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Continuous Probability Distributions in Concurrent Games

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Abstract

We present a model of concurrent games in which strategies are probabilistic and support both discrete and continuous distributions. This is a generalisation of the probabilistic concurrent strategies of Winskel, based on event structures. We first introduce *measurable event structures*, discrete fibrations of event structures in which each fibre is turned into a measurable space. We then construct a bicategory of *measurable games* and *measurable strategies* based on measurable event structures, and add probability to measurable strategies using standard techniques of measure theory. We illustrate the model by giving semantics to an affine, higher-order, probabilistic language with a type of real numbers and continuous distributions.

Keywords: Game semantics, concurrency, event structures, probability, measure theory.

1 Introduction

In the 25 years since its conception, game semantics [18,1] has developed into a powerful framework for modelling programming languages with computational effects such as state, control, concurrency, or nondeterminism. This range of applicability is due to the highly intensional nature of games models, where programs are interpreted as *strategies* specifying their behaviour in all possible evaluation contexts.

Another use of game semantics is in probabilistic computation. As first shown by Danos and Harmer [12] in a probabilistic version of the original Hyland-Ong game model [18], programs with random features can be interpreted as *probabilistic strategies* carrying the extra quantitative information. This works particularly well for probabilistic programs with state: the model is *fully abstract* for Probabilistic Algol, an extension of PCF [21] with ground type references and probability.

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Fig. 1. Two event structures.

Recently, *concurrent games* [22] were introduced as an alternative framework for game semantics, based on *event structures*, a fundamental model for concurrent processes. The framework has been used to model concurrent primitives in functional programs [6,10,11], and has been particularly successful at modelling nondeterminism in languages without state [5] (a problem known to be difficult in game semantics [16]).

In [28] the second author enriched concurrent games with probability, by introducing a notion of probabilistic event structures, extending previous work on probabilistic models for concurrency [24]. This made possible an analysis of Probabilistic PCF via games [7], including an intensional full abstraction result.

However, in all of the above models, all probability distributions are forced to be *discrete*. This makes those models unsatisfactory for practical probabilistic languages, in which *continuous* distributions are essential. Vákár and Ong have recently announced [20] a generalisation of the Danos-Harmer model supporting continuous distributions, which they apply to a stateful language to obtain a definability result, following [12].

In this paper, we propose a new probabilistic concurrent games model in which strategies are equipped to support arbitrary distributions on the real numbers. We rely on methods of measure theory and introduce *measurable event structures*. These generalise event structures and form the basis for a model of *measurable concurrent games*, to which one may adjoin probability. We illustrate the model by giving semantics to a higher-order, affine probabilistic language called $\mathrm{PPCF}^{\mathbb{R}}_{\mathrm{aff}}$, for which we prove an adequacy theorem.

Outline of the paper. In the next section, we introduce measurable event structures, independently of their application to concurrent games. Then, in Section 3, we define $\mathrm{PPCF}^{\mathbb{R}}_{\mathrm{aff}}$ and use it to motivate the development of a bicategory of measurable games and measurable strategies, which we enrich with probability in Section 4. Finally, in Section 5, we return to $\mathrm{PPCF}^{\mathbb{R}}_{\mathrm{aff}}$ and prove adequacy.

2 Measurable Event Structures

Our model is based on a generalisation of event structures supporting arbitrary probability measures, including continuous distributions. We start by recalling some elements of the theory of event structures, and introduce our notion of *fibred event structures*.

2.1 Fibred event structures

2.1.1 Event structures

Event structures are a model of concurrent processes in which occurrences of computational events are *partially ordered* following the causal constraints between them.

Figure 1a displays an event structure in which two initial events a_1 and a_2 occur in parallel, followed by a third event b. Here the partial order has $a_1 \leqslant b$ and $a_2 \leqslant b$, with the understanding that an event can only occur after each of its predecessors has occurred.

Processes modelled by event structures are potentially non-deterministic. An event structure carries information about which subsets of events are *consistent*, in which case they may occur together in an execution. For instance the diagram in Figure 1b represents a process in which an initial signal is followed by a boolean value chosen non-deterministically: in the corresponding event structure, $\{\mathbf{tt}, \mathbf{ff}\}$ is *not* a consistent subset. Causality and consistency are subject to some axioms. Following [26]:

Definition 2.1 An **event structure**[3] is a tuple $(E, \leqslant, \mathrm{Con})$ where E is a set of **events**, $\leqslant$ a partial order on E representing **dependency**, and Con a non-empty set of finite subsets of E called **consistent**, such that

$$\begin{aligned}
&[e) = \{e' \mid e' < e\} \text{ is finite for all } e \in E \\
&\{e\} \in \mathrm{Con} \text{ for all } e \in E \\
&Y \subseteq X \in \mathrm{Con} \implies Y \in \mathrm{Con} \\
&X \in \mathrm{Con} \text{ and } e \leqslant e' \in X \implies X \cup \{e\} \in \mathrm{Con}.
\end{aligned}$$

The diagrams of Figure 1 do not display $\leqslant$ and Con directly, but rather **immediate causality** $e \twoheadrightarrow e'$, defined as $e < e'$ with no events in between, and **immediate conflict** $e \rightsquigarrow e'$, defined as $[e) \cup \{e'\} \in \mathrm{Con}$, $[e') \cup \{e\} \in \mathrm{Con}$ and $\{e, e'\} \notin \mathrm{Con}$.

A **configuration** of E is a finite subset $x \subseteq E$ which is consistent and downwards-closed. The set of all configurations is denoted $\mathcal{C}(E)$ and throughout the paper it is considered as a partial order under inclusion. For $x, y \in \mathcal{C}(E)$, we say that y is a *covering* of x, written $x \multimap\!\!\subset y$, if there is $e \in E$ such that $e \notin x$ and $y = x \cup \{e\}$.

2.1.2 Fibres

We propose making the event structure E measurable by turning $\mathcal{C}(E)$ into a measurable space whose structure reflects that of E. We will review the basics of measure theory in the next section — for now let us introduce a crucial object of our approach: a form of *fibration* of event structures which we call a *fibred event structure.*

Consider a process outputting two real numbers ε and δ consecutively, each chosen non-deterministically in $\mathbb{R}$. An event structure representation of it is pictured

[3] Specifically we use *prime event structures with general consistency.*

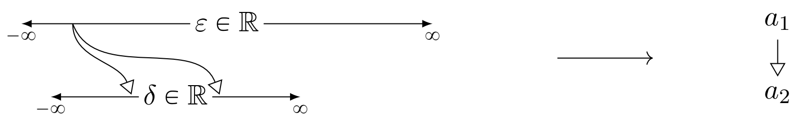


Fig. 2. A fibred event structure $f : E \to \mathfrak{B}$.

as E on the left of Figure 2. Each 'real line' represents an uncountable set of events, all pairwise in immediate conflict. Only one ε-branch is displayed — there are in fact uncountably many such "δ" real lines, one for each $\varepsilon \in \mathbb{R}$.

Configurations of E can have one of three forms: $\varnothing$, $\{\varepsilon\}$, or $\{\varepsilon, \delta\}$ where $\varepsilon, \delta \in E$ and $\varepsilon \rightarrowtriangle \delta$. Our approach involves projecting them to the configurations of a *base event structure* $\mathfrak{B}$, displayed on the right of the figure. The goal is to *encapsulate* the uncountable non-deterministic branching in E in *fibres* over the configurations of $\mathfrak{B}$: $\varnothing$, $\{a_1\}$ and $\{a_1, a_2\}$. The projection map $f : E \to \mathfrak{B}$ is an instance of a *map of event structures*:

Definition 2.2 A function $f : E \to E'$ is a **map of event structures** if

- it preserves configurations: for every $x \in \mathcal{C}(E)$, $fx \in \mathcal{C}(E')$, and
- it is locally injective: if $e, e' \in x$ are such that $f(e) = f(e')$, then $e = e'$.

We say f is **rigid** if additionally $e \leqslant e'$ implies $f(e) \leqslant f(e')$. Rigid maps are appropriate in this context: as the next lemma shows they provide a well-behaved notion of *fibres*:

Lemma 2.3 *If $f : E \to E'$ is a map of event structures, then f is rigid if and only if the induced map $\mathcal{C}(E) \to \mathcal{C}(E')$ is a discrete fibration of partial orders,* i.e. *for every $x \in \mathcal{C}(E)$, if $y \subseteq fx$ for some $y \in \mathcal{C}(E')$, then there exists a unique $x' \in \mathcal{C}(E)$ such that $x' \subseteq x$ and $fx' = y$.*

Proof. (Only if). Suppose that $f : E \to E'$ is rigid and that we have $x \in \mathcal{C}(E)$ and $y \in \mathcal{C}(E')$ such that $y \subseteq fx$. The restriction of f to x is injective by assumption, and $(f|_x)^{-1}y$ is necessarily the only $x' \subseteq x$ such that $fx' = y$. It is a configuration, since it is consistent (as a subset of x) and down-closed (f preserves and reflects causal dependency, and y is down-closed).

(If). Suppose now that f is not rigid, so there are $e, e' \in E$ such that $e \rightarrowtriangle e'$ but $f(e) \not\rightarrowtriangle f(e')$. Consider $x = [e']$ and $y = [f(e')]$. Then $y \subseteq fx$, but since $f(e') \in y$ and $f(e) \notin y$, there can be no $x' \in \mathcal{C}(E)$ such that $x' \subseteq x$ and $fx' = y$ (such an x' would not be down-closed). $\square$

Definition 2.4 A **fibred event structure** consists of a pair of event structures E and $\mathfrak{B}_E$, and a rigid map $f_E : E \to \mathfrak{B}_E$.

We use the same symbol to denote a map of event structures and the induced map on configurations. Accordingly, given a fibred event structure $f_E : E \to \mathfrak{B}_E$ and a configuration $p \in \mathcal{C}(\mathfrak{B}_E)$, the **fibre over** p is the preimage $f_E^{-1}\{p\} = \{x \in \mathcal{C}(E) \mid f_E x = p\}$. If $p \subseteq q \in \mathcal{C}(\mathfrak{B}_E)$, we write $r_{p,q} : f_E^{-1}\{q\} \to f_E^{-1}\{p\}$ for the **restriction map** determined by Lemma 2.3.

2.2 Measurable event structures

In the example of Figure 2, the fibre structure is as follows: $f_E^{-1}\{\varnothing\} = \{*\}$, $f_E^{-1}\{\{a_1\}\} \cong \mathbb{R}$, and $f_E^{-1}\{\{a_1, a_2\}\} \cong \mathbb{R} \times \mathbb{R}$, with the restriction map $r_{\{a_1\},\{a_1,a_2\}}$ acting as the first projection. It is this structure that we leverage in order to make E *measurable*. First we recall some definitions, for which a standard reference is *e.g.* [15].

2.2.1 Measure theory

Given a set X, a **σ-algebra** on X is a set Σ_X of subsets of X, containing X itself and closed under countable unions, and complements. (Any such Σ_X is also closed under countable intersections.) A **measurable space** is a pair (X, Σ_X) with X a set and Σ_X a σ-algebra on it. A **measurable function** $f : (X, \Sigma_X) \to (Y, \Sigma_Y)$ is a function $X \to Y$ such that for any $U \in \Sigma_Y$, $f^{-1}U \in \Sigma_X$.

