Counterexamples to a Conjecture of Dombi in Additive Number Theory

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Abstract

We disprove a 2002 conjecture of Dombi from additive number theory. More precisely, we find examples of sets $A \subset \mathbb{N}$ with the property that $\mathbb{N} \setminus A$ is infinite, but the sequence $n \to |\{(a,b,c): n=a+b+c \text{ and } a,b,c\in A\}|$, counting the number of 3-compositions using elements of A only, is strictly increasing.

1 Introduction

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ denote the natural numbers. A *k*-composition of an integer $n \in \mathbb{N}$ is a *k*-tuple of natural numbers (c_1, c_2, \ldots, c_k) such that $n = c_1 + c_2 + \cdots + c_k$. In contrast with partitions, the order of the summands matters in a composition.

Let $A \subseteq \mathbb{N}$, and define r(k, A, n) to be the number of k-compositions of n where the summands are chosen from A. The study of the function r was initiated by Erdős and Turán [16], who proved that if A is infinite then the sequence $(r(2, A, n))_{n\geq 0}$ cannot be eventually

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constant. Erdős and his co-authors returned to the study of r(k, A, n) in many papers; for example, [10, 11, 12, 14, 15, 19] just to name a few.

In a 1985 paper [13], Erdős, Sárközy, and Sós proved that $(r(2, A, n))_{n\geq 0}$ is eventually increasing¹ if and only if A is co-finite; that is, if A omits only finitely many integers. Their original proof was rather complicated and a considerable simplification was later found by Balasubramanian [5].

This result on r(2, A, n) prompted Dombi [9] to study r(k, A, n) for larger k. He conjectured that there is no set $A \subset \mathbb{N}$ with A co-infinite and $(r(3, A, n))_{n\geq 0}$ eventually increasing. In this note we refute Dombi's conjecture by constructing examples of co-infinite sets A for which $(r(3, A, n))_{n\geq 0}$ is eventually increasing; even eventually *strictly* increasing. We can also give explicit examples where $(r(3, A, n))_{n\geq 0}$ is strictly increasing right from the start.

A novelty in our approach for constructing explicit examples is the use of automatic sequences and the Walnut theorem-prover. For other applications of these ideas to additive number theory, see [6, 18, 7, 4].

2 A general construction

Let A be an infinite subset of N. There are two other ways to view A, both useful. We can consider its characteristic sequence $(a(n))_{n\geq 0}$, where a(n) is defined to be 1 if $n\in A$ and 0 otherwise. We can also consider its associated formal power series A(X), defined by $\sum_{n\geq 0} a(n)X^n$.

Then it is easy to see that for integers $k \geq 1$ we have

$$\sum_{i>0} r(k, A, i)X^i = A(X)^k.$$

Thus, the claim that $(r(k, A, n))_{n\geq 0}$ is eventually strictly increasing is equivalent to the claim that $(1-X)A(X)^k$ has coefficients that are eventually positive.²

It is easy to see that if $(r(k, A, n))_{n\geq 0}$ is strictly increasing right from the start, then $0 \in A$. Thus, the claim that $(r(k, A, n))_{n\geq 0}$ is strictly increasing right from the start is equivalent to the claim that $(1-X)A(X)^k$ has all positive coefficients.

Here is our first main result. We employ the usual asymptotic notation, where f = O(g) means $f(n) \leq Cg(n)$ for some C > 0 and all sufficiently large n. Also $f = \Theta(g)$ means f = O(g) and g = O(f). Finally, f = o(g) means $\lim_{n \to \infty} f(n)/g(n) = 0$.

Theorem 1. Let $k \geq 3$ be an integer. Let $F \subseteq \mathbb{N}$ and assume $0 \notin F$. Let $(f(n))_{n\geq 0}$ be its associated characteristic sequence and F(X) its associated power series $\sum_{i\geq 0} f(i)X^i$. Define $f'(n) = \sum_{0\leq i\leq n} f(i)$. Suppose $f'(n) = o(n^{\alpha})$ for some $\alpha < (k-2)/k$ and $A = \mathbb{N} \setminus F$. Then $(r(k,A,n))_{n\geq 0}$ is eventually strictly increasing.

¹By "increasing" in this paper we mean "(not necessarily strictly) increasing".

²To be clear, by "positive" we mean strictly greater than 0.

