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THREE USES OF THE HERBRAND-GENTZEN THEOREM IN RELATING MODEL THEORY AND PROOF THEORY

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1. Introduction. One task of metamathematics is to relate suggestive but nonelementary modeltheoretic concepts to more elementary proof-theoretic concepts, thereby opening up modeltheoretic problems to proof-theoretic methods of attack. Herbrand's Theorem (see [8] or also [9], vol. 2) or Gentzen's Extended Hauptsatz (see [5] or also [10]) was first used along these lines by Beth [1]. Using a modified version he showed that for all first-order systems a certain modeltheoretic notion of definability coincides with a certain proof theoretic notion. [1] In the present paper the Herbrand-Gentzen Theorem will be applied to generalize Beth's results from primitive predicate symbols to arbitrary formulas and terms.

This may be interpreted as showing that (apart from some relatively minor exceptions which will be made apparent below) the expressive power of each first-order system is rounded out, or the system is *functionally complete*, in the following sense: Any functional relationship which obtains between concepts that are expressible in the system is itself expressible and provable in the system.

A second application is concerned with the hierarchy of second-order formulas. A certain relationship is shown to hold between first-order formulas and those second-order formulas which are of the form $(\exists T_1)\ldots(\exists T_k)A$ or $(T_1)\ldots(T_k)A$ with A being a first-order formula. Modeltheoretically this can be regarded as a relationship between the class **AC** and the class $\mathbf{PC}_\Delta$ of sets of models, [2] investigated by Tarski in [12] and [13].

A third application is concerned with a problem of axiomatizability. For a system with the extralogical constants R, S, and T, it may sometimes be desirable, for example, to have an axiomatization in which each axiom involves only R and S or only S and T. More generally, given a system and given certain proper subsets of the set of extralogical constants of the system, the question arises whether or not there is an axiomatization such that each extralogical axiom involves only the constants of one of the subsets. It will be shown that for a first-order system an axiom system of the desired kind exists if and only if a certain modeltheoretic condition is satisfied.

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[1] For another interesting proof, more along modeltheoretic lines, see A. Robinson [11]. I am grateful to him for oral and written suggestions regarding several points, especially regarding (*iii*) of end § 3.

[2] See also 1 and 2 of [3]. In these and other matters, I owe an appreciation of the modeltheoretic approach to J. R. Büchi.

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In all three cases, a fundamental lemma (Lemma 1) is used which is derived from the Herbrand-Gentzen Theorem. The lemma concerns valid first-order formulas $A \supset A'$ in which both A and A′ may contain predicate symbols not occurring in the other. Several modeltheoretic properties are such that possession of the property by a first-order system is equivalent to the validity of a formula $A \supset A'$ of this kind. The lemma therefore seems a useful tool for further investigations. In particular, it may lend itself to questions of this kind: How is a certain modeltheoretic property of a system reflected by theorems in the system?

2. Lemma and extensions. We shall consider a system *PCI* of first-order predicate calculus without identity and a system *PCI*= of first-order predicate calculus with identity. *PCI* shall contain individual variables and constants, and predicate variables and constants of $n \geq 0$ arguments. *PCI* shall *not* contain a symbol for identity and shall *not* contain symbols for functions of $n \geq 1$ individual arguments. (Most results of this paper do not hold for first-order predicate calculus with function symbols but without axioms for identity.) The result of adding to *PCI* these further symbols, changing the formation rules accordingly, and adding also axioms for identity shall be *PCI*=. $\vdash$ and $\vdash_=$ shall stand for derivability in *PCI* or *PCI*= respectively. The letters A, B, C, etc. shall refer to the formulas of the system concerned. Those formulas in which no individual variable occurs free shall be *sentences*. (Hence any 0-place predicate symbol is a sentence.) The individual constants and the individual variables occurring free in a formula shall be the *individual parameters* of that formula. Likewise, the function symbols in a formula and the predicate symbols other than the identity sign shall be its *function* or *predicate parameters* [3] respectively. The identity sign shall not be a parameter but a logical constant, since its interpretation cannot vary (except for the range of definition). Thus among the parameters of a formula of *PCI* are all its predicate symbols, no function symbols, and none or more individual symbols. Among the parameters of a formula of *PCI*= are all its predicate symbols except the identity sign, all its function symbols, and none or more individual symbols.

LEMMA 1. *If* $\vdash A \supset A'$ *and if* A *and* A′ *have a predicate parameter in common, then there is an "intermediate" formula* B *such that* $\vdash A \supset B$, $\vdash B \supset A'$, *and all parameters of* B *are parameters of both* A *and* A′. *Also, if* $\vdash A \supset A'$ *and if* A *and* A′ *have no predicate parameter in common,* [4] *then either* $\vdash \neg A$ *or* $\vdash A'$.

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[3] This usage is taken from [1].

[4] This case was first called to my attention by P. C. Gilmore. For the special case where in addition A and A′ are sentences, he has found a much simpler argument in terms of satisfiability.

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PROOF. The results of [4] hold a fortiori for *PCI* in place of first-order predicate calculus with function symbols. Hence by Theorem 5 of [4], which is derived from the Herbrand-Gentzen Theorem, there is a B* which satisfies all the requirements of the lemma except perhaps that B* may contain individual parameters which are not parameters of both A and A′. Now take each individual parameter of B* which is not a parameter of A, replace all its free occurrences in B* by a new individual variable, and then universally quantify this variable over the entire formula. In the resulting formula similarly replace by an existentially quantified variable any individual parameter which is not a parameter of A′. The formula B which thus finally results satisfies the lemma.

The methods of [4] allow a more detailed study than is needed here of how the structure of A and B are related. For example, if A is a formula in prenex normal form containing only universal quantifiers and containing all the individual parameters of A′, then B can easily be shown to be a formula of the same kind.

Replacing A′ by $\neg$C, one can also state the lemma in terms of consistency, where a formula of *PCI* shall be *consistent* (relative to *PCI*) if and only if its negation is not a theorem of *PCI*.

LEMMA 1. (ALTERNATIVE VERSION.) *If* A *and* C *are each a consistent formula of PCI, and if there is no* B *whose parameters are parameters of both* A *and* C *such that* $\vdash A \supset B$ *and* $\vdash C \supset \neg B$, *then* $A \,.\, C$ *is also consistent.*

In this form, the lemma readily yields 2.6 and hence also 2.9 of [11]. Whether it is stronger than 2.6 is not clear. It is stronger than 2.9, since it does not require that the set of those formulas B such that $\vdash A \supset B$, $\vdash C \supset B$, and the parameters of B are parameters of both A and C, be complete.

Given formulas $A_1, \ldots, A_l$, a parameter of a formula A_i shall be *isolated* (with respect to $A_1, \ldots, A_l$) if and only if it is not a parameter of any other A_j, $1 \leq j \leq l$. Lemma 1 can then be generalized. The first part of the earlier version then becomes:

LEMMA 2. *If* $\vdash A_1 \,.\, \ldots \,.\, A_k \,.\supset.\, A_{k+1} \vee \ldots \vee A_l$ *and if each* A_i *has at least one nonisolated predicate parameter, then there are "intermediate" formulas* $B_1, \ldots, B_l$ *such that* $\vdash B_1 \,.\, \ldots \,.\, B_k \,.\supset.\, B_{k+1} \vee \ldots \vee B_l$, *such that* $\vdash A_1 \supset B_1, \ldots, \vdash A_k \supset B_k$ *and* $\vdash B_{k+1} \supset A_{k+1}, \ldots, \vdash B_l \supset A_l$, *and such that any parameter of any* B_i *is a nonisolated parameter of* A_i.

PROOF. By Lemma 1 and induction. Isolated parameters are removed from one A_i at a time, the other formulas having been transferred to the consequent of the conditional.

An analogue for *PCI*=, which includes the case of function parameters, is the following.

