Regular Separability in Büchi VASS

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- Abstract

We study the $(\omega$ -)regular separability problem for Büchi VASS languages: Given two Büchi VASS with languages L_1 and L_2 , check whether there is a regular language that fully contains L_1 while remaining disjoint from L_2 . We show that the problem is decidable in general and PSPACE-complete in the 1-dimensional case, assuming succinct counter updates. The results rely on several arguments. We characterize the set of all regular languages disjoint from L_2 . Based on this, we derive a (sound and complete) notion of inseparability witnesses, non-regular subsets of L_1 . Finally, we show how to symbolically represent inseparability witnesses and how to check their existence.

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1 Introduction

The separability problem asks, given languages L_1 and L_2 , whether there exists a language R that separates L_1 and L_2 , meaning $L_1 \subseteq R$ and $R \cap L_2 = \emptyset$. Here, R is constrained to be from a particular class S of admitted separators. Since safety verification of systems with concurrent components is usually phrased as an intersection problem for finite-word languages, and separators certify disjointness, deciding separability can be viewed as synthesizing safety certificates. Analogously, deciding separability for infinite-word languages is a way of certifying liveness. If S is the class of $(\omega$ -)regular languages, we speak of regular separability.

Separability problems have been studied intensively over the last few years. If the input languages are themselves regular and S is a subclass [42, 41, 40, 39, 43, 44, 35, 15], then separability generalizes the classical subclass membership problem. Moreover, separability for languages of infinite-state systems has received a significant amount of attention [17, 16, 14, 13, 10, 9, 12, 1, 51, 48, 11, 8]. Let us point out two prominent cases.

First, one of the main open problems in this line of research is whether regular separability is decidable for (reachability) languages of vector addition systems with states (VASS): A VASS consist of finitely many control states and a set of counters that can be incremented and decremented, but not tested for zero. Moreover, each transition is labeled by a word over the input alphabet. Here, a run is accepting if it reaches a final state with all counters being zero. While there have been several decidability results for subclasses of the VASS languages [17, 14, 13, 10, 9], the general case remains open. Second, a surprising result is that if K and L are coverability languages of well-structured transition systems (WSTS), then K and L are separable by a regular language if and only if they are disjoint [14]. As VASS are one example of WSTS, this result also applies to their coverability languages.

Regular separability in Büchi VASS In this paper, we study the regular separability problem for Büchi VASS. These are VASS that accept languages of *infinite words*. A run is accepting if it visits some final state infinitely often. Since no condition is placed on the counter values, Büchi VASS languages are an infinite-word analogue of finite-word coverability languages, where acceptance is defined by the reached state (not the counters). The regular separability problem is to decide, given Büchi VASS \mathcal{V}_1 and \mathcal{V}_2 , whether there exists an ω -regular language R such that $L(\mathcal{V}_1) \subseteq R$ and $L(\mathcal{V}_2) \cap R = \emptyset$.

Our main results are that (i) regular separability for Büchi VASS is decidable, and that (ii) for one-dimensional Büchi VASS (i.e. those with a single counter) the problem is PSPACE-complete. Here, we assume that the counter updates are encoded in binary.

Given that Büchi VASS accept using final states and their transition systems are WSTS, one may suspect that there is an analogue of the aforementioned result for WSTS: Namely, that two languages of Büchi VASS are separable by an ω -regular language if and only if they are disjoint. We show that this is not the case: There are Büchi VASS \mathcal{V}_1 and \mathcal{V}_2 such that $L(\mathcal{V}_1)$ and $L(\mathcal{V}_2)$ are disjoint, but not separable by an ω -regular language. In fact, we show an even larger disparity between these two problems for WSTS in the infinite-word case: We exhibit a natural class of WSTS for which intersection is decidable but regular separability is not. Thus, regular separability for Büchi VASS requires significantly new ideas and involves several phenomena that do not occur for finite-word languages of VASS.

New phenomena and key ingredients We first observe that we can assume one input language to be fixed, namely an infinite-word version D_n of the Dyck language. Then, following the *basic separator* approach from [17], we identify a small class \mathcal{B} of ω -regular

languages such that L is separable from D_n if and only if L is included in a finite union of sets from \mathcal{B} . Here, a crucial insight is that a Büchi automaton can guarantee disjointness from D_n without knowing exactly when the letter balance crosses zero. Note that a negative letter balance is the exact condition for non-membership in D_n . In contrast, in the finite word case, there are always separating automata that can tell when zero is crossed [17]. This insight is also key to the example differentiating disjointness and separability in Büchi VASS, and to the undecidability proof for certain WSTS despite decidable disjointness.

We then develop a decomposition of Büchi VASS languages into *finitely many* pieces, which are induced by what we call *profiles*. Inspired by Büchi automata, the idea of a profile is to fix the set of transitions that can and have to be taken infinitely often in a run. Finding the right generalization to Büchi VASS, however, turned out to be non-trivial. Our formulation refers to edges in the Karp-Miller graph, augmented by constraints that guarantee the existence of an accepting run. The resulting decomposition has properties similar to the decomposition of VASS languages into run ideals [33], which has been useful for previous separability procedures [17, 12].

We associate to each profile a system of linear inequalities and show that separability holds if and only if each of these systems is feasible. While this yields decidability, checking feasibility is not sufficient to obtain a PSPACE-upper bound in the one-dimensional case. Instead, we use Farkas' Lemma to obtain a dual system of inequalities so that separability fails if and only if one dual system is satisfiable. A solution to a dual system yields a pattern in the Karp-Miller graph, called *inseparability flower*, which witnesses inseparability. Compared to prior witnesses for deciding properties of VASS languages (e.g. regularity [18], language boundedness [7], and other properties [3]), inseparability flowers are quite unusual: they contain a non-linear condition, requiring one vector to be a scalar multiple of another.

For one-dimensional Büchi VASS, the condition degenerates into a linear one. This allows us to translate inseparability flowers into particular runs in a two-dimensional VASS subject to additional linear constraints. Using methods from [5], this yields a PSPACE procedure.

Related work It was already shown in 1976 that regular separability is undecidable for context-free languages [47, 30]. Over the last decade, there has been intense interest in deciding regular separability for subclasses of finite-word VASS reachability languages: The problem is decidable for (i) reachability languages of one-dimensional VASS [13], (ii) coverability languages of VASS [14], (iii) reachability languages of Parikh automata [9], and (iv) commutative reachability languages of VASS [10]. Moreover, decidability still holds if one input language is an arbitrary VASS language and the other is as in (i)-(iii) [17]. As discussed above, for finite-word coverability languages of WSTS, regular separability is equivalent to disjointness [14]. Moreover, the aforementioned undecidability for context-free languages has been strengthened to visibly pushdown languages [32]. To our knowledge, for languages of infinite words, separability has only been studied for regular input languages [38, 29].

Our result makes use of Farkas' Lemma to demonstrate the absence of what can be understood as a linear ranking function (on letter balances). There are precursors to this. In liveness verification [45], Farkas' Lemma has been used to synthesize, in a complete way, linear ranking functions proving the termination of while programs over integer variables. In the context of separability for finite words, Farkas' Lemma was used to distinguish separable from non-separable instances [17], similar to our approach. The novelty here is the combination of Farkas' Lemma with the new notion of profiles needed to deal with infinite runs.

The languages of Büchi VASS have first been studied by Valk [49] and (in the deterministic case) Carstensen [6]. Some complexity results (such as EXPSPACE-complexity of the

4 Regular Separability in Büchi VASS

emptiness problem) were shown by Habermehl [28]. More recently, there have been several papers on the topological complexity of Büchi VASS languages (and restrictions) [26, 20, 27]. See the recent article by Finkel and Skrzypczak [27] for an overview.

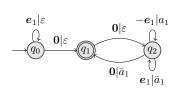
2 Preliminaries

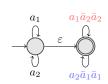
Dyck Language We use an infinite-word version of the Dyck language over n pairs of matching letters a_i, \bar{a}_i . We denote the underlying alphabet by $\Sigma_n := \bigcup_{i=1}^n \{a_i, \bar{a}_i\}$. The Dyck language contains those infinite words where every occurrence of \bar{a}_i has a matching occurrence of a_i to its left: $D_n := \{w \in \Sigma_n^\omega \mid \forall v \in \operatorname{prefix}(w) : \forall i \in [1, n] : \varphi_i(v) \geq 0\}$. Here, $\varphi_i : \Sigma_n^* \to \mathbb{Z}$ is the ith (letter) balance function that computes for a given word w the difference $|w|_{a_i} - |w|_{\bar{a}_i}$. We also use $\varphi(w)$ for the vector $(\varphi_1(w), \ldots, \varphi_n(w)) \in \mathbb{Z}^n$.

Büchi VASS and Automata A Büchi vector addition system with states (Büchi VASS) of dimension $d \in \mathbb{N}$ over alphabet Σ is a tuple $\mathcal{V} = (Q, q_0, T, F)$ consisting of a finite set of states Q, an initial state $q_0 \in Q$, a set of final states $F \subseteq Q$, and a finite set of transitions $T \subseteq Q \times \Sigma^* \times \mathbb{Z}^d \times Q$. The size of the Büchi VASS is $|\mathcal{V}| := |Q| + 1 + |F| + \sum_{(q, w, \delta, q') \in T} |w| + \sum_{i=1}^d \max\{\log |\delta(i)|, 1\}$. If d = 0, we call \mathcal{V} a Büchi automaton.

The semantics of the Büchi VASS is defined over configurations, which are elements of $Q \times \mathbb{N}^d$. We call the second component in a configuration the counter valuation and refer to the entry in dimension i as the value of counter i. The initial configuration is $(q_0, \mathbf{0})$. We lift the transitions of the Büchi VASS to a relation over configurations $\to \subseteq Q \times \mathbb{N}^d \times \Sigma^* \times Q \times \mathbb{N}^d$ as follows: $(q, \boldsymbol{m}) \xrightarrow{w} (q', \boldsymbol{m}')$ if there is $(q, w, \delta, q') \in T$ so that $\boldsymbol{m}' = \boldsymbol{m} + \delta$. A run of the Büchi VASS is an infinite sequence of transitions of the form $(q_0, \mathbf{0}) \xrightarrow{w_1} (q_1, \boldsymbol{m}_1) \xrightarrow{w_2} \cdots$ Thus, the sequence starts in the initial configuration and makes sure the target of one transition is the source of the next. The run is accepting if it visits final states infinitely often, meaning there are infinitely many configurations (q, \boldsymbol{m}) with $q \in F$. The run is said to be labeled by the word $w = w_0 w_1 \cdots$ in Σ^ω . The language $L(\mathcal{V})$ of the Büchi VASS consists of all infinite words that label an accepting run. Note that we can always ensure that every accepting run has an infinite-word label, by tracking in the state whether a non- ε -transition has occurred since the last visit to a final state. An infinite-word language is (ω) -regular, if it is the language of a Büchi automaton. As we only consider infinite-word languages, we just call them languages.

Karp-Miller Graphs We work with the Karp-Miller graph KM(\mathcal{V}) associated with a Büchi VASS \mathcal{V} [31]. Since we are interested in infinite runs, we define the Karp-Miller graph as a Büchi automaton. Its state set is a finite set of extended configurations, which are elements of $Q \times (\mathbb{N} \cup \{\omega\})^d$. The initial state is the initial configuration in the Büchi VASS. The final states are those extended configurations (q, m) with $q \in F$. The transitions are labeled by T, so instead of letters they carry full Büchi VASS transitions. An entry ω in an extended configuration denotes the fact that a prefix of a run can be repeated to produce arbitrarily high counter values. More precisely, the Karp-Miller graph is constructed as follows. From an extended configuration (q, m) we have a transition labeled by (q_1, a, δ, q_2) , if $q = q_1$ and $m + \delta$ remains non-negative. The latter addition is defined componentwise and assumes $\omega + k := \omega =: k + \omega$ for all $k \in \mathbb{Z}$. The result of taking the transition is the extended configuration (q_2, m_2) , where m_2 is constructed from $m + \delta$ as follows. We raise to ω all counters i for which there is an earlier configuration (q_2, m_1) with $m_1 \leq m + \delta$ and $m_1(i) < [m + \delta](i)$, earlier meaning on some path from $(q_0, \mathbf{0})$ to (q, m). If this is the case,





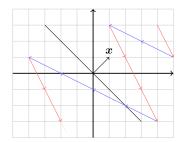


Figure 1 Left: A Büchi VASS accepting a language S with $S \cap D_1 = \emptyset$ but $S \not\mid D_1$. Here, $e_1 \in \mathbb{Z}$ is the one-dim. vector with entry 1. Right: A regular language that is not included in a finite union of languages $P_{i,k}$ and $S_{i,k}$, but that is included in $S_{x,k}$ for x = (1,1), k = 1. The horizontal and vertical dimensions denote the balance for a_1 resp. a_2 .

the path from (q_2, \mathbf{m}_1) to $(q_2, \mathbf{m} + \delta)$ can be repeated indefinitely to produce arbitrarily high values for counter *i*. We refer to the repetition of such a path in a run as *pumping*.

The Karp-Miller graph over-approximates the language of the Büchi VASS in the following sense. Every infinite sequence of transitions that leads to a run of the Büchi VASS is the labeling of an infinite run in the Karp-Miller graph. Moreover, if the run of the Büchi VASS is accepting, so is the run in the Karp-Miller graph. In the other direction, every finite transition sequence in the Karp-Miller graph represents a transition sequence in the Büchi VASS. The sequence in the Büchi VASS, however, may be longer to compensate negative effects on ω -entries by pumping.

3 Problem, Main Result, and Proof Outline

A language R is a regular separator for a pair of languages L_1, L_2 , if R is regular, $L_1 \subseteq R$, and $R \cap L_2 = \emptyset$. We write $L_1 \mid L_2$ for the fact that a regular separator exists. We consider here languages of Büchi VASS, and formulate the regular separability problem as follows. Given Büchi VASS \mathcal{V}_1 , \mathcal{V}_2 , check whether $L(\mathcal{V}_1) \mid L(\mathcal{V}_2)$ holds. Our main result is the following.

▶ **Theorem 3.1.** The regular separability problem for Büchi VASS is decidable.

It should be noted that our procedure is non-primitive recursive, as it explicitly constructs the Karp-Miller graph of an input Büchi VASS, which can be of Ackermannian size [36, Theorem 2]. In the case of VASS coverability languages (and even for more general WSTS), it is known that regular separability is equivalent to disjointness [14]. Thus, for finite words, separability reduces to the much better understood problem of disjointness. For the infinite-word languages considered here, the situation is different.

▶ Theorem 3.2. There are Büchi VASS languages L_1 , L_2 with $L_1 \cap L_2 = \emptyset$ and $L_1 \not\mid L_2$. There are classes of WSTS where intersection is decidable but separability is not.

For the second statement, we introduce the class of weak Büchi reset VASS, which are VASS with reset instructions, with the additional constraint that each run can only use resets a finite number of times. Details can be found in Appendix F.

For the first statement of Theorem 3.2, we give an intuition and refer to Appendix A for details. We choose $L_1 = L(\mathcal{V})$, where \mathcal{V} is the Büchi VASS in Figure 1(left), and $L_2 = D_1$, the Dyck language. To show $L(\mathcal{V}) \not\mid D_1$, suppose there is a Büchi automaton \mathcal{A} with n states such that $L(\mathcal{V}) \subseteq L(\mathcal{A})$ and $L(\mathcal{A}) \cap D_1 = \emptyset$. Then \mathcal{A} has to accept $(a_1^n \bar{a}_1^{n+1})^{\omega} \in L(\mathcal{V})$. However, pumping yields that for some m > n the word $(a_1^m \bar{a}_1^{n+1})^{\omega} \in D_1$ also has to be

accepted by \mathcal{A} , contradiction. Moreover, to show $L(\mathcal{V}) \cap D_1 = \emptyset$ we observe that in accepting runs of \mathcal{V} , almost every visit (meaning: all but finitely many) to the final state drops the letter balance by 1. Therefore on any accepting run this balance eventually becomes negative, yielding a word outside of D_1 .

