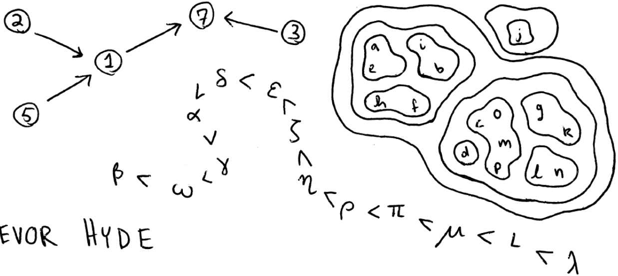
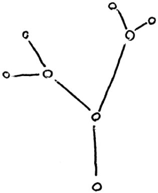
Combinatorial Species and Generating Functions



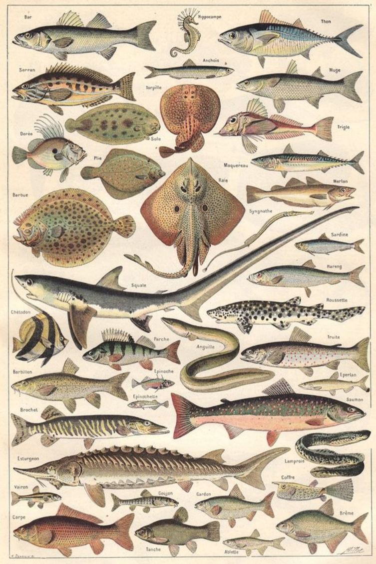


TREVOR HYDE

Combinatorial Species

Combinatorial species S is any sort of labelled structure built from a finite set A which does not depend on the names or properties of the elements of A.

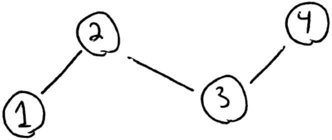
Let's see some examples.

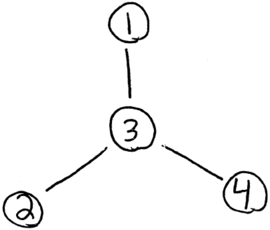


Example: **Trees**

Let $A = \{1, 2, 3, 4\}$.

Members of the species **Trees** built from A:





Example: **Linear Orders**

Let $A = \{a, b, c, d, e\}$.

Structures of type **Linear Orders** put on A:

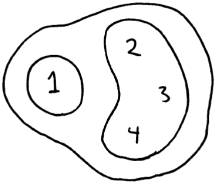
$b < e < a < c < d$

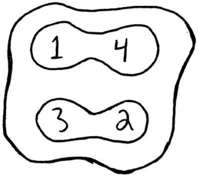
$e < d < c < b < a$

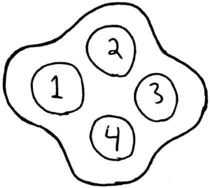
Example: **Partitions**

Let $A = \{1, 2, 3, 4\}$.

Members of **Partitions** constructed from *A*:



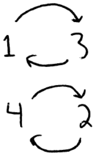


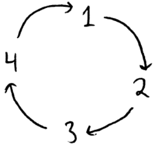


Example: **Permutations**

Let $A = \{1, 2, 3, 4\}$.

Structures of type **Permutations** built from A:





Combinatorial Species, again

Combinatorial species S is any sort of **labelled structure** built from a finite set A which does not depend on the **names** or **properties** of the elements of A.

More concretely, S is a function which sends a finite set A to $S(A)$ the set of all structures of S built from A.

For the technocrats: a combinatorial species is a functor $S : \mathrm{Fin}_0 \to \mathrm{Fin}_0$ from the groupoid of finite sets with bijections to itself.

