

# Infinite-Duration All-Pay Bidding Games

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## Abstract

A graph game is a two-player zero-sum game in which the players move a token throughout a graph to produce an infinite path, which determines the winner or payoff of the game. In *bidding games*, in each turn, we hold an “auction” (bidding) to determine which player moves the token. The players simultaneously submit bids and the higher bidder moves the token. Several different payment schemes have been considered. In *first-price* bidding, only the higher bidder pays his bid, while in *all-pay* bidding, both players pay their bids. Bidding games were largely studied with variants of first-price bidding. In this work, we study, for the first time, infinite-duration all-pay bidding games, and show that they exhibit the elegant mathematical properties of their first-price counterparts. This is in stark contrast with reachability all-pay bidding games, which were recently shown to be technically much more complicated than reachability first-price bidding games. Another orthogonal distinction between the bidding rules is in the recipient of the payments: in *Richman* bidding, the bids are paid to the other player, and in *poorman* bidding, the bids are paid to the “bank”. We focus on strongly-connected games. We completely solve all-pay Richman games. We first show that deterministic strategies cannot guarantee anything in this model. The main technical challenge is showing that with probabilistic strategies and mean-payoff objectives, the optimal expected payoff under all-pay bidding equals the optimal payoff under first-price bidding. We also construct almost-sure winning strategies for parity games. Under poorman all-pay bidding, in contrast to Richman bidding, deterministic strategies are useful and guarantee a payoff that is only slightly lower than the optimal payoff under first-price poorman bidding. This gives rise to winning strategies in parity games. Our proofs are constructive. For both Richman and poorman bidding we revisit the constructions for first-price bidding and significantly simplify them.

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## 1 Introduction

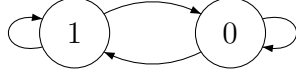
Graph games are two-player zero-sum games with deep connections to foundations of logic [25] as well as numerous practical applications, e.g., verification [15], reactive synthesis [23], and reasoning about multi-agent systems [2]. The game proceeds by placing a token on one of the vertices and allowing the players to move it throughout the graph to produce an infinite trace, which determines the winner or payoff of the game. Traditionally, the players alternate turns when moving the token. In *bidding games* [19, 18], however, the players have



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■ **Figure 1** The mean-payoff game  $\mathcal{G}_{\text{bowtie}}$  with the weights in the vertices.

Max \ Min	0	$b$
0	$B' = B$ Min wins	$B' = B + b$ Min wins
$b$	$B' = B - b$ Max wins	$B' = B$ Min wins

■ **Figure 2** Max's budget updates in four bidding outcomes under AP-Rich.

budgets, and in each turn, we hold an “auction” (bidding) to determine which player moves the token.

Four concrete bidding mechanisms have been previously defined. In all mechanisms, in each turn, both players simultaneously submit a bid that does not exceed their available budget, and the higher bidder moves the token. The mechanisms differ in their payment schemes, which we classify according to two orthogonal properties. First, in *first-price bidding* only the higher bidder pays his bid, whereas in *all-pay bidding*, both players pay their bids. Second, in *Richman bidding* (named after David Richman) the payment(s) are paid to the other player whereas in *poorman bidding* the payments are paid to the “bank”, thus the money is lost. We refer to the mechanisms using abbreviations in  $\{\text{FP}, \text{AP}\} \times \{\text{poor}, \text{Rich}\}$ . For example, FP-poor refers to first-price poorman and AP-Rich refers to all-pay Richman. The central quantity in bidding games is the *budget ratio*; for  $i \in \{1, 2\}$ , when Player  $i$ 's budget is  $B_i$ , then his budget ratio is  $B_i/(B_1 + B_2)$ .

*Reachability* FP-Rich and FP-poor games were studied in [19, 18]; each player has a target vertex, and the game ends once one of the targets is reached. It was shown that each vertex of the game has a *threshold ratio*, which is a necessary and sufficient initial budget for winning the game. Moreover, if a player has a sufficient budget for winning, he has a *deterministic* winning strategy. Only under FP-Rich bidding do reachability games have an intriguing equivalence with a class of games called *random-turn games* [22], in which in each turn, we toss a coin to determine which player moves the token (see more details in [18, 4]).

Reachability AP-poor games were only recently studied [7]. Technically, these games are significantly harder than first-price bidding: e.g., optimal strategies randomize over biddings, and already in very simple games, infinite support is required. Moreover, fundamental questions about this model are open.

Infinite-duration bidding games were studied under FP-Rich [4] and FP-poor [5] bidding. For qualitative objectives, e.g., *parity*, bidding games reduce to reachability first-price bidding games by showing that one of the players wins a strongly-connected game with any positive initial ratio. Things get more interesting with *mean-payoff* objectives, which are quantitative (see Fig. 1 for an example): an infinite play has a payoff which is Player 1's reward and Player 2's cost, thus in these games the players are called Max and Min, respectively. The central question is identifying the optimal payoff Max can guarantee in a mean-payoff game  $\mathcal{G}$  with an initial budget ratio  $r$ .<sup>1</sup>

Mean-payoff first-price bidding exhibit intriguing equivalences with random-turn games. For a bias  $p \in [0, 1]$ , we denote by  $\text{RT}(\mathcal{G}, p)$ , the random-turn game in which in each turn, we toss a coin that selects Max to move with probability  $p$  and Min with probability  $1 - p$ . The game  $\text{RT}(\mathcal{G}, p)$  is a mean-payoff *stochastic game* [13], its *mean-payoff value*, denoted  $\text{MP}(\text{RT}(\mathcal{G}, p))$ , is a well-known concept [24]. Under FP-Rich bidding, the initial budget

<sup>1</sup> Technically,  $c$  is optimal when Max can guarantee a payoff that is greater than  $c - \varepsilon$ , for every  $\varepsilon > 0$ .

ratio does not matter, and the optimal payoff Max can guarantee is  $\text{MP}(\text{RT}(\mathcal{G}, 0.5))$ . Thus, similar to reachability FP-Rich games, mean-payoff FP-Rich games are equivalent to fair random-turn games. Since such an equivalence is not known for reachability FP-poor games, it is surprising that mean-payoff FP-poor games are equivalent to random-turn games and the equivalence is in fact richer than for FP-Rich bidding: with an initial ratio of  $r$ , the optimal payoff Max can guarantee under FP-poor bidding is  $\text{MP}(\text{RT}(\mathcal{G}, r))$ . Thus, the initial ratio matters and coincides with the bias of the coin in the random-turn game. For example, in the game depicted in Fig. 1, with a ratio of  $r = 0.75$ , Max prefers FP-poor over FP-Rich, since with the first he can guarantee a payoff of 0.75 and with the second, only 0.5.

We study, for the first time, infinite-duration all-pay bidding games. The starting point of this research is inspired by the results for FP-poor bidding: the moral of these results is that as we “go to the infinity”, bidding games become cleaner and exhibit a more elegant mathematical structure. We ask: Does this phenomenon also hold for all-pay bidding, where reachability games are highly complex? Would infinite-duration all-pay bidding games reveal a clean mathematical structure like their first-price counterparts? We answer both of these questions positively.

Before surveying our results, we note that in terms of applications, all-pay bidding is often better suited for modelling practical applications than first-price bidding. Applications arise from viewing the players’ budgets as resources with little or no inherent value, e.g., time or strength, and a strategy as a recipe to invest resources with the goal of maximizing the expected utility. In many settings, invested resources are lost, thus all-pay bidding is more appropriate than first-price bidding. Reachability AP-poor games [7] can be seen as a dynamic variant of *Colonel Blotto* games [9], which have been extensively studied. Applications of Colonel Blotto games, which carry over to all-pay bidding games, include political lobbying and campaigning, rent seeking [27], and modelling biological processes [12]. In fact, due to their dynamic nature, bidding games are a better model for these applications.

Another application of bidding games is reasoning about systems in which the scheduler accepts payment in exchange for priority. *Blockchain technology* is one such example. Simplifying the technology, a blockchain is a log of transactions issued by clients and maintained by *miners*, who accept transaction fees from clients in exchange for writing transactions to the blockchain. In *Ethereum*, the blockchain consists of snippets of code (called *smart contracts*). Verification of Ethereum programs is both challenging and important since bugs can cause loss of money (e.g., [11]). Bidding games, and specifically AP-poor games, can model Ethereum programs: we associate players with clients and, as is standard in model checking, we associate the states of the program with the vertices of the graph. AP-poor bidding is the most appropriate bidding mechanism since in Ethereum, the transaction fees are always paid to the miners.

We start by studying all-pay Richman bidding and showing a complete picture for this bidding mechanism. We focus on games played on strongly-connected graphs. A simple argument shows that a deterministic Max strategy cannot guarantee anything under AP-Rich bidding. We turn to study probabilistic strategies, which is technically more challenging. We show that mean-payoff AP-Rich games are essentially equivalent to their first-price counterparts: in a game  $\mathcal{G}$ , with any positive initial ratio, Max’s optimal probabilistic strategy guarantees an expected payoff that equals the value of  $\mathcal{G}$  under FP-Rich, which in turn equals  $\text{MP}(\text{RT}(\mathcal{G}, 0.5))$ ; namely, the value of the fair random-turn game.

In parity AP-Rich games, we show that one of the players has a randomized strategy that guarantees winning with probability 1, and the winner depends only on the highest parity index in the game. Showing, again, an equivalence between parity AP-Rich and FP-Rich

games. Our almost-sure winning strategy in a parity AP-Rich game  $\mathcal{P}$  is obtained by evoking carefully and repeatedly an optimal strategy in a mean-payoff AP-Rich game  $\mathcal{G}$  so that even though the guarantee in  $\mathcal{G}$  is on the expected payoff, we obtain almost-sure guarantees in  $\mathcal{P}$ .

Our solution of mean-payoff AP-Rich games is based on a new significantly-simpler construction of optimal strategies in mean-payoff FP-Rich games. The previous constructions of optimal FP-Rich strategies [4, 6] use the history of the play to determine the next bid; namely, the bid is roughly a function of the difference between the number of Max wins and loses during the play. It is technically not possible to use such a strategy in AP-Rich, as we illustrate in the example below. We devise new FP-Rich strategies that are history independent: the bid depends only the current budget and vertex, hence the name *budget-based* strategies, which are interesting in their own right.

► **Example 1.** We describe a simple Max strategy for the mean-payoff game  $\mathcal{G}_{\text{bowtie}}$  that is depicted in Fig. 1, which achieves an expected payoff of 0.25 under AP-Rich bidding; still not optimal, but better than any deterministic strategy can achieve. We start with the following observation. Suppose Max chooses a bid uniformly random from  $\{0, b\}$ , for some  $b > 0$ . We assume Min wins ties. Thus, knowing Max’s strategy, Min chooses between deterministically bidding 0 or  $b$ . There are four possible outcomes (see Table 2). The “bad” outcomes for Max are  $\langle 0, 0 \rangle$  and  $\langle b, b \rangle$  since Min wins without any budget penalty. The two other outcomes are “good” since they are similar to FP-Rich bidding: Max pays  $b$  for winning and gains  $b$  when losing. To obtain a probabilistic strategy for AP-Rich, we use an optimal strategy  $f_{\text{FP}}$  for FP-Rich as in [4], which achieves a payoff of 0.5 by guaranteeing that Max wins as many biddings as Min (up to a constant). We input to  $f_{\text{FP}}$  the difference between Max wins and losses when restricting to good outcomes to obtain  $b = f_{\text{FP}}(\pi)$ . Intuitively, we expect half the outcomes in a play to be good, out of these,  $f_{\text{FP}}$  guarantees that Max roughly wins half the biddings, for a total payoff of 0.25.

The construction above relies on a classification of outcomes to “good” and “bad”. To obtain an optimal strategy for AP-Rich, we would like decrease to a minimum the probability mass of the “bad” outcomes by bidding uniformly at random in  $[0, b]$ . But then, it is not clear how to use  $f_{\text{FP}}$  to obtain  $b$ . For example, when Min bids 0, both when Max bids  $\varepsilon > 0$  and  $2\varepsilon$ , he wins the bidding. However, Max is “luckier” in the first outcome since he pays less for winning. To overcome this issue, we devise a new optimal budget-based bidding strategy  $f'_{\text{FP}}$  for FP-Rich. Intuitively, the current budget  $B$  reflects precisely how “lucky” Max was in the previous biddings. We construct an optimal strategy under AP-Rich that bids uniformly at random from  $[0, f'_{\text{FP}}(B)]$ . ◀

Finally, we study mean-payoff all-pay poorman bidding. Poorman bidding games tend to be unpredictable and technically more challenging than Richman bidding. AP-poor is no exception. We show that, contrary to AP-Rich, deterministic strategies are useful under AP-poor bidding. Consider a game  $\mathcal{G}$  and suppose Max’s ratio is  $r > 0.5$ . We show that the optimal payoff Max can guarantee with a deterministic strategy in  $\mathcal{G}$  is  $\text{MP}(\text{RT}(\mathcal{G}, (2r-1)/r))$ . Not too far from the optimal payoff under FP-poor, which is  $\text{MP}(\text{RT}(\mathcal{G}, r))$ . The result immediately implies that in a parity AP-poor game, one of the players wins with a ratio greater than 0.5. Here too, we first revisit mean-payoff FP-poor games and construct optimal budget-based strategies, which are significantly simpler than previous constructions. We leave open the problem of finding optimal probabilistic strategies for AP-poor bidding.

