FINITENESS CONDITIONS FOR SOLUBLE GROUPS

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1. Introduction

THE object of this paper is to make deductions connecting certain finiteness properties of a group, particularly *the maximal condition for normal subgroups*, with some of its algebraic properties, as expressed by identical relations in the sense of B. H. Neumann. Owing to our ignorance† about infinite simple groups, we confine attention mainly to *soluble* groups, that is, groups whose derived series terminates in the unit subgroup 1 after a finite number of steps.‡

Section 1 contains a statement and discussion of our main results, most of which are proved in section 2. Section 3 is devoted to the construction of examples of soluble groups with two generators which do not satisfy the maximal condition for normal subgroups; these illustrate some points referred to in section 1.3.

1.1. *Finiteness conditions.* It is convenient to use abbreviations for certain properties of groups. Those most often required are as follows:

(i) FG: the property of being finitely generated.

(ii) Max: the maximal condition for subgroups. (A group satisfies Max if and only if all its subgroups are FG.)

(iii) Max-n: the maximal condition for normal subgroups.

(iv) Max-Ω, where Ω is a set of distributive operators acting on a group: the maximal condition for Ω-subgroups. (A group G satisfies Max-n if and only if it satisfies Max-G, the elements of G acting as operators in the usual way, viz. to produce inner automorphisms.)

(v) Max-r: the maximal condition for right ideals in the group-ring. (A group G satisfies Max-r if and only if the group-ring $R = R(G)$ satisfies Max-R, the elements of R acting as multipliers on the right.)

† This ignorance has been considerably diminished in recent years. Finitely generated simple groups of infinite order were shown to exist by Graham Higman (**2**), and in a recent paper (**1**) Mrs. R. Camm has shown by construction that there are $2^{\aleph_0}$ distinct groups of this kind.

‡ This is the usually accepted definition, cf. Zassenhaus (**8**). Similarly, a *nilpotent* group is one whose lower central series terminates in 1 after a finite number of steps. Some Russian authors use these terms in a much wider sense.

If a_1, a_2,... are elements of the Ω-group G, we denote by

$$\mathrm{Gp}_{\Omega}\{a_1, a_2,...\}$$

the smallest Ω-subgroup of G containing a_1, a_2,...; and by

$$d_{\Omega}(G)$$

the smallest number of elements a_1, a_2,..., that can be chosen so that $\mathrm{Gp}_{\Omega}\{a_1, a_2,...\} = G$. By a well-known principle, G satisfies Max-Ω if and only if $d_{\Omega}(H)$ is finite for all Ω-subgroups H of G.

It is easy to deduce that *a soluble group cannot satisfy the maximal condition for normal subgroups unless it is finitely generated.* For if H is a normal Abelian subgroup of any group G, we have obviously

$$d(G) \leqslant d(G/H)+d_G(H).$$

If G satisfies Max-n, then $d_G(H)$ is finite. If G is soluble, we can choose for H the last term $\neq 1$ of the derived series of G. The result stated now follows by induction on the length of the derived series, since an Abelian group satisfying Max-n is finitely generated.

Clearly, a countable group which satisfies Max-n can only have a countable number of normal subgroups. But, as we shall see, there exist finitely generated soluble groups possessing an uncountable infinity of normal subgroups. So, for soluble groups, the condition Max-n is definitely stronger than FG. On the other hand, for any group, Max implies Max-n. But the group defined by two generators a and b and the single equation

$$a^{-1}ba = b^2$$

(we could take a and b to be the transformations $x \to 2x$ and $x \to x+1$ of the real axis) is metabelian and therefore soluble, but does not satisfy Max since the cyclic subgroups generated by

$$b,\ aba^{-1},\ a^2ba^{-2},...$$

form an infinite tower. We shall show, however, that finitely generated metabelian groups always satisfy Max-n. Thus, *for soluble groups, the condition* Max-n *comes in generality strictly between* Max *and* FG.

1.2. *Polycyclic groups.* Soluble groups which satisfy Max have been the subject of a number of papers, notably four by K. A. Hirsch (**3**), and may be regarded as fairly well known. Groups of this class we call *polycyclic* groups, for they are precisely those groups which can be obtained from the unit subgroup by a finite succession of cyclic extensions. Alternatively, they are those soluble groups G for which the successive factors

$$G/G',\ G'/G'',...$$

of the derived series are all finitely generated.

Some important classes of polycyclic groups are as follows: (i) all soluble subgroups of the unimodular groups, (ii) more generally, all soluble groups of matrices whose coefficients are integers of a given algebraic number field, and (iii) in a different direction of generality, all soluble subgroups of the group of automorphisms of a polycyclic group. That all these kinds of group are polycyclic is implicit in the paper (**6**) of Mal'cev. It may also be easily deduced from Zassenhaus (**9**), particularly Satz 6 of page 293.

Mal'cev proves also the remarkable result that *a soluble group which is not polycyclic must contain a non-finitely generated Abelian subgroup.* Thus if a soluble group does not satisfy Max, some of its Abelian subgroups do not even satisfy FG (and therefore not Max-n either).

On the other hand, as is well known, FG implies Max not only for Abelian groups but for all nilpotent groups. Thus *the finitely generated nilpotent groups form a special class of polycyclic groups*, and the distinctions necessary for general soluble groups do not arise. We observe in passing that the class of finitely generated nilpotent groups is the smallest class of groups which (i) includes all cyclic groups and (ii) includes the group $G = G_1 G_2$ whenever it includes both of the normal subgroups G_1 and G_2 of G. For this reason it might perhaps be appropriate to call groups of this important class *multicyclic.*

It is easy to see that, for any group G, Max-r implies Max. For if $G_1 < G_2 < \dots$ is a strictly infinite tower of subgroups of G and we define the right ideals R_i of the group-ring R of G by

$$R_i = \mathrm{Gp}_G\{\text{all } 1-x \text{ with } x \in G_i\},$$

we have a strictly infinite tower $R_1 < R_2 < \dots$ of right ideals in R. One of our main results (Theorem 1) states that, for soluble groups, Max implies Max-r. Hence *a soluble group satisfies* Max-r *if and only if it is polycyclic.*

It is at this point that another finiteness property of groups becomes relevant:

(vi) FR: the property of being *finitely related.* (A group G is finitely related if it can be defined by a finite number of generators subject to a finite number of relations. By the method of Tietze for passing from one system of generators and defining relations to another, we see that, whenever H is finitely generated and H/K finitely related, we necessarily have $d_H(K)$ finite as well.)