Any set S of subsets of X generates a σ-algebra $\mathrm{alg}(S)$, as the smallest σ-algebra containing S. The **Borel σ-algebra** on the set $\mathbb{R}$ of real numbers is generated by the set of all intervals: we write $\Sigma_{\mathbb{R}} = \mathrm{alg}\left(\{(a, b] \mid a \leqslant b \in \mathbb{R}\}\right)$.

The category **Meas** of measurable spaces and measurable functions has finite products: in the binary case $(X, \Sigma_X) \times (Y, \Sigma_Y) = (X \times Y, \Sigma_X \otimes \Sigma_Y)$, where the **product σ-algebra** is generated by the 'measurable rectangles': $\Sigma_X \otimes \Sigma_Y = \mathrm{alg}\left(\{U_X \times U_Y \mid U_X \in \Sigma_X, U_Y \in \Sigma_Y\}\right)$. It also has countable coproducts [4]: $\coprod_{i\in I}(X_i, \Sigma_{X_i}) = (\coprod_{i\in I} X_i, \Sigma_{\coprod_{i\in I} X_i})$, where $\coprod_{i\in I} X_i = \bigcup_{i\in I}\{i\} \times X_i$ and $\Sigma_{\coprod_{i\in I} X_i} = \mathrm{alg}\left(\{\{i\} \times U \mid i \in I \text{ and } U \in \Sigma_{X_i}\}\right)$.

Finally, given a measurable space (X, Σ_X) and a subset $S \subseteq X$, we can turn S into a measurable space (S, Σ_S) with the **subspace σ-algebra** $\Sigma_S = \{S \cap U \mid U \in \Sigma_X\}$.

2.2.2 Measurable fibres

Definition 2.5 A **measurable event structure** $\mathcal{E}$ consists of a fibred event structure $f_E : E \to \mathfrak{B}_E$ and, for each $p \in \mathcal{C}(\mathfrak{B}_E)$, a σ-algebra $\Sigma_{f_E^{-1}\{p\}}$ on the fibre over p, such that for every $p \subseteq q \in \mathcal{C}(\mathfrak{B}_E)$, $r_{p,q}$ is measurable.

The fibred event structure in Figure 2 is naturally turned into a measurable event structure by setting $\Sigma_{f_E^{-1}\{\{a_1\}\}} = \Sigma_{\mathbb{R}}$ and $\Sigma_{f_E^{-1}\{\{a_1,a_2\}\}} = \Sigma_{\mathbb{R}} \otimes \Sigma_{\mathbb{R}}$. Note that in any measurable event structure, the fibre over $\varnothing$ is a singleton and necessarily equipped with the trivial σ-algebra.

Remark 2.6 Via the standard correspondence between discrete fibrations and presheaves, a fibred event structure $f_E : E \to \mathfrak{B}_E$ yields a functor $\mathcal{C}(\mathfrak{B}_E)^{\mathrm{op}} \to \mathbf{Set}$, where the partial order $(\mathcal{C}(\mathfrak{B}_E), \subseteq)$ is seen as a category. Likewise, a measurable event structure induces a 'measurable presheaf' $\mathcal{C}(\mathfrak{B}_E)^{\mathrm{op}} \to \mathbf{Meas}$. Not all presheaves on $\mathcal{C}(\mathfrak{B}_E)$ are representable by fibred event structures in this way (see

[4] In fact **Meas** is a *topological category* [4] and has all small limits and colimits induced from those in **Set**. We only give explicit constructions for those needed in this paper.

[27] for a precise connection), but this presentation is more operationally intuitive and will facilitate the development of a game model in the next section.

2.2.3 A category of measurable event structures

To define maps of fibred event structures we adapt the standard notion of maps between discrete fibrations of categories:

Definition 2.7 A **map of fibred event structures** $(f_E : E \to \mathfrak{B}_E) \to (f_{E'} : E' \to \mathfrak{B}_{E'})$ is a pair of (not necessarily rigid) maps $\alpha : E \to E'$ and $\alpha_{\mathfrak{B}} : \mathfrak{B}_E \to \mathfrak{B}'_E$ of event structures, making the diagram

$$\begin{array}{ccc} E & \xrightarrow{f_E} & \mathfrak{B}_E \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha_{\mathfrak{B}}} \\ E' & \xrightarrow{f_{E'}} & \mathfrak{B}_{E'} \end{array}$$

commute. If $\mathcal{E}$, $\mathcal{E}'$ are measurable event structures with underlying fibration f_E and $f_{E'}$, respectively, $(\alpha, \alpha_{\mathfrak{B}})$ is a **measurable map** if for each $p \in \mathcal{C}(\mathfrak{B}_E)$, the map $f_E^{-1}\{p\} \to f_{E'}^{-1}\{\alpha_{\mathfrak{B}} p\} : x \mapsto \alpha x$ is measurable *w.r.t.* the σ-algebra on each fibre.

We will give examples of such maps in the next section, when introducing *measurable strategies*. We call **MES** the category of measurable event structures and measurable maps, with the obvious identities and composition.

Observe that the usual category **ES** of event structures embeds fully and faithfully into **MES**: the embedding **disc** : **ES** → **MES** sends E to the unique object of **MES** whose underlying fibration is the identity map $\mathrm{id} : E \to E$. A measurable event structure of this form is said to be **discrete**.

In the rest of the paper we use $\mathcal{E}, \mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{T}, \ldots$ to denote measurable event structures with underlying event structures $E, A, B, S, T, \ldots$ respectively. When making use of the underlying data (base event structures $\mathfrak{B}_E$, fibration maps f_E, *etc.*) we use subscripts to avoid ambiguity. Similarly, we write $\alpha : \mathcal{E} \to \mathcal{E}'$ for the pair $(\alpha, \alpha_{\mathfrak{B}})$, and for $p \in \mathfrak{B}_E$, α_p refers to the restriction of α to the fibre $f^{-1}\{p\}$, a measurable function $f_E^{-1}\{p\} \to f_{E'}^{-1}\{\alpha_{\mathfrak{B}} p\}$.

3 Measurable Games and Strategies

We proceed to give a presentation of our *measurable games* model, in which measurable event structures occupy a central place: once enriched with *polarity* they play the roles of both processes and types.

We aim in the rest of the paper to give an interpretation to a higher-order, affine probabilistic language called $\mathrm{PPCF}^{\mathbb{R}}_{\mathrm{aff}}$. We start by importing a few additional concepts from measure theory, to do with probability.

A **sub-probability measure** on a measurable space (X, Σ_X) is a map $\mu : \Sigma_X \to [0, 1]$ such that $\mu(\varnothing) = 0$ and such that for any countable family $\{U_i\}_{i \in I} \subseteq \Sigma_X$ with $U_i \cap U_j = \varnothing$ for every $i \neq j$, we have $\mu(\biguplus_i U_i) = \sum_i \mu(U_i)$. For $x \in X$, the **Dirac**

measure δ_x is defined as $\delta_x(U) = 1$ if $x \in U$, and 0 otherwise. Finally, given a sub-probability measure μ on X and a non-negative measurable function $g : X \to \mathbb{R}$, the integral $\int_{x \in X} g(x)\mu(\mathrm{d}x)$ is a well-defined element of $[0, \infty)$.

A **stochastic kernel** [14] from (X, Σ_X) to (Y, Σ_Y) is a map $k : X \times \Sigma_Y \to [0, 1]$ such that for every $x \in X$ the map $k(x, -)$ is a sub-probability measure, and for every $U \in \Sigma_Y$ the map $k(-, U)$ is measurable with respect to $\Sigma_{[0,1]}$, the subspace σ-algebra of $\Sigma_{\mathbb{R}}$. Such a map provides a notion of probability measure on the space Y parametrised by elements of X. Stochastic kernels can be composed: given $k : X \times \Sigma_Y \to [0, 1]$ and $h : Y \times \Sigma_Z \to [0, 1]$, their composition is the map $h \circ k : X \times \Sigma_Z \to [0, 1]$ defined as $(x, U) \mapsto \int_{y \in Y} h(y, U)k(x, dy)$.

3.1 A probabilistic language with continuous distributions

We introduce our main language of study, $\mathrm{PPCF}^{\mathbb{R}}_{\mathrm{aff}}$, an affine version of PCF enriched with a real number type and both discrete and continuous probabilistic primitives. It has types and terms defined as

$$A, B ::= \mathbf{Real} \mid \mathbf{Bool} \mid A \multimap B \quad M, N ::= x \mid \lambda x.M \mid MN \mid \bot \mid \mathbf{tt} \mid \mathbf{ff} \mid \mathbf{if}\, M\; N\; P$$
$$\mathbf{coin} \mid \underline{r} \mid M \leqslant 0 \mid \underline{d}$$

where r ranges over real numbers and d over a countable set $\mathcal{D}$ of stochastic kernels $\mathbb{R} \times \Sigma_{\mathbb{R}} \to [0, 1]$.

Elements of $\mathcal{D}$ may be thought of as families of distributions with one real parameter. In an example below we use $r \mapsto \mathrm{normal}(r, 1)$, the normal distributions with standard deviation 1. Note that $\mathrm{PPCF}^{\mathbb{R}}_{\mathrm{aff}}$ is designed to support a proof of concept for measurable game semantics. It lacks some features desirable for practical probabilistic programming, such as more general families of distributions, and primitives for observing data and performing inference.

The language is given an affine type system in the standard way, so that in a term of the form $\lambda x.M$ the variable x may appear at most once in M. We give some of the typing rules, with **Gnd** standing for either of the ground types **Real** or **Bool**:

$$\frac{}{\Gamma \vdash \bot : \mathbf{Real}} \qquad \frac{\Gamma \vdash M : \mathbf{Bool} \qquad \Delta \vdash N : \mathbf{Gnd} \qquad \Psi \vdash P : \mathbf{Gnd}}{\Gamma, \Delta, \Psi \vdash \mathbf{if}\, M\; N\; P : \mathbf{Gnd}}$$

$$\frac{\Gamma \vdash M : \mathbf{Real}}{\Gamma \vdash M \leqslant 0 : \mathbf{Bool}} \qquad \frac{}{\Gamma \vdash \mathbf{coin} : \mathbf{Bool}} \qquad \frac{\underline{d} \in \mathcal{D}}{\Gamma \vdash \underline{d} : \mathbf{Real} \multimap \mathbf{Real}}$$

$$\frac{}{\Gamma \vdash \underline{r} : \mathbf{Real}}$$

To define operational semantics, we follow [3,13,23] and first turn the set of terms into a measurable space. We write $\mathcal{T}^{\Gamma \vdash A}$ for the set of terms M for which the typing judgment $\Gamma \vdash M : A$ is derivable. Observe that every $M \in \mathcal{T}^{\Gamma \vdash A}$ can be canonically written as $S[\underline{r_1}/x_1, \ldots, \underline{r_n}/x_n]$, where the $\underline{r_i}$ are real number constants, and S is a term without any sub-term of the form $\underline{r}$, such that $\Gamma, x_1 : \mathbf{Real}, \ldots, x_n : \mathbf{Real} \vdash S : A$.

