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On systems of word equations over three unknowns with at most six occurrences of one of the unknowns

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ABSTRACT

In this paper, we investigate the open question, formulated in 1983 by Culik II and Karhumäki, asking whether there exist independent systems of three word equations over three unknowns admitting non-periodic solutions. In particular, we answer negatively the above mentioned question for systems in which one of the unknowns occurs at most six times. That is, we show that such systems admit only periodic solutions or they are not independent.

1. Introduction

The theory of *Word Equations* was initiated by Markov, [14], in 1954, when he proposed the satisfiability problem for word equations: decide whether or not a given word equation has solutions. This problem was solved by Makanin, who proved it to be decidable for free semigroups in [11], and for free groups in [12] and [13], see also [4] for a recent survey. However, his decision procedure is considered to be one of the most complicated algorithms in the literature. More recently, see [15] and [16], another algorithm for the satisfiability problem has appeared in the literature, having polynomial space complexity.

In this paper we consider word equations in a very simple setting, namely assuming that the equations are constant-free and over only three unknowns. Even in this simple case some problems prove to be extremely hard. For instance, the question whether there exist independent systems of three equations over three unknowns possessing non-periodic solutions, formulated by Culik II and Karhumäki in 1983 in [2], proves to be an intricate and demanding open problem, see also [1]. Although the Ehrenfeucht compactness property guarantees that any independent system of equations over a finite set of unknowns is finite, it is still open whether there can be such systems unboundedly large. Some non-trivial asymptotic lower bounds for the size of independent systems were achieved in [7] and [8]. However, if the number of unknowns is small, then not even such lower bounds are reported for the maximal size of independent systems of equations. A nontrivial step in this direction was achieved in [5], by proving that an independent system of at least two equations over three unknowns possessing a non-periodic solution contains only *balanced equations*, i.e., equations where, each unknown occurs equally many times on the left and right sides. Furthermore, in [3], the question formulated in [2] was restricted to a well-specified class of systems, i.e., in all other cases it was proved that systems of three equations over three unknowns either are dependent or admit only periodic solutions. In this paper, we continue this investigation by taking the case left open in [3] on which we impose one more constraint, i.e., one of the unknowns occurs at most six times in the system. Then, we show that, any system fulfilling these constraints admits only periodic solutions or it is not independent, i.e., we answer negatively the question from [2] for this subclass of systems. The framework in which we prove this result is that of a free group generated by a finite alphabet.

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The paper is organized as follows. In Section 2 we fix our terminology and introduce some basic notions and results. In Section 3, depending on the structure of the equation, we characterize what kind of *quasi-periodic solutions* it can admit, where by quasi-periodic we mean solutions where two unknowns are powers of a common word. Then, we look also at what happens with this type of solutions when, instead of only one we have a system of two or three word equations. Finally, in Section 4 we show that systems of three equations over three unknowns with at most six occurrences of one of the unknowns admit only periodic solutions or they are dependent.

2. Preliminaries

Let Σ be a finite alphabet, Σ^* the set of all finite words over Σ, 1 the empty word, and Σ^+ the set of all nonempty finite words over Σ. A word u is a *factor* (resp. *prefix*, *suffix*) of w if there are words x, y such that $w = xuy$ (resp. $w = uy$, $w = xu$). Then, we can write $u = x^{-1}wy^{-1}$ (resp. $u = wy^{-1}$, $u = x^{-1}w$). We use the notation $pref_k(w)$ to denote the prefix of length k of the word w. For $w \in \Sigma^*$ we denote by $|w|$ its *length*, i.e., the number of letters in w, and by $|w|_a$ the number of occurrences of the letter a in w for any $a \in \Sigma$. An integer $p \geq 1$ is a *period* of a word $w = a_1 \dots a_n$, with $a_i \in \Sigma$ for all $1 \leq i \leq n$, if $a_i = a_{i+p}$ for all $1 \leq i \leq n - p$. A word $w \in \Sigma^+$ is called *primitive* if it cannot be written as a power of another word. For a given word w, the unique primitive word u such that $w = u^n$ for some $n \geq 1$ is called the *primitive root* of w. The following result illustrates a fundamental property of words.

Lemma 2.1. *Two words $u, v \in \Sigma^*$ satisfy a nontrivial relation if and only if they both have the same primitive root, i.e., they are powers of a common word.*

Moreover, the previous result holds also when the two words u and v and the relation satisfied by them are considered in the free group generated by Σ, that is when we can use the operation of inversion. We can do it since if two elements of a free group satisfy nontrivial identity in a free group, then they are powers of the same element, see [10].

If $\Sigma = \{a_1, \dots, a_n\}$ and $w \in \Sigma^*$, then the *Parikh vector associated to* w is defined as $\psi(w) = (|w|_{a_1}, \dots, |w|_{a_n})$. For more details on all these notions we refer to, e.g., [1,9].

Denote by $u \wedge v$ the longest common prefix of the words u and v. Denote by u^ω right infinite word which consists of infinite repetition of the word u. In our proofs we use the following lemma heavily.

Lemma 2.2. *Let u and v be two words such that they are not powers of the same word. Let $w \in u(u \cup v)^\omega$ and $t \in v(u \cup v)^\omega$. Then $w \wedge t = u^\omega \wedge v^\omega$. In particular $w \wedge t$ does not depend on the choice of w and t.*

Now, let Σ be a finite alphabet and $\Xi = \{x_1, \dots, x_n\}$ a set of unknowns, with $\Sigma \cap \Xi = \emptyset$. An *equation* over the alphabet Σ, with Ξ as the set of unknowns is a pair $(u, v) \in (\Sigma \cup \Xi)^* \times (\Sigma \cup \Xi)^*$, usually written as $u = v$. We say that an equation is *constant-free* if both u and v contain only elements from Ξ. An equation $u = v$ is called *reduced* if $pref_1(u) \neq pref_1(v)$ and $suf_1(u) \neq suf_1(v)$ and *balanced* if $|u|_x = |v|_x$ for all unknowns $x \in \Xi$. Throughout this paper we will consider only reduced constant-free equations over three unknowns. A *solution* of an equation $u = v$ is a morphism $\varphi : (\Xi \cup \Sigma)^* \to \Sigma^*$ such that $\varphi(u) = \varphi(v)$ and $\varphi(a) = a$ for every $a \in \Sigma$. Thus, a solution is a $|\Xi|$-tuple of words over the alphabet Σ. We say that a solution φ is *periodic* if there exists a word $u \in \Sigma^*$ such that $\varphi(x) \in u^*$ for any $x \in \Xi$. If $\Xi = \{x, y, z\}$, then we say that φ is *quasi-periodic with respect to x and z* if there exists $u \in \Sigma^*$ such that $\varphi(x), \varphi(z) \in u^*$. We say that a solution is *purely non-periodic* if the images of no two unknowns are powers of a common word. Note that for equations over three unknowns the sets of periodic, quasi-periodic (which are not periodic), and purely non-periodic solutions form a partition of the solution set.

A *system of equations* is a non-empty set of equations. A *solution* of a system is a morphism $\varphi : (\Xi \cup \Sigma)^* \to \Sigma^*$ satisfying all of its equations. We say that two systems $\mathcal{S}$ and $\mathcal{S}'$ are *equivalent* if they have the same set of solutions. Moreover, we say that a system $\mathcal{S}$ is *independent* if it is not equivalent to any of its proper subsystems.

Let us recall now the following well-known result, see e.g., [1].

Proposition 2.3. *If a three elements set $X = \{x, y, z\} \subseteq \Sigma^+$ satisfies the relations*

$$\begin{cases} x\alpha = z\beta \\ x\gamma = y\delta \end{cases} \text{ with } \alpha, \beta, \gamma, \delta \in X^*,$$

then x, y, and z are powers of a common word.

Thus, without loss of generality, from this moment on we assume that the left-hand side of each equation starts with x and the right-hand side of each equation starts with z. The third unknown, y, will be also called the *hidden unknown*. Also, by quasi-periodic solution we mean quasi-periodic with respect to x and z.

3. On the quasi-periodic solutions of equations over three unknowns

Let us consider next the following constant-free equation with an equal number of y's in the two sides:

$$\alpha_1(x, z)y\alpha_2(x, z)y \dots y\alpha_n(x, z) = \beta_1(x, z)y\beta_2(x, z)y \dots y\beta_n(x, z), \quad (1)$$

where $\alpha_l(x, z), \beta_l(x, z) \in \{x, z\}^*$ for all $1 \leq l \leq n$, $pref_1(\alpha_1(x, z)) = x$, and $pref_1(\beta_1(x, z)) = z$.

Depending on the structure of $\alpha_l(x, z)$ and $\beta_l(x, z)$, for all $1 \le l \le n$, we can put the Eq. (1) in one of the following classes.

Class 1: For every $1 \le l \le n$, the Parikh vectors of $\alpha_l(x, z)$ and $\beta_l(x, z)$ coincide, i.e.,

$$|\alpha_l(x, z)|_x = |\beta_l(x, z)|_x \text{ and } |\alpha_l(x, z)|_z = |\beta_l(x, z)|_z.$$

Then, for any $i, k \ge 0$ and $u, y \in \Sigma^*$, (u^i, y, u^k) is a solution of (1). Consequently, we say that *Eq.* (1) *admits independently quasi-periodic solutions with respect to x and z*.

Class 2: There exists some $1 \le l \le n$ such that the Parikh vectors of $\alpha_l(x, z)$ and $\beta_l(x, z)$ differ and, moreover, for all such l's we have

$$|\alpha_l(x, z)|_x = |\beta_l(x, z)|_x \text{ and } |\alpha_l(x, z)|_z \ne |\beta_l(x, z)|_z,$$

or the symmetric case when for all such l's the roles of x and z are interchanged. Then, the only quasi-periodic solutions of (1) with respect to x and z, which are not periodic, are of the form $(u^i, y, 1)$, or symmetrically $(1, y, u^k)$; other triples, when substituted into (1), do not yield the graphical identity. From now on, we call triples of the form $(u^i, y, 1)$ or $(1, y, u^k)$, 1-*limited quasi-periodic with respect to x and z*. So, in this case, we say that *Eq.* (1) *admits only* 1-*limited quasi-periodic solutions with respect to x and z*.

Class 3: There exist some indices l and l', $l \ne l'$ such that

$$|\alpha_l(x, z)|_x \ne |\beta_l(x, z)|_x,\ |\alpha_l(x, z)|_z = |\beta_l(x, z)|_z,\ \text{and } |\alpha_{l'}(x, z)|_z \ne |\beta_{l'}(x, z)|_z,$$

or the symmetric case with x and z interchanged. Then, when we substitute in the initial equation a quasi-periodic solution of the form (u^i, y, u^k), which is not periodic, we obtain a nontrivial relation on u and y. Thus, any quasi-periodic solution with respect to x and z is actually periodic. So, in this case, we say that *the quasi-periodicity induces the periodicity* among solutions of the Eq. (1).

Class 4: Otherwise, for any $1 \le l \le n$ we have either

$$|\alpha_l(x, z)|_x \ne |\beta_l(x, z)|_x \text{ and } |\alpha_l(x, z)|_z \ne |\beta_l(x, z)|_z,\ \text{ or } |\alpha_l(x, z)|_x = |\beta_l(x, z)|_x \text{ and } |\alpha_l(x, z)|_z = |\beta_l(x, z)|_z.$$

In this case, for all $1 \le l \le n$ such that $\alpha_l(x, z)$ and $\beta_l(x, z)$ have distinct Parikh vectors, let $|\alpha_l(x, z)|_x - |\beta_l(x, z)|_x \ne 0$ be the *l-th exceed of x's* and $|\beta_l(x, z)|_z - |\alpha_l(x, z)|_z \ne 0$ be the *l-th exceed of z's*. For every such $1 \le l \le n$, we define the *l-th ratio* of this equation, denoted by R_l, as follows:

$$R_l = |\alpha_l(x, z)|_x - |\beta_l(x, z)|_x \ :\ |\beta_l(x, z)|_z - |\alpha_l(x, z)|_z.$$

If there are two indices $l \ne l'$ such that R_l and $R_{l'}$ are defined and $R_l \ne R_{l'}$, then any quasi-periodic solution with respect to x and z is actually periodic since, otherwise, after substituting it in (1) we obtain a non-trivial relation on two words. So, also in this case the quasi-periodicity of the Eq. (1) induces the periodicity.

We say that *Eq.* (1) *has ratio* $R = p{:}q$ if, for every $1 \le l \le n$ for which R_l is defined we have that $R_l = R$. Moreover, in this case, the quasi-periodic solutions with respect to x and z are completely characterized by this ratio in the sense that a triple $(x, y, z) = (u^i, y, u^k)$ is solution of Eq. (1) if and only if $ip = kq$.

Now, we can make the following observations.

Observation 3.1. *Let $\mathcal{S}$ be a system of equations over three unknowns having only quasi-periodic solutions. Suppose also that the system has a non-periodic solution. Then its set of non-periodic solutions is either independently quasi-periodic, or* 1-*limited quasi-periodic, or is characterized by a ratio R.*

Observation 3.2. *Let $\mathcal{S}$ be an independent system of equations. Assume that the set of non-periodic quasi-periodic solutions of $\mathcal{S}$ is independently quasi-periodic. Then $|\mathcal{S}| = 1$ and the system has only quasi-periodic solutions.*

Proof. Take an arbitrary equation $e \in \mathcal{S}$. Since (u^l, y, u^k) is a solution of e for any u, y and $l, k \ge 0$, the equation e is balanced. Hence, the number of y in both sides of e is the same. Moreover, the equation e is of the form

$$e : x\alpha(x, z)yt_1(x, y, z) = z\beta(x, z)yt_2(x, y, z)$$

and $|\alpha(x, z)|_z = |\beta(x, z)|_z + 1$ and $|\alpha(x, z)|_x + 1 = |\beta(x, z)|_x$. Hence, we have $|\alpha(x, z)|_z \ge 1$ and $|\beta(x, z)|_x \ge 1$ and we can write

$$e : x^i z\alpha'(x, z)yt_1(x, y, z) = z^j x\beta'(x, z)yt_2(x, y, z)$$

for some $i, j \ge 1$. Thus, the equation e has only quasi-periodic solutions, see e.g., [6]. Consequently, the set of non-periodic solutions of e is the set (u^l, y, u^k) for any u, y, and $l, k \ge 0$. Since e is balanced, the set of all solutions of e is the set (u^l, y, u^k) for any u, y, and $l, k \ge 0$. Moreover, this is true for each equation in $\mathcal{S}$. Since $\mathcal{S}$ is independent, this immediately implies that $|\mathcal{S}| = 1$. △

As an immediate consequence of the previous observation we can also note the following.

Observation 3.3. *Suppose that an equation has different numbers of the unknown y on its left side and on its right side. Then each of its quasi-periodic solution is actually periodic.*

The following result summarizes some results from [3] for which we give a shorter, alternative proof.

