GEORGE BOOLOS

DON'T ELIMINATE CUT[1]

The method of trees, as presented in e.g. Jeffrey's *Formal Logic: its Scope and Limits*[2] and Smullyan's *First-order Logic*,[3] and standard systems of natural deduction, like the one given in Mates's *Elementary Logic*,[4] are sound and complete methods of logic in the usual sense: they mark an inference as valid if and only if it is valid. There is, however, a significant difference between them in the *manner* in which they can demonstrate validity, a difference that sometimes results in a striking disparity in the *efficiency* or *speed* with which an inference can be shown to be valid. Although a tree that demonstrates the validity of an inference, i.e., a closed tree with the premisses and denial of the conclusion at its top, can be transformed into a natural deduction of the conclusion from the premisses that it requires approximately the same amount of time to write down, the converse, as we shall see, is emphatically not the case.

It is immediate from the presentation of a standard system of natural deduction such as Mates's that if $(A \rightarrow B)$ and A are derivable in the system, then so is B. On the other hand, it cannot be seen without a considerable amount of work that if there are closed trees for both $(A \rightarrow B)$ and A, then there is also one for B. Thus *modus ponens*, or *cut*, is obviously a valid derived rule of standard natural deduction systems, but *not obviously* a valid derived rule of the method of trees. It is well known, in a general way, that the elimination of cuts from derivations in a system in which cuts are always eliminable can greatly increase the length of derivations. But in view of the efficacy of the method of trees when applied to the usual sorts of examples and exercises found in logic texts, one might think that the danger of encountering a valid inference whose validity cannot feasibly be demonstrated by the method of trees is rather remote.

Not so. There is a simple inference that can be shown valid by means of a deduction in Mates's system whose every symbol can be written down in one or two pages of normally sized type or handwriting, but for which the smallest closed tree contains more symbols than there are nanoseconds between Big Bangs.

Journal of Philosophical Logic **13** (1984) 373–378. 0022–3611/84/0134–0373$00.60.

In fact, let H_n ('H' for 'heap') be the inference whose premisses are

$$(x)(y)(z)+x+yz = ++xyz$$

$$(x)\,dx = +xx$$

$$L1$$

$$(x)(Lx \rightarrow L+x1)$$

and whose conclusion is the sentence consisting of L, followed by $2^\wedge n$ consecutive occurrences of d, followed by 1. ('^' denotes exponentiation.) Thus, for example, the conclusion of H_3 is: *Ldddddddd*1. We shall show that the shortest tree-method proof of the validity of H_n contains $> 2^\wedge(2^\wedge n)$ characters and that the shortest natural deduction of the conclusion of H_n from the premisses contains $< 16(2^\wedge n + 8n + 21)$ characters. Thus the smallest closed tree for H_7 contains $> 2^\wedge 128 > 10^\wedge 38$ characters, but the smallest natural deduction for H_7 contains < 3280 characters, or, at 5 characters per word and 400 words per page, a bit more than a page and a half.

The extent to which this result provides a reason for favoring natural deduction over trees is an issue that we shall discuss after we have proved our claim about H_n. We first show that any closed tree for H_n contains more than $2^\wedge(2^\wedge n)$ symbols.

Let T be a tree with the premisses and denial of the conclusion of H_n at its top. Define an interpretation I as follows: The domain of I is the set of positive integers. I assigns one to 1, the function $x \mapsto 2x$ to d, the addition function to + (and an arbitrary n-ary function to any other n-place function sign). Thus the first two premisses of H_n are true under I.

Let den(s) be the denotation under I of the (closed) term s. We call a positive integer i *instanced* if for some term s, den(s) = i and the sentence $(Ls \rightarrow L+s1)$ occurs in T.

Finally, I specifies that L applies to a positive integer j iff all positive integers *less than* j are instanced.

Thus the premiss $L1$ is (trivially) true under I.

We now want to see that every sentence $(Lt \rightarrow L+t1)$ *that occurs in* T is also true under I. Assume that $(Lt \rightarrow L+t1)$ occurs in T. Let j = den(t). Suppose Lt true under I. Then all $i < j$ are instanced. But since $(Lt \rightarrow L+t1)$ occurs in T, j is also instanced. Thus all $i < j + 1$ are instanced. And since den($+t1$) = j+1, $L+t1$ is also true under I. Thus if $(Lt \rightarrow L+t1)$ occurs in T, it is true under I.

