On the Determinization of Weighted Automata

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Abstract

In the paper, we generalize an algorithm and some related results by MOHRI [25] for determinization of weighted finite automata (WFA) over the tropical semiring. We present the underlying mathematical concepts of his algorithm in a precise way for arbitrary semirings. We define a class of semirings in which we can show that the twins property is sufficient for the termination of the algorithm. We also introduce single-valued WFA and give a partial correction of a claim by MOHRI [25] by showing several characterizations of single-valued WFA, e.g., the formal power series computed by a single-valued WFA is subsequential iff it has bounded variation. Also, it is decidable in polynomial time whether a given WFA over the tropical semiring is single-valued.

1   Introduction

Weighted finite automata are of great theoretical and practical interest in computer science. They play a crucial role in the structure theory of recognizable languages in free monoids and trace monoids. However, weighted finite automata also have practical applications in speech recognition and image compression [6, 9, 14, 17, 18, 25]. The behaviour of a weighted finite automaton, for short WFA, can be described as a formal power series, i.e., a mapping from a free monoid into some semiring.

Although WFA were already studied in the seventies [4, 7, 11], there are many recent articles which focus mainly on two streams: WFA over the min–plus (tropical) and max–plus semirings and string-to-string transducers [2, 3, 8, 20].

In contrast to unweighted automata, there are WFA which do not admit a subsequential (deterministic) equivalent. However, MOHRI developed an algorithm which determinizes WFA over the tropical semiring [25] and which is implemented within the AT&T FSM Library™. This algorithm is not perfect, e.g., there are WFA on which MOHRI's algorithm does not terminate despite there are subsequential equivalents. Nevertheless, his algorithm is very successful on WFA which occur in speech recognition. MOHRI proves that in the tropical semiring the twins property is a sufficient condition for the termination of his algorithm [25].

MOHRI develops his ideas for WFA over the tropical semiring. Here we wish to investigate generalizations of his results to other semirings.

In Section 3, we generalize MOHRI's algorithm to an abstract notion of determinization of WFA over arbitrary semirings. This determinization utilizes so-called factorizations which depend on the semiring. We introduce the notion of a maximal factorization and show that in zero-divisor free semirings, maximal factorizations are optimal in comparison to arbitrary factorizations. Moreover, we prove that the twins property is a sufficient condition for the termination of our abstract determinization for a certain class of semirings which we call commutative min-semirings. MOHRI's determinization turns out as the particular case of our approach in the tropical semiring with a maximal factorization. Consequently, we achieve several results by MOHRI as specific cases of our approach, but we also inherit the problem that our approach does not always terminate even if the given WFA admits a subsequential equivalent.

In Section 4, we deal with unambiguous and single-valued WFA. A WFA is single-valued if all accepting paths of the same word have the same weight. This is a natural generalization of unambiguous WFA. By applying EILENBERG's cross section theorem, we show that every single-valued WFA admits an unambiguous equivalent. We also discuss a different proof of this result which relies on a construction due to SCHÜTZENBERGER and SAKAROVITCH [27]. Similar results are known for transducers over the tropical semiring and for certain string-to-string transducers [29, 15, 19, 31]. However these approaches rely on the cancellativity of the multiplication in the semiring while our approach just requires idempotency of the semiring addition. We achieve the following characterization: the behaviour $|T|$ of some single-valued WFA T is subsequential iff $|T|$ has bounded variation iff MOHRI's algorithm terminates on T iff T has the twins property. An example shows that the assumption that T is single-valued cannot be omitted. This partly corrects a claim by MOHRI [25][1]. Finally, we use techniques from [1, 3] to show that it is decidable in polynomial time whether a given WFA over the tropical semiring is single-valued.

2   Prerequisites

2.1   Basic Definitions and Notations

A *semiring* $(\mathbb{K}, \oplus, \odot, 0, 1)$ consists of a set $\mathbb{K}$ together with two binary operations $\oplus$, and $\odot$ such that $(\mathbb{K}, \oplus, 0)$ is a commutative monoid, $(\mathbb{K}, \odot, 1)$ is a monoid which distributes over $(\mathbb{K}, \oplus, 0)$, and 0 acts as a zero for all elements. Some $k \in \mathbb{K}$, $k \neq 0$ is called a *zero-divisor* if there is some $l \in \mathbb{K}$, $l \neq 0$ such that $k \odot l = 0$ or $l \odot k = 0$. If $\mathbb{K}$ does not have zero-divisors, then $\mathbb{K}$ is called *zero-divisor-free*. A semiring $\mathbb{K}$ is called *idempotent* if $k \oplus k = k$ for every $k \in \mathbb{K}$.

We call $K \subseteq \mathbb{K}$ a *subsemiring* of $\mathbb{K}$ if K is closed under $\oplus$ and $\odot$ and $0, 1 \in K$. For every $K \subseteq \mathbb{K}$, we call the closure of the set $K \cup \{0, 1\}$ under $\oplus$ and $\odot$ the *subsemiring generated by* K and denote it by $\langle K \rangle$. If $\langle K \rangle$ is finite for every finite set $K \subseteq \mathbb{K}$, then $\mathbb{K}$ is called *locally finite* [10]. Important examples of locally finite semirings are the semirings of the form $(\mathbb{K}, \min, \max, 0, 1)$ where min and max are defined by some total order on a set $\mathbb{K}$.

By $(\mathbb{N}, +, \cdot, 0, 1)$ we mark the semiring over $\mathbb{N} = \{0, 1, \dots\}$ with addition and multiplication of natural numbers. A semiring often used is the *tropical semiring* $\mathbb{T} = (\mathbb{R}_+ \cup \{\infty\}, \min, +, \infty, 0)$. We denote by $(\mathbb{B}, \vee, \wedge, \text{false}, \text{true})$ the *Boolean semiring*. It consists of the set $\mathbb{B} = \{\text{true}, \text{false}\}$ with logical disjunction and conjunction.

Let Δ be some finite alphabet with a total order $\leq$. We extend $\leq$ to Δ^* as follows. Firstly, for $u, v \in \Delta^*$, we set $u \leq v$ if u is shorter than v. Secondly, for $a, b \in \Delta$ and $u, v, w \in \Delta^*$, satisfying $a \lneq b$ and $|u| = |v|$, we set $wau \leq wbv$. By defining an operation min on the free monoid Δ^* by $\leq$, we obtain a semiring $(\Delta^* \cup \{0\}, \min, \cdot, 0, \varepsilon)$, where 0 denotes a new maximal element.

Mappings from a free monoid to some semiring $\mathbb{K}$ are called *formal power series*. For a semiring $\mathbb{K}$ and a finite set Q, we denote by $\mathbb{M}_{Q \times Q}(\mathbb{K})$ the set of all $Q \times Q$-matrices over $\mathbb{K}$. We

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[1]This claim is corrected in the electronic version of [25] on MOHRI's homepage.

denote by $\mathbb{M}_{1\times Q}(\mathbb{K})$ and $\mathbb{M}_{Q\times 1}(\mathbb{K})$ the set of all row resp. column matrices. For a row matrix (correspondingly for column matrices) $\lambda \in \mathbb{M}_{1\times Q}(\mathbb{K})$, we write $\lambda[q]$ for the q-th entry of λ. For $a \in \mathbb{K}$, an element $(a,\ldots,a) \in \mathbb{M}_{1\times Q}(\mathbb{K})$ we abbreviate by a^Q. We define the multiplication of matrices by the operations of $\mathbb{K}$, as usual. We use $\dashrightarrow$ for partial and $\rightarrow$ for total mappings. As usual, we define the sum over the empty set, $\bigoplus \emptyset$, as the zero 0 of $\mathbb{K}$.

2.2   Weighted finite automata

We recall some notions on weighted automata and recommend [4, 5, 22, 26, 28] for overviews.

Let $\mathbb{K}$ be an arbitrary semiring and let Σ be an alphabet.

A *weighted finite automaton* (for short *WFA*) over $\mathbb{K}$ is a tuple $[Q, E, \lambda, \varrho]$, where

- Q is a non-empty, finite set of *states*,

- E is a finite subset of $Q \times \Sigma \times \mathbb{K} \times Q$, and

- $\lambda \in \mathbb{M}_{1\times Q}(\mathbb{K})$, $\varrho \in \mathbb{M}_{Q\times 1}(\mathbb{K})$.

We call the tuples in E *transitions*.

There are two equivalent ways to define the behaviour of WFA. At first, we define the behaviour using matrices. Let $T = [Q, E, \lambda, \varrho]$ be a WFA. The set of transitions E defines a homomorphism $\theta : \Sigma^* \to \mathbb{M}_{Q\times Q}(\mathbb{K})$ in the following way: for every $a \in \Sigma$, let $\theta(a)$ be the matrix $\mathbb{M}_{Q\times Q}(\mathbb{K})$ such that for every $p, q \in Q$, we have

$$\theta(a)[p, q] := \bigoplus_{(p,a,k,q)\in E} k.$$

This mapping $\theta : \Sigma \to \mathbb{M}_{Q\times Q}(\mathbb{K})$ induces a unique homomorphism $\theta : \Sigma^* \to \mathbb{M}_{Q\times Q}(\mathbb{K})$.

The WFA T *computes* a formal power series $|T| : \Sigma^* \to \mathbb{K}$ by letting

$$|T|(w) := \lambda \odot \theta(w) \odot \varrho$$

for every $w \in \Sigma^*$. We call two WFA T and T' *equivalent* if $|T| = |T'|$, i.e. if $|T|(w) = |T'|(w)$ for every $w \in \Sigma^*$. If some formal power series $f : \Sigma^* \to \mathbb{K}$ is computed by a WFA, then we call f *recognizable*.

Now we give an equivalent method to define the behaviour of a WFA. Again let $T = [Q, E, \lambda, \varrho]$ be a WFA. We can regard λ and ϱ as mappings $\lambda, \varrho : Q \to \mathbb{K}$ such that for every $q \in Q$, we have $\lambda(q) = \lambda[q]$ and $\varrho(q) = \varrho[q]$.

