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An Intuitionistic Completeness Theorem for Classical Predicate Logic

Abstract. This paper presents an intuitionistic proof of a statement which under a classical reading is *logically* equivalent to Gödel's completeness theorem for classical predicate logic.

Keywords: classical predicate logic, Gödel's completeness theorem, intuitionistic completeness proof.

An intuitionistic structure $\mathfrak{M}$ is defined exactly as a standard Tarski structure, except that the condition $\mathfrak{M} \not\models \bot$ is weakened to $\mathfrak{M} \models \bot \Rightarrow \mathfrak{M} \models P$ (for P atomic), and all the clauses in the definition of $\models$ are read intuitionistically.

For intuitionistic predicate logic completeness with respect to validity in structures can be shown via a purely intuitionistic argument establishing a connection between Beth forcing and intuitionistic validity. The proof is carried out within a suitable fragment of the theory of lawless sequences, see [4, Ch. 13, Sections 1.7 and 1.10]. (For the completeness sake a variant of such a kind of proof is presented in the Appendix below.)

Our aim in this paper is to show that in the case of classical predicate logic a similar argument leads to an intuitionistic version of Gödel's completeness theorem. In contrast to the intuitionistic case, the proof uses essentially only a form of the fan theorem which is classically a variant of König's lemma.

Note on notation

We assume a standard primitive recursive coding $<>$ of all finite sequences of natural numbers onto $\mathbb{N}$ (the set of natural numbers); thus $<n_0, \ldots, n_m>$ is the code number of the sequence $n_0, \ldots, n_m$. The standard ordering of finite sequences is denoted by $\preceq$; thus $k \preceq k'$ (or equivalently $k' \succeq k$) means

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that k is an initial segment of k'. The length of a finite sequence k is written $\mathrm{lth}(k)$. We use $k' \succeq_n k$ as an abbreviation for $k' \succeq k \wedge \mathrm{lth}(k') = \mathrm{lth}(k) + n$.

If α is a choice sequence, then $\bar{\alpha}n$ is the initial segment of α of length n, so that $\bar{\alpha}0 = {<>}$ (the code of the empty sequence) and $\bar{\alpha}n = {<}\alpha(0), \ldots, \alpha(n-1){>}$ for $n > 0$.

For the sake of simplicity we restrict ourselves to the first-order language $\mathcal{L}$ of pure predicate logic, without equality, function, or constant symbols. There are countable lists $x_0, x_1, \ldots$ of (individual) variables and $P_0, P_1, \ldots$ of predicate symbols. The arity of P_i is denoted by $\tau(i)$; every atomic formula of $\mathcal{L}$ is therefore an expression of the form $P_i(x_{n_1}, \ldots, x_{n_{\tau(i)}})$. The basic logical constants are $\wedge, \vee, \rightarrow, \forall, \exists$, and $\perp$ (absurdity), with $\neg A$ defined as $A \rightarrow \perp$. For technical reasons we identify $\perp$ with P_0, so that $\tau(0) = 0$.

We use $\vdash_c$ and $\vdash_i$ to denote classical and intuitionistic derivability, respectively.

Given a set D, we write $\mathcal{L}(D)$ for the language obtained from $\mathcal{L}$ by adding constant symbols $\bar{d}$ for the elements d of D.

1. Generalized Beth models

We shall use some standard facts concerning generalized Beth models[1].

Definition 1.1. A *generalized Beth model* for the language $\mathcal{L}$ is a quadruple $\mathfrak{B} = \langle T_{01}, \preceq, D, \Vdash \rangle$, where:

(i) T_{01} is the full binary tree consisting of all finite 0-1-sequences, $\preceq$ the standard ordering of the nodes;

(ii) D is an inhabited set[2], the domain of $\mathfrak{B}$;

(iii) $\Vdash$, the forcing relation, is a binary relation between elements of T_{01} and atomic sentences $P_i(\bar{d}_1, \ldots, \bar{d}_{\tau(i)})$ of $\mathcal{L}(D)$ such that

(B1.1) $\exists n \, \forall k' \succeq_n k \, (k' \Vdash P_i(\ldots)) \Rightarrow k \Vdash P_i(\ldots)$;
(B1.2) $(k \Vdash P_i(\ldots)$ and $k' \succeq k) \Rightarrow k' \Vdash P_i(\ldots)$;
(B1.3) $k \Vdash \perp \Rightarrow k \Vdash P_i(\ldots)$.

