Profinite lambda-terms and parametricity

Sam van Gool^a Paul-André Melliès^b Vincent Moreau^c

^a Université Paris Cité
 ^b CNRS, Université Paris Cité, Inria
 ^c Université Paris Cité, Inria

Abstract

Combining ideas coming from Stone duality and Reynolds parametricity, we formulate in a clean and principled way a notion of profinite λ -term which, we show, generalizes at every type the traditional notion of profinite word coming from automata theory. We start by defining the Stone space of profinite λ -terms as a projective limit of finite sets of usual λ -terms, considered modulo a notion of equivalence based on the finite standard model. One main contribution of the paper is to establish that, somewhat surprisingly, the resulting notion of profinite λ -term coming from Stone duality lives in perfect harmony with the principles of Reynolds parametricity. In addition, we show that the notion of profinite λ -term is compositional by constructing a cartesian closed category of profinite λ -terms, and we establish that the embedding from λ -terms modulo $\beta\eta$ -conversion to profinite λ -terms is faithful using Statman's finite completeness theorem. Finally, we prove that the traditional Church encoding of finite words into λ -terms can be extended to profinite words, and leads to a homeomorphism between the space of profinite words and the space of profinite λ -terms of the corresponding Church type.

Keywords: higher-order automata, semantics of lambda-calculus, profinite monoids, Stone duality, regular languages

1 Introduction

In this paper, we formulate a notion of *profinite* λ -term which, as we will show, extends in a principled way, related to Reynolds parametricity, the important notion of profinite word found at the heart of automata theory.

Our starting point is provided by the Church encoding of finite words on a given finite alphabet $\Sigma = \{a_1, \dots, a_n\}$ into simply typed λ -terms. The idea of the encoding is to view every letter $a_i \in \Sigma$ as a variable a_i of type $\mathfrak{o} \Rightarrow \mathfrak{o}$ where \mathfrak{o} is an arbitrary base type. Once a variable $a_i : \mathfrak{o} \Rightarrow \mathfrak{o}$ has been declared in the context for each letter of Σ , a finite word $w = a_{w_1} \cdots a_{w_k} \in \Sigma^*$ can be naturally viewed as the composite $a_{w_k} \circ \cdots \circ a_{w_1}$ of type $\mathfrak{o} \Rightarrow \mathfrak{o}$. This composite is represented by the λ -term $\lambda c.a_{w_k}(\cdots(a_{w_1}c))$ where c is a variable of type \mathfrak{o} . The finite word w is thus encoded as the λ -term W defined as $\lambda a_1 \dots \lambda a_n.\lambda c.a_{w_k}(\cdots(a_{w_1}c))$ which is of type Church Σ , defined as

$$\underbrace{({ \hspace{-.6cm} \scriptscriptstyle 0} \hspace{-.6cm} \Rightarrow { \hspace{-.6cm} \scriptscriptstyle 0})}_{\text{type of } a_1} \Rightarrow \cdots \Rightarrow \underbrace{({ \hspace{-.6cm} \scriptscriptstyle 0} \hspace{-.6cm} \Rightarrow { \hspace{-.6cm} \scriptscriptstyle 0})}_{\text{type of } a_n} \Rightarrow \underbrace{ { \hspace{-.6cm} \scriptscriptstyle 0} \hspace{-.6cm} }_{\text{type of } c} \Rightarrow { \hspace{-.6cm} \scriptscriptstyle 0} \hspace{-.6cm} ,$$

where we have n occurrences of $\mathfrak{o} \Rightarrow \mathfrak{o}$, one for each letter $a_i \in \Sigma$, and one occurrence of \mathfrak{o} for the variable c, on the left of the base type \mathfrak{o} . Given a simple type A generated by the base type \mathfrak{o} , we write $\Lambda_{\beta\eta}\langle A\rangle$ for the set of closed λ -terms of type A, considered modulo β - and η -conversion. The Church encoding induces

a one-to-one correspondence

$$\Sigma^* \cong \Lambda_{\beta n} \langle \mathrm{Church}_{\Sigma} \rangle$$

between finite words on the alphabet Σ and simply typed λ -terms of type Church Σ up to $\beta\eta$ -equivalence. The correspondence allows us to think of finite words on the finite alphabet Σ as simply typed λ -terms of that specific type.

The finite set interpretation and deterministic automata

The connection between the Church encoding of finite words and automata theory has been considered in syntactic [17,32,24] and semantic [32,15,14,22] contexts.

Here, we follow the semantic track and focus on the finitary interpretation of the simply typed λ -calculus in the cartesian closed category **FinSet** of finite sets and functions between them, which, we claim, corresponds to deterministic finite state automata. In order to define this interpretation, we start by choosing a finite set Q which lets us define, for any simple type A, a finite set A in which we will interpret λ -terms of type A. This set A is inductively defined by

$$[\![\mathfrak{o}]\!]_Q \; := \; Q \quad \text{and} \quad [\![A \Rightarrow B]\!]_Q \; := \; [\![A]\!]_Q \Rightarrow [\![B]\!]_Q$$

where we interpret the functional type $A \Rightarrow B$ as the finite set of set-theoretic functions from the set $[\![A]\!]_Q$ to the set $[\![B]\!]_Q$. The interpretation then transports every simple type A to a finite set $[\![A]\!]_Q$ and every simply typed λ -term M of type

$$a_1:A_1,\ldots,a_n:A_n \vdash M:B$$

to a function between finite sets

$$\llbracket M \rrbracket_Q : \llbracket A_1 \rrbracket_Q \times \ldots \times \llbracket A_n \rrbracket_Q \longrightarrow \llbracket B \rrbracket_Q$$
.

This interpretation in **FinSet** induces a function $[\![-]\!]_Q:\Lambda_{\beta\eta}\langle A\rangle\longrightarrow [\![A]\!]_Q$ which is called the semantic bracket and transports every closed λ -term M of type A to its interpretation $[\![M]\!]_Q\in [\![A]\!]_Q$. In order to understand the connection with finite automata, it is instructive to examine how the interpretation acts on the open λ -term W encoding the finite word $w=a_{w_1}\dots a_{w_k}\in \Sigma^*$. By construction, the λ -term W is of type

$$a_1: \mathbb{O} \Rightarrow \mathbb{O}, \ldots, a_n: \mathbb{O} \Rightarrow \mathbb{O} \vdash W : \mathbb{O} \Rightarrow \mathbb{O}$$

where each letter $a_1, \ldots, a_n \in \Sigma$ appears as a variable of type $\mathfrak{o} \Rightarrow \mathfrak{o}$ in the context. The λ -term W is then interpreted as the functional

$$\llbracket W \rrbracket_Q : (Q \Rightarrow Q) \times \cdots \times (Q \Rightarrow Q) \longrightarrow (Q \Rightarrow Q)$$

which transports an *n*-tuple f_1, \ldots, f_n of endofunctions on the finite set Q to the composite endofunction $f_{w_k} \circ \cdots \circ f_{w_1}$ on the same finite set, that is,

$$[W]_O = f_1, \dots, f_n \mapsto f_{w_k} \circ \dots \circ f_{w_1} . \tag{1}$$

A deterministic finite state automaton on the alphabet $\Sigma = \{a_1, \ldots, a_n\}$ is defined as a tuple $\mathcal{A} = (Q, \delta, q_0, \mathsf{Acc})$ consisting of a finite set Q of states, a transition function $\delta : \Sigma \times Q \to Q$, an initial state $q_0 \in Q$ and a set of accepting states $\mathsf{Acc} \subseteq Q$. The transition function δ gives rise to a family of transition functions

$$\delta_{a_1} = \delta(a_1, -), \ldots, \delta_{a_n} = \delta(a_n, -) : Q \longrightarrow Q$$

where $\delta_a(q) = q'$ means that the automaton \mathcal{A} in state q transitions to the state q' when it encounters the letter $a \in \Sigma$.

Now observe that, if we apply the interpretation (1) of the simply typed λ -term W in **FinSet** to these transition functions $\delta_{a_1}, \ldots, \delta_{a_n}$, then we obtain the endofunction

$$\delta_w = [W]_Q(\delta_{a_1}, \dots, \delta_{a_n}) : Q \longrightarrow Q$$

which transforms each input state $q_0 \in Q$ into the output state $q_f = \delta_w(q_0) \in Q$ obtained by running the deterministic automaton \mathcal{A} on the finite word w encoded by the simply typed λ -term W. This simple observation establishes the connection between deterministic automata and the interpretation of simply typed λ -terms in **FinSet**.

In this way, if $\mathcal{A} = (Q, \delta, q_0, \mathsf{Acc})$ is a deterministic finite state automaton, then the tuple (Q, δ, q_0) induces an evaluation function

$$\operatorname{eval}_{(\delta,q_0)}$$
 : $[\![\operatorname{Church}_\Sigma]\!]_Q \longrightarrow Q$

which transports every functional $F \in [\![\operatorname{Church}_{\Sigma}]\!]_Q$ to the state $F(\delta_{a_1}, \ldots, \delta_{a_k})(q_0)$ in Q. Precomposing the evaluation function $\operatorname{eval}_{(\delta,q_0)}$ with the semantic bracket $[\![-]\!]_Q$ induces a composite function

$$\Sigma^* \cong \Lambda_{\beta\eta} \langle \operatorname{Church}_{\Sigma} \rangle \longrightarrow [\![\operatorname{Church}_{\Sigma}]\!]_Q \longrightarrow Q$$
 (2)

which associates a finite word $w \in \Sigma^*$ with the final state $q_f = \delta_w(q_0)$ returned by the automaton. The inverse image of the set $Acc \subseteq Q$ under this composite function is, by definition, the regular language L_A of finite words recognized by the deterministic automaton A.

The Boolean algebra $\operatorname{Reg}\langle A\rangle$ of regular languages

The regular language L_A described above is an element of the Boolean algebra $\operatorname{Reg}_Q\langle\operatorname{Church}_\Sigma\rangle$ of regular languages of λ -terms of type $\operatorname{Church}_\Sigma$ recognized by the finite set Q. This algebra may be defined as the image of the Boolean algebra homomorphism $[-]_Q^{-1}$ from $\wp([\operatorname{Church}_\Sigma]_Q)$ to $\wp(\Sigma^*)$, obtained by applying the contravariant power set functor $\wp: \operatorname{FinSet}^{\operatorname{op}} \longrightarrow \operatorname{BA}$ to the semantic bracket $[-]_Q$. In the theory of regular languages of simply typed λ -terms developed by Salvati [32,33], this point of view is extended to any type. The Boolean algebra $\operatorname{Reg}_Q\langle A\rangle$ of regular languages of λ -terms of higher-order type A recognizable by a finite set Q of states is defined as the image of the Boolean algebra homomorphism

$$\llbracket - \rrbracket_Q^{-1} \quad : \quad \wp(\llbracket A \rrbracket_Q) \quad \longrightarrow \quad \wp(\Lambda_{\beta\eta} \langle A \rangle) \ .$$

In other words, a set L of λ -terms of type A is recognizable by the finite set Q precisely when it is of the form $\llbracket - \rrbracket_Q^{-1}(\mathsf{Acc}) = \{M \in \Lambda_{\beta\eta}\langle A \rangle \mid \llbracket M \rrbracket_Q \in \mathsf{Acc}\}$ for some choice $\mathsf{Acc} \subseteq \llbracket A \rrbracket_Q$ of a set of accepting elements. Now, letting Q range over all finite sets, the collection $\mathrm{Reg}\langle A \rangle \subseteq \wp(\Lambda_{\beta\eta}\langle A \rangle)$ of regular languages of λ -terms of type A is defined in [32, Def. 1] as

$$\operatorname{Reg}\langle A \rangle = \bigcup \{ \operatorname{Reg}_Q \langle A \rangle \mid Q \text{ a finite set} \}.$$

Salvati [32, Thm. 8] then establishes that $\text{Reg}\langle A\rangle$ is a Boolean algebra, which boils down to the fact that $\text{Reg}\langle A\rangle$ is closed under intersection. The proof relies on a presentation of higher-order automata based on intersection types, and on the construction of a product higher-order automaton.