Given such an S, let $\mathcal{T}_S^{\Gamma\vdash A}$ be the subset of $\mathcal{T}^{\Gamma\vdash A}$ containing terms of the form $M = S[\underline{r_1}/x_1,\ldots,\underline{r_n}/x_n]$ for some $\underline{r_1},\ldots,\underline{r_n}$. There is a bijection $\mathcal{T}_S^{\Gamma\vdash A} \cong \mathbb{R}^n$, and we define $\Sigma_S^{\Gamma\vdash A}$ to be the (unique) σ-algebra which makes it an isomorphism $(\mathcal{T}_S^{\Gamma\vdash A}, \Sigma_S^{\Gamma\vdash A}) \cong (\mathbb{R}^n, \Sigma_{\mathbb{R}^n})$ in **Meas**. We then take $\Sigma_{\mathcal{T}^{\Gamma\vdash A}}$ to be the σ-algebra induced by seeing $\mathcal{T}^{\Gamma\vdash A}$ as the coproduct $\coprod_S \mathcal{T}_S^{\Gamma\vdash A}$, where S ranges over the terms containing no sub-terms of the form $\underline{r}$ and such that $\Gamma, x_1 : \mathbf{Real}, \ldots, x_n : \mathbf{Real} \vdash S : A$ for some $n \in \mathbb{N}$.

We then define a call-by-name, *deterministic* reduction relation $\to$ as

$$(\lambda x.M)N \to M[N/x] \qquad \textbf{if}\,\mathbf{tt}\;N\;P \to N \qquad \underline{r} \leqslant 0 \to \mathbf{tt} \quad (\text{if } r \leqslant 0)$$
$$\textbf{if}\,\mathbf{ff}\;N\;P \to P \qquad \underline{r} \leqslant 0 \to \mathbf{ff} \quad (\text{if } r > 0).$$

We also define evaluation contexts:

$$C[\,] ::= [\,] \mid \textbf{if } C[\,]\,N\,P \mid C[\,] \leqslant 0 \mid C[\,]\,N \mid d\,C[\,]$$

The one-step reduction relation between terms is then expressed as a stochastic kernel $\textsc{red}^{\Gamma\vdash A} : \mathcal{T}^{\Gamma\vdash A} \times \Sigma_{\mathcal{T}^{\Gamma\vdash A}} \to [0,1]$ defined for each $\mathrm{PPCF}^{\mathbb{R}}_{\mathrm{aff}}$ term M and $U \in \Sigma_{\mathcal{T}^{\Gamma\vdash A}}$ as follows:

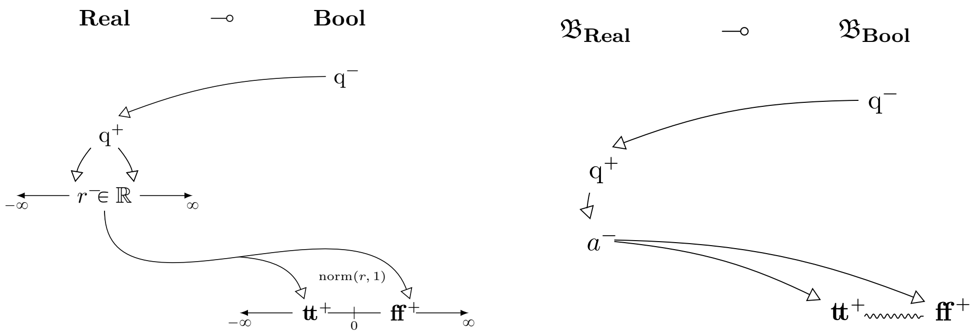
$$\textsc{red}^{\Gamma\vdash A}(M,U) = \begin{cases} \delta_{C[N]}(U) & \text{if } M = C[R] \text{ and } R \to N \\ \frac{1}{2}\delta_{C[\mathbf{tt}]}(U) + \frac{1}{2}\delta_{C[\mathbf{ff}]}(U) & \text{if } M = C[\mathbf{coin}] \\ d(r, \{r' \in \mathbb{R} \mid C[\underline{r'}] \in U\}) & \text{if } M = C[\underline{d}\,\underline{r}] \\ 0 & \text{if } M = \bot \\ \delta_M(U) & \text{otherwise.} \end{cases}$$

That $\textsc{red}$ is a stochastic kernel is a straightforward adaptation of [13]. Finally, for $U \in \Sigma_{\mathcal{T}^{\Gamma\vdash A}}$, the many-step **probability of reduction** is $\Pr(M \to U) = \sup_{n\in\mathbb{N}} \textsc{red}^n(M,U)$. Note that if $V \in \Sigma_{\mathbb{R}}$, we write $\underline{V}$ for $\{\underline{r} \mid r \in V\}$, an element of $\Sigma_{\mathcal{T}\vdash\mathbf{Real}}$.

3.2 Games and strategies as event structures

3.2.1 Terms as probabilistic strategies

In the concurrent games model presented here, the term $M = \lambda r.\ \mathrm{normal}(r,1) \leqslant 0$ will be interpreted as the *strategy* in Figure 3a, which combines all possible *execution traces* of M. Each trace is recorded a dialogue between **Player**, representing M, and **Opponent**, representing the execution environment. In this example every maximal trace is of the form $\mathrm{q}^- \rightarrowtriangle \mathrm{q}^+ \rightarrowtriangle r^- \rightarrowtriangle b^+$ for some $r \in \mathbb{R}$ and $b \in \{\mathbf{tt}, \mathbf{ff}\}$, and the *polarity* ($+$ or $-$) indicates which of the two players is responsible for a move, with the convention that Player $= +$, Opponent $= -$. We read such a dialogue as follows: the initial q^- is an external call to the program, the following q^+ is a call by the program to its argument, whose value is then supplied by the environment as r^-. Finally b^+ is the output of the function for this particular execution.



(a) (b)

Fig. 3. An interpretation for $M = \lambda r.\ \mathrm{normal}(r, 1) \leqslant 0$.

In our model the (uncountable) set of traces will arise as the *configurations* of a measurable event structure, as defined in Section 2, which we will further enrich with *probability* in Section 4. Figure 3b shows the corresponding *base* event structure, which acts as a discrete representation of the control flow of the program. We omit the details of the projection map from the event structure of 3a to that of 3b, as it is clear from the labelling of moves. To formalise this we must equip measurable event structures with the extra data of a *polarity function*, as in *e.g.* [22]:

Definition 3.1 An **event structure with polarity** (*esp* for short) is an event structure E together with a polarity function $\mathrm{pol}_E : E \to \{+, -\}$. A **map of esps** is a polarity-preserving map of event structures.

Accordingly, a **measurable esp** is a measurable event structure $\mathcal{E}$, where in addition E and $\mathfrak{B}_E$ have polarity, and f_E preserves it. A map $\alpha : \mathcal{E} \to \mathcal{E}'$ is a **map of measurable esps** whenever both α and $\alpha_{\mathfrak{B}}$ preserve polarity.

Configurations of an esp E are ordered by inclusion, as usual. In addition, we write $x \subseteq^+ y$ (*resp.* $x {-\!\!\subset}^+ y$) when $x \subseteq y$ (*resp.* $x {-\!\!\subset} y$) and every $e \in y \backslash x$ has $\mathrm{pol}(e) = +$; the relations $\subseteq^-$ and ${-\!\!\subset}^-$ are defined similarly.

3.2.2 Measurable games

As usual in game semantics, strategies are constrained by the *games* they play on. In this setting a **measurable game** is simply a measurable esp. We will eventually build a bicategory [2] with measurable games as objects, the soon-to-be-introduced *measurable strategies* as morphisms, and a suitable notion of 2-cells.

First we give the interpretation of $\mathrm{PPCF}^{\mathbb{R}}_{\mathrm{aff}}$ ground types as measurable games. The game $[\![\mathbf{Bool}]\!]$ is a discrete measurable esp, defined as the image under the functor **disc** of the event structure of Figure 1b, with polarity defined as $\mathrm{pol}(\mathrm{q}) = -$ and $\mathrm{pol}(\mathbf{tt}) = \mathrm{pol}(\mathbf{ff}) = +$. The measurable game $[\![\mathbf{Real}]\!]$ is defined as

with the only non-trivial fibre, that over the configuration $\{\mathrm{q}, a\}$, defined to be the measurable space $(\mathbb{R}, \Sigma_{\mathbb{R}})$.

3.3 Measurable strategies

We forget about probability for now and until Section 4. The rest of this section is dedicated to the development of our framework for game semantics in a measurable setting. We define measurable strategies on measurable games and describe their composition and organisation as a bicategory.

As mentioned earlier, a strategy in this framework is a measurable esp which is constrained by the game it plays on. In the same way as in [22], this constraint is expressed via a *labelling map* relating the two esps, subject to some conditions.

Definition 3.2 A **measurable strategy** on a measurable game $\mathcal{A}$ is a measurable esp $\mathcal{S}$ together with a measurable map $\sigma : \mathcal{S} \to \mathcal{A}$ (explicitly: two maps $\sigma : S \to A$ and $\sigma_{\mathfrak{B}} : \mathfrak{B}_S \to \mathfrak{B}_A$), such that:

- **courtesy**: If $e, e' \in \mathfrak{B}_S$ are such that $e \rightarrowtriangle e'$ and $\sigma_{\mathfrak{B}}(e) \not\rightarrowtriangle \sigma_{\mathfrak{B}}(e')$, then $\mathrm{pol}(e) = -$ and $\mathrm{pol}(e') = +$.
- **measurable receptivity**: If $p \in \mathcal{C}(\mathfrak{B}_S)$, and $\sigma_{\mathfrak{B}}\, p \subseteq^- q$ for some $q \in \mathcal{C}(\mathfrak{B}_A)$, then there is a unique $q' \in \mathcal{C}(\mathfrak{B}_S)$ such that $p \subseteq q'$ and $\sigma_{\mathfrak{B}}\, q' = q$, and furthermore the diagram

$$\begin{tikzcd} f_S^{-1}\{q'\} \arrow[r, "r^S_{p,q'}"] \arrow[d, "\sigma_{q'}"'] & f_S^{-1}\{p\} \arrow[d, "\sigma_p"] \\ f_A^{-1}\{q\} \arrow[r, "r^A_{\sigma_{\mathfrak{B}} p, q}"] & f_A^{-1}\{\sigma_{\mathfrak{B}} p\} \end{tikzcd}$$

  is a pullback in **Meas** (where recall the horizontal arrows are restriction maps induced by the fibred structure, and the vertical ones are restrictions of σ to the respective fibres).

We will see later that both conditions serve to ensure a well-behaved interaction with *copycat*, the identity strategy on a game. Informally, they prevent Player from constraining Opponent's behaviour further than is allowed by the game.

