

On the Decidability of Reachability in Continuous Time Linear Time-Invariant Systems*

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ABSTRACT

We consider the decidability of state-to-state reachability in linear time-invariant control systems over continuous time. We analyze this problem with respect to the allowable control sets, which are assumed to be the image under a linear map of the unit hypercube (i.e. zonotopes). This naturally models bounded (sometimes called saturated) controls. Decidability of the version of the reachability problem in which control sets are affine subspaces of \mathbb{R}^n is a fundamental result in control theory. Our first result is decidability in two dimensions ($n = 2$) if matrix A satisfies some spectral conditions and conditional decidability in general. If the transformation matrix A is diagonal with rational entries (or rational multiples of the same algebraic number) then the reachability problem is decidable. If the transformation matrix A only has real eigenvalues, the reachability problem is conditionally decidable. The time-bounded reachability problem is conditionally decidable and unconditionally decidable in two dimensions. Some of our results rely on the decidability of certain logical theories — namely the theory of the reals with exponential ($\mathfrak{R}_{\text{exp}}$) and with bounded sine ($\mathfrak{R}_{\text{exp},\sin}$) — which have been proven decidable conditional on Schanuel’s Conjecture — a unifying conjecture in transcendence theory. We also obtain a hardness result for a mild generalization of the problem where the target is a simple set (hypercube of dimension $n - 1$ or hyperplane) instead of a point. In this case, we show that the problem is at least as hard as the *Continuous Positivity problem* if the control set is a singleton, or the *Nontangential Continuous Positivity problem* if the control set is $[-1, 1]$.

CCS CONCEPTS

• **Computing methodologies** → **Computational control theory**.

KEYWORDS

LTI systems, control theory, reachability, decidability, linear differential equation, theory of the reals, exponential

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1 INTRODUCTION

This paper is concerned with *linear time-invariant (LTI) systems*. LTI systems are one of the most basic and fundamental models in control theory and have applications in circuit design, signal processing, and image processing, among many other areas. LTI systems have both discrete-time and continuous-time variants; here we are concerned solely with the continuous-time version.

A (continuous-time) LTI system in dimension n is specified by a transition matrix $A \in \mathbb{Q}^{n \times n}$, a control matrix $B \in \mathbb{Q}^{m \times n}$ and a set of controls $U \subseteq \mathbb{R}^m$. The evolution of the system is described by the differential equation $x'(t) = Ax(t) + Bu(t)$ where $u : \mathbb{R} \rightarrow U$ is a measurable function. Here we think of u as an input (or control) applied to the system. Note that the number of inputs is independent of the dimension: it is possible to have only one input ($m = 1$) in dimension n , or many inputs in small dimension ($m > n$).

Given such an LTI system, we say that state $x_0 \in \mathbb{R}^n$ can *reach* state $y \in \mathbb{R}^n$ if there exists $T \geq 0$ and a control¹ $u : [0, T] \rightarrow U$ such that the unique solution to the differential system $x'(t) = Ax(t) + Bu(t)$ for $t \in (0, T)$ with initial condition $x(0) = x_0$ satisfies $x(T) = y$. Similarly, given $t \geq 0$, we say that state x_0 can reach state y *in time at most t* if it can reach y at a time $T \leq t$. The problem of computing the set of all states reachable from a given state has been an active topic of research for several decades. Almost exclusively, the emphasis is typically on efficient and scalable methods to over- and under-approximate the reachable set [11, 19, 20, 30, 50]. Furthermore, this problem has numerous practical applications and is a fundamental basic block for the analysis of more complicated models, such as hybrid systems [1, 15, 53]. By contrast, there are relatively few results concerning the *decidability* of the reachable set—the focus of the present paper.

We consider the *LTI Reachability problem*: given an LTI system, and target state y , decide whether 0 can reach y . Specifically, we primarily focus on the case where the inputs are *saturated*, that is $U = [-1, 1]^m$. Equivalently, one can think of Bu as being a zonotope [42]. This naturally leads us to study the impact of the control matrix (B) on the LTI Reachability problem. We will also

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¹From now on, all controls are necessarily measurable functions, we omit it most of the time.

consider the *Bounded Time LTI Reachability problem* where we are given an upper bound on the time allowed to reach the target: given an LTI system, and target state y and a time bound T , decide whether 0 can reach y in time at most T . Finally, we consider the *LTI Set Reachability problem* where the target y becomes a set and we ask whether there exists a reachable point within this set.

Close variants of the LTI Reachability problem include the Controllability problem (set of points that can reach 0). It is also possible to consider the set of points reachable from a given source x_0 . Both problems are equivalent to the Reachability problem either in backward time and/or with a modification of the control matrix and set.

The decidability of point-to-point reachability for linear systems is open although for many different extensions and generalizations of the basic LTI model point-to-point reachability has been shown undecidable (see discussion of related work). While there are a number of classical results on decidability of reachability for LTI systems in the literature, these talk about “universal” reachability properties with almost the same names: *null reachability* (can one reach all states from the origin?) and *null controllability* (can one reach the origin from all states?) [7]. However these “universal” reachability problems are very different from the point-to-point version that we study. In particular, both null reachability and null controllability are decidable in polynomial time using linear algebra.

One of the first results about (continuous-time) LTI Reachability problems is that of Kalman [29] where the control sets are linear subspaces of \mathbb{R}^n . An important particular case is that of the *Orbit problem*: given a matrix A , an initial state x_0 and a state y , decide whether y is in the orbit of x under A , i.e. whether $y = e^{At}x_0$ for some² $t \geq 0$. This corresponds to the case when the control is a singleton (or equivalently with non-zero $x(0)$ and zero control set). This problem was shown to be decidable in polynomial time [12, 23]. These results yield (polynomial-time) decidability when the control sets are affine subspaces of \mathbb{R}^n . An exact description of the null controllable regions for general linear systems with saturating actuators was obtained [26], however, this formula does not immediately yield an algorithm (see Section 3).

In this paper, we study the decidability of several special instances of the LTI Reachability problem with saturated inputs. Specifically, we show the following results, all conditional on Schanuel’s Conjecture:

- In two dimensions ($n = 2$), the reachability problem is decidable.
- If A has real spectrum then the reachability problem is decidable.
- The time-bounded reachability problem is decidable.

These results are conditional in that they rely on the decidability of certain mathematical theories, namely the theory of the reals with exponential ($\mathfrak{R}_{\text{exp}}$) and with bounded sine ($\mathfrak{R}_{\text{exp,sin}}$). Both theories are known to be decidable assuming Schanuel’s conjecture [39], a major conjecture in transcendental number theory that is widely believed to be true. We also manage to find some class of LTI with *unconditionally* decidable reachability problem:

- In two dimensions, when the A has real spectrum and there is only one input ($m = 1$).
- When A is diagonalizable with rational eigenvalues (or rational multiples of the same algebraic number).
- When A is real diagonal, there is only one input and it has at most two nonzero entries.
- When A only has one eigenvalue, which is real, and there is only one input.

While those subclasses look ad-hoc, they all correspond to specific forms of the boundary of the reachable set and the study of the transcendental points on this boundary. In particular, some of those cases require some nontrivial theorems in transcendental number theory (Gelfond–Schneider, Lindemann–Weierstrass). See Section 3 for more details.

We also obtain a hardness result for a mild generalization of the problem where the target is a simple set (either a hyperplane or compact convex set of dimension $n-1$) instead of a point, and the control set is either $U = \{u\}$ or $U = [-1, 1]$. In this case, we show that the problem is at least as hard as the *Continuous Skolem problem* or the *Nontangential Continuous Skolem problem* which asks whether the first component $x_1(t)$ of the solution to a linear differential equation $x'(t) = Ax(t)$ has a zero (resp. nontangential zero). Showing decidability of any of these problems would entail a major new effectiveness result in Diophantine approximation, which suggests that the problem is very challenging.

