Math. Z. 215, 347–365 (1994)

Mathematische
Zeitschrift

Free hyperplane arrangements between A_{n-1} and B_n

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Received 8 December 1992; in final form 26 January 1993

1 Introduction

Much of the motivation for the study of arrangements of hyperplanes comes from the study of Coxeter arrangements. The sub-arrangements of Coxeter arrangements are less well understood. In the case of the root system A_{n-1} the sub-arrangements are the graphic arrangements, and using this connection to graphs one can analyze these arrangements completely from the perspective of free-ness (e.g. Theorem 3.3). Zaslavsky has developed a theory of signed graphs [Za] which, in principle, should allow one to analyze the sub-arrangements of B_n in a similar way. No one has yet been able to put that into practice.

For the class of sub-arrangements of B_n that contain A_{n-1} there is again a natural connection to graphs. It is this class of arrangements that we study. In this paper we describe completely which of these arrangements are free in a manner analogous to the situation for sub-arrangements of A_{n-1}. We were partly motivated by the recent work of Józefiak and Sagan [JS] on a certain sub-class of these free arrangements.

The paper is organized as follows. In the next section we collect the necessary known results concerning arrangements and establish graph-theoretic terminology. In §3 we characterize the sub-arrangements of A_{n-1} that are free, namely those parameterized by chordal graphs. For the most part these results are known, but they are not collected anywhere. The techniques employed in this section also serve to foreshadow those in §4. In §4 we classify the sub-arrangements of B_n that contain A_{n-1} and are free. They are parameterized by threshold graphs. We also show which of these free arrangements are supersolvable. A by-product of our analysis is an infinite collection of arrangements that are counterexamples to Orlik's conjecture. In addition we are able to enumerate all of the induction tables from A_{n-1} to B_n.

* Partially supported by NSF grant DMS-9201490
** Partially supported by an NSF Postdoctoral Fellowship

2 Arrangements and graphs

In this section, we review the necessary definitions, notation, and theorems from the theory of hyperplane arrangements and graphs. An excellent reference on hyperplane arrangements is the book by Orlik and Terao [OT], and we will mostly use their terminology.

A (*real*) *hyperplane arrangement* $\mathscr{A}$ is a collection of hyperplanes (codimension one vector subspaces) in an n-dimensional real vector space. A hyperplane H is defined as the kernel of a linear form l_H in the variables $x_1, \ldots, x_n$, and we will often identify the hyperplane H with this linear form. The *rank* of $\mathscr{A}$ is the dimension of the vector space spanned by these linear forms.

The *intersection lattice* $L(\mathscr{A})$ is the poset of subspaces X obtained as intersections of the hyperplanes in $\mathscr{A}$, ordered under *reverse* inclusion. Given a subspace $X \in L(\mathscr{A})$, there are two smaller arrangements associated to it. The *restriction* $\mathscr{A}^X$ is the arrangement of hyperplanes

$$\mathscr{A}^X = \{X \cap H : H \in \mathscr{A}\}$$

within the vector space X. The *localization* $\mathscr{A}_X$ is the sub-arrangement

$$\mathscr{A}_X = \{H \in \mathscr{A} : X \subseteq H\}.$$

The *characteristic polynomial* $\chi(\mathscr{A}, t)$ is defined by

$$\chi(\mathscr{A}, t) = \sum_{X \in L(\mathscr{A})} \mu(X)\, t^{\dim(X)}$$

where $\mu(X)$ denotes the *Möbius function* (see [OT, §2.2]) of $L(\mathscr{A})$.

The *module of $\mathscr{A}$-derivations* $D(\mathscr{A})$ is defined as follows. Let $S = \mathbb{R}[x_1, \ldots, x_n]$ be the polynomial ring, Q the product of all defining linear forms l_H for $H \in \mathscr{A}$, and QS the principal ideal generated by Q in S. Then $D(\mathscr{A})$ is the set of all derivations $\theta\colon S \to S$ with the property that $\theta(Q) \in QS$. $D(\mathscr{A})$ is actually a module over the polynomial ring S. The arrangement $\mathscr{A}$ is called a *free* arrangement if $D(\mathscr{A})$ is a free S-module (necessarily of rank n). If $\mathscr{A}$ is free, one can always find a basis for $D(\mathscr{A})$ as a free S-module consisting of derivations $\theta_i = \sum f_{ij} \frac{\partial}{\partial x_j}$ in which the polynomials f_{ij} are homogeneous of degree b_i for each i. The degrees b_i are then called *exponents* of the arrangement $\mathscr{A}$, and we write

$$\exp \mathscr{A} = (b_1, \ldots, b_n).$$

We list some consequences of free-ness that will be used later.

Theorem 2.1 (Factorization Theorem) [OT, Theorem 4.6.21] *If $\mathscr{A}$ is a free arrangement with* $\exp \mathscr{A} = (b_1, \ldots, b_n)$, *then*

$$\chi(\mathscr{A}, t) = \prod_{i=1}^{n} (t - b_i).$$

Theorem 2.2 (Localization Theorem) [OT, Theorem 4.2.23] *If $\mathscr{A}$ is a free arrangement, then any localization $\mathscr{A}_X$ is also free.*

A sequence $(\mathscr{A}, \mathscr{A}', \mathscr{A}'')$ of arrangements is called a *triple* if there exists a hyperplane $H \in \mathscr{A}$ such that $\mathscr{A}' = \mathscr{A} - \{H\}$ and $\mathscr{A}'' = \mathscr{A}^H$.

Theorem 2.3 (Addition Theorem) [OT, Theorem 4.3.13] *If $(\mathscr{A}, \mathscr{A}', \mathscr{A}'')$ is a triple of arrangements, in which $\mathscr{A}'$ and $\mathscr{A}''$ are both free and* $\exp \mathscr{A}'' \subset \exp \mathscr{A}'$, *then $\mathscr{A}$ is free. Furthermore, if* $\exp \mathscr{A}' = (b_1, \ldots, b_{n-1}, b_n - 1)$ *and* $\exp \mathscr{A}'' = (b_1, \ldots, b_{n-1})$ *then* $\exp \mathscr{A} = (b_1, \ldots, b_{n-1}, b_n)$.

Theorem 2.4 (Deletion Theorem) [OT, Theorem 4.3.12] *If $(\mathscr{A}, \mathscr{A}', \mathscr{A}'')$ is a triple of arrangements, in which $\mathscr{A}$ and $\mathscr{A}''$ are both free and* $\exp \mathscr{A}'' \subset \exp \mathscr{A}$, *then $\mathscr{A}'$ is free. Furthermore, if* $\exp \mathscr{A} = (b_1, \ldots, b_{n-1}, b_n)$ *and* $\exp \mathscr{A}'' = (b_1, \ldots, b_{n-1})$ *then* $\exp \mathscr{A}' = (b_1, \ldots, b_{n-1}, b_n - 1)$.

If $\mathscr{A}$ and $\hat{\mathscr{A}}$ are free arrangements then we say that $\mathscr{A} \prec \hat{\mathscr{A}}$ if $|\mathscr{A}| + 1 = |\hat{\mathscr{A}}|$ and if their exponents can be indexed such that

$$\exp \hat{\mathscr{A}} = (b_1, \ldots, b_{n-1}, b_n)$$
$$\exp \mathscr{A} = (b_1, \ldots, b_{n-1}, b_n - 1).$$

An *induction table* between two free arrangements $\mathscr{A}$ and $\mathscr{B}$ is a sequence of free arrangements

$$\mathscr{A} = \mathscr{A}_0 \prec \mathscr{A}_1 \prec \ldots \prec \mathscr{A}_k = \mathscr{B}.$$

Note that this is a slightly different notion than the induction tables given in [OT, §4.3].

We next define supersolvable arrangements. Given a sub-arrangement $\mathscr{A}'$ of an arrangement $\mathscr{A}$, we say $\mathscr{A}'$ is a *modular coatom* of $\mathscr{A}$ if

(1) For all pairs of hyperplanes $H_1, H_2 \in \mathscr{A} - \mathscr{A}'$ there exists a hyperplane $H_3 \in \mathscr{A}'$ containing their intersection $H_1 \cap H_2$.
(2) rank $\mathscr{A}'$ = rank $\mathscr{A} - 1$.

