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# Finite Automata, Definable Sets, and Regular Expressions over $\omega^n$-Tapes*

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The theory of finite automata and regular expressions over a finite alphabet $\Sigma$ is here generalized to infinite tapes $X = X_1 \dots X_k$, where $X_i$ are themselves tapes of length $\omega^n$, for some $n \geqslant 0$. Closure under the usual set-theoretical operations is established, and the equivalence of deterministic and nondeterministic automata is proved. A Kleene-type characterization of the definable sets is given and finite-length generalized regular expressions are developed for finitely denoting these sets. Decision problems are treated; a characterization of regular tapes by multiperiodic sets is specified. Characterization by equivalence relations is discussed while stressing dissimilarities with the finite case.

## 1. INTRODUCTION

Since the pioneering works of Büchi [1] and McNaughton [11], the theory of finite automata on infinite tapes has been developed quite extensively in varied directions, and has found many interesting applications. Thus, Büchi himself, in a series of papers (see, e.g., [2, 3]) has used automata on transfinite tapes to study the monadic second order theory of countable ordinals, while Rabin [13] has developed a powerful and rather complex theory of automata on infinite trees and has used it to solve a great number of decision problems. More recently, Linna [9, 10] and Cohen and Gold [6, 7] have begun building a theory of context-free languages with infinite words and of $\omega$-computations on Turing machines.

In this paper we generalize the classical theory of finite automata and regular expressions over a finite alphabet $\Sigma$ to the case of infinite tapes on this alphabet, specifically to tapes $X$ of the form $X = X_1X_2 \cdots X_k$ where $X_i$ are themselves tapes of length $\omega^n$ on $\Sigma$, for some $n \geqslant 0$. The material presented here is a natural generalization of the theory developed in [5], and relies heavily on some of the techniques given there. Although, as mentioned above, the possibility of building a theory of automata on tapes of denumerable (even nondenumerable) length has been already established by Büchi, our approach here is nevertheless markedly different, and our emphasis is on the finite automata, the regular sets defined by them, and the regular expressions denoting them, per se.

* Most of the results of this paper appear (some of them in slightly different form) in the author's dissertation [4] (in Hebrew; English summary), done under the supervision of Professor M. Rabin at the Hebrew University, Jerusalem.



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Following are some highlights of the results and of their presentation:

1. The classical Rabin–Scott and Kleene theories are obtained from this general theory as the special case $n = 0$.

2. The finiteness of the alphabet and of the set of states is necessary (in our approach at least) not only to ensure the solvability of the decision problem but also to ensure the validity of the "product table" technique, and thus also the Boolean closure properties of the collection of regular sets.

3. Although most of the theorems of the classical theory do not use the finiteness of the alphabet, we cannot apply them directly to the tapes of $\Sigma_n^*$, where $\Sigma_n$ is the (infinite) alphabet consisting of all tapes of length $\omega^n$, since the automata with which we deal are finite entities that have to be defined on the original finite $\Sigma$. Nevertheless, apart from some technical lemmas, the proofs in many cases are almost identical to those given in [5], and a reference is given where appropriate to the corresponding theorem in [5]. Many of the theorems are most easily established by a two-step induction scheme that allows us to pass from "restricted $n$-automata" that acts on tapes from $\Sigma_n$ only for $n = 0$ these are just the letters of $\Sigma$) to automata that act on tapes from $\Sigma_n^*$, and from these to restricted $(n + 1)$-automata.

4. The regular expressions of type $n \geqslant 0$ (which are formulas of finite length) are introduced and used to finitely and effectively represent the regular sets $V \subseteq \Sigma_n^*$. This is achieved by introducing one more binary operation symbol and adequately interpreting it in terms of operations on tapes.

5. Regular sets can be also characterized as finite unions of equivalence classes of some "regular" equivalence relation. There are, however, some dissimilarities with the case $n = 0$; for example, regular $V \subseteq \Sigma_n^*$ exist that are not definable by any $n$-automaton without "superfluous" states.

6. Unlike the case with finite tapes, not for every tape $X \in \Sigma_n$ is the set $\{X\}$ regular. A characterization of such "regular" tapes is given through the notion of an ultimately periodic set of type $n$, and the methods of Elgot and Rabin [8], for deciding the regularity of tapes $X$ in certain cases, are generalized to the infinite case.

Although some familiarity with automata theory is assumed, and a prior knowledge of [5] will be helpful, the paper is self-contained. All the basic terminology is defined in the next section.

## 2. Notation, Terminology, and Basic Definitions

*Notation.* $N$ is the set of nonnegative integers; $i, j, k, l, m, n$, will be variables ranging over $N$; $\omega$ is the first infinite ordinal; $\alpha, \beta, \lambda, \mu, \ldots$ will range over ordinals less than $\omega^\omega$. Let $\alpha < \omega^\omega$; as is well known, $\alpha$ can be written in a unique way as: $\alpha = \beta + \omega^m$ where $m \geqslant 0$ and $\beta \geqslant \omega^m$ or $\beta = 0$. We say that $\alpha$ is of *type m*. For convenience, we agree to say that 0 is of type 0. (Thus $\alpha$ is of type 0 if $\alpha = 0$ or $\alpha$ is a non-limit ordinal.)

In the sequel, $\Sigma$ will always denote a finite nonempty alphabet, and $A$, $B$, $R$, $S$ finite nonempty sets. $\Phi$ is the empty set; $P(A)$ is the power-set of $A$.

*Sequences.* An *$\alpha$-sequence on $S$* is a function $\varphi\colon \alpha \to S$; for $\beta < \alpha$, $\varphi(\beta)$ will be also denoted by $\varphi_\beta$, and the sequence itself by $(\varphi_\beta)_{\beta<\alpha}$, the explicit reference to $\alpha$ being generally omitted. The restriction of $\varphi$ to $\beta < \alpha$ will be denoted by $\varphi \mid \beta$. For an $\omega$-sequence $\varphi$ on $S$, we denote by $I(\varphi)$ the set of elements of $S$ that appear infinitely often in $\varphi$. Let $\varphi$ be an $\alpha$-sequence on $S$, where $\alpha = \beta + \omega^m$ is a limit ordinal ($m > 0$). *The trail representative* of $\varphi$ is the $\omega$-sequence $\psi = t(\varphi)$ defined by $\psi(i) = \varphi(\beta + \omega^{m-1}i)$ for $0 < i < \omega$. The *limit* of $\varphi$, $L(\varphi)$, is $I(t(\varphi))$.

Given $S$, we put $[S]^0 = S$, $[S]^{n+1} = P([S]^n) - \Phi$ and $[S]_0^n = \bigcup_0^n [S]^i$. Elements of $[S]^n$ will be called elements of *type $n$*. For $s \in S$, we also put $\{s\}^0 = s$, $\{s\}^{n+1} = \{\{s\}^n\}$ ($\{s\}^n \in [S]^n$).

Definition 2.1. An $\alpha$-sequence $\varphi$ is a *continuous sequence over $S$* if $\varphi(\beta) \in S$ for every $\beta < \alpha$ of type 0, and $\varphi(\beta) = L(\varphi \mid \beta)$ for every limit $\beta < \alpha$.

It is obvious that such a sequence is uniquely determined by its values on 0 and on nonlimit ordinals; moreover, type $\beta$ = type $\varphi_\beta$ for all $\beta < \alpha$.

Definition 2.2. Given $A_1, \dots, A_k$, let $B = \prod_1^k A_i$ be their Cartesian product. For $n \geqslant 0$ and $1 \leqslant j \leqslant k$ we define the $j$th *projection function* $p_j^n\colon [B]^n \to [A_j]^n$ inductively as follows: for $b = (a_1, \dots, a_k) \in [B]^0$, $p_j^0(b) = a_j$: for $q \in [B]^{n+1}$, $p_j^{n+1}(q) = \{p_j^n(c)\colon c \in q\}$. Obviously $p_j^n(\{b\}^n) = \{a_j\}^n$, and (by induction) $p_j^n(q) \in [A_j]^n$. $p_j^n(q)$ is the $j$th *component* of $q$ and will be denoted simply by $p_j(q)$.

Definition 2.3. Let $\varphi_j$ be continuous $\alpha$-sequences over $A_j$, $1 \leqslant j \leqslant k$. Their *product* $\varphi = \prod_1^k \varphi_j$ is the unique continuous $\alpha$-sequence over $B = \prod_1^k A_j$ that satisfies $\varphi(\lambda) = (\varphi_1(\lambda), \dots, \varphi_k(\lambda))$ for $\lambda < \alpha$ of type 0. Conversely if $\varphi$ is a continuous $\alpha$-sequence over $B$, then its *$j$-component* $p_j(\varphi)$ is the $\alpha$-sequence $\varphi_j$ on $A_j$ defined by $\varphi_j(\lambda) = p_j(\varphi(\lambda))$ for all $\lambda < \alpha$.

