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Rewriting preserving recognizability of finite tree languages ☆

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ABSTRACT

We show that left-linear generalized semi-monadic TRSs effectively preserve recognizability of finite tree languages (are EPRF-TRSs). We show that reachability, joinability, and local confluence are decidable for EPRF-TRSs.

1. Introduction

The notion of preservation of recognizability through rewriting is a widely studied concept in term rewriting [2–6,8,11–23]. Let Σ be a ranked alphabet, let R be a term rewrite system (TRS) over Σ, and let L be a tree language over Σ. Then $R^*_\Sigma(L)$ denotes the set of descendants of trees in L. A TRS R over Σ preserves Σ-recognizability (is a PΣR-TRS), if for each recognizable tree language L over Σ, $R^*_\Sigma(L)$ is recognizable. A TRS R over Σ preserves Σ-recognizability of finite tree languages (is a PΣRF-TRS), if for each finite tree language L over Σ, $R^*_\Sigma(L)$ is recognizable.

Let R be a TRS over Σ. Then its signature, $sign(R) \subseteq \Sigma$ is the ranked alphabet consisting of all symbols appearing in the rules of R. A TRS R over $sign(R)$ preserves recognizability (is a PR-TRS), if for each ranked alphabet Σ with $sign(R) \subseteq \Sigma$, R, as a TRS over Σ, preserves Σ-recognizability. A TRS R over $sign(R)$ preserves recognizability of finite tree languages (is a PRF-TRS), if for each ranked alphabet Σ with $sign(R) \subseteq \Sigma$, R, as a TRS over Σ, preserves Σ-recognizability of finite tree languages.

A TRS R over Σ effectively preserves Σ-recognizability (is an EPΣR-TRS), if for a given a bottom-up tree automaton (bta) $\mathcal{B}$ over Σ, we can effectively construct a bta $\mathcal{C}$ over Σ such that $L(\mathcal{C}) = R^*_\Sigma(L(\mathcal{B}))$. A TRS R over Σ effectively preserves Σ-recognizability of finite tree languages (is an EPΣRF-TRS), if for a given finite tree language L over Σ, we can effectively construct a bta $\mathcal{C}$ over Σ such that $L(\mathcal{C}) = R^*_\Sigma(L)$. A TRS R over $sign(R)$ effectively preserves recognizability of finite tree languages (is an EPRF-TRS), if for a given ranked alphabet Σ with $sign(R) \subseteq \Sigma$ and a given finite tree language L over Σ, we can effectively construct a bta $\mathcal{C}$ over Σ such that $L(\mathcal{C}) = R^*_\Sigma(L(\mathcal{B}))$.

Gyenizse and Vágvölgyi [13] presented a linear TRS R over $sign(R)$ such that R is an EP$sign(R)$R-TRS and R is not a PR-TRS. A trs R is murg if R is a union of a monadic trs and a right-ground trs. Vágvölgyi [23] showed that it is not decidable for a murg TRS R over Σ whether R is a PΣRF-TRS. Let R be a TRS over $sign(R)$, and let $\Sigma = \{f, \sharp\} \cup sign(R)$, where $f \in \Sigma_2 - sign(R)$ and $\sharp \in \Sigma_0 - sign(R)$. Gyenizse and Vágvölgyi [13] showed that R is an EPΣR-TRS if and only if R is an EPR-TRS. Gyenizse and Vágvölgyi [14] improved this result for left-linear TRSs. They showed the following. Let R be a left-linear TRS over $sign(R)$, and let $\Sigma = \{g, \sharp\} \cup sign(R)$, where $g \in \Sigma_1 - sign(R)$ and $\sharp \in \Sigma_0 - sign(R)$. Then R is an EPΣR-TRS if and only if R is an EPR-TRS.

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In [11] Gilleron showed that for a TRS R over Σ it is not decidable whether R is a PΣR-TRS. We may naturally introduce the above concepts for string rewrite systems as well. Otto [17] has proved that a string rewrite system R over the alphabet $alph(R)$ of R preserves $alph(R)$-recognizability if and only if R preserves recognizability. Otto [17] showed that it is not decidable for a string rewrite system R whether R preserves $alph(R)$-recognizability, and whether R preserves recognizability. Hence it is not decidable for a linear TRS R whether R is a PR-TRS [17].

In spite of the undecidability results of Gilleron [11] and Otto [17], we know several classes of EPR-TRSs. Gyenizse and Vágvölgyi [13] generalized the concept of a semi-monadic TRS [2] introducing the concept of a generalized semi-monadic TRS (GSM-TRS for short). They showed that each linear GSM-TRS R is an EPR-TRS. Takai et al. [19] introduced finite path overlapping TRS's (FPO-TRSs). They [19] showed that each right-linear FPO-TRS R is an EPR-TRS. They [19] also showed that each GSM-TRS R is an FPO-TRS. Thus we get that each right-linear GSM-TRS R is an EPR-TRS. Vágvölgyi [21] introduced the concept of a half-monadic TRS. A trs R over Σ is half-monadic if, for every rule $l \to r$ in R, either $height(r) = 0$ or $r = \sigma(y_1, \dots, y_k)$, where $\sigma \in \Sigma_k$, $k \geq 1$, and for each $i \in \{1, \dots, k\}$, either y_i is a variable (i.e., $y_i \in X$) or y_i is a ground term (i.e., $y_i \in T_\Sigma$). Each right-linear half-monadic TRS is an FPO-TRS. Hence each right-linear half-monadic TRS is an EPR-TRS. Using this result, Vágvölgyi [21] showed that termination and convergence are decidable properties for right-linear half-monadic term rewrite systems. Takai et al. [20] presented an EPR-TRS which is not an FPO-TRS, see Example 1 in [20]. Takai et al. [20] introduced layered transducing term rewriting systems (LT-TRSs). They [20] showed that each I/O separated LT-TRS R is an EPR-TRS.

We show that each terminating TRS is an EPRF-TRS. We adopt the construction of Salomaa [18], Coquidé et al. [2], and Gyenizse and Vágvölgyi [13], when showing that any left-linear GSM-TRS R is an EPRF-TRS. We slightly modify the proofs of the decision results of Gyenizse and Vágvölgyi [13] when we show the following decidability results.

(1) Let R be an EPRF-TRS over Σ, and let $p, q \in T_\Sigma(X)$. Then it is decidable whether $p \to_R^* q$. That is, reachability is decidable.

(2) Let R be an EPRF-TRS over Σ, and let $p, q \in T_\Sigma(X)$. Then it is decidable whether there exists a tree $r \in T_\Sigma(X)$ such that $p \to_R^* r$ and $q \to_R^* r$. That is, joinability is decidable.

(3) Let R be a confluent EPRF-TRS over Σ, and let $p, q \in T_\Sigma(X)$. Then it is decidable whether $p \leftrightarrow_R^* q$. That is, the word problem is decidable.

(4) For an EPRF-TRS R, it is decidable whether R is locally confluent.

(5) Let R be an EPRF-TRS, and let S be a TRS over Σ. Then it is decidable whether $\to_S^* \subseteq \to_R^*$.

(6) Let R and S be EPRF-TRSs. Then it is decidable which one of the following four mutually excluding conditions holds.
(i) $\to_R^* \subset \to_S^*$,
(ii) $\to_S^* \subset \to_R^*$,
(iii) $\to_R^* = \to_S^*$,
(iv) $\to_R^* \bowtie \to_S^*$,
where "$\bowtie$" stands for the incomparability relationship.

(7) Let R be an EPRF-TRS. Then it is decidable whether R is left-to-right minimal. (A TRS R is left-to-right minimal if for each rule $l \to r$ in R, $\to_{R-\{l\to r\}}^* \subset \to_R^*$.)

(8) Let R and S be TRSs such that $R \cup R^{-1}$ and $S \cup S^{-1}$ are EPRF-TRSs. Then it is decidable which one of the following four mutually excluding conditions holds.
(i) $\leftrightarrow_R^* \subset \leftrightarrow_S^*$,
(ii) $\leftrightarrow_S^* \subset \leftrightarrow_R^*$,
(iii) $\leftrightarrow_R^* = \leftrightarrow_S^*$,
(iv) $\leftrightarrow_R^* \bowtie \leftrightarrow_S^*$.

Fülöp's [6] undecidability results on deterministic top-down tree transducers simply imply the following. Each of the following questions is undecidable for any convergent left-linear EPRF-TRSs R and S over a ranked alphabet Ω, for any recognizable tree language $L \subseteq T_\Omega$ given by a tree automaton over Ω recognizing L. Here $\Gamma \subseteq \Omega$ is the smallest ranked alphabet for which $NF_R(L) \subseteq T_\Gamma$. Furthermore, the set of R-normal forms of the trees in L is denoted by $NF_R(L)$.
(i) Is $NF_R(L) \cap NF_S(L)$ empty?
(ii) Is $NF_R(L) \cap NF_S(L)$ infinite?
(iii) Is $NF_R(L) \cap NF_S(L)$ recognizable?
(iv) Is $T_\Gamma - NF_R(L)$ empty?
(v) Is $T_\Gamma - NF_R(L)$ infinite?
(vi) Is $T_\Gamma - NF_R(L)$ recognizable?
(vii) Is $NF_R(L)$ recognizable?
(viii) Is $NF_R(L) = NF_S(L)$?
(ix) Is $NF_R(L) \subseteq NF_S(L)$?

Fülöp and Gyenizse [7] showed that it is undecidable for a tree function induced by a deterministic homomorphism whether it is injective. Hence for any convergent left-linear EPRF-TRS R over a ranked alphabet Σ, and any recognizable tree language $L \subseteq T_\Sigma$, it is undecidable whether the tree function $\to_R^* \cap (L \times NF_R(L))$ is injective.

Finally we show the following. Let R be a linear collapse-free EPRF-TRS and S be a linear collapse-free EPR-TRS over the disjoint ranked alphabets $sign(R)$ and $sign(S)$, respectively. Then the disjoint union $R \oplus S$ of R and S is a linear collapse-free EPR-TRS.

This paper is divided into eight sections. In Section 2, we recall the necessary notions and notations. In Section 3, we study PR-TRSs. In Section 4, we study four classes of EPR-TRSs: murg TRSs, generalized semi-monadic TRSs, finite path overlapping TRSs, and layered transducing TRSs. In Section 5, we show that left-linear GSM-TRSs are EPRF-TRSs. We compare the proof with that of the main result in [13]. In Section 6, we illustrate the constructions presented in Section 5 by an example. In Section 7, we show various decidability and undecidability results on PRF-TRSs and EPRF-TRSs. Finally, in Section 8, we present our concluding remarks, and some open problems.

2. Preliminaries

We recall and invent some notations, basic definitions and terminology which will be used in the rest of the paper. Nevertheless the reader is assumed to be familiar with the basic concepts of term rewrite systems and of tree language theory [1,9,10].

2.1. Basic definitions

The composition of binary relations ρ and τ is denoted by $\rho \circ \tau$. The cardinality of a set A is denoted by $card(A)$.

2.2. Abstract reduction systems

An abstract reduction system is a pair $(A, \rightarrow)$, where the reduction $\rightarrow$ is a binary relation on the set A. $\rightarrow^{-1}$, $\leftrightarrow$, $\rightarrow^*$, and $\leftrightarrow^*$ denote the inverse, the symmetric closure, the reflexive transitive closure, and the reflexive transitive symmetric closure of the binary relation $\rightarrow$, respectively.

- $x \in A$ is reducible if there is y such that $x \rightarrow y$.
- $x \in A$ is irreducible if it is not reducible.
- $y \in A$ is a normal form of $x \in A$ if $x \rightarrow^* y$ and y is irreducible. If $x \in A$ has a unique normal form, the latter is denoted by $x{\downarrow}$.
- $y \in A$ is a descendant of $x \in A$ if $x \rightarrow^* y$.
- $x \in A$ and $y \in A$ are joinable if there is a z such that $x \rightarrow^* z \leftarrow^* y$, in which case we write $x{\downarrow}y$.

The reduction $\rightarrow$ is called

- confluent if for all $x, y_1, y_2 \in A$, if $y_1 \leftarrow^* x \rightarrow^* y_2$, then $y_1{\downarrow}y_2$;
- locally confluent if for all $x, y_1, y_2 \in A$, if $y_1 \leftarrow x \rightarrow y_2$, then $y_1{\downarrow}y_2$;
- terminating if there is no infinite chain $x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots$;
- convergent if it is both confluent and terminating.

If $\rightarrow$ is convergent, then each $x \in A$ has a unique normal form [1].

2.3. Terms

The set of nonnegative integers is denoted by N, and N^* stands for the free monoid generated by N with empty word λ as identity element. For a word $\alpha \in N^*$, $length(\alpha)$ stands for the length of α. Consider the words $\alpha, \beta, \gamma \in N^*$ such that $\alpha = \beta\gamma$. Then we say that β is a prefix of α. If $\gamma \neq \lambda$, then β is a proper prefix of α.

A ranked alphabet is a finite set Σ in which every symbol has a unique rank in N. For $m \geq 0$, Σ_m denotes the set of all elements of Σ which have rank m. The elements of Σ_0 are called constants. We assume that all ranked alphabets Σ and Δ that we consider have the following property: if $\sigma \in \Sigma_i$, and $\sigma \in \Delta_j$, then $i = j$. In other words, σ has the same rank in Σ as in Δ.

For a set of variables Y and a ranked alphabet Σ, $T_\Sigma(Y)$ denotes the set of Σ-terms (or Σ-trees) over Y. $T_\Sigma(\emptyset)$ is written as T_Σ. A term $t \in T_\Sigma$ is called a ground term. A tree $t \in T_\Sigma(Y)$ is linear if any variable of Y occurs at most once in t. We specify a countable set $X = \{x_1, x_2, \dots\}$ of variables which will be kept fixed in this paper. Moreover, we put $X_m = \{x_1, \dots, x_m\}$, for $m \geq 0$. Hence $X_0 = \emptyset$.

For any $m \geq 0$, we distinguish a subset $\overline{T}_\Sigma(X_m)$ of $T_\Sigma(X_m)$ as follows: a tree $t \in T_\Sigma(X_m)$ is in $\overline{T}_\Sigma(X_m)$ if and only if each variable in X_m appears exactly once in t.

For a term $t \in T_\Sigma(X)$, the height $height(t)$ of t is defined by tree induction.

(i) If $t \in \Sigma_0 \cup X$, then $height(t) = 0$.
(ii) If $t = f(t_1, \dots, t_n)$ with $f \in \Sigma_n$, $n > 0$, then $height(t) = 1 + \max\{height(t_i) \mid 1 \leq i \leq n\}$.

For a term $t \in T_\Sigma(X)$, the set of variables $var(t)$ of t and the set of positions $POS(t) \subseteq N^*$ are defined in the usual way.

For each $t \in T_\Sigma(X)$ and $\alpha \in POS(t)$, we introduce the subterm $t/\alpha \in T_\Sigma(X)$ of t at α and define the label $lab(t, \alpha) \in \Sigma \cup X$ in t at α as follows:

(a) for $t \in \Sigma_0 \cup X$, $t/\lambda = t$ and $lab(t, \lambda) = t$;
(b) for $t = f(t_1, \dots, t_m)$ with $m \geq 1$ and $f \in \Sigma_m$, if $\alpha = \lambda$ then $t/\alpha = t$ and $lab(t, \alpha) = f$, otherwise, if $\alpha = i\beta$ with $1 \leq i \leq m$, then $t/\alpha = t_i/\beta$ and $lab(t, \alpha) = lab(t_i, \beta)$.

For a tree $t \in T_\Sigma(X)$, $sub(t) = \{ t/\alpha \mid \alpha \in POS(t) \}$ is the set of subtrees of t. For a tree language $L \subseteq T_\Sigma$, the set $sub(L)$ of subtrees of all elements of L is defined by the equality $sub(L) = \bigcup(sub(t) \mid t \in L)$.

For $t \in T_\Sigma$, $\alpha \in POS(t)$, and $r \in T_\Sigma$, we define $t[\alpha \leftarrow r] \in T_\Sigma$ as follows.

(i) If $\alpha = \lambda$, then $t[\alpha \leftarrow r] = r$.

(ii) If $\alpha = i\beta$, for some $i \in N$ and $\beta \in N^*$, then $t = f(t_1, \dots, t_m)$ with $f \in \Sigma_m$ and $1 \leq i \leq m$. Then $t[\alpha \leftarrow r] = f(t_1, \dots, t_{i-1}, t_i[\beta \leftarrow r], t_{i+1}, \dots, t_m)$.

Let Σ be a ranked alphabet. For any $f \in \Sigma_1$, $k \geq 0$, and $t \in T_\Sigma$, the tree $f^k(t) \in T_\Sigma$, is defined by recursion: $f^0(t) = t$ and $f^{k+1}(t) = f(f^k(t))$ for $k \geq 0$.