Let $n \geq 1$. A path π of length n is a sequence

$$(q_0, a_0, k_0, q_1)\,(q_1, a_1, k_1, q_2)\,\ldots\,(q_{n-1}, a_{n-1}, k_{n-1}, q_n)$$

of transitions in E. The word $a_0 \ldots a_{n-1}$ is called the *label* of π. We say that π starts at q_0 and ends at q_n. We define $\sigma(\pi) := k_0 \odot k_1 \odot \cdots \odot k_{n-1}$, the *weight* of π. We assume that for every $q \in Q$ there is a path of length 0 which starts and ends at q, is labeled with ε and weighted with 1. For every $p, q \in Q$ and every $w \in \Sigma^*$, we denote by $p \overset{w}{\rightsquigarrow} q$ the set of all paths with label w which start at p and end at q. Then for every $w \in \Sigma^*$ one can show

$$|T|(w) = \bigoplus_{p,q\in Q,\, \pi \in p \overset{w}{\rightsquigarrow} q} \Big(\lambda(p) \odot \sigma(\pi) \odot \varrho(q)\Big) = \bigoplus_{p,q\in Q} \Big(\lambda(p) \odot \Big(\bigoplus_{\pi \in p \overset{w}{\rightsquigarrow} q} \sigma(\pi) \Big) \odot \varrho(q)\Big).$$

For every $q \in Q$, we call $\lambda(q)$ resp. $\varrho(q)$ the *initial weight* resp. *terminal weight* of q. Let $I := \{\, q \in Q \mid \lambda(q) \neq 0 \,\}$ and $F := \{\, q \in Q \mid \varrho(q) \neq 0 \,\}$. We call the states in I resp. F the *initial*

states resp. *accepting states* of T. Let $\pi = (q_0, a_0, k_0, q_1)(q_1, a_1, k_1, q_2) \ldots (q_{n-1}, a_{n-1}, k_{n-1}, q_n)$ be a path. Then π is *successful* if $q_0 \in I$ and $q_n \in F$. For every $0 \le i \le j \le n$, we let

$$\pi(i,j) := (q_i, a_i, k_i, q_{i+1}) \ldots (q_{j-1}, a_{j-1}, k_{j-1}, q_j)$$

be the subpath of π from q_i to q_j. If π and π' are paths such that π' starts at the same state where π ends, then we denote by $\pi\pi'$ the concatenation of π and π'. For every $p, q, r \in Q$ and $u, v \in \Sigma^*$, we write the concatenation of $p \overset{u}{\rightsquigarrow} q$ and $q \overset{v}{\rightsquigarrow} r$ as $p \overset{u}{\rightsquigarrow} q \overset{v}{\rightsquigarrow} r$. For subsets $P, R \subseteq Q$, we denote by $P \overset{w}{\rightsquigarrow} R$ the union of the sets $p \overset{w}{\rightsquigarrow} r$ for every $p \in P$, $r \in R$.

A *subsequential WFA* is a tuple $T = [Q, \delta, \sigma, q_0, k_0, \varrho]$ such that:

- Q is a finite set of states,

- $\delta : Q \times \Sigma \dashrightarrow Q$ and $\sigma : Q \times \Sigma \dashrightarrow \mathbb{K}$ are partial mappings such that for every $q \in Q$, $a \in \Sigma : \delta(q, a)$ is defined iff $\sigma(q, a)$ is defined,

- $q_0 \in Q$, $k_0 \in \mathbb{K}$,

- $\varrho : Q \to \mathbb{K}$ is a mapping.

We extend δ and σ to words $w \in \Sigma^*$ as follows: for every $q \in Q$, we set $\delta(q, \varepsilon) := q$ and $\sigma(q, \varepsilon) := 1$. For every $q \in Q$, $w \in \Sigma^*$, and $a \in \Sigma$, we set $\delta(q, wa) := \delta(\delta(q, w), a)$ and $\sigma(q, wa) := \sigma(q, w) \odot \sigma(\delta(q, w), a)$ provided that $\delta(q, w)$ and $\delta(\delta(q, w), a)$ are defined.

The formal power series recognized by T is defined for every $w \in \Sigma^*$, by

$$|T|(w) := \begin{cases} k_0 \odot \sigma(q_0, w) \odot \varrho(\delta(q_0, w)) & , \quad \text{if } \delta(q_0, w) \text{ is defined} \\ 0 & , \quad \text{otherwise.} \end{cases}$$

A *subsequential formal power series* is a formal power series which is recognized by a subsequential WFA. It is easy to transform a subsequential WFA T into a WFA computing $|T|$.

A formal power series from Σ^* to $\mathbb{B}$ can be considered as a subset of Σ^* or as a formal language over Σ. In the same way, recognizable formal power series from Σ^* to $\mathbb{B}$ can be considered as recognizable languages. We call a subsequential WFA over the Boolean semiring $\mathbb{B}$ a *deterministic finite automaton*.

3   Determinization

In this section, we deal with several approaches to determinize WFA. We explain a straightforward idea, MOHRI's algorithm, and a generalization of MOHRI's algorithm to arbitrary semirings.

This generalization utilizes so-called factorizations which depend on the semiring. We introduce the notion of a maximal factorization and show that maximal factorizations are in some sense optimal in comparison to arbitrary factorizations. This result is of practical importance: whenever one implements the determinization algorithm, one should prefer a maximal factorization. In Section 3.6, we will define min-semirings and show that for min-semirings the twins property is sufficient for the existence of a determinization of a WFA by our generalization using a maximal factorization.

Let $\mathbb{K}$ be a semiring and let Σ be an alphabet during this section.

3.1   A straightforward idea

Let $T = [Q, E, \lambda, \varrho]$ be a WFA. At first, we describe a straightforward idea to construct an equivalent, subsequential WFA $T' = [Q', \delta, \sigma, q_0, k_0, \varrho']$. We just sketch this method, because it is a very particular case of a more general approach, which we will discuss below.

The set of states Q' is the least subset of $\mathbb{M}_{1 \times Q}(\mathbb{K})$ which contains λ, and is closed under multiplication with matrices $\theta(a)$ for every $a \in \Sigma$. We use λ from T as initial state of T' and let $k_0 := 1$. For every letter $a \in \Sigma$ and every $u \in Q'$, we set $\delta(u,a) := u \odot \theta(a)$, $\sigma(u,a) := 1$, and $\varrho'(u) = u \odot \varrho$.

The lack of this idea is that Q' might be infinite. If $\mathbb{K}$ is finite, or more generally, if $\mathbb{K}$ is locally finite, the construction shows that for every WFA there exists a subsequential equivalent.

In the Boolean semiring, every matrix in $\mathbb{M}_{1 \times Q}(\mathbb{B})$ represents a subset of Q. Now, it is quite clear that in the Boolean semiring, this idea is just the classical determinization of WFA by a power set construction.

One can slightly modify this idea by leaving $\delta(u,a)$ and $\sigma(u,a)$ undefined if $u \odot \theta(a) = 0^Q$.

3.2 MOHRI's algorithm

In [25], MOHRI gives an improvement of the above approach. In comparison to the above approach, MOHRI's algorithm produces WFA's with fewer states as we will prove in Section 3.4. We sketch his approach for the tropical semiring $\mathbb{T} = (\mathbb{R}_+ \cup \{\infty\}, \min, +, \infty, 0)$. Let $T = [Q, E, \lambda, \varrho]$ be a WFA over $\mathbb{T}$. As before, we want to construct an equivalent, subsequential WFA $T' = [Q', \delta, \sigma, q_0, k_0, \varrho']$. Again, the states Q' are a subset of $\mathbb{M}_{1 \times Q}(\mathbb{T})$.

For every $u \in \mathbb{M}_{1 \times Q}(\mathbb{T})$, let $\min(u) := \min_{q \in Q} u[q]$. There is a matrix u' such that $u = \min(u) + u'$. If $u \neq \infty^Q$, then u' is uniquely determined and in this case, we denote u' in a rather sloppy way by $-\min(u) + u$.

We construct T'. We may assume $\lambda \neq \infty^Q$ since otherwise, a determinization of T is obvious.

Let $u \in \mathbb{M}_{1 \times Q}(\mathbb{T})$ and $a \in \Sigma$ with $u + \theta(a) \neq \infty^Q$. We abbreviate $u + \theta(a)$ by v. If the WFA is in state u and reads the letter a, then it does not change the state to v. It rather factorizes v into $\min(v)$ and $-\min(v) + v$, and changes the inner state to $-\min(v) + v$. Hence, the transition should be weighted with $\min(v)$. For every $u \in \mathbb{M}_{1 \times Q}(\mathbb{T})$ and every $a \in \Sigma$ with $u + \theta(a) \neq \infty^Q$, we define

- $\delta(u,a) := -\min\big(u + \theta(a)\big) + \big(u + \theta(a)\big)$ and

- $\sigma(u,a) := \min(u + \theta(a))$.

If $u + \theta(a) = \infty^Q$, then $\delta(u,a)$ and $\sigma(u,a)$ remain undefined. We define $k_0 := \min(\lambda)$ and $q_0 := -\min(\lambda) + \lambda$. As above, we set $\varrho'(u) = u + \varrho$.

We define the set of states Q' as the least subset of $\mathbb{M}_{1 \times Q}(\mathbb{T})$ which contains q_0 and is closed under δ, i.e., for every $u \in Q'$ and every $a \in \Sigma$, we have $\delta(u,a) \in Q'$ if $\delta(u,a)$ is defined.

The set Q' is not necessarily finite, even if there is some subsequential WFA which is equivalent to T. If Q' is finite, then we define $T' = [Q', \delta|_{Q' \times \Sigma}, \sigma|_{Q' \times \Sigma}, q_0, k_0, \varrho'_{Q'}]$.

In [25], MOHRI gives an algorithm which computes the WFA T'. This algorithm terminates iff Q' is finite. Besides other results, he shows that T' is indeed equivalent to T, provided that Q' is finite. We continue the examination of MOHRI's approach in Section 4.2.

3.3 A generalization of MOHRI's algorithm

If one tries to generalize MOHRI's algorithm to arbitrary semirings, then one encounters the problem of adapting the factorization of matrices $v \in \mathbb{M}_{1 \times Q}(\mathbb{T})$ into $\min(v)$ and $-\min(v) + v$ to arbitrary semirings. By the great diversity of semirings, we have to choose a rather abstract method.

Let $\mathbb{K}$ be an arbitrary semiring and Q be a nonempty, finite set. We call two mappings $f : \mathbb{M}_{1 \times Q}(\mathbb{K}) \setminus \{0^Q\} \to \mathbb{M}_{1 \times Q}(\mathbb{K})$ and $g : \mathbb{M}_{1 \times Q}(\mathbb{K}) \setminus \{0^Q\} \to \mathbb{K}$ a *factorization of dimension* Q if for every $u \in \mathbb{M}_{1 \times Q}(\mathbb{K}) \setminus \{0^Q\}$ we have

$$u = g(u) \odot f(u).$$

Let (f, g) be a factorization. For every $u \in \mathbb{M}_{1 \times Q}(\mathbb{K}) \setminus \{0^Q\}$, we have $f(u) \neq 0^Q$ and $g(u) \neq 0$. If for every $u \in \mathbb{M}_{1 \times Q}(\mathbb{K}) \setminus \{0^Q\}$ we have $f(u) = u$ and $g(u) = 1$, then we call (f, g) the *trivial factorization*. Every semiring $\mathbb{K}$ admits a factorization, namely the trivial factorization.

