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Finitary $\mathcal{M}$-adhesive categories

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Finitary $\mathcal{M}$-adhesive categories are $\mathcal{M}$-adhesive categories with finite objects only, where $\mathcal{M}$-adhesive categories are a slight generalisation of weak adhesive high-level replacement (HLR) categories. We say an object is finite if it has a finite number of $\mathcal{M}$-subobjects. In this paper, we show that in finitary $\mathcal{M}$-adhesive categories we not only have all the well-known HLR properties of weak adhesive HLR categories, which are already valid for $\mathcal{M}$-adhesive categories, but also all the additional HLR requirements needed to prove classical results including the Local Church-Rosser, Parallelism, Concurrency, Embedding, Extension and Local Confluence Theorems, where the last of these is based on critical pairs. More precisely, we are able to show that finitary $\mathcal{M}$-adhesive categories have a unique $\mathcal{E}$-$\mathcal{M}$ factorisation and initial pushouts, and the existence of an $\mathcal{M}$-initial object implies we also have finite coproducts and a unique $\mathcal{E}'$-$\mathcal{M}$ pair factorisation. Moreover, we can show that the finitary restriction of each $\mathcal{M}$-adhesive category is a finitary $\mathcal{M}$-adhesive category, and finitarity is preserved under functor and comma category constructions based on $\mathcal{M}$-adhesive categories. This means that all the classical results are also valid for corresponding finitary $\mathcal{M}$-adhesive transformation systems including several kinds of finitary graph and Petri net transformation systems. Finally, we discuss how some of the results can be extended to non-$\mathcal{M}$-adhesive categories.

1. Introduction

The field of algebraic graph transformation, that is, applying algebraic methods to the rule-based transformation of graphs and graph-like structures, dates back to the 1970s

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(Ehrig 1979). Since then, many theoretical results on the analysis of graph transformation systems have been proved. The main results include:

- The Local Church-Rosser and Parallelism Theorems:
  These are concerned with independent transformations. In the case of parallel or sequential independence, two rules can be applied in arbitrary order, and even in parallel, and still lead to the same result.
- The Concurrency Theorem:
  This is concerned with dependent transformations. In the case of sequential dependence, the rules can be combined by overlapping the dependent parts to give a concurrent rule whose application leads to the same result as before.
- The Embedding and Extension Theorems:
  These handle the extension of transformations into a larger context. A transformation sequence of a graph can be replayed in a larger graph if the embedding of the given graph into the larger graph is consistent.
- Completeness of critical pairs:
  This describes the fact that we can find a set of critical pairs that completely describe all conflicts that occur in a given transformation system.
- The Local Confluence Theorem:
  This states that a transformation system is locally confluent if all its critical pairs are strictly confluent, where the strictness condition enhances standard local confluence in a specific way.

All these results can be instantiated for a large variety of models, including various kinds of graph and Petri net transformation systems.

1.1. *Categorical frameworks*

The concepts of various adhesive (Lack and Sobociński 2004; Lack and Sobociński 2005) and weak adhesive high-level replacement (HLR) (Ehrig *et al.* 2006a) categories were a break through for the double pushout approach (DPO) to algebraic graph transformations (Rozenberg 1997). Almost all of the main results have been formulated and proved in at least one of these categorical frameworks.

On the one hand, the proofs are based on the following well-known HLR properties, which are valid in all adhesive and weak adhesive HLR categories:

(1) pushouts along $\mathcal{M}$-morphisms are pullbacks;
(2) $\mathcal{M}$ pushout–pullback decomposition lemma;
(3) cube pushout–pullback lemma; and
(4) uniqueness of pushout complements.

In fact, these properties are already valid in a slight generalisation of weak adhesive HLR categories, which are called $\mathcal{M}$-adhesive categories in Ehrig *et al.* (2010).

On the other hand, the following additional HLR requirements were needed in Ehrig *et al.* (2006a) to prove the main results:

(5) finite coproducts compatible with $\mathcal{M}$;

(6) an $\mathcal{E}'$-$\mathcal{M}'$ pair factorisation usually based on a suitable $\mathcal{E}$-$\mathcal{M}$ factorisation of morphisms; and

(7) initial pushouts.

While requirement (5) can be shown to hold in any weak adhesive HLR category with finite coproducts or $\mathcal{M}$-initial objects, finding general conditions under which requirements (6) and (7) are valid so that we can avoid an explicit verification for each instantiating category is still an open question. Prange *et al.* (2008) investigated this problem for comma and functor category constructions, but the results only hold under strong preconditions.

1.2. *Finite objects and the main results*

For various applications it is sufficient to assume that the models to be transformed are finite. In particular, this is a reasonable restriction when using tool support since most tools can only handle finite models. Moreover, for most applications, such as modelling system models or model transformations, we implicitly assume that transformations preserve finiteness because we do not want to consider infinite models. So, in a way, although the theory is developed for arbitrary graphs, the restriction to finite ones is not really a limitation, but just a more suitable setting.

For this reason, we want to analyse transformations in $\mathcal{M}$-adhesive categories with only finite objects. Formally, an object A in an $\mathcal{M}$-adhesive category is said to be finite if A has only a finite number of $\mathcal{M}$-subobjects, that is, there exist only finitely many $\mathcal{M}$-morphisms $m: A' \to A$ up to isomorphism. The category $\mathbf{C}$ is said to be finitary if it has only finite objects. Note that the notion of being 'finitary' depends on the class $\mathcal{M}$ of monomorphisms and the statement '**C** is finitary' must not be confused with '**C** is finite' in the sense of a finite number of objects and morphisms. In the standard cases of **Sets** and **Graphs**, where $\mathcal{M}$ is the class of all monomorphisms, finite objects are exactly finite sets and finite graphs, respectively.

In the current paper, we show that all the additional HLR requirements are valid in finitary $\mathcal{M}$-adhesive categories for suitable classes $\mathcal{E}$ and $\mathcal{E}'$, and with $\mathcal{M}' = \mathcal{M}$. Note that for $\mathcal{M}' = \mathcal{M}$, the $\mathcal{M}$-$\mathcal{M}'$ pushout–pullback decomposition property is the $\mathcal{M}$ pushout–pullback decomposition property, which is already valid in general $\mathcal{M}$-adhesive categories. The reason for the existence of an $\mathcal{E}$-$\mathcal{M}$ factorisation of morphisms in finitary $\mathcal{M}$-adhesive categories is the fact that we only need finite intersections of $\mathcal{M}$-subobjects and not infinite intersections, as would be required in general $\mathcal{M}$-adhesive categories. Moreover, we fix the choice of the class $\mathcal{E}$ to extremal morphisms with respect to $\mathcal{M}$. We are able to show that the finitary restriction $(\mathbf{C}_{\mathrm{fin}},\mathcal{M}_{\mathrm{fin}})$ of any $\mathcal{M}$-adhesive category $(\mathbf{C},\mathcal{M})$ is a finitary $\mathcal{M}$-adhesive category. Moreover, finitarity is preserved under functor and comma category constructions based on $\mathcal{M}$-adhesive categories.

The dependencies are shown in Figure 1, where the additional assumptions of finitarity and $\mathcal{M}$-initial objects are shown in the top row, the additional HLR requirements (5)–(7) shown in this paper in the centre and the classical theorems in the bottom row.

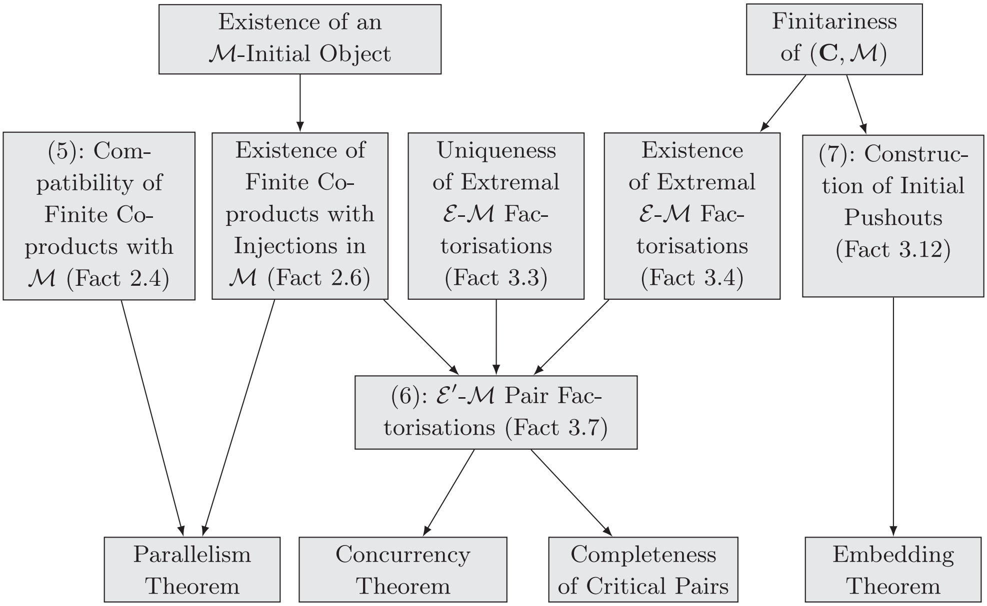


Fig. 1. Dependency graph

1.3. *Structure of the paper*

In Section 2, we introduce some basic notions related to $\mathcal{M}$-adhesive and finitary $\mathcal{M}$-adhesive categories including finite coproducts compatible with $\mathcal{M}$, $\mathcal{M}$-initial objects, finite objects and finite intersections, which are essential for the theory of finitary $\mathcal{M}$-adhesive categories. The first main result, which shows that all the additional HLR requirements mentioned above are valid for finitary $\mathcal{M}$-adhesive categories, is presented in Section 3. In Section 4, we show as our second main result that the finitary restriction of an $\mathcal{M}$-adhesive category is a finitary $\mathcal{M}$-adhesive category such that the results of Section 3 are applicable. In Section 5, we show that functorial constructions, including functor and comma categories, applied to finitary $\mathcal{M}$-adhesive categories are again finitary $\mathcal{M}$-adhesive categories under suitable conditions. In Section 6, we carry out an analysis to show how some of the results in Section 3 can be shown in a weaker form for (finitary) non-$\mathcal{M}$-adhesive categories, such as the category of simple graphs with all monomorphisms $\mathcal{M}$. In particular, we consider the construction of weak initial pushouts, which are the basis for the gluing condition required to construct (unique) minimal pushout complements in such categories, while initial pushouts are also the basis for the construction of (unique) pushout complements in (finitary) $\mathcal{M}$-adhesive categories. In Section 7, we compare the results valid for (finitary) $\mathcal{M}$-adhesive categories with those for (finitary) $\mathcal{M}$-PO-PB categories. In Sections 8 and 9, we discuss related work, summarise the main results and discuss some open problems for future research.

A short version of the current paper was published as Braatz *et al.* (2010), where we used the term '$\mathcal{M}$-adhesive' as shorthand for 'weak adhesive HLR'. In the current paper, we show all the results of Braatz *et al.* (2010) for the slightly more general notion of

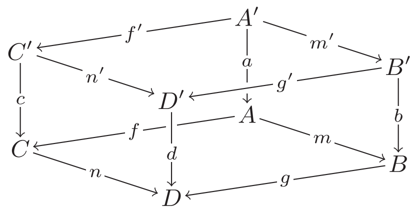


Fig. 2. Cube based on an $\mathcal{M}$-VK square

$\mathcal{M}$-adhesive categories. Moreover, we also give illustrative examples and full proofs of all results. As a new result, we show that not only can initial pushouts and $\mathcal{E}$-$\mathcal{M}$ factorisations in finitary $\mathcal{M}$-adhesive categories be constructed by finite $\mathcal{M}$-intersections, but also, in general, $\mathcal{M}$-adhesive categories can be constructed by general $\mathcal{M}$-intersections, provided the corresponding constructions exist. This result is valid for several kinds of graph and Petri net categories.

2. Basic notions related to finitary $\mathcal{M}$-Adhesive categories

Adhesive categories were introduced in Lack and Sobociński (2004) and generalised to (weak) adhesive HLR and $\mathcal{M}$-adhesive categories in Ehrig *et al.* (2006a), Ehrig *et al.* (2006b) and Ehrig *et al.* (2010) as a categorical framework for various kinds of graph and net transformation systems. Since $\mathcal{M}$-adhesive categories are the most general variant, we will use them in the current paper.

Definition 2.1 ($\mathcal{M}$-adhesive category). An *$\mathcal{M}$-adhesive category* $(\mathbf{C}, \mathcal{M})$ consists of a category $\mathbf{C}$ and a class $\mathcal{M}$ of monomorphisms in $\mathbf{C}$, which is closed under[†] isomorphisms and composition, such that $\mathbf{C}$ has pushouts and pullbacks along $\mathcal{M}$-morphisms, $\mathcal{M}$-morphisms are closed under pushouts and pullbacks, and pushouts along $\mathcal{M}$-morphisms are $\mathcal{M}$-van Kampen (VK) squares.

An *$\mathcal{M}$-VK square* is a pushout as at the bottom of the cube in Figure 2 with $m \in \mathcal{M}$ that satisfies the *(vertical) weak VK property*, that is, for any commutative cube, where in Figure 2 the back faces are pullbacks and $b, c, d \in \mathcal{M}$, the following statement holds: the top face is a pushout if and only if the front faces are pullbacks.

Remark 2.2.

(1) In contrast, the ‘horizontal’ weak VK property assumes $f \in \mathcal{M}$ instead of $b, c, d \in \mathcal{M}$, while the (standard) VK property does not require any additional $\mathcal{M}$-morphisms.
(2) A weak adhesive HLR category as defined in Ehrig *et al.* (2006a) is required to satisfy both the horizontal and vertical VK property, that is, for $f \in \mathcal{M}$ or $b, c, d \in \mathcal{M}$ in Figure 2.
(3) The fact that $\mathcal{M}$ is also closed under decomposition ($g \circ f \in \mathcal{M}$ and $g \in \mathcal{M}$ imply $f \in \mathcal{M}$) follows from the requirement that $\mathcal{M}$-morphisms are closed under pullbacks.

[†] We use the term ‘closed under’ synonymously with ‘stable under’ – both terms are used in the literature.

$$\begin{tikzcd} E_1 \arrow[r, shift left, "s_1"] \arrow[r, shift right, "t_1"'] \arrow[d, "f_E"] & V_1 \arrow[d, "f_V"] \\ E_2 \arrow[r, shift left, "s_2"] \arrow[r, shift right, "t_2"'] & V_2 \end{tikzcd}$$

(a) Graph morphism

$$\begin{tikzcd} G_1 \arrow[rr, "f"] \arrow[dr, "type_1"'] & & G_2 \arrow[dl, "type_2"] \\ & TG & \end{tikzcd}$$

(b) Typed graph morphism

Fig. 3. Graph and typed graph morphisms

We will now review some well-known examples of $\mathcal{M}$-adhesive categories – see Ehrig *et al.* (2006a) for a comprehensive overview and examples on the graph and Petri net categories presented below.

Example 2.3 ($\mathcal{M}$-adhesive categories).

(1) The category **Sets** of sets is given by the class of all sets as objects and all functions $f : A \to B$ as morphisms. The category $(\mathbf{Sets}, \mathcal{M}_S)$ with the class $\mathcal{M}_S$ of all injective functions forms an $\mathcal{M}$-adhesive category.

(2) The category **Graphs** of graphs is given by:

— **objects**:
$G = (V, E, s, t)$ consisting of a set V of nodes (also called vertices), a set E of edges and the source and target functions $s, t : E \to V$;

— **morphisms**:
$f : G_1 \to G_2$ with $f = (f_V, f_E)$ consisting of two functions

$$f_V : V_1 \to V_2$$
$$f_E : E_1 \to E_2$$

that preserve the source and target functions, that is,

$$f_V \circ s_1 = s_2 \circ f_E$$
$$f_V \circ t_1 = t_2 \circ f_E$$

(see Figure 3a).

