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Power series with coefficients from a finite set ☆

Jason P. Bell [a], Shaoshi Chen [b]

[a] *Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada*

[b] *Key Laboratory of Mathematics Mechanization, AMSS, Chinese Academy of Sciences, 100190 Beijing, China*

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We prove in this paper that a multivariate D-finite power series with coefficients from a finite set is rational. This generalizes a rationality theorem of van der Poorten and Shparlinski in 1996.

1. Introduction

In his thesis [16], Hadamard began the study of the relationship between the coefficients of a power series and the properties of the function it represents, especially its singularities and natural boundaries. Two special cases of the problem have been extensively studied: one is on power series with integer coefficients and the other is on power series with finitely many distinct coefficients.

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E-mail addresses: jpbell@uwaterloo.ca (J.P. Bell), schen@amss.ac.cn (S. Chen).

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In the first case, Fatou [13] in 1906 proved a lemma on rational power series with integer coefficients, which is now known as Fatou's lemma [33, p. 275]. The next celebrated result is the Pólya–Carlson theorem, which asserts that a power series with integer coefficients and of radius of convergence 1 is either rational or has the unit circle as its natural boundary. This theorem was first conjectured in 1915 by Pólya [25] and later proved in 1921 by Carlson [7]. Several extensions of the Pólya–Carlson theorem have been presented in [26,24,14,31,22,35,2].

In the second case, Fatou [13] was also the first to investigate power series with coefficients from a finite set by showing that such power series are either rational or transcendental. The study was continued by Pólya [25] in 1916, Jentzsch [17] in 1917, Carlson [6] in 1918 and finally Szegő [36,37] in 1922 settled the question by proving the following beautiful theorem (see [27, Chap. 11] and [10, Chap. 10] for its proof and related results).

Theorem 1 *(Szegő, 1922). Let $F = \sum f(n)x^n$ be a power series with coefficients from a finite values of $\mathbb{C}$. If F is continuable beyond the unit circle then it is a rational function of the form $F = P(x)/(1 - x^m)$, where P is a polynomial and m a positive integer.*

Szegő's theorem was generalized in 1945 by Duffin and Schaeffer [11] by assuming a weaker condition that f is bounded in a sector of the unit circle. In 2008, P. Borwein et al. in [5] gave a shorter proof of Duffin and Schaeffer's theorem. By using Szegő's theorem, van der Poorten and Shparlinski proved the following result [38].

Theorem 2 *(van der Poorten and Shparlinski, 1996). Let $F = \sum f(n)x^n$ be a power series with coefficients from a finite values of $\mathbb{Q}$. If $f(n)$ satisfies a linear recurrence equation with polynomial coefficients, then F is rational.*

A univariate sequence $f : \mathbb{N} \to K$ is *P-recursive* if it satisfies a linear recurrence equation with polynomial coefficients in $K[n]$. A power series $F = \sum f(n)x^n$ is *D-finite* if it satisfies a linear differential equation with polynomial coefficients in $K[x]$. By [32, Theorem 1.5], a sequence $f(n)$ is P-recursive if and only if the power series $F := \sum f(n)x^n$ is D-finite. The notion of D-finite power series can be generalized to the multivariate case (see Definition 4). Our main result is the following multivariate generalization of Theorem 2.

Theorem 3. *Let K be a field of characteristic zero, and let Δ be a finite subset of K. Suppose that $f : \mathbb{N}^d \to \Delta$ with $d \geq 1$ is such that*

$$F(x_1, \ldots, x_d) := \sum_{(n_1,\ldots,n_d)\in\mathbb{N}^d} f(n_1, \ldots, n_d)x_1^{n_1} \cdots x_d^{n_d} \in K[[x_1, \ldots, x_d]]$$

is D-finite. Then F is rational.