Example: **Linear Orders**

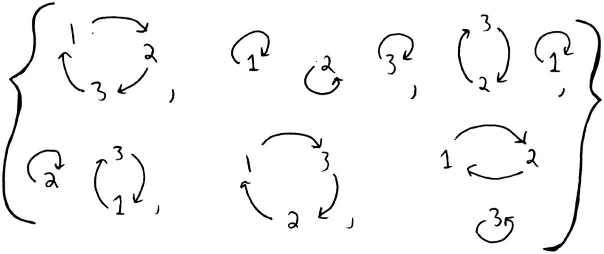
L is the species of **Linear Orders**.

$$L(\{a,b,c\}) = \left\{ \begin{array}{ll} a < b < c & b < a < c \\ a < c < b & b < c < a \\ c < a < b & c < b < a \end{array} \right\}$$

Example: **Permutations**

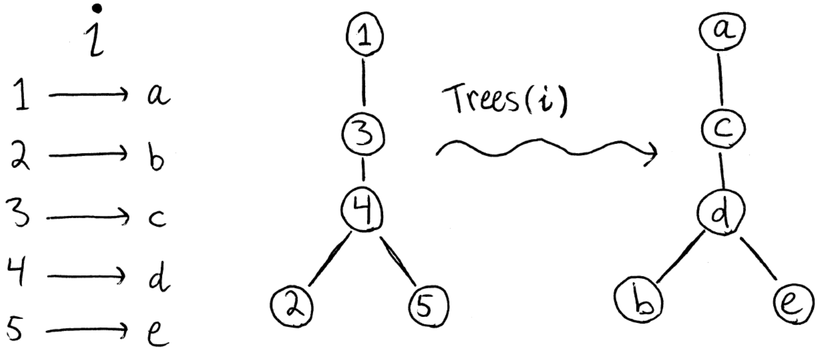
P is the species of **Permutations**.

$P(\{1,2,3\}) =$



What's in a name?

Members of a species S built from A "can't interpret" the names of the elements of A.
For example, if $A = \{1, 2, 3, 4, 5\}$ and $B = \{a, b, c, d, e\}$, then **Trees** "knows" how to use i to transform structures built from A into structures built from B.



To Specify a Species

Let $[n] = \{1, 2, 3, \ldots, n\}$.

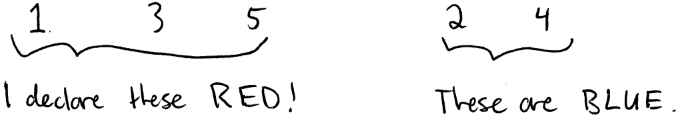
We only need to know $S[n]$ the members of S built from $[n]$ for each $n \geq 0$.

Say species S and T are **equivalent** and write $S \approx T$ if there is a "natural" way to get a correspondence between members of S and T.

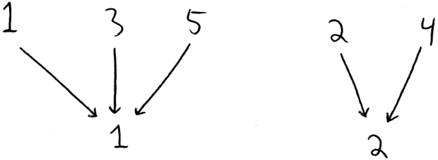
Technocrats: S and T are equivalent if S and T are naturally isomorphic as functors.

Example

The species of **Red and Blue Colorings of a Set**.



The species of **Functions to** [2].



The species of **Subsets**.

Counting Members of a Species

Write $|S[n]|$ for the number S structures on $[n]$.

If $S \approx T$, then $|S[n]| = |T[n]|$ for all $n \geq 0$.

Counting Members of a Species

Write $|S[n]|$ for the number S structures on $[n]$.

If $S \approx T$, then $|S[n]| = |T[n]|$ for all $n \geq 0$.

Caution: $|S[n]| = |T[n]|$ for each $n \geq 0$ does not imply $S \approx T$!

Non-Example

Linear Orders and **Permutations** have the same number of members $|L[n]| = |P[n]| = n!$, but they are not equivalent!

Question: How would you show two species were inequivalent? Hint: consider the symmetries of members of the two species.

Generating Function of a Species

To any species S we associate the **generating function**

$$S(x) = \sum_{n \geq 0} |S[n]| \frac{x^n}{n!}.$$

"$S(x)$ is an algebraic manifestation of the *entire* species S."
I hope to explain this idea through examples.

Example: **Permutations**

Let P be the species of **Permutations**.
Since $|P[n]| = n!$ we have

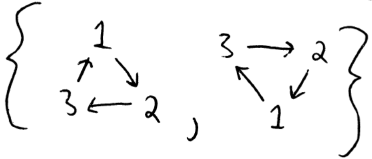
$$P(x) = \sum_{n \geq 0} |P[n]| \frac{x^n}{n!} = \sum_{n \geq 0} n! \frac{x^n}{n!} = \sum_{n \geq 0} x^n = \frac{1}{1-x}$$

The same calculation shows $L(x) = \frac{1}{1-x}$ when L is the species of **Linear Orders**.

