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COMBINING MODAL LOGICS

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1 INTRODUCTION

When can we say that a logic is a *combination* of others? In general, *any* logical system having more than one connective can be considered as a combination of logical systems having fewer connectives. In particular, *any* multimodal logic can be considered as a combination of, say, unimodal logics. So, in this general sense, *any* result on multimodal logics can be considered as a result on combining modal logics. What makes this chapter special among other ones studying multimodal logics is that here we investigate the following kind of problems:

> *Given a family* L *of modal logics and a combination method* C*, do certain properties of the 'component logics'* $L \in \mathsf{L}$ *transfer to their 'combination'* $\mathsf{C}(\mathsf{L})$*?*

Most of the *combination methods* considered in this chapter satisfy the following three criteria:

(C1) They are *finitary*, that is, C is defined only on *finite* families L of modal logics.

(C2) The combination $\mathsf{C}(\mathsf{L})$ of (multi)modal logics from L is a (multi)modal logic itself.

(C3) The combined logic $\mathsf{C}(\mathsf{L})$ is an extension of each component logic $L \in \mathsf{L}$.

For each considered combination method, we discuss in detail the possible transfer of the following two kinds of properties:

- **Axiomatisation/completeness.**
  There are two versions, depending on whether the combination method results in a syntactically or semantically defined logic. In the former case, the question is whether the combination of recursively (finitely) axiomatisable components remains recursively (finitely) axiomatisable, and in the latter, whether the Kripke completeness of the components transfers to their combination.

- **Decidability/complexity of the validity/satisfiability problem.**
  We study whether decidability of the validity problem transfers from the components to their combination and if so, what is the change in complexity. We also discuss the possible transfer of the finite model property.

For transfer results about several other properties (like versions of interpolation, decidability of various consequence relations, etc.) see [23] and the references therein. Combinations of deductive calculi (such as combined tableaux) are not considered either, see Chapter 2 of this handbook for some examples.

Combination methods not satisfying **(C1)**–**(C3)** are in general out of our scope, though see Section 5 for a discussion.

Notation and terminology. We will mainly consider possible world (or Kripke) semantics. *Kripke models* are pairs $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ that are based on relational structures $\mathfrak{F} = \langle W, R_1, \ldots, R_n \rangle$, where $n > 0$ is a natural number, W is a non-empty set and the R_i are binary relations on it. Such structures are called *n-frames* (or *frames*, for short). We say that an n-frame $\mathfrak{G} = \langle U, S_1, \ldots, S_n \rangle$ is a *subframe* of an n-frame $\mathfrak{F}$ ($\mathfrak{G} \subseteq \mathfrak{F}$, in symbols) if $U \subseteq W$ and $S_i = R_i \cap (U \times U)$, for $i = 1, \ldots, n$. A *path of length* k from point x to point y in an n-frame $\mathfrak{F}$ is a sequence $\langle x_0, \ldots, x_k \rangle$ of points such that $x_0 = x$, $x_k = y$ and $x_i R_j x_{i+1}$, for each $i < k$ and some j, $1 \le j \le n$. We call an n-frame $\mathfrak{F}$ *rooted* if there exists some $x \in W$ such that for every $y \in W$, $y \neq x$ there is a path from x to y. Such an x is called a *root of* $\mathfrak{F}$. We say that $\mathfrak{F}$ is of *depth* k if k is the length of the longest path in $\mathfrak{F}$. If such a longest path does not exist, then we say that $\mathfrak{F}$ is of *infinite depth.* An n-frame $\mathfrak{F}$ is called *tree-like* if it is rooted and $R = \bigcup_{i=1}^{n} R_i$ is weakly connected on the set $\{y \in W \mid yRx\}$ for every $x \in W$. If a tree-like frame is well-founded (i.e., there are no infinite descending R-chains $\ldots Rx_2 Rx_1 Rx_0$ of points) then we call $\mathfrak{F}$ a *tree.* The *depth* $d^{\mathfrak{F}}(x)$ of a point x in a tree $\mathfrak{F}$ is defined to be the length of the unique path from the root to x. If for no $n < \omega$ the point x is of depth n, then we say that x is of *infinite depth.* By the *co-depth* of a point x in a tree $\mathfrak{F}$ we understand the depth of the subtree of $\mathfrak{F}$ with root x.

Given a natural number n, the *n-modal language* $\mathcal{ML}_n$ has propositional variables $p, q, s, \ldots$, Boolean connectives $\neg$, $\wedge$, $\vee$, $\to$, $\leftrightarrow$, $\top$, $\bot$, and (unary) modal operators $\Box_1, \ldots, \Box_n$ and $\Diamond_1, \ldots, \Diamond_n$. $\mathcal{ML}_n$-formulas are formed inductively in the usual way. Given an $\mathcal{ML}_n$-formula φ, we let $sub\,\varphi$ denote the set of all subformulas of φ, and $md(\varphi)$ denote the *modal depth* of φ. We will also use the following abbreviations. For every formula φ, let

$$\Box^0\varphi = \varphi \qquad \text{and, for } n < \omega, \quad \Box^{n+1}\varphi = \Box\Box^n\varphi, \quad \Box^{\le n}\varphi = \bigwedge_{k \le n} \Box^k\varphi.$$

The *truth-relation* '$(\mathfrak{M}, w) \models \varphi$' connecting syntax and semantics is defined by induction on the construction of φ as usual. We say that φ is *true in* $\mathfrak{M}$ ($\mathfrak{M} \models \varphi$, in symbols), if $\mathfrak{M}, w \models \varphi$ for all $x \in W$. A formula φ is said to be *valid in a frame* $\mathfrak{F}$ ($\mathfrak{F} \models \varphi$, in symbols), if $\mathfrak{M} \models \varphi$ for every model $\mathfrak{M}$ that is based on $\mathfrak{F}$. Given a set Σ of formulas, we set

$$\mathsf{Fr}\,\Sigma \ = \ \{\mathfrak{F} \mid \mathfrak{F} \models \varphi, \text{ for all } \varphi \in \Sigma\}.$$

If $\mathfrak{M} \models \varphi$ for all $\varphi \in \Sigma$ then we say that $\mathfrak{M}$ is a *model for* Σ. Similarly, φ is said to be a *frame for* Σ, if $\mathfrak{F} \in \mathsf{Fr}\,\Sigma$.

By an *n-modal logic* (or *modal logic*[1], for short) we mean any set L of $\mathcal{ML}_n$-formulas that contains all valid formulas of classical propositional logic, the formulas

(K) $\qquad \Box_i(p \to q) \to (\Box_i p \to \Box_i q)$,

and is closed under the rules of Substitution, Modus Ponens and Necessitation, for $i = 1, \dots, n$ (see Chapter 2 of this handbook).

Let us briefly discuss two of the most common ways of defining a modal logic: the 'syntactical' way (via axioms) and the 'semantical' way (via a class of intended frames). First, given a set Σ of $\mathcal{ML}_n$-formulas and an n-modal logic L, we say that L *is axiomatised by* Σ, if L is the smallest n-modal logic containing Σ. If Σ can be chosen a recursive (or finite) subset of all $\mathcal{ML}_n$-formulas, then we say that L is *recursively* (*finitely*) *axiomatisable.* And second, given a class $\mathcal{C}$ of n-frames, the set

$$\mathsf{Log}\,\mathcal{C} = \{\varphi \mid \mathfrak{F} \models \varphi, \text{ for all } \mathfrak{F} \in \mathcal{C}\}$$

is always an n-modal logic. An n-modal logic L is called *Kripke complete* if $L = \mathsf{Log}\,\mathcal{C}$ for some class $\mathcal{C}$ of n-frames. In this case we also say that L is *characterised* (or *determined*) by $\mathcal{C}$. As is well-known, there exist incomplete modal logics, and similarly, there are Kripke complete logics that are not recursively axiomatisable (see Section 3.4 for some examples).

The *validity problem* for an n-modal logic L is the problem of deciding whether a given $\mathcal{ML}_n$-formula belongs to L or not. If this problem is decidable (or recursively enumerable) then we also say that the logic L is *decidable* (or *recursively enumerable*). A related problem is the *satisfiability problem* for L: given φ, decide whether φ is L-*satisfiable*, that is, whether there exists a model $\mathfrak{M}$ for L and a world w in $\mathfrak{M}$ such that $\mathfrak{M}, w \models \varphi$ holds. It is easy to see the connection between the two: $\varphi \in L$ iff $\neg\varphi$ is not L-satisfiable. Given a recursively enumerable logic L, we can have a decision algorithm for L if we can enumerate those formulas that are not in L. Clearly, this can be done if:

- the class of finite frames for L is recursively enumerable (up to isomorphism, of course), and
- L has the *finite model property*[2], that is

$$L = \mathsf{Log}\,\{\mathfrak{F} \in \mathsf{Fr}\,L \mid \mathfrak{F} \text{ is finite}\}.$$

This chapter is not self-contained in the sense that we discuss well-known modal logics like **K**, **S5**, **KD45**, **K4**, **S4**, **K4.3**, **GL**, **Grz**, **Alt**, etc. without defining them. We also use without explicit reference standard notions and results from basic modal logic,

[1]We consider here what are usually called *normal* modal logics only.

