A zero-test for series defined in terms of solutions to partial differential equations

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Consider an effective differential ring $\mathbb{A}$ of computable power series in $\mathbb{K}[[z_1, \ldots, z_k]]$ for some field $\mathbb{K}$ and assume that we have an effective zero-test for elements in $\mathbb{A}$. Consider a system of algebraic partial differential equations $P_1(f) = \cdots = P_p(f) = 0$ with $P_1, \ldots, P_p \in \mathbb{A}\{F\}$. If $f \in \mathbb{K}[[z_1, \ldots, z_k]]$ is the unique solution to this system of equations with suitable initial conditions, then we obtain a new effective differential ring $\mathbb{A}\{f\}$ of computable power series when adjoining f to $\mathbb{A}$. Under a mild additional hypothesis, we will describe a zero-test for elements in $\mathbb{A}\{f\}$.

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1. Introduction

There exist essentially two approaches in computer algebra for the computation with formal solutions of systems of ordinary or partial differential equations.

The first approach is based on the theory of differential algebra [Rit50, Sei56, Ros59, Kap57, Kol73, Bui92]. We start with an effective differential field $\mathbb{F}$ with derivations $d_1, \ldots, d_k$ and enrich $\mathbb{F}$ with the formal solution to a system $P_1, \ldots, P_p \in \mathbb{F}\{F\}$ of differential polynomials. From an algebraic point of view, this formal solution lies in the differential quotient ring $\mathbb{F}\{F\}/\{P_1, \ldots, P_p\}$ of $\mathbb{F}\{F\}$ by the perfect ideal $\{P_1, \ldots, P_p\}$ generated by $P_1, \ldots, P_p$. The theory of differential algebra is well-established and admits an effective counterpart: see [Bou94] for an overview. In particular, there exists a zero-test in $\mathbb{F}\{F\}/\{P_1, \ldots, P_p\}$. i.e., independently of initial conditions; is the complexity of zero testing lower?

In practice however, a simple function such as $f(x) = \mathrm{e}^x$ is not just the solution to the differential equation $f' = f$, but it also satisfies the initial condition $f(0) = 1$. When taking into account initial conditions, non-zero elements in $\mathbb{F}\{F\}/\{P_1, \ldots, P_p\}$ may vanish when we consider them as functions. For instance, assume that we redefine $f(x) = \mathrm{e}^x$ to be the unique solution to $f'' = f$ with $f(0) = 1$ and $f'(0) = 1$. Then $f' - f$ is non-zero as an element of $\mathbb{F}\{f\}/(f'' - f)$, even though $f' - f$ vanishes as a function.

In certain cases, it may therefore be necessary to consider a second approach, in which the new functions are defined to be solutions to systems of differential equations with initial conditions. Since most functions which arise in this way are analytic, it is natural to systematically work with computable power series.

More precisely, given an effective field $\mathbb{K}$, a multivariate power series $P \in \mathbb{K}[[z_1, \ldots, z_k]]$ is said to be *computable*, if there exists an algorithm which takes $\boldsymbol{i} \in \mathbb{N}^k$ on input and which outputs the corresponding coefficient $P_{\boldsymbol{i}}$ of $\boldsymbol{z}^{\boldsymbol{i}}$ in P. The set $\mathbb{K}[[z_1, \ldots, z_k]]^{\mathrm{com}}$ of such series forms a differential ring, even though $\mathbb{K}[[z_1, \ldots, z_k]]^{\mathrm{com}}$ does not come with a zero-test.

Now assume that we are given an effective subring $\mathbb{A}$ of $\mathbb{K}[[z_1, \ldots, z_k]]^{\mathrm{com}}$ and let $\mathbb{F}$ be the quotient field of $\mathbb{A}$. We wish to extend $\mathbb{A}$ with power series solutions $f \in \mathbb{K}[[z_1, \ldots, z_k]]^{\mathrm{com}}$ to systems of partial differential equations $P_1, \ldots, P_p \in \mathbb{F}\{F\}$. In the ordinary differential case, the coefficients of f can generally be computed as a function of the equations $P_1, \ldots, P_p$ and a finite number of initial conditions in $\mathbb{K}$. In the case of partial differential equations, the initial conditions are taken in $\mathbb{A} \cap \mathbb{K}[[z_1, \ldots, z_{k'}]]^{\mathrm{com}}$ with $k' < k$.

When extending $\mathbb{A}$ with the solution to a system of differential equations which satisfies explicit initial conditions, an important problem is to decide whether $Q(f) = 0$ for a given differential polynomial $Q \in \mathbb{F}\{F\}$. In the case of ordinary differential equations, there exists a variety of algorithms to solve this problem [DL84, DL89, Sha89, Sha93, PG95, PG97, vdH01, vdHS06, vdH02]; see [vdH02] for a discussion. Progress in the case of partial differential equations has been slower and the best currently known result [PG02] reduces the zero-test problem to the Ritt problem concerning the distribution of the singular zeros of a differential system among its irreducible components. In fact, questions related to the zero-test problem quickly tend to be undecidable:

THEOREM 1. *[DL84, Theorem 4.12] There does not exist an algorithm to decide whether a linear partial differential equation over* $\mathbb{Q}[z_1, \ldots, z_k]$ *has a power series solution in* $\mathbb{C}[[z_1, \ldots, z_k]]$.

promise problem!

In this paper, we propose a zero-test in the partial differential case, which is based on a generalization of the technique from [vdH02, vdHS06]. In a nutshell, assume that f is the unique solution to $P_1(f) = \cdots = P_p(f)$ with $P_1, \ldots, P_p \in \mathbb{F}\{F\}$ and explicit initial conditions. In order to test whether $Q(f) = 0$ for $Q \in \mathbb{F}\{F\}$, we first simplify the system of equations $P_1(g) = \cdots = P_p(g) = Q(g)$ using the theory of differential algebra. At the end, we obtain a new system of equations $R_1(g) = \cdots = R_r(g) = 0$. If we can show that this system admits a solution g which satisfies the same initial conditions as f, then the fact that $P_1(g) = \cdots = P_p(g) = 0$ implies $f = g$ and $Q(f) = 0$. If $Q(f) \neq 0$, then it suffices to expand $Q(f)$ sufficiently far in order to find a non-zero coefficient.

The main difficulty is to ensure the existence of a suitable solution g to the reduced system $R_1(g) = \cdots = R_r(g) = 0$, which may be highly singular. The main part of this paper is devoted to further reduce the system $R_1(g) = \cdots = R_r(g) = 0$ together with the initial conditions into an asymptotic normal form (called an asymptotic basis). If this asymptotic normal form is non-contradictory, then we will guarantee the existence of a solution in a suitable field $\mathbb{L}$ of power series with logarithmic coefficients. In order to make our zero-test work, we thus have to assume that f is also the unique solution in $\mathbb{L}$ to $P_1(g) = \cdots = P_p(g) = 0$ with the prescribed initial conditions. Fortunately, this is only a mild additional hypothesis. why?

Let us outline the structure of the paper. In section 2, we start with some quick reminders on so called grid-based power series. For a more detailed treatment, we refer to [vdH06, Chapter 2].

In section 3, we review the elementary theory of differential algebra. In order to lighten notations, we will only consider equations in one differential indeterminate. In section 3.6, we also introduce Ritt co-reduction in the "dual setting" where differential operators in $\mathbb{F}[d_1, \ldots, d_k]$ are replaced by operators in $\mathbb{F}[[d_1, \ldots, d_k]]$. This dual setting was considered before in the linear case [vdH07].

In section 4, we next switch to the case when we work over a field $\mathbb{F}$ of grid-based power series $\mathbb{K}[\![z_1^{\mathbb{R}} \cdots z_k^{\mathbb{R}}]\!] \supseteq \mathbb{K}[[z_1, \ldots, z_k]]$ with a suitable monomial ordering on $z_1^{\mathbb{R}} \cdots z_k^{\mathbb{R}}$. This field comes with natural valuation preserving derivations $\delta_i = z_i\, \partial / \partial z_i$. More generally, for suitable $\lambda_1, \ldots, \lambda_k \in \mathbb{R}^{>} = \{x \in \mathbb{R}: x > 0\}$, we will consider more general derivations $d_i = \boldsymbol{z}^{\lambda_i \boldsymbol{\alpha}}\, \delta_i$, where $\boldsymbol{z}^{\boldsymbol{\alpha}}$ is an infinitesimal monomial. Since $\mathbb{F}$ is a field of series, it is naturally to study the dominant parts of differential equations, which gives rise to a theory of "asymptotic differential algebra".

Although a general theory of asymptotic differential algebra is somewhat difficult to design due to the possible presence of infinite powers of initials and separants in the asymptotic analogue of Ritt reduction, a systematic theory can be developed in the case of "quasi-linear differential ideals". We will define and outline the basic properties of such ideals in section 5 and prove the existence of zeros in $\mathbb{K}[\log z_1, \ldots, \log z_k][\![z_1^{\mathbb{R}} \cdots z_k^{\mathbb{R}}]\!]$ in the case when the derivations are $\delta_1, \ldots, \delta_k$.

Starting with the equations $R_1(g) = \cdots = R_r(g) = 0$, we next have to put the equations in quasi-linear form. This is automatic for the derivations $d_i = \boldsymbol{z}^{\lambda_i \boldsymbol{\alpha}}\, \delta_i$ if $\boldsymbol{z}^{\boldsymbol{\alpha}}$ is sufficiently small. Using a suitable process of asymptotic dualizations and changes of derivations, we will show in section 6 how to end up with a system of quasi-linear equations w.r.t. $\delta_1, \ldots, \delta_k$. In section 7, we will give algorithmic counterparts of this construction and show how this can be used to elaborate our zero-test.

When applying the algorithm from this paper in the ordinary differential case, we notice a small difference with [vdH02, vdHS06]. In the present paper, we expand sufficiently far so as to make the equations in the reduced system quasi-linear. This simplifies the existence proof of solutions, but may require expansions up to a higher order than what was necessary before. Indeed, the previous algorithms relied on a theoretical result [vdH02, Theorem 3] for the existence proof (see also [vdH06, Chapter 8]).

2. Grid-based series

For convenience of the reader, we will briefly recall the definition of grid-based power series, as well as some notational conventions. For a more detailed exposition, we refer to [vdH06, Chapter 2].

Let $\mathbb{K}$ be an arbitrary ring of coefficients ($\mathbb{K}$ is usually a field) and $\mathfrak{M}$ a totally ordered multiplicative group (one may also consider partially ordered monoids). The ordering $\preccurlyeq$ on $\mathfrak{M}$ is called an *asymptotic dominance relation* and we call $\mathfrak{M}$ a *monomial group*. A subset $\mathfrak{G} \subseteq \mathfrak{M}$ is said to be *grid-based* if $\mathfrak{G} \subseteq \{\mathfrak{m}_1, \ldots, \mathfrak{m}_m\}^*\, \mathfrak{n}$ for some infinitesimal monomials $\mathfrak{m}_1, \ldots, \mathfrak{m}_m \in \mathfrak{M}^{\prec} = \{\mathfrak{m} \in \mathfrak{M}: \mathfrak{m} \prec 1\}$ and an arbitrary monomial $\mathfrak{n} \in \mathfrak{M}$. Here $\mathfrak{S}^* = \{\mathfrak{m}_1 \cdots \mathfrak{m}_k: \mathfrak{m}_1, \ldots, \mathfrak{m}_k \in \mathfrak{S}, k \in \mathbb{N}\}$ for any $\mathfrak{S} \subseteq \mathfrak{M}^{\prec}$.

A *grid-based series* is a formal sum $f = \sum_{\mathfrak{m} \in \mathfrak{M}} \mathfrak{m}\, f_{\mathfrak{m}}$ with $f_{\mathfrak{m}} \in \mathbb{K}$ and grid-based support $\operatorname{supp} f = \{\mathfrak{m} \in \mathfrak{M}: f_{\mathfrak{m}} \neq 0\}$. For reasons which will become clear later in the paper (when $\mathbb{K}$ will be allowed to be a non-commutative ring of operators), we multiplied $\mathfrak{m}$ on the right with its corresponding coefficient $f_{\mathfrak{m}}$. The set $\mathbb{K}[\![\mathfrak{M}]\!]$ of grid-based series forms a ring (and a field if $\mathbb{K}$ is a field).

Any non-zero grid-based series $f \in \mathbb{K}[\![\mathfrak{M}]\!]$ admits a unique *dominant monomial* $\mathfrak{d}_f \in \mathfrak{M}$ (the maximal element in its support) and a corresponding *dominant coefficient* $c_f = f_{\mathfrak{d}_f}$. By convention, $c_0 = 0$. The dominance relation $\preccurlyeq$ may be extended to $\mathbb{K}[\![\mathfrak{M}]\!]$ by setting $f \preccurlyeq g$ if $f = 0$ or $f \neq 0$ and $g \neq 0$ and $\mathfrak{d}_f \preccurlyeq \mathfrak{d}_g$. This notation further extends to the case when $f \in \mathbb{K}[\![\mathfrak{M}]\!]$ and $g \in \mathbb{L}[\![\mathfrak{M}]\!]$ for different rings $\mathbb{K}$ and $\mathbb{L}$. We also write $f \asymp g$ if $f \preccurlyeq g \preccurlyeq f$ and $f \prec g$ if $f \preccurlyeq g$ but $g \not\preccurlyeq f$. Sometimes, it is convenient to use Landau's notation and write $f = O(g)$ or $f = o(g)$ instead of $f \preccurlyeq g$ resp. $f \prec g$.

A family $(f_i)_{i\in I}\in\mathbb{K}[[\mathfrak{M}]]^I$ is said to be a *grid-based* or *summable*, if $\bigcup_{i\in I}\operatorname{supp} f_i$ is grid-based and $\{i\in I: f_{i,\mathfrak{m}}\neq 0\}$ is finite for each $\mathfrak{m}\in\mathfrak{M}$ (notice the double index convention $f_{i,\mathfrak{m}}=(f_i)_{\mathfrak{m}}$). In that case, its sum $F=\sum_{i\in I} f_i$ with $F_{\mathfrak{m}}=\sum_{i\in I} f_{i,\mathfrak{m}}$ is again in $\mathbb{K}[[\mathfrak{M}]]$.

3. Differential algebra

3.1. Selection of the admissible ordering

We assume that the reader is familiar with basic results from differential algebra. Let $\mathbb{F}$ be a field with a finite number of derivations $d_1,\ldots,d_k$. We assume that the space $\mathbb{F}\,d_1+\cdots+\mathbb{F}\,d_k$ is closed under the Lie bracket; the d_i are not required to commute. We will denote

$$\boldsymbol{D}=\{\boldsymbol{d}^{\boldsymbol{i}}=d_1^{i_1}\cdots d_k^{i_k}: i_1,\ldots,i_k\in\mathbb{N}\}.$$

Let $\lambda_1,\ldots,\lambda_k$ be fixed $\mathbb{Q}$-linearly independent numbers in $\mathbb{R}^>$ with

$$\lambda_p<\lambda_i+\lambda_j \qquad (1\leqslant p,i,j\leqslant k). \tag{1}$$

It is easy to construct such λ_1, ..., λ_k: starting with any vector $\boldsymbol{\mu}=(1,\mathrm{e},\ldots,\mathrm{e}^{k-1})$ of $\mathbb{Q}$-linearly independent real numbers, it suffices to produce a sufficiently precise rational approximation $\boldsymbol{\nu}=(\nu_1,\ldots,\nu_k)\in\mathbb{Q}^k$ of $\boldsymbol{\mu}$ and take $\lambda_i=\mu_i/\nu_i$. We define a partial ordering $\trianglelefteq$ and a total ordering $\leqslant$ on $\boldsymbol{D}$ by

$$\begin{aligned}\boldsymbol{d}^{\boldsymbol{i}}\trianglelefteq\boldsymbol{d}^{\boldsymbol{j}} &\iff i_1\leqslant j_1\wedge\cdots\wedge i_k\leqslant j_k\\ \boldsymbol{d}^{\boldsymbol{i}}\leqslant\boldsymbol{d}^{\boldsymbol{j}} &\iff \boldsymbol{\lambda}\cdot\boldsymbol{i}\leqslant\boldsymbol{\lambda}\cdot\boldsymbol{j}.\end{aligned}$$

The total ordering $\leqslant$ is admissible in Ritt's sense.