Lemma 3.4. *Let $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ be an independent system of equations over x, y, and z. Suppose that $\mathcal{S}_1$ has only quasi-periodic solutions and the system $\mathcal{S}_2$ consists of balanced equations.*

1. *If the set of non-periodic solutions of the system $\mathcal{S}_1$ is independently quasi-periodic and $|\mathcal{S}_2| \geq 2$, then the system $\mathcal{S}$ has only periodic solutions.*
2. *If the set of non-periodic solutions of the system $\mathcal{S}_1$ is not independently quasi-periodic and $|\mathcal{S}_2| \geq 1$, then the system $\mathcal{S}$ has only periodic solutions.*

Proof. Suppose first that the set of non-periodic solutions of $\mathcal{S}_1$ is independently quasi-periodic and $\{e_2, e_3\} \subseteq \mathcal{S}_2$ with $e_2 \neq e_3$. Since e_2 and e_3 are balanced, the set of non-periodic quasi-periodic solutions of each of them is either independently quasi-periodic or 1-limited quasi-periodic or characterized by a ratio R.

Let the set of non-periodic quasi-periodic solutions of one of the equations e_2 or e_3, say e_2, be independently quasi-periodic. Then, since e_2 is balanced, each periodic triple is a solution of e_2. Hence e_2 is a consequence of $\mathcal{S}_1$ and thus the system $\mathcal{S}$ is not independent.

If the sets of non-periodic quasi-periodic solutions of e_2 and e_3 are not independently quasi-periodic but are of different types, then the system $\mathcal{S}$ has only periodic solutions. Moreover, this is also true if the sets of non-periodic quasi-periodic solutions of the two equations are characterized by different ratios or if both e_2 and e_3 admit only 1-limited quasi-periodic solutions but one with $x = 1$ while the other with $z = 1$.

Let the sets of non-periodic quasi-periodic solutions of e_2 and e_3 both be 1-limited quasi-periodic with $x = 1$; the case when $z = 1$ is similar. Then, since e_3 is balanced, each periodic triple is a solution of e_3. Hence e_3 is a consequence of $\mathcal{S}_1 \cup \{e_2\}$ and thus $\mathcal{S}$ is not independent.

Let the sets of non-periodic quasi-periodic solutions of e_2 and e_3 both be characterized by a ratio R. Then, since e_3 is balanced, each periodic triple is a solution of e_3. Hence e_3 is a consequence of $\mathcal{S}_1 \cup \{e_2\}$ and thus $\mathcal{S}$ is not independent.

For the second part of the lemma, suppose that the set of non-periodic solutions of $\mathcal{S}_1$ is not independently quasi-periodic and $e \in \mathcal{S}_2$. Since e is balanced, then the set of non-periodic quasi-periodic solutions of e is independently quasi-periodic, or 1-limited quasi-periodic or characterized by a ratio R.

Let the set of non-periodic quasi-periodic solutions of e be independently quasi-periodic. Since e is balanced, then, each periodic triple is a solution of e. Hence, e is a consequence of $\mathcal{S}_1$ and thus the system $\mathcal{S}$ is not independent.

If the sets of non-periodic quasi-periodic solutions of e and the system $\mathcal{S}_1$ are not independently quasi-periodic but are of different types, then the system $\mathcal{S}$ has only periodic solutions.

Let the sets of non-periodic quasi-periodic solutions of e and $\mathcal{S}_1$ both be 1-limited quasi-periodic. Moreover for both equations either we have $x = 1$ or $z = 1$, otherwise, the system $\mathcal{S}$ admits only periodic solutions. Since e is balanced, then, each periodic triple is a solution of e. Hence, e is a consequence of $\mathcal{S}_1$ and thus $\mathcal{S}$ is not independent.

Let the sets of non-periodic quasi-periodic solutions of e and $\mathcal{S}_1$ both be characterized by some ratios R_1 and R_2. If $R_1 \neq R_2$, then the system $\mathcal{S}$ admits only periodic solutions. So, let us suppose $R_1 = R_2$. Then, since e is balanced, each periodic triple is a solution of e. Hence, e is a consequence of $\mathcal{S}_1$ and thus $\mathcal{S}$ is not independent. $\triangle$

4. Systems of three word equations with at most six occurrences of one of the unknowns

It is already known from [5] that an independent system of at least two equations over three unknowns which admits a non-periodic solution only contains balanced equations. Thus, if we investigate systems of three equations with at most six occurrences of one of the unknowns, then the only interesting case is when we have exactly one occurrence of this particular unknown in each side of every equation. In all the other cases, there is at least one equation which is either unbalanced or it is a non-trivial identity over only two unknowns and thus the system is not independent or it admits only periodic solutions. Moreover, from [3] we already know that the only class of systems of three equations over three unknowns which remains to be investigated is of the form:

$$\begin{cases} x^i y p_1(x,y,z) = z q_1(x,z) y r_1(x,y,z) \\ x^i y p_2(x,y,z) = z q_2(x,z) y r_2(x,y,z) \\ x^i y p_3(x,y,z) = z q_3(x,z) y r_3(x,y,z) \end{cases},$$

for some $i \geq 1$. For all the other classes of such systems it was proved in [3] that they admit only periodic solutions or they are not independent. Moreover, if we reverse all three equations, i.e., we read them from right to left, then the obtained system is of the same type, up to renaming the unknowns. Otherwise, due to [3], the obtained reversed system admits only periodic solutions or it is dependent which immediately implies that the same holds also for the initial one.

Depending on the structure of the constituent equations and also on which of the unknowns occurs six times, we divide our analysis into five cases. First, we consider the case when the unknown x occurs once on each side of every equation; the case when z does that is symmetric so we do not consider it here explicitly. Then, in the other four cases we investigate different classes of systems where the unknown y occurs once in each side of every equation. Thus, we distinguish the following five cases, which are separately investigated in the following subsections:

- Case 1: Systems of the form: $\begin{cases} xp_1(y,z) = q_1(y,z)xr_1(y,z) \\ xp_2(y,z) = q_2(y,z)xr_2(y,z) \\ xp_3(y,z) = q_3(y,z)xr_3(y,z) \end{cases}$. That is, the unknown x occurs once in each side of every equation. The case when this holds for the unknown z is symmetric.
- Case 2: Systems of the form: $\begin{cases} x^iyz^j = zv'_1(x,z)yv'_2(x,z)x \\ x^iyz^j = zu'_1(x,z)yu'_2(x,z)x, \\ x^iyz^j = zt'_1(x,z)yt'_2(x,z)x \end{cases}$ with $i, j \geq 1$. That is, the unknown y occurs once in each side of every equation and, when we reverse the equations, the left sides start with z^jy for some $j \geq 1$ while the right sides start with some expressions over x and z.
- Case 3: Systems of the form: $\begin{cases} x^iyv_1(x,z)x = zv'_1(x,z)yz^j \\ x^iyu_1(x,z)x = zu'_1(x,z)yz^j, \\ x^iyt_1(x,z)x = zt'_1(x,z)yz^j \end{cases}$ with $i, j \geq 1$. That is, the unknown y occurs once in each side of every equation and when we reverse the equations, the right sides start with z^jy for some $j \geq 1$ while the left sides start with some expressions over x and z.
- Case 4: Systems of the form: $\begin{cases} x^iyp_1(x,z) = zq_1(x,z)y \\ x^iyp_2(x,z) = zq_2(x,z)y, \\ x^iyp_3(x,z) = zq_3(x,z)y \end{cases}$ with $i \geq 1$. That is, the unknown y occurs once in each side of every equation and when we reverse the equations, the right sides start with y while the left sides start with expressions over x and z.
- Case 5: Systems of the form: $\begin{cases} x^iyv_1(x,z)z = zv'_1(x,z)yx^j \\ x^iyu_1(x,z)z = zu'_1(x,z)yx^j, \\ x^iyt_1(x,z)z = zt'_1(x,z)yx^j \end{cases}$ with $i, j \geq 1$. That is, the unknown y occurs once in each side of every equation and when we reverse the equations, the right sides start with x^jy for some $j \geq 1$ while the left sides start with expressions over x and z.

4.1. Case 1

If the unknown x occurs once in each side of every equation, then the system looks as follows:

$$\begin{cases} xp_1(y,z) = q_1(y,z)xr_1(y,z) \\ xp_2(y,z) = q_2(y,z)xr_2(y,z) \\ xp_3(y,z) = q_3(y,z)xr_3(y,z) \end{cases}.$$

Let us look first at the case when for all $1 \leq i \leq 3$ we have $r_i(y,z) = 1$, i.e., the right side of all three equations ends with x. We start by proving the following intermediate result.

Theorem 4.1. *Consider a system of two equations*

$$\begin{cases} xp_1(y,z)^{l_1} = q_1(y,z)^{r_1}x \\ xp_2(y,z)^{l_2} = q_2(y,z)^{r_2}x \end{cases}$$

where $p_1(y,z)$, $p_2(y,z)$, $q_1(y,z)$ and $q_2(y,z)$ are primitive words over y, z and $l_1, r_1, l_2, r_2 \geq 1$. Then

(i) *either all solutions of the system are periodic,*
(ii) *or the system is dependent and $p_1(y,z) = p_2(y,z)$ and $q_1(y,z) = q_2(y,z)$ and $l_1/gcd(l_1,r_1) = l_2/gcd(l_2,r_2)$ and $r_1/gcd(l_1,r_1) = r_2/gcd(l_2,r_2)$,*
(iii) *or the set of non-periodic solutions is the set of non-periodic solutions of equation $x = v(y,z)$ and, consequently, the equations $v(y,z)p_1(y,z)^{l_1} = q_1(y,z)^{r_1}v(y,z)$ and $v(y,z)p_2(y,z)^{l_2} = q_2(y,z)^{r_2}v(y,z)$ are identities over y, z. In particular $l_1 = r_1$, $l_2 = r_2$, $p_1(y,z)$ and $q_1(y,z)$ are conjugates, and $p_2(y,z)$ and $q_2(y,z)$ are conjugates.*

Proof. We will use the following well-known fact:

Fact 4.2. *For all $i \geq 1$, the equation $xy = yz$ is equivalent to the equation $x^iy = yz^i$.*

As a consequence of Fact 4.2 we have that, for all $i, j \geq 1$, our system is equivalent to

$$\begin{cases} xp_1(y,z)^{l_1i} = q_1(y,z)^{r_1i}x \\ xp_2(y,z)^{l_2j} = q_2(y,z)^{r_2j}x \end{cases}. \tag{2}$$

Consider two infinite words over y and z: $q_1(y,z)^\omega$ and $q_2(y,z)^\omega$. Either they are the same and, consequently, $q_1(y,z) = q_2(y,z)$ (case A), or there is a finite common prefix $w(y,z)$ of these two infinite words (case B).

In case A we take $i = r_2$ and $j = r_1$. Then the last system turns into

$$\begin{cases} xp_1(y,z)^{l_1r_2} = q_1(y,z)^{r_1r_2}x \\ xp_2(y,z)^{l_2r_1} = q_1(y,z)^{r_2r_1}x. \end{cases}$$

The right sides of the equations in last system are the same. Consequently, $p_1(y,z)^{l_1r_2} = p_2(y,z)^{l_2r_1}$. Then, either this equation is nontrivial and, consequently, two words y and z which satisfy it are powers of a common word, say p, or the equation is trivial. In the first possibility, since the equations in the last system are nontrivial, if x, z and y satisfy it, then they are all powers of p. In the second possibility $p_1(y,z) = p_2(y,z)$ and $l_1r_2 = l_2r_1$. The last equation is equivalent to

$$(l_1/gcd(l_1,r_1)) \cdot (r_2/gcd(l_2,r_2)) = (l_2/gcd(l_2,r_2)) \cdot (r_1/gcd(l_1,r_1)).$$

Hence, $l_1/gcd(l_1,r_1) = l_2/gcd(l_2,r_2)$ and $r_1/gcd(l_1,r_1) = r_2/gcd(l_2,r_2)$. By Fact 4.2, in such a case, the last system is dependent. Hence in case A either (i) or (ii) holds.

Next, we consider case B and we take the left infinite words $p_1(y,z)^{-\omega}$ and $p_2(y,z)^{-\omega}$. If they are the same then $p_1(y,z) = p_2(y,z)$ and using symmetric reasoning as the one in case A we conclude that either (i) or (ii) holds. If the left infinite words are not the same then $p_1(y,z) \neq p_2(y,z)$ and, consequently, there is a finite longest common prefix $u(y,z)$ of $p_1(y,z)^{\omega}$ and $p_2(y,z)^{\omega}$.

Let the words x, y, z be a non-periodic solution of the system in the theorem.

Assume first that y and z are powers of the same primitive word say p. Then the first equation turns into a nontrivial equation on p and x. Hence, x is a power of p raising a contradiction.

Assume now that y and z are not powers of the same word. Then, for sufficiently large i and j the longest common prefix of the left sides of the equations in the system (2) is equal to $xu(y,z)t$ and the longest common prefix of the right sides of (2) is $w(y,z)t$ where t is the longest common prefix of yz and zy. Hence, $xu(y,z) = w(y,z)$. Considering the longest common suffixes we obtain $u'(y,z)x = w'(y,z)$ for some $u'(y,z)$ and $w'(y,z)$. Hence, $w(y,z)u(y,z)^{-1} = u'(y,z)^{-1}w'(y,z)$. Either this equation is nontrivial, and then y and z and, consequently, x are powers of the same word, or it is trivial. Then $x = v(y,z)$, for $v(y,z) = w(y,z)u(y,z)^{-1} = u'(y,z)^{-1}w'(y,z)$. Putting $v(y,z)$ instead of x in the system in the theorem we obtain either a nontrivial identity on y and z, or trivial equations. In the first case x, y, and z are powers of the same word. In the latter case (iii) holds. △

Using the previous result we prove now that any system of three balanced equations such that one of the unknowns occurs only twice in each equation, once at the beginning of one side and once at the end of the other side, either is dependent or admits only periodic solutions.

Theorem 4.3. *Consider a system of three balanced word equations of the form*

$$\begin{cases} xp_1(y,z) = q_1(y,z)x \\ xp_2(y,z) = q_2(y,z)x. \\ xp_3(y,z) = q_3(y,z)x \end{cases}$$

where $p_1(y,z)$, $p_2(y,z)$, $p_3(y,z)$, $q_1(y,z)$, $q_2(y,z)$, $q_3(y,z)$ are nonempty. Then either the system admits only periodic solutions or is dependent.

