Higher-Order Probabilistic Programming

A Tutorial at POPL 2019

Part IV

Ugo Dal Lago

(Based on joint work with *Flavien Breuvart*, *Raphaëlle Crubillé*, *Charles Grellois*, *Davide Sangiorgi*,...)

ALMA MATER STUDIORUM UNIVERSITÀ DI BOLOGNA

Inria informatiques mathématiques

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The Landscape: *Type* Theory

Simple Types

$$\tau ::= \iota \;\;\Big|\;\; \tau \to \tau$$

The Landscape: *Type* Theory

Simple Types

$$\tau ::= \iota \;\mid\; \tau \to \tau$$

- Sound for termination, in absence of recursion.
- Poor expressive power.
- Intuitionistic Logic.

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Simple Types

$$\tau ::= \iota \mid \tau \to \tau$$

Polymorphic Types

$$\tau ::= \cdots \mid \alpha \mid \forall \alpha . \tau$$

The Landscape: *Type* Theory

Simple Types

- Second-order Intuistionistic Logic.
- Very expressive, extensionally.
- Still poor, intensionally.

Polymorphic Types

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Intersection Types

$$\tau ::= \cdots \mid \tau \wedge \tau$$

The Landscape: *Type* Theory

Simple Types

- Motivated by Semantics.
- *Complete* for termination.
- Type inference is undecidable.

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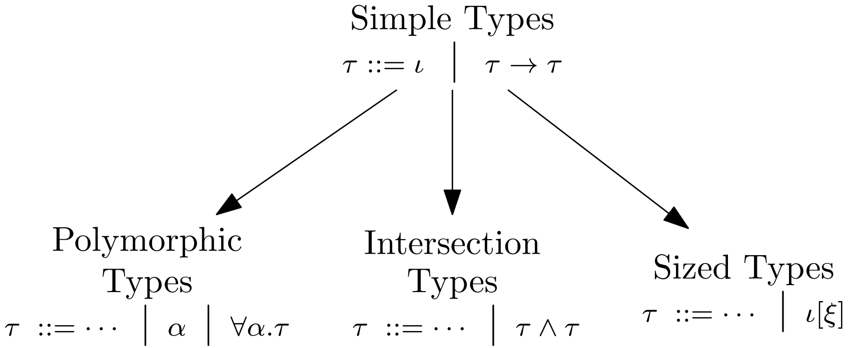

The Landscape: *Type* Theory

Simple Types

- Reasonably expressive, intensionally.
- Type inference remains decidable

Polymorphic Types

$$\tau \ ::= \ \cdots \ \mid \ \alpha \ \mid \ \forall \alpha . \tau$$

Intersection Types

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Sized Types

$$\tau \ ::= \ \cdots \ \mid \ \iota[\xi]$$

Determinism

$M\overline{s} \to^* N_s$

The Landscape: *Recursion* Theory

Determinism

$M\overline{s} \to^* N_s$

Probabilism

$[\![M\overline{s}]\!] = \mathcal{D}_s$

The Landscape: *Recursion* Theory

$\sum \mathcal{D}_s$ can be smaller than 1.

Determinism	**Probabilism**
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The Landscape: *Recursion* Theory

	Determinism	**Probabilism**
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Termination	$\exists N_s \in NF$	

The Landscape: *Recursion* Theory

Undecidable;
Σ^0_1-complete.

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The Landscape: *Recursion* Theory

D

Almost-Sure Termination
Π^0_2-complete.

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Π^0_2-complete.

Section 1

Sized Types

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  - But useless as a programming language.

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- For every type τ, define a set of reducible terms Red_τ.
- Prove that all reducible terms are normalizing…
- …and that all typable terms are reducible.

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$$\boxed{(\texttt{fix}\, x.M)V \to M\{\texttt{fix}\, x.M/x\}V}$$

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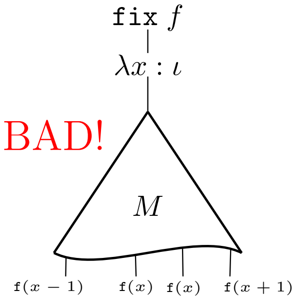

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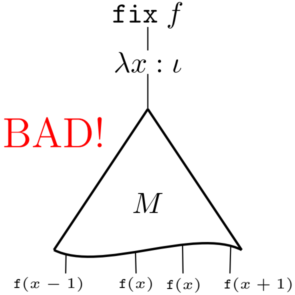

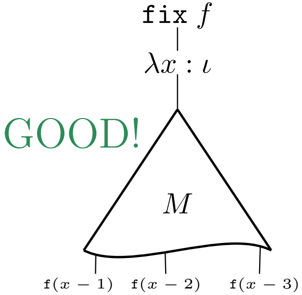

Deterministic Sized Types, Technically

- **Types.**

$$\xi ::= a \;\Big|\; \omega \;\Big|\; \xi + 1; \qquad\qquad \tau ::= \iota[\xi] \;\Big|\; \tau \to \tau.$$

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Index Terms

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> - Reducibility sets are of the form Red^{θ}_{τ}.
> - θ is an environment for index variables.
> - Proof of reducibility for `fix` $x.M$ is rather delicate.

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  - . . . but of an indexed form.
- **Type Inference.**
  - It is indeed *decidable*.
  - But *nontrivial*.

Probabilistic Termination

- **Examples**:

$$\texttt{fix}\ f.\lambda x.\texttt{if}\ x > 0\ \texttt{then}\ \texttt{if}\ \mathit{FairCoin}\ \texttt{then}\ f(x-1)\ \texttt{else}\ f(x+1);$$

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Biased Random Walk, the “wrong” way.

