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Fixed-point operations on ccc's. Part I

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Abstract

Most studies of fixed points involve their existence or construction. Our interest is in their equational properties. We study certain equational properties of the fixed-point operation in computationally interesting cartesian closed categories. We prove that in most of the poset categories that have been used in semantics, the least fixed-point operation satisfies four identities we call the Conway identities. We show that if $\mathscr{C}_0$ is a sub-ccc of any ccc $\mathscr{C}$ with a fixed-point operation satisfying these identities, then there is a simple normal form for the morphisms in the least sub-ccc of $\mathscr{C}$ containing $\mathscr{C}_0$ closed under the fixed-point operation. In addition, the standard functional completeness theorem is extended to Conway ccc's.

Contents

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1. Introduction

In many categories of posets, for each morphism $f: A \times B \to A$, and each $b \in B$, there is a least $a \in A$ satisfying $a = f(a, b)$, and furthermore, the function

$$f^\dagger : B \to A$$

$$b \mapsto \text{the least } a\text{: } a = f(a, b)$$

is also a morphism in the category.

It will come as no surprise that in such categories which are cartesian closed, the fixed-point operation $f \mapsto f^\dagger$ is itself determined by a morphism $\dagger_{A,B}: [A \times B \to A] \to [B \to A]$ in the category. We call such a fixed-point operation *internal*.

In previous work, the authors have studied the equational properties of the fixed-point operation in (enriched) algebraic theories. These categories do not have the structure provided by cartesian closed categories, and the fixed-point operation in these categories is *external*. One example is the category **Mat**$_S$ whose objects are the nonnegative integers. A morphism $n \to p$ is an $n \times p$ matrix whose entries are in the semiring S of all subsets of words on the alphabet A. The composition in the category is given by matrix multiplication. If $f: n \to n + p$, write

$$f = [a \;\; b],$$

where a is $n \times n$ and b is $n \times p$. Then, defining

$$a^* := \mathsf{id}_n + a + a^2 + \cdots$$

$$f^\dagger := a^* b$$

it is well known that $f^\dagger$ is the least solution to the fixed-point equation

$$\xi = a\xi + b.$$

(This equation may be written $\xi = f \cdot \langle \xi, \mathsf{id}_p \rangle$ in cartesian categories, see below.)

Another class of examples is determined by the ω-complete posets. If $(A, \leqslant)$ is such a poset, the category $\mathbf{Pow}_A$ has objects the finite powers of A; a morphism $A^n \to A^p$ is a continuous order preserving function. For any continuous

$$f : A^{n+p} \to A^n$$

there is a least such function $f^\dagger : A^p \to A^n$ such that

$$f^\dagger = A^p \xrightarrow{\langle f^\dagger, \mathrm{id}\rangle} A^{n+p} \xrightarrow{f} A^n.$$

Here, if $f, g : A^n \to A^p$, then $f \leqslant g$ if $xf \leqslant xg$, all $x \in A^n$. In each of these two examples, the operation taking a morphism f to $f^\dagger$ is not itself a morphism in the category. The fixed-point operations are external.

In the setting of algebraic theories enriched with an external fixed-point operation, the notion of an *iteration theory* seems to axiomatize the equational properties of all of the computationally interesting structures of this kind. These structures include the strong behaviors of flowchart algorithms, the regular sets of finite and infinite words, the sequacious functions, synchronization trees, the ω-continuous functors, and others. One may regard algebraic theories as cartesian (or cocartesian) categories, i.e., categories with all finite products (or coproducts). It is not difficult to formulate the identities defining iteration theories for cartesian categories with an external dagger. Most of the examples of iteration theories can be generalized accordingly.

We asked ourselves whether in the richer framework provided by cartesian closed categories, ccc's for short, there were other identities needed to capture all of the equational properties of the fixed-point operation. While we do not have the complete answer, we offer here what we consider an interesting part of the story.

The axioms for iteration theories fall naturally into two groups. The first group we have called the Conway identities, due to the form these identities take in the theory of regular sets [5]. The second group consists of an equation schema called the commutative identity. We formulate analogs of the Conway identities which are meaningful in cartesian categories, and add a new identity to this group: the "abstraction identity", which is meaningful only in ccc's.

If $\mathscr{C}$ is a ccc with fixed-point morphisms, and $\mathscr{C}_0$ is a sub-ccc of $\mathscr{C}$, using only the (ccc-version of the) Conway identities we give a normal form for the morphisms in the least sub-ccc containing $\mathscr{C}_0$ closed under the fixed-point morphisms (Theorem 7.12).

Secondly, using these same identities, we extend the standard functional completeness theorem to ccc's satisfying the Conway identities (see Theorem 8.6).

We show why the Conway identities hold in all order-enriched ccc's which have least prefixed points (Theorem 6.1). A generalization of this result to 2-ccc's will be given elsewhere [6].

We expect that Part II will contain a precise explanation of why the Conway identities should hold in all ccc's in which the fixed-point operation is in some sense constructive.

2. Preliminaries

Although we assume familiarity with cartesian closed categories, we introduce our notation here. A cartesian closed category $\mathscr{C}$ is a category with all finite products and an exponential for each pair of objects A, B in $\mathscr{C}$. We will assume that a terminal object 1 is fixed. For each pair A, B of objects, we assume that a fixed product object $A \times B$ and projections $\pi_A^{A\times B}: A \times B \to A$, and $\pi_B^{A\times B}: A \times B \to B$ are specified. Similarly, we assume a fixed choice of an exponential object $[A \to B]$ and evaluation morphism $e_{A,B}: A \times [A \to B] \to B$. (We frequently omit super and/or subscripts on the projection and evaluation morphisms.)

If $\mathscr{C}$ and $\mathscr{C}'$ are ccc's, a ccc-morphism $\varphi: \mathscr{C} \to \mathscr{C}'$ is a functor which preserves the distinguished terminal object, product projections, exponential objects and evaluation maps.

We assume products are associative 'on the nose', so that, e.g., $A \times (B \times C) = (A \times B) \times C$ and diagrams such as

$$\begin{tikzcd} A \times (B \times C) \arrow[r, "\pi_{B\times C}^{A\times(B\times C)}"] \arrow[d, "\mathrm{id}"'] & B \times C \arrow[d, "\pi_C^{B\times C}"] \\ (A \times B) \times C \arrow[r, "\pi_C^{(A\times B)}"'] & C \end{tikzcd}$$

commute.

The *composite* of $\alpha: A \to B$ and $\beta: B \to C$ is usually written in diagrammatic order[1] as $\alpha \cdot \beta: A \to C$ or just $\alpha\beta$. If $\alpha: A \to B$ and $\beta: A \to C$ are morphisms, we write

$$\langle \alpha, \beta \rangle : A \to B \times C$$

for the *target tupling of* α, β, the unique morphism $f: A \to B \times C$ such that $f \cdot \pi_B = \alpha$ and $f \cdot \pi_C = \beta$. Further, if $\alpha: A \to B$ and $\beta: C \to D$, we write

$$\alpha \times \beta: A \times C \to B \times D$$

for the morphism $\langle \pi_A \alpha, \pi_C \beta \rangle$. We note here two fundamental properties that will be used frequently without further comment.

$$h \cdot \langle \alpha, \beta \rangle = \langle h \cdot \alpha, h \cdot \beta \rangle,$$

$$\langle \alpha, \beta \rangle \cdot (\alpha' \times \beta') = \langle \alpha \cdot \alpha', \beta \cdot \beta' \rangle,$$

when sources and targets are appropriate. We will abuse notation and write

$$\pi_{A\times C}: A \times B \times C \to A \times C$$

to mean the appropriate target tupling of the projections π_A, π_C.

[1] We strongly prefer the diagrammatic order for composition, despite many objections from our friends.

If $\alpha : A \times B \to C$, we write

$$\Lambda(\alpha) : B \to [A \to C]$$

for the unique morphism f with the property

$$(\mathrm{id}_A \times f) \cdot e_{A,C} = \alpha.$$

The function Λ is a bijection

$$\Lambda : \mathrm{Hom}(A \times B, C) \to \mathrm{Hom}(B, [A \to C]), \tag{1}$$

with inverse

$$\mathrm{Hom}(B, [A \to C]) \to \mathrm{Hom}(A \times B, C)$$

$$g \mapsto (\mathrm{id}_A \times g) \cdot e_{A,C},$$

and should be decorated with A, B, C. (Of course $\mathrm{Hom}(A, B)$ is the set of morphisms $A \to B$.)

2.1. Base morphisms

Relative to the given collection of choices for products and exponentials, we define a *base morphism* in $\mathscr{C}$ as a morphism which is in the smallest sub-ccc of $\mathscr{C}$ with the same objects, products and exponentials. Thus, all projections $A_1 \times A_2 \to A_i$, $i = 1, 2$, the morphisms $A \to 1$, and all evaluation morphisms $A \times [A \to B] \to B$, are base; further, if α, β are base, so are

$$\alpha \cdot \beta, \qquad \langle \alpha, \beta \rangle \quad \text{and} \quad \Lambda(\alpha),$$

when the sources and targets are appropriate.

2.2. Hom functors

A cartesian closed category $\mathscr{C}$ has two hom-functors. The familiar 'external' set-valued functor

$$\mathrm{Hom} : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathbf{Set}$$

as well as an *internal* hom-functor

$$\mathrm{hom} : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{C}.$$

For $(f, g) : (A, B) \to (C, D)$ in $\mathscr{C}^{\mathrm{op}} \times \mathscr{C}$, $\mathrm{hom}(f, g)$ is defined by the equation

$$\mathrm{hom}(f, g) = [A \to B] \xrightarrow{\mathrm{hom}(f, \mathrm{id})} [C \to B] \xrightarrow{\mathrm{hom}(\mathrm{id}, g)} [C \to D],$$

where we write hom(id, g) or $[-.g]$ for the unique morphism $[C \to B] \to [C \to D]$ such that

$$\begin{tikzcd}
C \times [C \to B] \arrow[r, "\mathrm{id}_C \times [-.g]"] \arrow[d, "e"'] & C \times [C \to D] \arrow[d, "e"] \\
B \arrow[r, "g"'] & D
\end{tikzcd}$$

Similarly, we write hom(f, id) or $[f.-]$ for the unique morphism $[A \to B] \to [C \to B]$ such that

$$\begin{tikzcd}
C \times [A \to B] \arrow[r, "\mathrm{id}_C \times [f.-]"] \arrow[d, "f \times \mathrm{id}"'] & C \times [C \to B] \arrow[d, "e"] \\
A \times [A \to B] \arrow[r, "e"'] & B
\end{tikzcd}$$

Note that if f is base, so are the morphisms $[f.-]$ and $[-.f]$. (Barr and Wells [3] write $[C \to g]$ for $[-.g]$ and $[f \to B]$ for $[f.-]$.)

There is a fundamental connection between the two hom-functors.

