THE DECIDABILITY OF THE REACHABILITY PROBLEM FOR VECTOR ADDITION SYSTEMS

(Preliminary Version)

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Introduction:

Let V be a finite set of integral vectors in Euclidian N-space, and let $\underline{a}$ be an integral point in the first orthant of N-space. The *reachability set* $R(\underline{a},V)$ is the set of integral points $\underline{b}$ in the first orthant such that there is a polygonal path γ from $\underline{a}$ to $\underline{b}$ satisfying (i) all of γ lies in the first orthant, and (ii) the edges of γ are translates of the vectors in V. The *reachability problem* for the vector addition system $(\underline{a},V)$ asks for an algorithm to decide which integral points $\underline{b}$ are in $R(\underline{a},V)$. In this paper we give an algorithm to solve this problem.

Vector addition systems (or equivalently, Petri nets) have been considered by many authors in the last decade. The reachability problem is first stated in Karp and Miller [1969] but appears to have been known to Rabin and to Hopcroft already in 1966 (see Holt [1971]). This problem has arisen in several contexts: context-free matrix grammars [Ginsburg, 1966], conjugacy problems for certain infinite groups [Anshel, 1976], parallel program schemata [Karp and Miller, 1969], uniform recurrence equations [Karp, Miller, and Winograd, 1967]. An equivalent formulation of this problem is the liveness problem for Petri nets [Holt, 1971; Hack, 1975A; Landweber and Robertson, 1975].

The following special cases have been solved previously: (a) vector addition systems of dimension $N \leq 3$ [van Leeuwen, 1974], (b) V is symmetric about the origin [folklore], (c) conflict-free Petri nets [Crespi-Reghizzi and Mandrioli, 1974].

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The related problem of whether or not, given $\underline{a}$, V, $\underline{a}'$ and V', $R(\underline{a},V) \subseteq R(\underline{a}',V')$ was shown unsolvable by Rabin [1969, unpublished]. The question of whether or not $R(\underline{a},V) = R(\underline{a}',V')$ has also been shown unsolvable [Hack, 1975A].

There have been numerous complexity results, which show that even special cases of this problem require exponential space [Hack, 1975B; Cardoza, Lipton, and Meyer, 1975; Rackoff, 1976].

We solve the reachability problem by considering a somewhat more general problem. Let S be a closed convex subset of E^N such that (i) S is bounded by a finite set $H_1,\ldots,H_K$ of rational (N-1)-dimensional hyperplanes, (ii) for each H_i, there is a non-zero normal vector $\underline{u}_i$ pointing into S, (iii) the set $\{\underline{u}_1,\ldots,\underline{u}_K\}$ spans E^N. If $\underline{a}$ and $\underline{b}$ are integral points in S and V is a finite set of integral vectors, *$\underline{b}$ is reachable from $\underline{a}$ under motions in V while remaining in S*, $V: \underline{a} \rightsquigarrow \underline{b}$ in S, if there is a polygonal path γ from $\underline{a}$ to $\underline{b}$ which is entirely contained in S, whose edges are translates of vectors in V. The *generalized reachability problem* asks for an algorithm to decide which integral points are in the set $R(\underline{a},V,S)$ of points $\underline{b}$ such that $V: \underline{a} \rightsquigarrow \underline{b}$ in S. Observe that the reachability problem is the special case in which $\underline{u}_1,\ldots,\underline{u}_N$ are the unit vectors along the positive axes, and $H_1,\ldots,H_N$ are the (N-1)-dimensional hyperplanes normal to these vectors. We assume that $\underline{a}$, $\underline{b}$, S, K, $H_1,\ldots,H_K$ and $\underline{u}_1,\ldots,\underline{u}_K$ are fixed for the rest of this paper.

The *dimension of a vector addition system* $(\underline{a},V)$ is the dimension of the subspace of E^N spanned by V. Note that if V spans a space, S_0

of dimension $N_0 < N$, then $R(\underline{a},V,S)$ is contained in a translate S_1 of S_0, and indeed, $R(\underline{a},V,S) \subseteq S \cap S_1$ which is a closed, convex set bounded by rational (N_0-1)-dimensional hyperplanes $H_i \cap S_1$ (where this intersection is non-empty). Hence we may replace the first vector addition problem by one of dimension N_0. *Henceforth we assume that this reduction has taken place and* N *is the dimension of the vector addition problem.* The algorithm which we will provide will proceed by recursion on the dimension of the vector addition problem. One should note that in the dimension $N = 0$ case, V contains only the zero vector or else is empty, and $V: \underline{a} \rightsquigarrow \underline{b}$ in S if and only if $\underline{b} = \underline{a}$.

In order to describe our algorithm we need several further definitions. Let $\underline{d}$ be any point of E^N. The $\underline{u}$-coordinates of $\underline{d}$ are the numbers $\underline{d}\cdot\underline{u}_i$, $i = 1,2,\ldots,K$. Since the $\underline{u}_i$ span E^N, each point $\underline{d}$ is uniquely determined by its $\underline{u}$-coordinates. Also, there exist numbers $C_1,\ldots,C_K$ such that S consists of those integral points $\underline{d}$ with $\underline{d}\cdot\underline{u}_i \geq C_i$ for $i = 1,2,\ldots,K$.

Let Δ be any subset of $\{1,2,\ldots,K\}$, and let $\underline{c}$ and $\underline{d}$ be any points of E^N. We write $\underline{c} <_\Delta \underline{d}$ if $\underline{c}\cdot\underline{u}_i \leq \underline{d}\cdot\underline{u}_i$ for all i, and $\underline{c}\cdot\underline{u}_i < \underline{d}\cdot\underline{u}_i$ for those i in Δ (and only those). We write $\underline{c} <_w \underline{d}$ if $\underline{c} <_\Delta \underline{d}$ for some $\Delta \neq \emptyset$, and $\underline{c} <_s \underline{d}$ if $\underline{c} <_{\{1,2,\ldots,K\}} \underline{d}$. We will prove that these orderings are all well-founded on the integral points of S, and that any infinite sequence of distinct points of S contains an infinite subsequence which is strictly increasing in the ordering $<_w$.

Throughout this paper we will consider V to be a finite set of names for vectors, though in general we will identify the names with the corresponding vectors; we explicitly allow V to contain two names for the same vector. A *proper sum* of the vectors in V is a positive integral linear combination of the vectors in V. The *positive span of* V is the set of integral vectors $\underline{w}$, some positive integral multiple of which is a proper sum of vectors in V. These are simply the integral vectors in the same direction as rational convex combinations of vectors in V. We shall prove that the positive span of V is convex and closed under proper sum. In particular, if some integral vector $\underline{w}$ is not in the positive span of V, then all of V lies to one side of a rational hyperplane H, whose defining equation may be effectively calculated (lemma 1.5).

Finally, we note that $\underline{b} \in R(\underline{a},V,S) \iff \underline{a} \in R(\underline{b},-V,S) \iff R(\underline{a},V,S) \cap R(\underline{b},-V,S) \neq \emptyset$.

We now can sketch the construction of our algorithm. At several points in the algorithm, we will make non-deterministic choices from among a finite number of possibilities. The algorithm is said to succeed if some sequence of choices leads to a termination that reports success, otherwise it fails. This could be made into a deterministic algorithm in the usual fashion (i.e. one that conducts a search through the finite tree of possible choices).

(1) Test if $\underline{b}$ is representable as $\underline{a}$ plus a proper sum of vectors in V. This may be done effectively since this question is equivalent to the solvability of a system of linear equations in the natural numbers, a question of Presburger arithmetic. If $\underline{b}$ is not so representable, $\underline{b}$ is not in the reachability set, and no more need be done. An algorithm is given in §1 to answer this question.

(2) Test if V positively spans Z^N. If it does not, then as noted above, there is a hyperplane H such that V lies to one side of H. Let $\underline{u}$ be a normal vector to H such that $\underline{u}\cdot\underline{v} \geq 0$ for $\underline{v} \in V$. Add two boundary hyperplanes, $H_{\underline{a}}$ and $H_{\underline{b}}$, to $H_1,\ldots,H_K$, where $H_{\underline{a}}$ is the hyperplane through $\underline{a}$ parallel to H and $H_{\underline{b}}$ is the hyperplane through $\underline{b}$ parallel to H. The normals $\underline{u}_{\underline{a}}$ and $\underline{u}_{\underline{b}}$ are chosen to be $\underline{u}$ and $-\underline{u}$ respectively. Clearly, it is impossible simultaneously to increase all $\underline{u}$-coordinates in this new system, because any motion which increases the $\underline{u}_{\underline{a}}$-coordinate must decrease the $\underline{u}_{\underline{b}}$-coordinate. Henceforth we assume that $\{H_1,\ldots,H_K\}$ includes $H_{\underline{a}}$ and $H_{\underline{b}}$ if V does not positively span Z^N. An algorithm in §1 can be used to perform this test.

(3) Test if there exist $\underline{a}'$ and $\underline{b}'$ such that $\underline{a}' \in R(\underline{a},V,S)$, $\underline{b}' \in R(\underline{b},-V,S)$, $\underline{a} <_s \underline{a}'$, and $\underline{b} <_s \underline{b}'$. If so, $\underline{b} \in R(\underline{a},V,S)$, and the algorithm terminates and reports success. This test could be done using a very slight modification of the tree construction in Karp and Miller [1969]. However, in

§4 we give a somewhat more complicated tree from which we will derive this and further information.
(4) Again using the tree of step 3, it is possible to determine all maximal subsets $\Delta \subseteq \{1,\ldots,K\}$ of $\underline{u}$-coordinates which can simultaneously be increased unboundedly (the coordinates that are marked with ω in the tree of Karp and Miller). It is also possible to determine a finite list of cones such that all travel which increases the $\underline{u}$-coordinates in Δ is in directions within these cones. The cone construction is given in §3 and is applied repeatedly in sections 4-7.
(5) Non-deterministically choose such a Δ, and let $\Gamma = \{1,\ldots,K\} - \Delta$. Note that while the $\underline{u}$-coordinates indexed by Δ are unbounded, the $\underline{u}$-coordinates indexed by Γ are bounded. This gives rise to a finite set $\Sigma = \{\sigma_1,\ldots,\sigma_s\}$ of spaces of dimension less than N, spaces which are indexed by the components of $\underline{u}$-coordinates indexed in Γ. Each of these spaces is parallel to all the hyperplanes indexed by Γ. Partition $V = V_I \cup V_J$, where $V_I = \{\underline{v} \epsilon V \mid \underline{v} \cdot \underline{u}_j = 0 \text{ for } j \epsilon \Gamma\}$, and $V_J = V - V_I$. The vectors in V_I cause motions within the spaces in Σ, and the vectors in V_J cause motions among the spaces in Σ; i.e. vectors in V_J are "jumps" between the spaces.

