Nondeterminism and the Size of Two Way Finite Automata

William J. Sakoda *

and

Michael Sipser †

Computer Science Division
University of California
Berkeley, California 94720

§1.1 INTRODUCTION

An important goal of the theory of computation is the classification of languages according to computational difficulty. Classes such as P, NP, and LOGSPACE provide a natural framework for this, though it is a fundamental open problem to demonstrate languages distinguishing them. The complete languages of Cook, Karp, and others [1-7] are candidates for such languages in the sense that, if the classes are in fact different, these languages witness the difference.

We consider two questions on regular languages resembling these open problems. One of these questions concerns 2-way non-deterministic (2n) and 2-way deterministic (2d) finite automata:

For every 2nfa M, is there an equivalent 2dfa with only polynomially more states than M?

Let $s_{2n \to 2d}(n)$ be the least integer such that for every n-state 2nfa there is an equivalent $s_{2n \to 2d}(n)$ state 2dfa. The question above can then be restated: Is $s_{2n \to 2d}(n)$ bounded above by a polynomial $p(n)$?

Here is a summary of our results. In section 2.2 we present a sequence of languages $< C_1, C_2, \cdots >$ which is complete in the sense that in order to settle the question of polynomial boundedness of $s_{2n \to 2d}$, it suffices to determine the size of 2dfa required by each language C_n. $s_{2n \to 2d}(n)$ is bounded above by a polynomial $p(n)$ iff the size of 2dfa required by C_n is bounded above by some (other) polynomial $p'(n)$.

In section 2.3 we present languages $< B_1, B_2, \cdots >$ which are complete with respect to 1-way non-deterministic (1n) to 2-way deterministic conversion: B_n is an n-state 1n language, and requires the largest 2dfa of any n-state 1n language. We conjecture that the size of 2d acceptors for B_n is not bounded above by any polynomial $p(n)$. This conjecture is the starting point for the investigation described in section 4.

In section 4, we consider certain restricted forms of two way automata. Techniques are developed which yield tight lower bounds on the size of such acceptors of the complete languages.

Section 3 presents a convenient notation for comparing succinctness of description of different models of automata. This notation, together with a reducibility, is used to state an analogy between the problems studied in this paper and the *P =? NP* question.

§1.2 A MAP

2: Completeness Results.

2.1: Definitions of the languages C_n, B_n, and remarks about our models of finite automata.

2.2: Completeness of $< C_1, C_2, \cdots >$ for $2n \to 2d$.

2.3: Completeness of $< B_1, B_2, \cdots >$ for $1n \to 2d$.

3: An analogy to $P =? NP$.

4: Lower Bounds.

4.1: One way automata.

4.2: Restricted two way automata (parallel machines).

4.3: Restricted two way automata (series machines).

5: Related Work

* Supported by NSF grant MCS75-21760-A01.

† Supported by NSF grant MCS75-21760-A01 and an IBM Graduate Fellowship.

Presented at the Tenth Annual Symposium on Theory of Computing, May 1978.

§2.1 DEFINITIONS OF LANGUAGES C_n and B_n

In this section we present the languages $<C_1, C_2, \cdots>$ which are complete with respect to $2n \rightarrow 2d$ conversion, and the languages $<B_1, B_2, ...>$ which are complete for $1n \rightarrow 2d$ conversion. Statements and proofs of the completeness of C_n and B_n follow in sections 2.2 and 2.3. A comment about the models of finite automata we use appears at the end of this section.

Definition of language C_n:
(1) Let the alphabet Γ_n be the graphs consisting of n left nodes and n right nodes. Directed arcs may join any distinct pair of nodes. Figure 1 shows three members of Γ_5.

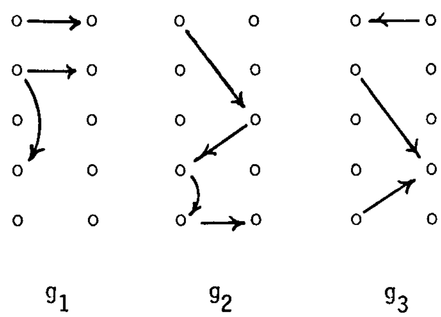


Figure 1

Given a sequence of graphs $g_1 g_2 \cdots g_k \epsilon \Gamma_n^*$, the *catenated graph* $g_1|g_2| \cdots |g_k$ is obtained by identifying adjacent left and right nodes in the sequence. For example, the graphs in Figure 1 catenate to yield the solid arcs in Figure 2.

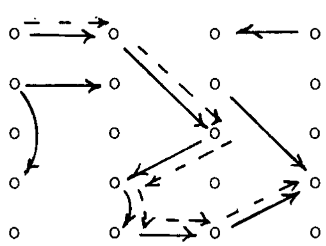


Figure 2

Solid arcs show the catenated graph $g_1|g_2|g_3$

(2) The language C_n consists of all sequences of graphs $g_1 g_2 \cdots g_k \epsilon \Gamma_n^*$ such that $g_1|g_2| \cdots |g_k$ has a directed path from a node in the first column to a node in the last column. In Figure 2, the dotted path witnesses the fact that $g_1 g_2 g_3$ is in C_5.

Definition (of language B_n):
(1) The alphabet Σ_n is a subset of Γ_n. A graph g from Γ_n is in Σ_n if every arc in g is directed left to right.

(2) The string $g_1 g_2 \cdots g_k$ is in B_n if the catenated graph $g_1|g_2| \cdots |g_k$ has a path from the leftmost column to the rightmost column.

Definition: A 2*-way non-deterministic finite automaton M* is defined by a 7-tuple $<Q, \Sigma, \vdash, \dashv, \delta, Q_0, F>$. An input string $a_1 a_2 \cdots a_k$ from Σ is presented delimited by left and right endmarkers $\vdash$ and $\dashv$. The automaton is started on the symbol a_1 in one of the states from the initial state set Q_0. The transition function $\delta: (\Sigma \cup \{\vdash, \dashv\}) \rightarrow P(Q \times \{L, R\})$ (P denoting power set) defines the moves of M; for example if $<q', L> \epsilon \delta(q, a)$ then M, scanning input symbol a in state q may move left (right, if R appears instead of L) and transfer to state q'. M accepts if from some initial configuration there is a sequence of moves which causes it to move onto the right endmarker into a final state $f \epsilon F$. A 2*-way deterministic finite automaton* (*2dfa*) is a 2nfa where the transition function is never multiply defined; that is $\delta: (\Sigma \cup \{\vdash, \dashv\}) \rightarrow (Q \times \{L, R\}) \cup \{\phi\}$.

Definition: A 1*-way non-deterministic finite automaton* (1*nfa*) is defined by a 5-tuple $<Q, \Sigma, \delta, Q_0, F>$. Q_0 is the set of initial states. $\delta: Q \times \Sigma \rightarrow P(Q)$ is the transition function.

Definition: A 1*-way deterministic finite automaton* (1*dfa*) is defined by a 5-tuple $<Q, \Sigma, \delta, q_0, F>$. q_0 is the initial state. $\delta: Q \times \Sigma \rightarrow Q$ is the (everywhere defined) transition function.

§2.2 $2N \rightarrow 2D$: $<C_1, C_2, \cdots>$ is COMPLETE

Theorem 2.2: The size of 2d acceptors for the languages C_n grows polynomially iff state expansion for $2n \rightarrow 2d$ conversion is polynomial. More specifically: Let $c(n)$ denote the size of 2dfa required by C_n.

(i): $c(n) \leqslant s_{2n \rightarrow 2d}(2n)$.

(ii): $s_{2n \rightarrow 2d}(n) \leqslant c(2n)$.