3.2. Ritt reduction

We will restrict our attention to the ring $\mathbb{P}=\mathbb{F}\{F\}=\mathbb{F}[\boldsymbol{D}F]$ of differential polynomials in one single indeterminate F. Given $P\in\mathbb{P}\setminus\mathbb{F}$, we will write $v_P\in\boldsymbol{D}F$ for its leader, I_P for its initial and S_P for its separant. The relation (1) ensures that $v_{d_i d_j P}=v_{d_j d_i P}$ for all i,j. Let $\boldsymbol{A}\in(\mathbb{P}\setminus\mathbb{F})^p$ a vector of differential polynomials and denote

$$\begin{aligned}\mathcal{H}_{\boldsymbol{A}} &= \{I_{A_1}^{i_1}S_{A_1}^{j_1}\cdots I_{A_p}^{i_p}S_{A_p}^{j_p}: i_1,j_1,\ldots,i_p,j_p\in\mathbb{N}\}.\\ H_{\boldsymbol{A}} &= I_{A_1}S_{A_1}\cdots I_{A_p}S_{A_p}\in\mathcal{H}_{\boldsymbol{A}}.\end{aligned}$$

Given $P\in\mathbb{P}$, Ritt reduction of P w.r.t. $\boldsymbol{A}$ yields a relation

$$H\,P=\boldsymbol{\Theta}\cdot\boldsymbol{A}+R, \tag{2}$$

where $H\in\mathcal{H}_{\boldsymbol{A}}$, R is reduced w.r.t. $\boldsymbol{A}$ and $\boldsymbol{\Theta}\in\mathbb{P}[\boldsymbol{d}]^p$ is a vector of linear differential operators. We understand that $\boldsymbol{\Theta}\cdot\boldsymbol{A}=\Theta_1 A_1+\cdots+\Theta_p A_p$. In particular, if $\boldsymbol{A}$ is a *Rosenfeld basis* (i.e. if $\{A_1,\ldots,A_p\}$ is a coherent auto-reduced set), then Ritt reduction of the Δ-polynomial of A_i and A_j ($i<j$) gives rise to a relation

$$H\,(S_{A_j}\boldsymbol{d}^{\boldsymbol{\alpha}}A_i-S_{A_i}\boldsymbol{d}^{\boldsymbol{\beta}}A_j)=\boldsymbol{\Theta}\cdot\boldsymbol{A}. \tag{3}$$

We will call (3) a *critical relation*. It can be rewritten in the form $\boldsymbol{\Psi}_{i,j}\cdot\boldsymbol{A}=0$.

3.3. Additive and multiplicative conjugation

Given a $P\in\mathbb{P}$ and $\varphi\in\mathbb{F}$, there exist unique differential polynomials $P_{+\varphi},P_{\times\varphi}\in\mathbb{P}$, called *additive* and *multiplicative conjugates* of P, with the properties that

$$\begin{aligned}P_{+\varphi}(f) &= P(\varphi+f)\\ P_{\times\varphi}(f) &= P(\varphi f)\end{aligned}$$

for all $\varphi \in \mathbb{F}$. These notions naturally extend to vectors and operators $\Omega \in \mathbb{P}[\boldsymbol{d}]$; for instance, $(\sum_{\boldsymbol{\alpha} \in \mathbb{N}^p} \Omega_{\boldsymbol{\alpha}} \boldsymbol{d}^{\boldsymbol{\alpha}})_{+\varphi} = \sum_{\boldsymbol{\alpha} \in \mathbb{N}^p} \Omega_{\boldsymbol{\alpha},+\varphi} \boldsymbol{d}^{\boldsymbol{\alpha}}$. We have

$$\begin{aligned} v_{P_{+\varphi}} &= v_P & v_{P_{\times\varphi}} &= v_P \\ I_{P_{+\varphi}} &= I_{P,+\varphi} & I_{P_{\times\varphi}} &= I_{P,\times\varphi} \\ S_{P_{+\varphi}} &= S_{P,+\varphi} & S_{P_{\times\varphi}} &= S_{P,\times\varphi} \end{aligned}$$

Consequently, if P Ritt reduces to R modulo $\boldsymbol{A}$, as in (2), then $P_{+\varphi}$ and $P_{\times\varphi}$ Ritt reduce to $R_{+\varphi}$ and $R_{\times\varphi}$ modulo $\boldsymbol{A}_{+\varphi}$ resp. $\boldsymbol{A}_{\times\varphi}$, with

$$\begin{aligned} H_{+\varphi} P_{+\varphi} &= \boldsymbol{\Theta}_{+\varphi} \cdot \boldsymbol{A}_{+\varphi} + R_{+\varphi} \\ H_{\times\varphi} P_{+\varphi} &= \boldsymbol{\Theta}_{\times\varphi} \cdot \boldsymbol{A}_{\times\varphi} + R_{\times\varphi} \end{aligned}$$

In particular, if $\boldsymbol{A}$ is a Rosenfeld basis with critical relations $\boldsymbol{\Psi}_{i,j} \cdot \boldsymbol{A} = 0$, then so are $\boldsymbol{A}_{+\varphi}$ and $\boldsymbol{A}_{\times\varphi}$, with critical relations $\boldsymbol{\Psi}_{i,j,+\varphi} \cdot \boldsymbol{A}_{+\varphi} = 0$ resp. $\boldsymbol{\Psi}_{i,j,\times\varphi} \cdot \boldsymbol{A}_{\times\varphi} = 0$.

3.4. Change of derivations

Another important construction is to change the derivations. Assume that

$$\begin{aligned} d_i &= \sigma_i d_i' \\ d_i' &= \tau_i d_i \end{aligned}$$

for certain σ_i, $\tau_i \in \mathbb{F}^{\neq}$. Then any differential polynomial $P \in \mathbb{P}$ can be rewritten into a differential polynomial $P' \in \mathbb{P}' = \mathbb{F}[\boldsymbol{D}' F]$ with respect to the derivations $d_1', \ldots, d_k'$ and *vice versa*. Similarly, any operator $\Theta \in \mathbb{P}[\boldsymbol{d}]$ can be rewritten as an operator in $\Theta' \in \mathbb{P}'[\boldsymbol{d}']$. We again have

$$\begin{aligned} v_{P'} &= v_P \\ I_{P'} &= I_P' \\ S_{P'} &= S_P' \end{aligned}$$

Consequently, if P Ritt reduces to R modulo $\boldsymbol{A}$, as in (2), then P' Ritt reduces to R' modulo $\boldsymbol{A}'$, with

$$H' P' = \boldsymbol{\Theta}' \cdot \boldsymbol{A}' + R'.$$

In particular, if $\boldsymbol{A}$ is a Rosenfeld basis with critical relations $\boldsymbol{\Psi}_{i,j} \cdot \boldsymbol{A} = 0$, then so are $\boldsymbol{A}'$, with critical relations $\boldsymbol{\Psi}_{i,j}' \cdot \boldsymbol{A}' = 0$.

3.5. Linear systems

Let $\mathbb{P}_{\text{lin}} = \mathbb{F}[\boldsymbol{d}] F$ be the set of linear homogeneous differential polynomials. A differential ideal $I \subseteq \mathbb{P}$ is said to be *linear*, if it is generated by $I \cap \mathbb{P}_{\text{lin}}$. Given $P \in \mathbb{P}$, we will denote by P_{lin} and P_{cst} its linear and constant parts. We also denote $P_{\text{hi}} = P - P_{\text{lin}} - P_{\text{cst}}$.

Given $P \in \mathbb{P}_{\text{lin}}^{\neq}$, its initial and separant $I_P = S_P \in \mathbb{K}^{\neq}$ are equal and invertible. Ritt reduction of $P \in \mathbb{P}_{\text{lin}}$ with respect to a system $\boldsymbol{A} \in (\mathbb{P}_{\text{lin}}^{\neq})^p$ yields a relation

$$P = \boldsymbol{\Theta} \cdot \boldsymbol{A} + R \tag{4}$$

with $\boldsymbol{\Theta} \in \mathbb{F}[\boldsymbol{d}]$ and $R \in \mathbb{P}_{\text{lin}}$. Similarly, the Δ-polynomial of two polynomials in $\mathbb{P}_{\text{lin}}^{\neq}$ is again in $\mathbb{P}_{\text{lin}}$. In fact, $\mathbb{P}_{\text{lin}}$ as a $\mathbb{F}[\boldsymbol{d}]$-module is isomorphic to the skew-ring $\mathbb{F}[\boldsymbol{d}]$. This leads to the alternative interpretation of (4) as the reduction of P w.r.t. $\boldsymbol{A}$ in the theory of non-commutative Groebner bases. Similarly, Δ-polynomials correspond to S-polynomials and Rosenfeld bases to Groebner bases.

Applying Buchberger's algorithm to a linear system $\boldsymbol{A} \in (\mathbb{P}_{\text{lin}}^{\neq})^p$, we thus obtain a Rosenfeld basis $\boldsymbol{B} \in (\mathbb{P}_{\text{lin}}^{\neq})^q$ with $[\boldsymbol{A}] = [\boldsymbol{B}] = \{\boldsymbol{B}\}$. Any linear differential ideal I may be represented as $I = [\boldsymbol{B}] = \{\boldsymbol{B}\}$ for such a basis $\boldsymbol{B}$.

From now on, we will freely use the concepts of Ritt reduction, critical relations, etc. for operators in $\mathbb{F}[\boldsymbol{d}]$. Given a Rosenfeld basis $\boldsymbol{L} \in \mathbb{F}[\boldsymbol{d}]^p$ and $i < j$, let $\boldsymbol{\Psi}_{i,j} \cdot \boldsymbol{A} = 0$ be the critical relation for L_i and L_j, with $\boldsymbol{\Psi}_{i,j} \in \mathbb{F}[\boldsymbol{d}]$. Let $\boldsymbol{\Sigma}$ be the matrix whose rows are the vectors $\boldsymbol{\Psi}_{i,j}$. It is well-known that these rows generate the module of $\boldsymbol{\Psi} \in \mathbb{F}[\boldsymbol{d}]^p$ with $\boldsymbol{\Psi} \cdot \boldsymbol{A} = 0$. In other words, we have a natural exact sequence

$$\mathbb{F}[\boldsymbol{d}]^{\boldsymbol{\Sigma}} \xrightarrow{\varphi} \mathbb{F}[\boldsymbol{d}]^{\boldsymbol{L}} \xrightarrow{\psi} \mathbb{F}[\boldsymbol{d}] \longrightarrow \mathbb{F}[\boldsymbol{d}]/(\boldsymbol{A}) \longrightarrow 0, \tag{5}$$

where

$$\begin{aligned} \varphi(A\colon \boldsymbol{\Psi}_{i,j} \to A_{i,j}) &= \sum_{i,j} A_{i,j}\, \boldsymbol{\Psi}_{i,j}. \\ \psi(A\colon L_i \to A_i) &= \sum_i A_i\, L_i. \end{aligned}$$

3.6. Ritt co-reduction

Assuming that $\mathbb{F}$ is a field of constants for $\boldsymbol{d}$, we can also consider the natural power series analogue $\mathbb{F}[[\boldsymbol{d}]]$ of $\mathbb{F}[\boldsymbol{d}]$, and apply the theory of standard bases. It will be convenient to regard this as a dual theory.

More precisely, given $L \in \mathbb{F}[[\boldsymbol{d}]]$, we will call the smallest variable $\boldsymbol{d}^{\boldsymbol{i}}$ occurring in L its *co-leader* v_L^*. The corresponding coefficient is called the *co-initial* I_L^* or *co-separant* S_L^*. Ritt *co-reduction* of L w.r.t. $\boldsymbol{A} \in (\mathbb{F}[[\boldsymbol{d}]]^{\neq})^p$ yields a relation

$$P = \boldsymbol{\Theta} \cdot \boldsymbol{A} + R \tag{6}$$

with $\boldsymbol{\Theta} \in \mathbb{F}[[\boldsymbol{d}]]$ and $R \in \mathbb{P}[[\boldsymbol{d}]]$ such that $v_{A_i}^* \ntrianglelefteq v_R^*$ for all i. If $\boldsymbol{A}$ is a Rosenfeld co-basis, then Ritt co-reduction of the *co-Δ-polynomial* of A_i and A_j with $i < j$ yields a *co-critical relation*

$$\boldsymbol{\Psi}_{i,j} \cdot \boldsymbol{A} = 0$$

with $\boldsymbol{\Psi}_{i,j} \in \mathbb{F}[[\boldsymbol{d}]]$. Denoting by $\boldsymbol{\Sigma}$ the matrix of co-critical relations, we again have a natural exact sequence

$$\mathbb{F}[[\boldsymbol{d}]]^{\boldsymbol{\Sigma}} \xrightarrow{\varphi} \mathbb{F}[[\boldsymbol{d}]]^{\boldsymbol{L}} \xrightarrow{\psi} \mathbb{F}[[\boldsymbol{d}]] \longrightarrow \mathbb{F}[[\boldsymbol{d}]]/(\boldsymbol{A}) \longrightarrow 0.$$

The process of Ritt co-reduction can be generalized slightly beyond the linear case. More precisely, let $\mathbb{P}^* = \mathbb{F}[[\boldsymbol{D}F]]_{\text{pol}}$ be the set of series P in the infinite number of variables $\boldsymbol{D}F$, such that P is polynomial in each particular variable in $\boldsymbol{D}F$. Now let $\boldsymbol{A} \in (\mathbb{F}[\boldsymbol{d}]F)^p$ be a linear system. We say that $P \in \mathbb{P}^*$ is *co-reduced* w.r.t. $\boldsymbol{A}$ if its co-leader $V = v_P^*$ does not satisfy $v_{A_i}^* \trianglelefteq V$ for any i. In the contrary case, we may consider P as a polynomial $P_d V^d + \cdots + P_0$ in V and write

$$P = (I_{A_i}^*)^{-1} P_d V^{d-1} \boldsymbol{d}^{\boldsymbol{\alpha}} A_i + \tilde{P}, \tag{7}$$

for the unique $\boldsymbol{\alpha}$ with $V = \boldsymbol{d}^{\boldsymbol{\alpha}} v_{A_i}^*$, with $\deg_V \tilde{P} < d$. After a finite number of such steps, we obtain a relation

$$P = \Phi\, \boldsymbol{d}^{\boldsymbol{\alpha}} A_i + Q, \tag{8}$$

where Q does no longer depend on V and $v_Q^* > V$. Continuing the same process on Q as long as Q is not reduced w.r.t. $\boldsymbol{A}$ results in an infinite sequence of eliminations which converges to a relation

$$P = \boldsymbol{\Theta} \cdot \boldsymbol{A} + R, \tag{9}$$

where $\boldsymbol{\Theta} \in \mathbb{P}^*[[\boldsymbol{d}]]^p$ and R is co-reduced w.r.t. $\boldsymbol{A}$.