The following clauses extend $\Vdash$ to compound sentences of $\mathcal{L}(D)$:

(B2) $k \Vdash A \wedge B \; := \; k \Vdash A$ and $k \Vdash B$;
(B3) $k \Vdash A \vee B \; := \; \exists n \, \forall k' \succeq_n k \, (k' \Vdash A$ or $k' \Vdash B)$;
(B4) $k \Vdash A \rightarrow B \; := \; \forall k' \succeq k \, (k' \Vdash A \Rightarrow k' \Vdash B)$;
(B5) $k \Vdash \exists x \, A(x) \; := \; \exists n \, \forall k' \succeq_n k \, \exists d \in D \; (k' \Vdash A(\bar{d}))$;

[1] These are also known as *exploding* or *fallible* Beth models.

[2] That is, $\exists d \, (d \in D)$.

(B6) $k \Vdash \forall x\, A(x) := \forall d \in D\, (k \Vdash A(\bar{d}))$.

We write $\mathfrak{B} \Vdash A$ for $<> \Vdash A$. If Δ is a set of sentences of $\mathcal{L}$, then $\mathfrak{B}$ is a *model* of Δ (written $\mathfrak{B} \Vdash \Delta$) if $\mathfrak{B} \Vdash B$ for every $B \in \Delta$.

Given a binary choice sequence γ, we write $\gamma \Vdash A$ for $\exists n\, (\bar{\gamma} n \Vdash A)$. If Δ is a set of sentences of $\mathcal{L}(D)$, then $\gamma \Vdash \Delta$ means '$\gamma \Vdash B$ for every $B \in \Delta$'.

NOTE. In what follows α, β and γ are supposed to range over *binary* choice sequences, and k, k' over elements of T_{01}.

Each of the following properties is an easy consequence of Definition 1.1:

(a) $\gamma \Vdash A \wedge B \Leftrightarrow \gamma \Vdash A$ and $\gamma \Vdash B$;
(b) $\gamma \Vdash A \vee B \Leftrightarrow \gamma \Vdash A$ or $\gamma \Vdash B$;
(c) $(\gamma \Vdash A$ and $\gamma \Vdash A \to B) \Rightarrow \gamma \Vdash B$;
(d) $\gamma \Vdash \exists x\, A(x) \Leftrightarrow \exists d \in D\, (\gamma \Vdash A(\bar{d}))$;
(e) $\gamma \Vdash \forall x\, A(x) \Rightarrow \forall d \in D\, (\gamma \Vdash A(\bar{d}))$;
(f) $\gamma \Vdash \perp \Rightarrow \gamma \Vdash A$ for all sentences A of $\mathcal{L}(D)$;
(g) $(k' \succeq k$ and $k \Vdash A) \Rightarrow k' \Vdash A$ (monotonicity);
(h) for any $n \in \mathbb{N}$, $\forall k' \succeq_n k\, (k' \Vdash A) \Leftrightarrow k \Vdash A$.

DEFINITION 1.2. Let Δ be a set of sentences of $\mathcal{L}$. A generalized Beth model $\mathfrak{B}$ for the language $\mathcal{L}$ is said to be *universal* for Δ if for all sentences A of $\mathcal{L}$, $\mathfrak{B} \Vdash A \Leftrightarrow \Delta \vdash_i A$.

The following basic result holds both intuitionistically and classically, cf. [4, Ch. 13, Theorem 2.2.8 and Exercise 13.2.5]:

LEMMA 1.3. *Every enumerable* [3] *set* Δ *of sentences of* $\mathcal{L}$ *has a generalized Beth model* $\langle T_{01}, \preceq, D, \Vdash \rangle$, *with* $D = \mathbb{N}$, *which is universal for* Δ.

