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The Parikh counting functions of sparse context-free languages are quasi-polynomials[☆]

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ABSTRACT

Let L be a sparse context-free language over an alphabet of t letters and let $f_L : \mathbb{N}^t \to \mathbb{N}$ be its Parikh counting function. We prove the following two results:

1. There exists a partition of $\mathbb{N}^t$ into a finite family of polyhedra such that the function f_L is a quasi-polynomial on each polyhedron of the partition.
2. There exists a partition of $\mathbb{N}^t$ into a finite family of rational subsets such that the function f_L is a polynomial on each set of the partition.

1. Introduction

In this paper, we study some combinatorial and decision problems concerning the *Parikh counting functions* of formal languages. Given an alphabet $A = \{a_1, \ldots, a_t\}$ and a word w over A, the *Parikh vector* of w is the vector $(n_1, \ldots, n_t)$, such that, for every i with $1 \leq i \leq t$, n_i is the number of occurrences of the letter a_i in w. Given a language L over A, we can associate with L a function, called the *Parikh counting function* of L,

$$f_L : \mathbb{N}^t \longrightarrow \mathbb{N},$$

which returns, for every vector $(n_1, \ldots, n_t) \in \mathbb{N}^t$, the number $f_L(n_1, \ldots, n_t)$ of all words in L having Parikh vector equal to $(n_1, \ldots, n_t)$.

If we impose restrictions on the growth rate of f_L, we obtain different classes of languages. In [15–17], for instance, it has been studied the notion of *Parikh slender* language. A language is termed Parikh slender if there exists a positive integer r such that, for every vector $(n_1, \ldots, n_t) \in \mathbb{N}^t, f_L(n_1, \ldots, n_t) \leq r$ holds. A notion that generalizes Parikh slenderness is that of *sparsity*. A language L is termed *sparse* if its counting function, that is, the function that maps every integer $n \geq 0$ into the number of words of L of length n, is polynomially upper bounded. One can prove that this property is equivalent to say that the Parikh counting function f_L of L is upper bounded by a polynomial on t variables. Sparse languages play a meaningful

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[1] Stefano Varricchio suddenly passed away on August 20th 2008, shortly after this paper has been completed. We will remember Stefano as our best friend and as an outstanding researcher. Working with Stefano was always an enthusiastic experience both for his beautiful and original ideas and for his scientific *rigueur*.

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role both in Computer science and in Mathematics and have been widely investigated in the past. The interest in this class of languages is due to the fact that, in the context-free case, it coincides with that one of *bounded* languages [19,25,26] (see also [4,22]). A language L is termed bounded if there exist n words $u_1, \ldots, u_n$ such that

$$L \subseteq u_1^* \cdots u_n^*.$$

Bounded context-free languages and their properties have been extensively studied by Ginsburg in [12], where, in particular, it is proved the decidability of the property of boundedness for context-free languages.

In this paper we present a combinatorial tool to give an exact description of the Parikh counting function of a sparse context-free language. This tool is based upon the notion of *quasi-polynomial*. A map $F : \mathbb{N}^t \to \mathbb{N}$ is a quasi-polynomial if there exists a positive integer d such that, for every $(n_1, \ldots, n_t) \in \mathbb{N}^t$, the value of F computed at $(n_1, \ldots, n_t)$ is given by:

$$F(n_1, \ldots, n_t) = p_{(d_1,d_2,\ldots,d_t)}(n_1, \ldots, n_t),$$

where, for every $i = 1, \ldots, t$, $d_i = n_i \bmod d$, and $p_{(d_1,d_2,\ldots,d_t)}$ is a polynomial with rational coefficients in t variables $x_1, \ldots, x_t$. One of the main results of this paper is that the Parikh counting function of a sparse context-free language can be *exactly* calculated using a finite number of quasi-polynomials. More precisely, if L is a sparse context-free language, then there exist a partition of $\mathbb{N}^t$ into a finite number of polyhedra $P_1, \ldots, P_s$, determined by hyperplanes with rational equations, and a finite number of quasi-polynomials $p_1, \ldots, p_s$, such that for any $(n_1, \ldots, n_t) \in \mathbb{N}^t$ one has:

$$f_L(n_1, \ldots, n_t) = p_j(n_1, \ldots, n_t),$$

where j is the index of the polyhedron P_j such that $(n_1, \ldots, n_t) \in P_j$. We remark that the polyhedra $P_1, \ldots, P_s$ as well as the quasi-polynomials $p_1, \ldots, p_s$ that give a description of the Parikh counting function can be effectively computed from an effective presentation of the language L. The latter theorem can be reformulated in a *"language-theoretic"* way. More precisely, one can prove that there exist a partition of $\mathbb{N}^t$ into a finite number of rational subsets $R_1, \ldots, R_s$ and a finite number of polynomials $p_1, \ldots, p_s$, such that for any $(n_1, \ldots, n_t) \in \mathbb{N}^t$ one has:

$$f_L(n_1, \ldots, n_t) = p_j(n_1, \ldots, n_t),$$

where j is the index of the rational set R_j such that $(n_1, \ldots, n_t) \in R_j$.

The decidability of some problems on the Parikh counting function of context-free languages is an easy consequence of our main results. In particular, one can decide whether a context-free language is Parikh slender or not, a result proved in [15].

A crucial step in the proof of our result is the computation of the non-negative solutions of a system of Diophantine linear equations. This problem deserves a special mention. Let $A \in \mathbb{Z}^{t\times n}$ be a matrix of integers, with $n \geq t$. One can prove that, under suitable conditions on the matrix A, for every vector $b \in \mathbb{Z}^t$, the system $AX = b$ of t Diophantine linear equations in the n unknowns $X = (x_1, \ldots, x_n)$, has a finite number of non-negative solutions. One can associate with the system a map $\mathcal{S} : \mathbb{Z}^t \longrightarrow \mathbb{N}$ such that, for every vector $b \in \mathbb{Z}^t$, $\mathcal{S}(b)$ is the number of all non-negative solutions of $AX = b$. This function is called the *partition function* of the vector b [6,7]. The computation of the partition function has been first considered in the context of Numerical Analysis where, in a celebrated paper by Dahmen and Micchelli [5], it has been proved that such function can be described by a set of quasi-polynomials. The result is obtained by making use of notions and techniques of the theory of *box splines*. Further investigations are in [6,7,28], where important theorems on the algebraic and combinatorial structure of the partition function have been obtained. In this paper, we focus our attention on systems such that the entries of A and b are non-negative numbers. In this case, for every vector b, the number of non-negative solutions of the system $AX = b$ is obviously finite so that the partition function associated with the matrix A is well defined. Here we present a combinatorial proof of the theorem by Dahmen and Micchelli in such a case. This proof, which appears to be new, is of elementary character, effective and makes the paper completely self-contained.

The above mentioned theorem is important in our construction because we will prove that, given a bounded context-free language, one can effectively construct a family of partition functions that describes the Parikh counting function of the language. Because of this fact, in the sequel of the paper, the partition function will be called the *counting function* of the Diophantine system defined by the matrix A.

In the proof of our main results, some important and deep theorems of formal language theory have been used. More precisely, the combinatorial method developed to describe the Parikh counting function of a context-free language is based upon the representation of such languages as semi-linear sets. In particular, two important results have been used: the celebrated Cross-Section theorem by Eilenberg and the theorems, by Ginsburg and Spanier and by Eilenberg and Schützenberger, which characterize the rational subsets of $\mathbb{N}^t$ [9,11]. These theorems provide a crucial tool in order to cope with the ambiguity of context-free languages. In this context, we also recall another important recent result that gives a characterization of sparse context-free languages in terms of finite unions of *Dyck loops* [20,21]. However, this latter result cannot be used to compute any counting function because of the ambiguity of the representation of such a language as a finite union of Dyck loops. Indeed, consider the language $L = \{a^n b^m c^m d^n \mid n, m \geq 0\} \cup \{a^n b^n c^m d^m \mid n, m \geq 0\}$. The language L is sparse context-free but it cannot be represented unambiguously as a finite union of Dyck loops.

We complete our analysis by proving that the Parikh counting function of a sparse context-free language is rational. This result is remarkable since, as it is well known [10], the Parikh counting function of a context-free language may be transcendental.

The paper is organized as follows. In Section 2, we give a combinatorial proof of the fact that the counting function of a system of Diophantine linear equations with coefficients in $\mathbb{N}$ can be described by quasi-polynomials. In Section 3, we will introduce the above mentioned results on the algebraic structure of bounded context-free languages. In Section 4, we will prove our main result on the Parikh counting function of bounded context-free languages. Finally, in Section 5, we will prove that the Parikh counting function of a sparse context-free language is rational.

2. On systems of diophantine equations

Given a system of Diophantine linear equations with positive coefficients, we can define a map that returns, for any vector of non-homogeneous terms, the number of non-negative solutions of the system. In this section, we develop a combinatorial tool to deal with such maps. For this purpose, we recall some preliminary definitions and results.

Lemma 1. *Let $q(x_1, \ldots, x_t, x)$ be a polynomial in $t + 1$ variables with rational coefficients and let*

$$F : \mathbb{N}^t \times \{\{-1\} \cup \mathbb{N}\} \longrightarrow \mathbb{Q}$$

be the map defined as:

$$F(x_1, \ldots, x_t, x) = \begin{cases} \sum_{\lambda=0,\ldots,x} q(x_1, \ldots, x_t, \lambda) & x \geq 0, \\ 0 & x = -1. \end{cases}$$

There exists a polynomial $p(x_1, \ldots, x_t, x)$ in $t + 1$ variables with rational coefficients such that, for every $(x_1, \ldots, x_t, x) \in \mathbb{N}^t \times \{\{-1\} \cup \mathbb{N}\}$, one has:

$$F(x_1, \ldots, x_t, x) = p(x_1, \ldots, x_t, x).$$

Proof. Write $q(x_1, \ldots, x_t, x)$ as:

$$a_0 + a_1 x + \cdots + a_n x^n, \tag{1}$$

where, for every $i = 0, \ldots, n$, a_i is a suitable polynomial in the variables $x_1, \ldots, x_t$ with rational coefficients. By (1), if $x \geq 0$, for every $x_1, \ldots, x_t \in \mathbb{N}$, one has:

$$F(x_1, \ldots, x_t, x) = \sum_{\lambda=0,\ldots,x} q(x_1, \ldots, x_t, \lambda) = \sum_{j=0,\ldots,n} \left(a_j \cdot \sum_{\lambda=0,\ldots,x} \lambda^j \right). \tag{2}$$

On the other hand, by using a standard argument (*cf* Lemma 21 of the Appendix), one can prove that, for any $j \in \mathbb{N}$, there exists a polynomial $p_j(x)$ with rational coefficients in one variable x such that:
(3.1) for any $x \in \mathbb{N}$, $p_j(x) = \sum_{\lambda=0,\ldots,x} \lambda^j$.
(3.2) $p_j(-1) = 0$.
For any $j = 0, \ldots, n$, let p_j be the polynomial defined above and let $p = p(x_1, \ldots, x_t, x)$ be the polynomial defined as:

$$p = \sum_{j=0,\ldots,n} a_j p_j.$$

Then by (3.2), one has $p(x_1, \ldots, x_t, -1) = 0$. Moreover, for every $x \geq 0$, by (2) and (3.1), one has:

$$F(x_1, \ldots, x_t, x) = \sum_{j=0,\ldots,n} \left(a_j \cdot \sum_{\lambda=0,\ldots,x} \lambda^j \right) = \sum_{j=0,\ldots,n} a_j p_j(x) = p(x_1, \ldots, x_t, x).$$

The proof is thus complete. □

Corollary 1. *Let t be a positive integer and let $q_t : \mathbb{N} \longrightarrow \mathbb{N}$ be the map that returns, for every $n \in \mathbb{N}$, the number of all distinct ways of writing n as sum $n_1 + \cdots + n_t$ of t non-negative integers. Then q_t is a polynomial.*

Proof. We proceed by induction on t. We remark that $q_1(n) = 1$, $q_2(n) = n + 1$, and $q_3(n) = (n + 1)(n + 2)/2$. We prove now the inductive step. A simple counting argument shows that, for every $n \in \mathbb{N}$, one has:

$$q_{t+1}(n) = \sum_{k=0,\ldots,n} q_t(n - k). \tag{3}$$

On the other hand, by the inductive hypothesis applied to the map q_t, one has that q_t is a polynomial. By (3), the claim follows by applying Lemma 1 to the map q_{t+1}. □

Remark 1. We observe that Corollary 1 also immediately follows from the well known combinatorial interpretation (see [13]):

$$q_t(n) = \binom{n + t - 1}{t - 1}.$$

Definition 1. A map $F : \mathbb{N}^t \longrightarrow \mathbb{N}$ is said to be a *quasi-polynomial* if there exist $d \in \mathbb{N}$, $d \geq 1$, and a family of polynomials in t variables with rational coefficients:

$$\{p_{(d_1,d_2,\cdots,d_t)} \mid d_1,\ldots,d_t \in \mathbb{N},\ 0 \leq d_i < d\},$$

where, for every $(x_1,\ldots,x_t) \in \mathbb{N}^t$, if d_i is the remainder of the division of x_i by d, one has:

$$F(x_1,\ldots,x_t) = p_{(d_1,d_2,\ldots,d_t)}(x_1,\ldots,x_t).$$

The number d is called the *period* of F.

To simplify the notation, the polynomial $p_{(d_1,d_2,\ldots,d_t)}$ is denoted $p_{d_1d_2\cdots d_t}$.

Definition 2. Let $F : \mathbb{N}^t \longrightarrow \mathbb{N}$ be a map. Given a subset C of $\mathbb{N}^t$, F is said to be a *quasi-polynomial over* C if there exists a quasi-polynomial q, such that $F(x) = q(x)$, for any $x \in C$.