The following lemma is basic for all subsequent development:

Lemma 2.4. (a) *If $\varphi$ is a continuous $\alpha$-sequence over $B = \prod_1^k A_j$, then $\varphi_j = p_j(\varphi)$ is a continuous $\alpha$-sequence over $A_j$ for $1 \leqslant j \leqslant k$, and $\varphi = \prod_1^k \varphi_j$.* (b) *If $\varphi_j$ are continuous $\alpha$-sequences over $A_j$ for $1 \leqslant j \leqslant k$ and $\varphi = \prod_1^k \varphi_j$ then $p_j(\varphi) = \varphi_j$.*

*Proof.* By [5, Fact 2.1], if $\alpha = \omega$ then $p_j(I(\varphi)) = I(p_j(\varphi))$. To prove (a) take a limit $\lambda < \alpha$, then $\varphi_j(\lambda) = p_j(\varphi(\lambda)) = p_j(L(\varphi \mid \lambda)) = p_j(I(t(\varphi \mid \lambda))) = I(p_j(t(\varphi \mid \lambda))) = I(t(p_j(\varphi \mid \lambda))) = I(t(\varphi_j \mid \lambda)) = L(\varphi_j \mid \lambda)$. As for (b), since $\varphi$ is a continuous $\alpha$-sequence, so, by (a), is $p_j(\varphi)$. $\varphi_j$ is, however, also a continuous $\alpha$-sequence, and since both sequences have the same values for $\lambda < \alpha$ of type 0, they are equal. Q.E.D.

We remark that the result is true only for finite $A_j$.

*Tapes.* Let $\Sigma$ be a given alphabet. A *tape of length $\alpha$* on $\Sigma$ is an $\alpha$-sequence on $\Sigma$. $X, Y, Z, \dots$ will range over tapes; $\Lambda$ is the unique tape of length zero. The *product* $XY$

of $X$ and $Y$ whose lengths are $\alpha$ and $\beta$, is the tape $Z$ of length $\alpha + \beta$ defined by $Z(\gamma) = X(\gamma)$ for $\gamma < \alpha$, $Z(\alpha + \gamma) = Y(\gamma)$ for $\gamma < \beta$. If $(X_i)$ is an $\omega$-sequence of tapes of lengths $\alpha_i > 0$, then their *$\omega$-product* is the tape $X = \prod_0^\omega X_i$ defined by $X(\gamma) = X_0(\gamma)$ for $\gamma < \alpha_0$, $X(\alpha_{i-1} + \gamma) = X_i(\gamma)$ for $\gamma < \alpha_i$, $i > 0$. If $U$, $V$ are sets of tapes then their *product* $UV$ is $\{XY: X \in U, Y \in V\}$. We also define as usual $V^0 = \{\Lambda\}$, $V^{n+1} = V^n V$, and the *star* of $V$, $V^*$ as $\bigcup_{i \in N} V^i$; the *$\omega$-power* of $V$ is $V^\omega = \{\prod_{i \in N} X_i: X_i \in V, X_i \neq \Lambda\}$. The *limit-product* of $U$ and $V$ (limprod) is $U \odot V = UV^\omega$.

Now given $\Sigma$ and $n \geqslant 0$, we let $\Sigma_n$ be the set of all tapes on $\Sigma$ of length $\omega^n$, so that $\Sigma_n^*$ is the set of all tapes of lengths $\omega^n k$, $0 \leqslant k < \omega$, i.e., $\Sigma_n^* = \{X: X = X_0 \cdots X_k, k \geqslant 0, X_i \in \Sigma_n\} \cup \{\Lambda\}$. Obviously $P(\Sigma_n^*)$ is closed under union, intersection, complementation, product, and star operations, while $P(\Sigma_n)$ is closed under the first three operations only. Given $V \subseteq \Sigma_n^*$, we define the *limit* of $V$ (lim $V$) as the set of all $X \in \Sigma_{n+1}$ that have an infinite number of initial sections in $V$.

*Tables and automata.* Assume a given $\Sigma$ and a fixed $n \geqslant 0$.

An *$n$-table* on $\Sigma$ is a triple $T = \langle S, M, s^* \rangle$ where $S$ is a finite set of *states*, $s^* \in S$ is the *initial* state and $M$ is the *transition* function: $M$: $[S]_0^n \times \Sigma \to S$. The *run* of $T$ on any $X \in \Sigma_n^*$ of length $\alpha$ is the unique *continuous* sequence $(s_\lambda)$ over $S$ of length $\alpha + 1$ that satisfies: $s_0 = s^*$, $s_{\lambda+1} = M(s_\lambda, x_\lambda)$ for $\lambda < \alpha$. Note that if $\lambda$ is of type $k$, then so is $s_\lambda$; so that, e.g., $s_{\omega^k \cdot m} \in [S]^k$ for all $0 \leqslant k \leqslant n$ and all $m \geqslant 1$. $M$ can be naturally extended to a function from $[S]_0^n \times \Sigma_n^*$ to $[S]^n$ by letting $M(q, X)$ be the last state in the unique run of $T$ on $X$ whose first state is $q$.

An *$n$-automaton* $\mathscr{A}$ on $\Sigma$ is a quadruple $\langle S, M, s^*, F \rangle$ where $T = \langle S, M, s^* \rangle$ is an $n$-table and $F \subseteq [S]^n$ is the set of *accepted* states. A tape $X \neq \Lambda$ in $\Sigma_n^*$ is *accepted* by $\mathscr{A}$ if $M(s^*, X) \in F$ ($\Lambda$ is accepted by $\mathscr{A}$ if $\{s^*\}^n \in F$). The set $T(\mathscr{A}) \subseteq \Sigma_n^*$ of all the tapes accepted by $\mathscr{A}$ is the set *defined* by $\mathscr{A}$. A set $V \subseteq \Sigma_n^*$ is *definable* if $V = T(\mathscr{A})$ for some $n$-automaton $\mathscr{A}$. The collection of all these sets will be denoted by $\mathscr{D}_n(\Sigma)$, the explicit reference to $\Sigma$ being generally omitted.

A *restricted* $n$-automaton $\mathscr{A}$ is like an $n$-automaton except that it acts only on tapes $X \in \Sigma_n$ (instead of $X \in \Sigma_n^*$) so that $T(\mathscr{A}) \subseteq \Sigma_n$; thus we can assume in this case that $M$ is defined only on $[S]_0^{n-1} \times \Sigma$. (For $n = 0$, $M$ may be viewed as a function from $\Sigma$ to $S$ and in this case too $T(\mathscr{A}) \subseteq \Sigma_0(=\Sigma)$). The collection of sets defined by restricted $n$-automata will be denoted by $\mathscr{R}\mathscr{D}_n(\Sigma)$.

## 3. Closure Properties

Assume a fixed $\Sigma$ and $n \geqslant 0$.

Definition 3.1. Given the $n$-tables $T_j = \langle S_j, M_j, s_j^* \rangle$ on $\Sigma$, for $1 \leqslant j \leqslant k$, we define the *product* $T = \prod_1^k T_j$, as the table $T = \langle S, M, s^* \rangle$ where $S = \prod_1^k S_j$, $s^* = (s_1^*, \ldots, s_k^*)$ and $M$ is defined "by coordinates," i.e., $M(q, \sigma) = (M_1(p_1(q), \sigma), \ldots, M_k(p_k(q), \sigma))$ for all $q \in [S]_0^n$.

LEMMA 3.2. *Let $T$ and $T_j$ for $1 \leqslant j \leqslant k$ be as in Definition* 3.1, *and $X \in \Sigma_n^*$.* (a) *If $\varphi$ is the run of $T$ on $X$, then $p_j(\varphi) = \varphi_j$ is the run of $T_j$ on $X$.* (b) *If $\varphi_j$ is the run of $T_j$ on $X$ for $1 \leqslant j \leqslant k$ then $\varphi = \prod_1^k \varphi_j$ is the run of $T$ on $X$.* (c) $p_j(M(s^*, X)) = M_j(s_j^*, X)$.

The proof is immediate by Lemma 2.4.

The product technique is needed for the closure under unions; in order to deal with the product and star operations, we adapt the "flag-table" technique of [5] to our case. We first give an informal discussion of the way this flag-table is constructed.