2.4. Substitutions

A substitution is a mapping $\sigma : X \to T_\Sigma(X)$ such that $\sigma(x_i) \neq x_i$ for only finitely many x_is. The finite set of variables that σ does not map to themselves is denoted by $Dom(\sigma)$. That is, $Dom(\sigma) = \{ x_i \mid \sigma(x_i) \neq x_i \}$. Let $Ran(\sigma) = \{ \sigma(x_i) \mid x_i \in Dom(\sigma) \}$. Any substitution $\sigma : X \to T_\Sigma(X)$ can be extended to a mapping $\hat{\sigma} : T_\Sigma(X) \to T_\Sigma(X)$ [1]. For any term $t \in T_\Sigma(X_m)$, $m \geq 0$, the term $\hat{\sigma}(t)$ is produced from t by replacing in parallel each occurrence of x_i with $\sigma(x_i)$ for $1 \leq i \leq m$. A renaming is an injective substitution σ such that $Ran(\sigma) \subseteq X$. It is well-known that a renaming is a bijective function [1]. The composition $\sigma\tau$ of two substitutions σ and τ is defined as $\sigma\tau(x_i) = \hat{\sigma}(\tau(x_i))$. $\sigma\tau$ is again a substitution. Clearly, the extension of $\sigma\tau$ is equal to the composition of the extensions of σ and τ, i.e., $\widehat{\sigma\tau} = \hat{\tau} \circ \hat{\sigma}$. To simplify notation, from now on, σ will denote the extension $\hat{\sigma} : T_\Sigma(X) \to T_\Sigma(X)$ as well.

For any trees $t \in \overline{T}_\Sigma(X_k)$, $t_1, \dots, t_k \in T_\Sigma(X)$ and for a substitution σ with $\sigma(x_i) = t_i$ for $i = 1, \dots, k$, we denote the term $\sigma(t)$ by $t[t_1, \dots, t_k]$ as well. Moreover, for any m, n with $1 \leq m \leq n$, for any tree $t \in T_\Sigma(\{ x_m, \dots, x_n \})$ and for any substitution σ with $\sigma(x_m) = t_m, \dots, \sigma(x_n) = t_n$, we denote $\sigma(t)$ also by $t[x_m \leftarrow t_m, \dots, x_n \leftarrow t_n]$.

Let Σ be a ranked alphabet and let $u, v \in T_\Sigma(X)$. The tree u is called a pattern of v and v is called an instance of u if there is a substitution σ such that $v = \sigma(u)$. Moreover, if u is linear, then we say that u is a linear pattern of v. We illustrate the concept of a pattern by an example. Let $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$, $\Sigma_0 = \{ \sharp \}$, $\Sigma_1 = \{ f \}$, $\Sigma_2 = \{ g \}$. Trees $f(x_2)$, $f(g(x_2, x_1))$, $f(g(\sharp, x_2))$ are linear patterns of $f(g(\sharp, \sharp))$. On the other hand, $f(f(x_1))$ is not a pattern of $f(g(\sharp, \sharp))$, since there is no substitution σ such that $\sigma(f(f(x_1))) = f(g(\sharp, \sharp))$. Observe that $f(g(x_1, x_1))$ is a pattern of $f(g(\sharp, \sharp))$. However, $f(g(x_1, x_1))$ is not linear.

Let Σ be a ranked alphabet and let $s, t \in T_\Sigma(X)$. A unifier of s and t is a substitution θ such that $\theta(s) = \theta(t)$. A most general unifier of s and t is a unifier θ of s and t with the following property: for each unifier σ of s and t, there is a substitution η such that $\eta\theta = \sigma$. It is decidable if s and t are unifiable [1]. If s and t are unifiable, then one can effectively construct a most general unifier of s and t [1]. It is well known that a most general unifier of s and t is unique up to renaming of variables [1].

Let $s, t \in T_\Sigma(X)$ be linear terms. Let $\alpha \in POS(s)$ and $s/\alpha \in X$. Then we say that s is *adjacent* to t at position α in the term $s[\alpha \leftarrow t]$.

2.5. Term rewrite systems

Let Σ be a ranked alphabet. Then a term rewrite system (TRS) R over Σ is a finite subset of $(T_\Sigma(X) - X) \times T_\Sigma(X)$ such that for each $(l, r) \in R$, each variable of r also occurs in l. Elements (l, r) of R are called rules and are denoted by $l \to r$. $sign(R) \subseteq \Sigma$ is the ranked alphabet consisting of all symbols appearing in the rules of R.

$$lhs(R) = \{ l \in T_\Sigma(X) \mid l \text{ is the left-hand side of some rule } l \to r \text{ in } R \}$$

is the set of left-hand sides of the rules in R.

$$rhs(R) = \{ r \in T_\Sigma(X) \mid r \text{ is the right-hand side of some rule } l \to r \text{ in } R \}$$

is the set of right-hand sides of the rules in R.

A TRS R is left-linear (resp. right-linear) if each element of $lhs(R)$ (resp. $rhs(R)$) is linear. A left-linear and right-linear TRS R is called linear. A ground TRS is one of which all rules are ground (i.e., elements of $T_\Sigma \times T_\Sigma$). We say that a TRS R is collapse-free if there is no rule $l \to r$ in R such that $r \in X$.

Let R be a TRS over Σ. Given any two terms s and t in $T_\Sigma(X)$ and a position $\alpha \in POS(s)$, we say that s rewrites to t at α and denote this by $s \to_R t$ if there is some pair $(l, r) \in R$ and a substitution σ such that $s/\alpha = \sigma(l)$ and $t = s[\alpha \leftarrow \sigma(r)]$. Here we also say that R rewrites s to t applying the rule $l \to r$ at α.

We say that a TRS R is confluent, locally confluent, terminating, or convergent, if $\to_R$ has the corresponding property.

Let R be a TRS over Σ.

(a) R is left-to-right minimal if for each rule $l \to r$ in R, $\to^*_{R-\{l\to r\}} \subset \to^*_R$.

(b) R is left-to-right ground minimal if for each rule $l \to r$ in R, $\to^*_{R-\{l\to r\}} \cap (T_\Sigma \times T_\Sigma) \subset \to^*_R \cap (T_\Sigma \times T_\Sigma)$.

The set of all ground terms that are irreducible for a TRS R is denoted by $IRR(R)$. Let $L \subseteq T_\Sigma$. The set of $\to_R$-normal forms of the trees in the tree language L is denoted by $NF_R(L)$. It should be clear that $NF_R(L) = R^*(L) \cap IRR(R)$.

We adopt the concept of a critical pair from [1]. Let R be a TRS and let $l_1 \to r_1, l_2 \to r_2$ be two rules whose variables have been renamed such that $var(l_1) \cap var(l_2) = \emptyset$. Let $\alpha \in POS(l_1)$ be such that $l_1/\alpha \notin X$ and $l_1/\alpha, l_2$ are unifiable. Let σ be a most general unifier of l_1/α and l_2. Then we call $\sigma(r_1), \sigma(l_1)[\alpha \leftarrow \sigma(r_2)]$ a critical pair of R.

Proposition 2.1 [1]. *Let R be a TRS over Σ. Then R is locally confluent if and only if for every critical pair (v_1, v_2) of R, v_1 and v_2 are joinable.*

The following problems have attracted attention from researchers.

Reachability problem:
Instance: A TRS R and terms $u, v \in T_\Sigma(X)$.
Question: Does $u \to^*_R v$ hold?

Joinability problem:
Instance: A TRS R and terms $u, v \in T_\Sigma(X)$.
Question: Is there a term $r \in T_\Sigma(X)$ such that $u \to^*_R r$ and $v \to^*_R r$.

Word problem:
Instance: A TRS R and terms $u, v \in T_\Sigma(X)$.
Question: Does $u \leftrightarrow^*_R v$ hold?

Let R and S be TRSs over disjoint ranked alphabets Σ and Δ, respectively. Then the disjoint union $R \oplus S$ of R and S is the TRS $R \cup S$ over the ranked alphabet $\Sigma \cup \Delta$. Let $\mathbf{C}$ be a class of TRSs, let $\mathbf{C}$ be closed under disjoint union. A property P is modular for $\mathbf{C}$ if for any $R, S \in \mathbf{C}$ over disjoint ranked alphabets, $R \oplus S$ has the property $\mathcal{P}$ if and only if both R and S have the property $\mathcal{P}$.

2.6. Tree languages

Let Σ be a ranked alphabet, a bottom-up tree automaton (bta) over Σ is a quadruple $\mathcal{A} = (\Sigma, A, R, A_f)$, where A is a finite set of states of rank 0, $\Sigma \cap A = \emptyset$, $A_f (\subseteq A)$ is the set of final states, R is a finite set of rules of the following two types:

(i) $\delta(e_1, \ldots, e_n) \to a$ with $n \geq 0$, $\delta \in \Sigma_n$, $a_1, \ldots, a_n, a \in A$ (reading rules).

(ii) $a \to a'$ with $a, a' \in A$ (λ-rules).

We consider R as a ground TRS over $\Sigma \cup A$. The tree language recognized by $\mathcal{A}$ is $L(\mathcal{A}) = \{ t \in T_\Sigma \mid (\exists a \in A_f)\ t \to^*_R a\}$. A tree language L is recognizable if there exists a bta $\mathcal{A}$ such that $L(\mathcal{A}) = L$ [9].

The bta $\mathcal{A} = (\Sigma, A, R, A_f)$ is deterministic if R has no λ-rules and R has no two rules with the same left-hand side.

Definition 2.2. *Let Σ be a ranked alphabet, and let $L \subseteq T_\Sigma$ be a finite tree language. We define the fundamental bta $\mathcal{B} = (\Sigma, B, R, B_f)$ of L as follows.*

$B = \{ \langle p \rangle \mid p \in sub(L) \}$.

$R = \{ \delta(\langle p_1 \rangle, \ldots, \langle p_m \rangle) \to \langle p \rangle \mid \delta \in \Sigma_m, m \geq 0, \langle p_1 \rangle, \ldots, \langle p_m \rangle, \langle p \rangle \in B, \text{ and } p = \delta(p_1, \ldots, p_m) \}$.

$B_f = \{ \langle p \rangle \mid p \in L \}$.

By direct inspection of Definition 2.2, we get the following.

Lemma 2.3. *Let Σ be a ranked alphabet, let $M \subseteq T_\Sigma$ be a finite tree language, and let $K \subseteq M$. Let $\mathcal{B} = (\Sigma, B, R, B_f)$ be the fundamental bta of M. Let $\beta\mathcal{A} = (\Sigma, B, R, \{ \langle p \rangle \mid p \in K \})$. Then statements (a)–(c) hold.*

(a) For any $t \in sub(M)$ and $b \in B$, $t \to^*_R b$ if and only if $b = \langle t \rangle$.

(b) $\mathcal{A}$ and $\mathcal{B}$ are deterministic.

(c) $L(\mathcal{B}) = M$ and $L(\mathcal{A}) = K$.

Proof. By direct inspection of Definition 2.2, we get (a) and (b). By (a) and the definition of $\mathcal{B}_f$, we obtain that $L(\mathcal{B}) = M$. Similarly, we get that $L(\mathcal{A}) = K$. $\square$

Statement 2.4. Let $\mathcal{A} = (\Sigma, A, R, A_f)$ be a deterministic bta. Let $t \in T_\Sigma$ be such that $t \to^*_R a$ for some $a \in A$, and that there is a position $\alpha \in POS(t)$ with $length(\alpha) > card(A)$. Then there are prefixes β, γ of α, and $u, v \in \overline{T}_\Sigma(X_1)$ and $z \in T_\Sigma$ such that

(*i*) $length(\beta) > length(\gamma)$,

(*ii*) $t = u[v[z]]$, $u[v]/\beta = x_1$, and $u/\gamma = x_1$,

(*iii*) $height(v) \geq 1$, and
(*iv*) $u[v[v[z]]] \to_R^* a$.

Proof. Hint. Similarly to the proof of the pumping lemma for recognizable tree languages [9], one can show that there are prefixes β, γ of α, and trees $u, v \in \overline{T}_\Sigma(X_1)$, $z \in T_\Sigma$, and a state $b \in A$ such that (i)–(iii) hold and $z \to_R^* b$, $v[b] \to_R^* b$, and $u[b] \to_R^* a$. Hence (iv) holds as well. □

3. TRSs preserving recognizability

We study TRSs preserving recognizability.

Let Σ be a ranked alphabet, let R be a TRS over Σ, and let L be a tree language over Σ. Then $R_\Sigma^*(L) = \{ p \mid q \to_R^* p \text{ for some } q \in L \}$ is the set of descendants of trees in L. When Σ is apparent from the context, we simply write $R^*(L)$ rather than $R_\Sigma^*(L)$. For the concept of a PΣR-TRS, a PΣRF-TRS, a PR-TRS, a PRF-TRS, an EPΣR-TRS, an EPΣRF-TRS, an EPR-TRS, and an EPRF-TRS, see the Introduction.

Observation 3.1. Let R be a TRS over a ranked alphabet Σ such that for each $t \in T_\Sigma$, the set $R^*(\{ t \})$ is finite. Then for each finite tree language $L \subseteq T_\Sigma$, $R^*(L)$ is finite and we can effectively construct it.

Proof. Clearly, $R^*(L) = \bigcup(R^*(\{ t \}) \mid t \in L)$. Thus $R^*(L)$ is finite. We compute $R^*(L)$ as follows. Let $W = L$. While there is $q \in T_\Sigma - W$ such that $p \to_R q$ for some $p \in W$ we add q to W. Since $R^*(L)$ is finite, we stop. When we stop we have $W = R^*(L)$. □

Observation 3.2. Each terminating TRS is an EPRF-TRS.

Proof. Let R be a terminating TRS over Σ and let $t \in T_\Sigma$ be arbitrary. We now show that $R^*(\{ t \})$ is finite. On the contrary, assume that $R^*(\{ t \})$ is infinite. Then t starts an infinite reduction sequence $t = t_0 \to_R t_1 \to_R t_2 \to_R t_3 \to_R \cdots$ by König's lemma. Hence R is not terminating, which is a contradiction. By Observation 3.1, for each finite tree language $L \subseteq T_\Sigma$, $R^*(L)$ is finite and we can effectively construct it. □

A TRS is monadic if each left-hand side is of height at least 1 and each right-hand side is of height at most 1.

Statement 3.3. There is a left-linear monadic TRS R over a ranked alphabet Σ such that R is an EPΣRF-TRS, and that R is not a PΣR-TRS.

Proof. Let $\Sigma = \Sigma_0 \cup \Sigma_2$, $\Sigma_0 = \{ \sharp \}$, and $\Sigma_2 = \{ f \}$. Let R over Σ consist of the rule $f(x_1, \sharp) \to f(x_1, x_1)$. Observe that $\Sigma = sign(R)$. We obtain by direct inspection that R is a left-linear monadic TRS R over Σ.

For any trees $p, q \in T_\Sigma$, if $p \to_R q$, then $height(p) = height(q)$. Hence for each $t \in T_\Sigma$, $R^*(\{ t \})$ is finite. By Observation 3.1, for each finite tree language $L \subseteq T_\Sigma$, $R^*(L)$ is finite and we can effectively construct it. By Lemma 2.3, we can construct a bta $\mathcal{C}$ over Σ such that $L(\mathcal{C}) = R^*(L)$. Consequently, R is an EPΣRF-TRS.

It remains to show that R is not a PΣR-TRS. To this end, we define the nth left comb left_n for $n \geq 0$, as follows.
(i) $\text{left}_0 = \sharp$, and
(ii) for each $n \geq 0$, $\text{left}_{n+1} = f(\text{left}_n, \sharp)$.
Let

$$L = \{ \text{left}_n \mid n \geq 0 \}.$$

Then L is a recognizable tree language. For each $n \geq 0$, $f(\text{left}_n, \text{left}_n) \in R^*(L)$. Furthermore, we have the following.

Claim 3.4. *For any $p \in R^*(L)$ and $s \in sub(p)$, if $s = f(t_1, t_2)$, then $t_2 = \sharp$ or $height(t_1) = height(t_2)$.*

Assume that $R^*(L)$ is a recognizable tree language. Then there is a deterministic bta $\mathcal{A} = (\Sigma, A, R, A_f)$ such that $L(\mathcal{A}) = R^*(L)$. Let $n > card(A)$. Consider the term $f(\text{left}_n, \text{left}_n) \in R^*(L)$. Then

$$\text{left}_n \overset{*}{\underset{R}{\to}} a \tag{1}$$

and $f(\text{left}_n, \text{left}_n) \to_R^* f(\text{left}_n, a) \to_R^* b$ for some $a \in A$ and $b \in A_f$. By (1) and Statement 2.4, there is $k > n$ such that $\text{left}_k \to_R^* a$. Consequently, $f(\text{left}_n, \text{left}_k) \to_R^* f(\text{left}_n, a) \to_R^* b$. Thus, $f(\text{left}_n, \text{left}_k) \in R^*(L)$. However, $height(\text{left}_n) = n < k = height(\text{left}_k)$. This contradicts Claim 3.4. □

Theorem 3.5. *There is a ranked alphabet Σ and there is a linear EPΣRF-TRS R such that R is not a PRF-TRS.*

Proof. Let $\Sigma = \Sigma_1 \cup \Sigma_0$, $\Sigma_1 = \{f, g\}$, $\Sigma_0 = \{\sharp\}$. Let R consist of the following five rules.
$f(g(x_1)) \to f(f(g(g(x_1))))$,
$f(\sharp) \to \sharp$, $\quad g(\sharp) \to \sharp$,
$\sharp \to f(\sharp)$, $\quad \sharp \to g(\sharp)$.
It should be clear that for each tree $t \in T_\Sigma$, $t \to_R^* \sharp$ and $\sharp \to_R^* t$. Hence for each nonempty tree language $L \subseteq T_\Sigma$, $R^*(L) = T_\Sigma$. Thus R is an EPΣRF-TRS.