Lemma 2. *The sum of a finite family of quasi-polynomials is a quasi-polynomial.*

Proof. It suffices to prove the claim for two quasi-polynomials. Let $f_1, f_2 : \mathbb{N}^t \longrightarrow \mathbb{N}$ be quasi-polynomials of periods d_1, d_2 respectively and let

$$\{p_{a_1\cdots a_t} \mid \forall\, i = 0,\ldots,t,\ 0 \leq a_i \leq d_1 - 1\}, \quad \text{and}$$
$$\{q_{b_1\cdots b_t} \mid \forall\, i = 0,\ldots,t,\ 0 \leq b_i \leq d_2 - 1\}$$

be the families of polynomials that define f_1 and f_2 respectively. Define a new quasi-polynomial f as follows. Take $d = d_1d_2$ as the period of f and, for every $(c_1,\ldots,c_t) \in \{0,1,\ldots,d-1\}^t$, take

$$f_{c_1\cdots c_t} = p_{a_1\cdots a_t} + q_{b_1\cdots b_t},$$

where, for any $i = 1,\ldots,t$, a_i and b_i are the remainders of the division of c_i by d_1 and d_2 respectively. It is easily checked that the quasi-polynomial f is the sum of f_1 and f_2. Indeed, if $x = (x_1,\ldots,x_t) \in \mathbb{N}^t$ and, for every $i = 1,\ldots,t$, $x_i \equiv c_i \mod d$, then one has

$$c_i \equiv a_i \mod d_1 \iff x_i \equiv a_i \mod d_1$$
$$c_i \equiv b_i \mod d_2 \iff x_i \equiv b_i \mod d_2.$$

Therefore, if $x = (x_1,\ldots,x_t) \in \mathbb{N}^t$ and $x_i \equiv c_i \bmod d$, then we have:

$$f(x) = f_{c_1\cdots c_t}(x) = p_{a_1\cdots a_t}(x) + q_{b_1\cdots b_t}(x) = f_1(x) + f_2(x).$$

The claim is thus proved. □

Lemma 3. *Let $F : \mathbb{N}^t \longrightarrow \mathbb{N}$ be a map, d be a positive integer, and C be a subset of $\mathbb{N}^t$. If there exists a family of quasi-polynomials $\{F_{d_1d_2\cdots d_t} \mid d_1,\ldots,d_t \in \mathbb{N},\ 0 \leq d_i < d\}$, such that , for every $(x_1,\ldots,x_t) \in C$, with $x_i \equiv d_i \bmod d$, one has: $F(x_1,\ldots,x_t) = F_{d_1d_2\cdots d_t}(x_1,\ldots,x_t)$, then F is a quasi-polynomial over C.*

Proof. Let k be the least common multiple of d and of the periods of the quasi-polynomials of the set $\{F_{d_1d_2\cdots d_t} \mid d_1,\ldots,d_t \in \mathbb{N},\ 0 \leq d_i < d\}$. Let $(r_1,\ldots,r_t)$ be a tuple of $\{0,1,\ldots k-1\}^t$ and let $(x_1,\ldots,x_t) \in C$ be such that, for every $i = 1,\ldots,t$, $x_i \equiv r_i \bmod k$. Then one can check that $F(x_1,\ldots,x_t) = q(x_1,\ldots,x_t)$ where q is a polynomial uniquely determined by $(r_1,\ldots,r_t)$. Indeed, one can first observe that the tuple $(r_1,\ldots,r_t)$ uniquely determines, for every $i = 1,\ldots,t$, the remainder d_i of the division of x_i by d since $d_i \equiv r_i \bmod d$. By hypothesis, one has $F(x_1,\ldots,x_t) = F_{d_1d_2\cdots d_t}(x_1,\ldots,x_t)$. Since k is a multiple of the period of the quasi-polynomial $F_{d_1d_2\cdots d_t}$, the tuple $(r_1,\ldots,r_t)$ also determines a polynomial q in the family of polynomials associated with $F_{d_1d_2\cdots d_t}$, such that $F(x_1,\ldots,x_t) = F_{d_1d_2\cdots d_t}(x_1,\ldots,x_t) = q(x_1,\ldots,x_t)$. The proof is thus complete. □

Lemma 4. *Let $\lambda : \mathbb{N}^t \longrightarrow \mathbb{Q}$ be a map such that, for any $(x_1,\ldots,x_t) \in \mathbb{N}^t$,*

$$\lambda(x_1,\ldots,x_t) = b_1x_1 + \cdots + b_tx_t,$$

where $b_1,\ldots,b_t$ are given rational coefficients. Let C, C' be subsets of $\mathbb{N}^t$ and let $a_1,\ldots,a_t$ be non-negative integers such that the following properties are satisfied: for any $(x_1,\ldots,x_t) \in C$, one has

- $\lambda(x_1,\ldots,x_t) \geq 0$,
- *if $\lambda \in \mathbb{N}$ and $\lambda < \lambda(x_1,\ldots,x_t)$, then $(x_1 - \lambda a_1, x_2 - \lambda a_2,\ldots,x_t - \lambda a_t) \in C'$.*

Let p be a quasi-polynomial over C' and define the map F as:

$$F(x_1,\ldots,x_t) = \sum_{0 \leq \lambda < \lambda(x_1,\ldots,x_t)} p(x_1 - \lambda a_1, x_2 - \lambda a_2,\ldots,x_t - \lambda a_t).$$

Then F is a quasi-polynomial over C.

Proof. Let $d \geq 1$ be the period of the quasi-polynomial p. Let $p_{d_1 d_2 \cdots d_t}$, with $0 \leq d_i \leq d-1$, be the polynomials defining p. Consider the set of integers μ:

$$0 \leq \mu \leq \lceil \lambda(x_1, \ldots, x_t) \rceil - 1,$$

and consider on it the partition:

$$F_0(x_1, \ldots, x_t) \cup F_1(x_1, \ldots, x_t) \cup \cdots \cup F_{d-1}(x_1, \ldots, x_t), \tag{4}$$

defined as: for any $j = 0, \ldots, d-1$:

$$\mu \in F_j(x_1, \ldots, x_t) \iff 0 \leq \mu \leq \lceil \lambda(x_1, \ldots, x_t) \rceil - 1 \text{ and } \mu \equiv j \pmod{d}.$$

By (4), for every $(x_1, \ldots, x_t) \in C$, we have:

$$F(x_1, \ldots, x_t) = \sum_{j=0}^{d-1} S_j(x_1, \ldots, x_t), \tag{5}$$

where, for any $j = 0, \ldots, d-1$:

$$S_j(x_1, \ldots, x_t) = \sum_{\mu \in F_j(x_1, \ldots, x_t)} p(x_1 - \mu a_1, x_2 - \mu a_2, \ldots, x_t - \mu a_t).$$

Now we prove that, for any $j = 0, \ldots, d-1$, S_j is a quasi-polynomial over C.

Let us fix a tuple $(d_1, \ldots, d_t) \in \{0, 1, \ldots d-1\}^t$. Let $(x_1, \ldots, x_t)$ be such that $x_i \equiv d_i \bmod d$. Then for any $\mu \in F_j(x_1, \ldots, x_t)$ one has $x_i - \mu a_i \equiv d_i - j a_i \bmod d$. Now set $q = p_{c_1 c_2 \cdots c_t}$, where $(c_1, \ldots, c_t) \in \{0, 1, \ldots d-1\}^t$ and $c_i \equiv d_i - j a_i \bmod d$. One has

$$S_j(x_1, \ldots, x_t) = \sum_{\mu \in F_j(x_1, \ldots, x_t)} q(x_1 - \mu a_1, x_2 - \mu a_2, \ldots, x_t - \mu a_t).$$

On the other side, by (4), one easily checks:

$$F_j(x_1, \ldots, x_t) = \left\{ j + d\mu \in \mathbb{N} \;\middle|\; 0 \leq \mu \leq \left\lfloor \frac{\lceil \lambda(x_1, \ldots, x_t) \rceil - 1 - j}{d} \right\rfloor \right\}.$$

It is important to remark that, in the formula above, if $\lambda(x_1, \ldots, x_t) = 0$, then

$$\frac{\lceil \lambda(x_1, \ldots, x_t) \rceil - 1 - j}{d} < 0.$$

In this case, since for any $0 \leq j \leq d-1$, $|-1-j| \leq d$, one has

$$\left\lfloor \frac{-1-j}{d} \right\rfloor = -1.$$

Hence $\lambda(x_1, \ldots, x_t) = 0$ implies that $F_j(x_1, \ldots, x_t)$ is the empty set and thus S_j is the null map. Let us consider the map

$$S : \mathbb{N}^t \times \{\{-1\} \cup \mathbb{N}\} \longrightarrow \mathbb{N},$$

where $S(x_1, \ldots, x_t, x)$ is defined as:

$$\begin{cases} \sum_{\mu=0}^{x} q(x_1 - (j + \mu d)a_1, x_2 - (j + \mu d)a_2, \ldots, x_t - (j + \mu d)a_t) & x \geq 0 \\ 0 & x = -1. \end{cases}$$

By the definition of the map S, for every $(x_1, \ldots, x_t) \in C$, with $x_i \equiv d_i \bmod d$, one has:

$$S_j(x_1, \ldots, x_t) = S\left(x_1, \ldots, x_t, \left\lfloor \frac{\lceil \lambda(x_1, \ldots, x_t) \rceil - 1 - j}{d} \right\rfloor \right). \tag{6}$$

Therefore, by applying Lemma 1 to S, there exists a polynomial $Q(x_1, \ldots, x_t, x)$ such that, for any $(x_1, \ldots, x_t, x) \in \mathbb{N}^t \times \{\{-1\} \cup \mathbb{N}\}$,

$$S(x_1, \ldots, x_t, x) = Q(x_1, \ldots, x_t, x), \tag{7}$$

hence, from Eqs. (6) and (7), for every $(x_1, \ldots, x_t) \in C$, with $x_i \equiv d_i \bmod d$, one has

$$S_j(x_1, \ldots, x_t) = Q\left(x_1, \ldots, x_t, \left\lfloor \frac{\lceil \lambda(x_1, \ldots, x_t) \rceil - 1 - j}{d} \right\rfloor \right). \tag{8}$$

By Lemma 22 of the Appendix, one has that:

$$\left\lfloor \frac{\lceil \lambda(x_1, \ldots, x_t) \rceil - 1 - j}{d} \right\rfloor \tag{9}$$

is a quasi-polynomial in the variables $x_1, \ldots, x_t$. Therefore, by (8), S_j coincides with a quasi-polynomial on the set of points $(x_1, \ldots, x_t) \in C$, with $x_i \equiv d_i \bmod d$. Obviously this fact holds for any $(d_1, \ldots, d_t) \in \{0, 1, \ldots d-1\}^t$ and, by Lemma 3, S_j is a quasi-polynomial over C. Finally, the fact that F is a quasi-polynomial over C follows from (5) by using Lemma 2. $\square$

Lemma 5. *Let $\lambda : \mathbb{N}^t \longrightarrow \mathbb{Q}$ be a map such that, for any $(x_1, \dots, x_t) \in \mathbb{N}^t$,*

$$\lambda(x_1, \dots, x_t) = b_1x_1 + \cdots + b_tx_t,$$

where $b_1, \dots, b_t$ are given rational coefficients.

Let C, C' be subsets of $\mathbb{N}^t$ and let $a_1, \dots, a_t$ be non-negative integers such that the following properties are satisfied: for any $(x_1, \dots, x_t) \in C$, one has

- $\lambda(x_1, \dots, x_t) \geq 0$,
- *for any $\lambda \in \mathbb{N}$ such that $\lambda < \lambda(x_1, \dots, x_t)$, $(x_1 - \lambda a_1, x_2 - \lambda a_2, \dots, x_t - \lambda a_t) \in C'$.*

Let p be a quasi-polynomial over C' and define the map F as:

$$F(x_1, \dots, x_t) = \sum_{0 \,\leq\, \lambda \,\leq\, \lambda(x_1,\dots,x_t)} p(x_1 - \lambda a_1, x_2 - \lambda a_2, \dots, x_t - \lambda a_t).$$

Then F is a quasi-polynomial over C.

Proof. The proof of Lemma 5 is the same as that of Lemma 4 except the point we describe now. In the sum above that defines the map F, the index λ runs over the set of integers of the closed interval $[0, \lambda(x_1, \dots, x_t)]$ so that:

$$\lambda \leq \lfloor \lambda(x_1, \dots, x_t) \rfloor.$$

Therefore, in order to prove the claim, one has to prove a slightly modified version of (9) of Lemma 4, that is: for any $j = 0, \dots, d - 1$,

$$\left\lfloor \frac{\lfloor \lambda(x_1, \dots, x_t) \rfloor - j}{d} \right\rfloor,$$

is a quasi-polynomial with rational coefficients in the variables $x_1, \dots, x_t$. This can be done by using an argument very similar to that one adopted in the proof of (9). □

Lemma 6. *Let $\lambda_1, \lambda_2 : \mathbb{N}^t \longrightarrow \mathbb{Q}$ be two maps such that, for any $(x_1, \dots, x_t) \in \mathbb{N}^t$,*

$$\lambda_1(x_1, \dots, x_t) = b_1x_1 + \cdots + b_tx_t, \qquad \lambda_2(x_1, \dots, x_t) = c_1x_1 + \cdots + c_tx_t$$

where $b_1, \dots, b_t$ and $c_1, \dots, c_t$ are given rational coefficients.