Profinite words in automata theory

The monoid $\widehat{\Sigma}^*$ of profinite words on a finite alphabet Σ plays an important role in automata theory, where profinite words encode the limiting behaviour of finite words with respect to deterministic finite automata [25]. The monoid $\widehat{\Sigma}^*$ is the free profinite monoid generated by Σ and can be constructed as the limit, computed in the category **Mon** of monoids, of the codirected (also known as *projective*) system of finite monoid homomorphisms

$$\left(\Sigma^*/\phi \longrightarrow \Sigma^*/\phi'\right)_{\phi \subset \phi'} \tag{3}$$

where ϕ and ϕ' range over the finite index congruences on Σ^* , subject to the condition that $\phi \subseteq \phi'$. Note that every such finite index congruence ϕ can be seen equivalently as a surjective homomorphism $h: \Sigma^* \to M$ to the finite monoid $M = \Sigma^*/\phi$ whose elements are the equivalence classes of the congruence ϕ . The surjectivity condition on h can be relaxed in order to show that the monoid $\widehat{\Sigma}^*$ of profinite words is in fact the codirected limit of the composite functor

$$\Sigma^*/\text{FinMon} \xrightarrow{\pi} \text{FinMon} \hookrightarrow \text{Mon}$$
 (4)

where **FinMon** denotes the category of finite monoids. Here, we use the notation $\Sigma^*/$ **FinMon** for the slice category whose objects (M, h) are the pairs consisting of a finite monoid M and of a (not necessarily surjective) homomorphism of the form $h: \Sigma^* \to M$, and whose morphisms $(M, h) \to (M', h')$ are the homomorphisms $f: M \to M'$ making the triangle

$$M \xrightarrow{f} M'$$

commute. The projection functor π in (4) transports (M,h) to the underlying finite monoid M. One obtains in this way $\widehat{\Sigma}^*$ as the limit of a codirected diagram of finite monoid homomorphisms

$$\left(M \longrightarrow M'\right)_{(M,h)\to(M',h')} \tag{5}$$

which extends the diagram (3) from finite index congruences ϕ on Σ^* to all homomorphisms $h: \Sigma^* \to M$ to a finite monoid M.

To explain the relationship with automata, recall that every homomorphism $h: \Sigma^* \to M$ to a finite monoid (M, \cdot_M, e_M) induces a deterministic finite automaton, by letting Q := M be the set of states, and defining $\delta(a,q) := q \cdot_M h(a)$ for every letter $a \in \Sigma$ and state $q \in M$. This establishes that every monoid homomorphism $h: \Sigma^* \to M$ to a finite monoid $M = \{q_1, \ldots, q_m\}$ induces a decomposition of Σ^* into m components $L_{q_i} = h^{-1}(q_i)$ for $1 \le i \le m$ where each L_{q_i} is a regular language. We will denote by $\operatorname{Reg}_{(M,h)}\langle \Sigma \rangle$ the Boolean algebra of languages generated by the regular languages of the form $L_q = h^{-1}(q)$, as q ranges over the elements of M. One obtains in this way a functor

$$\operatorname{Reg}_{(-)}\langle \Sigma \rangle : (\Sigma^*/\mathbf{FinMon})^{\operatorname{op}} \longrightarrow \mathbf{BA}$$
 (6)

to the category **BA** of Boolean algebras, which maps every pair (M,h) to the Boolean algebra $\operatorname{Reg}_{(M,h)}\langle\Sigma\rangle$. Note that this Boolean algebra $\operatorname{Reg}_{(M,h)}\langle\Sigma\rangle$ coincides with the image of the Boolean algebra homomorphism $h^{-1}:\wp(M)\longrightarrow\wp(\Sigma^*)$ obtained by applying the contravariant powerset functor \wp to the map $h\colon\Sigma^*\to M$. An important insight of [11, Sec. 4.2] is that the following directed diagram in **BA**, associated to the functor (6),

$$\left(\operatorname{Reg}_{(M',h')}\langle \Sigma \rangle \longrightarrow \operatorname{Reg}_{(M,h)}\langle \Sigma \rangle\right)_{(M,h)\to(M',h')}$$

may be obtained more directly by applying \wp to the codirected diagram of finite sets underlying (4) and (5). Since the colimit in **BA** of this diagram coincides with $\text{Reg}(\Sigma)$, one establishes in this way that the monoid of profinite words is in fact the Stone dual of the Boolean algebra $\text{Reg}(\Sigma)$ of regular sets, see [11] as well as §2 below for details.

From profinite words to profinite λ -terms

In order to define the notion of profinite λ -term at an arbitrary simple type A, we combine this general scheme with ideas coming from Reynolds parametricity. We have seen that, given a finite set Q, we can interpret any simple type A as a finite set $\llbracket A \rrbracket_Q$. To relate elements belonging to two different interpretations $\llbracket A \rrbracket_Q$ and $\llbracket A \rrbracket_{Q'}$ one can construct, given a relation $R \subseteq Q \times Q'$ between the finite sets used for the interpretation, a relation $\llbracket A \rrbracket_R \subseteq \llbracket A \rrbracket_Q \times \llbracket A \rrbracket_{Q'}$ between the two interpretations of the simple type A. Such inductively-defined relations are called logical relations. A fundamental fact is that λ -terms

are parametric, that is, for any λ -term M of type A and any relation $R \subseteq Q \times Q'$,

$$([\![M]\!]_Q, [\![M]\!]_{Q'}) \in [\![A]\!]_R$$
.

In particular, we will recall in Proposition B.4 the well-known fact that every partial surjection $f: Q \to Q'$, seen as a relation, induces a partial surjection $[\![A]\!]_f: [\![A]\!]_Q \to [\![A]\!]_{Q'}$ such that for every λ -term M of type A, its interpretation $[\![M]\!]_Q$ is in the domain of $[\![A]\!]_f$ and the partial surjection $[\![A]\!]_f$ sends the interpretation of M in the finite set $[\![A]\!]_Q$ to its interpretation in $[\![A]\!]_{Q'}$, that is,

$$[\![A]\!]_f([\![M]\!]_Q) = [\![M]\!]_{Q'}.$$
 (7)

An easy argument, given in Lemma 2.2 below, shows that, as a consequence, every partial surjection $f: Q \to Q'$ induces an inclusion of Boolean algebras $\operatorname{Reg}_{Q'}\langle A \rangle \subseteq \operatorname{Reg}_Q\langle A \rangle$. This leads us to the first main result of the paper, established in §2.

Theorem A. The diagram of Boolean algebras

$$\Big(\operatorname{Reg}_{Q'}\langle A\rangle \longrightarrow \operatorname{Reg}_{Q}\langle A\rangle\Big)_{f:Q\to Q'\in\mathbf{FinPSurj}}$$

indexed by the category **FinPSurj** of partial surjections between finite sets, is directed, and its colimit in **BA** coincides with the Boolean algebra $\text{Reg}\langle A \rangle$ of regular languages of higher-order type A.

At this stage, a key observation coming from Stone duality is that, for each finite set Q, the finite Boolean algebra $\text{Reg}_Q\langle A\rangle$ is join-generated by its finite set of atoms, which, as we will show in Proposition 3.1 below, is in bijection with the set

$$\llbracket A \rrbracket_Q^{\bullet} \ = \ \Big\{ \, \llbracket M \rrbracket_Q \mid M \in \Lambda_{\beta\eta} \langle A \rangle \, \Big\} \ \subseteq \ \llbracket A \rrbracket_Q$$

of definable elements in $[\![A]\!]_Q$. Moreover, using (7), we see that, for every partial surjection $f:Q \to Q'$, there exists a unique (total) surjection $[\![A]\!]_f^{\bullet}: [\![A]\!]_Q^{\bullet} \to [\![A]\!]_{Q'}^{\bullet}$ making the following diagram commute

in the category **FinPSet** of finite sets and partial functions. We are now ready to define the set $\widehat{\Lambda}_{\beta\eta}\langle A\rangle$ of profinite λ -terms of type A as the limit in the category **Set** of the codirected diagram of finite sets

$$\left(\ \llbracket A \rrbracket_f^{\bullet} \ : \ \llbracket A \rrbracket_Q^{\bullet} \longrightarrow \!\!\!\! \longrightarrow \!\!\!\! \longrightarrow \!\!\!\! \prod A \rrbracket_{Q'}^{\bullet} \right)_{f:Q \to Q' \in \mathbf{FinPSurj}}$$

indexed by partial surjections between finite sets. This diagram is dual to the directed diagram in **BA** defining the Boolean algebra $\operatorname{Reg}\langle A\rangle$ in Theorem A. Moreover, by Stone duality, the set $\widehat{\Lambda}_{\beta\eta}\langle A\rangle$ of profinite λ -terms of type A is not just a set, but a Stone space, dual to the Boolean algebra $\operatorname{Reg}\langle A\rangle$.

The conceptual definition of profinite λ -term which we have just given is nice but probably a little bit abstract to a reader with expertise in the λ -calculus but not necessarily in Stone duality. A more pedestrian way to understand it is to think of a profinite λ -term $\theta \in \widehat{\Lambda}_{\beta\eta}\langle A \rangle$ of type A as a family of definable elements $\theta_Q \in [\![A]\!]_Q^{\bullet}$ indexed by finite sets Q, such that the family θ is moreover natural with respect to finite partial surjections, in the expected sense that the equality $[\![A]\!]_f^{\bullet}(\theta_Q) = \theta_{Q'}$ holds for every partial surjection $f: Q \to Q'$ between finite sets.

Profinite λ -terms and Reynolds parametricity

The pedestrian definition of profinite λ -terms just given requires that the family of definable elements $\theta_Q \in \llbracket A \rrbracket_Q^{\bullet}$ is natural with respect to finite partial surjections, instead of asking the stronger property that the family θ is parametric in the traditional sense of Reynolds. We establish in §4 the important property that every profinite λ -term may be equivalently defined using parametricity instead of partial surjections, as follows:

Theorem B. A profinite λ -term $\theta \in \widehat{\Lambda}_{\beta\eta}\langle A \rangle$ of type A may be equivalently defined as a family of definable elements $\theta_Q \in [\![A]\!]_Q^{\bullet}$ indexed by finite sets Q, such that the family θ is moreover parametric with respect to any logical relation, in the sense that $(\theta_Q, \theta_{Q'}) \in [\![A]\!]_R$ for every relation $R \subseteq Q \times Q'$.

As we will see in §4, the fact that the notion based on parametricity is stronger than the notion based on naturality is easy to show. What is more difficult to establish that the two notions are in fact equivalent.

The cartesian closed category **ProLam** of profinite lambda-terms

We establish that the resulting notion of profinite λ -term is compositional by constructing a cartesian closed category **ProLam** of profinite λ -terms. There is a functor

idonobj :
$$Lam \longrightarrow ProLam$$

faithful functor by Statman's theorem, which embeds the simply typed λ -terms into profinite λ -terms. It associates to a simply typed λ -term M the profinite λ -term whose component at the finite set Q is the interpretation $[\![M]\!]_Q$.

Another interesting fact is that there exists, for every simple type A, a profinite λ -term defining a fixpoint operator

$$\Omega_A \in \widehat{\Lambda}_{\beta\eta}\langle (A \Rightarrow A) \Rightarrow (A \Rightarrow A) \rangle$$

which thus defines a morphism

$$\Omega_A : (A \Rightarrow A) \longrightarrow (A \Rightarrow A)$$

in the category **ProLam** of profinite λ -terms. The fixpoint operator Ω_A is similar in spirit but different in practice from the usual fixpoint operators $Y_A: (A \Rightarrow A) \Rightarrow A$ of Scott domain semantics, and one interesting direction for future work will be to understand how the two fixpoint operators Ω_A and Y_A are related.

We also establish at the end of the paper (see §7) that profinite λ -terms of type Church_{Σ} are the same thing as profinite words over the alphabet Σ in the traditional sense.

Theorem C. For every finite set Σ , there is a homeomorphism between the space of profinite λ -terms of type Church_{Σ} and the space of profinite words over Σ , that is,

$$\widehat{\Lambda}_{\beta\eta}\langle \mathrm{Church}_{\Sigma}\rangle \cong \widehat{\Sigma^*}$$
.

Related works

As explained in the introduction, our present definition of profinite λ -term relies on the notion of regular language of simply typed λ -terms introduced by Salvati [32]. Interestingly, the notion of regular language is formulated by Salvati in two different but equivalent ways. The first definition of regular language is based on the interpretation of λ -terms in the finite standard model **FinSet** of the simply typed λ -calculus. This is the definition which we recall and develop in the introduction and in the paper. The second equivalent definition given by Salvati relies on the construction of an intersection type system in direct correspondence with the finite monotone model of the simply typed λ -calculus constructed in the category **FinScott** of finite lattices and monotone maps between them, see [33] for a discussion. Aware of this correspondence with Scott semantics, Salvati and Walukiewicz actively promoted a semantic approach to higher-order model checking [34] which would complement the intersection type approach developed by Kobayashi and Ong [20,21]. However, besides the fascinating connections to Krivine environment machines and collapsible pushdown automata [35,16,7], it took several years for developing a precise connection between

Scott semantics and intersection type systems for higher-order model checking, with the emergence of a notion of higher-order parity automaton [22] founded on the discovery of an unexpected relationship with linear logic [8,9,15,14] combined with a comonadic translation designed by Melliès of the simply typed λY -calculus into a $\lambda Y_{\mu\nu}$ -calculus with inductive and coinductive fixpoints [22], or into a λY -calculus with priorities [38].

One fundamental idea which emerged from these works, also apparent in the work by Colcombet and Petrişan [10], is that there exists a correspondence between the specific category used for the semantic interpretation and a specific class of automata of interest. Typically, the interpretation of the simply typed λ -calculus in **FinSet** corresponds to the class of deterministic automata, while the interpretation in **FinScott** corresponds to the class of non-deterministic automata. In the present paper, we focus on the finite standard model in **FinSet**, and leave the investigation of the finite monotone lattice model in **Scott** for future works.