Example: **Cyclic Permutations**

Let C be the species of **Cyclic Permutations**.

$$C(\{1,2,3\}) =$$



Note that $|C[n]| = (n-1)!$.

$$C(x) = \sum_{n\geq 0} |C[n]| \frac{x^n}{n!} = \sum_{n \geq 0} (n-1)! \frac{x^n}{n!} = \sum_{n\geq 0} \frac{x^n}{n} = \log\left(\frac{1}{1-x}\right).$$

Example: **Finite Sets**

Let *Exp* be the species of **Finite Sets**.

$$E(\{1,2,3,4\}) = \{\{1,2,3,4\}\}$$

Then $|Exp[n]| = 1$ for every n.

$$Exp(x) = \sum_{n\geq 0} |Exp[n]| \frac{x^n}{n!} = \sum_{n\geq 0} \frac{x^n}{n!} = e^x.$$

You could view this as a “reason” why e^x shows up so often.
It is also related to why some people say things like

$$|\textbf{Finite Sets}| = e.$$

Examples: m **Element Sets**

Let E_m be the species of m **Element Sets**. Then

$$|E_m[n]| = \begin{cases} 1 & m = n \\ 0 & m \neq n. \end{cases}$$

Hence

$$E_m(x) = \sum_{n \geq 0} |E_m[n]| \frac{x^n}{n!} = \frac{x^m}{m!}.$$

We write X for the species E_1 of singletons.

Operations on Species: Sum

Let S and T be species.

Sum: Structures of $S + T$ on A are either structures of S or of T.

$$(S + T)(A) = S(A) \sqcup T(A).$$

$$(C + L)(\{1, 2\}) = \left\{ \begin{tikzcd} 1 \arrow[r, bend left] & 2 \arrow[l, bend left] \end{tikzcd} ,\quad 1 < 2, \quad 2 < 1 \right\}$$

Operations on Species: Product

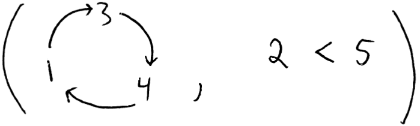
Let S and T be species.

Product: $S \cdot T$ is trickier to define.
A structure of $(S \cdot T)(A)$ is formed by partitioning $A = B \sqcup C$, then putting an S structure on B and a T structure on C.

$$(S \cdot T)(A) = \bigsqcup_{A = B \sqcup C} S(B) \times T(C).$$

How to build a member of $(C \cdot L)(A)$:

1. Say $A = \{1, 2, 3, 4, 5\}$
2. Split A into two pieces $B = \{1, 3, 4\}$, $C = \{2, 5\}$.
3. Put a cyclic structure on B and a linear structure on C.



Ops on Species = Ops on Generating Functions

There's some justice in this world:

$$(S + T)(x) = S(x) + T(x)$$
$$(S \cdot T)(x) = S(x)T(x).$$

Example

Consider the identity

$$\frac{1}{1-x} = \frac{1-x+x}{1-x} = 1 + x \cdot \frac{1}{1-x}.$$

Its fun to try “lifting” algebraic identities to the level of species.

$$L \overset{?}{\approx} 1 + X \cdot L.$$

Example

Consider the identity

$$\frac{1}{1-x} = \frac{1-x+x}{1-x} = 1 + x \cdot \frac{1}{1-x}.$$

Its fun to try "lifting" algebraic identities to the level of species.

algebraic species

$$L \approx 1 + X \cdot L.$$

"A linear order is either empty or it may be written as a first element followed by a linear order."

$$b < d < a < c \rightsquigarrow (b,\ d < a < c)$$

Composition

Let S and T be species and suppose T has no structures on the empty set ($|T[0]| = 0$.)