Let C be a subset of $\mathbb{N}^t$ and let $a_1, \dots, a_t$ be non-negative integers. Suppose that, for any $(x_1, \dots, x_t) \in C$, one has:

$$0 \leq \lambda_1(x_1, \dots, x_t) \leq \lambda_2(x_1, \dots, x_t),$$

and, for any $\lambda \in \mathbb{N}$ such that $\lambda_1(x_1, \dots, x_t) \leq \lambda \leq \lambda_2(x_1, \dots, x_t)$, one has:

$$(x_1 - \lambda a_1, x_2 - \lambda a_2, \dots, x_t - \lambda a_t) \in C',$$

where C' is a given subset of $\mathbb{N}^t$. Let p be a quasi-polynomial over C' and define the map F as:

$$F(x_1, \dots, x_t) = \sum_{\lambda_1 \,\leq\, \lambda \,\leq\, \lambda_2} p(x_1 - \lambda a_1, x_2 - \lambda a_2, \dots, x_t - \lambda a_t),$$

where $\lambda_1 = \lambda_1(x_1, \dots, x_t)$ and $\lambda_2 = \lambda_2(x_1, \dots, x_t)$. Then F is a quasi-polynomial over C. The same result holds whenever the index λ runs in the set of integers of the intervals:

$$(\lambda_1, \lambda_2), \quad (\lambda_1, \lambda_2], \quad [\lambda_1, \lambda_2).$$

Proof. Let us prove the case when λ runs in the interval $[\lambda_1, \lambda_2]$. Write

$$F(x_1, \dots, x_t) = S_1(x_1, \dots, x_t) - S_2(x_1, \dots, x_t), \tag{10}$$

where:

$$S_1(x_1, \dots, x_t) = \sum_{0 \,\leq\, \lambda \,\leq\, \lambda_2} p(x_1 - \lambda a_1, x_2 - \lambda a_2, \dots, x_t - \lambda a_t),$$

and

$$S_2(x_1, \dots, x_t) = \sum_{0 \leq \lambda < \lambda_1} p(x_1 - \lambda a_1, x_2 - \lambda a_2, \dots, x_t - \lambda a_t).$$

By applying Lemma 5 to S_1 and Lemma 4 to S_2, we have that S_1 and S_2 are quasi-polynomial and by Lemma 2, so is $S_1 - S_2$. The claim now follows from (10). The other three cases are similarly proved. □

Lemma 7. *Assuming the same hypotheses as in Lemma 4, the function*

$$S(x_1,\ldots,x_t) = \sum_{\lambda(x_1,\ldots,x_t)\,\le\,\lambda\,\le\,\lambda(x_1,\ldots,x_t)} p(x_1-\lambda a_1, x_2-\lambda a_2,\ldots,x_t-\lambda a_t).$$

is a quasi-polynomial over C.

Proof. It is a direct consequence of Lemma 6, assuming $\lambda_1(x_1,\ldots,x_t) = \lambda_2(x_1,\ldots,x_t) = \lambda(x_1,\ldots,x_t)$. □

Now we want to define some suitable regions of $\mathbb{R}^t$. More precisely, our regions will be polyhedral cones determined by a family of hyperplanes through the origin. We proceed as follows. Let π be a hyperplane of $\mathbb{R}^t$. Let us fix an equation for π denoted by $\pi(x) = 0$. We associate with π a map

$$f_\pi : \mathbb{R}^t \longrightarrow \{+,-,0\}$$

defined as: for any $x \in \mathbb{R}^t$,

$$f_\pi(x) = \begin{cases} + & \text{if } \pi(x) > 0,\\ 0 & \text{if } \pi(x) = 0,\\ - & \text{if } \pi(x) < 0.\end{cases}$$

We remark that the map defined above depends upon the hyperplane π and its equation in the obvious geometrical way. We can now give the following important two definitions.

Definition 3. Let $\Pi = \{\pi_1,\ldots,\pi_m\}$ be a family of hyperplanes of $\mathbb{R}^t$ with rational coefficients that satisfy the following property:

- Π includes the coordinate hyperplanes, that is, the hyperplanes defined by the equations $x_\ell = 0,\ \ell = 0,\ldots,t$;
- every hyperplane of Π passes through the origin.

Let $\sim$ be the equivalence defined over the set $\mathbb{N}^t$ as: for any $x,\ x' \in \mathbb{N}^t$,

$$x \sim x' \iff \forall\, i = 1,\ldots,m, \quad f_{\pi_i}(x) = f_{\pi_i}(x').$$

A subset C of $\mathbb{N}^t$ is called a *region (with respect to Π)* if it is a coset of $\sim$.

It may be useful to keep in mind that the singleton composed by the origin is a region. Moreover if $t = 2$, the set of all points of $\mathbb{N}^t \setminus \{0\}$ of every line of Π is a region also.

Definition 4. Let $F : \mathbb{N}^t \longrightarrow \mathbb{N}$ be a map. Then F is said to be a *a quasi-polynomial on polyhedral conic region* (*prc-quasi-polynomial*, for short) in $\mathbb{N}^t$ if there exists a partition $\mathcal{C} = \{C_1,\ldots,C_s\}$ of regions of $\mathbb{N}^t$ – defined by a family of hyperplanes satisfying Definition 3 – and a family $p_1,\ldots,p_s$ of quasi-polynomials, every one of which is associated with exactly one region of $\mathcal{C}$, such that, for any $(x_1,\ldots,x_t) \in \mathbb{N}^t$, one has:

$$F(x_1,\ldots,x_t) = p_j(x_1,\ldots,x_t),$$

where j is the index of the region C_j that contains $(x_1,\ldots,x_t)$.

Lemma 8. *The sum of a finite family of prc-quasi-polynomials is a prc-quasi-polynomial.*

Proof. It suffices to prove the claim for two prc-quasi-polynomials. Let F_1 and F_2 be two prc-quasi-polynomials and let $\mathcal{C} = \{C_1,\ldots,C_s\}$ and $\mathcal{D} = \{D_1,\ldots,D_r\}$ be the families of regions of F_1 and F_2 respectively. Moreover, let $\{p_1,\ldots,p_s\}$ and $\{q_1,\ldots,q_r\}$ be the families of quasi-polynomials of F_1 and F_2 respectively.

Consider the prc-quasi-polynomial defined as follows. Let $\mathcal{E}$ be the partition of regions of $\mathbb{N}^t$ given by the intersection of $\mathcal{C}$ and $\mathcal{D}$ respectively. It is worth noticing that $\mathcal{E}$ is determined by the union of the two families of hyperplanes that define $\mathcal{C}$ and $\mathcal{D}$ respectively. Then we associate the map $r_{lm} = p_l + q_m$ with every region E_{lm} of $\mathcal{E}$. By Lemma 2, r_{lm} is a quasi-polynomial. For any $x \in \mathbb{N}^t$ we have

$$F_1(x) = p_l(x), \qquad F_2(x) = q_m(x),$$

where l and m are the indices of the regions C_l and D_m that contain x. Hence we have

$$F_1(x) + F_2(x) = r_{lm}(x),$$

while x belongs to the region E_{lm}. Since r_{lm} is the quasi-polynomial associated with E_{lm}, this proves that $F_1 + F_2$ is equal to the prc-quasi-polynomial defined above. □

The following lemma is a crucial tool in the proof of the main result of this section.

Lemma 9. *Let $G : \mathbb{N}^t \longrightarrow \mathbb{N}$ be a prc-quasi-polynomial and let $a_1, \ldots, a_t \in \mathbb{N}$ with $(a_1, \ldots, a_t) \neq (0, \ldots, 0)$. Consider the map $\Lambda : \mathbb{N}^t \longrightarrow \mathbb{N}$ that associates with every $(x_1, \ldots, x_t) \in \mathbb{N}^t$, the value*

$$\Lambda(x_1, \ldots, x_t) = \min \left\{ \frac{x_i}{a_i} \mid a_i \neq 0 \right\}.$$

Let $\mathcal{S} : \mathbb{N}^t \longrightarrow \mathbb{N}$ be the map defined as: for every $(x_1, \ldots, x_t) \in \mathbb{N}^t$,

$$\mathcal{S}(x_1, \ldots, x_t) = \sum_{0 \leq \lambda \leq \Lambda(x_1, \ldots, x_t)} G(x_1 - \lambda a_1, x_2 - \lambda a_2, \ldots, x_t - \lambda a_t). \tag{11}$$

Then $\mathcal{S}$ is a prc-quasi-polynomial.

Proof. In order to prove the claim, we first associate with the map $\mathcal{S}$ a new family of regions that we define now. Let Π be the family of hyperplanes associated with the prc-quasi-polynomial G. For any $(x_1, x_2, \ldots, x_t) \in \mathbb{N}^t$ consider the line defined by the equation parameterized by λ:

$$(x_1 - \lambda a_1, x_2 - \lambda a_2, \ldots, x_t - \lambda a_t). \tag{12}$$

Let π be a hyperplane of the family Π and let $\pi(x) = \sum_{i=1,\ldots,t} \beta_i x_i = 0$ be its equation. The value of λ that defines the point of meeting of the line (12) with π is easily computed. Indeed, λ is such that

$$\sum_{i=1,\ldots,t} \beta_i (x_i - \lambda a_i) = 0,$$

so that

$$\sum_{i=1,\ldots,t} \beta_i x_i = \lambda \cdot \sum_{i=1,\ldots,t} \beta_i a_i \tag{13}$$

which gives

$$\lambda = \sum_{i=1,\ldots,t} \frac{\beta_i}{\gamma} x_i, \tag{14}$$

where

$$\gamma = \sum_{i=1,\ldots,t} \beta_i a_i.$$

It is worth to remark that (14) is not defined whenever

$$\gamma = \sum_{i=1,\ldots,t} \beta_i a_i = 0. \tag{15}$$

Let us first treat (15). Here, either the line of (12) belongs to π or such a line is parallel to π. Therefore, for every point x of the line of (12), the value of $f_\pi(x)$ is constant so that π is not relevant in determining a change of region when a point is moving on the line of (12). Because of this remark, we shall consider only hyperplanes of Π for which (15) does not hold. Denote Π' this set of hyperplanes. For any $\pi \in \Pi'$, with equation $\pi(x) = \sum_{i=1,\ldots,t} \beta_i x_i = 0$, consider the homogeneous linear polynomial

$$\lambda_\pi(x_1, \ldots, x_t) = \sum_{i=1,\ldots,t} \frac{\beta_i}{\gamma} x_i,$$

where $\gamma = \sum_{i=1,\ldots,t} \beta_i a_i$. We remark that for any $(x_1, x_2, \ldots, x_t) \in \mathbb{N}^t$, the line parameterized by (12) meets the hyperplane π in the point corresponding to the parameter $\lambda = \lambda_\pi(x_1, \ldots, x_t)$.

Consider an enumeration of the hyperplanes of the set Π and denote by $<$ the linear order on Π defined by such enumeration. Consider the new family $\widehat{\Pi}$ of hyperplanes defined by the following sets of equations:

1. $\pi(x) = 0, \pi \in \Pi$
2. $\lambda_{\pi\pi'}(x_1, \ldots, x_k) = 0$, with $\pi, \pi' \in \Pi', \pi < \pi'$, and $\lambda_{\pi\pi'}(x_1, \ldots, x_k) = \lambda_\pi(x_1, \ldots, x_k) - \lambda_{\pi'}(x_1, \ldots, x_k)$.

Call $\widehat{\mathcal{C}}$ the family of regions of $\mathbb{N}^t$ defined by $\widehat{\Pi}$.

We now associate with every region of $\widehat{\mathcal{C}}$ a quasi-polynomial. In order to do this, we need to establish some preliminary facts. Let us fix now a region C of $\widehat{\mathcal{C}}$ and let $x = (x_1, \ldots, x_t)$ be a point of $\mathbb{N}^t$ that belongs to C. Let i be such that

$$\Lambda(x) = \frac{x_i}{a_i}.$$

Observe that, for any other point $x' = (x'_1, \dots, x'_t)$ in C, one has

$$\Lambda(x') = \frac{x'_i}{a_i}.$$

Indeed, it is enough to prove that, for any given pair of distinct indices i, j, we have:

$$\frac{x_i}{a_i} \le \frac{x_j}{a_j} \iff \frac{x'_i}{a_i} \le \frac{x'_j}{a_j}.$$

This is equivalent to say that:

$$\lambda_{\pi\pi'}(x_1, \dots, x_t) \le 0 \iff \lambda_{\pi\pi'}(x'_1, \dots, x'_t) \le 0,$$

where π, π' are the hyperplanes $x_i = 0$ and $x_j = 0$ respectively. The previous equivalence is true because x and x' belong to the same region of $\widehat{\mathcal{C}}$.
Another important fact is the following. Let us consider any point x of the region C of $\widehat{\mathcal{C}}$. Consider the subset of hyperplanes of Π':

$$\{\pi_1, \dots, \pi_m\} = \{\pi \in \Pi' \mid 0 \le \lambda_\pi(x) \le \Lambda(x)\}.$$

We can always assume, possibly changing the enumeration of the above hyperplanes, that

$$0 \le \lambda_{\pi_1}(x) \le \cdots \le \lambda_{\pi_m}(x) \le \Lambda(x).$$

Remark. Observe that, for any other point x' of C, one has

$$\{\pi_1, \dots, \pi_m\} = \{\pi \in \Pi' \mid 0 \le \lambda_\pi(x') \le \Lambda(x)\},$$

and

$$0 \le \lambda_{\pi_1}(x') \le \cdots \le \lambda_{\pi_m}(x') \le \Lambda(x').$$

The remark above can be proved by using an argument very similar to that used to prove the previous condition. We suppose that the above inequalities are strict, i.e. $0 < \lambda_{\pi_1}(x) < \cdots < \lambda_{\pi_m}(x) < \Lambda(x)$. In this case, as before, one proves that the same inequalities are strict for any other point x' of the region C. The case when the inequalities are not all strict can be treated similarly.

From now on, by the sake of clarity, for any $x = (x_1, \dots, x_t) \in \mathbb{N}^t$ we set $y_\lambda(x) = (x_1 - \lambda a_1, \dots, x_t - \lambda a_t)$.
Consider the following sets:

- $Y_0(x) = \{y_\lambda(x) \mid \lambda \in \mathbb{N} \cap [0, \lambda_{\pi_1}(x))\}$,
- $Y_m(x) = \{y_\lambda(x) \mid \lambda \in \mathbb{N} \cap (\lambda_{\pi_m}, \Lambda(x))\}$,
- $Y_i(x) = \{y_\lambda(x) \mid \lambda \in \mathbb{N} \cap (\lambda_{\pi_i}(x), \lambda_{\pi_{i+1}}(x))\}$, $i = 1, \dots, m-1$.
- $Z_i(x) = \{y_\lambda(x) \mid \lambda \in \mathbb{N} \cap \{\lambda_{\pi_i}(x)\}\}$, $i = 1, \dots, m$.
- $Z_{m+1}(x) = \{y_\lambda(x) \mid \lambda \in \mathbb{N} \cap \{\Lambda(x)\}\}$.