(6) The set of possible paths between spaces is infinite, but it is possible to classify these into a finite number of types by analyzing loop structure. Roughly speaking, two paths in the same type differ from one another only in the number of times that some looping subpaths are traversed. The classification of these paths is given in §5.
(7) In §6 we use this classification to reduce the case in which there do not exist points $\underline{a}'$ and $\underline{b}'$ as in step (3) to a finite number of reachability problems for generalized lower-dimensional vector addition systems. In these lower-dimensional problems, we will admit sums of paths from a space back to itself as additional vectors, provided that they can be used arbitrarily often without destroying the possibility of reaching $\underline{b}$. Those which can be used at most finitely often we apply as many times as possible. To determine which sums of loops can only be used finitely often we must make a careful analysis of the directions in which it is possible to travel when increasing components in Δ unboundedly. This analysis is made in §6.4 using the cones of §3; the essential test to make is whether it is possible from given points $\underline{a}' \in R(\underline{a},V,S)$ and $\underline{b}' \in R(\underline{b},-V,S)$ to increase a set of $\underline{u}$-coordinates by moving $\underline{a}'$ to a point $\underline{a}''$ and $\underline{b}'$ to a point $\underline{b}''$ with $\underline{a}'' - \underline{a}' = \underline{b}'' - \underline{b}'$. (Note that the moving of $\underline{b}'$ to $\underline{b}''$ effectively reverses the motion of $\underline{a}'$ to $\underline{a}''$.)

To prove that this algorithm works entails two parts. First we show that if the algorithm reports success, then $b \in R(\underline{a},V,S)$. This follows from the fact that at any point in the construction, any vector $\underline{w}$ that has been added to the original set V of vectors is the sum of a sequence of vectors that it was possible to apply arbitrarily often at the time $\underline{w}$ was added to V.

Consequently, we may take each use of such a vector $\underline{w}$ in the path that we construct and replace it at a suitable point in the path with the sequence that gave rise to that $\underline{w}$. This gives rise to the path from $\underline{a}$ to $\underline{b}$ in S using motions in V.

On the other hand, if there is some path γ from $\underline{a}$ to $\underline{b}$ in S using motions in V, we must show that the algorithm produces some path δ. If V positively spans Z^N, then the algorithm clearly produces a path. Assume that V does not positively span Z^N. Then, in §7, we show that γ can be effectively modified to a path γ' which resembles one of the paths δ constructed by our algorithm. Specifically, γ' differs from δ in that certain motions are performed in a different order (which does not change the end point) and certain motions and their effective reversals (see step (7)) may be added or deleted.

§1. The Initial Tests

1.1 Theorem: There is an algorithm to decide whether $\underline{b}$ can be expressed as $\underline{a}$ plus some proper sum of vectors in V. Moreover, there is an algorithm to decide whether $k\underline{b}$ may be so expressed for some positive integer k.
Proof: We must show how to find non-negative integers $k_1,\ldots,k_m$ such that $\underline{b} = \underline{a} + \Sigma k_i \underline{v}_i$. This vector equation decomposes into N scalar equations, for which we must decide the existence of natural

numbers $k_1, \ldots, k_m$ such that

$$b_1 = a_1 + k_1 v_{11} + \ldots + k_m v_{m1}$$
$$\vdots$$
$$b_N = a_N + k_1 v_{1N} + \ldots + k_m v_{mN}.$$

By transposing negative terms to the opposite side of the equations, we may rewrite these equations in the form

$$\alpha_{01} + \Sigma\alpha_{i1}k_i = \beta_{01} + \Sigma\beta_{i1}k_i$$
$$\vdots$$
$$\alpha_{N1} + \Sigma\alpha_{iN}k_i = \beta_{N1} + \Sigma\beta_{iN}k_i$$

where all of the α_{ij} and β_{ij} are natural numbers. The desired k_i's exist if and only if

$$\exists k_1 \ldots \exists k_m [\bigwedge_j (\alpha_{0j} + \Sigma\alpha_{ij}k_i = \beta_{0j} + \Sigma\beta_{ij}k_i)]$$

is a theorem of Presburger arithmetic. This can be decided effectively.

For the second part of the theorem we must find natural numbers $k_0, k_1, \ldots, k_m$ such that

$$k_0\underline{b} = \underline{a} + \Sigma k_i\underline{v}_i.$$

The above algorithm can be used to solve this problem too. □

Remark: One certainly does not use the full strength of Presburger arithmetic in the preceding algorithm.

1.2 Note: *For the remainder of this paper we assume that the algorithm of Theorem 1.1 has been applied and that* $\underline{b}$ *can be expressed as* $\underline{a}$ *plus a proper sum of the vectors in* V. If this condition fails to hold, then $\underline{b}$ is certainly not reachable from $\underline{a}$.

Theorem 1.1 can also be used to study the positive span of V.

1.3 Lemma: The positive span Q of V is convex in the following sense: given $\underline{w}_1, \underline{w}_2 \in Q$ and a rational number $0 \le \frac{p}{q} \le 1$, the integral vector

$$q[\tfrac{p}{q}\underline{w}_1 + (1 - \tfrac{p}{q})\underline{w}_2] \in Q.$$ (I.e. Q is the set of integer points in a cone generated by V.)

Proof: There exist positive integers k_1 and k_2 such that $k_1\underline{w}_1$ and $k_2\underline{w}_2$ are proper sums of vectors in V. Thus $k_1 \cdot k_2 \cdot q[\frac{p}{q}w_1 + (1-\frac{p}{q})w_2]$ is a proper sum of vectors in V. □

1.4 Lemma: There is an algorithm to decide whether the positive span of V is Z^N.

Proof: Apply Theorem 1.1 to test if the unit vectors along the positive and negative axes are in the positive span. □

1.5 Lemma: If V does not positively span Z^N, then one can effectively construct an (N-1)-dimensional rational hyperplane H and an integral vector $\underline{u}$, normal to H, such that $\underline{u}\cdot\underline{v} \ge 0$ for each $\underline{v} \in V$; i.e. all of V lies to one side of H.

Proof: Pick a maximal-dimensional subspace W which is positively spanned by a subset of V. Then pick an integral basis B_0 for $W^\perp$, the orthogonal complement of W. Since the dimension of W is maximal, it follows that for each $\underline{x} \in B_0$, the dot products $\underline{x}\cdot\underline{v} \ge 0$ for $\underline{v} \in V$, or else all $\underline{x}\cdot\underline{v} \le 0$. Choose a new basis B for $W^\perp$ such that $\underline{x}'\cdot\underline{v} \ge 0$ for all $\underline{x}' \in B$ and $\underline{v} \in V$. Let the required $\underline{u} = \sum_{x' \in B} \underline{x}'$ and let H be the hyperplane through the origin, orthogonal to $\underline{u}$. □

1.6 Lemma: If V positively spans Z^N then $-\underline{v}$ is a proper sum of vectors in V.

Proof: Since $-\underline{v}$ is in the positive span of V, there exist non-negative integers $p, q_1, \ldots, q_m$ such that $p(-\underline{v}) = \sum_{i=1}^{m} q_i\underline{v}_i$. Thus

$$-\underline{v} = (p-1)\underline{v} + p(-v) = (p-1)\underline{v} + \sum_{i=1}^{m} q_i\underline{v}_i. \quad \square$$

§2. The Orderings $<_\Delta$

In order to prove Theorem 2.6, we need some basic facts about the orderings we use.

2.1 Lemma: The orderings $<_w$, $<_s$, and $<_\Delta$ are each well-founded on S.

Proof: We give only the proof for $<_w$; the others are similar and easier. Assume $<_w$ is not well-founded. Let $\underline{c}_1, \underline{c}_2, \ldots$ be an infinite sequence of points in S such that $\underline{c}_{i+1} <_w \underline{c}_i$. Clearly at least one $\underline{u}$-coordinate, say the j'th, must be decreased infinitely often by this sequence. Thus there must be an infinite subsequence $\underline{d}_1, \underline{d}_2, \ldots$ such that $\underline{d}_{i+1}\cdot\underline{u}_j < \underline{d}_i\cdot\underline{u}_j$. But since each $\underline{d}_i \in S$, $\{\underline{d}_i\cdot\underline{u}_j\}$ is a sequence of integers bounded below by some constant. □

2.2 Lemma: An infinite sequence $\underline{c}_1, \underline{c}_2, \ldots$ of distinct points of S contains an infinite subsequence $\underline{c}_{i_1}, \underline{c}_{i_2}, \underline{c}_{i_3}, \ldots$ such that

$$\underline{c}_{i_1} <_w \underline{c}_{i_2} <_w \underline{c}_{i_3} <_w \cdots .$$

Proof (following [Karp and Miller, 1969]): Extract an infinite subsequence $\underline{c}_{i_1}, \underline{c}_{i_2}, \ldots$ for which the numbers $\underline{u}_1 \cdot \underline{c}_{i_1}, \underline{u}_2 \cdot \underline{c}_{i_2}, \ldots$ are non-decreasing. Then extract a sub-subsequence, and so on, until one has all of the sequences

$$\underline{c}_{i_1} \cdot \underline{u}_1, \underline{c}_{i_1} \cdot \underline{u}_2, \ldots$$
$$\underline{c}_{i_1} \cdot \underline{u}_2, \underline{c}_{i_2} \cdot \underline{u}_2, \ldots$$
$$\vdots$$
$$\underline{c}_{i_1} \cdot \underline{u}_k, \underline{c}_{i_2} \cdot \underline{u}_k, \ldots$$

are all non-decreasing. Since the initially given points are all distinct,

$$\underline{c}_{i_1} <_w \underline{c}_{i_2} <_w \underline{c}_{i_3} <_w \cdots . \qquad \square$$

2.3 Corollary: Any set of points of S which are pairwise incomparable under $<_w$ is finite.

2.4 Lemma: If there exists a point $\underline{a}'$ in the reachability set $R(\underline{a},V,S)$ with $\underline{a} <_\Delta \underline{a}'$, for some Δ, then $\underline{a}'' = \underline{a}' + (\underline{a}' - \underline{a}) \in R(\underline{a}',V,S)$ and $\underline{a}' <_\Delta \underline{a}''$.

