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Induced subdivisions and bounded expansion

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ABSTRACT

We prove that for every graph H and for every integer s, the class of graphs that do not contain K_s, $K_{s,s}$, or any subdivision of H as induced subgraphs has bounded expansion; this strengthens a result of Kühn and Osthus (2004). The argument also gives another characterization of graph classes with bounded expansion and of nowhere-dense graph classes.

For a non-negative integer k, a $(\leq k)$-*subdivision* of a graph H is a graph obtained from H by subdividing each of its edges by at most k vertices (not necessarily the same number on each edge). For a graph G, let $\widetilde{\nabla}_k(G)$ denote the maximum of average degrees of graphs H such that some $(\leq k)$-subdivision of H appears as a subgraph in G; in particular, $\widetilde{\nabla}_0(G)$ is the maximum average degree of a subgraph of G. We say that a class of graphs $\mathcal{G}$ has *bounded expansion* if there exists a function $f : \mathbf{Z}_0^+ \to \mathbf{Z}_0^+$ such that for every $G \in \mathcal{G}$ and every non-negative integer k, $\widetilde{\nabla}_k(G) \leq f(k)$. We say that $\mathcal{G}$ is *nowhere-dense* if there exists a function $h : \mathbf{Z}_0^+ \to \mathbf{Z}_0^+$ such that for every $G \in \mathcal{G}$ and every non-negative integer k, G does not contain a $(\leq k)$-subdivision of $K_{h(k)}$ as a subgraph.

The notions of bounded expansion and nowhere-denseness formalize in a robust way the concept of sparseness of a graph class. Such graph classes have a number of important algorithmic and structural properties, including fixed-parameter tractability of model checking first-order logic when restricted to a class with bounded expansion [2] or a nowhere-dense class [3] and existence of low tree-depth colorings [6]. Also importantly, many naturally defined graph classes have bounded expansion, including proper minor closed classes, classes with bounded maximum degree, and more generally proper classes closed under topological minors, graphs drawn in a fixed surface with a bounded number of crossings on each edge, all graph classes with strongly sublinear separators, and many others. We refer the reader to the book of Nešetřil and Ossona de Mendez [7] for a more thorough treatment of the subject.

There are many equivalent definitions of nowhere-dense graph classes and classes of graphs with bounded expansion [7]. For example, to clarify the relationship to bounded expansion classes, it is also possible to define nowhere-dense classes in the terms of the density of graphs whose bounded depth

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subdivisions appear in the graphs of the class, as follows. A function $g : \mathbf{Z}_0^+ \to \mathbf{Z}_0^+$ is *subpolynomial* if $\lim_{n\to\infty} \frac{\log g(n)}{\log n} = 0$, and a function $f : \mathbf{Z}_0^+ \times \mathbf{Z}_0^+ \to \mathbf{Z}_0^+$ is *subpolynomial in the second argument* if for every non-negative integer k, the function $f(k, \cdot)$ is subpolynomial.

Lemma 1. *A class $\mathcal{G}$ is nowhere-dense if and only if there exists a function $f : \mathbf{Z}_0^+ \times \mathbf{Z}_0^+ \to \mathbf{Z}_0^+$ subpolynomial in the second argument such that for every non-negative integer k and graph $G \in \mathcal{G}$, each subgraph G' of G satisfies $\widetilde{\nabla}_k(G') \le f(k, |V(G')|)$.*

Most of the previously mentioned examples of graph classes with bounded expansion are closed under subgraphs. Graph classes that are only closed under induced subgraphs, and in particular graph classes characterized by forbidden induced minors or induced subdivisions, have been less studied in the context. The major issue is that such classes typically contain arbitrarily large cliques or bicliques (balanced complete bipartite graphs $K_{s,s}$), which have unbounded minimum degree. However, Kühn and Osthus [5] showed that at least with regard to the maximum average degree $\widetilde{\nabla}_0$, this is the only obstruction.

Theorem 2 (*Kühn and Osthus [5, Theorem 1]*)**.** *For every graph H and a positive integer s, there exists an integer d such that every graph with average degree at least d contains either $K_{s,s}$ as a subgraph or a subdivision of H as an* induced *subgraph.*

Note that if G contains a large biclique as a subgraph, applying Ramsey's theorem to each part of the biclique gives either a large clique or a large biclique as an induced subgraph. Hence, the previous theorem can be reformulated as follows.

