta_0 = \emptyset$  (which is satisfied by no words) and k = 0.

The main loop processes the positive examples  $z_i$  for  $i = 1, 2, ..., \ell$ . If  $z_i$  is accepted by  $(\mathcal{M}, \beta_k)$ , then the algorithm goes on to the next positive example. Otherwise, we say that the example  $z_i$  causes the update of  $\beta_k$ . Let  $G = \inf_{\mathcal{M}}(z_i)$  and  $B = Q \setminus G$  and let  $S_k$  be the set of pairs (q, B) such that  $q \in G$  and there is no negative example (w, 0) in T such that w satisfies (q, B). Then  $\beta_{k+1}$  is set to  $\beta_k \cup S_k$  and k is set to k+1.

When the positive examples  $z_1, z_2, ..., z_\ell$  have been processed, let  $\beta = \beta_k$  for the final value of k. If the Rabin acceptor  $(\mathcal{M}, \beta)$  is consistent with T, then it is returned. If not, the learning algorithm returns the default acceptor of type DRA for T.

PROPOSITION 8.2. Let  $\mathcal{R}$  be an IRA in singleton normal form with acceptance condition  $\alpha$  and assume that the input  $\mathcal{M}$  is an automaton isomorphic to the automaton of  $\mathcal{R}$ . Assume the sample T is consistent with  $\mathcal{R}$ . For each k, if  $z_i$  is the example that causes the update of  $\beta_k$ , then  $(\mathcal{M}, \beta_k)$  accepts  $z_j$  for all j < i. Moreover,  $(\mathcal{M}, \beta)$  is consistent with T.

PROOF. No pair (q, B) is added to the acceptance condition if there is a negative example in T that satisfies it, so  $(\mathcal{M}, \beta)$  is consistent with all the negative examples in T.

Consider any positive example (w, 1) from T. There exists i with  $\inf_{\mathcal{M}}(w) = \inf_{\mathcal{M}}(z_i)$ . When  $z_i$  is processed by the algorithm, if it is already accepted by the current  $(\mathcal{M}, \beta_k)$ , then it (and the word w) is also accepted by every subsequent hypothesis, including  $(\mathcal{M}, \beta)$ , because pairs are not removed from  $\beta_k$ .

Otherwise,  $z_i$  causes the update to  $\beta_k$ , and among the pairs in  $S_k$  that are added to  $\beta_k$  is at least one that  $z_i$  satisfies. To see this, note that  $z_i$  must satisfy some pair in  $\alpha$ , say  $(q, B_j)$ . Thus,  $q \in inf(z_i)$  and  $B_j \cap inf(z_i) = \emptyset$ . The pair (q, B) where  $B = Q \setminus inf(z_i)$  has  $B_j \subseteq B$ . Thus any  $\omega$ -word that satisfies (q, B) will also satisfy  $(q, B_j)$ . Because T is consistent with R, there can be no negative example in T satisfying (q, B), so pair (q, B) is part of  $S_k$  and is added to  $\beta_k$ . The word  $z_i$  (and the word w) satisfies (q, B) and is therefore accepted by  $(M, \beta_{k+1})$  and every subsequent hypothesis, including  $(M, \beta)$ .

# 8.4 Constructing $T_{Acc}^{IRA}$

In this section we describe the construction of the sample  $T_{Acc}^{IRA}$ , which conveys the acceptance condition of an IRA. Let  $\mathcal{R} = \langle \Sigma, Q, q_t, \delta, \alpha \rangle$  be a deterministic complete IRA of n states in singleton normal form, and let  $\mathcal{M}$  be its automaton. The construction of the sample  $T_{Acc}^{IRA}$  proceeds in stages, simulating the learning algorithm  $\mathbf{L}_{Acc}^{IRA}$  on the portion of the sample constructed so far to determine what examples still need to be added.

Initially,  $\gamma_0 = \emptyset$  and k = 0. The acceptance condition  $\gamma_k$  tracks the learning algorithm's  $\beta_k$ . The set of words accepted by  $(\mathcal{M}, \gamma_k)$  is always a subset of the set of words accepted by  $(\mathcal{M}, \alpha)$ .

The main loop is as follows. If  $(\mathcal{M}, \gamma_k)$  is equivalent to  $(\mathcal{M}, \alpha)$  then the construction of  $T_{Acc}^{IRA}$  is complete.

Otherwise, let  $D_k$  be the set of  $\omega$ -words that satisfy  $\alpha$  but not  $\gamma_k$ . Let C be the  $\leq$ -largest set in  $\{inf(w) \mid w \in D_k\}$ , and let  $w_{k+1} = \mathbf{Witness}(C, \mathcal{M})$ , an ultimately periodic word of length  $O(n^2)$ . Then  $w_{k+1}$  is added as a positive example to  $T_{Acc}^{IRA}$ .

Let  $B = Q \setminus inf(w_{k+1})$ . Define  $P_k$  to be the set of all (q, B) such that  $q \in inf(w_{k+1})$  and there is no  $\omega$ -word w' that satisfies (q, B) but not  $\alpha$ . Set  $\gamma_{k+1} = \gamma_k \cup P_k$ .

For each  $q \in inf(w_{k+1})$  such that there is some  $\omega$ -word w' that satisfies (q, B) but not  $\alpha$ , let  $u(v)^{\omega} = \mathbf{Witness}(inf(w'), \mathcal{M})$  for some such w' and include  $(u(v)^{\omega}, 0)$  as a negative example in  $T_{Acc}$ . The example  $u(v)^{\omega}$  is of length  $O(n^2)$ . Then set k to k+1 and continue with the main loop.

We prove a polynomial bound on the number of examples added to the sample  $T_{Acc}^{IRA}$ , thus showing that its length is bounded by a polynomial in the size of  $\mathcal{R}$ .

Proposition 8.3. If the acceptance condition  $\alpha$  is in singleton normal form and has m pairs, then at most m positive examples and at most m|Q| negative examples are added to  $T_{Acc}$ .

PROOF. We say an acceptance condition  $\gamma$  covers a pair (q,B) iff every  $\omega$ -word w that satisfies (q,B) also satisfies  $\gamma$ . We will show that after each positive example  $w_{k+1}$  is added to  $T_{Acc}$ , the condition  $\gamma_{k+1}$  covers at least one pair in  $\alpha$  that was not covered by  $\gamma_k$ .

Suppose not, and let k+1 be the least index for which  $\gamma_{k+1}$  does not cover a pair of  $\alpha$  that was not covered by  $\gamma_k$ . Because  $w_{k+1}$  is an example that satisfies  $\alpha$  but not  $\gamma_k$ , there must be a pair  $(q, B_j)$  of  $\alpha$  that is satisfied by  $w_{k+1}$ . Note that  $\gamma_k$  does not cover the pair  $(q, B_j)$ .

Then  $q \in inf(w_{k+1})$  and letting  $B = Q \setminus inf(w_{k+1})$ ,  $B_j \subseteq B$ . The pair (q, B) will be added to  $\gamma_k$  in constructing  $\gamma_{k+1}$  because every word that satisfies (q, B) also satisfies  $(q, B_j)$ .

If  $\gamma_{k+1}$  does not cover  $(q, B_j)$ , there must be a word w' that satisfies  $(q, B_j)$  but not (q, B). So  $q \in inf(w')$  and  $B_j \cap inf(w') = \emptyset$  but  $B \cap inf(w') \neq \emptyset$ . Let  $B' = Q \setminus inf(w')$ , so  $B_j \subseteq B'$ . We have  $B \cap B' \subseteq B$ .

Because  $inf(w_{k+1})$  and inf(w') are SCCs that overlap in q, their union is an SCC as well. Let w'' be an  $\omega$ -word such that inf(w'') is the union of  $inf(w_{k+1})$  and inf(w'). Then  $Q \setminus inf(w'') = B \cap B'$ . Note that w'' satisfies  $(q, B_j)$  because  $B_j \subseteq B \cap B'$  and thus is a positive example of  $\alpha$ .

Because  $B \cap B'$  is a proper subset of B, inf(w'') is a proper superset of  $inf(w_{k+1})$  and the positive example w'' would have been considered before  $w_{k+1}$  in the construction of  $T_{Acc}$ . (We can imagine all the positive examples of  $\alpha$  being considered in order to find a maximum positive counterexample at each stage.) At that time, it was either passed over because (1) the current  $\gamma_r$  already covered it, or (2) it contributed a new pair to the current  $\gamma_r$  to yield  $\gamma_{r+1}$ .

In case (1), there is some pair (q'', B'') in  $\gamma_r$  that is satisfied by w''. Then  $q'' \in inf(w'')$  and  $B'' \cap inf(w'') = \emptyset$ . Recall inf(w'') is the union of  $inf(w_{k+1})$  and inf(w'). Thus,  $B'' \cap inf(w_{k+1}) = \emptyset$  and  $B'' \cap inf(w') = \emptyset$ . Note that  $q'' \in inf(w_{k+1})$  or  $q'' \in inf(w')$ . If  $q'' \in inf(w_{k+1})$ ,  $w_{k+1}$  satisfies the pair (q'', B'') in  $\gamma_f$ , a contradiction, because  $w_{k+1}$  is not accepted by  $\gamma_k$  and  $r \leq k$ . And if  $q'' \in inf(w')$  then w' satisfies the pair (q'', B'') in  $\gamma_r$ , a contradiction, because w' is not accepted by  $\gamma_{k+1}$  and  $r \leq k$ .

In case (2), the positive example w'' contributes at least one term (q'', B'') to  $\gamma_{r+1}$ . In this case  $B'' = B \cap B'$  and  $q'' \in inf(w'')$ . Thus,  $q'' \in inf(w_{k+1})$  or  $q'' \in inf(w')$ , so  $w_{k+1}$  or w' satisfies the term (q'', B'') of  $\gamma_{r+1}$ , a contradiction because  $r+1 \le k$  and neither  $w_{k+1}$  nor w' is covered by  $\gamma_k$ .

Thus, each positive example added to  $T_{Acc}$  covers a new pair of  $\alpha$ , and at most m positive examples can be added. Each positive example added requires at most |Q| negative examples to avoid adding incorrect pairs, so at most m|Q| negative examples are added.

# 8.5 Correctness of $L_{Acc}^{IRA}$

We prove the correctness of the learning algorithm  $L_{Acc}^{IRA}$  and show that the classes  $\mathbb{R}$  and  $\mathbb{S}$  are identifiable in the limit using polynomial time and data.

THEOREM 8.4. Algorithm  $\mathbf{L}_{Acc}^{IRA}$  runs in time polynomial in the sizes of the inputs  $\mathcal{M}$  and T. Let  $\mathcal{R}$  be an IRA. If the input automaton  $\mathcal{M}$  is isomorphic to the automaton of  $\mathcal{R}$ , and the sample T is consistent with  $\mathcal{R}$  and subsumes  $T_{Acc}^{IRA}$ , then algorithm  $\mathbf{L}_{Acc}^{IRA}$  returns an IRA  $(\mathcal{M}, \beta)$  equivalent to  $\mathcal{R}$ .

PROOF. By Prop. 4.3,  $\mathbf{L}_{Acc}^{IRA}$  can construct the sequence  $z_1, z_2, \dots, z_\ell$  and the successive acceptance conditions  $\beta_k$  in time polynomial in the size of  $\mathcal{M}$  and the length of T.

Assume  $\mathcal{R}$  is an IRA, that  $\mathcal{M}$  is isomorphic to the automaton of  $\mathcal{R}$ , and that the sample T is consistent with  $\mathcal{R}$  and subsumes  $T_{Acc}^{IRA}$ . For ease of notation, we assume that the isomorphism is the identity.

We show by induction that for each k, the acceptance condition  $\beta_k$  in the learning algorithm  $\mathbf{L}_{Acc}^{\mathrm{IRA}}$  is the same as the acceptance condition  $\gamma_k$  in the construction of  $T_{Acc}^{\mathrm{IRA}}$ . This is true for k=0 because  $\beta_0=\gamma_0=\emptyset$ .

Assume that  $\beta_k = \gamma_k$  for some  $k \ge 0$ . If  $(\mathcal{M}, \gamma_k)$  is equivalent to  $\mathcal{R} = (\mathcal{M}, \alpha)$ , then also  $(\mathcal{M}, \beta_k)$  is equivalent to  $\mathcal{R}$ , and none of the remaining positive examples cause any additions to  $\beta_k$ . Thus this is the final value of k, so  $\beta = \beta_k$  and  $(\mathcal{M}, \beta)$  is equivalent to  $\mathcal{R}$ .

