Planar Graphs: Logical Complexity and Parallel Isomorphism Tests

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Abstract. We prove that every triconnected planar graph on n vertices is definable by a first order sentence that uses at most 15 variables and has quantifier depth at most $11 \log_2 n + 45$. As a consequence, a canonic form of such graphs is computable in AC^1 by the 14-dimensional Weisfeiler-Lehman algorithm. This gives us another AC^1 algorithm for the planar graph isomorphism.

1 Introduction

Let Φ be a first order sentence about graphs in terms of the adjacency and the equality relations. We say that Φ *distinguishes* a graph G from a graph H if Φ is true on G but false on H. We say that Φ *defines* G if it distinguishes G from every H non-isomorphic to G. The *logical depth* of a graph G, denoted by $D(G)$, is the minimum quantifier depth of a Φ defining G.

The k*-variable logic* consists of those first order sentences which use at most k variables (each of the k variables can occur a number of times). The *logical width* of a graph G, denoted by $W(G)$, is the minimum k such that G is definable by a Φ in the k-variable logic. If $k \geq W(G)$, let $D^k(G)$ denote the logical depth of G in the k-variable logic. Similarly, for non-isomorphic graphs G and H we let $D^k(G, H)$ denote the minimum quantifier depth of a k-variable sentence Φ distinguishing G from H.

The latter parameter is relevant to the Graph Isomorphism problem, namely, to the k*-dimensional Weisfeiler-Lehman algorithm* (see [1,5] for the description and history). Cai, Fürer, and Immerman [1] prove that, if $k \geq W(G) - 1$, then the output of this algorithm is correct for all input pairs (G, H). Furthermore, this condition on k is necessary if we consider the width of G in the logic with *counting quantifiers*. The latter parameter of G, as shown in [1], can be linear in the number of vertices.

Note that the k-dimensional Weisfeiler-Lehman algorithm is polynomial-time only if k is constant. Thus, the algorithm can be successful only for classes of graphs whose width in the logic with counting quantifiers is bounded by a constant. Cai, Fürer, and Immerman ask if this is the case for planar graphs. An affirmative answer is given by Grohe [3].

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In [5] we extend the approach to Graph Isomorphism suggested in [1] by taking into consideration not only the dimension but also the number of rounds performed by the Weisfeiler-Lehman algorithm. It turns out that the logarithmic-round k-dimensional Weisfeiler-Lehman algorithm is implementable in TC^1 and its count-free version even in AC^1. We apply this fact in [5] to show that the isomorphism problem for graphs of bounded treewidth is in TC^1 (earlier Grohe and Marino [4] proved that such graphs have bounded width in the logic with counting).

According to [5], to put the isomorphism problem for a class of graphs C in AC^1, it suffices to prove that, for a constant k, we have $D^k(G, G') = O(\log n)$ for all G and G' in C. We now apply this approach to planar graphs. Due to the efficient decomposability of graphs into triconnected components [8], it is enough to treat the class of triconnected planar graphs.[1]

Theorem 1. *Let G and G' be non-isomorphic triconnected planar graphs and let G have n vertices. Then $D^{15}(G, G') < 11 \log_2 n + 45$.*

Corollary 2. *The isomorphism problem for triconnected planar graphs is solvable in AC^1 by the logarithmic-round 14-dimensional Weisfeiler-Lehman algorithm.*

The seminal polynomial-time algorithm for this problem is designed by Hopcroft and Tarjan [6,7]. The first AC^1 algorithm follows from a work of Miller and Reif [11]. Another AC^1 algorithm is suggested in [5]. Both [11] and [5] start with AC^1 embedding of input graphs (as in [14]) and then use different methods to test isomorphism of the plane drawings. The new algorithm of Corollary 2 is combinatorially much simpler and more direct. In particular, we now do not need any embedding procedure. Curiously, the Weisfeiler-Lehman approach to Graph Isomorphism appeared a bit earlier even than [6,7] (cf. [15]), but only now we are able to establish that this method, and even its parallel version, works correctly for triconnected planar graphs.

With not so much extra work, we are able to strengthen Theorem 1.

Theorem 3. *For a triconnected planar graph G on n vertices we have $D^{15}(G) < 11 \log_2 n + 45$.*

In the framework of [5], this means that an appropriate modification of the logarithmic-round 14-dimensional Weisfeiler-Lehman algorithm computes a canonic form of a triconnected planar input graph, putting this problem in the class AC^1. Miller and Reif [11] show that the canonization of planar graphs AC^1-reduces to the triconnected case. Using this reduction, we hence obtain a new AC^1-algorithm for the planar graph isomorphism problem.

[1] Theorem 1 cannot be extended to biconnected planar graphs. For example, to distinguish between two complete bipartite graphs $K_{2,n-1}$ and $K_{2,n}$, we need to use n first order variables. We could try to extend Theorem 1 to all planar graphs by allowing counting quantifiers but this would require a further delicate analysis (and anyway would not lead us to Corollary 4 directly, since introducing counting quantifiers weakens an AC^1 bound to a TC^1 bound).

