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THOMAS EHRHARD

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THOMAS EHRHARD

Fédération de Recherche des Unités de Mathématiques de Marseille, FR 2291
Institut de Mathématiques de Luminy, UMR 6206
Campus de Luminy, Case 907, 13288 Marseille Cedex 9, France
Email: ehrhard@iml.univ-mrs.fr

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We investigate a new denotational model of linear logic based on the purely relational model. In this semantics, webs are equipped with a notion of 'finitary' subsets satisfying a closure condition and proofs are interpreted as finitary sets. In spite of a formal similarity, this model is quite different from the usual models of linear logic (coherence semantics, hypercoherence semantics, the various existing game semantics...). In particular, the standard fix-point operators used for defining the *general* recursive functions *are not* finitary, although the primitive recursion operators are. This model can be considered as a discrete analogue of the Köthe space semantics introduced in a previous paper: we show how, given a field, each finiteness space gives rise to a vector space endowed with a *linear topology*, a notion introduced by Lefschetz in 1942, and we study the corresponding model where morphisms are linear continuous maps (a version of Girard's quantitative semantics with coefficients in the field). In this way we obtain a new model of the recently introduced *differential lambda-calculus.*

1. Notation and introduction

Notation

If S is a set, we use $\mathscr{M}(S) = \mathbf{N}^S$ to denote the set of all multi-sets over S. If $\mu \in \mathscr{M}(S)$, we use $|\mu|$ to denote the support of μ, which is the set of all $a \in S$ such that $\mu(a) \neq 0$. A multi-set is finite if it has a finite support. If $a_1, \dots, a_n$ are elements of some given set S, we use $[a_1, \dots, a_n]$ to denote the corresponding multi-set over S. The usual operations on natural numbers are extended to multi-sets pointwise.

If $(S_i)_{i \in I}$ are sets, we use π_i to denote the i-th projection $\pi_i : \prod_{j \in I} S_j \to S_i$.

Introduction

In the purely relational model of linear logic, which is certainly the simplest denotational model of linear logic, formulae are interpreted as sets and proofs as relations between these sets. Additive connectives are interpreted as disjoint unions, multiplicative connectives as cartesian products and exponentials as the operation that maps a set S to the set of all finite multi-sets with domain included in S (the *finite multi-powerset* of S). In the category of finite-dimensional vector spaces over a given field, direct product, tensor product and linear function space give rise to similar operations on *bases*, for example, one obtains a basis of the tensor product of two vector spaces E and F by taking the cartesian product of a basis of E and a basis of F. Remember also that, given a basis of a vector space, there is a canonical way of defining a basis of the same cardinality for the dual of the space, and this is compatible with the fact that in the purely relational model, the linear negation of a set S is S itself.

Bearing these observations in mind, it is quite natural to think of the sets interpreting formulae in the purely relational model as the bases of some vector spaces. However, due to the exponentials, the dimension of these spaces cannot be restricted to be finite (the finite multi-powerset of a non-empty set is always infinite). Fortunately, by endowing these sets with an additional simple structure, we can preserve the interpretation of the sets interpreting formulae as 'bases': we present several aspects of this idea.

Let R be a field (or a unitary ring), given once and for all. We want to interpret formulae as R-vector spaces (or R-modules), and these spaces should admit as 'bases' the sets interpreting the corresponding formulae in the purely relational setting. So if A is a formula and $|A|$ is its interpretation in this relational model, the space $[A]$ interpreting A should be a subspace of $R^{|A|}$. Similarly, the space $[A^\perp]$ should be a subspace of $R^{|A|}$, which is isomorphic to the dual of the space $[A]$. Therefore, given $x \in [A]$ and $x' \in [A^\perp]$, we should be able to define a scalar $\langle x, x' \rangle \in R$, which is the application of the linear form x' to the vector x. If we bear in mind the fact that $|A|$ should be a kind of basis of $[A]$ and should also represent the dual of this basis in $[A^\perp]$, it appears that the formula giving $\langle x, x' \rangle$ should be

$$\langle x, x' \rangle = \sum_{a \in |A|} x_a x'_a \,.$$

However, the set $|A|$ is, in general, infinite, so we must manage to keep this sum finite (or absolutely converging if R is the field of real or complex numbers, which leads to the Köthe space approach – in the present setting, we make no topological assumption about R, so we shall require this sum to have only finitely many non-zero terms).

A simple way to fulfill this requirement is to interpret formulae as sets equipped with a notion of *finitary* subsets. Such a pair $(I, \mathscr{F})$ (where $\mathscr{F} \subseteq \mathscr{P}(I)$) will then have $(I, \mathscr{F}^\perp = \{u' \subseteq I \mid \forall u \in \mathscr{F}\ u \cap u' \text{ is finite}\})$ as orthogonal (linear negation). Since linear negation should be an involutive operation, we must require $\mathscr{F} = \mathscr{F}^{\perp\perp}$, and this will be our sole constraint on $\mathscr{F}$. So we define a *finiteness space* as a pair $X = (|X|, \mathsf{F}(X))$ where $|X|$ is a set, the *web* of X, and $\mathsf{F}(X)$ is a collection of subsets of $|X|$ satisfying $\mathsf{F}(X)^{\perp\perp} = \mathsf{F}(X)$. These subsets of $|X|$ will be called the *finitary sets* of X. The vector space associated with such a finiteness space X will be the collection $R\langle X \rangle$ of all vectors $x \in R^{|X|}$ whose support (the set of elements a of $|X|$ such that $x_a \neq 0$) is finitary in X. Therefore, by definition, for $x \in R\langle X \rangle$ and $x' \in R\langle X^\perp \rangle$ (where $X^\perp$ is obviously defined by $|X^\perp| = |X|$ and $\mathsf{F}(X^\perp) = \mathsf{F}(X)^\perp$), the sum $\sum_{a \in |X|} x_a x'_a$ will always have only finitely many non-zero terms, although both x and x' will, in general, have infinite supports.

Using finiteness spaces, in Section 2 we first build a new relational model of first-order propositional linear logic, a *linear category* (Bierman 1995), which is at first sight similar to the model of coherence spaces (Girard 1987): for instance, the notion of clique is replaced here by the notion of finitary set. However, finiteness spaces have very different properties from the properties of coherence spaces: in a coherence space, the whole structure is known when all finite cliques (and even, all two elements cliques) are known, whereas here, finite sets are always finitary, so that a finiteness space is described by the collection of all its *infinite* finitary sets and the notion of finiteness space is highly non ... finitary, in sharp contrast with the usual notions of denotational semantics (with the concept of *totality* (Girard 1986; Loader 1994) as a noticeable exception). In particular, finitary sets are not closed under directed unions (unless $\mathsf{F}(X)$ contains all subsets of $|X|$), so the situation is in some sense opposite to the usual domain-theoretic one. A finiteness space is not order-theoretically complete, but is complete in another topological sense. For this reason, interpreting recursion in finiteness spaces becomes a delicate issue; we shall show in Section 3 that tail-recursive iteration and thus, a tail-recursive version of primitive recursion, can be interpreted in finiteness spaces, so finiteness spaces provide a model for (a version of) Gödel's system $\mathscr{T}$, but *a priori* it does not give a model of PCF, as the standard interpretation of the fix-point operator Y is not a finitary set.

The observation that the fix-point is not finitary suggests some links between finiteness semantics and the normalisation properties of logical systems, and, seemingly, opens a new line of research in that direction. In this respect, the status of the empty set in finiteness spaces is interesting: usually (in coherence spaces for instance), the empty set is thought of as representing the ever-looping program. But here, since the typical ever-looping program, which is $(Y)\lambda x\, x$, cannot be interpreted, one should perhaps consider the empty set as a kind of 'daemon', in the sense of Girard's ludics (Girard 2001), that is, a pure termination, without resulting information.

One particularity of this model is that the finiteness space associated with a formula A of linear logic has as web the set interpreting A in the purely relational model of linear logic, and the interpretation of proofs in the two models coincide. The finiteness space structure is just a structure added to the purely relational model, and this structure is respected by the interpretation of proofs, which gives rise only to finitary sets. Due to the exponentials, the same cannot be said of the usual coherence semantics. The situation is identical in the models introduced in Bucciarelli and Ehrhard (2001).

In Section 4, we develop the algebraic theory of finiteness spaces, considering the R-vector spaces associated with them as explained above. Given a finiteness space X, we endow $R\langle X\rangle$ with a linear topology in the sense of Lefschetz (1942). This is a notion of topology for vector spaces or modules where basic neighbourhoods of 0 are *linear subspaces* and which is therefore quite different from the usual notions considered in functional analysis, for instance, such as Banach spaces and their locally convex generalisations. Linear topologies have a more algebraic flavour, and, in particular, make no topological assumptions on the underlying ring or field R, which will always be endowed with the discrete topology. For $R\langle X\rangle$, we choose the following basis of neighbourhoods of 0: a subset U of $R\langle X\rangle$ belongs to this basis if for some $u' \in \mathsf{F}(X^\perp)$, U is the set of all the elements of $R\langle X\rangle$ whose support does not meet u'. We then prove that $R\langle X \multimap Y\rangle$ is linearly isomorphic to the R-module of all linear and continuous functions from $R\langle X\rangle$ to $R\langle Y\rangle$ (in particular, $R\langle X^\perp\rangle$ is just the topological dual of $R\langle X\rangle$ equipped with the linear topology described above). In this way we obtain a model of multiplicative linear logic in which morphisms are linear and continuous functions on these topological vector spaces. We also exhibit the additive structure: unsurprisingly, $R\langle X \mathbin{\&} Y\rangle$ is both the direct product and the direct sum of X and Y.

We then interpret the exponential connectives of linear logic and retrieve the familiar structures: the exponential ! is an endofunctor on the category of finiteness spaces and linear continuous maps, and this functor has the canonical structure of a comonad, which allows us to interpret full first-order linear logic (in particular, there is a natural isomorphism between $!(X \mathbin{\&} Y)$ and $!X \otimes !Y$). We also show that the linear continuous functions from $R\langle !X\rangle$ to $R\langle Y\rangle$ can be seen as *entire* functions from $R\langle X\rangle$ to $R\langle Y\rangle$, that is, functions that are defined by a power series that converges on the whole space $R\langle X\rangle$. Finally, we exhibit the categorical structure of the exponential that corresponds to the differential operations on these entire functions, showing that we have obtained in this way a model of the recently introduced *differential lambda-calculus* (Ehrhard and Regnier 2003), which categorically is completely similar to the model of Köthe spaces presented in Ehrhard (2002).

All these constructions can be seen as rephrasing Girard's *quantitative semantics* of the lambda-calculus presented in Girard (1988) (see also Hasegawa (2002)) where lambda-terms are interpreted as normal functors that are power series whose coefficients are (possibly infinite) sets; the role of our additional finiteness space structure is to keep these coefficients finite and recast the quantitative approach in a completely standard algebraic setting. The price to pay is the impossibility of interpreting fix-point operators, but we tend to consider this to be a rather interesting feature: after all we do not have many simple denotational models introducing a natural divide between computational primitives.

2. The finitary relational model

Let I be a set and let $u, u' \subseteq I$. In this paper, we say that u and u' are *in duality*[†] if $u \cap u'$ is a *finite* set[‡]. Let $\mathscr{F} \subseteq \mathscr{P}(I)$. We use $\mathscr{F}^\perp$ to denote the set

$$\mathscr{F}^\perp = \{u' \subseteq I \mid \forall u \in \mathscr{F} \quad u \cap u' \text{ is finite}\} \subseteq \mathscr{P}(I).$$

Obviously, if u' finite, $u' \in \mathscr{F}^\perp$. It is also clear that if $u' \subseteq v' \in \mathscr{F}^\perp$, then $u' \in \mathscr{F}^\perp$, and $\mathscr{F}^\perp$ is closed under finite unions.

[†] One could say 'orthogonal', according to the tradition of linear logic, but this is a misleading terminology when one deals with vector spaces, so we shall avoid it.

[‡] Other natural definitions of duality, giving rise to other models of linear logic, are:

- $u \cap u'$ has at most one element, which gives rise to the standard model of coherence spaces;
- $u \cap u'$ is not empty, which gives rise to a quite simple model of non-uniform *totality*;
- $u \cap u'$ has exactly one element, which gives rise to Loader's totality spaces (Loader 1994);
- another natural choice, suggested by one of the referees of this paper, might be to require $u \cap u'$ to be co-finite. We have no idea what the resulting model might be, if any.

Due, maybe, to the logical complexity of the condition that the class of 'good' sets should be equal to its bi-dual (see the definition of finiteness spaces given later), it seems that these different choices lead to quite different interpretations. For instance, finiteness spaces are very different from coherence spaces. It is not clear whether these various cases can be handled within a common framework. Interestingly enough, the hypercoherence model (Ehrhard 1993) does not seem to admit such a synthetic description.

Moreover, this duality operation on subsets of $\mathscr{P}(I)$ has the following immediate properties, which we shall use implicitly. These are just 'abstract non-sense' properties, which do not use the particular definition of duality between elements of $\mathscr{P}(I)$ (here, having a finite intersection).

— $\mathscr{F} \subseteq \mathscr{G} \Rightarrow \mathscr{G}^\perp \subseteq \mathscr{F}^\perp$;
— $\mathscr{F} \subseteq \mathscr{F}^{\perp\perp}$;
— $\mathscr{F}^{\perp\perp\perp} = \mathscr{F}^\perp$.

The following statement is not particularly useful, but just shows that the closure $\mathscr{F}^{\perp\perp}$ of $\mathscr{F}$ admits a 'direct' characterisation.

Proposition 1. Let $\mathscr{F} \subseteq \mathscr{P}(I)$ be downward closed (that is, if $u \subseteq v \in \mathscr{F}$, then $u \in \mathscr{F}$). Let $u' \subseteq I$. Then $u' \in \mathscr{F}^\perp$ iff there is no infinite subset v of u' such that $v \in \mathscr{F}$ (that is $\mathscr{P}(u') \cap \mathscr{F} \subseteq \mathscr{P}_{\mathrm{fin}}(I)$). Let $u \subseteq I$. Then $u \in \mathscr{F}^{\perp\perp}$ iff, for any infinite subset v of u, there exists an infinite subset w of v such that $w \in \mathscr{F}$.

The proof is straightforward.

A *finiteness space* is a pair $X = (|X|, \mathsf{F}(X))$ where $|X|$ is a set (which can be assumed to be countable, since this property is preserved by all the constructions we consider) and $\mathsf{F}(X)$ is a subset of $\mathscr{P}(|X|)$ satisfying $\mathsf{F}(X)^{\perp\perp} = \mathsf{F}(X)$. The elements of $\mathsf{F}(X)$ are called the *finitary* sets of X. Observe that if $|X|$ is finite, $\mathsf{F}(X)$ must be the powerset of $|X|$: in finite dimension, there is only one possible finiteness structure.

The finiteness semantics of linear logic associates with any formula G a finiteness space $[G]$ and to any proof π of G, an element $[\pi]$ of $\mathsf{F}([G])$. More precisely, $|[G]|$ will be the interpretation of G in the purely relational semantics of first-order propositional linear logic (this interpretation is simply denoted by $|G|$), and $[\pi]$ will be the interpretation of π in this purely relational semantics (see the appendix, Section A).