It is easy to show that, if A is a normal Abelian subgroup of the finitely generated group G and if G/A satisfies both FR and Max-r, then G satisfies Max-n. But polycyclic groups are always finitely related. Since, according to the result mentioned above, they also satisfy Max-r, we conclude that *every finitely generated extension of an Abelian group by a polycyclic group* satisfies Max-n.

1.3. *Identical relations and* Max-n. We now turn to the question: for what varieties of soluble groups do all finitely generated groups of the variety satisfy Max-n? By a *variety* of groups we here mean the class of all groups which satisfy a certain identical relation.

Let
$$G = \gamma_1(G) \geqslant \gamma_2(G) \geqslant \dots$$
be the *lower central series* of a group G. If ϕ and ψ are two functions defined for all groups G and whose values for the argument G are subgroups of G, we define $[\phi, \psi]$ to be the function whose value for G is the subgroup $[\phi(G), \psi(G)]$ generated by all commutators $[x,y] = x^{-1}y^{-1}xy$ for which $x \in \phi(G)$, $y \in \psi(G)$. We write $\phi = \psi$ and $\phi \leqslant \psi$ to mean that $\phi(G) = \psi(G)$ and $\phi(G) \leqslant \psi(G)$ for all G, respectively. With this convention, the functions γ_k are defined inductively by the rule
$$\gamma_{k+1} = [\gamma_k, \gamma_1]$$
and we have $[\gamma_m, \gamma_n] \leqslant \gamma_{m+n}$ for all m, $n = 1, 2, \dots$. The groups G for which $\gamma_{k+1}(G) = 1$ form a variety, the variety of nilpotent groups of class at most k; when finitely generated, they all satisfy Max.

The class $\mathfrak{C}$ of *commutator-subgroup functions* we define to be the smallest class of subgroup functions which (i) contains the identity function γ_1 and (ii) with ϕ and ψ also contains $[\phi, \psi] = [\psi, \phi]$. A group G is soluble if and only if $\phi(G) = 1$ for some $\phi \in \mathfrak{C}$.

We define
$$\delta_k = [\gamma_k, \gamma_k] \quad (k = 1, 2, \dots),$$
so that $\delta_1(G) = G'$ and $\delta_2(G) = G''$ are the first and second derived groups of G. The groups G for which $\delta_k(G) = 1$, for given k, form a variety, the variety of extensions of an Abelian group by a nilpotent group of class less than k. Such groups, if finitely generated, all satisfy Max-n. It follows that *there are only a countable number of different types of group which are finitely generated extensions of Abelian groups by nilpotent groups*. For they are all homomorphic images of some $F/\delta_k(F)$, where F is a finitely generated free group, and each $F/\delta_k(F)$ has only a countable number of normal subgroups.

The equation $\delta_2(G) = 1$ means that G is metabelian, since $\delta_2(G) = G''$. There are, therefore, in particular, only a countable number of non-isomorphic finitely generated metabelian groups. A striking contrast is obtained if we replace the equation $G'' = 1$ by the equation
$$[G'', G] = 1,$$
which expresses that the central quotient group G/C (or group of inner automorphisms of G) is metabelian. For we show in section 3 that *this*

variety includes an uncountable infinity of different types of groups G, all having two generators and all having for centre C an arbitrarily assigned countable Abelian group $\neq 1$.

It is easy to see that the functions $\phi \in \mathfrak{C}$ fall into two mutually exclusive classes,

(i) those for which $\phi \geqslant \delta_k$ for some k,

and (ii) those for which $\phi \leqslant [\delta_2, \gamma_1]$.

Our result shows that this dichotomy is precisely the one which distinguishes whether the variety $\phi(G) = 1$ contains a countable or a non-countable number of different types of finitely generated groups G. For the varieties of the first kind, FG implies Max-n. For those of the second kind, FG does not imply Max-n; and in particular $F/\phi(F)$, where F is a free group with two generators, does not satisfy Max-n.

1.4. *Wreath groups.* Simple examples of soluble groups satisfying Max-n but not Max, as well as others satisfying FG but not Max-n, may be constructed by the process of *wreathing* two permutation groups G and Γ together, to produce a wreath group (Kranzgruppe) which we denote by

$$K = G \wr \Gamma.$$

This process is well known from the theory of imprimitive permutation groups, of monomial groups, and of induced matrix representations. We recall the essential facts.†

$K = G \wr \Gamma$ is a permutation group whose degree (possibly infinite) is the product of the degrees of G and Γ, and may be defined as follows. Let G permute a set of symbols ξ and Γ a set of symbols α. Then K is to permute the set of all ordered pairs (ξ, α). Associate with each α a copy G_α of G and form the ordinary (not the complete) direct product $\bar{G}$ of the G_α. If $u \in G$ and u maps ξ into η, the corresponding element u_α of G_α is to map (ξ, α) into (η, α) but is to leave invariant all pairs (ξ, β) for which $\beta \neq \alpha$. K is the split (or complemented) extension of $\bar{G}$ by Γ, so that $K = \bar{G}\Gamma_0$ and $\bar{G} \cap \Gamma_0 = 1$, where Γ_0 is a copy of Γ and permutes the direct factors G_α of $\bar{G}$ in the same way that Γ permutes the symbols α. (If $v \in \Gamma$ and maps α into β, we require that the corresponding element v_0 of Γ_0 shall map (ξ, α) into (ξ, β) for all ξ.)

It will be clear how one should define the *complete wreath group*

$$\bar{K} = G \bar{\wr} \Gamma$$

in an analogous manner, using for $\bar{G}$ the complete instead of the ordinary direct product of the G_α.