Corollary 4. *The canonization problem for planar graphs is solvable in* AC^1.

Theorem 3 is also a contribution in a recent line of research [10,12,13] devoted to a general study of the logical depth $D(G)$ as a mysterious graph invariant.

Sections 2 and 3 contain the necessary preliminaries. The proof of Theorem 1 takes Sections 4 and 5. Theorem 3 is proved in Section 6.

2 Ehrenfeucht-Fraïssé Games

Here we introduce the main technical tool for establishing first order definability properties of finite structures. Let G and G' be graphs with disjoint vertex sets. The *r-round k-pebble Ehrenfeucht-Fraïssé game on G and G'*, denoted by $\mathrm{Ehr}_r^k(G, G')$, is played by two players, Spoiler and Duplicator, with k pairwise distinct pebbles $p_1, \ldots, p_k$, each given in duplicate. Spoiler starts the game. A *round* consists of a move of Spoiler followed by a move of Duplicator. At each move Spoiler takes a pebble, say p_i, selects one of the graphs G or G', and places p_i on a vertex of this graph. In response Duplicator should place the other copy of p_i on a vertex of the other graph. It is allowed to move previously placed pebbles to another vertex and place more than one pebble on the same vertex.

After each round of the game, for $1 \le i \le k$ let x_i (resp. x'_i) denote the vertex of G (resp. G') occupied by p_i, irrespectively of who of the players placed the pebble on this vertex. If p_i is off the board at the moment, x_i and x'_i are undefined. If after each of r rounds the component-wise correspondence $(x_1, \ldots, x_k)$ to $(x'_1, \ldots, x'_k)$ is a partial isomorphism from G to G', this is a win for Duplicator; Otherwise the winner is Spoiler.

Let $\bar{v} = (v_1, \ldots, v_m)$ and $\bar{v}' = (v'_1, \ldots, v'_m)$ be sequences of vertices in, respectively, G and G' and let $m \le k$. We write $\mathrm{Ehr}_r^k(G, \bar{v}, G', \bar{v}')$ to denote the game that begins from the position where, for every $i \le m$, the vertices v_i and v'_i are already pebbled by p_i.

Proposition 5. (Immerman, Poizat, see [9, Theorem 6.10]) $D^k(G, G')$ *equals the minimum r such that Spoiler has a winning strategy in* $\mathrm{Ehr}_r^k(G, G')$.

All the above definitions and statements have a perfect sense for any kind of structures. Say, in Section 5 we deal with structures having ternary and quaternary relations. The notion of a partial isomorphism for such structures should be understood appropriately.

For our convenience, everywhere below it is assumed that vertex names correspond to pebbling; for example, vertices v in G and v' in G' are always under the same pebbles. Furthermore, we will write *Spoiler wins* with meaning that *Spoiler has a strategy winning against any Duplicator's strategy.*

3 Graph-Theoretic Notation and Definitions

The vertex set of a graph G is denoted by $V(G)$. The distance between vertices u and v is denoted by $d(u, v)$. If u and v are in different connected components, we set $d(u, v) = \infty$. The set $\Gamma(v) = \{ u : d(u, v) = 1\}$ is called the *neighborhood* of a vertex v in G and $\deg v = |\Gamma(v)|$ is the *degree* of v.

A graph is *k-connected* if it has at least $k+1$ vertices and remains connected after removal of any $k-1$ vertices. *Biconnected* and *triconnected* graphs correspond to $k = 2, 3$.

A *sphere graph* is a graph drawn in a sphere with no edge crossing. A *spherical embedding* of a graph G is an isomorphism from G to a sphere graph $\tilde{G}$. As very well known, a graph G is planar iff it has a spherical embedding. Two spherical embeddings $\sigma : G \to \tilde{G}$ and $\tau : G \to \hat{G}$ are equivalent if the isomorphism $\tau \circ \sigma^{-1}$ is induced by a homeomorphism of a sphere taking $\tilde{G}$ onto $\hat{G}$. A classical theorem of Whitney says that all spherical embeddings of a triconnected planar graph G are equivalent (see, e.g., [2]).

Throughout the paper $\log n$ denotes the binary logarithm. Unless stated otherwise, n will denote the number of vertices in a graph G.

4 Capturing Unique Embeddability by First Order Formalism

To prove Theorem 1, we have to design a strategy allowing Spoiler to win the Ehrenfeucht-Fraïssé game on non-isomorphic triconnected planar graphs G and G' with 15 pebbles in less than $11 \log n + 45$ rounds. A crucial fact on which the strategy will be based is the rigidity of triconnected planar graphs as stated in the Whitney theorem. In this section we aim at developing an important ingredient of the strategy forcing Duplicator to respect this rigidity at least locally.