LEMMA 3. *If* $\vdash_= A_1 \,.\, \ldots \,.\, A_k \,.\supset.\, A_{k+1} \vee \ldots \vee A_l$, *then there are "intermediate" formulas* $B_1, \ldots, B_l$ *such that* $\vdash_= B_1 \,.\, \ldots \,.\, B_k \,.\supset.\, B_{k+1} \vee \ldots \vee B_l$,

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such that $\vdash_= \mathrm{A}_1 \supset \mathrm{B}_1, \ldots, \vdash_= \mathrm{A}_k \supset \mathrm{B}_k$ *and* $\vdash_= \mathrm{B}_{k+1} \supset \mathrm{A}_{k+1}, \ldots, \vdash_= \mathrm{B}_l \supset \mathrm{A}_l$, *and such that any parameter of any* B_i *is a nonisolated parameter of* A_i.

PROOF. Consider any formulas $\mathrm{A}_1, \ldots, \mathrm{A}_l$. Let $\mathrm{f}_1, \ldots, \mathrm{f}_t$ $(\mathrm{t} \geq 0)$ be the function parameters which occur in at least one A_i, let $n_1, \ldots, n_t$ be their number of argument places, and let $\mathrm{F}_1, \ldots, \mathrm{F}_t$ be distinct predicate parameters not occurring in $\mathrm{A}_1, \ldots, \mathrm{A}_l$ with $n_1+1, \ldots, n_t+1$ argument places respectively. For each formula E let $U(\mathrm{E})$ be a conjunction of those sentences $(\mathrm{x}_1)\ldots(\mathrm{x}_{n_q})(\exists \mathrm{w})(\mathrm{F}_q(\mathrm{x}_1, \ldots, \mathrm{x}_{n_q}, \mathrm{w}) \,.\, (\mathrm{z})(\mathrm{F}_q(\mathrm{x}_1, \ldots, \mathrm{x}_n, \mathrm{z}) \supset \mathrm{w} = \mathrm{z}))$, if any, such that $1 \leq q \leq t$ and f_q occurs in E; if there are no such sentences, let $U(\mathrm{E})$ be $(\mathrm{x})(\mathrm{x} = \mathrm{x})$. Now consider all those formulas E such that any parameter of E occurs in at least one A_i and consider all those formulas D such that any parameter of D either is one of the symbols $\mathrm{F}_1, \ldots, \mathrm{F}_t$ or else is an individual or predicate (but not a function) parameter of at least one A_i. By Theorem 43 of [10] and induction on t, one can associate with each of these formulas E and D a formula E′ or D* respectively such that the following conditions are satisfied:

(1a) The parameters of E′ are those of E except that in place of any f_q there is F_q.

(1b) The parameters of D* are those of D except that in place of any F_q there is f_q.

(2a) If $\vdash_= \mathrm{E}_1 \,.\, \ldots \,.\, \mathrm{E}_r \,.\!\supset\!.\, \mathrm{E}_{r+1} \vee \ldots \vee \mathrm{E}_s$, then $\vdash_= U(\mathrm{E}'_1) \,.\, \mathrm{E}'_1 \,.\, \ldots \,.\, U(\mathrm{E}'_r) \,.\, \mathrm{E}'_r \,.\!\supset\!.\, \neg U(\mathrm{E}'_{r+1}) \vee \mathrm{E}'_{r+1} \vee \ldots \vee \neg U(\mathrm{E}'_s) \vee \mathrm{E}'_s$.

(2b) If $\vdash_= U(\mathrm{D}_1) \,.\, \mathrm{D}_1 \,.\, \ldots \,.\, U(\mathrm{D}_r) \,.\, \mathrm{D}_r \,.\!\supset\!.\, \neg U(\mathrm{D}_{r+1}) \vee \mathrm{D}_{r+1} \vee \ldots \vee \neg U(\mathrm{D}_s) \vee \mathrm{D}_s$, then $\vdash_= \mathrm{D}^*_1 \,.\, \ldots \,.\, \mathrm{D}^*_r \,.\!\supset\!.\, \mathrm{D}^*_{r+1} \vee \ldots \vee \mathrm{D}^*_s$.

(3) $\vdash_= \mathrm{E} \equiv \mathrm{E}'^*$.

Now suppose that $\vdash_= \mathrm{A}_1 \,.\, \ldots \,.\, \mathrm{A}_k \,.\!\supset\!.\, \mathrm{A}_{k+1} \vee \ldots \vee \mathrm{A}_l$. Then by (2a) also $\vdash_= U(\mathrm{A}'_1) \,.\, \mathrm{A}'_1 \,.\, \ldots \,.\, U(\mathrm{A}'_k) \,.\, \mathrm{A}'_k \,.\!\supset\!.\, \neg U(\mathrm{A}'_{k+1}) \vee \mathrm{A}'_{k+1} \vee \ldots \vee \neg U(\mathrm{A}'_l) \vee \mathrm{A}'_l$. For each A'_i, let $I(\mathrm{A}'_i)$ be a conjunction of the sentences $(\mathrm{x})(\mathrm{x} = \mathrm{x})$ and $(\mathrm{x})(\mathrm{y})(\mathrm{z}_1)\ldots(\mathrm{z}_n)(\mathrm{x} = \mathrm{y} \supset\!.\, \mathrm{G}(\mathrm{z}_1, \ldots, \mathrm{z}_{m-1}, \mathrm{x}, \mathrm{z}_{m+1}, \ldots, \mathrm{z}_n) \supset \mathrm{G}(\mathrm{z}_1, \ldots, \mathrm{z}_{m-1}, \mathrm{y}, \mathrm{z}_{m+1}, \ldots, \mathrm{z}_n))$ where G is an n-place predicate parameter of A'_i and $1 \leq m \leq n$. It follows from [9] vol. 1 pp. 373–379 that $\vdash I(\mathrm{A}'_1) \,.\, U(\mathrm{A}'_1) \,.\, \mathrm{A}'_1 \,.\, \ldots \,.\, I(\mathrm{A}'_k) \,.\, U(\mathrm{A}'_k) \,.\, \mathrm{A}'_k \,.\!\supset\!.\, \neg I(\mathrm{A}'_{k+1}) \vee \neg U(\mathrm{A}'_{k+1}) \vee \mathrm{A}'_{k+1} \vee \ldots \vee \neg I(\mathrm{A}'_l) \vee \neg U(\mathrm{A}'_l) \vee \mathrm{A}'_l$. Since all individual parameters and all predicate symbols other than the identity sign occurring in $I(\mathrm{A}'_i)$ or $U(\mathrm{A}'_i)$ also occur in A'_i, it follows from Lemma 2 that there are formulas $\mathrm{D}_1, \ldots, \mathrm{D}_l$, such that any individual parameter and any predicate symbol other than the identity sign occurring in a D_i is a nonisolated parameter of A'_i, such that $\vdash \mathrm{D}_1 \,.\, \ldots \,.\, \mathrm{D}_k \,.\!\supset\!.\, \mathrm{D}_{k+1} \vee \ldots \vee \mathrm{D}_l$, and such that $\vdash I(\mathrm{A}'_i) \,.\, U(\mathrm{A}'_i) \,.\, \mathrm{A}'_i \,.\!\supset \mathrm{D}_i$ if $1 \leq i \leq k$ and $\vdash \mathrm{D}_i \supset\!.\, \neg I(\mathrm{A}'_i) \vee \neg U(\mathrm{A}'_i) \vee \mathrm{A}'_i$ if $k < i \leq l$. Then also $\vdash_= U(\mathrm{A}'_i) \,.\, \mathrm{A}'_i \,.\!\supset \mathrm{D}_i$ if $1 \leq i \leq k$ and $\vdash_= \mathrm{D}_i \supset\!.\, \neg U(\mathrm{A}'_i) \vee \mathrm{A}'_i$ if $k < i \leq l$. It follows from (2b) that $\vdash_= \mathrm{A}'^*_i \supset \mathrm{D}^*_i$ if $1 \leq i \leq k$ and $\vdash_= \mathrm{D}^*_i \supset \mathrm{A}'^*_i$ if $k < i \leq l$. Then, by (3), $\vdash_= \mathrm{A}_i \supset \mathrm{D}^*_i$ if $1 \leq i \leq k$ and $\vdash_= \mathrm{D}^*_i \supset \mathrm{A}_i$ if $k < i \leq l$. From $\vdash \mathrm{D}_1 \,.\, \ldots \,.\, \mathrm{D}_k \,.\!\supset\!.\, \mathrm{D}_{k+1} \vee \ldots \vee \mathrm{D}_l$ and (2b) it also follows that $\vdash_= \mathrm{D}^*_1 \,.\, \ldots \,.\, \mathrm{D}^*_k \,.\!\supset\!.$

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$D^*_{k+1} \vee \ldots \vee D^*_l$. Finally, from (1a), (1b), and the construction of $D_1, \ldots, D_l$ it follows that any parameter of any D^*_i is a nonisolated parameter of A_i. Hence if $B_1, \ldots, B_l$ are $D^*_1, \ldots, D^*_l$ respectively, the lemma is satisfied. Q.E.D.