† For a precise and detailed account, with many new results, cf. M. Krasner and L. Kaloujnine, *Acta. Sci. Math. Szeged*, 13 (1950), 208–30 and 14 (1951), 39–66, 69–82. There is a slight difference in the notation we use here from that adopted in these papers.

K and $\bar{K}$ can also be expressed equivalently by (possibly infinite) monomial matrices whose coefficients are either 0 or elements of G; $\bar{G}$ is then represented by diagonal matrices and Γ_0 by permutation matrices. However, the permutation form of definition is more convenient when one has in mind the possibility of iterating the wreath operation. It is easy to see that, if G, Γ, and H are any three permutation groups, then $(G \wr \Gamma) \wr H$ and $G \wr (\Gamma \wr H)$ are not only isomorphic as abstract groups but also equivalent as permutation groups. A similar remark applies to the operation $\bar{\wr}$.

If G or Γ or both are not given as permutation groups but as abstract groups, it is convenient to interpret the symbols $G \wr \Gamma$ and $G \bar{\wr} \Gamma$ by regarding G (or Γ, or both, as the case may be) as replaced by their regular representations. With this convention, for example, as Kaloujnine (**5**) has shown, the Sylow p-subgroups of the symmetric group of degree p^n have the form

$$C_p \wr C_p \wr \dots \wr C_p$$

with n factors, where C_p is the cyclic group of order p.

Finally, let

$$H_0 < H_1 < \dots < H_m = G$$

be any chain of subgroups of G and suppose that H_0 contains no normal subgroup $\neq 1$ of G. Let Γ_i be the transitive permutation group which represents H_i by permutations of the cosets of H_{i-1}. Then the usual method of induced representations allows us to map G isomorphically into

$$\Gamma_1 \bar{\wr} \Gamma_2 \bar{\wr} \dots \bar{\wr} \Gamma_m.$$

For finite groups there is, of course, no distinction between $\wr$ and $\bar{\wr}$, but for infinite groups it becomes of importance that the above embedding of G is into the complete and not the ordinary wreath product of the Γ_i.

1.5. *Wreath groups and* Max-n. We show (Theorem 5) that, *if G satisfies* Max-n *and if Γ is finitely intransitive* (has only a finite number of transitive components) *and satisfies* Max-r, *then $G \wr \Gamma$ satisfies* Max-n. In particular, if $\Gamma_1, \Gamma_2, \dots, \Gamma_m$ are finitely intransitive polycyclic permutation groups, then $\Gamma_1 \wr \Gamma_2 \wr \dots \wr \Gamma_m$ is soluble and satisfies Max-n. Except in trivial cases these groups are not themselves polycyclic or even Abelian-by-polycyclic.

The result italicized remains true if the assumption that Γ satisfies Max-r is replaced by the following, which for soluble Γ is certainly weaker: Γ satisfies Max-n and each of the stabilizers H (that is, the subgroups leaving invariant one of the symbols permuted by Γ) can be connected to Γ by a series

$$H = H_0 < H_1 < \dots < H_n = \Gamma$$

such that the successive quotient groups H_i/H_{i-1} are all cyclic. The class of groups Γ which have a faithful finitely intransitive representation satisfying this condition on the stabilizers forms a natural generalization of the polycyclic groups. But the groups of this class, though obviously soluble, are not necessarily finitely generated or even countable and so do not have to satisfy Max-n, as is shown for example by the complete wreath group $C_\infty \bar{\wr} C_\infty$, where C_∞ is an infinite cyclic group. This explains why we have to require that Γ satisfies Max-n above, in addition to the stabilizer condition.

Another sufficient condition for $G \wr \Gamma$ to satisfy Max-n is that both G and Γ satisfy Max-n, that Γ is finitely intransitive, and that G has no non-trivial *central factors*. (H/K is a central factor of G if $[H, G] \leqslant K$, H being a normal subgroup of G; it is trivial if $H = K$.) Since a soluble $G \neq 1$ always has the non-trivial central factor G/G', however, this result is not relevant to the theory of soluble groups and is in any case very elementary.

A simple example of a soluble group satisfying FG but not Max-n is the group
$$K = C_\infty \wr G,$$
where G is any soluble group satisfying FG but not Max, taken in its regular representation in accordance with the convention explained above. This K may be regarded as the split extension of the group-ring R of G, considered as an additive group, by G itself, the elements of G acting on R as right-hand multipliers. The right ideals of R then coincide with the normal subgroups of K contained in R. Thus K satisfies Max-n only if G satisfies Max-r, which for soluble G is, as we have seen, equivalent to Max.

1.6. We mention finally two *unsolved problems*.

(i) Do there exist soluble groups satisfying FR but not Max-n ? Graham Higman (2) has shown that a finitely related group can be isomorphic with a proper quotient group of itself and it is clear that such a group does not satisfy Max-n. Our question is whether a soluble group can behave in a similar manner. Clearly, a group G satisfies both FR and Max-n if and only if all its quotient groups G/H satisfy FR. So the question may be put in the following way: Can a finitely related soluble group have homomorphic images which are not finitely related ? If the answer is negative, then for soluble groups the condition FR would come in generality strictly between Max and Max-n. For all polycyclic groups satisfy FR; the group defined by the equation $a^{-1}ba = b^2$ satisfies FR but not Max; and the group $C_\infty \wr C_\infty$ (or alternatively F/F'', where F is a free group with $1 < d(F) < \infty$) satisfies Max-n but not FR.

(ii) If a group G satisfies Max-n, all its central factors H/K must satisfy Max, since every subgroup between H and K is normal in G. Do there exist finitely generated soluble groups whose central factors all satisfy Max, but which do not themselves satisfy Max-n ? What can be said of the central factors of $C_\infty \wr G$, where G is soluble and finitely generated but not polycyclic?

2. Proofs

2.1. *'Poly' properties of groups.* We say that a property P of groups is a 'poly' property if, whenever H and G/H have the property P, so has G. Here H is, of course, a normal subgroup of G.