Given an $n$-automaton $\mathscr{A}$ and an $n$-table $T$, let $m$ be the number of $n$-type states of $T$. Take $m + 2$ copies of $T$, connect them in parallel with $\mathscr{A}$, and add a "control unit" $C$. Each one of $T$ copies can be in a *dormant* state in which case it is insensitive to the input and remains so until it is switched On by $C$, or in *active* state in which case it acts according to $T$, and remains so until it is switched Off and made dormant by $C$. In the initial state, $\mathscr{A}$ is in its initial state and all $T$ copies are dormant. At each point $\lambda$ of type $n$, $C$ checks for all active $T$ copies which are in the same state, and switches them Off except for the one with the least index. It then checks the state of $\mathscr{A}$ (*the driving automaton*) to see whether it is an accepted state, and if so, switches On one of the dormant copies of $T$.

The formalization of this structure is now quite obvious (see [5, Def. 3.9]; the treatment in [5] is more general since it deals with non-deterministic tables $T$, too, but this is not needed here). For completeness, we give it in the following:

DEFINITION 3.3. Let $\mathscr{A} = \langle S, M, s^*, F\rangle$ be an $n$-automaton, and $T = \langle R, L, r^*\rangle$ be an $n$-table, where $[R]^n$ has $m$ elements. The *flag-table of $\mathscr{A}$ relative to $T$* is an $n$-table $\mathscr{C} = \langle Q, P, q^*\rangle$ defined as follows. Let $R' = R \cup \{0\}$ where 0 is a new *dormant* state, and extend $L$ on $[R']_0^n$ by putting $L(\{0\}^i, \sigma) = 0$ for all $0 \leqslant i \leqslant n$ and $\sigma \in \Sigma$. (The value of $L$ on other elements of $[R']_0^n - [R]_0^n$ is irrelevant.) Let $G = (R')^k - R^k$ where $k = m + 2$ and put $Q = S \times G$. For $q \in [Q]_0^n$, $p_0(q)$ is the *state of the driving automaton* $\mathscr{A}$ and $p_j(q)$, for $1 \leqslant j \leqslant k$, is the *state of copy $j$* of $T$. The initial state of $\mathscr{C}$ is $q^* = (s^*, 0,..., 0)$. Coming now to the transition function $P$ of $\mathscr{C}$, we define (for $q \in [Q]_0^n$) $P(q, \sigma) = (s, r_1',..., r_k')$ by first putting $s = M(p_0(q), \sigma)$. If $q \in [Q]_0^{n-1}$, then $r_j' = L(p_j(q), \sigma)$ for $1 \leqslant j \leqslant k$. For $q \in [Q]^n$, $r_j' = L(p_j(q), \sigma)$ unless there is some $i < j$ for which $p_i(q) = p_j(q)$, in which case $r_j' = 0$. If, moreover, $p_0(q) \in F$, then we modify the definition by putting $r_{j_0}' = L(r^*, \sigma)$, where $j_0$ is the first index for which $p_0(q) = \{0\}^n$. (It is easy to see that there is always such an index.)

THEOREM 3.4. *$\mathscr{D}_n$ is closed under complementation, union, intersection, product, and star operations. Also if $U \in \mathscr{D}_n$ and $V \in \mathscr{RD}_{n+1}$ then $UV \in \mathscr{RD}_{n+1}$.*

*Proof.* If $U$ is defined by $\langle S, M, s^*, F\rangle$, so is $\Sigma_n^* - U$ by $\langle S, M, s^*, [S]^n - F\rangle$. If $V_j = T(\mathscr{A}_j)$ where $\mathscr{A}_j = \langle T_j ; F_j\rangle$, then $\bigcup_1^k V_j$ is defined by $\langle T; F\rangle$ and $\bigcap_1^k V_j$ by $\langle T; G\rangle$, where $T = \prod_1^k T_j$, $q \in F$ iff $p_j(q) \in F_j$ for some $1 \leqslant j \leqslant k$, $q \in G$ iff $p_j(q) \in G_j$ for all $1 \leqslant j \leqslant k$.

Suppose now $U$ is defined by $\mathscr{A}$, and $V$ by $\mathscr{B} = \langle T; F\rangle$ and assume first that $\Lambda \notin U$, $\Lambda \notin V$. Let $\mathscr{C} = \langle T'; G\rangle$ where $T'$ is the flag-table of $\mathscr{A}$ relatively to $T$ and $G$ is the set of all states of type $n$ of $\mathscr{C}$ which have one projection at least which is in $F$. One easily shows that $T(\mathscr{C}) = UV$. If $\Lambda \in U$ or $\Lambda \in V$, then $UV$ is the union of $T(\mathscr{C})$ with $V$ or with $U$ respectively so that again $UV \in \mathscr{D}_n$. Exactly the same proof holds also for $V \in \mathscr{R}\mathscr{D}_{n+1}$, except that the accepted states of $\mathscr{C}$ are now of course of type $n + 1$.

Turning finally to the star operation, let $U = T(\mathscr{A})$, $\mathscr{A} = \langle T, F\rangle$ and $m$ be the number of $n$-type states of $\mathscr{A}$. We build a slight variation $\mathscr{C}'$ of the flag-table construction as follows: connect $m + 2$ copies of $T$ in parallel (without any "driving automaton") and let the control unit switch On a dormant copy every time that *any* one of $T$ copies is in an accepted state of $\mathscr{A}$. In the initial state, the first copy is in the initial state of $\mathscr{A}$, all other copies are dormant. It is easy to verify that $T(\mathscr{C}') \cup \{\Lambda\} = U^*$ (see [5, Theorem 4.2]). Q.E.D.

THEOREM 3.5. *If* $U \in \mathscr{D}_n$, *then* $\lim U \in \mathscr{R}\mathscr{D}_{n+1}$.

*Proof.* If $U = T(\mathscr{A})$, $\mathscr{A} = \langle S, M, s^*F\rangle$, then $\lim U$ is defined by $\mathscr{B} = \langle S, M, s^*, G\rangle$ where $G \subseteq [S]^{n+1}$ and $q \in G$ iff $q \cap F \neq \Phi$. Q.E.D.

Turning now to the $\omega$-power operation, we first state:

THEOREM 3.6. *For every* $U \in \mathscr{D}_n$ *we can effectively find some* $\tilde{U} \in \mathscr{D}_n$ *such that* $U^\omega = U^*(\lim \tilde{U})$.

*Proof.* This was proved in [5, Lemma 5.2] for the case $n = 0$. Since, however, the finiteness of the original alphabet $\Sigma$ was not used in the proof, we can apply it to our case too, in the following way. Take $\Sigma_n$ as the new alphabet $\Sigma$. If $U$ was defined by $\mathscr{A} = \langle S, M, s^*, F\rangle$, then consider the derived 0-automaton on $\Sigma_n$, $\mathscr{B} = \langle [S]^n \cup \{s^*\}, M', s^*, F\rangle$ where $M'(q, X) = M(q, X)$ for all $q \in [S]^n \cup \{s^*\}$ and all $X \in \Sigma_n$. The set $\tilde{U}$ can be then defined, as in [5], by applying set-theoretical operations dealt with in 3.4 above, on sets from $\mathscr{D}_n$, so that $\tilde{U} \in \mathscr{D}_n$ too. Q.E.D.

THEOREM 3.7. *If* $U \in \mathscr{D}_n$ *then* $U^\omega \in \mathscr{D}_{n+1}$.

*Proof.* Let $\tilde{U}$ be such that $U^\omega = U^*(\lim \tilde{U})$; then $U^* \in \mathscr{D}_n$ (by 3.4), $\lim \tilde{U} \in \mathscr{R}\mathscr{D}_{n+1}$ (by 3.5), and so $U^\omega \in \mathscr{R}\mathscr{D}_{n+1}$ (by 3.4). Q.E.D.

THEOREM 3.8. $\mathscr{R}\mathscr{D}_n = \mathscr{D}_n \cap P(\Sigma_n)$.

*Proof.* That $\mathscr{D}_n \cap P(\Sigma_n) \subseteq \mathscr{R}\mathscr{D}_n$ is obvious, since any $n$-automaton is easily transformed into a restricted one, by restricting its transition function. On the other hand, if $U \in \mathscr{R}\mathscr{D}_n$, $U = T(\mathscr{A})$ where $\mathscr{A} = \langle S, M, s^*, F\rangle$ is a restricted $n$-automaton then take $\mathscr{A}' = \langle S \cup \{f\}, M', s^*, F\rangle$, where $f \notin S$ and $M'(q, \sigma) = M(q, \sigma)$ for $q \in [S]_0^{n-1}$, $M'(q, \sigma) = f$ otherwise. Clearly $U = T(\mathscr{A}')$, so that $U \in \mathscr{D}_n$ (and of course $U \subseteq \Sigma_n$).