If  $(\mathcal{M}, \gamma_k)$  accepts a proper subset of the language accepted by  $(\mathcal{M}, \alpha)$ , then in the construction of sample  $T_{Acc}^{IRA}$ ,  $D_k$  is equal to the  $\omega$ -words accepted by  $(\mathcal{M}, \alpha)$  but not by  $(\mathcal{M}, \gamma_k)$ . This causes the positive example  $(w_{k+1}, 1)$  to be added to  $T_{Acc}^{IRA}$ , where  $\inf_{\mathcal{M}}(w_{k+1})$  is  $\leq$ -largest in the set  $\{\inf_{\mathcal{M}}(w) \mid w \in D_k\}$ .

In the learning algorithm, because  $(\mathcal{M}, \beta_k)$  does not accept  $w_{k+1}$ , Prop. 8.2 implies that there must be an example  $z_i$  that causes the update to  $\beta_k$ , and all of the examples  $z_1, \ldots, z_{i-1}$  are accepted by  $(\mathcal{M}, \beta_k)$ . Because for every positive example (w, 1) in T there exists j such that  $\inf_{\mathcal{M}}(w) = \inf_{\mathcal{M}}(z_j)$ , there must be some r such that  $\inf_{\mathcal{M}}(w_{k+1}) = \inf_{\mathcal{M}}(z_r)$ . Moreover,  $i \leq r$ .

If i < r, then  $\inf_{\mathcal{M}}(w_i)$  is strictly  $\leq$ -larger than  $\inf_{\mathcal{M}}(w_r)$ , which contradicts the choice of  $w_{k+1}$  by the sample construction procedure, because  $z_i$  is accepted by  $(\mathcal{M}, \alpha)$  but not  $(\mathcal{M}, \gamma_k)$ . Thus i = r and the example  $z_i = w_{k+1}$  is the Manuscript submitted to ACM

element that causes the update to  $\beta_k$ . The negative examples included in  $T_{Acc}^{IRA}$  for the positive example  $w_{k+1}$  ensure that the update to  $\beta_k$  is the same as the update to  $\gamma_k$ , and  $\beta_{k+1} = \gamma_{k+1}$ .

Because  $\beta_k$  and  $\gamma_k$  are equal for all k, for the final value of k,  $\beta = \beta_k = \gamma_k$ , and therefore  $(\mathcal{M}, \beta)$  is equivalent to  $\mathcal{R}$ . Because the IRA  $(\mathcal{M}, \beta)$  is consistent with T, it is the acceptor returned by  $\mathbf{L}_{Acc}^{IRA}$ .

Theorem 8.5. The classes  $\mathbb{IR}$  and  $\mathbb{IS}$  are identifiable in the limit using polynomial time and data.

PROOF. By Prop. 4.1 it suffices to prove this for  $\mathbb{IR}$ . Let  $\mathcal{R}$  be an IRA in singleton normal form accepting the language L. The characteristic sample  $T_L = T_{Aut} \cup T_{Acc}^{IRA}$  is of size polynomial in the size of  $\mathcal{R}$ .

The combined learning algorithm  $\mathbf{L}^{\text{IRA}}$  with a sample T as input first runs  $\mathbf{L}_{Aut}$  on T to get a deterministic complete automaton  $\mathcal{M}$  and then runs  $\mathbf{L}_{Acc}^{\text{IRA}}$  on inputs  $\mathcal{M}$  and T and returns the resulting acceptor.  $\mathbf{L}^{\text{IRA}}$  runs in time polynomial in the length of T and returns a DRA consistent with T.

Now assume that the sample T is consistent with  $\mathcal{R}$  and subsumes  $T_L$ . Then by Theorem 5.2, the automaton  $\mathcal{M}$  is isomorphic to the automaton of  $\mathcal{R}$ . By Theorem 8.4, the acceptor returned by  $T^{IRA}$  is an IRA  $(\mathcal{M}, \beta)$  is equivalent to  $\mathcal{R}$ , and this is the acceptor also returned by  $L^{IRA}$ .

#### 9 CHARACTERISTIC SAMPLES IN POLYNOMIAL TIME

The definition of identification in the limit using polynomial time and data requires that a characteristic sample exist and be of polynomial size, but says nothing about the cost of computing it. An additional desirable property is that a characteristic sample be computable in polynomial time given an acceptor  $\mathcal A$  as input. We now address the question of polynomial time algorithms to compute the characteristic samples we have defined.

# 9.1 Computing $T_{Aut}$

For  $T_{Aut}$ , we need to be able to decide for two states  $q_1$  and  $q_2$  of an acceptor  $\mathcal{A}$  whether there exists an  $\omega$ -word that distinguishes them, and if so, to return one such word. We are thus led to consider the problems of inclusion and equivalence.

#### 9.2 The problems of inclusion and equivalence

The *inclusion problem* is the following. Given as input two  $\omega$ -acceptors  $\mathcal{A}_1$  and  $\mathcal{A}_2$  over the same alphabet, determine whether the language accepted by  $\mathcal{A}_1$  is a subset of the language accepted by  $\mathcal{A}_2$ , that is, whether  $[\![\mathcal{A}_1]\!] \subseteq [\![\mathcal{A}_2]\!]$ . If so, the answer should be "yes"; if not, the answer should be "no" and a *witness*, that is, an ultimately periodic  $\omega$ -word  $u(v)^\omega$  accepted by  $\mathcal{A}_1$  but rejected by  $\mathcal{A}_2$ .

The equivalence problem is similar: the input is two  $\omega$ -acceptors  $\mathcal{A}_1$  and  $\mathcal{A}_2$  over the same alphabet, and the problem is to determine whether they are equivalent, that is, whether  $[\![\mathcal{A}_1]\!] = [\![\mathcal{A}_2]\!]$ . If so, the answer should be "yes"; if not, the answer should be "no" and a witness, that is, an ultimately periodic  $\omega$ -word  $u(v)^{\omega}$  that is accepted by exactly one of  $\mathcal{A}_1$  and  $\mathcal{A}_2$ .

If we have a procedure to solve the inclusion problem, at most two calls to it will solve the equivalence problem. We describe polynomial time algorithms to solve the inclusion problem for DBAs, DCAs, and DPAs in Section 10 and 11, for DRAs and DSAs in Section 12, and for DMAs in Section 13. Referring to those sections, we obtain polynomial time algorithms to solve the equivalence problem for DBAs, DCAs, and DPAs from Theorem 10.1 and Theorem 11.2, for DRAs and DSAs from Theorem 12.2, and for DMAs from Theorem 13.6. Thus, we have the following.

THEOREM 9.1. Given an acceptor  $\mathcal{A}$  of type IBA, ICA, IPA, IRA, ISA, or IMA, the sample  $T_{Aut}$  for the automaton portion of  $\mathcal{A}$  may be computed in polynomial time.

PROOF. Given an acceptor  $\mathcal{A}$  and two states  $q_1$  and  $q_2$ , to determine whether there is an  $\omega$ -word that distinguishes them, we call the relevant polynomial time equivalence algorithm on the acceptors  $\mathcal{A}^{q_1}$  and  $\mathcal{A}^{q_2}$ , which returns a distinguishing word  $u(v)^{\omega}$  if they are not equivalent.

# 9.3 Computing T<sub>Acc</sub>

For  $T_{Acc}$ , the requirements depend on the type of acceptor.

Proposition 9.2. Given an IBA  $\mathcal{B}$ , the sample  $T_{Acc}^{IBA}$  can be computed in time polynomial in the size of  $\mathcal{B}$ .

PROOF. Given an IBA  $\mathcal{B} = \langle \Sigma, Q, q_t, \delta, F \rangle$  with automaton  $\mathcal{M}$ , the sample  $T_{Acc}^{\text{IBA}}$  described in Section 6.2 is computed as follows. For each SCC  $C \in maxSCCs(Q \setminus F)$ , let  $u(v)^{\omega} = \mathbf{Witness}(C, \mathcal{M})$  and include the negative example  $(u(v)^{\omega}, 0)$  in  $T_{Acc}^{\text{IBA}}$ .

To see that these examples are sufficient, suppose that  $q \in Q$  and  $w \in \Sigma^{\omega}$  are such that  $\mathcal{B}$  rejects w and  $q \in inf(w)$ . Then D = inf(w) is an SCC of  $\mathcal{B}$  contained in  $Q \setminus F$ , so it is contained in some  $C \in maxSCCs(Q \setminus F)$ , and there is a negative example  $(u(v)^{\omega}, 0)$  in  $T_{Acc}^{IBA}$  such that  $inf(u(v)^{\omega}) = C$ . Because  $D \subseteq C$ , we have  $q \in C$ .

Proposition 9.3. Given an IPA  $\mathcal{P}$ , the sample  $T_{Acc}^{IPA}$  can be computed in time polynomial in the size of  $\mathcal{P}$ .

PROOF. Given an IPA  $\mathcal{P}$  with automaton  $\mathcal{M}$ , the computation of  $T_{Acc}^{\text{IPA}}$  proceeds as described in Section 7.2. That is, the canonical forest  $\mathcal{F}^*(\mathcal{P})$  is computed in polynomial time, and for each node C in the forest,  $u(v)^\omega = \text{Witness}(C, \mathcal{M})$  is computed and  $(u(v)^\omega, l)$  is added to  $T_{Acc}^{\text{IPA}}$ , where l is the label of node C in the canonical forest.

Proposition 9.4. Given an IRA R, the sample  $T_{Acc}^{IRA}$  can be computed in time polynomial in the size of R.

PROOF. Given an IRA  $\mathcal{R} = (\mathcal{M}, \alpha)$ , the computation of  $T_{Acc}$  proceeds as described in Section 8.4. At each stage of the computation, it is necessary to find an  $\omega$ -word  $u(v)^{\omega}$  with the  $\leq$ -largest  $\inf_{\mathcal{M}}(u(v)^{\omega})$  that is accepted by  $(\mathcal{M}, \alpha)$  and rejected by  $(\mathcal{M}, \gamma_k)$ . Theorem 12.2 gives a polynomial time algorithm that not only tests the inclusion of two DRAs, but returns a witness  $u(v)^{\omega}$  with the  $\leq$ -largest  $\inf_{\mathcal{M}}(u(v)^{\omega})$  in the case of non-inclusion, because  $\mathcal{M}$  is isomorphic to  $\mathcal{M} \times \mathcal{M}$ .

Proposition 9.5. Given an IMA  $\mathcal{A}$ , the sample  $T_{Acc}^{IMA}$  can be computed in time polynomial in the size of  $\mathcal{A}$ .

PROOF. Given an IMA  $\mathcal{A} = (\mathcal{M}, \mathcal{F})$ , the sample  $T_{Acc}^{\text{IMA}}$  described in Section 6.1 is computed as follows. For each  $F \in \mathcal{F}$  determine whether F is a reachable SCC of  $\mathcal{M}$ , and if so, compute  $u(v)^{\omega} = \text{Witness}(F, \mathcal{M})$  and add  $(u(v)^{\omega}, 1)$  to the sample  $T_{Acc}^{\text{IMA}}$ .

Theorem 9.6. Let  $\mathcal{A}$  be an IBA, IPA, IRA, or IMA accepting the  $\omega$ -language L. Then the characteristic sample  $T_L$  for  $\mathcal{A}$  may be computed in polynomial time in the size of  $\mathcal{A}$ .

PROOF. By Theorem 9.1, the sample  $T_{Aut}$  can be computed in polynomial time in the size of  $\mathcal{A}$ , and by Prop. 9.2, 9.3, 9.4, or 9.5 the sample  $T_{Acc}^{IBA}$ ,  $T_{Acc}^{IPA}$ , or  $T_{Acc}^{IMA}$  can also be computed in polynomial time in the size of  $\mathcal{A}$ .

#### 10 INCLUSION ALGORITHMS

We show that there are polynomial time algorithms for the inclusion problem for DBAs, DCAs, DPAs, DRAs, DSAs, and DMAs. Recall that two calls to an inclusion algorithm suffice to solve the equivalence problem. By Claim 2.5, the inclusion and equivalence problems for DCAs are efficiently reducible to those for DBAs, and vice versa. Also, by Claim 2.2, the inclusion and equivalence problems for DBAs are efficiently reducible to those for DPAs. Thus it suffices to consider the inclusion problem for DPAs, DRAs, and DMAs.

*Remark.* In the case of DFAs, a polynomial algorithm for the inclusion problem can be obtained using polynomial algorithms for complementation, intersection and emptiness (since for any two languages  $L_1 \subseteq L_2$  if and only if  $L_1 \cap \overline{L_2} = \emptyset$ ). However, a similar approach does not work in the case of DPAs; although complementation and emptiness for DPAs can be computed in polynomial time, intersection cannot [13, Theorem 9].

For the inclusion problem for DBAs, DCAs, and DPAs, Schewe [35] gives the following result.

THEOREM 10.1 ([35]). The inclusion problems for DBAs, DCAs, and DPAs are in NL.

Because NL (nondeterministic logarithmic space) is contained in polynomial time, this implies the existence of polynomial time inclusion and equivalence algorithms for DBAs, DCAs, and DPAs. Because Schewe does not give a proof or reference for Theorem 10.1, and does not address the problem of returning a witness, for completeness we include a proof sketch.