2.1. *Object constructions*

As usual, we define various operations on finiteness spaces, which will interpret the corresponding logical operations on formulae. We first deal with the additive and multiplicative connectives. Let X and Y be finiteness spaces.

2.1.1. *Orthogonal.* The space $X^\perp$ is defined by $|X^\perp| = |X|$ and $\mathsf{F}(X^\perp) = \mathsf{F}(X)^\perp$.

2.1.2. *Additives.* The space 0 is defined by $|0| = \varnothing$ and $\mathsf{F}(0) = \{\varnothing\}$. The space $\top$ is its dual: $\top = 0^\perp = 0$, clearly.

The space $X \oplus Y$ is given by $|X \oplus Y| = |X| + |Y|$, and $\mathsf{F}(X \oplus Y) = \{u + v \mid u \in \mathsf{F}(X)$ and $v \in \mathsf{F}(Y)\}$ (where we use '+' to denote the disjoint union of sets). Indeed, one can easily check that $\mathsf{F}(X \oplus Y) = \mathsf{F}(X^\perp \oplus Y^\perp)^\perp$, and, therefore, $\mathsf{F}(X \oplus Y)^{\perp\perp} = \mathsf{F}(X \oplus Y)$ and the operation $\oplus$ coincides with its De Morgan dual &, that is $X \mathbin{\&} Y = X \oplus Y$.

2.1.3. *Multiplicatives.* We set $|1| = \{*\}$ and $\mathsf{F}(1) = \{\varnothing, \{*\}\}$ (the only possible choice), and its dual $\perp$ satisfies $\perp = 1^\perp = 1$.

The space $X \otimes Y$ is defined by $|X \otimes Y| = |X| \times |Y|$ and

$$\mathsf{F}(X \otimes Y) = \{u \times v \mid u \in \mathsf{F}(X) \text{ and } v \in \mathsf{F}(Y)\}^{\perp\perp}.$$

The next lemma is important as it shows that the bi-duality closure used in the definition of $\mathsf{F}(X \otimes Y)$ is essentially useless: in this particular case, it boils down to the $\subseteq$-downward closure of $\{u \times v \mid u \in \mathsf{F}(X) \text{ and } v \in \mathsf{F}(Y)\}$.

Lemma 2. Let $w \subseteq |X \otimes Y|$. Then $w \in \mathsf{F}(X \otimes Y)$ iff $\pi_1(w) \in \mathsf{F}(X)$ and $\pi_2(w) \in \mathsf{F}(Y)$.

Proof. Assume first that $\pi_1(w) \in \mathsf{F}(X)$ and $\pi_2(w) \in \mathsf{F}(Y)$. Then $w \subseteq \pi_1(w) \times \pi_2(w) \in \mathsf{F}(X \otimes Y)$, so $w \in \mathsf{F}(X \otimes Y)$.

Conversely, assume that $w \in \mathsf{F}(X \otimes Y)$. Let $u' \in \mathsf{F}(X)^\perp$. It will be sufficient to show that $u_0 = \pi_1(w) \cap u'$ is finite. Let $f : u_0 \to \pi_2(w)$ be a function such that for all $a \in u_0$, $(a, f(a)) \in w$ (that is $f \subseteq w$, identifying a function with its graph). We show that $f \in \mathsf{F}(X \otimes Y)^\perp = \{u \times v \mid u \in \mathsf{F}(X) \text{ and } v \in \mathsf{F}(Y)\}^\perp$. So let $u \in \mathsf{F}(X)$ and $v \in \mathsf{F}(Y)$. Since $u_0 \subseteq u' \in \mathsf{F}(X)^\perp$, the set $u_0 \cap u$ is finite and hence, since f is a function, $f \cap (u \times v)$ is finite. So $f \in \mathsf{F}(X \otimes Y)^\perp$, and, therefore, $f \cap w$ is finite, but $f \subseteq w$, so f is finite, and hence u_0 is finite, since $u_0 = \pi_1(f)$. $\square$

Observe that this proof uses the Axiom of Choice, and we do not know if this can be avoided.

The other multiplicative operations are defined as usual: the *par* is given by $X \parr Y = (X^\perp \otimes Y^\perp)^\perp$ and the *linear implication* by $X \multimap Y = (X \otimes Y^\perp)^\perp$. Tensor and par do not coincide in general (in contrast with what happened for the additives).

Lemma 3. Let $t \subseteq |X \multimap Y| = |X| \times |Y|$. Then $t \in \mathsf{F}(X \multimap Y)$ iff the following two conditions hold:

1 For any $u \in \mathsf{F}(X)$, we have $t(u) = \{b \in |Y| \mid \exists a \in u\ (a, b) \in t\} \in \mathsf{F}(Y)$.
2 For any $v' \in \mathsf{F}(Y)^\perp$, we have $t^\perp(v') \in \mathsf{F}(X)^\perp$ where $t^\perp = \{(b, a) \mid (a, b) \in t\} \subseteq |Y| \times |X| = |Y^\perp \multimap X^\perp|$ is the transpose of t.

Moreover, condition 2 can be weakened to:

2′ For any $b \in |Y|$, we have $t^\perp(\{b\}) \in \mathsf{F}(X)^\perp$.

Proof. Assume first that $t \in \mathsf{F}(X \multimap Y)$. Let $u \in \mathsf{F}(X)$. We show that $t(u) \in \mathsf{F}(Y)$. So let $v' \in \mathsf{F}(Y)^\perp$. We have $u \times v' \in \mathsf{F}(X \otimes Y^\perp)$, so $t \cap (u \times v')$ is finite, so $t(u) \cap v' = \pi_2(t \cap (u \times v'))$ is finite. Hence $t(u) \in \mathsf{F}(Y)$. So condition 1 holds for t, as does condition 2 since, clearly, $t^\perp \in \mathsf{F}(Y^\perp \multimap X^\perp)$.

Conversely, assume that t satisfies conditions 1 and 2′. Let $u \in \mathsf{F}(X)$ and $v' \in \mathsf{F}(Y)^\perp$. The set $t_0 = t \cap (u \times v')$ is finite, since $\pi_2(t_0) = t(u) \cap v'$ is finite by 1, and $\pi_1(t_0) = \bigcup_{b \in \pi_2(t_0)} (t^\perp(\{b\}) \cap u)$ is finite, as a finite union of sets, which are finite by 2′. $\square$

2.1.4. *Exponentials.* These are the constructions that really introduce the infinite into logic, and here they create infinite finitary sets. We take for $|!X|$ the set $\mathscr{M}_{\mathrm{fin}}(|X|)$ of all finite multi-sets over $|X|$ and set

$$\mathsf{F}(!X) = \{u^! \mid u \in \mathsf{F}(X)\}^{\perp\perp} \tag{1}$$

where $u^! = \{\mu \in |!X| \mid |\mu| \subseteq u\} = \mathscr{M}_{\mathrm{fin}}(u)$ (a set that is infinite as long as u is non-empty). If $U \subseteq |!X|$, we define $|U| = \bigcup\{|\mu| \mid \mu \in U\}$ and call this set the *global support* of U.

The next lemma is an analogue of Lemma 2, so it is very important.

Lemma 4. Let $U \subseteq |!X|$. Then $U \in \mathsf{F}(!X)$ iff $|U| \in \mathsf{F}(X)$.

Proof. If $|U| \in \mathsf{F}(X)$, then $U \subseteq |U|^! \in \mathsf{F}(!X)$, and we conclude immediately.

Assume conversely that $U \in \mathsf{F}(!X)$. Let $u' \in \mathsf{F}(X)^\perp$. It will be sufficient to show that $u_0 = |U| \cap u'$ is finite. Let $f : u_0 \to U$ be a function such that, for each $a \in u_0$, we have $a \in |f(a)|$. Such a function exists, since $u_0 \subseteq |U|$. Let $U' = f(u_0)$. We contend that $U' \in \mathsf{F}(!X)^\perp$. Let $u \in \mathsf{F}(X)$. It will be sufficient to show that $U' \cap u^!$ is finite. But $f^{-1}(U' \cap u^!) = f^{-1}(U') \cap f^{-1}(u^!) = u_0 \cap f^{-1}(u^!) \subseteq u' \cap u$ and $u' \cap u$ is finite, so $U' \cap u^!$ is finite because f is surjective onto U'. Therefore $U' \in \mathsf{F}(!X)^\perp$. Hence $U \cap U' = U'$ is finite, and therefore, so is $|U'|$. But clearly $u_0 \subseteq |U'|$, so u_0 is finite, as desired. □

This proof also uses the Axiom of Choice.

2.2. *The category of finiteness spaces and finitary relations*

We define the category of finiteness spaces and finitary relations **Fin**: its objects are the finiteness spaces, and if X and Y are finiteness spaces, a morphism from X to Y in **Fin** is an element of $\mathsf{F}(X \multimap Y)$, a *finitary relation* from X to Y. Identity and composition of morphisms are defined as usual (as in the category of sets and relations). It follows from Lemma 3 that in this way we have defined a category.

It is straightforward to check that **Fin** is cartesian, & being the cartesian product and $\top$ the terminal object (it is also co-cartesian, and finite sums coincide with finite products). The pairing operation on morphisms and the projections are defined exactly as in the category of sets and relations.

2.2.1. *Monoidal structure.*

Let $t_1 \in \mathbf{Fin}(X_1, Y_1)$ and $t_2 \in \mathbf{Fin}(X_2, Y_2)$. We define, as in the purely relational model,

$$t_1 \otimes t_2 = \{((a_1, a_2), (b_1, b_2)) \mid (a_1, b_1) \in t_1 \text{ and } (a_2, b_2) \in t_2\}.$$

We prove that $t_1 \otimes t_2 \in \mathbf{Fin}(X_1 \otimes X_2, Y_1 \otimes Y_2)$. So, let $u_1 \in \mathsf{F}(X_1)$, $u_2 \in \mathsf{F}(X_2)$ and $w' \in \mathsf{F}(Y_1 \otimes Y_2)^\perp$. It will be sufficient to show that $w_0 = (t_1 \otimes t_2) \cap ((u_1 \times u_2) \times w')$ is finite; indeed, by Lemma 2, any element of $\mathsf{F}(X_1 \otimes X_2)$ is included in a set of the shape $v_1 \times v_2$, with $v_1 \in \mathsf{F}(X_1)$ and $v_2 \in \mathsf{F}(X_2)$. But $\pi_2(w_0) = (t_1(u_1) \times t_2(u_2)) \cap w'$, so $\pi_2(w_0)$ is finite. Let $(b_1, b_2) \in \pi_2(w_0)$. To conclude, it will be sufficient to show that the set $u = \{(a_1, a_2) \mid ((a_1, a_2), (b_1, b_2)) \in w_0\} \subseteq |X_1| \times |X_2|$ is finite. But $\pi_1(u) \subseteq u_1 \cap {t_1}^\perp(\{b_1\})$, and this latter intersection is finite, since $\{b_1\} \in \mathsf{F}({Y_1}^\perp)$. Similarly, $\pi_2(u)$ is finite, so u is finite.

The category **Fin**, equipped with the tensor product $\otimes$, is symmetric monoidal. The tensor product is indeed associative, as can be checked easily using Lemma 2 again (the isomorphism between $X \otimes (Y \otimes Z)$ and $(X \otimes Y) \otimes Z$ being the obvious bijection between the webs of these spaces). Symmetry of the tensor product is obvious. The tensor unit is 1 and the various required coherence diagrams (see, for example, Mac Lane (1971) and Bierman (1995)) commute simply because they commute in the category of sets and relations. This category is monoidal closed, the objects of morphisms from X to Y being $X \multimap Y$; this is due to the associativity of the tensor product, to our definition of $X \multimap Y$ as $(X \otimes Y^{\perp})^{\perp}$ and to the fact that the operation $X \mapsto X^{\perp}$ is an involutive (contravariant) endofunctor, defined on morphisms as the transpose operation $t \mapsto t^{\perp}$ mentioned in Lemma 3. Moreover, by Lemma 3, the obvious bijection between $|X^{\perp}|$ and $|X \multimap \perp|$ is an isomorphism between $X^{\perp}$ and $X \multimap \perp$, so **Fin** is $\star$-autonomous.

2.2.2. *Functorial and comonadic structure of the exponential.* We first define the functorial promotion of morphisms. Let $t \in \mathbf{Fin}(X, Y)$, and set, as in the purely relational model,

$$!t = \{([a_1, \dots, a_n], [b_1, \dots, b_n]) \mid (a_i, b_i) \in t \text{ for each } i = 1, \dots, n\}.$$

We prove that $!t \in \mathbf{Fin}(!X, !Y)$. Let $u \in \mathsf{F}(X)$ and $V' \in \mathsf{F}(!Y)^{\perp}$. It will be sufficient to show that $W = !t \cap (u^! \times V')$ is finite; indeed, by Lemma 4, any element of $\mathsf{F}(!X)$ is included in a set of the shape $v^!$ with $v \in \mathsf{F}(X)$. But $\pi_2(W) = !t(u^!) \cap V' = t(u)^! \cap V'$ is finite by Lemma 3 applied to t. Now let $b_1, \dots, b_n \in |Y|$ be such that $v = [b_1, \dots, b_n] \in \pi_2(W)$. It will be sufficient to show that the set $U = \{\mu \mid (\mu, v) \in W\}$ is finite. But if $\mu \in U$, we can write μ as $\mu = [a_1, \dots, a_n]$ with $(a_i, b_i) \in t$ and $a_i \in u$ for $i = 1, \dots, n$. So U is contained in the image of the set $(t^{\perp}(\{b_1\}) \cap u) \times \cdots \times (t^{\perp}(\{b_n\}) \cap u)$ under the mapping $(a_1, \dots, a_n) \mapsto [a_1, \dots, a_n]$. But each of the sets $t^{\perp}(\{b_i\}) \cap u$ is finite by Lemma 3 (as $\{b_i\} \in \mathsf{F}(Y)^{\perp}$), so U is finite.

There is a canonical bijection between $|!(X \mathbin{\&} Y)|$ and $|!X \otimes !Y|$: associate with each $\mu \in |!(X \mathbin{\&} Y)|$ the pair of multi-sets (λ, ρ), where λ is the restriction of μ to $|X|$ and ρ is the restriction of μ to $|Y|$ (multi-sets being considered as integer-valued functions). This bijection is an isomorphism of finiteness spaces from $!(X \mathbin{\&} Y)$ to $!X \otimes !Y$ (this is actually an immediate consequence of Lemmas 2 and 4):

$$!(X \mathbin{\&} Y) \simeq !X \otimes !Y. \tag{2}$$

It is also obvious that $!\top$ and 1 are isomorphic.

It remains to check that the structure maps d^X (dereliction) and p^X (digging) of the purely relational model are morphisms in the category **Fin**. Remember that

$$\mathrm{d}^X = \{([a], a) \mid a \in |X|\} \subseteq |!X \multimap X|.$$

Let $u \in \mathsf{F}(X)$ and $u' \in \mathsf{F}(X)^{\perp}$. Then $\mathrm{d}^X \cap (u^! \times u') = \{([a], a) \mid a \in u \cap u'\}$ is finite since $u \cap u'$ is finite. Therefore, $\mathrm{d}^X \in \mathsf{F}(!X \multimap X)$.