In the remainder of the section, we outline the proof of Theorem 3.1. Assume we are given $L_1 = L(\mathcal{V}_1)$ and $L_2 = L(\mathcal{V}_2)$ and this is a non-trivial instance of separability, meaning L_1, L_2 are not regular and $L_1 \cap L_2 = \emptyset$. For proving separability, we could enumerate regular languages until we find a separator. The difficult part is disproving separability. Inseparability of L_1 and L_2 is witnessed by a set of words $W \subseteq L_1$ so that every regular language R containing them already intersects L_2 , formally: $W \subseteq R$ implies $R \cap L_2 \neq \emptyset$. Showing the existence of such a set W is difficult for two reasons. First, it is unclear which sets of words ensure the universal quantification over all regular languages. Second, as we have a non-trivial instance of separability, W (if it exists) will be a non-regular language. So it is unclear how to represent it in a finite way and how to check its existence.

To address the first problem and understand the sets of words that disprove separability, we use diagonalization. Call an $(L_2$ -)separator candidate a regular language that is disjoint from L_2 . Let R_1, R_2, \ldots be an enumeration of the separator candidates. If L_1 is not separable from L_2 , for every R_i there is a word $w_i \in L_1$ with $w_i \notin R_i$. We call such a set of words $W = \{w_1, w_2, \ldots\}$ that escapes every separator candidate an inseparability witness.

▶ **Observation 3.3.** $L_1 \not\mid L_2$ if and only if there is an inseparability witness.

Our decision procedure will check the existence of an inseparability witness. We obtain the procedure in four steps: the first is a simplification, the second is devoted to understanding the separator candidates, the third is another simplification, and the last characterizes the inseparability witnesses and checks their existence.

- **Step 1: Fixing L_2** We first reduce general regular separability to regular separability from the Dyck language. The reduction is simple and works just as for finite words [17].
- ▶ Lemma 3.4. Given Büchi VASS V_1 and V_2 , we can compute a Büchi VASS V over Σ_n so that $L(\mathcal{V}_1) \mid L(\mathcal{V}_2)$ if and only if $L(\mathcal{V}) \mid D_n$, where n is the dimension of \mathcal{V}_2 .
- Step 2: Understanding the Separator Candidates To understand the regular languages that are disjoint from D_n , we will define basic separators, sets $P_{i,k}$ and $S_{x,k}$, on which we elaborate in a moment. The following theorem says that finite unions of basic separators are sufficient for regular separability. This is our first technical result and shown in Section 4.
- ▶ **Theorem 3.5.** If $R \subseteq \Sigma_n^{\omega}$ is regular and $R \cap D_n = \emptyset$, then R is included in a finite union of basic separators.

For the definition of $P_{i,k}$, we note that the words outside D_n have, for some index $i \in [1, n]$, an earliest moment in time where the balance between a_i and \bar{a}_i falls below zero. To turn this into a regular language, we impose an upper bound $k \in \mathbb{N}$ on the (positive) balance between the letters a_i and \bar{a}_i that is maintained until the earliest moment is reached. This yields the regular language

$$P_{i,k} \ := \ \{w \in \Sigma_n^\omega \mid \exists v \in \operatorname{prefix}(w) \colon \varphi_i(v) < 0 \land \forall u \in \operatorname{prefix}(v) \colon \varphi_i(u) \leq k\}.$$

The family of languages $P_{i,k}$ already captures the complement of D_n . The problem is that we may need infinitely many such languages to cover the language R of interest. For every bound k, a regular R with $R \cap D_1 = \emptyset$ may contain a word with a higher balance before falling below zero, take for example $R = a_1^* \bar{a}_1^w$. The first insight is that if R can fall below zero from arbitrarily high values, then the underlying Büchi automaton has to contain loops with a negative balance. The R thus contains words uv with an unconstrained prefix and a suffix that decomposes into $v = v_1 v_2 \cdots$ so that every infix $w = v_\ell$ has a negative balance on letter a_i . The observation suggests the definition of a language that contains precisely the words u.v. To make the language regular, we impose a bound k on the positive balance that can be used during the infixes w. Call the resulting language $S_{i,k}$. Unfortunately, taking the $P_{i,k}$ and the $S_{i,k}$ as basic separators is still not enough: Figure 1(right) exhibits a regular language, disjoint from D_1 , that is not included in a finite union of $P_{i,k}$ and $S_{i,k}$, because it contains infixes where the balance on each letter exceeds all bounds in each coordinate.

The second insight is that we can catch the remaining words with a version of $S_{i,k}$ that weights coordinates with some $\boldsymbol{x} \in \mathbb{N}^n$. Let us give some intuition on this. The words from R that we cannot catch with a $P_{i,k}$ must come across, for each i that becomes negative, a loop with positive balance on i (otherwise, the balance on those i would be bounded). But then, the only way such words can avoid D_1 is by ending up in a strongly connected component where every loop (with a final state) makes progress towards crossing 0, i.e. is negative in some coordinate. One can then conclude that even all $\mathbb{Q}_{\geq 0}$ -linear combinations of loops (a convex set) must avoid the positive orthant $\mathbb{Q}^n_{\geq 0} \subset \mathbb{Q}^n$. By the Hyperplane Separation Theorem (we use it in the form of Farkas' Lemma), this is certified by a hyperplane that separates all loop effects from $\mathbb{Q}^n_{\geq 0}$. This hyperplane is given by some orthogonal vector $\boldsymbol{x} \in \mathbb{N}^n$, meaning that every loop balance must have negative scalar product with \boldsymbol{x} . Hence, we can catch these words by:

$$S_{\boldsymbol{x},k} := \left\{ u.v \in \Sigma_n^{\omega} \, \middle| \, \text{a.)} \, \forall f \in \text{infix}(v) \colon \langle \boldsymbol{x}, \varphi(f) \rangle \leq k, \text{ and} \\ \text{b.)} \, v = v_0.v_1.v_2 \dots \wedge \forall \ell \in \mathbb{N} \colon \langle \boldsymbol{x}, \varphi(v_\ell) \rangle < 0 \right\}.$$

Coming back to Figure 1(right), the weight vector $\mathbf{x} = (1, 1)$ guarantees that the weighted balance decreases indefinitely and also the weighted balances of all infixes stay bounded. In [17], a similar argument has been used to show sufficiency of basic separators.

Step 3: Pumpable Languages With the basic separators at hand, the task is to understand the sets of words witnessing inseparability. While studying this problem, we observed that the argumentation for the $P_{i,k}$ was always similar to the one for the $S_{x,k}$. This led us to the question of whether we can get rid of the $P_{i,k}$ in separators. The answer is positive, and hinges on a new notion of pumpability for languages over Σ_n .

Call infinite words u and v equivalent, written $u \sim v$, if v can be obtained from v by removing and inserting finitely many letters: There are $u_0, v_0 \in \Sigma^*$ and $v \in \Sigma^\omega$ such that $v = u_0 v$ and $v = v_0 v$. We say that a language $v \in \Sigma^\omega$ is pumpable if for every $v \in L$ and every $v \in \mathbb{N}$, there exists a decomposition $v = w_0 v_1$ and a word $v_0 \in \Sigma^*$ that is a prefix of a word in $v \in \mathbb{N}$, there exists a decomposition $v = v_0 v_1$ and a word $v_0 \in \Sigma^*$ that is a prefix of a word in $v \in \mathbb{N}$, the indices $v \in [1, n]$ where $v \in \mathbb{N}$ becomes negative on some prefix of $v \in \mathbb{N}$, we have $v \in \mathbb{N}$ and $v \in \mathbb{N}$ there is a word $v \in \mathbb{N}$ with $v \in \mathbb{N}$ there is a word $v \in \mathbb{N}$ with $v \in \mathbb{N}$ where the letter balance exceeds $v \in \mathbb{N}$ before becoming negative, and thus $v \in \mathbb{N}$ with the previous characterization of separator candidates, what is left to separate $v \in \mathbb{N}$ from $v \in \mathbb{N}$ are the languages $v \in \mathbb{N}$.

▶ **Lemma 3.6.** If $L \subseteq \Sigma_n^{\omega}$ is pumpable, then $L \mid D_n$ if and only if $L \mid_{\lim} D_n$, where $L \mid_{\lim} D_n$ means $L \subseteq \bigcup_{\boldsymbol{x} \in X} S_{\boldsymbol{x},k}$ for some finite set $X \subseteq \mathbb{N}^n$ and some $k \in \mathbb{N}$.

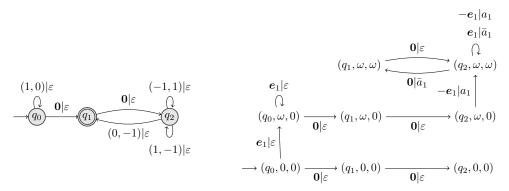


Figure 2 Left: The Büchi VASS $\bar{\mathcal{V}}$ constructed from the Büchi VASS \mathcal{V} found in Figure 1(left). Note how the added second counter tracks the letter balance of the now removed transition labels, incrementing on letter a_1 and decrementing on letter \bar{a}_1 . Right: The Büchi VASS $\mathcal{V}_{\mathsf{pump}}$ corresponding to \mathcal{V} as given by Theorem 3.7. Here we did not mark the final states to reduce visual clutter; every state that includes q_1 is considered final. Similarly, the two labels above the loop in the top right correspond to two distinct transitions. Note that $\mathcal{V}_{\mathsf{pump}}$ essentially looks like $\mathsf{KM}(\bar{\mathcal{V}})$, just with different transition labels.

In our context, pumpability is interesting because we can turn every Büchi VASS language into a pumpable language without affecting separability.

- ▶ Theorem 3.7. Let V be a d-dim. Büchi VASS over Σ_n . We can compute a d-dim. Büchi VASS V_{pump} that satisfies the following:
- 1. $L(\mathcal{V}_{pump})$ is pumpable,
- **2.** there is a $k \in \mathbb{N}$ so that $L(\mathcal{V}_{pump}) \subseteq L(\mathcal{V}) \subseteq L(\mathcal{V}_{pump}) \cup P_k$, and
- **3.** $L(\mathcal{V}) \mid D_n \text{ if and only if } L(\mathcal{V}_{pump}) \mid D_n.$

The construction of $\mathcal{V}_{\mathsf{pump}}$ employs the Karp-Miller graph in an original way, namely to track the unboundedness of letter balances. Let $\bar{\mathcal{V}}$ be the (d+n)-dimensional Büchi VASS obtained from \mathcal{V} by tracking the effect of the letters from Σ_n in n additional counters. For $\bar{\mathcal{V}}$, we construct the Karp-Miller graph. The relationship between the languages of $\mathsf{KM}(\bar{\mathcal{V}})$ and \mathcal{V} is as follows. For all words where every letter balance stays non-negative, their runs in \mathcal{V} can be mimicked in $\mathsf{KM}(\bar{\mathcal{V}})$. For all other words, where the balance eventually becomes negative, this only holds if the corresponding counter in $\bar{\mathcal{V}}$ has been raised to ω beforehand. Essentially, the new Büchi VASS $\mathcal{V}_{\mathsf{pump}}$ restricts \mathcal{V} to those runs that have counterparts in $\mathsf{KM}(\bar{\mathcal{V}})$. This is achieved with a simple product construction of \mathcal{V} and $\mathsf{KM}(\bar{\mathcal{V}})$. The thing to note is that every word from $L(\mathcal{V})$ that does not make it into $L(\mathcal{V}_{\mathsf{pump}})$ belongs to P_k , where k is the maximum concrete number in $\mathsf{KM}(\bar{\mathcal{V}})$: A run in \mathcal{V} that cannot be mimicked in $\mathsf{KM}(\bar{\mathcal{V}})$ will at some point have a negative letter balance, before reaching ω in $\mathsf{KM}(\bar{\mathcal{V}})$ in that component; thus all counter values had been at most k until that point.

An example on how to construct $\bar{\mathcal{V}}$ and \mathcal{V}_{pump} can be found in Figure 2, where both were constructed for the Büchi VASS found in Figure 1(left).

In the proof of Theorem 3.5, we make use of Theorem 3.7 (recall that a regular language is the language of a 0-dimensional Büchi VASS). This may look like cyclic reasoning, but it is not: We will show Theorem 3.7(1)+(2) directly, using the arguments above. With this, we prove Theorem 3.5, which in turn is used to derive Lemma 3.6 and Theorem 3.7(3).

Step 4: Non-Separability Witnesses and Decidability Because of pumpability, it remains to decide whether a Büchi VASS language $L(\mathcal{V})$ is included in a finite union $\bigcup_{x \in X} S_{x,k}$ for

some k. Part of the difficulty is that we have no bound on the cardinality of X. To circumvent this, we decompose $L(\mathcal{V})$ into a finite union $\bigcup_{\pi} L_{\pi}(\mathcal{V})$, where π is a *profile*, meaning a set of edges in $\mathsf{KM}(\mathcal{V})$ seen infinitely often during a run of \mathcal{V} . We then show that each $L_{\pi}(\mathcal{V})$ is either (i) included in a single separator $S_{x,k}$ or (ii) escapes every finite union $\bigcup_{x \in X} S_{x,k}$.

Here, it is key to show an even stronger fact: In case (i), not only $L_{\pi}(\mathcal{V})$ is included in some $S_{\boldsymbol{x},k}$, but the entire set of runs in $\mathsf{KM}(\mathcal{V})$ that eventually remain in π . The advantage of strengthening is that finiteness of $\mathsf{KM}(\mathcal{V})$ allows us to express inclusion in $S_{\boldsymbol{x},k}$, for some k, as a *finite* system of linear inequalities over \boldsymbol{x} : We say that (1) the balance of every primitive cycle, weighted by \boldsymbol{x} , is at most zero and (2) the balance, weighted by \boldsymbol{x} , of some cycle containing all edges from π is negative. Here, (1) and (2) correspond to Conditions a.) and b.) of $S_{\boldsymbol{x},k}$. If they are met, then the runs of $\mathsf{KM}(\mathcal{V})$ along π are included in $S_{\boldsymbol{x},k}$ for some k.

We then prove that if the system is not feasible, then \mathcal{V} has runs that escape every finite union $\bigcup_{x \in X} S_{x,k}$. To this end, we employ Farkas' Lemma: It tells us that if there is no solution, then the dual system has a solution. The solution of the dual system can be interpreted as an executable linear combination of primitive cycles with non-negative balances. We show that these cycles can be arranged in a pattern in $\mathsf{KM}(\mathcal{V})$ we call inseparability flower. Such an inseparability flower then yields a sequence of runs ρ_1, ρ_2, \ldots in $\mathsf{KM}(\mathcal{V})$ such that ρ_k escapes $S_{x,k}$ for every vector x. Finally, pumpability allows us to lift these runs of $\mathsf{KM}(\mathcal{V})$ to runs of \mathcal{V} and thus conclude inseparability.

This equips us with two possible decision procedures: We can either check solvability of each system of inequalities, or detect inseparability flowers in $KM(\mathcal{V})$.

4 Basic Separators

We prove Theorem 3.5, that any regular language R over Σ_n with $R \cap D_n = \emptyset$ is contained in a finite union of languages $P_{i,k}$ and $S_{\boldsymbol{x},k}$. Note that a single value of k is sufficient, since we have $P_{i,k} \subseteq P_{i,k+1}$ and $S_{\boldsymbol{x},k} \subseteq S_{\boldsymbol{x},k+1}$ for each i,\boldsymbol{x},k . The proof decomposes the Büchi automaton for R in a way that allows us to forget about connectedness issues and reason over cycles (and their letter balances) using techniques from linear algebra. We make use of the following basic fact from linear programming [46, Corollary 7.1f].

▶ Theorem 4.1 (Farkas' Lemma (variant), [46]). Let $A \in \mathbb{Q}^{m \times n}$ be a matrix and let $b \in \mathbb{Q}^m$ be a vector. Then the system $Ax \leq b$ has a solution $x \in \mathbb{Q}^n_{\geq 0}$ if and only if $y^\top b \geq 0$ for each vector $y \in \mathbb{Q}^m_{\geq 0}$ with $y^\top A \geq 0$.