Let u be the term in the denial of the conclusion and let $k = \text{den}(u) = 2^\wedge(2^\wedge n)$. We want to show that if some $j < k$ is not instanced, then T is open. It will then follow that if T is closed, for every $j < k$ there is a term t such that $\text{den}(t) = j$ and $(Lt \rightarrow L{+}t1)$ occurs in T. As it is clear that if $\text{den}(s) = i$, $\text{den}(t) = j$ and $i \neq j$, then $s \neq t$ and $(Ls \rightarrow L{+}s1) \neq (Lt \rightarrow L{+}t1)$, it will also follow that if T is closed, at least $k - 1$ sentences of the form $(Lt \rightarrow L{+}t1)$ occur in T, and therefore T contains more than k symbols.

Accordingly, suppose that some $j < k$ is not instanced, but that T is closed. We shall obtain a contradiction.

Since some $j < k$ is not instanced, the denial $-Lu$ of the conclusion is true under I, as are $(x)(y)(z) +x{+}yz = {+}{+}xyz$, $(x)\, dx = +xx$, and $L1$.

And since T is closed, each of its branches contains some sentence and its denial. Now the only sentences that can occur in any branch of T are the premisses, the denial of the conclusion, and sentences of the forms $(y)(z)s''(y,z) = t''(y,z)$, $(z)s'(z) = t'(z)$, $s = t$, $(Ls \rightarrow Lt)$, Lt, and $-Lt$, for the tree rules (i.e., the standard tree rules, which do *not* include the rule *XM* discussed below) do not lead out of this collection of sentences. Thus as T is closed, each of its branches must contain some sentence Lt and its denial $-Lt$.

This is impossible, however, for at every stage of the construction of T, there is at least one branch in which all sentences *other than* $(x)(Lx \rightarrow L{+}x1)$ are true under I. (Cf. the usual soundness proof for the method of trees.) This is certainly the case at the beginning of the construction of T (when there is only one branch, consisting of the premisses and denial of the conclusion), and, inductively, remains the case throughout the construction of T, since the only tree rules relevant to T are *UI*, equals for equals, and the rule for the undenied conditional, and these all preserve truth under I. (The statement that the rule for the undenied conditional, which is a branching rule, preserves truth under I means, of course, that if the premiss $(A \rightarrow B)$ of the rule is true under I, then either the left conclusion $-A$ or the right conclusion B is true under I). The only problematical case in the induction step of the argument is that in which the sentence $(x)(Lx \rightarrow L{+}x1)$ is a premiss of a rule of inference. But if an identity $s = t$ occurs in a branch, then the term s begins with $+$ or d, as does the term t, and thus neither s nor t is the term 1. Therefore the only rule relevant to $(x)(Lx \rightarrow L{+}x1)$ is *UI*, which when applied to this sentence yields a conclusion $(Lt \rightarrow L{+}t1)$. And as we saw earlier, any sentence of this form that occurs in T is automatically true under I.

Therefore if T is closed, T contains more than $2^\wedge(2^\wedge n)$ symbols.

The other half of our task is to bound the number of symbols in the shortest deduction in Mates's system of the conclusion of H_n from its premisses. Let '$d[n]$' denote the sequence consisting of n occurrences of d. Then the conclusion of H_n may be written: $Ld[2^\wedge n]1$. And let '$M(y)$' abbreviate the formula: $(Ly \,\&\, (x)(Lx \rightarrow L{+}xy))$. Thus $M(1)$ is the conjunction of the last two premisses.

One deduction for H_n begins with a subdeduction from the premisses of the two sentences

$$(M(a) \rightarrow M(da))$$

$$(y)(M(y) \rightarrow M(dy))$$

The cost so far is slightly more than 300 symbols. Then follow n trios of lines

$$(M(d[2^\wedge(i-1)]a) \rightarrow M(d[2^\wedge i]a))$$

$$(M(a) \rightarrow M(d[2^\wedge i]a))$$

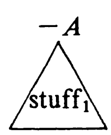
$$(y)(M(y) \rightarrow M(d[2^\wedge i]y)) \quad (1 \leqslant i \leqslant n),$$

costing in all $14 \times 2^\wedge n + 114n - 14$ symbols. $(M(1) \rightarrow M(d[2^\wedge n]1))$ is then inferred by US $(= UI)$, and the conclusion $Ld[2^\wedge n]1$ then follows by T. The total number of symbols in this deduction, according to my count, is $16 \times 2^\wedge n + 114n + 329$. If $n = 7$, this number is 3175.