Another important line of work at the interface of automata theory and λ -calculus was initiated by Hillebrand et Kanellakis [17] with a purely syntactic description of regular languages of finite words using the Church encoding in the simply typed λ -calculus. This alternative approach is extremely promising and has seen a recent revival with the works by Nguyên and Pradic on implicit automata theory [24,23]. Our definition of profinite λ -term is formulated using the finite standard model, but it is largely independent of it, and it would thus be interesting to recast our definition of profinite λ -term in this purely syntactic framework.

In the study of regular languages and profinite monoids, the potential role of Stone duality was identified early on by Pippenger [26], and can also already be recognized in the "implicit operations" which were introduced by Reiterman [29] and play a role in Almeida's important work on profinite semigroups [3]. It is also interesting to note in this context that monoidal relations, under the name of "relational morphisms", have long played an important role in (pro)finite semigroup theory, as exemplified for example by Rhodes and Steinberg [31], and our crucial use of logical relations in this paper opens up potential new connections with that theory.

The specific methodology of understanding profinite algebraic structure by applying Stone duality to a lattice of regular languages that we closely follow in Sections 2 and 3 of this paper emerged from an influential series of works by Gehrke, Grigorieff and Pin [12,13], culminating in Gehrke's [11], which contains the most general account to date of that line of research. In a direction that is related to, but different from, the one pursued in this paper, Bojańczyk [6] generalized these profinite ideas to the category of algebras given by an arbitrary monad, also see the more recent work by Adámek et al. [1] pursuing a similar direction. While these works were always based in an algebraic setting, a novel contribution of this paper is to show how these ideas extend to the setting of the simply typed λ -calculus and cartesian closed categories.

Overview of the paper

We start by recalling in §2 the notion of regular language of λ -terms induced by the finite standard model of the simply typed λ -calculus. Then, as explained in the introduction, we establish in §2 that the Boolean algebra $\operatorname{Reg}\langle A\rangle$ of regular languages of simply typed λ -terms of type A formulated by Salvati can be equivalently expressed (Theorem A) as a colimit in **BA** of a specific directed diagram of finite Boolean algebras $\operatorname{Reg}_Q\langle A\rangle$. This leads us to introduce in §3 the set $\widehat{\Lambda}_{\beta\eta}\langle A\rangle$ of profinite λ -terms of type A, which we define as the limit in **Set** of a specific codirected diagram of finite sets $[\![A]\!]_Q^\bullet\subseteq [\![A]\!]_Q$. We also show that, by construction, the set of profinite λ -terms can be equipped with a natural topology which turns $\widehat{\Lambda}_{\beta\eta}\langle A\rangle$ into the Stone space dual to the Boolean algebra $\operatorname{Reg}_Q\langle A\rangle$. We establish in the next section §4 that profinite λ -terms can be defined in an alternative and more direct way as families of definable elements $\theta_Q\in [\![A]\!]_Q^\bullet$ satisfying a parametricity property with respect to any binary relation $R\subseteq Q\times Q'$. This is the essence of Theorem B mentioned in the introduction. We then show in §5 that the resulting notion of profinite λ -term is compositional in the technical sense that it defines a cartesian closed category **ProLam** whose objects are the simply types and whose morphisms are profinite λ -terms. Using Statman's theorem, we establish in §6 that the canonical functor from the category **Lam** of simply typed λ -terms to the category **ProLam** of profinite λ -terms is a faithful embedding. Finally, we establish in §7 our theorem (Theorem C) that given a finite alphabet Σ of letters, the notion of profinite λ -terms of type Church Σ coincides with the

usual notion of profinite words over Σ . We conclude and give a number of perspectives for future work in §8.

2 Regular languages of λ -terms

In this section, we define the collection $\operatorname{Reg}\langle A\rangle$ of regular languages at an arbitrary type A, and establish Theorem A of the introduction, showing how $\operatorname{Reg}\langle A\rangle$ can be built as the colimit of a directed diagram in the category of Boolean algebras.

Definition 2.1 Let Q be a finite set and A a type. We say that a subset $L \subseteq \Lambda_{\beta\eta}\langle A \rangle$ is a *regular language* of type A recognized by Q if there exists a subset Acc of $[\![A]\!]_Q$ such that $L = [\![-]\!]_Q^{-1}(A$ cc), that is,

for any
$$M \in \Lambda_{\beta\eta}\langle A \rangle$$
, $M \in L \iff [\![M]\!]_Q \in \mathsf{Acc}$.

We denote the Boolean algebra of regular languages of type A recognized by Q by $\operatorname{Reg}_Q\langle A\rangle\subseteq\wp(\Lambda_{\beta\eta}\langle A\rangle)$ and we write

$$\operatorname{Reg}\langle A \rangle = \bigcup \left\{ \operatorname{Reg}_Q\langle A \rangle \mid Q \text{ a finite set} \right\}$$

for the collection of regular languages of type A.

While it is clear that, for each individual finite set Q, the set $\operatorname{Reg}_Q\langle A\rangle$ is closed under the Boolean operations, since it is defined as the image of the Boolean homomorphism $[-]_Q^{-1}$, it is not immediately apparent that the union $\operatorname{Reg}\langle A\rangle$ is also closed under the Boolean operations. The following lemma contains the crucial argument needed to prove this fact.

Lemma 2.2 Let f: Q woheadrightarrow Q' be a partial surjection. Then, for any type A, we have an inclusion of Boolean algebras $\operatorname{Reg}_{Q'}\langle A \rangle \subseteq \operatorname{Reg}_Q\langle A \rangle$.

Proof. Let A be any type and let $\operatorname{Acc}' \subseteq \llbracket A \rrbracket_{Q'}$, recognizing the language $L = \llbracket - \rrbracket_{Q'}^{-1}(\operatorname{Acc}')$ in $\operatorname{Reg}_{Q'}\langle A \rangle$. We define the subset Acc of $\llbracket A \rrbracket_Q$ as $\llbracket A \rrbracket_f^{-1}(\operatorname{Acc}')$, that is $\{x \in \llbracket A \rrbracket_Q \mid x \llbracket A \rrbracket_f y \text{ for some } y \in \operatorname{Acc}'\}$. Then, for any term M of type A, we have $\llbracket M \rrbracket_Q \in \operatorname{Acc}$ if and only if $\llbracket M \rrbracket_{Q'} \in \operatorname{Acc}'$ as follows by Proposition B.5. We conclude that L is equal to $\llbracket - \rrbracket_Q^{-1}(\operatorname{Acc})$, so that L is also recognized by Q.

In order to prove that the union $\operatorname{Reg}\langle A\rangle$ of the Boolean algebras $\operatorname{Reg}_Q\langle A\rangle$ is again a Boolean algebra, we will apply the following general principle from universal algebra in the case where **V** is the variety of Boolean algebras. We give a proof in Appendix C; also see, for example, [2, Rem. 3.4.4(iii) on p. 136].

Proposition 2.3 For any finitary variety of algebras V, the forgetful functor $V \to \mathbf{Set}$ creates directed colimits.

We are now ready to prove our first main result, Theorem A of the introduction.

Theorem 2.4 (Theorem A) The diagram of Boolean algebras

$$\left(\operatorname{Reg}_{Q'}\langle A \rangle \longrightarrow \operatorname{Reg}_{Q}\langle A \rangle \right)_{f:Q \to Q' \in \mathbf{FinPSurj}}$$

indexed by the category **FinPSurj** of partial surjections between finite sets, is directed, and its colimit in **BA** coincides with the Boolean algebra $\text{Reg}\langle A \rangle$ of regular languages of higher-order type A.

Proof. We first show that the diagram of inclusions of Boolean algebras is directed. Indeed, for any finite sets Q_1 and Q_2 , we have, for i=1,2, the partial surjection $f_i\colon Q_1+Q_2 \twoheadrightarrow Q_i$ defined by $q\,f_i\,q'$ if and only if $q\in Q_i$ and q=q'. Thus, Lemma 2.2 gives that $\mathrm{Reg}_{Q_i}\langle A\rangle\subseteq\mathrm{Reg}_{Q_1+Q_2}\langle A\rangle$. Now, by Proposition 2.3, applied in the case $\mathbf{V}=\mathbf{B}\mathbf{A}$, the union $\mathrm{Reg}\langle A\rangle$ of the sets in the diagram is again a Boolean algebra, and it is the colimit of the diagram in $\mathbf{B}\mathbf{A}$.

We end this section by showing explicitly how we recover in this context the result of [32, Thm. 8] that $\text{Reg}\langle A\rangle$ is closed under binary intersection.

Proposition 2.5 For any type A, the set of regular languages $\operatorname{Reg}\langle A\rangle \subseteq \wp(\Lambda_{\beta\eta}\langle A\rangle)$ is closed under binary intersection.

Proof. Suppose that $L_1 \in \operatorname{Reg}_{Q_1}\langle A \rangle$ and $L_2 \in \operatorname{Reg}_{Q_2}\langle A \rangle$. By the argument given in the proof of Theorem A, both L_1 and L_2 are in $\operatorname{Reg}_{Q_1+Q_2}\langle A \rangle$ which is a Boolean algebra, so their intersection $L_1 \cap L_2$ is also in $\operatorname{Reg}_{Q_1+Q_2}\langle A \rangle$.

3 The space of profinite λ -terms

The aim of this section is to define profinite λ -terms of an arbitrary simple type A as special parametric families of semantic elements, and to show that they form a Stone space dual to the Boolean algebra $\text{Reg}\langle A \rangle$; for a recap of the basics of Stone duality that we will use below, see Appendix D.

Throughout this section, we fix a simple type A. We saw in the previous section that $\operatorname{Reg}\langle A \rangle$ is a Boolean algebra which is the colimit of a directed diagram of inclusions between the Boolean algebras $\operatorname{Reg}_Q\langle A \rangle$. As $\operatorname{Reg}_Q\langle A \rangle$ is finite for every Q, it is isomorphic to $\wp(X_Q(A))$, where $X_Q(A)$ is the set of atoms of $\operatorname{Reg}_Q\langle A \rangle$. Applying discrete Stone duality to the directed diagram of inclusions of finite Boolean algebras, we thus obtain a codirected diagram of maps $X_Q(A) \twoheadrightarrow X_{Q'}(A)$, still indexed by partial surjections $Q \twoheadrightarrow Q'$. We now first give a more concrete description of that diagram.

Proposition 3.1 For every finite set Q, the set of atoms $X_Q(A)$ of $\operatorname{Reg}_Q\langle A\rangle$ is in a bijection with the set $[\![A]\!]_Q^{\bullet}$ of definable elements of type A, given by the function

$$\begin{bmatrix} A \end{bmatrix}_Q^{\bullet} \longrightarrow X_Q(A)
q \longmapsto \llbracket - \rrbracket_Q^{-1}(\{q\}) .$$

Proof. The Boolean algebra $\operatorname{Reg}_Q\langle A\rangle$ is, by definition, the image of the Boolean algebra homomorphism $\llbracket - \rrbracket_Q^{-1} : \wp(\llbracket A \rrbracket_Q) \to \wp(\Lambda_{\beta\eta}\langle A\rangle)$ Thus, $\operatorname{Reg}_Q\langle A\rangle$ arises as the following epi-mono factorization of Boolean algebras

$$\wp(\llbracket A \rrbracket_Q) \xrightarrow{\llbracket - \rrbracket_Q^{-1}} \wp(\Lambda_{\beta\eta} \langle A \rangle)$$

$$\mathbb{R}eg_Q \langle A \rangle$$

Applying the discrete duality functor $At: CABA \rightarrow Set$ to this diagram, we get the dual epi-mono factorization of sets

$$[\![A]\!]_Q \xleftarrow{[\![-]\!]_Q} \Lambda_{\beta\eta}\langle A \rangle$$

$$X_Q(A)$$

Since $[\![A]\!]_Q^{\bullet}$ is by definition the image of $[\![-]\!]_Q$ in $[\![A]\!]_Q$, the result follows by the uniqueness up to isomorphism of epi-mono factorizations in **CABA**.