Decomposing with profiles We decompose R = L(A) into a (not necessarily disjoint) union of several languages, each linked to a so-called *profile*. We will later see that for pumpable R, every such profile language already has to be contained in a single $S_{x,k}$.

▶ Definition 4.2. Let \mathcal{A} be a Büchi automaton. A profile of \mathcal{A} is a set π of transitions of \mathcal{A} for which there exists a cycle σ_{π} in \mathcal{A} such that (a) σ_{π} contains exactly the transitions in π , and (b) σ_{π} starts (and ends) in a final state q_{π} .

We denote by $\Pi(A)$ the finite set of profiles of A. Moreover, we associate to every accepting run ρ of A its profile $\Pi(\rho)$, which contains exactly the transitions appearing infinitely often in ρ . This definition is sound, as the infinitely occurring transitions of an accepting run must form a cycle due to repetition, which visits a final state due to acceptance.

Given a profile π of \mathcal{A} , we define $L_{\pi}(\mathcal{A}) \subseteq L(\mathcal{A})$ to be the language of all words that have an accepting run ρ of \mathcal{A} with $\Pi(\rho) = \pi$. Note that this language is still regular: From \mathcal{A} one can construct a Büchi automaton that guesses a point after which only transitions

from π can occur, and once this point is reached it keeps a list of already used transitions from π in each state. Then only once all transitions of π have been used the state becomes final and the list is set back to empty.

This now allows us to view R as the union of the languages $L_{\pi}(A)$ with $\pi \in \Pi(A)$. We show that each language $L_{\pi}(A)$ is either contained in $S_{\boldsymbol{x},k}$ for some \boldsymbol{x},k , or there is a cycle that, assuming the pumpability from the previous section, makes $L_{\pi}(A)$ intersect D_n .

- ▶ **Lemma 4.3.** Let A be a Büchi automaton over Σ_n and let π be one of its profiles. Then one of the following conditions holds:
 - (i) There is a number $k \in \mathbb{N}$ and a vector $\mathbf{x} \in \mathbb{N}^n$ such that $L_{\pi}(A) \subseteq S_{\mathbf{x},k}$, or
- (ii) there is a cycle σ' in \mathcal{A} over w' with $\varphi(w') \geq \mathbf{0}$, and σ' contains all transitions from π . Assume $L_{\pi}(\mathcal{A}) \neq \emptyset$, otherwise Condition (i) trivially holds. We build a system $\mathbf{A}_{\pi} \mathbf{x} \leq \mathbf{b}$ of linear inequalities as follows. It contains one inequality $\langle \mathbf{x}, \varphi(v) \rangle \leq 0$ for each word v read by a primitive cycle of transitions in π . By primitive cycle we mean a cycle that does not repeat a state. Moreover, the system contains the inequality $\langle \mathbf{x}, \varphi(v_{\pi}) \rangle \leq -1$ for the cycle σ_{π} over v_{π} that justifies the profile π . Let us quickly remark that the solution space of the system $\mathbf{A}_{\pi} \mathbf{x} \leq \mathbf{b}$ is independent of the precise choice of the justifying cycle σ_{π} : To see this, we claim that $\mathbf{A}_{\pi} \mathbf{x} \leq \mathbf{b}$ holds if and only if all primitive cycles in π have an \mathbf{x} -weighted balance at most zero, and at least one primitive cycle in π has a strictly negative \mathbf{x} -weighted balance. For the "if" direction, note that a sufficiently long repetition of σ_{π} will contain each primitive cycle as a (possibly non-contiguous) subsequence. This means, the repetition, and thus σ_{π} , must have a strictly negative \mathbf{x} -weighted balance. For the converse, we observe that σ_{π} can be decomposed into primitive cycles. Thus, if σ_{π} has strictly negative \mathbf{x} -weighted balance, then so must at least one of its constituent primitive cycles.

Applying Farkas' Lemma to $A_{\pi}x \leq b$ either yields a solution $x \in \mathbb{Q}^n_{\geq 0}$ or a vector $y \in \mathbb{Q}^m_{\geq 0}$ with $y^{\top}A_{\pi} \geq 0$ and $y^{\top}b < 0$. In both cases we assume wlog, that the given vector has entries in \mathbb{N} , as we can always multiply with the lcm of the denominators.

Suppose we have a solution \boldsymbol{x} . We claim that then $L_{\pi}(\mathcal{A}) \subseteq S_{\boldsymbol{x},k}$, where $k = |Q_{\pi}| \cdot h$ and h is the maximal length of a transition label of \mathcal{A} . This is because \boldsymbol{x} weights primitive cycles non-positively, and k is chosen such that for any infix v of a word in $L_{\pi}(\mathcal{A})$, if |v| > k, then v's associated transition sequence has to contain a primitive cycle. Thus, infixes at almost all start positions of a word in $L_{\pi}(\mathcal{A})$ must have \boldsymbol{x} -weighted balance $\leq k$.

If we obtain a vector $\mathbf{y} = (y_1, \dots, y_m)$, then we can view it as a selection of rows in the matrix \mathbf{A}_{π} , where the jth row is being selected y_j many times. Since each row corresponds to a cycle, this is also a selection of cycles. Then by $\mathbf{y}^{\top}\mathbf{b} < 0$ we selected σ_{π} , where we can insert the other selected cycles. By $\mathbf{y}^{\top}\mathbf{A}_{\pi} \geq \mathbf{0}$ this forms a cycle σ' as required, with non-negative letter balance for all letter pairs. A detailed proof can be found in Appendix C.

Here, we used a system of linear inequalities $A_{\pi}x \leq b$, which was solely dependent on \mathcal{A} and π . We reasoned that if this system has a solution, then Condition (i) has to hold. This is a fact that we want to refer to in a later proof, and therefore we formalize it here.

▶ Corollary 4.4. If \mathcal{A} is a Büchi automaton with a profile π for which there is an $\mathbf{x} \in \mathbb{N}^n$ with $\mathbf{A}_{\pi}\mathbf{x} \leq \mathbf{b}$, then $L_{\pi}(\mathcal{A}) \subseteq S_{\mathbf{x},k}$ for some $k \in \mathbb{N}$.

With Theorem 3.7 and Lemma 4.3, we can now show Theorem 3.5. Suppose $R = L(\mathcal{A})$ for some Büchi automaton \mathcal{A} . First, applying Theorem 3.7 with d = 0 yields a Büchi automaton $\mathcal{A}_{\text{pump}}$ such that $L(\mathcal{A}) \subseteq L(\mathcal{A}_{\text{pump}}) \cup P_{\ell}$ for some $\ell \in \mathbb{N}$ and $L(\mathcal{A}_{\text{pump}}) \cap D_n = \emptyset$. Therefore, it suffices to show that $L(\mathcal{A}_{\text{pump}})$ is included in a finite union of languages $S_{\boldsymbol{x},k}$. Suppose not. Then the set $L(\mathcal{A}_{\text{pump}})$ decomposes into the sets $L_{\pi}(\mathcal{A}_{\text{pump}})$ for $\pi \in \Pi(\mathcal{A}_{\text{pump}})$. By Lemma 4.3, we know that for some π , Condition (ii) must hold: Otherwise, each $L_{\pi}(\mathcal{A}_{\text{pump}})$ would be

included in some $S_{x,k}$. But if (ii) holds for π , then there is a cycle σ' in $\mathcal{A}_{\mathsf{pump}}$ that contains π (and thus visits a final state) and reads a word v with $\varphi(v) \geq \mathbf{0}$. Now for some finite prefix u, the word uv^{ω} belongs to $L(\mathcal{A}_{\mathsf{pump}})$. Since $\varphi(v) \geq \mathbf{0}$, there is some lower bound $B \in \mathbb{Z}$ such that for each $i \in [1, n]$ and every prefix p of uv^{ω} , we have $\varphi_i(p) \geq B$. Finally, since $L(\mathcal{A}_{\mathsf{pump}})$ is pumpable, we can exchange a prefix in $w = uv^{\omega}$ to obtain another word $w' \in L(\mathcal{A}_{\mathsf{pump}})$ where every prefix p has $\varphi(p) \geq \mathbf{0}$. Hence $w' \in D_n$ and thus $L(\mathcal{A}_{\mathsf{pump}}) \cap D_n \neq \emptyset$, a contradiction.

5 Deciding Regular Separability

We now present the algorithm to decide, given a Büchi VASS \mathcal{V} whether $L(\mathcal{V}) \mid D_n$. We first employ Theorem 3.7, because for pumpable languages we only have to deal with one type of basic separators. The next step is to generalize the notion of profiles from Büchi automata to Büchi VASS. Recall that for a sequence χ of transitions in \mathcal{V} , $\delta(\chi)$ denotes its effect on the counters of \mathcal{V} . If χ is a transition sequence in $\mathsf{KM}(\mathcal{V})$, then χ is labeled with a transition sequence of \mathcal{V} , so we define $\delta(\chi)$ accordingly. Since we consider Büchi VASS with input alphabet Σ_n , we write $\varphi(\chi)$ for the image of the input word under φ . Again, this notation is used for transition sequences in $\mathsf{KM}(\mathcal{V})$. We also write $\Delta(\chi) = (\delta(\chi), \varphi(\chi))$.

▶ **Definition 5.1.** Let V be a Büchi VASS. A profile for V is a set π of edges in KM(V) for which there exists a cycle σ in KM(V) such that (i) σ contains exactly the edges in π , (ii) σ starts (and ends) in a final state, and (iii) $\delta(\sigma) \geq \mathbf{0}$.

Clearly, every Büchi VASS has a finite set of profiles, which we denote by $\Pi(\mathcal{V})$. Moreover, $\Pi(\mathcal{V})$ can be constructed effectively: Given a set of edges, a simple reduction to checking unboundedness of a counter can be used to check if it is a profile. Furthermore, to every run ρ of \mathcal{V} , we can associate a profile: The run ρ must have a corresponding run in $\mathsf{KM}(\mathcal{V})$, which has a finite set $\Pi(\rho)$ of edges that are used infinitely often. Thus, ρ decomposes as $\rho_0 \rho_1$ such that ρ_1 only contains edges from π . Then, ρ_1 decomposes into $\sigma_1 \sigma_2 \cdots$ such that each σ_i uses every edge from $\Pi(\rho)$ at least once and starts (and ends) in a final state. Since \leq is a well-quasi ordering on \mathbb{N}^n , there are r < s such that $\delta(\sigma_r \cdots \sigma_s) \geq \mathbf{0}$. Thus, $\sigma = \sigma_r \cdots \sigma_s$ is our desired transition sequence showing that $\Pi(\rho)$ is a profile. For each $\pi \in \Pi(\mathcal{V})$, we denote by $L_{\pi}(\mathcal{V})$ the set of all words accepted by runs ρ of \mathcal{V} for which $\Pi(\rho) = \pi$. Then clearly:

▶ Lemma 5.2. $L(\mathcal{V}) = \bigcup_{\pi \in \Pi(\mathcal{V})} L_{\pi}(\mathcal{V}).$

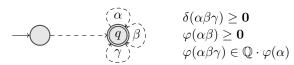
A system of inequalities for each profile Our next step is to associate with each profile $\pi \in \Pi(\mathcal{V})$ a system of linear inequalities. We need some terminology. A π -cycle is a cycle σ in KM(\mathcal{V}) that only contains edges in π . If in addition, σ visits each state of KM(\mathcal{V}) at most once, except for the initial state, which is visited twice, then σ is a primitive π -cycle. Clearly, a primitive π -cycle has length $\leq |\pi|$. Moreover, from every π -cycle σ , one can successively cut out primitive π -cycles until it is empty. Therefore, if τ_1, \ldots, τ_m are the primitive π -cycles of KM(\mathcal{V}), then there are numbers $r_1, \ldots, r_m \in \mathbb{N}$ such that $\Delta(\sigma) = r_1 \cdot \Delta(\tau_1) + \cdots + r_m \cdot \Delta(\tau_m)$. We call σ a complete π -cycle if this holds for some $r_1, \ldots, r_m \geq 1$. Observe that if π is a profile, then this is always witnessed by a complete π -cycle: Take any cycle σ witnessing that π is a profile. Then $\sigma^{|\pi|}$ contains each primitive π -cycle as a subsequence. Hence, the cycle $\sigma^{m\cdot|\pi|}$ is complete: We can carry out the cutting in each factor $\sigma^{|\pi|}$ so as to cut some τ_i at least once. Moreover, $\sigma^{m\cdot|\pi|}$ still witnesses that π is a profile, since $\delta(\sigma^{m\cdot|\pi|}) = m \cdot |\pi| \cdot \delta(\sigma) \geq \mathbf{0}$.

Let us now construct the system of inequalities associated with π . Let σ be a complete π -cycle witnessing that π is a profile and let τ_1, \ldots, τ_m be the primitive π -cycles. Let $\mathbf{A}_{\pi} \in \mathbb{Z}^{(m+1)\times n}$ be the matrix with rows $\varphi(\tau_1), \ldots, \varphi(\tau_m), \varphi(\sigma)$, and let $\mathbf{b} \in \mathbb{Z}^{m+1}$ be the

column vector (0, ..., 0, -1). Then clearly, $\mathbf{A}_{\pi} \mathbf{x} \leq \mathbf{b}$ is equivalent to $\langle \mathbf{x}, \varphi(\sigma) \rangle < 0$ and $\langle \mathbf{x}, \varphi(\tau) \rangle \leq 0$ for each primitive π -cycle τ .

Inseparability flowers An inseparability flower is a structure in the Karp-Miller graph

 $\mathsf{KM}(\mathcal{V})$ as depicted to the right. It consists of a final state q and three cycles α, β, γ that all start in q and that meet the given conditions.



Let us give some intuition on why such a flower is the relevant structure to look for. True to its name, an inseparability flower guarantees the existence of an inseparability witness, i.e. a family of words accepted by the pumpable Büchi VASS $\mathcal V$ that escape every basic separator $S_{\boldsymbol x,k}$. Such a family of words therefore needs an accepting run for each member, and the three conditions of the flower provide such runs: The first condition ensures that the three cycles actually correspond to a transition sequence enabled in $\mathcal V$. The second condition guarantees that for every $\boldsymbol x \in \mathbb N^n$, the $\boldsymbol x$ -weighted letter balance of α or of β is positive; unless they are both zero, in which case the third condition ensures that $\alpha\beta\gamma$ has $\boldsymbol x$ -weighted balance zero. This allows us to construct, for each k, a run that escapes $S_{\boldsymbol x,k}$ for all $\boldsymbol x$: By sufficiently repeating each cycle α , β , and γ , we obtain a run that for each $\boldsymbol x \in \mathbb N^n$, will either (i) have infixes with $\boldsymbol x$ -weighted balance > k, or (ii) attain some $\boldsymbol x$ -weighted balance infinitely often. Each of these properties rules out membership in $S_{\boldsymbol x,k}$. Proposition 5.5 proves this formally.

▶ Theorem 5.3. Let V be a Büchi VASS such that L(V) is pumpable. Then the following are equivalent: (i) $L(V) \not\mid D_n$. (ii) There is a profile $\pi \in \Pi(V)$ such that the system $A_{\pi}x \leq b$ has no solution $x \in \mathbb{N}^n$. (iii) There exists an inseparability flower in $\mathsf{KM}(V)$.