LEMMA 1. FG, Max, Max-n, Max-Ω *(for Ω-groups) and* FR *are 'poly' properties.*

For FG the result is immediate. For the three maximal properties it follows from the fact that, if H is a normal subgroup of G and if L and M are subgroups of G such that

$$L \leqslant M, \qquad LH = MH, \qquad \text{and} \qquad L \cap H = M \cap H,$$

then $L = M$; so that an infinite tower in G implies an infinite tower in either H or G/H if not both. For FR, suppose H generated by $a_1,\ldots, a_r$ with defining relations $h_1(a) = \ldots = h_\rho(a) = 1$; and suppose G/H generated by $\bar{b}_1,\ldots, \bar{b}_s$ with defining relations $k_1(\bar{b}) = \ldots = k_\sigma(\bar{b}) = 1$. In each coset $\bar{b}_i$ choose an element b_i. Then $k_\alpha(b) = f_\alpha(a)$ $(\alpha = 1,\ldots, \sigma)$ and $b_i^{-1}a_j b_i = g_{ij}(a)$ $(i = 1,\ldots, s;\ j = 1,\ldots, r)$, with suitable words f_α, g_{ij}. It is clear that the a's and b's together generate G and that the $\rho+\sigma+rs$ relations

$$h_\beta(a) = 1 \quad (\beta = 1,\ldots, \rho),$$
$$k_\alpha(b) = f_\alpha(a) \quad (\alpha = 1,\ldots, \sigma),$$
$$b_i^{-1}a_j b_i = g_{ij}(a) \quad (i = 1,\ldots, s;\ j = 1,\ldots, r)$$

together constitute a set of defining relations for G.

If P is any property of groups, we define *poly-P* to be the least extensive 'poly' property of groups which belongs to every group having property P. If P is itself a 'poly' property, then poly-P is clearly the same as P. In general, then, G is a poly-P group if and only if it has a series of subgroups of finite length,

$$G = G_0 \geqslant G_1 \geqslant \ldots \geqslant G_m = 1,$$

such that each of the successive quotient groups G_{i-1}/G_i has property P. (Here the word *series* implies as usual that each G_i is normal in G_{i-1} though not necessarily in G itself.)

For example, G is polysimple if and only if it has a composition series. Polyabelian groups are the same as soluble groups, and so on.†

From Lemma 1 we have the immediate and no doubt well-known result:

COROLLARY. *All polycyclic groups satisfy* Max *and* FR.

A rather more general result will be useful.

† The prefix *meta* used to have the force we are here ascribing to *poly*. Unfortunately, *metabelian* and *metacyclic* have now come to mean polyabelian and polycyclic, respectively, *in two steps* (with $m = 2$ in the appropriate series).

LEMMA 2. *Let G be finitely generated and let H be a normal subgroup contained in the hypercentre of G. If G/H is finitely related, then H is polycyclic (and hence H and G are also finitely related). In particular, finitely generated nilpotent groups are polycyclic.*†

To say that H is contained in the hypercentre of G means that the series defined by

$$H = H_0, \qquad H_{k+1} = [H_k, G] \quad (k = 0, 1,...)$$

terminates with $H_m = 1$ for some finite m. An inductive argument shows that we may suppose $[H, G] = 1$ without loss of generality, so that we assume H is contained in the centre of G.

Let $\bar{a}_1, \bar{a}_2,..., \bar{a}_r$ generate G/H subject to the defining relations

$$k_1(\bar{a}) = k_2(\bar{a}) = ... = k_\sigma(\bar{a}) = 1;$$

and let $a_1, a_2,..., a_r$ be chosen from the corresponding cosets of H in G. Since G is finitely generated, we can find a finite number of elements $b_1, b_2,..., b_s$ in H such that the a's and b's together generate G, because the a's alone already generate G modulo H. Then the $s+\sigma$ elements $k_i(a)$, b_j ($i = 1,..., \sigma$; $j = 1,..., s$) generate H. Hence H is a finitely generated Abelian group and therefore polycyclic.

The final remark of Lemma 2 may be obtained by taking $H = G$, a finitely generated nilpotent group; for then G is its own hypercentre.

2.2. *A lemma on modules.* We now show that, for soluble groups, Max is equivalent to Max-r. To do this it is sufficient to show that for these groups Max implies Max-r, since (as was mentioned in section 1) the converse implication is trivial. Thus we have to prove that the group-ring R of a polycyclic group satisfies Max-R. We shall actually prove a rather more general fact (Lemma 3) about groups with operators.

An additive group M which admits the elements of a multiplicative group G as right-hand multipliers subject to the usual rules $ux \in M$, $(u+v)x = ux+vx$, $u(xy) = (ux)y$, and $u1 = u$, where $u, v \in M$ and $x, y \in G$ and 1 is the unit element of G, we shall call a *G-module*. We adopt the convention that, if N is a subgroup of M and H a subgroup of G, then NH is the smallest subgroup of M containing all the products vy, where $v \in N$ and $y \in H$.

LEMMA 3. *Let H be a normal subgroup of the group G, such that G/H is either finite or cyclic. Let N be an H-module contained in the G-module M and such that $M = NG$. Then, if N satisfies* Max-H, *M satisfies* Max-G.

† That finitely generated nilpotent groups satisfy FR is shown by Lyndon (4). That they are polycyclic is also well known and easy to prove directly.

Proof. (i) Let $(G:H) = n$ be finite and suppose that $G = \bigcup_{i=1}^{n} Hs_i$. Then

$$M = Ns_1+Ns_2+\ldots+Ns_n.$$

Each summand Ns_i is clearly an H-module, since H is normal in G. The H-submodules of Ns_i are all of the form $N_0 s_i$, where N_0 is an H-submodule of N. It follows that each of the Ns_i satisfies Max-H. Since Max-H is a poly property of H-modules, we conclude that M satisfies Max-H. *A fortiori*, M satisfies Max-G.