The category $(\mathbf{Graphs}, \mathcal{M}_G)$ with the class $\mathcal{M}_G$ of all injective graph morphism forms an $\mathcal{M}$-adhesive category.

(3) Graphs can also be typed over a given type graph TG, leading to a category of typed graphs. A type graph is a distinguished graph TG. For a type graph TG the category $\mathbf{Graphs_{TG}}$ of typed graphs is given by:

— **objects**:
$(G, type)$ consisting of a graph G and a graph morphism $type : G \to TG$;

— **morphisms**:
$f : (G_1, type_1) \to (G_2, type_2)$ given by a graph morphism $f : G_1 \to G_2$ such that

$$type_2 \circ f = type_1$$

(see Figure 3b).

Note that $\mathbf{Graphs_{TG}} = (\mathbf{Graphs} \setminus TG)$ is the slice category over **Graphs**.

The category $(\mathbf{Graphs_{TG}}, \mathcal{M}_{TG})$ with the class $\mathcal{M}_{TG}$ of all injective typed graph morphisms forms an $\mathcal{M}$-adhesive category.

(4) Ehrig *et al.* (2006a) introduced attributed graphs (G, D) to model graphs with attributes for nodes and edges. They consist of a graph part G modelling the graphical structure and a data type part D with respect to a data type signature $DSIG$ for the attributes. Morphisms are compatible pairs of graph morphisms and data type homomorphisms. This leads to the category **AGraphs** of attributed graphs and $\mathbf{AGraphs}_{ATG}$ of typed attributed graphs.

The categories $(\mathbf{AGraphs}, \mathcal{M}_{AG})$ and $(\mathbf{AGraphs}_{ATG}, \mathcal{M}_{TAG})$ with classes $\mathcal{M}_{AG}$ and $\mathcal{M}_{TAG}$ of injective (typed) attributed graph morphisms with isomorphic data type component form $\mathcal{M}$-adhesive categories.

(5) The category **ElemNets** of elementary Petri nets is given by

— **objects**:
$N = (P, T, pre, post : T \to \mathcal{P}(P))$ with a set P of places, a set T of transitions, and predomain and postdomain functions $pre, post : T \to \mathcal{P}(P)$, where $\mathcal{P}(P)$ is the power set of P,

— **morphisms**:
$f : N_1 \to N_2$ with $f = (f_P, f_T)$, consisting of functions

$$f_P : P_1 \to P_2$$
$$f_T : T_1 \to T_2$$

compatible with the predomain and postdomain functions.

The category **PTNets** is defined in a similar way to **ElemNets**, but with the power set functor $\mathcal{P}$ replaced by the free commutative monoid functor $\_^{\oplus}$.

The categories $(\mathbf{ElemNets}, \mathcal{M}_{EN})$ and $(\mathbf{PTNets}, \mathcal{M}_{PT})$, where $\mathcal{M}_{EN}$ and $\mathcal{M}_{PT}$ are the classes of all injective elementary Petri net morphisms and injective place/ transition net morphisms, respectively, are $\mathcal{M}$-adhesive categories.

Ehrig *et al.* (2006a) shows that all the above examples are $\mathcal{M}$-adhesive categories.

Note that Ehrig *et al.* (2006a) considered weak adhesive HLR categories, but all the results are also valid for $\mathcal{M}$-adhesive categories – each weak adhesive HLR category is also an $\mathcal{M}$-adhesive category. In fact, the main results in Ehrig *et al.* (2006a) concerning adhesive high-level replacement systems are based on the HLR properties (1)–(4) of (weak) adhesive HLR categories (see Ehrig *et al.* (2006a, Theorem 4.26)), and since the proof of Ehrig *et al.* (2006a, Theorem 4.26) only uses the vertical weak VK property, and not the horizontal one, these HLR properties are also valid in $\mathcal{M}$-adhesive categories.

We will now consider the additional HLR requirements mentioned in the introduction, such as finite coproducts compatible with $\mathcal{M}$. The compatibility of the morphism class $\mathcal{M}$ with (finite) coproducts was required for the construction of parallel rules in Ehrig *et al.* (2006a), but, in fact, finite coproducts (if they exist) are always compatible with $\mathcal{M}$ in $\mathcal{M}$-adhesive categories.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "in_A"'] \arrow[dr, phantom, "(1)"] & A' \arrow[dr, "in_{A'}"] & & B \arrow[dl, "in_B"'] \arrow[r, "g"] \arrow[dr, phantom, "(2)"] & B' \arrow[d, "in_{B'}"] \\
A+B \arrow[rr, "f+\mathrm{id}_B"] & & A'+B \arrow[rr, "\mathrm{id}_{A'}+g"] & & A'+B'
\end{tikzcd}$$

Fig. 4. Finite coproducts compatible with $\mathcal{M}$

Fact 2.4 (finite coproducts compatible with $\mathcal{M}$). For each $\mathcal{M}$-adhesive category $(\mathbf{C}, \mathcal{M})$ with finite coproducts, finite coproducts are compatible with $\mathcal{M}$, that is, $f_i \in \mathcal{M}$ for $i = 1, \ldots, n$ implies that $f_1 + \cdots + f_n \in \mathcal{M}$.

Proof. It suffices to show this for the binary case $n = 2$. For $f : A \to A' \in \mathcal{M}$, we have pushout (1) in Figure 4 with $(f + \mathrm{id}_B) \in \mathcal{M}$ since $\mathcal{M}$-morphisms are closed under pushouts. Similarly, we have $(\mathrm{id}_{A'} + g) \in \mathcal{M}$ in pushout (2) for $g : B \to B' \in \mathcal{M}$. Hence,

$$(f + g) = (\mathrm{id}_{A'} + g) \circ (f + \mathrm{id}_B) \in \mathcal{M}$$

by composition of $\mathcal{M}$-morphisms. □

It often makes sense when constructing coproducts to use pushouts over $\mathcal{M}$-initial objects in the following sense.

Definition 2.5 ($\mathcal{M}$-initial object). An initial object I in $(\mathbf{C}, \mathcal{M})$ is said to be $\mathcal{M}$-*initial* if for each object $A \in \mathbf{C}$ the unique morphism $i_A : I \to A$ is in $\mathcal{M}$.

Note that if $(\mathbf{C}, \mathcal{M})$ has an $\mathcal{M}$-initial object, then all initial objects are $\mathcal{M}$-initial since $\mathcal{M}$ is closed under isomorphisms and composition.

The $\mathcal{M}$-initial objects in the $\mathcal{M}$-adhesive categories $(\mathbf{Sets}, \mathcal{M}_S)$, $(\mathbf{Graphs}, \mathcal{M}_G)$, $(\mathbf{Graphs}_{TG}, \mathcal{M}_{TG})$, $(\mathbf{PTNets}, \mathcal{M}_{PT})$ and $(\mathbf{ElemNets}, \mathcal{M}_{EN})$ are defined by the empty set, empty graphs, empty graphs, empty nets and empty nets, respectively. But in $(\mathbf{AGraphs}, \mathcal{M}_{AG})$, there is no $\mathcal{M}$-initial object. An initial object in this category is the attributed graph $(\emptyset, T_{DSIG})$, which consists of an empty graph part and the term algebra T_{DSIG} of the data type signature $DSIG$ as data type part. Then, the unique morphism $(\emptyset, T_{DSIG}) \to (G, D)$ contains the evaluation homomorphism from T_{DSIG} to D as data type part, which is, in general, not an isomorphism. Hence, the unique morphism is not necessarily in $\mathcal{M}_{AG}$, so $(\emptyset, T_{DSIG})$ is not $\mathcal{M}$-initial.

In order to satisfy the additional HLR requirement (5), we need finite coproducts, which can be constructed by pushouts over an initial object if it exists. Moreover, we can show that the injections into a coproduct are in $\mathcal{M}$ if it is constructed under an $\mathcal{M}$-initial object.

Fact 2.6 (existence of finite coproducts). For each $\mathcal{M}$-adhesive category $(\mathbf{C}, \mathcal{M})$ with $\mathcal{M}$-initial object, $(\mathbf{C}, \mathcal{M})$ has finite coproducts, where the injections into coproducts are in $\mathcal{M}$.

Proof. It suffices to show this for the binary case. The coproduct $A + B$ of A and B can be constructed by the pushout (1) in Figure 5a, which exists because of $i_A, i_B \in \mathcal{M}$. This

$$\begin{tikzcd} I \arrow[r, "i_A"] \arrow[d, "i_B"'] \arrow[dr, phantom, "(1)"] & A \arrow[d, "in_A"] \\ B \arrow[r, "in_B"] & A + B \end{tikzcd}$$

(a) Coproducts by $\mathcal{M}$-initial object

$$\begin{tikzcd} A_1' \arrow[dr, "m_1"] \arrow[dd, "i"'] & \\ {} \arrow[r, phantom, "(=)"] & A \\ A_2' \arrow[ur, "m_2"'] & \end{tikzcd}$$

(b) $\mathcal{M}$-subobject

Fig. 5. Coproducts and $\mathcal{M}$-subobjects

also implies $in_A, in_B \in \mathcal{M}$ since $\mathcal{M}$-morphisms are closed under pushouts in $\mathcal{M}$-adhesive categories. $\square$

Example 2.7 (finite coproduct in Graphs$_{\text{fin}}$). The category **Graphs**$_{\text{fin}}$ of finite graphs has an initial object I, *viz*. the empty graph. The unique morphism $i_A : I \to A$ to a graph A is an inclusion, so $i_A \in \mathcal{M}_G$, that is, I is $\mathcal{M}$-initial. The coproduct of two graphs is given by the componentwise disjoint union.

Note that an $\mathcal{M}$-adhesive category may still have coproducts even if it does not have an $\mathcal{M}$-initial object. For example, the $\mathcal{M}$-adhesive category $(\mathbf{AGraphs}_{ATG}, \mathcal{M}_{TAG})$ has finite coproducts, as shown in Ehrig *et al.* (2006a), but the coproduct injections are, in general, not in $\mathcal{M}_{TAG}$ since they are not necessarily isomorphic on the data part. However, coproducts in $\mathbf{AGraphs}_{ATG}$ are compatible with $\mathcal{M}_{TAG}$, that is, the coproduct of $\mathcal{M}_{TAG}$-morphisms is an $\mathcal{M}_{TAG}$-morphism..

We will now consider finite objects in $\mathcal{M}$-adhesive categories. Intuitively, we are interested in those objects where the graph or net part is finite. This can be expressed in a general $\mathcal{M}$-adhesive category by the fact that we have only a finite number of $\mathcal{M}$-subobjects. An $\mathcal{M}$-*subobject* of an object A is an isomorphism class of $\mathcal{M}$-morphisms $m : A' \to A$, where $\mathcal{M}$-morphisms

$$m_1 : A_1' \to A$$
$$m_2 : A_2' \to A$$

belong to the same $\mathcal{M}$-subobject of A if there is an isomorphism

$$i : A_1' \xrightarrow{\sim} A_2'$$

with

$$m_1 = m_2 \circ i$$

(*cf.* Figure 5b).

Definition 2.8 (finite object and finitary $\mathcal{M}$-adhesive category). An object A in an $\mathcal{M}$-adhesive category $(\mathbf{C}, \mathcal{M})$ is said to be *finite* if A has finitely many $\mathcal{M}$-subobjects.
An $\mathcal{M}$-adhesive category $(\mathbf{C}, \mathcal{M})$ is said to be *finitary* if each object $A \in \mathbf{C}$ is finite.

Remark 2.9. In the case where $\mathcal{M}$ is the class of all monomorphisms, finitarity of $(\mathbf{C}, \mathcal{M})$ coincides with the fact that $\mathbf{C}$ is finitely (well-)powered.

In $(\mathbf{Sets}, \mathcal{M}_S)$, the finite objects are the finite sets. Graphs in $(\mathbf{Graphs}, \mathcal{M}_G)$ and $(\mathbf{Graphs}_{TG}, \mathcal{M}_{TG})$ are finite if the node and edge sets have finite cardinality, while TG

$$
\begin{tikzcd}
 & & A_i \arrow[dr, "m_i"] & \\
A' \arrow[urr, bend left, "n'_i"] \arrow[drr, bend right, "n'_j"'] \arrow[r, "\exists! a"] \arrow[ur, phantom, "(=)"] \arrow[dr, phantom, "(=)"] & A \arrow[ur, "n_i"] \arrow[dr, "n_j"'] & (=) & B \\
 & & A_j \arrow[ur, "m_j"'] &
\end{tikzcd}
$$

Fig. 6. Finite $\mathcal{M}$-intersection

itself may be infinite. Petri nets in $(\textbf{ElemNets}, \mathcal{M}_{EN})$ and $(\textbf{PTNets}, \mathcal{M}_{PT})$ are finite if the number of places and transitions is finite. A typed attributed graph $AG = ((G, D), t)$ in $(\textbf{AGraphs}_{ATG}, \mathcal{M}_{TAG})$ with typing $t: (G, D) \to ATG$ is finite if the graph part of G, that is, all vertex and edge sets except the set V_D of data vertices generated from D, is finite, while the attributed type graph ATG or the data type part D may be infinite, because $\mathcal{M}$-morphisms are isomorphisms on the data type part. The restrictions of the categories $(\textbf{Sets}, \mathcal{M}_S)$, $(\textbf{Graphs}, \mathcal{M}_G)$, $(\textbf{Graphs}_{\textbf{TG}}, \mathcal{M}_{TG})$, $(\textbf{AGraphs}_{ATG}, \mathcal{M}_{TAG})$, $(\textbf{ElemNets}, \mathcal{M}_{EN})$ and $(\textbf{PTNets}, \mathcal{M}_{PT})$ to finite objects are finitary $\mathcal{M}$-adhesive categories (see Section 4).

In the following, we will use finite $\mathcal{M}$-intersections in various constructions. Finite $\mathcal{M}$-intersections are a generalisation of pullbacks to an arbitrary, but finite, number of $\mathcal{M}$-subobjects and, thus, a special case of limits.

Definition 2.10 (finite $\mathcal{M}$-intersection). Given an $\mathcal{M}$-adhesive category $(\mathbf{C}, \mathcal{M})$ and morphisms $m_i : A_i \to B \in \mathcal{M}$ ($i \in \mathcal{I}$ for finite $\mathcal{I}$) with the same codomain object B, a *finite $\mathcal{M}$-intersection* of m_i ($i \in \mathcal{I}$) is an object A with morphisms $n_i : A \to A_i$ ($i \in \mathcal{I}$), such that

$$m_i \circ n_i = m_j \circ n_j \qquad (i, j \in \mathcal{I})$$

and for each other object A' and morphisms

$$n'_i : A' \to A_i \qquad (i \in \mathcal{I})$$

with

$$m_i \circ n'_i = m_j \circ n'_j \qquad (i, j \in \mathcal{I})$$

there is a unique morphism $a : A' \to A$ with

$$n_i \circ a = n'_i \qquad (i \in \mathcal{I}).$$

Note that finite $\mathcal{M}$-intersections can be constructed by iterated pullbacks and, hence, always exist in $\mathcal{M}$-adhesive categories. Moreover, since pullbacks preserve $\mathcal{M}$-morphisms, the morphisms n_i are also in $\mathcal{M}$. In Section 6, we will use general $\mathcal{M}$-intersections where $\mathcal{I}$ is a general set instead of a finite one.

3. Additional HLR requirements for finitary $\mathcal{M}$-adhesive categories

In order to prove the main classical results for adhesive HLR systems based on (weak) adhesive HLR categories, the additional HLR requirements (5)–(7) have to be valid. In the case of finitary $\mathcal{M}$-adhesive categories $(\mathbf{C}, \mathcal{M})$, we are able to show that these additional HLR requirements are valid for suitable classes $\mathcal{E}$ and $\mathcal{E}'$, where we fix the choice of the class $\mathcal{E}$ to extremal morphisms with respect to $\mathcal{M}$.