Proof: Apply the sequence of vectors which carries $\underline{a}$ to $\underline{a}'$, to $\underline{a}'$. $\square$

2.5 Lemma: If $R(\underline{a},V,S)$ contains a point $\underline{a}'$ such that for some point $\underline{a}'' \in R(\underline{a}',V,S)$ $\underline{a}' <_s \underline{a}''$, then $R(\underline{a},V,S)$ contains a points $\underline{a}^*$ with $\underline{a} <_s \underline{a}^*$.

Proof: Let $\underline{c} = \underline{a}'' - \underline{a}'$. For some natural number j, $\underline{a} <_s \underline{a}' + j\underline{c}$. Let $\underline{a}^* = \underline{a}' + j\underline{c}$. $\square$

2.6 Theorem: If V positively spans Z^N, and if $R(\underline{a},V,S)$ contains a point $\underline{a}'$ with $\underline{a} <_s \underline{a}'$, and if $R(\underline{b},-V,S)$ contains a point $\underline{b}'$ with $\underline{b} <_s \underline{b}'$, then $\underline{b}$ is reachable from $\underline{a}$.

Proof: By Note 1.2, there is a polygonal path P_1 from $\underline{a}$ to $\underline{b}$ all of whose edges are in V, but which need not remain within S. Choose an integer m_1 so that the ball of radius m_1 about $\underline{a}$ contains P_1.

Next find a point $\underline{b}'$ in $R(\underline{b},-V,S)$ with $\underline{b} <_s \underline{b}'$. By Lemma 1.6, there is a polygonal path P_2 from $\underline{b}'$ to $\underline{b}$, all of whose edges are in V, but which need not remain within S. Choose an integer m_2 such that the ball of radius m_2 about $\underline{b}$ entirely contains P_2.

Similarly, we may find a point $\underline{a}'$ in $R(\underline{a},V,S)$ with $\underline{a} <_s \underline{a}'$. Let P_3 be a polygonal path in S from $\underline{a}$ to $\underline{a}'$ all of whose edges are in V. We may suppose that P_3 is entirely contained within a ball about $\underline{a}$ of some integral radius m_3.

Find an integer n_1 such that if $\underline{b}_1 = \underline{b} + n_1(\underline{b}' - \underline{b})$, $U_i(\underline{b}_1)$, the distance from $\underline{b}_1$ to the ith bounding hyperplane, is at least m_3, for $i = 1,2,\ldots,k$. By Lemma 2.6, $\underline{b}_1 \in R(\underline{b},-V,S)$. Observe that the path P_3 can now be applied to $\underline{b}_1$ to carry $\underline{b}_1$ to $\underline{b}_1 + (\underline{a}' - \underline{a})$; moreover $\underline{b}_1 <_s \underline{b}_1 + (\underline{a}' - \underline{a})$.

Next find an integer n_2 such that if $\underline{b}_2 = \underline{b}_1 + n_2(\underline{a}' - \underline{a})$, then $U_i(\underline{b}_2) \geq n_1 m_2$ for $i = 1,2, \ldots,k$. Then the path P_2 may be applied n_1 times to $\underline{b}_2$, giving a point $\underline{b}_3 = \underline{b}_2 + n_1(\underline{b} - \underline{b}')$. Next observe that $\underline{a}_3 = \underline{a} + n_2(\underline{a}' - \underline{a})$ and $\underline{b}_3 = \underline{b} + n_2(\underline{a}' - \underline{a})$ so that $\underline{b}_3$ and $\underline{a}_3$ are in the same relative positions as $\underline{b}$ and $\underline{a}$. Hence we may travel from $\underline{a}_3$ to $\underline{b}_3$ along P_1 and remain in S. $\square$

§3. Cones

Suppose we are given a point $\underline{p}$ in S and two sets R and W of vectors such that $\underline{r} \cdot \underline{u}_i \geq 0$ for all $\underline{r} \in R$ and $i = 1,2,\ldots,K$; and $\underline{w} \cdot \underline{u}_i \neq 0$ only if there is an $\underline{r} \in R$ with $\underline{r} \cdot \underline{u}_i > 0$ for $\underline{w} \in W$ and $i = 1,2,\ldots,K$. In other words, if $\underline{p}' \in R(\underline{p},R,S)$, $\underline{p} <_w \underline{p}'$, and every vector in W is parallel to all the bounding hyperplanes which are parallel to all vectors in R. In the next few paragraphs we describe a procedure to find a finite set $\underline{p}_1,\ldots,\underline{p}_n$ of points, with $\underline{p} <_w \underline{p}_i$, $i = 1,2,\ldots,n$, and n finite sets of vectors $R_1,\ldots,R_n$, with $\underline{r} \cdot \underline{u}_i \geq 0$ for all $\underline{r} \in \cup R_j$ and $i = 1,2,\ldots,K$, such that for any point $\underline{p}'$, with $\underline{p} <_w \underline{p}'$, which is reachable from $\underline{p}$ using the vectors in $R \cup W$, $\underline{p}'$ is reachable from one of the points $\underline{p}_i$ and the direction of $\underline{p}' - \underline{p}_i$ is the same as the direction of a proper sum of vectors in R_i. Informally, if $\underline{p}$ can be moved (using $R \cup W$) arbitrarily far from some of the bounding hyperplanes and parallel to the others, then this motion, modulo a step from $\underline{p}$ to one of the $\underline{p}_i$'s, is in the direction of a vector in the cone generated by one of the R_i's.

We begin with two examples in which we take S to be the first orthant of E^4. Let $\underline{p}_1$ be $(4,3,1,1)$, $R = \{\underline{r} = (1,2,1,5)\}$ and $W = \{\underline{u} = (1,-1,3,0),\ \underline{v} = (-1,1,2,0)\}$. In this case it is possible to move $\underline{p}_1$ to a point $\underline{p}_1' \;{}_w{>}\; \underline{p}_1$ in the direction of all rational convex combinations (a rational convex combination of $\underline{w}_1,\ldots,\underline{w}_k$ is any sum $\Sigma \frac{m_i}{n_i} \underline{w}_i$ where $0 \le \frac{m_i}{n_i} \le 1$ and $\Sigma \frac{m_i}{n_i} = 1$) of the following vectors, each of which is ${}_w{>}\ 0$: $\underline{r}$, $\underline{r}+\underline{u}$, $\underline{r}+2\underline{u}$, $\underline{r}+\underline{v}$, $\underline{u}+\underline{v}$. For if $\underline{p}_1$ plus some proper sum of $\underline{r}$, $\underline{u}$, and $\underline{v}$ were such a $\underline{p}_1'$, then that proper sum could be decomposed as a proper sum of the five vectors listed; specifically, if $k\underline{r} + m\underline{u} + n\underline{v} + \underline{p}_1 \;{}_w{>}\; \underline{p}_1$ then we can decompose the left-hand sum:

$$k\underline{r} + m\underline{u} + n\underline{v} = \begin{cases} k\underline{r} + (m-n)\underline{u} + n(\underline{u+v}) & \text{if } m \ge n \\ k\underline{r} + (n-m)\underline{v} + m(\underline{u+v}) & \text{if } n < m\ . \end{cases}$$

In the first case, $2k \ge (m-n)$, since the second components must add up to be non-negative. Here if $m-n$ is even, $m-n = 2q$, $k\underline{r} + m\underline{u} + n\underline{v} = (k-q)\underline{r} + q(\underline{r+2u}) + n(\underline{u+v})$, and if $m-n$ is odd, $m-n = 2q+1$, $k\underline{r} + m\underline{u} + n\underline{v} = (k-q-1)\underline{r} + q(\underline{r+2u}) + (\underline{r+u}) + n(\underline{u+v})$. A similar analysis applies in the case $m < n$.

For the second example, we take $\underline{p}_2 = (0,0,1,1)$, and the same R and W. Then the points $\underline{p}_2' \;{}_w{>}\; \underline{p}_2$ reachable from $\underline{p}_2$ using vectors in $R \cup W$ are those pairs reachable from $\underline{p}_2$ using the vectors $\underline{r}$, $\underline{r}+\underline{u}$, $\underline{r}+2\underline{u}$, and $\underline{r}+\underline{v}$, together with the points reachable from $\underline{p}_2 + \underline{r}$ using the above vectors together with $\underline{u} + \underline{v}$. This is just the sum of two cones with vertices at $\underline{p}_2$ and $\underline{p}_2 + \underline{r}$. Two cones are required here because while it is impossible to travel along $\underline{u} + \underline{v}$ from $\underline{p}_2$ and remain within S, there is no violation of the restriction to S if we first shift to $\underline{p}_2 + \underline{r}$.