Proof of (i): C_n is accepted by a $2n$-state 2nfa. □

Proof of (ii), *Overview:* An n-state non-deterministic machine N is given. We can assume that there is a $c(2n)$-state 2dfa G for the language C_{2n}. We will show that there is a $c(2n)$ – state 2dfa D which is equivalent to N.

There are 2 parts to this proof. The first part shows a way of encoding computations of the machine N as Γ_{2n}–graphs (i.e., graphs from Γ_{2n}). Figure 3 illustrates this encoding.

To each string $x = \vdash a_1 a_2 \cdots a_k \dashv$ we associate a catenated sequence of Γ_{2n} – graphs
$g(x) = g(\vdash) \mid g(a_1) \mid g(a_2) \mid \cdots \mid g(a_k) \mid g(\dashv)$ in such a way that x is accepted by N iff $g(x) \epsilon C_{2n}$. A possible stra-

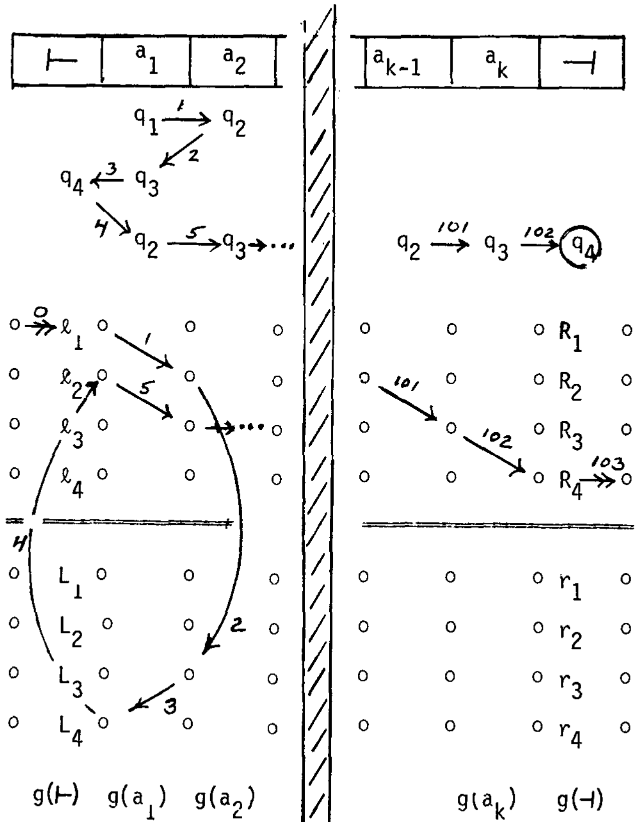


Figure 3

q_1 is initial
q_4 is final

tegy for recognizing $L(N)$ is then: "On input x, test the catenated sequence of graphs $g(x)$ for membership in C_{2n}; accept if and only if $g(x)\epsilon C_{2n}$." The second part of the proof shows how to obtain the required 2dfa D by using this strategy.

Proof of (ii), *Part 1:* An n-state 2nfa N is given. We will give a catenated sequence of Γ_{2n} – graphs
$g = \quad g(\vdash) \mid g(a_1) \mid g(a_2) \mid \cdots \mid g(a_k) \mid g(\dashv)$ which encodes the computation of N on input $\vdash a_1 a_2 \cdots a_k \dashv$. The encoding we wish to establish is:

Lemma: There is a path from the leftmost to the rightmost column in g (that is, $g\epsilon C_{2n}$) iff N accepts input $\vdash a_1 a_2 \cdots a_k \dashv$.

We now define g to satisfy the Lemma. Let $q_1, q_2, \cdots, q_n$ enumerate the states of N.

Definition of g(b): With each symbol $b\epsilon(\Sigma \cup \{\vdash, \dashv\})$, associate a graph $g(b)\epsilon\Gamma_{2n}$. Let the nodes of $g(b)$ be named as in Figure 4.

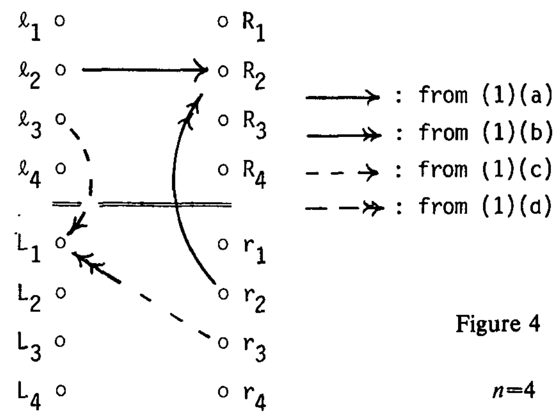


Figure 4

n=4

(1): For real input symbols $b\epsilon\Sigma$, the arcs of $g(b)$ consist of:

$$\left.\begin{array}{ll}(a) & (l_i \rightarrow R_j)\\ (b) & (r_i \rightarrow R_j)\end{array}\right\} (q_j, R)\epsilon\delta(q_i, a)$$

$$\left.\begin{array}{ll}(c) & (l_i \rightarrow L_j)\\ (d) & (r_i \rightarrow L_j)\end{array}\right\} (q_j, L)\epsilon\delta(q_i, a)$$

(2): For the left endmarker, define $g(\vdash)$ by:

$$(a) \quad (r_i \rightarrow R_j)\Big\} \; (q_j, R)\epsilon\delta(q_i, \vdash)$$

$$(b) \quad (l_i \rightarrow R_i)\Big\} \; q_i \text{ is an initial state of } N.$$

(*cf* Fig. 5)

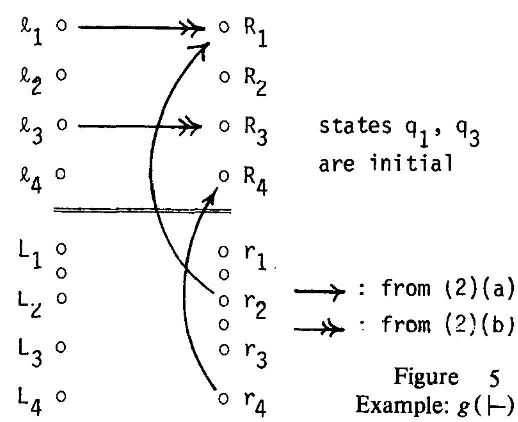


Figure 5
Example: $g(\vdash)$

(3): For the right endmarker, define $g(\dashv)$ by:

$$(a) \quad (l_i \rightarrow L_j)\Big\} \; (q_j, L)\epsilon\delta(q_i, \dashv)$$

$$(b) \quad (l_i \rightarrow R_i)\Big\} \; q_i \text{ is a final state of } N.$$

(*cf* Fig. 6)

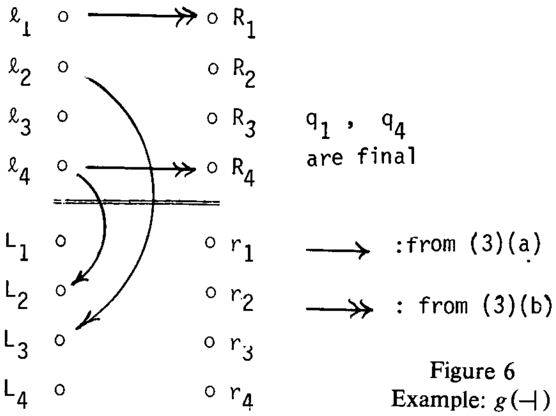


Figure 6
Example: $g(\dashv)$

This concludes the definition of g.