Related Work. It is well-known that besides linear systems, most control problems are undecidable [8, 9]. For example, point-to-point reachability is hard or undecidable for *piecewise linear systems* [2, 3, 5, 7, 10, 31–33], for *saturated linear systems* [48] and point-to-set reachability is undecidable for polynomial systems, a consequence of [22]. However, to the best of our knowledge, there are no (un)decidability results within the class of LTI systems, except when the control sets are affine subspaces. An exact description of the null controllable regions for general linear systems with saturating actuators was obtained [26], but it does not immediately translate into a decidability result. On the other hand, the reachability problem is well-known to be challenging in practice, even for LTI systems. There is a vast literature on efficient and scalable methods to over- and under-approximate the reachable set [11, 19, 20, 30, 50]. However those methods, by construction, cannot lead to any decidability results on their own. In fact, one can observe that a corollary of these methods is that the “only” difficult part of the problem, in terms of decidability, is the boundary of the reachable set.

A range of different control problems for discrete- and continuous-time LTI systems under constraints on the set of controls have been studied in the literature [14, 16, 19, 21, 25–28, 46, 47, 49, 50, 54, 55]. Kalman showed that when the control set is $U = \mathbb{R}^m$, the system is globally null-controllable (every point can be controlled to 0) if and only if (A, B) is controllable (see Definition 4.1) [29]. Lee and Markus considered U such that $0 \in U \subset \mathbb{R}^m$ and showed that if (A, B) is controllable and all eigenvalues have negative real parts, then the system is globally null-controllable [38]. Sontag considered the problem of asymptotic null-controllability which asks if there is a control that reaches the origin in the limit [49]. Summers discussed over estimation of the reachable set (from origin)

²Although this problem is known as the “Orbit” problem, it really is a semi-orbit problem. If one considers $t \in \mathbb{R}$, the problem reduces to two semi-orbit problems.

by n -dimensional ellipsoids when $U = [-1, 1]^m$ [50]. Schmitendorf considered time varying matrices $A(t)$ and $B(t)$ and gives a characterisation for a given point to be controllable when U is compact [46], however this does not immediately yield an algorithm (see Section 3). Lafferriere considered a different reachability problem where the inputs are expressible in the first-order theory of the reals with some unknown coefficients [35, 36], and it was generalized by [17, 18] but our problem is of a very different nature because we do not require the input to have a closed-form.

2 EXAMPLES

The idea of using an external input to manipulate the state of some system to achieve a certain goal is fundamental and everywhere in our lives. In order to give a better idea of the problem we are trying to solve, we will informally explain the theory and its goals via some examples.

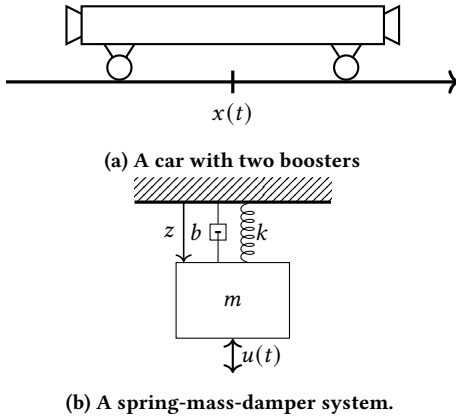


Figure 1: Two examples of dynamical systems with controls.

Consider the toy example in Figure 1a (taken from Chapter 1 of [41]). A car has two boosters, one at the front and one at the back. At time $t = 0$, it starts at some position $x_0 \in \mathbb{R}$ on the real line, with velocity v_0 . The objective is to reach the origin and stay there indefinitely, that is to reach the origin with a speed of 0. The external input in the above problem is the effects of the boosters that affect the acceleration directly, thereby affecting the velocity and the position. We model the state of the system by its position and velocity $S(t) = (x(t), v(t)) \in \mathbb{R}^2$. Assuming the front and rear boosters are similar and give a max acceleration of M units, we can model the dynamics by

$$x'(t) = v(t), \quad v'(t) = u(t)$$

where $u(t) \in [-M, M]$: the acceleration is positive when the rear booster is on, and negative when the front booster is on. We call $U = [-M, M]$ the control set. Combining both equations and writing it in matrix form gives us

$$S(0) = \begin{bmatrix} x_0 \\ v_0 \end{bmatrix}, \quad S'(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} S(t) + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} u(t) = AS(t) + Bu(t).$$

Here, the problem is to find or “synthesize” a control u such that we reach $(0, 0)$ from the initial point. Note that in real life, we cannot change the control (booster output) arbitrarily fast, *i.e.* not

all functions u are to be considered. In this work, we neglect this aspect and allow any function $u : \mathbb{R} \rightarrow U$ that is *measurable*, which is essentially the minimum mathematical condition for the problem to make sense. Observe that already in this toy example, it is natural to consider a bound on the acceleration: the control set is therefore bounded.

In the example of the car, we viewed the input as something under our control that we used to achieve some objective. A dual view is to consider certain safety problems and check whether the input, now controlled by an adversary, can be used to steer the system to a bad state. Consider the system in fig. 1b, a spring-mass-damper system with an external force acting on it. This typically models a vehicle’s suspension. For example, consider a bike travelling on a road and encountering a speed breaker or hump. We are interested in the vertical movement of the tires, after they cross the hump. Here, the tire acts as the mass, the damper and spring form the bike suspension, which provides shock absorption and the recoil force upon hitting the ground, the road is modelled by the external force u . A bad state is one when the tire’s vertical movement is higher than certain admissible value which we want to avoid (for it could damage or even break the suspension). Here the problem is to decide whether it is possible via some external input to reach a bad state. Similar to the previous example, we could model the above system by an LTI system with a bounded control set.

3 CHALLENGES

A major challenge in solving the continuous-time reachability problem is the fact that there is no simple formula, or more exactly, no formula that is immediately computable. Given a LTI system $x'(t) = Ax(t) + Bu(t)$, there is a general expression for $x(t)$ given u that involves an integral and exponential of matrix (see Section 4):

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-s)} Bu(s) ds. \quad (1)$$

Therefore, the reachability problem is equivalent to checking whether there exists a u such that (1) is equal to the target y . Unfortunately, it seems impossible to obtain a more useful closed-form formula without knowing more about the shape of u . This is why we now assume that $x(0) = 0$ and $u : \mathbb{R} \rightarrow U$ for some convex set U . The former is without loss of generality³ and the latter is very common in the literature.

The simplest case is when $U = \mathbb{R}^m$: the reachable set can be shown to be a linear subspace, the image of the so-called controllability matrix $[B \ AB \ \dots \ A^{n-1}B]$ and therefore the reachability problem reduces to an orbit problem (if the image of the controllability matrix is not the whole space, what happens on the remaining space is exactly an orbit problem).

A more interesting case, and the subject of this paper, is when U is a compact convex polytope and in particular a hypercube: $U = [-1, 1]^m$. This is known as the *saturated input* case. It is not hard to see that when $x(0) = 0$ and U is convex, the reachable set \mathcal{R} is strictly convex. Unfortunately, the set \mathcal{R} can still be very complicated and checking whether a single point lies inside it turns out to be a challenging problem. We are aware of two distinct but similar results in this direction. Schmitendorf et al. [46] have some

³One can always shift the control set U to ensure that $x(0) = 0$.

general conditions under which a control can steer a point to the origin. In the particular case at hand, they turn out to be equivalent to another formulation by [26] where the boundary of the reachable region is described by the sets of the form

$$\int_0^\infty e^{At} b \operatorname{sgn}(c^T e^{At} b) dt$$

where $c \in \mathbb{R}^n \setminus \{0\}$ is a parameter and b is a column of B . The main challenge is that evaluating this integral is potentially a hard problem. In particular, if A has a complex eigenvalue whose argument is not rational multiple of π , then the sign of $c^T e^{At} b$ will follow a completely irregular pattern. In fact, the a priori simpler problem of deciding whether $c^T e^{At} b$ will change sign *at all* is exactly the continuous Skolem problem. This problem is open and has been shown to be related to difficult number theoretical questions (see Section 4.2). Note, however, that computing this integral, or rather deciding if this integral is less than some prescribed number, does not necessarily require solving the continuous Skolem problem. In fact, a solution to Skolem would not help per se (there could be infinitely many changes, whose values are not even algebraic), and conversely, computing this integral does not necessarily help deciding the existence of a sign change.

