Typed λ-calculi and superclasses of regular transductions

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Abstract

We propose to use Church encodings in typed λ-calculi as the basis for an automata-theoretic counterpart of implicit computational complexity, in the same way that monadic second-order logic provides a counterpart to descriptive complexity. Specifically, we look at transductions i.e. string-to-string (or tree-to-tree) functions – in particular those with superlinear growth, such as polyregular functions, HDT0L transductions and Sénizergues's "k-computable mappings".

Our first results towards this aim consist showing the inclusion of some transduction classes in some classes defined by λ-calculi. In particular, this sheds light on a basic open question on the expressivity of the simply typed λ-calculus. We also encode regular functions (and, by changing the type of programs considered, we get a larger subclass of polyregular functions) in the elementary affine λ-calculus, a variant of linear logic originally designed for implicit computational complexity.

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1 Introduction

The main goal of this paper is to provide some evidence for connections between:

- automata theory, in particular transducers (loosely defined as devices which compute string-to-string (or tree-to-tree) functions and are "finite-state" in some way);
- programming language theory, in particular the expressive power of some *typed λ-calculi*, i.e. some (minimalistic) statically typed functional programming languages.

Our first concrete result is:

▶ **Theorem 1.1.** *The functions from strings to strings that can be expressed (in a certain way) in the* simply-typed λ-calculus *(STλ) – we shall call these the* λ-definable string functions *(Definition 1.7) – enjoy the following properties:*

- *they are closed under composition;*
- *they are* regularity-preserving*: the inverse image of a* regular language *is regular;*
- *they contain all the transductions defined by* HDT0L systems (see [27, 11])*, a variant of the* L-systems *originally introduced by Lindenmayer for mathematical biology [21].*

We believe that this is conceptually interesting for the study of both λ-calculi and automata:

- It is directly relevant to a basic and natural open problem about the functions $\mathbb{N} \to \mathbb{N}$ definable (in some way) in STλ. This problem is simple enough to be presented without assuming any background in programming language theory; we shall do this in §1.1.
- Another corollary is that the simply typed λ-calculus subsumes all the natural classes of regularity-preserving functions that we know of. We indeed prove in this paper that

the *closure by composition of HDT0L transductions* (a class that we shall abbreviate as "HDT0L+composition") contains the *polyregular functions* recently introduced by Bojańczyk [5]; therefore, it includes *a fortiori* the well-known classes of *regular*, *rational* and *sequential* string functions (see e.g. [10, 23], or the introduction to [5]).

The above-mentioned classes can be defined using transducers, and they admit alternative characterizations which attest to their robustness. For instance, regular functions can be also characterized by Monadic Second-Order Logic [9].

The general pattern: encoding transductions More generally, several results in this paper consist in considering, on one hand, some class $\mathcal{A}$ of automata with output, and on the other hand, some typed λ-calculus $\mathcal{P}$ with a type T. The programs of type T in $\mathcal{P}$ must be able to take (encodings of) strings as inputs and output (encodings of) strings. Then "compiling" the automata in $\mathcal{A}$ into programs in $\mathcal{P}$, we get:

$$\left\{\begin{matrix}\text{functions computed}\\ \text{by transducers in } \mathcal{A}\end{matrix}\right\} \subseteq \left\{\begin{matrix}\text{functions computed}\\ \text{by programs in } \mathcal{P} \text{ of type } T\end{matrix}\right\}$$

Of course, an equality sign here would be more satisfying. But we do not know whether all the λ-definable string functions (in STλ) are in HDT0L+composition. In contrast, for our next result, even though we only claim and prove an inclusion in this paper, we are actually fairly confident that the converse holds. But it appears to be significantly more difficult than the direction treated here: our tentative proof[1] for this converse – which has not been thoroughly checked – requires the development of new tools in denotational semantics.

Linear logic vs streaming string transducers This next result involves the *elementary affine λ-calculus* (EAλ) introduced by Baillot, De Benedetti and Ronchi Della Rocca [4].

▶ **Theorem 1.2.** *The programs of a certain type in* EAλ *compute all* regular functions*, and compute only* linear time *and* regularity-preserving *functions.*

See Theorem 1.11 for a precise statement. EAλ is mainly inspired by Girard's *linear logic* [14], a "resource-sensitive" constructive logic that has already been used to characterize complexity classes (see §1.2). In programming languages, *linearity* refers to the prohibition of *duplication*: a function is linear if it uses its argument at most[2] once. Linearity appears in automata theory under the name[3] "copyless assignment". This refers to a technical condition in the definition of *streaming string transducers* (SSTs), a machine model introduced by Alur and Černý [3]. Hence the relevance of the elementary affine λ-calculus to regular functions:

- the functions computed by SSTs are exactly the regular functions;
- without the linearity condition, the class obtained is instead the HDT0L transductions, as proved recently by Filiot and Reynier [11].

Thus, our work gives a precise technical contents to this analogy between linearity in λ-calculi and copyless assignments in automata theory: what makes "copyful SSTs" impossible to encode in EAλ is their non-linearity.

[1] A joint work with Paolo Pistone, Thomas Seiller and Lorenzo Tortora de Falco. We intend to present this work in a future paper – hence the numbering in the title.
[2] Strictly speaking, such a function is *affine*; a linear function uses its argument *exactly* once. But we follow here a widespread abuse of language.
[3] The term "linearity" itself has also been used, e.g. in [10]: "updates should make a linear use of registers".

String functions of superlinear growth In the above theorem, instead of the converse inclusion, we have merely stated an upper bound on the definable string functions in EAλ, in terms of time complexity. This already means that we capture a much smaller class of functions than in our previous result on STλ. Indeed, since HDT0L transductions can grow exponentially, when one composes them, the rates of growth can become towers of exponentials. However, the other classes that we mentioned contain only tractable functions: not only do they grow polynomially, they are also computable in polynomial time. In particular the regular functions are computed in linear time – and we match this bound.

Another instance of our pattern takes place in the same language EAλ. By changing the type of programs considered, we manage to code a larger subclass of polyregular functions – it contains functions whose output length may grow polynomially with arbitrary exponent. As an added benefit, this partially answers a natural question concerning EAλ (we discuss this further in §1.2).

With this last result, together with Theorem 1.1, we hope to contribute to the recent surge of interest in superlinear transductions, exemplified by the introduction of polyregular functions [5] – whose slogan is “the polynomial growth finite state transducers” – and the study of non-linear streaming string transducers. Concerning the latter, HDT0L systems were mostly used to describe *languages* previously; in fact, before Filiot and Reynier’s work [11], their semantics as transductions seems to have been considered only once: in an invited paper without proofs [27], Sénizergues claims to characterize the HDT0L+composition class using iterated pushdown automata.

Tree transductions Finally, we shall also see that in all the above programming languages, there is a type of functions from binary trees to binary trees, and all *regular tree functions* can be encoded as programs of this type. This relies on their characterization by *bottom-up ranked tree transducers* [1] generalizing SSTs with a relaxed and more subtle linearity condition – closely related, as we shall see, to the *additive conjunction* of linear logic (while the linearity of SSTs is purely *multiplicative*). We do not investigate superlinear tree transducers here.

Plan of the paper In the remainder of this introduction, we first briefly present the simply typed λ-calculus, and state our motivating problem on the functions $\mathbb{N} \to \mathbb{N}$ that it can express (§1.1). The other introductory subsection (§1.2) situates our work in the conceptual landscape, and surveys related work and inspirations. Section 2 introduces the automata-theoretic classes of functions studied here, and proves some inclusions between them. Section 3 and 4 are dedicated respectively to STλ and EAλ.

Intended audience We have attempted to make the parts involving the simply typed λ-calculus accessible to a broad audience, since the arguments involved are rather elementary. However, the exposition of the results on EAλ assumes some familiarity with linear logic.

1.1 Motivation: λ-definable numeric functions

1.1.1 Introduction to the λ-calculus and to Church encodings

The untyped λ-calculus is a naive syntactic theory of functions. Its terms are generated by the grammar[4] $t, u ::= x \mid t\ u \mid \lambda x.\ t$ (where x is taken in a countable set of “variables”), which

[4] The terms must actually be considered up to renaming of bound variables, just as usual mathematical practice dictates that x is bound in t in the expression $x \mapsto t$. The details of this renaming equivalence,

mirrors the basic operations of function application ($t\ u \approx t(u)$) and function formation ($\lambda x.\ t \approx x \mapsto t$). The equational theory on these λ-terms is the congruence generated by

$$(\lambda x.\ t)\ u =_\beta t\{x := u\} \qquad \text{where } t\{x := u\} \text{ is the substitution of } x \text{ by } u \text{ in } t$$

which corresponds to the usual way of computing a function, e.g. $(x \mapsto x^2 + 1)(42) = 42^2 + 1$.

This example cannot be directly expressed in the λ-calculus since it does not have primitive integers among its terms. Instead, we use *Church encodings* to represent natural numbers: morally, $n \in \mathbb{N}$ is encoded as the n-fold iteration functional $\overline{n} : f \mapsto f^n = f \circ \ldots \circ f$. For instance, $\overline{2} = \lambda f.\ (\lambda x.\ f\ (f\ x))$. Using this encoding, the untyped λ-calculus can represent any computable function $f : \mathbb{N} \to \mathbb{N}$: for some term t, $t\ \overline{n} =_\beta \overline{f(n)}$ for all $n \in \mathbb{N}$.

To avoid the pitfalls of Turing-completeness (e.g. to obtain only total functions), one technique is to add a *type system*: a way of annotating terms with *types* specifying some of their behavior. In the simply typed λ-calculus (STλ), we use the *simple types* defined as $A, B := o \mid A \to B$, where o is the single *base type*. We write $t : A$ when the term t can be given the type A. The meaning of $t : A \to B$ is morally that t is a function taking inputs of type A and returning outputs of type B.

The rules of STλ allow us for example to show that assuming $f : o \to o$ and $x : o$, we have $f\ (f\ x) : o$; from this, one can then deduce that $\overline{2} = \lambda f.\ (\lambda x.\ f\ (f\ x)) : (o \to o) \to (o \to o)$ (without assumption). In general, one can show that the terms of type $\mathtt{Nat} = (o \to o) \to (o \to o)$, quotiented by $=_\beta$, are in bijection[5] with $\mathbb{N}$ via $n \mapsto \overline{n}$. So $\mathtt{Nat}$ can legitimately be seen as the type of natural numbers in STλ.

1.1.2 A question: expressible functions in the simply typed λ-calculus

At this point, we may ask: *what are the functions* $\mathbb{N} \to \mathbb{N}$ *definable in* STλ? As hinted in the introduction, this kind of question depends heavily on the *type* of the programs (i.e. λ-terms) that we use to code these functions. A classical result is:

▶ **Theorem 1.3** (Schwichtenberg 1975 [26]). *Let $f : \mathbb{N} \to \mathbb{N}$. There exists $t : \mathtt{Nat} \to \mathtt{Nat}$ such that $t\ \overline{n} =_\beta \overline{f(n)}$ for all $n \in \mathbb{N}$ if and only if f is an* extended polynomial, *i.e. a function generated from 0, 1, $+$, $\times$ and a conditional* $\mathtt{if}\ n = 0\ \mathtt{then}\ p\ \mathtt{else}\ q$.

So the λ-terms of type $\mathtt{Nat} \to \mathtt{Nat}$ have a rather low expressivity. One trick to allow more functions to be defined is to perform a substitution of the input type.

▶ **Notation 1.4.** For types A and B, we abbreviate the substitution $A\{o := B\}$ as $A[B]$.

We shall consider λ-terms of type $\mathtt{Nat}[A] \to \mathtt{Nat}$ (by expanding the definitions, $\mathtt{Nat}[A] = (A \to A) \to (A \to A)$) where A is an arbitrary simple type. Terms of this type still define numeric functions, thanks to a simple "substitution lemma": $\overline{n} : \mathtt{Nat}$ entails that $\overline{n}$ can also be given the type $\mathtt{Nat}[A]$ for all types A and all $n \in \mathbb{N}$. Typically, one can check that $(\lambda x.\ x\ \overline{2}) : \mathtt{Nat}[o \to o] \to \mathtt{Nat}$ represents the function $n \mapsto 2^n$.

To our knowledge, there is only one characterization of the class of functions $\mathbb{N} \to \mathbb{N}$ thus obtained, due to Joly [19]. It is formulated in terms of untyped λ-terms subject to a kind of complexity constraint in an unrealistic (by Joly's own admission) cost model. Therefore, it would be of obvious interest to describe this class without reference to the λ-calculus.

called "α-conversion", are uninteresting and can be found in any textbook on the λ-calculus. Similarly, in the substitution $t\{x := u\}$ introduced later, only the free occurrences of x, i.e. not appearing under a $\lambda x.$, must be substituted.