2. Intuitionistic structures

DEFINITION 2.1. An *intuitionistic structure* for the language $\mathcal{L}$ is an ordered couple $\mathfrak{M} = \langle D, I \rangle$, where D (the domain of $\mathfrak{M}$) is an inhabited set, and I (the valuation mapping) is a function that associates with every predicate symbol P_i a subset of the Cartesian power $D^{\tau(i)}$ (in particular $I(P_0)$ is a subset of $\{\emptyset\}$) such that

$$\emptyset \in I(P_0) \Rightarrow (d_1, \ldots, d_{\tau(i)}) \in I(P_i),$$

for all $d_1, \ldots, d_{\tau(i)} \in D$ and $i > 0$.

[3] Here 'X is enumerable' means 'there exists a *lawlike* surjective mapping from $\mathbb{N}$ to X'.

Now, given an intuitionistic structure $\mathfrak{M} = \langle D, I\rangle$, the inductive definition of the relation $\mathfrak{M} \models A$ (A is *valid* in $\mathfrak{M}$, where A is a sentence of $\mathcal{L}(D)$) is completely similar to that in the case of the usual Tarski semantics:

(M1) $\mathfrak{M} \models P_i(\bar{d}_1, \dots, \bar{d}_{\tau(i)}) := (d_1, \dots, d_{\tau(i)}) \in I(P_i)$;
(M2) $\mathfrak{M} \models A \wedge B := (\mathfrak{M} \models A$ and $\mathfrak{M} \models B)$;
(M3) $\mathfrak{M} \models A \vee B := (\mathfrak{M} \models A$ or $\mathfrak{M} \models B)$;
(M4) $\mathfrak{M} \models A \to B := (\mathfrak{M} \models A \Rightarrow \mathfrak{M} \models B)$;
(M5) $\mathfrak{M} \models \exists x\, A(x) := \exists d \in D\, (\mathfrak{M} \models A(\bar{d}))$;
(M6) $\mathfrak{M} \models \forall x\, A(x) := \forall d \in D\, (\mathfrak{M} \models A(\bar{d}))$.

Given a set Δ of sentences of $\mathcal{L}$, $\mathfrak{M}$ is a *model* of Δ (or $\mathfrak{M} \models \Delta$) if $\mathfrak{M} \models B$ for every $B \in \Delta$.

3. The completeness theorem

We use **PEM** to denote the set of sentences of $\mathcal{L}$ of the form $\forall x_1 \dots x_n\, (B \vee \neg B)$, that is, the set of all closed instances (in $\mathcal{L}$) of the *tertium non datur*.

Given a set Δ of sentences and a sentence A of $\mathcal{L}$, we write $\Delta \models_c A$ to mean 'for every intuitionistic structure $\mathfrak{M}$ with $\mathfrak{M} \models$ **PEM**: if $\mathfrak{M} \models \Delta$, then $\mathfrak{M} \models A$'.

THEOREM 3.1. *Let A be a sentence and Δ an enumerable set of sentences of $\mathcal{L}$. Then*

$$\Delta \vdash_c A \Leftrightarrow \Delta \models_c A.$$

PROOF. That $\Delta \vdash_c A$ implies $\Delta \models_c A$ is evident. We shall prove the converse.

Let $\mathbf{Un}\ (\Delta_1) = \langle T_{01}, \leq, \mathbb{N}, \Vdash\rangle$ be a generalized Beth model which is universal for the set $\Delta_1 = \Delta \cup \mathbf{PEM}$. Then we have, for all sentences A of $\mathcal{L}$, $\mathbf{Un}(\Delta_1) \Vdash A \Leftrightarrow \Delta \vdash_c A$. Furthermore, since $\mathbf{Un}(\Delta_1) \Vdash \mathbf{PEM}$, it follows, for any γ and any sentence B of $\mathcal{L}(\mathbb{N})$,