[2]It would be more precise to call this *finite frame property.* However, as is well-known, it is equivalent to saying that $L = \{\varphi \mid \mathfrak{M} \models \varphi, \text{ for all finite models } \mathfrak{M} \text{ for } L\}$.

such as p-morphisms and disjoint unions, generated subframes, unravelling, results on Sahlqvist formulas and canonicity, etc. For notions and statements not defined or proved here, see other chapters in this handbook or [12, 10].

2 FUSION OF MODAL LOGICS

Within the constraints **(C1)**–**(C3)** above, the formation of *fusions* (also known as *independent joins*), is the simplest and perhaps the most natural way of combining modal logics:

DEFINITION 1. Let L_1 and L_2 be two modal logics formulated in languages $\mathcal{ML}_n$ and $\mathcal{ML}_m$ in such a way that they have disjoint sets of modal operators (say, $\Box_1, \dots, \Box_n$ and $\Box_{n+1}, \dots, \Box_{n+m}$, respectively). Then the *fusion*

$$L_1 \otimes L_2$$

of L_1 *and* L_2 is the smallest $(n+m)$-modal logic L containing both L_1 and L_2.

It is easy to see that if each L_i is axiomatised by a set Σ_i of axioms (written in the respective languages) then $L_1 \otimes L_2$ is axiomatised by the union $\Sigma_1 \cup \Sigma_2$. This means that *no* axiom containing modal operators from both of the languages of L_1 and L_2 is required to axiomatise $L_1 \otimes L_2$. In other words, in fusions the modal operators of the component logics are kind of 'independent,' they 'do not interact'.

The formation of fusions is clearly an associative binary operation on modal logics. Therefore, one can define the fusion

$$L_1 \otimes L_2 \otimes \cdots \otimes L_n$$

of n modal logics in a straightforward way, for any natural number $n \geq 2$. Observe that well-known multimodal logics like $\mathbf{K}_n$ or $\mathbf{S5}_n$ are the fusions of their unimodal 'counterparts':

$$\mathbf{K}_n = \underbrace{\mathbf{K} \otimes \cdots \otimes \mathbf{K}}_{n}, \qquad \mathbf{S5}_n = \underbrace{\mathbf{S5} \otimes \cdots \otimes \mathbf{S5}}_{n}.$$

The formation of fusions as a combination method does satisfy criterion **(C3)**, as the following result of Thomason [78] shows:

THEOREM 2. *The fusion of consistent modal logics is a conservative extension of the components.*

2.1 Transfer results

We begin with the following result of Kracht and Wolter [48], and Fine and Schurz [16] stating that *Kripke completeness* of the components transfers to their fusion:

THEOREM 3. *If modal logics* L_1 *and* L_2 *are characterised by classes of frames* $\mathcal{C}_1$ *and* $\mathcal{C}_2$*, respectively, and if* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *are closed under the formation of disjoint unions and isomorphic copies, then the fusion* $L_1 \otimes L_2$ *of* L_1 *and* L_2 *is characterised by the class*

$$\mathcal{C}_1 \otimes \mathcal{C}_2 = \big\{\langle W, R_1, \dots, R_n, S_1, \dots, S_m\rangle \mid \langle W, R_1, \dots, R_n\rangle \in \mathcal{C}_1,\ \langle W, S_1, \dots, S_m\rangle \in \mathcal{C}_2\big\}.$$

It should be clear that if $\mathcal{C}_1$ and $\mathcal{C}_2$ determine logics L_1 and L_2, respectively, then all frames in $\mathcal{C}_1 \otimes \mathcal{C}_2$ are frames for the fusion $L_1 \otimes L_2$. Let us outline the proof of the converse statement, i.e., that $\mathcal{C}_1 \otimes \mathcal{C}_2$ actually characterises $L_1 \otimes L_2$. To simplify notation, we assume that L_1 and L_2 are unimodal logics with the boxes $\Box_1$ and $\Box_2$, respectively. The fusion $L = L_1 \otimes L_2$ is then a bimodal logic in the language $\mathcal{ML}_2$.

With each $\mathcal{ML}_2$-formula φ of the form $\Box_i \psi$ $(i = 1, 2)$ we associate a new variable q_φ which will be called the *surrogate* of φ. For an $\mathcal{ML}_2$-formula φ containing no surrogate variables, denote by $\varphi^{\mathbf{1}}$ the formula that results from φ by replacing all its subformulas of the form $\Box_2 \psi$, which are not within the scope of other $\Box_2$, with their surrogate variables $q_{\Box_2 \psi}$. So $\varphi^{\mathbf{1}}$ is a unimodal formula containing only $\Box_1$. Let

$$\Theta^{\mathbf{1}}(\varphi) = \{p \mid p \text{ is a variable in } \varphi\} \cup \{\chi \in sub\, \Box_2 \psi \mid \Box_2 \psi \in sub\, \varphi\}.$$

The formula $\varphi^{\mathbf{2}}$ and the set $\Theta^{\mathbf{2}}(\varphi)$ are defined symmetrically.

Suppose now that φ is satisfiable in a model based on a frame for L. We need to construct a frame in $\mathcal{C}_1 \otimes \mathcal{C}_2$ satisfying φ. As we know only how to build frames for the unimodal fragments of L, the frame is constructed step-by-step alternating between $\Box_1$ and $\Box_2$.

Note first that since L_1 is characterised by $\mathcal{C}_1$, there is a model $\mathfrak{M}$ based on a frame in $\mathcal{C}_1$ and satisfying $\varphi^{\mathbf{1}}$ at a point r. Our aim now is to ensure that the formulas of the form $\Box_2 \psi$ have the same truth-values as their surrogates $q_{\Box_2 \psi}$. To do this, with each point x in $\mathfrak{M}$ we can associate the formula

$$\varphi_x = \bigwedge \{\psi \in \Theta^{\mathbf{1}}(\varphi) \mid (\mathfrak{M}, x) \models \psi^{\mathbf{1}}\} \wedge \bigwedge \{\neg \psi \mid \psi \in \Theta^{\mathbf{1}}(\varphi),\ (\mathfrak{M}, x) \not\models \psi^{\mathbf{1}}\},$$

construct a model $\mathfrak{M}_x$ based on a frame in $\mathcal{C}_2$ and satisfying $\varphi_x^{\mathbf{2}}$ in a world y, and then hook $\mathfrak{M}_x$ to $\mathfrak{M}$ by identifying x and y. After that we can switch to $\Box_1$ and in the same manner ensure that formulas $\Box_1 \psi$ have the same truth-values as $q_{\Box_1 \psi}$ at all points in every $\mathfrak{M}_x$, and so on. In this construction we use the fact that $\mathcal{C}_1$ and $\mathcal{C}_2$ are closed under isomorphic copies and disjoint unions: the $\mathfrak{M}_x$ should be mutually disjoint and the final model is the union of the models constructed at each step. Note that this construction is a special case of *fibring semantics* that is called iterated *dovetailing* [19, 20].

However, to realise this quite obvious scheme, we must be sure that $\varphi_x^{\mathbf{2}}$ is really satisfiable in a frame for L_2, which may impose some restrictions on the models we choose. First, in the construction above it is enough to deal with points x accessible from r in at most $md(\varphi)$ steps; no other point has any influence on the truth of φ at r. Let X be the set of all such points. Now, a sufficient and necessary condition for φ_x to be satisfiable in a frame for L (and so for $\varphi_x^{\mathbf{2}}$ to be satisfiable in a frame for L_2) can be formulated using the following general description of formulas of type φ_x.

Suppose Γ is a finite set of formulas closed under subformulas. Define the *consistency-set* $C(\Gamma)$ *of* Γ by taking

$$C(\Gamma) = \{\psi_\Delta \mid \Delta \subseteq \Gamma\},$$

where for $\Delta \subseteq \Gamma$,

$$\psi_\Delta = \bigwedge \{\chi \mid \chi \in \Delta\} \wedge \bigwedge \{\neg \chi \mid \chi \in \Gamma - \Delta\}.$$

In particular, for all $x \in X$, we have $\varphi_x \in C(\Theta^{\mathbf{1}}(\varphi))$. Given a formula φ, define

$$\Sigma_1(\varphi) = \{\psi \in C(\Theta^{\mathbf{1}}(\varphi)) \mid \neg \psi \notin L\}, \qquad \Sigma_2(\varphi) = \{\psi \in C(\Theta^{\mathbf{2}}(\varphi)) \mid \neg \psi \notin L\}.$$

The formulas in $\Sigma_i(\varphi)$ can be regarded as 'state descriptions' of the points in the possible models with respect to the formulas in $\Theta^{\mathbf{i}}(\varphi)$. In particular, for all $x \in X$, φ_x is satisfiable in a frame for L iff $\varphi_x \in \Sigma_1(\varphi)$. In other words, we should start with a model $\mathfrak{M}$ satisfying $\varphi^{\mathbf{1}} \wedge \Box_1^{\leq md(\varphi)}(\bigvee \Sigma_1(\varphi))^{\mathbf{1}}$ at a point r. Of course, the subsequent models $\mathfrak{M}_x$ must satisfy $\varphi_x^{\mathbf{2}} \wedge \Box_2^{\leq md(\varphi)}(\bigvee \Sigma_2(\varphi_x))^{\mathbf{2}}$ at all points $x \in X$, and so on. The interested reader may find more details in [48], [16].