PROOF SKETCH. For i=1,2, let  $\mathcal{P}_i=\langle \Sigma,Q_i,(q_t)_i,\delta_i,\kappa_i\rangle$  be a DPA. It suffices to guess two states  $q_1\in Q_1$  and  $q_2\in Q_2$ , and two words  $u\in \Sigma^*$  and  $v\in \Sigma^+$ , and to check that for i=1,2,  $\delta_i((q_t)_i,u)=q_i$  and  $\delta_i(q_i,v)=q_i$ , and also, that the smallest value of  $\kappa_1(q)$  in the loop in  $\mathcal{P}_1$  from  $q_1$  to  $q_1$  on input v is odd, while the smallest value of  $\kappa_2(q)$  in the loop in  $\mathcal{P}_2$  from  $q_2$  to  $q_2$  on input v is even. If these checks succeed, then  $[\![\mathcal{P}_1]\!]$  is not a subset of  $[\![\mathcal{P}_2]\!]$ , and the ultimately periodic word  $u(v)^\omega$  is a witness.

Logarithmic space is enough to record the two guessed states  $q_1$  and  $q_2$  as well as the current minimum values of  $\kappa_1$  and  $\kappa_2$  as the loops on v are traversed in the two automata. The words u and v need only be guessed symbol-by-symbol, using a pointer in each automaton to keep track of its current state.

This approach does not seem to work in the case of testing DRA or DMA inclusion, because the acceptance conditions would seem to require keeping track of more information than would fit in logarithmic space. To supplement the proof sketch for Schewe's theorem, in the next section we give an explicit polynomial time algorithm for testing DPA inclusion. In the following sections, we give polynomial time algorithms for testing inclusion for DRAs and DMAs, which are novel results.

# 11 INCLUSION AND EQUIVALENCE FOR DPAS, DBAS, DCAS

In this section we describe an explicit polynomial time algorithm for the inclusion problem for two DPAs, which yields algorithms for DBAs and DCAs. If  $\mathcal{P} = \langle \Sigma, Q, q_t, \delta, \kappa \rangle$  is a complete DPA and  $w \in \Sigma^{\omega}$ , we let  $\mathcal{P}(w)$  denote the minimum color visited by  $\mathcal{P}$  infinitely often on input w, that is,  $\mathcal{P}(w) = \kappa(\inf(w))$ .

#### 11.1 Searching for w with given minimum colors in two acceptors

We first describe an algorithm that searches for an  $\omega$ -word that yields specified minimum colors in two different DPAs over the same alphabet.

For i = 1, 2, let  $\mathcal{P}_i = \langle \Sigma, Q_i, (q_t)_i, \delta_i, \kappa_i \rangle$  be a DPA, and let  $\mathcal{M}_i$  be the automaton of  $\mathcal{P}_i$ . Given inputs of  $\mathcal{P}_1$  and two nonnegative integers  $k_1$  and  $k_2$ , the Colors algorithm constructs the product automaton  $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$  and the set  $Q' = \{(q_1, q_2) \in Q_1 \times Q_2 \mid \kappa_1(q_1) \geq k_1 \wedge \kappa_2(q_2) \geq k_2\}$ .

The algorithm then computes S = maxSCCs(Q') for the automaton  $\mathcal{M}$ , and loops through the SCCs  $C \in S$  checking whether C is reachable in  $\mathcal{M}$ ,  $\min(\kappa_1(\pi_1(C))) = k_1$ , and  $\min(\kappa_2(\pi_2(C))) = k_2$ . If so, it returns the ultimately periodic word  $u(v)^{\omega} = \text{Witness}(C, \mathcal{M})$ . If none of the elements  $C \in S$  satisfies this condition, then the answer "no" is returned.

THEOREM 11.1. The algorithm Colors takes as input two DPAs  $\mathcal{P}_1$  and  $\mathcal{P}_2$  over the same alphabet and two nonnegative integers  $k_1$  and  $k_2$ , runs in polynomial time, and determines whether there exists an  $\omega$ -word w such that  $\mathcal{P}_1(w) = k_1$  and  $\mathcal{P}_2(w) = k_2$ . If not, it returns the answer "no". If so, it returns an ultimately periodic  $\omega$ -word  $u(v)^{\omega}$  such that  $\mathcal{P}_1(u(v)^{\omega}) = k_1$  and  $\mathcal{P}_2(u(v)^{\omega}) = k_2$ .

PROOF. The polynomial running time of the algorithm follows from Props. 4.4 and 4.7. To see the correctness of the algorithm, suppose first that it returns an ultimately periodic word  $u(v)^{\omega}$ . This occurs only if it finds an SCC C of M such that C is reachable in M,  $\kappa_1(\pi_1(C)) = k_1$ , and  $\kappa_2(\pi_2(C)) = k_2$ . Then for  $i = 1, 2, \pi_i(C)$  is the set of states visited infinitely often by  $M_i$  on the input  $u(v)^{\omega}$ , which has minimum color  $k_i$ .

To see that the algorithm does not incorrectly answer "no", suppose w is an  $\omega$ -word such that for  $i=1,2, \mathcal{P}_i(w)=k_i$ . Let  $D_i=\inf_{\mathcal{M}_i}(w)$ , an SCC of  $\mathcal{M}_i$ . No state in  $D_i$  has a color less than  $k_i$ , so if  $D=\inf_{\mathcal{M}}(w)$ , then  $D\subseteq \mathcal{Q}'$ . Also, D is a reachable SCC in  $\mathcal{M}$ .

Then D is contained in some element C of maxSCCs(Q'). Because there are no states  $(q_1, q_2)$  in C with  $\kappa_1(q_1) < k_1$  or  $\kappa_2(q_2) < k_2$ , we must have  $\kappa_i(\pi_i(C)) = k_i$  for i = 1, 2. Also, C is reachable in  $\mathcal{M}$  because D is. Thus, the algorithm will find at least one such C and return  $u(v)^\omega$  such that  $inf_{\mathcal{M}}(u(v)^\omega) = C$ .

# 11.2 An inclusion algorithm for DPAs

The inclusion problem for DPAs  $\mathcal{P}_1$  and  $\mathcal{P}_2$  over the same alphabet can be solved by looping over all odd  $k_1$  in the range of  $\kappa_1$  and all even  $k_2$  in the range of  $\kappa_2$ , calling the Colors algorithm with inputs  $\mathcal{P}_1$ ,  $\mathcal{P}_2$ ,  $k_1$ , and  $k_2$ . If the Colors algorithm returns any witness  $u(v)^{\omega}$ , then  $u(v)^{\omega} \in \llbracket \mathcal{P}_1 \rrbracket \setminus \llbracket \mathcal{P}_2 \rrbracket$ , and  $u(v)^{\omega}$  is returned as a witness of non-inclusion. Otherwise, by Theorem 11.1, there is no  $\omega$ -word w accepted by  $\mathcal{P}_1$  and not accepted by  $\mathcal{P}_2$ , and the answer "yes" is returned for the inclusion problem. Note that for i=1,2, the range of  $\kappa_i$  has at most  $|Q_i|$  distinct elements. Thus we have the following.

THEOREM 11.2. There are polynomial time algorithms for the inclusion and equivalence problems for two DPAs over the same alphabet.

From Claims 2.2 and 2.5, we have the following.

THEOREM 11.3. There are polynomial time algorithms for the inclusion and equivalence problems for two DBAs (or DCAs) over the same alphabet.

#### 12 AN INCLUSION ALGORITHM FOR DRAS

In this section we describe a polynomial time algorithm to solve the inclusion problem for two DRAs. The algorithm returns a  $\leq$ -largest witness in the case of non-inclusion.

The algorithm **SubInc**<sup>DRA</sup> takes as input two DRAs  $\mathcal{R}_1 = (\mathcal{M}_1, \alpha_1)$  and  $\mathcal{R}_2 = (\mathcal{M}_2, \alpha_2)$  in singleton normal form, where  $\alpha_1$  consists of a single pair (q', B'). It also takes as input a subset S of the state set of the product automaton  $\mathcal{M} = \mathcal{M}_{A}$ 

# Algorithm 1 SubInc $^{DRA}$

```
Input: Two DRAs \mathcal{R}_1 = (\mathcal{M}_1, \alpha_1) and \mathcal{R}_2 = (\mathcal{M}_2, \alpha_2) in singleton normal form, where \alpha_1 = \{(q', B')\} and \alpha_2 = (\mathcal{M}_1, \alpha_2)
    \{(q_1'', B_1''), \dots, (q_k'', B_k'')\}, and a set S of states of \mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2
Output: u(v)^{\omega} \in [\![\mathcal{R}_1]\!] \setminus [\![\mathcal{R}_2]\!] with \inf_{\mathcal{M}}(u(v)^{\omega}) \subseteq S if such exists, else "none".
   \mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2
   W \leftarrow \emptyset
   S' \leftarrow S \setminus \{(q_1, q_2) \in S \mid q_1 \in B'\}
   C \leftarrow maxSCCs(S')
   for each reachable C \in C such that q' \in \pi_1(C) do
           if for no j \in [1..k] is q_i'' \in \pi_2(C) and B_i'' \cap \pi_2(C) = \emptyset then
                    W \leftarrow W \cup \{ \mathbf{Witness}(C, \mathcal{M}) \}
                                                                                                                                                     ▶ A new candidate witness
                    J = \{q_j'' \mid j \in [1..k], B_j'' \cap \pi_2(C) = \emptyset\}
                    S'' \leftarrow C \setminus \{(q_1, q_2) \in C \mid q_2 \in J\}
                    Call SubInc<sup>DRA</sup> recursively with \mathcal{R}_1, \mathcal{R}_2, and S''
                    if the returned value is u(v)^{\omega} then
                            W \leftarrow W \cup \{u(v)^{\omega}\}
   if W is \emptyset then
           return "none"
   else
           Let u(v)^{\omega} \in W have the \leq-largest value of inf_{\mathcal{M}}(u(v)^{\omega})
           return u(v)^{\omega}
```

 $\mathcal{M}_1 \times \mathcal{M}_2$ . The problem it solves is to determine whether there exists an  $\omega$ -word  $u(v)^{\omega}$  with  $\inf_{\mathcal{M}}(u(v)^{\omega}) \subseteq S$  such that  $u(v)^{\omega} \in [\![\mathcal{R}_1]\!] \setminus [\![\mathcal{R}_2]\!]$ . If there is such a word, the algorithm returns one with the  $\leq$ -largest value of  $\inf_{\mathcal{M}}(u(v)^{\omega})$ , and otherwise, it returns "none".

PROPOSITION 12.1. For i=1,2 let  $\mathcal{R}_i=(\mathcal{M}_i,\alpha_i)$  be a DRA in singleton normal form. Assume  $\alpha_1=\{(q',B')\}$  and  $\alpha_2=\{(q''_1,B''_1),\ldots,(q''_k,B''_k)\}$ . Let  $\mathcal{M}=\mathcal{M}_1\times\mathcal{M}_2$ , and let S be a subset of the states of  $\mathcal{M}$ . Then with inputs  $\mathcal{R}_1,\mathcal{R}_2$ , and S, the algorithm SubInc DRA runs in polynomial time and returns  $u(v)^{\omega}\in [\mathcal{R}_1]\setminus [\mathcal{R}_2]$  with the  $\leq$ -largest value of  $\inf_{\mathcal{M}}(u(v)^{\omega})$  contained in S, if such exists, else it returns "none".

PROOF. When the element  $u(v)^{\omega} = \text{Witness}(\mathcal{M}, C)$  is added to W, we have that C is a reachable SCC of  $\mathcal{M}$  contained in  $S, q' \in \pi_1(C)$ , and  $B' \cap \pi_1(C) = \emptyset$  (because S' contains no elements  $(q_1, q_2)$  with  $q_1 \in B'$ ), so  $(\mathcal{M}_1, \{(q', B')\})$  accepts  $u(v)^{\omega}$ . Also, we have that for no  $j \in [1..k]$  do we have  $q''_j \in \pi_2(C)$  and  $B''_j \cap \pi_2(C) = \emptyset$ , so  $\mathcal{R}_2$  rejects  $u(v)^{\omega}$ . Thus, any returned  $u(v)^{\omega}$  is a witness to the non-inclusion of  $[(\mathcal{M}_1, \{(q', B')\})]$  in  $[\mathcal{R}_2]$  with  $\inf_{\mathcal{M}} (u(v)^{\omega}) \subseteq S$ .