COROLLARY 3.9. $\mathscr{R}\mathscr{D}_n$ *is closed under union, intersection, complementation (relatively to* $\Sigma_n$*), and left product by elements of* $\mathscr{D}_n$. *It also contains all limits and* $\omega$*-powers of sets in* $\mathscr{D}_n$.

We remark that all the closure properties dealt with above are effective; that is, given the automata (which are finite entities) that define the constituent sets, we can effectively construct the required automata that define the composite ones.

## 4. Regular Sets and Regular Expressions

THEOREM 4.1. *$\mathscr{D}_n$ is the closure of $\mathscr{R}\mathscr{D}_n$ under union, product, and star operations (thus, $\mathscr{D}_n$ is the Kleene closure of $\mathscr{R}\mathscr{D}_n$).*

*Proof.* Let $C(\mathscr{R}\mathscr{D}_n)$ be the closure of $\mathscr{R}\mathscr{D}_n$ under union, product, and star operations. Since $\mathscr{R}\mathscr{D}_n \subseteq \mathscr{D}_n$, and $\mathscr{D}_n$ is closed under these operations, we obviously have $C(\mathscr{R}\mathscr{D}_n) \subseteq \mathscr{D}_n$. Suppose now $V \in \mathscr{D}_n$, $V = T(\mathscr{A})$, $\mathscr{A} = \langle S, M, s^*, F \rangle$, $Q = [S]^n \cup \{s^*\} = \{q_1 ,..., q_m\}$ where $q_1 = s^*$. For $1 \leqslant i, j \leqslant m$, let $V_{ij}^0 = \{X \in \Sigma_n : M(q_i, X) = q_j\}$ and for $1 \leqslant k \leqslant m$, put $V_{ij}^k = V_{ij}^{k-1} \cup V_{ik}^{k-1}(V_{kk}^{k-1})^* V_{kj}^{k-1}$. It is immediate by induction on $k$, that $V_{ij}^k$ consists of all tapes $X \in \Sigma_n^*$ such that $M(q_i, X) = q_j$ and all states of type $n$ in the run of $\mathscr{A}$ on $X$ beginning with $q_i$ are from $\{q_1 ,..., q_k\}$. By induction, $V_{ij}^k \in C(\mathscr{R}\mathscr{D}_n)$ for all $1 \leqslant k \leqslant m$; if $F = \{q_{i_1} ,..., q_{i_r}\}$ we immediately get $V = \bigcup_1^r V_{1i_j}^m$, which shows that $V \in C(\mathscr{R}\mathscr{D}_n)$. (If $\Lambda \in V$, we add to this union the set $\{\Lambda\} = \Phi^*$, where of course $\Phi \in \mathscr{R}\mathscr{D}_n$). Q.E.D.

We remark that this theorem is true for $n = 0$ too, where $\mathscr{R}\mathscr{D}_0 = P(\Sigma)$.

THEOREM 4.2. *$\mathscr{R}\mathscr{D}_{n+1}$ is the closure under union of the* limprod *of sets in $\mathscr{D}_n$.*

*Proof.* We already know that the limprod of sets in $\mathscr{D}_n$ are in $\mathscr{R}\mathscr{D}_{n+1}$, and that $\mathscr{R}\mathscr{D}_{n+1}$ is closed under union. Suppose now $V \in \mathscr{R}\mathscr{D}_{n+1}$, $V = T(\mathscr{A})$, $\mathscr{A} = \langle S, M, s^*, F \rangle$, where we can assume without loss of generality that $F$ contains only one $(n+1)$-type state: $q$. Let $q = \{r_1 ,..., r_k\}$ $(r_i \in [S]^n)$ and put $\mathscr{B} = \langle S, M, s^*, r_1 \rangle$, and for all $1 \leqslant i \leqslant k$, $\mathscr{C}_i = \langle S', M_i', t, r_{i+1} \rangle$ $(r_{k+1} = r_1)$ where $S' = S \cup \{t, f\}$, $M_i'(t, \sigma) = M(r_i, \sigma)$, $M_i'(q, \sigma) = M(q, \sigma)$ for all $q \in [S]_0^{n-1}$, $M_i'(r_j, \sigma) = M(r_j, \sigma)$ for all $1 \leqslant j \leqslant k$, $M_i'(q, \sigma) = f$ for all other $q \in [S']_0^n$. Putting $W = T(\mathscr{B})$, $V_i = T(\mathscr{C}_i)$, it is clear that $V = W \odot (V_1 V_2 \cdots V_k)$. Q.E.D.

DEFINITION 4.3. Given a finite alphabet $\Sigma$, we define inductively the collection $\mathscr{K}_n$ of *n-type regular sets* on $\Sigma$ $(n \geqslant 0)$ as follows:

1. $\Phi \in \mathscr{K}_n$ for all $n \geqslant 0$.
2. $\{\sigma\} \in \mathscr{K}_0$ for all $\sigma \in \Sigma$.
3. If $U, V \in \mathscr{K}_n$, then so are $U \cup V$, $U \cdot V$, and $V^*$.
4. If $U, V \in \mathscr{K}_n$, then $U \odot V \in \mathscr{K}_{n+1}$.

The intersection $\mathscr{K}_n \cap P(\Sigma_n)$ (that is, the collection of all sets in $\mathscr{K}_n$ that contain tapes from $\Sigma_n$) will be denoted by $\mathscr{R}\mathscr{K}_n$. We immediately remark that $\mathscr{K}_n$ is the closure of $\mathscr{RK}_n$ under union, product and star operations, while $\mathscr{RK}_{n+1}$ is the closure under union of the limprod of elements from $\mathscr{RK}_n$.

THEOREM 4.4. *$\mathscr{RK}_n = \mathscr{RD}_n$, $\mathscr{K}_n = \mathscr{D}_n$ for all $n \geqslant 0$.*

*Proof.* By a "two-step" induction on $n$, which is typical of induction proofs in this context. First we remark that $\mathscr{RK}_0 = P(\Sigma) = \mathscr{RD}_0$. Assuming that $\mathscr{RK}_n = \mathscr{RD}_n$, we immediately get by the preceding remark and Theorem 4.1 that $\mathscr{K}_n = \mathscr{D}_n$. But then, using again the same remark and Theorem 4.2 we get that $\mathscr{RK}_{n+1} = \mathscr{RD}_{n+1}$.
Q.E.D.

This Kleene characterization of sets definable by $n$-type automata allows us to give still another finite representation for such sets through a generalization of the mechanism of regular expressions, which we now define.

DEFINITION 4.5. Given a finite alphabet $\Sigma$, let $\bar{\Sigma}$ be the finite set of symbols given by $\bar{\Sigma} = \{\bar{\sigma}: \sigma \in \Sigma\} \cup \{\mp, \bar{\cdot}, \overline{\odot}, \bar{*}, (\ , \ ), \bar{\Phi}\}$. We define the collection $\mathscr{E}_n$ of *$n$-type regular expressions on $\Sigma$* (which are in fact *finite strings* from $\bar{\Sigma}$), inductively as follows:

1. $\bar{\sigma} \in \mathscr{E}_0$.
2. $\bar{\Phi} \in \mathscr{E}_n$ for all $n \geqslant 0$.
3. If $\varphi_1, \varphi_2 \in \mathscr{E}_n$, then so are $(\varphi_1 \mp \varphi_2)$, $(\varphi_1 \bar{\cdot} \varphi_2)$ and $(\varphi_1 \bar{*})$.
4. If $\varphi_1, \varphi_2 \in \mathscr{E}_n$, then $(\varphi_1 \overline{\odot} \varphi_2) \in \mathscr{E}_{n+1}$.

Needless to say the 0-type regular expressions are just the usual Kleene regular expressions.

If $\varphi$ is an $n$-type regular expression on $\Sigma$, then the set $|\varphi|$ *denoted* by $\varphi$ is defined inductively as follows:

$$|\bar{\sigma}| = \{\sigma\};\ |\bar{\Phi}| = \Phi;\ |(\varphi_1 \mp \varphi_2)| = |\varphi_1| \cup |\varphi_2|;\ |(\varphi_1 \bar{\cdot} \varphi_2)| = |\varphi_1| \cdot |\varphi_2|;$$
$$|(\varphi_1 \bar{*})| = |\varphi_1|^*;\ |(\varphi_1 \overline{\odot} \varphi_2)| = |\varphi_1| \odot |\varphi_2|.$$

All the preceding development proves then the following:

THEOREM 4.6. *A set $V \subseteq \Sigma_n^*$ is definable by an $n$-type automaton iff it is denoted by an $n$-type regular expression. Moreover, the transfer from one formalism to the other can be done effectively.*

*Remark.* Using Theorem 3.6, we see that the preceding theorem remains true also when we interpret $|\varphi_1 \overline{\odot} \varphi_2|$ as $|\varphi_1| \cdot \lim |\varphi_2|$.