Proposition 2.1. *Suppose that $f : D \to A$, $g : E \to B$ and $h : C \to F$ in $\mathscr{C}$. The following diagram commutes:*

$$\begin{tikzcd}
\mathrm{Hom}(A \times B, C) \arrow[r, "\Lambda"] \arrow[d, "\mathrm{Hom}(f \times g, h)"'] & \mathrm{Hom}(B, [A \to C]) \arrow[d, "\mathrm{Hom}(g, \mathrm{hom}(f, h))"] \\
\mathrm{Hom}(D \times E, F) \arrow[r, "\Lambda"'] & \mathrm{Hom}(E, [D \to F])
\end{tikzcd}$$

The bijections Λ in (1) can also be internalized as base isomorphisms

$$\lambda : [A \times B \to C] \to [B \to [A \to C]]. \tag{2}$$

Proposition 2.2. *The following diagram commutes.*

$$\begin{tikzcd}
\mathrm{Hom}(A \times B \times C, D) \arrow[r, "\Lambda"] \arrow[d, "\Lambda"'] & \mathrm{Hom}(C, [A \times B \to D]) \arrow[d, "\mathrm{Hom}(\mathrm{id}, \lambda)"] \\
\mathrm{Hom}(B \times C, [A \to D]) \arrow[r, "\Lambda"'] & \mathrm{Hom}(C, [B \to [A \to D]])
\end{tikzcd}$$

(This fact might be taken as the definition of λ.)

3. The external Conway identities

3.1. External dagger operations

Our interest is in fixed-point operations on ccc's. A preliminary notion is that of a dagger operation. Suppose that $\mathscr{C}$ is a ccc. An *external dagger in product form* is a family of functions $d = (d_{B,A})$, indexed by all pairs of $\mathscr{C}$-objects, where

$$d_{B,A} : \mathrm{Hom}(A \times B, A) \to \mathrm{Hom}(B, A).$$

An *external dagger in functional form* $d = (d_{B,A})$ in $\mathscr{C}$ is a family of functions

$$d_{B,A} : \mathrm{Hom}(B, [A \to A]) \to \mathrm{Hom}(B, A).$$

(We will usually omit the subscripts on d.)

Due to the bijections $\Lambda : \mathrm{Hom}(A \times B, A) \to \mathrm{Hom}(B, [A \to A])$, for any dagger operation d in product form there is a corresponding dagger operation d^f in functional form. In fact, Λ determines a bijective correspondence between the two types of daggers:

$$d_{B,A} := Hom(A \times B, A) \xrightarrow{\Lambda} \mathrm{Hom}(B, [A \to A]) \xrightarrow{d^f_{B,A}} \mathrm{Hom}(B, A). \tag{3}$$

Remark 3.1. When we want to apply an external dagger operation to a morphism $f : A \to A$, we apply it to $\pi_A^{A \times 1} \cdot f : A \times 1 \to A$ instead. We write

$$d_A : \mathrm{Hom}(A, A) \to \mathrm{Hom}(1, A),$$

for this function.

Notation. If d is an external dagger in either functional or product form, we frequently write the value $d_{B,A}(f)$ as

$$f^\dagger : B \to A.$$

An external dagger d in product form satisfies the *fixed-point identity* if for all $g : A \times B \to A$, the following diagram commutes:

$$\begin{tikzcd} B \arrow[r, "\langle g^\dagger, \mathrm{id}_B \rangle"] \arrow[dr, "g^\dagger"'] & A \times B \arrow[d, "g"] \\ & A \end{tikzcd} \tag{4}$$

We say an external dagger operation in functional form satisfies the functional fixed-point identity if the following diagram commutes, for all $f : B \to [A \to A]$:

$$\begin{tikzcd} B \arrow[r, "\langle f^\dagger, f \rangle"] \arrow[dr, "f^\dagger"'] & A \times [A \to A] \arrow[d, "e"] \\ & A \end{tikzcd} \tag{5}$$

Of course, the two versions are equivalent.

Proposition 3.2. *An external dagger in functional form satisfies the identity* (5) *iff the corresponding dagger in product form satisfies the identity* (4).

3.2. The cartesian Conway identities

In this section, we list analogs of the 'Conway identities' for an external dagger in product form on a ccc [4]. In fact, these identities require only the cartesian structure, explaining the terminology. (In cartesian closed categories, these identities can be given either for dagger operations in product form or in functional form.) In Section 4, we introduce an identity meaningful only in cartesian closed categories.

(1) An external dagger d in product form satisfies the *parameter identity* if

$$d_{C,A}((\mathrm{id}_A \times g)\cdot f) = g \cdot d_{B,A}(f), \tag{6}$$

or, using the †-notation,

$$((\mathrm{id}_A \times g)\cdot f)^\dagger = g \cdot f^\dagger.$$

for all morphisms $g: C \to B$ and $f: A \times B \to A$.

(2) An external dagger in product form satisfies the *composition identity* if for any morphisms $f: M \times P \to N$ and $g: N \times P \to M$, the following equation holds:

$$d_{P,M}(\langle f, \pi_P^{M\times P}\rangle \cdot g) = \langle d_{P,N}(\langle g, \pi_P^{N\times P}\rangle \cdot f), \mathrm{id}_P\rangle \cdot g, \tag{7}$$

or, using the †-notation,

$$(\langle f, \pi_P^{M\times P}\rangle \cdot g)^\dagger = \langle (\langle g, \pi_P^{N\times P}\rangle \cdot f)^\dagger, \mathrm{id}_P\rangle \cdot g.$$

Note that

$$M \times P \xrightarrow{\langle f, \pi_P^{M\times P}\rangle} N \times P \xrightarrow{g} M$$

and

$$N \times P \xrightarrow{\langle g, \pi_P^{N\times P}\rangle} M \times P \xrightarrow{f} N.$$

(3) An external dagger in product form satisfies the *simplified composition identity* if for any morphisms $f: M \to N$ and $g: N \times P \to M$, the following equation holds:

$$d_{P,N}(g\cdot f) = d_{P,M}((f \times \mathrm{id}_P)\cdot g)\cdot f,$$

or, using the †-notation,

$$(g\cdot f)^\dagger = ((f \times \mathrm{id}_P)\cdot g)^\dagger \cdot f.$$

Note that

$$M \times P \xrightarrow{f \times \mathrm{id}_P} N \times P \xrightarrow{g} M$$

and

$$N \times P \xrightarrow{g} M \xrightarrow{f} N.$$

(4) An external dagger in product form satisfies the *double dagger identity* if the square

$$\begin{array}{ccc}
\mathrm{Hom}(A \times A \times B, A] & \xrightarrow{d_{A \times B, A}} & \mathrm{Hom}(A \times B, A) \\
{\scriptstyle \mathrm{Hom}(\Delta_A \times \mathrm{id}, \mathrm{id})} \downarrow & & \downarrow {\scriptstyle d_{B,A}} \\
\mathrm{Hom}(A \times B, A) & \xrightarrow[d_{B,A}]{} & \mathrm{Hom}(B, A)
\end{array}$$

commutes; i.e., for all morphisms $f: A \times A \times B \to A$,

$$f^{\dagger\dagger} = ((\Delta_A \times \mathrm{id}_B) \cdot f)^\dagger,$$

where

$$\Delta_A := \langle \mathrm{id}_A, \mathrm{id}_A \rangle : A \to A \times A.$$

Definition 3.3. The *cartesian Conway identities* are the parameter, composition and double dagger identities. A cartesian category equipped with an external dagger d in product form is a *Conway* cartesian category, or Conway cc for short, if d satisfies the cartesian Conway identities.

Remark 3.4. It is known that the cartesian Conway identities, together with the 'commutative identity' axiomatize the class of iteration theories (see [4]). The composition identity may be replaced by the simplified composition identity if the fixed-point identity is added.

Remark 3.5. It is interesting to consider the 'unparameterized' versions of the last two Conway identities, since these seem to occur occasionally in the literature [13].

Composition identity

$$(f \cdot g)^\dagger = (g \cdot f)^\dagger \cdot g,$$

all $f: M \to N$, $g: N \to M$. We may express this form of the composition identity by saying that the diagram given in Fig. 1 commutes (recall Remark 3.1).

Double dagger identity

$$f^{\dagger\dagger} = (\Delta_A \cdot f)^\dagger,$$

all $f: A \times A \to A$.

$$\begin{tikzcd}
 & \mathrm{Hom}(M,N) \arrow[dl, "\mathrm{Hom}(g,\mathrm{id})"'] \arrow[dr, "\mathrm{Hom}(\mathrm{id},g)"] & \\
\mathrm{Hom}(N,N) \arrow[d, "d"'] & & \mathrm{Hom}(M,M) \arrow[d, "d"] \\
\mathrm{Hom}(1,N) \arrow[rr, "\mathrm{Hom}(\mathrm{id},g)"'] & & \mathrm{Hom}(1,M)
\end{tikzcd}$$

Fig. 1.

3.3. Some consequences of the cartesian Conway identities

Using the arguments in [4], it can be shown that the cartesian Conway identities imply each of the following identities, some of which will be used in Section 7.

1. *Fixed-point identity*

$$g^\dagger = \langle g^\dagger, \mathrm{id}_B\rangle \cdot g,$$

all $g: A \times B \to A$. (Note: This is a special case of the composition identity (7): let $f = \pi_N^{N\times P}$.)

(2) *Left zero identity*

$$(A \times B \xrightarrow{\pi_B} B \xrightarrow{f} A)^\dagger = B \xrightarrow{f} A, \tag{8}$$

all morphisms $f: B \to A$.

(3) Pairing identity. Suppose that

$$f: A \times B \times P \to A,$$

$$g: A \times B \times P \to B,$$

so that

$$\langle f, g\rangle : A \times B \times P \to A \times B$$

$$f^\dagger : B \times P \to A.$$

Define h by

$$h := B \times P \xrightarrow{\langle f^\dagger, \mathrm{id}\rangle} A \times B \times P \xrightarrow{g} B.$$

The pairing identity is

$$\langle f, g\rangle^\dagger = \langle \langle h^\dagger, \mathrm{id}_P\rangle \cdot f^\dagger, h^\dagger\rangle. \tag{9}$$

Remark 3.6. The pairing identity is sometimes called the 'Bekic identity', e.g., in [14].

4. *Transposition identity*. Let $\tau := \langle \pi_B, \pi_A \rangle : A \times B \to B \times A$ be the base isomorphism. Then, if

$$f: A \times B \times P \to A \times B,$$

the transposition identity is

$$f^\dagger \cdot \tau = ((\tau^{-1} \times \mathsf{id}_P) \cdot f \cdot \tau)^\dagger. \tag{10}$$

5. For the next identity, assume that

$$f: S \times P \to S,$$
$$\beta: S \to Q,$$
$$g: R \times Q \to R,$$

so that

$$S \times R \times P \xrightarrow{\pi_{S \times P}} S \times P \xrightarrow{f} S$$

and

$$S \times R \times P \xrightarrow{(\beta \times \mathsf{id})} Q \times R \times P \xrightarrow{\pi_{R \times Q}} R \times Q \xrightarrow{g} R$$

and h is defined by

$$h := S \times R \times P \xrightarrow{\langle \pi_{S \times P} \cdot f,\ (\beta \times \mathsf{id}_{R \times P}) \cdot \pi_{R \times Q} \cdot g \rangle} S \times R.$$

Then, the following is true in all Conway cc's.

$$h^\dagger = \langle f^\dagger, f^\dagger \cdot \beta \cdot g^\dagger \rangle. \tag{11}$$

6. Now assume that $f: S \times P \to S$ and $g: R \times P \to R$, so that

$$\langle \pi_{S \times P} \cdot f,\ \pi_{R \times P} \cdot g \rangle : S \times R \times P \to S \times R.$$

Then the following identity holds in all Conway cc's.

$$\langle \pi_{S \times P} \cdot f,\ \pi_{R \times P} \cdot g \rangle^\dagger = \langle f^\dagger, g^\dagger \rangle. \tag{12}$$

7. Last, assume that

$$f: R \times S \times P \to R,$$
$$\alpha: R \to S.$$

Define h by

$$h = R \times P \xrightarrow{\langle \mathsf{id}_R, \alpha \rangle \times \mathsf{id}_P} R \times S \times P \xrightarrow{f} R.$$

Using the above notation, the following identity holds in all Conway cc's.

$$h^\dagger \cdot \alpha = (f^\dagger \cdot \alpha)^\dagger. \tag{13}$$

3.4. Natural dagger

It would be natural, to overuse a word, to ask how the dagger operations $d_{B,A}$ on the ccc $\mathscr{C}$ are related to one another. One can give several answers to this question by making use of the following concept, described in [12].