We now show that g satisfies the Lemma. Moves made by N correspond to paths through the catenated graph $g = g(\vdash) \mid g(a_1) \mid \cdots \mid g(a_k) \mid g(\dashv)$, and conversely. Figure 3 (at the beginning of this proof) shows an example of this correspondence. The top level of Figure 3 shows a string presented to N with endmarkers. The second level shows the initial and final segments of an accepting computation of N on the input. The third level shows those arcs in the catenated graph $g(\vdash) \mid g(a_1) \mid g(a_2) \mid \cdots \mid g(a_k) \mid g(\dashv)$ which correspond to steps in the computation. Let us discuss this correspondence.

N's first step is right from a_1 to a_2. By (1a) in the definition of $g(a_1)$, arc 1 is present in the catenated graph. Similarly, arc 2 is present by clause (1c), arc 3 by (1d), arc 4 by (2a), and arcs 5, 101, and 102 by (1a). By pursuing this correspondence it can be shown in general that:

SUBLEMMA 1: There is a path in $g = g(\vdash) \mid g(a_1) \mid \cdots \mid g(a_k) \mid g(\dashv)$, from l_i to R_j iff N, starting on symbol a_1 of input $\vdash a_1 a_2 \cdots a_k \dashv$ in state q_i, reaches the right endmarker in state q_j.

Now by (2b), arc 0 connects the leftmost column of g to l_1 because q_1 is an initial state of N. By (3b), arc 103 connects R_4 to the rightmost column of g because q_4 is a final state of N.

In general, it follows from (2b) that:

SUBLEMMA 2: There is an arc from the leftmost column of g to l_i iff q_i is an initial state of N.

From (3b), we get:

SUBLEMMA 3: There is an arc from R_j to the rightmost column of g iff q_j is a final state of N.

The three sublemmas together yield the main lemma. This concludes Part 1 of the proof.

PROOF of (ii), *Part 2:*

In Part 1 of the proof we have shown how to encode the computation of any given n-state non-deterministic machine $N = \langle Q, \Sigma, \vdash, \dashv, \delta, q_0, F \rangle$ as $2n$-graphs. Now assume we have a 2-way deterministic finite automaton $G = \langle Q_G, \Gamma_{2n}, \vdash, \dashv, \delta_G, q_G, F_G \rangle$ for the language C_{2n}. G will be used to construct a deterministic machine D which is equivalent to N. On input $x = \vdash a_1 a_2 \cdots a_k \dashv$, D will simulate the computation of G on the input $\vdash g(\vdash)\, g(a_1)\, g(a_2) \cdots g(a_k)\, g(\dashv) \dashv$. By the Lemma above, this will cause D to accept x iff N accepts x. We now indicate the correspondence between the computations of G and D, and define the machine D by specifying its initial state, final states, and transition function δ_D. D will have the same state set as G.

We will analyze the moves of G in 4 different situations, arranging in each case to have D simulate the action of G.

(1) Main segment of the computation (cf. Fig. 7) .

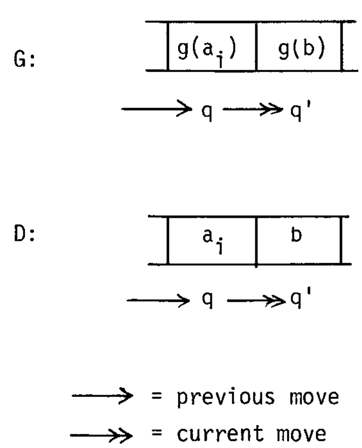


Figure 7

G's previous move: Moves onto symbol $g(a_i)$ into state q.

Possible next moves for G: Move one square left, or right, or remain forever on this symbol, according to $\delta_G(q, g(a_i))$.

D's previous move: Move onto symbol a_i into state q.

D's simulation: imitate the move of G. We set $\delta_D(q, a_i) = \delta_G(q, g(a_i))$.

(2) Left endmarker (cf. Fig. 8).

G's previous move: Move left onto the symbol $g(\vdash)$ into state q.

Consider G's computation on the 2-symbol string $\vdash\ g(\vdash)$ starting on $g(\vdash)$ in state q. Possible outcomes of G's subsequent moves:

Case (a): Falls off the right end of the 2-symbol string $\vdash\ g(\vdash)$ into state q'.

Case (b): Fails to do so, thereby rejecting the input.

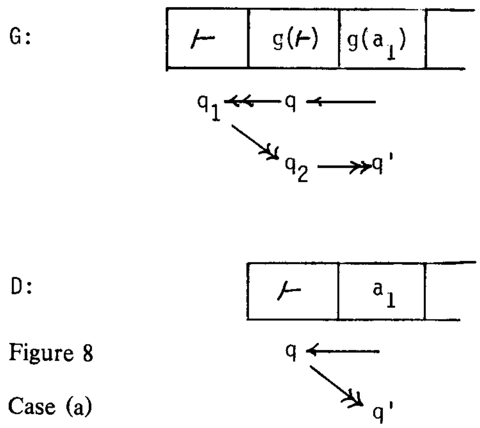


Figure 8

Case (a)

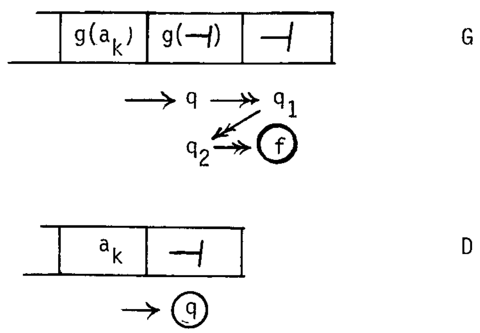
D's previous move: Moves left onto the symbol $\vdash$ into state q.

D's simulation:

Case (a): Compress the moves made by G into a single move. Set $\delta_D(q, \vdash)=\langle q',R\rangle$.

Case (b): G has rejected its input. D should reject also. Set $\delta_D(q, \vdash)=\phi$.

(3) Right endmarker and final states (cf. Fig. 9).



Make q a final state of D

Case (a)
Figure 9

G's previous move: Moves right onto $g(\dashv)$ in state q.

Outcomes for *G*: Consider the ensuing computation of G on the 2-symbol string $g(\dashv)\ \dashv$. We distinguish three possible results:

Case (a): G passes through a final state f, accepting the input.

Case (b): G does not pass through a final state, and

(i) falls off the left end of $g(\dashv)\ \dashv$ into some state q'.

(ii) does not fall off the left end of this string, rejecting the input.

D's previous move: Move right onto $\dashv$ in state q.

D's simulation:

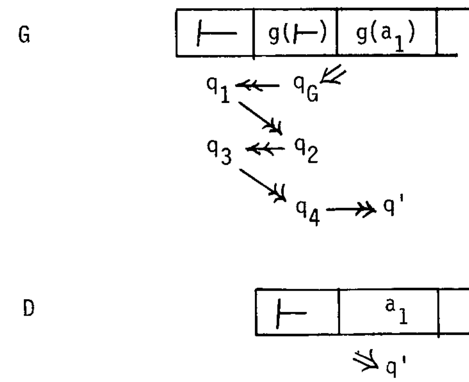
Case (a): Since G has accepted, D should accept also. Make q a final state of D, and set $\delta_D(q,\dashv)=\phi$.

Case (b):

(i): D, in a single move, simulates *G*'s eventual move left into state q'. We set $\delta_D(q,\dashv)=\langle q',L\rangle$.

(ii): G has rejected its input. D should reject also, so we set $\delta_D(q,\dashv)=\phi$.

(4) Initial states (cf. Fig. 10).