We note that a multivariate rational power series

$$F(x_1, \cdots, x_d) = \sum_{(n_1,\ldots,n_d)\in\mathbb{N}^d} f(n_1, \ldots, n_d) x_1^{n_1} \cdots x_d^{n_d}$$

with all coefficients in $\{0, 1\}$ has a very restricted form. In particular, the set E of $(n_1, \ldots, n_d) \in \mathbb{N}^d$ for which $f(n_1, \ldots, n_d) \neq 0$ is *semilinear*; that is there exist $n \in \mathbb{N}$ and finite subsets $V_0, \ldots V_n$ of $\mathbb{N}^d$, and $b_1, \ldots, b_n \in \mathbb{N}^d$ such that

$$E = V_0 \bigcup \left\{ \bigcup_{i=1}^{n} \left(b_i + \sum_{v\in V_i} v \cdot \mathbb{N} \right) \right\}. \tag{1}$$

Although this result is known, we are unaware of a reference and give a proof of this fact in Proposition 11.

The remainder of this paper is organized as follows. The basic properties of D-finite power series are recalled in Section 2. The proof of Theorem 3 is given in Section 3. In Section 4, we present several applications of our main theorem on generating functions over nonnegative points on algebraic varieties.

2. D-finite power series

Throughout this paper, we let $\mathbb{N}$ denote the set of all nonnegative integers. Let K be a field of characteristic zero and let $K(\mathbf{x})$ be the field of rational functions in several variables $\mathbf{x} = x_1, \ldots, x_d$ over K. By $K[[\mathbf{x}]]$ we denote the ring of formal power series in $\mathbf{x}$ over K and by $K((\mathbf{x}))$ we denote the field of fractions of $K[[\mathbf{x}]]$. For two power series $F = \sum f(n_1, \ldots, n_d) x_1^{n_1} \cdots x_d^{n_d}$ and $G = \sum g(n_1, \ldots, n_d) x_1^{n_1} \cdots x_d^{n_d}$, the Hadamard product of F and G is defined by

$$F \odot G = \sum f(n_1, \ldots, n_d) g(n_1, \ldots, n_d) x_1^{n_1} \cdots x_d^{n_d}.$$

Let $D_{x_1}, \ldots, D_{x_d}$ denote the derivations on $K((\mathbf{x}))$ with respect to $x_1, \ldots, x_d$, respectively.

Definition 4 *([19])*. A formal power series $F(x_1, \ldots, x_d) \in K[[\mathbf{x}]]$ is said to be *D-finite* over $K(\mathbf{x})$ if the set of all derivatives $D_{x_1}^{i_1} \cdots D_{x_d}^{i_d}(F)$ with $i_j \in \mathbb{N}$ span a finite-dimensional $K(\mathbf{x})$-vector subspace of $K((\mathbf{x}))$. Equivalently, for each $i \in \{1, \ldots, d\}$, F satisfies a nontrivial linear partial differential equation of the form

$$\left\{ p_{i,m_i} D_{x_i}^{m_i} + p_{i,m_1-1} D_{x_i}^{m_i-1} + \cdots + p_{i,0} \right\} F = 0 \quad \text{with } p_{i,j} \in K[\mathbf{x}].$$

The notion of D-finite power series was first introduced in 1980 by Stanley [32], and has since become ubiquitous in algebraic combinatorics as an important part of the study of generating functions (see [34, Chap. 6]). We recall some closure properties of this class of power series.

Proposition 5 *([20]). Let $\mathcal{D}$ denote the set of all D-finite power series in $K[[\mathbf{x}]]$. Then*

(i) $\mathcal{D}$ forms a subalgebra of $K[[\mathbf{x}]]$, i.e., if $F, G \in \mathcal{D}$ and $\alpha, \beta \in K$, then $\alpha F + \beta G \in \mathcal{D}$ and $FG \in \mathcal{D}$.
(ii) $\mathcal{D}$ is closed under the Hadamard product, i.e., if $F, G \in \mathcal{D}$, then $F \odot G \in \mathcal{D}$.
(iii) If $F(x_1, \ldots, x_d)$ is D-finite, and

$$\alpha_1(y_1, \ldots, y_d), \ldots, \alpha_d(y_1, \ldots, y_d) \in K[[y_1, \ldots, y_d]]$$

are algebraic over $K(y_1, \ldots, y_d)$ and the substitution makes sense, then $F(\alpha_1, \ldots, \alpha_d)$ is also D-finite over $K(y_1, \ldots, y_d)$.
In particular, if $F(x_1, \ldots, x_d)$ is D-finite and the evaluation of F at $x_d = 1$ makes sense, then $F(x_1, \ldots, x_{d-1}, 1)$ is D-finite.