Definition 3.2 Let A be any simple type. We define the set of *profinite* λ -terms as the limit $\widehat{\Lambda}_{\beta\eta}\langle A\rangle$ in **Set** of the diagram

$$\left(\ \llbracket A \rrbracket_f^{\bullet} \ : \ \llbracket A \rrbracket_Q^{\bullet} \longrightarrow \!\!\!\! \longrightarrow \!\!\!\! \longrightarrow \!\!\!\! \prod A \rrbracket_{Q'}^{\bullet} \right)_{f:Q \to Q' \in \mathbf{FinPSurj}}$$

It follows from the way that one calculates limits in **Set** that, concretely, a profinite λ -term in $\widehat{\Lambda}_{\beta\eta}\langle A\rangle$ is a family θ of definable elements $\theta_Q \in [\![A]\!]_Q^{\bullet}$ where Q ranges over all finite sets such that

for every partial surjection
$$f: Q \to Q'$$
, we have $[A]_f^{\bullet}(\theta_Q) = \theta_{Q'}$. (8)

Note that, by Proposition B.5, the condition (8) on the family θ is equivalent to the condition that, for any term M of type A and any finite sets Q and Q',

if
$$\theta_Q = [\![M]\!]_Q$$
 and $|Q| \ge |Q'|$ then $\theta_{Q'} = [\![M]\!]_{Q'}$. (9)

We conclude this section by equipping the set $\widehat{\Lambda}_{\beta\eta}\langle A\rangle$ with a natural topology, and showing that this topology turns $\widehat{\Lambda}_{\beta\eta}\langle A\rangle$ into the Stone dual space of the Boolean algebra $\operatorname{Reg}\langle A\rangle$. The easiest way to define the topology of $\widehat{\Lambda}_{\beta\eta}\langle A\rangle$ is to say that it is the subspace topology inherited from the inclusion

$$\widehat{\Lambda}_{\beta\eta}\langle A\rangle \longrightarrow \prod_{Q} \llbracket A \rrbracket_{Q}^{\bullet}$$

into the product space $\prod_Q \llbracket A \rrbracket_Q^{\bullet}$ computed in the category **Top** of topological spaces, where each component $\llbracket A \rrbracket_Q^{\bullet}$ is considered as a topological space equipped with the discrete topology. More concretely, for any finite set Q and $q \in \llbracket A \rrbracket_Q^{\bullet}$, let us write $U_{Q,q}$ for the set of profinite λ -terms that take value q at Q, that is,

$$U_{Q,q} := \left\{ \quad \theta \in \widehat{\Lambda}_{\beta\eta} \langle A \rangle \mid \theta_Q = q \quad \right\} .$$

The topology on $\widehat{\Lambda}_{\beta\eta}\langle A\rangle$ is now defined by taking the collection of sets $U_{Q,q}$ as a basis, where Q ranges over all finite sets and q ranges over all the elements of Q. The following result is proved via an argument similar to the one given in [11, Sec. 4.2] for profinite algebras.

Proposition 3.3 The space $\widehat{\Lambda}_{\beta\eta}\langle A \rangle$ is the Stone dual space of the Boolean algebra $\operatorname{Reg}\langle A \rangle$. In particular, $\operatorname{Reg}\langle A \rangle$ is isomorphic to the Boolean algebra of clopen sets of $\widehat{\Lambda}_{\beta\eta}\langle A \rangle$.

Proof. As recalled in Appendix D, Stone duality arises from the dual equivalence between **FinSet** and **FinBA** by taking the projective and inductive completions, respectively. Therefore, we have in particular that $\widehat{\Lambda}_{\beta\eta}\langle A\rangle$, which is defined as the codirected limit of the diagram of finite discrete spaces $[\![A]\!]_Q^{\bullet}$ in **Top**, is the dual space of the directed colimit of the diagram of finite Boolean algebras $\operatorname{Reg}_Q\langle A\rangle$, which is the Boolean algebra $\operatorname{Reg}_Q\langle A\rangle$ by Theorem A. The second statement now follows because any Boolean algebra is isomorphic to the collection of clopen sets of its dual space.

4 Profinite λ -terms and parametricity

Let A be any simple type. A parametric family is a family of points $\theta_Q \in [\![A]\!]_Q$, where Q ranges over all finite sets, such that for any relation $R \subseteq Q \times Q'$, we have $\theta_Q [\![A]\!]_R \theta_{Q'}$.

Every parametric family θ is in particular parametric with respect to partial surjections. Therefore, a parametric family whose component are definable elements is a profinite λ -term. We now show that the converse holds.

Theorem 4.1 (Theorem B) A profinite λ -term $\theta \in \widehat{\Lambda}_{\beta\eta}\langle A \rangle$ of type A may be equivalently defined as a parametric family of definable elements $\theta_Q \in \llbracket A \rrbracket_Q^{\bullet}$.

Proof. Let θ be a profinite λ -term, viewed as a family of definable elemens which is parametric with respect to every partial surjection, or equivalently, satisfying condition (9). Let Q_1 and Q_2 be any two finite sets and let $R \subseteq Q_1 \times Q_2$ be any relation. Pick any finite set Q of cardinality $\max(|Q_1|, |Q_2|)$. Since θ_Q is in particular definable, pick a λ -term M in $\Lambda_{\beta\eta}\langle A\rangle$ such that θ_Q is $[M]_Q$. Since $|Q| \ge |Q_i|$ for i = 1, 2, by (9) we now also have θ_{Q_i} is equal to $[M]_{Q_i}$. By Proposition B.1, we obtain that $[M]_{Q_1}$ $[A]_R$ $[M]_{Q_2}$ which proves that θ is a parametric family.

5 The cartesian closed category of profinite λ -terms

We now show that profinite λ -terms assemble into a cartesian closed category **ProLam** which thus provides an interpretation of the simply typed λ -calculus. In order to construct the category **ProLam**, we find it

convenient to use a general construction introduced by Jacq and Melliès [18] in a more general monoidal and 2-categorical setting. Suppose given a cartesian closed category C and a functor

$$\mathcal{P} : \mathbf{C} \longrightarrow \mathbf{Set}$$

which is *cartesian product preserving* in the sense that the canonical functions

$$\begin{array}{cccc} \langle \mathcal{P}(\pi_1), \mathcal{P}(\pi_2) \rangle & : & \mathcal{P}(A \times B) & \longrightarrow & \mathcal{P}(A) \times \mathcal{P}(B) \\ & !_{\mathcal{P}(1)} & : & \mathcal{P}(1) & \longrightarrow & 1 \end{array}$$

are bijections for all objects A and B of the category \mathbf{C} . We denote by

$$m_{A,B}$$
 : $\mathcal{P}(A) \times \mathcal{P}(B) \longrightarrow \mathcal{P}(A \times B)$
 m_1 : $1 \longrightarrow \mathcal{P}(1)$

the inverse functions. In that situation, one defines the category $\mathbf{C}[\mathcal{P}]$ whose objects are the objects of \mathbf{C} and whose hom-sets are defined as follows:

$$\mathbf{C}[\mathcal{P}](A,B) := \mathcal{P}(A \Rightarrow B)$$

using the internal hom-object $A \Rightarrow B$ of the cartesian closed category C. One establishes that

Proposition 5.1 The category C[P] is cartesian closed and comes equipped with a cartesian closed identity-on-object functor

$$idonobj_{\mathbf{C},\mathcal{P}} : \mathbf{C} \longrightarrow \mathbf{C}[\mathcal{P}]$$

which strictly preserves the cartesian product as well as the internal hom.

The interested reader will find the proof of Proposition 5.1 in Appendix E. Now, in order to obtain the category **ProLam** of profinite λ -terms using this categorical construction, we start by recalling the definition of the cartesian closed category **Lam** freely generated by the terminal category.

Definition 5.2 The category Lam has as objects the simple types of the λ -calculus and its hom-sets are defined as

$$\mathbf{Lam}(A,B) := \Lambda_{\beta\eta} \langle A \Rightarrow B \rangle$$

for all pairs A and B of simple types.

At this stage, we are ready to consider the functor $\mathcal{P}: \mathbf{Lam} \longrightarrow \mathbf{Set}$ which transports every simple type A to the set of profinite λ -terms

$$\mathcal{P}(A) = \widehat{\Lambda}_{\beta\eta} \langle A \rangle \tag{10}$$

and every λ -term M of type $A \Rightarrow B$ to the set-theoretic function sending a profinite λ -term θ of type A on the family ($[\![M]\!]_Q(\theta_Q)$) which is a profinite λ -term of type B. It is interesting to observe that the functor $\mathcal P$ is cartesian product preserving and that we have canonical bijections

$$\mathfrak{P}(A \times B) \cong \mathfrak{P}(A) \times \mathfrak{P}(B)$$
 $\mathfrak{P}(1) \cong 1$

for every pair of simple types A and B. By applying the construction, we obtain a cartesian closed category

$$\mathbf{ProLam} := \mathbf{Lam}[\mathfrak{P}]$$

whose objects are the simple types of the λ -calculus and whose hom-sets are defined as follows:

$$\mathbf{ProLam}(A,B) := \widehat{\Lambda}_{\beta\eta} \langle A \Rightarrow B \rangle$$
.

Remark 5.3 Note that the functors \mathcal{P} are chosen to be valued in **Set**, but we could choose any cartesian category **S** as long as \mathcal{P} still is a cartesian product preserving relatively to the cartesian structure of **S**. The construction will then yield a cartesian closed category $\mathbf{C}[\mathcal{P}]$ enriched over **S**. As a matter of fact, the functor $\mathcal{P}: \mathbf{Lam} \to \mathbf{Set}$ used in (10) to construct $\mathbf{ProLam} = \mathbf{Lam}[\mathcal{P}]$ happens to factor through the category **Stone** of Stone spaces, in the following way:

$$\mathbf{Lam} \xrightarrow{\widehat{\Lambda}_{\beta\eta}\langle -\rangle} \mathbf{Stone} \xrightarrow{\quad \text{forget} \quad} \mathbf{Set}$$

This shows that the cartesian closed category **ProLam** of profinite λ -terms may be also considered as enriched over the category **Stone** of Stone spaces.

6 A faithful embedding from λ -terms to profinite λ -terms

By construction, the category **ProLam** comes equipped with a cartesian closed identity-on-object functor

$$idonobj : Lam \longrightarrow ProLam$$
 (11)

which may also be derived from the fact that **Lam** is the free cartesian closed category. We now establish that

Proposition 6.1 The functor idonobj is faithful.

Towards proving Proposition 6.1, we first claim that the category **ProLam** can be obtained as the limit of a codirected diagram of cartesian closed categories, described in the following way. Given a finite set Q and a simple type A, consider the equivalence relation

$$\sim_O^A \subseteq \Lambda_{\beta\eta}\langle A \rangle \times \Lambda_{\beta\eta}\langle A \rangle$$

on the set of simply typed λ -terms of type A modulo $\beta\eta$ -conversion, defined as:

$$M \sim_Q^A N \iff [M]_Q = [N]_Q$$
.

When we fix the finite set Q, the family \sim_Q of equivalence relations \sim_Q^A parametrized by simple types A defines a congruence relation on the category **Lam**, in the expected sense that

$$\text{if } f \sim_Q^{A \Rightarrow B} f' \quad \text{and} \quad g \sim_Q^{B \Rightarrow C} g', \quad \text{then} \quad g \circ f \sim_Q^{A \Rightarrow C} g' \circ f'$$

for any tuple of morphisms f, f', g, g' of the form:

$$A \xrightarrow{f} B \xrightarrow{g} C \quad .$$

From this, it follows that we can define the category

$$\mathbf{Lam}_Q := \mathbf{Lam} / \sim_Q$$

obtained by considering the morphisms of the free cartesian closed category Lam modulo the congruence relation \sim_O in the expected sense that

$$\mathbf{Lam}_{Q}(A,B) = \Lambda_{\beta\eta}\langle A \Rightarrow B \rangle / \sim_{Q}^{A \Rightarrow B} . \tag{12}$$

We then establish (see Appendix E for the proof) that

Proposition 6.2 For every finite set Q, the category \mathbf{Lam}_Q is cartesian closed and comes equipped with a cartesian closed identity-on-object functor

$$\pi_O$$
: Lam \longrightarrow Lam_O.

Moreover, every partial surjection $f:Q \rightarrow Q'$ in the category **FinPSurj** induces a cartesian closed identity-on-object functor

$$\operatorname{Lam}_f : \operatorname{Lam}_Q \longrightarrow \operatorname{Lam}_{Q'}$$

making the diagram of cartesian closed functors commute:

$$egin{aligned} egin{aligned} \pi_Q & \mathbf{Lam}_Q \ & \mathbf{Lam}_f \ & \mathbf{Lam}_{Q'} \end{aligned}$$

From this observation, it is not too difficult to show that

Proposition 6.3 The category **ProLam** is the codirected limit of the diagram of cartesian closed categories Lam_Q indexed by finite sets and partial surjections. The projection functor

$$\pi_Q$$
: **ProLam** \longrightarrow **Lam**_Q

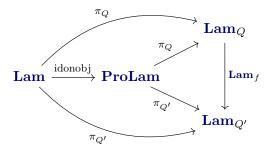
is defined by transporting every morphism $\theta: A \to B$ defined as a family θ of definable elements in

$$\mathbf{ProLam}(A,B) = \widehat{\Lambda}_{\beta\eta} \langle A \Rightarrow B \rangle$$

to the instance θ_Q in

$$\mathbf{Lam}_Q(A,B) \,=\, \Lambda_{\beta\eta}\langle A\Rightarrow B\rangle\,/\sim_Q^{A\Rightarrow B} \,\,\cong\, \llbracket A\Rightarrow B\rrbracket_Q^{\bullet} \,\,.$$

One also establishes that the canonical functor (11) is also characterized by the fact that it is the unique cartesian closed functor making the diagram below commute:



This observation provides us with a clean proof that the functor idonobj is faithful. Indeed, by Statman's finite completeness theorem [36], for every pair of morphisms $f, g : A \rightrightarrows B$ in the category **Lam** which is (by definition) a pair of λ -terms M and N modulo $\beta\eta$ -conversion, either M and N are equal modulo $\beta\eta$ -conversion or there exists a finite set Q such that the interpretations $[\![M]\!]_Q = \pi_Q(M)$ and $[\![N]\!]_Q = \pi_Q(N)$ are different. In particular, idonobj(f) and idonobj(g) differ in the second case. This establishes that the canonical functor idonobj : **Lam** \to **ProLam** is faithful, as claimed in Proposition 6.1.

