

From combinatorial species to general differential 2-rigs

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Goals : in the context of general differential 2-rigs (Loregian, Trimble, [5]) :

- Can we solve differential equations using the same techniques as for combinatorial species ?
- Can some theorems about combinatorial species be extended ?

- 1 Background
- 2 Resolution of some differential equations
- 3 The number of solutions
- 4 Virtual differential 2-rigs

Summary

- 1 Background
 - Differential 2-rigs
 - Combinatorial species
- 2 Resolution of some differential equations
- 3 The number of solutions
- 4 Virtual differential 2-rigs

Definition, Loregian, [5].

A **2-rig** is a category \mathcal{C} with :

- finite coproducts $+$, called the addition,
- an other monoidal structure \otimes , called the multiplication,
- natural isomorphisms :

$$X \otimes Y + X \otimes Z \underset{\sim}{\xrightarrow{\delta^L}} X \otimes (Y + Z)$$

$$Y \otimes X + Z \otimes X \underset{\sim}{\xrightarrow{\delta^R}} (Y + Z) \otimes X$$

Example

- $(\text{Set}, +, \times, 1)$.
- If R is a ring, then $(\text{Mod}_R, \oplus, \otimes, R)$ is an example.
- If (\mathcal{A}, \oplus, j) is a monoidal category, then $([\mathcal{A}^{\text{op}}, \text{Set}], +, *, I = \mathcal{A}(j, -))$ is an example, where $*$ is the **Day convolution** :

$$F * G = \int^{U, V \in \mathcal{A}} F U \times G V \times \mathcal{A}(U \oplus V, -)$$

- If \mathcal{C} is a 2-rig, then the category $\mathcal{C}[Y]$ with objects finite families of objects of \mathcal{C} noted $(A_1, \dots, A_n) = \sum_{i=0}^n A_i \otimes Y^i$ with component-wise sum and Cauchy product :

$$\left(\sum_{i=0}^n A_i \otimes Y^i \right) \otimes \left(\sum_{j=0}^m B_j \otimes Y^j \right) = \left(\sum_{k=0}^{m+n} \left(\sum_{i+j=k} A_i \otimes B_j \right) \otimes Y^k \right)$$

Definition, Loregian, [5].

A **differential** 2-rig is a 2-rig \mathcal{C} with :

- an endofunctor ∂ , called the derivation,
- natural isomorphisms :

$$\partial X + \partial Y \xrightarrow[\sim]{\partial i_X + \partial i_Y} \partial(X + Y)$$

$$\partial X \otimes Y + X \otimes \partial Y \xrightarrow[\sim]{l} \partial(X \otimes Y)$$

such that : naturality, compatibility with the left-/right-distributors, compatibility with the \otimes -associator, compatibility with the left-/right- \otimes -unitors.

Ex for naturality : for all morphisms $u : X \rightarrow X'$, $v : Y \rightarrow Y'$, we want the following diagram to commute :

$$\begin{array}{ccc}
 \partial(X \otimes Y) & \xrightarrow{\partial(u \otimes v)} & \partial(X' \otimes Y') \\
 \uparrow l_{X,Y} & & \uparrow l_{X',Y'} \\
 \partial X \otimes Y + X \otimes \partial Y & \xrightarrow{\partial u \otimes v + u \otimes \partial v} & \partial X' \otimes Y' + X' \otimes \partial Y'
 \end{array}$$

Example

- If we want to endow $(\mathcal{C}[Y], +, \otimes, I)$ with a derivation satisfying $\partial Y = I$, the Leibniz rule impose to set :

$$\partial \sum_{i=0}^n A_i \otimes Y^i = \sum_{i=0}^{n-1} (i+1) A_{i+1} \otimes Y^i$$

where $(i+1)A_{i+1}$ is the sum of $(i+1)$ copies of A_{i+1} .

Definition 1, Joyal, [3].

Let \mathcal{B} be the category of finite sets, with morphisms being the bijections. Define the category of combinatorial species $\text{Spc} = [\mathcal{B}, \text{FinSet}]$, which is equivalent to $[\mathcal{B}, \mathcal{B}]$.

Remark.