Corollary 3. *For every graph H and a positive integer s, there exists an integer d as follows. If $\mathcal{G}$ is a class of graphs that do not contain K_s, $K_{s,s}$, or any subdivision of H as induced subgraphs, then $\widetilde{\nabla}_0(G) \le d$ for all $G \in \mathcal{G}$.*

In this note, we show that the result of Kühn and Osthus [5] can be easily extended to prove that such graph classes actually have bounded expansion.

Theorem 4. *For every graph H and a positive integer s, if $\mathcal{G}$ is a class of graphs that do not contain K_s, $K_{s,s}$, or any subdivision of H as an induced subgraph, then $\mathcal{G}$ has bounded expansion.*

Theorem 4 is a consequence of the following new characterization of graph classes with bounded expansion. For a graph G, let $\widetilde{\nabla}_k^{\circ}(G)$ denote the maximum of average degrees of graphs H such that some $(\le k)$-subdivision of H appears as an *induced* subgraph in G. The *k-subdivision* of a graph H is the graph obtained from H by subdividing each edge exactly k times. Let $\widetilde{\nabla}_k^{e}(G)$ denote the maximum of average degrees of graphs H whose k-subdivision appears as an induced subgraph in G.

Theorem 5. *For a class of graphs $\mathcal{G}$, the following statements are equivalent.*

(a) *$\mathcal{G}$ has bounded expansion.*
(b) *There exists a function $f : \mathbf{Z}_0^+ \to \mathbf{Z}_0^+$ such that for every $G \in \mathcal{G}$ and every non-negative integer k, $\widetilde{\nabla}_k^{\circ}(G) \le f(k)$.*
(c) *There exists a function $f : \mathbf{Z}_0^+ \to \mathbf{Z}_0^+$ such that for every $G \in \mathcal{G}$ and every non-negative integer k, $\widetilde{\nabla}_k^{e}(G) \le f(k)$.*

The same argument gives a characterization of nowhere-dense graph classes.

Theorem 6. *For a class of graphs $\mathcal{G}$, the following statements are equivalent.*

(a) *$\mathcal{G}$ is nowhere-dense.*
(b) *There exists a function $f : \mathbf{Z}_0^+ \times \mathbf{Z}_0^+ \to \mathbf{Z}_0^+$ subpolynomial in the second argument such that for every non-negative integer k and graph $G \in \mathcal{G}$, every induced subgraph G' of G satisfies $\widetilde{\nabla}_k^{\circ}(G') \le f(k, |V(G')|)$.*

(c) *There exists a function $f : \mathbf{Z}_0^+ \times \mathbf{Z}_0^+ \to \mathbf{Z}_0^+$ subpolynomial in the second argument such that for every non-negative integer k and graph $G \in \mathcal{G}$, every induced subgraph G' of G satisfies $\widetilde{\nabla}_k^e(G') \le f(k, |V(G')|)$.*
(d) *There exists a positive integer k_0 and a function $h : \mathbf{Z}_0^+ \to \mathbf{Z}_0^+$ such that for every $G \in \mathcal{G}$, G does not contain K_{k_0,k_0} as an induced subgraph, and for every non-negative integer k, G does not contain a $(\le k)$-subdivision of $K_{h(k)}$ as an induced subgraph.*

1. The proofs

Let G be a bipartite graph with bipartition (A, B)—note that the bipartition is an ordered pair, since the roles of A and B in the following definitions and arguments differ. A *hat* over this bipartition is a 3-vertex path in G with endpoints in B (and the midpoint in A). We say that a set $\mathcal{P}$ of hats is *uncrowded* if any two hats in $\mathcal{P}$ join distinct pair of vertices and have distinct midpoints. We say that it is *induced* if the midpoint of each hat has exactly two neighbors in B, i.e., $\bigcup \mathcal{P}$ is an induced subgraph of G.

Kühn and Osthus [5] proved the following.