We now show by induction on the recursive calls that if w is any  $\omega$ -word such that  $\inf_{\mathcal{M}}(w) \subseteq S$ ,  $(\mathcal{M}_1, \{(q', B')\})$  accepts w, and  $\mathcal{R}_2$  rejects w, then  $\operatorname{SubInc}^{DRA}$  returns a witness  $u(v)^\omega$  such that  $\inf_{\mathcal{M}}(u(v)^\omega)$  is at least as large as  $\inf_{\mathcal{M}}(w)$  in the  $\leq$ -ordering. Let  $D = \inf_{\mathcal{M}}(w)$ . Then D is a reachable SCC of  $\mathcal{M}$  such that  $q' \in \pi_1(D)$ ,  $B' \cap \pi_1(D) = \emptyset$ , and for no  $j \in [1..k]$  do we have  $q''_j \in \pi_2(D)$  and  $B''_j \cap \pi_2(D) = \emptyset$ . Then  $D \subseteq S'$  because  $B' \cap \pi_1(D) = \emptyset$ . Thus, D must be a subset of exactly one of the elements C of  $\max SCCs(S')$ . Then C is reachable,  $q' \in \pi_1(C)$ , and  $B' \cap \pi_1(C) = \emptyset$  (because C is a subset of S').

If C is such that for no  $j \in [1..k]$  do we have  $q_j'' \in \pi_2(C)$  and  $B_j'' \cap \pi_2(C) = \emptyset$ , then a witness  $u(v)^\omega = \text{Witness}(C, \mathcal{M})$  is added to W, and we have that  $C = \inf_{\mathcal{M}} (u(v)^\omega)$  is at least as large in the  $\leq$ -ordering as  $D = \inf_{\mathcal{M}} (w)$ , because  $D \subseteq C$ . Manuscript submitted to ACM

Otherwise, the set  $J = \{q_j'' \mid j \in [1..k], B_j'' \cap \pi_2(C) = \emptyset\}$  is non-empty, and the algorithm removes from C all the states  $(q_1, q_2)$  such that  $q_2 \in J$  to form the set S''. Because  $D \subseteq C$ , if  $B_j'' \cap \pi_2(C) = \emptyset$ , then also  $B_j'' \cap \pi_2(D) = \emptyset$ . Thus, if for any  $q_j'' \in J$  we have  $q_j'' \in \pi_2(D)$ , this would violate the assumption that  $\mathcal{R}_2$  rejects w. Hence,  $D \subseteq S''$ , and by the inductive assumption on the recursive calls, the recursive call to  $\mathbf{SubInc}^{DRA}$  returns a witness  $u(v)^\omega$  such that  $\inf_{\mathcal{M}}(u(v)^\omega)$  is at least as large in the  $\leq$ -ordering as  $\inf_{\mathcal{M}}(w)$ . Because the top-level algorithm returns  $u(v)^\omega$  to maximize  $\inf_{\mathcal{M}}(u(v)^\omega)$  with respect to  $\leq$ , it will be at least as large as  $\inf_{\mathcal{M}}(w)$ .

For the polynomial running time, we note that all the SCCs C considered are distinct elements of a decreasing forest of SCCs for the automaton  $\mathcal{M}$ , and so there can be at most as many as the number of states of  $\mathcal{M}$ .

THEOREM 12.2. There are polynomial time algorithms to solve the inclusion and equivalence problems for two DRAs (resp. DSAs)  $\mathcal{R}_1$  and  $\mathcal{R}_2$ . In the case of non-inclusion or non-equivalence, these algorithms return a witness  $u(v)^{\omega}$  with the  $\leq$ -largest value of  $\inf_{\mathcal{M}}(u(v)^{\omega})$ , where  $\mathcal{M}$  is the product of the automata of  $\mathcal{R}_1$  and  $\mathcal{R}_2$ .

PROOF. It suffices to consider just DRAs, by Claim 2.6. Given two DRAs  $\mathcal{R}_1 = (\mathcal{M}_1, \alpha_1)$  and  $\mathcal{R}_2 = (\mathcal{M}_2, \alpha_2)$ , we may assume they are in singleton normal form. Then for each pair  $(q_i, B_i)$  in  $\alpha_1$ , we call **SubInc**<sup>DRA</sup> with inputs  $(\mathcal{M}_1, \{(q_i, B_i)\}), \mathcal{R}_2$ , with S equal to the whole state set of  $\mathcal{M}$ . If all of these calls return "none", then  $[\![\mathcal{R}_1]\!]$  is a subset of  $[\![\mathcal{R}_2]\!]$ , and the answer returned is "yes". Otherwise, one or more calls return a witness, and  $u(v)^\omega$  is returned such that  $\inf_{\mathcal{M}} (u(v)^\omega)$  is  $\leq$ -largest among the witnesses returned by the calls. The running time and correctness follow from the running time and correctness guarantees of **SubInc**<sup>DRA</sup>.

#### 13 AN INCLUSION ALGORITHM FOR DMAS

In this section we develop a polynomial time algorithm to solve the inclusion problem for two DMAs over the same alphabet. The proof proceeds in two parts: (1) a polynomial time reduction of the inclusion problem for two DMAs to the inclusion problem for a DBA and a DMA, and (2) a polynomial time algorithm for the inclusion problem for a DBA and a DMA.

# 13.1 Reduction of DMA inclusion to DBA/DMA inclusion

We first reduce the problem of inclusion for two arbitrary DMAs to the inclusion problem for two DMAs where the first one has just a single final state set. For i = 1, 2, define the DMA  $\mathcal{U}_i = \langle Q_i, \Sigma, (q_i)_i, \delta_i, \mathcal{F}_i \rangle$ , where  $\mathcal{F}_i$  is the set of final state sets for  $\mathcal{U}_i$ . Let the elements of  $\mathcal{F}_1$  be  $\{F_1, \ldots, F_k\}$ , and for each  $j \in [1..k]$ , let

$$\mathcal{U}_{1,j} = \langle Q_1, \Sigma, (q_i)_1, \delta_1, \{F_j\} \rangle,$$

that is,  $\mathcal{U}_{1,j}$  is  $\mathcal{U}_1$  with  $F_j$  as its only final state set. Then by the definition of DMA acceptance,

$$\llbracket \mathcal{U}_1 \rrbracket = \bigcup_{i=1}^k \llbracket \mathcal{U}_{1,j} \rrbracket,$$

which implies that to test whether  $\llbracket \mathcal{U}_1 \rrbracket \subseteq \llbracket \mathcal{U}_2 \rrbracket$ , it suffices to test for all  $j \in \llbracket 1..k \rrbracket$  that  $\llbracket \mathcal{U}_{1,j} \rrbracket \subseteq \llbracket \mathcal{U}_2 \rrbracket$ .

PROPOSITION 13.1. Suppose **A** is a procedure that solves the inclusion problem for two DMAs over the same alphabet, assuming that the first DMA has a single final state set. Then there is an algorithm that solves the inclusion problem for two arbitrary DMAs over the same alphabet, say  $\mathcal{U}_1$  and  $\mathcal{U}_2$ , which simply makes  $|\mathcal{F}_1|$  calls to **A**, where  $\mathcal{F}_1$  is the family of final state sets of  $\mathcal{U}_1$ .

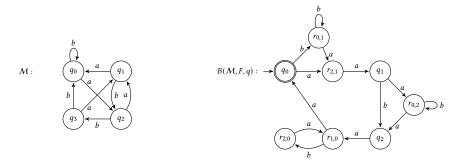


Fig. 4. Example of the construction of  $B(\mathcal{M}, F, q)$  with  $F = \{q_0, q_1, q_2\}$  and  $q = q_0$ .

Next we describe a procedure SCCtoDBA that takes as inputs a deterministic automaton  $\mathcal{M}$ , an SCC F of  $\mathcal{M}$ , and a state  $q \in F$ , and returns a DBA  $B(\mathcal{M}, F, q)$  that accepts exactly  $L(\mathcal{M}, F, q)$ , where  $L(\mathcal{M}, F, q)$  is the set of  $\omega$ -words w that visit only the states of F when processed by  $\mathcal{M}$  starting at state q, and visits each of them infinitely many times.

Assume the states in F are  $\{q_0, q_1, \ldots, q_{m-1}\}$ , where  $q_0 = q$ . The DBA  $B(\mathcal{M}, F, q)$  is  $\langle \mathcal{Q}', \Sigma, q_0, \delta', \{q_0\}\rangle$ , where we define Q' and  $\delta'$  as follows. We create new states  $r_{i,j}$  for  $i, j \in [0..m-1]$  such that  $i \neq j$ , and denote the set of these by R. We also create a new dead state  $d_0$ . Then the set of states Q' is  $Q \cup R \cup \{d_0\}$ .

For  $\delta'$ , the dead state  $d_0$  behaves as expected: for all  $\sigma \in \Sigma$ ,  $\delta'(d_0, \sigma) = d_0$ . For the other states in Q', let  $\sigma \in \Sigma$  and  $i \in [0..m-1]$ . If  $\delta(q_i, \sigma)$  is not in F, then in order to deal with runs that would visit states outside of F, we define  $\delta'(q_i, \sigma) = d_0$  and, for all  $j \neq i$ ,  $\delta'(r_{i,j}, \sigma) = d_0$ .

Otherwise, for some  $k \in [0..m-1]$  we have  $q_k = \delta(q_i, \sigma)$ . If  $k = (i+1) \mod m$ , then we define  $\delta'(q_i, \sigma) = q_k$ , and otherwise we define  $\delta'(q_i, \sigma) = r_{k,(i+1) \mod m}$ . For all  $j \in [0..m-1]$  with  $j \neq i$ , if k = j, we define  $\delta'(r_{i,j}, \sigma) = q_k$ , and otherwise we define  $\delta'(r_{i,j}, \sigma) = r_{k,j}$ .

Intuitively, for an input from  $L(\mathcal{M}, F, q)$ , in  $B(\mathcal{M}, F, q)$  the states  $q_i$  are visited in a repeating cyclic order:  $q_0, q_1, \ldots, q_{m-1}$ , and the meaning of the state  $r_{i,j}$  is that at this point in the input,  $\mathcal{M}$  would be in state  $q_i$ , and the machine  $B(\mathcal{M}, F, q)$  is waiting for a transition that would arrive at state  $q_j$  in  $\mathcal{M}$ , in order to proceed to state  $q_j$  in  $B(\mathcal{M}, F, q)$ . An example of the construction is shown in Fig. 4; the dead state and unreachable states are omitted for clarity.

LEMMA 13.2. Let  $\mathcal{M}$  be a deterministic automaton with alphabet  $\Sigma$  and states Q, and let F an SCC of  $\mathcal{M}$  and  $q \in F$ . With these inputs, the procedure SCCtoDBA runs in polynomial time and returns the DBA  $B(\mathcal{M}, F, q)$ , which accepts the language  $L(\mathcal{M}, F, q)$  and has  $|F|^2 + 1$  states.

PROOF. Suppose w is in  $L(\mathcal{M}, F, q)$ . Let  $q = s_0, s_1, s_2, \ldots$  be the sequence of states in the run of  $\mathcal{M}$  from state q on input w. This run visits only states in F and visits each one of them infinitely many times. We next define a particular increasing sequence  $i_{k,\ell}$  of indices in s, where k is a positive integer and  $\ell \in [0, m-1]$ . These indices mark particular visits to the states  $q_0, q_1, \ldots, q_{m-1}$  in repeating cyclic order. The initial value is  $i_{1,0} = 0$ , marking the initial visit to  $q_0$ . If  $i_{k,\ell}$  has been defined and  $\ell < m-1$ , then  $i_{k,\ell+1}$  is defined as the least natural number j such that  $j > i_{k,\ell}$  and  $s_j = q_{\ell+1}$ , marking the next visit to  $q_{\ell+1}$ . If  $\ell = m-1$ , then  $i_{k+1,0}$  is defined as the least natural number j such that  $j > i_{k,\ell}$  and  $s_j = q_0$ , marking the next visit to  $q_0$ .

 $<sup>^3</sup>$ This construction is reminiscent of the construction transforming a generalized Büchi into a Büchi automaton [17, 42], by considering each state in F as a singelton set of a generalized Büchi, but here we need to send transitions to states outside F to a sink state.

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There is a corresponding division of w into a concatenation of finite segments  $w_{1,1}, w_{1,2}, \ldots, w_{1,m-1}, w_{2,0}, \ldots$  between consecutive elements in the increasing sequence of indices. An inductive argument shows that in  $B(\mathcal{M}, F, q)$ , the prefix of w up through  $w_{k,\ell}$  arrives at the state  $q_{\ell}$ , so that w visits  $q_0$  infinitely often and is therefore accepted by  $B(\mathcal{M}, F, q)$ .

Conversely, suppose  $B(\mathcal{M}, F, q)$  accepts the  $\omega$ -word w. Let  $s_0, s_1, s_2, \ldots$  be the run of  $B(\mathcal{M}, F, q)$  on w, and let  $t_0, t_1, t_2, \ldots$  be the run of  $\mathcal{M}$  starting from q on input w. An inductive argument shows that if  $s_n = q_i$  then  $t_n = q_i$ , and if  $s_n = r_{i,j}$  then  $t_n = q_i$ . Because the only way the run  $s_0, s_1, \ldots$  can visit the final state  $q_0$  infinitely often is to progress through the states  $q_0, q_1, \ldots, q_{m-1}$  in repeating cyclic order, the run  $t_0, t_1, \ldots$  must visit only states in F and visit each of them infinitely often, so  $w \in L(\mathcal{M}, F, q)$ .