Let $\Delta = \Sigma \cup \{h\}$, where $h \in \Delta_1$. Then $R^*(\{f(g(h(\sharp)))\}) = \{f^n(g^n(h(t))) \mid n \geq 0, t \in T_\Sigma\}$ is not recognizable. $\square$

Theorem 3.6. *Let R be a TRS, and let $\Sigma = \{f, \sharp\} \cup sign(R)$, where $f \in \Sigma_2 - sign(R)$ and $\sharp \in \Sigma_0 - sign(R)$. R is an EPΣRF-TRS if and only if R is an EPRF-TRS.*

Proof. ($\Leftarrow$) Trivial.

($\Rightarrow$) Let Γ be an arbitrary ranked alphabet with $sign(R) \subseteq \Gamma$. To each symbol $g \in \Gamma_k - sign(R)$, $k \geq 0$, we assign a tree $t_g \in T_\Sigma(X_k)$. To this end, we number the symbols in $\Gamma - sign(R)$ from 1 to $|\Gamma - sign(R)|$. Furthermore, we define the nth right comb right_n for $n \geq 0$, as follows.

(i) $\mathrm{right}_0 = \sharp$,

(ii) for each $n \geq 0$, $\mathrm{right}_{n+1} = f(x_{n+1}, \mathrm{right}_n)$.

For the definition of the nth left comb left_n, see the proof of Statement 3.3. To any symbol $g \in \Gamma_k - sign(R)$, $k \geq 0$, with number m, we assign the tree $t_g = f(\mathrm{left}_m, \mathrm{right}_k)$. Recall that the number m uniquely determines the symbol g. Hence for any symbols $f, g \in \Gamma - sign(R)$, if $t_f = t_g$, then $f = g$.

Consider the TRS

$$S = \{g(x_1, \ldots, x_k) \to t_g \mid k \geq 0, g \in \Gamma_k - sign(R)\}$$

over $\Gamma \cup \Sigma$. It should be clear that S is a convergent TRS.

Claim 3.7. *(a) For any $r, w \in T_\Gamma$, if $r \to_R w$, then $r\downarrow_S \to_R w\downarrow_S$.*
(b) For any $r \in T_\Gamma$ and $t \in T_\Sigma$, if $r\downarrow_S \to_R t$, then there is a $w \in T_\Gamma$ such that $t = w\downarrow_S$ and $r \to_R w$.
(c) For any $r \in T_\Gamma$ and $t \in T_\Sigma$, if $r\downarrow_S \to_R^* t$, then there is a $w \in T_\Gamma$ such that $t = w\downarrow_S$ and $r \to_R^* w$.

Proof. Clearly, Statements (a) and (b) hold. Statement (c) simply follows from (b). $\square$

Claim 3.8. *A tree language L over Γ is finite if and only if the tree language $NF_S(L)$ over Σ is finite. A tree language L over Γ is recognizable if and only if the tree language $NF_S(L)$ over Σ is recognizable. Given a bta $\mathcal{A} = (\Gamma, A, R_\mathcal{A}, A_f)$ recognizing L we can construct a bta $\mathcal{B} = (\Sigma, A, R_\mathcal{B}, B_f)$ recognizing $NF_S(L)$. Conversely, given a bta $\mathcal{B} = (\Sigma, A, R_\mathcal{B}, B_f)$ recognizing $NF_S(L)$, we can construct a bta $\mathcal{A} = (\Gamma, A, R_\mathcal{A}, A_f)$ recognizing L.*

Let L be any finite tree language over Γ. By Claim 3.8, $NF_S(L)$ is a finite tree language over Σ. First we show that

$$NF_S(R_\Gamma^*(L)) \subseteq R_\Sigma^*(NF_S(L)) \,. \tag{2}$$

Let $u \in L$ and let $u \to_R^* v$. Then by Claim 3.7 (a), $u\downarrow_S \to_R^* v\downarrow_S$. Hence $v\downarrow_S \in R_\Sigma^*(NF_S(L))$. Second we show that

$$R_\Sigma^*(NF_S(L)) \subseteq NF_S(R_\Gamma^*(L)) \,. \tag{3}$$

Let $u \in L$. Assume that $u\downarrow_S \to_R^* t$ for some $t \in T_\Sigma$. Then by Claim 3.7 (c), there is a $w \in T_\Gamma$ such that $t = w\downarrow_S$ and $u \to_R^* w$. Hence $t \in NF_S(R_\Gamma^*(L))$.

By (2) and (3),

$$NF_S(R_\Gamma^*(L)) = R_\Sigma^*(NF_S(L)) \,.$$

Consequently, by Claim 3.8, $R_\Gamma^*(L)$ is recognizable if and only if $R_\Sigma^*(NF_S(L))$ is recognizable. Moreover, given a bta $\mathcal{A} = (\Gamma, A, R_\mathcal{A}, A_f)$ recognizing $R_\Gamma^*(L)$ we can construct a bta $\mathcal{B} = (\Sigma, B, R_\mathcal{B}, B_f)$ recognizing $R_\Sigma^*(NF_S(L))$. Conversely, given a bta $\mathcal{B} = (\Sigma, B, R_\mathcal{B}, B_f)$ recognizing $R_\Sigma^*(NF_S(L))$, we can construct a bta $\mathcal{A} = (\Gamma, A, R, A_f)$ recognizing $R_\Gamma^*(L)$. Hence if R is an EPΣRF-TRS, then R is an EPΓRF-TRS. As Γ is an arbitrary ranked alphabet with $sign(R) \subseteq \Gamma$, R is an EPRF-TRS. $\square$

The proof of the following result is similar to the proof of Theorem 3.6.

Theorem 3.9. *Let R be any TRS over $sign(R)$, and let $\Sigma = \{f, \sharp\} \cup sign(R)$, where $f \in \Sigma_2 - sign(R)$ and $\sharp \in \Sigma_0 - sign(R)$. Then R is a PΣRF-TRS if and only if R is a PRF-TRS.*

Consequence 3.10. Let R be a TRS over Σ such that there is a symbol $f \in \Sigma_2 - sign(R)$ and there is a constant $\sharp \in \Sigma_0 - sign(R)$. Then R is a PRF-TRS if and only if R is a PΣRF-TRS. Moreover, R is an EPRF-TRS if and only if R is an EPΣRF-TRS.

4. Classes of EPR-TRSs

We study four classes of EPR-TRSs: murg TRSs, generalized semi-monadic TRSs (GSM-TRSs), finite path overlapping TRSs (FPO-TRSs), and layered transducing TRSs (LT-TRSs).

4.1. Murg TRSs

A TRS is called right-ground if each right-hand side is ground. A TRS R over Σ is *murg* if R is the union of a monadic TRS and a right-ground TRS over Σ. Obviously, each monadic TRS is murg, and each right-ground TRS is murg.

Theorem 4.1. *There is a ranked alphabet Σ and a murg TRS R over Σ such that R is not a PΣRF-TRS.*

Proof. Let $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$, $\Sigma_0 = \{\sharp, \$, \flat\}$, $\Sigma_1 = \{f\}$, and $\Sigma_2 = \{g, h\}$. Let the TRS R over Σ consist of the rules
$\sharp \to f(\sharp)$, $\quad \sharp \to \flat$,
$\$ \to f(\$)$, $\quad \$ \to \flat$,
$g(x_1, x_1) \to h(x_1, x_1)$.
Consider a tree $h(t_1, t_2) \in R^*(g(\sharp, \$))$ where $t_1, t_2 \in T_\Sigma$. Then

$$g(\sharp, \$) \xrightarrow[R]{*} g(f^k(\sharp), f^k(\$)) \xrightarrow[R]{*} g(f^k(\flat), f^k(\flat)) \xrightarrow[R]{} h(f^k(\flat), f^k(\flat)) = h(t_1, t_2)$$

holds for some $k \geq 0$. Hence $R^*(\{g(\sharp, \$)\}) \cap \{h(t_1, t_2) \mid t_1, t_2 \in T_\Sigma\} = \{h(f^k(\flat), f^k(\flat)) \mid k \geq 0\}$.

It is well known that the intersection of any two recognizable tree languages is a recognizable tree language. Observe that $\{h(t_1, t_2) \mid t_1, t_2 \in T_\Sigma\}$ is a recognizable tree language, and $\{h(f^k(\flat), f^k(\flat)) \mid k \geq 0\}$ is not a recognizable tree language. Thus $R^*(\{g(\sharp, \$)\})$ is not a recognizable tree language. $\square$

With an arbitrary Post Correspondence System (PCS) $\langle \mathbf{w}, \mathbf{z} \rangle$, Vágvölgyi [23] associated a ranked alphabet Σ, containing the distinguished nullary symbol $\# \in \Sigma_0$, and a murg TRS R over Σ. Vágvölgyi [23] showed the following results.

Statement 4.2. If PCS $\langle \mathbf{w}, \mathbf{z} \rangle$ has a solution, then R is an EPΣR-TRS.

Statement 4.3. If PCS $\langle \mathbf{w}, \mathbf{z} \rangle$ has no solution, then $R^*(\{\#\})$ is not a recognizable tree language over Σ.

Statement 4.3 implies the following statement.

Statement 4.4. If PCS $\langle \mathbf{w}, \mathbf{z} \rangle$ has no solution, then R is not a PΣRF-TRS.

The following result is a simple consequence of Statements 4.2 and 4.4.

Statement 4.5. PCS $\langle \mathbf{w}, \mathbf{z} \rangle$ has a solution if and only if R is an EPΣRF-TRS if and only if TRS R is a PΣRF-TRS.

Statement 4.5 implies the following result.

Consequence 4.6. The following problem is undecidable:
Instance: A murg TRS R over a ranked alphabet Σ.
Question: Is R a PΣRF-TRS?

4.2. GSM-TRSs

We now recall the notion of a GSM-TRS, and overview the main results on GSM-TRSs [13].

Definition 4.7. *Let R be a TRS over Σ. We say that R is a GSM-TRS if the following holds. For any rules $l_1 \to r_1$ and $l_2 \to r_2$ in R, for any positions $\alpha \in POS(r_1)$ and $\beta \in POS(l_2)$, and for any linear pattern $l_3 \in T_\Sigma(X)$ of l_2/β with $var(l_3) \cap var(l_1) = \emptyset$, if*
(i) $\alpha = \lambda$ or $\beta = \lambda$,
(ii) r_1/α and l_3 are unifiable, and
(iii) σ is a most general unifier of r_1/α and l_3,
then
(a) $l_2/\beta \in X$ or
(b) for each $\gamma \in POS(l_3)$, if $l_2/\beta\gamma \in X$, then $\sigma(l_3/\gamma) \in X \cup T_\Sigma$.

Notice that Condition (a) implies that $l_3 \in X$.

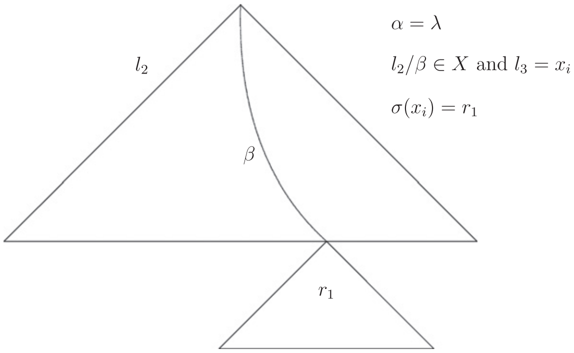

Fig. 1. The unification of r_1/α and the pattern l_3 of l_2/β by a most general unifier σ, when Condition (a') holds.

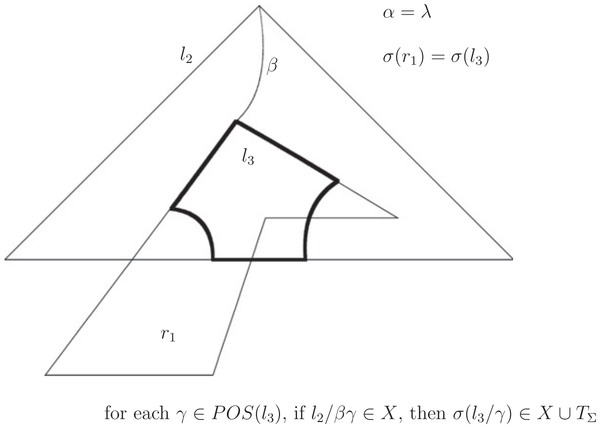

Fig. 2. The unification of r_1/α and the pattern l_3 of l_2/β by a most general unifier σ, when Condition (b') holds.

Example 4.8. Let $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \Sigma_3$, $\Sigma_0 = \{\sharp\}$, $\Sigma_1 = \{f\}$, and $\Sigma_3 = \{g\}$. Let the TRS R over Σ consist of the rule

$$g(x_1, x_2, \sharp) \to f(g(x_1, \sharp, x_1)) .$$

We obtain by direct inspection that R is left-linear GSM-TRS.

Each right-linear GSM-TRS R is an EPR-TRS [19].
To grasp the key ideas of the paper, we adopt the following concept from [13].

Definition 4.9. *A TRS R over Σ is restricted right-left overlapping if the following holds. For any rules $l_1 \to r_1$ and $l_2 \to r_2$ in R, for any positions $\alpha \in POS(r_1)$ and $\beta \in POS(l_2)$, and for any pattern $l_3 \in T_\Sigma(X)$ of l_2/β with $var(l_3) \cap var(l_1) = \emptyset$, if (i), (ii), and (iii) in Definition 4.7 holds, then (a'), (b'), or (c') hold.*
(a') $\alpha = \lambda$, $l_2/\beta \in X$.
(b') $\alpha = \lambda$ and for each $\gamma \in POS(l_3)$, if $l_2/\beta\gamma \in X$, then $\sigma(l_3/\gamma) \in X \cup T_\Sigma$.
(c') $\beta = \lambda$ and for each $\gamma \in POS(l_3)$, if $l_2/\gamma \in X$, then $\sigma(l_3/\gamma) \in X \cup T_\Sigma$.

We visualize the unification of r_1/α and the pattern l_3 of l_2/β by a most general unifier σ, when Condition (a') (Condition (b') and Condition (c'), respectively) holds on Fig. 1 (Fig. 2 and Fig. 3, respectively). Assume that Condition (a') holds. Then $l_3 \in X$, $Dom(\sigma) = \{x_i\}$, and $\sigma(x_i) = r_1$. Furthermore, l_2 is adjacent to r_1 at position β in the term $\sigma(l_2)$. Assume that Condition (b') holds and that $l_3 \notin X$. Then a right-hand side r_1 and a nonvariable linear pattern of a subterm of a left-hand side l_2 are unified. Assume that Condition (c') holds and that $l_3 \notin X$. Then a subterm of a right-hand side r_1 and a nonvariable linear pattern of a left-hand side l_2 are unified.
The proofs of the following two results are straightforward.

Observation 4.10. A TRS R is a GSM-TRS if and only if R is restricted right-left overlapping.

Observation 4.11. Each murg TRS is a GSM-TRS as well.

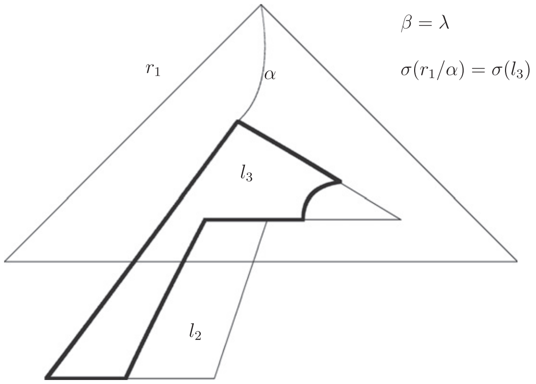

Fig. 3. The unification of r_1/α and the pattern l_3 of l_2 by a most general unifier σ, when Condition (c') holds.

By Theorem 4.1 and Observation 4.11, there is a ranked alphabet Σ and a GSM-TRS R over Σ such that R is not a PΣRF-TRS. Gyenizse and Vágvölgyi [13] observed that Fülöp's [6] undecidability results on deterministic top-down tree transducers simply imply the following.