**Further related work:** All the results surveyed above highly depend on the fact that the players’ bids can be arbitrarily small. This is a problematic assumption for practical applications. To address this limitation *discrete bidding games* were studied in [14], where the budgets are given in “cents” and the minimal positive bid is one cent. Their motivation came

from recreational play like bidding chess [8, 17]. AP-poor is not suited for discrete bidding since the budgets run out quickly. Discrete AP-Rich bidding has been studied in [21] (we encourage the reader to try playing AP-Rich tic-tac-toe online: <http://tiny.cc/hqbgzoz>). While the issue of tie breaking does not play a key role in continuous bidding, it is important in discrete bidding [1]. Non-zero-sum FP-Rich games were studied in [20].

## 2 Preliminaries

A bidding game is a two-player game that is played on a directed graph  $G = \langle V, E \rangle$ , where  $V$  is a finite set of vertices and  $E \subseteq V \times V$  is a set of directed edges. The *neighbors* of a vertex  $v \in V$  is the set of vertices  $\{u \in V : \langle v, u \rangle \in E\}$ . A *path* in  $G$  is a finite or infinite sequence of vertices  $v_1, v_2, \dots$  such that for every  $i \geq 1$ , we have  $\langle v_i, v_{i+1} \rangle \in E$ . We denote by  $\text{cycles}(G)$ , the set of simple cycles in  $G$ . We call a bidding game *strongly-connected* when the graph  $G$  is strongly-connected. For  $i \in \{1, 2\}$ , we use  $-i$  as short for  $3 - i$  when referring to the “other player”.

### 2.1 Bidding games, strategies, and plays.

The game proceeds as follows. We place a token on one of the vertices in the graph. In each turn, we hold a bidding to determine which player moves the token. Formally, a *strategy* is a recipe for how to play a game. It is a function that, given a finite *history* of the game, prescribes to a player a probability distribution over *actions* to take, where we define these two notions below. A strategy is *deterministic* when it always prescribes one action with probability 1.

Actions in bidding games are pairs of the form  $\langle b, v \rangle$ , where  $b \in \mathbb{R}_{\geq 0}$  is a bid and  $v \in V$  is the vertex to move the token to upon winning the bidding. Suppose Player  $i$ , for  $i \in \{1, 2\}$ , bids  $b_i$  in a bidding. The *winner* of the bidding is Player 1 if  $b_1 > b_2$  and otherwise it is Player 2. That is, we resolve ties in favor of Player 2. This is an arbitrary decision and our results are not affected by the tie-breaking mechanism that is used. A history in a bidding game is  $\pi = \langle v_1, b_{1,1}, b_{1,2}, \dots, \langle v_n, b_{n,1}, b_{n,2} \rangle, v_{n+1} \in (V \times \mathbb{R}_{\geq 0}^2)^* \cdot V$ , where for  $1 \leq j \leq n+1$ , the token is placed on vertex  $v_j$  at round  $j$ , and  $b_{j,i}$ , denotes Player  $i$ ’s bid at round  $j$ , for  $i \in \{1, 2\}$ . An action  $\langle b, v \rangle$  is *legal* following  $\pi$  if (1)  $v$  is a neighbor of  $v_{n+1}$ , thus a history gives rise to a path in the graph, which we denote by  $\text{path}(\pi)$ , and (2) the bid  $b$  does not exceed the available budget as we define below.

For  $i \in \{1, 2\}$ , we denote by  $W_i(\pi), L_i(\pi) \subseteq \{1, \dots, n\}$  the indices in which Player  $i$  wins and loses biddings, respectively. Suppose Player  $i$ ’s initial budget is  $B_i^I \in \mathbb{R}_{\geq 0}$ . The available budget of Player  $i$  following  $\pi$ , denoted  $B_i(\pi)$ , depends on the bidding mechanism, and we consider four bidding mechanisms that differ in the payment scheme in each bidding:

- **First-price Richman:** only the winner pays the loser, thus  $B_i(\pi) = B_i^I + \sum_{j \in L_i(\pi)} b_{j,-i} - \sum_{j \in W_i(\pi)} b_{j,i}$ .
- **First-price poorman:** only the winner pays the “bank”, thus  $B_i(\pi) = B_i^I - \sum_{j \in W_i(\pi)} b_{j,i}$ .
- **All-pay Richman:** both players pay each other, thus  $B_i(\pi) = B_i^I + \sum_{1 \leq j \leq n} b_{j,-i} - \sum_{1 \leq j \leq n} b_{j,i}$ .
- **All-pay poorman:** both players pay the “bank”, thus  $B_i(\pi) = B_i^I - \sum_{1 \leq j \leq n} b_{j,i}$ .

► **Definition 2. (Budget-based strategy).** A strategy  $f$  is called budget-based if it is independent of the history of the play and it is a function from the current vertex and budget to a vertex and a probability distribution on bids.

An initial vertex  $v_0$  and two strategies  $f$  and  $g$  for the players give rise to a probability distribution over infinite *plays*, which we denote by  $\text{dist}(f, g)$ , where we omit the initial vertex since it usually does not play a role in our results. When  $f$  and  $g$  are deterministic, there is a unique play that gets probability 1 in  $\text{dist}(f, g)$ , and we denote it by  $\text{play}(f, g)$ . It is defined inductively as follows. The initial vertex is  $v_0$ . Let  $\pi$  be a finite play that ends in  $v$ , we define its continuation  $\pi'$  as follows. Let  $\langle b_1, u_1 \rangle = f(\pi)$  and  $\langle b_2, u_2 \rangle = g(\pi)$ . Then, if  $b_1 > b_2$ , we define  $\pi' = \pi, \langle v, b_1, b_2 \rangle, u_1$  and otherwise  $\pi' = \pi, \langle v, b_1, b_2 \rangle, u_2$ . Since we consider probabilistic strategies with continuous support, the definition requires us to define a probability space using a *cylinder construction* [3, Theorem 2.7.2], which is technical but standard and we do not present it here. When  $f$  and  $g$  are clear from the context, we omit them and simply write  $\mathbb{P}$  and  $\mathbb{E}$  instead of  $\mathbb{P}_{\text{dist}(f, g)}$  and expectation  $\mathbb{E}_{\text{dist}(f, g)}$ .

## 2.2 Objectives and values

The central quantity in bidding games is the ratio between the players' budgets.

► **Definition 3. (Budget ratio).** For  $i \in \{1, 2\}$ , let  $B_i$  be Player  $i$ 's budget. Then, Player  $i$ 's ratio is  $B_i / (B_1 + B_2)$ . In Richman bidding, since the money only exchanges hands, the sum of budgets is constant and we normalize it to 1. In poorman bidding, we often normalize Player 2's budget to 1.

A qualitative *objective*  $O$  is a set of infinite paths. The central question in qualitative bidding games concerns the necessary and sufficient initial ratio for guaranteeing that an objective is satisfied. The qualitative objectives that we consider are:

- **Reachability:** Player 1 has a target vertex  $t$  and an infinite play is winning iff it visits  $t$ .
- **Parity:** Each vertex is labeled with an index in  $\{1, \dots, d\}$ . An infinite path is winning for Player 1 iff the parity of the maximal index that is visited is odd.

► **Definition 4. (Winning strategies).** Consider a Player 1 strategy  $f$ . We say that  $f$  surely wins if it is deterministic and for every deterministic strategy  $g$ , we have  $\text{play}(f, g) \in O$ . We say that  $f$  almost-surely wins if for every strategy  $g$ , we have  $\mathbb{P}_{\pi \sim \text{dist}(f, g)}[\pi \in O] = 1$ .

The quantitative objective we focus on is *mean-payoff*. Each play in a mean-payoff game has a *payoff*, which is Player 1's reward and Player 2's cost, thus in mean-payoff games, we refer to Player 1 as Max and to Player 2 as Min. The central question in mean-payoff bidding games concerns the optimal payoff a player can guarantee with an initial ratio, called the *mean-payoff value*. Technically, a mean-payoff game is played on a weighted directed graph  $\langle V, E, w \rangle$ , where  $w : V \rightarrow \mathbb{Q}$ . Consider an infinite path  $\eta = v_1, v_2 \dots \in V^\omega$ . For  $n \in \mathbb{N}$ , the prefix of length  $n$  of  $\eta$  is  $\eta^n = v_1, \dots, v_n$ . The *energy* of  $\eta^n$  is  $\text{energy}(\eta^n) = \sum_{1 \leq i < n} w(v_i)$ . We define the payoff of an infinite and finite paths. For an infinite path  $\eta$  and  $n \in \mathbb{N}$ , we define  $\text{payoff}(\eta^n) = \text{energy}(\eta^n) / (n - 1)$  and  $\text{payoff}(\eta) = \liminf_{n \rightarrow \infty} \text{payoff}(\eta^n)$ . Note that the definition gives Min an advantage. For  $i \in \{1, 2\}$ , let  $\Pi_i^D(r)$  and  $\Pi_i^P(r)$  respectively denote the set of deterministic and probabilistic strategies for Player  $i$  with an initial ratio of  $r$ .

► **Definition 5. (Mean-payoff value).** Consider a mean-payoff game  $\mathcal{G}$  and a ratio  $r$ .

- The expected-value of  $\mathcal{G}$  w.r.t.  $r$ , denoted  $\text{eMP}(\mathcal{G}, r)$ , is  $c \in \mathbb{R}$  if for every  $\varepsilon > 0$  and no matter where the game starts, Max has a probabilistic strategy  $f \in \Pi_{\text{Max}}^P(r)$  s.t. for every  $g \in \Pi_{\text{Min}}^P(1-r)$ , we have  $\liminf_{n \rightarrow \infty} \mathbb{E}_{\pi^n \in \text{dist}_n(f, g)}[\text{payoff}(\pi^n)] > c - \varepsilon$ , and dually, there is  $g \in \Pi_{\text{Min}}^P(1-r)$  s.t. for every  $f \in \Pi_{\text{Max}}^P(r)$ , we have  $\liminf_{n \rightarrow \infty} \mathbb{E}_{\pi^n \in \text{dist}_n(f, g)}[\text{payoff}(\pi^n)] < c + \varepsilon$ .



- The sure-value of  $\mathcal{G}$  w.r.t.  $r$ , denoted  $\text{smp}(\mathcal{G}, r)$ , is  $c \in \mathbb{R}$  if Max can deterministically guarantee a payoff of  $c$ : for every  $\varepsilon > 0$  and no matter where the game starts, there is  $f \in \Pi_{\text{Max}}^D(r)$  s.t. for every  $g \in \Pi_{\text{Min}}^D(1-r)$ , we have  $\text{payoff}(\text{play}(f, g)) > c - \varepsilon$ . And, Max cannot do better: for every Max strategy  $f \in \Pi_{\text{Max}}^D(r)$ , there is  $g \in \Pi_{\text{Min}}^D(1-r)$  that guarantees  $\text{payoff}(\text{play}(f, g)) < c + \varepsilon$ .

We sometimes write  $\text{MP}_\gamma(\mathcal{G}, r)$ , for  $\gamma \in \{\text{FP-Rich}, \text{AP-Rich}, \text{FP-poor}, \text{AP-poor}\}$  to highlight the bidding mechanism that is used. We note that in order to show that  $f$  guarantees an expected payoff of  $c$ , it suffices to show that it guarantees an expected payoff of  $c$  against every deterministic strategy  $g$ .

### 2.3 Random-turn games, strengths, and potentials

In this section, we describe the tool that, intuitively, allows us to extend a solution for the game  $\mathcal{G}_{\text{bowtie}}$  (Fig. 1) to a general strongly-connected game. In such games, we need a measure of how “important” it is to choose the successor of a vertex, which we call the *strength*. The higher the strength of  $v \in V$ , the higher a player should bid while the token is in  $v$ . To define strengths, we need several definitions.