7 Profinite λ -terms and profinite words

The higher-order language theory on simply typed λ -terms is designed to extend the traditional language theory on words on a given finite alphabet Σ . The idea is that a finite word on the alphabet Σ is the same thing as a λ -term of type Church Σ modulo $\beta\eta$ -conversion. In particular, we recall below a folklore result which states that the Boolean algebra Reg $\langle \text{Church}_{\Sigma} \rangle$ of regular higher-order languages on Church Σ coincides with the Boolean algebra Reg $\langle \Sigma \rangle$ of regular languages on the finite alphabet Σ .

Proposition 7.1 For every finite alphabet Σ , one has an isomorphism of Boolean algebra

$$\operatorname{Reg}\langle \operatorname{Church}_{\Sigma} \rangle \cong \operatorname{Reg}\langle \Sigma \rangle$$

given by the Church encoding.

Proof. The Church encoding provides a one-to-one correspondence between subsets $L \subseteq \Sigma^*$ of words over the alphabet Σ and subsets $\mathcal{L} \subseteq \Lambda_{\beta\eta}\langle \mathrm{Church}_{\Sigma} \rangle$ of λ -terms of type Church_{Σ} closed modulo $\beta\eta$ -conversion. We show that a subset $L \subseteq \Sigma^*$ is regular if and only if the associated subset $\mathcal{L} \subseteq \Lambda_{\beta\eta}\langle \mathrm{Church}_{\Sigma} \rangle$ is an element of $\mathrm{Reg}\langle \mathrm{Church}_{\Sigma} \rangle$.

In one direction, suppose that $L \subseteq \Sigma^*$ is a language of words recognized by a DFA $\mathcal{A} = (Q, \delta, q_0, \mathsf{Acc})$. We recall from the introduction that the associated set $\mathcal{L} \subseteq \Lambda_{\beta\eta}\langle \mathsf{Church}_{\Sigma} \rangle$ of λ -terms is the inverse image by the semantic bracket

$$\llbracket - \rrbracket_Q : \Lambda_{\beta\eta} \langle \operatorname{Church}_{\Sigma} \rangle \longrightarrow \llbracket \operatorname{Church}_{\Sigma} \rrbracket_Q$$

of the set of functionals in $[\![\operatorname{Church}_{\Sigma}]\!]_Q$ defined as follows

$$\operatorname{eval}_{(\delta,q_0)}^{-1}(\operatorname{Acc}) = \{ F \in [\![\operatorname{Church}_{\Sigma}]\!]_Q \mid F(\delta_{a_1},\ldots,\delta_{a_n})(q_0) \in \operatorname{Acc} \} \ .$$

By definition, $\mathcal{L} \subseteq \Lambda_{\beta\eta}\langle \mathrm{Church}_{\Sigma} \rangle$ is thus an element of $\mathrm{Reg}_{Q}\langle \mathrm{Church}_{\Sigma} \rangle$ and thus an element of $\mathrm{Reg}\langle \mathrm{Church}_{\Sigma} \rangle$.

Conversely, by definition of $\operatorname{Reg}\langle\operatorname{Church}_{\Sigma}\rangle$, it is sufficient to establish, for every finite set Q, that every subset $\mathcal{L} \in \operatorname{Reg}_Q\langle\operatorname{Church}_{\Sigma}\rangle$ has its corresponding subset $L \subseteq \Sigma^*$ a regular language. By definition of $\operatorname{Reg}_Q\langle\operatorname{Church}_{\Sigma}\rangle$, we may suppose without loss of generality that $\mathcal{L} \in \operatorname{Reg}_Q\langle\operatorname{Church}_{\Sigma}\rangle$ is of the form

$$\mathcal{L} = \llbracket - \rrbracket_Q^{-1}(\{F\}) = \{ M \in \Lambda_{\beta\eta} \langle \mathrm{Church}_{\Sigma} \rangle \mid \llbracket M \rrbracket_Q = F \}$$

where F is a functional in $[\![Church_{\Sigma}]\!]_Q$. The corresponding set $L \subseteq \Sigma^*$ is the finite intersection of all the regular languages recognized by the DFAs of the form $\mathcal{A} = (Q, \delta, q_0, \{q_f\})$ where the unique final state q_f is equal to $q_f = F(\delta_{a_1}, \ldots, \delta_{a_n})(q_0)$. As a finite intersection of regular languages, the set $L \subseteq \Sigma^*$ is itself regular.

We use this result in order to establish our Theorem C.

Theorem 7.2 (Theorem C) For every finite alphabet Σ , there is a homeomorphism

$$\widehat{\Lambda}_{\beta\eta}\langle \mathrm{Church}_{\Sigma}\rangle \cong \widehat{\Sigma}^*$$

between the space $\widehat{\Lambda}_{\beta\eta}\langle \mathrm{Church}_{\Sigma}\rangle$ of profinite λ -terms of type Church_{Σ} and the space $\widehat{\Sigma}^*$ of profinite words.

Proof. By Proposition 3.3, the space $\widehat{\Lambda}_{\beta\eta}\langle \text{Church}_{\Sigma}\rangle$ is the Stone dual of the Boolean algebra $\text{Reg}\langle \text{Church}_{\Sigma}\rangle$ which is isomorphic to the Boolean algebra $\text{Reg}\langle \Sigma\rangle$ by Proposition 7.1. From this follows that the space $\widehat{\Lambda}_{\beta\eta}\langle \text{Church}_{\Sigma}\rangle$ is homeomorphic to the Stone dual of $\text{Reg}\langle \Sigma\rangle$ which coincides with the space $\widehat{\Sigma}^*$ of profinite words by an important result of Stone duality, see [25].

One main benefit of extending finite words into profinite words is that a new class of implicit operations become available [4]. In particular, there exists an *idempotent power operator* $u \mapsto u^{\omega}$ which turns every profinite word u into another profinite word noted u^{ω} , and defines a continuous function

$$u \longmapsto u^{\omega} : \widehat{\Sigma}^* \longrightarrow \widehat{\Sigma}^*$$
 (13)

see for example [25, Prop. 2.5]. The construction is based on the observation that for any element x of a finite monoid M, there exists a unique power x^n of x, for $n \ge 1$, which is idempotent. This unique power is obtained when n is the factorial of the cardinality of M, and is also usually written x^{ω} . The continuous function (13) is obtained by taking the profinite limit of this operation on monoids. We show the construction generalizes from profinite words to profinite λ -terms at every type A.

Proposition 7.3 For every simple type A, there exists a profinite λ -term

$$\Omega_A : (A \Rightarrow A) \Rightarrow A \Rightarrow A$$
 (14)

which, given any $M \in \widehat{\Lambda}_{\beta\eta}\langle A \Rightarrow A \rangle$, satisfies the idempotency equation

$$(\Omega_A M) \circ (\Omega_A M) = \Omega_A M$$

between profinite λ -terms, where $g \circ f$ is notation for $\lambda x.f(gx)$ where f and g are profinite λ -terms.

We have seen in Theorem C that one recovers the traditional notion of profinite words on a finite alphabet Σ by considering the profinite λ -terms of type Church $_{\Sigma}$. Accordingly, the continuous operation (13) can be recovered as the profinite λ -term

$$\lambda u.\lambda f_1...f_n.\Omega_{\odot}(u f_1...f_n)$$
 : Church_{\Sigma} \Rightarrow Church_{\Sigma}

where $\Omega_{\mathfrak{o}}: (\mathfrak{o} \Rightarrow \mathfrak{o}) \Rightarrow (\mathfrak{o} \Rightarrow \mathfrak{o})$ denotes the idempotent power operator Ω_A at type $A = \mathfrak{o}$. Note that we use the compositional calculus provided by the cartesian closed category **ProLam** in order to see the expression $\lambda u.\lambda f_1...f_n.\Omega_{\mathfrak{o}}(u f_1...f_n)$ as a profinite λ -term.

8 Conclusion

In this paper, we introduce the notion of profinite λ -term of a given simple type which we define in a clean and principled way by establishing in Theorem A and Theorem B that the definitions based on duality theory and on parametricity coincide. We also establish in Theorem C that the Church encoding of finite words as λ -terms extends to profinite words, in the sense that the usual notion of profinite word on a finite alphabet Σ coincides with the notion of profinite λ -term on the type Church Σ encoding the alphabet Σ . We also construct a cartesian closed category **ProLam** of profinite λ -terms, and construct a cartesian closed functor

idonobj :
$$Lam \longrightarrow ProLam$$

from the cartesian closed category of usual simply typed λ -terms. We also show that the embedding functor from simply typed λ -terms to profinite λ -terms is faithful, using Statman's theorem. The construction shows that simply typed λ -terms can be considered as particular profinite λ -terms, and that profinite λ -terms can be manipulated in the same compositional way as usual simply typed λ -terms.

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A Background on syntax and semantics

Syntax

We consider in this paper the simply typed λ -calculus with product types over a single base type \mathfrak{o} . That is, a *simple type* is an expression generated by the grammar

$$A, B ::= 0 \mid A \Rightarrow B \mid A \times B \mid 1$$

and a λ -term is an expression generated by the grammar

$$M, N ::= x \mid \lambda(x : A).M \mid MN \mid \langle M, N \rangle \mid M.i \mid ()$$

where x is a variable, A is a simple type and i = 1, 2. Finally, a typing judgment is an expression of the form

$$x_1:A_1,\ldots,x_n:A_n\vdash M:B$$

where each x_i is a variable, M is a λ -term, and each A_i and B are simple types.

The inductive rules defining a *valid typing judgment*, where $\Gamma = (x_1 : A_1, \dots, x_n : A_n)$, are the following:

$$\frac{\Gamma \vdash x_i : A_i \text{ for } 1 \leq i \leq n}{\Gamma \vdash x_i : A_i} \frac{\Gamma \vdash M : A \Rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B} \frac{\Gamma \vdash N : A}{\Gamma \vdash \lambda(x : A).M : A \Rightarrow B}$$

$$\frac{\Gamma \vdash M_1 : A_1 \quad \Gamma \vdash M_2 : A_2}{\Gamma \vdash \langle M_1, M_2 \rangle : A_1 \times A_2} \frac{\Gamma \vdash M : A_1 \times A_2}{\Gamma \vdash M.i : A_i} \text{ for } i = 1, 2$$

$$\frac{\Gamma \vdash M : A_1 \times A_2}{\Gamma \vdash M.i : A_i} \text{ for } i = 1, 2$$

Note that for any λ -term M, there is at most one type A such that $\varnothing \vdash M : A$ is a valid typing judgment. When it exists, we say that the λ -term M has type A, and call this type A the type of the λ -term M.

We define inductively a family of congruences depending on a simple type and a context. The relation $=_{\beta\eta}$ associated to a simple type A and a context Γ is written as a judgment

$$\Gamma \vdash M =_{\beta \eta} N : A$$

which is inductively defined using the following rules, divided into the four following categories.

• β -reductions, i.e. computations

$$\frac{\Gamma, x: A \vdash M: B \quad \Gamma \vdash N: A}{\Gamma \vdash (\lambda(x:A).M)N =_{\beta\eta} M[x:=N]: B} \qquad \frac{\Gamma \vdash M_1: A_1 \quad \Gamma \vdash M_2: A_2}{\Gamma \vdash \langle M_1, M_2 \rangle. i =_{\beta\eta} M_i: A_i} \text{ for } i = 1, 2$$

where M[x := N] is the substitution of N for x in M, without capture of free variables.