It will be useful to have a characterisation of pullbacks in **Meas**. Suppose X, Y, Z are measurable spaces and $g : X \to Y$ and $h : Z \to Y$ are measurable functions. The pullback

$$\begin{tikzcd} P \arrow[r, "\Pi_2"] \arrow[d, "\Pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, "h"] \\ X \arrow[r, "g"] & Y \end{tikzcd}$$

exists and has underlying set the pullback in **Set**: $P = \{(x, z) \in X \times Z \mid g(x) = h(z)\}$, with Π_1 and Π_2 the usual projections. The associated Σ_P is the subspace σ-algebra induced by $\Sigma_X \otimes \Sigma_Z$, using that $P \subseteq X \times Z$.

We take a closer look at the two conditions of Definition 3.2 in turn. The *courtesy* axiom says that a strategy may only specify additional causal dependencies of Player moves on Opponent moves. It is a constraint on $\mathfrak{B}_S$, and indeed on S: using that f_A and f_S are rigid maps of esps, it is easy to see that the condition still holds replacing $\mathfrak{B}_S, \sigma_{\mathfrak{B}}$ with S, σ. The purpose of the *receptivity* axiom is twofold. Restricted to the base $\mathfrak{B}_S$, it is the receptivity axiom of [22], stating that at any stage Player must be prepared to let Opponent play the moves that $\mathfrak{B}_A$ makes available to them. In addition, the pullback condition is a way of encoding the same axiom for the $S \to A$ component of the strategy, while enforcing that for any such *Opponent extension* the fibre structure of S reflects that of A.

3.3.1 Morphisms of measurable strategies

Often it is not appropriate to compare measurable strategies up to strict equality, so (as in [22]) we introduce a notion of morphism between them. Such morphisms play the role of 2-cells in the bicategory we define below.

Definition 3.3 For measurable strategies $\sigma : \mathcal{S} \to \mathcal{A}$ and $\tau : \mathcal{T} \to \mathcal{A}$, a **morphism of measurable strategies** is a measurable map $\alpha : \mathcal{S} \to \mathcal{T}$ which commutes with the labelling maps, *i.e.* $\tau \circ \alpha = \sigma$. When α is an isomorphism, we write $\sigma \cong \tau$.

3.4 Interaction of measurable strategies

We introduce two fundamental constructions on measurable esps:

Definition 3.4 Given esps E and E', we define $E \parallel E'$ to be the event structure with events $E + E'$, causality and polarity induced from E and E', and consistent subsets those of the form $X + X'$ (the disjoint union, often written $X \parallel X'$) for $X \in \mathrm{Con}_E$ and $X' \in \mathrm{Con}_{E'}$. Thus, the **parallel composition** $\mathcal{E} \parallel \mathcal{E}'$ of measurable esps $\mathcal{E}$ and $\mathcal{E}'$ is defined to be the fibration $f_E \parallel f_{E'} : E \parallel E' \to \mathfrak{B}_E \parallel \mathfrak{B}_{E'}$ where the fibres are obtained as product spaces: $(f_E \parallel f_{E'})^{-1}\{p \parallel p'\} = f_E^{-1}\{p\} \times f_{E'}^{-1}\{p'\}$. This makes all restriction maps measurable.

Next, the esp $E^\perp$ is defined as having events, consistency and causality those of E, and the opposite polarity: $\mathrm{pol}_{E'}(e) = -\mathrm{pol}_E(e)$ for all $e \in E$. Given a measurable esp $\mathcal{E}$, its **dual** $\mathcal{E}^\perp$ is given by $f_E^\perp = f_E : E^\perp \to \mathfrak{B}_E^\perp$, with fibres the same as in $\mathcal{E}$.

Note the above use of $\parallel$ as an operation on maps of esps. We observe that this operation lifts to maps of fibred and measurable esps. Furthermore, it is functorial and makes $(\mathbf{MES}, \parallel, \mathbf{1})$ a symmetric monoidal category, where $\mathbf{1}$ is the empty measurable event structure. We will make use of this functorial action, and write $\mathfrak{B}_{E \parallel E'}$ and $f_{E \parallel E'}$ for $\mathfrak{B}_E \parallel \mathfrak{B}_{E'}$ and $f_E \parallel f_{E'}$.

We define a **measurable strategy from $\mathcal{A}$ to $\mathcal{B}$** to be one on the game $\mathcal{A}^\perp \parallel \mathcal{B}$. Aiming for a notion of composition, our goal is now to investigate the *interaction* of measurable strategies $\sigma : \mathcal{S} \to \mathcal{A}^\perp \parallel \mathcal{B}$ and $\tau : \mathcal{T} \to \mathcal{B}^\perp \parallel \mathcal{C}$. Traditionally in concurrent games, this is done via a pullback construction in the category of event structures, which must be adapted to the fibred setting. Consider the diagram

$$\begin{tikzcd}
 & T \circledast S \arrow[dl, "\Pi_1"'] \arrow[dr, "\Pi_2"] \arrow[rrr, dashed, "f_{T\circledast S}"] \arrow[d, phantom, "\vee"] & & & \mathfrak{B}_{T\circledast S} \arrow[dl, "(\Pi_1)_{\mathfrak{B}}"'] \arrow[dr, "(\Pi_2)_{\mathfrak{B}}"] \arrow[d, phantom, "\vee"] & \\
S \parallel C \arrow[dr, "\sigma\parallel C"'] \arrow[rrr, bend left] & & A \parallel T \arrow[dl, "A\parallel\tau"] \arrow[rrr, bend right] & \mathfrak{B}_{S\parallel C} \arrow[dr, "(\sigma\parallel C)_{\mathfrak{B}}"'] & & \mathfrak{B}_{A\parallel T} \arrow[dl, "(A\parallel\tau)_{\mathfrak{B}}"] \\
 & A \parallel B \parallel C \arrow[rrr] & & & \mathfrak{B}_{A\parallel B\parallel C} &
\end{tikzcd}$$

in the category of event structures (without polarity), where $T \circledast S$ and $\mathfrak{B}_{T\circledast S}$ are obtained as pullbacks as indicated on the diagram (pullbacks always exist in **ES**, see *e.g.* [8]). For readability we have left out labels for horizontal fibration maps; and $f_{T\circledast S}$ is the canonical map induced by the universal property of $\mathfrak{B}_{T\circledast S}$. We write $\tau \circledast \sigma : T \circledast S \to A \parallel B \parallel C$ and $(\tau \circledast \sigma)_{\mathfrak{B}} : \mathfrak{B}_{T\circledast S} \to \mathfrak{B}_{A\parallel B\parallel C}$ for the composite maps through the diagram.

Standard reasoning (using properties of pullbacks in **ES**) shows that $f_{T\circledast S}$ is rigid, so that $\mathcal{T} \circledast \mathcal{S} = (T \circledast S \xrightarrow{f_{T\circledast S}} \mathfrak{B}_{T\circledast S})$ is a fibred event structure. Moreover, given $p \in \mathcal{C}(\mathfrak{B}_{T\circledast S})$, the fibre $f^{-1}_{T\circledast S}\{p\}$ corresponds to the following pullback diagram in **Set**:

$$\begin{tikzcd}
 & f^{-1}_{T\circledast S}\{p\} \arrow[dl, "\Pi_1"'] \arrow[dr, "\Pi_2"] \arrow[d, phantom, "\vee"] & \\
f^{-1}_{S\parallel C}\{(\Pi_1)_{\mathfrak{B}}\, p\} \arrow[dr, "\sigma"] & & f^{-1}_{A\parallel T}\{(\Pi_2)_{\mathfrak{B}}\, p\} \arrow[dl, "\tau"'] \\
 & f^{-1}_{A\parallel B\parallel C}\{(\tau \circledast \sigma)_{\mathfrak{B}}\, p\} &
\end{tikzcd}$$

We define $\Sigma_{f^{-1}_{T\circledast S}\{p\}}$ so that the above is also a pullback diagram in **Meas**. The induced map $\tau \circledast \sigma : \mathcal{T} \circledast \mathcal{S} \to \mathcal{A} \parallel \mathcal{B} \parallel \mathcal{C}$ is measurable and it is the appropriate notion of **interaction of σ and τ**.

Lemma 3.5 *The tuple $(\mathcal{T} \circledast \mathcal{S}, \Pi_1, \Pi_2)$ is the pullback of $\sigma \parallel \mathcal{C}$ and $\mathcal{A} \parallel \tau$ in* **MES**.

3.5 A bicategory of measurable strategies

Finally we organise measurable games and strategies into a bicategory. We start with composition.

3.5.1 Composition via hiding

We have seen that the map $\tau \circledast \sigma : \mathcal{T} \circledast \mathcal{S} \to \mathcal{A} \parallel \mathcal{B} \parallel \mathcal{C}$ describes the outcome of the interaction of $\mathcal{S}$ and $\mathcal{T}$, which synchronise via moves of the game $\mathcal{B}$. In order to obtain from this a measurable strategy *from $\mathcal{A}$ to $\mathcal{C}$* we *hide* the synchronisation events of $\mathcal{T} \circledast \mathcal{S}$.

Specifically, we define $T \odot S$ to be the event structure with events those $e \in T \circledast S$ whose image under $\tau \circledast \sigma$ lies in either the A or the C component of $A \parallel B \parallel C$, and with all the data of an event structure induced from $T \circledast S$. The base event structure $\mathfrak{B}_{T\odot S}$ is obtained from $\mathfrak{B}_{T\circledast S}$ analogously with respect to $\mathfrak{B}_{A\parallel B\parallel C}$.

Lemma 3.6 *The restriction of $f_{T\circledast S}$ to $T \odot S$ is a well-defined rigid map $f_{T\odot S} : T \odot S \to \mathfrak{B}_{T\odot S}$.*

Proof. We check that f maps events of $T \odot S$ to events of $\mathfrak{B}_{T\odot S}$. By definition, for any $e \in T \odot S$ we have $(\tau \circledast \sigma)e \in A$ or C, and therefore $(f_{A\|B\|C} \circ (\tau \circledast \sigma))e \in \mathfrak{B}_A$ or $\mathfrak{B}_C$. Since $f_{A\|B\|C} \circ (\tau \circledast \sigma) = (\tau \circledast \sigma)_{\mathfrak{B}} \circ f_{T\circledast S}$, the event $f_{T\odot S}e$ is an element of $\mathfrak{B}_{T\circledast S}$ mapped to $\mathfrak{B}_A$ or $\mathfrak{B}_C$ by $(\tau \circledast \sigma)_{\mathfrak{B}}$. Therefore, by definition, $f_{T\odot S}e \in \mathfrak{B}_{T\odot S}$.

