Lambda Calculus and Probabilistic Computation (Extended Version)

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Abstract—We introduce two extensions of the $\lambda\text{-}\mathrm{calculus}$ with a probabilistic choice operator, $\Lambda_\oplus^\mathrm{cbv}$ and $\Lambda_\oplus^\mathrm{cbn}$, modeling respectively call-by-value and call-by-name probabilistic computation. We prove that both enjoys confluence and standardization, in an extended way: we revisit these two fundamental notions to take into account the asymptotic behaviour of terms. The common root of the two calculi is a further calculus based on Linear Logic, $\Lambda_\oplus^!$, which allows us to develop a unified, modular approach.

I. INTRODUCTION

The pervasive role of stochastic models in a variety of domains (such as machine learning, natural language, verification) has prompted a vast body of research on probabilistic programming languages; such a language supports at least discrete distributions by providing an operator which models sampling. In particular, the functional style of probabilistic programming, pioneered by [28], attracts increasing interest because it allows for higher-order computation, and offers a level of abstraction well-suited to deal with mathematical objects. Early work [18], [24], [22], [26], [23] has evolved in a growing body of software development and theoretical research. In this context, the λ -calculus has often been used as a core language.

In order to model higher-order probabilistic computation, it is a natural approach to take the λ -calculus as general paradigm, and to enrich it with a probabilistic construct. The most simple and concrete way to do so ([10], [8], [13]) is to equip the untyped λ -calculus with an operator \oplus , which models flipping a fair coin. This suffices to have universality, as proved in [8], in the sense that the calculus is sound and complete with respect to *computable* probability distributions. The resulting calculus is however *non-confluent*, as it has been observed early (see [8] for an analysis). We revise the issue in Example 1. The problem with confluence is handled in the literature by fixing a deterministic reduction strategy, typically the leftmost-outermost strategy. This is not satisfactory both for theoretical and practical reasons, as we discuss later.

In this paper, we propose a more general point of view. Our goal is a foundational calculus, which plays the same role as the λ -calculus does for deterministic computation. More precisely, taking the point of view propounded by Plotkin in [25], we discriminate between a *calculus* and a *programming language*. The former defines the reduction rules, independently from any reduction strategy, and enjoys confluence and standardization, the latter is specified by a deterministic strategy (an abstract machine). *Standardization* is what relates

the two: the programming language implements the standard strategy associated to the calculus. Indeed, standardization implies the existence of a strategy (the standard strategy) which is guaranteed to reach the result, if it exists.

In this spirit, we consider a probabilistic *calculus* to be characterized by a specific calling mechanism; the reduction is otherwise only constrained by the need of discriminating between duplicating a function which samples from a distribution, and duplicating the result of sampling. Think of tossing a coin and duplicating the result, versus tossing the coin twice, which is indeed the issue at the core of confluence failure, as the following examples (adapted from [9], [8]) show.

Example 1 (Confluence). Let us consider the untyped λ -calculus extended with a binary operator \oplus which models fair, binary probabilistic choice: $M \oplus N$ reduces to either M or N with equal probability 1/2; we write this as $M \oplus N \to \{M^{\frac{1}{2}}, N^{\frac{1}{2}}\}$. Intuitively, the result of evaluating a probabilistic term is a distribution on its possible values.

- 1) Consider the term PQ, where $P = (\lambda z.z \text{ XOR } z)$, and $Q = (T \oplus F)$; XOR is the standard construct for exclusive OR, $T = \lambda xy.x$ and $F = \lambda xy.y$ code the boolean values.

 If we first reduce Q, we obtain $(\lambda z.z \text{ XOR } z)T$ or $(\lambda z.z \text{ XOR } z)F$, with equal probability 1/2. This way, PQ evaluates to $\{F^1\}$, i.e. F with probability 1.
 - If we reduce the outermost redex first, PQ reduces to $(T \oplus F)$ XOR $(T \oplus F)$, and the term evaluates to the distribution $\{T^{\frac{1}{2}}, F^{\frac{1}{2}}\}$.
 - The two resulting distributions are not even comparable.
- 2) The same phenomenon appears even if we restrict ourselves to call-by-value. Consider for example the reductions of PN with P as in 1), and $N = (\lambda xy.x \oplus y)$. We obtain the same two different distributions as above.

In this paper, we define two probabilistic λ -calculi, respectively based on the call-by-value (CbV) and call-by-name (CbN) calling mechanism. Both enjoy confluence and standardization, in an extended way: indeed we revisit these two fundamental notions to take into account the asymptotic behaviour of terms. The common root of the two calculi is a further calculus based on Linear Logic, which is an extension of Simpson's linear λ -calculus [30], and which allows us to develop a unified, modular approach.

Content and Contributions: In Section IV, we introduce a call-by-value calculus, denoted $\Lambda_{\oplus}^{\text{cbv}}$, as a probabilistic

extension of the call-by-value λ -calculus of Plotkin (where the β -reduction fires only in case the argument is a value, *i.e.* either a variable or a λ -abstraction). We choose to study in detail call-by-value for two main reasons. First, it is the most relevant mechanism to probabilistic programming (most of the abstract languages we cited are call-by-value, but also realworld stochastic programs such as Church [16]). Second, callby-value is a mechanism in which dealing with functions, and duplication of functions, is clean and intuitive, which allows us to address the issue at the core of confluence failure. The definition of value (in particular, a probabilistic choice is not a value) together with a suitable restriction of the evaluation context for the probabilistic choice, allow us to recover key results: confluence and a form of standardization (Section V). Let us recall that, in the classical λ -calculus, standardization means that there is a strategy which is complete for all reduction sequences, i.e., for every reduction sequence $M \to^* N$ there is a standard reduction sequence from M to N. A standard reduction sequence with the same property exists also here. An unexpected result is that strategies which are complete in the classical case, are not so here, notably the leftmost strategy.

In Section VI we study the asymptotic behavior of terms. Our leading question is how the asymptotic behaviour of different sequences starting from the same term compare. We first analyze if and in which sense confluence implies that the result of a *probabilistically terminating* computation is unique. We formalize the notion of *asymptotic result* via *limit distributions*, and establish that there is a *unique* maximal one.

In Section VII we address the question of how to find such greatest limit distribution, a question which arises from the fact that evaluation in $\Lambda_{\oplus}^{\text{cbv}}$ is non-deterministic, and different sequences may terminate with different probability. With this aim, we extend the notion of standardization to limits; this extension is non-trivial, and demands the development of new sophisticated proof methods.

We prove that the new notion of standardization supplies a family of *complete reduction strategies* which are guaranteed to reach the unique maximal result. Remarkably, we are able to show that, when evaluating programs, i.e., closed terms, this family *does include* the leftmost strategy. As we have already observed, this is the deterministic strategy which is typically adopted in the literature, in either its call-by-value ([18], [7]) or its call-by-name version ([10], [13]), but without any completeness result with respect to *probabilistic* computation. Our result offers an "a posteriori" justification for its use!

The study of $\Lambda_{\oplus}^{\text{cbv}}$ allows us to develop a crisp approach, which we are then able to use in the study of different probabilistic calculi. Because the issue of duplication is central, it is natural to expect a benefit from the fine control over copies which is provided by Linear Logic. In Section IX we use our tools to introduce and study a probabilistic *linear* λ -calculus, Λ_{\oplus}^{l} . The linear calculus provides not only a finer control on duplication, but also a modular approach to confluence and standardization, which allow us to formalize a call-by-name version of our calculus, namely $\Lambda_{\oplus}^{\text{cbn}}$, in Section X. We prove that $\Lambda_{\oplus}^{\text{cbn}}$ enjoys properties analogous to those of $\Lambda_{\oplus}^{\text{cbv}}$, in

particular confluence and standardization.

In Section II we provide the reader with some background and motivational observations. Basic notions of discrete probability and rewriting are reviewed in Section III.

Related Work: The idea of extending the λ -calculus with a probabilistic construct is not new; without any ambition to be exhaustive, let us cite [22], [26], [10], [13], [8], [5]. In all these cases, a specific reduction strategy is fixed; they are indeed languages, not calculi, according to Plotkin's distinction.

The issue about confluence appears every time the λ -calculus is extended with a choice effect: quantum, algebraic, non-deterministic. The ways of framing the same problem in different settings are naturally related, and we were inspired by them. Confluence for an algebric calculus is dealt with in [1] for the call-by-value, and in [31] for the call-by-name. In the quantum case we would like to cite [7], [6], which are based on Simpson's calculus [30]. A probabilistic extension of Simpson's calculus was first proposed in [11]. The language is similar to that of Λ^1_{\oplus} ; however in [11] (as also in [7], [6]) no reduction (not even β) is allowed in the scope of a !-operator. The reduction there hence corresponds to surface reduction, which in Sec. IX we show to be the *standard strategy* for Λ^1_{\oplus} .

To our knowledge, the only proposal of a probabilistic λ -calculus in which the reduction is independent from a *specific strategy* is for *call-by-name*, namely the calculus of [19], in the line of work of differential [14] and algebric [31] λ -calculus. The focus in [19] is essentially semantical, as the author want to study an equational theory for the λ -calculus, based on an extension of Böhm trees. [19] develops results which in their essence are similar to those we obtain for call-by-name in Sec. X, in particular confluence and standardization, even if his calculus –which internalizes the probabilistic behavior– is quite different from ours, and so are the proof techniques.

Finally, we wish to mention that proposals of a probabilistic λ -calculus could also be extracted from semantical models, such as the one in [3], which develops an idea earlier presented in [29], and in which the notion of graph models for λ -calculus has been extended with a probabilistic construct.

II. BACKGROUND AND MOTIVATIONAL OBSERVATIONS

In this section, we first review -in a non-technical waythe specific features of probabilistic programs, and how they differ from classical ones. We then focus on some motivational observations which are relevant to our work. First, we give an example of features which are lost if a programming language is characterized by a strategy which is not rooted in a more general calculus. Then, we illustrate some of the issues which appear when we study a general calculus, instead of a specific reduction strategy. Addressing these issues will lead us to develop new notions and tools.

A. Classical vs. Probabilistic Programs

A classical program defines a deterministic input-output relation; it terminates (on a given input), or does not; if it terminates, the program only runs for a finite number of

steps. Instead, a *probabilistic* program generates a probability *distribution over possible outputs*; it *terminates* (on a given input) with a certain probability; it may have runs which take *infinitely many steps* even when termination has probability 1.

A probabilistic program is a stochastic model. The intuition is that the probabilistic program P is executed, and random choices are made by sampling; this process defines a distribution over all the possible outputs of P. Even if the termination probability is 1 (almost sure termination), that degree of certitude is typically not reached in any finite number of steps, but it appears as a limit. A standard example is a term M which reduces to either the normal form T or M itself, with equal probability 1/2. After n steps, M reduces to T with probability $\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}$. Only at the limit this computation terminates with probability 1.

Probabilistic vs. Quantitative: The notion of probabilistic termination is what sets apart probabilistic λ -calculus from other quantitative calculi such as those in [1], [14], [31], and from the non-deterministic λ -calculus [9]. For this reason, the asymptotic behaviour of terms will be the focus of this paper.

B. Confluence of the calculus is relevant to programming

Functional languages have their foundation in the λ -calculus and its properties, and such properties (notably, confluence and standardization) have theoretical and practical implications. A strength of classical functional languages -which is assuming growing importance- is that they are inherently parallel (we refer e.g. to [21] for discussion on deterministic parallel programming): every sub-expression can be evaluated in parallel, because of referential transparency; still, we can perform reasoning, testing and debugging on a program using a sequential model, because the result of the calculus is independent from the evaluation order. Not to force a sequential strategy impacts the implementation of the language, but also the conception of programs. As advocated by Harper, the parallelism of functional languages exposes the "dependency structure of the computation by not introducing any dependencies that are not forced on us by the nature of the computation itself."

This feature of functional languages is rooted in the *confluence* of the λ -calculus, and is an example of what is lost in the probabilistic setting, if we give-up either confluence, or the possibility of non-deterministic evaluation.

C. The result of probabilistic computation

A ground for our approach is the distinction between calculus and language. Some of the issues which we will need to address do not appear when working with probabilistic languages, because they are based on a simplification of the λ -calculus. Programming languages only evaluate programs, i.e., closed terms (without free variables). A striking simplification appears from another crucial restriction, weak evaluation, which does not evaluate function bodies (the scope of λ -abstractions). In weak call-by-value (base of the ML/CAML family of probabilistic languages) values are normal forms.

What is the result of a probabilistic computation is well understood only in the case of *programming languages*: the result of a program is a distribution on its possible outcomes, which are *normal forms* w.r.t. a chosen strategy. In the literature of probabilistic λ -calculus, two main deterministic strategies have been studied: weak left strategy in CbV [8] and head strategy in CbN [13], whose normal forms are respectively the closed values and the head normal forms.

When considering a calculus instead of a language, the identity between normal forms and results does not hold anymore, with important consequences in the definition of limit distributions. We investigate this issue in Sec. VI. The approach we develop is general and uniform to all our calculi.

III. TECHNICAL PRELIMINARIES

We review basic notions on discrete probability and rewriting which we use through the paper. We assume that the reader has some familiarity with the λ -calculus.

A. Basics on Discrete Probability

A discrete probability space is given by a pair (Ω,μ) , where Ω is a countable set, and μ is a discrete probability distribution on Ω , i.e. is a function from Ω to $[0,1] \subset \mathbb{R}$ such that $\|\mu\| := \sum_{\omega \in \Omega} \mu(\omega) = 1$. In this case, a probability measure is assigned to any subset $A \subseteq \Omega$ as $\mu(A) = \sum_{\omega \in A} \mu(\omega)$. In the language of probability theory, a subset of Ω is called an *event*.

Let (Ω,μ) be as above. Any function $F:\Omega\to\Delta$, where Δ is another countable set, induces a probability distribution μ^F on Δ by composition: $\mu^F(d\in\Delta):=\mu(F^{-1}(d))$ i.e. $\mu\{\omega\in\Omega:F(\omega)=d\}$. In the language of probability theory, F is called a discrete random variable on (Ω,μ) .

- **Example 2** (Die). 1) Consider tossing a die once. The space of possible outcomes is the set $\Omega = \{1,2,3,4,5,6\}$. The probability measure μ of each outcome is 1/6. The event "result is odd" is the subset $\mathcal{O} = \{1,3,5\}$, whose probability measure is $\mu(\mathcal{O}) = 1/2$.
 - 2) Let Δ be a set with two elements {Even,Odd}, and F the obvious function from Ω to Δ . F induces a distribution on Δ , with $\mu^F(\text{Even}) = 1/2$ and $\mu^F(\text{Odd}) = 1/2$.

B. Subdistributions and $DST(\Omega)$

Given a countable set Ω , a function $\mu:\Omega\to[0,1]$ is a probability subdistribution if $\|\mu\|\leq 1$. We write $\mathrm{DST}(\Omega)$ for the set of subdistributions on Ω . With a slight abuse of language, we will use the term distribution also for subdistribution. Subdistributions allow us to deal with partial results and non-successful computations.

Order: $\mathrm{DST}(\Omega)$ is equipped with the standard order relation of functions : $\mu \leq \rho$ if $\mu(\omega) \leq \rho(\omega)$ for each $\omega \in \Omega$.

Support: The support of μ is $Supp(\mu) = \{\omega : \mu(\omega) > 0\}$.

Representation: We represent a distribution by explicitly indicating the support, and (as superscript) the probability assigned to each element by μ . We write $\mu = \{a_0^{p_0},...,a_n^{p_n}\}$ if $\mu(a_0) = p_0,...,\mu(a_n) = p_n$ and $\mu(a_j) = 0$ otherwise.

C. Multidistributions

To syntactically represent the global evolution of a probabilistic system, we rely on the notion of multidistribution [2].

A multiset is a (finite) list of elements, modulo reordering, i.e. $[a,b,a] = [a,a,b] \neq [a,b]$; the multiset [a,a,b] has three elements. Let $\mathcal X$ be a countable set and m a multiset of pairs of the form pM, with $p \in]0,1]$, and $M \in \mathcal{X}$. We call m = $[p_i M_i | i \in I]$ (where the index set I ranges over the elements of m) a multidistribution on \mathcal{X} if $\sum_{i \in I} p_i \leq 1$. We denote by $\mathtt{MDST}(\mathcal{X})$ the set of all multidistributions on $\mathcal{X}.$

We write the multidistribution [1M] simply as [M]. The sum of multidistributions is denoted by +, and it is the concatenation of lists. The product $q \cdot m$ of a scalar q and a multidistribution m is defined pointwise: $q \cdot [p_1 M_1, ..., p_n M_n] =$ $[(qp_1)M_1,...,(qp_n)M_n].$

Intuitively, a multidistribution $m \in MDST(\mathcal{X})$ is a syntactical representation of a discrete probability space where at each element of the space is associated a probability and a term of \mathcal{X} . To the multidistribution $\mathbf{m} = [p_i M_i \mid i \in I]$ we associate a probability distribution $\mu \in DST(\mathcal{X})$ as follows:

$$\mu(M)\!=\!\left\{ \begin{array}{l} p \quad \text{if } p\!=\!\sum_{i\in I} p_i \text{ s.t. } M_i\!=\!M \\ 0 \text{ otherwise;} \end{array} \right.$$
 and we call μ the probability distribution associated to m.

