Exponential Space Computation of Gröbner Bases

Klaus Kühnle Ernst W. Mayr
Institut für Informatik
Technische Universität München
D-80290 München, Germany
{kuehnle|mayr}@informatik.tu-muenchen.de
http://wwwmayr.informatik.tu-muenchen.de/

Abstract

Given a polynomial ideal and a term order, there is a unique reduced Gröbner basis and, for each polynomial, a unique normal form, namely the smallest (w.r.t. the term order) polynomial in the same coset. We consider the problem of finding this normal form for any given polynomial without prior computation of the Gröbner basis. This is done by transforming a representation of the normal form into a system of linear equations and solving this system. Using the ability to find normal forms, we show how to obtain the Gröbner basis in exponential space.

1 Introduction

Let us first fix some of the notation used in the remainder of this paper. Let $\mathbb{Q}[x_1,\ldots,x_n]$ be the polynomial ring in the indeterminates $x_1,\ldots,x_n$ over the rationals. Let T be the set of terms or power products in these indeterminates and let $\prec$ be some term order on T, i.e. a total order on the power products extending the divisibility relation and respecting multiplication. We extend this order in the usual way to a partial order on monomials by just ignoring the coefficients and, further, to a partial order on polynomials by the following rule: p_1 is greater than p_2 if the largest monomial occurring in p_1 but not in p_2 is greater than the largest monomial occurring in p_2 but not in p_1. We will consider an ideal I in the polynomial ring $\mathbb{Q}[x_1,\ldots,x_n]$ that is generated by s polynomials $f_1,\ldots,f_s$, i.e. $I=\{\sum_{i=1}^{s} c_i f_i \mid c_i \in \mathbb{Q}[x_1,\ldots,x_n]\}$. With respect to this ideal I and the term order $\prec$ we then have, for any polynomial h, a unique normal form, denoted by $\mathrm{NF}(h)$, with the following defining property: $\mathrm{NF}(h)$ is the smallest, w.r.t. the term order, monic polynomial in the I-coset of h. In other words: $\mathrm{NF}(h)$ is the outcome of a complete reduction of h w.r.t. the Gröbner basis for I and $\prec$. (Of course, $\mathrm{NF}(h)$ depends on the ideal I and the term order $\prec$, which we disregard in our notation, since there will be no ambiguity.)

This paper presents an algorithm for finding the normal form of a given polynomial h and for computing the unique reduced Gröbner basis for a given ideal, both using exponential space. It is structured as follows: First, we fix a way to represent the term order, allowing us to talk about its representation size and to bound the length of a reduction w.r.t. a Gröbner basis in terms of the size of the representation of the term order; this will yield a bound on the degree of the normal form of h. As a consequence, it is possible to bound the degrees of the coefficients of a representation of $h-\mathrm{NF}(h)$ as a linear combination of the given generators $f_1,\ldots,f_s$ and to transform this representation into a system of linear equations over the scalar field $\mathbb{Q}$, whose solution is just the vector of coefficients of the normal form of h. We will solve this system of linear equations efficiently and thus obtain the normal form of h. Using the calculation of normal forms as a subroutine, we will compute the Gröbner basis of the given ideal by enumerating all terms up to the known bound on the degree of the Gröbner basis and calculating their normal forms; we will add the difference of a term and its normal form to the Gröbner basis if this term is not irreducible (i.e. is not equal to its normal form) but all its divisors are. Finally, we will express the computational effort needed for all these computations in terms of the input size, ending up with the main results of this paper: Normal form calculations as well as computations of Gröbner bases can be done in exponential space.

2 A bound on the degree of the normal form

A term order $\prec$ on the set T of all terms of $\mathbb{Q}[x_1,\ldots,x_n]$ is defined to be a total order with the additional properties $\forall t\in T\colon 1\prec t$ and $\forall t,u,v\in T\colon u\prec v\Rightarrow tu\prec tv$. Robbiano showed in [30] that any such term order can be represented by at most n weight functions $W_1,\ldots,W_n$, mapping from the set of terms into the set of real numbers as follows:

$$W_k({x_1}^{u_1}{x_2}^{u_2}\ldots{x_n}^{u_n}) := \sum_{i=1}^{n} w_{k,i}u_i$$

where $w_{k,i}$ are real numbers, specifying the term order. A term u is greater than another term v in this term order if

$$\exists k\colon W_k(u)>W_k(v)\wedge\forall j>k\colon W_j(u)=W_j(v)$$

A constructive proof of this was given by Dubé, Mishra and Yap in [9]; they also show that the weights $w_{k,i}$ can be assumed to be nonnegative.

Example: In the polynomial ring $\mathbb{Q}[x,y,z]$, the term order $\prec$ given by the 9 rational weights

$$\begin{pmatrix} w_{1,x} & w_{1,y} & w_{1,z} \\ w_{2,x} & w_{2,y} & w_{2,z} \\ w_{3,x} & w_{3,y} & w_{3,z} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 3 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

implies for example $xyz \prec x^2y^2 \prec x^3y \prec z^3 \prec x^2yz$.

Because it is not easy to represent numbers in a way such that all reals are finitely describable, we will restrict ourselves to term orders that are representable by weight functions with rational weights. This is a proper restriction, i.e. there are term orders that cannot be represented with rational weights; however, every term order can be arbitrarily tightly approximated with rational weights in the following sense: Given a term order and a natural number, there is a term order representable with rational weights that agrees with the given term order on all terms whose degrees are bounded by the given number. (We will come back to such approximations at the end of section 6.)

Let the weights $w_{k,i}$ be given as integer numbers (this is no restriction compared with rational weights) and let A be the maximum of all the $w_{k,i}$. We are going to recall parts of [9]: We combine the characterizing weight functions W_k to a single weight function W by simulating their lexicographic interrelation by means of a combination of them reminiscent of the B-adic representation of numbers:

$$W(u) = \sum_{k=1}^{n} B^{k-1} W_k(u)$$

For any given finite set of terms, it is possible to choose B large enough, so that W will properly represent the term order on these terms, i.e. $W(u) < W(v)$ if and only if $u \prec v$. We will now find an appropriate value for B.