Next, remember that

$$\mathrm{p}^X = \{(\mu_1 + \cdots + \mu_n, [\mu_1, \dots, \mu_n]) \mid \mu_1, \dots, \mu_n \in |!X|\} \subseteq |!X \multimap !!X|.$$

Let $u \in \mathsf{F}(X)$ and $\mathscr{U}' \in \mathsf{F}(!!X)^{\perp}$. We will prove that $\mathrm{p}^X \cap (u^! \times \mathscr{U}')$ is finite. Observe that $(u^!)^! = \{[\mu_1, \dots, \mu_n] \mid \mu_1, \dots, \mu_n \in |!X| \text{ and } \mu_1 + \cdots + \mu_n \in u^!\}$, and therefore $\mathrm{p}^X \cap (u^! \times \mathscr{U}') = \{(\mu_1 + \cdots + \mu_n, [\mu_1, \dots, \mu_n]) \mid [\mu_1, \dots, \mu_n] \in (u^!)^! \cap \mathscr{U}'\}$, but $(u^!)^! \cap \mathscr{U}'$ is finite, and hence $\mathrm{p}^X \cap (u^! \times \mathscr{U}')$ is finite.

2.2.3. *Infinitary additives.* We have seen that $\oplus$ and $\&$ are interpreted in the same way in finiteness spaces and $X \oplus Y = X \mathbin{\&} Y$ is both the cartesian product and the direct sum of X and Y (biproduct). This is no longer the case for the infinitary versions of these operations. Let $(X_l)_{l\in L}$ be a family of finiteness spaces, whose webs are assumed to be pairwise disjoint for notational convenience. Their direct product, denoted by $\&_{l\in L} X_l$, has $\sum_{l\in L} |X_l|$ as web, and a subset w of this web will belong to $\mathsf{F}(\&_{l\in L} X_l)$ iff each of its 'projections' $w \cap |X_l|$ belongs to $\mathsf{F}(X_l)$. Indeed, it is easily seen that, with this definition of $\mathsf{F}(\&_{l\in L} X_l)$, a subset w' of $\sum_{l\in L} |X_l|$ belongs to $\mathsf{F}(\&_{l\in L} X_l)^\perp$ iff $w' \cap |X_l|$ is empty for almost all values of l, and $w' \cap |X_l| \in \mathsf{F}(X_l)^\perp$ for each $l \in L$. Therefore, $\mathsf{F}(\&_{l\in L} X_l)^{\perp\perp} = \mathsf{F}(\&_{l\in L} X_l)$. These considerations also lead us to define infinitary sum $\oplus_{l\in L} X_l$ as follows: its web is $\sum_{l\in L} |X_l|$, and a subset w of this web belongs to $\mathsf{F}(\oplus_{l\in L} X_l)$ iff $w \cap |X_l|$ is empty for almost all values of l, and $w \cap |X_l| \in \mathsf{F}(X_l)$ for each $l \in L$.

It is easy to check that the space $\&_{l\in L} X_l$ so defined is indeed the product of the spaces X_l in the category **Fin** (and, therefore, $\oplus_{l\in L} X_l$ is their sum).

Typically, it is reasonable to consider the type of natural numbers as a solution to the fix-point equation $N = 1 \oplus N$, and for this reason, the finiteness space of natural numbers N is defined as follows: $|\mathsf{N}| = \mathbf{N}$, the set of natural numbers, and a subset u of $\mathbf{N}$ belongs to $\mathsf{F}(\mathsf{N})$ iff u is finite. The successor and predecessor functions (which define the natural bijection between the webs of the spaces $1 \oplus \mathsf{N}$ and N) are easily seen to be isomorphisms between these finiteness spaces, so they are finitary. The dual is given by $|\mathsf{N}^\perp| = \mathbf{N}$ and $\mathsf{F}(\mathsf{N}^\perp) = \mathscr{P}(\mathbf{N})$.

2.3. *A finiteness property of the relational interpretation of linear logic*

We have defined a semantics of linear logic in which formulae are interpreted as finiteness spaces and proofs as finitary subsets. But the operations on webs and morphisms used for defining this semantics are just the usual corresponding operations in the pure relational semantics. Therefore, we can derive from our model construction some properties of the purely relational semantics of linear logic. Typically, we have the following result.

Proposition 5. Let π be a proof of the sequent $\vdash \Gamma, G$ and let σ be a proof of the sequent $\vdash G^\perp, \Delta$. Let ρ be the proof of $\vdash \Gamma, \Delta$ obtained by applying a cut rule. Then, for any element (φ, ψ) of $[\rho]$, there is only a finite number of elements a of $[G]$ such that $(\varphi, a) \in [\pi]$ and $(a, \psi) \in [\sigma]$.

Bearing in mind the usual coherence space semantics, one might think that this result is completely trivial and that the finite set in question is always a singleton. But this would be forgetting the essential difference between the purely relational model and the coherence space model concerning the issue of *uniformity*. The point is that, in coherence spaces, when one builds $!X$, one takes for $|!X|$ the collection of all finite multi-sets μ of elements of $|X|$ such that $|\mu|$ is a *clique* of X. It is only for this reason that, in the setting of the theorem, the 'intermediate set' is a singleton. Without this uniformity restriction when building the exponential, the intermediate sets can have arbitrarily large finite cardinalities.

The following typical example is formulated in a lambda-calculus style for notational convenience only. Consider the term t, representing a normal proof of $\vdash (!\mathbf{Bool})^\perp, \mathbf{Bool}$

where **Bool** $= 1 \oplus 1$ (with exactly two normal proofs denoted by **t** and **f**),

$$t(x) = \mathtt{if}\ x\ \mathtt{then}\,(\mathtt{if}\ x\ \mathtt{then}\,\mathbf{t}\,\mathtt{else}\,\mathbf{f})\,\mathtt{else}\,(\mathtt{if}\ x\ \mathtt{then}\,\mathbf{t}\,\mathtt{else}\,\mathbf{f}).$$

Consider, on the other hand, the following term u, representing a normal proof of $\vdash \mathbf{Bool}^{\perp}, \mathbf{Bool}$,

$$u(y) = \mathtt{if}\ y\ \mathtt{then}\,\mathbf{t}\,\mathtt{else}\,\mathbf{t}\,.$$

Then, with the notations of the theorem, taking $\varphi = [\mathbf{t}, \mathbf{f}] \in |!\mathbf{Bool}|$ and $\psi = \mathbf{t} \in |\mathbf{Bool}|$, we get $\{\mathbf{t}, \mathbf{f}\}$ as the set of intermediate points. In coherence spaces, this phenomenon does not occur, simply because $[\mathbf{t}, \mathbf{f}]$ is not an element of the web of the coherence space associated with $!\mathbf{Bool}$ since **t** and **f** are not coherent, due to the definition of $\oplus$.

3. Iteration and fix-points

One particularly interesting feature of this semantics is that it does not admit the usual fix-point operators as finitary objects. However, it is reasonably expressive in computational terms since there are finitary iteration operators, as we shall see.

We extend linear logic with a type **N** of natural numbers, which is intuitively subject to the equation $\mathbf{N} = 1 \oplus \mathbf{N}$. This can be represented in sequent calculus by the following rules and axioms. We do not claim that this particular presentation has any good proof-theoretic properties, and introduce it only to give precise definitions of a possible extension of propositional linear logic with a basic type of natural numbers, and of the denotational semantics of such a system.

$$\frac{}{\vdash \mathbf{N}}\ (\mathit{Zero})$$

is an axiom with denotation the finitary set $\{0\} \subseteq \mathbf{N}$.

$$\frac{\begin{array}{c}\vdots\,\pi\\ \vdash \mathbf{N}, \Gamma\end{array}}{\vdash \mathbf{N}, \Gamma}\ (\mathit{Successor})$$

is a proof with denotation $\{(n+1, \gamma) \mid (n, \gamma) \in [\pi]\}$, which is finitary as long as $[\pi]$ is.

And, finally,

$$\frac{\begin{array}{c}\vdots\,\zeta\\ \vdash \Gamma\end{array} \qquad \begin{array}{c}\vdots\,\sigma\\ \vdash \mathbf{N}^{\perp}, \Gamma\end{array}}{\vdash \mathbf{N}^{\perp}, \Gamma}\ (\mathit{Case})$$

is a proof, with denotation $\{(0, \gamma) \mid \gamma \in [\zeta]\} \cup \{(n+1, \gamma) \mid (n, \gamma) \in [\sigma]\}$, which is finitary as long as $[\zeta]$ and $[\sigma]$ are.

For each set U, we define an iteration operator $\mathsf{It}_U \subseteq U^+$ where

$$U^+ = \mathbf{N} \multimap\ !(!U \multimap U) \multimap\ !U \multimap U$$

(with connectives interpreted in the purely relational model of linear logic) as a union $\mathsf{It}_U = \bigcup_{n\in\mathbf{N}} \mathsf{It}_U^{(n)}$ of increasing approximations. We set $\mathsf{It}_U^{(0)} = \{(0, [], [a], a) \mid a \in U\}$ and

$$\mathsf{It}_U^{(n+1)} = \mathsf{It}_U^{(0)} \cup \left\{ \left(m+1, \varphi + [(\mu_1, b_1), \dots, (\mu_k, b_k)], \sum_{i=1}^{k} \mu_i, b \right) \mid (m, \varphi, [b_1, \dots, b_k], b) \in \mathsf{It}_U^{(n)} \right\}.$$

Consider the operator $\mathsf{Case}_U \subseteq \mathbf{N} \multimap\, !(\mathbf{N} \multimap U) \multimap\, !U \multimap U$ given by

$$\mathsf{Case}_U = \{(0, [], [a], a) \mid a \in U\} \cup \{(m+1, [(m, b)], [], b)\}.$$

This case operator is definable in linear logic as follows:

$$\dfrac{\dfrac{\dfrac{\vdash U^\perp, U}{\vdash ?U^\perp, U}\,(\mathit{Dereliction})}{\vdash ?(\mathbf{N}\otimes U^\perp), ?U^\perp, U}\,(\mathit{Weakening}) \qquad \dfrac{\dfrac{\dfrac{\vdash \mathbf{N}^\perp, \mathbf{N} \qquad \vdash U^\perp, U}{\vdash \mathbf{N}^\perp, \mathbf{N}\otimes U^\perp, U}\,(\mathit{Tensor})}{\vdash \mathbf{N}^\perp, ?(\mathbf{N}\otimes U^\perp), U}\,(\mathit{Dereliction})}{\vdash \mathbf{N}^\perp, ?(\mathbf{N}\otimes U^\perp), ?U^\perp, U}\,(\mathit{Weakening})}{\vdash \mathbf{N}^\perp, ?(\mathbf{N}\otimes U^\perp), ?U^\perp, U}\,(\mathit{Case})$$

Consider the definable operator $\Phi : !U^+ \multimap U^+$ given by

$$\Phi = \lambda\mathscr{I}\,\lambda n\,\lambda f\,\lambda x\,(((\mathsf{Case}_U)n)\lambda m\,(\mathscr{I})mf(f)x)x$$

(we adopt Krivine's notational conventions for the λ-calculus: the application of M to N is written $(M)N$ and $(\dots((M)N_1)\dots)N_k$ is written simply $(M)N_1\dots N_k$).

One can check that $\mathsf{It}_U^{(0)} = \Phi(\varnothing)$ and $\mathsf{It}_U^{(n+1)} = \Phi(\mathsf{It}_U^{(n)})$ in the purely relational model. It follows that It_U is indeed a (tail-recursive) iteration operator.

Proposition 6. Iteration is a finitary operation: that is, if X is a finiteness space, then

$$\mathsf{It}_{|X|} \in \mathsf{F}(\mathsf{N} \multimap\, !(!X \multimap X) \multimap\, !X \multimap X).$$

Proof. Observe first that if $(m, \varphi, \mu, a) \in \mathsf{It}_{|X|}^{(n)}$, then $m \leqslant n$. Now let $u \in \mathsf{F}(\mathsf{N})$: that is, u is a finite subset of $\mathbf{N}$. Let $n \in \mathbf{N}$ be greater than all the elements of u. Then by the observation above, $\mathsf{It}_{|X|}(u) = \mathsf{It}_{|X|}^{(n)}(u)$. But $\mathsf{It}_{|X|}^{(n)} = \Phi^{n+1}(\varnothing)$ and Φ is finitary (since it is definable in linear logic), and, therefore, $\mathsf{It}_{|X|}^{(n)}$ is finitary. So, by Lemma 3, $\mathsf{It}_{|X|}(u) \in \mathsf{F}(!(!X \multimap X) \multimap\, !X \multimap X)$.

To conclude, it suffices to show that

$$H \in \mathsf{F}(!(!X \multimap X) \multimap\, !X \multimap X)^\perp \Rightarrow {\mathsf{It}_{|X|}}^\perp(H) \in \mathsf{F}(\mathsf{N})^\perp.$$

But this is trivial since $\mathsf{F}(\mathsf{N})^\perp = \mathscr{P}(\mathbf{N})$. $\square$

We have defined It_U as the least fix-point of the operator Φ, and in this way obtained a finitary operation. One might expect therefore that the least fix-point operator itself is finitary. We now show that this is not the case.

In the purely relational semantics, the fix-point operator at type U is the set $\mathsf{Y}_U \subseteq\, !(!U \multimap U) \multimap U$ given by $\mathsf{Y}_U = \bigcup_{n\in\mathbf{N}} \mathsf{Y}_U^{(n)}$ where $\mathsf{Y}_U^{(0)} = \varnothing$ and

$$\mathsf{Y}_U^{(n+1)} = \left\{ \left([([a_1, \dots, a_k], a)] + \sum_{i=1}^{k} \varphi_i, a \right) \mid (\varphi_i, a_i) \in \mathsf{Y}_U^{(n)} \text{ for } i = 1, \dots, k \right\}.$$

The reader can check that this is just the usual definition of the least fix-point operator in the Kleisli category of the purely relational model of linear logic, which is a model of PCF (a cpo-enriched cartesian closed category, the order on morphisms being inclusion).

Our negative argument is based on a PCF term that has already been considered in Danos and Harmer (2000). Using the least fix-point operator, we construct $f \subseteq !\mathbf{N} \multimap \mathbf{N}$ as follows:

$$f = (\mathsf{Y}_{!\mathbf{N}\multimap\mathbf{N}})\lambda g\,\lambda n\,(((\mathsf{Case}_{\mathbf{N}})n)\lambda m\,(1 + (g)(m+1)))0.$$

In other words, f is defined recursively as

$$f \;=\; \lambda n\,\mathtt{if}\,n = 0\,\mathtt{then}\,0\,\mathtt{else}\,(1 + (f)n)\,.$$

Then $\{(k[1] + [0], k) \mid k \in \mathbf{N}\} \subseteq f$, and, therefore, $f(\{0,1\}^!) = \mathbf{N}$. But $\{0,1\}^! \in \mathsf{F}(!\mathsf{N})$ and $\mathbf{N} \notin \mathsf{F}(\mathsf{N})$. So $f \notin \mathsf{F}(!\mathsf{N} \multimap \mathsf{N})$. Therefore the fix-point operator $\mathsf{Y}_{!\mathbf{N}\multimap\mathbf{N}}$ is not finitary in the finiteness space $!(!X \multimap X) \multimap X$ where $X = !\mathsf{N} \multimap \mathsf{N}$.