Let $T = [Q, E, \lambda, \varrho]$ be a WFA over $\mathbb{K}$, and let (f, g) be a factorization of dimension Q. We may assume that $\lambda \neq 0^Q$. We define:

- $\tilde{\delta} : \mathbb{M}_{1 \times Q}(\mathbb{K}) \times \Sigma \dashrightarrow \mathbb{M}_{1 \times Q}(\mathbb{K})$,
  $\tilde{\delta}(u, a) := f\big(u \odot \theta(a)\big)$, for every $u \in \mathbb{M}_{1 \times Q}(\mathbb{K})$, $a \in \Sigma$ with $u \odot \theta(a) \neq 0^Q$,

- $\tilde{\sigma} : \mathbb{M}_{1 \times Q}(\mathbb{K}) \times \Sigma \dashrightarrow \mathbb{K}$,
  $\tilde{\sigma}(u, a) := g\big(u \odot \theta(a)\big)$, for every $u \in \mathbb{M}_{1 \times Q}(\mathbb{K})$, $a \in \Sigma$ with $u \odot \theta(a) \neq 0^Q$, and

- $\tilde{\varrho} : \mathbb{M}_{1 \times Q}(\mathbb{K}) \to \mathbb{K}$,
  $\tilde{\varrho}(u) := u \odot \varrho$, for every $u \in \mathbb{M}_{1 \times Q}(\mathbb{K})$.

Let $\tilde{Q}$ be the least subset of $\mathbb{M}_{1 \times Q}(\mathbb{K})$ which contains $f(\lambda)$ and is closed under $\tilde{\delta}$, i.e., for every $u \in \tilde{Q}$ and $a \in \Sigma$, we have $\tilde{\delta}(u, a) \in \tilde{Q}$ provided that $\tilde{\delta}(u, a)$ is defined. If $\tilde{Q}$ is finite, then we call

$$\tilde{T} := \left[\tilde{Q}, \ \tilde{\delta}|_{\tilde{Q} \times \Sigma}, \ \tilde{\sigma}|_{\tilde{Q} \times \Sigma}, \ f(\lambda), \ g(\lambda), \ \tilde{\varrho}|_{\tilde{Q}}\right]$$

the *determinization of T by (f, g)*. If $\tilde{Q}$ is infinite, then the determinization of T by (f, g) is not defined. To avoid some inconvenient technical details, we say that the determinization of T by (f, g) is not defined if $\lambda = 0^Q$.

Let us mention that MOHRI defines his algorithm using the factorization

$$g(u) := \bigoplus_{q \in Q} u[q] \quad \text{and} \quad f(u) = \big(g(u)\big)^{-1} \odot u$$

for every $u \in \mathbb{M}_{1 \times Q}(\mathbb{K}) \setminus \{0^Q\}$ (cf. [25, p. 285]). The main problem is that it is left open how to interpret the power $^{-1}$ in general semirings. For example, consider the semiring $(\mathbb{N}, +, \cdot, 0, 1)$, let $Q = \{q_1, q_2\}$ and $u = (4, 10)$, i.e., $g(u) = 14$. Regardless of how we interpret 14^{-1}, we cannot define $f(u)$ in a way that the key property $g(u) \cdot f(u) = u$, i.e., $14 \cdot f(u) = (4, 10)$ is satisfied.

If (f, g) is the trivial factorization, then the determinization of T by (f, g) is the WFA which we obtain by the idea sketched in Section 3.1. If $\mathbb{K}$ is locally finite then the determinization of T with respect to the trivial factorization is defined.

If $\mathbb{K}$ is the tropical semiring $\mathbb{T}$, and we set $g(u) = \min(u)$ and $f(u) = -\min(u) + u$ for $u \in \mathbb{M}_{1 \times Q}(\mathbb{T}) \setminus \{\infty^Q\}$, then the determinization of T by (f, g) yields the same as in Section 3.2.

Theorem 3.1. *Let $\mathbb{K}$ be an arbitrary semiring, $T = [Q, E, \lambda, \varrho]$ be a WFA over $\mathbb{K}$ and (f, g) be a factorization of dimension Q. If the determinization of T by (f, g) is defined, then it is equivalent to T.*

Proof. Let $\tilde{T}$ be the determinization of T by (f, g). We denote the initial state of $\tilde{T}$ by $\tilde{u}$, i.e., $\tilde{u} := f(\lambda)$. Let $w \in \Sigma^*$ be arbitrary. We show the following two assertions by an induction on the length of $w \in \Sigma^*$.

1. If $\tilde{\delta}(\tilde{u}, w)$ is defined, then $\lambda \odot \theta(w) = g(\lambda) \odot \tilde{\sigma}(\tilde{u}, w) \odot \tilde{\delta}(\tilde{u}, w)$.

2. If $\tilde{\delta}(\tilde{u}, w)$ is not defined, then $\lambda \odot \theta(w) = 0^Q$.

For $w = \varepsilon$, we have $\tilde{\delta}(\tilde{u}, \varepsilon) = \tilde{u}$ and $\tilde{\sigma}(\tilde{u}, \varepsilon) = 1$. Moreover, $\theta(\varepsilon)$ is the identity matrix. Thus, the equation in (1) reduces to $\lambda = g(\lambda) \odot \tilde{u}$. This is true, because (f, g) is a factorization. Further, (2) is obviously true for $w = \varepsilon$.

Now, let $w \in \Sigma^*$ satisfy (1) and (2), and let $a \in \Sigma$ be arbitrary.

Case 1: $\tilde{\delta}(\tilde{u}, wa)$ is defined. This implies that $\tilde{\sigma}(\tilde{u}, wa)$, $\tilde{\delta}(\tilde{u}, w)$ and $\tilde{\sigma}(\tilde{u}, w)$ are defined. We only have to consider (1), because (2) is obviously true. Applying the definition of $\tilde{\delta}$ and $\tilde{\sigma}$, that (f, g) is a factorization and the inductive hypothesis, we obtain

$$
\begin{aligned}
g(\lambda) \odot \tilde{\sigma}(\tilde{u}, wa) \odot \tilde{\delta}(\tilde{u}, wa) &= g(\lambda) \odot \tilde{\sigma}(\tilde{u}, w) \odot \tilde{\sigma}\big(\tilde{\delta}(\tilde{u}, w), a\big) \odot \tilde{\delta}\big(\tilde{\delta}(\tilde{u}, w), a\big) \\
&= g(\lambda) \odot \tilde{\sigma}(\tilde{u}, w) \odot g\big(\tilde{\delta}(\tilde{u}, w) \odot \theta(a)\big) \odot f\big(\tilde{\delta}(\tilde{u}, w) \odot \theta(a)\big) \\
&= g(\lambda) \odot \tilde{\sigma}(\tilde{u}, w) \odot \tilde{\delta}(\tilde{u}, w) \odot \theta(a) ~=~ \lambda \odot \theta(w) \odot \theta(a) ~=~ \lambda \odot \theta(wa).
\end{aligned}
$$

This proves (1).

Case 2: $\tilde{\delta}(\tilde{u}, wa)$ is not defined. We only have to consider (2), because (1) is obviously true.

Case 2.1: $\tilde{\delta}(\tilde{u}, w)$ is defined. It implies that $\tilde{\delta}\big(\tilde{\delta}(\tilde{u}, w), a\big)$ is not defined, i.e., $\tilde{\delta}(\tilde{u}, w) \odot \theta(a) = 0^Q$. In combination with the inductive hypothesis, we obtain:

$$
\lambda \odot \theta(wa) = \lambda \odot \theta(w) \odot \theta(a) = g(\lambda) \odot \tilde{\sigma}(\tilde{u}, w) \odot \tilde{\delta}(\tilde{u}, w) \odot \theta(a) = g(\lambda) \odot \tilde{\sigma}(\tilde{u}, w) \odot 0^Q = 0^Q.
$$

Case 2.2: $\tilde{\delta}(\tilde{u}, w)$ is not defined. By the inductive hypothesis, we have $\lambda \odot \theta(w) = 0^Q$. Thus, we conclude

$$
\lambda \odot \theta(wa) ~=~ \lambda \odot \theta(w) \odot \theta(a) ~=~ 0^Q \odot \theta(a) ~=~ 0^Q.
$$

Now, it is easy to show that T and $\tilde{T}$ are equivalent. Let $w \in \Sigma^*$ be arbitrary. If $\tilde{\delta}(\tilde{u}, w)$ is not defined, then $|\tilde{T}|(w) = 0$. By (2), we have $\lambda \odot \theta(w) = 0^Q$, and thus, $|T|(w) = 0$.

If $\tilde{\delta}(\tilde{u}, w)$ is defined, then by (1) we obtain

$$
|\tilde{T}|(w) ~=~ g(\lambda) \odot \tilde{\sigma}(\tilde{u}, w) \odot \tilde{\varrho}\big(\tilde{\delta}(\tilde{u}, w)\big) ~=~ g(\lambda) \odot \tilde{\sigma}(\tilde{u}, w) \odot \tilde{\delta}(\tilde{u}, w) \odot \varrho ~=~ \lambda \odot \theta(w) \odot \varrho ~=~ |T|(w).
$$

$\square$

3.4 Maximal factorizations

The existence of the determinization of some WFA T depends on the choice of f and g. Thus, we examine factorizations. In particular, we are interested in knowing which factorizations are more suitable than others.

We call a factorization (f, g) *maximal* if for every $u \in \mathbb{M}_{1 \times Q}(\mathbb{K})$ and every $k \in \mathbb{K}$ with $k \odot u \neq 0^Q$ we have $f(u) = f(k \odot u)$. The restriction "$k \odot u \neq 0^Q$" is not just due to the fact that $f(0^Q)$ is not defined. Even if we modify the notion of a factorization in a way that $f(0^Q)$ and $g(0^Q)$ are defined, it is definitely necessary to keep the restriction "$k \odot u \neq 0^Q$" in the notion of a maximal factorization. Otherwise, we could show for every $u \in \mathbb{M}_{1 \times Q}(\mathbb{K})$ that $f(u) = f(0 \odot u) = f(0^Q)$ and the notion of a maximal factorization becomes meaningless.

If (f, g) is a maximal factorization, then for every $u \in \mathbb{M}_{1 \times Q}(\mathbb{K}) \setminus \{0^Q\}$ we have $f(f(u)) = f\big(g(u) \odot f(u)\big) = f(u)$.