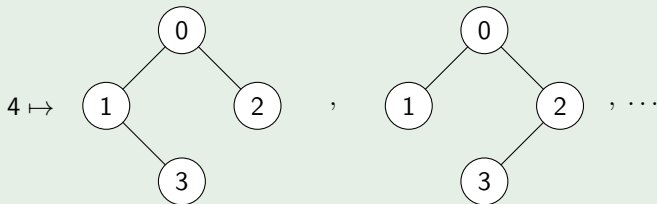
Decompose : $\mathcal{B} \simeq \coprod_{n=0}^{\infty} S_n$

So $X : \mathcal{B} \rightarrow \text{FinSet}$ can be decomposed as :

- a sequence of finite sets X_n , $n \geq 0$,
- a sequence of left actions of S_n on X_n , $n \geq 0$.

Example (species of trees)

Define the species of trees, by assigning to a finite set E the set of trees on E :



and the action of S_n on X_n permutes the vertices of a tree chosen in the set X_n .

Structure of differential 2-rig on Spc : for species X, Y and a finite set E :

- Sum :

$$(X + Y)(E) = X(E) + Y(E) = X(E) \amalg Y(E)$$

- Multiplication :

$$(X \otimes Y)(E) = \sum_{E_1 + E_2 = E} X(E_1) \times Y(E_2)$$

- Derivation :

$$(\partial X)(E) = X(E + 1) = X(E + \{*\})$$

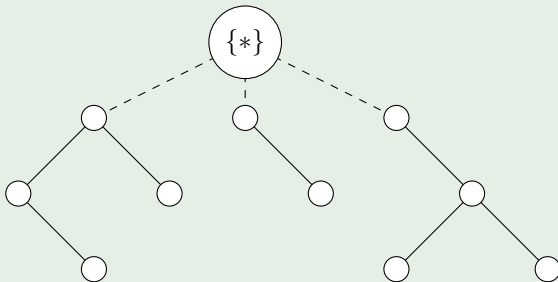
Additional structure on Spc : for species X, Y and a finite set E :

- Substitution :

$$(X \circ Y)(E) = \sum_{\pi \text{ partition of } E} X(\pi) \times \prod_{p \in \pi} Y(p)$$

Example (derivative of the species of trees, Bergeron, [2])

If X is the species of trees, the species ∂X assigns to a finite set E the set of trees on $E + \{*\}$:



So ∂X is the species of disjoint sets of rooted trees.

Summary

- 1 Background
- 2 Resolution of some differential equations
 - Former results to find fixed points of functors
 - Examples of equations
- 3 The number of solutions
- 4 Virtual differential 2-rigs

First goal : can we solve (some) differential equations in general 2-rigs ?

Polynomial differential equations : finding fixed points of :

$$X \mapsto A_0 + A_1 \otimes \partial X + A_2 \otimes (\partial^2)X + \cdots + A_n \otimes (\partial^n)X$$

For instance :

$$X \mapsto \partial X$$

Technique : use initial algebras and terminal coalgebras to find fixed points of functors.

Example

Take a set A . What are the fixed points of the following functor ?

$$\begin{aligned} T_A : \text{Set} &\rightarrow \text{Set} \\ S &\mapsto 1 + (A \times S) \end{aligned}$$

Start from the initial object \emptyset or the terminal object 1 , and recursively apply T_A to the unique morphisms $\emptyset \xrightarrow{!_1} T_A(\emptyset)$ and $1 \xleftarrow{!_2} T_A(1)$:

Example

$$\emptyset \xrightarrow{!_1} 1 \xrightarrow{T_A!_1} 1 + A \xrightarrow{T_A^2!_1} 1 + A + A^2 \xrightarrow{T_A^3!_1} 1 + A + A^2 + A^3 \rightarrow \dots$$

$$1 \xleftarrow{!_2} 1 + A \xleftarrow{T_A!_2} 1 + A + A^2 \xleftarrow{T_A^2!_2} 1 + A + A^2 + A^3 \leftarrow \dots$$

Taking :

- the colimit of the first equation gives A^* , ie the initial algebra of T_A ,
- the limit of the second equation gives $A^* + A^{\mathbb{N}}$, ie the terminal coalgebra of T_A ,

and they give solutions to $T_A(X) \simeq X$.