• η -expansions, i.e. extensions

$$\frac{\Gamma \vdash M : A \Rightarrow B}{\Gamma \vdash M =_{\beta\eta} \lambda(x : A).M \ x : A \Rightarrow B} \ (x : A) \not \in \Gamma \qquad \frac{\Gamma \vdash M : A_1 \times A_2}{\Gamma \vdash M =_{\beta\eta} \langle M.1, M.2 \rangle : A_1 \times A_2} \qquad \frac{\Gamma \vdash M : 1}{\Gamma \vdash M =_{\beta\eta} () : 1}$$

• Congruence rules

$$\frac{\Gamma \vdash X_{i} =_{\beta\eta} X_{i} : A}{\Gamma \vdash X_{i} =_{\beta\eta} X_{i} : A} \text{ for } 1 \leq i \leq n \qquad \frac{\Gamma \vdash M =_{\beta\eta} M' : A \Rightarrow B \qquad \Gamma \vdash N =_{\beta\eta} N' : A}{\Gamma \vdash MN =_{\beta\eta} M'N' : B}$$

$$\frac{\Gamma, X : A \vdash M =_{\beta\eta} M' : B}{\Gamma \vdash \lambda(X : A) \cdot M =_{\beta\eta} \lambda(X : A) \cdot M' : A \Rightarrow B} \qquad \frac{\Gamma \vdash M_{1} =_{\beta\eta} M'_{1} : A_{1} \qquad \Gamma \vdash M_{2} =_{\beta\eta} M'_{2} : A_{2}}{\Gamma \vdash M : A_{1} \times A_{2}}$$

$$\frac{\Gamma \vdash M =_{\beta\eta} M' : A_{1} \times A_{2}}{\Gamma \vdash M : A_{1} \times A_{2}} \text{ for } i = 1, 2$$

$$\frac{\Gamma \vdash M =_{\beta\eta} M' : A \Rightarrow B}{\Gamma \vdash M : A_{1} \times A_{2}} \text{ for } i = 1, 2$$

• Equivalence rules

$$\frac{\Gamma \vdash M =_{\beta\eta} N : A}{\Gamma \vdash N =_{\beta\eta} M : A} \qquad \frac{\Gamma \vdash M =_{\beta\eta} M' : A}{\Gamma \vdash M =_{\beta\eta} M'' : A}$$

Note that if the judgment

$$\Gamma \vdash M =_{\beta n} N : A$$

is derivable, then both typing judgments

$$\Gamma \vdash M : A$$
 and $\Gamma \vdash N : A$

are valid. The reflexivity of $=_{\beta\eta}$ is derivable, because one can always construct a derivation of $\Gamma \vdash M =_{\beta\eta} M$: A from one of $\Gamma \vdash M : A$. Thus, $=_{\beta\eta}$ is an equivalence relation. Furthermore, it is a congruence by construction.

The set $\Lambda_{\beta\eta}\langle A\rangle$ is defined as the quotient $\{M \ \lambda\text{-term} \mid \varnothing \vdash M : A \text{ is derivable}\}/=_{\beta\eta}$.

Finite standard semantics

We recall what are *finite standard models* of the simply typed λ -calculus and how to interpret the simply typed λ -calculus inside them. This amounts to working in the cartesian closed category **FinSet** of finite sets and functions. For any finite set Q, we define the *standard model based on* Q in the following way.

As already indicated in the introduction for the base and arrow cases, for any type A, the interpretation $[\![A]\!]_O$ is the finite set inductively defined by

$$\llbracket \mathbb{o} \rrbracket_Q \; := \; Q \qquad \llbracket A \Rightarrow B \rrbracket_Q \; := \; \llbracket A \rrbracket_Q \Rightarrow \llbracket B \rrbracket_Q \qquad \llbracket A \times B \rrbracket_Q \; := \; \llbracket A \rrbracket_Q \times \llbracket B \rrbracket_Q \qquad \llbracket 1 \rrbracket_Q \; := \; \{*\}$$

where the symbols \times and \Rightarrow on the right hand sides refer to the cartesian product and set of functions between finite sets, respectively.

In the same way, for any finite set Q, the interpretation of any context $\Gamma := (x_1 : A_1, \dots, x_n : A_n)$ of typed variables is defined as

$$\llbracket \Gamma \rrbracket_Q \quad := \quad \llbracket A_1 \rrbracket_Q \times \cdots \times \llbracket A_n \rrbracket_Q .$$

Note that these constructions preserve finiteness, and that the sets $[\![A]\!]_Q$ and $[\![\Gamma]\!]_Q$ of interpretations are thus finite sets for any type A and context Γ .

Let Q be any finite set. We define the *semantic bracket* $\llbracket - \rrbracket_Q$ to be the function that transports any valid typing judgment

$$\Gamma \vdash M : A$$

to a function between finite sets

$$\llbracket\Gamma \vdash M : A \rrbracket_Q \quad : \quad \llbracket\Gamma \rrbracket_O \ \longrightarrow \ \llbracket A \rrbracket_O \ .$$

The inductive definition is recalled in Figure A.1 and uses crucially the fact that, given a context Γ and a λ -term M, there is at most one simple type A such that the $\Gamma \vdash M : A$ is derivable, which is due to the fact that the λ -abstractions are annotated with types.

```
\begin{split} & [\![\Gamma \vdash x_i : A_i]\!]_Q(\bar{q}) \ := \ q_i \\ & [\![\Gamma \vdash MN : B]\!]_Q(\bar{q}) \ := \ [\![\Gamma \vdash M : A \Rightarrow B]\!]_Q(\bar{q}) \, ([\![\Gamma \vdash N : A]\!]_Q(\bar{q})) \\ & [\![\Gamma \vdash \lambda(x : A).M : A \Rightarrow B]\!]_Q(\bar{q}) \ := \ q_{n+1} \in [\![A]\!]_Q \mapsto [\![\Gamma, x : A \vdash M : B]\!]_Q(\bar{q}, q_{n+1}) \\ & [\![\Gamma \vdash \langle M_1, M_2 \rangle : B_1 \times B_2]\!]_Q(\bar{q}) \ := \ \langle [\![\Gamma \vdash M_1 : B_1]\!]_Q(\bar{q}), [\![\Gamma \vdash M_2 : B_2]\!]_Q(\bar{q}) \rangle \\ & [\![\Gamma \vdash M.i : B_i]\!]_Q(\bar{q}) \ := \ \pi_i \, ([\![\Gamma \vdash M : B_1 \times B_2]\!]_Q(\bar{q})) \quad \text{for } i = 1, 2 \\ & [\![\Gamma \vdash () : 1]\!]_Q(\bar{q}) \ := \ \ast \quad \text{the unique element of } [\![1]\!]_Q \end{split}
```

Fig. A.1. Definition of the semantic bracket on valid typing judgments, by induction on the complexity of the term on the right of the typing judgment. In this definition, we use the following notations: $\Gamma = (x_1 : A_1, \dots, x_n : A_n); \bar{q} = (q_1, \dots, q_n)$ is an arbitrary element of $\llbracket \Gamma \rrbracket_Q = \llbracket A_1 \rrbracket_Q \times \dots \times \llbracket A_n \rrbracket_Q$; A, B, B_1 and B_2 are types; x is a variable not appearing in Γ ; M, N, M_1, M_2 are terms.

The semantic bracket $\llbracket - \rrbracket_Q$ preserves the relation $=_{\beta\eta}$ in the sense that if the judgment $\Gamma \vdash M =_{\beta\eta} N$: A is derivable, then the interpretations $\llbracket \Gamma \vdash M : A \rrbracket_Q$ and $\llbracket \Gamma \vdash N : A \rrbracket_Q$ are equal functionals. Therefore, the semantic bracket lifts to a function $\llbracket - \rrbracket_Q : \Lambda_{\beta\eta} \langle A \rangle \longrightarrow \llbracket A \rrbracket_Q$.

B Logical relations

Definition of logical relations

Logical relations [37,27,28,36,30] provide a well-known method in denotational semantics for establishing important properties of models of the λ -calculus, and exhibiting instructive connections between them. As explained in the introduction, we make crucial use of logical relations in the case of finite standard models, in order to relate the interpretations of a given type A at different finite sets Q and Q', and to construct from there our profinite semantics of the simply typed λ -calculus, see §5.

We define the category **FinRel** whose objects are triples $(X, Y, R \subseteq X \times Y)$ consisting of two finite sets X and Y and a binary relation $R \subseteq X \times Y$ between X and Y, and whose morphisms from (X, Y, R) to (X', Y', R') are pairs (f, g) of functions $f: X \to X'$ and $g: Y \to Y'$ which preserve binary relations in

the sense that for all $x \in X, y \in Y$, if x R y then f(x) R' g(y), where we use the notation x R y to mean that $(x, y) \in R$. The category **FinRel** is cartesian closed, with the following constructions for cartesian product and internal hom-object. Given two objects of $(X_i, Y_i, R_i \subseteq X_i \times Y_i)$ of **FinRel** for i = 1, 2, their cartesian product is defined as the relation

$$R_1 \times R_2 \subseteq (X_1 \times X_2) \times (Y_1 \times Y_2)$$

relating the finite sets $X_1 \times X_2$ and $Y_1 \times Y_2$ (cartesian products computed in **FinSet**) in the following way:

$$\langle x_1, x_2 \rangle (R_1 \times R_2) \langle y_1, y_2 \rangle \stackrel{\text{def}}{\Longrightarrow} x_1 R_1 y_1 \text{ and } x_2 R_2 y_2.$$

Their internal hom-object is defined as the relation

$$R_1 \Rightarrow R_2 \subseteq (X_1 \Rightarrow X_2) \times (Y_1 \Rightarrow Y_2)$$

relating the sets of functions $X_1 \Rightarrow X_2$ and $Y_1 \Rightarrow Y_2$ (internal hom-objects computed in **FinSet**) in the following way:

$$f(R_1 \Rightarrow R_2) g \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad \text{for every } x_1 \in X_1 \text{ and } y_1 \in Y_1,$$

 $\text{if } x_1 R_1 y_1 \text{ then } f(x_1) R_2 g(y_1) .$

Using the cartesian product and internal hom-object in **FinRel**, we associate to any type A and to any binary relation $R \subseteq X \times Y$ the *logical relation* $[\![A]\!]_R \subseteq [\![A]\!]_X \times [\![A]\!]_Y$ by induction over the simple type A as

$$[\![\emptyset]\!]_R \ := \ R \qquad [\![A \Rightarrow B]\!]_R \ := \ [\![A]\!]_R \Rightarrow [\![B]\!]_R \qquad [\![A \times B]\!]_R \ := \ [\![A]\!]_R \times [\![B]\!]_R \qquad [\![1]\!]_R \ := \ \operatorname{id}_{\{*\}}$$

Fundamental lemma of logical relations

The following proposition, known as the fundamental lemma of logical relations, says that the semantic bracket of any term preserves any logical relation, see e.g. [5, Lem. 4.5.3].

Proposition B.1 Suppose that

$$x_1:A_1,\ldots,x_n:A_n\vdash M:B$$

is a valid typing judgment, and consider any binary relation $R \subseteq X \times Y$ between finite sets. In that case, every pair of n-tuples

$$(x_1, \dots, x_n) \in [\![A_1]\!]_X \times \dots \times [\![A_n]\!]_X$$
$$(y_1, \dots, y_n) \in [\![A_1]\!]_Y \times \dots \times [\![A_n]\!]_Y$$

which satisfies that

for each
$$1 \le i \le n$$
, $x_i [A_i]_R y_i$,

also satisfies that

$$[\![M]\!]_X(x_1,\ldots,x_n)$$
 $[\![B]\!]_R$ $[\![M]\!]_Y(y_1,\ldots,y_n)$.

Proof. We prove the fundamental theorem for logical relations. For this, we will be using the following graphical representation. Let $g: X \to X'$ and $h: Y \to Y'$ be two functions and let $R \subseteq X \times Y$ and $R' \subseteq X' \times Y'$ be two relations. When we have the property that

for all
$$x \in X$$
 and $y \in Y$,
if $(x, y) \in R$, then $(g(x), h(y)) \in R'$.

we draw a square with a vertical double arrow

$$\begin{array}{ccc} X & & \xrightarrow{R} & Y \\ g \downarrow & & \downarrow h \\ X' & & \xrightarrow{R'} & Y' \end{array}$$

In this setting, what we have to prove is that if $\Gamma \vdash M : A$ is a valid typing judgment, then for any relation

$$R \subset X \times Y$$

we have a square

The proof is by induction on valid typing judgments

$$\Gamma \vdash M : A$$

whose formation rules are syntax-directed.

