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Pseudo-commutativity of KZ 2-monads

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Abstract

In this paper we prove that KZ 2-monads (also known as lax-idempotent 2-monads) are pseudo-commutative. The main examples of KZ 2-monads for us will be 2-monads whose algebras are $\mathscr{V}$-categories with chosen colimits of a given class; this provides a large family of examples of pseudo-commutative 2-monads. In order to achieve this we characterise pseudo-commutativities on a 2-monad in terms of extra structure on its 2-category of algebras and pseudomorphisms. We also consider tensor products associated to pseudo-closed structures and show some results on preservation of colimits. To cover the general case of $\mathscr{V}$-enriched categories and not only ordinary categories we are led to consider monads enriched in a 2-category, and some of the associated two-dimensional monad theory.

Keywords: Limit; 2-Monad; KZ 2-monad; Pseudo-commutative 2-monad; 2-Category

1. Introduction

This paper studies categories equipped with extra structure satisfying a uniqueness condition that could be phrased as *if the extra structure exists then it arises in a specified way, unique up to isomorphism.* A typical example of this kind of category is a category with finite colimits: if these colimits exist they arise in a unique (up to isomorphism) way, given by the definition of colimit. If one thinks of extra structure imposed on a category as a family of operations satisfying

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some axioms, our main result roughly states that if the extra structure satisfies this uniqueness condition then the operations *commute* with each other up to isomorphism.

The fact that certain colimits commute with certain limits is fundamental in many areas of mathematics. The most common manifestation of this phenomenon is the commutation of *filtered colimits* with *finite limits* in many common categories; this is the fundamental in the theory of pro-finite objects and its variations (e.g., pro-finite groups); in the theory of sheaves on topological spaces (where the stack on a point is a filtered colimit) innumerable aspects of the theory depend upon this commutation, and the same applies to algebraic geometry. Another example is the commutation of *pullbacks* with some colimits (e.g., with all colimits the case of **Set** or any topos) that ensures a good behaviour under *change of base*.

The present paper could be considered a step zero in a program aimed to obtain an algebraic formulation and understanding of the commutation of some colimits with some limits. We say step zero because here we are concerned with commutation of colimits with colimits, or indeed of any other structure on a category that satisfies the uniqueness condition referred to at the beginning of this introduction. The mixed case of colimits and limits, although more involved, should fit in the same abstract framework.

Let's first recall the lower dimensional case of sets with extra structure. One way of thinking of categories of sets with extra algebraic structure is by means of monads on the category of sets; this includes the usual intuition of sets equipped with operations satisfying equations, and more besides. The idea of operations commuting with each other is encapsulated in the notion of a *commutative monad* introduced by Kock [23–25].

Similarly, but in one dimension up, categories with extra algebraic structure can be thought of in terms of 2-monads; a few examples are monoidal categories and their braided and symmetric variants, categories equipped with a monad, or two monads with a distributive law between them, categories with (finite or otherwise) chosen (co)products, or finite biproducts. When the algebraic structures are "commutative" the 2-monad is called pseudo-commutative, and this is the case studied in detail in [10], where the main example is provided by symmetric strict monoidal categories. Observe that the braided strong monoidal functors between two symmetric strict monoidal categories are the objects of a category that is also symmetric strict monoidal. This is true in general: for a pseudo commutative 2-monad T, the pseudomorphisms of T-algebras $A \to B$ are the objects of a T-algebra $[\![A, B]\!]$, or more precisely, the 2-category of T-algebras and pseudomorphisms T-Alg is *pseudo-closed* [10].

The main example in this paper, in fact a family of examples, are categories with chosen colimits of a given class, and the associated 2-monads. These are examples of *KZ* or *lax-idempotent* 2-monads. Our main result states that any such 2-monad is pseudo-commutative in a canonical way. The 2-monads corresponding to a class of colimits are different from other examples in that, although we know their algebras, there is no easy description of the 2-monad itself. Indeed, if one tries to fashion a direct proof of the pseudo-commutativity of these 2-monads, one quickly finds numerous obstacles. We avoid them by considering the wider class of KZ 2-monads, obtaining at the same time cleaner statements and proofs.

The notion of colimit makes sense in the context of enriched categories and indeed in the examples our categories can be enriched in vector spaces, simplicial sets, the category $\mathbb{2} = \{* \to \bullet\}$ (so $\mathbb{2}$-categories are partially ordered sets), and many other possibilities. This makes us consider the 2-monads T_Φ on the 2-category $\mathscr{V}$-**Cat** of $\mathscr{V}$-enriched categories, whose algebras are $\mathscr{V}$-categories with chosen colimits of the class Φ, as studied in [19]. In order to be able to even express the results on the pseudo-commutativity of T_Φ in this enriched setting, we are forced to consider T_Φ not only as a 2-monad, that is a monad enriched in **Cat**, but as a monad enriched in

$\mathscr{V}$-**Cat**. Working with $\mathscr{V}$-**Cat** presents no advantages over working with a general 2-category $\mathscr{W}$ (required to be symmetric monoidal closed, complete and cocomplete); because of this we opt for using $\mathscr{W}$ if only because of the notational clarity it provides. To accommodate the existing theory of 2-monads to this enrichment in $\mathscr{W}$ we define the $\mathscr{W}$-category T-Alg, whose enriched homs are "objects of pseudomorphisms," and provide easy extensions of some of the results in [3] to the $\mathscr{W}$-enriched framework. We emphasise that enriching in $\mathscr{V}$-**Cat** would not have shorten or simplified any of the paper's material.

The connection between the existence of a pseudo-commutativity on T and the KZ condition is made via a characterisation of the former in terms of data in the (enriched) 2-category T-Alg (that will also give rise to a pseudo-closed structure). More concretely, we require that a certain family of 1-cells in $\mathscr{W}$ with domain and codomain T-algebras lift to 1-cells in T-Alg; this will automatically be the case for KZ 2-monads because the 1-cells in question will be left adjoints in $\mathscr{W}$, and for KZ 2-monads these are always pseudomorphisms.

The paper is organised as follows.

After this introduction, Section 2 recalls some of the necessary background on two-dimensional monad theory. In Section 3 we describe the $\mathscr{W}$-category T-Alg of algebras (and pseudomorphisms) of a $\mathscr{W}$-monad T. The necessary adaptations of the pseudo-closed 2-categories and pseudo-commutative 2-monads of [10] to the $\mathscr{W}$-enriched context are described in Section 4. Section 5 proves one of the key results of this work, characterising pseudo-commutativities in terms of data in T-Alg. The pseudo-closed structure of the $\mathscr{W}$-category T-Alg for a pseudo-commutative T together with the induced tensor product can be found in Section 6; this is largely an easy adaptation of the 2-categorical case, but we add some results on preservation of colimits by the tensor product that will be of use in a forthcoming paper. In Section 7 we prove our main result stating that KZ $\mathscr{W}$-monads are pseudo-commutative, while in Section 8 we look at the example of monads given by completion under a class of chosen colimits. Finally, there is Appendix A where we confine some standard extensions of the existence of flexible replacements to $\mathscr{W}$-enriched monads, and the proof that the 2-monad for chosen finite colimits is finitary.

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2. Background on 2-monads

In this section we summarise the concepts of two-dimensional monad theory necessary throughout the rest of the paper. The basic references on 2-categories are [22,2] and for 2-monad theory [3]; [29] provides a good survey of both.

A 2-monad (T, η, μ) on a 2-category $\mathscr{K}$ is a 2-functor $T : \mathscr{K} \to \mathscr{K}$ together with 2-natural transformations $\mu : T^2 \Rightarrow T$, $\eta : 1 \Rightarrow T$ satisfying the usual monad axioms: $\mu_X \mu_{TX} = \mu_X (T\mu_X)$ and $\mu_X(T\eta_X) = 1_{TX} = \mu_X \eta_{TX}$.

Given a 2-monad T on $\mathscr{K}$, a *pseudo-T-algebra* is an object A of $\mathscr{K}$ with a 1-cell $a : TA \to A$ and invertible 2-cells

$$a(Ta) \cong a\mu_A, \qquad 1_a \cong a\eta_X \tag{1}$$

satisfying two axioms (see [3]). When these 2-cells are identities we say that (A, a) is a *strict T-algebra*, or simply a T-algebra. We will usually denote a T-algebra (A, a) simply by its object part A, omitting the explicit mention of the action, which will then be denoted by the lowercase of the letter we use for the object part. The paper will centre on strict algebras, although pseudoalgebras will be mentioned in some examples.

For the benefit of the reader unfamiliar with 2-monads, we provide the following standard example.

Example 2.1. A monoidal category can be identified with a pseudoalgebra for a 2-monad T on **Cat**. The category TC has objects and arrows, respectively, finite sequences of objects and finite sequences of arrows of C. Concatenation of lists endows TC with the structure of a strict monoidal category. The 1-cell $a : TC \to C$ is a functor that can be thought as providing the tensor product of a list of objects

$$(x_1, x_2, \dots, x_n) \mapsto x_1 \otimes x_2 \otimes \cdots \otimes x_n$$

while its value on the empty list can be thought as the unit object for the monoidal structure. The isomorphisms (1) provide the associativity and unit constraints. Observe that a pseudo-T-algebra is not exactly the same as a monoidal category but rather an *unbiased* monoidal category; see [31] for a full explanation. Monoidal categories are algebras for a 2-monad whose description is related to the original Mac Lane's proof of the coherence theorem for monoidal categories [32] and Kelly's notion of a *club* – see [14] and the references therein.

Example 2.2. We will later refer to the following 2-monad S on **Cat**, that is the main example of a pseudo-commutative 2-monad in [10]. For a category X, SX has objects the lists of objects of X and arrows $(x_1, \dots, x_n) \to (y_1, \dots, y_n)$ the $n+1$-tuples $(f_1, \dots, f_n, s)$ where s is a permutation of n elements and $f_i : x_i \to y_{s(i)}$ is an arrow in X. Composition is induced by the multiplication of permutations and the composition in X. Note that there are no arrows between lists of different length. The category SX is not only strict (unbiased) monoidal via concatenation of lists (definition on arrows should be obvious) but is moreover *symmetric*. If $\underline{x} = (x_1, \dots, x_n)$, $\underline{y} = (y_1, \dots, y_m)$, the component of the symmetry $\underline{x} \otimes \underline{y} \to \underline{y} \otimes \underline{x}$ is the arrow $(1, \dots, 1, s)$, where s is the permutation of $n+m$ elements: $s(i) = i+n$ if $1 \leqslant i \leqslant n$, $s(i) = i-n$ if $n+1 \leqslant i \leqslant n+m$. S-algebras can be identified with strict (unbiased) symmetric monoidal categories.

The examples of 2-monads we are more interested in are 2-monads whose algebras are categories with chosen colimits of a certain class [19]. These examples are discussed in Section 8. Other structures that can be presented as algebras for a 2-monad on **Cat** are: braided monoidal categories; categories equipped with an endo-functor, a pointed endo-functor or a monad; a category equipped with two monads and a distributive law between them (e.g., distributive categories).

Given a pseudomonad $T : \mathscr{K} \to \mathscr{K}$ and two pseudo-T-algebras A and B, a *lax morphism* from A to B is a 1-cell $f : A \to B$ in $\mathscr{K}$ together with a 2-cell

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "a"'] \arrow[dr, phantom, "\swarrow \bar{f}"] & TB \arrow[d, "b"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

that satisfies two axioms of compatibility with the pseudoalgebra structures. We could have chosen the 2-cell $\bar{f}$ in the opposite direction, and the resulting notion is called an *oplax morphism*.

When $\bar{f}$ is invertible we say that $(f, \bar{f})$ is a *pseudomorphism*, and when $\bar{f} = 1$ we say that f is a *strict morphism*. A 2-cell between two lax morphisms $(f, \bar{f}), (g, \bar{g}) : A \to B$ is just a 2-cell $\alpha : f \Rightarrow g$ in $\mathscr{K}$ compatible with $\bar{f}$, $\bar{g}$.

Now we can combine algebras and morphisms to form 2-categories. For a given 2-monad T on $\mathscr{K}$ we denote by Ps-T-Alg the 2-category with objects pseudo-T-algebras, 1-cells pseudomorphisms and 2-cells the ones defined above. We denote by T-Alg the 2-category of (strict) T-algebras and pseudomorphisms between them, and by T-Alg_s the 2-category of (strict) T-algebras and strict morphisms between them. In all instances the 2-cells are the same, all these 2-categories have a forgetful 2-functor into $\mathscr{K}$, and there is an inclusion 2-functor $J : T\text{-}\mathrm{Alg}_\mathrm{s} \to T\text{-}\mathrm{Alg}$. Full details can be found in [3].

Example 2.3. For the 2-monad T on **Cat** of Example 2.1, it is not hard to see that a pseudomorphism corresponds to a strong monoidal functor (in the terminology of [12]), and in fact Ps-T-Alg is (equivalent) to the 2-category of monoidal categories, strong monoidal functors and monoidal natural transformations (see for example [31, Chapter 3]). Similarly, for 2-monad S of Example 2.2, Ps-S-Alg can be identified with the 2-category of symmetric monoidal categories, braided functors and monoidal natural transformations. By methods developed in [3] with [13] as a predecessor, one can find 2-monads T', S' and equivalences between Ps-T-Alg and T'-Alg and Ps-S-Alg and S'-Alg.

3. Monads enriched in a monoidal 2-category

Enriched categories provide a framework in which to study categories whose hom sets have more structure, for example, hom sets that are abelian groups; or even categories whose homs are not simply sets with extra structure but some other type of objects, as for example chain complexes or non-negative real numbers. The notion of enriched category we consider is the classical one due to Eilenberg and Kelly [9]. Thus we will enrich in categories that are symmetric monoidal closed, complete and cocomplete. However, in some parts of the paper we only use the closed structure. The power of the theory of enriched categories is exemplified by [15].

When we are dealing with $\mathscr{V}$-enriched categories with extra algebraic structure, usually we are not contemplating an ordinary 2-monad on $\mathscr{V}$-**Cat** but actually a $\mathscr{V}$-**Cat**-enriched monad. The case of most interest for us will be the 2-monads on $\mathscr{V}$-**Cat** whose algebras are $\mathscr{V}$-categories with chosen colimits of a certain class, considered in [19].

Unless we impose some restrictive conditions on $\mathscr{V}$, there is no advantage in working with $\mathscr{V}$-**Cat** instead of a more general 2-category $\mathscr{W}$, so we will follow this second option, assuming that $\mathscr{W}$ is a complete and cocomplete symmetric monoidal closed **Cat**-enriched category. Sometimes we consider $\mathscr{W}$ to be just closed – in the sense of [9] – with “composition” transformations

$$L^X_{Y,Z} : [Y, Z] \to \big[[X, Y], [X, Z]\big].$$

Before proceeding to the kernel of this paper we need to say some words on the two-dimensional monad theory associated to a $\mathscr{W}$-enriched monad. Denote by I the unit object of $\mathscr{W}$. As usual, the functor $\mathscr{W}(I, -) : \mathscr{W} \to$ **Set** induces a 2-functor $(-)_0 : \mathscr{W}$-**Cat** $\to$ **Cat**. But taking into account that $\mathscr{W}$ is a 2-category, and so $\mathscr{W}(I, -)$ is in fact a 2-functor into **Cat**, we get a 2-functor $(-)_1 : \mathscr{W}$-**Cat** $\to$ 2-**Cat**. Extending the usual notation of A_0 for the underlying (ordinary) category of a $\mathscr{W}$-category A, we call A_1 its *underlying 2-category,* and similarly for functors and transformations.

3.1. Strength and enrichment

Suppose $\mathscr{K}$ is a $\mathscr{W}$-category that admits cotensor products. Recall that a cotensor of an object B of $\mathscr{K}$ with an object X of $\mathscr{W}$, denoted by $\{X, B\}$, is defined by the existence of a $\mathscr{W}$-natural isomorphism $\mathscr{K}(A, \{X, B\}) \cong [X, \mathscr{K}(A, B)]$. Cotensor products are a particular instance of weighted limits (see [15]). As such, for any $\mathscr{W}$-functor $T : \mathscr{K} \to \mathscr{K}$ there is a canonical comparison $\mathscr{W}$-natural transformation

$$\bar{t}_{X,B} : T\{X, B\} \to \{X, TB\}$$

whose component $\bar{t}_{X,B}$ can be characterised as the unique arrow making the following diagram commutative.

$$\begin{tikzcd}
{[X, \mathscr{K}(A, B)]} \arrow[rr, "{[X,\mathsf{T}]}"] \arrow[d, "\cong"'] & & {[X, \mathscr{K}(TA, TB)]} \arrow[d, "\cong"] \\
\mathscr{K}(A, \{X, B\}) \arrow[r, "\mathsf{T}"] & \mathscr{K}(TA, T\{X, B\}) \arrow[r, "{\mathscr{K}(TA,\bar{t})}"] & \mathscr{K}(TA, \{X, TB\})
\end{tikzcd} \tag{2}$$

Observe that the T-algebra structure on the cotensor product $\{X, A\}$ of $X \in \mathscr{W}$ with a T-algebra A can be written in terms of $\bar{t}$ and $a : TA \to A$ as

$$T\{X, A\} \xrightarrow{\bar{t}_{X,A}} \{X, TA\} \xrightarrow{\{X,a\}} \{X, A\}.$$

When $\mathscr{K} = \mathscr{W}$, cotensor products are just internal homs, and a further transformation is associated to the enrichment of T in $\mathscr{W}$, namely the *strength*

$$t_{X,Y} : X \otimes TY \to T(X \otimes Y).$$

This transformation satisfies $t_{X,Y\otimes Z}.(X \otimes t_{Y,Z}) = t_{X\otimes Y,Z}$, and the composition of $t_{I,X}$ with the canonical unit isomorphisms is the identity arrow of TX. We denote by $t'_{X,Y} : TX \otimes Y \to T(X \otimes Y)$ the transformation obtained from t and the symmetry of $\mathscr{W}$ in the obvious way. An ordinary endo-functor equipped with a strength is called a *strong functor*. There is a bijection between strengths on $T : \mathscr{W} \to \mathscr{W}$ and enrichments of T in $\mathscr{W}$: if the enrichment is given by arrows $\mathsf{T}_{X,Y} : [X, Y] \to [TX, TY]$, then $\mathsf{T}_{X,Y}$ and $t_{X,Y}$ are related by

$$\begin{aligned}
&\bigl([X, Y] \otimes TX \xrightarrow{1\otimes\mathsf{T}_{X,Y}} [TX, TY] \otimes TX \xrightarrow{\mathrm{ev}} TY\bigr) \\
&\quad = \bigl([X, Y] \otimes TX \xrightarrow{t_{[X,Y],X}} T\bigl([X, Y] \otimes X\bigr) \xrightarrow{T\mathrm{ev}} TY\bigr).
\end{aligned}$$

When (T, η, μ) is a $\mathscr{W}$-monad, the associated strength of T satisfies additional properties, equivalent to the $\mathscr{W}$-naturality of η and μ; namely,

$$\mu_{X\otimes Y}.T(t_{X,Y}).t_{X,TY} = t_{X,Y}.(X \otimes \mu_X) \quad \text{and} \quad t_{X,Y}.(X \otimes \eta_Y) = \eta_{X\otimes Y}.$$

These equalities translate in terms of $\bar{t}_{X,Y} : T[X, Y] \to [X, TY]$ as

$$[X, \mu_Y].\bar{t}_{X,TY}.(T\bar{t}_{X,Y}) = \bar{t}_{X,Y}.\mu_{[X,Y]} \quad \text{and} \quad \bar{t}_{X,Y}.\eta_{[X,Y]} = [X, \eta_Y].$$

Ordinary monads equipped with a strength satisfying the aforementioned equalities are called *strong monads*, and appear in the series [23–25]. Then, to give a strong monad on $\mathscr{W}$ is equivalent to giving a $\mathscr{W}$-monad.

3.2. *The $\mathscr{W}$-category T*-Alg

As in the rest of the section, $\mathscr{W}$ will be a complete and cocomplete monoidal closed 2-category. The theory of monads on categories and their algebras can be generalised to the enriched context [8]. For a $\mathscr{W}$-monad (T, η, μ) on a $\mathscr{W}$-category $\mathscr{K}$, the Eilenberg–Moore $\mathscr{W}$-category of algebras of T, which we will denote by $T\text{-Alg}_\mathrm{s}$, has objects the T_0-algebras, where T_0 is the ordinary monad on $\mathscr{K}_0$ underlying T. If A, B are T-algebras, the enriched hom $T\text{-Alg}_\mathrm{s}(A, B)$ is defined by the equalizer of the following pair:

$$\mathscr{K}(A,B) \xrightarrow{\mathsf{T}} \mathscr{K}(TA,TB) \xrightarrow{\mathscr{K}(1,b)} \mathscr{K}(TA,B), \qquad \mathscr{K}(A,B) \xrightarrow{\mathscr{K}(a,1)} \mathscr{K}(TA,B).$$

The corresponding forgetful $\mathscr{W}$-functor will be denoted by U_s. Observe that the 2-category $T\text{-Alg}_{\mathrm{s},1}$ is the 2-category of algebras and strict morphisms of algebras for the 2-monad T_1 on $\mathscr{K}_1$, and $U_{\mathrm{s},1}$ the corresponding forgetful 2-functor. Henceforth we shall identify the 2-categories $T\text{-Alg}_{\mathrm{s},1}$ and $T_1\text{-Alg}_\mathrm{s}$.

For each pair of T-algebras A, B, we have 1-cells in $\mathscr{W}$

$$\sigma_{A,B} : \mathscr{K}(A,B) \xrightarrow{\mathsf{T}} \mathscr{K}(TA,TB) \xrightarrow{\mathscr{K}(TA,b)} \mathscr{K}(TA,B) \tag{3}$$

that form a $\mathscr{W}$-natural transformation $\sigma : \mathscr{K}(U_s-, U_s-) \Rightarrow \mathscr{K}(TU_s-, U_s-) : T\text{-Alg}_\mathrm{s}^{\mathrm{op}} \otimes T\text{-Alg}_\mathrm{s} \to \mathscr{W}$. Observe that σ satisfy the following equations:

$$\sigma_{TA,B}\sigma_{A,B} = \mathscr{K}(\mu_A, B)\sigma_{A,B}, \qquad \mathscr{K}(\eta_A, B)\sigma_{A,B} = 1. \tag{4}$$

This transformation will play a central role in later sections.