Theorem, Trnokvá et al.

A set functor has an initial algebra if and only if it has a fixed point.

First Adámek's theorem, [6].

If \mathcal{C} has an initial object 0 and ω -composition, and $F : \mathcal{C} \rightarrow \mathcal{C}$ preserves colimits of ω -chains, then the initial algebra of F is the colimit of :

$$0 \xrightarrow{!} F0 \xrightarrow{F!} F^2 0 \rightarrow \dots$$

Second Adámek's theorem, [1].

If \mathcal{C} has colimits and $F : \mathcal{C} \rightarrow \mathcal{C}$ preserves colimits of λ -chains for some infinite ordinal λ , then the initial algebra of F is $F^\lambda 0 \xrightarrow{F^\lambda !} F^{\lambda+1} 0$.

Lambek's theorem.

If $F : \mathcal{C} \rightarrow \mathcal{C}$ has an initial algebra $\alpha : F(X) \rightarrow X$, then α is an isomorphism.

Remark.

Dual versions also work.

Remark.

If they exist :

- the initial algebra is the smallest fixed point,
- the terminal coalgebra is the largest fixed point.

Difficult :

- comodules : no
- linear species $([GL(p), Vect_k], \oplus, \otimes)$: no
- etc.

Idea : $(\mathbb{N}, +)$ and (\mathbb{N}, \cdot) are monoidal categories.

Structure on $[(\mathbb{N}, +), \text{Vect}_k]$.

Consider $[(\mathbb{N}, +), \text{Vect}_k]$ with 'Day convolution'. That is, for objects F, G :

$$\begin{aligned}
 F + G &= (F_n \oplus G_n)_{n \in \mathbb{N}} \\
 F * G &= \left(\int^{p, q \in \mathbb{N}} (F(p) \otimes G(q)) \odot \mathbb{N}(p + q, n) \right)_{n \in \mathbb{N}} \\
 &= \left(\sum_{p+q=n} F(p) \otimes G(q) \right)_{n \in \mathbb{N}} \\
 I &= (k, 0, 0, \dots)
 \end{aligned}$$

Derivation ? Copy polynomials :

$$\partial F = ((n+1)F_{n+1})_{n \in \mathbb{N}} = \left(\bigoplus_{1 \leq k \leq n+1} F_{n+1} \right)_{n \in \mathbb{N}}$$

Structure on $[(\mathbb{N}, \cdot), \text{Vect}_k]$.

Consider $[(\mathbb{N}, \cdot), \text{Vect}_k]$ with 'Day convolution'. That is, for objects F, G :

$$\begin{aligned} F + G &= (F_n \oplus G_n)_{n \in \mathbb{N}} \\ F * G &= \left(\int^{p, q \in \mathbb{N}} (F(p) \otimes G(q)) \odot \mathbb{N}(p \cdot q, n) \right)_{n \in \mathbb{N}} \\ &= \left(\sum_{p \cdot q = n} F(p) \otimes G(q) \right)_{n \in \mathbb{N}} \\ I &= (0, k, 0, 0, \dots) \end{aligned}$$

Derivation ? For a prime number r :

$$\partial F = \partial_r F = 0 \oplus (\delta_n F_{r \cdot n})_{n \geq 1}$$

for some coefficients δ_n . Only choice of coefficients :

$$\partial F = \left(0, ((v_r(n) + 1) F_{r \cdot n})_{n \geq 1} \right)$$

Can we use the initial algebra or coalgebra techniques to solve the differential equation $\partial V \simeq V$ in our two examples of structures ?

$0 = (0, 0, \dots)$ is both initial and terminal. We want to study :

$$0 \xrightarrow{!} \partial 0 \xrightarrow{\partial !} \partial^2 0 \rightarrow \dots$$

$$0 \xleftarrow{!} \partial 0 \xleftarrow{\partial !} \partial^2 0 \leftarrow \dots$$

Issue : in our two structures we have $\partial 0 = 0$.

We even have $\partial I = 0$.

Let's completely solve the differential equation $\partial V \simeq V$ in our two examples of structures.