Lemma 7 (*Kühn and Osthus [5, Lemma 18]*)**.** *Let r be a positive integer and let G be a bipartite graph with bipartition (A, B), such that each vertex of A has degree at most $4r$. If G contains an uncrowded set of at least $\frac{r^{11}}{2^8}|B|$ hats, then G has an induced subgraph G' with bipartition $(A', B') = (A \cap V(G'), B \cap V(G'))$ such that $B' \neq \emptyset$ and the set of all 3-vertex paths in G' with midpoints in A' forms an induced uncrowded set of at least $\frac{r^9}{2^{15}}|B'|$ hats over (A', B').*

We also need another lemma of Kühn and Osthus [5] (the *branch vertices* of a subdivision of a graph H are the vertices of the subdivision corresponding to the original vertices of H).

Lemma 8 (*Kühn and Osthus [5, Lemma 20]*)**.** *Let $r \ge 2^{25}$ be an integer. Let (A, B) be a partition of vertices of a graph G such that A is an independent set of G, $\chi(G[B]) \le r$ and $\widetilde{\nabla}_0(G[B]) \le r^3$. Let G' be the spanning bipartite subgraph of G containing exactly the edges of G with one end in A and the other end in B. If G' contains an induced uncrowded set of at least $\frac{r^9}{2^{15}}|B|$ hats over (A, B), then G contains an induced subgraph G'' such that G'' is the 1-subdivision of a graph of average degree at least r, with all branch vertices contained in B.*

Suppose a $(\le k)$-subdivision of a graph H is a subgraph of another graph G. Then there exists a function φ that assigns to vertices of H distinct vertices of G (the branch vertices of the subdivision) and to edges of H paths of length at most $k + 1$ in G, such that for every $uv \in E(H)$, the path $\varphi(uv)$ has endpoints $\varphi(u)$ and $\varphi(v)$ and its internal vertices do not belong to $\varphi(V(H))$, and for distinct edges $e_1, e_2 \in E(H)$, the paths $\varphi(e_1)$ and $\varphi(e_2)$ do not intersect except possibly in their endpoints. Without loss of generality, we can assume that $\varphi(e)$ is an induced path in G for every $e \in E(H)$. In these circumstances, we say that φ is a *model of a $(\le k)$-subdivision of H in G*.

Combining Lemmas 7 and 8 with Theorem 2 gives the following result.

Lemma 9. *For all integers $r, k, s \ge 1$, there exists an integer $d_{r,k,s} = O((rsk)^{12})$ such that for every graph G, if $\widetilde{\nabla}_0(G) \le s$ and $\widetilde{\nabla}_k^e(G) < r$, then $\widetilde{\nabla}_k(G) < \widetilde{\nabla}_{k-1}(G) + d_{r,k,s}$.*

Proof. Let $r_0 = \max(r, 2^{25}, s + 1, sk/2)$ and $d_{r,k,s} = \frac{r_0^{11}(sk+1)}{2^6}$. Since $r_0 = O(rsk)$, we have $d_{r,k,s} = O((rsk)^{12})$.

Suppose for a contradiction that $\widetilde{\nabla}_k(G) \ge \widetilde{\nabla}_{k-1}(G) + d_{r,k,s}$. Let H be a graph of average degree at least $\widetilde{\nabla}_{k-1}(G)+d_{r,k,s}$ such that some $(\le k)$-subdivision of H appears as a subgraph of G. Let φ be a model of a $(\le k)$-subdivision of H in G. Let $B = \varphi(V(H))$. Let H_1 be the subgraph of H consisting of the edges such that the path $\varphi(e)$ has length exactly $k + 1$. Since H has average degree at least $\widetilde{\nabla}_{k-1}(G) + d_{r,k,s}$ and a $(\le k - 1)$-subdivision of $H - E(H_1)$ is a subgraph of G, we conclude that the graph H_1 has average degree at least $d_{r,k,s}$.

Let G_1 be an auxiliary graph with vertex set $E(H_1)$, such that distinct $e_1, e_2 \in E(H_1)$ are adjacent in G_1 if and only if there exists an edge of G with one end in an internal vertex of $\varphi(e_1)$ and the other end in an internal vertex of $\varphi(e_2)$. Let C be any subset of $E(H_1) = V(G_1)$, and let D be the union of internal vertices of the paths $\varphi(e)$ for $e \in C$; we have $|D| = k|C|$, and $|E(G_1[C])| \le |E(G[D])| \le \frac{\widetilde{\nabla}_0(G)}{2}|D| \le \frac{sk}{2}|C|$. Hence, $\widetilde{\nabla}_0(G_1) \le sk$, and in particular $\chi(G_1) \le sk+1$. Hence, G_1 contains an independent set S of size at least $\frac{|E(H_1)|}{sk+1}$. Let H_2 be the spanning subgraph of H_1 with edge set S, and note that the average degree of H_2 is at least $\frac{d_{r,k,s}}{sk+1} = \frac{r_0^{11}}{2^6}$. By the choice of S, for any distinct $e_1, e_2 \in E(H_2)$, there are no edges between the internal vertices of $\varphi(e_1)$ and $\varphi(e_2)$ in G.