Remark 3.1. When $\mathscr{K}$ admits cotensor products, $\sigma_{A,B}$ and $\sigma_{A,\{X,B\}}$ are related by the commutativity of the following square (a consequence of the commutativity of (2)).

$$\begin{tikzcd}
{[X, \mathscr{K}(A,B)]} \arrow[r, "{[X,\sigma_{A,B}]}"] \arrow[d, "\cong"'] & {[X, \mathscr{K}(TA,B)]} \arrow[d, "\cong"] \\
\mathscr{K}(A, \{X,B\}) \arrow[r, "\sigma_{A,\{X,B\}}"] & \mathscr{K}(TA, \{X,B\})
\end{tikzcd} \tag{5}$$

The 2-categories $T_1\text{-Alg}_\ell$ and $T_1\text{-Alg}$ of algebras and, respectively, lax morphisms and pseudomorphisms are the underlying 2-categories of two $\mathscr{W}$-categories, $T\text{-Alg}_\ell$ and $T\text{-Alg}$. For a pair of T-algebras A, B, the object $T\text{-Alg}_\ell(A, B)$ comes equipped with a 2-cell depicted in Fig. 1, universal with respect to the equalities in Fig. 2. The one-dimensional part of the universal property says that given any other 2-cell $\delta : \sigma_{A,B}.p \Rightarrow \mathscr{K}(a, B).p : L \to \mathscr{K}(TA, B)$ satisfying the same equations, there exists a unique 1-cell $\hat{p} : L \to T\text{-Alg}_\ell(A, B)$ such that $\delta = \gamma.\hat{p}$. The two-dimensional part of the universal property says that given a 2-cell δ as above and another

$$\begin{tikzcd}
 & \mathscr{K}(A,B) \arrow[dr, "\sigma_{A,B}"] & \\
T\text{-Alg}_\ell(A,B) \arrow[ur, "U_{\ell,A,B}"] \arrow[dr, "U_{\ell,A,B}"'] & \Downarrow\gamma & \mathscr{K}(TA,B) \\
 & \mathscr{K}(A,B) \arrow[ur, "\mathscr{K}(a,1)"'] &
\end{tikzcd}$$

Fig. 1. Universal 2-cell defining $T\text{-Alg}_\ell(A, B)$.

$\epsilon : \sigma_{A,B}.q \Rightarrow \mathscr{K}(a, B).q : L \to \mathscr{K}(TA, B)$, and a 2-cell $\alpha : p \Rightarrow q$ compatible with δ, ϵ in the sense that $\delta(\sigma_{A,B}.\alpha) = (\mathscr{K}(a, B).\alpha)\epsilon$, then $\alpha = U_{\ell,A,B}.\hat{\alpha}$ for a unique $\hat{\alpha} : \hat{p} \Rightarrow \hat{q}$.

If we further require the 2-cell of Fig. 1 to be invertible, we obtain another object that we denote by $T\text{-Alg}(A, B)$; the object of pseudomorphisms.

Remark 3.2. The 2-cell of Fig. 1 can be constructed by considering an inserter of the pair of 1-cells $\sigma_{A,B}, \mathscr{K}(a, B) : \mathscr{K}(A, B) \to \mathscr{K}(TA, B)$ and then two equifiers to impose the equations of Fig. 2. Hence it can also be constructed as a limit on one step: there exists a small 2-category $\mathscr{H}_\ell$, a weight $\chi_\ell : \mathscr{H}_\ell \to \mathbf{Cat}$ and a 2-functor $H_{\ell,A,B} : \mathscr{H}_{\ell,A,B} \to \mathscr{W}$ such that $\lim(\chi_\ell, H_{\ell,A,B})$ is $T\text{-Alg}_\ell(A, B)$. The same applies to $T\text{-Alg}(A, B)$, by using an iso-inserter instead of an inserter.

Now it is routine to see that the objects $T\text{-Alg}_\ell(A, B)$ and $T\text{-Alg}(A, B)$ are the enriched homs of two $\mathscr{W}$-categories, that we write $T\text{-Alg}_\ell$ and $T\text{-Alg}$ respectively, both with objects the T-algebras in $\mathscr{K}$. For example, the composition $T\text{-Alg}_\ell(A, B) \otimes T\text{-Alg}_\ell(B, C) \to T\text{-Alg}_\ell(A, C)$ and identity $I \to T\text{-Alg}_\ell(A, A)$ correspond to the 2-cells in Fig. 3 (where we omit the tensor product symbol as a space saving measure).

The 1-cells $U_{\ell,A,B} : T\text{-Alg}_\ell(A, B) \to \mathscr{K}(A, B)$ and $U_{A,B} : T\text{-Alg}(A, B) \to \mathscr{K}(A, B)$ provide the effect on enriched homs of forgetful $\mathscr{W}$-functors $U_\ell : T\text{-Alg}_\ell \to \mathscr{K}$ and $U : T\text{-Alg} \to \mathscr{K}$. There are obvious identity on objects inclusions $J : T\text{-Alg}_s \to T\text{-Alg}$ and $T\text{-Alg} \to T\text{-Alg}_\ell$. The first exists simply because in the definition of the homs of T-Alg we used an iso-inserter, and identities are trivially invertible, or in other words, strict morphisms are pseudomorphisms. The second exists because iso-inserters factor through the respective inserters, or in other words, pseudomorphisms are also lax morphisms.

Because $\mathscr{W}\text{-}\mathbf{Cat} \to 2\text{-}\mathbf{Cat}$ is induced by $\mathscr{W}_1(I, -) : \mathscr{W}_1 \to \mathbf{Cat}$, and representable 2-functors preserve limits, it is easy to see that $T\text{-Alg}_{\ell,1}$ is the usual 2-category of algebras and lax morphisms $T_1\text{-Alg}_\ell$; similarly, $T\text{-Alg}_1$ is $T_1\text{-Alg}$.

4. Pseudo-closed $\mathscr{W}$-categories and pseudo-commutative $\mathscr{W}$-monads

In this section and the next we give an outline of the main constructions and results of [10], where Hyland and Power give structures on a 2-monad T that ensure that the 2-category T-Alg is pseudo-closed in a suitable sense. Here we recall the definition of pseudo-closed structures, leaving the structures on the 2-monad for the next section.

$$
\begin{tikzcd}
 & \mathscr{K}(A,B) \arrow[r, "\sigma_{A,B}"] \arrow[d, phantom, "\Downarrow\gamma"] & \mathscr{K}(TA,B) \arrow[dr, "\sigma_{TA,B}"] & \\
T\text{-Alg}_\ell(A,B) \arrow[ur, "U_\ell"] \arrow[r, "U_\ell"] \arrow[dr, "U_\ell"'] & \mathscr{K}(A,B) \arrow[ur, "\mathscr{K}(a,1)"'] \arrow[dr, "\sigma A,B"] \arrow[d, phantom, "\Downarrow\gamma"] & & \mathscr{K}(T^2A,B) \\
 & \mathscr{K}(A,B) \arrow[r, "\mathscr{K}(a,1)"'] & \mathscr{K}(TA,B) \arrow[ur, "\mathscr{K}(Ta,1)"'] &
\end{tikzcd}
$$

$$\|$$

$$
\begin{tikzcd}
 & \mathscr{K}(A,B) \arrow[dr, "\sigma_{A,B}"] \arrow[dd, phantom, "\Downarrow\gamma"] & & \\
T\text{-Alg}_\ell(A,B) \arrow[ur, "U_\ell"] \arrow[dr, "U_\ell"'] & & \mathscr{K}(TA,B) \arrow[r, "\mathscr{K}(\mu_A,1)"] & \mathscr{K}(T^2A,B) \\
 & \mathscr{K}(A,B) \arrow[ur, "\mathscr{K}(a,1)"'] & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \mathscr{K}(A,B) \arrow[dr, "\sigma_{A,B}"] \arrow[dd, phantom, "\Downarrow\gamma"] & & \\
T\text{-Alg}_\ell(A,B) \arrow[ur, "U_\ell"] \arrow[dr, "U_\ell"'] & & \mathscr{K}(TA,B) \arrow[r, "\mathscr{K}(\eta_A,1)"] & \mathscr{K}(A,B) = 1 \\
 & \mathscr{K}(A,B) \arrow[ur, "\mathscr{K}(a,1)"'] & &
\end{tikzcd}
$$

Fig. 2. Equalities for $T\text{-Alg}(A,B)_\ell$.

4.1. *Pseudo-closed structures*

Closed categories arose in the early days of category theory [9], and although in many examples a closed structure is accompanied by a monoidal structure, most of the time the former is easier to describe (e.g., the category of k-modules for a commutative ring k). In the case of the 2-categories of algebras something similar takes place: in order to construct a tensor product, if possible, it is usually simpler to first consider a pseudo-closed structure.

We take the definition of a pseudo-closed structure from Hyland and Power [10], changing **Cat** by $\mathscr{W}$.

Definition 4.1. (See [10].) A *pseudo-closed $\mathscr{W}$-category* is a $\mathscr{W}$-category $\mathscr{K}$ equipped with the following data: $\mathscr{W}$-functors $V : \mathscr{K} \to \mathscr{W}$ and $[-,-] : \mathscr{K}^{\mathrm{op}} \otimes \mathscr{K} \to \mathscr{K}$, an object $I \in \mathscr{K}$, $\mathscr{W}$-(extraordinary) natural transformations $j_A : I \to [A,A]$, $e_A : [I,A] \to A$, $i_A : A \to [I,A]$,

$$\begin{tikzcd}
 & {[A,C]} \arrow[r, "\mathsf{T}"] & {[TA,TC]} \arrow[ddr, "{[1,c]}"] & \\
{[B,C][A,B]} \arrow[ur, "\mathrm{comp}"] \arrow[r, "\mathsf{TT}"] & {[TB,TC][TA,TB]} \arrow[ur, "\mathrm{comp}"] \arrow[r, "{[1,c]1}"] \arrow[d, phantom, "\Downarrow\gamma\otimes 1"] & {[TB,C][TA,TB]} \arrow[dr, "\mathrm{comp}"] & \\
T\text{-}\mathrm{Alg}_\ell(B,C)T\text{-}\mathrm{Alg}_\ell(A,B) \arrow[u, "U_\ell U_\ell"] \arrow[r, "U_\ell U_\ell"] \arrow[dr, "U_\ell U_\ell"'] & {[B,C][A,B]} \arrow[r, "1\mathsf{T}"] \arrow[d, phantom, "\Downarrow 1\otimes\gamma"] & {[B,C][TA,TB]} \arrow[u, "{[b,1]1}"] \arrow[d, "{1[1,b]}"] & {[TA,C]} \\
 & {[B,C][A,B]} \arrow[r, "{1[a,1]}"'] & {[B,C][TA,B]} \arrow[ur, "\mathrm{comp}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & {[A,A]} \arrow[dr, "\sigma_{A,A}"] & \\
I \arrow[ur, "\mathrm{id}"] \arrow[rr, "a"] \arrow[dr, "\mathrm{id}"'] & & {[TA,A]} \\
 & {[A,A]} \arrow[ur, "{[a,1]}"'] &
\end{tikzcd}$$

Fig. 3. 2-cells corresponding to compositions and identities of T-Alg_ℓ.

$k^A_{B,C} : [B,C] \to [[A,B],[A,C]]$. This data must satisfy the commutativity of the diagrams in $\mathscr{K}_1$ in Fig. 4 and

- $V[-,-] = \mathscr{K}(-,-) : \mathscr{K}^{\mathrm{op}} \otimes \mathscr{K} \to \mathscr{W}$;
- the 1-cell $I \xrightarrow{j_A} \mathscr{K}(I,[A,A]) = V[I,[A,A]] \xrightarrow{Ve_{[A,A]}} V[A,A] = \mathscr{K}(A,A)$ is the identity of A;
- there are equivalences $i_A \dashv e_A$ in the 2-category $\mathscr{K}_1$ whose units are identity 2-cells, i.e., retracts equivalences;
- the 1-cell $\mathscr{W}_1(I, V(i_A e_A)) : \mathscr{K}_1(I,A) \to \mathscr{K}_1(I,A)$ in **Cat** takes each $f : I \to A$ in $\mathscr{K}_1$ to $e_A[f,A]j_A : I \to [A,A] \to [I,A] \to A$.

When $\mathscr{W}$ is **Cat** we recover the definition of a pseudo-closed 2-category in [10]. It is also clear that if $\mathscr{K}$ is a pseudo-closed $\mathscr{W}$-category, then its underlying 2-category $\mathscr{K}_1$ is pseudo-closed.

Example 4.1. A candidate to a pseudo-closed 2-category is, for example, the 2-category of symmetric strict monoidal categories, braided monoidal functors and monoidal transformations: given two such categories A, B, the category of braided monoidal functors $A \to B$ and monoidal transformations between them has a canonical structure of a symmetric strict monoidal category. This example is studied in great detail in [10]. Another possible example of a pseudo-closed 2-category is the 2-category finitely cocomplete categories, finitely cocontinuous functors and natural transformations. Here again, given two such categories A, B, finitely cocontinuous functors $A \to B$ and transformations between them form a finitely cocontinuous category. However,

$$\begin{tikzcd}
I \arrow[r, "j_B"] \arrow[dr, "j_{[A,B]}"'] & {[B,B]} \arrow[d, "k^A"] \\
& {[[A,B],[A,B]]}
\end{tikzcd}
\qquad
\begin{tikzcd}
{[A,C]} \arrow[r, "k^A_{A,C}"] \arrow[d, equal] & {[[A,A],[A,C]]} \arrow[d, "{[j_A,1]}"] \\
{[A,C]} & {[I,[A,C]]} \arrow[l, "e_{[A,C]}"]
\end{tikzcd}$$

$$\begin{tikzcd}
{[C,D]} \arrow[r, "k^A_{C,D}"] \arrow[d, "k^B_{C,D}"'] & {[[A,C],[A,D]]} \arrow[r, "k^{[A,B]}"] & {[[[A,B],[A,C]],[[A,B],[A,D]]]} \arrow[d, "{[k^A_{B,C},1]}"] \\
{[[B,C],[B,D]]} \arrow[rr, "{[1,k^A_{B,D}]}"'] & & {[[B,C],[[A,B],[A,D]]]}
\end{tikzcd}$$

$$\begin{tikzcd}
{[A,B]} \arrow[r, "k^I_{A,B}"] \arrow[dr, "{[e_A,1]}"'] & {[[I,A],[I,B]]} \arrow[d, "{[1,e_B]}"] \\
& {[[I,A],B]}
\end{tikzcd}$$

Fig. 4. Some of the axioms of a pseudo-closed 2-category.

this 2-category is not quite pseudo-closed; to obtain a pseudo-closed structure one has to move to categories with *chosen colimits*. This example is studied in Section 8.

4.2. Pseudo-commutativities

The series [23–25] studies the structures on a strong monad (defined on a closed category) that induce a closed structure on the corresponding category of Eilenberg–Moore algebras in such a way that the associated adjunction is a closed adjunction. The main result is that these closed structures correspond to a property of this monad that was named *commutativity*. One basic example is the free abelian group monad on the category of sets. The category of algebras for this monad is (isomorphic to) the category of abelian groups, which is manifestly closed. The commutativity of this monad is an expression of the fact that the addition in an abelian group is commutative. Hyland and Power [10] deal with the higher dimensional problem of defining pseudo-commutativity for 2-monads, and finding the right level of generality that allows for a large number of interesting examples but at the same time remains manageable.

Definition 4.2. (See [10].) A *pseudo-commutativity* for a $\mathscr{W}$-monad $T:\mathscr{W}\to\mathscr{W}$ is an invertible modification depicted in (6) of Fig. 5, satisfying the axioms resulting from replacing in [10, Definition 5] the Cartesian product of **Cat** by the tensor product $\otimes$ of $\mathscr{W}$:

1. $\gamma_{X\otimes Y,Z}.(t_{X,Y}\otimes TZ)=t_{X,Y\otimes Z}.(X\otimes\gamma_{Y,Z})$.
2. $\gamma_{X,Y\otimes Z}.(TX\otimes t_{Y,Z})=\gamma_{X\otimes Y,Z}.(t'_{X,Y}\otimes TZ)$.
3. $\gamma_{X,Y\otimes Z}.(TX\otimes t'_{Y,Z})=t'_{X\otimes Y,Z}.(\gamma_{X,Y}\otimes Z)$.
4. $\gamma_{X,Y}.(\eta_X\otimes TY)=1$.

$$
\begin{tikzcd}
TX \otimes TY \arrow[r, "t'_{X,TY}"] \arrow[d, "t_{TX,Y}"'] & T(X \otimes TY) \arrow[r, "Tt_{X,Y}"] \arrow[d, phantom, "\Downarrow \gamma_{X,Y}"] & T^2(X \otimes Y) \arrow[d, "\mu_{X\otimes Y}"] \\
T(TX \otimes Y) \arrow[r, "Tt'_{X,Y}"'] & T^2(X \otimes Y) \arrow[r, "\mu_{X\otimes Y}"'] & T(X \otimes Y)
\end{tikzcd}
\tag{6}
$$

$$
\begin{tikzcd}
T[X,Y] \arrow[r, "T(\mathsf{T})"] \arrow[d, "\bar{t}_{X,Y}"'] & T[TX,TY] \arrow[r, "\bar{t}"] \arrow[d, phantom, "\Downarrow \bar{\gamma}_{X,Y}"] & {[TX,T^2Y]} \arrow[d, "{[1,\mu_Y]}"] \\
{[X,TY]} \arrow[r, "\mathsf{T}"'] & {[TX,T^2Y]} \arrow[r, "{[1,\mu_Y]}"'] & {[TX,TY]}
\end{tikzcd}
\tag{7}
$$

Fig. 5. Pseudo-commutativity.

$$
\begin{tikzcd}
T^2X \otimes TY \arrow[r, "t'"] \arrow[d, "t"'] & T(TX \otimes TY) \arrow[r, "Tt'"] \arrow[d, "Tt"] & T^2(X \otimes TY) \arrow[r, "T^2t"] \arrow[d, phantom, "\Downarrow T\gamma_{X,Y}"] & T^3(X \otimes Y) \arrow[d, "T\mu"] \\
T(T^2X \otimes Y) \arrow[d, "Tt'"'] \arrow[dr, phantom, "\Downarrow \gamma_{TX,Y}"] & T^2(TX \otimes Y) \arrow[r, "T^2t'"'] \arrow[d, "\mu"] & T^3(X \otimes Y) \arrow[r, "T\mu"] \arrow[d, "\mu"] & T^2(X \otimes Y) \arrow[d, "\mu"] \\
T^2(TX \otimes Y) \arrow[r, "\mu"'] & T(TX \otimes Y) \arrow[r, "Tt'"'] & T^2(X \otimes Y) \arrow[r, "\mu"'] & T(X \otimes Y)
\end{tikzcd}
\tag{8}
$$

$$
\begin{tikzcd}
TX \otimes T^2Y \arrow[r, "t'"] \arrow[d, "t"'] & T(X \otimes T^2Y) \arrow[r, "Tt"] \arrow[d, phantom, "\Downarrow \gamma_{X,TY}"] & T^2(X \otimes TY) \arrow[d, "\mu"] \\
T(TX \otimes TY) \arrow[r, "Tt'"] \arrow[d, "Tt"'] & T^2(X \otimes TY) \arrow[r, "\mu"'] \arrow[d, "T^2t"] & T(X \otimes TY) \arrow[d, "Tt"] \\
T^2(TX \otimes Y) \arrow[d, "T^2t'"'] \arrow[dr, phantom, "\Downarrow T\gamma_{X,Y}"] & T^3(X \otimes Y) \arrow[r, "\mu"] \arrow[d, "T\mu"] & T^2(X \otimes Y) \arrow[d, "\mu"] \\
T^3(X \otimes Y) \arrow[r, "T\mu"] & T^2(X \otimes Y) \arrow[r, "\mu"'] & T(X \otimes Y)
\end{tikzcd}
\tag{9}
$$

Fig. 6. Pasting diagrams for Definition 4.2-6 and 7.

5. $\gamma_{X,Y}.(TX \otimes \eta_Y) = 1$.
6. $\gamma_{X,Y}.(\mu_X \otimes TY)$ is equal to the pasting (8) in Fig. 6.
7. $\gamma_{X,Y}.(TX \otimes \mu_Y)$ is equal to the pasting (9) in Fig. 6.

A $\mathscr{W}$-monad equipped with a pseudo-commutativity will be called a *pseudo-commutative* $\mathscr{W}$-monad.

The main example of a pseudo-commutative 2-monad in [10] is the 2-monad on **Cat** whose algebras are symmetric strict monoidal categories (see Example 2.2); in this case the modification (6) is the canonical isomorphism mediating between two lexicographic orders. We refer the reader to the detailed exposition in Section 3 of the cited paper.

Remark 4.2. In [10, p. 161] it is observed that any two of the three first axioms in Definition 4.2 – the so-called strength axioms – imply the third. This is a fact that we shall use later.

One could translate the definition of a pseudo-commutativity, which is given in terms of the monoidal structure, into data and conditions that refer exclusively to the closed structure. This translation provides the notion of a pseudo-commutativity for a $\mathscr{W}$-monad when $\mathscr{W}$ is a closed 2-category, not necessarily monoidal. Showing that Definitions 4.2 and 4.3 are equivalent when $\mathscr{W}$ is monoidal closed is only a matter of a routine verification.