4. ASYMPTOTIC DIFFERENTIAL ALGEBRA

4.1. The function field and its derivations

Given formal indeterminates $z_1, \ldots, z_k$, let $\delta_1, \ldots, \delta_k$ denote the valuation-preserving partial derivatives

$$\delta_i = z_i \frac{\partial}{\partial z_i}.$$

Let $\mathbb{U} = \mathbb{Q}(\lambda_1, \ldots, \lambda_k)$ and let $\boldsymbol{\mu}$ be a vector of $\mathbb{U}$-linearly independent positive real numbers. We give $\mathfrak{Z} = z_1^{\mathbb{U}} \cdots z_k^{\mathbb{U}}$ the structure of a monomial group by

$$\boldsymbol{z}^{\boldsymbol{i}} \prec \boldsymbol{z}^{\boldsymbol{j}} \Leftrightarrow \boldsymbol{\mu} \cdot \boldsymbol{i} > \boldsymbol{\mu} \cdot \boldsymbol{j}.$$

Notice that $\mathfrak{Z}$ is isomorphic to an additive subgroup of $\mathbb{R}$. From now on, we will assume that $\mathbb{F}$ is a field of grid-based series of the form

$$\mathbb{F} = \mathbb{K} [\![\mathfrak{Z}]\!],$$

where $\mathbb{K}$ is a differential field for the derivations $\delta_1, \ldots, \delta_k$.

REMARK 2. Most results in this paper still hold if we replace $\mathbb{U}$ by any subgroup of $\mathbb{R}$ which contains $\mathbb{Q}(\lambda_1, \ldots, \lambda_k)$ and for any dominance relation $\preccurlyeq$ on $\mathfrak{Z} = z_1^{\mathbb{U}} \cdots z_k^{\mathbb{U}}$ with $z_1, \ldots, z_k \prec 1$. However, proofs by induction over grid-based supports in $\mathfrak{Z}$ generally have to be replaced by transfinite induction proofs.

Given $\boldsymbol{\delta}^{\boldsymbol{i}} < \boldsymbol{\delta}^{\boldsymbol{j}}$ in $\boldsymbol{\Delta} = \delta_1^{\mathbb{N}} \cdots \delta_k^{\mathbb{N}}$ and $\boldsymbol{z}^{\boldsymbol{\alpha}} \preccurlyeq 1$ in $\mathfrak{Z}$, we will have to consider generalized Newton slopes between terms $\boldsymbol{\delta}^{\boldsymbol{i}} F$ and $\boldsymbol{z}^{\boldsymbol{\alpha}} \boldsymbol{\delta}^{\boldsymbol{j}} F$. Both terms can be "equalized" modulo the change of derivations $\boldsymbol{d} = (\boldsymbol{z}^{\lambda_1 \boldsymbol{\kappa}} \delta_1, \ldots, \boldsymbol{z}^{\lambda_k \boldsymbol{\kappa}} \delta_k)$, where

$$\boldsymbol{\kappa} = \frac{\boldsymbol{\alpha}}{\boldsymbol{\lambda} \cdot (\boldsymbol{j} - \boldsymbol{i})}.$$

If $\boldsymbol{d} \neq \boldsymbol{\delta}$, then the monomials $\boldsymbol{z}^{\lambda_1 \boldsymbol{\kappa}}, \ldots, \boldsymbol{z}^{\lambda_k \boldsymbol{\kappa}}$ are all infinitesimal, and we have

$$[d_i, d_j] = \lambda_j \kappa_i \boldsymbol{z}^{\lambda_i \boldsymbol{\kappa}} d_j - \lambda_i \kappa_j \boldsymbol{z}^{\lambda_j \boldsymbol{\kappa}} d_i \prec 1 \tag{10}$$

for all i, j. In other words, although the d_i do not commute, they do commute from the asymptotic point of view. Similarly, if $\boldsymbol{d} \neq \boldsymbol{\delta}$, then each derivation d_i asymptotically commutes with elements $a \in \mathbb{K}$ in the operator ring $\mathbb{F}[\boldsymbol{d}] = \mathbb{F}[d_1, \ldots, d_k]$:

$$d_i a - a d_i = \delta_i(a) \boldsymbol{z}^{\lambda_i \boldsymbol{\kappa}} \prec 1. \tag{11}$$

4.2. Dominant differential ideals

Let $\boldsymbol{D} = \{\boldsymbol{d}^{\boldsymbol{i}} : \boldsymbol{i} \in \mathbb{N}^k\}$ be as before. Any differential polynomial $P \in \mathbb{P} = \mathbb{K} [\![\mathfrak{Z}]\!] [\boldsymbol{D} F]$ can also be considered as a series $P = \sum_{\mathfrak{m} \in \mathfrak{Z}} \mathfrak{m} P_{\mathfrak{m}}$ in the ring

$$\mathbb{S} = \mathbb{K}[\boldsymbol{D} F] [\![\mathfrak{Z}]\!] \supsetneq \mathbb{P}.$$

Similarly, an operator $\Theta \in \mathbb{S}[\boldsymbol{d}]$ can be considered as a series $\Theta = \sum_{\mathfrak{m} \in \mathfrak{M}} \mathfrak{m} \Theta_{\mathfrak{m}}$ in the ring

$$\mathbb{D} = \mathbb{K}[\boldsymbol{D} F] [\boldsymbol{d}] [\![\mathfrak{Z}]\!].$$

Recall our convention to multiply monomials in the series on the right with the corresponding coefficients.

Because of (10), the set $\mathbb{K}[\boldsymbol{D}F]$ is not necessarily stable under differentiation. It will be convenient to introduce the commutative differential ring $\mathbb{K}[\bar{\boldsymbol{D}}F]$ obtained through formal substitution of d_1, ..., d_k by pairwise commuting variants $\bar{d}_1$, ..., $\bar{d}_k$. The corresponding bijection $P\mapsto\bar{P}$ between $\mathbb{K}[\boldsymbol{D}F]$ and $\mathbb{K}[\bar{\boldsymbol{D}}F]$, with inverse $P\mapsto\underline{P}$, satisfies

$$\underline{\bar{d}_i\,\bar{P}} = d_i\,P + o(1),$$

for all $i\in\{1,\ldots,k\}$ and $P\in\mathbb{K}[\boldsymbol{D}F]$. If $\boldsymbol{d}\neq\boldsymbol{\delta}$, then (11) implies that $\mathbb{K}$ is a field of constants for the derivations $\bar{\boldsymbol{d}}$. Given $P\in\mathbb{S}$, we will denote $D_P=\overline{c_P}$. This notation is extended to differential ideals I of $\mathbb{S}$ using

$$D_I=\{D_P\colon P\in I\}.$$

The set D_I is clearly an ideal of $\mathbb{K}[\boldsymbol{D}F]$. Assume that $D_I\not\ni 1$. Given $p\in D_I$, there exists a $P\in I$ with $p=D_P$. Replacing P by $\mathfrak{d}_P^{-1}P$, we may assume without loss of generality that $P\asymp 1$, whence $P=\underline{p}+E$ with $E\prec 1$. Now we have $\bar{d}_i\,p\neq 0$ and $d_i\,E\prec 1$ for all i, whence $d_i\,P=\underline{\bar{d}_i\,p}+o(1)$ and $\bar{d}_i\,p=D_{d_i\,P}\in D_I$. This proves that D_I is a differential ideal, called the *dominant differential ideal* of I.

REMARK 3. If I is a radical differential ideal, then D_I is not necessarily radical. For instance, if we take $k=1$, $\boldsymbol{d}=\boldsymbol{\delta}$ and $I=\{(F-z_1^{-1})\,(F-z_1^{-2})\}$, then $D_I=[F^2]$.

A differential ideal I of $\mathbb{S}$ is said to be *strong*, if it is closed under grid-based summation. Since $\mathfrak{Z}$ is archimedian, the strong differential ideal $[[I]]$ generated by a differential ideal $I\subseteq\mathbb{S}$ is just the completion of I:

$$[[I]]=\{\hat{f}\in\mathbb{S}\colon\forall\mathfrak{m}\in\mathfrak{Z},\exists f\in\mathbb{S},\hat{f}-f\prec\mathfrak{m}\}. \tag{12}$$

It follows that $D_{[[I]]}=D_I$. Any strong differential ideal I of $\mathbb{S}$ is clearly a $\mathbb{D}$-module. The contrary is not always true: taking $k=1$, the $\mathbb{D}$-module generated by f^2, $(\delta f)^2$, $(\delta^2 f)^2$, ... does not contain $f^2+z\,(\delta f)^2+z^2\,(\delta^2 f)^2+\cdots$.

PROPOSITION 4. *Let $I\subseteq\mathbb{S}$ be a $\mathbb{D}$-module such that D_I is finitely generated. Then I is a strong differential ideal.*

PROOF. Let $P_1,\ldots,P_p\in I$ with $P_1\asymp\cdots\asymp P_p\asymp 1$ be such that D_I is generated by $D_{P_1},\ldots,D_{P_p}$. Given $Q\in I$, we claim that there exists an operator $\boldsymbol{\Theta}\in\mathbb{D}^p$ with

$$\begin{aligned}\operatorname{supp}\boldsymbol{\Theta} &\subseteq \mathfrak{G}=\mathfrak{E}\operatorname{supp}Q\\ \mathfrak{E} &= (\operatorname{supp}P_1\cup\cdots\cup\operatorname{supp}P_p\cup\{\boldsymbol{z}^{\lambda_1\boldsymbol{\kappa}},\ldots,\boldsymbol{z}^{\lambda_k\boldsymbol{\kappa}}\})^*\end{aligned}$$

such that $Q=\boldsymbol{\Theta}\cdot\boldsymbol{P}$. We compute the coefficients $\boldsymbol{\Theta}_{\mathfrak{m}}$ of $\boldsymbol{\Theta}$ by induction over $\mathfrak{m}\in\mathfrak{G}$. Assume that $\boldsymbol{\Theta}_{\succ\mathfrak{m}}$ with $\tilde{Q}=Q-\boldsymbol{\Theta}_{\succ\mathfrak{m}}\cdot\boldsymbol{P}\preccurlyeq\mathfrak{m}$ has been computed. Then $\operatorname{supp}\tilde{Q}\subseteq\mathfrak{G}$ and $\tilde{Q}\in I$. Since D_I is generated by $D_{P_1},\ldots,D_{P_p}$, there exists an operator $\boldsymbol{\Omega}\in\mathbb{K}[\boldsymbol{D}F][\boldsymbol{d}]$ with $Q_{\mathfrak{m}}=\boldsymbol{\Omega}\cdot\boldsymbol{P}_1$. Taking $\boldsymbol{\Theta}_{\mathfrak{m}}=\boldsymbol{\Omega}$, it follows that $Q-\boldsymbol{\Theta}_{\succcurlyeq\mathfrak{m}}\cdot\boldsymbol{P}\prec\mathfrak{m}$, so the induction hypothesis is satisfied for the next monomial $\tilde{\mathfrak{m}}\succ\mathfrak{m}$ in $\mathfrak{G}$. By induction, we thus compute an operator $\boldsymbol{\Theta}\in\mathbb{D}^p$ which satisfies $Q-\boldsymbol{\Theta}_{\succcurlyeq\mathfrak{m}}\cdot\boldsymbol{P}\prec\mathfrak{m}$ for all $\mathfrak{m}\in\mathfrak{G}$. It follows that $Q=\boldsymbol{\Theta}\cdot\boldsymbol{P}$.

Having shown our claim, let $(Q_j)_{j\in J}\in I^J$ be a grid-based family. Then there exists a family of operators $(\boldsymbol{\Theta}_j)_{j\in J}\in(\mathbb{D}^p)^J$ with $Q_j=\boldsymbol{\Theta}_j\cdot\boldsymbol{P}$ and $\operatorname{supp}\boldsymbol{\Theta}_j\subseteq(\operatorname{supp}Q_j)\,\mathfrak{E}$ for all $j\in J$. For each $\mathfrak{m}\in\mathfrak{Z}$, and using the facts that $\{\mathfrak{n}\in\mathfrak{G}\colon\mathfrak{n}\succcurlyeq\mathfrak{m}\}$ is finite and $(Q_j)_{j\in J}$ is grid-based, there are only a finite number of indices $j\in J$ with $Q_j\succcurlyeq\mathfrak{m}$. Consequently, $(\boldsymbol{\Theta}_j)_{j\in J}$ is a grid-based family and $\sum_{j\in J}Q_j=(\sum_{j\in J}\boldsymbol{\Theta}_j)\cdot\boldsymbol{P}\in I$. □

REMARK 5. We do not know yet whether the dominant differential ideal of a finitely generated $\mathbb{D}$-module is necessarily finitely generated as well.

4.3. Asymptotic reduction

Let us consider a vector $\boldsymbol{A}=(A_1,\ldots,A_p)\subseteq \mathbb{S}^p$ with $D_{A_i}\notin\mathbb{K}$ for each i. A series $P\in\mathbb{S}$ is said to be *asymptotically reduced* w.r.t. A if D_P is Ritt reduced w.r.t. $D_{\boldsymbol{A}}=(D_{A_1},\ldots,D_{A_p})$.

Due to the presence of H in (2), a similar division technique as for standard bases will not always work in the case of asymptotic reduction: infinite sequences of partial reductions may give rise to infinite powers of initials and separants. Nevertheless, we will now show that the mechanism does work in the special but important case when all initials and separants are 1.