Make q' the
initial state of D

Figure 10

G: Begins on $g(\vdash)$ in state q_G.

Outcome: Consider the computation of G on the 2-symbol string $\vdash\ g(\vdash)$ starting on $g(\vdash)$ in state q_G. If the machine G is to ever accept anything, it must fall off the right end of this string into a state q'. We want to arrange the initial state of D to correspond to this situation.

D: Set the initial state of D to be q'. D begins on symbol a_1 in state q'.

This completes our construction of the machine D. In the course of the construction we have considered 4 different kinds of moves made by G, and have arranged in each case to have D simulate the movė. Using this correspondence, it is straightforward to show that D executes the required simulation of G. This concludes Part 2 of the proof.

§2.2 $1N \rightarrow 2D$: $\langle B_1, B_2, \cdots \rangle$ is COMPLETE

In this section we show that the languages B_n are complete for $1n \rightarrow 2d$ conversion.

Theorem 2.3:

(i): B_n is accepted by an n-state 1-way non-deterministic fa.

(ii): Among all languages accepted by n-state 1nfa, B_n requires the largest 2-way deterministic fa.

Proof of (i): Easy. □

Proof of (ii): This proof is similar in structure to the proof of part (ii) of Theorem 2.2. Let n-state 1nfa $N=<Q, \Delta, \delta, Q_0, F>$ be given. Let $G=<Q_G, \Sigma_n, \vdash, \dashv, q_G, F_G>$ be a 2dfa accepting B_n. We will demonstrate a 2dfa $D=<Q_G, \Delta, \vdash, \dashv, \delta_D, q_D, F_D>$ (with the same state set as G) such that D accepts $L(N)$.

To each $a \epsilon \Delta$ we associate a graph $g(a) \epsilon \Sigma_n$. In addition, we pick graphs $s, f \epsilon \Sigma_n$ to be associated with the initial and final states of N, respectively. We will choose the $g(a)$, s, and f to satisfy:

Lemma: N accepts input $a_1 a_2 \cdots a_k$ iff the catenated graph $s \mid g(a_1) \mid g(a_2) \mid \cdots \mid g(a_k) \mid f$ is in B_n.

Let $q_1, q_2, \ldots, q_n$ enumerate the states of N.

Definition (of $g(a)$, s, f):

(1) For $a \epsilon \Delta$, $g(a) \epsilon \Sigma_n$ consists of the arcs

$$(i \rightarrow j) \mid q_j \epsilon \delta(q_i, a)$$

[$(i \rightarrow j)$ indicating a directed arc from left node i to right node j].

(2) The graph $s \epsilon \Sigma_n$ consists of the arcs

$$(i \rightarrow i) \mid q_i \epsilon Q_0.$$

(3) The graph $f \epsilon \Sigma_n$ consists of the arcs

$$(i \rightarrow i) \mid q_i \epsilon F.$$

Let the nodes of the catenated graph $g(a_1) \mid g(a_2) \mid \cdots \mid g(a_k)$ be named as in Figure 11.

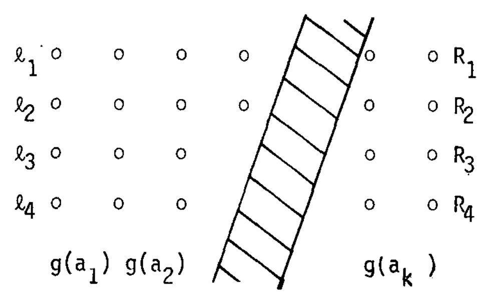


Figure 11 n = 4

It is easy to verify:

Sublemma: There is a path from l_i to R_j in the catenated graph $g(a_1) \mid g(a_2) \mid \cdots \mid g(a_k)$ iff N, started on the left end of the string $a_1 a_2 \cdots a_k$ in state q_i, can reach symbol a_k in state q_j.

The Lemma then follows from the Sublemma and (2) and (3) in the definition above.

The proof now proceeds exactly as in Part 2 of the theorem for C_n : On input $\vdash a_1\, a_2 \cdots a_k \dashv$, 2dfa D is constructed to simulate the computation of G on input $\vdash s\; g(a_1)\; g(a_2)\; \cdots g(a_k)\; f \dashv$. By the Sublemma, D will be equivalent to N. □

§3 AN ANALOGY TO P =? NP

This section presents a convenient notation for comparing the succinctness of description of different models of automata.

Define a ***language sequence*** L to be an infinite sequence of languages $<L_1, L_2, \ldots>$. We consider objects of this type because we are interested in the *rate of growth* of the state complexities of sequences of languages. Given any automata model, say 1nfa, we wish to classify those language sequences which have succinct representations using this model. Define 1N to be the class of language sequences $\mathrm{L}=<L_1, L_2, \ldots>$ with the property that some polynomial $p(i)$ bounds the size of the smallest 1nfa accepting the language L_i. Analogously, define the classes 1D, 2D, and 2N corresponding to the models 1dfa, 2dfa, and 2nfa. The primary questions we address can now be reformulated as: "Is 2N equal to 2D?" and: "Is 1N contained within 2D?"

It is interesting to consider closure properties of these classes. For example, all four classes are closed under union and intersection. By the union of two language sequences L and L′ we mean the pairwise union of the component languages $<L_1 \cup L_1', L_2 \cup L_2', \ldots>$. Closure under complement is another matter, however. The one way deterministic class, 1D, is clearly closed under complementation, but as we show in the next section, 1N is not closed. We do not know the status of 2D or 2N.

We now consider closure under a certain reducibility.

Definition: For alphabets Δ_1 and Δ_2 and languages $L_1 \subseteq \Delta_1$ and $L_2 \subseteq \Delta_2$, we say that L_1 ***homomorphically reduces to*** L_2 ($L_1 \leqslant_h L_2$), if there is a map g: $\Delta_1 \rightarrow \Delta_2^*$ and $i, f \epsilon \Delta_2^*$ such that for any string $s = s_1 s_2 \cdots s_k$ $(s_i \epsilon \Delta_1)$, $s \epsilon L_1$ iff $i\, g(s_1) g(s_2) \cdots g(s_k) f \epsilon L_2$.

Informally, this says that L_1 homomorphically reduces to L_2 if by adding end-markers to L_1, every string in L_1 can be homomorphically mapped to L_2. We extend the notion of homomorphic reduction to language sequences. Given two language sequences L and L′, say that $\mathrm{L} \leqslant_h \mathrm{L}'$ if there is a polynomial p such that each L_i is h reducible to L_j' for some $j \leqslant p(i)$.

In this notation, it is the case that for regular languages L_1 and L_2, if $L_1 \leqslant_h L_2$, then L_1 requires a 2dfa at most twice as large as that required by L_2. This can be proved by a generalization of Theorem 2.2 (ii), part 2. From this it follows that 2D is closed under $\leqslant_h$, i.e., for any language sequence L in 2D, if $\mathrm{K} \leqslant_h \mathrm{L}$ then K is in 2D. Similarly, we can show that 1D, 1N, and 2N are closed under $\leqslant_h$. The language sequence C has the special property that for every L in 2N, L $\leqslant_h$ C. This is a generalization of the lemma to Theorem 2.2 (ii), part 1 stating that if L_1 is accepted by an n state 2nfa then $L_1 \leqslant_h C_{2n}$. The proof given supports the more general statement.

This discussion indicates a close analogy between the classes 2D and 2N and the classes P and NP. Here, $\leqslant_h$ plays the role of polynomial time reducibility and the complete sequence C corresponds to an NP complete set such as 3SAT. Similarly, B is complete for the class 1N with respect to the $\leqslant_h$ reducibility.