Example 3 (Distribution vs. multidistribution). If $\mathbf{m} = \begin{bmatrix} \frac{1}{2}a, \frac{1}{2}a \end{bmatrix}$, then $\mu = \{a^1\}$. Please observe the difference between distribution and multidistribution: if m' = [1a], then $m \neq m'$, but $\mu = \mu'$.

D. Binary relations (notations and basic definitions)

Let \rightarrow_r be a binary relation on a set \mathcal{X} . We denote \rightarrow_r^* its reflexive and transitive closure. We denote $=_r$ the reflexive, symmetric and transitive closure of \rightarrow_r . If $u \in \mathcal{X}$, we write $u \not\to_r$ if there is no $t \in \mathcal{X}$ such that $u \to_r t$; in this case, uis in \rightarrow_r -normal form. Figures convention: as is standard, in the figures we depict \rightarrow^* as \rightarrow ; solid arrows are universally quantified, dashed arrows are existentially quantified.

Confluence and Commutation: Let $r, s, t, u \in \mathcal{X}$. The relations \rightarrow_1 and \rightarrow_2 on \mathcal{X} commute if $(r \rightarrow_1^* s \text{ and } r \rightarrow_2^* t)$ imply there is u such that $(s \rightarrow_2^* u \text{ and } r_3 \rightarrow_1^* u)$; they diamondcommute (\diamond -commute) if $(r \rightarrow_1 s \text{ and } r \rightarrow_2 t)$ imply there is u such that $(s \rightarrow_2 u \text{ and } t \rightarrow_1 u)$. The relation \rightarrow is confluent (resp. diamond) if it commutes (resp. \diamond -commutes) with itself. It is well known that \diamond -commutation implies commutation, and diamond implies confluence.

IV. CALL-BY-VALUE CALCULUS
$$\Lambda_{\oplus}^{\mathsf{cbv}}$$

We define $\Lambda_{\oplus}^{\text{cbv}}$, a CbV probabilistic λ -calculus.

A. Syntax of Λ_{\oplus}^{cbv}

1) The language: Terms and values are generated respectively by the grammars:

$$\begin{array}{lll} M,N,P,Q & ::= x \,|\, \lambda x.M \,|\, MM \,|\, M \oplus M & (\text{terms } \Lambda_{\oplus}) \\ V,W & ::= x \,|\, \lambda x.M & (\text{values } \mathcal{V}) \end{array}$$

where x ranges over a countable set of variables (denoted by x,y,...). Λ_{\oplus} and \mathcal{V} denote respectively the set of terms and of values. Free variables are defined as usual. M[N/x] denotes

the term obtained by capture-avoiding substitution of N for each free occurrence of x in M.

Contexts (\mathbf{C}) and surface contexts (\mathbf{S}) are generated by the grammars:

$$\mathbf{C} ::= \Box | M\mathbf{C} | \mathbf{C}M | \lambda x. \mathbf{C} | \mathbf{C} \oplus M | M \oplus \mathbf{C} \text{ (contexts)}$$

$$\mathbf{S} ::= \Box | M\mathbf{S} | \mathbf{S}M \text{ (surface contexts)}$$

where \Box denotes the *hole* of the term context. Given a term context C, we denote by C(M) the term obtained from C by filling the hole with M, allowing the capture of free variables. All surface contexts are contexts. Since the hole will be filled with a redex, *surface contexts* formalize the fact that the redex (the hole) is not in the scope of a λ -abstraction, nor of a \oplus .

 $\mathtt{MDST}(\Lambda_{\oplus})$ denotes the set of *multi-distributions* on Λ_{\oplus} .

2) Reductions: We first define reduction rules on terms (Fig. 1), and one-step reduction from terms to multidistributions (Fig. 2). We then lift the definition of reduction to a binary relation on $MDST(\Lambda_{\oplus})$.

Observe that, usually, a reduction step is given by the closure under context of the reduction rules. However, to define a reduction from term to term is not informative enough, because we still have to account for the probability. The meaning of $M \oplus N$ is that this term reduces to either M or N, with equal probability $\frac{1}{2}$. There are various way to formalize this fact; here, we use multidistributions.

a) Reduction Rules and Steps: The reduction rules on the terms of Λ_{\oplus} are defined in Fig. 1.

$$\begin{array}{|c|c|c|c|c|}\hline & \beta_v\text{-rule} & Probabilistic rules \\ (\lambda x.M)V \mapsto_{\beta_v} M[V/x] & \text{if } V \in \mathcal{V} & M \oplus N \mapsto_{l \oplus} M & M \oplus N \mapsto_{r \oplus} N \\ \hline \end{array}$$

Figure 1: Reduction Rules

The (one-step) reduction relations \rightarrow_{β_v} , $\rightarrow_{\oplus} \subseteq \Lambda_{\oplus} \times$ $\mathtt{MDST}(\Lambda_{\oplus})$ are defined in Fig. 2. Observe that the probabilistic rules $\mapsto_{r\oplus,l\oplus}$ are closed only under surface contexts, while the reduction rule \mapsto_{β_n} is closed under general context C (hence $\Lambda_{\oplus}^{\mathsf{cbv}}$ is a conservative extension of Plotkin's CbV λ -calculus, see IV-B). We denote by \rightarrow the union $\rightarrow_{\beta_n} \cup \rightarrow_{\oplus}$.

$$\frac{(\lambda x.M)V \mapsto_{\beta_{v}} M[V/x] \quad V \in \mathcal{V}}{\mathbf{C}((\lambda x.M)V) \to_{\beta_{v}} [\mathbf{C}(M[V/x])]}$$

$$\frac{M \oplus N \mapsto_{l \oplus} M \quad M \oplus N \mapsto_{r \oplus} N}{\mathbf{S}(M \oplus N) \to_{\oplus} [\frac{1}{2} \mathbf{S}(M), \frac{1}{2} \mathbf{S}(N)]}$$

Figure 2: Reduction Steps

b) Lifting: We lift the reduction relation $\rightarrow \subseteq \Lambda_{\oplus} \times$ $\mathtt{MDST}(\Lambda_{\oplus})$ to a relation $\Rightarrow\subseteq \mathtt{MDST}(\Lambda_{\oplus}) \times \mathtt{MDST}(\Lambda_{\oplus})$, as defined in Fig. 3. Observe that \Rightarrow is a reflexive relation.

We define in the same way the lifting of any relation $\rightarrow_r \subseteq$ $\Lambda_{\oplus} \times \mathtt{MDST}(\Lambda_{\oplus})$ to a binary relation \Rightarrow_r on $\mathtt{MDST}(\Lambda_{\oplus})$. In particular, we lift $\rightarrow_{\beta_v}, \rightarrow_{\oplus}$ to $\Rightarrow_{\beta_v}, \Rightarrow_{\oplus}$.

c) Reduction sequences: A \Rightarrow -sequence (reduction sequence) from m is a sequence $m = m_0, ..., m_i, m_{i+1}, ...$ such that $\mathtt{m}_i \Rightarrow \mathtt{m}_{i+1} \ (\forall i)$. We write $\mathtt{m} \Rightarrow^* \mathtt{n}$ to indicate that there is a *finite* sequence from \mathtt{m} to \mathtt{n} , and $\langle \mathtt{m}_n \rangle_{n \in \mathbb{N}}$ for an *infinite sequence*.

- d) β_v equivalence: We write $=_{\beta_v}$ for the transitive, reflexive and symmetric closure of \Rightarrow_{β_v} ; abusing the notation, we will write $M =_{\beta_v} N$ for $[M] =_{\beta_v} [N]$.
- e) Normal Forms: $\mathcal N$ denotes the set of \to -normal forms. Given $\to_r \in \Lambda_\oplus \times \mathtt{MDST}(\Lambda_\oplus)$, a term M is in \to_r -normal form if $M \not\to_r$, i.e. there is no m such that $M \to_r \mathfrak m$. It is easy to check that all closed \to -normal forms are values, however a value is not necessarily a \to -normal form.
- 3) Full Lifting: The definition of lifting allows us to apply a reduction step \rightarrow to any number of M_i in the multidistribution $\mathtt{m} = [p_i M_i \mid i \in I]$. If no M_i is reduced, then $\mathtt{m} \Rightarrow \mathtt{m}$ (the relation \Rightarrow is reflexive). Another important case is when all M_i for which a reduction step is possible are indeed reduced. This notion of full reduction, denoted by \Rightarrow , is defined as follows.

$$\frac{M \not\to}{[M] \rightrightarrows [M]} \qquad \frac{M \to \mathtt{m}}{[M] \rightrightarrows \mathtt{m}} \qquad \frac{([M_i] \rightrightarrows \mathtt{m}_i)_{i \in I}}{[p_i M_i \, | \, i \in I] \rightrightarrows \sum_{i \in I} p_i \cdot \mathtt{m}_i}$$

Obviously, $\rightrightarrows \subset \Rightarrow$. Similarly to lifting, also the notion of full lifting can be extended to any reduction. For any $\to_r \subseteq \Lambda_\oplus \times \texttt{MDST}(\Lambda_\oplus)$, its full lifting is denoted by $\rightrightarrows_r \subseteq \texttt{MDST}(\Lambda_\oplus) \times \texttt{MDST}(\Lambda_\oplus)$. The relation \rightrightarrows plays an important role in VII.

B. $\Lambda_{\oplus}^{\text{cbv}}$ and the λ -calculus

A comparison between $\Lambda_{\oplus}^{\mathsf{cbv}}$ and the λ -calculus is in order. Let Λ be the set of λ -terms; we denote by Λ^{cbn} the CbN λ -calculus, equipped with the reduction \to_{β} [4], and by Λ^{cbv} the CbV λ -calculus, equipped with the reduction \to_{β_v} [25].

 $\Lambda_{\oplus}^{\mathrm{cbv}}$ is a conservative extension of Λ^{cbv} . A translation $(\cdot)_{\lambda}$: $\Lambda_{\oplus} \to \Lambda$ can be defined as follows, where z is a fresh variable which is used by no term:

$$\begin{array}{lll} (x)_{\lambda} &= x \\ (M \oplus N)_{\lambda} &= z(M)_{\lambda}(N)_{\lambda} \end{array} \Big| (MN)_{\lambda} &= (M)_{\lambda}(N)_{\lambda} \\ (\lambda x.M)_{\lambda} &= \lambda x.(M)_{\lambda} \end{array}$$

The translation is injective (if $(M)_{\lambda} = (N)_{\lambda}$ then M = N) and preserves values.

Proposition 4 (Simulation). The translation is sound and complete. Let $M, N \in \Lambda_{\oplus}$.

- 1) $M \rightarrow_{\beta_v} N$ implies $(M)_{\lambda} \rightarrow_{\beta_v} (N)_{\lambda}$;
- 2) $(M)_{\lambda} \rightarrow_{\beta_v} Q$ implies there is a (unique) N, with $Q = (N)_{\lambda}$ and $M \rightarrow_{\beta_v} N$.

C. Discussion (Surface Contexts)

The notion of surface context which we defined is familiar in the setting of λ -calculus: it corresponds to *weak evaluation*, which we discussed in II-C. In $\Lambda_{\oplus}^{\text{cbv}}$, the \to_{β_v} -reduction is *unrestricted*. Closing the \oplus -rules under surface context S expresses the fact that the \oplus -redex is not reduced under λ -abstraction, nor in the scope of another \oplus . The former is fundamental to confluence: it means that a function which samples from a distribution can be duplicated, but we cannot pre-evaluate the sampling. The latter is a technical simplification, which we adopt to avoid unessential burdens with associativity. To require no reduction in the scope of \oplus is very similar to allow no reduction in the branches of an if-then-else.

V. CONFLUENCE AND STANDARDIZATION

A. Confluence

We prove that Λ_{\oplus}^{cbv} is confluent. We modularize the proof using the Hindley-Rosen lemma. The notions of commutation and \diamond -commutation which we use are reviewed in Sec. III-D.

Lemma (Hindley-Rosen). Let \rightarrow_1 and \rightarrow_2 be binary relations on the same set \mathcal{R} . Their union $\rightarrow_1 \cup \rightarrow_2$ is confluent if both \rightarrow_1 and \rightarrow_2 are confluent, and \rightarrow_1 and \rightarrow_2 commute.

The following criterion allows us to *work pointwise* in proving commutation and confluence of binary relations on *multidistributions*, namely \Rightarrow_{β_n} and \Rightarrow_{\oplus} .

Lemma 5 (Pointwise Criterion). Let \rightarrow_o , $\rightarrow_b \subseteq \Lambda_{\oplus} \times MDST(\Lambda_{\oplus})$ and \Rightarrow_o , \Rightarrow_b their lifting (as defined in IV-A2b). Property (*) below implies that \Rightarrow_o , \Rightarrow_b \diamond -commute.

(*) If $M \rightarrow_b n$ and $M \rightarrow_o s$, then $\exists r \text{ s.t. } n \Rightarrow_o r \text{ and } s \Rightarrow_b r$.

Proof. We prove that (**) $\mathbf{m} \Rightarrow_b \mathbf{n}$ and $\mathbf{m} \Rightarrow_o \mathbf{s}$ imply exists \mathbf{r} s.t. $\mathbf{n} \Rightarrow_o \mathbf{r}$ and $\mathbf{s} \Rightarrow_b \mathbf{r}$. Let $\mathbf{m} = [p_i M_i \mid i \in I]$. By definition of lifting, for each M_i , we have $[M_i] \Rightarrow_b \mathbf{n}_i$ and $[M_i] \Rightarrow_o \mathbf{s}_i$, with $\mathbf{n} = \sum p_i \cdot \mathbf{n}_i$ and $\mathbf{s} = \sum p_i \cdot \mathbf{s}_i$. It is easily checked, that for each M_i , it exists \mathbf{r}_i s.t. $\mathbf{n}_i \Rightarrow_o \mathbf{r}_i$ and $\mathbf{s}_i \Rightarrow_b \mathbf{r}_i$. If either $[M_i] \Rightarrow_b \mathbf{n}_i$ or $[M_i] \Rightarrow_o \mathbf{s}_i$ uses reflexivity (rule L1), it is immediate to obtain r_i . Otherwise, \mathbf{r}_i is given by property (*). Hence $\mathbf{r} = \sum_i p_i \cdot \mathbf{r}_i$ satisfies (**).

We derive confluence of \Rightarrow_{β_v} from the same property in the CbV λ -calculus [25], [27], using the simulation of Prop. 4.

Lemma 6. The reduction \Rightarrow_{β_v} is confluent.

Proof. Assume $\mathbf{m} \Rightarrow_{\beta_v}^* \mathbf{n}$ and $\mathbf{m} \Rightarrow_{\beta_v}^* \mathbf{s}$. We first observe that if $\mathbf{m} = [p_i M_i \mid i \in I]$, then \mathbf{n} and \mathbf{s} are respectively of the shape $[p_i N_i \mid i \in I]$, $[p_i S_i \mid i \in I]$, with $M_i \to_{\beta_v}^* N_i$ and $M_i \to_{\beta_v}^* S_i$. By Prop. 4, we can project such reduction sequences on Λ^{cbv} , obtaining that for each $i \in I$, $(M_i)_{\lambda} \to_{\beta_v}^* (N_i)_{\lambda}$ and $(M_i)_{\lambda} \to_{\beta_v}^* (S_i)_{\lambda}$. Since \to_{β_v} in CbV λ -calculus is confluent, there are $R_i \in \Lambda$ such that $(N_i)_{\lambda} \to_{\beta_v}^* R_i$ and $(S_i)_{\lambda} \to_{\beta_v}^* R_i$. By Prop. 4.2, for each $i \in I$ there is a unique $T_i \in \Lambda_{\oplus}$ such that $(T_i)_{\lambda} = R_i$, and the proof is given.

We prove that the reduction \Rightarrow_{\oplus} is diamond, i.e., the reduction diagram closes in one step.

Lemma 7. The reduction \Rightarrow_{\oplus} is diamond.

Proof. We prove that if $M \to_{\oplus} \mathbf{n}$ and $M \to_{\oplus} \mathbf{s}$, then $\exists \mathbf{r}$ such that $\mathbf{n} \Rightarrow_{\oplus} \mathbf{r}$ and $\mathbf{s} \Rightarrow_{\oplus} \mathbf{r}$. The claim then follows by Lemma 5, by taking $\to_o = \to_b = \to_{\oplus}$. Let $M = \mathbf{S}(P \oplus Q) = \mathbf{S}'(P' \oplus Q')$, $\mathbf{n} = \left[\frac{1}{2}\mathbf{S}(P), \frac{1}{2}\mathbf{S}(Q)\right]$ and $\mathbf{s} = \left[\frac{1}{2}\mathbf{S}'(P'), \frac{1}{2}\mathbf{S}'(Q')\right]$. Because of definition of surface context, the two \oplus -redexes do not overlap: $P' \oplus Q'$ is a subterm of \mathbf{S} and $P \oplus Q$ is a subterm of \mathbf{S}' . Hence we can reduce those redexes in \mathbf{S} and \mathbf{S}' , to obtain \mathbf{r} . □

We prove commutation of \Rightarrow_{\oplus} and \Rightarrow_{β_v} by proving a stronger property: they \diamond -commute.