The DBA  $B(\mathcal{M}, F, q)$  has a dead state, and |F| states for each element of F, for a total of  $|F|^2 + 1$  states. The running time of the procedure **SCCtoDBA** is linear in the size of  $\mathcal{M}$  and the size of the resulting DBA, which is polynomial in the size of  $\mathcal{M}$ .

We now show that this construction may be used to reduce the inclusion of two DMAs to the inclusion of a DBA and a DMA. Recall that if  $\mathcal{A}$  is an acceptor and q is a state of  $\mathcal{A}$ , then  $\mathcal{A}^q$  denotes the acceptor  $\mathcal{A}$  with the initial state changed to q.

LEMMA 13.3. Let  $\mathcal{U}_1$  be a DMA with automaton  $\mathcal{M}_1$  and a single final state set  $F_1$ . Let  $\mathcal{U}_2$  be an arbitrary DMA over the same alphabet as  $\mathcal{U}_1$ , with automaton  $\mathcal{M}_2$  and family of final state sets  $\mathcal{F}_2$ . Let  $\mathcal{M}$  denote the product automaton  $\mathcal{M}_1 \times \mathcal{M}_2$  with unreachable states removed. Then  $[\![\mathcal{U}_1]\!] \subseteq [\![\mathcal{U}_2]\!]$  iff for every state  $(q_1, q_2)$  of  $\mathcal{M}$  with  $q_1 \in F_1$  we have  $[\![\mathcal{B}(\mathcal{M}_1, F_1, q_1)]\!] \subseteq [\![\mathcal{U}_2]\!]$ .

PROOF. Suppose that for some state  $(q_1, q_2)$  of  $\mathcal{M}$  with  $q_1 \in F_1$ , we have  $w \in [\![B(\mathcal{M}_1, F_1, q_1)]\!] \setminus [\![\mathcal{U}_2^{q_2}]\!]$ . Let  $C_1$  be the set of states visited infinitely often in  $B(\mathcal{M}_1, F_1, q_1)$  on input w, and let  $C_2$  be the set of states visited infinitely often in  $\mathcal{U}_2^{q_2}$  on input w. Then  $C_1 = F_1$  and  $C_2 \notin \mathcal{F}_2$ . Let u be a finite word such that  $\mathcal{M}(u) = (q_1, q_2)$ . Then  $\inf_{\mathcal{M}_1} (uw) = C_1 = F_1$  and  $\inf_{\mathcal{M}_2} (uw) = C_2$ , so  $uw \in [\![\mathcal{U}_1]\!] \setminus [\![\mathcal{U}_2]\!]$ .

Conversely, suppose that  $w \in \llbracket \mathcal{U}_1 \rrbracket \setminus \llbracket \mathcal{U}_2 \rrbracket$ . For i = 1, 2 let  $C_i = \inf_{\mathcal{M}_i}(w)$ . Note that  $C_1 = F_1$  and  $C_2 \notin \mathcal{F}_2$ . Let w = xw', where x is a finite prefix of w that is sufficiently long that the run of  $\mathcal{M}_1$  on w does not visit any state outside  $C_1$  after x has been processed, and for i = 1, 2 let  $q_i = \mathcal{M}_i(x)$ . Then  $(q_1, q_2)$  is a (reachable) state of  $\mathcal{M}_1 \in F_1$ , and the  $\omega$ -word w', when processed by  $\mathcal{M}_1$  starting at state  $q_1$  visits only states of  $C_1 = F_1$  and visits each of them infinitely many times, that is,  $w' \in \llbracket B(\mathcal{M}_1, F_1, q_1) \rrbracket$ . Moreover, when w' is processed by  $\mathcal{M}_2$  starting at state  $q_2$ , the set of states visited infinitely often is  $C_2$ , which is not in  $\mathcal{F}_2$ . Thus,  $w' \in \llbracket B(\mathcal{M}_1, F_1, q_1) \rrbracket \setminus \llbracket \mathcal{U}_2^{q_2} \rrbracket$ .

To turn this into an algorithm to test inclusion for two DMAs,  $\mathcal{U}_1$  with automaton  $\mathcal{M}_1$  and a single final state set  $F_1$  that is an SCC of  $\mathcal{M}_1$  and  $\mathcal{U}_2$  with automaton  $\mathcal{M}_2$ , we proceed as follows. Construct the product automaton  $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$  with unreachable states removed, and for each state  $(q_1, q_2)$  of  $\mathcal{M}$ , if  $q_1 \in F_1$ , construct the DBA  $B(\mathcal{M}_1, F_1, q_1)$  and the DMA  $\mathcal{U}_2^{q_2}$  and test the inclusion of language accepted by the DBA in the language accepted by the DMA. If all of these tests return "yes", then the algorithm returns "yes" for the inclusion question for  $\mathcal{U}_1$  and  $\mathcal{U}_2$ . Otherwise, for the first test that returns "no" and a witness  $u(v)^{\omega}$ , the algorithm finds by breadth-first search a minimum length finite word u' such that  $\mathcal{M}(u') = (q_1, q_2)$ , and returns the witness  $u'u(v)^{\omega}$ .

Combining this with Prop. 13.1, we have the following.

Theorem 13.4. Let A be an algorithm to test inclusion for an arbitrary DBA and an arbitrary DMA over the same alphabet. There is an algorithm to test inclusion for an arbitrary pair of DMAs  $U_1$  and  $U_2$  over the same alphabet whose Manuscript submitted to ACM

running time is linear in the sizes of  $\mathcal{U}_1$  and  $\mathcal{U}_2$  plus the time for at most  $k \cdot |Q_1| \cdot |Q_2|$  calls to the procedure A, where k is the number of final state sets in  $\mathcal{U}_1$ , and  $Q_i$  is the state set of  $\mathcal{U}_i$  for i = 1, 2.

# 13.2 A DBA/DMA inclusion algorithm

In this section, we give a polynomial time algorithm **DBAinDMA** to test inclusion for an arbitrary DBA and an arbitrary DMA over the same alphabet.

Assume the inputs are a DBA  $\mathcal{B}=(\mathcal{M}_1,F_1)$  and a DMA  $\mathcal{U}=(\mathcal{M}_2,\mathcal{F})$ . The overall strategy of the algorithm is to seek an SCC C of  $\mathcal{M}=\mathcal{M}_1\times\mathcal{M}_2$  such that  $\pi_1(C)\cap F_1\neq\emptyset$  and  $\pi_2(C)\notin\mathcal{F}$ . If such a C is found, the algorithm calls Witness $(C,\mathcal{M})$ , which returns  $u(v)^\omega$  such that  $\inf_{\mathcal{M}}(u(v)^\omega)=C$ . Because  $\inf_{\mathcal{M}_1}(u(v)^\omega)=\pi_1(C)$  and  $\pi_1(C)\cap F_1\neq\emptyset$ ,  $u(v)^\omega\in[\![\mathcal{B}]\!]$ , and because  $\inf_{\mathcal{M}_2}(u(v)^\omega)=\pi_2(C)$  and  $\pi_2(C)\notin\mathcal{F}$ ,  $u(v)^\omega\notin[\![\mathcal{U}]\!]$ . The details are given in Algorithm 2.

# Algorithm 2 DBAinDMA

```
Input: A DBA \mathcal{B} = (\mathcal{M}_1, F_1) and a DMA \mathcal{U} = (\mathcal{M}_2, \mathcal{F}), where Q_i is the state set of \mathcal{M}_i for i = 1, 2.

Output: u(v)^\omega \in \llbracket \mathcal{B} \rrbracket \setminus \llbracket \mathcal{U} \rrbracket if such exists, else "yes".

\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2
C \leftarrow \max SCCs(Q_1 \times Q_2)
for each reachable C \in C such that \pi_1(C) \cap F_1 \neq \emptyset do

if \pi_2(C) \notin \mathcal{F} then

return Witness(C, \mathcal{M})
else

for each F \in \mathcal{F} such that F \subseteq C and each F \in \mathcal{F} do

S \leftarrow \{(q_1, q_2) \in Q_1 \times Q_2 \mid q_1 \in \pi_1(C) \land q_2 \in F \setminus \{q\}\}
\mathcal{D} \leftarrow \max SCCs(S)

for each F \in \mathcal{F} and F \in \mathcal{F} and F \in \mathcal{F} then

return WitnessF \in \mathcal{F} and F \in \mathcal{F} then
```

Theorem 13.5. The DBAinDMA algorithm runs in polynomial time and solves the inclusion problem for an arbitrary DBA  $\mathcal B$  and an arbitrary DMA  $\mathcal U$  over the same alphabet.

PROOF. Suppose the returned value is a witness  $u(v)^{\omega}$ . Then the algorithm found an SCC E with  $\pi_1(E) \cap F_1 \neq \emptyset$  and  $\pi_2(E) \notin \mathcal{F}$  and returned Witness $(E, \mathcal{M})$ . In this case, the returned value is correct.

Suppose for the sake of contradiction that the algorithm incorrectly returns the answer "yes", that is, there exists an  $\omega$ -word w such that  $w \in [\![\mathcal{B}]\!]$  and  $w \notin [\![\mathcal{U}]\!]$ . Let C' denote  $\inf_{\mathcal{M}}(w)$ . Then because  $w \in [\![\mathcal{B}]\!]$ ,  $\pi_1(C') \cap F_1 \neq \emptyset$ , and because  $w \notin [\![\mathcal{U}]\!]$ ,  $\pi_2(C') \notin \mathcal{F}$ .

Then C' is a subset of a unique SCC  $C \in maxSCCs(Q_1 \times Q_2)$  and  $\pi_1(C) \cap F_1 \neq \emptyset$ . It must be that  $\pi_2(C) \in \mathcal{F}$ , because otherwise the algorithm would have returned Witness $(C, \mathcal{M})$ . Consider the collection

$$R = \{ F \in \mathcal{F} \mid \pi_2(C') \subseteq F \subseteq \pi_2(C) \},\$$

of all the  $F \in \mathcal{F}$  contained in  $\pi_2(C)$  that contain  $\pi_2(C')$ . The collection R is nonempty because  $C' \subseteq C$ , and therefore  $\pi_2(C') \subseteq \pi_2(C)$ , and  $\pi_2(C) \in \mathcal{F}$ , so at least  $\pi_2(C)$  is in R. Let F' denote a minimal element of R in the subset ordering. Manuscript submitted to ACM

Then  $\pi_2(C') \subseteq F'$  but because  $\pi_2(C') \notin \mathcal{F}$ , it must be that  $\pi_2(C') \neq F'$ . Thus, there exists some  $q \in F'$  that is not in  $\pi_2(C')$ . When the algorithm considers this F' and q, then because  $\pi_2(C') \subseteq F' \setminus \{q\}$ , C' is contained R and therefore is a subset of a unique SCC D in maxSCCs(R).

Because  $C' \subseteq D$ , and  $\pi_1(C') \cap F_1 \neq \emptyset$ , we have  $\pi_1(D) \cap F_1 \neq \emptyset$ . Also,  $\pi_2(C') \subseteq \pi_2(D) \subseteq F'$ , but because  $q \notin \pi_2(D)$ ,  $\pi_2(D)$  is a proper subset of F'. When the algorithm considers this D, because  $\pi_1(D) \cap F_1 \neq \emptyset$ , it must find that  $\pi_2(D) \in \mathcal{F}$ , or else it will return **Witness** $(D, \mathcal{M})$ . But then  $\pi_2(D)$  is in R and is a proper subset of F', contradicting our choice of F' as a minimal element of R. Thus, if the algorithm outputs "yes", this is a correct answer.

Combining Theorem 13.4, Theorem 13.5, and the reduction of equivalence to inclusion, we have the following.

THEOREM 13.6. There are polynomial time algorithms to solve the inclusion and equivalence problems for two arbitrary DMAs over the same alphabet.

# 14 COMPUTING THE AUTOMATON $\mathcal{M}_{\sim_I}$

In this section we use polynomial time algorithms to construct the automaton  $\mathcal{M}_{\sim_L}$  of the right congruence relation  $\sim_L$  of the language L accepted by an acceptor  $\mathcal{A}$  of one of the types DBA, DCA, DPA, DRA, DSA, or DMA. This gives a polynomial time algorithm to test whether a given DBA (resp., DCA, DPA, DRA, DSA, DMA) is of type IBA (resp., ICA, IPA, IRA, ISA, IMA).