Definition 4.3. Assume $\mathscr{W}$ is a closed 2-category (not necessarily monoidal). A pseudo-commutativity for a $\mathscr{W}$-enriched monad T is a modification $\bar{\gamma}$ as in (7) subject to the following conditions.

1. The following 2-cells are equal:

$$[Y,Z] \xrightarrow{L^X} \big[[X,Y],[X,Z]\big] \xrightarrow{\mathsf{T}} \big[T[X,Y],T[X,Z]\big] \underset{\Downarrow[1,\bar{\gamma}_{X,Z}]}{\overset{}{\rightrightarrows}} \big[T[X,Y],[TX,TZ]\big],$$

$$[Y,Z] \xrightarrow{\mathsf{T}} [TY,TZ] \xrightarrow{L^{TX}} \big[[TX,TY],[TX,TZ]\big] \underset{\Downarrow[\bar{\gamma}_{X,Y},1]}{\overset{}{\rightrightarrows}} \big[T[X,Y],[TX,TZ]\big].$$

2. The following 2-cells are equal:

$$T[Y,Z] \xrightarrow{TL^X} T\big[[X,Y],[X,Z]\big] \underset{\Downarrow\bar{\gamma}_{[X,Y],[X,Z]}}{\overset{}{\rightrightarrows}} \big[T[X,Y],T[X,Z]\big] \xrightarrow{[1,\bar{t}]} \big[T[X,Y],[X,TZ]\big],$$

$$T[Y,Z] \underset{\Downarrow\bar{\gamma}_{Y,Z}}{\overset{}{\rightrightarrows}} [TY,TZ] \xrightarrow{L^X} \big[[X,TY],[X,TZ]\big] \xrightarrow{[\bar{t},1]} \big[T[X,Y],[X,TZ]\big].$$

3. The following 2-cells are equal:

$$T[Y,Z] \xrightarrow{TL^X} T\big[[X,Y],[X,Z]\big] \xrightarrow{\bar{t}} \big[[X,Y],T[X,Z]\big] \underset{\Downarrow[1,\bar{\gamma}_{X,Z}]}{\overset{}{\rightrightarrows}} \big[[X,Y],[TX,TZ]\big],$$

$$T[Y,Z] \underset{\Downarrow\bar{\gamma}_{Y,Z}}{\overset{}{\rightrightarrows}} [TY,TZ] \xrightarrow{L^{TX}} \big[[TX,TY],[TX,TZ]\big] \xrightarrow{[\mathsf{T},1]} \big[[X,Y],[TX,TZ]\big]$$

4. $\bar{\gamma}_{X,Y}.\eta_{[X,Y]}$ is an identity.
5. $[\eta_X, TY].\bar{\gamma}_{X,Y}$ is an identity.
6. $\bar{\gamma}_{X,Y}.\mu_{[X,Y]}$ is equal to the pasting (10) in Fig. 7.
7. $[\mu_X, TY].\bar{\gamma}_{X,Y}$ is equal to the pasting (11) in Fig. 7.

Conditions 4 to 7 appear in [10, Proposition 8]. Conditions 1, 2, 3 correspond respectively to the axioms 1, 2, 3 of Definition 4.2.

$$
\begin{tikzcd}
T^2[X,Y] \arrow[r, "T^2(\mathsf{T})"] \arrow[d, "T\bar{t}"'] & T^2[TX,TY] \arrow[r, "T\bar{t}"] \arrow[d, phantom, "\Downarrow T\bar{\gamma}_{X,Y}"] & T[TX,T^2Y] \arrow[d, "T[1,\mu_Y]"] \\
T[X,TY] \arrow[r, "T(\mathsf{T})"] \arrow[d, "\bar{t}"'] & T[TX,T^2Y] \arrow[r, "T[1,\mu_Y]"] \arrow[d, "\bar{t}"'] & T[TX,TY] \arrow[d, "\bar{t}"] \\
{[X,T^2Y]} \arrow[d, "\mathsf{T}"'] \arrow[dr, phantom, "\Downarrow \bar{\gamma}_{X,TY}"] & {[TX,T^3Y]} \arrow[r, "{[1,T\mu_Y]}"] \arrow[d, "{[1,\mu_{TY}]}"] & {[TX,T^2Y]} \arrow[d, "{[1,\mu_Y]}"] \\
{[TX,T^3Y]} \arrow[r, "{[1,\mu_{TY}]}"'] & {[TX,T^2Y]} \arrow[r, "{[1,\mu_Y]}"'] & {[TX,TY]}
\end{tikzcd}
\tag{10}
$$

$$
\begin{tikzcd}
T[X,Y] \arrow[r, "T(\mathsf{T})"] \arrow[d, "\bar{t}"'] \arrow[dr, phantom, "\bar{\gamma}_{X,Y}\Downarrow"] & T[TX,TY] \arrow[r, "T(\mathsf{T})"] \arrow[d, "\bar{t}"] & T[T^2X,T^2Y] \arrow[r, "\bar{t}"] \arrow[d, phantom, "\Downarrow \bar{\gamma}_{TX,TY}"] & {[T^2X,T^3Y]} \arrow[d, "{[1,\mu_{TY}]}"] \\
{[X,TY]} \arrow[d, "\mathsf{T}"'] & {[TX,T^2Y]} \arrow[r, "\mathsf{T}"] \arrow[d, "{[1,\mu_Y]}"] & {[T^2X,T^3Y]} \arrow[r, "{[1,\mu_{TY}]}"] \arrow[d, "{[1,T\mu_Y]}"] & {[T^2X,T^2Y]} \arrow[d, "{[1,\mu_Y]}"] \\
{[TX,T^2Y]} \arrow[r, "{[1,\mu_Y]}"'] & {[TX,TY]} \arrow[r, "\mathsf{T}"'] & {[T^2X,T^2Y]} \arrow[r, "{[1,\mu_Y]}"'] & {[T^2X,TY]}
\end{tikzcd}
\tag{11}
$$

Fig. 7. Pasting diagrams for Definition 4.3-6 and 7.

Remark 4.3. When $\mathscr{W}$ is monoidal closed, conditions 2, 3 of Definition 4.3 are respectively equivalent to the following two conditions that appear in [10, Proposition 8].

1. $[TX, \bar{t}_{Y,Z}].\bar{\gamma}_{X,[Y,Z]}$ is the closed transpose of $[t'_{X,Y}, TZ].\bar{\gamma}_{X\otimes Y,Z}$.
2. $[X, \bar{\gamma}_{Y,Z}].\bar{t}_{X,[Y,Z]}$ is the closed transpose of $[t_{X,Y}, TZ].\bar{\gamma}_{X\otimes Y,Z}$.

There is a similar equivalent formulation for Definition 4.3-1 but it involves both $\bar{\gamma}$ and γ and we choose to ignore it here. We are allowed to do this by Remark 4.2: any two of the strength axioms for $\bar{\gamma}$ (equivalently, γ) imply the third [10, Proposition 1].

Example 4.4. To illustrate Definition 4.3, and for the benefit of the reader unfamiliar with [10], we exhibit the canonical pseudo-commutativity, in its form $\bar{\gamma}$, for the 2-monad S on **Cat** whose algebras are symmetric strict monoidal categories. We already described S in Example 2.2. An object of $T[X,Y]$ is an n-tuple $(f_1,\dots,f_n)$ of functors $f_i : X \to Y$. The domain of the component of $\bar{\gamma}_{X,Y}$ corresponding to this object has as domain the functor $TX \to TY$ given on objects by

$$(x_1,\dots,x_m) \mapsto (f_1x_1,\dots,f_1x_m, f_2x_1,\dots,f_2x_m,\dots,f_nx_1,\dots,f_nx_m) \tag{12}$$

while the codomain is the functor $TX \to TY$ given on objects by

$$(x_1,\dots,x_m) \mapsto (f_1x_1,\dots,f_nx_1, f_1x_2,\dots,f_nx_2,\dots,f_1x_m,\dots,f_nx_m). \tag{13}$$

So domain and codomain are given by the two different lexicographic orderings of the objects $f_i x_j$. The component $((\bar{\gamma}_{X,Y})_{(f_1,\dots,f_n)})_{(x_1,\dots,x_m)}$ is the unique isomorphism between (12) and (13) induced by the symmetry of TY.

5. A characterisation of pseudo-commutativity

Although almost all the section's material remains valid when the base 2-category $\mathscr{W}_1$ is closed, for simplicity we shall assume that it is in fact monoidal closed. In the absence of interesting examples when $\mathscr{W}_1$ is not monoidal, the lost of generality is outweighted by the simplification in the exposition of the sections that follow.

As mentioned at the beginning of the previous section, the main point of [24] is the correspondence between the commutativity of a monad and the existence of a closed structure on its category of Eilenberg–Moore algebras such that the induced adjunction is closed. In the case of pseudo-commutativities something similar happens, but the correspondence is not so clean; this reflects the fact that Definition 4.1 is not as "weak" as possible but as "strict" as examples allow.

The commutativity of an enriched monad T can be reinterpreted as saying that each component of the natural transformation

$$\xi_{X,Y} : [X, TY] \xrightarrow{\mathsf{T}} \left[TX, T^2Y\right] \xrightarrow{[TX,\mu_Y]} [TX, TY]$$

is a morphism of T-algebras. One can hope that a similar result will hold for pseudo-commutativities, and as we shall see it does. However, it will not be enough to consider this transformation only for T-algebras of the form TY but we will need the transformation to be defined for a wider class T-algebras. Then, we shall consider the transformation

$$\sigma_{X,B} : [X, B] \xrightarrow{\mathsf{T}} [TX, TB] \xrightarrow{[TX,b]} [TX, B] \tag{14}$$

defined for all T-algebras B. An explanation for this can be found in a simple difference between the definitions of a commutativity and of a pseudo-commutativity: the number of objects involved in the axioms. The former states that two arrows $TX \otimes TY \to T(X \otimes Y)$ should be equal. The latter includes conditions that speak about the equality of certain 2-cells between 1-cells with domain, for example, $TX \otimes TY \otimes Z$ and codomain $T(X \otimes Y \otimes Z)$. See Definition 4.2. When written in terms of a transformation of the form (14), this condition will involve a T-algebra of the form $[Y, TZ]$, which need not to be free. This can be seen more clearly in Definition 4.3 and it is explained in detail below.

A key observation of [10] is that a pseudo-commutativity on a $\mathscr{W}$-monad T on $\mathscr{W}$ induces a pseudomorphism structure on the composite (14) for every T-algebra B, and these arrows form a pseudonatural transformation in the following way. Consider the 2-functors $[-,-], [T-,-] : \mathscr{W}_1^{\mathrm{op}} \times T\text{-}\mathrm{Alg}_1 \to T\text{-}\mathrm{Alg}_1$ and observe that the 1-cells (14) are the components of a pseudonatural transformation

$$U[-,-] \Rightarrow U[T-,-] : \mathscr{W}_1^{\mathrm{op}} \times T\text{-}\mathrm{Alg}_1 \to \mathscr{W}_1. \tag{15}$$

Indeed, if $f : B \to C$ is a 1-cell in T-Alg, the structural 2-cell σ_f corresponding to f is the 2-cell below.

$$\begin{tikzcd}
{[X,B]} \arrow[r, "\mathsf{T}"] \arrow[d, "{[1,f]}"'] & {[TX,TB]} \arrow[r, "{[1,b]}"] \arrow[d, "{[1,Tf]}"'] \arrow[dr, phantom, "{[1,\bar{f}^{-1}]\Downarrow}"] & {[TX,B]} \arrow[d, "{[1,f]}"] \\
{[X,C]} \arrow[r, "\mathsf{T}"] & {[TX,TC]} \arrow[r, "{[1,c]}"] & {[TX,C]}
\end{tikzcd} \tag{16}$$

The pseudonatural transformation obtained by precomposing σ with the inclusion

$$1 \times J_1 : \mathscr{W}_1^{\mathrm{op}} \times T\text{-}\mathrm{Alg}_{\mathrm{s},1} \to \mathscr{W}_1^{\mathrm{op}} \otimes T\text{-}\mathrm{Alg}_1$$

is in fact 2-natural. In other words, σ is 2-natural on *strict* morphisms.

Further properties of σ are provided in [10, Propositions 13 and 16]. This section is devoted to prove that to give such a transformation satisfying such conditions is actually equivalent to giving a pseudo-commutativity.

Theorem 5.1. *To give a pseudo-commutativity for a $\mathscr{W}$-monad T is equivalent to giving an invertible modification with components*

$$\begin{tikzcd}
T[X,B] \arrow[r, "T\sigma_{X,B}"] \arrow[d, "{[X,b].\bar{t}_{X,B}}"'] \arrow[dr, phantom, "\swarrow \bar{\sigma}_{X,B}"] & T[TX,B] \arrow[d, "{[TX,b].\bar{t}_{TX,B}}"] \\
{[X,B]} \arrow[r, "\sigma_{X,B}"'] & {[TX,B]}
\end{tikzcd} \tag{17}$$

for each $X \in \mathscr{W}$, $B \in T\text{-}\mathrm{Alg}_{\mathrm{s},1}$ making $(\sigma_{X,B}, \bar{\sigma}_{X,B})$ a pseudomorphism of T-algebras and satisfying:

1. $$\begin{tikzcd}
{[X,B]} \arrow[r, "\sigma_{X,B}"] \arrow[dr, "1"'] & {[TX,B]} \arrow[d, "{[\eta_X,1]}"] \\
& {[X,B]}
\end{tikzcd}$$

2. $$\begin{tikzcd}
{[X,B]} \arrow[r, "\sigma_{X,B}"] \arrow[d, "\sigma_{X,B}"'] & {[TX,B]} \arrow[d, "\sigma_{TX,B}"] \\
{[TX,B]} \arrow[r, "{[\mu_X,1]}"'] & {[T^2X,B]}
\end{tikzcd}$$

3. *The following 2-cells are equal, for X, Y in $\mathscr{W}$ and B a T-algebra:*

$$\begin{tikzcd}
T[Y,B] \arrow[r, "TL^X"] & T\big[[X,Y],[X,B]\big] \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\bar{\sigma}_{[X,Y],[X,B]}"] & \big[T[X,Y],[X,B]\big],
\end{tikzcd}$$

$$\begin{tikzcd}
T[Y,B] \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\bar{\sigma}_{Y,B}"] & {[TY,B]} \arrow[r, "L^X"] & \big[[X,TY],[X,B]\big] \arrow[r, "{[\bar{t},1]}"] & \big[T[X,Y],[X,B]\big].
\end{tikzcd}$$

4. *The following* 2*-cells are equal, for* X, Y *in* $\mathscr{W}$ *and* B *a* T*-algebra:*

$$T[Y,B] \xrightarrow{TL^X} T\big[[X,Y],[X,B]\big] \xrightarrow{\bar{t}} \big[[X,Y],T[X,B]\big] \underset{\Downarrow[1,\bar{\sigma}_{X,B}]}{\rightrightarrows} \big[[X,Y],[TX,B]\big],$$

$$T[Y,B] \underset{\Downarrow\bar{\sigma}_{Y,B}}{\rightrightarrows} [TY,B] \xrightarrow{L^{TX}} \big[[TX,TY],[TX,B]\big] \xrightarrow{[\mathsf{T},1]} \big[[X,Y],[TX,B]\big].$$

Moreover, in each condition it is enough to consider free T*-algebras* $B = TZ$.

We split the proof of the theorem in several lemmas.

Lemma 5.2. *Let* $T : \mathscr{W} \to \mathscr{W}$ *be a* $\mathscr{W}$*-enriched monad. There is a bijection between modifications* $\bar{\gamma}$ *as depicted in* (7) *and modifications as depicted in* (17)*. Furthermore, the following are equivalent.*

1. $\bar{\sigma}_{X,B}$ *equips* $\sigma_{X,B} : T[X,B] \to [TX,B]$ *with a structure of a pseudomorphism of* T*-algebras, for all* X *in* $\mathscr{W}$ *and* T*-algebra* B.
2. $\bar{\sigma}_{X,TY}$ *equips* $\sigma_{X,TY}$ *with a structure of a pseudomorphism of* T*-algebras for all* X, Y *in* $\mathscr{W}$.
3. $\bar{\gamma}$ *satisfies conditions* 4 *and* 6 *of Definition* 4.3.

Proof. The bijection between $\bar{\gamma}$ and $\bar{\sigma}$ is depicted in Fig. 8, where each one of the modifications is given in terms of the other. The two conditions $\bar{\sigma}$ must satisfy in order to be a pseudomorphism structure are

$$\big(\bar{\sigma}_{X,B}.T[X,b].T\bar{t}_{X,B}\big)\big([TX,b].\bar{t}_{TX,B}.T\bar{\sigma}_{X,B}\big) = \bar{\sigma}_{X,B}.\mu_{[X,B]}, \tag{18}$$

$$\bar{\sigma}_{X,B}.\eta_{[X,B]} = 1. \tag{19}$$

By rewriting the left-hand side of (18) in terms of $\bar{\gamma}$ according to Fig. 8 it is not hard to show that it corresponds to the pasting (10), for $Y = B$. The right-hand side corresponds to $\bar{\gamma}_{X,B}.\mu_{[X,B]}$. This shows that the pseudomorphism condition (18) follows from condition 6 of Definition 4.3. Similarly, (19) follows from condition 4. This shows that 3 implies 1.

It is clear that 1 implies 2, so it remains to prove that 2 implies 3. Expressing (10) and $\bar{\gamma}_{X,Y}.\mu_{[X,Y]}$ in terms of $\bar{\sigma}$ one can easily see that (18) for $B = TY$ implies condition 6 of Definition 4.3. Similarly, (19) for $B = TY$ implies condition 4 of the aforementioned definition. □

Lemma 5.3. *Let* $T : \mathscr{W} \to \mathscr{W}$ *be a* $\mathscr{W}$*-enriched monad and the two modifications* $\bar{\gamma}$ *and* $\bar{\sigma}$ *as in the previous lemma.*

1. *The following conditions are equivalent, where* X, Y *are objects of* $\mathscr{W}$ *and* B *is a T-algebra:*

$$[\eta_X, TY].\bar{\sigma}_{X,TY} = 1, \qquad [\eta_X, TY].\bar{\gamma}_{X,Y} = 1, \qquad [\eta_X, B].\bar{\sigma}_{X,B} = 1.$$

2. *The following conditions are equivalent.*
   (a) *Condition* 7 *of Definition* 4.3.
   (b) $(\bar{\sigma}_{TX,TY}.\sigma_{X,TY})(\sigma_{TX,TY}.T\bar{\sigma}_{X,TY}) = [\mu_X, TY].\bar{\sigma}_{X,TY}$ *for* X, Y *in* $\mathscr{W}$.
   (c) $(\bar{\sigma}_{TX,B}.\sigma_{X,B})(\sigma_{TX,B}.T\bar{\sigma}_{X,B}) = [\mu_X, B].\bar{\sigma}_{X,B}$ *for* X *in* $\mathscr{W}$ *and* B *a* T*-algebra.*

$$\begin{tikzcd}
T[X,B] \arrow[rr, "T(\mathsf{T})"] \arrow[dd, "\bar{t}"'] \arrow[ddrr, phantom, "\Downarrow \bar{\gamma}_{X,B}"] & & T[TX,TB] \arrow[r, "T[1,b]"] \arrow[d, "\bar{t}"] & T[TX,B] \arrow[d, "\bar{t}"] \\
 & & {[TX,T^2B]} \arrow[r, "{[TX,Tb]}"] \arrow[d, "{[1,\mu_B]}"] & {[TX,TB]} \arrow[dd, "{[1,b]}"] \\
{[X,TB]} \arrow[r, "\mathsf{T}"] \arrow[d, "{[1,b]}"'] & {[TX,T^2B]} \arrow[r, "{[1,\mu_B]}"] \arrow[d, "{[1,Tb]}"] & {[TX,TB]} \arrow[dr, "{[1,b]}"'] & \\
{[X,B]} \arrow[r, "\mathsf{T}"'] & {[TX,TB]} \arrow[rr, "{[1,b]}"'] & & {[TX,B]}
\end{tikzcd}$$

$$\begin{tikzcd}
T[X,Y] \arrow[r, "T(\mathsf{T})"] \arrow[dr, "{T[1,\eta_Y]}"] \arrow[dd, "\bar{t}"'] & T[TX,TY] \arrow[dr, "{T[1,T\eta_Y]}"] \arrow[drr, bend left, "1"] & & \\
 & T[X,TY] \arrow[r, "T(\mathsf{T})"] \arrow[d, "\bar{t}"'] \arrow[ddrr, phantom, "\Downarrow \bar{\sigma}_{X,TY}"] & T[TX,T^2Y] \arrow[r, "{T[1,\mu_Y]}"] & T[TX,TY] \arrow[d, "\bar{t}"] \\
{[X,TY]} \arrow[r, "{[1,T\eta_Y]}"] \arrow[dr, bend right, "1"'] & {[X,T^2Y]} \arrow[d, "{[1,\mu_Y]}"] & & {[TX,T^2Y]} \arrow[d, "{[1,\mu_Y]}"] \\
 & {[X,TY]} \arrow[r, "\mathsf{T}"'] & {[TX,T^2Y]} \arrow[r, "{[1,\mu_Y]}"'] & {[TX,TY]}
\end{tikzcd}$$

Fig. 8. $\bar{\gamma}$ and $\bar{\sigma}$ one in terms of one another.

Proof. 1. This is a simple consequence of the bijection between $\bar{\gamma}$ and $\bar{\sigma}$ as depicted in Fig. 8.