Lemma 8. The reductions \Rightarrow_{β_v} and $\Rightarrow_{\oplus} \diamond$ -commute.

Proof. By using Lemma 5, we only need to prove that if $M \to_{\beta_v}$ n and $M \to_{\oplus}$ s, then $\exists r$ such that $n \Rightarrow_{\oplus} r$ and $s \Rightarrow_{\beta_v} r$. The proof is by induction on M. Cases M = x and $M = \lambda x.P$ are not possible given the hypothesis.

- 1) Case $M = P \oplus Q$. M is the only possible \oplus -redex. Assume the β_v -redex is inside P (the other case is similar), and that $P \oplus Q \to_{\beta_v} [P' \oplus Q]$, $P \oplus Q \to_{\oplus} [\frac{1}{2}P, \frac{1}{2}Q]$. It is immediate that $\mathbf{r} = [\frac{1}{2}P', \frac{1}{2}Q]$ satisfies the claim.
- 2) Case M = PQ. M cannot have the form $(\lambda x.P')V$ because neither P nor Q could contain a \oplus -redex.
 - a) Assume that the β_v -redex is inside P, and the \oplus -redex inside Q. We have $PQ \to_{\beta_v} [P'Q]$ (with $P \to_{\beta_v} P'$), $PQ \to_{\oplus} \left[\frac{1}{2}PQ', \frac{1}{2}PQ''\right]$ (with $Q \to_{\oplus} \left[\frac{1}{2}Q', \frac{1}{2}Q''\right]$). It is immediate that $\mathbf{r} = \left[\frac{1}{2}P'Q', \frac{1}{2}P'Q''\right]$ satisfies the claim. The symmetric case is similar.
 - b) Assume that both redexes are inside Q. Let us write M as S(Q). Assume $Q \to_{\beta_v} [N]$, $Q \to_{\oplus} [\frac{1}{2}Q', \frac{1}{2}Q'']$, therefore $S(Q) \to_{\beta_v} [S(N)] = \mathbf{n}$ and $S(Q) \to_{\oplus} [\frac{1}{2}S(Q'), \frac{1}{2}S(Q'')] = \mathbf{s}$. We use the inductive hypothesis on Q to obtain $\mathbf{r}' = [\frac{1}{2}R', \frac{1}{2}R'']$ such that $[N] \Rightarrow_{\oplus} [\frac{1}{2}R', \frac{1}{2}R'']$, $[Q'] \Rightarrow_{\beta_v} [R']$, $[Q''] \Rightarrow_{\beta_v} [R'']$. We conclude that for $\mathbf{r} = [\frac{1}{2}S(R'), \frac{1}{2}S(R'')]$, it holds that $\mathbf{n} \Rightarrow_{\oplus} \mathbf{r}$ and $\mathbf{s} \Rightarrow_{\beta_v} \mathbf{r}$.

Theorem 9. The reduction \Rightarrow is confluent.

Proof. By Hindley-Rosen, from Lemmas 8, 6, and 7.

Let us call n an \mathcal{N} -multidistribution if $n \in MDST(\mathcal{N})$ *i.e.* $n = [p_i M_i]$ and all M_i are \rightarrow -normal forms. The following fact is an immediate consequence of confluence:

Fact. The \mathcal{N} -multidistribution to which m reduces, if any, is unique.

1) Discussion: While immediate, the above fact is hardly useful, for two reasons. First, we know that probabilistic termination is not necessarily reached in a finite number of steps; the relevant notion is not that $\mathtt{m} \Rightarrow^* \mathtt{n} \in \mathtt{MDST}(\mathcal{N})$, but rather that of a distribution which is defined as limit by the sequence $\langle \mathtt{m}_n \rangle_{n \in \mathbb{N}}$. Secondly, in Plotkin's CbV calculus the result of computation is formalized by the notion of value, and considering normal forms as values is unsound ([25], page 135). In Section VI-B we introduce a suitable notion of limit distribution, and study the implications of confluence on it.

B. A Standardization Property

In this section, we first introduce surface and left reduction as strategies for \Rightarrow . In the setting of the CbV λ -calculus, the former corresponds to weak reduction, the latter to the standard strategy originally defined in [25]. We then establish a standardization result, namely that every *finite* \Rightarrow -sequence can be partially ordered as a sequence in which all surface reductions are performed first. A counterexample shows that in $\Lambda_{\oplus}^{\rm cbv}$, a standardization result using left reduction fails.

1) Surface and Left Reduction: We remind the reader that in the λ -calculus, a deterministic strategy defines a function from terms to redexes, associating to every term the next redex to be reduced. More generally, we call reduction strategy for \rightarrow a reduction relation \rightarrow_a such that $\rightarrow_a \subseteq \rightarrow$. The notion of strategy can be easily formalized through the notion of context. With this in mind, let us consider surface and left contexts.

- Surface contexts S have been defined in Sec.IV-A1.
- Left contexts ${f L}$ are defined by the following grammar: ${f L} := \Box \, | \, {f L} M \, | \, V {f L}$

Note that in particular a left contexts is a surface context.

- We call *surface reduction*, denoted by $\stackrel{s}{\rightarrow}$ (with lifting $\stackrel{s}{\rightarrow}$) and *left reduction*, denoted by $\stackrel{i}{\rightarrow}$ (with lifting $\stackrel{i}{\rightarrow}$), the closure of the reduction rules in Fig. 1 under surface contexts and left contexts, respectively. It is clear that $\stackrel{s}{\rightarrow} = \stackrel{s}{\rightarrow} \beta_v \cup \rightarrow_{\oplus}$. Observe that $\stackrel{i}{\rightarrow} \subsetneq \stackrel{s}{\rightarrow}$.
- A reduction step $M \to m$ is *deep*, written $M \stackrel{d}{\to} m$, if it is *not* a surface step. A reduction step is *internal* (written $M \stackrel{int}{\to} m$) if it is *not* a left step. Observe that $\stackrel{d}{\to} \subset \stackrel{int}{\to}$.

Example 10. • $(\stackrel{l}{\rightarrow} \subsetneq \stackrel{s}{\rightarrow})$ Let M = x(II)(II), where $I = \lambda x.x.$ Then $M \stackrel{s}{\rightarrow} [xI(II)]$ and $M \stackrel{s}{\rightarrow} [x(II)I]$; instead, $M \stackrel{l}{\rightarrow} [xI(II)]$, $M \not \rightarrow [x(II)I]$.

• $(\stackrel{d}{\rightarrow} \subsetneq \stackrel{int}{\rightarrow})$ Let $M = (\lambda x.II)(II)$. Then $M \stackrel{int}{\rightarrow} (\lambda x.I)(II)$ and $M \stackrel{int}{\rightarrow} (\lambda x.II)I$, while $M \stackrel{d}{\rightarrow} (\lambda x.I)(II)$ and $M \not\stackrel{g}{\rightarrow} (\lambda x.II)I$

Intuitively, left reduction chooses the leftmost of the surface redexes. More precisely, this is the case for closed terms (for example, the term (xx)(II) has a $\stackrel{\$}{\rightarrow}$ -step, but no $\stackrel{|}{\rightarrow}$ -step).

Surface Normal Forms: We denote by \mathcal{S}^{cbv} the set of $\stackrel{\$}{\rightarrow}$ normal forms. We observe that all values are surface normal forms (but the converse does not hold): $\mathcal{V} \subsetneq \mathcal{S}^{\text{cbv}}$ (and $\mathcal{N} \subsetneq \mathcal{S}^{\text{cbv}}$). The situation is different if we restrict ourselves to close term, in fact the following result holds, which is easy to check.

Lemma 11. If M is a closed term, the following three are equivalent:

- 1) M is $a \stackrel{s}{\rightarrow}$ -normal form;
- 2) M is a $\xrightarrow{\prime}$ -normal form;
- 3) M is a value.

2) Finitary Surface Standardization: The next theorem proves a standardization result, in the sense that every finite reduction sequence can be (partially) ordered in a sequence of surface steps followed by a sequence of deep steps.

Theorem 12 (Finitary Surface Standardization). In $\Lambda_{\oplus}^{\text{cbv}}$, if $m \Rightarrow^* n$ then exists r such that $m \stackrel{\$}{\Rightarrow}^* r$ and $r \stackrel{d}{\Rightarrow}^* n$.

Proof. We build on an analogous result for CbV λ -calculus, which is folklore and is proved explicitly in Appendix V-B. We then only need to check that deep steps commute with \oplus -steps, which is straightforward technology (the full proof is in Appendix V-B).

a) Finitary Left Standardization does not hold: The following statement is false for $\Lambda_{\oplus}^{\text{cbv}}$.

"If $m \Rightarrow^* n$ then there exists r such that $m \stackrel{\downarrow}{\Rightarrow}^* r$ and $r \stackrel{int}{\Rightarrow}^* n$."

Example 13 (Counter-example). Let us consider the following sequence, where $I = \lambda x.x$ and $M = (II)((\lambda x.y \oplus z)I)$. $[M] \stackrel{\text{int}}{\Rightarrow} [(II)(y \oplus z)] \Rightarrow_{\oplus} [\frac{1}{2}(II)y, \frac{1}{2}(II)z] \Rightarrow_{\beta_v} [\frac{1}{2}Iy, \frac{1}{2}(II)z]$. If we anticipate the reduction of (II), we have $M \stackrel{\leftarrow}{\to}_{\beta_v} [I((\lambda x.y \oplus z)I)]$, from where we cannot reach $[\frac{1}{2}Iy, \frac{1}{2}(II)z]$. Observe that the sequence is already surface-standard!

VI. ASYMPTOTIC EVALUATION

The specificity of probabilistic computation is to be concerned with asymptotic behavior; the focus is not what happens after a finite number n of steps, but when n tends to infinity. In this section, we study the asymptotic behavior of \Rightarrow -sequences with respect to evaluation. The intuition is that a reduction sequence defines a distribution on the possible outcomes of the program. We first clarify what is the outcome of evaluating a probabilistic term, and then we formalize the idea of result "at the limit" with the notion of *limit distribution* (Def. 18). In Sec. VI-B we investigate how the asymptotic result of different sequences starting from the same m compare.

We recall that to each multidistribution m on Λ_{\oplus} is associated a probability distribution $\mu \in DST(\Lambda_{\oplus})$ (see Sec.III-C). We use the following **letter convention**: given a multidistribution m,n,r,... we denote the associated distribution by the corresponding Greek letter $\mu,\nu,\rho,...$ If $\langle m_n \rangle_{n\in\mathbb{N}}$ is a \Rightarrow -sequence, then $\langle \mu_n \rangle_{n\in\mathbb{N}}$ is the sequence of associated distributions.

A. Probabilistic Evaluation

We start by studying the property of being valuable (VI-A1) and by analyzing some examples (VI-A2). This motivates the more general approach we introduce in VI-A3.

1) To be valuable: In the CbV λ -calculus, the key property of a term M is to be valuable, i.e., M can reduce to a value. To be valuable is a yes/no property, whose probabilistic analogous is the probability to reduce to a value. If m describes the result of a computation step, the probability that such a result is a value is simply $\mu(\mathcal{V}) := \sum_{V \in \mathcal{V}} \mu(V)$, i.e. the probability of the event $\mathcal{V} \subset \Lambda_{\oplus}$. Since the set of values is closed under reduction, the following property holds:

Fact 14. If $V \in \mathcal{V}$ and $V \to m$, then m = [W], with $W \in \mathcal{V}$, and $V \to_{\beta_v} [W]$.

Let $\langle \mathbf{m}_n \rangle_{n \in \mathbb{N}}$ be a \Rightarrow -sequence, and $\langle \mu_n \rangle_{n \in \mathbb{N}}$ the sequence of associated distributions. The sequence of reals $\langle \mu_n(\mathcal{V}) \rangle_{n \in \mathbb{N}}$ is nondecreasing and bounded, because of Fact 14. Therefore the limit exists, and is the supremum: $\lim_{n \to \infty} \mu_n(\mathcal{V}) = \sup_n \{\mu_n(\mathcal{V})\}$. This fact allows us the following definition.

- The sequence $\langle \mathbf{m}_n \rangle_{n \in \mathbb{N}}$ evaluates with probability p if $p = \sup_n \mu_n(\mathcal{V})$, written $\langle \mathbf{m}_n \rangle_{n \in \mathbb{N}} \Rightarrow^{\infty} p$.
- m is *p-valuable* if p is the greatest probability to which a sequence from m can evaluate.

Example 15. Let $T = \lambda xy.x$ and $F = \lambda xy.y$.

- 1) Consider the term PP where $P=(\lambda x.(xx\oplus T))$. Then $PP\to [(PP)\oplus T]\Rightarrow [\frac{1}{2}PP,\frac{1}{2}T]\Rightarrow^{2n}[\frac{1}{2^n}PP,\frac{1}{2}T,...,\frac{1}{2^n}T]$. Since $\lim_{n\to\infty}\sum_1^n\frac{1}{2^n}=1$, PP is 1-valuable.

 2) Consider the term QQ, where $Q=\lambda x.(xx\oplus (T\oplus F))$.
- 2) Consider the term QQ, where $Q = \lambda x.(xx \oplus (T \oplus F))$. Then $QQ \to_{\beta_v} [(QQ) \oplus (T \oplus F)] \Rightarrow^* [\frac{1}{2}QQ, \frac{1}{4}T, \frac{1}{4}F] \Rightarrow^* ...$ It is immediate that QQ is 1-valuable.
- 3) Let $\Delta = \lambda x.xx$, so that $\Delta\Delta$ is a divergent term, and let $N = \lambda x.(xx) \oplus (T \oplus (\Delta\Delta))$. Then $NN \to_{\beta_v} [(NN) \oplus (T \oplus (\Delta\Delta))] \Rightarrow^* [\frac{1}{2}NN, \frac{1}{4}T, \frac{1}{4}(\Delta\Delta)] \Rightarrow^* ... NN$ is $\frac{1}{2}$ -valuable.
- 2) Result of a CbV computation: The notion of being p-valuable allows for a simple definition, but it is too coarse. Consider Example 15; both 1) and 2) give examples of 1-valuable term. However, in 1) the probability is concentrated in the value T, while in 2) T and F have equal probability $\frac{1}{2}$. Observe that T and F are different normal forms, and are not β_v -equivalent. To discriminate between T and F, we need a finer notion of evaluation. Since the calculus is CbV, the result "at the limit" is intuitively a distribution on the possible values that the term can reach. Some care is needed though, as the following example shows.

Example 16. Consider Plotkin's CbV λ -calculus. Let $\omega_3 = \lambda x.xxx$; the term $M = (\lambda x.x)\lambda x.\omega_3\omega_3$ has the following \rightarrow_{β_v} -reduction: $M = (\lambda x.x)(\lambda x.\omega_3\omega_3) \rightarrow_{\beta_v} M_1 = \lambda x.\omega_3\omega_3 \rightarrow_{\beta_v} M_2 = \lambda x.\omega_3\omega_3\omega_3 \rightarrow_{\beta_v} \cdots$. We obtain a reduction sequence where $\forall n \geq 1$, $M_n = \lambda x.\omega_3\underline{\omega_3...\omega_3}$. Each M_i is a

value, but there is not a "final" one in which the reduction ends. Transposing this to $\Lambda_{\oplus}^{\text{cbv}}$, let $\mathtt{m}_0 = [M]$, $\mathtt{m}_i = [M_i]$. The \Rightarrow -sequence $\langle \mathtt{m}_n \rangle_{n \in \mathbb{N}}$ is 1-valuable, but the distribution on values is different at every step. In other words, $\forall V \in \mathcal{V}$, the sequence $\langle \mu_n(V) \rangle$ has limit 0. Observe that however all the values M_i are β_v -equivalent.

3) Observations and Limit Distribution: Example 16 motivates the approach that we develop now: the result of probabilistic evaluation is not a distribution on values, but a distribution on some events of interest. In the case of $\Lambda_{\oplus}^{\text{cbv}}$, the most informative events are equivalence classes of values.

We first introduce the notion of observation, and then that of limit distribution.

Definition 17. A set of *observations* for $(\Lambda_{\oplus}, \Rightarrow)$ is a set $0bs \subseteq \mathcal{P}(\Lambda_{\oplus})$ such that $\forall \mathbf{U}, \mathbf{Z} \in 0bs$, if $\mathbf{U} \neq \mathbf{Z}$ then $\mathbf{U} \cap \mathbf{Z} = \emptyset$, and if $m \Rightarrow m'$ then $\mu(\mathbf{U}) \leq \mu'(\mathbf{U})$.

Note that, given $\mu \in DST(\Lambda_{\oplus})$, $\mathbf{U} \in Obs$ has probability $\mu(\mathbf{U})$ (similarly to the event "the result is Odd" in Example 2).