An arrangement $\mathscr{A}$ is called *supersolvable* if there exists a chain of arrangements (called an M-chain)

$$\emptyset = \mathscr{A}_0 \subset \mathscr{A}_1 \subset \ldots \subset \mathscr{A}_r = \mathscr{A}$$

in which $\mathscr{A}_i$ is a modular coatom of $\mathscr{A}_{i+1}$ for $0 \leqq i \leqq r-1$. This definition is different than the one given in [OT, Definition 2.1.32], but is known to be equivalent [Te, Corollary 2.17].

Theorem 2.5 [OT, Theorem 4.3.21] *If $\mathscr{A}$ is supersolvable then $\mathscr{A}$ is free. Furthermore, $\mathscr{A}$ has exponents $(b_1, \ldots, b_r)$ where b_i is the cardinality of $\mathscr{A}_i - \mathscr{A}_{i-1}$.*

Theorem 2.6 [St1, Proposition 3.2] *If $\mathscr{A}$ is a supersolvable arrangement then any localization $\mathscr{A}_X$ is supersolvable.*

Lastly we collect some terminology and notation from graph theory. A *graph* $G = (V, E)$ is an ordered pair where

$$V = V(G) = \{1, 2, \ldots, n\} = [n]$$

and $E = E(G)$ is a collection of two-element subsets of V. The set V is called the *vertex set* and E is called the *edge set*. Let $E_n = ([n], \emptyset)$ be the *empty graph* on n vertices and K_n be the *complete graph* on n vertices.

Associated with each vertex $v \in V$ is its *neighborhood*

$$N(v) = \{j \in V \mid \{v, j\} \in E\}.$$

The cardinality of the set $N(v)$ is called the *degree* of v and is denoted $\deg(v)$. The *degree sequence* $d(G) = (d_1, \ldots, d_n)$ is the n-tuple of degrees of vertices in G in non-increasing order.

If $U \subseteq V$ and $F \subseteq E$ then we call $H = (U, F)$ a *subgraph* of G. Given $U \subseteq V$ let

$$E(U) = \{\{i, j\} \mid i, j \in U \text{ and } \{i, j\} \in E\}.$$

The graph $G_U = (U, E(U))$ is called the *(vertex)-induced subgraph of* U. If $v \in V$ then by $G - v$ we mean the induced subgraph G_{V-v}. If $e \in E$ then by $G - e$ we mean the graph whose vertex set is V and whose edge set is $E - e$.

A *cycle* C in a graph G is a sequence of vertices $v_0, v_1, \ldots, v_l$ such that $\{v_i, v_{i+1}\} \in E$ for $0 \leqq i \leqq l-1$ and $v_i \neq v_j$ for all i and j except that $v_0 = v_l$. A *chord* of C is an edge $\{v_i, v_j\}$ where v_i and v_j are not consecutive vertices on the cycle.

3 Subarrangements of A_n

In this section we discuss the sub-arrangements of A_n. We will show that all the free sub-arrangements of A_n are in fact supersolvable, and that they are parameterized by chordal graphs. We will also discuss what the induction tables for A_n look like. Most of what we prove in this section is already known, the main exception being Theorem 3.6 in which we characterize the induction tables from $\emptyset$ to A_{n-1}. We include the other results here for two reasons. First, some of these results are hard to find explicitly stated or proven in the literature. Second, the proofs in this section will act as a model for what follows. We begin with a brief review of chordal graphs and their relevant properties.

A graph $G(V, E)$ is said to be *chordal* if every cycle of G of length greater than 3 has a chord. Chordal graphs have been studied at great length. For a clear introduction see [Go].

Let $G = (V, E)$ be an arbitrary graph and $v \in V$. We will call v a *simplicial* vertex if $G_{N(v)}$ is a complete graph. A *vertex elimination order* for G is an ordering $v_1, \ldots, v_n$ of the vertices with the property that v_i is a simplicial vertex in the graph $G - \{v_1, \ldots, v_{i-1}\}$ for all $1 \leqq i \leqq n-1$. The next theorem characterizes exactly when a graph has a vertex elimination ordering.

Theorem 3.1 [FG] (see also [Go, Theorem 4.1]) *Let $G = (V, E)$ be a graph. The following are equivalent*
(1) *G is a chordal graph,*
(2) *G has a vertex elimination ordering. Moreover, any simplicial vertex of G can be used to start such an ordering.*

We will also require the following lemma of Dirac [Di] (see also [Go, Lemma 4.2]).

Lemma 3.2 *Every chordal graph G that is not a complete graph has at least two non-adjacent simplicial vertices.*

A sub-arrangement $\mathscr{A}$ of A_n is completely defined by a graph $G=(V, E)$ where

$$x_i - x_j \in \mathscr{A} \text{ if and only if } \{i, j\} \in E.$$

Given a graph G we will let $\mathscr{A}(G)$ be the arrangement that it defines (see also [OT, Sect. 2.4]). The main theorem in this section is due to R. Stanley:

Theorem 3.3 *Let $G=(V, E)$ be a graph. The following are equivalent:*
(1) G is a chordal graph with vertex elimination order $v_1, v_2, \ldots, v_n$,
(2) $\mathscr{A}(G)$ is a supersolvable arrangement with M*-chain*

$$\emptyset \subset \mathscr{A}(G-\{v_1, \ldots, v_{n-1}\}) \subset \ldots \subset \mathscr{A}(G-\{v_1\}) \subset \mathscr{A}(G)$$

(3) $\mathscr{A}(G)$ is a free arrangement.

Proof. The implication (1) implies (2) follows from [St1, Proposition 2.8], and a proof is given in [BZ, Corollary 2.10]. The implication that (2) implies (3) follows from Theorem 2.5. So we are left to show that (3) implies (1).

Suppose that $\mathscr{A}(G)$ is free but G is not chordal. Let $C=\{v_1, \ldots, v_k\}$ be a cycle in G without a chord, where $k \geqq 4$. Then the arrangement $\mathscr{A}(G_C)$ is a localization of $\mathscr{A}(G)$ to the subspace where all coordinates x_{v_i} for $v_i \in C$ are equal, and hence must be free by Theorem 2.2. We will now show that $\mathscr{A}(G_C)$ cannot be free by computing its characteristic polynomial.

It is easy to see that $\mathscr{A}(G_C)$ is a *general position* arrangement (see [OT, Definition 5.1.19]) of rank $k-1$. It then follows that its characteristic polynomial is

$$\chi(t) = \chi(\mathscr{A}(G_C), t) = \frac{(t-1)^k - (-1)^k(1-kt)}{t} + (-1)^k(1-k).$$

Note that the coefficient of t^{k-2} in $\chi(t)$ is k. If $\mathscr{A}(G_C)$ is free then $\chi(t)$ has positive integer roots $0 < b_1 \leqq b_2 \leqq \ldots \leqq b_{k-1}$. Since the coefficient of t^{k-2} in $\chi(t)$ is k we have that

$$b_1 = b_2 = \ldots = b_{k-2} = 1 \quad \text{and} \quad b_{k-1} = 2.$$

Since the constant term $b_1 b_2 \ldots b_{k-1} = \pm(k-1)$ we conclude that $k=3$ which contradicts that C had size $k \geqq 4$. □

It will be useful for us to give the following combinatorial interpretation of the exponents of $\mathscr{A}(G)$ when G is chordal. The proof follows from the M-chain given in (2) of the previous theorem.

Lemma 3.4 *Let G be a chordal graph with vertex elimination order $\{v_1, \ldots, v_n\}$. For $1 \leqq i \leqq n-1$ let b_i be the degree of v_i in the graph $G-\{v_1, \ldots, v_{i-1}\}$. Then $\{b_1, \ldots, b_{n-1}\}$ are the exponents of the free arrangement $\mathscr{A}(G)$.*

Lemma 3.5 *Let G be a chordal graph and $e \in E$ such that $G' = G - e$ is also chordal. Then* $\exp \mathscr{A}(G') \prec \exp \mathscr{A}(G)$.