We will first consider a special kind of $\mathcal{E}$-$\mathcal{M}$ factorisation using these extremal morphisms.

Definition 3.1 (extremal $\mathcal{E}$-$\mathcal{M}$ factorisation). Given an $\mathcal{M}$-adhesive category $(\mathbf{C}, \mathcal{M})$, the class $\mathcal{E}$ of all *extremal morphisms with respect to* $\mathcal{M}$ is defined by

$$\mathcal{E} := \{\, e \text{ in } \mathbf{C} \,|\, \text{for all } m, f \text{ in } \mathbf{C} \text{ with } m \circ f = e : m \in \mathcal{M} \text{ implies } m \text{ isomorphism} \,\}.$$

For a morphism $f : A \to B$ in $\mathbf{C}$, an *extremal $\mathcal{E}$-$\mathcal{M}$ factorisation* of f is given by an object $\bar{B}$ and morphisms

$$e : A \to \bar{B} \in \mathcal{E}$$
$$m : \bar{B} \to B \in \mathcal{M}$$

such that $m \circ e = f$.

Remark 3.2. Although the class $\mathcal{E}$ in several example categories consists of all epimorphisms, we will show below that the class $\mathcal{E}$ of extremal morphisms with respect to $\mathcal{M}$ is not necessarily a class of epimorphisms. However, if we require $\mathcal{M}$ to be the class of all monomorphisms and e in the definition of $\mathcal{E}$ in Definition 3.1 to be an epimorphism, then $\mathcal{E}$ is the class of all extremal epimorphisms in the sense of Adámek *et al.* (1990).

Fact 3.3 (uniqueness of extremal $\mathcal{E}$-$\mathcal{M}$ factorisations). Given an $\mathcal{M}$-adhesive category $(\mathbf{C}, \mathcal{M})$, the extremal $\mathcal{E}$-$\mathcal{M}$-factorisations are unique up to isomorphism, that is, for each morphism $f : A \to B$ in $\mathbf{C}$ with extremal $\mathcal{E}$-$\mathcal{M}$-factorisations

$$m \circ e = f \quad \textit{via} \quad \bar{B}$$
$$m' \circ e' = f \quad \textit{via} \quad \bar{B}',$$

we have an isomorphism

$$i : \bar{B} \to \bar{B}'$$

with

$$i \circ e = e'$$
$$m' \circ i = m.$$

Proof. Since $m \in \mathcal{M}$ and $m' \in \mathcal{M}$ and $\mathcal{M}$-adhesive categories have pullbacks along $\mathcal{M}$-morphisms, we can construct the pullback (1) in Figure 7a, where $p \in \mathcal{M}$ and $p' \in \mathcal{M}$, because $\mathcal{M}$-morphisms are closed under pullbacks. Since

$$m \circ e = f = m' \circ e',$$

the universal property of the pullback induces a unique morphism $q : A \to P$ with

$$p \circ q = e$$
$$p' \circ q = e'.$$

$$
\begin{tikzcd}
 & & \bar{B} \arrow[dr, "m"] & \\
A \arrow[urr, bend left, "e"] \arrow[drr, bend right, "e'"'] \arrow[r, "\exists! q"] \arrow[ur, phantom, "(=)"] \arrow[dr, phantom, "(=)"] & P \arrow[ur, "p"] \arrow[dr, "p'"'] \arrow[rr, phantom, "(1)"] & & B \\
 & & \bar{B}' \arrow[ur, "m'"'] &
\end{tikzcd}
$$

(a) Uniqueness of $\mathcal{E}$-$\mathcal{M}$-factorisations

$$
\begin{tikzcd}
 & & A \arrow[dll, bend right, "e"'] \arrow[dl, "e_i"'] \arrow[dd, "f"] \\
\bar{B} \arrow[r, "\bar{m}_i"] \arrow[drr, bend right, "m"'] \arrow[ur, phantom, "(=)"] \arrow[dr, phantom, "(=)"] & B_i \arrow[dr, "m_i"] \arrow[r, phantom, "(=)"] & {} \\
 & & B
\end{tikzcd}
$$

(b) $\mathcal{M}$-intersection factorisation

Fig. 7. Factorisations

Now, because $e \in \mathcal{E}$ and $e' \in \mathcal{E}$ with factorisations

$$
\begin{aligned}
p \circ q &= e \\
p' \circ q &= e'
\end{aligned}
$$

with $p \in \mathcal{M}$ and $p' \in \mathcal{M}$, we have that p and p' are isomorphisms with corresponding inverses $p^{-1} : \bar{B} \to P$ and $(p')^{-1} : \bar{B}' \to P$.
Finally, the required isomorphism can be constructed by

$$i := p' \circ p^{-1}$$

with

$$
\begin{aligned}
i \circ e &= p' \circ p^{-1} \circ p \circ q \\
&= p' \circ q \\
&= e'
\end{aligned}
$$

and

$$
\begin{aligned}
m' \circ i &= m' \circ p' \circ p^{-1} \\
&= m \circ p \circ p^{-1} \\
&= m.
\end{aligned}
$$

This completes the proof. □

Fact 3.4 (existence of extremal $\mathcal{E}$-$\mathcal{M}$ factorisations). Given a finitary $\mathcal{M}$-adhesive category $(\mathbf{C}, \mathcal{M})$, we can construct an extremal $\mathcal{E}$-$\mathcal{M}$ factorisation $m \circ e = f$ for each morphism $f : A \to B$ in $\mathbf{C}$.

Construction: The morphism $m : \bar{B} \to B$ is constructed as the finite $\mathcal{M}$-intersection of all $\mathcal{M}$-subobjects $m_i : B_i \to B$ for which there exists $e_i : A \to B_i$ with

$$f = m_i \circ e_i.$$

This leads to a suitable finite index set $\mathcal{I}$, and $e : A \to \bar{B}$ is the induced unique morphism with

$$\bar{m}_i \circ e = e_i$$

for all $i \in \mathcal{I}$.

Proof. The $\mathcal{M}$-subobjects $m_i : B_i \to B$ with $e_i : A \to B_i$ and $f = m_i \circ e_i$ contain at least the trivial subobject given by

$$\begin{aligned} B_i &= B \\ m_i &= \mathrm{id}_B \in \mathcal{M} \\ e_i &= f. \end{aligned}$$

Since each object B is finite in $(\mathbf{C}, \mathcal{M})$, the intersection of all $\mathcal{M}$-subobjects $m_i : B_i \to B$ as defined above exists and is finite. This also shows that $\bar{m}_i \in \mathcal{M}$ and $m \in \mathcal{M}$ because $\mathcal{M}$ is closed under pullbacks and composition.

It remains to show that $e \in \mathcal{E}$. Let

$$e = m' \circ e'$$

be a factorisation of e with $m' \in \mathcal{M}$. So we have that $m \circ m'$ is an $\mathcal{M}$-subobject of B and

$$m \circ m' \circ e' = f,$$

and since $\mathcal{M}$-subobjects are equivalence classes, there exists, without loss of generality, an $i \in \mathcal{I}$ such that

$$\begin{aligned} B' &= B_i \\ m \circ m' &= m_i \\ e' &= e_i. \end{aligned}$$

This implies that there exists

$$\bar{m}_i : \bar{B} \to B_i = B'$$

with

$$m_i \circ \bar{m}_i = m.$$

Now,

$$m_i \circ \bar{m}_i \circ m' = m \circ m' = m_i$$

and the fact that $m_i \in \mathcal{M}$ is a monomorphism implies that

$$\bar{m}_i \circ m' = \mathrm{id}_{B'}.$$

Moreover,

$$m \circ m' \circ \bar{m}_i = m_i \circ \bar{m}_i = m$$

and the fact that $m \in \mathcal{M}$ is a monomorphism implies that

$$m' \circ \bar{m}_i = \mathrm{id}_{\bar{B}}.$$

Hence, m' and $\bar{m}_i$ are mutually inverse isomorphisms and $e \in \mathcal{E}$. □

Example 3.5 (extremal $\mathcal{E}$-$\mathcal{M}$ factorisation in $\mathbf{Graphs}_{\mathrm{fin}}$). Consider the morphism $f : A \to B$ in $\mathbf{Graphs}_{\mathrm{fin}}$ shown in Figure 9. There are three different subobjects $m_i : B_i \to B$ of B

$$\begin{tikzcd}
A \arrow[rr, "e"] \arrow[rrrr, bend left, "f"] \arrow[dr, "e' = e_i"'] & & \bar{B} \arrow[rr, "m"] \arrow[dl, bend left, "\bar{m}_i"] & (=) & B \\
& B' = B_i \arrow[ur, bend left, "m'"] \arrow[urrr, bend right, "m_i"'] & & &
\end{tikzcd}$$

Fig. 8. An $\mathcal{M}$-intersection factorisation is an extremal $\mathcal{E}$-$\mathcal{M}$-factorisation

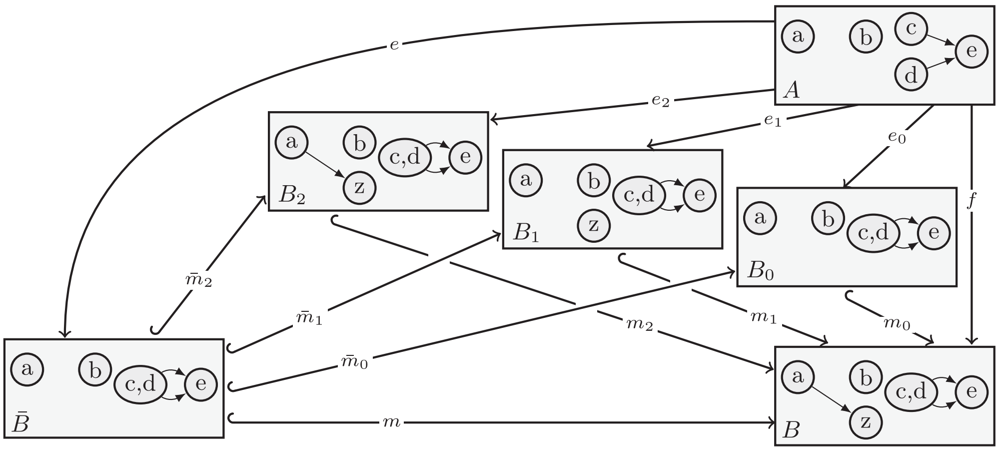


Fig. 9. Construction of extremal $\mathcal{E}$-$\mathcal{M}$-factorisation in $\mathbf{Graphs}_{\mathrm{fin}}$

containing the image $f(A)$, which ensures that there are morphisms $e_i : A \to B_i$ such that

$$f = m_i \circ e_i.$$

Note that the graphs B_1 and B_2 also contain the node z, and B_2 contains the edge between a and z.

The extremal $\mathcal{E}$-$\mathcal{M}$ factorisation $(\bar{B}, e, m)$ is constructed by computing the $\mathcal{M}$-intersection $\bar{B}$ of $(m_i)_{i\in\{0,1,2\}}$. Then the universal property of the $\mathcal{M}$-intersection induces a unique morphism $e : A \to \bar{B}$, and the morphism $m : \; : \bar{B} \to B$ is obtained by composition

$$m = m_i \circ \bar{m}_i.$$

Note that we have

$$(\bar{B}, m) = (B_0, m_0).$$

In the categories $(\mathbf{Sets}, \mathcal{M}_S)$, $(\mathbf{Graphs}, \mathcal{M}_G)$, $(\mathbf{Graphs_{TG}}, \mathcal{M}_{TG})$, $(\mathbf{ElemNets}, \mathcal{M}_{EN})$ and $(\mathbf{PTNets}, \mathcal{M}_{PT})$, the extremal $\mathcal{E}$-$\mathcal{M}$ factorisation $f = m \circ e$ for $f : A \to B$ with finite A and B is just the well-known epi-mono factorisation of morphisms, which also works for infinite objects A and B because these categories have not only finite but also general intersections. For $(\mathbf{AGraphs_{ATG}}, \mathcal{M}_{TAG})$, the extremal $\mathcal{E}$-$\mathcal{M}$ factorisation of

$$(f_G, f_D) : (G, D) \to (G', D')$$

with finite (or infinite) G and G' is given by

$$(f_G, f_D) = (m_G, m_D) \circ (e_G, e_D)$$

with

$$\begin{aligned}(e_G, e_D)&: (G, D) \to (\bar{G}, \bar{D})\\ (m_G, m_D)&: (\bar{G}, \bar{D}) \to (G', D'),\end{aligned}$$

where e_G is an epimorphism, m_G is a monomorphism and m_D is an isomorphism. In general, e_D (and thus (e_G, e_D) also) is not an epimorphism since m_D is an isomorphism, so e_D has to be essentially the same as f_D. This means that the class $\mathcal{E}$, which depends on $\mathcal{M}$, is not necessarily a class of epimorphisms.

For various results we will need an additional class $\mathcal{E}'$, which contains pairs of morphisms with the same codomain. In a category with coproducts, a suitable definition for morphisms e, f to be in $\mathcal{E}'$ is to require that the corresponding induced morphism from the coproduct of their domains is in $\mathcal{E}$. Given an $\mathcal{E}$-$\mathcal{M}$ factorisation and binary coproducts, we are able to construct an $\mathcal{E}'$-$\mathcal{M}$ pair factorisation in a standard way – see Ehrig *et al.* (2006a), where the more general notion of $\mathcal{E}'$-$\mathcal{M}'$ pair factorisations for some morphism class $\mathcal{M}'$ is considered. We will begin by recalling the definition of $\mathcal{E}'$-$\mathcal{M}$ pair factorisation.

Definition 3.6 ($\mathcal{E}'$-$\mathcal{M}$ pair factorisation). Given a morphism class $\mathcal{M}$ and a class $\mathcal{E}'$ of morphism pairs with common codomain in a category **C**, we say **C** has an $\mathcal{E}'$-$\mathcal{M}$ *pair factorisation* if for each pair of morphisms

$$\begin{aligned}f_A&: A \to D\\ f_B&: B \to D,\end{aligned}$$

there are, unique up to isomorphism, an object C and morphisms

$$\begin{aligned}e_A&: A \to C\\ e_B&: B \to C\\ m&: C \to D\end{aligned}$$

with

$$\begin{aligned}(e_A, e_B) &\in \mathcal{E}'\\ m &\in \mathcal{M}\end{aligned}$$

and

$$\begin{aligned}m \circ e_A &= f_A\\ m \circ e_B &= f_B.\end{aligned}$$

In other words, there is a unique subobject such that f_A and f_B factor through its elements and the respective morphisms belong to $\mathcal{E}'$.

(a) $\mathcal{E}'$-$\mathcal{M}$ pair factorisation

(b) Construction of $\mathcal{E}'$-$\mathcal{M}$ pair factorisation

Fig. 10. Pair factorisation

In the following, we will consider an $\mathcal{E}$-$\mathcal{M}$ factorisation that is either an extremal $\mathcal{E}$-$\mathcal{M}$ factorisation (Definition 3.1) or a classical $\mathcal{E}$-$\mathcal{M}$ factorisation with suitable classes $\mathcal{E}$ of epimorphisms and $\mathcal{M}$ of monomorphisms.