[5] Except for the term $\lambda f.\ f$, but it may be identified with $\overline{1} = \lambda f.\ (\lambda x.\ f\ x)$ by extending the equational theory with the innocuous "η-rule": if $t : A \to B$ for some A, B then $t =_\eta \lambda y.\ t\ y$.

▶ **Open question 1.5.** Characterize the functions $f : \mathbb{N} \to \mathbb{N}$ definable in STλ in the following way: there exists a type A and a term $t : \mathtt{Nat}[A] \to \mathtt{Nat}$ such that for all $n \in \mathbb{N}$, $t\,\overline{n} =_\beta \overline{f(n)}$.

It might seem surprising that this problem is still open despite the central role that the simply typed λ-calculus has played in programming language theory and in proof theory for the past few decades. We believe that this is due in part by some well-known facts (cf. [12]) that suggest that there might be no satisfying answer: while any tower of exponentials $2 \uparrow^h n$ of fixed height h can be expressed by a term of type $\mathtt{Nat}[A_h] \to \mathtt{Nat}$ (A_h becoming increasingly complicated as $h \to +\infty$), many simple functions of tame growth are inexpressible. If we look at functions of two variables, there is a striking example: *subtraction* cannot be defined by any term of type[6] $\mathtt{Nat}[A] \to (\mathtt{Nat}[B] \to \mathtt{Nat})$, no matter what simple types A, B are chosen.

One aim of the present paper – starting with the subsection below, which lends a new significance to old results – is to argue that this pessimism is perhaps unwarranted.

1.1.3 The relevance of automata to λ-definability

To gain some insight on this problem, let us both generalize and (temporarily) specialize it:

- We replace natural numbers by *strings* over a finite alphabet Σ. There exists a simple type $\mathtt{Str}_\Sigma$ of *Church-encoded strings* and an encoding $t \in \Sigma^* \rightsquigarrow \overline{t} : \mathtt{Str}_\Sigma$ inducing a bijection $\Sigma^* \cong (\{t \mid t : \mathtt{Str}_\Sigma\}/ =_\beta)$. We recover Church numerals as the special case of Church-encoded strings over *unary alphabets*: $\mathtt{Nat} = \mathtt{Str}_{\{1\}}$. Schwichtenberg's result on $\mathtt{Nat} \to \mathtt{Nat}$ (Theorem 1.3) can be suitably generalized to $\mathtt{Str}_\Gamma \to \mathtt{Str}_\Sigma$ (see [28, 20]).
- We shall start by looking at predicates $\mathtt{Str}[A] \to \mathtt{Bool}$ – i.e. at *languages* – instead of functions $\mathtt{Str}[A] \to \mathtt{Str}$, with the usual type $\mathtt{Bool} = o \to (o \to o)$ of booleans in STλ.

Fortunately, the languages definable in STλ are known:

▶ **Theorem 1.6** (Hillebrand & Kanellakis 1995 [18]). *A language $L \subseteq \Sigma^*$ can be expressed as $L = \mathcal{L}(t) = \{w \in \Sigma^* \mid t\,\overline{w} =_\beta \mathtt{true}\}$ for some λ-term $t : \mathtt{Str}_\Sigma[A] \to \mathtt{Bool}$ and some simple type A if and only if it is a* regular language.

Furthermore, Joly stated his result [19] for arbitrary free algebras, and the above theorem also generalizes to a characterization of *regular tree languages* for such free algebras (using the right definition of Church encoding). As for the specialization to $\mathbb{N}$, it tells us that a subset of $\mathbb{N}$ can be decided by a term of type $\mathtt{Nat}[A] \to \mathtt{Bool}$ if and only if it is *ultimately periodic* – this fact is generalized in Joly's paper to ultimately periodic subsets of $\mathbb{N}^k$.

We deduce from the above theorem the regularity preservation claimed in Theorem 1.1:

▶ **Definition 1.7.** *A* λ-definable string function *is a function $f : \Gamma^* \to \Sigma^*$ that can be expressed by some term $t : \mathtt{Str}_\Gamma[A] \to \mathtt{Str}_\Sigma$, in the sense that $t\,\overline{w} = \overline{f(w)}$ for all $w \in \Sigma^*$.*

▶ **Corollary 1.8.** *The preimage of a regular language by a λ-definable function is regular.*

Proof. Let $f : \Gamma^* \to \Sigma^*$ be defined by some term $t : \mathtt{Str}_\Gamma[A] \to \mathtt{Str}_\Sigma$, and let $L \subseteq \Sigma^*$ be a regular language. Then $L = \mathcal{L}(u)$ for some $u : \mathtt{Str}_\Sigma[B] \to \mathtt{Bool}$. By the substitution lemma, t can be given the type $\mathtt{Str}_\Gamma[A[B]] \to \mathtt{Str}_\Sigma[B]$, so one can define the term $\lambda x.\ u\ (t\ x) : \mathtt{Str}[A[B]] \to \mathtt{Bool}$ in STλ. To conclude, observe that $f^{-1}(L) = \mathcal{L}(\lambda x.\ u\ (t\ x))$. ◀

The same substitution lemma can be used to establish that the λ-definable string functions are closed under composition. To prove Theorem 1.1, it remains only to show that HDT0L transductions are λ-definable – which is the subject of Section 3.1.

[6] We see a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ as the function $x \in \mathbb{N} \mapsto (y \mapsto f(x, y)) \in \mathbb{N}^{\mathbb{N}}$; cf. §3.

A final word about the relevance of the HDT0L+composition class to numeric functions (i.e. the unary case). We have already mentioned that Sénizergues claims (without giving a proof) this class to be equivalent to his *k-computable mappings* [27], defined in terms of a variant of iterated pushdown automata. The unary version of these mappings, called the *k-computable sequences*, had been previously studied in detail by Fratani and Sénizergues [13], who showed that they generalized some integer sequences of interest in number theory. Thus, an optimistic scenario could be: $\texttt{Nat}[A] \to \texttt{Nat}$ in STλ, unary HDT0L+composition and k-computable sequences all define the same class of functions $\mathbb{N} \to \mathbb{N}$, making this class a canonical mathematical object.

Generalizing this to strings, we propose a concrete question related to our open problem:

▶ **Open question 1.9.** Are the λ-definable string functions of Definition 1.7, the closure by composition of HDT0L transductions, and Sénizergues's k-computable mappings all the same class of functions from strings to strings?

1.2 Other motivations and related work

An analogy with machine-free complexity Beyond the very concrete goal stated above, the present work is an attempt to transpose to the context of automata some ideas from *implicit computational complexity* (ICC) – a field whose aim is to characterize complexity classes without reference to a particular machine model. Another field which fits the description just given is *descriptive complexity*, which establishes correspondences of the form "the predicates in the complexity class $\mathcal{C}$ are exactly those expressible in the logic $\mathcal{L}_{\mathcal{C}}$". Its very successful automata-theoretic counterpart is the use of *Monadic Second-Order Logic* (MSO) over various structures, ranging from finite words to graphs, infinite trees, ordinals…Concerning transductions in MSO, see [9, 7]. In contrast, the methods of ICC – e.g. term rewriting, function algebras, or λ-calculi – have a more computational flavor. To sum up:

	declarative programming	functional programming
complexity	Descriptive Complexity	Implicit Complexity (ICC)
automata	Monadic Second-Order Logic (MSO)	**this paper: Church encodings**

We should mention that there already exists some ICC-like work on transduction classes, for example function algebras for regular functions using combinators [2, 8, 6]. The closest to ours is perhaps Bojańczyk's characterization of polyregular functions by a variant of STλ [5], which we discuss in §2.3. The main difference is that both these works use primitive data types for strings, whereas we encode strings as higher-order functions (i.e. functions taking functions as arguments) inside "purely logical" calculi.

A few words about verification (and linear logic) The bottom row of the above table is also related to the field of *formal verification.* For instance, MSO over infinite words – whose decidability was proved by Büchi using automata – subsumes Linear Temporal Logic. The relevance of Church encodings in typed λ-calculi has been demonstrated in the context of *higher-order model checking*, an active field of research concerned with verifying functional programs: see Grellois's PhD thesis [16] and references therein. By generalizing this use of Church encodings, Melliès was led to introduce higher-order parity automata [22]. The introduction to [22] is particularly instructive: it proposes a "dictionary between automata theory and the simply typed λ-calculus" via Church encodings.

We should mention that linear logic plays an important role in some of Grellois and Melliès's work (e.g. [17]). For MSO over infinite words, there is also a recent application of linear logic, namely Pradic and Riba's approach to the synthesis problem [25].

Implicit complexity in $\mathrm{EA}\lambda$ Linear logic and its byproducts have also been used for ICC: one of the first works of this kind is the characterization of elementary recursive functions in Girard's Elementary Linear Logic (ELL) [15]. ELL later inspired the elementary affine λ-calculus [4] – or rather a variant that we baptized *a posteriori* $\mu\mathrm{EA}\lambda$ in [24] – which refines this by giving types of programs corresponding to each level of the k-EXPTIME hierarchy:

▶ **Theorem 1.10** (Baillot et al. [4]). *In $\mu\mathrm{EA}\lambda$, a predicate can be decided by a term of type* $!\mathtt{Str}_\Sigma \multimap !^{k+2}\mathtt{Bool}$ *iff it is in* k-EXPTIME. *In particular,* $!\mathtt{Str}_\Sigma \multimap !!\mathtt{Bool}$ *corresponds to* P.

We overload notation: here $\mathtt{Str}_\Sigma$ and $\mathtt{Bool}$ are respectively the $\mathrm{EA}\lambda$ types of strings over Σ and of booleans; they differ from the $\mathrm{ST}\lambda$ types of the same name. The unary connective '!' is the *exponential modality* of linear logic, which marks a duplicable resource, and plays a role in controlling complexity in $\mu\mathrm{EA}\lambda$; $\multimap$ is the linear function arrow.

We recently showed [24] that by replacing $\mu\mathrm{EA}\lambda$ by $\mathrm{EA}\lambda$ (i.e. by removing *type fixpoints* from $\mu\mathrm{EA}\lambda$) we get *regular languages* instead of polynomial time for the case $k = 0$. Just as we saw for $\mathrm{ST}\lambda$ in the previous section, that means that $!\mathtt{Str}_\Gamma \multimap !\mathtt{Str}_\Sigma$ is a type of regularity-preserving functions, closed under composition (whereas in $\mu\mathrm{EA}\lambda$, it corresponds to the class FP of polynomial time functions, cf. [24]). Another type for regular languages in $\mathrm{EA}\lambda$ is $\mathtt{Str}_\Sigma \multimap !\mathtt{Bool}$, so functions of type $\mathtt{Str}_\Gamma \multimap \mathtt{Str}_\Sigma$ are also of interest. We prove:

▶ **Theorem 1.11.** *The terms of type* $\mathtt{Str}_\Gamma \multimap \mathtt{Str}_\Sigma$ *(resp.* $!\mathtt{Str}_\Gamma \multimap !\mathtt{Str}_\Sigma$*)* $\mathrm{EA}\lambda$ *can only express* linear (resp. polynomial) time *and* regularity-preserving *functions; on the other hand:*

- *all* regular functions *can be defined by* $\mathrm{EA}\lambda$ *terms of both types;*
- *furthermore, the functions expressible with* $!\mathtt{Str}_\Gamma \multimap !\mathtt{Str}_\Sigma$ *are closed under* composition by substitution *(Definition 2.11), and therefore may have* $\Omega(n^k)$ *growth for any* $k \in \mathbb{N}$.

2 Several classes of transductions

This section recalls the HDT0L, regular, and polyregular transductions, and introduces our new "composition by substitution" operation. Along the way, we prove the inclusions

$$\{\text{regular + comp. by subst.}\} \subseteq \{\text{polyregular functions}\} \subsetneq \{\text{HDT0L + composition}\}$$

as we promised in the introduction. Finally, we (almost) define the regular *tree* functions.

A preliminary remark for this section on automata: recall that for a finite alphabet Σ, the set of words over Σ, denoted by Σ^*, is the free monoid over the set of generators Σ. Therefore, any function $\Sigma \to M$ to a monoid M uniquely extends to a morphism $\Sigma^* \to M$.

2.1 Register transducers and HDT0L systems

Our first machine model for string-to-string functions has been mentioned in the introduction: it is the non-linear version of streaming string transducers. Basically, we enrich finite automata with some memory: a finite number of *string-valued registers*. At each transition, the contents of the registers can be recombined by concatenation. After the input has been entirely read, an output function is invoked to determine the final result from the registers.