We are now able to associate a quasi-polynomial with the region C of $\widehat{\mathcal{C}}$. For this purpose, take two points x, x' in C. By the facts discussed before, one has that the lines of (12) associated with x and x' respectively, meet the hyperplanes of Π' in the same order. We recall, that a change of region on the generic line of (12) happens only when the line meets a hyperplane of Π'. Therefore, since $\widehat{\mathcal{C}}$ is a refinement of $\mathcal{C}$ and x and x' are in a same region with respect to $\mathcal{C}$, the above conditions imply that, for every $i = 0, \dots, m$, the two sets of points $Y_i(x)$ and $Y_i(x')$ are subsets of a same common region of $\mathcal{C}$. Hence there exists a quasi-polynomial p_i, depending on i and on the region C, such that, for any $y \in Y_i(x)$ and for any $y' \in Y_i(x')$, $G(y) = p_i(y)$, $G(y') = p_i(y')$. By the previous remark and by Lemma 6, one has that, for any $i = 0, \dots, m$, there exists a quasi-polynomial q_i, depending on i, and on C, such that for any $x \in C$

$$q_i(x) = \sum_{y \in Y_i(x)} G(y).$$

Observe that, since x and x' are in the same region C, as before one derives that $Z_i(x)$ and $Z_i(x')$ are in the same region with respect to $\mathcal{C}$. Therefore, as before, by applying Lemma 7 there exists a quasi-polynomial r_i, depending on i and on C, such that for any $x \in C$

$$r_i(x) = \sum_{y \in Z_i(x)} G(y).$$

On the other hand, by (11), we have that, for any $(x_1, \dots, x_t) \in C$, $\mathcal{S}(x_1, \dots, x_t)$ is equal to:

$$q_0(x) + r_1(x) + q_1(x) + r_2(x) + q_2(x) + \cdots r_m(x) + q_m(x) + r_{m+1}(x). \tag{16}$$

Thus, $\mathcal{S}(x_1, \dots, x_t)$ on the region C is represented as a sum of quasi-polynomials. This, together with Lemma 2 applied to (16) imply that the map $\mathcal{S}$ is a quasi-polynomial over every region of $\widehat{\mathcal{C}}$. The proof of the claim is thus complete. □

Theorem 1. *Let*

$$\mathcal{S} : \mathbb{N}^t \longrightarrow \mathbb{N}$$

be the map which counts, for any vector $(n_1, \ldots, n_t) \in \mathbb{N}^t$, the number of distinct non-negative solutions of a given Diophantine system:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1k}x_k = & n_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2k}x_k = & n_2 \\ \quad\quad\quad\quad \cdot & \cdot \\ \quad\quad\quad\quad \cdot & \cdot \\ \quad\quad\quad\quad \cdot & \cdot \\ a_{t1}x_1 + a_{t2}x_2 + \cdots + a_{tk}x_k = & n_t \end{cases} \tag{17}$$

where the numbers $a_{ij} \in \mathbb{N}$ and, for every $i = 1, \ldots, k$, there exists $j = 1, \ldots, t$ such that $a_{ij} \neq 0$.

Then there exists a partition $\mathcal{C} = \{C_1, \ldots, C_s\}$ of polyhedral conic regions of $\mathbb{N}^t$ – defined by a family of hyperplanes satisfying Definition 3 – and a family $p_1, \ldots, p_s$ of quasi-polynomials with rational coefficients, every one of which is associated with exactly one region of $\mathcal{C}$, such that, for any $(x_1, \ldots, x_t) \in \mathbb{N}^t$, one has:

$$\mathcal{S}(x_1, \ldots, x_t) = p_j(x_1, \ldots, x_t),$$

where j is the index of the region C_j that contains $(x_1, \ldots, x_t)$. Moreover such regions and quasi-polynomials can be effectively constructed starting from the coefficients of the system.

Proof. For any vector $(n_1, \ldots, n_t) \in \mathbb{N}^t$, let $\mathrm{Sol}(n_1, \ldots, n_t)$ be the set of the non-negative solutions of the Diophantine system (17) and denote by $\mathcal{S} : \mathbb{N}^t \longrightarrow \mathbb{N}$, the map defined as: for any vector $(n_1, \ldots, n_t) \in \mathbb{N}^t$,

$$\mathcal{S}(n_1, \ldots, n_t) = \mathrm{Card}(\mathrm{Sol}(n_1, \ldots, n_t)),$$

that is, it associates with every vector $(n_1, \ldots, n_t)$ the number of non-negative distinct solutions of the system (17). Let us prove that the map $\mathcal{S}$ is a prc-quasi-polynomial. For this purpose, we proceed by induction on the number of unknowns of the system (17). We start by proving the basis of the induction. In this case, our system has one unknown, say x, and it can be written as:

$$\begin{cases} a_1x = & n_1 \\ a_2x = & n_2 \\ \quad \cdot & \\ \quad \cdot & \\ \quad \cdot & \\ a_tx = & n_t. \end{cases}$$

The system has solutions (and, in this case, it is unique) if and only if there exists $\lambda \in \mathbb{N}$ such that:

$$\lambda(a_1, \ldots, a_t) = (\lambda a_1, \ldots, \lambda a_t) = (n_1, \ldots, n_t). \tag{18}$$

Let us consider the line ℓ (through the origin) defined by the parametric equation (18). The line ℓ can be determined as the intersection of suitable hyperplanes through the origin. Let us consider the family of regions defined by the set of these hyperplanes together with the coordinate hyperhyperplanes. One can easily associate with every region a quasi-polynomial. For this purpose, we remark that the set of points of the line ℓ with integral coordinates, without the origin, is a region. On this region, the counting function of the system takes the value 0 or 1. Therefore this map coincides with the quasi-polynomial given by $p = 0,\ q = 1$ with the periodical rule $d = lcm\{a_1, \ldots, a_t\}$. To any other region, we associate p. The basis of the induction is thus proved.

Let us now prove the inductive step. If $(x_1, \ldots, x_k) \in \mathrm{Sol}(n_1, \ldots, n_t)$, the system (17) can be written as:

$$\begin{cases} a_{12}x_2 + \cdots + a_{1k}x_k = & n_1 - a_{11}x_1 \\ a_{22}x_2 + \cdots + a_{2k}x_k = & n_2 - a_{21}x_1 \\ \quad\quad \cdot & \quad \cdot \\ \quad\quad \cdot & \quad \cdot \\ \quad\quad \cdot & \quad \cdot \\ a_{t2}x_2 + \cdots + a_{tk}x_k = & n_t - a_{t1}x_1. \end{cases} \tag{19}$$

This implies that:

$$n_1 - a_{11}x_1 \geq 0, \qquad n_2 - a_{21}x_1 \geq 0, \qquad n_t - a_{t1}x_1 \geq 0,$$

so that, since x_1 must be an integer ≥ 0, one has:

$$0 \leq x_1 \leq \frac{n_1}{a_{11}}, \qquad 0 \leq x_1 \leq \frac{n_2}{a_{21}}, \qquad \ldots, \qquad 0 \leq x_1 \leq \frac{n_t}{a_{t1}},$$

and thus:

$$0 \le x_1 \le \Lambda(n_1, \ldots, n_t),$$

where the map $\Lambda : \mathbb{N}^t \longrightarrow \mathbb{N}$ is defined as:

$$\Lambda(x_1, \ldots, x_t) = \min\left\{ \frac{x_i}{a_{i1}} \;\mid\; a_{i1} \neq 0 \right\}. \tag{20}$$

We remark that, since the vector $(a_{11}, a_{21}, \ldots, a_{t1}) \neq (0, 0, \ldots, 0)$, the map Λ is well defined. Set $K = \lfloor \Lambda(x_1, \ldots, x_t) \rfloor$. We can write $\mathrm{Sol}(n_1, \ldots, n_t)$ as:

$$\mathrm{Sol}(n_1, \ldots, n_t) = (0 \times \mathrm{Sol}_0) \cup (1 \times \mathrm{Sol}_1) \cup \cdots \cup (K \times \mathrm{Sol}_K), \tag{21}$$

where, for every $i = 0, \ldots, K$, Sol_i denotes the set of non-negative solutions of the Diophantine system:

$$\begin{cases} a_{12}x_2 + \cdots + a_{1k}x_k = & n_1 - a_{11}i \\ a_{22}x_2 + \cdots + a_{2k}x_k = & n_2 - a_{21}i \\ \quad\vdots & \quad\vdots \\ a_{t2}x_2 + \cdots + a_{tk}x_k = & n_t - a_{t1}i. \end{cases} \tag{22}$$

By (21), for any $(n_1, \ldots, n_t) \in \mathbb{N}^t$, we have:

$$\mathcal{S}(n_1, \ldots, n_t) = \sum_{i=0,\ldots,K} \mathrm{Card}(\mathrm{Sol}_i). \tag{23}$$

By applying the inductive hypothesis to the system (22), we have that there exists a prc-quasi-polynomial $G : \mathbb{N}^t \longrightarrow \mathbb{N}$ such that, for any $(n_1, \ldots, n_t) \in \mathbb{N}^t$, if $0 \le i \le K$,

$$\mathrm{Card}(\mathrm{Sol}_i) = G(n_1 - a_{11}i, n_2 - a_{21}i, \ldots, n_t - a_{t1}i), \tag{24}$$

so that, by (23) and (24), one has:

$$\mathcal{S}(n_1, \ldots, n_t) = \sum_{0 \le \lambda \le \Lambda(n_1,\ldots,n_t)} G(n_1 - \lambda a_{11}, n_2 - \lambda a_{21}, \ldots, n_t - \lambda a_{t1}). \tag{25}$$

By (25), the fact that $\mathcal{S}$ is a prc-quasi-polynomial follows from Lemma 9. Finally we remark that the proof gives an effective procedure to construct the claimed prc-quasi-polynomial that describes the map $\mathcal{S}$. □

Corollary 2. *Let*

$$\mathcal{S} : \mathbb{N}^t \longrightarrow \mathbb{N}$$

be the map which counts, for any vector $(n_1, \ldots, n_t) \in \mathbb{N}^t$, the number of distinct non-negative solutions of a given Diophantine system:

$$\begin{cases} a_{10} + a_{11}x_1 + a_{12}x_2 + \cdots + a_{1k}x_k = & n_1 \\ a_{20} + a_{21}x_1 + a_{22}x_2 + \cdots + a_{2k}x_k = & n_2 \\ \quad\vdots & \vdots \\ a_{t0} + a_{t1}x_1 + a_{t2}x_2 + \cdots + a_{tk}x_k = & n_t, \end{cases} \tag{26}$$

where the numbers $a_{ij} \in \mathbb{N}$ and, for every $i = 1, \ldots, k$, there exists $j = 1, \ldots, t$ such that $a_{ij} \neq 0$. Set $a_0 = (a_{10}, a_{20}, \ldots, a_{t0})$. Then there exist a partition $\mathcal{C} = \{C_1, \ldots, C_s\}$ of polyhedral conic regions of $\mathbb{N}^t$ – defined by a family of hyperplanes satisfying Definition 3 – and a family $p_1, \ldots, p_s$ of quasi-polynomials with rational coefficients, every one of which is associated with exactly one region of $\mathcal{C}$, such that the following condition holds: for any $\eta = (n_1, \ldots, n_t) \in \mathbb{N}^t$ with $\eta \ge a_0$, one has:

$$\mathcal{S}(\eta) = p_j(\eta - a_0),$$

where j is the index of the region C_j that contains $\eta - a_0$. Moreover such regions and quasi-polynomials can be effectively constructed starting from the coefficients of the system.

Proof. First consider the system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1k}x_k = & n_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2k}x_k = & n_2 \\ \qquad\qquad\vdots & \vdots \\ a_{t1}x_1 + a_{t2}x_2 + \cdots + a_{tk}x_k = & n_t. \end{cases} \tag{27}$$

According to Theorem 1, there exists a prc-quasi-polynomial F that counts, for every $(n_1, \ldots, n_t) \in \mathbb{N}^t$, the number of the solutions of the diophantine system (27). Let $\mathcal{C} = \{C_1, \ldots, C_s\}$ be the partition of $\mathbb{N}^t$ in regions and let $\{p_1, \ldots, p_s\}$ be the family of quasi-polynomials that define F.

Let $a_0 = (a_{10}, \ldots, a_{t0})$ be the vector whose components are the entries of the first column of the matrix of the system (26). For every $\eta \in \mathbb{N}^t$ with $\eta \geq a_0$, the components of the vector $\eta - a_0$ are non-negative integers so that:

$$\mathcal{S}(\eta) = p_j(\eta - a_0),$$

where j is the index of the region of the family $\mathcal{C}$ that contains the vector $\eta - a_0$. This concludes the proof. □

Example 1. For the sake of clarity, we find useful to show the proof of Theorem 1 on the following very simple example. Consider the Diophantine system:

$$\begin{cases} x_1 + 2x_2 = & n_1 \\ 2x_1 + 3x_2 = & n_2, \end{cases} \tag{28}$$

where $n_1, n_2 \in \mathbb{N}$ and let $F : \mathbb{N}^2 \longrightarrow \mathbb{N}$ be the counting function of the system (28). By following the proof of Theorem 1, together with that of Lemma 9, we construct the partition of $\mathbb{N}^2$ in regions and the family of quasi-polynomials that describe the function F. In the sequel, the following notation is adopted: (x_1, x_2) and (n_1, n_2) are respectively the vector of the unknowns and the vector of the non-homogeneous terms of the system, while x, y are free variables over the set $\mathbb{N}$. Observe that x_1, x_2 gives a solution of (28) if and only if $(n_1 - x_1, n_2 - 2x_1) = (2t, 3t),\ t \geq 0$.