Then we check that $f_{T\odot S}$ is a rigid map of event structures. For any $x \in \mathcal{C}(T\odot S)$, $f_{T\odot S}x$ is a configuration, since it is the restriction of $f_{T\circledast S}[x]$ to events of $\mathfrak{B}_{T\odot S}$. The restriction of $f_{T\odot S}$ to any configuration x coincides with the restriction of $f_{T\circledast S}$ to x, so it must be locally injective. Finally, $f_{T\odot S}$ preserves order because $f_{T\circledast S}$ does. □

After this step of hiding, every configuration $p \in \mathcal{C}(\mathfrak{B}_{T\odot S})$ has a unique **witness** $[p] \in \mathcal{C}(\mathfrak{B}_{T\circledast S})$, and similarly every $x \in \mathcal{C}(T \odot S)$ induces $[x] \in \mathcal{C}(T \circledast S)$, satisfying $f^{-1}_{T\odot S}\{p\} \cong f^{-1}_{T\circledast S}\{[p]\}$ via $x \mapsto [x]$. It is therefore natural to define $\Sigma_{f^{-1}_{T\odot S}\{p\}} = \Sigma_{f^{-1}_{T\circledast S}\{[p]\}}$, modulo the iso; we get a measurable esp $\mathcal{T} \odot \mathcal{S}$ and a measurable map $\tau \odot \sigma : \mathcal{T} \odot \mathcal{S} \to \mathcal{A}^\perp \parallel \mathcal{C}$.

Lemma 3.7 *The map $\tau \odot \sigma : \mathcal{T} \odot \mathcal{S} \to \mathcal{A}^\perp \parallel \mathcal{C}$ is a measurable strategy, called the* ***composition of** σ **and** τ.*

Proof. We rely on the corresponding arguments in standard concurrent games – this is detailed in [8]. Here, because $(\tau \circ \sigma)_{\mathfrak{B}}$ is the composition, in the traditional sense, of strategies $\tau_{\mathfrak{B}}$ and $\sigma_{\mathfrak{B}}$, it is necessarily courteous. The composition $\tau \odot \sigma$ is receptive because the interaction $\tau \circledast \sigma$ is, which can be verified via a simple diagram chase. □

3.5.2 Measurable copycat

For a measurable game $\mathcal{A}$, the identity strategy on $\mathcal{A}$ is the **measurable copycat** strategy $\text{æ}_{\mathcal{A}} : \mathbb{C}_{\mathcal{A}} \to \mathcal{A}^\perp \parallel \mathcal{A}$, which acts as a forwarder of information from one copy of $\mathcal{A}$ to the other.

The components $\mathbb{C}_A$ and $\mathfrak{B}_{\mathbb{C}_A}$ of the measurable esp $\mathbb{C}_{\mathcal{A}}$ are instances of the same construction. Formally, the events, polarity and consistency of $\mathbb{C}_A$ are those of $A^\perp \parallel A$, and the causality is that of $A^\perp \parallel A$ enriched with the pairs $\{((a,1),(a,2)) \mid a \in A \text{ and } \mathrm{pol}_A(a) = +\} \cup \{((a,2),(a,1)) \mid \mathrm{pol}_A(a) = -\}$. The base $\mathfrak{B}_{\mathbb{C}_A}$ is defined as $\mathbb{C}_{\mathfrak{B}_A}$; the maps æ_A and $(\text{æ}_A)_{\mathfrak{B}}$ are identities on events, and $f_{\mathbb{C}_A}$ has the same action as $f_A \parallel f_A$.

Given $p \in \mathcal{C}(\mathfrak{B}_{\mathbb{C}_A})$, the fibre over p is equipped with the smallest σ-algebra making the map $\text{æ}_A : f^{-1}_{\mathbb{C}_A}\{p\} \to f^{-1}_{A^\perp\|A}\{(\text{æ}_A)_{\mathfrak{B}}\, p\}$ measurable.

Lemma 3.8 *The map $\text{æ}_{\mathcal{A}} : \mathbb{C}_{\mathcal{A}} \to \mathcal{A}^\perp \parallel \mathcal{A}$ is a measurable strategy. Furthermore, if σ is a measurable strategy from $\mathcal{A}$ to $\mathcal{B}$, then $\sigma \odot \text{æ}_{\mathcal{A}} \cong \sigma$.*

The proof relies on existing composition results for concurrent strategies [8], along with an analysis of the fibre structure in the interaction $\sigma \circledast \text{æ}_A$. Similarly, we can show that composition is only associative up to isomorphism, in such a way that:

Theorem 3.9 *There is a bicategory* **MG** *with measurable games as objects, measurable strategies as morphisms, and morphisms of measurable strategies as 2-cells.*

4 Probabilistic Strategies

We add probability to measurable strategies by introducing the notion of *valuation* on a measurable esp. Although the framework of the previous section works in full generality, valuations are only well-defined on a restriction of the model where esps satisfy two additional conditions.

First, say a measurable space (X, Σ_X) is a **standard Borel space** (see *e.g.* [19]) if it is measurably isomorphic to $(\mathbb{R}, \Sigma_{\mathbb{R}})$, or if X is countable and $\Sigma_X = \mathcal{P}X$, the powerset of X. A measurable esp is said to be a **standard Borel esp** if all its fibres are standard Borel spaces.

The next condition is usually required in probabilistic concurrent games [28,7] and forbids any *races* (*i.e.* minimal conflicts) between Player and Opponent moves in a measurable esp:

Definition 4.1 A measurable esp $\mathcal{E}$ is **race-free** if for every $p \in \mathcal{C}(\mathfrak{B}_E)$, if $p \subseteq^+ q$ and $p \subseteq^- p'$, then $p' \cup q \in \mathcal{C}(\mathfrak{B}_E)$ and moreover the diagram

$$\begin{tikzcd} f_E^{-1}\{p' \cup q\} \arrow[r, "r_{p',p'\cup q}"] \arrow[d, "r_{q,p'\cup q}"'] & f_E^{-1}\{p'\} \arrow[d, "r_{p,p'}"] \\ f_E^{-1}\{q\} \arrow[r, "r_{p,q}"] & f_E^{-1}\{p\} \end{tikzcd}$$

is a pullback in **Meas**. Say an esp E is race-free if the measurable esp $\mathbf{disc}(E)$ is race-free.

Because standard Borel, race-free esps are closed under the various constructions of Section 3, there is a sub-bicategory of **MG** involving only such esps. We shall assume from now on that all measurable esps are race-free and standard Borel; in particular, we regularly make use of the property that in a standard Borel space all singleton subsets are measurable. We first introduce valuations on discrete esps.

4.1 Probabilistic esps: the discrete case

We are interested in representing the uncertainty with which some configurations of an esp E occur in an execution. The *probabilistic event structures with polarity* of [28] take a *global* approach: a *configuration-valuation* is a function $v : \mathcal{C}(E) \to [0, 1]$, satisfying certain axioms, where for $x \in \mathcal{C}(E)$ the coefficient $v(x)$ is the probability that the process will reach x, given that Opponent plays all the negative moves in x.

Here we instead adopt a more *local* (and marginally more general) approach, and for each $x \in \mathcal{C}(E)$ we assign coefficients to **positive extensions** of x, *i.e.* configurations $y \in \mathcal{C}(E)$ such that $x \subseteq^+ y$. We write $v(x, y)$ for this coefficient, representing the conditional probability that y will occur given than x has. If $v(-,-)$ is to make sense as a form of conditional probability, we must have $v(x, x) = 1$, and a *chain rule*: $v(x, z) = v(x, y)v(y, z)$, when $x \subseteq^+ y \subseteq^+ z$.

We must also ensure that $v(x,-)$ is a probability distribution on the positive extensions of x. If those extensions are pairwise incompatible, then indeed the sum $\sum_{x \subseteq^+ y} v(x, y)$ must be $\leqslant 1$; if instead extensions $y_1, \dots, y_n$ are not pairwise mutually exclusive then we must account for any overlap, using the inclusion-exclusion principle. This is condition (3) in the definition below, called **drop condition** in [28]; condition (4) formalises the requirement that Player and Opponent, whenever they are causally independent, are also probabilistically independent.

Definition 4.2 A **(discrete) valuation** on an esp E is a family of coefficients $(v(x, y))_{x \subseteq^+ y \in \mathcal{C}(E)}$ indexed by positive extensions, and satisfying:

(1) for every $x \in \mathcal{C}(E)$, $v(x, x) = 1$;

(2) if $x \subseteq^+ y \subseteq^+ z$, $v(x, z) = v(x, y)v(y, z)$;

(3) if $x \subseteq^+ y_1, \dots, y_n$, then

$$\sum_I (-1)^{|I|+1} v(x, \bigcup_{i \in I} y_i) \leqslant 1,$$

where I ranges over nonempty subsets of $\{1, \dots, n\}$ such that $\bigcup_{i \in I} y_i$ is consistent;

(4) if $x \subseteq^+ y$ and $x \subseteq^- x'$, then $v(x, y) = v(x', y \cup x')$ (recall that $y \cup x' \in \mathcal{C}(E)$ by race-freeness).

4.2 Probabilistic measurable esps: the general case

Suppose now that $\mathcal{E}$ is a measurable esp. We generalise Definition 4.2 by considering a family of stochastic kernels $k^E_{p,q}$ from $f_E^{-1}\{p\}$ to $f_E^{-1}\{q\}$, indexed by positive extensions $p \subseteq^+ q$ in $\mathcal{C}(\mathfrak{B}_E)$. Informally, for $x \in f_E^{-1}\{p\}$, the sub-probability measure $k^E_{p,q}(x,-)$ represents the conditional distribution on those positive extensions of x lying in the fibre over q — note that all such extensions are necessarily incompatible. More formally, the *support* of $k^E_{p,q}(x,-)$ should be included in $r^{-1}_{p,q}\{x\}$, the set of extensions of x, so we ask that $k^E_{p,q}(x, f_E^{-1}\{q\} \backslash r^{-1}_{p,q}\{x\}) = 0$.

4.2.1 Lifting stochastic kernels through pullback squares

The following technical lemma will be crucial, both for generalising condition (4) and when we study the interaction of probabilistic strategies in 4.3.