▶ **Definition 2.1.** *A* register transducer *over input and output alphabets* Γ *and* Σ *consists of:*

- *a finite set* Q *of* states, *with an initial state* $q_I \in Q$
- *a finite set* R *of* registers *(or variable names), disjoint from* Γ *and* Σ
- *a transition function* $\delta : Q \times \Gamma \to Q \times (R \to (\Sigma \cup R)^*)$ *(i.e.* $Q \times \Gamma \to Q \times ((\Sigma \cup R)^*)^R$*)*
- *an output function* $F : Q \to (\Sigma \cup R)^*$

A configuration *of this register transducer*[7] *is a pair* (q,s) *with* $q \in Q$ *and* $s : R \to \Sigma^*$. *For* $c \in \Gamma$, *we write* $(q,s) \longrightarrow_c (q',s')$ *when* $(q',u) = \delta(q,c)$ *and* $s' = s^* \circ u$, *where* $s^* : (\Sigma \cup R)^* \to \Sigma^*$ *is the monoid morphism taking* $a \in \Sigma$ *to itself and* $r \in R$ *to* $s(r)$.

The image *of a string* $w = w_1 \dots w_n \in \Gamma^*$ *by this register transducer is* $s^*(F(q))$ *where* $q \in Q$ *and* $s : R \to \Sigma^*$ *are uniquely determined by* $(q_I, (x \in X \mapsto \varepsilon)) \longrightarrow_{w_1} \dots \longrightarrow_{w_n} (q,s)$. *This defines a function* $\Gamma^* \to \Sigma^*$.

For example, let $Q = \{q\}$, $R = \{X, Y\}$ and $\delta(q,c) = (q, (X \mapsto Xc, Y \mapsto cY))$ for all $c \in \Gamma = \Sigma = \{a,b\}$. Then for $w = w_1 \dots w_n \in \Gamma^*$,

$$(q,(x \in X \mapsto \varepsilon)) \longrightarrow_{w_1} \dots \longrightarrow_{w_n} (q,s) \qquad \text{with } s(X) = w \text{ and } s(Y) = \texttt{reverse}(w)$$

So, if we take as output function $F(q) = XY$, the function defined is $w \mapsto w \cdot \texttt{reverse}(w)$.

Alternatively, these functions can be specified using monoid morphisms:

▶ **Definition 2.2.** *A* HDT0L system *consists of:*
- *an input alphabet* Γ, *an output alphabet* Σ, *and a working alphabet* Δ*;*
- *an initial word* $d \in \Delta^*$*;*
- *for each* $c \in \Gamma$, *a* monoid morphism $h_c : \Delta^* \to \Delta^*$*;*
- *a final morphism* $h' : \Delta^* \to \Sigma^*$.

It defines the transduction taking $w = w_1 \dots w_n \in \Gamma^*$ *to* $h' \circ h_{w_1} \circ \dots \circ h_{w_n}(d) \in \Sigma^*$.

The family $(h_c)_{c\in\Gamma}$ may be equivalently given as a morphism $H : \Gamma^* \to \mathrm{Hom}(\Delta^*, \Delta^*)$ (the latter is a monoid for function composition); the image of the word w is then $h'((H(w))(d))$.

▶ **Theorem 2.3** (Filiot & Reynier [11]). *A string function* $\Gamma^* \to \Sigma^*$ *can be computed by a register transducer iff it can be specified by a HDT0L system.*

2.2 (Poly)regular functions vs HDT0L(+composition)

We shall take *linear* register transducers as our definition of regular functions. Enriching this class with a "squaring with underlining" operation yields Bojańczyk's polyregular functions.

▶ **Definition 2.4** (Alur & Černý [3]). *A* streaming string transducer (SST) *is a register transducer satisfying the* copyless assignment *conditions: for all* $r \in R$,
- *for any register update in the transducer – i.e. any* $u : R \to (\Sigma \cup R)^*$ *such that* $(q',u) = \delta(q,c)$ *for some* $q, q' \in Q$ *and* $c \in \Gamma$ *–* r *appears at most once among all* $u(r')$ *for* $r' \in R$*;*
- *for all* $q \in Q$, r *appears at most once in the string* $F(q)$.

A function $\Gamma \to \Sigma^*$ *is* regular *if it is computed by some SST.*

▶ Remark 2.5. The important part is the first item; the condition on output functions can be removed without increasing the expressivity of streaming string transducers.

▶ **Definition 2.6.** *Let* Γ *be a finite alphabet. We write* $\underline{\Gamma} = \{\underline{c} \mid c \in \Gamma\}$ *for a disjoint copy of* Γ *made of "underlined" letters. The function* $\texttt{squaring}_\Gamma : \Gamma^* \to (\Gamma \cup \underline{\Gamma})^*$ *is illustrated by the following example for* $\Gamma = \{1,2,3,4\}$*:* $\texttt{squaring}_\Gamma(1234) = \underline{1}2341\underline{2}3412\underline{3}4123\underline{4}$.

▶ **Definition 2.7.** *The class of* polyregular functions *is the smallest class closed under composition containing the regular functions and the functions* $\texttt{squaring}_\Gamma$ *for all finite* Γ.

[7] We borrow the name from `https://www.mimuw.edu.pl/~bojan/papers/toolbox.pdf`.

Bojańczyk's original definition [5] is the closure by composition of *sequential* functions, squaring and an additional "iterated reverse" function. Ours is equivalent because all regular functions are polyregular, all sequential functions are regular, and iterated reverse is a regular function (for this last point, we invite the reader to consult the definition of iterated reverse in [5] and check that a SST with two registers suffices to compute it).

Let us now compare these classes to the HDT0L transductions (+ composition).

▶ **Proposition 2.8.** *All regular functions can be specified by HDT0L systems.*

Proof. Streaming string transducers are special cases of register transducers. ◀

▶ **Theorem 2.9.** *All polyregular functions are compositions of HDT0L transductions.*

Proof sketch. Thanks to the previous proposition, it suffices to show that the functions $\texttt{squaring}_\Gamma$ can be computed by composing register transducers. We decompose them as

$$\mathsf{1234} \mapsto \underline{\mathsf{1}}\mathsf{1}\underline{\mathsf{2}}\mathsf{12}\underline{\mathsf{3}}\mathsf{123}\underline{\mathsf{4}} \mapsto \underline{\mathsf{4}}\mathsf{321}\underline{\mathsf{3}}\mathsf{21}\underline{\mathsf{2}}\mathsf{1}\underline{\mathsf{1}} \mapsto \underline{\mathsf{4}}\mathsf{321(4)}\underline{\mathsf{3}}\mathsf{21(43)}\underline{\mathsf{2}}\mathsf{1(432)}\underline{\mathsf{1}} \mapsto \underline{\mathsf{1}}\mathsf{2341}\underline{\mathsf{2}}\mathsf{3412}\underline{\mathsf{3}}\mathsf{4123}\underline{\mathsf{4}}$$

where the parentheses are not part of the string, they only serve to help readability.

- The 1st step uses two registers, one for the output and one containing the current prefix.
- The 3rd step uses one register for the output and another register keeping track of the underlined characters seen thus far, by concatenating their non-underlined counterparts.
- The 2nd and 4th steps just apply the reverse function, which is regular. ◀

▶ **Proposition 2.10.** *There exists a HDT0L transduction which is not polyregular.*

Proof. Polyregular functions have polynomial growth [5], while HDT0L transductions may grow exponentially. Take e.g. a HDT0L system with $h_a(b) = bb$ for all $a \in \Gamma, b \in \Sigma$. ◀

2.3 Composition by substitutions vs polynomial list functions

We come to our new operation on functions which allows increasing the exponent of polynomial growth. It preserves polyregular functions, but this is not easy to establish from the definition using the squaring function. We shall instead rely on another characterization: an enriched variant of the simply typed λ-calculus, called the "polynomial list functions" formalism in [5].

▶ **Definition 2.11.** *Let $f : \Gamma^* \to I^*$, and for each $i \in I$, let $g_i : \Gamma^* \to \Sigma^*$. The* composition by substitutions *of f with the family $(g_i)_{i \in I}$ is the function*

$$\mathrm{CbS}(f, (g_i)_{i\in I}) : w \mapsto g_{i_1}(w) \dots g_{i_n}(w) \text{ where } f(w) = i_1 \dots i_k$$

That is, we first apply f to the input, then every letter i in the result of f is substituted by the image of the original input by g_i. Thus, $\mathrm{CbS}(f, (g_i)_{i\in I})$ is a function $\Gamma^* \to \Sigma^*$.

As an example, this can be used to define the "squaring without underlining" function[8] $w \mapsto w^{|w|}$, which can be expressed as $\mathrm{CbS}(f : w \mapsto a^{|w|}, (g_a : w \mapsto w))$ with f and g_a regular. Its growth rate is quadratic, while regular functions have at most linear growth.

▶ Remark 2.12. More generally, the smallest class containing regular functions and closed by both CbS and usual function composition contains, for all $k \in \mathbb{N}$, some f with $|f(w)| = \Theta(|w|^k)$. However, we conjecture that $\texttt{squaring}_{\{1\}}$ (with underlining) is not in this class.

[8] It is a classic exercise in formal languages to prove that if L is a regular language, then $\{w \mid w^{|w|} \in L\}$ is also regular. Our study of superlinear transduction classes provides a wider context for this fact.

We now recall how polynomial list functions are defined. They enrich the grammar of λ-terms with constants whose meaning can be specified by extending the β-rule of §1.1, e.g.

$$\mathtt{is}_a\ b =_\beta \mathtt{true} \quad \text{if } a = b \qquad \mathtt{is}_a\ b =_\beta \mathtt{false} \quad \text{if } a \neq b$$

The grammar of types is also extended accordingly. For instance, any finite set τ induces a type also written τ, such that the elements $a \in \tau$ correspond to the terms $a : \tau$ of this type. There are also operations expressing the cartesian product ($\times$) and disjoint union ($+$) of two types; and a type of lists (A^* is the type of lists over the type A). So we actually consider

$$\mathtt{is}^\tau_a : \tau \to \{\mathtt{true}\} + \{\mathtt{false}\} \qquad \text{for any finite set } \tau$$

and in the expression $\mathtt{is}^\tau_a\ b$, one therefore requires b to be part of a finite set τ specified in advance which also contains a. See [5, Section 4] for the other primitive operations that are added to STλ; we make use of `is`, `case`, `map` and `concat` here. Bojańczyk's result is that if Γ and Σ are finite sets, then the polynomial list functions of type $\Gamma^* \to \Sigma^*$ correspond exactly the polyregular functions.

▶ Remark 2.13. There is no substitution in the input type, and this is why our λ-definable string functions are still more expressive than polynomial list functions. On the other hand, this shows that primitive data types provide an alternative way of going beyond the poor expressive power (cf. [28, 20]) of the functions defined by $\mathtt{Str}_\Gamma \to \mathtt{Str}_\Sigma$ in STλ.

▶ **Lemma 2.14.** *Let $I = \{i_1, \dots, i_{|I|}\}$. Then the function $\mathtt{match}^{I,\tau} : I \to \tau \to \dots \to \tau \to \tau$ which returns its $(k+1)$-th argument[9] when its 1st argument is i_k is a polynomial list function.*

Proof sketch. By induction on $|I|$, it is definable from $\mathtt{is}^I_i$ ($i \in I$) & $\mathtt{case}^{\{\mathtt{true}\},\{\mathtt{false}\},\tau}$. ◀

▶ **Theorem 2.15.** *Polyregular functions are closed under composition by substitutions.*

Proof. Let $f : \Gamma^* \to I^*$, and for $i \in I$, $g_i : \Gamma^* \to \Sigma^*$ be polyregular functions. Assuming that f and g_i ($i \in I$) are defined by polynomial list functions of the same name, $\mathrm{CbS}(f, (g_i)_{i \in I})$ can be expressed as $\lambda w.\ \mathtt{concat}^\Sigma\ (\mathtt{map}^{I,\Sigma^*}\ (\lambda i.\ \mathtt{match}^{I,\Sigma^*}\ i\ (g_{i_1}\ w)\ \dots\ (g_{i_{|I|}}\ w))\ (f\ w))$. ◀

2.4 Register tree transducers

To define the regular tree functions, the first step is to consider the tree version of register transducers. We shall restrict ourselves to binary trees, as in [1, §3.7].