Composition: $S \circ T$ takes some explaining.
To construct a member of $(S \circ T)(A)$:
1. Partition $A = B_1 \sqcup B_2 \sqcup \ldots \sqcup B_k$ into non-empty sets.
2. Put a T structure on each B_i.
3. Put an S structure on $\{B_1, B_2, \ldots, B_k\}\big)$.

$$(S \circ T)(A) = \bigsqcup_{A = \sqcup_{i \le k} B_i} S[k] \times \prod_{i \le k} T(B_i).$$

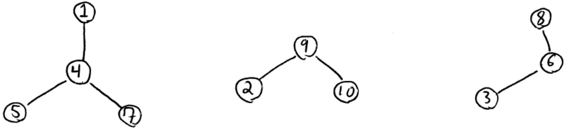
Example: **Cyclic Perms ∘ Trees**

C and T are the species of **Cyclic Permutations** and **Trees**.
Let's build a member of $(C \circ T)[10]$:

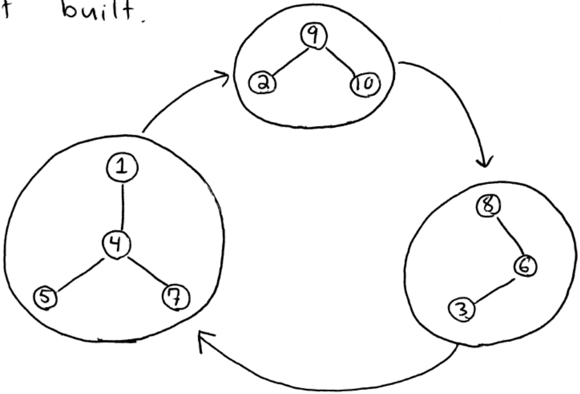
1. Split [10] into some number of non-empty sets

$B_1 = \{1, 4, 5, 7\}$ $B_2 = \{2, 9, 10\}$ $B_3 = \{3, 6, 8\}$

2. Choose a Tree structure for each B_i



3. Choose a Cyclic structure on the set of Trees just built.



Exponentiation

Useful to compose with *Exp* the species of **Finite Sets**.

Members of $(Exp \circ S)(A)$ have a simpler description:
1. Partition $A = B_1 \sqcup B_2 \sqcup \ldots \sqcup B_k$ into non-empty sets.
2. Put an S structure on each B_i.

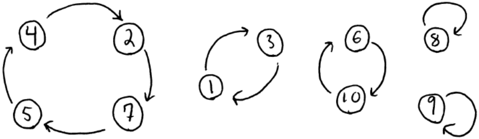
Comp of Species = Comp of Generating Functions

$$(S \circ T)(x) = S(T(x))$$
$$(Exp \circ S)(x) = e^{S(x)}.$$

Example: $Exp \circ C$

C is the species of **Cyclic Permutations**.

Here is a member of the species $(Exp \circ C)[10]$.



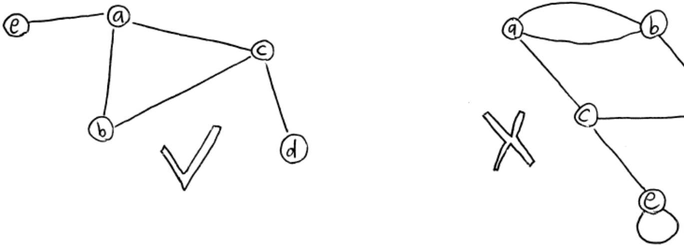
$$\therefore P \approx Exp \circ C,$$

$$\textbf{Permutations} = Exp(\textbf{Cyclic Permutations})$$

$$\frac{1}{1-x} = \exp\left(\log\left(\frac{1}{1-x}\right)\right)$$

Simple graphs

A simple graph is a graph with no edges from a point to itself and at most one edge between a pair of points.



Question: Can you express **Simple Graphs** as the composition of two species? Hint: Try counting the number of simple graphs built from [*n*] and then work backwards.

Bernoulli Polynomials

Bernoulli polynomials $B_n(q)$ arise in the study of the Riemann Zeta function, p-adic integration, classic summation formulas, etc.

Their coefficients are the mysterious **Bernoulli numbers**.