Statement 4.12 [13]. Each of the following questions is undecidable for any convergent left-linear GSM-TRSs R and S over a ranked alphabet Ω, for any recognizable tree language $L \subseteq T_\Omega$ given by a tree automaton over Ω recognizing L, where $\Gamma \subseteq \Omega$ is the smallest ranked alphabet for which $NF_R(L) \subseteq T_\Gamma$.
(i) Is $NF_R(L) \cap NF_S(L)$ empty?
(ii) Is $NF_R(L) \cap NF_S(L)$ infinite?
(iii) Is $NF_R(L) \cap NF_S(L)$ recognizable?
(iv) Is $T_\Gamma - NF_R(L)$ empty?
(v) Is $T_\Gamma - NF_R(L)$ infinite?
(vi) Is $T_\Gamma - NF_R(L)$ recognizable?
(vii) Is $NF_R(L)$ recognizable?
(viii) Is $NF_R(L) = NF_S(L)$?
(ix) Is $NF_R(L) \subseteq NF_S(L)$?

4.3. FPO-TRSs

We now adopt the concept of an FPO-TRS from [19]. Let $s, t \in T_\Sigma(X) - X$, and let $\gamma \in POS(s) \cap POS(t)$ such that $\gamma \neq \lambda$. We say that the term s sticks out of t at γ if Conditions (a)–(c) hold.
(a) $\gamma \neq \lambda$, and $t/\gamma \in X$.
(b) s/γ is not a ground term.
(c) For any proper prefix δ of γ, $lab(s, \delta) = lab(t, \delta)$.
Assume that s sticks out of t and $s/\gamma \notin X$, i.e., $s/\gamma \notin T_\Sigma \cup X$. Then we say that s properly sticks out of t at γ.

Example 4.13. Let $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$, $\Sigma_0 = \{\sharp, \$\}$, $\Sigma_1 = \{g\}$, $\Sigma_2 = \{f\}$. The term $f(g(x_1), \sharp)$ sticks out of $f(g(x_2), \$)$ at the position 11, and $f(g(g(x_1)), \sharp)$ properly sticks out of $f(g(x_2), \$)$ at the position 11. On the other hand, $f(g(\sharp), \sharp)$ does not stick out of $f(g(x_2), \$)$ at the position 11, because $f(g(\sharp), \sharp)/11 = \sharp$ is a ground term. Furthermore, $\sharp$ does not stick out of x_1 because x_1 is a variable.

The sticking-out graph of a TRS R is a directed graph, where the rewrite rules of R are the vertices. The set E of directed edges is defined as follows. Let $l_1 \to r_1$ and $l_2 \to r_2$ be arbitrary elements of R.
(i) If r_2 properly sticks out of a subterm of l_1, then E contains a directed edge from $l_2 \to r_2$ to $l_1 \to r_1$ with weight 1.
(ii) If a subterm of r_2 properly sticks out of l_1, then E contains a directed edge from $l_2 \to r_2$ to $l_1 \to r_1$ with weight 1.
(iii) If a subterm of l_1 sticks out of r_2, then E contains a directed edge from $l_2 \to r_2$ to $l_1 \to r_1$ with weight 0.
(iv) If l_1 sticks out of a subterm of r_2, then E contains a directed edge from $l_2 \to r_2$ to $l_1 \to r_1$ with weight 0.
An FPO-TRS is a TRS R such that the sticking-out graph of R does not have a cycle of weight 1 or more [19].

Proposition 4.14 [19]. *A TRS R is generalized semi-mondic if and only if the sticking-out graph of R has no edge with weight one.*

Example 4.15. Let $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$, $\Sigma_0 = \{\sharp, \$\}$, $\Sigma_1 = \{g\}$, $\Sigma_2 = \{f\}$. Let $R = \{f(x_1, \sharp) \to f(h(\$), x_1),\ g(x_1) \to f(g(x_1, \$)\}$. The right-hand side of the second rule properly sticks out of the left-hand side of the first rule at the position 1.

Hence there is a directed edge of weight 1 from the second rule to the first rule. The sticking-out graph also has a directed edge of weight 0 from the second rule to itself. The right-hand side of the first rule does not stick out of its left-hand side, because $f(h(\$), x)/1 = h(\$)$ is a ground term. There are no other directed edges since there are no other sticking-out relations between the subterms of these rewrite rules. The sticking-out graph has a cycle of weight 0, but does not have a cycle of weight 1 or more. Hence R is an FPO-TRS.

Proposition 4.16 [19]. *Each right-linear FPO-TRS R is an EPR-TRS.*

Theorem 4.17. *There is an FPO-TRS R such that R is not a PRF-TRS.*

Proof. Let $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$, $\Sigma_0 = \{ \$ \}$, $\Sigma_1 = \{ d, g \}$, $\Sigma_2 = \{ f \}$. Let the TRS R over Σ consist of the following rules.
$\$ \to d(\$)$, $g(d(x_1)) \to f(g(x_1), d(x_1))$, $g(\$) \to \$$,
$f(\$, x_1) \to \$$, $f(\$, x_1) \to h(x_1, x_1)$.
The sticking out graph contains two directed edges. An edge with weight 0 points from the second rule to itself. Because the left-hand side $g(d(x_1))$ of the second rule sticks out of the subterm $g(x_1)$ of the right-hand side of the second rule at position 1. An edge with weight 0 points from the third rule to the second rule. Because the left-hand side $g(\$)$ of the third rule sticks out of the subterm $g(x_1)$ of the right-hand side of the second rule at position 1.

The sticking-out graph has a cycle of weight 0, but does not have a cycle of weight 1 or more. Hence R is an FPO-TRS.

By direct inspection of R, we get that R is an FPO-TRS. We now study the set $R^*(\{ g(\$) \}) \cap \{ h(t_1, t_2) \mid t_1, t_2 \in T_\Sigma \}$. Assume that
$g(\$) = u_0 \to_R u_1 \to_R u_2 \to_R \cdots \to_R u_{k-1} \to_R u_k = h(t_1, t_2)$
for some $k \geq 1$ and $t_1, t_2 \in T_\Sigma$. We iterate application of the first and the second rules. We can change the order of applications of the first and second rules. Then we apply the third rule. Then we apply the fourth rule finitely many times. We apply the fifth rule in the kth step, and hence $u_{k-1} = f(\$, t_1)$, and $t_1 = t_2$. Thus we obtain the following reduction sequence for some $n \geq 1$:
$g(\$) \to_R g(d(\$)) \to_R g(d^2(\$)) \to_R \cdots \to_R g(d^n(\$)) \to_R$
$f(g(d^{n-1}(\$)), d^n(\$)) \to_R f(f(g(d^{n-2}(\$)), d^{n-1}(\$)), d^n(\$)) \to_R$
$f(f(f(g(d^{n-3}(\$)), d^{n-2}(\$)), d^{n-1}(\$)), d^n(\$)) \to_R \cdots \to_R$
$f(\ldots f(f(f(g(\$), d(\$)), d^2(\$)), d^3(\$)), \ldots, d^n(\$)) \to_R$
$f(\ldots f(f(f(\$, d(\$)), d^2(\$)), d^3(\$)), \ldots, d^n(\$)) \to_R$
$f(\ldots f(f(\$, d^2(\$)), d^3(\$)), \ldots, d^n(\$)) \to_R$
$f(\ldots f(\$, d^3(\$)), \ldots, d^n(\$)) \to_R \cdots \to_R f(\$, d^n(\$)) \to_R h(d^n(\$), d^n(\$))$.
In the light of the above reduction sequence, one can show that
$R^*(\{ g(\$) \}) \cap \{ h(t_1, t_2) \mid t_1, t_2 \in T_\Sigma \} = \{ h(d^n(\$), d^n(\$)) \mid n \geq 1 \}$.
Since $\{ h(t_1, t_2) \mid t_1, t_2 \in T_\Sigma \}$ is a recognizable tree language, and $\{ h(d^n(\$), d^n(\$)) \mid n \geq 1 \}$ is not a recognizable tree language, we get that $R^*(\{ g(\$) \})$ is not a recognizable tree language. □

4.4. LT-TRSs

We now adopt the concept of an LT-TRS from [20]. Let $\Sigma = F \cup Q$ be a ranked alphabet, where $F \cap Q = \emptyset$ and $Q \subseteq \Sigma_1$. An LT-TRS is a linear TRS over Σ consisting of rules of the form

$$f(t_1, \ldots, t_n) \to r \text{ or } t \to r \ . \qquad (4)$$

Here
(a) $f \in F_n$, $n \geq 1$,
(b) $t_i, t \in T_\Sigma \cup \{ q(u) \mid q \in Q \text{ and } u \in X \cup T_\Sigma \}$ for $1 \leq i \leq n$, and
(c) $r \in X \cup \{ q(s) \mid q \in Q \text{ and } s \in T_F(X) \}$.

Example 4.18. Let $\Sigma = \Sigma_0 \cup \Sigma_1 = F \cup Q$, $\Sigma_0 = \sharp$, $\Sigma_1 = \{f, q\}$, $F = \{ f \}$, $Q = \{ q \}$. Let $R = \{ q(x_1) \to q(f(x_1)) \}$. Then R is an LT-TRS. Note that $q(f(x_1))$ properly sticks out of $q(x_1)$ at position 1. Hence the sticking out graph of R has a directed edge with weight 1 pointing from the rule $q(x_1) \to q(f(x_1))$ to itself. Consequently R is not an FPO-TRS.

Example 4.19. Let $\Sigma = \Sigma_0 \cup \Sigma_1$, $\Sigma_0 = \sharp$, $\Sigma_1 = Q = \{ q, q_1, q_2 \}$, $\Sigma_2 = F = \{f, h\}$. Let R consist of the rules
$\sharp \to q_1(\sharp)$,
$f(\sharp, q_1(x_1)) \to q_2(f(\sharp, x_1))$,
$f(q_1(x_1), q_2(x_2)) \to q(f(h(x_2), x_1))$,
$q(x_1) \to x_1$.
Then R is an LT-TRS.

We say that an LT-TRS R is I/O separated if the following holds.

- $F = F_I \cup F_O$, $F_I \cap F_O = \emptyset$. A function symbol in F_I (respectively, F_O) is called an input symbol (respectively, output symbol).
- For each rule $l \to r$, if $l/\lambda = f \in F$, then $f \in F_I$.
- For each rule $l \to r$, no input symbol appears in r. That is, $r \in X \cup \{ q(s) \mid q \in Q \text{ and } s \in T_{F_O}(X) \}$.

Example 4.20. Let $F = \{ f, g, h \}$, $F_I = \{ f \}$, $F_O = \{ g, h \}$, and $Q = \{ q_1, q_2, q \}$. TRS R consists of the rules
$f(q_1(x_1), q_2(x_2)) \to q(g(h(x_2), x_1))$, $\quad q_1(x_1) \to q(h(x_1))$.
Observe that R is an I/O separated LT-TRS.

Proposition 4.21 [20]**.** *Each I/O separated LT-TRS R is an EPR-TRS.*

Statement 4.22. There is an LT-TRS R over a ranked alphabet Σ such that R is not a PΣRF-TRS.

Proof. Let $\Sigma = \Sigma_0 \cup \Sigma_1$, $\Sigma_0 = \{ \sharp \}$, $\Sigma_1 = \{ f, q \}$, $F = \{ \sharp, f \}$, and $Q = \{ q \}$. Let TRS R consist of the rules
$f(q(x_1)) \to q(f(x_1))$, $\quad \sharp \to q(f(\sharp))$.
Then R is an LT-TRS. It is not hard to see that

$$R^*(\{ \sharp \}) \cap \{ q^k(f^m(\sharp)) \mid k, m \geq 0 \} = \{ q^k(f^k(\sharp)) \mid k \geq 0 \} .$$

Here $\{ q^k(f^m(\sharp)) \mid k, m \geq 0 \}$ is a recognizable tree language, and $\{ q^k(f^k(\sharp)) \mid k \geq 0 \}$ is not a recognizable tree language. The intersection of two recognizable tree languages is also a recognizable tree language. Hence $R^*(\{ \sharp \})$ is not a recognizable tree language. $\square$

5. Main results

We now show that each left-linear GSM-TRS is an EPRF-TRS.

Theorem 5.1. *Each left-linear GSM-TRS is an EPRF-TRS.*

Proof. Let R be a left-linear GSM-TRS over some ranked alphabet Σ. Without loss of generality we may assume that for each rule $l \to r$ in R, $l \in \overline{T}_\Sigma(X_n)$ for some $n \geq 0$. Moreover, let L be a finite tree language over Σ. Via a series of lemmas we show that $R^*(L)$ is recognizable. To this end, we construct a bta $\mathcal{C}$ over Σ such that $L(\mathcal{C}) = R^*(L)$. Our construction is illustrated by an example in Section 6.

Let E be the set of all ground terms u over Σ such that there are rules $l_1 \to r_1$ and $l_2 \to r_2$ in R, and there are positions $\alpha \in POS(r_1)$ and $\beta \in POS(l_2)$, and there is a linear pattern $l_3 \in T_\Sigma(X) - X$ of l_2/β with $var(l_3) \cap var(l_1) = \emptyset$ such that
(i) $\alpha = \lambda$ or $\beta = \lambda$,
(ii) r_1/α and l_3 are unifiable,
(iii) σ is a most general unifier of r_1/α and l_3, and
(iv) there is a position $\gamma \in POS(l_3)$ such that $l_2/\beta\gamma \in X$ and $\sigma(l_3/\gamma) \in T_\Sigma$ and $u = \sigma(l_3/\gamma)$.
It should be clear that E is finite and is effectively constructable. Let

$$D = sub(L \cup E) \cup$$
$$sub(\{ r[e_1, \ldots, e_n] \mid n \geq 0, r \in rhs(R) \cap T_\Sigma(X_n), e_1, \ldots, e_n \in sub(L \cup E) \} .$$

Observe that

$$D = sub(D) . \tag{5}$$

Let $\mathcal{B} = (\Sigma, B, S_B, B')$ be the fundamental bta of D, and let $\mathcal{A} = (\Sigma, B, S_B, \{ \langle p \rangle \mid p \in L \})$. By Definition 2.2 and (5)

$$B = \{ \langle p \rangle \mid p \in D \} . \tag{6}$$

By Lemma 2.3, $\mathcal{B}$ and $\mathcal{A}$ are deterministic btas over Σ and $L(\mathcal{B}) = D$ and

$$L(\mathcal{A}) = L . \tag{7}$$

For each $i \geq 0$, consider the bta $\mathcal{C}_i = (\Sigma, B, S_i, \{ \langle p \rangle \mid p \in L \})$, where S_i is defined by recursion on i (for an example see Section 6). Let $S_0 = S_B$. Then

$$\mathcal{C}_0 = \mathcal{A} . \tag{8}$$

Let us assume that $i \geq 1$ and we have defined the set S_{i-1}. Then we define S_i as follows.
(a) $S_{i-1} \subseteq S_i$.
(b) For any rule $l \to r$ in R with $n \geq 0$, $l \in \overline{T}_\Sigma(X_n)$, for all $e_1, \ldots, e_n \in sub(L \cup E)$, if $l[\langle e_1 \rangle, \ldots, \langle e_n \rangle] \to^*_{S_{i-1}} c$ for some $c \in B$, then we put the λ-rule $\langle r[e_1, \ldots, e_n] \rangle \to c$ in S_i.

It should be clear that there is an integer $m \geq 0$ such that $S_m = S_{m+1}$. Let m be the least integer such that $S_m = S_{m+1}$. Let $\mathcal{C} = \mathcal{C}_m$. Let $S = S_m$, and from now on we write $\mathcal{C} = (\Sigma, B, S, \{ \langle p \rangle \mid p \in L \})$, rather than $\mathcal{C}_m = (\Sigma, B, S_m, \{ \langle p \rangle \mid p \in L \})$.

Intuitively, we construct a sequence of bottom-up tree automata $\mathcal{C}_i = (\Sigma, B, S_i, \{ \langle p \rangle \mid p \in L \})$, $0 \leq i \leq m$, having the same input ranked alphabet, state set, and final state set. $\mathcal{C}_0$ recognizes L and evaluates each element p of D to the corresponding state $\langle p \rangle$. For each $i \geq 0$, S_{i+1} contains S_i, and for each rule $l \to r$ in R, $\mathcal{C}_{i+1}$ simulates, on the right-hand side r, the computation of $\mathcal{C}_i$ on the left-hand side l.

Conversely, consider a piece of computation of $\mathcal{C}_{i+1}$, $i \geq 0$, on the right-hand side r of a rule $l \to r$ in R. We can simulate this piece of computation of $\mathcal{C}_{i+1}$ in the following way.

- We apply the rule $l \to r$ for the input tree in a reverse way, i.e., we substitute the left-hand side for the right-hand side.
- For each state $\langle p \rangle \in B$ appearing in a variable position of r, p is substituted for the corresponding variable in l.
- $\mathcal{C}_i$ simulates on the left-hand side l in the resulting new input tree the computation of $\mathcal{C}_{i+1}$ on the right-hand side r.

We iterate this simulation on any input tree to $\mathcal{C}$. In this way, we obtain another input tree, which can be rewritten to the original input tree by R and is evaluated by $\mathcal{C}_0$ to the same state as the original input tree is evaluated by $\mathcal{C}$.

