NORTH-HOLLAND

Sequence Operators From Groups

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Dedicated to J. J. Seidel

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ABSTRACT

This paper is inspired by the paper *Some Canonical Sequences of Integers* by Bernstein and Sloane. The main observation is that seven of the operators defined in that paper have natural interpretations in terms of counting orbits of groups, providing a pattern which is completed by five further operators. Some of their eigen-sequences also have group-theoretic meaning.

1. INTRODUCTION

A permutation group G on an infinite set Ω is *oligomorphic* if, for each positive integer n, there are only finitely many G-orbits on Ω^n. These groups have close connections with logic: the first-order theory of a countable structure is $\aleph_0$-categorical if and only if $\mathrm{Aut}(M)$ is oligomorphic (the Engeler-Ryll-Nardzewski–Svenonious theorem). There are also strong links with enumerative combinatorics (see below). More information can be found in Cameron [3].

Associated with an oligomorphic permutation group, there are three obvious sequences of integers:

- $F_n^*(G)$, the number of G-orbits on Ω^n;
- $F^n(G)$, the number of G-orbits on ordered n-tuples of distinct points;
- $f_n(G)$, the number of G-orbits on n-sets.

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The first two sequences carry equivalent information since

$$F_n^*(G) = \sum_{k=1}^{n} S(n,k) F_k(G),$$

that is, the first is the Stirling transform of the second (in the terminology of [1]). Sometimes the first is more natural, e.g., $F_n^*(\mathrm{Aut}(M))$ is the number of n-types in the $\aleph_0$-categorical theory of the countable structure M. But I will consider only the second and third sequences below. When I speak of generating functions, I will always mean the ordinary generating function $f_G(x) = \sum f_n(G)x^n$ of the third sequence, and the exponential generating function $F_G(x) = \sum F_n(G)x^n/n!$ of the second. (By convention, $f_0 = F_0 = 1$.)

Now suppose that M is a countable structure such that $G = \mathrm{Aut}(M)$ is oligomorphic. Without loss of generality, we may suppose that M is *homogeneous*, that is, any isomorphism between finite substructures of M extends to an automorphism of M. The *age* of M, $\mathrm{Age}(M)$, is the class of finite substructures of M. Now

- $f_n(G)$ is the number of unlabelled n-element structures in $\mathrm{Age}(M)$;
- $F_n(G)$ is the number of labelled n-element structures in $\mathrm{Age}(M)$.

A theorem of Fraïssé gives a necessary and sufficient condition for a class of finite structures to be the age of a countable homogeneous structure. For any class satisfying this condition (of which many exist!), the unlabelled and labelled enumeration problems are thus equivalent to calculating $f_n(G)$ and $F_n(G)$, respectively, for an appropriate oligomorphic group G.

In fact, for any oligomorphic group G, there is a power series in infinitely many variables, the *modified cycle index*, of which both $f_G(x)$ and $F_G(x)$ are specializations. The series is itself a special case of Joyal's cycle index of a species [4]. (Joyal would presumably regard an age as a species.)

2. OPERATORS FROM GROUPS

Let H and K be permutation groups on sets Γ and Δ, respectively. Then the direct product $H \times K$ is a permutation group on the disjoint union $\Gamma \cup \Delta$ and the wreath product $H \operatorname{Wr} K$ is a permutation group on the Cartesian product $\Gamma \times \Delta$ (regarded as a cover of Δ with fibers isomorphic to Γ). The point stabilizer H_α in a transitive group H acts on the points different from α.

These group-theoretic constructions can be used to define operators on sequences. Once we have understood how these operators work, they can be extended to all sequences (a_n) of positive integers with $a_0 = 1$. In terms of generating functions, we have

$$F_{H\times K}(x) = F_H(x)\cdot F_K(x),$$

$$f_{H\times K}(x) = f_H(x)\cdot f_K(x),$$

$$F_{H\,\mathrm{Wr}\,K}(x) = F_K(F_H(x) - 1).$$

The sequence $(f_n(H \,\mathrm{Wr}\, K))$ is not uniquely determined by the sequences $(f_n(H))$ and $(f_n(K))$; we require $(f_n(H))$ and the modified cycle index of K. This is the most interesting case! Also, $F_n(H_\alpha) = F_{n+1}(H)$ if H is transitive; so $F_{H_\alpha}(x) = F_H'(x)$. (The sequence $(f_n(H_\alpha))$ is not uniquely determined by $(f_n(H))$; it requires knowledge of the modified cycle index of H.)

Two particular groups we require are:

- S, the symmetric group on a countable set;
- A, the group of order-preserving permutations of the rationals.

We have $F_S(x) = e^x$, $F_A(x) = f_S(x) = f_A(x) = 1/(1-x)$.

Now we consider some operators. Where these occur in [1], I have used the same names. I have also used **SUM** for the operator which maps a sequence to its sequence of partial sums. The other new operators are defined below. The columns give, respectively, the map of groups and the effect on the sequences (f_n) and (F_n). The first row requires that G is transitive. L is the left shift.