The decision procedure Before we prove Theorem 5.3, let us see how to use it to decide separability. Given Büchi VASS \mathcal{V}_1 and \mathcal{V}_2 , we can compute \mathcal{V} so that $L(\mathcal{V}_1) \mid L(\mathcal{V}_2)$ if and only if $L(\mathcal{V}) \mid D_n$, by Lemma 3.4. Then Theorem 3.7 tells us that $L(\mathcal{V}_{pump})$ is pumpable and we have $L(\mathcal{V}) \mid D_n$ if and only if $L(\mathcal{V}_{pump}) \mid D_n$. Finally, by Theorem 5.3, we can check whether $L(\mathcal{V}_{pump}) \mid D_n$ by checking the systems $A_{\pi}x \leq b$ for satisfiability: If there is a solution for every $\pi \in \Pi(\mathcal{V}_{pump})$, then we have separability; otherwise, we have inseparability. Since the systems $A_{\pi}x \leq b$ are constructed directly from $\mathsf{KM}(\mathcal{V}_{pump})$, we need to explicitly construct the latter. Therefore our procedure may take Ackermann time, because Karp-Miller graphs can be Ackermann large [36, Theorem 2].

▶ Example 5.4. Consider the instance of regular separability where our two inputs are the Büchi VASS \mathcal{V} found in Figure 1(left), and another Büchi VASS accepting the language D_1 . Since we are already in the case of wanting to decide $L(\mathcal{V}) \mid D_1$, we can skip the first step of applying Lemma 3.4. The second step is to apply Theorem 3.7 and construct \mathcal{V}_{pump} , which we have already done for this case in Figure 2(right).

Now we have to construct $\mathsf{KM}(\mathcal{V}_{\mathsf{pump}})$, which can be found in Figure 3. There are two relevant parts of $\mathsf{KM}(\mathcal{V}_{\mathsf{pump}})$, where we can find cycles involving a final state: (1) the part on the right, where the state tuples contain ω twice and the counter value is 0, and (2) the part at the top with triple ω s. In the following we will only write down the states, as the counter values and the other contents of the state tuples will be clear from context.

For part (1), the Büchi VASS $\mathcal{V}_{\mathsf{pump}}$ has only a single profile π_1 containing only the two edges between q_1 and q_2 . Since each π_1 -cycle σ only consists of repetitions of the primitive cycle $q_1 \xrightarrow{\mathbf{0}|\varepsilon} q_2 \xrightarrow{\mathbf{0}|\bar{a}_1} q_1$, we have $\varphi(\sigma) < 0$. Therefore the system $\mathbf{A}_{\pi_1} \mathbf{x} \leq \mathbf{b}$ trivially has a solution $\mathbf{x} = 1$.

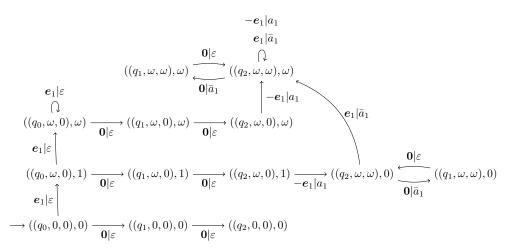


Figure 3 The Karp-Miller graph $\mathsf{KM}(\mathcal{V}_{\mathsf{pump}})$ of the Büchi VASS $\mathcal{V}_{\mathsf{pump}}$ from Figure 2(left). Here we did not mark the final states to reduce visual clutter; every state that includes q_1 is considered final. For similar reasons, we also only labelled the edges of the graph with letters and counter effects. The proper edge labels would be full transitions of $\mathcal{V}_{\mathsf{pump}}$, including source and target state.

Regarding part (2), $\mathcal{V}_{\mathsf{pump}}$ has exactly two more profiles: profile π_2 containing only the two edges between q_1 and q_2 , and profile π_3 , which additionally contains the two loop edges on q_2 . The cycles of π_2 look almost exactly like the cycles of π_1 with only the counter values of the nodes in the graph being different. Thus, the system $\mathbf{A}_{\pi_2}\mathbf{x} \leq \mathbf{b}$ is the exact same system as $\mathbf{A}_{\pi_1}\mathbf{x} \leq \mathbf{b}$ and also trivially has a solution $\mathbf{x} = 1$.

For π_3 , we have as primitive cycles both the loop edges on q_2 as well as the primitive cycle of π_2 . To obtain a complete π_3 -cycle, we simply insert both loops into the π_2 -cycle at q_2 forming the cycle $\sigma = q_1 \xrightarrow{\mathbf{0}|\varepsilon} q_2 \xrightarrow{-e_1|a_1} q_2 \xrightarrow{e_1|\bar{a}_1} q_2 \xrightarrow{\mathbf{0}|\bar{a}_1} q_1$. Since σ contains all primitive cycles exactly once without overlap, it is automatically complete. We also have $\delta(\sigma) = 0$, meaning σ is a cycle witnessing π_3 as a profile. Thus these cycles lead to the following system of inequalities $A_{\pi_3} \mathbf{x} \leq \mathbf{b}$:

$$\begin{array}{ll} 1 \cdot x_1 \leq 0 & \quad \text{loop 1} \\ -1 \cdot x_1 \leq 0 & \quad \text{loop 2} \\ -1 \cdot x_1 \leq 0 & \quad \pi_2\text{-cycle} \\ -1 \cdot x_1 \leq -1 & \quad \text{complete π_3-cycle} \end{array}$$

Clearly this system has no solution; the first and last inequality are contradictory. Therefore we conclude regular inseparability for $L(\mathcal{V})$ and D_1 .

While not part of the decision procedure, for an inseparable instance of the problem as we have here, we can also find an inseparability flower in $\mathsf{KM}(\mathcal{V}_{\mathsf{pump}})$. In this case we have $\alpha = q_1 \xrightarrow{\mathbf{0}|\varepsilon} q_2 \xrightarrow{\mathbf{0}|\bar{a}_1} q_1$, $\beta = q_1 \xrightarrow{\mathbf{0}|\varepsilon} q_2 \xrightarrow{-e_1|a_1} q_2 \xrightarrow{-e_1|a_1} q_2 \xrightarrow{\mathbf{0}|\bar{a}_1} q_1$, and $\gamma = q_1 \xrightarrow{\mathbf{0}|\varepsilon} q_2 \xrightarrow{e_1|\bar{a}_1} q_2 \xrightarrow{\mathbf{0}|\bar{a}_1} q_1$. This selection of cycles meets all the requirements of a flower: $\delta(\alpha\beta\gamma) = 0$, $\varphi(\alpha\beta) = 0$, and $\varphi(\alpha\beta\gamma) = -3 = 3 \cdot \varphi(\alpha)$.

Inseparability flowers disprove separability The remainder of this section is devoted to proving Theorem 5.3. The implication "(i) \Rightarrow (ii)" follows by applying Corollary 4.4 to $\mathsf{KM}(\mathcal{V})$, viewed as a Büchi automaton; see Lemma D.1. For "(iii) \Rightarrow (i)", we employ Lemma 3.6:

▶ **Proposition 5.5.** If L(V) is pumpable and KM(V) has an insep. flower, then $L(V) \not\mid D_n$.

Proof. Suppose there is an inseparability flower α, β, γ in $\mathsf{KM}(\mathcal{V})$ and also $L(\mathcal{V}) \mid D_n$. By Lemma 3.6, there is a $k \in \mathbb{N}$ and a finite set $X \subseteq \mathbb{N}^n$ such that $L(\mathcal{V}) \subseteq \bigcup_{x \in X} S_{x,k}$. We claim that for every $x \in \mathbb{N}^n$, at least one of the following holds:

$$\langle \boldsymbol{x}, \varphi(\alpha) \rangle > 0,$$
 or $\langle \boldsymbol{x}, \varphi(\beta) \rangle > 0,$ or $\langle \boldsymbol{x}, \varphi(\alpha\beta\gamma) \rangle = 0.$ (1)

Indeed, if $\langle \boldsymbol{x}, \varphi(\alpha) \rangle \leq 0$ and $\langle \boldsymbol{x}, \varphi(\beta) \rangle \leq 0$, then $\varphi(\alpha\beta) \geq \mathbf{0}$ implies that $\langle \boldsymbol{x}, \varphi(\alpha) \rangle = \langle \boldsymbol{x}, \varphi(\beta) \rangle = 0$. Since $\varphi(\alpha\beta\gamma) = N \cdot \varphi(\alpha)$ for some $N \in \mathbb{Q}$, we have $\varphi(\alpha\beta\gamma) = \mathbf{0}$. This proves the claim. Because of (1), the sequence $\alpha^{k+1}\beta^{k+1}\gamma^{k+1}$ either has an infix χ with $\langle \boldsymbol{x}, \varphi(\chi) \rangle > k$ or we have $\langle \boldsymbol{x}, \varphi(\alpha^{k+1}\beta^{k+1}\gamma^{k+1}) \rangle = 0$. Since $\delta(\alpha^{k+1}\beta^{k+1}\gamma^{k+1}) \geq \mathbf{0}$, there is a run ρ such that $\rho\alpha^{k+1}\beta^{k+1}\gamma^{k+1}$ is a run in \mathcal{V} . Hence, $\rho(\alpha^{k+1}\beta^{k+1}\gamma^{k+1})^{\omega}$ is a run in \mathcal{V} whose word cannot belong to $S_{\boldsymbol{x},k}$ for any $\boldsymbol{x} \in \mathbb{N}^n$, contradicting $L(\mathcal{V}) \subseteq \bigcup_{\boldsymbol{x} \in X} S_{\boldsymbol{x},k}$.

Constructing inseparability flowers It remains to show the implication "(ii) \Rightarrow (iii)". Suppose there is a profile $\pi \in \Pi(\mathcal{V})$ whose associated system of inequalities $A_{\pi}x \leq b$ is unsatisfiable. By Farkas' Lemma, there exists a $\mathbf{y} \in \mathbb{N}^{m+1}$ such that $\mathbf{y}^{\top}A_{\pi} \geq \mathbf{0}$ and $\mathbf{y}^{\top}b < 0$. From this vector \mathbf{y} , we now construct an inseparability flower in $\mathsf{KM}(\mathcal{V})$.

Let σ be the complete π -cycle in $\mathsf{KM}(\mathcal{V})$ that was chosen to construct A_{π} . Let τ_1, \ldots, τ_m be the primitive π -cycles. Since σ is complete, there is a vector $\mathbf{r} = (r_1, \ldots, r_m) \in \mathbb{N}^m$ so that $r_1, \ldots, r_m \geq 1$ and $\Delta(\sigma) = r_1 \cdot \Delta(\tau_1) + \cdots + r_m \cdot \Delta(\tau_m)$. Moreover, since σ contains every edge of π , we can wlog. write $\sigma = \sigma_0 \cdots \sigma_m$ such that between σ_{i-1} and σ_i , σ arrives in the initial state of τ_i . The decomposition allows us to insert further repetitions of the primitive cycles. For $\mathbf{z} = (z_1, \ldots, z_m) \in \mathbb{N}^m$ with $\mathbf{z} \geq \mathbf{r}$, we define $\sigma^{\mathbf{z}}$ as $\sigma_0 \tau_1^{z_1 - r_1} \sigma_1 \cdots \tau_m^{z_m - r_m} \sigma_m$. Then $\Delta(\sigma^{\mathbf{z}}) = z_1 \cdot \Delta(\tau_1) + \cdots + z_m \cdot \Delta(\tau_m)$. In particular, for $\mathbf{s}, \mathbf{t} \geq \mathbf{r}$, we have $\Delta(\sigma^{\mathbf{s}} \sigma^{\mathbf{t}}) = \Delta(\sigma^{\mathbf{s}+\mathbf{t}})$.

Recall that every transition in a Karp-Miller graph is labeled by a VASS transition, and so every transition sequence χ in $\mathsf{KM}(\mathcal{V})$ is labeled by a transition sequence in \mathcal{V} , which we denote by $\mathsf{trans}(\chi)$. We now define the transition sequences α , β , and γ as $\mathsf{trans}(\sigma^z)$ for suitable vectors z. For α , we take $\mathsf{trans}(\sigma)$, the transitions labeling the complete π -cycle. Observe that $\sigma = \sigma^r$. We proceed to define $\beta = \mathsf{trans}(\sigma^s)$ and $\gamma = \mathsf{trans}(\sigma^t)$. The choice of the vectors s and t has to meet the requirements on an inseparability flower: $\varphi(\alpha\beta) \geq 0$, $\delta(\alpha\beta\gamma) \geq 0$, and $\varphi(\alpha\beta\gamma) \in \mathbb{Q} \cdot \varphi(\alpha)$.

Step I: Building β . We will define s so that $\varphi(\alpha\beta) = \varphi(\sigma^r\sigma^s) = \varphi(\sigma^{r+s}) \geq 0$. The remaining two requirements (i.e. $\delta(\alpha\beta\gamma) \geq 0$ and $\varphi(\alpha\beta\gamma) \in \mathbb{Q} \cdot \varphi(\alpha)$) will be ensured with an appropriate choice of t in Step II. Let us now describe how to pick s. Recall that s is the vector from the application of Farkas' Lemma. It can be understood as assigning a repetition count s in the every primitive cycle s in the profile and a repetition count s in the complete s cycle s. Since s ince s include s

$$\boldsymbol{y}^{\top} \boldsymbol{A}_{\pi} = \sum_{i=1}^{m} y_i \cdot \varphi(\tau_i) + y_{m+1} \cdot \varphi(\sigma) = \sum_{i=1}^{m} (y_i + y_{m+1} r_i) \varphi(\tau_i) = \varphi(\sigma^{\hat{\boldsymbol{y}}}).$$

We now choose $M \in \mathbb{N}$ such that $\mathbf{s} = M \cdot \hat{\mathbf{y}} - \mathbf{r} \geq \mathbf{r}$. This is possible since all entries in $\hat{\mathbf{y}}$ are positive, due to $y_{m+1} > 0$ by $\mathbf{y}^{\top} \mathbf{b} < 0$, and $r_i > 0$ for all i by definition. Then we have $\varphi(\alpha\beta) = \varphi(\sigma^{\mathbf{r}}\sigma^{\mathbf{s}}) = \varphi(\sigma^{\mathbf{r}+\mathbf{s}}) = \varphi(\sigma^{M \cdot \hat{\mathbf{y}}}) = M \cdot \varphi(\sigma^{\hat{\mathbf{y}}}) \geq \mathbf{0}$.

Step II: Building γ . It remains to define t so that $\gamma = \operatorname{trans}(\sigma^t)$ satisfies $\delta(\alpha\beta\gamma) =$ $\delta(\sigma^{r+s+t}) \geq 0$ and $\varphi(\alpha\beta\gamma) \in \mathbb{Q} \cdot \varphi(\alpha)$. The idea is to choose t so that r+s+t is a positive multiple of r. Such a choice is possible, because r has positive entries everywhere: We pick $N \in \mathbb{N}$ such that $t := N \cdot r - s - r \ge r$. Then indeed $\delta(\alpha\beta\gamma) = \delta(\sigma^{r+s+t}) = \delta(\sigma^{N\cdot r}) = \delta(\sigma^{N\cdot r})$ $N \cdot \delta(\sigma^{r}) = N \cdot \delta(\sigma) \ge \mathbf{0} \text{ and } \varphi(\alpha\beta\gamma) = \varphi(\sigma^{r+s+t}) = \varphi(\sigma^{N \cdot r}) = N \cdot \varphi(\sigma^{r}) = N \cdot \varphi(\alpha).$

6 One-dimensional Büchi VASS

Our second contribution is the precise complexity of separability for the 1-dimensional case.

▶ Theorem 6.1. Regular separability for 1-dimensional Büchi VASS with binary encoded updates is PSPACE-complete.

For the lower bound, we use a simple reduction from the disjointness problem $L_1 \cap L_2 \stackrel{f}{=} \emptyset$ for finite-word languages of 1-dim. VASS [24]. However, we also show that separability is PSPACE-hard even if the input languages are promised to be disjoint. See Appendix E.1.

For the upper bound, we rely on the results in Section 5, but need a modification. There, to simplify the exposition, we first make the input language pumpable, which may incur an Ackermannian blowup. A closer look at the results, however, reveals that we can also check separability directly on the Karp-Miller graph of $\bar{\mathcal{V}}$ as defined in Section 3.

▶ Proposition 6.2. Let V be a Büchi VASS with $L(V) \subseteq \Sigma_n^{\omega}$. Then $L(V) \not\mid D_n$ if and only if $KM(\overline{\mathcal{V}})$ has an inseparability flower.