The coefficients of a D-finite power series are highly structured. In the univariate case, a power series $f = \sum a(n)x^n$ is D-finite if and only if the sequence $a(n)$ is P-recursive, i.e., it satisfies a linear recurrence equation with polynomial coefficients in n [32]. The structure in the multivariate case is much more profound, which was explored by Lipshitz in [20]. We continue this exploration to study the position of nonzero coefficients. To this end, we recall a notion of size in the semigroup $(\mathbb{N}, +)$. A subset $S \subseteq \mathbb{N}$ is *syndetic* if there is some positive integer C such that if $n \in S$ then $n+i \in S$ for some $i \in \{1, \ldots, C\}$. Note that a syndetic subset of $\mathbb{N}$ has nonzero density. The term "syndetic" comes from the study of topological dynamics [15, Chapter 2] and further used by Bergelson et al. [3] for studying general semigroups. Syndetic sets are also closely related to the Cobham's theorem on automatic sequences [1, Chapter 11].

Example 6. The subset of all even numbers in $\mathbb{N}$ is syndetic, but the subset $S := \{p_1^{m_1} \cdots p_n^{m_n} \mid m_1, \ldots, m_n \in \mathbb{N}\}$ with $p_1, \ldots, p_n$ being prime numbers is not syndetic since the difference between two successive integers $a_i, a_{i+1} \in S$ tends to infinity as i tends to infinity.

Lemma 7. *Let K be a field of characteristic zero and let*

$$G(x_1, \ldots, x_d) = \sum_{(n_1, \ldots, n_d) \in \mathbb{N}^d} g(n_1, \ldots, n_d) x_1^{n_1} \cdots x_d^{n_d} \in K[[\mathbf{x}]]$$

be a D-finite power series over $K(\mathbf{x})$. Then the set

$$\{n \in \mathbb{N} \mid \exists (n_1, \ldots, n_{d-1}) \in \mathbb{N}^{d-1} \text{ such that } g(n_1, \ldots, n_{d-1}, n) \neq 0\}$$

is either finite or syndetic.

Proof. We let L denote the field of fractions of $K[[x_1, \ldots, x_{d-1}]]$. Then we may regard G as a power series in $L[[x_d]]$ and it is D-finite in x_d over $L(x_d)$ and it is straightforward to see that the lemma reduces to the univariate case. Thus we now assume that $G(x) = \sum g(n)x^n \in L[[x]]$ is D-finite. Then there exist $m \geq 1$, distinct nonnegative integers $a_1 = 0, a_2, \ldots, a_m$, and nonzero polynomials $P_1, \ldots, P_m \in L[z]$ such that

$$\sum_{j=1}^{m} P_j(n)g(n + a_j) = 0$$

for all sufficiently large n. Then there is some M such that $P_1(n) \cdots P_m(n) \neq 0$ for $n > M$. If $m = 1$ then we see that $g(n) = 0$ for $n > M$. Thus we assume that $m > 1$. Then if $n > M$ and $g(n)$ is nonzero then $g(n + a_j)$ is nonzero for some $1 < j \leq m$ and so the set of n for which $g(n)$ is nonzero is syndetic. □

3. Proof of the main theorem

The proof of Theorem 2 by van der Poorten and Shparlinski is based on the fact that any univariate D-finite power series represents an analytic function with only finitely many poles [32], so it is impossible to have the unit circle as its natural boundary. Then their result follows from Szegő's theorem. The singularities of analytic functions represented by multivariate D-finite power series are much more involved. It is not known how to extend Szegő's theorem to the multivariate case. Thus new ideas are needed in order to generalize Theorem 2 to the multivariate case.