Proof. We will show that there is some vertex elimination order for G that is also a vertex elimination order for G'. The lemma will then follow from Lemma 3.4.

Suppose that $e=\{x, y\}$. Since $G-e$ is chordal it follows that $K = N(x) \cap N(y)$ induces a complete graph. Otherwise if v, $w \in K$ with v not adjacent to w then there would be a 4-cycle x, v, y, w without a chord. Moreover, we claim that K is a *cutset* that disconnects x from y in G'. If not, let $P = x, v_1, v_2, \ldots, y$ be the shortest path from x to y that avoids K. In G, however, P is a cycle of length at least 4 without a chord.

Let $\bar{G}$ be the induced subgraph of $G - e$ consisting of the vertices in the connected component of x in $G - K$ plus the vertices in K. It is clear that $\bar{G}$ is still chordal. If x is simplicial in $\bar{G}$ then it is simplicial in $G - e$ and G so we can choose x first in the vertex elimination order and proceed by induction. If x is not simplicial then by Lemma 3.2 there are two non-adjacent simplicial vertices in $\bar{G}$. At least one of these is not in $K \cup \{x\}$. Call this vertex v. Then v is simplicial in $G - e$ and also in G and so by Theorem 3.1 (2), we can choose v to be the first vertex in an elimination order and then proceed by induction. □

Theorem 3.6 *A sequence*

$$\emptyset = \mathscr{A}_0 \prec \mathscr{A}_1 \prec \ldots \prec \mathscr{A}_L = A_{n-1}$$

where $L = \binom{n}{2}$ *is an induction table from* $\emptyset$ *to* A_{n-1} *if and only if for all* $0 \leqq i \leqq L$, $\mathscr{A}_i = \mathscr{A}(G_i)$ *where* $G_i = (V, E_i)$ *is a chordal graph with* $|E_i| = i$ *and* G_i *is a subgraph of* G_{i+1}.

Proof. The necessity of this condition follows from Theorem 3.3. The sufficiency follows from Lemma 3.5. □

Unfortunately we are not able to further enumerate the induction tables in this case. In the next section, when we consider induction tables from A_{n-1} to B_n, we will be able to count explicitly the number of such induction tables.

4 Threshold graphs and arrangements between A_{n-1} and B_n

In this section we characterize all of the arrangements containing A_{n-1} and contained in B_n that are free. We also characterize which of those are supersolvable. There are also two important corollaries to our characterization. First, we are able to show which of these free arrangements satisfy Orlik's property of having their restriction to a hyperplane be free. In particular we are able to produce infinite classes of counterexamples to Orlik's conjecture [Or]. Second, we are able to enumerate the number of induction tables from A_{n-1} to B_n.

The basis of our characterization is a class of graphs called *threshold graphs.* These graphs were introduced by Chvátal and Hammer in [CH]. They will play the role in this theory that the chordal graphs played in the previous section. We begin this section with a discussion of these graphs and a summary of the properties that we will require.

Let G be a graph. A vertex v of G is said to be an *isolated* vertex if it is adjacent to no other vertex. It is said to be a *cone* vertex if it is adjacent to every other vertex. Obviously no graph can have both an isolated vertex and a cone vertex. Let $\mathscr{G}$ be the collection of graphs defined by the following recursive condition:

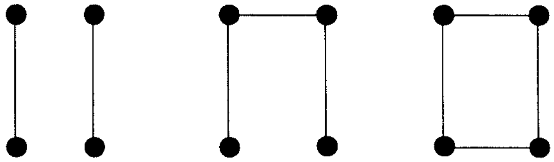

Fig. 1

(1) The graph consisting of one vertex is in $\mathscr{G}$.
(2) If G is a graph that has a vertex v that is either isolated or a cone and $G-v\in\mathscr{G}$ then $G\in\mathscr{G}$.

This collection $\mathscr{G}$ will be called *threshold* graphs. We should note that this is not the standard definition of threshold graphs, but it is known to be equivalent [CH]. Associated with a threshold graph are two kinds of orderings on its vertices. First is the *decomposition order* $v_1, \ldots, v_n$ where v_i is the isolated or cone vertex of $G-\{v_1, \ldots, v_{i-1}\}$ guaranteed by the definition. The second kind of order is a *degree order* where $\deg(v_i)\geqq\deg(v_j)$ for all $i\leqq j$. Note that there is more than one degree order, since two vertices can have the same degree, however there is only one degree order up to an automorphism of the graph [HIS].

Theorem 4.1 *If G is an n-vertex graph then the following are equivalent:*

(1) $G\in\mathscr{G}$.
(2) *G does not contain any of the graphs in Fig. 1 as induced subgraphs.*
(3) *For all vertices $i, j\in V(G)$, if $\deg(i)\geqq\deg(j)$ then $N(j)\subseteq N(i)$.*

Proof. [CH]. See also [Go, Theorem 10.7] and [Go, Exercise 3]. □

The degree sequences of threshold graphs have been completely characterized [HIS]. Moreover, there is a unique unlabelled threshold graph associated with each such degree sequence. We call a degree sequence of a threshold graph a *threshold sequence.* We partially order the threshold sequences of length n by the product order, i.e.,

$$(d_1, \ldots, d_n)\leqq(e_1, \ldots, e_n)\Leftrightarrow d_i\leqq e_i \quad \text{for all } 1\leqq i\leqq n.$$

Call this partial order $\mathscr{T}_n$. $\mathscr{T}_n$ was first investigated in [HIS] where it was shown to be a lattice.

Lemma 4.2 *If G and H are unlabelled threshold graphs on n vertices with degree sequences $d(G)$ and $d(H)$ respectively, then $d(G)\leqq d(H)$ if and only if H is a subgraph of G.*

Proof. See [HIS]. □

Let $U_n=\{(i,j): 1\leqq i<j\leqq n\}$ under the product order. By $J(U_n)$ we mean the *lattice of order ideals* of U_n ordered by containment (see [St2, Chap. 3]).

Theorem 4.3 *The posets $\mathscr{T}_n$ and $J(U_n)$ are isomorphic.*

Proof. If $d=(d_1, \ldots, d_n)$ is a threshold sequence then let

$$\Phi(d)=\{(i,j): 1 \leqq i \leqq n-1 \quad \text{and} \quad i+1 \leqq j \leqq d_i+1\}.$$

It follows from [HIS, Lemma 9] that this is the necessary bijection. $\square$

Corollary 4.4 *The number of sequences of unlabelled threshold graphs*

$$E_n = G_0 < G_1 < \ldots < G_{\binom{n}{2}} = K_n$$

in which $|E(G_i)|=i$ *is equal to the number of shifted Young tableaux* [Th] *of shape* $(n-1, n-2, \ldots, 2, 1)$ *i.e.*

$$\frac{\binom{n}{2}!}{(n-1)_{n-1}(n)_{n-2}\ldots(2n-3)_1}.$$

Proof. The number of such chains by Lemma 4.2 is equal to the number of maximal chains in $\mathcal{T}_n$. That is the same as the number of maximal chains in $J(U_n)$, by Theorem 4.3, which is the number of linear extensions of U_n [St, Proposition 3.5.2]. This is well-known to be the number of shifted Young tableaux of shape $(n-1, n-2, \ldots, 2, 1)$ which, by a hook-length formula [Th] is the number claimed in the corollary. $\square$

We now return to our discussion of arrangements. Recall the defining forms for the Coxeter arrangements A_{n-1}, D_n, and B_n:

$$\begin{aligned} A_{n-1} &= \{x_i - x_j : 1 \leqq i < j \leqq n\} \\ D_n &= A_{n-1} \cup \{x_i + x_j : 1 \leqq i < j \leqq n\} \\ B_n &= D_n \cap \{x_i : 1 \leqq i \leqq n\}. \end{aligned}$$

It is clear then that an arrangement $\mathcal{A}$ with $A_{n-1} \subseteq \mathcal{A} \subseteq B_n$ is completely specified by two pieces of data:

(1) a graph G on vertex set $[n]$ defined by $G=\{\{i,j\}: x_i + x_j \in \mathcal{A}\}$,
(2) a subset L of $[n]$ defined by $L=\{i: x_i \in \mathcal{A}\}$.