Let d be a bound on the degrees of the generating polynomials $f_1, \ldots, f_s$ of the ideal I. DUBÉ showed in [8] the existence of a Gröbner basis G for I where the degrees of all polynomials in G are bounded by $2(\frac{d^2}{2}+d)^{2^{n-1}}$ (remember that n was the number of indeterminates in the polynomial ring). Hence, the maximal weight (i.e. the result of applying some weight function W_k) of any term occurring in a polynomial in G is bounded by $2A(\frac{d^2}{2}+d)^{2^{n-1}}$. The difference between the weights of two terms is either zero or at least 1. Setting $B := 2A(\frac{d^2}{2}+d)^{2^{n-1}}+1$ in the above definition of the unified weight function therefore guarantees that W will properly represent the term order on G. It should be mentioned that the term order is not in general properly represented by this unified weight function on terms whose degrees exceed $2(\frac{d^2}{2}+d)^{2^{n-1}}$.

Example: Let us refer to the previous example, where $A = 3$. For a proper representation of the term order on terms whose degrees do not exceed 4 we use $B = 4A+1 = 13$. We then get $W(xyz) = 395$, $W(x^2y^2) = 422$, $W(x^3y) = 551$, $W(z^3) = 552$ and $W(x^2yz) = 565$. In contrast, compare the term y^5 whose degree exceeds the bound assumed for the determination of B with the term x. We have $y^5 \prec x$ but $W(y^5) = 205 \geq 170 = W(x)$.

Though we do not know any Gröbner basis as yet, we consider the process of reducing h w.r.t. G according to BUCHBERGER's algorithm ([3]). In such a reduction, a step consists of deleting some monomial in the current polynomial by subtracting a suitable multiple of a suitable polynomial of the Gröbner basis in such a way that the largest monomial of the multiple of the Gröbner basis element and the monomial chosen for deletion cancel out. Here, we always choose the largest possible monomial for deletion; this strategy implies that the monomials that are deleted during the reduction process strictly decrease w.r.t. the term order. We want to bound the degrees and therefore the number of these monomials in order to get a bound on the length (i.e. the number of steps) of the reduction.

The fact that the unified weight function W represents the term order on G immediately implies that in any reduction step, the weights (i.e. the results of applying the unified weight function) of the monomials that are inserted into the polynomial are strictly smaller than the weight of the monomial that is deleted. (The weight of a monomial is the weight of the corresponding term.) Hence, the weight of a polynomial, defined to be the maximal weight of its monomials, will not increase in any step of the reduction w.r.t. G. The weight of h is therefore a bound on the weights of all monomials that are deleted during the reduction of h.

Let u be a monomial of h with maximal weight. Then

$$\begin{aligned} W(u) &= \sum_{k=1}^{n} B^{k-1} W_k(u) \\ &\leq \sum_{k=0}^{n-1} B^k A \deg(u) \\ &\leq B^n A \deg(h) \ . \end{aligned}$$

The weight of any monomial is at least its degree, because 1 is the smallest possible nonzero value for the weights $w_{k,i}$ and, for any i, we must have some nonzero $w_{k,i}$. Thus, the above bound on the weights is also a bound on the degrees of the monomials that are deleted during the reduction. These monomials are all distinct; consequently, the number of them and with it the length of the reduction of h is bounded by $(B^n A \deg(h))^n$.

In every reduction step, the degree of the polynomial to be reduced increases by at most the maximal degree of the polynomials in the Gröbner basis. The product of this maximal degree and the number of reduction steps is clearly a bound on the increase of the degree of h during its reduction. Hence, the degree of the normal form of h is bounded by

$$\begin{aligned} & \deg(h) + 2(\frac{d^2}{2}+d)^{2^{n-1}} (B^n A \deg(h))^n \\ \leq\ & 2(\frac{d^2}{2}+d)^{2^{n-1}} B^{n^2} A^n \deg(h)^{n+1} \\ =\ & 2(\frac{d^2}{2}+d)^{2^{n-1}} (2A(\frac{d^2}{2}+d)^{2^{n-1}}+1)^{n^2} A^n \deg(h)^{n+1} \\ \leq\ & ((2A(\frac{d^2}{2}+d)^{2^{n-1}}+1)^n \deg(h))^{n+1} \\ =:\ & N \end{aligned}$$

and we have the result of this section:

Proposition: In $\mathbb{Q}[x_1, \ldots, x_n]$, let a polynomial ideal I be given by generators whose degrees are bounded by d and let a term order be given by n weight functions, as described at the beginning of this section, where the integer weights are bounded by A. Then the degree of the unique normal form of a given polynomial h w.r.t. the given ideal and term order is bounded by $((2A(\frac{d^2}{2}+d)^{2^{n-1}}+1)^n \deg(h))^{n+1}$.

3 Reducing the membership problem to a matrix equation

We will now exploit the upper bound on the degrees of normal forms for actually finding them. The difference of h and its normal form is certainly in the ideal and thus representable as a linear combination of the generators. (Note that we take the original ideal basis rather than the Gröbner basis G, whose existence we only used for bounding the degree of the normal form.) HERMANN showed in [13] that there is a representation

$$h - \mathrm{NF}(h) = \sum_{i=1}^{s} f_i c_i \ ,$$

where the degrees of the coefficients c_i are bounded by $D := \deg(h-\mathrm{NF}(h)) + (sd)^{2^n} \leq N + (sd)^{2^n}$ (as before, d is the bound on the degrees of the f_i and N is the bound on the degree of $\mathrm{NF}(h)$ developed in the last section).

Expanding all polynomials to sums of monomials, viz. the ideal generators to $f_i = \sum\{f_{i,t}t \mid t \in T \wedge \deg(t) \leq d\}$, the unknown factors to $c_i = \sum\{c_{i,t}t \mid t \in T \wedge \deg(t) \leq D\}$ and, finally, the normal form of h to an unknown polynomial $r = \sum\{r_t t \mid t \in T \wedge \deg(t) \leq N\}$, we get

$$\begin{aligned}
h &= r + \sum_{i=1}^{s} f_i c_i \\
&= \sum_{\substack{t\in T \\ \deg(t)\leq N}} r_t t + \sum_{i=1}^{s} \left(\sum_{\substack{t\in T \\ \deg(t)\leq d}} f_{i,t} t \right) \left(\sum_{\substack{t\in T \\ \deg(t)\leq D}} c_{i,t} t \right) \\
&= \sum_{\substack{t\in T \\ \deg(t)\leq N}} r_t t + \sum_{i=1}^{s} \sum_{\substack{t\in T \\ \deg(t)\leq d+D}} \left(\sum_{\substack{u,v\in T \\ uv=t}} f_{i,u} c_{i,v} \right) t \\
&= \sum_{\substack{t\in T \\ \deg(t)\leq N}} r_t t + \sum_{\substack{t\in T \\ \deg(t)\leq d+D}} \left(\sum_{i=1}^{s} \sum_{\substack{u,v\in T \\ uv=t}} f_{i,u} c_{i,v} \right) t
\end{aligned}$$