Before the proof of our main theorem, we first prove a lemma about finitely generated $\mathbb{Z}$-algebras.

Lemma 8. *Let R be a finitely generated $\mathbb{Z}$-algebra that is an integral domain of characteristic zero. Then there is only a finite set of prime numbers that divide a given nonzero element of R; i.e., for any $x \in R \setminus \{0\}$, there exists finitely many prime numbers $p_1, \ldots, p_m$ such that $n \in \{p_1^{i_1} \cdots p_m^{i_m} \mid i_1, \ldots, i_m \in \mathbb{N}\}$ if $x \in nR$.*

Proof. Let U denote the group of units of R. By a result of Roquette [28] (or see [18, page 39, Corollary]) we have that U is a finitely generated abelian group and so U_0, the subgroup of U generated by the rational numbers in U is a finitely generated subgroup of $\mathbb{Q}^*$. In particular, there exist prime numbers $q_1, \ldots, q_t$ such that every positive rational number in U is in the multiplicative subgroup of $\mathbb{Q}^*$ generated by $\pm 1, q_1, \ldots, q_t$. Thus if x is a unit and $x \in nR$ then n is an integer unit of R and hence in the semigroup generated by $\pm 1, q_1, \ldots, q_t$.

For the general case, we let $S = R[1/x]$, which is still a finitely generated $\mathbb{Z}$-algebra that is an integral domain of characteristic zero. We observe that if $x \in nS$ then n is necessarily a unit in S and by the above remarks we have that n lies in a semigroup generated by ± 1 along with a finite set of prime numbers. We note that if $x \in nR$ then $x \in nS$ and so we obtain the desired result. □

Proof of Theorem 3. We prove this by induction on d. When $d = 0$, F is constant and there is nothing to prove. We now suppose that the result holds whenever $d < k$ and we consider the case when $d = k$. Since F is D-finite, we have that $F(x_1, \ldots, x_k)$ satisfies a nontrivial linear differential equation of the form

$$\sum_{j=0}^{\ell} P_j(x_1, \ldots, x_k) D_{x_k}^j F = 0,$$

where $P_0, \ldots, P_\ell$ are polynomials in $K[x_1, \ldots, x_k]$. Translating this into a relation for the coefficients of F, we see that there exists some positive integer N and polynomials $Q_{a_1,\ldots,a_k}(t) \in K[t]$ for $(a_1, \ldots, a_k) \in \{-N, \ldots, N\}^k$, not all zero, such that

$$\sum_{-N \le a_1,\ldots,a_k \le N} Q_{a_1,\ldots,a_k}(n_k) f(n_1 - a_1, \ldots, n_k - a_k) = 0 \tag{2}$$

for all $(n_1, \ldots, n_k) \in \mathbb{N}^k$, where we take $f(i_1, \ldots, i_k) = 0$ if some i_j is negative. By dividing our polynomials $Q_{a_1,\ldots,a_k}(t)$ by t^a for some nonnegative integer a if necessary, we may assume that $q(a_1, \ldots, a_k) := Q_{a_1,\ldots,a_k}(0)$ is nonzero for some $(a_1, \ldots, a_k) \in \{-N, \ldots, N\}^k$. We now let R denote the $\mathbb{Z}$-subalgebra of K generated by Δ and by the coefficients of $Q_{a_1,\ldots,a_k}(t) \in K[t]$ with $(a_1, \ldots, a_k) \in \{-N, \ldots, N\}^k$. Then R is finitely generated. By construction, we have

$$\sum_{-N \le a_1,\ldots,a_k \le N} q(a_1, \ldots, a_k) f(n_1 - a_1, \ldots, n_k - a_k) \in n_k R$$

for all $(n_1, \ldots, n_k) \in \mathbb{N}^k$. Now let Γ denote the set of all numbers of the form