## 5. CLOSURE UNDER PROJECTION AND NONDETERMINISTIC AUTOMATA

Closure under projection is one of the main topics of interest when dealing with automata, since it is closely tied to existential quantification on the one hand (when everything is interpreted in terms of an appropriate monadic second-order language) and to nondeterministic automata and their equivalence to the deterministic ones on the other hand.

DEFINITION 5.1. Given two alphabets $\Sigma$ and $\Sigma'$, a function $f$ from $\Sigma$ to $\Sigma'$ is called a *projection*. Such a function can be naturally extended to the collection of corresponding tapes by defining $f(X) = (f(x_\lambda))$ for any tape $X$ on $\Sigma$, and to sets of such tapes by defining $f(V) = \{f(X): X \in V\}$.

It is obvious that $f$ is preserved by concatenation, that is $f(XY) = f(X) \cdot f(Y)$. From which the following immediately follows: $f(U \cup V) = f(U) \cup f(V)$, $f(UV) = f(U) \cdot f(V)$; $f(U^*) = f(U)^*$; $f(U^\omega) = (f(U))^\omega$; $f(U \odot V) = f(U) \odot f(V)$.

THEOREM 5.2. *If $V \in \mathscr{D}_n(\Sigma)$ and $f: \Sigma \to \Sigma'$ is a projection, then $f(V) \in \mathscr{D}_n(\Sigma')$.*

*Proof.* The theorem is trivial for $\mathscr{RD}_0$. By a two-step induction, and using the characterization theorems of the preceding sections and the remark above, we immediately conclude that if it is true for $\mathscr{RD}_n$ then it is also true for $\mathscr{D}_n$, and if it is true for $\mathscr{D}_n$ then it is also true for $\mathscr{RD}_{n+1}$. Q.E.D.

We turn now to nondeterministic automata.

An *n-type nondeterministic* automaton $\mathscr{A} = \langle S, M, s^*, F\rangle$ is similar to a deterministic one, except that $M$ is a function from $[S]_0^n \times \Sigma$ to $P(S)$ rather than to $S$. Thus in the definition of a run of $\mathscr{A}$ on $X$, the condition $s_{\lambda+1} = M(s_\lambda, x_\lambda)$ should be replaced by $s_{\lambda+1} \in M(s_\lambda, x_\lambda)$, so that now the run of $\mathscr{A}$ on $X$ is not uniquely defined, and $\mathscr{A}$ *accepts* $X$ if there is *some* run of $\mathscr{A}$ on $X$ which ends by an accepted state.

Still, it is crucial to note that even for the nondeterministic case, the state of $\mathscr{A}$ at some limit point $\lambda$ of $X$ is uniquely and deterministically defined by the sequence of states up to $\lambda$ (since the run must be a *continuous* sequence).

DEFINITION 5.3. Given a nondeterministic $n$-type automaton $\mathscr{A} = \langle S, M, s^*, F\rangle$ on $\Sigma$, we define its *deterministic image* as the automaton $\mathscr{A}' = \langle S \cup \{f\}, M', s^*, F\rangle$ on $\Sigma' = \Sigma \times S$, where $M'(q, (\sigma, s)) = s$ if $s \in M(q, \sigma)$ and $M'(q, (\sigma, s)) = f$ otherwise, for all states $q$, all $\sigma \in \Sigma$, and all $s \in S$.

LEMMA 5.4. *Given $\mathscr{A}$ and $\mathscr{A}'$ as in Definition 5.3, let $X'$ be an $\alpha$-tape on $\Sigma'$ ($\alpha < \omega^{n+1}$), and $\varphi = (v_\beta)$ be the (unique) run of $\mathscr{A}'$ on $X'$. If $v_\beta \in [S]_0^n$ for all $\beta \leqslant \alpha$, then $\varphi$ is a run of $\mathscr{A}$ on $X = p_1(X')$. Also, if $X$ is an $\alpha$-tape on $\Sigma$, $\varphi = (v_\beta)$ is a run of $\mathscr{A}$ on $X$ and $X' = ((x_\beta, v_{\beta+1}))$, then $\varphi$ is* the *run of $\mathscr{A}'$ on $X'$.*

*Proof.* By induction on $\alpha$. For $\alpha = 0$ this is trivial; the transition from $\alpha$ to $\alpha + 1$ is simple, while the case of limit $\alpha$ is immediate using the remark above.

THEOREM 5.5. *Every set $V$ defined by a nondeterministic $n$-type automaton is a definable set.*

*Proof.* Given $V = T(\mathscr{A})$, let $\mathscr{A}'$ be the deterministic image of $\mathscr{A}$, and $V' = T(\mathscr{A}')$, where $V' \subseteq \Sigma_n'^*$, $\Sigma' = \Sigma \times S$. $V' \in \mathscr{D}_n(\Sigma')$ and by 4.4 $V = p_1(V')$, so that by 5.2 $V \in \mathscr{D}_n(\Sigma)$. Q.E.D.

## 6. Characterization of Regular Tapes

Definition 6.1. A tape $X \in \Sigma_n^*$ is *regular* if $\{X\} \in \mathscr{D}_n$.

As is well known (and follows from the preceding sections), every tape $X \in \Sigma_0^*$ is regular. This is not true of course for $n > 0$. Thus the problem arises of characterizing the collection of regular tapes. The case of tapes in $\Sigma_1$ has been already dealt with by Büchi [1], who showed that such a tape is an ultimately multiperiodic tape.

This result is generalized in the following discussion to tapes in $\Sigma_n^*$, $n > 0$. From now on we assume for simplicity that $\Sigma = \{0, 1\}$. We begin by dealing with tapes in $\Sigma_n$.

Definition 6.2. For $n \geqslant 1$ we denote by $\mathscr{P}_n$ the set of all $n$-tuples of nonnegative integers ($\mathscr{P}_1 = N$). For such an $n$-tuple $v = (k_{n-1}, ..., k_0)$, we put $d(v) = \omega^{n-1}k_{n-1} + \cdots + \omega^0 k_0$ $(0 \leqslant d(v) < \omega^n)$. For $X \in \Sigma_n$, we put $K(X) = \{v \in \mathscr{P}_n : X(d(v)) = 1\}$.

We characterize the regular tapes $X \in \Sigma_n$, by characterizing the corresponding subsets $K(X)$ of $\mathscr{P}_n$.

Definition 6.3. An *arithmetic progression* is a set $\{a + nb : n = 0, 1, ...\}$ where $a, b$ are nonnegative integers; this set is a *proper* arithmetic progression if $b \neq 0$. A set $A \subseteq \mathscr{P}_n$ is a *periodic set of order* $k$ $(0 \leqslant k \leqslant n)$ if it is the Cartesian product of $n$ arithmetic progressions of which exactly $k$ are proper ones. A set $B \subseteq \mathscr{P}_n$ is an *ultimately $n$-periodic set* if it is a finite union of periodic sets of order $k$ for arbitrary $0 \leqslant k \leqslant n$.

Ultimately $n$-periodic sets have an interesting geometrical interpretation. Let us picture $\mathscr{P}_n$ as the nonnegative $1/2^n$ part of the euclidian $n$-dimensional space with integral coordinates. A periodic set of order 0 is just a point in this space. A periodic set of order $k$ is obtained by choosing a $k$-dimensional box, and translating it up to infinity in the $k$ dimensions parallel to the positive axes. An ultimately $n$-periodic set is a finite union of such patterns for different $k$.

The collection of all ultimately $n$-periodic sets will be denoted by $\mathscr{U}\mathscr{P}_n$.

Before giving our main theorem for this section, we state the following lemma whose proof is almost immediate by induction.

