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Nondeterministic Extensions of Untyped λ-Calculus*

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The main concern of this paper is the interplay between functionality and nondeterminism. We ask whether the analysis of parallelism in terms of sequentiality and nondeterminism, which is usual in the algebraic treatment of concurrency, remains correct in the presence of functional application and abstraction. We argue in favour of a distinction between nondeterminism and parallelism, due to the conjunctive nature of the former in contrast to the disjunctive character of the latter. This is the basis of our analysis of the operational and denotational semantics of the nondeterministic λ-calculus, which is the classical calculus plus a choice operator, and of our election of bounded indeterminacy as the semantic counterpart of conjunctive nondeterminism. This leads to operational semantics based on the idea of *must* preorder, coming from the classical theory of solvability and from the theory of process algebras. To characterize this relation, we build a model using the inverse limit construction over nondeterministic algebras, and we prove it fully abstract using a generalization of Böhm trees. We further prove conservativity theorems for the equational theory of the model and for other theories related to nondeterministic λ-calculus with respect to classical λ-theories.

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1. INTRODUCTION

Multivalued functions are commonly regarded as natural candidates to model the input–output behaviour of nondeterministic computing devices. Motivations for developing a theory of computable multifunctions usually come from the study of parallelism. This is clear from the early studies by Plotkin and Smyth leading to the theory of powerdomains [48, 53] and from applications of this theory to defining resumption semantics for languages equipped with a parallel evaluation operator [29].

This also led to the consideration of nondeterministic extensions of functional languages, as in [5, 6], which concern Plotkin's PCF [49], a typed lambda calculus with constants and fixed point combinator, extended by adding an erratic choice operator. A first study of pure λ-calculus with erratic choice is [52]. Recent proposals of both typed and untyped lambda calculi extended with choice and parallel operators are [14, 12, 15, 33, 44, 45, 51].

As a matter of fact the study of nondeterminism in a functional setting is further motivated by research on higher order process algebras, which are equipped with functional application and abstraction, as pursued by Thomsen's CHOCS [54] and, at least implicity, by Milner's π-calculus. Further results in this direction are reported in [27, 28].

Nondeterminism and parallelism are often related concepts. As an example, all approaches to concurrency theory which are based on process algebras (see, e.g., [40, 24, 31, 7]) rely on the analysis of parallelism in terms of sequentiality and nondeterminism, which is at the basis of interleaving semantics. The question we ask here is whether such analysis remains valid in the context of applicative languages. As a first step we need to clarify the concept of nondeterminism in the framework of rewriting systems, which usually are the basis for the operational semantics of these languages.

In automata theory nondeterminism arises when more than one transition from the same state is possible. If terms are considered as states, then even Church–Rosser systems, which admit a functional interpretation, could be nondeterministic. This sounds odd, and some authors [41] call this phenomenon "indeterminacy" to keep it distinct from nondeterminism.

If a term rewriting system as well as a combinatory rewriting system [34] is not Church–Rosser, however, we cannot conclude that it is nondeterministic. Consider for

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example the type-free λ-calculus with surjective pairing: it still admits a functional interpretation, but it is not Church–Rosser (see [34, Theorem 1.2.10 of Chap. III]).

Term rewriting systems are usually seen as the asymmetric versions of equational calculi, which can be recovered by taking the symmetric closure of the reduction relation: let us call such an equational theory the derived theory of the system. But, as is argued in [37], the rewriting paradigm is more general than the equational one, in that there are nontrivial rewriting systems whose derived equational theories are inconsistent. We believe this inconsistency to be a typical feature of nondeterminism, which intuitively consists in the possibility of getting different values starting from a single datum: equating values which are actually different naturally leads to inconsistency.

It seems too strong, however, to assume the inconsistency of the derived theory as part of the definition of nondeterminism (as pointed out to us by readers of the draft version of this paper, including an anonymous referee). A more intentional criterion can be based on the idea of redexes, instead of terms, as states: in this perspective a term is a complex object and there is a distributed control acting on it, one control unit per redex occurrence in the term. If we accept this idea, a system is nondeterministic if it provides a term which is a redex for two different reduction rules. It can easily be checked that this does not imply the inconsistency of the derived equation theory.

Two kinds of operators, considered in the literature, are related to nondeterminism and parallelism. The first one ($\oplus$) is erratic choice (see, e.g., [5, 45]), coming with the following reduction rule:

$$M \oplus N \to M \qquad \text{and} \qquad M \oplus N \to N.$$

The second one is an operator we shall denote by $\|$ whose behaviour is described by the rule (see, e.g. [15]):

$$M \to M \quad \text{implies} \quad M \parallel N \to M' \parallel N$$
$$\text{and} \quad N \parallel M \to N \parallel M'.$$

If we apply the previous definition of nondeterminism, only the choice operator is truly nondeterministic, since $M \parallel N$ is not a redex. It is useful however, to compare these two operators with respect to their intended semantics and to termination.

In the case of nonconfluent and nondeterministic rewriting system one can say that a term converges if there exists a terminating reduction leading to a value: this is a *may* convergency predicate. Or one can ask for termination of all possible reductions out of a term to say that it is convergent: this is a *must* convergency predicate. The intended meaning of the parallel operator is to be convergent if either of its arguments converges; that is (see [15]),

$$(M \parallel N){\downarrow} \Leftrightarrow M{\downarrow} \vee N{\downarrow},$$

where $M{\downarrow}$ means that M converges. To reduce parallelism to nondeterminism we have to elect the may convergency criterion so that the choice operator satisfies the same property as $\|$ with respect to the convergency predicate. In this case we speak of *disjunctive* nondeterminism.

What is the operational idea behind this? If we meet a choice $M \oplus N$, we have to start parallel computations of M and N, deferring the choice between them until either from M or from N a value is reached. Of course it would not be effective to decide in advance which term would converge, but it seems harmless to suppose that we will be able to detect that we have actually reached a value on one side or the other. But it is possible to delay the choice in all cases? The answer is surely not. Let M be such that $M \xrightarrow{*} \lambda x.x\Delta\Delta$, and N such that $N \xrightarrow{*} \lambda x.x\mathbf{I}$, where $\mathbf{I} \equiv \lambda x.x$ and $\Delta \equiv \lambda x.xx$; then the value of $M \oplus N$ will be $\lambda x.x\Delta\Delta$ or $\lambda x.x\mathbf{I}$ depending on the reduction strategy we used, that is, say, on the relative speed with which M and N converge. But in the reduction of $(M \oplus N)\mathbf{I}$ a choice must occur before the reduct of M or on N is applied to $\mathbf{I}$. On the other hand, we expect that $(M \oplus N)\mathbf{I}$ evaluates to $(\lambda x.x\mathbf{I})\mathbf{I}$, that is to $\mathbf{I}$, and not to $(\lambda x.x\Delta\Delta)\mathbf{I}$ and hence to $\Delta\Delta$ (that is, to a typical divergent combinator). Indeed, when dealing with disjunctive nondeterminism, we ask for an effective method of picking up a value if any, as happens with deterministic simulations of nondeterministic Turing machines. Now there is no effective way to guarantee that the right choice will be the actual one since the convergency predicate is not recursive. We conclude this informal argument by observing that no effective normalizing strategy exists when we allow choice operators.

We also remark that choosing convergency as the termination predicate leads to unbounded indeterminacy. In fact, with disjunctive nondeterminism even a finitely branching reduction calculus has terms which reduce to infinitely many values. For example, consider the term $H\mathbf{0}$ of a λ-calculus extended with $\oplus$, where

$$H \equiv \mathbf{\Theta}(\lambda hx.x \oplus h(\mathbf{Succ}\ x)).$$

$\mathbf{\Theta} \equiv (\lambda xy.y(xxy))(\lambda xy.y(xxy))$ is a fixpoint combinator and $\mathbf{0}$, $\mathbf{Succ}$ represent zero an successor in some numerical system (see [9]) whose numerals have normal form; then $H\mathbf{0}$ reduces to an infinite countable set of head normal forms. $H\mathbf{0}$ is of course may-convergent, but clearly must-divergent, having the infinite reduction

$$H\mathbf{0} \xrightarrow{*} \mathbf{0} \oplus H(\mathbf{Succ}\ \mathbf{0}) \to H(\mathbf{Succ}\ \mathbf{0})$$
$$\xrightarrow{*} \mathbf{Succ}\ \mathbf{0} \oplus H(\mathbf{Succ}(\mathbf{Succ}\ \mathbf{0})) \to \cdots.$$

Observe that the example shows an instance of head reduction, extending the classical λ-calculus notion in an evident way (see [9] and 2.3 below for formal definitions).

Consequently, we feel that the answer to our question about reducibility of parallelism to nondeterminism inside the framework of applicative languages has to be negative. If not for parallelism, what is nondeterminism useful for? It seems unreasonable to think of erratic choice as an operator which actually "does" something for us. Our idea is to look at $\oplus$ as a kind of declarative operator. Its meaning is that we are willing to abstract away from some unpredictable events (as relative reduction speeds above) in the computation. We will accept whichever of two (or more) alternatives will be the actual one, provided that a minimum is assured to us: that in any case the computation will eventually give a meaningful value, that is, it will converge.

From our discussion it should be clear that the semantics of erratic choice should not be *angelic*, but rather *demonic* (or *conjunctive*) non determinism, so that the following holds:

$$(M \oplus N)\downarrow \Leftrightarrow M\downarrow \wedge N\downarrow.$$

This is clearly achieved by taking must convergency as the termination predicate.

The main concern of this paper is the analysis of the interplay between functionality and nondeterminism. To this aim, we introduce the nondeterministic λ-calculus to be the classical calculus extended with a choice operator. The mentioned conjunctive nature of nondeterminism provides the basis for the analysis of the operational and denotational semantics of our calculus, for which bounded indeterminacy suffices. This has technical advantages, too; from [4] we know that unbounded countable nondeterminism leads to loss of continuous least fix point, fully abstract semantics using standard domain theory (but see [46] for continuous models of unbounded indeterminacy in Lehman's categorical setting). We are able to build a model which is fully abstract with respect to operational semantics based on our notion of must-solvability and head reduction; moreover the theory of this model turns out to be a conservative extension of the maximally consistent λ-theory equating all unsolvable terms, called $\mathscr{H}^*$ in [9].

Our results improve on [5, 6], in that we do not commit ourselves with additive semantics of functional application, facing the more general problem of finding a model of non deterministic type free λ-calculus. For purely syntactical aspects we get inspiration from [52], which to our knowledge remains the first work on non deterministic extensions of type free λ-calculus. Our main improvement on this work is that we base our study entirely on the reduction relation; we do not postulate any equation which could be considered reasonable among terms of our calculus: we justify Sharma's theories on the basis of our operational, algebraic and denotational semantics, giving a clean perspective to look at the relation between the non deterministic theories and the classical ones, culminating in the conservativity theorems.

The paper is organized as follows. In Section 2 we define the term syntax and the reduction relation. In Section 3 we introduce the operational semantics of the calculus, based on a generalization of the classical notion of solvability. To study this semantics we consider nondeterministic unfolding trees which generalize classical Böhm trees and show that the preorder on which the operational semantics is based is included in the preorder induced by a suitable tree inclusion (semiseparability theorem). Such a notion is further investigated in Section 4 from a denotational point of view, leading us to the introduction of a notion of model of nondeterministic type free λ-calculus and (Section 5) to a full abstraction theorem equating orders induced by the operational and tree preorders to the semantical order. We finally investigate in Section 6 nondeterministic theories with respect to their conservativity property on λ-theories. To avoid detour through complicated details, more technical proofs are collected in Section 7.

2. THE NONDETERMINISTIC λ-CALCULUS

Let us introduce the syntax of our calculus and the relative notion of reduction.

DEFINITION 2.1. Let Var be any denumerable set of variables. The set $\Lambda_\oplus$ of the terms of the nondeterministic λ-calculus is the least set s.t.

(i) $x \in \Lambda_\oplus$ for all $x \in \mathrm{Var}$;
(ii) $M, N \in \Lambda_\oplus \Rightarrow (MN) \in \Lambda_\oplus$;
(iii) $M \in \Lambda_\oplus$, $x \in \mathrm{Var} \Rightarrow (\lambda x.M) \in \Lambda_\oplus$;
(iv) $M, N \in \Lambda_\oplus \Rightarrow M \oplus N \in \Lambda_\oplus$.

The set of closed terms will be denoted by $\Lambda^0_\oplus$.

The terms are considered up to renaming of bound variables; clearly the set Λ of classical λ-terms is a subset of $\Lambda_\oplus$.

DEFINITION 2.2. Let $M[N/X]$ denote the simultaneous substitution of each occurrence of x by an occurrence of N in M, up to renaming of bound variables in M to avoid variable clashes; then $\rightarrow \subseteq \Lambda_\oplus \times \Lambda_\oplus$ is the least relation such that

(β) $(\lambda x.M)N \rightarrow M[N/x]$;
(μ) $N \rightarrow N' \Rightarrow MN \rightarrow MN'$;
(ν) $M \rightarrow M' \Rightarrow MN \rightarrow M'N$;
(ξ) $M \rightarrow M' \Rightarrow \lambda x.M \rightarrow \lambda x.M'$;
($\oplus 1$) $M \oplus N \rightarrow M$, $M \oplus N \rightarrow N$;
($\oplus 2$) $M \rightarrow M' \Rightarrow M \oplus N \rightarrow M' \oplus N$, $N \oplus M \rightarrow N \oplus M'$.

As usual $\xrightarrow{*}$ is the reflexive and transitive closure of $\rightarrow$.

Notation. We will write $\to_\beta$ when a one step β-contraction occurs; similarly we write $\to_\oplus$ when only one $\oplus$-contraction occurs. $\xrightarrow{*}_\beta$ and $\xrightarrow{*}_\oplus$ are their reflexive and transitive closures, respectively.

The following definition extends to the present calculus the notions of head reduction and of head normal form.

DEFINITION 2.3.

(i) A term M is a *head normal form* iff $M \equiv \lambda x_1 \cdots x_m.xM_1 \cdots M_n$, where $n, m \geqslant 0$; HNF is the set of head normal forms.

(ii) If $M \equiv \lambda x_1 \cdots x_n.\underline{(\lambda y.P)\,Q}M_1 \cdots M_n$ or $M \equiv \lambda x_1 \cdots x_n.\underline{(P \oplus Q)}\,M_1 \cdots M_n$, then the underlined subterm is called the *head redex* of M.

(iii) If R is the head redex of M, then $M \to_h N$ iff N results from M by contracting R (*head reduction*).

(iv) $M \to_i N \Leftrightarrow M \to N \wedge M \not\to_h N$ (*internal reduction*).

(v) N is a *head normal form* of M iff $N \in$ HNF and $M \xrightarrow{*} N$; N is a *principal head normal form* of M iff N is a head normal form of M and $M \xrightarrow{*}_h N$.

Inspired by the extensional equivalence of Morris [43] and its analogue due to Wadsworth [55] and by the idea of testing given by De Nicola and Hennessy [18] for process algebras, we define the following notions (see also [33]):

DEFINITION 2.4. For $M \in \Lambda_\oplus$ define

(i) $M\downarrow_{\text{must}} \Leftrightarrow M$ has no infinite head reduction,

(ii) $M \sqsubseteq_{\text{must}} N \Leftrightarrow \forall C[\,].\ C[M]\downarrow_{\text{must}} \Rightarrow C[N]\downarrow_{\text{must}}$,

(iii) $M \simeq_{\text{must}} N \Leftrightarrow M \sqsubseteq_{\text{must}} N \sqsubseteq_{\text{must}} M$.

We abbreviate $M\downarrow_{\text{must}}$ as $M\downarrow$ and we write $M\uparrow$ to mean not $M\downarrow$.

It should be noted that $\sqsubseteq_{\text{must}}$ is by definition a precongruence, so that $\simeq_{\text{must}}$ is a congruence relation. As for classical λ-calculus one says that M is solvable if and only if it has a head normal form, we say that M is *must-solvable* if and only if $M\downarrow$. Must-solvability implies that each convergent term has a finite set of *principal* head normal forms.

LEMMA 2.5. *If* $M\downarrow$, *then the set* $\{N \mid N \in \text{HNF} \wedge M \xrightarrow{*}_h N\}$ *is finite and nonempty.*

Proof. By Definition 2.4 and the König lemma, the tree of head reductions being finitary. ∎

This is not a limitation in the expressive power of a theory based on must-solvability, because being must-solvable does not imply having a finite set of normal forms: consider, e.g., the term $H'\mathbf{0}$, where

$$H' \equiv \mathbf{\Theta}(\lambda hx.x \oplus \mathbf{Succ}(hx)),$$

which has the same set of normal forms as $H\mathbf{0}$ in the Introduction. But $H'\mathbf{0}$ is must-solvable, while $H\mathbf{0}$ is not.

The reason to focus on head reductions in classical λ-calculus is that any normalizing reduction can be factored into two parts, an exhaustive head reduction followed by internal reductions:

$$M \xrightarrow{*}_h L \xrightarrow{*}_i N.$$

This remains true even if we do not reach a normal form: i.e., any finite reduction can be rearranged in this way (just the head reduction may not be exhaustive, terminating in a term which is not in HNF), and, moreover, the internal part will have recursively the same structure on subterms. This still holds in our calculus, and can be obtained as a consequence of the standardization theorem.

To define the notion of standard reduction some machinery is needed, basically to keep track of the redexes and of the order in which they are contracted.

Let $\Lambda'_\oplus$ be defined as $\Lambda_\oplus$, adding the clauses:

(a) $M, N \in \Lambda'_\oplus,\ x \in Var \Rightarrow (\lambda_0 x.M)\,N \in \Lambda'_\oplus$;

(b) $M, N \in \Lambda'_\oplus \Rightarrow M \oplus_0 N \in \Lambda'_\oplus$.

Let $\to'$ be the obvious extension of $\to$ to $\Lambda'_\oplus$. Finally, let $|\cdot|: \Lambda'_\oplus \to \Lambda_\oplus$ be the map erasing all indexes 0 decorating λ or $\oplus$. Assume that $\sigma': M' \xrightarrow{*}{}' N'$ is a finite $\to'$ reduction sequence; then it has the form

$$\sigma': M' \equiv M'_0 \xrightarrow{R'_0} M'_1 \xrightarrow{R'_1} M'_2 \cdots \xrightarrow{R'_{n-1}} M'_n \equiv N',$$

where $M'_i \xrightarrow{R'_i} M'_{i+1}$ means that the redex occurrence R'_i has been contracted in the reduction step. The erasure map can be extended to σ' as follows:

$$|\sigma'| : |M'| \equiv |M'_0| \xrightarrow{|R'_0|} |M'_1| \xrightarrow{|R'_2|} |M'_2| \cdots$$
$$\xrightarrow{|R'_{n-1}|} |M'_n| \equiv |N'|.$$

Vice versa, given $\sigma: M \xrightarrow{*} N$ and $M' \in \Lambda'_\oplus$ such that $M \equiv |M'|$, there exist a (unique) σ' and a (unique) N' such that $\sigma': M' \xrightarrow{*}{}' N'$, $\sigma = |\sigma'|$ and $N \equiv |N'|$.

Given $M \in \Lambda_\oplus$, we write $R \in M$ to mean that R is a redex occurrence in M; similarly, if $\mathcal{F} = \{R_1, ..., R_n\}$, then $\mathcal{F} \subseteq M$ means that $R_i \in M$, for all $1 \leqslant i \leqslant n$.

DEFINITION 2.6.

(i) Let $\mathcal{F} \subseteq M$ be a set of redex occurrences in $M \in \Lambda_\oplus$; then $(M, \mathcal{F}) \in \Lambda'_\oplus$ is obtained from M by decorating with 0 the main λ or $\oplus$ of each redex occurrence $R \in \mathcal{F}$.

(ii) Let $M, N \in \Lambda_\oplus$, $\mathcal{F} \subseteq M$, and $\sigma: M \xrightarrow{*} N$ be any finite reduction sequence; then the set of *residuals* of $\mathcal{F}$ modulo σ, written $\mathcal{F}/\sigma$, is the unique set of redex occurrences $\mathcal{F}' \subseteq N$ such that $\sigma': (M, \mathcal{F}) \xrightarrow{*}{}' (N, \mathcal{F}')$ and $\sigma = |\sigma'|$.

Let $M \in \Lambda_{\oplus}$ and $R, R' \in M$ be two redex occurrences in M; then we write $R \leqslant_M R'$ if and only if either R' is a subterm of R or the main operator of R is to the left of the main operator of R'. If σ has the form

$$\sigma\colon M \equiv M_0 \xrightarrow{R_0} M_1 \xrightarrow{R_1} M_2 \cdots \xrightarrow{R_{n-1}} M_n \equiv N$$

and $0 \leqslant i < j \leqslant n$ then

$$\sigma_{i,j}\colon M_i \xrightarrow{R_i} \cdots \xrightarrow{R_{j-1}} M_j.$$

Finally, let $\mathscr{F}_i = \{R' \in M_i \mid R' \leqslant_{M_i} R_i\} \subseteq M_i$ for $0 \leqslant i \leqslant n$.