Proof. Take the subsystem of the first two equations. Then, according to Theorem 4.1, either the subsystem admits only periodic solutions, or is dependent or the set of non-periodic solutions is the set of non-periodic solutions of one equation of the form $x = v(y,z)$. In the first case the whole system admits only periodic solutions. In the second case the whole system is dependent. In the third case we substitute $x = v(y,z)$ to the third equation obtaining $v(y,z)p_3(y,z) = q_3(y,z)v(y,z)$. If the last equation is a nontrivial identity on y and z, then y and z are powers of the same primitive word, and consequently, x, y, z are powers of the same word. Hence, the system admits only periodic solutions. If the last equation is a trivial identity on y and z, then the third equation is a consequence of the first two equations. Hence, the system is dependent. △

Let us look next at the case when on the right sides of the three equations the unknown x might be followed by some occurrences of y and z. Then, we reverse all three equations, i.e., we read them from right to left, and consider the obtained system. If there exist two indices $i \neq j$ such that $r_i(y,z) = 1$ and $r_j(y,z) \neq 1$, then we apply Proposition 2.3 for this obtained reversed system to conclude that also the initial one admits only periodic solutions. Otherwise, the two sides of all equations of this obtained system begin with y and z respectively and, moreover, we have exactly one occurrence of the third unknown x in each side of every equation. Thus, this system is of one of the types investigated in the following four cases presented below, for which we will show that they admit only periodic solutions or they are not independent.

4.2. Case 2

Let us look now at systems of the following type:

$$\begin{cases} x^iyz^j = zv_1'(x,z)yv_2'(x,z)x \\ x^iyz^j = zu_1'(x,z)yu_2'(x,z)x, \\ x^iyz^j = zt_1'(x,z)yt_2'(x,z)x \end{cases}$$

for some $i, j \geq 1$. First, we give the following technical result, which proves to be useful for our analysis of this case.

Lemma 4.4. *Let x, y, z satisfy an equation of the form*

$$x^lt_1(x,z)yf_1(x,y,z) = zt_2(x,z)yf_2(x,y,z)$$

and, moreover, z is a suffix of x or x is a suffix of z. Then either x and z are powers of the same word, or $|z| > l|x|$ or $t_2(x,z) = z^k$ with $k \geq 0$.

Proof. Assume contrarily that $|z| \leq l|x|$ and x and z are not powers of the same word. If $|x| \leq |z| \leq l|x|$, then x is a period of z. But x is also a suffix of z. Hence, we immediately obtain that x and z are powers of the same word, see e.g. [1], which is a contradiction. If $|z| < |x|$ and $t_2(x,z) = z^k x t_2'(x,z)$, for some $k \geq 0$, then z is a period of x and z is a suffix of x. Hence we obtain again that x and z are powers of the same word. Again a contradiction. Hence, $t_2(x,z) = z^k$. △

Now, we show that any system containing two distinct equations of the type considered in this case admits at most quasi-periodic solutions.

Theorem 4.5. *Let the words x, y, z satisfy a system of two distinct equations of the following type:*

$$\begin{cases} x^i y z^j = z v_1'(x,z) y v_2'(x,z) x \\ x^i y z^j = z u_1'(x,z) y u_2'(x,z) x \end{cases}$$

for some $i, j \geq 1$. Then x and z are powers of the same word.

Proof. Assume that x and z are not powers of a common word. Then, we have the following two possibilities, depending on the lengths of x and z.

If $|z| \leq |x|$, then we can apply Lemma 4.4 with $l = 1$ to each equation and obtain $v_1'(x,z) = z^{k_1}$ and $u_1'(x,z) = z^{k_2}$ for some $k_1, k_2 \geq 0$. Hence our system is of the form

$$\begin{cases} x^i y z^j = z^{k_1+1} y v_2'(x,z) x \\ x^i y z^j = z^{k_2+1} y u_2'(x,z) x \end{cases}$$

for some $k_1, k_2 \geq 0$. Since the system is symmetric, we may assume without loss of generality, that $k_1 \geq k_2$. Then, from the two equations, we obtain $z^{k_1-k_2} y v_2'(x,z) = y u_2'(x,z)$ with $k_1 - k_2 \geq 0$. If $k_1 > k_2$, then by applying Proposition 2.3 to this equation and to the first equation in our system we obtain that the system has only periodic solutions. Assume now that $k_1 = k_2$. Consequently, $v_2'(x,z) = u_2'(x,z)$. Since the two equations are distinct, we may assume that $v_2'(x,z)$ is not graphically identical to $u_2'(x,z)$. Then $v_2'(x,z) = u_2'(x,z)$ is a nontrivial identity on x and z. Hence x and z are powers of the same word.

If $|z| > |x|$, then we apply the same reasoning as above but we change the roles of x and z and apply a lemma which is symmetric to Lemma 4.4. △

Next, we show that whenever we add a third equation to the previous system with the two sides starting with $x^i y$ and z, respectively, then the obtained system admits only periodic solutions or it is dependent.

Theorem 4.6. *Let the words x, y, z satisfy a system of three distinct balanced equations of the following type:*

$$\begin{cases} x^i y z^j = z v_1'(x,z) y v_2'(x,z) x \\ x^i y z^j = z u_1'(x,z) y u_2'(x,z) x, \\ x^i y t_1(x,y,z) = z t_1'(x,y,z) \end{cases}$$

for some $i, j \geq 1$. Then, the system is not independent or admits only periodic solutions.

Proof. Suppose that the system is independent. We already know, due to Theorem 4.5, that the system containing the first two equations admits at most quasi-periodic solutions with respect to x and z. Since the two sides of the equations start with $x^i y$ and $z v_1'(x,z) y$, they cannot admit independently quasi-periodic solutions. But then, due to Lemma 3.4, the initial system of three equations admits only periodic solutions. △

The following result, solving this case of our analysis, is an immediate consequence of the previous theorem.

Corollary 4.7. *The system of balanced equations*

$$\begin{cases} x^i y z^j = z v_1'(x,z) y v_2'(x,z) x \\ x^i y z^j = z u_1'(x,z) y u_2'(x,z) x, \\ x^i y z^j = z t_1'(x,z) y t_2'(x,z) x \end{cases}$$

for some $i, j \geq 1$, is not independent or admits only periodic solutions.

4.3. Case 3

Let us look next at systems of the following type:

$$\begin{cases} x^i y v_1(x,z) x = z v_1'(x,z) y z^j \\ x^i y u_1(x,z) x = z u_1'(x,z) y z^j \\ x^i y t_1(x,z) x = z t_1'(x,z) y z^j \end{cases}$$

for some $i, j \geq 1$. Then, we show that in this case, one equation is enough to force the solutions to be at most quasi-periodic.

Theorem 4.8. *Let the words x, y, z satisfy a balanced equation*

$$x^i y v_1(x,z) x = z v_1'(x,z) y z^j.$$

Then x and z are powers of the same word.

Proof. Assume that x and z are not powers of the same word. Then, we have two cases depending on the lengths of x and z.

If $|x| \geq |z|$, then we can apply Lemma 4.4 with $l = 1$. We obtain $v_1'(x,z) = z^k$, for some $k \geq 0$. Hence, the equation is of the form

$$x^i y u_1(x,z) x = z^{k+1} y z^j,$$

for some $k \geq 0$. But this equation is not balanced, which is a contradiction.

If $|x| < |z|$, then we apply the same reasoning as above but we change the roles of x and z and apply a lemma which is symmetric to Lemma 4.4. △

Then, we show that if we add another equation with the two sides starting with $x^i y$ and z, respectively, then the obtained system admits only periodic solutions or it is dependent.

Theorem 4.9. *Let the words x, y, z satisfy two balanced equations*

$$\begin{cases} x^i y v_1(x,z) x = z v_1'(x,z) y z^j \\ x^i y u_1(x,y,z) = z u_1'(x,y,z) \end{cases}.$$

Then this system is not independent or admits only periodic solutions.

Proof. Suppose that the system is independent. We already know, due to Theorem 4.8, that the first equation of this system admits at most quasi-periodic solutions with respect to x and z. Since its two sides start with $x^i y$ and $z v_1'(x,z) y$ it cannot admit independently quasi-periodic solutions. But then, due to Lemma 3.4, the initial system of two equations admits only periodic solutions. △

As an immediate consequence, we obtain the following corollary which solves completely the third case of our analysis.

Corollary 4.10. *The system of balanced equations*

$$\begin{cases} x^i y v_1(x,z) x = z v_1'(x,z) y z^j \\ x^i y u_1(x,z) x = z u_1'(x,z) y z^j, \\ x^i y t_1(x,z) x = z t_1'(x,z) y z^j \end{cases}$$

for some $i, j \geq 1$, is not independent or admits only periodic solutions.

4.4. Case 4

Let us look next at systems of the following type:

$$\begin{cases} x^i y p_1(x,z) = z q_1(x,z) y \\ x^i y p_2(x,z) = z q_2(x,z) y \\ x^i y p_3(x,z) = z q_3(x,z) y \end{cases}$$

for some $i \geq 1$. We start our investigation by considering first only solutions consisting of **nonempty** words. Also, in what follows by a quasi-periodic solution we mean a triple x, y, z such that x and z are powers of the same primitive word p and y is not a power of p.

First, we give some intermediary results which will be very useful for our considerations of this case. That is, we look at systems where the unknown y occurs only twice in each equation: once exactly after the first block of x's on one side and once somewhere in the middle of the other side.

Lemma 4.11. *Let x, y, z, such that x and z are not powers of a common word, be a non-periodic solution of the following system*

$$\begin{cases} x^i y p_1(x,z) = z q_1(x,z) y r_1(x,z) \\ x^i y p_2(x,z) = z q_2(x,z) y r_2(x,z) \end{cases}$$

where $p_1(x,z)$, $p_2(x,z)$, $r_1(x,z)$, $r_2(x,z)$, $q_1(x,z)$, $q_2(x,z)$ are words over x, z, $q_1(x,z)$ contains x, and $i \geq 2$.

1. *If $q_2(x,z)$ is nonempty, then there is a non-periodic solution x', y, z'' satisfying $x = (x'z'')^k x'$ and $z = ((x'z'')^k x')^{i-1} x'z''$, for some $k \geq 0$, of the system*

$$\begin{cases} y p_1(x,z) = z''x'x^{-1} q_1(x, x^{i-1}x'z'') y r_1(x,z) \\ y p_2(x,z) = z''x'x^{-1} q_2(x, x^{i-1}x'z'') y r_2(x,z) \end{cases}$$

where x stands for $(x'z'')^k x'$ and z stands for $((x'z'')^k x')^{i-1} x'z''$.

2. *If $q_2(x,z)$ is empty, then there is a non-periodic solution x', y, z'' satisfying $x=(x'z'')^k x'$ and $z=((x'z'')^k x')^{i-1}x'z''$, for some $k\geq 0$, of the system*

$$\begin{cases} yp_1(x,z)=z''x'x^{-1}q_1(x,x^{i-1}x'z'')yr_1(x,z)\\ (x'z'')^k x'yp_2(x,z)=x'z''yr_2(x,z)\end{cases}$$

where x stands for $(x'z'')^k x'$ and z stands for $((x'z'')^k x')^{i-1}x'z''$.

Proof. Note first that the expression $x^{-1}q_j(x,x^{i-1}x'z'')$, with $j\in\{1,2\}$, has sense since q_j is nonempty and $i\geq 2$, and, consequently, $q_j(x,x^{i-1}x'z'')$ starts with x.

The words x,z are not powers of the same word. This means that the set $\{x,z\}$ is a two-element code. We consider the following cases depending on the lengths of x and z.

Case $|z|\leq(i-1)|x|$. Then $|x^i|\geq|xz|$. Since $q_1(x,z)$ contains x, then $|zq_1(x,z)|\geq|xz|$. So, we have two expressions over x and z sharing a common prefix of length larger that $|xz|$. Consequently, x and z are powers of a common word, which is a contradiction.

Case $(i-1)|x|<|z|<i|x|$. Then there is a nonempty z' such that $z=x^{i-1}z'$ and $|z'|<|x|$, $z'\in \mathit{Pref}(x)$. Observe here that $\{x,z\}$ is a two-element code if and only if $\{x,z'\}$ is a two-element code. Then the first equation turns into $xyp_1(x,z)=z'q_1(x,z)yr_1(x,z)$. Since $q_1(x,z)$ contains x, then $z'q_1(x,x^{i-1}z')$ contains x. Moreover, since $i\geq 2$ the right side of this equation starts with $z'x$. Consequently, $x=z'^k x'$, for some $k\geq 1$ and $|x'|<|z'|$, $x'\in \mathit{Pref}(z')$. Hence, $x'yp_1(x,z)=z'x'x^{-1}q_1(x,x^{i-1}z')yr_1(x,z)$. We have that $\{x',z'\}$ is a two-element code if and only if $\{x,z'\}$ is a two-element code.

Since x' is a prefix of z', then we can write $z'=x'z''$ and finally

$$yp_1(x,z)=z''x'x^{-1}q_1(x,x^{i-1}x'z'')yr_1(x,z)$$

where $x=(x'z'')^k x'$ and $z=((x'z'')^k x')^{i-1}x'z''$. Observe that $\{x,z\}$ is a code if and only if $\{x',z''\}$ is a code.

Substituting $x=(x'z'')^k x'$ and $z=((x'z'')^k x')^{i-1}x'z''$ in the second equation of the system in the lemma we obtain

$$yp_2(x,z)=z''x'x^{-1}q_2(x,x^{i-1}x'z'')yr_2(x,z),$$

if $q_2(x,z)$ is nonempty, and

$$(x'z'')^k x'yp_2(x,z)=x'z''yr_2(x,z),$$

if $q_2(x,z)$ is empty.

Case $|z|=i|x|$. Then $z=x^i$, i.e., x and z are powers of the same word, which is a contradiction.

Case $|z|>i|x|$. Then $z=x^i z''$, for some nonempty z''. Observe here that $\{x,z\}$ is a two-element code if and only if $\{x,z''\}$ is a two-element code. Hence, if $q_2(x,z)$ is nonempty, then x',z'',y satisfying $x=(x'z'')^0x'$ and $z=((x'z'')^0x')^{i-1}x'z''$ is a non-periodic solution of the system

$$\begin{cases} yp_1(x,z)=z''x'x^{-1}q_1(x,x^{i-1}x'z'')yr_1(x,z)\\ yp_2(x,z)=z''x'x^{-1}q_2(x,x^{i-1}x'z'')yr_2(x,z)\end{cases}.$$

If $q_2(x,z)$ is empty, then x',z'',y satisfying $x=(x'z'')^0x'$ and $z=((x'z'')^0x')^{i-1}x'z''$ is a non-periodic solution of the system

$$\begin{cases} yp_1(x,z)=z''x'x^{-1}q_1(x,x^{i-1}x'z'')yr_1(x,z)\\ (x'z'')^0x'yp_2(x,z)=x'z''yr_2(x,z)\end{cases}. \quad \triangle$$

Next we consider systems of the same type as in the previous result, except that now the left sides start with only one occurrence of x.