Lemma 6.4. *$A \in \mathscr{U}\mathscr{P}_{n+1}$ iff it is a finite union of sets of the form $\{(b + mc, d) : m \in N, d \in B\}$ for some $b, c \in N$ and $B \in \mathscr{U}\mathscr{P}_n$.*

Theorem 6.5. *$X \in \Sigma_n$ is regular iff $K(X) \in \mathscr{U}\mathscr{P}_n$.*

*Proof.* By induction. (For $n = 1$ some trivial notational modifications should be made in the proof.) Suppose $X \in \Sigma_n$ and $\{X\} \in \mathscr{D}_n$, so that $\{X\} \in \mathscr{R}\mathscr{D}_n$. By Theorem 4.2, there are $Y, Z \in \Sigma_{n-1}^*$ such that $X = Y \odot Z$. If $Y = Y_0 \cdots Y_k$, $Z = Z_0 \cdots Z_q$, where

$Y_i, Z_j \in \Sigma_{n-1}$, then it is trivial that $\{Y_i\}, \{Z_j\} \in \mathscr{RD}_{n-1}$. Thus, by induction, $K(Y_i)$ and $K(Z_j)$ are in $\mathscr{UP}_{n-1}$. Clearly, however, $K(Y)$ is the union of the sets $\{(i, c): c \in K(Y_i)\}$ for $0 \leqslant i \leqslant k$ and $K(Y_i) \neq \Phi$; $K(Z^\omega)$ is on the other hand the union of the sets $\{(q+1)m + j; c): m \in N, c \in K(Z_j)\}$ for $0 \leqslant j \leqslant q$, and $K(Z_j) \neq \Phi$. Thus $K(Y)$ and $K(Z^\omega)$ are in $\mathscr{UP}_n$ and so is $K(X) = K(Y) \cup K'$ where $K'$ is $K(Z^\omega)$ with the first coordinate translated by $k+1$.

Suppose now that $K(X) \in \mathscr{UP}_n$, and observe first that $K(X) = \{(a + mb, c): m \in N, c \in K'\}$ where $K' \in \mathscr{UP}_{n-1}$. Let $Y \in \Sigma_{n-1}$ be such that $K(Y) = K'$ and denote by $\bar{0}_k$ an $\omega^k$-tape consisting wholly of 0's. Then $X = Z_0 \cdots Z_{a-1} Y Z^\omega$ where $Z_i = \bar{0}_{n-1}$, and $Z = \bar{0}_{n-1}$ if $b = 0$, $Z = Z_0 \cdots Z_{b-2} Y$ if $b \neq 0$ (for $b = 1$, $Z = Y$). Thus, using induction, $X$ is regular. To finish the proof we show that if for some $X, Y, Z \in \Sigma_n$, $K(X) = K(Y) \cup K(Z)$ where $Y$ and $Z$ are regular, then so is $X$. Let $\Sigma' = \{0, 1\}^3$, $U_1 = \{X' \in \Sigma_n': p_1(X') = Y\}$, $U_2 = \{X' \in \Sigma_n': p_2(X') = Z\}$, $U_3 = \{X' \in \Sigma_n': p_3(x_\alpha') = 1$ iff $p_1(x_\alpha') = 1$ or $p_2(x_\alpha') = 1$ for all $\alpha < \omega^n\}$. $U_1, U_2, U_3$ are definable and so is $X = p_3(U_1 \cap U_2 \cap U_3)$. Q.E.D.

THEOREM 6.6. *$X \in \Sigma_n^*$ iff $K(X) \in \mathscr{UP}_{n+1}$.*

*Proof.* It is clear that $X \in \Sigma_n^*$ is regular iff $Z = X\bar{0}_{n+1}$ is regular. Since $K(X) = K(Z)$, the result follows immediately from the preceding theorem. Q.E.D.

## 7. DECISION PROBLEMS

Since $n$-automata are finite entities, one may ask if there is an algorithm for deciding for any such given entity $\mathscr{A}$ whether it has a certain property (such as $T(\mathscr{A}) = \Phi$) or not. Moreover many decision problems (such as: "is $X$ accepted by $\mathscr{A}$"? "Does $\mathscr{A}$ accept any tape of length $k$?" etc.) which are trivial in the case of finite tapes because they consist of a finite number of questions, are not so in our case. This section is devoted to such problems.

If $X \in \Sigma_n^*$, $X = X_1 \cdots X_k$ we say that the *macro-length* of $X$ is $k$.

LEMMA 7.1. *Suppose we can decide for every restricted n-automaton $\mathscr{A}$ whether $T(\mathscr{A}) = \Phi$. Then the same can be decided for any n-automaton.*

*Proof.* We give two proofs. (a) Let $\mathscr{A}$ be an $n$-automaton. In Theorem 4.1 we defined certain sets $V_{ij}^k$ such that $V_{ij}^0$ are defined by some (effectively constructed) restricted $n$-automata, $V_{ij}^k = V_{ij}^{k-1} \cup V_{ik}^{k-1}(V_{kk}^{k-1})^* V_{kj}^{k-1}$, and $T(\mathscr{A})$ is the union of some of these sets. By assumption, we can decide whether $T(V_{ij}^0) = \Phi$ or not; also, assuming by induction that we can decide it for $V_{ij}^{k-1}$, then it is clear that $V_{ij}^k \neq \Phi$ iff either $V_{ij}^{k-1} \neq \Phi$ or both $V_{ik}^{k-1}$ and $V_{kj}^{k-1}$ are not empty; which gives immediately the decision algorithm for $T(\mathscr{A})$.

(b) Assuming that $\mathscr{A}$ has $m$ states of type $n$, it is clear that $T(\mathscr{A}) \neq \Phi$ iff it contains a tape with macro-length $\leqslant m$. (The bound here is $m$ rather than $m-1$, since $s^*$ is in $S$ and not in $[S]^n$.) Thus, assuming the lemma hypothesis, we show how to decide for any given $n$-automaton $\mathscr{A}$ and any given $k$, whether $\mathscr{A}$ accepts tapes with macro-length $k$. For $k = 1$ this is trivial since we have only to build $\mathscr{B}$ such that $T(\mathscr{B}) = T(\mathscr{A}) \cap \Sigma_n$, and check whether $T(\mathscr{B}) \neq \Phi$. Continuing by induction, suppose $\mathscr{A} = \langle S, M, s^*, F\rangle$, $[S]^n = \{q_1, ..., q_m\}$ and let $\mathscr{B}_i = \langle S, M, s^*, \{q_i\}\rangle$, $\mathscr{C}_i = \langle S, M, q_i, F\rangle$ for $1 \leqslant i \leqslant m$. Then $\mathscr{A}$ accepts tapes with macro-length $k$ iff there is some $i$ for which $\mathscr{B}_i$ accepts tapes with macro-length $k - 1$, and $\mathscr{C}_i$ accepts tapes with macro-length 1. Since the question $\Lambda \in T(\mathscr{A})$ is trivially decidable, the theorem is proved. Q.E.D.

THEOREM 7.2. *The question "Is $T(\mathscr{A}) = \Phi$?" is decidable for any n-automaton $\mathscr{A}$.*

*Proof.* By a two-step induction; the claim is immediate for a restricted 0-automaton. Assuming it to be true for restricted $n$-automata, it will be true for $n$-automata by the preceding lemma. Assuming it now to be true for $n$-automata, let $\mathscr{A}$ be a restricted $(n + 1)$-automaton. By Theorem 4.2, $T(\mathscr{A})$ is a finite union of sets of the form $U \odot V$, where $U$, $V$ are defined by some explicitly given $n$-automata. Obviously $U \odot V \neq \Phi$ iff $U \neq \Phi$ and $V - \{\Lambda\} \neq \Phi$, both questions being decidable. Q.E.D.

As a corollary we get of course the decidability of various related questions, such as: $T(\mathscr{A}) = T(\mathscr{B})$, $T(\mathscr{A}) \subseteq T(\mathscr{B})$, $T(\mathscr{A}) \cap T(\mathscr{B}) = \Phi$, $T(\mathscr{A})$ contains tapes of macro-length $k$, etc. We also remark that we can decide for any given $n$-type regular expressions $\varphi$, $\psi$ whether $|\varphi| = |\psi|$ or not.

The question "Given $X \in \Sigma_n^*$ and an $n$-automaton $\mathscr{A}$, is $X \in T(\mathscr{A})$?" is trivial for $n = 0$ but not so for $n > 0$. In fact, in its most general form it is undecidable. The problem was first studied by Elgot and Rabin [8], who found a particular class of tapes for which this question is decidable. We show here (without proofs) how their methods can be adapted to the general case.

First we note that if $X \in \Sigma_n$ is "effectively regular," that is, if some automaton $\mathscr{A}$ that defines $\{X\}$ is given (either directly or via a regular expression denoting $X$ or an ultimately $n$-periodic set describing $K(X)$) then the question "Is $X \in T(\mathscr{B})$?" is decidable for any $n$-automaton $\mathscr{B}$. One has only to build $\mathscr{C}$ such that $T(\mathscr{C}) = T(\mathscr{A}) \cap T(\mathscr{B})$ and check whether $T(\mathscr{C}) = \Phi$ or not. We turn then to the study of tapes that can be "reduced" to regular tapes.

We assume again that $\Sigma = \{0, 1\}$, and deal only with tapes from $\Sigma_n$, $n \geqslant 1$.