§4 LOWER BOUNDS.

We now turn our attention to the state complexity of the complete languages B_n and C_n under various machine models. We will concentrate on B_n. All of our lower bounds on the size of recognizers for B_n extend directly to C_n, because a recognizer for C_n can be converted to one for B_n by deleting transitions for symbols from $\Gamma_n - \Sigma_n$. In some cases, we have tighter lower bounds for C_n.

Let s be a string over Σ_n. Say that s is ***live*** if it is a member of B_n, i.e., if there is a path from any left node to any right node. Otherwise, s is ***dead***. In addition, ***node m in s is live (dead)*** means that there is (is not) a path from any left node to the m^{th} from the top right node in s. Given any 1dfa, $M = < Q, \Sigma_n, \delta, q_0, F >$ and arbitrary state q and input string s, we abbreviate $\delta(q,s)$ by $q(s)$ and $\delta(q_0,s)$ by $M(s)$.

§4.1 One way automata

We begin by considering deterministic and nondeterministic one way automata. The computations of these machines are relatively easy to analyze and consequently our results are fairly strong.

There is an n state 1n acceptor for B_n. Thus by the subset construction there is a 2^n state 1d acceptor. The following result shows this to be optimal.

Theorem 4.1.1: Any 1dfa accepting B_n has at least 2^n states.

Proof: By counting information. □

There is a $2^{(n+1)^2}$ state 1d acceptor for C_n. This is close to optimal.

Theorem 4.1.2: Any 1dfa accepting C_n has at least $2^{(n-2)^2}$ states.

So we see that nondeterminism is extremely helpful to machines accepting B_n. Curiously, however, it is not of any use at all to machines accepting the complement of B_n, $\overline{B_n}$.

Theorem 4.1.3: Any 1nfa accepting $\overline{B_n}$ has at least 2^n states.

Proof: Assume to the contrary that M is a 1nfa accepting $\overline{B_n}$ with fewer than 2^n states. For every state q we wish to let r_q be the set of nodes which M thinks are live when it is in state q. Formally, let $r_q = \{m \mid$ for any y in Σ_n containing $(m \rightarrow m)$, $q(y)$ enters only reject states$\}$. For each $a \subseteq [1,n]$ we let x_a be $\{(1 \rightarrow m) \mid m \in a\}$.

Fact 1: For every $a \subseteq [1,n]$ and any $q \in M(x_a)$, $a \subseteq r_q$.

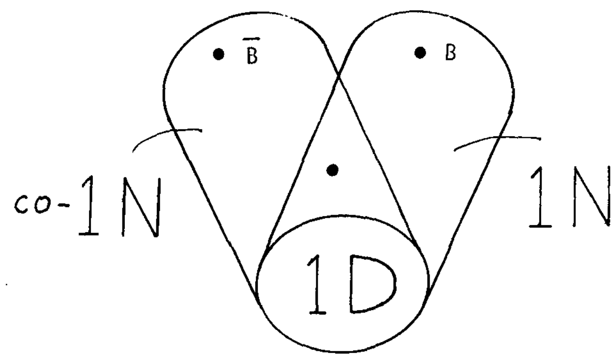
Otherwise there would be a live string $x_a y$ which would be accepted by M, a contradiction. ∘

Fact 2: For some $a \subseteq [1,n]$, every $q \in M(x_a)$ has the property that $r_q \neq a$.

Otherwise, each subset a would have a distinct state q associated with it, implying that there are at least 2^n states in M, a contradiction. ∘

Let a be as in fact 2. Let $z \in \Sigma_n$ be $\{(m \rightarrow m) \mid m \notin a\}$. the string $x_a z$ is in $\overline{B_n}$ yet we claim that M rejects it. To see this, note that for all $q \in M(x_a)$, there is an $m \in r_q - a$ (by facts 1 and 2). For each such m, $(m \rightarrow m) \in z$ and thus each $q \in M(x_a)$ is driven to reject states by z (by the definition of r_q). Hence all branches of the computation of M on $x_a z$ terminate in reject states. □

A corollary to this theorem is that the class 1N is not closed under complement. This is indicated in the following diagram.



This diagram presents the question of whether 1D is equal to the intersection of 1N and co-1N. The languages $< (0+1)^*1(0+1)^n \mid N = 1, 2, \ldots >$ witness a negative answer to this.

We also have enough machinery to easily solve the problem of whether 2D is contained within 1N. Restrict the graphs of Σ_n to not have any right nodes with more than one arc. The strings over this new alphabet our thus restricted to be forests. Define T_n to be B_n restricted to such strings. It is straightforward to accept T_n with an $O(n^2)$ state 2dfa, and it is not any harder to accept $\overline{T}_n$. However, as a corollary to the above proof, we see that $\overline{T}_n$ requires 2^n states on a 1nfa. In fact, by taking the *join* of T_n and $\overline{T}_n$ in some reasonable way, i.e., $\hat{T}_n = \{s \mid s \in T_n$ iff the length of s is even$\}$, we obtain languages with succinct 2d acceptors yet neither they nor their complements have succinct 1n descriptions. Thus 2D is not contained within 1N ∪ co-1N.

§4.2 Restricted two way automata (parallel machines)

Theorems 2.1 and 2.2 reduce the question of whether 2n→2d (1n→2d) conversion is polynomially bounded to the question of whether for some polynomial $p(n)$, C_n (B_n) is accepted by a $p(n)$ state 2dfa. We conjecture that polynomial conversion is not possible. In order to gain insight into this we consider 2d acceptors whose behavior has been restricted.

Restrictions that are placed on the 2d automata are of two forms: limiting head behavior and limiting communication. We limit the head behavior by permitting the machine to only make a series of one-way passes over the input. The communication is limited by restricting the exchange of information between sweeps. One version of this is the parallel union finite automaton.

DEFINITION: A ***parallel union finite automaton (pufa)*** P is a set $\{M_1, \ldots, M_k\}$ of 1dfa. The language accepted by P is the union of the languages accepted by its component machines.

For some languages, pufa can be exponentially more succinct than 1dfa. For example $\{x\#y \mid x,y\epsilon\{0,1\}^n$ and $x \neq y\}$ is the union of n $O(n)$ state 1dfa, yet requires a 2^n state 1dfa. However, we can show that pufa are not more succinct than 1dfa for B_n.

THEOREM 4.2.1: In any parallel union finite automaton accepting B_n, one of the component machines must have at least 2^n states.

Proof sketch: Assume to the contrary that there is a pufa accepting B_n, and containing component machines all with fewer than 2^n states. Choose any component machine. Having too few states to distinguish all possible subsets of the n nodes, it occasionally gets "confused" as to whether some node is actually live. The key point is that whenever the machine is uncertain as to the status of a particular node, it must assume that it dead. Otherwise, if it were to wrongly assume that the node was live and accepted the input based upon that assumption, then the entire pufa would wrongly accept. Knowing this, our procedure is to construct a string in which some node is live yet which fools this component machine into assuming that it is dead. We then extend the string, continuing the path from that node in such a way as to fool a second component machine. Ultimately we get a live string which fools all of the component machines into believing it is dead. This string causes the pufa to err.

A more formal proof follows.

PROOF: We perform an induction on k, the number of machines in P.

Basis, $k=1$:

This follows from theorem 4.1.1, since a pufa with only one component machine is in fact a 1dfa.