DEFINITION 2.7. A reduction $\sigma\colon M_0 \xrightarrow{R_0} \cdots \xrightarrow{R_{n-1}} M_n$ is *standard* if and only if

$$\forall i, j \leqslant n-1 \,.\, i < j \Rightarrow R_j \notin \mathscr{F}_i/\sigma_{i,j}.$$

We write $M \xrightarrow{*}_s N$ if there is a standard reduction $\sigma\colon M \xrightarrow{*} N$.

In words, a reduction is standard if and only if a residual of a redex which either includes a contracted redex or whose main operator occurs to the left of the main operator of a contracted redex, is never contracted thereafter.

THEOREM 2.8. (Standardization Theorem). $\forall M, N \in \Lambda_{\oplus}$. $M \xrightarrow{*} N \Rightarrow M \rightarrow_s N$.

Proof. The proof is strightforward adaption of Klop's proof in [34] for classical λ-calculus (see also [9, Sect. 12.3]).

Even if the proof of standardization easily adapts from the analogous proof for the classical λ-calculus, Theorem 2.8 cannot be obtained as a corollary of its classical counterpart. This is due to the fact that we cannot delay all choices after β-contractions or viceversa, as it is shown in the following figure:

$$\begin{tikzcd}
(\lambda x.xx)(P \oplus Q) \arrow[r, "\beta"'] \arrow[d, "\oplus"'] & (P \oplus Q)(P \oplus Q) \arrow[d, "\oplus", "*"'] & ((\lambda x.P) \oplus Q)R \arrow[d, "\oplus"'] & ?? \\
(\lambda x.xx)P & ?? \quad PQ & (\lambda x.P)R \arrow[r, "\beta"'] & P[R/x]
\end{tikzcd}$$

The diagrams do not commute. Indeed a β-contraction can multiply ⊕-redexes (left); hence the number of possible choices, and ⊕-contractions may create new β-redexes (right).

COROLLARY 2.9. *For any reduction* $\sigma\colon M \xrightarrow{*} N$, *there exists a reduction* $\sigma'\colon M \xrightarrow{*} N$ *such that, for some* $L \in \Lambda_{\oplus}$,

$$\sigma' = M \xrightarrow{*}_h L \xrightarrow{*}_i N.$$

Proof. Immediate by the standardization theorem. ∎

3. ALGEBRAIC SEMANTICS

In order to introduce the reader to the study of the functional behaviour of terms, let us take into account the terms

$$M \equiv \lambda x.x(P \oplus Q) \quad \text{and} \quad N \equiv (\lambda x.xP) \oplus (\lambda x.xQ).$$

If we assume the *immediate deterministic structure* of a term to be the set of terms obtained from it by eliminating the choices in all possible ways, we have

$$\forall P, Q \in \Lambda. \{T \in \Lambda \mid M \xrightarrow{*}_{\oplus} T\} = \{T \in \Lambda \mid N \xrightarrow{*}_{\oplus} T\};$$

i.e., M and N have the same immediate deterministic structure. However, let us apply M and N to the term $\Delta \equiv \lambda y.yy$; as shown in Fig. 1, where the head reductions of $M\Delta$ and $N\Delta$ are illustrated, $M\Delta$ can produce PP or QQ or PQ or QP, while $N\Delta$ can produce PP or QQ only.
Furthermore, if we take

$$P = \lambda x.x\mathbf{U}_3^3\Delta \quad \text{and} \quad Q \equiv \lambda xy.yy,$$

where $\mathbf{U}_3^3 \equiv \lambda x_1x_2x_3.x_3$, a simple (head) reduction shows that

$$PP, QQ, QP \xrightarrow{*} \Delta, \quad \text{while} \quad PQ \xrightarrow{*} \Delta\Delta \rightarrow \Delta\Delta \rightarrow \cdots,$$

that is, $M\Delta\uparrow$ while $N\Delta\downarrow$; hence $N \not\sqsubseteq_{\text{must}} M$.

It comes out that the immediate deterministic structure of terms is not sufficient to characterize their applicative behaviour. Hence, we are led to consider not only the possibility of eliminating choices from terms, but also *how* this can be performed, thus arguing that M and N must have a different "choice structure." Informally speaking, the

$$\begin{tikzcd}
(\lambda x.x(P \oplus Q))(\lambda y.yy) \arrow[r, "\beta"'] & (\lambda y.yy)(P \oplus Q) \arrow[r, "\beta"'] & (P \oplus Q)(P \oplus Q) \arrow[d, "\oplus"', "*"] \arrow[dd, "\oplus", "*"'] \arrow[dr, "\oplus", "*"'] \arrow[drr, "\oplus", "*"'] & & \\
& (\lambda y.yy)P \arrow[r, "\beta"'] & PP & PQ & QP \\
(\lambda x.xP \oplus \lambda x.xQ)(\lambda y.yy) \arrow[ur, "\oplus\beta"] \arrow[r, "\oplus\beta"'] & (\lambda y.yy)Q \arrow[r, "\beta"'] & QQ & &
\end{tikzcd}$$

FIG. 1. Different behaviours of $M\Delta$ and $N\Delta$.

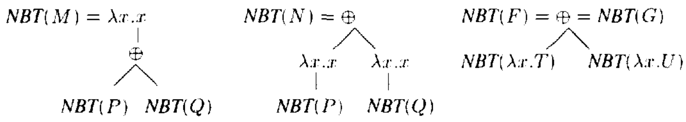

FIG. 2. Examples of nondeterministic Böhm trees

choice structure is responsible for different behaviors of two (or more) terms which have the same immediate deterministic structure.

Indeed, the distinction in behaviours of M and N is due to the fact that the choice $P \oplus Q$ in M is duplicated by Δ in $M\Delta$, thus giving the opportunity to obtaining every possible combination of P and Q. This is not possible in N, where duplication comes only after the choice. It seems then that some difference can be recognized in M and N in the relative position of abstracted variables with respect of the choice operator.

Aiming at a formalization of the notion of choice structure of a term, it has to be noted that it cannot be immediately deduced from the syntactical tree of the term itself. Consider, as a counterexample, the syntactically different terms

$$F \equiv \lambda x.(T \oplus U) \quad \text{and} \quad G \equiv (\lambda x.T) \oplus (\lambda x.U);$$

to obtain an intuition that F and G have indeed the same functional behaviour, we observe that, for any H, FH and GH can produce by head reduction exactly the same objects, namely those yielded from $T[H/X]$ and $U[H/x]$.

We will give a formalization of the notion of choice structure of a term by means of *nondeterministic Böhm trees*, which extend Böhm trees of classical λ-calculus to the nondeterministic case; in such a perspective, turning back to the previous examples, we will obtain different trees for the terms M and N and the same tree for F and G, as shown in Fig. 2.

3.1. Nondeterministic Böhm Trees

We will now introduce a kind of unfolding trees, generalizing the notion of Böhm trees of the classical λ-calculus (for this notion see [9]), and a suitable notion of approximation, to be considered as a cut of the tree. This will give us what is called the algebraic semantics of the calculus, after, e.g., [3, 22].

The difficulty here is that, since the Church–Rosser property does not hold, we cannot consider our trees as a representation of the directed set of "approximated" reducts of a term: instead, we have to take into account all possible reductions, without making choices, but simply representing them in the tree.

Definition 3.1. We define, by mutual induction, two sets $\mathscr{S}_0$ and $\mathscr{S}_1$ of *approximants*:

(i) $\Omega \in \mathscr{S}_0$, where Ω is a new constant representing divergency;

(ii) $M \in \mathscr{S}_0 - \{\Omega\} \Rightarrow \lambda x.M \in \mathscr{S}_0$;

(iii) $\{\Omega\} \in \mathscr{S}_1$;

(iv) $M_1, ..., M_n \in \mathscr{S}_0 - \{\Omega\} \Rightarrow \{M_1, ..., M_n\} \in \mathscr{S}_1$ for $n > 0$;

(v) $\mathscr{M}_1, ..., \mathscr{M}_m \in \mathscr{S}_1$, $x \in \mathrm{Var} \Rightarrow x\mathscr{M}_1 \cdots \mathscr{M}_m \in \mathscr{S}_0$ for $m \geqslant 0$.

The idea behind this notation is that an approximation is a finite set in $\mathscr{S}_1$ of elements in $\mathscr{S}_0$; This derives from the fact that, if $M\downarrow$, then there is a finite set of head normal forms we derive from it; this settles the first level, and we go on revursively with the bodies of these terms.

We consider in clause (v) the application of a variable rather than a set since we have in mind head normal forms: a set is a sum, and any term with a sum in head position is not a head normal form. Similar remarks apply to clause (ii).

For every approximant, there is a term in $\Lambda_\oplus \Omega$ (i.e., $\Lambda_\oplus$ extended with the constant Ω) which corresponds to it in natural way; we define, by mutual induction, $\vartheta_0 : \mathscr{S}_0 \to \Lambda_\oplus \Omega$ and $\vartheta_1 : \mathscr{S}_1 \to \Lambda_\oplus \Omega$:

$$\vartheta_0(\Omega) = \vartheta_1(\{\Omega\}) = \Omega, \qquad \vartheta_0(\lambda x.M) = \lambda x.\vartheta_0(M),$$

$$\vartheta_0(x\mathscr{M}_1 \cdots \mathscr{M}_m) = x\vartheta_1(\mathscr{M}_1) \cdots \vartheta_1(\mathscr{M}_m);$$

$$\vartheta_1(\{M_1, ..., M_n\}) = \vartheta_0(M_1) \oplus \cdots \oplus \vartheta_0(M_n).$$

To simplify the notation, we will identify $\mathscr{M} \in \mathscr{S}_1$ with $\vartheta_1(\mathscr{M}) \in \Lambda_\oplus \Omega$.

Definition 3.2. Let $\mathscr{M} \in \mathscr{S}_1$; we define

$$NBT(\mathscr{M}) = NBT_1(\mathscr{M}),$$

where

$$NBT_0(\Omega) = \Omega$$

and

$$NBT_0(\lambda\vec{x}.\xi\mathscr{M}_1 \cdots \mathscr{M}_m) = \begin{array}{c} \lambda\vec{x}.\xi \\ \diagup \quad \diagdown \\ NBT_1(\mathscr{M}_1) \ \cdots \ NBT_1(\mathscr{M}_m) \end{array}$$

$$NBT_1(\{M_1, ..., M_n\}) = \begin{array}{c} \oplus \\ \diagup \quad \diagdown \\ NBT_0(M_1) \cdots NBT_0(M_n) \end{array}$$

Remark 3.3. Note that the symbol $\oplus$ is here just an object to be used as a label in some of the nodes of the tree, and it must not be confused with the same symbol as it has been introduced into the syntax of $\Lambda_{\oplus}$. Note also that the order of the subnodes of a node labelled by $\lambda\vec{x}.\xi$ is relevant, as well as multiple occurrences of the same subtree; this is not the case for sons of nodes labelled by $\oplus$.

*NBT*s may be seen as infinite $\mathscr{S}_1$-terms, that is as elements of the completion of $\mathscr{S}_1$ under the order induced on $\mathscr{S}_1$ by the relation freely generated by $\Omega \preccurlyeq M$, for all M, on $\mathscr{S}_0$ (see for similar constructions involving the extension of algebraic semantics to powerdomains [1, 26]). More precisely, they are the limits of those directed subsets of $\mathscr{S}_1$ generated by the following family of maps:

DEFINITION 3.4. For each natural number k, we define a map $\omega^k: \Lambda_{\oplus} \to \mathscr{S}_1$ by:

(i) $\omega^0(M) = \{\Omega\}$,

(ii) $\omega^{k+1}(M)$

$$= \begin{cases} \{\Omega\} & \text{if } M\uparrow; \\ \{\lambda\vec{x}.\xi\omega^k(M_1)\cdots\omega^k(M_m) \mid \lambda\vec{x}.\xi M_1\cdots M_m \\ \quad \text{is a principal } hnf \text{ of } M\} & \text{otherwise.} \end{cases}$$

Furthermore, we denote

$$M^{[k]} = \vartheta_1 \circ \omega^k(M).$$

Remark 3.5. We note that $M^{[k]}$ is always a β-Ω-normal form in the sense of [9, Chap. 14]; indeed, in any term of this shape, no $\oplus$-redex can create new β-Ω-redexes. We denote by $N^{\Omega}_{\oplus}$ the set of such terms.

EXAMPLE 3.6. Let $M \equiv \lambda x.x(yx) \oplus \lambda x.xz \in \Lambda_{\oplus}$. Then we have

$$\omega^1(M) = \{\lambda x.x\omega^0(yx), \lambda x.\omega^0(z)\} = \{\lambda x.x\{\Omega\}\},$$
$$\omega^2(M) = \{\lambda x.x\omega^1(yx), \lambda x.x\omega^1(z)\}$$
$$= \{\lambda x.x\{y\omega^0(x)\}, \lambda x.x\{z\}\},$$
$$= \{\lambda x.x\{y\{\Omega\}\}, \lambda x.x\{z\}\},$$
$$M^{[1]} = \lambda x.x\Omega,$$
$$M^{[2]} = \lambda x.x(y\Omega) \oplus \lambda x.xz.$$

The trees $NBT(\omega^1(M))$ and $NBT(\omega^2(M))$ are shown in Fig. 3.

To compare two terms, that is their trees, simple inclusion does not suffice even in the classical λ-calculus. What we need is a generalization of the relation "$\sqsubseteq$" in [9], or, equivalently, of $<^{\prime}_{k}$ in [32]. This will be achieved in several steps.

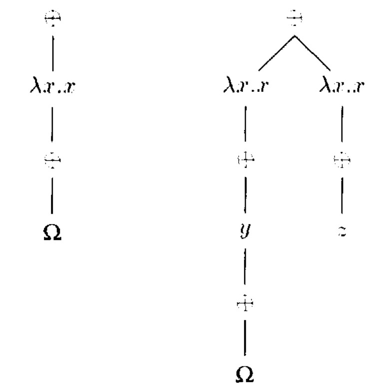

FIG. 3. Nondeterministic Böhm trees.

We first recall the notion of *equivalence* ($\sim$) for head normal forms [13, 9] (called *similarity* in [32]), adapting it to the set $\mathscr{S}_0$.

DEFINITION 3.7.

(i) Let $M \equiv \lambda x_1 \cdots x_n.\xi.\mathscr{M}_1 \cdots \mathscr{M}_m \in \mathscr{S}_0$, then head$(M) = \xi$, ord$(M) = n$, deg$(M) = m$;

(ii) for $M, N \in \mathscr{S}_0 - \{\Omega\}$,

$$M \sim N \Leftrightarrow \text{deg}(M) - \text{ord}(M) = \text{deg}(N) - \text{ord}(N)$$
$$\text{and} \qquad \text{head}(M) \equiv \text{head}(N).$$

The role of this relation in discriminating head normal forms is illustrated in [13, 16] for the classical λ-calculus: in our setting, sets of head normal forms must be compared, and we will consider two sets $\mathscr{M}$ and $\mathscr{N}$ equivalent iff

$$\forall X \in \mathscr{M}\ \exists Y \in \mathscr{N}.X \sim Y \quad \text{and} \quad \forall Y \in \mathscr{N}\ \exists X \in \mathscr{M}.X \sim Y.$$

EXAMPLE 3.8. Let $M \equiv \lambda x_1x_2x_3.x_1x_3(x_2x_3) \oplus \lambda x_1x_2.x_1x_2x_2 \oplus \lambda x_1x_2x_3.x_1x_3x_2$ and $N \equiv \lambda x_1x_2.x_1x_1 \oplus \lambda x_1x_2x_3.x_1x_2$.

We have

$$\omega^2(M) = \{\lambda x_1x_2x_3.x_1\{x_3\}\{x_2\{\mathbf{\Omega}\}\}, \lambda x_1x_2.x_1\{x_2\}\{x_2\},$$
$$\lambda x_1x_2x_3.x_1\{x_3\}\{x_2\}\};$$
$$\omega^2(N) = \{\lambda x_1x_2.x_1\{x_1\}, \lambda x_1x_2x_3.x_1\{x_2\}\};$$
$$M^{[2]} = \lambda x_1x_2x_3.x_1x_3(x_2\mathbf{\Omega}) \oplus \lambda x_1x_2.x_1x_2x_2$$
$$\oplus \lambda x_1x_2x_3.x_1x_3x_2;$$
$$N^{[2]} = N.$$

It comes out that there exists a $\sim$-equivalence class of $\omega^2(M) \cup \omega^2(N)$ which does not contain any element of

$\omega^2(M)$. In this case, we can immediately find a context such that $C[M]$ converges while $C[N]$ does not. Indeed, take $C[\ ] \equiv [\ ](\lambda a_1 a_2 a_3.a_1)\, x_2 x_3 x_4(\Delta\Delta)$; then

$$\omega^2(C[M^{[2]}]) = \{x_2\{x_4\}\{\mathbf{\Omega}\}, x_3\{\mathbf{\Omega}\}\},$$
$$\omega^2(C[N^{[2]}]) = \{\mathbf{\Omega}\}.$$

The previous example shows that nonequivalent sets of head normal forms can be easily separated by looking at the first level of their trees only. In the general case, however, while comparing sets of head normal forms, it is necessary to analyze the internal structure of their elements, which in turn encapsulate other sets of terms.

LEMMA 3.9. *Define the binary relation* $\sim_\eta \subseteq \mathcal{S}_0 \times \mathcal{S}_0$ *by*

$$M \sim_\eta \lambda x.M\{x\} \qquad \textit{if} \quad x \notin FV(M) \qquad \textit{and} \qquad M \doteq \mathbf{\Omega}.$$

Let $\mathcal{M} \in \mathcal{S}_1$ *and*

$$\eta(\mathcal{M}) = \{\bar{M} \in \mathcal{S}_0 \mid \exists M \in \mathcal{M}.\bar{M} \sim_\eta M \wedge \mathrm{ord}(\bar{M}) = \max\{\mathrm{ord}(M') \mid M' \in \mathcal{M}\}\}.$$

Then $\sim_\eta \subseteq \sim$ *and* $\eta(\mathcal{M}) \in \mathcal{S}_1$ *is a finite set of* $\sim$*-equivalent objects of* $\mathcal{S}_0$.

Proof. Immediate by Definition 3.7 from the fact that for each M and $n \geqslant \mathrm{ord}(M)$ there exists (up to α-equivalence) exactly one $\bar{M}$ s.t. $\bar{M} \sim_\eta M$ and $\mathrm{ord}(\bar{M}) = n$. ▌

Remark 3.10. The relation $\sim_\eta$ is clearly reminiscent of η conversion even if here we do not define it as a congruence. It turns out that, using this relation, we can obtain from $\sim$-equivalent objects terms having the same order and degree. This is useful as a technical tool to simplify the construction we are carrying out.

The following definitions enable us to select those parts of $\mathcal{M}$ and $\mathcal{N}$ which must be collated to establish the relative functional behaviour of $\mathcal{M}$ and $\mathcal{N}$.

DEFINITION 3.11. Let $\mathcal{M}, \mathcal{N} \in \mathcal{S}_1$; then the set $\mathsf{Pair}(\mathcal{M}, \mathcal{N})$ is defined as follows. Suppose that

$$\mathcal{M} = \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_k \quad \text{and} \quad \mathcal{N} = \mathcal{N}_1 \cup \cdots \cup \mathcal{N}_h,$$

where the $\mathcal{M}_i$ and $\mathcal{N}_j$ are the $\sim$-equivalence classes of the elements of $\mathcal{M}$ and $\mathcal{N}$, respectively. Furthermore, suppose that, for some i and j, $\mathcal{M}_i \cup \mathcal{N}_j$ consists of $\sim$-equivalent objects. Take

$$\eta(\mathcal{M}_i \cup \mathcal{N}_j) = \{\bar{M}^1, ..., \bar{M}^m, \bar{N}^1, ..., \bar{N}^n\},$$

where the $\bar{M}^u$, $\bar{N}^v$ come from $\mathcal{M}_i$ and $\mathcal{N}_j$, respectively; these objects will have the shapes

$$\bar{M}^u \equiv \lambda\vec{y}.x\mathcal{M}_1^u \cdots \mathcal{M}_l^u, \text{ for } 1 \leqslant u \leqslant m \qquad \text{and}$$
$$\bar{N}^v \equiv \lambda\vec{y}.x\mathcal{N}_1^v \cdots \mathcal{N}_l^v, \quad \text{for } 1 \leqslant v \leqslant n.$$

Then for each $1 \leqslant p \leqslant l$,

$$\langle \{\mathcal{M}_p^1, ..., \mathcal{M}_p^m\}, \{\mathcal{N}_p^1, ..., \mathcal{N}_p^n\} \rangle \in \mathsf{Pair}(\mathcal{M}, \mathcal{N}).$$

Note that in the previous definition we indeed compare the following "matrices" having the same number of columns and different numbers of rows:

$$\begin{pmatrix} \mathcal{M}_1^1 & \cdots & \mathcal{M}_l^1 \\ \mathcal{M}_1^m & \cdots & \mathcal{M}_l^m \end{pmatrix} \begin{pmatrix} \mathcal{N}_1^1 & \cdots & \mathcal{N}_l^1 \\ \mathcal{N}_1^n & \cdots & \mathcal{N}_l^n \end{pmatrix}.$$

We take as elements of $\mathsf{Pair}(\mathcal{M}, \mathcal{N})$ the pairs of corresponding columns. Henceforth we shall use U, V, possibly decorated by indices and primes, to denote these columns.