Group	Unlabelled	Labelled
$G \mapsto G_\alpha$	undefined	L
$G \mapsto G \times G$	**CONV**	**EXP-CONV**
$G \mapsto G\,\mathrm{Wr}\,G$	undefined	**O1**
$G \mapsto G \times S$	**SUM**	**BINOMIAL**
$G \mapsto G \times A$	**SUM**	**O2**
$G \mapsto G\,\mathrm{Wr}\,S$	**EULER**	**EXP**
$G \mapsto G\,\mathrm{Wr}\,A$	**INVERT**	**O3**
$G \mapsto S\,\mathrm{Wr}\,G$	undefined	**STIRLING**
$G \mapsto A\,\mathrm{Wr}\,G$	undefined	**O4**

The operators **O1–O4** are defined by their effect on e.g.f.'s as follows. As in [1], $A(x)$ and $\mathscr{A}(x)$ denote the ordinary and exponential generating functions of a sequence (a_n), and $B(x)$ and $\mathscr{B}(x)$ those of its transform by the operator under consideration:

O1: $\mathscr{B}(x) = \mathscr{A}(\mathscr{A}(x) - 1)$;
O2: $\mathscr{B}(x) = \mathscr{A}(x)/(1 - x)$;
O3: $\mathscr{B}(x) = 1/(2 - \mathscr{A}(x))$;
O4: $\mathscr{B}(x) = \mathscr{A}(x/(1 - x))$.

Other groups define further operators. For example, let C be the group preserving the cyclic order of the complex roots of unity. We have $f_C(x) = 1/(1 - x)$ and $F_C(x) = 1 - \log(1 - x)$; this determines all the operators except the entry under "Unlabelled" corresponding to $G \mapsto G \operatorname{Wr} C$, which is defined on the ordinary generating function by

$$B(x) = 1 - \sum_{n=1}^{\infty} \frac{\phi(n)}{n} \log(2 - A(x^n)).$$

where ϕ is Euler's totient, and the entry under "Unlabelled" for $G \mapsto C \operatorname{Wr} G$, which is undefined.

3. EIGEN-SEQUENCES

It seems to happen quite often that eigen-sequences of these operators are associated with interesting groups. In most cases, there is on obvious reason; but a few can be explained by structural properties of various groups.

Suppose that Φ is some operator on groups, and **O** the operator on sequences corresponding to $(F_n(G)) \mapsto (F_n(\Phi(G)))$. If there exists a group G which satisfies the "differential equation" $G_\alpha \cong \Phi(G)$, then the sequence $(F_n(G))$ is shifted left by the operator **O**.

For example, the group $G = A$ satisfies $A_\alpha \cong A \times A$ (the two factors acting on the left and the right of α). So **EXP-CONV** $\circ\, F(A) = L \circ F(A)$, in agreement with a property of $F_n(A) = n!$ noted in [1] (sequence S15 in the table). We see that the same equation shows that the factorials are also shifted left by the operator **O2** defined in the preceding section.

Similarly, $G = S \operatorname{Wr} S$ satisfies $G_\alpha \cong G \times S$, so **BINOMIAL** $\circ\, F(G) = L \circ F(G)$. Now $F_n(G) = F_n^*(S)$ is the nth Bell number (sequence S1 in [1]) giving a known characterization of multiple transitivity: a group G is n-transitive if and only if $F_n^*(G)$ is the nth Bell number. (G is n-transitive if and only if $F_m(G) = 1$ for $m \leqslant n$.)

There is a group G satisfying $G_\alpha \cong G \operatorname{Wr} A$, defined in Cameron [2] (and called ∂T_3 there); it is the automorphism group of a ternary relational structure associated with the leaves of a binary tree. The sequence $F(G)$, where $F_1(G) = 1$ and $F_n(G) = (2n-3)!! = 1 \cdot 3 \cdot 5 \cdots (2n-3)$ for $n > 1$, is shifted left by the operator **O3**.

It can be shown that no transitive permutation group G can satisfy $G_\alpha \cong G \operatorname{Wr} S$. However, the group C of the preceding section does satisfy $F_n(C_\alpha) = F_n(C \operatorname{Wr} S)$ for all n, corresponding to the decomposition of permutations into disjoint cycles. So $\mathbf{EXP} \circ F(C) = L \circ F(C)$ (see sequence S15 in [1]).

The sequences shifted left by **O1** and **O4**, commencing

$$1, 1, 2, 7, 37, 269, 2535, \ldots$$

and

$$1, 1, 3, 15, 111, 1131, 15081, \ldots$$

respectively, appear to be new. I know no groups satisfying the corresponding "differential equations" $G_\alpha \cong G \operatorname{Wr} G$ or $G_\alpha \cong A \operatorname{Wr} G$.

Similar group-theoretic facts sometimes translate into identities between the operators. For example, for any transitive group G, we have

$$(S \operatorname{Wr} G)_\alpha \cong S \times (S \operatorname{Wr} G_\alpha),$$

whence

$$L \circ \mathbf{STIRLING} = \mathbf{BINOMIAL} \circ \mathbf{STIRLING} \circ L.$$

This identity translates into the familiar relation

$$\sum_{l=k}^{n} \binom{n-1}{l} S_{l,k} = S_{n,k+1}.$$

It also shows that **STIRLING** maps the all-one sequence (the eigen-sequence of L) to the Bell numbers S1 (the solution of $\mathbf{BINOMIAL} \circ a = L \circ a$).

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