It should be emphasized that all three lemmas apply not only to sentences expressing assertions but also to formulas containing free individual variables and thus expressing conditions.

3. Theorems on definability. Consider an arbitrary set Σ of sentences and arbitrary parameters R, $S_1, \ldots, S_p, \ldots$ of sentences of Σ. We shall say that the set of (standard) models of Σ *defines* the values of R *in terms of* (the underlying domain and) the values of $S_1, \ldots, S_p, \ldots$ if and only if any two models of Σ which agree in the underlying domain **D** and in the interpretation of $S_1, \ldots, S_p, \ldots$ also agree in the interpretation of R. For example, if R; $S_1, \ldots, S_p, \ldots$; $T_1, \ldots, T_q, \ldots$ are the only parameters which occur in a sentence of Σ, then the set of models of Σ defines the values of R in terms of the values of $S_1, \ldots, S_p, \ldots$ if and only if, whenever $\langle \mathbf{D}; \mathbf{R}; \mathbf{S}_1, \ldots, \mathbf{S}_p, \ldots; \mathbf{T}_1, \ldots, \mathbf{T}_q, \ldots \rangle$ and $\langle \mathbf{D}; \mathbf{R}'; \mathbf{S}_1, \ldots, \mathbf{S}_p, \ldots; \mathbf{T}'_1, \ldots, \mathbf{T}'_q, \ldots \rangle$ are both models of Σ, then $\mathbf{R} = \mathbf{R}'$. Then the set of models of Σ may be regarded as a *function* yielding for any $\mathbf{D}$; $\mathbf{S}_1, \ldots, \mathbf{S}_p, \ldots$ at most one $\mathbf{R}$.

This notion can be extended beyond the case of primitive parameters. For our purposes it is sufficient to restrict the discussion throughout to parameters and other expressions of $PCI{=}$. In the case of a predicate parameter S, we considered interpretations of S. These are also interpretations of $\hat{x}_1 \ldots \hat{x}_n(S(x_1, \ldots, x_n))$, where $n \geq 0$ and where $S(x_1, \ldots, x_n)$ is an atomic formula. We may now generalize by taking an arbitrary formula G and considering interpretations of $\hat{x}_1 \ldots \hat{x}_m(G(x_1, \ldots, x_m))$, where $x_1, \ldots, x_m$ are the individual variables, if any, which have a free occurrence in G. Similarly, in the case of an individual or function parameter S with n arguments, $n \geq 0$, we considered interpretations of $\hat{x}_1 \ldots \hat{x}_n \hat{y}(S(x_1, \ldots, x_n) = y)$. We may now generalize by taking an arbitrary *individual term* $F(x_1, \ldots, x_l)$ of $l \geq 0$ variables and considering interpretations of $\hat{x}_1 \ldots x_l \hat{y}(F(x_1, \ldots, x_l) = y)$. In either case, we shall call $\hat{x}_1 \ldots \hat{x}_m(G(x_1, \ldots, x_m))$ and $x_1 \ldots \hat{x}_l \hat{y}$ $(F(x_1, \ldots, x_l) = y)$ the *abstract* of the formula G or term F respectively (the y being chosen in some unique manner and distinct from $x_1, \ldots, x_l$).

Given arbitrary formulas or terms H, $G_1, \ldots, G_p, \ldots$ we shall therefore say that the set of (standard) models of Σ *defines* the values of the abstract of H *in terms of* (the underlying domain and) the values of the abstracts of $G_1, \ldots, G_p, \ldots$ if and only if any two models of Σ which agree in the underlying domain and in the interpretation of the abstracts of $G_1, \ldots, G_p, \ldots$ also agree in the interpretation of the abstract of H. This is a model-theoretic notion of definability.

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Stricter notions of definability result when further requirements are imposed. In the case of parameters, for example, one may require not only that the values of R are a function of those of $S_1, \ldots, S_p, \ldots$ but also that this fact can be expressed in the formal language in which Σ is embedded. One may further require that this cannot only be expressed but also be proved. One may also require that the values of R are determined by those of $S_1, \ldots, S_p, \ldots$ (and by the underlying domain) in a manner which can be expressed in the same way for all values of $S_1, \ldots, S_p, \ldots$ and hence is independent of these values. All these further requirements are met by the notion of explicit definability, which we shall now discuss.

We begin with the case of parameters. The underlying logic is always assumed to be formalized and to be such that a formal system results when Σ is added to it as axiom set. Let $R, S_1, \ldots, S_p, \ldots$ be parameters of sentences of Σ. We shall then say that the system obtained by adding Σ as axiom set to the underlying logic *defines* R *explicitly in terms of* $S_1, \ldots, S_p, \ldots$ [5] if and only if from Σ one can derive in the underlying logic respectively a sentence $(x_1)\ldots(x_n)(R(x_1, \ldots, x_n) \equiv B)$, if R is an n-place predicate parameter ($n \geq 0$), or a sentence $(x_1)\ldots(x_n)(R(x_1, \ldots, x_{n-1}) = x_n \,.\!\equiv B)$, if R is an $(n-1)$-place individual or function parameter ($n \geq 1$), such that B is a formula containing no free individual variables other than $x_1, \ldots, x_n$ and no further parameters other than $S_1, \ldots, S_p, \ldots$. [6]

The notion of explicit definability also can be generalized from primitive parameters to abstracts of arbitrary terms or formulas. For convenience, our phrasing will refer to the terms and formulas themselves rather than their abstracts. We first require an analogue for (abstracts of) terms and formulas in place of parameters of the notion that a formula B contains only certain parameters. Let each of $G_1, \ldots, G_p, \ldots$ be an arbitrary formula or term. We shall then say that a formula B is *composed of* $G_1, \ldots, G_p, \ldots$, if and only if there are atomic symbols $G'_1, \ldots, G'_p, \ldots$ of a type corresponding to that of $G_1, \ldots, G_p, \ldots$ respectively, and there is a formula B' containing no parameters other than $G'_1, \ldots, G'_p, \ldots$ or individual variables, such that B is obtained from B' by substitution (without collision of variables) of $G_1, \ldots, G_p, \ldots$ for $G'_1, \ldots, G'_p, \ldots$ respectively. (Sub-

[5] This terminology is patterned after [11]. The terminology for the modeltheoretic notion was chosen to be similar.

[6] Note that if there is a formula B as described, then there is also one which contains all the variables $x_1, \ldots, x_n$ as free individual variables. For example, if x_i has no free occurrence in B, then we can replace B by $B \,.\, (P(x_i, \ldots, x_i) \vee \neg P(x_i, \ldots, x_i))$ where P is a predicate symbol occurring in B. Also note that for the case where R and $S_1, \ldots, S_p, \ldots$ are all individual or function parameters it would be of considerable interest to study the stricter requirement that $(x_1)\ldots(x_{n-1})(R(x_1, \ldots, x_{n-1}) = G)$ be deducible from Σ for some term G containing no parameters other than $x_1, \ldots, x_{n-1}, S_1, \ldots, S_p, \ldots$. The methods of this paper do not seem sufficient for a study of this kind.

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stitution for a predicate symbol is described for example in [10], p. 156.)