Proposition 6.2 allows us to phrase inseparability as the existence of a run in $\overline{\mathcal{V}}$ that satisfies certain constraints. Recall that if \mathcal{V} is 1-dimensional and over Σ_1 , then $\overline{\mathcal{V}}$ has two counters the second of which tracks the letter balance.

▶ Corollary 6.3. Let V be a 1-dimensional Büchi VASS with $L(V) \subseteq \Sigma_1^{\omega}$ and $L(V) \cap D_1 = \emptyset$. Then $L(\mathcal{V}) \not\mid D_1$ if and only if there exist states p, q, r with r final, and a run in $\overline{\mathcal{V}}$ as follows:

$$(q_{0},0,0) \xrightarrow{*} \overbrace{(p,x_{1},y_{1})}^{\sigma_{1}} \xrightarrow{*} (p,x_{2},y_{2}) \xrightarrow{*} \overbrace{(q,x_{3},y_{3})}^{\ast} \xrightarrow{*} (q,x_{4},y_{4})$$

$$\xrightarrow{*} \overbrace{(r,x_{5},y_{5})}^{\alpha} \xrightarrow{*} \underbrace{(r,x_{6},y_{6})}^{\ast} \xrightarrow{*} \overbrace{(r,x_{7},y_{7})}^{\ast} \xrightarrow{*} (r,x_{8},y_{8})$$

$$(1) \quad y_{3} < y_{4} \text{ and also } (a) \quad x_{3} \leq x_{4}$$
or $(b) \quad x_{1} < x_{2} \text{ and } y_{1} \leq y_{2}$

$$(2) \quad y_{5} \leq y_{7}$$

$$(3) \quad x_{5} \leq x_{8}$$

$$(4) \quad \text{if } y_{5} = y_{6}, \text{ then } y_{5} = y_{8}.$$

Observe that an inseparability flower in $\mathsf{KM}(\bar{\mathcal{V}})$ must carry ω in the second coordinate, meaning the letter balance is unbounded. Otherwise, it would yield an accepting run of \mathcal{V} , which cannot exist because $L(\mathcal{V}) \cap D_1 = \emptyset$. If the flower has ω in the second coordinate, we can construct a finite run as above. The cycles σ_1 and σ_2 plus Condition 1 ensure that indeed the second coordinate becomes ω . Condition 2 is $\varphi(\alpha\beta) \geq 0$. Condition 3 says $\delta(\alpha\beta\gamma) \geq 0$. Finally, to express $\varphi(\alpha\beta\gamma) \in \mathbb{Q} \cdot \varphi(\alpha)$, note that for integers $a \in \mathbb{Q} \cdot b$ iff b = 0 implies a = 0. Condition 4 expresses that $y_6 - y_5 = 0$ implies $y_8 - y_5 = 0$.

In order to apply Corollary 6.3 for deciding $L(\mathcal{V}_1) \mid L(\mathcal{V}_2)$ for 1-dim. Büchi VASS $\mathcal{V}_1, \mathcal{V}_2$ with binary counter updates, we would like to follow the approach for the general case and use Lemma 3.4 to first construct \mathcal{V} so that $L(\mathcal{V}_1) \mid L(\mathcal{V}_2)$ if and only if $L(\mathcal{V}) \mid D_1$. From \mathcal{V} , we would then construct the 2-dimensional Büchi VASS $\bar{\mathcal{V}}$ that tracks the letter balance, and on $\overline{\mathcal{V}}$ we would then check the conditions of Corollary 6.3. The problem is that, under binary updates, the intermediary $\mathcal V$ may become exponentially large. We use the fact that also $\mathcal V$ has binary counters available. This allows us to directly construct a compact variant of \mathcal{V} :

▶ Lemma 6.4. Given 1-dim. Büchi VASS V_1, V_2 with binary updates, there is a a 1-dim. Büchi VASS V with $L(V_1) \cap L(V_2) = \emptyset$ iff $L(V) \cap D_1 = \emptyset$, $L(V_1) \mid L(V_2)$ iff $L(V) \mid D_1$, and we can construct in time polynomial in $|V_1| + |V_2|$ the 2-dim. Büchi VASS \bar{V} (binary updates).

Detecting constrained runs in 2-VASS It remains to check for the existence of runs in $\bar{\mathcal{V}}$ as described in Corollary 6.3, and to check whether $L(\mathcal{V}_1) \cap L(\mathcal{V}_2) = \emptyset$. Both of these problems reduce to what we call the *constrained runs problem for 2-VASS*. Recall that *Presburger arithmetic* is the first-order theory of $(\mathbb{N}, +, <, 0, 1)$. We will use the existential fragment to express conditions on counter values of VASS like the ones from Corollary 6.3. The *constrained runs problem* is the following:

Given A 2-dim. VASS \mathcal{V} (with updates encoded in binary), a number $m \in \mathbb{N}$, states q_1, \ldots, q_m in \mathcal{V} , a quantifier-free Presburger formula $\psi(x_1, y_1, \ldots, x_m, y_m)$, and $s, t \in [1, m], s \leq t$. Question Does there exist a run $(q_0, 0, 0) \stackrel{*}{\to} (q_1, x_1, y_1) \stackrel{*}{\to} \cdots \stackrel{*}{\to} (q_m, x_m, y_m)$ that visits a final state between (q_s, x_s, y_s) and (q_t, x_t, y_t) and satisfies $\psi(x_1, y_1, \ldots, x_m, y_m)$? Lemma 6.4 and Corollary 6.3 imply that if $L(\mathcal{V}_1) \cap L(\mathcal{V}_2) = \emptyset$, then $L(\mathcal{V}_1) \mid L(\mathcal{V}_2)$ reduces to the constrained runs problem on $\bar{\mathcal{V}}$. Moreover, checking $L(\mathcal{V}_1) \cap L(\mathcal{V}_2) = \emptyset$ reduces via a product construction to checking emptiness of a 2-VASS. Such a 2-VASS has an accepting run iff $(q_0, 0, 0) \stackrel{*}{\to} (q, x, y) \stackrel{*}{\to} (q, x', y')$ with $(x, y) \leq (x', y')$ and q final. Hence, this problem also reduces to the constrained runs problem for 2-VASS. We thus need to show:

▶ **Proposition 6.5.** The constrained runs problem for 2-VASS is solvable in PSPACE.

For Proposition 6.5, we show that if there is a constrained run, then there is one with at most exponential counter values along the way. For this, we use methods from [5].

Complexity in higher dimension We leave open two natural questions: (i) What is the complexity of regular separability for Büchi d-VASS, for each $d \geq 2$? (ii) What is the complexity of regular separability for Büchi VASS (where the dimension is part of the input)?

Given that the regular separability and the disjointness problem usually (but not always [32, 48]) coincide regarding decidability, we expect the complexity of regular separability to be PSPACE in every fixed dimension d and EXPSPACE in general. The lower bounds follow from Theorem 6.1 for fixed d and from [14] (because disjointness is EXPSPACE-complete [22, 34]). However, it is not clear how to show the upper bounds.

The clearest obstacle is that inseparability flowers involve a non-linear condition: The requirement $\varphi(\alpha\beta\gamma)\in\mathbb{Q}\cdot\varphi(\alpha)$ is not expressible in Presburger arithmetic. There are several generic results providing EXPSPACE upper bounds for detecting particular types of runs in VASS [18, 3, 4]. However, the numerical properties directly expressible there are confined to Presburger arithmetic. The only reason we could obtain the PSPACE upper bound for d=1 is that the non-linear condition degenerates into a linear condition in dimension one: It is equivalent to " $\varphi(\alpha\beta\gamma)=0$ or $\varphi(\alpha)\neq 0$ ".

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A Proof Details for Overview

A.1 Proof of Part 1 of Theorem 3.2

Here we proof the first part of Theorem 3.2, which is the following:

▶ **Theorem A.1.** There are Büchi VASS languages L_1 , L_2 with $L_1 \cap L_2 = \emptyset$ and $L_1 \not\mid L_2$. **Proof.** We choose $L_1 = L(\mathcal{V})$, where \mathcal{V} is the Büchi VASS in Figure 1(left), and $L_2 = D_1$

Proof. We choose $L_1 = L(\mathcal{V})$, where \mathcal{V} is the Büchi VASS in Figure 1(left), and $L_2 = D_1$, the Dyck language. We claim that each $w \in L(\mathcal{V})$ can be written as $w = uv_1v_2\cdots$ with $\varphi_1(v_\ell) < 0$ for every $\ell \in \mathbb{N}$. This clearly implies $L(\mathcal{V}) \cap D_1 = \emptyset$. Suppose $w \in L(\mathcal{V})$. Note that on q_2 , reading a_1 decrements the counter and reading \bar{a}_1 increments the counter. Thus, from a configuration (q_2, x) , a word v read in q_2 can have balance $\varphi_1(v)$ at most x. And moreover, if $\varphi_1(v) > 0$, then this decreases the counter. Furthermore, in order to visit q_1 , the balance has to drop once. Therefore, between any two (not necessarily successive) visits to the final state q_1 , one of the following holds: (i) the counter strictly decreases or (ii) the input word v satisfies $\varphi_1(v) < 0$. Since q_1 is visited infinitely often, we can decompose $w = uv_1v_2\dots$ such that after reading v_ℓ , we are in (q_1, x_ℓ) and we have $x_1 \leq x_2 \leq \cdots$. Then "(i)" cannot happen for any v_ℓ and thus we have $\varphi_1(v_\ell) < 0$ for every ℓ . Hence, the claim is proven.

It remains to show $L(\mathcal{V}) \not\mid D_1$. Towards a contradiction, suppose there is a Büchi automaton \mathcal{A} with n states such that $L(\mathcal{V}) \subseteq L(\mathcal{A})$ and $L(\mathcal{A}) \cap D_1 = \emptyset$. Note that \mathcal{V} accepts $(a_1^n \bar{a}_1^{n+1})^{\omega}$: We drive up the counter to n in q_0 and then read each $a_1^n \bar{a}_1^{n+1}$ in a loop from q_1 to q_1 . However, a run of \mathcal{A} must cycle on some non-empty infix of a_1^n and thus, for some m > n, also accept $w = (a_1^m \bar{a}_1^{n+1})^{\omega}$. Since $w \in D_1$, that is a contradiction.

The second part of Theorem 3.2 is proven in Appendix F.

A.2 Proof of Lemma 3.4

▶ **Lemma 3.4.** Given Büchi VASS V_1 and V_2 , we can compute a Büchi VASS V over Σ_n so that $L(V_1) \mid L(V_2)$ if and only if $L(V) \mid D_n$, where n is the dimension of V_2 .

For the proof, we need the concept of rational transductions of infinite words.

Rational transductions A finite state Büchi transducer is a tuple $\mathcal{T} = (Q, \Sigma, \Gamma, E, q_0, Q_f)$ consists of a finite set of states Q, an input alphabet A, an initial state $q_0 \in Q$, a set of final states $Q_f \subseteq Q$, and a transition relation $E \subseteq Q \times (\Sigma \cup \{\varepsilon\}) \times (\Gamma \cup \{\varepsilon\}) \times Q$. For a transition $(q, a, b, q') \in E$, we also write $q \xrightarrow{(a,b)} q'$. The transducer \mathcal{T} recognizes the binary relation $T(\mathcal{T}) \subseteq \Sigma^{\omega} \times \Gamma^{\omega}$ containing precisely those pairs $(u, v) \in \Sigma^{\omega} \times \Gamma^{\omega}$, for which there is a transition sequence

$$q_0 \xrightarrow{(a_1,b_1)} q_1 \xrightarrow{(a_2,b_2)} \dots$$

such that $u = a_1 a_2 \cdots$, $v = b_1 b_2 \cdots$, and for infinitely many $i \in \mathbb{N}$, we have $q_i \in Q_f$. We say that a relation $T \subseteq \Sigma^{\omega} \times \Gamma^{\omega}$ is *rational* if there is a finite-state Büchi transducer \mathcal{T} with $T = T(\mathcal{T})$. For a language $L \subseteq \Gamma^{\omega}$ and a relation $T \subseteq \Sigma^{\omega} \times \Gamma^{\omega}$, we define

$$TL = \{ u \in \Sigma^{\omega} \mid \exists v \in L \colon (u, v) \in T \}.$$

Moreover, for relations $T \subseteq \Sigma^{\omega} \times \Gamma^{\omega}$ and $S \subseteq \Theta^{\omega} \times \Sigma^{\omega}$, we define

$$S \circ T = \{(u, w) \in \Theta^{\omega} \times \Gamma^{\omega} \mid \exists v \in \Sigma^{\omega} \colon (u, v) \in S, \ (v, w) \in T\}.$$

Using a simple product construction, we observe that for rational transductions S and T, the relation $S \circ T$ is (effectively) rational as well. By simply exchanging the two input coordinates, one can also show that if $T \subseteq \Sigma^{\omega} \times \Gamma^{\omega}$ is rational, then so is

$$T^{-1} = \{(u, v) \in \Gamma^{\omega} \times \Sigma^{\omega} \mid (v, u) \in T\}.$$

The following is also entirely straightforward.

▶ Lemma A.2. A language $L \subseteq \Sigma^{\omega}$ is a Büchi VASS language if and only if there exists a rational transduction T and a number $n \in \mathbb{N}$ such that $L = TD_n$. Moreover, the translation can be performed in exponential time.

Here, the automaton underlying an n-dim. Büchi VASS is translated into a transducer with input in D_n and vice-versa. More precisely, for $h \in \mathbb{N}$ an operation of +h on the ith counter is translated into the string $(a_i)^h$, whereas -h is translated into $(\bar{a}_i)^h$. The **0**-vector is hereby translated into $a_1\bar{a}_1$ instead of ε , to ensure that every infinite run of the Büchi VASS actually corresponds to an infinite word in D_n . The only reason why this construction is not feasible in polynomial time, is because we assume that counter operations of Büchi VASS are encoded in binary. In particular, the string $(a_i)^h$ mentioned above takes h steps to write down, whereas the size of the Büchi VASS is only dependent on $\log h$. However, the construction only takes polynomial time, if counter updates are encoded in unary, or if strings such as $(a_i)^h$ are subjected to some exponential compression.

We also need the following lemma. The proof is exactly the same as the corresponding proof in [17]. The only difference is that we have infinite instead of finite words.

▶ **Lemma A.3.** Let $T \subseteq \Sigma^{\omega} \times \Gamma^{\omega}$ be rational and $L \subseteq \Sigma^{\omega}$ and $K \subseteq \Gamma^{\omega}$. Then $L \mid TK$ if and only if $T^{-1}L \mid K$.

Proof. Suppose $L \subseteq R$ and $R \cap TK = \emptyset$ for some regular R. Then clearly $T^{-1}L \subseteq T^{-1}R$ and $T^{-1}R \cap K = \emptyset$. Therefore, the regular set $T^{-1}R$ witnesses $T^{-1}L \mid K$. Conversely, if $T^{-1}L \mid K$, then $K \mid T^{-1}L$ and hence, by the first direction, $(T^{-1})^{-1}K \mid L$. Since $(T^{-1})^{-1} = T$, this reads $TK \mid L$ and thus $L \mid TK$.

We are now ready to prove Lemma 3.4.

Proof of Lemma 3.4. Given Büchi VASS V_1 and V_2 , where V_2 is *n*-dimensional, Lemma A.2 allows us to compute in exponential time a rational transduction T such that $L(\mathcal{V}_2) = TD_n$. We apply Lemma A.2 again to construct a Büchi VASS \mathcal{V} for $T^{-1}L(\mathcal{V}_1)$. Then we have

$$L(\mathcal{V}_1) \mid L(\mathcal{V}_2) \iff L(\mathcal{V}_1) \mid TD_n \iff T^{-1}L(\mathcal{V}_1) \mid D_n \iff L(\mathcal{V}) \mid D_n,$$

where the second equivalence is due to Lemma A.3.