A *configuration* C in a graph G is a set of labeled vertices of G. In fact, labels will be the pebbles in an Ehrenfeucht-Fraïssé game. At the same time we will often use a label as a name of a vertex. By an *X-configuration* we mean 5 pairwise distinct vertices labeled by x, y, u, v, and w such that $x, y, u, v \in \Gamma(w)$. By an *H-configuration* we mean 6 pairwise distinct vertices labeled by x, y, z, u, v, and w such that z and w are adjacent, $x, y \in \Gamma(z)$, and $u, v \in \Gamma(w)$. Thus, contraction of the edge $\{z, w\}$ makes an H-configuration an X-configuration. Suppose that G is a triconnected planar graph and consider its unique spherical embedding. We call an X-configuration C *collocated* if u, x, y, v occur around w exactly in this order (up to cyclic shifts and the direction of a roundabout way). We call an H-configuration C *collocated* if $xzwu$ and $yzwv$ are segments of the two facial cycles containing the edge $\{z, w\}$. A configuration obtained from a collocated X- or H-configuration by interchanging the labels x and y will be called a *twisted* configuration.

We will treat X- and H-configurations uniformly, setting $z = w$ for X-configurations. Whenever we use the term *configuration* alone, it will refer to any X- or H-configuration.

Lemma 6. *Let G and G' be triconnected planar graphs, G having n vertices. Let $C = \{x, y, z, u, v, w\}$ and $C' = \{x', y', z', u', v', w'\}$ be sets of pebbled vertices in, respectively, G and G' such that C is a collocated configuration and C' is a* *twisted configuration. Starting with this position, Spoiler wins the Ehrenfeucht-Fraïssé game on G and G' with 15 pebbles in less than* $6\log n + 26$ *moves.*

The proof of Lemma 6 is omitted due to space limitation and can be found in [16].

5 Proof of Theorem 1

Lemma 6 allows us to reduce the Ehrenfeucht-Fraïssé game on G and G' to the game on their spherical embeddings (which are unique by the Whitney theorem). We use two combinatorial specifications for the concept of an embedding. One is a standard notion of a *rotation system*. The other is a related, but in a sense "poorer", notion of a *layout system* (see Subsections 5.1 and 5.3 for the definitions). Denote the rotation and the layout systems for G and G' by R and R' and by L and L' respectively. As we proved in [5], every rotation system is succinctly definable, in particular, Spoiler has an efficient winning strategy in the Ehrenfeucht-Fraïssé game on R and R'. In Subsection 5.2 we will see that Spoiler can win the game on L and L' by emulating the game on R and R'. In its turn, Lemma 6 allows Spoiler to win the game on G and G' by emulating the game on L and L'. This emulation is presented in Subsection 5.4. After all these steps are made, the proof of Theorem 1 in Subsection 5.5 follows easily.

5.1 Two Specifications of a Graph Embedding

The following definitions are introduced for a connected graph G with minimum vertex degree at least 3.

A *rotation system* $R = \langle G, T\rangle$ is a structure consisting of a graph G and a ternary relation T on $V(G)$ satisfying the following conditions:

(1) If $T(a,b,c)$, then b and c are in $\Gamma(a)$, the neighborhood of a in G.

(2) For every a the binary relation $T_a(b,c) = T(a,b,c)$ is a directed cycle on $\Gamma(a)$ (i.e., for every b there is exactly one c such that $T_a(b,c)$, for every c there is exactly one b such that $T_a(b,c)$, and the digraph T_a is connected).

If G is embedded in a surface, it is supposed that T_a describes the circular order in which the edges of G incident to a occur if we go around a clockwise.

Given a rotation system $R = \langle G, T\rangle$, we define another rotation system $R^* = \langle G, T^*\rangle$ by $T^*_a(b,c) = T_a(c,b)$ and call it the *conjugate* of R. Geometrically, R^* is a variant of R if we look at R from the other side of the surface. Obviously, $(R^*)^* = R$.

A *layout system* $L = \langle G, T, Q\rangle$ is a structure consisting of a graph G and two relations on $V(G)$, ternary T and quaternary Q, satisfying the following conditions:

(1) If $T(a,b,c)$, then b and c are in $\Gamma(a)$. Furthermore, for every a the binary relation $T_a(b,c) = T(a,b,c)$ is an undirected cycle on $\Gamma(a)$ (that is, T_a is symmetric, irreflexive, and connected).

(2) If $Q(b_1,a_1,a_2,b_2)$, then b_1,a_1,a_2,b_2 is a path in G or, if $b_1 = b_2$, it is a cycle. Every pair (a_1,a_2) with a_1 and a_2 adjacent in G extends to exactly two

quadruples (b_1, a_1, a_2, b_2) and (c_1, a_1, a_2, c_2) satisfying Q. Moreover, for both $i = 1, 2$, the b_i and c_i are the neighbors of a_{3-i} in the cycle T_{a_i}, that is, $T(a_i, a_{3-i}, b_i)$ and $T(a_i, a_{3-i}, c_i)$ are both true.