The set of points $\underline{p}_1'$ reachable from $\underline{p}_1$ with $\underline{p}_1' \;{}_w{>}\; \underline{p}_1$ (respectively $\underline{p}_2'$ and $\underline{p}_2$) in the previous two examples are examples of positive cones, which we describe below and give an algorithm to compute their bases. Let $\underline{p}$, R, and W be as in the first paragraph of this section; we suppose further that W contains no vectors all of whose $\underline{u}$-components are non-negative, for we could put such a vector into R and delete it from W without changing the situation. From $\underline{p}$, R, and W, we will construct a finite set of vertex points $P = \{\underline{p}_1,\ldots,\underline{p}_n\}$ and finite sets of vectors $R_1,\ldots,R_n$, all of whose $\underline{u}$-components are ≥ 0, such that (i) each $\underline{p}_i \;{}_w{>}\; \underline{p}$, (ii) the directions in which one can travel unboundedly from $\underline{p}_i$ are rational convex combinations of the vectors in R_i, and (iii) if $R \cup W$: $\underline{p} \rightsquigarrow \underline{q}$ in S and $\underline{p} <_w \underline{q}$, then $q \in \cup\{C_{\underline{p}_i} \mid \underline{p}_i \in P\}$, where $C_{\underline{p}_i}$ is the cone with vertex at $\underline{p}_i$ of points given by

$$C_{\underline{p}_i} = \{\underline{q} \mid R_i\colon \underline{p}_i \rightsquigarrow \underline{q} \ \text{ in } S\}.$$

The construction goes by induction on the number components i such that $\underline{u}_i \cdot \underline{w} < 0$ for some $\underline{w} \in W$. If none, W is empty and then we obtain as the only cone the one with vertex $\underline{p}_1 = \underline{p}$ and we let $R_1 = R$. Otherwise, choose the least index i such that $\underline{u}_i \cdot \underline{w} < 0$ for some $\underline{w} \in W$, and partition W into $W_< = \{\underline{w} \in W \mid \underline{w} \cdot \underline{u}_i < 0\} \cup W_\ge = \{\underline{w} \in W \mid \underline{w} \cdot \underline{u}_i \ge 0\}$. For each $\underline{w} \in W_<$, let $\Lambda_0(\underline{w})$ be the set of minimal proper sums L of $\underline{w}$ and vectors in $R \cup W_\ge$ such that (a) the coefficient k_L of $\underline{w}$ is > 0 and (b) $\underline{L} \cdot \underline{u}_i = 0$. Let $\Lambda_1(\underline{w}) = \Lambda_0(\underline{w}) \cup \{L - \underline{w}, L - 2\underline{w}, \ldots, L - k_L \underline{w} \mid L \in \Lambda_0(\underline{w})\}$, and let $\Lambda = \cup\{\Lambda_1(\underline{w}) \mid \underline{w} \in W_<\}$.

Next partition Λ into $\Lambda_2 = \{\lambda \in \Lambda \mid$ all $\underline{u}$-components of λ are $\ge 0\} \cup \Lambda_W = \{\lambda \in \Lambda \mid$ some $\underline{u}$-component of $\lambda < 0\}$. For all the cone vertices $\underline{p}'$ constructed below, the set of vectors $W_{\underline{p}'}$ (playing the role of W above) will be $W_\ge \cup \Lambda_W$; note that the number of indices i such that $\underline{u}_i \cdot \underline{w} < 0$ for some $\underline{w} \in W_{\underline{p}}$, has been reduced by 1.

For each element $\lambda \in \Lambda_R$, test if it has a decomposition as a sum of a sequence of the vectors which make it up as a proper sum such that each of the partial sums, when added to $\underline{p}$, results in a point in S. If yes, add λ to R. $R_{\underline{p}}$ is the result of all these additions. If no, take each decomposition of λ and add to it all minimal proper sums $\underline{u}$ of vectors in $R_{\underline{p}}$ which will keep the $\underline{p}$ plus all the partial sums in S. Then a new vertex point is $\underline{u} + \underline{p}$, and $R_{\underline{u+p}}$ receives all of $R_{\underline{p}}$ and the vector λ. The construction of the paragraph is repeated until Λ_R is exhausted.

<u>3.1 Theorem</u>: There is an algorithm to compute a

finite set of points $P = \{\underline{p}_1,\ldots,\underline{p}_n\}$ and finite sets of vectors $R_1,\ldots,R_n$, which enjoy properties (i) to (iii) above.

Proof: To verify (i) to (iii) on the points and vectors constructed above, proceed by induction on the number of indices i such that $\underline{u}_i \cdot \underline{w} < 0$ for some $\underline{w} \in W$. The inductive step follows by a calculation analogous to that for the first example. □

Let R be a finite set of integral vectors which spans N-space. An *interior point* of the cone C_R positively spanned by R is an integral point $\underline{p}$ such that for some rational $\varepsilon > 0$, all rational points $\underline{p}'$ with $|\underline{p}' - \underline{p}| < \varepsilon$, $\in R$.

3.2 Lemma: Let Q and R be finite sets of integral vectors such that R spans N-space; then either (i) or (ii) below:

(i) $C_R \cap C_Q$ contains no interior points and we can effectively compute an (N-1)-dimensional rational hyperplane H containing this intersection and a vector $\underline{u}$ normal to H such that either $\underline{u}\cdot\underline{r} \le 0$ for all $\underline{r} \in R$ and $\underline{u}\cdot\underline{s} \ge 0$ for all $\underline{s} \in Q$, or else $\underline{u}\cdot\underline{s} = 0$ for all $\underline{s} \in Q$.

(ii) $C_R \cap C_Q$ has an interior point $\underline{p}$ and for all vectors $\underline{v} \in C_R \cup C_Q$ there is an integer n such that $\underline{v} + n\underline{p} \subset C_R \cap C_Q$.

Moreover, we can effectively determine which case applies.

§4. Trees

Given an initial point $\underline{a}$, a set V of vectors, and a space S, we wish to describe the construction of a tree $T(\underline{a},V,S)$ which will yield us much information about the reachability set. In particular, we will learn if the reachability set is finite. If the reachability set is infinite, $T(\underline{a},V,S)$ will be used to determine whether or not there is a point $\underline{a}' \in R(\underline{a},V,S)$ with $\underline{a} <_S \underline{a}'$; if there is no such $\underline{a}'$, $T(\underline{a},V,S)$ will enable us to decompose the reachability problem into finitely many problems of lower dimension.

$T(\underline{a},V,S)$ is a tree with nodes labelled by points in S and edges labelled by vectors in V. This tree will be constructed by induction on the dimension of the vector addition system $(\underline{a},V,S)$. For dimension zero, V contains at most the zero vector. $T(\underline{a},V,S)$ is simply $\underline{a}$.

Suppose the construction has been given for all vector addition systems of dimension less than n. We give the construction for dimension n in phases.

Phase I. The first phase is done by stages. At stage zero, the root of $T(\underline{a},V,S)$ is constructed and labelled $\underline{a}$. At any later stage we consider the nodes constructed at the immediately preceding stage. For each such node, labelled e.g. $\underline{p}$, we ask if $\underline{q} \le_\Delta \underline{p}$ for some ancestor $\underline{q}$ of $\underline{p}$ in the tree and some (possibly empty) Δ. If yes, $\underline{p}$ is a "repeating point" and will be dealt with in Phase II. If no, $\underline{p} + \underline{w}$ is calculated for each $\underline{w} \in V$. For each of these that lies in S, a node labelled $\underline{p} + \underline{w}$ is attached to $\underline{p}$ by an edge labelled $\underline{w}$. Let T_I be the tree constructed by Phase I.

Phase II. This phase deals with the repeating points discovered in Phase I. Consider all permutations of all repeating nodes. For each such permutation, we extend T_I in the manner described below; and at the end, we join all those trees at their roots (which are identical, since they are all extensions of T_I) and merge them as far as possible.

Fix some permutation of the repeating nodes. Assume that the extension of T_I has been formed for the first i repeating nodes, ordered by the permutation. Let $\underline{q}$ be the (i+1)-st node, and let $\underline{q}'$ be its earliest ancestor such that $\underline{q}' <_\Delta \underline{q}$. To $\underline{q}$ attach the subtree $T_{\underline{q}'}$, rooted at $\underline{q}'$, labelling the edges as in $T_{\underline{q}'}$, and adjusting the node labels by adding $(\underline{q} - \underline{q}')$ to each. Note that each of the adjusted nodes is a point in S, since adding $(\underline{q} - \underline{q}')$ does not decrease any $\underline{u}$-coordinate. Each point in the subtree is a repeating point, and must have repeating vectors calculated for it. This calculation will be described in the next paragraph.

Let T_{II} be the finite tree obtained by joining all the extensions obtained for all the permutations at their root. Consider any repeating point $\underline{q}$ in T_{II} and assume that repeating vectors have been calculated for the ancestors of $\underline{q}$. The repeating vectors of $\underline{q}$ are those of its ancestors as well as the following. For each ancestor $\underline{q}'$ of $\underline{q}$ such that $\underline{q}' <_\Delta \underline{q}$, $(\underline{q} - \underline{q}')$ is a repeating vector for $\underline{q}$.

Phase III. Let $\underline{q}$ be any repeating point in T_{II}, for which the set Δ, of indices of $\underline{u}$-coordinates that can be increased using the set R of repeating vectors associated with $\underline{q}$, is not empty, and let $\Gamma = \{1,\ldots,k\} - \Delta$. Γ indexes the hyperplanes which $\underline{q}$ parallels under motions by its repeating vectors, and thus determines a vector space C the dimension of which is lower than the dimension of the space spanned by the vectors in V (since $\Delta \neq \emptyset$). Let W be the set of vectors in V which are parallel to all of the hyperplanes in Γ. Compute the finite list of cones and vertices given by applying Theorem 3.1 to $\underline{q}$, R, and W. We may extend the tree so far constructed so that it reaches each of these vertices, and assume that we are now at one of them, say $\underline{q}'$, with corresponding set $R_{\underline{q}'}$ of basis vectors for its cone. Let $\hat{\underline{w}}$ be the projection of $\underline{w}$ into C, and let $\hat{V} = \{\hat{\underline{v}} \mid \underline{v} \in V\}$. Note that the dimension of the space spanned by the vectors in $\hat{V}$ is less than the dimension of the space spanned by the vectors in V. Let $\hat{S}$ be the space obtained by translating C to $\underline{q}'$ and intersecting the resultant space with S. The space $\hat{S}$ is bounded by some hyperplanes $\hat{H}_1,\ldots,\hat{H}_K$. Note that with a suitable change of basis, $\hat{\underline{w}}$ and $\underline{q}$ may be considered to be integral. Now construct the tree $T(\hat{\underline{q}}',\hat{V},\hat{S})$, and its lift T^* (which as an unlabelled tree is the same as $T(\hat{\underline{q}}',\hat{V},\hat{S})$) whose root is $\underline{q}'$, whose edge labels are $\underline{w}$ for each edge label $\hat{\underline{w}}$ in $T(\hat{\underline{q}}',\hat{V},\hat{S})$, and whose label at any node is $\underline{q}'$ plus the sum of the vectors between $\underline{q}'$ and that node. Note that some node labels in T^* will not be in S, for they may be labels of points to negative sides of the hyperplanes indexed in Δ. We use T^* to define a new tree $T^{**}(\underline{q}')$, all of whose labels will be in S. $T_{III} = T(\underline{q}',V,S)$ will be obtained from T_{II} by adjoining to each repeating point of T_{II}, $T^{**}(\underline{q}')$.