Proposition 7. In general, the fix-point operator $\mathsf{Y}_{|X|}$ is not finitary in $!(!X \multimap X) \multimap X$.

4. Module associated with a finiteness space

It is possible to interpret linear logic and the simply typed differential lambda-calculus in finiteness space, each proof or differential lambda-term being mapped to a finitary set in the finiteness space interpreting its type. This might be called a *qualitative* interpretation (this terminology is due to Girard (Girard 1986; Girard 1988)). We are, however, more interested here in a *quantitative* semantics where proofs and terms are interpreted as linear, or, more generally, analytic functions, with coefficients taken in a field, or more generally a ring. The coefficients appearing in the interpretations of proofs and terms will be non-negative integers (not only 0's and 1's), but it will make sense to apply the corresponding functions to arbitrary vectors.

We associate a vector space, or more generally a module, with any finiteness space. The web of the finiteness space will be a kind of (generally infinite) 'basis' for this vector space and the finitary structure will tell us which are the acceptable vectors (linear combinations of vectors taken in this basis).

Let R be a fixed ring (or even a semi-ring; in the case $R = \{0,1\}$ with the semi-ring structure defined by $1 + 1 = 1$, we retrieve the relational 'qualitative' model presented in Section 2.2). When convenient, we shall assume that R is a field, but this is not necessary in general.

Given a finiteness space X, we define an R-module $R\langle X\rangle$ as follows. An element x of $R\langle X\rangle$ is an $|X|$-indexed family of elements of R whose support $|x| = \{a \in |X| \mid x_a \neq 0\}$ belongs to $\mathsf{F}(X)$. Module operations are defined componentwise. If $a \in |X|$, we use e_a to denote the element of $R\langle X\rangle$ given by $(e_a)_b = \delta_{a,b}$. The family $(e_a)_{a\in|X|}$ is a kind of 'canonical basis' of $R\langle X\rangle$ (although it is not, in general, a generating system in the standard algebraic sense). If $\mathsf{F}(X) = \mathscr{P}_{\mathrm{fin}}(|X|)$ is the minimal finitary structure over $|X|$, then $R\langle X\rangle$ is just the free R-module generated by $|X|$.

We can endow the module $R\langle X\rangle$ with a *linear topology* in the sense of Lefschetz (Lefschetz 1942; Barr 1976; Blute 1996). This topology, which we denote by $\lambda(X)$, is defined as follows. For $u' \in \mathsf{F}(X^\perp)$, let us set

$$\mathrm{V}_X(u') = \{x \in R\langle X\rangle \mid |x| \cap u' = \varnothing\}$$

and observe that, for $u', v' \in \mathsf{F}(X^\perp)$, we have $\mathrm{V}_X(u') \cap \mathrm{V}_X(v') = \mathrm{V}_X(u' \cup v')$. We say that a subset U of $R\langle X\rangle$ is open if, for any $x \in U$, there exists $u' \in \mathsf{F}(X^\perp)$ such that $x + \mathrm{V}_X(u') \subseteq U$. In other words, $(\mathrm{V}_X(u'))_{u' \in \mathsf{F}(X)^\perp}$ is a basis of neighbourhood of 0 for this topology that is invariant by translation. Observe first the following easy facts (which, by the way, can be generalised to all linearly topologised vector spaces).

— If $\mathsf{F}(X) = \mathscr{P}_{\mathrm{fin}}(|X|)$, then $\lambda(X)$ is the discrete topology. This is, in particular, the case when $|X|$ is finite (all finite dimensional spaces have the discrete topology).
— If $\mathsf{F}(X) = \mathscr{P}(|X|)$, then $\lambda(X)$ is the product topology (R being endowed with the discrete topology and $R\langle X\rangle$ being considered as the product $R^{|X|}$). In general, the topology of $R\langle X\rangle$ will be finer than the product topology and coarser than the discrete topology on $R^{|X|}$.
— If a sequence $x(n)$ of elements of $R\langle X\rangle$ tends to 0 when $n \to \infty$, so does any sequence of the shape $\lambda_n x(n)$ where the $\lambda_n \in R$ are arbitrary scalars.
— Each basic neighbourhood $\mathrm{V}_X(u')$ of 0 is obviously open, but also closed. Indeed, if $x \in R\langle X\rangle \setminus \mathrm{V}_X(u')$, let $a \in |x| \cap u'$, and we have $\mathrm{V}_X(u') \cap (x + \mathrm{V}_X(\{a\})) = \varnothing$. So the topology $\lambda(X)$ is always totally disconnected.

This linear topology $\lambda(X)$ is always Hausdorff, and complete, as we shall see immediately. Endowing R with the *discrete* topology, it is easy to see that addition and multiplication by a scalar are continuous operations.

A Cauchy sequence (in full generality, we should consider nets, and not just sequences, but this would not change our reasoning) in $R\langle X\rangle$ is a sequence $(x(n))_{n\in\mathbf{N}}$ of elements of $R\langle X\rangle$ such that, for each neighbourhood U of 0, there exists $n \in \mathbf{N}$ such that, for all $p, q \in \mathbf{N}$, if $p, q \geqslant n$, then $x(p) - x(q) \in U$. In other words: for each $u' \in \mathsf{F}(X)^\perp$ there exists $n \in \mathbf{N}$ such that the restriction of $x(p)$ to u' does not depend on p for $p \geqslant n$; or, for each $u' \in \mathsf{F}(X)^\perp$, the sequence $(x(n)|_{u'})_{n\in\mathbf{N}}$ is ultimately constant.

Lemma 8. The space $R\langle X\rangle$ is complete.

Proof. Let $(x(n))_{n\in\mathbf{N}}$ be a Cauchy sequence in $R\langle X\rangle$. Let $u = \bigcup_{n\in\mathbf{N}} |x(n)|$. We will first show that $u \in \mathsf{F}(X)$. Let $u' \in \mathsf{F}(X^\perp)$. Let n be such that $x(p)|_{u'} = x(n)|_{u'}$ for all $p \geqslant n$. Let $a \in u \cap u'$. Let $m \in \mathbf{N}$ be such that $a \in |x(m)|$. If $m \geqslant n$, since $x(m)|_{u'} = x(n)|_{u'}$, we have $a \in |x(n)|$, and, therefore, $a \in u' \cap \bigcup_{m=0}^{n} |x(m)|$. But $\bigcup_{m=0}^{n} |x(m)| \in \mathsf{F}(X)$ since $\mathsf{F}(X)$ is closed under finite unions. Hence $u \cap u'$ is finite, so $u \in \mathsf{F}(X)$. For each $a \in u$, $\{a\} \in \mathsf{F}(X^\perp)$, so there exists $n_a \in \mathbf{N}$ such that $x(n)_a = x(n_a)_a$ for all $n \geqslant n_a$. Let $x \in R^{|X|}$ be defined by $x_a = x(n_a)_a$. It is clear that $|x| \subseteq u$, so $x \in R\langle X\rangle$ and $\lim_{n\to\infty} x(n) = x$. □

Because of completeness, we have a very simple criterion for the convergence of a series.

Lemma 9. Let $(x(n))_{n\in\mathbf{N}}$ be a family of elements of $R\langle X\rangle$. The series $\sum_{n=0}^{\infty} x(n)$ converges in $R\langle X\rangle$ iff $\lim_{n\to\infty} x(n) = 0$.

Proof. The condition is clearly necessary. It is sufficient because there is a basis of neighbourhoods of 0 that consists of linear subspaces of $R\langle X\rangle$. Indeed, let V be such a neighbourhood (typically, $V = \mathrm{V}_X(u')$ for some $u' \in \mathsf{F}(X^\perp)$). Let $n \in \mathbf{N}$ be such that $x(p) \in V$ whenever $p \geqslant n$. Then, if $p, q \in \mathbf{N}$ are such that $n \leqslant p \leqslant q$, we have $\sum_{i=p+1}^{q} x(i) \in V$ since V is a linear subspace. Therefore, the series considered satisfies the Cauchy criterion and converges by completeness of $R\langle X\rangle$. □

It is clear that when $(x(n))_{n\in\mathbf{N}}$ converges to 0, so does $(x(\sigma(n)))_{n\in\mathbf{N}}$, where σ is any permutation of $\mathbf{N}$, and that $\sum_{n=0}^{\infty} x(\sigma(n)) = \sum_{n=0}^{\infty} x(n)$, so we can speak of summable families indexed over an arbitrary countable set. In particular, if $x \in R\langle X\rangle$, the family $(x_a e_a)_{a\in|X|}$ is summable and its sum is equal to x.

Observe also that if φ is an isomorphism from X to Y in the category **Fin**, then the function $\varphi_* : R\langle X\rangle \to R^{|Y|}$ defined by $\varphi_*(x)_b = x_{\varphi^{-1}(b)}$ takes its values in $R\langle Y\rangle$ and is a linear homeomorphism between $R\langle X\rangle$ and $R\langle Y\rangle$. Of course, not all linear homeomorphisms between the vector spaces associated to finiteness spaces are of this particular shape.

If X is a finiteness space, remember that we use $X^\perp$ to denote the finiteness space $(|X|, \mathsf{F}(X)^\perp)$. Given $x \in R\langle X\rangle$ and $x' \in R\langle X^\perp\rangle$, the sum $\sum_{a\in|X|} x_a x'_a$ has only finitely many non-zero terms, and, therefore, defines an element of R, which we denote by $\langle x, x'\rangle$.

Let X and Y be finiteness spaces. If $x \in R\langle X\rangle$ and $y \in R\langle Y\rangle$, we define $x \otimes y$ by $(x \otimes y)_{a,b} = x_a y_b$, which is clearly an element of $R\langle X \otimes Y\rangle$.

Next remember that we have defined the finiteness space $X \multimap Y$ as $(X \otimes Y^\perp)^\perp$. A morphism from X to Y is an element A of $R\langle X \multimap Y\rangle$, to be considered as a matrix indexed over $|X| \times |Y|$, with coefficients in R. To such a matrix A (called a *finitary* matrix from X to Y), we can associate a linear map $\widehat{A} : R\langle X\rangle \to R^{|Y|}$ by setting $\widehat{A}(x)_b = \sum_{a\in|X|} A_{a,b} x_a \in R$ for each $b \in |Y|$. This sum is indeed finite by Lemma 3.

Conversely, let $f : R\langle X\rangle \to R\langle Y\rangle$ be a function. Define $\mathsf{M}(f) \in R^{|X|\times|Y|}$, the matrix of f, by $\mathsf{M}(f)_{a,b} = f(e_a)_b$.

Lemma 10. The linear map $\widehat{A}$ takes its values in $R\langle Y\rangle$, is continuous and satisfies $\mathsf{M}(\widehat{A}) = A$.

Proof. We have $|\widehat{A}(x)| \subseteq |A|(|x|) \in \mathsf{F}(Y)$ by Lemma 3 so $\widehat{A}$ takes its values in $R\langle Y\rangle$. To prove continuity, we must show that for each $v' \in \mathsf{F}(Y^\perp)$, there exists $u' \in \mathsf{F}(X^\perp)$ such that $\widehat{A}(\mathrm{V}_X(u')) \subseteq \mathrm{V}_Y(v')$. Simply take $u' = \{a \in |X| \mid \exists b \in v'\ (a, b) \in |A|\}$. The last statement of the lemma is trivial. □

Lemma 11. Let $f : R\langle X\rangle \to R\langle Y\rangle$ be a linear and continuous function. Then $\mathsf{M}(f) \in R\langle X \multimap Y\rangle$ and, moreover, $\widehat{\mathsf{M}(f)} = f$.

Proof. Let $u \in \mathsf{F}(X)$ and $v' \in \mathsf{F}(Y^\perp)$, we must show that $w = |\mathsf{M}(f)| \cap (u \times v')$ is finite. Since f is continuous, we know that there exists $u' \in \mathsf{F}(X^\perp)$ such that $f(\mathrm{V}_X(u')) \subseteq \mathrm{V}_Y(v')$. Let $a \in |X|$. If $a \notin u'$, then $e_a \in \mathrm{V}_X(u')$ and hence $|f(e_a)| \cap v' = \emptyset$. So, setting $w(a) = \{b \mid (a, b) \in w\}$, we have

$$w = \bigcup_{a\in u\cap u'} (\{a\} \times w(a)).$$

But for each a, $w(a)$ is a subset of $|f(e_a)| \cap v'$ and therefore is finite. So w itself is finite since $u \cap u'$ is finite. The last part of the lemma follows from the continuity and linearity of f and from the fact that $x = \sum_{a \in |X|} x_a e_a$ for all $x \in R\langle X \rangle$. $\square$

We summarise these simple observations.

Proposition 12. There is a linear isomorphism between $R\langle X \multimap Y \rangle$ and the R-module of linear continuous functions from $R\langle X \rangle$ to $R\langle Y \rangle$.

Although not technically essential, this proposition is important as it means that, in spite of the fact that the modules we consider are given together with a 'basis' (the web of the underlying finiteness space), the notion of morphism between these modules is defined independently of these webs. This fact, together with the functoriality of the various operations on objects, shows that, at least in principle, the category of modules we consider could be presented in an intrinsic way. Mimicking what we did in Ehrhard (2002), we could say, for instance, that a 'finitary R-module' is an R-module M equipped with a linear topology (in the sense of Lefschetz (1942)) *such that there exists* (and *not* equipped with) a finiteness space X and a linear homeomorphism between M and the module $R\langle X \rangle$, equipped with the topology $\lambda(X)$; then all the constructions we perform on finiteness spaces can be transferred to finitary R-modules, that is, can be expressed in a web-independent way.

We now want to give a functional account of the linear topology of $R\langle X \multimap Y \rangle$, in terms of the linear topologies of $R\langle X \rangle$ and $R\langle Y \rangle$ (we were unable to do the same for Köthe spaces in Ehrhard (2002); finiteness spaces are much simpler). For this purpose, we shall first characterise the compact subsets of $R\langle X \rangle$ in terms of finitary sets.

Lemma 13. A subset K of $R\langle X \rangle$ is compact for the topology $\lambda(X)$ iff:

1 K is closed;
2 the set $|K| = \bigcup\{|x| \mid x \in K\}$ belongs to $\mathsf{F}(X)$;
3 for each $a \in |X|$, the set $\{x_a \mid x \in K\}$ is a finite subset of R.

Proof. Assume first that K is compact. Condition 1 holds because $R\langle X \rangle$ is Hausdorff. Condition 3 holds because, for each $a \in |X|$, the projection $x \mapsto x_a$ from $R\langle X \rangle$ to R is continuous, and because the compact subsets for the discrete topology are just the finite ones.