The approach we follow in this paper is to study the membership in the boundary. Indeed, it is not hard to see (see Lemma 5.2) that if we can decide membership into the boundary, then we can decide reachability. Consider the example illustrated on Figure 2: already in dimension 2, the boundary of the reachable set from $x(0) = 0$ when $U = [-1, 1]$, $B = \begin{bmatrix} b_1 & b_2 \end{bmatrix}$ and A is stable is not trivial; it consists of two smooth curves joining at singular points. In this case, the boundaries are exactly (see full version for the details) the sets

$$\begin{aligned} \partial\mathcal{R}(A, b_1) &= \left\{ \pm \begin{bmatrix} 2 - 4t^3 \\ 3 - 6t^2 \end{bmatrix} : t \in [0, 1] \right\}, \\ \partial\mathcal{R}(A, B) &= \left\{ \pm \begin{bmatrix} 4 - 4|t|^3 \\ 6t^2 \operatorname{sgn}(t) \end{bmatrix} : t \in [-1, 1] \right\}. \end{aligned}$$

Hence deciding membership of a point (x, y) is exactly deciding

$$\exists t \in [-1, 1]. 4 - 4|t|^3 = \pm x \wedge 6t^2 \operatorname{sgn}(t) = \pm y \quad (2)$$

which can be shown, after a bit of rewriting, to be in \mathfrak{R}_0 , the first-order theory of the reals. Therefore, membership is decidable in this case by Tarski's Theorem. This example turns out to be easy because A is diagonal and all eigenvalues of A are rational and therefore, the boundary can be described by polynomial equations (see Proposition 5.3).

A small modification of this example already turns out to be much more challenging, if we consider the case where

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -\sqrt{2} \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Then a similarly computation (see the full version) shows that

$$\partial\mathcal{R}(A, B) = \left\{ \pm \begin{bmatrix} -2|t| \\ \sqrt{2} (1 - |t|\sqrt{2}) \operatorname{sgn}(t) \end{bmatrix} : t \in [-1, 1] \right\}.$$

Hence deciding membership of a point (x, y) is exactly deciding

$$\exists t \in [-1, 1]. -2|t| = \pm x \wedge \sqrt{2} (1 - |t|\sqrt{2}) \operatorname{sgn}(t) = \pm y$$

which can be shown, after a bit of rewriting, to be in \mathfrak{R}_{\exp} , the first-order theory of the reals with exponential. However, this formula is not in \mathfrak{R}_0 because we need to raise t to some irrational power ($\sqrt{2}$) which leaves its decidability open a priori. This example is a particular case of Proposition 5.4: when the eigenvalues of the matrix are real, the formulas will only involve real exponentials and the boundary can be expressed in the theory of reals with exponential (\mathfrak{R}_{\exp}). This theory is known to be decidable subject to Schanuel's conjecture (see Section 4.4). Unfortunately, the formulas may further involve sine and cosine when A has complex eigenvalues, and the theory of reals with sine and cosine is undecidable for example by Richardson's theorem [45] and its improvements [34]. This suggests that the reachability problem is hard, but surprisingly, we have only been able to show some hardness results in the case of set reachability.

It turns out that the second example is decidable by a similar argument to the proof of Proposition 5.3 using Theorem 4.6 (Gelfond-Schneider). Indeed, we can remove the absolute value and sgn in (2) by doing a case distinction on the sign of t . Then the first equation in (2) implies that $2t = x$ and hence the second that $\sqrt{2}(1 - (x/2)^{\sqrt{2}}) = y$. But since x is assumed to be algebraic, $(x/2)^{\sqrt{2}}$ must be transcendental hence y cannot be algebraic, a contradiction. It follows that the boundary contains no algebraic points which shows decidability of the problem. This reasoning, however, does not seem to apply in the general case of a system of dimension $n = 2$ with $m = 2$ controls.

The subclasses of linear systems that we identified for our decidability results, while ad-hoc at first sight, really correspond to subclasses of exponential polynomial equations that we can solve systematically.

4 PRELIMINARIES

We denote the usual Euclidean norm of vectors $x \in \mathbb{C}^n$ by $\|x\|$ and $\|A\|$ the induced norm on matrices $A \in \mathbb{C}^{n \times n}$. Any induced norm is consistent ($\|Ax\| \leq \|A\|\|x\|$) and therefore submultiplicative ($\|AB\| \leq \|A\|\|B\|$). Given a matrix $A \in \mathbb{C}^{n \times n}$, e^A denotes the matrix exponential of A . In particular, we have that $\|e^A\| \leq e^{\|A\|}$. We denote the boundary of set S by ∂S and its closure by \bar{S} . We denote the transpose of a vector or matrix A by A^T and its spectrum by $\sigma(A)$. Additional preliminaries are available in the full version.

4.1 Control Theory

The following definition is standard in the literature of control theory.

Definition 4.1 (Controllable). A pair of matrices (A, B) is called controllable if the rank of $[B, AB, \dots, A^{n-1}B]$ is n , where A is an $n \times n$ matrix.

Consider two vectors c and b in \mathbb{R}^n and define a function $f_{c,A,b}(t) = c^T e^{At} b$. Then it follows from the Jordan decomposition that $f_{c,A,b}(t) = \sum_{j=1}^m P_j(t) e^{\theta_j t}$ where each θ_j is an eigenvalue of A and P_j a polynomial. The following properties of f are well-known (see e.g. [24]).

LEMMA 4.2. *Let $f_{c,A,b}(t)$ be the function defined above. Then*

- *if $f_{c,A,b} \neq 0$, then the number of zeros of $f_{c,A,b}$ in any bounded interval is finite,*

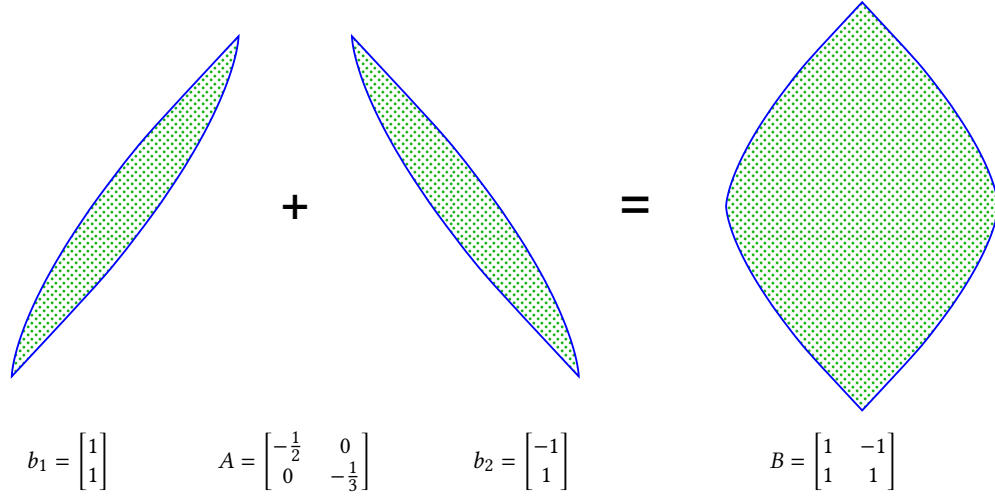


Figure 2: Example of a simple LTI system: the reachable set is depicted in three cases. The two pictures on the left correspond to the case of one control ($B = b_1$ and b_2 respectively). The picture on the right corresponds to the case where $B = [b_1, b_2]$: the reachable set is then the Minkowski sum of the two reachable sets.

- (A, b) is controllable \iff for all c , $f_{c,A,b} \neq 0$,
- if the eigenvalues of A are real and b, c are nonzero then $f_{c,A,b}$ has at most $n - 1$ zeros.

Given a matrix A , a time bound $\tau \in \mathbb{R} \cup \{\infty\}$ and a control set U , define the null-controllable set C and the reachable set \mathcal{R} as

$$C_\tau(A, B, U) = \bigcup_{T=0}^{\tau} \left\{ - \int_0^T e^{-At} B u(t) dt \mid u : [0, T] \rightarrow U \text{ measurable} \right\},$$

$$\mathcal{R}_\tau(A, B, U) = \bigcup_{T=0}^{\tau} \left\{ \int_0^T e^{At} B u(t) dt \mid u : [0, T] \rightarrow U \text{ measurable} \right\}.$$

When $\tau = \infty$, we simply write $C(A, B, U)$ and $\mathcal{R}(A, B, U)$. It follows immediately from those definitions that $\mathcal{R}_\tau(A, B, U) = C_\tau(-A, B, -U)$ and $U = -U$ for a hypercube (or any symmetric set). It is customary in the literature to express results about the null-controllable sets. However, since we are interested in reachability questions, we find it more convenient to state all results using the reachable set.