Since the closure under *finite* disjoint unions is enough when we work with finite frames, we obtain the following:

THEOREM 4. *If both L_1 and L_2 are modal logics having the finite model property, then their fusion $L_1 \otimes L_2$ has the finite model property as well.*

As is shown by Wolter [82], *decidability* of the components also transfers to their fusion:

THEOREM 5. *If L_1 and L_2 are both decidable modal logics then $L_1 \otimes L_2$ is decidable as well.*

Further results showing that other important properties (such as Halldén completeness, decidability of the global consequence relation, uniform interpolation property) of modal logics are preserved under fusions were obtained in [48, 82].

As is discussed in Chapter 6 of this handbook, from the algebraic point of view every modal logic L can be regarded as the equational theory of modal algebras generated by the equations $\{`\varphi = 1' \mid \varphi \in L\}$. Thus, the problem of whether decidability is preserved under the formation of fusions of modal logics is an instance of the more general question: under which conditions does the decidability of two equational theories T_1 and T_2 imply the decidability of the union $T_1 \cup T_2$. The shared Boolean connectives impose special conditions on these equational theories; see the results of Ghilardi [29] that put the fusion construction to this more general context. Other extensions of Theorem 5 to fusions sharing not only the Booleans but also a universal modality and nominals are discussed in [30], and to fusions of non-normal modal logics in [6, 4].

2.2 Complexity of fusions

Unlike the properties considered above, upper complexity bounds do not always transfer under the formation of fusions (the lower bounds are inherited by Theorem 2 as long as we take fusions of consistent logics). The known decision procedures provide a time complexity bound for the fusion that is non-deterministic and one exponent higher than the maximal time complexity of the components. However, in general it is not known whether this increase in complexity is unavoidable. In particular, it is not known whether PSPACE- or EXPTIME-completeness transfers under the formation of fusions (see Theorem 7 below for some special cases when it actually does).

The following characterisation of the transfer of coNP-completeness was given by Spaan [77]. In order to formulate her theorem, we require the following notion. Say that a frame $\langle W', R' \rangle$ is a *skeleton subframe* of a frame $\langle W, R \rangle$ if $W' \subseteq W$ and $R' \subseteq R$. We use $\circ$ to denote reflexive points and $\bullet$ for irreflexive ones.

THEOREM 6. *Suppose that the unimodal logics L_1 and L_2 are characterised by classes $\mathcal{C}_1$ and $\mathcal{C}_2$ of frames, resprectively, that are closed under the formation of isomorphic copies and disjoint unions. Then there are the following three cases for the complexity of $L_1 \otimes L_2$ (below $\{i, j\} = \{1, 2\}$):*

(1) *$L_1 \otimes L_2$ is* coNP*-complete.*

(2) *$\mathcal{C}_i$ consists of disjoint unions of singleton frames. In this case $L_1 \otimes L_2$ is polynomially reducible to* $\mathsf{Log}(\mathcal{C}_j)$.

(3) *$L_1 \otimes L_2$ is* PSPACE*-hard, whenever one of the following six cases holds:*

(i) $\bullet\!\leftarrow\!\bullet\!\rightarrow\!\bullet$ *and* $\bullet\!\rightarrow\!\bullet$ *are skeleton subframes of some frames in $\mathcal{C}_i$ and $\mathcal{C}_j$, respectively;*

(ii) $\circ\!\rightarrow\!\bullet\!\rightarrow\!\bullet$ *and* $\bullet\!\rightarrow\!\bullet$ *are skeleton subframes of some frames in $\mathcal{C}_i$ and $\mathcal{C}_j$, respectively;*

(iii) $\bullet\!\rightarrow\!\circ\!\rightarrow\!\bullet$ *and* $\bullet\!\rightarrow\!\bullet$ *are skeleton subframes of some frames in $\mathcal{C}_i$ and $\mathcal{C}_j$, respectively;*

(iv) $\bullet\!\rightarrow\!\bullet\!\rightarrow\!\bullet$ *and* $\circ\!\rightarrow\!\bullet$ *are skeleton subframes of some frames in $\mathcal{C}_i$ and $\mathcal{C}_j$, respectively;*

(v) $\bullet\!\rightarrow\!\bullet\!\rightarrow\!\bullet$ *and* $\circ\!\rightarrow\!\bullet$ *are skeleton subframes of a frame in $\mathcal{C}_i$ and* $\bullet\!\leftrightarrow\!\bullet$ *is a skeleton subframe of a frame in $\mathcal{C}_j$;*

(vi) $\bullet\!\rightarrow\!\bullet\!\rightarrow\!\bullet$ *and* $\circ\!\rightarrow\!\bullet$ *are skeleton subframes of a frame in $\mathcal{C}_i$ and* $\bullet\!\rightarrow\!\circ$ *is a skeleton subframe of a frame in $\mathcal{C}_j$.*

A close inspection of this result shows that almost all interesting fusions are PSPACE-hard. (An exception is the fusion **Alt** $\otimes$ **Alt** of two **Alt** logics that is coNP-complete by Theorem 6. We remind the reader that **Alt** is the coNP-complete logic determined by all *functional* frames.) In fact, the proof of Halpern and Moses [35] can be easily modified to obtain the following result on a matching upper bound for several 'standard' fusions:

THEOREM 7. *Let $n > 1$ and $L_i \in \{\mathbf{K}, \mathbf{T}, \mathbf{K4}, \mathbf{S4}, \mathbf{KD45}, \mathbf{S5}\}$, for all $1 \leq i \leq n$. Then $L_1 \otimes \cdots \otimes L_n$ is* PSPACE*-complete.*

Note that while **K**, **T**, **K4** and **S4** are PSPACE-complete themselves, **KD45** and **S5** are coNP-complete.

3 PRODUCT OF MODAL LOGICS

The formation of Cartesian products of various structures—vector and topological spaces, algebras, etc.—is a standard mathematical way of capturing the multidimensional character of our world. In modal logic, products of Kripke frames are natural constructions allowing us to reflect interactions between modal operators representing time, space, knowledge, actions, etc. The product construction as a combination method on modal logics was introduced in [74, 75, 24] and has been used in applications in computer science and artificial intelligence ever since (see, e.g., [68, 15, 7, 69, 18], and [23] and references therein).

DEFINITION 8. The *product* of two n-frames frames $\mathfrak{F}_1 = \langle W_1, R_1^1, \ldots, R_1^n \rangle$ and $\mathfrak{F}_2 = \langle W_2, R_2^1, \ldots, R_2^m \rangle$ is the $(n+m)$-frame

$$\mathfrak{F}_1 \times \mathfrak{F}_2 = \langle W_1 \times W_2, R_h^1, \ldots, R_h^n, R_v^1, \ldots, R_v^m \rangle$$

where $W_1 \times W_2 = \{\langle u, v \rangle \mid u \in W_1,\ v \in W_2\}$ and, for all $u_1, u_2 \in W_1$ and $v_1, v_2 \in W_2$,

$$\begin{array}{lll} \langle u_1, v_1 \rangle \, R_h^i \, \langle u_2, v_2 \rangle & \text{iff} & u_1 R_1^i u_2 \ \text{ and } \ v_1 = v_2 \quad (1 \leq i \leq n), \\ \langle u_1, v_1 \rangle \, R_v^j \, \langle u_2, v_2 \rangle & \text{iff} & u_1 = u_2 \ \text{ and } \ v_1 R_2^j v_2 \quad (1 \leq j \leq m). \end{array}$$

Such a frame will be called a *product frame*. The subscripts h and v appeal to the geometrical intuition of considering the R_h^i as 'horizontal' accessibility relations in $\mathfrak{F}_1 \times \mathfrak{F}_2$ and the R_v^j as 'vertical' ones; see Fig. 1 for an illustration.

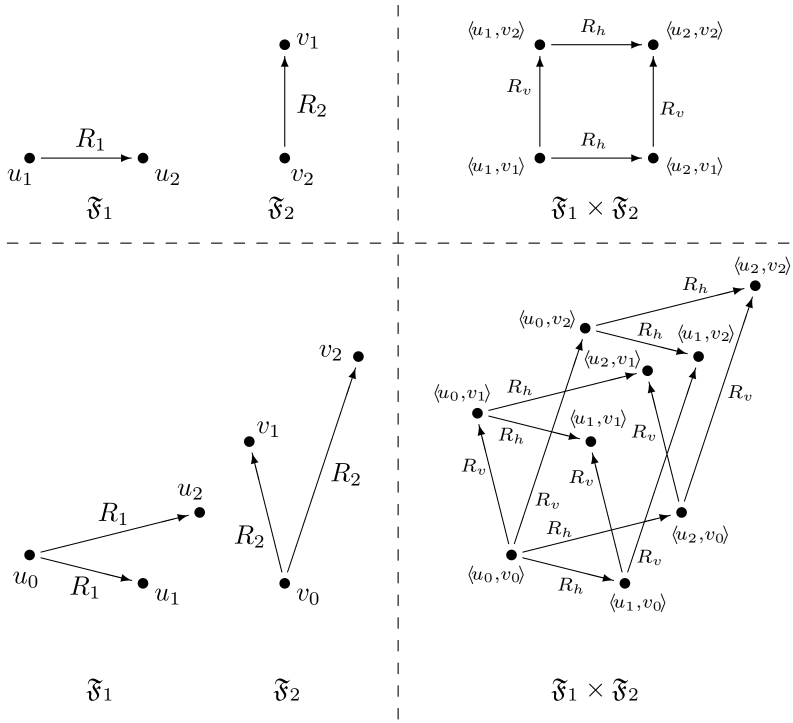


Figure 1. Product frames.