To prove property 2, let $u' \in \mathsf{F}(X^\perp)$. Since $\mathrm{V}_X(u')$ is a neighbourhood of 0 and since K is compact, there is a finite subset M of K such that $K \subseteq M + \mathrm{V}_X(u')$. Let $a \in |K| \cap u'$. Let $x \in K$ be such that $a \in |x|$. Let $y \in M$ be such that $x \in y + \mathrm{V}_X(u')$. Since $a \in u'$, we have $x_a = y_a$ and hence $a \in |y|$. We have shown that $u' \cap |K| \subseteq u' \cap |M|$, and we can conclude since $|M| \in \mathsf{F}(X)$ as M is finite.

Assume now that K satisfies the three conditions of the lemma and let us prove that K is compact. For $a \in |X|$, let $N_a = \{x_a \mid x \in K\}$. This set is finite, reduced to $\{0\}$ for $a \notin |K| \in \mathsf{F}(X)$, and will be considered as a discrete topological space. The topological product space $\prod_{a \in |X|} N_a$ is a topological subspace of $R\langle X \rangle$ (indeed, $\mathscr{P}_{\mathrm{fin}}(|K|) = \{|K| \cap u' \mid u' \in \mathsf{F}(X^\perp)\}$, so the topology induced by $R\langle X \rangle$ on its subspace $\{x \in R\langle X \rangle \mid |x| \subseteq |K|\}$ is just the product topology, this subspace being identified with the product $R^{|K|}$) and is compact by the Tychonov theorem. Therefore, K is compact as a closed subset of a compact space. □

Let us say that a subspace F of $R\langle X\rangle$ is *linearly compact* if F is the closure of the linear span of a compact subset of $R\langle X\rangle$. This notion coincides with the notion of linear compactness introduced in Lefschetz (1942). More precisely, assuming that R is a field (which we will do until the end of this subsection), the following holds.

Proposition 14. Let F be a linear subspace of $R\langle X\rangle$. The following conditions are equivalent:

1 F is linearly compact;
2 F is closed and $|F| \in \mathsf{F}(X)$;
3 (the original Lefschetz definition) for any filter $\mathcal{G}$ of closed affine subspaces of $R\langle X\rangle$ such that $G \cap F \neq \varnothing$ for each $G \in \mathcal{G}$, we have $\cap\mathcal{G} \cap F \neq \varnothing$.

Proof. That 1 implies 2 is straightforward (observe that, given any $u \subseteq |X|$, the set $\{x \in R\langle X\rangle \mid |x| \subseteq u\}$ is closed). To prove the converse, we proceed as follows (a kind of pivot method), assuming that $|X|$ is countable, which, as already mentioned, is a reasonable restriction. First, enumerate $|F| = \{a_1, a_2, \dots\}$ (if this set is finite, F is finite-dimensional and the result is trivial). Then choose one element $x(1) \in F$ such that $x(1)_{a_1} \neq 0$. Since R is a field, we can assume that $x(1)_{a_1} = 1$. Next we can linearly project F onto $R \cdot x(1)$ by the map $p : x \mapsto x_1 \cdot x(1)$, and we have $F = R \cdot x(1) \oplus F_1$ where $F_1 = (\mathrm{Id} - p)(F)$ is a closed subspace of F such that $a_1 \notin |F_1|$. We can iterate this process, producing a sequence $x(i)$ of elements of F such that $x(i)_{a_j} = 0$ for $j < i$ (this process can stop at some finite rank N, producing $F_N = 0$, and in that case again, F is finite-dimensional; in the following we use $N \in \mathbf{N} \cup \{\infty\}$ to deal with both cases). Using Lemma 13, it is easy to check that the collection $\{x(i) \mid i < N\}$ is a compact subset of F whose linear span is dense in F (indeed, due to the fact that $|F| \in \mathsf{F}(X)$, the topology of F is simply the induced product topology of $R^{|F|}$). More precisely, any element x of F can be written in exactly one way as a converging sum $x = \sum_{i=1}^{\infty} \lambda_i x(i)$ with $\lambda_i \in R$ and, conversely, each such sum converges to an element of F. This establishes a linear homeomorphism between F and R^N equipped with the product topology. We retrieve the fact, mentioned in Lefschetz (1942), that a linearly compact subspace is linearly homeomorphic to a power of R.

We leave the equivalences for 3 to the reader, as they are not essential to our purpose. □

As in Lefschetz (1942), let us say that a linearly topologised vector space is *locally linearly compact* if its topology admits a sub-basis of neighbourhoods of 0 that consists of linearly compact subspaces only. We have a straightforward characterisation of the finiteness spaces that give rise to locally linearly compact spaces $R\langle X\rangle$.

Proposition 15. The space $R\langle X\rangle$ is locally linearly compact if and only if there exist $u \in \mathsf{F}(X)$ and $u' \in \mathsf{F}(X^\perp)$ such that $u \cup u' = |X|$. In that case we shall simply say that X is locally linearly compact.

Proof. If F is locally linearly compact, let F be a linearly compact neighbourhood of 0 and set $u = |F| \in \mathsf{F}(X)$. There must exist $u' \in \mathsf{F}(X^\perp)$ such that $\mathrm{V}_X(u') \subseteq F$, and for such a u' we have $u \cup u' = |X|$.

Conversely, if we have two such subsets u and u' of $|X|$, then for any $v' \in \mathsf{F}(X^\perp)$, the 0-neighbourhood $\mathrm{V}_X(v' \cup u') = \{x \in R\langle X\rangle \mid |x| \subseteq u \setminus (u' \cup v')\}$ is a linearly compact subspace of $R\langle X\rangle$, and since these neighbourhoods generate the topology $\lambda(X)$, the space $R\langle X\rangle$ is locally linearly compact. □

Saying that X is locally linearly compact means intuitively that $\mathsf{F}(X)$ has a greatest element. More precisely, it means that the quotient order associated to the following preorder $\sqsubseteq$ on $\mathsf{F}(X)$

$$u \sqsubseteq v \quad \text{if} \quad u \setminus v \text{ is finite}$$

has a greatest element. This property is easily seen to be preserved by all the space constructions we consider (including linear negation) *apart from the exponentials*, as we shall see soon.

Let X and Y be finiteness spaces. If F is a linearly compact subspace of $R\langle X\rangle$ and V is a neighbourhood of 0 in $R\langle Y\rangle$ (we can of course assume that V is also a linear subspace of $R\langle Y\rangle$), we define

$$\mathscr{W}(F, V) = \{A \in R\langle X \multimap Y\rangle \mid \widehat{A}(F) \subseteq V\}.$$

As an immediate corollary of Proposition 14, we obtain the following characterisation of the linear topology of $R\langle X \multimap Y\rangle$.

Proposition 16. The subsets $\mathscr{W}(F, V)$ constitute a basis of neighbourhoods of 0 for the topology $\lambda(X \multimap Y)$.

When restricted to the case where $|Y|$ is a singleton, this is exactly the topology prescribed by Lefschetz for the topological dual of a linearly topologised vector space.

4.1. *Monoidal structure*

Given two matrices $A \in R\langle X \multimap Y\rangle$ and $B \in R\langle Y \multimap Z\rangle$, we define their product $C = BA$ indexed over $|X| \times |Z|$ by $C_{a,c} = \sum_{b\in|Y|} B_{b,c}A_{a,b}$. It is easy to check that this sum is finite, and that the resulting matrix belongs to $R\langle X \multimap Z\rangle$. Moreover, one can check that $\widehat{BA} = \widehat{B} \circ \widehat{A}$. The identity matrix $\mathrm{I} \in R\langle X \multimap X\rangle$ is defined by $\mathrm{I}_{a,b} = \delta_{a,b}$. In that way, we have defined a category whose objects are the finiteness spaces and whose morphisms are the finitary matrices (or equivalently, the linear continuous functions).

We use $\mathbf{Fin}(R)$ to denote this category.

In $\mathbf{Fin}(R)$, the operation $\otimes$ defines a tensor product, whose object part has been defined above. Given $A \in R\langle X \multimap Y\rangle$ and $A' \in R\langle X' \multimap Y'\rangle$, we define $A \otimes A' \in R^{(|X|\times|X'|)\times(|Y|\times|Y'|)}$ by $(A \otimes A')_{(a,a'),(b,b')} = A_{a,b}A'_{a',b'}$. Then it is easy to check that $A \otimes A' \in R\langle(X \otimes X') \multimap (Y \otimes Y')\rangle$, and that this operation $\otimes$ on morphisms is functorial. If $x \in R\langle X\rangle$ and $x' \in R\langle X'\rangle$, we have, in particular, $\widehat{A \otimes A'}(x \otimes x') = \widehat{A}(x) \otimes \widehat{A'}(x')$.

It is then routine to check that $(\mathbf{Fin}(R), \otimes)$ is a symmetric monoidal category, the unit of the tensor being the finiteness space 1 given by $|1| = \{\star\}$ (so that $R\langle 1\rangle = R$). This symmetric monoidal category is closed (with $X \multimap Y$ as objects of morphisms from X to Y), and is actually a $\star$-autonomous category, $\perp = 1$ being the dualising object. If $f : R\langle X \otimes Y\rangle \to R\langle Z\rangle$ is linear and continuous, the corresponding linear and continuous function $f' : R\langle X\rangle \to R\langle Y \multimap Z\rangle$ is given by $f'(x)(y) = f(x \otimes y)$ (considering $f'(x)$ as a continuous linear function). The evaluation function $\mathrm{ev} : R\langle (Y \multimap Z) \otimes Y\rangle \to R\langle Z\rangle$ is given by $\mathrm{ev}(f \otimes x) = f(x)$ (again, identifying $R\langle Y \multimap Z\rangle$ with the space of continuous and linear functions from $R\langle Y\rangle$ to $R\langle Z\rangle$).

4.1.1. *Universal property of the tensor product.* We assume again in this subsection that R is a field (because under this hypothesis we have a simple characterisation of linearly compact subspaces: see Proposition 14). Of course, the function $\tau : R\langle X\rangle \times R\langle Y\rangle \to R\langle X \otimes Y\rangle$ defined by $\tau(x, y) = x \otimes y$ is bilinear, so any continuous linear function $g : R\langle X \otimes Y\rangle \to R\langle Z\rangle$ determines a bilinear function $f = g \circ \tau : R\langle X\rangle \times R\langle Y\rangle \to R\langle Z\rangle$. These bilinear mappings can be characterised as the *hypocontinuous* ones, a mild adaptation of a standard notion, which is strictly weaker in general than continuity with respect to the product topology (standard hypocontinuity involves *boundedness*, a notion that does not really make sense in the present setting).

We first argue that such a bilinear map cannot be required to be continuous. For this, it will clearly be enough to show that the bilinear map $e : R\langle X\rangle \times R\langle X^\perp\rangle \to R$ given by $e(x, x') = \langle x, x'\rangle$ is not continuous in general. It is easy to check that this function is continuous if and only if there exists $u' \in \mathsf{F}(X^\perp)$ and $u \in \mathsf{F}(X)$ such that $u \cup u' = |X|$, that is, if and only if X is locally linearly compact.

We show that some of our space constructions give rise to non-locally linearly compact spaces. Let $X = !\mathsf{N}$ (the space N was defined at the end of Section 2.2; its web is $\mathbf{N}$ and a subset of $\mathbf{N}$ is finitary if it is finite). Let $U \in \mathsf{F}(X)$ and $U' \in \mathsf{F}(X^\perp)$, we want to show that $U \cup U'$ cannot be equal to $|X|$. Let $u = |U| = \bigcup\{|\mu| \mid \mu \in U\} \in \mathsf{F}(N) = \mathscr{P}_{\mathrm{fin}}(\mathbf{N})$. Let $a \in \mathbf{N} \setminus u$. Then for each non-zero integer n we have $n[a] \notin U$. On the other hand, the set $\{n[a] \in U' \mid n > 0\}$ must be finite. So there is a non-zero integer n such that $n[a] \notin U \cup U'$, and thus $U \cup U' \neq |X|$. So $!N$ is not locally linearly compact and the corresponding bilinear evaluation map is not continuous.

Let X, Y and Z be finiteness spaces. A bilinear function $f : R\langle X\rangle \times R\langle Y\rangle \to R\langle Z\rangle$ is hypocontinuous if, for any linear neighbourhood W of 0 in $R\langle Z\rangle$:

— for any linearly compact subspace F of $R\langle X\rangle$ there is a linear neighbourhood V of 0 in $R\langle Y\rangle$ such that $f(F \times V) \subseteq W$; and

— for any linearly compact subspace G of $R\langle Y\rangle$ there is a linear neighbourhood U of 0 in $R\langle X\rangle$ such that $f(U \times G) \subseteq W$.

Of course, if both spaces X and Y are locally linearly compact, then a bilinear map on $R\langle X\rangle \times R\langle Y\rangle$ is hypocontinuous if and only if it is continuous.

The tensor product we have defined has the standard universal property with respect to this notion of bilinear mappings.

Proposition 17. The bilinear function $\tau : R\langle X\rangle \times R\langle Y\rangle \to R\langle X \otimes Y\rangle$ defined by $\tau(x, y) = x \otimes y$ is hypocontinuous, and, moreover, for any hypocontinuous bilinear function f :

$R\langle X\rangle \times R\langle Y\rangle \to R\langle Z\rangle$, there is a unique continuous linear function $g : R\langle X \otimes Y\rangle \to R\langle Z\rangle$ such that $f = g \circ \tau$.

This is a direct corollary of Proposition 12, Proposition 16 and of the monoidal closeness of the category $\mathbf{Fin}(R)$.

We can also characterise $R\langle X \otimes Y\rangle$ as the topological dual of the space of all hypocontinuous bilinear functions from $R\langle X\rangle \times R\langle Y\rangle$ to R equipped with the topology of uniform convergence on all linearly compact subspaces of $R\langle X\rangle \times R\langle Y\rangle$ (a basis of neighbourhood for this space is given by the sets $\{g \mid g(F \times G) = \{0\}\}$ for F and G linearly compact subspaces of $R\langle X\rangle$ and $R\langle Y\rangle$, respectively.).

4.2. *Metrisability*

We have seen that the modules associated to finiteness spaces cannot be assumed to be locally linearly compact. The next natural question to ask is whether they can be assumed to be metrisable. The answer again is no and can be obtained as follows: show first that $R\langle X\rangle$ is metrisable iff there is a non-decreasing sequence $(u_n)_{n\in\mathbf{N}}$ of elements of $\mathsf{F}(X)$ such that for each $u \in \mathsf{F}(X)$ there exists n such that $u \subseteq u_n$. Then show that the finiteness space $(!N)^\perp$ does not have this property (this boils down to a Cantor diagonal argument). So logical constructs, and more specifically exponentials, oblige us to consider a fairly general class of finiteness spaces.