Fact 3.7 (construction of $\mathcal{E}'$-$\mathcal{M}$ pair factorisation). Given a category **C** with an $\mathcal{E}$-$\mathcal{M}$ factorisation and binary coproducts, **C** also has an $\mathcal{E}'$-$\mathcal{M}$ pair factorisation for the class

$$\mathcal{E}' = \{(e_A : A \to C, e_B : B \to C) \mid e_A, e_B \in \mathbf{C} \text{ with induced } e : A + B \to C \in \mathcal{E}\}.$$

Proof. Given

$$f_A : A \to D$$
$$f_B : B \to D$$

with induced

$$f : A + B \to D,$$

we consider the $\mathcal{E}$-$\mathcal{M}$ factorisation

$$f = m \circ e$$

of f with $e \in \mathcal{E}$ and $m \in \mathcal{M}$, and define

$$e_A = e \circ in_A$$
$$e_B = e \circ in_B.$$

Then $(e_A, e_B) \in \mathcal{E}'$ and $m \in \mathcal{M}$ defines an $\mathcal{E}'$-$\mathcal{M}$ pair factorisation of (f_A, f_B) that is unique up to isomorphism since all other $\mathcal{E}'$-$\mathcal{M}$ pair factorisations also lead to an $\mathcal{E}$-$\mathcal{M}$ factorisation through the induced morphism in $\mathcal{E}$, and $\mathcal{E}$-$\mathcal{M}$ factorisations are unique up to isomorphism. $\square$

Remark 3.8. With the previous facts, we now have extremal $\mathcal{E}$-$\mathcal{M}$ factorisations and corresponding $\mathcal{E}'$-$\mathcal{M}$ pair factorisations for all finitary $\mathcal{M}$-adhesive categories with $\mathcal{M}$-initial objects, and these factorisations are unique up to isomorphism.

Example 3.9 ($\mathcal{E}'$-$\mathcal{M}$ pair factorisation in Graphs). Consider the morphisms

$$f_A : A \to D$$
$$f_B : B \to D$$

in **Graphs** shown in Figure 11. Since **Graphs** has binary coproducts and $\mathcal{E}$-$\mathcal{M}$ factorisations, we obtain an $\mathcal{E}'$-$\mathcal{M}$ pair factorisation of f_A and f_B by computation of the coproduct

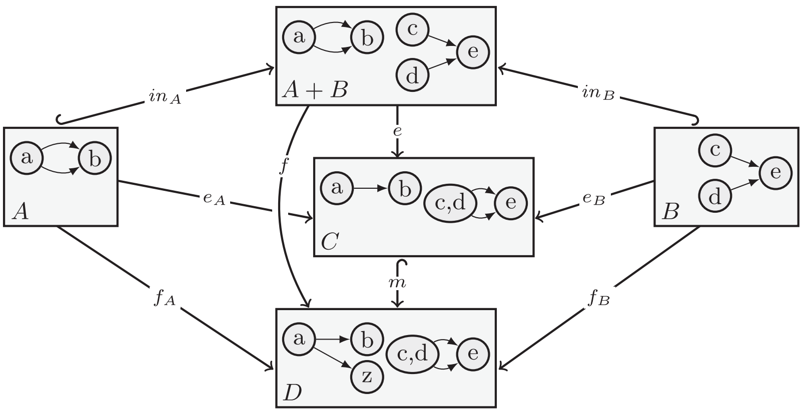


Fig. 11. $\mathcal{E}'$-$\mathcal{M}$ pair factorisation in **Graphs**

$$\begin{tikzcd}
B \arrow[r, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "(1)"] & L \arrow[d, "m"] \\
C \arrow[r, "c"] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "b^*"] \arrow[rr, bend left, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "(3)"] & B' \arrow[r, "b'"] \arrow[d, "a'"] \arrow[dr, phantom, "(2)"] & L \arrow[d, "m"] \\
C \arrow[r, "c^*"] \arrow[rr, bend right, "c"'] & C' \arrow[r, "c'"] & G
\end{tikzcd}$$

(a) Initial pushout

$$\begin{tikzcd}
B \arrow[r, "b"] \arrow[rr, bend left, "b^*"] \arrow[d, "a"'] \arrow[dr, phantom, "(1)"] & L \arrow[d, "m"] \arrow[dr, phantom, "(2)"] & K \arrow[l, "l"'] \arrow[d, "k"] \\
C \arrow[r, "c"] \arrow[rr, bend right, "c^*"'] & G & D \arrow[l, "f"']
\end{tikzcd}$$

(b) Pushout complement construction

Fig. 12. Initial pushout and pushout complement construction

$A + B$ and the construction of an $\mathcal{E}$-$\mathcal{M}$ factorisation of the induced unique morphism

$$f : A + B \to D$$

as shown in Figure 11.

Finally, let us consider the construction of initial pushouts in finitary $\mathcal{M}$-adhesive categories. As with the extremal $\mathcal{E}$-$\mathcal{M}$ factorisation, we are able to construct initial pushouts by finite $\mathcal{M}$-intersections of $\mathcal{M}$-subobjects in finitary $\mathcal{M}$-adhesive categories. First we will recall the definition.

Definition 3.10 (initial pushout). A pushout (1) over a morphism $m : L \to G$ as shown in Figure 12a with $b, c \in \mathcal{M}$ in an $\mathcal{M}$-adhesive category $(\mathbf{C}, \mathcal{M})$ is said to be *initial* if the following condition holds: for all pushouts (2) over m with $b', c' \in \mathcal{M}$ there exist unique morphisms $b^*, c^* \in \mathcal{M}$ such that

$$b' \circ b^* = b$$
$$c' \circ c^* = c,$$

and (3) is a pushout.

Remark 3.11. As shown in Ehrig *et al.* (2006a), the initial pushout allows us to define a gluing condition, which is necessary and sufficient for the construction of pushout

$$
\begin{tikzcd}
B \arrow[r, "u_i"] \arrow[rr, bend left, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "(\mathrm{Q}_i)"] & B_i \arrow[r, "b_i"] \arrow[d, "a_i"] \arrow[dr, phantom, "(\mathrm{P}_i)"] & L \arrow[d, "m"] \\
C \arrow[r, "v_i"] \arrow[rr, bend right, "c"'] & C_i \arrow[r, "c_i"] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "(1)"] & L \arrow[d, "m"] \\
C \arrow[r, "c"] & G
\end{tikzcd}
$$

(a) Construction of initial pushout

$$
\begin{tikzcd}
L \arrow[r, "\mathrm{id}_L"] \arrow[d, "m"'] \arrow[dr, phantom, "(4)"] & L \arrow[d, "m"] \\
G \arrow[r, "\mathrm{id}_G"] & G
\end{tikzcd}
$$

(b) Trivial pushout over m

Fig. 13. Construction of initial pushout over m

complements. Given $m: L \to G$ with initial pushout (1) as shown in Figure 12b and

$$l: K \to L \in \mathcal{M},$$

which can be considered as the left-hand side of a rule. The gluing condition is satisfied if there exists $b^*: B \to K$ with

$$l \circ b^* = b.$$

In this case, the pushout complement object D in (2) can be constructed as a pushout object of a and b^*.

We will now show the construction of initial pushouts by finite $\mathcal{M}$-intersections.

Fact 3.12 (initial pushouts in finitary $\mathcal{M}$-adhesive categories). Each finitary $\mathcal{M}$-adhesive category has initial pushouts.

Construction: Given $m: L \to G$, we consider all those $\mathcal{M}$-subobjects $b_i: B_i \to L$ of L and $c_i: C_i \to G$ of G such that there is a pushout (P_i) over m. Since L and G are finite, this leads to a finite index set $\mathcal{I}$ for all (up to isomorphism) (P_i) with $i \in \mathcal{I}$. We now construct $b: B \to L$ as the finite $\mathcal{M}$-intersection of $(b_i)_{i\in\mathcal{I}}$ and $c: C \to G$ as the finite $\mathcal{M}$-intersection of $(c_i)_{i\in\mathcal{I}}$. Then there is a unique $a: B \to C$ such that (Q_i) commutes for all $i \in \mathcal{I}$ and the outer diagram (1) is the initial pushout over m.

Proof. We have to show that (1) is the initial pushout over m. As mentioned earlier, $\mathcal{I}$ is finite, but we also have that $\mathrm{card}(\mathcal{I}) \geqslant 1$ using the trivial pushout (4) over m in Figure 13b.

Since finite $\mathcal{M}$-intersections can be constructed by iterated pullbacks, we will start with $\mathcal{I} = \{1, 2\}$ and pushouts (P_i) and show that (Q_i) and, thus, $(\mathrm{Q}_i) + (\mathrm{P}_i)$ are pushouts for $i \in \mathcal{I}$.

In the cube in Figure 14a, the top and bottom faces are pullbacks by the $\mathcal{M}$-intersection construction. The right back and right front faces are pushouts (P_1) and (P_2) with $b_1, b_2 \in \mathcal{M}$, so they are also pullbacks in $\mathcal{M}$-adhesive categories. By pullback composition and decomposition, the left back and left front faces, and hence all squares, are also pullbacks. Since the right back face is a pushout (P_1) along $b_1 \in \mathcal{M}$, and $u_1, v_1, b_2, c_2 \in \mathcal{M}$, the vertical weak VK property implies that the left front square (Q_{12}) is also a pushout. Similarly, the left back square (Q_{11}) also becomes a pushout, so the compositions $(\mathrm{Q}_{11}) + (\mathrm{P}_1)$ and $(\mathrm{Q}_{12}) + (\mathrm{P}_2)$ are pushouts too.

$$\begin{tikzcd}
{} & B_1 \arrow[dr, "b_1"] \arrow[ddd, "a_1"] & {} \\
B_{12} \arrow[ur, "u_1"] \arrow[dr, "u_2"] \arrow[ddd, "a_{12}"] & {} & L \arrow[ddd, "m"] \\
{} & B_2 \arrow[ur, "b_2"] \arrow[ddd, "a_2"] & {} \\
{} & C_1 \arrow[dr, "c_1"] & {} \\
C_{12} \arrow[ur, "v_1"] \arrow[dr, "v_2"] & {} & G \\
{} & C_2 \arrow[ur, "c_2"] & {}
\end{tikzcd}$$

(a) Cube for $n = 2$

$$\begin{tikzcd}
{} & B_{12} \arrow[dr, "b_{12}"] \arrow[ddd, "a_{12}"] & {} \\
B_{123} \arrow[ur] \arrow[dr, "u_3"] \arrow[ddd, "a_{123}"] & {} & L \arrow[ddd, "m"] \\
{} & B_3 \arrow[ur, "b_3"] \arrow[ddd, "a_3"] & {} \\
{} & C_{12} \arrow[dr, "c_{12}"] & {} \\
C_{123} \arrow[ur] \arrow[dr, "v_3"] & {} & G \\
{} & C_3 \arrow[ur, "c_3"] & {}
\end{tikzcd}$$

(b) Cube for $n = 3$

Fig. 14. Van Kampen cubes

$$\begin{tikzcd}
B \arrow[r, "b"] \arrow[d, "a"] \arrow[dr, phantom, "(1)"] & L \arrow[d, "m"] \arrow[dr, phantom, "(1')"] & B' = B_{i_0} \arrow[l, "b'"'] \arrow[d, "a'"] \\
C \arrow[r, "c"] & G & C' = C_{i_0} \arrow[l, "c'"']
\end{tikzcd}$$

Fig. 15. Initiality of pushout (1)

For the case $n = 3$, the $\mathcal{M}$-intersections B_{123} and C_{123} are given by the top and bottom faces of the cube in Figure 14b, where the right back square is the pushout $(Q_{11}) + (P_1)$ along $b_{12} = b_1 \circ u_1 : B_{12} \to L$ constructed above, and the right front face is pushout (P_3). Hence, we have the same assumptions as above for the case $n = 2$ and can conclude that the left back and left front faces are also pushouts.

For $\mathrm{card}(\mathcal{I}) = n$, we can use this iterated construction to obtain $B = B_{1\ldots n}$ and $C = C_{1\ldots n}$ with pushouts (Q_i) defined by composition. Now $(1) = (Q_1) + (P_1)$ is a pushout along $b \in \mathcal{M}$ and also initial because every other pushout $(1')$ over m with $b' \in \mathcal{M}$ is equal to (P_{i_0}) for some $i_0 \in \mathcal{I}$. Hence, the initiality property is given by the pushout (Q_{i_0}) as constructed above. □

Example 3.13 (initial pushout in $\mathbf{Graphs}_{\mathrm{fin}}$). Consider the morphism $m\colon L \to G$ in the category $\mathbf{Graphs}_{\mathrm{fin}}$ of finite graphs shown in Figure 16. There are five different pushouts $(P_i)_{i\in\{0,\ldots,4\}}$ over m. The interface B_i of each pushout contains at least the nodes c and d identified by the morphism m. Moreover, each B_i consists of the nodes a and b that are mapped by m to the source and target, respectively, of an edge in the graph G. Some of the pushouts also contain the node e and one or both of the edges to the node e. Furthermore, the corresponding graphs C_i contain the node z and the edges from a to b and z. Moreover, the nodes c and d are identified in the graphs C_i.

The so-called boundary B is obtained as the $\mathcal{M}$-intersection of $(b_i)_{i\in\{0,\ldots,4\}}$ and the context C as the $\mathcal{M}$-intersection of $(c_i)_{i\in\{0,\ldots,4\}}$. The graphs B and C consist of the nodes and edges that all graphs B_i and C_i, respectively, have in common. Note that the initial pushout (B, C, a, b, c) over m coincides with (P_0).

The following theorem summarises the fact that the additional HLR requirements mentioned above are valid for all finitary $\mathcal{M}$-adhesive categories.

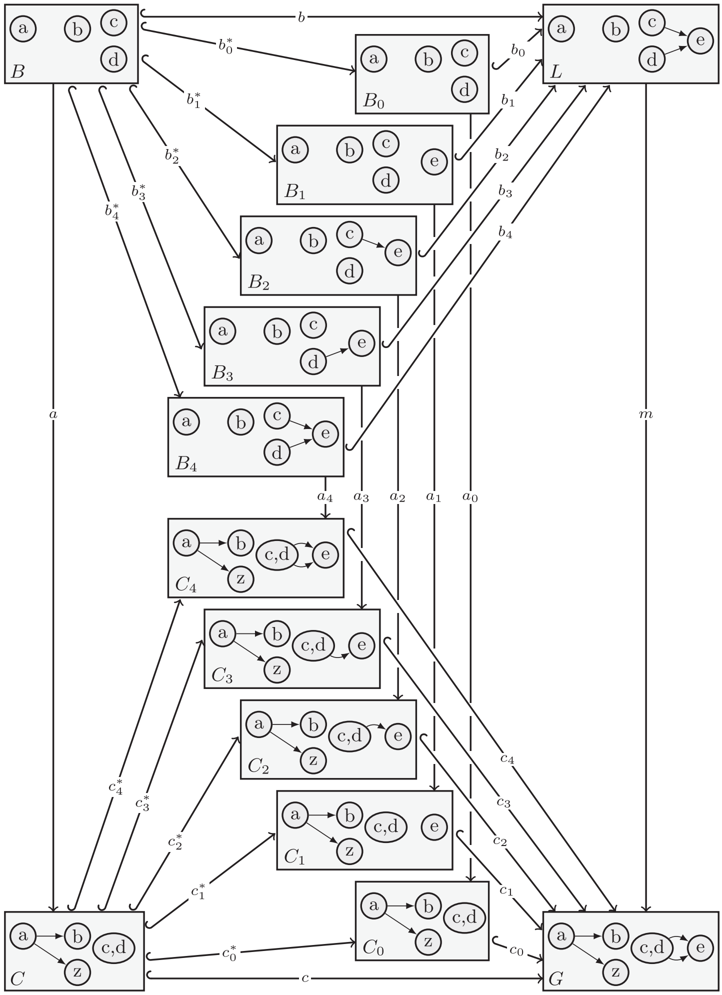


Fig. 16. Construction of initial pushout in $\mathbf{Graphs}_{\mathrm{fin}}$

Theorem 3.14 (additional HLR requirements in finitary $\mathcal{M}$-adhesive categories). Given a finitary $\mathcal{M}$-adhesive category $(\mathbf{C}, \mathcal{M})$, the following additional HLR requirements are valid:

(1) $(\mathbf{C}, \mathcal{M})$ has initial pushouts.
(2) $(\mathbf{C}, \mathcal{M})$ has a unique extremal $\mathcal{E}$-$\mathcal{M}$ factorisation, where $\mathcal{E}$ is the class of all extremal morphisms with respect to $\mathcal{M}$.