Suppose that $F, G : \mathscr{C}^{op} \times \mathscr{C} \to \mathscr{D}$ are functors. A family of maps $u_B : F(B,B) \to G(B,B)$ indexed by the $\mathscr{C}$-objects is a *dinatural transformation* between F, G if for each $\mathscr{C}$-morphism $f : B \to C$, the hexagon of Fig. 2 commutes.

It is easy to see that a natural transformation is a special case of a dinatural transformation. We show that a fixed-point operation that satisfies two of the Conway identities is a dinatural transformation. We define the functors F, G: $(\mathscr{C}^{op})^2 \times \mathscr{C}^2 \to \mathbf{Set}$ as follows. If $(f,g) : (A, B) \to (A', B')$ in $(\mathscr{C}^{op})^2$, and if $(h, k) : (C, D) \to (C', D')$ in $\mathscr{C}^2$,

$$F(f,g;\, h,k) := \mathrm{Hom}(A \times B, C) \xrightarrow{\mathrm{Hom}(f\times g,\, h)} \mathrm{Hom}(A' \times B', C'),$$

$$G(f,g;\, h,k) := \mathrm{Hom}(B, C) \xrightarrow{\mathrm{Hom}(g,\, h)} \mathrm{Hom}(B', C').$$

Note that if $\mathbf{A} := (A, B)$ then $F(\mathbf{A}, \mathbf{A}) = \mathrm{Hom}(A \times B, A)$, $G(\mathbf{A}, \mathbf{A}) = \mathrm{Hom}(B, A)$, and

$$d_{B,A} : F(\mathbf{A}, \mathbf{A}) \to G(\mathbf{A}, \mathbf{A}).$$

Proposition 3.7. *The family $d_{B,A}$ determines a dinatural transformation between F and G iff d satisfies both the parameter and simplified composition identity.*

Proof. d determines a dinatural transformation between F and G iff the diagram of Fig. 3 commutes, for all morphisms $f : A \to C$, $g : B \to D$ in $\mathscr{C}$.

Assume first the diagram commutes. When $f = \mathsf{id}$, this identity becomes the parameter identity. When $g = \mathsf{id}$, it becomes the simplified composition identity. Conversely, the simplified composition identity implies that for all $h : C \times D \to A$,

$$(h \cdot f)^\dagger = ((f \times \mathsf{id}_D) \cdot h)^\dagger \cdot f.$$

The parameter identity now implies

$$\begin{aligned} g \cdot (h \cdot f)^\dagger &= ((\mathsf{id} \times g) \cdot (f \times \mathsf{id}) \cdot h)^\dagger \cdot f \\ &= ((f \times g) \cdot h)^\dagger \cdot f, \end{aligned}$$

which is exactly what Fig. 3 asserts. □

3.5. The parameter identity

The parameter identity is fundamental, due, in part, to the following fact.

$$
\begin{tikzcd}
 & F(B,B) \arrow[r, "u_B"] & G(B,B) \arrow[dr, "G(\mathrm{id},f)"] & \\
F(C,B) \arrow[ur, "F(f,\mathrm{id})"] \arrow[dr, "F(\mathrm{id},f)"'] & & & G(B,C) \\
 & F(C,C) \arrow[r, "u_C"'] & G(C,C) \arrow[ur, "G(f,\mathrm{id})"'] &
\end{tikzcd}
$$

Fig. 2.

$$
\begin{tikzcd}
 & \mathrm{Hom}(C\times D,C) \arrow[r, "d"] & \mathrm{Hom}(D,C) \arrow[dr, "\mathrm{Hom}(g,\mathrm{id})"] & \\
\mathrm{Hom}(C\times D,A) \arrow[ur, "\mathrm{Hom}(\mathrm{id},f)"] \arrow[dr, "\mathrm{Hom}(f\times g,\mathrm{id})"'] & & & \mathrm{Hom}(B,C) \\
 & \mathrm{Hom}(A\times B,\ A) \arrow[r, "d"'] & \mathrm{Hom}(B,A) \arrow[ur, "\mathrm{Hom}(\mathrm{id},f)"'] &
\end{tikzcd}
$$

Fig. 3.

Proposition 3.8. *An external dagger on a ccc satisfies the parameter identity iff for each object A, there is a morphism*

$$\dagger_A : [A \to A] \to A$$

such that the function

$$d_{B,A} : \mathrm{Hom}(A \times B, A) \to \mathrm{Hom}(B, A)$$

is given by the equation

$$d_{B,A}(g) = \Lambda(g) \cdot \dagger_A.$$

Lemma 3.9. *The following are equivalent, for an external dagger d in product form on a cartesian closed category.*

1. *d satisfies the parameter identity.*
2. *The corresponding dagger d^f in functional form satisfies the identity*

$$d^f(g \cdot f) = g \cdot d^f(f), \tag{14}$$

for all $g : C \to B$, and all $f : B \to [A \to A]$.

3. *For each object A, d determines a natural transformation between the functors*

$$\mathrm{Hom}(A \times -, A) \overset{\bullet}{\rightarrow} \mathrm{Hom}(-, A).$$

4. *For each object A, d^f determines a natural transformation between the two Hom-functors*

$$\mathrm{Hom}(-, [A \to A]) \overset{\bullet}{\rightarrow} \mathrm{Hom}(-, A).$$

Proof of Proposition 3.8. By the lemma, d satisfies the parameter identity iff d^f determines a natural transformation between certain hom-functors. By the Yoneda Lemma, [12], such natural transformations are determined by morphisms

$$\dagger_A : [A \to A] \to A,$$

as claimed. The converse direction is obvious. □

Remark 3.10. In fact, when d satisfies the parameter identity, the morphism $\dagger_A$ is $d^f(\mathsf{id}_{[A \to A]}) = d(e_{A,A})$, by (3). Thus, we have the equations

$$\dagger_A = d(e_{A,A}) = e^{\dagger}_{A,A};$$

$$d(f) = \Lambda(f) \cdot e^{\dagger}_{A,A},$$

all $f: A \times B \to A$.

Example. Suppose that $\mathscr{C} = CPO$ is the category of cpo's (see Section 6) and $d_{B,A}$ assigns to each morphism $g : A \times B \to A$ the function $d(g) : B \to A$ whose value on $b \in B$ is the least $a \in A$ such that

$$a = g(a, b).$$

Then, if $f: C \to B$ is any continuous function,

$$f \cdot d(g) = d((\mathsf{id}_A \times f) \cdot g),$$

since for each $c \in C$, $d((\mathsf{id}_A \times f) \cdot g)(c)$ is the least $a \in A$ such that $a = g(a, f(c))$, which is just $(f \cdot d(g))(c)$. Hence, the least fixed-point operation satisfies the parameter identity. Of course, this function is determined by the morphism

$$[A \to A] \to A$$

$$g \mapsto \mu g,$$

where μg is the least $a \in A$ such that $a = g(a)$.

Recall that a category $\mathscr{C}$ is *well-pointed*, or *has enough points*, if, for all $f, g : A \to B$, $f = g$ when $y \cdot f = y \cdot g$ for all $y : 1 \to A$.

Lemma 3.11. *Suppose that $\mathscr{C}$ is a well-pointed ccc equipped with an external dagger d. Then d satisfies the parameter identity iff d is a pointwise dagger, i.e.,*

$$y \cdot f^{\dagger} = ((\mathsf{id}_A \times y) \cdot f)^{\dagger}, \tag{15}$$

for all $f: A \times B \to A$ and $y: 1 \to B$.

Proof. Suppose that (15) holds universally. Then, for all $f: A \times B \to A$, $g: C \to B$ and $u: 1 \to C$,

$$\begin{aligned} y \cdot ((\mathsf{id}_A \times g) \cdot f)^{\dagger} &= ((\mathsf{id}_A \times y \cdot g) \cdot f)^{\dagger} \\ &= y \cdot g \cdot f^{\dagger}. \end{aligned}$$

Since $\mathscr{C}$ is well-pointed, it follows that

$$((\mathsf{id}_A \times g) \cdot f)^{\dagger} = g \cdot f^{\dagger}.$$

The converse is a special case of the parameter identity. □

4. The abstraction identity

In cartesian closed categories of known computational interest, e.g., *CPO*, the fixed-point operation satisfies an identity involving lambda abstraction which apparently does not follow from the cartesian Conway identities. We may express this identity by saying the diagram of Fig. 4 commutes where α is the base morphism

$$\langle e_{P,R}, \pi_P \rangle : P \times [P \to R] \to R \times P.$$

In more elementary terms, suppose that

$$f: R \times P \times S \to R,$$

$$\begin{tikzcd}
\mathrm{Hom}(R \times P \times S,\ R) \arrow[rr, "\mathrm{Hom}(\alpha \times \mathrm{id},\ \mathrm{id})"] \arrow[d, "d"'] & & \mathrm{Hom}(P \times [P \to R] \times S,\ R) \arrow[d, "\Lambda"] \\
\mathrm{Hom}(P \times S, R) \arrow[dr, "\Lambda"'] & & \mathrm{Hom}([P \to R] \times S,\ [P \to R]) \arrow[dl, "d"] \\
& \mathrm{Hom}(S,\ [P \to R]) &
\end{tikzcd}$$

Fig. 4.

so that

$$f^\dagger : P \times S \to R,$$

$$\Lambda(f^\dagger) : S \to [P \to R].$$

Define $g : P \times [P \to R] \times S \to R$ by

$$\begin{tikzcd}
P \times [P \to R] \times S \arrow[r, "g"] \arrow[d, "\langle e_{P,R}, \pi_P \rangle \times \mathrm{id}_S"'] & R \\
R \times P \times S \arrow[ur, "f"'] &
\end{tikzcd}$$

Lastly, define $h : [P \to R] \times S \to [P \to R]$ by

$$h := \Lambda(g) : [P \to R] \times S \to [P \to R].$$

Thus, both $h^\dagger$ and $\Lambda(f^\dagger)$ are morphisms $S \to [P \to R]$. With this notation the *abstraction identity* becomes

$$h^\dagger = \Lambda(f^\dagger). \tag{16}$$

In the category *CPO* (see Section 6), given cpo's S, P, R and a continuous function $f : R \times P \times S \to R$, we have

$$f^\dagger(p, s) = \mu_r(r = f(r, p. s)).$$

Thus,

$$\Lambda(f^\dagger)(s) = \lambda_p \mu_r(r = f(r, p, s)).$$

Now the function $g : P \times [P \to R] \times S \to R$ is

$$g(p, \beta, s) = f(\beta(p), p, s),$$

so that

$$h(\beta, s) = \lambda_p f(\beta(p), p, s).$$

It follows that

$$h^\dagger(s) = \mu_\beta(\forall p \in P \ \beta(p) = f(\beta(p), p, s)),$$

and for a fixed $s \in S$, this least function β is given by

$$\beta(p) = \mu_r(r = f(r, p, s)),$$

i.e., $$\begin{aligned} \beta &= \lambda_p(f^\dagger(p, s)) \\ &= \Lambda(f^\dagger). \end{aligned}$$

5. Conway ccc's

We define a *Conway ccc* as a ccc having an external dagger which satisfies the abstraction identity as well as the cartesian Conway identities, i.e., the parameter, composition and double dagger identities. It follows that such an operation is a fixed-point operation which is determined by a family of $\mathscr{C}$-morphisms

$$\dagger_A : [A \to A] \to A.$$

Further, it satisfies all of the other identities in Section 3.3.