Conversely, any graph G on vertices $[n]$ and subset L of $[n]$ specifies an arrangement which we will call $\mathcal{A}^*(G,L)$ lying between A_{n-1} and B_n. We will draw the *diagram* of G, L as the usual diagram of the graph G, and add loops on the vertices which lie in L.

Examples.

$$\begin{aligned} \mathcal{A}^*(E_n, \emptyset) &= A_{n-1} \\ \mathcal{A}^*(K_n, \emptyset) &= D_n \\ \mathcal{A}^*(K_n, [n]) &= B_n \end{aligned}$$

$$\mathcal{A}^*(\{\{1,2\},\{1,3\}\},\{2\}) = \{x_1 - x_2, x_1 - x_3, x_2 - x_3, x_1 + x_2, x_1 + x_3, x_2\}$$

and the diagram for this last example is shown in Fig. 2.

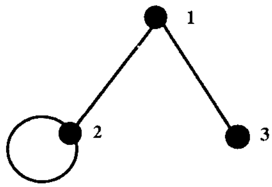

Fig. 2

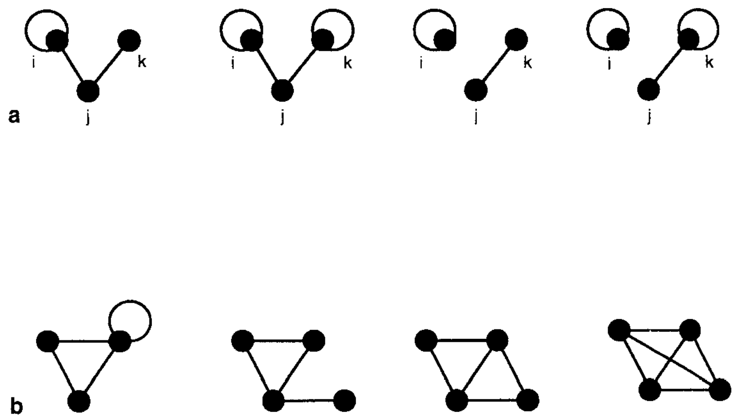

Fig. 3

Note that a permutation of the labels on the diagram corresponds to acting on the arrangement $\mathscr{A}^*(G, L)$ by a permutation matrix, which does not affect any of the properties in which we are interested, such as freeness or supersolvability. Therefore it makes sense to say that $\mathscr{A}^*(G, L)$ has one of these properties even if we only specify G, L as an *unlabelled* diagram.

Prior to proving the main results, we present a lemma cataloguing information about some arrangements $\mathscr{A}^*(G, L)$ that will be useful in the proofs.

Lemma 4.5 (a) *If G is a graph in Fig.* 1 *then $\mathscr{A}^*(G, L)$ is not free for any set L.*
(b) *The diagrams in Fig.* 3(a) *represent arrangements that are not free.*
(c) *The diagrams in Fig.* 3(b) *represent arrangements $\mathscr{A}^*(G, L)$ which are* not *supersolvable (although we will see later that they are free).*

Proof. (a), (b) If these arrangements were free, their characteristic polynomials would factor completely over the integers by Theorem 2.1. It has been checked (using the program "Matroid" available from the first author) that these characteristic polynomials all contain an irreducible quadratic factor.
(c) Assume one of these arrangements $\mathscr{A}$ is supersolvable, and let b be its largest exponent, i.e. the largest root of its characteristic polynomial. Then $\mathscr{A}$ must contain a modular coatom having at least $\# \mathscr{A} - b$ hyperplanes by Theorem 2.5.

However, one can check by a tedious (but finite) enumeration that for each of these arrangements, every sub-arrangement with at least $\#\mathscr{A}-b$ hyperplanes has the same rank as $\mathscr{A}$, and therefore cannot be a modular coatom. □

We come now to the first main result.

Theorem 4.6 *$\mathscr{A}^*(G,L)$ is free if and only if G is a threshold graph and L is an initial segment for some degree order on the vertices of G.*

Proof. ($\Rightarrow$) Assume that $\mathscr{A}^*(G,L)$ is free. By Theorem 2.2, every localization of $\mathscr{A}^*(G,L)$ is free. This implies that for any subset $U\subseteq[n]$ that contains an edge of G or that intersects L, the arrangement $\mathscr{A}^*(G_U, L\cap U)$ must also be free, since it is the localization of $\mathscr{A}^*(G,L)$ to the subspace $\bigcap_{u\in U}\{x_u=0\}$. In particular, G cannot have any of the graphs in Fig. 1 as induced subgraphs by Lemma 4.5(a). Therefore G is a threshold graph by Theorem 4.1(2). Now suppose that L violates the condition of the theorem, which means that there must be some vertices $i\in L$ and $j\notin L$ with $\deg(j)>\deg(i)$. By Theorem 4.1(3), there must exist a vertex k which is adjacent to j but not to i. There are four cases depending upon whether $\{i,j\}$ is an edge of G and whether $k\in L$ for what the subdiagram on vertices i, j, k can look like, and they are exactly those pictured in Fig. 3(b). Since these localizations are not free, we have reached a contradiction, and L must satisfy the condition of the theorem.
($\Leftarrow$) This direction of the theorem will be proven in two steps.

Step 1 We show that if G is a threshold graph, then under certain extra hypotheses on L, the arrangement $\mathscr{A}^*(G,L)$ is supersolvable (and hence free by Theorem 2.5).

Lemma 4.7 *Let G be a threshold graph with vertices $1, 2, \ldots, n$ in some degree order for which L is an initial segment. Suppose further that L contains at least one endpoint from every edge of G. Then $\mathscr{A}^*(G,L)$ is supersolvable with* M*-chain*

$$\emptyset\subset\mathscr{A}^*(G_{[1]}, L\cap[1])\subset\ldots\subset\mathscr{A}^*(G_{[n]}, L\cap[n])=\mathscr{A}^*(G,L).$$

Proof. Since G is a threshold graph, we know that either n is an isolated vertex or 1 is a cone vertex of G. The proof of the lemma will then follow by induction on n from the following two sublemmas.

Sublemma 4.8 *Assume n is an isolated vertex of G. If $\mathscr{A}^*(G_{[n-1]}, L\cap[n-1])$ satisfies the lemma and has exponents $(b_1, \ldots, b_{n-1})$, then $\mathscr{A}^*(G,L)$ satisfies the lemma and has exponents*

$$\begin{cases}(b_1,\ldots,b_{n-1},n-1) & \text{if } n\notin L\\ (b_1,\ldots,b_{n-1},n) & \text{if } n\in L.\end{cases}$$

Proof. We wish to show that $\mathscr{A}^*(G_{[n-1]}, L\cap[n-1])$ is a modular coatom in $\mathscr{A}^*(G,L)$. Note that

$$\mathscr{A}^*(G,L)-\mathscr{A}^*(G_{[n-1]}, L\cap[n-1])=\{x_j-x_n : 1\leq j<n\}\cup\{x_n:\ \text{if}\, n\in L\},$$

a set of cardinality

$$\begin{cases} n-1 & \text{if } n \notin L \\ n & \text{if } n \in L \end{cases}$$

so that the sublemma will follow. Checking the intersections of pairs of hyperplanes in this set, we find that

$$\{x_j - x_n = 0\} \cap \{x_k - x_n = 0\} \subset \{x_j - x_n = 0\}$$

and the latter hyperplane is always in $\mathscr{A}^*(G_{[n-1]}, L \cap [n-1])$. Meanwhile

$$\{x_j - x_n = 0\} \cap \{x_n = 0\} \subset \{x_j = 0\}$$

and the latter hyperplane will be in $\mathscr{A}^*(G_{[n-1]}, L \cap [n-1])$ if $n \in L$, since L is an initial segment of $1, 2, \ldots, n$. Note also that

$$\operatorname{rank} \mathscr{A}^*(G_{[n-1]}, L \cap [n-1]) = \begin{cases} n-2 & \text{if } G, L = \emptyset \\ n-1 & \text{else} \end{cases}$$

$$\operatorname{rank} \mathscr{A}^*(G, L) = \begin{cases} n-1 & \text{if } G, L = \emptyset \\ n & \text{else} \end{cases}$$

so that $\mathscr{A}^*(G_{[n-1]}, L \cap [n-1])$ is a modular coatom of $\mathscr{A}^*(G, L)$, and the proof is complete. $\square$

Sublemma 4.9 *Assume* 1 *is an cone vertex of* G, *and let* $[\hat{i}]$ *denote the set* $[i] - \{1\}$. *If* $\mathscr{A}^*(G_{[\hat{n}]}, L \cap [\hat{n}])$ *satisfies the lemma and has exponents* $(b_1, \ldots, b_{n-1})$ *then* $\mathscr{A}^*(G, L)$ *satisfies the lemma and has exponents* $(1, b_1 + 2, \ldots, b_{n-1} + 2)$.