Given $\mathcal{M}, \mathcal{N} \in \mathcal{S}_1$, $\mathsf{Pair}(\mathcal{M}, \mathcal{N})$ selects the subterms to be compared during the first step of the analysis of the internal structure of $\mathcal{M}$ and $\mathcal{N}$. As in [32], this notation has to be extended to each level of the tree.

DEFINITION 3.12. Given $\mathcal{M}, \mathcal{N} \in \mathcal{S}_1$, *define* $\mathsf{Pair}_k(\mathcal{M}, \mathcal{N})$, for each natural number k, as follows:

(i) $\mathsf{Pair}_1(\mathcal{M}, \mathcal{N}) = \mathsf{Pair}(\mathcal{M}, \mathcal{N})$.

(ii) Let $\mathsf{Pair}(\mathcal{M}, \mathcal{N}) = \{\langle \mathsf{U}_1, \mathsf{V}_1 \rangle, ..., \langle \mathsf{U}_l, \mathsf{V}_l \rangle\}$ and

$$\mathcal{M}'_i = \bigcup \{\mathcal{M}' \mid \mathcal{M}' \in \mathsf{U}_i\}, \quad \mathcal{N}'_i = \bigcup \{\mathcal{N}' \mid \mathcal{N}' \in \mathsf{V}_i\},$$

where $1 \leqslant i \leqslant l$; then

$$\mathsf{Pair}_{k+1}(\mathcal{M}, \mathcal{N}) = \{\langle \mathsf{A}, \mathsf{B} \rangle \mid \exists i \leqslant l.\ \langle \mathsf{A}, \mathsf{B} \rangle \in \mathsf{Pair}_k(\mathcal{M}'_i, \mathcal{N}'_i)\}.$$

Remark 3.13. Note that in (ii), for $1 \leqslant i \leqslant l$, U_i and V_i are finite nonempty sets of objects in $\mathcal{S}_1$, hence families of finite nonempty sets of objects in $\mathcal{S}_0$; it follows that their unions $\mathcal{M}'_i$ and $\mathcal{N}'_i$ are again elements of $\mathcal{S}_1$.

EXAMPLE 3.14. Let $M \equiv x(y(a \oplus b)) \oplus xcd$ and $N \equiv x(ya \oplus yb) \oplus xcd$ (see Fig. 4); we have

$$\omega^3(M) = \{x\{y\{a, b\}\}, x\{c\}\{d\}\}$$
$$\omega^3(N) = \{x\{y\{a\}, y\{b\}\}, x\{c\}\{d\}\}.$$

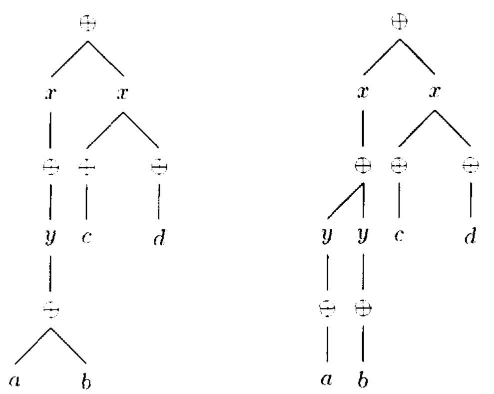

FIG. 4. Respectively the trees of $\omega^3(M)$ and $\omega^3(N)$.

From this compute:

$$\mathsf{Pair}_1(\omega^3(M), \omega^3(N)) = \{\langle\{\{y\{a,b\}\}\}, \{\{y\{a\}, y\{b\}\}\}\rangle, \langle\{\{c\}\}, \{\{c\}\}\rangle, \langle\{\{d\}\}, \{\{d\}\}\rangle\};$$

$$\begin{aligned}\mathsf{Pair}_2(\omega^3(M), \omega^3(N)) &= \mathsf{Pair}_1(\{y\{a,b\}\}, \{y\{a\}, y\{b\}\}) \cup \mathsf{Pair}_1(\{c\}, \{c\}) \cup \mathsf{Pair}_1(\{d\}, \{d\}) \\ &= \{\langle\{\{a,b\}\}, \{\{a\}, \{b\}\}\rangle\}.\end{aligned}$$

We are now ready to introduce the ordering relation $\leqslant$ over trees.

DEFINITION 3.15. Given $\mathcal{M}, \mathcal{N} \in \mathcal{S}_1$ we define a relation $\leqslant_k$ for each k, by:

$$\mathcal{M} \leqslant_1 \mathcal{N} \Leftrightarrow \mathcal{M} = \{\mathbf{\Omega}\} \vee \forall N \in \mathcal{N}\ \exists M \in \mathcal{M}.\ M \sim N,$$

$$\mathcal{M} \leqslant_{k+1} \mathcal{N} \Leftrightarrow \mathcal{M} \leqslant_k \mathcal{N} \wedge \forall \langle \mathsf{U}, \mathsf{V}\rangle \in \mathsf{Pair}_k(\mathcal{M}, \mathcal{N}).\, \mathsf{U} \sqsubseteq^{\#} \mathsf{V},$$

where

$$\mathsf{U} \sqsubseteq^{\#} \mathsf{V} \Leftrightarrow \forall \mathcal{N}_j \in \mathsf{V}\ \exists \mathcal{M}_i \in \mathsf{U}.\, \mathcal{M}_i \leqslant_1 \mathcal{N}_j.$$

From this we can define

$$\mathcal{M} \leqslant \mathcal{N} \Leftrightarrow \forall k.\, \mathcal{M} \leqslant_k \mathcal{N}.$$

Finally, for any $M, N \in \Lambda_{\oplus}$,

$$M \leqslant_k N \Leftrightarrow \omega^k(M) \leqslant_k \omega^k(N),$$
$$M \leqslant N \Leftrightarrow \forall k.\, M \leqslant_k N.$$

EXAMPLE 3.16. Consider the M and N of Example 3.14. Now the following relations hold:

$$M \leqslant_i N \leqslant_i M \quad \text{for } i = 0, 1, 2.$$

Indeed

$$\omega^1(M) = \{x\{\mathbf{\Omega}\}, x\{\mathbf{\Omega}\}\{\mathbf{\Omega}\}\} = \omega^1(N)$$

and

$$\omega^2(M) = \{x\{y\{\mathbf{\Omega}\}\}, x\{c\}\{d\}\} = \omega^2(N).$$

But, looking at Example 3.14, we see that in $\mathsf{Pair}_2(\omega^3(M), \omega^3(N))$ we have a (unique) pair formed by a $\mathsf{U} = \{\{a, b\}\}$ and by a $\mathsf{V} = \{\{a\}, \{b\}\}$, such that

$$\mathsf{U} \sqsubseteq^{\#} \mathsf{V} \quad \text{and} \quad V \not\sqsubseteq^{\#} \mathsf{U},$$

hence $M \leqslant_3 N \not\leqslant_3 M$.

3.2. Discriminability and Tree Inclusion

Let $M, N \in \Lambda_{\oplus}$. In order to give a sufficient condition for the existence of a discriminating context $C[\ ]$ such that $C[M]\downarrow \wedge C[N]\uparrow$, we will distinguish three cases:

$M \not\leqslant_1 N$. In such case we are considering terms encoding nonequivalent sets of head normal forms: the existence of a discriminating context will be guaranteed by the application of the separability technique of classical λ-calculus [16], as exemplified in 3.8.

$M \not\leqslant_2 N$ and the previous case does not hold. In such case we have terms which encode equivalent sets of head normal forms, but the existence of $\langle \mathsf{U}, \mathsf{V}\rangle \in \mathsf{Pair}_2(\omega^h(M), \omega^h(N))$ (for some $h \geqslant 2$) such that $\mathsf{U} \not\sqsubseteq^{\#} \mathsf{V}$ implies that different sets (i.e., choices) appearing at the second level in $NBT(M)$ and $NBT(N)$ are indeed "arguments" of the same variables, namely those labelling the first level of the trees: substituting suitable combinators for these variables, we will succeed in discriminating among sets. This is the case where we can say that M and N exhibit a different *choice structure* at the second level of their NBTs.

$M \not\leqslant_k N$, where $k > 2$ and the previous cases do not hold. In such case M and N have an uniform structure "up to" the kth level of their NBTs, where they eventually encode different information. Using such uniformity we shall prove that this information can be lifted up until one of the previous cases hold.

DEFINITION 3.17. A context is a *head context* iff it is of the form

$$C[\ ] \equiv (\lambda x_1 \cdots x_n.[\ ])\, X_1 \cdots X_n U_1 \cdots U_m.$$

LEMMA 3.18. *Let $\mathcal{M} = \omega^k(M)$ for $k \geqslant 1$ and $M\downarrow$; assume that $\mathcal{M}/_{\sim} = \{[M_1], ..., [M_h]\}$; then*

(i) *for each* $i \in \{1, ..., h\}$ *there exists a head context* $C_i[\ ]$ *and an integer* r_i *s.t. for each* $L \in \mathscr{M}$

$$\omega^k(C_i[L]) = \begin{cases} \{x_i \mathscr{L}_1 \cdots \mathscr{L}_{r_i}\} & \text{if } L \in [M_i] \\ \{y\} & \text{otherwise,} \end{cases}$$

where y *is any fixed variable;*

(ii) *there exists a head context* $C[\ ]$ *s.t.*

$$\omega^k(C[M]) = \{z_1, ..., z_h\},$$

where $\{z_1, ..., z_h\}$ *are new, pairwise distinct variables, and for each* $i \leqslant h$

$$\omega^k(C[L]) = \{z_i\} \Leftrightarrow L \in [M_i].$$

Proof. By standard separability techniques (see [16]). ∎

λ-calculus encodes all recursive functions; this can be done in many different ways, choosing a suitable *numeral system*. For technical reasons, we will make use of Church's numeral system,

$$\mathbf{c}_0, \mathbf{c}_1 \cdots \qquad \text{where} \quad \mathbf{c}_n \equiv \lambda fx.\underbrace{f(\cdots(fx)\cdots)}_{n},$$

defining $\mathbf{Succ} \equiv \lambda xyz.y(xyz)$ we have $\mathbf{Succ}\,\mathbf{c}_n =_\beta \mathbf{c}_{n+1}$, for any n; such system is *adequate* [9] in that all recursive functions are representble using it. In the sequel we shall write $\mathbf{n}$ for $\mathbf{c}_n$.

It is well known that the test for equality for Church numerals is λ-definable; however, we need to represent such a test with a combinator of a special shape. The proofs of the next lemmas and corollary are given in Section 7.

LEMMA 3.19. *There exists a combinator* $\mathbf{H} \in \Lambda$ *of the shape*

$$\mathbf{H} \equiv \lambda xy.xH_1 \cdots H_l,$$

with $x \notin FV(H_1) \cup \cdots \cup FV(H_l)$, *such that, for all non-negative integers* n, m,

$$\mathbf{Hnm} =_{\beta\eta} \begin{cases} \mathbf{1} & \text{if } n = m \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

COROLLARY 3.20. *If* $N \equiv \mathbf{n}_1 \oplus \cdots \oplus \mathbf{n}_r$, *with* $r \geqslant 1$, *then, for all* m,

$$\omega^1(\mathbf{H}N\mathbf{m}) = \{\mathbf{0} \mid \exists i \leqslant r.n_i = m\} \cup \{\mathbf{1} \mid \exists j \leqslant r.n_j \neq m\}.$$

LEMMA 3.21. *For* $M, N \in \Lambda_\oplus$,

$$M \not\leqslant_2 N \Rightarrow \exists C[\ ].C[M]\downarrow \wedge C[N]\uparrow.$$

EXAMPLE 3.22. We exhibit an example of the case where $M \leqslant_1 N$ but $M \not\leqslant_2 N$. Take $M \equiv xy \oplus xz$ and $N \equiv x(y \oplus z)$; now $\omega^2(M) = \{x\{y\}, x\{z\}\}$, while $\omega^2(N) = \{x\{y, z\}\}$. Computing $\mathsf{Pair}_1(\omega^2(M), \omega^2(N))$, we get

$$\{\langle\{\{y\}, \{z\}\}, \{\{y, z\}\}\rangle\}.$$

Clearly $\{\{y\}, \{z\}\} \not\sqsubseteq^\sharp \{\{y, z\}\}$. We take

$$C_0[\ ] \equiv (\lambda xyz.[\ ])(\lambda w.aww)\,\mathbf{1}\,\mathbf{2}.$$

Simple calculations give us

$$\omega^2(C_0[M]) = \{a\{\mathbf{1}\}\{\mathbf{1}\}, a\{\mathbf{2}\}\{\mathbf{2}\}\}$$
$$\text{and} \quad \omega^2(C_0[N]) = \{a\{\mathbf{1}, \mathbf{2}\}\{\mathbf{1}, \mathbf{2}\}\}.$$

Taking

$$C_1[\ ] \equiv (\lambda a.[\ ])(\lambda uv.\mathbf{P}(\mathbf{H}u\mathbf{1})(\mathbf{H}v\mathbf{2})),$$

where $\mathbf{P}$ λ-defines multiplication, we have

$$\omega^2(C_1[C_0[M]]) = \{\mathbf{0}\} \quad \text{and} \quad \omega^2(C_1[C_0[N]]) = \{\mathbf{0}, \mathbf{1}\}.$$

Now, taking $C_2[\ ] \equiv [\ ](\mathbf{K}(\Delta\Delta))\,\mathbf{I}$, we have

$$\omega^2(C_2[C_1[C_0[M]]]) = \{\mathbf{I}\} \quad \text{while}$$
$$\omega^2(C_2[C_1[C_0[N]]]) = \{\mathbf{\Omega}\}.$$

The following lemma, whose detailed proof is deferred to Section 7 extends to the present calculus the Böhm out lemma of the classical λ-calculus (see [9]).

LEMMA 3.23. *For* $M, N \in \Lambda_\oplus$ *and* $k \geqslant 2$,

$$M \not\leqslant_k N \Rightarrow \exists C[\ ].C[M] \not\leqslant_2 C[N].$$

We are finally in place to prove the main theorem of the present section.

THEOREM 3.24 (Semiseparability). *For any* $M, N \in \Lambda_\oplus$,

$$M \sqsubseteq_{\text{must}} N \Rightarrow M \leqslant N.$$

Proof. By contraposition, we prove (see Definition 2.4)

$$\exists k.M \not\leqslant_k N \Rightarrow \exists C[\ ].C[M]\downarrow \wedge C[N]\uparrow.$$

Indeed,

$$M \not\leqslant N \Rightarrow \exists k.M \not\leqslant_k N$$

$$\Rightarrow \begin{cases} (k=1) & \exists C[\ ].C[M]\downarrow \wedge C[N]\uparrow & \text{substitute } \Delta\Delta \text{ for } y \text{ in Lemma 3.18} \\ (k>1) & \exists C[\ ].C[M] \not\leqslant_2 C[N] & \text{by Lemma 3.23} \\ & \Rightarrow \exists C[\ ], C'[\ ].C'[C[M]]\downarrow \wedge C'[C[N]]\uparrow & \text{by Lemma 3.21.} \end{cases}$$

∎

4. MODELS OF NONDETERMINISTIC λ-CALCULI

λ-calculus studies functions under their applicative behaviour; consequently models of this calculus and of its variants are applicative structures, that is, sets equipped with a binary operation whose intended meaning is functional application. In the present case the structure we are looking for is an applicative structure with an extra operator modeling $\oplus$. To achieve such a structure, we shall work in the framework of *nondeterministic algebras*.

4.1. Nondeterministic Algebras

To fix notation and to keep the treatement of denotational semantics self-contained, we briefly summarize the relevant definitions and facts about nondeterministic algebras and powerdomain functors (see [23]).

DEFINITION 4.1. A nondeterministic algebra is a structure $\langle E, +, \sqsubseteq \rangle$, where $\langle E, \sqsubseteq \rangle$ is a *cpo* and $+$ is binary continuous function on E satisfying

(i) $x + x = x$;

(ii) $x + y = y + x$;

(iii) $(x + y) + z = x + (y + z)$.

A nondeterministic algebra is a *Smyth algebra* iff it satisfies (i)–(iii) and $x + y \sqsubseteq x$; it is a *Hoare algebra* iff it satisfies (i)–(iii) and $x \sqsubseteq x + y$.

Nondeterministic algebras, henceforth **NDA**, form a category, whose morphisms are Scott continuous functions $f: D \to E$ such that

$$(\mathrm{Lin}) \quad f(x + y) = f(x) + f(y)$$

for all $x, y \in D$. We shall write $D \to_{\mathrm{lin}} E$ as a shorthand for $\mathrm{Hom}_{\mathbf{NDA}}[D, E]$. Smyth and Hoare algebras are the objects of the categories **SNDA** and **HNDA** respectively, which are full subcategories of **NDA**.

NDA has the trivial as its terminal object, and all products $\langle D, +_D \rangle \times \langle E, +_E \rangle = \langle D \times E, + \rangle$ where $D \times E$ is the product of D and E as cpo's, and

$$\langle x, y \rangle + \langle x', y' \rangle = \langle x +_D x', y +_E y' \rangle;$$

hence it is cartesian.

By continuity of $+$, the space $D \to_{\mathrm{lin}} E$ of linear continuous functions between two **NDA** objects is a cpo with respect to pointwise ordering. Moreover, it is itself an **NDA** object, defining summation of functions by pointwise summation:

$$(f + g)(x) = f(x) +_E g(x).$$

However, the category **NDA** is not cartesian closed. This is due to the fact that, if we restrict the evaluation morphism $eval(f, x) = f(x)$ of the category **CPO** to the linear functions, it comes out that it is not linear (hence it is not a morphism of **NDA**): $\langle f + g, x + y \rangle = \langle f, x \rangle + \langle g, y \rangle$ but $eval(f + g, x + y) \neq eval(f, x) + eval(g, y)$; the best we have is the linearity of *eval* in each argument separately, that is, *eval* is bilinear in the terminology of [29]. As a matter of fact **NDA** is only monoidal closed with respect to the free tensor product constructed in [29].

By definition **NDA** embeds into the category **CPO**, so that free algebras are built from cpos by a functor $(\cdot)^\natural$: $\mathbf{CPO} \to \mathbf{NDA}$ called a *powerdomain functor*, which is the right adjoint of the forgetful functor. The same is true of **SNDA** and **HNDA**, so that we have the powerdomain functors $(\cdot)^\sharp$ and $(\cdot)^\flat$ respectively.

In case of algebraic cpos, this functors admits and explicit definition as follows (see [48, 53]):

DEFINITION 4.2. Let $(D, \sqsubseteq)$ be any algebraic cpo and $M(D) = \{u \subseteq \mathscr{K}(D) \mid u \text{ is finite}, \neq \emptyset\}$, where $\mathscr{K}(D)$ is the subset of Scott compact elements of D. Then define the following preorders on $M(D)$:

(i) $u \sqsubseteq^\flat v \Leftrightarrow \forall x \in u\ \exists y \in v.\ x \sqsubseteq y$;

(ii) $u \sqsubseteq^\sharp v \Leftrightarrow \forall y \in v\ \exists x \in u.\ x \sqsubseteq y$;

(iii) $u \sqsubseteq^\natural v \Leftrightarrow u \sqsubseteq^\flat v \wedge u \sqsubseteq^\sharp v$.

For $* \in \{\flat, \sharp, \natural\}$, the object part of each of these functors is defined by $D^* = \mathrm{Idl}(M(D), \sqsubseteq^*)$, namely the ideal completion of $(M(D), \sqsubseteq^*)$, ordered by subset inclusion, which is an **NDA** object with respect to the continuous function $\uplus: D^* \times D^* \to D^*$ defined by $I \uplus J = \{u \cup v \mid u \in I \wedge v \in J\}$.