If $(\mathbf{C}, \mathcal{M})$ has an $\mathcal{M}$-initial object, we also have that:

(3) $(\mathbf{C}, \mathcal{M})$ has finite coproducts compatible with $\mathcal{M}$.
(4) $(\mathbf{C}, \mathcal{M})$ has a unique $\mathcal{E}'$-$\mathcal{M}$ pair factorisation, where the class $\mathcal{E}'$ is induced by $\mathcal{E}$.

Proof. Requirement (1) follows from Fact 3.12. Requirement (2) follows from Facts 3.3 and 3.4. Requirement (3) from Facts 2.6 and 2.4. Finally, Requirement (4) follows from Fact 3.7. □

4. Finitary restriction of $\mathcal{M}$-adhesive categories

In order to construct $\mathcal{M}$-adhesive categories, it is important to know that $(\mathbf{Sets}, \mathcal{M}_S)$ is an $\mathcal{M}$-adhesive category and that, like weak adhesive HLR categories, $\mathcal{M}$-adhesive categories are also closed under product, slice, coslice, functor and comma category constructions, provided suitable conditions are satisfied – see Ehrig *et al.* (2006a). This allows us to show that $(\mathbf{Graphs}, \mathcal{M}_G)$, $(\mathbf{Graphs_{TG}}, \mathcal{M}_{TG})$, $(\mathbf{ElemNets}, \mathcal{M}_{EN})$ and $(\mathbf{PTNets}, \mathcal{M}_{PT})$ are $\mathcal{M}$-adhesive categories also. However, it is more difficult to show similar results for the additional HLR requirements considered in Section 3, especially since we only have weak results for the existence and construction of initial pushouts (Prange *et al.* 2008).

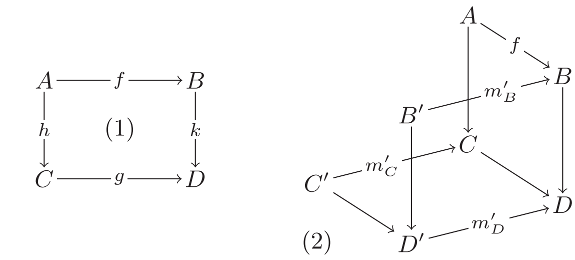
We have already shown that these additional HLR requirements are valid in finitary $\mathcal{M}$-adhesive categories under weak assumptions. It remains to show how to construct finitary $\mathcal{M}$-adhesive categories. In the main result of this section, we will show that for any $\mathcal{M}$-adhesive category $(\mathbf{C}, \mathcal{M})$, the restriction $(\mathbf{C}_{\text{fin}}, \mathcal{M}_{\text{fin}})$ to finite objects is a finitary $\mathcal{M}$-adhesive category for the morphism class $\mathcal{M}_{\text{fin}}$, where $\mathcal{M}_{\text{fin}}$ is the corresponding restriction of $\mathcal{M}$. Moreover, we know how to construct pushouts and pullbacks in $\mathbf{C}_{\text{fin}}$ along $\mathcal{M}_{\text{fin}}$-morphisms because the inclusion functor $I_{\text{fin}} : \mathbf{C}_{\text{fin}} \to \mathbf{C}$ creates and preserves pushouts and pullbacks along $\mathcal{M}_{\text{fin}}$ and $\mathcal{M}$, respectively.

Definition 4.1 (finitary restriction of $\mathcal{M}$-adhesive category). Given an $\mathcal{M}$-adhesive category $(\mathbf{C}, \mathcal{M})$ the restriction to all finite objects of $(\mathbf{C}, \mathcal{M})$ defines the full subcategory $\mathbf{C}_{\text{fin}}$ of $\mathbf{C}$, and $(\mathbf{C}_{\text{fin}}, \mathcal{M}_{\text{fin}})$ with $\mathcal{M}_{\text{fin}} = \mathcal{M} \cap \mathbf{C}_{\text{fin}}$ is said to be a *finitary restriction* of $(\mathbf{C}, \mathcal{M})$.

Remark 4.2. Note that an object A in $\mathbf{C}$ is finite in $(\mathbf{C}, \mathcal{M})$ if and only if A is finite in $(\mathbf{C}_{\text{fin}}, \mathcal{M}_{\text{fin}})$. If $\mathcal{M}$ is the class of all monomorphisms in $\mathbf{C}$, then $\mathcal{M}_{\text{fin}}$ is not necessarily the class of all monomorphisms in $\mathbf{C}_{\text{fin}}$. It is an open question whether for an adhesive category $\mathbf{C}$, which is based on the class of all monomorphisms, there may be monomorphisms in $\mathbf{C}_{\text{fin}}$ that are not monomorphisms in $\mathbf{C}$, so it is not clear whether the finite objects in $\mathbf{C}$ and $\mathbf{C}_{\text{fin}}$ are the same. This problem is avoided for $\mathcal{M}$-adhesive categories, where finitarity depends on $\mathcal{M}$.

In order to prove that if $(\mathbf{C}, \mathcal{M})$ is an $\mathcal{M}$-adhesive category, then $(\mathbf{C}_{\text{fin}}, \mathcal{M}_{\text{fin}})$ is an $\mathcal{M}$-adhesive category too, we have to analyse the construction and preservation of pushouts and pullbacks in $(\mathbf{C}, \mathcal{M})$ and $(\mathbf{C}_{\text{fin}}, \mathcal{M}_{\text{fin}})$. This corresponds to the following creation and preservation properties of the inclusion functor $I_{\text{fin}} : \mathbf{C}_{\text{fin}} \to \mathbf{C}$.

Definition 4.3 (creation and preservation of pushout and pullback). Given an $\mathcal{M}$-adhesive category $(\mathbf{C}, \mathcal{M})$, the inclusion functor $I_{\text{fin}} : \mathbf{C}_{\text{fin}} \to \mathbf{C}$ *creates pushouts along* $\mathcal{M}$ if for each pair of morphisms f and h in $\mathbf{C}_{\text{fin}}$ with $f \in \mathcal{M}_{\text{fin}}$ and pushout (1) (see Figure 17a) in $\mathbf{C}$ we already have $D \in \mathbf{C}_{\text{fin}}$ such that (1) is a pushout in $\mathbf{C}_{\text{fin}}$ along $\mathcal{M}_{\text{fin}}$.
Similarly, I_{fin} *creates pullbacks along* $\mathcal{M}$ if, for each pullback (1) in $\mathbf{C}$ with $g \in \mathcal{M}_{\text{fin}}$ and $B, C, D \in \mathbf{C}_{\text{fin}}$, we also have $A \in \mathbf{C}_{\text{fin}}$ such that (1) is a pullback in $\mathbf{C}_{\text{fin}}$ along $\mathcal{M}_{\text{fin}}$.



(a) Pushout or pullback (b) Creation by pullbacks

Fig. 17. Creation and preservation of pushout and pullback

I_{fin} *preserves pushouts (pullbacks) along* $\mathcal{M}_{\text{fin}}$ if each pushout (pullback) (1) in $\mathbf{C}_{\text{fin}}$ with $f \in \mathcal{M}_{\text{fin}}$ ($g \in \mathcal{M}_{\text{fin}}$) is also a pushout (pullback) in $\mathbf{C}$ with $f \in \mathcal{M}$ ($g \in \mathcal{M}$).

Fact 4.4 (creation and preservation of pushout and pullback). Given an $\mathcal{M}$-adhesive category $(\mathbf{C}, \mathcal{M})$, the inclusion functor $I_{\text{fin}} : \mathbf{C}_{\text{fin}} \to \mathbf{C}$ creates pushouts and pullbacks along $\mathcal{M}$ and preserves pushouts and pullbacks along $\mathcal{M}_{\text{fin}}$.

Proof.

(1) I_{fin} creates pullbacks along $\mathcal{M}$ because, given pullback (1) in $\mathbf{C}$ with $B, C, D \in \mathbf{C}_{\text{fin}}$ and $g \in \mathcal{M}$, we also have $f \in \mathcal{M}$. Moreover, each $\mathcal{M}$-subobject of A is also an $\mathcal{M}$-subobject of B because $f \in \mathcal{M}$. Hence, $B \in \mathbf{C}_{\text{fin}}$ implies that $A \in \mathbf{C}_{\text{fin}}$ and (1) is also a pullback in $\mathbf{C}_{\text{fin}}$ with $f \in \mathcal{M}_{\text{fin}}$.

(2) I_{fin} creates pushouts along $\mathcal{M}$ because given pushout (1) in $\mathbf{C}$ with $A, B, C \in \mathbf{C}_{\text{fin}}$ and $f \in \mathcal{M}$, we also have $g \in \mathcal{M}$. It remains to show that $D \in \mathbf{C}_{\text{fin}}$.
Given morphism $m'_D : D' \to D \in \mathcal{M}$, we obtain morphisms

$$m'_B : B' \to B \in \mathcal{M}$$
$$m'_C : C' \to C \in \mathcal{M}$$

by pullback constructions as in (2) in Figure 17b. By Lemma 4.5 below, for

$$m'_D : D' \to D \in \mathcal{M}$$
$$m''_D : D'' \to D \in \mathcal{M}$$

with corresponding m'_B, m'_C, m''_B and m''_C, we have that

$$m'_B \cong m''_B$$
$$m'_C \cong m''_C$$

implies that

$$m'_D \cong m''_D$$

too. This is equivalent to the injectivity of the $\mathcal{M}$-subobject function

$$\Phi : \mathrm{MSub}(D) \to \mathrm{MSub}(B) \times \mathrm{MSub}(C)$$

defined by

$$\Phi([m'_D]) = ([m'_B], [m'_C]).$$

Here, $\mathrm{MSub}(X)$ is the set of all $\mathcal{M}$-subobjects of $X = D, B, C$ and m'_B, m'_C are constructed by pullbacks of m'_D as discussed above. Note that $[m'_D]$ is the subobject corresponding to m'_D. Now $B, C \in \mathbf{C}_{\text{fin}}$ implies that $\mathrm{MSub}(B)$ and $\mathrm{MSub}(C)$ are finite. Hence, $\mathrm{MSub}(B) \times \mathrm{MSub}(C)$ is finite, and the injectivity of Φ implies that $\mathrm{MSub}(D)$ is finite too, and, therefore, $D \in \mathbf{C}_{\text{fin}}$.

(3) I_{fin} preserves pushouts along $\mathcal{M}_{\text{fin}}$ because, given pushout (1) in $\mathbf{C}_{\text{fin}}$ with $f \in \mathcal{M}_{\text{fin}}$, we also have $f \in \mathcal{M}$. Since I_{fin} creates pushouts along $\mathcal{M}$ by part (2), the pushout (1′) of $f \in \mathcal{M}$ and h in $\mathbf{C}$ is also a pushout in $\mathbf{C}_{\text{fin}}$. By the uniqueness of pushouts, this means that (1) and (1′) are isomorphic and, thus, (1) is also a pushout in $\mathbf{C}$.

(4) Similarly, we can show that I_{fin} preserves pullbacks along $\mathcal{M}_{\text{fin}}$ using the fact that I_{fin} creates pullbacks along $\mathcal{M}$ as shown in part (1). □

We still need to show the following lemma to complete part (2) in the previous proof.

Lemma 4.5. Given m'_D, m''_D and derived m'_B, m''_B, m'_C, m''_C as above, then

$$m'_B \stackrel{\sim}{=} m''_B$$
$$m'_C \stackrel{\sim}{=} m''_C$$

implies

$$m'_D \stackrel{\sim}{=} m''_D.$$

Proof. From

$$m'_B \stackrel{\sim}{=} m''_B$$
$$m'_C \stackrel{\sim}{=} m''_C,$$

and the uniqueness of pullback constructions up to isomorphism, we can assume without loss of generality that

$$m'_B = m''_B$$
$$m'_C = m''_C$$

in diagrams (3) and (4) in Figure 18, corresponding to (2) in Figure 17b for m'_D and m''_D, respectively. We now let A' be the pullback in the top faces of the cubes (3) and (4). Hence, the top, bottom, right front and left back faces are pullbacks along $\mathcal{M}$-morphisms. Note, that the top and back faces are equal in (3) and (4), and the right back square is the pushout (1) along $f \in \mathcal{M}$.

Moreover, we have $m'_A, m'_B, m'_C, m'_D, m''_D \in \mathcal{M}$ such that the vertical weak VK property (Definition 2.1) implies that the left front squares in (3) and (4) are pushouts. Since pushouts are unique up to isomorphism, it follows that

$$D' \stackrel{\sim}{=} D''$$
$$m'_D \stackrel{\sim}{=} m''_D,$$

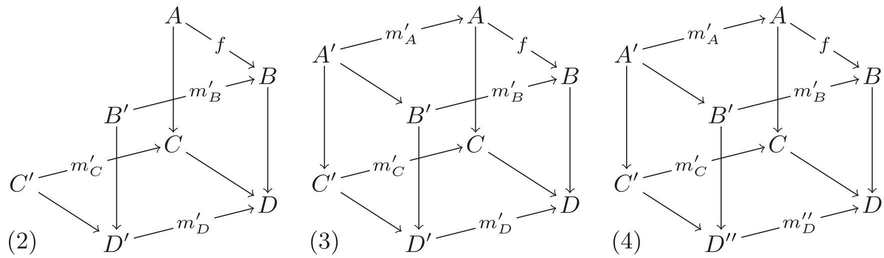


Fig. 18. Cubes in Lemma 4.5

which completes the proof. □

We are now able to show the second main result.

Theorem 4.6. The finitary restriction $(\mathbf{C}_{\text{fin}}, \mathcal{M}_{\text{fin}})$ of any $\mathcal{M}$-adhesive category $(\mathbf{C}, \mathcal{M})$ is a finitary $\mathcal{M}$-adhesive category.

Proof. According to Remark 4.2, an object A in $\mathbf{C}$ is finite in $(\mathbf{C}, \mathcal{M})$ if and only if it is finite in $(\mathbf{C}_{\text{fin}}, \mathcal{M}_{\text{fin}})$. Hence, all objects in $(\mathbf{C}_{\text{fin}}, \mathcal{M}_{\text{fin}})$ are finite.
Moreover, $\mathcal{M}_{\text{fin}}$ is closed under isomorphisms, composition and decomposition because this is true for $\mathcal{M}$. $(\mathbf{C}_{\text{fin}}, \mathcal{M}_{\text{fin}})$ has pushouts along $\mathcal{M}_{\text{fin}}$ because $(\mathbf{C}, \mathcal{M})$ has pushouts along $\mathcal{M}$ and I_{fin} creates pushouts along $\mathcal{M}$ by Fact 4.4. This also implies that $\mathcal{M}_{\text{fin}}$ is preserved by pushouts along $\mathcal{M}_{\text{fin}}$ in $\mathbf{C}_{\text{fin}}$. Similarly, $(\mathbf{C}_{\text{fin}}, \mathcal{M}_{\text{fin}})$ has pullbacks along $\mathcal{M}_{\text{fin}}$ and $\mathcal{M}_{\text{fin}}$ is preserved by pullbacks along $\mathcal{M}_{\text{fin}}$ in $\mathbf{C}_{\text{fin}}$.
Finally, the vertical weak VK property of $(\mathbf{C}, \mathcal{M})$ implies the vertical weak VK property of $(\mathbf{C}_{\text{fin}}, \mathcal{M}_{\text{fin}})$ using the fact that I_{fin} preserves pushouts and pullbacks along $\mathcal{M}_{\text{fin}}$ and creates pushouts and pullbacks along $\mathcal{M}$. □

One direct consequence of Theorem 4.6 is the fact that the finitary restrictions of $(\mathbf{Sets}, \mathcal{M}_S)$, $(\mathbf{Graphs}, \mathcal{M}_G)$, $(\mathbf{Graphs_{TG}}, \mathcal{M}_{TG})$, $(\mathbf{ElemNets}, \mathcal{M}_{EN})$, $(\mathbf{PTNets}, \mathcal{M}_{PT})$ and $(\mathbf{AGraphs_{ATG}}, \mathcal{M}_{TAG})$ are all finitary $\mathcal{M}$-adhesive categories satisfying not only the axioms of $\mathcal{M}$-adhesive categories, but also the additional HLR requirements stated in Theorem 3.14. However, we may observe that the existence of finite coproducts in $(\mathbf{AGraphs_{ATG}}, \mathcal{M}_{TAG})$ is valid (Ehrig *et al.* 2006a), but cannot be concluded from the existence of $\mathcal{M}$-initial objects as required in Theorem 3.14 (3) and (4). Moreover, I is an $\mathcal{M}_{\text{fin}}$-initial object in $(\mathbf{C}_{\text{fin}}, \mathcal{M}_{\text{fin}})$ if I is an $\mathcal{M}$-initial object in $(\mathbf{C}, \mathcal{M})$.