Therefore consider the Diophantine system:

$$\begin{cases} 2x_2 = & n_1 \\ 3x_2 = & n_2 \end{cases} \tag{29}$$

where $n_1, n_2 \in \mathbb{N}$. Let $G : \mathbb{N}^2 \longrightarrow \mathbb{N}$ be the counting function of the system (29). Let $\Pi = \{\pi_1, \pi_2, \pi_3\}$ be the set of the lines defined by the equations:

$$\pi_1(x, y) \equiv x = 0, \qquad \pi_2(x, y) \equiv y = 0, \qquad \pi_3(x, y) \equiv 3x - 2y = 0.$$

Let $\mathcal{R}$ be the partition of $\mathbb{N}^2$ determined by Π. Then $\mathcal{R}$ is formed by the following 6 regions:

$$R_0 = \{(0, 0)\}, \qquad R_1 = \{(x, 0) \ :\ x > 0\}, \qquad R_2 = \{(x, y) \ :\ 3x > 2y, x, y > 0\},$$
$$R_3 = \{(x, y) \ :\ 3x = 2y,\ x, y > 0\}, \qquad R_4 = \{(x, y) \ :\ 3x < 2y,\ x, y > 0\},$$
$$R_5 = \{(0, y) \ :\ y > 0\}.$$

Let $\mathcal{P}$ be the family of polynomials given by:

$$p_0(x, y) = p_3(x, y) \equiv 1, \qquad p_1(x, y) = p_2(x, y) = p_4(x, y) = p_5(x, y) \equiv 0.$$

One can check that the prc-quasi-polynomial determined by $\mathcal{R}$ and $\mathcal{P}$ is the function G.

Let (x, y) be a given point of $\mathbb{N}^2$ and let $\ell(\lambda)$ be the line represented by the equation parameterized by λ:

$$(x - \lambda, y - \lambda).$$

For every $i = 1, 2, 3$, let $\lambda_{\pi_i}(x, y)$ be the value of λ that defines the point of meeting of the line $\ell(\lambda)$ with π_i. Then one has:

$$\lambda_{\pi_1}(x, y) = x, \qquad \lambda_{\pi_2}(x, y) = y/2, \qquad \lambda_{\pi_3}(x, y) = 2y - 3x.$$

Consider the new family $\widehat{\Pi}$ of lines defined by the following sets of equations:

1. $\pi_i(x, y) = 0,\ i = 1, 2, 3,$ (that is, the lines in Π),
2. For every pair i, j of indices with $1 \leq i < j \leq 3$, let

$$\lambda_{\pi_{ij}}(x, y) \equiv \lambda_{\pi_i}(x, y) - \lambda_{\pi_j}(x, y) = 0.$$

It easily checked that there exists exactly one line of the previous form (2) which is represented by the equation $y - 2x = 0$. Thus one has:

$$\widehat{\Pi} = \{x = 0,\ y = 0,\ 2x - 3y = 0,\ 2x - y = 0\}.$$

If we denote by $\widehat{\mathcal{R}}$ the family of regions of $\mathbb{N}^2$ defined by $\widehat{\Pi}$, we have:

$$\widehat{R}_0 = \{(0,0)\},\quad \widehat{R}_1 = \{(x,0)\ :\ x > 0\},\quad \widehat{R}_2 = \{(x,y)\ :\ 3x > 2y,\ y > 0\},$$
$$\widehat{R}_3 = \{(x,y)\ :\ 3x = 2y,\ y > 0\},\quad \widehat{R}_4 = \{(x,y)\ :\ 3x < 2y,\ 2x > y,\ x,y > 0\},$$
$$\widehat{R}_5 = \{(x,y)\ :\ 2x = y,\ y > 0\},\quad \widehat{R}_6 = \{(x,y)\ :\ 2x < y,\ x,y > 0\},$$
$$\widehat{R}_7 = \{(0,y)\ :\ y > 0\}.$$

Now we associate with every region $\widehat{R}_i$ of $\widehat{R}$ a quasi-polynomial $\hat{q}_i$. Actually, set:

$$\hat{q}_0 = \hat{q}_3 = \hat{q}_4 = \hat{q}_5 \equiv 1 \quad \text{and} \quad \hat{q}_1 = \hat{q}_2 = \hat{q}_6 = \hat{q}_7 \equiv 0.$$

One can check that the prc-quasi-polynomial determined by $\widehat{\mathcal{R}}$ together with the list of polynomials above is the function F.

3. Preliminaries on context-free languages

The aim of this section is to present some results concerning bounded context-free languages. We assume that the reader is familiar with the basic notions of rational, context-free and semi-linear languages. The reader is referred to [1,3,8,12,18,27].

3.1. Semi-linear and semi-simple sets

Now we recall some results about semi-linear sets of the free commutative monoid and the free commutative group. For this purpose, we follow [3]. The free abelian monoid and the free abelian group on k generators are respectively identified with $\mathbb{N}^k$ and $\mathbb{Z}^k$ with the usual additive structure. The operation of addition is extended from elements to subsets: if $X, Y \subseteq \mathbb{N}^k$ (resp. $X, Y \subseteq \mathbb{Z}^k$), $X + Y \subseteq \mathbb{N}^k$ (resp. $X + Y \subseteq \mathbb{Z}^k$) is the set of all sum $x + y$, where $x \in X$, $y \in Y$. It might be convenient to consider the elements of $\mathbb{N}^k$ and $\mathbb{Z}^k$ as vectors of the vector space $\mathbb{Q}^k$. Given v in $\mathbb{N}^k$ or in $\mathbb{Z}^k$, the expression $\mathbb{N}v$ stands for the subset of all elements nv, where $n \in \mathbb{N}$. This expression can be extended to $\mathbb{Z}v$, whenever v is in $\mathbb{Z}^k$. Let $B = \{b_1, \ldots, b_\ell\}$ be a finite subset of $\mathbb{Z}^k$. Then we denote by $B^{\oplus}$ the submonoid of $\mathbb{N}^k$ generated by B, that is

$$B^{\oplus} \;=\; b_1^{\oplus} + \cdots + b_\ell^{\oplus} \;=\; \{n_1 b_1 + \cdots + n_k b_\ell \;\mid\; n_i \in \mathbb{N}\}.$$

In the sequel, the symbol $\mathbb{K}$ stands for $\mathbb{N}$ when it concerns the free abelian monoid $\mathbb{N}^k$ and for $\mathbb{Z}$ when it concerns the free abelian group $\mathbb{Z}^k$. The following definitions are useful.

Definition 5. Let X be a subset of $\mathbb{Z}^k$ (resp. $\mathbb{N}^k$). Then

1. X is $\mathbb{K}$-*linear* if it is of the form

$$a + \sum_{i=1}^{\ell} \mathbb{K} b_i,\ a, b_i \in \mathbb{Z}^k,\ (\text{resp.}\mathbb{N}^k),\quad i = 1, \ldots, \ell,\ \text{for some } \ell \geq 0;$$

2. X is $\mathbb{K}$-*simple* if the vectors b_i are linearly independent in $\mathbb{Q}^k$,
3. X is $\mathbb{K}$-*semi-linear* if X is a finite union of $\mathbb{K}$-linear sets;
4. X is *semi-simple* if X is a finite disjoint union of $\mathbb{K}$-simple sets.

Remark 2. In the definition of simple set, the vector a and those of the set $\{b_1, \ldots, b_\ell\}$ shall be called a *representation* of X.

There exists a classical and important connection between the concept of semi-linear set and the *Presburger arithmetic*. Denote by $\mathcal{Z} = \langle \mathbb{Z}; =; <; +; 0; 1\rangle$ and by $\mathcal{N} = \langle \mathbb{N}; =; +; 0; 1\rangle$ respectively the *standard* and the *positive Presburger arithmetic*. Given a subset X of $\mathbb{N}^k$ (resp. $\mathbb{Z}^k$), we say that X is *first-order definable* in $\mathbb{N}^k$ (resp. $\mathbb{Z}^k$), or a *Presburger set* of $\mathbb{N}^k$ (resp. $\mathbb{Z}^k$), if

$$X = \{(x_1, \ldots, x_k) \mid P(x_1, \ldots, x_k) \text{ is true}\},$$

where P is a Presburger formula (with at most k free variables) over $\mathbb{N}$ (resp. $\mathbb{Z}$). Ginsburg and Spanier, in [11], and Eilenberg and Schützenberger, in [9], proved the following result.

Theorem 2 (*Ginsburg and Spanier, 1966; Eilenberg and Schützenberger, 1969*). *Given a subset X of $\mathbb{N}^k$ (resp. $\mathbb{Z}^k$), the following assertions are equivalent:*

1. *X is first-order definable in $\mathcal{N}$ (resp. $\mathcal{Z}$);*
2. *X is $\mathbb{N}$-semi-linear in $\mathbb{N}^k$ (resp. $\mathbb{Z}^k$);*
3. *X is $\mathbb{N}$-semi-simple in $\mathbb{N}^k$ (resp. $\mathbb{Z}^k$).*

Remark 3. A $\mathbb{Z}$-semi-linear set of $\mathbb{Z}^k$ is always $\mathbb{N}$-semi-linear in $\mathbb{Z}^k$.

Remark 4. Theorem 2 is effective. Indeed, one can effectively represent a $\mathbb{N}$-semi-linear set X as a semi-simple set. More precisely, one can effectively construct a finite family $\{V_i\}$ of finite sets of vectors such that the vectors in V_i form a representation of a simple set X_i and X is the disjoint union of the sets X_i.

Remark 5. Given a monoid M, a subset of M is *rational* if it is obtained from finite subsets of M by applying finitely many times the rational operations, that is, the set union, the product, and the *Kleene closure* operator. Obviously, a semi-linear set of $\mathbb{N}^k$ or $\mathbb{Z}^k$ is rational but one can prove that the opposite is true (see [27]). Therefore, Conditions 2 and 3 of Theorem 2 and the property of rationality are equivalent in $\mathbb{N}^k$ or $\mathbb{Z}^k$. We recall that the previous equivalence has been proven in the larger context of finitely generated commutative monoids by Eilenberg and Schützenberger [9].

Now we recall the celebrated *Cross-Section theorem* by Eilenberg.

Theorem 3. *Let $\alpha : A^* \longrightarrow B^*$ be a morphism and let L be a rational language of A^*. Then one can effectively construct a rational subset L' of L such that α maps bijectively L' onto $\alpha(L)$.*

Let $A = \{a_1, \dots, a_t\}$ be a finite alphabet and $u \in A^*$ be a word. Then the *Parikh vector* of u is defined as

$$\psi(u) = (|u|_{a_1}, \dots, |u|_{a_t}),$$

and the function

$$\psi : A^* \longrightarrow \mathbb{N}^t,$$

defined above is the canonical epimorphism associated with the free commutative monoid $\mathbb{N}^t$. In the sequel, ψ will be also called the *Parikh function*. Now we state the following well known theorem due to Parikh.

Theorem 4. *The image of any context-free language under the Parikh function is an effective semi-linear set.*

3.2. Bounded languages

The aim of this paragraph is to present some results concerning bounded context-free languages. Let us first introduce the notion of bounded language.

Definition 6. Let L be a language of A^*. Then, for any positive integer k, L is called *k-bounded* if there exist nonempty words $u_1, \dots, u_k \in A^*$ such that

$$L \subseteq u_1^* \cdots u_k^*.$$

Moreover, we say that L is *bounded* if there exists an integer k such that L is k-bounded.

Theorem 5 (*Ginsburg, [12]*). *It is decidable whether a context-free language is bounded or not.*

Remark 6. The procedure involved in the test of Theorem 5 allows one to construct, from a given bounded context-free language L, a finite set $\{u_1, \dots, u_k\}$ of words such that $L \subseteq u_1^* \cdots u_k^*$.

Consider a bounded language L and suppose that, for some words $u_1, \dots, u_k \in A^*$, $L \subseteq u_1^* \cdots u_k^*$. We set

$$Ind(L) = \{(l_1, \dots, l_k) \in \mathbb{N}^k \mid u_1^{l_1} \cdots u_k^{l_k} \in L\}.$$

The following result was proven in [12]. For the sake of completeness, we give a simple constructive proof using Theorem 4.

Theorem 6. *Let L be a bounded context-free language. Then $Ind(L)$ is a semi-linear set. Moreover, one can effectively construct $Ind(L)$.*

Proof. Let Σ be the alphabet of L and let $L \subseteq u_1^* \cdots u_k^*$. Let $A = \{a_1, \dots, a_k\}$ be a new alphabet with k letters. Consider the morphism

$$\zeta : A^* \longrightarrow \Sigma^*, \tag{30}$$

generated by the map,

$$\forall\, i = 1, \dots, k, \quad a_i \longrightarrow u_i.$$

Since L is context-free, the language

$$X = \zeta^{-1}(L) \cap a_1^* \cdots a_k^*,$$

is also context-free and by Theorem 4, $\psi(X)$ is semi-linear. Finally it is easily seen that $Ind(L) = \psi(X)$. Indeed, for every vector $x = (l_1, \dots, l_k) \in \mathbb{N}^k$, we have,

$$x \in Ind(L) \implies u_1^{l_1} \cdots u_k^{l_k} \in L \implies a_1^{l_1} \cdots a_k^{l_k} \in X \implies \psi(a_1^{l_1} \cdots a_k^{l_k}) = x \in \psi(X),$$

so that $Ind(L) \subseteq \psi(X)$. The inverse inclusion is similarly proved. Hence $Ind(L)$ is semi-linear.

Since every step of this proof and Theorem 4 are effective, one has that $Ind(L)$ can be effectively computed starting from L. $\square$

If $L \subseteq u_1^* \cdots u_k^*$, then we define the map:

$$\phi : \mathbb{N}^k \longrightarrow u_1^* \cdots u_k^*, \tag{31}$$

such that, for every vector $(l_1, \ldots, l_k) \in \mathbb{N}^k$,

$$\phi(l_1, \ldots, l_k) = u_1^{l_1} \cdots u_k^{l_k}.$$

The following result proved in [16] is a consequence of Theorems 3 and 4.

Lemma 10. *Let $L \subseteq u_1^* \cdots u_k^*$ be a bounded context-free language. Then there exists a semi-linear set B of $\mathbb{N}^k$ such that $\phi(B) = L$ and ϕ is injective on B. Moreover, B can be effectively constructed.*

Proof. Let Σ be the alphabet of L and $A = \{a_1, \ldots, a_k\}$ be an alphabet with k letters. Consider now the morphism $\zeta : A^* \longrightarrow \Sigma^*$ as defined in (30). Since

$$\zeta(a_1^* \cdots a_k^*) = u_1^* \cdots u_k^*,$$

by Theorem 3, there exists a regular subset R of $a_1^* \cdots a_k^*$ such that ζ maps bijectively R onto $u_1^* \cdots u_k^*$. Let L' be the language defined as

$$L' = \zeta^{-1}(L) \cap R. \tag{32}$$

Since L' is context-free, by Theorem 6, the set $Ind(L')$ is a semi-linear set of $\mathbb{N}^k$. Set $B = Ind(L')$. As shown in [16], one can easily prove that $L = \phi(B)$ and, moreover, ϕ is injective on B.