It is not hard to see that the product construction commutes with the three basic operations on frames:

PROPOSITION 9. *For all frames $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$, $\mathfrak{H}_i$, $i \in I$, the following hold:*

(i) *If $\mathfrak{F}$ is a p-morphic image of $\mathfrak{H}$, then $\mathfrak{F} \times \mathfrak{G}$ is a p-morphic image of $\mathfrak{H} \times \mathfrak{G}$.*

(ii) *If $\mathfrak{F}$ is a generated subframe of $\mathfrak{H}$, then $\mathfrak{F} \times \mathfrak{G}$ is a generated subframe of $\mathfrak{H} \times \mathfrak{G}$.*

(iii) *If $\mathfrak{F}$ is a disjoint union of $\mathfrak{H}_i$, $i \in I$, then $\mathfrak{F} \times \mathfrak{G}$ is isomorphic to the disjoint union of $\mathfrak{H}_i \times \mathfrak{G}$, $i \in I$.*

Products of Kripke frames can be used to define a natural combination method on modal logics:

DEFINITION 10. Let L_1 and L_2 be two Kripke complete modal logics formulated in languages $\mathcal{ML}_n$ and $\mathcal{ML}_m$ in such a way that they have disjoint sets of modal operators (say, $\Box_1, \dots, \Box_n$ and $\Box_{n+1}, \dots, \Box_{n+m}$, respectively). Then the *product of* L_1 *and* L_2 is the modal logic

$$L_1 \times L_2 \;=\; \mathsf{Log}\{\mathfrak{F}_1 \times \mathfrak{F}_2 \mid \mathfrak{F}_i \in \mathsf{Fr}L_i,\ i = 1, 2\}.$$

For example, $\mathbf{K}_n \times \mathbf{K}_m$ is the $(n+m)$-modal logic determined by all product frames $\mathfrak{F}_1 \times \mathfrak{F}_2$, where $\mathfrak{F}_1$ is an n-frame and $\mathfrak{F}_2$ an m-frame; $\mathbf{S4} \times \mathbf{S5}$ is the bimodal logic determined by all product frames $\mathfrak{F}_1 \times \mathfrak{F}_2$ such that $\mathfrak{F}_1$ is reflexive and transitive, and $\mathfrak{F}_2$ is an equivalence frame.

Note that the product of Kripke complete modal logics is always Kripke complete by definition. It is important to emphasise that in order to make the product construction a well-defined combination method on Kripke complete modal logics, we have to consider products of *all* possible Kripke frames for L_1 and L_2. The reason is that even if $\mathsf{Log}\,\mathcal{C}_1 = \mathsf{Log}\,\mathcal{C}_1'$ and $\mathsf{Log}\,\mathcal{C}_2 = \mathsf{Log}\,\mathcal{C}_2'$, then we can have

$$\mathsf{Log}\{\mathfrak{F}_1 \times \mathfrak{F}_2 \mid \mathfrak{F}_i \in \mathcal{C}_i,\ i=1,2\} \ \neq\ \mathsf{Log}\{\mathfrak{F}_1 \times \mathfrak{F}_2 \mid \mathfrak{F}_i \in \mathcal{C}_i',\ i=1,2\},$$

see [23] for examples.

There are several attempts for extending the product construction from Kripke complete logics to arbitrary modal logics, mainly by considering product-like constructions on Kripke models, see [37, 23]. All the suggested methods so far result in sets of formulas that are not closed under the rule of Substitution, thus do not satisfy our criterion **(C2)**. Van Benthem *et al*. [79] show that by defining a product-like operator on their topological semantics, one can get back the *fusion* of modal logics determined by transitive frames.

Once the two-dimensional definition is given, there are essentially two ways of defining products of three or more modal logics. First, we can generalise in a straightforward way the definitions above. To simplify notation, from now on we will mostly consider products of unimodal frames and logics only. (However, we will discuss the multimodal versions in those cases when it does make a difference.)

DEFINITION 11. Given a natural number $n > 1$, the *product* of frames $\mathfrak{F}_1 = \langle W_1, R_1\rangle$, $\mathfrak{F}_2 = \langle W_2, R_2\rangle$, ..., $\mathfrak{F}_n = \langle W_n, R_n\rangle$ is the n-frame

$$\mathfrak{F}_1 \times \cdots \times \mathfrak{F}_n \ =\ \big\langle W_1 \times \cdots \times W_n, \bar{R}_1, \ldots, \bar{R}_n\big\rangle$$

where, for each $i = 1, \ldots, n$, $\bar{R}_i$ is a binary relation on $W_1 \times \cdots \times W_n$ such that

$$\langle u_1, \ldots, u_n\rangle\, \bar{R}_i\, \langle v_1, \ldots, v_n\rangle \quad \text{iff} \quad u_i R_i v_i \ \text{ and } \ u_k = v_k, \ \text{ for } k \neq i.$$

Then, given Kripke complete (uni)modal logics L_i formulated in the language having $\Box_i$ $(i = 1, \ldots, n)$, the *product of* $L_1, \ldots, L_n$ is the n-modal logic

$$L_1 \times \cdots \times L_n \ =\ \mathsf{Log}\{\mathfrak{F}_1 \times \cdots \times \mathfrak{F}_n \mid \mathfrak{F}_i \in \mathsf{Fr}L_i,\ i = 1, \ldots, n\}.$$

For example, $\mathbf{K}^n = \overbrace{\mathbf{K} \times \cdots \times \mathbf{K}}^{n}$ is the logic determined by all n-dimensional product frames; $\mathbf{S5}^n$ is the logic determined by all product frames $\mathfrak{F}_1 \times \cdots \times \mathfrak{F}_n$, where each $\mathfrak{F}_i$ is a (possible different) equivalence frame.

The second way would be to define $L_1 \times \cdots \times L_n$ as $(((L_1 \times L_2) \times L_3) \times \cdots \times L_{n-1}) \times L_n$. The easily established fact that the frame $\mathfrak{F}_1 \times \cdots \times \mathfrak{F}_n$ is isomorphic to

$$(((\mathfrak{F}_1 \times \mathfrak{F}_2) \times \mathfrak{F}_3) \times \cdots \times \mathfrak{F}_{n-1}) \times \mathfrak{F}_n$$

might seem to suggest that the two definitions are equivalent. However, the situation is not that simple. For example, it is not known whether the equalities

$$\mathbf{K}^4 \stackrel{?}{=} \mathbf{K}^3 \times \mathbf{K} \qquad \text{and} \qquad \mathbf{S5}^4 \stackrel{?}{=} \mathbf{S5}^3 \times \mathbf{S5}$$

hold. The problem here is that $\mathbf{K}^4$ is characterised by the class of products of four 1-frames, while $\mathbf{K}^3 \times \mathbf{K}$ by the class of products of *arbitrary* (that is, not necessarily product) 3-frames for $\mathbf{K}^3$ and 1-frames for $\mathbf{K}$. Now, the thing is that these arbitrary $\mathbf{K}^3$-frames are not necessarily isomorphic to product frames (in fact, we do not even know what they look like; see Theorem 25).

For this reason, we take Definition 11 above as the 'official' definition of higher dimensional product logics. Note, however, that in Section 3.3 we provide a characterisation of arbitrary (countable) frames for $\mathbf{K}\times\mathbf{K}$ and $\mathbf{S5}\times\mathbf{S5}$ (among many other two-dimensional product logics), and prove—with the help of this characterisation—that for many three-dimensional products the two definitions actually coincide. For instance,

$$\mathbf{K}^3 \;=\; (\mathbf{K}\times\mathbf{K})\times\mathbf{K} \qquad \text{and} \qquad \mathbf{S5}^3 \;=\; (\mathbf{S5}\times\mathbf{S5})\times\mathbf{S5},$$

see Corollary 23.

3.1 General transfer results

Compared to fusions, there are very few general transfer results for products. In fact, as we shall see in Sections 3.3 and 3.4, for many cases the lack of transfer of finite axiomatisability and decidability is the 'norm'.