The root of $T^{**}(\underline{q}')$ is $\underline{q}'$. This is in S, and needs no further attention; the corresponding set of repeating vectors has already been calculated in Phase II. Suppose that q^* is a node in T^* and we have already dealt with all the ancestors in T^* of $\underline{q}^*$. If $\underline{q}^*$ is not a repeating point of T^* we consider all minimal proper sums $\underline{y}$ of vectors in $R_{\underline{q}'}$ (minimal in the sense that no coefficient can be decreased) such that $\underline{y} + \underline{q}^* \in S$. Let $\underline{p}^*$ be the predecessor $\underline{q}^*$ in T^*, and let $\underline{p}^{**}$ be the lift of $\underline{p}^*$ to T^{**}. We attach to $\underline{p}^{**}$ branches whose edges are labelled by all permutations of all such minimal proper sums $\underline{y}$, whose end nodes are labelled $\underline{p} + \underline{y}$ and whose intermediate nodes are labelled in the obvious way. At the ends of these branches we attach the lift $\underline{q}^{**}$ of $\underline{q}^*$ by adjoining an edge labelled $\underline{w}$ to the node $\underline{p} + \underline{y}$, where $\hat{\underline{w}} = \underline{q}^* - \underline{p}^*$, and labelling its endpoint $\underline{p} + \underline{y} + \underline{w}$. The repeating vectors associated to all these nodes are the same as those associated to $\underline{q}'$.

If $\underline{q}^*$ is a repeating point of T^*, and the edge leading to $\underline{q}^*$ is labelled by $\underline{w}$, consider all minimal proper sums $\underline{y}$ of the repeating vectors in $R_{\underline{q}'}$ such that $\underline{q}^* + \underline{y}$ is in S, and for each predecessor $\underline{p}^*$ of $\underline{q}^*$ in T^* such that $\underline{p}^* <_W \underline{q}^*$, if $\underline{p}^{**}$ is an image of $\underline{p}^*$ in T^{**}, $\underline{p}^{**} <_W \underline{q}^* + \underline{y}$. Attach branches to the images of the immediate predecessor of $\underline{q}^*$ in T^{**} whose edges are labelled by the successive edges in all permutations of all such $\underline{y} \cup \{\underline{w}\}$, such that all node labels (calculated in the usual fashion) are in S. Note in each case the end node is labelled $\underline{q}^* + \underline{y}$. The repeating vectors associated to any of these points, e.g. q''', are calculated as follows: First compute the basis R for the positive cone at $\underline{q}$ (the repeating point selected at the beginning of Phase II). Second, compute the set W of vectors $\underline{q}''' - \underline{q}''$, where $\hat{q}'' <_\Delta \hat{q}'''$ for some Δ, and $\underline{q}''$ predeces $\underline{q}'''$ in T^{**}. The repeating vectors associated with $\underline{q}'''$ then are a basis for the positive cone at $\underline{q}'''$ computed using R and W.

By induction on dimension, one can prove that $T(\underline{a},V,S)$ is finite.

4.1 Theorem: $R(\underline{a},V,S)$ is finite if and only if $T(\underline{a},V,S)$ has no repeating point with a non-zero associated repeating vector.

Proof: If $\underline{q}$ is such a repeating point and $\underline{w} \neq 0$ is an associated repeating vector, then all the poins $\underline{q} + k\underline{w}$ (for $k = 0,1,2,\ldots$) are in $R(\underline{a},V,S)$.

4.2 Theorem: There is an algorithm to determine if $R(\underline{a},V,S)$ or $R(\underline{b},-V,S)$ is finite, and if so to decide if $\underline{b} \in R(\underline{a},V,S)$.

Proof: If $R(\underline{a},V,S)$ is finite then all points in this set are node labels in $T(\underline{a},V,S)$. Note that in this case the tree construction terminates at the end of Phase I, and case (3) is never encountered. The dual argument applies if $R(\underline{b},-V,S)$ is finite.

4.3 Remark: From $T(\underline{a},V,S)$ we may compute those maximal Δ such that there exist points $\underline{a}'\in R(\underline{a},V,S)$ and $\underline{a}''$ in $R(\underline{a}',V,S)$ with $\underline{a}' <_\Delta \underline{a}''$. If one of these sets Δ is $\{1,2,\ldots,K\}$ and one of the corresponding sets Δ for $R(\underline{b},-V,S)$ is also $\{1,2,\ldots,K\}$, then we are done by Theorem 2.6 and Remark 2.7, since if V does not positively span Z^N, we add two new parallel bounding hyperplanes, and it is impossible to move away from both of them simultaneously. One should note that the distance between these hyperplanes is not critical, so long as it is "sufficiently large". By examining the occurrences of all such $\underline{a}'$ together with their ancestors in $T(\underline{a},V,S)$, we can determine a bound $B(\Delta)$ on the values of $\underline{u}$-components indexed in $\Gamma = \{1,2,\ldots,K\} - \Delta$ of points $\underline{a}''$ reachable from $\underline{a}$ from which such points $\underline{a}'$ are reachable.

For each boundary hyperplane H_i for $i \in \Gamma$, we may construct a parallel boundary hyperplane, H_i^*, $B(\Delta)$ units away from H_i, on the positive side of H_i. (Note: Any hyperplane at least $B(\Delta)$ units away from H_i will do.) The positive side of this new hyperplane H_i^* is toward H_i, and thus its normal vector $\underline{u}$ is $-\underline{u}_i$. Let $S(\Delta) = \{\underline{s} \in S \mid \underline{s}$ lies between the new hyperplanes$\}$. If there is a path from $\underline{b}$ to $\underline{a}$ using vectors in $-V$, such a path must in fact remain within some $S(\Delta)$ we have just constructed.

§5. τ-Simple Paths

As we have already seen (Remark 4.3), we may restrict our attention to the case in which either $\underline{a}$ cannot be moved to a point $\underline{a}'$, $\underline{a} \leq_s \underline{a}'$ under V or $\underline{b}$ cannot be moved to a point $\underline{b}'$, $\underline{b} <_s \underline{b}'$ under $-V$. Without loss of generality, we assume the former case. Then the set of maximal, non-empty Δ such that V: $\underline{a} \rightsquigarrow \underline{a}' \rightsquigarrow \underline{a}''$ in S, $\underline{a}' <_\Delta \underline{a}''$, can be computed from $T(\underline{a},V,S)$. The important point for us is that all of these Δ are proper subsets of $\{1,2,\ldots,K\}$. For each of these maximal Δ's the possible values of $\underline{u}$-coordinates indexed in $\Gamma = \{1,2,\ldots,K\} - \Delta$ of points reachable from $\underline{a}$ are bounded, and these bounds can be computed from $T(\underline{a},V,S)$. Thus, for any path from $\underline{a}$ to $\underline{b}$ this path must satisfy the bounds on $\underline{u}$-coordinates of intermediate points imposed by some one of these Δ's. That is, this path consists of a sequence of motions among and within a finite set of lower-dimensional subspaces indexed by the permissible values of $\underline{u}$-components indexed in $\Gamma = \{1,2,\ldots,K\} - \Delta$.

Let $\Sigma = \{\sigma_1,\ldots,\sigma_s\}$ be the finite list of spaces corresponding to a fixed maximal Δ. Partition V into $V_J \cup V_I$ where V_I consists of the vectors V parallel to the spaces σ_i, and $V_J = V - V_I$ consists of the vectors which can be used to jump among them. Let $\tau \subseteq \Sigma$. We define the τ-simple paths among those spaces as follows: The τ-simple paths from σ_i to σ_j of rank 0 consist of (i) a finite set of vertices, labelled $\sigma_{i_0} = \sigma_i$, σ_{i_1}, $\ldots$, $\sigma_{i_n} = \sigma_j$, where all σ_i's except possibly σ_{i_0} and σ_{i_n} are distinct members of τ; and (ii) edges connecting successive vertices are labelled by elements of V_J in such a way that if $\underline{v}$ is the label on the edge connecting σ_{i_k} and $\sigma_{i_{k+1}}$, then for any point $\underline{p}$ in σ_{i_k}, $\underline{p} + \underline{v} \in \sigma_{i_{k+1}}$. Now suppose that the τ-simple maps of rank $< m$ have been defined. A τ-simple map of rank m from σ_i to σ_j consists of (i) a τ-simple rank 0 path $\sigma_{i_0},\ldots,\sigma_{i_n}$ from σ_i to σ_j such that at vertex σ_{i_k} a finite ordered set of distinct $(\tau - \{\sigma_{i_0},\ldots,\sigma_{i_k}\})$-simple, rank $< m$, paths from σ_{i_k} to σ_{i_k} are adjoined. By a straightforward induction, there are only finitely many Σ-simple maps from any space in Σ to any other, and one can effectively compute a list of them.

§6. The Reduction To Lower Dimension

We continue the algorithm for the case in which V: $\underline{a} \rightsquigarrow \underline{a}' \rightsquigarrow \underline{a}''$ in S where $\underline{a}' <_\Delta \underline{a}''$ for some nondeterministic choice of a maximal Δ, $\emptyset \neq \Delta \neq \{1,2,\ldots,K\}$, and we assume that bounds on the components in $\Gamma = \{1,2,\ldots,K\} - \Delta$ have been computed, that additional (N-1)-dimensional bounding hyperplanes have been erected, and that the finite set Σ of lower-dimensional subspaces has been determined.

6.1 Compute $T_{\underline{a}} = T(\underline{a},V,S \cap \cup\{\sigma | \sigma \in \Sigma\})$. If $\underline{b}$ is a node label on this tree the algorithm reports success. Otherwise any possible path $V: \underline{a} \rightsquigarrow \underline{b}$ in S must pass through one of the repeating points $\underline{p}$ of the given tree. Non-deterministically choose one of these repeating points $\underline{p}$ and a τ-simple path α from $\sigma(\underline{p}) \in \Sigma$ to $\sigma(\underline{b})$ (the spaces containing $\underline{p}$ and $\underline{b}$ respectively). We will now describe a procedure to travel from $\underline{p}$ to $\sigma(\underline{b})$ using α to produce a family of lower-dimensional vector addition systems in $\sigma(\underline{b})$ (6.2, 6.3). As we travel along α, we will produce some auxiliary vector addition systems. We will alternate this routine with one which checks to make sure that the vector addition systems so constructed are legitimate.