(ii) Let G/H be an infinite cyclic group and suppose that $G = \mathrm{Gp}\{H, x\}$. Then every element $f \neq 0$ of M is expressible in at least one way in the form

$$f = \sum_{r \leqslant n \leqslant s} c_n x^n,$$

where each $c_n \in N$ and $c_s \neq 0$. We call such an expression an x-polynomial, although negative powers of x may be involved; and we call c_s the leading coefficient and $c_s x^s$ the leading term.

We have to show that, if M_0 is any G-submodule of M, then there exists a finite number of elements $f_1,\ldots, f_k$ of M_0 such that $M_0 = \mathrm{Gp}_G\{f_1,\ldots,f_k\}$. We may suppose $M_0 \neq 0$. It is clear that the set N_0 of all leading coefficients of x-polynomial expressions of elements of M_0, taken together with 0, forms an H-submodule of N. For if f and g have leading terms $c_s x^s$ and $d_t x^t$, then $f \pm g x^{s-t}$ will have leading coefficient $c_s \pm d_t$, provided this is not zero; and if y is any element of H, then $f x^{-s} y x^s$ will have leading coefficient $c_s y$. By hypothesis, N satisfies Max-H. Hence there exist elements $a_1,\ldots, a_l$ of N_0 such that $N_0 = \mathrm{Gp}_H\{a_1,\ldots, a_l\}$. Choose $f_i \in M_0$, such that f_i is expressible as an x-polynomial with leading coefficient a_i. Without loss of generality, we may assume that none of the f_i involves negative powers of x and that the leading term of f_i is $a_i x^m$, where m is independent of i. Now let f be any element of M_0. For a suitable value of n, $fx^n = g$ will be expressible as a polynomial involving no negative powers of x. If its leading term is ax^p, where $p \geqslant m$, then the element a of N_0 will be expressible as a sum of terms of the form $\pm a_i y$, where $i = 1,\ldots, l$ and $y \in H$. The corresponding sum of terms $\pm f_i x^{-m} y x^p$ will then have no negative powers of x, will belong to $M_1 = \mathrm{Gp}_G\{f_1,\ldots,f_l\}$, and will have the same leading term as g. We deduce that $g \equiv g_1 \bmod M_1$, where g_1 is expressible as a polynomial involving only the powers $x^0, x^1,\ldots, x^{m-1}$. Also $g_1 \in M_0$, since $M_1 \leqslant M_0$. Now the elements of M expressible in the form $\sum_{i=0}^{m-1} a_i x^i$, with $a_i \in N$, form an H-module which, as in case (i), satisfies Max-H. Thus the set M_2 of elements M_0 which are expressible in this form may be written $M_2 = \mathrm{Gp}_H\{f_{l+1},\ldots,f_k\}$. Since $f = gx^{-n}$, it is now clear that $M_0 = \mathrm{Gp}_G\{f_1,\ldots,f_k\}$ as required. Thus the lemma is proved.

Let us say that a subgroup H of G is *polycyclically contained in* G if there exists a series of subgroups

$$H = H_0 < H_1 < \dots < H_m = G$$

such that m is finite and each H_i/H_{i-1} is cyclic.

We then have immediately:

COROLLARY. *The conclusion of Lemma 3 still holds if H, instead of being normal in G with G/H cyclic or finite, is polycyclically contained in G.*

Another easy consequence is the following.

THEOREM 1. *The group ring of a polycyclic group satisfies the maximal condition for right ideals.*

For if G is polycyclic we may take $H = 1$ in the above corollary, and in Lemma 3 we may take M to be the group-ring of G and N to be the additive group of integer multiples of the unit element of G. The conclusion of Theorem 1 also holds for groups which are poly-cyclic-or-finite, but it is easy to see that such groups are merely finite extensions of polycyclic groups.

2.3. *Some criteria for* Max-n. Let A be a normal Abelian subgroup of any group G, let $\Gamma = G/A$ and let R be the group-ring of Γ. For $x \in G$, $a \in A$, and $\xi = xA$, we may define $a^{\xi} = x^{-1}ax$, which is independent of the choice of x in the coset ξ. For a general element $\eta = m_1 \xi_1 + \dots + m_r \xi_r$ of R, where the m_i are integers and the $\xi_i \in \Gamma$, we define

$$a^{\eta} = \prod_{i=1}^{r} (a^{\xi_i})^{m_i}.$$

In this way, we may regard A (apart from the irrelevant multiplicative notation) as an R-module. And it is then obvious that the R-submodules of A coincide with the normal subgroups of G which are contained in A. Note that $d_R(A) = d_{\Gamma}(A) = d_G(A)$.

LEMMA 4. *In order that the extension G of the Abelian group A shall satisfy* Max-n, *the following three conditions are together necessary and sufficient:*

(i) $\Gamma = G/A$ *satisfies* Max-n;

(ii) $d_R(A) = m$ *is finite, where R is the group-ring of* Γ;

(iii) *if* $A = \mathrm{Gp}_R\{a_1, a_2, \dots, a_m\}$, *each of the R-modules* R/N_i $(i = 1, 2, \dots, m)$ *satisfies* Max-R, *where N_i consists of all elements η of R such that* $a_i^{\eta} = 1$.

For if $A_i = \mathrm{Gp}_R\{a_i\}$, we have, by (iii), $A = A_1 A_2 \dots A_m$ and A_i is R-isomorphic with R/N_i. Since Max-R is a 'poly' property of R-modules, it follows that A satisfies Max-R, or, what is the same, Max-G. By (i), G/A satisfies Max-G. Hence, again by Lemma 1, G satisfies Max-G, or, what is the same, Max-n. Thus the conditions (i)–(iii) are sufficient. Their necessity is clear.

Theorem 2. *If G is an extension of the Abelian group A such that* (i) G/A *satisfies* Max-n, (ii) $A = \mathrm{Gp}_G\{a_1, a_2, \ldots, a_m\}$ *with finite m, and* (iii) *for each a_i there is a subgroup H_i of the centralizer of a_i which is polycyclically contained in G; then G satisfies* Max-n.