Induction, proving case k from case $k-1$:

Our induction hypothesis is that the theorem holds for pufa having fewer than k components. Suppose the theorem fails for pufa $P=\{M_1, \ldots, M_k\}$ all of whose components have fewer than 2^n states. In particular, component machine M_1 has fewer than 2^n states. For every state q of M_1, we say that q is ***dead*** if there is no string s which drives q to an accept state, otherwise we say that q is ***live***. Any string which drives M_1 to a dead state is said to ***kill*** M_1.

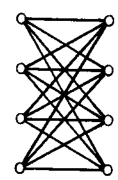
Fact 1: Any dead string s kills M_1.

Otherwise s drives M_1 to a live state q which then can be driven via some string t to an accept state. The string st is dead yet M_1 and therefore P accept it, a contradiction. ∘

Fact 2: There is a live string v which kills M_1.

Otherwise, all live strings drive M_1 to live states. On the other hand, fact 1 states that all dead strings drive M_1 to dead states. By then designating all the live states of M_1 to be accept states and all the dead states to be reject states, we obtain a 1dfa accepting exactly the live strings. This machine recognizes B_n with fewer than 2^n states, contradicting theorem 4.1.1. ∘

Let r (reset) be the complete bipartite graph on 2^n nodes, i.e., $r=\{(i\rightarrow j) \mid i,j\epsilon[1,n]\}$ and let $u=vr$.



Example: r with $n=4$

For each component machine M_i define a new machine M_i' obtained by changing the start state of M_i to be $M_i(u)$. Let R_i and R_i' be the languages accepted respectively by the machines M_i and M_i'. By assumption P accepts B_n; in other words $B_n=R_1' \cup \cdots \cup R_k'$. Since $\{t \mid ut\epsilon B_n\}=B_n$, $B_n=R_1' \cup \cdots \cup R_k'$. However, u kills M_1 and thus $R_1'=\phi$. This allows us to conclude that $B_n=R_2' \cup \cdots \cup R_k'$.

Now we construct a pufa $P'=\{M_2', \ldots, M_k'\}$ which accepts B_n and yet has only $k-1$ machines, all with fewer than 2^n states. This contradicts the induction hypothesis. □

Remarks: The existing techniques for proving lower bounds by counting information vs. crossing sequences are insufficient to give this result, as it is possible for a large number of small machines to have enough states among them to carry the necessary information across. This result shows that their inability to communicate prevents them from successfully doing so. Also, this proof provides a bound of *exactly* 2^n states. Thus we know that the best pufa is actually the obvious single 2^n state 1dfa.

A similar proof shows:

THEOREM 4.2.2: In any pufa accepting C_n, one of the component machines must have at least $2^{(n-2)^2}$ states.

DEFINITION: A ***parallel intersection finite automaton (pifa)*** P is a set $\{M_1, \ldots, M_k\}$ of 1dfa. The language accepted by P is the intersection of the languages accepted by its component machines M_i.

Theorem 4.2.3: If $P=\{M_1,\ldots,M_k\}$ is a pifa accepting B_n, then one of its component machines has at least 2^n states.

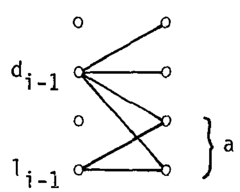
Proof: Assume that a pifa P exists which contradicts the theorem. We further assume wlog that each component machine has at least one reject state. Our goal is to construct a string $s\notin B_n$ which deceives P into accepting it. The construction is in stages, the i^{th} stage deceiving M_i. This is done by convincing M_i that some node is live when in fact it isn't. That node is then connected to the end of the string, causing M_i to wrongly accept.

Three variables are used: l denotes the current live node, d denotes the node which the current machine believes is live but which actually is dead, and s denotes the current partial string.

Stage 0: Let $s_0=\{(1\rightarrow1)\}$, $l_0=1$, and $d_0=2$.

Stage i: We are assuming from the previous stage that in string s_{i-1}, node l_{i-1} is live and node d_{i-1} is dead. Our goal for this stage is to preserve these properties for s_i, l_i, and d_i. In addition, we must connect node d_i to node d_{i-1} of the previous stage.

Consider M_i. For each state q of M_i we let r_q be the set of nodes which M_i believes are live when it is in state q. Formally, $r_q=\{m \mid$ all strings which contain a path from left node m to any right node, drive q to an accept state$\}$. For every $a\subseteq[1,n]$ let $x_a\in\Sigma_n$ be the graph $\{(d_{i-1}\rightarrow m) \mid m\in[1,n]\}\cup\{(l_{i-1}\rightarrow m) \mid m\in a\}$



Example: x_a with $n=4$, $d_{i-1}=2$, $l_{i-1}=4$, and $a=\{3,4\}$

Fact 1: For every $a\subseteq[1,n]$, the set of live nodes of $s_{i-1}x_a$ is a.

This follows from the stated properties of s_{i-1}, l_{i-1}, and d_{i-1}. ∘

Fact 2: For every $a\subseteq[1,n]$, if $M_i(s_{i-1}x_a)=q$ then $a\subseteq r_q$.

Otherwise, suppose there is some $m\in a-r_q$. By fact 1 and the definition of r_q, there is some t such that $M_i(s_{i-1}x_at)$ is rejecting yet $s_{i-1}x_at$ is in B_n, a contradiction. ∘

Fact 3: There are distinct, nonempty subsets $a,b\subseteq[1,n]$ and a state q such that $M_i(s_{i-1}x_a)=M_i(s_{i-1}x_b)=q$.

This is true because, for all nonempty subsets a, $s_{i-1}x_a$ is live (by fact 1) and therefore $M_i(s_{i-1}x_a)$ is accepting. There are 2^n-1 non empty subsets and at most 2^n-2 accepting states, thus by counting, the desired a and b exist. ∘

Let a, b, and q be as in fact 3. Since a and b are distinct, we assume wlog that there is some $m\in b-a$. By fact 2 we know that $a,b\subseteq r_q$ and thus $m\in r_q$. Let $s_i=s_{i-1}x_a$, $d_i=m$, and $l_i=$ any node in a.

Verifying the desired relationships between s_i, d_i, and l_i, we see by fact 1 that node d_i is dead because $d_i\notin a$ and l_i is live because $l_i\in a$. Furthermore, d_i is connected to d_{i-1} of the previous stage. Subsequent stages will ensure that d_i is connected to a right node of s and therefore, since $d_i\in r_q$, M_i will accept s.

End of stage i.

Finally we let $s=s_ky$ where $y=\{(d_k\rightarrow1)\}$. Straightforward inductions show that for each stage the node at l_i connects to the start but not the end of s, and the node at d_i connects to the end but not the start. From this we conclude that $s\notin B_n$ and that each M_i accepts s, a contradiction. □

A similar proof shows that:

Theorem 4.2.4: In any pifa accepting C_n, one of the component machines must have at least $2^{(n-3)^2}$ states.

§4.3 Restricted two way automata (series machines)

The automata considered in the previous section are very limited in the sense that no communication is permitted between the component machines. In the next model under consideration, the i^{th} component machine M_i may pass information to the $i+1^{st}$, M_{i+1}. This done by using the result of the computation of M_i to determine the starting state of M_{i+1}.

Definition: A ***series finite automaton (sfa)*** S is an ordered collection of 1dfa, $(M_1,\ldots,M_k)$ together with functions $(f_1,\ldots,f_{k-1})$ where each f_i is a mapping of the states of M_i into the states of M_{i+1}. On input t, S runs these machines one at a time, in order, over the input tape. The ending state of M_i on t determines, via f, the starting state of M_{i+1}. The final state of M_k determines the acceptance of t.

These machines are more complex than parallel machines and correspondingly, our results are weaker. Even the case where the sfa has only two components appears difficult to analyze. We conjecture that in this case, one of the component machines must have at least 2^n states in order that the series fa accept B_n. The main result of this section is in support of this.