$$\gamma \Vdash B \text{ or } \gamma \Vdash \neg B. \tag{$*$}$$

Now to each γ we associate an intuitionistic structure $\mathfrak{M}_\gamma = \langle \mathbb{N}, I\rangle$ by letting

$$(d_1, d_2, \dots, d_{\tau(i)}) \in I(P_i) := \gamma \Vdash P_i(\bar{d}_1, \bar{d}_2, \dots, \bar{d}_{\tau(i)}),$$

for $i \geq 0$ and $d_1, d_2, \dots, d_{\tau(i)} \in \mathbb{N}$.

We need the following

LEMMA 3.2. *For all γ and all sentences A of $\mathcal{L}(\mathbb{N})$:*

$$\mathfrak{M}_\gamma \models A \Leftrightarrow \gamma \Vdash A.$$

PROOF. The proof is by induction on the logical complexity of A. For A atomic the assertion holds by the definition of $\mathfrak{M}_\gamma$. The cases $A \equiv B \wedge C$, $A \equiv B \vee C$, and $A \equiv \exists x\, B(x)$ are trivial.

Case $A \equiv B \to C$. Assume $\mathfrak{M}_\gamma \models B \to C$, that is, $\mathfrak{M}_\gamma \models B \Rightarrow \mathfrak{M}_\gamma \models C$. In view of $(*)$, there are two possibilities, $\gamma \Vdash B \to C$ or $\gamma \Vdash \neg(B \to C)$. In case $\gamma \Vdash B \to C$ there is nothing to prove, so assume $\gamma \Vdash \neg(B \to C)$. Then $\gamma \Vdash B \wedge \neg C$ (for any $k \in T_{01}$ forces any classical tautology), which implies $\gamma \Vdash B$ and hence (by the induction hypothesis, IH) $\mathfrak{M}_\gamma \models B$. So $\mathfrak{M}_\gamma \models C$, and thus (again by IH) $\gamma \Vdash C$. On the other hand, $\gamma \Vdash B \wedge \neg C$ implies $\gamma \Vdash \neg C$. This shows that $\gamma \Vdash \bot$, hence $\gamma \Vdash B \to C$.

Conversely, assume $\gamma \Vdash B \to C$, and suppose that $\mathfrak{M}_\gamma \models B$. Then, by IH, $\gamma \Vdash B$ and hence $\gamma \Vdash C$, which implies (again by IH) $\mathfrak{M}_\gamma \models C$. Therefore, $\mathfrak{M}_\gamma \models B \to C$.

Case $A \equiv \forall x\, B(x)$. Assume $\mathfrak{M}_\gamma \models \forall x\, B(x)$, that is, $\forall d \in \mathbb{N}\, (\mathfrak{M}_\gamma \models B(\bar{d}))$. Then by IH $\forall d \in \mathbb{N}\, (\gamma \Vdash B(\bar{d}))$. In view of $(*)$, there are two possibilities, $\gamma \Vdash \forall x\, B(x)$ or $\gamma \Vdash \neg\forall x\, B(x)$. In case $\gamma \Vdash \forall x\, B(x)$ there is nothing to prove, so assume $\gamma \Vdash \neg\forall x\, B(x)$. Then $\gamma \Vdash \exists x\, \neg B(x)$, which implies, for some $d \in \mathbb{N}$, $\gamma \Vdash \neg B(\bar{d})$. On the other hand, $\gamma \Vdash B(\bar{d})$. This shows that $\gamma \Vdash \bot$, hence $\gamma \Vdash \forall x\, B(x)$.