In the appendix, we find equations involving the internal morphisms $\dagger_A$ which are equivalent to the Conway identities for an external dagger.

6. Examples and nonexamples of Conway ccc's

We give several categories of posets which are Conway ccc's. First, we review some basic definitions. A poset $X = (X, \leqslant)$ is a *cpo* if X has a least element $\perp_X$ and every directed subset D has a sup. An element $x \in X$ is *compact* if whenever $D \subseteq X$ is directed and $x \leqslant \sup D$ then for some $d \in D$, $x \leqslant d$. Let X_0 denote the set of compact elements in X. A cpo is *algebraic* if for each $x \in X$, the set

$$x\!\downarrow\; = \{y \in X_0 : y \leqslant x\}$$

is directed and $x = \sup x\!\downarrow$. A cpo is *bounded complete* if every bounded subset (as well as every directed subset) has a sup. A *Scott domain* is a cpo which is bounded complete and algebraic. A function $f : X \to Y$ between cpo's is *monotone* if $x \leqslant x' \Rightarrow xf \leqslant x'f$, all $x, x' \in X$; a function $f : X \to Y$ is *continuous* if it preserves sup's of nonempty directed sets, i.e.,

$$f(\sup D) = \sup f(D),$$

for all nonempty directed $D \subseteq X$.

An explanation of why the following poset-categories satisfy the Conway identities is given in Section 6.2.

6.1. Some poset ccc's

(1) The category CPO_m has all cpo's as objects, and all monotone functions as morphisms. This category is a ccc, where the exponential object $[A \to B]$ is the set of all monotone maps $A \to B$ with the pointwise ordering. Binary products are defined as usual. When $f : A \times B \to A$, we write f_b for the function $A \to A$ defined by $a \mapsto f(a, b)$. For each morphism $f : A \times B \to A$ the function $f^\dagger : B \to A$ is defined as follows. Given $b \in B$, we define

$$\begin{aligned} f^\dagger(b) &:= \text{the least } a \in A : a = f_b(a) \\ &= \mu_a(a = f(a, b)). \end{aligned}$$

Further, the function

$$\dagger_{B,A} : [A \times B \to A] \to [B \to A]$$
$$f \mapsto f^\dagger$$

is monotone.

(2) The category *CPO* has all cpo's as objects, and all continuous functions as morphisms. This category is a ccc whose exponential object $[A \to B]$ is the set of all continuous functions $A \to B$. It is well-known that if for each pair A, B of cpo's we define

$$\dagger_{B,A} : [A \times B \to A] \to [B \to A]$$

by

$$f \mapsto f^\dagger$$
$$f^\dagger(b) = \mu_a(a = f_b(a)),$$

then $\dagger_{A,B}$ is also a continuous function, and hence a morphism in *CPO*.

(3) The category *SD* is the full subcategory of *CPO* whose objects are the Scott domains. It is well-known that this category is a ccc [1].

(4) *Effective Scott domains.* A Scott domain $(X, \leqslant)$ is effective if there is a surjection e_0 from the nonnegative integers onto the compact elements in X such that
- the relation which holds iff the elements $e_0(n)$, $e_0(m)$ have a common upper bound, is recursive;
- the relation

  $$e_0(n) = e_0(m) \sqcup e_0(p)$$

  is recursive;
- $e_0(0) = \bot$.

It is known [1] that the full subcategory *ED* of *SD* determined by the effective Scott domains forms a sub-ccc of *SD*.

(5) the category of dI domains and stable functions is a ccc, which is a subcategory of *CPO*, although not a full subcategory, in which the least fixed-point operation is also stable, see [9].

6.2. *Why?*

Each of the poset categories in Section 6.1 is an *order enriched* ccc; i.e., there is a partial order on the hom sets such that the operations of composition, pairing and lambda abstraction are monotone. We will use Lemma 3.11 above together with the corollary of following theorem to show that each of the examples is a Conway ccc.

For a morphism $f : A \times B \to A$ in an order enriched ccc, we say that a morphism $\xi : B \to A$ is a *prefixed point* of f if

$$\langle \xi, \mathrm{id}_B \rangle \cdot f \leqslant \xi.$$

Theorem 6.1. *Suppose that $\mathscr{C}$ is an order enriched ccc such that for each $f: A \times B \to A$ there is a least prefixed point $f^\dagger: B \to A$ of f. Then, if the dagger operation $f \mapsto f^\dagger$ satisfies the parameter identity, $\mathscr{C}$ is a Conway ccc.*

Our proof is based on the following lemma.

Lemma 6.2. *Suppose that $f: A \times B \times C \to A$ and $\xi: B \times C \to A$ in a ccc. Define the morphism g by*

$$\begin{tikzcd} B \times [B \to A] \times C \arrow[r, "g"] \arrow[d, "\langle e_{B,A}, \pi_B\rangle \times \mathrm{id}_C"'] & A \\ A \times B \times C \arrow[ur, "f"'] & \end{tikzcd}$$

Then,

$$\langle A\xi, \mathrm{id}_C\rangle \cdot Ag = A(\langle \xi, \mathrm{id}_{B \times C}\rangle \cdot f): C \to A.$$

Proof. For any $\xi: B \times C \to A$, the triangle

$$\begin{tikzcd} B \times C \arrow[d, "\mathrm{id}_B \times \langle A\xi, \mathrm{id}_C\rangle"'] \arrow[dr, "\langle \xi, \mathrm{id}_{B\times C}\rangle"] & \\ B \times [B \to A] \times C \arrow[r, "\langle e_{B,A}, \pi_B\rangle \times \mathrm{id}_C"'] & A \times B \times C \end{tikzcd}$$

commutes. Also, the following square commutes by the definition of g:

$$\begin{tikzcd} B \times [B \to A] \times C \arrow[r, "\langle e_{B,A}, \pi_B\rangle \times \mathrm{id}_C"] \arrow[d, "\mathrm{id}_B \times Ag"'] & A \times B \times C \arrow[d, "f"] \\ B \times [B \to A] \arrow[r, "e_{B,A}"'] & A \end{tikzcd}$$

Thus, pasting together these two diagrams, we see the following commutes.

$$\begin{tikzcd} B \times C \arrow[d, "\mathrm{id}_B \times (\langle A\xi, \mathrm{id}_C\rangle \cdot Ag)"'] \arrow[dr, "\langle \xi, \mathrm{id}_{B\times C}\rangle \cdot f"] & \\ B \times [B \to A] \arrow[r, "e_{B,A}"'] & A \end{tikzcd}$$

The proof of the lemma if complete. $\square$

Proof of Theorem 6.1. It is known that the cartesian Conway identities hold in $\mathscr{C}$, cf. [4, Ch. 8]. Thus, we need to show only that the abstraction identity holds in $\mathscr{C}$. But by Lemma 6.2, since Λ creates an order isomorphism between the posets $\mathrm{Hom}(B \times C, A)$ and $\mathrm{Hom}(C, [B \to A])$,

$$\langle \Lambda\xi, \mathsf{id}_C \rangle \cdot \Lambda g \leqslant \Lambda\xi \quad \text{iff } \langle \xi, \mathsf{id}_{B \times C} \rangle \cdot f \leqslant \xi.$$

In words, ξ is a prefixed point of f iff $\Lambda(\xi)$ is a prefixed point of $\Lambda(g)$. It follows that

$$\Lambda(f^\dagger) = (\Lambda g)^\dagger. \quad \Box$$

The following corollary answers the question: Why do the poset categories in Section 6.1 form Conway ccc's?

Corollary 6.3. *Let $\mathscr{C}$ be a well-pointed, order enriched ccc. Suppose that the order on the hom sets is pointwise, i.e., if f, $g: A \to B$ are such that $x \cdot f \leqslant x \cdot g$ for all $x: 1 \to A$ then $f \leqslant g$. Suppose that for each $f: A \times B \to A$ we are given a morphism $f^\dagger: B \to A$ such that for each $y: 1 \to B$, the value $yf^\dagger$ is the least morphism $x: 1 \to A$ with*

$$\langle x, y \rangle \cdot f \leqslant x.$$

Then $\mathscr{C}$ is a Conway ccc.

Remark 6.4. By well-pointedness, $f^\dagger$ is unique with the above property.

Proof. Clearly, $f^\dagger$ is the least prefixed point of f. Further, by Lemma 3.11, the parameter identity holds in $\mathscr{C}$. The result follows from Theorem 6.1. $\Box$

Remark 6.5. It is well-known that the least fixed-point operation in *CPO* satisfies (at least the simple version of) the cartesian Conway identities (see, e.g., [14, pp. 21]). The proof in [4] of the pairing identity follows the argument in [11]. A 2-categorical generalization of Theorem 6.1 will be given in [6].

6.3. *Nonexamples*

We give two examples of ccc's which either do not have fixed-point operations, or the operations do not satisfy the Conway identities.

(1) The category *PER* (see, e.g., [1]) of partial equivalence relations on ω is a ccc which has both an initial object and binary coproducts. It follows that *PER* does not have fixed-point morphisms. Indeed, it is well-known [1, 15] that if a nontrivial ccc either has an initial object or has binary coproducts, then it cannot have fixed-point morphisms $[A \to A] \to A$, for every object A.

(2) Let $\mathscr{C}$ be the category whose objects are complete lattices and whose morphisms are the monotone maps. If $f: A \times B \to A$, for each $b \in B$ there is a least $a \in A$ with $a = f(a, b)$ and there is also a greatest such a. Write $\mu f: B \to A$ for the function

returning the least fixed point, and $\nu f: B \to A$ for the function returning the greatest. We now define an external dagger operation as follows. Suppose that $f: A \times B \to A$.

$$d_{B,A}(f) := \begin{cases} \mu f & \text{if } A \times B \text{ infinite,} \\ \nu f & \text{otherwise.} \end{cases}$$

Then d satisfies the fixed-point identity, but does not satisfy the parameter identity. Indeed, suppose that $A = B = \{0 < 1\}$ and let $C = \{0 < 1 < \cdots < \infty\}$, the non-negative integers with a 'top'. Let $g: C \to B$ be $\lambda_c\, 1$, the constant function with value 1, and let $h = \pi_A^{A \times B}$. Then

$$\begin{aligned} d_{B,A}(h) &= \lambda_b\, 1, \\ d_{C,A}((\mathsf{id} \times g) \cdot h) &= \lambda_c\, 0, \\ g \cdot d_{B,A}(h) &= \lambda_c\, 1. \end{aligned}$$

This operation also fails to satisfy the composition identity. For the same lattices A, C, let $h: A \to C$ be the inclusion, and let $g: C \to A$ be the unique monotone map such that $h \cdot g = \mathsf{id}_A$. Then

$$\begin{aligned} (h \cdot g)^\dagger &= 1, \\ (g \cdot h)^\dagger &= 0, \end{aligned}$$

and $(g \cdot h)^\dagger \cdot g = 0g \neq (h \cdot g)^\dagger = 1$.