Solutions in $[(\mathbb{N}, +), \text{Vect}_k]$.

The solutions of $\partial V \simeq V$ are, up to isomorphism, the \mathbb{N} -graded vector spaces of the form $V = (k^\alpha)_{n \geq 0}$ for an infinite cardinal α , and the trivial space.

Proof.

$$\begin{aligned} \partial V \simeq V &\Leftrightarrow \forall n, V_n \simeq (n+1)V_{n+1} \\ &\Rightarrow V_0 \simeq V_1 \simeq 2V_2 \simeq 3!V_3 \simeq \dots \simeq n!V_n \simeq \dots \end{aligned}$$

3 steps :

- except the trivial solution, the dimensions must be infinite,
- assume $V = (k^{\alpha_n})_n$,
- equation on the dimensions α_n :

$$\forall n, \alpha_n \simeq (n+1)\alpha_{n+1}$$

Remark.

Imposing $V_0 = \Lambda$ for some infinite dimensional vector space Λ , we get exactly one solution up to isomorphism :

$$V = (\Lambda, \Lambda, \dots)$$

Remark.

If Λ is a non-trivial finite dimensional vector space, there is no solution.

Is $V_0 = \Lambda$ a nice initial condition ? Like $X[\emptyset] = \emptyset$ for species used by Labelle in [4], in

$$\begin{cases} \partial X = X \\ X[\emptyset] = \emptyset \end{cases}$$

Remark.

Similarly we can solve :

$$\begin{cases} \partial V \simeq A \otimes V + B \\ V_0 = \Lambda \end{cases}$$

but only under some conditions on A, B, Λ .

Definition.

For $n \in \mathbb{N}$, write the decomposition

$$n = w_r(n) r^{v_r(n)}$$

Solutions in $[(\mathbb{N}, \cdot), \text{Vect}_k]$.

The solutions of $\partial V \simeq V$ are, up to isomorphism, the \mathbb{N} -graded vector spaces of the form $V = (0, (U_{w_r(n)})_{n \geq 1})$, where, for w prime to r , U_w is the trivial space or of the form k^{α_w} for an infinite cardinal α_w .

Proof.

$$\partial V \simeq V \Leftrightarrow V_0 = 0 \quad \text{and} \quad \forall n \geq 1, V_n \simeq (v_r(n) + 1)V_{rn}$$

$$\Leftrightarrow V_0 = 0 \quad \text{and} \quad \forall w \text{ prime to } r, \forall v \geq 0, V_{wr^v} \simeq (v + 1)V_{wr^{v+1}}$$

$$\Leftrightarrow \begin{cases} V_0 = 0 \\ V_1 \simeq V_r, & V_r \simeq 2V_{r^2}, & V_{r^2} \simeq 3V_{r^3}, & V_{r^3} \simeq 4V_{r^4} & \dots \\ V_2 \simeq V_{2r}, & V_{2r} \simeq 2V_{2r^2}, & V_{2r^2} \simeq 3V_{2r^3}, & V_{2r^3} \simeq 4V_{2r^4} & \dots \\ V_3 \simeq V_{3r}, & V_{3r} \simeq 2V_{3r^2}, & V_{3r^2} \simeq 3V_{3r^3}, & V_{3r^3} \simeq 4V_{3r^4} & \dots \\ \dots \\ V_w \simeq V_{wr}, & V_{wr} \simeq 2V_{wr^2}, & V_{wr^2} \simeq 3V_{wr^3}, & V_{wr^3} \simeq 4V_{wr^4} & \dots \\ \dots \end{cases}$$

Set $U_v^{(w)} = V_{wr^v}$ for w prime to r , and use the fact that each $n \in \mathbb{N}$ has a unique decomposition $n = wr^v$ with w prime to r .

Remark.

Imposing $V_w = \Lambda^{(w)}$ for some infinite dimensional vector spaces $\Lambda^{(w)}$ for w prime to r , we get exactly one solution up to isomorphism.

Is $V_w = \Lambda^{(w)}$ for w prime to r a nice initial condition ?