Concerning the adjointness, consider the continuous function $\{\!|\cdot|\!\}: D \to D^*$, defined by $\{\!|x|\!\} = \{u \in M(D) \mid u \sqsubseteq^* \{x\}\}$; then, given any algebraic cpo D, nda E, and $f \in Hom_{\mathbf{CPO}}[D, E]$, there exists exactly one morphism $\mathbf{ext}(f): D^* \to_{\mathrm{lin}} E$ such that $f = \mathbf{ext}(f) \circ \{\!|\cdot|\!\}$, i.e.,

$$\begin{tikzcd} D \arrow[d, "\{\!|\cdot|\!\}"'] \arrow[dr, "f"] & \\ D^* \arrow[r, dashed, "\mathbf{ext}(f)"'] & E \end{tikzcd}$$

where $\mathbf{ext}(f)$ is defined as

$$\mathbf{ext}(f)(I) = \bigsqcup \{\bar{f}(u) \mid u \in I\},$$

which is the unique continuous extension of the function $\bar{f}(u) = f(x_1) + \cdots + f(x_n)$, for $u = \{x_1, ..., x_n\}$.

From this the morphism part of $(\cdot)^*$ can be defined by $f^* = \mathsf{ext}(\{\!|\cdot|\!\} \circ f)$, which implies the commutativity of the diagram

$$\begin{tikzcd} D \arrow[r, "f"] \arrow[d, "\{\!|\cdot|\!\}"'] & E \arrow[d, "\{\!|\cdot|\!\}"] \\ D^* \arrow[r, dashed, "f^*"'] & E^* \end{tikzcd}$$

The following proposition says that $(\cdot)^*$ is actually a functor, and moreover an $\mathcal{C}$-functor in terms of [29].

Proposition 4.3. *Let $(\cdot)^*$ be an operator among $(\cdot)^\flat$, $(\cdot)^\sharp$, and $(\cdot)^\natural$; then*

(i) $(Id_D)^* = Id_{D^*}$;

(ii) $f \circ g \colon D \to_{\text{cont}} E \Rightarrow (f \circ g)^* = f^* \circ g^*$;

(iii) $f \sqsubseteq g \Rightarrow f^* \sqsubseteq g^*$ (*$\mathcal{C}$-functoriality property*),

where $\sqsubseteq$ is the pointwise ordering.

It is worth to stress two relevant facts: the first one is that algebraic cpos are in general not closed under powerdomain functors. More precisely bounded complete cpos are closed under $(\cdot)^\sharp$ and $(\cdot)^\flat$, but not under $(\cdot)^\natural$, for which one has to consider more complex structures such as **SFP** domains (see [48]). The second fact, which is connected to the construction to be illustrated in the sequel is pointed out by the following proposition.

Proposition 4.4. *For any algebraic* **CPO** *D and* **NDA** *E,*

$$D \to_{\text{cont}} E \simeq D^* \to_{\text{lin}} E \qquad \textit{for} \quad * \in \{\flat, \sharp, \natural\}$$

where $\to_{\text{cont}}$ refers to continuous functions and $\simeq$ is an isomorphism in the category of **CPO**.

Proof. The isomorphism is given by ext and $\lambda g.g \circ \{\!|\cdot|\!\}$: from the universal property claimed above we immediately have that $\mathsf{ext}(f) \circ \{\!|\cdot|\!\} = f$. On the other hand, if $h = g \circ \{\!|\cdot|\!\}$ for $g\colon D^* \to_{\text{lin}} E$, then, for $u = \{x_1, ..., x_n\} \in M(D)$,

$$\begin{aligned} \mathsf{ext}(h)(\downarrow u) &= \bar{h}(u) \\ &= h(x_1) + \cdots + h(x_n) \\ &= g(\{\!|x_1|\!\}) + \cdots + g(\{\!|x_n|\!\}) \\ &= g(\{\!|x_1|\!\} \uplus \cdots \uplus \{\!|x_n|\!\}) \\ &= g(\downarrow u) \end{aligned}$$

where $\downarrow u = \{y \mid \exists x \in u . y \sqsubseteq x\}$. It follows that $\mathsf{ext}(g \circ \{\!|\cdot|\!\}) = g$. That these functions are order preserving and the reverse is routine. ∎

4.2. Semilinear Applicative Structures

We are now able to introduce the special kind of applicative structure we need.

Defintion 4.5. A *semilinear applicative structure* is a triple $\langle X, \cdot, + \rangle$ such that

(i) $\langle X, \cdot \rangle$ is an applicative structure,

(ii) $+\colon X^2 \to X$ is an idempotent, commutative, and associative operation,

(iii) $\forall x, y, z \in X . (x + y) \cdot z = (x \cdot z) + (y \cdot z)$.

A *linear applicative structure* is a semilinear applicative structure satisfying:

(iv) $\forall x, y, z \in X . x \cdot (y + z) = (x \cdot y) + (x \cdot z)$.

Both semilinear and linear applicative structures are *extensional* if they are such as applicative structures, i.e.,

(v) $\forall x, y \in X . (\forall z \in X . x \cdot z = y \cdot z) \Rightarrow x = y$.

The word *semilinear* is used since in general the application is not right distributive with respect to the sum: i.e., it is not linear. This is due to the fact that the application will be used to model continuous functions whose arguments are "sets," that is, sums, and it is not true in general that the value of these functions is the set of their values on the "elements" of the argument.

To model our calculus we construct an applicative structure whose domain is an **SNDA** object, in consequence of the fact that, operationally and algebraically, we have that $M \oplus N$ is less then M.

The usual method to construct applicative structures in the category of cpo's is solving the equation $D = D \to D$, say by an inverse limit construction. The trouble is that the applicative structure arising from such a construction is surely linear. On the contrary the study of the operational and algebraic semantics in the previous sections suggests that the equation $(M \oplus N)\, L = ML \oplus NL$ should hold in the model, but $M(N \oplus L) = MN \oplus ML$ should not. Therefore we have to find a semilinear applicative structure which is not a linear one.

The solution we propose is as follows: first we solve the equation $D = D^\sharp \to_{\text{lin}} D$ in the category **SNDA** via an isomorphism Φ (recall that powerdomain functors are $\mathcal{C}$-functors and locally continuous in the sense of [50], so that the projection method works); second, we define a binary continuous operation $\cdot\colon D \times D \to D$ as $d \cdot e = \Phi(d)\{\!|e|\!\}$. Now since Φ is an isomorphism of **SNDA** it is linear itself, that is, $\Phi(d + d') = \Phi(d) + \Phi(d')$, hence $(d + d') \cdot e = (d \cdot e) + (d' \cdot e)$. On the other hand, $\{\!|e + e'|\!\}$ being strictly included in $\{\!|e|\!\} \uplus \{\!|e'|\!\}$ in $D^\sharp$ as soon as e and e' are distinct in D, we have

$$\begin{aligned} d \cdot (e + e') = \Phi(d)\{\!|e + e'|\!\} &\leqslant \Phi(d)(\{\!|e|\!\} \uplus \{\!|e'|\!\}) \\ &= \Phi(d)\{\!|e|\!\} + \Phi(d)\{\!|e'|\!\} = (d \cdot e) + (d \cdot e'), \end{aligned}$$

i.e., an inequality which depends on d and in general is strict. Indeed this implies that representable functions will be continuous, but in general not linear (see Proposition 4.4).

DEFINITION 4.6. Take D_0 as any nontrivial Smyth algebra (e.g., $(\mathbf{2})^\sharp$), and $D_{n+1}=[(D_n)^\sharp \to_{\text{lin}} D_n]$; then inductively define $\varphi_n: D_n \to_{\text{lin}} D_{n+1}$ and $\psi_n: D_{n+1} \to_{\text{lin}} D_n$ as follows:

(i) $\varphi_0(x)=\lambda y.x$, $\psi_0(y)=y(\perp)$,

(ii) $\varphi_{n+1}(x)=\varphi_n \circ x \circ (\psi_n)^\sharp$, $\psi_{n+1}(y)=\psi_n \circ y \circ (\varphi_n)^\sharp$.

PROPOSITION 4.7. *The mappings φ_n, ψ_n are well defined, that is, they are linear. Furthermore for each natural number n $\langle \varphi_n, \psi_n \rangle$ is an embedding–projection pair; that is,*

(i) $\psi_n \circ \varphi_n = Id_n$;

(ii) $\varphi_n \circ \psi_n \sqsubseteq Id_n$.

Proof. Linearity is proved by induction on n. For $n=0$,

$$\varphi_0(x+y)(z)=z=z+z=\varphi_0(x)(z)+\varphi_0(y)(z),$$

and

$$\psi_0(f+g)=(f+g)(\perp)=f(\perp)+g(\perp)=\psi_0(f)+\psi_0(g).$$

If $n>0$ then the thesis follows from the inductive hypothesis, since for any (continuous) f, $f^\sharp$ is always linear and composition of linear functions is linear. To prove that $\langle \varphi_n, \psi_n \rangle$ is an embedding–projection pair we again make induction on n. If $n=0$,

$$\psi_0 \circ \varphi_0(x)=\psi_0(\lambda y.x)=(\lambda y.x)(\perp)=x$$

and

$$\varphi_0 \circ \psi_0(f)=\varphi_0(f(\perp))=\lambda z.f(\perp) \sqsubseteq f.$$

For the inductive step,

$$\begin{aligned}
\psi_{n+1} \circ \varphi_{n+1}(x) &= \psi_{n+1}(\varphi_n \circ x \circ (\psi_n)^\sharp) \\
&= \psi_n \circ (\varphi_n \circ x \circ (\psi_n)^\sharp) \circ (\varphi_n)^\sharp \\
&= (\psi_n \circ \varphi_n) \circ x \circ (\psi_n \circ \varphi_n)^\sharp \\
&\qquad \text{by Proposition 4.3 (ii)} \\
&= Id_n \circ x \circ (Id_n)^\sharp \\
&\qquad \text{by the inductive hypothesis} \\
&= Id_n \circ x \circ Id_{D_n^\sharp} \\
&\qquad \text{by Proposition 4.3 (i)} \\
&= x,
\end{aligned}$$

and (ii) is proved similarly, this time using the $\mathscr{C}$-functoriality of $(\cdot)^\sharp$. ∎

DEFINITION 4.8. (i) $D_*=\lim_{\leftarrow}(D_n, \psi_n)=\{x \in \prod_n D_n \mid \forall n \in \omega . \psi_n(x_{n+1})=x_n\}$ with coordinate-wise ordering;

(ii) $\Phi_{m,n}: D_m \to_{\text{lin}} D_n$ is defined as

$$\Phi_{m,n}=\begin{cases} \varphi_{n-1} \circ \cdots \circ \varphi_m & \text{if } m<n \\ Id & \text{if } m=n \\ \psi_n \circ \cdots \circ \psi_{m-1} & \text{if } n<m; \end{cases}$$

(iii) $\Phi_{*,n}: D_* \to_{\text{lin}} D_n$ and $\Phi_{n,*}: D_n \to_{\text{lin}} D_*$ are defined as

$$\Phi_{*,n}(x)=x_n$$
$$\Phi_{n,*}(y)=\langle \Phi_{n,m}(y) \rangle_{m \in \omega}$$

As usual with inverse limit constructions, each D_n embeds into D_*, by the $\langle \Phi_{n,*}, \Phi_{*,n} \rangle$ embedding-projection pair; we will write $x_n=\Phi_{n,*} \circ \Phi_{*,n}(x)$ for $x \in D_*$ and $a_n=(\Phi_{n,*} \circ \Phi_{*,n})^\sharp(a)=\Phi^\sharp_{n,*} \circ \Phi^\sharp_{*,n}(a)$ for $a \in D^\sharp_*$.

Once we have the family of functions of the above definition we can explicitly define the desired isomorphism, together with the application.

DEFINITION 4.9. (i) The maps $F: D_* \to [D^\sharp_* \to_{\text{lin}} D_*]$ and $\tilde{F}: D_* \to [D_* \to_{\text{cont}} D_*]$ are defined by

$$F(x)=\lambda a \in D^\sharp_* . \bigsqcup_n x_{n+1}(a_n) \quad \text{and} \quad \tilde{F}=(\lambda g.g \circ \{\!|\cdot|\!\}) \circ F.$$

(ii) The maps $G: [D^\sharp_* \to_{\text{lin}} D_*] \to D_*$ and $\tilde{G}: [D_* \to_{\text{cont}} D_*] \to D_*$ are defined by

$$G(f)=\bigsqcup_n (\lambda a \in D^\sharp_n . (f(\Phi^\sharp_{n,*}(a)))_n) \quad \text{and} \quad \tilde{G}=G \circ \textbf{ext}.$$

(iii) The operation of application $\cdot: D_* \times D_* \to D_*$ is defined by

$$x \cdot y=\tilde{F}(x)(y)=F(x)\{\!|y|\!\}.$$

We list in the following lemma some relevant properties of the domain D_*:

LEMMA 4.10. *For any $x, y, z \in D_*$ and $a \in D^\sharp_*$,*

(i) $(x_m)_n=x_{\min(n,m)}$;

(ii) $x=\bigsqcup_n x_n$, $a=\bigsqcup_n a_n$;

(iii) $x_{n+1} \cdot y_n=x_{n+1}(\{\!|y_n|\!\})$;

(iv) $x_{n+1} \cdot y=x_{n+1} \cdot y_n=(x \cdot y_n)_n$;

(v) $x_0 \cdot y=x_0=(x \cdot \perp)_0$;

(vi) $(x+y)_n=x_n+y_n=(x_n+y_n)_n$.

Proof.

(i) Consequence of the fact that $\langle \Phi_{n,*}, \Phi_{*,n} \rangle$ is an embedding–projection pair.

(ii) $x = \bigsqcup_n x_n$ is standard in inverse limit constructions; to see that $a = \bigsqcup_n a_n$:

(a) Let $\langle \varphi, \psi \rangle$ be an injection–projection pair for some D to some E: then

$$\varphi(\mathscr{K}(D)) \subseteq \mathscr{K}(E).$$

Indeed, let $x \in \mathscr{K}(D)$; then for any directed $Y \subseteq E$

$$\varphi(x) \sqsubseteq \bigsqcup Y \Rightarrow x \sqsubseteq \psi\left(\bigsqcup Y\right) = \bigsqcup \psi(Y)$$

$$\Rightarrow \exists y \in Y. x \sqsubseteq \psi(y)$$

$$\Rightarrow \exists y \in Y. \varphi(x) \sqsubseteq \varphi \circ \psi(y) \sqsubseteq y.$$

(b) $\mathscr{K}(D_*) = \bigcup_n \mathscr{K}(D_n)$: indeed if $d \in \mathscr{K}(D_n)$, then $d \in \mathscr{K}(D_*)$ follows from (a) and the fact that $\langle \Phi_{n,*}, \Phi_{*,n} \rangle$ is an embedding–projection pair; on the other hand, from $d \in \mathscr{K}(D_*)$, it follows that

$$d = \bigsqcup_n d_n \Rightarrow \exists m. d = d_m.$$

Now, given any directed $S \subseteq D_m$,

$$\Phi_{*,m}(d) \sqsubseteq \bigsqcup S \Rightarrow d = \Phi_{m,*} \circ \Phi_{*,m}(d) \sqsubseteq \Phi_{m,*}\left(\bigsqcup S\right)$$

$$\Rightarrow d \sqsubseteq \bigsqcup \Phi_{m,*}(S)$$

$$\Rightarrow \exists s \in S. d \sqsubseteq \Phi_{m,*}(s)$$

$$\Rightarrow \exists s \in S. \Phi_{*,m}(d) \sqsubseteq s.$$

That $a_n \sqsubseteq a$ for all n is immediate. Vice versa, let $u \in a$; then $u = \{d^1, ..., d^r\} \in M(D_*)$. Using (b) we know that each d_i is compact in some D_{m_i}; then we choose $m = \max\{m_i \mid 1 \leqslant i \leqslant r\}$. By (a), $u \in M(D_m)$. On the other hand

$$\Phi^{\#}_{*,m}(a) = \bigcup \{\bar{\Phi}_{*,m}(v) \mid v \in a\},$$

but

$$\bar{\Phi}_{*,m}(u) = \{\!|\Phi_{*,m}(d^1)|\!\} \uplus \cdots \uplus \{\!|\Phi_{*,m}(d^r)|\!\}$$

$$= \{\!|d^1|\!\} \uplus \cdots \uplus \{\!|d^r|\!\}$$

$$= \downarrow u,$$

so that $u \in \downarrow u \subseteq a_m$, from which we conclude that $a \sqsubseteq \bigsqcup_n a_n$.

(iii) Let us note preliminarly that, by the very definition of $(\cdot)^{\#}$,

$$\{\!|y|\!\}_n = \Phi^{\#}_{*,n}\{\!|y|\!\} = \{\!|\Phi_{*,n}(y)|\!\} = \{\!|y_n|\!\}.$$

Now

$$\begin{aligned} x_{n+1} \cdot y_n &= F(x_{n+1})\{\!|y_n|\!\} \\ &= \bigsqcup_m (x_{n+1})_{m+1} \{\!|y_n|\!\}_m \\ &= \bigsqcup_m (x_{n+1})_{m+1} (\{\!|y|\!\}_n)_m \\ &= x_{n+1}\{\!|y|\!\}_n \\ &= x_{n+1}\{\!|y_n|\!\}. \end{aligned}$$

(iv)–(v) Similar to the proof of the corresponding properties for Scott D_∞ models.

(vi) By the linearity of $\Phi^{\#}_{*,n}$ we immediately have $(x+y)_n = x_n + y_n$. On the other hand

$$\begin{aligned} (x_n + y_n)_n &= (x_n)_n + (y_n)_n \\ &= x_n + y_n. \quad \blacksquare \end{aligned}$$

LEMMA 4.11. *The mappings F and G are continuous and linear, that is, they are* **NDA** *morphisms. Furthermore, the structure*

$$\langle D_*, \cdot, + \rangle$$

is a semilinear applicative structure.

Proof. Let $x, y \in D_*$ and $a \in D^{\#}_*$; then

$$\bigsqcup_n (x+y)_{n+1}(a_n) = \bigsqcup_n (x_{n+1} + y_{n+1})(a_n)$$

by Lemma 4.10 (vi)

$$= \bigsqcup_n (x_{n+1}(a_n) + y_{n+1}(a_n))$$

$$= \bigsqcup_n x_{n+1}(a_n) + \bigsqcup_n y_{n+1}(a_n)$$

by continuity of $+$;

hence

$$\begin{aligned} F(x+y) &= \lambda a \in D^{\#}_*. \bigsqcup_n (x+y)_{n+1}(a_n) \\ &= \lambda a \in D^{\#}_*. \bigsqcup_n x_{n+1}(a_n) + \bigsqcup_n y_{n+1}(a_n) \\ &= \left(\lambda a \in D^{\#}_*. \bigsqcup_n x_{n+1}(a_n)\right) + \left(\lambda a \in D^{\#}_*. \bigsqcup_n y_{n+1}(a_n)\right) \\ &= F(x) + F(y). \end{aligned}$$

Let $f, g \in [D_*^\sharp \to_{\text{lin}} D_*]$ and $a \in D_*^\sharp$; then

$$((f+g)(a))_n = (f(a)+g(a))_n = (f(a))_n + (g(a))_n$$

by Lemma 4.10 (vi); it follows that

$$\begin{aligned}G(f+g) &= \bigsqcup_n (\lambda b \in D_*^\sharp.((f+g)(\Phi_{n,*}^\sharp(b)))_n)\\ &= \bigsqcup_n (\lambda b \in D_n^\sharp.(f(\Phi_{n,*}^\sharp(b)))_n + (g(\Phi_{n,*}^\sharp(b)))_n)\\ &= \bigsqcup_n ((\lambda b \in D_n^\sharp.(f(\Phi_{n,*}^\sharp(b)))_n)\\ &\quad + (\lambda b \in D_n^\sharp.(g(\Phi_{n,*}^\sharp(b)))_n))\\ &= \bigsqcup_n (\lambda b \in D_n^\sharp.(f(\Phi_{n,*}^\sharp(b)))_n)\\ &\quad + \bigsqcup_n (\lambda b \in D_n^\sharp.(g(\Phi_{n,*}^\sharp(b)))_n)\\ &= G(f) + G(g).\end{aligned}$$

This establishes the linearity property; the continuity property is proved in the same way as in the category **CPO**.

Finally, let $x, y, z \in D_*$:

$$\begin{aligned}(x+y)\cdot z &= \tilde{F}(x+y)(z)\\ &= F(x+y)(\{\!|z|\!\})\\ &= (F(x)+F(y))(\{\!|z|\!\}) \qquad \text{by linearity of } F\\ &= F(x)(\{\!|z|\!\}) + F(y)(\{\!|z|\!\})\\ &= (x\cdot z) + (y\cdot z). \quad \blacksquare\end{aligned}$$

THEOREM 4.12. *The domain D_* satisfies the equation*

$$D \simeq [D^\sharp \to_{\text{lin}} D]$$

in the category of **NDA** *and consequently in that of* **SNDA**; *it satisfies also the equation*

$$D \simeq [D \to_{\text{cont}} D]$$

in the category **CPO** *as pictured in the diagram*

$$D_* \underset{F}{\overset{G}{\rightleftarrows}} [D_*^\sharp \to_{\text{lin}} D_*] \underset{\lambda g.g \circ \{\!|\cdot|\!\}}{\overset{\text{ext}}{\rightleftarrows}} [D_* \to_{\text{cont}} D_*].$$

We conclude that $\langle D_*, \cdot, + \rangle$ is an *extensional* semilinear applicative structure.