Remark 4.7. We can conclude from Theorems 3.14 and 4.6 that the main results for the DPO approach, including the Local Church-Rosser, Parallelism, Concurrency, Embedding, Extension, and Local Confluence Theorems, are valid in all finitary restrictions of $\mathcal{M}$-adhesive categories. This also includes the corresponding results with nested application conditions (Habel and Pennemann 2009) because shifts along morphisms and rules preserve the finiteness of the objects occurring in the application conditions.

5. Functorial constructions of finitary $\mathcal{M}$-adhesive categories

Like (weak) adhesive HLR categories, $\mathcal{M}$-adhesive categories and finitary $\mathcal{M}$-adhesive categories are also closed under product, slice, coslice, functor and comma categories under suitable conditions (Ehrig *et al.* 2006a). However, it is enough to show this just for functor and comma categories because all the others are special cases.

Fact 5.1 (finitary functor categories). Given a finitary $\mathcal{M}$-adhesive category $(\mathbf{C}, \mathcal{M})$ and a category $\mathbf{X}$ with a finite class of objects, the functor category $(\mathbf{Funct}(\mathbf{X}, \mathbf{C}), \mathcal{M}_F)$ is a finitary $\mathcal{M}$-adhesive category also, where $\mathcal{M}_F$ is the class of all $\mathcal{M}$-functor transformations $t : F' \to F$, that is, natural transformations $t : F' \to F$ such that

$$t(X) : F'(X) \to F(X) \in \mathcal{M}$$

for all objects X in $\mathbf{X}$.

Proof. By Ehrig *et al.* (2006a, Theorem 4.15.3), $(\mathbf{Funct}(\mathbf{X}, \mathbf{C}), \mathcal{M}_F)$ is a (weak) adhesive HLR category and hence also an $\mathcal{M}$-adhesive category because the horizontal weak VK property is not needed in the proof of the theorem. It remains to show that each $F : \mathbf{X} \to \mathbf{C}$ is finite. Since $\mathrm{Obj}_{\mathbf{X}}$ is finite, we have objects $X_1, \ldots, X_n$ in $\mathbf{X}$. We want to show that there are only finitely many $\mathcal{M}$-functor transformations $t : F' \to F$ up to isomorphism. In each case, we have

$$t(X_k) : F'(X_k) \to F(X_k) \in \mathcal{M},$$

with, say, $i_k \in \mathbb{N}$ different choices using $F(X_k) \in \mathbf{C}$ and $\mathbf{C}$ is a finitary $\mathcal{M}$-adhesive category. Hence, altogether we have at most $i = i_1 \cdot \ldots \cdot i_n \in \mathbb{N}$ different $t : F' \to F$ up to isomorphism. □

Remark 5.2. For infinite (discrete) $\mathbf{X}$, we have

$$\mathbf{Funct}(\mathbf{X}, \mathbf{C}) \cong \prod_{i \in \mathbb{N}} \mathbf{C}.$$

With

$$\mathbf{C} = \mathbf{Sets}_{\mathrm{fin}},$$

the object $(2_i)_{i \in \mathbb{N}}$ with $2_i = \{1, 2\}$ has an infinite number of subobjects $(1_i)_{i \in \mathbb{N}}$ of $(2_i)_{i \in \mathbb{N}}$ with $1_i = \{1\}$, because in each component $i \in \mathbb{N}$, we have two choices of injective functions

$$f_{1/2} : \{1\} \to \{1, 2\}.$$

Hence, $\mathbf{Funct}(\mathbf{X}, \mathbf{C})$ is not finitary because $(2_i)_{i \in \mathbb{N}}$ in $\prod_{i \in \mathbb{N}} \mathbf{C}$ is not finite.

In the following, we will use $F \downarrow G$ to denote the (standard) comma category and

$$\mathbf{ComCat}(F, G; \mathcal{I})$$

to denote the version of comma categories used in Ehrig *et al.* (2006a). Given functors

$$F : \mathbf{A} \to \mathbf{C}$$
$$G : \mathbf{B} \to \mathbf{C}$$

an object in $F \downarrow G$ is a **C**-morphism

$$op\colon F(A) \to G(B)$$

for objects A in **A** and B in **B**, while we have a family

$$op = \left[op^k : F(A) \to G(B)\right]_{k\in\mathcal{I}}$$

in **ComCat**$(F, G; \mathcal{I})$. For

$$card(\mathcal{I}) = 1,$$

we have $F \downarrow G$ is a special case of **ComCat**$(F, G; \mathcal{I})$. But **ComCat**$(F, G; \mathcal{I})$ can also be represented as a standard comma category $F\mathcal{I} \downarrow G\mathcal{I}$, where

$$F\mathcal{I} = D^{\mathcal{I}} \circ F$$
$$G\mathcal{I} = D^{\mathcal{I}} \circ G,$$

and the functor

$$D^{\mathcal{I}} : \mathbf{C} \to \mathbf{Funct}(\mathcal{I}, \mathbf{C})$$

produces $\mathcal{I}$-fold duplicates – this was pointed out by one of the referees. Hence, it is sufficient to formulate the following fact for the standard comma category $F \downarrow G$, though it is also valid for **ComCat**$(F, G; I)$ (see Remark 5.4).

Fact 5.3 (finitary comma categories). Given finitary $\mathcal{M}$-adhesive categories $(\mathbf{A}, \mathcal{M}_1)$ and $(\mathbf{B}, \mathcal{M}_2)$ and functors $F : \mathbf{A} \to \mathbf{C}$ and $G : \mathbf{B} \to \mathbf{C}$, where F preserves pushouts along $\mathcal{M}_1$ and G preserves pullbacks along $\mathcal{M}_2$, then the comma category $F \downarrow G$ with

$$\mathcal{M} = (\mathcal{M}_1 \times \mathcal{M}_2) \cap F \downarrow G$$

is a finitary $\mathcal{M}$-adhesive category.

Proof. By Ehrig *et al.* (2006a, Theorem 4.15.4), **ComCat**$(F, G; \mathcal{I})$ and, in particular, $F \downarrow G$ is a (weak) adhesive HLR category, and also an $\mathcal{M}$-adhesive category because the horizontal weak VK property is not needed.

It remains to show that each object

$$(A, B, op : F(A) \to G(B))$$

is finite. By assumption, A and B are finite with a finite number of subobjects

$$m_{1,i} : A_i \to A \in \mathcal{M}_1 \qquad (i \in \mathcal{I}_1)$$
$$m_{2,j} : B_j \to B \in \mathcal{M}_2 \qquad (j \in \mathcal{I}_2).$$

Hence, we have at most $|\mathcal{I}_1| \cdot |\mathcal{I}_2|$ $\mathcal{M}$-subobjects of (A, B, op) of the form where for each i, j, there is at most one $op_{i,j}$ such that (1) in Figure 19 commutes because G preserves pullbacks along $\mathcal{M}_2$ such that $G(m_{2,j})$ is a monomorphism in **C** according to the mono-characterisation by pullbacks. □

Remark 5.4. Note that Fact 5.3 is also valid for **ComCat**$(F, G; \mathcal{I})$, the comma category according to Ehrig *et al.* (2006a), which is the standard comma category $F \downarrow G$ with an

$$\begin{array}{ccc} F(A_i) & \xrightarrow{op_{i,j}} & G(B_j) \\ \downarrow F(m_{1,i}) & (1) & \downarrow G(m_{2,j}) \\ F(A) & \xrightarrow{op} & G(B) \end{array}$$

Fig. 19. Finite comma categories

$\mathcal{I}$-indexed set of morphisms instead of a single one, and where a direct proof is obtained by replacing op by

$$\left[op^k : F(A) \to G(B)\right]_{k\in\mathcal{I}}$$

in the proof of Fact 5.3.

6. Extension to non-$\mathcal{M}$-adhesive categories

There are some relevant categories in computer science which are not $\mathcal{M}$-adhesive for sensible choices of the monomorphism class $\mathcal{M}$. As a running example, we will consider simple graphs.

Definition 6.1 (category SGraphs). The category **SGraphs** of simple graphs is given by:

— **objects**:
$S = (S_V, S_E)$ consisting of sets S_V of vertices and

$$S_E \subseteq S_V \times S_V$$

of edges $(s, t) \in S_E$ with source $s \in S_V$ and target $t \in S_V$.

— **morphisms**:
$f : S \to S'$ with $f = (f_V)$ consisting of a function $f_V : S_V \to S'_V$ on vertices that satisfies $(f_V(s), f_V(t)) \in S'_E$ for all $(s, t) \in S_E$.

— **composition and identities**:
These are given by the composition and identities on the vertex functions.

In contrast to the category **Graphs**, edges do not have identities in **SGraphs**. Hence, there is at most one edge per direction between two vertices, and the edges constitute a relation on the vertices. In fact, an isomorphic category is treated under the name **Rel**, *viz.* the category of relations, in Adámek *et al.* (1990).

There is a wide variety of categories that are similar to simple graphs in the sense that their objects contain some kind of relational structure. Since relational structures are omnipresent in computer science – for example, in databases, non-deterministic automata and logical structures – the study of transformations in these categories is also highly relevant. For example, the Resource Description Framework (RDF) (Klyne and Carroll 2004), which constitutes the formal underpinning of the emerging Semantic Web, defines RDF graphs to be sets of sentences about resources. This also leads to a category **RDFGraphs** with relational edge structure, the details of which can be found in Braatz and Brandt (2008) and Braatz (2009).

Figure 20 shows a counter-example for the vertical weak VK property in **SGraphs**, where $\mathcal{M}$ is chosen to be the class of all monomorphisms. The squares at the bottom

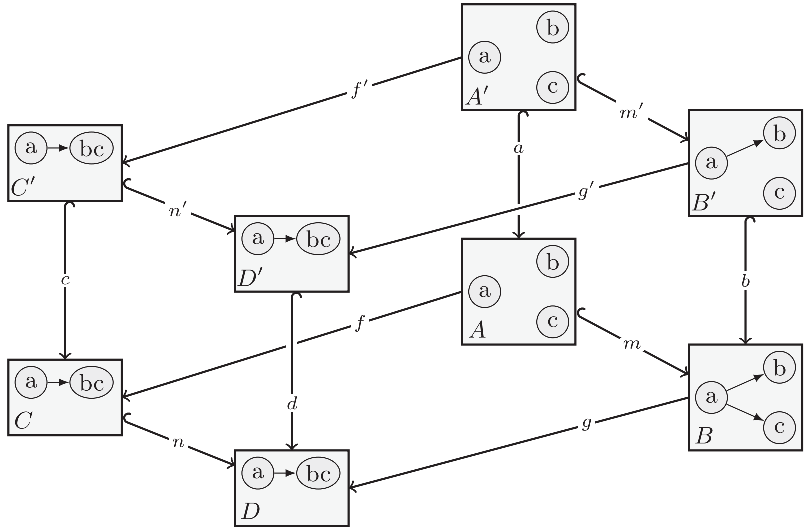
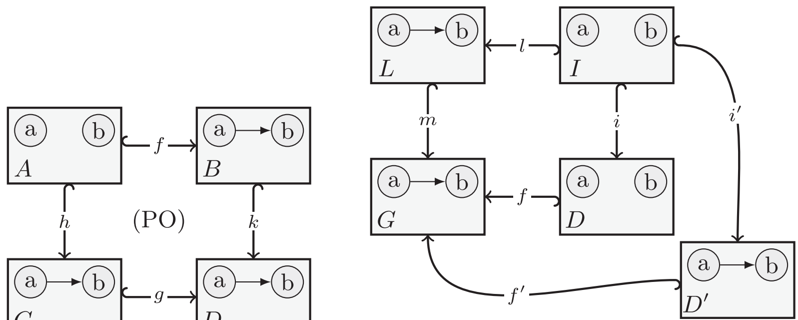


Fig. 20. Counter-example for $\mathcal{M}$-VK square in **SGraphs**



(a) Pushout that is not a pullback

(b) Non-unique pushout complements

Fig. 21. Pushout that is not a pullback and non-unique pushout complements in **SGraphs**

and top are pushouts and the back faces are pullbacks, but the right front face is not a pullback. In fact, we could only obtain $\mathcal{M}$-adhesiveness if we chose the $\mathcal{M}$-morphisms to be bijective on edges, but this would not be satisfactory since transformations should be able to add and delete edges.

Furthermore, pushouts along monomorphisms in **SGraphs** are not necessarily pullbacks, as shown in Figure 21a. In fact, we have already seen instances of pushouts along monomorphisms that are not pullbacks in the top and bottom faces of Figure 20. Moreover, pushout complements are not unique in **SGraphs** and similar categories, as shown in Figure 21b. The problem arises from the edge that is deleted between L and I. Since edges do not have identities in **SGraphs**, this edge can either be deleted, as in D, or preserved, as in D'. Both cases lead to G being a pushout of L and D or D', respectively, under I.

This leads to double-pushout transformations being non-deterministic even for determined rule and match in such categories. However, we can canonically choose a minimal pushout complement (MPOC), which is the approach taken in Braatz and Brandt (2008) and Braatz (2009) for the category **RDFGraphs**. Intuitively, we delete as much as possible and choose the smallest of the pushout complements, where all edges are deleted rather than preserved. This leads to a new variant of the double-pushout transformation framework that is applicable to relational structures.

Therefore, it is interesting to explore the extent to which the results for finitary $\mathcal{M}$-adhesive categories presented in this paper are also valid in such non-$\mathcal{M}$-adhesive categories so that we can transfer a significant portion of the extensive theoretical results from $\mathcal{M}$-adhesive categories to the MPOC framework, and possibly to other approaches as well.

In the following, we introduce $\mathcal{M}$-PO-PB categories, which are based on a class $\mathcal{M}$ compatible with pushouts and pullbacks, but no VK property is required.

Definition 6.2 ($\mathcal{M}$-PO-PB category). A category $\mathbf{C}$ together with a class $\mathcal{M}$ of monomorphisms is said to be a *$\mathcal{M}$-PO-PB category* $(\mathbf{C}, \mathcal{M})$ if $\mathcal{M}$ is closed under composition, decomposition and isomorphisms, pushouts and pullbacks along $\mathcal{M}$ exist and $\mathcal{M}$ is closed under pushouts and pullbacks.

An object A in $(\mathbf{C}, \mathcal{M})$ is said to be *finite* if the number of $\mathcal{M}$-subobjects of A is finite and the $\mathcal{M}$-PO-PB category is said to be *finitary* if each object A in $(\mathbf{C}, \mathcal{M})$ is finite. The $\mathcal{M}$-PO-PB category $(\mathbf{C}, \mathcal{M})$ is said to be *well-powered* if the class of all $\mathcal{M}$-subobjects of each object A is a set.

Note that our notion of $\mathcal{M}$-PO-PB categories is more restricted than the notion of $\mathcal{M}$-categories found in Cockett and Lack (2002) and elsewhere since we are also concerned with pushouts, so we require compatibility of the class $\mathcal{M}$ with pushouts.