Summary

- 1 Background
- 2 Resolution of some differential equations
- 3 **The number of solutions**
 - Labelle's result about the number of solutions for combinatorial species
 - A conjecture which would extend Labelle's result
 - Examples of equations in the context of our conjecture
- 4 Virtual differential 2-rigs

Definition 2.1, Labelle [4].

Given species $F_{i,j}$, a solution of the differential problem

$$\begin{cases} \partial Y_i = F_{i,j}(X_1, \dots, X_k, Y_1, \dots, Y_p), & 1 \leq i \leq p, 1 \leq j \leq k \\ Y_i[\emptyset, \dots, \emptyset] = \emptyset, & 1 \leq i \leq p \end{cases}$$

is a family of species $A = (A_i(X_1, \dots, X_k))_{1 \leq i \leq p}$ and natural isomorphisms

$$\theta_{i,j} : \partial A_i / \partial X_j \xrightarrow{\sim} F_{i,j}(X_1, \dots, X_k, A_1, \dots, A_p)$$

such that

$$A_i[\emptyset, \dots, \emptyset] = \emptyset, \quad 1 \leq i \leq p$$

Example

$$\begin{cases} \partial X = A \otimes X + B \\ X[\emptyset] = \emptyset \end{cases}$$

Part of theorem A, Labelle [4].

If m is a finite (possibly null) cardinal number or $m = 2^{\aleph_0}$, then there exists a normalized compatible differential problem having exactly m non-isomorphic combinatorial solutions. Moreover, no differential problem can have exactly $m = \aleph_0$ or $m > 2^{\aleph_0}$ non-isomorphic combinatorial solutions.

Lemma 2.6, Labelle [4].

For $n = (n_1, \dots, n_k) \in \mathbb{N}^k$, there exists only a finite number $\mu_n > 0$ of non-isomorphic molecular species

$$M_n^{(i)} = M_n^{(i)}(X_1, \dots, X_k)$$

supported by multisets having multcardinality n .

Every species $H = H(X_1, \dots, X_k)$ has a unique molecular decomposition of the form

$$H = \sum_{n \in \mathbb{N}^k, 1 \leq i \leq \mu_n} C_n^{(i)}(H) M_n^{(i)}$$

where $C_n^{(i)}(H)$ are natural integers.

Moreover, for any pair H, K of species we have

$$H \simeq K \quad \Leftrightarrow \quad \forall n, \forall i, C_n^{(i)}(H) = C_n^{(i)}(K)$$

Conjecture.

If \mathcal{C} is a monoidal category with initial object 0, such that the cardinality of \mathcal{C}_0 is κ , and such that the 2-rig $[\mathcal{C}^{op}, \text{Set}]$ can be endowed with a derivation ∂ , then the differential problem :

$$\begin{cases} \partial X & \simeq & X \\ X[0] & = & \{*\} \end{cases}$$

has at most 2^κ solutions.

We want to replace $[(\mathbb{N}, +), \text{Vect}_k]$ with something of the form $[C^{op}, \text{Set}]$:

- Replace $(\mathbb{N}, +)$ by $(\mathbb{N}, \geq, \min) = (\mathbb{N}, \leq, \max)^{op}$.
- We want to replace Vect_k by Set : same properties :

$$k^\alpha \oplus k^\beta = k^{\alpha+\beta}$$

$$k^\alpha \otimes k^\beta = k^{\alpha \times \beta}$$

Define the differential 2-rig $[(\mathbb{N}, \geq, \min), \text{Set}]$:

- Sum :

$$F + G = (F_n + G_n)_{n \in \mathbb{N}}$$

- Multiplication :

$$\begin{aligned} F * G &= \left(\int^{p, q \in \mathbb{N}} F(p) \times G(q) \times \mathbb{N}(n, \min(p, q)) \right)_{n \in \mathbb{N}} \\ &= \left(\sum_{n \leq p, q} F(p) \times G(q) \right)_{n \in \mathbb{N}} \end{aligned}$$

- Derivation :

$$\partial F = \left(\coprod_{k \in \mathbb{N}_0} F_n \right)_{n \in \mathbb{N}} = (\mathbb{N}_0 F_n)_{n \in \mathbb{N}}$$

Is ∂ really Leibniz ? For example for naturality.