Proof. For G itself, Max-n is equivalent to Max-G. By (i), G/A satisfies Max-G. If $A_i = \mathrm{Gp}_G\{a_i\}$, we have $A = A_1 A_2 \ldots A_m$. Hence, since Max-G is a 'poly' property of G-groups, it will be sufficient to prove that each A_i satisfies Max-G. But if B_i is the cyclic group generated by a_i, then B_i is an H_i-module satisfying Max-H_i, since H_i acts identically on B_i. Also $A_i = B_i^G$, in the sense of module theory. Thus the conditions of the corollary to Lemma 3 are fulfilled, with $H = H_i$ and $M = A_i$ and we have the required conclusion that A_i satisfies Max-G.

An immediate consequence is

Theorem 3. *Every finitely generated extension G of an Abelian group A by a polycyclic group $\Gamma = G/A$ satisfies* Max-n.

For (i) Γ, being polycyclic, satisfies Max and hence Max-n. Also, by the corollary to Lemma 1, Γ is finitely related. Since G is finitely generated, it follows that $d_G(A) = m$ is finite and so (ii) $A = \mathrm{Gp}_G\{a_1, \ldots, a_m\}$ for suitable $a_i \in A$. The condition (iii) of Theorem 2 can now be satisfied by taking each $H_i = A$, and Theorem 3 follows.

Corollary 1. *Every finitely generated group G such that $\delta_k(G) = 1$ for some value of $k = 1, 2, \ldots$ satisfies* Max-n.

Here $\delta_k(G)$ is the derived group of the term $A = \gamma_k(G)$ of the lower central series of G. Hence, if $\delta_k(G) = 1$, A is Abelian. And in any case G/A is a finitely generated nilpotent group. But, by the case $H = G$ of Lemma 2, every finitely generated nilpotent group is polycyclic. Thus Corollary 1 follows as a special case of Theorem 3.

Corollary 2. *There are only a countable number of non-isomorphic finitely generated Abelian-by-polycyclic groups.*

For every such group G is isomorphic with some F/N, where F is a finitely generated free group. Suppose A is a normal Abelian subgroup of G such that G/A is polycyclic and let A correspond to the subgroup M/N of F/N in the above isomorphism. Then F/M is isomorphic with G/A and is therefore polycyclic. Since polycyclic groups are finitely related by Lemma 1 and F is finitely generated, we have $d_F(M)$ finite. Hence there are only a countable number of choices for $M = \mathrm{Gp}_F\{a_1, \ldots, a_m\}$ in the given F. But $N \geqslant M'$ since M/N is Abelian, and F/M' satisfies Max-n by Theorem 3. Therefore $N = \mathrm{Gp}_F\{M', b_1, b_2, \ldots, b_n\}$ with suitable $b_i \in M$ and n finite. Thus there are only a countable number of choices for N and the result follows.

In particular, *there are only a countable number of different types of finitely generated group G such that $\delta_k(G) = 1$ for some k.*

2.4. *The case of wreath groups.* In order that $K = G \wr \Gamma$ shall satisfy Max-n, it is clearly necessary that both G and Γ shall satisfy Max-n. To obtain a sufficient condition for K to satisfy Max-n, we merely need to impose enough further restrictions to ensure that the base group $\bar{G}$ shall satisfy Max-K. Here $\bar{G}$ is the direct product of copies G_α of G in one-to-one correspondence with the symbols α permuted by the permutation group Γ. A further necessary condition is that Γ shall be finitely intransitive, since for each transitive component of Γ the product of the corresponding G_α will be normal in K. Let Γ have m transitive components and let the corresponding products of the G_α be $\bar{G}_1,\ldots, \bar{G}_m$, so that $\bar{G}$ is the direct product of $\bar{G}_1,\ldots, \bar{G}_m$. Since Max-$K$ is a 'poly' property of K-groups, it will be sufficient if each $\bar{G}_i$ satisfies Max-K. Thus we need only discuss the case when Γ is transitive.

Now let H be a normal subgroup of K contained in $\bar{G}$. We define two normal subgroups L, M of G which are uniquely determined by H. L consists of all elements $u \in G$ such that, for some fixed α, we have $u_\alpha \in H$. M consists of all elements $v \in G$ for which there exists in H an element whose α-component is equal to v. Since H is normal in K and, by hypothesis, Γ is transitive, it follows that the G_α are all conjugate in K and hence L and M depend only on H, not on the choice of α. By forming commutators of elements of H with elements of G_α, we see that L and M are normal in G and that $[M, G] \leqslant L$. Thus M/L is a central factor of G.

Now let $H_1 \leqslant H_2 \leqslant \ldots$ be a formally infinite tower of normal subgroups of K, all contained in $\bar{G}$; and let L_i, M_i be the subgroups of G derived as described above from H_i. Clearly $L_i \leqslant L_{i+1}$ and $M_i \leqslant M_{i+1}$ for all $i = 1, 2,\ldots$. Since, by hypothesis, G satisfies Max-n, we have

$$L_i = L_{i+1} = \ldots = L \quad \text{and} \quad M_i = M_{i+1} = \ldots = M$$

from a certain i onwards. Without loss of generality, we may assume this $i = 1$, so that $L_j = L$ and $M_j = M$ for all j.