В **Proof Details for Pumpability**

Let us formally define the constructions of $\bar{\mathcal{V}}$ and $\mathcal{V}_{\text{pump}}$.

▶ **Definition B.1.** Let $V = (Q, q_0, T, F)$ be a d-dimensional Büchi VASS over Σ_n . Then $\overline{\mathcal{V}}=(Q,q_0,\overline{T},F)$ is the (d+n)-dimensional Büchi VASS over Σ_n with transitions constructed as follows: $(q, \varepsilon, (\delta, \varphi(w)), q') \in \overline{T}$ if and only if $(q, w, \delta, q') \in T$.

Furthermore, $\mathcal{V}_{\mathsf{pump}} = (Q_{\mathsf{pump}}, q_{\mathsf{pump}}, T_{\mathsf{pump}}, F_{\mathsf{pump}})$ is the d-dimensional Büchi VASS over Σ_n constructed as follows:

- $Q_{\mathsf{pump}} = Q \times (\mathbb{N} \cup \{\omega\})^{d+n}, i.e. \ the \ states \ of \ \mathsf{KM}(\bar{\mathcal{V}}),$
- $q_{\mathsf{pump},0} = (q_0, \mathbf{0}),$
- $F_{\mathsf{pump}} = F \times (\mathbb{N} \cup \{\omega\})^{d+n}, \ and$
- $(q_{\mathsf{pump}}, w, \delta, q'_{\mathsf{pump}}) \in T_{\mathsf{pump}} \ if \ and \ only \ if \ there \ is \ a \ transition \ (q_{\mathsf{pump}}, t, q'_{\mathsf{pump}}) \ in \ \mathsf{KM}(\mathcal{V})$ labelled by $t = (q, \varepsilon, (\delta, \varphi(w)), q') \in \bar{T}$.

We split the first two parts of Theorem 3.7 into Lemmas B.2 and B.3, which we prove separately. The third part later follows from Lemma 3.6, which, in turn, follows from Theorem 3.5.

▶ Lemma B.2. $L(V_{pump})$ is pumpable.

Proof. Consider some $w \in L(\mathcal{V}_{pump})$ and some $k \in \mathbb{N}$. Let ρ be an accepting run of \mathcal{V} over w. By construction of L_{pump} , there exists a corresponding run $\bar{\rho}$ in $\mathsf{KM}(\bar{\mathcal{V}})$. Let $\Omega \subseteq [d+1,d+n]$ be the set of coordinates where the states of $\mathsf{KM}(\overline{\mathcal{V}})$ carry ω eventually during $\bar{\rho}$. Then at some point, $\bar{\rho}$ visits an extended configuration $(q, \bar{m}) \in Q \times \mathbb{N}^{d+n}_{\omega}$ where all coordinates from Ω in $\bar{\boldsymbol{m}}$ are ω . Decompose $\bar{\rho} = \bar{\rho}_0 \bar{\rho}_1$ so that $\bar{\rho}_0$ reaches $(q, \bar{\boldsymbol{m}})$. Let $\rho = \rho_0 \rho_1$ and $w = w_0 w_1$ be the corresponding decompositions of ρ and w. Then ρ_0 reaches a configuration $(q, \boldsymbol{m}) \in Q \times \mathbb{N}^d \text{ in } V.$

Let $\ell = \max_{i \in [1,n]} \{0, -\varphi_i(w_0)\} + k$. By the construction of Karp-Miller graphs, there exists a run $\bar{\rho}'_0$ in $\bar{\mathcal{V}}$ that reaches a configuration $(q, \bar{\boldsymbol{m}}') \in \mathbb{N}^{d+n}$ such that $\bar{\boldsymbol{m}}'(i) \geq \boldsymbol{m}(i)$ for $i \in [1, d+n] \setminus \Omega$, and $\bar{\boldsymbol{m}}'(i) > \boldsymbol{m}(i) + \ell$ for $i \in \Omega$. Then $\bar{\rho}'_0$ corresponds to a run ρ'_0 in \mathcal{V} . It reaches a configuration (q, m') with $m' \geq m$ and thus $\rho'_0 \rho_1$ is a run of \mathcal{V} . It reads a word $w'_0w_1 \in L_{\mathsf{pump}}$, where w'_0 is the prefix read by ρ'_0 . Since w'_0 was also read by $\bar{\rho}'_0$ in $\bar{\mathcal{V}}$, it is a prefix of some word in D_n , as mandated by the additional counters of $\bar{\mathcal{V}}$.

We claim that now $\varphi(w'_0) \geq \varphi(w_0)$ and for every $i \in [1,n]$ where w ever becomes negative, we have $\varphi_i(w_0) \geq \max\{\varphi_i(w_0), 0\} + k$. The first condition follows from the fact that $\varphi_i(w_0') = \boldsymbol{m}'(d+i) \geq \boldsymbol{m}(d+i) = \varphi_i(w_0)$. For the second condition, note that if w ever becomes negative in coordinate i, then $\bar{\rho}$ must necessarily visit a configuration where in coordinate i, there is an ω . In particular, we have $d+i \in \Omega$ and thus $\varphi_i(w_0') = \boldsymbol{m}'(d+i) \geq \boldsymbol{m}(d+i) + \ell = \varphi_i(w_0) + \max_{i \in [1,n]} \{0, -\varphi_i(w_0)\} + k \geq \max\{\varphi_i(w_0), 0\} + k$.

▶ **Lemma B.3.** There exists a $k \in \mathbb{N}$ such that $L(\mathcal{V}_{pump}) \subseteq L(\mathcal{V}) \subseteq L(\mathcal{V}_{pump}) \cup P_k$.

Proof. The inclusion $L(\mathcal{V}_{\mathsf{pump}}) \subseteq L(\mathcal{V})$ is obvious from the construction. For the second inclusion, define $k \in \mathbb{N}$ to be the largest number occurring in the states of $\mathsf{KM}(\bar{\mathcal{V}})$. We claim that then $L(\mathcal{V}) \subseteq P_k \cup L(\mathcal{V}_{\mathsf{pump}})$. Let $w \in L$ be accepted by a run ρ in \mathcal{V} and suppose $w \notin P_{i,k}$ for some $i \in [1,n]$. If u is a prefix of w, then we say that $i \in [1,n]$ is crossing at u if $\varphi_i(u) < 0$ and $\varphi_i(v) \geq 0$ for every prefix v of u. Observe that whenever i is crossing at u, then $\varphi_i(v) > k$ for some prefix v of u: Otherwise, w would belong to $P_{i,k}$. This implies that ρ has a corresponding run in $\mathsf{KM}(\bar{\mathcal{V}})$: Whenever a counter in [d+1,d+n] drops below zero, it must have been higher than k before and thus been set to ω . Therefore, w is also accepted by $\mathsf{KM}(\bar{\mathcal{V}})$ and thus $w \in L(\mathcal{V}_{\mathsf{pump}})$.

C Proof Details for Basic Separators

Proof of Lemma 4.3. First of all, if $L_{\pi}(A)$ is empty, then Condition (i) trivially holds. Thus, in the following we assume that $L_{\pi}(A) \neq \emptyset$ and in particular that the final state q_{π} associated with the profile π is reachable from A's initial state.

We want to set up a system of linear inequalities that has a solution x if and only if there is a k such that $L_{\pi}(A) \subseteq S_{x,k}$. Therefore let us talk about some requirements that are necessary for the above inclusion to hold. These requirements will be on cycles of transitions in π , and we make sure that they can be expressed as linear inequalities.

The cycle σ_{π} that contains exactly the transitions in π has to be over a word $v_{\pi} \in \Sigma_{n}^{*}$ with $\langle \boldsymbol{x}, \varphi(v_{\pi}) \rangle \leq -1$. Otherwise, we can just repeat σ_{π} infinitely often and prepend any prefix leading to q_{π} from \mathcal{A} 's initial state, yielding a word that violates requirement b.) of $S_{\boldsymbol{x},k}$. Any primitive cycle σ in \mathcal{A} of transitions in π has to be over a word v with $\langle \boldsymbol{x}, \varphi(v) \rangle \leq 0$. Otherwise we repeat σ_{π} infinitely often from some arbitrary prefix reaching q_{π} like before, and then perform k+1 insertions of the cycle σ into each copy of σ_{π} . This yields a word that violates requirement a.) of $S_{\boldsymbol{x},k}$, and we can do this for any k.

We can now use these requirements on cycles to construct a linear system of inequalities $\mathbf{A}_{\pi}\mathbf{x} \leq \mathbf{b}$ for \mathbf{x} . For the cycle σ_{π} corresponding to word $v_{\pi} \in \Sigma_{n}^{*}$, we add the inequality

$$x_1\varphi_1(v_{\pi}) + x_2\varphi_2(v_{\pi}) + \ldots + x_n\varphi_n(v_{\pi}) \le -1,$$

and for each primitive cycle σ of transitions in π over a word $v \in \Sigma_n^*$, we add the inequality

$$x_1\varphi_1(v) + x_2\varphi_2(v) + \ldots + x_n\varphi_n(v) \le 0.$$

Let us argue that the precise choice of the justifying cycle σ_{π} does not affect the satisfiability of the system $\mathbf{A}_{\pi}\mathbf{x} \leq \mathbf{b}$. To this end we argue that $\mathbf{x} \in \mathbb{N}^n$ is a valid solution to the system if and only if (1) all primitive cycles have an \mathbf{x} -weighted balance at most zero, and (2) at least one primitive cycle has a strictly negative \mathbf{x} -weighted balance. Constraint (1) is clearly equivalent to the inequalities added for each primitive cycle.

For constraint (2), assume that the inequality $\langle \boldsymbol{x}, \varphi(v_{\pi}) \rangle \leq -1$ holds. Now observe that any valid choice of σ_{π} is a cycle and therefore can be constructed by inserting primitive cycles into each other a finite number of times. If all primitive cycles had non-negative \boldsymbol{x} -weighted balance, then the \boldsymbol{x} -weighted balance for σ_{π} could not be negative.

For the other implication direction, assume that constraint (2) holds, and let the primitive cycle with negative x-weighted balance be σ' . Since any valid choice of σ_{π} contains each transition in π , its $|\pi|$ -fold repetition $\sigma_{\pi}^{\dagger}\pi$ contains each primitive cycle as a (possibly non-contiguous) subsequence. Now, if we delete σ' from $\sigma_{\pi}^{\dagger}\pi$, the remaining (possibly not connected) transition sequences still combine to form a collection of cycles, since σ' is a cycle. Thus, the summed-up x-weighted balance of this collection is the sum of x-weighted balances of primitive cycles, and can therefore be at most zero by Condition (1). Then adding σ' back in gives us that $\langle x, \varphi(v_{\pi}^{\dagger}\pi) \rangle$ is negative, and therefore $\langle x, \varphi(v_{\pi}) \rangle$ is as well. Since the letter balance can only have integer-values, and weighting by $x \in \mathbb{N}$ does not change this, it follows that $\langle \boldsymbol{x}, \varphi(v_{\pi}) \rangle \leq -1$.

The characterization of solutions x via constraints (1) and (2) is clearly independent of σ_{π} , meaning the precise choice of the latter does not affect satisfiability of the system $A_{\pi}x \leq b$. Furthermore, the restriction of $x \in \mathbb{N}^d$ is not a meaningful one, as we can always compute a solution in \mathbb{Q}^d from one in \mathbb{N}^d , as we explain below.

Applying Farkas' Lemma (Theorem 4.1) to this system of equations, we either obtain a vector $\boldsymbol{x} \in \mathbb{Q}_{\geq 0}^n$ as a suitable solution, or we obtain a vector $\boldsymbol{y} \in \mathbb{Q}_{\geq 0}^m$ with $\boldsymbol{y}^{\top} \boldsymbol{A}_{\pi} \geq \boldsymbol{0}^{\top}$ and $\mathbf{y}^{\top}\mathbf{b} < 0$, where m is the number of rows of \mathbf{A}_{π} .

In the first case, we can multiply the entries of x by their denominators' least common multiple, say ℓ , to yield a suitable vector $\ell \cdot x = x' \in \mathbb{N}^n$. Furthermore we set $k = |Q_{\pi}| \cdot h$, where $|Q_{\pi}| \supseteq \{q_{\pi}\}$ is the set of all states of \mathcal{A} adjacent to transitions in π , and h is the length of the longest word appearing as a transition label of A. With this we can show that $L_{\pi}(\mathcal{A}) \subseteq S_{x',k}$: Each word $w \in L_{\pi}(\mathcal{A})$ decomposes into uv with $v = v_0v_1v_2\cdots$ such that u leads to q_{π} from A's initial state and each v_i corresponds to some cycle σ_i on q_{π} , that contains each transition of π at least once. Then we have $\langle \boldsymbol{x}', \varphi(v_i) \rangle = \ell \cdot \langle \boldsymbol{x}, \varphi(v_i) \rangle < \ell \cdot 0 = 0$ as required by $S_{x,k}$: each cycle σ_j can be obtained by starting with σ_{π} , which contributes at most -1 to this value, and inserting finitely many primitive cycles, which all add at most 0. Moreover, we need to show $\langle x', \varphi(f) \rangle \leq k$ for every infix f of v. Towards a contradiction assume there is at least one infix f of v, for which this does not hold. Since f fulfils $\langle x', \varphi(f) \rangle > |Q_{\pi}| \cdot h$, and h is the maximum length of a transition label, the transition sequence corresponding to f has to be longer than $|Q_{\pi}|$. Thus this sequence repeats a state and therefore has to contain a primitive cycle. However, all such primitive cycles add at most 0 to the value $\langle x', \varphi(f) \rangle$, meaning one could delete the word corresponding to this cycle from f and still fulfil the aforementioned requirement. One can repeatedly remove primitive cycles until one obtains a word f' of length $|f'| \leq |Q_{\pi}| \cdot h$ with $\langle x', \varphi(f') \rangle > |Q_{\pi}| \cdot h$. This is a contradiction, therefore infixes such as f cannot exist.

In the other case we also multiply y with the least common multiple of its entries, say ℓ , to yield $\ell \cdot y = y' \in \mathbb{N}^m$. Furthermore, each row of the matrix A_{π} essentially contains the φ -values of its corresponding cycle. The requirement ${y'}^{\top} A_{\pi} = \ell y^{\top} A_{\pi} \geq \ell \cdot 0^{\top} = 0^{\top}$ can then be seen as a selection of cycles, whose combined φ -values are all 0 or above. Moreover, the requirement $y'^{\top}b = \ell y^{\top}b < \ell \cdot 0 = 0$ ensures that σ_{π} is selected at least once, because all other entries of **b** are 0, meaning we would have $\mathbf{y}^{\top}\mathbf{b} = 0$ if σ_{π} was not selected. This means we can combine all the selected cycles into one large cycle σ' via matching states, which is possible because σ_{π} visits all states in Q_{π} . Since the combined φ -values of all the cycles selected by y are 0 or above, we have that σ' corresponds to a word w' with $\varphi(w') \geq 0$. Finally, σ' also contains all transitions of π as required, because it contains the cycle σ_{π} .

Regarding Theorem 3.5 We mentioned in 4 that a single value of k is sufficient for a finite union of basic separators $P_{i,k}$ and $S_{x,k}$. This is because we have $P_{i,k} \subseteq P_{i,k+1}$ and $S_{\boldsymbol{x},k} \subseteq S_{\boldsymbol{x},k+1}$ for each $i \in [1,n], \boldsymbol{x} \in \mathbb{N}^n, k \in \mathbb{N}$. Therefore it suffices to show the following: Let \mathcal{A} be a Büchi automaton with $L(\mathcal{A}) = R \subseteq \Sigma_n^{\omega}$ and $R \cap D_n = \varnothing$. Then there is a finite set $X \subseteq \mathbb{N}^n$ and a number $k \in \mathbb{N}$ such that $R \subseteq \bigcup_{i \in [1,n]} P_{i,k} \cup \bigcup_{\boldsymbol{x} \in X} S_{\boldsymbol{x},k}$.

Here R is a separator candidate in the sense of the original phrasing of the theorem, because it is ω -regular and disjoint from D_n .