2. A rewriting of diagram (11) in terms of $\bar{\sigma}$, using Fig. 8 together with the fact that $\bar{\sigma}$ is a modification, shows that 2(b) implies 2(a). Explicitly, the aforementioned rewriting would yield the 2-cell

$$\begin{aligned}
&\big(\big[T^2X,\mu_Y\big].\bar{\sigma}_{TX,T^2Y}.T[TX,\eta_{TY}].\sigma_{X,TY}.T[X,\eta_Y]\big)\big(\sigma_{TX,TY}.\bar{\sigma}_{X,TY}.T[X,\eta_X]\big)\\
&\quad=\big(\big[T^2X,\mu_Y\big].\big[T^2X,\eta_{TY}\big].\bar{\sigma}_{TX,TY}.\sigma_{X,TY}.T[X,\eta_Y]\big)\big(\sigma_{TX,TY}.\bar{\sigma}_{X,TY}.T[X,\eta_X]\big)\\
&\quad=\big((\bar{\sigma}_{TX,TY}.\sigma_{X,TY})(\sigma_{TX,TY}.\bar{\sigma}_{X,TY})\big).T[X,\eta_X].
\end{aligned}$$

Obviously, 2(b) is a particular instance of 2(c), so it only rests to prove that 2(c) implies 2(a). First we rewrite the left-hand side of 2(c) in terms of $\bar{\gamma}$ to obtain the following 2-cell:

$$\begin{aligned}
&\big(\big[T^2X,b\big].\mathsf{T}.[TX,b].\bar{\gamma}_{X,B}\big)\big(\big[T^2X,b\big].\bar{\gamma}_{TX,B}.T[TX,b].T(\mathsf{T})\big)\\
&\quad=\big(\big[T^2X,b\big].[TX,Tb].\mathsf{T}.\bar{\gamma}_{X,B}\big)\big(\big[T^2X,b\big].T[TX,Tb].\bar{\gamma}_{TX,TB}.T(\mathsf{T})\big)\\
&\quad=\big[T^2X,b\big].\big(\big([TX,\mu_B].\mathsf{T}.\bar{\gamma}_{X,B}\big)\big(\big[T^2X,\mu_B\big].\bar{\gamma}_{TX,TB}.T(\mathsf{T})\big)\big).
\end{aligned} \tag{20}$$

This 2-cell is simply the composition of the pasting (11) with the 1-cell $[T^2X, b]$. By 2(a), the 2-cell (20) is equal to

$$\big[T^2X, b\big].[\mu_X, TB].\bar{\gamma}_{X,B} = [\mu_X, B].\big[T^2X, b\big].\bar{\gamma}_{X,B} = [\mu_X, B].\bar{\sigma}_{X,B}. \qquad \square$$

Lemma 5.4.

1. *The following conditions are equivalent.*
   (a) *Condition* 2 *of Definition* 4.3.
   (b) *Condition* 3 *of Theorem* 5.1.
   (c) *Condition* 3 *of Theorem* 5.1 *for* $B = TZ$.
2. *The following conditions are equivalent.*
   (a) *Condition* 3 *of Definition* 4.3.
   (b) *Condition* 4 *of Theorem* 5.1.
   (c) *Condition* 4 *of Theorem* 5.1 *for* $B = TZ$.

Proof. 1. We start by showing that 1(a) implies 1(b). We rewrite the latter in terms of $\bar{\gamma}$ using Fig. 8 to obtain – recall that the T-algebra structure of $[X, B]$ is given by $[X, b].\bar{t}_{X,B}$ –

$$\begin{aligned}
&\big[T[X, Y], [X, b].\bar{t}_{X,B}\big].\bar{\gamma}_{[X,Y],[X,B]}.TL^X_{Y,B} \\
&\quad = \big[\bar{t}_{X,Y}, [X, B]\big].L^X_{TY,B}.[TY, b].\bar{\gamma}_{Y,B} \\
&\quad = \big[T[X, Y], b\big].\big[\bar{t}_{X,Y}, [X, TB]\big].L^X_{TY,TB}.\bar{\gamma}_{Y,B}
\end{aligned}$$

where in the second equality we used the naturality of the composition L. Now it is clear that the resulting equation is simply condition 2 of Definition 4.3 post-composed with $[T[X, Y], b]$.

It is clear that 1(c) is a particular instance of 1(b), so it only rests to prove that 1(c) implies 1(a). Rewriting condition 2 of Definition 4.3 in terms of $\bar{\sigma}$ yields the following equality that we must verify:

$$\begin{aligned}
&\big[T[X, Y], \bar{t}_{X,Z}\big].\bar{\sigma}_{[X,Y],T[X,Z]}.T\big[[X, Y], \eta_{[X,Z]}\big].TL^X_{Y,Z} \\
&\quad = \big[\bar{t}_{X,Y}, [X, TZ]\big].L^X_{TY,TZ}.\bar{\sigma}_{Y,TZ}.T[X, \eta_Z].
\end{aligned} \tag{21}$$

The left-hand side of (21) can be written as (22) below using the fact that $\bar{t}_{X,Z} : T[X, Z] \to [X, TZ]$ is always a strict morphism of T-algebras:

$$\bar{\sigma}_{[X,Y],[X,TZ]}.T\big[[X, Y], \bar{t}_{X,Z}\big].T[X, Y], \eta_{[X,Z]}.TL^X_{Y,Z} \tag{22}$$

$$= \bar{\sigma}_{[X,Y],[X,TZ]}.T\big[[X, Y], [X, \eta_Z]\big].TL^X_{Y,Z} \tag{23}$$

$$= \bar{\sigma}_{[X,Y],[X,TZ]}.TL^X_{Y,TZ}.T[Y, \eta_Z]. \tag{24}$$

In (23) we used the compatibility between η and the strength $\bar{t}$, and in (24) the naturality of L^X. Now it is clear that the right-hand side of (21) equals (24) because we are assuming condition 3 of Theorem 5.1 for $B = TZ$.

2. The proof of the second part of the lemma is direct and left to the reader. $\square$

The following proposition – where the monoidal structure appears explicitly – is a counterpart of Remark 4.3.

Proposition 5.5. *When $\mathscr{W}$ is monoidal closed conditions* 3 *and* 4 *of Theorem* 5.6 *are respectively equivalent to the following two other.*

1. $\bar{\sigma}_{X,[Y,B]}$ *is the closed transpose of* $[t'_{X,Y}, B].\bar{\sigma}_{X\otimes Y,B}$.
2. $[X, \bar{\sigma}_{Y,B}].\bar{t}_{X,[Y,B]}$ *is the closed transpose of* $[t_{X,Y}, B].\bar{\sigma}_{X\otimes Y,B}$.

Proof. Once one uses the fact that the canonical isomorphism $[X \otimes Y, B] \cong [X, [Y, B]]$ can be written as the composition

$$[X \otimes Y, B] \xrightarrow{L^Y} \big[[Y, X \otimes Y], [Y, B]\big] \xrightarrow{[c,1]} \big[X, [Y, B]\big]$$

where $c : X \to [Y, X \otimes Y]$ is the unit of the adjunction $(- \otimes Y) \dashv [Y, -]$, the proof is routine. $\square$

Theorem 5.1 can be restated in terms of data in $T\text{-Alg}_1$.

Theorem 5.6. *To give a pseudo-commutativity for a $\mathscr{W}$-enriched monad $T : \mathscr{W} \to \mathscr{W}$ is equivalent to giving a* 2*-natural transformation*

$$\sigma : [-,-] \Rightarrow [T-,-] : \mathscr{W}_1^{\mathrm{op}} \times T\text{-Alg}_{\mathrm{s},1} \to T\text{-Alg}_1$$

that has components with underlying 1*-cell $\sigma_{X,B}$, and satisfying the following conditions in* $T\text{-Alg}_1$*:*

1. $[\eta_X, B].\sigma_{X,B} = 1$.
2. $\sigma_{TX,B}.\sigma_{X,B} = [\mu_X, B]$.
3. *The following* 1*-cells in* $T\text{-Alg}_1$ *are equal, for X in $\mathscr{W}$ and B a T-algebra:*

$$[Y, B] \xrightarrow{L^X} \big[[X, Y], [Y, B]\big] \xrightarrow{\sigma_{[X,Y],[X,B]}} \big[T[X, Y], [X, B]\big],$$

$$[Y, B] \xrightarrow{\sigma_{Y,B}} [TY, B] \xrightarrow{L^X} \big[[X, TY], [X, B]\big] \xrightarrow{[\bar{t},1]} \big[T[X, Y], [X, B]\big].$$

4. *The following* 1*-cells in* $T\text{-Alg}_1$ *are equal, for X in $\mathscr{W}$ and B a T-algebra:*

$$[Y, B] \xrightarrow{L^X} \big[[X, Y], [X, B]\big] \xrightarrow{[1,\sigma_{X,B}]} \big[[X, Y], [TX, B]\big],$$

$$[Y, B] \xrightarrow{\sigma_{Y,B}} [TY, B] \xrightarrow{L^{TX}} \big[[TX, TY], [TX, B]\big] \xrightarrow{[\mathrm{T},1]} \big[[X, Y], [TX, B]\big].$$

Moreover, it is enough for these conditions to be satisfied for free algebras $B = TZ$.

Proof. This is a direct consequence of Theorem 5.1. Conditions 3 and 4 are restatements of conditions 3 and 4 of that theorem. $\square$

Proposition 5.7. *Conditions* 3 *and* 4 *of Theorem* 5.6 *are respectively equivalent to*

1. $\sigma_{X,[Y,B]}: [X, [Y, B]] \to [TX, [Y, B]]$ *is the closed transpose – in* T-Alg_1 *– of*

$$\big[t'_{X,Y}, B\big].\sigma_{X\otimes Y,B}: [X \otimes Y, B] \to [TX \otimes Y, B].$$

2. $[X, \sigma_{X,B}]: [X, [Y, B]] \to [X, [TY, B]]$ *is the closed transpose – in* T-Alg_1 *– of*

$$[t_{X,Y}, B].\sigma_{X\otimes Y,B}: [X \otimes Y, B] \to [X \otimes TY, B].$$

Remark 5.8. Theorem 5.6 remains valid if one replaces the 2-natural transformation σ by a *pseudonatural* transformation

$$\sigma: [-,-] \Rightarrow [T-,-]: \mathscr{W}_1^{\mathrm{op}} \times T\text{-}\mathrm{Alg}_1 \to T\text{-}\mathrm{Alg}_1$$

whose components have underlying 1-cells $\sigma_{X,B}$, and adding the requirement that this pseudonatural transformation should be 2-natural when restricted to $\mathscr{W}_1^{\mathrm{op}} \times T\text{-}\mathrm{Alg}_{\mathrm{s},1}$.

6. T-Alg as a pseudo-closed $\mathscr{W}$-category

The present section exhibits the canonical pseudo-closed structure of the $\mathscr{W}$-category T-Alg for a pseudo-commutative $\mathscr{W}$-monad T on $\mathscr{W}$. The construction in the case of $\mathscr{W} = \mathbf{Cat}$ is due to Hyland and Power [10] and we largely follow the same lines as them, modifying the arguments to the enriched case. As in the previous section, the results are valid for in the case when $\mathscr{W}$ is closed and the pseudo-commutativity is given in the form of Definition 4.3. However, we prefer to treat the less general case of a monoidal closed 2-category $\mathscr{W}$ as it considerably simplifies the argumentation, especially when dealing with the composition – see Lemma 6.6. At the present this mild assumption does not seem to exclude any interesting example.

6.1. *The T-algebra structure on the object of pseudomorphisms*

In this section we briefly explain how a pseudo-commutativity on T induces a T-algebra $[\![A, B]\!]$ with underlying $\mathscr{W}$-category T-Alg(A, B), yielding the following $\mathscr{W}$-enriched version of [10, Section 6].

Theorem 6.1. *A pseudo-commutativity on a $\mathscr{W}$-monad T on $\mathscr{W}$ induces a pseudo-closed structure on T-*Alg.

A fact we need to recall from [3] is that T_1-Alg admits certain limits and the forgetful 2-functor $U_1: T_1\text{-}\mathrm{Alg} \to \mathscr{W}_1$ preserves them. The limits we are referring to are products, inserters and equifiers, and therefore limits that can be constructed from these. See [3, Section 2]. Moreover, the components of the projections from these limits are strict morphisms of algebras.

We want a T-algebra $[\![A,B]\!]$ with underlying object $T\text{-Alg}(A,B)$ in $\mathscr{W}_1$, and so there will be a universal invertible 2-cell

$$\begin{tikzcd}
& {[A,B]} \arrow[dr, "\sigma_{A,B}"] & \\
{[\![A,B]\!]} \arrow[ur, "U_{A,B}"] \arrow[dr, "U_{A,B}"'] \arrow[rr, phantom, "\Downarrow\gamma"] & & {[TA,B]} \\
& {[A,B]} \arrow[ur, "{[a,1]}"'] &
\end{tikzcd} \tag{25}$$

in $\mathscr{W}_1$ satisfying the universal property described in Section 3.2. This 2-cell can be constructed from $\sigma_{A,B}$ and $[a,B]$ by considering an iso-inserter and two equifiers, and therefore, by the observations on the existence of limits in T_1-Alg of the previous paragraph, the diagram (25) itself will be a limit in T_1-Alg if $\sigma_{A,B}, [a,B] : [A,B] \to [TA,B]$ is a pseudomorphism of T-algebras.

Both $[A,B]$ and $[TA,B]$ are T-algebras as described in Section 3.1, and $[a,1]$ is always a strict morphism of algebras. Finally, the fact that $\sigma_{A,B}$ is a pseudomorphism is precisely what happens when T is pseudo-commutative by Theorem 5.1. The components $U_{A,B}$ of the forgetful $\mathscr{W}$-functor $U : T\text{-Alg} \to \mathscr{W}$ are strict morphisms of algebras, and the T-algebra structure $T(T\text{-Alg}(A,B)) \to T\text{-Alg}(A,B)$ is the unique 1-cell in $\mathscr{W}_1$ whose post-composition with $U_{A,B}$ equals

$$T\big(T\text{-Alg}(A,B)\big) \xrightarrow{T(U_{A,B})} T[A,B] \xrightarrow{\bar{t}} [A,TB] \xrightarrow{[A,b]} [A,B].$$

Remark 6.2. Since σ is 2-natural on *strict* morphisms of algebras, one easily deduces that $[\![A,f]\!]$ is a strict morphism of algebras for any strict morphism f. The fact that $[\![f,B]\!]$ is a strict morphism for any pseudomorphism f is easily verified.

6.2. Multilinear maps

Before describing the pseudo-closed structure on T-Alg we will briefly mention its closed multicategory structure, which in this case seems to arise more naturally. Later we shall use these multilinear maps to describe the composition of the pseudo-closed structure and to describe a tensor product of T-algebras.

Multilinear maps in the case of a pseudo-commutative 2-monad on **Cat** are explained at length in [10]. We choose, then, to keep the details to a minimum as our own contribution is only marginal.

Given T-algebras A, B and an object X of $\mathscr{W}$, a 1-cell $f : X \otimes A \to B$ is a *left parametrised morphism of T-algebras* when it is equipped with an invertible 2-cell

$$\begin{tikzcd}
X \otimes TA \arrow[r, "t"] \arrow[d, "1\otimes a"'] & T(X\otimes A) \arrow[r, "f"] \arrow[d, phantom, "\Downarrow\bar{f}"] & TB \arrow[d, "b"] \\
X \otimes A \arrow[rr, "f"] & & B
\end{tikzcd} \tag{26}$$

satisfying the obvious equations, analogous to the axioms of a pseudomorphism of T-algebras but this time involving the strength $t_{X,A}: X \otimes TA \to T(X \otimes A)$. We say that this parametrised morphism is *strict* when the 2-cell $\bar{f}$ is an identity. Right parametrised morphisms can be defined in the same way, now using $t': TA \otimes X \to T(X \otimes A)$ instead of t, and combining t, t' morphisms parametrised on both sides are easily described.

In fact, there is a universal object $T\text{-Alg}(X, A; B)$ in $\mathscr{W}$ that classifies parametrised morphisms; in particular (but not equivalently) 1-cells $I \to T\text{-Alg}(X, A; B)$ are in bijection with parametrised morphisms as described above. Indeed, we will have

$$T\text{-Alg}(X, A; B) \cong \big[X, T\text{-Alg}(A, B)\big] \tag{27}$$

and we could take this as defining the left-hand side. Alternatively, we can transpose along the adjunction $(X \otimes -) \dashv [X, -]$ the universal 2-cell defining the right-hand side of (27) (recall that $T\text{-Alg}(A, B)$ is defined as a certain limit, and then so is the right-hand side of (27)), to obtain a universal invertible 2-cell (after using Proposition 5.7-2)

$$\begin{tikzcd}
 & {[X \otimes A, B]} \arrow[dr, "{[t,1].\sigma_{A,B}}"] & \\
T\text{-Alg}(X, A; B) \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\Downarrow\cong"] & & {[X \otimes TA, B]} \\
 & {[A, B]} \arrow[ur, "{[a,1]}"'] &
\end{tikzcd} \tag{28}$$

satisfying two conditions that correspond to the equations in Fig. 2. The equation involving the multiplication of T uses Theorem 5.6-2, while the condition involving the unit of T uses Theorem 5.6-1.

A multilinear map $f: A \otimes B \to C$ will have two structures: one of a parametrised morphism $UA \otimes B \to C$ and another of a parametrised morphism $A \otimes UB \to C$, and both will commute in the sense that the two pastings in Fig. 9 must be equal (see also [10, p. 169]). Observe that to express this condition one requires the existence of a pseudo-commutativity on T (in the form (6)). There is a bijection between multilinear maps $A \otimes B \to C$ and pseudomorphisms $A \to [\![B, C]\!]$; if we denote by

$$\bar{f}_2 : c.(Tf).t \Rightarrow f.(A \otimes b) : A \otimes TB \to C,$$

$$\bar{f}_1 : c.(Tf).t' \Rightarrow f.(a \otimes B) : TA \otimes B \to C$$

the left and right parametrised morphisms structures, these are related to the corresponding pseudomorphism $g: A \to [\![B, C]\!]$ in the following way. The 1-cell g in T_1-Alg corresponds, by definition of $[\![B, C]\!]$ (see Section 6.1), to an invertible 2-cell in T_1-Alg

$$\hat{g} : \sigma_{B,C}.U.g \Rightarrow [b, C].U.g. \tag{29}$$

$$
\begin{tikzcd}
 & T(A\otimes TB) \arrow[r, "Tt"] \arrow[dr, phantom, "\Downarrow\gamma"] & T^2(A\otimes B) \arrow[dr, "\mu"] & \\
TA\otimes TB \arrow[ur, "t'"] \arrow[r, "t"] \arrow[d, "a\otimes 1"'] & T(TA\otimes B) \arrow[r, "Tt'"] \arrow[d, "T(a\otimes 1)"'] & T^2(A\otimes B) \arrow[r, "\mu"] \arrow[d, "T^2 f"] & T(A\otimes B) \arrow[d, "Tf"] \\
A\otimes TB \arrow[r, "t"] \arrow[dd, "1\otimes b"'] & T(A\otimes B) \arrow[r, phantom, "\Downarrow T\bar{f}_1"] \arrow[dr, "Tf"'] & T^2C \arrow[r, "\mu"] \arrow[d, "Tc"] & TC \arrow[dd, "c"] \\
 \arrow[r, phantom, "\Downarrow\bar{f}_2"] & & TC \arrow[dr, "c"] & \\
A\otimes B \arrow[rrr, "f"] & & & C
\end{tikzcd}
$$

$$\|$$

$$
\begin{tikzcd}
TA\otimes TB \arrow[r, "t'"] \arrow[d, "1\otimes b"'] & T(A\otimes TB) \arrow[r, "Tt"] \arrow[d, "T(1\otimes b)"'] & T^2(A\otimes B) \arrow[r, "\mu"] \arrow[d, "Tf"] & T(A\otimes B) \arrow[d, "f"] \\
TA\otimes B \arrow[r, "t'"] \arrow[dd, "a\otimes 1"'] & T(A\otimes B) \arrow[r, phantom, "\Downarrow T\bar{f}_2"] \arrow[dr, "Tf"'] & T^2C \arrow[r, "\mu"] \arrow[d, "Tc"] & TC \arrow[dd, "c"] \\
 \arrow[r, phantom, "\Downarrow\bar{f}_1"] & & TC \arrow[dr, "c"] & \\
A\otimes B \arrow[rrr, "f"] & & & C
\end{tikzcd}
$$

Fig. 9. Commutation axiom of a multilinear map.

This means that $U.g : A \to [B, C]$ is a pseudomorphism with two-dimensional structure

$$
\begin{tikzcd}
TA \arrow[r, "T(U.g)"] \arrow[dd, "a"'] & T[B,C] \arrow[d, "\bar{t}"] \\
 \arrow[r, phantom, "\swArrow_{\bar{g}}"] & {[B,TC]} \arrow[d, "{[1,c]}"] \\
A \arrow[r, "U.g"] & {[B,C]}
\end{tikzcd}
\tag{30}
$$

that satisfies a condition that states that $\hat{g}$ is a 2-cell in T_1-Alg; namely, a compatibility condition involving $\bar{g}$, $\hat{g}$ and the 2-cell $\bar{\sigma}_{B,C}$ of Section 5.

The 1-cell $U.g$ and the 2-cell (29) have as closed transpose in $\mathscr{W}_1$ respectively the 1-cell $f : A \otimes B \to C$ and the 2-cell $\bar{f}_2$, while $\bar{g}$ (30) has as closed transpose $\bar{f}_1$. The compatibility condition that makes $\hat{g}$ a 2-cell between pseudomorphisms corresponds to the commutation condition between $\bar{f}_1$, $\bar{f}_2$ in Fig. 9 (this condition involves the pseudo-commutativity). The details of this bijection between multilinear maps $A \otimes B \to C$ and pseudomorphisms $A \to [\![B, C]\!]$ are analogous to the case $\mathscr{W} = \mathbf{Cat}$ found in [10].

Remark 6.3. Observe that on the bijection between multilinear maps $f : A \otimes B \to C$ and pseudomorphisms $g : A \to [\![B, C]\!]$ the following can be added. The multilinear morphism f is strict in the second variable ($\bar{f}_2$ is an identity) if and only if g factors through $T\text{-Alg}_s(B, C)$, and f is strict in the first variable ($\bar{f}_1$ is an identity) if and only if g is a strict morphism of T-algebras.