If we also expand h to $h = \sum\{h_t t \mid t \in T \wedge \deg(t) \leq \deg(h)\}$ and compare coefficients in our polynomial equation, we get, for every term t involved, an equation in $\mathbb{Q}$ of the form

$$h_t = r_t + \sum_{i=1}^{s} \sum_{\substack{u,v\in T \\ uv=t}} f_{i,u} c_{i,v} \ ,$$

where we assume all coefficients whose indices are too high to be zero.

Example: In the polynomial ring $\mathbb{Q}[x,y]$ with term order given by

$$\begin{pmatrix} w_{1,x} & w_{1,y} \\ w_{2,x} & w_{2,y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(lexicographic), let the ideal I be generated by $f = x^2 + 3xy$ and $g = xy - 1$. We will calculate the normal form of $h = x$ by solving the equation $h = r + bf + cg$ for r. The proposition of the last section yields a bound of over 10^9 on the degree of $\mathrm{NF}(h)$ and the considerations of this section imply a similar bound for the coefficient polynomials b and c. We will instead assume a bound of 1 for all these degrees in order to make the example fit on the page. With this bound, we can write out our equation as follows:

$$\begin{aligned}
x = \quad & r_1 + r_y y + r_x x \\
+ \ & (b_1 + b_y y + b_x x)(x^2 + 3xy) \\
+ \ & (c_1 + c_y y + c_x x)(xy - 1)
\end{aligned}$$

Multiplying out and comparing coefficients yields, for every term (indicated in parentheses), an equation involving the coefficients of that term. Thus we get the following system of equations (those for the terms y^2 and y^3 are omitted, since they are void):

$$\begin{array}{lllllllll}
(1) & 0 = & r_1 & & & & -c_1 & & \\
(y) & 0 = & & r_y & & & & -c_y & \\
(x) & 1 = & & & r_x & & & & -c_x \\
(xy) & 0 = & & & 3b_1 & & +c_1 & & \\
(xy^2) & 0 = & & & & 3b_y & & +c_y & \\
(x^2) & 0 = & & & b_1 & & & & \\
(x^2 y) & 0 = & & & & b_y & +3b_x & & +c_x \\
(x^3) & 0 = & & & & & b_x & &
\end{array}$$

After obvious simplifications, we are left with the system

$$\begin{array}{llllll}
0 = & r_y & & -c_y & & \\
1 = & & r_x & & & -c_x \\
0 = & & 3b_y & +c_y & & \\
0 = & & b_y & & +c_x &
\end{array}$$

whose solution is not unique. Because we are interested in the solution where $r_1 + r_y y + r_x x$ is minimal with respect to the term order, we just set $r_x = 0$ and get $c_x = -1$, $b_y = 1$, $c_y = -3$ and finally $r_y = -3$. Hence, the normal form of x w.r.t. the given ideal and term order is $-3y$.

The bound on the degrees of the terms involved in the polynomial equation was $\max(N, d+D)$; therefore, we get at most $(\max(N, d+D))^n$ equations in $\mathbb{Q}$. We can write all these equations as a single matrix equation by forming a vector H of the coefficients of h, a vector C of unknowns consisting of the r_t and the $c_{i,t}$ and a matrix $\mathcal{F}$ whose entries are the coefficients $f_{i,t}$ of the generating polynomials f_i (the same coefficient may occur quite often in this matrix), some 1's and a lot of 0's. The matrix equation is then just

$$H = \mathcal{F} C \ .$$

This matrix equation is a direct translation of the polynomial equation $h = r + \sum_{i=1}^{s} f_i c_i$; thus, the solutions are just the vectors of coefficients r_t of polynomials r in the coset of h whose degrees do not exceed N and the corresponding coefficients $c_{i,t}$ of representations of $h - r$ as linear combinations of the f_i. The solution we are aiming for is one with minimal r w.r.t. the term order.

We already mentioned that the number of equations or the number of rows of $\mathcal{F}$ is bounded by $(\max(N, d+D))^n$. Now we will bound the number of columns of $\mathcal{F}$ or the number of unknowns in the vector C. We get at most N^n unknowns r_t for the normal form of h and at most D^n unknowns $c_{i,t}$ from every c_i; hence, there are altogether not more than $N^n + sD^n$ unknowns. The format of the matrix, i.e. the maximum of its height and width, is therefore bounded by $N^n + s(d+D)^n =: M$.

For the calculation of $\mathcal{F}$, we have to determine the places in $\mathcal{F}$ where the coefficients of the generating polynomials f_i go. The entry in the matrix $\mathcal{F}$ in the row corresponding

to the term t and the column corresponding to the unknown r_v is one if $v = t$ and zero otherwise. The entry in the matrix $\mathcal{F}$ in the row corresponding to the term t and the column corresponding to the unknown $c_{i,v}$ is the coefficient $f_{i,\frac{t}{v}}$ if t is divisible by v and zero otherwise. So, the required coefficient is determined by computing the place where to look it up in the table containing the coefficients of the f_i. The space required for that is the space needed for the division of a term by another one. Such a division is merely a subtraction of the corresponding exponent vectors. Hence, the space requirement is essentially that for writing down two terms. The degrees of the terms involved are bounded by $\max(N, d+D)$; thus, the space needed is at most $2n(\log(\max(N, d+D))+1)$.

The space for writing down the entire matrix is far too large; therefore, we do not determine the matrix in advance but compute each entry when it is needed during the further treatment of $\mathcal{F}$.

4 Finding the normal form

Now we are going to find an algorithm for solving the matrix equation $H = \mathcal{F}C$ such that the polynomial $r = \sum_t r_t t$ built from the entries of the solution vector is minimal w.r.t. the term order. It is clear that this r is the normal form of h we are looking for.