Our aim is to show that $R^*(L) = L(\mathcal{C})$. To this end, first we show five preparatory lemmas, then the inclusion $L(\mathcal{C}) \subseteq R^*(L)$, then again five preparatory lemmas, and finally the inclusion $R^*(L) \subseteq L(\mathcal{C})$.

Lemma 5.2. $L = L(\mathcal{C}_0)$.

Proof. By Lemma 2.3, $L(\mathcal{A}) = L$. By the definition of $\mathcal{B}$ and $\mathcal{C}_0$, we have $L(\mathcal{C}_0) = L(\mathcal{A}) = L$. $\square$

Lemma 5.3. *For any $p \in T_\Sigma$, $r \in RHS(R) \cap T_\Sigma(X_n)$, $n \geq 0$, and $e_1, \ldots, e_n \in sub(L \cup E)$, if $p \to^*_{S_0} \langle r[e_1, \ldots, e_n] \rangle$, then $p = r[e_1, \ldots, e_n]$.*

Proof. Recall that $S_0 = S_B$ and $\mathcal{B} = (\Sigma, B, S_B, B')$ is the fundamental bta of D. By definition, for any $r \in T_\Sigma(X_n)$ and $e_1, \ldots, e_n \in sub(L \cup E)$, $r[e_1, \ldots, e_n] \in D$. Consequently, by Lemma 2.3 (a), $p = r[e_1, \ldots, e_n]$. $\square$

Lemma 5.4. *For any $i \geq 1$, $p \in T_\Sigma$, $q, t \in T_{\Sigma \cup B}$, $k \geq 1$, and $v_1, \ldots, v_k \in T_{\Sigma \cup B}$, if*

$$p = v_1 \underset{S_0}{\to} v_2 \underset{S_0}{\to} \cdots \underset{S_0}{\to} v_k = q \underset{S_i - S_{i-1}}{\to} t \,, \tag{9}$$

then there exists an $s \in T_\Sigma$ such that

$$s \underset{R}{\to} p \text{ and } s \overset{*}{\underset{S_{i-1}}{\to}} t \,. \tag{10}$$

Proof. Let α be the position of q where $\mathcal{C}_i$ applies an $(S_i - S_{i-1})$-rule

$$\langle r[e_1, \ldots, e_n] \rangle \to c \tag{11}$$

with $r \in T_\Sigma(X_n)$, $n \geq 0$, and $e_1, \ldots, e_n \in sub(L \cup E)$ in the last step $q \to_{S_i - S_{i-1}} t$ of (9). Here r is the right-hand side of a rule $l \to r$ used to add the transition (11) with item (b) of the construction of S_i. Then

$$q = u[\langle r[e_1, \ldots, e_n] \rangle] \,,$$

where $u \in \overline{T}_\Sigma(X_1)$, $u/\alpha = x_1$. Hence $p = u[s]$ for some $s \in T_\Sigma$ and $s \to^*_{S_0} \langle r[e_1, \ldots, e_n] \rangle$. By Lemma 5.3,

$$p = u[r[e_1, \ldots, e_n]] \,.$$

Finally, $t = u[c]$. By (b) of the definition of S_i, $i \geq 1$, there is a rule $l \to r$ in R with $l \in \overline{T}_\Sigma(X_n)$, $n \geq 0$ such that

$$l[\langle e_1 \rangle, \ldots, \langle e_n \rangle] \overset{*}{\underset{S_{i-1}}{\to}} c \,.$$

Let

$$s = u[l[e_1, \ldots, e_n]] \,.$$

Then

$$s \underset{R}{\to} p \tag{12}$$

and

$$s = u[l[e_1, \ldots, e_n]] \overset{*}{\underset{S_0}{\to}} u[l[\langle e_1 \rangle, \ldots, \langle e_n \rangle]] \overset{*}{\underset{S_{i-1}}{\to}} u[c] = t \,. \tag{13}$$

Hence (10) holds. $\square$

Lemma 5.5. *For any $i \geq 0$, $p \in T_\Sigma$, and $q \in T_{\Sigma \cup B}$, if $p \to^*_{S_i} q$, then there is an $s \in T_\Sigma$ such that*

$$s \xrightarrow[R]{*} p \text{ and } s \xrightarrow[S_0]{*} q\,.$$

Proof. We proceed by induction on i. For $i = 0$ the statement is trivial. Let us suppose that $i \geq 1$ and that we have shown the statement for $1, 2, \ldots, i-1$. Let

$$p \xrightarrow[S_i]{*} q\,, \tag{14}$$

and let m be the number of $(S_i - S_{i-1})$-rules applied by $\mathcal{C}_i$ along (14). We show by induction on m that

$$\text{there is } s \in T_\Sigma \text{ such that } s \xrightarrow[R]{*} p \text{ and } s \xrightarrow[S_0]{*} q\,. \tag{15}$$

If $m = 0$, then $p \to^*_{S_{i-1}} q$ and hence by the induction hypothesis on i, (15) holds.

Let us suppose that $m \geq 1$ and that for $0, 1, \ldots, m-1$, we have shown (15). Let $p \to^*_{S_i} q$ where $\mathcal{C}_i$ applies m $(S_i - S_{i-1})$-rules. Then there are integers n, k, $1 \leq k \leq n$, and there are trees $t_1, t_2, u_1, u_2, \ldots, u_n \in T_{\Sigma \cup B}$ such that
$p = u_1 \to_{S_{i-1}} \cdots \to_{S_{i-1}} u_k = t_1 \to_{S_i - S_{i-1}} u_{k+1} = t_2 \to_{S_i} \cdots \to_{S_i} u_n = q$.
Then along the reduction subsequence $t_2 = u_{k+1} \to_{S_i} \cdots \to_{S_i} u_n = q$ $\mathcal{C}_i$ applies $m-1$ $(S_i - S_{i-1})$-rules.
By the induction hypothesis on i, there is a tree $s_1 \in T_\Sigma$ such that

$$s_1 \xrightarrow[R]{*} p \text{ and } s_1 \xrightarrow[S_0]{*} t_1\,. \tag{16}$$

Hence

$$s_1 \xrightarrow[S_0]{*} t_1 \xrightarrow[S_i - S_{i-1}]{} t_2\,.$$

By Lemma 5.4, there is a tree $s_2 \in T_\Sigma$ such that

$$s_2 \xrightarrow[R]{} s_1 \text{ and } s_2 \xrightarrow[S_{i-1}]{*} t_2\,. \tag{17}$$

Hence there is $j \geq 0$ and there are $w_1, \ldots, w_j \in T_{\Sigma \cup B}$ such that

$$s_2 = w_1 \xrightarrow[S_{i-1}]{} w_2 \xrightarrow[S_{i-1}]{} \cdots \xrightarrow[S_{i-1}]{} w_j = t_2 = u_{k+1} \xrightarrow[S_i]{} \cdots \xrightarrow[S_i]{} u_n = q\,, \tag{18}$$

and along (18), $\mathcal{C}_i$ applies $m-1$ $(S_i - S_{i-1})$-rules. By the induction hypothesis on m, there is a tree $s_3 \in T_\Sigma$ such that

$$s_3 \xrightarrow[R]{*} s_2 \text{ and } s_3 \xrightarrow[S_0]{*} q\,.$$

Hence by (16) and (17),

$$s_3 \xrightarrow[R]{*} s_2 \xrightarrow[R]{} s_1 \xrightarrow[R]{*} p\,.$$

Thus (15) holds. □

Lemma 5.6. $L(\mathcal{C}) \subseteq R^*(L)$.

Proof. Let $p \in L(\mathcal{C})$. Then $p \to^*_S b$ for some $b \in \{\, \langle p \rangle \mid p \in L \,\}$. Hence by Lemma 5.5, there is an $s \in T_\Sigma$ such that

$$s \xrightarrow[R]{*} p \text{ and } s \xrightarrow[S_0]{*} b\,. \tag{19}$$

Hence $s \in L(\mathcal{C}_0)$. By Lemma 5.2, $s \in L$. Thus by (19), $p \in R^*(L)$. □

Now we show the inclusion $R^*(L) \subseteq L(\mathcal{C})$. To this end, we prove five lemmas. We formulate the first one in the light of our observations on Conditions (a'), (b'), and (c') right after Definition 4.9. It describes the following phenomenon. We unify a right-hand side r_1 and a nonvariable linear pattern of a subterm of a left-hand side l_2 or a subterm of a right-hand side r_1 and a nonvariable linear pattern of a left-hand side l_2. In both cases, we substitute elements of $sub(L \cup E)$ for the variables of r_1. Then we show that we also substitute elements of $sub(L \cup E)$ for the variables appearing in the left-hand side l_2. The unified term becomes an element of D. This membership and (6) justify the choice of B as the state set of $\mathcal{B}$.

Lemma 5.7. *Let $l_1 \to r_1$ and $l_2 \to r_2$ be rules in R. Let $\alpha \in POS(r_1)$, where $r_1/\alpha \in T_\Sigma(X_j)$, $j \geq 0$. Let $\beta \in POS(l_2)$, where $l_2/\beta \in T_\Sigma(X) - X$, and let $s \in \overline{T}_\Sigma(X_k) - X$, $k \geq 1$, be a linear pattern of l_2/β. Let $\alpha = \lambda$ or $\beta = \lambda$. Let*

$$(r_1/\alpha)[e_1, \ldots, e_j] = s[z_1, \ldots, z_k], \tag{20}$$

where $e_1, \ldots, e_j \in sub(L \cup E)$, $z_1, \ldots, z_k \in T_\Sigma$. Let $\gamma \in POS(s)$ be such that $l_2/\beta\gamma \in X$, and $s/\gamma = x_\nu$, for some $1 \leq \nu \leq k$. Then $z_\nu \in sub(L \cup E)$.

Proof. Let $l_1 \in T_\Sigma(X_m)$ for some $m \geq 0$. Let $l_3 = s[x_{m+1}, \ldots, x_{m+k}]$. Then $l_3 \in T_\Sigma(\{x_{m+1}, \ldots, x_{m+k}\})$ is a linear pattern of l_2/β, for each $m + 1 \leq i \leq m + k$, x_i appears exactly once in l_3. Moreover, $var(l_1) \cap var(l_3) = \emptyset$, and by (20),

$$(r_1/\alpha)[e_1, \ldots, e_j] = l_3[x_{m+1} \leftarrow z_1, \ldots, x_{m+k} \leftarrow z_k]. \tag{21}$$

Let $\sigma_1 : X \to T_\Sigma(X)$ be a most general unifier of r_1/α and l_3. By (21), there is a substitution $\sigma_2 : X \to T_\Sigma(X)$ such that

$$\sigma_2(\sigma_1(r_1/\alpha)) = (r_1/\alpha)[e_1, \ldots, e_j] = l_3[x_{m+1} \leftarrow z_1, \ldots, x_{m+k} \leftarrow z_k] = \sigma_2(\sigma_1(l_3)),$$

where $\sigma_2(\sigma_1(x_i)) = e_i$ for $1 \leq i \leq j$ and $\sigma_2(\sigma_1(x_{m+i})) = z_i$ for $1 \leq i \leq k$. Let $\gamma \in POS(s)$ be such that $l_2/\beta\gamma \in X$, and $s/\gamma = x_\nu$, for some $1 \leq \nu \leq k$. By Definition 4.7 and by the definition of E, $\sigma_1(x_{m+\nu}) \in X \cup E$. If $\sigma_1(x_{m+\nu}) \in X$, then $\sigma_2(\sigma_1(x_{m+\nu}))$ is a subtree of e_μ for some $\mu \in \{1, \ldots, j\}$. Hence by the definition of $e_1, \ldots, e_j$, $z_\nu = \sigma_2(\sigma_1(x_{m+\nu})) \in sub(L \cup E)$. If $\sigma_1(x_{m+\nu}) \in E$, then $z_\nu = \sigma_2(\sigma_1(x_{m+\nu})) = \sigma_1(x_{m+\nu}) \in E \subseteq sub(E)$. □

Intuitively, the following lemma states that along a reduction sequence of S we can reverse the order of the consecutive application of an S_0-rule at $\alpha \in N^*$ and the application of an $(S - S_0)$-rule at $\beta \in N^*$ if α is not a prefix of β and β is not a prefix of α.

Lemma 5.8. *Let*

$$u_1 \underset{S}{\to} u_2 \underset{S}{\to} u_3$$

be a reduction sequence of $\mathcal{C}$, where $u_1, u_2, u_3 \in T_{\Sigma \cup B}$. Let $\alpha \in POS(u_1)$, and $\beta \in POS(u_2)$ be such that $u_1 \to_S u_2$ applying a rule $\mathbf{rule_1}$ *of S_0 at α, and that $u_2 \to_S u_3$ applying an $(S - S_0)$-rule* $\mathbf{rule_2}$ *at β. If α is not a prefix of β and β is not a prefix of α, then there is a tree $v \in T_{\Sigma \cup B}$ such that $u_1 \to_S v$ applying* $\mathbf{rule_2}$ *at β, and $v \to_S u_3$ applying* $\mathbf{rule_1}$ *at α.*

Proof. Straightforward. □

Lemma 5.9. *For any $n \geq 0$, $u \in \overline{T}_\Sigma(X_n)$, $v_1, \ldots, v_n, v \in D$, $m \geq 1$, and $w_1, \ldots, w_m \in T_{\Sigma \cup B}$, if*

$$u[\langle v_1 \rangle, \ldots, \langle v_n \rangle] = w_1 \underset{S_0}{\to} w_2 \underset{S_0}{\to} \cdots \underset{S_0}{\to} w_m = \langle v \rangle, \tag{22}$$

then $u[v_1, \ldots, v_n] = v$.

Proof. We proceed by induction on $height(u)$. The basis $height(u) = 0$ of the induction is trivial. The induction step is a simple consequence of the definition of S_0. □

Lemma 5.10. *Let $i \geq 0$, $t \in \overline{T}_{\Sigma \cup B}(X_1)$, $\alpha \in POS(t)$, $t/\alpha = x_1$, $p \in D - sub(L)$, and $w \in sub(L)$. Let*

$$t[\langle p \rangle] = u_1 \underset{S_i}{\to} u_2 \underset{S_i}{\to} \cdots \underset{S_i}{\to} u_n = \langle w \rangle \tag{23}$$

with $n \geq 1$, $u_1, \ldots, u_n \in T_{\Sigma \cup B}$. Then along (23), $\mathcal{C}_i$ applies a rule in $S_i - S_0$ at some prefix β of α.

Proof. By contradiction. Assume that

$$t[\langle p \rangle] = u_1 \underset{S_0}{\to} u_2 \underset{S_0}{\to} \cdots \underset{S_0}{\to} u_n = \langle w \rangle. \tag{24}$$

By Lemma 5.9, we have p is a subterm of w. Since $w \in sub(L)$, we have $p \in sub(L)$. On the other hand, $p \in D - sub(L)$. This is a contradiction. □

We now consider a reduction $t = t_1 \to_S t_2 \to_S t_3 \to_S \cdots \to_S t_m = b$ ending in a final state b of $\mathcal{C}$. We assume that an instance $l[\langle v_1 \rangle, \ldots, \langle v_n \rangle]$ of a left-hand side $l \in lhs(R)$ is a subtree of t_j. Here $\langle v_1 \rangle, \ldots, \langle v_n \rangle$ are any states in B, that is, $v_1, \ldots, v_n \in D$. Furthermore, we suppose that S has already applied all reading rules and λ-rules at the variable positions of l along the prefix $t = t_1 \to_S t_2 \to_S t_3 \to_S \cdots \to_S t_j$. Consequently, S does not rewrite at the variable positions of l along the suffix $t_j \to_S t_{j+1} \to_S \cdots \to_S t_m = b$. Then we show that $v_1, \ldots, v_n \in sub(L \cup E)$. This observation justifies that in item (b) of the definition of S_i, the terms $e_1, \ldots, e_n$ are taken as the elements of $sub(L \cup E)$. Consequently, it also justifies the definition of D and B.

Lemma 5.11. *Let $t \in L(\mathcal{C})$, $m \geq 1$, $t_1, \ldots, t_m \in T_{\Sigma \cup B}$, $b \in \{ \langle p \rangle \mid p \in L \}$, and let*

$$t = t_1 \underset{S}{\rightarrow} t_2 \underset{S}{\rightarrow} t_3 \underset{S}{\rightarrow} \cdots \underset{S}{\rightarrow} t_m = b\,. \tag{25}$$

Let $l \rightarrow r$ be a rule in R, where $l \in \overline{T}_\Sigma(X_n)$ and $n \geq 1$. Moreover, let $1 \leq j \leq m$, and let

$$t_j/\alpha = l[\langle v_1 \rangle, \ldots, \langle v_n \rangle]\,, \tag{26}$$

where $n \geq 1$, $v_1, \ldots, v_n \in D$, $\alpha \in POS(t_j)$. Let $\alpha_1, \ldots, \alpha_n \in POS(l)$ be such that

$$l/\alpha_i = x_i \text{ for } 1 \leq i \leq n\,. \tag{27}$$

Consider the reduction subsequence

$$t_j \underset{S}{\rightarrow} t_{j+1} \underset{S}{\rightarrow} \cdots \underset{S}{\rightarrow} t_m = b \tag{28}$$

of (25). If $\mathcal{C}$ does not apply any rules at the positions $\alpha\alpha_1, \ldots, \alpha\alpha_n$ along (28), then $v_1, \ldots, v_n \in sub(L \cup E)$.