Define a matrix A to be *stable* if all its eigenvalues have negative real part, *antistable* if all its eigenvalues have positive real part, *weakly-stable* (also called *semi-stable* in [26]) if all its eigenvalues have nonpositive real parts and *weakly-antistable* if all its eigenvalues have nonnegative real parts. Clearly A is stable (resp. weakly-stable) if and only if $-A$ is antistable (resp. weakly-antistable).

In some cases, it is well-known that it is possible to decompose the system into its stable and weakly-antistable parts (or dually into its antistable and semistable parts). In particular, this is possible when the control set is a hypercube:

PROPOSITION 4.3 ([26]). *Let (A, B) be controllable, $U = [-1, 1]^m$.*

- If A is weakly-antistable, then $\mathcal{R} = \mathbb{R}^n$
- If A is stable, then \mathcal{R} is a bounded convex open set containing the origin

- If $A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ where $A_1 \in \mathbb{R}^{n_1 \times n_1}$ stable and $A_2 \in \mathbb{R}^{n_2 \times n_2}$ weakly-antistable and B is partitioned as $\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ accordingly, then $\mathcal{R} = \mathcal{R}(A_1, B_1, U) \times \mathbb{R}^{n_2}$.

This fact, suggests that we need to study the reachable region of stable systems. Decompose B as $[b_1, \dots, b_m]$ and assume that $U = [-1, 1]^m$, then it is not hard to check that

$$\mathcal{R}(A, B, [-1, 1]^m) = \mathcal{R}(A, b_1, [-1, 1]) + \dots + \mathcal{R}(A, b_m, [-1, 1]) \quad (3)$$

is a Minkowski sum of reachable regions in which $m = 1$, i.e. B is a column vector. In this case, one can obtain an explicit description of \mathcal{R} . Note however that (3) does not, by itself, allows for a reduction to this simpler case: even if we have an algorithm to decide membership in each smaller control set, deciding membership in the Minkowski sum is nontrivial.

THEOREM 4.4 ([26]). *Let A be stable, $b \in \mathbb{R}^{n \times 1}$ such that (A, b) is controllable and $U = [-1, 1]$. Then $\mathcal{R}(A, b, U)$ is an open convex set containing 0 and its boundary is given by*

$$\partial \mathcal{R}(A, b, U) = \left\{ \int_0^\infty e^{At} b \operatorname{sgn}(c^T e^{At} b) dt : c \in \mathbb{R}^n \setminus \{0\} \right\}$$

which is a strictly convex set.

4.2 Continuous Skolem Problem

The *Continuous Skolem problem* is a fundamental decision problem concerning the reachability of linear continuous-time dynamical system [6]. Given an initial point and a system of linear differential equation, the problem asks whether the orbit ever intersects a given hyperplane. More precisely, given a matrix $A \in \mathbb{R}^{n \times n}$, two vectors $c, b \in \mathbb{R}^n$, with rational (or algebraic coefficients), the question is whether there exists $t \geq 0$ such that $c^T e^{At} b = 0$. One can also consider the bounded time version of this problem, where one asks

about the existence of a zero at time $t \leq T$ for some prescribed rational number T . While similar in spirit to the Orbit problem (does the orbit reach a given point?), it is of a very different nature. In fact, decidability of this problem is still open, even when restricting to the case of a bounded time interval [13]. The Continuous Skolem problem admits several reformulations, notably whether a linear differential equation or an exponential polynomial admits a zero [6].

The Continuous Skolem problem can be seen as the continuous analog of the *Skolem problem*, which asks whether a linear recurrent sequence has a zero. The Skolem problem is a famously open problem in number theory and computer science, which is known to be decidable up to dimension 4 and not known to be either decidable or undecidable starting from dimension 5. We refer the reader to [43] for a recent survey on the Skolem problem.

The Continuous Skolem problem is only known to be decidable in very specific cases: in low dimension, with a dominant real eigenvalue or a particular spectrum [6, 13]. Recent developments suggest that the Continuous Skolem problem is a very challenging problem. Indeed, decidability of the problem in the case of two (or more) rationally linearly independent frequencies would imply a new effectiveness result in Diophantine approximation that seem far off at the moment [13]. Even decidability in the bounded case is nontrivial because of *tangential zeros*, and has only been shown recently subject to Schanuel's Conjecture, a unifying conjecture in transcendental number theory. While Schanuel's Conjecture is widely believed to be true, its far-reaching consequences suggest that any proof is a long way off. For instance, it easily implies that $\pi + e$ is transcendental, but meanwhile the much weaker fact that $\pi + e$ is irrational is still unknown! It should be noted however that from a complexity theoretic perspective, the Continuous Skolem problem is only known to be at least NP-hard [6].

We now introduce the *Continuous Nontangential Skolem problem*, a variant of this problem where only zero-crossings are considered. Given a matrix $A \in \mathbb{R}^{n \times n}$, two vectors $c, b \in \mathbb{R}^n$, with rational (or algebraic coefficients), the question is whether there exists $t \geq 0$ such that $f(t) = 0$ and $f'(t) \neq 0$, where $f(t) = c^T e^{At} b = 0$. We call such a time t a *nontangential zero*, as opposed to *tangential zero* that would satisfy $f(t) = f'(t) = 0$. Clearly any nontangential zero is a zero but some systems admit tangential zeros.

We believe that this problem is essentially as hard as the Continuous Skolem problem. Indeed, one of the reasons why the Continuous Skolem problem is believed to be hard is a Diophantine hardness proof [13]. In short, this reduction shows that decidability would entail some major new effectiveness result in Diophantine approximation, namely computability of the Diophantine-approximation types of all real algebraic numbers. But one can observe that the reduction of [13] only relies on nontangential zeros, hence decidability of the Nontangential Skolem problem would also entail those Diophantine effectiveness results.

We note that there is a subtlety in the definition of the Nontangential Skolem problem: one needs to decide the existence of nontangential zeros but it is entirely possible that it also has some tangential zeros. Hence, even over a bounded interval, it is not clear that the problem is decidable. For instance, a Newton-based method would not be able to distinguish between a tangential or a nontangential zero using a finite number of iterations. The problem

easily becomes decidable, *over a bounded interval* under the premise that there are no tangential zeros. We also believe that a variant of the decidability argument in [13] would show that the problem is decidable over bounded interval, assuming Schanuel's conjecture. As we have seen before, the problem is hard for Diophantine-approximation types over unbounded intervals.

4.3 First-order theory of the reals

A *sentence* in the first-order theory of the reals is (although one can allow more general expression that interleave quantifiers and connectives) an expression of the form $\phi = Q_1 x_1 \cdots Q_n x_n. \psi(x_1, \dots, x_n)$ where each Q_1, \dots, Q_n is one of the quantifiers \exists or \forall , and ψ is a Boolean combinations (built from connectives \wedge, \vee and \neg) of *atomic predicates* of the form $P(x) \sim 0$ where P is a polynomial with integer coefficients and \sim is one of the relations $<, \leq, =, >, \geq, \neq$. A theory is said to be decidable if there is an algorithm that, given a sentence, can determine if it is true or false. A famous result by Tarski is that first-order theory of reals admits quantifier elimination and is decidable. In this paper, we denote by \mathfrak{R}_0 this theory, formally this is the first-order theory of the structure $(\mathbb{R}, 0, 1, <, +, \cdot)$.

THEOREM 4.5 (TARSKI'S THEOREM [51]). *The first-order theory \mathfrak{R}_0 of reals is decidable.*

We note that although the theory only allows integer coefficients, one can easily introduce algebraic coefficients by creating new variables and express that they are the roots on some polynomial. See also [4, 44] for more efficient decision procedures for the first-order theory of reals.

4.4 Transcendental number theory

A complex number is said to be *algebraic* if it is a root of a nonzero polynomial with integer coefficients. We denote by $\overline{\mathbb{Q}}$ the field of all algebraic numbers. A non-algebraic number is called *transcendental*. We will use that all field operations on algebraic numbers (including comparisons) are effective, see e.g. [4]. We will use transcendence theory in our proofs, essentially to argue that some equalities between two numbers are impossible. A classical results concerns powers of algebraic numbers.