4.3. *Products and coproducts*

This category has all denumerable products and coproducts. Indeed, let $(X_i)_{i\in I}$ be an at most countable family of finiteness spaces. Then $\&_{i\in I}\ X_i$ is the product of the spaces X_i in the category $\mathbf{Fin}(R)$. It is clear that $R\langle \&_{i\in I}\ X_i\rangle$ is canonically isomorphic to the product module $\prod_{i\in I} R\langle X_i\rangle$ and that its topology is the product of the topologies $\lambda(X_i)$. The finiteness space $\oplus_{i\in I} X_i$ is the sum of the spaces X_i in this category and $R\langle \oplus_{i\in I} X_i\rangle$ is the sub-module of $\prod_{i\in I} R\langle X_i\rangle$ whose elements are the families that vanish in almost all components of the product. Of course, when I is finite, one has $\oplus_{i\in I} X_i = \&_{i\in I}\ X_i$, a property that is completely standard in this kind of category, where morphisms can be added and where composition commutes to these sums (a category enriched over commutative monoids, see Mac Lane (1971)).

4.4. *Exponentials*

Let us first introduce some additional notation concerning finite multi-sets. If μ is an element of $\mathscr{M}_{\mathrm{fin}}(I)$, we define its *size* (or cardinality) as $\#\mu = \sum_{i\in I} \mu(i) \in \mathbf{N}$. We also define its *factorial* as $\mu! = \prod_{i\in I} \mu(i)! \in \mathbf{N}$. If $\mu, \nu \in \mathscr{M}_{\mathrm{fin}}(I)$ are such that $\nu \leqslant \mu$, we define the *binomial coefficient*

$$\binom{\mu}{\nu} = \frac{\mu!}{\nu!(\mu-\nu)!} = \prod_{i\in I} \binom{\mu(i)}{\nu(i)}\,.$$

For $x \in R^I$ and $\mu \in \mathscr{M}_{\mathrm{fin}}(I)$, we define $x^\mu \in R$ as $x^\mu = \prod_{i \in I} x_i^{\mu(i)}$. Since the multi-set μ is finite, this product makes sense (we adopt the usual convention that $z^0 = 1$ for all $z \in R$). It is essential to observe that

$$x^\mu \neq 0 \Rightarrow |\mu| \subseteq |x|\,. \tag{3}$$

With this notation, the usual binomial equation immediately generalises as follows: for $x, y \in R^I$ and $\mu \in \mathscr{M}_{\mathrm{fin}}(I)$,

$$(x+y)^\mu = \sum_{v \leqslant \mu} \binom{\mu}{v} x^v y^{\mu - v}\,. \tag{4}$$

Now let S be a commutative monoid (with additive notations for the operations). If $\mu \in \mathscr{M}_{\mathrm{fin}}(S)$, we use $\Sigma(\mu)$ to denote the element of S given by $\Sigma(\mu) = \sum_{s \in S} \mu(s)s$.

Next we introduce *multinomial coefficients* for multi-sets. Let J be another index set. Let $\mu \in \mathscr{M}_{\mathrm{fin}}(I)$ and let $\sigma \in \mathscr{M}_{\mathrm{fin}}(I \times J)$ (to be considered here as a J-indexed collection of multi-sets over I). If the property

$$\forall i \in I \quad \sum_{j \in J} \sigma(i,j) = \mu(i)$$

holds, we define the multinomial coefficient

$$\begin{bmatrix} \mu \\ \sigma \end{bmatrix} = \frac{\mu!}{\sigma!} \in \mathbf{N}\,.$$

The binomial coefficient $\binom{\mu}{v}$ corresponds to the particular case $J = \{1,2\}$, $\sigma(i,1) = v(i)$ and $\sigma(i,2) = \mu(i) - v(i)$.

Let $x \in R\langle X \rangle$. We define $x^! \in R^{|!X|}$ by $x^!_\mu = x^\mu$. It follows from property (3) that $|x^!| \subseteq |x|^! \in \mathsf{F}(!X)$, so $x^! \in R\langle !X \rangle$. We now describe the action of ! on morphisms. Let $A \in R\langle X \multimap Y \rangle$, we define $!A \in R^{|!X| \times |!Y|}$ by setting, for $\mu \in |!X|$ and $v \in |!Y|$,

$$(!A)_{\mu,v} = \sum_{\sigma \in L(\mu,v)} \begin{bmatrix} v \\ \sigma \end{bmatrix} A^\sigma\,, \tag{5}$$

where $L(\mu, v)$ is the (finite) set of all multi-sets σ over $|X| \times |Y|$ such that $\sum_{b \in |Y|} \sigma(a,b) = \mu(a)$ for each $a \in |X|$ and $\sum_{a \in |X|} \sigma(a,b) = v(b)$ for each $b \in |Y|$.

The first thing to observe is that $!A \in R\langle !X \multimap !Y \rangle$ as, if $(\mu, v) \in |!A|$, then μ and v must have the same cardinality n, and must be of the shape $[a_1, \ldots, a_n]$ and $[b_1, \ldots, b_n]$, respectively, with $(a_i, b_i) \in |A|$ for each i. In other words $|!A| \subseteq !|A|$ where the exponential in the right-hand side of this equation is taken in the relational category **Fin** of Section 2.2. But then, since $|A| \in \mathsf{F}(X \multimap Y)$, it follows that $|!A| \in \mathsf{F}(!X \multimap !Y)$.

Next we claim that this operation is functorial. That $!\mathrm{Id} = \mathrm{Id}$ is fairly clear. Let $A \in R\langle X \multimap Y \rangle$ and $B \in R\langle Y \multimap Z \rangle$, we must check now that $(!B)(!A) = !(BA)$. The proof is based on the following simple identity.

Lemma 18. Let I and J be sets and let $\alpha \in \mathscr{M}_{\mathrm{fin}}(I)$ and $\beta \in \mathscr{M}_{\mathrm{fin}}(J)$ be such that $\#\alpha = \#\beta = n$. Then

$$\begin{bmatrix} n \\ \alpha \end{bmatrix} \begin{bmatrix} n \\ \beta \end{bmatrix} = \sum_{\gamma \in L(\alpha,\beta)} \begin{bmatrix} n \\ \gamma \end{bmatrix}.$$

Proof. We can assume without loss of generality that I and J are finite. Let U_i ($i \in I$) and V_j ($j \in J$) be pairwise distinct formal indeterminates. In the algebra $\mathscr{P}$ of polynomials of indeterminates (U_i) and (V_j) (over any field with characteristic 0), we compute the expression $(\sum_{i\in I} U_i)^n(\sum_{j\in J} V_j)^n$ in two different ways. First, we have

$$\begin{aligned}\left(\sum_{i\in I} U_i\right)^n\left(\sum_{j\in J} V_j\right)^n &= \left(\sum_{\substack{\alpha\in\mathscr{M}_{\mathrm{fin}}(I)\\ \#\alpha=n}} \begin{bmatrix} n \\ \alpha \end{bmatrix} U^\alpha\right)\left(\sum_{\substack{\beta\in\mathscr{M}_{\mathrm{fin}}(J)\\ \#\beta=n}} \begin{bmatrix} n \\ \beta \end{bmatrix} V^\beta\right)\\ &= \sum_{\substack{\alpha\in\mathscr{M}_{\mathrm{fin}}(I),\,\beta\in\mathscr{M}_{\mathrm{fin}}(J)\\ \#\alpha=\#\beta=n}} \begin{bmatrix} n \\ \alpha \end{bmatrix}\begin{bmatrix} n \\ \beta \end{bmatrix} U^\alpha V^\beta .\end{aligned}$$

On the other hand,

$$\begin{aligned}\left(\sum_{i\in I} U_i\right)^n\left(\sum_{j\in J} V_j\right)^n &= \left(\sum_{(i,j)\in I\times J} U_i V_j\right)^n\\ &= \sum_{\substack{\gamma\in\mathscr{M}_{\mathrm{fin}}(I\times J)\\ \#\gamma=n}} \begin{bmatrix} n \\ \gamma \end{bmatrix} \prod_{(i,j)\in I\times J} (U_i V_j)^{\gamma(i,j)}\\ &= \sum_{\substack{\gamma\in\mathscr{M}_{\mathrm{fin}}(I\times J)\\ \#\gamma=n}} \begin{bmatrix} n \\ \gamma \end{bmatrix} \prod_{i\in I} U_i^{\sum_{j\in J}\gamma(i,j)} \prod_{j\in J} V_j^{\sum_{i\in I}\gamma(i,j)}\\ &= \sum_{\substack{\alpha\in\mathscr{M}_{\mathrm{fin}}(I),\,\beta\in\mathscr{M}_{\mathrm{fin}}(J)\\ \#\alpha=\#\beta=n}} \left(\sum_{\gamma\in L(\alpha,\beta)} \begin{bmatrix} n \\ \gamma \end{bmatrix}\right) U^\alpha V^\beta\end{aligned}$$

and we conclude. □

We can now prove that $(!B)(!A) = !(BA)$, as previously stated. Given $\mu \in |!X|$ and $\rho \in |!Z|$, we have

$$\begin{aligned}!(BA)_{\mu,\rho} &= \sum_{\varphi\in L(\mu,\rho)} \begin{bmatrix} \rho \\ \varphi \end{bmatrix} \prod_{(a,c)\in|X|\times|Z|} \left(\sum_{b\in|Y|} B_{b,c}A_{a,b}\right)^{\varphi(a,c)}\\ &= \sum_{\varphi\in L(\mu,\rho)} \begin{bmatrix} \rho \\ \varphi \end{bmatrix} \prod_{(a,c)\in|X|\times|Z|} \left(\sum_{\substack{\nu\in|!Y|\\ \#\nu=\varphi(a,c)}} \begin{bmatrix} \varphi(a,c) \\ \nu \end{bmatrix} \prod_{b\in|Y|} (B_{b,c}A_{a,b})^{\nu(b)}\right)\\ &= \sum_{\varphi\in L(\mu,\rho)} \begin{bmatrix} \rho \\ \varphi \end{bmatrix} \sum_{\psi\in L'(\varphi)} \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \prod_{(a,b,c)\in|X|\times|Y|\times|Z|} (B_{b,c}A_{a,b})^{\psi(a,b,c)}\\ &= \sum_{\substack{\varphi\in L(\mu,\rho)\\ \psi\in L'(\varphi)}} \begin{bmatrix} \rho \\ \psi \end{bmatrix} \prod_{(a,b,c)\in|X|\times|Y|\times|Z|} (B_{b,c}A_{a,b})^{\psi(a,b,c)}\end{aligned}$$

where $L'(\varphi)$ is the set of all $\psi \in \mathscr{M}_{\mathrm{fin}}(|X| \times |Y| \times |Z|)$ such that $\sum_{b\in|Y|} \psi(a,b,c) = \varphi(a,c)$, for all $(a,c) \in |X| \times |Y|$. Given $\nu \in |!Y|$, let $L(\mu,\nu,\rho)$ be the set of all $\psi \in \mathscr{M}_{\mathrm{fin}}(|X| \times |Y| \times |Z|)$ such that

$$\begin{aligned}\forall a \in |X| \quad & \sum_{\substack{b\in|Y|\\ c\in|Z|}} \psi(a,b,c) = \mu(a)\,,\\ \forall b \in |Y| \quad & \sum_{\substack{a\in|X|\\ c\in|Z|}} \psi(a,b,c) = \nu(b) \quad \text{and}\\ \forall c \in |Z| \quad & \sum_{\substack{a\in|X|\\ b\in|Y|}} \psi(a,b,c) = \rho(c)\,.\end{aligned}$$

Using this notation, we get

$$!(BA)_{\mu,\rho} = \sum_{\nu\in|!Y|} \sum_{\psi\in L(\mu,\nu,\rho)} \begin{bmatrix}\rho\\ \psi\end{bmatrix} \prod_{(a,b,c)\in|X|\times|Y|\times|Z|} (B_{b,c}A_{a,b})^{\psi(a,b,c)},$$

so it will be sufficient to show that, for all $\nu \in |!Y|$,

$$\sum_{\psi\in L(\mu,\nu,\rho)} \begin{bmatrix}\rho\\ \psi\end{bmatrix} \prod_{(a,b,c)\in|X|\times|Y|\times|Z|} (B_{b,c}A_{a,b})^{\psi(a,b,c)} = \sum_{\substack{\sigma\in L(\mu,\nu)\\ \tau\in L(\nu,\rho)}} \begin{bmatrix}\rho\\ \tau\end{bmatrix}\begin{bmatrix}\nu\\ \sigma\end{bmatrix} B^{\tau}A^{\sigma}\,.$$

But given $\psi \in L(\mu,\nu,\rho)$, we can define $\psi_1 \in \mathscr{M}_{\mathrm{fin}}(|X| \times |Y|)$ and $\psi_2 \in \mathscr{M}_{\mathrm{fin}}(|Y| \times |Z|)$ by $\psi_1(a,b) = \sum_{c\in|Z|} \psi(a,b,c)$ and $\psi_2(b,c) = \sum_{a\in|X|} \psi(a,b,c)$. It is quite clear that $\psi_1 \in L(\mu,\nu)$ and that $\psi_2 \in L(\nu,\rho)$, and also that

$$\prod_{(a,b,c)\in|X|\times|Y|\times|Z|} (B_{b,c}A_{a,b})^{\psi(a,b,c)} = B^{\psi_2}A^{\psi_1}\,.$$

Therefore, we are reduced to showing that, for any $\sigma \in L(\mu,\nu)$ and $\tau \in L(\nu,\rho)$,

$$\begin{bmatrix}\rho\\ \tau\end{bmatrix}\begin{bmatrix}\nu\\ \sigma\end{bmatrix} = \sum_{\substack{\psi\in L(\mu,\nu,\rho)\\ \psi_1=\sigma,\,\psi_2=\tau}} \begin{bmatrix}\rho\\ \psi\end{bmatrix}.$$

Let L be the set of all $\psi \in \mathscr{M}_{\mathrm{fin}}(|X| \times |Y| \times |Z|)$ such that $\psi_1 = \sigma$ and $\psi_2 = \tau$. It is clear that $L \subseteq L(\mu,\nu,\rho)$. Multiplying both sides of the equation above by $\nu!/\rho!$, we are left with showing that

$$\begin{bmatrix}\nu\\ \tau\end{bmatrix}\begin{bmatrix}\nu\\ \sigma\end{bmatrix} = \sum_{\psi\in L} \begin{bmatrix}\nu\\ \psi\end{bmatrix},$$

which follows from Lemma 18 and from the fact that L can be considered as the set of all families $(\psi_b)_{b\in|Y|}$ such that, for each $b \in |Y|$, $\psi_b \in L(\sigma_b, \tau^b)$, where $\sigma_b \in |!X|$ is defined by $\sigma_b(a) = \sigma(a,b)$ and $\tau^b \in |!Z|$ is defined by $\tau^b(c) = \tau(b,c)$.

From the functoriality of !, we can deduce that, for all $A \in R\langle X \multimap Y\rangle$ and $x \in R\langle X\rangle$, the following essential equation holds:

$$(A \cdot x)^! = !A \cdot x^!\,. \tag{6}$$

Indeed, we can define a linear morphism $\tilde{x} \in R\langle 1 \multimap X\rangle$ by $\tilde{x}_{*,a} = x_a$ (for each $a \in |X|$), and it is clear that $x^! = !\tilde{x} \cdot 1^!$ where $1 \in R\langle 1\rangle$ is the unit of R. Equation (6) was, by the way, our starting point for getting to expression (5).