As an intermediate step, we want to have a maximal regular minor $\mathcal{F}'$ of $\mathcal{F}$ such that the solution of the corresponding matrix equation $H' = \mathcal{F}'C'$ represents the normal form of h. To this end, we have to remove rows and columns of $\mathcal{F}$ that are linearly dependent on the others. The deletion of a column is the same as setting the corresponding unknown to zero while the deletion of a row just removes redundancy. Note that we do not have enough space to write down the matrix $\mathcal{F}$ and therefore can not actually remove anything. What we will develop is a method for determining with reasonable space requirements whether a given row or column is in $\mathcal{F}'$. This method will consist of two rank computations as follows: If we fix an arbitrary order on the rows (resp., columns) of $\mathcal{F}$, then the k-th row (resp., column) is in $\mathcal{F}'$ if the ranks of the minor of $\mathcal{F}$ consisting of the first $k-1$ rows (resp., columns) and the one consisting of the first k rows (resp., columns) differ, whereas that row (resp., column) will be dispensable if those two ranks are equal. It is clear that the maximal regular minor resulting from this method depends on the order of the rows and columns chosen. It is also clear that the solution of the matrix equation with this maximal regular minor will represent the normal form of h if and only if the order of the columns is chosen such that the columns corresponding to the r_t come last (i.e. after the ones corresponding to the $c_{i,t}$) and in ascending term order of their indices (i.e. the column corresponding to r_t comes before the column corresponding to r_u if $t \prec u$). Incidentally, the order of the rows does not influence the solution.

The tools needed for determining this maximal regular minor of $\mathcal{F}$ are an efficient way of enumerating, in the given term order, all terms up to the given degree bound and an efficient method for calculating the rank of a matrix.

For the comparison of two terms $u = {x_1}^{u_1}{x_2}^{u_2}\ldots{x_n}^{u_n}$ and v, we just compare $\sum_{i=1}^{n} w_{k,i}u_i$ and the same for v. The space for writing down these two numbers does not exceed $2(\log(nA\max(N, d+D))+1)$. Note that the same space can be reused for all n weight functions. For the enumeration of the terms, we need, in essence, space for writing down three terms, namely the last one output and two others for an exhaustive search of the next one to output, and for the comparison of two terms; this space does not exceed $3n(\log(\max(N, d+D))+1)+2(\log(nA\max(N, d+D))+1)$. The exhaustive search is possible since we can enumerate all terms up to the given degree bound in lexicographic order.

As to the rank calculation, we will adopt a method of IBARRA, MORAN and ROSIER, described in [15], which runs in parallel as follows: First, we multiply every row of the matrix (and the right hand side) by the lcm (or even simpler, by the product) of the denominators of this row's entries to get an integer matrix and then multiply this integer matrix by its transpose. From the resulting matrix we calculate the characteristic polynomial. Finally, the rank of the original matrix is the difference of the degree of the characteristic polynomial and the highest power of the indeterminate that divides the characteristic polynomial.

Let K be a bound on the numerators and denominators of the coefficients of the polynomial h to be reduced and the generating polynomials $f_1, \ldots, f_s$. Then, K is a bound on the numerators and denominators of the entries of H and $\mathcal{F}$ and, consequently, the lcm (or the product) of the denominators is at most K^{M+1} and the entries of the integer matrix are bounded by K^{M+2} (remember that M was a bound on the format of the matrix $\mathcal{F}$). The calculation of these integer values as well as the multiplication of the integer matrix by its transpose can be done in $O(\log M \log\log(M(K^{M+2})^2))$ parallel time, since $\log(M(K^{M+2})^2)$ is an upper bound on the number of bits required for writing down any number occurring during that calculation. As has been shown by GALIL and PAN in [11], the characteristic polynomial of an integer matrix can be computed in $O(\log^2(Mp))$ parallel time where p is an upper bound on the number of bits required for writing down any number occurring during the calculation. In our case, $p = \log((M(K^{M+2})^2)^M M!) \leq 2M(M+2)\log(2K)$ will suffice. The determination of the smallest index of the nonvanishing coefficients of the characteristic polynomial can be done in $O(\log M)$ parallel time. Summing up, we use $O(\log^2(M \log K))$ time for the complete rank calculation, i.e. scaling of the rows for getting an integer matrix, multiplication of the integer matrix by its transpose, computation of the characteristic polynomial and finding the smallest index of the nonvanishing coefficients.

By the parallel computation thesis, shown by FORTUNE and WYLLIE in [10], we can do all this sequentially using no more space than the square of the time required by a parallel algorithm. Thus, for the rank calculation we use space amounting to $O(\log^4(M \log K))$ in addition to the space for the subroutines used, namely the enumerations of the rows and columns and the calculations of the places where to look for the entries of the original matrix. As shown above, the space needed for these subroutines does not exceed $O(n\log(A(N+(sd)^{2^n})))$; hence, the complete decision whether a given row or column of $\mathcal{F}$ is in our maximal regular minor requires no more space than $O(\log^4(M \log K)+n\log(A(N+(sd)^{2^n})))$.

Next, we have to compute the unique solution of the matrix equation $H' = \mathcal{F}'C'$ where $\mathcal{F}'$ is the maximal regular minor of $\mathcal{F}$, determined above, and H' and C' are the corresponding shortened versions of H and C.

In [11], GALIL and PAN also showed that the inverse of a regular $M \times M$-matrix can be computed in $O(\log^2(Mp))$ parallel time where p is an upper bound on the number of bits required for writing down any number occurring during the calculation. From the inverse of the matrix we get

the solution of the matrix equation within essentially the same time. Again by the parallel computation thesis of [10], we can solve this matrix equation sequentially using space $O(\log^4(M \log K))$. Again, we need additional space for the enumerations of the rows and columns and the calculations of the places where to look for the entries of the matrix.

Once more, note that we do never write down the entire matrix for all that has been described, because this would take too much space. Instead, we calculate each entry when it is needed and we determine, for each row or column, whether it is in our maximal regular minor at the time when we consider the entries of that row or column.