▶ **Definition 2.16.** *The set* $\mathrm{BinTree}(\Sigma)$ *of* binary trees *over the alphabet Σ, and the set* $\partial\mathrm{BinTree}(\Sigma)$ *of* one-hole binary trees[10]*, are generated by the respective grammars*

$$T, U ::= \langle\rangle \mid a\langle T, U\rangle \quad (a \in \Sigma) \qquad\qquad T' ::= \square \mid a\langle T', T\rangle \mid a\langle T, T'\rangle \quad a \in \Sigma)$$

That is, $\mathrm{BinTree}(\Sigma)$ *consists of binary trees whose leaves are all equal to $\langle\rangle$ and whose nodes are labeled with letters in Σ. As for* $\partial\mathrm{BinTree}(\Sigma)$*, it contains trees with exactly one leaf labeled $\square$ instead of $\langle\rangle$. This "hole" $\square$ is intended to be substituted by a tree: for $T' \in \partial\mathrm{BinTree}(\Sigma)$ and $U \in \mathrm{BinTree}(\Sigma)$, $T'[U]$ denotes T' where $\square$ has been replaced by U.*

[9] See the beginning of §3 for an explanation of functions with multiple arguments in STλ.
[10] Our choice of notation is motivated by the fact that in enumerative combinatorics, the derivative of a generating function or species of structures corresponds to taking one-hole contexts.

▶ **Definition 2.17.** *The* binary tree (resp. one-hole binary tree) expressions *over the variable sets V and V' are generated by the grammar (with $x \in V$, $x' \in V'$ and $a \in \Sigma$)*

$$E, F ::= \langle\rangle \mid x \mid a\langle E, F\rangle \mid E'[E] \qquad \text{(resp. } E', F' := \square \mid x' \mid a\langle E', E\rangle \mid a\langle E, E'\rangle \mid E'[F']\text{)}$$

The sets of such expressions is denoted by $\mathrm{ExprBT}(\Sigma, V, V')$ (resp. $\mathrm{Expr}\partial\mathrm{BT}(\Sigma, V, V')$).

Given $\rho : V \to \mathrm{BinTree}(\Sigma)$ and $\rho' : V' \to \partial\mathrm{BinTree}(\Sigma)$, one defines $E(\rho, \rho') \in \mathrm{BinTree}(\Sigma)$ for $E \in \mathrm{ExprBT}(\Sigma)$ and $E'(\rho, \rho') \in \partial\mathrm{BinTree}(\Sigma)$ for $E \in \mathrm{Expr}\partial\mathrm{BT}(\Sigma)$ in the obvious way.

▶ **Definition 2.18.** *A* register tree transducer (RTT) $\Gamma^* \to \Sigma^*$ *consists of: a finite set Q of* states *with an initial state $q_I \in Q$; two disjoint finite sets R, R' of* registers*; an output function $F : Q \to \mathrm{ExprBT}(\Sigma, R, R')$; and a transition function (where $R_{\lhd\rhd} = R \times \{\lhd, \rhd\}$)*

$$\delta : Q \times Q \times \Gamma \to Q \times (R \to \mathrm{ExprBT}(\Sigma, R_{\lhd\rhd}, R'_{\lhd\rhd})) \times (R' \to \mathrm{Expr}\partial\mathrm{BT}(\Sigma, R_{\lhd\rhd}, R'_{\lhd\rhd}))$$

The set of configurations of a RTT is $Q \times \mathrm{BinTree}(\Sigma)^R \times \partial\mathrm{BinTree}(\Sigma)^{R'}$. It processes its input tree in a single bottom-up traversal, computing for each subtree a configuration, starting with $(q_I, (r \mapsto \langle\rangle), (r' \mapsto \square))$ at the leaves. The configuration at $a\langle T, U\rangle$ is obtained from the one at T and the one at U by applying δ to the pair of states and to a, and using each expression E in the image to determine the value $E(\rho, \rho')$ of the corresponding register, where ρ maps $(r, \lhd)$ (resp. $(r, \rhd)$) to the value of r in the configuration of the left subtree T (resp. right subtree U), and similarly for ρ'. See [1, §3.7] for a more precise definition.

Regular tree functions are actually characterized by Alur and D'Antoni's *bottom-up ranked tree transducers* [1]. They are register tree transducers with a kind of linearity condition, whose statement is more complicated than in the case of SSTs. We give the full definition – which involves a "conflict relation" over registers – in Appendix B.4.

3 Transductions in the simply typed λ-calculus

3.1 HDT0L+composition string functions are λ-definable

After these long preliminaries, at last, it is time to encode transductions in STλ.

First, we need to state precisely the definitions of Church encodings beyond `Nat`. For a finite alphabet Σ, we take $\mathtt{Str}_\Sigma = (o \to o)^{|\Sigma|} \to o \to o$. This requires some explanations:

- The function arrow is left-associative, so this is the same as $(o \to o)^{|\Sigma|} \to (o \to o)$. In general, a term of type $A_1 \to \ldots \to A_n \to B = A_1 \to (\ldots(A_{n-1} \to (A_n \to B))\ldots)$ should be thought of as a function with n inputs of type $A_1, \ldots, A_n$ and one output of type B (this is analogous to the set-theoretic isomorphism $B^{A_1 \times A_2} \cong (B^{A_2})^{A_1}$).
- For the same reasons we abbreviate $A \to \ldots \to A \to B$ with n times A as $A^n \to B$ (and at the level of terms, $(\ldots((f\ x_1)\ x_2)\ldots)\ x_n$ as $f\ x_1\ \ldots\ x_n$).

Observe that $\mathtt{Str}_{\{1\}} = (o \to o) \to o \to o = \mathtt{Nat}$ as we claimed in the introduction.

Given an enumeration $\Sigma = \{a_1, \ldots, a_{|\Sigma|}\}$, a string $w = a_{i_1} \ldots a_{i_n} \in \Sigma^*$ is encoded as

$$\overline{w} = \lambda f_1.\ \ldots\ \lambda f_{|\Sigma|}.\ \lambda x.\ f_{i_1}\ (\ldots\ (f_{i_n}\ x)\ldots) \quad \text{(morally, } \overline{w}(f_1, \ldots, f_{|\Sigma|}) = f_{i_1} \circ \ldots \circ f_{i_n}\text{)}$$

With this, the definition of λ-definable string functions (Definition 1.7) is now fully rigorous.

The first two items in the statement of Theorem 1.1 have already been established in the introduction (more precisely §1.1.3), so let us prove the last one.

▶ **Lemma 3.1.** *Any monoid morphism $\Gamma^* \to \Sigma^*$ can be defined by a term in* STλ *of type* $\mathtt{Str}_\Gamma \to \mathtt{Str}_\Sigma$ *– there is no need for a substitution in the input type.*

Proof. For $\Gamma = \{g_1, \dots, g_k\}$, $\Sigma = \{s_1, \dots, s_l\}$, let $\varphi : \Gamma^* \to \Sigma^*$ be a morphism. We define

$$t = \lambda z.\ \left(\lambda f_1.\ \dots\ \lambda f_l.\ z\ \left(\overline{\varphi(g_1)}\ f_1\ \dots\ f_l\right)\ \dots\ \left(\overline{\varphi(g_k)}\ f_1\ \dots\ f_l\right)\right)$$

One can check that $t : \mathtt{Str}_\Gamma \to \mathtt{Str}_\Sigma$ represents φ. Morally, the reason is that for $u = g_{i_1} \dots g_{i_n} \in \Gamma^*$, $\overline{\varphi(g_{i_1})}(f_1, \dots, f_n) \circ \dots \circ \overline{\varphi(g_n)}(f_1, \dots, f_n) = \overline{\varphi(g_{i_1}) \dots \varphi(g_{i_n})}(f_1, \dots, f_n)$. ◀

▶ **Theorem 3.2.** *HDT0L transductions are λ-definable string functions.*

Proof. Consider a HDT0L system defining a function $f : \Gamma^* \to \Sigma^*$ with alphabets $\Gamma = \{g_1, \dots, g_k\}$, Σ and Δ, initial word $d \in \Delta^*$, morphisms $h_i : \Delta^* \to \Delta^*$ for $i \in \{1, \dots, k\}$ corresponding to $c_i \in \Gamma$, and final morphism $h' : \Delta^* \to \Sigma^*$. By the above lemma, each h_i (resp. h') can be represented by $u_i : \mathtt{Str}_\Delta \to \mathtt{Str}_\Delta$ (resp. $u' : \mathtt{Str}_\Delta \to \mathtt{Str}_\Sigma$).

It is important to note that the input and output types of the u_i are equal. This allows us to define the term $t = \lambda z.\ u'\ (z\ u_1\ \dots\ u_k\ \overline{d}) : \mathtt{Str}_\Gamma[\mathtt{Str}_\Delta] \to \mathtt{Str}_\Sigma$ which expresses f. ◀

3.2 Regular tree functions are λ-definable

In the case of tree-to-tree functions, we also prove that register tree transducers (RTTs) can be encoded in STλ – and consequently, their closure under composition also can. However, we are not aware of any alternative characterization of this class – we only know that it it is a (strict) superclass of the regular tree functions. So we must work directly with RTTs.

The type of *Church encodings of binary trees* is $\mathtt{BT}_\Sigma = (o \to o \to o)^{|\Sigma|} \to o \to o$ (where Σ is the alphabet of node labels). Given an enumeration $\Sigma = \{a_1, \dots, a_n\}$, each $T \in \mathrm{BinTree}(\Sigma)$ is encoded as a λ-term $\overline{T} = \lambda f_1.\ \dots\ \lambda f_n.\ \lambda x.\ \widehat{T} : \mathtt{BT}_\Sigma$, where we define inductively $\widehat{\langle\rangle} = x$ and $\widehat{a_i\langle T, U\rangle} = f_i\ \widehat{T}\ \widehat{U}$. Morally, $\overline{T}(f_1, \dots, f_n, x)$ is the result of a single-pass bottom-up traversal of T, starting with x at the leaves and combining the results of subtrees with T.

▶ Remark 3.3. Analogously, $\mathtt{Str}_\Sigma$ can be seen as an encoding of "unary trees" whose bottom-up traversals correspond to right-to-left traversals of the corresponding strings (think of the `fold_right` / `foldr` functions in some functional programming languages).

▶ **Lemma 3.4.** *Any $T' \in \partial\mathrm{BinTree}(\Sigma)$ can be compiled to a term $\mathcal{C}(T')$ in* STλ *of type $\partial\mathtt{BT}_\Sigma = \mathtt{BT}_\Sigma \to \mathtt{BT}_\Sigma$ such that for all $U \in \mathrm{BinTree}(\Sigma)$, $\mathcal{C}(T')\ \overline{U} =_\beta \overline{T'[U]}$. Similarly:*

- *any $E \in \mathrm{ExprBT}(\Sigma, V = \{x_1, \dots, x_n\}, V' = \{x'_1, \dots, x'_m\})$ can be compiled to a λ-term $\mathcal{C}(E) : (\mathtt{BT}_\Sigma)^n \to (\partial\mathtt{BT}_\Sigma)^m \to \partial\mathtt{BT}_\Sigma \to \mathtt{BT}_\Sigma$ such that for all $\rho : V \to \mathrm{BinTree}(\Sigma)$ and $\rho' : V' \to \partial\mathrm{BinTree}(\Sigma)$, $\overline{E(\rho, \rho')} =_\beta \mathcal{C}(E)\ \overline{\rho(x_1)}\ \dots\ \overline{\rho(x_n)}\ \mathcal{C}(\rho'(x'_1))\ \dots\ \mathcal{C}(\rho'(x'_m))$.*
- *any $E' \in \mathrm{Expr}\partial\mathrm{BT}(\Sigma)$ can be compiled to a term $\mathcal{C}(E') : (\mathtt{BT}_\Sigma)^n \to (\partial\mathtt{BT}_\Sigma)^m \to \partial\mathtt{BT}_\Sigma$ enjoying the analogous property.*

▶ **Theorem 3.5.** *Any function from* $\mathrm{BinTree}(\Gamma)$ *to* $\mathrm{BinTree}(\Sigma)$ *computed by a register tree transducer can be expressed by a λ-term of type $\mathtt{BT}_\Gamma[A] \to \mathtt{BT}_\Sigma$ for some simple type A.*

Proof sketch. As discussed above, the kind of bottom-up traversal done by a register tree transducer corresponds exactly to the "fold function" embodied by the Church encoding of a tree. One would want to directly encode the RTT by setting A to be its type of configurations; the main obstacle to defining such an A is the lack of product and sum types in STλ (unlike in polynomial list functions, cf. §2.3). To overcome this, we use a continuation-passing-style transformation with return type $\mathtt{BT}_\Sigma$. Cf. Appendix A for details of the proof. ◀

▶ **Corollary 3.6.** *Any regular tree function is definable in* STλ.