$$\sum_{-N \le a_1,\ldots,a_k \le N} q(a_1, \ldots, a_k) s(a_1, \ldots, a_k)$$

with $s(a_1, \ldots, a_k) \in \Delta \cup \{0\}$. Then Γ is a finite set. By Lemma 8, there is a finite set of prime numbers $p_1, \ldots, p_m$ such that for each nonzero $x \in \Gamma$ we have that if n is a positive integer with $x \in nR$ then n is in the semigroup generated by $p_1, \ldots, p_m$. In particular,

$$\sum_{-N \le a_1,\ldots,a_k \le N} q(a_1, \ldots, a_k) f(n_1 - a_1, \ldots, n_k - a_k) = 0$$

whenever n_k is not in the multiplicative semigroup generated by $p_1, \ldots, p_m$. Equivalently,

$$G(x_1, \ldots, x_k) := F(x_1, \ldots, x_k) \left(\sum_{0 \le a_1,\ldots,a_k \le N} q(a_1, \ldots, a_k) x_1^{a_1} \cdots x_k^{a_k} \right) x_1^N \cdots x_k^N$$

has the property that $g(n_1, \ldots, n_k) = 0$ whenever $n_k \ge N$ and $n_k - N$ is not in the semigroup generated by $p_1, \ldots, p_m$, where $g(n_1, \ldots, n_k)$ denotes the coefficient of $x_1^{n_1} \cdots x_k^{n_k}$

in $G(x_1, \ldots, x_k)$. Since G is just F multiplied by a polynomial, $G(x_1, \ldots, x_k)$ is D-finite by Proposition 5 (i); moreover, all coefficients of G lie in the finite set Γ. Note that any translate of the multiplicative semigroup generated by $p_1, \ldots, p_m$ cannot be syndetic by the same argument as in Example 6. Therefore, Lemma 7 implies that there is some positive integer M such that $g(n_1, \ldots, n_k) = 0$ whenever $n_k > M$. Thus we have

$$G = \sum_{i=0}^{M} G_i(x_1, \ldots, x_{k-1}) x_k^i$$

for some power series $G_0, \ldots, G_M \in K[[x_1, \ldots, x_{k-1}]]$. Then for $i \in \{0, \ldots, M\}$, we have that $G_i x_k^i$ is the Hadamard product of G with $x_k^i \prod_{j=1}^{k-1}(1 - x_j)^{-1}$ and so each $G_i x_k^i$ is D-finite by Proposition 5 (ii). Then specializing $x_k = 1$ gives each G_i is D-finite by Proposition 5 (iii). Since each G_i has coefficients in a finite set, we see by the induction hypothesis that each G_i is rational and so G is rational. But this now gives that F is rational by our definition of G, completing the proof. □

4. Generating functions over nonnegative integer points on algebraic varieties

Let $V \subseteq \mathbb{A}_K^d$ be an affine algebraic variety over an algebraically closed field K of characteristic zero. We define the *generating function over nonnegative integer points* on V by

$$F_V(x_1, \ldots, x_d) := \sum_{(n_1, \ldots, n_d) \in V \cap \mathbb{N}^d} x_1^{n_1} \cdots x_d^{n_d}.$$

Then one can ask the following questions about the properties of F_V that often reflect the global geometric structure of V:

1. When F_V is zero? This is Hilbert Tenth Problem when K is the field of rational numbers. In 1970, Matiyasevich [23,9] proved that this problem is undecidable.
2. When F_V is a polynomial? If so, V has only finitely many nonnegative integer points. Siegel's theorem on integral points answers this question for a smooth algebraic curve C of genus $g \geq 1$ defined over a number field K [4, Chap. 7].
3. When F_V is a rational function? This is always true when the variety V is defined by linear polynomials with integer coefficients [33, Chap. 4].
4. When F_V is D-finite? By our main theorem, we see that this question is the same as question (3), by taking $f(n_1, \ldots, n_d) = 1$ if $(n_1, \ldots, n_d) \in V \cap \mathbb{N}^d$ and $f(n_1, \ldots, n_d) = 0$ otherwise (see Corollary 9).
5. When F_V satisfies an algebraic differential equation? More precisely, we say that a power series $F(x_1, \ldots, x_d) \in K[[x_1, \ldots, x_d]]$ is *differentially algebraic* if the transcendence degree of the field generated by all of the derivatives $D_{x_1}^{i_1} \cdots D_{x_d}^{i_d}(F)$ with $i_j \in \mathbb{N}$