A *random-turn* game [22] parameterized by  $p \in [0, 1]$  is played on a graph as a bidding game, only that instead of bidding for moving, we throw a (biased) coin to determine which player moves the token: Player 1 is chosen with probability  $p$  and Player 2 with probability  $1 - p$ . Formally, a random-turn game is a stochastic game [13]. The objective in  $\text{RT}(\mathcal{G}, p)$  matches that of  $\mathcal{G}$ , and we focus on strongly-connected mean-payoff random-turn games. The mean-payoff value of  $\text{RT}(\mathcal{G}, p)$ , denote  $\text{MP}(\text{RT}(\mathcal{G}, p))$ , is defined as the expected payoff that a player can guarantee. It is well-known that it is achievable using *positional* strategies [24]. Such a strategy takes as input the vertex the token is placed on, and returns which vertex to move the token to upon being selected by the coin toss. Let  $\sigma_{\text{Max}}$  and  $\sigma_{\text{Min}}$  respectively denote optimal positional strategies for Max and Min. For  $v \in V$ , we denote  $v^+ = \sigma_{\text{Max}}(v)$  and  $v^- = \sigma_{\text{Min}}(v)$ . The concept of *potentials* was originally defined in the context of the strategy iteration algorithm [16]. We denote the potential of a vertex  $v \in V$  by  $\text{Pot}_p(v)$  and the strength of  $v$  by  $\text{St}_p(v)$ , and we define them as solutions to the following equations.

$$\begin{aligned} \text{Pot}_p(v) &= p \cdot \text{Pot}_p(v^+) + (1 - p) \cdot \text{Pot}_p(v^-) + w(v) - \text{MP}(\text{RT}(\mathcal{G}, p)) \text{ and} \\ \text{St}_p(v) &= p \cdot (1 - p) \cdot (\text{Pot}_p(v^+) - \text{Pot}_p(v^-)) \end{aligned}$$

There are optimal strategies for which  $\text{Pot}_p(v^-) \leq \text{Pot}_p(v') \leq \text{Pot}_p(v^+)$ , for every neighbor  $v'$  of  $v$ , which can be found, for example, using the strategy iteration algorithm. Note that  $\text{St}(v) \geq 0$ , for every  $v \in V$ . We denote the maximal strength by  $S_{\text{max}} = \max_{v \in V} \text{St}_p(v)$  and we assume  $S_{\text{max}} > 0$  otherwise the game is trivial as all weights are equal.

Consider a finite path  $\eta = v_1, \dots, v_n$  in  $\mathcal{G}$ . We intuitively think of  $\eta$  as a play, where for every  $1 \leq i < n$ , the bid of Max in  $v_i$  is  $\text{St}(v_i)$  and he moves to  $v_i^+$  upon winning. Thus, when  $v_{i+1} = v_i^+$ , we think of Max as *investing*  $\text{St}_p(v_i)$  and when  $v_{i+1} \neq v_i^+$ , we think of Min winning the bid thus Max *gains*  $\text{St}_p(v_i)$ . We denote by  $I^+(\eta)$  and  $G^+(\eta)$  the sum of investments and gains, respectively. Note that  $I^+(\eta)$  and  $G^+(\eta)$  are defined w.r.t  $\text{RT}(\mathcal{G}, p)$  and  $p$  will be clear from the context. Recall that the energy of  $\eta$  is  $\text{energy}(\eta) = \sum_{1 \leq i < n} w(v_i)$ . The following lemma connects the energy, potentials, and strengths.

► **Lemma 6.** [6] *Consider a strongly-connected game  $\mathcal{G}$ , and  $p \in [0, 1]$ . For every finite path  $\eta = v_1, \dots, v_n$  in  $\mathcal{G}$ , we have  $\text{Pot}_p(v_1) - \text{Pot}_p(v_n) + (n - 1) \cdot \text{MP}(\text{RT}(\mathcal{G}, p)) \leq \text{energy}(\eta) + G^+(\eta)/(1 - p) - I^+(\eta)/p$ . In particular, when  $p = \nu/(\mu + \nu)$  for  $\nu, \mu > 0$ , there is a constant*

$P = \min_v \text{Pot}_p(v) - \max_v \text{Pot}_p(v)$  such that  $\frac{\nu \cdot \mu}{\nu + \mu} \cdot (\text{energy}(\eta) - P - (n-1) \cdot \text{MP}(\text{RT}(\mathcal{G}, \frac{\nu}{\mu + \nu}))) \geq \mu \cdot I^+(\eta) - \nu \cdot G^+(\eta)$ .

The proof of the following corollary can be found in App. A.

► **Corollary 7.** *Consider a strongly-connected game  $\mathcal{G}$ , let  $\pi$  be an infinite play, let  $\mu, \nu \in \mathbb{R}_{>0}$ , and  $M \in \mathbb{R}$ . If  $\mu \cdot I^+(\pi) - \nu \cdot G^+(\pi) \geq M$  for every finite prefix  $\pi^n$  of  $\pi$ , then  $\text{payoff}(\pi) \geq \text{MP}(\text{RT}(\mathcal{G}, \frac{\nu}{\mu + \nu}))$ .*

### 3 All-Pay Richman Bidding Games

In this section we completely solve infinite-duration all-pay Richman bidding games. We focus on strongly-connected games. We start with the following negative result that shows that deterministic strategies are useless in all-pay Richman bidding.

► **Theorem 8.** *Let  $\mathcal{G}$  be a strongly-connected game. For any initial ratio  $r \in (0, 1)$  and a deterministic Player 1 strategy, Player 2 has a strategy that wins all but a constant number of biddings. Specifically,*

1. *When  $\mathcal{G}$  is a mean-payoff game, the sure-value of  $\mathcal{G}$  is the lowest possible; namely,  $\text{sMP}_{\text{AP-Rich}}(\mathcal{G}, r) = \min_{\eta \in \text{cycles}(\mathcal{G})} \text{MP}(\eta)$ ;*
2. *When  $\mathcal{G}$  is a parity game that has a cycle in  $\text{cycles}(\mathcal{G})$  with an even maximal parity index, then no deterministic Player 1 strategy can guarantee winning with a positive probability.*

**Proof.** Let  $2\varepsilon > 0$  be Player 2's initial budget. Consider a deterministic Player 1 strategy that, for  $j \geq 1$ , bids  $b_{j,1}$  in the  $i$ -th bidding. Player 2 bids  $b_{j,2} = b_{j,1} + 2^{-i} \cdot \varepsilon$  when  $b_{j,2} \leq B_2$ , and otherwise he bids 0. Suppose Player 1's initial budget is  $B_1^I$ . Since whenever Player 2 loses a bidding, Player 1 bids at least  $\varepsilon$ , and the sum of Player 2's payments to Player 1 in a finite play is less than  $\varepsilon$ , Player 1 can win at most  $\lceil B_1^I / \varepsilon \rceil$  times. ◀

#### 3.1 Revisiting mean-payoff first-price Richman games

Our construction of optimal probabilistic strategies in mean-payoff AP-Rich games is based on a new significantly-simpler construction of budget-based strategies for mean-payoff FP-Rich games. Our constructions throughout the paper use the following definition:

► **Definition 9. (Shift function).** *The shift function  $\lambda : (0, 1) \rightarrow (1, +\infty)$  is defined as  $\lambda(x) = -\frac{\log(1-x)}{\log(1+x)}$ .*

The proof of the following lemma can be found in App. C.

► **Lemma 10.** *The shift function has the following properties:*

- *For every  $y \in (1, +\infty)$ , there exists  $x \in (0, 1)$  such that  $\lambda(x) = y$ .*
- *For every  $x \in (0, 1)$ , we have  $(1 - x) = (1 + x)^{-\lambda(x)}$ .*

Previous constructions of optimal strategies for mean-payoff FP-Rich games can be found in [4, 6]. Below we revisit mean-payoff FP-Rich games and devise optimal budget-based strategies. For ease of presentation, we illustrate the construction on the simple game  $\mathcal{G}_{\text{bowtie}}$ , and it can easily be extended to general SCCs, as we note in the proof.

► **Proposition 11.** *Consider the game  $\mathcal{G}_{\text{bowtie}}$ , depicted in Fig. 1. Under first-price Richman bidding, for every initial ratio  $r \in (0, 1)$  and  $\varepsilon > 0$ , Max has a deterministic budget-based strategy that guarantees a payoff of at least  $0.5 - \varepsilon$ , thus  $\text{sMP}_{\text{FP-Rich}}(\mathcal{G}_{\text{bowtie}}, r) = \text{MP}(\text{RT}(\mathcal{G}_{\text{bowtie}}, 0.5)) = 0.5$ .*



**Proof.** Let  $B_{\text{Max}}^I > 0$  be Max's initial budget. It is convenient to set the weights in the game to  $-1$  and  $c$ , where  $c > 1$  depends on  $\varepsilon$ , and setting Max's goal to keep the energy bounded from below by a constant. Then, as the length of the play tends to infinity, the ratio of Max's wins tends to  $\frac{1}{1+c}$ , thus the payoff in the original game is  $0.5 - \varepsilon$ . Let  $\alpha$  such that  $\alpha = \lambda(c)$  (see Lem. 10). Max maintains the invariant that when the energy is  $k \in \mathbb{N}$ , his budget exceeds  $(1 + \alpha)^{-k}$ . We choose an initial  $k^I \in \mathbb{N}$  such that the invariant holds for  $B_{\text{Max}}^I$ , which is possible since  $\lim_{k \rightarrow \infty} (1 + \alpha)^{-k} = 0$ . Max keeps the energy above 1.

Max plays as follows. When Max's budget is  $B$ , he bids  $\alpha \cdot B$ . Note that the strategy is budget based since the bid depends only on the current budget.<sup>2</sup> To conclude the proof, we show that the invariant is maintained and that it guarantees that  $k > 1$ . We distinguish between the two outcomes of a bidding. If Max loses, the energy decreases to  $k - 1$ . Moreover, Min overbids Max, thus Max's new budget  $B'$  is at least  $B + \alpha B \geq \frac{1}{(1+\alpha)^k} + \frac{\alpha}{(1+\alpha)^k} = (1 + \alpha)^{k-1}$ . On the other hand, if Max wins, the energy increases to  $k + c$  and his new budget  $B'$  is at least  $B - \alpha B = \frac{1-\alpha}{(1+\alpha)^k}$ . By plugging in  $\alpha$  and  $c$  in Lemma 10, we obtain  $(1 - \alpha) = (1 + \alpha)^{-c}$ , thus  $B' = (1 + \alpha)^{-(k+c)}$ , as required. To conclude the proof, we have  $k > 1$ , since otherwise, by the invariant, Max's budget would be  $(1 + \alpha)^{-1} > 1$ , which is impossible since the budgets sum up to 1 in Richman bidding. ◀

### 3.2 Mean-payoff all-pay Richman games with probabilistic strategies

This section consists of our main technical contribution on all-pay Richman bidding. We find optimal probabilistic strategies and show that they do not depend on the initial ratio, thereby showing equivalence between mean-payoff all-pay and first-price Richman games.

► **Lemma 12.** *Let  $\mathcal{G}$  be a strongly-connected mean-payoff all-pay Richman bidding game. For every initial ratio  $r \in (0, 1)$  of Max, for every  $\varepsilon > 0$ , Max has a probabilistic budget-based strategy in  $\mathcal{G}$  that guarantees an expected payoff of at least  $\text{MP}(\text{RT}(\mathcal{G}, \frac{1}{2+\varepsilon}))$ .*

**Proof.** Let  $c = 1 + \varepsilon$ , and let  $p = \frac{1}{2+\varepsilon} = \frac{1}{1+c}$ . We find vertex strengths using  $\text{RT}(\mathcal{G}, p)$  as in Section 2.3. Let  $\alpha \in (0, 1)$  s.t.  $\lambda(\alpha) = 1 + \varepsilon$  (see Lemma 10). We define Max's strategy  $f$  as follows.

- When the token is on vertex  $v$  with strength  $s = \text{St}_p(v)$ , and Max's budget is  $B$ , Max bids  $x \sim \text{Unif}[0, \alpha B \frac{s}{S_{\max}}]$ .
- Upon winning, Max moves the token to  $v^+$ .

We claim that no matter which deterministic strategy  $g$  Min chooses, we have  $\liminf_{n \rightarrow \infty} \mathbb{E}_{\pi^n \in \text{dist}_n(f, g)}[\text{payoff}(\pi^n)] > \text{MP}(\text{RT}(\mathcal{G}, p))$ . We assume Wlog that Min bids  $y \in [0, \alpha B \frac{s}{S_{\max}}]$ , since he has the tie-breaking advantage and does not profit from bidding higher.

Proofs in first-price bidding show an invariant between changes in energy and changes in budget (see Prop. 11). For all-pay bidding, however, such an invariant is not possible. Our invariant uses a new component, which we refer to as “luck”. Intuitively, a Max bid of  $x$  is “unlucky” when  $x$  is much larger than  $y$  or when  $x$  is slightly smaller than  $y$ . In the first case, Max pays a lot for winning and in the second, Min pays little for winning. The top-left to bottom-right diagonal in Tab. 2 take these scenarios to the extreme. Dually, when  $x$  is just above  $y$  or way below  $y$ , Max is “lucky”. See the other diagonal in the table. The key idea of the proof is that on average, the unlucky and lucky cases cancel out.

<sup>2</sup> For general SCCs, at a vertex  $v \in V$ , we bid  $\alpha \cdot B \cdot \text{St}_{1/(1+c)}(v)/S_{\max}$ , see more details in Lem. 12.

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To formalize the notion of luck, we define a quantity  $L$  as follows. Initially, we set  $L_0 = \log_{1+\alpha} B$ . We define the change in luck following a bidding

$$\Delta L = \begin{cases} c(\frac{s}{2S_{\max}} + \frac{y-x}{\alpha B}), & \text{if } x > y, \\ (-\frac{s}{2S_{\max}} + \frac{y-x}{\alpha B}), & \text{if } x \leq y. \end{cases} \quad (3.1)$$

We formalize the intuition that lucky and unlucky events cancel out with the following claim, proved in App. D.