Relations T and Q also have clear geometric meaning. Namely, T_a determines the (undirected) circular order in which the edges of G incident to a are embedded. Note that now we specify no clockwise (or counter-clockwise) direction around a. This is the point where a layout system deviates from a rotation system. Thus, if a vertex a_1 and its neighborhood are already embedded and a_2 is adjacent to a_1, we have still two different ways to embed the neighborhood of a_2. The proper choice is determined by Q. Namely, it is supposed that the facial cycle going via b_1, a_1, a_2 goes further via b_2 and the facial cycle going via c_1, a_1, a_2 goes further via c_2.

Given a rotation system $R = \langle G, T_R \rangle$, we associate with it a layout system $\mathsf{L}(R) = \langle G, T_L, Q \rangle$ according to the geometric meaning. Namely, T_L is defined by $T_L(a, b, c) = T_R(a, b, c) \vee T_R(a, c, b)$. To define Q, we first introduce the successor and the predecessor functions on $\Gamma(a)$, s_a and p_a, by the equalities $c = s_a(b)$ and $b = p_a(c)$ if $T_R(a, b, c) = 1$. Now we set the following two relations true: $Q(p_{a_1}(a_2), a_1, a_2, s_{a_2}(a_1))$ and $Q(s_{a_1}(a_2), a_1, a_2, p_{a_2}(a_1))$. As easily seen, $\mathsf{L}(R) = \mathsf{L}(R^*)$.

Let $L = \mathsf{L}(R)$. The following simple lemma says that the pair $\{R, R^*\}$ is reconstructible from L.

Lemma 7. *If $\mathsf{L}(R') = \mathsf{L}(R)$, then either $R' = R$ or $R' = R^*$.*

In fact, Lemma 7 is essentially strengthened below, see Lemma 10. The following result is obtained in [5, Theorem 10].

Theorem 8. *For a rotation system $R = \langle G, T \rangle$, we have $D^5(R) < 3 \log n + 8$.*

5.2 Reducing the Play on Layout Systems to the Play on Rotation Systems

Lemma 9. *Let $R = \langle G, T \rangle$ and $R' = \langle G, T' \rangle$ be rotation systems. Let $L = \mathsf{L}(R)$ and $L' = \mathsf{L}(R')$. Suppose that, while $T(a_1, b_1, c_1) = T(a_2, b_2, c_2) = 1$ in R, in R' we have $T'(a'_1, b'_1, c'_1) = T'(a'_2, c'_2, b'_2) = 1$. Then Spoiler wins $\mathrm{Ehr}^9_{2 \log n + 4}(L, a_1, b_1, c_1, a_2, b_2, c_2, L', a'_1, b'_1, c'_1, a'_2, b'_2, c'_2)$.*

Proof. Case 1: $a_1 = a_2 = a$. Correspondingly, suppose that $a'_1 = a'_2 = a'$. The case that $\{b_1, c_1\}$ and $\{b_2, c_2\}$ intersect is simple; we hence suppose that all these vertices are pairwise distinct. Spoiler restricts play to the graphs $T_a \setminus \{b_1, b_2\}$ and $T'_{a'} \setminus \{b'_1, b'_2\}$, where T_a and $T'_{a'}$ denote undirected cycles in the structures L and L'. In these graphs $d(c_1, c_2) = \infty$ while $d(c'_1, c'_2) < \infty$ and hence Spoiler wins in less than $\log \deg a + 1$ moves using the standard halving strategy.

Case 2: a_1 and a_2 are adjacent. It suffices to consider a special subcase where $b_1 = a_2$ and $b_2 = a_1$. Spoiler can force either this subcase or Case 1 in 2 extra moves. By the definition of $\mathsf{L}(R)$, we have $Q(c_1, a_1, a_2, c_2) = 0$ whereas $Q'(c'_1, a'_1, a'_2, c'_2) = 1$, which is a win for Spoiler.

Case 3: $d(a_1, a_2) \geq 2$. Spoiler reduces this case to Case 2 in $\lceil \log d(a_1, a_2) \rceil$ moves. He first pebbles a vertex a_3 on the midway between a_1 and a_2 and then two more vertices b_3, c_3 so that $T(a_3, b_3, c_3) = T(a_i, b_i, c_i)$, $i = 1, 2$. For Duplicator's response a'_3, b'_3, c'_3, assume that one of the relations $T'(a'_3, b'_3, c'_3)$ or $T'(a'_3, c'_3, b'_3)$ is true for else Spoiler has already won. We have either $T'(a'_3, b'_3, c'_3) \neq T'(a'_1, b'_1, c'_1)$ or $T'(a'_3, b'_3, c'_3) \neq T'(a'_2, b'_2, c'_2)$. In either case, one of the tuples $(a_i, b_i, c_i, a_3, b_3, c_3)$, for $i = 1$ or $i = 2$, is similar to the initial position, while the distance between the two a-vertices has decreased. Spoiler just iterates this trick sufficiently many times.

Let $W(S, S')$ denote the minimum k such that non-isomorphic structures S and S' are distinguishable in the k-variable logic.