It follows immediately from the definition that, given a sequence $\langle \mathbf{m}_n \rangle_{n \in \mathbb{N}}$, then for each $\mathbf{U} \in \mathtt{Obs}$ the sequence $\langle \mu_n(\mathbf{U}) \rangle_{n \in \mathbb{N}}$ is nondecreasing and bounded, and therefore has a limit, the sup. Moreover, monotony implies the following

$$\sup_{n} \{ \sum_{\mathbf{U} \in 0bs} \mu_n(\mathbf{U}) \} = \sum_{\mathbf{U} \in 0bs} \sup_{n} \{ \mu_n(\mathbf{U}) \}.$$
 (1)

which guarantees that the distribution ρ in Def. 18 is well defined, because $\sup_n \|\mu_n\| \le 1$ and (1) gives $\sup_n \|\mu_n\| = \|\rho\|$.

Definition 18. Let 0bs be a set of observations. The sequence $(m_n)_{n\in\mathbb{N}}$ defines a distribution $\rho \in DST(0bs)$, where $\forall U \in 0bs$,

$$\rho(\mathbf{U}) := \sup_{n} \{ \mu_n(\mathbf{U}) \}.$$

- We call such a ρ the *limit distribution* of $\langle m_n \rangle_{n \in \mathbb{N}}$. Letter convention: greek bold letters denote limit distributions.
- The sequence $\langle m_n \rangle_{n \in \mathbb{N}}$ converges to (or evaluates to) the limit distribution $\boldsymbol{\rho}$, written

$$\langle \mathbf{m}_n \rangle_{n \in \mathbb{N}} \Downarrow_{\mathtt{Obs}} \boldsymbol{\rho}.$$

• If m has a sequence which converges to ρ , we write $\underset{n}{\mathbb{m}} \Longrightarrow_{nbs} \rho$.

• Given m, we denote by $\text{Lim}_{\text{Obs}}(m)$ the set $\{ \rho \mid m \Longrightarrow_{\text{Obs}}^{\infty} \rho \}$ of all limit distributions of m. If $\text{Lim}_{\text{Obs}}(m)$ has a greatest element, we indicate it by $[\![m]\!]_{\text{Obs}}$.

If Obs is clear from the context, we omit the index which specifies it, and simply write $\langle m_n \rangle_{n \in \mathbb{N}} \psi \rho$, $m \Rightarrow^{\infty} \rho$, Lim(m).

The notion of limit distribution formalizes what is the *result* of evaluating a probabilistic term, once we choose the set Obs of observations which interest us. In VI-B we prove that confluence implies that Lim(m) has a unique maximal element.

a) Sets of Observations for $\Lambda_{\oplus}^{\mathrm{cbv}}$: Let us consider two partitions of the set $\mathcal{V} \subset \Lambda_{\oplus}$, the trivial one $\{\mathcal{V}\}$, and the set \mathcal{V}_{\sim} of values up to the equivalence $=_{\beta_v}$, i.e. the collection of all events $\{W \in \mathcal{V} \mid W =_{\beta_v} V\}$. For the set \mathcal{N} of \rightarrow -normal forms (see IV-A2e), interesting partitions are $\{\mathcal{N}\}$ and the set of singletons $\mathcal{N}_{\{\}} := \{\{M\}, M \in \mathcal{N}\}$.

Proposition 19. $\{V\}$, V_{\sim} , $\{\mathcal{N}\}$ and $\mathcal{N}_{\{\}}$ are each a set of observations for $(\Lambda_{\oplus}, \Rightarrow)$.

Proof. Clearly, any partition of \mathcal{N} satisfies the conditions in Def. 17. For $\{\mathcal{V}\}$ and \mathcal{V}_{\sim} , the result follows from Fact 14. \square

Notice that convergence w.r.t. $\{\mathcal{V}\}$ corresponds to the notion of being p-valuable. Instead $\{\mathcal{N}\}$ and $\mathcal{N}_{\{\}}$ correspond to normalization and reaching a specific normal form, respectively; however these are events which are *not significant* in a CbV perspective, as we already discussed in V-A1. For this reason, in Sec. VII we will focus on the study of $0bs := \mathcal{V}_{\sim}$ (Sec. VII).

Example 20. • Let Obs be either V_{\sim} or $\mathcal{N}_{\{\}}$.

- 1) Let $\langle \mathfrak{m}_n \rangle_{n \in \mathbb{N}}$ be the sequence in Example 15.1, starting from [PP]. Then $\langle \mathfrak{m}_n \rangle_{n \in \mathbb{N}} \downarrow_{\text{Obs}} \{\mathbf{T}^1\}$.
- 2) Let $\langle \mathbf{m}_n \rangle_{n \in \mathbb{N}}$ be the computation in Example 15.2, starting from [QQ]. Then $\langle \mathbf{m}_n \rangle_{n \in \mathbb{N}} \Downarrow_{\text{obs}} \{\mathbf{T}^{\frac{1}{2}}, \mathbf{F}^{\frac{1}{2}}\}.$
- 3) Let $\langle m_n \rangle_{n \in \mathbb{N}}$ be the computation in Example 15.3, starting from [NN]. Then $\langle m_n \rangle_{n \in \mathbb{N}} \downarrow_{\text{obs}} \{\mathbf{T}^{\frac{1}{2}}\}$.
- Let $\langle m_n \rangle_{n \in \mathbb{N}}$ be the reduction sequence in Example 16, starting with $[(\lambda x.x)\lambda x.\omega_3\omega_3]$. By taking as set of observations \mathcal{V}_{\sim} , the sequence converges to $\{\lambda x.\omega_3\omega_3^1\}$.
- b) Discussion: Each observation expresses a result of interest for the evaluation of the term M. To better understand this, let us examine what become our notions of observation in the case of usual (non-probabilistic) CbV λ -calculus. Let $M \to^* N \in \mathbf{U}$ and $\mathbf{U} \in \mathsf{Obs}$; if $\mathbf{U} \in \{\mathcal{V}\}$ then M is valuable, if $\mathbf{U} \in \mathcal{V}_\sim$, then M reduces to the value N up to β_v -equivalence,

if $\mathbf{U} \in \{\mathcal{N}\}$, then M normalizes, finally $\mathbf{U} = \{N\} \in \mathcal{N}_{\{\}}$ means that M has normal form N. We say that $\mathbf{U} \in \mathsf{Obs}$ is a result of evaluating M, if $M \to^* N \in \mathbf{U}$. Clearly, fixed Obs , confluence implies that the result of evaluating M, if any, is unique.

c) Sets of observations for Surface Reduction: It is interesting to examine the set of observations for surface reduction $\stackrel{s}{\Rightarrow}$. When considering $\stackrel{s}{\rightarrow}$, values are $\stackrel{s}{\rightarrow}$ -normal forms (the converse does not hold!). Therefore $\{\{V\} \mid V \in \mathcal{V}\}$ (where $\{V\}$ is a singleton) is a set of observations for $(\Lambda_{\oplus}, \stackrel{s}{\Rightarrow})$. In other words, when restricting oneself to surface reduction, the result of a probabilistic computation (i.e. the limit distribution) is a distribution on the possible values of the term. Observe that all set of observations for \Rightarrow (Prop. 19) are also set of observations for $\stackrel{s}{\Rightarrow}$.

B. Uniqueness and Adequacy of the Evaluation

In this section, we adapt similar results from [15], to which we refer for details. We assume a set 0bs to be fixed, hence we omit the index. For concreteness, think of \mathcal{V}_{\sim} , but the results only depend on the properties in Def. 17, and on confluence.

How do different reduction sequences from the same initial m compare? More precisely, assume m $\Rightarrow^{\infty} \rho$ and m $\Rightarrow^{\infty} \mu$, how do ρ and μ compare? Intuitively, the limit distributions of m (which are the result of a *probabilistically terminating* sequence) play the role of normal forms in finitary termination. As confluence implies uniqueness of normal forms, a similar property holds when considering *probabilistic termination* and limits, in the sense that each m has a *unique maximal limit distribution* (Thm. 22). While the property is similar, the proof is not as immediate as in the finitary case. The key result is Lemma 21 which implies both that Lim(m) has a greatest element (Thm. 22), and adequacy of the evaluation (Thm. 23).

Recall that the order \leq on distributions is defined pointwise (Sec. III-A).

Lemma 21 (Main Lemma). $\Lambda_{\oplus}^{\text{cbv}}$ has the following property: $\forall \mathtt{m}, \mathtt{s}$, if $\mu \in \text{Lim}(\mathtt{m})$, and $\mathtt{m} \Rightarrow^* \mathtt{s}$, then $\mathtt{s} \Rightarrow^{\infty} \sigma$ with $\mu \leq \sigma$. Moreover, if μ is maximal in $\text{Lim}(\mathtt{m})$ then $\sigma = \mu$.

Proof. Let $\mu \in \text{Lim}(m)$, and $\langle m_n \rangle_{n \in \mathbb{N}}$ be a sequence from $m = m_0$ which converges to μ . Assume $m \Rightarrow^* s$. As illustrated in Fig. 4, from s we build a sequence $s = s_{m_0} \Rightarrow^* s_{m_1} \Rightarrow^* s_{m_2}...$, where each segment $s_{m_i} \Rightarrow^* s_{m_{i+1}}$ ($i \geq 0$) is given by confluence from $m_i \Rightarrow^* s_{m_i}$ and $m_i \Rightarrow m_{i+1}$. Let $\langle s_n \rangle_{n \in \mathbb{N}}$ be the concatenation of all such segments and let σ be its limit distribution. Clearly, $\sigma \in \text{Lim}(m)$. Since by construction $m_i \Rightarrow^* s_{m_i}$, then for each $\mathbf{V} \in \text{Obs}$, $\mu_i(\mathbf{V}) \leq \sigma(\mathbf{V})$ (because $\mu_i(\mathbf{V}) \leq \sigma_{m_i}(\mathbf{V})$ by definition of observation). Therefore $\sup_n \{\mu_n(\mathbf{V})\} = \mu(\mathbf{V}) \leq \sigma(\mathbf{V})$. If μ is maximal, then $\sigma = \mu$.

Theorem 22 (Greatest Limit Distribution). Lim(m) has a greatest element, which we indicate by [m].

Proof. The proof of both existence and uniqueness of maximal elements relies on Lemma 21. Let us explicitly show uniqueness. Let $\mu \in \text{Lim}(m)$ be maximal. Given any $\rho \in \text{Lim}(m)$, we prove that $\rho \leq \mu$. Let $\langle \mathbf{r}_n \rangle_{n \in \mathbb{N}}$ be a sequence from m such that $\langle \mathbf{r}_n \rangle_{n \in \mathbb{N}} \Downarrow \rho$. By Lemma 21, $\forall \mathbf{r}_n$ there is a \Rightarrow -sequence from

 \mathbf{r}_n which has limit $\boldsymbol{\mu}$. Therefore $\forall \mathbf{V} \in \mathcal{V}, \forall n, \rho_n(\mathbf{V}) \leq \boldsymbol{\mu}(\mathbf{V})$, hence $\boldsymbol{\rho}(\mathbf{V}) \leq \boldsymbol{\mu}(\mathbf{V})$. If $\boldsymbol{\rho}$ is maximal, $\boldsymbol{\rho} = \boldsymbol{\mu}$.

Theorem 23 (Adequacy of evaluation). *If* $m \Rightarrow^* s$, *then* $[\![m]\!] = [\![s]\!]$.

Proof. Observe first that $[\![s]\!] \in \text{Lim}(m)$, hence $[\![s]\!] \leq [\![m]\!]$. Indeed, if $\langle s_n \rangle_{n \in \mathbb{N}} \Downarrow [\![s]\!]$, by concatenanting $m \Rightarrow^* s$ with $\langle s_n \rangle_{n \in \mathbb{N}}$, we have $m \Rightarrow^* [\![s]\!]$. By Lemma 21, it holds that $[\![m]\!] \in \text{Lim}(s)$, hence $[\![m]\!] \leq [\![s]\!]$. Therefore $[\![m]\!] = [\![s]\!]$.

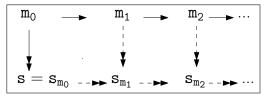


Figure 4: Proof of Main Lemma

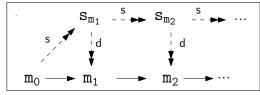


Figure 5: Surface evaluation

VII. ASYMPTOTIC STANDARDIZATION

In this section, we focus on V_{\sim} as set of observations, which is the most natural choice in a CbV setting, in particular if we want to evaluate *programs*, i.e., closed terms.

We proved, in Thm. 22, that each m has a unique maximal limit distribution [m]. Now we address the question: is there a reduction strategy which is guaranteed to converge to [m]? We show that surface evaluation provides such a strategy; indeed, any limit distribution in Lim(m) can be reached by surface evaluation (Thm. 26). This result of asymptotic completeness is the main technical contribution of the section.

Following the notation introduced in VI-A3, we denote by **V** the set $\{W \in \mathcal{V} | W =_{\beta_n} V\}$. We observe that:

Fact 24. Let $M \stackrel{d}{\to} m$, then m has form [P] and $M =_{\beta_v} P$; M is a value if and only if P is a value.

As a consequence of the previous fact, we have

Lemma 25. If $m \stackrel{d}{\Rightarrow} s$ then $\mu(V) = \sigma(V)$, and $\mu(V) = \sigma(V)$, for each $V \in \mathcal{V}_{\sim}$.

We write $\mathtt{m} \stackrel{s}{\Rightarrow} \boldsymbol{\mu}$ (resp. $\mathtt{m} \stackrel{l}{\Rightarrow} \boldsymbol{\mu}$) if there is a sequence $(\mathtt{m}_n)_{n \in \mathbb{N}}$ such that all steps $\mathtt{m}_i \Rightarrow \mathtt{m}_{i+1}$ are surface (resp. left) reductions and $(\mathtt{m}_n)_{n \in \mathbb{N}} \Downarrow \boldsymbol{\mu}$. Remember that given \mathtt{m} , we write $[\![\mathtt{m}]\!]$ for the unique maximal element of $\mathtt{Lim}(\mathtt{m})$, and $\mathtt{m} \stackrel{\Rightarrow}{\Rightarrow} \boldsymbol{\mu}$ if there *exists* $\mathtt{a} \Rightarrow$ -sequence from \mathtt{m} which converges to $\boldsymbol{\mu}$.

We now prove asymptotic completeness for surface evaluation. We exploit finitary standardization (Thm. 12) and *extend* it to the limit. In the proof, it is essential the fact that $\stackrel{d}{\Rightarrow}$ -steps preserve the distributions (Lemma 25).

Theorem 26 (Asymptotic Completeness of Surface Reduction). $m \Rightarrow^{\infty} \mu$ if and only if $m \Rightarrow^{\infty} \mu$.

Proof. We prove that $\mathbf{m} \Rightarrow \boldsymbol{\mu}$ implies $\mathbf{m} \stackrel{\$}{\Rightarrow} \boldsymbol{\mu}$ (the other direction holds by definition). Assume $\langle \mathbf{m}_n \rangle_{n \in \mathbb{N}} \Downarrow \boldsymbol{\mu}$, with $\mathbf{m} = \mathbf{m}_0$. As illustrated in Fig. 5, we build a sequence $\langle \mathbf{s}_{\mathbf{m}_n} \rangle$ such that $\mathbf{m}_0 = \mathbf{s}_{\mathbf{m}_0}$ and $\forall i$ ($\mathbf{s}_{\mathbf{m}_i} \stackrel{\$}{\Rightarrow}^* \mathbf{s}_{\mathbf{m}_{i+1}}$ and $\mathbf{s}_{\mathbf{m}_{i+1}} \stackrel{\$}{\Rightarrow}^* \mathbf{m}_{i+1}$). If i=0, by Thm. 12 it exists $\mathbf{s}_{\mathbf{m}_1}$ such that $\mathbf{m}_0 = \mathbf{s}_{\mathbf{m}_0} \stackrel{\$}{\Rightarrow}^* \mathbf{s}_{\mathbf{m}_1} \stackrel{\$}{\Rightarrow}^* \mathbf{m}_1$. We then procede by induction: for each i>0, we apply Thm. 12 to the sequence $\mathbf{s}_{\mathbf{m}_i} \stackrel{\$}{\Rightarrow}^* \mathbf{m}_i \Rightarrow \mathbf{m}_{i+1}$, and obtain the multidistribution $\mathbf{s}_{\mathbf{m}_{i+1}}$ such that $\mathbf{s}_{\mathbf{m}_i} \stackrel{\$}{\Rightarrow}^* \mathbf{s}_{\mathbf{m}_{i+1}}$, and $\mathbf{s}_{\mathbf{m}_{i+1}} \stackrel{\$}{\Rightarrow}^* \mathbf{m}_{i+1}$, as wanted. The concatenation of all segments $\mathbf{s}_{\mathbf{m}_0} \stackrel{\$}{\Rightarrow}^* \mathbf{s}_{\mathbf{m}_1}, \dots, \mathbf{s}_{\mathbf{m}_i} \stackrel{\$}{\Rightarrow}^* \mathbf{s}_{\mathbf{m}_{i+1}}, \dots$ is a $\stackrel{\$}{\Rightarrow}$ -sequence. Let $\boldsymbol{\sigma}$ be its limit. By Lemma 25 and the fact that $\mathbf{s}_{\mathbf{m}_i} \stackrel{\$}{\Rightarrow}^* \mathbf{m}_i$, we have $\sigma_{\mathbf{m}_i}(\mathbf{V}) = \mu_i(\mathbf{V})$, for each $\mathbf{V} \in \mathcal{V}_{\sim}$. We conclude $\boldsymbol{\sigma} = \boldsymbol{\mu}$ because $\forall i$:

- 1) $\sigma_{\mathbf{m}_i}(\mathbf{V}) = \mu_i(\mathbf{V}) \leq \boldsymbol{\mu}(\mathbf{V})$, therefore $\boldsymbol{\sigma}(\mathbf{V}) \leq \boldsymbol{\mu}(\mathbf{V})$.
- 2) $\mu_i(\mathbf{V}) = \sigma_{m_i}(\mathbf{V}) \le \sigma(\mathbf{V})$, therefore $\mu(\mathbf{V}) \le \sigma(\mathbf{V})$.