Lemma 4.12. *Let x,y,z, such that x and z are not powers of a common word, be a non-periodic solution of the following system*

$$\begin{cases} xyp_1(x,z)=zq_1(x,z)yr_1(x,z)\\ xyp_2(x,z)=zq_2(x,z)yr_2(x,z)\end{cases}$$

where $p_1(x,z)$, $p_2(x,z)$, $r_1(x,z)$, $r_2(x,z)$, $q_1(x,z)$, $q_2(x,z)$ are words over x,z, and $q_1(x,z)$ contains x.

1. *If $q_2(x,z)$ contains x, then $q_1(x,z)=z^{j_1}xq_1'(x,z)$, $q_2(x,z)=z^{j_2}xq_2'(x,z)$, for some $j_1,j_2\geq 0$ and there is a non-periodic solution x',y,z' satisfying $x=(x'z')^k x'$ and $z=x'z'$, for some $k\geq 0$, of the system*

$$\begin{cases} yp_1(x,z)=z'z^{j_1}x'q_1'(x,z)yr_1(x,z)\\ yp_2(x,z)=z'z^{j_2}x'q_2'(x,z)yr_2(x,z)\end{cases}.$$

2. *If $q_2(x,z)$ does not contain x, then $q_1(x,z)=z^{j_1}xq_1'(x,z)$, $q_2(x,z)=z^{j_2}$, for some $j_1,j_2\geq 0$, and there is a non-periodic solution x',y,z' satisfying $x=(x'z')^k x'$ and $z=x'z'$, for some $k\geq 0$, of the system*

$$\begin{cases} yp_1(x,z)=z'z^{j_1}x'q_1'(x,z)yr_1(x,z)\\ (x'z')^k x'yp_2(x,z)=(x'z')^{j_2+1}yr_2(x,z)\end{cases}.$$

Proof. Since the words x and z are not powers of a common word, the set $\{x, z\}$ is a two-element code. Again, we discuss several cases depending on the lengths of x and z.

Case $|z| > |x|$. Then $z = x'z'$ and $x = x' = (x'z')^0x'$. Observe that $\{x, z\}$ is a two-element code if and only if $\{x', z'\}$ is a two-element code. Hence, if $q_2(x, z)$ contains x, then x', z', y is a non-periodic solution of the system

$$\begin{cases} yp_1(x,z) = z'z^{j_1}x'q'_1(x,z)yr_1(x,z) \\ yp_2(x,z) = z'z^{j_2}x'q'_2(x,z)yr_2(x,z) \end{cases}.$$

If $q_2(x, z)$ does not contain x, then x', z', y is a non-periodic solution of the system

$$\begin{cases} yp_1(x,z) = z'z^{j_1}x'q'_1(x,z)yr_1(x,z) \\ (x'z'')^0x'yp_2(x,z) = (x'z')^{j_2+1}yr_2(x,z) \end{cases}.$$

Case $|z| = |x|$. Then $x = z$, i.e., x and z are powers of the same word, which is a contradiction.

Case $|z| < |x|$. Then, since $q_1(x, z)$ contains x, we obtain that $x = z^kx'$, for some $k \geq 1$ and $|x'| < |z|$, $x' \in \mathit{Pref}(z)$. Note that $\{x, z\}$ is a two-element code if and only if $\{x', z\}$ is a two-element code. Hence, the first equation becomes $x'yp_1(x, z) = z^{j_1+1}x'q'_1(x, z)yr_1(x, z)$. Hence, $z = x'z'$, and, consequently, $yp_1(x, z) = z'z^{j_1}x'q_1(x, z)yr_1(x, z)$. Observe that $\{x, z\}$ is a two-element code if and only if $\{x', z'\}$ is a two-element code. Substituting $z = x'z'$ and $x = z^kx' = (x'z')^kx'$ in the second equation in the system in the lemma we obtain

$$yp_2(x,z) = z'z^{j_2}x'q'_2(x,z)yr_2(x,z),$$

if $q_2(x, z)$ contains x and

$$(x'z')^kx'yp_2(x,z) = (x'z')^{j_2+1}yr_2(x,z),$$

if $q_2(x, z)$ does not contain x. Hence, the result. △

Now we are ready to start analyzing systems of the type considered in our Case 4, i.e., when the unknown y occurs once exactly after the first block of x's on one side and once at the end of the other side of each equation.

Lemma 4.13. *Consider a system of two balanced equations*

$$\begin{cases} xyp_1(x,z) = zq_1(x,z)y \\ xyp_2(x,z) = zq_2(x,z)y \end{cases}$$

where $p_1(x, z)$, $p_2(x, z)$ are nonempty words on x and z, and $q_1(x, z)$ and $q_2(x, z)$ are words over x, z. Then

(i) *either all solutions of the system are periodic,*
(ii) *or the system is dependent and $p_1(x, z) = p_2(x, z)$ and $q_1(x, z) = q_2(x, z)$*
(iii) *or $q_1(x, z) = q'_1(x, z)x$ and $q_2(x, z) = q'_2(x, z)x$ for some $q'_1(x, z)$, $q'_2(x, z)$.*

Proof. Observe first that, since the equations are balanced, then both $q_1(x, z)$ and $q_2(x, z)$ contain x.

Let x, z, y be a non-periodic solution of the system in the lemma. Assume that x and z are nonempty powers of the same primitive word p. Then the first equation is a nontrivial identity on p and y. Hence y is a power of p and, consequently, the solution x, y, z is periodic, which is a contradiction.

Hence, we may apply point 1 from Lemma 4.12. Let $q_1(x, z) = z^{j_1}xq''_1(x, z)$ with $j_1 \geq 0$ and let $q_2(x, z) = z^{j_2}xq''_2(x, z)$ with $j_2 \geq 0$. If $q''_1(x, z) = 1$, then $q_1(x, z) = z^{j_1}x$ ends by x. Similarly, if $q''_2(x, z) = 1$, then $q_2(x, z) = z^{j_2}x$ ends by x.

By Lemma 4.12 there is a non-periodic solution x', y, z' satisfying $x = (x'z')^kx'$ and $z = x'z'$, for some $k \geq 0$, of the system

$$\begin{cases} yp_1(x,z) = z'z^{j_1}x'q''_1(x,z)y \\ yp_2(x,z) = z'z^{j_2}x'q''_2(x,z)y \end{cases}$$

where x stands for $(x'z')^kx'$ and z stands for $x'z'$.

We may thus apply Theorem 4.1. Since the solution x', y, z' is non-periodic, in Theorem 4.1 either (ii) or (iii) holds.

Assume first that (iii) holds. Hence, the expressions $p_1((x'z')^kx', x'z')$ and $z'(x'z')^{j_1}x'q''_1((x'z')^kx', x'z')$ are conjugates as words over x' and z'. In the word $p_1((x'z')^kx', x'z')$ each letter z' is preceded by x'. Hence, in its conjugate each letter z' is cyclically preceded by x'. Hence, x' is the last letter of $z'(x'z')^{j_1}x'q''_1((x'z')^kx', x'z')$. We may assume that $q''_1(x, z)$ is nonempty. Then, it has to end by x since the replacement for z ends by z'. This means that $q(x, z)$ ends by x. Similarly, we can also prove that $q_2(x, z)$ ends by x.

Assume now that point (ii) from Theorem 4.1 holds. Then, we can write $z'(x'z')^{j_1}x'q''_1((x'z')^kx', x'z') = q''(x', z')^{l_1}$ and $z'(x'z')^{j_2}x'q''_2((x'z')^kx', x'z') = q''(x', z')^{l_2}$ with $q''(x', z')$ primitive.

If we have $l_1 = l_2 = 1$, then we obtain that $z'(x'z')^{j_1}x'q''_1((x'z')^kx', x'z') = z'(x'z')^{j_2}x'q''_2((x'z')^kx', x'z')$. Let $j_2 \geq j_1$, the other case being symmetric. Then

$$q''_1((x'z')^kx', x'z') = (z'x')^{j_2-j_1}q''_2((x'z')^kx', x'z').$$

For each $k \geq 0$, $q_1''((x'z')^kx', x'z')$ starts with x'. Hence $j_2 = j_1$, and, consequently, $w = q_1''((x'z')^kx', x'z') = q_2''((x'z')^kx', x'z')$. Since, for each $k \geq 0$ the set $\{(x'z')^kx', x'z'\}$ is a suffix code, the factorization of w in these words is unique. Hence, $q_1''(x, z) = q_2''(x, z)$. This means that the right-hand sides of the equations in the system in the lemma are the same. If we use it we obtain $p_1(x, z) = p_2(x, z)$. Now, either this last equation is identity and then (ii) holds, or it is a nontrivial equation on x and z. In the latter case x and z are powers of the same primitive word say p. Then the first equation of our system is a nontrivial identity on p and y. Hence, y is a power of p. Hence, x, y, z is a periodic solution, which is a contradiction.

It remains to consider the cases when $l_1 \geq 2$ or $l_2 \geq 2$. Assume $l_1 \geq 2$, the proof for $l_2 \geq 2$ being analogous. Since $z'(x'z')^{j_1}x'q_1''((x'z')^kx', x'z') = q''(x', z')^{l_1}$, then the word $q''(x', z')$ starts with z'. Suppose that $q_1''(x, z)$ or $q_2''(x, z)$ ends by z. Then $q''(x', z')$ ends by z'. Since $l_1 \geq 2$, then $z'z'$ is a subword of $q''(x', z')^{l_1} = z'(x'z')^{j_1}x'q_1''((x'z')^kx', x'z')$. On the right-hand side, however, each z' except the first one is preceded by x'. Thus, we obtain a contradiction. Hence, $q_1''(x, z)$ and $q_2''(x, z)$, and consequently, $q_1(x, z)$ and $q_2(x, z)$ end by x. $\triangle$

Let us look next at the case when the left sides of the equations start with at least two x's.

Lemma 4.14. *Consider a system of two balanced equations*

$$\begin{cases} x^iyp_1(x, z) = zq_1(x, z)y \\ x^iyp_2(x, z) = zq_2(x, z)y \end{cases}$$

where $p_1(x, z)$, $p_2(x, z)$ are nonempty words over x, z, and $q_1(x, z)$, $q_2(x, z)$ are words over x, z, and $i \geq 2$. Then

(i) *either all solutions of the system are periodic,*
(ii) *or the system is dependent and $p_1(x, z) = p_2(x, z)$ and $q_1(x, z) = q_2(x, z)$*
(iii) *or $q_1(x, z) = q_1'(x, z)x^i$ and $q_2(x, z) = q_2'(x, z)x^i$ for some $q_1'(x, z)$, $q_2'(x, z)$.*

Proof. Let x, y, z be a non-periodic solution of the system in the lemma.

If x and z are powers of a common word, say p, then, since $p_1(x, z)$ is nonempty, the first equation is a nontrivial identity on p and y. Hence, y is a power of p, and, consequently, the solution x, y, z is periodic; a contradiction.

We may thus assume that x and z are not powers of a common word. Since the system is balanced, then both $q_1(x, z)$ and $q_2(x, z)$ contain x. Hence, we may apply point 1 of Lemma 4.11 and obtain that there is a non-periodic solution x', y, z'' satisfying $x = (x'z'')^kx'$ and $z = ((x'z'')^kx')^{i-1}x'z''$, for some $k \geq 0$, of the system

$$\begin{cases} yp_1(x, z) = z''x'x^{-1}q_1(x, x^{i-1}x'z'')y \\ yp_2(x, z) = z''x'x^{-1}q_2(x, x^{i-1}x'z'')y \end{cases}$$

where x stands for $(x'z'')^kx'$ and z stands for $((x'z'')^kx')^{i-1}x'z''$.

We may now apply Theorem 4.1. Since x', y, z'' is a non-periodic solution, then in Theorem 4.1, either (ii) or (iii) holds. Denote, for $i \in \{1, 2\}$,

$$p_i'(x', z'') = p_i((x'z'')^kx', ((x'z'')^kx')^{i-1}x'z'')$$

and

$$q_i'(x,' z'') = z''x'((x'z'')^kx')^{-1}q_i((x'z'')^kx', ((x'z'')^kx')^{i-1}x'z'').$$

Assume first that (iii) holds in Theorem 4.1. Then $p_1'(x', z'')$ and $q_1'(x', z'')$ are conjugates. In $p_1'(x', z'')$ each letter z'' is preceded by x'. Hence, in $q_1'(x', z'')$ each z'' is cyclically preceded by x'. Hence, the last letter of $q_1'(x', z'')$ is x'. If, in $q_1(x, z)$, the last letter were z, then, in $q_1'(x', z'')$ the last letter would be z''. Hence, the last letter of $q_1(x, z)$ is x.

Both equations from the system in the lemma are balanced. Hence, $q_1(x, z)$ contains at least $i \geq 2$ letters x. Consequently, $q_1(x, z) = wx$, for a nonempty w. Hence, $q_1'(x', z'')$ ends by $(x'z'')^kx'$. Hence, the word $w' = (x'z'')^{k+1}x'$ is a cyclical subword of $p_1'(x', z'') = p_1(x, z)$ where x stands for $(x'z'')^kx'$ and z stands for $((x'z'')^kx')^{i-1}x'z''$. For any, $k \geq 0$, the word w' is not a subword of x or z. It is not a subword of xz or xx since x ends with x' and z starts with x', but w' is a subword of zx and zz and just before the occurrence in these words there is a word $x^{i-1} = ((x'z'')^kx')^{i-1}$. This means that $x^i = ((x'z'')^kx')^i$ is a suffix of $q_1'(x', z'')$.

If z occurs in $q_1(x, z)$, then zx^l, for some $l \geq 1$, is a suffix of $q_1(x, z)$. Hence, $z''((x'z'')^kx')^l$ is a suffix of $q_1'(x', z'') = z''x'x^{-1}q_1(x, z)$ where $x = (x'z'')^kx'$ and $z = ((x'z'')^kx')^{i-1}x'z''$. Since $((x'z'')^kx')^i$ is also a suffix of this word, then $l \geq i$ and consequently x^i is a suffix of $q_1(x, z)$. If $q_1(x, z) = x^l$, for some $l \geq 1$, then, since it contains at least i letters x, we have $l \geq i$, and, consequently it ends in x^i. Similarly, we prove that $q_2(x, z)$ ends in x^i. Hence, (iii) holds.

Assume now that (ii) holds in Theorem 4.1. Then, $q_1'(x', z'') = q''(x', z'')^{l_1}$ and $q_2'(x', z'') = q''(x', z'')^{l_2}$, for some $l_1, l_2 \geq 1$. Since $q_1'(x', z'')$ starts with z'', then $q''(x', z'')$ starts with z''.