**Claim:** Suppose that following a finite play, Max bids  $x \sim \text{Unif}[0, \alpha B \frac{s}{S_{\max}}]$ , then for every Min bid  $y \in [0, \alpha B \frac{s}{S_{\max}}]$ , we have

$$\mathbb{E}_{x \sim \text{Unif}[0, \alpha B \frac{s}{S_{\max}}]}[\Delta L] \geq 0. \quad (3.2)$$

For any finite play  $\pi$ , recall that  $I^+(\pi)$  and  $G^+(\pi)$  are the sum of the strengths of the vertices where Max wins, respectively loses. We prove in App. E that the budget  $B(\pi)$  of Max after the play  $\pi$  satisfies the following invariant, by induction on the length of the play and using Bernoulli's inequality.

**Claim:** For every finite play  $\pi$  coherent with the strategy  $f$  of Max, we have

$$B(\pi) \geq (1 + \alpha)^{L - \frac{1}{2S_{\max}}(c \cdot G^+(\pi) - I^+(\pi))}. \quad (3.3)$$

Since the sum of budgets is 1, we have  $B(\pi) \leq 1 = (1 + \alpha)^0$ . Comparing the exponents, we obtain  $\frac{1}{S_{\max}}(c \cdot G^+(\pi) - I^+(\pi)) \geq 2L$ . Since  $\mathbb{E}[\Delta L] \geq 0$  at each turn and for each  $y$ , we conclude that for any deterministic strategy  $g$  of Min we have  $E[L] \geq L_0$  where the expectation is taken over the probability space of all plays defined by  $f$  and  $g$ . Combining with Lemma 6, by plugging  $\nu = c$  and  $\mu = 1$ , we finally obtain

$$\mathbb{E}[\text{energy}(\pi)] \geq \frac{(c+1)S_{\max}}{c} L_0 + P + (n-1)\text{MP}(\text{RT}(\mathcal{G}, p)). \quad (3.4)$$

Dividing by  $n$ , we obtain  $\liminf_{n \rightarrow \infty} \text{payoff}(\pi^n) \geq \text{MP}(\text{RT}(\mathcal{G}, p))$ , and we are done.  $\blacktriangleleft$

Since  $\text{MP}(\text{RT}(\mathcal{G}, p))$  is continuous in  $p$  [10, 26], it follows from Lemma 12 that  $\text{eMP}(\mathcal{G}, r) \geq \text{MP}(\text{RT}(\mathcal{G}, 0.5))$ , for every  $r > 0$ . Note that since the definition of payoff favors Min, Min can follow Max's strategy above to show that Max cannot achieve a payoff of  $\text{MP}(\text{RT}(\mathcal{G}, 0.5)) + \varepsilon$ . We thus have the following.

► **Theorem 13.** *Let  $\mathcal{G}$  be a strongly-connected mean-payoff all-pay Richman bidding game. The expected mean-payoff value in  $\mathcal{G}$  equals the surely mean-payoff value in  $\mathcal{G}$  under first-price bidding, which in turn equals the mean-payoff value of the random-turn game  $\text{RT}(\mathcal{G}, 0.5)$  in which the player who moves is chosen uniformly at random, thus for every budget ratio  $r \in (0, 1)$ , we have  $\text{eMP}_{\text{AP-Rich}}(\mathcal{G}, r) = \text{sMP}_{\text{FP-Rich}}(\mathcal{G}, r) = \text{MP}(\text{RT}(\mathcal{G}, 0.5))$ .*

### 3.3 Qualitative all-pay Richman games

In this section we consider strongly-connected parity AP-Rich games. Recall that under FP-Rich, one of the players wins with any initial budget, and the winner depends on the parity of the highest parity index. Analogously, we show that in parity AP-Rich games, one of the players wins with probability 1 with any positive initial budget.

Let  $\mathcal{P}$  be a parity game in which the maximal parity index is  $d \in \mathbb{N}$ . We construct a mean-payoff game  $\mathcal{G}$  by setting the weight of a vertex  $v \in V$  to be 1 if the parity of  $v$  is  $d$ , and otherwise  $w(v) = 0$ . The key property of this weight function is that any path  $\eta$  with  $\text{payoff}(\eta) > 0$  must visit a vertex with index  $d$  infinitely many times, and thus satisfies the parity objective. The proof of the following lemma can be found in App. F.

► **Lemma 14.** *Let  $\mathcal{G}$  be a strongly-connected mean-payoff game with non-negative weights and at least one strictly positive weight. Then, for every  $p \in (0, 1)$ , we have  $MP(RT(\mathcal{G}, p)) > 0$ .*

The challenge in using an optimal strategy  $f$  for  $\mathcal{G}$  in  $\mathcal{P}$  is that the guarantees in  $\mathcal{G}$  are on the expected payoff, and they do not immediately imply almost-sure guarantees in  $\mathcal{P}$ . We manage to recover these guarantees by carefully and repeatedly invoking  $f$  to obtain the following result.

► **Theorem 15.** *Let  $\mathcal{G}$  be a strongly-connected parity all-pay Richman bidding game in which the highest parity is  $d \in \mathbb{N}$ . Then, the player corresponding to the parity of  $d$  has a randomized almost-sure winning strategy, no matter where the game starts and with which positive initial ratio.*

**Proof.** We assume WLog that the highest parity index  $d$  is odd and denote by  $S \subseteq V$ , the set of vertices with parity index  $d$ . We construct  $\mathcal{G}$  as in the above. Since  $\mathcal{G}$  is strongly-connected, for each  $r \in (0, 1)$  by Thm. 13, we have  $MP(\mathcal{G}, r) = MP(RT(\mathcal{G}, \frac{1}{2}))$ . Moreover, the proof of Lemma 12 shows that there exists a strategy  $\sigma$  which guarantees a payoff of at least  $MP(RT(\mathcal{G}, \frac{1}{3}))$  independent of the initial vertex and the initial budget. By Lemma 14, we have  $MP(RT(\mathcal{G}, \frac{1}{3})) = 2\delta$  for some  $\delta > 0$ .

Next, we define a function  $\tau : \mathbb{R}_{\geq 0} \rightarrow \mathbb{N}$  that takes an initial ratio  $r$ , and returns an index such that for every  $n \geq \tau(r)$ , for every strategy of the opponent we have  $\mathbb{E}[\text{energy}(\pi^n)] > n \cdot \delta$ . It follows from eq. (3.4), that  $\mathbb{E}[\text{energy}(\pi^n)] \geq (n-1) \cdot MP(RT(\mathcal{G}, \frac{1}{3})) + C_r \geq 2(n-1) \cdot \delta + C_r$ , where  $C_r$  is a constant depending on  $r$ . We define  $\tau(r) = \lceil |C_r|/\delta \rceil + 3$  hence, as required, for every  $n \geq \tau(r)$  we have

$$\mathbb{E}^\sigma[\text{energy}(\pi^n)] \geq 2(n-1) \cdot \delta + C_r \geq n \cdot \delta + (\tau(r) - 2) \cdot \delta + C_r > n \cdot \delta + |C_r| + C_r \geq n \cdot \delta.$$

The proof of the following claim can be found in App. G.

**Claim:** If Player 1 follows the strategy  $\sigma$  with an initial budget  $r$ , then the probability of visiting a vertex in  $S$  during the first  $\tau(r)$  steps is at least  $\delta$ .

We are now ready to define the Player 1 strategy  $f$  that almost-surely guarantees the parity objective. Let  $r \in (0, 1)$  be the initial budget of Player 1. We partition the budget into portions  $r/2, r/4, r/8, \dots$ , and the time over which the game evolves into time periods  $T_1, T_2, T_3, \dots$ , where each  $T_k$  is of length  $\tau(r/2^k)$ . Then for each  $k \in \mathbb{N}$ , the strategy uses portion  $r/2^k$  of the initial budget to follow  $\sigma$  during the time period  $T_k$ .

We claim that  $f$  guarantees almost-surely visiting  $S$  infinitely often. We prove this by contradiction and suppose that there exists a strategy  $g$  for the opponent such that  $S$  is visited only finitely many times with positive probability (over the probability space defined by  $f$  and  $g$ ). This implies that

$$\begin{aligned} 0 &< \mathbb{P}[\exists k \in \mathbb{N} \text{ s.t. } S \text{ not visited after } T_k] = \mathbb{P}[\cup_{k \in \mathbb{N}} \{S \text{ not visited after } T_k\}] \\ &\leq \sum_{k=0}^{\infty} \mathbb{P}[\{S \text{ not visited after } T_k\}], \end{aligned}$$

where the last inequality follows by the union bound. Hence, there exists  $K \in \mathbb{N}$  such that  $\mathbb{P}[\{S \text{ not visited after } T_K\}] > 0$ . This event is equivalent to  $S$  not being visited in time periods  $T_{K+1}, T_{K+2}, \dots$ , which in turn implies that for every  $L \in \mathbb{N}$  the probability of  $S$  not being visited in between time  $T_K$  and  $T_{K+L}$  is also positive. But for a fixed  $L$  this

probability can be factorized as

$$\begin{aligned}
 0 &< \mathbb{P}[\{S \text{ not visited in } T_{K+1}, T_{K+2}, \dots\}] \leq \mathbb{P}[\{S \text{ not visited in } T_{K+1}, T_{K+2}, \dots, T_{K+L}\}] \\
 &= \prod_{l=1}^{L-1} \mathbb{P}[\{S \text{ not visited in } T_{K+l+1}\} \mid \{S \text{ not visited in } T_{K+1}, \dots, T_{K+l}\}] \\
 &\quad \cdot \mathbb{P}[\{S \text{ not visited in } T_{K+1}\}].
 \end{aligned} \tag{3.5}$$

A key point in the proof of the following claim is that the construction of  $f$  is budget-based. As the budget used by the strategy  $f$  at the beginning of time period  $T_{K+l}$  is  $r/2^{K+l}$ , by expanding each of the factors in eq. 3.5 as a sum over the initial vertex of the time period, one can deduce the claim.

**Claim:** For each  $l \in \mathbb{N}$  we have

- $\mathbb{P}[\{S \text{ not visited in } T_{K+l+1}\} \mid \{S \text{ not visited in } T_{K+1}, \dots, T_{K+l}\}] \leq 1 - \delta$ , and
- $\mathbb{P}[\{S \text{ not visited in } T_{K+1}\}] \leq 1 - \delta$ . ◀

## 4 All-Pay Poorman Bidding Games

In this section, we construct optimal deterministic strategies in all-pay poorman games. We find it surprising that, contrary to all-pay Richman bidding games (Theorem 8), deterministic strategies are useful in all-pay poorman bidding games. Throughout this section it is technically convenient to keep Min's budget normalized to 1, thus for a budget  $B$  of Max, we consider the ratio  $B/(B+1)$ .

### 4.1 Revisiting mean-payoff first-price poorman games

The value of mean-payoff first-price poorman games was first identified in [5].

► **Theorem 16** ([5]). *Let  $\mathcal{G}$  be a strongly-connected mean-payoff all-pay poorman bidding game. For every initial ratio  $r = \frac{B_0}{B_0+1} \in (0, 1)$  of Max, we have  $\text{sMP}_{FP\text{-poor}}(\mathcal{G}, r) = \text{MP}(\text{RT}(\mathcal{G}, r))$ .*

We revisit this result and provide an alternative proof by constructing new and significantly simpler optimal budget-based bidding strategies. We then base our solution to all-pay bidding on the budget-based strategy.

► **Lemma 17.** *Let  $\mathcal{G}$  be a strongly-connected mean-payoff all-pay poorman bidding game. For every initial ratio  $r = \frac{B_0}{B_0+1} \in (0, 1)$  of Max, for every  $\varepsilon > 0$ , Max has a deterministic budget-based strategy that guarantees a payoff of  $\text{MP}(\text{RT}(\mathcal{G}, \frac{B_0-\varepsilon}{B_0+1}))$ .*

**Proof.** Let  $B_0 \in \mathbb{R}$  be Max's initial budget. As usual, we keep Min's budget normalized to 1. Let  $\varepsilon > 0$ , and let  $W = B_0 - \varepsilon$ . We design a deterministic strategy for Max that maintains his budget  $B$  above  $W$ . Then, the value of the updated budget  $B'$  of Max after a bidding where Max bids  $b \in [0, 1)$  and Min bids  $a \in [0, 1)$  is as follows: If Max wins the bidding ( $b > a$ ), we have  $B' = B - b$ . The key new insight is that when Max loses the bidding ( $a \geq b$ ), we have

$$B' = \frac{B}{1-a} > \frac{B(1-a^2)}{1-a} = B(1+a) > B + Wa. \tag{4.1}$$

Intuitively, the property states that every cent is  $W$  times more valuable to Min than it is to Max. For example, if Max's budget is 2 and Min's budget is 1, then paying 0.1 is twice as

painful for Min as it is for Max. Roughly, on average, this means that Max wins  $W \sim B_0$  times more biddings than Min, thus he guarantees a payoff close to  $\text{MP}(\text{RT}(\mathcal{G}, \frac{B_0}{B_0+1}))$ .