Recall that  $\mathcal{A}^q$  is the acceptor  $\mathcal{A}$  with the initial state changed to q. If  $q_1$  and  $q_2$  are two states of  $\mathcal{A}$ , testing the equivalence of  $\mathcal{A}^{q_1}$  to  $\mathcal{A}^{q_2}$  determines whether these two states have the same right congruence class, and, if not, returns a witness  $u(v)^\omega$  that is accepted from exactly one of the two states. The following is a consequence of Theorems 11.2, 12.2, and 13.6.

PROPOSITION 14.1. There is a polynomial time procedure to test whether two states of an arbitrary DBA, DCA, DPA, DRA, DSA or DMA  $\mathcal A$  have the same right congruence class, returning the answer "yes" if they do, and returning "no" and a witness  $u(v)^{\omega}$  accepted from exactly one of the states if they do not.

We now describe an algorithm **RightCon** that takes as input a DBA (or DCA, DPA, DRA, DSA, or DMA)  $\mathcal{A}$  accepting a language L and returns a deterministic automaton  $\mathcal{M}$  isomorphic to the right congruence automaton of L, i.e.,  $\mathcal{M}_{\sim L}$ .

# Algorithm 3 RightCon

```
Input: An acceptor \mathcal{A} = \langle \Sigma, Q, q_l, \delta, \alpha \rangle of type DBA, DCA, DPA, DRA, DSA, or DMA Output: A deterministic automaton \mathcal{M} isomorphic to \mathcal{M}_{\sim_L}, where L = [\![\mathcal{A}]\!] Q' \leftarrow \{\varepsilon\} q'_l \leftarrow \varepsilon \delta' is initially undefined while there exists x \in Q' and \sigma \in \Sigma such that \delta'(x, \sigma) is undefined do q_1 \leftarrow \delta(q_l, x\sigma) if there exists y \in Q' such that [\![\mathcal{A}^{q_1}]\!] = [\![\mathcal{A}^{q_2}]\!] for q_2 = \delta(q_l, y) then Define \delta'(x, \sigma) = y else Q' \leftarrow Q' \cup \{x\sigma\}Define \delta'(x, \sigma) = x\sigma return \mathcal{M} = \langle \Sigma, Q', q'_l, \delta' \rangle
```

Assume the input acceptor is  $\mathcal{A} = \langle \Sigma, Q, q_t, \delta, \alpha \rangle$ . The **RightCon** algorithm constructs a deterministic automaton  $\mathcal{M} = \langle \Sigma, Q', q'_t, \delta' \rangle$  in which the states are elements of  $\Sigma^*$  and  $q'_t = \varepsilon$ . The set Q' initially contains just  $\varepsilon$ , and  $\delta'$  is completely undefined.

While there exists a word  $x \in Q'$  and a symbol  $\sigma \in \Sigma$  such that  $\delta'(x,\sigma)$  has not yet been defined, loop through the words  $y \in Q'$  and ask whether the states  $\delta(q_t,x\sigma)$  and  $\delta(q_t,y)$  have the same right congruence class in  $\mathcal{A}$ . If so, then define  $\delta'(x,\sigma)$  to be y. If no such y is found, then the word  $x\sigma$  is added as a new state to Q', and the transition  $\delta'(x,\sigma)$  is defined to be  $x\sigma$ .

This process must terminate because the elements of Q' represent distinct right congruence classes of L, and  $\mathcal{M}_{\sim_L}$  cannot have more than |Q| states. When it terminates, the automaton  $\mathcal{M} = \langle \Sigma, Q', q'_l, \delta' \rangle$  is isomorphic to the right congruence automaton of  $\mathcal{A}$ ,  $\mathcal{M}_{\sim_L}$ .

Theorem 14.2. The RightCon algorithm with input an acceptor  $\mathcal{A}$  (a DBA, DCA, DPA, DRA, DSA, or DMA) accepting L, runs in polynomial time and returns  $\mathcal{M}$ , a deterministic automaton isomorphic to  $\mathcal{M}_{\sim L}$ ,

To test whether a given DBA (resp., DCA, DPA, DRA, DSA, DMA)  $\mathcal{A}$  is an IBA (resp., ICA, IPA, IRA, ISA, IMA), we run the **RightCon** algorithm on  $\mathcal{A}$  and test the returned automaton  $\mathcal{M}$  for isomorphism with the automaton of  $\mathcal{A}$ . If they isomorphic, then  $\mathcal{A}$  is an IBA (resp. ICA, IPA, IRA, ISA, IMA), otherwise it is not. This proves the following.

THEOREM 14.3. There is a polynomial time algorithm to test whether a given DBA (resp., DCA, DPA, DRA, DSA, DMA) is an IBA (resp., ICA, IPA, IRA, ISA, IMA).

### 15 TESTING MEMBERSHIP IN IX

In the previous section we showed that there is a polynomial time algorithm to test whether a given DBA  $\mathcal{B}$  is an IBA. However, we can also ask the following harder question. Given a DBA  $\mathcal{B}$  that is not an IBA, is  $\llbracket \mathcal{B} \rrbracket \in \mathbb{B}$ , that is, does there exist an IBA  $\mathcal{B}'$  such that  $\llbracket \mathcal{B}' \rrbracket = \llbracket \mathcal{B} \rrbracket$ ? This section shows that there are such polynomial time algorithms for DBAs, DCAs, DPAs, DRAs, DSAs, and DMAs. The algorithms first compute the right congruence automaton  $\mathcal{M} = \mathcal{M}_{\sim_L}$ , where  $L = \llbracket \mathcal{A} \rrbracket$ , and then attempt to construct an acceptance condition  $\alpha$  of the appropriate type such that  $\llbracket (\mathcal{M}, \alpha) \rrbracket = L$ .

# 15.1 Testing membership in IB

We describe the algorithm **TestInIB** that takes as input a DBA  $\mathcal{B}$  and returns an IBA accepting  $[\![\mathcal{B}]\!]$  if  $[\![\mathcal{B}]\!] \in \mathbb{B}$ , and otherwise returns "no". By Claim 2.5, the case of a DCA is reduced to that of a DBA.

THEOREM 15.1. The algorithm **TestInIB** takes a DBA  $\mathcal{B}$  as input, runs in polynomial time, and returns an IBA accepting  $[\![\mathcal{B}]\!]$  if  $[\![\mathcal{B}]\!]$   $\in$   $[\![\mathcal{B}]\!]$ , and otherwise returns "no".

PROOF. The algorithm calls the **RightCon** algorithm, and also the inclusion and equivalence algorithms from Theorem 11.3, which run in polynomial time in the size of  $\mathcal{B}$ . If the algorithm returns an acceptor, it is an IBA accepting  $[\![\mathcal{B}]\!]$ .

To see that the algorithm does not incorrectly return the answer "no", suppose  $\mathcal{B}'$  is an IBA accepting  $[\![\mathcal{B}]\!]$ . Then because  $\mathcal{M}$  is isomorphic to  $\mathcal{M}_{\sim_L}$ , we may assume that  $\mathcal{B}' = (\mathcal{M}, F')$ . For every state  $q \in F'$ , the inclusion query with  $(\mathcal{M}, \{q\})$  will answer "yes", so q will be added to F. Thus,  $F' \subseteq F$ , and  $[\![(\mathcal{M}, F)]\!]$  subsumes  $[\![(\mathcal{M}, F')]\!]$ . Every state q added to F preserves the condition that  $[\![(\mathcal{M}, F)]\!]$  is a subset of  $[\![\mathcal{B}]\!]$ , so the final equivalence check will pass, and  $(\mathcal{M}, F)$  will be returned.

# Algorithm 4 TestInIB

```
Input: A DBA \mathcal{B}
Output: If [\![\mathcal{B}]\!] \in \mathbb{B} then return an IBA accepting [\![\mathcal{B}]\!], else return "no"

\mathcal{M} \leftarrow \text{RightCon}(\mathcal{B})
F \leftarrow \emptyset

for each state q of \mathcal{M} do

if [\![(\mathcal{M}, \{q\})]\!] \subseteq [\![\mathcal{B}]\!] then

F \leftarrow F \cup \{q\}

if [\![(\mathcal{M}, F)]\!] = [\![\mathcal{B}]\!] then

return (\mathcal{M}, F)
else

return "no"
```

### 15.2 Testing membership in $\mathbb{P}$

We describe the algorithm **TestInIP** that takes as input a DPA  $\mathcal{P}$  and returns an IPA accepting  $\llbracket \mathcal{P} \rrbracket$  if  $\llbracket \mathcal{P} \rrbracket \in \mathbb{P}$ , and otherwise returns "no".

### Algorithm 5 TestInIP

```
Input: A DPA \mathcal{P}
Output: If [\![\mathcal{P}]\!] \in \mathbb{P} then return an IPA accepting [\![\mathcal{P}]\!], else return "no"
    \mathcal{M} = \langle \Sigma, Q, q_i, \delta \rangle \leftarrow \text{RightCon}(\mathcal{P})
    Define \kappa(q) = 0 for all states q \in Q
    for k = 1 to |Q| do
            if [\![(\mathcal{M},\kappa)]\!] = [\![\mathcal{P}]\!] then
                     return (\mathcal{M}, \kappa)
            else if k is odd then
                     while \llbracket \mathcal{P} \rrbracket is not a subset of \llbracket (\mathcal{M}, \kappa) \rrbracket do
                             Let u(v)^{\omega} be the returned witness
                             Define \kappa(q) = k for all q \in inf_{\mathcal{M}}(u(v)^{\omega})
            else
                     while \llbracket \mathcal{P} \rrbracket is not a superset of \llbracket (\mathcal{M}, \kappa) \rrbracket do
                             Let u(v)^{\omega} be the returned witness
                             Define \kappa(q) = k for all q \in inf_{\mathcal{M}}(u(v)^{\omega})
    return "no"
```

THEOREM 15.2. The algorithm **TestInIP** takes a DPA  $\mathcal{P}$  as input, runs in polynomial time, and returns an IPA accepting  $[\![\mathcal{P}]\!]$  if  $[\![\mathcal{P}]\!]$   $\in \mathbb{P}$ , and otherwise returns "no".

PROOF. The algorithm calls the **RightCon** algorithm and the inclusion and equivalence algorithms for DPAs from Theorem 11.2, which run in polynomial time in the size of  $\mathcal{P}$ . Below we show that each while loop terminates after at most |Q| iterations. If the algorithm returns an acceptor, then the acceptor is an IPA accepting  $[\![\mathcal{P}]\!]$ .

To see that the algorithm does not incorrectly return the answer "no", suppose  $\mathcal{P}'$  is an IPA accepting  $\llbracket \mathcal{P} \rrbracket$ . We may assume that  $\mathcal{P}' = (\mathcal{M}, \kappa^*)$ , where  $\kappa^*$  is the canonical coloring of  $\mathcal{P}'$ . We prove inductively that the final coloring  $\kappa$  is equal to  $\kappa^*$ . To do so, we consider the conditions after the for loop has been completed  $\ell$  times: (1) if  $\ell$  is even then  $\llbracket (\mathcal{M}, \kappa) \rrbracket \subseteq \llbracket \mathcal{P} \rrbracket$ , and if  $\ell$  is odd, then  $\llbracket \mathcal{P} \rrbracket \subseteq \llbracket (\mathcal{M}, \kappa) \rrbracket$ , and (2) for all  $q \in \mathcal{Q}$ , if  $\kappa^*(q) \leq \ell$  then  $\kappa(q) = \kappa^*(q)$ , and if  $\kappa^*(q) > \ell$  then  $\kappa(q) = \ell$ .

The initialization of  $\kappa(q) = 0$  for all  $q \in Q$  implies that these two conditions hold for  $\ell = 0$ . Suppose the conditions hold for some  $\ell \geq 0$ . If the equivalence check at the start of the next iteration returns "yes" then the correct IPA  $(\mathcal{M}, \kappa)$  is returned. Otherwise,  $k = \ell + 1$ ; we consider the cases of odd and even k.

If k is odd, then by condition (1),  $[\![(\mathcal{M},\kappa)]\!]\subseteq [\![\mathcal{P}]\!]$  and at least one witness  $u(v)^\omega$  accepted by  $\mathcal{P}$  and rejected by  $[\![(\mathcal{M},\kappa)]\!]$  will be processed in the while loop. Consider such a witness  $u(v)^\omega$  and let  $C=\inf_{\mathcal{M}}(u(v)^\omega)$ . Then because  $\kappa(q)=\kappa^*(q)$  if  $\kappa(q)\leq \ell$ , it must be that  $\kappa^*(C)>\ell$  and  $\kappa(C)=\ell$ . For all  $q\in C$ ,  $\kappa(q)$  is set to  $k=\ell+1$ , so at least one state changes  $\kappa$ -color from  $\ell$  to  $\ell+1$ . This can happen at most |Q| times, so the while loop for this k must terminate after at most |Q| iterations. No state q with  $\kappa^*(q)\leq \ell$  has its  $\kappa$ -value changed, so when the while loop is terminated, we have that  $\kappa^*(q)\leq \ell$  implies  $\kappa(q)=\kappa^*(q)$ .