The space used for solving the matrix equation $H = \mathcal{F}C$ is

$$\begin{aligned} & O(\log^4(M \log K) + n \log(A(N+(sd)^{2^n}))) \\ = \; & O(\log^4((N^n + s(d+D)^n) \log K) + n \log(A(N+(sd)^{2^n}))) \\ \subseteq \; & O(\log^4(((N+(sd)^{2^n})^n) \log K) + n \log(A(N+(sd)^{2^n}))) \\ \subseteq \; & O(n^4 \log^4(A(N+(sd)^{2^n}) \log K)) \end{aligned}$$

It is already clear how the solution of the matrix equation represents the normal form of the given polynomial. Thus, we can calculate this normal form within the space bound just stated and have the result of this section:

Proposition: In $\mathbb{Q}[x_1, \ldots, x_n]$, let the term order $\prec$ be given by n^2 integer weights bounded by A, as described in section 2. Let the polynomial ideal I be given by s generators with degrees bounded by d and numerators and denominators of the coefficients bounded by K. Then, the normal form of a given polynomial h w.r.t. $\prec$ and I can be computed using $O(n^4 \log^4(A(N+(sd)^{2^n}) \log K))$ space where N is the bound on the degree of the normal form of h stated in the proposition at the end of section 2.

5 Computing the Gröbner basis

We will now use the calculation of normal forms as a subroutine for the computation of Gröbner bases. To this end, let us introduce some more terminology: We will call a term u a direct divisor of another term $t \neq u$ if u divides t but there is no term $v \notin \{u, t\}$ such that u divides v and v divides t; in other words, the direct divisors of a term are just those terms where the exponent vector is smaller by 1 in exactly one coordinate and equal in all others. It is obvious that any term has at most n direct divisors (remember that n was the number of indeterminates). If a monic monomial m is reducible (i.e. is different from its normal form) but all its direct divisors are irreducible, then we will call m *minimally reducible*. Clearly, if all direct divisors of a monomial are irreducible, then so are all its (not necessarily direct) proper divisors.

It is not hard to see that the unique reduced Gröbner basis of a given ideal is just the set G of all the polynomials $m - \mathrm{NF}(m)$ where m is a minimally reducible monic monomial. Indeed, on the one hand, every polynomial that is not minimal in its coset (i.e. is different from its normal form) will be reducible w.r.t. G; on the other hand, any such minimally reducible monic monomial m could not be reduced w.r.t. $G \setminus \{m - \mathrm{NF}(m)\}$.

Example: Let us refer to the example of section 3, where we had in the polynomial ring $\mathbb{Q}[x, y]$ with lexicographic term order setting $y \prec x$ an ideal generated by the polynomials $f = x^2 + 3xy$ and $g = xy - 1$. We already calculated $\mathrm{NF}(x) = -3y$; in the same way, the normal forms of other monic monomials can be determined. It turns out that 1 and y are irreducible, i.e. equal to their normal forms, and that $\mathrm{NF}(y^2) = -\frac{1}{3}$. All other monic monomials cannot be minimally reducible, since they are multiples of x or y^2. Hence, the Gröbner basis of the given ideal consists of the two polynomials $x + 3y$ and $y^2 + \frac{1}{3}$.

For generating G, we enumerate all monic monomials (= terms) up to the degree bound on Gröbner bases, shown by DUBÉ in [8]. For every such monomial m, we calculate its normal form and also the normal forms of all its direct divisors. If m turns out to be minimally reducible, then we output $m - \mathrm{NF}(m)$ as an element of the Gröbner basis. It is clear that the overall output we will produce by this method is the unique reduced Gröbner basis.

The space needed for the enumeration of all monic monomials is essentially that for writing down a term. For the direct divisors we need space for another term; and for the normal form calculations, the space bound from the preceding section applies. This gives a space requirement of $O(n^4 \log^4(A(N+(sd)^{2^n}) \log K))$ altogether. However, note that one of the parameters in this space bound is the degree of the input polynomial h (hidden in the parameter N) but here we do not have such a polynomial as part of the input. Therefore, we have to replace this parameter in the space bound by the bound on the degrees of the polynomials whose normal form we calculate. This new parameter is just the degree bound of DUBÉ ([8]) on the Gröbner basis. We will take this into account in the next section; let us now briefly summarize the calculation of the Gröbner basis.

The outermost loop of our algorithm is an enumeration of all monic monomials up to DUBÉ's degree bound. In every pass through the loop, we treat the current monic monomial m in the following way: We call $n+1$ times the subroutine for the normal form calculation applied to the n direct divisors of m and to m itself. As the result of these $n+1$ calls of the normal form calculating subroutine, we get the information whether m is minimally reducible in addition to the normal form of m. In case m is minimally reducible, we output $m - \mathrm{NF}(m)$ as an element of the Gröbner basis and proceed by taking the next monic monomial in the outermost loop.

Let us also summarize the subroutine for the calculation of normal forms. Let h denote the monic monomial that is the input of the subroutine. We consider (but do not write down) the matrix equation $H' = \mathcal{F}'C'$ where $\mathcal{F}'$ is our maximal regular minor of $\mathcal{F}$ and $H = \mathcal{F}C$ is the matrix equation representing the polynomial equation $h = \mathrm{NF}(h) + \sum_{i=1}^{s} f_i c_i$ in the way described in section 3. We calculate, one by one, those entries of the solution vector C' which are the coefficients of the normal form of h (in case h is a direct divisor, we actually need only the coefficient r_h in order to know whether h is in normal form) by performing the necessary parts of the multiplication $\mathcal{F}'^{-1} \cdot H'$. We never write down the matrix $\mathcal{F}'^{-1}$ but compute each entry when it is needed. The computation of the entries of $\mathcal{F}'^{-1}$ can be done within the required space bound by virtue of [11] and [10]. Note that the entries of $\mathcal{F}'$ used in this computation are also determined from scratch each time they are used. As has been shown in the previous sections, the decision whether a row or column is in the maximal regular minor as well as the determination of any entry of $\mathcal{F}$ can be done within the required space bounds.

The effort necessary for the normal form subroutine does not increase substantially if we also calculate the entries $c_{i,t}$ of the solution vector C' and thus obtain the coefficients c_i

of the representation of $h-\mathrm{NF}(h)$ as a linear combination of the ideal generators $f_1, \ldots, f_s$. This means that we can, for each element of the Gröbner basis, compute, in addition, the coefficients of a representation as a linear combination of the original basis within the same space bound.

6 Complexity considerations

In this section, we will summarize the statements about the space requirements of the methods described so far.