Given arbitrary formulas or terms $H, G_1, \ldots, G_p, \ldots$ we shall then say that the system obtained by adding Σ as axiom set to the underlying logic *defines* H *explicitly in terms of* $G_1, \ldots, G_p, \ldots$, if and only if from Σ one can derive in the underlying logic respectively a sentence $(x_1)\ldots(x_n)(H \equiv B)$, if H is a formula with $n \geq 0$ free individual variables, or a sentence $(x_1)\ldots(x_n)(H = x_n .\equiv B)$, if H is a term with $n-1 \geq 0$ variables, such that B is a formula which is composed of $G_1, \ldots, G_p, \ldots$ and contains no free individual variables other than $x_1, \ldots, x_n$. This is a prooftheoretic notion of definability.

Definability in the prooftheoretic sense usually implies definability in the modeltheoretic sense. The implication holds for any of the usual systems of logic provided that in derivations from Σ substitution into a sentence of Σ is not permitted. For in that case any sentence derivable from Σ, including any sentence $(x_1)\ldots(x_n)(H \equiv B)$ or $(x_1)\ldots(x_n)(H = x_n .\equiv B)$, is true for any model of Σ. We shall now show that conversely definability in the modeltheoretic sense implies definability in the prooftheoretic sense whenever the underlying logic is either *PCI=* or (disregarding minor exceptions) *PCI*. Now ordinarily a first-order system can be represented by adding a suitable set Σ as axiom set to *PCI=* or *PCI* respectively. [7] Moreover this can be done in a way which excludes substitution into any sentence of Σ. Hence ordinarily the prooftheoretic and the modeltheoretic notion of definability for a first-order system coincide.

This result was first proved by Beth [1] for the case where the parameter R to be defined and the parameters $S_1, \ldots, S_p, \ldots$ in terms of which it is to be defined are predicate parameters of one or more arguments, and where, moreover, R is to be defined in terms of *all* the other predicate parameters of the system. Besides extending this to other parameters $S_1, \ldots, S_p, \ldots$, we shall now strengthen this result and show that if the values of R are a function of the values of only certain parameters of the system then R is explicitly definable in terms of these parameters alone. [8] Thus a definition of R in a first-order system, if possible at all, is possible without "auxiliary" parameters (without "Umwege").

THEOREM 1. *Let* Σ *be a set of sentences of PCI= (of PCI) and let* R, $S_1, \ldots, S_p, \ldots$ *be parameters of sentences of* Σ. *Suppose that* R *and, in the case of PCI, at least one* S_p *is a predicate parameter. Then if the set of models of* Σ *defines the values of* R *in terms of the values of* $S_1, \ldots, S_p, \ldots$, *then also the system obtained by adding* Σ *as axiom set to PCI= (to PCI) defines* R *explicitly in terms of* $S_1, \ldots, S_p, \ldots$.

[7] An exception are those unusual first-order systems which contain function symbols but no identity sign.

[8] This way of arranging the presentation is due independently to E. W. Beth and J. R. Büchi. Some strengthening along these lines also occurs in [11] Theorem 4.9.

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PROOF. We consider first the case of *PC1*. Let $T_1, \ldots, T_q, \ldots$ be those parameters of a sentence of Σ, if any, which are distinct from $R, S_1, \ldots, S_p, \ldots$ and distinct from each other. Let $R', T'_1, \ldots, T'_q, \ldots$ be parameters which are of the same kind as $R, T_1, \ldots, T_q, \ldots$ respectively and which are distinct from each other and from any parameter of a sentence of Σ. Let Σ' be the set of sentences which are obtained from a sentence of Σ by substituting $R', T'_1, \ldots, T'_q, \ldots$ for $R_1, T_1, \ldots, T_q, \ldots$ respectively. Now suppose that the set of models of Σ defines the values of R in terms of the values of $S_1, \ldots, S_p, \ldots$. Then any model satisfying $\Sigma \cup \Sigma'$ also satisfies $(x_1)\ldots(x_n)(R(x_1, \ldots, x_n) \equiv R'(x_1, \ldots, x_n))$, where $n \geq 0$ is the number of arguments of R. Then by the completeness of *PC1*, $\Sigma \cup \Sigma' \vdash (x_1)\ldots(x_n)$ $(R(x_1, \ldots, x_n) \equiv R'(x_1, \ldots, x_n))$. Then there is a conjunction $A(R; S_1, \ldots, S_l; T_1, \ldots, T_m)$ with no parameters other than $R, S_1, \ldots, S_l, T_1, \ldots, T_m$ of sentences of Σ, and a conjunction $A(R'; S_1, \ldots, S_l; T'_1, \ldots, T'_m)$ obtained from it by substitution of $R', T'_1, \ldots, T'_m$ for $R, T_1, \ldots, T_m$ respectively, such that

(1) $\vdash A(R; S_1, \ldots, S_l; T_1, \ldots, T_m) \,.\, A(R'; S_1, \ldots, S_l; T'_1, \ldots, T'_m) \,.\!\supset$ $(x_1)\ldots(x_n)(R(x_1, \ldots, x_n) \equiv R'(x_1, \ldots, x_n))$.[9]

Then also

(2) $\vdash A(R; S_1, \ldots, S_l; T_1, \ldots, T_m) \,.\, R(x_1, \ldots, x_n) \,.\!\supset\!.$ $A(R'; S_1, \ldots, S_l; T'_1, \ldots, T'_m) \supset R'(x_1, \ldots, x_n)$.

Then by Lemma 1 there is a formula $B(S_1, \ldots, S_l; x_1, \ldots, x_n)$ containing no free individual variables other than $x_1, \ldots, x_n$ and no further parameters other than $S_1, \ldots, S_l$, such that

(3) $\vdash A(R; S_1, \ldots, S_l; T_1, \ldots, T_m) \,.\, R(x_1, \ldots, x_n) \,.\!\supset B(S_1, \ldots, S_l; x_1, \ldots, x_n)$,

and

(4) $\vdash B(S_1, \ldots, S_l; x_1, \ldots, x_n) \supset\!.$ $A(R'; S_1, \ldots, S_l; T'_1, \ldots, T'_m) \supset R'(x_1, \ldots, x_n)$.

Substitution into (4) of $R, T_1, \ldots, T_m$ for $R', T'_1, \ldots, T'_m$ respectively yields

(5) $\vdash B(S_1, \ldots, S_l; x_1, \ldots, x_n) \supset\!. A(R; S_1, \ldots, S_l; T_1, \ldots, T_m) \supset R(x_1, \ldots, x_n)$.

Now (3) and (5) yield

(6) $\vdash A(R; S_1, \ldots, S_l; T_1, \ldots, T_m) \supset\!. R(x_1, \ldots, x_n) \equiv B(S_1, \ldots, S_l; x_1, \ldots, x_n)$.

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[9] This part of the proof alone shows that if the set of models of Σ defines the values of R in terms of the values of $S_1, \ldots, S_p, \ldots$, then it also defines the values of R in terms of the values of a finite subset $S_1, \ldots, S_l$. Certain other (modeltheoretic) relationships between the values of the parameters of a first-order system can be shown by a similar argument to be of "finite character" in the same sense.

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Hence also $\Sigma \vdash (x_1) \ldots (x_n)(R(x_1, \ldots, x_n) \equiv B(S_1, \ldots, S_l; x_1, \ldots, x_n))$. From the construction of B it follows that the system obtained by adding Σ as axiom set to *PCI* defines R explicitly in terms of $S_1, \ldots, S_p, \ldots$. For the case of *PCI*= the proof is similar except that Lemma 3 is used in place of Lemma 1. Q.E.D.

In the case of *PCI* the restriction that at least one S_p be a predicate parameter is needed since otherwise there would be no formula whose only parameters are $x_1, \ldots, x_n$ and $S_1, \ldots, S_p, \ldots$. If the set of models of Σ defines the values of R in terms of the values of $S_1, \ldots, S_p, \ldots$, and if none of $S_1, \ldots, S_p, \ldots$ are predicate parameters, then all predicate parameters in the antecedent of (2) and also those in the consequent of (2) are isolated. Then by Lemma 1 either $\vdash \neg(A(R; S_1, \ldots, S_l; T_1, \ldots, T_m) . R(x_1, \ldots, x_n))$ and hence $\Sigma \vdash (x_1) \ldots (x_n) \neg R(x_1, \ldots, x_n)$, or else $\vdash A(R'; S_1, \ldots, S_l; T_1, \ldots, T_m) \supset R'(x_1, \ldots, x_n)$ and hence $\Sigma \vdash (x_1) \ldots (x_n) R(x_1, \ldots, x_n)$. This is the special case considered in § 5 of [1].