THEOREM 4.6 (GELFOND-SCHNEIDER). *If a and b are algebraic numbers with $a \neq 0, 1$ and b irrational, then any value⁴ of a^b is transcendental.*

An important generalization of this result is the Lindemann-Weierstrass Theorem. In particular, we will use the following reformulation by Baker:

THEOREM 4.7 (LINDEMANN-WEIERSTRASS, BAKER'S REFORMULATION). *If $\alpha_1, \dots, \alpha_k$ are distinct algebraic numbers, then $e^{\alpha_1}, \dots, e^{\alpha_k}$ are linearly independent over the algebraic numbers.*

Our results in some cases depend on Schanuel's conjecture, a unifying conjecture in transcendental number theory [37] that generalizes many of the classical results in the field (including Theorems 4.6 and 4.7). The conjecture states that if $\alpha_1, \dots, \alpha_k \in \mathbb{C}$

⁴In general, a^b is defined by $e^{b \log a}$ and can have several values depending on the branch of the logarithm.

are linearly independent over \mathbb{Q} then some k -element subset of $\{\alpha_1, \dots, \alpha_k, e^{\alpha_1}, \dots, e^{\alpha_k}\}$ is algebraically independent.

Assuming Schanuel's Conjecture, MacIntyre and Wilkie [40] have shown decidability of the first-order theory of the expansion of the real field with the exponentiation function and the sin and cos functions restricted to bounded intervals.

THEOREM 4.8 (WILKIE AND MACINTYRE). *If Schanuel's conjecture is true, then, for each $n \in \mathbb{N}$, the first-order theory of the structure $(\mathbb{R}, 0, 1, <, +, \cdot, \exp, \cos \upharpoonright_{[0,n]}, \sin \upharpoonright_{[0,n]})$ is decidable.*

In the rest of the paper, we denote by \mathfrak{R}_{\exp} the first-order theory of the reals with the exponential, and $\mathfrak{R}_{\exp, \sin}$ the first-order theory of the reals with the exponential and the sin and cos functions restricted to a bounded interval.

5 DECIDABILITY

The goal of this section is to study the decidability of the LTI Reachability problem in various special cases. We will always restrict ourselves to the case where the control set is a hypercube $U = [-1, 1]^m$. Surprisingly, and despite the explicit description given by Theorem 4.4, this problem remains challenging (see Section 6 for some hardness results). A well-known observation, already made in Section 4.1 is that we can simplify the problem when the input lies in a hypercube.

LEMMA 5.1 (SEE FULL VERSION). *For any $A \in \mathbb{R}^{n \times n}$, $\tau \in \mathbb{R} \cup \{\infty\}$ and $B \in \mathbb{R}^{n \times m}$, if A is stable or $\tau < \infty$ then there exists computable real matrices⁵ $C_1, \dots, C_m, P_1, \dots, P_m$ such that*

$$\mathcal{R}_\tau(A, B, [-1, 1]^m) = \sum_{i=1}^m P_i \mathcal{R}_\tau(C_i, b_i, [-1, 1])$$

where the b_i are the columns of B , $\sigma(C_i) \subseteq \sigma(A)$ and (C_i, b_i) is controllable for all i . In particular, if for every i , the membership in $\mathcal{R}_\tau(C_i, b_i, [-1, 1])$ or in $\partial \mathcal{R}_\tau(C_i, b_i, [-1, 1])$ is expressible in a theory \mathfrak{R} that contains \mathfrak{R}_0 , then membership in $\mathcal{R}_\tau(A, B, [-1, 1]^m)$ is expressible in that theory \mathfrak{R} .

A second observation is that, for the purpose of decidability, we can focus on the boundary of the reachable set. Indeed, we can compute arbitrarily good approximations of the boundary and hence, solve the problem if we know that the target is not on the boundary. This only gives a semi-decision for the problem however because the algorithm will never conclude when the target is exactly on the boundary. If we can decide if an algebraic point is on the boundary, then we can either immediately conclude (target on the boundary) or make sure that the semi-decision procedure will finish (target not on the boundary). The following result is well-known:

LEMMA 5.2 (SEE FULL VERSION). *There is an algorithm that, given⁶ $A \in \mathbb{Q}^{n \times n}$ stable and $B \in \mathbb{Q}^{n \times k}$ and $p \in \mathbb{N}$, computes two convex polytopes P_- and P_+ such that $P_- \subseteq \partial \mathcal{R}(A, b, [-1, 1]) \subseteq P_+$ and the Hausdorff distance⁷ between P_- and P_+ is less than 2^{-p} .*

⁵Note that the C_i can be of lower dimension than n and the P_i are not necessarily square.

⁶In fact, this is still true for algebraic and even computable coefficients. A real is computable if one can produce arbitrary precise rational approximations of it.

⁷Recall that the Hausdorff distance, which measures how far two sets are from each other, between two sets X and Y is defined by $d(X, Y) = \max(\sup_{x \in X} \inf_{y \in Y} \|x - y\|, \sup_{y \in Y} \inf_{x \in X} \|x - y\|)$.

We start with the simplest case where A is already diagonal. In fact, this seemingly easy case is already difficult and we only manage to solve it unconditionally in some cases.

PROPOSITION 5.3. *The LTI Reachability problem is decidable when $U = [-1, 1]^m$ and one of the following conditions holds:*

- A is real diagonal, B is a column (i.e. $m = 1$) and it has at most 2 nonzero entries,
- A is real diagonalizable and its eigenvalues are rational, or a rational multiple of the same algebraic number,
- A only has one eigenvalue, which is real, and B is a column (i.e. $m = 1$).

PROOF. We start by observing that in the second case, we can reduce to the case where A is diagonal. Indeed, since A is real diagonalizable, we can write $A = P^{-1}MP$ where P is real and M is diagonal. Note that M satisfies all the assumptions since it contains only the eigenvalues of M . Furthermore, the reachable set is easily observed to be $\mathcal{R}(A, B, U) = P^{-1}\mathcal{R}(M, PB, U)$ hence it is equivalent to decide if $P_y \in \mathcal{R}(M, PB, U)$.

Assume that A is diagonal (this covers the first two cases of the theorem with the above remark). Write $A = \text{diag}(\lambda_1, \dots, \lambda_n)$ with $\lambda_i \geq \lambda_{i+1}$ without loss of generality. Decompose A into $A = \text{diag}(A_1, A_2)$ where A_1 contains the nonnegative λ_i and A_2 the negative ones. Then A_1 is weakly antistable and A_2 is stable. Decompose B into B_1 and B_2 accordingly. Then by Proposition 4.3, $\mathcal{R}(A, B, U) = \mathbb{R}^{n_1} \times \mathcal{R}(A_2, B_2, U)$. Then by Lemma 5.1, we have that $\mathcal{R}(A_2, B_2, [-1, 1]^m) = \sum_{i=1}^m P_i \mathcal{R}(C_i, b_i, [-1, 1])$ where the b_i are the columns of B_2 and (C_i, b_i) is controllable for all i .

We now assume that A is diagonal with negative eigenvalues, $B = b$ is a column vector, (A, b) is controllable and $U = [-1, 1]$. Write $A = \text{diag}(\mu_1, \dots, \mu_k)$ where the μ_i are negative. Then by Theorem 4.4, we have that $\partial \mathcal{R}(A, b, [-1, 1]) = \{\beta_c : c \in \mathbb{R}^n \setminus \{0\}\}$, where

$$\beta_c := \int_0^\infty e^{At} b \operatorname{sgn}(c^T e^{At} b) dt$$

And observe that

$$f_c(t) := c^T e^{At} b = c^T \text{diag}(e^{\mu_1 t}, \dots, e^{\mu_k t}) b = \sum_{i=1}^k c_i e^{\mu_i t} b_i.$$

If all entries of A are a rational multiple of the same algebraic number: write⁸ $\mu_i = p_i \alpha$ where $p_i \in \mathbb{Z}$ and $\alpha \in \mathbb{Q}$. Then

$$f_c(t) = \sum_{i=1}^k c_i b_i (e^{\alpha t})^{p_i} = Q(c, e^{\alpha t})$$

where Q is a polynomial with algebraic coefficients. Let d be the degree of $Q(c, \cdot)$ (that does not depend on c but only on the μ_i), then $Q(c, \cdot)$ has at most d nontangential⁹ zeros, call them $z_1 < z_2 < \dots < z_k$. Each gives rise to some unique t_i satisfying $z_i = e^{\alpha t_i}$. It

⁸We can put the least common denominator in α , hence they become integer multiples.