Proof. Let $A(X) = \sum_{i \geq 0} a(i)X^i$. It suffices to show that the coefficients of $(1 - X)A(X)^k$ are eventually positive. Now by the binomial theorem we have

$$(1-X)A(X)^{k} = (1-X)\left(\frac{1}{1-X} - F(X)\right)^{k}$$

$$= (1-X)\left(\sum_{0 \le i \le k} (-1)^{i} {k \choose i} \frac{1}{(1-X)^{k-i}} F(X)^{i}\right)$$

$$= \frac{1}{(1-X)^{k-1}} + \sum_{1 \le i \le k-2} (-1)^{i} {k \choose i} \frac{1}{(1-X)^{k-i-1}} F(X)^{i}$$

$$+ (-1)^{k-1} k F(X)^{k-1} + (-1)^{k} (1-X) F(X)^{k}. \tag{1}$$

Now, examining the coefficient of X^n of both sides for $n \ge 0$, we see that on the left it is r(k, A, n) - r(k, A, n - 1). (Here by convention r(k, A, -1) = 0.) On the other hand, the coefficient of X^n in

- $1/(1-X)^{k-1}$ is $\binom{n+k-2}{k-2} = \Theta(n^{k-2})$,
- $\binom{k}{i} \frac{1}{(1-X)^{k-i-1}} F(X)^i$ is $O(n^{k-i-2+i\alpha})$ for $1 \le i \le k-2$,
- $kF(X)^{k-1}$ is $O(n^{(k-1)\alpha})$,
- $(1-X)F(X)^k$ is $O(n^{k\alpha})$.

It is now easy to see that the $\Theta(n^{k-2})$ term dominates the remaining terms, since $\alpha < (k-2)/k$ by hypothesis. Hence the coefficient of X^n on the right-hand side is eventually positive, and the result is proved.

3 Results for automatic sets

In some cases we can find a good exact formula for the difference sequence

$$d(n) := r(k, A, n) - r(k, A, n - 1).$$

Here by "good" we mean that one can compute the n'th term in time polynomial in $\log n$. In particular, this is possible if A is a b-automatic set. We now explain how this can be done.

Recall that a set A is b-automatic if its characteristic sequence $(a(n))_{n\geq 0}$ can be computed by a DFAO (deterministic finite automaton with output) reading the base-b representation of n as input. Then we can use known enumeration techniques to show that $(d(n))_{n\geq 0}$ is a b-regular sequence [1, 2, 3] and even compute a linear representation for it. By a linear representation for a sequence x(n) we mean a triple (v, γ, w) consisting of a row vector v, a column vector w, and a matrix-valued morphism γ with domain $\Sigma_b = \{0, 1, \ldots, b-1\}$ such that

$$x(n) = v\gamma(z)w$$

for all $z \in \Sigma_b^*$ such that $[z]_b = n$. (Here $[z]_b$ is the integer represented by the string z, represented in base b, with most significant digit at the left.) This gives us our desired good formula.

More precisely, since we can write a first-order logical formula specifying that (x_1, x_2, \ldots, x_k) is a k-composition of elements of A, we can obtain a linear representation for r(k, A, n) from the DFAO for A. Then, using a simple construction based on block matrices, as in [8, p. 14] we can find a linear representation for the difference d(n). This can be carried out using the Walnut software system, originally designed by Hamoon Mousavi [17], and available for free at

https://cs.uwaterloo.ca/~shallit/walnut.html .

See [20] for more details.

The rank of a linear representation is the dimension of the vector v. Linear representations can be minimized (and thus we can find a linear representation of minimum rank) using an algorithm of Schützenberger, as described in [8, Chap. 2]. We used an implementation written in Maple.

If a b-regular sequence takes only finitely many values, then it is in fact automatic, and an automaton for it can be computed explicitly in terms of its linear representation (using the so-called "semigroup trick"; see [20, §4.11]).

Let $F \subseteq \mathbb{N}$ be a *b*-automatic set, let $(f(n))_{n\geq 0}$ be its characteristic sequence, and define $f'(n) = \sum_{0\leq i\leq n} f(i)$. By the classification of automatic sequences in [6, Lemmas 2.1–2.3] we know that either $f'(n) = O((\log n)^c)$ for some constant c, or $f'(n) \geq n^d$ infinitely often, for some constant d > 0. Furthermore, given a DFAO computing $(f(n))_{n\geq 0}$, it is decidable which of the two alternatives hold. If $f'(n) = O((\log n)^c)$, then we know from Theorem 1 that $(r(k, A, n))_{n\geq 0}$ is eventually strictly increasing for $k \geq 3$, where $A = \mathbb{N} \setminus F$.

4 An explicit example

Now we turn to our particular example. Choose the "forbidden set" F to be

$${2^{n+2}-1: n \ge 0} = {3,7,15,31,\ldots},$$

and define $A = \mathbb{N} \setminus F$. By Theorem 1 we know $(r(k, A, n))_{n \geq 0}$ is eventually strictly increasing for $k \geq 3$. We now show it is strictly increasing right from the start, and find an explicit formula for the difference r(k, A, n) - r(k, A, n - 1).