Remark 27. We observe that completeness of surface evaluation (Thm. 26) is specific to convergence w.r.t. V_{\sim} and $\{V\}$ (the most natural set of observations in CbV). Surface evaluation is not necessarily complete if we evaluate w.r.t. other sets of observations, such as normal forms, where deep steps may be needed. Consider, for example, the term $\lambda z.II \xrightarrow{d} \lambda z.I$. To define a complete strategy w.r.t. $\mathcal{N}_{\{\}}$ demands to refine the approach.

A. Surface and Left Evaluation

We are now equipped to tackle the goal of this section, namely the existence of a strategy to find the greatest limit distribution of a program.

Since our aim is to reach the greatest limit, it makes sense to reduce "whenever is possible", and use the full lifting \Rightarrow (Def. IV-A3). The reason is easy to see. Consider for example $\mathbf{m} = \left[\frac{1}{2}\Delta\Delta, \frac{1}{2}II\right]$, which has greatest limit $[\![\mathbf{m}]\!] = \{\mathbf{I}^{\frac{1}{2}}\}$. We observe that a \Rightarrow -sequence from \mathbf{m} may very well keep reducing only the diverging term $\Delta\Delta$ and never reach $[\![\mathbf{m}]\!]$. The reduction \Rightarrow , instead, forces the reduction of each term which is not in normal form for \rightarrow .

Lemma 28. Let ρ be maximal among the limit distribution of all \rightrightarrows -sequences from m. Then $\rho = [m]$.

Proof. Obviously, $\rho \in \text{Lim}(m)$. It is straightforward to check that if μ is the limit of a \Rightarrow -sequence, then there is a \Rightarrow -sequence, whose limit is greater or equal to μ .

We write $\stackrel{\mathfrak{s}}{\rightrightarrows}$ (resp. $\stackrel{\perp}{\rightrightarrows}$) for the full lifting of $\stackrel{\mathfrak{s}}{\Longrightarrow}$ (resp. $\stackrel{\perp}{\hookrightarrow}$). Observe that given \mathfrak{m} , there is only one $\stackrel{\perp}{\rightrightarrows}$ -sequence. We use the letters $\mathfrak{l}=\langle l_n\rangle_{n\in\mathbb{N}},\ \mathfrak{s}=\langle s_n\rangle_{n\in\mathbb{N}},\ \mathfrak{t}=\langle t_n\rangle_{n\in\mathbb{N}}$ to indicate (infinite) reduction sequences. We say that \mathfrak{m} is *closed* if it is a multidistribution on closed terms $i.e.\ \mathfrak{m}=[p_iM_i\,|\,i\!\in\!I]$ with M_i closed $\forall i\!\in\!I$.

Proposition 29 (Left Evaluation). Let m be closed.

1) Let $\mathfrak{s},\mathfrak{t}$ be $\stackrel{s}{\Longrightarrow}$ -sequences from \mathfrak{m} ; $\mathfrak{s} \Downarrow \boldsymbol{\mu}$ if and only if $\mathfrak{t} \Downarrow \boldsymbol{\mu}$.

2) Let \mathfrak{s} be any $\stackrel{\mathfrak{s}}{\Rightarrow}$ -sequence from \mathfrak{m} , and \mathfrak{l} the $\stackrel{\mathfrak{l}}{\Rightarrow}$ -sequences from \mathfrak{m} . Then $\mathfrak{s} \ \psi \ \mu$ if and only if $\mathfrak{l} \ \psi \ \mu$.

Proof. [15], Sec.6, studies a CbV probabilistic λ -calculus with surface reduction $(\Lambda_{\oplus}^{\text{weak}})$ and proves using a diamond property that if $\mathbf{m} \stackrel{\mathtt{s}}{=} {}^k \mathbf{m}_k$ and $\mathbf{m} \stackrel{\mathtt{s}}{=} {}^k \mathbf{r}_k$ (both sequence have k steps) then $\forall V \in \mathcal{V}, \ \mu_k(V) = \rho_k(V)$. Hence claim (1.) follows.

Claim (2.) follows from (1.) and from Lemma 11, which implies that if $M \stackrel{s}{\to} n$ is *closed*, we can always choose a surface step which is a $\stackrel{1}{\to}$ -step.

Putting all elements together, we have proved that the limit distribution of $any \stackrel{s}{\Longrightarrow}$ -sequence from m is [m]. In particular, [m] is also the limit distribution of the $\stackrel{l}{\Longrightarrow}$ -sequence from m.

Theorem 30. For m closed, the following hold.

- 1) Let \mathfrak{s} be any $\stackrel{\mathfrak{s}}{\Longrightarrow}$ -sequence from \mathfrak{m} . Then $\mathfrak{s} \Downarrow \llbracket \mathfrak{m} \rrbracket$.
- 2) Let \mathfrak{l} be the $\stackrel{\prime}{\Longrightarrow}$ -sequence from \mathfrak{m} . Then $\mathfrak{l} \Downarrow \llbracket \mathfrak{m} \rrbracket$.
- 3) The sets $\{\rho \mid m \Rightarrow^{\infty} \rho\}, \{\rho \mid m \Rightarrow^{\infty} \rho\}, \text{ and } \{\rho \mid m \Rightarrow^{\infty} \rho\}$ have the same greatest element, which is $[\![m]\!]$.

While left reduction is not standard for finite sequences (as Example 13 shows), still is able to reach $[\![m]\!]$, if we only evaluate programs, i.e., closed terms. Thm. 30 justifies (a posteriori!) the use of the leftmost-outermost strategy in the literature of probabilistic λ -calculus: left evaluation actually produces the best asymptotic result. However, it is not the only strategy to achieve this: any $\stackrel{\hspace{0.1em}\mathsf{\scriptscriptstyle\$}}{=}$ -sequence will.

VIII. SUMMING-UP AND OVERVIEW

The definition of reduction in $\Lambda_{\oplus}^{\text{cbv}}$ is based on two components: the β_v -rule and the \oplus -rule. We stress that only the \oplus -step is constrained, while β_v is inherited "as is" from the λ -calculus. The β_v -rule is allowed in *all* contexts, while the \oplus -rule is disabled in a function body. This avoids confusion between duplicating a function which performs a choice, and duplicating the choice, that is the core of confluence failure. It is then natural to expect that the fine control on duplication which is offered by linear logic could be beneficial.

In Sec. IX we apply the methods and tools which we have developed to study $\Lambda_\oplus^{\text{cbv}}$ to define a probabilistic linear calculus $\Lambda_\oplus^!$ which extends with a probabilistic choice Simpson's linear λ -calculus [30]. This is a result of interest in its own, but also evidence that our approach is robust, as it transfers well to other probabilistic calculi. In Sec. X we then define a call-by-name probabilistic calculus, $\Lambda_\oplus^{\text{cbn}}$, and we show that similar results to the ones we have established for $\Lambda_\oplus^{\text{cbv}}$ hold.

As we will see, the three calculi follow the same pattern: the \oplus -reduction (and *only this* reduction) is restricted to surface contexts. In Sec. XI we discuss how the three calculi relate.

IX. PROBABILISTIC LINEAR LAMBDA CALCULUS

 $\Lambda^!$ [30] is an untyped linear λ -calculus which is closely based on linear logic. Abstraction is refined into linear abstraction $\lambda x.M$ and non-linear abstraction $\lambda!x.M$, which allows duplication of the argument. The argument of $\lambda!x.M$ is required to be suspended as thunk !N, that corresponds to the !-box of linear logic. In this section, we define a probabilistic

linear λ -calculus $\Lambda^!_\oplus$ by extending $\Lambda^!$ with an operator \oplus . We demand that probabilistic choice is not reduced under the scope of a ! operator, while the β -reduction is unrestricted. We show that this suffices to preserve confluence; we then study the properties of the calculus.

A. Syntax of $\Lambda^!_{\oplus}$

1) The language: Raw terms M,N,... are built up from a countable set of variables x,y,... according to the grammar:

$$M ::= x \mid !M \mid \lambda x.M \mid \lambda !x.M \mid MM \mid M \oplus N \text{ (terms } \Lambda^!_{\oplus})$$

We say that x is affine (resp. linear) in M if x occurs free at most (resp. exactly) once in M, and moreover, the free occurrence of x does not lie within the scope of a ! operator. A term M is affine (resp. linear) if for every subterm $\lambda x.P$ of M, x is so in P. Henceforth, we consider affine terms only.

It is immediate to observe that if M is affine (linear) and $M \rightarrow N$, then N is affine (linear).

Contexts (C) and surface contexts (S) are generated by the grammars:

where \Box denotes the *hole* of the term context. Observe that a *surface* context is defined in a different way than in IV-A. Here it expresses the fact that a *surface redex cannot occur in the scope of a*! *operator* (nor in the scope of a \oplus).

2) Reductions: We follow the same pattern as for $\Lambda_{\oplus}^{\text{cbv}}$. The beta rules $\mapsto_{l\oplus}$, are given in Fig. 7. The probabilistic rules $\mapsto_{l\oplus}$, $\mapsto_{r\oplus}$ are as in Fig. 1. The reduction steps are in Fig. 6: the β -rule is closed under general context, while the \oplus -rules are closed under surface contexts. The β -rules also can be restricted to the closure under surface contexts, as shown in Fig. 6. A \rightarrow -step is deep (written $\stackrel{d}{\rightarrow}$) if it is not surface. The lifting of the relation \rightarrow : $\Lambda_{\oplus}^{!} \times \text{MDST}(\Lambda_{\oplus}^{!})$ to a binary relation on $\Rightarrow \text{MDST}(\Lambda_{\oplus}^{!})$ is defined as in Fig. 3.

Beta Step \rightarrow_{β}	Surface Beta Step $\overset{\mathtt{s}}{\rightarrow}_{\beta}$
$M \mapsto_{\beta} M'$	$M \mapsto_{\beta} M'$
$\overline{\mathbf{C}(M) \to_{\beta} [\mathbf{C}(M')]}$	$S(M) \xrightarrow{s}_{\beta} [S(M')]$
	$(Surface) \oplus -Step \rightarrow_{\oplus} := \xrightarrow{s}_{\oplus}$
	$M \oplus N \mapsto_{l \oplus} M M \oplus N \mapsto_{r \oplus} N$
	$S(M \oplus N) \xrightarrow{s} \oplus \left[\frac{1}{2}S(M), \frac{1}{2}S(N)\right]$
Reduction Step \rightarrow	Surface Reduction Step $\stackrel{s}{\rightarrow}$
$\rightarrow := \rightarrow_{\beta} \cup \rightarrow_{\oplus}$	$\stackrel{\mathtt{s}}{\rightarrow} := \stackrel{\mathtt{s}}{\rightarrow}_{\beta} \cup \rightarrow_{\oplus}$

Figure 6: Reduction Steps

$$(\lambda x.M)N \mapsto_{\beta} M[N/x] \qquad (\lambda!x.M)!N \mapsto_{\beta} M[N/x]$$

Figure 7: β reduction rules for $\Lambda^!_{\oplus}$

Remark 31. To limit notations for reductions and contexts, we use the same as for $\Lambda_{\oplus}^{\text{cbv}}$, clearly the meaning is different.

B. $\Lambda_{\oplus}^{!}$ is a conservative extension of $\Lambda^{!}$

As in IV-B, we denote by \rightarrow_{β} both the reduction in $\Lambda^!$ and the β reduction in $\Lambda^!_{\oplus}$; we prove that $(\Lambda^!_{\oplus}, \Rightarrow_{\beta})$ is a conservative extension of $(\Lambda^!, \rightarrow_{\beta})$.

Definition 32 (Translation). $(\cdot)_! : \Lambda^!_{\oplus} \to \Lambda^!$ is defined in the following way, where z is a fixed fresh variable

$$\begin{array}{lll} (x)_! &= x \\ (M \oplus N)_! &= z \; !(M)_! \; !(N)_! \\ (MN)_! &= (M)_! (N)_! \\ \end{array} \begin{array}{ll} (\lambda x.M)_! &= \lambda x.(M)_! \\ (\lambda !x.M)_! &= \lambda !x.(M)_! \\ (!M)_! &= !(M)_! \end{array}$$

Note that the translation of terms of the form $M \oplus N$ is designed so to preserves surface reduction.

Proposition 33 (Simulation). Let $M \in \Lambda^!_{\oplus}$.

- 1) $M \rightarrow_{\beta} [N]$ implies $(M)_! \rightarrow_{\beta} (N)_!$.
- 2) $(M)_! \to_{\beta} P$ implies that exists (unique) $N \in \Lambda_{\oplus}^!$, with $N = (P)_!$ and $M \to_{\beta} [N]$.
- 3) $M \xrightarrow{s}_{\beta} [N]$ implies $(M)_! \xrightarrow{s}_{\beta} (N)_!$.
- 4) $(M)_! \xrightarrow{s}_{\beta} P$ implies exists (unique) $N \in \Lambda^!_{\oplus}$, s.t. $N = (P)_!$ and $M \xrightarrow{s}_{\beta} [N]$.

The translation tells us that the reduction $[M] \Rightarrow_{\beta} [N]$ on $\Lambda^!_{\oplus}$ behaves as the reduction $(M)_! \rightarrow_{\beta} (N)_!$ on $\Lambda^!$.

C. Confluence and Finitary Standardization for $\Lambda^!_{\oplus}$ The following properties hold for $\Lambda^!$ [30].

Theorem (Simpson 05). The following hold in $\Lambda^!$.

- 1) **Confluence**. \rightarrow_{β} is confluent.
- 2) Surface Standardization. If $M \to_{\beta}^* N$ then exists R such that $M \xrightarrow{s}_{\beta}^* R$ and $R \xrightarrow{d}^* N$.

We show, using the methods developed for $\Lambda_{\oplus}^{\text{cbv}}$ and the translation in Def. 32, that the same properties hold for $\Lambda_{\oplus}^!$.

1) Confluence: We follow the same approach as in Sec. V-A. In fact, we already have most of the building blocks for the proof. Observe that Lemma 5 is general enough to apply also to binary relations on $MDST(\Lambda^!_{\oplus})$.

Lemma 34. 1) The reduction \Rightarrow_{\oplus} is diamond.

- 2) The reduction \Rightarrow_{β} is confluent.
- 3) The reductions \Rightarrow_{β} and \Rightarrow_{\oplus} commute.

Proof. The details of the proof are in Appendix C1. The proof of 1) and 2) is as for Lemmas 7 and 6; 3) is proved using Lemma 5, by induction on the term. \Box

By Hindley-Rosen Lemma, we obtain

Theorem 35. The reduction \Rightarrow of $\Lambda^!_{\oplus}$ is confluent.

2) Surface standardization:

Proposition 36 (Finitary Surface Standardization). In $\Lambda^!_{\oplus}$, if $m \Rightarrow^* n$ then exists r such that $m \stackrel{s}{\Rightarrow}^* r$ and $r \stackrel{d}{\Rightarrow}^* n$.

Proof. The proof is given in Appendix C2. \Box

D. Asymptotic behaviour

Normal forms are defined as in IV-A2e; we denote by $\mathcal{N}^!$ the set of \rightarrow -normal forms, and by $\mathcal{S}^!$ the set of the *surface normal forms* (i.e. the $\stackrel{s}{\rightarrow}$ -normal forms). Clearly $\mathcal{N}^! \subsetneq \mathcal{S}^!$. We define $\mathcal{N}^!_{\{\}} := \{\{M\}, M \in \mathcal{N}^!\}$, and $\mathcal{S}^!_{\sim}$ as the set of all events $\mathbf{R} := \{S \in \mathcal{S}^! \mid S =_{\beta} R\}$.

a) Observations: A set of observations for $(\Lambda_{\oplus}^!, \Rightarrow)$ is defined in the same way as that for $(\Lambda_{\oplus}, \Rightarrow)$ (Def. 17).

Proposition 37. Each of the following sets $\{\mathcal{N}^!\}$, $\{\mathcal{S}^!\}$, $\mathcal{N}^!_{\{\}}$, $\mathcal{S}^!_{\sim}$, is a set of observations for $(\Lambda^!_{\oplus}, \Rightarrow)$.