Let us finally prove that B is constructible. Indeed, by Theorem 3, the set R is effectively constructible. On the other hand, by applying standard results, the set L' defined in (32) is an effective context-free language. By using Theorem 6, we can effectively construct the set $B = Ind(L')$ which is semi-linear. □

We recall that, given a language L, the *counting function* of L is a function $c_L : \mathbb{N} \longrightarrow \mathbb{N}$ that returns, for every $n \in \mathbb{N}$, the number of all words of L of length n. A language is termed *sparse* if its counting function is polynomially upper bounded. The following remarkable result states that sparsity and boundedness are equivalent for context-free languages (*see* [19,25,26]).

Theorem 7. *Let L be a context-free language. Then L is sparse if and only if L is bounded.*

For the sake of completeness, we end this paragraph by proving a useful characterization of sparse languages based upon the notion of the Parikh counting function.

Lemma 11. *Let L be a language. Then L is sparse if and only if its Parikh counting function f_L is polynomially upper bounded.*

Proof. Let us prove the necessity condition. Assume that $p(x)$ is a polynomial such that, for any $n \in \mathbb{N}$, $c_L(n) \le p(n)$. For any $(n_1, \ldots, n_t) \in \mathbb{N}^t$, the set of all words of L having $(n_1, \ldots, n_t)$ as the Parikh vector is included in the set of all words of L of length $n_1 + \cdots + n_t$. This implies that, for any $(n_1, \ldots, n_t) \in \mathbb{N}^t$,

$$f_L(n_1, \ldots, n_t) \le c_L(n_1 + \cdots + n_t) \le q(n_1, \ldots, n_t),$$

where q is the polynomial $q(n_1, \ldots, n_t) = p(n_1 + \cdots + n_t)$.

Let us prove the sufficiency. Assume that p is a polynomial such that, for any $(n_1, \ldots, n_t) \in \mathbb{N}^t$,

$$f_L(n_1, \ldots, n_t) \le p(n_1, \ldots, n_t). \tag{33}$$

We can suppose that all the coefficients of the polynomial p are non-negative. This implies that, for every $(n_1, \ldots, n_t) \in \mathbb{N}^t$,

$$p(n_1, \ldots, n_t) \ \le \ p(\underbrace{n_1 + \cdots + n_t, \ldots, n_1 + \cdots + n_t}_{t\text{-times}}). \tag{34}$$

Let $q_t : \mathbb{N} \longrightarrow \mathbb{N}$ be the map that returns, for every $n \in \mathbb{N}$, the number of all distinct ways of writing n as sum $n_1 + \cdots + n_t$ of t non-negative integers. By Corollary 1, q_t is a polynomial. For any $n \in \mathbb{N}$, one has:

$$c_L(n) = \sum_{n = n_1 + \cdots + n_t} f_L(n_1, \ldots, n_t), \quad n_1, \ldots, n_t \in \mathbb{N},$$

so that, by (33) and (34),

$$c_L(n) \le q_t(n) p(\underbrace{n, \ldots, n}_{t\text{-times}}).$$

This concludes the proof. □

4. On the Parikh counting function of bounded context-free languages

The first result we prove, concerns the structure of the Parikh counting function of a bounded context-free language. We recall that, given a language L on an alphabet $A = \{a_1, \ldots, a_t\}$, the Parikh counting function of L is the function $f_L : \mathbb{N}^t \longrightarrow \mathbb{N}$ such that, for any vector $(n_1, \ldots, n_t) \in \mathbb{N}^t$:

$$f_L(n_1, \ldots, n_t) = \mathrm{Card}(\{u \in A^* \ : \ \psi(u) = (n_1, \ldots, n_t)\}).$$

Assumption. In the sequel, we assume that L is a bounded context-free language and words $u_1, \ldots, u_k$ are such that

$$L \subseteq u_1^* \cdots u_k^*.$$

According to Lemma 10 and Theorem 2, there exists a semi-simple set B such that $L = \phi(B)$ and ϕ is injective on B. Set

$$B = \bigcup_{i=1,\ldots,s} B_i, \tag{35}$$

where, for every $i = 1, \ldots, s$, B_i is simple and let

$$L = \bigcup_{i=1,\ldots,s} L_i, \tag{36}$$

where, for every $i = 1, \ldots, s$, $L_i = \phi(B_i)$.
We need some preliminary results.

Lemma 12. *Let L_i and L_j be two languages of (36) with $i \neq j$. Then L_i and L_j are disjoint.*

Proof. By contradiction, suppose that $L_i \cap L_j \neq \emptyset$ and let $x \in L_i \cap L_j$.Then there exist $c_i \in B_i$ and $c_j \in B_j$ such that

$$x = \phi(c_i) = \phi(c_j).$$

By the injectivity of ϕ on B, we have

$$c_i = c_j$$

and thus

$$B_i \cap B_j \neq \emptyset,$$

which is a contradiction. This proves the claim. $\square$

Lemma 13. *Let B_i be a simple set of (35) and*

$$B_i = b_0 + b_1^{\oplus} + \cdots + b_n^{\oplus},$$

where $b_0, \ldots, b_n$ are the vectors of the representation of B_i. Then, for every vector $v = (v_1, \ldots, v_t) \in \mathbb{N}^t$, the number of words of L_i whose Parikh vector is v, equals the number of non-negative solutions of the Diophantine system:

$$\begin{cases} \lambda_0^1 + \lambda_1^1 x_1 + \lambda_2^1 x_2 + \cdots + \lambda_n^1 x_n = & v_1 \\ \lambda_0^2 + \lambda_1^2 x_1 + \lambda_2^2 x_2 + \cdots + \lambda_n^2 x_n = & v_2 \\ \vdots & \vdots \\ \lambda_0^t + \lambda_1^t x_1 + \lambda_2^t x_2 + \cdots + \lambda_n^t x_n = & v_t, \end{cases} \tag{37}$$

where, for every $i = 0, \ldots, n$ and for every $j = 1, \ldots, t$,

$$\lambda_i^j = |\phi(b_i)|_{a_j}.$$

Moreover, for every $i = 1, \ldots, n$, there exists j with $1 \leq j \leq t$, such that $\lambda_i^j \neq 0$.

Proof. It is easily proved that, for every $i = 1, \ldots, n$ there exists j with $1 \leq j \leq t$, such that $\lambda_i^j \neq 0$. Indeed, denying the latter condition implies that, for some s with $1 \leq s \leq n$, the vector $(\lambda_s^1, \ldots, \lambda_s^t)$ is null. This implies that the word $|\phi(b_s)|$ is the empty word so that b_s is the null vector. Since $b_s \in B_i$, this contradicts the fact that B_i is simple.

For any vector $v = (v_1, \ldots, v_t) \in \mathbb{N}^t$, let $\mathrm{Sol}(v)$ be the subset of vectors $x = (x_1, \ldots, x_n) \in \mathbb{N}^n$ which are solutions of the system (37). Define the map:

$$\theta : \mathrm{Sol}(v) \longrightarrow L_i,$$

as

$$\theta(x) = \theta(x_1, \ldots, x_n) = \phi(b_0 + b_1 x_1 + \cdots + b_n x_n).$$

One can check that, for every $\ell = 1, \ldots, t$:

$$|\phi(b_0 + b_1x_1 + \cdots + b_nx_n)|_{a_\ell} = \lambda_0^\ell + \lambda_1^\ell x_1 + \cdots + \lambda_n^\ell x_n = v_\ell,$$

so that the codomain of θ is a subset of the set $L_i \cap \psi^{-1}(v)$ of all words of L_i whose Parikh vector is v.
Now we prove that θ is a bijection of $\mathrm{Sol}(v)$ onto the language $L_i \cap \psi^{-1}(v)$. The map θ is injective on its domain. Indeed, let $x = (x_1, \ldots, x_n),\ y = (y_1, \ldots, y_n) \in \mathrm{Sol}(v)$. If $\theta(x) = \theta(y)$ then

$$\phi(b_0 + b_1x_1 + \cdots + b_nx_n) = \phi(b_0 + b_1y_1 + \cdots + b_ny_n),$$

and, by the injectivity of ϕ on B_i, we have

$$b_0 + b_1x_1 + \cdots + b_nx_n = b_0 + b_1y_1 + \cdots + b_ny_n.$$

Since B_i is simple, the latter gives

$$\forall\, i = 1, \ldots, n, \quad x_i = y_i,$$

thus obtaining $x = y$.

We prove that the map θ is surjective. Indeed, let $u \in L_i \cap \psi^{-1}(v)$ and let $x \in B_i$ be such that $\phi(x) = u$. Write x as $x = b_0 + b_1x_1 + \cdots + b_nx_n$. One has $v = \psi(u)$. It is easily checked that, for every $\ell = 1, \ldots, t$:

$$\lambda_0^\ell + \lambda_1^\ell x_1 + \cdots + \lambda_n^\ell x_n = v_\ell.$$

Hence $x \in \mathrm{Sol}(v)$ and $\theta(x) = u$. Thus θ is surjective and, therefore, $\mathrm{Card}(L_i \cap \psi^{-1}(v)) = \mathrm{Card}(\mathrm{Sol}(v))$. The proof of the lemma is thus complete. □

Lemma 14. *Let L_i be a language of (36). Then there exist a prc-quasi-polynomial $F_i : \mathbb{N}^t \longrightarrow \mathbb{N}$ and a vector $\beta_i \in \mathbb{N}^t$ such that, for any vector $\eta \in \mathbb{N}^t$ with $\eta \geq \beta_i$ one has:*

$$f_{L_i}(\eta) \;=\; F_i(\eta - \beta_i).$$

Proof. The claim follows by applying Corollary 2 to the Diophantine system obtained by applying Lemma 13 to the language L_i. □

Theorem 8. *Let L be a sparse context-free language and let $f_L : \mathbb{N}^t \longrightarrow \mathbb{N}$ be its Parikh counting function. Then there exist vectors $\beta, \beta_1, \ldots, \beta_s \in \mathbb{N}^t$ and prc-quasi-polynomials $F_i : \mathbb{N}^t \longrightarrow \mathbb{N}$, with $i = 1, \ldots, s$, such that the following property holds: for any vector $\eta \in \mathbb{N}^t$ with $\eta \geq \beta$, one has:*

$$f_L(\eta) \;=\; F_1(\eta - \beta_1) + \cdots + F_1(\eta - \beta_s).$$

Proof. We can suppose that the language L is written as in (36). For the sake of simplicity, assume that $L = L_1 \cup L_2$, the proof in the general case being completely similar. By Lemma 12, we have that, for every $\eta = (n_1, \ldots, n_t) \in \mathbb{N}^t$,

$$f_L(\eta) = f_{L_1}(\eta) + f_{L_2}(\eta), \tag{38}$$

where

$$f_{L_1}(\eta) = \mathrm{Card}(\{u \in L_1 \;\mid\; \psi(u) = \eta\}), \qquad f_{L_2}(\eta) = \mathrm{Card}(\{u \in L_2 \;\mid\; \psi(u) = \eta\}).$$

By Lemma 14, there exist two vectors $\beta_1,\ \beta_2$ of $\mathbb{N}^t$ and two prc-quasi-polynomials F_1, F_2 such that:

$$\forall\, \eta \in \mathbb{N}^t, \eta \geq \beta_1, \quad f_{L_1}(\eta) = F_1(\eta - \beta_1),$$

and

$$\forall\, \eta \in \mathbb{N}^t, \eta \geq \beta_2, \quad f_{L_2}(\eta) = F_2(\eta - \beta_2).$$

Let β be the vector of $\mathbb{N}^t$ such that, for every $i = 1, \ldots, t$, the ith component of β is the largest between those of β_1 and β_2 at position i, respectively. For any $\eta \in \mathbb{N}^t$ with $\eta \geq \beta$, we therefore have

$$f_L(\eta) = f_{L_1}(\eta) + f_{L_2}(\eta) \;=\; F_1(\eta - \beta_1) + F_2(\eta - \beta_2),$$

and this concludes the proof. □

Theorem 8 holds for "sufficiently large" vectors. Now we prove a stronger version of this theorem in order to have a complete description of the Parikh counting function. For this purpose, we need to refresh a definition. If π is a hyperplane of $\mathbb{R}^t$, the map

$$f_\pi : \mathbb{R}^t \longrightarrow \{+, -, 0\}$$

is defined as: for any $x \in \mathbb{R}^t$,

$$f_\pi(x) = \begin{cases} + & \text{if } \pi(x) > 0, \\ 0 & \text{if } \pi(x) = 0, \\ - & \text{if } \pi(x) < 0. \end{cases}$$

Definition 7. Let $\Pi = \{\pi_1, \dots, \pi_m\}$ be a family of hyperplanes of $\mathbb{R}^t$ with rational coefficients. Let $\sim_\Pi$ be the equivalence defined over the set $\mathbb{N}^t$ as: for any $x,\ x' \in \mathbb{N}^t$,

$$x \sim_\Pi x' \iff \forall\, k = 1, \dots, m, \quad f_{\pi_k}(x) = f_{\pi_k}(x').$$

A subset C of $\mathbb{N}^t$ is called a *region (with respect to Π)* if it is a coset of $\sim_\Pi$.

Remark 7. The previous construction is the same as Definition 3, the only difference being that we are now considering hyperplanes not necessarily through the origin. It is useful to observe that every coset of the equivalence $\sim_\Pi$ is uniquely determined by a sequence of m components in $\{+, 0, -\}$. On the other hand, a sequence of such kind defines a region only if the corresponding set of points of $\mathbb{N}^t$ is not empty. It is easily seen that, in general, the latter condition may be very well false, that is, the set of points defined by the sequence is empty.