Let $A_2 = E(H_2)$ and let G_2 be an auxiliary bipartite graph with bipartition (A_2, B), where ev with $e \in E(H_2)$ and $v \in B$ is an edge of G_2 if and only if there exists an edge of G between an internal vertex of $\varphi(e)$ and v. Let D_2 be the union of internal vertices of the paths $\varphi(e)$ for $e \in A_2$. Since the average degree of H_2 is greater than 2, we have $|B| < |A_2| \le |D_2|$. Note that $|E(G_2)| \le |E(G[D_2 \cup B])| \le \frac{\widetilde{\nabla}_0(G)}{2}|D_2 \cup B| \le s|D_2| = sk|A_2| \le 2r_0|A_2|$. Hence, less than half of the vertices of A_2 have degree more than $4r_0$ in G_2. Let $A_2' \subseteq A_2$ consist of all vertices of A_2 whose degree in G_2 is at most $4r_0$, and let G_2' be the induced subgraph of G_2 with bipartition (A_2', B). We have $|A_2'| \ge |A_2|/2 = |E(H_2)|/2 \ge \frac{r_0^{11}}{2^8}|B|$. Let $\mathcal{P}$ be the set of paths uev in G_2, where $e \in A_2'$ and u and v are the endpoints of $\varphi(e)$; then $\mathcal{P}$ is an uncrowded set of at least $\frac{r_0^{11}}{2^8}|B|$ hats over (A_2', B). Let G_3 with bipartition (A_3, B_3) be the induced subgraph of G_2' obtained by applying Lemma 7, and let H_3 be the subgraph of H_2 with vertex set $\varphi^{-1}(B_3)$ and edge set A_3. We have $|E(H_3)| = |A_3| \ge \frac{r_0^9}{2^{15}}|B_3|$, and for each $e \in E(H_3)$, the internal vertices of $\varphi(e)$ have no neighbors in B_3 other than the endpoints of $\varphi(e)$, in addition to the previously established property that they are non-adjacent to the internal vertices of $\varphi(e')$ for any edge $e' \in E(H_3)$ distinct from e.

Finally, we consider an auxiliary graph $G_4 = G_3 \cup G[B_3]$. Note that $\widetilde{\nabla}_0(G_3[B_3]) = \widetilde{\nabla}_0(G[B_3]) \le s \le r_0^3$ and $\chi(G_3[B_3]) \le \widetilde{\nabla}_0(G[B_3]) + 1 \le s + 1 \le r_0$. Let G_5 be the induced bipartite subgraph of G_4 obtained by applying Lemma 8, with bipartition $(A_5, B_5) = (A_3 \cap V(G_5), B_3 \cap V(G_5))$. Then φ restricted to $A_5 \cup B_5$ shows that G contains the k-subdivision of a graph of average degree at least $r_0 \ge r$ as an induced subgraph, and thus $\widetilde{\nabla}_k^e(G) \ge r$; this is a contradiction. □

The new characterizations of classes with bounded expansion and nowhere-dense classes now readily follow.

Proof of Theorem 5. For every graph G and a non-negative integer k, we have $\widetilde{\nabla}_k(G) \ge \widetilde{\nabla}_k^{\circ}(G) \ge \widetilde{\nabla}_k^e(G)$, and thus (a) ⇒ (b) ⇒ (c).