We say that $\boldsymbol{A}$ is *normal* if $A_i\asymp 1$ and $I_{D_{A_i}}=S_{D_{A_i}}=1$ for all i. Given $P\in\mathbb{S}$, we claim that there exists an expression

$$P=\boldsymbol{\Theta}\cdot\boldsymbol{A}+R, \tag{13}$$

where $\boldsymbol{\Theta}\in\mathbb{D}^p$ satisfies $\boldsymbol{\Theta}\preccurlyeq P$ and R is reduced w.r.t. $\boldsymbol{A}$. We will call $\boldsymbol{\Theta}$ and R the quotient and remainder of the *asymptotic reduction* of P w.r.t. $\boldsymbol{A}$. The coefficients of $\boldsymbol{\Theta}$ and R are computed by induction over the grid-based set

$$\mathfrak{G}=(\operatorname{supp}\boldsymbol{A}\cup\{\boldsymbol{z}^{\lambda_1\boldsymbol{\kappa}},\ldots,\boldsymbol{z}^{\lambda_k\boldsymbol{\kappa}}\})^*\operatorname{supp}P.$$

So let $\mathfrak{m}\in\mathfrak{G}$ and assume that we computed $\boldsymbol{\Theta}_{\mathfrak{n}}$ and $R_{\mathfrak{n}}$ for all $\mathfrak{n}\succ\mathfrak{m}$, in such a way that

$$P=\boldsymbol{\Theta}_{\succ\mathfrak{m}}\cdot\boldsymbol{A}+\tilde{P}, \tag{14}$$

with $\operatorname{supp}\tilde{P}\subseteq\mathfrak{G}$ and $\tilde{P}\preccurlyeq\mathfrak{m}$. We may assume without loss of generality that $\tilde{P}\asymp\mathfrak{m}$ (if $\tilde{P}\prec\mathfrak{m}$, then we may simply take $\boldsymbol{\Theta}_{\mathfrak{m}}=0$). Ritt reduction of $D_{\tilde{P}}$ w.r.t. $D_{\boldsymbol{A}}$ yields a relation

$$D_{\tilde{P}}=\boldsymbol{\Omega}\cdot D_{\boldsymbol{A}}+S,$$

with $\boldsymbol{\Omega}\in\mathbb{K}[\bar{\boldsymbol{D}}F][\bar{\boldsymbol{d}}]^p$ and $S\in\mathbb{K}[\bar{\boldsymbol{D}}F]$. In view of (10) and (11), we get

$$\tilde{P}_{\mathfrak{m}}=\underline{\boldsymbol{\Omega}}\cdot\boldsymbol{A}_1+\underline{S}+\tilde{S} \tag{15}$$

with $\underline{\boldsymbol{\Omega}}\in\mathbb{K}[\boldsymbol{D}F][\boldsymbol{d}]^p$, $\underline{S}\in\mathbb{K}[\boldsymbol{D}F]$ and $\tilde{S}\prec 1$ with $\operatorname{supp}\tilde{S}\subseteq\{\boldsymbol{z}^{\lambda_1\boldsymbol{\kappa}},\ldots,\boldsymbol{z}^{\lambda_k\boldsymbol{\kappa}}\}^*$. Taking $\boldsymbol{\Theta}_{\mathfrak{m}}=\underline{\boldsymbol{\Omega}}$, it follows that

$$\begin{aligned}
P &= \boldsymbol{\Theta}_{\succcurlyeq\mathfrak{m}}\cdot\boldsymbol{A}+\tilde{\tilde{P}}\\
\tilde{\tilde{P}} &= \tilde{P}-\mathfrak{m}\,\boldsymbol{\Theta}_{\mathfrak{m}}\cdot\boldsymbol{A}\\
&= \mathfrak{m}\,\underline{S}+\tilde{P}_{\prec\mathfrak{m}}+\mathfrak{m}\,\underline{\boldsymbol{\Omega}}\cdot\boldsymbol{A}_{\prec 1}+\mathfrak{m}\,\tilde{S}.
\end{aligned}$$

If $\underline{S}\neq 0$, then $\tilde{\tilde{P}}$ is reduced w.r.t. $\boldsymbol{A}$ and the construction stops. Otherwise, we have $\operatorname{supp}\tilde{\tilde{P}}\subseteq\mathfrak{G}$ and $\tilde{\tilde{P}}\prec\mathfrak{m}$ and the induction hypothesis is satisfied for the next monomial $\tilde{\mathfrak{m}}\in\mathfrak{G}$.

We will say that $\boldsymbol{A}$ is *quasi-normal* if $I_{D_{A_i}}=S_{D_{A_i}}\in\mathbb{K}^{\neq}$ for all i. In that case, let $B_i=A_i/([\boldsymbol{d}^{\boldsymbol{r}_i}F]A_i)$ for each i, where $\bar{\boldsymbol{d}}^{\boldsymbol{r}_i}F$ is the leader of D_{A_i} and $[\boldsymbol{d}^{\boldsymbol{r}_i}F]A_i$ the coefficient of $\boldsymbol{d}^{\boldsymbol{r}_i}F$ in A_i. Then $\boldsymbol{B}=(B_1,\ldots,B_p)$ is normal and we call it the *normalization* of $\boldsymbol{A}$.

REMARK 6. If $\boldsymbol{A}$ is not quasi-normal, then the same argument still leads to a relation

$$HP=\boldsymbol{\Theta}\cdot\boldsymbol{A}+R,$$

with $H\in c_{I_{A_1}}^{\mathbb{N}}c_{S_{A_1}}^{\mathbb{N}}\cdots c_{I_{A_p}}^{\mathbb{N}}c_{S_{A_p}}^{\mathbb{N}}$ whenever the reduction process stops. However, we no longer obtain a nice relation in the case when P reduces to 0.

4.4. Asymptotic co-reduction

As we did in section 3.6 for Ritt reduction, it is natural to introduce asymptotic co-reduction, the dual notion of asymptotic reduction. Since $1+\delta+\delta^2+\cdots$ cannot be applied to z, we will assume $\boldsymbol{d}\neq\boldsymbol{\delta}$. The set

$$\mathbb{S}^* = \mathbb{K}[[\boldsymbol{D}F]]_{\text{pol}}[\![\mathfrak{Z}]\!]$$

plays the role of $\mathbb{S}$ in the dual setting. Similarly, operator algebra

$$\mathbb{D}^* = \mathbb{K}[[\boldsymbol{D}F]]_{\text{pol}}[[\boldsymbol{d}]]\,[\![\mathfrak{Z}]\!]$$

is the natural counterpart of $\mathbb{D}$. In a similar way as before, one introduces the commutative variant $\mathbb{K}[[\bar{\boldsymbol{D}}\ F]]_{\text{pol}}$ of $\mathbb{K}[[\boldsymbol{D}F]]_{\text{pol}}$. Again, the dominant part $D_I \subseteq \mathbb{K}[[\bar{\boldsymbol{D}}\ F]]_{\text{pol}}$ of a differential ideal $I\subseteq\mathbb{S}^*$ is a differential ideal.

Any series $P\in\mathbb{S}^*$ can also be regarded as a series in the variables $\boldsymbol{D}F$ and $\boldsymbol{z}$, with a support $\operatorname{supp}' P$ which is a subset of $\mathfrak{F}=\{\boldsymbol{d}^{\boldsymbol{\alpha}_1}F\cdots\boldsymbol{d}^{\boldsymbol{\alpha}_r}F\,\mathfrak{z}: \boldsymbol{\alpha}_1,\ldots,\boldsymbol{\alpha}_r\in\mathbb{N},\mathfrak{z}\in\mathfrak{Z}\}$. A subset $\mathfrak{G}\subseteq\mathfrak{F}$ is said to be admissible if it occurs as the support of a series in $\mathbb{S}^*$. A family $(P_i)_{i\in I}\in(\mathbb{S}^*)^I$ is said to be *summable* if $\bigcup_{i\in I}\operatorname{supp}' P_i$ is admissible and $\{i\in I: P_{\mathfrak{f}}\neq 0\}$ is finite for each $\mathfrak{f}\in\mathfrak{F}$. In that case, $\sum_{i\in I}P_i\in\mathbb{S}^*$. A differential ideal I of $\mathbb{S}^*$ is said to be *strong* if it is closed under infinite summation.

Let $\mathbb{S}^*_{[\geqslant w]}$ be the set of series in $\mathbb{S}^*$ which are strong linear combinations of monomials $\boldsymbol{d}^{\boldsymbol{\alpha}_1}F\cdots\boldsymbol{d}^{\boldsymbol{\alpha}_r}F\,\mathfrak{z}\in\mathfrak{F}$ of weight $\alpha_{1,1}+\cdots+\alpha_{1,k}+\cdots+\alpha_{r,1}+\cdots+\alpha_{r,k}$ at least w. The strong differential ideal $[[I]]$ generated by a differential ideal $I\subseteq\mathbb{S}^*$ coincides with a suitable completion of I:

$$[[I]]=\{\hat{f}\in\mathbb{S}: \forall\mathfrak{m}\in\mathfrak{Z}, \forall w\in\mathbb{N}, \exists f\in\mathbb{S}^*, \exists\varepsilon\in\mathbb{S}^*_{[\geqslant w]}, \hat{f}-f-\varepsilon\prec\mathfrak{m}\}. \tag{16}$$

It follows that $D_{[[I]]}=[[D_I]]$. Any strong differential ideal of $\mathbb{S}^*$ is a $\mathbb{D}^*$-module. Any $\mathbb{D}^*$-module $I\subseteq\mathbb{S}^*$, such that D_I is strong and finitely generated as a strong differential ideal, is a strong differential ideal.

Consider a vector $\boldsymbol{A}\in(\mathbb{S}^*)^p$ with $D_{A_i}\notin\mathbb{S}$ for all i. We will say that $\boldsymbol{A}$ is co-normal, if $A_i\asymp 1$, $I^*_{D_{A_i}}=S^*_{D_{A_i}}=1$ and D_{A_i} is linear for all i. Given $P\in\mathbb{S}^*$, we say that P is *asymptotically co-reduced* w.r.t. $\boldsymbol{A}$ if $D_P\in\mathbb{K}$ or $v^*_{A_i}\not\trianglelefteq v^*_P$ for all i. In a similar way as above, *asymptotic co-reduction* of P w.r.t. $\boldsymbol{A}$ yields a relation

$$P=\boldsymbol{\Theta}\cdot\boldsymbol{A}+R,$$

where $\boldsymbol{\Theta}\in(\mathbb{D}^*)^p$ satisfies $\boldsymbol{\Theta}\preccurlyeq P$ and R is co-reduced w.r.t. $\boldsymbol{A}$.

5. Quasi-linear differential ideals

5.1. Asymptotic bases

Let $\boldsymbol{A}\in\mathbb{S}^p$ be normal in the sense of section 4.3 and assume that D_{A_i} is linear for each i. We say that $\boldsymbol{A}$ is an *asymptotic basis*, if $D_{\boldsymbol{A}}$ is a Rosenfeld basis and if each pair (A_i,A_j) with $i<j$ satisfies an *asymptotic critical relation*

$$\boldsymbol{d}^{\boldsymbol{\alpha}}A_i-\boldsymbol{d}^{\boldsymbol{\beta}}A_j=\boldsymbol{\Theta}\cdot\boldsymbol{A}, \tag{17}$$

where $\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbb{N}^k$ and $\boldsymbol{\Theta}\in\mathbb{D}$ are such that $\bar{\boldsymbol{d}}^{\boldsymbol{\alpha}}D_{A_i}-\bar{\boldsymbol{d}}^{\boldsymbol{\beta}}D_{A_j}=D_{\boldsymbol{\Theta}}\cdot D_{\boldsymbol{A}}$ is a critical relation for D_{A_i} and D_{A_j}. We call $\boldsymbol{d}^{\boldsymbol{\alpha}}A_i-\boldsymbol{d}^{\boldsymbol{\beta}}A_j$ the *asymptotic Δ-polynomial* of A_i and A_j. We will denote by $\boldsymbol{\Sigma}$ the matrix formed by the $q=p\,(p-1)/2$ row vectors $\boldsymbol{\Psi}_{i,j}\in\mathbb{D}^p$, such that $\boldsymbol{\Psi}_{i,j}\cdot\boldsymbol{A}=0$ is an asymptotic critical relation.

More generally, we say that $\boldsymbol{A} \in \mathbb{S}^p$ is an *asymptotic basis*, if $\boldsymbol{A}$ is quasi-normal and if its normalization is an asymptotic basis in the above sense.

LEMMA 7. *Let $\boldsymbol{A} \in \mathbb{S}^p$ be a normal asymptotic basis and $\boldsymbol{\Psi} \in \mathbb{D}^p$. Then there exists an operator $\boldsymbol{\Phi} \in \mathbb{D}^q$ such that $\boldsymbol{\Psi} - \boldsymbol{\Phi} \cdot \boldsymbol{\Sigma} \preccurlyeq \boldsymbol{\Psi} \cdot \boldsymbol{A}$.*

PROOF. We will construct $\boldsymbol{\Phi}$ by induction over the grid-based set

$$\mathfrak{G} = (\operatorname{supp} \boldsymbol{A} \cup \{\boldsymbol{z}^{\lambda_1 \boldsymbol{\kappa}}, \ldots, \boldsymbol{z}^{\lambda_k \boldsymbol{\kappa}}\})^* \operatorname{supp} \boldsymbol{\Psi}.$$

Let $\mathfrak{m} \in \mathfrak{G}$ be such that $\boldsymbol{\Phi}_{\succ \mathfrak{m}}$ has been computed. As the induction hypothesis, assume that $\mathfrak{m} \succ \boldsymbol{\Psi} \cdot \boldsymbol{A}$, $\operatorname{supp} \boldsymbol{\Phi}_{\succ \mathfrak{m}} \subseteq \mathfrak{G}$ and

$$\tilde{\boldsymbol{\Psi}} = \boldsymbol{\Psi} - \boldsymbol{\Phi}_{\succ \mathfrak{m}} \cdot \boldsymbol{\Sigma} \preccurlyeq \mathfrak{m}.$$

Since $\tilde{\boldsymbol{\Psi}} \cdot \boldsymbol{A} \prec \mathfrak{m}$, we have $\tilde{\boldsymbol{\Psi}}_{\mathfrak{m}} \cdot D_{\boldsymbol{A}} = 0$. Hence, the exactness of (5) implies the existence of a vector $\boldsymbol{V}$ with entries in $\mathbb{K}[\boldsymbol{D}F][\boldsymbol{d}]$, such that $\tilde{\boldsymbol{\Psi}}_{\mathfrak{m}} = \boldsymbol{V} \cdot \boldsymbol{\Sigma}_1$. Taking $\boldsymbol{\Phi}_{\mathfrak{m}} = \boldsymbol{V}$, we have

$$\tilde{\tilde{\boldsymbol{\Psi}}} = \boldsymbol{\Psi} - \boldsymbol{\Phi}_{\succcurlyeq \mathfrak{m}} \cdot \boldsymbol{\Sigma} \prec \mathfrak{m}$$

and $\operatorname{supp} \tilde{\tilde{\boldsymbol{\Psi}}} \subseteq \mathfrak{G}$. By induction over $\mathfrak{G}$, we obtain a vector $\boldsymbol{\Phi}$ with $\boldsymbol{\Psi} - \boldsymbol{\Phi} \cdot \boldsymbol{\Sigma} \preccurlyeq \boldsymbol{\Psi} \cdot \boldsymbol{A}$. □

COROLLARY 8. *The strong $\mathbb{D}$-module of operators $\boldsymbol{\Psi} \in \mathbb{D}^p$ with $\boldsymbol{\Psi} \cdot \boldsymbol{A} = 0$ is generated by the rows of $\boldsymbol{\Sigma}$.* □

COROLLARY 9. *Let $\boldsymbol{A} \in \mathbb{S}^p$ be a normal asymptotic basis. Then the strong differential ideal generated by $\boldsymbol{A}$ is given by $[[\boldsymbol{A}]] = \mathbb{D}^p \cdot \boldsymbol{A}$ and coincides with the set of $P \in \mathbb{S}$ which reduce asymptotically to 0 w.r.t. $\boldsymbol{A}$.*

PROOF. Given $P = \boldsymbol{\Psi} \cdot \boldsymbol{A} \in \mathbb{D}^p \cdot \boldsymbol{A}$ with $P \neq 0$, the lemma implies that there exists an operator $\tilde{\boldsymbol{\Psi}} = \boldsymbol{\Psi} - \boldsymbol{\Phi} \cdot \boldsymbol{\Sigma}$ with $P = \tilde{\boldsymbol{\Psi}} \cdot \boldsymbol{A}$ and $\tilde{\boldsymbol{\Psi}} \preccurlyeq P$. In particular, $D_P \in [D_{\boldsymbol{A}}]$, so that $[[\boldsymbol{A}]] = \mathbb{D}^p \cdot \boldsymbol{A}$, by proposition 4. Furthermore, $D_P \in [D_{\boldsymbol{A}}]$ implies that P is not asymptotically reduced w.r.t. $\boldsymbol{A}$. Now let R be the remainder of the asymptotic reduction of P w.r.t. $\boldsymbol{A}$. Since $R \in [[\boldsymbol{A}]]$, the above argument shows that $R = 0$. □

A differential ideal I of $\mathbb{S}$ is said to be *quasi-linear* if D_I is linear.