We have the following lemma for maximal factorizations:

Lemma 3.2. *Let $\mathbb{K}$ be a semiring, $T = [Q, E, \lambda, \varrho]$ be a WFA and (f, g) be a maximal factorization. Let $u \in \mathbb{M}_{1 \times Q}(\mathbb{K})$ and $w \in \Sigma^+$ be a non-empty word satisfying $u \odot \theta(w) \neq 0^Q$. Then, $\tilde{\delta}(u, w)$ is defined and we have $\tilde{\delta}(u, w) = f(u \odot \theta(w)) = \tilde{\delta}(f(u), w)$.*

Proof. At first, we show by an induction on the length of w that $\tilde{\delta}(u, w)$ is defined and the first equation is true. If w is a letter, then this equation is the definition of $\tilde{\delta}$. So let $w \in \Sigma^+$ and $a \in \Sigma$ satisfy $u \odot \theta(wa) \neq 0^Q$, i.e., $u \odot \theta(w) \odot \theta(a) \neq 0^Q$. By induction, $\tilde{\delta}(u, w)$ is defined and the left equation is true. Consequently,

$$
\tilde{\delta}(u, wa) = \tilde{\delta}\big(\tilde{\delta}(u, w), a\big) = \tilde{\delta}\big(f(u \odot \theta(w)), a\big) = f\big(f(u \odot \theta(w)) \odot \theta(a)\big).
$$

Because (f, g) is maximal, we can multiply the argument of the outermost occurrence of f by $g(u \odot \theta(w))$.

$$= f\big(g(u \odot \theta(w)) \odot f(u \odot \theta(w)) \odot \theta(a)\big) = f\big(u \odot \theta(w) \odot \theta(a)\big) = f\big(u \odot \theta(wa)\big).$$

We show the second equation. Because $u \odot \theta(w) = g(u) \odot f(u) \odot \theta(w)$ and $u \odot \theta(w) \neq 0^Q$, we have $f(u) \odot \theta(w) \neq 0^Q$, i.e., $\tilde{\delta}(f(u), w)$ is defined. By the first equation,

$$\tilde{\delta}(u, w) = f(u \odot \theta(w)) = f\big(g(u) \odot f(u) \odot \theta(w)\big) = f\big(f(u) \odot \theta(w)\big) = \tilde{\delta}(f(u), w).$$

$\square$

If we consider Lemma 3.2 for $w = \varepsilon$, we see $\tilde{\delta}(u, \varepsilon) = u$ but $f(u \odot \theta(\varepsilon)) = \tilde{\delta}(f(u), \varepsilon) = f(u)$. Hence, both equations in Lemma 3.2 hold for $w = \varepsilon$ iff $u = f(u)$.

The following theorem shows that maximal factorizations are optimal in comparison to other factorizations if the semiring $\mathbb{K}$ is zero-divisor free, or more generally, if for every $u \in \mathbb{M}_{1 \times Q}(\mathbb{K}) \setminus \{0^Q\}$, $g(u)$ is not a zero-divisor.

Theorem 3.3. *Let $T = [Q, E, \lambda, \varrho]$ be a WFA over $\mathbb{K}$ satisfying $\lambda \neq 0^Q$. Let (f, g) be a maximal factorization of dimension Q and assume that for every $u \in \mathbb{M}_{1 \times Q}(\mathbb{K}) \setminus \{0^Q\}$, $g(u)$ is not a zero-divisor. Let (f', g') be an arbitrary factorization of dimension Q.*

1. *If the determinization of T by (f', g') exists, then the determinization of T by (f, g) exists.*

2. *Assume that the determinization of T by (f', g') exists, and let Q' be the set of its states. Let $\tilde{Q}$ be the set of states of the determinization of T by (f, g). We have $\tilde{Q} = f(Q')$, and thus, $|\tilde{Q}| \leq |Q'|$.*

Proof. At first, note that for every $u \in \mathbb{M}_{1 \times Q}(\mathbb{K}) \setminus \{0^Q\}$ we have $g'(u) \odot f'(u) = u = g(u) \odot f(u)$ and since (f, g) is maximal, $f(f'(u)) = f(u) = f(f(u))$. Consider the following mappings:

- $\delta' : \mathbb{M}_{1 \times Q}(\mathbb{K}) \times \Sigma \dashrightarrow \mathbb{M}_{1 \times Q}(\mathbb{K})$,
  $\delta'(u, a) := f'\big(u \odot \theta(a)\big)$, for every $u \in \mathbb{M}_{1 \times Q}(\mathbb{K})$, $a \in \Sigma$ satisfying $u \odot \theta(a) \neq 0^\infty$, and

- $\tilde{\delta} : \mathbb{M}_{1 \times Q}(\mathbb{K}) \times \Sigma \dashrightarrow \mathbb{M}_{1 \times Q}(\mathbb{K})$,
  $\tilde{\delta}(u, a) := f\big(u \odot \theta(a)\big)$, for every $u \in \mathbb{M}_{1 \times Q}(\mathbb{K})$, $a \in \Sigma$ satisfying $u \odot \theta(a) \neq 0^\infty$.

Let Q' be the least subset of $\mathbb{M}_{1 \times Q}(\mathbb{K})$ which contains $f'(\lambda)$ and is closed under δ', i.e., for every $u \in Q'$ and every $a \in \Sigma$, we have $\delta'(u, a) \in Q$ provided that $\delta'(u, a)$ is defined. Let $\tilde{Q}$ be the least subset of $\mathbb{M}_{1 \times Q}(\mathbb{K})$ which contains $f(\lambda)$ and is closed under $\tilde{\delta}$ in the same sense. To prove (1) and (2), we show $\tilde{Q} = f(Q')$. As mentioned above, we have $f(f'(\lambda)) = f(\lambda)$. To show $\tilde{Q} = f(Q')$, we derive the following two assertions for every $u \in \mathbb{M}_{1 \times Q}(\mathbb{K}) \setminus \{0^Q\}$ and every $a \in \Sigma$.

a. $\delta'(u, a)$ is defined iff $\tilde{\delta}\big(f(u), a\big)$ is defined.

b. If $\delta'(u, a)$ is defined, then we have $\tilde{\delta}\big(f(u), a\big) = f\big(\delta'(u, a)\big)$.

We have $u \odot \theta(a) = g(u) \odot f(u) \odot \theta(a)$. Because $g(u) \neq 0$, and $g(u)$ is not a zero-divisor, we have $u \odot \theta(a) = 0^Q$ iff $f(u) \odot \theta(a) = 0^Q$. This proves (a).

Now, assume that $\delta'(u, a)$ and $\tilde{\delta}(f(u), a)$ are defined, i.e., $u \odot \theta(a) \neq 0^Q$. We show:

$$f\big(\delta'(u, a)\big) = f\big(f'(u \odot \theta(a))\big) = f\big(u \odot \theta(a)\big) = f\big(g(u) \odot f(u) \odot \theta(a)\big)$$

$$= f\big(f(u) \odot \theta(a)\big) = \tilde{\delta}\big(f(u), a\big).$$

$\square$

If both (f,g) and (f',g') are maximal factorizations, then f and f' are mutually inverse "bijections" between the determinizations of T by (f,g) and by (f',g'). Hence, all determinizations of some WFA T by maximal factorizations yield the same subsequential WFA up to renaming the states.

Note that by Theorem 3.3 the determinization by some maximal factorization is optimal among determinizations by arbitrary factorizations. It is possible that the determinization of a WFA T by some maximal factorization (f,g) does not exist, even if there is an equivalent subsequential WFA. If the determinization of T by some maximal factorization exists, then it is not necessarily the smallest subsequential WFA equivalent to T. It is quite possible that there are better approaches to determinize WFA which are entirely different from our generalization of MOHRI's approach. For example, in the semiring $(\mathbb{Z} \cup \{\infty\}, \min, +, \infty, 0)$, where $\mathbb{Z}$ denotes the set of integers, an algorithm can enumerate an infinite list of all subsequential WFA's and check whether there is an equivalent WFA in the list.[2] This algorithm is far from being practical, but it terminates iff an equivalent WFA exists.

3.5 Examples of maximal factorizations

In the previous section, we have seen that maximal factorizations are optimal in some sense, and thus, maximal factorizations are of practical interest. In this section, we present maximal factorizations for several semirings. Let Q be a nonempty, finite set.

In the Boolean semiring $\mathbb{B}$, for every factorization (f,g) and every $u \in \mathbb{M}_{1\times Q}(\mathbb{B}) \setminus \{0^Q\}$, we have $f(u) = u$ and $g(u) = 1$. Hence, there is only the trivial factorization. The trivial factorization in the Boolean semiring is a maximal factorization.

We consider the semiring $(\mathbb{N}, +, \cdot, 0, 1)$. For every $u \in \mathbb{M}_{1\times Q}(\mathbb{N})\setminus\{0^Q\}$, let $g(u)$ be the greatest common divisor of the entries of u. There is a mapping $f : \mathbb{M}_{1\times Q}(\mathbb{N}) \setminus \{0^Q\} \to \mathbb{M}_{1\times Q}(\mathbb{N})$ such that $u = g(u) \cdot f(u)$. Clearly, (f,g) is maximal. For $u \in \mathbb{M}_{1\times Q}(\mathbb{N}) \setminus \{0^Q\}$ and $n \geq 1$, we have $g(n \cdot u) = n \cdot g(u)$ and $f(n \cdot u) = f(u)$. A generalization to the semiring $(\mathbb{Z}, +, \cdot, 0, 1)$ is obvious.

For the tropical semiring $\mathbb{T}$, MOHRI's factorization $g(u) = \min(u)$ and $f(u) = -\min(u) + u$ for $u \in \mathbb{M}_{1\times Q}(\mathbb{T}) \setminus \{\infty^Q\}$, as explained in Section 3.2, is maximal. We call this factorization *Mohri's factorization*.

Now, let $\mathbb{K}$ be some semiring such that $(\mathbb{K} \setminus \{0\}, \odot)$ is a group. We may assume that $Q = \{q_1, \ldots, q_{|Q|}\}$. Let $u \in \mathbb{M}_{1\times Q}(\mathbb{K}) \setminus \{0^Q\}$ and i be the smallest integer with $u[q_i] \neq 0$. We set $g(u) := u[q_i]$ and $f(u) := g(u)^{-1} \odot u$. This pair of mappings (f,g) obviously forms a factorization. Let $u \in \mathbb{M}_{1\times Q}(\mathbb{K}) \setminus \{0^Q\}$ and $k \in \mathbb{K} \setminus \{0\}$ be arbitrary. The least integer i such that $u[q_i] \neq 0$ is also the least integer i such that $(k \odot u)[q_i] \neq 0$. We have $(k \odot u)[q_i] = k \odot u[q_i]$, i.e., $g(k \odot u) = k \odot g(u)$, and obtain

$$f(k \odot u) = g(k \odot u)^{-1} \odot k \odot u = \big(k \odot g(u)\big)^{-1} \odot k \odot u = g(u)^{-1} \odot k^{-1} \odot k \odot u = f(u).$$

Hence, (f,g) is a maximal factorization.

Now, consider the string semiring $(\Delta^* \cup \{0\}, \min, \cdot, 0, \varepsilon)$ over an alphabet Δ with some ordering $\leq$. For every $u \in \mathbb{M}_{1\times Q}(\mathbb{N})\setminus\{0^Q\}$, let $g(u)$ be the longest common prefix of the non-zero entries of u. There is a unique "residual" mapping $f : \mathbb{M}_{1\times Q}(\Delta^*)\setminus\{0^Q\} \to \mathbb{M}_{1\times Q}(\Delta^*)$ such that (f,g) is a factorization. This factorization is maximal.