Suppose now that $f : \mathbf{Z}_0^+ \to \mathbf{Z}_0^+$ is a function such that for every $G \in \mathcal{G}$ and every non-negative integer k, $\widetilde{\nabla}_k^e(G) \le f(k)$. Let $g(0) = f(0)$ and for every positive integer k, let $d_{r,k,s}$ be the constant from Lemma 9, where $r = f(k) + 1$ and $s = f(0)$, and let $g(k) = g(k-1) + d_{r,k,s}$. Consider a graph $G \in \mathcal{G}$. By induction, we will show that $\widetilde{\nabla}_k(G) \le g(k)$ for every non-negative integer k. For $k = 0$, we have $\widetilde{\nabla}_0(G) = \widetilde{\nabla}_0^e(G) \le f(0) = g(0)$; hence, we can assume that $k \ge 1$ and that $\widetilde{\nabla}_{k-1}(G) \le g(k-1)$. However, then Lemma 9 implies $\widetilde{\nabla}_k(G) < \widetilde{\nabla}_{k-1}(G) + d_{f(k)+1,k,f(0)} \le g(k-1) + d_{f(k)+1,k,f(0)} = g(k)$. We conclude that $\mathcal{G}$ has bounded expansion, and thus (c) ⇒ (a). □

To prove the implication (d) ⇒ (a) of Theorem 6, we use Ramsey's theorem in the following form.

Theorem 10. *For all positive integers k, c, and s, there exists a (minimum) positive integer $R_{k,c}(s)$ such that for any coloring of the edges of the complete k-uniform hypergraph on $R_{k,c}(s)$ vertices using c colors, there exists an induced complete subhypergraph with s vertices all of whose edges receive the same color.*

For a non-negative integer k, let $\mathcal{T}_k$ denote the class of graphs with at most $6k+4$ vertices, with each edge assigned red or blue color, and with four distinct vertices assigned labels $1, \ldots, 4$. For $G_1, G_2 \in \mathcal{G}$, we write $G_1 \cong G_2$ if G_1 and G_2 are isomorphic via an isomorphism that preserves the edge colors and labels. Note that the equivalence $\cong$ has only finitely many classes; let c_k denote their number.

Suppose a graph G contains a $(\le k)$-subdivision of the clique K with vertex set $\{1, \ldots, m\}$ as a subgraph, and let φ be the corresponding model. For a 4-tuple $I = \{i_1, i_2, i_3, i_4\}$ such that $1 \le i_1 < i_2 < i_3 < i_4 \le m$, let $G_{\varphi,I}$ denote the subgraph of G induced by the vertices of $\varphi(I)$ and the vertices of the paths $\varphi(e)$ for the edges e joining the vertices of I, with the edges of these six paths colored red and all other edges colored blue, and with the vertices $\varphi(i_1), \ldots, \varphi(i_4)$ given labels 1, …, 4 in order. Note that $G_{\varphi,I}$ belongs to $\mathcal{T}_k$. The *φ-type* of the tuple I is the equivalence class of $\cong$ containing $G_{\varphi,I}$.

Proof of Theorem 6. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are trivial (using the alternative characterization of nowhere-denseness from Lemma 1). Additionally, the implication (a) $\Rightarrow$ (d) follows from the original definition of nowhere-denseness.

Let us now consider the implication (c) $\Rightarrow$ (a). Suppose that $f : \mathbf{Z}_0^+ \times \mathbf{Z}_0^+ \to \mathbf{Z}_0^+$ is a function subpolynomial in the second argument such that for every non-negative integer k and graph $G \in \mathcal{G}$, every induced subgraph G' of G satisfies $\widetilde{\nabla}_k^e(G') \le f(k, |V(G')|)$. We can without loss of generality assume that f is non-decreasing in the second argument. Let us define $g(0, n) = f(0, n)$ for all n. For any positive integer k, let $d_{r(k,n),k,s(n)}$ be the constant from Lemma 9, where $r(k, n) = f(k, n) + 1$ and $s(n) = f(0, n)$, and let $g(k, n) = g(k-1, n) + d_{r(k,n),k,s(n)}$. As in the proof of Theorem 5, induction on k shows that $\widetilde{\nabla}_k(G') \le g(k, |V(G')|)$ for every (induced) subgraph G' of a graph $G \in \mathcal{G}$. Furthermore, for any fixed k,

$$g(k, n) = f(0, n) + \sum_{t=1}^{k} d_{r(t,n),t,s(n)} = f(0, n) + \sum_{t=1}^{k} O((f(t, n)f(0, n)t)^{12}),$$

and thus g is subpolynomial in the second argument. It follows that $\mathcal{G}$ is nowhere-dense, and thus (c) $\Rightarrow$ (a).