4 Streaming transducers in the elementary affine λ-calculus

The grammar of terms of EAλ and its equational theory are given by

$$t, u ::= x \mid \lambda x.\, t \mid \lambda !x.\, t \mid t\, u \mid !t \qquad (\lambda x.\, t)\, u =_\beta t\{x := u\} \quad (\lambda !x.\, t)\, (!u) =_\beta t\{x := u\}$$

where x is taken in a countable set of variables; we take $=_\beta$ to be the smallest congruence generated by the two rules above. The type system of EAλ is given in Appendix B.1. It enforces two important constraints on terms. The first means that one must use $\lambda!$ to define non-linear functions – in other words, a subterm must be marked by '!' to be duplicable:

(linearity) in any subterm of the form $\lambda x.\, t$, x appears *at most once* in t

An additional constraint related specifically to Elementary Linear Logic [15] is

(stratification) in any subterm of the form $\lambda x.\, t$ (resp. $\lambda !x.\, t$),
the *depth* of each occurrence of x in t is 0 (resp. 1)

By depth we mean the number of !'s in t surrounding x. Stratification entails that in the two rules above generating $=_\beta$, the depth of the subterm u is the same on both sides; thus, we have an invariant for $=_\beta$. In particular one cannot define type-cast functions taking any $!t$ to t (*dereliction*) or to $!!t$ (*digging*). ('!' is called the *exponential modality*.)

4.1 Encoding streaming string transducers

The Church-encoded strings over $\Sigma = \{a_1, \dots, a_n\}$ are defined in EAλ as:

$$\text{for } w = a_{i_1} \dots a_{i_n} \in \Sigma^*, \quad \overline{w} = \lambda !f_1.\, \dots \lambda !f_n.\, !(\lambda x.\, f_{i_1}\, (\dots (f_{i_n}\, x) \dots))$$

and they are given the type $\mathtt{Str}_\Sigma = \forall \alpha.\, \mathtt{Str}_\Sigma[\alpha]$ where $\mathtt{Str}_\Sigma[\alpha] = (!(\alpha \multimap \alpha))^{|\Sigma|} \multimap !(\alpha \multimap \alpha)$. (As we did for ST$\lambda$, we abbreviate $A \multimap \dots \multimap A \multimap B$ with k times A as $A^k \multimap B$.) The $\forall \alpha$ is a second-order quantifier – the type system of EAλ indeed supports polymorphism.

Another encoding in EAλ is that of the finite set $\{1, \dots, k\}$, represented by the type $\mathtt{Fin}(n) = \forall \alpha.\, \alpha^n \multimap \alpha$: the encoding of $i \in \{1, \dots, k\}$ is $\lambda x_1.\, \dots\, \lambda x_k.\, x_i$. For instance the type $\mathtt{Bool}$ mentioned in §1.2 is $\mathtt{Fin}(2) = \forall \alpha.\, \alpha \multimap \alpha \multimap \alpha$ – this mirrors the STλ booleans.

As we discussed in §3.2, it is most natural to process a string right-to-left using its Church encoding. But register transducers work in a left-to-right fashion. To compensate for that, we shall *propagate output functions backwards* instead.

▶ **Definition 4.1.** *Let (Q, q_I, R, δ, F) be a register transducer with input alphabet Γ. We define $\delta^O : \Gamma \times (Q \to (\Sigma \cup R)^*) \to (Q \to (\Sigma \cup R)^*)$ by $\delta^O(a, G) = (q \in Q \mapsto s^*_{a,q}(G(q'_{a,q})))$, where $(q'_{a,q}, s_{a,q}) = \delta(q, a)$ and $s^*_{a,q}$ is the unique extension of $s_{a,q} : R \to (\Sigma \cup R)^*$ to a monoid morphism $(\Sigma \cup R)^* \to (\Sigma \cup R)^*$ taking each letter of Σ to itself.*

▶ **Proposition 4.2.** *Let $w = w_1 \dots w_n \in \Gamma^*$. The image of w by the register transducer (Q, q_I, R, δ, F) is $\varphi(G(q_I))$ where $G = \delta^O(w_1, (\dots \delta^O(w_n, F) \dots))$ and $\varphi : (\Sigma \cup R)^* \to \Sigma^*$ erases all letters from R in its input.*

We must now implement this idea as a term of type $\mathtt{Str}_\Gamma \multimap \mathtt{Str}_\Sigma$ in EAλ. *This is the same thing as a term of type* $\mathtt{Str}_\Gamma[A] \multimap \mathtt{Str}_\Sigma[\alpha]$, where α is a free type variable and A may contain α: by linearity, the quantified type variable in the input is instantiated only once.

To implement this, one would want to iterate over the type of output functions; naively, one would set A to be $\mathtt{Fin}(|Q|) \multimap \mathtt{Str}_{\Sigma \cup R}$. However this type contains an exponential (inside $\mathtt{Str}_{\Sigma\cup R}$) and so, *because of the stratification property, it is useless to produce an output of type* $\mathtt{Str}[\alpha]$ *since* α *is exponential-free.* (This can be made rigorous using the *truncation* operation for EAλ introduced in [24].) Instead, we shall iterate over the *purely linear* type $A = \mathtt{Fin}(|Q|) \multimap (\alpha \multimap \alpha)^{|R|} \multimap (\alpha \multimap \alpha)$. It differs from the previous candidate by the absence of exponentials and of $|\Sigma|$ arguments of type $\alpha \multimap \alpha$. This reflects the fact that, if F is a *copyless* output function, then for all $q \in Q$, $\overline{F(q)}$ is *linear* in all arguments corresponding to register names. As for those corresponding to Σ, they will be somehow replaced with non-linear variables provided by the context.

We illustrate the construction on the register transducer computing $w \mapsto w \cdot \mathtt{reverse}(w)$ given in §2.1, which is actually a streaming string transducer (that is, it is copyless). The general proof is given in Appendix B.2. We make a further simplication: since this transducer has a single state, we drop the $\mathtt{Fin}(|Q|)$ argument in the type A. There are 2 registers, so our term has type $\mathtt{Str}_{\{a,b\}}[A] \multimap \mathtt{Str}_{\{a,b\}}[\alpha]$ for $A = (\alpha \multimap \alpha) \multimap (\alpha \multimap \alpha) \multimap (\alpha \multimap \alpha)$.

First, we define EAλ terms corresponding to each $\delta^O(c, -)$ (Definition 4.1) for $c \in \{a, b\}$:

$$d_c = \lambda G.\ \lambda r_X.\ \lambda r_Y.\ G\ (\lambda z.\ r_X\ (f_c\ z))\ (\lambda z.\ f_c\ (r_Y\ y)) : A \multimap A$$

These terms use non-linearly the free variables $f_a, f_b : \alpha \multimap \alpha$. Observe that the linearity condition of EAλ (r_X and r_Y occur at most once) is satisfied precisely because the corresponding register update is copyless! Next, we define $t : \mathtt{Str}_{\{a,b\}}[A] \multimap \mathtt{Str}_{\{a,b\}}[\alpha]$ as

$$t = \lambda u.\ \lambda !f_a.\ \lambda !f_b.\ (\lambda !h.\ !(h\ (\lambda r_X.\ \lambda r_Y.\ (\lambda z.\ r_X\ (r_Y\ z)))\ (\lambda x.\ x)\ (\lambda y.\ y)))\ (u\ !d_a\ !d_b)$$

Note that $!d_a$ contains f_a at depth 1, bound by $\lambda !f_a$. Let $w = w_1 \dots w_n \in \{a, b\}^*$. Then $\overline{w}\ !d_a\ !d_b =_\beta\ !(\lambda x.\ d_{w_1}\ (\dots (d_{w_n}\ x) \dots))$. Passing this as argument to $(\lambda !h.\ \dots)$ unpacks this exponential: $h = \lambda x.\ d_{w_1}\ (\dots (d_{w_n}\ x) \dots)$. Next, h is applied to a representation of the output function $F(q) = XY$; so what we obtain represents $\delta^O(w_1, \dots, \delta^O(w_n, F))$. Indeed,

$$(d_{w_1} \circ \dots \circ d_{w_n})\ (\lambda r_X.\ \lambda r_Y.\ r_X \circ r_Y) =_\beta \lambda r_X.\ \lambda r_Y.\ r_X \circ f_{w_1} \dots \circ f_{w_n} \circ f_{w_n} \circ \dots \circ f_{w_1} \circ r_Y$$

where $g_1 \circ \dots \circ g_m$ is an abbreviation for $\lambda x.\ g_1\ (\dots (g_m\ x) \dots)$. By applying the above to two identity functions, we erase r_X and r_Y; thus, in the end, we get $t\ \overline{w} =_\beta \overline{w \cdot \mathtt{reverse}(w)}$.

In general, since streaming string transducers can compute all regular functions:

▶ **Theorem 4.3** (proved in Appendix B.2)**.** *Any regular function* $\Gamma^* \to \Sigma^*$ *can be computed by an* EAλ *term of type* $\mathtt{Str}_\Gamma \multimap \mathtt{Str}_\Sigma$ *or* $!\mathtt{Str}_\Gamma \multimap !\mathtt{Str}_\Sigma$*.*

The last part is because any term of type $A \multimap B$ in EAλ can be type-cast into a term of type $!A \multimap !B$ [4, Proposition 28].

We have done the hard part in proving Theorem 1.11. There remains only:

▶ **Proposition 4.4.** *The expressible functions for the type* $!\mathtt{Str}_\Gamma \multimap !\mathtt{Str}_\Sigma$ *in* EAλ *are closed under composition by substitution.*

Proof. See Appendix B.3. ◀

▶ **Theorem 4.5.** *Any* EAλ *term of type* $\mathtt{Str}_\Gamma \multimap \mathtt{Str}_\Sigma$ *(resp.* $!\mathtt{Str}_\Gamma \multimap !\mathtt{Str}_\Sigma$*) defines a function computable in linear (resp. polynomial) time.*

Proof sketch. Let us start with $!\mathtt{Str}_\Gamma \multimap !\mathtt{Str}_\Sigma$. We proved in [24] (building on work in [4]) that, in a larger system called μEAλ, this type corresponds exactly to polynomial time functions. In particular, when we restrict to the subsystem EAλ, the polynomial time upper bounds still hold. For $\mathtt{Str}_\Gamma \multimap \mathtt{Str}_\Sigma$, we can routinely adapt the arguments in [24, 4] to obtain a linear time bound for μEAλ. The algorithm is to perform β-reduction with a particular "stratified" reduction strategy. ◀

4.2 Bottom-up ranked tree transducers and the two linear conjunctions

The EAλ type of Church-encoded binary trees with node labels in $\Sigma = \{a_1, \dots, a_{|\Sigma|}\}$ is

$$\mathtt{BT}_\Sigma = \forall\alpha.\ \mathtt{BT}_\Sigma[\alpha] \qquad \text{where } \mathtt{BT}_\Sigma[\alpha] = (!(\alpha \multimap \alpha \multimap \alpha))^{|\Sigma|} \multimap !\alpha \multimap !\alpha$$

To each $T \in \mathrm{BinTree}(\Sigma)$ we associate $\overline{T} : \mathtt{BT}_\Sigma$ in the obvious way.

▶ **Theorem 4.6.** *Any regular tree function can be expressed by some $t : \mathtt{BT}_\Gamma \multimap \mathtt{BT}_\Sigma$ in EAλ.*

Proof sketch. We give only the main ideas here; a more detailed proof is provided in Appendix B.4. As before, this amounts to translating any bottom-up ranked tree transducer (BRTT) to some term $t : \mathtt{BT}_\Gamma[A] \multimap \mathtt{BT}_\Sigma[\alpha]$, where A may contain the type variable α. Here, the natural direction of processing for a Church-encoded binary tree is bottom-up, and this coincides with the way a BRTT works, unlike the case of strings in the previous subsection.

First, let us consider the case of a register tree transducer $(Q, q_I, R, R', F, \delta)$ enjoying a linearity condition directly analogous to streaming string transducers (SSTs). Then we take

$$A = \mathtt{Fin}(|Q|) \otimes \alpha^{\otimes|R|} \otimes (\alpha \multimap \alpha)^{\otimes|R'|} \qquad \text{representing } \textit{configurations} \text{ of the BRTT}$$

the use of $\otimes$ denoting the second-order encoding of the *multiplicative conjunction*

$$A_1 \otimes \ldots \otimes A_m = \forall\beta.\ (A_1 \multimap \ldots \multimap A_m \multimap \beta) \multimap \beta \qquad B^{\otimes m} = B \otimes \ldots \otimes B$$

An element of $\mathrm{BinTree}(\Sigma)$ (resp. $\partial\mathrm{BinTree}(\Sigma)$) contained in a register is therefore represented as a term of type α (resp. $\alpha \multimap \alpha$), using non-linearly the free variables $f_i : \alpha \multimap \alpha \multimap \alpha$ ($i \in \{1, \dots, |\Sigma|\}$) and $x : \alpha$. To compare with the encoding of SSTs, a string which supports concatenation *on both sides* can be seen as a one-hole unary tree, hence its type $\alpha \multimap \alpha$. (The uniqueness of the hole in $\partial\mathrm{BinTree}(\Sigma)$ turns out to be a linearity condition as well!) It is then possible to encode the transitions and output function of the BRTT.