We now proceed to define formally a budget-based bidding strategy  $f$  for Max that guarantees a payoff of at least  $\text{MP}(\text{RT}(\mathcal{G}, p))$ , where  $p = (B_0 - \varepsilon)/(B_0 + 1)$ . We pick  $\alpha \in (0, 1)$  satisfying  $\lambda(\alpha) = 1 + \varepsilon$  (see Lemma 10). We find vertex strengths using  $\text{RT}(\mathcal{G}, p)$  as in Section 2.3. Let  $N = \max(W, 1) \cdot S_{\max}$ . The strategy  $f$  is defined as follows.

- When the token is placed on a vertex  $v$  with strength  $s = \text{St}_p(v)$  and Max's budget is  $B$ , Max bids  $f(B, s) = \frac{\alpha \cdot s}{N}(B - W)$ .
- Upon winning, Max moves the token to  $v^+$ .

We first show that Max's bidding strategy  $f$  is legal, by showing that we always have  $B > W$ . Indeed, initially, we have  $B_0 > W$ , and then whenever Max loses a bidding his budget increases, and when Max wins a bidding his updated budget is  $B - f(B, s) = B - \frac{\alpha \cdot s}{N}(B - W)$ , which is still greater than  $W$  since  $\frac{\alpha \cdot s}{N} < 1$ .

Next, for any finite play  $\pi$ , let  $H(\pi) = (1 + \varepsilon) \cdot I^+(\pi) - (B_0 - \varepsilon) \cdot G^+(\pi) - N \cdot \log_{1+\alpha}(\varepsilon)$ . Recall that  $I^+(\pi)$  and  $G^+(\pi)$  denote the sum of the strengths of the vertices of  $\pi$  where Max wins, respectively loses. We prove in App. H that the budget  $B(\pi)$  of Max after the play  $\pi$  satisfies the following invariant, using induction on the length of  $\pi$  and Bernoulli's inequality.

**Claim:** For every finite play  $\pi$  coherent with the strategy  $f$  of Max, we have

$$(B(\pi) - W)^N \geq (1 + \alpha)^{-H(\pi)}. \quad (4.2)$$

As a consequence, we show the existence of a bound  $M$  such that  $H(\pi) \geq M$  for every finite play  $\pi$  coherent with  $f$ .

**Claim:** There exists  $M \in \mathbb{R}$  such that for every finite play  $\pi$  coherent with  $f$ , we have

$$(1 + \varepsilon)I^+(\pi) - (B_0 - \varepsilon)G^+(\pi) \geq M. \quad (4.3)$$

The proof of the claim, found in App I, can be summarised as follows. Since the left-hand side of the equation is equal to  $H(\pi) + N \cdot \log_{1+\alpha}(\varepsilon)$ , proving the claim is equivalent to proving a lower bound for  $H(\pi)$ . To do so, we show that  $H(\pi)$  cannot get too low, as past some threshold Equation (4.2) guarantees that the budget  $B(\pi)$  of Max is so high that his next bid according to the strategy  $f$  will be above 1 (the whole budget of Min). Thus Max is guaranteed to win the next bidding, which results in  $H(\pi)$  going back up.

Combining Claim 4.3 with Corollary 7 (plugging  $\nu = B_0 - \varepsilon$  and  $\mu = 1 + \varepsilon$ ), we finally obtain that any infinite play coherent with the strategy  $f$  has a mean-payoff greater than  $\text{MP}(\text{RT}(\mathcal{G}, (B_0 - \varepsilon)/(B_0 + 1)))$ . ◀

## 4.2 Infinite-duration all-pay poorman-bidding games

This section consists of our main technical contribution for all-pay poorman bidding. We first solve mean-payoff games, and then use our solution to solve parity games. Recall that, the value of a strongly-connected mean-payoff game  $\mathcal{G}$  w.r.t. an initial ratio  $r \in (0, 1)$  under FP-poor is  $\text{MP}(\text{RT}(\mathcal{G}, r))$ . The following theorem shows that the sure-value under all-pay poorman bidding is not far from the sure-value under first-price poorman bidding.

► **Theorem 18.** *Let  $\mathcal{G}$  be a strongly-connected mean-payoff all-pay poorman bidding game. For every initial ratio  $r = \frac{B_0}{B_0+1} \in (0, 1)$  of Max, we have  $\text{sMP}_{AP\text{-poor}}(\mathcal{G}, r) = \text{MP}(\text{RT}(\mathcal{G}, \frac{B_0-1}{B_0}))$ .*

Since  $\text{MP}(\text{RT}(\mathcal{G}, p))$  is continuous in  $p$  [10, 26], we demonstrate the theorem by proving the following lemma.

► **Lemma 19.** *Let  $\mathcal{G}$  be a strongly-connected mean-payoff all-pay poorman bidding game. For every initial ratio  $r = \frac{B_0}{B_0+1} \in (0, 1)$  of Max, for every  $\varepsilon > 0$ ,*

1. *Max can deterministically guarantee a payoff of  $MP(RT(\mathcal{G}, \frac{B_0-1-\varepsilon}{B_0}))$ ;*
2. *For any fixed deterministic strategy of Max, Min can guarantee a payoff of  $MP(RT(\mathcal{G}, \frac{B_0-1+\varepsilon}{B_0}))$ .*

**Proof.** In App. J, we show that the strategy that we constructed in Prop. 16 for FP-poor guarantees a payoff of  $MP(RT(\mathcal{G}, \frac{B_0-1-\varepsilon}{B_0}))$  under AP-poor. The analysis needs to be adjusted to AP-poor. The main difference is that the change in Max's budget following losing a bidding to a bid  $a \in (0, 1)$  of Min is now

$$B' = \frac{B-b}{1-a} > \frac{(B-b)(1-a^2)}{1-a} = (B-b)(1+a) > B-b+Wa.$$

For Item 2, consider a deterministic strategy  $f$  of Max. We describe a Min strategy  $g$  as follows. Let  $W = B_0 - 1 + \varepsilon$ , let  $N = B_0 S_{\max}$ , and we choose  $\alpha \in (0, 1)$  satisfying  $\lambda(\alpha) = \frac{1}{1-\varepsilon}$ . For each vertex  $v \in V$  with strength  $s$  and a Max budget  $B$ , Min fixes a threshold  $t(v, B) = \frac{s\alpha}{N}(W + 1 - B)$ . When the token is placed on  $v$ , Min computes the bid  $b$  that Max will bid according to  $f$ . If  $b \leq t(v, B)$ , Min bids  $b$ , wins the bidding and moves to  $v^-$ . If  $b > t(v, B)$ , Min judges that the budget required to win the bidding is not worth it, and stays out by bidding 0. In App. K, we show that  $g$  guarantees a payoff of  $MP(RT(\mathcal{G}, \frac{B_0-1+\varepsilon}{B_0}))$ . Intuitively, technically, the threshold  $t(v, B)$  is where the invariant on Max's budget is (close to) an equality no matter the bid of Min. Thus, if Max bids higher than  $t(v, B)$ , he is paying “too much” for a win and eventually his budget will run out. ◀

In App. L, we show that a construction of a mean-payoff game from a parity game as in Sec. 3.3 together with Lem. 19 provides the following.

► **Corollary 20.** *Consider a strongly-connected parity game  $\mathcal{G}$  in which the highest priority is odd. For every initial ratio  $r > 0.5$ , Player 1 has a surely winning strategy in  $\mathcal{G}$ .*

## 5 Discussion

We study, for the first time, infinite-duration all-pay bidding games. We show a complete picture for all-pay Richman bidding: deterministic strategies cannot guarantee anything, and with probabilistic strategies, AP-Rich coincides with FP-Rich both for mean-payoff and parity objectives. For all-pay poorman bidding, we show that, surprisingly, deterministic strategies can guarantee a payoff not too far from the optimal payoff under FP-poor bidding.

We leave open the problem of classifying the expected value in a mean-payoff AP-poor bidding game  $\mathcal{G}$ . It is tempting to conjecture that as in AP-Rich bidding, the expected value coincides with the sure-value under first-price bidding. However, this might not be true due to the following difference between the bidding rules. When both players bid  $b > 0$  (see the bottom-right cell in Table 2), the situation under AP-Rich bidding is worse than under AP-poor bidding. Indeed, with AP-Rich, Max's updated budget is  $B' = B - b + b = B$ , whereas with AP-poor, his budget is  $B' = (B-b)/(1-b)$ , and we have  $B' > B$  when  $r > 0.5$ . This slight difference might lead to  $\text{eMP}_{\text{AP-poor}}(\mathcal{G}, r) > \text{sMP}_{\text{FP-poor}}(\mathcal{G}, r)$ .

*Taxman* bidding spans the spectrum between Richman and poorman bidding: for a constant  $\tau \in [0, 1]$ , portion  $\tau$  is paid to the other player and portion  $(1 - \tau)$  is paid to the bank. Mean-payoff first-price taxman bidding games were studied in [6] with the motivation to better understand the differences between FP-Rich and FP-poor bidding. The same motivation applies to all-pay bidding, and we find it interesting to study all-pay



taxman bidding. Unlike first-price bidding, with all-pay bidding classifying both the sure- and expected-value for all-pay taxman bidding are interesting problems.

In terms of computational complexity, since solving stochastic games is in NP and coNP, the equivalences between strongly-connected mean-payoff bidding games and random-turn games implies the same upper bound on the problem of finding the sure or expected value of the corresponding bidding game. Finding a lower bound, e.g., that bidding games are harder than general stochastic games, is an open problem also for first-price bidding.

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## A Proof of Corollary 7

Given an infinite play  $\pi$  and  $n \in \mathbb{N}$ , let us denote by  $\pi^n$  the prefix of  $\pi$  of length  $n$ . Suppose that  $\mu \cdot I^+(\pi^n) - \nu \cdot G^+(\pi^n) \geq M$  for every  $n \in \mathbb{N}$ . Then applying Lemma 6 yields

$$\text{payoff}(\pi) = \liminf_{n \rightarrow \infty} \text{payoff}(\pi^n) \geq \liminf_{n \rightarrow \infty} \left( \text{MP}(\text{RT}(\mathcal{G}, \frac{\nu}{\mu + \nu})) + \frac{P}{n-1} + \frac{(\mu + \nu)M}{\mu\nu(n-1)} \right).$$

This concludes the proof, as both fractions tend towards 0 as  $n$  approaches  $\infty$ .

## B Bernoulli's inequality

Bernoulli inequality is stated as follows.

$$1 + rx \geq (1 + x)^r \text{ for all } r \in [0, 1] \text{ and } x \geq -1. \quad (\text{B.1})$$

## C Properties of the shift function

The shift function is surjective as

- (a)  $\lim_{x \rightarrow 0} \lambda(x) = 1$  (l'Hôpital rule);
- (b)  $\lim_{x \rightarrow 1} \lambda(x) = +\infty$ ;
- (c)  $\lambda$  is continuous as its denominator is strictly positive over the domain, and  $\log$  is continuous.

As a consequence, for every  $y \in [1, \infty)$ , there exists  $x \in [0, 1]$  such that  $\lambda(x) = y$ . The second item is a direct consequence of the definition of the shift function.

## D Proof of Equation 3.2

Let  $\beta$  denote  $\alpha B \frac{s}{S_{\max}}$ , let us fix a finite play  $\pi$  and a bid  $y \in [0, \alpha B \frac{s}{S_{\max}}]$  of Min. We obtain:

$$\begin{aligned} \mathbb{E}_{x \sim \text{Unif}[0, \beta]}[\Delta L] &= \frac{1}{\beta} \int_0^\beta \Delta L(x) dx \\ &= \frac{1}{\beta} \left( \int_0^y \Delta L(x) dx + \int_y^\beta \Delta L(x) dx \right) \\ &= \frac{1}{\beta} \left( \int_0^y \left( -\frac{s}{2S_{\max}} + \frac{y-x}{\alpha B} \right) dx + \int_y^\beta c \left( \frac{s}{2S_{\max}} + \frac{y-x}{\alpha B} \right) dx \right) \\ &= \frac{1}{\beta} \left( y \left( -\frac{s}{2S_{\max}} + \frac{y}{\alpha B} \right) - \frac{y^2}{2\alpha B} + c(\beta - y) \left( \frac{s}{2S_{\max}} + \frac{y}{\alpha B} \right) - \frac{c\beta^2}{2\alpha B} + \frac{cy^2}{2\alpha B} \right). \end{aligned}$$

Since  $\beta = \alpha B \frac{s}{S_{\max}}$ , we may substitute  $\frac{s}{S_{\max}} = \frac{\beta}{\alpha B}$  above to get

$$\begin{aligned} \mathbb{E}_{x \sim \text{Unif}[0, \beta]}[\Delta L] &= \frac{1}{\beta} \left( y \left( -\frac{\beta}{2\alpha B} + \frac{y}{\alpha B} \right) - \frac{y^2}{2\alpha B} + c(\beta - y) \left( \frac{\beta}{2\alpha B} + \frac{y}{\alpha B} \right) - \frac{c\beta^2}{2\alpha B} + \frac{cy^2}{2\alpha B} \right) \\ &= \frac{1}{\beta} \left( \frac{y(y-\beta)}{2\alpha B} + \frac{c\beta y}{2\alpha B} - \frac{cy^2}{2\alpha B} \right) \\ &= \frac{1}{\beta} \left( \frac{y(y-\beta)}{2\alpha B} + \frac{cy(\beta-y)}{2\alpha B} \right) \\ &= \frac{(c-1)y(\beta-y)}{2\beta\alpha B} \geq 0, \end{aligned}$$

where the last inequality follows since  $c > 1$  and  $y \in [0, \beta]$ .