In this section we discuss some basic properties of the product construction and the very few general transfer results about it. To begin with, observe that in the definition of product logics it is enough to consider only *rooted* frames for the component logics. Indeed, the inclusion

$$L_1\times\cdots\times L_n \;\subseteq\; \mathsf{Log}\{\mathfrak{F}_1\times\cdots\times\mathfrak{F}_n \mid \mathfrak{F}_i \text{ is a rooted frame for } L_i,\ i=1,\dots,n\}$$

should be clear. To show the converse, suppose $\varphi\notin L_1\times\cdots\times L_n$, i.e., φ is refuted at a point $\langle u_1,\dots,u_n\rangle$ in some model based on a product frame $\mathfrak{F}_1\times\cdots\times\mathfrak{F}_n$, where $\mathfrak{F}_i$ is a frame for L_i, $i=1,\dots,n$. For each i, let $\mathfrak{G}_i$ be the subframe of $\mathfrak{F}_i$ generated by u_i. Then $\mathfrak{G}_i$ is also a frame for L_i, for $i=1,\dots,n$. On the other hand, it is readily checked that $\mathfrak{G}_1\times\cdots\times\mathfrak{G}_n$ is isomorphic to the subframe of $\mathfrak{F}_1\times\cdots\times\mathfrak{F}_n$ generated by $\langle u_1,\dots,u_n\rangle$. Thus we obtain the following:

PROPOSITION 12. *For all Kripke complete modal logics $L_1,\dots,L_n$,*

$$L_1\times\cdots\times L_n \;=\; \mathsf{Log}\{\mathfrak{F}_1\times\cdots\times\mathfrak{F}_n \mid \mathfrak{F}_i \textit{ is a rooted frame for } L_i,\ i=1,\dots,n\}.$$

For instance, $\mathbf{S5}^n$ is determined by products of universal frames $\langle W_i, W_i\times W_i\rangle$, $i=1,\dots,n$. Moreover, each such 'universal product frame' is a p-morphic image of a *cubic universal product frame*, i.e., the nth power of the same universal frame $\langle W, W\times W\rangle$. Indeed, it is easy to see that if a set W is such that there are surjections $f_i : W\to W_i$, for $i=1,\dots,n$, then the map f defined by

$$f(w_1,\dots,w_n)=\langle f_1(w_1),\dots,f_n(w_n)\rangle$$

is a p-morphism from the frame $\langle W, W\times W\rangle^n$ onto

$$\langle W_1, W_1\times W_1\rangle\times\cdots\times\langle W_n, W_n\times W_n\rangle\,.$$

Such a set and surjections can be found, for example, by taking the disjoint union of the W_i as W and defining f_i so that it is the identity map on W_i and arbitrary otherwise. Therefore, we obtain:

PROPOSITION 13. $\mathbf{S5}^n$ *is determined by the class of all cubic universal product frames.*

The formation of products as a combination method satisfies criterion **(C3)**, as the following proposition shows:

PROPOSITION 14. *For all Kripke complete modal logics* $L_1, \dots, L_n$*,*

$$L_1 \otimes \cdots \otimes L_n \ \subseteq\ L_1 \times \cdots \times L_n.$$

Proof. Given a product frame $\mathfrak{F}_1 \times \cdots \times \mathfrak{F}_n = \left\langle W_1 \times \cdots \times W_n, \bar{R}_1, \dots, \bar{R}_n \right\rangle$ such that each $\mathfrak{F}_i = \langle W_i, R_i \rangle$ is a frame for L_i ($i = 1, \dots, n$), fix some $1 \leq i \leq n$. For every $n-1$-tuple $\bar{u}_i = \langle u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n \rangle$ with $u_j \in W_j$, for $j \neq i$, we take the set

$$W_{\bar{u}_i} = \{\langle u_1, \dots, u_n \rangle \mid u_i \in W_i,\ \langle u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n \rangle = \bar{u}_i\},$$

and let $S_{\bar{u}_i}$ be the restriction of $\bar{R}_i$ to $W_{\bar{u}_i}$, i.e., $S_{\bar{u}_i} = \bar{R}_i \cap (W_{\bar{u}_i} \times W_{\bar{u}_i})$. Then we have:

- $\langle W_{\bar{u}_i}, S_{\bar{u}_i} \rangle$ is isomorphic to $\langle W_i, R_i \rangle$;
- $\left\langle W_1 \times \cdots \times W_n, \bar{R}_i \right\rangle$ is the disjoint union of the frames $\langle W_{\bar{u}_i}, S_{\bar{u}_i} \rangle$, for all $n-1$-tuples $\bar{u}_i$.

❑

As we shall see in Section 3.3, the inclusion in Proposition 14 is proper: product logics always include certain *interactions* between the modal operators of their components. Note, however, that the modal operators within each component are not affected by these interactions, that is, the product $L_1 \times \cdots \times L_n$ of consistent Kripke complete logics $L_1, \dots, L_n$ is a *conservative extension* of each of them. One can even show a slightly stronger statement:

PROPOSITION 15. *Let* $L_1, \dots, L_n, L_{n+1}$ *be consistent Kripke complete unimodal logics. Then the logic* $L_1 \times \cdots \times L_n \times L_{n+1}$ *is a conservative extension of* $L_1 \times \cdots \times L_n$*, i.e., for every* $\mathcal{ML}_n$*-formula* φ*,*

$$\varphi \in L_1 \times \cdots \times L_n \quad \textit{iff} \quad \varphi \in L_1 \times \cdots \times L_n \times L_{n+1}.$$

Proof. We prove this only for the case $L_1 = \cdots = L_n = L$; the general case is considered in a similar way. First, it is readily checked that for any $n+1$-dimensional product frame

$$\mathfrak{F} = \left\langle W_1 \times \cdots \times W_n \times W_{n+1}, \bar{R}_1, \dots, \bar{R}_n, \bar{R}_{n+1} \right\rangle,$$

the projection map $f(w_1, \dots, w_n, w_{n+1}) = \langle w_1, \dots, w_n \rangle$ is a p-morphism from the 'n-reduct'

$$\mathfrak{F}_{(n)} = \left\langle W_1 \times \cdots \times W_n \times W_{n+1}, \bar{R}_1, \dots, \bar{R}_n \right\rangle$$

of $\mathfrak{F}$ onto the n-dimensional product frame $\mathfrak{F}^- = \left\langle W_1 \times \cdots \times W_n, \bar{R}_1, \dots, \bar{R}_n \right\rangle$.

Now suppose that $\varphi \in L^{n+1}$ and $\mathfrak{G}$ is an n-dimensional product frame for L^n. As L is consistent and Kripke complete, there exists a frame $\mathfrak{H}$ for L. Then the product $\mathfrak{F} = \mathfrak{G} \times \mathfrak{H}$ is a frame for L^{n+1}, and so $\mathfrak{F} \models \varphi$. Since $\mathfrak{F}^- = \mathfrak{G}$, we finally obtain $\mathfrak{G} \models \varphi$.

Conversely, suppose that $\varphi \in L^n$, and let $\mathfrak{F}$ be an $n+1$-dimensional product frame for L^{n+1}. Then clearly $\mathfrak{F}^-$ is a frame for L^n, and so $\mathfrak{F} \models \varphi$. ❑

A useful property of certain product logics that sometimes they are determined by their *countable* product frames:

THEOREM 16. *Let L_i be a Kripke complete unimodal logic such that $\mathsf{Fr}L_i$ is first-order definable in the language having equality and a binary predicate symbol R_i, for each $i = 1, \dots, n$. Then $L_1 \times \cdots \times L_n$ is determined by the class of its countable product frames.*

Proof. For each i, let Γ_i denote the first-order theory defining $\mathsf{Fr}L_i$ in the language $\mathcal{L}_n$ having equality and binary predicate symbols $R_1, \dots, R_n$. Now let $\mathcal{L}_n^\times$ be the $n+1$-sorted extension of $\mathcal{L}_n$ that has the binary predicate symbols $R_1, \dots, R_n$ of sort 0, countably many unary predicate symbols $P_0, P_1, \dots$ of sort 0, and for each sort i ($i = 1, \dots, n$) a unary function symbol f_i taking an argument of sort 0 and returning a value of sort i. For each $\phi \in \Gamma_i$, denote by ϕ' the formula obtained by substituting $f_i(x)$ for all occurrences of each variable x in ϕ ($i = 1, \dots, n$). Let

$$\Sigma = \{\phi' \mid \phi \in \Gamma_i,\ i = 1, \dots, n\} \cup \{\pi\},$$

where π is the following sentence:

$$\begin{aligned}\forall x \forall y\ \big(f_1(x) = f_1(y) \wedge \cdots \wedge f_n(x) = f_n(y) \to x = y\big) \\ \wedge \forall x_1 \dots \forall x_n \exists y\ \big(f_1(y) = x_1 \wedge \cdots \wedge f_n(y) = x_n\big) \\ \wedge \bigwedge_{i=1}^{n} \forall x \forall y\ \Big(x R_i y\ \leftrightarrow\ \big(f_i(x) R_i f_i(y) \wedge \bigwedge_{\substack{j=1 \\ j \neq i}}^{n} f_j(x) = f_j(y)\big)\Big)\end{aligned}$$