Proof. We must show that $\mathscr{A}^*(G_{[i]}, L \cap [i])$ is a modular coatom in $\mathscr{A}^*(G_{[i+1]}, L \cap [i+1])$ for $1 \leqq i \leqq n-1$ (the case $i = 0$ is trivial). We have

$$\begin{aligned} &\mathscr{A}^*(G_{[i+1]}, L \cap [i+1]) - \mathscr{A}^*(G_{[i]}, L \cap [i]) \\ &\quad = \mathscr{A}^*(G_{[\widehat{i+1}]}, L \cap [\widehat{i+1}]) - \mathscr{A}^*(G_{[\hat{i}]}, L \cap [\hat{i}]) \cup \{x_1 - x_{i+1}, x_1 + x_{i+1}\} \end{aligned}$$

which is a set of cardinality $b_{i+1} + 2$. Note that the set $\mathscr{A}^*(G_{[\widehat{i+1}]}, L \cap [\widehat{i+1}]) - \mathscr{A}^*(G_{[\hat{i}]}, L \cap [\hat{i}])$ only contains hyperplanes of the form $x_{i+1} - x_j$ or $x_{i+1} + x_j$ or x_{i+1} for $j \leqq i$. Since $\mathscr{A}^*(G_{[\hat{i}]}, L \cap [\hat{i}])$ is a modular coatom in $\mathscr{A}^*(G_{[\widehat{i+1}]}, L \cap [\widehat{i+1}])$ by hypothesis, we need only check intersections of pairs of hyperplanes that contain at least one of $\{x_1 - x_{i+1}, x_1 + x_{i+1}\}$. We have

$$\begin{aligned} &\{x_1 - x_{i+1} = 0\} \cap \{x_1 + x_{i+1} = 0\} \subset \{x_1 = 0\} \\ &\{x_1 - x_{i+1} = 0\} \cap \{x_{i+1} - x_j = 0\} \subset \{x_1 - x_j = 0\} \\ &\{x_1 - x_{i+1} = 0\} \cap \{x_{i+1} + x_j = 0\} \subset \{x_1 + x_j = 0\} \\ &\{x_1 - x_{i+1} = 0\} \cap \{x_{i+1} = 0\} \subset \{x_1 = 0\} \\ &\{x_1 + x_{i+1} = 0\} \cap \{x_{i+1} - x_j = 0\} \subset \{x_1 + x_j = 0\} \\ &\{x_1 + x_{i+1} = 0\} \cap \{x_{i+1} + x_j = 0\} \subset \{x_1 - x_j = 0\} \\ &\{x_1 + x_{i+1} = 0\} \cap \{x_{i+1} = 0\} \subset \{x_1 = 0\}. \end{aligned}$$

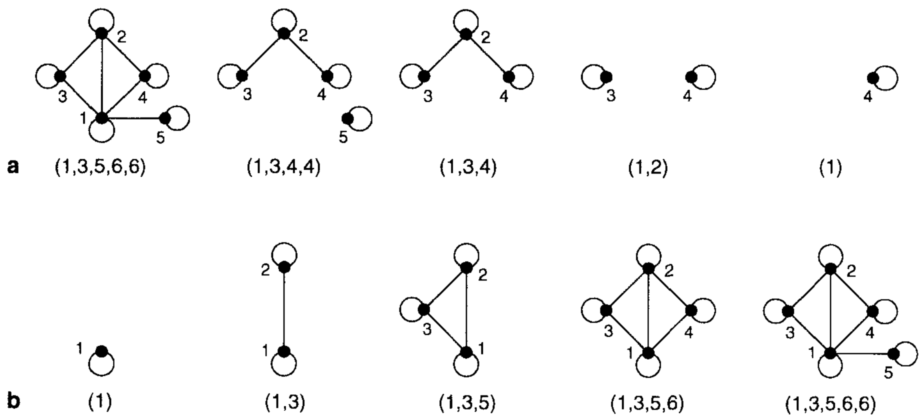

Fig. 4

All of the hyperplanes of the form x_1+x_j or x_1-x_j on the right hand sides are in $\mathscr{A}^*(G_{[i]}, L\cap[i])$, since 1 is a cone vertex of G. The hyperplane x_1 is also in $\mathscr{A}^*(G_{[i]}, L\cap[i])$, since the assumption on L implies that $L\neq\emptyset$, and hence $1\in L$ since L is an initial segment of $1, 2, \ldots, n$. Note also that

$$\operatorname{rank} \mathscr{A}^*(G_{[i]}, L\cap[i])=i$$

$$\operatorname{rank} \mathscr{A}^*(G_{[i+1]}, L\cap[i+1])=i+1.$$

Therefore, $\mathscr{A}^*(G_{[i]}, L\cap[i])$ is indeed a modular coatom in $\mathscr{A}^*(G_{[i+1]}, L\cap[i+1])$, and the sublemma follows. $\square$

This completes the proof of the lemma, and Step 1. $\square$

Example. Let $G=\{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,4\}\}$ and $L=[5]$. The diagram for $\mathscr{A}^*(G, L)$ is shown in Fig. 4(a) with its vertices labelled in a degree order. A decomposition order is given, along with the recursively calculated exponents. Figure 4(b) shows the M-chain for $\mathscr{A}^*(G, L)$ guaranteed by the lemma.

Step 2 We now treat the general case, where G is a threshold graph with vertices $1, 2, \ldots, n$ in some degree order, and L is initial segment of this order. Let $\mathscr{A}_i=\mathscr{A}^*(G,[i])$, $\mathscr{A}_i^{\text{con}}=\mathscr{A}_i^{\{x_i=0\}}$, i.e. $\mathscr{A}_i^{\text{con}}$ is the restriction of $\mathscr{A}_i$ to the hyperplane x_i. An easy calculation shows that $\mathscr{A}_i^{\text{con}}=\mathscr{A}^*(G_{[n]-\{i\}}, [n]-\{i\})$, i.e. restricting $\mathscr{A}_i$ to the hyperplane x_i simply removes the vertex i and puts a loop on all other vertices in the diagram. Consequently, $\mathscr{A}_i^{\text{con}}$ is always free (and in fact, supersolvable) by Step 1, since $G_{[n]-\{i\}}$ is still a threshold graph by Theorem 4.1(2).