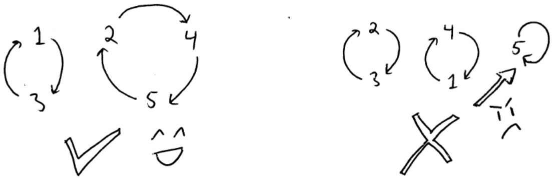
$1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, -\frac{1}{30}, 0, \frac{5}{66}, 0, -\frac{691}{2730}, 0, \frac{7}{6}, 0, -\frac{3617}{510}, 0, \frac{43867}{798}, \ldots$

$B_n(q)$ is defined by the following identity:

$$xe^{qx} = (e^x - 1)\sum_{n\geq 0} B_n(q)\frac{x^n}{n!}.$$

Question: Can you lift this to a species identity?

A permutation with no fixed points is called a **derangement**.



Counting derangements of $[n]$ is a classic problem.

Let’s count with species!

Derangements

$$\textbf{Permutations} = \mathit{Exp}(\textbf{Cyclic Permutations})$$

$$\textbf{Derangements} = \mathit{Exp}(\textbf{Cyclic Permutations of Size} \geq \mathbf{2})$$

$$C_{\geq 2}(x) = C(x) - x = \log\left(\frac{1}{1-x}\right) - x.$$

$$D(x) = e^{\log(\frac{1}{1-x}) - x} = \frac{1}{1-x} e^{-x}.$$

How to multiply a power series by $\frac{1}{1-x}$:

Say $f(x) = \sum_{n \geq 0} a_n x^n$ is a power series.

$f(x)$

$\frac{1}{1-x}$	a_0 +	$a_1 x$ +	$a_2 x^2$ +	$a_3 x^3 + \cdots$
1 +	a_0	$a_1 x$	$a_2 x^2$	$a_3 x^3$
x +	$a_0 x$	$a_1 x^2$	$a_2 x^3$	$a_3 x^4$
x^2 +	$a_0 x^2$	$a_1 x^3$	$a_2 x^4$	
x^3 +	$a_0 x^3$	$a_1 x^4$		
x^4 +	$a_0 x^4$			

$$\frac{1}{1-x} f(x) = a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + (a_0 + a_1 + a_2 + a_3)x^3 + \cdots$$

Derangements

$$
\begin{aligned}
D(x) &= \frac{1}{1-x}e^{-x} = \frac{1}{1-x}\sum_{n\geq 0}\frac{(-1)^n}{n!}x^n \\
&= \sum_{n\geq 0}\left(\sum_{m\leq n}\frac{(-1)^m}{m!}\right)x^n = \sum_{n\geq 0}\left(n!\sum_{m\leq n}\frac{(-1)^m}{m!}\right)\frac{x^n}{n!}
\end{aligned}
$$

Derangements

$$D(x) = \frac{1}{1-x}e^{-x} = \frac{1}{1-x}\sum_{n\geq 0}\frac{(-1)^n}{n!}x^n$$

$$= \sum_{n\geq 0}\left(\sum_{m\leq n}\frac{(-1)^m}{m!}\right)x^n = \sum_{n\geq 0}\left(n!\sum_{m\leq n}\frac{(-1)^m}{m!}\right)\frac{x^n}{n!}$$

Therefore,

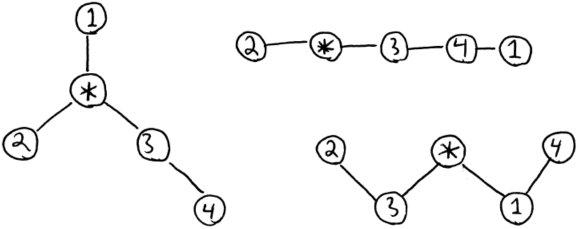
$$\boxed{|D[n]| = n!\sum_{m\leq n}\frac{(-1)^m}{m!}}$$

Differentiation

The **derivative** of S is the species:

$$DS(A) = S(A \sqcup \{*\})$$

S structures on A together with a **distinguished point** $*$.
Here are members of $DT[4]$ where T is the species of **Trees**.



Example: Derivative of **Linear Orders**

Let L be the species of **Linear Orders**.