7. A normal form theorem

In this section, suppose that $\mathscr{C}$ is a Conway ccc, and that $\mathscr{C}_0$ is a sub-ccc of $\mathscr{C}$. We say that a sub-ccc $\mathscr{D}$ is *closed under the dagger operation* if $f^\dagger = \Lambda(f) \cdot \dagger_A$ is in $\mathscr{D}$ whenever $f: A \times B \to A$ is. Clearly, $\mathscr{D}$ is closed under the dagger operation iff each morphism $\dagger_A$ is a $\mathscr{D}$-morphism, for each object A in $\mathscr{D}$. We give a simple, concrete description of the least sub-ccc of $\mathscr{C}$ containing all $\mathscr{C}_0$-morphisms which is closed under the dagger operation.

7.1. Normal descriptions

A *$\mathscr{C}_0$-normal description* $D = (f; \beta)$ is a pair consisting of a $\mathscr{C}_0$-morphism $f: P \times S \to P$ and a base morphism $\beta: P \to R$. We write

$$D: S \xrightarrow[P]{} R,$$

and say D has *source* S, *target* R and *weight* P, in analogy with the normal descriptions in [7]. We define the *behavior* $|D|$ of $D = (f; \beta)$ as the morphism

$$|D| := S \xrightarrow{f^\dagger} P \xrightarrow{\beta} R.$$

Definition 7.1. Let ND denote the collection of all morphisms $|D|$, for normal descriptions D.

Remark. Normal descriptions of a different kind were used in several places in the study of iteration theories [4, Ch. 5; 7].

7.2. The theorem

In this section we prove that ND is the least sub-ccc of $\mathscr{C}$ which contains $\mathscr{C}_0$ is closed under the dagger operation.

Definition 7.2. For $f: S \to P$ in $\mathscr{C}_0$, let $D_f: S \xrightarrow[P]{} P$ be the following normal description:

$$D_f := (\pi_S^{P \times S} \cdot f;\ \mathsf{id}_P).$$

Proposition 7.3. *With the above notation, $|D_f| = f$, so that $\mathscr{C}_0 \subseteq ND$.*

Proof. For any $f: S \to P$ in $\mathscr{C}_0$,

$$\begin{aligned} |D_f| &= (\pi_S^{P \times S} \cdot f)^{\dagger} \cdot \mathsf{id}_P \\ &= f, \end{aligned}$$

by the left zero identity (8). $\square$

We turn now to composition. If $D = (f;\ \beta): P \xrightarrow[S]{} Q$ and $E = (g; \gamma): Q \xrightarrow[R]{} T$ are normal descriptions, then

$$\begin{aligned} f &: S \times P \to S, \\ \beta &: S \to Q, \\ g &: R \times Q \to R, \\ \gamma &: R \to T, \end{aligned}$$

where f, $g \in \mathscr{C}_0$ and β, γ are base. Thus, the morphism h belongs to $\mathscr{C}_0$ where

$$h := S \times R \times P \xrightarrow{\langle \pi_{S \times P} \cdot f,\, (\beta \times \mathsf{id}_{R \times P}) \cdot \pi_{R \times Q} \cdot g \rangle} S \times R. \tag{17}$$

Definition 7.4. For normal descriptions $D = (f; \beta): P \xrightarrow[S]{} Q$, $E = (g; \gamma): Q \xrightarrow[R]{} T$, define the normal description $D \cdot E: P \xrightarrow[S \times R]{} T$ by

$$D \cdot E := (h;\ \delta),$$

where h is the morphism in (17) and $\delta := \pi_R \cdot \gamma$.

Proposition 7.5. *With the above notation,* $|D \cdot E| = |D| \cdot |E|$. *Thus, the collection ND is closed under composition.*

Proof. Suppose that $D = (f; \beta): P \xrightarrow[S]{} Q$ and $E = (g; \gamma): Q \xrightarrow[R]{} T$. Then,

$$\begin{aligned} |D \cdot E| &= h^\dagger \cdot \delta \\ &= \langle f^\dagger, f^\dagger \cdot \beta \cdot g^\dagger \rangle \cdot \pi_R \cdot \gamma, \quad \text{by identity (11)}, \\ &= f^\dagger \cdot \beta \cdot g^\dagger \cdot \gamma \\ &= |D| \cdot |E|. \qquad \square \end{aligned}$$

In order to show that ND is closed under target tupling, assume that $D = (f; \alpha): P \xrightarrow[S]{} N$ and $E = (g; \beta): P \xrightarrow[R]{} M$ are normal descriptions, so that $f: S \times P \to S$, $g: R \times P \to R$, $\alpha: S \to N$, and $\beta: R \to M$. Thus, the morphism

$$h := \langle \pi_{S \times P} \cdot f, \pi_{R \times P} \cdot g \rangle : S \times R \times P \to S \times R \tag{18}$$

belongs to $\mathscr{C}_0$, since f and g do.

Definition 7.6. Given normal descriptions $D = (f; \alpha): P \xrightarrow[S]{} N$ and $E = (g; \beta): P \xrightarrow[R]{} M$, define the normal description $\langle D, E \rangle$ by

$$\langle D, E \rangle := (h; \gamma): P \to N \times M,$$

where h is defined in (18), and where γ is defined by

$$\gamma := \alpha \times \beta : S \times R \to N \times M.$$

Proposition 7.7. *With the above notation,* $|\langle D, E \rangle| = \langle |D|, |E| \rangle$, *so that the collection ND is closed under target tupling.*

Proof. Given the normal descriptions $D = (f; \alpha): P \xrightarrow[S]{} N$ and $E = (g; \beta): P \xrightarrow[R]{} M$,

$$\begin{aligned} |\langle D, E \rangle| &= h^\dagger \cdot \gamma \\ &= \langle f^\dagger, g^\dagger \rangle \cdot (\alpha \times \beta) \quad \text{by identity (12)}, \\ &= \langle f^\dagger \cdot \alpha, g^\dagger \cdot \beta \rangle \\ &= \langle |D|, |E| \rangle. \quad \square \end{aligned}$$

In order to show that ND is closed under the dagger operation, suppose that D is a normal description $D = (f; \alpha): S \times P \xrightarrow[S]{} S$, so that

$$f: R \times S \times P \to R$$

$$\alpha: R \to S.$$

Then

$$h := R \times P \xrightarrow{\langle \mathrm{id}_R, \alpha\rangle \times \mathrm{id}_P} R \times S \times P \xrightarrow{f} R \tag{19}$$

belongs to $\mathscr{C}_0$, since α is base and $f \in \mathscr{C}_0$.

Definition 7.8. Given $D = (f;\alpha): S \times P \xrightarrow[R]{} S$, define the normal description $D^\dagger$ by

$$D^\dagger := (h;\alpha),$$

where h is defined in (19).

Proposition 7.9. *With the above notation, $|D^\dagger| = |D|^\dagger$, so that the collection ND is closed under the dagger operation.*

Proof. Suppose that $D = (f;\alpha): S \times P \xrightarrow[R]{} S$. Then,

$$\begin{aligned} |D^\dagger| &= |(h;\alpha)| \\ &= h^\dagger \cdot \alpha \\ &= (f^\dagger \cdot \alpha)^\dagger \quad \text{by identity (13),} \\ &= |D|^\dagger. \quad \square \end{aligned}$$

In order to show that ND is closed under lambda abstraction, we use the abstraction identity, of course.

Given $D = (f;\beta): P \times S \xrightarrow[R]{} T$, so that

$$f: R \times P \times S \to R,$$
$$\beta: R \to T,$$

we define the morphisms g and h as in Section 4 above.

$$\begin{tikzcd} P \times [P \to R] \times S \arrow[r, "g"] \arrow[d, "\langle e_{P,R}, \pi_P\rangle \times \mathrm{id}_S"'] & R \\ R \times P \times S \arrow[ur, "f"'] & \end{tikzcd}$$

The morphism $h: [P \to R] \times S \to [P \to R]$ is

$$h := \Lambda(g): [P \to R] \times S \to [P \to R].$$

Note that g and h are in $\mathscr{C}_0$ if f is.

Definition 7.10. Given $D = (f;\beta): P \times S \xrightarrow[R]{} T$, define the normal description ΛD by

$$\Lambda D := (h;\ [-.\beta]),$$

where h is defined in (20).

Note that $[-.\beta]$ is base since β is.

Proposition 7.11. *Suppose that* $|D| = (f;\beta): P \times S \xrightarrow[R]{} T$. *Then*

$$|\Lambda D| = \Lambda(|D|),$$

so that ND is closed under lambda abstraction.

Proof

$$\begin{aligned}|\Lambda D| &= h^{\dagger}\cdot[-.\beta]\\ &= \Lambda(f^{\dagger})\cdot[-.\beta] \quad \text{by the abstraction identity (16),}\\ &= \Lambda(f^{\dagger}\cdot\beta) \quad \text{as is easy to check,}\\ &= \Lambda(|D|). \quad \square\end{aligned}$$

We have proved the following theorem.

Theorem 7.12. *The collection ND of* $\mathscr{C}$*-morphisms which are behaviors of normal descriptions is the least sub-ccc of* $\mathscr{C}$ *which contains* $\mathscr{C}_0$ *and is closed under the dagger operation.*

Proof. We have shown that ND forms a sub-ccc which contains $\mathscr{C}_0$ and is closed under the dagger operation. Clearly any sub-ccc which contains $\mathscr{C}_0$ and is closed under the dagger operation must contain

$$f^{\dagger}\cdot\beta,$$

when f is in $\mathscr{C}_0$ and β is base. $\square$

Since the abstraction identity is only used in one part, the above argument proves the following result.

Corollary 7.13. *Suppose that* $\mathscr{C}$ *is a Conway cartesian category containing the sub-cartesian category* $\mathscr{C}_0$. *Then the least sub-cartesian category of* $\mathscr{C}$ *which contains* $\mathscr{C}_0$ *and which is closed under the dagger operation is the collection of all morphisms of the form*

$$f^{\dagger}\cdot\beta,$$

where f *is in* $\mathscr{C}_0$ *and* β *is base.*

(Of course, the notion of 'base' morphism must be modified for cartesian categories.)

8. Functional completeness

Suppose that $\mathscr{C}$ is a ccc. It follows from general facts [10] that there exists a ccc $\mathscr{C}[x]$ with the same objects as $\mathscr{C}$ which is obtained by adding a morphism $x:1 \to A$ freely to $\mathscr{C}$. The ccc $\mathscr{C}[x]$ is determined up to isomorphism by the following properties. First, $\mathscr{C}$ is a sub-ccc of $\mathscr{C}[x]$. Second, for each ccc $\mathscr{D}$, each ccc morphism $\varphi:\mathscr{C} \to \mathscr{D}$, and for each $\mathscr{D}$-morphism $y:1 \to A\varphi$, there exists a unique ccc morphism $\psi:\mathscr{C}[x] \to \mathscr{D}$ such that

$$\mathscr{C} \hookrightarrow \mathscr{C}[x] \xrightarrow{\psi} \mathscr{D} = \mathscr{C} \xrightarrow{\varphi} \mathscr{D},$$

$$x\psi = y.$$

(Recall that a ccc is given with explicit products, exponentials and a distinguished terminal object that are preserved by ccc morphisms.)