### E Proof of Claim 3.3

Consider a finite play  $\pi$  in  $\mathcal{G}$  coherent with the strategy  $f$ . To ease the notation, we write  $H = c \cdot G^+(\pi) - I^+(\pi)$  and omit references to  $\pi$ . We show that the following invariant holds:

$$B \geq (1 + \alpha)^{L - \frac{H}{2S_{\max}}}.$$

We proceed by induction on the number of turns, where the base case follows from our choice of  $L_0$ . Suppose by induction that the equation holds for the values  $B$  and  $H$  obtained after some prefix of  $\pi$ , and that in the next round Min bids  $a$  and Max bids  $b$ . Let  $B' = B + a - b = B + \Delta B$  and  $L' = L + \Delta L$ . We want to show that  $B' \geq (1 + \alpha)^{L' - \frac{H'}{2S_{\max}}}$ , where  $H' = H - s$  if Min wins, and  $H' = H + cs$  if Max wins. By definition of  $\Delta L$ , we get

$$\Delta B = \begin{cases} (\Delta L + \frac{s}{2S_{\max}})\alpha B, & \text{if } a \geq b; \\ (\frac{\Delta L}{c} - \frac{s}{2S_{\max}})\alpha B, & \text{if } a < b. \end{cases} \quad (\text{E.1})$$

To conclude, we distinguish between the case in which Min wins and Max wins:

1. If Min wins the bidding, i.e.,  $a \geq b$ , then  $H' = H - s$ , and we get:

$$\begin{aligned} B' &= B + \Delta B \stackrel{(\text{E.1})}{=} B + (\Delta L + \frac{s}{2S_{\max}})\alpha B \\ &\stackrel{(\text{B.1})}{=} B \cdot (1 + (\Delta L + \frac{s}{2S_{\max}})\alpha) \geq B \cdot (1 + \alpha)^{\Delta L + \frac{s}{2S_{\max}}} \\ &\stackrel{\text{ind. hyp.}}{\geq} (1 + \alpha)^{L + \Delta L - \frac{H - s}{2S_{\max}}} \\ &= (1 + \alpha)^{L' - \frac{H'}{2S_{\max}}}. \end{aligned}$$

Here, Bernoulli's inequality could be used since  $\alpha > -1$  and  $\Delta L + \frac{s}{2S_{\max}} = \frac{a-b}{\alpha B} \in [0, 1]$ .

2. If Max wins the bidding, i.e.,  $a < b$ , then  $H' = H + cs$ , and we get:

$$\begin{aligned} B' &= B + \Delta B \stackrel{(\text{E.1})}{=} B + (\frac{\Delta L}{c} - \frac{s}{2S_{\max}})\alpha B \\ &\stackrel{(\text{B.1})}{=} B \cdot (1 - \alpha(\frac{\Delta L}{c} - \frac{s}{2S_{\max}})) \geq B \cdot (1 - \alpha)^{-\frac{\Delta L}{c} + \frac{s}{2S_{\max}}} \\ &\stackrel{10}{\geq} B \cdot (1 + \alpha)^{\Delta L - \frac{cs}{2S_{\max}}} \stackrel{\text{ind. hyp.}}{\geq} (1 + \alpha)^{L + \Delta L - \frac{H + cs}{2S_{\max}}} \\ &= (1 + \alpha)^{L' - \frac{H'}{2S_{\max}}}. \end{aligned}$$

Here, Bernoulli's inequality could be used since  $-\alpha > -1$  and  $-\frac{\Delta L}{c} + \frac{s}{2S_{\max}} = \frac{b-a}{\alpha B} \in [0, 1]$ .

We also used Lemma 10:  $1 - \alpha = (1 + \alpha)^{-c}$  since  $\lambda(\alpha) = c$ .

### F Proof of Lemma 14

Let  $v_0 \in V$  be a vertex whose weight is positive. Since  $\mathcal{G}$  is strongly-connected, every vertex  $v \in V$  admits a shortest path to  $v_0$ . Fix one such path for each  $v$  and let  $v'$  be the successor of  $v$  along this path (for  $v = v_0$ , let  $v'$  be any of its neighbors). Define the strategy  $\sigma$  for Max via  $\sigma(v) = v'$ , so Max moves the token along the edge  $\langle v, v' \rangle$  upon winning the coin toss. We show that this strategy guarantees a positive mean-payoff with probability 1.

Let  $|V| = n$ . The length of a shortest path from any vertex in  $\mathcal{G}$  to  $v_0$  is at most  $n - 1$ . Thus if Max follows the strategy  $\sigma$  and wins  $n - 1$  consecutive coin tosses, the token will reach  $v_0$  at least once in those  $n - 1$  turns. As the coin tosses are pairwise independent, the probability of Max winning  $n - 1$  times in a row is  $p^{n-1}$ . We will use this observation to show that  $\sigma$  ensures positive mean-payoff.

For an infinite game play  $\pi$ , let  $\pi^m$  be its finite prefix of length  $m$ . Moreover, let  $v_i(\pi)$  denote the  $i$ -th vertex along  $\pi$ . If we write  $m = k \cdot (n - 1) + r$  with  $0 \leq r < n - 1$ , the expected energy of  $\pi^m$  under  $\sigma$  and any fixed strategy of the opponent is

$$\begin{aligned}
\mathbb{E}[\pi^m] &\geq \mathbb{E}^\sigma[\pi^{k \cdot (n-1)}] \geq w(v_0) \cdot \mathbb{E}[\#\{1 \leq j \leq k \cdot (n-1) \mid v_j(\pi) = v_0\}] \\
&= w(v_0) \cdot \sum_{i=0}^{k-1} \mathbb{E}[\#\{1 \leq j \leq n-1 \mid v_{i \cdot (n-1)+j}(\pi) = v_0\}] \\
&\geq w(v_0) \cdot \sum_{i=0}^{k-1} \mathbb{E}[\mathbb{I}(v_{i \cdot (n-1)+j}(\pi) = v_0 \text{ for some } 1 \leq j \leq n-1)] \\
&= w(v_0) \cdot \sum_{i=0}^{k-1} \mathbb{P}[v_{i \cdot (n-1)+j}(\pi) = v_0 \text{ for some } 1 \leq j \leq n-1] \\
&\geq w(v_0) \cdot \sum_{i=0}^{k-1} b^{n-1} = w(v_0) \cdot k \cdot p^{n-1},
\end{aligned}$$

where the last inequality follows from the above observation. Since  $k = (m - r)/(n - 1)$  and  $r < n - 1$ , we have  $k > m/(n - 1) - 1$ . Thus,

$$\liminf_{m \rightarrow \infty} \frac{\mathbb{E}[\pi^m]}{m} \geq w(v_0) \cdot \liminf_{m \rightarrow \infty} \frac{(m/(n-1) - 1) \cdot p^{n-1}}{m} = \frac{w(v_0) \cdot p^{n-1}}{n-1} > 0,$$

and  $\sigma$  ensures positive mean-payoff as claimed.

## G Proof of Theorem 15

**Claim:** If Player 1 follows the strategy  $\sigma$  with an initial budget  $r$ , then the probability of visiting a vertex in  $S$  during the first  $\tau(r)$  steps is at least  $\delta$ .

To see why this holds, note that for every strategy of the opponent we know that  $\mathbb{E}[\text{energy}(\pi^{\tau(r)})] > \tau(r) \cdot \delta$ . Expanding the left hand side,

$$\begin{aligned}
\tau(r) \cdot \delta &< \mathbb{E}[\text{energy}(\pi^{\tau(r)})] \\
&= \mathbb{E}[\text{energy}(\pi^{\tau(r)}) \cdot \mathbb{I}(S \text{ visited in } \tau(r) \text{ steps})] + \mathbb{E}[\text{energy}(\pi^{\tau(r)}) \cdot \mathbb{I}(S \text{ not visited in } \tau(r) \text{ steps})] \\
&= \mathbb{E}[\text{energy}(\pi^{\tau(r)}) \cdot \mathbb{I}(S \text{ visited in } \tau(r) \text{ steps})] + 0 \leq \mathbb{E}[\tau(r) \cdot \mathbb{I}(S \text{ visited in } \tau(r) \text{ steps})] \\
&\leq \tau(r) \cdot \mathbb{P}[S \text{ visited in } \tau(r) \text{ steps}],
\end{aligned}$$

which proves the claim. Here,  $\mathbb{I}(\cdot)$  is the indicator function and the last inequality holds since all weights in the graph are either 0 or 1 and thus  $\text{energy}(\pi^{\tau(r)}) \leq \tau(r)$  for any path  $\pi$ . We refer to this claim as the “important property” in the rest of this proof.

**Claim:** For each  $l \in \mathbb{N}$  we have

- $\mathbb{P}[\{S \text{ not visited in } T_{K+l+1}\} \mid \{S \text{ not visited in } T_{K+1}, \dots, T_{K+l}\}] \leq 1 - \delta,$
- $\mathbb{P}[\{S \text{ not visited in } T_{K+1}\}] \leq 1 - \delta.$

Fix  $1 \leq l \leq L - 1$ . We show that the corresponding conditional probability factor in eq. (3.5) is bounded from above by  $1 - \delta$ . For each  $m \in \mathbb{N}$  let  $v_m(\pi)$  denote the vertex of path  $\pi$  at the end of time period  $T_m$ . To make the notation more succinct, let

$A = \{S \text{ not visited in } T_{K+l+1}\}$  and  $B = \{S \text{ not visited in } T_{K+1}, \dots, T_{K+l}\}$ . By the law of total probability,

$$\begin{aligned} \mathbb{P}[A \mid B] &= \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} = \sum_{v \in V} \frac{\mathbb{P}[A \cap B \cap \{v_{K+l}(\pi) = v\}]}{\mathbb{P}[B]} \\ &= \sum_{v \in V} \frac{\mathbb{P}[A \mid B \cap \{v_{K+l}(\pi) = v\}] \cdot \mathbb{P}[B \cap \{v_{K+l}(\pi) = v\}]}{\mathbb{P}[B]} \\ &= \sum_{v \in V} \mathbb{P}[A \mid B \cap \{v_{K+l}(\pi) = v\}] \cdot \mathbb{P}[\{v_{K+l}(\pi) = v\} \mid B]. \end{aligned} \quad (\text{G.1})$$

Now, note that the strategy  $\sigma$  is budget-based. Since the strategy  $f$  is restarted after the time period  $T_{K+l}$  ends and follows strategy  $\sigma$  using the portion  $r/2^{K+l+1}$  of the initial budget, this implies that the probability of  $S$  being visited in time period  $T_{K+l+1}$  depends only on  $v_{K+l}(\pi)$  and not on the prior game history. Therefore, we conclude that  $\mathbb{P}[A \mid B \cap \{v_{K+l}(\pi) = v\}] = \mathbb{P}[A \mid \{v_{K+l}(\pi) = v\}]$  for each  $v \in V$ . By the previous Claim this probability is at most  $1 - \delta$ . Thus, plugging this back into equation (G.1) we conclude that

$$\begin{aligned} \mathbb{P}[A \mid B] &= \sum_{v \in V} \mathbb{P}[A \mid B \cap \{v_{K+l}(\pi) = v\}] \cdot \mathbb{P}[\{v_{K+l}(\pi) = v\} \mid B] \\ &\leq \sum_{v \in V} (1 - \delta) \cdot \mathbb{P}[\{v_{K+l}(\pi) = v\} \mid B] \\ &= (1 - \delta) \cdot \sum_{v \in V} \mathbb{P}[\{v_{K+l}(\pi) = v\} \mid B] \\ &= (1 - \delta) \cdot \mathbb{P}[v_{K+l}(\pi) \in V \mid B] = 1 - \delta. \end{aligned}$$

This shows that each conditional probability factor in eq. (3.5) is at most  $(1 - \delta)$ . Similarly as above, by using the law of total probability to write

$$\mathbb{P}[\{S \text{ not visited in } T_{K+1}\}] = \sum_{v \in V} \mathbb{P}[\{S \text{ not visited in } T_{K+1}\} \mid \{v_K(\pi) = v\}] \cdot \mathbb{P}[\{v_K(\pi) = v\}]$$

and applying the important property we conclude that  $\mathbb{P}[\{S \text{ not visited in } T_{K+1}\}] < 1 - \delta$ . This concludes the proof.