Now let L_α and M_α be the subgroups of G_α which correspond to the subgroups L and M of G and let $\bar{L}$ and $\bar{M}$ be the direct products of the L_α and of the M_α respectively. Then $\bar{L} \leqslant H_j \leqslant \bar{M}$ for all j. But M/L is a central factor of G. Since G satisfies Max-n, it follows that M/L is a finitely generated Abelian group A. Hence $\bar{M}/\bar{L}$ is the direct product $\bar{A}$ of copies A_α of A, and these copies are permuted transitively by Γ. We conclude that K will certainly satisfy Max-n provided that the groups $A \wr \Gamma$ do so, where A runs through the various central factors of G. This will clearly be the case if G has no non-trivial central factors, since then every A is a unit group and $\bar{L} = \bar{M}$. For the general case we can obtain a sufficiency condition from the corollary to Lemma 3. Suppose H is a subgroup of Γ leaving one of the symbols α invariant (that is, suppose H is contained in one of the stabilizers of Γ) and suppose further that H is polycyclically contained in Γ. Then H acts identically on A_α. But $A_\alpha \cong A$ is a finitely generated Abelian group. Hence A_α, when regarded as an H-module, satisfies Max-H. Moreover, in the sense of module theory, $\bar{A} = A_\alpha^\Gamma$. The corollary referred to then tells us that $\bar{A}$ satisfies Max-Γ, and hence $A \wr \Gamma$ satisfies Max-n. Summing up, we have the theorem:

THEOREM 4. *Let G and Γ be permutation groups both satisfying* Max-n *and let Γ be finitely intransitive. Then $G \wr \Gamma$ will also satisfy* Max-n *provided that either of the two following conditions holds*:

(i) *G has no non-trivial central factors.*

(ii) *The stabilizers of the permutation group Γ each contain a subgroup which is polycyclically contained in Γ.*

We remark that (ii) of Theorem 4 amounts to one condition for each transitive component of Γ since the stabilizers belonging to any given transitive component are all conjugate in Γ.

A slight modification of the above argument yields

THEOREM 5. *If G satisfies* Max-n *and Γ is finitely intransitive and satisfies* Max-r, *then $G \wr \Gamma$ satisfies* Max-n.

We may suppose Γ transitive. Using the same notation as before, let $A_\alpha = \mathrm{Gp}\{a_1, \ldots, a_m\}$ and $\bar{A}_i = \mathrm{Gp}_\Gamma\{a_i\}$ for $i = 1, 2, \ldots, m$. Then

$$\bar{A} = \bar{A}_1 \bar{A}_2 \ldots \bar{A}_m$$

and each $\bar{A}_i$ is a Γ-homomorphic image of the group-ring of Γ. Since Γ satisfies Max-r, it follows that the $\bar{A}_i$ and hence $\bar{A}$ also all satisfy Max-Γ. Further, Max-r implies Max-n; so Γ and therefore $A \wr \Gamma$ satisfies Max-n. Here A is any central factor of G. The conclusion that $G \wr \Gamma$ satisfies Max-n now follows as before.

COROLLARY. *If $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ are finitely intransitive polycyclic permutation groups, then*

$$\Gamma_1 \wr \Gamma_2 \wr \ldots \wr \Gamma_m$$

satisfies Max-n.

This follows by induction on m from the fact that polycyclic groups satisfy Max-r.

Since Max-n is a 'poly' property of groups, the properties P which have been shown in the preceding sections to imply Max-n can always be replaced by the corresponding properties poly-P. We have not thought it worth while to state this explicitly for each particular case.

3. Examples

3.1. We shall now consider the question, *for what varieties* $\mathfrak{B}$ *of soluble group is it true that* $\mathfrak{B}$ *contains an uncountable infinity of different types of finitely generated group?* By the remark made at the end of section 2.3, the variety of all groups G such that $\delta_k(G) = 1$ contains only a countable number of non-isomorphic finitely generated groups. We shall prove that, *on the contrary*, the variety of all groups G such that $[G'', G] = 1$ contains an uncountable number of non-isomorphic finitely generated groups.

Let
$$\zeta(G)$$
be the centre of the group G. We recall that $d(G)$ is the minimum number of elements which suffice to generate G. Then we have the following more precise result:

THEOREM 6. *Let C be a given countable Abelian group $\neq 1$. Then there exists an uncountable infinity of different types of group G such that*
$$d(G) = 2, \quad \zeta(G) \cong C, \quad \textit{and} \quad [G'', G] = 1.$$

We shall begin by deducing Theorem 6 from

THEOREM 7. *There exists a group G whose centre Z is the free Abelian group with $\aleph_0$ generators and such that*
$$d(G) = 2, \quad \zeta(G/Z) = Z/Z, \quad \textit{and} \quad [G'', G] = 1.$$

For, since $C \neq 1$, there will exist $2^{\aleph_0}$ different subgroups K of Z such that $Z/K \cong C$. We could, for instance, map 'half' the free generators of Z onto a set of generators of the countable group C and with the remaining 'half' still at our disposal we could map an arbitrary selection of these onto 1 and the remainder onto some other chosen element of C. In this way we would obtain $2^{\aleph_0}$ homomorphisms of Z onto C, no two of which have the same kernel K. By hypothesis, Z is the centre of G, so these K will all be normal in G. Since $\zeta(G/Z) = Z/Z$, we have *a fortiori* $\zeta(G/K) = Z/K \cong C$. Further, G/K will belong like G to the variety defined by $[\delta_2, \gamma_1] = 1$ and $d(G/K) = 2$ since G/K is not Abelian and therefore not cyclic. Let Γ be one of these groups G/K, considered as an abstract group. Since Γ is countable and G is finitely generated, there are only a countable number of different homomorphisms of G onto Γ. Hence only a countable number of the kernels K will make G/K isomorphic with a given Γ. But the total number of kernels K is uncountable. Hence the number of non-isomorphic groups Γ is also uncountable. Thus Theorem 6 follows from Theorem 7.

That a finitely generated group may have $2^{\aleph_0}$ distinct types of homomorphic image was proved by B. H. Neumann (7). However, the specially simple structure of the groups we are here considering may perhaps justify the present treatment.

We shall also consider the question: to what extent it is possible to specialize the variety of groups G defined by $[G'', G] = 1$ by imposing further identical relations, giving a smaller variety $\mathfrak{V}$ which still has an uncountable infinity of non-isomorphic finitely generated groups? An additional identity of the form $x^m \equiv 1$ with $m > 0$ would clearly be too strong, because a finitely generated soluble group of positive exponent must be of finite order. But it is possible to impose such an identity not on the group itself but on its derived group. We shall in fact prove the theorem:

THEOREM 8. *Let $\epsilon_m(G)$ be the group generated by the m-th powers of the elements of the derived group G' of G. If $m > 2$, the variety of all groups G such that*

$$[G'', G] = \epsilon_m(G) = 1$$

contains $2^{\aleph_0}$ non-isomorphic groups with two generators each.