The notion of a multilinear map "in two variables" can be easily extended to allow any number of variables. Together with an obvious notion of morphism between multilinear maps, these form a category what we shall denote by $T_1\text{-Alg}(A_1, \ldots, A_n; C)$; accordingly to the paragraph above there is an isomorphism

$$T_1\text{-Alg}(A_1, \ldots, A_{n+1}; C) \cong T_1\text{-Alg}\big(A_1, \ldots, A_n; [\![A_{n+1}, C]\!]\big). \tag{31}$$

Defining $T_1\text{-Alg}(; C) = \mathscr{W}_1(I, C)$ we obtain a closed **Cat**-enriched multicategory T_1-Alg such that the usual forgetful 2-functor into $\mathscr{W}_1$ is a morphism of **Cat**-enriched multicategories.

Example 6.4. A first but nonetheless important example of a multilinear map is the evaluation $\mathrm{ev} : [\![A, B]\!] \otimes A \to B$; by definition, this is the multilinear map associated to the identity pseudomorphism of $[\![A, B]\!]$. According to Remark 6.3 (taking the evaluation as f and the identity as g) we can deduce a couple of properties that will be needed later:

1. Since the identity is a strict morphism of T-algebras (in the notation above, $\bar{g} = 1$), then the parametrised morphism $\mathrm{ev} : [\![A, B]\!] \otimes UA \to B$ (this is, the action of T is on the first variable) is strict ($\bar{f}_1 = 1$).
2. Since $J : T\text{-Alg}_s(A, B) \to [\![A, B]\!]$ trivially factors through $T\text{-Alg}_s(A, B)$, we deduce that the associated parametrised morphism is strict (action on the first variable). Moreover, this parametrised morphism is the composite

$$T\text{-Alg}_s(A, B) \otimes A \xrightarrow{J \otimes B} U[\![A, B]\!] \otimes A \xrightarrow{\mathrm{ev}} B.$$

Example 6.5. The main example of a multilinear map for us will be the composition

$$\mathrm{comp} : [\![B, C]\!] \otimes [\![A, B]\!] \to [\![A, C]\!]. \tag{32}$$

The reason why (32) is a multilinear map is explained in [10]: any multilinear map (32) corresponds to a unique multilinear map $[\![B, C]\!] \otimes [\![A, B]\!] \otimes A \to C$; (32) will correspond to the composite of multilinear maps

$$[\![B, C]\!] \otimes [\![A, B]\!] \otimes A \xrightarrow{1 \otimes \mathrm{ev}} [\![B, C]\!] \otimes B \xrightarrow{\mathrm{ev}} C \tag{33}$$

where ev is the multilinear map of the preceding Example 6.4.

The following lemma says that the endo-$\mathscr{W}$-functor $[\![A, -]\!]$ of T-Alg restricts to the sub-$\mathscr{W}$-category $T\text{-Alg}_s$.

Lemma 6.6. *Let $k : [\![B, C]\!] \to [\![[\![A, B]\!], [\![A, C]\!]]\!]$ be the pseudomorphism of T-algebras associated to the composition multilinear map of Example* 6.5*. Then, kJ factors through* $T\text{-Alg}_s([\![A, B]\!], [\![A, C]\!])$.

Proof. An equivalent condition to the thesis is that the parametrised morphism

$$T\text{-Alg}_s(B, C) \otimes [\![A, B]\!] \xrightarrow{J\otimes 1} T\text{-Alg}(B, C) \otimes [\![A, B]\!] \xrightarrow{\text{comp}} [\![A, C]\!]$$

be strict. This will happen exactly when the parametrised morphism below (T-action on the middle variable) is strict,

$$T\text{-Alg}_s(B, C) \otimes [\![A, B]\!] \otimes UA \xrightarrow{J\otimes 1\otimes 1} T\text{-Alg}(B, C) \otimes [\![A, B]\!] \otimes UA$$
$$\xrightarrow{1\otimes \text{ev}} T\text{-Alg}(B, C) \otimes B \xrightarrow{\text{ev}} C.$$

This follows immediately from the two observations in Example 6.4 after rewriting the composite in the following way:

$$T\text{-Alg}_s(B, C) \otimes [\![A, B]\!] \otimes UA \xrightarrow{1\otimes \text{ev}} T\text{-Alg}_s(B, C) \otimes B$$
$$\xrightarrow{J\otimes 1} T\text{-Alg}(B, C) \otimes B \xrightarrow{\text{ev}} C. \qquad \square$$

The following observation will be used in Corollary 7.6 and re-interpreted in Section 8 as a familiar fact about functors that are right exact in each variable.

Proposition 6.7. *If the forgetful* 2*-functor $T_1\text{-Alg} \to \mathscr{W}_1$ is full on invertible* 2*-cells, then every partial map in each variable $f : A_1 \otimes \cdots \otimes A_n \to C$ is automatically a multilinear map.*

Proof. We briefly provide the proof in the case of $n = 2$. Given a partial map in each variable $f : A \otimes B \to C$ and the corresponding pseudomorphism $h : A \to [B, C]$, the commutation condition between both left and right structures of Fig. 9 is equivalent to the 2-cell corresponding to the right structure $h_2 : c.Tf.t_{A,B} \Rightarrow f.(A \otimes b)$

$$\hat{h} : \sigma_{B,C}.h \Rightarrow [b, C].h$$

being a 2-cell in T_1-Alg (this is (29)). This condition is automatic from our assumption that the forgetful 2-functor is full on invertible 2-cells. $\square$

The condition in the proposition above that the forgetful 2-functor $T_1\text{-Alg} \to \mathscr{W}_1$ be full on invertible 2-cells was shown in [18, Proposition 5.1] to be equivalent to requiring that any 1-cell in $\mathscr{W}_1$ have at most one lax morphism structure. In particular, KZ 2-monads satisfy this condition.

6.3. Pseudo-closed structure on T-Alg

Now we exhibit the pseudo-closed structure on T-Alg induced by a pseudo-commutativity on T, keeping the details to a minimum as this description is completely analogous to the case of 2-monads considered in [10].

The internal hom $[\![-,-]\!]$ is the $\mathscr{W}$-functor described in Section 6.1, with unit object FI, the free T-algebra on the neutral object I of $\mathscr{W}$. The 1-cell $j_A : FI \to [\![A,A]\!]$ is the unique strict morphism corresponding to the identity $I \to T\text{-Alg}(A,A)$. Next we have to provide the retract equivalence $i_A \dashv e_A : [\![FI,A]\!] \to A$ in $T\text{-Alg}_1$:

$$e_A : [\![FI,A]\!] \xrightarrow{U_{FI,A}} [FI,A] \xrightarrow{[\eta_I,A]} [I,A] \xrightarrow{\cong} A, \tag{34}$$

$$i_A : A \xrightarrow{\cong} [I,A] \xrightarrow{F_{I,A}} [\![FI,FA]\!] \xrightarrow{[\![FI,a]\!]} [\![FI,A]\!]. \tag{35}$$

Since U_1 reflects adjoint equivalences and retract equivalences, it is enough to show that there is a retract equivalence $U_1(i_A) \dashv U_1(e_A)$ in $\mathscr{W}_1$ and that e_A is a pseudomorphism of T-algebras. The latter is trivial, e_A being a strict morphism, while the existence of the retract equivalence in $\mathscr{W}_1$ is a particular case of Corollary A.3.

The composition $k : [\![B,C]\!] \to [\![[\![A,B]\!],[\![A,C]\!]]\!]$ will be the pseudomorphism of T-algebras associated (see Section 6.2) to the multilinear map (32) of Example 6.5.

The verification of the axioms of a pseudo-closed $\mathscr{W}$-category, almost identical to the 2-categorical version in [10], is left to the reader; it is mostly straightforward, and uses Corollary A.3.

Remark 6.8. For a pseudo-commutative $\mathscr{W}$-monad T, the 2-category T_1-Alg inherits a pseudo-closed structure from T-Alg.

6.4. Tensor products

Given a pseudo-commutative 2-monad T, under a mild assumption on T [10, Theorem 14] ensures the existence of an induced tensor product on T-Alg. However, can obtain more information than simply that.

We shall assume that T is a $\mathscr{W}$-monad with a rank on $\mathscr{W}$; e.g., T is *finitary*. The 2-monads of Examples 2.1 and 2.2 are finitary; see also Lemma 8.2.

The construction of the tensor product, always following [10], proceeds in the following manner. The assumption that T has a rank ensures that the $\mathscr{W}$-category $T\text{-Alg}_s$ is cocomplete (see Lemma A.4) and in particular $T\text{-Alg}_s$ will admit *tensor* products with objects of $\mathscr{W}$: given X in $\mathscr{V}$-**Cat** and A, B in $T\text{-Alg}_s$, there is a T-algebra $X * A$ and a $\mathscr{W}$-natural isomorphism

$$T\text{-Alg}_s(X * A, B) \cong \bigl[X, T\text{-Alg}_s(A,B)\bigr].$$

In Section A.1 we will see that the 2-adjoint $(-)'$ to the inclusion $J : T\text{-Alg}_s \to T\text{-Alg}$ lifts to a $\mathscr{W}$-enriched adjoint, and the counit of this adjunction, with components strict morphisms

$$q_A : A' \to A \tag{36}$$

is an equivalence (in $T\text{-Alg}_1$) and a retract. The existence of this left adjoint was further studied and clarified in [28]. Some of the fundamental facts about it are recalled in Section A.1.

As we saw in Lemma 6.6, for each T-algebra A the $\mathscr{W}$-functor $[\![A,-]\!]:T\text{-Alg}\to T\text{-Alg}$ restricts to a $\mathscr{W}$-functor $T\text{-Alg}_{\mathrm{s}}\to T\text{-Alg}_{\mathrm{s}}$, and we have a commutative diagram

$$\begin{tikzcd} T\text{-Alg}_{\mathrm{s}} \arrow[r, "{[\![A,-]\!]}"] \arrow[d, "J"'] & T\text{-Alg}_{\mathrm{s}} \arrow[d, "U_{\mathrm{s}}"] \\ T\text{-Alg} \arrow[r, "{T\text{-Alg}(A,-)}"'] & \mathscr{W} \end{tikzcd} \tag{37}$$

Now we show that $[\![A,-]\!]:T\text{-Alg}_{\mathrm{s}}\to T\text{-Alg}_{\mathrm{s}}$ has a left adjoint. The composition $U_s[\![A,-]\!]$ preserves cotensor products because $T\text{-Alg}(A,-)$ and J do so (see Lemma A.2); as U_{s} creates cotensor products, this means that $[\![A,-]\!]$ preserves cotensor products. This implies, by a basic fact of enriched category theory, that $[\![A,-]\!]$ has a left adjoint precisely when its underlying ordinary functor has one. This observation, together with the adjoint triangle theorem [7] and the fact that U_{s} is monadic and $T\text{-Alg}_{\mathrm{s}}$ cocomplete (Lemma A.4), implies that $[\![A,-]\!]$ has a left adjoint if and only if $T\text{-Alg}(A,J-)$ does. And it indeed does, the left adjoint being $-*A':\mathscr{W}\to T\text{-Alg}_{\mathrm{s}}$. So we have a 2-functor $-\oslash A:T\text{-Alg}_{\mathrm{s}}\to T\text{-Alg}_{\mathrm{s}}$ and $\mathscr{W}$-natural isomorphisms

$$T\text{-Alg}_{\mathrm{s}}(-\oslash A,C)\cong T\text{-Alg}_{\mathrm{s}}\big(-,[\![A,C]\!]\big). \tag{38}$$

As usual, (38) combines all the $\mathscr{W}$-functors $-\oslash A$ into a $\mathscr{W}$-functor $\oslash:T\text{-Alg}_{\mathrm{s}}\otimes T\text{-Alg}_{\mathrm{s}}\to T\text{-Alg}_{\mathrm{s}}$.

Lemma 6.9. *The $\mathscr{W}$-functor $\oslash$ preserves all colimits in the first variable. It preserves ϕ-colimits on the second variable, for a weight ϕ, if T preserves ϕ-colimits.*

Proof. The first assertion is obvious as each $-\oslash A$ has a right adjoint. To prove the second assertion, observe that $A\oslash -$ preserves a certain colimit for all A if and only if for each T-algebra C the 2-functor $[\![-,C]\!]$ sends this colimit into a limit, if and only if $U_{\mathrm{s}}[\![-,C]\!]$ have the same property, since U_{s} creates limits. The isomorphism

$$U_{\mathrm{s}}[\![-,C]\!]=T\text{-Alg}(J-,C)\cong T\text{-Alg}_{\mathrm{s}}\big((J-)',C\big)$$

transforms the problem into showing that $T\text{-Alg}_{\mathrm{s}}((J-)',C)$ sends colimits that are preserved by T into limits. This holds by Lemma A.1 of Appendix A. □

For the rest of this section we will work upon the pseudo-closed 2-category T_1-Alg (see Remark 6.8). We do no attempt to obtain a $\mathscr{W}$-enriched version of the monoidal structure on this 2-category, which is *pseudo* or *weak* in nature.

After obtaining the functor $\oslash$, [10] constructs a tensor product $\boxtimes$ in T_1-Alg by applying [3, Theorem 5.1]. To summarise the details needed here,

$$A\boxtimes B=J\big(A'\oslash B\big) \tag{39}$$

with unit object FI the free T-algebra on the unit object of $\mathscr{W}$. The relationship between monoidal and pseudo-closed structure can be expressed as the existence of pseudonatural equivalences

$$T_1\text{-Alg}(A \boxtimes B, C) \simeq T_1\text{-Alg}\big(A, [\![B, C]\!]\big). \tag{40}$$

These equivalences are given by the composite

$$\begin{aligned} T_1\text{-Alg}\big(J(A' \oslash B), C\big) &\xrightarrow{[\![B,-]\!]} T_1\text{-Alg}\big(\big[\!\big[B, J\big(A' \oslash B\big)\big]\!\big], [\![B, C]\!]\big) \\ &\xrightarrow{T_1\text{-Alg}(s_{A,B},1)} T_1\text{-Alg}\big(A, [\![B, C]\!]\big) \end{aligned} \tag{41}$$

where $s_{A,B} : A \to [\![B, J(A' \oslash B)]\!]$ is the unit of the adjunction $(-)' \oslash B \dashv [\![B, J-]\!]$.

The observations above are the basic ingredients of Hyland–Power's result:

Theorem 6.10. *(See [10].) A pseudo-commutativity on a $\mathscr{W}$-monad T with a rank on $\mathscr{W}$ induces a monoidal structure on* T_1-Alg. *Moreover, the biadjunction $F \dashv_{\mathrm{b}} U : T\text{-Alg} \to \mathscr{W}$ is monoidal.*

The last assertion that $F \dashv_{\mathrm{b}} U$ is monoidal means the following. Firstly, U is, in the terminology of [5], *weak monoidal*. This means that it is equipped a pseudonatural transformation

$$\chi_{A,B} : U(A) \otimes U(B) \to U(A \boxtimes B) \tag{42}$$

and a 1-cell $I \to UFI$ (in this case the unit of T) satisfying a higher version of the usual axioms of a monoidal functor. See [5, Definition 2]. Secondly, F is *strong monoidal*; that is, it is weak monoidal and the morphisms $F(X) \boxtimes F(Y) \to F(X \otimes Y)$, $FI \to FI$ are equivalences (the latter can be taken to be the identity). The unit $n : 1 \Rightarrow UF$ and the counit $e : FU \Rightarrow 1$ are monoidal pseudonatural transformations [5, Definition 3], and the invertible modifications $Ue.nU \cong 1$ and $eF.Fn \cong 1$ are monoidal [5, Definition 3].

Remark 6.11. The equivalences (40) show that the tensor product $A \boxtimes B$ classifies multilinear maps with domain A, B, in the sense that the 1-cell (42) induces equivalences

$$T_1\text{-Alg}(A, B; C) \simeq T_1\text{-Alg}(A \boxtimes B, C). \tag{43}$$

Next we show that the constructed tensor product pseudofunctor is 2-natural when restricted to strict morphisms of algebras. This result will be useful in a forthcoming paper, allowing us to speak of the preservation by the tensor product $\boxtimes$ of certain 2-categorical colimits in $T_1\text{-Alg}_{\mathrm{s}}$.

Lemma 6.12. *The* 1*-cells* (41) *are* 2*-natural not only in $A, C \in T_1$*-Alg *but also in $B \in T_1\text{-Alg}_s$.*

Proof. The result is obtained by setting in Theorem A.5: $\mathscr{P} = T_1\text{-Alg}_{\mathrm{s}}$, $\mathscr{L} = T_1\text{-Alg}$, $G(B, C) = [\![B, C]\!]$, $H(B, A) = A' \oslash B$. The isomorphisms (38) exhibit H as a left parametrised left adjoint of G. $\square$

Corollary 6.13. *The restriction of the tensor product pseudofunctor to strict morphisms in the second variable*

$$T_1\text{-Alg} \times T_1\text{-Alg}_s \xrightarrow{1\times J} T_1\text{-Alg} \times T_1\text{-Alg} \xrightarrow{\boxtimes} T_1\text{-Alg}$$

is (isomorphic to) the 2*-functor*

$$T_1\text{-Alg} \times T_1\text{-Alg}_s \xrightarrow{(-)'\times 1} T_1\text{-Alg}_s \times T_1\text{-Alg}_s \xrightarrow{\oslash} T_1\text{-Alg}_s \xrightarrow{J} T_1\text{-Alg}.$$

Proof. By Lemma 6.12 above, for any strict morphism of T-algebras $f : B \to D$ we have a commutative diagram

$$\begin{tikzcd}
T_1\text{-Alg}(J(A' \oslash B), C) \arrow[r] \arrow[d, "T_1\text{-Alg}(J(A'\oslash f), C)"'] & T_1\text{-Alg}(A, [\![B, C]\!]) \arrow[d, "T_1\text{-Alg}(A, [\![f, C]\!])"] \\
T_1\text{-Alg}(J(A' \oslash D), C) \arrow[r] & T_1\text{-Alg}(A, [\![D, C]\!])
\end{tikzcd}$$

that is 2-natural on $A, C \in T_1$-Alg, where the horizontal functors are the retract equivalences (41). This means that the strict morphism $J(A' \oslash f)$ satisfies the defining condition of $A \boxtimes f$. Thus the pseudofunctor $(A \boxtimes J-)$ is isomorphic to the 2-functor $J(A' \oslash -)$, and letting A vary, $(? \boxtimes J-)$ is isomorphic to the 2-functor $J(?' \oslash -) : T_1\text{-Alg} \times T\text{-Alg}_s \to T_1\text{-Alg}$. □

Corollary 6.14. *If T preserves conical colimits of a certain class then the restriction of the tensor product $\boxtimes$ to $T_1\text{-Alg}_s$ does so too, in each variable.*

Proof. Combine Lemma 6.9, Corollary 6.13 and Lemma A.1. □

The corollary holds for a more general kind of colimits, namely any 2-categorical colimit preserved by T_1. By considering only conical colimits, that is the only case we will be interested in, we avoid the detour that would entail explaining the relationship between 2-categorical weights and their induced $\mathscr{W}$-enriched weights.

We postpone the examples for the next and subsequent sections.

7. KZ 2-monads

With cocomplete categories as an example, Kock [26] (published in the form of [27]) introduced a special kind of *doctrine*, called *KZ doctrine*, as certain set of data and coherence conditions in a 2-category, as well as their algebras. So for example, there is a KZ doctrine whose algebras are finitely cocomplete categories. KZ doctrines are easier to work with than pseudomonads, having less data and less coherence conditions. However, these can be regarded as part of the data and coherence conditions of a pseudomonad, a fact that is mentioned in [27] and proved in full in [33]. Independently from Kock, Zöberlein [36] discovered the same concept, and therefrom the name of a KZ doctrine. Later Street [35], Marmolejo [33] and Kelly and Lack [18] made contributions to the subject. As we are mostly interested in 2-monads and not in more general pseudomonads, we roughly follow [18].

We will call a 2-monad (T, η, μ) on a 2-category $\mathscr{K}$ *Kock–Zöberlein*, abbreviated *KZ*, or *lax-idempotent*, when any 1-cell $f : A \to B$ in $\mathscr{K}$ between T-algebras has a unique structure of a lax morphism of T-algebras. This is equivalent to the condition that a 1-cell $a : TA \to A$ is a T-algebra structure if and only if there exists an adjunction $a \dashv \eta_A$ whose counit is an identity. Another equivalent condition is the existence of a modification $\delta : T.\eta \Rrightarrow \eta.T : T \Rightarrow T^2$ satisfying

$$\delta.\eta = 1 \quad \text{and} \quad \mu.\delta = 1. \tag{44}$$

Further equivalent conditions are given in [18, Theorem 6.2]. If T is a KZ 2-monad the forgetful 2-functor $U_\ell : T\text{-Alg} \to \mathscr{K}$ is locally fully faithful.

If A, B are T-algebras, the unique lax morphism structure on a 1-cell $f : A \to B$ in $\mathscr{K}$ is given by the following pasting, where the unlabelled 2-cell denotes the unit of the adjunction $a \dashv \eta_A$.

$$\begin{tikzcd}
TA \arrow[r, equal] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow", near start] & TA \arrow[r, "Tf"] & TB \arrow[d, "b"] \\
A \arrow[ur, "\eta_A"'] \arrow[r, "f"'] & B \arrow[ur, "\eta_B"] \arrow[r, equal] & B
\end{tikzcd} \tag{45}$$

It follows that a 1-cell $f : A \to B$ has a (unique) structure of a pseudomorphism of T-algebras if and only if (45) is invertible. Also, the forgetful 2-functor $U : T\text{-Alg} \to \mathscr{K}$ is injective on 1-cells and locally fully faithful.