⁹We are only interested in zero-crossings, since tangential zeros do not change the integral. In doing so, we also get for free that all nontangential zeros have multiplicity 1, hence they are all distinct.

follows that, up to a sign,

$$\begin{aligned} \pm\beta_c &= \int_0^{t_1} e^{At} b \, dt - \int_{t_1}^{t_2} e^{At} b \, dt + \dots + (-1)^k \int_{t_k}^{\infty} e^{At} b \, dt \\ &= A^{-1}((e^{At_1} - I) - (e^{At_2} - e^{At_1}) + \dots - (-1)^k e^{At_k})b \\ &= A^{-1} \left(2 \sum_{i=1}^k (-1)^{i-1} e^{At_i} - I \right) b. \end{aligned}$$

In particular, since A is diagonal, the j^{th} component of β_c is

$$\begin{aligned} \beta_{c,j} &= \pm \frac{1}{\mu_j} \left(2 \sum_{i=1}^k (-1)^{i-1} e^{-\mu_j t_i} - 1 \right) b_j \\ &= \pm \frac{1}{\mu_j} \left(2 \sum_{i=1}^k (-1)^{i-1} (e^{-\alpha t_i} - 1)^{p_i} \right) b_j \\ &= \pm \frac{1}{\mu_j} \left(2 \sum_{i=1}^k (-1)^{i-1} z_i^{p_i} - 1 \right) b_j \\ &= R_{k,j}(z_1, \dots, z_k) \end{aligned}$$

where $R_{k,j}$ is a polynomial with algebraic coefficients that does not depend on c . Note that the sign can be determined easily: it is the sign of $f_c(0) = \sum_{i=1}^k b_i c_i$. We can now write a formula in the first-order theory of the reals to express that a target y is on the border¹⁰:

$$\Phi(y) := \exists c. c \neq 0 \bigwedge \bigvee_{k=0}^d \Phi_k(y, c)$$

to check for a point on a border in direction c ,

$$\Phi_k(y, c) := \exists z_1, \dots, z_k. \Psi_k(c, z) \bigwedge \Psi'_k(c, z) \bigwedge \bigwedge_{j=1}^n (y_j = R_{k,j}(z))$$

to match the target with some parameters z_1, \dots, z_k ,

$$\Psi_k(c, z) := \bigwedge_{i=1}^k (Q(c, z_i) = 0 \wedge Q'(c, z_i) \neq 0) \bigwedge 0 < z_1 < \dots < z_k$$

to check that the z_i are zeros of $Q(c, \cdot)$,

$$\Psi'_k(c, z) := \forall u. (u > 0 \wedge Q(c, u) = 0 \wedge Q'(c, u) \neq 0) \Rightarrow \bigvee_{i=1}^k u = z_i$$

to check the z_i are the only zeros of $Q(c, \cdot)$.

We have shown that membership in the border is expressible in \mathfrak{R}_0 , which shows that membership in the entire reachable set is expressible in \mathfrak{R}_0 by Lemma 5.1, and hence decidable.

If b has at most two nonzero entries: let $i \neq j$ be those two entries, then

$$\begin{aligned} f_c(t) = 0 &\Leftrightarrow c_i e^{\mu_i t} b_i + c_j e^{\mu_j t} b_j = 0 \\ &\Leftrightarrow 1 + \frac{c_j b_j}{c_i b_i} e^{(\mu_i - \mu_j)t} = 0 \\ &\Leftrightarrow t = t_1 := \frac{1}{\mu_i - \mu_j} \ln \frac{-c_i b_i}{c_j b_j}. \end{aligned}$$

We assume that we are not in the previous case, so in particular μ_i and μ_j must be distinct and \mathbb{Q} -linearly independent. If $c_i = 0$ then

¹⁰We write Q' for $\frac{\partial Q(c, z)}{\partial z}$ which is also a polynomial.

f_c has no zero unless $c_j = 0$, in which case it is constant equal to zero and $\beta_c = 0$. When $c_i = 0$ and $c_j \neq 0$, the sign of f_c is constant. If $c_i b_i = -c_j b_j$ then the only zero of f_c is at $t = 0$ and the sign is then constant once again. In all case where the sign is constant on $(0, \infty)$, we have that

$$\beta_c = \pm \int_0^{\infty} e^{At} b \, dt = -A^{-1}b$$

which is algebraic and hence can easily be checked against the target. In all other cases, we have

$$\begin{aligned} \beta_c &= \int_0^{t_1} e^{At} b \, dt - \int_{t_1}^{\infty} e^{At} b \, dt \\ &= A^{-1}(2e^{At_1} - I_n)b = 2A^{-1}e^{At_1}b - A^{-1}b. \end{aligned}$$

Recall that b has two nonzero entries and A is diagonal, hence $A^{-1}e^{At_1}b$ also has two nonzero entries: $\mu_i^{-1}e^{\mu_i t_1}b_i$ and $\mu_j^{-1}e^{\mu_j t_1}b_j$ respectively. We now argue that one or both of those values are transcendental which prevents the target from being on the border. Observe that $\mu_i^{-1}e^{\mu_i t_1}b_i$ is algebraic if and only if $e^{\mu_i t_1}$ is algebraic. But

$$e^{\mu_i t_1} = e^{\frac{\mu_i}{\mu_i - \mu_j} \ln \frac{-c_j b_j}{c_i b_i}} = \left(\frac{-c_j b_j}{c_i b_i} \right)^{\frac{\mu_i}{\mu_i - \mu_j}}$$

which is transcendental by Theorem 4.6 if $\frac{\mu_i}{\mu_i - \mu_j}$ is irrational, since we assumed that $\frac{-c_j b_j}{c_i b_i}$ is not 0 or 1. Therefore, β_c is algebraic only when $\frac{\mu_i}{\mu_i - \mu_j}, \frac{\mu_j}{\mu_i - \mu_j} \in \mathbb{Q}$. This would imply that $\frac{\mu_i}{\mu_j} \in \mathbb{Q}$, a contradiction since we assume that they are \mathbb{Q} -linearly independent. In summary, the only two possible algebraic points on the border are 0 and $A^{-1}b$ and it is easy to check (i) if they are indeed on the border, (ii) if they are equal to the target. Clearly one can write a formula in \mathfrak{R}_0 to decide if this is the case.

At this point, we can conclude for the general case because we assume that B only consist of one column. Note that we would not be able to conclude if B had several columns because we can only write a formula for $\partial\mathcal{R}(A_i, b_i, [-1, 1]) \cap \overline{\mathcal{Q}}^n$. Indeed, if we have two convex sets C and D , then $\partial(C + D) \subseteq \partial C + \partial D$ but in general we do not have $\partial(C + D) \cap \overline{\mathcal{Q}}^n \subseteq (\partial C \cap \overline{\mathcal{Q}}^n) + (\partial D \cap \overline{\mathcal{Q}}^n)$. A simple counter-example is $C = [0, \pi]$ and $D = [0, 4 - \pi]$: then $C + D = [0, 4]$, $\partial(C + D) \cap \overline{\mathcal{Q}} = \{0, 4\}$ but $\partial C \cap \overline{\mathcal{Q}} = \{0\}$ and $\partial D \cap \overline{\mathcal{Q}} = \{0\}$. It is unclear whether such a counter-example can be built with actual reachable sets however.

If A only has one eigenvalue which is real: by using Lemma 5.1 as before, it suffices to show that membership is expressible in \mathfrak{R}_0 for some controllable pairs (A_i, b_i) . The crucial point here is that the spectrum of A_i is included in that of A . Since A has a unique eigenvalue λ , A_i also has a unique eigenvalue λ . If $\lambda > 0$, the reachable set is \mathbb{R}^{n_1} by Proposition 4.3, hence expressible in \mathfrak{R}_0 .