Next we present a semiring which does not admit a maximal factorization. Let $\mathbb{K}'$ be the set of all natural numbers which admit a factorization into an even number of primes, e.g., $4 = 2 \cdot 2$ and $126 = 2 \cdot 3 \cdot 3 \cdot 7$ belong to $\mathbb{K}'$, but 2 and $18 = 2 \cdot 3 \cdot 3$ do not belong to $\mathbb{K}'$. Let $\mathbb{K} = \mathbb{K}' \cup \{1, \infty\}$ and consider the semiring $(\mathbb{K}, \min, \cdot, \infty, 1)$ where min is defined by the usual ordering of natural

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[2]The equivalence between an arbitrary and a subsequential WFA over $(\mathbb{Z} \cup \{\infty\}, \min, +, \infty, 0)$ is decidable by [21, Prop. 5.3] since every subsequential WFA over $(\mathbb{Z} \cup \{\infty\}, \min, +, \infty, 0)$ is also a subsequential WFA over $(\mathbb{Z} \cup \{-\infty\}, \max, +, -\infty, 0)$.

numbers, $\cdot$ is the usual multiplication of natural numbers and ∞ denotes a new maximal element. Let $Q = \{1, 2\}$. By contradiction, assume that $\mathbb{K}$ admits a maximal factorization (f, g) of dimension Q. Let $u = (2 \cdot 3 \cdot 5 \cdot 7, 2 \cdot 3 \cdot 5 \cdot 11)$. Since $2 \cdot 3 \cdot (5 \cdot 7, 5 \cdot 11) = u = 2 \cdot 5 \cdot (3 \cdot 7, 3 \cdot 11)$, we have $f((5 \cdot 7, 5 \cdot 11)) = f(u) = f((3 \cdot 7, 3 \cdot 11))$. However, $g((5 \cdot 7, 5 \cdot 11))$ is a common divisor of $5 \cdot 7$ and $5 \cdot 11$ in $\mathbb{K}$, i.e., $g((5 \cdot 7, 5 \cdot 11)) = 1$, and hence, $f((5 \cdot 7, 5 \cdot 11)) = (5 \cdot 7, 5 \cdot 11)$. In the same way, we can derive $f((3 \cdot 7, 3 \cdot 11)) = (3 \cdot 7, 3 \cdot 11)$ which is a contradiction.

3.6   A generalization of the twins property

The twins property was introduced by CHOFFRUT in 1977 [7] and studied in various papers, e.g., [1, 4, 25]. In [25], MOHRI proves that the twins property is a sufficient condition for the termination of his determinization algorithm on WFA over the tropical semiring. We want to generalize this result to a larger class of semirings.

Let $T = [Q, E, \lambda, \varrho]$ be a WFA over some semiring $\mathbb{K}$. Following [25], we say that T has the *twins property* if we have for every $u, v \in \Sigma^*$ and for every $p, q \in Q$

$$I \overset{u}{\leadsto} p \overset{v}{\leadsto} p \neq \emptyset \ \wedge \ I \overset{u}{\leadsto} q \overset{v}{\leadsto} q \neq \emptyset \quad \Longrightarrow \quad \theta(v)[p, p] = \theta(v)[q, q].$$

In his proof, MOHRI uses several conditions which cannot be applied to arbitrary semirings. Following [20, 30], some path $\pi = (q_0, a_0, k_0, q_1) \ldots (q_{n-1}, a_{n-1}, k_{n-1}, q_n)$ is called *victorious* if we have

$$\sigma(\pi) = \theta(a_0 \ldots a_{n-1})[q_0, q_n].$$

MOHRI's conditions in [25] are:

1. Let $w \in \Sigma^*$ and $p, q \in Q$. If $\theta(w)[p, q] \neq 0$, then there is a victorious path in $p \overset{w}{\leadsto} q$.

2. Let π be a victorious path. For every $0 \leq i \leq j \leq |\pi|$ the path $\pi(i, j)$ is victorious.

3. The commutativity of the tropical semiring.[3]

If we want to generalize MOHRI's result to a larger class of semirings, then we have to take care of these conditions. Just assume that there are exactly two paths $\pi, \pi' \in p \overset{w}{\leadsto} q$ in condition (1), and $\sigma(\pi) = k$ and $\sigma(\pi') = l$. Then, $\theta(w)[p, q] = k \oplus l$. To assume the existence of a victorious path, we need either $\theta(w)[p, q] = \sigma(\pi)$ or $\theta(w)[p, q] = \sigma(\pi')$, i.e., we need $k \oplus l \in \{k, l\}$. Hence, we restrict ourselves to semirings $\mathbb{K}$ satisfying $k \oplus l \in \{k, l\}$ for every $k, l \in \mathbb{K}$.

This property has an important consequence.

Lemma 3.4. *Let $\mathbb{K}$ be a semiring. The following three conditions are equivalent.*

1. *For every $k, l \in \mathbb{K}$, we have $k \oplus l \in \{k, l\}$.*

2. *There is a linear order relation $\leq$ on $\mathbb{K}$ such that $\oplus$ is the minimum with respect to $\leq$.*

3. *There is a linear order relation $\leq$ on $\mathbb{K}$ such that:*

    (a) The operation $\oplus$ is the minimum with respect to $\leq$.

    (b) The ordering $\leq$ is stable with respect to $\odot$, i.e., for every $k_1, k_2, l \in \mathbb{K}$, $k_1 \leq k_2$ implies $l \odot k_1 \leq l \odot k_2$ and $k_1 \odot l \leq k_2 \odot l$.

---

[3]For instance, he uses the commutativity of the tropical semiring to obtain $\sigma(\pi_0) = \sigma(\pi_0') + \theta_1(p_0, u_2, p_0)$ in the last part of the proof of Theorem 11 in [25].

Proof. (3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1) are obvious. We show (1) $\Rightarrow$ (3). For every $k, l \in \mathbb{K}$ let

$$k \leq l \quad :\Longleftrightarrow \quad k \oplus l = k.$$

By (1), $\leq$ is reflexive and total, and it is antisymmetric by the commutativity of $\oplus$.
Let $k_1, k_2, k_3 \in \mathbb{K}$ satisfying $k_1 \leq k_2 \leq k_3$. Then, we have

$$k_1 \oplus k_3 = (k_1 \oplus k_2) \oplus k_3 = k_1 \oplus (k_2 \oplus k_3) = k_1 \oplus k_2 = k_1,$$

i.e., $k_1 \leq k_3$. Thus, $\leq$ is an ordering, and it is easy to see that $\oplus$ is the minimum with respect to $\leq$. Assertion (3b) follows from the distributivity of $\odot$ over $\oplus$. $\qquad\square$

If a semiring $\mathbb{K}$ satisfies assertions (1,2,3) in Lemma 3.4, then we call $\mathbb{K}$ a *min-semiring*. In [23] these semirings are called extremal

The tropical semiring and the semiring $(\mathbb{N} \cup \{\infty\}, \min, \cdot, \infty, 1)$ (where $\cdot$ denotes integer multiplication) are min-semirings. Moreover, every ordered monoid $(\mathbb{M}, \leq, \oplus, 1_{\mathbb{M}})$ admits an extension to a zero-divisor-free min-semiring $(\mathbb{M} \cup \{\infty\}, \min, \oplus, \infty, 1_{\mathbb{M}})$ by setting $m \leq \infty$ for every $m \in \mathbb{M}$ and defining min with respect to $\leq$. Thus, min-semirings are naturally related to ordered algebraic structures which are an important field in mathematics [12, 13, 16].

If $\mathbb{K}$ is a min-semiring, then we have $k \leq 0$ for every $k \in \mathbb{K}$. Every min-semiring is idempotent. However, there are idempotent semirings which are not min-semirings, e.g., $(2^M, \cup, \cap, \emptyset, M)$ for some non-empty set M.

It is well-known in the theory of semirings, that a semiring $\mathbb{K}$ is idempotent iff $\oplus$ is the infimum over some partial ordering which is stable with respect to $\odot$ [13]. Lemma 3.4 is a variant of this result for min-semirings.

Let $\mathbb{K}$ be a min-semring. By induction, one can easily show that for every $n \geq 1$, and every $k_1, \ldots, k_n \in \mathbb{K}$, there is some $1 \leq i \leq n$ such that $\oplus_{1 \leq j \leq n} k_j = k_i$.

Now, consider MOHRI's first condition for a min-semiring $\mathbb{K}$. Since

$$\theta(w)[p, q] = \bigoplus_{\pi \in p \overset{w}{\rightsquigarrow} q} \sigma(\pi),$$

we can assume that there is some path $\pi \in p \overset{w}{\rightsquigarrow} q$ such that $\sigma(\pi) = \theta(w)[p, q]$ provided that $\theta(w)[p, q] \neq 0$ and $\mathbb{K}$ is a min-semiring.

Now, we take care of MOHRI's second condition. Unfortunately, this condition is not true in general, even if $\mathbb{K}$ is a min-semiring. We can avoid this condition in our proof of Theorem 3.5, below. However, due to this problem, we cannot prove Theorem 3.5 by slight adjustments to MOHRI's proof.

Theorem 3.5. *Let $\mathbb{K}$ be a commutative min-semiring and let $T = [Q, E, \lambda, \varrho]$ be a WFA satisfying $\lambda \neq 0^Q$. Let (f, g) be a maximal factorization. If T has the twins property, then the determinization of T by (f, g) is defined.*

Proof. Let $\tilde{\delta} : \mathbb{M}_{1 \times Q}(\mathbb{K}) \times \Sigma \dashrightarrow \mathbb{M}_{1 \times Q}(\mathbb{K})$ and $\tilde{Q}$ be defined as in Section 3.3. We prove the theorem by showing that $\tilde{Q}$ is finite. Our strategy is to construct a finite subset $K' \subseteq \mathbb{K}$ such that for every $w \in \Sigma^*$ we have $\tilde{\delta}(f(\lambda), w) \in f(\mathbb{M}_{1 \times Q}(K'))$ if $\tilde{\delta}(f(\lambda), w)$ is defined.

Let $w = a_0 \ldots a_{|w|-1} \in \Sigma^*$ such that $\tilde{\delta}(f(\lambda), w)$ is defined. We derive information on $\lambda \odot \theta(w)$.

Let $Q' := \{ p \in Q \mid (\lambda \odot \theta(w))[p] \neq 0 \}$. We have $Q' \neq \emptyset$. For every $p \in Q'$, let $q_{0,p} \in Q$ and $\pi_p \in q_{0,p} \overset{w}{\rightsquigarrow} p$ be a path such that

$$\lambda(q_{0,p}) \odot \sigma(\pi_p) = (\lambda \odot \theta(w))[p]. \tag{1}$$

Note that $q_{0,p}$ and π_p exist, because $\mathbb{K}$ is a min-semiring.

For $p \in Q'$, we denote $\pi_p = (q_{p,0}, a_0, k_{p,0}, q_{p,1}) \ldots (q_{p,|w|-1}, a_{|w|-1}, k_{p,|w|-1}, q_{p,|w|})$.