In [27] it is shown that left adjoint morphisms between algebras are pseudomorphisms. If A, B are T-algebras and $f \dashv f^* : B \to A$ is an adjunction in $\mathscr{K}$, then f^*, just as any 1-cell, is a lax morphism and hence f has a structure of an oplax (or colax) morphism of T-algebras given by

$$\begin{tikzcd}
TA \arrow[d, equal] \arrow[dr, "Tf"] \arrow[d, phantom, shift left=3, "\Rightarrow"] & \\
TA \arrow[d, "a"'] & TB \arrow[l, "Tf^*"'] \arrow[d, "b"] \arrow[dl, phantom, "\Swarrow"] \\
A \arrow[dr, "f"'] & B \arrow[l, "f^*"'] \arrow[d, equal] \arrow[d, phantom, shift right=3, "\Rightarrow"] \\
& B
\end{tikzcd} \tag{46}$$

It follows from [18, Lemma 6.5] that the oplax structure $fa \Rightarrow bTf$ is invertible and its inverse is (45).

Definition 7.1. We say that a $\mathscr{W}$-monad T on $\mathscr{K}$ is a *KZ* or *lax-idempotent* $\mathscr{W}$-monad if its underlying 2-monad T_1 on the 2-category $\mathscr{K}_1$ is a KZ 2-monad in the usual sense.

One can ask whether this definition is the right one or we need a $\mathscr{W}$-enriched definition that makes no reference to the underlying 2-monad. Luckily there is no real difference between the two approaches, as explained in the following proposition.

Proposition 7.1. *A $\mathscr{W}$-monad T on $\mathscr{K}$ is KZ if and only if the forgetful $\mathscr{W}$-functor $U_\ell : T\text{-Alg}_\ell \to \mathscr{K}$ is fully faithful.*

Proof. This proof is just a reinterpretation of the proof of (vi) $\Rightarrow$ (i) in [18, Proposition 6.1]. If T_1 is a KZ 2-monad in the usual sense then for each T-algebra A we have an adjunction $a \dashv \eta_A$ with unit $\theta_a : 1 \Rightarrow \eta_A.a$ and counit an identity. It is not hard two show that the 2-cell (47) given by (48) satisfies the equations in Fig. 2.

$$\begin{tikzcd}
& \mathscr{K}(A,B) \arrow[dr, "\sigma_{A,B}"] & \\
\mathscr{K}(A,B) \arrow[ur, "1"] \arrow[dr, "1"'] \arrow[rr, phantom, "\Downarrow"] & & \mathscr{K}(TA,B) \\
& \mathscr{K}(A,B) \arrow[ur, "\mathscr{K}(a,1)"'] &
\end{tikzcd} \tag{47}$$

$$\begin{tikzcd}
\mathscr{K}(A,B) \arrow[r, "\sigma_{A,B}"] & \mathscr{K}(TA,B) \arrow[rr, "1"] \arrow[dr, "\mathscr{K}(\eta_A,1)"'] & \arrow[d, phantom, "\Downarrow \mathscr{K}(\theta_a,1)"] & \mathscr{K}(TA,B) \\
& & \mathscr{K}(A,B) \arrow[ur, "\mathscr{K}(a,1)"'] &
\end{tikzcd} \tag{48}$$

We shall show that this 2-cell has the one-dimensional part of the universal property of the 2-cell that defines $T\text{-Alg}_\ell(A,B)$ (Fig. 1), and hence $U_{\ell,A,B}$ is an isomorphism. Given $p : L \to \mathscr{K}(A,B)$ and a 2-cell $\alpha : \sigma_{A,B}.p \Rightarrow \mathscr{K}(a,B).p$ we have

$$\begin{aligned}
\alpha &= \big(\mathscr{K}(\theta_a, B).\mathscr{K}(a, B).p\big)\alpha \\
&= \big(\mathscr{K}(a, B).\mathscr{K}(\eta_A, B).\alpha\big)\big(\mathscr{K}(\theta_a, B).\sigma_{A,B}.p\big) \\
&= \mathscr{K}(\theta_a, B).\sigma_{A,B}.p
\end{aligned}$$

where the first equality holds by one of the triangular equalities of the adjunction $(\theta_a, 1) : a \dashv \eta_A$, and the last because $\mathscr{K}(\eta_A, B).\alpha = 1$. So far we saw that α can be written as the composition of p with (47); it only rests to prove that p is the unique 1-cell with this property. This is easy because $\mathscr{K}(\eta_A).\mathscr{K}(\theta_a, B).\sigma_{A,B}$ is the identity 2-cell of the identity 1-cell of $\mathscr{K}(A,B)$. This finishes one half of the proof.

On the other direction, if U_ℓ is fully faithful as a $\mathscr{W}$-functor, its underlying 2-functor is too fully faithful, and this is the forgetful 2-functor $T_1\text{-Alg}_\ell \to \mathscr{K}_1$. This means that T_1 is KZ. □

Lemma 7.2. *Let $T : \mathscr{W} \to \mathscr{W}$ be a KZ $\mathscr{W}$-monad. Then the* 1*-cell*

$$\sigma_{X,B} : [X, B] \xrightarrow{\mathsf{T}} [TX, TB] \xrightarrow{[TX,b]} [TX, B]$$

is part of a coretract adjunction with right adjoint $[\eta_X, B] : [TX, B] \to [X, B]$. In particular, it is a pseudomorphism.

Proof. We have $[\eta_X, B].[TX, b].\mathsf{T} = [X, b].[\eta_X, TB].\mathsf{T} = [X, b].[X, \eta_X] = 1$ by 2-naturality of η, so indeed we can define the unit of our adjunction as the identity. Now define the counit as the following 2-cell

$$\begin{tikzcd}
& {[X,B]} \arrow[drr, bend left, "\mathsf{T}"] & & & \\
{[TX,B]} \arrow[ur, "{[\eta_X,1]}"] \arrow[r, "\mathsf{T}"] \arrow[rrr, bend right, "{[1,\eta_B]}"'] & {[T^2X,TB]} \arrow[rr, shift left, "{[T\eta_X,1]}"] \arrow[rr, shift right, "{[\eta_{TX},1]}"'] \arrow[rr, phantom, "\Downarrow"] & & {[TX,TB]} \arrow[r, "{[1,b]}"] & {[TX,B]}
\end{tikzcd} \tag{49}$$

where the unlabelled 2-cell is $[\delta_X, 1]$. Now we check the axioms of an adjunction. First, $[\eta_X, B].[TX, b].[\delta_X, TB].\mathsf{T} = [X, b].[\delta_X\eta_X, TB].\mathsf{T} = 1$ by (44). The other triangular identity of an adjunction follows from (44):

$$\begin{aligned}
[TX, b].[\delta_X, TB].\mathsf{T}.[TX, b].\mathsf{T} &= [\delta_X, B].\big[T^2X, b\big].\big[T^2X, Tb\big].\mathsf{T}.\mathsf{T} \\
&= [\delta_X, B].\big[T^2X, b\big].\big[T^2X, \mu_B\big].\mathsf{T}.\mathsf{T} \\
&= [\delta_X, B].\big[T^2X, b\big].[\mu_X, TB].\mathsf{T} \\
&= [\delta_X, B].[\mu_X, B].[TX, b].\mathsf{T} \\
&= 1. \qquad \square
\end{aligned}$$

Theorem 7.3. *Every KZ $\mathscr{W}$-monad $T : \mathscr{W} \to \mathscr{W}$ is pseudo-commutative. Moreover, the pseudo-commutativity is unique.*

Proof. We have to check the conditions in Theorem 5.6. By Lemma 7.2 σ lifts to a pseudonatural transformation $[-,-] \Rightarrow [T-,-] : \mathscr{W}_1^{\mathrm{op}} \times T_1\text{-Alg} \to T_1\text{-Alg}$. Moreover this lifting is unique because $U_1 : T_1\text{-Alg} \to \mathscr{W}_1$ is injective on 1-cells and locally fully faithful. Conditions 1 to 4 in Theorem 5.6 hold trivially, because U_1 is injective in 1-cells; in other words, these conditions hold if and only if they hold in $\mathscr{W}_1$. The uniqueness of the pseudo-commutativity is equivalent to the uniqueness of the pseudomorphism structure on each $\sigma_{X,B}$, which holds by the properties of U_1 already mentioned. $\square$

Corollary 7.4. *If $T : \mathscr{W} \to \mathscr{W}$ is a KZ $\mathscr{W}$-monad, then T-*Alg *has a canonical structure of a pseudo-closed $\mathscr{W}$-category. Moreover, if T has a rank, the induced pseudo-closed structure on the* 2*-category T_1-*Alg *has an associated monoidal structure with unit object FI and whose tensor product satisfies* (40).

Proof. It is a consequence of Theorem 7.3 together with Section 6. $\square$

Example 7.5. There are pseudo-commutative 2-monads which are *not* KZ. For example, the 2-monad T on **Cat** whose algebras are the symmetric strict monoidal categories. See [10] for a detailed description of the pseudo-commutativity for this 2-monad. One of the several possible ways of seeing that this T is not lax-idempotent is to show that there cannot be a 2-natural transformation $\delta_X : T\eta_X \Rightarrow \eta_{TX} : TX \to T^2X$.

We record the following easy consequence of Proposition 6.7 that will be re-interpreted in the next section as the familiar fact about colimits of functors of several variables.

Corollary 7.6. *In the case of KZ $\mathscr{W}$-monads there is no distinction between partial maps in each variable and multilinear maps.*

Remark 7.7. When $\mathscr{W}$ is locally a preorder any pseudo-commutativity is just a commutativity in the sense of [23,25,24], but the monad could still be KZ and not an idempotent monad. See Example 8.8.

The pseudo-commutativity of a KZ 2-monad can be explicitly computed in terms of its KZ structure, for example the modification $\delta : T.\eta \Rrightarrow \eta.T$.

Corollary 7.8. *The pseudo-commutativity of a KZ $\mathscr{W}$-monad $T : \mathscr{W} \to \mathscr{W}$ – respectively its inverse – can be written in the form depicted on the left-hand side – respectively on the right-hand side – of Fig.* 10.

Proof. First observe that the domain and codomain of the 2-cell on the left-hand side of Fig. 10 match the domain and codomain of (6). Checking this entails a very simple calculation that holds for any strong monad. The pseudo-commutativity of T in its form (6) corresponds to a pseudomorphism structure for $\sigma_{X,TY}$ as depicted in (17). This 2-cell is simply the mate (45) for $f = \sigma_{X,TY}$, $A = [X, TY]$, $B = [TX, TY]$. The unit of the adjunction between the T-algebra structure $[X, \mu_Y].\bar{t}_{X,TY} : T[X, TY] \to [X, TY]$ and $\eta_{[X,TY]}$ can be written in the following form:

$$\begin{tikzcd} T[X,TY] \arrow[r, bend left, "T\eta_{[X,TY]}"] \arrow[r, bend right, "\eta_{T[X,TY]}"'] \arrow[r, phantom, "\Downarrow\delta"] & T^2[X,TY] \arrow[r, "T\bar{t}_{X,TY}"] & T\big[X, T^2Y\big] \arrow[r, "T[1,\mu_Y]"] & T[X,TY]. \end{tikzcd} \tag{50}$$

The pseudomorphism structure of $\sigma_{X,TY}$ is thus the 2-cell obtained by post-composing (50) with the 1-cell

$$[TX, \mu_Y].\bar{t}_{TX,TY}.T\sigma_{X,TY} : T[X, TY] \to [TX, TY].$$

By Lemma 5.2, following the rule depicted in Fig. 8, there is a 2-cell $\bar{\gamma}_{X,Y}$ that corresponds to the pseudomorphism structure obtained above. Using that δ is a modification and that T is a strong monad one readily produces $\bar{\gamma}_{X,Y}$ as

$$\begin{tikzcd} T[X,Y] \arrow[r, bend left, "T\eta"] \arrow[r, bend right, "\eta T"'] \arrow[r, phantom, "\Downarrow\delta"] & T^2[X,Y] \arrow[r, "T\bar{t}_{X,Y}"] & T[X,TY] \arrow[r, "T\sigma_{X,TY}"] & T[TX,TY] \arrow[r, "{[1,\mu].\bar{t}}"] & {[TX,TY]} \end{tikzcd} \tag{51}$$

(where we suppressed some subscripts to save space). Finally, the pseudo-commutativity of T is the exponential transpose of (51), which can be checked to be the 2-cell on the left-hand side of Fig. 10.

Now, to compute the inverse of the pseudo-commutativity in terms of δ we must perform the same steps as above but starting with the inverse of the pseudomorphism structure of $\sigma_{X,TY}$. By the comments in the beginning of the section, this inverse is the oplax morphism structure given

$$
\begin{tikzcd}
TX \otimes TY \arrow[d, bend right=60, "\eta_{TX}\otimes 1"'] \arrow[d, bend left=60, "T\eta_X\otimes 1"] \arrow[d, phantom, "\overset{\Leftarrow}{\delta_X\otimes 1}"] \\
T^2X \otimes TY \arrow[d, "t'_{TX,TY}"] \\
T(TX \otimes TY) \arrow[d, "Tt_{TX,Y}"] \\
T^2(TX \otimes Y) \arrow[d, "T^2t'_{X,Y}"] \\
T^3(X \otimes Y) \arrow[d, "\mu_{T(X\otimes Y)}"] \\
T^2(X \otimes Y) \arrow[d, "\mu_{X\otimes Y}"] \\
T(X \otimes Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
TX \otimes TY \arrow[d, bend right=60, "1\otimes\eta_{TY}"'] \arrow[d, bend left=60, "1\otimes T\eta_Y"] \arrow[d, phantom, "\overset{\Leftarrow}{1\otimes\delta_Y}"] \\
TX \otimes T^2Y \arrow[d, "t_{TX,TY}"] \\
T(TX \otimes TY) \arrow[d, "Tt'_{X,TY}"] \\
T^2(X \otimes TY) \arrow[d, "T^2t_{X,Y}"] \\
T^3(X \otimes Y) \arrow[d, "\mu_{T(X\otimes Y)}"] \\
T^2(X \otimes Y) \arrow[d, "\mu_{X\otimes Y}"] \\
T(X \otimes Y)
\end{tikzcd}
$$

Fig. 10. Pseudo-commutativity and its inverse in terms of δ.

by the pasting (46) for $f = \sigma_{X,TY}$, $A = [X, TY]$, $B = [TX, TY]$. This oplax structure can be written in terms of δ because it is the pasting of three 2-cells that can be written in that form: the lax morphism structure of $\sigma^*_{X,TY} = [\eta_X, TY]$ is the identity (i.e., it is a strict morphism), the unit of the adjunction $\sigma_{X,TY} \dashv [\eta_X, TY]$ is an identity and the counit is the pasting (49). From here is not hard to obtain the 2-cell in the right-hand side of Fig. 10. $\square$

A priori it is not obvious that either of the 2-cells in Fig. 10 is invertible. We showed above that the invertibility follows from *doctrinal adjunction*.

Recall from [10] that a pseudo-commutativity γ is symmetric when

$$\gamma_{Y,X} . c_{TX,TY} = Tc_{X,Y} . \gamma_{X,Y}$$

where c denotes the symmetry of $\mathscr{W}$. Corollary 7.8 immediately yields:

Corollary 7.9. *The pseudo-commutativity of a KZ $\mathscr{W}$-monad is always symmetric.*

One can deduce from the symmetry condition that $(T_1\text{-Alg}, FI, \boxtimes, \ldots)$ is a *symmetric monoidal bicategory*, a fact that is mentioned – without proof – in [10]. An explicit definition of symmetric Gray monoids can be found in [5], and of the general symmetric monoidal bicategories in [34].

Remark 7.10. In [10] it was observed that when a pseudo-commutativity is symmetric some of the axioms in the definition of a pseudo-commutativity (Definition 4.2) are redundant. Indeed, it is enough to require only one out of the three strength axioms, one out of the two unit axioms and one out of the two multiplication axioms. Using this fact, it is not hard to show directly that the 2-cell on the left-hand side of Fig. 10 satisfies the pseudo-commutativity axioms. However, the author does not see a direct way of proving that this 2-cell is an isomorphism – which is the missing condition required of a pseudo-commutativity – other than mimicking the proof given here.

We end the section with an example of how to construct the pseudo-commutativity from the KZ structure.

Example 7.11. We shall work with the category **Ab** of abelian groups, but one can choose a slightly more general setup like commutative monoids if wished. The 2-monad D on **Ab**-**Cat** whose algebras are **Ab**-categories with chosen n-ary coproducts for each $n \in \mathbb{N}$ is a KZ **Ab**-**Cat**-monad, as any other colimit-completion 2-monad. An explicit description of D can be given as follows. If X is a category, DX has objects finite sequences of objects of X, which we shall write as $(x_1 + x_2 + \cdots + x_m)$; the empty sequence will be denoted by 0. An arrow from such a sequence to another $(y_1 + \cdots + y_n)$ is an $m \times n$ matrix $\bar{f} = (f_{i,j})$ with entries $f_{i,j} : x_i \to y_j$ an arrow in X. Composition in DX is given by matrix multiplication: given $\bar{f}$ as above and $\bar{g} : (y_1 + \cdots + y_n) \to (z_1 + \cdots + z_r)$, the composition $\bar{g}\bar{f}$ has components

$$(\bar{g}\bar{f})_{i,j} = \sum_k g_{j,k} f_{i,j} \in X(x_i, z_j).$$

Identity arrows are provided by identity matrices. The obvious definition of addition of matrices gives DX its structure of an **Ab**-category. Next we describe the monad structure. The unit is simply given by components $\eta_X : X \to DX$ that send and object x to the string of length one (x). The components of the multiplication $\mu_X : D^2X \to DX$ are **Ab**-functors given by removing parentheses:

$$\begin{aligned}&\big(\big(x_1^1 + \cdots + x_{n_1}^1\big) + \big(x_1^2 + \cdots + x_{n_1}^2\big) + \cdots \big(x_1^m + \cdots + x_{n_m}^m\big)\big)\\ &\quad\mapsto \big(x_1^1 + \cdots + x_{n_1}^1 + x_1^2 + \cdots + x_{n_1}^2 + \cdots x_1^m + \cdots + x_{n_m}^m\big).\end{aligned}$$

Now that we have an explicit description of D we may go on and find its KZ structure in the form of the modification $\delta : D.\eta \Rrightarrow \eta.D$. The component of **Ab**-natural transformation $\delta_X : D\eta_X \Rightarrow \eta_{DX} : DX \to D^2X$ corresponding to $\bar{x} = (x_1 + \cdots + x_n) \in DX$ is

$$(\delta_X)_{\bar{x}} : \big((x_1) + \cdots + (x_n)\big) \to \big((x_1 + \cdots + x_n)\big)$$

given by an $n \times 1$ matrix with entries arrows $M_{i,1} : (x_i) \to (x_1 + \cdots + x_n)$ in DX. So each $M_{i,1}$ is itself a $1 \times n$ matrix with entries $f_j^i : x_i \to x_j$, for $1 \leqslant j \leqslant n$. It is not hard to see that by setting $f_i^i = 1 : x_i \to x_i$ and equal to 0 otherwise, the resulting transformation δ_X satisfies the axioms (44) making it into a KZ structure for D.

Now we compute the pseudo-commutativity induced by δ as stipulated in Corollary 7.8, that is, we compute the 2-cell on the left-hand side of Fig. 10. We begin by observing that the domain of this 2-cell is the **Ab**-functor that sends an object $((x_1 + \cdots + x_m), (y_1 + \cdots + y_n)) \in DX \otimes DY$ to

$$\big((x_1, y_1) + (x_1, y_2) + \cdots + (x_1, y_n) + (x_2, y_1) + \cdots + (x_m, y_1) + \cdots + (x_m, y_n)\big). \tag{52}$$

Observe that the order of the objects is given by a lexicographic ordering. The codomain of the pseudo-commutativity will be the functor that sends the same object to

$$\big((x_1, y_1) + (x_2, y_1) + \cdots + (x_m, y_1) + (x_1, y_2) + \cdots + (x_1, y_n) + \cdots + (x_m, y_n)\big) \tag{53}$$

given by the other lexicographic ordering. The corresponding component of the pseudo-commutativity will be an arrow from (52) to (53) in $D(X \otimes Y)$, and turns out to be just the permutation matrix that relates one lexicographic order with the other.

8. Categories with finite colimits

We now turn to our main example of KZ 2-monads, and thus pseudo-commutative 2-monads; namely, monads on $\mathscr{V}$-**Cat** whose algebras are $\mathscr{V}$-categories with a given class of *chosen* colimits. These monads are enriched in $\mathscr{V}$-**Cat**, which is essential in order to endow the $\mathscr{V}$-categories of pseudomorphisms with an algebra structure, as shown in the previous sections. This is just the familiar fact that given $\mathscr{V}$-categories A, B admitting colimits of a certain class Φ, Φ-cocontinuous $\mathscr{V}$-functors $A \to B$ form not only an ordinary category but a Φ-cocomplete $\mathscr{V}$-category Φ-**Cocts**$[A, B]$; this $\mathscr{V}$-category inherits a choice of colimits when B is equipped with such a choice. This family of monads expands the examples of pseudo-commutative 2-monads provided in [10]. When the class of colimits in question is a class of finite colimits, the corresponding monad is finitary and thus we can construct an associated tensor product.

Let Φ be a small class of colimits, by which we understand a small class of weights $\phi : D \to \mathscr{V}$. Recall from [15, Section 5.5] that the free completion of a (small) $\mathscr{V}$-category A under Φ-colimits, denoted by ΦA, can be obtained as the closure under Φ-colimits of the representables in $[A^{\mathrm{op}}, \mathscr{V}]$. The Yoneda embedding $\mathsf{y}_A : A \to \Phi A$ induces equivalences of $\mathscr{V}$-categories Φ-**Cocts**$[\Phi A, B] \simeq [A, B]$ for all Φ-cocomplete $\mathscr{V}$-category B, with pseudoinverse given by left Kan extension along y_A. Here Φ-**Cocts**$[C, D]$ denotes the $\mathscr{V}$-category of Φ-cocontinuous $\mathscr{V}$-functors $C \to D$; these are the enriched homs of a $\mathscr{V}$-**Cat**-category Φ-**Cocts** with objects the Φ-cocomplete small $\mathscr{V}$-categories.