Theorem: Given S, a series automaton accepting B_n with only two component machines M_1 and M_2. If either component has fewer than $\sqrt{2^{n-1}}$ states, then the other one has at least 2^n states. The square root function in our discussions always rounds up to an integer.

The proof of this depends upon an analysis of the degree to which a 1dfa can be "confused", as a function of the number of states it contains. We formalize this notion of confusion as follows.

Given any string s over Σ_n, we define the ***accessibility set*** of s to be the set of live nodes of s. Call the two strings ***equivalent*** if their accessibility sets are equal. Let q be any state in a 1dfa M which operates over Σ_n. We say that accessibility set a ***is associated with*** state q if there is some string with accessibility set a which drives M to q. The state q is called ***k confused*** if there are at least k different accessibility sets associated with it. Machine M is ***k confusable*** if it has a k confused state. For example, the 2^n state acceptor for B_n is only 1 confusable, and it follows from lemma 1, that any machine with fewer than 2^n states is at least 2 confusable. Trivially, the 1 state 1dfa is 2^n confusable. It seems reasonable to expect that the smaller a machine is, the more confusable it must get. The following table summarizes or knowledge of this phenomenon.

# states	degree of confusion
$< n$	total (2^n) confusion
$\geqslant n$	$< 2^n$ confusion possible
$\leqslant 2^n-2$	$\geqslant \sqrt{2^{n-1}}$ confusion
$\leqslant 2^n-1$	$\geqslant 2$ confusion
$\geqslant 2^n$	1 confusion possible

Of the following four theorems, 4.3.1 through 4.3.4, only the last is requisite to the main result of this section. The others present related aspects of confusion.

Theorem 4.3.1: There is a 2^n-1 state 1dfa which is not more than 2 confusable.

Proof: We construct M as follows. Each state of M is assigned one of the 2^n-1 nonempty accessibility sets. As long as the input string is live, M has no trouble staying in the assigned state. However, there are no states left to assign to the empty accessibility set and therefore the dead strings must drive M into the same states as do the live ones. It follows that there are two accessibility sets associated with each state, namely one nonempty set and the empty set. Hence each state is 2 confused and M is only 2 confusable. □

For the next theorem, it is useful to note that there is a four element subset of Σ_n which preserves all of the descriptive power. Let $\Sigma_n' = \{p_1, p_2, x_a, x_d\}$ where

$$p_1 = \{(1 \to 2), (2 \to 1)\} \cup \{(i \to i) \mid i \epsilon [3,n]\}$$
$$p_2 = \{(i \to j) \mid j = i+1 \bmod n\}$$
$$x_a = \{(i \to i) \mid i \epsilon [1,n]\} \cup \{(1 \to 2)\}$$
$$x_d = \{(i \to i) \mid i \epsilon [2,n]\}$$

The languages obtained by restricting B_n to strings over Σ_n' still possess the completeness property.

Theorem 4.3.2: There is an n state 1dfa which is not 2^n confusable.

Proof: Let $q_1, \ldots, q_n$ be the states of 1dfa M. M will be constructed so that every accessibility set is associated with q_i except for the singleton $\{i\}$. We only need to define M over the restricted alphabet $\Sigma_n' = \{p_1, p_2, x_a, x_d\}$ discussed above.

Arbitrarily, we let M start in q_1. If M is in q_i and reads input symbol p_1 or p_2 then M enters q_j where j is the node that i is carried to under the permutation. If M is in q_i and reads either x_a or x_d then M enters q_1. □

Theorem 4.3.3: Any 1dfa with fewer than n states is 2^n confusable.

Proof: The intuition behind this proof is that partial confusion can be used to induce greater confusion. We are given a 1dfa M with k states, where $k < n$. We will focus our attention upon certain accessibility sets, namely the singletons: $\{1\}, \{2\}, \ldots, \{n\}$. First, we show that M get confused on the singletons alone.

Fact: There are k singletons which are all associated with the same state.

We prove this fact inductively by showing that for every $i \leqslant k$, there is some set, A_i of i singletons and a state q_i with which all are associated.

Basis, $i = 1$:

Trivially true since the singleton $\{1\}$ must be associated with some state q_1.

Induction, proving case $i+1$ from case i:

The induction hypothesis gives us a set A_i containing i singletons and a state q_i such that every singleton in A_i is associated with q_i. We first show that every set of i singletons can be assigned a state q' such that each member of the set is associated q'. Choose any set A' containing i singletons. Say $A_i = \{\{m_1\}, \ldots, \{m_i\}\}$ and $A' = \{\{m_1'\}, \ldots, \{m_i'\}\}$. Let $x \epsilon \Sigma_n$ be the graph $\{(m_j \to m_j') \mid j \epsilon [1,n]\}$. It is not hard to see that $q_i(x)$ is the state q' that we wish to assign to A'. Thus every set A' can be assigned a state q'.

Now we show that some state q is assigned to *two* distinct sets, A' and A''. This is because there are more sets than states, i.e., there are $\binom{n}{i}$ different sets of i singletons and since $1 \leqslant i \leqslant k < n$ we know that $\binom{n}{i} \geqslant n > k =$ the number of states. Hence q exists.

Finally we construct A_{i+1} and q_{i+1}. The sets A' and A'' each contain i singletons thus between them there must be at least $i+1$ singletons. Since both A' and A'' were assigned to q, each of these $i+1$ singletons is associated with q. Consequently, we let these singletons constitute A_{i+1} and we let q be q_{i+1}.

Fact proved.

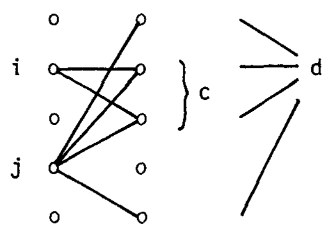
Let $m_1, \ldots, m_k$ and q_k be the singletons and state shown to exist by the above fact. If M is not 2^n confusable then for each state q_j there is some accessibility set, b_j, which is not associated with q_j. Let $x \epsilon \Sigma_n$ be $\{(m_j \to m) \mid m \epsilon b_j, j \epsilon [1,k]\}$. Let $\hat{q} = q_k(x)$. Since each $\{m_j\}$ is associated with q_k, each b_j is associated with $\hat{q}$. However, b_j is not associated with q_j for every j, and so $\hat{q}$ is not equal to any q_j, a contradiction. □

The following result is used in theorem 4.3.5.

Theorem 4.3.4: Any 1dfa with fewer than 2^n-1 states is at least $\sqrt{2^{n-1}}$ confusable.

Proof: Given M with fewer than 2^n-1 states. As a first step we demonstrate that there are two accessibility sets a and b associated with the same state, where $a \not\subseteq b$ and $b \not\subseteq a$. Since M

has fewer than 2^n-1 states and there are 2^n-1 nonempty accessibility sets, there must be two distinct sets, a' and b', which are associated with the same state q. If a' and b' meet the desired conditions for a and b we are done, otherwise assume wlog that $a \subseteq b$. Let i be a node in a and j a node in $b-a$. For each pair of accessibility sets c and d where $c \subseteq d$, we assign that pair to some state r with which they are both associated, as follows. Let $x \epsilon \Sigma_n$ be $\{(i \rightarrow m) \mid m \epsilon c\} \cup \{(j \rightarrow m) \mid m \epsilon d\}$.



Example: x with $n=5$, $i=2$, $j=4$, $c=\{2,3\}$, $d=\{1,2,3,5\}$

It is not hard to see that $q'(x)$ is the desired state r.