(here x and y are variables of sort 0, and x_i is of sort i, for $i = 1, \dots, n$). Now suppose that $\varphi \notin L_1 \times \cdots \times L_n$, for some $\mathcal{ML}_n$-formula φ. Then φ is not true in a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ based on the product $\mathfrak{F}_1 \times \cdots \times \mathfrak{F}_n$ of frames $\mathfrak{F}_i = \langle W_i, S_i \rangle$ such that $\mathfrak{F}_i \models \Gamma_i$ for $i = 1, \dots, n$. Define a first-order $\mathcal{L}_n^\times$-structure I by taking

$$I = \big\langle W_1 \times \cdots \times W_n, W_1, \dots, W_n; \bar{S}_1, \dots, \bar{S}_n, \mathfrak{V}(p_0), \mathfrak{V}(p_1), \dots, pr_1, \dots, pr_n \big\rangle,$$

where $pr_i : W_1 \times \cdots \times W_n \to W_i$ are the projection functions. It is easy to see that $I \models \Sigma$. Since without the extra sorts and the projections I is nothing but the modal model $\mathfrak{M}$ considered as a first-order structure, we also have $I \not\models \forall x \varphi^\star(x)$ (where $\varphi^\star$ is the *standard translation* of φ). In other words, $\Sigma' = \Sigma \cup \{\exists x \neg \varphi^\star(x)\}$ is true in I. By the downward Löwenheim–Skolem–Tarski theorem, there is a countable first-order $\mathcal{L}_n^\times$-structure

$$J = \big\langle U, U_1, \dots, U_n; , R_1^J, \dots, R_n^J, P_0^J, P_1^J, \dots, f_1^J, \dots, f_n^J \big\rangle$$

such that $J \models \Sigma'$. For each $i = 1, \dots, n$, define

$$Q_i = \{\big\langle f_i^J(u), f_i^J(v) \big\rangle \mid \langle u, v \rangle \in R_i^J\},$$

and for each $j < \omega$,

$$P_j^{I'} = \{\big\langle f_1^J(w), \dots, f_n^J(w) \big\rangle \mid w \in P_j^J\}.$$

Since $J \models \pi$, the map $h(w) = \big\langle f_1^J(w), \dots, f_n^J(w) \big\rangle$ is an isomorphism between J and the first-order $\mathcal{L}_n^\times$-structure

$$I' = \Big\langle U_1 \times \cdots \times U_n, U_1, \dots, U_n; \bar{Q}_1, \dots, \bar{Q}_n, P_0^{I'}, P_1^{I'}, \dots, pr_1, \dots, pr_n \Big\rangle.$$

Thus, $I' \models \Sigma$ and $I' \not\models \forall x \varphi^\star(x)$. Let $\mathfrak{G}_i = \langle U_i, Q_i \rangle$, $i = 1, \dots, n$. Define a valuation $\mathfrak{W}$ in the (countable) product frame $\mathfrak{G} = \mathfrak{G}_1 \times \cdots \times \mathfrak{G}_n$ by taking $\mathfrak{W}(p_j) = P_j^{I'}$ for $j < \omega$. As without the extra sorts and the projections I' is just a (countable) modal model $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{W} \rangle$ considered as a first-order structure, this means that φ is not true in $\mathfrak{N}$.

Note that in fact we have also proved that

$$\varphi \in L_1 \times \cdots \times L_n \qquad \text{iff} \qquad \Sigma \models \forall x \varphi^\star(x), \tag{1}$$

for any $\mathcal{ML}_n$-formula φ. ❑

In many cases recursive enumerability of the components transfers to their product:

THEOREM 17. *Let L_i be a Kripke complete unimodal logic such that $\mathsf{Fr}L_i$ is definable by a recursive set of first-order sentences in the language having equality and a binary predicate symbol R_i, for each $i = 1, \dots, n$. Then the product logic $L_1 \times \cdots \times L_n$ is recursively enumerable.*

Proof. We use the notation of the proof of Theorem 16. Since now the sets Γ_i are recursive, Σ is recursive as well. And since the consequence relation of first-order logic is recursively enumerable, it follows from (1) that $L_1 \times \cdots \times L_n$ is recursively enumerable. ❑

3.2 Connections with other formalisms

The product construction shows up in various disguises, here we discuss three examples: first-order logics, 'interpreted systems' for temporal epistemic logics, and modal extensions of description logics.

First-order classical and modal logics

Let us fix a natural number $n > 0$ and consider the fragment of classical first-order logic that

- uses n individual variables $x_1, \dots, x_n$,
- contains neither equality, nor individual constants, nor function symbols, and
- whose atomic formulas are of the form $P(x_1, \dots, x_n)$, where P is an n-ary predicate symbol.

This fragment can be regarded as the '*n-variable substitution- and equality-free fragment*' of classical first-order logic. The following map $\cdot^\bullet$ provides a one-to-one correspondence between formulas of this fragment and $\mathcal{ML}_n$-formulas:

$$\begin{array}{rclcrcl} P_i(x_1, \dots, x_n)^\bullet & = & p_i & \qquad & (\varphi \wedge \psi)^\bullet & = & \varphi^\bullet \wedge \psi^\bullet, \\ (\neg\varphi)^\bullet & = & \neg\varphi^\bullet, & & (\exists x_i \psi)^\bullet & = & \Diamond_i \psi^\bullet \qquad (1 \le i \le n). \end{array}$$

It is not hard to see that, for every first-order formula of the fragment,

$$\varphi \text{ is first-order valid} \quad \text{iff} \quad \varphi^\bullet \in \mathbf{S5}^n.$$

Indeed, every first-order structure $I = \left\langle D^I, \dots, P_i^I, \dots \right\rangle$ can be considered as a modal model $\mathfrak{M}(I) = \langle \langle W, \dots, R_i, \dots \rangle, \mathfrak{V} \rangle$, where

- W is the set of all variable assignments in I, i.e., the set of all functions from the variables $x_1, \dots, x_n$ into D^I;
- $\mathfrak{a} R_i \mathfrak{b}$ iff $\mathfrak{a}(x_j) = \mathfrak{b}(x_j)$ for all variables x_j different from x_i, $1 \le i \le n$;
- $\mathfrak{V}(p_i) = P_i^I$.

The set W of all assignments in I can be regarded as the n^{th} Cartesian power of the domain D^I. The underlying frame of $\mathfrak{M}(I)$ then turns into a product frame for $\mathbf{S5}^n$: the nth power of the universal $\mathbf{S5}$-frame $\langle D^I, D^I \times D^I \rangle$. On the other hand, $\mathbf{S5}^n$ is determined by such cubic universal product frames by Proposition 13.

The idea of such a 'modal approach' to classical first-order logic was suggested by Quine [66] and Kuhn [51] and fully realised by Venema [80]. 'Approximating' first-order logic with logical systems of propositional character was an important motive in the *algebraic* treatment of classical first-order logic; see the work of Tarski and his school [38, 39, 1, 11, 13, 34, 62]. The modal algebras (see Chapter 6 of this handbook) corresponding to the product logic $\mathbf{S5}^n$ are known in the algebraic logic literature as *diagonal-free cylindric set algebras of dimension n*.

As is shown in [23], a similar connection can be established between n-variable fragments of quantified modal logics L (with constant domains) and $n+1$-dimensional product logics of the form

$$L \times \overbrace{\mathbf{S5} \times \cdots \times \mathbf{S5}}^{n}.$$

Temporal epistemic logics

Here we briefly discuss the connections to the 'interpreted systems' approach proposed by Fagin *et al*. [15] which gives rise to various combinations of propositional temporal and epistemic logics ranging from fusions to products of these logics.

Suppose S is a non-empty set (of 'states') and $\mathfrak{F} = \langle T, < \rangle$ is a strict linear order (the 'flow of time'). Suppose also that $\mathcal{R}$ is a non-empty set of functions from T to S (the available 'runs of events' over $\mathfrak{F}$), and let $R_1, \dots, R_n$ be binary relations on $T \times \mathcal{R}$. Then the tuple

$$\mathfrak{S} = \langle T, \mathcal{R}, <, R_1, \dots, R_n \rangle$$

is called a *interpreted system.* A *valuation* $\mathfrak{V}$ *in* $\mathfrak{S}$ is a function from the set of propositional variables into the set 2^S of all subsets of S. The pair $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{V} \rangle$ is called a *model based on* $\mathfrak{S}$.

We interpret the modal language $\mathcal{ML}_{n+1}$ at $\langle$timepoint,run$\rangle$ pairs in these models. $\Box_1$ represents the temporal operator 'always in the future', while $\Box_2, \dots, \Box_{n+1}$ represent the respective knowledge of n agents:

- $(\mathfrak{M}, \langle t, f \rangle) \models p$ iff $f(t) \in \mathfrak{V}(p)$,
- $(\mathfrak{M}, \langle t, f \rangle) \models \varphi \wedge \psi$ iff $(\mathfrak{M}, \langle t, f \rangle) \models \varphi$ and $\langle t, f \rangle \models \psi$,
- $(\mathfrak{M}, \langle t, f \rangle) \models \neg \varphi$ iff not $(\mathfrak{M}, \langle t, f \rangle) \models \varphi$,
- $(\mathfrak{M}, \langle t, f \rangle) \models \Box_1 \varphi$ iff $(\mathfrak{M}, \langle t', f \rangle) \models \varphi$ whenever $f' > f$,
- $(\mathfrak{M}, \langle t, f \rangle) \models \Box_i \varphi$ iff $(\mathfrak{M}, \langle t', f' \rangle) \models \varphi$ whenever $\langle t, f \rangle \, R_{i-1} \, \langle t', f' \rangle$ $(i = 2, \dots, n+1)$.