over $K(x_1, \ldots, x_d)$ is finite. If a power series is not differentially algebraic, then it is called *transcendentally transcendental*. For a nice survey on transcendentally transcendental functions, see Rubel [29].

Corollary 9. *Let $V \subseteq \mathbb{A}^d_K$ be an affine variety over an algebraically closed field K of characteristic zero. Then the power series*

$$F_V(x_1, \ldots, x_d) := \sum_{(n_1,\ldots,n_d)\in V\cap\mathbb{N}^d} x_1^{n_1} \cdots x_d^{n_d}$$

is D-finite if and only if it is rational.

To show an application of this corollary, let us consider the linear system $A\mathbf{x} = 0$, where A is a $d \times m$ matrix with integer entries. Let E be the set of all vectors $(n_1, \ldots, n_d) \in \mathbb{N}^d$ such that $A\mathbf{x} = 0$. We now give a proof of the following classical theorem in enumerative combinatorics.

Theorem 10 *(Theorem 4.6.11 in [33]). The generating function*

$$F_E(x_1, \ldots, x_d) := \sum_{(n_1,\ldots,n_d)\in E} x_1^{n_1} \cdots x_d^{n_d}$$

represents a rational function of $x_1, \ldots, x_d$.

Proof. By Corollary 9, it suffices to show that F_E is D-finite. We first recall a fact proved by Lipshitz in [19, p. 377] that if the power series $G(\mathbf{x}) = \sum g(n_1, \ldots, n_d)x_1^{n_1} \cdots x_d^{n_d}$ is D-finite and $C \subseteq \mathbb{N}^d$ is the set of elements of $\mathbb{N}^d$ satisfying a finite set of inequalities of the form $\sum a_i n_i + b \geq 0$, where the $a_i, b \in \mathbb{Z}$, then the power series

$$H(\mathbf{x}) := \sum_{(n_1,\ldots n_d)\in C} g(n_1, \ldots, n_d)x_1^{n_1} \cdots x_d^{n_d}$$

is D-finite. Note that $R(x_1, \ldots, x_d) := \sum x_1^{n_1} \cdots x_d^{n_d} = 1/\prod_{i=1}^d (1 - x_i)$ is D-finite and any equality $\sum a_i n_i = 0$ is equivalent to two inequalities $\sum a_i n_i \geq 0$ and $\sum(-a_i)n_i \geq 0$. Then the D-finiteness of F_E follows from the fact. □

We now derive some properties of an algebraic variety E from the generating function F_E when $d = 2$. We first prove a basic result that is probably well-known, but for which we are unaware of a reference.

Proposition 11. *Let*

$$F(x_1, \cdots, x_d) = \sum_{(n_1,\ldots,n_d)\in\mathbb{N}^d} f(n_1, \ldots, n_d)x_1^{n_1} \cdots x_d^{n_d} \in \mathbb{Q}[[x_1, \ldots, x_d]]$$

with $f(n_1,\dots,n_d) \in \{0,1\}$ for all $(n_1,\dots,n_d) \in \mathbb{N}^d$. Then F is rational if and only if the support set $E := \{(n_1,\dots,n_d) \in \mathbb{N}^d \mid f(n_1,\dots,n_d) \neq 0\}$ of F is semilinear *(see Equation (1) for the definition of semilinearity).*