PROPOSITION 10. *Let I be a strong differential ideal of $\mathbb{S}$. Then the following conditions are equivalent:*

a) I is quasi-linear.

b) I admits a normal asymptotic basis.

PROOF. Assume that I is quasi-linear. By definition, there exists $\boldsymbol{A} \in \mathbb{S}^p$ with $A_i \asymp 1$, $D_{A_i} \in \mathbb{K}[\boldsymbol{d}]F$, $I_{A_i} = S_{A_i} = 1$ for each i, and such that $D_{\boldsymbol{A}}$ is a Rosenfeld basis. Given $i < j$, we claim that asymptotic Ritt reduction of the asymptotic Δ-polynomial of A_i and A_j yields an asymptotic critical relation (17). Indeed, the remainder R of the asymptotic reduction must vanish, since $D_R \in D_I = D_{[\boldsymbol{A}]}$ and D_R is reduced w.r.t. $\boldsymbol{A}$. We thus get a relation (17). Furthermore, Ritt reduction of $\bar{\boldsymbol{d}}^{\boldsymbol{\alpha}} D_A - \bar{\boldsymbol{d}}^{\boldsymbol{\beta}} D_B$ w.r.t. $D_{\boldsymbol{A}}$ yields $\bar{\boldsymbol{d}}^{\boldsymbol{\alpha}} D_A - \bar{\boldsymbol{d}}^{\boldsymbol{\beta}} D_B = D_{\boldsymbol{\Theta}} \cdot D_{\boldsymbol{A}}$, which is thereby a critical relation for D_{A_i} and D_{A_j}.

Inversely, assume that I admits a normal asymptotic basis $\boldsymbol{A}$. Given $P \in I$, corollary 9 implies that P reduces to 0 modulo $\boldsymbol{A}$. In particular, $D_P \in [D_{\boldsymbol{A}}]$ and $[D_{\boldsymbol{A}}]$ is a linear differential ideal. □

5.2. Asymptotic co-bases

It is rather straightforward to dualize the theory from the previous section. Let us briefly state the dual versions of the main results, while omitting proofs.

Let $\boldsymbol{A} \in (\mathbb{S}^*)^p$ be co-normal in the sense of section 4.4. In particular, D_{A_i} is linear for each i. We say that $\boldsymbol{A}$ is an *asymptotic co-basis*, if $D_{\boldsymbol{A}}$ is a Rosenfeld co-basis and if for each pair $i < j$, we have an *asymptotic co-critical relation*

$$\boldsymbol{d}^{\boldsymbol{\alpha}} A_i - \boldsymbol{d}^{\boldsymbol{\beta}} A_j = \boldsymbol{\Theta} \cdot \boldsymbol{A}, \tag{18}$$

where $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^k$ and $\boldsymbol{\Theta} \in \mathbb{D}^*$ are such that $\bar{\boldsymbol{d}}^{\boldsymbol{\alpha}} D_A - \bar{\boldsymbol{d}}^{\boldsymbol{\beta}} D_B = D_{\boldsymbol{\Theta}} \cdot D_{\boldsymbol{A}}$ is a co-critical relation for D_{A_i} and D_{A_j}. We will denote by $\boldsymbol{\Sigma}$ the matrix formed by the $q = p\,(p-1)/2$ row vectors $\boldsymbol{\Psi}_{i,j} \in (\mathbb{D}^*)^p$, such that $\boldsymbol{\Psi}_{i,j} \cdot \boldsymbol{A} = 0$ is an asymptotic critical relation. A differential ideal I of $\mathbb{S}^*$ is said to be *quasi-linear* if D_I is linear.

PROPOSITION 11. *Let $\boldsymbol{A}$ be a co-normal asymptotic co-basis. The strong $\mathbb{D}^*$-module of operators $\boldsymbol{\Psi} \in (\mathbb{D}^*)^p$ with $\boldsymbol{\Psi} \cdot \boldsymbol{A} = 0$ is generated by the rows of $\boldsymbol{\Sigma}$.* □

PROPOSITION 12. *Let $\boldsymbol{A} \in (\mathbb{S}^*)^p$ be a co-normal asymptotic co-basis. Then $[[\boldsymbol{A}]] = (\mathbb{D}^*)^p \cdot \boldsymbol{A}$ coincides with the set of $P \in \mathbb{S}^*$ which co-reduce asymptotically to* 0 *w.r.t. $\boldsymbol{A}$.*

PROPOSITION 13. *Let I be a strong differential ideal of $\mathbb{S}^*$. Then the following conditions are equivalent:*

a) I is quasi-linear.

b) I admits a co-normal asymptotic co-basis.

5.3. Solving quasi-linear equations with respect to $\boldsymbol{\delta}$

In this section, we consider an asymptotic basis $\boldsymbol{A}$ in the special case when $\boldsymbol{\delta} = \boldsymbol{d}$ and $\mathbb{K}$ is a field of constants for $\boldsymbol{\delta}$. We will show the existence of solutions to the equation $\boldsymbol{A}(f) = \boldsymbol{0}$ in the ring $\mathbb{K}[\log \boldsymbol{z}][[\mathfrak{Z}]]$, where $\mathbb{K}[\log \boldsymbol{z}] = \mathbb{K}[\log z_1, \ldots, \log z_k]$.

LEMMA 14. *Let $\boldsymbol{A} \in (\mathbb{K}[\boldsymbol{\delta}]\, F \oplus \mathbb{K}[\log\, \boldsymbol{z}])^p$ be a Rosenfeld basis. Then there exists an $f \in \mathbb{K}[\log \boldsymbol{z}]$ with $\boldsymbol{A}(f) = \boldsymbol{0}$.*

PROOF. Applying the tangent cone algorithm to $\boldsymbol{A}_{\text{lin}}$, we obtain a matrix $\boldsymbol{\Theta}$ with coefficients in $\mathbb{K}[[\boldsymbol{\delta}]]$ such that $\boldsymbol{\Theta} \cdot \boldsymbol{A}_{\text{lin}} \in (\mathbb{K}[\boldsymbol{\delta}]\, F)^q$ is a Rosenfeld co-basis. We claim that $\boldsymbol{g} = -\boldsymbol{\Theta} \cdot \boldsymbol{A}_{\text{cst}}$ is compatible with $\boldsymbol{\Theta} \cdot \boldsymbol{A}_{\text{lin}}$ in the sense of [vdH07, Section 3.2], i.e. that $\boldsymbol{\Psi} \cdot \boldsymbol{g} = 0$ for any $\boldsymbol{\Psi} \in \mathbb{K}[[\boldsymbol{\delta}]]^q$ with $\boldsymbol{\Psi} \cdot \boldsymbol{\Theta} \cdot \boldsymbol{A}_{\text{lin}} = 0$. Indeed, let $\boldsymbol{\Sigma}$ be the matrix of critical relations for the Rosenfeld basis $\boldsymbol{A}_{\text{lin}}$. Since $\mathbb{K}[[\boldsymbol{\delta}]]$ is a flat ring over $\mathbb{K}[\boldsymbol{\delta}]$, the exact sequence

$$\mathbb{K}[\boldsymbol{d}]^{\boldsymbol{\Sigma}} \longrightarrow \mathbb{K}[\boldsymbol{d}]^{\boldsymbol{A}_{\text{lin}}} \longrightarrow \mathbb{K}[\boldsymbol{d}] \longrightarrow \mathbb{K}[\boldsymbol{d}]/(\boldsymbol{A}_{\text{lin}}) \longrightarrow 0$$

transforms into an exact sequence

$$\mathbb{K}[[\boldsymbol{d}]]^{\boldsymbol{\Sigma}} \longrightarrow \mathbb{K}[[\boldsymbol{d}]]^{\boldsymbol{A}_{\text{lin}}} \longrightarrow \mathbb{K}[[\boldsymbol{d}]] \longrightarrow \mathbb{K}[[\boldsymbol{d}]]/(\boldsymbol{A}_{\text{lin}}) \longrightarrow 0.$$

In other words, given $\boldsymbol{\Psi} \in \mathbb{K}[[\boldsymbol{\delta}]]^q$ with $\boldsymbol{\Psi} \cdot \boldsymbol{\Theta} \cdot \boldsymbol{A}_{\text{lin}} = 0$, there exists a $\boldsymbol{\Phi} \in \mathbb{K}[[\boldsymbol{d}]]^{\boldsymbol{\Sigma}}$ with $\boldsymbol{\Psi} \cdot \boldsymbol{\Theta} = \boldsymbol{\Phi} \cdot \boldsymbol{\Sigma}$. It follows that

$$\boldsymbol{\Psi} \cdot \boldsymbol{\Theta} \cdot \boldsymbol{A}_{\text{lin}} - \boldsymbol{\Psi} \cdot \boldsymbol{g} = \boldsymbol{\Psi} \cdot \boldsymbol{\Theta} \cdot \boldsymbol{A} = \boldsymbol{\Phi} \cdot \boldsymbol{\Sigma} \cdot \boldsymbol{A} = 0.$$

In [vdH07, Section 4.2], we have shown how to compute a solution in $\mathbb{K}[\log \boldsymbol{z}]$ to

$$\boldsymbol{\Theta} \cdot \boldsymbol{A}(f) = \boldsymbol{\Theta} \cdot \boldsymbol{A}_{\text{lin}}(f) - \boldsymbol{g} = \boldsymbol{0}.$$

Now $\boldsymbol{A}_{\text{lin}}$ is in the strong ideal generated by $\boldsymbol{\Theta} \cdot \boldsymbol{A}_{\text{lin}}$, so there exists a matrix $\tilde{\boldsymbol{\Theta}}$ with coefficients in $\mathbb{K}[[\boldsymbol{\delta}]]$ with $\tilde{\boldsymbol{\Theta}} \cdot \boldsymbol{\Theta} \cdot \boldsymbol{A}_{\text{lin}} = \boldsymbol{A}_{\text{lin}}$. Consequently, there exists a matrix $\tilde{\boldsymbol{\Phi}}$ with entries in $\mathbb{K}[[\boldsymbol{d}]]$ and $\tilde{\boldsymbol{\Theta}} \cdot \boldsymbol{\Theta} - \text{Id} = \tilde{\boldsymbol{\Phi}} \cdot \boldsymbol{\Sigma}$. We conclude that $\tilde{\boldsymbol{\Theta}} \cdot \boldsymbol{\Theta} \cdot \boldsymbol{A} - \boldsymbol{A} = \tilde{\boldsymbol{\Phi}} \cdot \boldsymbol{\Sigma} \cdot \boldsymbol{A} = \boldsymbol{0}$ and $\boldsymbol{A}(f) = \tilde{\boldsymbol{\Theta}} \cdot \boldsymbol{\Theta} \cdot \boldsymbol{A}(f) = \boldsymbol{0}$. □

PROPOSITION 15. *Let $\boldsymbol{A} \in \mathbb{S}^p$ be an asymptotic basis. Then there exists an $f \in \mathbb{K}[\log \boldsymbol{z}][[\mathfrak{Z}]]$ with $\boldsymbol{A}(f) = \boldsymbol{0}$ and $f \prec 1$.*

PROOF. We may assume that $\boldsymbol{A}$ is normal. We compute the coefficients $f_{\mathfrak{m}}$ of a solution $f \in \mathbb{K}[\log \boldsymbol{z}][[\mathfrak{Z}]]$ with $\operatorname{supp} f \subseteq \mathfrak{G} = (\operatorname{supp} \boldsymbol{A})^*$ by induction over $\mathfrak{m}$. We have $f_1 = 0$. Given $\mathfrak{m} \in \mathfrak{G}$, assume that $f_{\mathfrak{n}}$ has been computed for all $\mathfrak{n} \succ \mathfrak{m}$, in such a way that $\boldsymbol{A}_{+\varphi, \times \mathfrak{m}, \text{cst}} \preccurlyeq \boldsymbol{A}_{+\varphi, \times \mathfrak{m}} \asymp \mathfrak{m}$ for $\varphi = f_{\succ \mathfrak{m}}$. We have

$$\begin{aligned} \boldsymbol{A}_{+\varphi, \times \mathfrak{m}, \text{cst}} &\preccurlyeq \mathfrak{m} \\ \boldsymbol{A}_{+\varphi, \times \mathfrak{m}, \text{lin}} &\asymp \mathfrak{m} \\ \boldsymbol{A}_{+\varphi, \times \mathfrak{m}, \text{hi}} &\prec \mathfrak{m} \end{aligned}$$

and each asymptotic critical relation

$$\boldsymbol{\delta}^{\boldsymbol{\alpha}} A_i - \boldsymbol{\delta}^{\boldsymbol{\beta}} A_j = \boldsymbol{\Theta} \cdot \boldsymbol{A}$$

gives rise to a relation

$$\boldsymbol{\delta}^{\boldsymbol{\alpha}} A_{i, +\varphi, \times \mathfrak{m}} - \boldsymbol{\delta}^{\boldsymbol{\beta}} A_{j, +\varphi, \times \mathfrak{m}} = \boldsymbol{\Theta} \cdot \boldsymbol{A}_{+\varphi, \times \mathfrak{m}}.$$

Extracting dominant parts, it follows that $D_{\boldsymbol{A}_{+\varphi, \times \mathfrak{m}}}$ is a Rosenfeld basis. Moreover, $\boldsymbol{A}_{+\varphi, \times \mathfrak{m}, \text{lin}} = \boldsymbol{A}_{\text{lin}, \times \mathfrak{m}} + o(\mathfrak{m})$, whence $D_{\boldsymbol{A}_{+\varphi, \times \mathfrak{m}}} \in (\mathbb{K}[\boldsymbol{\delta}] F \oplus \mathbb{K}[\log \boldsymbol{z}])^p$. By lemma 14, it follows that $D_{\boldsymbol{A}_{+\varphi, \times \mathfrak{m}}}$ admits a solution $g \in \mathbb{K}[\log \boldsymbol{z}]$. Taking $f_{\mathfrak{m}} = g$ and $\tilde{\varphi} = f_{\succcurlyeq \mathfrak{m}}$, we have $\operatorname{supp} \boldsymbol{A}_{+\tilde{\varphi}} \subseteq \mathfrak{G}$ and $\boldsymbol{A}_{+\tilde{\varphi}, \times \mathfrak{m}, \text{cst}} \prec \boldsymbol{A}_{+\tilde{\varphi}, \times \mathfrak{m}}$, so the induction hypothesis is verified for the next monomial in $\mathfrak{G}$. □

6. PROVING QUASI-LINEARITY

6.1. Initial quasi-linearity

LEMMA 16. *Let $\boldsymbol{A} \in \mathbb{P}^p$ be a Rosenfeld basis w.r.t. $\boldsymbol{d} = (\boldsymbol{z}^{\lambda_1 \boldsymbol{\kappa}} \delta_1, \ldots, \boldsymbol{z}^{\lambda_k \boldsymbol{\kappa}} \delta_k)$ and let $\boldsymbol{d}^{\boldsymbol{r}_i} F$ be the leader of A_i for each i. Assume that $A_{i,\text{cst}} = 0$ and $A_i = \boldsymbol{d}^{\boldsymbol{r}_i} F + o(1)$ for all i. Consider the critical relation*

$$H\,(S_{A_j} \boldsymbol{d}^{\boldsymbol{\alpha}} A_i - S_{A_i} \boldsymbol{d}^{\boldsymbol{\beta}} A_j) = \boldsymbol{\Theta} \cdot \boldsymbol{A} \tag{19}$$

obtained by Ritt reduction of $S_{A_j} \boldsymbol{d}^{\boldsymbol{\alpha}} A_i - S_{A_i} \boldsymbol{d}^{\boldsymbol{\beta}} A_j$ modulo $\boldsymbol{A}$. Then $\boldsymbol{\Theta} \preccurlyeq H$. Moreover, if $I_{A_i} = I_{A_{i,\text{cst}}} + o(I_{A_i})$ for each i, then $\boldsymbol{A}$ is an asymptotic basis.