• If M is the ith variable x_i of the context Γ , then the projections π_i form a square

• If M is NN', then by induction we have the two squares

from which we can deduce the square

• If M is $\lambda(a:A).N$, then by induction we have the square

from which we deduce the square

• If M is $\langle M_1, M_2 \rangle$, then by induction we have the two squares, for i = 1, 2

$$\begin{bmatrix} \Gamma \end{bmatrix}_X & \xrightarrow{ \llbracket \Gamma \rrbracket_R } & \llbracket \Gamma \rrbracket_Y \\
 \llbracket M_i \rrbracket_X \downarrow & \downarrow & \downarrow \llbracket M_i \rrbracket_Y \\
 \llbracket A_i \rrbracket_X & \xrightarrow{ \llbracket A_i \rrbracket_R } & \llbracket A_i \rrbracket_Y$$

from which we can deduce the square

• If M is N.i for i = 1, 2, then by induction we have a square

$$\begin{bmatrix} \Gamma \end{bmatrix}_X \xrightarrow{\llbracket \Gamma \rrbracket_R} & \llbracket \Gamma \rrbracket_Y \\
 \llbracket N \rrbracket_X \downarrow & \downarrow \llbracket N \rrbracket_Y \\
 \llbracket A_1 \times A_2 \rrbracket_X \xrightarrow{\llbracket A_1 \times A_2 \rrbracket_R} & \llbracket A_1 \times A_2 \rrbracket_Y$$

from which we can deduce the square

$$\begin{bmatrix} \Gamma \end{bmatrix}_X & \xrightarrow{ \llbracket \Gamma \rrbracket_R } & \llbracket \Gamma \rrbracket_Y \\
 \downarrow & \downarrow \pi_i \circ \llbracket N \rrbracket_Y \\
 \llbracket A_i \rrbracket_X & \xrightarrow{ \llbracket A_i \rrbracket_R } & \llbracket A_i \rrbracket_Y
 \end{bmatrix}$$

• If M is (), then we have the square

given the fact that $[\![1]\!]_R$ is $[\![1]\!]_P \times [\![1]\!]_Q = \{(*,*)\}.$

There is an abstract explanation of the fundamental theorem using cartesian closed categories (CCC) for which a general version of the semantic bracket $[\![-]\!]$ can be defined. The definition given in Figure A.1 is the instantiation of this general construction to the particular CCC **FinSet**. For **C** any CCC, one can build an associated CCC denoted \mathbf{C}_{\rightarrow} whose objects are relations

$$R \subseteq \mathbf{C}(1,X) \times \mathbf{C}(1,Y)$$

where 1 is the terminal object of \mathbb{C} , and whose morphisms are the squares made of pairs of morphisms whose images by the functor $\mathbb{C}(1,-)$ form a pair preserving the relations. We then have the diagram of

CCCs

$$\mathbf{C}_{ op} \stackrel{\Pi_l}{\Longrightarrow} \mathbf{C}$$

where Π_l and Π_r are the two functors $\mathbf{C}_{\to} \to \mathbf{C}$ sending $R \subseteq \mathbf{C}(1,X) \times \mathbf{C}(1,Y)$ on X and Y respectively. These two functors respect the CCC structure on the nose. Therefore, the interpretation $[\![M]\!]_R$ of M at the relation R, seen as an object of \mathbf{C}_{\to} , is the square whose vertical sides are the interpretations $[\![M]\!]_X$ and $[\![M]\!]_Y$ at the objects X and Y of \mathbf{C} .

Partial surjections

To finish this section of the appendix, we concentrate on logical relations that are given by partial surjections. Let $f \subseteq X \times Y$ be a relation between finite sets. We will say that f is *surjective* if for every $y \in Y$, there exists $x \in X$ such that x f y, and we will call f a *partial function* if for all $x \in X$, $y, y' \in Y$, if x f y and x f y' then y = y'. Finally, we call f a *partial surjection* if it is both surjective and a partial function. In this case, we use the notation $f: X \to Y$. Note that a partial surjection f is, up to isomorphism, the same thing as a span in **FinSet** of the form

$$X \stackrel{\pi_1}{\longleftrightarrow} f \stackrel{\pi_2}{\longrightarrow} Y$$

where the fact that the function π_1 is an injection corresponds to partiality while the fact that the function π_2 is a surjection corresponds to surjectivity. We show that the property of being a partial surjection is stable under binary products and arrows.

Lemma B.2 If $f: X \rightarrow Y$ and $f': X' \rightarrow Y'$ are partial surjections, then so is $f \times f'$.

Proof. We first show that $f \times f'$ is partial. Let $\langle x, x' \rangle$ in $X \times X'$ and $\langle y_i, y_i' \rangle$ in $Y \times Y'$ for i = 1, 2 such that

$$(\langle x, x' \rangle, \langle y_i, y_i' \rangle) \in f \times f' \text{ for } i = 1, 2.$$

By partiality of f and f', we obtain respectively that $y_1 = y_2$ and $y'_1 = y'_2$. Therefore, $f \times f'$ is partial.

We now show that $f \times f'$ is surjective. Let $\langle y, y' \rangle$ be any element of $Y \times Y'$. By surjectivity of f and f', there exists respectively x in X and x' in X' such that $(x, y) \in f$ and $(x', y') \in f'$. Therefore, $(\langle x, x' \rangle, \langle y, y' \rangle) \in f \times f'$ which proves that $f \times f'$ is surjective.

Lemma B.3 If $f: X \to Y$ and $f': X' \to Y'$ are partial surjections, then so is the relation $f \Rightarrow f'$.

Proof. Let $f: X \to Y$ and $f': X' \to Y'$ be two partial surjections. Note that two functions $g: X \to X'$ and $h: Y \to Y'$ are related by $f \Rightarrow f'$ if and only if there exists a function $c: f \to f'$ such that the following diagram commutes:

$$X \xleftarrow{\pi_1} f \xrightarrow{\pi_2} Y$$

$$g \downarrow \qquad \downarrow c \qquad \downarrow h \qquad (B.1)$$

$$X' \xleftarrow{\pi'_1} f' \xrightarrow{\pi'_2} Y'$$

First, we show that the relation $f \Rightarrow f'$ is a partial function. Let $g: X \to X'$ and $h_1, h_2: Y \to Y'$ such that $(g, h_i) \in f \Rightarrow f'$ for i = 1, 2. We thus have two maps $c_i: f \to f'$ for i = 1, 2 such that

$$g \circ \pi_1 = \pi'_1 \circ c_i$$
 and $h_i \circ \pi_2 = \pi'_2 \circ c_i$ for $i = 1, 2$.

The maps all fit in the diagram

$$X \stackrel{\pi_1}{\longleftarrow} f \xrightarrow{\pi_2} Y$$

$$g \downarrow c_1 \downarrow c_2 \quad h_1 \downarrow h_2 \cdot X' \stackrel{\pi'_1}{\longleftarrow} f' \xrightarrow{\pi'_2} Y'$$

By injectivity of π'_1 , we get that $c_1 = c_2$. Therefore,

$$h_1 \circ \pi_2 = \pi_2' \circ c_1 = \pi_2' \circ c_2 = h_2 \circ \pi_2$$
.

By surjectivity of π_2 , we get that $h_1 = h_2$. This proves that the relation $f \Rightarrow f'$ is a partial function.

We now show that the relation $f \Rightarrow f'$ is surjective. Let $h: Q \to Q'$ be any function. As π'_2 is surjective, it has a section $s: Q' \to f'$, that is, $\pi'_2 \circ s = \mathrm{id}_{Q'}$. We then define the function $c: f \to f'$ as $s \circ h \circ \pi_2$. Note that $\pi'_2 \circ c = h \circ \pi_2$, so we obtain the commuting diagram

$$X \stackrel{\pi_1}{\longleftarrow} f \stackrel{\pi_2}{\longrightarrow} Y$$

$$\downarrow c \qquad \downarrow \qquad \downarrow h \qquad \downarrow \qquad \downarrow$$

$$X' \stackrel{\pi'_1}{\longleftarrow} f' \stackrel{\pi'_2}{\longleftarrow} Y'$$

As π_1 is injective, it has a retraction $r: X \to f$, that is, $r \circ \pi_1 = \mathrm{id}_f$. We define the function $g: X \to X'$ as $\pi'_1 \circ c \circ r$ and note that

$$g \circ \pi_1 = \pi'_1 \circ c .$$

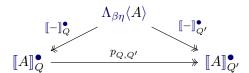
Thus, the diagram (B.1) commutes for this choice of g, c, and h, which means that (g,h) is in the relation $f \Rightarrow f'$, as required.

Using Lemmas B.2 and B.3, we get a proof of the following proposition, by induction on the structure of the simple type.

Proposition B.4 If $f: X \to Y$ is a partial surjection, then the relation $[\![A]\!]_f \subseteq [\![A]\!]_X \times [\![A]\!]_Y$ is a partial surjection for any type A.

The next proposition results from Propositions B.1 and B.4.

Proposition B.5 For any partial surjection $f: Q \to Q'$ and for any type A, the set $[\![A]\!]_Q^{\bullet} \subseteq [\![A]\!]_Q$ of definable elements is contained in the domain of $[\![A]\!]_f$, and the restriction of $[\![A]\!]_f$ to $[\![A]\!]_Q^{\bullet}$ is the unique function $p_{Q,Q'}$ that makes the following diagram commute:



Proof. By Proposition B.1, for any term M of type A, we have $[\![M]\!]_Q$ $[\![A]\!]_f$ $[\![M]\!]_{Q'}$, so that any definable element is in the domain of $[\![A]\!]_f$. By Proposition B.4, $[\![A]\!]_f$ is in particular a partial function, so that it makes the diagram commute. For the uniqueness, simply note that any $q \in [\![A]\!]_Q^{\bullet}$ can by definition be written as $[\![M]\!]_Q$ for some term M of type A, and must therefore be sent to $[\![M]\!]_{Q'}$ by any function making the diagram commute.

Note that Proposition B.5 in particular implies that, if $f,g:Q \Rightarrow Q'$ are two partial surjections, then, while their semantic interpretations $[\![A]\!]_f, [\![A]\!]_g: [\![A]\!]_Q \Rightarrow [\![A]\!]_{Q'}$ are in general distinct, their restrictions to the set $[\![A]\!]_Q^{\bullet}$ of definable elements must both be equal to the function $p_{Q,Q'}$.

C On the creation of directed colimits (Section 2)

Proof of Proposition 2.3

Let $(B_i)_{i\in I}$ be a directed diagram of algebras in **V**. First note that the colimit in **Set** of the underlying sets $|B_i|$ may be computed as the union

$$|B| := \bigcup_{i \in I} |B_i|$$
.

We now explain how to construct the unique algebra structure on |B| such that each inclusion function $\iota_i \colon |B_i| \hookrightarrow |B|$ becomes a homomorphism. Let \star be an operation in the signature for the variety \mathbf{V} of arity

n, say. For any $b_1, \ldots, b_n \in |B|$, the fact that the diagram $(B_i)_{i \in I}$ is directed implies that we can pick $i \in I$ such that $b_1, \ldots, b_n \in |B_i|$. Therefore, for ι_i to be a homomorphism, we must define

$$\star^B(b_1,\ldots,b_n) := \iota_i(\star^{B_i}(b_1,\ldots,b_n)) .$$

An easy verification shows that this gives a well-defined algebra structure on the set |B| for which the algebra B becomes a colimit of the diagram in V.

D Basics of Stone duality (Section 3)

In this section, we give a brief recap of the basic notions of Stone duality that we used in this paper. All of these results are standard and we refer to, for example, [19] for proofs.

A basic observation that underlies Stone duality for Boolean algebras, which was already made by Birkhoff, is that any finite Boolean algebra is determined up to isomorphism by its set of atoms. Moreover, homomorphisms between finite Boolean algebras are completely described as the inverse image mappings of functions in the other direction between the atom sets. More formally, we have the following *finite* version of Stone duality.

Proposition D.1 The contravariant power set functor

$$\wp : \mathbf{FinSet}^{\mathrm{op}} \longrightarrow \mathbf{FinBA}$$

is part of a dual equivalence of categories, whose converse is the functor

$$\mathbf{At} : \mathbf{FinBA} \longrightarrow \mathbf{FinSet}^{\mathrm{op}}$$

which sends a finite Boolean algebra to its set of atoms, and a homomorphism $h: B \to B'$ between finite Boolean algebras to the restriction of its lower adjoint to the set of atoms.

Proposition D.1 may be used to derive two further equivalences of categories, by taking either inductive or projective completions of the categories on both sides of this duality.

To this end, recall that, if \mathbf{C} is a small category with finite colimits, then its inductive completion $\mathbf{Ind}(\mathbf{C})$ is characterized up to equivalence as a cocomplete category that admits a full embedding $I: \mathbf{C} \to \mathbf{Ind}(\mathbf{C})$ which preserves finite colimits, sends objects of \mathbf{C} to finitely presentable objects of $\mathbf{Ind}(\mathbf{C})$, and such that every object of $\mathbf{Ind}(\mathbf{C})$ is a colimit of objects in the image of I [19, VI.1.8].

Concretely, the category $\mathbf{Ind}(\mathbf{C})$ may be constructed as the category of functors $\mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ that are filtered colimits of representables, or equivalently that preserve finite colimits (still under the assumption that finite colimits exist in \mathbf{C}). The dual notion to the inductive completion is the *projective completion* $\mathbf{Pro}(\mathbf{C})$, which is equivalent to $\mathbf{Ind}(\mathbf{C}^{\mathrm{op}})^{\mathrm{op}}$.

By purely formal considerations, we now obtain from Proposition D.1 a dual equivalence of categories

$$Ind(FinBA) \leftrightarrows Pro(FinSet)^{op}$$
. (D.1)

Let us recall in more concrete terms what are the categories on either side of this dual equivalence, and how the functors act. On the left hand side, the inductive completion of the category **FinBA** of finite Boolean algebras is the category **BA** of Boolean algebras; this is an instance of the general fact [19, VI.2.2] that any finitary variety of algebras is the inductive completion of its full subcategory of finitely presentable objects. On the right hand side, the projective completion of the category **FinSet** is the category of profinite sets. A convenient way to realize this category is by first embedding the category **FinSet** into the category **Top** of topological spaces and continuous functions, using the discretization functor

$$D : \mathbf{FinSet} \longrightarrow \mathbf{Top}$$

which sends a finite set X to the discrete topological space based on the set X. Because the category **Top** is complete and the functor D preserves finite limits and sends objects **FinSet** to finitely co-presentable objects, the category **Pro(FinSet)** may be realized as the subcategory **Stone** of **Top** that consists of the

codirected limits of objects in the image of D, that is, finite discrete spaces. Such spaces are called *Stone spaces* (or also *Boolean spaces*), and may be characterized as compact Hausdorff spaces in which the clopen subsets form a basis for the topology [19, VI.2.3].