6.2 The initial auxiliary vector addition systems at $\underline{p}$ have the vertices $\{\underline{p}'\}$ of the positive cones at $\underline{p}$ (constructed using as initial point $\underline{p}$, as R the set of repeating vectors at $\underline{p}$, and as W the set of vectors in V_I which affect only those components in which some vector $\underline{r} \in R$ has a positive value). The vector sets for these initial vector addition systems will consist of $V_I \cup V^*$ where V^* is a basis for the positive cone with the given vertex. Given an auxiliary vector addition system $(\underline{a}',V_{\underline{a}'})$, let U be the set of sums of τ-simple loops from $\sigma(\underline{a}')$ to $\sigma(\underline{a}')$ already traversed in α. Form the tree $T(\underline{a}',V_{\underline{a}'} \cup U, S \cap \sigma(\underline{a}'))$. Beginning at the root $\underline{a}'$ of this tree examine all points $\underline{a}''$ at which vectors $\underline{u} \in U$ are applied; delete all branches which extend $\underline{a}''$ in which both (i) $\underline{q} = \underline{a}''$ + some partial sum of $\underline{u}$ is not in S, and (ii) $\underline{q}$ can not be made to lie in S by adding a proper sum of basis vectors for the positive cone at $\underline{a}''$. If a proper sum of basis vectors for the positive cone at $\underline{a}''$ needs to be added (as in ii), actually add all minimal proper sums which keep the path from $\underline{a}''$ to $\underline{a}'' + \underline{u}$ within S.

6.2.1 <u>Remark</u>: The reader should note that if the path from $\underline{a}''$ to $\underline{a}'' + \underline{u}$ can be made to lie in S only by interleaving vectors in the set X of those vectors which affect components other than those which can be increased using the vectors in the cone at $\underline{a}''$, then a subsequence $\underline{u}$ in which some of the τ-simple loops are omitted, with the relevant vectors from X, will appear in a branch of the tree $T_{\underline{a}}$ which extends from the point $\underline{p}$. The omitted loops will then be picked up in making the construction of 6.2 from a repeating point on this branch. In §7 we shall return to this issue.

6.3 Let $\underline{w}$ be the next vector along α. For any non-repeating point $\underline{p}_1$ of the tree constructed in 6.2, if $\underline{p}_1 + \underline{w} \in S$, $\underline{p}_1 + \underline{w}$ is the origin of a new auxiliary vector addition system, whose vectors are $V_{\underline{a}'}$. For each repeating point $\underline{p}_1$, if $\underline{p}_1' + \underline{w} + \underline{x} \in S$ where $\underline{p}_1'$ is a vertex of a positive cone constructed for $\underline{p}_1$ (from 3.1) and $\underline{x}$ is some minimal proper sum of the basis vectors $R_{\underline{p}_1'}$ at $\underline{p}_1'$, then $\underline{p}_1' + \underline{w} + \underline{x}$ is the origin of a vector addition system whose vectors consist of $V_{\underline{a}'} \cup R_{\underline{p}_1'}$. The cones (from 3.1) are constructed by taking as R the repeating vectors at $\underline{p}_1$, and as W the vectors in $V_{\underline{a}'} \cup U$ where non-zero $\underline{u}$-components are among the $\underline{u}$-components which can be improved using vectors in R. (In the algorithm of that theorem, when proper sums of vectors are decomposed as sums of sequences, one must be sure also to decompose the vectors $\underline{u} \in U$.)

The points in $\sigma(\underline{b})$ obtained from the end of the construction of 6.1-6.3 will be the initial points of the lower-dimensional vector addition systems in $\sigma(\underline{b})$ which we have promised.

At $\underline{b}$ construct $T_{\underline{b}} = T(\underline{b},-V, S \cap \cup\{\sigma | \sigma \in \Sigma\})$ and consider the set of points $B = \{\underline{b}' \in \sigma(\underline{b}) \mid \underline{b}' \in T_{\underline{b}}\}$. If one of the initial points is in B, then $\underline{b} \in R(\underline{a},V,S)$, and our algorithm reports success. Non-deterministically choose a point $\underline{b}$ in B, and an initial point $\underline{a}'$ as produced by §6.1 - 6.3. These are used in the checking procedure of the next section.

6.4 Next we need a checking routine to determine which vectors included among the vectors in the auxiliary vector addition system at $\underline{a}'$ can be used arbitrarily often. Consider the bases (possibly empty) $R_{\underline{a}'}$ and $R_{\underline{b}'}$ for the cones at $\underline{a}'$ and $\underline{b}'$. ($R_{\underline{a}'}$ is constructed taking for R (of Theorem 3.1) the set of auxiliary vectors at $\underline{a}'$ which have non-negative $\underline{u}$-coordinates and for W the set of auxiliary vectors which are non-zero in some $\underline{u}$-component only if some vector in R is non-zero in this component.) Translate these two sets of vectors to the origin, and use them as the vector sets R and Q of Lemma 3.2. The dimension of the space mentioned in that lemma will simply be the maximum of the dimensions of the spaces spanned by R and Q. If the intersection of the cones contains an interior point $\underline{p}$, then any motion from $\underline{a}'$ using the vectors in $R_{\underline{a}'} = R$ can be effectively reversed by a motion from $\underline{b}'$ using $R_{\underline{b}'} = Q$ by simply translating $\underline{a}'$ to $\underline{a}' + n\underline{p}$, and $\underline{b}'$ to $\underline{b}' + n\underline{p}$ for a suitable n as well as making the desired motion. In this case we take as vector set $V_{\underline{a}'}$ at $\underline{a}'$, {auxiliary vectors at $\underline{a}'$} $\cup$ $\{-r \mid r \in R_{\underline{b}'}\}$.

Now suppose the intersection of the cones does not contain an interior point $\underline{p}$. If R is contained in a hyperplane H_0 of dimension less than $\sigma(\underline{b})$, we take as vector set at $\underline{a}'$ {auxiliary vectors at $\underline{a}'$} $\cup$ $\{-r \mid r$ is a basis vector for the cone $C_{R_{\underline{b}'}} \cap H_0\}$.

If Q is contained in such a hyperplane H_0, then we distinguish two cases: In the first H_0 cannot be taken parallel to any H_i, $i \in \Delta$, and in the second, H_0 can be so taken. In the first case motions in Q can move $\underline{b}'$ to points which are farther from all the H_i, $i \in \Delta$, simultaneously. Then we take as vector set $V_{\underline{a}'}$ at $\underline{a}'$, {auxiliary vectors $\underline{v}$ at $\underline{a}'$ | $\underline{v}$ is a summand of a proper sum $\underline{r}$ in R, $r \in H_0$} $\cup$ $\{-\underline{r} \mid r \in R_{\underline{b}'}\}$. Note that the vectors which we put into $V_{\underline{a}'}$ can be effectively reversed, by suitably translating $\underline{a}'$ and $\underline{b}'$, as above. The remaining auxiliary vectors $\underline{v}$ which are not in $V_{\underline{a}'}$ can only be used finitely often; the bound on the number of applications of $\underline{v}$ can be computed by finding the length $L_{\underline{v}}$ of the projection of $\underline{v}$ onto the unit normal of the hyperplane H_0. Specifically, $\underline{v}$ cannot be applied more than $|\underline{b}' - \underline{a}'|/L_{\underline{v}}$ times beyond the uses already made of $\underline{v}$ in constructing $\underline{a}'$.

Consequently, we can find a pair of parallel (N-1)-dimensional hyperplanes M_0 and M_1 which are parallel to H_0, whose positive sides face each other, such that all points in all of the trees constructed so far in the spaces $\sigma \in \Sigma$ are at least sup $\{|\underline{b}' - \underline{a}'|/L_{\underline{v}} \mid$ for all omitted auxiliary vectors $\underline{v}\}$ units from each of them, as new bounding hyperplanes.

In the case that Q is contained in a hyperplane H_0 parallel to one of the H_i, $i \in \Delta$, then motions from $\underline{b}'$ using Q are all parallel to H_i. We will take as vector set $V_{\underline{a}'}$ the set {auxiliary vectors $\underline{v}$ at $\underline{a}'$ which are summands of some proper sum $\underline{r} \in R$ such that $\underline{r} \in H_0\}$ $\cup$ $\{-\underline{r} \mid \underline{r} \in R_{\underline{b}'}\}$. As the bounding hyperplane M_0, we take H_i, and as M_1 we take a hyperplane parallel to and on the positive side of H_i, which is "sufficiently far away". To determine how far M_1 must be from M_0 we must first note that the values of $\underline{u}_i$-coordinates of points reachable from $\underline{a}'$ using the vector set $V_{\underline{a}'}$ must differ from the $\underline{u}$-coordinate of $\underline{u}'$ by a multiple of the greatest common divisor of the set $D = \{m \mid m$ is a non-zero $\underline{u}_i$-component of a vector in $V_{\underline{a}'}\}$. Moreover, the value m_0 of the $\underline{u}_i$-component of $\underline{b}'$ can be achieved by applying the vectors in $V_{\underline{a}'}$ no more than ($\Pi|m|$, for $m \in D$) times each. We certainly want M_0 and M_1 sufficiently far apart so that each vector in $V_{\underline{a}'}$ can be used this often from any point in the trees which were used to construct $\underline{a}'$. Moreover, since it may be necessary to repeat this construction for all of the bounding hyperplanes H_i, $i \in \Delta$, we had better allow enough space between M_0 and M_1 to apply the vectors $\Pi|e|$ times, where e ranges over non-zero $\underline{u}_j$-components, $j \in \Delta$, of vectors in $V_{\underline{a}'}$.

In the case that C_R and C_Q can be separated by hyperplane H_0, then all auxiliary vectors not actually in H_0 can only be used finitely often, and we take as the set of vectors $V_{\underline{a}'}$, those auxiliary vectors in H_0 together with $\{-\underline{r} \mid \underline{r}$ is a basis vector for the cone $C_{R_{\underline{b}}} \cap H_0\}$. We then proceed to construct an additional pair of bounding hyperplanes, as in the last paragraph.