We shall now generalize from parameters to arbitrary formulas and terms.

THEOREM 2. *Let Σ be a set of sentences of PCI= (of PCI) and let $G_0, G_1, \ldots, G_p, \ldots$ be arbitrary formulas or terms of PCI= (of PCI). In the case of PCI, suppose that G_0 and at least one other G_r are formulas. Then if the set of models of Σ defines the values of the abstract of G_0 in terms of the values of the abstracts of $G_1, \ldots, G_p, \ldots$, then also the system obtained by adding Σ as axiom set to PCI= (to PCI) defines G_0 explicitly in terms of $G_1, \ldots, G_p, \ldots$.*

PROOF. We consider first the case of *PCI*=. Let $G_0, G_1, \ldots, G_p, \ldots$ be distinct. For each formula G_r let $n_r \geq 0$ be the number of distinct free individual variables in G_r and for each term G_r let $n_r - 1 \geq 0$ be the number of individual variables in G_r. Let $S_0, S_1, \ldots, S_p, \ldots$ be distinct predicate parameters not occurring in any sentence of Σ with $n_0, n_1, \ldots, n_p, \ldots$ arguments respectively. For each formula G_r, let E_r be the sentence $(x_1) \ldots (x_{n_r})(G_r \equiv S_r(x_1, \ldots, x_{n_r}))$, where $x_1, \ldots, x_{n_r}$ are the free individual variables in G_r, if any, and let $G_r(y_1, \ldots, y_{n_r})$ be the result of substituting $y_1, \ldots, y_{n_r}$ for $x_1, \ldots, x_{n_r}$ respectively in G_r. For each term G_r, let E_r be the sentence $(x_1) \ldots (x_{n_r})(G_r = x_{n_r} . \equiv S_r(x_1, \ldots, x_{n_r}))$, where $x_1, \ldots x_{n_r - 1}$ are the individual variables in G_r, if any, and x_{n_r} is another variable chosen in some unique manner, and let $G_r(y_1, \ldots, y_{n_r - 1})$ be the result of substituting $y_1, \ldots, y_{n_r - 1}$ for $x_1, \ldots, x_{n_r - 1}$ respectively in G_r. To Σ add the sentences $E_0, E_1, \ldots, E_p, \ldots$ and let Σ^* be the resulting set. The axiom set Σ^* may be thought of as a definitional extension of the axiom set Σ. Now suppose that the set of models of Σ defines the values of the abstract of G_0 in terms of the values of the abstracts of $G_1, \ldots, G_p, \ldots$. Then also the set of models of Σ^* defines the values of S_0 in terms of the values of $S_1, \ldots, S_p, \ldots$. Then by Theorem 1 (see also (6) of the proof), there exists a conjunction A of sentences of Σ^* and a formula $B(S_1, \ldots, S_p, \ldots; z_1, \ldots, z_{n_0})$ containing no free individual variables other than arbitrarily chosen

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$z_1, \ldots, z_{n_0}$ and no further parameters other than $S_1, \ldots, S_p, \ldots$, such that

(7) $\vdash_= A \supset (z_1)\ldots(z_{n_0})(S_0(z_1, \ldots, z_{n_0}) \equiv B(S_1, \ldots, S_p, \ldots; z_1, \ldots, z_{n_0}))$.

We assume that any individual variable in the consequent of (7) is distinct from any individual variable in any G_r. This can be done by relettering, if necessary. Let $A' \supset (z_1)\ldots(z_{n_0})(D' \equiv B')$ be the result of substituting into (7) for each $S_r(y_1, \ldots, y_{n_r})$ the formula $G_r(y_1, \ldots, y_{n_r})$ or $G_r(y_1, \ldots, y_{n_r-1}) = y_{n_r}$ respectively, depending on whether G_r is a formula or a term. Since collision of variables is avoided,

(8) $\vdash_= A' \supset (z_1)\ldots(z_{n_0})(D' \equiv B')$.

Any sentence in A which is a sentence of Σ appears unchanged in A′. Any sentence E_r in A appears in A′ as a provable sentence $(x_1)\ldots(x_{n_r})(G_r \equiv G_r)$ or $(x_1)\ldots(x_{n_r})(G_r = x_{n_r} .\equiv. G_r = x_{n_r})$. Removal of these provable sentences from A′ in (8) yields another theorem of *PC1=*, whose antecedent is a conjunction of sentences of Σ. Hence $\Sigma \vdash_= (z_1)\ldots(z_{n_0})(D' \equiv B')$. Now D′ is either $G_0(z_1, \ldots, z_{n_0})$ or $G_0(z_1, \ldots, z_{n_0-1}) = z_{n_0}$, depending on whether G_0 is a formula or a term. Also B′ is a formula composed of $G_1, \ldots, G_p, \ldots$ and containing no free individual variables other than $z_1, \ldots, z_{n_0}$. Hence relettering of $(z_1)\ldots(z_{n_0})(D' \equiv B')$, if necessary, will yield a sentence $(x_1)\ldots(x_{n_0})(G_0 \equiv B'')$ or $(x_1)\ldots(x_{n_0})(G_0 = x_{n_0} .\equiv B'')$ respectively, where B″ contains no free individual variables other than $x_1, \ldots, x_{n_0}$ and where B″ is also a formula composed of $G_1, \ldots, G_p, \ldots$. For *PC1=* this establishes the theorem. In the case of *PC1*, any term G_r is an individual parameter. A proof can then be carried out similarly except that S_r and E_r are used only for those G_r which are not terms but formulas. Q.E.D.

In the case of *PC1* the restriction that at least one G_r, $r \geq 1$, be a formula is needed since otherwise there would be no formula composed of $G_1, \ldots, G_p, \ldots$. The further restriction for *PC1* that G_0 be a formula is needed since explicit definability of a term in our sense requires the presence of the identity sign. This restriction is not serious since for purposes of defining a term an identity sign, if not present, can be introduced by definition (see [6]).

A further extension is possible to those second-order systems whose extralogical axioms are of the form $(\exists T^1)\ldots(\exists T^m)A$, where $m \geq 0$ and A is a first-order sentence, provided that the expression to be defined and the expressions in terms of which it is to be defined are themselves of first order.[10] We shall therefore consider a system *PC2* (*PC2=*) of second-order predicate calculus without (with) a symbol for identity, obtained from *PC1* (*PC1=*) by adding predicate (predicate and function) quantifiers together with the more basic axioms governing their use.

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[10] This extension is due to J. R. Büchi.

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THEOREM 3. *Let* Σ *be a set of sentences of PC2*$=$ *(of PC2) of the form* $(\exists T^1)\ldots(\exists T^m)A$, *where* $m \geq 0$ *and where* A *is a sentence of PC1*$=$ *(of PC1). Let* $G_0, G_1, \ldots, G_p, \ldots$ *be arbitrary formulas or terms of PC1*$=$ *(of PC1), such that, in the case of PC1,* G_0 *and at least one other* G_r *are formulas. Then if the set of models of* Σ *defines the values of the abstract of* G_0 *in terms of the values of the abstracts of* $G_1, \ldots, G_p, \ldots$, *then also the system obtained by adding* Σ *as axiom set to PC2*$=$ *(to PC2) defines* G_0 *explicitly in terms of* $G_1, \ldots, G_p, \ldots$.