If $l_1 = l_2$, then $q_1'(x', z'') = q_2'(x', z'')$. Hence, we have

$$z''x'((x'z'')^kx')^{-1}q_1((x'z'')^kx', ((x'z'')^kx')^{i-1}x'z'') = z''x'((x'z'')^kx')^{-1}q_2((x'z'')^kx', ((x'z'')^kx')^{i-1}x'z'').$$

Consequently, $w(x', z'') = q_1(x, z) = q_2(x, z)$ where x stands for $(x'z'')^kx'$ and z stands for $((x'z'')^kx')^{i-1}x'z''$. Since the set $\{(x'z'')^kx', ((x'z'')^kx')^{i-1}x'z''\}$ is a suffix code, then the factorization of $w(x', z')$ in these words is unique. Hence,

$q_1(x, z) = q_2(x, z)$. Consequently, the right-hand sides of equations in the system in the lemma are the same. We use it to prove that, then, $p_1(x, z) = p_2(x, z)$. Now, either this last equation is an identity on x and z and, consequently, (ii) holds, or it is a nontrivial identity on the words x and z. In the latter case the words x and z are powers of the same primitive word say p. Then the first equation of the system in the lemma is a nontrivial identity on p and y. Hence the word y is also a power of p. Hence, the solution x, y, z is periodic, which is a contradiction.

If $l_1 \neq l_2$, then $l_1 > l_2$ or $l_2 > l_1$. Assume that $l_1 > l_2$ the other case being symmetric. Then $l_1 \geq 2$. Suppose that $q_1(x, z)$ or $q_2(x, z)$ ends by z. Then $q_1'(x', z'')$ or $q_2'(x', z'')$ ends by z'' and, consequently, $q''(x', z'')$ ends by z''. Then $z''z''$ is a subword of $q''(x', z'')^2$ and, hence, a subword of $q_1(x', z'')$. However each letter z'' in $q_1(z', x'')$, except the first one, is preceded by x', so we reached a contradiction.

This means that $q_1(x, z)$ and $q_2(x, z)$ end by x. Since $i \geq 2$ and the equations in the system in the lemma are balanced, then $q_1(x, z) = w_1 x$ and $q_2(x, z) = w_2 x$ and w_1, w_2 are nonempty words. Hence, $q''(x', z'')$ ends by $(x'z'')^k x'$, and, consequently, $w' = (x'z'')^{k+1}x'$ is a subword of $q''(x', z'')^2$, and, hence, a subword of $q_1'(x', z'')$.

The word $q_1'(x', z'')$ is in the form $z''x'u_1(x, z)$, if $q_1(x, z)$ starts with x, or $z''x'((x'z'')^k x')^{i-2}x'z''u_1(x, z)$, if $q_1(x, z)$ starts with z, where x stands for $(x'z'')^k x'$ and z stands for $((x'z'')^k x')^{i-1}x'z''$. Let us denote now $u_1'(x', z'') = u_1((x'z'')^k x', ((x'z'')^k x')^{i-1}x'z'')$.

In the first case the letter x' starts $u_1'(x', z'')$. Hence, w' is a subword of $u_1'(x', z'')$. As we already proved, this is possible only if w' is preceded by $x^{i-1} = ((x'z'')^k x')^{i-1}$ and, consequently, when $x^i = ((x'z'')^k x')^i$ is a suffix of $q''(x', z'')^l$, for some $l < l_2$. Hence, $x^i = ((x'z'')^k x')^i$ is a suffix of $q_1'(x', z'')$. We repeat the earlier reasoning to prove that $q_1(x, z)$ ends by x^i.

In the other case either w' occurs in $u_1(x, z)$ where x stands for $(x'z'')^k x'$ and z stands for $((x'z'')^k x')^{i-1}x'z''$ or it does not occur in $u_1(x, z)$ and its occurrence starts before $u_1(x, z)$. In the first case we proceed as in the previous case concluding that $q_1(x, z)$ ends by x^i. In the second case w' is preceded by a prefix $z''x'((x'z'')^k x')^{i-2}$ of $q_1'(x', z'')$ and it is the only occurrence of w' in $q_1'(x', z'')$. Hence, $l_1 = 2$ and the occurrence of w' contains the border between $q''(x'z'')$ in $q''(x', z'')^{l_1} = q''(x', z'')^2$. In this case, $q''(x', z'') = z''x'((x'z'')^k x')^{i-1} = w''$. Since $q''(x', z'')$ is longer than $z''x'((x'z'')^k x')^{i-2}$ and $l_1 = 2$, $q''(x', z'')$ is a suffix of $u_1'(x', z'')$. We have $q''(x', z'') = z''x'x^{i-1}$ where x stands for $(x'z'')^k x'$. The word $u_1'(x', z'')$ can be factorized in words $x = (x'z'')^k x'$ and $z = ((x'z'')^k x')^{i-1}x'z''$. Since the set $\{(x'z'')^k x', ((x'z'')^k x')^{i-1}x'z''\}$ is a suffix code, the factorization is unique and can be obtained by reading the word $u_1'(x', z'')$ backwards and cutting off the words which stand for x and z. Since the letter just before x^{i-1} in $q''(x', z'')$ is x', the word $u_1'(x', z'')$ ends in $x^i = ((x'z'')^k x')^i$, and, consequently, $u_1(x, z)$ ends by x^i. Hence, $q_1(x, z)$ ends in x^i.

We proved that in both cases $q_1(x, z)$ ends in x^i. We will prove next that also $q_2(x, z)$ ends in x^i.

Suppose that $q_2(x, z)$ contains z. Then it ends in zx^l, for some $l \geq 1$. Hence, $q_2'(x', z'')$ ends in $z''((x'z'')^k x')^l$. Since $l_1 > l_2$, then $q_2'(x', z'')$ is a proper suffix of $q_1'(x', z'')$. Since, as we already proved, $((x'z'')^k x')^i$ is also a suffix of $q_1'(x', z'')$, we have $l \geq i$ and, consequently $q_2(x, z)$ ends in x^i.

Suppose that $q_2(x, z)$ does not contain z. Since the second equation in the system in the lemma is balanced, we have $q_2(x, z) = x^l$ and $l \geq i$. Hence, $q_2(x, z)$ ends in x^i. This completes the proof. △

As a consequence of Lemmata 4.13 and 4.14 we have the following result.

Corollary 4.15. *Consider a system of two balanced equations*

$$\begin{cases} x^i y p_1(x, z) = z q_1(x, z) y \\ x^i y p_2(x, z) = z q_2(x, z) y \end{cases}$$

where $p_1(x, z)$, $p_2(x, z)$ are nonempty words over x and z, and $q_1(x, z)$ and $q_2(x, z)$ are words over x, z and $i \geq 1$. Then

(i) *either all solutions of the system are periodic,*
(ii) *or the system is dependent and $p_1(x, z) = p_2(x, z)$ and $q_1(x, z) = q_2(x, z)$*
(iii) *or $q_1(x, z) = q_1'(x, z)x^i$ and $q_2(x, z) = q_2'(x, z)x^i$ for some $q_1'(x, z)$, $q_2'(x, z)$.*

Moreover, as a deeper consequence of Lemmata 4.13 and 4.14 we have the following.

Theorem 4.16. *Consider a system of two balanced equations*

$$\begin{cases} x^i y p_1(x, z) = z q_1(x, z) y \\ x^i y p_2(x, z) = z q_2(x, z) y \end{cases}$$

where $p_1(x, z)$, $p_2(x, z)$ are nonempty words on x and z, and $q_1(x, z)$, $q_2(x, z)$ are words over x, z and $i \geq 1$. Then

(i) *either all solutions of the system are periodic,*
(ii) *or the system is dependent and $p_1(x, z) = p_2(x, z)$ and $q_1(x, z) = q_2(x, z)$*
(iii) *or $q_1(x, z)$ and $q_2(x, z)$ end in x^i and the set of non-periodic solutions is the set of non-periodic solutions of one equation $x^i y = v(x, z)$ where $v(x, z)$ is nonempty. Consequently, the equations $v(x, z)p_1(x, z) = zq_1(x, z)x^{-i}v(x, z)$ and $v(x, z)p_2(x, z) = zq_2(x, z)x^{-i}v(x, z)$ are identities over x, z. In particular $p_1(x, z)$ and $zq_1(x, z)x^{-i}$ are conjugates, and $p_2(x, z)$ and $zq_2(x, z)x^{-i}$ are conjugates.*

(iv) *the system is dependent,* $zq_1(x,z) = (q'(x,z))^{r_1}x^i$, $zq_2(x,z) = (q'(x,z))^{r_2}x^i$, $p_1(x,z) = (p'(x,z))^{l_1}$ *and* $p_2(x,z) = (p'(x,z))^{l_2}$, *for some primitive* $q'(x,z)$, $p'(x,z)$ *and* $l_1, l_2, r_1, r_2 \geq 1$ *satisfying* $l_1/gcd(l_1,r_1) = l_2/gcd(l_2,r_2)$ *and* $r_1/gcd(l_1,r_1) = r_2/gcd(l_2,r_2)$.

Proof. It is enough to prove that point (iii) in Corollary 4.15 implies (i)–(iii). If the point (iii) in Corollary 4.15 is satisfied then the system in the theorem looks like

$$\begin{cases} x^iyp_1(x,z) = zq'_1(x,z)x^iy \\ x^iyp_2(x,z) = zq'_2(x,z)x^iy \end{cases}.$$

If we substitute $y' = x^iy$ and apply Theorem 4.1 to the obtained system, then we obtain the result. △

Let us note now that, as a consequence of Theorem 4.16 we obtain the following result which solves Case 4 for nonempty solutions, i.e., the ones containing only non-empty words.

Theorem 4.17. *Consider nonempty solutions of the following system of balanced equations*

$$\begin{cases} x^iyp_1(x,z) = zq_1(x,z)y \\ x^iyp_2(x,z) = zq_2(x,z)y \\ x^iyp_3(x,z) = zq_3(x,z)y \end{cases}.$$

Then either the system admits only periodic solutions or it is dependent.

Proof. According to Theorem 4.16, the subsystem consisting of the two first equations either admits only periodic solutions, or is dependent or both $q_1(x,z)$ and $q_2(x,z)$ end with x^i and the set of non-periodic solutions of this subsystem is the set of non-periodic solutions of one equation of the form $x^iy = v_1(x,z)$. In the first case the whole system admits only periodic solutions. In the second case the whole system is dependent. So, let us consider the third case now and apply again Theorem 4.16 to the last two equations of the system from the theorem. Then, either this subsystem admits only periodic solutions, or is dependent or both $q_2(x,z)$ and $q_3(x,z)$ end with x^i and the set of non-periodic solutions of this subsystem is the set of non-periodic solutions of one equation of the form $x^iy = v_2(x,z)$. Just as above, the first two cases imply that the whole system admits only periodic solutions or it is dependent, respectively. Thus, the only remaining case is when $x^iy = v_1(x,z) = v_2(x,z)$. If this is a nontrivial identity over x and z then they are powers of a common word, which immediately implies that so is y, i.e., the system admits only periodic solutions. Otherwise, i.e., $v_1(x,z) = v_2(x,z)$ is an identity over $\{x,z\}$, the system from the theorem is dependent since its first two equations are equivalent with its last two equations. △

Next, we look at the set of solutions where at least one of the words x, z, or y is the empty word.

Theorem 4.18. *Consider the system of balanced equations*

$$\begin{cases} x^iyp_1(x,z) = zq_1(x,z)y \\ x^iyp_2(x,z) = zq_2(x,z)y. \\ x^iyp_3(x,z) = zq_3(x,z)y \end{cases}$$

If at least one of the words x, z, or y is the empty word, then either the system admits only periodic solutions or all of its constituent equations accept exactly the same subset of solutions.

Proof. Now, if $x = 1$ or $y = 1$ then all three equations of the system become non-trivial identities over z and y or over x and z, respectively, which immediately implies that they are powers of a common word. Thus, the system admits only periodic solutions. So, let us look now at the case when $z = 1$. If at least one of the expressions $p_i(x,z)$ with $i \in \{1,2,3\}$ also contains an x, then the coresponding equation is a non-trivial identity over x and y, and thus they must be powers of a common word. Thus, also in this case the system admits only periodic solutions. Otherwise, since the three equations of the system are balanced, they all become $x^iy = x^iy$, i.e., they all accept the same subset of solutions with $z = 1$. △

As an immediate consequence of Theorems 4.17 and 4.18 we have a solution of Case 4 of our analysis.

Corollary 4.19. *The system of balanced equations*

$$\begin{cases} x^iyp_1(x,z) = zq_1(x,z)y \\ x^iyp_2(x,z) = zq_2(x,z)y \\ x^iyp_3(x,z) = zq_3(x,z)y \end{cases}$$

either admits only periodic solutions or is dependent.

4.5. Case 5

The last case we have to consider consists of systems of the type

$$\begin{cases} x^iyv_1(x,z)z = zv'_1(x,z)yx^j \\ x^iyu_1(x,z)z = zu'_1(x,z)yx^j \\ x^iyt_1(x,z)z = zt'_1(x,z)yx^j \end{cases}$$

with $i, j \geq 1$.

As in the previous section we first consider only **nonempty** solutions. Now, we give some intermediate results which will be useful later on for our considerations.

Lemma 4.20. *A system of the form*

$$\begin{cases} x^iyz^{l_1} = z^{l_1}yx^j \\ x^iyz^{l_2} = z^{l_2}yx^j \end{cases}$$

where $l_1 > l_2 \geq 1$ and $i, j \geq 1$, admits only periodic solutions.

Proof. Since the right-hand side of the second equation is a proper suffix of the right-hand side of the first one, we replace this suffix by the left-hand side of the second equation and obtain $x^iyz^{l_1} = z^{l_1-l_2}x^iyz^{l_2}$. Hence, $x^iyz^{l_1-l_2} = z^{l_1-l_2}x^iy$. Consequently, x^iy and $z^{l_1-l_2}$ commute. There is a primitive word p such that $x^iy, z^{l_1-l_2} \in p^*$. Hence, $x^iy, z \in p^*$ and, consequently, $x^iyz^{l_1}, z^{l_1} \in p^*$. Using the first equation we have $yx^j \in p^*$. Hence, $x^iyyx^j = yx^jx^iy$, and consequently, $x, y \in p^*$. Hence, x, y, z are powers of p. Hence, all solutions of the system are periodic. △

Let us look next at systems where at least in one equation to the left or to the right of the unknown y we have an expression over both x and z.

Lemma 4.21. *Consider a system of the form*

$$\begin{cases} x^iyp_1(x,z)z = zq_1(x,z)yx^j \\ x^iyx^{j-i}z = zyx^j \end{cases}$$

where $q_1(x, z)$ contains x, $p_1(x, z)$ is a word on x and z, $i \geq 2$, $j \geq 1$. Then,

- *either all solutions of the system are periodic*
- *or each of its non-periodic solutions satisfies $x^iyx^j = z$.*

Proof. Assume that x, y, z is a non-periodic solution of the system in lemma. Suppose that x and z are powers of the same primitive word p. Since y is not a power of this word, both equations are trivial identities on y and p. Hence, from the second equation, we obtain $z = x^i$. Putting it to the first equation we obtain $yp_1(x, x^i)x^i = q_1(x, x^i)yx^j$. Since $q_1(x, z)$ contains x, it is a nontrivial identity on x and y. Hence, y is a power of p, which is a contradiction.