PROOF. Let $R_0 = S_{A_j} \boldsymbol{d}^{\boldsymbol{\alpha}} A_i - S_{A_i} \boldsymbol{d}^{\boldsymbol{\beta}} A_j$. The process of Ritt reduction yields a sequence of relations

$$H_p R_p = Q_p \boldsymbol{d}^{\boldsymbol{\gamma}_p} A_{i_p} + R_{p+1}, \tag{20}$$

obtained by pseudo-division of R_p by $\boldsymbol{d}^{\boldsymbol{\gamma}_p} A_{i_p}$, considered as polynomials in the common leader $V = \boldsymbol{d}^{\boldsymbol{\gamma}_p + \boldsymbol{r}_{i_p}} F$ of R_p and $\boldsymbol{d}^{\boldsymbol{\gamma}_p} A_{i_p}$. In particular, R_{p+1} is free from V. At the end of the reduction process, we have $R_q = 0$ for some q.

Let us proof by induction over p that $Q_p \preccurlyeq H_p\, R_{i_p}$ and thus $R_{p+1} \preccurlyeq H_p\, R_{i_p}$. From $\operatorname{val}_{\bar{V}}(D_{\boldsymbol{d}^{\boldsymbol{\gamma}_p} A_{i_p}}) = 1$, we deduce $v = \operatorname{val}_{\bar{V}}(D_{Q_p \boldsymbol{d}^{\boldsymbol{\gamma}_p} A_{i_p}}) \geqslant 1$. Let C be the coefficient of V^v in $Q_p\, \boldsymbol{d}^{\boldsymbol{\gamma}_p}\, A_{i_p}$. Then C is also the coefficient of V^v in $H_p\, R_p$, since R_{p+1} is free from V. In particular, $Q_p\, \boldsymbol{d}^{\boldsymbol{\gamma}_p}\, A_{i_p} \asymp C \preccurlyeq H_p\, R_p$. Telescoping the relations (20) for $p = 0, \ldots, q-1$, we get $\boldsymbol{\Theta} \preccurlyeq H$.

The hypothesis $A_i = \boldsymbol{d}^{\boldsymbol{r}_i} F + o(1)$ implies in particular that $S_{A_i} = 1 + o(1)$. If we also have $I_{A_i} = I_{A_{i,\mathrm{cst}}} + o(I_{A_i})$ for each i, then $H = H_{\mathrm{cst}} + o(H)$, whence (19) can be divided by $\mathfrak{d}_H$ and rewritten into an asymptotic critical relation. □

Proposition 17. *Let $\boldsymbol{A}$ be a Rosenfeld basis w.r.t. $\boldsymbol{\delta}$ and assume that $f \in \mathbb{F}$ satisfies $\boldsymbol{A}(f) = \boldsymbol{0}$, but $H_{\boldsymbol{A}}(f) \neq 0$. Then there exist $\mathfrak{m}, \mathfrak{n} \in \mathfrak{Z}$ and $\boldsymbol{d} = (\boldsymbol{z}^{\lambda_1 \boldsymbol{\kappa}} \delta_1, \ldots, \boldsymbol{z}^{\lambda_k \boldsymbol{\kappa}} \delta_k)$, such that $\boldsymbol{A}_{+f_{\succ \mathfrak{m}}, \times \mathfrak{n}}$ is an asymptotic basis w.r.t. $\boldsymbol{d}$.*

Proof. Consider the Rosenfeld basis $\boldsymbol{B} = \boldsymbol{A}_{+f}$ w.r.t. $\boldsymbol{\delta}$, which satisfies $B_{i,\mathrm{cst}} = 0$ for all i. Let $\boldsymbol{\delta}^{\boldsymbol{r}_i} F$ be the leader of B_i for each i. Since $H_{\boldsymbol{B},\mathrm{cst}} \neq 0$, the coefficient $[\boldsymbol{\delta}^{\boldsymbol{r}_i} F] B_i \in \mathbb{F}$ of $\boldsymbol{\delta}^{\boldsymbol{r}_i} F$ in B_i does not vanish. Taking $\boldsymbol{z}^{\boldsymbol{\kappa}}$ sufficiently small, we have $[\boldsymbol{d}^{\boldsymbol{r}_i} F] B_i \succ [\boldsymbol{d}^{\boldsymbol{s}} F] B_i$ for all $\boldsymbol{s} \neq \boldsymbol{r}_i$. From now on, let us consider the entries of $\boldsymbol{A}$ and $\boldsymbol{B}$ as differential polynomials in $\boldsymbol{d}$. Modulo a multiplicative conjugation by a sufficiently small $\mathfrak{n} \in \mathfrak{Z}$, we may assume without loss of generality that $B_i = B_{i,\mathrm{lin}} + o(B_i)$ for all i, while preserving the fact that $[\boldsymbol{d}^{\boldsymbol{r}_i} F] B_i \succ [\boldsymbol{d}^{\boldsymbol{s}} F] B_i$ for all $\boldsymbol{s} \neq \boldsymbol{r}_i$. Modulo a division of each B_i by $[\boldsymbol{d}^{\boldsymbol{r}_i} F] B_i$, we are thus in a position where $\boldsymbol{B}$ satisfies the conditions of lemma 16. We conclude that $\boldsymbol{B}$ is an asymptotic basis w.r.t. $\boldsymbol{d}$, and so is $\boldsymbol{B}_{-f_{\preccurlyeq \mathfrak{m}}} = \boldsymbol{A}_{+f_{\succ \mathfrak{m}}}$ for any $\mathfrak{m} \prec 1$. □

6.2. Dualization

In this section, we assume that $\boldsymbol{d} \neq \boldsymbol{\delta}$. Given a strong differential ideal $I \subseteq \mathbb{S}$, the *dualization* of I is the smallest strong differential ideal $I^* \subseteq \mathbb{S}^*$ which contains I.

Lemma 18. *Given a strong quasi-linear differential ideal $I \subseteq \mathbb{S}$, its dualization is again quasi-linear and we have $D_{I^*} = D_I^*$.*

Proof. Let $\boldsymbol{A} \in \mathbb{S}^p$ be an asymptotic basis for I and let $\boldsymbol{\Sigma}$ be the vector of critical relations for $\boldsymbol{A}$, giving rise to an exact sequence

$$\mathbb{K}[\bar{\boldsymbol{d}}]^{D_{\boldsymbol{\Sigma}}} \longrightarrow \mathbb{K}[\bar{\boldsymbol{d}}]^{D_{\boldsymbol{A}}} \longrightarrow \mathbb{K}[\bar{\boldsymbol{d}}] \longrightarrow \mathbb{K}[\bar{\boldsymbol{d}}]/D_I \longrightarrow 0. \tag{21}$$

Since $\mathbb{K}[[\bar{\boldsymbol{d}}]]$ is a flat ring over $\mathbb{K}[\bar{\boldsymbol{d}}]$, we also get an exact sequence

$$\mathbb{K}[[\bar{\boldsymbol{d}}]]^{D_{\boldsymbol{\Sigma}}} \longrightarrow \mathbb{K}[[\bar{\boldsymbol{d}}]]^{D_{\boldsymbol{A}}} \longrightarrow \mathbb{K}[[\bar{\boldsymbol{d}}]] \longrightarrow \mathbb{K}[[\bar{\boldsymbol{d}}]]/D_I \longrightarrow 0. \tag{22}$$

Now consider an element $P = \boldsymbol{\Psi} \cdot \boldsymbol{A} \neq 0$ with $\boldsymbol{\Psi} \in (\mathbb{D}^*)^p$. We claim that we may rewrite $\boldsymbol{\Theta} = \boldsymbol{\Omega} \cdot \boldsymbol{\Sigma} + \boldsymbol{\Xi}$ in such a way that $\boldsymbol{\Xi} \preccurlyeq P$. Indeed, using the exactness of (22) instead of the exactness of (5), this is proved in a similar way as lemma 7. Our claim implies that $D_P = D_{\boldsymbol{\Xi}} \cdot D_{\boldsymbol{A}} \in D_I^*$. This shows that $D_{(\mathbb{D}^*)^p \cdot \boldsymbol{A}} = D_I^*$.

Now consider a matrix $\boldsymbol{\Omega}$ with entries in $\mathbb{K}[[\boldsymbol{d}]]$ such that $\boldsymbol{B} = \boldsymbol{\Omega} \cdot \boldsymbol{A}$ is normal and $D_{\boldsymbol{B}}$ is a Rosenfeld co-basis. We claim that $\boldsymbol{B}$ is an asymptotic basis for $(\mathbb{D}^*)^p \cdot \boldsymbol{A}$. Indeed, as in the first part of the proof of proposition 10, asymptotic co-reduction of the asymptotic co-Δ-polynomial of B_i and B_j with $i < j$ yields an asymptotic co-critical relation for B_i and B_j. Our claim and proposition 12 imply that $I^* = (\mathbb{D}^*)^p \cdot \boldsymbol{B} = (\mathbb{D}^*)^p \cdot \boldsymbol{A}$ is quasi-linear. We conclude that $D_{I^*} = D_{(\mathbb{D}^*)^p \cdot \boldsymbol{A}} = D_I^*$. □

Lemma 19. *If $I \subseteq \mathbb{S}$ is a strong quasi-linear differential ideal, then so is I^*.*

PROOF. Assume that D_I is generated by $L_1, \ldots, L_p \in \mathbb{K}[\bar{\boldsymbol{d}}] F$. Then D_I^* is again generated by $L_1, \ldots, L_p$. □

6.3. Changing derivations

We will now study what happens to quasi-linear differential ideals when changing the derivations. Given two systems of derivations

$$\begin{aligned} \boldsymbol{d} &= (\boldsymbol{z}^{\lambda_1 \boldsymbol{\kappa}} \delta_1, \ldots, \boldsymbol{z}^{\lambda_k \boldsymbol{\kappa}} \delta_k) \\ \boldsymbol{d}' &= (\boldsymbol{z}^{\lambda_1 \boldsymbol{\kappa}'} \delta_1, \ldots, \boldsymbol{z}^{\lambda_k \boldsymbol{\kappa}'} \delta_k) \end{aligned}$$

with $\boldsymbol{z}^{\boldsymbol{\kappa}}, \boldsymbol{z}^{\boldsymbol{\kappa}'} \preccurlyeq 1$ we will write $\boldsymbol{d}' < \boldsymbol{d}$ if $\boldsymbol{z}^{\boldsymbol{\kappa}} \prec \boldsymbol{z}^{\boldsymbol{\kappa}'}$. In that case, we have

$$\mathbb{S} \subsetneq \mathbb{S}^* \subsetneq \mathbb{S}' \subsetneq (\mathbb{S}')^*,$$

where $\mathbb{S}' = \mathbb{K}[\boldsymbol{D}' F] [\![\mathfrak{Z}]\!]$ and $(\mathbb{S}')^* = \mathbb{K}[[\boldsymbol{D}' F]]_{\mathrm{pol}} [\![\mathfrak{Z}]\!]$ denote the natural counterparts of $\mathbb{S}$ and $\mathbb{S}^*$ for $\boldsymbol{d}'$. More generally, we will use primes when working with respect to $\boldsymbol{d}'$.

From now on, let $\boldsymbol{A} \in (\mathbb{S}^*)^p$ be an asymptotic co-basis. For each i, let $\bar{\boldsymbol{d}}^{\boldsymbol{r}_i} F$ be the co-leader of D_{A_i}. The largest $\boldsymbol{d}' < \boldsymbol{d}$ for which D'_{A_i} is not reduced to a single term for some i, is called the *next critical derivation*. If $\boldsymbol{A}$ is also quasi-normal w.r.t. $\boldsymbol{d}'$, then it follows in particular that $(\bar{\boldsymbol{d}}')^{\boldsymbol{r}_i} F$ is the leader of D_{A_i} for each i.

LEMMA 20. *Let $\boldsymbol{A} \in (\mathbb{S}^*)^p$ be an asymptotic co-basis w.r.t. $\boldsymbol{d}$ and let $\boldsymbol{d}' < \boldsymbol{d}$ be the next critical derivation. Assume that D'_{A_i} is linear for each i. Then $\boldsymbol{A} \in (\mathbb{S}')^p$ is also an asymptotic basis w.r.t. $\boldsymbol{d}'$.*

PROOF. Without loss of generality, we may assume that $\boldsymbol{A}$ is co-normal w.r.t. $\boldsymbol{d}$. We will denote by $\boldsymbol{A}'$ the normalization of $\boldsymbol{A}$ w.r.t. $\boldsymbol{d}'$. Let us prove that any element $P \in [\boldsymbol{A}]$ asymptotically reduces to 0 w.r.t. $\boldsymbol{A}'$. In particular, this will imply that the asymptotic Δ-polynomials of the form $(\boldsymbol{d}')^{\boldsymbol{\alpha}} A_i' - (\boldsymbol{d}')^{\boldsymbol{\beta}} A_j'$ asymptotically reduce to 0 w.r.t. $\boldsymbol{A}'$, and thereby give rise to the required asymptotic critical relations for $\boldsymbol{A}'$.

Assume for contradiction that there exists a $P \in [\boldsymbol{A}]$, such that D'_P is reduced w.r.t. $D'_{\boldsymbol{A}'}$. We may even assume that D'_P is totally reduced w.r.t. $D'_{\boldsymbol{A}'}$, i.e. that each variable $V \in \bar{\boldsymbol{D}}' F$ with $v_{D'_{A'_i}} \trianglelefteq V$ for some i has been eliminated from D'_P. Since $\boldsymbol{A}$ is an asymptotic co-basis w.r.t. $\boldsymbol{d}$, asymptotic co-reduction of P yields a relation $P = \boldsymbol{\Theta} \cdot \boldsymbol{A}$. Let us prove by induction over $\mathfrak{m} \in \operatorname{supp} \boldsymbol{\Theta}$ that $\mathfrak{m}\, \boldsymbol{\Theta}_{\mathfrak{m}} \cdot \boldsymbol{A} \prec' P$. This yields the desired contradiction, since this would imply $P = \boldsymbol{\Theta} \cdot \boldsymbol{A} \prec' P$, by strong linearity.