Now, let $n \geq 0$ and let $0 \leq i_1 < i_2 \leq i_3 < i_4 \leq i_5 < \cdots < i_{2n} \leq |w|$ such that we have for every $1 \leq l \leq n$ and every $p \in Q' : q_{p,i_{2l-1}} = q_{p,i_{2l}}$. By the pigeon hole principle, we can assume

$$|w| - \sum_{1 \leq l \leq n} (i_{2l} - i_{2l-1}) \quad \leq \quad |Q|^{|Q'|} \quad \leq \quad |Q|^{|Q|}. \tag{2}$$

We denote by $\leq_\oplus$ the ordering of $\mathbb{K}$ given by Lemma 3.4.

Now let $p \in Q'$ and $1 \leq l \leq n$ be arbitrary. We have

$$\theta(a_{i_{2l-1}} \ldots a_{i_{2l}-1})[q_{p,i_{2l-1}}, q_{p,i_{2l}}] \quad \leq_\oplus \quad \sigma\big(\pi_p(i_{2l-1}, i_{2l})\big). \tag{3}$$

Since $\mathbb{K}$ is a min-semiring, there is a path ν which starts and ends at the same states as $\pi_p(i_{2l-1}, i_{2l})$, has the same label as $\pi_p(i_{2l-1}, i_{2l})$ and $\theta(a_{i_{2l-1}} \ldots a_{i_{2l}-1})[q_{p,i_{2l-1}}, q_{p,i_{2l}}] = \sigma(\nu)$. Let π'_p be the path which we obtain by replacing in π_p the part $\pi_p(i_{2l-1}, i_{2l})$ by ν.

We have $\sigma(\pi'_p) \leq_\oplus \sigma(\pi_p)$ by (3), the choice of ν, and the stability of $\leq_\oplus$ under $\odot$. In combination with the choice of π_p, we obtain

$$\lambda(q_{p,0}) \odot \sigma(\pi'_p) \quad \leq_\oplus \quad \lambda(q_{p,0}) \odot \sigma(\pi_p) \quad = \quad (\lambda \odot \theta(w))[p]. \tag{4}$$

Consequently, we have $\lambda(q_{p,0}) \odot \sigma(\pi'_p) = (\lambda \odot \theta(w))[p]$ in (4).

We perform such a replacement for every $1 \leq l \leq n$ and every $p \in Q'$. To avoid technical overhead, we assume that we were lucky, i.e, we assume that for $p \in Q'$ and $1 \leq l \leq n$ line (3) is an equation:

$$\theta(a_{i_{2l-1}} \ldots a_{i_{2l}-1})[q_{p,i_{2l-1}}, q_{p,i_{2l}}] \quad = \quad \sigma\big(\pi_p(i_{2l-1}, i_{2l})\big). \tag{5}$$

Because T has the twins property, we have for arbitrary $p, q \in Q'$ and $1 \leq l \leq n$ the equality $\sigma\big(\pi_p(i_{2l-1}, i_{2l})\big) = \sigma\big(\pi_q(i_{2l-1}, i_{2l})\big)$. Thus, we can define for some $p \in Q'$

$$k := \bigodot_{1 \leq l \leq n} \sigma\big(\pi_p(i_{2l-1}, i_{2l})\big).$$

We are now able to derive information on $\lambda \odot \theta(w)[p]$ using (1) and k. Let $\hat{\pi}_p$ be the path[4] which we obtain by "erasing" the parts $\pi_p(i_{2l-1}, i_{2l})$ from π_p for every $1 \leq l \leq n$. We define for every $p \in Q'$

$$k_p := \lambda(q_{p,0}) \odot \sigma(\hat{\pi}_p).$$

By the commutativity of $\odot$ we have

$$(\lambda \odot \theta(w))[p] = k \odot k_p. \tag{6}$$

If we set $k_p := 0$ for $p \in Q \setminus Q'$, then (6) holds for every $p \in Q$. We define $k' \in \mathbb{M}_{1 \times Q}(\mathbb{K})$ by setting $k'[p] = k_p$. We can state (6) as $\lambda \odot \theta(w) = k \odot k'$. Let

$$K' := \big\{ \lambda(q) \odot \sigma(\pi) \,\big|\, q, p \in Q, \ v \in \Sigma^*, \ |v| \leq |Q|^{|Q|}, \ \pi \in q \overset{v}{\leadsto} p \big\}.$$

By (2), we have $k_p \in K'$ for every $p \in Q$, and thus $k' \in \mathbb{M}_{1 \times Q}(K')$. Note that K' and $\mathbb{M}_{1 \times Q}(K')$ are finite. By Lemma 3.2, we obtain

$$\tilde{\delta}\big(f(\lambda), w\big) = \tilde{\delta}(\lambda, w) = f(\lambda \odot \theta(w)) = f(k \odot k') = f(k') \in f(\mathbb{M}_{1 \times Q}(K')).$$

Thus, $\tilde{Q}$ is finite. $\square$

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[4]Let $\hat{w} \in \Sigma^*$ be the label of $\hat{\pi}_p$. Please note that we do NOT necessarily have $\lambda(q_{p,0}) \odot \sigma(\hat{\pi}_p) = \lambda \odot \theta(\hat{w})[p]$. (There are easy counterexamples.)

Next we show that we cannot prove Theorem 3.5 for arbitrary semirings: Both the assumption that $\mathbb{K}$ is commutative and that $\mathbb{K}$ is a min-semiring cannot be left out

Example 3.6. *Let* $\Sigma = \{a\}$. *Consider the WFA* T_1 *over the semiring* $(\mathbb{N}, +, \cdot, 0, 1)$ *defined by* $Q = \{q_0, q_1\}$, $\lambda(q_0) = \lambda(q_1) = \varrho(q_0) = \varrho(q_1) = 1$ *and* $E = \{(q_0, a, 1, q_0), (q_0, a, 1, q_1), (q_1, a, 1, q_1)\}$.
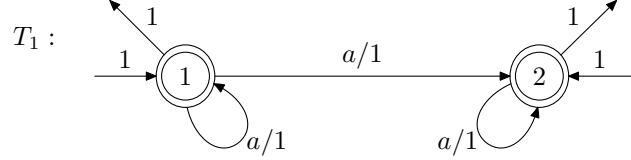
For every $n \geq 0$ *we have* $\lambda \cdot \theta(a^n) = (1, n+1)$. *It is obvious that every maximal factorization* (f, g) *satisfies* $f((1, n+1)) = (1, n+1)$. *Thus, the determinization of* T_1 *by some maximal factorization does not exist, although* T_1 *has the twins property and* $(\mathbb{N}, +, \cdot, 0, 1)$ *is commutative.*

Example 3.7. *Let* $\Sigma = \{a\}$ *and* $\Delta = \{a, b\}$ *with* $a < b$. *Consider the string semiring* $(\Delta^*, \min, \cdot, 0, \varepsilon)$ *and the WFA* T_2 *defined by* $Q = \{q_0, q_1\}$, $\lambda(q_0) = \varrho(q_0) = \varrho(q_1) = \varepsilon$, $\lambda(q_1) = b$ *with transitions* $E = \{(q_0, a, a, q_0), (q_1, a, a, q_1)\}$.
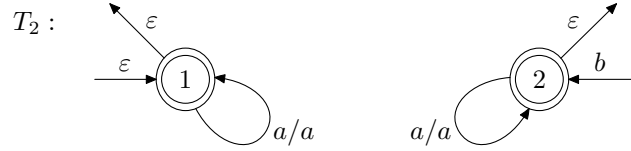
For every $n \geq 0$ *we have* $\lambda \cdot \theta(a^n) = (a^n, ba^n)$. *As above, it is obvious that every maximal factorization* (f, g) *satisfies* $f((a^n, ba^n)) = (a^n, ba^n)$. *Thus, the determinization of* T_2 *by some maximal factorization does not exist, although* T_2 *has the twins property and* $(\Delta^*, \min, \cdot, 0, \varepsilon)$ *is a min-semiring.*

It is an open question whether one can generalize Theorem 3.5 to idempotent commutative semirings. If $\mathbb{K}$ is an idempotent semiring, then $\oplus$ is the infimum over some partial ordering, i.e., we have a slightly weaker condition than in a min-semiring. However, in an idempotent semiring we can no longer assume the existence of victorious paths which is of crucial importance in the proof of Theorem 3.5 (e.g., for the existence of ν). On the other hand, we do not have a counterexample which shows that Theorem 3.5 cannot be generalized to idempotent semirings.

4 Unambiguous and Single-valued WFA

Subsequentially, we fix an alphabet Σ and an arbitrary semiring $\mathbb{K}$. We define here two classes of weighted finite automata and consider their relationships and properties. A WFA T is called *unambiguous* if every word $w \in \Sigma^*$ is the label of at most one successful path in T.

This is a proper extension of the strength of computability defined by subsequential WFA. However, not every formal power series is computable by an unambiguous WFA. It is known that there are recognizable formal power series which are not unambiguous, see e.g. [24, 20].

Example 4.1. *Let* $\Sigma = \{a, b\}$ *and set* $f(w) := \min\{|w|_a, |w|_b\}$ $(w \in \Sigma^*)$. *It defines a recognizable formal power series* $f : \Sigma^* \to \mathbb{T}$ *which cannot be computed by an unambiguous WFA.*

We still need some definitions and a normal form. Let $T = [Q, E, \lambda, \varrho]$ be a WFA. We call some state $q' \in Q$ *accessible* if there are $p, q \in Q$ and $u, v \in \Sigma^*$ satisfying $\lambda(p) \neq 0$, $\varrho(q) \neq 0$, and $p \overset{u}{\leadsto} q' \overset{v}{\leadsto} q \neq \emptyset$. We call T *trim* if every state in Q is accessible.

Remark 4.2. *Let $T = [Q, E, \lambda, \varrho]$ be a WFA such that we have $|T|(w) \neq 0$ for at least one nonempty word $w \in \Sigma^+$. There is a WFA $T' = [Q', E', \lambda', \varrho']$ such that:*

1. *We have $|T'|(w) = |T|(w)$ for every nonempty word $w \in \Sigma^+$, and $|T'|(\varepsilon) = 0$.*

2. *For every $p, q \in Q'$ and every $a \in \Sigma$, there is at most one $k \in \mathbb{K}$ with $(p, a, k, q) \in E'$.*

3. *We have $|Q'| \leq |Q| + 2$.*

4. *There are $p', q' \in Q'$ such that*

   (a) *$\lambda'[p'] = \varrho'[q'] = 1$.*
   (b) *For every $r \in Q$ with $r \neq p'$, we have $\lambda'[r] = 0$.*
   (c) *For every $r \in Q$ with $r \neq q'$, we have $\varrho'[r] = 0$.*
   (d) *$E' \subseteq (Q' \setminus \{q'\}) \times \Sigma \times (\mathbb{K} \setminus \{0\}) \times (Q' \setminus \{p'\})$.*

5. *T' is trim.*

Note that T' in Remark 4.2 is not necessarily subsequential, even if T is subsequential.