Let us first consider the problem of computing the normal form of a polynomial. We are given an ideal I, a term order $\prec$ and a polynomial h to be reduced. Let *size* be the number of bits needed to write down this input. Here, we assume that I is given by a collection $f_1, \ldots, f_s$ of polynomials whose degrees are bounded by d and whose coefficients' numerators and denominators are bounded by K. The term order is given by a collection of n^2 integer weights (n is the number of indeterminates) which are bounded by A.

It is clear that d, K and A are bounded by 2^{size} and that n and s are bounded by $size$. The degree of h is also bounded by 2^{size} but in view of the problem of calculating Gröbner bases we will only use a bound of $2^{2^{O(size)}}$ on this degree.

The bound N on the degree of the normal form of h is

$$\begin{aligned} N &= ((2A(\frac{d^2}{2}+d)^{2^{n-1}}+1)^n \deg(h))^{n+1} \\ &\in (2^{2^{O(size)}} \deg(h))^{O(size)} \\ &\subseteq 2^{2^{O(size)}} \end{aligned}$$

Note that, in this estimate, we used $2^{2^{O(size)}}$ as a bound on the degree of h.

Since $O(n^4 \log^4(A(N+(sd)^{2^n}) \log K)) \subseteq 2^{O(size)}$ the considerations of section 4 imply the first main result of this paper.

Theorem: The calculation of the normal form of a given polynomial w.r.t. a given ideal and term order can be done in exponential space.

Let us now turn to the calculation of Gröbner bases. Here, we are given an ideal and a term order in the same way as described above (only the polynomial h is missing this time). We enumerate all monic monomials up to DUBÉ's degree bound ([8]), which is $2^{2^{O(size)}}$, and use these monic monomials and their direct divisors as the polynomial h to be reduced in the normal form calculation. Note that, in the consideration of the normal form calculation, we took $2^{2^{O(size)}}$ as bound on the degree of h; therefore, we can adopt the results from there. After each such normal form calculation, the space for that calculation can be freed completely, because, in the case of a direct divisor, the result only influences whether we proceed testing or interrupt the work on the current monomial and, in the case of the monomial itself, we will immediately output the difference of the monomial and its normal form, if nonzero, as an element of the Gröbner basis. Thus, we need space for enumerating all monic monomials, negligible space for the control of the order in which the direct divisors are tested and space for the normal form calculation. The second main result follows:

Theorem: The unique reduced Gröbner basis of a given ideal w.r.t a given term order can be computed in exponential space.

It is not hard to see that the two main results remain valid if we assume a fixed term order that is not part of the input, i.e. if the input consists only of the ideal basis and possibly the polynomial to be reduced.

Another variant to be considered is the case of a fixed term order and a fixed ideal, both not being part of the input. It is clear that it does not make sense to look at the complexity of the Gröbner basis calculation in this case. But if we take a polynomial h as input, we can consider the complexity of its normal form calculation. Here, n, s, d and A are constant, because we consider the ideal as well as the term order as fixed. Only $\deg(h)$ and K, as a bound on the numerators and denominators of the coefficients of h, are bounded by $2^{O(size)}$. Note that we do not have to use a more generous bound on $\deg(h)$ here since there is no Gröbner basis calculation we want to provide for. It turns out that, in this case, the space needed for the calculation of the normal form of h is bounded by $O(size^4)$; in other words, the normal form calculation can be done in polynomial space.

Finally, let us look at the problem of computing a *universal Gröbner basis*, i.e. a basis that is a Gröbner basis w.r.t. every term order. The universal Gröbner basis of an ideal is the union of the Gröbner bases for all the possible term orders (note that there is a continuum of them). For the sake of feasibility, we need to bound the number of term orders to be considered. DUBÉ's bound on the degrees of the elements in a Gröbner basis is independent of the term order and hence applies also to the universal Gröbner basis. Relevant for the Gröbner basis is only the behaviour of the term order on polynomials within this bound. Results from linear programming (cf. e.g. [27],[32]) imply that we need only consider term orders with integer weights which are doubly exponentially bounded. Thus, we can enumerate and temporarily write down all relevant term orders within our exponential space bound. The following algorithm, outputting the universal Gröbner basis, will therefore run in exponential space: In the outermost loop, we enumerate all monic monomials within DUBÉ's degree bound ([8]). In the next to outermost loop, we enumerate all relevant term orders. Within that, we calculate the normal form as usual, check whether the same normal form has already been calculated with another term order (several recomputations are necessary for that) and, in case it has not, output the difference of the term and its normal form, as usual. Note that no element of the Gröbner basis is repeatedly output by this algorithm. We can thus state:

Theorem: The universal Gröbner basis of a given ideal can be computed in exponential space.

7 Bibliographic notes

The representation of term orders by a collection of weight functions is due to ROBBIANO. His presentation [30], where he showed that every term order can be represented by at most n weight functions (n is the number of indeterminates), is essentially an excerpt from his more comprehensive treatment [31] on generalized standard bases. DUBÉ, MISHRA and YAP took up this idea, gave a constructive proof of the representability of term orders by weight functions and used this representation for bounding the length of a reduction of a polynomial with respect to some basis in [9]. By a reduction we always mean a process like the one appearing in BUCHBERGER's algorithm which was originally described in [3].

The doubly exponential bound on the degrees of the elements in Gröbner bases has been proved by DUBÉ in [8]. HERMANN proved in her dissertation [13] the doubly exponential bound on the degrees of the coefficients in the representation of a polynomial as a linear combination of the ideal generators. MAYR and MEYER showed in [22] that this bound is asymptotically tight. The first application of this complexity bound to the polynomial ideal membership problem was given in [20]. HUYNH pointed out in [14] that the semigroup construction of [22] yields a doubly exponential lower bound on the degrees of (the elements of) Gröbner bases. This, again, implies that any algorithm computing Gröbner bases requires at least exponential space in the worst case.

Efficient parallel matrix calculations as used here for the solution of our system of linear equations have been widely studied since the appearance of CSANKY's paper [7]. Many researchers contributed improvements, cf. [28], [15], [2], [1], [25] and [26]. The work of GALIL and PAN ([11]) gives a better account for the Boolean complexity, which is important in our case of very large numbers. The easy transition between parallel time complexity and sequential space requirements is justified by the parallel computation thesis of FORTUNE and WYLLIE in [10]. The books of PAPADIMITRIOU and STEIGLITZ ([27]) and SCHRIJVER ([32]) are a rich source for topics concerned with linear programming and the like.