In general, a BRTT is a register transducer equipped with a reflexive and symmetric *conflict relation* $\asymp$ over $R \cup R'$, and it satisfies a relaxed linearity condition formulated in terms of $\asymp$. Following [1], we say that $P \subseteq R \cup R'$ is *non-conflicting* if $\forall x, y \in P,\ x = y \lor x \not\asymp y$. We take A to be the following, *where P ranges over non-conflicting subsets*:

$$A = \mathtt{Fin}(|Q|) \otimes \mathop{\&}_{P} \left(\alpha^{\otimes|P\cap R|} \otimes (\alpha \multimap \alpha)^{\otimes|P\cap R'|}\right)$$

using the second-order encoding of the *additive conjunction*

$$A_1 \mathbin{\&} \ldots \mathbin{\&} A_m := \forall\gamma.\ (\forall\beta.\ \beta \multimap (\beta \multimap A_1) \multimap \ldots \multimap (\beta \multimap A_m) \multimap \gamma) \multimap \gamma$$

Further explanations of this choice and the role of $\asymp$ are given in Appendix B.4. ◀

5 Conclusion

We exhibited some relationships between the functions between Church-encoded strings (or trees) in two typed λ-calculi and those computed by variants of finite-state transducers. On the automata-theoretic side, we showed that the closure under composition of HDT0L transductions is a superclass of many pre-existing transduction classes. By showing that this large transduction class is included in the λ-definable string functions, we advanced our understanding of the latter. As for EAλ, the results here are still preliminary; hopefully, the sequel to this paper should prove the converse inclusions to Theorems 4.3 and 4.6, giving a characterization of regular (tree) functions quite different from the already existing ones.

Aside from that, there are many imaginable perspectives around the theme “implicit complexity for automata”. For instance, is it possible to characterize star-free languages in some λ-calculus, analogously to their algebraic characterization by aperiodic monoids?

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A Register tree transducers in $\mathrm{ST}\lambda$

This section is dedicated to the proof of Theorem 3.5.

First, let us sketch the proof of Lemma 3.4. Let $\Sigma = \{a_1, \dots, a_n\}$. We define $\mathcal{C}$ over $\partial\mathrm{BinTree}(\Sigma)$ by induction:

- $\mathcal{C}(\Box) = \lambda z.\ z$,
- $\mathcal{C}(a_i\langle T', U\rangle) = \lambda z.\ \lambda f_1.\ \dots\ \lambda f_n.\ \lambda x.\ f_i\ (\mathcal{C}(T')\ z\ f_1\ \dots\ f_n\ x)\ (\overline{U}\ f_1\ \dots\ f_n\ x)$,
- $\mathcal{C}(a_i\langle U, T'\rangle) = \lambda z.\ \lambda f_1.\ \dots\ \lambda f_n.\ \lambda x.\ f_i\ (\overline{U}\ f_1\ \dots\ f_n\ x)\ (\mathcal{C}(T')\ z\ f_1\ \dots\ f_n\ x)$.

The compilation of expressions follows a similar scheme, with more cases. In particular function application plays the main role in the translation of $E[E']$ and $E'[F']$ to λ-terms.

Next, let (Q, R, R', F, δ) be a register tree transducer. We may assume without loss of generality that $Q = \{1, \dots, |Q|\}$. Our goal is to encode this transducer into a simply typed λ-term of type $\mathtt{BT}_\Gamma[A] \to \mathtt{BT}_\Sigma$. We take

$$A = B^{|Q|} \to \mathtt{BT}_\Sigma \quad \text{where } B = \mathtt{BT}_\Sigma^{|R|} \to (\mathtt{BT}_\Sigma \to \mathtt{BT}_\Sigma)^{|R'|} \to \mathtt{BT}_\Sigma$$

(Recall that $C^m \to D$ is merely an abbreviation for $C \to \dots \to C \to D$.)

A down-to-earth explanation[11] of these types is as follows. The functions of B take as input the contents of the registers, and uses this to produce a result of type $\mathtt{BT}_\Sigma$. In particular, recall that when the transducer has finished visiting the entire tree, an *output function* (depending on the final state) is called to determine the result from the final contents of the registers; this function can be expressed as a λ-term u of type B.

As for A, the terms of type A include (among others) all the terms of the form (for $q \in Q$, $T_k \in \mathrm{BinTree}(\Sigma)$ and $T'_l \in \partial\mathrm{BinTree}(\Sigma)$)

$$\mathtt{Conf}(q, (T_k)_{k\in R}, (T'_l)_{l\in R'}) = \lambda f_1.\ \dots\ \lambda f_{|Q|}.\ f_q\ \overline{T_1}\ \dots\ \overline{T_{|R|}}\ \mathcal{C}(T'_1)\ \dots\ \mathcal{C}(T'_{|R'|})$$

Thanks to this, we can use A to represent $Q \times \mathrm{BinTree}(\Sigma)^R \times \partial\mathrm{BinTree}(\Sigma)^{R'}$, that is, the set of *configurations* of the register tree transducer (assuming that we are in the middle of a computation whose final result will be of type $\mathtt{BT}_\Sigma$). When, at some point, the transducer is at state $q \in Q = \{1, \dots, |Q|\}$, and its registers contain $(T_k)_{k\in R}$ and $(T'_l)_{l\in R'}$, the λ-term associated to its current configuration takes the q-th input function and gives it as arguments these register contents. Of course, the encoding depends of a fixed enumeration of the registers: $R = \{\hat{r}_1, \dots, \hat{r}_{|R|}\}$ and $R' = \{\hat{r}'_1, \dots, \hat{r}'_{|R'|}\}$.

The above discussion suggests that our register tree transducer be translated to a λ-term of the following form, for some $L : A$ and $N_1, \dots, N_{|\Gamma|} : A \to A \to A$:

$$\lambda z.\ (z\ N_1\ \dots\ N_{|\Gamma|}\ L)\ u_1\ \dots\ u_{|Q|} : \mathtt{BT}_\Gamma[A] \to \mathtt{BT}_\Sigma$$

where u_q encodes the output function at the state $q \in Q$, in such a way that for all $T \in \mathrm{BinTree}(\Sigma)$, $\overline{T}\ L\ N_1\ \dots\ N_{|\Gamma|} : A$ is (up to $=_\beta$) the representation of the final configuration reached by the transducer when it reads T.

The remaining task is to define L and $N_1, \dots, N_{|\Gamma|}$. Obviously L should represent the initial configuration: writing q_I for the initial state, $L = \mathtt{Conf}(q_I, (\langle\rangle)_{k\in R}, (\Box)_{l\in R'})$. Concerning N_i for $i \in \{1, \dots, |\Gamma|\}$, the property we want is that

$$N_i\ \mathtt{Conf}(q_\triangleleft, (T_k)_{k\in R}, (T'_l)_{l\in R'})\ \mathtt{Conf}(q_\triangleright, (U_k)_{k\in R}, (U'_l)_{l\in R'}) =_\beta \mathtt{Conf}(q, (V_k)_{k\in R}, (V'_l)_{l\in R'})$$

for some $q, (V_k), (V'_l)$ determined by the variable-update and state-update rules of the transducer for the i-th letter g_i of $\Gamma = \{g_1, \dots, g_{|\Gamma|}\}$.

To define these configuration-update terms, we first define the terms $M_{i,q_\triangleleft,q_\triangleright} : A$ containing the free variables $r_{k,\triangleleft}, r_{k,\triangleright}$ of type $\mathtt{BT}_\Sigma$ for $k \in \{1, \dots, R\}$, and $r'_{l,\triangleleft}, r'_{l,\triangleright}$ of type $\mathtt{BT}_\Sigma \to \mathtt{BT}_\Sigma$ for $l \in R'$. For $q_\triangleleft, q_\triangleright \in Q$ and $g_i \in \Gamma$, if $\delta(q_\triangleleft, q_\triangleright, g_i) = (q, \psi, \psi')$ then

$$M_{i,q_\triangleleft,q_\triangleright} = \lambda f_1.\ \dots\ \lambda f_{|Q|}.\ f_q\ (\mathcal{C}(\psi(\hat{r}_1))\ r_{1,\triangleleft}\ \dots\ r'_{|R'|,\triangleright})\ \dots\ (\mathcal{C}(\psi'(\hat{r}'_{|R'|}))\ r_{1,\triangleleft}\ \dots\ r'_{|R'|,\triangleright})$$

with $|R| + |R'|$ arguments passed to f_q.

Then the following choice for N_i works: $N_i = \lambda c_\triangleleft.\ \lambda c_\triangleright.\ c_\triangleleft\ H_{i,1}\ \dots\ H_{i,|Q|}$ where

- $H_{i,q_\triangleleft} = \lambda \vec{r_\triangleleft} \vec{r'_\triangleleft}.\ c_\triangleright\ (\lambda \vec{r_\triangleright} \vec{r'_\triangleright}.\ M_{i,q_\triangleleft,1})\ \dots\ (\lambda \vec{r_\triangleright} \vec{r'_\triangleright}.\ M_{i,q_\triangleleft,|Q|})$;
- for any term t, $\lambda \vec{r_\triangleleft} \vec{r'_\triangleleft}.\ t$ is an abbreviation for $\lambda r_{1,\triangleleft}.\ \dots\ \lambda r_{|R|,\triangleleft}.\ \lambda r'_{1,\triangleleft}.\ \dots\ \lambda r'_{|R'|,\triangleleft}.\ t$, and similarly for $\lambda \vec{r_\triangleright} \vec{r'_\triangleright}.\ t$.

[11] For the reader familiar with programming language theory, a more conceptual explanation is that this type is isomorphic to $\neg'\neg'((1 + \ldots + 1) \times \mathtt{BT}_\Sigma^{|R|} \times (\mathtt{BT}_\Sigma \to \mathtt{BT}_\Sigma)^{|R'|})$, where $\neg' D = D \to \mathtt{BT}_\Sigma$. This relativized double negation is used to eliminate the $\times$ and $+$ type constructors, which do not exist in our version of STλ. As stated before, we are indeed using a continuation-passing-style transformation.

B Details on transductions in EAλ (Section 4)

B.1 The type system of EAλ

The following is mostly copied from our previous work [24].

The grammar of types for EAλ is

$$A ::= \alpha \mid S \qquad S ::= \sigma \multimap \tau \mid \forall\alpha.\ S \qquad \sigma,\tau ::= A \mid !\sigma$$

The two first classes of types are called respectively *linear* and *strictly linear*. (We follow the terminology of [4]; "linear" does not mean exponential-free, it merely means that the head connective is not an exponential.)

The typing judgements involve a context split into three parts: they are of the form $\Gamma \mid \Delta \mid \Theta \vdash t : \sigma$. The idea is that the partial assignements Γ, Δ and Θ of variables to types correspond respectively to linear, non-linear and "temporary" variables; accordingly, Γ maps variables to linear types (denoted A above), Δ maps variables to types of the form $!\sigma$, while Θ maps variables to arbitrary types. The domains of Γ, Δ and Θ are required to be pairwise disjoint. The derivation rules for EAλ are:

variable rules $$\frac{}{\Gamma, x : A \mid \Delta \mid \Theta \vdash x : A} \qquad \frac{}{\Gamma \mid \Delta \mid \Theta, x : \sigma \vdash x : \sigma}$$

abstraction rules $$\frac{\Gamma, x : A \mid \Delta \mid \Theta \vdash t : \tau}{\Gamma \mid \Delta \mid \Theta \vdash \lambda x.\ t : A \multimap \tau} \qquad \frac{\Gamma \mid \Delta, x : !\sigma \mid \Theta \vdash t : \tau}{\Gamma \mid \Delta \mid \Theta \vdash \lambda !x.\ t : !\sigma \multimap \tau}$$

application rule[12] $$\frac{\Gamma \mid \Delta \mid \Theta \vdash t : \sigma \multimap \tau \quad \Gamma' \mid \Delta \mid \Theta \vdash u : \sigma}{\Gamma \uplus \Gamma' \mid \Delta \mid \Theta \vdash t\ u : \tau}$$

quantifier rules[13] $$\frac{\Gamma \mid \Delta \mid \Theta \vdash t : S}{\Gamma \mid \Delta \mid \Theta \vdash t : \forall\alpha.\ S} \qquad \frac{\Gamma \mid \Delta \mid \Theta \vdash t : \forall\alpha.\ S}{\Gamma \mid \Delta \mid \Theta \vdash t : S\{\alpha := A\}}$$

functorial promotion rule $$\frac{\varnothing \mid \varnothing \mid \Theta \vdash t : \sigma}{\Gamma \mid !\Theta, \Delta \mid \Theta' \vdash !t : !\sigma}$$

In these rules, following the conventions established above, A stands for a linear type, S stands for a strictly linear type and σ and τ stand for arbitrary types. In particular, in the quantifier elimination rule, α can only be instantiated by a linear type. So, for instance, one cannot give the type $!\beta \multimap !\beta$ to $\lambda x.\ x$ through a quantifier introduction followed by a quantifier elimination; indeed, as one would expect, the only normal term of this type is $\lambda !x.\ !x$. (Despite this, the polymorphism is still impredicative.)