DEFINITION 7.3. Let $X \in \Sigma_n$, and $0 \leqslant i < n$. *A block of order $i$* of $X$ is a subtape of $X$ of the form $(x_\lambda)_{\alpha \leqslant \lambda < \alpha + \omega^i}$ for some $\alpha$ of type $\geqslant i$, or $\alpha = 0$. Such a block is *homogeneous* if it consists entirely of 0's or entirely of 1's. The *index* of $X$ is the maximal $0 \leqslant i < n$ such that all blocks of order $i$ are homogeneous. In this case every subtape $Z$ of length $\omega^{i+1}$ of $X$ can be written as $Z = \prod_0^\omega Z_j$ where for each $j$, $Z_j = \bar{0}_i$ or $Z_j = \bar{1}_i$; we then denote by $\hat{Z}$ the $\omega$-tape defined by $z_i = 0$ or 1, respectively, for $0 \leqslant i < \omega$.

DEFINITION 7.4. For $X \in \Sigma_n$ and $d \geqslant 1$, we define the *d-contraction* of $X$, $X'$, to be denoted by $D(X, d)$, as follows. Assume first $n = 1$ and let $(k_i)$ be the sequence of indices for which $x_{k_i} = 1$. The corresponding sequence $(k_i')$ for $X'$ is defined inductively: $k_1' = k_1$; for $i \geqslant 1$, $k_{i+1}' = k_{i+1}$ if $k_{i+1} - k_i \leqslant d + 1$; otherwise, $k_{i+1}'$ is the unique number satisfying $k'_{i+1} - k'_i \equiv k_{i+1} - k_i \pmod{d!}$ and $d + 1 < k'_{i+1} - k'_i \leqslant d + 1 + d!$. For the general case suppose $X \in \Sigma_n$, $n > 1$, has index $i$. We define $X' = D(X, d)$ by defining all its $(i+1)$-blocks $\zeta'$, by the formula $\zeta' = D(\zeta, d)$ where $\zeta$ is the corresponding $(i+1)$-block in $X$. $X$ is *d-contractible* if $D(X, d)$ is effectively regular; it is *contractible* if it is $d$-contractible for any $d > 1$.

DEFINITION 7.5. Let $X \in \Sigma_n$ be of index $i$ and $\mathscr{A}$ be an $n$-automaton with $k$ 0-type states. The *contraction coefficient* of $X$ relative to $\mathscr{A}$, $C(X, \mathscr{A})$, is $d_i^3 d_{i+1}$, where $d_0 = k$, $d_{j+1} = 2^{d_j}$ for $j \geqslant 0$.

LEMMA 7.6. *For* $X \in \Sigma_n$ *and* $\mathscr{A} = \langle S, M, s^*, F\rangle$ *we have,* $M(s^*, X) = M(s^*, D(X, C(X, \mathscr{A})))$.

THEOREM 7.7. *There is an algorithm for deciding for any n-automaton* $\mathscr{A}$ *and any* $X \in \Sigma_n$ *which is* $C(X, \mathscr{A})$ *contractible, whether* $X \in T(\mathscr{A})$ *or not.*

COROLLARY 7.8. *If* $X \in \Sigma_n$ *is contractible, then the question "Is* $X \in T(\mathscr{A})$*?" is decidable for any n-automaton* $\mathscr{A}$.

The membership decision problem has been thus essentially reduced to the case $n = 1$. For this case, it has been shown in [8] that tapes $X \in \Sigma_1$ for which $K(X)$ is the range of a function like $f(m) = m!$, $f(m) = a^m$ or $f(m) = m^a$ (where $a$ is a constant) are contractible. The collection of such functions has been extensively studied and much enlarged in [14], and independently in [4].

## 8. CHARACTERIZATION BY EQUIVALENCE RELATIONS

An important characterization of regular sets in the case of finite tapes is achieved through the notion of a "regular" relation. A binary relation $R$ on $Z^*$ is *regular* if it is right-invariant (that is $R(X, Y) \to R(XZ, YZ)$ for all $X, Y, Z \in \Sigma^*$) and has a finite index (= number of equivalence classes). In fact, the following theorem due to Nerode (see [12, Theorem 2]), holds:

THEOREM. *For* $U \subseteq \Sigma^*$, *the following three conditions are equivalent*: (1) *U is regular.* (2) *U is a union of some of the equivalence classes of a regular relation on* $\Sigma^*$. (3) *The binary relation E on* $\Sigma^*$ *defined by*: *"*$E(X, Y)$ *holds if and only if for all* $Z \in \Sigma^*$, $XZ \in U \Leftrightarrow YZ \in U$*" is a regular relation.*

One cannot expect this theorem to be true with $\Sigma_n^*$ in place of $\Sigma^*$ (even with an appropriate definition of "regular relation"), since the definition of $E$ in (3) does not take into account the situation "in the limit." In fact we shall define a notion of "regular relation on $\Sigma_n^*$" for which the first two conditions are equivalent, but are not equivalent to the third one. Following this we will state some restricting conditions under which the equivalence of the three conditions can be indeed assumed.

First we would like, however, to somewhat generalize the concept of $n$-automaton in such a way that it will accept tapes of *any* length $\alpha < \omega^{n+1}$, and not only of length $\omega^n k$. The set of all tapes of length $<\omega^{n+1}$ will be denoted by $\Sigma_{(n)}^{(*)}$.

DEFINITION 8.1. A *hybrid* $n$-automaton $\mathscr{A}$ is like an $n$-automaton except that now the set of accepted states $F$ is a subset of $[S]_0^n$. In this case $T(\mathscr{A}) \subseteq \Sigma_{(n)}^{(*)}$. The collection of sets definable by hybrid deterministic $n$-automata will be denoted by $\mathscr{D}_{(n)}$. The following facts can be easily proved:

THEOREM 8.2. (1) $\mathscr{D}_{(0)} = \mathscr{D}_0$ ; $\mathscr{D}_n \subseteq \mathscr{D}_{(n)}$ ; $\mathscr{D}_{(n)} \subseteq \mathscr{D}_{(n+1)}$ .

(2) $\mathscr{D}_{(n)}$ *is closed under complement, union, intersection, and product operations.*

(3) $\mathscr{D}_{(n)} = \mathscr{D}_n \cdot \mathscr{D}_{(n-1)}$ .

(4) *If* $U \subseteq \Sigma_{(n)}^{(*)}$ *is defined by a hybrid (deterministic/nondeterministic) n-automaton, then* $U = \bigcup_1^m V_i W_i$ *where* $V_i \in \mathscr{D}_n$ , *and* $W_i$ *are defined by hybrid (deterministic/nondeterministic)* $(n-1)$*-automata.*

(5) *If* $U \subseteq \Sigma_{(n)}^{(*)}$ *is defined by a hybrid nondeterministic n-automaton then* $U \in \mathscr{D}_{(n)}$ .

We only remark that (5) follows immediately, by induction, from (2), (3), and (4).

DEFINITION 8.3. Let $X \in \Sigma_{(n)}^{(*)}$ be of length $\alpha$. The *type* of $X$ — $Ty(x)$⁻ is the type of $\alpha$. If $R$ is an equivalence relation on $\Sigma_{(n)}^{(*)}$, we choose a symbol for each equivalence class of $R$, and the symbol denoting the equivalence class of $X$ will be denoted by $e_R(X)$. $\varphi_R(X)$ is the $\alpha$-sequence $(\varphi_\lambda)$ defined by $\varphi_\lambda = e_R(X \mid \lambda)$. We also recall the notions of a trail representative $t(\varphi)$ of an $\alpha$-sequence $\varphi$, and its limit $L(\varphi) = I(t(\varphi))$ as defined at the beginning of Section 2.

DEFINITION 8.4. An equivalence relation $R$ on $\Sigma_{(n)}^{(*)}$ is *regular* if the following holds:

(1) If $R(X, Y)$ then the $Ty(X) = Ty(Y)$.

(2) $R$ has finite index.

(3) $R$ is right invariant.

(4) For any $X, Y \in \Sigma_{(n)}^{(*)}$ of type $m > 0$, $L(\varphi_R(X)) = L(\varphi_R(Y))$ entails $e_R(X) = e_R(Y)$ (i.e., $R$ is "right-invariant in the limit").

THEOREM 8.5. $U \in \mathscr{D}_{(n)}$ *if and only if it is the union of some of the equivalence classes of a regular relation* $R$ *on* $\Sigma_{(n)}^{(*)}$.