Lemma 10. *Let $R = \langle G, T' \rangle$ and $R' = \langle G, T \rangle$ be rotation systems such that neither $R' \cong R$ nor $R' \cong R^*$. Suppose that $m \geq \max\{W(R, R'), W(R^*, R')\}$ and set $k = 3 + \max\{m, 6\}$. Let $L = \mathsf{L}(R)$ and $L' = \mathsf{L}(R')$. Then*

$$D^k(L, L') \leq \max\{D^m(R, R'), D^m(R^*, R')\} + 2 \log n + 7.$$

Proof. We design a strategy for Spoiler in $\mathrm{Ehr}^k(L, L')$. In the first three rounds he pebbles vertices a_0, b_0, c_0 in $V(G)$ so that $T(a_0, b_0, c_0) = 1$. Denote Duplicator's responses by a'_0, b'_0, c'_0 and suppose that either $T'(a'_0, b'_0, c'_0) = 1$ or $T'(a'_0, c'_0, b'_0) = 1$ (otherwise Spoiler has won). Without loss of generality, suppose the former (otherwise just interchange b_0 and c_0 and consider R^* and T^* instead of R and T). Starting from the 4-th round, Spoiler emulates $\mathrm{Ehr}^m(R, R')$ keeping the pebbles on a_0, b_0, c_0. His win in this game means that either the equality, or the adjacency in G, or the ternary relation is violated. The former two cases imply also Spoiler's win in $\mathrm{Ehr}^k(L, L')$. In the latter case we arrive at the conditions of Lemma 9 and Spoiler needs no more than $2 \log n + 4$ extra moves to win.

5.3 The Layout and the Rotation System of a Triconnected Planar Graph

Let σ be an embedding of a connected graph G with minimum degree at least 3 in a sphere. Recall that, by definition, σ is an isomorphism from G to a sphere graph $\tilde{G}$. We define the rotation system $R_\sigma = \langle G, T_\sigma \rangle$ according to a natural geometric meaning. Namely, for $a \in V(G)$ and $b, c \in \Gamma(a)$ we have $T_\sigma(a, b, c) = 1$ if, looking at the neighborhood of $\sigma(a)$ in $\tilde{G}$ from the standpoint at the sphere center, $\sigma(b)$ is followed by $\sigma(c)$ in the clockwise order. Note that R^*_σ corresponds to the view on $\tilde{G}$ from the outside. We can define the layout system L_σ also geometrically, as described in Subsection 5.1. Equivalently, we set $L_\sigma = \mathsf{L}(R_\sigma)$.

Let $\sigma : G \to \tilde{G}$ and $\tau : G \to \hat{G}$ be two spherical embeddings of G. Suppose that they are equivalent, that is, $\tau \circ \sigma^{-1}$ is induced by a homeomorphism from the sphere where $\tilde{G}$ is drawn onto the sphere where $\hat{G}$ is drawn. Since $\tau \circ \sigma^{-1}$ takes a facial cycle to a facial cycle, we have $L_\sigma = L_\tau$. By Lemma 7, we also have $\{R_\sigma, R^*_\sigma\} = \{R_\tau, R^*_\tau\}$.

Given a triconnected planar graph G, we define $L_G = L_\sigma$ and $R_G = R_\sigma$ for σ being an arbitrary embedding of G in a sphere. By the Whitney theorem, the definition does not depend on a particular choice of σ if we agree that R_G is defined up to taking the conjugate.

5.4 Reducing the Play on Graphs to the Play on Layout Systems

Lemma 11. *Suppose that G and G' are non-isomorphic triconnected planar graphs. Let $L_G = \langle G, T, Q\rangle$ and $L_{G'} = \langle G', T', Q'\rangle$.*

1. *If $T(a,b,c) \neq T'(a',b',c')$, then Spoiler wins $\mathrm{Ehr}^{15}_{6\log n+28}(G, a, b, c, G', a', b', c')$.*
2. *If $Q(b_1, a_1, a_2, b_2) \neq Q'(b'_1, a'_1, a'_2, b'_2)$, then Spoiler wins $\mathrm{Ehr}^{15}_{6\log n+30}(G, b_1, a_1, a_2, b_2, G', b'_1, a'_1, a'_2, b'_2)$.*

Proof. **1.** Let $b, c \in \Gamma(a)$ and $b', c' \in \Gamma(a')$. Suppose that $T(a,b,c) = 0$ while $T'(a',b',c') = 1$ (the other case is symmetric). The former condition implies that $\deg a \geq 4$ and in the embedding of G the vertices b and c are separated by vertices $s, t \in \Gamma(a) \setminus \{b, c\}$. Spoiler pebbles such s and t. Let Duplicator respond with $s', t' \in \Gamma(a') \setminus \{b', c'\}$. Without loss of generality, suppose that, if in the embedding of G' we go around a' in the order b', c' and so on, then we meet s' before t' (otherwise just change the notation by transposing s and t).