Proof. To prove the theorem it remains to show that F and G are mutually inverse: actually the second isomorphism will follow from this one and from Corollary 4.4, which applies to the $(\cdot)^\sharp$ functor as well.

(a) $G \circ F = Id$: by definition $(G \circ F)(x) = G(f)$, where

$$f = \lambda a \in D_*^\sharp. \bigsqcup_n x_{n+1}(a_n);$$

now we observe that if y is in (the image of) D_n then $y = y_n$, and similarly if a is in the image of $D_n^\sharp$; now given such an a

$$\begin{aligned}(f(a))_n &= \left(\bigsqcup_m x_{m+1}(a_m)\right)_n\\ &= (x_{n+1}(a))_n\\ &= x_{n+1}(a).\end{aligned}$$

It follows that

$$\begin{aligned}G(f) &= \bigsqcup_n (\lambda a \in D_n^\sharp.x_{n+1}(a))\\ &= \bigsqcup_n x_{n+1}\\ &= x.\end{aligned}$$

(b) $F \circ G = Id$: we note that

$$\begin{aligned}G(f)_{n+1}(a_n) &= (f(\Phi_{n,*}^\sharp(a_n)))_n\\ &= (f(a))_n\end{aligned}$$

so that

$$\begin{aligned}(F \circ G)(f)(a) &= \bigsqcup_n (f(a))_n\\ &= f(a),\end{aligned}$$

that is, $(F \circ G)(f) = (f)$.

To prove extensionality:

$$\begin{aligned}\forall z.x\cdot z = y\cdot z &\Rightarrow \tilde{F}(x)(z) = \tilde{F}(y)(z)\\ &\Rightarrow \tilde{F}(x) = \tilde{F}(y)\\ &\Rightarrow x = \tilde{G} \circ \tilde{F}(x) = \tilde{G} \circ \tilde{F}(y) = y. \quad \blacksquare\end{aligned}$$

4.3. Syntactical Models

We present a notion of *model*, which actually does not directly interpret the relation $\to$, but the equivalence relation $\simeq_{\text{must}}$. More precisely, we extend the classical notion of syntactical model of lambda calculus (see [30]) to the case of the nondeterministic calculus and show that the algebra D_*, together with a suitable interpretation map, is an instance of such a model.

DEFINITION 4.13. A *syntactical model* is a semilinear applicative structure $\mathscr{M} = \langle X, \cdot, + \rangle$ equipped with a map $[\![\cdot]\!]: \Lambda_{\oplus} \to (Env \to X)$, such that the triple $\langle X, \cdot, [\![\cdot]\!] \rangle$, for any $\rho \in Env = Var \to X$, satisfies

(i) $[\![x]\!]_\rho = \rho(x)$;

(ii) $[\![MN]\!]_\rho = [\![M]\!]_\rho \cdot [\![N]\!]_\rho$;

(iii) $[\![\lambda x.M]\!]_\rho \cdot d = [\![M]\!]_{\rho[d/x]}$ for all $d \in X$;

(iv) $\rho \restriction FV(M) = \rho' \restriction FV(M) \Rightarrow [\![M]\!]_\rho = [\![M]\!]_{\rho'}$;

(v) $[\![\lambda x.M]\!]_\rho = [\![\lambda y.M[y/x]]\!]_\rho$ if $y \notin FV(M)$;

(vi) $(\forall d \in X. [\![M]\!]_{\rho[d/x]} = [\![N]\!]_{\rho[d/x]}) \Rightarrow [\![\lambda x.M]\!]_\rho = [\![\lambda x.N]\!]$;

which are the clauses of the classical definition of the syntactical λ-model of [30], and furthermore

(vii) $[\![M \oplus N]\!]_\rho = [\![M]\!]_\rho + [\![N]\!]_\rho$.

Finally, we call *extensional* any syntactical model whose underlying semilinear applicative structure is extensional.

LEMMA 4.14. *If* $\langle X, \cdot, +, [\![\cdot]\!] \rangle$ *is an extensional syntactical model, then for any* $M, N \in \Lambda_{\oplus}$ *and for all* $\rho \in Env$,

$$[\![\lambda x.M \oplus N]\!]_\rho = [\![(\lambda x.M) \oplus (\lambda x.N)]\!]_\rho.$$

Proof. Let $d \in X$ be an arbitrary element; then, for any $\rho \in Env$,

$$\begin{aligned}
&[\![\lambda x.M \oplus N]\!]_\rho \cdot d \\
&\quad = [\![M \oplus N]\!]_{\rho[d/x]} && \text{by Definition 4.13 (iii)} \\
&\quad = [\![M]\!]_{\rho[d/x]} + [\![N]\!]_{\rho[d/x]} && \text{by Definition 4.13 (vii)} \\
&\quad = [\![\lambda x.M]\!]_\rho \cdot d + [\![\lambda x.N]\!]_\rho \cdot d && \text{by Definition 4.13 (iii)} \\
&\quad = ([\![\lambda x.M]\!]_\rho + [\![\lambda x.N]\!]_\rho) \cdot d && \text{by semilinearity.}
\end{aligned}$$

Since d is arbitrary, it follows that

$$\begin{aligned}
&[\![\lambda x.M \oplus N]\!]_\rho \\
&\quad = [\![\lambda x.M]\!]_\rho + [\![\lambda x.N]\!]_\rho && \text{by Definition 4.13 (vi)} \\
&\quad = [\![(\lambda x.M) \oplus (\lambda x.N)]\!]_\rho && \text{by Definition 4.13 (vii).} \quad \blacksquare
\end{aligned}$$

DEFINITION 4.15. Given the structure $\langle D_*, \cdot, + \rangle$ and $\rho \in Env = Var \to D_*$, we define the map $[\![\cdot]\!]: \Lambda_{\oplus} \to (Env \to D_*)$ as follows:

(i) $[\![x]\!]_\rho = \rho(x)$,

(ii) $[\![MN]\!]_\rho = [\![M]\!]_\rho \cdot [\![N]\!]_\rho$,

(iii) $[\![\lambda x.M]\!]_\rho = \tilde{G}(\lambda d \in D_*. [\![M]\!]_{\rho[d/x]})$,

(iv) $[\![M \oplus N]\!]_\rho = [\![M]\!]_\rho + [\![N]\!]_\rho$.

This is a good definition, since in (ii) the continuity and linearity of application, abstraction, and $+$ ensure that the function $\lambda d \in D_*. [\![M]\!]_{\rho[d/x]}$ is continuous and linear as well.

PROPOSITION 4.16. *The quadruple* $\langle D_*, \cdot, +, [\![\cdot]\!] \rangle$ *is a syntactical model; furthermore it is extensional.*

Proof. By Theorem 4.12 the structure $\langle D_*, \cdot, + \rangle$ is an extensional semilinear applicative structure. The rest is routine; e.g.,

$$\begin{aligned}
[\![\lambda x.M]\!]_\rho \cdot d &= \tilde{F}(\tilde{G}(\lambda d'. [\![M]\!]_{\rho[d'/x]}))(d) \\
&= (\lambda d'. [\![M]\!]_{\rho[d'/x]})(d) \\
&= [\![M]\!]_{\rho[d/x]},
\end{aligned}$$

hence Definition 4.13 (iii) is verified. $\blacksquare$

5. FULL ABSTRACTION

The main result of this section is a theorem stating that the operational and denotational semantics constructed so far coincide. Such a result is an instance of that property which is called in the literature "full abstraction" (see [49, 38]).

In effect it could be questioned whether our reduction relation is actually "operational" since it does not formalize an evaluation mechanism; that is, it does not force any reduction strategy.

In such a case one should refer to the main result of this section as a characterization theorem of the local structure of the model D_* (see [9] where the local structure of a λ-model is its equational theory, as opposed to the global structure of properties which are not equationally expressible).

However, in view of the fact that we put the emphasis of our treatement on the notion of head reduction, which actually is an evaluation mechanism, we feel entitled to speak of full abstraction.

THEOREM 5.1 (Full Abstraction Theorem). *For all* $M, N \in \Lambda_{\oplus}$, *if* $[\![\cdot]\!]: \Lambda_{\oplus} \to (Env \to D_*)$ *is the interpretation map of* 4.15, *then*

$$M \sqsubseteq_{must} N \Leftrightarrow \forall \rho. [\![M]\!]_\rho \sqsubseteq [\![N]\!]_\rho.$$

To prove the theorem we shall use ideas from classical λ-calculus.

Following Wadsworth, Hyland, and Lévy (see [32, 36, 55]), we introduce an indexing notion $\mathscr{I}$ assigning a natural number to each term. This corresponds semantically to the projection of the denotation of any term M in the algebra $D_{\mathscr{I}(M)}$, so that the denotation of M is the limit of all its denotations under indexing function $\mathscr{I}$.

It turns out that the set of interpretations of the terms $M^{[k]}$ has a cofinality property with respect to the set of interpretations of $M^{\mathscr{I}}$, giving us the main lemma needed to establish the theorem.

In the sequel the intended interpretation is D_*.

DEFINITION 5.2. Let $\mathscr{I}$ be an *indexing function*, that is, a map $\mathscr{I}: \Lambda_{\oplus}\Omega \to \mathbb{N}$; then, writing $M^{\mathscr{I}}$ to denote the term M labelled with the index $\mathscr{I}(M)$, $[\![M^{\mathscr{I}}]\!]$ is defined as follows:

(i) $[\![\Omega^{\mathscr{I}}]\!]_\rho = \bot_{\mathscr{I}(\Omega)} = \bot$;

(ii) $[\![x^{\mathscr{I}}]\!]_\rho = (\rho(x))_{\mathscr{I}(x)}$;

(iii) $[\![(MN)^{\mathscr{I}}]\!]_\rho = ([\![M^{\mathscr{I}}]\!]_\rho \cdot [\![N^{\mathscr{I}}]\!]_\rho)_{\mathscr{I}(MN)}$;

(iv) $[\![(\lambda x.M)^{\mathscr{I}}]\!]_\rho = (\tilde{G}(\lambda d.[\![M^{\mathscr{I}}]\!]_{\rho[d/x]}))_{\mathscr{I}(\lambda x.M)}$;

(v) $[\![(M \oplus N)^{\mathscr{I}}]\!]_\rho = ([\![M^{\mathscr{I}}]\!]_\rho + [\![N^{\mathscr{I}}]\!]_\rho)_{\mathscr{I}(M \oplus N)}$.

LEMMA 5.3. *For any $M \in \Lambda_{\oplus}$ and all $\rho \in Env$,*

$$[\![M]\!]_\rho = \bigsqcup_{\mathscr{I}} [\![M^{\mathscr{I}}]\!]_\rho.$$

Proof. By induction on M using the equation $x = \bigsqcup_n x_n$ of Lemma 4.10. ∎

In the sequel we call terms together with their indexes modulo some indexing function *indexed terms*. For the sake of symbolic manipulation we allow multiple indexed terms $(M^n)^m$, whose intended meaning is $M^{\min(m,n)}$, and make this minimalization over indexes into an explicit reduction step (see [9, Chap. 14]).

DEFINITION 5.4. First extend the definition of substitution to indexed terms inductively from the base clause $x^m[N^n/x] \equiv (N^n)^m$. Now define the following binary relation $\rhd$ over indexed terms:

(i) $(\lambda x.M)^{n+1} N \rhd (M[N^n/x])^n$;

(ii) $(\lambda x.M)^0 N \rhd (M[\mathbf{\Omega}^0/x])^0$;

(iii) $\mathbf{\Omega}^n \rhd \mathbf{\Omega}^0$;

(iv) $\lambda x.\mathbf{\Omega}^n \rhd \mathbf{\Omega}^0$;

(v) $\mathbf{\Omega}^n M \rhd \mathbf{\Omega}^0$;

(vi) $\mathbf{\Omega}^n \oplus M \rhd \mathbf{\Omega}^0$;

(vii) $M \oplus \mathbf{\Omega}^n \rhd \mathbf{\Omega}^0$;

(viii) $(M \oplus N)^{n+1} L \rhd (ML^n \oplus NL^n)^n$;

(ix) $(M \oplus N)^0 L \rhd (M\mathbf{\Omega}^0 \oplus N\mathbf{\Omega}^0)^0$;

(c) $(M^m)^n \rhd M^{\min(m,n)}$;

(xi) $M^m \rhd N^n \Rightarrow C[M^m] \rhd C[N^n]$.

LEMMA 5.5. *The relation $\rhd$ is strongly normalizing and Church–Rosser.*

Proof. An easy extension of the classical proof of strong normalization of the labelled λ-calculus (see [9]) establishes the strong normalizability property: just note the decreasing indexes in clauses (i) and (viii) of Definition 5.4, and that the length of the term decreases in the other cases.

Now by case inspection of overlapping left hand sides in Definition 5.4 one sees that $\rhd$ is weakly Church–Rosser; that is,

$$M^n \rhd N^p \wedge M^n \rhd L^q \Rightarrow \exists T^r.N^p \rhd^* T^r \wedge L^q \rhd^* T^r.$$

We illustrate two relevant cases:

Case 1. (i)–(iv).

$$\begin{tikzcd}
(\lambda x.\mathbf{\Omega}^m)^{n+1} N \arrow[rr, "\rhd"] \arrow[d, "\nabla"'] & & (\mathbf{\Omega}^m)^n \arrow[d, dashed, "\nabla"] \\
(\mathbf{\Omega}^0)^{n+1} N \arrow[r, dashed, "\rhd"] & \mathbf{\Omega}^0 N \arrow[r, dashed, "\rhd"] & \mathbf{\Omega}^0
\end{tikzcd}$$

Case 2. (iii)–(viii).

$$\begin{tikzcd}
(\mathbf{\Omega}^m \oplus M)^{n+1} L \arrow[rr, "\rhd"] \arrow[ddd, "\nabla"'] & & (\mathbf{\Omega}^m L^n \oplus ML^n)^n \arrow[d, dashed, "\nabla"] \\
& & (\mathbf{\Omega}^0 \oplus ML^n)^n \arrow[d, dashed, "\nabla"] \\
& & (\mathbf{\Omega}^0)^n \arrow[d, dashed, "\nabla"] \\
(\mathbf{\Omega}^0)^{n+1} L \arrow[r, dashed, "\rhd"] & \mathbf{\Omega}^0 L \arrow[r, dashed, "\rhd"] & \mathbf{\Omega}^0
\end{tikzcd}$$

Hence $\rhd$ is Church–Rosser by the Newman lemma. ∎

COROLLARY 5.6.

(i) $\mathbf{N}_{\oplus}^{\mathbf{\Omega}} = \{|M| \mid in \rhd\text{-}normal\ form\}$, *where* $|\cdot|$ *is the index erasing map*;

(ii) $\forall M \in \Lambda_{\oplus}\ \forall \mathscr{I}\ \exists N \in \mathbf{N}_{\oplus}^{\mathbf{\Omega}}\ \exists \mathscr{J}.M^{\mathscr{I}} \rhd N^{\mathscr{J}}$.

Proof. Recall that $\mathbf{N}_{\oplus}^{\mathbf{\Omega}} = \{M^{[k]} = \vartheta_1 \circ \omega^k(M) \mid M \in \Lambda_{\oplus}, k \in \mathbb{N}\}$. Let us observe that, after the very definition of ϑ_1, this set could be inductively defined by

(i) $\mathbf{\Omega} \in \mathbf{N}_{\oplus}^{\mathbf{\Omega}}$;

(ii) $M_1, ..., M_m \in \mathbf{N}_{\oplus}^{\mathbf{\Omega}} \wedge x_1, ..., x_n, x \in Var \Rightarrow \lambda x_1 \cdots x_n.xM_1 \cdots M_m \in \mathbf{N}_{\oplus}^{\mathbf{\Omega}}$;

(iii) $M, N \in \mathbf{N}_{\oplus}^{\mathbf{\Omega}} - \{\mathbf{\Omega}\} \Rightarrow M \oplus N \in \mathbf{N}_{\oplus}^{\mathbf{\Omega}}$.

Now to prove (i) is routine. (ii) follows from (i) and the previous lemma. ∎

LEMMA 5.7.

$$M \in \mathbf{N}_{\oplus}^{\mathbf{\Omega}} \Rightarrow \exists k \in \mathbb{N}\ \forall h \geq k.\omega^h(M) = \omega^k(M);$$

hence, defining height(M) *as the minimal k satisfying the above statement,*

$$M \leqslant N \Leftrightarrow M \leqslant_{\text{height}(M)} N \Leftrightarrow M \leqslant N^{[\text{height}(M)]}.$$

Proof. Note that, since $M \in \mathbf{N}_{\oplus}^{\Omega}$, $NBT(M)$ differs from the syntactical tree only in that the operator $\oplus$ is treated as a set constructor, and some abstractions are pushed into sums: e.g.,

$$\omega^k(\lambda x.P \oplus Q) = \omega^k(\lambda x.P) \cup \omega^k(\lambda x.Q) = \omega^k(\lambda x.P \oplus \lambda x.Q).$$

Now take as k the depth of the syntactical tree of M. ∎

We state here the main lemma to prove the full abstraction theorem; its proof is deferred to Section 7.

LEMMA 5.8. *For any $M \in \Lambda_{\oplus}$ and natural number k,*

$$\forall \rho \in Env.\, [\![M]\!]_\rho = \bigsqcup_k [\![M^{[k]}]\!]_\rho.$$

THEOREM 5.9. *For all $M, N \in \Lambda_{\oplus}$,*

$$\forall \rho \in Env.\, [\![M]\!]_\rho \sqsubseteq [\![N]\!]_\rho \Rightarrow M \sqsubseteq_{\text{must}} N.$$

Proof.

$$\begin{aligned} M^{[1]} = \mathbf{\Omega} &\Leftrightarrow \forall k.\, M^{[k]} = \mathbf{\Omega} \\ &\Leftrightarrow \forall k.\, [\![M^{[k]}]\!] = \bot \\ &\Leftrightarrow [\![M]\!] = \bigsqcup_k [\![M^{[k]}]\!] = \bot \quad \text{by Lemma 5.8,} \end{aligned}$$

since $[\![\mathbf{\Omega}]\!] = \bot$; hence

$$\begin{aligned} M \not\sqsubseteq_{\text{must}} N &\Rightarrow \exists C[\ ].\, C[M]\!\downarrow \wedge\ C[N]\!\uparrow \\ &\Rightarrow \exists C[\ ].\, \omega^1(C[M]) \neq \{\mathbf{\Omega}\} = \omega^1(C[N]) \\ &\Rightarrow \exists C[\ ].\, [\![C[M]]\!] \neq \bot = [\![C[N]]\!] \\ &\Rightarrow [\![M]\!] \not\sqsubseteq [\![N]\!], \end{aligned}$$

the context operation being the composition of abstraction, application, and +, that is, a monotonic function. ∎

COROLLARY 5.10. *For all $M, N \in \Lambda_{\oplus}$,*

(i) $M \leqslant N \Rightarrow [\![M]\!] \sqsubseteq [\![N]\!]$,

(ii) $M \sqsubseteq_{\text{must}} N \Leftrightarrow M \leqslant N \Leftrightarrow [\![M]\!] \sqsubseteq [\![N]\!]$.

Proof. To establish (i),

$$\begin{aligned} M \leqslant N &\Rightarrow \forall k.\, M^{[k]} \leqslant M \leqslant N \\ &\Rightarrow \forall k.\, [\![M^{[k]}]\!] \sqsubseteq [\![N]\!] \\ &\Rightarrow [\![M]\!] \sqsubseteq [\![N]\!]. \end{aligned}$$

Now (ii) follows from (i) and Theorems 3.24 and 5.9. ∎

6. NONDETERMINISTIC THEORIES AND CONSERVATIVITY

In this section we address the question of relationship between the nondeterministic calculus studied so far and the classical one, seen as its deterministic subcalculus.

In [5] a nondeterministic extension of Plotkin's PCF was introduced, consisting in a simply typed λ-calculus with recursion and a choice operator. The authors proposed an operational semantics based on a reduction relation forcing outermost evaluation, and a denotational semantics in which powerdomain functors were used just to interpret basic types, functional types being restricted to linear (that is additive) functions over non deteministic algebras.

As observed in [6], this results in a theory of the equivalence of terms which does not preserve the equivalences holding in the deterministic subcalculus. Indeed, to recover such a conservativity property while preserving an additive semantics, Astesiano and Costa were led in [6] to an operational semantics based on a rewriting system, which follows an outermost strategy with sharing mechanism for arguments.

It should be noted that this is a high cost solution, due not just to the complexity of both syntax and semantics, but to the strong limitation it imposes to the power of nondeterminism. Indeed, it corresponds to forcing uniform behaviour of multiplied instances of the same choice redexes.

This problem can be recast in the perspective of call-by-name and call-by-value λ-calculi, whose study was initiated in [47], and recently received a categorical foundation in [42].