*Proof.* We give a rapid sketch of the proof. If $U = T(\mathscr{A})$ where $\mathscr{A} = \langle S, M, s^*, F\rangle$, then we define $R$ by letting $R(X, Y)$ hold if and only if $M(s^*, X) = M(s^*, Y)$. Clearly the conditions of the theorem are satisfied. Conversely suppose $U$ is the union of some of the equivalence classes of a regular relation $R$ on $\Sigma_{(n)}^{(*)}$. Let $T_m = \{X \in \Sigma_{(n)}^{(*)}: Ty(X) = m\}$, and $S = \{e_R(X): X \in T_0\}$. Define inductively a function $f: T_m \to [S]^m$ as follows: For $X \in T_0$, $f(X) = e_R(X)$. For $X \in T_{m+1}$ of length $\alpha$, let $f_\lambda = f(X \mid \lambda)$

for $\lambda < \alpha$, and put $f(X) = L((f_\lambda))$. Define now the hybrid $n$-automaton $\mathscr{A} = \langle S, M, s^*, F\rangle$ as follows: $s^* = f(\Lambda)$; $F = \{q \in [S]_0^n\colon (\exists\, X \in U)\,(f(X) = q)\}$; for $q \in [S]_0^n$ and $\sigma \in \Sigma$, $M(q, \sigma) = f(X\sigma)$ where $X$ is any tape for which $f(X) = q$. It can easily be checked that $T(\mathscr{A}) = U$. Q.E.D.

We immediately remark that there is a $U \in \mathscr{D}_n$ for which the relation defined by "$E(X, Y)$ holds if and only if for all $Z \in \Sigma_{(n)}^{(*)} XZ \in U \Leftrightarrow YZ \in U$" is not regular according to 8.3. Indeed, take $n = 1$, $\Sigma = \{0, 1\}$, $U = \Sigma^* \cdot \{0^\omega\}$; $U \in \mathscr{D}_1$. We certainly have $E(0^n, 1^n)$ for all $n \geqslant 1$, but nevertheless $E(0^\omega, 1^\omega)$ is not true, thus contradicting condition (4) of 8.3.

Definition 8.6. Let $\mathscr{A} = \langle S, M, s^*, F\rangle$ be a hybrid $n$-automaton. We say that $q \in [S]_0^n$ is *accessible* if there is some $X \in \Sigma_{(n)}^{(*)}$ for which $M(s^*, X) = q$. $\mathscr{A}$ is *connected* if all 0-*type* states are accessible. Two states $q_1, q_2 \in [S]^m$, $m \geqslant 0$, are *indistinguishable* if for any $X \in \Sigma_{(n)}^{(*)}$, $M(q_1, X) \in F$ if and only if $M(q_2, X) \in F$. They are *equivalent* if they are accessible, of type $m \geqslant 0$, and for every member of the one there is a corresponding member of the other from which it is indistinguishable. $\mathscr{A}$ is *consistent* if every two equivalent states are indistinguishable.

Finally $\mathscr{A}$ is *normal* if: (1) it is connected, (2) any two 0-type states are distinguishable, (3) for any two equivalent states $q_1, q_2$ we have $M(q_1, \sigma) = M(q_2, \sigma)$ for all $\sigma \in \Sigma$.

Theorem 8.7. *For $U \subseteq \Sigma_{(n)}^{(*)}$, the following three conditions are equivalent:*

1. *$U$ is defined by some normal automaton.*
2. *$U$ is defined by some consistent automaton.*
3. *The relation $E$ on $\Sigma_{(n)}^{(*)}$ induced by $U$ is regular.*

*Proof.* That (1) entails (2) is trivial, since any normal automaton is also consistent. Assume now that $U$ is defined by some consistent $n$-automaton $\mathscr{A} = \langle S, M, s^*, F\rangle$, we show that $E$ is regular. $E$ is certainly a right-invariant equivalence relation, and since $M(s^*, X) = M(s^*, Y)$ entails $E(X, Y)$, $E$ has a finite index. It remains to show that $E$ is right-invariant in the limit. Suppose that $X$, $Y$ are two tapes of type $m > 0$ for which $L(\varphi_R(X)) = L(\varphi_R(Y))$. Let $M(s^*, X) = q_1$, $M(s^*, Y) = q_2$, and let $(p_i^X)$ $[(p_i^Y)]$ be the trail representative of the run of $\mathscr{A}$ on $X$ [$Y$, resp.]. For $d \in q_1$ there is an infinite sequence $(i_j)$ for which $p_{i_j}^X = d$; let $(m_j)$ be an infinite subsequence of $(i_j)$ for which $t(\varphi_R(X))(m_j)$ is constant, say $e \in L(\varphi_R(X))$, so that $e \in L(\varphi_R(Y))$ and there is an infinite sequence $(k_j)$ for which $t(\varphi_R(Y))(k_j) = e$; for a certain infinite subsequence $(l_j)$ of $(k_j)$, we have that $p_{i_j}^Y$ is constant, say $d'$. It is easy to see that $d$ and $d'$ are distinguishable. By symmetry for every $d' \in q_2$ there is some $d \in q_1$ from which it is indistinguishable. Thus, $q_1$ and $q_2$ are equivalent and since $\mathscr{A}$ is consistent they are indistinguishable. Thus, for any $Z \in \Sigma_{(n)}^{(*)}$, $M(s^*, XZ) = M(M(s^*, X), Z) = M(q_1, Z)$ and $M(s^*, YZ) = M(q_2, Z)$ are both in $F$ or both not in $F$, so that $XZ \in U$ if and only if $YZ \in U$, which shows that $E(X, Y)$ holds. Suppose, finally, that the relation $E$ induced by $U$ is regular; $U$ is then the union of some of the equivalence classes of a regular relation $E$, and thus by 8.5

is definable by some $n$-type hybrid automaton $\mathscr{A}$. It is easy to see however, that the construction of $\mathscr{A}$ in 8.5 gives, in our case, a consistent automaton.

Now to every consistent $n$-automaton, one can build an equivalent normal one, using standard methods. Let $\mathscr{A} = \langle S, M, s^*, F\rangle$ be a consistent $n$-automaton, where we assume without loss of generality that $\mathscr{A}$ is connected, and let $S' \subseteq S$ be a maximal set of mutually distinguishable states of type 0. Define $f$: $[S]^m \to [S']^m$ for $0 \leqslant m \leqslant n$ as follows: for $s \in S$, $f(s)$ is the unique element of $S'$ from which it is indistinguishable, and for $q \in [S]^m$, $m > 0$, $f(q) = \{f(c): c \in q\}$. It is easily checked that $f(q_1) = f(q_2)$ entails that $q_1$ and $q_2$ are indistinguishable. Define now $\mathscr{A}' = \langle S', M', s'^*, F'\rangle$, where $s^* = f(s^*)$, $F' = \{f(q): q \in F\}$ and $M'(q, \sigma) = f(M(c, \sigma))$w here $c$ is such that $f(c) = q$. We leave it to the reader to check that $\mathscr{A}'$ is normal and $T(\mathscr{A}') = U$. Q.E.D.

We have already seen that the relation induced on $\Sigma_1^*$, by the set $\Sigma^*\{0^\omega\}$ is not regular. This shows that there are regular sets which are not defined by any consistent or normal automaton. In fact this set cannot be defined by any automaton whose all 0-type states are distinguishable (because such an automaton would certainly be consistent since it would not have any equivalent states of type 1). Thus a certain amount of "redundancy" is unavoidable in the general case. This situation is in sharp contrast with the case of automata with finite tapes, where every regular set can be defined by a minimal automaton, all of whose states are distinguishable.

It might be interesting to note that this set $U$ does have a nondeterministic "normal" automaton that defines it. Let $\mathscr{A} = \langle\{s_0, s_1, s_2\}, M, s_0, \{\{s_1\}\}\rangle$ where $M(s_1, 0) = s_1$, $M(s_0, 0) = M(s_0, 1) = \{s_0, s_1\}$ and $M(s, \sigma) = s_2$ in all other cases. All 0-type states are accessible and "distinguishable": in particular, $10^\omega$ distinguishes between $s_0$ and $s_1$ (since there is an accepting run beginning with $s_0$, but no such run exists that begins with $s_1$), and so $\mathscr{A}$ is "normal." This adds a certain new "flavor" to nondeterministic automata in the infinite tapes case.

We end by remarking that all of the notions mentioned here are decidable. In fact, the following is easily proved.

THEOREM 8.8. *Given a hybrid n-automaton $\mathscr{A}$, the following questions are decidable*: (1) *Is $q \in [S]_0^n$ accessible*? (2) *Are $q_1, q_2 \in [S]^n$ distinguishable*? (3) *Are $q_1, q_2$ equivalent*? (4) *Is $\mathscr{A}$ connected*? (5) *Is $\mathscr{A}$ consistent*? (6) *Is $\mathscr{A}$ normal*?

On the other hand, finding a decision procedure for answering the question "Given $\mathscr{A}$, is $T(\mathscr{A})$ definable by a normal automaton?" is still an open problem.

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