B.2 Encoding regular functions (Theorem 4.3)

We fix an input alphabet $\Gamma = \{g_1, \dots, g_{|\Gamma|}\}$ and an output alphabet $\Sigma = \{s_1, \dots, s_{|\Sigma|}\}$.

A first important remark is that the Church encoding in EAλ consists of an exponential packaging around an exponential-free term with non-linear free variables.

▶ **Notation B.1.** We write $t ::_{\Sigma,\alpha} \sigma$, where σ is a type which may contain the type variable α, when the term t uses non-linearly the variables $f_i : \alpha \multimap \alpha$ for $i \in \{1, \dots, |\Sigma|\}$ and then has type A. Formally, using the typing judgment introduced in the previous subsection:

$$t ::_{\Sigma,\alpha} \sigma \iff \varnothing \mid \varnothing \mid f_1 : \alpha \multimap \alpha,\ \dots,\ f_{|\Sigma|} : \alpha \multimap \alpha \vdash t : \sigma$$

[12] $\Gamma \uplus \Gamma'$ means $\Gamma \cup \Gamma'$ with the assumption that the domains of Γ and Γ' are disjoint.
[13] In the introduction rule (left), α must not appear as a free variable in Γ, Δ and Θ.

Note that $t ::_{\Sigma,\alpha} \sigma \iff \lambda!f_1.\ \dots\ \lambda!f_{|\Sigma|}.\ !t : (!(\alpha \multimap \alpha))^{|\Sigma|} \multimap !\sigma$.

▶ **Definition B.2.** *For $w = a_{i_1} \dots a_{i_n} \in \Sigma^*$, we define $\widetilde{w}$ as follows:*

$$\widetilde{w} = \lambda x.\ f_{i_1}\ (\dots\ (f_{i_n}\ x)\ \dots) ::_{\Sigma,\alpha} \alpha \multimap \alpha \quad \textit{so that } \overline{w} = \lambda!f_1.\ \dots\ \lambda!f_{|\Sigma|}.\ !\widetilde{w} : \mathtt{Str}_\Sigma$$

$\widetilde{w}$ is a sort of Church encoding of w *relatively to representations of letters provided by the context in the form of non-linear variables* $f_i : \alpha \multimap \alpha$. This makes $\alpha \multimap \alpha$ a kind of *relative type of strings in* Σ^*, whose advantage over $\mathtt{Str}_\Sigma$ is that it contains no exponential.

Let us fix a streaming string transducer (Q, q_I, R, δ, F). We assume without loss of generality that $Q = \{1, \dots, |Q|\}$. Recall that $\mathtt{Fin}(|Q|) = \forall\beta.\ \beta^{|Q|} \multimap \beta$ represents the set of states: the state q corresponds to the q-th projection function $\pi_q = \lambda x_1.\ \dots\ \lambda x_{|Q|}.\ x_q$.

According to the discussion in Section 4.1, $A = \mathtt{Fin}(|Q|) \multimap (\alpha \multimap \alpha)^{|R|} \multimap (\alpha \multimap \alpha)$ (where α is a free type variable) should be seen as a type of *linear output functions*, that is, of maps $G : Q \to (\Sigma \cup R)^*$ such that for all $q \in Q$ and $r \in R$, $G(q)$ contains r at most once. (Again, this is a *relative* type depending on the use of non-linear variables $f_i : \alpha \multimap \alpha$ given externally.) To formalize this we generalize the operation $w \rightsquigarrow \widetilde{w}$ to words over $\Sigma \cup R$, given a fixed enumeration $R = \{r_1, \dots, r_{|R|}\}$:

▶ **Definition B.3.** *For $\omega = c_1 \dots c_n \in (\Sigma \cup R)^*$, we define*

$$\widetilde{\omega} = \lambda x.\ \chi(c_1)\ (\dots\ (\chi(c_n)\ x)\ \dots) \quad \textit{where } \chi(c) = \begin{cases} f_i \textit{ for } c = a_i \in \Sigma \\ p_j \textit{ for } c = r_j \in R \end{cases}$$

Note that then this is consistent with the previous definition of $\widetilde{\omega}$ for $\omega \in \Sigma^*$.

We also set $\widehat{\omega} = \lambda p_1.\ \dots \lambda p_{|R|}.\ \widetilde{\omega}$.

Given an output function $G : Q \to (\Sigma \cup R)^*$, we define $\widehat{G} = \lambda x.\ x\ \widehat{G(1)}\ \dots\ \widehat{G(|Q|)}$, so that $\widehat{G}$ applied to the i-th projection (representing the i-th state) yields $\widehat{G(i)}$.

▶ **Proposition B.4.** *Because of the linearity constraint for well-typed terms:*

- *for $\omega \in (\Sigma \cup R)^*$, each register name in R occurs* at most once *in ω if and only if* $p_1 : \alpha \multimap \alpha, \dots, p_{|R|} : \alpha \multimap \alpha \mid \varnothing \mid f_1 : \alpha \multimap \alpha, \dots, f_{|\Sigma|} : \alpha \multimap \alpha \vdash \widetilde{\omega} : \alpha \multimap \alpha$ *or equivalently* $\widehat{\omega} ::_{\Sigma,\alpha} (\alpha \multimap \alpha)^{|R|} \multimap \alpha \multimap \alpha$
- *for $G : Q \to (\Sigma \cup R)^*$, $\widehat{G} ::_{\Sigma,\alpha} A$ iff G is a* linear (i.e. copyless) *output function.*

What we are seeking is an EAλ term of type $\mathtt{Str}_\Gamma[A] \multimap \mathtt{Str}_\Sigma[\alpha]$ (with the above type A of linear output functions) computing the same string function as the SST. We will restrict our search to terms of the form

$$\lambda z.\ \lambda!f_1\ \dots\ \lambda!f_{|\Sigma|}.\ (\lambda!h.\ !u)\ (z\ !d_1\ \dots\ !d_{|\Gamma|})$$

where $d_i ::_{\Sigma,\alpha} A \multimap A$ for all $i \in \{1, \dots, |\Gamma|\}$, and $h : A \multimap A \mid \varnothing \mid \varnothing \vdash u : \alpha \multimap \alpha$. It is thanks to the presence of these outer $\lambda!f_i$ binding non-linearly the $f_i : \alpha \multimap \alpha$ that the various relative representations we are manipulate are meaningful. The idea is that, since

$$(\lambda!h.\ !u)\ (\overline{w}\ !d_1\ \dots\ !d_{|\Gamma|}) =_\beta\ !u\{h := (\lambda x.\ d_{i_1}\ (\dots (d_{i_n}\ x) \dots))\} \quad \text{for } w = g_{i_1} \dots g_{i_n} \in \Gamma^*,$$

we can take:

- d_i to be the representation of the action of the transition for the letter $g_i \in \Gamma$ on the output function, that is, what we called $\delta^O(g_i, -)$ (Definition 4.1);
- u to be a term applying h to a representation of the SST's original output function F, and then using the result to extract the image of the input word, following the recipe of Proposition 4.2.

Formally, what we want for d_i ($i \in \{1,\dots,|\Gamma|\}$) is $d_i ::_{\Sigma,\alpha} A \multimap A$ and $d_i\ \widehat{G} =_\beta \widetilde{\delta^O(g_i,G)}$. The typing condition tells us to look for a term of the form $d_i = \lambda z.\ \lambda y.\ y\ t_{i,1}\ \dots\ t_{i,q}$, where for all $q \in Q$, we have $z : A,\ y : \texttt{Fin}(|Q|) \mid \varnothing \mid f_1 : \alpha \multimap \alpha,\ \dots,\ f_{|\Sigma|} : \alpha \multimap \alpha \vdash t_{i,q} : A$.

Let $q \in Q$. We also want $t_{i,q}\{z := \widehat{G}\}$ to somehow represent $\delta^O(a,G)(q)$ for $a = g_i \in \Gamma$; by definition, this equals $s^*_{a,q}(G(q'_{a,q}))$, where $(q'_{a,q}, s_{a,q}) = \delta(q,a)$ (cf. Definition 4.1). We use a technique analogous to the proof of Lemma 3.1 to express the application of a monoid morphism on a Church-encoded string (here, the concerned string is $z\ \pi_{q'_{q,a}}$):

$$t_{i,q} = \lambda p_1.\ \dots\ \lambda p_{|R|}.\ z\ \pi_{q'_{q,a}}\ \widetilde{s_{a,q}(r_1)}\ \dots\ \widetilde{s_{a,q}(r_{|R|})}$$

In order for this to be well-typed, various linearity conditions must be satisfied. In particular, each p_j ($j \in \{1,\dots,|R|\}$) must occur at most once in all $\widetilde{s_{a,q}(r_1)},\dots,\widetilde{s_{a,q}(r_{|R|})}$. By definition of the encoding $\widetilde{(\,\cdot\,)}$, this is the case iff each r_j occurs at most once in all $s_{a,q}(r)$ for $r \in R$. This none other than the copyless assignment condition for transitions.

This concludes the definition of d_i. As for u, it is set to $u = h\ \widehat{F}\ \pi_{q_I}\ (\lambda x.\ x)\ \dots\ (\lambda x.\ x)$, with $|R|$ times $(\lambda x.\ x)$. Using Proposition 4.2, one can check that the term we get in the end – that is, $\lambda z.\ \lambda!f_1\ \dots\ \lambda!f_{|\Sigma|}.\ (\lambda!h.\ !u)\ (z\ !d_1\ \dots\ !d_{|\Gamma|})$ – computes the right function.

B.3 Encoding composition by substitution (Proposition 4.4)

We must show that if $f : \Gamma^* \to I^*$ and $g_i : \Gamma^* \to \Sigma^*$ are defined by some respective EAλ terms $t : !\texttt{Str}_\Gamma \multimap !\texttt{Str}_I$ and $u_i : !\texttt{Str}_\Gamma \multimap !\texttt{Str}_\Sigma$ (for $i \in I$, assuming w.l.o.g. that $I = \{1,\dots,|I|\}$), then their composition by substitution $\mathrm{CbS}(f,(g_i)_{i\in I})$ is also definable as a term of type $!\texttt{Str}_\Gamma \multimap !\texttt{Str}_\Sigma$. The term we use for that purpose is

$$\lambda!s.\ (\lambda!x.\ \lambda!y_1.\ \dots\lambda!y_{|I|}.\ !s')\ (t\ !s)\ (u_1\ !s)\ \dots\ (u_{|I|}\ !s)$$

$$\text{where } s' = \lambda!f_1.\ \dots\ \lambda!f_{|\Sigma|}.\ x\ (y_1\ !f_1\ \dots\ !f_{|\Sigma|})\ \dots\ (y_{|I|}\ !f_1\ \dots\ !f_{|\Sigma|})$$

B.4 Encoding regular tree functions (Theorem 4.6)

Regular tree functions are the functions computed by bottom-up ranked tree transducers, whose definition we now give in its entirety.

▶ **Definition B.5** ([1]). *A* conflict relation *is a binary reflexive and symmetric relation.*

Let $\asymp$ be a conflict relation over $V \cup V'$, and $E \in \mathrm{ExprBT}(\Sigma,V,V')$. The expression E is consistent with $\asymp$ *when*

- *each variable in $V \cup V'$ appears at most once in E;*
- *for all $x,y \in V \cup V'$, if $x \neq y$ and $x \asymp y$, then E does not contain both x and y.*

Consistency with $\asymp$ is defined in the same way for expressions in $\mathrm{Expr}\partial\mathrm{BT}(\Sigma,V,V')$.