Notice that $\mathscr{A}_i, \mathscr{A}_{i-1}, \mathscr{A}_i^{\text{con}}$ form a triple $(\mathscr{A}, \mathscr{A}', \mathscr{A}'')$ of arrangements in the sense of Section 2. We will say that G is *strippable* if for $1\leq i\leq n$ the Addition-Deletion Theorem (Theorems 2.2 and 2.3) can be applied to conclude that $\mathscr{A}_{i-1}$ is free, i.e. if $\mathscr{A}_i, \mathscr{A}_i^{\text{con}}$ are both free and the exponents of $\mathscr{A}_i^{\text{con}}$ are a subset of those for $\mathscr{A}_i$. To complete Step 2, it therefore suffices to prove the following

Lemma 4.10 *If G is a threshold graph, then G is strippable. Furthermore, if $\mathscr{A}_n$ has exponents $(1, b_2, \ldots, b_n)$ then $\mathscr{A}_0$ has exponents $(b_2-2, \ldots, b_n-2, n-1)$.*

Proof. Since G is a threshold graph, either n is an isolated vertex or 1 is a cone vertex of G. The proof of the lemma then follows by induction on n (the case $n=1$ being trivial) from the following two sublemmas:

Sublemma 4.11 *Assume n is an isolated vertex of G. If $G_{[n-1]}$ satisfies the lemma, then so does G.*

Proof. Let $\mathscr{A}_i$ and $\mathscr{A}_i^{\text{con}}$ be defined as before, and define

$$\tilde{\mathscr{A}}_i = \mathscr{A}^*(G_{[n-1]}, [i]), \qquad \tilde{\mathscr{A}}_i^{\text{con}} = \tilde{\mathscr{A}}_i^{\{x_i=0\}}$$

for $1 \leqq i \leqq n-1$. Since $G_{[n-1]}$ satisfies the lemma, $\tilde{\mathscr{A}}_{n-1}, \tilde{\mathscr{A}}_0$ are both free, say with exponents $(1, b_2, \ldots, b_{n-1})$ and $(b_2-2, \ldots, b_{n-1}-2, n-2)$ respectively. By Sublemma 4.8, $\mathscr{A}_n$ is free with exponents $(1, b_2, \ldots, b_{n-1}, n)$ and

$$\mathscr{A}_n^{\text{con}} = \mathscr{A}^*(G_{[n]-\{n\}}, [n]-\{n\}) = \tilde{\mathscr{A}}_{n-1}$$

is free with exponents $(1, b_2, \ldots, b_{n-1})$. Therefore by Theorem 2.4 we conclude that $\mathscr{A}_{n-1}$ is free with exponents $(1, b_2, \ldots, b_{n-1}, n-1)$. Let f be the map on sequences $s=(s_1, \ldots, s_{n-1})$ which takes s to $f(s)=(s_1, \ldots, s_{n-1}, n-1)$. Then we have just shown that the exponents for $\tilde{\mathscr{A}}_{n-1}$ and $\mathscr{A}_{n-1}$ are related by f. Furthermore, Sublemma 4.8 implies that the exponents for $\tilde{\mathscr{A}}_i^{\text{con}}$ and $\mathscr{A}_i^{\text{con}}$ are also related by the map f. It then follows by repeated application of Theorem 2.4 that $\mathscr{A}_i$ is free for $0 \leqq i \leqq n$ and that the exponents for $\tilde{\mathscr{A}}_i$ and $\mathscr{A}_i$ are related by f. Hence G is strippable. Furthermore, since $\tilde{\mathscr{A}}_0$ has exponents $s=(b_2-2, \ldots, b_{n-1}-2, n-2)$, $\mathscr{A}_0$ must have exponents $f(s)=(b_2-2, \ldots, b_{n-1}-2, n-2, n-1)$ which bear the desired relation to the exponents $(1, b_2, \ldots, b_{n-1}, n)$ of $\mathscr{A}_n$. $\square$

Sublemma 4.12 *Assume 1 is a cone vertex of G and let $[\hat{i}]=[i]-\{1\}$ as before. If $G_{[\hat{n}]}$ satisfies the lemma, then so does G.*

Proof. Let $\mathscr{A}_i$ and $\mathscr{A}_i^{\text{con}}$ be defined as before, and define

$$\hat{\mathscr{A}}_i = \mathscr{A}^*(G_{[\hat{n}]}, [\hat{i}]), \qquad \hat{\mathscr{A}}_i^{\text{con}} = \hat{\mathscr{A}}_i^{\{x_i=0\}}$$

for $1 \leqq i \leqq n-1$. Since $G_{[\hat{n}]}$ satisfies the lemma, $\hat{\mathscr{A}}_n$, $\hat{\mathscr{A}}_1$ are both free, say with exponents $(1, b_2, \ldots, b_{n-1})$ and $(b_2-2, \ldots, b_{n-1}-2, n-2)$ respectively. Let g be the map on sequences $s=(s_1, \ldots, s_{n-1})$ which takes s to $g(s)=(1, s_1+2, \ldots, s_{n-1}+2)$. Then Sublemma 4.9 implies that the exponents for $\hat{\mathscr{A}}_n$ and $\mathscr{A}_n$ are related by g, and so are the exponents for $\hat{\mathscr{A}}_i^{\text{con}}$ and $\mathscr{A}_i^{\text{con}}$, $2 \leqq i \leqq n$. It then follows by repeated application of Theorem 2.4 that $\mathscr{A}_i$ is free for $1 \leqq i \leqq n$ and that the exponents for $\hat{\mathscr{A}}_i$ and $\mathscr{A}_i$ are related by g. Furthermore, since $\hat{\mathscr{A}}_1$ has exponents $s=(b_2-2, \ldots, b_{n-1}-2, n-2)$, $\mathscr{A}_1$ must have exponents $g(s)=(1, b_2, \ldots, b_{n-1}, n)$. Meanwhile, $\mathscr{A}_1^{\text{con}} = \mathscr{A}^*(G_{[n]-\{1\}}, [n]-\{1\}) = \hat{\mathscr{A}}_n$ has exponents $(1, b_2, \ldots, b_{n-1})$, so a final application of Theorem 2.4 implies that $\mathscr{A}_0$ is free with exponents $(1, b_2, \ldots, b_{n-1}, n-1)$. Hence G is strippable. Since $\mathscr{A}_n$ has exponents $g(1, b_2, \ldots, b_n-1)=(1, 3, b_2+2, \ldots, b_{n-1}+2)$, the exponents of $\mathscr{A}_0$ and $\mathscr{A}_n$ also bear the desired relation to each other. $\square$

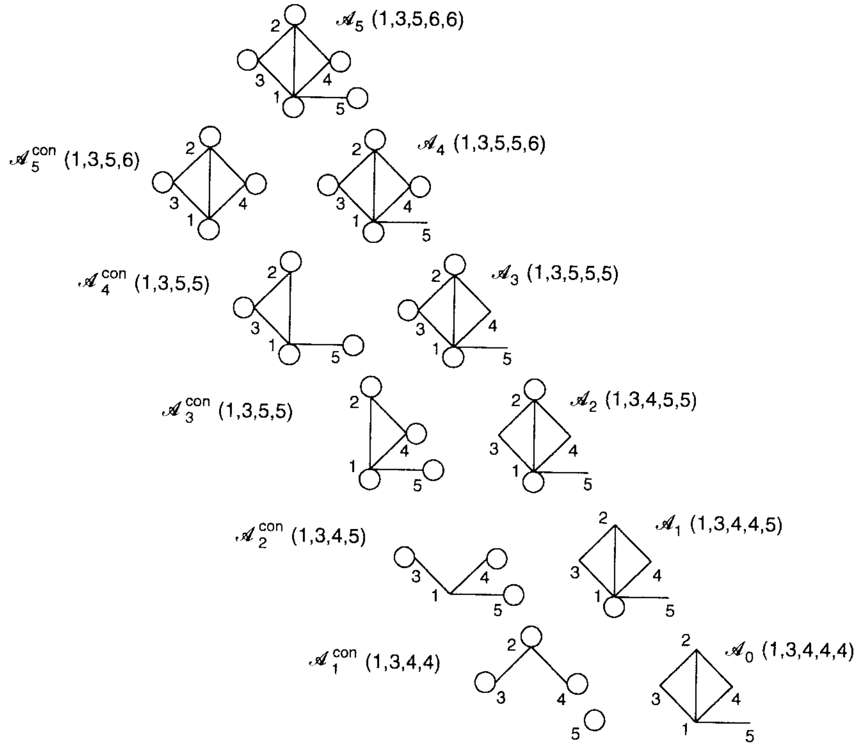

Fig. 5

This completes the proof of Lemma 4.10, and of Step 2, and hence Theorem 4.6. □

Example. Figure 5 shows the "stripping" for the same graph G as in the previous example.