4.4.1. *Comonadic structure.* This functor ! has a structure of comonad given by two natural transformations $\mathrm{d}^X \in R\langle !X \multimap X\rangle$ (dereliction) and $\mathrm{p}^X \in R\langle !X \multimap !!X\rangle$ (digging). The matrices of these natural transformations have only 0 and 1 coefficients and are given by

$$\mathrm{d}^X_{\mu,a} = \delta_{\mu,[a]} \quad \text{and} \quad \mathrm{p}^X_{\mu,M} = \delta_{\mu,\Sigma(M)}.$$

We have already checked in Section 2 that $|\mathrm{d}^X| \in \mathsf{F}(!X \multimap X)$ and $|\mathrm{p}^X| \in \mathsf{F}(!X \multimap !!X)$ (these supports are exactly the dereliction and digging morphisms of the finiteness spaces relational model presented in that section).

The naturality of these morphisms can be checked by simple computations. The following equations express the fact that these natural transformations endow the functor ! with a comonad structure, and they are checked similarly.

$$\mathrm{d}^{!X} \circ \mathrm{p}^X = \mathrm{Id}_{!X}\,, \quad !\mathrm{d}^X \circ \mathrm{p}^X = \mathrm{Id}_{!X}$$

and

$$\mathrm{p}^{!X} \circ \mathrm{p}^X = !\mathrm{p}^X \circ \mathrm{p}^X\,.$$

Let us just check that p^X is natural, so let $A \in R\langle X \multimap Y\rangle$, and let us check, for $\mu \in |!X|$ and $N \in |!!Y|$, that

$$(\mathrm{p}^Y \circ !A)_{\mu,N} = (!!A \circ \mathrm{p}^X)_{\mu,N}\,,$$

that is

$$(!A)_{\mu,\Sigma(N)} = \sum_{\substack{M\in|!!X|\\ \Sigma(M)=\mu}} (!!A)_{M,N}\,. \tag{7}$$

Let $x \in R\langle X\rangle$ and let $N \in |!!Y|$. Applying equation (6) twice, we have $(A \cdot x)^{!!} = !!A \cdot x^{!!}$. We obtain, therefore,

$$\begin{aligned}
\sum_{\mu\in|!X|} (!A)_{\mu,\Sigma(N)} x^\mu &= \sum_{\mu\in|!X|} (!A)_{\mu,\Sigma(N)} x^!_\mu \\
&= (!A \cdot x^!)_{\Sigma(N)} \\
&= (Ax)^!_{\Sigma(N)} \quad \text{by equation (6)} \\
&= (Ax)^{!!}_N \\
&= (!!A \cdot x^{!!})_N \quad \text{by equation (6) again} \\
&= \sum_{M\in|!!X|} (!!A)_{M,N} x^{!!}_M \\
&= \sum_{\mu\in|!X|} \left(\sum_{\substack{M\in|!!X|\\ \Sigma(M)=\mu}} (!!A)_{M,N} \right) x^\mu\,.
\end{aligned}$$

To summarise, the following equation holds in R:

$$\sum_{\mu\in|!X|}(!A)_{\mu,\Sigma(N)}x^\mu=\sum_{\mu\in|!X|}\left(\sum_{\substack{M\in|!!X|\\ \Sigma(M)=\mu}}(!!A)_{M,N}\right)x^\mu\,. \tag{8}$$

From this we deduce equation (7) by the following simple argument. Let $I\subseteq|X|$ be finite. Let $U=(U_a)_{a\in I}$ be a family of pairwise distinct formal indeterminates. Let $R'=R[U]$ be the ring of polynomials with coefficients in R and indeterminates U_a. Using the canonical embedding of R into R', we can consider A as an element of $R'\langle X\multimap Y\rangle$. Therefore, equation (8) holds, with scalars now taken in R' (we have only used the ring structure of R for proving this equation). Let $x\in R'\langle X\rangle$ be defined by $x_a=U_a$ if $a\in I$ and $x_a=0$ otherwise. In this particular case, equation (8) gives

$$\sum_{\mu\in\mathscr{M}_{\mathrm{fin}}(I)}(!A)_{\mu,\Sigma(N)}U^\mu=\sum_{\mu\in\mathscr{M}_{\mathrm{fin}}(I)}\left(\sum_{\substack{M\in|!!X|\\ \Sigma(M)=\mu}}(!!A)_{M,N}\right)U^\mu\,.$$

We can now conclude, since the monomials U^μ are linearly independent in R' (considered as an R-module).

4.4.2. *Fundamental isomorphism and the co-algebraic structure.* Given two finiteness spaces X and Y, recall from Section 2 that there is a canonical isomorphism of finiteness spaces (2)

$$!(X\ \&\ Y)\simeq\ !X\otimes\ !Y\,.$$

Given $x\in R\langle X\rangle$ and $y\in R\langle Y\rangle$, we have $x^!\otimes y^!\in R\langle !X\otimes !Y\rangle$. The corresponding element of $R\langle !(X\ \&\ Y)\rangle$ is easily seen to be $(x\oplus y)^!$.

Since $\&$ is the cartesian product in $\mathbf{Fin}(R)$, there is a diagonal linear map $\Delta^X : X\to X\ \&\ X$ whose matrix is given by $\Delta^X_{a,(i,b)}=\delta_{a,b}$. Let

$$\mathrm{contr}^X : !X\to\ !X\otimes\ !X$$

be obtained by composing $!\Delta^X$ with the isomorphism (2). Similarly, let

$$\mathrm{weak}^X : !X\to 1$$

be obtained by composing $!0$ (where 0 is the unique morphism $X\to 0$) with the isomorphism $!0\simeq 1$. Using (5), the matrices of these operators are easily seen to be given by

$$\mathrm{weak}^X_{\mu,*}=\delta_{\mu,[]}\quad\text{and}\quad\mathrm{contr}^X_{\mu,(\lambda,\rho)}=\delta_{\mu,\lambda+\rho}\,.$$

As is standard, these two morphisms define a structure of co-algebra on $!X$ (we are actually in a 'new-Seely' situation, following the terminology of Bierman (1995)). The first is used for interpreting the weakening rule of linear logic and the second is used for the contraction rule.

4.4.3. *Morphisms as power series.* The category whose objects are finiteness spaces, where a morphism from X to Y is a linear and continuous function from $R\langle !X\rangle$ to $R\langle Y\rangle$ (that is, a matrix in $R\langle !X \multimap Y\rangle$), with d^X as identity at X and composition of $\varphi \in R\langle !X \multimap Y\rangle$ with $\psi \in R\langle !Y \multimap Z\rangle$ defined as the product of matrices

$$\psi\ !\varphi\ \mathrm{p}^X \tag{9}$$

is the *Kleisli category* of the comonad ! (the category of co-free co-algebras of the co-monad), and is cartesian closed, essentially because of the fundamental isomorphism presented above. If we consider $|X|$ as a set of formal indeterminates, then morphisms in the Kleisli category can be considered as power series (this is the basic idea of Girard's quantitative semantics (Girard 1988)):

— An element of $R\langle X\rangle$ is a *valuation* in R for these indeterminates (subject to the restriction that its domain must belong to $\mathsf{F}(X)$).
— An element μ of $|!X|$ is a *multi-exponent* (or, equivalently, a primitive monomial, that is a pure monomial without coefficient) on the indeterminates of $|X|$: if $\mu = [\xi_1, \dots, \xi_n]$, the corresponding primitive monomial is the formal product of indeterminates $\xi_1 \dots \xi_n$ (here we will use ξ^μ to denote this formal product). The value of this monomial for a valuation $x \in R\langle X\rangle$ is just $x_{\xi_1} \times \cdots \times x_{\xi_n}$ (product computed in R), a value that we have already decided to denote as x^μ.
— An element φ of $R\langle !X \multimap Y\rangle$ is seen as the following power series with coefficients in $R\langle Y\rangle$:

$$\sum_{\mu\in|!X|} \xi^\mu \Big(\sum_{b\in|Y|} \varphi_{\mu,b} e_b\Big)\,.$$

This viewpoint is sensible, since the application of φ to $x \in R\langle X\rangle$ in the Kleisli category under consideration is just $\varphi \cdot x^! = \sum_{\mu\in|!X|} x^\mu \big(\sum_{b\in|Y|} \varphi_{\mu,b} e_b\big) \in R\langle Y\rangle$ (this sum can have infinitely many non-zero terms, but, nevertheless, makes sense because, for each particular b, there are only finitely many μ's such that $x^\mu \varphi_{\mu,b} \neq 0$).

One can check that, just as in Ehrhard (2002), the Kleisli composition (9) corresponds to the usual composition of power series (substitution), so that this Kleisli category is a cartesian closed category of power series. A matrix $\varphi \in R\langle !X \multimap Y\rangle$ will therefore be called a power series from X to Y. If $x \in R\langle X\rangle$, we use $\varphi(x)$ to denote the value $\varphi \cdot x^!$ of this power series at x.

We now give two concrete examples. In both cases we shall take a singleton for $|Y|$ so that $R\langle Y\rangle = R$ and our power series will be scalar-valued.

— If $|X|$ is also a singleton $\{\xi\}$, then $|!X| = \{\xi^n \mid n \in \mathbf{N}\}$ and $\mathsf{F}(!X) = \mathscr{P}(|!X|)$. In that case, $R\langle !X \multimap Y\rangle$ is the space $R[\xi]$ of all polynomials of one indeterminate ξ. Similarly, when $|X|$ is finite (so, as we have seen, there is only one finiteness structure for X), $R\langle !X \multimap Y\rangle$ is the space of polynomials of n indeterminates, where n is the cardinality of $|X|$. It is only when $|X|$ become infinite that true 'power series' come in, as shown by our second example.
— Consider now the case where $X = \mathsf{N}$, whose web will be considered as an infinite set of indeterminates $\{\xi_i \mid i \in \mathbf{N}\}$ rather than as $\mathbf{N}$. An element of $R\langle X\rangle$ is an R-valuation of these indeterminates whose support must be finite, that is, which must take the value 0 for all but a finite number of indeterminates, so that $R\langle \mathsf{N}\rangle$ can again be assimilated to the space $R[\chi]$ of all polynomials of one indeterminate χ:

the element $\sum a_n \xi_n$ of $R\langle \mathsf{N} \rangle$ corresponds to the polynomial $\sum a_n \chi^n$. An element ξ^μ of $|!X|$ is a primitive monomial on these indeterminates and a collection of such monomials is finitary (in $!X$) if it mentions only a finite number of indeterminates. A power series $\varphi = \sum_{\mu \in |!X|} \varphi_\mu \xi^\mu \in R\langle !X \multimap Y \rangle$ must therefore have only finitely many non-zero coefficients of monomials mentioning the elements of any given finite set of indeterminates. For that reason, the sum $\sum_{\mu \in |!X|} \varphi_\mu x^\mu \in R\langle !X \multimap Y \rangle$ will have only finitely non-zero terms, for any $x \in R\langle X \rangle$. Such a power series can perfectly well be infinite, and can even have an unbounded degree (total degree, or degree in a given indeterminate, as the next example shows); consider, for instance, the series $\sum_{n=0}^{\infty} \xi_0^n \xi_n$, which represents the map $P \mapsto P(P(0))$ from $R[\chi]$ to R (identifying $R\langle \mathsf{N} \rangle$ with $R[\chi]$). So these power series cannot be considered as polynomials, though, when computing their values on actual valuations of their indeterminates, only finite computations are required, in sharp contrast with the usual power series. They are infinite objects only when they mention infinitely many indeterminates.

4.4.4. *Algebraic structure.* Due to the fact that (finite) products and co-products coincide in $\mathbf{Fin}(R)$, there is also a co-diagonal linear morphism $\mathrm{a}^X : X \,\&\, X \to X$ (as a function $R\langle X \rangle \times R\langle X \rangle \to R\langle X \rangle$, this is simply addition: $\mathrm{a}^X(x, y) = x + y$), whose matrix is given by $\mathrm{a}^X_{(i,a),b} = \delta_{a,b}$. Similarly, there is a zero-map $0 \to X$. The map $!\mathrm{a}^X$ composed with the isomorphism (2) gives rise to a linear morphism

$$\mathrm{c}^X : !X \otimes !X \to !X$$

whose matrix is given by $\mathrm{c}^X_{(\lambda,\rho),\mu} = \binom{\mu}{\lambda} \delta_{\lambda+\rho,\mu}$. The value of this coefficient can be obtained by applying Formula (5) in the particular case where $A = \mathrm{a}^X$, or, more simply, by observing that we must have

$$\mathrm{c}^X(x^! \otimes y^!) = (x + y)^!$$

for each $x, y \in R\langle X \rangle$, and by applying the generalised binomial equation (4).

Applying the functor ! to the zero-map mentioned above, we obtain, similarly, a linear map $\mathrm{u}^X : 1 \to !X$, that is, an element u^X of $R\langle X \rangle$ that is defined by $\mathrm{u}^X_\mu = \delta_{\mu,[]}$.

The linear map c^X can be considered as defining a binary, bilinear and hypocontinuous commutative and associative multiplication on $R\langle !X \rangle$. Given $S, T \in R\langle !X \rangle$, we write $S * T$ or simply ST for $\mathrm{c}^X(S, T)$. This multiplication admits u^X as neutral element. So we have endowed $!X$ with a structure of commutative algebra whose multiplication can be interpreted as a kind of 'convolution product', if we remember that $R\langle !X \rangle$ is the topological dual of $R\langle (!X)^\perp \rangle$, which itself can be seen as a space of power series from $R\langle X \rangle$ to R (which play the role of test functions in the theory of distributions). From this viewpoint, the element $x^!$ of $R\langle !X \rangle$ (when $x \in R\langle X \rangle$) corresponds to the 'Dirac mass at x', which maps a test function φ to its value at x, that is $\varphi(x)$. The unit u^X corresponds to the Dirac mass at 0, and the convolution product is given by $(S * T)(\varphi) = S(\lambda x\, T(\lambda y\, \varphi(x+y)) = T(\lambda y\, S(\lambda x\, \varphi(x+y)))$ (using notation from the lambda-calculus). See Ehrhard (2002) for more details (the setting is different, but the analogy with the convolution product of distributions is preserved).

To summarise, $!X$ has a structure of co-algebra and of algebra, and is indeed a commutative and co-commutative Hopf algebra (an antipode can be defined by applying the functor ! to the linear map $x \mapsto -x$, from $R\langle X\rangle$ to itself, and one obtains in this way the matrix $S \in R\langle !X \multimap !X\rangle$ given by $S_{\mu,\nu} = (-1)^{\#\mu}\delta_{\mu,\nu}$). This kind of Hopf algebra seems to be known as a 'divided power algebra'.