Making the identifications $\operatorname{Ind}(\operatorname{FinBA}) \simeq \operatorname{BA}$ and $\operatorname{Pro}(\operatorname{FinSet}) \simeq \operatorname{Stone}$, we can now describe more concretely the equivalence functors appearing in (D.1) above, which give us a pair of functors

$$\mathbf{Spec}: \mathbf{BA} \hookrightarrow \mathbf{Stone}^{\mathrm{op}}: \mathbf{Clp}$$
.

On objects, if X is a Stone space, then $\mathbf{Clp}(X)$ is the Boolean algebra of clopen subsets of X, and on morphisms, if $f: X \to Y$ is a continuous function between Stone spaces, then $\mathbf{Clp}(f)$ is the Boolean algebra homomorphism

$$f^{-1} \colon \mathbf{Clp}(Y) \to \mathbf{Clp}(X)$$
.

If B is a Boolean algebra, then $\mathbf{Spec}(B)$ is defined as the Stone space of homomorphisms $B \to \mathbf{2}$, where **2** is the two element Boolean algebra, and the topology on $\mathbf{Spec}(B)$ is induced by seeing it as a subspace of the |B|-fold power $2^{|B|}$ of the discrete two-element space. The space $\mathbf{Spec}(B)$ is called the *spectrum* or *dual space* of the Boolean algebra B. If $h \colon B \to B'$ is a homomorphism of Boolean algebras, then

$$\mathbf{Spec}(h) \colon \mathbf{Spec}(B') \to \mathbf{Spec}(B)$$

is defined by pre-composing any $u: B' \to \mathbf{2}$ with h to obtain a homomorphism $u \circ h: B \to \mathbf{2}$.

Stone duality for Boolean algebras is the following result.

Proposition D.2 The pair of functors (Spec, Clp) defines a dual equivalence of categories.

An important part of Proposition D.2 is the fact that any Boolean algebra B is isomorphic to the Boolean subalgebra of $\wp(\mathbf{Spec}(B))$ that consists of the clopen sets. This may be proved explicitly by showing that the image of the natural embedding

$$\begin{split} B &\to \wp(\mathbf{Spec}(B)), \\ b &\mapsto \{u \in \mathbf{Spec}(B) \mid u(b) = 1\} \end{split}$$

consists exactly of the clopen subsets of $\mathbf{Spec}(B)$.

To finish this brief recap of Stone duality, we also recall [19, VI.3.2] that, conversely to (D.1), one may instead extend Proposition D.1 by taking the inductive completion of the category **FinSet** to get **Set**, and the projective completion of the category **FinBA** to obtain the category **CABA** of complete and atomic Boolean algebras [19, VII.1.16]. We thus have a different dual equivalence,

$$\wp \colon \mathbf{Set}^{\mathrm{op}} \leftrightarrows \mathbf{CABA} \colon \mathbf{At}$$

often referred to as *discrete Stone duality*. A consequence of this fact, which is also not hard to prove in a more direct way, is the following:

Proposition D.3 The contravariant power set functor

$$\wp : \mathbf{Set}^{\mathrm{op}} \longrightarrow \mathbf{CABA}$$

induces, for any sets X, Y, a bijection

$$\operatorname{Hom}_{\mathbf{Set}}(X,Y) \to \operatorname{Hom}_{\mathbf{CABA}}(\wp(Y),\wp(X))$$

between functions $X \to Y$ and complete Boolean algebra homomorphisms $\wp(Y) \to \wp(X)$, which is moreover natural in both X and Y.

E On cartesian closed categories (Sections 5 and 6)

This appendix contains the proofs of Proposition 5.1 and 6.2.

Proof of Proposition 5.1

The composition law of the category $\mathbf{C}[\mathcal{P}]$

$$\circ_{A,B,C}^{\mathbf{C}[\mathcal{P}]}: \mathbf{C}[\mathcal{P}](B,C) \times \mathbf{C}[\mathcal{P}](A,B) \longrightarrow \mathbf{C}[\mathcal{P}](A,C)$$

is defined by composing the bijection $m_{B\Rightarrow C,A\Rightarrow B}$

$$\mathcal{P}(B \Rightarrow C) \times \mathcal{P}(A \Rightarrow B) \longrightarrow \mathcal{P}((B \Rightarrow C) \times (A \Rightarrow B))$$

with the image by the functor \mathcal{P}

$$\mathcal{P}((B \Rightarrow C) \times (A \Rightarrow B)) \longrightarrow \mathcal{P}(A \Rightarrow C)$$

of the internal composition law in ${\bf C}$

$$\circ_{A.B.C}^{\mathbf{C}}\,:\,(B\Rightarrow C)\times(A\Rightarrow B)\longrightarrow A\Rightarrow C$$

The fact that the category $\mathbb{C}[\mathcal{P}]$ is cartesian may be derived from the fact that the family of bijections

$$\mathbf{C}[\mathcal{P}](A, B \times C) = \mathcal{P}(A \Rightarrow (B \times C))$$

$$\cong \mathcal{P}((A \Rightarrow B) \times (A \Rightarrow C))$$

$$\cong \mathcal{P}(A \Rightarrow B) \times \mathcal{P}(A \Rightarrow C)$$

$$= \mathbf{C}[\mathcal{P}](A, B) \times \mathbf{C}[\mathcal{P}](A, C)$$

is natural in A, B and C. Similarly, the fact that $\mathbf{C}[\mathcal{P}]$ is cartesian closed can be derived from the fact that the family of bijections

$$\mathbf{C}[\mathcal{P}](A \times B, C) = \mathcal{P}((A \times B) \Rightarrow C)$$

$$\cong \mathcal{P}(B \Rightarrow (A \Rightarrow C))$$

$$= \mathbf{C}[\mathcal{P}](B, A \Rightarrow C)$$

is natural in B and C. The functor idonobj $_{\mathbf{C},\mathcal{P}}$ transports every morphism $f:A\to B$ in the category \mathbf{C} to the element of $\mathcal{P}(A\Rightarrow B)$ defined as the image of $*\in 1$ by the function

$$1 \xrightarrow{m_1} \mathcal{P}(1) \xrightarrow{\mathcal{P}(\lceil f \rceil)} \mathcal{P}(A \Rightarrow B)$$

where the morphism $\lceil f \rceil : 1 \to A \Rightarrow B$ denotes in the category **C** the "currification" of the morphism $f : A \to B$.

Proof of Proposition 6.2

Let Q be any finite set. The category \mathbf{Lam}_Q is defined as the quotient

$$\mathbf{Lam}_Q \ = \ \mathbf{Lam}/\sim_Q$$

which means that, for any objects A and B of Lam_Q , i.e. simple types, we have

$$\mathbf{Lam}_Q(A, B) = \Lambda_{\beta\eta} \langle A \Rightarrow B \rangle / \sim_Q^{A \Rightarrow B}$$
.

Definition E.1 We write \mathcal{P}_Q for the functor

$$\mathcal{P}_Q$$
 : $\begin{array}{ccc} \mathbf{Lam} & \longrightarrow & \mathbf{Set} \\ A & \longmapsto \Lambda_{\beta\eta} \langle A \rangle / \sim_Q^A \ \cong \llbracket A
rbracket^{lacktright}_Q \end{array}$

which sends a morphism in the hom-set $\mathbf{Lam}(A, B)$, i.e. a λ -term M of type $A \Rightarrow B$, on the function

$$q \in \llbracket A \rrbracket_Q^{\bullet} \longmapsto \llbracket M \rrbracket_Q(q)$$
.

As \mathcal{P}_Q is cartesian product preserving and as the category \mathbf{Lam}_Q is obtained as $\mathbf{Lam}[\mathcal{P}_Q]$, we get from Proposition 5.1 that \mathbf{Lam}_Q is a CCC and that there is a cartesian closed identity-on-object functor

$$\pi_Q := \mathrm{idonobj}_{\mathbf{Lam}, \mathcal{P}_Q} : \mathbf{Lam} \longrightarrow \mathbf{Lam}_Q.$$

We now prove a first functoriality proposition, i.e. that partial surjections yield natural transformations between cartesian product preserving functors.

Proposition E.2 The assignment sending a finite set Q on the cartesian product preserving functor \mathcal{P}_Q is a functor from the category **FinPSurj** to the category of cartesian product preserving functors and natural transformations between them.

In particular, if $f: Q \rightarrow Q'$ is a partial surjection, then

$$\mathcal{P}_f : \mathcal{P}_O \longrightarrow \mathcal{P}_{O'}$$

is a natural transformation.

Proof. Let Q and Q' be finite sets and f:Q woheadrightarrow Q' be a partial surjection. We define the natural transformation \mathcal{P}_f as the family of maps

$$\mathcal{P}_{f,A} := \llbracket A \rrbracket_f^{\bullet} = p_{Q,Q'} : \llbracket A \rrbracket_Q^{\bullet} \longrightarrow \llbracket A \rrbracket_{Q'}^{\bullet}$$

with A ranging over all simple types. These maps have been described in Proposition B.5, and naturality of the whole is a consequence of Proposition B.1. Indeed, if M is a λ -term of type $A \Rightarrow B$, we have

$$\begin{bmatrix} A \end{bmatrix}_{Q}^{\bullet} \xrightarrow{ \begin{bmatrix} A \end{bmatrix}_{f}^{\bullet}} \begin{bmatrix} A \end{bmatrix}_{Q'}^{\bullet} \\
 \mathcal{P}_{Q}(M) \downarrow & \downarrow \mathcal{P}_{Q'}(M) \\
 \begin{bmatrix} B \end{bmatrix}_{Q}^{\bullet} \xrightarrow{ \begin{bmatrix} B \end{bmatrix}_{f}^{\bullet}} \begin{bmatrix} B \end{bmatrix}_{Q'}^{\bullet}$$

which proves that \mathcal{P}_f is a natural transformation.

We now show that the construction of the CCC $\mathbb{C}[\mathcal{P}]$ is functorial in \mathcal{P} .

Proposition E.3 Let C be a CCC. Then, the assignment

$$\mathbf{C}[-] \quad : \quad \mathcal{P} \ \longmapsto \ (\mathbf{C}[\mathcal{P}], \mathrm{idonobj}_{\mathbf{C},\mathcal{P}}) \ ,$$

from cartesian product preserving functors to CCCs with an identity-on-object inclusion of \mathbf{C} , extends to a functor.

In particular, if \mathcal{P} and \mathcal{P}' are cartesian product preserving and α is a natural transformation from \mathcal{P} to \mathcal{P}' , then there is a functor of CCCs $\mathbf{C}[\alpha]$ from $\mathbf{C}[\mathcal{P}]$ to $\mathbf{C}[\mathcal{P}']$.

Proof. The definition of $\mathbf{C}[-]$ on objects is in Proposition 5.1. We now treat the case of morphisms. Let α be a natural transformation from \mathcal{P} to \mathcal{P}' . We define an identity-on-object functor $\mathbf{C}[\alpha] : \mathbf{C}[\mathcal{P}] \longrightarrow \mathbf{C}[\mathcal{P}']$ as, for any two objects A and B of $\mathbf{C}[\mathcal{P}]$,

$$\mathbf{C}[\alpha] := \alpha_{A \Rightarrow B} : \mathcal{P}(A \Rightarrow B) \longrightarrow \mathcal{P}'(A \Rightarrow B)$$

and $\mathbf{C}[\alpha]$ is a functor of CCCs.

The equality

$$\mathbf{C}[\alpha] \circ \mathrm{idonobj}_{\mathbf{C},\mathcal{P}} = \mathrm{idonobj}_{\mathbf{C},\mathcal{P}'}$$

comes from the commutativity of the diagram

$$\begin{array}{cccc}
1 & \xrightarrow{m_1} & \mathcal{P}(1) & \xrightarrow{\mathcal{P}(\lceil f \rceil)} & \mathcal{P}(A \Rightarrow B) \\
\downarrow & & & \downarrow \\
m'_1 & & & \downarrow \\
\alpha_A \Rightarrow B \\
\mathcal{P}'(1) & \xrightarrow{\mathcal{P}'(\lceil f \rceil)} & \mathcal{P}'(A \Rightarrow B)
\end{array}$$

which is due to the naturality of α .

Ultimately, we remark that one can factorize the construction of \mathbf{Lam}_Q as the construction of the cartesian product preserving functor \mathcal{P}_Q followed by the construction of the CCC $\mathbf{Lam}[\mathcal{P}_Q]$. Proposition E.2 shows that the first step is functorial, and Proposition E.3 shows that the second one is also functorial. This finishes the proof of Proposition 6.2.