Theorem 38. For any choice of Obs, $\Lambda^!_{\oplus}$ has the properties:

- Lim(m) has a greatest element, which we indicate as [m].
- If $m \Rightarrow^* s$, then [m] = [s].

c) Asymptotic Standardization: For the rest of the section we focus on $0bs := \mathcal{S}^!_{\sim}$. Notice that if ρ is a limit distribution, $\rho \in \mathtt{MDST}(\mathcal{S}^!_{\sim})$. We have established that for each $\mathtt{m} \in \Lambda^!_{\oplus}$, $\mathtt{Lim}(\mathtt{m})$ has a unique maximal element $[\![\mathtt{m}]\!]$. We now want to have a strategy to find $[\![\mathtt{m}]\!]$. Surface reduction plays that role. We use the following fact, which is easy to verify.

Fact 39. Let $M \stackrel{d}{\rightarrow} n$. Then

- 1) n is of the form [N], and $M =_{\beta} N$;
- 2) $M \in S^!$ if and only if $N \in S^!$.

Theorem 40 (Asymptotic Completeness). In $\Lambda^!_{\oplus}$ it holds that $\mathtt{m} \Rightarrow^{\infty} \mu$ if and only if $\mathtt{m} \xrightarrow{s_{+}^{\infty}} \mu$.

Proof. As for Thm. 26, now using Fact 39 and Prop. 36. □

Similarly to Sec. VII, we can establish that any (infinitary) $\stackrel{\hspace{0.1em}\mathsf{\scriptscriptstyle\$}}{\Rightarrow}$ -sequences from m converges precisely to [m], where $\stackrel{\hspace{0.1em}\mathsf{\scriptscriptstyle\$}}{\Rightarrow}$ indicate the full lifting of the relation $\stackrel{\hspace{0.1em}\mathsf{\scriptscriptstyle\$}}{\Rightarrow}\subseteq\Lambda^!_{\scriptscriptstyle{\pitchfork}}\times \mathtt{MDST}(\Lambda^!_{\scriptscriptstyle{\pitchfork}})$.

Theorem 41 (Surface Evaluation). Let $\mathfrak{s} = \langle \mathfrak{s}_n \rangle_{n \in \mathbb{N}}$ be any $\stackrel{\mathfrak{s}}{\Longrightarrow}$ -sequences from \mathfrak{m} . It holds that $\mathfrak{s} \downarrow \llbracket \mathfrak{m} \rrbracket$.

X. Call-by-Name calculus $\Lambda_{\oplus}^{\mathtt{cbn}}$

We show that results similar to those for $\Lambda_\oplus^{\text{cbv}}$ hold for a CbN calculus, denoted $\Lambda_\oplus^{\text{cbn}}$. We could adapt all the proofs, but now we prefer to follow a different way. Once we take the point of view of linear logic, we have a roadmap to CbN via Girard's translation of intuitionistic into linear logic. More precisely, we rely on recent work [12], [17] which expresses those translations in $\textit{untyped }\lambda\text{-calculus}$. We exploit the faithful nature of the translation to transfer both confluence and standardization from $\Lambda_\oplus^!$ to $\Lambda_\oplus^{\text{cbn}}$, essentially for free.

A. Syntax of $\Lambda_{\oplus}^{\mathtt{cbn}}$

We write $\Lambda_{\oplus}^{\text{cbn}}$ for the set of terms Λ_{\oplus} equipped with the reduction relation \Rightarrow defined below.

1) The language: Terms and contexts (C) are the same as in $\Lambda_{\oplus}^{\text{cbv}}$. Surface contexts (S) are generated by the grammar:

 $S ::= \Box | \lambda x. S | SM \text{ (cbn surface contexts)}$

$$(\lambda x.M)N \mapsto_{\beta} M[N/x]$$

Figure 8: Beta Reduction Rule for $\Lambda_{\oplus}^{\text{cbr}}$

2) Reductions: The β -rule \mapsto_{β} is as in the CbN λ -calculus (Fig. 8). The probabilistic rules $\mapsto_{l\oplus}, \mapsto_{l\oplus}$ are as in Fig. 1.

Reduction steps $\rightarrow, \rightarrow_{\beta}, \rightarrow_{\oplus} \subseteq \Lambda_{\oplus} \times MDST(\Lambda_{\oplus})$ and surface reduction steps $\stackrel{s}{\to}, \stackrel{s}{\to}_{\oplus}, \stackrel{s}{\to}_{\beta} \subseteq \Lambda_{\oplus} \times MDST(\Lambda_{\oplus})$ are defined in Fig. 6, following the usual pattern. By definition of surface context, a reduction step is surface if it does not occur in argument position (nor in the scope of \oplus).

The *lifting* of $\rightarrow \subseteq \Lambda_{\oplus} \times MDST(\Lambda_{\oplus})$ to a binary relation \Rightarrow on MDST($\Lambda^{\text{cbn}}_{\oplus}$) is defined as in Fig. 3. The full lifting \rightrightarrows is defined as in IV-A3.

3) Normal Forms: We denote by $\mathcal{N}^{\mathtt{cbn}}$ the set of \rightarrow -normal forms, and by S^{cbn} the set of the surface normal forms (i.e. the $\stackrel{s}{\rightarrow}$ -normal forms). Clearly $\mathcal{N}^{cbn} \subsetneq \mathcal{S}^{cbn}$.

Let us extend to $\Lambda_{\oplus}^{\text{cbn}}$ the notion of *head normal form*. Head reduction $\stackrel{h}{\rightarrow}$ is the closure of both the β and the probabilistic rules under head context H, which is defined by the following grammar

$$\mathbf{H} ::= \lambda x. \mathbf{H} | \mathbf{K} \qquad \mathbf{K} ::= \Box | \mathbf{K} M \qquad (\text{ head contexts })$$

Remark. A common way to write head context H is as follows:

$$\mathbf{H} := \lambda x_1 ... \lambda x_k .\Box P_1 ... P_n$$
 (head contexts)

Observe that $\xrightarrow{h} \subsetneq \xrightarrow{s}$ (for example, the reduction $(\lambda x.(\lambda y.y)P)Q \xrightarrow{s} (\lambda x.P)Q$ is not a head reduction). However, the two relations have the same normal forms. Let us write \mathcal{H} for the set of head normal forms. If M is in surface normal form, it is also in head normal form. It is easy to verify that a head normal form has no $\stackrel{s}{\rightarrow}$ -redex, and conclude:

$$S^{cbn} = \mathcal{H}$$

4) $\Lambda_{\oplus}^{\text{cbn}}$ to $\Lambda_{\oplus}^!$.: In [17], the translation from Λ^{cbn} into a linear λ -calculus is proved sound and complete. We follow their work to define a similar translation $(\cdot)_{\mathbb{N}}: \Lambda^{cbn}_{\oplus} \to \Lambda^{!}_{\oplus}$:

$$\begin{aligned} &(x)_{\scriptscriptstyle{\mathbb{N}}} &= x \\ &(MN)_{\scriptscriptstyle{\mathbb{N}}} &= (M)_{\scriptscriptstyle{\mathbb{N}}}!(N)_{\scriptscriptstyle{\mathbb{N}}} & |(\lambda x.M)_{\scriptscriptstyle{\mathbb{N}}} &= \lambda!x.(M)_{\scriptscriptstyle{\mathbb{N}}} \\ &(M\oplus N)_{\scriptscriptstyle{\mathbb{N}}} &= (M)_{\scriptscriptstyle{\mathbb{N}}} \oplus (N)_{\scriptscriptstyle{\mathbb{N}}} \\ &([p_iM_i\,|\,i\!\in\!I])_{\scriptscriptstyle{\mathbb{N}}} &= [p_i(M_i)_{\scriptscriptstyle{\mathbb{N}}}\,|\,i\!\in\!I] \end{aligned}$$

The following extend to the probabilistic setting an analogous result proved in [17]. Observe that, with a slight abuse of notation, reductions in the two calculi are denoted in the same way, the meaning being clear from the context.

Proposition 42 (Simulation). The translation (.), is sound and complete; it preserves surface reduction and surface normal forms. Let $M \in \Lambda_{\oplus}^{\text{cbn}}$; the following hold:

- 1) if $M \rightarrow n$ then $(M)_{N} \rightarrow (n)_{N}$;
- 2) if $M \stackrel{s}{\to} n$ then $(M)_{\mathbb{N}} \stackrel{s}{\to} (n)_{\mathbb{N}}$;
- 3) if $(M)_{\mathbb{N}} \to s$ then $\exists ! n$ such that $s = (n)_{\mathbb{N}}$ and $M \to n$;
- 4) if $(M)_{\mathbb{N}} \xrightarrow{s} s$ then $\exists ! n$ such that $s = (n)_{\mathbb{N}}$ and $M \xrightarrow{s} n$;
- 5) $M \in \mathcal{H}$ if and only if $(M)_{\mathbb{N}} \in \mathcal{S}!$.

Proof. The proof is in Appendix X-A4.

B. Confluence and Finitary Standardization for $\Lambda_{\oplus}^{\mathtt{cbn}}$

The fact that surface reduction is preserved by $(.)_{N}$ is crucial to transfer the standardization result from $\Lambda_{\oplus}^{!}$ to Λ_{\oplus}^{cbn} . We show that via translation, $\Lambda_{\oplus}^{\mathtt{cbn}}$ inherits both the confluence and the surface standardization property from $\Lambda^!_{\oplus}$.

Theorem 43 (Confluence). The relation \Rightarrow_{cbn} is confluent.

Proof. From Thm. 35, using back-and-forth Thm 42.

Theorem 44 (Finitary Surface standardization). If $m \Rightarrow^* n$ then exists r such that $m \stackrel{s}{\Rightarrow}^* r$ and $r \stackrel{d}{\Rightarrow}^* n$.

Proof. From Thm. 36, by using back-and-forth Thm 42, and the fact that the translation preserves surface reduction.

In the classical λ -calculus, the standardization property (Barendregt, Th. 11.4.7) says that every reduction sequence can be ordered in such a way to perform first only left β redexes, reading the term from left to right, and then internal ones (a redex is internal if it is not the leftmost one).

In $\Lambda_{\oplus}^{\mathrm{cbn}}$ this notion of standardization fails, as the following example (which we take from [19]) shows.

Example 45. In each step, we underline the redex. Consider $[(\lambda x.I(y \oplus z))I] \Rightarrow [(\lambda x.y \oplus z)I] \Rightarrow [\frac{1}{2}(\lambda x.y)I, \frac{1}{2}(\lambda x.z)I] \Rightarrow$ $\left[\frac{1}{2}y, \frac{1}{2}(\lambda x.z)I\right]$, where only the last step reduces a left redex. If we perform the left redex first, we have $[(\lambda x.I(y\oplus z))I] \Rightarrow$ $[I(y \oplus z)]$, from which $[\frac{1}{2}y, \frac{1}{2}(\lambda x.z)I]$ cannot be reached.

A consequence of standardization is that M has a head normal form iff the \xrightarrow{h} -sequence from M terminates. In the following section we retrieve an analogue of this result.

C. Asymptotic behaviour

We denote by \mathcal{H}_{\sim} the set of head normal forms up to the equivalence $=_{\beta}$, and we define $\mathcal{N}^{\text{cbn}}_{\{\}} = \{\{M\}, M \in \mathcal{N}^{\text{cbn}}\}.$ a) Observations: Observations are defined as in Def. 17.

Proposition 46. Each of the following is a set of observations for $\Lambda_{\oplus}^{\text{cbn}}$: $\{\mathcal{H}\}$, $\{\mathcal{N}^{\text{cbn}}\}$, \mathcal{H}_{\sim} , $\mathcal{N}_{\{\}}^{\text{cbn}}$.

b) Convergence and Limit distributions: Once we fix a set of observations Obs for $\Lambda_{\oplus}^{\mathtt{cbn}},$ the definition of convergence and limit distribution are as in Def. 18. We observe that Theorems 22 and 23 both hold. Hence in particular

Theorem 47. For any choice of Obs, the following holds in $\Lambda^{\mathtt{cbn}}_{\oplus}$: given m, $\mathtt{Lim}(\mathtt{m})$ has a greatest element $[\![\mathtt{m}]\!]$.

We now study the notion of convergence induced by *choos*ing head normal forms as outcome, i.e. $Obs := \mathcal{H}_{\sim}$. Therefore, if $\rho \in Lim(m)$, it holds $\rho \in MDST(\mathcal{H}_{\sim})$. The following results match the analogous results in $\Lambda^{!}_{\oplus}$ (Thm. 40 and 41).

Theorem 48. Let $\mathtt{Obs} := \mathcal{H}_{\sim}$. For every multidistribution m:

- $m \Rightarrow^{\infty} \mu$ if and only if $m \Rightarrow^{\infty} \mu$.
- If $\langle \mathbf{s}_n \rangle_{n \in \mathbb{N}}$ is a $\stackrel{s}{\Longrightarrow}$ -sequences of full surface reductions from m, then $\langle s_n \rangle_{n \in \mathbb{N}} \Downarrow [m]$.

Similarly to Prop. 29, it is not hard to prove that in Λ^{cbn}_{\oplus} , $\stackrel{s}{\Longrightarrow}$ satisfies a diamond property in the sense of [15], and hence all $\stackrel{s}{\Rightarrow}$ -sequences from m converge to the same limit distribution. Since $\stackrel{h}{\Rightarrow} \subset \stackrel{s}{\Rightarrow}$ and since head reduction and surface reduction have the same normal forms, we can always choose a $\stackrel{l}{\rightarrow}$ step whenever a $\stackrel{s}{\Rightarrow}$ -step is possible. This allows us to retrieve a result of completeness for head reduction:

Let
$$\langle \mathbf{s}_n \rangle_{n \in \mathbb{N}}$$
 be the $\stackrel{h}{\Longrightarrow}$ -sequences of full head reductions from \mathbf{m} . It holds that $\langle \mathbf{s}_n \rangle_{n \in \mathbb{N}} \Downarrow [\![\mathbf{m}]\!]$.

Once again, this justifies a posteriori the choice of head reduction in probabilistic CbN (such as [13]). Observe that we follow the same reasoning as in the case of $\Lambda_\oplus^{\rm cbv}$ (with \mathcal{V}_\sim as set of observations). First we proved that surface reduction is sufficient to reach the greatest limit distribution, then we observed that in particular left reduction can be chosen. There is a close parallelism between $\Lambda_\oplus^{\rm cbv}$ and $\Lambda_\oplus^{\rm cbn}$: similar results hold if we consider as set of observations \mathcal{V}_\sim and \mathcal{H}_\sim respectively.

XI. CONCLUSION AND DISCUSSION

A. Summary

In this paper we design two probabilistic extensions of respectively the CbV and CbN λ -calculus, $\Lambda_{\oplus}^{\rm cbv}$ and $\Lambda_{\oplus}^{\rm cbn}$, which we propose as foundational calculi for probabilistic computation. Both calculi enjoy confluence and standardization, in an extended way. Namely, first we prove both properties for the finite sequences, exploiting classical methods, then we extend these properties to the limit, developing new sophisticated proof methods. In particular, we prove the uniqueness of the (maximal) result, parametrized by the notion of set of observations, and that the asymptotic extension of surface standardization supplies a family of complete reduction strategies which are guaranteed to reach the best result. The two calculi have a common root in the linear λ -calculus $\Lambda^!_{\oplus}$, which is both a technical tool and a calculus of interest in its own, in which a fine control of the interaction between copying and choice is possible.

In all three calculi, β -reduction is unconstrained; hence for each calculus, its restriction to only β -reduction exactly gives the usual corresponding (CbN, CbV, or linear) λ -calculus; this is not the case for extensions in which a strategy is fixed.

New proof methods include the asymptotic extension of surface standardization (Thm. 26), and the use of a translation to transfer standardization properties, namely from $\Lambda^!_{\oplus}$ to $\Lambda^{\rm cbn}_{\oplus}$. It is worth stressing a crucial element: the fact that the translation is sound, complete and preserves surface contexts is what allows us to transfer the results.

B. Discussion

a) Relating the calculi (Girard's Translations): The key to understand how $\Lambda_{\oplus}^{\text{cbv}}$, $\Lambda_{\oplus}^{\text{cbn}}$, and $\Lambda_{\oplus}^!$ relate are the two Girard's translations which embed intuitionistic logic into linear logic, and which are well known to respectively correspond to CbN and CbV computations. Let us clarify this. Let us start from $\Lambda_{\oplus}^!$: the natural constraint to avoid copying the result of a choice is "no \oplus -reduction in the scope of !" (i.e., inside a !-box). Using the intuition provided by Girard's translations

as a guide, the constraint above becomes respectively "no \oplus -reduction in the scope of a λ -abstraction" (in CbV) and "no \oplus -reduction in argument position" (in CbN). Our three notions of surface context express these three constraints.

The intuitive reasoning above can be formalized thanks to a recent line of work [12], [17], which internalizes the insights coming from linear logic and proof nets into a λ -syntax. The resulting calculus subsumes both CbN and CbV λ -calculi via Girard's translation. The idea of a system which subsumes both CbV and CbN had been already advocated and developed by Levy, via the Call-By-Push-Value paradigm [20]. And indeed, [12] can be seen as an untyped version of Levy's calculus. We leave to the future a comprehensive approach, where a probabilistic linear calculus is the metalanguage in which all the results are developed.