Consider any state q with  $\kappa^*(q) \geq \ell + 1$ . By property (2), at the start of this iteration of the for loop,  $\kappa(q) = \ell$ . Referring to the canonical forest  $\mathcal{F}^*$  for  $\mathcal{P}'$ , the state q is in  $\Delta(D)$  for some node D of  $\mathcal{F}^*$ . The node D is a descendant (or possibly equal to) some node C for which the states  $q \in \Delta(C)$  all have  $\kappa^*(q) = \ell + 1$ . Thus, as long as the value of  $\kappa(q)$  remains  $\ell$ , the SCC C will have  $\kappa(C)$  even and  $\kappa^*(C)$  odd, and the while loop cannot terminate. But we have shown that it does terminate, so after termination we must have  $\kappa(q) = k = \ell + 1$ . Thus, after this iteration of the for loop, property (2) holds for  $\ell + 1$ .

The case of even k is dual to the case of odd k. Because the range of  $\kappa^*$  is [0..j] for some  $j \leq |Q|$ , the equivalence test must return "yes" before the for loop completes, at which point the IPA  $(\mathcal{M}, \kappa)$  is returned.

### 15.3 Testing membership in IR

We describe the algorithm **TestInIR** that takes as input a DRA  $\mathcal{R}$  and returns an IRA accepting  $[\![\mathcal{R}]\!]$  if  $[\![\mathcal{R}]\!] \in \mathbb{R}$ , and otherwise returns "no". By Claim 2.6, the case of a DSA is reduced to that of a DRA.

We first show that given a DRA  $\mathcal{R}$  such that  $[\![\mathcal{R}]\!] \in \mathbb{R}$ , there is an IRA equivalent to  $\mathcal{R}$  whose size is bounded by a polynomial in the size of  $\mathcal{R}$ .

LEMMA 15.3. Let  $\mathcal{R}$  be a DRA in singleton normal form whose acceptance condition has m pairs, and assume  $[\![\mathcal{R}]\!] \in \mathbb{R}$ . Let  $\mathcal{M}$  be the right congruence automaton of  $[\![\mathcal{R}]\!]$  with state set  $\mathcal{Q}$  and assume  $|\mathcal{Q}| = n$ . Then there exists an acceptance condition  $\alpha$  in singleton normal form with at most m pairs such that  $(\mathcal{M}, \alpha)$  accepts  $[\![\mathcal{R}]\!]$ .

PROOF. Let  $\mathcal{R} = (\mathcal{M}_1, \alpha_1)$ , where all the states of  $\mathcal{M}_1$  are reachable, and let the function f map each state of  $\mathcal{M}_1$  to the state of its right congruence class in  $\mathcal{M}$ . It suffices to show that for each  $(q, B) \in \alpha_1$  there exists an acceptance condition  $\alpha'$  of  $\mathcal{M}$  containing at most n pairs such that  $[\![(\mathcal{M}_1, \{(q, B)\})]\!] \subseteq [\![(\mathcal{M}, \alpha')]\!] \subseteq [\![\mathcal{R}]\!]$ . Taking the union of these  $\alpha'$  conditions for all m pairs  $(q, B) \in \alpha_1$  yields the desired acceptance condition  $\alpha$  for  $\mathcal{M}$ .

Because we assume  $[\![\mathcal{R}]\!] \in \mathbb{R}$ , there exists an IRA  $(\mathcal{M}, \alpha_2)$  in singleton normal form that accepts  $[\![\mathcal{R}]\!]$ . Given any  $u(v)^\omega$  in  $[\![\mathcal{R}]\!]$ , let  $C = \inf_{\mathcal{M}_1} (u(v)^\omega)$ . Then  $f(C) = \inf_{\mathcal{M}} (u(v)^\omega)$  and there exists  $(q', B') \in \alpha_2$  such that  $q' \in f(C)$ , and  $f(C) \cap B' = \emptyset$ . Then also  $[\![(\mathcal{M}, \{(q', Q \setminus f(C))\})]\!] \subseteq [\![\mathcal{R}]\!]$ . To see this, consider any  $u'(v')^\omega$  with  $D = \inf_{\mathcal{M}} (u'(v')^\omega)$  and  $q' \in D$  and  $D \cap (Q \setminus f(C)) = \emptyset$ . Then D is a subset of f(C),  $D \cap B' = \emptyset$ ,  $u'(v')^\omega$  satisfies (q', B'), and  $u'(v')^\omega \in [\![\mathcal{R}]\!]$ .

Given a pair  $(q, B) \in \alpha_1$  the construction of the initially empty acceptance condition  $\alpha'$  proceeds as follows. Let  $C_0$  be the maximum SCC of  $\mathcal{M}_1$  that contains q and excludes B. If  $C_0$  is empty, then  $(\mathcal{M}_1, \{(q, B)\})$  does not accept any words, and the empty condition  $\alpha'$  suffices. If  $C_0$  is nonempty, then there is an element  $u(v)^\omega$  of  $[\![(\mathcal{M}_1, \{(q, B)\}]\!]]$  such that  $C_0 = \inf_{\mathcal{M}_1} (u(v)^\omega)$  and there is a pair  $(q_0, B_0)$  in  $\alpha_2$  such that  $q_0 \in f(C_0)$  and  $B_0 \cap f(C_0) = \emptyset$ . We add the pair  $(q_0, Q \setminus f(C_0))$  to  $\alpha'$  and note that by the argument in the preceding paragraph,  $[\![(\mathcal{M}, \alpha')]\!]] \subseteq [\![\mathcal{R}]\!]$ .

If  $\llbracket (\mathcal{M}_1, \{(q,B)\} \rrbracket \subseteq \llbracket (\mathcal{M}, \alpha') \rrbracket$  then  $\alpha'$  is the desired acceptance condition. If not, there exists a word  $u(v)^\omega$  such that for  $C = \inf_{\mathcal{M}_1} (u(v)^\omega)$  we have  $q \in C$  and  $C \cap B = \emptyset$ , but either  $q_0 \notin f(C)$  or  $f(C) \cap (Q \setminus f(C_0)) \neq \emptyset$ . Because  $C_0$  is the maximum SCC of  $\mathcal{M}_1$  containing q and excluding B, we have  $C \subseteq C_0$ , so  $f(C) \subseteq f(C_0)$  and therefore  $q_0 \notin f(C)$ . Let  $C_1$  be the maximum SCC C of  $\mathcal{M}_1$  such that  $C \subseteq C_0$ ,  $q \in C$ , and  $q_0 \notin f(C)$ . This is not empty, so there is a word  $u'(v')^\omega$  such that  $C_1 = \inf_{\mathcal{M}_1} (u'(v')^\omega)$ , which is in  $\llbracket \mathcal{R} \rrbracket$  because it satisfies (q, B). Thus there exists a pair  $(q_1, B_1)$  in  $\alpha_2$  that is satisfied by  $u'(v')^\omega$ , and we add the pair  $(q_1, Q \setminus f(C_1))$  to the acceptance condition  $\alpha'$ . As above, we have  $\llbracket (\mathcal{M}, \alpha') \rrbracket \subseteq \llbracket \mathcal{R} \rrbracket$ .

If now  $[\![(\mathcal{M}_1,\{(q,B)\})]\!] \subseteq [\![(\mathcal{M},\alpha')]\!]$ , then  $\alpha'$  is the desired acceptance condition. If not, we repeat this step again. In general, after k steps of this kind,  $\alpha'$  consists of k pairs of the form  $(q_i,Q\setminus f(C_i))$  for  $i\in[0..k-1]$ , where all of the states  $q_i$  are distinct and  $C_{i+1}\subseteq C_i$  for  $i\in[0,k-2]$ . Because Q has n states, there can only be n repetitions of this step before  $\alpha'$  satisfies the required condition, and thus  $\alpha'$  has at most n pairs.

The algorithm **TestInIR** is based on the algorithm to learn Horn sentences by Angluin, Frazier, and Pitt [11], using the analogy between singleton normal form for Rabin automata and propositional Horn clauses. **TestInIR** maintains for each state q of the right congruence automaton  $\mathcal{M}$  an ordered sequence  $S_q$  of SCCs of  $\mathcal{M}$ , each of which corresponds to a positive example of  $[\![\mathcal{R}]\!]$ . At each iteration, the algorithm uses these sequences and inclusion queries with  $[\![\mathcal{R}]\!]$  to construct an acceptance condition  $\alpha$  for a hypothesis  $(\mathcal{M}, \alpha)$ , which it tests for equivalence to  $\mathcal{R}$ . In the case of non-equivalence, the witness is a positive example of  $[\![\mathcal{R}]\!]$  that is used to update the sequences  $S_q$ .

In **TestInIR** the test of whether  $C \cup C_i$  is positive is implemented by calling **Witness** $(C \cup C_i)$  and testing the resulting word  $u(v)^{\omega}$  for membership in  $\llbracket \mathcal{R} \rrbracket$ .

Because pairs are only added to  $\alpha$  that preserve inclusion in  $[\![\mathcal{R}]\!]$ , it is clear that any witness  $u(v)^{\omega}$  returned in response to the test of equivalence of  $(\mathcal{M}, \alpha)$  and  $\mathcal{R}$  is a positive example of  $[\![\mathcal{R}]\!]$ . Note also that all elements of  $S_q$  are SCCs of  $\mathcal{M}$  that contain q. The proof of correctness and running time of **TestInIR** depends on the following two lemmas.

LEMMA 15.4. Assume that  $\mathcal{R}'$  is an IRA equivalent to the target DRA  $\mathcal{R}$ . Consider a positive example  $u(v)^{\omega}$  of  $[\![\mathcal{R}]\!]$  returned in response to the test of equivalence of  $(\mathcal{M}, \alpha)$  and  $\mathcal{R}$ , and let  $C = \inf_{\mathcal{M}} (u(v)^{\omega})$ . Let (q', B) be a pair of  $\mathcal{R}'$  such that  $q' \in C$  and  $C \cap B = \emptyset$ . If for some  $q \in C$  and some  $C_i$  in  $S_q$  we have  $C_i \cap B = \emptyset$  then for some  $j \leq i$ , the element  $C_j$  of  $S_q$  will be replaced by  $C_j \cup C$ .

PROOF. Assume that there is no such replacement for j < i. When i is considered, we have  $q \in C_i$  and  $q \in C$ , so  $C_i \cup C$  is the union of overlapping SCCs and therefore an SCC. Then  $C_i \cup C$  is positive because  $q' \in C$ , so  $q' \in C_i \cup C$ , and  $C \cap B = \emptyset$  and  $C_i \cap B = \emptyset$  by hypothesis, so  $(C_i \cup C) \cap B = \emptyset$  and the word **Witness** $(C_i \cup C)$  is a positive example of  $[\![R]\!]$  because it satisfies (q', B).

To see that  $C \nsubseteq C_i$ , we assume to the contrary. Then  $(q',Q \setminus C_i)$  is an element of  $\alpha$ . To see this, we show that  $[\![(\mathcal{M},\{(q',Q\setminus C_i)\}]\!]\subseteq [\![\mathcal{R}]\!]$ . Let  $u'(v')^\omega$  with  $D=\inf_{\mathcal{M}}(u'(v')^\omega)$  satisfy  $(q',Q\setminus C_i)$ . Then  $q'\in D$  and  $D\cap (Q\setminus C_i)=\emptyset$ , and therefore  $D\subseteq C_i$  and  $C_i\cap B=\emptyset$  by hypothesis. Thus  $u'(v')^\omega$  satisfies (q',B) and is in  $[\![\mathcal{R}]\!]$ . Because  $(q',Q\setminus C_i)$  is an element of  $\alpha$ ,  $u(v)^\omega$  is accepted by  $(\mathcal{M},\alpha)$  because  $q'\in C$  and  $C\cap (Q\setminus C_i)=\emptyset$  (because we assume  $C\subseteq C_i$ ). But this means that  $u(v)^\omega$  cannot be a witness to the non-equivalence of  $(\mathcal{M},\alpha)$  and  $\mathcal{R}$ , a contradiction.

Thus, the conditions for  $C_i$  to be replaced by  $C_i \cup C$  are satisfied.

LEMMA 15.5. Assume that  $\mathcal{R}' = (\mathcal{M}, \alpha')$  is an IRA in singleton normal form equivalent to the target DRA  $\mathcal{R}$ . The following two conditions hold throughout the algorithm **TestInIR**.