Lemma 4.3 *Let X, Y, Z be standard Borel spaces, and let $f : Z \to X$ and $r : Y \to X$ be measurable functions. Consider the pullback ($Y \xleftarrow{\Pi_1} W \xrightarrow{\Pi_2} Z$) of r along f, where W is seen as a subspace of $Y \times Z$ as described in Section 3.3. Then:*

- *For every* $y \in Y$, $z \in Z$, *and* $U \in \Sigma_W$, *the* sections $U_y = \{z \in Z \mid (y, z) \in U\}$ *and* $U_z = \{y \in Y \mid (y, z) \in U\}$ *are in* Σ_Z *and* Σ_Y, *respectively.*
- *If* $k : X \times \Sigma_Y \to [0, 1]$ *is a stochastic kernel satisfying* $k(x, Y \backslash r^{-1}\{x\}) = 0$ *for every* x, *then the map* $k^{\#} : Z \times \Sigma_W \to [0, 1]$ *defined by* $k^{\#}(z, U) = k(f(z), U_z)$ *is a stochastic kernel.*

Proof. The proof of the first statement is standard. Let $k : X \times \Sigma_Y \to [0, 1]$ be a stochastic kernel, and let $z \in Z$. Then $k^{\#}(z, -)$ is a sub-probability measure, because $k(z, -)$ is countably additive and $(-)_z$ commutes with countable disjoint union. Now, for each $U \in \Sigma_W$, we must show that $k^{\#}(-, U) : W \to [0, 1]$ is measurable. For any U of the form $E_Y \times E_Z$, and for any $V \in \Sigma_{[0,1]}$, we have $k^{\#}(-, U)^{-1}V = \{z \in Z \mid k(f(z), U_z) \in V\} = \{z \in E_z \mid k(f(z), E_Y \cap r^{-1}\{f(z)\}) \in V\} \cup \{z \in Z \backslash E_z \mid k(f(z), \varnothing) \in V\}$ but by assumption $k(f(z), E_Y \cap r^{-1}\{f(z)\}) = k(f(z), E_Y)$ for any z, so we get $f^{-1}(k(-, E_Y)^{-1}V \cap E_Z) \cup f^{-1}(k(-, \varnothing)^{-1}V \backslash E_Z)$, a measurable set. So the set $\mathcal{D}$ of $U \in \Sigma_W$ such that $k^{\#}(-, U)$ is measurable contains all generating elements. To show $\mathcal{D} = \Sigma_W$, by the λ-π theorem [19] it is enough to show that $\mathcal{D}$ is closed under complements and countable disjoint unions. This is easily checked using standard measure-theoretic arguments. $\square$

4.2.2 General valuations

We can now generalise Definition 4.2 from the discrete case, by rephrasing conditions (1)-(4) in this setting:

Definition 4.4 A **valuation** on a race-free measurable esp $\mathcal{E}$ consists of a family $\mathcal{K}^{\mathcal{E}} = (k^E_{p,q})_{p \subseteq^+ q \in \mathcal{C}(\mathfrak{B}_E)}$ of stochastic kernels

$$k^E_{p,q} : f_E^{-1}\{p\} \times \Sigma_{f_E^{-1}\{q\}} \to [0, 1]$$

such that for all $x \in f_E^{-1}\{p\}$, we have $k^E_{p,q}(x, f_E^{-1}\{q\} \backslash r^{-1}_{p,q}\{x\}) = 0$, satisfying the following conditions:

(1) $k^E_{p,p}(x, -) = \delta_x$ for every $x \in f_E^{-1}\{p\}$;

(2) if $p_1 \subseteq^+ p_2 \subseteq^+ p_3$, then $k^E_{p_1,p_3} = k^E_{p_2,p_3} \circ k^E_{p_1,p_2}$;

(3) if $q \subseteq^+ p_1, \ldots, p_n$ and $x \in f_E^{-1}\{q\}$, then

$$\sum_I (-1)^{|I|+1} k^E_{q, \bigcup_{i \in I} p_i}(x, f_E^{-1}\{\bigcup_{i \in I} p_i\}) \leqslant 1,$$

where I ranges over nonempty subsets of $\{1, \ldots, n\}$ such that $\bigcup_{i \in I} p_i$ is consistent;

(4) if $p \subseteq^+ q$ and $p \subseteq^- p'$ then $k^E_{p', q \cup p'} = (k^E_{p,q})^{\#}$, the lifting of $k^E_{p,q}$ through the pullback of the race-freeness condition for $\mathcal{E}$, as in Lemma 4.3.

Definition 4.5 A **probabilistic strategy** on a race-free measurable game $\mathcal{A}$ consists of a measurable strategy $\sigma : \mathcal{S} \to \mathcal{A}$, and a valuation $\mathcal{K}^S$ on $\mathcal{S}$.

Note that this is well-defined: if $\mathcal{A}$ is race-free then by receptivity, so is $\mathcal{S}$.

4.3 Interaction of probabilistic strategies

Probabilistic strategies $(\sigma : \mathcal{S} \to \mathcal{A}^\perp \parallel \mathcal{B},\ \mathcal{K}^\mathcal{S})$ and $(\tau : \mathcal{T} \to \mathcal{B}^\perp \parallel \mathcal{C},\ \mathcal{K}^\mathcal{T})$ interact and compose as measurable strategies; it remains to equip the composition $\mathcal{T} \odot \mathcal{S}$ with a valuation, and we start by making the interaction $\mathcal{T} \circledast \mathcal{S}$ probabilistic.

While doing this, we must account for the fact that $\mathcal{T} \circledast \mathcal{S}$ is *not* an esp: the polarity of synchronisation events is not well-defined. We say that an event $e \in \mathfrak{B}_{T\circledast S}$ is a **σ-action** if $(\Pi_1)_\mathfrak{B} e$ is a positive element of $\mathfrak{B}_S$, and that e is a **τ-action** if $(\Pi_2)_\mathfrak{B} e$ is a positive element of $\mathfrak{B}_T$ (no event is both a σ-action and a τ-action, but some events are neither of the two). For $p, q \in \mathcal{C}(\mathfrak{B}_{T\circledast S})$ with $p \subseteq q$, we write $p \subseteq^\sigma q$ (resp. $p \subseteq^\tau q$) if all events of $q \backslash p$ are σ-actions (resp. τ-actions). Whenever $p \subseteq^\sigma q$, we see that the fibre $f^{-1}_{T\circledast S}\{q\}$ *extends* $f^{-1}_{T\circledast S}\{p\}$ according to $\mathcal{S}$:

Lemma 4.6 *If $p \subseteq^\sigma q$ in $\mathcal{C}(\mathfrak{B}_{T\circledast S})$, with $(\Pi_1)_\mathfrak{B} q = q_S \parallel q_C$ and $(\Pi_1)_\mathfrak{B} p = p_S \parallel p_C$, then the diagram*

$$\begin{tikzcd} f^{-1}_{T\circledast S}\{q\} \arrow[r, "r_{p,q}"] \arrow[d, "\Pi_S"'] & f^{-1}_{T\circledast S}\{p\} \arrow[d, "\Pi_S"] \\ f_S^{-1}\{q_S\} \arrow[r, "r_{p_S,q_S}"] & f_S^{-1}\{p_S\} \end{tikzcd}$$

is a pullback, where Π_S is Π_1 composed with the projection $f^{-1}_{S\parallel C}\{q_S \parallel q_C\} \to f_S^{-1}\{q_S\}$ (and similarly for p).

Of course the corresponding result holds for τ-extensions, and therefore using Lemma 4.3 we can define a family of stochastic kernels $k^{T\circledast S}_{p,q}$ indexed by $p, q \in \mathcal{C}(\mathfrak{B}_{T\circledast S})$ such that $p \subseteq^\sigma q$ or $p \subseteq^\tau q$, by lifting through the pullback square the relevant kernel in $\mathcal{K}^\mathcal{S}$ or $\mathcal{K}^\mathcal{T}$.

For $p, q \in \mathcal{C}(\mathfrak{B}_{T\circledast S})$, write $p \subseteq^{+,0} q$ if $p \subseteq q$ and all events in $q \backslash p$ have either positive or neutral polarity. If $p \subseteq^{+,0} q$, then there exists a *covering chain* $p \longrightarrow\!\!\!\!\subset^{\lambda_1} u_1 \longrightarrow\!\!\!\!\subset^{\lambda_2} \dots \longrightarrow\!\!\!\!\subset^{\lambda_n} u_n \longrightarrow\!\!\!\!\subset^{\lambda_{n+1}} q$, with $\lambda_i \in \{\sigma, \tau\}$ for each i. With respect to this chain we can define a kernel $k^{T\circledast S}_{p,u_1,\dots,u_n,q} = k^{T\circledast S}_{u_n,q} \circ \dots \circ k^{T\circledast S}_{p,u_1}$. In fact, we will see that the properties of the valuations $\mathcal{K}^\mathcal{S}$ and $\mathcal{K}^\mathcal{T}$ ensure that this kernel does not depend on the particular choice of chain. We first prove two auxiliary lemmas.

Lemma 4.7 *Let $\lambda \in \{\sigma, \tau\}$ and suppose $p \longrightarrow\!\!\!\!\subset^\lambda u \longrightarrow\!\!\!\!\subset^\lambda q$ in $\mathcal{C}(\mathfrak{B}_{T\circledast S})$. Then $k^{T\circledast S}_{p,q} = k^{T\circledast S}_{u,q} \circ k^{T\circledast S}_{p,u}$.*

Proof. It is enough to show that the lifting $(-)^\#$ of Lemma 4.3 preserves composition. It is a straightforward verification. □

Lemma 4.8 *If $p \longrightarrow\!\!\!\!\subset^\sigma q$ and $p \longrightarrow\!\!\!\!\subset^\tau p'$ in $\mathcal{C}(\mathfrak{B}_{T\circledast S})$, then $p' \cup q \in \mathcal{C}(\mathfrak{B}_{T\circledast S})$ and $k^{T\circledast S}_{q,p'\cup q} \circ k^{T\circledast S}_{p,q} = k^{T\circledast S}_{p',p'\cup q} \circ k^{T\circledast S}_{p,p'}$.*

Proof. Using that $\mathcal{S}$ and $\mathcal{T}$ are race-free, it is easy to see that

$$\begin{tikzcd} f^{-1}_{T\circledast S}\{q \cup p'\} \arrow[r, "r_{q,q\cup p'}"] \arrow[d, "r_{p',q\cup p'}"'] & f^{-1}_{T\circledast S}\{q\} \arrow[d, "r_{p,q}"] \\ f^{-1}_{T\circledast S}\{p'\} \arrow[r, "r_{p,p'}"] & f^{-1}_{T\circledast S}\{p\} \end{tikzcd}$$

is a pullback in **Meas**. Then, for $x \in f^{-1}_{T\circledast S}$, we check that the sub-probability measures $k^{T\circledast S}_{q,p'\cup q} \circ k^{T\circledast S}_{p,q}(x, -)$ and $k^{T\circledast S}_{p',p'\cup q} \circ k^{T\circledast S}_{p,p'}(x, -)$ on $f^{-1}_{T\circledast}\{q \cup p'\}$ are the same. It is sufficient that they agree on all $U \in \Sigma_{f^{-1}_{T\circledast S}\{q\cup p'\}}$ such that $U \subseteq r^{-1}_{p,p'\cup q}\{x\}$, and an inspection of the pullback above shows that $r^{-1}_{p,p'\cup q}\{x\}$, viewed as a subspace of $f^{-1}_{T\circledast}\{q \cup p'\}$, is isomorphic to the product $r^{-1}_{p,p'}\{x\} \times r^{-1}_{p,q}\{x\}$.