Lemma 15. *Every coset of $\sim_\Pi$ is a polyhedron that can be effectively computed starting from Π.*

Proof. It is easily seen that every coset of $\sim_\Pi$ is a polyhedron (see [14]). Now set $\Pi = \{\pi_1, \dots, \pi_m\}$ and let us enumerate all the sequences of the form $(\epsilon_1, \dots, \epsilon_n) \in \{0, +, -\}^m$. Let υ be a sequence of this type and let $\mathrm{Sol}(\upsilon)$ be the set of all non-negative solutions of the system of Diophantine inequalities:

$$\{\pi_i(x) \ \bowtie_i \ 0\}_{i=1,\dots,m},$$

where

$$\bowtie_i = \begin{cases} > & \text{if } \epsilon_i = +, \\ = & \text{if } \epsilon_i = 0, \\ < & \text{if } \epsilon_i = -. \end{cases}$$

By the previous remark, the sequence υ determines a coset of $\sim_\Pi$ if and only if the set $\mathrm{Sol}(\upsilon)$ is not empty. On the other hand, the set $\mathrm{Sol}(\upsilon)$ is a rational subset of $\mathbb{N}^t$ that can be effectively computed starting from the entries of the system. Since $\mathrm{Sol}(\upsilon)$ is a rational subset of $\mathbb{N}^t$, one can effectively decide whether such a set is empty or not. □

Now, let $F_j : \mathbb{N}^t \longrightarrow \mathbb{N}$, with $j = 1, \dots, s$, be prc-quasi-polynomials and let $\beta_1, \dots, \beta_s$ be vectors of $\mathbb{N}^t$. For every $j = 1, \dots, s$, define a map $G_j : \mathbb{N}^t \longrightarrow \mathbb{N}$ as: for every $x \in \mathbb{N}^t$,

$$G_j(x) = \begin{cases} F_j(x - \beta_j) & \text{if } x \geq \beta_j, \\ 0 & \text{otherwise.} \end{cases}$$

Define a map $G : \mathbb{N}^t \longrightarrow \mathbb{N}$ as: for every $x \in \mathbb{N}^t$,

$$G(x) = \sum_{j=1}^{s} G_j(x).$$

The following lemma holds.

Lemma 16. *There exists a partition $\mathcal{P}$ of $\mathbb{N}^t$ into polyhedra such that the function G coincides with a quasi-polynomial on every coset of $\mathcal{P}$.*

Proof. For every prc-quasi-polynomials F_j, with $1 \leq j \leq s$, let Π_j be the family of hyperplanes through the origin associated to F_j. Let moreover $\beta_j = (b_{j1}, \dots, b_{jt})$ be the vector corresponding to F_j in the definition of G_j.
For every Π_j, we define a new family of hyperplanes Π_j' as follows:
if the hyperplane $\pi \equiv a_1x_1 + \cdots + a_tx_t = 0$ is in Π_j then we put in Π_j' the hyperplane $\pi' \equiv a_1(x_1 - b_{j1}) + \cdots + a_t(x_t - b_{jt}) = 0$.
Now consider the family of hyperplanes $\bar{\Pi}$, defined by:

$$\bar{\Pi} = \bigcup_{j=1}^{s} \Pi_j' \bigcup \Pi_{\mathbb{N}^t}$$

where by $\Pi_{\mathbb{N}^t}$ we have denoted the family of all coordinate hyperplanes $x_i = 0$, with $i = 1, \dots, t$.
Let $\sim_{\bar{\Pi}}$ be the equivalence relation determined, according to Definition 7, by hyperplanes in $\bar{\Pi}$. As remarked above, every coset is a polyhedron and we let $\mathcal{P}$ be the family of all cosets.
Now let $P \in \mathcal{P}$ be a coset. Let F_j, with $1 \leq j \leq s$, be one of prc-quasi-polynomials under consideration and let $\beta_j = (b_{j1}, \dots, b_{jt})$ be the vector corresponding to F_j in the definition of G_j. Observe now that, since all coordinate hyperplanes $x_i = 0$, with $1 \leq i \leq t$, are in Π_j, it follows that all the hyperplanes $x_i = b_{ji}$, with $1 \leq i \leq t$, are in Π_j'. It follows that if we put $X = P \bigcap \{x | x \geq \beta_j\}$ then either X is empty or X is P. Indeed, either the coset P has a minus $(-)$ value w.r.t. some hyperplane $x_i = b_{ji}$ or, respectively, for all the hyperplanes $x_i = b_{ji}$, with $1 \leq i \leq t$, P has a $+$ or a ϵ value.
Assume first $X = P \neq \emptyset$. Let x be any element of X. Since $x \geq \beta_j$, $x - \beta_j$ is in $\mathbb{N}^t$. Since the position of x w.r.t. the hyperplanes in Π_j' is completely specified by P, it follows that the position of $x - \beta_j$ w.r.t. the hyperplanes in Π_j of F_j is completely specified. It follows that there exists a unique region C of F_j such that if x is in X then $x - \beta_j$ is in C. Let $q(x)$ be the unique quasi-polynomial corresponding, in F_j, to the region C. So, w.r.t. to F_j, we assign to the coset P the function $q_j(x)$ defined by $q_j(x) = q(x - \beta_j)$. It is easily seen that $q_j(x)$ is a quasi-polynomial.
In case the set $X = P \bigcap \{x | x \geq \beta_j\}$ turns out to be empty, we assign the quasi-polynomial $q(x) \equiv 0$.
We repeat this for all F_j, with $j = 1, \dots, s$.
Since the sum of quasi-polynomials is a quasi-polynomial the result follows. □

Theorem 9. *Let L be a sparse context-free language and let $f_L : \mathbb{N}^t \longrightarrow \mathbb{N}$ be its Parikh counting function. Then there exists a partition of $\mathbb{N}^t$ into a finite family of polyhedra such that the function f_L is a quasi-polynomial on each polyhedron of the partition. Formally, there exist a finite family of polyhedra $P_1, \ldots, P_s$ – every one of which is defined by hyperplanes with rational coefficients – and a family $p_1, \ldots, p_s$ of quasi-polynomials with rational coefficients such that for, any $\eta = (n_1, \ldots, n_t) \in \mathbb{N}^t$, the following holds:*

$$f_L(\eta) = p_j(\eta),$$

where j is the index of the polyhedron P_j that contains η.

Proof. The statement follows by applying Theorem 8 and Lemma 16. $\square$

Now we want to reformulate Theorem 9 in a *"language-theoretic"* way. For this purpose, the following results are needed.

Lemma 17. *Let π be a hyperplane and assume that the coefficients of the equation $\pi(x) = 0$ that define π are rational. Then the set of all points of $\mathbb{N}^t$ such that $\pi(x) \leq 0$ is a rational set of $\mathbb{N}^t$.*

Proof. Denote by X_π the set given by

$$X_\pi = \{x \in \mathbb{N}^t \ : \ \pi(x) \leq 0\}.$$

Let $\pi(x) \equiv a_0 + a_1x_1 + \cdots + a_tx_t$ and let m be the *l.c.m.* of the denominators of the numbers a_i, for $i = 0, \ldots, t$. We can assume that $m \geq 1$. Then the inequality $a_0 + a_1x_1 + \cdots + a_tx_t \leq 0$ is equivalent to

$$ma_0 + ma_1x_1 + \cdots + ma_tx_t \leq 0. \tag{39}$$

Let $a_{i_1}, \ldots, a_{i_r}$ and $a_{j_1}, \ldots, a_{j_s}$ be enumerations of all non-negative coefficients and, respectively, of all positive coefficients of the set $\{a_0, a_1, \ldots, a_t\}$ of the coefficients of $\pi(x)$. Then (39) is equivalent to:

$$ma_{j_1}x_{j_1} + \cdots + ma_{j_s}x_{j_s} < -ma_{i_1}x_{i_1} - \cdots - ma_{i_r}x_{i_r} - ma_0, \tag{40}$$

where we have assumed $a_0 < 0$. Eq. (40) is expressible by a formula in the Presburger arithmetic over $\mathbb{N}$. Call this formula $P(x_1, \ldots, x_t)$. Then the set X_π is defined by the formula $P(x_1, \ldots, x_t)$, that is:

$$X_\pi = \{(x_1, \ldots, x_t) \ : \ P(x_1, \ldots, x_t) \text{ is true}\}.$$

By Theorem 2, X_π is a rational subset of $\mathbb{N}^t$. $\square$

Lemma 18. *Let P be a polyhedron defined by hyperplanes represented by equations with rational coefficients. Then the set of points of $\mathbb{N}^t$ contained in P is a rational set of $\mathbb{N}^t$.*

Proof. Let $\Pi = \{\pi_1, \ldots, \pi_m\}$ be the family of hyperplanes that define P and let X_P be the set of all points of $\mathbb{N}^t$ that belong to P. Then X_P is the set of all non-negative solutions of the system of Diophantine inequalities:

$$\begin{cases} \pi_1(x) \leq & 0 \\ \quad \cdot & \cdot \\ \quad \cdot & \cdot \\ \quad \cdot & \cdot \\ \pi_m(x) \leq & 0, \end{cases}$$

where, for every $i = 1, \ldots, m$, $\pi_i(x) = 0$ is the fixed equation with rational coefficients that represents π_i. So the statement follows from Lemma 17 and the fact that the family of rational sets of $\mathbb{N}^t$ is a Boolean algebra. $\square$

Let d be a positive integer and let $(d_1, \ldots, d_t) \in \mathbb{N}^t$ be a vector whose components are not larger than $d - 1$. Let

$$X_{(d_1,\ldots,d_t)} \tag{41}$$

be the set of all vectors $(x_1, \ldots, x_t)$ of $\mathbb{N}^t$ such that, for every $i = 1, \ldots, t$ $x_i \equiv d_i \mod d$. The following lemma is easily proven.

Lemma 19. *Every set of the form (41) is rational in $\mathbb{N}^t$.*

Theorem 10. *Let L be a sparse context-free language and let $f_L : \mathbb{N}^t \longrightarrow \mathbb{N}$ be its Parikh counting function. Then there exists a partition of $\mathbb{N}^t$ into a finite family of rational sets such that the function f_L is a polynomial on each rational set of the partition. Formally, there exist a partition of $\mathbb{N}^t$ into a finite family of rational sets $R_1, \ldots, R_s$ and a family of polynomials with rational coefficients $p_1, \ldots, p_s$ such that, for any vector $\eta \in \mathbb{N}^t$ one has:*

$$f_L(\eta) = p_j(\eta),$$

where j is the index of the rational set R_j that contains η.

Proof. By Theorem 9, there exist a partition $\mathscr{P} = \{P_1, \ldots, P_s\}$ of polyhedra and a family $p_1, \ldots, p_s$ of quasi-polynomials with rational coefficients, every one of which is associated with exactly one polyhedron of $\mathscr{P}$, such that the following condition holds: for any $\eta = (n_1, \ldots, n_t) \in \mathbb{N}^t$, one has:

$$f_L(\eta) = p_j(\eta),$$

where j is the index of the polyhedron P_j that contains η. Let P be a polyhedron of $\mathscr{P}$ and let p be its corresponding quasi-polynomial. We show that there exists a partition of P into a finite family of rational sets of $\mathbb{N}^t$ such that, on each set of this partition, p coincides with a polynomial. Let d be the period of p and let $(d_1, \ldots, d_t)$ be a vector of $\mathbb{N}^t$, with $d_i \le d - 1$. The set P admits a partition into a finite family of sets of the form

$$X_{(d_1,\ldots,d_t)} \cap P,$$

where $X_{(d_1,\ldots,d_t)}$ is defined as in (41). By Lemmas 18 and 19 and the fact that the family of rational sets of $\mathbb{N}^t$ is a Boolean algebra, it follows that the set $X_{(d_1,\ldots,d_t)} \cap P$ is rational in $\mathbb{N}^t$. On the other hand, for every $x \in X_{(d_1,\ldots,d_t)} \cap P$, one has $p(x) = p_{(d_1,\ldots,d_t)}(x)$. This concludes the proof. □

Theorem 11. *The rational sets $R_1, \ldots, R_s$ and the polynomials $p_1, \ldots, p_s$, defined in the statement of Theorem 10 can be effectively constructed starting from an effective presentation of the language L.*

Proof. The proof is a walk through the results, each one being effective, we gathered so far. It is useful to divide the proof into the following subsequent steps.

Step 1. Starting from L, one can effectively construct a finite set $\{u_1, \ldots, u_k\}$ of nonempty words such that $L \subseteq u_1^* \cdots u_k^*$. This is done by executing the procedure involved in Theorem 5 (*cf.* Remark 6).

Step 2. One can effectively construct a semi-linear set $B \subseteq \mathbb{N}^k$ such that $L = \phi(B)$ and ϕ is injective on B. This is done in Lemma 10.

Step 3. One can effectively represent B as a semi-simple set. More precisely, one can construct a finite family of finite sets of vectors, say $\{V_i\}$, where the vectors of V_i form a representation of a simple set B_i and such that B is the disjoint union of the sets B_i. This is done according to Theorem 2 and Remark 4.

Step 4. For every $n \ge 0$ and for every set V_i, one can effectively construct the Diophantine system (37) stated in Lemma 13. This is done by using the sets of words $\{u_1, \ldots, u_k\}$ and the vectors of V_i computed in Step 1 and in Step 3, respectively.

Step 5. Since the proof of Corollary 2 is constructive, for every language $L_i = \phi(B_i)$, one can effectively construct a prc-quasi-polynomial $F_i : \mathbb{N}^t \longrightarrow \mathbb{N}$ and a vector β_i of $\mathbb{N}^t$ such that, for any vector $\eta \ge \beta_i$, one has:

$$f_{L_i}(\eta) = F_i(\eta - \beta_i).$$

Step 6. By making use of Lemma 15, one can see that the proof of Lemma 16 is constructive. So the partition of $\mathbb{N}^t$ into polyhedra and the corresponding family of quasi-polynomials defined in the statement of Theorem 9, can be effectively constructed.

Step 7. Since the proofs of Lemmas 17–19, together with the proof of Theorem 10, are constructive, the result follows. □

We end this section by proving an interesting application of the results we got so far. Let L be a language over an alphabet of t letters and let f_L be its Parikh counting function. Recall that L is termed *Parikh slender* if there exists a positive integer r such that, for every $(n_1, \ldots, n_t) \in \mathbb{N}^t$,

$$f_L(n_1, \ldots, n_t) \le r.$$

The family of Parikh slender context-free languages has been investigated in [15–17]. The following result of [15] can be proved easily as a consequence of Theorems 10 and 11.