Consider X-configurations $C = \begin{smallmatrix} u & x & y & v & w \\ s & b & t & c & a \end{smallmatrix}$ and $C' = \begin{smallmatrix} u' & x' & y' & v' & w' \\ s' & b' & t' & c' & a' \end{smallmatrix}$. Here the bottom row consists of vertices and the top row of their labels. Clearly, C is collocated. Since the configuration $\tilde{C}' = \begin{smallmatrix} u' & x' & y' & v' & w' \\ s' & t' & b' & c' & a' \end{smallmatrix}$ is collocated, the C' is twisted. By Lemma 6, Spoiler wins having made at most $6\log n + 28$ moves in total.

2. Let, say, $Q(b_1, a_1, a_2, b_2) = 0$ and $Q'(b'_1, a'_1, a'_2, b'_2) = 1$. Assume that we are not in the conditions of Item 1 and that the equality relation is always respected by Duplicator. In particular, $T(a_2, a_1, b_2) = T(a_1, a_2, b_1) = 1$. It easily follows that $b'_1 \neq b'_2$ and that neither of the facial cycles going through a_1a_2 is a triangle. Spoiler pebbles the vertices c_1 and c_2 in G such that $C = \begin{smallmatrix} x & y & z & u & v & w \\ c_2 & b_2 & a_2 & b_1 & c_1 & a_1 \end{smallmatrix}$ is a collocated H-configuration. Denote Duplicator's responses by c'_1 and c'_2. Unless we arrive at the conditions of Item 1, the configuration $C' = \begin{smallmatrix} x' & y' & z' & u' & v' & w' \\ c'_2 & b'_2 & a'_2 & b'_1 & c'_1 & a'_1 \end{smallmatrix}$ is twisted and Spoiler wins by Lemma 6.

Lemma 12. *Suppose that G and G' are non-isomorphic triconnected planar graphs. Denote $L = L_G$ and $L' = L_{G'}$. Let $m \geq W(L, L')$ and $k = \max\{m, 15\}$. Then*

$$D^k(G, G') \leq D^m(L, L') + 6\log n + 30.$$

Proof. We have to design a strategy for Spoiler in $\mathrm{Ehr}^k(G, G')$. He emulates $\mathrm{Ehr}^m(L, L')$ following an optimal strategy for this game. His victory in $\mathrm{Ehr}^m(L, L')$ means that one of the conditions of Lemma 11 is met and hence Spoiler needs $6\log n + 30$ extra moves to win $\mathrm{Ehr}^k(G, G')$.

5.5 Finishing the Proof of Theorem 1

Let $L = L_G$ and $L' = L_{G'}$. Let $R = R_G$ and $R' = R_{G'}$ (any of the two conjugated variants can be taken). Applying successively Lemmas 12, 10, and 8, we get

$$\begin{aligned} D^{15}(G, G') &\le D^{15}(L, L') + 6 \log n + 30 \\ &\le \max\{D^5(R, R'), D^5(R^*, R')\} + 8 \log n + 37 \\ &< 11 \log n + 45. \end{aligned}$$

6 Defining a Triconnected Planar Graph (Proof of Theorem 3)

We now prove Theorem 3. It differs from Theorem 1, which we already proved, by allowing G' to be an *arbitrary* graph non-isomorphic to G. Luckily, the proof techniques we used for Theorem 1 are still applicable. The idea is to show that for every G' one of two possibilities must be the case: Either G' even locally is far from being triconnected planar and Spoiler can efficiently exploit this difference or G' is locally indistinguishable from a triconnected planar graph, in particular, with G' we can naturally associate a rotation system, and hence Spoiler can apply the strategy of Theorem 1 designed for triconnected planar graphs.

Let G be a triconnected planar graph on n vertices. We use the tight connection between logical distinguishability of two structures and the Ehrenfeucht-Fraïssé game on these structures. Lemma 6 for X-configurations can be rephrased as follows: For every collocated X-configuration C in G and every twisted X-configuration T in a triconnected planar graph H (a possibility that $H \cong G$ is not excluded), there is a first order formula $\Phi_{C,T}(w, x, y, v, u)$ of quantifier depth less than $6 \log n + 26$ with 15 variables, of which the variables w, x, y, v, u are free, such that $G, C \models \Phi_{C,T}$ and $H, T \not\models \Phi_{C,T}$. Similar formulas $\Psi_{C,T}(z, w, x, y, v, u)$ exist for H-configurations.

Given a collocated X-configuration C in G, define Φ_C to be the conjunction of $\Phi_{C,T}$ over all twisted configurations T. A problem with this definition is that there are infinitely many triconnected planar graphs H and twisted X-configurations T in them. However, every $\Phi_{C,T}$ has quantifier depth at most $6 \log n + 26$ and, as well known, over a finite vocabulary there are only finitely many inequivalent first order formulas of a bounded quantifier depth. If Φ_{C,T_1} and Φ_{C,T_2} are logically equivalent, then we put in Φ_C only one of these formulas thereby making Φ_C well-defined. Furthermore, we define $\Phi(w, x, y, v, u)$ to be the disjunction of Φ_C over all collocated X-configurations C in G. We also suppose that Φ explicitly says that x, y, v, u are pairwise distinct and all adjacent to w.