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# Algorithm 6 TestInIR

```
Input: A DRA \mathcal{R} = (\mathcal{M}_1, \alpha_1) in singleton normal form with |\alpha_1| = m
Output: If [\![\mathcal{R}]\!] \in \mathbb{R} then return an IRA accepting [\![\mathcal{R}]\!], else return "no"
   \mathcal{M} \leftarrow \text{RightCon}(\mathcal{R})
   Let Q be the states of \mathcal{M} and n = |Q|
   For each q \in Q initialize a sequence S_q to be empty
   for k = 1, mn^3 do
           for all q \in Q do
                  \alpha_q = \emptyset
                  for all C \in S_q do
                          for all q' \in C do
                                 if \llbracket (\mathcal{M}, \{(q', Q \setminus C)\} \rrbracket \subseteq \llbracket \mathcal{R} \rrbracket then
                                         \alpha_q = \alpha_q \cup \{(q', Q \setminus C)\}\
           \alpha \leftarrow \bigcup_{q \in Q} \alpha_q
           if [\![(\mathcal{M},\alpha)]\!] = [\![\mathcal{R}]\!] then
                  return (\mathcal{M}, \alpha)
           else
                  Let u(v)^{\omega} be the witness returned
                  Let C = inf_{\mathcal{M}}(u(v)^{\omega})
                  for all q \in C do
                         if there is some C_i \in S_q such that C \not\subseteq C_i and C \cup C_i is positive then
                                 Let i be the least such i and replace C_i by C_i \cup C
                                 Add C to the end of the sequence S_q
   return "no"
```

- (1) For all  $q \in Q$ , elements  $C_i$  of  $S_q$ , and  $(q', B) \in \alpha'$ , if  $C_i$  satisfies (q', B) then for no j < i do we have  $C_j \cap B = \emptyset$ .
- (2) For all  $q \in Q$ , elements  $C_i$  and  $C_j$  of  $S_q$  with j < i, and  $(q', B) \in \alpha'$ , if  $C_i$  satisfies (q', B) then  $C_j$  does not satisfy (q', B).

PROOF. We first show that condition (1) implies condition (2). Let  $q \in Q$ ,  $C_i$  and  $C_j$  be in  $S_q$  with j < i and  $(q', B) \in \alpha'$ . If  $C_i$  satisfies (q', B) then by condition (1), we have  $C_j \cap B \neq \emptyset$ , so  $C_j$  does not satisfy (q', B).

We now prove condition (1) by induction on the number of witnesses to non-equivalence. The condition holds of the empty sequences  $S_q$ . Suppose conditions (1) and (2) hold of the sequences  $S_q$ , and the witness to non-equivalence is  $u(v)^\omega$  with  $C = \inf_{\mathcal{M}} (u(v)^\omega)$ . If  $q \notin C$  then  $S_q$  is not modified, so assume  $q \in C$ . We consider two cases, depending on whether C is added to the end of  $S_q$  or causes some  $C_\ell$  to be replaced by  $C_\ell \cup C$ .

Assume that C is added to the end of  $S_q$  and property (1) fails to hold. Then it must be that for  $C_i = C$ , some  $C_j$  in  $S_q$  with j < i and some  $(q', B) \in \alpha'$ ,  $C_i$  satisfies (q', B) and  $C_j \cap B = \emptyset$ . By Lemma 15.4, because  $C_j \cap B = \emptyset$ , C should cause  $C_\ell$  for some  $\ell \le j$  to be replaced by  $C \cup C_\ell$  rather than being added to the end of  $S_q$ , a contradiction.

Assume that C causes  $C_{\ell}$  in  $S_q$  to be replaced by  $C_{\ell} \cup C$  and property (1) fails to hold. Then it must be for some  $C_i$  in  $S_q$  and pair (q', B) in  $\alpha'$ , either (i)  $i > \ell$  and  $C_i$  satisfies (q', B) and  $(C_{\ell} \cup C) \cap B = \emptyset$ , or (ii)  $i < \ell$  and  $C_{\ell} \cup C$  satisfies (q', B) and  $C_i \cap B = \emptyset$ . In case (i), it must be that  $C_i$  satisfies (q', B) and  $C_{\ell} \cap B = \emptyset$ , which contradicts the assumption that property (1) holds before C is processed, because  $\ell < i$ . In case (ii),  $q' \in C_{\ell} \cup C$  and  $(C_{\ell} \cup C) \cap B = \emptyset$ . Thus  $C \cap B = \emptyset$  and  $C_{\ell} \cap B = \emptyset$ . If  $q' \in C_{\ell}$ , then  $C_{\ell}$  satisfies (q', B), violating the assumption that property (2) holds before C is processed. If  $q' \in C$  then C satisfies (q', B) and because  $C_i \cap B = \emptyset$ , by Lemma 15.4, for some  $j \leq i$  we have Manuscript submitted to ACM

 $C_j$  replaced by  $C_j \cup C$ , a contradiction because  $\ell > i$ . Thus, in either case property (1) holds after  $C_\ell$  is replaced by  $C_\ell \cup C$ .

THEOREM 15.6. The algorithm **TestInIR** takes as input a DRA  $\mathcal{R}$ , runs in polynomial time, and returns an IRA accepting  $\|\mathcal{R}\|$  if  $\|\mathcal{R}\| \in \mathbb{R}$ , and otherwise returns "no".

PROOF. Assume the input is DRA  $\mathcal{R} = (\mathcal{M}_1, \alpha_1)$  in singleton normal form with  $|\alpha_1| = m$ . The algorithm **TestInIR** computes the right congruence automaton  $\mathcal{M}$  of  $[\![\mathcal{R}]\!]$ , which has n states, at most the number of states of  $\mathcal{R}$ . The main loop of the algorithm is executed at most  $mn^3$  times, and each execution makes calls to the inclusion and equivalence algorithms for DRAs, and runs in time polynomial in the size of  $\mathcal{R}$ , so the overall running time of **TestInIR** is polynomial in the size of  $\mathcal{R}$ .

Clearly, if  $[\![R]\!] \notin [\![R]\!]$ , then the test of equivalence between  $(\mathcal{M}, \alpha)$  and  $\mathcal{R}$  will not succeed, and the value returned will be "no". Assume that  $[\![R]\!] \in [\![R]\!]$ . Then by Lemma 15.3, there is an IRA  $\mathcal{R}' = (\mathcal{M}, \alpha')$  in singleton normal form equivalent to  $\mathcal{R}$  such that  $|\alpha'| \leq mn$ . For every state  $q \in Q$ , each member of the sequence  $S_q$  satisfies some pair (q', B) in  $\alpha'$ , and by Lemma 15.5, no two members of  $S_q$  can satisfy the same pair, so the length of each  $S_q$  is bounded by mn. Each positive counterexample must either add another member to at least one sequence  $S_q$  or cause at least one member of some sequence  $S_q$  to increase in cardinality by 1. The maximum cardinality of any member of any  $S_q$  is n, and the total number of sequences  $S_q$  is n, so no more than  $mn^3$  positive counterexamples can be processed before the test of equivalence between  $(\mathcal{M}, \alpha)$  and  $\mathcal{R}$  succeeds and  $(\mathcal{M}, \alpha)$  is returned.

# 15.4 Testing membership in IM

We describe the algorithm **TestInIM** that takes as input a DMA  $\mathcal{U}$  and returns an IMA accepting  $\llbracket \mathcal{U} \rrbracket$  if  $\llbracket \mathcal{U} \rrbracket \in \mathbb{IM}$ , and otherwise returns "no".

# Algorithm 7 TestInIM

```
Input: A DMA \mathcal{U} = \langle \Sigma, Q, q_i, \delta, \mathcal{F} \rangle
Output: If [\![\mathcal{U}]\!] \in \mathbb{I}\!M then return an IMA accepting [\![\mathcal{U}]\!], else return "no"

\mathcal{M} \leftarrow \text{RightCon}(\mathcal{U})
\mathcal{F}' \leftarrow \emptyset
while [\![(\mathcal{M}, \mathcal{F}')]\!] \neq [\![\mathcal{U}]\!] do

Let u(v)^\omega be the witness returned

Let C = \inf_{\mathcal{M}} (u(v)^\omega)
if u(v)^\omega \in [\![\mathcal{U}]\!] then

\mathcal{F}' \leftarrow \mathcal{F}' \cup \{C\}
else

return "no"

return (\mathcal{M}, \mathcal{F}')
```

THEOREM 15.7. The algorithm TestInIM takes as input a DMA  $\mathcal{U}$ , runs in polynomial time, and returns an IMA accepting  $[\![\mathcal{U}]\!]$  if  $[\![\mathcal{U}]\!]$   $\in$   $[\![\mathcal{M}]\!]$ , and otherwise returns "no".

PROOF. The algorithm calls the **RightCon** algorithm and also the DMA equivalence algorithm from Theorem 13.6, which run in polynomial time. When there is a witness  $u(v)^{\omega}$  accepted by  $\mathcal{U}$ , there is a set  $F \in \mathcal{F}$  whose image in  $\mathcal{M}$  is added to  $\mathcal{F}'$ , so there can be no more such witnesses than the number of sets in  $\mathcal{F}$ . After this, there must be a Manuscript submitted to ACM

successful equivalence test or a witness rejected by  $\mathcal{U}$ , either of which terminates the while loop. Thus, the overall running time is polynomial in the size of  $\mathcal{U}$ .

If the algorithm returns an acceptor  $(\mathcal{M}, \mathcal{F}')$ , then the acceptor is an IBA that accepts  $[\![\mathcal{U}]\!]$ . To see that the algorithm does not incorrectly return the answer "no", assume that  $\mathcal{U}'$  is an IBA accepting  $[\![\mathcal{U}]\!]$ . We may assume that  $\mathcal{U}' = (\mathcal{M}, \mathcal{F}'')$ , where  $\mathcal{F}''$  contains no redundant sets. Then the first witness will be a word accepted by  $\mathcal{U}$  that will add an element of  $\mathcal{F}''$  to  $\mathcal{F}'$ . This continues until all the elements of  $\mathcal{F}''$  have been added to  $\mathcal{F}'$ , at which point the while loop terminates with equivalence.

# 15.5 Variants of the testing algorithms

A variant of the task considered above is the following. Given the right congruence automaton  $\mathcal{M}$  of a language  $[\![\mathcal{B}]\!]$  in  $\mathbb{B}$ , and access to information from certain queries about  $[\![\mathcal{B}]\!]$ , learn an acceptance condition  $\alpha$  such that  $(\mathcal{M}, \alpha)$  accepts  $[\![\mathcal{B}]\!]$ . In the case of  $[\![\mathcal{B}]\!]$ , the algorithm **TestInIB** could be modified to perform this task using just equivalence queries with respect to  $[\![\mathcal{B}]\!]$ . Similarly, equivalence queries would suffice in the case of  $[\![\mathcal{M}]\!]$ . For  $[\![\mathcal{B}]\!]$ , subset and superset queries with respect to the target language would suffice. And for  $[\![\mathcal{R}]\!]$ , **TestInIR** could be modified to use membership and equivalence queries with respect to the target language, relying on negative examples to remove incorrect pairs rather than using subset queries.

#### 16 DISCUSSION

We have shown that the non-deterministic classes of  $\omega$ -automata NBA, NCA, NPA, NRA, NSA, and NMA cannot be identified in the limit using polynomial data. A negative result regarding query learning of NBA, NPA, and NMA was obtained by Angluin and Fisman [4]. That result makes a plausible assumption of cryptographic hardness, which is not required here. On the positive side we have shown that the classes IB, IC, IP, IR, IS, and IM can be identified in the limit using polynomial time and data. Moreover, we have shown that in each case a characteristic sample can be constructed in polynomial time. We have given new polynomial time algorithms to test inclusion and equivalence for DBAs, DCAs, DPAs, DRAs, DSAs, and DMAs. We have given a polynomial time algorithm to compute the right congruence automaton  $\mathcal{M}_{\sim L}$  for a language L specified by a DBA, DCA, DPA, DRA, DSA, or DMA. This yields a polynomial time algorithm to test whether an acceptor  $\mathcal{A}$  of type DBA is of type IBA, and similarly for acceptors of types DCA, DPA, DRA, DSA, and DMA. Moreover, we have given a polynomial time algorithm to test whether an acceptor  $\mathcal{A}$  of type DBA accepts a language in the class IB, and similarly for acceptors of types DCA, DPA, DRA, DSA and DMA.

The question of whether the full deterministic classes  $\mathbb{DBA}$ ,  $\mathbb{DCA}$ ,  $\mathbb{DPA}$ ,  $\mathbb{DRA}$ , and  $\mathbb{DMA}$  can be learned in the limit using polynomial time and data remains open. Another intriguing open question is whether the classes  $\mathbb{IB}$ ,  $\mathbb{IC}$ ,  $\mathbb{IP}$ ,  $\mathbb{IR}$ ,  $\mathbb{IS}$ , and  $\mathbb{IM}$  can be learned by polynomial time algorithms using membership and equivalence queries. However, this question is not easier than whether the corresponding deterministic classes can be learned by polynomial time algorithms using membership and equivalence queries, by the result of Bohn and Löding [12].

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