Proof. Let $1 \leq i \leq n$ be arbitrary. By (26) and (27),

$$t_j/\alpha\alpha_i = \langle v_i \rangle\,. \tag{29}$$

By Lemma 5.10, $\mathcal{C}$ applies a rule in $S - S_0$ at some prefix of $\alpha\alpha_i$ along (28). Let $\beta \in POS(t_j)$ be the longest prefix of $\alpha\alpha_i$ such that $\mathcal{C}$ applies a rule **rule** in $S - S_0$ at β along (28). Then **rule** is of the form $\langle r_1[e_1, \ldots, e_\kappa] \rangle \rightarrow c$, where $\kappa \geq 0$, $r_1 \in T_\Sigma(X_\kappa)$, $e_1, \ldots, e_\kappa \in sub(L \cup E)$, and there is a rule $l_1 \rightarrow r_1$ in R. Moreover there exists $\xi, j < \xi \leq m$, such that

$$t_j/\beta \overset{*}{\underset{S}{\rightarrow}} t_{j+1}/\beta \overset{*}{\underset{S}{\rightarrow}} \cdots \overset{*}{\underset{S}{\rightarrow}} t_\xi/\beta = \langle r_1[e_1, \ldots, e_\kappa] \rangle\,,$$

where for each $\pi, j \leq \pi \leq \xi - 1$, $t_\pi/\beta = t_{\pi+1}/\beta$ or $t_\pi/\beta \rightarrow_S t_{\pi+1}/\beta$. We lose no generality by assuming that

$$t_j/\beta \underset{S}{\rightarrow} t_{j+1}/\beta \underset{S}{\rightarrow} \cdots \underset{S}{\rightarrow} t_\xi/\beta = \langle r_1[e_1, \ldots, e_\kappa] \rangle\,. \tag{30}$$

By Lemma 5.8 we may assume that there exists $\nu, j \leq \nu \leq \xi$ such that
(a) along the reduction subsequence

$$t_j/\beta \underset{S}{\rightarrow} \cdots \underset{S}{\rightarrow} t_\nu/\beta \tag{31}$$

of (30) no rule is applied at any prefix of $\alpha\alpha_i$, that
(b) along the reduction subsequence

$$t_\nu/\beta \underset{S}{\rightarrow} \cdots \underset{S}{\rightarrow} t_\xi/\beta = \langle r_1[e_1, \ldots, e_\kappa] \rangle$$

of (30), S applies only rules of S_0.
Then

$$t_\nu/\beta = s[\langle z_1 \rangle, \ldots, \langle z_k \rangle] \tag{32}$$

for some $k \geq 1$, $s \in \overline{T}_\Sigma(X_k)$, and $z_1, \ldots, z_k \in D$. By (32), (b) of the definition of ν, and Lemma 5.9,

$$s[z_1, \ldots, z_k] = r_1[e_1, \ldots, e_\kappa]\,. \tag{33}$$

The word α is a prefix of β or β is a prefix of α. Hence we can distinguish two cases.
Case 1 α is a prefix of β, see Fig. 4. In this case,

$$\beta = \alpha\gamma \tag{34}$$

for some $\gamma \in N^*$, and hence t_ν/β is a subtree of t_ν/α. Now by (26), the definition of ν, and (32),

$$s \text{ is a linear pattern of } l/\gamma\,. \tag{35}$$

Let ω be the prefix of $\alpha\alpha_i$ with $length(\omega) = length(\alpha\alpha_i) - 1$. Observe that $\mathcal{C}$ applies a rule of S_0 at the position ω along (28). Hence

$$s \notin X\,. \tag{36}$$

We define $\delta \in N^*$ by the equation $\gamma\delta = \alpha_i$. Then

$$\beta\delta = \alpha\alpha_i\,, \tag{37}$$

and by (a) of the definition of ν,

$$\delta \in POS(s), \quad \delta \in POS(l/\gamma), \text{ and } (l/\gamma)/\delta = x_i\,. \tag{38}$$

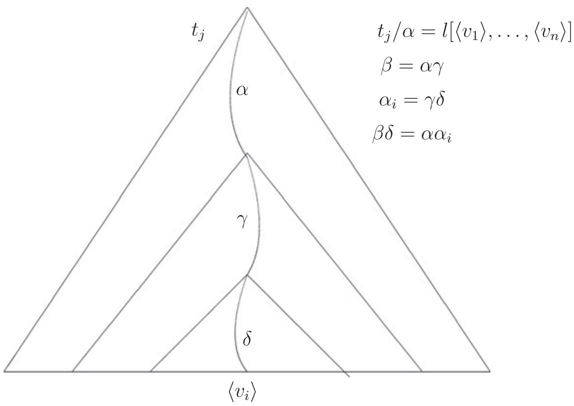

Fig. 4. Case 1.

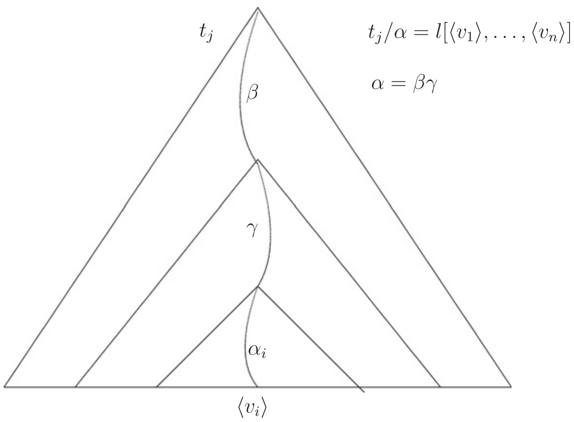

Fig. 5. Case 2.

By (37) and by (a) of the definition of ν,

$$\beta\delta \in POS(t_\nu)\,.$$

By (29), (37), (a) of the definition of ν, and (32),

$$\langle v_i\rangle = (t_j/\beta)/\delta = (t_\nu/\beta)/\delta = s[\langle z_1\rangle, \ldots, \langle z_k\rangle]/\delta = \langle z_\mu\rangle \tag{39}$$

for some $1 \le \mu \le k$. As R is a GSM-TRS, by (35), (36), (38), (33), and Lemma 5.7, $z_\mu \in sub(L \cup E)$. By (39), $v_i = z_\mu$. Thus $v_i \in sub(L \cup E)$.

Case 2 β is a prefix of α, see Fig. 5. In this case

$$\alpha = \beta\gamma \tag{40}$$

for some $\gamma \in N^*$, and hence t_j/α is a subtree of t_j/β. Now by (26), the definition of ν, and (32),

$$s/\gamma \text{ is a linear pattern of } l\,. \tag{41}$$

Moreover, by (a) of the definition of ν,

$$\alpha_i \in POS(s/\gamma),\quad l/\alpha_i \in X,\ \text{ and } (s/\gamma)/\alpha_i \in X\,. \tag{42}$$

Let ω be the prefix of $\alpha\alpha_i$ with $length(\omega) = length(\alpha\alpha_i) - 1$. Observe that $\mathcal{C}$ applies a rule of S_0 at the position ω along (28). Hence

$$s/\gamma \notin X\,. \tag{43}$$

By (40) and by (a) of the definition of ν,

$$\beta\gamma\alpha_i = \alpha\alpha_i \in POS(t_\nu)\,. \tag{44}$$

Then by (29), (44), (a) of the definition of ν, and (32),

$$\langle v_i \rangle = (t_j/\beta)/\gamma\alpha_i = (t_\nu/\beta)/\gamma\alpha_i = s[\langle z_1 \rangle, \ldots, \langle z_k \rangle]/\gamma\alpha_i = \langle z_\mu \rangle \tag{45}$$

for some $1 \leq \mu \leq k$. By (33),

$$(s/\gamma)[z_1, \ldots, z_k] = s[z_1, \ldots, z_k]/\gamma = r_1[e_1, \ldots, e_\kappa]/\gamma\,. \tag{46}$$

As R is a GSM-TRS, by (41), (43), (42), (45), (46), and Lemma 5.7, $z_\mu \in sub(L \cup E)$. By (45), $v_i = z_\mu$. Thus $v_i \in sub(L \cup E)$. $\square$

Lemma 5.12. $R^*(L) \subseteq L(\mathcal{C})$.

Proof. By (7) and (8), $L = L(\mathcal{C}_0)$. As $S_{i-1} \subseteq S_i$ for $i \geq 1$, we have $L \subseteq L(\mathcal{C}_i)$ for $i \geq 0$. Hence $L \subseteq L(\mathcal{C})$. Thus it is sufficient to show that for each $t \in L(\mathcal{C})$, if $t \rightarrow_R t'$, then $t' \in L(\mathcal{C})$. To this end, let us suppose that $t \rightarrow_R t'$, applying the rule $l \rightarrow r$ in R at $\alpha \in POS(t)$. Here $l \in \overline{T}_\Sigma(X_n)$ for some $n \geq 0$. Let $\alpha_1, \ldots, \alpha_n \in POS(l)$ be such that

$$l/\alpha_i = x_i \text{ for } 1 \leq i \leq n\,.$$

Then

$$t = s[l[u_1, \ldots, u_n]]\,,$$

where $s \in \overline{T}_\Sigma(X_1)$, $\alpha \in POS(s)$, $s/\alpha = x_1$, and $u_1, \ldots, u_n \in T_\Sigma$. Moreover,

$$t' = t[\alpha \leftarrow r[u_1, \ldots, u_n]] = s[r[u_1, \ldots, u_n]]\,.$$

As $t \in L(\mathcal{C})$, there is a reduction sequence

$$t = t_1 \underset{S}{\rightarrow} t_2 \underset{S}{\rightarrow} t_3 \underset{S}{\rightarrow} \cdots \underset{S}{\rightarrow} t_m = b, \tag{47}$$

where $m \geq 1$, $b \in \{\, \langle p \rangle \mid p \in L \,\}$, $t_1, \ldots, t_m \in T_{\Sigma \cup B}$. Furthermore, by (6), there are integers j, k with $1 \leq j \leq k \leq m$ such that

(i) $t_j = s[l[\langle v_1 \rangle, \ldots, \langle v_n \rangle]]$, where $v_i \in D$ and $u_i \rightarrow_S^* \langle v_i \rangle$ for $1 \leq i \leq n$,
(ii) $t_k = s[c_0]$, for some $c_0 \in B$, where $l[\langle v_1 \rangle, \ldots, \langle v_n \rangle] \rightarrow_S^* c_0$, and that
(iii) along the reduction subsequence $t_j \rightarrow_S t_{j+1} \rightarrow_S \cdots \rightarrow_S t_k$ of (47), $\mathcal{C}$ does not apply any rules at the positions $\alpha\alpha_1, \ldots, \alpha\alpha_n$.

Intuitively, Condition (iii) says the following. Along the prefix
$t = t_1 \rightarrow_S t_2 \rightarrow_S \rightarrow_S t_3 \rightarrow_S \cdots \rightarrow_S t_j$
of the reduction sequence (47), for each $i = 1, \ldots, n$, $\mathcal{C}$ might have applied a reading rule and then λ-rules at the position $\alpha\alpha_i$. However, along the suffix
$t_j \rightarrow_S t_{j+1} \rightarrow_S \cdots \rightarrow_S t_k$
of (47), $\mathcal{C}$ does not apply any rules at the position $\alpha\alpha_i$, it applies a reading rule above the position $\alpha\alpha_i$, for $1 \leq i \leq n$.
By Lemma 5.11, $v_1, \ldots, v_n \in sub(L \cup E)$. Hence by item (b) in the definition of S_i, $i \geq 1$, and by the definition of $\mathcal{C}$, the rule

$$\langle r[v_1, \ldots, v_n] \rangle \rightarrow c_0 \tag{48}$$

is in S. Thus we get

$$t' = s[r[u_1, \ldots, u_n]] \overset{*}{\underset{S}{\rightarrow}} s[r[\langle v_1 \rangle, \ldots, \langle v_n \rangle]] \overset{*}{\underset{S}{\rightarrow}} s[\langle r[v_1, \ldots, v_n] \rangle] \underset{S}{\rightarrow} s[c_0] \overset{*}{\underset{S}{\rightarrow}} b\,. \tag{49}$$

As $b \in \{\, \langle p \rangle \mid p \in L \,\}$, we have $t' \in L(\mathcal{C})$. $\square$

By Lemmas 5.6 and 5.12, we get that $R^*(L) = L(\mathcal{C})$. $\square$

Note that we used the assumption that R is left-linear along the proof of Lemma 5.12. At the beginning of the proof, we assumed that R applies a rule $l \rightarrow r$, where $l \in \overline{T}_\Sigma(X_n)$ for some $n \geq 0$. Let us drop the assumption that R is left-linear and that $l \in \overline{T}_\Sigma(X_n)$ for some $n \geq 0$. We generalize in a natural way the construction of S_i, $i \geq 1$, and $\mathcal{C}$ in the proof of Theorem 5.1. Assume that

- $l = l_1[x_{i_1}, \ldots, x_{i_{m_1}}]$, $r = r_1[x_{j_1}, \ldots, x_{j_{m_2}}]$ for some $l_1 \in \overline{T}_\Sigma(X_{m_1})$, $r_1 \in \overline{T}_\Sigma(X_{m_2})$, and $m_1, m_2 \geq 0$ and that
- $x_{i_{k_1}} = x_{i_{k_2}} = x_{j_{k_3}} = x_{j_{k_4}}$ for some $k_1, k_2 \in \{\, 1, \ldots, m_1 \,\}$ and $k_3, k_4 \in \{\, 1, \ldots, m_2 \,\}$.

In Statement (i) in the proof we observed that $u_{i_{k_1}} \rightarrow_S^* v_{i_{k_1}}$ and $u_{i_{k_2}} \rightarrow_S^* v_{i_{k_2}}$ for some $v_{i_{k_1}}, v_{i_{k_2}} \in D$. In Statement (ii) we observed that
$l[\langle v_1 \rangle, \ldots, \langle v_n \rangle] \rightarrow_S^* c_0$.
However, in general, $v_{i_{k_1}} \neq v_{i_{k_2}}$. Hence $l_1[\langle v_{i_1} \rangle, \ldots, \langle v_{i_{k_1}} \rangle, \ldots, \langle v_{i_{k_2}} \rangle, \ldots, \langle v_{i_{m_1}} \rangle]$ is not an instance of l. Consequently, there are no states $\langle v_1 \rangle, \ldots, \langle v_n \rangle$ such that $l[\langle v_1 \rangle, \ldots, \langle v_n \rangle] = l_1[\langle v_{i_1} \rangle, \ldots, \langle v_{i_{k_1}} \rangle, \ldots, \langle v_{i_{k_2}} \rangle, \ldots, \langle v_{i_{m_1}} \rangle]$. Hence we cannot write that

$l[\langle v_1\rangle, \ldots, \langle v_n\rangle] = l_1[\langle v_{i_1}\rangle, \ldots, \langle v_{i_{k_1}}\rangle, \ldots, \langle v_{i_{k_2}}\rangle, \ldots, \langle v_{i_{m_1}}\rangle] \to_S^* c_0$.
Consequently the condition of item (b) in the definition of S_i, $i \geq 1$, does not hold. Hence we cannot add the rule (48) to S_i for any $i \geq 1$. Thus (48) is not in S. Hence $s[\langle r[v_1, \ldots, v_n]\rangle] \to_S s[c_0]$ does not hold in general. Thus (49) does not hold either.

We illustrate the above discussion by the following example. We drop the assumption that R is left-linear, and we generalize in a natural way the construction of $\mathcal{C}$ in the proof of Theorem 5.1. Consider the ranked alphabet Σ and the TRS R in the proof of Theorem 4.1. Let $L = \{ g(\sharp, \$) \}$. We construct the bta $\mathcal{C}$ applying the generalized construction to R and L. We obtain the bta
$\mathcal{C} = (\Sigma, \{ \langle\sharp\rangle, \langle\$\rangle, \langle g(\sharp,\$)\rangle, \langle\flat\rangle, \langle h(\sharp,\sharp)\rangle, \langle h(\$,\$)\rangle, \langle h(g(\sharp,\$), g(\sharp,\$))\rangle\}, S, \{ \langle g(\sharp,\$)\rangle \})$. Here S consists of the following rules.
$\sharp \to \langle\sharp\rangle, \quad \$ \to \langle\$\rangle, \quad g(\langle\sharp\rangle, \langle\$\rangle) \to \langle g(\sharp,\$)\rangle,$
$\flat \to \langle\sharp\rangle, \quad f(\langle\sharp\rangle) \to \sharp, \quad \flat \to \langle\$\rangle, \quad f(\langle\$\rangle) \to \$,$
$h(\langle\sharp\rangle, \langle\sharp\rangle) \to \langle h(\sharp,\sharp)\rangle, \quad h(\langle\$\rangle, \langle\$\rangle) \to \langle h(\$,\$)\rangle.$
Observe that $L(\mathcal{C}) = L \subset R_\Sigma^*(L)$.