## **H**    **Proof of Claim 4.2**

Consider a finite play  $\pi$  in  $\mathcal{G}$  coherent with the strategy  $f$ . We show by induction on the length of  $\pi$  that the following invariant holds:

$$(B(\pi) - W)^N \geq (1 + \alpha)^{-H(\pi)}.$$

At the start of the game, the equation holds by definition of  $H$  since  $B_0 - W = \varepsilon$ . For the induction step, suppose that the equation holds for the values  $B$  and  $H$  obtained after prefix of  $\pi$ , and that in the next round Min bids  $a$  and Max bids  $f(B, s)$ . We show that the equation still holds for the updated values  $B'$  and  $H'$ :

1. If  $a \geq f(B, s)$ , then  $H' \geq H - (B_0 - \varepsilon)s$ , and

$$\begin{aligned} (B' - W)^N &\stackrel{(4.1)}{>} (B - W + Wa)^N \geq (B - W + W \cdot f(B, s))^N \\ &= (B - W)^N (1 + \frac{Ws}{N} \alpha)^N \stackrel{\text{ind. hyp.}}{\geq} (1 + \alpha)^{-H} (1 + \frac{Ws}{N} \alpha)^N \\ &\stackrel{(B.1)}{\geq} (1 + \alpha)^{-H+Ws} = (1 + \alpha)^{-H+(B_0-\varepsilon)s} \\ &\geq (1 + \alpha)^{-H'}. \end{aligned}$$



Bernoulli's inequality could be used since  $\alpha > -1$  and  $\frac{Ws}{N} = \min(\frac{Ws}{S_{max}}, \frac{s}{S_{max}}) \in [0, 1]$ . We also used the fact that  $W = B_0 - \varepsilon$ .

2. If  $a < f(B, s)$ , then  $H' = H + (1 + \varepsilon)s$ , and

$$\begin{aligned} (B' - W)^N &= (B - W - f(B, s))^N \\ &= (B - W)^N (1 - \frac{s}{N}\alpha)^N \stackrel{\text{ind. hyp.}}{\geq} (1 + \alpha)^{-H} (1 - \frac{s}{N}\alpha)^N \\ &\stackrel{\text{(B.1)}}{\geq} (1 + \alpha)^{-H} (1 - \alpha)^s \stackrel{10}{=} (1 + \alpha)^{-H - \lambda(\alpha)s} = (1 + \alpha)^{-H - (1 + \varepsilon)s} \\ &\geq (1 + \alpha)^{-H'}. \end{aligned}$$

Bernoulli's inequality could be used since  $-\alpha > -1$  and  $\frac{s}{N} = \min(\frac{s}{S_{max}}, \frac{s}{WS_{max}}) \in [0, 1]$ . We also used the fact that  $\lambda(\alpha) = 1 + \varepsilon$ .

### I Proof of Claim 4.3

We show the existence of a bound  $M \in \mathbb{R}$  such that for every finite play  $\pi$  in  $\mathcal{G}$  coherent with the strategy  $f$ , we have

$$(1 + \varepsilon)I^+(\pi) - (B_0 - \varepsilon)G^+(\pi) \geq M.$$

Note that, by definition of  $H(\pi)$ , we can get the previously mentioned bound by showing a lower bound for  $H(\pi)$ . Let  $S_{\min}$  denote the minimal strength strictly greater than 0 that appears in the game. We show by induction on the length of  $\pi$  that the following holds:

$$H(\pi) \geq \min(-N \cdot \log_{1+\alpha}(\varepsilon), N \cdot \log_{1+\alpha}(\frac{S_{\min}\alpha}{N})) - (B_0 - \varepsilon)S_{\max}. \quad (\text{I.1})$$

The equation holds at the start of the game since  $H$  is initially equal to  $-N \cdot \log_{1+\alpha}(\varepsilon)$ . Now suppose that the equation is satisfied for the value  $H$  obtained at some prefix of the play  $\pi$ , and let us prove that it still holds for the value  $H'$  at the next step. If Max wins the corresponding round, we immediately get the result as  $H' > H$ . Similarly, if the strength of the current vertex equals 0, we immediately get the result as  $H' = H$ . Now let us suppose that Max loses the round, and that the strength  $s$  of the current vertex is strictly greater than 0 (hence  $s \geq S_{\min}$ ). Since Max loses, Min was able to outbid him, hence the bid  $f(B, s)$  of Max was smaller than 1. By definition of  $f$ , this yields  $B - W \leq \frac{N}{s\alpha} \leq \frac{N}{S_{\min}\alpha}$ . Therefore, applying Equation 4.2 yields  $H \geq N \cdot \log_{1+\alpha}(\frac{1}{B-W}) \geq N \cdot \log_{1+\alpha}(\frac{S_{\min}\alpha}{N})$ . Combining this with the fact that  $H' \geq H - (B_0 - \varepsilon)s \geq H - (B_0 - \varepsilon)S_{\max}$  immediately proves Equation I.1.

### J Lemma 19: proof of Item 1

► **Lemma 21.** *Let  $\mathcal{G}$  be a strongly-connected mean-payoff all-pay Poorman bidding game. For every initial ratio  $r = \frac{B_0}{B_0+1} \in (0, 1)$  of Max, for every  $\varepsilon > 0$ , Max can deterministically guarantee a payoff of  $MP(RT(\mathcal{G}, \frac{B_0-1-\varepsilon}{B_0}))$ .*

**Proof.** The proof is nearly identical to the one of Proposition 16. The main difference is that we initially set the value of  $p$  to  $\frac{B_0-1-\varepsilon}{B_0}$  instead of  $\frac{B_0-\varepsilon}{B_0}$ , and the value  $H(\pi)$  is set to  $(1 + \varepsilon) \cdot G^+(\pi) - (B_0 - 1 - \varepsilon) \cdot I^+(\pi) - N \cdot \log_{1+\alpha}(\varepsilon)$ , (whereas for Proposition 16 it was  $(1 + \varepsilon) \cdot G^+(\pi) - (B_0 - \varepsilon) \cdot I^+(\pi) - N \cdot \log_{1+\alpha}(\varepsilon)$ ). Then, using the same deterministic strategy for Max, we show that the invariant  $(B(\pi) - W)^N \geq (1 + \alpha)^{-H(\pi)}$  holds once again, which gives us the desired result through the use of Corollary 7.

**The strategy of Max** To begin, we define a deterministic strategy  $f$  for Max: Let  $W = B_0 - \varepsilon$  and let  $\alpha \in (0, 1)$  satisfying  $\lambda(\alpha) = 1 + \varepsilon$  (see Lemma 10). We find vertex strengths using  $\text{RT}(\mathcal{G}, p)$  as in Section 2.3. Let  $N = \max(W, 1) \cdot S_{\max}$ . The strategy  $f$  is defined as follows.

- If the token is on vertex  $v$  with strength  $s = \text{St}_p(v)$  and Max's budget is  $B$ , he bids  $f(B, s) = \frac{\alpha \cdot s}{N}(B - W)$ .
- Upon winning, Max moves the token to  $v^+$ .

We first show that Max's bidding strategy  $f$  is legal, by showing that we always have  $B > W$ . Indeed, initially, we have  $B_0 > W$ , and when Max bids  $f(B, s)$  against any bid  $a$  of Min, the updated budget of Max is  $B - f(B, s) + a \geq B - f(B, s) = B - \frac{\alpha \cdot s}{N}(B - W)$ , which stays greater than  $W$  since  $\frac{\alpha \cdot s}{N} < 1$ .

**The invariant** We show an invariant between budget and energy, that in turn allows us to prove the Lemma through the use of Corollary 7. For a finite play  $\pi$ , let  $B(\pi)$  denote the budget of Max after  $\pi$ , and let

$$H(\pi) = (1 + \varepsilon) \cdot G^+(\pi) - (B_0 - 1 - \varepsilon) \cdot I^+(\pi) - N \cdot \log_{1+\alpha}(\varepsilon).$$

**Claim:** For every finite play  $\pi$  coherent with the strategy  $f$  of Max, we have

$$(B(\pi) - W)^N \geq (1 + \alpha)^{-H(\pi)}. \quad (\text{J.1})$$

At the start of the game, the equation holds by definition of  $H$  since  $B_0 - W = \varepsilon$ . For the induction step, suppose that the equation holds for the values  $B$  and  $H$  corresponding to some prefix of the play  $\pi$ , and that in the next round Min bids  $a$  and Max bids  $f(B, s)$ . We show that the equation still holds for the updated values  $B'$  and  $H'$ . First, we can approximate the updated budget of Max as follows:

$$B' = \frac{B - f(B, s)}{1 - a} > \frac{(B - f(B, s))(1 - a^2)}{1 - a} = (B - f(B, s))(1 + a) > B - f(B, s) + Wa. \quad (\text{J.2})$$

Then we distinguish between the case where Min wins and Max wins.

1. If  $a \geq f(B, s)$ , then  $H' \geq H - (B_0 - \varepsilon)s$ , and

$$\begin{aligned} (B' - W)^N &\stackrel{(\text{J.2})}{\geq} (B - W - f(B, s) + Wa)^N \geq (B - W + (W - 1) \cdot f(B, s))^N \\ &= (B - W)^N (1 + \frac{(W-1)s}{N} \alpha)^N \stackrel{\text{ind. hyp.}}{\geq} (1 + \alpha)^{-H} (1 + \frac{(W-1)s}{N} \alpha)^N \\ &\stackrel{(\text{B.1})}{\geq} (1 + \alpha)^{-H + (W-1)s} = (1 + \alpha)^{-H + (B_0 - 1 - \varepsilon)s} \\ &\geq (1 + \alpha)^{-H'}. \end{aligned}$$

Bernoulli's inequality could be used since  $\alpha > -1$  and  $\frac{Ws}{N} = \min(\frac{Ws}{S_{\max}}, \frac{s}{S_{\max}}) \in [0, 1]$ .

We also used the fact that  $W = B_0 - \varepsilon$ .

2. If  $f(B, s) > a$ , then  $H' = H + (1 + \varepsilon)s$ , and

$$\begin{aligned} (B' - W)^N &= (B - W - f(B, s) + a)^N \geq (B - W - f(B, s))^N \\ &= (B - W)^N (1 - \frac{s}{N} \alpha)^N \stackrel{\text{ind. hyp.}}{\geq} (1 + \alpha)^{-H} (1 - \frac{s}{N} \alpha)^N \\ &\stackrel{(\text{B.1})}{\geq} (1 + \alpha)^{-H} (1 - \alpha)^s \stackrel{10}{=} (1 + \alpha)^{-H - \lambda(\alpha)s} = (1 + \alpha)^{-H - (1 + \varepsilon)s} \\ &\geq (1 + \alpha)^{-H'}. \end{aligned}$$

Bernoulli's inequality could be used since  $-\alpha > -1$  and  $\frac{s}{N} = \min(\frac{s}{S_{\max}}, \frac{s}{WS_{\max}}) \in [0, 1]$ .

We also used the fact that  $\lambda(\alpha) = 1 + \varepsilon$ .

Finally, since Min's budget is normalized to 1, Max's bids increase with his budget, and he gains budget whenever he wins a bidding, we obtain an upper bound on  $B(\pi)$ . Together with Corollary 7, we obtain the required result.

**Lower bound over the energy** Finally, we show that there exists a lower bound on the value of  $(1 + \varepsilon)I^+(\pi) - (B_0 - 1 - \varepsilon)I^+(\pi)$ , where  $\pi$  ranges over the plays coherent with the strategy  $f$  of Max. Then, Corollary 7 yields that the mean-payoff of any play coherent with the strategy  $f$  is at least  $\text{MP}(\text{RT}(\mathcal{G}, \frac{B_0 - 1 - \varepsilon}{B_0}))$ , which concludes the proof of Lemma 23.

Note that, by definition of  $H$ , we can get the previously mentioned lower bound by showing a lower bound for  $H$ . Let  $S_{\min}$  denote the minimal strength strictly greater than 0 that appears in the game. We prove that

$$H \geq \min(-N \cdot \log_{1+\alpha}(\varepsilon), N \cdot \log_{1+\alpha}(\frac{S_{\min}^\alpha}{N}) - (B_0 - 1 - \varepsilon)S_{\max}). \quad (\text{J.3})$$

This follows from the fact that if  $H$  gets too low, Equation J.1 implies that the budget of Max is so high that his next bid will be above 1, ensuring him a win, and an increase of  $H$ .

Formally, Equation J.3 holds at the start of the game since  $H$  is initially equal to  $-N \cdot \log_{1+\alpha}(\varepsilon)$ . Now suppose that the equation holds for the value  $H$  corresponding to some prefix of  $\pi$ , and let us prove that it still holds for the value  $H'$  at the next step. If Max wins the round, we immediately get the result as  $H' > H$ . Similarly, if the strength of the current vertex equals 0, we immediately get the result as  $H' = H$ . Now let us suppose that Max loses the round, and that the strength  $s$  of the current vertex is strictly greater than 0 (hence  $s \geq S_{\min}$ ). Since Max loses, Min was able to outbid him, hence the bid  $f(B, s)$  of Max was smaller than 1. By definition of  $f$ , this yields  $B - W \leq \frac{N}{s\alpha} \leq \frac{N}{S_{\min}\alpha}$ . Therefore, applying Equation J.1 yields  $H \geq N \cdot \log_{1+\alpha}(\frac{S_{\min}^\alpha}{N})$ . Combining this with the fact that  $H' \geq H - (B_0 - 1 - \varepsilon)s \geq H - (B_0 - 1 - \varepsilon)S_{\max}$  immediately proves Equation J.3. ◀

## K Lemma 19: proof of Item 2

Note that if we switch the signs of all of the weights in a game graph  $\mathcal{G}$ , the roles of Max and Min are swapped. Relying on this observation, we prove a result symmetric to Corollary 7, that we then use to prove the second item of Lemma 19.