Conversely, if $\gamma \Vdash \forall x\, B(x)$, then $\exists n \forall d \in \mathbb{N}\, (\bar{\gamma}n \Vdash B(\bar{d}))$ and hence, by IH, $\forall d \in \mathbb{N}\, (\mathfrak{M}_\gamma \models B(\bar{d}))$, that is, $\mathfrak{M}_\gamma \models \forall x\, B(x)$. ■

Now let **FAN** stand for the following version of the fan theorem (without choice parameters):

$$\forall \alpha\, \exists n\, \varphi(\bar{\alpha}n) \to \exists m\, \forall \alpha\, \exists n \leq m\, \varphi(\bar{\alpha}n),$$

and suppose that $\Delta \models_c A$ holds. Then we have $\forall \gamma\, (\mathfrak{M}_\gamma \models \Delta_1 \Rightarrow \mathfrak{M}_\gamma \models A)$ and so, in view of $(*)$ and Lemma 3.2, $\forall \gamma\, (\gamma \Vdash \Delta \Rightarrow \gamma \Vdash A)$. Since $\mathbf{Un}(\Delta_1)$ is a model of Δ, it follows $\forall \gamma\, (\gamma \Vdash \Delta)$ and thus $\forall \gamma\, (\gamma \Vdash A)$. The latter means $\forall \gamma\, \exists n\, (\bar{\gamma}n \Vdash A)$, which implies (by **FAN** and monotonicity) $\exists m\, \forall k\, (\mathrm{lth}(k) = m \Rightarrow k \Vdash A)$, that is, $\mathbf{Un}(\Delta_1) \Vdash A$. Therefore $\Delta \vdash_c A$.

This completes the proof of the theorem. ■

REMARKS. (i). Since under a classical reading **FAN** is an equivalent form of a variant of König's lemma, viz. $\forall m\, \exists \alpha\, \forall n \leq m\, \varphi(\bar{\alpha}n) \to \exists \alpha\, \forall n\, \varphi(\bar{\alpha}n)$, all the arguments involved in the proof of the above theorem are also acceptable from a classical point of view. Furthermore, by classical logic this theorem is equivalent to Gödel's completeness theorem, see [2, Theorem 1].

(ii). For another, entirely different intuitionistic version of completeness for classical logic we refer the reader to [3].

4. Appendix

Given a set Δ of sentences and a sentence A of $\mathcal{L}$, we write $\Delta \models_i A$ to mean 'for every intuitionistic structure $\mathfrak{M}$: if $\mathfrak{M} \models \Delta$, then $\mathfrak{M} \models A$'.

The following statement is an obvious extension of a completeness result presented in [1].

THEOREM 4.1. *Let A be a sentence and Δ an enumerable set of sentences of $\mathcal{L}$. Then*

$$\Delta \vdash_i A \Leftrightarrow \Delta \models_i A.$$

PROOF. That $\Delta \vdash_i A$ implies $\Delta \models_i A$ is evident. We shall prove the converse.

Assume the variables α, β, γ to range over binary choice sequences satisfying: (i) the density axiom $\forall k\, \exists \alpha\, (\alpha \in k)$, (ii) (the above version of) the fan theorem (**FAN**), and (iii) the principle of open data (**OD**) $\varphi(\alpha) \to \exists n\, \forall \beta \in \bar{\alpha} n\, \varphi(\beta)$, for φ not containing choice parameters besides α. [NOTE. It is easily shown that **FAN** implies its relativized versions $\forall \alpha \in k\, \exists n\, \varphi(\bar{\alpha} n) \to \exists m\, \forall \alpha \in k\, \exists n \le m\, \varphi(\bar{\alpha} n)$, for any $k \in T_{01}$.]

Let $\mathbf{Un}(\Delta) = \langle T_{01}, \le, \mathbb{N}, \Vdash \rangle$ be a generalized Beth model which is universal for Δ, and let $\mathfrak{M}_\alpha = \langle \mathbb{N}, I \rangle$ be the intuitionistic structure associated with α, so that

$$(d_1, d_2, \ldots, d_{\tau(i)}) \in I(P_i) \Leftrightarrow \alpha \Vdash P_i(\bar{d}_1, \bar{d}_2, \ldots, \bar{d}_{\tau(i)}),$$

for $i \ge 0$ and $d_1, d_2, \ldots, d_{\tau(i)} \in \mathbb{N}$.