A characterization of the ccc $\mathscr{C}[x]$ is known as the *functional completeness theorem for ccc's* that we recall now from [10]. We write x_B for the morphism $B \to 1 \xrightarrow{x} A$.

Theorem 8.1. *Each morphism $B \to C$ of $\mathscr{C}[x]$ can be written uniquely as*

$$\langle \mathsf{id}_B, x_B\rangle \cdot f,$$

for some $f: B \times A \to C$.

By Theorem 8.1, an explicit representation of the ccc $\mathscr{C}[x]$ is the following. The objects of $\mathscr{C}[x]$ are the same as those of $\mathscr{C}$. The morphisms $B \to C$ in $\mathscr{C}[x]$ are the morphisms $B \times A \to C$ in $\mathscr{C}$. We denote the correspondence by

$$\varphi: \mathrm{Hom}_{\mathscr{C}}(B \times A, C) \to \mathrm{Hom}_{\mathscr{C}[x]}(B, C),$$

$$f \mapsto f'.$$

The composite in $\mathscr{C}[x]$ of $f: X \times A \to Y$ and $g: Y \times A \to Z$ is, by definition

$$f' \cdot g' := \langle f, \pi_A\rangle \cdot g.$$

The idenity $B \to B$ in $\mathscr{C}[x]$ is the $\mathscr{C}$-morphism $\pi_B^{B\times A}$. The evaluation map $B \times [B \to C] \to C$ in $\mathscr{C}[x]$ is defined as the morphism

$$B \times [B \to C] \times A \xrightarrow{\pi} B \times [B \to C] \xrightarrow{e_{B,C}} C.$$

The category $\mathscr{C}$ may be considered to be a sub-ccc of $\mathscr{C}[x]$ via the inclusion functor

$$\iota: \mathscr{C} \to \mathscr{C}[x],$$

$$f: B \to C \mapsto B \times A \xrightarrow{\pi_B} B \xrightarrow{f} C,$$

which is a ccc-morphism. Thus, for all pairs of objects B, C,

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(B,C) \arrow[r, "\iota"] \arrow[d, "\mathrm{Hom}(\pi,\mathrm{id})"'] & \mathrm{Hom}_{\mathscr{C}[x]}(B,C) \\
\mathrm{Hom}_{\mathscr{C}}(B\times A, C) \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

commutes.

The morphism $x:1\to A$ in $\mathscr{C}[x]$ is the $\mathscr{C}$-morphism

$$\pi_A^{1\times A}:1\times A\to A.$$

We note several properties of this correspondence.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(X\times X\times Y\times A, Z) \arrow[r, "\varphi"] \arrow[d, "\mathrm{Hom}_{\mathscr{C}}(\Delta\times\mathrm{id},\mathrm{id})"'] & \mathrm{Hom}_{\mathscr{C}[x]}(X\times X\times Y, Z) \arrow[d, "\mathrm{Hom}_{\mathscr{C}[x]}(\Delta\times\mathrm{id},\mathrm{id})"] \\
\mathrm{Hom}_{\mathscr{C}}(X\times Y\times A, Z) \arrow[r, "\varphi"'] & \mathrm{Hom}_{\mathscr{C}[x]}(X\times Y, Z)
\end{tikzcd} \tag{20}$$

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(B\times C\times A, D) \arrow[r, "\varphi"] \arrow[d, "\Lambda"'] & \mathrm{Hom}_{\mathscr{C}[x]}(B\times C, D) \arrow[d, "\Lambda"] \\
\mathrm{Hom}_{\mathscr{C}}(C\times A,\ [B\to D]) \arrow[r, "\varphi"'] & \mathrm{Hom}_{\mathscr{C}[x]}(C, [B\to D])
\end{tikzcd} \tag{21}$$

Suppose now that $\mathscr{C}$ is equipped with an external dagger operation in product form,

$$d:\mathrm{Hom}(B\times P,\ B)\to\mathrm{Hom}(P,\ B),$$

$$f\mapsto f^{\dagger}.$$

We define an external dagger operation on $\mathscr{C}[x]$:

$$\langle \mathrm{id}_{B\times P},\ x_{B\times P}\rangle\cdot g:B\times P\to B \ \mapsto\ \langle \mathrm{id}_P,\ x_P\rangle\cdot g^{\dagger}:P\to B. \tag{22}$$

Thus, by definition, the following square commutes:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(C\times D\times A, C) \arrow[r, "d"] \arrow[d, "\varphi"'] & \mathrm{Hom}_{\mathscr{C}}(D\times A, C) \arrow[d, "\varphi"] \\
\mathrm{Hom}_{\mathscr{C}[x]}(C\times D, B) \arrow[r, "d"'] & \mathrm{Hom}_{\mathscr{C}[x]}(D, C)
\end{tikzcd} \tag{23}$$

Proposition 8.2. *Suppose the parameter identity holds in $\mathscr{C}$. Then the external dagger operation defined by (22) above extends the dagger operation on $\mathscr{C}$. Further, the parameter identity holds in $\mathscr{C}[x]$.*

Proof. First we show that the dagger operation defined on $\mathscr{C}[x]$ extends the operation given on $\mathscr{C}$. Since the dagger operation on $\mathscr{C}$ satisfies the parameter identity, the following square commutes.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(C \times D, C) \arrow[r, "d"] \arrow[d, "\mathrm{Hom}_{\mathscr{C}}(\mathrm{id} \times \pi_D^{D\times A}, \mathrm{id})"'] & \mathrm{Hom}_{\mathscr{C}}(D, C) \arrow[d, "\mathrm{Hom}_{\mathscr{C}}(\pi_D^{D\times A}, \mathrm{id})"] \\
\mathrm{Hom}_{\mathscr{C}}(C \times D \times A, C) \arrow[r, "d"'] & \mathrm{Hom}_{\mathscr{C}}(D \times A, C)
\end{tikzcd} \tag{24}$$

Placing (24) on top of (23), we obtain the fact that

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(C \times D, C) \arrow[r, "d"] \arrow[d, "\iota"'] & \mathrm{Hom}_{\mathscr{C}}(D, C) \arrow[d, "\iota"] \\
\mathrm{Hom}_{\mathscr{C}[x]}(C \times D, C) \arrow[r, "d"'] & \mathrm{Hom}_{\mathscr{C}[x]}(D, C)
\end{tikzcd}$$

commutes, which shows that the dagger in $\mathscr{C}[x]$ extends that in $\mathscr{C}$.

To see that the parameter identity holds in $\mathscr{C}[x]$, suppose that $g : E \times A \to D$ in $\mathscr{C}$, so that $g' : E \to D$ in $\mathscr{C}[x]$. It is easy to check that the squares

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(C \times D \times A, C) \arrow[r, "\mathrm{Hom}_{\mathscr{C}}(\mathrm{id} \times \langle g, \pi_A \rangle,\ \mathrm{id})"] \arrow[d, "\varphi"'] & \mathrm{Hom}_{\mathscr{C}}(C \times E \times A, C) \arrow[d, "\varphi"] \\
\mathrm{Hom}_{\mathscr{C}[x]}(C \times D, C) \arrow[r, "\mathrm{Hom}_{\mathscr{C}[x]}(\mathrm{id} \times g', \mathrm{id})"'] & \mathrm{Hom}_{\mathscr{C}[x]}(C \times E, C)
\end{tikzcd} \tag{25}$$

and

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(D \times A, C) \arrow[r, "\mathrm{Hom}_{\mathscr{C}}(\langle g, \pi_A \rangle,\ \mathrm{id})"] \arrow[d, "\varphi"'] & \mathrm{Hom}_{\mathscr{C}}(E \times A, C) \arrow[d, "\varphi"] \\
\mathrm{Hom}_{\mathscr{C}[x]}(D, C) \arrow[r, "\mathrm{Hom}_{\mathscr{C}[x]}(g', \mathrm{id})"'] & \mathrm{Hom}_{\mathscr{C}[x]}(E, C)
\end{tikzcd} \tag{26}$$

commute. In the following diagram, the smaller inner squares on the left and right are (25) and (26). To prove the parameter identity holds in $\mathscr{C}[x]$, we must show the inner square commutes. But the outer square commutes, since the parameter identity holds in $\mathscr{C}$, and the top and bottom squares commute, by (23). It follows that the inner square commutes, completing the argument (see Fig. 5).

Remark 8.3. Equation (24) is necessary in order that the dagger operation defined by (23) is an extension of the dagger operation on $\mathscr{C}$.

From now on, we assume that the parameter identity holds in $\mathscr{C}$, and hence that it holds in $\mathscr{C}[x]$.

$$
\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(C \times D \times A, C) \arrow[rrr, "d"] \arrow[dddd] \arrow[dr, "\varphi"] & & & \mathrm{Hom}_{\mathscr{C}}(D \times A, C) \arrow[dl, "\varphi"'] \arrow[dddd] \\
& \mathrm{Hom}_{\mathscr{C}[x]}(C \times D, C) \arrow[r, "d"] \arrow[dd] & \mathrm{Hom}_{\mathscr{C}[x]}(D, C) \arrow[dd] & \\
& & & \\
& \mathrm{Hom}_{\mathscr{C}[x]}(C \times E, C) \arrow[r, "d"] & \mathrm{Hom}_{\mathscr{C}[x]}(E, C) & \\
\mathrm{Hom}_{\mathscr{C}}(C \times E \times A, C) \arrow[rrr, "d"'] \arrow[ur, "\varphi"] & & & \mathrm{Hom}_{\mathscr{C}}(E \times A, C) \arrow[ul, "\varphi"']
\end{tikzcd}
$$

Fig. 5.

Proposition 8.4. *Suppose that $\mathscr{D}$ is a ccc with a dagger operation satisfying the parameter identity and suppose that $\varphi : \mathscr{C} \to \mathscr{D}$ is a ccc morphism that preserves dagger. Let $y : 1 \to A\varphi$ be a fixed morphism in $\mathscr{D}$. Then the unique ccc morphism $\psi : \mathscr{C}[x] \to \mathscr{D}$ which extends φ and maps x to y also preserves dagger.*

Proof. Suppose that $f : B \times P \to B$ in $\mathscr{C}[x]$, so that

$$f = \langle \mathsf{id}_{B \times P}, x_{B \times P} \rangle \cdot F,$$

for some $F : B \times P \times A \to B$ in $\mathscr{C}$. Then,

$$
\begin{aligned}
(f\psi)^\dagger &= ((\langle \mathsf{id}_{B \times P}, x_{B \times P} \rangle \cdot F)\psi)^\dagger \\
&= (\langle \mathsf{id}_{B\varphi \times P\varphi}, y_{B\varphi \times P\varphi} \rangle \cdot F\varphi)^\dagger \\
&= ((\mathsf{id}_{B\varphi} \times \langle \mathsf{id}_{P\varphi}, y_{P\varphi} \rangle) \cdot F\varphi)^\dagger \\
&= \langle \mathsf{id}_{P\varphi}, y_{P\varphi} \rangle \cdot F^\dagger \varphi,
\end{aligned}
$$

since the parameter identity holds in $\mathscr{C}$ and since dagger is preserved by the ccc morphism φ. Also

$$
\begin{aligned}
f^\dagger \psi &= (\langle \mathsf{id}_{B \times P}, x_{B \times P} \rangle \cdot F)^\dagger \psi \\
&= ((\mathsf{id}_B \times \langle \mathsf{id}_P, x_P \rangle) \cdot F)^\dagger \psi \\
&= (\langle \mathsf{id}_P, x_P \rangle \cdot F^\dagger)\psi \\
&= \langle \mathsf{id}_{P\varphi}, y_{P\varphi} \rangle \cdot F^\dagger \varphi. \quad \square
\end{aligned}
$$