It follows from Eq. (1) for k = 3 that

$$D(X) = \sum_{i \geq 0} d(i)X^i = \sum_{i \geq 0} (i+1)X^i - 3(1 + \sum_{i \geq 0} (\lfloor \log_2(i+1) \rfloor - 1)X^i) + 3F(X)^2 - (1-X)F(X)^3.$$

Hence, by considering the coefficient of X^n on both sides, we have

$$d(n) = (n+1) - 3\lfloor \log_2(n+1) \rfloor + 3 + 3g(n) - (h(n) - h(n-1)),$$

for $n \ge 1$ where $g(n) = [X^n]F(X)^2$ and $h(n) = [X^n]F(X)^3$.

Next we show that $e(n) := d(n) - (n+1) + 3\lfloor \log_2(n+1) \rfloor$ is 2-automatic and find an explicit automaton for it. We start with the following Walnut commands, which can be typed in verbatim into Walnut (without the line numbers):

- (1) reg f msd_2 "0*(11)1*":
- (2) reg power2 msd_2 "0*10*":
- (3) def log2 "\$power2(x) & 1<x & x<=n":
- (4) def a3n n "n=x+y+z & $^{s}f(x)$ & $^{s}f(y)$ & $^{s}f(z)$ ":
- (5) def a3n1 n "n=x+y+z+1 & $^{s}f(x)$ & $^{s}f(y)$ & $^{s}f(z)$ ":
- (6) def np1 n "x<=n":
- (7) def log2n1 n "\$log2(n+1,x)":

The explanation for these commands is as follows:

- (1) asserts that n has a base-2 expansion consisting of two 1's followed by any number of 1's, allowing leading zeros. Hence f(n) is true iff $n \in F$.
- (2) asserts that power2(n) is true iff n is a power of 2.
- (3) asserts that $1 < x \le n$ and x is a power of 2.
- (4) asserts that n = x + y + z and each of x, y, z belongs to $\mathbb{N} \setminus F$. It computes a linear representation for the number of such compositions, which turns out to have rank 100.
- (5) asserts that n-1=x+y+z and each of x,y,z belongs to $\mathbb{N}\setminus F$. It computes a linear representation for the number of such compositions, which turns out to have rank 81.
- (6) asserts that $x \leq n$. It computes a linear representation for the number of such x, which is n+1. It turns out to have rank 2.
- (7) asserts that $1 < x \le n+1$ and x is a power of 2; it computes the number of such x, which is $\lfloor \log_2(n+1) \rfloor$. It turns out to have rank 6.

From the linear representations above in parts (4)–(7), we can then use a simple block matrix construction to compute the linear representation for e(n). It has a linear representation of rank 100 + 81 + 2 + 6 = 189. Now we can minimize it, getting a linear representation (v, γ, w) of rank 10, as follows:

When we run the semigroup trick on this linear representation, we get a DFAO of 33 states. This DFAO can then be minimized, giving us the 28-state DFAO described in Table 2. Here $\delta(q, i)$ gives the automaton's transition from state q on input i, and $\tau(q)$ gives the output associated with state q.

It remains to see d(n) > 0 for all n. It is easy to see that $0 \le g(n) \le 2$ and $0 \le h(n) \le 6$ for all n. Hence for $n \ge 1$ we have $d(n) \le n + 1 - 3\lfloor \log_2(n+1)\rfloor - 3$, which by standard estimates is positive for $n \ge 12$. For $0 \le n < 12$ we can check from Table 1 that d(n) > 0.

Table 1: First few values of r(3, A, n) and d(n).

We can sum up these calculations as follows:

Theorem 2. Let $F = \{2^{n+2} - 1 : n \ge 0\}$ and define $A = \mathbb{N} \setminus F$. Then the sequence $(r(3,A,n))_{n\ge 0}$ is strictly increasing from the start, and the difference d(n) = r(3,A,n) - r(3,A,n-1) can be calculated in time polynomial in $\log_2 n$ using the formula

$$d(n) = n + 1 - 3\lfloor \log_2(n+1) \rfloor + e(n),$$

where e(n) is the sequence computed by the automaton in Table 2.