Moreover, using that the diagrams

$$\begin{tikzcd} f^{-1}_{T\circledast S}\{p'\} \arrow[r, "r_{p,p'}"] \arrow[d, "\Pi_T"'] & f^{-1}_{T\circledast S}\{p\} \arrow[d, "\Pi_T"] \\ f^{-1}_{T}\{p'_T\} \arrow[r, "r_{p_T,p'_T}"] & f^{-1}_{T}\{p_T\} \end{tikzcd} \qquad \begin{tikzcd} f^{-1}_{T\circledast S}\{q\} \arrow[r, "r_{p,q}"] \arrow[d, "\Pi_S"'] & f^{-1}_{T\circledast S}\{p\} \arrow[d, "\Pi_S"] \\ f^{-1}_{S}\{q_S\} \arrow[r, "r_{p_S,q_S}"] & f^{-1}_{S}\{p_S\} \end{tikzcd}$$

are pullbacks, we see that $r^{-1}_{p,p'}\{x\} \cong r^{-1}_{p_T,p'_T}\{x_T\}$ and $r^{-1}_{p,q}\{x\} \cong r^{-1}_{p_S,q_S}\{x_S\}$. So we let U be of the form $E \times E'$ for measurable sets $E \subseteq r^{-1}_{p_T,p'_T}\{x_T\}$ and $E' \subseteq r^{-1}_{p_S,q_S}\{x_S\}$. Then, we have

$$\begin{aligned} & k^{T\circledast S}_{q,p'\cup q} \circ k^{T\circledast S}_{p,q}(x, U) \\ &= \int_{y\in f^{-1}_{T\circledast S}\{q\}} k^{T\circledast S}_{q,p'\cup q}(y, U) k^{T\circledast S}_{p,q}(x,\, dy) \\ &= \int_{y\in r^{-1}_{p,q}\{x\}} k^{T}_{q_T,(p'\cup q)_T}(y_T, U_y) k^{S}_{p_S,q_S}(x_S,\, dy_S) \\ &= \int_{z\in E'} k^{T}_{p_T,p'_T}(r_{p,q}(z), E) k^{S}_{p_S,q_S}(x_S,\, dz) \\ &= \int_{z\in E'} k^{T}_{p_T,p'_T}(x_T, E) k^{S}_{p_S,q_S}(x_S,\, dz) \\ &= k^{S}_{p_S,q_S}(x_S, E') \times k^{T}_{p_T,p'_T}(x_T, E') \end{aligned}$$

and a symmetric calculation shows that $k^{T\circledast S}_{p',p'\cup q} \circ k^{T\circledast S}_{p,p'}(x, U)$ has the same value. $\square$

We can now show that any two parallel chains in $\mathcal{T} \circledast \mathcal{S}$ yield the same composite kernel:

Lemma 4.9 *If $p \subseteq^{+,0} q \in \mathcal{C}(\mathfrak{B}_{T\circledast S})$ and we have two chains*

$$\begin{aligned} & p \multimap^{\lambda_1} u_1 \multimap^{\lambda_2} \ldots \multimap^{\lambda_{n-1}} u_{n-1} \multimap^{\lambda_n} q \\ & p \multimap^{\rho_1} u'_1 \multimap^{\rho_2} \ldots \multimap^{\rho_{n-1}} u'_{n-1} \multimap^{\rho_n} q \end{aligned}$$

where $\lambda_i, \rho_i \in \{\sigma, \tau\}$ *for each* i, *then* $k^{T\circledast S}_{p,u_1,\ldots,u_n,q} = k^{T\circledast S}_{p,u'_1,\ldots,u'_n,q}$. *Thus we may write* $k^{T\circledast S}_{p,q}$ *for the kernel obtained via* any *chain.*

Proof. By induction on n. If $n = 0$, the result holds directly since there is only one possible chain from p to q. If $n > 0$, consider $v = u_1 \cup u'_1$. By the induction hypothesis, any chain from u_1 to q yields the same kernel, so in particular

$$k^{T\circledast S}_{u_1,u_2,\ldots,u_n,q} = k^{T\circledast S}_{v,q} \circ k^{T\circledast S}_{u_1,v}$$

and similarly we have

$$k^{T\circledast S}_{u'_1,u'_2,\ldots,u'_n,q} = k^{T\circledast S}_{v,q} \circ k^{T\circledast S}_{u'_1,v}.$$

Next, observe that $p {\longrightarrow\!\!\!\!\subset}^{\lambda_1} u_1 {\longrightarrow\!\!\!\!\subset}^{\rho_1} v$ and $p {\longrightarrow\!\!\!\!\subset}^{\rho_1} u'_1 {\longrightarrow\!\!\!\!\subset}^{\lambda_1} v$. If $\lambda_1 = \rho_1$, then it follows from Lemma 4.7 that $k^{T\circledast S}_{u_1,v} \circ k^{T\circledast S}_{p,u_1} = k^{T\circledast S}_{u'_1,v} \circ k^{T\circledast S}_{p,u'_1}$, since both are equal to $k^{T\circledast S}_{p,v}$. If instead $\lambda_1 \neq \rho_1$, then Lemma 4.8 shows that the same equality holds. Thus

$$\begin{aligned}
k^{T\circledast S}_{p,u_1,\ldots,u_n,q} &= k^{T\circledast S}_{u_1,\ldots,u_n,q} \circ k^{T\circledast S}_{p,u_1} \\
&= k^{T\circledast S}_{v,q} \circ k^{T\circledast S}_{u_1,v} \circ k^{T\circledast S}_{p,u_1} \\
&= k^{T\circledast S}_{v,q} \circ k^{T\circledast S}_{u'_1,v} \circ k^{T\circledast S}_{p,u'_1} \\
&= k^{T\circledast S}_{v,q} \circ k^{T\circledast S}_{u'_1,v} \circ k^{T\circledast S}_{p,u'_1} \\
&= k^{T\circledast S}_{u'_1,\ldots,u'_n,q} \circ k^{T\circledast S}_{p,u'_1} \\
&= k^{T\circledast S}_{p,u'_1,\ldots,u'_n,q}.
\end{aligned}$$

□

Following the above process, we obtain a family $\mathcal{K}^{\mathcal{T}\circledast\mathcal{S}} = (k^{T\circledast S}_{p,q})_{p\subseteq^{+,0}q}$ of kernels, indexed by the positive/neutral extensions in $\mathcal{C}(\mathfrak{B}_{T\circledast S})$. The following lemma will be central to proving that probabilistic strategies are closed under composition.

Lemma 4.10 *The family of kernels* $\mathcal{K}^{\mathcal{T}\circledast\mathcal{S}}$ *satisfies the following properties:*

(1) $k^{T\circledast S}_{p,p}(x, -) = \delta_x$ *for every* $x \in f^{-1}_{T\circledast S}\{p\}$;

(2) if $p_1 \subseteq^{+,0} p_2 \subseteq^{+,0} p_3$, *then* $k^{T\circledast S}_{p_1,p_3} = k^{T\circledast S}_{p_2,p_3} \circ k^{T\circledast S}_{p_1,p_2}$;

(3) if $q \subseteq^{+,0} p_1, \ldots, p_n$ *and* $x \in f^{-1}_{T\circledast S}\{q\}$, *then*

$$\sum_I (-1)^{|I|+1} k^{T\circledast S}_{q,\bigcup_{i\in I} p_i}(x, f^{-1}_{T\circledast S}\{\bigcup_{i\in I} p_i\}) \leqslant 1,$$

where I *ranges over nonempty subsets of* $\{1, \ldots, n\}$ *such that* $\bigcup_{i\in I} p_i$ *is consistent;*

(4) if $p \subseteq^{+,0} q$ *and* $p \subseteq^{-} p'$ *then* $k^{T\circledast S}_{p',q\cup p'} = (k^{T\circledast S}_{p,q})^{\#}$, *the lifting of* $k^{T\circledast S}_{p,q}$ *through the pullback of the race-freeness condition for* $\mathcal{T}\circledast\mathcal{S}$, *as in Lemma 4.3.*

Proof. Properties (1) and (2) hold by definition, (4) is a straightforward verification, and the argument for (3) is similar to that in the discrete case, see *e.g.* [25]. □

4.4 Composition and copycat

Recall from 3.5.1 that every configuration p of the composition $\mathfrak{B}_{T\odot S}$ has a unique interaction witness $[p] \in \mathfrak{B}_{T\odot S}$, and the fibre over p is defined as $f^{-1}_{T\odot S}\{p\} = f^{-1}_{T\circledast S}\{[p]\}$. Whenever $p \subseteq^+ q$ in $\mathcal{C}(\mathfrak{B}_{T\odot S})$, it must be the case that $[p] \subseteq^{+,0} [q]$, and therefore we may set $k^{T\odot S}_{p,q} = k^{T\circledast S}_{[p],[q]}$. This defines a family $\mathcal{K}^{\mathcal{T}\odot\mathcal{S}} = (k^{T\odot S}_{p,q})_{p\subseteq^+ q\in\mathcal{C}(\mathfrak{B}_{T\odot S})}$, and it is a direct consequence of Lemma 4.10 that this satisfies the axioms of Definition 4.4. Therefore:

Lemma 4.11 *The family $\mathcal{K}^{\mathcal{T}\odot\mathcal{S}}$ is a valuation on $\mathcal{T}\odot\mathcal{S}$, so that $(\tau\odot\sigma, \mathcal{K}^{\mathcal{T}\odot\mathcal{S}})$ is a probabilistic strategy, called the **composition of** $(\sigma, \mathcal{K}^{\mathcal{S}})$ **and** $(\tau, \mathcal{K}^{\mathcal{T}})$.*

Finally, for any measurable and race-free game $\mathcal{A}$, we make the copycat strategy $\text{cc}_{\mathcal{A}} : \mathbb{C}\mathbb{C}_{\mathcal{A}} \to \mathcal{A}^{\perp} \parallel \mathcal{A}$ probabilistic.

Lemma 4.12 *For $p, q \in \mathcal{C}(\mathfrak{B}_{\mathbb{C}\mathbb{C}_A})$ such that $p \subseteq^+ q$, for every $x \in f^{-1}_{\mathbb{C}\mathbb{C}_A}\{p\}$ there is a unique $y \in f^{-1}_{\mathbb{C}\mathbb{C}_A}\{q\}$ such that $r_{p,q}(y) = x$, and setting $k^{\mathbb{C}\mathbb{C}_A}_{p,q}(x, U) = \delta_y(U)$ defines a stochastic kernel.*

It is straightforward to check that the family $\mathcal{K}^{\mathbb{C}\mathbb{C}_{\mathcal{A}}}$ made up by the $k^{\mathbb{C}\mathbb{C}_A}_{p,q}$ is a valuation. So as before, we proceed to construct a bicategory with race-free measurable games as objects, and probabilistic strategies as morphisms, and where copycat is the identity morphism. If $\sigma : \mathcal{S} \to \mathcal{A}$ to $\tau : \mathcal{T} \to \mathcal{A}$ are probabilistic strategies, a map $\alpha : \mathcal{S} \to \mathcal{T}$ is a **morphism of probabilistic strategies** if it is a morphism of measurable strategies such that for all $p \subseteq^+ q \in \mathcal{C}(\mathfrak{B}_S)$, and for all $x \in f^{-1}_S\{p\}$ and $U \in \Sigma_{f^{-1}_T\{\alpha_{\mathfrak{B}}\, q\}}$, we have $k^S_{p,q}(x, \alpha^{-1}U) \leqslant k^T_{\alpha_{\mathfrak{B}}\, p, \alpha_{\mathfrak{B}}\, q}(\alpha x, U)$.

Theorem 4.13 *There is a bicategory* **PG** *with race-free standard Borel games as objects, probabilistic strategies as morphisms, and morphisms of probabilistic strategies as 2-cells.*

5 Game Semantics for $\mathbf{PPCF}^{\mathbb{R}}_{\mathbf{aff}}$

We define the semantics of $\mathrm{PPCF}^{\mathbb{R}}_{\mathrm{aff}}$ in **PG**, or rather in the quotiented *category* $\mathbf{PG}_{\cong}$ where probabilistic strategies are considered up to isomorphism, as usual in concurrent game semantics.