We now assume that $\lambda < 0$, $B = b$ is a column vector, (A, b) is controllable and $U = [-1, 1]$. Then by Theorem 4.4, we have that $\partial\mathcal{R}(A, b, [-1, 1]) = \{\beta_c : c \in \mathbb{R}^n \setminus \{0\}\}$, where

$$\beta_c := \int_0^{\infty} e^{At} b \operatorname{sgn}(c^T e^{At} b) \, dt$$

But since A has a unique real eigenvalue λ , $e^{At} = e^{\lambda t} P(t)$ where $P(t)$ is a matrix where each entry is a polynomial in t with real algebraic coefficients. It follows that $c^T e^{At} b = e^{\lambda t} Q(c, t)$ where Q is a polynomial with algebraic coefficients and $Q(c, \cdot)$ has at

most d nontangential zeros (tangential zeros do not change the integral) which are distinct, and where d is independent of c . We can write a formula in \mathfrak{R}_0 to express those zeros $t_1 < t_2 < \dots < t_k$. Furthermore, note that by integration by part, we have

$$\int_u^v e^{At} b \, dt = \int_u^v e^{\lambda t} P(t) \, dt = [e^{\lambda t} R(t)]_u^v$$

where R is some polynomial matrix with algebraic coefficients. It follows that, up to a sign,

$$\begin{aligned} \pm \beta_c &= \int_0^{t_1} e^{At} b \, dt - \int_{t_1}^{t_2} e^{At} b \, dt + \dots + (-1)^k \int_{t_k}^{\infty} e^{At} b \, dt \\ &= [e^{\lambda t} R(t)]_0^{t_1} b - [e^{\lambda t} R(t)]_{t_1}^{t_2} b + \dots + (-1)^k [e^{\lambda t} R(t)]_{t_k}^{\infty} b \\ &= \left(2 \sum_{i=1}^k (-1)^{i-1} e^{\lambda t_i} R(t_i) - R(0) \right) b. \end{aligned}$$

In particular, the j^{th} component of β_c is of the form

$$\beta_{c,j} = S_{j,0}(b) + \sum_{i=1}^k e^{\lambda t_i} S_{j,i}(t_1, \dots, t_k, b)$$

where the S_i are polynomials with algebraic coefficients. But now recall that the t_i are algebraic since they are the roots of $Q(c, \cdot)$ and they are distinct because they are in fact the nontangential zeros. Furthermore, the target β has algebraic coordinates. Hence, by Theorem 4.7 (take $\alpha_i = \lambda t_i \in \overline{\mathbb{Q}}$ and add $\alpha_0 = 0$), the only way this can happen is if $S_{j,i}(t_1, \dots, t_k, b) = 0$ for $1 \leq i \leq k$ and $\beta_{c,j} = S_{j,0}(b)$. Crucially, those conditions do not involve any exponentials so we can express all those conditions in \mathfrak{R}_0 . Here again, we can only conclude when B is a column, because we only have a formula for the algebraic point on the border of each controllable system. \square

The main obstacle to generalizing this result is that the Boolean formulas involved in description of the boundary become too complicated, either involving three distinct exponentials or a combination of exponentials and polynomials (exponential polynomial). In fact, deciding if an exponential polynomial has zero is exactly the Continuous Skolem problem, and is not known to be decidable, even for real eigenvalues. We can recover decidability if we assume that the first-order theory of the reals with exponential is decidable. This is known to be true if Schanuel's conjecture hold, see Theorem 4.8.

PROPOSITION 5.4 (SEE FULL VERSION). *The LTI Reachability problem when $U = [-1, 1]^m$ and A has real eigenvalues reduces to deciding $\mathfrak{R}_{\text{exp}}$. In particular, it is decidable if Schanuel's conjecture is true.*

One cannot easily generalize the previous result to any matrix because complex eigenvalues involve expression with \exp and \sin over unbounded domains. It is well-known that the first-order theory of reals with unbounded \sin is undecidable (by embedding of Peano arithmetic). This explains why very few results are known about the Continuous Skolem problem in the unbounded case, even assuming Schanuel's conjecture. Nevertheless, one can show that in dimension two, only bounded sine and cosine are necessary to solve the problem.

PROPOSITION 5.5 (SEE FULL VERSION). *The LTI Reachability problem in dimension $n = 2$ when $U = [-1, 1]^m$ reduces to deciding $\mathfrak{R}_{\text{exp}, \sin}$. In particular, it is decidable if Schanuel's conjecture is true.*

In fact, dimension 2 is special enough that we can show unconditional decidability of the reachability problem if A has real eigenvalues and B is a column (i.e. there is only one input).

PROPOSITION 5.6 (SEE FULL VERSION). *The LTI Reachability problem in dimension $n = 2$ when $U = [-1, 1]$, B is a column and A has real eigenvalues is decidable.*

Finally, another way to avoid the use of unbounded sine and cosine is to consider the Bounded Time LTI Reachability problem, which is also very natural in control theory.

PROPOSITION 5.7 (SEE FULL VERSION). *The Bounded Time LTI Reachability problem when $U = [-1, 1]^m$ and the time bound is algebraic reduces to deciding $\mathfrak{R}_{\text{exp}, \sin}$. In particular, it is decidable if Schanuel's conjecture is true.*

6 HARDNESS

We saw in the previous section that the LTI Reachability problem seems very challenging, requiring powerful tools like the first-order theory of the reals with exponential and Schanuel's conjecture. In this section, we give some evidence that the problem is indeed difficult. Our first observation is that, in some sense, the LTI Set Reachability problem trivially contains the Skolem problem when the input set is $\{0\}$, in other words, when there is no input.

THEOREM 6.1 (SEE FULL VERSION). *The Continuous Skolem problem reduces to the LTI Set Reachability problem with input set $U = \{z\}$, where $z = (1, \dots, 1)$, the matrix A is stable and the target set is a compact convex set of dimension $n - 1$.*

However, we are not really satisfied with this hardness result. Indeed, the problem is fundamentally different when U is a singleton. To see that, observe that if $U = \{z\}$ for some $z \in \mathbb{R}^n$, then the reachable set is just the orbit of z under $x' = Ax$. In particular, we can see that if A is stable then the orbit is a *closed set* minus¹¹ an algebraic point (0). In particular, we can trivially decide whether an algebraic point is 0 or not, so deciding reachability is really about **deciding membership in a closed set**. Now compare that with the situation when $U = [-1, 1]^n$: by Proposition 4.3, when A is stable, the reachable set is **open**. This topological difference can lead to some difficulty because deciding membership in the boundary may involve some difficult transcendence results. For this reason, it is important to study hardness when U is not a singleton.

We show that the problem remains hard when $U = [-1, 1]$, by reducing to the Continuous Nontangential Skolem problem. Recall that this problem, asks whether an exponential polynomial (or equivalently a linear differential equation) has a zero-crossing (nontangential zero). We argued in Section 4.2 that this problem is essentially as hard as the Skolem problem.

THEOREM 6.2. *The Continuous Nontangential Skolem problem reduces to the LTI Set Reachability problem with a single saturated input, i.e. $x' = Ax + bu$ with A stable, $b \in \mathbb{R}^n$ and $u(t) \in [-1, 1]$, and the target set can be chosen to be either a hyperplane, or a convex compact set of dimension $n - 1$.*

¹¹The orbit converges to 0 but never reaches it, hence the reachable set is the closure (closed set) minus 0.

PROOF. Let c, A, b be an instance of the Continuous Nontangential Skolem problem. Let $f_c(t) = c^T e^{At} b$, the problem asks whether f_c has any zero-crossing at $t \geq 0$. Note that for any $\alpha > 0$, $f_c(t)$ is zero-crossing if and only if $e^{-\alpha t} f_c(t)$ is zero-crossing. Furthermore, $e^{-\alpha t} f_c(t)$ is still an exponential polynomial, so without loss of generality we can assume that all eigenvalues of A have negative real parts, by taking α sufficiently large. In other words, we can assume that A is stable. In particular, A must be invertible.