After we add a new pair of (N-1)-dimensional bounding hyperplanes, it may be possible to extend the trees constructed in 6.1 (since the new bounding hyperplanes further restrict the notion $<_w$, certain points will no longer be repeating points, and further computations will be possible in Phase I of the construction of §4). The extensions of this tree will lead to new initial points $\underline{a}'$ (by use of 6.1 and 6.2) and new associated vector sets, from which we may make nondeterministic choices. However, none of the new branches of the tree can have a repeating point whose positive cone has a basis vector which involves one or more of the omitted auxiliary vectors. Since all proper sums of vectors which involve such a $\underline{v}$ non-trivially, point toward one of the new bounding hyperplanes M_0 and M_1, such proper sums cannot be in any positive cone.

6.5 Remark: The number of times that one introduces new boundary hyperplanes with the checking routine 6.4 is finite. One should note that for auxiliary vectors $\underline{v}$ not in $V_{\underline{a}}$, and all $i \in \Gamma$, $\underline{v}\cdot\underline{u}_i = 0$, and if $\underline{u}$ is a unit vector normal to M_0 (or M_1), $\underline{v}\cdot\underline{u} \neq 0$. Moreover, it is impossible to move any point $\underline{p}$ to a point $\underline{p}'$ which is simultaneously further from both M_0 and M_1 than $\underline{p}$. Consequently, if we then use the tree construction of §4, replacing S by the set $S_1 = S \cap \{$integral points between M_0 and $M_1\}$, to compute a finite set Σ' of spaces σ' which contain all the reachable points, the resulting spaces σ' have lower dimension than the spaces σ with which we have been computing in this section. (The space simultaneously parallel to the bounding hyperplanes from which we can move only boundedly far is the solution space of a system of linear equations containing a new independent equation; the new equation asserts $\underline{x}\cdot\underline{u} = 0$ where $\underline{u}$ is a unit normal to M_0). Hence, after a finite number of cycles, the construction of this section must stop. In the worst case, it continues, until all the lower-dimensional spaces have dimension 0, in which case we simply have a finite set of dimension zero vector addition systems.

6.6 Remark: If a path from $\underline{a}$ to $\underline{b}$ is constructed by our procedure (i.e. some run of our algorithm reports success), then V: $\underline{a} \rightsquigarrow \underline{b}$ in S. This follows from the fact that at any point in the construction, any vector that has been added to the original set V of vectors is the sum of a sequence of vectors that it was possible to apply arbitrarily often at some time. Consequently, we may take each use of such a vector $\underline{w}$ in the path that we construct and replace it at a suitable point in the path with the sequence that gave rise to that $\underline{w}$. The checking routine, 6.4, guarantees that if cone vectors are needed to apply a sum of a τ-simple loop, then these vectors can be effectively reversed. The result of this reordering is the required path V: $\underline{a} \rightsquigarrow \underline{b}$ in S. The converse to Remark 6.6 is verified in §7.

§7. Verification of the Algorithm

7.1 Theorem: If V: $\underline{a} \rightsquigarrow \underline{b}$ in S, then the algorithm halts and reports success.

Proof: Assume V: $\underline{a} \rightsquigarrow \underline{b}$ in S, and assume that the path γ actually followed consists of the points $\underline{c}_0, \underline{c}_1, \ldots, \underline{c}_m$, where $\underline{a} = \underline{c}_0$ and $\underline{b} = \underline{c}_m$, and uses the motions $\underline{w}_1, \ldots, \underline{w}_m$ (i.e., $\underline{c}_{i+1} - \underline{c}_i = \underline{w}_{i+1}$).

For the case in which the dimension is zero, the theorem is trivial: $\underline{a}$ must equal $\underline{b}$. Assuming the result for all dimensions below N, we show the result for dimension N.

Clearly the test in step one must show that $\underline{b}$ can be expressed as $\underline{a}$ plus a proper sum of vectors in V, otherwise the path γ could not exist.

Next note that if the test in step three shows that there exist $\underline{a}' \in R(\underline{a},V,S)$ and $\underline{b}' \in R(\underline{b},-V,S)$ with $\underline{a} <_s \underline{a}'$ and $\underline{b} <_s \underline{b}'$, then the algorithm terminates and reports success. So we consider the case in which this test fails.

The information in the tree constructed in step three offers several choices of sets Δ of $\underline{u}$-coordinates that are simultaneously unbounded, and corresponding to each Δ, a set $\Gamma = \{1,\ldots,K\} - \Delta$ of $\underline{u}$-coordinates that are bounded and for which maximum values can be determined by examining the tree. We select any Δ which is compatible with γ in the sense that for any $\underline{c}_j$ in γ, no $\underline{u}$-coordinate which is indexed in Γ exceeds the maximum given for that coordinate in the tree. Such must exist.

For the chosen Δ, the algorithm gives a set Σ of spaces of dimension less than N. Each of these spaces is parallel to all the hyperplanes H_j for $j \in \Gamma$. V is partitioned into $V_I = \{\underline{v} \in V \mid \underline{v}\cdot\underline{u}_j = 0$ for $j \in \Gamma\}$ and $V_J = V - V_I$. Note that the dimension of the space spanned by V_I is less than or equal to the dimension of any space $\sigma \in \Sigma$ which is strictly smaller than N.

The algorithm proceeds by constructing additional hyperplanes corresponding to the bounds on the components indexed in Γ, thus further restricting S to a space S_1. Using the resultant space, the algorithm calculates $T = T(\underline{a},V,S_1)$. If $\underline{b} \in T$, the algorithm terminates and reports success, so we must consider the case in which $\underline{b} \notin T$. Note that T must contain repeating points, for otherwise the set $R(\underline{a},V,S)$ would be finite and $\underline{b} \in T$.

Follow γ for as long as it remains in T. The last point $\underline{c}_i$ of γ which is in T must be a repeating point (otherwise $\underline{c}_{i+1} = \underline{c}_i + \underline{w}_{i+1} \in T$, since $\underline{c}_{i+1}$ is assumed to be in S_1). We assume that γ was chosen in such a way that i is maximal; i.e. we assume that no path γ' from $\underline{a}$ to $\underline{b}$ in S remains in T longer than γ does. A similar tree is constructed at $\underline{b}$, using -V. Let $\bar{\gamma}$ be $\underline{c}_m(-\underline{w}_m)\underline{c}_{m-1} \cdots (-\underline{w}_1)\underline{c}_0$. Note that $(\bar{\bar{\gamma}}) = \gamma$. Let $\bar{\gamma}_2$ be the longest subpath of $\bar{\gamma}$ starting at $\underline{b}$ that remains in the tree constructed at $\underline{b}$ and ends in $\sigma(\underline{b})$.

Let $\underline{c}_j$ be the endpoint of $\bar{\gamma}_2$. Break γ into three parts $\gamma_0 = \underline{c}_0 \cdots c_i$, $\gamma_1 = \underline{c}_i \cdots \underline{c}_j$ and $\gamma_2 = \underline{c}_j \cdots \underline{c}_m$.

At this point the algorithm non-deterministically chooses a τ-simple path α from $\sigma(\underline{c}_i)$, the subspace in Σ in which $\underline{c}_i$ lies, to $\sigma(\underline{b})$. We may assume that the α chosen corresponds to γ_1, where the correspondence comes from systematically deleting motions in V_I and repetitions of τ-simple loops, and replacing points $\underline{q}$ by the space $\sigma(\underline{q})$.

Then the computations of sections 6.2 and 6.3 are performed using α. In §6.4 an attempt is made to calculate new bounding hyperplanes for each initial point. At least one of these is compatible with γ. If this results in adding new hyperplanes, then the algorithm goes back to step 4 and this process is carried out again. As remarked in §6.5, this process ultimately terminates in a finite number of iterations, and when it does so, any basis vector in a cone can be effectively reversed using the cone at $\underline{c}_j$. We thus assume that this process has terminated.

<u>7.2 Lemma</u>: From the assumption that γ was chosen to maximize the number of motions that remain in the tree, we can show that any loop in γ_1 from a space σ back to itself must have the property that any V_I-motion in the loop can be moved out of the loop (either before the loop or after it), perhaps requiring the use of a proper sum of basis vectors for the cone based at $\underline{c}_i$ to do so.

<u>Proof</u>: Let $\underline{w} \in V_I$ be the first such embedded motion, and assume that moving $\underline{w}$ to follow the loop would cause some part of the resultant path to leave S. If $\underline{w}$ affects only those $\underline{u}$-coordinates that can be increased using vectors in the cone at $\underline{c}_i$, then by using such vectors before the loop, the result can be made to lie in S. Note that the extra vectors can be effectively reversed, thus we can build the required path. On the other hand, if $\underline{w}$ affects $\underline{u}$-coordinates that cannot be increased using vectors in the cone at $\underline{c}_i$, then either $\underline{w}$ appears explicitly in the subtree of T rooted at $\underline{c}_i$ or affects only $\underline{u}$-coordinates that can be increased using cone vectors at some node $\underline{w}'$ in this subtree. To see this, note that the projection of $\underline{w}$ which will be used at $\underline{c}_i$ cannot be $\underline{0}$, nor can the projections of the vectors in V_J. Take any path in T through $\underline{c}_i$ that leads to $\underline{w}$ or to such a $\underline{w}'$ which includes only vectors in the cone at $\underline{c}_i$ and motions in $\underline{w}_{i+1},\underline{w}_{i+2},\ldots,\underline{w}$. Specifically, this path can be chosen to be the path γ_0 to $\underline{c}_i$, followed by disjoint subpaths β of $\underline{c}_i\underline{w}_{i+1}\underline{c}_{i+1} \cdots \underline{w}$, consisting only of consecutive motions in V_J and with possible uses of vectors in the cone at $\underline{c}_i$ made between the β subpaths (to force the result to remain in S). (Cf. Remark 6.2.1.) This path gives rise to a path γ' from $\underline{a}$ to $\underline{b}$ in S which remains in T longer than γ did, which contradicts the choice of γ. □

By a similar argument, one can show that multiple uses of a subloop of a given loop can be moved outside that loop.

By this lemma, we may assume without loss of generality that the loops of γ_1 do not have embedded V_I-motions in them.