Remark 3. It is easy to see that if $(r(k, A, n))_{n\geq 0}$ is strictly increasing from the start, then this is also true of $(r(k', A, n))_{n\geq 0}$ for $k'\geq k$. So our example of Theorem 2 also works for all $k\geq 3$; in particular, for k=4. Dombi [9] conjectured there were examples of eventually strictly increasing sequences for k=4, but was not able to prove the existence of one.

q	$\delta(q,0)$	$\delta(q,1)$	$\tau(q)$	q	$\delta(q,0)$	$\delta(q,1)$	$\tau(q)$
0	0	1	0	14	23	24	9
1	2	3	3	15	16	17	0
2	4	5	3	16	12	20	12
3	6	7	3	17	25	17	3
4	8	9	3	18	21	22	-1
5	10	11	3	19	23	26	13
6	12	13	6	20	21	22	-3
7	14	7	3	21	23	27	9
8	8	15	3	22	21	22	3
9	16	17	2	23	23	23	3
10	12	18	10	24	23	23	0
11	19	11	3	25	23	27	15
12	12	20	3	26	23	23	-1
13	21	22	0	27	23	23	-3

Table 2: Automaton for e(n).

References

- [1] J.-P. Allouche and J. O. Shallit. The ring of k-regular sequences. Theoret. Comput. Sci. **98** (1992), 163–197.
- [2] J.-P. Allouche and J. Shallit. Automatic Sequences: Theory, Applications, Generalizations. Cambridge University Press, 2003.
- [3] J.-P. Allouche and J. O. Shallit. The ring of k-regular sequences, II. *Theoret. Comput. Sci.* **307** (2003), 3–29.
- [4] J.-P. Allouche and J. Shallit. Additive properties of the evil and odious numbers and similar sequences. Arxiv preprint arXiv:2112.13627 [math.NT]. Available at https://arxiv.org/abs/2112.13627, 2022.
- [5] R. Balasubramanian. A note on a result of Erdős, Sárközy and Sós. *Acta Arith.* **49** (1987), 45–53.
- [6] J. Bell, K. Hare, and J. Shallit. When is an automatic set an additive basis? *Proc. Amer. Math. Soc. Ser. B* **5** (2018), 50–63.
- [7] J. Bell, T. F. Lidbetter, and J. Shallit. Additive number theory via approximation by regular languages. *Internat. J. Found. Comp. Sci.* **31** (2020), 667–687.

- [8] J. Berstel and C. Reutenauer. *Noncommutative Rational Series With Applications*, Vol. 137 of *Encyclopedia of Mathematics and Its Applications*. Cambridge University Press, 2011.
- [9] G. Dombi. Additive properties of certain sets. Acta Arith. 103 (2002), 137–146.
- [10] P. Erdős. Problems and results in additive number theory. In *Colloque sur la Théorie des Nombres, Bruxelles, 1955*, pp. 127–137. Georges Thone and Masson & Cie, 1956.
- [11] P. Erdős, A. Sárközy, and V. T. Sós. Problems and results on additive properties of general sequences. I. *Pacific J. Math.* **118** (1985), 347–357.
- [12] P. Erdős and A. Sárközy. Problems and results on additive properties of general sequences. II. *Acta Math. Hung.* **48** (1986), 201–211.
- [13] P. Erdős, A. Sárközy, and V. T. Sós. Problems and results on additive properties of general sequences. IV. In *Number Theory*, Vol. 1122 of *Lecture Notes in Mathematics*, pp. 85–104. Springer-Verlag, 1985.
- [14] P. Erdős, A. Sárközy, and V. T. Sós. Problems and results on additive properties of general sequences. V. *Monatsh. Math.* **102** (1986), 183–197.
- [15] P. Erdős, A. Sárközy, and V. T. Sós. Problems and results on additive properties of general sequences. III. Stud. Sci. Math. Hung. 22 (1987), 53–63.
- [16] P. Erdős and P. Turán. On a problem of Sidon in additive number theory and on some related problems. J. London Math. Soc. 16 (1941), 212–215. Addendum, 19 (1944), 208.
- [17] H. Mousavi. Automatic theorem proving in Walnut. Arxiv preprint arXiv:1603.06017 [cs.FL], available at http://arxiv.org/abs/1603.06017, 2016.
- [18] A. Rajasekaran, J. Shallit, and T. Smith. Additive number theory via automata theory. Theoret. Comput. Sci. 64 (2020), 542–567.
- [19] A. Sárközy. On the number of additive representations of integers. In *More Sets, Graphs and Numbers*, Vol. 15 of *Bolyai Soc. Math. Stud.*, pp. 329–339. Springer-Verlag, 2006.
- [20] J. Shallit. The Logical Approach to Automatic Sequences: Exploring Combinatorics on Words with Walnut, Vol. 482 of London Math. Soc. Lecture Notes Series. Cambridge University Press, 2022.