Obviously it will be enough to prove Theorem 8 for the cases $m = 4$ and m an odd prime. We observe that the theorem does not hold if $m = 2$. For $\epsilon_2(G) = 1$ implies that G' is Abelian and hence G metabelian. And by Corollary 2 to Theorem 3 there is only a countable number of non-isomorphic finitely generated metabelian groups.

3.2. *Proofs of Theorems* 7 *and* 8. Let F be the free group with two generators a and b and let $B = \mathrm{Gp}_F\{b\}$. It is easy to see that B is freely generated by the conjugates

$$\ldots, b_{-1}, b_0, b_1, b_2, \ldots \qquad (1)$$

of b, where $\qquad b_j = a^{-j}ba^j \quad (j = 0, \pm 1, \pm 2, \ldots).$

F/B is an infinite cyclic group, so that $F' \leqslant B$. Thus every quotient group G of the group $F/[B', F]$ will satisfy $[G'', G] = 1$ and it is natural to look for the groups we require among these quotient groups. However, it is rather easier to proceed indirectly.

We define B to be the group generated by the elements (1) subject to the following system of defining relations.

First, we make B nilpotent of class 2 by imposing the relations

$$[b_i, b_j, b_k] = 1 \quad (i, j, k = 0, \pm 1, \pm 2, \ldots). \qquad (2)$$

If we left the matter here, B' would be the Abelian group freely generated by the commutators

$$c_{ij} = [b_j, b_i] \quad (i < j). \qquad (3)$$

Next, we make $\qquad b_j^p = 1 \quad (j = 0, \pm 1, \ldots), \qquad (4)$

where p is either a prime (including the possibility $p = 2$) or else 0 (in which case the relations (4) are negligible). When p is a prime, (4) implies that the c_{ij} are all of order p and, as a result of (2) and (4), both B/B' and B' become elementary p-groups with the independent generators (1) mod B'

and (3) respectively. When $p = 0$, B/B' and B' are the Abelian groups freely generated by these same sets of generators.

Finally, we pass from the group so far defined by (2) and (4) to a quotient group, still denoted by B, by adjoining the relations

$$c_{ij} = c_{i+k,j+k} \quad (i, j, k = 0, \pm 1, \ldots). \tag{5}$$

The effect of (5) is that, if we write

$$d_r = c_{i,i+r}, \tag{6}$$

which by (5) is independent of i, the elements

$$d_1, d_2, d_3, \ldots \tag{7}$$

form a set of independent generators of B'.

The group $B = B_p$ is now completely defined by (2), (4), and (5). It clearly possesses an automorphism mapping each b_j onto the next one, b_{j+1}; for this transformation permutes the generators and leaves the system of defining relations unaffected. Thus we may define G as the group generated by

$$a \quad \text{and} \quad b = b_0 \tag{8}$$

with the defining relations

$$a^{-1} b_j a = b_{j+1} \tag{9}$$

together with (2), (4), and (5). G is the split extension of B by the cyclic group of infinite order generated by a. Hence $G' \leqslant B$.

If p is an odd prime, every element $x \in B$ satisfies $x^p = 1$ because B is generated by elements of order p and is nilpotent of class $2 < p$. In this case, then, $\epsilon_p(G) = 1$. When $p = 2$, we can only say that $x^4 = 1$ for all $x \in B$, so that $\epsilon_4(G) = 1$.

The centre of G is B' and the centre of G/B' is the unit group B'/B'. For the d_r, which generate B', all commute with b by (2). They also all commute with a by (9), (3), and (5), together with the definition (6). Thus B' is certainly contained in the centre of G. That G/B' has unit centre is evident from (9). Hence *every subgroup K of B' is normal in G and the centre of G/K is B'/K.*

Theorems 7 and 8 now follow immediately. When $p = 0$, the group $B' = Z$, the centre of G, is a free Abelian group with the $\aleph_0$ free generators $d_1, d_2, \ldots$; so that G satisfies all the requirements of Theorem 7. When p is a prime, there is only a countable number of different groups C which are homomorphic images of B', viz. the elementary Abelian p-groups of finite order, together with groups isomorphic with B' itself. And if $C \neq 1$ and is finite, there is only a countable number of different subgroups K of B' such that $B'/K \cong C$. There are altogether, however, $2^{\aleph_0}$ subgroups K in B', all of them normal in G, so that 'most' of them must give $B'/K \cong B'$.

Since a given group Γ can be isomorphic with G/K for only a countable number of different choices of K, there will still be among these quotient groups an uncountable infinity of different types. This gives Theorem 8.

A more precise view of the possibilities for these quotient groups may be obtained as follows. Writing the Abelian group B' additively, define (for $k = 0, 1, 2, \ldots$)

$$e_{k+1} = d_{k+1} - 2kd_k + \left\{\binom{2k}{2} - 1\right\} d_{k-1} - \left\{\binom{2k}{3} - 2k\right\} d_{k-2} + \ldots + \\ + (-1)^k \left\{\binom{2k}{k} - \binom{2k}{k-2}\right\} d_1.$$

Then it is easy to show that, for $k > 1$,

$$e_k, e_{k+1}, e_{k+2}, \ldots$$

is a basis of $\delta_k(G)$. Therefore if K is the subgroup generated by any selection of the e's and if $\Gamma = G/K$, $\delta_k(\Gamma)/\delta_{k+1}(\Gamma)$ will be of order 1 or p according to whether e_k was selected or not. In other words, *there exist quotient groups* Γ *of* $G = G_p$ *for which an arbitrary selection of the* $\delta_k(\Gamma)/\delta_{k+1}(\Gamma)$ *are of order* p *and the rest all of order* 1. Since $\delta_k(G)$ is a characteristic subgroup of G, isomorphic Γ's must have corresponding δ-quotients also isomorphic.

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