Now, we apply point 2 in Lemma 4.11. Hence, there is a non-periodic solution x', y, z'' of the system

$$\begin{cases} yp_1(x,z)z = z''x'x^{-1}q_1(x, x^{i-1}x'z'')yx^j \\ (x'z'')^kx'yx^{j-i}z = x'z''yx^j \end{cases}$$

where x stands for $(x'z'')^kx'$, for some $k \geq 0$ and z stands for $((x'z'')^kx')^{i-1}x'z''$.

Case $k > 0$. Then the system is in the form

$$\begin{cases} yp_1(x,z)z = z''x'x^{-1}q_1(x, x^{i-1}x'z'')yx^j \\ (x'z'')^{k-1}x'yx^{j-i}z = yx^j \end{cases}.$$

By Proposition 2.3 this system admits only periodic solutions; a contradiction.

Case $k = 0$. Then the system is in the form

$$\begin{cases} yp_1(x,z)z = z''x'x^{-1}q_1(x, x^{i-1}x'z'')yx^j \\ yx^{j-i}z = z''yx^j \end{cases}$$

where x stand for x' and z stands for $x'^{i-1}x'z''$. Hence, the system is in the form

$$\begin{cases} yp_1(x, x^iz'')x^iz'' = z''q_1(x, x^iz'')yx^j \\ yx^{j-i}z'' = z''yx^j \end{cases}.$$

We may apply Theorem 4.16 to the system consisting of the reversed equations. Since the solution x, y, z'' is non-periodic, then one of the possibilities (ii)–(iv) holds. First, we exclude (ii) and (iv); neither (ii) nor (iv) can hold since $q_1(x, x^iz'')$ contains x.

Assume (iii) holds. Then $p_1(x, x^iz'')$ starts with x^j and the set of non-periodic solutions of the system satisfies $yx^j = v(x, z'')$. Using it in the second equation we have $z''v(x, z'') = v(x, z'')x^{-i}z''$. It is an identity on x and z''. Hence, $v(x, z'') = z''^l$, for some $l \geq 1$.

We have $p_1(x, x^iz'') = x^jp'_1(x, z'')$. Using it and $yx^j = z''^l$ in the first equation we obtain $z''^lp'_1(x, z'')x^iz'' = z''q_1(x, x^iz'')z''^l$ which is an identity on x and z''. Since $q_1(x, x^iz'')$ starts with x, then $l = 1$. We have $yx^j = z''$. Hence, the words x, y, z satisfy $x^iyx^j = z$. △

The next result gives some other useful properties which will be useful later.

Lemma 4.22. *Let $i, j \geq 1$ and $q_1(x, z)$, $q_2(x, z)$, $p_1(x, z)$, $p_2(x, z)$ be words over $\{x, z\}$. Let $p_1(x, x^iz')x^i = x^jp'_1(x, z')$, $p_2(x, x^iz')x^i = x^jp'_2(x, z')$, and $v(x, z')$ be a nonempty word over $\{x, z'\}$. Let $v(x, z')p'_1(x, z')z' = z'q_1(x, x^iz')v(x, z')$ and $v(x, z')p'_2(x, z')z' = z'q_2(x, x^iz')v(x, z')$ be identities on x, z'. Then*

(i) *either $q_1(x, z) = q_2(x, z)$ and $p_1(x, z) = p_2(x, z)$,*
(ii) *or $v(x, z')$ starts and ends by z' and each occurrence of z' in $v(x, z')$ except the first one is preceded by x^i.*

Proof. Clearly $v(x, z')$ starts and ends by z'.

Let $p_1(x, z)z = x^{l_1}zp_1''(x, z)$, for some $l_1 \geq 0$ and $p_2(x, z)z = x^{l_2}zp_2''(x, z)$, for some $l_2 \geq 0$. Then, we obtain $x^{l_1+i}z'p_1''(x, x^iz')x^iz' = x^jp_1'(x, z')z'$ and $x^{l_2+i}z'p_2''(x, x^iz')x^iz' = x^jp_2'(x, z')z'$. Hence, $l_1 + i \geq j$ and $l_2 + i \geq j$.

Case $l_1 \geq j$ or $l_2 \geq j$. Assume that $l_1 \geq j$, the other case being symmetric. Then $x^{l_1-j}x^iz'p_1''(x, x^iz')x^iz' = p_1'(x, z')z'$. Each occurrence of z' in the left side is preceded by x^i. Hence, each occurrence of z' in the right-hand side is preceded by x^i.

The words $p_1'(x, z')z'$ and $z'q_1(x, x^iz')$ are conjugates. Each occurrence of z' in $p_1'(x, z')z'$ is preceded by x^i. Hence, $p_1'(x, z')z'$ starts by x and, consequently, $p_1'(x, z')z' \neq z'q_1(x, x^iz')$. Moreover the first letter z' in $z'q_1(x, x^iz')$ is preceded cyclically by x^i. Hence, $q_1(x, x^iz')$ ends by x^i.

Since $p_1'(x, z')z' \neq z'q_1(x, x^iz')$, we have $v(x, z') = (z'q_1(x, x^iz'))^kz'q_1'''(x, z')$ where $k \geq 0$ and $q_1'''(x, z')$ is a prefix of $q_1(x, x^iz')$. Since $q_1(x, x^iz')$ ends by x^i and each occurrence of z' in $q_1(x, x^iz')$ is preceded by x^i, each occurrence of z' in $v(x, z')$ except that the first one is preceded by x^i. Hence, (ii) holds.

Case $l_1 < j$ and $l_2 < j$. We have $l_1 + i \geq j$. Hence, $x^{l_1+i-j}z'p_1''(x, x^iz')x^iz' = p_1'(x, z')z'$. Each occurrence of z' except the first one in the left side of this equality is preceded by x^i. The first one is cyclically preceded by $z'x^{l_1+i-j}$ where $l_1 + i - j < i$. Since each occurrence of z' in $q_1(x, x^iz')$ is preceded by x^i and $z'q_1(x, x^iz')$ and $p_1'(x, z')z'$ are conjugates, then $z'q_1(x, x^iz')$ ends in $z'x^{l_1+i-j}$. Similarly we conclude that $z'q_2(x, x^iz')$ ends in $z'x^{l_2+i-j}$. We have

$$v(x, z') = (z'q_1(x, x^iz'))^{k_1}q_1'''(x, z')$$

and

$$v(x, z') = (z'q_2(x, x^iz'))^{k_2}q_2'''(x, z')$$

where $k_1, k_2 \geq 0$, $q_1'''(x, z')$ and $q_2'''(x, z')$ are prefixes of $z'q_1(x, x^iz')$ and $z'q_2(x, x^iz')$, respectively.

Claim. *The words $z'q_1(x, x^iz')$ and $z'q_2(x, x^iz')$ are primitive.*

Proof of the claim. Assume that the word $z'q_1(x, x^iz')$ is not primitive. Then $z'q_1(x, x^iz') = q'(x, z')^l$, for some $l \geq 2$. Clearly, $q'(x, z')$ starts with z' and it is a suffix of $z'q_1(x, x^iz')$. Hence it ends by $z'x^{l_1+i-j}$. Hence, $z'x^{l_1+i-j}z'$ is a subword of $q'(x, z')^2$, and consequently a subword of $z'q_1(x, x^iz')$. But each occurrence of z' in $z'q_1(x, x^iz')$ except the first one is preceded by x^i. A contradiction. Hence, $z'q_1(x, x^iz')$ is primitive. Similarly we prove that $z'q_2(x, x^iz')$ is primitive. $\triangle$

Subcase $k_1 = 0$ or $k_2 = 0$. Assume $k_1 = 0$, the proof for the other being symmetric. Then $v(x, z') = q_1'''(x, z')$ where $q_1'''(x, z')$ is a prefix of $z'q_1(x, x^iz')$. Hence, in $v(x, z')$ each z' except the first one is preceded by x^i. Hence, (ii) holds.

Subcase $k_1 > 0$ and $k_2 > 0$. We have

$$(z'q_1(x, x^iz'))^{k_1}q_1'''(x, z') = (z'q_2(x, x^iz'))^{k_2}q_2'''(x, z').$$

Since $v(x, z')$ ends in z', then both sides of the last equation end in z'. Hence, in the left-hand side the first block of at most $i - 1$ letters x from the right is x^{l_1+i-j} which is a suffix of $z'q_1(x, x^iz')$. Similarly, in the right-hand side the first block of at most $i - 1$ letters x from the right is x^{l_2+i-j}. Hence, $q_1'''(x, z') = q_2'''(x, z')$ and $l_1 = l_2$ and

$$(z'q_1(x, x^iz'))^{k_1} = (z'q_2(x, x^iz'))^{k_2}.$$

But now, the claim implies that $z'q_1(x, x^iz') = z'q_2(x, x^iz')$ and $k_1 = k_2$. Since the set $\{x, x^iz'\}$ is a code, we immediately obtain that $q_1(x, z) = q_2(x, z)$.

Since $v(x, z')p_1'(x, z')z' = z'q_1(x, x^iz')v(x, z')$ and

$$v(x, z')p_2'(x, z')z' = z'q_2(x, x^iz')v(x, z') = z'q_1(x, x^iz')v(x, z')$$

we may equalize the left-hand sides and obtain $p_1'(x, z') = p_2'(x, z')$. Since additionally $l_1 = l_2$, we have $p_1''(x, x^iz) = p_2''(x, x^iz)$. Since $\{x, x^iz\}$ is a code, then $p_1''(x, z) = p_2''(x, z)$ and finally $p_1(x, z) = p_2(x, z)$. Hence, (i) holds. $\triangle$

Now we are finally ready to look at systems of two equations of the form given in Case 5, where, moreover, the left side of each equation starts with at least two x's and at least one right side starts with an expression over both x and z.

Lemma 4.23. *Consider a system of two balanced equations*

$$\begin{cases} x^iyp_1(x, z)z = zq_1(x, z)yx^j \\ x^iyp_2(x, z)z = zq_2(x, z)yx^j \end{cases}$$

where $p_1(x, z)$, $p_2(x, z)$, $q_1(x, z)$ and $q_2(x, z)$ are words over x, z, $q_1(x, z)$ contains x and $i \geq 2$, $j \geq 1$. Then

(i) *either all solutions of the system are quasi-periodic,*
(ii) *or the system is dependent and $p_1(x, z) = p_2(x, z)$ and $q_1(x, z) = q_2(x, z)$*
(iii) *or all of its nonquasi-periodic solutions satisfy one equation $x^iyx^j = v(x, z)$ where $v(x, z)$ starts and ends by z.*
(iv) *or $p_1(x, z)$ and $p_2(x, z)$ start with x^j and $q_1(x, z)$ and $q_2(x, z)$ end by x^i.*

Proof. Since the equations are balanced, if $q_2(x, z)$ is empty, then $j \geq i$ and the system is of the form

$$\begin{cases} x^i y p_1(x, z) z = z q_1(x, z) y x^j \\ x^i y x^{j-i} z = z y x^j \end{cases}.$$

By Lemma 4.21 all non-periodic solutions satisfy $x^i y x^j = z$. Hence, (iii) holds.

Let x, y, z be a nonquasi-periodic solution of the system in the lemma. Since x and z are not powers of the same word, $q_1(x, z)$ contains x, and $q_2(x, z)$ is nonempty, we may apply Lemma 4.11.

Then there is a non-periodic solution x', y, z'' satisfying $x = (x'z'')^k x'$, $z = ((x'z'')^k x')^{i-1} x'z''$, for some $k \geq 0$, of the system

$$\begin{cases} y p_1(x, z) z = z'' x' x^{-1} q_1(x, x^{i-1} x'z'') y x^j \\ y p_2(x, z) z = z'' x' x^{-1} q_2(x, x^{i-1} x'z'') y x^j \end{cases}$$

where x stands for $(x'z'')^k x'$ and z stands for $((x'z'')^k x')^{i-1} x'z''$.

If $k > 0$, then the left side of the first equation ends by $x'z''$ while the right side ends with $z''x'$. Hence, $x'z'' = z''x'$, i.e., x' and z'' are powers of the same word, say p. Hence, the first equation is a nontrivial identity on p and y. Consequently, y is a power of p, which is a contradiction.

If $k = 0$, then $x = x'$ and there is a non-periodic solution x, y, z'' satisfying $z = x^i z''$ of the system

$$\begin{cases} y p_1(x, z) z = z'' q_1(x, x^i z'') y x^j \\ y p_2(x, z) z = z'' q_2(x, x^i z'') y x^j \end{cases}$$

where z stands for $x^i z''$.

We may now apply Theorem 4.16 to the system consisting of the reversed equations. Since the last system has a non-periodic solution one of (ii)–(iv) holds in Theorem 4.16.

If (ii) holds in Theorem 4.16, then $p_1(x, z) = p_2(x, z)$ and $q_1(x, x^i z'') = q_2(x, x^i z'')$. The word $w(x, z'') = q_1(x, x^i z'') = q_2(x, x^i z'')$ has a double factorization in words from the two-element suffix code $\{x, z^i z''\}$ defined by $q_1(x, x^i z'')$ and $q_2(x, x^i z'')$. Hence, it is the same factorization, i.e., $q_1(x, z) = q_2(x, z)$ and thus (ii) holds.

If (iii) holds in Theorem 4.16, then we have $p_1(x, x^i z'') x^i = x^j p_1'(x, z'')$ and $p_2(x, x^i z'') x^i = x^j p_2'(x, z'')$ and $y x^j = v(x, z'')$. Moreover, when we replace these equalities in the two equations, we also obtain that $v(x, z'') p_1'(x, z'') z'' = z'' q_1(x, x^i z'') v(x, z'')$ and $v(x, z'') p_2'(x, z'') z'' = z'' q_2(x, x^i z'') v(x, z'')$ are identities on x, z''. We apply Lemma 4.22 obtaining either (ii) or in $v(x, z'')$ each z'' except the first one is preceded by x^i. Hence, in the right side of $x^i y x^j = x^i v(x, z'')$ each z'' is preceded by x^i. Hence, $x^i v(x, z'') = v'(x, x^i z'')$ for some unique v'. Since the set $\{x, x^i z''\}$ is a code, the words x, y, z satisfy $x^i y x^j = v'(x, z)$. Since z'' is a prefix and a suffix of $v(x, z'')$, the word $v'(x, z)$ starts and ends by z. Consequently, (iii) holds.