Similarly, for H-configurations we define a formula $\Psi(z, w, x, y, v, u)$ by $\Psi = \bigvee_C (\bigwedge_T \Psi_{C,T})$.

Notice that the order of variables we have chosen for $\Phi(w, x, y, v, u)$ plays some role. Namely, if the 5-tuple (w, x, y, v, u) is a collocated X-configuration as defined in Section 4, then in the embedding of G the vertices x, y, v, u occur around w in the order as written. Introduce two permutations $\sigma = (xyvu)$ and $\tau = (xu)(yv)$. The former corresponds to the cyclic shift of the four vertices around w, the latter corresponds to a reflection (changing the direction around w). Define

$$\hat{\Phi}(w, x, y, v, u) = \bigwedge_{i=0}^{1} \bigwedge_{j=0}^{3} \Phi(w, \tau^i\sigma^j(x), \tau^i\sigma^j(y), \tau^i\sigma^j(v), \tau^i\sigma^j(u)).$$

We now make an important observation: $\hat{\Phi}$ has a clear geometric meaning for 5-tuples of vertices of G.

Lemma 13. *Let $a \in V(G)$ and $b_j \in \Gamma(a)$ for all $j \le 4$. In the embedding of G, the vertices b_1, b_2, b_3, b_4 occur around a in the order as written if and only if $G, a, b_1, b_2, b_3, b_4 \models \hat{\Phi}$.*

Proof. Indeed, suppose that b_1, b_2, b_3, b_4 is a right order around a. Then the X-configuration $C = \begin{smallmatrix} x & y & v & u & w \\ b_1 & b_2 & b_3 & b_4 & a \end{smallmatrix}$ is collocated and remains so after reassigning the labels x, y, v, u with respect to the permutation $\tau^i\sigma^j$ for any i and j. It remains to notice that Φ is true for any collocated X-configuration by construction.

For the opposite direction, suppose that b_1, b_2, b_3, b_4 is a wrong order around a. Consistently with the previous notation, let $\sigma = (1234)$ and $\tau = (14)(23)$. A key observation here is that, for some permutation $\pi = \tau^i\sigma^j$, the X-configuration $T = \begin{smallmatrix} x & y & v & u & w \\ b_{\pi(1)} & b_{\pi(2)} & b_{\pi(3)} & b_{\pi(4)} & a \end{smallmatrix}$ is twisted. By the definition of $\Phi_{C,T}$, we have $G, a, b_{\pi(1)}, b_{\pi(2)}, b_{\pi(3)}, b_{\pi(4)} \models \neg\Phi_{C,T}$ for every collocated X-configuration C in G. It follows that $G, a, b_{\pi(1)}, b_{\pi(2)}, b_{\pi(3)}, b_{\pi(4)} \models \neg\Phi_C$ for every C and hence $G, a, b_{\pi(1)}, b_{\pi(2)}, b_{\pi(3)}, b_{\pi(4)} \models \neg\Phi$. Equivalently, we have $G, a, b_1, b_2, b_3, b_4 \models \neg\Phi(w, \pi(x), \pi(y), \pi(v), \pi(u))$. Thus, $G, a, b_1, b_2, b_3, b_4 \models \neg\hat{\Phi}(w, x, y, v, u)$, as required.

Let $\sim$ denote the adjacency relation. Define a first order statement

$$A_G = \forall x, y_1, y_2, y_3, y_4 \Bigg(\bigwedge_{i=1}^{4} y_i \sim x \wedge \bigwedge_{i \ne j} \neg(y_i = y_j) \to$$
$$\Big(\hat{\Phi}(x, y_1, y_2, y_3, y_4) \vee \hat{\Phi}(x, y_2, y_1, y_3, y_4) \vee \hat{\Phi}(x, y_1, y_3, y_2, y_4) \Big) \wedge$$
$$\Big(\hat{\Phi}(x, y_1, y_2, y_3, y_4) \to \neg\hat{\Phi}(x, y_2, y_1, y_3, y_4) \wedge \neg\hat{\Phi}(x, y_1, y_3, y_2, y_4) \Big) \Bigg),$$

The quantifier depth of A_G is at most $6 \log n + 31$. Note that $y_1y_2y_3y_4$, $y_2y_1y_3y_4$, and $y_1y_3y_2y_4$ are the three possible arrangements of four vertices up to the action of the dihedral group $D_4 = \{\tau^i\sigma^j\}_{i,j}$. Saying that exactly one of these arrangements corresponds to the geometric order around x, the A_G is true on G.