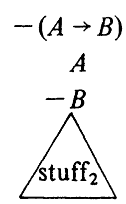
There are some annotative comments on the deduction that it may be helpful to make. The associativity of addition is needed to deliver $(x)(Lx \rightarrow L{+}xda)$ from $(x)(Lx \rightarrow L{+}xa)$; without it we could only obtain $(x)(Lx \rightarrow L{+}{+}xaa)$ (and not $(x)(Lx \rightarrow L{+}x{+}aa)$). Apart from this use of associativity, the subdeduction that ends with the line $(M(a) \rightarrow M(da))$ is a "self-proving" conditionalization on $M(a)$. The first line of the ith trio follows by US from the line $(y)(M(y) \rightarrow M(d[2^\wedge(i-1)]y))$ immediately above it. The middle line follows by T from the first line of the trio and the line $(M(a) \rightarrow M(d[2^\wedge(i-1)]a))$, which is two lines above the first line. Finally, the last line of the trio follows from UG from the middle line.

It should be emphasized that the contrast we have drawn is between *standard formulations* of the method of trees and those of natural deduction. We have been supposing that the method of trees is so formulated that the version of cut appropriate to trees, which Jeffrey calls XM (excluded middle),[5] and which allows one to split any open branch of a tree in two and append any sentence at all to the bottom of one of the new branches and the negation of the sentence to the bottom of the other, is not one of the (underived) rules of the method. Were *XM* present, one could write down a closed tree for H_n containing approximately the same number of symbols as the natural deduction for H_n that we have just given. It is somewhat ironic that it is the failure of the usual formulations of the tree method to permit a certain sort of ***branching*** that can be blamed for their inefficiency in treating inferences like H_n.

The most significant feature possessed by natural deduction but not the method of trees, a feature that can easily seem like a virtue, is not so much that natural deduction replicates ordinary reasoning rather more faithfully than the tree method, in which derivations are one and all given the unnatural shape of proofs by contradiction, quantifier stripping and cases, but that it permits the development and utilization within derivations of ***subsidiary*** conclusions, or, as they would be called in a more informal setting, *lemmas.* In criticizing a certain pair of systems, Feferman once wrote, ". . . nothing like sustained ordinary reasoning can be carried on in either logic."[6] Sustained ordinary reasoning cannot be carried on in a tree system unsupplemented by *XM*, where we are unable to appeal to previously established conclusions. This difficulty with the method of trees, if it is a difficulty, is one for which an obvious remedy exists: add *XM*. Then from closed trees

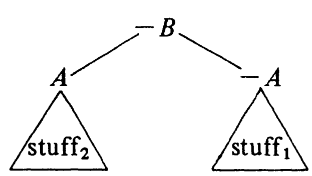


and



for A and $(A \to B)$

one can use *XM* to obtain, immediately, a closed tree for B:



Of course, whether one should favor, adopt, or teach systems in which sustained ordinary reasoning, or rather, a highly idealized version of it, can or cannot be carried out are practical or normative questions on which other features of the systems may bear. The result about feasibility given above hardly decides these issues.

NOTES AND REFERENCES

[1] The contents of this paper were presented at the 1983 AMS Special Session on Proof Theory.

[2] Richard Jeffrey, *Formal Logic: its Scope and Limits*, second edition, McGraw-Hill Book Company, New York (1981).

[3] Raymond M. Smullyan, *First-order Logic*, Springer-Verlag New York Inc., New York (1968).

[4] Benson Mates, *Elementary Logic*, second edition, Oxford University Press, New York (1972).

[5] Jeffrey, *op. cit.*, pp. 34–35.

[6] 'Toward useful type-free theories, I', to appear in *Journal of Symbolic Logic*.

Department of Linguistics and Philosophy,
Massachusetts Institute of Technology,
Cambridge,
MA 02139, U.S.A.