If (iv) holds in Theorem 4.16, then we have $p_1(x, x^i z'') x^i z'' = x^j p'(x, z'')^{r_1}$, $p_2(x, x^i z'') x^i z'' = x^j p'(x, z'')^{r_2}$, $z'' q_1(x, x^i z'') = q'(x, z'')^{l_1}$, $z'' q_2(x, x^i z'') = q'(x, z'')^{l_2}$, for some primitive $p'(x, z'')$, $q'(x, z'')$ and $l_1, l_2, r_1, r_2 \geq 1$ satisfying $l_1/gcd(l_1, r_1) = l_2/gcd(l_2, r_2)$ and $r_1/gcd(l_1, r_1) = r_2/gcd(l_2, r_2)$. Assume $l_1 > l_2$ the other case being symmetric. Then, we also have $r_1 > r_2$, and thus $l_1, r_1 \geq 2$.

Let $p_1(x, z) z = x^l z p_1'(x, z)$, for some $l \geq 0$. Consequently, $p'(x, z'')$ starts with $x^{l+i-j} z''$. Since $p'(x, z'')$ ends with z'' the word $z'' x^{l+i-j} z''$ is a subword of $p'(x, z'')^2$ and, consequently, of $p'(x, z'')^{r_1}$. In $p_1(x, x^i z'') x^i z''$ each z'' is preceded by x^i. Hence, in $p'(x, z'')^{r_1}$ each z'' except possibly the first one is preceded by x^i. Hence, $l + i - j \geq i$ and, consequently, $l \geq j$.

Since $p'(x, z'')$ starts with $x^{l+i-j} z''$, then $p_2(x, x^i z'') x^i z''$ starts with $x^{l+i} z''$, where $l \geq j$. Suppose that $p_2(x, z) z = x^m z p_2'(x, z)$, for some $m \geq 0$. Then $p_2(x, x^i z'') x^i z''$ starts with $x^{m+i} z''$. Hence, $m + i = l + i$ and, consequently, $m = l \geq j$.

By $z'' q_1(x, x^i z'') = q'(x, z'')^{r_1}$, $q'(x, z'')$ starts with z''. Since each z'' in $z'' q_1(x, x^i z'')$ except the first one is preceded by x^i, then $q'(x, z'')^{r_1 - 1}$ in $q'(x, z'')^{r_1}$ is preceded by x^i. Hence, $q'(x, z'')$ ends in x^i and, consequently, x^i ends $q_1(x, x^i z'')$ and $q_2(x, x^i z'')$. Then $q_1(x, z)$ and $q_2(x, z)$ end in x^i. Hence, (iv) holds. △

The following two results consider the case when the left sides of the two equations start with only one x, which is immediately followed by y. First, we look at systems where on the right side of one equation we have only z's occurring before y while on the other equation we have an expression over both x and z.

Lemma 4.24. *Consider the system of two balanced equations*

$$\begin{cases} x y p_1(x, z) z = z q_1(x, z) y x^j \\ x y p_2(x, z) z = z^{j_2+1} y x^j \end{cases}$$

where $q_1(x, z)$ contains x, $p_1(x, z)$, $p_2(x, z)$ are words over x and z, $j \geq 1$, $j_2 \geq 0$. Then,

- *either all solutions of the system are periodic*
- *or all of its non-periodic solutions satisfy $x y x^j = z^{j_2+1}$.*

Proof. Assume that x, y, z is a non-periodic solution of the system in lemma. Suppose that x and z are powers of the same primitive word p. Since y is not a power of this word, both equations are trivial identities on y and p. Hence, from the second equation, we obtain $x = z^{j_2+1}$. Putting it to the first equation we obtain $z^{j_2} y p_1(z^{j_2+1}, z) = q_1(z^{j_2+1}, z) y z^{(j_2+1)j-1}$. Since $q_1(x, z)$ contains x, $q_1(z^{j_2+1}, z)$ contains at least $j_2 + 1$ characters z and thus the equation is a nontrivial identity on z and y. Hence, y is a power of p, which is a contradiction.

Now, we apply point 2 in Lemma 4.12. Hence, $q_1(x, z) = z^{j_1}xq'_1(x, z)$ and there is a non-periodic solution x', y, z' of the system

$$\begin{cases} yp_1(x, z)z = z'z^{j_1}x'q'_1(x, z)yx^j \\ (x'z')^k x'yp_2(x, z)z = (x'z')^{j_2+1}yx^j \end{cases}$$

where x stands for $(x'z')^k x'$, for some $k \ge 0$ and z stands for $x'z'$.

Case $k \ge j_2 + 1$. Then the system looks like

$$\begin{cases} yp_1(x, z)z = z'z^{j_1}x'q'_1(x, z)yx^j \\ (x'z')^{k-j_2-1}x'yp_2(x, z)z = yx^j \end{cases}$$

where x stands for $(x'z')^k x'$ and z stands for $x'z'$. By Proposition 2.3 this system admits only periodic solutions, which is a contradiction.

Case $k \le j_2$. Then the system looks like

$$\begin{cases} yp_1(x, z)z = z'z^{j_1}x'q'_1(x, z)yx^j \\ yp_2(x, z)z = z'z^{j_2-k}yx^j \end{cases}.$$

where x stands for $(x'z')^k x'$ and z stands for $x'z'$.

Suppose that $k > 0$. Then $x'z'$ is a suffix of the left side while $z'x'$ is a suffix of the right side. Hence, x' and z' are powers of the same primitive word say p. Hence, the first equation is a nontrivial identity on p and y. Consequently, y is also a power of p, which is a contradiction.

Thus, $k = 0$ and our system looks like

$$\begin{cases} yp_1(x, z)z = z'z^{j_1}xq'_1(x, z)yx^j \\ yp_2(x, z)z = z'z^{j_2}yx^j \end{cases}$$

where z stands for xz'.

We may apply Theorem 4.16 to the system which consists of reversed equations. Since the solution x, y, z' is non-periodic, then one of the possibilities (ii-iv) holds.

Assume that (ii) holds. Then

$$z'(xz')^{j_2} = z'(xz')^{j_1}xq'_1(x, xz')$$

is an identity on x and z'. Notice now that either $q'_1(x, xz')$ is empty or it starts with x. In the former case the left side ends by z' while the right side ends by x. In the latter case xx is a subword of the right side while it is not a subword of the left side. In both cases we raise a contradiction.

Assume (iii) holds. Then $p_1(x, xz')x$ and $p_2(x, xz')x$ starts with x^j and the system is equivalent to $yx^j = v(x, z')$. Then $p_1(x, xz')x = x^j p'_1(x, z')$ and $p_2(x, xz')x = x^j p'_2(x, z')$. Hence, if $p'_2(x, z')$ is not empty, then it ends in x. Moreover in $p'_2(x, z')$ before each z' except possibly the first one we have an x. In particular $z'z'$ is not a subword of $p'_2(x, z')$.

We have that $v(x, z')p'_2(x, z')z' = z'(xz')^{j_2}v(x, z')$ is an identity on x and z'. Hence, $p'_2(x, z')z'$ and $z'(xz')^{j_2}$ are conjugates. So, either $p'_2(x, z')$ is empty or it ends in x. Since $z'z'$ is not a subword of $p'_2(x, z')$ we have $p'_2(x, z') = 1$, if $j_2 = 0$ or $p'_2(x, z') = z'(xz')^{j_2-1}x = (z'x)^{j_2}$, if $j_2 > 0$. In both cases $p'_2(x, z') = (z'x)^{j_2}$.

Putting the last dependencies to $v(x, z')p'_2(x, z')z' = z'(xz')^{j_2}v(x, z')$ we obtain in both cases $v(x, z')z'(xz')^{j_2} = z'(xz')^{j_2}v(x, z')$. Hence $v(x, z') = (z'(xz')^{j_2})^l$, for some $l \ge 1$. We put it into the first equation and obtain

$$(z'(xz')^{j_2})^l p'_1(x, z')z' = z'(xz')^{j_1}xq'_1(x, xz')(z'(xz')^{j_2})^l.$$

Assume first that $q'_1(x, xz')$ is empty. Then

$$(z'(xz')^{j_2})^l p'_1(x, z')z' = z'(xz')^{j_1}x(z'(xz')^{j_2})^l.$$

If $l > 1$, then the longest prefix of the left side which does not contain $z'z'$ is $z'(xz')^{j_2}$ while the longest prefix of the right side which does not contain $z'z'$ is $z'(xz')^{j_1}xz'(xz')^{j_2}$. The second prefix is longer than the first one, which raises a contradiction. Hence, $l = 1$.

Assume now that $q'_1(x, xz')$ is not empty. Then it starts with x and just before it, we have another occurrence of x. The first occurrence of xx in the left side is inside $p'_1(x, z')$. Hence, $(z'(xz')^{j_2})^l$ is a prefix of $(z'x)^{j_1+1}$, and, consequently $l = 1$.

We have $v(x, z') = z'(xz')^{j_2}$. Hence, the words x, y, z' satisfy $yx^j = z'(xz')^{j_2}$. Hence, the words x, y, z satisfy $xyx^j = z^{j_2+1}$. Hence, point 2 of the lemma holds.

Assume now that, in Theorem 4.16, (iv) holds. We have $z'(xz')^{j_2} = q'(x, z')^{l_2}$ and $z'(xz')^{j_1}xq'_1(x, xz') = q'(x, z')^{l_1}$, for some primitive $q'(x, z')$ and $l_1, l_2 \ge 1$. Since $z'(xz')^{j_2}$ is primitive, then $l_2 = 1$ and, consequently, $z'(xz')^{j_1}xq'_1(x, xz')$ is a power of $z'(xz')^{j_2}$.

Observe here that each power of $z'(xz')^{j_2}$ ends in z'. Hence, $q'_1(x, z')$ cannot be empty. Hence, $q'_1(x, xz')$ starts with x and, consequently, xx is a subword of $z'(xz')^{j_1}xq'_1(x, xz')$. However xx is not a subword of any power of $z'(xz')^{j_2}$ raising a contradiction. △

Next, we take the case when the right side of at least one equation starts with an expression over both x and z.

Lemma 4.25. *Consider a system of two balanced equations*

$$\begin{cases} xyp_1(x,z)z = zq_1(x,z)yx^j \\ xyp_2(x,z)z = zq_2(x,z)yx^j \end{cases}$$

where $p_1(x,z)$, $p_2(x,z)$, $q_1(x,z)$ and $q_2(x,z)$ are words over x and z, $q_1(x,z)$ contains x and $j \geq 1$. Then

(i) *either all solutions of the system are periodic,*
(ii) *or the system is dependent and $p_1(x,z) = p_2(x,z)$ and $q_1(x,z) = q_2(x,z)$*
(iii) *or all of its non-periodic solutions satisfy one equation $xyx^j = v(x,z)$ where $v(x,z)$ starts and ends by z.*
(iv) *or $p_1(x,z)$ and $p_2(x,z)$ start by x^j and $q_1(x,z)$ and $q_2(x,z)$ end by x.*

Proof. If $q_2(x,z)$ does not contain x, then we apply Lemma 4.24. Hence, (i) or (iii) holds.

If $q_2(x,z)$ contains x, then we can write $q_2(x,z) = z^{j_2}xq_2'(x,z)$, for some $j_2 \geq 0$. Since $q_1(x,z)$ contains x, we also have $q_1(x,z) = z^{j_1}xq_1'(x,z)$, for some $j_1 \geq 0$.

Assume that x, y, z is a non-periodic solution of the system in the lemma. Assume that x and z are powers of a primitive word p. Then, since $q_1(x,z)$ contains x, $|q_1(x,z)| > |x|$ and the first equation is a nontrivial identity on p and y. Hence, y is a power of p, which is a contradiction.

Hence, x and z are not powers of the same word. We apply point 1 of Lemma 4.12. Then, there is a nonperiodic solution x', y, z' of the system

$$\begin{cases} yp_1(x,z)z = z'z^{j_1}x'q_1'(x,z)yx^j \\ yp_2(x,z)z = z'z^{j_2}x'q_2'(x,z)yx^j \end{cases}$$

where x stands for $(x'z')^kx'$ and z stands for $x'z'$.

If $k > 0$, then $x'z'$ is a suffix of the left side of the first equation and $z'x'$ is a suffix of the right side of the first equation. Hence, x' and z' are powers of the same primitive word say p. Since the sides of the first equation start by z' and y, the first equation is a nontrivial identity on p and y. Hence, y is a power of p; a contradiction.

If $k = 0$, then $x = x'$ and the system is of the form

$$\begin{cases} yp_1(x,z)z = z'z^{j_1}xq_1'(x,z)yx^j \\ yp_2(x,z)z = z'z^{j_2}xq_2'(x,z)yx^j \end{cases}$$

where z stands for xz'.

We apply Theorem 4.16 to the system consisting of reversed equations. Since the last system has a non-periodic solution, one of (ii-iv) holds in Theorem 4.16.

Assume (ii) in Theorem 4.16 holds. Then $p_1(x,xz') = p_2(x,xz')$. Since the set $\{x, xz'\}$ is a code, then $p_1(x,z) = p_2(x,z)$. Hence, the left sides of the equations in the system in the lemma are the same. Consequently, $q_1(x,z) = q_2(x,z)$. Now, either this is an identity on x and z and thus (ii) holds, or it is a nontrivial equation. Hence, x and z are powers of the same word which raises a contradiction.

Assume (iii) in Theorem 4.16 holds. Then $p_1(x,xz')x = x^jp_1'(x,z')$ and $p_2(x,xz')x = x^jp_2'(x,z')$ and $yx^j = v(x,z')$. Moreover $v(x,z')p_1'(x,z')z' = z'q_1(x,xz')v(x,z')$ and $v(x,z')p_2'(x,z')z' = z'q_2(x,xz')v(x,z')$ are identities on x, z'. We apply Lemma 4.22 obtaining either (ii) or $yx^j = v(x,z')$ where $v(x,z')$ starts and ends by z' and each z' except the first one is preceded by x. Hence, on the right side of $xyx^j = xv(x,z')$ each z' is preceded by x. Hence, $xv(x,z') = v'(x,xz')$ for some unique v'. Since the set $\{x, xz'\}$ is a code, the words x, y, z satisfy $xyx^j = v'(x,z)$. Since z' is a prefix and a suffix of $v(x,z')$, the word $v'(x,z)$ starts and ends by z. Consequently, (iii) holds.

If (iv) holds in Theorem 4.16, then we have $p_1(x,xz')xz' = x^jp'(x,z')^{r_1}$, $p_2(x,xz')xz' = x^jp'(x,z')^{r_2}$, $z'q_1(x,xz') = q'(x,z')^{l_1}$, $z'q_2(x,xz') = q'(x,z')^{l_2}$, for some primitive $p'(x,z')$, $q'(x,z')$ and $l_1, l_2, r_1, r_2 \geq 1$ satisfying the relations $l_1/gcd(l_1,r_1) = l_2/gcd(l_2,r_2)$ and $r_1/gcd(l_1,r_1) = r_2/gcd(l_2,r_2)$. Assume $l_1 > l_2$ the other case being symmetric. Then $r_1 > r_2$. Hence, $l_1, r_1 \geq 2$.