We now show that we can assume that (A, b) is controllable. Let $V = \text{span}[b, Ab, \dots, A^{n-1}b]$ where n is the dimension of A , and assume that $\dim V < n$. Then $b \in V$ and $AV \subseteq V$ by Cayley–Hamilton theorem, so by an orthogonal change of basis P ,

$$P^{-1}c = \begin{bmatrix} c_V \\ * \end{bmatrix}, \quad P^{-1}AP = \begin{bmatrix} A_V & * \\ 0 & * \end{bmatrix}, \quad P^{-1}b = \begin{bmatrix} b_V \\ 0 \end{bmatrix}.$$

It then follows that $c^T e^{At} b = c_V e^{A_V t} b_V$, but note that

$$\text{span}[b_V, A_V b_V, \dots, A_V^{k-1} b_V] = V$$

by construction, therefore (A_V, b_V) is controllable.

We can now assume that A is stable and (A, b) controllable. Without loss of generality, we can assume that $f_c(0) = c^T b \geq 0$, by considering $-c$ instead of c if this is not the case. Also recall that $\mathcal{R} := \mathcal{R}(A, b, [-1, 1])$. Therefore, by Theorem 4.4, $\mathcal{R}(A, b, U)$ is an open convex set containing 0 and its boundary is given by $\partial\mathcal{R}(A, b, U) = \{\beta_v : v \in \mathbb{R}^n \setminus \{0\}\}$, where

$$\beta_v := \int_0^\infty e^{At} b \text{sgn}(v^T e^{At} b) dt$$

which is a strictly convex set. We start by claiming the following:

- (a) $x \in \overline{\mathcal{R}}$ if and only if $x = \int_0^\infty e^{At} b u(t) dt$ for some $u : \mathbb{R} \rightarrow [-1, 1]$,
- (b) β_c is the unique external point in direction c : $\overline{\mathcal{R}} \cap (\beta_c + H_c) = \{\beta_c\}$ where $H_c = \{x \in \mathbb{R}^n : c^T x = 0\}$ is the hyperplane of normal c ,
- (c) $-A^{-1}b \in \overline{\mathcal{R}}$,
- (d) $-A^{-1}b = \beta_c$ if and only if f_c has no zero-crossings (nontangential zeros),
- (e) $-A^{-1}b \neq \beta_c$,
- (f) $-A^{-1}b = \beta_c$ if and only if $(-A^{-1}b + H_c) \cap \mathcal{R} = \emptyset$

We now go through those claims one by one: claim (a) is direct consequence of (1) with the change of variable $\xi = t - s$. Claim (b) is essentially already proven in [26] but we give the gist of the proof. Pick $x \in \overline{\mathcal{R}}$, then by claim (a) there exists $u : \mathbb{R} \rightarrow [-1, 1]$ such that $x = \int_0^\infty e^{At} b u(t) dt$. Then check that, since $|u(t)| \leq 1$,

$$\begin{aligned} c^T x &= \int_0^\infty c^T e^{At} b u(t) dt \leq \int_0^\infty |c^T e^{At} b| dt \\ &= \int_0^\infty c^T e^{At} b \text{sgn}(c^T e^{At} b) dt = c^T \beta_c. \end{aligned}$$

Therefore, β_c is a maximizer in direction c . But by strict convexity of $\overline{\mathcal{R}}$, there can only be a unique one, for otherwise a whole line segment would be in $\partial\mathcal{R}$ which would be a contradiction. To show claim (c), simply observe that since $e^{At} \rightarrow 0$ as $t \rightarrow \infty$, we have, by claim (a),

$$-A^{-1}b = \int_0^\infty e^{At} b dt = \int_0^\infty e^{At} b u(t) dt \in \overline{\mathcal{R}} \quad \text{where } u(t) = 1.$$

Claim (d) follows from the computation above and the remark that

$$\begin{aligned} c^T (\beta_c + A^{-1}b) &= \int_0^\infty c^T e^{At} b (\text{sgn}(c^T e^{At} b) - 1) dt \\ &= \int_0^\infty |f_c(t)| - f_c(t) dt. \end{aligned}$$

Indeed, there are two cases (since we assumed that $f_c(0) \geq 0$):

- Either $f_c(t) \geq 0$ for all $t \geq 0$, then f_c has no zero-crossings and $c^T (\beta_c + A^{-1}b) = 0$. But $-A^{-1}b \in \overline{\mathcal{R}}$ and β_c is the maximizer in direction c , so by strict convexity, $\beta_c = -A^{-1}b$.
- Either f_c has at least one zero-crossing, then by continuity there exists an open interval (a, b) such that $|f_c(t)| - f_c(t) > 0$. But since $|f_c(t)| - f_c(t) \geq 0$ for all t , it follows that $c^T (\beta_c + A^{-1}b) > 0$ hence $\beta_c \neq -A^{-1}b$.

Claim (e) follows from a similar argument since $\beta_{-c} = -\beta_c$ and

$$\begin{aligned} c^T (\beta_c - A^{-1}b) &= \int_0^\infty c^T e^{At} b (\text{sgn}(c^T e^{At} b) + 1) dt \\ &= \int_0^\infty |f_c(t)| + f_c(t) dt. \end{aligned}$$

But now assume, for contradiction, that $\beta_{-c} = -A^{-1}b$. Then $c^T (\beta_c - A^{-1}b) = 0$ so $f_c(t) \leq 0$ for all t , hence f_c has no zero-crossings. But then $-A^{-1}b = \beta_c$ by claim (d). This implies that $c^T (\beta_c + A^{-1}b)$ and $f_c(t) \geq 0$ by the computation of claim (d). Therefore, $f_c \equiv 0$ and (A, b) is not controllable by Lemma 4.2, a contradiction.

Finally, for claim (f), observe that if $-A^{-1}b \in \mathcal{R}$ then $-A^{-1}b \neq \beta_c$ since $\beta_c \in \partial\mathcal{R}$ and \mathcal{R} is open, so the result is true. Otherwise, $-A^{-1}b \in \partial\mathcal{R}$ by claim (e). Hence, there are two cases: if $-A^{-1}b = \beta_c$ then the intersection is empty by claim (b). Otherwise, $-A^{-1}b \neq \beta_c$, but $-A^{-1}b \neq \beta_{-c}$ by claim (e) so it must be the case that $-A^{-1}b + H_c$ intersects \mathcal{R} . To show this formally, observe that by strict convexity $(\beta_{-c}, \beta_c) \subseteq \mathcal{R}$. Also, since β_c is the maximizer in direction c , $-A^{-1}b \neq \beta_c$ implies that $c^T x < c^T \beta_c$ where $x = -A^{-1}b$. A similar reasoning for β_{-c} shows that $c^T \beta_{-c} < c^T x < c^T \beta_c$. Define $y = \beta_{-c} + \alpha(\beta_c - \beta_{-c})$ where $\alpha = \frac{c^T x - c^T \beta_{-c}}{c^T \beta_c - c^T \beta_{-c}}$. Then $\alpha \in (0, 1)$ by the previous inequality, hence $y \in (\beta_{-c}, \beta_c) \subseteq \mathcal{R}$. But by construction $c^T y = c^T \beta_{-c} + \alpha c^T (\beta_c - \beta_{-c}) = c^T x$, therefore $y \in x + H_c$.

We can now describe an algorithm that solves this instance of Continuous Nontangential Skolem problem by reducing to the LTI Set Reachability problem. Consider the LTI with matrix A , control matrix b , input set $[-1, 1]$ and target set $Y = \{x \in \mathbb{R}^n : c^T x = -c^T A^{-1}b\}$. Note that Y is a hyperplane with algebraic coefficients, and check that $Y = -A^{-1}b + H_c$. Let \mathcal{R} be the reachable set of this LTI, then it follows by claims (c) and (f) above that $Y \cap \mathcal{R} = \emptyset$ if and only if $-A^{-1}b = \beta_c$ if and only if f_c has no zero-crossings. Hence, Y is reachable if and only if the Continuous Nontangential Skolem problem instance has a zero-crossing.

The set Y above is convex but not compact, but since we have chosen A to be stable, the reachable set is bounded by Proposition 4.3 and it is easy to compute a bound M such that $\mathcal{R} \subseteq [-M, M]^n$. Then we can define $\hat{Y} = Y \cap [-M, M]^n$ which is now compact convex, and clearly $\mathcal{R} \cap Y = \emptyset$ if and only if $\mathcal{R} \cap \hat{Y} = \emptyset$. \square

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