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- Probabilistic termination **is** thus:
  - Sensitive to *the actual distribution* from which we sample.
  - Sensitive to *how many recursive calls* we perform.

One-Counter Blind Markov Chains

- They are automata of the form (Q, δ) where
  - Q is a finite set of *states*.
  - $\delta : Q \to \mathsf{Dist}(Q \times \{-1, 0, 1\})$.
- They are a very special form of One-Counter Markov Decision Processeses [BBEK2011].
  - Everything is purely deterministic.
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  - Everything is purely deterministic.
  - The counter value is ignored.
- The probability of reaching a configuration where the counter is 0 can be approximated arbitrarily well *in polynomial time*.

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$$\Gamma \mid \Delta \vdash M : \mu$$

Probabilistic Sized Types [DLGrellois2017]

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  recursive structure by a OCBMC.
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A **distribution type**, i.e., a finite distribution of types.

$$\Gamma \mid \Delta \vdash M : \mu$$

Every higher-order variable occurs **at most once**.

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Probabilistic Sized Types [DLGrellois2017]

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Form σ, one can build a OCBMC:

- σ is a distribution type.
- It keeps track of the probability of each recursive call.

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This is sufficient for typing:

- Unbiased random walks;
- Biased random walks.

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- **Typing Probabilistic Choice**

$$\frac{\Gamma \mid \Delta \vdash M : \tau \quad \Gamma \mid \Omega \vdash N : \rho}{\Gamma \mid \frac{1}{2}\Delta + \frac{1}{2}\Omega \vdash M \oplus N : \frac{1}{2}\tau + \frac{1}{2}\rho}$$

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- Reducibility sets are now on the form $Red_{\tau}^{\theta,p}$
- p stands for the *probability* of being reducible.
- Reducibility sets are continuous:

$$Red_{\tau}^{\theta,p} = \bigcup_{q<p} Red_{\tau}^{\theta,q}$$

es.

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Section 2

Intersection Types

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- **Termination**
  - Again by reducibility.
- **Completeness**
  - By *subject expansion*, the dual of subject reduction.

Oracle Intersection Types [BreuvartDL2018]

- Probabilistic choice can be seen as a form of read operation:

$$M \oplus N = \texttt{if}\ \mathit{BitInput}\ \texttt{then}\ M\ \texttt{else}\ N$$

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$$M \oplus N = \texttt{if}\ \mathit{BitInput}\ \texttt{then}\ M\ \texttt{else}\ N$$

- **Types**

$$\tau ::= \star \;\Big|\; A \to s \cdot B \qquad A ::= \{\tau_1, \ldots, \tau_n\} \qquad s \in \{0,1\}^*$$

- **Typing Rules: Examples**

$$\frac{\Gamma \vdash M : s \cdot A}{\Gamma \vdash M \oplus N : 0s \cdot A} \qquad \frac{\Gamma \vdash M : r \cdot \{A \to s \cdot B\} \quad \Gamma \vdash N : q \cdot A}{\Gamma \vdash MN : (rqs) \cdot B}$$

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This is **unavoidable**, due to recursion theory.

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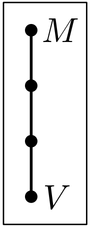
Intersection Types and Computations



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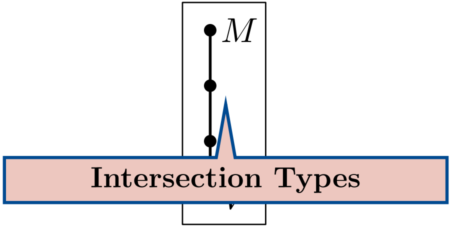

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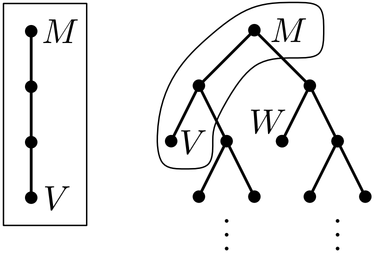

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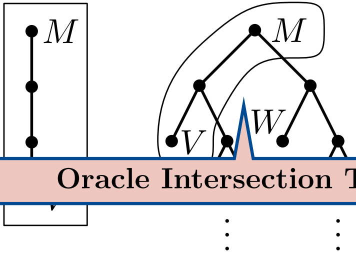

Oracle Intersection Types

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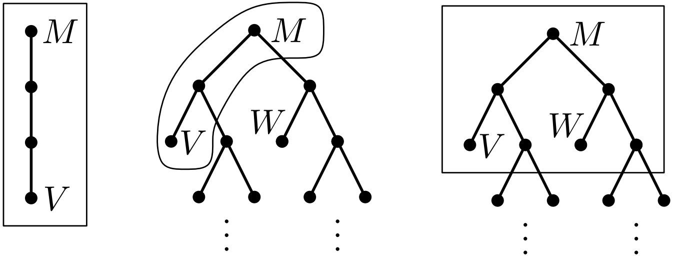
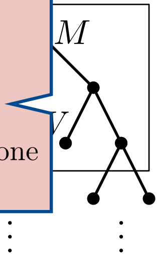


Intersection Types and Computations

Monadic Intersection Types [BDL2018]

- They are a combination of oracle and sized types.
- Intersections are needed for preciseness.
- Distributions of types allow to analyse more than one probabilistic branch in the same type derivation.



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  - In linear dependent types [DLG2011], one is (relatively complete) for deterministic first-order functions.
  - How about probabilism?
    - Monadic types becomes indexed:
      $$\mu ::= \{\sigma[i] : p[i]\}_{i \in I}$$
    - Subtyping is coupling-based.

Thank You!

Questions?