Suppose now that G' is an arbitrary graph non-isomorphic to G. We have to bound $D^{15}(G,G')$ from above. We assume that G' is connected and has minimum degree at least 3; otherwise Spoiler wins fast. If $G' \not\models A_G$, then G and G' are distinguished by A_G and hence $D^{15}(G,G') \le 6\log n + 31$.

Suppose that $G' \models A_G$. The A_G ensures that, for every vertex a in G' and $b_1, b_2, b_3, b_4 \in \Gamma(a)$, we have a unique (up to shifting and redirecting) ordering of b_1, b_2, b_3, b_4 satisfying $\hat{\Phi}(x, y_1, y_2, y_3, y_4)$. We use it to associate with G' a layout system $L' = \langle G', T', Q' \rangle$ (as if this ordering corresponds to some embedding of G'). Given $a \in V(G')$ of degree at least 4, we first want to define pairs $b, c \in \Gamma(a)$ such that b and c are neighboring in this "pseudo-embedding" of G'.

We let $N(a,b,c) = \neg\exists s, t\, \hat{\Phi}(a,b,s,c,t)$. Consider a first order sentence

$$B_G = \forall a, b\Big(\deg a \ge 4 \wedge b \sim a \to \exists_{=2}\, c\, N(a,b,c)\Big)$$

(written with harmless shorthands). This sentence has a clear geometric meaning and is true on G. If $G' \not\models B_G$, then G and G' are distinguished by B_G and we are done.

Suppose that $G' \models B_G$. We are now able to define a ternary relation T' on $V(G')$. Suppose that $b', c' \in \Gamma(a')$ and $b' \ne c'$. If $\deg a' = 3$, we set $T'(a', b', c') = 1$. Let $\deg a' \ge 4$. In this case we set $T'(a', b', c') = 1$ iff $N(a', b', c')$ is true.

The B_G ensures that, for every a', $T'_{a'}$ is a union of cycles. If $T'_{a'}$ is disconnected for some a', Spoiler wins fast. He first pebbles the a'. Denote Duplicator's response in G by a. Spoiler restricts further play to $\Gamma(a)$ and $\Gamma(a')$ and follows his winning strategy in the game on graphs $(T_G)_a$ and $T'_{a'}$, one of which is connected and the other is not. Spoiler's win in this game entails disagreement $N(a,b,c) \ne N(a', b', c')$ for some pebbled b, c in G and the corresponding b', c' in G'. In the next two moves Spoiler forces disagreement between the truth values of $\hat{\Phi}$ on some 5-tuples and wins in $6\log n + 26$ extra moves.

Suppose hence that $T'_{a'}$ is connected for every a', i.e, is a cycle on $\Gamma(a')$. Similarly to the above, we can use the formula Ψ to construct a sentence Λ_G of quantifier depth at most $6\log n + 32$ providing us with the following dichotomy. If $G' \not\models \Lambda_G$, the G and G' are distinguished by Λ_G and we are done. Otherwise Ψ in a natural way determines a quaternary relation Q' such that $L' = \langle G', T', Q' \rangle$ is a layout system.

We have to consider the latter possibility. In its turn, it splits into two cases. If $L' = \mathsf{L}(R')$ for no rotation system R', this means that, if we fix a triple a'_1, b'_1, c'_1 with $T'(a'_1, b'_1, c'_1) = 1$ and set $T'_{R'}(a'_1, b'_1, c'_1) = 1$, then there are a triple a'_2, b'_2, c'_2 and two a'_1-a'_2-paths P_1 and P_2 such that propagation of the truth value of $T'_{R'}(a'_1, b'_1, c'_1)$ along P_1 and P_2 gives different results, say, $T'_{R'}(a'_2, b'_2, c'_2) = 1$ for P_1 and $T'_{R'}(a'_2, c'_2, b'_2) = 1$ for P_2. Spoiler pebbles $a'_1, b'_1, c'_1, a'_2, b'_2, c'_2$. Let Duplicator respond with $a_1, b_1, c_1, a_2, b_2, c_2$ in G. Suppose that $T_R(a_1, b_1, c_1) = 1$ for $R \in \{R_G, R^*_G\}$. Spoiler wins similarly to the proof of Lemma 9, using P_1 if $T_R(a_2, c_2, b_2) = 1$ and P_2 if $T_R(a_2, b_2, c_2) = 1$. This argument works only if P_1 and P_2 are not too long. It is not hard to show that, if the diameter of G' is smaller than n, then we have a choice of such paths with P_1 of length less than

n and P_2 of length less than $2n$. The case of G and G' having different diameters is easy for Spoiler.

If $L' = \mathsf{L}(R')$ for some rotation system R', then Spoiler plays as if G' was a triconnected planar graph. Namely, he follows the strategy of Section 5 using L' for $L_{G'}$ and R' for $R_{G'}$. Spoiler's win in this simulation means that he forces pebbling some tuples of vertices in G and G' on which the formula Φ or the formula Ψ disagree, and hence logarithmically many extra moves suffice for Spoiler to have a win in $\mathrm{Ehr}^{15}(G, G')$. The proof is complete.

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