Call a measurable esp $\mathcal{E}$ **negative** if the initial moves in $\mathfrak{B}_E$ (and therefore in E) are negative. Let $\mathbf{PG}^{-}_{\cong}$ be the subcategory of $\mathbf{PG}_{\cong}$ consisting of negative games and negative (isomorphism classes of) strategies. We can show that $(\mathbf{PG}^{-}_{\cong}, \parallel, \mathbf{1})$ is symmetric monoidal closed and, in a special case sufficient for our purposes, the function space $\mathcal{A} \multimap \mathcal{B}$ can be characterised as follows:

Lemma 5.1 *If A, B are negative esps such that B has a unique initial move b_0, then $A \multimap B$ has events, polarity and consistent sets those of $A^{\perp} \parallel B$, and causality the transitive closure of $\leqslant_{A^{\perp}\parallel B} \cup \{(b_0, a) \mid a \text{ initial in } A\}$. If $\mathcal{A}$, $\mathcal{B}$ are measurable games where $\mathfrak{B}_A$ (and therefore A) has a unique initial move, then $f_{A\multimap B} : (A \multimap$*

$B) \to (\mathfrak{B}_A \multimap \mathfrak{B}_B)$ is defined to have the same action as $f_{A^\perp \| B}$. For any $p \in \mathcal{C}(\mathfrak{B}_A \multimap \mathfrak{B}_B)$, $f^{-1}_{A \multimap B}\{p\} = f^{-1}_{A^\perp \| B}\{p\}$.

The interpretation of $\mathrm{PPCF}^{\mathbb{R}}_{\mathrm{aff}}$ ground types was given in 3.2, and for higher types we set $[\![A \multimap B]\!] = [\![A]\!] \multimap [\![B]\!]$. The interpretation of the discrete part of $\mathrm{PPCF}^{\mathbb{R}}_{\mathrm{aff}}$ (that is, the PCF primitives and **coin**) follows the standard one (see *e.g.* [9,7]) and is made measurable via the functor **disc**. The strategy $[\![\bot]\!]$ is the unique (up to iso) strategy on $[\![\mathbf{Real}]\!]$ with no positive moves. The constant $\underline{r}$ is interpreted by $[\![\underline{r}]\!] : \mathcal{S} \to [\![\mathbf{Real}]\!]$, where $\mathcal{S}$ is the measurable sub-esp of $[\![\mathbf{Real}]\!]$ with unique maximal configuration $\{\mathrm{q}^-, r^+\}$, and with base $\mathfrak{B}_S = \mathfrak{B}_{[\![\mathbf{Real}]\!]}$. It remains to define the kernel for the extension $\{\mathrm{q}\} \subseteq^+ \{\mathrm{q}, a\}$: given the unique element of $f_S^{-1}\{\{\mathrm{q}\}\}$, it behaves like the Dirac measure on the singleton set $f_S^{-1}\{\{\mathrm{q}, a\}\}$.

Then, $[\![M \leqslant 0]\!]^\Gamma$ is defined as the composition of $[\![M]\!]^\Gamma$ with a strategy $\leqslant 0$ from $[\![\mathbf{Real}]\!]$ to $[\![\mathbf{Bool}]\!]$. Similarly for each $d \in \mathcal{D}$, $[\![d]\!]^\Gamma$ is a probabilistic strategy from $[\![\mathbf{Real}]\!]$ to $[\![\mathbf{Real}]\!]$ behaving like the kernel d. We omit their explicit definition, hoping that both of them are reconstructible from the example of Figure 3.

5.1 Probability of convergence of a strategy

We now aim to compare the probability of convergence of a closed term to that of its interpretation. The former was defined in 3.1, and we say here what we mean by the *probability of convergence* of a ground type strategy.

If $\sigma : \mathcal{S} \to [\![\mathbf{Bool}]\!]$ is a probabilistic strategy, then we observe that by receptivity, there is a unique $p_0 \in \mathcal{C}(\mathfrak{B}_S)$ such that $\sigma_{\mathfrak{B}} p_0 = \{\mathrm{q}^-\}$, and necessarily $f_S^{-1}\{p_0\}$ is a singleton, containing some $x \in \mathcal{C}(S)$. For $b \in \{\mathtt{tt}, \mathtt{ff}\}$, the **probability of convergence** $\Pr(\sigma \to b)$ is a sum, indexed by configurations of $\mathfrak{B}_S$ mapping to $\{\mathrm{q}, b\}$, of the total measure of the corresponding fibre. That is,

$$\Pr(\sigma \to b) = \sum_{\substack{p \in \mathcal{C}(\mathfrak{B}_S) \\ \sigma_{\mathfrak{B}}\, p = \{\mathrm{q}, b\}}} k^S_{p_0, p}(x, f_S^{-1}\{p\}).$$

Similarly, if $\tau : \mathcal{T} \to [\![\mathbf{Real}]\!]$, we write p_0 and x for the unique configurations over $\{q^-\}$. If $U \in \Sigma_{\mathbb{R}}$, viewed as a measurable subset of the fibre $f^{-1}_{[\![\mathbf{Real}]\!]}\{\{\mathrm{q}, a\}\}$, we set

$$\Pr(\tau \to U) = \sum_{\substack{p \in \mathcal{C}(\mathfrak{B}_T) \\ \tau_{\mathfrak{B}}\, p = \{\mathrm{q}, a\}}} k^T_{p_0, p}(x, \sigma_p^{-1} U),$$

where $\sigma_p^{-1} U$ is well-defined since σ_p is a measurable function $f_S^{-1}\{p\} \to f^{-1}_{[\![\mathbf{Real}]\!]}\{\{\mathrm{q}, a\}\}$.

5.2 Soundness

Next, we prove that our semantics is sound by relating, in Theorem 5.3 below, the denotation of a closed term to the denotations of its reducts. As a special case of the result, we obtain that for a term $\vdash M$ of type **Real**, if $U \in \Sigma_{\mathbb{R}}$, the following equality holds:

$$\Pr(\llbracket M \rrbracket \to U) = \int_{N \in \mathcal{T}^{\vdash \mathbf{Real}}} \Pr(\llbracket N \rrbracket \to U) \textsc{red}(M, dN).$$

The next lemma ensures that the integral on the right-hand side is well-defined:

Lemma 5.2 *Let A be a PPCF$_{\mathit{aff}}$ type of the form $A_1 \to \cdots \to A_n \to \mathbf{Gnd}$ for some ground type $\mathbf{Gnd}$. Suppose $\alpha_i : \llbracket A_i \rrbracket$, $i = 1, \ldots, n$, are probabilistic strategies. Then the function $h : \mathcal{T}^{\vdash A} \times \Sigma_{\llbracket \mathbf{Gnd} \rrbracket} \to [0, 1]$ defined as*

$$(M, U) \mapsto \Pr(\llbracket M \rrbracket (\alpha_i)_{i=1}^n \to U$$

is a stochastic kernel, where $\llbracket M \rrbracket (\alpha_i)_{i=1}^n$ is short for $(\Lambda^{-1})^n(\llbracket M \rrbracket) \odot (\|_{i=1}^n \alpha_i)$.

Proof. That $h(M, -)$ is a sub-probability measure for fixed M is straightforward, using the drop condition on probabilistic strategies. So we fix $U \in \Sigma_{\mathbf{Gnd}}$ and show that $h(-, U)$ is measurable. Recall that $\mathcal{T}^{\vdash A}$ is defined as a coproduct, so it is sufficient to show that for every term $y_1 : \mathbf{Real}, \ldots, y_m : \mathbf{Real} \vdash S : A$ containing no constants of the form $\underline{r}$, the map $\mathbb{R}^m \to [0, 1]$ defined as $(r_i)_{i=0}^m \mapsto h(S[\underline{r_i}/y_i], U)$ is measurable. This follows from a more general result, stating that for any strategy σ from $\llbracket \mathbf{Real} \rrbracket^m$ to $\llbracket \mathbf{Real} \rrbracket$, the map $(r_i)_{i=0}^m \mapsto \Pr(\sigma(\llbracket \underline{r_i} \rrbracket)_{i=0}^m \to U)$ is measurable.$\square$

Theorem 5.3 (Soundness) *Let $A = A_1 \to \cdots \to A_n \to \mathbf{Gnd}$ for some ground type $\mathbf{Gnd}$, let α_i be a probabilistic strategy on $\llbracket A_i \rrbracket$ for each $i = 1, \ldots, n$, and let $U \in \Sigma_{\mathbf{Gnd}}$. Then*

$$\Pr(\llbracket M \rrbracket (\alpha_i)_{i=1}^n \to U) = \int_{N \in \mathcal{T}^{\vdash A}} \Pr(\llbracket N \rrbracket (\alpha_i)_{i=1} \to U) \textsc{red}(M, dN).$$

Proof. We examine each case in the definition of RED. The details are standard.$\square$

5.3 Adequacy

Our final result is *adequacy*, showing that the probability of convergence of a term and that of its interpretation coincide:

Theorem 5.4 (Adequacy) *Let $\vdash M : \mathbf{Bool}$ be a PPCF$^{\mathbb{R}}_{\mathit{aff}}$ term. Then for $b \in \{\mathtt{tt}, \mathtt{ff}\}$, we have $\Pr(M \to b) = \Pr(\llbracket M \rrbracket \to b)$. Similarly, if $\vdash M : \mathbf{Real}$ is a PPCF$^{\mathbb{R}}_{\mathit{aff}}$ term and $U \in \Sigma_{\mathbb{R}}$, then $\Pr(M \to \underline{U}) = \Pr(\llbracket M \rrbracket \to U)$.*

The proof uses the soundness theorem and follows a standard argument based on logical relations. We omit the details.

6 Conclusion

The model we defined in this paper is strongly intensional. Compare, for instance, the primitive **coin** with the term $M = \mathbf{if}(\text{normal}(0, 1) \leqslant 0)\ \mathtt{tt}\ \mathtt{ff}$. Both give rise to the same probability distribution on $\{\mathtt{tt}, \mathtt{ff}\}$. But viewed as a strategy, $\llbracket \mathbf{coin} \rrbracket$ has only two positive events, one for **tt** and one for **ff**, whereas $[\![M]\!]$ has a continuum of such events, with the probability equally spread between those labelled **tt** and those labelled **ff**, according to normal$(0, 1)$. This level of intensionality informs our understanding of probabilistic programs and will facilitate the addition of computational effects to the language.

There are many opportunities for further work. A sequel paper will combine the present work with the *thin concurrent games* of [10], already enriched with discrete probability in [7]. This gives rise to a cartesian closed category which one can use to give semantics to a non-affine probabilistic language. Furthermore, by moving to a call-by-value setting using standard game semantics techniques [17], one could apply the framework to those probabilistic programming languages designed for Bayesian modelling and inference. It may in particular be useful in connection with inference algorithms involving an exploration of the space of execution traces (*e.g.* [29]).

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