Proof. The sufficiency follows from Theorem 10. For the other direction assume that $F(x_1,\dots,x_d)$ is rational. Since $f(n_1,\dots,n_d) \in \{0,1\}$ for all $(n_1,\dots,n_d) \in \mathbb{N}^d$, we have that F can be written in the form $F = P/Q$ with $P, Q \in \mathbb{Z}[x_1,\dots,x_d]$ and the gcd of the collection of coefficients of P and Q equal to 1. For any prime number $p \in \mathbb{N}$, the modulo p mapping $\phi_p : \mathbb{Z}[[x_1,\dots,x_d]] \to \mathbb{F}_p[[x_1,\dots,x_d]]$ is a homomorphism. Then $\phi_p(Q \cdot F) = \phi_p(Q) \cdot \phi_p(F) = \phi_p(P)$, which then implies that the sequence $f : \mathbb{N}^d \to \{0,1\}$ has a rational generating function over any finite field $\mathbb{F}_p$, where p is a prime number. By Salon's theorem [30], which is a multi-dimensional extension of the theorem by Christol, Kamae, Mendès France, and Rauzy [8], the sequence $f : \mathbb{N}^d \to \{0,1\}$ is p-automatic for every prime number p. Then the Cobham–Semenov theorem [12] implies that the support set E of f is semilinear. $\square$

We now use this result in the special case when $d = 2$.

Theorem 12. *Let $p(x,y) \in K[x,y]$ be a nonzero polynomial satisfying that the generating function*

$$F_p(x,y) := \sum_{\substack{(n,m)\in\mathbb{N}^2 \\ p(n,m)=0}} x^n y^m$$

is rational. Then $p = c \cdot f \cdot g$, where $c \in K$ is a constant, f is a product of linear polynomials in x and y with integer coefficients and g has only finite roots in $\mathbb{N}^2$.

Proof. Let $p = p_1 \cdots p_r$ with p_i irreducible over K. Assume that $p_1,\dots,p_m$ have only finitely many zeros in $\mathbb{N}^2$ and that p_i with $i > m$ has infinitely many roots in $\mathbb{N}^2$. Then let $g = p_1 \cdots p_m$. We show that $p_{m+1},\dots,p_r$ are, up to scalar multiplication, polynomials of the form $ax + by + c$ with $a, b, c \in \mathbb{Z}$. By Proposition 11, the set E of all nonnegative points (n,m) on the curve $p(x,y) = 0$ is semilinear. Now suppose that E is infinite. Then if the subset V_i in (1) is not contained in a line in $\mathbb{Z}^2$ through the origin, then the set

$$b_i + \sum_{v\in V_i} v \cdot \mathbb{N}$$

is Zariski dense in the plane, which is impossible since E is contained in the zero set of a nonzero polynomial. Thus we see that after refining our decomposition of E if necessary, we may assume that each $|V_i| = 1$ for $i > 0$. Let q be any irreducible factor of p having infinitely many zeros in $\mathbb{N}^2$. Then there is some $V_i = \{v\} \subseteq \mathbb{N}^2$ with $i > 0$, such that $q(b_i + vn) = 0$ for infinitely many $n \in \mathbb{N}$. Write $b_i = (c,d)$ and $v = (a,b)$. Then $q(c + an, d + bn) = 0$ for infinitely many $n \in \mathbb{N}$ and so $q(c + at, d + bt) = 0$ for all $t \in K$.

Hence the linear polynomial $ay - bx - (da - cb)$ divides q. Since q is irreducible over K, then $q = \lambda(ay - bx - (da - cb))$ for some constant $\lambda \in K$. This completes the proof. $\square$

The theorem as above cannot be extended to the case when $d > 2$ as shown in the following example.

Example 13. Let $p = x - y + 2z^2 + zy^2$. We claim that $E := \{(n, n, 0) \mid n \in \mathbb{N}\}$ is the set of all zeros of p in $\mathbb{N}^3$. Suppose that (a, b, c) is another $\mathbb{N}^3$-point with c nonzero. Then $a + 2c^2 + cb^2 = b$ and so $c(2c + b^2) = 2c^2 + cb^2 \leq b$ since a is nonnegative. But if c is strictly positive then we must have $2c + b^2 \leq c(2c + b^2) \leq b$, which gives $c \leq 0$, a contradiction.