Note also that all $\mathcal{M}$-adhesive categories are also $\mathcal{M}$-PO-PB categories and our standard examples are, like $(\mathbf{Sets}, \mathcal{M})$ and $(\mathbf{Graphs}, \mathcal{M})$, non-finitary, but well-powered. Facts 2.4–3.7 regarding coproducts and factorisations are already valid for (finitary) $\mathcal{M}$-PO-PB categories, where we need in addition $\mathcal{M}$-initial objects for Facts 2.6 and 3.7. Moreover, Facts 5.1 and 5.3 remain valid for finitary $\mathcal{M}$-PO-PB categories, but the problem is still open for the creation of pushouts in Fact 4.4 and, hence, also for Theorem 4.6.

By contrast, initial pushouts, as defined in Definition 3.10 and constructed in Fact 3.12, do not, in general, exist in finitary $\mathcal{M}$-PO-PB categories. The problem is that the squares between the initial pushout and the comparison pushouts have to be pushouts themselves. Therefore, we need to define a weaker variant of initial pushouts that does not require these squares to be pushouts but just to be commutative.

Definition 6.3 (weak initial pushout). Given an $\mathcal{M}$-PO-PB category $(\mathbf{C}, \mathcal{M})$, a pushout (1) as in Definition 3.10 over a morphism $m\colon L \to G$ with $b, c \in \mathcal{M}$ is said to be *weak initial* if for all pushouts (2) over m with $b', c' \in \mathcal{M}$, there exist unique morphisms $b^*, c^* \in \mathcal{M}$,

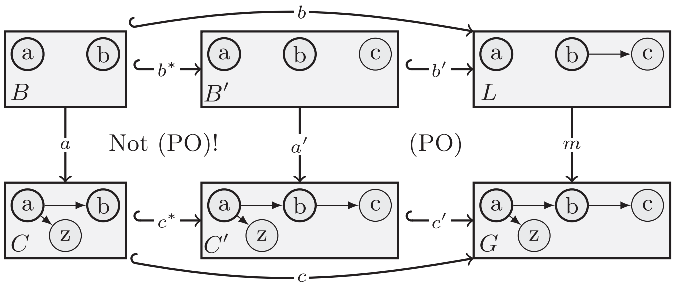


Fig. 22. Weak initial pushout in $\mathbf{SGraphs}_{\text{fin}}$

such that

$$b' \circ b^* = b$$
$$c' \circ c^* = c,$$

and (3) commutes.

Remark 6.4. Note that in $\mathcal{M}$-adhesive categories, each weak initial pushout is also an initial pushout since the initial pushout can be decomposed by $\mathcal{M}$-pushout–pullback-decomposition, which holds in $\mathcal{M}$-adhesive categories because the comparison pushout is also a pullback. However, this does not hold in general $\mathcal{M}$-PO-PB categories.

Example 6.5 (weak initial pushout in $\mathbf{SGraphs}_{\text{fin}}$). Figure 22 shows an example of a weak initial pushout in the category $\mathbf{SGraphs}_{\text{fin}}$ of finite simple graphs. The objects B and C and the corresponding morphisms a, b and c constitute a pushout over the morphism m from L to G, which, in fact, is a weak initial pushout for this morphism. Moreover, the objects B' and C' and the morphisms a', b' and c' constitute a comparison pushout over m because there is at most one edge between two nodes in simple graphs like G. However, the unique morphisms b^* and c^* do *not* give rise to a pushout in the left square since the edge between nodes b and c will not be constructed by a pushout. Note that the right square is a pushout in $\mathbf{SGraphs}_{\text{fin}}$, but not in $\mathbf{Graphs}_{\text{fin}}$.

We will now show the existence and construction of weak initial pushouts for finitary $\mathcal{M}$-PO-PB categories, provided $\mathcal{M}$-pushouts are closed under pullbacks in the following sense.

Definition 6.6 (closure of $\mathcal{M}$-pushouts under pullbacks). Given an $\mathcal{M}$-PO-PB category $(\mathbf{C}, \mathcal{M})$, we say that *$\mathcal{M}$-pushouts are closed under pullbacks* if for each morphism $m : L \to G$ and commutative diagram in Figure 23 with pushouts over m in the right squares, pullbacks in the top and bottom and $b_1, b_2 \in \mathcal{M}$ (so $c_1, c_2, u_1, u_2, v_1, v_2 \in \mathcal{M}$), then the diagonal square is a pushout.

Fact 6.7 (existence of weak initial pushouts). Finitary $\mathcal{M}$-categories have weak initial pushouts, which can be constructed by finite $\mathcal{M}$-intersections provided $\mathcal{M}$-pushouts are closed under pullbacks.

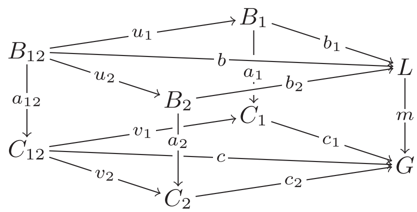


Fig. 23. Closure of $\mathcal{M}$-pushouts under pullbacks

Proof. Given $m: L \to G$, we consider, in the same way as for Fact 3.12, all pushouts P_i ($i \in \mathcal{I}$) over m with $b_i, c_i \in \mathcal{M}$ up to isomorphism. Since L and G are finite, we can assume that $\mathcal{I}$ is finite, and then construct, again in the same way as for Fact 3.12, the weak initial pushout by iterated pullbacks. The closure of $\mathcal{M}$-pushouts under pullbacks ensures that the constructed diagonal square is, in fact, a pushout. But we do not necessarily have pushouts in the left squares in the diagram in Figure 23. Moreover, each $\mathcal{M}$-pushout over m coincides up to isomorphism with P_i for some $i \in \mathcal{I}$. Hence, there are $u_i, v_i \in \mathcal{M}$ by construction that are unique because b_i and c_i are monomorphisms. Putting this all together, this means that the constructed square is a weak initial pushout. □

Note that the required closure of $\mathcal{M}$-pushouts under pullbacks already holds in $\mathcal{M}$-adhesive categories. Moreover, the closure property holds in the categories **SGraphs** and **RDFGraphs**, allowing us to construct weak initial pushouts in these categories according to Fact 6.7.

Example 6.8 (weak initial pushout in SGraphs$_{\text{fin}}$). In Figure 24 we have constructed all pushouts $(P_i)_{i \in \{0,\dots,3\}}$ over $m: L \to G$ up to isomorphism, where (P_0) corresponds to the outer diagram in Figure 22 and (P_2) to the right diagram, and the initial pushout (B, C, a, b, c) over m is the finite $\mathcal{M}$-intersection of all (P_i) $(i = 0, \dots, 3)$ and equal to (P_0). The weak initial pushout contains the dangling points a and b in the boundary and the additional node z and additional edges from a to b and from a to z in the context, which are exactly those elements that are common in all pushouts over m.

Remark 6.9. In a similar way to the observation in Remark 3.11, weak initial pushouts allow us to define a gluing condition, which in this case is necessary and sufficient for the existence and uniqueness of minimal pushout complements.

Given $m : L \to G$ with weak initial pushout (1) as shown in Figure 25 and $l : K \to L \in \mathcal{M}$, which can be considered as the left-hand side of a rule, the gluing condition for match m is satisfied if there exists $b^* : B \to K$ with $l \circ b^* = b$. In this case, the minimal pushout complement object D in (2) can be constructed as the pushout object of a and b^* (Braatz 2009). Braatz *et al.* (2011) examined two alternative constructions for (minimal) pushout complements in **Graphs** and **SGraphs** using (weak) initial pushouts and the other quasi-coproduct complements.

For several non-finitary categories, including **Sets**, **Graphs**, **SGraphs**, **RDFGraphs**, **AGraphs**$_{ATG}$, **PTINets** and **AHLINets**, we already know that these categories have initial pushouts (see Ehrig *et al.* (2006a), Braatz *et al.* (2011), Braatz, (2009) and

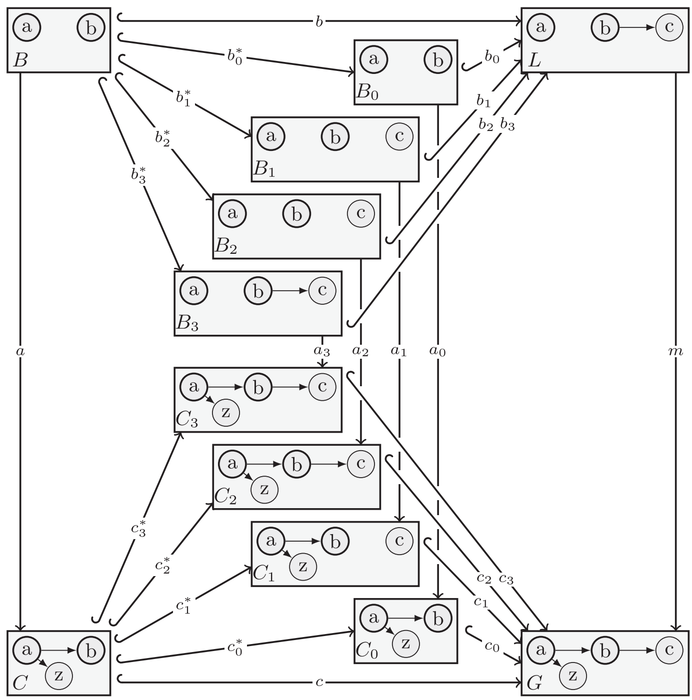


Fig. 24. Construction of weak initial pushout in $\mathbf{SGraphs}_{\mathrm{fin}}$

$$\begin{tikzcd}
B \arrow[r, "b"] \arrow[rr, bend left, "b^*"] \arrow[d, "a"] \arrow[dr, phantom, "(1)"] & L \arrow[d, "m"] \arrow[dr, phantom, "(2)"] & K \arrow[l, "l"'] \arrow[d, "k"] \\
C \arrow[r, "c"] \arrow[rr, bend right, "c^*"'] & G & D \arrow[l, "f"']
\end{tikzcd}$$

Fig. 25. Construction of minimal pushout complement

Modica *et al.* (2010) for explicit constructions). So it is interesting to ask if the initial pushouts in these categories can also be constructed by (general) $\mathcal{M}$-intersections. Fortunately, it is not necessary to check explicitly each construction of initial pushouts in these categories because the following fact states that the (weak) initial pushouts in all $\mathcal{M}$-PO-PB categories $(\mathbf{C}, \mathcal{M})$ fulfill the universal property of $\mathcal{M}$-intersections (see the note following Definition 2.10). To prove this fact, we have to assume that the class $(\mathrm{P}_i)_{i\in\mathcal{I}}$ of all pushouts over $m: L \to G$ in Figure 26a is already a set, because to get (general) $\mathcal{M}$-intersections of $(b_i)_{i\in\mathcal{I}}$ as a special kind of limit, we need $\mathcal{I}$ to be a set. The property of $(\mathbf{C}, \mathcal{M})$ being well-powered (Definition 6.2) makes sure that the class $(b_i)_{i\in\mathcal{I}}$ can be considered to be a set using only one representative $b_i \in \mathcal{M}$ for each $\mathcal{M}$-subobject of L.

Fact 6.10 (construction of (weak) initial pushouts in $\mathcal{M}$-PO-PB categories). If the (weak) initial pushout over a morphism $m: L \to G$ exists in some well-powered $\mathcal{M}$-PO-PB category $(\mathbf{C}, \mathcal{M})$, it can be constructed by (general) $\mathcal{M}$-intersections.

$$
\begin{tikzcd}
B \arrow[r, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "(1)"] & L \arrow[d, "m"] \\
C \arrow[r, "c"] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "b_i^*"] \arrow[d, "a"'] \arrow[rr, bend left, "b"] \arrow[dr, phantom, "(\mathrm{Q}_i)"] & B_i \arrow[r, "b_i"] \arrow[d, "a_i"] \arrow[dr, phantom, "(\mathrm{P}_i)"] & L \arrow[d, "m"] \\
C \arrow[r, "c_i^*"] \arrow[rr, bend right, "c"'] & C_i \arrow[r, "c_i"] & G
\end{tikzcd}
$$

(a) Initial pushout

$$
\begin{tikzcd}
 & & B_k \arrow[dr, "b_k"] & \\
B' \arrow[r, "x"] \arrow[urr, bend left, "b'_k"] \arrow[drr, bend right, "b'_i"'] \arrow[ur, phantom, "(=)"] \arrow[dr, phantom, "(=)"] & B \arrow[ur, "\mathrm{id}_B"] \arrow[dr, "b_i^*"'] \arrow[rr, phantom, "(=)"] & & L \\
 & & B_i \arrow[ur, "b_i"'] &
\end{tikzcd}
$$

(b) $\mathcal{M}$-intersection

Fig. 26. Initial pushout and $\mathcal{M}$-intersection

Construction: We assume we are given a morphism $m: L \to G$ in a well-powered $\mathcal{M}$-PO-PB category $(\mathbf{C}, \mathcal{M})$ and let $(\mathrm{P}_i)_{i \in \mathcal{I}}$ in Figure 26a be all pushouts over m (up to isomorphism) with $b_i \in \mathcal{M}$. If (1) in Figure 26a is a (weak) initial pushout over m, then B together with $(b_i^* : B \to B_i)_{i \in \mathcal{I}}$ induced by (weak) initiality of (1) is an $\mathcal{M}$-intersection of $(b_i)_{i \in \mathcal{I}}$, and C together with induced $(c_i^* : C \to C_i)_{i \in \mathcal{I}}$ is an $\mathcal{M}$-intersection of $(c_i)_{i \in \mathcal{I}}$.

Proof. Since $(\mathbf{C}, \mathcal{M})$ is well-powered, the class $\mathcal{I}$ can be considered to be a set. By (weak) initiality of (1) for $i \in \mathcal{I}$, there are induced morphisms b_i^*, $c_i^* \in \mathcal{M}$ such that

$$
\begin{aligned}
b_i \circ b_i^* &= b \\
c_i \circ c_i^* &= c.
\end{aligned}
$$

This implies that there are

$$b_i \circ b_i^* = b = b_j \circ b_j^*$$

and

$$c_i \circ c_i^* = c = c_j \circ c_j^*$$

for $i, j \in \mathcal{I}$.

Now, let

$$
\begin{gathered}
B' \in Ob_{\mathbf{C}} \\
b' : B' \to L \\
(b'_i : B' \to B_i)_{i \in \mathcal{I}}
\end{gathered}
$$

such that for $i \in \mathcal{I}$ there is

$$b_i \circ b'_i = b'.$$

Since the (weak) initial pushout over m is a pushout over m there is, without loss of generality, some $k \in \mathcal{I}$ such that $(\mathrm{P}_k) = (1)$. So we obtain a morphism $x: B' \to B$ by choosing

$$x := b'_k : B' \to B_k$$

$$\begin{tikzcd}
 & & A \arrow[dll, bend right, "e"'] \arrow[dl, "e_i"'] \arrow[dd, "f"] \\
\bar{B} \arrow[r, "\bar{m}_i"] \arrow[drr, bend right, "m"'] & B_i \arrow[dr, "m_i"] & \\
 & & B
\end{tikzcd}$$

Fig. 27. Extremal $\mathcal{E}$-$\mathcal{M}$ factorisation as $\mathcal{M}$-intersection

because $B_k = B$ (see Figure 26b). Then for all $i \in \mathcal{I}$ there is

$$\begin{aligned} b_i \circ b_i^* \circ x &= b \circ x \\ &= b_k \circ x \\ &= b_k \circ b'_k \\ &= b' \\ &= b_i \circ b'_i, \end{aligned}$$

which by monomorphism b_i implies

$$b_i^* \circ x = b'_i.$$

Let $y : B' \to B$ be such that

$$b_i^* \circ y = b'_i$$

for all $i \in \mathcal{I}$. Then we have

$$b_i^* \circ y = b'_i = b_i^* \circ x,$$

which implies that $x = y$ because b_i^* is a monomorphism. So we have that $x : B' \to B$ is the unique morphism with $b_i^* \circ x = b'_i$, so B together with $(b_i^*)_{i\in\mathcal{I}}$ is an $\mathcal{M}$-intersection of $(b_i)_{i\in\mathcal{I}}$.