PROOF. We consider first the case of *PC2*$=$. We assume that no variable which occurs bound in a sentence of Σ occurs in any other sentence of Σ or in $G_0, G_1, \ldots, G_p, \ldots$. This can be done by relettering, if necessary. Then let Σ' be the set of first-order sentences A for which there is a sentence $(\exists T^1)\ldots(\exists T^m)A$ in Σ, $m \geq 0$. Now suppose that the set of models of Σ defines the values of the abstract of G_0 in terms of the values of the abstracts of $G_1, \ldots, G_p, \ldots$. Then so does the set of models of Σ', since any model of Σ' is also a model of Σ. Then by Theorem 2, we can deduce from Σ' in *PC1*$=$ a sentence $(x_1)\ldots(x_n)(G_0 \equiv B)$ or $(x_1)\ldots(x_n)(G_0 = x_n \,.\!\equiv B)$ respectively, where B is a suitable definiens of G_0 in terms of $G_1, \ldots, G_p, \ldots$. Then, by our choice of letters for the bound variables, this sentence can also be deduced from Σ in *PC2*$=$. For *PC2* the proof is similar. Q.E.D.

For each of the three Theorems, the proof contains some instructions for finding a definiens B. It also provides some information on how the structure of B is related to that of the sentences in Σ. For example, in the case of Theorem 1, if all sentences in Σ are in prenex normal form and without existential quantifiers or function parameters, then B is also a formula of this kind. In general, however, the relationship is more complex.

Three generalizations of the notion of definability will now be discussed briefly. Σ and $G_0, G_1, \ldots, G_p, \ldots$ shall be as in Theorem 2 or 3.

(i) Let G_0 be a formula or term respectively and suppose that the set of models of Σ defines the values of $\hat{x}_1\ldots\hat{x}_{n_0}(G_0 \,.\, F)$ or $\hat{x}_1\ldots\hat{x}_{n_0}\hat{y}(G_0 = y \,.\, F)$ respectively in terms of the values of the abstracts of $G_1, \ldots, G_p, \ldots$, where F is a formula whose free individual variables $x_1, \ldots, x_{n_0}$ are those of G_0. Then the set of models of Σ may be said to define the values of the abstract of G_0 in terms of the abstracts of $G_1, \ldots, G_p, \ldots$ *within the range determined by* F. Then by Theorem 2 or 3 there is a formula B composed of $G_1, \ldots, G_p, \ldots$ and containing no free individual variables other than $x_1, \ldots, x_{n_0}$ or $x_1, \ldots, x_{n_0}, y$ respectively, such that from Σ one can deduce $(x_1)\ldots(x_{n_0})(G_0 \,.\, F \,.\!\equiv B)$ or $(x_1)\ldots(x_{n_0})(y)(G_0 = y \,.\, F \,.\!\equiv B)$ respectively, and therefore $(x_1)\ldots(x_{n_0})(F \supset\!. \, G_0 \equiv B)$ or $(x_1)\ldots(x_{n_0})(F \supset\!. \,(y)(G_0 = y \,.\!\equiv B)$ respectively.

(ii) Suppose for some positive integers $m_1, \ldots, m_k, \ldots$ all models of Σ of cardinality other than $m_1, \ldots, m_k, \ldots$ which agree in the underlying domain and in the interpretation of $G_1, \ldots, G_p, \ldots$ also agree in the

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interpretation of G_0. Then there is a formula B composed of $G_1, \ldots, G_p, \ldots$ and containing no free variables other than $x_1, \ldots, x_{n_0}$ and there are formulas $E_{m_1}, \ldots, E_{m_l}$ expressing respectively that there are exactly m_i individuals, $1 \leq i \leq l$, such that from Σ one can deduce $\neg E_{m_1} \centerdot \ldots \centerdot \neg E_{m_l} \centerdot \supset \centerdot (x_1) \ldots (x_{n_0})(G_0 \equiv B)$ or $\neg E_{m_1} \centerdot \ldots \centerdot \neg E_{m_l} \centerdot \supset \centerdot (x_1) \ldots (x_{n_0})(G_0 = x_{n_0} \centerdot \equiv B)$ respectively.

(iii) In the case of modeltheoretic definability the set of models of Σ yields the values of the abstract of G_0 as a single-valued function of (the underlying domain and) the values of the abstracts of $G_1, \ldots, G_p, \ldots$. A generalization of this is the notion of a k-valued function, where k is a preassigned cardinal number. More precisely, one may wish to consider those cases where, for some k, there are in any collection of models of Σ which all agree in the underlying domain and in the interpretation of the abstracts of $G_1, \ldots, G_p, \ldots$ at most k different interpretations of the abstract of G_0. Then by the usual completeness or finiteness type of argument (e.g. near end of § 4), this already holds for some finite k. The prooftheoretic notion of definability can be generalized in a related fashion. Again by a finiteness argument, no generality is lost by assuming that k is finite. Perhaps the simplest cases are then those where there are k formulas $B_1, \ldots, B_k$ composed of $G_1, \ldots, G_p, \ldots$ and containing no free individual variables other than $x_1, \ldots, x_{n_0}$ such that from Σ one can deduce $(x_1) \ldots (x_{n_0})(G_0 \equiv B_1) \vee \ldots \vee (x_1) \ldots (x_{n_0})(G_0 \equiv B_k)$ or $(x_1) \ldots (x_{n_0})(G_0 = x_{n_0} \centerdot \equiv B_1) \vee \ldots \vee (x_1) \ldots (x_{n_0})(G_0 = x_{n_0} \centerdot \equiv B_k)$ respectively. Whether, or under what conditions, a modeltheoretic notion and a prooftheoretic notion (or a variant of it) of the kind just described coincide is for any $k > 1$ an unsolved problem.

4. A theorem on the hierarchy of second-order formulas. In the hierarchy of second-order formulas, which largely remains to be investigated, there is a lowest class consisting of all formulas which are equivalent to a first-order formula and there are two immediate successor classes consisting of all formulas which do not belong to the lowest class and which are equivalent to a formula $(\exists T^1) \ldots (\exists T^n)A$ or $(T^1) \ldots (T^n)A$ respectively, A being a first-order formula. We shall now prove a theorem concerning the relationship between these three classes. The parameters of $(\exists T^1) \ldots (\exists T^n)A$ or of $(T^1) \ldots (T^n)A$, where A is a first-order formula, shall be those parameters of A which are distinct from $T^1, \ldots, T^n$. As before, a parameter of a formula of a given set shall be *isolated* (with respect to the formulas of the set) if and only if it is not the parameter of any other formula of the set.

THEOREM 4. *Let* $(\exists T_1^1) \ldots (\exists T_1^{n_1})A_1, \ldots, (\exists T_k^1) \ldots (\exists T_k^{n_k})A_k, (T_{k+1}^1) \ldots (T_{k+1}^{n_{k+1}})A_{k+1}, \ldots, (T_l^1) \ldots (T_l^{n_l})A_l$ *be formulas of PC2= (of PC2), such that in the case of PC2 each formula contains at least one nonisolated predicate parameter, and such that* $A_1, \ldots, A_l$ *are formulas of PC1=(of PC1). Suppose*

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that

$$(9)\quad (\exists T_1^1)\ldots(\exists T_1^{n_1})A_1 . \ldots . (\exists T_k^1)\ldots(\exists T_k^{n_k})A_k . \supset . (T_{k+1}^1)\ldots(T_{k+1}^{n_{k+1}})A_{k+1} \vee \ldots \vee (T_l^1)\ldots(T_l^{n_l})A_l$$

is valid. Then there are formulas $B_1, \ldots, B_l$ *of* $PC1{=}$ (*of* $PC1$) *whose parameters are nonisolated parameters of* $(\exists T_1^1)\ldots(\exists T_1^{n_1})A_1, \ldots, (T_l^1)\ldots(T_l^{n_l})A_l$ *respectively such that* $B_1 . \ldots . B_k . \supset . B_{k+1} \vee \ldots \vee B_l$, $(\exists T_i^1)\ldots(\exists T_i^{n_i})A_i \supset B_i$ $(1 \leq i \leq k)$ *and* $B_i \supset (T_i^1)\ldots(T_i^{n_i})A_i$ $(k < i \leq l)$ *are all valid.*