Proof of Theorem 3.5. We begin by invoking Theorem 3.7 on \mathcal{A} to obtain a Büchi automaton $\mathcal{A}_{\mathsf{pump}}$, whose language is pumpable, and a number ℓ such that $L(\mathcal{A}_{\mathsf{pump}}) \subseteq L(\mathcal{A}) \subseteq L(\mathcal{A}_{\mathsf{pump}}) \cup P_{\ell}$. Using the theorem this way is feasible, because Büchi automata can be seen as 0-dimensional Büchi VASS. Since $L(\mathcal{A}) \cap D_n = \emptyset$ and $L(\mathcal{A}_{\mathsf{pump}}) \subseteq L(\mathcal{A})$ we have $L(\mathcal{A}_{\mathsf{pump}}) \cap D_n = \emptyset$. It now suffices to show that the basic separators theorem holds for $L(\mathcal{A}_{\mathsf{pump}})$: If there are X, k such that $L(\mathcal{A}_{\mathsf{pump}}) \subseteq \bigcup_{i \in [1,n]} P_{i,k} \cup \bigcup_{x \in X} S_{x,k}$ then $L(\mathcal{A}) \subseteq L(\mathcal{A}_{\mathsf{pump}}) \cup P_{\ell} \subseteq \bigcup_{i \in [1,n]} P_{i,o} \cup \bigcup_{x \in X} S_{x,o}$, where $o = \max(k,\ell)$.

Now consider the decomposition $L(\mathcal{A}_{\mathsf{pump}}) = \bigcup_{\pi \in \Pi(\mathcal{A}_{\mathsf{pump}})} L_{\pi}(\mathcal{A}_{\mathsf{pump}})$. If we can show that each language $L_{\pi}(\mathcal{A}_{\mathsf{pump}})$ is contained in a finite union of basic separators, then we are done. In the following let us fix a profile π of $\mathcal{A}_{\mathsf{pump}}$.

We now invoke Lemma 4.3 on $\mathcal{A}_{\mathsf{pump}}$ and π . If Condition (i) holds, then this already yields x, k such that $L_{\pi}(\mathcal{A}_{\mathsf{pump}}) \subseteq S_{x,k}$, and we need not concern ourselves with the languages $P_{i,k}$. In the other case, Condition (ii) yields a cycle c' in $\mathcal{A}_{\mathsf{pump}}$ that contains all transitions in π and is over a word w' with $\varphi(w') \geq 0$. Since Condition (i) did not hold, we know that $L_{\pi}(\mathcal{A}_{\mathsf{pump}})$ is not empty, which means that all states adjacent to transitions of π are reachable from \mathcal{A} 's initial state, including the final state q_{π} associated with π . Let u' be a word that reaches q_{π} from $\mathcal{A}_{\mathsf{pump}}$'s initial state. Then $\tilde{w} = u'(w')^{\omega} \in L(\mathcal{A}_{\mathsf{pump}})$.

Let m be the lowest value of φ_i for any index i and prefix of \tilde{w} , formally $m = \min_{i \in [1,n], v \in \operatorname{prefix}(\tilde{w})} \varphi_i(v)$. Since $\varphi(w') \geq 0$ we know that $m \in \mathbb{Z}$ is well-defined. Moreover, since $L(\mathcal{A}_{\mathsf{pump}})$ is pumpable, there is a decomposition $\tilde{w} = u_0 w_1$ and a word $v_0 \in \Sigma_n^*$ such that $v_0 w_1 \in L(\mathcal{A}_{\mathsf{pump}})$, $\varphi(v_0) \geq \varphi(u_0)$, and $\varphi_i(v_0) \geq \varphi_i(u_0) + |m|$ for all indices i where there is a $v \in \operatorname{prefix}(\tilde{w})$ with $\varphi_i \geq 0$. Then swapping u_0 for v_0 in \tilde{w} can only increase the φ_i -values of its prefixes, and in fact all such values that fell below 0 are now raised above 0 by choice of |m|. This means that $v_0 w_1 \in L(\mathcal{A}_{\mathsf{pump}}) \cap D_n$, which is a contradiction.

D Proof Details for Decidability

▶ **Lemma 3.6.** If $L \subseteq \Sigma_n^{\omega}$ is pumpable, then $L \mid D_n$ if and only if $L \mid_{\lim} D_n$, where $L \mid_{\lim} D_n$ means $L \subseteq \bigcup_{\boldsymbol{x} \in X} S_{\boldsymbol{x},k}$ for some finite set $X \subseteq \mathbb{N}^n$ and some $k \in \mathbb{N}$.

Proof. The "if" direction is trivial. Conversely, let $L \mid D_n$. By Theorem 3.5, we have $L \subseteq \bigcup_{i \in [1,n]} P_{i,k} \cup \bigcup_{x \in X} S_{x,k}$ for some finite $X \subseteq \mathbb{N}^n$ and $k \in \mathbb{N}$. We claim that $L \subseteq \bigcup_{x \in X} S_{x,k}$, which yields $L \mid_{\lim} D_n$. Indeed, given $u \in L$, pumpability yields a word $u' \in L$ such that $u' \sim u$ and $u' \notin P_{i,k}$ for any $i \in [1,n]$. Since $u' \in L \subseteq P_k \cup \bigcup_{x \in X} S_{x,k}$, we conclude $u' \in S_{x,k}$. Finally, observe that membership in $S_{x,k}$ is not affected by changing a finite prefix of a word. Therefore, we also have $u \in S_{x,k}$.

▶ Lemma D.1. Let $\pi \in \Pi(\mathcal{V})$. If $A_{\pi}x \leq b$ for $x \in \mathbb{N}^n$, then $L_{\pi}(\mathcal{V}) \subseteq S_{x,k}$ for some $k \in \mathbb{N}$.

Proof. We regard $\mathsf{KM}(\mathcal{V})$ as a Büchi automaton. Then, π is in particular a profile for $\mathsf{KM}(\mathcal{V})$. Moreover, the cycle witnessing that π is a profile is also an admissible cycle for π in $\mathsf{KM}(\mathcal{V})$ as a Büchi automaton. Thus, Corollary 4.4 implies $L_{\pi}(\mathcal{V}) \subseteq L_{\pi}(\mathsf{KM}(\mathcal{V})) \subseteq S_{\boldsymbol{x},k}$ for some $k \in \mathbb{N}$.

E Proof Details for One-dimensional Büchi VASS

E.1 Theorem **6.1**: PSPACE-hardness

We begin with the straightforward reduction from intersection emptiness of finite-word languages of 1-dim. VASS. Suppose $L_1, L_2 \subseteq \Sigma^*$ are finite-word languages of 1-dim. VASS with succinct counter updates, and acceptance by final state. Checking whether the intersection $L_1 \cap L_2$ is empty is PSPACE-complete [24]. We construct Büchi VASS for $L_1 \#^{\omega}$ and $L_2 \#^{\omega}$, where # is a fresh letter. Since L_1 and L_2 are coverability languages of finitely-branching WSTS, we know from [14, Theorem 7] that $L_1 \cap L_2 = \emptyset$ if and only if $L_1 \mid L_2$. Furthermore, with a fresh letter #, it is easy to observe that $L_1 \mid L_2$ if and only if $L_1 \#^{\omega} \mid L_2 \#^{\omega}$.

Hardness for disjoint languages In the PSPACE-hardness proof above, one can notice that the languages $L_1\#^{\omega}$ and $L_2\#^{\omega}$ are regularly separable if and only if they are disjoint. In order to further highlight the disparity between the finite-word case of WSTS languages (where disjointness and separability coincide [14]) and the infinite-word case, we want to present a proof that PSPACE-hardness already holds if the input languages are promised to be disjoint: Note that with this promise, separability in the finite-word case becomes trivial.

Here, we reduce directly from configuration reachability in bounded one-counter automata, which was shown to be PSPACE-hard in [23].

A bounded one-counter automaton $B = (\mathcal{V}_B, b)$ consists of a 1-dim. VASS \mathcal{V}_B equipped with a bound $b \in \mathbb{N}$ on its counter values. This means transitions of B are enabled if and only if they meet the firing restrictions of a VASS and also lead to a configuration (q, m) with $m \leq b$. Here, counter values and the bound b are encoded in binary. In particular, the size of B is that of the underlying VASS plus $\log b$, and the size of a configuration (q, m) is $\log m$.

Now we want to construct two 1-dim. Büchi VASS \mathcal{V}_1 and \mathcal{V}_2 , whose languages are always disjoint, but are ω -regular separable if and only if (q,m) is not reachable from $(q_0,0)$ in $B=(\mathcal{V}_B,b)$. Let T be the set of transitions of B. We use $\Sigma=T\cup\{\#\}\cup\Sigma_1$ as the alphabet for \mathcal{V}_1 and \mathcal{V}_2 . Let \mathcal{V}_{D1} be the 1-dim. Büchi VASS accepting D_1 , i.e. \mathcal{V}_{D1} consists of a single state, both initial and final, with two loops $e_1|a_1$ and $-e_1|\bar{a}_1$. Furthermore let \mathcal{V}_S be the 1-dim. Büchi VASS from Figure 1(left) accepting the language S with $S\cap D_1=\emptyset$ but $S\not\upharpoonright D_1$, which we talked about in Section 3 (see the proof of the first statement in Theorem 3.2).

We start constructing \mathcal{V}_1 by using a copy of \mathcal{V}_B with all states being non-final and every transition $t \in T$ labelled with t itself. Then we add a copy of \mathcal{V}_{D1} with its only state still being final. To connect the two copies, we add the transition -m|# from state q of \mathcal{V}_B to the initial state of \mathcal{V}_{D1} .

For V_2 we also start with a copy of V_B with all non-final states and transitions labeled with themselves, but we also invert every transition effect, changing it from $z \in \mathbb{Z}$ to -z. Then we add a new initial state q'_0 with the same outgoing transitions as the initial state of V_B , except we change their original effects z to b-z. These new transitions of q'_0 are labelled with their original copies from T. Additionally, we add a copy of V_S with q_2 still being a final state. The two copies are then connected with a transition m-b|# from q to the initial state of V_S . If $(q,m)=(q_0,0)$, we also add a transition 0|# from q'_0 to the initial state of V_S .

Now let R_1 be the set of all transition sequences over T that cover (q, m) in \mathcal{V}_B and do not necessarily respect the bound b. Formally, $\rho \in R_1$ if ρ leads from $(q_0, 0)$ to (q, m') in \mathcal{V}_B for some $m' \in \mathbb{N}$ with $m' \geq m$. Moreover let R_2 be the set of all transition sequences over T that reach q with a counter value below m, when respecting the upper bound b, but not

necessarily the lower bound 0 of VASS counters. Formally, $\rho \in R_2$ if ρ leads from $(q_0, 0)$ to (q, z') in B' for some $z' \in \mathbb{Z}$ with $z' \leq m$, where $B' = (\mathcal{V}', b)$ and \mathcal{V}' is just \mathcal{V} interpreted as a \mathbb{Z} -VASS. We now want to argue that there are languages L_1, L_2 such that $L(\mathcal{V}_1) = R_1 \# L_1$ and $L(\mathcal{V}_2) = R_2 \# L_2$.

 $L(\mathcal{V}_1) = R_1 \# L_1$ is easy to see, since \mathcal{V}_1 simulates \mathcal{V}_B faithfully, and can only read # if a configuration (q,m) or greater is reached. For $L(\mathcal{V}_2) = R_2 \# L_2$ observe that before reading #, \mathcal{V}_2 essentially simulates \mathcal{V}_B with inverted counter values, starting with b instead of 0. Since \mathcal{V} can go above b, this essentially simulates going below 0 in B. The # can also only be read if in q the counter is valued at least m-b, which corresponds to at most m before inversion. Let us now show that $L(\mathcal{V}_1) \cap L(\mathcal{V}_2) = \emptyset$, and furthermore $L(\mathcal{V}_1) \mid L(\mathcal{V}_2)$ if and only if $(q_0,0) \to (q,m)$ in B.

The configuration (q, m) not being reachable in B is equivalent to R_1 and R_2 being disjoint. In this case $L(\mathcal{V}_1)$ and $L(\mathcal{V}_2)$ also have to be disjoint, since the prefixes before the '#' of their words cannot coincide. They are however ω -regular separable: With Q being the states of B, an exponential size Büchi automaton A with states $Q \times \{0, \ldots, b\}$ can simulate B. To make A accept all words with prefixes in R_1 , we add a final state with loops on all input letters, that is reachable by every transition that would make the counter value go above b. Now $L(\mathcal{V}_1) \subseteq L(A)$ is clear. Since transition sequences that do not respect the bound b cannot be prefixes of elements of R_2 , $L(A) \cap L(\mathcal{V}_2) = \emptyset$ immediately follows. Thus, L(A) is an ω regular separator for $L(\mathcal{V}_1)$ and $L(\mathcal{V}_2)$, which also means that they are disjoint.

If (q,m) is reachable in B, we have a finite transition sequence $\rho \in R_1 \cap R_2$. Reading ρ then leads to (q,m) in \mathcal{V}_1 , respectively to (q,b-m) in \mathcal{V}_2 . Therefore if # is read right after, the counter value of either Büchi VASS would be 0. This implies that $\rho \# D_1 \subseteq L(\mathcal{V}_1)$ and $\rho \# S \subseteq L(\mathcal{V}_1)$, as the second component of either VASS would be simulated faithfully after this prefix. A regular separator A for $L(N_1)$ and $L(N_2)$ would therefore have to accept all words in $\rho \# D_1$ but no words in $\rho \# S$. By adding a new initial state q_{init} to A and adding all outgoing transitions of states reachable via $\rho \#$ in the original A to q_{init} , we obtain an ω -regular separator for D_1 and S. This is a contradiction, since we established earlier that these languages are not ω -regular separable as shown in the proof of the first half of Theorem 3.2.

It remains to show that $L(\mathcal{V}_1) \cap L(\mathcal{V}_2) = \emptyset$ in the case where (q, m) is reachable in B. For transition sequences ρ over T, we know that $\rho \in R_1 \cap R_2$ if and only if ρ reaches (q, m) in B. Therefore two words $w_1 \in L(\mathcal{V}_1)$ and $w_2 \in L(\mathcal{V}_2)$ can only agree on a prefix $\rho \#$, if ρ has this property. However, in this case $w_1 = \rho \# w_1'$ for some $w_1' \in D_1$ and $w_2 = \rho \# w_2'$ for some $w_2' \in S$. This yields $w_1 \neq w_2$ since D_1 and S are disjoint.

E.2 Proof of Proposition 6.2

▶ Proposition 6.2. Let V be a Büchi VASS with $L(V) \subseteq \Sigma_n^{\omega}$. Then $L(V) \not\mid D_n$ if and only if $\mathsf{KM}(\bar{V})$ has an inseparability flower.

Proof. We first invoke Theorem 3.7 to obtain $\mathcal{V}_{\mathsf{pump}}$ with $L(\mathcal{V}_{\mathsf{pump}}) \not\mid D_n$ if and only if $L(\mathcal{V}) \not\mid D_n$. Recall that $\mathcal{V}_{\mathsf{pump}}$ was constructed as the product of \mathcal{V} and $\mathsf{KM}(\bar{\mathcal{V}})$, which means that every cycle of $\mathcal{V}_{\mathsf{pump}}$ is also a cycle of $\mathsf{KM}(\bar{\mathcal{V}})$.

For the if direction, we get that $\mathsf{KM}(\mathcal{V}_{\mathsf{pump}})$ contains an inseparability flower by Theorem 5.3. Its three cycles then correspond to three cycles of $\mathcal{V}_{\mathsf{pump}}$, which then also appear in $\mathsf{KM}(\bar{\mathcal{V}})$, where they still fulfill the requirements of an inseparability flower.