Proposition 8.5. *If $\mathscr{C}$ is a Conway ccc, then so is $\mathscr{C}[x]$. In particular:*
- *if the double dagger identity holds in $\mathscr{C}$, then it holds in $\mathscr{C}[x]$,*
- *if the composition identity holds in $\mathscr{C}$, then it holds in $\mathscr{C}[x]$,*
- *if the abstraction identity holds in $\mathscr{C}$, then it holds in $\mathscr{C}[x]$.*

Proof. Suppose that the double dagger identity holds in $\mathscr{C}$. Then

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{C}}(C\times C\times B\times A, C) \arrow[r, "d_{C\times B\times A, C}"] \arrow[d, "\mathrm{Hom}_{\mathscr{C}}(\Delta\times \mathrm{id}, \mathrm{id})"'] & \mathrm{Hom}_{\mathscr{C}}(C\times B\times A, C) \arrow[d, "d_{B\times A, C}"] \\ \mathrm{Hom}_{\mathscr{C}}(C\times B\times A, C) \arrow[r, "d_{B\times A, C}"'] & \mathrm{Hom}_{\mathscr{C}}(B\times A, C) \end{tikzcd} \tag{27}$$

commutes. We want to show

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{C}[x]}(C\times C\times B, C) \arrow[r, "d_{C\times B, C}"] \arrow[d, "\mathrm{Hom}_{\mathscr{C}[x]}(\Delta\times \mathrm{id}, \mathrm{id})"'] & \mathrm{Hom}_{\mathscr{C}[x]}(C\times B, C) \arrow[d, "d_{B, C}"] \\ \mathrm{Hom}_{\mathscr{C}[x]}(C\times B, C) \arrow[r, "d_{B, C}"'] & \mathrm{Hom}_{\mathscr{C}[x]}(B, C) \end{tikzcd} \tag{28}$$

commutes. We place square (28) inside the square (27), and label the connecting arrows φ as in the previous diagram, and use property (20).

We now assume that the composition identity holds in $\mathscr{C}$, so that if $F: B\times P\times A\to C$ and $G: C\times P\times A\to B$ in $\mathscr{C}$, then

$$(\langle F, \pi^{B\times P\times A}_{P\times A}\rangle\cdot G)^\dagger = \langle(\langle G, \pi^{C\times P\times A}_{P\times A}\rangle\cdot F)^\dagger, \mathrm{id}_{P\times A}\rangle\cdot G.$$

Now, in order to prove the composition identity holds in $\mathscr{C}[x]$, suppose that $f = \langle \mathrm{id}_{B\times P}, x_{B\times P}\rangle\cdot F: B\times P\to C$ and $g = \langle \mathrm{id}_{C\times P}, x_{C\times P}\rangle\cdot G: C\times P\to B$ in $\mathscr{C}[x]$. We show that

$$(\langle f, \pi^{B\times P}_P\rangle\cdot g)^\dagger = \langle(\langle g, \pi^{C\times P}_P\rangle\cdot f)^\dagger, \mathrm{id}_P\rangle\cdot g. \tag{29}$$

But, by definition,

$$\begin{aligned}(\langle f, \pi^{B\times P}_P\rangle\cdot g)^\dagger &= (\langle \mathrm{id}_{B\times P}, x_{B\times P}\rangle\cdot\langle F, \pi^{B\times P\times A}_{P\times A}\rangle\cdot G)^\dagger\\ &= \langle \mathrm{id}_P, \pi_P\rangle\cdot(\langle F, \pi^{B\times P\times A}_{P\times A}\rangle\cdot G)^\dagger,\end{aligned}$$

and

$$(\langle g, \pi^{C\times P}_P\rangle\cdot f)^\dagger = \langle \mathrm{id}_P, \pi_P\rangle\cdot(\langle G, \pi^{C\times P\times A}_{P\times A}\rangle\cdot F)^\dagger.$$

Thus, using the composition identity in $\mathscr{C}$,

$$\begin{aligned}\langle \mathrm{id}_P, x_P\rangle\cdot(\langle F, \pi^{B\times P\times A}_{P\times A}\rangle\cdot G)^\dagger &= \langle \mathrm{id}_P, x_P\rangle\cdot\langle(\langle G, \pi^{C\times P\times A}_{P\times A}\rangle\cdot F)^\dagger, \mathrm{id}_{P\times A}\rangle\cdot G\\ &= \langle\langle \mathrm{id}_P, x_P\rangle\cdot(\langle G, \pi^{C\times P\times A}_{P\times A}\rangle\cdot F)^\dagger, \mathrm{id}_P\rangle\\ &\quad\cdot\langle \mathrm{id}_{C\times P}, x_{B\times P}\rangle\cdot G,\end{aligned}$$

proving (29).

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(R \times P \times S \times A, R) \arrow[rr, "\mathrm{Hom}_{\mathscr{C}}(\alpha \times \mathrm{id},\ \mathrm{id})"] \arrow[d, "d"'] & & \mathrm{Hom}_{\mathscr{C}}(P \times [P \to R] \times S \times A, R) \arrow[d, "\Lambda"] \\
\mathrm{Hom}_{\mathscr{C}}(P \times S \times A,\ R) \arrow[dr, "\Lambda"'] & & \mathrm{Hom}_{\mathscr{C}}([P \to R] \times S \times A,\ [P \to R]) \arrow[dl, "d"] \\
& \mathrm{Hom}_{\mathscr{C}}(S \times A,\ [P \to R]) &
\end{tikzcd}$$

Fig. 6.

Assume now that the abstraction identity holds in $\mathscr{C}$. Then the diagram of Fig. 6 commutes, where α is the base morphism

$$\langle e_{P,R}, \pi_P \rangle : P \times [P \to R] \to R \times P.$$

In order to show the corresponding diagram commutes in $\mathscr{C}[x]$, we use (21), and the fact that the square

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(R \times P \times S \times A, R) \arrow[r, "\mathrm{Hom}_{\mathscr{C}}(\alpha \times \mathrm{id}_{S \times A},\ \mathrm{id})"] \arrow[d, "\varphi"'] & \mathrm{Hom}_{\mathscr{C}}(P \times [P \to R] \times S \times A, R) \arrow[d, "\varphi"] \\
\mathrm{Hom}_{\mathscr{C}[x]}(R \times P \times S,\ R) \arrow[r, "\mathrm{Hom}_{\mathscr{C}[x]}(\alpha \times \mathrm{id}_S,\ \mathrm{id})"'] & \mathrm{Hom}_{\mathscr{C}[x]}(P \times [P \to R] \times S, R)
\end{tikzcd}$$

commutes. We then give the same kind of argument as that for the double dagger identity. $\square$

The following theorem gives a summary of the results proved in this section.

Theorem 8.6. *Suppose that $\mathscr{C}$ is a ccc and that $\mathscr{C}[x]$ is the ccc obtained by freely adding a morphism $x : 1 \to A$ to $\mathscr{C}$, where A is a $\mathscr{C}$-object. Suppose that $\mathscr{C}$ is equipped with an external dagger operation which satisfies the parameter identity. Then there is a unique way to define an external dagger operation on $\mathscr{C}[x]$ which is preserved by the inclusion $\iota : \mathscr{C} \hookrightarrow \mathscr{C}[x]$.*

Suppose that $\mathscr{C}[x]$ is turned into a ccc with an external dagger in this way. Then:
- *The parameter identity holds in $\mathscr{C}[x]$.*
- *Suppose that $\mathscr{D}$ is another ccc with an external dagger operation that satisfies the parameter identity. If $\varphi : \mathscr{C} \to \mathscr{D}$ is any ccc morphism which preserves dagger, and if $y : 1 \to \varphi(A)$ is any $\mathscr{D}$-morphism, then the unique ccc morphism $\mathscr{C}[x] \to \mathscr{D}$ which extends φ and maps x to y also preserves dagger.*

- *If the fixed point, double dagger, composition, or abstraction identity holds in $\mathscr{C}$, then it holds in $\mathscr{C}[x]$ also. Thus, if $\mathscr{C}$ is a Conway ccc, then so is $\mathscr{C}[x]$.*

Conversely, if there is a way of defining dagger on $\mathscr{C}[x]$ such that the inclusion $\mathscr{C} \hookrightarrow \mathscr{C}[x]$ preserves the operation, then dagger satisfies the Eq. (23).

Corollary 8.7. *Suppose that $\mathscr{C}$ is a ccc equipped with an external dagger which satisfies the parameter identity. Suppose that $\mathscr{C}_0$ is a sub-ccc of $\mathscr{C}$ closed under dagger. If $x: 1 \to A$ is a $\mathscr{C}$-morphism between the $\mathscr{C}_0$-objects 1 and A, then the smallest sub-ccc $\mathscr{D}$ of $\mathscr{C}$ containing $\mathscr{C}_0$ and the morphism x is closed under dagger. Moreover, each morphism $B \to C$ of this sub-ccc can be written in the form $\langle \mathrm{id}_B, x_B \rangle \cdot f$, for some $\mathscr{C}_0$-morphism $f: B \times A \to C$.*

Proof. This is immediate from Theorem 8.6. The fact that the smallest sub-ccc containing $\mathscr{C}_0$ and x is closed under dagger can also be established by using the fact that dagger is internalizable in $\mathscr{C}$. Thus, $\mathscr{C}_0$, and hence $\mathscr{D}$ also, contains all the morphisms $\dagger_B: [B \to B] \to B$ for any C_0-object B. □

Remark 8.8. The parameter, composition, double dagger and abstraction identities are not the only ones which may be transferred from $\mathscr{C}$ to $\mathscr{C}[x]$. In particular, we mention the *power identities*. We recall that if $f: B \times P \to B$, then the powers of f are defined inductively as follows:

$$f^0 := \pi_B^{B \times P},$$

$$f^{k+1} := \langle f^k, \pi_P^{B \times P} \rangle \cdot f.$$

The power identities are

$$(f^k)^\dagger = f^\dagger,$$

all $k \geq 1$. If the power identities hold in $\mathscr{C}$, then they hold in $\mathscr{C}[x]$.

Appendix: Internal Conway identities

When an external dagger satisfies the parameter identity, it is determined by a family of internal morphisms. In this appendix, we indicate internal versions of the Conway identities equivalent to their external counterparts. Since the parameter identity is assumed, we will consider only the double dagger, composition and abstraction identities.

Suppose that $d = (d_{B,A})$ is an external dagger in the ccc $\mathscr{C}$ which satisfies the parameter identity. Thus, there are $\mathscr{C}$-morphisms

$$\dagger_A: [A \to A] \to A$$

for each object A such that for all pairs of objects A, B, the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}(A \times B, A) \arrow[r, "\Lambda"] \arrow[d, "d_{B,A}"'] & \mathrm{Hom}(B, [A \to A]) \arrow[dl, "\mathrm{Hom}(\mathrm{id}, \dagger_A)"] \\
\mathrm{Hom}(B, A) &
\end{tikzcd} \tag{A.1}$$

We define the internal morphisms $\dagger_{B,A} : [A \times B \to A] \to [B \to A]$ by

$$\begin{tikzcd}
(A \times B \to A) \arrow[r, "\lambda"] \arrow[d, "\dagger_{B,A}"'] & (B \to [A \to A]] \arrow[dl, "\mathrm{hom}(\mathrm{id}, \dagger_A)"] \\
[B \to A] &
\end{tikzcd} \tag{A.2}$$

where $\lambda : [A \times B \to A] \to [B \to [A \to A]]$ is the base isomorphism (2).