PROOF. We consider first the case of $PC2{=}$. Assume that no T_i^r occurs in any A_j, $j \neq i$. This can be done by relettering, if necessary. Now suppose that (9) is valid. Then also $A_1 . \ldots . A_k . \supset . A_{k+1} \vee \ldots \vee A_l$ is valid and therefore provable in $PC1{=}$. Then there are formulas $B_1, \ldots, B_l$ as described in Lemma 3. Now let S be any parameter of any B_i. Then by Lemma 3, S is a parameter of A_i. Moreover, by Lemma 3, S is a parameter of some A_p, $p \neq i$. Therefore S is distinct from $T_i^1, \ldots, T_i^{n_i}$. Hence S is also a parameter of $(\exists T_i^1)\ldots(\exists T_i^{n_i})A_i$ or $(T_i^1)\ldots(T_i^{n_i})A_i$ respectively. Similarly, S is a parameter of $(\exists T_p^1)\ldots(\exists T_p^{n_p})A_p$ or $(T_p^1)\ldots(T_p^{n_p})A_p$. It follows that S is a nonisolated parameter of $(\exists T_i^1)\ldots(\exists T_i^{n_i})A_i$ or $(T_i^1)\ldots(T_i^{n_i})A_i$. Finally, since any parameter of B_i is distinct from $T_i^1, \ldots, T_i^{n_i}$, it follows that $(\exists T_i^1)\ldots(\exists T_i^{n_i})A_i \supset B_i$ is valid, when $1 \leq i \leq k$ and therefore $\vdash_{=} A_i \supset B_i$, and that $B_i \supset (T_i^1)\ldots(T_i^{n_i})A_i$ is valid, when $k < i \leq l$ and therefore $\vdash_{=} B_i \supset A_i$. Hence $B_1, \ldots, B_l$ satisfy the Theorem. For $PC2$ the proof is similar, using Lemma 2 instead. Q.E.D.

In the special case where $(\exists T_1^1)\ldots(\exists T_1^{n_1})A_1 \equiv (T_2^1)\ldots(T_2^{n_2})A_2$ is valid, there is, by Theorem 4, a first-order formula B such that $(\exists T_1^1)\ldots(\exists T_1^{n_1})A_1 \equiv B$ is valid. In this respect therefore the lowest class is related to its two successor classes in the hierarchy of second-order formulas as the class of recursive sets is related to its two successor classes in the Kleene hierarchy ([10] § 57 Theorem V). In the Kleene hierarchy one often uses this relationship to prove that a set is not recursively enumerable by showing that its complement is recursively enumerable but not recursive. Similarly, Theorem 4 may be of use in proving that a formula cannot be expressed in the form $(\exists T^1)\ldots(\exists T^n)A$ (in the form $(T^1)\ldots(T^n)A$) by showing that its negation can be expressed in this form but not in the form B.

A difference between the two hierarchies is the following. According to Theorem 4, using A_2' in place of $\neg A_2$ and B_2' in place of $\neg B_2$, if $(\exists T_1^1)\ldots(\exists T_1^{n_1})A_1$ and $(\exists T_2^1)\ldots(\exists T_2^{n_2})A_2'$ are "disjoint" in the sense that $(\exists T_1^1)\ldots(\exists T_1^{n_1})A_1 \supset \neg(\exists T_2^1)\ldots(\exists T_2^{n_2})A_2'$ is valid, then they can always be "separated" by two first-order formulas B_1 and B_2' in the sense that $(\exists T_1^1)\ldots(\exists T_1^{n_1})A_1 \supset B_1$, $(\exists T_2^1)\ldots(\exists T_2^{n_2})A_2' \supset B_2'$, and $B_1 \supset \neg B_2'$ are all valid. In contrast, two disjoint recursively enumerable sets cannot always be separated by two recursive sets (see [10] pp. 308–312).

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The case where $n_i = 0$ for each $i > k$ is also of interest. The theorem then shows that instead of first forming conjunctions of formulas $(\exists T^1) \ldots (\exists T^n)A$ and then deriving first-order consequences one may interchange the two processes, first deriving first-order consequences and only then forming conjunctions.

Lemmas 2 and 3 and Theorem 4 can all be generalized, so to speak, to an infinite number of formulas in the antecedent or consequent. For example, to generalize Lemma 3, consider two arbitrary sets Φ and Ψ of formulas of $PC_{I=}$, such that any model satisfying all formulas of Φ satisfies at least one formula of Ψ. Let Ψ' be the set of negations of formulas of Ψ. Then $\Phi \cup \Psi'$ has no models. Then by the completeness of $PC_{I=}$ (see e.g. [7]) there are formulas $A_1, \ldots, A_k$ of Φ and $\neg A_{k+1}, \ldots, \neg A_l$ of Ψ' such that $\vdash_= \neg (A_1 \cdot \ldots \cdot A_k \cdot \neg A_{k+1} \cdot \ldots \cdot \neg A_l)$. Then Lemma 3 becomes applicable.

When thus generalized, and when restricted to sentences, the above consequences of Theorem 4 can be interpreted as relating the class $\mathbf{PC}_\Delta$ and the class $\mathbf{AC}$ of model sets (see Tarski [13]). The above separability, when generalized, can be expressed as follows: If $K_1 \in \mathbf{PC}_\Delta$ and $K_2 \in \mathbf{PC}_\Delta$ and if K_1 and K_2 are disjoint (and belong to the same similarity class), then there is an $L_1 \in \mathbf{AC}$ and an $L_2 \in \mathbf{AC}$ such that $K_1 \subseteq L_1$, $K_2 \subseteq L_2$, and L_1 and L_2 are disjoint. In particular, if $K_1 \in \mathbf{PC}_\Delta$ and $K_2 \in \mathbf{PC}_\Delta$ and if K_1 and K_2 are complements of each other (relative to their similarity class), then $K_1 \in \mathbf{AC}$ and $K_2 \in \mathbf{AC}$. The above interchangeability, when generalized, can be expressed as follows: For each set X of models, let $\mathbf{C}(X) = \bigcap (Y | X \subseteq Y \in \mathbf{AC})$. Let $\{K_\alpha\}_{\alpha<\mu}$ be any class of sets $K_\alpha \in \mathbf{PC}_\Delta$. Then $\mathbf{C}(\bigcap_{\alpha<\mu} (K_\alpha)) = \bigcap_{\alpha<\mu} (\mathbf{C}(K_\alpha))$.

5. Possible groupings of axioms. Consider a formal system with three extralogical constants R, S, and T, such that the system defines R explicitly in terms of S, the definiens being $B(S; x_1, \ldots, x_n)$. Then the extralogical axioms of the system can be rewritten so that they fall into two groups, where the only axiom of one group is $(x_1) \ldots (x_n)(R(x_1, \ldots, x_n) \equiv B(S; x_1, \ldots, x_n))$, which contains only R and S as parameters, while the axioms of the other group are obtained by substituting $B(S; x_1, \ldots, x_n)$ for $R(x_1, \ldots, x_n)$ into the original extralogical axioms and hence contain only S and T as parameters. This is an example where it is possible after suitable axiomatization to divide the extralogical axioms into one or more groups, each axiom in a certain group containing only certain of the extralogical constants. It raises the general question under what conditions a certain desired grouping of the extralogical axioms of a system is possible.

Given a set Π_i of parameters, a sentence shall be called a Π_i-*sentence* if and only if every parameter of it belongs to Π_i. Given a class $\{\Pi_1, \ldots, \Pi_i, \ldots\}$ of sets of parameters, a formal system shall be called $\{\Pi_1, \ldots,$

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$\Pi_i, \ldots\}$-*axiomatizable* if and only if the system can be axiomatized in a manner such that each extralogical axiom is a Π_j-sentence for some $\Pi_j \in \{\Pi_1, \ldots, \Pi_i, \ldots\}$. We shall now examine under what conditions for a given class $\{\Pi_1, \ldots, \Pi_i, \ldots\}$ a first-order system is $\{\Pi_1, \ldots, \Pi_i, \ldots\}$-axiomatizable. We shall restrict ourselves throughout to systems where each of the extralogical axioms contains no parameters that are variables. The underlying logic will be either *PCI=* or *PCI*.