For the only if direction, observe that $\mathsf{KM}(\mathcal{V}_{\mathsf{pump}})$ is essentially the product construction of $\mathsf{KM}(\mathcal{V})$ and $\mathsf{KM}(\bar{\mathcal{V}})$. Furthermore, any transition sequence permitted by $\bar{\mathcal{V}}$ is also permitted

by \mathcal{V} , as the former only added restrictions in the form of more counters, but did not remove any. Thus, each cycle of $\mathsf{KM}(\bar{\mathcal{V}})$ (including the ones that make up its inseparability flower) also appears as a cycle in $\mathsf{KM}(\mathcal{V}_{\mathsf{pump}})$. This implies that $\mathsf{KM}(\mathcal{V}_{\mathsf{pump}})$ also has an inseparability flower, and by Theorem 5.3 it follows that $L(\mathcal{V}_{\mathsf{pump}}) \not\mid D_n$.

E.3 Proof of Lemma 6.4

▶ Lemma 6.4. Given 1-dim. Büchi VASS V_1, V_2 with binary updates, there is a a 1-dim. Büchi VASS V with $L(V_1) \cap L(V_2) = \emptyset$ iff $L(V) \cap D_1 = \emptyset$, $L(V_1) \mid L(V_2)$ iff $L(V) \mid D_1$, and we can construct in time polynomial in $|V_1| + |V_2|$ the 2-dim. Büchi VASS \bar{V} (binary updates).

Proof. Recall that $\bar{\mathcal{V}}$ is constructed from a Büchi VASS \mathcal{V} over alphabet Σ_n by adding n additional counters and for each transition t replacing its label $w \in \Sigma_n^*$ with ε and instead adding to t an effect of $\varphi(w)$ on the n additional counters. A precise definition can be found in Appendix B. To now proof Lemma 6.4 we have to argue that we can modify Lemma 3.4 to directly construct $\bar{\mathcal{V}}$ instead of \mathcal{V} , and that the modified version is feasible in polynomial time.

If we analyze the proof of Lemma 3.4 in Appendix A then we obtain exponential time complexity for this construction. The bottleneck here is Lemma A.2. However, we already mentioned in the proof of Lemma A.2, that its associated complexity shrinks from exponential to polynomial time, if we can somehow compress the exponentially long transition labels that we end up with. An adequate compression for this is replacing an exponentially long string $w \in \Sigma_n^*$ by its effect on the letter balance $\delta(w)$, which is exactly what we do when going from \mathcal{V} to $\bar{\mathcal{V}}$. Since we encode $\delta(w)$ in binary, this is an exponential compression, and therefore the time complexity of constructing $\bar{\mathcal{V}}$ directly is only polynomial, as required.

Note that for two 1-dimensional Büchi VASS as input, we have n=1. But our proof shows that constructing $\bar{\mathcal{V}}$ in polynomial time would still be feasible for Büchi VASS of arbitrary dimension n.

E.4 Proof of Proposition 6.5

▶ **Proposition 6.5.** The constrained runs problem for 2-VASS is solvable in PSPACE.

Proof. We show that if there is a constrained run, then there is one where all counters have at most exponential values along the way. For this, we rely on a result from [5] about linear path schemes.

A linear path scheme (LPS) for a 2-dimensional VASS \mathcal{V} is a regular expression of the form $S = \sigma_0 \lambda_1 \sigma_1 \cdots \lambda_n \sigma_n$. Its alphabet is the set T of transition of \mathcal{V} , and each infix λ_i corresponds to a cycle of transitions in \mathcal{V} .

Each LPS S induces a reachability relation \to_S over configurations of \mathcal{V} , where $(q, x, y) \to_S (q', x', y')$ if and only if there are numbers $x_1, \ldots, x_n \in \mathbb{N}$ such that $\sigma_0 \lambda_1^{x_1} \sigma_1 \cdots \lambda_m^{x_n} \sigma_n$ is a run of \mathcal{V} from (q, x, y) to (q', x', y'). In [5, Theorem 3.1], it is shown that for any two states q, q' in a 2-VASS \mathcal{V} , there exists a set \mathcal{S} of LPSs, each of which is of polynomial length, such that for $x, y, x', y' \in \mathbb{N}$, (q', x', y') is reachable from (q, x, y) if and only if $(q, x, y) \to_S (q', x', y')$ for some S from \mathcal{S} .

In [5], this yields a PSPACE algorithm for configuration reachability in 2-dimensional VASS: If there is run reaching a certain configuration, then there is one of the form $\sigma_0 \lambda_1^{x_1} \sigma_1 \cdots \lambda_n^{x_n} \sigma_n$ for some LPS $\sigma_0 \lambda_1 \sigma_1 \cdots \lambda_n \sigma_n$ of polynomial length. Now the fact that $\sigma_0 \lambda_1^{x_1} \sigma_1 \cdots \lambda_n^{x_n} \sigma_n$ is a run between two given configurations can be expressed using a set of linear inequalities

over x_1, \ldots, x_n . Since each solvable polynomial-sized set of linear inequalities has a solution with at most exponential entries, this yields a run where all counters are at most exponential.

We only need to extend this argument from [5] slightly: First, we want to guess a system of linear inequalities, whose solutions would satisfy the Presburger formula ψ . To this end, we view ψ as a propositional formula by treating each atomic formula as a proposition. For Presburger arithmetic, an atomic formula is either an equality $t_1 = t_2$ or an inequality $t_1 < t_2$, where t_1, t_2 are additive terms over variables and/or the constants 0, 1. With this propositional view of ψ , we can guess an assignment to its propositions, and verify that its a satisfying assignment, feasible in polynomial space. If this assignment sets an atomic formula of the form $t_1 = t_2$ to false, this means that $t_1 < t_2$ or $t_2 < t_1$ has to hold. Similarly if $t_1 < t_2$ is set to false then $t_1 = t_2$ or $t_2 < t_1$ has to hold. In both cases, we simply guess one of the two atomic formulas that have to hold instead. With these guesses together with the unchanged atomic formulas that were set to true, we obtain a system of equalities and inequalities, whose solutions would satisfy ψ . Formally, this system is comprised of matrices $A \in \mathbb{Z}^{\ell \times m}$, $C \in \mathbb{Z}^{k \times m}$ and vectors $b \in \mathbb{Z}^{\ell}$, $d \in \mathbb{Z}^{k}$ with entries encoded in binary, such that $x \in \mathbb{N}^m$ is a solution if and only if Ax < b and Cx = d. In fact unary encodings would suffice for our definition of Presburger, since an entry of e.g. 3 would have come from a term of the form y + y + y for a variable y, meaning all entries are polynomial in the size of ψ . However, we do not require unary encodings and can also work with binary ones.

Now we only need to check that there is a constrained run $(q_0,0,0) \stackrel{*}{\to} (q_1,x_1,y_1) \stackrel{*}{\to} \cdots \stackrel{*}{\to} (q_m,x_m,y_m)$, whose counter values indeed satisfy these equalities and inequalities. This is the case if Az < b and Cz = d, where $z = (x_1,y_1,\ldots,x_m,y_m)$. If such a constrained run exists, then for each $i \in [1,m]$, there is an LPS for the part $(q_{i-1},x_{i-1},y_{i-1}) \stackrel{*}{\to} (q_i,x_i,y_i)$ such that said run conforms to each of these LPSs. By imposing (a) the linear inequalities of [5], which make sure that all counters stay non-negative, and (b) our linear inequalities Az < b and equalities Cx = d, we obtain a new (poynomial-size) system of linear inequalities over the exponents in the LPSs.

By [50] systems like these have minimal solutions with at most exponential entries, yielding an overall run with at most exponential counter values. Binary encoding then means that these solutions only take up polynomial space. More specifically, this implies that we can simply guess configurations (q_1, x_1, y_1) to (q_m, x_m, y_m) of the constrained run in PSPACE, and then check that equalities and inequalities of our system hold for them, i.e. that they are actual solutions to the system. This concludes the description of our decision procedure.

As a final remark, note that [50] assumes inequalities of the form $t_1 \leq t_2$ rather than $t_1 < t_2$. However, since we seek solutions in \mathbb{N}^m , we can simply express $t_1 < t_2$ as $t_1 + 1 \leq t_2$ to circumvent this issue.

F Regular Separability vs. Intersection

In this section we prove the second part of Theorem 3.2. To this end we present a class of WSTS such that, for their ω -languages, intersection is decidable whereas regular separability is not. A d-dimensional reset $B\ddot{u}chi\ VASS$ over alphabet Σ is a tuple $\mathcal{V}=(Q,q_0,T,F)$. The only difference to $B\ddot{u}chi\ VASS$ is in the finite set of transitions which, besides adding a vector, may reset a counter, $T\subseteq Q\times(\mathbb{Z}^d\cup\{\mathbf{r}_1,\ldots,\mathbf{r}_d\})\times\Sigma^*\times Q$. The configurations are defined like for $B\ddot{u}chi\ VASS$, but the transition relation has to be adapted. We have $(q,m)\xrightarrow{w}(q',m')$ if there is a transition (q,x,w,q') such that either (i) $x\in\mathbb{Z}^d$ and m'=m+x or (ii) $x=\mathbf{r}_i$ for some $i\in[1,d]$ and m'(j)=m(j) for $j\in[1,d]\setminus\{i\}$ and m'(i)=0. Acceptance is defined as before, and so is the language (of infinite words) $L(\mathcal{V})$.

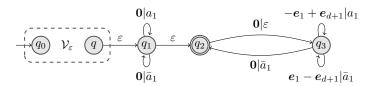


Figure 4 Weak Büchi reset VASS \mathcal{V}' in the proof of Theorem F.2.

For general Büchi reset VASS, emptiness and intersection are undecidable [37, Theorem 10]. We consider a slight restriction of the model that makes the problems decidable. A Büchi reset VASS is *weak* if there is no path from a final state to a reset transition. In particular, an accepting run can only perform finitely many resets. Note that the usual product construction of \mathcal{V}_1 and \mathcal{V}_2 to yield a Büchi reset VASS for $L(\mathcal{V}_1) \cap L(\mathcal{V}_2)$ preserves weakness.

▶ Theorem F.1. For weak Büchi reset VASS, emptiness (hence intersection) is decidable.

Here, emptiness can be decided using standard techniques. We order the configurations $Q \times \mathbb{N}^d$ in the usual way: We have $(q, \boldsymbol{m}) \leq (q', \boldsymbol{m}')$ if q = q' and $\boldsymbol{m} \leq \boldsymbol{m}'$. First, one observes that for any Büchi VASS \mathcal{V} , the set $U(\mathcal{V}) \subseteq Q \times \mathbb{N}^d$ of all configurations (q, \boldsymbol{m}) from which an infinite accepting run can start, is upward closed. Moreover, using a saturation procedure, we can effectively compute the finitely many minimal elements $(q_1, \boldsymbol{m}_1), \ldots, (q_\ell, \boldsymbol{m}_\ell)$ of $U(\mathcal{V})$. The details can be found in Lemma F.3 at the end of this section. Then, for a weak Büchi reset VASS \mathcal{V} , we do the following. We construct the Büchi VASS \mathcal{V}' , which is obtained from \mathcal{V} by deleting all reset transitions. Now $L(\mathcal{V})$ is non-empty if and only if \mathcal{V} , as a reset VASS, can cover any of the configurations $(q_1, \boldsymbol{m}_1), \ldots, (q_\ell, \boldsymbol{m}_\ell)$ of $U(\mathcal{V}')$. Whether the latter is the case can be decided because coverability is decidable in reset VASS [19, 25, 2].

▶ Theorem F.2. For weak B. reset VASS over Σ_1 , regular separability from D_1 is undecidable.

We reduce from the place boundedness problem for reset VASS. A reset VASS is a Büchi reset VASS without input words and without final states. For $k \in \mathbb{N}$, we say that a reset VASS \mathcal{V} is k-place bounded if for every reachable configurations (q, \mathbf{m}) , we have $\mathbf{m}(1) \leq k$. Moreover, we call \mathcal{V} place bounded if \mathcal{V} is k-place bounded for some $k \in \mathbb{N}$. The place boundedness problem then asks whether a given reset VASS is place bounded. The place boundedness problem (more generally, the boundedness problem) for reset VASS is known to be undecidable [19, Theorem 8] (for a simpler proof, see [37, Theorem 18]).

Given a d-dim. reset VASS \mathcal{V} , we build a (d+1)-dim. weak Büchi reset VASS \mathcal{V}' with

$$L(\mathcal{V}') = \{ w \in S_{1,k} \mid k \in \mathbb{N}, \ \mathcal{V} \text{ can reach some } (q, \mathbf{m}) \in Q \times \mathbb{N}^d \text{ with } \mathbf{m}(1) \ge k \}.$$

Before we describe \mathcal{V}' , observe that $L(\mathcal{V}') \mid D_1$ iff \mathcal{V} is place bounded. If \mathcal{V} is k-place bounded, then $L(\mathcal{V}') \subseteq S_{1,k}$ and thus $L(\mathcal{V}') \mid D_1$. On the other hand, if \mathcal{V} is not place bounded, then $L(\mathcal{V}') = \bigcup_{k \in \mathbb{N}} S_{1,k}$. As for the Büchi VASS in Figure 1 (left), one can show $L(\mathcal{V}') \not\mid D_1$.

The construction is depicted in Figure 4. The dashed box contains $\mathcal{V}_{\varepsilon}$, which is obtained from \mathcal{V} by changing every transition (p, \mathbf{u}, q) into $(p, (\mathbf{u}, 0), \varepsilon, q)$. In the figure, q stands for arbitrary states of $\mathcal{V}_{\varepsilon}$, meaning for every state q in $\mathcal{V}_{\varepsilon}$, we have a transition $(q, \mathbf{0}, \varepsilon, q_1)$. Observe that in the states $q_1, q_2, q_3, \mathcal{V}'$ behaves exactly like the Büchi VASS in Figure 1(left), except that the additional counter ensures that for each infix the balance on letter a_i is bounded by k from configurations $(q_1, (k, \mathbf{u}))$. Thus the accepted language from $(q_1, (k, \mathbf{u}))$ is exactly $S_{1,k}$. This shows that \mathcal{V}' accepts the language (2).

▶ **Lemma F.3.** Let V be a Büchi VASS. We can compute the set U(V) of minimal configurations from which there is an infinite accepting run.

Proof. It is decidable whether a given Büchi VASS has an accepting run [21, 28]. We strengthen this result to checking whether a given Büchi VASS \mathcal{V} has an accepting run starting in a downward-closed set of configurations. The downward-closed set is given as a finite union I of ideals, each represented by a generalized configuration $(q, m) \in Q \times (\mathbb{N} \cup \{\omega\})^d$. The algorithm is as follows. We construct an instrumented Büchi VASS \mathcal{V}^I from \mathcal{V} and I that starts in a gadget for I from which it moves to \mathcal{V} . This gadget selects one of the ideals, say (q, m), and increments each counter c to at most m(c). Note that m(c) may be ω , in which case we may put an arbitrary value to this counter. After this initial phase, \mathcal{V}^I moves to state q of \mathcal{V} . The states in the gadget are not accepting, so \mathcal{V}^I will eventually move to \mathcal{V} to obtain an infinite accepting run. To be precise, we have in \mathcal{V} an accepting run from a configuration in I if and only if \mathcal{V}^I has an accepting run.

With this, we can saturate a set of markings S, initially $S = \emptyset$. We repeatedly ask for an accepting run starting in a downward-closed set of configurations represented by a set of ideals I. Initially, we just ask for any run, $I = Q \times \{\omega^d\}$. If such a run does not exist, we return S. If such a run exists, we can reconstruct a configuration $(q, m) \in I$, $m \in \mathbb{N}^d$, with which \mathcal{V}^I moved from the gadget for I to \mathcal{V} . This can be done with an enumeration. We add (q, m) to S and refine the downward-closed set represented by I by subtracting the upward-closure of the new S. The subtraction can be computed effectively and yields a new set of ideals with which we repeat the check of an accepting run. The process terminates: the set S represents an upward-closed set of configurations, and every infinite sequence of such sets becomes stationary due to the wqo. In the moment when the set becomes stationary, we will no longer find an accepting run and return.