In [52] the task of finding correct nondeterministic extensions of the notion of λ-theory in the case of the untyped calculus has been carried out, distinguising two value-passing mechanisms (namely β-rules), which are viewed as relativizing call-by-name and call-by-value to the choice operator. The first rule allows unrestricted substitution, in such a way that choices can be multiplied and performed at any time during evaluation; this has been called run-time-choice in [25]. The second rule forces passed arguments to be "deterministic," that is, to perform the reduction $(\lambda x.M)\, N \to M[N/x]$, N needs to be free of any occurrence of the choice operator. This is called call-time-choice in [25].

DEFINITION 6.1. The binary relations $\to_{\text{r}}, \to_{\text{c}} \subseteq \Lambda_{\oplus}^2$ are obtained from the following clauses, closing under contexts:

(β_{r}) $(\lambda x.M)\, N \to_{\text{r}} M[N/x]$;

(β_{c}) $(\lambda x.M)\, N \to_{\text{c}} M[N/x]$ if $N \in \Lambda$;

($\oplus$) $M \oplus N \to_* M$, $M \oplus N \to_* N$, for $* = \text{r, c}$.

The relation $\to_{\text{r}}$ is clearly the reduction relation studied in the previous sections of this paper. The relation $\to_{\text{c}}$

induces a nondeterministic calculus which, as we will see, is equivalent to a calculus of finite sets of classical terms.

In the next subsections we shall consider Sharma's theories and give new proofs of their consistency, relying on our semantical results for the case of run-time-choice theory, here called λ_r. Moreover, we will prove that both λ_r and λ_c (that is, the call-time-choice theory) are conservative extensions of the theory λ, and that the theory of the model D_*, which we call $\mathscr{T}_{\text{must}}$, is conservative with respect to the maximal consistent λ-theory $\mathscr{H}^*$. Since λ_r is a subtheory of $\mathscr{T}_{\text{must}}$, this can be considered as an alternative proof of its consistency.

6.1. Call-Time Choice λ-Calculus

DEFINITION 6.2. The theory λ_c is the equational theory over $\Lambda_\oplus$ whose axioms and rules are as follows:

(β_c) $(\lambda x.M)\,N = M[N/x]$ if $N \in \Lambda$

(ρ) $M = M$

(σ) $M = M \Rightarrow N = M$

(τ) $M = N, N = L \Rightarrow M = L$

(μ) $M = N \Rightarrow LM = LN$

(ν) $M = N \Rightarrow ML = NL$

(ξ) $M = N \Rightarrow \lambda x.M = \lambda x.N$

(ζ_1) $M \oplus M = M$

(ζ_2) $M \oplus N = N \oplus M$

(ζ_3) $(M \oplus N) \oplus L = M \oplus (N \oplus L)$

(ε) $M = N \Rightarrow M \oplus L = N \oplus L$

(δ) $(M \oplus N)\,L = ML \oplus NL$

(θ) $L(M \oplus N) = LM \oplus LN$

(γ) $\lambda x.M \oplus N = \lambda x.M \oplus \lambda x.N$.

In [52] it was actually proved that a subtheory of λ_c was consistent, namely the theory obtained by deleting axiom (γ). The result was established by defining a notion of reduction (different from $\rightarrow_c$) essentially by orienting from left to right the axioms of λ_c, and then proving a Church–Rosser theorem.

The proof of the consistency theorem was, however, very long, and the difficulty with the axiom (γ) could not be overcome. In contrast, we give here a very short proof of the consistency of the whole theory, in a way that, in our opinion, enlightens the fact that the λ_c-calculus is nothing more than a calculus of (closures under β-conversion of) finite sets of classical terms.

As a first step we prove a simple property of the reduction relation $\rightarrow_c$, that fails in case of $\rightarrow_r$.

Notation. We will write $\rightarrow_{\beta_c}$ when a one step β_c-contraction occurs; similarly we write $\rightarrow_\oplus$ when only one $\oplus$-contraction occurs. $\xrightarrow{*}_{\beta_c}$ and $\xrightarrow{*}_\oplus$ are their reflexive and transitive closures, respectively.

LEMMA 6.3.

$\forall M,\ M_1,\ M_2 \in \Lambda_\oplus . M \rightarrow_{\beta_c} M_1 \rightarrow_\oplus M_2 \Rightarrow \exists M_3 \in \Lambda_\oplus . M \rightarrow_\oplus M_3 \rightarrow_{\beta_c} M_2$, *that is*

$$\begin{tikzcd} M \arrow[r, "\beta_c"'] \arrow[d, dashed, "\oplus"'] & M_1 \arrow[d, "\oplus"] \\ M_3 \arrow[r, dashed, "\beta_c"'] & M_2 \end{tikzcd}$$

Proof. By induction on M, and then by cases. The interesting case is when $M \equiv (\lambda x.M')\,M''$ and $M_1 \equiv M'[M''/x]$; in $M_1 \rightarrow_\oplus M_2$ the only possibility is that a (residual of a) $\oplus$ redex in M' is contracted, since it must be the case that $M'' \in \Lambda$. It follows that $M' \equiv C[P_1 \oplus P_2]$ for some P_1 and P_2 and $M_1 \equiv C'[P_1[M''/x] + P_2[M''/x]$ if x is not bounded above in $C[\ ]$ and $C'[\ ]$ results from $C[\ ]$ substituting M'' for all free occurrences of x; in this case $M_2 \equiv C'[P_i[M''/x]]$ for $i = 1$ or 2. Then

$$\begin{aligned}(\lambda x.M')\,M'' &\rightarrow_\oplus (\lambda x.C[P_i])\,M'' \rightarrow_{\beta_c} (C[P_i])[M''/x] \\ &\equiv C'[P_i[M''/x]]\end{aligned}$$

so that we take $M_3 \equiv (\lambda x.C[P_i])\,M''$. If x is bounded above the hole $[\ ]$ in the context $C[\ ]$ the proof is similar and easier. ∎

COROLLARY 6.4.

$$\forall M, N \in \Lambda_\oplus . M \xrightarrow{*}_c N \Rightarrow \exists L \in \Lambda_\oplus . M \xrightarrow{*}_\oplus L \xrightarrow{*}_{\beta_c} N.$$

Proof. The proof is illustrated in the following picture, where vertical arrows represent one-step $\oplus$-reductions, horizontal arrows represent one-step β_c-reductions, and each square is an application of Lemma 6.3:

$$\begin{tikzcd} M \arrow[d] & & & & & \\ M_1 \arrow[r] \arrow[d, dashed] & M_2 \arrow[d] & & & & \\ L_1 \arrow[r, dashed] \arrow[d, dashed] & M_3 \arrow[r] \arrow[d, dashed] & M_4 \arrow[r] \arrow[d, dashed] & M_5 \arrow[d] & & \\ L \arrow[r, dashed] & L_2 \arrow[r, dashed] & L_3 \arrow[r, dashed] & M_6 \arrow[r] & M_7 \arrow[r] & N \end{tikzcd}$$ ∎

DEFINITION 6.5. Let $\mathscr{A} \subseteq \Lambda$ and $M, N \in \Lambda_{\oplus}$, then

(i) $\mathscr{A}^{+} = \{M \mid \exists N \in \mathscr{A} . M =_{\beta} N\}$;

(ii) $\mathsf{det}(M) = \{L \in \Lambda \mid M \xrightarrow{*}_{c} L\}^{+}$;

(iii) $M \sqsubseteq_{c} N \Leftrightarrow \mathsf{det}(M) \subseteq \mathsf{det}(N)$;

(iv) $M =_{c} N \Leftrightarrow M \sqsubseteq_{c} N \sqsubseteq_{c} M$.

The operation $(\cdot)^{+}$ is the usual closure under β-conversion; the intuitive meaning of $\mathsf{det}(M)$ is "the set of deterministic values of M".

DEFINITION 6.6. Let $\mathscr{A} \subseteq \Lambda$; $\mathscr{A}$ is β-closed iff $\mathscr{A} = \mathscr{A}^{+}$. Furthermore if $\mathscr{A}$, $\mathscr{B}$ are β-closed then

(i) $\mathscr{A}\mathscr{B} = \{MN \mid M \in \mathscr{A}, N \in \mathscr{B}\}^{+}$;

(ii) $\lambda x . \mathscr{A} = \{\lambda x . M \mid M \in \mathscr{A}\}^{+}$;

(iii) $\mathscr{A}[\mathscr{B}/x] = \{M[N/x] \mid M \in \mathscr{A}, N \in \mathscr{B}\}^{+}$.

LEMMA 6.7. *For any* $M, N \in \Lambda_{\oplus}$

(i) $\mathsf{det}(M \oplus N) = \mathsf{det}(M) \cup \mathsf{det}(N)$;

(ii) $\mathsf{det}(\lambda x . M) = \lambda x . \mathsf{det}(M)$;

(iii) $\mathsf{det}(MN) = \mathsf{det}(M)\, \mathsf{det}(N)$.

Proof. Parts (i) and (ii) are clear. To see (iii):

$$L \in \mathsf{det}(MN) \Rightarrow \exists L' \in \Lambda . MN \xrightarrow{*}_{c} L' =_{\beta} L$$

be definition; by Corollary 6.4 there is a $P \in \Lambda_{\oplus}$ s.t.

$$MN \xrightarrow{*}_{\oplus} P \xrightarrow{*}_{\beta_c} L';$$

now $MN \xrightarrow{*}_{\oplus} P$ implies that $P \equiv M'N'$, where $M \xrightarrow{*}_{\oplus} M'$ and $N \xrightarrow{*}_{\oplus} N'$. On the other hand, we note that no β_c contraction can delete an occurrence of a $\oplus$; e.g.,

$$\mathbf{K}L(M \oplus N) \not\xrightarrow{*}_{\beta_c} L,$$

since $M \oplus N \notin \Lambda$. It follows that $P \xrightarrow{*}_{\beta_c} L'$ implies $P \in \Lambda$ being $L' \in \Lambda$. We conclude that $L \in \mathsf{det}(M)\, \mathsf{det}(N)$, that is $\mathsf{det}(MN) \subseteq \mathsf{det}(M)\, \mathsf{det}(N)$. The inverse inclusion is clear. ∎

LEMMA 6.8.

$$M \in \Lambda_{\oplus}, N \in \Lambda \Rightarrow \mathsf{det}(M[N/x]) = \mathsf{det}(M)[\mathsf{det}(N)/x].$$

Proof. By induction on M using Lemma 6.7. The only nontrivial case is when $M \equiv M_1 M_2$; now

$$\begin{aligned}
&\mathsf{det}((M_1 M_2)[N/x]) \\
&\quad = \mathsf{det}(M_1[N/x]\, M_2[N/x]) \\
&\quad = \mathsf{det}(M_1[N/x])\, \mathsf{det}(M_2[N/x]) \\
&\qquad \text{by Lemma 6.7 (iii)} \\
&\quad = \mathsf{det}(M_1)[\mathsf{det}(N)/x]\, \mathsf{det}(M_2)[\mathsf{det}(N)/x] \\
&\qquad \text{by the inductive hypothesis} \\
&\quad = \mathsf{det}(M_1 M_2)[\mathsf{det}(N)/x] \\
&\qquad \text{since } N \in \Lambda,
\end{aligned}$$

where in the last step above we observe that, since $N \in \Lambda$, $\mathsf{det}(N)$ is a set of β-convertible terms; now in the classical calculus we know that

$$Q_1 =_{\beta} Q_2 \Rightarrow P[Q_1/x] =_{\beta} P[Q_2/x],$$

from which it follows that

$$\begin{aligned}
(P_1[Q_1/x])(P_2[Q_2/x]) &=_{\beta} (P_1 P_2)[Q_1/x] \\
&=_{\beta} (P_1 P_2)[Q_2/x]. \quad ∎
\end{aligned}$$

THEOREM 6.9. *For any* $M, N \in \Lambda_{\oplus}$,

$$\lambda_c \vdash M = N \Rightarrow M =_{c} N.$$

Proof. Using Lemma 6.7 and 6.8, we check that axioms of λ_c are satisfied by the relation $=_c$; indeed the only interesting case is that of (β_c): let $M \in \Lambda_{\oplus}$ and $N \in \Lambda$, then using Lemma 6.7 we have

$$\begin{aligned}
\mathsf{det}((\lambda x . M)\, N) &= \mathsf{det}(\lambda x . M)\, \mathsf{det}(N) \\
&= (\lambda x . \mathsf{det}(M))\, \mathsf{det}(N) \\
&= \mathscr{A},
\end{aligned}$$

say, then

$$\begin{aligned}
P \in \mathscr{A} &\Leftrightarrow \exists Q \in \mathsf{det}(M) . P =_{\beta} (\lambda x . Q)\, N =_{\beta} Q[N/x] \\
&\Leftrightarrow P \in \mathsf{det}(M)[\mathsf{det}(N)/x] = \mathsf{det}(M[N/x]),
\end{aligned}$$

by Lemma 6.8. ∎

We conjecture that $M =_{c} N \Rightarrow \lambda_c \vdash M = N$.

COROLLARY 6.10. *The theory* λ_c *is consistent.*

Proof. For any $M, N \in \Lambda$

$$M =_{c} N \Leftrightarrow \mathsf{det}(M) = \mathsf{det}(N) \Leftrightarrow M =_{\beta} N,$$

that is $=_c$ restricted to Λ coincides with $=_{\beta}$. This implies that the theory induced by $=_c$ is a conservative extension of λ: hence it is consistent. Then, by the theorem, λ_c is consistent. ∎

In force of Corollary 6.4 it follows that $\mathsf{det}(M)$ is essentially the "immediate deterministic structure" of the discussion at the beginning of Section 3. Therefore the meaning of Theorem 6.9 is that the choice structure gets definitely lost in this calculus.

6.2. Run-Time Choice λ-Calculus

In previous section we studied the properties of the relation $\to_r$ and gave both operational and denotational characterizations of the equivalence it induces over terms. In this section our aim is to present an axiomatization of this equivalence (even if not a complete one), allowing to compare this relation with the β-convertibility relation of the classical λ-calculus. Here too, as in the case of the theory λ_c, we get inspiration from [52] and [15].

DEFINITION 6.11. The theory λ_r is the equational theory over $\Lambda_\oplus$ which results from λ_c substituting (β_c) with the unrestricted axiom

$$(\beta_r) \quad (\lambda x.M)\,N = M[N/x]$$

and eliminating axioms (θ) and (γ).

This theory was proved consistent in [52] with syntactical methods: for us it is actually a corollary of previous results.

THEOREM 6.12. *The syntactical model D_* is a nontrivial model of λ_r, hence λ_r is consistent.*

Proof. By Definition 4.13, Lemma 4.14, and Proposition 4.16 we know that D_* is a syntactical model (actually an extensional one); now the proof that the equations and rules up to (ξ) are valid runs in the classical way; the rest is an immediate consequence of the definition of the interpretation of $\oplus$ and of the semilinearity of D_*. ∎

In his proof Sharma did not prove consistency of the full theory, which in his formulation also had among its axioms

$$(\gamma) \quad \lambda x.M \oplus N = \lambda x.M \oplus \lambda x.N.$$

We do not include this axiom in the theory λ_r because of its special status, illustrated in the following proposition. Note, however, that (γ) holds in D_*; hence the previous theorem readily extends to the theory $\lambda_r + \gamma$.

Another equation which appeared in the literature (see [5]) is

$$(\iota) \quad M \oplus N = \lambda x.(Mx \oplus Nx) \quad \text{if } x \notin FV(M \oplus N).$$

This is clearly connected with (actually equivalent to) the classical axiom η.

PROPOSITION 6.13.

(i) $\lambda_r + \eta \vdash \gamma$;

(ii) $\lambda_r + \iota \vdash \eta$.

Proof. (i) Let $y \notin FV(M) \cup FV(N)$:

$$\begin{aligned}(\lambda x.M \oplus N)\,y &= M[y/x] \oplus N[y/x] && \text{by } (\beta_r)\\ &= (\lambda x.M)\,y \oplus (\lambda x.N)\,y && \text{by } (\beta_r)\\ &= (\lambda x.M \oplus \lambda x.N)\,y && \text{by } (\delta),\end{aligned}$$

from which $\lambda x.M \oplus N = \lambda x.M \oplus \lambda x.N$ follows by (η).

(ii) Let $x \notin FV(M)$:

$$\begin{aligned}M &= M \oplus M && \text{by } (\zeta_1)\\ &= \lambda x.Mx \oplus Mx && \text{by } (\iota)\\ &= \lambda x.Mx && \text{by } (\zeta_1).\end{aligned}$$

Remark 6.14. The independence of the axiom (γ) from the theory λ_r has been proved in [17]: this has been shown by building a model $N_* = N_*^\sharp$ of the theory λ_r in the category **CPO**, using Moggi's strong monads (see [42]); this model, which is nonextensional, does not satisfy (γ).

We leave as an open question whether the theory $\lambda_r + \gamma$ is extensional, but we conjecture it is not.

6.3. Conservativity of the Theory $\mathscr{T}_{\text{must}}$

In this section we finally compare the theory induced by the equivalence studied throughout this paper with the theories $\lambda + \eta$ and $\mathscr{H}^*$. The main theorem is a conservativity result; it is readily seen that this can be considered an alternative (syntactical) proof of the consistency for the theory $\mathscr{T}_{\text{must}}$ defined below.

DEFINITION 6.15. $\mathscr{T}_{\text{must}} = \{M = N \mid M, N \in \Lambda_\oplus^0, M \simeq_{\text{must}} N\}$.

We know from the semantic construction in the previous sections that

PROPOSITION 6.16. *The theory $\mathscr{T}_{\text{must}}$ is the theory of the model D_*; hence it is consistent.*

Proof. An immediate consequence of the full abstraction theorem. ∎

This fact is not illuminating, however, with respect to the question of conservativity. To establish this result we are going to prove a "simulation lemma" which says that contexts containing the $\oplus$ operator do not discriminate more w.r.t. must-convergency than classical contexts do.

DEFINITION 6.17. Let $M \in \Lambda_\oplus$, $\mathscr{F} \subseteq M$ a subset of the set of redexes occurring in M, and σ a head reduction starting with M; then

(i) σ is *finitely often* in $\mathscr{F}$ iff

$$\exists m\, \forall n \geq m.\sigma_{0,n+1}: M \xrightarrow{*}_r M_n \xrightarrow{\Delta}_r M_{n+1} \Rightarrow \Delta \notin \mathscr{F}/\sigma_{0,n};$$

let σ be any head reduction of M finitely often in $\mathscr{F}$; then

$$\deg(\mathscr{F}, \sigma) = \max\{n \mid \exists m \in \mathbb{N}\ \exists R \in \mathscr{F}/\sigma_{0,m} . M \xrightarrow{m}_h \lambda\vec{x}.RM_1 \cdots M_n\};$$

(iii) let $\sigma_1, ..., \sigma_n$ be head reductions of M finitely often in $\mathscr{F}$; then

$$\deg(\mathscr{F}, \sigma_1, ..., \sigma_n) = \max\{\deg(\mathscr{F}, \sigma_i) \mid 1 \leqslant i \leqslant n\}.$$

Remark 6.18. σ is *finitely often* in $\mathscr{F}$ iff it contracts at most a finite number of (residuals of) redexes in $\mathscr{F}$.

The next lemma is based on the idea that, if a context containing the operator $\oplus$ converges on a term M, while it diverges on a term N, the choices caused by the $\oplus$'s inside the context which are essential for this convergence–divergence property are bounded above by those which are necessary to converge on M. On the other hand, we know that all the reductions on M will converge, while there is a diverging reduction on N. The point is to simulate the choices of this last reduction, encoding them in a classical context. Because of its complexity we defer the proof of the next lemma to Section 7.

LEMMA 6.19 (Simulation Lemma). *Given $M, N \in \Lambda_\oplus$,*

$$\exists D[\ ] \in \Lambda_\oplus[\ ].D[M]\downarrow \wedge D[N]\uparrow \Rightarrow \exists C[\ ] \in \Lambda[\ ].C[M]\downarrow \wedge C[N]\uparrow.$$

We remind the reader that SOL is the set of classical terms reducing to a head normal form, and that

$$\mathscr{H}^* = \{M = N \mid M, N \in \Lambda^0, \forall C[\ ] \in \Lambda[\ ].C[M] \in \mathrm{SOL} \Leftrightarrow C[N] \in \mathrm{SOL}\}.$$

It is known that $\mathscr{H}^*$ is the theory of the models D_∞. We conclude the present section with the conservativity theorem for $\mathscr{T}_{\mathrm{must}}$.