We now compare the proof Theorem 3.19 in [13] with that of Theorem 5.1. In the proof of Theorem 3.19 in [13], for a linear GSM-TRS R and a recognizable tree language L, they constructed a bta $\mathcal{C}$ such that

$$R^*(L) = L(\mathcal{C})\,. \tag{50}$$

In the proof of Theorem 5.1, for a left-linear GSM-TRS R and a finite tree language L, we constructed a bta $\mathcal{C}$ such that (50) holds. The two constructions of $\mathcal{C}$ are slightly different. The construction of $\mathcal{C}$ in [13] can naturally be generalized to the case when R need not be right-linear. Then $R^*(L)$ is not recognizable in most cases, and (50) does not hold in general. However, one might think that in the general case the proof of Theorem 3.19 in [13] can be converted to a proof of Theorem 5.1 under the assumption that L is finite. We now show that this belief is not justified.

The proof of Theorem 3.19 in [13] and that of Theorem 5.1 have similar structures. Clearly, we can assign to each statement in the proof of Theorem 3.19 in [13] a parallel statement of the proof in Theorem 5.1. This statement is similar and also has a similar proof. Lemma 3.9 in [13], the first major step of the proof of Theorem 3.19 in [13] is parallel to Lemma 5.3, the first step of the proof of Theorem 5.1. Then in the two proofs, the proofs of the parallel lemmas refer to the previous results in a parallel way. For example, the proof of Lemma 3.10 in [13], the parallel of Lemma 5.4, refers to Lemma 3.9 in [13], the parallel of Lemma 5.3 and the proof of Lemma 5.4 refers to Lemma 5.3. Hence the proofs of the parallel statements are also parallel. Thus the proof of Theorem 3.19 in [13] and that of Theorem 5.1 are parallel. However, the above mentioned first steps of these two parallel proofs are different. That is, the proof of Lemma 3.9 in [13] is essentially different from that of Lemma 5.3. The proof of Lemma 3.9 in [13] uses that R is linear. The proof of Lemma 5.3 uses that L is finite. The difference prevails between the proofs of the parallel lemmas along the proof of Theorem 3.19 in [13] and that of Theorem 5.1. Thus if we were replacing the first step of the proof of Theorem 5.1 with that of Theorem 3.19 in [13], we would not get a proof of Theorem 5.1.

Statement 3.3 and Theorem 5.1 imply the following result.

Theorem 5.13. *There is a left-linear monadic TRS R over a ranked alphabet Σ such that R is an EPRF-TRS and that R is not a PΣR-TRS.*

6. An example

We illustrate the construction of $\mathcal{C}_j$, $j \geq 0$, appearing in the previous section by an example. Let $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \Sigma_3$, $\Sigma_0 = \{ \sharp \}$, $\Sigma_1 = \{ f \}$, $\Sigma_3 = \{ g \}$. Let the TRS R over Σ consist of the following two rules.

$$f(f(g(x_1, \sharp, \sharp))) \to f(f(x_1))\,,$$

$$g(x_1, x_2, \sharp) \to f(g(x_1, \sharp, x_1))\,.$$

By direct inspection we obtain that R is a left-linear GSM-TRS. Here $E = \{ \sharp \}$. Let $L = \{ g(\sharp, \sharp, \sharp) \}$. Then $sub(L \cup E) = \{ \sharp, g(\sharp, \sharp, \sharp) \}$. It is not hard to see that

$$R^*(L) = \{ f^n(g(\sharp, \sharp, \sharp)) \mid n \geq 0 \} \cup \{ f^n(\sharp) \mid n \geq 2 \}\,.$$

By direct inspection we obtain that the set of subterms of the right-hand sides of the rules of R is

$$\{ x_1, f(x_1), f(f(x_1)), \sharp, g(x_1, \sharp, x_1), f(g(x_1, \sharp, x_1)) \}\,.$$

Then
$D = \{ \sharp,\ f(\sharp),\ g(\sharp,\sharp,\sharp),\ f(f(\sharp)),\ f(g(\sharp,\sharp,\sharp)),\ f(f(g(\sharp,\sharp,\sharp))),\ g(g(\sharp,\sharp,\sharp),\sharp,g(\sharp,\sharp,\sharp)),$
$f(g(g(\sharp,\sharp,\sharp),\sharp,g(\sharp,\sharp,\sharp))) \}\,.$

$\mathcal{C}_0 = (\Sigma, B, S_0, \{ \langle g(\sharp,\sharp,\sharp)\rangle \})$, where $B = \{ \langle\sharp\rangle,\ \langle g(\sharp,\sharp,\sharp)\rangle,\ \langle f(\sharp)\rangle,\ \langle f(g(\sharp,\sharp,\sharp))\rangle,\ \langle f(f(\sharp))\rangle,\ \langle f(f(g(\sharp,\sharp,\sharp)))\rangle,$
$\langle g(g(\sharp,\sharp,\sharp),\sharp,g(\sharp,\sharp,\sharp))\rangle,\ \langle f(g(g(\sharp,\sharp,\sharp),\sharp,g(\sharp,\sharp,\sharp)))\rangle \}$.

Furthermore, S_0 consists of the following eight rules.
$\sharp \to \langle\sharp\rangle$,
$g(\langle\sharp\rangle, \langle\sharp\rangle, \langle\sharp\rangle) \to \langle g(\sharp, \sharp, \sharp)\rangle$,
$f(\langle\sharp\rangle) \to \langle f(\sharp)\rangle$,
$f(\langle g(\sharp, \sharp, \sharp)\rangle) \to \langle f(g(\sharp, \sharp, \sharp))\rangle$,
$f(\langle (f\sharp)\rangle) \to \langle f(f(\sharp))\rangle$,
$f(\langle f(g(\sharp, \sharp, \sharp))\rangle) \to \langle f(f(g(\sharp, \sharp, \sharp)))\rangle$,
$g(\langle g(\sharp, \sharp, \sharp)\rangle, \langle\sharp\rangle, \langle g(\sharp, \sharp, \sharp)\rangle) \to \langle g(g(\sharp, \sharp, \sharp), \sharp, g(\sharp, \sharp, \sharp))\rangle$,
$f(\langle g(g(\sharp, \sharp, \sharp), \sharp, g(\sharp, \sharp, \sharp))\rangle) \to \langle f(g(g(\sharp, \sharp, \sharp), \sharp, g(\sharp, \sharp, \sharp)))\rangle$.

Consider the bta $\mathcal{C}_1 = (\Sigma, B, S_1, \{\, \langle g(\sharp, \sharp, \sharp)\rangle \,\}$. Here S_1 contains all rules of S_0 and the rules
$\langle f(f(\sharp))\rangle \to \langle f(f(g(\sharp, \sharp, \sharp)))\rangle$,
$\langle f(g(\sharp, \sharp, \sharp))\rangle \to \langle g(\sharp, \sharp, \sharp)\rangle$.
We obtain the last two rules in the following way. First we consider the rule $f(f(g(x_1, \sharp, \sharp))) \to f(f(x_1))$ in R. We substitute $\langle\sharp\rangle \in sub(L \cup E)$ for x_1 in its left-hand side. We get the term $f(f(g(\langle\sharp\rangle, \sharp, \sharp)))$. Observe that $f(f(g(\langle\sharp\rangle, \sharp, \sharp))) \to^*_{S_0} \langle f(f(g(\sharp, \sharp, \sharp)))\rangle$. In this way, we put the rule $\langle f(f(\sharp))\rangle \to \langle f(f(g(\sharp, \sharp, \sharp)))\rangle$ in S_1. Second we consider the rule $g(x_1, x_2, \sharp) \to f(g(x_1, \sharp, x_1))$ in R. We substitute $\langle\sharp\rangle$ for x_1 and $\langle\sharp\rangle$ for x_2 in its left-hand side. Observe that $g(\langle\sharp\rangle, \langle\sharp\rangle, \sharp) \to^*_{S_0} \langle g(\sharp, \sharp, \sharp)\rangle$. In this way, we put the rule $\langle f(g(\sharp, \sharp, \sharp))\rangle \to \langle g(\sharp, \sharp, \sharp)\rangle$ in S_1.

Consider the bta $\mathcal{C}_2 = (\Sigma, B, S_2, \{\, \langle g(\sharp, \sharp, \sharp)\rangle \,\})$. Here S_2 contains all rules of S_1 and the rules
$\langle f(f(\sharp))\rangle \to \langle f(g(\sharp, \sharp, \sharp))\rangle$,
$\langle f(f(\sharp))\rangle \to \langle g(\sharp, \sharp, \sharp)\rangle$.
We obtain the last two rules in the following way. We consider the rule $f(f(g(x_1, \sharp, \sharp))) \to f(f(x_1))$ in R. We substitute $\langle\sharp\rangle \in sub(L \cup E)$ for x_1 in its left-hand side. We get the term $f(f(g(\langle\sharp\rangle, \sharp, \sharp)))$. Observe that
$f(f(g(\langle\sharp\rangle, \sharp, \sharp))) \to^*_{S_1} f(\langle f(g(\sharp, \sharp, \sharp))\rangle) \to_{S_1} f(\langle g(\sharp, \sharp, \sharp)\rangle) \to_{S_1} \langle f(g(\sharp, \sharp, \sharp))\rangle$ and
$f(f(g(\langle\sharp\rangle, \sharp, \sharp))) \to^*_{S_1} \langle f(g(\sharp, \sharp, \sharp))\rangle \to_{S_1} \langle g(\sharp, \sharp, \sharp)\rangle$.
In this way, we put the rules $\langle f(f(\sharp))\rangle \to \langle f(g(\sharp, \sharp, \sharp))\rangle$ and $\langle f(f(\sharp))\rangle \to \langle g(\sharp, \sharp, \sharp)\rangle$ in S_1.

The bta $\mathcal{C}_3 = (\Sigma, B, S_3, \{\, \langle g(\sharp, \sharp, \sharp)\rangle \,\})$ is equal to $\mathcal{C}_2$. By direct inspection we obtain that the states $\langle f(f(g(\sharp, \sharp, \sharp)))\rangle$, $\langle g(g(\sharp, \sharp, \sharp), \sharp, g(\sharp, \sharp, \sharp))\rangle$, $\langle f(g(g(\sharp, \sharp, \sharp), \sharp, g(\sharp, \sharp, \sharp)))\rangle$ are superfluous as the final state $\langle g(\sharp, \sharp, \sharp)\rangle$ cannot be reached from any of them. Hence we drop all of them and also omit all rules in which they appear. In this way we obtain the bta $\mathcal{B}_1 = (\Sigma, B_1, Q_1, \{\, \langle g(\sharp, \sharp, \sharp)\rangle \,\})$, where $B_1 = \{\, \langle\sharp\rangle,\ \langle g(\sharp, \sharp, \sharp)\rangle,\ \langle f(\sharp)\rangle,\ \langle f(g(\sharp, \sharp, \sharp))\rangle,\ \langle f(f(\sharp))\rangle \,\}$ and Q_1 consists of the following rules.
$\sharp \to \langle\sharp\rangle$,
$g(\langle\sharp\rangle, \langle\sharp\rangle, \langle\sharp\rangle) \to \langle g(\sharp, \sharp, \sharp)\rangle$,
$f(\langle\sharp\rangle) \to \langle f(\sharp)\rangle$,
$f(f\langle(\sharp)\rangle) \to \langle f(f(\sharp))\rangle$,
$\langle f(g(\sharp, \sharp, \sharp))\rangle \to \langle g(\sharp, \sharp, \sharp)\rangle$,
$\langle f(f(\sharp))\rangle \to \langle f(g(\sharp, \sharp, \sharp))\rangle$,
$\langle f(f(\sharp))\rangle \to \langle g(\sharp, \sharp, \sharp)\rangle$.
We obtain the bta $\mathcal{B}_2 = (\Sigma, B_2, Q_2, \{\langle\, \{g(\sharp, \sharp, \sharp)\rangle \,\})$ from $\mathcal{B}_1$ by eliminating the λ-rules. Here $B_2 = B_1$ and Q_2 consists of the following rules.
$\sharp \to \langle\sharp\rangle$,
$g(\langle\sharp\rangle, \langle\sharp\rangle, \langle\sharp\rangle) \to \langle g(\sharp, \sharp, \sharp)\rangle$,
$f(\langle\sharp\rangle) \to \langle f(\sharp)\rangle$,
$f(\langle f(\sharp)\rangle) \to \langle f(f(\sharp))\rangle$.
$f(\langle f(\sharp)\rangle)) \to \langle f(g(\sharp, \sharp, \sharp))\rangle$,
$f(\langle f(\sharp)\rangle) \to \langle g(\sharp, \sharp, \sharp)\rangle$,
$f(\langle g(\sharp, \sharp, \sharp)\rangle) \to \langle g(\sharp, \sharp, \sharp)\rangle$.

By direct inspection we obtain that the states $\langle f(f(\sharp))\rangle$ and $\langle f(g(\sharp, \sharp, \sharp))\rangle$
are superfluous as the final state $\langle g(\sharp, \sharp, \sharp)\rangle$ cannot be reached from any of them. Hence we drop all of them and also omit all rules in which they appear.
In this way we obtain the bta $\mathcal{B}_3 = (\Sigma, B_3, Q_3, \{\, \langle g(\sharp, \sharp, \sharp)\rangle \,\})$.

Here $B_3 = \{\, \langle\sharp\rangle,\ \langle g(\sharp, \sharp, \sharp)\rangle,\ \langle f(\sharp)\rangle \,\}$ and Q_3 consists of the following five rules.
$\sharp \to \langle\sharp\rangle$,
$g(\langle\sharp\rangle, \langle\sharp\rangle, \langle\sharp\rangle) \to \langle g(\sharp, \sharp, \sharp)\rangle$,
$f(\langle\sharp\rangle) \to \langle f(\sharp)\rangle$,
$f(\langle f(\sharp)\rangle) \to \langle g(\sharp, \sharp, \sharp)\rangle$,
$f(\langle g(\sharp, \sharp, \sharp)\rangle) \to \langle g(\sharp, \sharp, \sharp)\rangle$.
Then $L(\mathcal{C}_3) = L(\mathcal{B}_3)$. We obtain by direct inspection that $L(\mathcal{B}_3) = R^*(L)$.

7. PRF-TRSs

We show various decidability and undecidability results on PRF-TRSs and EPRF-TRSs. We show that reachability, joinability, and local confluence are decidable for EPRF-TRSs.

We now show that reachability is decidable for EPRF-TRSs.

Theorem 7.1. *Let R be an EPRF-TRS over Σ and let $p, q \in T_\Sigma(X)$. Then it is decidable whether $p \rightarrow_R^* q$.*

Proof. Let $m \geq 0$ be such that $var(p) \subseteq X_m$, $var(q) \subseteq X_m$. Let us introduce new constant symbols $Z = \{ z_1, \ldots, z_m \}$ with $Z \cap \Sigma = \emptyset$. For each $t \in T_\Sigma(X_m)$, we define $t_z \in T_{\Sigma \cup Z}$ as $t_z = t[z_1, \ldots, z_m]$. By direct inspection we obtain that for all $u, v \in T_\Sigma(X)$,

$$u \underset{R}{\rightarrow} v \text{ if and only if } u_z \underset{R}{\rightarrow} v_z\ ,$$

hence

$$u \overset{*}{\underset{R}{\rightarrow}} v \text{ if and only if } u_z \overset{*}{\underset{R}{\rightarrow}} v_z\ .$$

Consider the singleton set $\{ p_z \}$. As R is an EPRF-TRS, $R^*_{\Sigma \cup Z}(\{ p_z \})$ is a recognizable tree language over $\Sigma \cup Z$, and we can construct a bta over $\Sigma \cup Z$ which recognizes $R^*_{\Sigma \cup Z}(\{ p_z \})$. Hence we can decide whether $q_z \in R^*_{\Sigma \cup Z}(\{ p_z \})$ [9]. Clearly, $q_z \in R^*_{\Sigma \cup Z}(\{ p_z \})$ if and only if $p \rightarrow_R^* q$. □

We now show that joinability is decidable for EPRF-TRSs.

Theorem 7.2. *Let R be an EPRF-TRS over Σ, and let $p, q \in T_\Sigma(X)$. Then it is decidable whether there is a tree $r \in T_\Sigma(X)$ such that $p \rightarrow_R^* r$ and $q \rightarrow_R^* r$.*

Proof. For each $t \in T_\Sigma(X_m)$, we define $t_z \in T_{\Sigma \cup Z}$ as in the proof of Theorem 7.1.