Let us finally make some remarks on related work. The cardinalities and degrees of Gröbner bases as well as the effort needed for their computation has been dealt with in several more or less restrictive special cases. We will only mention a few of the many papers on these topics. BUCHBERGER ([4], [5]) and LAZARD ([18]) gave improved bounds on the degrees of the elements of a reduced Gröbner basis for the cases of polynomial rings with at most two indeterminates. MÖLLER and MORA required in [23] a graded (= degree compatible) term order as well as the knowledge of an H-basis in order to get a degree bound for the Gröbner basis that is doubly exponential in the dimension of the ideal (The notion of H-basis seems to have been introduced by MACAULAY in [19] and means that every polynomial in the ideal is representable as a linear combination of the basis such that no summand in the linear combination has degree greater than the polynomial to be represented). An exponential bound on the degrees of Gröbner bases for zero-dimensional ideals was shown by CANIGLIA, GALLIGO and HEINTZ in [6]. Extending this result, KRICK and LOGAR showed in [17] that Gröbner bases of zero- or one-dimensional ideals can be computed in time exponential in the number of indeterminates. (In the same paper, they asserted, but did not proof, that Gröbner bases of arbitrary ideals can be computed in time doubly exponential in the dimension of the ideal.) There, as well as in most other papers on this topic, unit costs for basic operations in the scalar field are supposed. But this assumption does not seem to be appropriate, as the coefficients of the polynomials may become triply exponentially large in the worst case.

The computation of universal Gröbner bases and the number of term orders in this context have been considered by WEISPFENNING in [33], by MORA and ROBBIANO in [24] and by RITTER and WEISPFENNING in [29].

KOPPENHAGEN and MAYR present an optimal (exponential space) Gröbner basis algorithm for binomial ideals (each generator is a difference of two monomials) in [16]. This algorithm is based on combinatorial principles and does not rely on the parallel computation thesis.

A survey on recent developments concerning the algorithmic aspects of polynomial ideal theory can be found in [21]. More emphasis on applications, mainly to elimination theory and algebraic geometry, was put in the survey [12] of HEINTZ and MORGENSTERN.

References

[1] BERKOWITZ, S. J. On computing the determinant in small parallel time using a small number of processors. *Information Processing Letters 18*, 3 (1984), 147–150.

[2] BORODIN, A., VON ZUR GATHEN, J., AND HOPCROFT, J. Fast parallel matrix and GCD computations. *Information and Control 52* (1982), 241–256.

[3] BUCHBERGER, B. *Ein Algorithmus zum Auffinden der Basiselemente des Restklassenringes nach einem nulldimensionalen Polynomideal.* PhD thesis, Universität Innsbruck, 1965.

[4] BUCHBERGER, B. A criterion for detecting unnecessary reductions in the construction of Gröbner bases. In *Proceedings of the 2nd International Symposium on Symbolic and Algebraic Computation, EUROSAM '79* (1979), LNCS 72, Springer-Verlag, pp. 3–21.

[5] BUCHBERGER, B. A note on the complexity of computing Gröbner-bases. In *Proceedings of the European Computer Algebra Conference, Eurocal'83* (1983), LNCS 162, Springer-Verlag, pp. 137–145.

[6] CANIGLIA, L., GALLIGO, A., AND HEINTZ, J. Some new effectivity bounds in computational geometry. In *Proceedings of the 6th International Conference on Applied Algebra, Algebraic Algorithms and Error-Correcting Codes* (1988), LNCS 357, Springer-Verlag, pp. 131–151.

[7] CSANKY, L. Fast parallel matrix inversion algorithms. *SIAM Journal on Computing 5*, 4 (1976), 618–623.

[8] DUBÉ, T. W. The structure of polynomial ideals and Gröbner bases. *SIAM Journal on Computing 19*, 4 (1990), 750–773.

[9] DUBÉ, T. W., MISHRA, B., AND YAP, C. K. Admissible orderings and bounds for Gröbner bases normal form algorithms. Technical Report 258, Department of Computer Science, Courant Institute of Mathematical Sciences, New York University, 1986.

[10] FORTUNE, S., AND WYLLIE, J. Parallelism in random access machines. In *Proceedings of the 10th Annual ACM Symposium on Theory of Computing* (1978), ACM Press, pp. 114–118.

[11] GALIL, Z., AND PAN, V. Parallel evaluation of the determinant and of the inverse of a matrix. *Information Processing Letters 30* (1989), 41–45.

[12] HEINTZ, J., AND MORGENSTERN, J. On the intrinsic complexity of elimination theory. *Journal of Complexity 9* (1993), 471–498.

[13] HERMANN, G. Die Frage der endlich vielen Schritte in der Theorie der Polynomideale. *Mathematische Annalen 95* (1926), 736–788.

[14] HUYNH, D. T. A superexponential lower bound for Gröbner bases and Church-Rosser commutative Thue systems. *Information and Control 68*, 1-3 (1986), 196–206.

[15] IBARRA, O. H., MORAN, S., AND ROSIER, L. E. A note on the parallel complexity of computing the rank of order n matrices. *Information Processing Letters 11* (1980), 162.

[16] KOPPENHAGEN, U., AND MAYR, E. W. An optimal algorithm for constructing the reduced Gröbner basis of binomial ideals. In *Proceedings of the International Symposium on Symbolic and Algebraic Computation, ISSAC '96 (these proceedings)* (1996), ACM Press.

[17] KRICK, T., AND LOGAR, A. Membership problem, representation problem and the computation of the radical for one-dimensional ideals. In *Effective Methods in Algebraic Geometry (MEGA'90), Progress in Mathematics 94* (1991), Birkhäuser Verlag, pp. 203–216.

[18] LAZARD, D. Gröbner bases, Gaussian elimination and resolution of systems of algebraic equations. In *Proceedings of the European Computer Algebra Conference, Eurocal'83* (1983), LNCS 162, Springer-Verlag, pp. 146–156.

[19] MACAULAY, F. S. *Algebraic Theory of Modular Systems.* Cambridge University Press, Cambridge, 1916.

[20] MAYR, E. W. Polynomial ideals and applications. *Mitteilungen der Mathematischen Gesellschaft in Hamburg XII*, 4 (1992), 1207–1215.