Let us denote by Φ-**Colim** be the 2-category of $\mathscr{V}$-categories with chosen Φ-colimits, $\mathscr{V}$-functors strictly preserving these and $\mathscr{V}$-natural transformations. The hom $\mathscr{V}$-category Φ-**Colim**(A, B) is the full sub-$\mathscr{V}$-category of $[A, B]$ determined by the $\mathscr{V}$-functors that strictly preserve Φ-colimits. There is an obvious forgetful 2-functor $U_s : \Phi$-**Colim** $\to \mathscr{V}$-**Cat**. The main result of [19] is the monadicity of U_{s} (as a 2-functor) in the strong sense that there is an adjunction $F_{\mathrm{s}} \dashv U_{\mathrm{s}}$ and the canonical comparison 2-functor Φ-**Colim** $\to T_\Phi$-$\mathrm{Alg}_{\mathrm{s}}$ is an isomorphism, where $T_\Phi = U_{\mathrm{s}} F_{\mathrm{s}}$. If $\eta : 1 \Rightarrow T_\Phi$ is the unit of the monad, there is an equivalence of $\mathscr{V}$-categories making the following diagram commutative.

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "\mathsf{y}_A"'] & T_\Phi A \arrow[d, "\simeq"] \\ & \Phi A \end{tikzcd}$$

Corollary 8.1. *The* 2*-monad* T_Φ *on* $\mathscr{V}$-**Cat** *whose algebras are* $\mathscr{V}$*-categories with chosen* Φ*-colimits is pseudo-commutative. Therefore, the* 2*-category* T_Φ-Alg *is pseudo-commutative.*

Proof. Theorem 6.3 of [19] asserts that the 2-monad T_Φ is a KZ 2-monad. The result follows from Theorem 7.3. For the last part apply Corollary 7.4. □

Still following [19], the canonical "inclusion" 2-functor from Φ-**Colim** to the 2-category Φ-**Cocts** of Φ-cocomplete $\mathscr{V}$-categories and Φ-cocontinuous $\mathscr{V}$-functors (this 2-functor is not injective on objects) can be factored as

$$\Phi\text{-}\mathbf{Colim} \to \Phi\text{-}\mathbf{Cocts}_c \to \Phi\text{-}\mathbf{Cocts}$$

where the first 2-functor is bijective on objects and the second is fully faithful. In other words, the 2-category in the middle has objects $\mathscr{V}$-categories with chosen Φ-colimits and 1-cells Φ-cocontinuous $\mathscr{V}$-functors. [19, Theorem 6.2] shows that there is a canonical isomorphism $T_\Phi\text{-Alg} \cong \Phi\text{-}\mathbf{Cocts}_c$ that commutes with the corresponding forgetful 2-functors into $\mathscr{V}$-**Cat**.

Although the following results hold for any class of finite colimits, for simplicity we restrict ourselves to the class of all finite colimits Fin. For this we assume that $\mathscr{V}$ is locally finitely presentable as a monoidal category, so the theory developed in [16] applies.

Lemma 8.2. *The* 2*-monad* R *on* $\mathscr{V}$-**Cat** *whose algebras are* $\mathscr{V}$*-categories with chosen finite colimits is finitary. Equivalently, the forgetful* 2*-functor* $U_s : \mathsf{Fin}\text{-}\mathbf{Colim} \to \mathscr{V}\text{-}\mathbf{Cat}$ *is finitary.*

For a proof of this lemma see Section A.3. From the results and remarks of Section 6.4 we deduce:

Corollary 8.3. R_1-Alg*, and hence* Fin-$\mathbf{Cocts}_c$*, are monoidal* 2*-categories, with the monoidal structure induced by the canonical pseudo-closed structure. Moreover, the biadjunction* $F \dashv_b U : T_1\text{-Alg} \to \mathscr{V}\text{-}\mathbf{Cat}$ *is monoidal.*

The tensor product $\boxtimes$ in R-Alg satisfies (40), which could be rewritten as

$$\mathrm{Rex}[A \boxtimes B, C] \simeq \mathrm{Rex}\big[A, \mathrm{Rex}[B, C]\big].$$

This universal property can be expressed in terms of the monoidal constraint (42) $\chi_{A,B} : A \otimes B \to A \boxtimes B$ that classifies multilinear maps. By Corollary 7.6 multilinear maps are just partial maps in each variable, that in the present case means simply $\mathscr{V}$-functors that are right exact in each variable. The one-dimensional part of the universal property of the tensor product asserts that every functor $A \otimes B \to C$ that is right exact in each variable factors as $\chi_{A,B}$ followed by a unique up to isomorphism right exact functor $A \boxtimes B \to C$.

$$\begin{tikzcd} A \otimes B \arrow[r, "\chi_{A,B}"] \arrow[dr] \arrow[r, phantom, shift right=4, "\cong"] & A \boxtimes B \arrow[d, dotted] \\ & C \end{tikzcd}$$

The fact that multilinear maps $f : A \otimes B \to C$ are simply the partial maps, i.e., the right exact $\mathscr{V}$-functors in each variable, can be rephrased as the familiar fact about commutation of colimits: if $\phi : D_\phi^{\mathrm{op}} \to \mathscr{V}$, $\psi : D_\psi^{\mathrm{op}} \to \mathscr{V}$ are finite weights and $g : D_\phi^{\mathrm{op}} \to A$, $h : D_\psi^{\mathrm{op}} \to B$ two $\mathscr{V}$-functors, then the following two isomorphisms are equal:

$$\begin{aligned}\operatorname{colim}\big(\psi, \operatorname{colim}\big(\phi, f(g \otimes h)\big)\big) &\xrightarrow{\cong} \operatorname{colim}\big(\psi, f\big(\operatorname{colim}(\phi, g) \otimes h\big)\big)\\ &\xrightarrow{\cong} f\big(\operatorname{colim}(\phi, g) \otimes \operatorname{colim}(\psi, h)\big),\end{aligned}$$

$$\begin{aligned}\operatorname{colim}\big(\psi, \operatorname{colim}\big(\phi, f(g \otimes h)\big)\big) &\xrightarrow{\cong} \operatorname{colim}\big(\phi, \operatorname{colim}\big(\psi, f(g \otimes h)\big)\big)\\ &\xrightarrow{\cong} \operatorname{colim}\big(\phi, f\big(g \otimes \operatorname{colim}(\psi, h)\big)\big)\\ &\xrightarrow{\cong} f\big(\operatorname{colim}(\phi, g) \otimes \operatorname{colim}(\psi, h)\big).\end{aligned} \tag{54}$$

The first isomorphism in the composite (54) is induced by the pseudo-commutativity of R.

Remark 8.4. The fact that the free R-algebra functor $F : \mathscr{V}\text{-}\mathbf{Cat} \to R\text{-Alg}$ is strong monoidal implies, as explained in the paragraph after Theorem 6.10, that $R(X) \boxtimes R(Y)$ is canonically equivalent to $R(X \otimes Y)$.

Example 8.5. Let k be a commutative ring and A a k-algebra and denote by $\Sigma\mathsf{A}$ the corresponding k-Mod-category with one object. Then $R(\Sigma\mathsf{A})$ is equivalent to $\mathsf{A}\text{-Mod}_f$, the category of finitely presented A-modules. Remark 8.4 above can be reinterpreted as the equivalence

$$\mathsf{A}\text{-Mod}_f \boxtimes \mathsf{B}\text{-Mod}_f \simeq \mathsf{A} \otimes \mathsf{B}\text{-Mod}_f.$$

This can be of course shown directly, and in fact is one of the most basic observations about $\boxtimes$ or any tensor product that might play its role, including Deligne's tensor product [6]. The universal functor

$$\mathsf{A}\text{-Mod}_f \otimes \mathsf{B}\text{-Mod}_f \to \mathsf{A} \otimes \mathsf{B}\text{-Mod}_f$$

is given by $\otimes_k$, the tensor product over k.

Remark 8.6. The last section of Kelly's book [15,17] uses the extensive machinery developed therein to describe tensor products of Φ-cocomplete enriched categories. If A, B are two such categories, $A \boxtimes B$ is (equivalent to) the closure under Φ-colimits of the representables in $\Phi\text{-}\mathbf{Cts}[A^{\mathrm{op}}, B^{\mathrm{op}}; \mathscr{V}]$, the $\mathscr{V}$-category of $\mathscr{V}$-functors $A^{\mathrm{op}} \otimes B^{\mathrm{op}} \to \mathscr{V}$ that are Φ-continuous in each variable. In particular, when $\Phi = \mathsf{Fin}$ the class of finite colimits, we have that $A \boxtimes B$ is equivalent to the closure under finite colimits of the representables in $\mathrm{Lex}[A^{\mathrm{op}}, B^{\mathrm{op}}; \mathscr{V}]$, and $A \otimes B \to A \boxtimes B$ is *dense* in the sense of [15,17, Chapter 5]. (Note that the inclusion functor from $\mathrm{Lex}[A^{\mathrm{op}}, B^{\mathrm{op}}; \mathscr{V}]$ to $[A^{\mathrm{op}} \otimes B^{\mathrm{op}}, \mathscr{V}]$ does *not* preserve colimits.)

Remark 8.7. Let A, B be two $\mathscr{V}$-categories with chosen finite colimits and $\chi_{A,B} : A \otimes B \to A \boxtimes B$ the corresponding universal multilinear $\mathscr{V}$-functor. Remark 8.6 above implies

1. $\chi_{A,B}$ is fully faithful.

2. $\chi_{A,B}$ is a dense $\mathscr{V}$-functor and each object of $A \boxtimes B$ is a finite colimit of a $\mathscr{V}$-functor $\chi_{A,B}.F$, for some $\mathscr{V}$-functor F into $A \otimes B$ (however this colimit is not necessarily $\chi_{A,B}$-absolute, i.e., preserved by all the representables $(A \boxtimes B)(\chi_{A,B}(a,b), -)$, so it is not necessarily part of a density presentation of $\chi_{A,B}$).

Example 8.8. Consider the example of $\mathscr{V} = \mathbb{2}$, the category with two objects $\top$ and $\bot$ and one non-identity arrow $\bot \to \top$. The name of the objects is chosen to make the Cartesian product behave like the meet $x \wedge y$, the coproduct as the meet $x \vee y$, and the internal hom $x \Rightarrow y$ as the implication of classical logic. It is well known that $\mathbb{2}$-**Cat** is isomorphic to the 2-category **POrd** of partially ordered sets: the partially ordered set corresponding to a $\mathbb{2}$-category A is $\operatorname{ob} A$ with ordering $a \leqslant b$ if and only if $A(a,b) = \top$. In fact, this partially ordered set is just the underlying category of the $\mathbb{2}$-enriched category A, and because $\top$ is a strong generator in $\mathbb{2}$ (this is $\mathbb{2}(\top, -) : \mathbb{2} \to \mathbf{Set}$ is conservative), there is no difference between $\mathbb{2}$-enriched conical colimits and ordinary conical colimits. Hence, A has finite coproducts when it has finite joins; coequalizers are trivial in the case of partially ordered sets. Tensor products of the form $\bot * a$ always give an initial object and the other case $\top * a \cong a$ is always trivial. From these observations we deduce that an A finitely cocomplete when it has finite joins and an initial object (a bottom element).

Given a $\mathbb{2}$-category A, $[A^{\mathrm{op}}, \mathbb{2}]$ can be identified with the partially ordered set of order-ideals of A (subsets I of A such that if $a \in I$ then any $b \leqslant a$ is also in I). The representable presheaf $A(-,a)$ is identified with the order-ideal $\downarrow(a) = \{b \in A\colon\ b \leqslant a\}$. The free completion of A under finite colimits can be identified with the partially ordered set of order-ideals of A of the form $\downarrow(a_1) \cup \cdots \cup \downarrow(a_n)$, for some finite subset $\{a_1, \ldots, a_n\}$ of A. The $\mathbb{2}$-enriched monad R on $\mathbb{2}$-**Cat** whose algebras are partially ordered sets with chosen finite joins (including bottom object) is the monad corresponding to completion under a class of finite colimits, and hence it is KZ and pseudo-commutative. Since $\mathbb{2}$ is a partially ordered set, as observed in Remark 7.7, R is in fact *commutative*. However, R is not idempotent.

Remark 8.9. We have decided to consider finite colimits to present the theory above, but we could have chosen finite limits instead and obtained the same results. There is a pseudo-commutative 2-monad L on $\mathscr{V}$-**Cat** whose algebras are $\mathscr{V}$-categories with chosen limits, and thus the induced internal homs and tensor product. The neutral object of this tensor product is (equivalent to) $\mathscr{V}_f^{\mathrm{op}}$, the opposite of the category of finitely presented objects of $\mathscr{V}$. The 2-monad L is related to R by $LX \cong (R(X^{\mathrm{op}}))^{\mathrm{op}}$, and is not KZ or lax-idempotent but the dual notion of *oplax-idempotent*.

Appendix A

A.1. Flexible replacement

In this section we provide the – for the most part routine – $\mathscr{W}$-enriched versions of some of results concerning the left adjoint of the inclusion $J : T\text{-}\mathrm{Alg}_{\mathrm{s}} \to T\text{-}\mathrm{Alg}$. The results that follow are largely simple modifications of ideas from [3,28].

Although the 2-categories of algebras and strict morphisms $T\text{-}\mathrm{Alg}_{\mathrm{s}}$ are of a theoretical importance, most of the examples of interesting 2-categories associated to a 2-monad appear in the form 2-categories of (sometimes pseudo or lax) algebras and lax or pseudomorphisms. For simplicity, and because it is the case relevant to this paper, we will only consider the 2-categories T-Alg of strict algebras and pseudomorphisms.

Thanks to the classical theory of monads and algebras, we have a great deal of control over the 2-categories $T\text{-Alg}_s$, so it is a good idea to try to transform this knowledge to the most interesting 2-categories $T\text{-Alg}$ via the inclusion 2-functor $J : T\text{-Alg}_s \to T\text{-Alg}$. This idea appears in [13] and was pushed on in [3], where conditions are given that guarantee the existence of a left adjoint to J, usually denoted by $(-)' : T\text{-Alg} \to T\text{-Alg}_s$. Later [28] gave necessary and sufficient conditions for the existence of this left adjoint, greatly clarifying the situation.

According to [28], given a 2-monad T on a 2-category $\mathscr{K}$, A' can be constructed as a codescent object of the (strict) codescent data in $T\text{-Alg}_s$

$$T^3A \underset{Ta}{\overset{\mu_{TA}}{\underset{T\mu_A}{\rightrightarrows}}\!\!\!\to} T^2A \underset{a}{\overset{\mu_A}{\underset{T\eta_A}{\leftarrow}}\!\!\!\rightrightarrows} TA. \tag{55}$$

So J has a left adjoint whenever $T\text{-Alg}_s$ admits codescent objects, which are a special class of colimits, and in particular when $\mathscr{K}$ is complete and cocomplete and T has a rank. A *strict coherence data* in a 2-category $\mathscr{K}$ is a diagram

$$X_3 \underset{r}{\overset{p}{\underset{q}{\rightrightarrows}}\!\!\!\to} X_2 \underset{c}{\overset{d}{\underset{e}{\leftarrow}}\!\!\!\rightrightarrows} X_1$$

satisfying equations: $d.e = 1_{X_1} = c.e$, $d.p = d.q$, $c.r = c.q$, $c.p = d.r$. There is a (locally discrete) 2-category $\mathscr{C}$ such that a 2-functor $\mathscr{C} \to \mathscr{K}$ is exactly a strict coherence data in $\mathscr{K}$. A *codescent object* of a coherence data is a certain colimit of the diagram above, weighted by a certain weight $\chi : \mathscr{C}^{\mathrm{op}} \to \mathbf{Cat}$; details can be found in [28].

Observe that the diagrams (55) define a 2-functor $\mathrm{Coh} : T\text{-Alg}_s \to [\mathscr{C}, T\text{-Alg}_s]$, whose composition with the colimit 2-functor $\mathrm{colim}(\chi, -) : [\mathscr{C}, T\text{-Alg}_s] \to T\text{-Alg}_s$ is the flexible replacement comonad $(-)'J$. The 2-category $T\text{-Alg}_s$ has and the 2-functor Coh preserves those ϕ-weighted colimits that are preserved by T, while the $\mathrm{colim}(\chi, -)$ is obviously cocontinuous; therefore, $(-)'J$ preserves ϕ-colimits if T does so.

We are not only interested in the 2-categorical case above but also the $\mathscr{W}$-enriched case. The underlying 2-category 2-functor $(-)_1 : \mathscr{W}\text{-}\mathbf{Cat} \to 2\text{-}\mathbf{Cat}$ has a left adjoint that we will denote by $(-)_{\mathscr{W}}$. A 2-categorical weight $\beta : \mathscr{Q}^{\mathrm{op}} \to \mathbf{Cat}$ has an associated $\mathscr{W}$-enriched weight $\hat{\beta} : \mathscr{Q}_{\mathscr{W}}^{\mathrm{op}} \to \mathscr{W}$ that is the $\mathscr{W}$-functor corresponding to the 2-functor

$$\mathscr{Q}^{\mathrm{op}} \xrightarrow{\beta} \mathbf{Cat} \xrightarrow{-*I} \mathscr{W}_1.$$

It is not hard to show that $\hat{\beta}$-colimits in a $\mathscr{W}$-category are *a fortiori* β-colimits in the corresponding underlying 2-category. Moreover, if $H : \mathscr{Q} \to \mathscr{K}_1$ is a 2-functor with associated $\mathscr{W}$-functor $\tilde{H} : \mathscr{Q}_{\mathscr{W}} \to \mathscr{K}$, $\mathrm{colim}(\beta, H)$ exists if and only if $\mathrm{colim}(\hat{\beta}, \tilde{H})$ exists. These observations are the ingredients we need to obtain the following result.

Lemma A.1. *Let T be a $\mathscr{W}$-enriched monad on $\mathscr{K}$ and suppose that the 2-monad T_1 admits flexible replacements (i.e. $T_1\text{-Alg}_s$ admits χ-colimits). Then:*

1. *The $\mathscr{W}$-enriched flexible replacement comonad $(-)'J$ can be obtained as the composite*

$$T\text{-Alg}_{\mathrm{s}} \to [\mathscr{C}_{\mathscr{W}}, T\text{-Alg}_{\mathrm{s}}] \xrightarrow{\mathrm{colim}(\hat{\chi},-)} T\text{-Alg}_{\mathrm{s}}$$

where $\hat{\chi} : \mathscr{C}_{\mathscr{W}}^{\mathrm{op}} \to \mathscr{W}$ *is the weight induced by* $\chi : \mathscr{C}^{\mathrm{op}} \to \mathbf{Cat}$.

2. *The comonad* $(-)'J$ *preserves any colimit that is preserved by the forgetful* $\mathscr{W}$*-functor* $U_{\mathrm{s}} : T\text{-Alg}_{\mathrm{s}} \to \mathscr{K}$.
3. *The inclusion* $\mathscr{W}$*-functor* $J : T\text{-Alg}_{\mathrm{s}} \to T\text{-Alg}$ *preserves any colimit that is preserved by* U_{s}.

The last part of the lemma follows from part 2 because the $\mathscr{W}$-category T-Alg constructed in Section 3.2 is isomorphic to the Kleisli $\mathscr{W}$-category of the comonad $(-)'J$, with universal $\mathscr{W}$-functor $J : T\text{-Alg}_{\mathrm{s}} \to T\text{-Alg}$.

We use repeatedly throughout the paper the fact that J preserves cotensor products.

Lemma A.2. *If* T *is a* $\mathscr{W}$*-monad on a cotensored* $\mathscr{W}$*-category* $\mathscr{K}$*, then* T-Alg *has and* $J : T\text{-Alg}_{\mathrm{s}} \to T\text{-Alg}$ *preserves cotensor products with objects of* $\mathscr{W}$.

Proof. It is standard that the $\mathscr{W}$-functor $U_{\mathrm{s}} : T\text{-Alg}_{\mathrm{s}} \to \mathscr{W}$ creates cotensor products. So the cotensor product of a T-algebra A with an object X of $\mathscr{W}$ is the object $\{X, A\}$ with algebra structure $\{X, a\}.\bar{t}_{X,A} : T\{X, A\} \to \{X, TA\} \to \{X, A\}$. The $\mathscr{W}$-natural isomorphism $T\text{-Alg}_{\mathrm{s}}(A, \{X, B\}) \cong [X, T\text{-Alg}_{\mathrm{s}}(A, B)]$ is induced by a projection $p_B : X \to T\text{-Alg}_{\mathrm{s}}(\{X, B\}, B)$ (that corresponds under the isomorphisms above to the identity of $\{X, B\}$).