- b) On non-deterministic λ -calculi: The finitary results we presented (namely, confluence and finitary surface standardization) also hold if the probabilistic choice is replaced by non-deterministic choice (just forget the coefficients). Asymptotic results, instead, are specific to probabilistic computation.
- c) $\Lambda^!_{\oplus}$ and quantum λ -calculi: The fine control of duplication which $\Lambda^!$ inherits from linear logic has made it an ideal base for quantum λ -calculi (such as [7], [6]). In those calculi, *surface reduction* is the key ingredient to allow for the coexistence of quantum bits with duplication and erasing. *No reduction* (not even β) is allowed in the scope of a ! operator. Our results show that β -reduction can be unrestricted, only measurement (the quantum analogue of \oplus) needs to be surface.

APPENDIX

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$$\frac{V \in \mathcal{V}}{(\lambda x. M) V \stackrel{\$}{\to} [M[V/x]]} \frac{M \oplus N \stackrel{\$}{\to} [\frac{1}{2}M, \frac{1}{2}N]}{M \oplus N \stackrel{\$}{\to} \mathbf{n}} \frac{M \stackrel{\$}{\to} \mathbf{n}}{M N \stackrel{\$}{\to} \mathbf{m@N}} \frac{N \stackrel{\$}{\to} \mathbf{n}}{M N \stackrel{\$}{\to} N@\mathbf{n}}$$

Figure 9: Surface Reduction

$$\frac{V \in \mathcal{V}}{(\lambda x.M)V \xrightarrow{\downarrow} [M[V/x]]} \frac{}{M \oplus N \xrightarrow{\downarrow} [\frac{1}{2}M, \frac{1}{2}N]}$$

$$\frac{M \xrightarrow{\downarrow} \mathbf{m}}{MN \xrightarrow{\downarrow} \mathbf{m}@N} \frac{V \in \mathcal{V} \quad N \xrightarrow{\downarrow} \mathbf{n}}{VN \xrightarrow{\downarrow} V@\mathbf{n}}$$

Figure 10: Left Evaluation

APPENDIX

A. Proofs of Section V-B

We prove Thm. 12, *i.e.* finitary Surface Standardization for $\Lambda_{\oplus}^{\text{cbv}}$. We start by establishing Surface Standardization for the (non probabilistic) call-by-value λ -calculus, Λ^{cbv} , in A2. This result is folklore, but we could not find it in the literature. In A3 we extend the result to $\Lambda_{\oplus}^{\text{cbv}}$.

- 1) Preliminary definitions:
- a) Surface and left reduction: Surface and left reduction have been defined in Sec. V-B1; Fig. 9 and Fig. 10 give explicitly the inference rules for surface and left steps; we use the notation defined below:

Notation. If $m = [p_i M_i | i \in I]$, we write m@Q for $[p_i(M_iQ) | i \in I]$, and Q@m for $[p_i(QM_i) | i \in I]$.

We recall that a reduction step \rightarrow is *deep*, written \xrightarrow{d} , (resp. *internal*, written \xrightarrow{int}) if it is not a surface step (a left step). We have already observed that $\xrightarrow{d} \subset \xrightarrow{int}$, and that since a \oplus -redex is always surface, a \xrightarrow{d} step is always a \rightarrow_{β_n} step.

- *b)* Parallel β_v -reduction:
- Parallel β_v -reduction is a standard definition, and is given in Fig. 11. We define its lifting \Rightarrow_{β_v} as usual (see Section IV-A).
- Deep parallel reduction ($\stackrel{d}{\Rightarrow}$, with lifting $\stackrel{d}{\Rightarrow}$) indicates that $M \Rightarrow [S]$ and $M \stackrel{d}{\Rightarrow}^* [S]$. We make the rules explicit in Fig. 12.

Fact 49. The following holds

$$\stackrel{d}{\Rightarrow}_{\beta_v} \subseteq \stackrel{d}{\Rightarrow}_{\beta_v} \subseteq \stackrel{d}{\Rightarrow}_{\beta_v}^*$$

c) Translation: We refine the translation given in in Sec. IV-B in order to preserves surface reduction.

Let z,w be fresh variables. $(\cdot)_{\lambda}:\Lambda_{\oplus}\to\Lambda$ is defined as follows:

$$\begin{array}{ll} (x)_{\lambda} &= x \\ (M \oplus N)_{\lambda} &= z(\lambda w.(M)_{\lambda})\lambda w.(N)_{\lambda} \left| (MN)_{\lambda} \right. = (M)_{\lambda}(N)_{\lambda} \\ (\lambda x.M)_{\lambda} &= \lambda x.(M)_{\lambda} \end{array}$$

The following is straightforward to check.

Lemma 50. Assume $M \in \Lambda_{\oplus}$.

1)
$$P \rightarrow_{\beta_v} [Q]$$
 and $(Q)_{\lambda} = S$ (in Λ_{\oplus}) \iff $(P)_{\lambda} \rightarrow_{\beta_v} S$ (in Λ).

$$\frac{M \not \mapsto_{\beta_{v}}[N]}{\lambda x.M \not \mapsto_{\beta_{v}}[\lambda x.N]}$$

$$\frac{M \not \mapsto_{\beta_{v}}[M'] \quad N \not \mapsto_{\beta_{v}}[N']}{MN \not \mapsto_{\beta_{v}}[M'N']}$$

$$\frac{M \not \mapsto_{\beta_{v}}[M'] \quad W \not \mapsto_{\beta_{v}}[W'] \quad W \quad \text{value}}{(\lambda x.M)W \not \mapsto_{\beta_{v}}[M'[W'/x]]}$$

$$\frac{M \not \mapsto_{\beta_{v}}[M'] \quad N \not \mapsto_{\beta_{v}}[N']}{M \oplus N \not \mapsto_{\beta_{v}}[M' \oplus N']}$$

Figure 11: β -Parallel Reduction

$$x \overset{\text{d}}{\leftrightarrow}_{\beta_{v}}[x] \qquad \frac{M \underset{\beta_{v}}{\leftrightarrow}_{\beta_{v}}[N]}{\lambda x.M \overset{\text{d}}{\leftrightarrow}_{\beta_{v}}[\lambda x.N]}$$

$$\frac{M \overset{\text{d}}{\leftrightarrow}_{\beta_{v}}[S] \quad N \overset{\text{d}}{\leftrightarrow}_{\beta_{v}}[T]}{MN \overset{\text{d}}{\leftrightarrow}_{\beta_{v}}[ST]}$$

$$\frac{M \underset{\beta_{v}}{\leftrightarrow}_{\beta_{v}}[S] \quad N \underset{\beta_{v}}{\leftrightarrow}_{\beta_{v}}[T]}{M \oplus N \overset{\text{d}}{\leftrightarrow}_{\beta_{v}}[S \oplus T]}$$

Figure 12: Deep Parallel Reduction

- 2) $P \xrightarrow{s}_{\beta_v} [Q]$ and $(Q)_{\lambda} = S$ (in Λ_{\oplus}) \iff $(P)_{\lambda} \xrightarrow{s}_{\beta_v} S$ (in Λ).
- 3) $P \not\mapsto_{\beta_v} [Q]$ and $(Q)_{\lambda} = S$ (in Λ_{\oplus}) $\iff (P)_{\lambda} \not\mapsto_{\beta_v} (Q)_{\lambda}$ (in Λ).
- 2) Λ^{cbv} and Surface Standardization: With the standard definition of left, internal, and parallel reduction (denoted $\xrightarrow{1}_{\beta_v}$, $\xrightarrow{\text{int}}_{\beta_v}$, $\xrightarrow{\text{respectively}}$) the following results are well known to hold (see [25], [27]).
- (a) If $M \to_{\beta_n}^* N$ then exists S such that

$$M \stackrel{\text{\tiny I}}{\to}^*_{\beta_v} S \stackrel{\text{\tiny int}}{\to}^*_{\beta_v} N.$$

- (c) If $M \stackrel{\text{int}}{+}_{\beta_n} M' \stackrel{1}{\rightarrow}_{\beta_n} N$, it exists S s.t. $M \stackrel{1}{\rightarrow}_{\beta_n}^* S \stackrel{\text{int}}{+}_{\beta_n} N$.

The results of the following lemma are immediately obtained from the previous ones, by observing that a left reduction is a surface reduction, a deep reduction is always an internal reduction, and that $\xrightarrow{\text{int}}_{\beta_v}$ does not modify the shape of a term (see [27]).

Lemma 51. 1) If $M \to_{\beta_v}^* N$ then exists S such that $M \xrightarrow{s}_{\beta_v}^* S \xrightarrow{d}_{\beta_v}^* N$.

- 2) If $M \mapsto_{\beta_v} N$ then exists S s.t. $M \stackrel{s}{\rightarrow}_{\beta_v}^* S \stackrel{d}{\leftrightarrow}_{\beta_v} N$.
- 3) If $M \stackrel{d}{\leftrightarrow}_{\beta_v} M' \stackrel{s}{\rightarrow}_{\beta_v} N$ then exists S s.t. $M \stackrel{s}{\rightarrow}_{\beta_v}^* S \stackrel{d}{\leftrightarrow}_{\beta_v} N$.

Proof. The first two are by induction on N. We recall that $\stackrel{1}{\to} \subset \stackrel{s}{\to}$.

- 1) By (a) $M \to_{\beta_v}^* N$ implies $M \xrightarrow{\downarrow}_{\beta_v}^* S \xrightarrow{\text{int}}_{\beta_v}^* N$. We examine N.
 - N = x. Then S = N and the result holds trivially.
 - $N = \lambda x.P$. Hence $S = \lambda x.Q \xrightarrow{\inf_{\beta_v}^*} \lambda x.P$. Then $M \xrightarrow{\downarrow_{\beta_v}^*} \lambda x.Q \xrightarrow{d_{\beta_v}^*} \lambda x.P$.

• N = PQ. Then S = P'Q', where $P' \to_{\beta_v}^* P$ and $Q' \to_{\beta_n}^* Q$. By induction $P' \stackrel{s}{\to}_{\beta_n}^* P'' \stackrel{d}{\to}_{\beta_n}^{*} P$ and $Q' \stackrel{\beta_v}{\Rightarrow_{\beta_v}^*} Q'' \stackrel{d}{\Rightarrow_{\beta_v}^*} Q$, and the desired sequence is $M \stackrel{1}{\rightarrow}_{\beta_v}^* P'Q' \stackrel{s}{\Rightarrow_{\beta_v}^*} P''Q'' \stackrel{d}{\Rightarrow_{\beta_v}^*} PQ$.

The result follows since $\xrightarrow{1}_{\beta_v} \subset \xrightarrow{d}_{\beta_v}$.

- 2) Similar to the previous one, using (b) , i.e., the fact that $M \mapsto_{\beta_v} N \text{ implies } M \xrightarrow{\downarrow}_{\beta_v}^* S \xrightarrow{\text{int}}_{\beta_v} N.$
 - $N=\lambda x.P.$ Then $S=\lambda x.Q$ and $Q \! \Rightarrow_{\beta_v} \! P.$ By
 - definition, $M \stackrel{\downarrow}{\to}^*_{\beta_v} \lambda x. Q \stackrel{\text{d}}{\mapsto}_{\beta_v} \lambda x. P.$ N = PQ. Then S = P'Q', with $P' \nleftrightarrow_{\beta_v} P$ and $Q' \nleftrightarrow_{\beta_v} Q$. By induction $P' \stackrel{\text{s}}{\to}^*_{\beta_v} P'' \stackrel{\text{d}}{\nleftrightarrow}_{\beta_v} P$ and
- 3) By induction on M.
 - M = x or $M = \lambda x.P$. Immediate.
 - $\begin{array}{lll} M &= & (\lambda x.P)V. & \text{Assume} \\ (\lambda x.P)V \overset{\mathrm{d}}{+\!\!\!+}_{\beta_v}(\lambda x.P')V' & \overset{\mathrm{s}}{-\!\!\!-}_{\beta_v} & N. & \text{Since} & \text{the} \end{array}$ deep step is an internal step, the surface step is a left step, we have $(\lambda x.P)V \stackrel{\text{int}}{+}_{\beta_v} (\lambda x.P')V' \stackrel{1}{\to}_{\beta_v} N$. From (c), it exists $S, M \stackrel{\text{int}}{\to}_{\beta_v} S \stackrel{\text{int}}{+}_{\beta_v} N$. The $\stackrel{\text{int}}{+}_{\beta_v}$ step is in particular a +++ β_v , hence from point 2 it holds that $S \stackrel{s}{\to}_{\beta_v}^* S' \stackrel{d}{+}_{\beta_v} N$, hence the claim.
 - M = PQ. By hypothesis, $PQ \stackrel{d}{\leftrightarrow}_{\beta_v} P'Q' \stackrel{s}{\rightarrow}_{\beta_v} N$; the surface redex is inside either P' or Q', say Q'. We have N=P'R, $Q \stackrel{\text{d}}{+}_{\beta_v} Q' \stackrel{\text{s}}{\to}_{\beta_v} R$ and by induction $Q \stackrel{\text{s}}{\to}_{\beta_v}^* R' \stackrel{\text{d}}{+}_{\beta_v} R$. Hence $PQ \stackrel{\text{s}}{\to}_{\beta_v}^* PR' \stackrel{\text{d}}{+}_{\beta_v} P'R$.
- 3) Surface Standardization in Λ_{\oplus}^{cbv} : In order to prove Theorem 12, we need a lemma.

Lemma 52. If $M \stackrel{d}{\leftrightarrow} [M']$ and $M' \stackrel{s}{\rightarrow} n$, then it exists s, such that $[M] \stackrel{s}{\Rightarrow}^* s$ and $s \stackrel{d}{\Rightarrow} n$.

Proof. If $M' \stackrel{s}{\to}_{\beta_n}$ n, the claim holds by simulation in Λ^{cbv} and Lemma 51, point (3). If $M' \stackrel{s}{\rightarrow}_{\oplus} n$, we procede by induction on M.

- 1) The case M = x and $M = \lambda x.P$ do not apply.
- 2) Let $M = P \oplus Q$. Assume $P \oplus Q \stackrel{d}{\leftrightarrow} [R \oplus S]$, so $P \leftrightarrow_{\beta_v} R$, $Q \mapsto_{\beta_v} S$, and $R \oplus S \stackrel{s}{\to}_{\oplus} [\frac{1}{2}R, \frac{1}{2}S]$. By Lemma 51, point 2 and simulation in Λ^{cbv} , it holds that $P \stackrel{\text{s}}{\to}^* P' \stackrel{\text{d}}{\mapsto} R$ and $Q \stackrel{s}{\to}^* Q' \stackrel{d}{\pitchfork} S$. Therefore $P \oplus Q \stackrel{s}{\to}_{\oplus} \left[\frac{1}{2}P, \frac{1}{2}Q\right] \stackrel{s}{\to}^*$ $\left[\frac{1}{2}P', \frac{1}{2}Q'\right] - \left[\frac{1}{2}R, \frac{1}{2}S\right]$
- 3) Let M = PQ. Assume $PQ \stackrel{d}{\leftrightarrow} [RT]$ (with $P \stackrel{d}{\leftrightarrow} R$ and $Q \stackrel{d}{\leftrightarrow} T$) and $RT \rightarrow_{\oplus} n$ with the \oplus -redex in either R or T, say is in R. Hence $R \to_{\oplus} \mathbf{r} = \left[\frac{1}{2}R_i \mid i \in \{1,2\}\right]$ and $RT \xrightarrow{s}$ $\left[\frac{1}{2}R_iT \mid i \in \{1,2\}\right] = n$. By induction, from $P \stackrel{d}{\Rightarrow} [R] \stackrel{s}{\Rightarrow} r$ we have $[P] \stackrel{s}{\Rightarrow}^* [\frac{1}{2}S_i | i \in \{1,2\}]$ and $S_i \stackrel{d}{\leftrightarrow} R_i$. Therefore $[PQ] \stackrel{s}{\to}^* [\frac{1}{2}S_iQ | i \in \{1,2\}] \stackrel{d}{\to} [\frac{1}{2}R_iT | i \in \{1,2\}] = n$.

Corollary 53. If $m \stackrel{d}{\Rightarrow} n$ and $n \stackrel{s}{\Rightarrow}^* r$, then exists s with $m \stackrel{s}{\Rightarrow}^* s$ and s ⇔r.

Proof. By induction on the length k of $n \stackrel{\$}{\Rightarrow}^{(k)} r$. If k = 0 the result is trivial. Otherwise, let $n \stackrel{\$}{\Rightarrow}^* r$ be $n \stackrel{\$}{\Rightarrow} n_1 \stackrel{\$}{\Rightarrow}^{(k-1)} r$. By

Lemma 52, from $\mathbf{m} \overset{d}{\oplus} \mathbf{n} \overset{s}{\Rightarrow} \mathbf{n}_1$ we have that $\mathbf{m} \overset{s}{\Rightarrow}^* \mathbf{s} \overset{d}{\oplus} \mathbf{n}_1 \overset{s}{\Rightarrow}^{(k-1)}$ r. By inductive hypothesis, $s \stackrel{s}{\Rightarrow}^* r' \stackrel{d}{\Rightarrow} r$, hence $m \stackrel{s}{\Rightarrow}^* s \stackrel{s}{\Rightarrow}^*$

Now we are able to prove the theorem:

Thm. 12: if $m \Rightarrow^* n$ then then exists r such that $m \stackrel{\$}{\Rightarrow}^* r$ and

Proof. By induction on the length k of the reduction $m \Rightarrow^* n$, using Corollary 53.