Using the τ-simple loop α from $\sigma(\underline{c}_i)$ to $\sigma(\underline{b})$, corresponding to γ_1 we now construct a sequence δ, the existence of which guarantees that the algorithm reports success. The conjoined sequence $\gamma_0\delta = \underline{a}_0\underline{x}_1\underline{a}_1 \cdots \underline{a}_k$ will satisfy several conditions which will lead to what we refer to as alignment with the conjoined sequence $\gamma_0\gamma_1$. The idea is that δ will be the result of making some of the motions that were used to form γ_1, while deferring other motions. From time to time it may be necessary to make motions in forming δ which are not used to form γ_1. These extra motions will be necessitated by the requirement that δ remain in S. We will then show that corresponding to these extra motions are other motions which can be applied that effectively reverse the extra motions. In addition, if a loop is used in forming γ_1, it will be replaced by its sum in forming δ. Recall that a vector $\underline{w} \in W_1$ can be effectively reversed using W_2 if there exist $\underline{w}_1$ and $\underline{w}_2$ which are proper sums of vectors in W_1 and W_2, respectively, such that $\underline{w} + \underline{w}_1 + \underline{w}_2 = \underline{0}$. (In practice, $\underline{w}$ and $\underline{w}_1$ will be vectors, all of whose $\underline{u}$-coordinates are non-negative, and they will be applied to a path emanating from $\underline{a}$, and $(-\underline{w}_2)$ will be a vector all of whose $\underline{u}$-coordinates are non-negative, and it will be applied to a path emanating from $\underline{b}$.) Recall that $\gamma = \underline{c}_0 \cdots \underline{c}_i \cdots \underline{c}_j \cdots \underline{c}_m$, and assume that $i \le s \le j$. Let $\underline{a}_r$ be any point, given by the algorithm, which extends γ_0. We may say that $\underline{a}_r$ *aligns with* $\underline{c}_s$ if and only if

(i) $\sigma(\underline{a}_r) = \sigma(\underline{c}_s)$;

(ii) the motions $\underline{w} \in V_I$ and the sums of τ-simple loops that are used to form $\underline{c}_s$ which are not used at least as often to form $\underline{a}_r$ affect only $\underline{u}$-coordinates that can be increased using proper sums of the vectors in the basis of the cone at $\underline{c}_i$;

(iii) the motions used to form $\underline{a}_r$ not used at least as often to form $\underline{c}_s$ can be effectively reversed using the basis for the cone at $\underline{c}_j$; in addition, none of these motions decreases any $\underline{u}$-component.

We prove three lemmas about alignment, which will be used to complete the proof of the theorem. Note that (γ_0, V_I) aligns with itself. The first two lemmas will enable us to show that if alignment can be maintained, then the resultant vector addition systems (which are of lower dimension than the original) cause the algorithm to report success. The third lemma will be used to show that alignment can be maintained.

7.3 Lemma: If $\underline{a}_r$ aligns with $\underline{c}_s$, then those motions used in forming $\underline{c}_s$ that were not used at least as often in forming $\underline{a}_r$ can be added to $\underline{a}_r$, perhaps requiring some proper sums of basis vectors of the cone at $\underline{c}_i$ to remain in S. Let $\underline{a}''$ be the result of adding the appropriate vectors.

Proof: Follows from the definition of alignment.□

7.4 Lemma: $\underline{a}''$ and $\underline{c}_j$ can effectively be moved to points $\underline{a}'$ and $\underline{c}'$ using proper sums of basis vectors at $\underline{a}_r$ and $\underline{c}_j$ (respectively), such that $\underline{a}'' <_w \underline{a}'$, $\underline{c}_j <_w \underline{c}'$, and $\underline{a}' - \underline{c}_s = \underline{c}' - \underline{c}_j$.

Proof: Note that $\underline{a}''$ differs from $\underline{c}_s$ only in that it has extra motions, all of which are sums of basis vectors of the cone at $\underline{a}_r$. These are effectively reversible using the basis vectors of the cone at $\underline{c}_j$ (by section 6.4 and lemma 3.2). Let $\underline{a}'$ be the result of adding the necessary sums from the cone at $\underline{a}_r$ to $\underline{a}''$ and let $\underline{c}'$ be the result of adding the necessary sums from the cone at $\underline{c}_j$. □

7.5 Lemma: Assume that $\underline{a}_r$ is not an initial point (in the sense of section 6.3) and that $\underline{a}_r$ is aligned with $\underline{c}_s$. Then there exist $\underline{a}'$ and $\underline{c}'$ such that $\underline{a}'$ is aligned with $\underline{c}'$ and $\underline{a}'$ is some descendent of a node with a label equal to $\underline{a}_r$ given by the algorithm.

Proof: Assume $\underline{a}_r$ aligns with $\underline{c}_s$. Consider the correspondence between the next move $\underline{w} = \underline{w}_{s+1}$ along γ and the next motion $\underline{y}$ to be made in α.

Case 1: $\underline{w} = \underline{y}$. In this case, the algorithm gives $\underline{a}' = \underline{a}_r + \underline{w} + \underline{x}$ as its next point along the constructed path, where $\underline{x} = \underline{0}$ unless $\underline{a}_{r+1} + \underline{w} \notin S$. In this case, $\underline{x}$ is a minimal proper sum of basis vectors of the cone based at $\underline{a}_r$. But the fact that these vectors can be effectively reversed is guaranteed by the construction in §6, thus $\underline{x}$ must be effectively reversible. The appropriate $\underline{c}'$ is clearly $\underline{c}_s + \underline{w}$.

Case 2: $\underline{w} \in V_I$. If $\underline{a}_r + \underline{w}$ extends $\underline{a}_r$ in the tree of section 6.2, then setting $\underline{a}' = \underline{a}_r + \underline{w}$ and $\underline{c}' = \underline{c}_s + \underline{w}$ clearly continues the alignment.

If $\underline{a}_r + \underline{w}$ does not extend $\underline{a}_r$ in the tree, there are three possibilities. The first is that $\underline{a}_r = \underline{q}$ for some predecessor $\underline{q}$ of $\underline{a}_r$ in the tree. In this case, merely use $\underline{q}$ instead of $\underline{a}_r$, giving $\underline{a}' = \underline{q} + \underline{w}$ and $\underline{c}' = \underline{c}_s + \underline{w}$.

The second possibility is that $\underline{a}_r$ is a repeating point and $\underline{w}$ affects only those $\underline{u}$-coordinates which can be increased using vectors in the basis of the cone associated with $\underline{a}_r$. Then the use of $\underline{w}$ can be deferred, for it is still in V_I. In this case, $\underline{a}' = \underline{a}_r$ and $\underline{c}' = \underline{c}_s + \underline{w}$.

The final possibility is that $\underline{a}_r + \underline{w} \notin S$. We may also assume that if $\underline{a}_r$ is a repeating point, then there is no $\underline{x}$ in the cone at $\underline{a}_r$ for which $\underline{a}_r + \underline{w} + \underline{x} \in S$, for that possibility was handled in the previous paragraph. Since $\underline{c}_s + \underline{w} \in S$, the construction of this lemma must have omitted some motions that were used to form $\underline{c}_s$, as well as perhaps making motions that were not used to form $\underline{c}_s$. First note that these extra motions could not have caused $\underline{a}_r + \underline{w} \notin S$, because no extra motion that is used decreases any $\underline{u}$-coordinate. Thus the difficulty arises because of omitted motions. For motions to have been omitted from the constructed path, the path must have passed through repeating points, in which case the omitted vectors can affect only those $\underline{u}$-coordinates that could be increased using proper sums of vectors in the basis of the cone associated with the repeating point.

These vectors are still available at $\underline{a}_r$, and since for no $\underline{x}$ in the cone at $\underline{a}_r$ is $\underline{a}_r + \underline{w} + \underline{x} \in S$, it must be the case that there is a subtree based at $\underline{a}_r$ which begins with one of the vectors that was in the basis of the cone associated with a repeating point encountered before $\underline{a}_r$. One of two things will happen: either some successor $\underline{a}''$ of $\underline{a}_r$ with $\underline{a}_r <_w \underline{a}''$ will be a repeating point with a cone in which can be found an $\underline{x}$ such that $\underline{a}'' + \underline{x} + \underline{w} \in S$, and $\underline{a}'' - \underline{a}_r$ can be effectively reversed (because all motions from $\underline{a}_r$ to $\underline{a}''$ were in some cone); or no such successor of $\underline{a}_r$ exists, in which case some successor $\underline{a}''$ (where $\underline{a}_r <_w \underline{a}''$, since only proper sums of basis vectors of cones will be used to move from $\underline{a}_r$ to $\underline{a}''$) will have an edge labelled $\underline{w}$. In either case, $\underline{a}' = \underline{a}''$ and $\underline{c}' = \underline{c}_s$ preserves alignment and leaves a previously considered situation for the construction to continue.

Case 3: $\underline{w} \notin V_I$ and $\underline{w} \neq \underline{y}$. In this case $\underline{w}$ must be the first motion of a loop β from $\sigma(\underline{c}_s)$ to $\sigma(\underline{c}_s)$ which corresponds to a τ-simple loop in α which has already been traversed in α. By the choice of γ, there are no V_I-motions in this loop. By induction on the rank of the τ-simple loop that corresponds to β, we can show that the sum of β is the sum of τ-simple loops which have already been traversed. We simply repeat the argument of case 2 using as $\underline{w}$ the sum of each τ-simple loop. $\square$

Now let δ be the path formed by repeatedly using the construction of lemma 7.4 until α is exhausted, and let $\underline{a}_r$ and $\underline{c}_s$ be the points that result. By lemmas 7.2 and 7.3, these points may be moved to $\underline{a}'$ and $\underline{c}'$ such that $\underline{c}' - \underline{a}' = \underline{c}_j - \underline{c}_s$. The existence of γ shows that there is a path from $\underline{c}_s$ to $\underline{c}_j$ in S using motions in V_I as well as sums of τ-simple loops along δ. This implies that there must be a path from $\underline{a}'$ to $\underline{c}'$ using the vectors associated with $\underline{a}_r$ by the algorithm, all of which lie in $\sigma(\underline{b})$. By induction, the algorithm reports success for this lower-dimensional problem and hence for the original problem.

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