Finally, let us prove the implication (d) $\Rightarrow$ (a). Let k_0 be a positive integer and $h : \mathbf{Z}_0^+ \to \mathbf{Z}_0^+$ a function such that for every $G \in \mathcal{G}$, G does not contain K_{k_0,k_0} as an induced subgraph, and for every non-negative integer k, G does not contain a $(\le k)$-subdivision of $K_{h(k)}$ as an induced subgraph. For a non-negative integer k, let $h'(k) = R_{4,c_k}(2s(k) + 2)$, where $s(k) = R_{2,2}(\max(k_0, h(k)))$. We claim that no graph in $\mathcal{G}$ contains a $(\le k)$-subdivision of $K_{h'(k)}$ as a subgraph, implying that $\mathcal{G}$ is nowhere-dense and thus (d) $\Rightarrow$ (a).

Indeed, suppose for a contradiction that $G \in \mathcal{G}$ contains a $(\le k)$-subdivision of the clique K with vertex set $\{1, \ldots, h'(k)\}$ as a subgraph, and let φ be the corresponding model. By Theorem 10, all 4-tuples contained in some $(2s(k) + 2)$-element subset Q of $V(K)$ have the same φ-type. Since G does not contain a $(\le k)$-subdivision of $K_{h(k)}$ as an induced subgraph and $h(k) < 2s(k)+2$, the graphs $G_{\varphi,I}$ for 4-tuples $I \subseteq Q$ cannot contain only red edges. However, any blue edge in these graphs gives $K_{s(k),s(k)}$ as a subgraph of G.

For example, if the blue edge is between the vertex with label 4 and the second vertex of the red path between vertices with labels 1 and 3, and $Q = \{1, \ldots, 2s(k) + 2\}$, then letting A be the set of second vertices of the paths $\varphi(1i)$ for $i \in \{3, \ldots, s(k)+2\}$ and letting $B = \{s(k)+3, \ldots, 2s(k+2)+2\}$, considering the 4-tuples $\{1, 2, i, j\}$ for $3 \le i \le s(k)+2$ and $s(k)+3 \le j \le 2s(k)+2$, we conclude that G contains all edges with one end in A and the other end in B. The other possibilities for blue edges of the graphs $G_{\varphi,I}$ are handled similarly.

Let A and B be disjoint subsets of $V(G)$ of size $s(n)$ such that G contains all edges with one end in A and the other end in B. Applying Theorem 10 to $G[A]$ and $G[B]$, with colors encoding the presence or absence of edges, we conclude either that $G[A]$ or $G[B]$ contains a clique of size at least $h(k)$, or both of them contain an independent set of size at least k_0, the latter implying that G contains K_{k_0,k_0} as an induced subgraph. This contradicts the choice of k_0 and h. $\square$

To derive Theorem 4, we also need the following result.

Theorem 11 (*Komlós and Szemerédi [4], Bollobás and Thomason [1]*)**.** *There exists $c > 0$ such that for every positive integer n, every graph of average degree at least cn^2 contains a subdivision of K_n as a subgraph.*

Proof of Theorem 4. Let $n = |V(H)|$, and let c be the constant from Theorem 11. Let d be the constant of Corollary 3 for H and s. Let $f(0) = d$ and let $f(k) = cn^2$ for every $k \ge 1$.

Consider any graph $G \in \mathcal{G}$. By assumptions and Corollary 3, every induced subgraph of G has average degree at most d, and thus $\widetilde{\nabla}_0^e(G) \le f(0)$. Consider any positive integer k and let H_1 be any graph whose k-subdivision appears in G as an induced subgraph H_1'. Since H_1' does not contain a subdivision of H as an induced subgraph, and each edge of H_1 is subdivided at least once to obtain H_1', we conclude that H_1 does not contain a subdivision of H as a subgraph. Consequently, H_1 does not contain a subdivision of K_n as a subgraph, and the average degree of H_1 is less than cn^2 by Theorem 11. It follows that $\widetilde{\nabla}_k^e(G) < cn^2 = f(k)$ for every positive integer k. Hence, $\mathcal{G}$ has bounded expansion by Theorem 5. $\square$

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