Let $p_1(x,z)z = x^lzp_1'(x,z)$, for some $l \geq 0$. Consequently, $p'(x,z')$ starts with $x^{l+1-j}z'$. Since $p'(x,z')$ ends with z' the word $z'x^{l+1-j}z'$ is a subword of $p'(x,z')^2$ and, consequently, of $p'(x,z')^{r_1}$. In $p_1(x,xz')xz'$ each z' is preceded by x. Hence, in $p'(x,z')^{r_1}$ each z' except possibly the first one is preceded by x. Hence, $l+1-j \geq 1$ and, consequently, $l \geq j$.

Since $p'(x,z')$ starts with $x^{l+1-j}z'$, then $p_2(x,xz')xz'$ starts with $x^{l+1}z'$, where $l \geq j$. Suppose that $p_2(x,z)z = x^mzp_2'(x,z)$, for some $m \geq 0$. Then $p_2(x,xz')xz'$ starts with $x^{m+1}z'$. Hence, $m+1 = l+1$ and, consequently, $m = l \geq j$.

By $z'q_1(x,xz') = q'(x,z')^{l_1}$, $q'(x,z')$ starts with z'. Since each z' in $z'q_1(x,xz')$ except the first one is preceded by x, then $q'(x,z')^{l_1-1}$ in $q'(x,z')^{l_1}$ is preceded by x. Hence, $q'(x,z')$ ends in x and, consequently, both $q_1(x,xz')$ and $q_2(x,xz')$ end in x. Thus $q_1(x,z)$ and $q_2(x,z)$ end in$x$, i.e., (iv) holds. △

Next, we consider a system of two balanced equations of the type given by our Case 5.

Lemma 4.26. *Consider a system of two balanced equations*

$$\begin{cases} x^i y p_1(x,z) z = z q_1(x,z) y x^j \\ x^i y p_2(x,z) z = z q_2(x,z) y x^j \end{cases}$$

where $p_1(x,z)$, $p_2(x,z)$, $q_1(x,z)$ and $q_2(x,z)$ are words over x, z and $i \geq 1$, $j \geq 1$. Then

(i) *either all solutions of the system are quasi-periodic,*
(ii) *or the system is dependent and $p_1(x,z) = p_2(x,z)$ and $q_1(x,z) = q_2(x,z)$,*
(iii) *or all of its nonquasi-periodic solutions satisfy one equation $x^i y x^j = v(x,z)$ where $v(x,z)$ starts and ends by z,*
(iv) *or the system is dependent and $zq_1(x,z) = (q'(x,z))^{r_1} x^i$, $zq_2(x,z) = (q'(x,z))^{r_2} x^i$, $p_1(x,z) = x^j (p'(x,z))^{l_1}$ and $p_2(x,z) = x^j (p'(x,z))^{l_2}$, for some primitive $q'(x,z)$, $p'(x,z)$ and $l_1, l_2, r_1, r_2 \geq 1$ satisfying $l_1/\gcd(l_1,r_1) = l_2/\gcd(l_2,r_2)$ and $r_1/\gcd(l_1,r_1) = r_2/\gcd(l_2,r_2)$.*

Proof. Suppose the system admits a nonquasi-periodic solution x, y, z. Then x and z are not powers of the same word.

Assume that $q_1(x,z)$, $q_2(x,z)$, $p_1(x,z)$, $p_2(x,z)$ do not contain x. Then, the system is of the form

$$\begin{cases} x^i y z^{l_1} = z^{l_1} y x^j \\ x^i y z^{l_2} = z^{l_2} y x^j. \end{cases}$$

We apply Lemma 4.20. Since the solution x, y, z is non-periodic, then $l_1 = l_2$, and consequently $p_1(x,z) = p_2(x,z)$ and $q_1(x,z) = q_2(x,z)$.

If $q_1(x,z)$ and $q_2(x,z)$ do not contain x, we consider the system of reversed equations. Hence, we may assume that $q_1(x,z)$ or $q_2(x,z)$ contains x. Assume $q_1(x,z)$ contains x, the other case being symmetric. We apply Lemmata 4.25 and 4.23. If in these lemmata (i)–(iii) hold, then (i)–(iii) hold.

If in these lemmata (iv) holds, then $p_1(x,z)$ and $p_2(x,z)$ start by x^j and $q_1(x,z)$ and $q_2(x,z)$ end by x^i. Hence, $p_1(x,z) = x^j p_1'(x,z)$ and $p_2(x,z) = x^j p_2'(x,z)$ and $q_1(x,z) = q_1'(x,z) x^i$ and $q_2(x,z) = q_2'(x,z) x^i$. Hence, the system from the lemma looks like

$$\begin{cases} x^i y x^j p_1'(x,z) z = z q_1'(x,z) x^i y x^j \\ x^i y x^j p_2'(x,z) z = z q_2'(x,z) x^i y x^j \end{cases}$$

where $i, j \geq 1$. If we substitute $y' = x^i y x^j$, then we may apply Theorem 4.1.

If (i) holds in Theorem 4.1, then, since $y' = x^i y x^j$, then all solutions of the system in the lemma are periodic. Hence, (i) holds.

If (ii) holds in Theorem 4.1, then (iv) holds.

If (iii) holds in Theorem 4.1, then (iii) holds. Observe additionally, that, since $v(x,z) p_1'(x,z) z = z q_1'(x,z) v(x,z)$ is an identity, $v(x,z)$ has to start and end by z. △

Next, we refine Lemma 4.26 by strengthening case (iii) and prove the following result for systems of two equations of the type considered in Case 5. Recall that, for now, we consider only nonempty solutions.

Theorem 4.27. *Consider a system of two balanced equations*

$$\begin{cases} x^i y p_1(x,z) z = z q_1(x,z) y x^j \\ x^i y p_2(x,z) z = z q_2(x,z) y x^j \end{cases}$$

where $p_1(x,z)$, $p_2(x,z)$, $q_1(x,z)$ and $q_2(x,z)$ are words over x, z and $i \geq 1$, $j \geq 1$. Then

(i) *either all solutions of the system are quasi-periodic,*
(ii) *or the system is dependent and $p_1(x,z) = p_2(x,z)$ and $q_1(x,z) = q_2(x,z)$,*
(iii) *or all of its non-periodic solutions satisfy one equation $x^i y x^j = v(x,z)$ where $v(x,z)$ starts and ends by z,*
(iv) *or the system is dependent and $zq_1(x,z) = (q'(x,z))^{r_1} x^i$, $zq_2(x,z) = (q'(x,z))^{r_2} x^i$, $p_1(x,z) = x^j (p'(x,z))^{l_1}$ and $p_2(x,z) = x^j (p'(x,z))^{l_2}$, for some primitive $q'(x,z)$, $p'(x,z)$ and $l_1, l_2, r_1, r_2 \geq 1$ satisfying the relations $l_1/\gcd(l_1,r_1) = l_2/\gcd(l_2,r_2)$ and $r_1/\gcd(l_1,r_1) = r_2/\gcd(l_2,r_2)$.*

Proof. We apply Lemma 4.26. It is enough to prove that if the system is independent and has nonquasi-periodic solution, then it has no non-periodic quasi-periodic solutions. Suppose x, y, z is nonquasi-periodic solution. Then x and z are not powers of the same word. By Lemma 4.26, the words x, y, z satisfy $x^i y x^j = v(x,z)$ where $v(x,z)$ starts and ends with z. Then $y = x^{-i} v(x,z) x^{-j}$. Observe that, since $v(x,z)$ starts and ends with z, the expression on the right is irreducible. We apply it to the first equation. We have $v(x,z) x^{-j} p_1(x,z) z = z q_1(x,z) x^{-i} v(x,z)$. Since x and z are not powers of the same word this equation should be a trivial identity on x and z. Let $p_1(x,z) z = x^{l_1} z p_1'(x,z)$, for some $l_1 \geq 0$, and let $z q_1(x,z) = q_1'(x,z) z x^{r_1}$, for some $r_1 \geq 0$. Let $p_2(x,z) z = x^{l_2} z p_2'(x,z)$, for some $l_2 \geq 0$, and let $z q_2(x,z) = q_2'(x,z) z x^{r_2}$, for some $r_2 \geq 0$.

Case $l_1 \geq j$ or $l_2 \geq j$. Assume $l_1 \geq j$, the other case being symmetric. Then, after reducing x^{-j}, the expression on the left side of the identity does not contain x^{-1}. Hence, the expression on the right side does not contain x^{-1}. Consequently, $r_1 \geq i$. We have $p_1(x,z) = x^j p_1''(x,z)$ and $q_1(x,z) = q_1''(x,z) x^i$. The first equation in the system in the theorem looks like

$$x^i y x^j p_1''(x,z) z = z q_1''(x,z) x^i y x^j.$$

Since, for all nonempty x, z, $|x^i| < |zq''(x, z)x^i|$, this equation has no non-periodic quasi-periodic solutions. Hence, all of its quasi-periodic solutions are periodic.

Case $l_1 < j$ and $l_2 < j$. We have $v(x, z)x^{l_1-j}zp'_1(x, z) = q'_1(x, z)zx^{r_1-i}v(x, z)$. Since the expression on the left is irreducible, the expression on the right has to contain a factor x^{-1}. Hence, $i > r_1$. Moreover, since the expressions on the right and on the left are the same we have $v(x, z) = q'_1(x, z)z$, $j - l_1 = i - r_1$, and $zp'_1(x, z) = v(x, z)$. Hence, $p_1(x, z)z = x^{l_1}v(x, z)$ and $zq_1(x, z) = v(x, z)x^{r_1}$ where $j - l_1 = i - r_1$.

Similarly we conclude that $p_2(x, z)z = x^{l_2}v(x, z)$ and $zq_2(x, z) = v(x, z)x^{r_2}$ where $j - l_2 = i - r_2$. Consequently, the system in the lemma looks like

$$\begin{cases} x^iyx^{l_1}v(x, z) = v(x, z)x^{r_1}yx^j \\ x^iyx^{l_2}v(x, z) = v(x, z)x^{r_2}yx^j \end{cases}$$

where $j - l_1 = i - r_1$ and $j - l_2 = i - r_2$.

If $l_1 = l_2$, then $r_1 = r_2$, and, consequently, (ii) holds.

If $l_1 \neq l_2$, then for any nonempty solution $|x^{l_1}v(x, z)| \neq |x^{l_2}v(x, z)|$. If the system has a quasi-periodic non-periodic solution, then $|x^{l_1}v(x, z)| = |x^j| = |x^{l_2}v(x, z)|$. Hence, the system has no quasi-periodic non-periodic solution. Hence, (iii) holds. △

We are now ready to give the following theorem which solves systems of the form given by our Case 5 for nonempty solutions.

Theorem 4.28. *Consider nonempty solutions of the system of balanced equations*

$$\begin{cases} x^iyp_1(x, z) = zq_1(x, z)yx^j \\ x^iyp_2(x, z) = zq_2(x, z)yx^j \\ x^iyp_3(x, z) = zq_3(x, z)yx^j \end{cases}.$$

Then either the system admits only periodic solutions or it is dependent.

Proof. Observe first that, since the equations in the system in the theorem are balanced, each periodic triple is a solution of the system.

Let $x = p^s$, y, $z = p^r$ be an arbitrary quasi-periodic nonperiodic solution of the the first equation. Then s and r satisfy two linear equations. Since the first equation is balanced, the linear equations are dependent. Take one of the linear equations e_1. Observe that $r = 0$ and $s = 0$ is a solution of it.

Similarly, for the two other equations we build linear equations e_2 and e_3. The solvable system of linear equations $\{e_1, e_2, e_3\}$ is over two unknowns s and r. Hence, it is dependent.

This means that we may choose a subsystem of two equations of the system in theorem such that each quasi-periodic solution of the subsystem is a solution of the third equation. We apply Theorem 4.27 to this subsystem.

If (i) holds for this subsystem, then the whole system is dependent.

If (ii) holds for this subsystem, then the subsystem is dependent and, consequently, the whole system is dependent.

If (iii) holds for this subsystem, then we apply $y = x^{-i}v(x, z)x^{-j}$ to the third equation. If we obtain a nontrivial identity, then all non-periodic solutions of the whole system are quasi-periodic and they satisfy $y = x^{-i}v(x, z)x^{-j}$. Hence, they are periodic raising a contradiction.

If we obtain trivial identity, then each non-periodic solution of the subsystem is a solution of the whole system. As we noticed each periodic solution of the subsystem is a solution of the whole system. Hence, the system is dependent.

If (iv) holds for this subsystem, then the subsystem is dependent and, consequently, the whole system is dependent. △

Finally, we look at the set of solutions where at least one of the words x, z, or y is the empty word.

Theorem 4.29. *Consider the system of balanced equations*

$$\begin{cases} x^iyp_1(x, z) = zq_1(x, z)yx^j \\ x^iyp_2(x, z) = zq_2(x, z)yx^j \\ x^iyp_3(x, z) = zq_3(x, z)yx^j \end{cases}.$$

If one of the words x, z, or y is the empty word, then either the system admits only periodic solutions or all of its constituent equations accept exactly the same subset of solutions.

Proof. Now, if $x = 1$ or $y = 1$ then all three equations of the system become non-trivial identities over z and y or over x and z, respectively, which immediately implies that they are powers of a common word. Thus, the system admits only periodic solutions. So, let us look now at the case when $z = 1$. Then, the system becomes

$$\begin{cases} x^iyx^{j_1} = x^{i_1}yx^j \\ x^iyx^{j_2} = x^{i_2}yx^j \\ x^iyx^{j_3} = x^{i_3}yx^j \end{cases}$$

with $i_k, j_k \geq 0$ and $i + j_k = i_k + j$ for all $1 \leq k \leq 3$. If for some $1 \leq k \leq 3$, $i \neq i_k$ or $j \neq j_k$, then the correspondent equation is a non-trivial identity over x and y, and thus they must be powers of a common word. Thus, also in this case the system admits only periodic solutions. Otherwise, all three equations become $x^iyx^j = x^iyx^j$, i.e., they all accept the same subset of solutions with $z = 1$. △

As an immediate consequence of Theorems 4.28 and 4.29 we solve Case 5 of our analysis.

Corollary 4.30. *The system of balanced equations*

$$\begin{cases} x^i y p_1(x, z) = z q_1(x, z) y x^j \\ x^i y p_2(x, z) = z q_2(x, z) y x^j \\ x^i y p_3(x, z) = z q_3(x, z) y x^j \end{cases}$$

either admits only periodic solutions or is dependent.

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