We say that φ *is true in* $\mathfrak{M}$ if $(\mathfrak{M}, \langle t, f\rangle) \models \varphi$ holds, for every $\langle t, f\rangle \in T \times \mathcal{R}$.

Given a propositional temporal logic $\mathsf{Log}\,\mathcal{C}_1$ determined by a class $\mathcal{C}_1$ of strict linear orders and an n-modal epistemic logic L determined by a class $\mathcal{C}_2$ of n-frames, we can obtain a 'combined' temporal-epistemic logic by considering all $\mathcal{ML}_{n+1}$-formulas that are true in all models that are based on interpreted systems of the form $\langle T, \mathcal{R}, <, R_1, \dots, R_n\rangle$ such that $\langle T, <\rangle \in \mathcal{C}_1$ and $\langle T \times \mathcal{R}, R_1, \dots, R_n\rangle \in \mathcal{C}_2$. By Theorem 3, this combined logic is just the *fusion* of $\mathsf{Log}\,\mathcal{C}_1$ and L.

By imposing various constraints on interpreted systems, we can reflect some interesting features of agents. An interpreted system $\mathfrak{S}$ models agents who *know the time* if, for all $t, t' \in T$, $f, f' \in \mathcal{R}$, and $i = 1, \dots, n$,

$$\langle t, f\rangle\, R_i\, \langle t', f'\rangle \text{ implies } t = t'.$$

In other words, if A_i believes that at moment t relative to an evolution f the pair $\langle t', f'\rangle$ represents a possible state of affairs, then $t = t'$. So at each moment t the agents are assumed to know that the clock is at t. Systems represented by structures of this type are known as *synchronous*.

An interpreted system models agents who *do not learn* if, for all agents A_i, $f, f' \in \mathcal{R}$ and $t, t' \in T$, we have

$$\langle t, f\rangle\, R_i\, \langle t', f'\rangle \text{ implies } \forall s \geq t\ \exists s' \geq t'\ \ \langle s, f\rangle\, R_i\, \langle s', f'\rangle\,.$$

Intuitively, an agent A_i does not learn if, whenever it regards w as a possible state of affairs at moment t, then it regards w as a possible state of affairs at every moment $s \geq t$ as well. Under the condition that agents know the time, this means that if agent A_i regards an evolution f' as possible at t then it regards f' as possible at every $s > t$. Similarly, an interpreted system models agents who *do not forget* if, for all A_i, $t, t' \in T$ and $f, f' \in \mathcal{R}$, we have

$$\langle t, f\rangle\, R_i\, \langle t', f'\rangle \text{ implies } \forall s \leq t\ \exists s' \leq t'\ \ \langle s, f\rangle\, R_i\, \langle s', f'\rangle\,.$$

Systems of this type are known also as systems with *perfect recall*.

If an interpreted system models agents who know time, do not forget and do not learn, then, for all agents A_i, $t, t' \in T$ and $f, f' \in \mathcal{R}$, we have

$$\langle t, f\rangle\, R_i\, \langle t', f'\rangle \text{ implies } t = t' \text{ and } \forall s\ \ \langle s, f\rangle\, R_i\, \langle s, f'\rangle\,.$$

Thus, the interpretation of $\mathcal{ML}_{n+1}$-formulas in $\mathfrak{S}$ corresponds to evaluating them in the *product* of frames $\mathfrak{F} = \langle T, <\rangle$ and $\langle \mathcal{R}, S_1, \dots, S_n\rangle$, where

$$f S_i f' \quad \text{iff} \quad \exists t, t' \in T\ \ \langle t, f\rangle\, R_i\, \langle t', f'\rangle \quad \text{iff} \quad \forall t \in T\ \ \langle t, f\rangle\, R_i\, \langle t, f'\rangle\,.$$

'Modal' description logics

As is discussed in Chapter 13 of this handbook, originally *description logics* have been designed and used as a formalism for knowledge representation and reasoning only in 'static' application domains. Later on, several attempts have been made in the literature in order to extend description logics with 'dynamic' features such as knowledge as time- or action-dependence, beliefs of different agents, etc. (see, e.g., [72, 71, 57, 32, 7, 3, 5, 84, 86, 88]). Here we briefly describe a simple 'modal' extension of the basic concept language $\mathcal{ALC}$ (see Chapter 13 of this handbook) and its connection to products.

Imagine, for instance, a car salesman John who, besides standard ABox and TBox knowledge bases (see Chapter 13 of this handbook), also wants to include 'modalised' concepts such as describing a $\mathsf{Customer}$ as

$$\mathsf{Homo\_sapiens} \sqcap \langle\text{sometime in the past}\rangle\, \exists \mathsf{buys.Car},$$

or a $\mathsf{Potential\_customer}$ as

$$[\text{John believes}]\, \langle\text{eventually}\rangle\, \mathsf{Customer}.$$

Concept descriptions in the extended concept language $\mathcal{ML}_n^{\mathcal{ALC}}$ that is able to express these can be formed according to the following rules:

$$C, D \quad \rightarrow \quad A \mid \top \mid \bot \mid \neg C \mid C \sqcap D \mid C \sqcup D \mid \forall r.C \mid \exists r.C \mid \Box_i C \mid \Diamond_i C,$$

where A ranges over concept names, r ranges over role names, and $i = 1, \dots, n$.

The intended semantics of $\mathcal{ML}_n^{\mathcal{ALC}}$ is defined as follows. An $\mathcal{ML}_n^{\mathcal{ALC}}$-*interpretation with constant domains and roles* is a pair $\mathfrak{M} = \langle \mathfrak{F}, I\rangle$ in which $\mathfrak{F} = \langle W, R_1, \dots, R_n\rangle$ is an n-frame and I is a function associating with each $w \in W$ a usual $\mathcal{ALC}$-interpretation

$$I(w) = \big\langle \Delta, \dots, A^{\mathfrak{M},w}, \dots, r, \dots \big\rangle$$

(that is, Δ is a nonempty set, $A^{\mathfrak{M},w} \subseteq \Delta$ for each concept name A, and $r \subseteq \Delta \times \Delta$ for each role name r). The (world-dependent) interpretation of concept names is inductively extended to arbitrary concept descriptions. Here we give the definition for the new 'modal' constructors only:

$$(\Box_i C)^{\mathfrak{M},w} = \bigcap_{wR_iv} C^{\mathfrak{M},v}, \qquad (\Diamond_i C)^{\mathfrak{M},w} = \bigcup_{wR_iv} C^{\mathfrak{M},v}.$$

Now given a Kripke complete n-modal logic L, we say that a concept description C is $L_{\mathcal{ALC}}$-*satisfiable* (with an empty knowledge base) if there is an $\mathcal{ML}_n^{\mathcal{ALC}}$-interpretation (with constant domains and roles) $\mathfrak{M} = \langle \mathfrak{F}, I\rangle$ and a world w in $\mathfrak{F}$ such that $\mathfrak{F}$ is a frame for L and $C^{\mathfrak{M},w} \neq \emptyset$.

Now, by extending the correspondence between $\mathcal{ALC}$ (with m role names) and the modal logic $\mathbf{K}_m$ (see Chapter 13 of this handbook), it is straightforward to see that $L_{\mathcal{ALC}}$-satisfiability coincides with $L \times \mathbf{K}_m$-satisfiability.

3.3 Axiomatising products

Product logics are defined in a semantical way: they are logics determined by classes of product frames, and so Kripke complete by definition. Therefore, the proper 'transfer' question to ask is how a possible axiomatisation for a product logic relates to axiomatisations of its components.

To begin with, observe that the following properties hold in every product frame of the form $\langle W, R_1, \dots, R_n\rangle$, for all $i, j = 1, \dots, n$, $i \neq j$:

- *left commutativity*: $\forall x, y, z \in W\ \big(xR_jy \wedge yR_iz \rightarrow \exists u \in W\ (xR_iu \wedge uR_jz)\big)$,

- *right commutativity*: $\forall x, y, z \in W\ \big(xR_iy \wedge yR_jz \to \exists u \in W\ (xR_ju \wedge uR_iz)\big)$,
- *Church–Rosser property*: $\forall x, y, z \in W\ \big(xR_jy \wedge xR_iz \to \exists u \in W\ (yR_iu \wedge zR_ju)\big)$,

see Fig. 2.

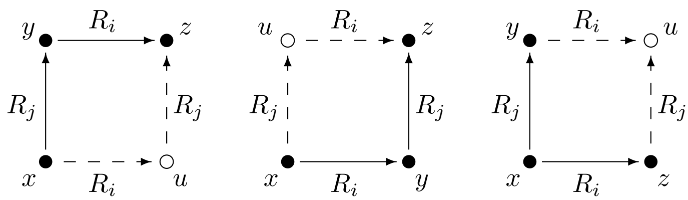


Figure 2. Left and right commutativity and Church–Rosser properties.