Consider a finite path  $\eta = v_1, \dots, v_n$  in a game graph  $\mathcal{G}$ . We denote by  $I^-(\eta)$  the investments of Min (i.e., the sum of the strengths  $\text{St}(v_i)$  of the vertices such that  $v_{i+1} = v_i^-$ ), and we denote by  $G^-(\eta)$  the gains of Min (i.e., the sum of the strengths  $\text{St}(v_i)$  of the vertices such that  $v_{i+1} \neq v_i^-$ ). We can derive the following Corollary from Corollary 7.

► **Corollary 22.** *Consider an infinite play  $\pi$  of a game  $\mathcal{G}$ , and let  $\mu, \nu \in \mathbb{R}^+$  and  $M \in \mathbb{R}$ . If  $\mu \cdot I^-(\pi) - \nu \cdot G^-(\pi) \geq M$  for every finite prefix  $\pi$  of  $\pi^\omega$ , then  $\text{MP}(\pi^\omega) \leq \text{MP}(\text{RT}(\mathcal{G}, \frac{\mu}{\mu+\nu}))$ .*

**Proof.** Let  $p$  denote  $\frac{\mu}{\mu+\nu}$ . Let  $\pi$  be an infinite play of a game  $\mathcal{G}$ , and assume that for every finite prefix  $\pi$  of  $\pi^\omega$  we have  $\mu \cdot I^-(\pi) - \nu \cdot G^-(\pi) \geq M$ . Consider the game graph  $\hat{\mathcal{G}}$  obtained by multiplying all the weights of  $\mathcal{G}$  by  $-1$ . First, note that  $\text{MP}(\text{RT}(\mathcal{G}, p)) = -\text{MP}(\text{RT}(\hat{\mathcal{G}}, 1 - p))$ . Moreover, the optimal strategies of Max and Min are swapped between  $\mathcal{G}$  and  $\hat{\mathcal{G}}$ : For every vertex  $v$  of  $\mathcal{G}$ , if we denote by  $\hat{v}$  the corresponding vertex of  $\hat{\mathcal{G}}$ , then the optimal successor  $v^-$  of  $v$  chosen by Min in  $\text{RT}(\mathcal{G}, p)$  corresponds to the optimal successor  $\hat{v}^+$  of  $\hat{v}$  chosen by Max in  $\text{RT}(\hat{\mathcal{G}}, 1 - p)$ . Therefore, the potential  $\text{Pot}_p(v)$  in  $\text{RT}(\mathcal{G}, p)$  is equal to minus the potential  $\text{Pot}_{1-p}(\hat{v})$  in  $\text{RT}(\hat{\mathcal{G}}, 1 - p)$ , thus the strengths  $\text{St}_p(v)$  and  $\text{St}_{1-p}(\hat{v})$  are equal. As a consequence, if we denote by  $\hat{\pi}^\omega$  the play in  $\hat{\mathcal{G}}$  that corresponds to the same sequence of vertices as  $\pi^\omega$ ,

then for every prefix  $\hat{\pi}$  of  $\hat{\pi}^\omega$ , we have that  $I^+(\hat{\pi}) = I^-(\pi)$  and  $G^+(\hat{\pi}) = G^-(\pi)$ . Thus our initial assumption becomes  $\mu \cdot I^+(\hat{\pi}) - \nu \cdot G^+(\hat{\pi}) \geq M$  for every finite prefix  $\hat{\pi}$  of  $\hat{\pi}^\omega$ , and we can apply Lem. 7 to get

$$\text{MP}(\pi^\omega) = -\text{MP}(\hat{\pi}^\omega) \leq -\text{MP}(\text{RT}(\hat{\mathcal{G}}, \frac{\nu}{\mu + \nu})) = -\text{MP}(\text{RT}(\hat{\mathcal{G}}, 1 - p)) = \text{MP}(\text{RT}(\mathcal{G}, p)).$$

This concludes the proof.  $\blacktriangleleft$

We now prove that  $\text{sMP}(\mathcal{G}, r) \leq \text{MP}(\text{RT}(\mathcal{G}, \frac{B_0 - 1}{B_0}))$ .

► **Lemma 23.** *Let  $\mathcal{G}$  be a strongly-connected mean-payoff all-pay poorman bidding game. For every initial ratio  $r = \frac{B_0}{B_0 + 1} \in (0, 1)$  of Max, for every  $\varepsilon > 0$ , for any fixed deterministic strategy of Max, Min can guarantee a payoff of  $\text{MP}(\text{RT}(\mathcal{G}, \frac{B_0 - 1 + \varepsilon}{B_0}))$ .*

**Proof.** Let  $\varepsilon > 0$ , and let us fix a deterministic strategy  $f$  of Max. To prove the lemma, we construct a strategy  $g$  of Min such that the infinite play  $\pi^\omega$  coherent with  $f$  and  $g$  satisfies  $\text{MP}(\pi^\omega) \leq \text{MP}(\text{RT}(\mathcal{G}, \frac{B_0 - 1 + \varepsilon}{B_0}))$ . The structure of the proof is similar to the one in our previous proofs: We begin by defining a strategy  $g$  for Min. Then, we show that this strategy guarantees an invariant relating the budget of Max with the gains and investments of Min. Finally, we use this invariant to show an upper bound over  $(B_0 - 1 + \varepsilon) \cdot I^-(\pi) - (1 - \varepsilon) \cdot G^-(\pi)$  for every finite play  $\pi$  coherent with the strategies  $f$  and  $g$ . This allows us to conclude through the use of Corollary 22.

**Min's strategy** We formally define Min's strategy and the intuition can be found in the body of the paper. The bidding function  $\beta^-$  of Min is defined as follows. Let  $W = B_0 - 1 + \varepsilon$  and let  $\alpha \in (0, 1)$  satisfying  $\lambda(\alpha) = \frac{1}{1 - \varepsilon}$  (see Lemma 10). We find vertex strengths using  $\text{RT}(\mathcal{G}, \frac{B_0 - 1 + \varepsilon}{B_0})$  as in Section 2.3. Let  $N = \max(W, 1) \cdot S_{\max}$ .

- If the token is on vertex  $v$  with strength  $s = \text{St}_p(v)$  and Max's budget is  $B$ , Min computes the bid  $b$  of Max according to the strategy  $f$ , and acts as follows:
  1. If  $b \leq \frac{s\alpha}{N}(W + 1 - B)$ , then Min bids  $b$ ;
  2. If  $b > \frac{s\alpha}{N}(W + 1 - B)$ , then Min bids 0.
- Upon winning, Min moves the token to  $v^-$ .

By definition of  $\alpha$  and  $N$ , we have that  $\frac{s\alpha}{N} < \frac{1}{W}$  for every strength  $s$  appearing in the game. Therefore, whenever Min wins the round by matching the bid of Max, we can approximate the updated budget  $B'$  of Max as follows.

$$B' = \frac{B - b}{1 - b} = B + \frac{B - 1}{1 - b}b \leq B + \frac{B - 1}{1 - \frac{1}{W}(W + 1 - B)}b = B + Wb. \quad (\text{K.1})$$

As a first consequence of this approximation, we get that Min's bidding strategy is legal: since  $Wb \leq \frac{Ws}{N}\alpha(W + 1 - B) < W + 1 - B$ , we obtain that  $B' < W + 1$  whenever  $B < W + 1$ . Since the initial budget  $B_0$  of Max is smaller than  $W + 1$ , this ensures that the budget of Max will always be smaller than  $W + 1$ , which guarantees that Min will never have to bid more than 1 (his budget) when he has to match the bid of Max according to the strategy  $g$ .

**The invariant** We show an invariant between budget and energy. For a finite play  $\pi$ , let  $B(\pi)$  denote the budget of Max after the play  $\pi$ , and let

$$H(\pi) = (B_0 - 1 - \varepsilon) \cdot I^-(\pi) - (1 - \varepsilon) \cdot G^-(\pi) + N \cdot \log_{1-\alpha}(\varepsilon).$$

**Claim:** For every finite play  $\pi$  coherent with the strategies  $f$  and  $g$ , we have

$$(W + 1 - B(\pi))^N \geq (1 - \alpha)^{H(\pi)}. \quad (\text{K.2})$$

At the start of the game, the equation holds by definition of  $H$  since  $W + 1 - B_0 = \varepsilon$ . For the induction step, suppose that the equation holds for the values  $B$  and  $H$  obtained after a prefix of  $\pi$ , and that in the next round Min bids  $a$  and Max bids  $b$ . We show that the equation still holds for the updated values  $B'$  and  $H'$ :

1. If  $b \leq \frac{s\alpha}{N}(W + 1 - B)$ , then Min matches the bid of Max and wins, hence  $a = b$  and  $H' = H + (B_0 - 1 + \varepsilon)s$ . Then

$$\begin{aligned} (W + 1 - B')^N &\stackrel{(\text{K.1})}{\geq} (W + 1 - B - Wb)^N \geq (W + 1 - B - W \frac{s\alpha}{N}(W + 1 - B))^N \\ &= (W + 1 - B)^N (1 - \frac{Ws}{N}\alpha)^N \stackrel{\text{ind. hyp.}}{\geq} (1 - \alpha)^H (1 - \frac{Ws}{N}\alpha)^N \\ &\stackrel{(\text{B.1})}{\geq} (1 - \alpha)^{H+Ws} = (1 - \alpha)^{H+(B_0-1+\varepsilon)s} \\ &= (1 - \alpha)^{H'}. \end{aligned}$$

Bernoulli's inequality could be used since  $\alpha > -1$  and  $\frac{Ws}{N} = \min(\frac{Ws}{s_{max}}, \frac{s}{s_{max}}) \in [0, 1]$ . We also used the fact that  $W = B_0 - 1 + \varepsilon$ .

2. If  $b > \frac{s\alpha}{N}(W + 1 - B)$ , then Min bids 0 and loses, hence  $a = 0$  and  $H' \geq H - (1 - \varepsilon)s$ . Then

$$\begin{aligned} (W + 1 - B')^N &= (W + 1 - B + b)^N > (W + 1 - B + \frac{s\alpha}{N}(W + 1 - B))^N \\ &= (W + 1 - B)^N (1 + \frac{s}{N}\alpha)^N \stackrel{\text{ind. hyp.}}{\geq} (1 - \alpha)^H (1 + \frac{s}{N}\alpha)^N \\ &\stackrel{(\text{B.1})}{\geq} (1 - \alpha)^H (1 + \alpha)^s \stackrel{10}{=} (1 - \alpha)^{H-s/\lambda(\alpha)} = (1 - \alpha)^{H-(1-\varepsilon)s} \\ &\geq (1 - \alpha)^{H'}. \end{aligned}$$

Bernoulli's inequality could be used since  $-\alpha > -1$  and  $\frac{s}{N} = \min(\frac{s}{s_{max}}, \frac{s}{Ws_{max}}) \in [0, 1]$ . We also used the fact that  $\lambda(\alpha) = \frac{1}{1-\varepsilon}$ .

**Lower bound over the energy** For every finite play  $\pi$  coherent with the strategies  $f$  and  $g$ , as the budget of Max never goes below zero ( $B(\pi) \geq 0$ ), Equation K.2 implies  $H(\pi) \geq N \cdot \log_{1-\alpha}(W + 1) = N \cdot \log_{1-\alpha}(B_0 + \varepsilon)$ . Then, by applying the definition of  $H$  we get

$$(B_0 - 1 + \varepsilon)I^-(\pi) - (1 - \varepsilon)I^-(\pi) = H - N \cdot \log_{1-\alpha}(\varepsilon) \geq N \cdot \log_{1-\alpha}\left(\frac{B_0 + \varepsilon}{\varepsilon}\right).$$

Therefore, by applying Corollary 22, we obtain that the mean-payoff of the infinite play coherent with the strategies  $f$  and  $g$  is at most  $\text{MP}(\text{RT}(\mathcal{G}, \frac{B_0-1+\varepsilon}{B_0}))$ . This concludes the proof of Lemma 23.  $\blacktriangleleft$

**L**    Proof of Corollary 20

Given a strongly-connected parity game  $\mathcal{G}$  in which the highest priority  $d$  is odd, we obtain a mean-payoff game as in Section 3.3: we set the weight of a vertex  $v \in V$  to 1 if its priority is  $d$ , and otherwise  $w(v) = 0$ . By Lem. 14, we have  $\text{MP}(\text{RT}(\mathcal{G}, p)) > 0$ , for every  $p \in (0, 1)$ . A deterministic strategy in  $\mathcal{G}$  that guarantees a positive mean-payoff is a winning strategy for Player 1 in  $\mathcal{P}$  since it guarantees visiting positive-valued vertices in  $\mathcal{G}$  infinitely often, which are the maximal-indexed vertices in  $\mathcal{P}$ . Lem. 19 shows that such a deterministic strategy exists when Player 1's ratio is  $r > 0.5$ , thus the corollary follows.