On objects F, G , at the level $n \geq 0$:

$$\left\{ \begin{array}{lcl} (\partial(F * G))_n & = & \coprod_{k \in \mathbb{N}_0} \coprod_{n \leq p, q} F(p) \times G(q) \\ (\partial F * G + F * \partial G)_n & = & \coprod_{n \leq p, q} (\coprod_{k \in \mathbb{N}_0} F(p)) \times G(q) \\ & & + \coprod_{n \leq p, q} F(p) \times (\coprod_{k \in \mathbb{N}_0} G(q)) \\ & \simeq & \coprod_{t \in \{0,1\}} \coprod_{k \in \mathbb{N}_0} \coprod_{n \leq p, q} F(p) \times G(q) \end{array} \right.$$

The above isomorphism is natural. If we fix a bijection $\mathbb{N}_0 \simeq \{0,1\} \times \mathbb{N}_0$, independently of F, G , we can show we have a natural isomorphism between the two above expressions, by reindexing.

Goal : solve $\partial V \simeq V$ in this structure.

Solutions in $[(\mathbb{N}, \geq, \min), \text{Set}]$.

The solutions of $\partial V \simeq V$ are, up to isomorphism, the objects $V = (V_n)_{n \in \mathbb{N}}$ such that each V_n is an infinite set or 0.

Proof.

$$\partial V \simeq V \quad \Leftrightarrow \quad \forall n \geq 0, V_n \simeq \aleph_0 V_n$$

So $V_0 = 0$ or even $V_0 = \Lambda$ doesn't fix a 'reasonable' number of solutions : (\mathbb{N}, \geq, \min) has \aleph_0 objects, but we have strictly more than 2^{\aleph_0} solutions even with the initial condition.

Summary

- 1 Background
- 2 Resolution of some differential equations
- 3 The number of solutions
- 4 Virtual differential 2-rigs
 - Virtual species
 - Generalization

Recall Labelle's decomposition of combinatorial species :

$$H = \sum_{n \in \mathbb{N}^k, 1 \leq i \leq \mu_n} C_n^{(i)}(H) M_n^{(i)}$$

where $C_n^{(i)}$ are natural integers and $M_n^{(i)}$ are molecular species.

If we :

- allow negative coefficients, writing $H = H_p - H_n$ for two species H_p, H_n ,
- quotient up to $H_p - H_n = H'_p - H'_n \Leftrightarrow H_p + H'_n \simeq H'_p + H_n$,

we get the virtual species.

It can give solutions to equations which otherwise wouldn't have any.

Definition.

A category \mathcal{C} is cancellative if for every objects A, B, C , the property $A + B \simeq A + C$ implies $B \simeq C$.

Consider a **cancellative** differential 2-rig $(\mathcal{C}, +, \otimes, \partial)$.

Definition.

Set $(\mathcal{C}^2, \boxplus, \boxtimes, \bar{\partial})$, where :

$$(A, B) \boxplus (C, D) = (A + C, B + D)$$

$$(A, B) \boxtimes (C, D) = (A \otimes C + B \otimes D, A \otimes D + B \otimes C)$$

$$\bar{\partial}(A, B) = (\partial A, \partial B)$$

Theorem.

$(\mathcal{C}^2, \boxplus, \boxtimes, \bar{\partial})$ is a differential 2-rig.

Definition.

The virtual category $\mathbb{V}(\mathcal{C})$ is \mathcal{C}^2 quotiented by $(A, B) \sim (C, D)$ if and only if $A + D \simeq C + B$, ie the category with :

- objects : \mathcal{C}_0^2 quotiented by \sim ,
- morphisms $[(A, B)] \rightarrow [(C, D)]$: the morphisms $(A', B') \rightarrow (C', D')$ for all $(A, B) \sim (A', B')$ and $(C, D) \sim (C', D')$.

Theorem.

The virtual category $\mathbb{V}(\mathcal{C})$ is a differential 2-rig.

Theorem.

\mathcal{C} quotiented by isomorphisms, can be embedded into $\mathbb{V}(\mathcal{C})$ as a differential 2-rig.



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