THEOREM 6.20. (i) $\lambda_r + \gamma \subseteq \mathscr{T}_{\mathrm{must}}$,

(ii) *$\mathscr{T}_{\mathrm{must}}$ is a conservative extension of $\mathscr{H}^*$,*

(iii) *$\mathscr{T}_{\mathrm{must}}$ is a conservative extension of $\lambda + \eta$.*

Proof. To prove the first part, simply note that $\mathscr{T}_{\mathrm{must}}$ is the theory of a model of λ_r, and that this model validates γ since it is extensional by Lemma 4.14 (alternatively one can use the algebraic semantics to prove that the axioms of λ_r are included in $\mathscr{T}_{\mathrm{must}}$). As to (ii): let $M, N \in \Lambda^0$ be such that $\mathscr{T}_{\mathrm{must}} \not\vdash M = N$; then there is a context $D[\ ] \in \Lambda_\oplus[\ ]$ such that, say, $D[M]\downarrow$ and $D[N]\uparrow$. By Lemma 6.19, there is a context $C[\ ] \in \Lambda[\ ]$ such that $C[\ ]\downarrow$, that is, $C[M] \in \mathrm{SOL}$ and $C[N] \notin \mathrm{SOL}$; hence $\mathscr{H}^* \not\vdash M = N$. It follows that $\mathscr{H}^* \subseteq \mathscr{T}_{\mathrm{must}}$. On the other hand, and *a fortiori*, if $M, N \in \Lambda^0$, then

$$\begin{aligned}\mathscr{T}_{\mathrm{must}} \vdash M = N &\Rightarrow \forall D[\ ] \in \Lambda_\oplus[\ ].D[M]\downarrow \Leftrightarrow D[N]\downarrow \\ &\Rightarrow \forall C[\ ] \in \Lambda[\ ].C[M] \in \mathrm{SOL} \Leftrightarrow C[N] \in \mathrm{SOL} \\ &\Rightarrow \mathscr{H}^* \vdash M = N,\end{aligned}$$

since the restriction of the predicate $\downarrow$ to Λ coincides with the set SOL.

Finally (iii) follows from (ii) and the fact that $\mathscr{H}^*$ is an extensional λ-theory. ∎

7. DETAILED PROOFS

In this section we present in detail those proofs which, for the sake of readability, have been omitted from the previous text.

7.1. Semi-Separability

7.1.1. *Head Contexts*

DEFINITION 7.1. (i) A context is a *head context* iff it is of the form

$$C[\ ] \equiv (\lambda x_1 \cdots x_n.[\ ])\, X_1 \cdots X_n U_1 \cdots U_m;$$

(ii) abbreviate $x_1 \cdots x_n$ with $\vec{x}$, $X_1 \cdots X_n$ with $\vec{X}$ and $U_1 \cdots U_m$ with $\vec{U}$; similarly consider a context $D[\ ] \equiv (\lambda y_1 \cdots y_h.[\ ])\, Y_1 \cdots Y_h V_1 \cdots V_k$, abbreviated $(\lambda\vec{y}.[\ ])\, \vec{Y}\vec{V}$; then define

$$D \bullet C[\ ] \equiv (\lambda\vec{x}\vec{z}.[\ ])\, \vec{X}^\circ \vec{Z} \vec{U}^\circ \vec{V},$$

where $\circ = [\vec{Y}/\vec{y}]$, and $\vec{z} \equiv z_1 \cdots z_r \equiv y_{i_1} \cdots y_{i_r}$, if $\{y_{i_1} \cdots y_{i_r}\} = \{y_1, ..., y_h\} - \{x_1, ..., x_n\}$ and $\vec{Z} \equiv Y_{i_1} \cdots Y_{i_r}$.

LEMMA 7.2. *Let $C[\ ], D[\ ]$ be head contexts; then, for any $M \in \Lambda_\oplus$ and $k \geqslant 1$: if $\omega^k(M) = \{M_1, ..., M_l\}$ then:*

(i) $\omega^k(C[M])$

$$= \begin{cases} \omega^k(C[M_1]) \cup \cdots \cup \omega^k(C[M_l]) & \text{if } C[M]\downarrow \\ \{\mathbf{\Omega}\} & \text{otherwise} \end{cases}$$

(ii) $\omega^k(D \bullet C[M]) = \omega^k(D[C[M]])$.

Proof. Let $C[\ ] \equiv (\lambda x_1 \cdots x_n.[\ ])\, X_1 \cdots X_n U_1 \cdots U_m$; then

$$\begin{aligned} C[M] &\xrightarrow{*}_h M^* U_1 \cdots U_m \\ &\xrightarrow{*}_h M_i^* U_1 \cdots U_m \\ &{}_h\!\xleftarrow{*} C[M_i] \end{aligned}$$

for $i = 1, ..., l$, where $* = [X_1/x_1, ..., X_n/x_n]$. It follows that L is a principal hnf of $C[M]$ iff for some $i \leqslant l$, L is a principal hnf of $C[M_i]$, establishing (i).

To prove (ii) let $D[\ ] \equiv (\lambda \vec{y}.[\ ])\ \vec{Y}\vec{V}$, and $C[\ ]$ as above; then, reasoning as for (i), we have

$$\begin{aligned} D[C[M]] &\equiv (\lambda \vec{y}.(\lambda \vec{x}.M)\ \vec{X}\vec{U})\ \vec{Y}\vec{V} \\ &\xrightarrow{*}_h (\lambda \vec{x}.M)^\circ\ \vec{X}^\circ \vec{U}^\circ \vec{V} \\ &\equiv (\lambda \vec{x}.M^\diamond)\ \vec{X}^\circ \vec{U}^\circ \vec{V} \\ &\xrightarrow{*}_h M^{\diamond\star}\ \vec{U}^\circ \vec{V} \\ &{}_h\xleftarrow{*} (\lambda \vec{x}\vec{z}.M)\ \vec{X}^\circ \vec{Z}\vec{U}^\circ \vec{V} \\ &\equiv D \bullet C[M], \end{aligned}$$

where $= [\vec{Y}/\vec{y}]$, $\diamond = [\vec{Z}/\vec{z}]$, $\star = [\vec{X}^\circ/\vec{x}]$, and $\vec{Z}$, $\vec{z}$ are as in Definition 7.1. ∎

7.1.2. *Test for Equality*

LEMMA 7.3 (See Lemma 3.19). *There exists a combinator* $\mathbf{H} \in \Lambda$ *of the shape*

$$\mathbf{H} \equiv \lambda xy.xH_1 \cdots H_l,$$

with $x \notin FV(H_1) \cup \cdots \cup FV(H_l)$, *such that, for all non-negative integers* n, m,

$$\mathbf{Hnm} =_{\beta\eta} \begin{cases} \mathbf{1} & \text{if } n = m \\ \mathbf{0} & \text{otherwise} \end{cases}$$

Proof. To build $\mathbf{H}$ we solve the following system of equations in the theory $\lambda\beta\eta$:

$$\begin{cases} \mathbf{H00} & = \mathbf{1} \\ \mathbf{H0(Succ}\ y) & = \mathbf{0} \\ \mathbf{H(Succ}\ x)\ \mathbf{0} & = \mathbf{0} \\ \mathbf{H(Succ}\ x)(\mathbf{Succ}\ y) & = \mathbf{H}xy. \end{cases}$$

Let $\mathbf{H} = \lambda uv.uPQ(vRT)$, where the combinators P, Q, R, T will be specified later. We compute

$$\begin{aligned} \mathbf{H00} &= \mathbf{0}PQ(\mathbf{0}RT) \\ &= QT; \\ \mathbf{H0(Succ}\ y) &= \mathbf{0}PQ(\mathbf{Succ}\ yRT) \\ &= Q(R(yRT)); \\ \mathbf{H(Succ}\ x)\ \mathbf{0} &= \mathbf{Succ}\ xPQ(\mathbf{0}RT) \\ &= P(xPQ)\ T; \\ \mathbf{H(Succ}\ x)(\mathbf{Succ}\ y) &= \mathbf{Succ}\ xPQ(\mathbf{Succ}\ yRT) \\ &= P(xPQ)(R(yRT)). \end{aligned}$$

Now we choose

$$\begin{aligned} P &\equiv \lambda ab.b\mathbf{O}a \\ Q &\equiv \lambda a.a\mathbf{K} \\ R &\equiv \lambda ab.b(\mathbf{K0})\ \mathbf{C}_* a \\ T &\equiv \lambda a.a\mathbf{1}(\mathbf{K0}) \end{aligned}$$

where $\mathbf{O} \equiv \lambda ab.b$ and $\mathbf{C}_* \equiv \lambda ab.ba$; it is straightforward to see that these choices give the desired result: in particular for the fourth equation we have

$$\begin{aligned} P(xPQ)(R(yRT)) &= R(yRT)\ \mathbf{O}(xPQ) \\ &= \mathbf{O}(\mathbf{K0})\ \mathbf{C}_*(yRT)(xPQ) \\ &= \mathbf{C}_*(yRT)(xPQ) \\ &= xPQ(yRT) \\ &= \mathbf{H}xy. \end{aligned}$$ ∎

COROLLARY 7.4 (See Corollary 3.20). *If* $N \equiv \mathbf{n}_1 \oplus \cdots \oplus \mathbf{n}_r$, *with* $r \geqslant 1$, *then, for all* m,

$$\omega^1(\mathbf{H}N\mathbf{m}) = \{\mathbf{0} \mid \exists i \leqslant r.n_i = m\} \cup \{\mathbf{1} \mid \exists j \leqslant r.n_j \neq m\}.$$

Proof. From the shape of $\mathbf{H}$ one sees that, when applied to closed terms, it behaves like the head context $\lambda y.[\ ]\ H_1 \cdots H_l$; on the other hand the $=_{\beta\eta}$ in the lemma is actually $\rightarrow_{\beta\eta}$, because the numerals $\mathbf{0}$ and $\mathbf{1}$ are normal forms. Since $\rightarrow_{\beta\eta} \subseteq \rightarrow$, and using the standardization theorem we have that each exhaustive head reduction of $\mathbf{H}N\mathbf{m}$ has to start with

$$\begin{aligned} \mathbf{H}N\mathbf{m} &\rightarrow NH_1[\mathbf{m}/y] \cdots H_l[\mathbf{m}/y] \\ &\rightarrow \mathbf{n}_i H_1[\mathbf{m}/y] \cdots H_l[\mathbf{m}/y] \end{aligned}$$

for some $1 \leqslant i \leqslant r$; now the corollary follows from Lemma 7.3 and Lemma 7.2. ∎

7.1.3. *Semi-separability Lemmas*

LEMMA 7.5 (See Lemma 3.21). *For* $M, N \in \Lambda_\oplus$,

$$M \not\leqslant_2 N \Rightarrow \exists C[\ ].C[M]\downarrow \wedge C[N]\uparrow.$$

Proof. By cases.

Case 1. When $M \not\leqslant_1 N$, then either $\omega^1(N) = \mathcal{N} = \{\Omega\}$ and $\omega^1(M) = \mathcal{M} \neq \{\Omega\}$, so that there is nothing to prove, or there exists $N' \in \mathcal{N}$ such that, for all $M' \in \mathcal{M}$, $M' \not\prec N'$. If $(\mathcal{M} \cup \mathcal{N})_{/\sim} = \{[L_1], ..., [L_h]\}$, then there is an i such that $N' \in [L_i]$ and $[L_i] \cap \mathcal{M} = \varnothing$. By Lemma 3.18 (ii) we know that there is a context $C[\ ]$ such that $\omega^1(C[N']) = \{z_i\}$ for some variable z_i, and for all $M' \in \mathcal{M}$ there is a $j \neq i$ s.t. $\omega^1(C[M']) = \{z_j\}$, where $z_i \doteq z_j$. Now let

$$C'[\ ] \equiv (\lambda z_1 \cdots z_h.[\ ])\, z_1 \cdots z_{i-1} (\Delta\Delta)\, z_{i+1} \cdots z_h.$$

This is a head context; hence by Lemma 7.2 and by the fact that head contexts are closed under composition we conclude that $\omega^1(C' \bullet C[M])$ is a subset of $\{z_1, ..., z_h\} - \{z_i\}$, and that $\omega^1(C' \bullet C[N]) = \omega^1(C' \bullet C[N']) = \{\Omega\}$.

Case 2. When $M \leqslant_1 N$ but $M \not\leqslant_2 N$, then by definition we have

$$M \leqslant_1 N \wedge M \not\leqslant_2 N \Rightarrow \exists \langle \mathsf{U}, \mathsf{V} \rangle \in \mathsf{Pair}_1(\mathcal{M}, \mathcal{N}) = \mathsf{Pair}(\mathcal{M}, \mathcal{N}).\mathsf{U} \not\sqsubseteq^{\#} \mathsf{V};$$

this means that, for some $[P] \in (\mathcal{M} \cup \mathcal{N})_{/\sim}$, letting $\mathcal{M}_{[P]} = [P] \cap \mathcal{M}$ and similarly letting $\mathcal{N}_{[P]} = [P] \cap \mathcal{N}$, we have

$$\langle \mathsf{U}, \mathsf{V} \rangle \in \mathsf{Pair}(\mathcal{M}_{[P]}, \mathcal{N}_{[P]}).$$

By Lemma 3.18 (i) there exist a context $C[\ ]$ and an integer r such that

$$\begin{aligned} \omega^2(C[M]) &= \omega^2(C[\mathcal{M} - \mathcal{M}_{[P]}]) \cup \omega^2(C[\mathcal{M}_{[P]}]) \\ &= \{y\} \cup \{x\mathcal{M}_1^1 \cdots \mathcal{M}_r^1, ..., x\mathcal{M}_1^n \cdots \mathcal{M}_r^n\} \end{aligned}$$

and

$$\begin{aligned} \omega^2(C[N]) &= \omega^2(C[\mathcal{N} - \mathcal{N}_{[P]}]) \cup \omega^2(C[\mathcal{N}_{[P]}]) \\ &= \{y\} \cup \{x\mathcal{N}_1^1 \cdots \mathcal{N}_r^1, ..., x\mathcal{N}_1^m \cdots \mathcal{N}_r^m\}, \end{aligned}$$

so that, for some $1 \leqslant i \leqslant r$, it must be the case that

$$\mathsf{U} = \{\mathcal{M}_i^1, ..., \mathcal{M}_i^n\} \quad \text{and} \quad \mathsf{V} = \{\mathcal{N}_i^1, ..., \mathcal{N}_i^m\}.$$

In the sequel we assume that

$$\forall j \leqslant n, k \leqslant m.x \notin FV(\mathcal{M}_i^j) \wedge x \notin FV(\mathcal{N}_i^k);$$

there is no theoretical loss, since as in the classical λ-calculus, one can use the technique of Böhm transformations to make these occurrences harmless (see [9, Sect. 10.3]).

Since $\mathsf{U} \not\sqsubseteq^{\#} \mathsf{V}$, there exists $1 \leqslant k \leqslant m$ such that, for all $1 \leqslant j \leqslant n$, we have $\mathcal{M}_i^j \not\leqslant_1 \mathcal{N}_i^k$: then $\mathcal{M}_i^j \neq \{\Omega\}$ for all j. Now we have two subcases.

Subcase 2.1. $\mathcal{N}_i^k = \{\Omega\}$: it follows that, taking $C'[\ ] \equiv (\lambda x.[\ ])(\lambda a_1 \cdots a_r.a_i)$, we have

$$\omega^2(C'[\{x\mathcal{M}_1^1 \cdots \mathcal{M}_r^1, ..., x\mathcal{M}_1^n \cdots \mathcal{M}_r^n\}]) = \{\mathcal{M}_i^1, ..., \mathcal{M}_i^n\},$$

while

$$\omega^2(C'[\{x\mathcal{N}_1^1 \cdots \mathcal{N}_r^1, ..., x\mathcal{N}_1^m \cdots \mathcal{N}_r^m\}]) = \{\Omega\};$$

it follows that $C' \bullet C[M] \downarrow$ and $C' \bullet C[N] \uparrow$.

Subcase 2.2. $\mathcal{N}_i^k \neq \{\Omega\}$: then for each $1 \leqslant j \leqslant n$ there is a $[Q_j] \in (\bigcup \mathsf{U} \cup \mathcal{N}_i^k)_{/\sim}$ s.t.

$$[Q_j] \subseteq \mathcal{N}_i^k - \mathcal{M}_i^j. \tag{1}$$

Using Lemma 3.18 (ii) we know that there exists a head context $\tilde{C}[\ ]$ transforming $\bigcup \mathsf{U} \cup \mathcal{N}$ into a set of variables having the same cardinality as $(\bigcup \mathscr{U} \cup \mathcal{N}_i^k)_{/\sim}$, say $\{z_1, ..., z_h\}$. (1) now implies that

$$\forall j \exists l \leqslant h.z_l \in \omega^1(\tilde{C}[\mathcal{N}_i^k]) - \omega^1(\tilde{C}[\mathcal{M}_i^j]). \tag{2}$$

We define $\bar{C}[\ ]$ as the composition $((\lambda z_1 \cdots z_h.[\ ])\, \mathbf{1} \cdots \mathbf{h}) \bullet \tilde{C}[\ ]$. Say that $|(\mathcal{N}_i^k)_{/\sim}| = l$: then we take

$$C'[\ ] \equiv (\lambda x.[\ ])(\lambda y_1 \cdots y_l.v\underbrace{\bar{C}[y_i] \cdots \bar{C}[y_i]}_{l}).$$

Set $\mathscr{X}^j = \omega^h(\bar{C}[\mathcal{M}_i^j])$ and $\mathscr{Y} = \omega^h(\bar{C}[\mathcal{N}_i^k])$; they are sets of Church numerals, and, by (2), no $\mathscr{X}^j$ contains all the numerals in $\mathscr{Y}$. Now

$$\omega^h(C' \bullet C[M]) = \{y, v\underbrace{\mathscr{X}^1 \cdots \mathscr{X}^1}_{l}, ..., v\underbrace{\mathscr{X}^n \cdots \mathscr{X}^n}_{l}\},$$

$$\omega^h(C' \bullet C[N]) = \{y, v\underbrace{\mathscr{Y} \cdots \mathscr{Y}}_{l}\}.$$

Using the combinator $\mathbf{H}$ of Lemma 3.19, we finally define

$$C''[\ ] \equiv (\lambda v.[\ ])(\lambda v_1 \cdots v_l.\mathbf{P}_l(\mathbf{H}v_1\mathbf{1}) \cdots (\mathbf{H}v_l\mathbf{l}))(\mathbf{K}(\Delta\Delta))\,\mathbf{I},$$

where $\mathbf{P}_l$ λ-defines the numeric function $\prod_{i=1}^{l} n_i$. It follows that, by Corollary 3.20,

$$\omega^1(C'' \bullet C' \bullet C[M]) = \{\mathbf{0}\}(\mathbf{K}(\Delta\Delta))\,\mathbf{I} = \{\mathbf{I}\},$$

while

$$\begin{aligned} \omega^1(C'' \bullet C' \bullet C[N]) &= \begin{cases} \{\mathbf{0}, \mathbf{1}\}(\mathbf{K}(\Delta\Delta))\,\mathbf{I} \\ \{\mathbf{1}\}(\mathbf{K}(\Delta\Delta))\,\mathbf{I} \end{cases} \text{or} \\ &= \{\Omega\} \end{aligned}$$

according to the numerals in $\mathscr{Y}$. ∎

LEMMA 7.6 (See Lemma 3.23). *For* $M, N \in \Lambda_{\oplus}$ *and* $k \geqslant 2$,

$$M \not\leqslant_k N \Rightarrow \exists C[\ ].C[M] \not\leqslant_2 C[N].$$

Proof. By induction on k. Suppose $k>2$: then, letting $\mathcal{M}=\omega^k(M)$ and $\mathcal{N}=\omega^k(N)$, there are two subcases: either $\mathcal{M}\not\leqslant_{k-1}\mathcal{N}$, in which case the thesis follows directly from the inductive hypothesis, or $\mathcal{M}\leqslant_{k-1}\mathcal{N}$ and $\mathcal{M}\not\leqslant_k\mathcal{N}$. In the last case by definition we have that

$$\exists\langle \mathsf{U},\mathsf{V}\rangle\in\mathsf{Pair}_{k-1}(\mathcal{M},\mathcal{N}).\,\mathsf{U}\not\sqsubseteq^{\sharp}\mathsf{V}.$$

This implies the existence of $\langle \mathsf{U}',\mathsf{V}'\rangle\in\mathsf{Pair}_1(\mathcal{M},\mathcal{N})$ s.t. $\langle \mathsf{U},\mathsf{V}\rangle\in\mathsf{Pair}_{k-2}(\bigcup \mathsf{U}',\ \bigcup \mathsf{V}')$. It follows that, for some $[P]\in(\mathcal{M}\cup\mathcal{N})_{/\sim}$ the sets $\mathcal{M}_{[P]}$ and $\mathcal{N}_{[P]}$, defined as in the proof of the previous lemma, are nonempty and

$$\langle \mathsf{U},\mathsf{V}\rangle\in\mathsf{Pair}_{k-2}(\mathcal{M}_{[P]},\mathcal{N}_{[P]}).$$

Using Lemma 3.18, we can find a context $C[\ ]$ "selecting" P; thus we get

$$\omega^k(C[M])=\{y,x\mathcal{M}_1^1\cdots\mathcal{M}_r^1,\ldots,x\mathcal{M}_1^n\cdots\mathcal{M}_r^n\}$$
$$\omega^k(C[N])=\{y,x\mathcal{N}_1^1\cdots\mathcal{N}_r^1,\ldots,x\mathcal{N}_1^m\cdots\mathcal{N}_r^m\}$$

and, for some $1\leqslant i\leqslant r$,

$$\mathsf{U}'=\{\mathcal{M}_i^1,\ldots,\mathcal{M}_i^n\},\quad \mathsf{V}'=\{\mathcal{N}_i^1,\ldots,\mathcal{N}_i^m\}.$$

Again, w.l.o.g. we suppose that x does not occur free in any term in U' or in V', and we choose $C'[\ ]\equiv(\lambda x.[\ ])(\lambda a_1\cdots a_r.a_i)$, so that by Lemma 7.2,

$$\omega^k(C'\bullet C[M])=\{y\}\cup\mathcal{M}_i^1\cup\cdots\cup\mathcal{M}_i^n=_{\text{def}}\bar{\mathcal{M}}$$
$$\omega^k(C'\bullet C[M])=\{y\}\cup\mathcal{N}_i^1\cup\cdots\cup\mathcal{N}_i^m=_{\text{def}}\bar{\mathcal{N}};$$

hence $\langle \mathsf{U},\mathsf{V}\rangle\in\mathsf{Pair}_{k-2}(\bar{\mathcal{M}},\bar{\mathcal{N}})$, from which we conclude that $C'\bullet C[M]\not\leqslant_{k-1}C'\bullet C[M]$. The inductive hypothesis now applies. ∎

7.2. Full Abstraction Lemmas

In what follows we sometime omit the environment ρ from $[\![M]\!]_\rho$, when it is not necessary.