We shall first consider the case where the given set of extralogical axioms is finite. Let A be a conjunction of the given extralogical axioms and let Π be the set of parameters of A. Let C be any extralogical axiom of any axiomatization of the system and let Π_j be the set of parameters of C. Suppose first that the underlying logic is *PCI*. If A and C have no predicate parameter in common, then by Lemma 1 either $\vdash$ C, so that C can be dropped from the set of extralogical axioms, or $\vdash \neg$A, so that the system is $\{\Pi_i\}$-axiomatizable for any set Π_i containing at least one predicate parameter. If A and C have a predicate parameter in common, then by Lemma 1 there is a sentence B such that $\vdash \mathrm{A} \supset \mathrm{B}$ and $\vdash \mathrm{B} \supset \mathrm{C}$, and therefore such that C can be replaced by B as extralogical axiom, and such that the parameters of B are in $\Pi \cap \Pi_j$. It follows that in the case of *PCI* we can restrict ourselves to sets Π_i which contain only parameters of A and which, moreover, contain at least one predicate parameter. Similarly Lemma 3 allows us in the case of *PCI=* to consider only sets Π_i of parameters of A. [11]

THEOREM 5. *Let* A *be any sentence of PCI= (of PCI) containing no predicate or function variables, and let* $\Pi_1, \ldots, \Pi_k$ *be sets of parameters of* A, *where* $k \geq 1$. *In the case of PCI suppose that each* Π_i *contains at least one predicate parameter. For each* i, *let* $\mathrm{S}_i^1, \ldots, \mathrm{S}_i^{n_i}$ *be those parameters of* A *(if any) which are not in* Π_i, *let* $\mathrm{T}_i^1, \ldots, \mathrm{T}_i^{n_i}$ *be distinct variables not occurring in* A *which are of the same kind as the constants* $\mathrm{S}_i^1, \ldots, \mathrm{S}_i^{n_i}$ *respectively, and let* A_i *be the result of substituting* $\mathrm{T}_i^1, \ldots, \mathrm{T}_i^{n_i}$ *for* $\mathrm{S}_i^1, \ldots, \mathrm{S}_i^{n_i}$ *respectively in* A. *Then the system obtained by adding* A *as axiom to PCI= (to PCI) is* $\{\Pi_1, \ldots, \Pi_k\}$-*axiomatizable if and only if the conditional*

$$(10) \quad (\exists \mathrm{T}_1^1)\ldots(\exists \mathrm{T}_1^{n_1})\mathrm{A}_1 \,.\, \ldots \,.\, (\exists \mathrm{T}_k^1)\ldots(\exists \mathrm{T}_k^{n_k})\mathrm{A}_k \,.\!\supset \mathrm{A}$$

is valid. [12]

PROOF. We consider first the case of *PCI=*. Let A, $\Pi_1, \ldots, \Pi_k$ and, for each i, A_i and $\mathrm{T}_i^1, \ldots, \mathrm{T}_i^{n_i}$ be as described. Suppose first that the system obtained by adding A as axiom to *PCI=* is $\{\Pi_1, \ldots, \Pi_k\}$-axiomatizable. Then by the completeness of *PCI=* there is a conjunction B such that

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[11] There also would result no loss of generality from the further restriction that no Π_i consists of all the parameters of A and that no Π_i is included in any other Π_j.

[12] The converse conditional $\mathrm{A} \supset. (\exists \mathrm{T}_1^1)\ldots(\exists \mathrm{T}_1^{n_1})\mathrm{A}_1 \,.\, \ldots \,.\, (\exists \mathrm{T}_k^1)\ldots(\exists \mathrm{T}_k^{n_k})\mathrm{A}_k$ is always valid. Hence the system is $\{\Pi_1, \ldots, \Pi_k\}$-axiomatizable if and only if the set of models of A is the intersection of the sets of models of $(\exists \mathrm{T}_i^1)\ldots(\exists \mathrm{T}_i^{n_i})\mathrm{A}_i$, $1 \leq i \leq k$.

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$\vdash_= B \supset A$ and $\vdash_= A \supset B$ and such that each term D of B is a Π_i-sentence for some i, $1 \leq i \leq k$.[13] Then for each term D of B, the conditional $(\exists T_i^1)\ldots(\exists T_i^{n_i})A_i \supset D$ is valid for some i, $1 \leq i \leq k$. Hence (10) is valid.

Conversely, suppose that (10) is valid. Then by Theorem 4 there are sentences $B_1, \ldots, B_k$ of *PCI*= such that any parameter of B_i, $1 \leq i \leq k$, is a parameter of $(\exists T_i^1)\ldots(\exists T_i^{n_i})A_i$ and therefore in Π_i, and such that $B_1 . \ldots . B_k .\supset A$ and $(\exists T_i^1)\ldots(\exists T_i^{n_i})A_i \supset B_i$, $1 \leq i \leq k$, are all valid. Then $\vdash_= B_1 . \ldots . B_k .\supset A$ and $\vdash_= A \supset B_i$, $1 \leq i \leq k$. Hence $B_1, \ldots, B_k$ added as extralogical axioms to *PCI*= provide a $\{\Pi_1, \ldots, \Pi_k\}$-axiomatization of the system obtained by adding A to *PCI*=.

For *PCI* the proof is similar. Since each Π_i contains at least one predicate parameter of A, there is in (10) at least one nonisolated predicate parameter in each $(\exists T_i^1)\ldots(\exists T_i^{n_i})A_i$. Q.E.D.

We shall now briefly consider the case where the given set of extralogical axioms is infinite. Let $\{\Pi_1, \ldots, \Pi_i, \ldots\}$ be the given class of sets of parameters. Let $C_1, \ldots, C_r, \ldots$ be the conjunctions formed from the given axioms. For each C_r and each Π_i, replace as in Theorem 5 all parameters of C_r which are not in Π_i by existentially quantified variables, and let $Q_i(C_r)$ be the resulting second-order sentence. Any parameter in $Q_i(C_r)$ belongs to Π_i. Then the given system is $\{\Pi_1, \ldots, \Pi_i, \ldots\}$-axiomatizable if and only if for each C_r there are sentences $Q_{i_1}(C_{r_1}), \ldots, Q_{i_m}(C_{r_m})$ such that $Q_{i_1}(C_{r_1}) . \ldots . Q_{i_m}(C_{r_m}) .\supset C_r$ is valid.

A class $\{\Pi_1, \ldots, \Pi_k\}$ of sets of parameters shall be called *finer* than a class $\{\Pi_1', \ldots, \Pi_l'\}$ if and only if each Π_i in $\{\Pi_1, \ldots, \Pi_k\}$ is a subset of some Π_j' in $\{\Pi_1', \ldots, \Pi_l'\}$ but not conversely. The following example shows that for some systems there is no class $\{\Pi_1, \ldots, \Pi_k\}$ such that the system is $\{\Pi_1, \ldots, \Pi_k\}$-axiomatizable which is finer than all other such classes. Let $C = C(f, g)$ be a conjunction of axioms for lattice theory, postulating the commutative, associative, and absorptive laws for the "meet" f and the "join" g (see L2 to L4, § 24, of [2]). Let $D = D(f, R)$ be the axiom $(x)(y)(R(x, y) \equiv. f(x, y) = y)$, so that $R(x, y)$ may be interpreted as $x \geq y$. Then the system obtained by adding C . D to *PCI*= is $\{\{f, g\}, \{f, R\}\}$-axiomatizable. The same system is also $\{\{f, g\}, \{g, R\}\}$-axiomatizable, since D can be replaced by $(x)(y)(R(x, y) \equiv. g(x, y) = x)$. Yet the system is not $\{\{f, g\}, \{R\}\}$-axiomatizable. To show this it is sufficient by Theorem 5 to find an interpretation for which the conditional $(\exists R)(C . D) . (\exists f)(\exists g)(C . D) .\supset. C . D$ is false. Now consider any lattice and interpret f as "meet", g as "join", and $R(x, y)$ as $x \leq y$. Then $(\exists R)(C . D)$ is true (letting R be $\geq$) and $(\exists f)(\exists g)(C . D)$ is true (letting f be the "join" and g the "meet"), yet D and therefore C . D is false.

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[13] This part of the proof alone shows that there is a finite $\{\Pi_1, \ldots, \Pi_k\}$-axiomatization.

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