There are 3^n-2^n such pairs and fewer than 2^n states. Thus there must be a state to which at least $(3^n-2^n)/2^n=(3/2)^n-1$ pairs are assigned. From this we can conclude that there must be at least $\sqrt{(3/2)^n-1}$ accessibility sets associated with that state. For sufficiently large n (see note following proof) $\sqrt{(3/2)^n-1} > n+1$ and thus we have at least $n+2$ distinct sets associated with that state. However there cannot exist a chain $a_1 \subset a_2 \subset \cdots \subset a_{n+2}$ of length $n+2$ if each of the a_i is a subset of $[1,n]$. Hence, there must be two accessibility sets, a and b, associated with the same state, such that $a \not\subseteq b$ and $b \not\subseteq a$. This completes the first step of the proof.

Now we essentially repeat this idea using the sets a and b rather than a' and b'. Choose $i \epsilon a-b$ and $j \epsilon b-a$ and define x as before, except we now permit arbitrary pairs of sets c and d. Again, each pair is assigned a state with which each member of the pair is associated. This time there are $2^n(2^n-1)/2$ pairs. Since there are fewer than 2^n states, at least 2^{n-1} pairs are assigned to the same state, implying that there must be at least $\sqrt{2^{n-1}}$ accessibility sets associated with that state. □

Note: We have assumed in the first step of this proof that we are dealing with sufficiently large n. Nevertheless, the theorem holds for all n by examining the small n individually and applying more careful counting arguments. The details are omitted.

Theorem 4.3.5: Given S, a series automaton accepting B_n with only 2 component machines M_1 and M_2. If either component has fewer than $\sqrt{2^{n-1}}$ states, then the other one has at least 2^n states.

We will actually prove that the other component has at least 2^n-1 states, a slightly weaker result. The stronger result holds by an additional argument, which we omit.

Proof: Suppose sfa S violates these conditions. Our plan is construct a live string t_L and a dead string t_D, which are jointly accepted or rejected by S. These strings will be constructed in three parts; a common live beginning string s_B, a common live ending string s_E, and middle strings s_L and s_D. That is, t_L will be $s_B s_L s_E$ and t_D will be $s_B s_D s_E$. The actual deception of P occurs during the middle strings. The beginning and ending strings are merely to enable us to predict the starting state of M_2.

First we analyze the structure of M_1. Call a string q ***internal*** if for every live s there is a string t such that st is live and $q(st)=q$. Let $x \epsilon \Sigma_n$ be the complete bipartite graph $\{(i \rightarrow j) \mid i,j \epsilon [1,n]\}$.

Fact 1: There is some live string s which drives M_1 to an internal state.

If the starting state q_0 of M_1 is internal then we are done, otherwise there is some live string s_1 such that any live extension of s_1 will not carry M_1 back to q_0. Let $q_1=q_0(s_1x)$. This means that q_0 is unreachable from q_1 via live strings. If q_1 is not internal we can repeat the above to obtain a new state q_2 from which it is impossible to reach either q_0 or q_1 via live strings. Since M_1 is finite state this process cannot go on forever and so eventually we will obtain an internal state q_i. The string s which drives M to q_i is $s_1 x s_2 x \cdots s_i x$. ∘

We let s_B be the string s constructed in fact 1.

Our next step is to construct the strings s_L and s_D. For this step we need to know the starting state of M_2 which depends upon the ending state of M_1 on inputs t_B and t_L which in turn depend upon the strings we are currently constructing. To get around this circularity we assume for this step that $M_1(t_L)=M_1(t_D)=M_1(s_B)$, the internal state of fact 1. This determines r_0, the starting state of M_2. The construction of s_E in the final step of our proof ensures that this assumption is valid. We let q_1 and r_1 be $M_1(s_B)$ and $r_0(s_B)$ respectively. We assume wlog that M_2 has fewer than $\sqrt{2^{n-1}}$ states, and thus M_1 has fewer than 2^n-1 states.

Consider the automaton M_1' obtained by changing the start state of M_1 to be q_1. Since M_1 has fewer than 2^n-1 states, M_1' is $\sqrt{2^{n-1}}$ confusable (by theorem 4.3.4). In other words, there are $\sqrt{2^{n-1}}$ pairwise inequivalent strings, $s_1, \ldots, s_k$, where $k=\sqrt{2^{n-1}}$, which all drive M_1 from state q_1 to the same state q. Now M_2 has fewer than $\sqrt{2^{n-1}}$ states, thus two of these strings, s_i and s_j must drive M_2 from r_1 to the same state r. Since s_i and s_j are inequivalent, there is some node m which is live in, say s_i, and dead in s_j. We adjoin the symbol $x=\{(m \rightarrow 1)\}$ to both strings obtaining a live string s_L and a dead string s_D. Let state q_2 and r_2 be states $q(x)$ and $r(x)$. Note that both s_L and s_D drive M_1 from q_1 to q_2 and M_2 from r_1 to r_2.

At this point, all that remains is to construct s_E, a live string which drives M_1 from q_2 back to q_1. This is easy since q_1 is an internal state and thus the live string s_L can be extended to a live string $s_L s_E$ which drives q_1 to q_1.

So, no matter whether S receives t_L or t_D as input, M_2 will be driven into the same state, a contradiction. □

§5 RELATED WORK

Meyer and Fischer [8] first considered the relative succinctness of various kinds of descriptions of regular sets.

Joel Seiferas [9] has investigated 1n→2d conversion. He demonstrates lower bounds for a restricted 2dfa model and considers several interesting regular languages. For example, let alpahbet Δ_n be the power set of $\{1,...,n\}$ and let L_n be:

$$\left(\bigcup_{\substack{a \in \Delta_n \\ i \in a}} a \, (\Delta_n)^i \right)^*$$

We can show L_n to be complete for 1n→2d conversion by demonstrating that for any n state 1n language K, $K \leqslant_{\mathrm{h}} L_n$.

Acknowledgements

This research began with a study of the language B_n. We wish to thank Dana Angluin for posing the questions which initiated this study.

We are deeply indebted to Manuel Blum for many valuable suggestions.

REFERENCES

[1] Cook, S. A., The complexity of theorem proving procedures, *Third Annual ACM Symposium on Theory of Computing,* May 1971, 151-158.

[2] Karp, R. M., Reducibility among combinatorial problems, in, "Complexity of Computer Computations", R. E. Miller and J. W. Thatcher, eds., Plenum Press, N. Y. (1972), 85-104.

[3] Savitch, W. J., Relationships between nondeterministic and deterministic tape complexities, *J. Comput. Syst. Sci. 4* (1970), 177-192.

[4] Cook, S. and R. Sethi, Storage Requirements for Deterministic polynomial time recognizeable languages, *Sixth Annual ACM Symposium on Theory of Computing,* May 1974, 33-39.

[5] Jones, N. D. and W. T. Laaser, Complete problems for deterministic polynomial time, *Sixth Annual ACM Symposium on Theory of Computing,* May 1974, 40-46.

[6] Even, S. and R. E. Tarjan, A combinatorial problem which is complete in polynomial space, *Seventh Annual ACM Symposium on Theory of Computing,* May 1975, 66-71.

[7] Schaeffer, T. J., Complexity of decision problems based on finite two-person perfect-information games, *Eighth Annual ACM Symposium on Theory of Computing,* May 1976, 41-49.

[8] Meyer, A. R. and M. J. Fischer, Economy of description by automata, grammers, and formal systems, *Twelfth Annual Symposium an Switching and Automata Theory,* October 1971, 188-191.

[9] Seiferas, J., unpublished manuscript, (1973).