Let $\tilde{P} = P - \boldsymbol{\Theta}_{\succ \mathfrak{m}} \cdot \boldsymbol{A} \sim' P$. Then $\overline{\boldsymbol{\Theta}_{\mathfrak{m}}}$ is the quotient of the co-reduction of $\overline{\tilde{P}_{\mathfrak{m}}}$ w.r.t. $D_{\boldsymbol{A}}$. Consequently, starting with $R_i = \tilde{P}$, there exists an infinite sequence of equations

$$R_i = \mathfrak{m}\, \boldsymbol{\Phi}_i \cdot \boldsymbol{A} + R_{i+1} \qquad (\boldsymbol{\Phi}_i \in \mathbb{K}[[\boldsymbol{D} F]]_{\mathrm{pol}}[[\boldsymbol{d}]]^p),$$

each of which is induced by a partial co-reduction of the type (7), and such that

$$\boldsymbol{\Theta}_{\mathfrak{m}} = \sum_i \boldsymbol{\Phi}_i.$$

More precisely, let $V = \boldsymbol{d}^{\boldsymbol{\alpha}} F$ be the co-leader of $R_{i,\mathfrak{m}}$ and write $R_{i,\mathfrak{m}} = R_{i,\mathfrak{m},d} V^d + \cdots + R_{i,\mathfrak{m},0}$. Then there exists an index j with $V = \boldsymbol{d}^{\boldsymbol{\beta}} v^*_{A_{j,1}}$, $\Phi_{i,j} = (R_{i,\mathfrak{m},d} V^{d-1} / I^*_{A_{j,1}})\, \boldsymbol{d}^{\boldsymbol{\beta}}$ and $\Phi_{i,j'} = 0$ for $j' \neq j$. Assuming that $R_i \sim' P$, we must have $\mathfrak{m}\, R_{i,\mathfrak{m},d} V^d \prec' P$ since $D'_{R_i} = D'_P$ is totally reduced w.r.t. $D'_{\boldsymbol{A}}$ and $D'_{R_{i,\mathfrak{m},d} V^d}$ involves $(\bar{\boldsymbol{d}}')^{\boldsymbol{\alpha}} F \trianglerighteq ((\bar{\boldsymbol{d}}')^{\boldsymbol{\alpha} - \boldsymbol{\beta}} F) = v_{D_{A_j}}$. Combined with the fact that $\boldsymbol{d}^{\boldsymbol{\beta}} A_j \asymp' V$, this yields $\mathfrak{m}\, \boldsymbol{\Phi}_i \cdot \boldsymbol{A} \preccurlyeq' \mathfrak{m}\, R_{i,\mathfrak{m},d} V^d \prec' P$ and $R_{i+1} \sim' P$. In other words, our hypothesis $\tilde{P} \sim' P$ implies $\mathfrak{m}\, \boldsymbol{\Phi}_i \cdot \boldsymbol{A} \prec' P$ for all i, whence $\mathfrak{m}\, \boldsymbol{\Theta}_{\mathfrak{m}} \prec' P$. □

REMARK 21. The lemma admits an alternative proof based on the fact that the asymptotic co-critical relations for $\boldsymbol{A}$ w.r.t. $\boldsymbol{d}$ obtained by asymptotic co-reduction of the asymptotic co-Δ-polynomials continue to provide asymptotic critical relations for $\boldsymbol{A}$ w.r.t. $\boldsymbol{d}'' < \boldsymbol{d}$ which are sufficiently close to $\boldsymbol{d}$. Using lemma 10, it follows that $\boldsymbol{A}$ is an asymptotic basis w.r.t. $\boldsymbol{d}''$. If $\boldsymbol{d}' < \boldsymbol{d}''$, then asymptotic reduction of the asymptotic Δ-polynomials w.r.t. $\boldsymbol{d}'$ yield new asymptotic critical and co-critical relations w.r.t. $\boldsymbol{d}''$, which allows us to continue the process with $\boldsymbol{d}''$ instead of $\boldsymbol{d}$. It can be shown that $\boldsymbol{d}'$ is reached after a finite number of steps.

7. ALGORITHMS

Given an effective ring $\mathbb{K}$, we recall that a multivariate power series $P \in \mathbb{K}[[z_1, \ldots, z_k]]$ is *computable*, if there exists an algorithm which takes $\boldsymbol{i} \in \mathbb{N}^k$ on input and which outputs the corresponding coefficient $P_{\boldsymbol{i}}$ of $\boldsymbol{z}^{\boldsymbol{i}}$ in P. The set $\mathbb{K}[[z_1, \ldots, z_k]]^{\text{com}}$ of such series forms a differential ring. Although $\mathbb{K}$ is usually a field, it is also allowed to take a non-commutative ring of operators for $\mathbb{K}$.

Given an effective monomial group $\mathfrak{M}$, a grid-based series $P \in \mathbb{K}[[\mathfrak{M}]]$ is said to be *computable* if there exist infinitesimal $\mathfrak{m}_1, \ldots, \mathfrak{m}_m \in \mathfrak{M}$ and $\mathfrak{n} \in \mathfrak{M}$, as well as a computable series $\check{P} \in \mathcal{R}[[z_1, \ldots, z_k]]$, such that $P = \check{P}(\mathfrak{m}_1, \ldots, \mathfrak{m}_m)\,\mathfrak{n}$. We denote by $\mathbb{K}[[\mathfrak{M}]]^{\text{com}}$ the ring of such series.

The definition of computable power series applies in the obvious way to differential operators $\Phi \in \mathbb{K}[[d_1, \ldots, d_k]]$. More generally, a series $P \in \mathbb{K}[[\boldsymbol{D}F]]_{\text{pol}}$ is said to be computable, if there exists an algorithm which takes a monomial $M = (\boldsymbol{d}^{\boldsymbol{\alpha}_1} F) \cdots (\boldsymbol{d}^{\boldsymbol{\alpha}_r} F)$ on input and which outputs the corresponding coefficient P_M. We denote by $\mathbb{K}[[\boldsymbol{D}F]]_{\text{pol}}$ the ring of such series.

Throughout this section, we will assume that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ as in sections 3.1 and 4.1 have been fixed and that $\mathbb{U}$ is an effective field with an effective zero-test. For instance, we may take $\mathbb{U} = \mathbb{Q}(\mathrm{e})$ and $\mu_i = \pi^i$.

7.1. Asymptotic bases for linear differential ideals

Assume that $\mathbb{K}$ is an effective field with an effective zero-test. Let $\mathfrak{Z} = z_1^{\mathbb{U}} \cdots z_k^{\mathbb{U}}$, $\mathbb{F} = \mathbb{K}[[\mathfrak{Z}]]^{\text{com}}$ and $\mathbb{P} = \mathbb{F}[\boldsymbol{\Delta} F]$. Consider a Rosenfeld basis $\boldsymbol{A} \in \mathbb{P}_{\text{lin}}^p$, such that the leaders of the A_i are known (remind that we do not have a zero-test in $\mathbb{F}$). Then we may apply the theory from section 6 in order to find an asymptotic basis for $[\boldsymbol{A}]$ w.r.t. $\boldsymbol{\delta}$.

Algorithm `asymptotic_basis`$(\boldsymbol{A})$
Input: a Rosenfeld basis $\boldsymbol{A} \in \mathbb{P}_{\text{lin}}^p$ with known leaders.
Output: an asymptotic basis for $[\boldsymbol{A}]$.

Step 1. [Initial asymptotic basis]
Let $\boldsymbol{d}^{\boldsymbol{r}_i} F$ be the leader of A_i for each i.
Take $\boldsymbol{d}$ sufficiently large, such that $[\boldsymbol{d}^{\boldsymbol{r}_i} F] A_i \succ [\boldsymbol{d}^{\boldsymbol{s}} F] A_i$ for all i and $\boldsymbol{s} \neq \boldsymbol{r}_i$.
By lemma 16, $\boldsymbol{A}$ is an asymptotic basis w.r.t. $\boldsymbol{d}$.
Go to step 4.

Step 2. [Dualization]
Normalize $\boldsymbol{A}$.
Compute $D_{\boldsymbol{A}} \in (\mathbb{K}[\boldsymbol{d}] F)^p$ with respect to $\boldsymbol{d}$.
Using the tangent cone algorithm, find a Rosenfeld co-basis $\boldsymbol{B} \in (\mathbb{K}[[\bar{\boldsymbol{d}}]]^{\text{com}} F)^q$ for $D_{\boldsymbol{A}}$.
For each B_i, let $\boldsymbol{\Phi}_i \in (\mathbb{K}[[\boldsymbol{d}]]^{\text{com}})^p$ be such that $B_i = \bar{\boldsymbol{\Phi}}_i \cdot D_{\boldsymbol{A}}$.
Replace $\boldsymbol{A}$ by $(\boldsymbol{\Phi}_1 \cdot \boldsymbol{A}, \ldots, \boldsymbol{\Phi}_q \cdot \boldsymbol{A})$.
By lemma 18, $\boldsymbol{A}$ is an asymptotic co-basis w.r.t. $\boldsymbol{d}$.

Step 3. [Change derivations]
Compute the next critical derivation $\boldsymbol{d}'$ and replace $\boldsymbol{d}$ by $\boldsymbol{d}'$.
By lemma 20, $\boldsymbol{A}$ is an asymptotic basis w.r.t. $\boldsymbol{d}$.

Step 4. [Done?]
If $\boldsymbol{d} = \boldsymbol{\delta}$, then return $\boldsymbol{A}$.
Otherwise, go to step 2.

REMARK 22. It is readily checked that none of the computations with the series requires the use of a zero-test in $\mathbb{F}$. For instance, in step 1, it suffices to compute the coefficient $[\boldsymbol{\delta}^{\boldsymbol{s}} F] A_i$ up to the order of $[\boldsymbol{\delta}^{\boldsymbol{r}_i} F] A_i$: if $[\boldsymbol{\delta}^{\boldsymbol{s}} F] A_i \prec [\boldsymbol{\delta}^{\boldsymbol{r}_i} F] A_i$, then $[\boldsymbol{d}^{\boldsymbol{r}_i} F] A_i \succ [\boldsymbol{d}^{\boldsymbol{s}} F] A_i$ for all $\boldsymbol{d}$. Similarly, for the computation of the next critical derivation $\boldsymbol{d}'$ in step 3, it suffices to evaluate $[\boldsymbol{\delta}^{\boldsymbol{s}} F] A_i$ up to the order of $[\boldsymbol{\delta}^{\boldsymbol{r}_i} F] A_i$, where $\boldsymbol{\delta}^{\boldsymbol{r}_i} F$ stands for the co-leader of A_i.

THEOREM 23. *The algorithm* `asymptotic_basis` *is correct and terminates.*

PROOF. The correctness being ensured by lemmas 16, 18 and 20, it suffices to prove the termination. Let $\boldsymbol{d}^1, \boldsymbol{d}^2, \ldots$ and $\boldsymbol{A}^1, \boldsymbol{A}^2, \ldots$ be the successive values of $\boldsymbol{d}$ and $\boldsymbol{A}$ at the start of step 2. For each $\boldsymbol{d}^i$ and $j \in \{1, \ldots, p^i\}$, let $(\boldsymbol{d}^i)^{\boldsymbol{r}_j^i} F$ be the leader of A_j^i, and consider the anti-chain $S^i = (\boldsymbol{r}_1^i, \ldots, \boldsymbol{r}_{p^i}^i) \in (\mathbb{N}^k)^{p^i}$. Then $S^1, S^2, \ldots$ forms a decreasing sequence of anti-chains, which implies the finiteness of the sequence. □

7.2. Ensuring ultimate quasi-linearity

The argument from the proof of proposition 17 can be generalized so as to provide asymptotic bases w.r.t. $\boldsymbol{\delta}$. More precisely, let us consider a Rosenfeld basis $\boldsymbol{A} \in \mathbb{P}^p$ with $H_{\boldsymbol{A}} \neq 0$ and whose leaders are known. If $[\boldsymbol{A}_{\times \mathfrak{m}}]$ is quasi-linear for a sufficiently small $\mathfrak{m} = \boldsymbol{z}^{\boldsymbol{\alpha}} \preccurlyeq 1$, then we will show how to compute such an $\mathfrak{m}$ and an asymptotic basis $[\boldsymbol{A}_{\times \mathfrak{m}}]$. If the algorithm fails, then we will provide a proof that $\boldsymbol{A}_{\text{cst}} \neq \boldsymbol{0}$.

Algorithm `ql_asymptotic_basis`($\boldsymbol{A}$)
Input: a Rosenfeld basis $\boldsymbol{A} \in \mathbb{P}^p$ with $H_{\boldsymbol{A}} \neq 0$ and known leaders.
Output: an asymptotic basis for $[\boldsymbol{A}_{\times \mathfrak{m}}]$ for some $\mathfrak{m} = \boldsymbol{z}^{\boldsymbol{\alpha}} \preccurlyeq 1$ or
fail and a certificate that $\boldsymbol{A}_{\text{cst}} \neq \boldsymbol{0}$.

Step 1. [Initial asymptotic basis]
Let $\boldsymbol{d}^{\boldsymbol{r}_i} F$ be the leader of A_i for each i.
Take $\boldsymbol{d}$ sufficiently large, such that $[\boldsymbol{d}^{\boldsymbol{r}_i} F] A_i \succ [\boldsymbol{d}^{\boldsymbol{s}} F] A_i$ for all i and $\boldsymbol{s} \neq \boldsymbol{r}_i$.
Replace $\boldsymbol{A}$ by $\boldsymbol{A}_{\times \mathfrak{m}}$, where $\mathfrak{m} \preccurlyeq 1$ is sufficiently small such that $D_{\boldsymbol{A}}$ is affine.
Go to step 4.

Step 2. [Dualization]
Normalize $\boldsymbol{A}$.
Compute $D_{\boldsymbol{A}} \in (\mathbb{K}[\boldsymbol{d}] F)^p$ with respect to $\boldsymbol{d}$.
Using the tangent cone algorithm, find a Rosenfeld co-basis $\boldsymbol{B} \in (\mathbb{K}[[\bar{\boldsymbol{d}}]]^{\text{com}} F)^q$ for $D_{\boldsymbol{A}}$.
For each B_i, let $\boldsymbol{\Phi}_i \in (\mathbb{K}[[\boldsymbol{d}]]^{\text{com}})^p$ be such that $B_i = \bar{\boldsymbol{\Phi}}_i \cdot D_{\boldsymbol{A}}$.
Replace $\boldsymbol{A}$ by $(\boldsymbol{\Phi}_1 \cdot \boldsymbol{A}, \ldots, \boldsymbol{\Phi}_q \cdot \boldsymbol{A})$.

Step 3. [Change derivations]
Compute the next critical derivation $\boldsymbol{d}'$ and replace $\boldsymbol{d}$ by $\boldsymbol{d}'$.
Replace $\boldsymbol{A}$ by $\boldsymbol{A}_{\times \mathfrak{m}}$, where $\mathfrak{m} \preccurlyeq 1$ is sufficiently small such that $D_{\boldsymbol{A}}$ is affine.

Step 4. [Done?]
If $D_{\boldsymbol{A},\text{cst}} \neq \boldsymbol{0}$, then return **fail**.
If $\boldsymbol{d} = \boldsymbol{\delta}$, then return $\boldsymbol{A}$.
Otherwise, go to step 2.