Claim 7.3. *For any $p, q \in T_\Sigma(X)$, there is a tree $r \in T_\Sigma(X)$ such that $p \rightarrow_R^* r$ and $q \rightarrow_R^* r$ if and only if $R^*_{\Sigma \cup Z}(\{ p_z \}) \cap R^*_{\Sigma \cup Z}(\{ q_z \}) \neq \emptyset$.*

Proof. Assume that $R^*_{\Sigma \cup Z}(\{ p_z \}) \cap R^*_{\Sigma \cup Z}(\{ q_z \}) \neq \emptyset$. Then there is a tree $s \in R^*_{\Sigma \cup Z}(\{ p_z \}) \cap R^*_{\Sigma \cup Z}(\{ q_z \})$. We define r from s by replacing each occurrence of z_i by x_i for $1 \leq i \leq m$. Then $p \rightarrow_R^* r$ and $q \rightarrow_R^* r$.

Assume that there is a tree $r \in T_\Sigma(X)$ such that $p \rightarrow_R^* r$ and $q \rightarrow_R^* r$. Hence $r_z \in R^*_{\Sigma \cup Z}(\{ p_z \})$ and $r_z \in R^*_{\Sigma \cup Z}(\{ q_z \})$. Thus $R^*_{\Sigma \cup Z}(\{ p_z \}) \cap R^*_{\Sigma \cup Z}(\{ q_z \}) \neq \emptyset$. □

As R is an EPRF-TRS, $R^*_{\Sigma \cup Z}(\{ p_z \})$ and $R^*_{\Sigma \cup Z}(\{ q_z \})$ are recognizable, and we can construct two btas over $\Sigma \cup Z$ which recognize $R^*_{\Sigma \cup Z}(\{ p_z \})$ and $R^*_{\Sigma \cup Z}(\{ q_z \})$, respectively. Hence we can decide whether $R^*_{\Sigma \cup Z}(\{ p_z \}) \cap R^*_{\Sigma \cup Z}(\{ q_z \}) = \emptyset$ [9]. By Claim 7.3, if $R^*_{\Sigma \cup Z}(\{ p_z \}) \cap R^*_{\Sigma \cup Z}(\{ q_z \}) \neq \emptyset$, then there is a tree $r \in T_\Sigma(X)$ such that $p \rightarrow_R^* r$ and $q \rightarrow_R^* r$. Otherwise, there is no tree $r \in T_\Sigma(X)$ such that $p \rightarrow_R^* r$ and $q \rightarrow_R^* r$. □

We now show that the word problem is decidable for confluent EPRF-TRSs.

Theorem 7.4. *Let R be a confluent EPRF-TRS over Σ, and let $p, q \in T_\Sigma(X)$. Then it is decidable whether $p \leftrightarrow_R^* q$.*

Proof. $p \leftrightarrow_R^* q$ if and only if there is a tree $r \in T_\Sigma(X)$ such that $p \rightarrow_R^* r$ and $q \rightarrow_R^* r$. By Theorem 7.2, we can decide whether there is a tree $r \in T_\Sigma(X)$ such that $p \rightarrow_R^* r$ and $q \rightarrow_R^* r$. □

We now show that local confluence is decidable for EPRF-TRSs.

Theorem 7.5. *Let R be an EPRF-TRS over Σ. Then it is decidable whether R is locally confluent.*

Proof. We construct a finite set CP of critical pairs such that for each critical pair (s, t) of R, there is an element (u, v) of CP and a renaming $\sigma : X \rightarrow X$ such that $(s, t) = (\sigma(u), \sigma(v))$. Thus the theorem follows from Proposition 2.1 and Theorem 7.2. □

Theorem 7.6. *Let R be an EPRF-TRS and S be a TRS over Σ. Then it is decidable whether $\rightarrow_S^* \subseteq \rightarrow_R^*$.*

Proof. Let $m \geq 0$ be such that for all variables x_i occurring on the left-hand side of some rule in S, $x_i \in X_m$, that is, $i \leq m$. From now on, for each $t \in T_\Sigma(X_m)$, we define $t_z \in T_{\Sigma \cup Z}$ as in the proof of Theorem 7.1.

Claim 7.7. $\rightarrow_S^* \subseteq \rightarrow_R^*$ *if and only if for each rule $l \rightarrow r$ in S, $r_z \in R^*_{\Sigma \cup Z}(\{ l_z \})$.*

Proof. ($\Rightarrow$) Let $l \to r$ be an arbitrary rule in S. Clearly, $l \to_R^* r$. Thus $r_z \in R^*_{\Sigma\cup Z}(\{ l_z \})$.

($\Leftarrow$) Let us suppose that $t_1, t_2 \in T_\Sigma(X)$, and that $t_1 \to_S t_2$ applying the rule $l \to r$. As $r_z \in R^*_{\Sigma\cup Z}(\{ l_z \})$, $l_z \to_R^* r_z$ holds. Hence $l \to_R^* r$ implying that $t_1 \to_R^* t_2$ as well. □

Let $l \to r$ be an arbitrary rule in S. We can construct a bta over $\Sigma \cup Z$ recognizing the singleton set $\{ l_z \}$. As R is an EPRF-TRS, $R^*_{\Sigma\cup Z}(\{ l_z \})$ is recognizable, and we can construct a bta over $\Sigma \cup Z$ recognizing $R^*_{\Sigma\cup Z}(\{ l_z \})$. Hence we can decide whether $r_z \in R^*_{\Sigma\cup Z}(\{ l_z \})$. Thus by Claim 7.7, we can decide whether $\to_S^* \subseteq \to_R^*$. □

Consequence 7.8. Let R and S be EPRF-TRS over Σ. Then it is decidable which one of the following four mutually excluding conditions holds.
(i) $\to_R^* \subset \to_S^*$,
(ii) $\to_S^* \subset \to_R^*$,
(iii) $\to_R^* = \to_S^*$,
(iv) $\to_R^* \bowtie \to_S^*$,
where "$\bowtie$" stands for the incomparability relationship.

Observation 7.9. If one omits a rule from a left-linear GSM-TRS, then the resulting TRS still remains a left-linear GSM-TRS.

One can easily show the following result applying Theorem 5.1, Consequence 7.8, and Observation 7.9.

Consequence 7.10. For a left-linear GSM-TRS R, it is decidable whether R is left-to-right minimal.

Consequence 7.8 also implies the following.

Consequence 7.11. Let R and S be TRSs over Σ such that $R \cup R^{-1}$ and $S \cup S^{-1}$ are EPRF-TRSs. Then it is decidable which one of the following four mutually excluding conditions holds.
(i) $\leftrightarrow_R^* \subset \leftrightarrow_S^*$,
(ii) $\leftrightarrow_S^* \subset \leftrightarrow_R^*$,
(iii) $\leftrightarrow_R^* = \leftrightarrow_S^*$,
(iv) $\leftrightarrow_R^* \bowtie \leftrightarrow_S^*$.

Theorem 7.12. *Let R be an EPRF-TRS and S be a TRS over a ranked alphabet Σ. Let $g \in \Sigma - (sign(R) \cup \Sigma_0)$. Let $\sharp \in \Sigma_0$ be irreducible for R. Then it is decidable whether $\to_S^* \cap (T_\Sigma \times T_\Sigma) \subseteq \to_R^* \cap (T_\Sigma \times T_\Sigma)$.*

Proof. We assume that $g \in \Sigma_1$. One can easily modify the proof of this case when proving the more general case $g \in \Sigma_k$, $k \geq 1$. For each $t \in T_\Sigma(X)$, we define $t_g \in T_\Sigma$ from t by substituting $g^i(\sharp)$ for all occurrences of the variable x_i for $i \geq 1$.

Claim 7.13. $\to_S^* \cap (T_\Sigma \times T_\Sigma) \subseteq \to_R^* \cap (T_\Sigma \times T_\Sigma)$ *if and only if for each rule $l \to r$ in S, $r_g \in R_1^*(\{ l_g \})$.*

Proof. ($\Rightarrow$) Let $l \to r$ be an arbitrary rule in S. Clearly, $l_g \to_S r_g$. Thus by our assumption $l_g \to_R^* r_g$.

($\Leftarrow$) Let us suppose that $t_1, t_2 \in T_\Sigma$, and that $t_1 \to_S t_2$ applying the rule $l \to r$. As $r_g \in R_1^*(\{ l_g \})$, $l_g \to_R^* r_g$ holds. Hence $l \to_R^* r$ implying that $t_1 \to_R^* t_2$ as well. □

For each rule $l \to r$ in S, the tree language $\{ l_g \}$ is recognizable, and we can construct a bta over Σ recognizing $\{ l_g \}$. As R is an EPRF-TRS, $R^*(\{ l_g \})$ is also recognizable, and we can construct a bta over Σ recognizing $R^*(\{ l_g \})$. Hence for each rule $l \to r$ in S, we can decide whether or not $r_g \in R^*(\{ l_g \})$. Thus by Claim 7.13, we can decide whether $\to_S^* \cap (T_\Sigma \times T_\Sigma) \subseteq \to_R^* \cap (T_\Sigma \times T_\Sigma)$. □

One can easily show the following result applying Theorem 5.1, Observation 7.9, and Theorem 7.12.

Consequence 7.14. Let R be a left-linear GSM-TRS over Σ. Let $g \in \Sigma - (sign(R) \cup \Sigma_0)$, and let $\sharp \in \Sigma_0$ be irreducible for R. Then it is decidable whether R is left-to-right ground minimal.

Consequence 7.15. Let R_1 and R_2 be EPRF-TRSs over Σ. Moreover, let $g_1, g_2 \in \Sigma - \Sigma_0$ be such that for each $i \in \{ 1, 2 \}$, g_i does not occur on the left-hand side of any rule in R_i. Let $\sharp_1, \sharp_2 \in \Sigma_0$ be such that for each $i \in \{ 1, 2 \}$, $\sharp_i$ is irreducible for R_i. Then it is decidable which one of the following four mutually excluding conditions holds.
(i) $\to_{R_1}^* \cap (T_\Sigma \times T_\Sigma) \subset \to_{R_2}^* \cap (T_\Sigma \times T_\Sigma)$,
(ii) $\to_{R_2}^* \cap (T_\Sigma \times T_\Sigma) \subset \to_{R_1}^* \cap (T_\Sigma \times T_\Sigma)$,
(iii) $\to_{R_1}^* \cap (T_\Sigma \times T_\Sigma) = \to_{R_2}^* \cap (T_\Sigma \times T_\Sigma)$,
(iv) $\to_{R_1}^* \cap (T_\Sigma \times T_\Sigma) \bowtie \to_{R_2}^* \cap (T_\Sigma \times T_\Sigma)$.

By Statement 4.12 and Theorem 5.1 we have the following.

Theorem 7.16. *Each of the following questions is undecidable for any convergent left-linear EPR-TRSs R and S over a ranked alphabet Ω, for any recognizable tree language $L \subseteq T_\Omega$ given by a tree automaton over Ω recognizing L, where $\Gamma \subseteq \Omega$ is the smallest ranked alphabet for which $NF_R(L) \subseteq T_\Gamma$.*
(i) Is $NF_R(L) \cap NF_S(L)$ empty?
(ii) Is $NF_R(L) \cap NF_S(L)$ infinite?
(iii) Is $NF_R(L) \cap NF_S(L)$ recognizable?
(iv) Is $T_\Gamma - NF_R(L)$ empty?
(v) Is $T_\Gamma - NF_R(L)$ infinite?
(vi) Is $T_\Gamma - NF_R(L)$ recognizable?
(vii) Is $NF_R(L)$ recognizable?
(viii) Is $NF_R(L) = NF_S(L)$?
(ix) Is $NF_R(L) \subseteq NF_S(L)$?

Lemma 7.17. *Let R and S be linear collapse-free TRSs over the disjoint ranked alphabets Σ and Δ, respectively. Let Γ be a ranked alphabet with $\Sigma \cup \Delta \subseteq \Gamma$. Consider R and S as TRSs over Γ. Then*
(i) $\to_S \circ \to_R \subseteq \to_R \cup (\to_R \circ \to_S)$, and
(ii) $\to^*_{R\cup S} = \to^*_R \circ \to^*_S$.

Proof. The proof of (i) is straightforward. Condition (ii) is a simple consequence of (i). □

Theorem 7.18. *Let R be a linear collapse-free EPRF-TRS and S be a linear collapse-free EPR-TRS over the disjoint ranked alphabets $sign(R)$ and $sign(S)$, respectively. Then $R \oplus S$ is a linear collapse-free EPR-TRS.*

Proof. Clearly, $R \oplus S$ is a linear collapse-free TRS. Let L be a recognizable tree language over some ranked alphabet Γ, where $sign(R) \cup sign(S) \subseteq \Gamma$. By Lemma 7.17, $(R \oplus S)^*_\Gamma(L) = S^*_\Gamma(R^*_\Gamma(L))$. As R is an EPRF-TRS, $R^*_\Gamma(L)$ is recognizable. Moreover, since S preserves recognizability, $S^*_\Gamma(R^*_\Gamma(L))$ is also recognizable. □

The proof of the following result is similar to the proof of Theorem 7.18.

Theorem 7.19. *Let R be a linear collapse-free PRF-TRS and S be a linear collapse-free PR-TRS over the disjoint ranked alphabets Σ and Δ, respectively. Then $R \oplus S$ is a PR-TRS.*

Theorem 7.20. *Let R and S be TRSs over the disjoint ranked alphabets Σ and Δ, respectively. If $R \oplus S$ is an EPRF-TRS, then R and S are also EPRF-TRSs.*

Proof. Let L be a finite recognizable tree language over some ranked alphabet Γ, where $\Sigma \subseteq \Gamma$. It is sufficient to show that $R^*_\Gamma(L)$ is recognizable. Without loss of generality we may rename the symbols of Γ such that $\Gamma \cap \Delta = \emptyset$. Thus $R^*_\Gamma(L) = (R \oplus S)^*_{\Gamma\cup\Delta}(L)$. Since $\Sigma \cup \Delta \subseteq \Gamma \cup \Delta$ and $R \oplus S$ is an EPRF-TRS, we get that $R^*_\Gamma(L)$ is recognizable and we can effectively construct a bta recognizing $R^*_\Gamma(L)$. □

8. Conclusion and open problems

We showed that each left-linear GSM-TRS is an EPRF-TRS. We showed that reachability, joinability, and local confluence are decidable for EPRF-TRSs. We showed that the following problem is undecidable:
Instance: A murg TRS R over a ranked alphabet Σ.
Question: Is R a PΣRF-TRS?
Our results give rise to several open problems.

- What is the time and space complexity of constructing the bta $\mathcal{C}$ in the proof of Theorem 5.1?
- Generalize the notion of a left-linear GSM-TRS such that the obtained TRS is still an EPRF-TRS. A possible way is to use the sticking out graph of a TRS R.
- Show the following conjecture. Let R be a right-linear TRS over $sign(R)$, and let $\Sigma = \{g, \sharp\} \cup sign(R)$, where $g \in \Sigma_1 - sign(R)$ and $\sharp \in \Sigma_0 - sign(R)$. Then R is an EPΣRF-TRS if and only if R is an EPRF-TRS. Show the corresponding conjectures when R is left-linear or R is linear.
- Show that a string rewrite system R over the alphabet $alph(R)$ of R preserves $alph(R)$-recognizability of finite string languages if and only if R preserves recognizability of finite string languages. Show that it is not decidable for a string rewrite system R whether R preserves $alph(R)$-recognizability of finite string languages, and whether R preserves recognizability of finite string languages. Hence it is not decidable for a linear TRS R whether R is a PΣRF-TRS and whether R is a PRF-TRS.

• Show that the property of preserving recognizability of finite tree languages and the property of effectively preserving recognizability of finite tree languages are modular for the class of all left-linear collapse-free TRSs, for the class of all right-linear collapse-free TRSs, for the class of all linear collapse-free TRSs, and for the class of all collapse-free TRSs.

• Let R be a TRS over Σ. An external normal form of R is a ground term $t \in T_\Sigma$ such that t is not an instance of any term in $\cup(sub(l) \mid l \in lhs(R))$ [3]. At first glance one might believe that R is a PRF-TRS if and only if R has external normal forms. We now show that this belief is unjustified. Let Σ be such that $\Sigma = \Sigma_0 \neq \emptyset$ and that $card(\Sigma) \geq 2$. We define the TRS R over Σ as follows. We take a constant $\zeta \in \Sigma$. For each constant $\delta \in \Sigma$, we put the rule $\delta \to \zeta$ in R. Then $sign(R) = \Sigma$. Observe that R has no external forms. Let Δ be a ranked alphabet such that $\Sigma \subseteq \Delta$. Then for each $t \in T_\Delta$, the set $R^*_\Delta(\{t\})$ is finite. By Observation 3.1, R is an EPΔRF-TRS. Thus R is an EPRF-TRS. Next, we define the TRS S from TRS R by dropping the rule $\zeta \to \zeta$. Then $sign(S) = sign(R) = \Sigma$. Clearly, ζ is an external normal form of R. As above, we can show that S is an EPRF-TRS. Although our guess above turned out to be wrong, we conjecture that there is some subtle connection between external normal forms and the property of preserving recognizability of finite tree languages.

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