Example member of $DL[5]$:

$$1 < 5 < 2 < * < 4 < 3$$

Notice there's a natural correspondence between members of DL and members of $L^2 = L \cdot L$:

$$1 < 5 < 2 < * < 4 < 3 \leftrightsquigarrow (1 < 5 < 2, 4 < 3)$$

Thus $DL \approx L^2$.

Differentiation

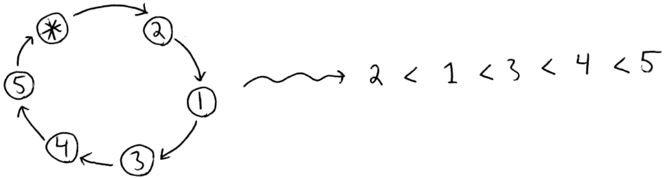
Believe It or Not!

$$DS(x) = \frac{d}{dx} S(x).$$

$$DL \approx L^2 \Longrightarrow \frac{d}{dx} \frac{1}{1-x} = \frac{1}{(1-x)^2}.$$

Example: Derivative of **Cyclic Permutations**

Let C be the species of **Cyclic Permutations**.



"Linear orders are the derivative of cyclic permutations."

$$DC \approx L$$
$$\frac{d}{dx}\log\left(\frac{1}{1-x}\right) = \frac{1}{1-x}.$$

Example: Derivative of **Finite Sets**

Let *Exp* be the species of **Finite Sets**.

Adding an extra point to a set yields a set.

$$\therefore DExp \approx Exp \implies \frac{d}{dx}e^x = e^x.$$

Application

What are the chances of a random permutation of $[2n]$ having all even length cycles?

$C_2 =$ **Even Cyclic Permutations**
$P_2 =$ **Permutations with all Even Cycles**

$$P_2 \approx Exp(C_2)$$

$$C_2(x) = \sum_{n \geq 1} \frac{x^{2n}}{2n} = \frac{1}{2} \sum_{n \geq 1} \frac{\left(x^2\right)^n}{n} = \frac{1}{2} \log\left(\frac{1}{1-x^2}\right) = \log\left(\frac{1}{\sqrt{1-x^2}}\right).$$

$$P_2(x) = \exp\left(\log\left(\frac{1}{\sqrt{1-x^2}}\right)\right) = \frac{1}{\sqrt{1-x^2}}.$$

Application

$P_2 =$ **Permutations with all Even Cycles**

$$P_2(x) = \frac{1}{\sqrt{1-x^2}}.$$

Expand with binomial theorem:

$$\frac{1}{\sqrt{1-x^2}} = \left(1-x^2\right)^{-1/2} = \sum_{n\geq 0} \binom{-1/2}{n} (-1)^n x^{2n}$$

$$= \sum_{n\geq 0} (2n)! \binom{-1/2}{n} (-1)^n \frac{x^{2n}}{(2n)!}$$

Application

Therefore, the probability of a random permutation of $[2n]$ having all even length cycles is:

$$\frac{|P_2[2n]|}{(2n)!} = (-1)^n \binom{-\frac{1}{2}}{n}$$

$$\begin{aligned}
(-1)^n \binom{-\frac{1}{2}}{n} &= (-1)^n \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\cdots\left(-\frac{1}{2}-n+1\right)}{n!} \\
&= \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+2\right)\cdots\left(\frac{1}{2}+n-1\right)}{n!} \\
&= \frac{1\cdot 3\cdot 5\cdots 2n-1}{2^n n!} = \frac{1\cdot 3\cdot 5\cdots 2n-1}{2^n n!}\cdot\frac{2\cdot 4\cdot 6\cdots 2n}{2^n n!} \\
&= \frac{1}{2^{2n}}\frac{(2n)!}{n!n!} = \frac{1}{2^{2n}}\binom{2n}{n}
\end{aligned}$$

Application

$$\frac{|P_2[2n]|}{(2n)!} = \frac{1}{2^{2n}}\binom{2n}{n}$$

Probability that a random permutation of $[2n]$ has all even length cycles is the same as the probability of getting exactly n heads in $2n$ coin tosses.

Thanks.