4.4.5. *Derivatives.* There is, moreover, a linear 'anti-dereliction' map $\partial_0^X : X \to !X$, simply given by the matrix $(\partial_0^X)_{a,\mu} = \delta_{[a],\mu}$, so that $\mathrm{d}^X \circ \partial_0^X = \mathrm{Id}_X$. Let φ be a power series from X to Y, that is, a linear map from $!X$ to Y. Then $A = \varphi\partial_0^X \in R\langle X \multimap Y\rangle$ is given by $A_{a,b} = \varphi_{[a],b}$ and so is the 'linear part' of φ, which is precisely what a derivative at 0 of φ should be. Indeed, recall that when $f : E \to F$ is a function between two Banach spaces (for instance), the derivative of f at 0 (when it exists) is the (necessarily unique) linear continuous function $h : E \to F$ such that $(f(x)-f(0)-h(x))/\|x\| \to 0$ when $x \to 0$, meaning that $h(x)$ is the best possible linear approximation of $f(x) - f(0)$. Similarly, in the present setting, we have

$$\varphi(x) - \varphi(0) - A \cdot x = \sum_{b\in|Y|}\left(\sum_{\substack{\mu\in|!X|\\ \#\mu\geqslant 2}} \varphi_\mu x^\mu\right) e_b,$$

which means that all the terms in $\varphi(x) - \varphi(0) - A \cdot x$ have a total degree $\geqslant 2$. We will use $\varphi'_0 = A = \varphi\partial_0^X$ to denote this derivative.

Let $x \in R\langle X\rangle$. The derivative of φ at x is the derivative at 0 of the power series $\psi : R\langle X\rangle \to R\langle Y\rangle$ defined by $\psi(u) = \varphi(x+u)$: $\varphi'_x = \psi'_0$. The map $\varphi' : R\langle X\rangle \to R\langle (X \multimap Y)\rangle$ defined by $\varphi'(x) = \varphi'_x$ is itself 'analytic', that is, can be defined by a power series as follows: composing $\mathrm{Id}_{!X} \otimes \partial_0^X : !X \otimes X \to !X \otimes !X$ and $\mathrm{c}^X : !X \otimes !X \to !X$, we obtain a linear map

$$\partial^X : !X \otimes X \to !X,$$

therefore $\varphi\partial^X$ is a linear map $!X \otimes X \to !Y$, which can be transposed (using monoidal closeness) into a map $!X \to (X \multimap Y)$, which turns out to be φ'.

This derivation process can, therefore, be iterated: to $\varphi \in R\langle !X \multimap Y\rangle$, we can associate $\varphi^{(n)} \in !X \multimap (X^{\otimes n} \multimap Y)$, the n-the derivative of φ. This derivative can also be obtained by precomposing φ with a morphism

$$\partial_n^X : !X \otimes X^{\otimes n} \to !X$$

defined by induction over n as follows: $\partial_0^X = \mathrm{Id}_{!X}$ and $\partial_{n+1}^X = \partial_n^X(\partial^X \otimes \mathrm{Id}_{X^{\otimes n}})$. The matrix of this operator is given by

$$(\partial_n^X)_{\mu,(a_1,\dots,a_n),\nu} = \frac{\nu!}{\mu!}\delta_{\mu+[a_1,\dots,a_n],\nu}\,.$$

So $\varphi^{(n)}(x)$, seen as an n-linear map from $R\langle X\rangle^n$ to $R\langle X\rangle$, is symmetrical, as is standard.

4.4.6. *The Taylor formula and the exponential.* Let X be a finiteness space. The linear and continuous map ∂_0^X defines an embedding of $R\langle X\rangle$ into $R\langle !X\rangle$, with retraction d^X, so that $R\langle X\rangle$ can be considered canonically as a subspace of $R\langle !X\rangle$, which we do now. Given $x \in R\langle X\rangle$, the corresponding element of $R\langle !X\rangle$, still denoted by x, is $\sum_{a\in|X|} x_a e_{[a]}$. If $n \in \mathbf{N}$, we write x^n for the n-th power of x (multiplication being the convolution product $*$ on $R\langle !X\rangle$). By definition of this product, for each $\mu \in |!X|$, we have

$$x^n_\mu = \delta_{n,\#\mu} \sum_{\substack{(a_1,\dots,a_n)\in|X|^n \\ [a_1,\dots,a_n]=\mu}} \mu! \prod_{i=1}^n x_{a_i} = \delta_{n,\#\mu} n! x^\mu$$

since there are exactly $n!/\mu!$ tuples $(a_1,\dots,a_n) \in |X|^n$ such that $[a_1,\dots,a_n] = \mu$ (when $n = \#\mu$). Observe, in particular, that $|x^n| = \{\mu \in |x|^! \mid \#\mu = n\}$.

Lemma 19. The series $\sum_{n=0}^\infty \frac{x^n}{n!}$ converges to $x^!$ in $R\langle !X\rangle$.

Proof. For convergence it suffices to prove that $\lim_{n\to\infty} x^n = 0$ in $R\langle !X\rangle$. Let $U' \in \mathsf{F}((!X)^\perp)$. Since $|x| \in \mathsf{F}(X)$, the set $U' \cap |x|^!$ is finite. Then, if $n \in \mathbf{N}$ is such that $n > \#\mu$ for all $\mu \in U' \cap |x|^!$, we have $x^n \in \mathrm{V}_{!X}(U')$ and so $\lim_{n\to\infty} x^n = 0$.

Let $\mu \in |!X|$ and $m = \#\mu$. We have

$$\begin{aligned}\left(\sum_{n=0}^\infty \frac{x^n}{n!}\right)_\mu &= \sum_{n=0}^\infty \frac{x^n_\mu}{n!} \\ &= \frac{1}{m!} m! x^\mu = x^!_\mu\end{aligned}$$

and hence $\sum_{n=0}^\infty \frac{x^n}{n!} = x^!$. □

Let φ be a power series from X to Y, so that φ can be seen as a linear function from $!X$ to Y : $\varphi \in R\langle !X \multimap Y\rangle$. Then we have, by continuity of φ considered as a linear map (using a 'dot notation' for linear application):

$$\begin{aligned}\varphi(x) &= \varphi \cdot x^! \\ &= \sum_{n=0}^\infty \frac{1}{n!} \varphi \cdot x^n\end{aligned}$$

for all $x \in R\langle X\rangle$. For all such x, we have

$$\varphi^{(n)}(0) \cdot x^{\otimes n} = (\varphi \partial^X_n) \cdot (0^! \otimes x^{\otimes n}) = \varphi \cdot x^n$$

where $x^{\otimes n} = \underbrace{x \otimes \cdots \otimes x}_{n\times} \in X^{\otimes n}$, and hence the Taylor formula holds

$$\varphi(x) = \sum_{n=0}^\infty \frac{1}{n!} \varphi^{(n)}(0) \cdot x^{\otimes n}.$$

5. Concluding remarks

Among several problems raised by this interpretation, let us point out the impossibility of defining *in a standard way* a finitary model of the pure lambda-calculus. Such a model would be a finiteness space U together with an embedding retraction pair from $!U \multimap U$

into U. But (under mild hypotheses, for example, the assumption that the space N is a retract of U) this would induce fix-point operators whose existence was disproved in Section 3.

This situation is embarrassing for two reasons. First, as we know from the work of Hasegawa (Hasegawa 1997), all coefficients in the quantitative interpretation of pure lambda-terms are finite, and the purpose of finiteness spaces being to keep all coefficients finite, pure lambda-terms should admit a finitary interpretation. Second, the finiteness space model presented here is perfectly adapted for interpreting the simply typed differential lambda-calculus of Ehrhard and Regnier (2003), but this calculus admits a natural untyped version whose denotational semantics would require something like a finitary model of the pure lambda-calculus.

The solution might be a non-standard interpretation, which would not use a directed limit or co-limit (like the construction of D_∞ in Scott domains or coherence spaces) because these infinitary constructions are not available in the category of finiteness spaces and finitary relations (or in the category of finiteness spaces and linear continuous functions, a semi-ring of coefficients being given); this is due to the fact that finitary sets are not closed under directed unions in general.

Appendix A. The interpretation of proofs in the category of sets and relations

To each formula G of first-order propositional linear logic (without atoms, only logical constants), we associate a set $|G|$ as follows:

— $|0| = |\top| = \emptyset$ and $|G \oplus H| = |G \mathbin{\&} H| = |G| + |H|$ (where $+$ denotes the disjoint union on sets, which can be defined, for instance, by $S + T = \{1\} \times S \cup \{2\} \times T$);
— $|1| = |\bot| = \{*\}$ and $|F \otimes G| = |F \mathbin{⅋} G| = |F| \times |G|$ (where $*$ is a distinguished element);
— $!F = ?F = \mathscr{M}_{\mathrm{fin}}(|F|)$ where $\mathscr{M}_{\mathrm{fin}}(S)$ is the set of finite multi-sets over S.

If $\Gamma = G_1, \ldots, G_n$ is a list of formulae, then $|\Gamma| = |G_1 ⅋ \cdots ⅋ G_n| = |G_1| \times \cdots \times |G_n|$. Given a formula G, the formula $G^\perp$ is defined by induction using the usual De Morgan identities of linear logic. It is clear then that $|G^\perp| = |G|$.

To each proof π of a sequent in first-order propositional linear logic $\vdash \Gamma$, we associate a subset of $[\pi]$ of the set $|\Gamma|$ by induction on π.

Tensor unit: If the proof π is

$$\frac{}{\vdash 1}$$

then $[\pi] = \{*\}$.

With unit: If the proof π is

$$\frac{}{\vdash \Gamma, \top}$$

then $[\pi] = \emptyset$.

With: If the proof π is

$$\frac{\begin{matrix} \vdots\, \pi_1 \\ \vdash \Gamma, G \end{matrix} \qquad \begin{matrix} \vdots\, \pi_2 \\ \vdash \Gamma, H \end{matrix}}{\vdash \Gamma, G \mathbin{\&} H}$$

then $[\pi] = \{(c,(1,a)) \mid (c,a) \in [\pi_1])\} \cup \{(c,(2,b)) \mid (c,b) \in [\pi_2])\}$.

Left plus: If the proof π is

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Gamma, S\end{array}}{\vdash \Gamma, G \oplus H}$$

then $[\pi] = \{(c,(1,a)) \mid (c,a) \in [\pi_1])\}$. And similarly if π ends with a right plus rule.

Par unit: If the proof π is

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Gamma\end{array}}{\vdash \Gamma, \bot}$$

then $[\pi] = \{(c,*) \mid c \in [\pi_1]\}$.

Par: If the proof π is

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Gamma, G, H\end{array}}{\vdash \Gamma, G \parr H}$$

then $[\pi] = \{(c,(a,b)) \mid (c,a,b) \in [\pi_1]\}$.

Tensor: If the proof π is

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Gamma, G\end{array} \qquad \begin{array}{c}\vdots\,\pi_2\\ \vdash \Delta, H\end{array}}{\vdash \Gamma, \Delta, G \otimes H}$$

then $[\pi] = \{(c,d,(a,b)) \mid (c,a) \in [\pi_1])$ and $(d,b) \in [\pi_2]\}$.

Weakening: If the proof π is

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Gamma\end{array}}{\vdash \Gamma, ?G}$$

then $[\pi] = \{(c,[]) \mid c \in [\pi_1]\}$.

Contraction: If the proof π is

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Gamma, ?G, ?G\end{array}}{\vdash \Gamma, ?G}$$

then $[\pi] = \{(c,x+y) \mid (c,x,y) \in [\pi_1]\}$ where $x+y$ denotes the sum of the multi-sets x and y.

Dereliction: If the proof π is

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Gamma, G\end{array}}{\vdash \Gamma, ?G}$$

then $[\pi] = \{(c,[a]) \mid (c,a) \in [\pi_1]\}$.

Promotion: If the proof π is

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash ?G^1,\ldots,?G^k,G\end{array}}{\vdash ?G^1,\ldots,?G^k,!G}$$

then $[\pi]$ is the set of all $k+1$-tuples of the shape $(\sum_{j=1}^n x_j^1,\ldots,\sum_{j=1}^n x_j^k,[a_1,\ldots,a_n])$ where $((x_j^1,\ldots,x_j^k,a_j))_{j=1,\ldots,n}$ is any finite family of elements of $[\pi_1]$.

The exchange rule does not deserve particular mention.

Cut: If the proof π is

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Gamma,G\end{array}\qquad\qquad\begin{array}{c}\vdots\,\pi_2\\ \vdash \Delta,G^\perp\end{array}}{\vdash \Gamma,\Delta}$$

then $[\pi]=\{(c,d)\mid \exists a\,(c,a)\in[\pi_1]\text{ and }(d,a)\in[\pi_2]\}$.

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References

Barr, M. (1976) Duality of vector spaces. *Cahiers de Topologie et Géométrie Différentielles* **17** 3–14.

Bierman, G. (1995) What is a categorical model of intuitionistic linear logic? In: Dezani-Ciancaglini, M. and Plotkin, G. D. (eds.) Proceedings of the second Typed Lambda-Calculi and Applications conference. *Springer-Verlag Lecture Notes in Computer Science* **902** 73–93.

Blute, R. (1996) Hopf algebras and linear logic. *Mathematical Structures in Computer Science* **6** (2) 189–217.

Bucciarelli, A. and Ehrhard, T. (2001) On phase semantics and denotational semantics: the exponentials. *Annals of Pure and Applied Logic* **109** (3) 205–241.

Danos, V. and Harmer, R. (2000) Probabilistic game semantics. In: *Proceedings of the fifteenth Symposium on Logic in Computer Science*, IEEE Computer Society Press.

Ehrhard, T. (1993) Hypercoherences: a strongly stable model of linear logic. *Mathematical Structures in Computer Science* **3** 365–385.

Ehrhard, T. (2002) On Köthe sequence spaces and linear logic. *Mathematical Structures in Computer Science* **12** 579–623.

Ehrhard, T. and Regnier, L. (2003) The differential lambda-calculus. *Theoretical Computer Science* **309** (1–3) 1–41.

Girard, J.-Y. (1986) The system F of variable types, fifteen years later. *Theoretical Computer Science* **45** 159–192.

Girard, J.-Y. (1987) Linear logic. *Theoretical Computer Science* **50** 1–102.

Girard, J.-Y. (1988) Normal functors, power series and the λ-calculus. *Annals of Pure and Applied Logic* **37** 129–177.

Girard, J.-Y. (2001) Locus Solum. *Mathematical Structures in Computer Science* **11** (3) 301–506.

Hasegawa, R. (1997) The generating functions of lambda terms (extended abstract). In: *Combinatorics, Complexity and Logic*, (Proceedings of the first international conference on discrete mathematics and theoretical computer science, DMTCS'96, Auckland, New Zealand, December 9–13, 1996.), Discrete Mathematics and Theoretical Computer Science Series, Springer-Verlag 253–263.

Hasegawa, R. (2002) Two applications of analytic functors. *Theoretical Computer Science* **272** (1–2) 113–175.

Lefschetz, S. (1942) *Algebraic topology*, American mathematical society colloquium publications **27**, American Mathematical Society.

Loader, R. (1994) Linear logic, totality and full completeness. In: *Proceedings of the ninth Symposium on Logic in Computer Science*, IEEE Computer Science Press 292–298.

Mac Lane, S. (1971) *Categories for the Working Mathematician*, Graduate Texts in Mathematics **5**, Springer-Verlag.