In light of the previous theorem, we will call (G, L) a *free pair* if G is a threshold graph and L is an initial segment of some degree order on its vertices, i.e. if $\mathscr{A}^*(G, L)$ is free.

Remark 1 The above proof shows that all free arrangements between A_{n-1} and B_n are *recursively free*, and more precisely *free by supersolvable resolution* in the sense of Ziegler [Zi].

Remark 2 As a corollary to the above proof we have

Porism 4.13 *All free arrangements between A_{n-1} and D_n have $n-1$ occurring as one of their exponents.*

Is there an algebraic proof or significance to this porism?

This characterization of the free arrangements between A_{n-1} and B_n allows us to determine when the restriction of one of these arrangements to one of its hyperplanes is free.

Theorem 4.14 *Let $\mathscr{A}^*(G, L)$ be a free arrangement between A_{n-1} and B_n and H one of its hyperplanes. Then the restriction $\mathscr{A}^*(G, L)^H$ is not free if and only if*
(1) *H is of the form $x_i - x_j$.*
(2) *$\{i,j\} \in G$ i.e. $x_i + x_j \in \mathscr{A}^*(G, L)$.*
(3) *There exists a vertex $h \notin L$ which has strictly higher degree than i or j.*

Proof. Since $\mathscr{A}^*(G, L)$ is free, we know by the previous theorem that G is a threshold graph and L is an initial segment of some degree order $1, 2, \ldots, n$ on its vertices. There are three possible forms for H: $x_i, x_i + x_j, x_i - x_j$.

If H is of the form x_i, it is easy to check that

$$\mathscr{A}^*(G, L)^H = \mathscr{A}^*(G_{[n]-\{i\}}, [n] - \{i\}),$$

which is free (and even supersolvable according to Lemma 4.7) since $G_{[n]-\{i\}}$ is still a threshold graph by Theorem 4.1.

If H is of the form $x_i + x_j$, it easy to check that

$$\mathscr{A}^*(G, L)^H = \mathscr{A}^*(v * G_{[n]-\{i,j\}}, \{v\} \cup (L - \{i,j\})),$$

where $v * G$ denotes the graph obtained from G by adding a new cone vertex v. Note that $G_{[n]-\{i,j\}}$ is still a threshold graph by Theorem 4.1, and v goes at the beginning of the degree order so that $\{v\} \cup ([n] - \{i,j\})$ is still an initial segment in some degree order. Therefore $\mathscr{A}^*(G, L)^H$ is still free in this case.

This only leaves the case where H is $x_i - x_j$ for some i, j. Assume without loss of generality that $i < j$, so that $\deg(i) \geq \deg(j)$ and hence $N(j) \subseteq N(i)$ by Theorem 4.1(3). It is easy to check that

$$\mathscr{A}^*(G, L)^H = \begin{cases} \mathscr{A}^*(G_{[n]-\{j\}}, L - \{j\}) & \text{if } \{i,j\} \notin G \quad \text{or} \quad i \in L \\ \mathscr{A}^*(G_{[n]-\{j\}}, L \cup \{i\}) & \text{if } \{i,j\} \in G \quad \text{and} \quad i \notin L. \end{cases}$$

Note that $G_{[n]-\{j\}}$ is a threshold graph by Theorem 4.1(2) in either case. Since $L - \{j\}$ is an initial segment in a degree order for $G_{[n]-\{j\}}$, $\mathscr{A}^*(G, L)^H$ is free in the first case where $\{i,j\} \notin G$ or $i \in L$. In the other case, we need to check whether $L \cup \{i\}$ is an initial segment in some degree order for $G_{[n]-\{j\}}$. This fails to be true exactly when there is some vertex h with $\deg(h) > \deg(i)$ and $h \notin L$, so we are done. □

It had been conjectured by Orlik [Or] that the restriction of a free arrangement to any of its hyperplanes is still free, and proven for many special cases. However, an isolated counterexample was found in [ER], namely a 5-dimensional arrangement having 21 hyperplanes. The previous theorem allows one to generate an infinite class of counterexamples, including one 4-dimensional counterexample having 10 hyperplanes, described in the example below. Note that this is the minimum dimension possible for such a counterexample, since every 3-dimensional arrangement has only 2-dimensional restrictions which are always supersolvable and hence free.

Example. Let $G = \{\{1,2\}, \{1,3\}, \{2,3\}, \{3,4\}\}$, $L = \emptyset$ and H the hyperplane $x_1 - x_2$. The diagram for $\mathscr{A}^*(G, L)$ is the second one in Fig. 3(b), hence it is free. $\mathscr{A}^*(G, L)^H$ has the diagram pictured in Fig. 3(a), hence it is not free. One can

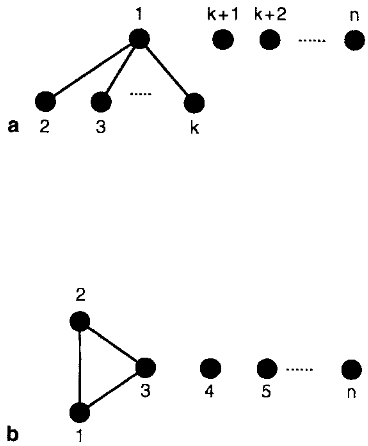

Fig. 6

easily check that the previous theorem implies this is the only 4-dimensional counterexample to Orlik's conjecture between A_{n-1} and B_n.

We now turn to the question of which arrangements between A_{n-1} and B_n are supersolvable. Lemma 4.7 shows that a large class of the free arrangements are actually supersolvable, and the next result asserts that these are "almost all" of the supersolvable arrangements between A_{n-1} and B_n.

Theorem 4.15 *$\mathscr{A}^*(G, L)$ is supersolvable if and only if*
(1) (G, L) is a free pair, and
(2) either L contains at least one endpoint from every edge of G, or the diagram for $\mathscr{A}^*(G, L)$ is one of those pictured in Fig. 6.

Proof. ($\Leftarrow$) Lemma 4.7 asserts that if (G, L) is a free pair and L contains at least one endpoint from every edge of G then $\mathscr{A}^*(G, L)$ is supersolvable. Therefore it only remains to show that the arrangements in Fig. 6 are supersolvable. To show that Fig. 6(a) is supersolvable, note that the A_{n-1} arrangement is a modular coatom of this arrangement, and A_{n-1} is known to be supersolvable [OT, Example 2.1.33]. To show that Fig. 6(b) is supersolvable, note that its localization $\mathscr{A}^*(G_{[3]}, L\cap[3])$ is the D_3 arrangement which is well-known to be equivalent to the A_3 arrangement, and hence supersolvable. It is then easy to check that

$$\mathscr{A}^*(G_{[3]}, L\cap[3]) \subset \mathscr{A}^*(G_{[4]}, L\cap[4]) \subset \ldots \subset \mathscr{A}^*(G_{[n]}, L\cap[n])$$

completes an M-chain for $\mathscr{A}^*(G, L)$.
($\Rightarrow$) Assume $\mathscr{A}^*(G, L)$ is supersolvable. By Theorem 2.6, every localization of $\mathscr{A}^*(G, L)$ is supersolvable, and hence every subset $U \subseteq [n]$ whose vertex-induced subgraph G_U contains an edge or which intersects L must have $\mathscr{A}^*(G_U, L\cap U)$ supersolvable. This fact will be our main tool in the proof.