Lemma 9.1. *With the above notation, when the external dagger d satisfies the parameter identity, the following diagram commutes, for all objects R, P, S:*

$$\begin{tikzcd}
\mathrm{Hom}(R \times P \times S, R] \arrow[r, "\Lambda"] \arrow[d, "d_{P \times S, R}"'] & \mathrm{Hom}(S, [R \times P \to R]) \arrow[d, "\mathrm{Hom}(\mathrm{id}_S, \dagger_{P,R})"] \\
\mathrm{Hom}(P \times S, R) \arrow[r, "\Lambda"'] & \mathrm{Hom}(S, [P \to R])
\end{tikzcd} \tag{A.3}$$

Proof. Consider the diagram given in Fig. 7.

$$\begin{tikzcd}
\mathrm{Hom}(R \times P \times S, R) \arrow[r, "\Lambda"] \arrow[d, "\Lambda"'] & \mathrm{Hom}(S, [R \times P \to R]) \arrow[d, "\mathrm{Hom}(\mathrm{id}, \lambda)"] \\
\mathrm{Hom}(P \times S, [R \to R]) \arrow[r, "\Lambda"] \arrow[d, "\mathrm{Hom}(\mathrm{id}, \dagger_R)"'] & \mathrm{Hom}(S, [P \to [R \to R]]) \arrow[d, "\mathrm{Hom}(\mathrm{id}, \mathrm{hom}(\mathrm{id}, \dagger_R))"] \\
\mathrm{Hom}(P \times S, R) \arrow[r, "\Lambda"'] & \mathrm{Hom}(S, [P \to R])
\end{tikzcd}$$

Fig. 7.

The top square commutes by Proposition 2.2. The bottom square commutes by Proposition 2.1. Thus, the outside of the diagram commutes. But the composite of the left-hand sides is $d_{P \times S, R}$, and the composite of the right-hand sides is Hom(id, $\dagger_{P,R}$), by (33). Thus, the Lemma is proved. □

A.1. Double dagger

Suppose that $d = (d_{B,A})$ is an external dagger determined as above by the internal dagger $\dagger = (\dagger_A)$.

The internal dagger satisfies the *weak internal double dagger identity* if the following square commutes, for all objects A:

$$\begin{tikzcd}
{[A \times A \to A]} \arrow[r, "\dagger_{A,A}"] \arrow[d, "{[\Delta_A \cdot -]}"'] & {[A \to A]} \arrow[d, "\dagger_A"] \\
{[A \to A]} \arrow[r, "\dagger_A"'] & A
\end{tikzcd} \tag{A.4}$$

When (A.4) commutes, since Hom is a functor, it follows that

$$\begin{tikzcd}
\mathrm{Hom}(B, [A \times A \to A]) \arrow[r, "{\mathrm{Hom}(\mathrm{id}, \dagger_{A,A})}"] \arrow[d, "{\mathrm{Hom}(\mathrm{id}, [\Delta_A \cdot -])}"'] & \mathrm{Hom}(B, [A \to A]) \arrow[d, "{\mathrm{Hom}(\mathrm{id}, \dagger_A)}"] \\
\mathrm{Hom}(B, [A \to A]) \arrow[r, "{\mathrm{Hom}(\mathrm{id}, \dagger_A)}"'] & \mathrm{Hom}(B, A)
\end{tikzcd} \tag{A.5}$$

commutes. Applying the fact that the diagram

$$\begin{tikzcd}
\mathrm{Hom}(A \times A \times B, A]) \arrow[r, "\Lambda"] \arrow[d, "{\mathrm{Hom}(\Delta_A \times \mathrm{id}, \mathrm{id})}"'] & \mathrm{Hom}(B, [A \times A \to A]) \arrow[d, "{\mathrm{Hom}(\mathrm{id}, [\Delta_A \cdot -])}"] \\
\mathrm{Hom}(A \times B, A) \arrow[r, "\Lambda"'] & \mathrm{Hom}(B, [A \to A])
\end{tikzcd} \tag{A.6}$$

always commutes, as well as the definition (A.1), we obtain from (A.5) the fact that the diagram

$$\begin{tikzcd}
\mathrm{Hom}(A \times A \times B, A) \arrow[r, "d_{A \times B, A}"] \arrow[d, "{\mathrm{Hom}(\Delta_A \times \mathrm{id}, \mathrm{id})}"'] & \mathrm{Hom}(A \times B, A) \arrow[d, "d_{B,A}"] \\
\mathrm{Hom}(A \times B, A) \arrow[r, "d_{B,A}"'] & \mathrm{Hom}(B, A)
\end{tikzcd}$$

commutes. Thus, d satisfies the double dagger identity.

As for the converse, suppose that d satisfies the double dagger identity. By combining (A.6) and Definition 32 with the fact that the square

$$\begin{tikzcd} \mathrm{Hom}(A \times A \times B, A) \arrow[r, "d_{A \times B, A}"] \arrow[d, "A"'] & \mathrm{Hom}(A \times B, A) \arrow[d, "A"] \\ \mathrm{Hom}(B, [A \times A \to A]) \arrow[r, "\mathrm{Hom}(\mathrm{id}, \dagger_{A,A})"'] & \mathrm{Hom}(B, [A \to A]) \end{tikzcd} \tag{A.7}$$

commutes, a little diagram chasing shows that (A.5) also commutes. It then follows that the internal dagger satisfies Eq. (A.6), since for a given object object A we may take $B = [A \times A \to A]$. Thus, we have proved most of the following proposition.

Proposition 9.2. *Suppose that d is an external dagger satisfying the parameter identity. Then the following are equivalent.*
- *d satisfies the double dagger identity.*
- *The corresponding internal dagger $\dagger$ satisfies the weak internal double dagger identity, Eq.* (*A*.4).

When $\mathscr{C}$ has enough points, each of the two previous conditions is equivalent to d satisfies the weak double dagger identity, i.e., the following square commutes:

$$\begin{tikzcd} \mathrm{Hom}(A \times A, A) \arrow[r, "d_{A \times A}"] \arrow[d, "\mathrm{Hom}(\Delta, \mathrm{id})"'] & \mathrm{Hom}(A \times A) \arrow[d, "d_A"] \\ \mathrm{Hom}(A, A) \arrow[r, "d_A"'] & \mathrm{Hom}(1, A) \end{tikzcd} \tag{A.8}$$

where d_A was defined in Remark 3.1.

Proof. We need only prove the last statement. It is clear that the double dagger identity implies the weak double dagger identity. Conversely, it is clear that Eq. (A.8) is equivalent to the commutativity of the square (A.5) with $B = 1$. Thus, if $\mathscr{C}$ has enough points, and if (A.5) commutes with $B = 1$, then (A.4) commutes. □

A.2. Composition

Using no new ideas, it can be shown that if d is an external dagger determined by the internal dagger $\dagger$, then the following are equivalent:
- d satisfies the composition identity.
- $\dagger$ satisfies Fig. 8: When $\mathscr{C}$ has enough points, each of the above is equivalent to
- d satisfies following weak composition identity:

$$d(f \cdot g) = d(g \cdot f) \cdot g, \tag{A.9}$$

for all $f: M \to N$ and $g: N \to M$.

$$\begin{tikzcd}
 & {[M \to N]} \arrow[dl, "\mathrm{hom}(g,\mathrm{id})"'] \arrow[dr, "\mathrm{hom}(\mathrm{id},g)"] & \\
{[N \to N]} \arrow[d, "\dagger_N"'] & & {[M \to M]} \arrow[d, "\dagger_M"] \\
N \arrow[rr, "g"'] & & M
\end{tikzcd}$$

Fig. 8.

Remark 9.3. Note that the condition (A.8) on † says that it is a dinatural transformation from the internal hom-functor to the identity functor on $\mathscr{C}$ [2, 13]. In [16], this property is shown to characterize the least fixed-point operator in certain algebraic poset categories.

A.3. Abstraction

Suppose that d is an external dagger satisfying the parameter identity. Recall that d satisfies the abstraction identity iff diagram of Fig. 4 commutes. The weak abstraction identity is obtained by letting $S = 1$. We will prove the following.

Proposition 9.4. *For an external dagger d determined by the internal dagger † the following are equivalent.*

- *d satisfies the abstraction identity.*
- *† satisfies the following weak internal abstraction identity.*

$$\begin{tikzcd}
{[R \times P \to R]} \arrow[r, "\mathrm{hom}(\langle e_{P,R}, \pi_P \rangle, \mathrm{id})"] \arrow[d, "\dagger_{P,R}"'] & {[P \times [P \to R] \to R]} \arrow[d, "\lambda"] \\
{[P \to R]} \arrow[r, "\dagger_{[P \to R]}"'] & {[[P \to R] \to [P \to R]]}
\end{tikzcd} \qquad (A.10)$$

When $\mathscr{C}$ has enough points, each of the preceding conditions is equivalent to:

- *d satisfies the weak abstraction identity.*

Proof. The idea is to apply Λ to the diagram of Fig. 4 defining the abstraction identity in order to translate this diagram into others involving just $\mathrm{Hom}(S, -)$. Peruse the diagram given in Fig. 9.

The top square commutes by Proposition 2.1. The right-hand square commutes, by Proposition 2.2. The left-hand square is (A.3). The lower triangle commutes since d is

$$
\begin{tikzcd}
\mathrm{Hom}(R\times P\times S,R) \arrow[rrr, "\mathrm{Hom}(\alpha\times \mathrm{id},\mathrm{id})"] \arrow[dr, "\Lambda"] \arrow[ddd, "d"'] & & & \mathrm{Hom}(P\times [P\to R]\times S,R) \arrow[dl, "\Lambda"'] \arrow[ddd, "\Lambda"] \\
& \mathrm{Hom}(S,[R\times P\to R]) \arrow[r, "\mathrm{Hom}(\mathrm{id},\mathrm{hom}(\alpha,\mathrm{id}))"] \arrow[d, "\mathrm{Hom}(\mathrm{id},\dagger)"'] & \mathrm{Hom}(S,[P\times [P\to R]\to R]) \arrow[d, "\mathrm{Hom}(\mathrm{id},\lambda)"] & \\
& \mathrm{Hom}(S,[P\to R]) & \mathrm{Hom}(S,[[P\to R]\to [P\to R]]) \arrow[l, "\mathrm{Hom}(\mathrm{id},\dagger)"'] & \\
\mathrm{Hom}(P\times S,R) \arrow[ur, "\Lambda"] & & & \mathrm{Hom}([P\to R]\times S,[P\to R]) \arrow[ul, "\Lambda"'] \arrow[ull, "d"]
\end{tikzcd}
$$

Fig. 9.

determined by †. Thus, since each of the morphisms Λ is a bijection, the outside of the diagram commutes iff the inside square commutes. But the outside commutes iff d satisfies the abstraction identity. Hece, if d does satisfy this identity, since S is arbitrary, it follows that (A.10) commutes. Conversely, if (A.10) commutes, the inside square commutes, for each S. Thus, so does the outside of the diagram and d satisfies the abstraction identity.

The last statement follows from the above argument, letting $S = 1$ and using the extra hypothesis. □

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