4.1   A cross-section construction

We introduce the following concept. We call a WFA $T = [Q, E, \lambda, \varrho]$ *single-valued* if for every $w \in \Sigma^*$ and any two successful paths $\pi \in i \overset{w}{\leadsto} f$ and $\pi' \in i' \overset{w}{\leadsto} f'$ of T we have

$$\lambda(i) \odot \sigma(\pi) \odot \varrho(f) = \lambda(i') \odot \sigma(\pi') \odot \varrho(f').$$

That is to say, for successful paths with identical labels, the weights including initial and final weights, coincide. In the case of the tropical or a min-semiring it simply means that the weight of a successful path together with the respective initial and final weight is equal to the value for the considered label defined by $|T|$.

This provides an extension of the notion of unambiguous WFA. However, the construction in this section is used to show that these WFA admit equivalent unambiguous ones. Hence the power of computability of the two classes of formal power series coincide. To prove Theorem 4.4 we will need the cross section theorem due to EILENBERG [11] (cf. also [4]).

Proposition 4.3 (Eilenberg). *Let Σ and Δ be alphabets. For a morphism $\alpha : \Sigma^* \to \Delta^*$ and any recognizable language $A \subseteq \Sigma^*$ there exists a recognizable language $B \subseteq A$ such that α maps bijectively B onto $\alpha(A)$.*

Given a single-valued WFA T we use Proposition 4.3 to construct a cross section of the set of all successful paths in T that contains for every label exactly one of the original successful paths. However, we have to restrict ourselves to idempotent semirings.

Theorem 4.4. *Let $\mathbb{K}$ be an idempotent semiring and $T = [Q, E, \lambda, \varrho]$ a single-valued WFA over $\mathbb{K}$. There exists an equivalent unambiguous WFA for T.*

Proof. Firstly, we treat the case where $|T|(\varepsilon) = 0$. Let $T = [Q, E, \lambda, \varrho]$ be a single-valued WFA. We may assume that T has the normal form of Remark 4.2, in particular 4.(d), and only has accessible states. Let $\Sigma, \{i\}$ and $\{f\}$ be the considered alphabet, initial and accepting states of T, respectively. We define

$$R := \{k \in \mathbb{K} \mid \exists\, p, q \in Q, a \in \Sigma,\ (p, a, k, q) \in E\},$$

the set of all weights of transitions in E. Then $E \subseteq Q \times \Sigma \times R \times Q$ can be seen as an alphabet. We obtain the morphism $\alpha : E^* \to \Sigma^*$ needed for Proposition 4.3 by the natural extension of

$$\alpha(p, a, k, q) := a, \ \alpha(\varepsilon) := \varepsilon$$

to words. Thus, a path in T simply is mapped to its label. Let

$$S := \big\{ (p, a, k, p')(q, a', k', q') \in E^2 \mid p' \neq q \big\} \subseteq E^*$$

be the set of all words over E with length 2 which are not successive. The set of all successful paths in T

$$P := \Big[\big(\{i\} \times \Sigma \times R \times Q \big) E^* \cap E^* \big(Q \times \Sigma \times R \times \{f\} \big) \Big] \setminus E^* S E^*,$$

is recognizable in E. Then $\alpha(P)$ contains the labels of successful paths.

Due to Proposition 4.3 there exists a recognizable language $B \subseteq P$, such that α maps B bijectively onto $\alpha(P)$. Now, let $A = [Q_A, E_A, i_A, F_A]$ be a deterministic finite automaton for B over the alphabet E. We define a WFA $T' = [Q_A, E', \lambda', \varrho']$, where $\lambda'(i_A) = 1$, $\lambda'(q) = 0$ for $q \in Q_A \setminus \{i_A\}$, $\varrho'(q) = 1$ for $q \in F_A$, and $\varrho'(q) = 0$ for $q \in Q_A \setminus F_A$. We define E' by:

$$E' := \big\{ (p, a, k, q) \mid \exists \, z_1, z_2 \in Q_A \text{ such that } (p, (z_1, a, k, z_2), q) \in E_A \big\}. \tag{7}$$

Since α maps B bijectively onto $\alpha(P)$ the constructed WFA T' is unambiguous. We have to show that T' and T are equivalent.

Let $w = a_1 a_2 \ldots a_n$ be an element of $\alpha(P)$. Since T is single-valued and $\mathbb{K}$ is an idempotent semiring there exists a path

$$\pi_T = (z_0, a_1, k_1, z_1)(z_1, a_2, k_2, z_2) \ldots (z_{n-1}, a_n, k_n, z_n) \in B$$

which satisfies $\alpha(\pi_T) = w$ und $|T|(w) = \bigodot_{1 \leq l \leq n} k_l$. There is a path

$$\pi_A = (q_0, (z_0, a_1, k_1, z_1), q_1)(q_1, (z_1, a_2, k_2, z_2), q_2) \ldots (q_{n-1}, (z_{n-1}, a_n, k_n, z_n), q_n)$$

in A with label π_T. Because of the construction of E' in (7) we find

$$\pi_{T'} = (q_0, a_1, k_1, q_1)(q_1, a_2, k_2, q_2) \ldots (q_{n-1}, a_n, k_n, q_n)$$

as a path in T' such that $|T'|(w) = \bigodot_{1 \leq l \leq n} k_l = |T|(w)$.

Conversely, if $w = a_1 a_2 \ldots a_n$ is a label of a successful path in T', there exists a path

$$\pi_{T'} = (q_0, a_1, k_1, q_1)(q_1, a_2, k_2, q_2) \ldots (q_{n-1}, a_n, k_n, q_n)$$

in T'. The word w is mapped to $\bigodot_{1 \leq l \leq n} k_l$ by T'. Again following the construction of E', there are states $z_0, z_1, \ldots, z_n$ in Q and a path

$$\pi_A = (q_0, (z_0, a_1, k_1, z_1), q_1)(q_1, (z_1, a_2, k_2, z_2), q_2) \ldots (q_{n-1}, (z_{n-1}, a_n, k_n, z_n), q_n)$$

in A for $\pi_{T'}$. Since A recognizes $B \subseteq P$, we have $(z_0, a_1, k_1, z_1) \ldots (z_{n-1}, a_n, k_n, z_n) \in B$ and this is a successful path in T. It follows that $|T|(a) = \bigodot_{1 \leq l \leq n} k_l = |T'|(a)$, because T is single-valued and $\mathbb{K}$ is idempotent.

If we consider the second case $|T|(\varepsilon) \neq 0$, we can construct a normal form T_1 from T using Remark 4.2, so, it holds $|T_1|(w) = |T|(w)$ for every nonempty word $w \in \Sigma^+$ and $|T_1|(\varepsilon) = 0$. Following the proof in the first case above, we obtain an equivalent unambiguous WFA $T_2 = [Q_2, E_2, \lambda_2, \varrho_2]$ for T_1. Now we define a WFA T' by adding a new state q and extending $\lambda_2(q) := 1$ and $\varrho_2(q) := |T|(\varepsilon)$. $\qquad\square$

Note that Theorem 4.4 is already known for transducers over the tropical semiring and for certain string-to-string transducers [29, 15, 19] even in a stronger fashion [31]. However, these approaches rely on the cancellativity of the multiplication in the semiring while our approach just requires idempotency of the semiring addition.

Also, note that every min-semiring is idempotent. So especially for the min-semirings, introduced in Section 3.6, one can apply Theorem 4.4.

As already mentioned, the proof of Theorem 4.4 relies on the cross section theorem. We could give an alternative proof for this theorem which utilizes a so-called immersion (morphism with special conditions) of an (non-weighted) automaton due to SCHÜTZENBERGER and SAKAROVITCH [27]. We briefly describe the main steps of this proof. For a single-valued WFA S we consider the underlying (non-weighted) automaton $\mathcal{A}$. With [27, Theorem 3], for $\mathcal{A}$ there exists an equivalent unambiguous automaton $\mathcal{B}$ and an immersion from $\mathcal{B}$ into $\mathcal{A}$, mapping edges in $\mathcal{B}$ to edges in $\mathcal{A}$ and states in $\mathcal{B}$ to states in $\mathcal{A}$. This morphism has certain compatibility properties. We want to dispose weights along $\mathcal{B}$. For this, to an edge in $\mathcal{B}$ we associate the weight of the image of this edge under the immersion considered not in $\mathcal{A}$ but in the originally given weighted automaton S. We call the resulting WFA S'. Now, very similar to the part of the first proof showing $|T| = |T'|$, using also the idempotency of the semiring and properties of the immersion, it is possible to prove the equivalence of the constructed WFA S' and S.

4.2   Weighted finite automata over the tropical semiring

In this section we consider WFA over the tropical semiring $\mathbb{T}$. We always consider determinization with respect to the maximal factorization by MOHRI, as explained in Section 3.5. If not indicated otherwise, the underlying alphabet and the set of transitions are Σ and E, respectively. For the class of single-valued and trim WFA we show that the determinization is optimal in the sense that for a given single-valued and trim WFA T the determinization is defined iff T is determinizable, i.e., there exists an equivalent subsequential WFA for T. This extends MOHRI [25] from unambiguous to single-valued WFA. Furthermore, we show that it is decidable whether a single-valued WFA over $\mathbb{T}$ computes a subsequential formal power series.

On Σ^* one can define a metric d by setting

$$\forall u, v \in \Sigma^* : d(u, v) = |u| + |v| - 2|u \wedge v|.$$

For two reals r_1, r_2 the euclidean metric is denoted by $|r_1 - r_2|$.

A partial function $\alpha : \Sigma^* \dashrightarrow \mathbb{R}_+$ has *bounded variation* if for all $k \geq 0$ there is a $K \geq 0$ such that for all $u, v \in \Sigma^*$ with $\alpha(u), \alpha(v) \in \mathbb{R}_+$ the following holds:
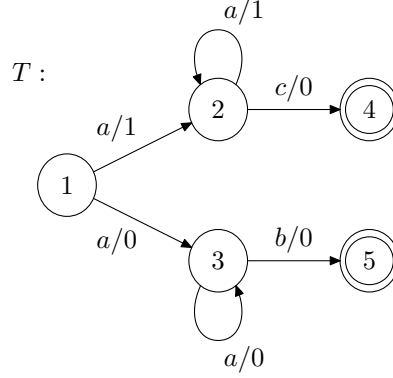
$$d(u, v) \leq k \Rightarrow |\alpha(u) - \alpha(v)| \leq K.$$

Lemma 4.5 ([25])**.** *Every subsequential formal power series has bounded variation.*

On the other hand, the formal power series of an unambiguous WFA does not necessarily have bounded variation, as the following example shows:

Example 4.6. *The formal power series computed by the following unambiguous WFA T does*

not have bounded variation:

In [25, p. 283, Th. 9] Mohri claims a characterization of recognizable power series which are subsequential by saying a recognizable series is subsequential if and only if it has bounded variation. Lemma 4.7 shows the incorrectness of his claim.