We need the following

LEMMA 4.2. *For all sentences A of $\mathcal{L}(\mathbb{N})$:*

$$\mathfrak{M}_\alpha \models A \Leftrightarrow \alpha \Vdash A.$$

PROOF. The proof is by induction on the logical complexity of A. For A atomic the assertion holds by the definition of $\mathfrak{M}_\alpha$. The cases $A \equiv B \wedge C$, $A \equiv B \vee C$, and $A \equiv \exists x\, B(x)$ are trivial.

Case $A \equiv B \to C$. Assume $\mathfrak{M}_\alpha \models B \to C$. Then, by **OD**, there is $n \in \mathbb{N}$ such that $\forall \beta \in \bar{\alpha}n\, (\mathfrak{M}_\beta \models B \to C)$, that is, $\forall \beta \in \bar{\alpha}n\, (\mathfrak{M}_\beta \models B \Rightarrow \mathfrak{M}_\beta \models C)$, which implies (by IH) $\forall \beta \in \bar{\alpha}n\, (\beta \Vdash B \Rightarrow \beta \Vdash C)$. Assume now that $k \succeq \bar{\alpha}n$ and $k \Vdash B$; then $\forall \gamma \in k\, (\gamma \Vdash B)$, from which it follows $\forall \gamma \in k\, (\gamma \Vdash C)$ and thus (by **FAN**, monotonicity, and density) $k \Vdash C$. This holds for all $k \succeq \bar{\alpha}n$, therefore $\bar{\alpha}n \Vdash B \to C$ and hence $\alpha \Vdash B \to C$.

Conversely, assume $\alpha \Vdash B \to C$, and suppose that $\mathfrak{M}_\alpha \models B$. Then, by IH, $\alpha \Vdash B$ and hence $\alpha \Vdash C$, which implies (again by IH) $\mathfrak{M}_\alpha \models C$. Therefore, $\mathfrak{M}_\alpha \models B \to C$.

Case $A \equiv \forall x\, B(x)$. We have: $\mathfrak{M}_\alpha \models \forall x\, B(x) \Leftrightarrow$ (by **OD**) $\exists n \forall \beta \in \bar{\alpha}n\, (\mathfrak{M}_\beta \models \forall x\, B(x)) \Leftrightarrow \exists n \forall \beta \in \bar{\alpha}n \forall d \in \mathbb{N}\, (\mathfrak{M}_\beta \models B(\bar{d})) \Leftrightarrow \exists n \forall d \in \mathbb{N} \forall \beta \in \bar{\alpha}n\, (\mathfrak{M}_\beta \models B(\bar{d})) \Leftrightarrow$ (by IH) $\exists n \forall d \in \mathbb{N} \forall \beta \in \bar{\alpha}n\, (\beta \Vdash B(\bar{d})) \Leftrightarrow$ (by **FAN**, monotonicity, and density) $\exists n \forall d \in \mathbb{N}\, (\bar{\alpha}n \Vdash B(\bar{d})) \Leftrightarrow \exists n\, (\bar{\alpha}n \Vdash \forall x\, B(x)) \Leftrightarrow \alpha \Vdash \forall x\, B(x)$. ■

Now suppose $\Delta \models_i A$. Then we have $\forall \alpha\, (\mathfrak{M}_\alpha \models \Delta \Rightarrow \mathfrak{M}_\alpha \models A)$ and so, in view of Lemma 4.2, $\forall \alpha\, (\alpha \Vdash \Delta \Rightarrow \alpha \Vdash A)$. Since $\mathbf{Un}(\Delta)$ is a model of Δ, it follows $\forall \alpha\, (\alpha \Vdash \Delta)$ and thus $\forall \alpha\, (\alpha \Vdash A)$. The latter means $\forall \alpha \exists n\, (\bar{\alpha}n \Vdash A)$, which implies (by **FAN**, monotonicity, and density) $<> \Vdash A$, that is, $\mathbf{Un}(\Delta) \Vdash A$. Therefore $\Delta \vdash_i A$.

This completes the proof of the theorem. ■

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