We will show that the 1-cell

$$X \xrightarrow{p_B} T\text{-Alg}_{\mathrm{s}}\big(\{X, B\}, B\big) \xrightarrow{J} T\text{-Alg}\big(\{X, B\}, B\big) \tag{56}$$

induces isomorphisms $T\text{-Alg}(A, \{X, B\}) \cong [X, T\text{-Alg}(A, B)]$. The existence of these isomorphisms follows easily from the definition of $T\text{-Alg}(A, B)$ as a limit in $\mathscr{W}_1$ (Section 3.2) and the fact that $\{X, -\}$ preserves limits and Remark 3.1. It is not hard to see that these isomorphisms are $\mathscr{W}$-natural in A. As such, by Yoneda, they are induced by an arrow $q_B : X \to T\text{-Alg}(\{X, B\}, B)$. To finish the proof we must show that q_B is the arrow (56). The square (57) commutes by definition of J (which is just a comparison 1-cell resulting from the universal property of the objects of pseudomorphisms) and the fact that $[X, -]$ preserves limits and Remark 3.1. Considering the case $A = \{X, B\}$, the arrow q_B is the result of applying the right vertical arrow of (57) to the identity $\mathrm{id} : I \to T\text{-Alg}(\{X, B\}, \{X, B\})$. Because identities are strict morphisms of algebras, id factors through J, yielding $Jp_B = q_B$.

$$\begin{tikzcd} T\text{-Alg}_{\mathrm{s}}(A, \{X, B\}) \arrow[r, "J"] \arrow[d, "\cong"'] & T\text{-Alg}(A, \{X, B\}) \arrow[d, "\cong"'] \\ {[X, T\text{-Alg}_{\mathrm{s}}(A, B)]} \arrow[r, "{[X,J]}"] & {[X, T\text{-Alg}(A, B)]} \end{tikzcd} \qquad \square \tag{57}$$

Corollary A.3. *If* J *has a left adjoint then there are canonical retract equivalences in* $\mathscr{W}_1$

$$T\text{-Alg}(FX, B) \simeq \mathscr{K}(X, UB). \tag{58}$$

Proof. Consider the following 1-cell in $\mathscr{W}_1$:

$$e_{X,B} : T\text{-Alg}(FY, B) \xrightarrow{U} \mathscr{K}(TA, UB) \xrightarrow{\mathscr{K}(\eta_Z, UB)} \mathscr{K}(Z, UB). \tag{59}$$

The functor $\mathscr{W}_1(I, e_{Z,B}) : T_1\text{-Alg}(F_1 Z, B) \to \mathscr{K}_1(Z, U_1 B)$ is a retract equivalence by Theorem 5.1 and Corollary 5.6 of [3]. Since U preserves cotensor products with objects of $\mathscr{W}$, we deduce that $\mathscr{W}_1(X, e_{Z,B})$ is an equivalence for all X in $\mathscr{W}$, as this functor is, up to composing with canonical isomorphisms, $\mathscr{W}_1(I, e_{Z,\{X,B\}})$. It follows that $e_{Z,B}$ is an equivalence in $\mathscr{W}_1$. To prove that it is a retract equivalence, it is enough to show that it has a right inverse (see [11, A.1.1.1]), which will be provided by the composite

$$\mathscr{K}(Z, UB) \xrightarrow{F_s} T\text{-Alg}_s(FZ, FUB) \xrightarrow{T\text{-Alg}_s(FZ,b)} T\text{-Alg}_s(FZ, B) \xrightarrow{J} T\text{-Alg}(FZ, B).$$

The fact that this is a right inverse is just a consequence of the adjunction $F_s \dashv U_s$. □

We finish the section with an enriched version of [3, Theorem 3.8]. An examination of the proof of the mentioned theorem reveals that it can be carried over the enriched context with virtually no modifications. The enrichment can be in any complete and cocomplete symmetric monoidal closed category $\mathscr{V}$, not necessarily a 2-category.

Lemma A.4. *Let T be a $\mathscr{V}$-enriched monad with a rank on a $\mathscr{V}$-category cocomplete $\mathscr{K}$. If $\mathscr{K}$ has cotensor products, then $T\text{-Alg}_s$ is cocomplete.*

A.2. A parametrised biadjunction

A consequence of the existence of flexible replacements, and one of the main results of [3] is its Theorem 5.1. This is exactly the result used to show the existence of a tensor product associated to a pseudo-closed structure on T-Alg, i.e. the existence of pseudonatural equivalences $T\text{-Alg}(A \boxtimes B, C) \to T\text{-Alg}(A, [\![B, C]\!])$. However, to show that the restriction of the tensor product to strict morphisms is (isomorphic to) a 2-functor (Corollary 6.13) we need to show the 2-naturality of this transformation with respect to strict morphisms in the variable B. The way to obtain this is via a parametrised version of Theorem 5.1 in [3].

Recall that given 2-categories $\mathscr{P}, \mathscr{L}, \mathscr{M}$ and 2-functors $H : \mathscr{P} \times \mathscr{L} \to \mathscr{M}$ and $G : \mathscr{P}^{\mathrm{op}} \times \mathscr{M} \to \mathscr{L}$, a *left parametrised adjunction* is a 2-natural isomorphism

$$\pi_{*,?,-} : \mathscr{M}\big(H(*, ?), -\big) \cong \mathscr{L}\big(?, G(*, -)\big). \tag{60}$$

For each object P of $\mathscr{P}$ we obtain an adjunction

$$\pi_{P,?,-} : \mathscr{M}\big(H(P, ?), -\big) \cong \mathscr{L}\big(?, G(P, -)\big).$$

We shall work under the blanket assumptions of [3]: T is a 2-monad with a rank on a complete and cocomplete 2-category $\mathscr{K}$.

Theorem A.5. *Let $G : \mathscr{P}^{\mathrm{op}} \times T\text{-Alg} \to \mathscr{L}$ be a 2-functor and assume that the composite $G(\mathscr{P}^{\mathrm{op}} \times J) : \mathscr{P}^{\mathrm{op}} \times T\text{-Alg}_s \to \mathscr{L}$ has a left parametrised left adjoint $H : \mathscr{P} \times \mathscr{L} \to T\text{-Alg}_s$, with unit $s_P : 1 \Rightarrow G(P, H(P, -))$. Then the 2-natural transformation*

$$T\text{-Alg}\big(JH(*, ?), -\big) \xrightarrow{G(*,-)} \mathscr{L}\big(G\big(*, JH(*, ?)\big), G(*, -)\big) \xrightarrow{\mathscr{L}(s_*,1)} \mathscr{L}\big(?, G(*, -)\big) \tag{61}$$

is a retract equivalence.

A.3. *The monad for finite colimits is finitary*

In this section we prove that the 2-monad on $\mathscr{V}$-**Cat** whose algebras are $\mathscr{V}$-categories with chosen finite colimits is finitary. The observation that the category of (small) categories with certain chosen (co)limits is monadic over **Cat** is attributed in [1] to C. Lair [30]. The fact that the monad for certain finite (co)limits is finitary can be considered to be present in [4], modulo the subtleties mentioned in [1]. As the case of enriched categories seems to be missing from the literature we feel necessary to provide a complete proof that the 2-monad constructed in [19] associated to a class of finite colimits is finitary. In addition to the usual hypotheses on $\mathscr{V}$, in order to have a good theory of finite enriched limits and colimits one has to require $\mathscr{V}$ to be locally finitely presentable as a monoidal category [16].

A.3.1. *Colimits in* $\mathscr{V}$-**Gph**

Denote by $\mathscr{V}$-**Gph** the category of $\mathscr{V}$-graphs and $\mathscr{V}$-graphs morphisms. By a $\mathscr{V}$-graph G we mean a family of objects $\operatorname{ob} G$ and a family of objects in $\mathscr{V}$, $\{G(x,y)\}_{x,y\in \operatorname{ob} G}$. Colimits in $\mathscr{V}$-**Gph** have the following simple description. If $D: J \to \mathscr{V}\text{-}\mathbf{Gph}_0$ is a functor with J small, write $G_j = D(j)$. Define $\operatorname{ob} G$ as $\operatorname{colim}_j \operatorname{ob} G_j$, with universal cocone $q_j : \operatorname{ob} G_j \to \operatorname{ob} G$. Define $G(x,y)$ as the colimit in $\mathscr{V}$ of the functor $G: J \to \mathscr{V}$ defined on objects by sending $j \in J$ to $\sum_{q_j(u)=x,\, q_j(v)=y} G_j(u,v)$ and on arrows in the obvious way. We obtain morphisms of $\mathscr{V}$-graphs $q_j : G_j \to G$ forming a colimiting cocone. Details, along with a more conceptual description using the bicategory $\mathscr{V}$-**Mat** of $\mathscr{V}$-matrices, can be found in [20].

A.3.2. *Filtered colimits in* $\mathscr{V}$-**Cat**

To describe filtered colimits of $\mathscr{V}$-categories will be enough to describe filtered colimits of the corresponding underlying $\mathscr{V}$-graphs, as the forgetful functor $\mathscr{V}\text{-}\mathbf{Cat}_0 \to \mathscr{V}\text{-}\mathbf{Gph}$ is finitarily monadic [20]. Let $D: J \to \mathscr{V}\text{-}\mathbf{Cat}_0$ be an ordinary functor with J filtered. We shall also denote by D the functor $J \to \mathscr{V}$-**Gph** resulting from composing with the forgetful functor. To abbreviate, we denote $D(j)$ by C_j. We know that D has a colimit since the 2-category $\mathscr{V}\text{-}\mathbf{Cat}_0$ is cocomplete; that is, there exists a $\mathscr{V}$-category C and a natural transformation $q_j : C_j \to C$ inducing an isomorphism $\mathscr{V}\text{-}\mathbf{Cat}_0(C,B) \cong \lim_j \mathscr{V}\text{-}\mathbf{Cat}_0(C_j,B)$ natural in B.

As J is filtered, the $\mathscr{V}$-enriched homs $C(x,y)$ have a simpler description than in the general case. Pick $j \in J$ and define a functor

$$H^j : (j \downarrow J) \to \left[C_j^{\mathrm{op}} \otimes C_j, \mathscr{V}\right]_0 \tag{62}$$

by $H^j(\alpha : j \to k) = C_k((D\alpha)-, (D\alpha)?)$; H^j is defined on an arrow $\gamma : (\alpha : j \to k) \to (\beta : j \to \ell)$ by the effect on enriched homs of the $\mathscr{V}$-functor $D\gamma : C_k \to C_\ell$.

Lemma A.6.

$$\operatorname{colim} H^j \cong C(q_j-, q_j?) : C_j^{\mathrm{op}} \otimes C_j \to \mathscr{V}.$$

Proof. The category $(j \downarrow J)$ is filtered because J is so, and the projection functor $P : (j \downarrow J) \to J$ is final. Since the forgetful $\mathscr{V}$-**Cat** $\to$ $\mathscr{V}$-**Gph** is finitary, as previously mentioned $C(x,y)$ is the colimit of a functor $G_{x,y} : J \to \mathscr{V}_0$

$$G_{x,y}(k) = \sum_{q_k(u)=x,\, q_k(v)=y} C_k(u,v) \tag{63}$$

and using the fact that P is final, $C(x,y) \cong \operatorname{colim} G_{x,y}P$. Now we show that

$$\bigl(\operatorname{colim} H^j\bigr)(x,y) \cong C\bigl(q_j(x), q_j(y)\bigr) \tag{64}$$

for all $x, y \in C_j$ by exhibiting a bijection between cocones $\rho : H^j(-)(x,y) \Rightarrow z$ and cocones $\tau : G_{q_j(x),q_j(y)}P \Rightarrow z$. To give τ is to give for each object $\alpha : j \to k$ in $(j \downarrow J)$ and $u, v \in C_k$ such that $q_k(u) = q_j(x)$, $q_k(v) = q_j(y)$, arrows in $\mathscr{V}_0$

$$\tau_\alpha^{u,v} : C_k(u,v) \to z. \tag{65}$$

The naturality of τ with respect to α means that for any $\beta : k \to \ell$ in J we have

$$\begin{tikzcd} C_k(u,v) \arrow[rr, "D\beta"] \arrow[dr, "\tau_\alpha^{u,v}"'] & & C_\ell((D\beta)u,(D\beta)v) \arrow[dl, "\tau_{\beta\alpha}^{(D\beta)u,(D\beta)v}"] \\ & z & \end{tikzcd} \tag{66}$$

On the other hand to give ρ is equivalent to giving for each $\alpha : j \to k$ in $(j \downarrow J)$ an arrow

$$\rho_\alpha : C_k\bigl((D\alpha)x, (D\alpha)y\bigr) \to z \tag{67}$$

satisfying the following naturality condition for each arrow $\gamma : (\alpha\colon\ j \to k) \to (\beta\colon\ j \to \ell)$ in J.

$$\begin{tikzcd} C_k((D\alpha)x,(D\alpha)y) \arrow[rr, "D\gamma"] \arrow[dr, "\rho_\alpha"'] & & C_\ell((D\beta)x,(D\beta)y) \arrow[dl, "\rho_\beta"] \\ & z & \end{tikzcd} \tag{68}$$

Given ρ define (65) in the following way. Choose an arrow $\beta : k \to k'$ in J such that $(D\beta)u = D(\beta\alpha)x$ and $(D\beta)v = D(\beta\alpha)y$ and set

$$\tau_\alpha^{u,v} : C_k(u,v) \xrightarrow{D\beta} C_{k'}\bigl((D\beta)u, (D\beta)v\bigr) = C_{k'}\bigl(D(\beta\alpha)x, D(\beta\alpha)y\bigr) \xrightarrow{\rho_{\beta\alpha}} z. \tag{69}$$

Using the fact that J is filtered and the naturality of ρ (made explicit in (68)) its routine to verify that (69) does not depend on the choice of $\beta : k \to k'$. Conversely, given τ we can define ρ_α (67) as $\tau_\alpha^{(D\alpha)x,(D\alpha)y} : C_k((D\alpha)x,(D\alpha)y) \to z$. The naturality condition (68) is immediately implied by the naturality of τ (66). The correspondence between τ and ρ just described is a bijection, yielding an isomorphism (64) that is induced by the cocone

$$H^j(\alpha)(x,y) = C_k\bigl((D\alpha)x, (D\alpha)y\bigr) \xrightarrow{q_k} C\bigl(q_k(D\alpha)x, q_k(D\alpha)y\bigr) = C(q_jx, q_jy). \tag{70}$$

Now we turn our attention to the matter of the $\mathscr{V}$-naturality (in x, y) of the isomorphisms (64). We shall show the $\mathscr{V}$-naturality on one of the two variables, namely the commutativity of the square

$$\begin{tikzcd}
(\operatorname{colim} H^j)(x,y)\otimes C_j(y,y') \arrow[r] \arrow[d] & C(q_jx,q_jy)\otimes C_j(y,y') \arrow[d] \\
(\operatorname{colim} H^j)(x,y') \arrow[r] & C(q_jx,q_jy')
\end{tikzcd} \tag{71}$$

with horizontal arrows induced by the isomorphism (64) and vertical arrows given by the respective $\mathscr{V}$-functor structures. The case of the other variable is completely analogous. The diagram (71) commutes if for each $\alpha : j \to k$ in J the following diagram commutes:

$$\begin{tikzcd}
C_k((D\alpha)x,(D\alpha)y)\otimes C_j(y,y') \arrow[r, "q_k\otimes 1"] \arrow[d, "1\otimes D\alpha"'] & C(q_jx,q_jy)\otimes C_j(y,y') \arrow[d, "1\otimes D\alpha"] \\
C_k((D\alpha)x,(D\alpha)y)\otimes C_k((D\alpha)y,(D\alpha)y') \arrow[r, "q_k\otimes 1"] \arrow[d, "\mathrm{comp}"'] & C(q_jx,q_jy)\otimes C_k((D\alpha)y,(D\alpha)y') \arrow[d, "\mathrm{comp}"] \\
C_k((D\alpha)x,(D\alpha)y') \arrow[r, "q_k"] & C(q_jx,q_jy')
\end{tikzcd}$$

which does because q_k is a $\mathscr{V}$-functor. This finishes the proof of the lemma. □

A.3.3. Filtered colimits of categories with chosen finite colimits

Theorem A.7. *For any class of finite weights Φ, the forgetful $\mathscr{V}$-**Cat**-functor $U_s : \Phi$-**Colim** $\to$ $\mathscr{V}$-**Cat** creates filtered colimits. Equivalently, the $\mathscr{V}$-**Cat**-monad T_Φ on $\mathscr{V}$-**Cat** whose algebras are $\mathscr{V}$-categories with chosen Φ-colimits is finitary.*

Proof. The theorem can be equivalently expressed as asserting that the ordinary functor U_s creates filtered colimits; for Φ-**Colim** has cotensor products and hence any ordinary conical colimit in it is automatically an enriched colimit (see [15, Section 3.8]).

We follow the notation employed in Lemma A.6: J will be a filtered category, $D : J \to \Phi\text{-}\mathbf{Colim}_0$ a functor whose composition with $(U_s)_0$ will be also denoted by D, $D(j)$ will be abbreviated by C_j and $\operatorname{colim} D \in \mathscr{V}\text{-}\mathbf{Cat}_0$ by C, with colimiting cocone $q_j : C_j \to C$.

First we must equip C with chosen Φ-colimits. Let $\phi : P^{\mathrm{op}} \to \mathscr{V}$ be a weight in Φ, and in particular a finite weight, and $G : P \to C$ a $\mathscr{V}$-functor. The $\mathscr{V}$-category P is finite, and then finitely presented in $\mathscr{V}\text{-}\mathbf{Cat}_0$, so G factors as $G_j : P \to C_j$ followed by $q_j : C_j \to C$, for some $j \in J$. Consider the chosen colimit $\operatorname{colim}(\phi, G_j)$ in C_j, with unit $\eta_j : \phi \Rightarrow C_j(G-, \operatorname{colim}(\phi, G_j))$. We shall show that $q_j(\operatorname{colim}(\phi, G_j))$ is a colimit of G weighted by ϕ, or in other words that there exists an isomorphism in $\mathscr{V}\text{-}\mathbf{Cat}_1(C, \mathscr{V})$

$$\big[P^{\mathrm{op}}, \mathscr{V}\big]\big(\phi-, C(G-, ?)\big) \cong C\big(q_j\big(\operatorname{colim}(\phi, G_j)\big), ?\big). \tag{72}$$

Since C is a colimit of the functor D into $\mathscr{V}$-$\mathbf{Cat}_0$ and $P:(j\downarrow J)\to J$ is final, C is a colimit of DP. This is also a 2-categorical colimit of the associated 2-functor into the 2-category $\mathscr{V}$-$\mathbf{Cat}_1$. It will be enough to exhibit $\mathscr{V}$-natural isomorphism between functors $C_k\to\mathscr{V}$

$$\big[P^{\mathrm{op}},\mathscr{V}\big]\big(\phi-,C(G-,q_k?)\big)\cong C\big(q_j\big(\mathrm{colim}(\phi,G_j),q_k?\big)\big) \tag{73}$$

for each $\alpha:j\to k$ in J, and natural with respect to arrows in $j\downarrow J$. We define (73) by the following string of isomorphisms:

$$\begin{aligned}
\big[P^{\mathrm{op}},\mathscr{V}\big]\big(\phi-,C(G-,q_k?)\big) &= \big[P^{\mathrm{op}},\mathscr{V}\big]\big(\phi-,C\big(q_k(D\alpha)G_j-,q_k?\big)\big)\\
&\cong \big[P^{\mathrm{op}},\mathscr{V}\big]\Big(\phi-,\operatorname*{colim}_{\beta:k\to\ell}C_\ell\big(D(\beta\alpha)G_j-,(D\beta)?\big)\Big) &(74)\\
&\cong \operatorname*{colim}_{\beta:k\to\ell}\big[P^{\mathrm{op}},\mathscr{V}\big]\big(\phi-,C_\ell\big(D(\beta\alpha)G_j-,(D\beta)?\big)\big) &(75)\\
&\cong \operatorname*{colim}_{\beta:k\to\ell}C_\ell\big(\mathrm{colim}\big(\phi,D(\beta\alpha)G_j\big),(D\beta)?\big) &(76)\\
&\cong \operatorname*{colim}_{\beta:k\to\ell}C_\ell\big(D(\beta\alpha)\big(\mathrm{colim}(\phi,G_j)\big),(D\beta)?\big) &(77)\\
&\cong C\big(q_\ell D(\beta\alpha)\big(\mathrm{colim}(\phi,G_j)\big),q_\ell(D\beta)?\big) &(78)\\
&= C\big(q_j\big(\mathrm{colim}(\phi,G_j)\big),q_k?\big). &(79)
\end{aligned}$$

We briefly explain each isomorphism: (74) is an application of Lemma A.6; ϕ is finitely presented in $[P^{\mathrm{op}},\mathscr{V}]$ because it is a finite weight (see [21, Section 3]), hence the isomorphism (75); (76) is just the definition of colimit and (77) is the isomorphism resulting from using the fact that $D(\beta\alpha)$ (strictly) preserves colimits; (78) is another application of Lemma A.6 and finally the equality (79) holds by naturality of the cocone q_k. This shows that $q_j(\mathrm{colim}(\phi,G_j))$ is a colimit of G weighted by ϕ. To find the unit $\eta:\phi\Rightarrow C(G-,q_j(\mathrm{colim}(\phi,G_j)))$ of this colimit it is enough to take the $\alpha=1:j\to j$ and from the identity morphism of $q_j(\mathrm{colim}(\phi,G_j))$ in (79) work our way up through the isomorphisms to obtain

$$\eta:\phi\xrightarrow{\eta_j}C_j\big(G_j-,\mathrm{colim}(\phi,G_j)\big)\xrightarrow{q_j}C\big(G-,q_j\big(\mathrm{colim}(\phi,G_j)\big)\big).$$

A standard argument using the fact that J is filtered proves that neither $q_j(\mathrm{colim}(\phi,G_j))$ nor η depend on the choice of j. So we can now stipulate this object with the named unit as the chosen colimit in C of G weighted by ϕ, and furthermore, these choices make the $\mathscr{V}$-functors $q_j:C_j\to C$ strictly preserve colimits.

The definition colimits in the previous paragraph makes $q_j:C_j\to C$ a colimiting cocone in Φ-$\mathbf{Colim}_0$. Indeed, given another cocone $t_j:C_j\to B$ the respective induced $\mathscr{V}$-functor $t:C\to B$ strictly preserves Φ-colimits. For, any such colimit in C is of the form $q_j(\mathrm{colim}(\phi,G_j))$ as above, and then

$$\begin{aligned}
t\big(\mathrm{colim}(\phi,G)\big)&=tq_j\big(\mathrm{colim}(\phi,G_j)\big)=t_j\big(\mathrm{colim}(\phi,G_j)\big)\\
&=\mathrm{colim}(\phi,t_jG_j)=\mathrm{colim}(\phi,tq_jG_j)=\mathrm{colim}(\phi,tG)
\end{aligned}$$

shows that t strictly preserves chosen colimits. □

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