Lemma 4.7. *The function f of Example 4.1 is a recognizable formal power series, has bounded variation, but it is not subsequential.*

Proof. By Example 4.1, it remains to show that f has bounded variation. Let $w \in \Sigma^*$ be arbitrary. By distinguishing the cases $|w|_a < |w|_b$ and $|w|_a \geq |w|_b$, we obtain $f(w) \leq f(wa) \leq f(w) + 1$ and $f(w) \leq f(wb) \leq f(w) + 1$. By an induction on the length of $w' \in \Sigma^*$, we have

$$f(w) \leq f(ww') \leq f(w) + |w'|, \quad \text{i.e.,} \quad 0 \leq f(ww') - f(w) \leq |w'|.$$

Let $w, u, v \in \Sigma^*$ be arbitrary such that $w = wu \wedge wv$, and thus, $d(wu, wv) = |u| + |v|$. We have

$$|f(wu) - f(wv)| \leq |f(wu) - f(w)| + |f(wv) - f(w)| \leq |u| + |v| = d(wu, wv).$$

Hence, f satisfies the definition of bounded variation for $K := k$.   □

On the other hand, if the considered WFA is single-valued, trim and has the property of bounded variation, then it has the twins property and hence its determinization is defined. Therefore, we have the following propositions.

Proposition 4.8. *Every trim and single-valued WFA computing a formal power series with bounded variation has the twins property.*

Proof. Let $T = [Q, E, \lambda, \varrho]$ be a trim and single-valued WFA such that $|T|$ has bounded variation and let I, F in T be the set of initial and accepting states, respectively.

Consider $q_1, q_2 \in Q, i_1, i_2 \in I$ and $u, v \in \Sigma^*$ such that

$$i_1 \overset{u}{\leadsto} q_1 \neq \emptyset \ \wedge \ i_2 \overset{u}{\leadsto} q_2 \neq \emptyset \ \wedge \ q_1 \overset{v}{\leadsto} q_1 \neq \emptyset \ \wedge \ q_2 \overset{v}{\leadsto} q_2 \neq \emptyset.$$

Since T is trim and has bounded variation, there are $w_1, w_2 \in \Sigma^*, f_1, f_2 \in F$ and a $K \geq 0$ such that

$$q_1 \overset{w_1}{\leadsto} f_1 \neq \emptyset \ \wedge \ q_2 \overset{w_2}{\leadsto} f_2 \neq \emptyset \ \wedge \ \Big[\forall k \geq 0 : \big| |T|(uv^k w_1) - |T|(uv^k w_2) \big| \leq K \Big].$$

T is single-valued and trim, hence all paths between two states over identical words have the same weight:

$$|T|(uv^k w_1) = \lambda(i_1) + \theta(u)[i_1, q_1] + \theta(w_1)[q_1, f_1] + k \cdot \theta(v)[q_1, q_1] + \varrho(f_1)$$
$$=: k \cdot \theta(v)[q_1, q_1] + C_1$$
$$|T|(uv^k w_2) = \lambda(i_2) + \theta(u)[i_2, q_2] + \theta(w_2)[q_2, f_2] + k \cdot \theta(v)[q_2, q_2] + \varrho(f_2)$$
$$=: k \cdot \theta(v)[q_2, q_2] + C_2$$

$$\Longrightarrow \forall k \geq 0 : \left| C_1 - C_2 + k\big(\theta(v)[q_1, q_1] - \theta(v)[q_2, q_2]\big) \right| \leq K$$
$$\Longrightarrow \left| \theta(v)[q_1, q_1] - \theta(v)[q_2, q_2] \right| = 0 \qquad \qquad \square$$

Now, for the considered class of formal power series computed by single-valued WFA, we can show that the properties of bounded variation and subsequentiality coincide.

Proposition 4.9. *Let f be a formal power series. The following two conditions are equivalent:*

1. *f is subsequential.*

2. *f has bounded variation and is computable by a single-valued WFA.*

Proof. Every subsequential (and therefore single-valued) WFA for f computes a formal power series with bounded variation (Lemma 4.5). Conversely, if f is computable by a single-valued WFA, by Proposition 4.8 this WFA has the twins property. Applying Theorem 3.5, we get $(2) \Rightarrow (1)$. $\qquad \square$

Theorem 4.10. *Let T be a single-valued and trim WFA. The following conditions are equivalent:*

1. *$|T|$ is subsequential.*

2. *The determinization of T is defined.*

3. *T has the twins property.*

4. *$|T|$ has bounded variation.*

Proof. The equivalence follows from Proposition 4.8, Theorems 3.5 and 3.1, and Lemma 4.5. $\qquad \square$

4.3   Decidable properties

It is decidable in polynomial time whether a transducer has the twins property [1, 3]. We apply techniques from [1, 3] to show that it is decidable in polynomial time whether a given WFA over the tropical semiring is single-valued.

Throughout this section, let $T = [Q, E, \lambda, \varrho]$ be a WFA over the tropical semiring and denote by I resp. F the sets of initial resp. accepting states of T.

We call two states $q, q' \in Q$ *co-twins* if there exists a word $w \in \Sigma^*$ such that both $q \overset{w}{\leadsto} F$ and $q' \overset{w}{\leadsto} F$ are non-empty. We denote the set of all co-twins of T by $\mathcal{C}$.

Now, let $\mathcal{F}$ be the least subset of $\mathcal{C} \times \mathbb{R}$ such that

1. for every $q, q' \in I$, $(q, q') \in \mathcal{C}$, we have $\big(q, q', \lambda(q) - \lambda(q')\big) \in \mathcal{F}$ and

2. for every $(p, p', k) \in \mathcal{F}$, $(q, q') \in \mathcal{C}$, (p, a, l, q), $(p', a, l', q') \in E$, we have $(q, q', k+l-l') \in \mathcal{F}$.

Lemma 4.11. *With the preceding terminology, the following two assertions are equivalent:*

1. *T is not single-valued.*

2. *There is some $(q, q', k) \in \mathcal{F}$ such that $q, q' \in F$ and $\varrho(q') - \varrho(q) \neq k$.*

Moreover, these conditions are satisfied if

$$(*) \;\; \text{there are } (q, q', k_1) \in \mathcal{F} \text{ and } (q, q', k_2) \in \mathcal{F} \text{ such that } k_1 \neq k_2.$$

Proof. (1) $\Rightarrow$ (2) By (1), there is some word $w \in \Sigma^*$ and there are $q_0, q_0' \in I$, $q_{|w|}, q_{|w|}' \in F$, and paths $\pi \in q_0 \overset{w}{\leadsto} q_{|w|}$, $\pi' \in q_0' \overset{w}{\leadsto} q_{|w|}'$ such that

$$\lambda(q_0) + \theta(\pi) + \varrho(q_{|w|}) \;\neq\; \lambda(q_0') + \theta(\pi') + \varrho(q_{|w|}'),$$

and hence,

$$\lambda(q_0) + \theta(\pi) - \lambda(q_0') - \theta(\pi') \;\neq\; \varrho(q_{|w|}') - \varrho(q_{|w|}). \tag{8}$$

For $0 \le i < |w|$, we denote the i-th transition of π resp. π' by (q_i, a_i, k_i, q_{i+1}) resp. $(q_i', a_i, k_i', q_{i+1}')$.

For every $0 \le i \le |w|$, we have $(q_i, q_i') \in \mathcal{C}$. Moreover, $(q_0, q_0', \lambda(q_0) - \lambda(q_0')) \in \mathcal{F}$, and by an induction on i, we can show that for every $0 \le i \le |w|$,

$$\big(q_i,\, q_i',\, \lambda(q_0) + \theta(\pi(0,i)) - \lambda(q_0') - \theta(\pi'(0,i))\big) \;\in\; \mathcal{F}.$$

The last statement for $i = |w|$, $q := q_{|w|}$, and $q' := q_{|w|}'$ together with equation (8) proves (2).

(2) $\Rightarrow$ (1) By induction, we can show that for every triple $(q, q', k) \in \mathcal{F}$ there are states $p, p' \in I$, some word $w \in \Sigma^*$, and paths $\pi \in p \overset{w}{\leadsto} q$, $\pi' \in p' \overset{w}{\leadsto} q'$ such that

$$\lambda(p) + \theta(\pi) - \lambda(p') - \theta(\pi') \;=\; k.$$

Let (q, q', k) as in (2). By considering w, p, p', π, and π' as above, we conclude

$$\lambda(p) + \theta(\pi) + \varrho(q) - \lambda(p') - \theta(\pi') - \varrho(q') \;\neq\; 0,$$

and hence, T is not single-valued.

Finally, assume that $(*)$ is satisfied and let $(q, q', k_1) \in \mathcal{F}$ and $(q, q', k_2) \in \mathcal{F}$ such that $k_1 \neq k_2$. Since $(q, q') \in \mathcal{C}$, there are some word $w \in \Sigma^*$, states $p, p' \in F$, and paths $\pi \in q \overset{w}{\leadsto} p$, $\pi' \in q' \overset{w}{\leadsto} p'$. By an induction on these paths, we can show that both $(p, p', k_1 + \theta(\pi) - \theta(\pi'))$ and $(p, p', k_2 + \theta(\pi) - \theta(\pi'))$ belong to $\mathcal{F}$. Since $k_1 \neq k_2$, one of these triples proves (2). $\qquad\square$

Condition $(*)$ in Lemma 4.11 will be crucial for the complexity of an algorithm which explores the set $\mathcal{F}$ to decide whether T is single-valued.

Theorem 4.12. *Let T be a WFA over the tropical semring. It is decidable in polynomial time whether T is single-valued.*

Proof. At first, the algorithm computes the set $\mathcal{C}$. This is possible in polynomial time since $\mathcal{C}$ is the least subset of $Q \times Q$ which contains $F \times F$ and is closed as follows: for every $(q, q') \in \mathcal{C}$ and transitions $(p, a, k, q), (p', a, l, q') \in E$, we have $(p, p') \in \mathcal{C}$.

Then, the algorithm generates the set $\mathcal{F}$. Whenever the algorithm produces a new triple in $\mathcal{F}$, it checks whether condition (2) or $(*)$ in Lemma 4.11 is satisfied. If so, then T is not single-valued. If the algorithm computes the entire set $\mathcal{F}$, and both condition (2) and $(*)$ in Lemma 4.11 are not satisfied, then T is single-valued.

The set $\mathcal{F}$ is possibly infinite. However, in every subset $\mathcal{F}$ which consists of more that $|\mathcal{C}|$ triples, there are two triples as in condition $(*)$. Hence, the algorithm generates at most $|\mathcal{C}| + 1 \le |Q|^2 + 1$ triples, i.e., it terminates in polynomial time. $\qquad\square$

Acknowledgment

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