LEMMA 7.7. *For any $M,N\in\Lambda_\oplus$ and any indexing functions $\mathcal{I}$ and $\mathcal{J}$:*

(i) $M^{\mathcal{I}}\rhd^* N^{\mathcal{J}}\in\mathbf{N}_\oplus^\Omega\Rightarrow N\leqslant M$;

(ii) $M^{\mathcal{I}}\rhd N^{\mathcal{J}}\Rightarrow\forall\rho.[\![M^{\mathcal{I}}]\!]_\rho=[\![N^{\mathcal{J}}]\!]_\rho$.

Proof. (i) $NBT(M)$ is the same as $NBT(N)$ with the possible exception of some nodes labelled with Ω; since $N\in\mathbf{N}_\oplus^\Omega$ the thesis follows by induction on the height of N.

To prove (ii) one checks the clauses in Definition 5.4 against the equations of Lemma 4.10; e.g.,

$$\begin{aligned}[\![(M\oplus N)^{n+1}L]\!]&=[\![(M\oplus N)^{n+1}]\!]\cdot[\![L]\!]\\&=[\![M\oplus N]\!]_{n+1}\cdot[\![L]\!]\\&=([\![M]\!]+[\![N]\!])_{n+1}\cdot[\![L]\!].\end{aligned}$$

Now call $x=[\![M]\!]$, $y=[\![N]\!]$, $z=[\![L]\!]$:

$$\begin{aligned}(x+y)_{n+1}\cdot z&=(x_{n+1}+y_{n+1})\cdot z\\&=x_{n+1}\cdot z+y_{n+1}\cdot z\\&=(x\cdot z_n)_n+(y\cdot z_n)_n\\&=(x\cdot z_n+y\cdot z_n)_n\\&=[\![(ML^n\oplus NL^n)^n]\!].\end{aligned}$$ ∎

LEMMA 7.8. *For any $M\in\Lambda_\oplus$, if $L\in\mathbf{N}_\oplus^\Omega$ and $L\leqslant M$, then $[\![L]\!]\sqsubseteq[\![M]\!]$.*

Proof. Using the inductive definition of $\mathbf{N}_\oplus^\Omega$.

Case 1. $L\equiv\Omega$; then

$$[\![L]\!]=\bot\sqsubseteq[\![M]\!].$$

In the sequel, since $L\not\equiv\Omega$, $L\leqslant M$ implies that $M\downarrow$: let $\{M_1,\ldots,M_r\}$ be the principal hnfs of M.

Case 2. $L\equiv x$; now

$$x\leqslant_1 M\Rightarrow x\leqslant_1 M_1\wedge\cdots\wedge x\leqslant_1 M_r;$$

hence, for $i=1,\ldots,r$,

$$M_i\equiv\lambda y_1\cdots y_{n_i}.xM_1^i\cdots M_{n_i}^i.$$

By Lemma 5.3 we know that $[\![x]\!]=\bigsqcup_{\mathcal{I}}[\![x^{\mathcal{I}}]\!]$; hence we proceed by induction on $q=\mathcal{I}(x)$.

Subcase 2.1. $q=0$; then, by Lemma 4.10 (v), for $i=1,\ldots,r$,

$$\begin{aligned}[\![x^0]\!]&=[\![x]\!]_0\\&=[\![\lambda y_1\cdots y_{n_i}.x^0\underbrace{\Omega\cdots\Omega}_{n_i}]\!]\\&\sqsubseteq[\![M_i]\!].\end{aligned}$$

We conclude that $[\![x^0]\!]\sqsubseteq[\![M_1]\!]+\cdots+[\![M_r]\!]=[\![M]\!]$.

Subcase 2.2. $q>0$; then, by Lemma 4.10 (iv), for $i=1,\ldots,r$,

$$[\![x^q]\!]=[\![\lambda y_1\cdots y_{n_i}.x^q y_1^{q-1}\cdots y_{n_i}^{q-n_i}]\!].$$

Now each pair in $\mathsf{Pair}_1(\omega^2(L), \omega^2(M_i))$ will have the shape

$$\langle \{\{y_j^{q-j}\}\}, \{\mathcal{M}_1, ..., \mathcal{M}_s\} \rangle$$

and $\{y_j^{q-j}\} \leqslant \mathcal{M}_h$ for $h = 1, ..., s$. By the inductive hypothesis,

$$[\![y_j^{q-j}]\!] \sqsubseteq [\![N_1]\!] + \cdots + [\![N_t]\!] = [\![\mathcal{M}_h]\!],$$

given that $\mathcal{M}_h = \{N_1, ..., N_t\}$. This means that again $[\![x^q]\!] \sqsubseteq [\![M_i]\!]$ for each i, so that $[\![x^q]\!] \sqsubseteq [\![M]\!]$.

Case 3. $L \equiv \lambda x_1 \cdots x_m . xL_1 \cdots L_n$; then the pairs in $\mathsf{Pair}_1(\omega^k(L), \omega^k(M))$, where $k = \mathsf{height}(L)$, are of the form

$$\langle \{\mathcal{L}_j\}, \{\mathcal{M}_j^1, ..., \mathcal{M}_j^r\} \rangle,$$

where $\mathcal{L}_j = \omega^{k-1}(L_j)$. By the inductive hypothesis, $[\![\mathcal{L}_j]\!] \sqsubseteq [\![\mathcal{M}_j^h]\!]$ for $h = 1, ..., r$; we conclude, as at the end of Subcase 2.2, that $[\![L]\!] \sqsubseteq [\![M]\!]$.

Case 4. $L \equiv L_1 \oplus \cdots \oplus L_n$, where we can suppose that the L_i are not sums. Again let $k = \mathsf{height}(L) > 0$. From the definition of $\leqslant$ we know that

(a) $\forall [P] \in (\mathcal{L} \cup \mathcal{M})_{/\sim} . [P] \cap \mathcal{L} \neq \varnothing$,

(b) $\forall \langle \mathsf{U}, \mathsf{V} \rangle \in \mathsf{Pair}(\mathcal{L}, \mathcal{M}) . \mathsf{U} \sqsubseteq^\sharp \mathsf{V}$,

where $\mathcal{L} = \omega^k(L)$ and $\mathcal{M} = \omega^k(M)$. For sake of simplicity suppose that, e.g.,

$$\mathcal{L} = \{x\mathcal{L}_1, x\mathcal{L}_2\} \quad \text{and} \quad \mathcal{M} = \{x\mathcal{M}_1, x\mathcal{M}_2, x\mathcal{M}_3\}; \tag{3}$$

then $\mathsf{Pair}_1(\mathcal{L}, \mathcal{M})$ contains only the pair

$$\langle \{\mathcal{L}_1, \mathcal{L}_2\}, \{\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3\} \rangle.$$

We know that

$$\forall j \leqslant 3\ \exists i \leqslant 2 . \mathcal{L}_i \leqslant_{k-1} \mathcal{M}_j,$$

so that the inductive hypothesis applies, giving that

$$\forall j \leqslant 3\ \exists i \leqslant 2 . [\![\mathcal{L}_i]\!] \sqsubseteq [\![\mathcal{M}_j]\!];$$

that is

$$\forall j \leqslant 3\ \exists i \leqslant 2 . [\![x\mathcal{L}_i]\!] \sqsubseteq [\![x\mathcal{M}_j]\!].$$

Since in any Smyth algebra D, for any $a_1, ..., a_n$, $b_1, ..., b_m \in D$,

$$\forall j \leqslant m\ \exists i \leqslant n . a_i \sqsubseteq b_j \Rightarrow a_1 + \cdots + a_n \sqsubseteq^\sharp b_1 + \cdots + b_m,$$

we get that

$$[\![L]\!] = [\![x\mathcal{L}_1]\!] + [\![x\mathcal{L}_2]\!] \sqsubseteq [\![x\mathcal{M}_1]\!] + [\![x\mathcal{M}_2]\!] + [\![x\mathcal{M}_3]\!] = [\![M]\!].$$

The general case is a trivial extension of (3). ∎

LEMMA 7.9 (See Lemma 5.8). *For any* $M \in \Lambda_\oplus$ *and natural number* k,

$$\forall \rho \in \mathit{Env} . [\![M]\!]_\rho = \bigsqcup_k [\![M^{[k]}]\!]_\rho .$$

Proof. For any $k \in \mathbb{N}$, $M^{[k]} \in \mathbf{N}_\oplus^\Omega$ and $M^{[k]} \leqslant M$; hence by Lemma 7.8,

$$[\![M^{[k]}]\!] \sqsubseteq [\![M]\!],$$

that is,

$$\bigsqcup_k [\![M^{[k]}]\!] \sqsubseteq [\![M]\!].$$

Let $\mathcal{I}$ be any indexing map; then by Corollary 5.6 there exist $\mathcal{J}$ and $L \in \mathbf{N}_\oplus^\Omega$ such that $M^\mathcal{I} \rhd^* L^\mathcal{J}$. By Lemma 7.7, $[\![M^\mathcal{I}]\!] = [\![L^\mathcal{J}]\!]$ and $L \leqslant M$; now $[\![L^\mathcal{J}]\!] \sqsubseteq [\![L]\!]$ and, since $L \in \mathbf{N}_\oplus^\Omega$, $L \leqslant M^{[k]}$, where $k = \mathsf{height}(L)$, by Lemma 5.7; again by Lemma 7.8 it follows that

$$[\![L]\!] \sqsubseteq [\![M^{[k]}]\!];$$

hence

$$[\![M^\mathcal{I}]\!] = [\![L^\mathcal{J}]\!] \sqsubseteq [\![L]\!] \sqsubseteq [\![M^{[k]}]\!].$$

From this we conclude, by Lemma 5.3, that

$$[\![M]\!] = \bigsqcup_\mathcal{I} [\![M^\mathcal{I}]\!] = \bigsqcup_k [\![M^{[k]}]\!]. \quad ∎$$

7.3. The Simulation Lemma

LEMMA 7.10 (See Lemma 6.19). *Given* $M, N \in \Lambda_\oplus$,

$$\exists D[\ ] \in \Lambda_\oplus[\ ] . D[M]\downarrow \wedge D[N]\uparrow \Rightarrow \exists C[\ ] \in \Lambda[\ ] . C[M]\downarrow \wedge C[N]\uparrow .$$

Proof. W.l.o.g. let us suppose that $M, N \in \Lambda_\oplus^0$; then there exists $F \in \Lambda_\oplus$ s.t. $FM\downarrow$ and $FN\uparrow$. Let F' be the term in Λ obtained from F by substituting all occurrences of a subterm of the form $P \oplus Q$ with an occurrence of xPQ, where x is a fresh variable. For any $r \in \mathbb{N}$ define

$$T_r \equiv \lambda xyz_1 \cdots z_r w . w(xz_1 \cdots z_r)(yz_1 \cdots z_r).$$

We show that there exist an $F'' \in \Lambda$ and a vector $\vec{L} \in (Var \cup \{\mathbf{K}, \mathbf{O}, \Delta\Delta\})^*$ s.t.

$$F''M\vec{L}\downarrow \quad \text{and} \quad F''N\vec{L}\uparrow.$$

Let $\tau_1, ..., \tau_m$ be the set of the head reductions of FM, and let σ be any divergent head reduction of FN; let $\mathscr{F}$ be the set of $\oplus$ redexes of F.

Case 1. σ does not contract any redex in $\mathscr{F}$: then choose $F'' \equiv F'$ and $\vec{L}$ is empty.

Case 2. $\tau_1, ..., \tau_m$ do not contract any redex in $\mathscr{F}$: then choose $F'' \equiv F'[\Delta\Delta/x]$.

Case 3. both σ and $\tau_1, ..., \tau_m$ contract redexes in $\mathscr{F}$. Since $\tau_1, ..., \tau_m$ are finite, they are finitely often in $\mathscr{F}$; hence $k = \deg(\mathscr{F}, \tau_1, ..., \tau_m)$ for some k. We proceed as follows:

— we choose an $r \geq k$ and take $F'' \equiv F'[T_r/x]$;

— we perform all possible head reductions of $F''M$ until either a head normal form is reached, or a term with T_r in head position;

— we reduce $F''N$ until a term with T_r in head position is reached: this must happen, since σ reduces some redex in $\mathscr{F}$ and no head normal form can be reached, otherwise we would have $FN\downarrow$.

Suppose that the term obtained in the reduction of $F''N$ is

$$\lambda\vec{x}.(T_rPQ)\,N_1 \cdots N_m$$

and, supposing r chosen greater than m, the next steps in the head reduction of $F''N$ will give

$$U_0 \equiv \lambda\vec{x}z_{m+1} \cdots z_r w.\, w(PN_1 \cdots N_m z_{m+1} \cdots z_r)$$
$$(QN_1 \cdots N_m z_{m+1} \cdots z_r).$$

We note that $w \notin FV(PN_1 \cdots N_m) \cup FV(QN_1 \cdots N_m)$. Correspondingly from the reductions of $F''M$ we get

$$U_1 \equiv \lambda\vec{x}_1.(T_rP_1Q_1)\,M_{1,1} \cdots M_{1,m_1}$$
$$\cdots$$
$$U_q \equiv \lambda\vec{x}_q.(T_rP_qQ_q)\,M_{q,1} \cdots M_{q,m_q}$$
$$U_{q+1} \equiv \lambda\vec{x}_{q+1}.\xi_{q+1} M_{q+1,1} \cdots M_{q+1,m_{q+1}}$$
$$\cdots$$
$$U_p \equiv \lambda\vec{x}_p.\xi_p M_{p,1} \cdots M_{p,m_p}$$

and from these, for $1 \leq i \leq q$, the head reductions proceed giving certain U'_i of the form

$$\lambda\vec{x}_i z_{m_i+1} \cdots z_r w.\, w(P_iM_{i,1} \cdots M_{i,m_i} z_{m_i+1} \cdots z_r)$$
$$(Q_iM_{i,1} \cdots M_{i,m_i} z_{m_i+1} \cdots z_r),$$

where we make a similar remark about the w as for U_0. Because of our assumptions all head variables appearing in the terms above are bound variables; hence they must occur in the prefixed string of abstractions of the respective terms. For any closed term in head normal form define its "head distance" as follows:

$$\text{hd}(\lambda v_1 \cdots v_n.\xi R_1 \cdots R_m) = i \quad \text{if } \xi \equiv v_i.$$

Now we can always assume that for all $q+1 \leq i \leq p$,

$$\text{hd}(U_0) \neq \text{hd}(U_i),$$

because we simply suppose the r to be chosen suitably large. If this condition is also satisfied for $1 \leq i \leq q$ then we take $F'' \equiv F'[T_r/x]$ and $\vec{L} \equiv y_1 \cdots y_{h-1}(\Delta\Delta)$, where $h = \text{hd}(U_0)$, and we are done.

However, nothing prevents us from having some U'_j, where $1 \leq j \leq q$, s.t. $\text{hd}(U'_j) = \text{hd}(U_0)$, and of course this cannot be settled with a choice of r, since both head distances will depend on it. In this case suppose that the original reduction σ has, after $\lambda\vec{x}(P \oplus Q)\,N_1 \cdots N_m$, a choice to the left, namely, it continues with $\lambda\vec{x}.PN_1 \cdots N_m$. In this case, if $l = \max\{|\vec{x}|, |\vec{x}_{q+1}|, ..., |\vec{x}_p|\}$, then take

$$\vec{L} \equiv y_1 \cdots y_{h-1}\mathbf{K}y_{h+1} \cdots y_l\vec{L}',$$

where $\vec{L}'$ remains to be determined. (Clearly, if the choice is to the right, we take $\mathbf{O}$ instead of $\mathbf{K}$.) By the way,

$$U'_i\vec{L}\downarrow \quad \text{if } i \neq 0 \quad \text{and} \quad \text{hd}(U'_i) \neq \text{hd}(U_0),$$

since the head variable will be replaced by some y, and the rest can be ignored; otherwise

$$U_0\vec{L} \xrightarrow{*}_h PN_1 \cdots N_m\vec{y},$$

and

$$U'_i\vec{L} \xrightarrow{*}_h P_iM_{i,1} \cdots M_{i,m_i}\vec{y}',$$

where $\vec{y}, \vec{y}' \subseteq y_1 \cdots y_{h-1}y_{h+1} \cdots y_l$.

If either σ or $\tau_1, ..., \tau_m$ do not contract any other redex in $\mathscr{F}$, we are in a case similar to Case 1 or to Case 2: consequently we shall choose the $\vec{L}'$ accordingly. Otherwise the present case applies, and we repeat the same reasoning. This process, however, is bounded because the $\tau_1, ..., \tau_m$ were finitely often in $\mathscr{F}$. This implies that we must reach a point in which either (the simulation of) σ definitely diverges, or all the reducts obtained from (the simulation of) $\tau_1, ..., \tau_m$ are similar to the U_i above, when $q+1 \leq i \leq p$, that is we can suppose that they all have a different head distance from that of the term coming from σ. In the former case we add nothing to the $\vec{L}$ constructed up to that point; in the latter case we add

$$w_1 \cdots w_s(\Delta\Delta),$$

supposing s to be the head distance of the term coming from σ. ∎

8. CONCLUSIONS AND FURTHER WORK

We have studied nondeterministic extensions of pure λ-calculus in their operational, denotational and axiomatic aspects. We have given some evidence of the conjunctive nature of the nondeterminism in the functional setting in contrast to the disjunctive nature of parallelism; furthermore, we have shown the possibility of grasping the former in a framework which is not disturbing, but rather consistently extending what has been known for the classical calculus.

The point of a consistent treatement of both concepts of nondeterminism and parallelism has been pursued to some extent in [44, 45] in the perspective of the lazy λ-calculus.

To take a step forward, let us recall the idea underlying our treatement of the convergency predicate. We considered as convergent any term such that any (head) reduction starting with it gives something which can be considered as a value; for "sums" of terms this implies that they satisfy the convergency property if and only if all of the summands do. This can be generalized to any property, and formalized in a system which uses types to express properties of terms so that one could say that $M \oplus N$ has type σ if and only if both M and N have type σ.

On the other hand, we have argued for the disjunctive nature of parallelism, so that, if $M \parallel N$ represents the parallelization of M and N, we say that $M \parallel N$ has type σ provided that either M or N (or both) does.

In a system like Curry's simple types this does not seem to be expressive; we have instead considered in [19, 20], as a further development of the present work, a system like that presented in [8], where one has both conjunctive and disjunctive types. Consistently with our perspective the following rules are sound:

$$\frac{M:\sigma \quad N:\tau}{M \oplus N : \sigma \vee \tau} \quad \text{and} \quad \frac{M:\sigma \quad N:\tau}{M \parallel N : \sigma \wedge \tau}$$

where the exchange between conjunctive/disjunctive operators and disjunctive/conjunctive types is the effect of the usual duality.

It seems that this provides a framework in which the denotational semantics of the calculus can be assessed, along the lines of filter model technique introduced in [10], giving us the right place where questions about the semantical analysis of nondeterminism and parallelism can be profitably investigated.

The next step is to extend these techniques to cope with communication. Insights in this direction can be found in [27, 28], while a direct extension of the present approach is [21].

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