Since $\mathscr{A}^*(G, L)$ is also free, (G, L) must be a free pair. If L contains an endpoint of every edge of G then we are done, so assume not, i.e. there is an edge e of G both of whose endpoints are not in L. Let $1, 2, \ldots, n$ be a

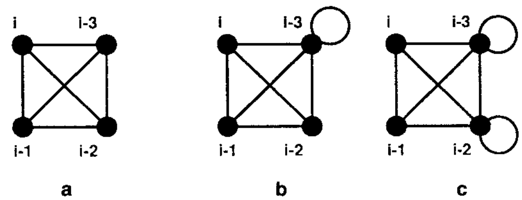

Fig. 7

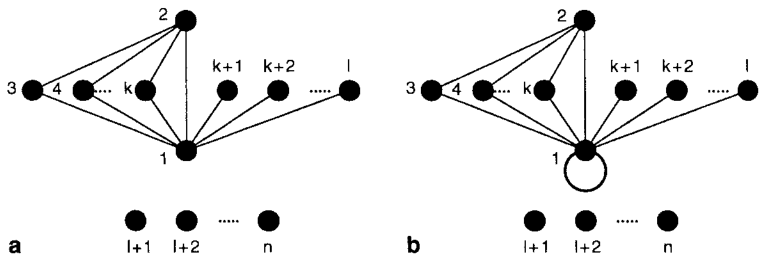

Fig. 8

degree order on the vertices such that L is an initial segment, and without loss of generality (see [HIS, Lemma 9]), we may assume there exists an i such that $[i-1]$ are the cone vertices in the reverse of their decomposition order, $[n]-[i]$ are the isolated vertices in their decomposition order, and i is the initial vertex in the decomposition. Now, since $G_{[n]-[i]}$ contains no edges, the edge e must contain a vertex $j<i$. Since L is an initial segment, $\{i-1, i\}$ must also be an edge neither of whose endpoints are in L. We claim that this forces $i \leqq 3$. To see this, assume $i \geqq 4$, and note that $\mathscr{A}^*(G, L)$ would then have a localization with diagram like one of those pictured in Fig. 7.

Figure 7(a) appears as the fourth diagram in Fig. 3(b), while Figs. 7(b), (c) contain vertex-induced subdiagrams $G_{\{i,i-1,i-3\}}$ that look like the first diagram in Fig. 3(b). Since these diagrams are not supersolvable, we have a contradiction and the claim is proven.

Therefore $i=1, 2$, or 3. If $i=1$ then G would have no edges, but we know $e \in G$. If $i=2$ then the diagram for $\mathscr{A}^*(G, L)$ looks like Fig. 6(a) and we are done. If $i=3$ then the diagram for $\mathscr{A}^*(G, L)$ looks either Fig. 8(a) or (b).

However, the fact that it cannot contain any of the first three diagrams in Fig. 3(b) as a vertex-induced subdiagram forces it to look like Fig. 8(a) with $l=k=3$, i.e. the diagram in Fig. 6(b). The proof is now complete. □

This characterization of the supersolvable arrangements between A_{n-1} and B_n shows that in contrast to the sub-arrangements of A_n, there are many free but *not* supersolvable arrangements in this class. For example, if $L=\emptyset$ then $\mathscr{A}^*(G, L)$ is free but not supersolvable for "almost all" threshold graphs G. The Coxeter arrangement D_n is a well-known special case of this phenomenon.

We now lay the groundwork for enumerating the induction tables from A_{n-1} to B_n. In the manner of §3 we will show that all chains of diagrams

G, L such that (G, L) is a free pair give rise to an induction table. We begin with a lemma that is purely numerical in nature. If $S=\{b_1, \ldots, b_n\}$ and $T=\{e_1, \ldots, e_n\}$ are sets of integers we say that $S \gtrdot T$ if for some index i we have

$$T=\{b_1, \ldots, b_{i-1}, b_i-1, b_{i+1}, \ldots, b_n\}.$$

Lemma 4.16 *Let S, S', S'' and T, T', T'' be lists of exponents related to each other as in the Addition-Deletion Theorem (Theorem 2.3 and 2.4). In addition suppose that $S \gtrdot T$ and either $S'=T'$ or $S' \gtrdot T'$. Then $S'' \gtrdot T''$.*

Proof. The proof is easy by counting the necessary multiplicities of the exponents. □

We will also require the following easy lemma whose proof is left to the reader.

Lemma 4.17 *If (G, L) is a free pair and $e \in E(G)$ such that $G-e$ is a threshold graph then $(G-e, L)$ is a free pair.*

The basis of our enumeration of induction sequences is the following lemma,

Lemma 4.18 *Suppose that (G, L) is a free pair and $e \in E(G)$ is an edge such that $G-e$ is threshold. Then $\mathscr{A}^*(G-e, L)$ is free and*

$$\exp \mathscr{A}^*(G, L) \gtrdot \exp \mathscr{A}^*(G-e, L).$$

Proof. By Lemma 4.17 and Theorem 4.6 we know that $\mathscr{A}^*(G-e, L)$ is free. We prove the fact about the exponents by induction on $n-|L|$. If $|L|=n$ then both $\mathscr{A}^*(G, L)$ and $\mathscr{A}^*(G-e, L)$ are supersolvable and the theorem follows easily from Lemma 4.7.

If neither are supersolvable then $L=[i]$ for some $i<n$. Using an induction as in Sublemma 4.11, let $\mathscr{A}_{i+1}=\mathscr{A}^*(G,[i+1])$, $\hat{\mathscr{A}}_{i+1}=\mathscr{A}^*(G-e,[i+1])$ with $\mathscr{A}_{i+1}^{\text{con}}=\mathscr{A}_{i+1}^{\{x_{i+1}=0\}}$ and $\hat{\mathscr{A}}_{i+1}^{\text{con}}=\hat{\mathscr{A}}_{i+1}^{\{x_{i+1}=0\}}$. By induction $\exp \mathscr{A}_{i+1} \gtrdot \exp \hat{\mathscr{A}}_{i+1}$ and $\exp \mathscr{A}_{i+1}^{\text{con}} \gtrdot \exp \hat{\mathscr{A}}_{i+1}^{\text{con}}$. Applying Lemma 4.16 we see that $\exp \mathscr{A}_i \gtrdot \exp \hat{\mathscr{A}}_i$ and the lemma is proved. □

Lemma 4.19 *If $(G,[i])$ is a free pair and $i>1$ then*

$$\exp \mathscr{A}^*(G,[i]) \gtrdot \exp \mathscr{A}^*(G,[i-1]).$$

Proof. This follows from Lemma 4.10. □

We are now ready to present the main enumeration theorem for induction tables.

Theorem 4.20 *The number of induction tables*

$$A_{n-1}=\mathscr{A}_0 \lessdot \mathscr{A}_1 \lessdot \ldots \lessdot \mathscr{A}_{\binom{n}{2}}=B_n$$

is

$$n!\left(\begin{matrix}\binom{n+1}{2} \\ n\end{matrix}\right) \frac{\binom{n}{2}!}{(n-1)_{n-1}(n)_{n-2} \ldots (2n-3)_1}.$$

Proof. We begin by considering the number of such induction tables up to the action of permuting the coordinates. Then we can assume that the threshold graphs are labelled in degree order (there is a unique one up to the permutation action) and $L=[i]$ for some $0 \leqq i \leqq n$. From Theorem 4.6 $\mathscr{A}_i$ is free if and only if $\mathscr{A}_i = \mathscr{A}^*(G_i, L_i)$ for some free pair G_i, L_i. Moreover, by Lemmas 4.18 and 4.19, $\mathscr{A}_i \lessdot \mathscr{A}_{i+1}$ if and only if G_i, L_i and G_{i+1}, L_{i+1} are free pairs and
(1) $G_i = G_{i+1} - e$ and $L_i = L_{i+1}$ or
(2) $G_i = G_{i+1}$ and if $L_i = [j]$ then $L_{i+1} = [j+1]$.

It then follows that the number of such induction tables is equal to the number of chains in $\mathscr{T}_n$ times the number ways of putting on the loops, i.e.,

$$\binom{\binom{n+1}{2}}{n} \frac{\binom{n}{2}!}{(n-1)_{n-1}(n)_{n-2} \ldots (2n-3)_1}.$$

Now, if we wish to take into account the labeling, note that every permutation of $[n]$ gives rise to a unique labeling by the order in which the vertices receive their loops. This concludes the proof. □

Acknowledgements. We would like thank Uri Peled, Bruce Sagan, Doug West, and Günter Ziegler for helpful discussions.

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