A bottom-up ranked tree transducer (BRTT) *is a register tree transducer (Q,q_I,R,R',F,δ) endowed with a conflict relation $\asymp$ on $R \cup R'$, such that:*

- *for all $q \in Q$, the expression $F(q)$ is consistent with $\asymp$;*
- *for all $\varepsilon : R \to \mathrm{ExprBT}(\Sigma,R_{\triangleleft\triangleright},R'_{\triangleleft\triangleright})$ and $\varepsilon' : R' \to \mathrm{Expr}\partial\mathrm{BT}(\Sigma,R_{\triangleleft\triangleright},R'_{\triangleleft\triangleright})$, if there exist $q,q_\triangleleft,q_\triangleright \in Q$ and $a \in \Gamma$ such that $(q,\varepsilon,\varepsilon') = \delta(q_\triangleleft,q_\triangleright,a)$, then*
  - *all $\varepsilon(r)$ for $r \in R$ and all $\varepsilon'(r')$ for $r' \in R'$ are consistent with $\asymp$;*
  - *if $x_1,x_2,y_1,y_2 \in R \cup R'$, $x_1 \asymp x_2$ and, for some $z \in \{\triangleleft,\triangleright\}$, (x_1,z) appears in*[14] *$(\varepsilon \cup \varepsilon')(y_1)$ and (x_2,z) appears in $(\varepsilon \cup \varepsilon')(y_2)$, then $y_1 \asymp y_2$.*

[14] By $\varepsilon \cup \varepsilon'$ we mean the map $R \cup R' \to \mathrm{ExprBT}(\Sigma,R_{\triangleleft\triangleright},R'_{\triangleleft\triangleright}) \cup \mathrm{Expr}\partial\mathrm{BT}(\Sigma,R_{\triangleleft\triangleright},R'_{\triangleleft\triangleright})$ induced in the obvious way by ε and ε' – recall that $R \cup R'$ is a disjoint union.

This is indeed a kind of generalized linearity condition: when $\asymp = \{(x,x) \mid x \in R \cup R'\}$, we recover the notion of copyless assignment used for streaming string transducers. Before we start translating BRTTs into EAλ terms, we first reformulate this relaxed linearity using non-conflicting subsets.

▶ **Definition B.6.** *Let X be a set endowed with a conflict relation $\asymp$. A subset $P \subseteq X$ is said to be* non-conflicting *if $\forall x, y \in P$, $x = y \vee x \not\asymp y$. We write $P \sqsubseteq X$.*

▶ Remark B.7. As the reader might have noticed, the notations are meant to draw parallels to the structure of coherence spaces (a simple semantics of linear logic).

▶ **Notation B.8.** For $E \in \mathrm{ExprBT}(\Sigma, V, V') \cup \mathrm{Expr}\partial\mathrm{BT}(\Sigma, V, V')$, we write $\mathcal{V}(E)$ for the set of variables occurring in E, so that $\mathcal{V}(E) \subseteq V \cup V'$.

▶ **Proposition B.9.** *An expression $E \in \mathrm{ExprBT}(\Sigma, V, V') \cup \mathrm{Expr}\partial\mathrm{BT}(\Sigma, V, V')$ is consistent with a conflict relation over $V \cup V'$ if and only if it is* linear *(each variable appears at most once) and $\mathcal{V}(E) \sqsubseteq V \cup V'$ (that is, $\mathcal{V}(E)$ is* non-conflicting*).*

A register tree transducer $(Q, q_I, R, R', F, \delta)$ endowed with a conflict relation $\asymp$ on $R \cup R'$ is a BRTT if and only if (recall that $R_{\triangleleft\triangleright} = R \times \{\triangleleft, \triangleright\}$):

- *for all $q \in Q$, $F(q)$ is linear and $\mathcal{V}(F(q))$ is non-conflicting;*
- *for all $\varepsilon : R \to \mathrm{ExprBT}(\Sigma, R_{\triangleleft\triangleright}, R'_{\triangleleft\triangleright})$ and $\varepsilon' : R' \to \mathrm{Expr}\partial\mathrm{BT}(\Sigma, R_{\triangleleft\triangleright}, R'_{\triangleleft\triangleright})$, if there exist $q, q_\triangleleft, q_\triangleright \in Q$ and $a \in \Gamma$ such that $(q, \varepsilon, \varepsilon') = \delta(q_\triangleleft, q_\triangleright, a)$, then*
  - *all $\varepsilon(r)$ for $r \in R$ and all $\varepsilon'(r')$ for $r' \in R'$ are linear;*
  - *for all non-conflicting $P \sqsubseteq R \cup R'$, the sets $\mathcal{V}((\varepsilon \cup \varepsilon')(y))$ for $y \in P$ are pairwise disjoint, and their union $\bigcup_{y \in P} \mathcal{V}((\varepsilon \cup \varepsilon')(y))$ is non-conflicting in $R_{\triangleleft\triangleright} \cup R'_{\triangleleft\triangleright}$ – where the conflict relation of the latter is defined so that[15] there is never a conflict between $(x_1, \triangleleft)$ and $(x_2, \triangleright)$ for $x_1, x_2 \in R \cup R'$.*

The moral of the story until now is that, while it is not true that the transition function performs copyless assignments, one can say instead that:

- for every non-conflicting set of register names $P \sqsubseteq R \cup R'$, the contents of the registers in P after a transition are obtained linearly (by copyless assignment) from the contents of a non-conflicting subset of $R_{\triangleleft\triangleright} \cup R'_{\triangleleft\triangleright}$;
- in the end, depending on the final state, one such $P \sqsubseteq R \cup R'$ is used linearly to produce the output.

We must now show that EAλ is expressive enough to accomodate this variation on linearity.

Let $(Q, q_I, R, R', F, \delta, \asymp)$ be a BRTT. Analogously to the encoding of SSTs in EAλ, we encode this as a term of the form $\lambda z.\ \lambda! f_1.\ \ldots\ \lambda! f_{|\Gamma|}.\ \lambda! x.\ (\ldots)$. Thus, we will be able to manipulate representations of data relatively to non-linear variables $f_i : \alpha \multimap \alpha \multimap \alpha$ (representing $g_i\langle -, - \rangle$ for $g_i \in \Gamma$) and $x : \alpha$ (representing $\langle\rangle$): abbreviating $\vec{f} : \alpha \multimap \alpha \multimap \alpha$ for $f_1 : \alpha \multimap \alpha \multimap \alpha$, ..., $f_{|\Sigma|} : \alpha \multimap \alpha \multimap \alpha$, there are natural encodings

$$
\begin{aligned}
T \in \mathrm{BinTree}(\Sigma) &\rightsquigarrow \varnothing \mid \varnothing \mid \vec{f} : \alpha \multimap \alpha \multimap \alpha,\ x : \alpha \vdash \widetilde{T} : \alpha \\
T' \in \partial\mathrm{BinTree}(\Sigma) &\rightsquigarrow \varnothing \mid \varnothing \mid \vec{f} : \alpha \multimap \alpha \multimap \alpha,\ x : \alpha \vdash \widetilde{T'} : \alpha \multimap \alpha \\
E \in \mathrm{ExprBT}(\Sigma, V, V') &\rightsquigarrow \varnothing \mid \varnothing \mid \vec{f} : \alpha \multimap \alpha \multimap \alpha,\ x : \alpha \vdash \widetilde{E} : \alpha^{|V|} \multimap (\alpha \multimap \alpha)^{|V'|} \multimap \alpha \\
E' \in \mathrm{Expr}\partial\mathrm{BT}(\Sigma, V, V') &\rightsquigarrow \varnothing \mid \varnothing \mid \ldots \vdash \widetilde{E'} : \alpha^{|V|} \multimap (\alpha \multimap \alpha)^{|V'|} \multimap (\alpha \multimap \alpha)
\end{aligned}
$$

[15] Pursuing the analogy with coherence spaces, we have, morally, $R_{\triangleleft\triangleright} \cong (R \otimes \{\triangleleft\}) \;\&\; (R \otimes \{\triangleright\})$.

For a BRTT with $\asymp = \{(x,x) \mid x \in R \cup R'\})$, i.e. with truly copyless assignments, the relative type of configurations would be

$$A = \mathtt{Fin}(|Q|) \otimes \alpha^{\otimes|R|} \otimes (\alpha \multimap \alpha)^{\otimes|R'|} \qquad \text{where } B^{\otimes m} = B \otimes \ldots \otimes B$$

The transition after reading some label $a \in \Sigma$ in a node must be of the type $A \multimap A \multimap A$. Morally, this is isomorphic to $(A \otimes A) \multimap A$, and since $|R_{\lhd\rhd}| = 2|R|$,

$$A \otimes A \cong \mathtt{Fin}(|Q|) \otimes \mathtt{Fin}(|Q|) \otimes \alpha^{\otimes|R_{\lhd\rhd}|} \otimes (\alpha \multimap \alpha)^{|R'_{\lhd\rhd}|}$$

These isomorphisms of linear logic are not quite reflected as actual type isomorphisms in EAλ, since the multiplicative conjunction $\otimes$ does not exist as a primitive, and we use instead a second-order encoding already exploited in [4, 24]. But they illustrate the reason why $\delta(-,-,a)$ $(a \in \Gamma)$ can be turned into a term of type $A \multimap A \multimap A$; in particular the type

$$\alpha^{\otimes|R_{\lhd\rhd}|} \otimes (\alpha \multimap \alpha)^{\otimes|R'_{\lhd\rhd}|} \multimap \alpha^{\otimes|R|} \otimes (\alpha \multimap \alpha)^{\otimes|R'|}$$

corresponds to the $(\varepsilon \cup \varepsilon') : R \cup R' \to \mathrm{ExprBT}(\Sigma, R_{\lhd\rhd}, R'_{\lhd\rhd}) \cup \mathrm{Expr}\partial\mathrm{BT}(\Sigma, R_{\lhd\rhd}, R'_{\lhd\rhd})$ that was mentioned in the definition of BRTTs. And the function arrow can be linear because the register update $(\varepsilon \cup \varepsilon')$ is copyless.

We now come to the case of a BRTT with an arbitrary conflict relation. The relaxed linearity of $(\varepsilon \cup \varepsilon')$ is then manifested as the fact that for all non-conflicting $P \sqsubseteq R \cup R'$, one can represent (relatively to f_i and x) its action to produce the new contents of P as an EAλ term of type

$$\alpha^{\otimes|S \cap R_{\lhd\rhd}|} \otimes (\alpha \multimap \alpha)^{\otimes|S \cap R'_{\lhd\rhd}|} \multimap \alpha^{\otimes|P \cap R|} \otimes (\alpha \multimap \alpha)^{\otimes|P \cap R'|} \quad S = \bigcup_{y \in P} \mathcal{V}((\varepsilon \cup \varepsilon')(y))$$

It is important to observe that S is also non-conflicting, thanks to a previous proposition. The entirety of $(\varepsilon \cup \varepsilon')$ can therefore be faithfully represented by an EAλ term of type

$$\mathop{\&}_{S \sqsubseteq R_{\lhd\rhd} \cup R'_{\lhd\rhd}} \left(\alpha^{\otimes|S \cap R_{\lhd\rhd}|} \otimes (\alpha \multimap \alpha)^{\otimes|S \cap R'_{\lhd\rhd}|}\right) \multimap \mathop{\&}_{P \sqsubseteq R \cup R'} \left(\alpha^{\otimes|P \cap R|} \otimes (\alpha \multimap \alpha)^{\otimes|P \cap R'|}\right)$$

using the encoding of the additive conjunction in EAλ

$$A_1 \mathbin{\&} \ldots \mathbin{\&} A_m := \forall\gamma.\ (\forall\beta.\ \beta \multimap (\beta \multimap A_1) \multimap \ldots \multimap (\beta \multimap A_m) \multimap \gamma) \multimap \gamma$$

This explains the use of the type of configurations

$$A = \mathtt{Fin}(|Q|) \otimes \mathop{\&}_{P \sqsubseteq R \cup R'} \left(\alpha^{\otimes|P \cap R|} \otimes (\alpha \multimap \alpha)^{\otimes|P \cap R'|}\right)$$

for general BRTTs. (To recover the left side of the previous type from $A \otimes A$, use the canonical function $(A_1 \mathbin{\&} \ldots \mathbin{\&} A_m) \otimes (B_1 \mathbin{\&} \ldots \mathbin{\&} B_m) \multimap \mathop{\&}_{1 \le i,j \le m} (A_i \otimes B_j)$.)

At the end, one must extract the output from the final configuration. Fortunately, for any state, the corresponding output expression in $\mathrm{ExprBT}(\Sigma, R, R')$ only involves a non-conflicting set $P \sqsubset R \cup R'$ of variables. Thus, one can project the final configuration to retrieve a datum of type $\alpha^{\otimes|P \cap R|} \otimes (\alpha \multimap \alpha)^{\otimes|P \cap R'|}$ for this specific P; this is sufficient to determine the output tree (represented by an element of type α).