Now the corresponding generating function is equal to $1/((1 - x)(1 - y))$ which is rational, but the polynomial p is not of the integer-linear form up to scalar multiplication.

As in the first question, we can show that it is undecidable to test whether the generating function F_V for an arbitrary algebraic variety V is D-finite or not. Let $P \in \mathbb{Q}[x_1, \dots, x_d]$ be any polynomials over $\mathbb{Q}$ in $x_1, \dots, x_d$ and let V be the algebraic variety defined by

$$V := \{(a_1, \dots, a_d, b, c) \in \overline{\mathbb{Q}} \mid P(a_1, \dots, a_d)^2 + (b - c^2)^2 = 0\}.$$

The undecidability follows from the equivalence that the generating function F_V is D-finite if and only if P has no root in $\mathbb{N}^d$. Clearly, $F_V = 0$ if P has no root in $\mathbb{N}^d$ and then it is D-finite. Now suppose that P has at least one root in $\mathbb{N}^d$. Then the generating function F_V is of the form

$$F_V = \sum_{\substack{(n_1,\dots,n_d,m)\in\mathbb{N}^{d+1} \\ P(n_1,\dots,n_d)=0}} x_1^{n_1} \cdots x_d^{n_d} y^{m^2} z^m.$$

It is sufficient to show that $G_V(x_1, \dots, x_d, y) := F_V(x_1, \dots, x_d, y, 1)$ is not D-finite. Clearly, the set

$$\{m \mid \exists (n_1, \dots, n_d) \in \mathbb{N}^d \text{ such that } g(n_1, \dots, n_d, m) \neq 0\}$$

is the set of square numbers, which is neither finite nor syndetic. Thus G_V is not D-finite by Lemma 7.

Example 14. Let $p = x^2 - y \in K[x, y]$. Then the associated generating function is $F(x, y) = \sum_{m \geq 0} x^m y^{m^2}$. Since p is not of the integer-linear form, $F(x, y)$ is not D-finite by Theorem 12. Actually, we can show that $F(x, y)$ is transcendentally transcendental. Suppose that $F(x, y)$ is differentially algebraic. Then it satisfies a nontrivial algebraic differential equation $Q(x, y, F, D_x(F), \dots, D_x^r(F)) = 0$, where $r \in \mathbb{N}$

and $Q \in K[z_1, z_2, \dots, z_{r+3}]$. Note that the evaluation of a power series at $y = 2$ gives a ring homomorphism $e_2 : K[[x, y]] \to K[[x]]$ and we have a commuting square

$$\begin{array}{ccc} K[[x,y]] & \xrightarrow{e_2} & K[[x]] \\ \downarrow & & \downarrow \\ K[[x,y]] & \xrightarrow{e_2} & K[[x]], \end{array}$$

where both vertical maps are differentiation with respect to x. It follows that $F(x, 2) = \sum_{m\geq 0} 2^{m^2} x^m$ is also differentially algebraic. This leads to a contradiction with the fact proved by Mahler in [21, p. 200, Theorem 16] on the rate of coefficient growth of a differentially algebraic power series, since $2^{m^2} \gg (m!)^c$ for any positive constant c.

This example motivates us to formulate the following conjecture, which can be viewed as an analogue of the Pólya–Carlson theorem in the context of algebraic geometry and differential algebra.

Conjecture 15. *Let $V \subseteq \mathbb{A}^d_K$ be an affine variety over an algebraically closed field K of characteristic zero. Then the power series*

$$F_V(x_1, \dots, x_d) := \sum_{(n_1,\dots,n_d)\in V\cap\mathbb{N}^d} x_1^{n_1} \cdots x_d^{n_d}$$

is either rational or transcendentally transcendental.

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