If k = 0, the result is trivial (r = m). Otherwise, $m \Rightarrow m_1 \Rightarrow^* n$. By induction, we have $m_1 \stackrel{\$}{\Rightarrow}^* r \stackrel{d}{\Rightarrow}^* n$. We can separate the first step in two: $m \stackrel{s}{\Rightarrow} m' \stackrel{d}{\Rightarrow} m_1$, by reducing first only the elements of m which have a surface reduction, and then only the elements which have a deep reduction. The step $m' \stackrel{d}{\Rightarrow} m_1$ can be regarded as a parallel step. By Corollary 53, from $m' \stackrel{d}{\Rightarrow} m_1 \stackrel{s}{\Rightarrow}^* r$ we obtain $m' \stackrel{s}{\Rightarrow}^* s \stackrel{d}{\Rightarrow} r$, hence it holds that $m \stackrel{s}{\Rightarrow} m' \stackrel{s}{\Rightarrow}^* s \stackrel{d}{\Rightarrow}^* r \stackrel{d}{\Rightarrow} n$.

B. Proofs of Section VI

a) Monotone Convergence.: We recall the following standard result.

 \Box

Theorem (Monotone Convergence for Sums). Let \mathcal{X} be a countable set, $f_n: \mathcal{X} \to [0,\infty]$ a non-decreasing sequence of functions, such that $f(x) := \lim_{n \to \infty} f_n(x) = \sup_n f_n(x)$ exists for each $x \in \mathcal{X}$. Then

$$\lim_{n \to \infty} \sum_{x \in \mathcal{X}} f_n(x) = \sum_{x \in \mathcal{X}} f(x)$$

Hence, given $\mu_n: \mathtt{Obs} \to [0,1]$ and $\rho(\mathbf{U}) = \lim_{n \to \infty} \mu_n(\mathbf{U})$, the following holds:

$$\underset{n \to \infty}{\lim} \sum_{\mathbf{U} \in \mathtt{Obs}} \mu_n(\mathbf{U}) \, = \, \sum_{\mathbf{U} \in \mathtt{Obs}} \! \boldsymbol{\rho}(\mathbf{U})$$

b) Existence of maximals.: We recall the definition of norm $\|\mu\| = \sum_{x \in \mathcal{X}} \mu(x)$.

Lemma 54 (Existence of maximals). *Confluence implies that:*

- 1) Norms(m) = { $\|\boldsymbol{\mu}\| \mid \boldsymbol{\mu} \in Lim(m)$ } has a greatest
- 2) Lim(m) has maximal elements.

Proof. (1.) Let $p = \sup \text{Norms}(m)$. We show that $p \in$ Norms(m), by providing a rewrite sequence $\langle m_n \rangle_{n \in \mathbb{N}}$ from m such that $\langle \mathbf{m}_n \rangle_{n \in \mathbb{N}} \Rightarrow^{\infty} \boldsymbol{\tau}$ and $\|\boldsymbol{\tau}\| = p$.

The following facts are all easy to check:

a. If $\alpha < \beta$ then $\|\alpha\| < \|\beta\|$.

- b. If $p \notin Norms(m)$, then for each ϵ , there exists $\mu \in Lim(m)$ such that $\|\boldsymbol{\mu}\| \ge p - \epsilon$.
- c. The Main Lemma implies that, fixed ϵ , if $m \Rightarrow^{\infty} \mu$ with $\|\boldsymbol{\mu}\| \ge (p-\epsilon)$, and $\mathbf{m} \Rightarrow^* \mathbf{s}$, then there exists \mathbf{s}' , such that $s \Rightarrow^* s'$ and $||\sigma'|| > (p-2\epsilon)$.

(Proof: Main Lemma implies that there is a rewrite sequence $\langle s_n \rangle_{n \in \mathbb{N}}$ from s which converges to $\sigma \geq \mu$. Therefore $\langle s_n \rangle_{n \in \mathbb{N}} \Rightarrow^{\infty} \boldsymbol{\sigma}$ where $\|\boldsymbol{\sigma}\| \geq (p - \epsilon)$. For the same ϵ , there is an index N such that $s \Rightarrow^* s_N$ and $\|\sigma_N\| \ge (\|\boldsymbol{\sigma}\| - \epsilon)$, hence $\|\sigma_N\| \ge p - 2\epsilon$.)

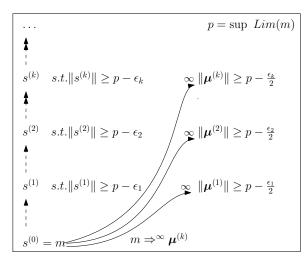


Figure 13: A sequence whose limit distribution is a maximal element of Lim(m)

d. $\forall \delta \in \mathbb{R}^+$ there exists k such that $\frac{p}{2^k} \leq \delta$.

For each $k \in \mathbb{N}$, let $\epsilon_k = \frac{p}{2^k}$. Let $\mathfrak{s}^{(0)} = \mathfrak{m}$. From here, we build a sequence of reductions $\mathfrak{m} \Rightarrow \mathfrak{s}^{(1)} \Rightarrow \mathfrak{s}^{(2)} \Rightarrow \mathfrak{s}$... whose limit has norm p, as illustrated in Fig. 13. For each k > 0, we observe that:

- By (b.) there exists $\boldsymbol{\mu}^{(k)} \in Lim(\mathbf{m})$ such that $\|\boldsymbol{\mu}^{(k)}\| \ge (p \frac{1}{2} \frac{p}{2k})$.
- From $m \Rightarrow^* s^{(k-1)}$, we use (c.) to establish that there exists $s^{(k)}$ such that $s^{(k-1)} \Rightarrow^* s^{(k)}$ and $\|\sigma^{(k)}\| \ge (p \frac{p}{2^k})$. Observe that $\mu^{(k)}, s^{(k-1)}, s^{(k)}$ resp. instantiate μ, s, s' of (c.).

Let $\langle \mathbf{s}_n \rangle_{n \in \mathbb{N}}$ be the concatenation of all the finite sequences $\mathbf{s}^{(k-1)} \Rightarrow^* \mathbf{s}^{(k)}$. By construction, $\langle \mathbf{s}_n \rangle_{n \in \mathbb{N}} \Rightarrow^{\infty} \boldsymbol{\tau}$ such that $\|\boldsymbol{\tau}\| = p$. Hence $p \in \text{Norms}(\mathbf{m})$.

(1. \Rightarrow 2.) We observe that if $\langle m_n \rangle_{n \in \mathbb{N}} \Rightarrow^{\infty} \mu$ and $\|\mu\|$ is maximal in Norms(m), then μ is maximal in Lim(m), because of (a.).

C. Proofs of Section IX

1) Confluence of $\Lambda_{\oplus}^!$: We prove Theorem 35. First, we need to prove some preliminary results.

Lemma 55. If $M \rightarrow_{\beta} n$ and $M \rightarrow_{\oplus} s$, then exists r such that $n \Rightarrow_{\oplus} r$ and $s \Rightarrow_{\beta} r$

Proof. We reason by induction on M. The key case is case 5. Case M = x and M = !P are not possible given the hypothesis.

- 1) Case $M = P \oplus Q$. Similar to Lemma 8, case (1)
- 2) Case M = S(Q), and both redexes are inside Q. Similar to Lemma 8, case (2.2b).
- 3) Case M = PQ, with the β -redex inside P, and the \oplus -redex inside Q. Similar to Lemma 8, case (2.2a).
- 4) Case $M=(\lambda!x.P)!Q$, where M is the β -redex. The \oplus -redex needs to be inside P. Assume $P\to_{\oplus} \left[\frac{1}{2}P_1,\frac{1}{2}P_2\right]$. We have $M\to_{\oplus} \left[\frac{1}{2}(\lambda!x.P_1)!Q,\,\frac{1}{2}(\lambda!x.P_2)!Q\right]$, and $M\to_{\beta} \left[P[Q/x]\right]$. It is immediate that the multidistribution $\mathbf{r}=\left[\frac{1}{2}P_1[Q/x],\frac{1}{2}P_2[Q/x]\right]$ satisfies the claim.
- 5) Case $M = (\lambda x. P)Q$, where M is the β -redex. If the \oplus -redex is inside P, we reason as above. Assume

that the \oplus -redex is inside Q, and we have $Q \to_{\oplus} [\frac{1}{2}Q_1,\frac{1}{2}Q_2]$. The key observation is that in P there is at most one occurrence of x. Let assume there is exactly one occurrence (the case of none is easy). Let \mathbf{C} be the context such that $P = \mathbf{C}(x)$ (i.e., \mathbf{C} is P, with a hole in the place of x). Observe that $P[Q/x] = \mathbf{C}(Q)$. We have $M \to_{\beta} [P[Q/x] = \mathbf{C}(Q)]$, and $M \to_{\oplus} [\frac{1}{2}(\lambda x.P)Q_1,\frac{1}{2}(\lambda x.P)Q_2]$. The multidistribution $\mathbf{r} = [\frac{1}{2}\mathbf{C}(Q_1),\frac{1}{2}\mathbf{C}(Q_2)]$ satisfies the claim.

Lem. 34:

- 1) The reduction \Rightarrow_{\oplus} is diamond.
- 2) The reduction \Rightarrow_{β} is confluent.
- 3) The reductions \Rightarrow_{β} and \Rightarrow_{\oplus} commute.

Proof. 1) Same proof as for Lemma 7.

- 2) Inherited from $\Lambda^!$ via the translation (.)_! and Prop. 33.
- 3) We prove that \Rightarrow_{β} and \Rightarrow_{\oplus} \diamond -commute, by using Lemma 5 and Lemma 55.

Thm. 35 The reduction \Rightarrow of $\Lambda^!_{\oplus}$ is confluent.

Proof. By Hindley-Rosen Lemma, from Lemma 34.

2) Surface Standardization in $\Lambda_{\oplus}^{!}$: In order to prove Proposition 36, first we prove a lemma.

Lemma 56. If $M \stackrel{d}{\leftrightarrow} M'$ and $M' \stackrel{s}{\rightarrow} n$, then $[M] \stackrel{s}{\Rightarrow}^* s$ and $s \stackrel{d}{\Rightarrow} n$.

Proof. If $M' \xrightarrow{s}_{\beta} n$, the claim holds by *simulation* in Λ !. If $M' \xrightarrow{s}_{\oplus} n$, we procede by induction on M.

- 1) The case M = x and M = !P do not apply.
- 2) Let $M = P \oplus Q$. Similar to Lemma 52, Point (2.).
- 3) Let M = PQ, $\lambda!x.P$ or $\lambda x.P$. Similar to Lemma 52, Point (3.).

Corollary 57. If $m \stackrel{d}{\Rightarrow} n$ and $n \stackrel{s}{\Rightarrow}^* r$, then exists s with $m \stackrel{s}{\Rightarrow}^* s$ and $s \stackrel{d}{\Rightarrow} r$.

Proof. Same as Lemma 53, using Lemma 56. □

Then we can prove:

Prop. 36 If $m \Rightarrow^* n$ then there exists r such that $m \stackrel{\$}{\Rightarrow}^* r$ and $r \stackrel{\texttt{d}}{\Rightarrow}^* n$.

Proof. Same as the proof of Thm. 12, by induction on the length of the reduction $m \Rightarrow^* n$, using this time Corollary 57.

D. Proofs of Section X

1) $\Lambda_{\oplus}^!$ is a conservative extension of $\Lambda_{\oplus}^{\mathtt{cbn}}$ (X-A4): To prove Proposition 42, we first prove that $(\cdot)_{\mathtt{N}}$ preserves both surface contexts and \oplus -redexes.

Lemma 58. Given $M \in \Lambda_{\oplus}^{\text{cbn}}$, (S1) holds \iff (S2) holds, where:

_ * s

- S1: in $\Lambda_{\oplus}^{\text{cbn}}$, there exists S surface context and a redex $r = R_1 \oplus R_2$ such that M = S(r);
- S2: in $\Lambda^{!}_{\oplus}$ there exists T surface context and a redex $u = U_1 \oplus U_2$ such that $(M)_{\mathbb{N}} = T(u)$;

and moreover $(S(R_i))_{y} = T(U_i)$, for $i \in \{1,2\}$.

Proof. \Longrightarrow . By induction on the form of the surface context.

- \square . Since M=r, then $(M)_{\scriptscriptstyle N}=(R_1)_{\scriptscriptstyle N}\oplus (R_2)_{\scriptscriptstyle N}$. Hence $u=(R_1)_{\scriptscriptstyle N}\oplus (R_2)_{\scriptscriptstyle N}$ and $T=\square$ satisfy the claim.
- SQ. We have that M = SQ(r) = S(r)Q. Hence $(S(r)Q)_{\scriptscriptstyle \rm N} = (S(r))_{\scriptscriptstyle \rm N}!(Q)_{\scriptscriptstyle \rm N}$. By inductive hypothesis, there exist T' and u such that $(S(r))_{\scriptscriptstyle \rm N} = T'(u)$, and $(S(R_i))_{\scriptscriptstyle \rm N} = T'(U_i)$. By definition of surface context in $\Lambda^!_{\oplus}$, the claim hold with $T = T'!(Q)_{\scriptscriptstyle \rm N}$, and the same u.
- $\lambda x.S. (\lambda x.S(r))_{\text{\tiny N}} = \lambda! x.(S'(r))_{\text{\tiny N}}$, and the claim holds by inductive hypothesis.

 \Leftarrow . We examine the possible form of T, given that $(M)_{N} = T(u)$; we prove that M = S(r) and that $(S(R_i))_{N} = T(U_i)$.

- □. Immediate.
- T'Q. We have that (T'Q)(u) = T'(u)Q, and $T'(u)Q = (LN)_{\scriptscriptstyle N} = (L)_{\scriptscriptstyle N}!(N)_{\scriptscriptstyle N}$ with M = LN. Therefore $T'(u) = (L)_{\scriptscriptstyle N}$, and the claim holds by inductive hypothesis and definition of surface context.
- $\lambda!x.T'$. We have that $(\lambda!x.T')(u) = \lambda!x.T'(u)$ and $\lambda!x.T'(u) = \lambda!x.(M')_{\text{\tiny N}}$, with $M = \lambda x.M'$. The claim holds by inductive hypothesis.

Proposition. 42. [Simulation] The translation $(.)_{N}$ is sound and complete; it preserves surface reduction and surface normal forms. Let $M \in \Lambda_{\oplus}^{\text{cbn}}$; the following hold:

- 1) if $M \rightarrow n$ then $(M)_{\scriptscriptstyle N} \rightarrow (n)_{\scriptscriptstyle N}$;
- 2) if $M \stackrel{s}{\rightarrow} n$ then $(M)_{N} \stackrel{s}{\rightarrow} (n)_{N}$;
- 3) if $(M)_{\mathbb{N}} \to \mathbb{S}$ then $\exists ! \mathbb{n}$ such that $\mathbb{S} = (\mathbb{n})_{\mathbb{N}}$ and $M \to \mathbb{n}$;
- 4) if $(M)_{\mathbb{N}} \stackrel{s}{\to} s$ then $\exists ! n$ such that $s = (n)_{\mathbb{N}}$ and $M \stackrel{s}{\to} n$;
- 5) $M \in \mathcal{H}$ if and only if $(M)_{\mathbb{N}} \in \mathcal{S}^!$.

Proof. We prove (1.)-(4.); since $\rightarrow = \rightarrow_{\beta} \cup \rightarrow_{\oplus}$, we deal separately with the two reductions. Point (5.) is an immediate consequence of the other points.

- \rightarrow_{β} We deal with \rightarrow_{β} via simulation in $\Lambda^{\rm cbn}$ and $\Lambda^!$, since the analogous result is proved in [17]. We have defined a translation $(-)_!:\Lambda^!_{\oplus}\to\Lambda^!$ which is sound and complete, and preserves surface reduction. It is straightforward to define a similar translation from $\Lambda^{\rm cbn}_{\oplus}$ into $\Lambda^{\rm cbn}$. Therefore, if in $\Lambda^{\rm cbn}_{\oplus}$ it holds $M\to_{\beta}N$, we translate in $\Lambda^{\rm cbn}$, use the result in [17] and conclude (via simulation) that $(M)_{\mathbb{N}}\to_{\beta}(N)_{\mathbb{N}}$ in $\Lambda^!_{\oplus}$. Similarly for (2.)-(3.)-(4.).
- \rightarrow_{\oplus} Immediate consequence of Lemma 58, which proves that that $(\cdot)_{_{\mathbb{N}}}$ preserves both surface contexts and \oplus -redexes.