Bisimilarity of Pushdown Automata is Nonelementary

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Highlights 2013, Paris
September 19th

Pushdown automata

A pushdown automaton P is given by

- a finite set of control states $p, q, \dots$
- a finite set of stack symbols $\perp, A, B, C, \dots$
- labeled rules of the kind
  - $pA \xrightarrow{a} q$ (pop).
  - $pA \xrightarrow{b} qB$ (internal), where $A = \perp$ iff $B = \perp$.
  - $pA \xrightarrow{c} qBC$ (push), where $A = \perp$ iff $\perp \in \{B, C\}$.

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An example for a pushdown automaton P

Rules: $qA \xrightarrow{a} qAA \quad qA \xrightarrow{b} qBA \quad qB \xrightarrow{a} qAB \quad qB \xrightarrow{b} qBB$

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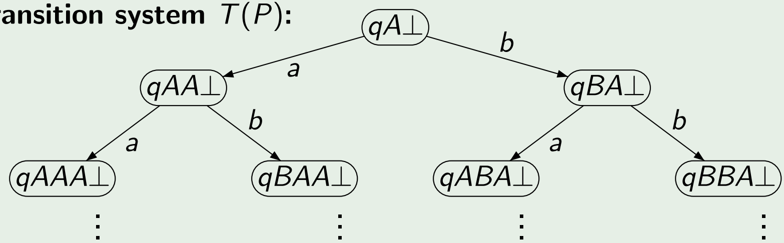
Transition system $T(P)$:



Bisimilarity checking of pushdown processes: Results

Theorem (Sénizergues 1998)

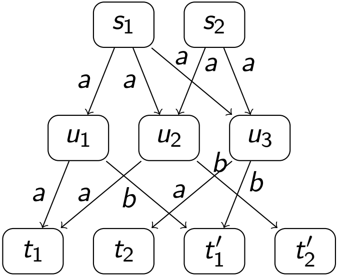
Bisimilarity of pushdown processes is decidable.

(two semi-decision procedures, no prim. rec. upper bound known)

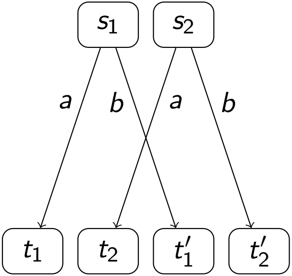
Theorem (Kučera, Mayr 2002)

Bisimilarity of pushdown processes is EXP*-hard.*

Two useful gadgets from Jančar/Srba and Chen/v. Breugel/Worrell



(a) **OR**-gadget



(b) **AND**-gadget

Proposition

We have

- in the **left** picture: $s_1 \sim s_2$ iff $(t_1 \sim t_2$ **OR** $t_1' \sim t_2')$.
- in the **right** picture: $s_1 \sim s_2$ iff $(t_1 \sim t_2$ **AND** $t_1' \sim t_2')$.

Proving EXPSPACE-hardness

Fix an input of length n.

A **0-counter** is a sequence $c = b_0 \cdots b_{n-1} \in \{0,1\}^n$.
Its **value** is $\mathbf{val}(c) \stackrel{\text{def}}{=} \sum_{i=0}^{n-1} 2^i \cdot b_i \in \{0, \ldots, 2^n - 1\}$.

Convention: For each 0-counter write c_i we assume $\mathbf{val}(c_i) = i$.

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Proposition

$\exists$ gadget with control states q, q' such that the following holds:

Assume the stack content

$$\sigma \ = \ \boxed{c_j b c_i}\,\mathbf{w}\perp,$$

where c_j and c_i are the 0-counters of value j and i, resp. We have

$$q(\sigma) \sim q'(\sigma) \quad \text{iff} \quad j = i - 1$$

Proving EXPSPACE-hardness

$$q(\boxed{c_j b c_i}\mathbf{w}\bot) \sim q'(\boxed{c_j b c_i}\mathbf{w}\bot) \quad \text{iff} \quad j = i - 1$$

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Idea:

1. Lead the game to $p(\boxed{c_0 0}c_j b c_i \mathbf{w}\perp) \overset{?}{\sim} p'(\boxed{c_0 0}c_j b c_i \mathbf{w}\perp)$ by pushing bit 0 followed by the 0-counter c_0.

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2. De-synchronize the two stacks by
   - popping the **two top-most** 0-counters from the **left stack** and
   - and popping **the top-most** 0-counter from the **right stack**.

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Proving EXPSPACE-hardness

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   leading us to $p(\boxed{c_i}\,\mathbf{w}\bot) \overset{?}{\sim} p'(\boxed{c_j b c_i}\,\mathbf{w}\bot)$
3. Check if $j = i - 1$ deterministically as follows:
   3.1 **Left automaton** outputs c_i
   3.2 **Right automaton** outputs $T(c_j)$, where T is a deterministic letter-to-letter transducer that outputs the successor function.

Proving EXPSPACE-hardness

How to push

$$c_0 1 \cdots c_1 1 c_{2^n-1} 1$$

onto both stacks?

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1. First, we explicitly push $c_{2^n-1} 1$ onto the stacks.

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$$q(c_i 1 c_{i+1} 1 \cdots c_{2^n-1} 1 \bot) \overset{?}{\sim} q(c_i 1 c_{i+1} 1 \cdots c_{2^n-1} 1 \bot)$$

   Duplicator's job is to push $c_j 1$ with $j = i - 1$ onto the stacks.

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   Duplicator's job is to push $c_j 1$ with $j = i - 1$ onto the stacks.
5. She pushes some c_j onto the stacks (via n **OR**-gadgets)
   - If Spoiler believes $j = i - 1$, then goto 4.
   - If Spoiler does not believe $j = i - 1$, then play subgame!

Proving EXPSPACE-hardness

We now know how Duplicator can push $\#d_{2^{2^n}-1}$ onto the stacks.

$\Rightarrow$ We now know how Duplicator can push any $\#d_i$ onto the stacks.

$\Rightarrow$ Duplicator can push

$$d_0\#d_1\cdots\#d_{2^{2^n}-1}$$

onto both stacks by using the same ideas!

Open questions

There are (too) many of them!

- Primitive recursive lower/upper bounds for PDA bisimilarity
- Complexity of DPDA language equivalence
- Decidability of equivalence of Deterministic Higher-Order PDA
- Decidability of bisimilarity of PA-processes
- Decidability of bisimilarity of Ground Tree Rewrite Systems
- Decidability of weak bisimilarity of Single-State PDA
- Decidability of weak bisimilarity of Basic Parallel Processes
- ...

Thank you for your attention!