Corollary 3. *It is decidable whether a context-free language is Parikh slender or not.*

Proof. Let L be a context-free language. We can suppose that L is bounded. Indeed, by Lemma 11, the Parikh counting function of L is upper bounded by a polynomial if and only if L is sparse. By Theorem 7, L is sparse if and only if L is bounded. By Theorem 5, the property of boundedness is decidable for context-free languages. By Theorem 10, there exist a partition of $\mathbb{N}^t$ into a finite family of rational sets $R_1, \ldots, R_s$ and a family of polynomials with rational coefficients $p_1, \ldots, p_s$ such that, for any vector $\eta \in \mathbb{N}^t$, one has:

$$f_L(\eta) = p_j(\eta),$$

where j is the index of the rational set R_j that contains η. By Theorem 2, up to a refinement of the partition of $\mathbb{N}^t$ mentioned above, we can suppose that the sets R_j are simple. Let R and p be respectively a simple set of the partition and its corresponding polynomial. Let $b_0, b_1, \ldots, b_k$ be the vectors of a representation of R. Hence we have:

$$R = \{(b_0^1 + n_1 b_1^1 + \cdots + n_k b_k^1, \ldots, b_0^t + n_1 b_1^t + \cdots + n_k b_k^t) \; : \; n_1, \ldots, n_k \in \mathbb{N}^t\}, \tag{42}$$

where, for every $i = 0, \ldots, k$, $b_i = (b_i^1, b_i^2, \ldots, b_i^t)$. Let us associate with R the following polynomial in k variables $n_1, \ldots, n_k$:

$$q(n_1, \ldots, n_k) = p(b_0^1 + n_1 b_1^1 + \cdots + n_k b_k^1, \ldots, b_0^t + n_1 b_1^t + \cdots + n_k b_k^t). \tag{43}$$

Let $\mathcal{Q}$ be the family of all the polynomials that can be obtained, as above, from every set of the partition of $\mathbb{N}^t$ and let δ be the maximum of the set of the degrees of all the polynomials of $\mathcal{Q}$. Now we prove the following equivalence:

$$L \text{ is slender if and only if } \delta = 0. \tag{44}$$

Sufficiency is obvious. We prove now the necessity. Assume $\delta \geq 1$. Hence there exists a polynomial q in $\mathcal{Q}$ of degree δ. Let R be a rational set associated with q. Since $\delta \geq 1$, by (42) and (43), one has that R is an infinite subset of $\mathbb{N}^t$ and the image, under the map q, of the set R is infinite. This implies that the image of the map f_L is infinite as well. Hence L is not slender. Thus Condition (44) is proved.

Finally, the main claim follows from (44), by remarking that, by Theorem 11 and by Remark 4, all the polynomials of $\mathcal{Q}$ can be effectively constructed from an effective presentation of the language. □

5. On the rationality of the Parikh counting function of a sparse context-free language

The aim of this section is to prove that the Parikh counting function of a sparse context-free language is rational. In order to prove this result, some preliminary notions and results concerning formal power series have to be recalled. We follow the classical reference [24]. Let $\mathbb{K}$ be a commutative semiring and let $X = \{x_1, \ldots, x_t\}$ be a set of t commutative variables. We identify the set of all commutative monomials over X with $\mathbb{N}^t$. We denote by $\mathbb{K}[X]$ and by $\mathbb{K}[[X]]$ respectively the semiring of polynomials and the semiring of formal power series on the set of commutative variables X and with coefficients taken in $\mathbb{K}$. A power series is a map $\mathbb{N}^t \longrightarrow \mathbb{K}$. Any power series r of $\mathbb{K}[[X]]$ can be written as a formal sum

$$r = \sum_{n_1,\ldots,n_t \in \mathbb{N}} (r, x_1^{n_1} \cdots x_t^{n_t}) x_1^{n_1} \cdots x_t^{n_t},$$

where $(r, x_1^{n_1} \cdots x_t^{n_t})$ is the coefficient of $\mathbb{K}$ associated with the monomial $x_1^{n_1} \cdots x_t^{n_t}$ by the series r. We recall that the family of rational power series of $\mathbb{K}[[X]]$, denoted $Rat(\mathbb{K}[[X]])$, is the smallest subset of $\mathbb{K}[[X]]$ that contains $\mathbb{K}[X]$ and that is closed with respect to the rational operations, that is, the operations that, given series $s, t \in \mathbb{K}[[X]]$, associate with them, the sum $s + t$, the (Cauchy) product st and the star $s^* = \sum_{i=0}^{\infty} s^i$. We will prove the following statement.

Theorem 12. *Let us consider a Diophantine system defined as:*

$$\begin{cases} a_{10} + a_{11}x_1 + a_{12}x_2 + \cdots + a_{1k}x_k = & n_1 \\ a_{20} + a_{21}x_1 + a_{22}x_2 + \cdots + a_{2k}x_k = & n_2 \\ \quad\quad\quad\quad\quad \vdots & \vdots \\ a_{t0} + a_{t1}x_1 + a_{t2}x_2 + \cdots + a_{tk}x_k = & n_t, \end{cases} \tag{45}$$

where the numbers $a_{ij} \in \mathbb{N}$ and, for every $i = 1, \ldots, k$, there exists $j = 1, \ldots, t$ such that $a_{ij} \neq 0$. Let $\mathcal{S} : \mathbb{N}^t \longrightarrow \mathbb{N}$ be the counting function of the system, that is the map that associates with every $(n_1, \ldots, n_t) \in \mathbb{N}^t$ the number of non-negative solutions of the system. Then $\mathcal{S}$ is $\mathbb{N}$-rational.

Let us associate with the system (45) a formal power series $S \in \mathbb{N}[[X]]$ defined as:

$$S = S_0 S_1 \cdots S_k,$$

with

$$S_0 = x_1^{a_{10}} \cdots x_t^{a_{t0}},$$

and, for every $\ell = 1, \ldots, k$,

$$S_\ell = (x_1^{a_{1\ell}} \cdots x_t^{a_{t\ell}})^*,$$

where the numbers a_{ij} are the coefficients of the system (45).
Since, for every $\ell = 1, \ldots, k$, the column $(a_{1\ell}, \ldots, a_{t\ell})$ of the matrix of the system (45) is non-null, the series $S_\ell \in Rat(\mathbb{N}[[X]])$. Hence, since $Rat(\mathbb{N}[[X]])$ is closed under the rational operations, one has that $S \in Rat(\mathbb{N}[[X]])$.

Lemma 20. *The series S is equal to the map $\mathcal{S}$.*

Proof. By developing the formal series $S = S_0 S_1 \cdots S_k$, we have that S is equal to:

$$x_1^{a_{10}} \cdots x_t^{a_{t0}} \cdot \left(\sum_{\mu_1 \geq 0} (x_1^{a_{11}} \cdots x_t^{a_{t1}})^{\mu_1} \right) \cdots \left(\sum_{\mu_k \geq 0} (x_1^{a_{1k}} \cdots x_t^{a_{tk}})^{\mu_k} \right).$$

By the formula above, one can check that, for any $x_1^{n_1}\cdots x_t^{n_t}$, the coefficient $(S, x_1^{n_1}\cdots x_t^{n_t})$ of $x_1^{n_1}\cdots x_t^{n_t}$ is the cardinal number of the set:

$$\{(\mu_1,\ldots,\mu_k)\in\mathbb{N}^t \mid x_1^{n_1}\cdots x_t^{n_t} = x_1^{a_{10}}\cdots x_t^{a_{t0}} \cdot (x_1^{a_{11}}\cdots x_t^{a_{t1}})^{\mu_1}\cdots(x_1^{a_{1k}}\cdots x_t^{a_{tk}})^{\mu_k}\}).$$

The number above is clearly equal to the value of the map $\mathcal{S}$ computed at $(n_1,\ldots,n_t)$ and this concludes the proof. □

Now we can prove Theorem 12.

Proof of Theorem 12. By Lemma 20, the counting function $\mathcal{S}$ of the Diophantine system (45) coincides with the formal power series S. The claim follows by recalling that S is $\mathbb{N}$-rational. □

We can now prove the announced result.

Corollary 4. *The Parikh counting function of a bounded context-free language is $\mathbb{N}$-rational. Moreover, it can be effectively computed.*

Proof. As shown in Section 4, given a bounded context-free language, one can effectively construct a finite family of systems of Diophantine linear equations of the form (45) such that the Parikh counting function of the language coincides with the sum of the counting functions of such systems. Then the claim follows by applying Theorem 12. □

Remark 8. From the previous result, we get a different proof of Corollary 3. Indeed, L is slender if and only if the series f_L has finite image. On the other hand, by Corollary 2.7 of [2], it is decidable whether a rational series has a finite image.

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Appendix of Section 2

The following lemma can be proved by following [23], Ch. 1.

Lemma 21. *Let $m \in \mathbb{N}$. There exists a polynomial p in one variable x with rational coefficients such that:*
(1) for any $n \in \mathbb{N}$, $p(n) = \sum_{\lambda=0,\ldots,n} \lambda^m$,
(2) p factorizes as $p(x) = (x+1)p'(x)$ where p' is a polynomial in one variable x with rational coefficients.

Proof. Let $m \geq 1$. There exist numbers $b_0,\ldots,b_m$ such that, for every $k \in \mathbb{N}$, the number k^m can be expressed as:

$$k^m = b_0\binom{k}{0} + \cdots + b_m\binom{k}{m}.$$

Therefore, the previous equation gives:

$$\sum_{k=0,\ldots,n} k^m = b_0 \cdot \sum_{k=0,\ldots,n}\binom{k}{0} + b_1 \cdot \sum_{k=0,\ldots,n}\binom{k}{1} + \cdots + b_m \cdot \sum_{k=0,\ldots,n}\binom{k}{m}. \tag{46}$$

On the other hand, one has that:

$$\sum_{k=0,\ldots,n}\binom{k}{r} = \binom{n+1}{r+1}. \tag{47}$$

By applying (47) to every addendum of the sum of (46), one obtains a polynomial p that satisfies the claim of the lemma. □

Example. Taking $m = 2$,we express as a polynomial the sum of the squares of the first k non-negative integers. Let us recall that, for any $k \in \mathbb{N}$:

$$k^2 = 2\binom{k}{2} + \binom{k}{1}.$$

Then, for any $n \in \mathbb{N}$, by applying (47), one has:

$$\begin{aligned}\sum_{k=0,\ldots,n} k^2 &= 2\sum_{k=0,\ldots,n}\binom{k}{2} + \sum_{k=0,\ldots,n}\binom{k}{1} = 2\binom{n+1}{3} + \binom{n+1}{2}\\ &= 2\frac{(n+1)n(n-1)}{6} + \frac{(n+1)n}{2}\end{aligned}$$

which finally gives the claimed polynomial.

Lemma 22. *Let $d \in \mathbb{N}$ and let $\lambda : \mathbb{N}^t \longrightarrow \mathbb{Q}$ be the map such that, for any $(x_1, \ldots, x_t) \in \mathbb{N}^t$,*

$$\lambda(x_1, \ldots, x_t) = \sum_1^t b_i x_i,$$

where $b_1, \ldots, b_t$ are non-negative rational numbers. Let k be a constant integer, then the map

$$\phi(x_1, \ldots, x_t) = \left\lfloor \frac{\lceil \lambda(x_1, \ldots, x_t) \rceil + k}{d} \right\rfloor.$$

is a quasi-polynomial.

Proof. We can represent the rational numbers $b_1, \ldots, b_t$ as $b_1 = f_1/g, \ \ldots, b_t = f_t/g$, where $f_1, \ldots, f_t$ and g are non-negative integers.

Let $(x_1, \ldots, x_t) \in \mathbb{N}^t$. For every $i = 1, \ldots, t$, let $r_i, \alpha_i \in \mathbb{N}$ such that:

$$x_i = \alpha_i g d + r_i, \quad 0 \le r_i < gd. \tag{48}$$

By using (48), one derives

$$\begin{aligned}
\phi(x_1, \ldots, x_t) &= \left\lfloor \frac{\left\lceil d \sum_{i=1,\ldots,t} f_i \alpha_i + \sum_{i=1,\ldots,t} \frac{f_i r_i}{g} \right\rceil + k}{d} \right\rfloor \\
&= \left\lfloor \frac{d \left(\sum_{i=1,\ldots,t} f_i \alpha_i \right) + \left\lceil \sum_{i=1,\ldots,t} \frac{f_i r_i}{g} \right\rceil + k}{d} \right\rfloor = \left\lfloor \left(\sum_{i=1,\ldots,t} f_i \alpha_i \right) + \frac{\left\lceil \sum_{i=1,\ldots,t} \frac{f_i r_i}{g} \right\rceil + k}{d} \right\rfloor \\
&= \left(\sum_{i=1,\ldots,t} f_i \alpha_i \right) + \left\lfloor \frac{\left\lceil \sum_{i=1,\ldots,t} \frac{f_i r_i}{g} \right\rceil + k}{d} \right\rfloor = \left(\sum_{i=1,\ldots,t} \frac{b_i}{d} (x_i - r_i) \right) + \left\lfloor \frac{\left\lceil \sum_{i=1,\ldots,t} \frac{f_i r_i}{g} \right\rceil + k}{d} \right\rfloor.
\end{aligned}$$

For any $r_1, \ldots, r_t$, with $0 \le r_i \le gd - 1$, consider the polynomial

$$p_{(r_1,\ldots,r_t)}(x_1, \ldots, x_t) = \left(\sum_{i=1,\ldots,t} \frac{b_i}{d} (x_i - r_i) \right) + \left\lfloor \frac{\left\lceil \sum_{i=1,\ldots,t} \frac{f_i r_i}{g} \right\rceil + k}{d} \right\rfloor.$$

We have just proved that for any non-negative integers $x_1, \ldots, x_t$, if $x_i \equiv r_i \mod gd$, then

$$\phi(x_1, \ldots, x_t) = p_{(r_1,\ldots,r_t)}(x_1, \ldots, x_t).$$

Therefore $\phi(x_1, \ldots, x_t)$ is a quasi-polynomial. □

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