[21] MAYR, E. W. On polynomial ideals, their complexity, and applications. In *Fundamentals of Computation Theory, 10th International Conference, FCT '95* (1995), LNCS 965, Springer-Verlag, pp. 89–105.

[22] MAYR, E. W., AND MEYER, A. R. The complexity of the word problems for commutative semigroups and polynomial ideals. *Advances in Mathematics 46*, 3 (1982), 305–329.

[23] MÖLLER, H. M., AND MORA, F. Upper and lower bounds for the degree of Groebner bases. In *Proceedings of the 3rd International Symposium on Symbolic and Algebraic Computation EUROSAM'84* (1984), LNCS 174, Springer-Verlag, pp. 172–183.

[24] MORA, T., AND ROBBIANO, L. The Gröbner fan of an ideal. In *Computational aspects of commutative algebra*, L. Robbiano, Ed. Academic Press, 1989, pp. 49–74.

[25] MULMULEY, K. A fast parallel algorithm to compute the rank of a matrix over an arbitrary field. In *Proceedings of the 18th Annual ACM Symposium on Theory of Computing* (1986), ACM Press, pp. 338–339.

[26] PAN, V. Complexity of parallel matrix computations. *Theoretical Computer Science 54*, 1 (1987), 65–85.

[27] PAPADIMITRIOU, C. H., AND STEIGLITZ, K. *Combinatorial optimization: Algorithms and complexity.* Prentice-Hall, Englewood Cliffs, 1982.

[28] PREPARATA, F. P., AND SARWATE, D. V. An improved parallel processor bound in fast matrix inversion. *Information Processing Letters 7*, 2 (1978), 148–150.

[29] RITTER, G., AND WEISPFENNING, V. On the number of term orders. *Applicable Algebra in Engineering, Communication and Computing 2* (1991), 55–79.

[30] ROBBIANO, L. Term orderings on the polynomial ring. In *Proceedings of the 10th European Conference on Computer Algebra, EUROCAL '85. Vol. 2: Research contributions* (1985), LNCS 204, Springer-Verlag, pp. 513–517.

[31] ROBBIANO, L. On the theory of graded structures. *Journal of Symbolic Computation 2*, 2 (1986), 139–170.

[32] SCHRIJVER, A. *Theory of linear and integer programming.* John Wiley & Sons, Chichester-New York-Brisbane-Toronto-Singapore, 1986.

[33] WEISPFENNING, V. Constructing universal Gröbner bases. In *Proceedings of the 5th International Conference on Applied Algebra, Algebraic Algorithms and Error-Correcting Codes (AAECC-5)* (1987), LNCS 356, Springer-Verlag, pp. 408–417.

Index of notations

$\prec$	the term order
A	maximum of all the weights $w_{k,i}$
B	basis in the definition of the unified weight function W $B := 2A(\frac{d^2}{2}+d)^{2^{n-1}}+1$
C	vector consisting of the unknowns r_t and $c_{i,t}$
C'	part of C corresponding to the regular minor $\mathcal{F}'$
c_i	i-th coefficient polynomial in the representation $h-\mathrm{NF}(h) = \sum_{i=1}^{s} f_i c_i$
$c_{i,t}$	coefficient of c_i belonging to the term t
D	degree bound (from [13]) on the coefficients c_i of the representation of $h-\mathrm{NF}(h) = \sum_{i=1}^{s} f_i c_i$ $D = \deg(h-\mathrm{NF}(h))+(sd)^{2^n} \leq N+(sd)^{2^n}$
d	bound on the degrees of the ideal generators
$\mathcal{F}$	matrix consisting essentially of the coefficients $f_{i,t}$
$\mathcal{F}'$	maximal regular minor of $\mathcal{F}$
f_i	i-th generating polynomial
$f_{i,t}$	coefficient of f_i belonging to the term t
G	unique reduced Gröbner basis for I w.r.t. $\prec$
H	vector of the coefficients of h
H'	part of H corresponding to the regular minor $\mathcal{F}'$
h	the polynomial to be reduced
h_t	coefficient of h belonging to the term t
I	the ideal generated by $f_1, \ldots, f_s$
K	bound on the numerators and denominators of the coefficients of h and the f_i
M	bound on the format of the matrix $\mathcal{F}$ $M = N^n+s(d+D)^n$
N	bound on the degree of the normal form of h $N = ((2A(\frac{d^2}{2}+d)^{2^{n-1}}+1)^n \deg(h))^{n+1}$
n	number of indeterminates in the polynomial ring
$\mathrm{NF}(f)$	normal form (w.r.t. $\prec$ and I) of a polynomial f
r_t	coefficient of $\mathrm{NF}(h)$ belonging to the term t
s	number of generators of I
$size$	number of bits required to write down the input
T	the set of all terms or power products
W	unified weight function, defined by $W(u) = \sum_{k=1}^{n} B^{k-1} W_k(u)$
W_k	k-th weight function, defined by $W_k(x_1{}^{u_1} \ldots x_n{}^{u_n}) = \sum_{i=1}^{n} w_{k,i} u_i$
$w_{k,i}$	i-th weight of the k-th weight function

x_i i-th indeterminate of the polynomial ring

KLAUS KÜHNLE is a Ph.D. student in the Lehrstuhl für Effiziente Algorithmen at the Institut für Informatik, in Technische Universität München, Germany. He holds a Diplom from the Friedrich-Alexander-Universität Erlangen, Germany. His research interests are in Commutative Algebra and Algebraic Geometry.

ERNST W. MAYR is a full Professor at the Lehrstuhl für Effiziente Algorithmen in the institut für Informatik at the Technische Universität München, Germany. Prof. Mayr obtained Dr. and Diplom from the Technische Universität Müchen, and an M.Sc. degree from M.I.T. He was an Assistant Professor at Stanford Computer Science Department (1982-1989), Professor for Theoretical Computer Science at Johann Wolfgang Goethe-University in Frankfurt am Main (1988-1993) and Department Chair (1990-1991). Since 1993, he has held the Chair for Efficient Algorithms at Technische Universität München. His research interests are in Algorithms, Complexity Theory, Parallel Computation, Petri nets and Computer Algebra. He is a winner of an IBM faculty development award (1983) and the National Science Foundation (USA)'s Presidential Young Investigator award(1984) and is a member of ACM, IEEE-CS, SIAM, EATCS, GI, SigmaXi.