The proof for C and $(c_i^*)_{i\in\mathcal{I}}$ works analogously. □

In the same way, extremal $\mathcal{E}$-$\mathcal{M}$ factorisations in $\mathcal{M}$-PO-PB categories, if they exist, can also be constructed by $\mathcal{M}$-intersection.

Fact 6.11 (construction of extremal $\mathcal{E}$-$\mathcal{M}$ factorisations in $\mathcal{M}$-PO-PB categories). If the extremal $\mathcal{E}$-$\mathcal{M}$ factorisation

$$(\bar{B}, e : A \to \bar{B}, m : \bar{B} \to B)$$

of a morphism $f : A \to B$ exists in some well-powered $\mathcal{M}$-PO-PB category $(\mathbf{C}, \mathcal{M})$, then it can be constructed by a (general) $\mathcal{M}$-intersection.

Construction: We assume we are given a morphism $f : A \to B$ in a well-powered $\mathcal{M}$-PO-PB category $(\mathbf{C}, \mathcal{M})$ and let $(m_i : B_i \to B)_{i\in\mathcal{I}}$ in Figure 27 be (representatives of) all subobjects of B such that there are $e_i : A \to B_i$ with $m_i \circ e_i = f$. If $\bar{B}$ together with $e : A \to \bar{B}$ and $m : \bar{B} \to B$ is an extremal $\mathcal{E}$-$\mathcal{M}$ factorisation of $f : A \to B$, then there exist $(\bar{m}_i : \bar{B} \to B_i)_{i\in\mathcal{I}}$ such that $\bar{B}$ together with $(\bar{m}_i)_{i\in\mathcal{I}}$ is an $\mathcal{M}$-intersection of $(m_i)_{i\in\mathcal{I}}$.

$$\begin{tikzcd}
& & B_i \arrow[dr, "m_i"] & \\
A \arrow[r, "a_i"] \arrow[urr, bend left, "e_i"] \arrow[drr, bend right, "e"'] & C_i \arrow[ur, "c_i"] \arrow[dr, "d_i"'] \arrow[r, phantom, "(\mathrm{PB}_i)"] & {} & B \\
& & \bar{B} \arrow[ur, "m"'] &
\end{tikzcd}$$

(a) Pullback (PB_i)

$$\begin{tikzcd}
& & B_k \arrow[dr, "m_k"] & \\
B' \arrow[r, "x"] \arrow[urr, bend left, "m'_k"] \arrow[drr, bend right, "m'_i"'] & \bar{B} \arrow[ur, "\mathrm{id}_{\bar{B}}"] \arrow[dr, "\bar{m}_i"'] \arrow[r, phantom, "(=)"] & {} & B \\
& & B_i \arrow[ur, "m_i"'] &
\end{tikzcd}$$

(b) $\mathcal{M}$-intersection

Fig. 28. Extremal $\mathcal{E}$-$\mathcal{M}$ factorisation is $\mathcal{M}$-intersection

Proof. Since $(\mathbf{C}, \mathcal{M})$ is well-powered, the class $\mathcal{I}$ can be considered to be a set. For all $i \in \mathcal{I}$, we construct the pullback (PB_i) of m and m_i in Figure 28a. Then, since

$$m \circ e = f = m_i \circ e_i$$

from the universal property of the pullback, we obtain a unique morphism $a_i : A \to C_i$ with

$$d_i \circ a_i = e$$
$$c_i \circ a_i = e_i.$$

From $m_i \in \mathcal{M}$ and pullback (PB_i), it follows that we also have $d_i \in \mathcal{M}$, which together with

$$d_i \circ a_i = e$$

implies that d_i is an isomorphism because $e \in \mathcal{E}$ and $\mathcal{E}$ is the class of extremal morphisms (see Definition 3.1). So we have a morphism $\bar{d}_i : \bar{B} \to C_i$ with

$$\bar{d}_i \circ d_i = \mathrm{id}_{C_i}$$
$$d_i \circ \bar{d}_i = \mathrm{id}_{\bar{B}}.$$

Defining

$$\bar{m}_i := c_i \circ \bar{d}_i,$$

we have

$$\begin{aligned} m_i \circ \bar{m}_i &= m_i \circ c_i \circ \bar{d}_i \\ &= m \circ d_i \circ \bar{d}_i \\ &= m \circ \mathrm{id}_{\bar{B}} \\ &= m, \end{aligned}$$

which implies that for $i, j \in \mathcal{I}$, we have

$$m_i \circ \bar{m}_i = m = m_j \circ \bar{m}_j.$$

Moreover, $m, m_i \in \mathcal{M}$ implies $\bar{m}_i \in \mathcal{M}$.

We now let B' be an object in $\mathbf{C}$ and $(m'_i : B' \to B_i)_{i \in \mathcal{I}}$ with

$$m_i \circ m'_i = m' : B' \to B$$

for all $i \in \mathcal{I}$. We have to show that there is a unique $x : B' \to B$ with

$$\bar{m}_i \circ x = m'_i$$

for all $i \in \mathcal{I}$. Indeed, there is an index $k \in \mathcal{I}$ such that

$$\begin{aligned} B_k &= \bar{B} \\ e_k &= e \\ m_k &= m. \end{aligned}$$

We can then obtain a morphism $x : B' \to \bar{B}$ by choosing $x := m'_k$ and we have

$$\begin{aligned} m_i \circ \bar{m}_i \circ x &= m \circ x \\ &= m_k \circ x \\ &= m_k \circ m'_k \\ &= m' \\ &= m_i \circ m'_i, \end{aligned}$$

which by the fact that m_i is a monomorphism implies that

$$\bar{m}_i \circ x = m'_i$$

in Figure 28b.

The uniqueness of x then follows from the fact that $\bar{m}_i$ is a monomorphism. □

7. Comparison of results for (finitary) $\mathcal{M}$-adhesive categories and (finitary) $\mathcal{M}$-PO-PB categories

The main difference between $\mathcal{M}$-PO-PB categories (Definition 6.2) and $\mathcal{M}$-adhesive categories (Definition 2.1) is the fact that pushouts and pullbacks in $\mathcal{M}$-adhesive categories are compatible in the sense that pushouts along $\mathcal{M}$-morphisms are $\mathcal{M}$-Van Kampen squares. As shown in Ehrig *et al.* (2006a), this compatibility allows us to show the following well-known HLR properties:

(1) pushouts along $\mathcal{M}$-morphisms are pullbacks;
(2) $\mathcal{M}$ pushout–pullback decomposition lemma;
(3) the cube pushout–pullback lemma;
(4) uniqueness of pushout complements.

This compatibility is not valid for $\mathcal{M}$-PO-PB categories in general, as shown in Figure 20 for simple graphs. Hence, it cannot be expected that the HLR-properties (1)–(4) above are valid for $\mathcal{M}$-PO-PB categories, and, indeed, Figure 21 shows explicit counter-examples for properties (1) and (4) for simple graphs.

For finitary $\mathcal{M}$-adhesive categories, we have shown the following additional HLR-requirements

(5) finite coproducts compatible with $\mathcal{M}$ based on the existence of an $\mathcal{M}$-initial object;
(6) an $\mathcal{E}'$-$\mathcal{M}$ pair factorisation based on extremal $\mathcal{E}$-$\mathcal{M}$ factorisation of morphisms; and
(7) initial pushouts.

Requirement (5) is also valid for (finitary) $\mathcal{M}$-PO-PB categories in a similar way to $\mathcal{M}$-adhesive categories. In finitary $\mathcal{M}$-PO-PB categories, extremal $\mathcal{E}$-$\mathcal{M}$-factorisations can be constructed as in Fact 3.4 for finitary $\mathcal{M}$-adhesive categories. In $\mathcal{M}$-PO-PB categories, extremal $\mathcal{E}$-$\mathcal{M}$-factorisations can be constructed by $\mathcal{M}$-intersections, provided they exist (Fact 6.11). If we have binary coproducts and extremal $\mathcal{E}$-$\mathcal{M}$-factorisations, we also have $\mathcal{E}'$-$\mathcal{M}$ pair factorisation in $\mathcal{M}$-PO-PB categories according to Fact 3.7. In general, we do not have initial pushouts in finitary $\mathcal{M}$-PO-PB categories, but we do have weak initial pushouts constructed by finite $\mathcal{M}$-intersections, provided $\mathcal{M}$-pushouts are closed under pullbacks (Fact 6.7).

Moreover, if weak initial pushouts exist in $\mathcal{M}$-PO-PB categories, they can be constructed by $\mathcal{M}$-intersections (Fact 6.10). Like initial pushouts (Remark 3.11), weak initial pushouts allow us to define a gluing condition that is necessary and sufficient for the existence and uniqueness of (minimal) pushout complements, which are required for the construction of direct transformations (Remark 6.9).

The functorial constructions shown for finitary $\mathcal{M}$-adhesive categories in Section 5 are also valid for finitary $\mathcal{M}$-PO-PB categories, because we do not need compatibility of pushouts and pullbacks in the case of $\mathcal{M}$-PO-PB categories.

8. Related work

In the classical double pushout (DPO) approach, rules are defined by a span of monomorphisms, and applications are constructed by two pushouts describing the deletion and creation of elements (Ehrig 1979). Various modifications of this concept have been studied in the literature – for example: the double-pullback approach, where the transformation is done by constructing pullbacks (Heckel *et al.* 2001); the sesqui-pushout approach, where rule morphisms may be non-injective and the deletion is done by a certain pullback, which is not necessarily a pushout (Corradini *et al.* 2006); and the single pushout approach (Löwe and Ehrig 1990), where partial morphisms are used, which allows us to describe the rule by a single partial morphism, which is applied using a pushout. Löwe (2010) lifts the concepts directly to span categories, where all other involved morphisms are also spans.

While the original DPO approach was defined on graphs, it was later lifted to a categorical setting using a distinguished morphism class $\mathcal{M}$ as rule morphisms, and with various instantiations. In particular, adhesive and weak adhesive HLR categories are a suitable concept providing many of the required properties. The literature contains various versions of adhesive (Lack and Sobociński 2004), quasiadhesive (Lack and Sobociński 2005), weak adhesive HLR (Ehrig *et al.* 2006a), partial map adhesive (Heindel 2010) and $\mathcal{M}$-adhesive (Ehrig *et al.* 2010). In adhesive categories, the class $\mathcal{M}$ of morphisms is fixed to all monomorphisms, while in quasiadhesive the class of all regular monomorphisms is considered. With slightly different requirements concerning the existence of pushouts and pullbacks along or over $\mathcal{M}$-morphisms and requirements of $\mathcal{M}$-morphisms in the van Kampen property, they are basically special weak adhesive HLR categories. In contrast, partial map adhesive categories are based on hereditary pushouts, which are pushouts that have to be preserved by the inclusion functor from the category **C** into the category of partial maps over **C**. As shown in Ehrig *et al.* (2010), partial map adhesive categories are also $\mathcal{M}$-adhesive ones. Since all the main properties are valid in $\mathcal{M}$-adhesive categories, we have chosen to work with them in the current paper.

9. Conclusions and future work

We have introduced finite objects in $\mathcal{M}$-adhesive categories, which are a slight generalisation of weak adhesive HLR categories (Ehrig *et al.* 2006a). This leads to finitary $\mathcal{M}$-adhesive categories, such as the category $\mathbf{Sets}_{\text{fin}}$ of finite sets and $\mathbf{Graphs}_{\text{fin}}$ of finite graphs with the class $\mathcal{M}$ of all monomorphisms.

In order to prove the main results, including the Local Church-Rosser, Parallelism, Concurrency, Embedding, Extension, and Local Confluence Theorems, we have not only used the well-known HLR properties, but also the additional HLR requirements listed in the introduction. In particular, initial pushouts are required, and are important for defining the gluing condition and pushout complements, though constructing them explicitly is often tedious. In the current paper, we have shown that for finitary $\mathcal{M}$-adhesive categories, initial pushouts can be constructed by finite $\mathcal{M}$-intersections. Moreover, the other additional HLR requirements are also valid in finitary $\mathcal{M}$-adhesive categories so the main results are valid for all $\mathcal{M}$-adhesive transformation systems with finite objects, which are especially important in most application domains.

In order to construct finitary $\mathcal{M}$-adhesive categories, we can either restrict $\mathcal{M}$-adhesive categories to all finite objects or apply suitable functor and comma category constructions, which are already known for (weak) adhesive HLR categories (Ehrig *et al.* 2006a). In addition, we have shown that for non-finitary categories, initial pushouts, if they exist, can also be constructed by general $\mathcal{M}$-intersections. This is valid in $\mathcal{M}$-adhesive categories and some kinds of more general $\mathcal{M}$-PO-PB categories, where no VK properties are required. Finally, we have extended some of the results to non-$\mathcal{M}$-adhesive categories, such as the category of simple graphs.

Although in most areas where the theory of graph transformations is applied, only finite graphs are considered, we have developed the theory for general graphs, including infinite graphs, and it is implicitly assumed that the results can be restricted to finite graphs and to attributed graphs with a finite graph part, but where the data algebra may be infinite.

Although adhesive categories (Lack and Sobociński 2004) are special cases of $\mathcal{M}$-adhesive categories for the class $\mathcal{M}$ of all monomorphisms, we have to be careful in specialising our results to finitary adhesive categories. While an object is finite in an $\mathcal{M}$-adhesive category **C** if and only if it is finite in the finitary restriction $\mathbf{C}_{\text{fin}}$ (with $\mathcal{M}_{\text{fin}} = \mathcal{M} \cap \mathbf{C}_{\text{fin}}$), this is valid in adhesive categories if the inclusion functor $I : \mathbf{C}_{\text{fin}} \to \mathbf{C}$ preserves monomorphisms. This means that for an adhesive category **C** based on the class of all monomorphisms, there may be monomorphisms in $\mathbf{C}_{\text{fin}}$ that are not monomorphisms in **C**, so it is not clear whether the finite objects in **C** and $\mathbf{C}_{\text{fin}}$ are the same. It is not known whether there exists an adhesive category where this property fails, or whether this can be shown in general. This problem is avoided for $\mathcal{M}$-adhesive categories, where finitarity depends on $\mathcal{M}$. In fact, we consider $\mathcal{M}$-adhesive categories ($\mathbf{C}$,$\mathcal{M}$) with restriction to finite objects ($\mathbf{C}_{\mathrm{fin}}$,$\mathcal{M}_{\mathrm{fin}}$), where $\mathcal{M}_{\mathrm{fin}}$ is the restriction of $\mathcal{M}$ to morphisms between finite objects. In this case, the inclusion functor $I : \mathbf{C}_{\mathrm{fin}} \to \mathbf{C}$ preserves $\mathcal{M}$-morphisms such that finite objects in $\mathbf{C}_{\mathrm{fin}}$ with respect to $\mathcal{M}_{\mathrm{fin}}$ are exactly the finite objects in $\mathbf{C}$ with respect to $\mathcal{M}$.

In the case of $\mathcal{M}$-PO-PB categories, it is not known whether the finitary restriction $(\mathbf{C}_{\mathrm{fin}}, \mathcal{M}_{\mathrm{fin}})$ becomes a finitary $\mathcal{M}$-PO-PB category. Moreover, it would be interesting to find a variant of the Van-Kampen-property that allows us to prove at least weak versions of the main results known for $\mathcal{M}$-adhesive systems. The closure of $\mathcal{M}$-pushouts under pullbacks is a first step in this direction because it allows us to construct weak initial pushouts for finitary $\mathcal{M}$-PO-PB categories.

Moreover, we still need to compare our notion of finite objects in $\mathcal{M}$-PO-PB categories with similar notions in category theory (MacLane 1971; Adámek *et al.* 1990), and to investigate other examples of $\mathcal{M}$-PO-PB categories. Also, the relationships for working on (finite) subobject lattices in adhesive categories in Baldan *et al.* (2008) and Baldan *et al.* (2011) are a valuable line of further research.

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