THEOREM 24. *The algorithm* `ql_asymptotic_basis` *is correct and terminates.*

PROOF. The algorithm is a perturbation of the algorithm `asymptotic_basis` from the previous section. In steps 1 and 3, lemmas 16 and 20 can only be applied under the condition that $D_{\boldsymbol{A},\mathrm{cst}} \neq 0$, in which case the previous correction and termination proof generalizes. Whenever $D_{\boldsymbol{A},\mathrm{cst}} \neq 0$ in step 4, this implies in particular that $\boldsymbol{A}_{\mathrm{cst}} \neq 0$ for the input basis $\boldsymbol{A}$. □

7.3. Existence of zeros revisited

Until now, we have not really used the fact that the derivations are allowed to act in a non-trivial way on $\mathbb{K}$. In the next section, we will work over $\mathbb{K}[\![z_1^{\mathbb{U}} \cdots z_{k-1}^{\mathbb{U}}]\!] [\![z_k^{\mathbb{U}}]\!]$ instead of $\mathbb{K}[\![\mathfrak{Z}]\!]$. This requires some adaptations of proposition 15.

PROPOSITION 25. *Let* $\mathbb{K}$ *be a field of constants for* $\boldsymbol{\delta}$ *and consider an asymptotic basis* $\boldsymbol{A} \in ((\mathbb{K}[\boldsymbol{\delta}]F)[\![\mathfrak{Z}]\!] \oplus \mathbb{K}[\log \boldsymbol{z}][\![\mathfrak{Z}]\!])^p$. *Then* $\boldsymbol{A}(f) = \boldsymbol{0}$ *admits a solution in* $\mathbb{K}[\log \boldsymbol{z}][\![\mathfrak{Z}]\!]$.

PROOF. This is proved using a straightforward adaptation of the proof of proposition 15. □

PROPOSITION 26. *Let* $\mathbb{K}$ *be a field of constants for* $\boldsymbol{\delta}$ *and consider a Rosenfeld basis* $\boldsymbol{A} \in (\mathbb{K}[\![\mathfrak{Z}]\!][\boldsymbol{\delta}]F \oplus \mathbb{K}[\log \boldsymbol{z}][\![\mathfrak{Z}]\!])^p$. *Then* $\boldsymbol{A}(f) = \boldsymbol{0}$ *admits a solution in* $\mathbb{K}[\log \boldsymbol{z}][\![\mathfrak{Z}]\!]$.

PROOF. Using a procedure similar to `ql_asymptotic_basis`, we may compute a monomial $\mathfrak{m} \succcurlyeq 1$ in $\mathfrak{Z}$ such that $[[\boldsymbol{A}_{\times \mathfrak{m}}]]$ is quasi-linear w.r.t. $\boldsymbol{\delta}$. Indeed, since $\boldsymbol{A}$ admits no non-linear terms, instead of choosing $\mathfrak{m} \preccurlyeq 1$ such that $D_{\boldsymbol{A}}$ is affine in steps 1 and 3, we may now choose $\mathfrak{m} \succcurlyeq 1$ sufficiently large such that $D_{\boldsymbol{A}}$ is linear. By proposition 25, $[[\boldsymbol{A}_{\times \mathfrak{m}}]]$ admits a zero g in $\mathbb{K}[\log \boldsymbol{z}][\![\mathfrak{Z}]\!]$. It follows that $f = g/\mathfrak{m}$ is a solution to $\boldsymbol{A}(f) = \boldsymbol{0}$. □

Assume now that we wish to work over $\mathbb{K}[\![z_1^{\mathbb{U}} \cdots z_{k-1}^{\mathbb{U}}]\!] [\![z_k^{\mathbb{U}}]\!]$ instead of $\mathbb{K}[\![\mathfrak{Z}]\!]$. This fits in our setting by taking $\mathbb{K} = \Bbbk[\![\tilde{\mathfrak{Z}}]\!]$ for a constant field $\Bbbk$ and a copy $\tilde{\mathfrak{Z}} = \tilde{z}_1^{\mathbb{U}} \cdots \tilde{z}_k^{\mathbb{U}}$ of $\mathfrak{Z}$ with $\delta_i = \tilde{z}_i \frac{\partial}{\partial \tilde{z}_i}$ on $\mathbb{K}$. The variables $\tilde{z}_k$ and $z_1, \ldots, z_{k-1}$ will simply be ignored.

PROPOSITION 27. *Let* $\Bbbk$ *be a field of constants for* $\boldsymbol{\delta}$. *Denote* $\mathbb{K} = \Bbbk[\![\tilde{z}_1^{\mathbb{U}} \cdots \tilde{z}_{k-1}^{\mathbb{U}}]\!]$ *and* $\mathbb{L} = \Bbbk[\log \tilde{\boldsymbol{z}}][\![\tilde{z}_1^{\mathbb{U}} \cdots \tilde{z}_{k-1}^{\mathbb{U}}]\!]$. *Consider an asymptotic basis* $\boldsymbol{A} \in \mathbb{K}[\![z_k^{\mathbb{U}}]\!]^p$. *Then there exists an* $f \in \mathbb{L}[\![z_k^{\mathbb{U}}]\!]$ *with* $\boldsymbol{A}(f) = \boldsymbol{0}$ *and* $f \prec 1$ *as a series in* z_k.

PROOF. The proof is analogous to the proof of proposition 15. We first notice that the absence of the variables $z_1, \ldots, z_{k-1}$ from $\boldsymbol{A}$ implies their absence in the asymptotic critical relations and during all computations in the proof. This time, the Rosenfeld basis $\boldsymbol{B} = D_{\boldsymbol{A}_{+\varphi, \times \mathfrak{m}}}$ lies in $(\mathbb{K}[\boldsymbol{\delta}]F \oplus \mathbb{L})^p$ instead of $(\mathbb{K}[\boldsymbol{\delta}]F \oplus \mathbb{K}[\log \boldsymbol{z}])^p$. By proposition 26, the equation $\boldsymbol{B}(g) = \boldsymbol{0}$ admits a solution in $\Bbbk[\log \tilde{\boldsymbol{z}}][\![\tilde{z}_1^{\mathbb{U}} \cdots \tilde{z}_k^{\mathbb{U}}]\!]$. Since $\tilde{z}_k$ does not occur in $\boldsymbol{B}$ (regarding $\log \tilde{z}_k$ and $\tilde{z}_k$ as separate variables), the constant term $[\tilde{z}_k^0] g \in \mathbb{L}$ of g w.r.t. $\tilde{z}_k$ is again a zero of $\boldsymbol{B}$. This ensures that the analogue of the proof of proposition 15 goes through. □

7.4. A zero-test in extension rings with solutions to PDEs

Let $\Bbbk$ be an effective field with an effective zero-test. Assume that we are given an effective differential subring $\mathbb{A}$ of $\Bbbk[[z_1, \ldots, z_k]]^{\mathrm{com}}$ for $\boldsymbol{\delta}$ with an effective zero-test. Assume also that the coefficients of $f \in \mathbb{A}$ as a series in z_k are still in $\mathbb{A}$ and that there exists an algorithm to compute them. These coefficients lie in the effective differential ring $\mathbb{A}^\flat = \mathbb{A} \cap \Bbbk[[z_1, \ldots, z_k]]^{\mathrm{com}} \subseteq \Bbbk[\![z_1^{\mathbb{U}} \cdots z_{k-1}^{\mathbb{U}}]\!]$. We will denote by $\mathbb{K} \subseteq \Bbbk[\![z_1^{\mathbb{U}} \cdots z_{k-1}^{\mathbb{U}}]\!]$ the quotient field of $\mathbb{A}^\flat$. In view of the previous section, we may apply the theory from this paper for series in the field $\mathbb{F} = \mathbb{K}[\![\mathfrak{Z}]\!]^{\mathrm{com}}$, where $\mathfrak{Z} = z_k^{\mathbb{U}}$.

Let $P \in \mathbb{A}[\mathbf{\Delta} F]$ be a partial differential equation with a solution $f \in \Bbbk[[z_1, \ldots, z_k]]^{\mathrm{com}}$ and coefficients $[z_k^n] f \in \mathbb{A}^\flat$. Assume that there exists an order n_0 such that the equation

$$P_{+f}(\varepsilon) = 0 \qquad (\varepsilon \prec z_k^{n_0}) \tag{23}$$

admits only $\varepsilon = 0$ as a solution in $\Bbbk[\log \boldsymbol{z}] [\![z_1^{\mathbb{U}} \cdots z_{k-1}^{\mathbb{U}}]\!] [\![z_k^{\mathbb{U}}]\!]$. This is typically the case when P is sufficiently non-singular, so that the coefficients of f in z_k are given by a recurrence relation. The differential ring $\mathbb{A}[\mathbf{\Delta} f]$ is clearly a subring of $\Bbbk[[z_1, \ldots, z_k]]^{\mathrm{com}}$. We will now give an effective zero-test in this ring.

Algorithm `zero_test`(Q)
Input: a polynomial $P \in \mathbb{A}[\mathbf{\Delta} F]$.
Output: **true** if $P(f) = 0$ and **false** otherwise.

Step 1. [Initial order]
Set $n := n_0 + 1$ and $\mathcal{E} := \{P, Q\}$.

Step 2. [Rosenfeld basis]
If $Q(f) \succcurlyeq z_k^n$ then return **false**.
Compute a Rosenfeld basis $\boldsymbol{A} \in \mathbb{A}[\mathbf{\Delta} f]^p$ for $\mathcal{E}$.
Let $\mathcal{Z}$ be the subset of $Z \in \{I_{A_1}, S_{A_1}, \ldots, I_{A_p}, S_{A_p}\}$ with $Z(f) \prec z_k^n$.
If $\mathcal{Z} \neq \varnothing$ then set $\mathcal{E} = \{A_1, \ldots, A_p\} \cup \mathcal{Z}$ and repeat step 2.

Step 3. [Certification]
If `ql_asymptotic_basis`$(\boldsymbol{A}_{+f, \times z_k^{n_0}})$ does not return **fail**, then return **true**.

Step 4. [Increase order]
Set $n := 2n$ and $\mathcal{E} := \{P, Q\}$.
Return to step 2.

THEOREM 28. *The algorithm* `zero_test` *is correct and terminates.*

PROOF. Let us first prove the correctness. If the algorithm returns **false**, then we clearly have $Q(f) \neq 0$. If the algorithm returns **true**, then the ideal $[\boldsymbol{A}_{+f, \times z_k^{n_0}}]$ is quasi-linear in step 3. By proposition 27, this implies the existence of a zero $\delta \in \mathbb{K}[\log \boldsymbol{z}] [\![z_1^{\mathbb{U}} \cdots z_{k-1}^{\mathbb{U}}]\!] [\![z_k^{\mathbb{U}}]\!]$ with $\delta \prec 1$. Since $H_{\boldsymbol{A}}(f) \neq 0$, any solution of $\boldsymbol{A}_{+f}$ is also a solution of P_{+f} and of Q_{+f}. It follows that $P_{+f}(\varepsilon) = 0$ for $\varepsilon = z_k^{n_0} \delta \prec z_k^{n_0}$. But we assumed that $\varepsilon = 0$ is the only solution to (23). It follows that $Q_{+f}(\varepsilon) = Q(f) = 0$.

Let us now prove the termination. For a fixed order n, the loop in step 2 terminates because the successive rankings of $\boldsymbol{A}$ are lower and lower. If $Q(f) \neq 0$, then we clearly have $Q(f) \succcurlyeq z_k^n$ for a sufficiently large n. If $Q(f) = 0$, then, for a sufficiently large n, we have $Z(f) \neq 0 \Rightarrow Z(f) \succcurlyeq z_k^n$ for all Z considered in step 2. Consequently, the successive values of $\boldsymbol{A}$ in step 2 all satisfy $\boldsymbol{A}(f) = 0$. When entering step 3, we therefore have $\boldsymbol{A}_{+f, \times z_k^{n_0}, \mathrm{cst}} = \mathbf{0}$, whence `ql_asymptotic_basis` does not return **fail**. □

8. FINAL NOTES

Several remarks can be made about the generality of our zero-test. In order to avoid unnecessary complications of the notations, we have presented our algorithm in the case when we adjoin a single solution to a partial differential equation with initial conditions. Of course, the theory of differential algebra also works for differential polynomials in a finite number of indeterminates. It should be straightforward to generalize our zero-test to the case when we adjoin several solutions at the same time.

Similarly, we may replace the single equation $P(f) = 0$ by a system of equations $P_1(f) = \cdots = P_p(f) = 0$. In that case, the initial conditions generally lay in the ring $\mathbb{A}^\flat = \mathbb{A} \cap \mathbb{k}[[z_1, \ldots, z_{k'}]]^{\mathrm{com}}$ for some $k' < k$. In particular, this requires to take $\mathbb{F} = \mathbb{K}[[z_{k'+1}^{\mathbb{U}} \cdots z_k^{\mathbb{U}}]]$, where $\mathbb{K}$ is the quotient field of $\mathbb{A}^\flat$.

We assumed that $\varepsilon = 0$ is the unique solution in $\mathbb{k}[\log \boldsymbol{z}][[z_1^{\mathbb{U}} \cdots z_{k-1}^{\mathbb{U}}]][[z_k^{\mathbb{U}}]]$ to (23). This condition is usually met in practice and in particular when Cauchy-Kovalevskaya's theorem applies. Nevertheless, the condition can probably be weakened to the existence of a unique solution in $\mathbb{k}[\log \boldsymbol{z}][[\boldsymbol{z}]]$ or some ring in between. In order to prove such a result, one would have to carefully study the supports of the series which arise during our computations. We notice that the existence of a unique solution in $\mathbb{k}[[\boldsymbol{z}]]$ is a minimal requirement. However, in view of results such as theorem 1, it might be hard to escape from the intrusion of logarithms. As a matter of fact, the proof of this theorem fails if we replace $\mathbb{K}[[\boldsymbol{z}]]$ by $\mathbb{K}[\log \boldsymbol{z}][[\boldsymbol{z}]]$.

It is appropriate to notice one important advantage of our setting: we only required $\mathbb{A}$ to be an effective subring of $\mathbb{k}[[z_1, \ldots, z_k]]^{\mathrm{com}}$ with an effective zero-test. Such rings can be constructed via a tower of extensions of $\mathbb{k}$ of the type considered in section 7.4. However, $\mathbb{A}$ does not necessarily have to be of this type. For instance, in the case of ordinary differential equations, we know that the series f in z occurring in Stirling's series for $\Gamma(z^{-1})$ is differentially transcendental over $\mathbb{Q}$. We may thus take $\mathbb{A} = \mathbb{Q}\{f\}$.

Even though the proof of the new zero-test might seem complex, we notice that the actual algorithm is actually quite simple and can expected to be reasonably efficient. Indeed, apart from the usual (inevitable?) step of computing Rosenfeld bases over $\mathbb{F}$, we only have to apply the tangent cone algorithm a few times for series over the simpler field $\mathbb{K}$.

Several points of a more theoretical interest might deserve further study. First of all, we made a few artificial hypothesis on $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, which one may try to remove as much as possible. More importantly, it would be nice to have a genuine theory of asymptotic differential algebra, which goes beyond the quasi-linear case. In the continuation of [AvdD04], such a theory might even work for more general "asymptotic functions fields" than our fields of grid-based series $\mathbb{K}[[\mathfrak{Z}]]$.

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