These properties can also be expressed by modal formulas. One can easily check that an arbirtary (not necessarily product) n-frame is left commutative iff it validates the formulas

$$\boldsymbol{com}^{\boldsymbol{l}}_{ij} \ = \ \Diamond_j\Diamond_i p \to \Diamond_i\Diamond_j p,$$

it is right commutative iff it validates

$$\boldsymbol{com}^{\boldsymbol{r}}_{ij} \ = \ \Diamond_i\Diamond_j p \to \Diamond_j\Diamond_i p,$$

and it is Church–Rosser iff it validates

$$\boldsymbol{chr}_{ij} \ = \ \Diamond_i\Box_j p \to \Box_j\Diamond_i p.$$

The corresponding left and right commutativity axioms can be combined into a single *commutativity* axiom

$$\boldsymbol{com}_{ij} \ = \ \boldsymbol{com}^{\boldsymbol{l}}_{ij} \wedge \boldsymbol{com}^{\boldsymbol{r}}_{ij}.$$

DEFINITION 18. Given modal logics L_i formulated in the language having $\Box_i$ ($i = 1, \dots, n$), the *commutator*

$$[L_1, \dots, L_n]$$

of $L_1, \dots, L_n$ is the smallest n-modal logic containing all the L_i and the axioms $\boldsymbol{com}_{ij}$ and $\boldsymbol{chr}_{ij}$, for all $i, j = 1, \dots, n$, $i \neq j$.

Note that the commutator of (finitely) axiomatisable modal logics is always (finitely) axiomatisable by definition. Moreover, since the axioms $\boldsymbol{com}_{ij}$ and $\boldsymbol{chr}_{ij}$ are Sahlqvist formulas, we also have:

PROPOSITION 19. *The commutator of canonical logics is canonical, and so Kripke complete.*

It is worth noting that even if all components are Kripke complete, their commutator is not necessarily so: non-examples are $[\mathbf{K4}, \mathbf{GL.3}]$ and $[\mathbf{GL}, \mathbf{Grz.3}]$, see Section 3.4.

Commutators are natural candidates for axiomatising products. As $\boldsymbol{com}_{ij}$ and $\boldsymbol{chr}_{ij}$ are valid in every product frame, by Proposition 14 we always have that

$$[L_1, \dots, L_n] \ \subseteq \ L_1 \times \cdots \times L_n, \tag{2}$$

whenever $L_1, \dots, L_n$ are Kripke complete modal logics. Those tuples of logics $L_1, \dots, L_n$ for which the converse inclusion also holds are called *product-matching*.

Axiomatising two-dimensional product logics

We begin with a general result of Gabbay and Shehtman [24] stating that certain pairs of modal logics are always product-matching.

Consider the first-order language with equality and a binary predicate R. A formula ψ in this language is called *positive* if it is built up from atoms using only $\wedge$ and $\vee$. A sentence of the form

$$\forall x \forall y \forall \bar{z}\, \big(\psi(x, y, \bar{z}) \to R(x, y)\big)$$

is said to be a *universal Horn sentence* if $\psi(x, y, \bar{z})$ is a positive formula. We call an $\mathcal{ML}$-formula φ a *Horn formula*, if there is a universal Horn sentence φ_H such that, for all frames $\mathfrak{F}$,

$$\mathfrak{F} \models \varphi \qquad \text{iff} \qquad \mathfrak{F} \models \varphi_H.$$

An $\mathcal{ML}$-formula is called *variable free* if it contains no propositional variables, i.e., all its atomic subformulas are constants $\bot$ or $\top$.

DEFINITION 20. A modal logic is called *Horn axiomatisable* if it is axiomatisable by only Horn and variable-free formulas.

It is not hard to see that if L is a Kripke complete and Horn axiomatisable logic then $\mathsf{Fr}L$ is defined by the set

$$\Gamma_L = \{\varphi_H \mid \varphi \text{ is a Horn axiom of } L\} \ \cup\ \{\varphi^\star \mid \varphi \text{ is a variable-free axiom of } L\} \qquad (3)$$

of first-order formulas (here $\varphi^\star$ is the *standard translation* of φ). Examples of Kripke complete Horn axiomatisable logics are **K**, **D**, **K4**, **S4**, **KD45**, **T**, **S5**.

THEOREM 21. *Let L_1 and L_2 be Kripke complete and Horn axiomatisable modal logics. Then*

$$L_1 \times L_2 \ = \ [L_1, L_2].$$

Proof. The heart of the proof is the following lemma that can be proved by constructing the necessary p-morphism in a step-by-step manner, see [23, Lemmas 5.2 and 5.8].

LEMMA 22. *Let L_1 and L_2 be Kripke complete and Horn axiomatisable unimodal logics. Then every countable rooted 2-frame for $[L_1, L_2]$ is a p-morphic image of a product frame for $L_1 \times L_2$.*

Now, by Proposition 19, $[L_1, L_2]$ is determined by the class of commutative and Church–Rosser frames from $\mathsf{Fr}(L_1 \otimes L_2)$. By (3), this class is first-order definable in the language with equality and two binary predicate symbols. Let $\varphi \notin [L_1, L_2]$. Then, using the standard translation $\varphi^\star$ of φ and the downward Löwenheim–Skolem–Tarski theorem, it is not hard to see that we can have a countable rooted 2-frame $\mathfrak{F}$ for $[L_1, L_2]$ refuting φ. Now, using Lemma 22, we can find a product frame $\mathfrak{G}$ for $L_1 \times L_2$ having $\mathfrak{F}$ as its p-morphic image. By Proposition 9, it follows that $\mathfrak{G} \not\models \varphi$, and so $\varphi \notin L_1 \times L_2$. Therefore, $L_1 \times L_2 \subseteq [L_1, L_2]$. The converse inclusion has already been shown as (2). ❑

As a corollary of Theorem 21 we obtain that finite axiomatisability of two Kripke complete and Horn axiomatisable logics transfers to their product. An interesting corollary of Lemma 22 is the following:

COROLLARY 23. *Let L_1, L_2 and L_3 be Kripke complete and Horn axiomatisable unimodal logics. Then*

$$L_1 \times L_2 \times L_3 \;=\; (L_1 \times L_2) \times L_3 \;=\; L_1 \times (L_2 \times L_3).$$

Unfortunately, no other general result is known about axiomatisations of two-dimensional products. In Section 3.4 we shall see several examples of pairs of finitely axiomatisable modal logics whose products are not even recursively enumerable. Such are, for instance, $\mathsf{Log}\{\langle \mathbb{N}, <\rangle\} \times \mathsf{Log}\{\langle \mathbb{N}, <\rangle\}$, **K4** × **GL.3** and **S4** × **Grz.3**.

Moreover, Theorem 21 cannot be generalised even to logics whose classes of frames are definable by *universal* first-order formulas. As the following theorem shows, for many transitive logics L, the pairs of '**K4.3** and L' and '**Grz.3** and L' are not product-matching:

THEOREM 24. (i) [23] *Let L be any Kripke complete logic containing* **K4** *and having the two-element reflexive chain as its frame. Then* $\mathbf{K4.3} \times L \neq [\mathbf{K4.3}, L]$.

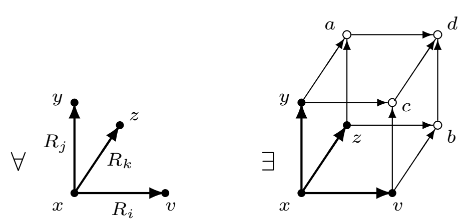
(ii) [24] *Let L_1 be any Kripke complete logic containing* **Grz** *and having the two-element reflexive chain as its frame. Let L_2 be any Kripke complete logic containing* **S4** *and having either* (a) *the two-element reflexive chain or* (b) *the two-element cluster as its frame. Then* $L_1 \times L_2 \neq [L_1, L_2]$.

There are many open questions in the area. For instance, it is not known whether such 'standard' products like **K4.3**×**K** or **K4.3**×**S5** or **K4.3**×**K4.3** are product-matching, or even finitely axiomatisable (they are recursively enumerable by Theorem 17). In general, no examples for pairs of logics are known that are not product-matching, but whose product is finitely axiomatisable.

Axiomatising higher dimensional product logics

Tuples of more than two modal logics are almost never product-matching. To begin with, it is straightforward to see that all n-dimensional product frames $\langle W, R_1, \dots, R_n\rangle$ satisfy the following 'cubifying' properties whenever $n \geq 3$ and $i, j, k = 1, \dots, n$ are distinct:

$$\begin{aligned}\Phi_{ijk} \;=\; & \forall x, y, z, v \in W \big(xR_iv \wedge xR_jy \wedge xR_kz \;\rightarrow\; \exists a, b, c, d \in W \\ & (vR_jc \wedge vR_kb \wedge yR_ic \wedge yR_ka \wedge zR_ib \wedge zR_ja \wedge aR_id \wedge bR_jd \wedge cR_kd)\big).\end{aligned}$$



It is not hard to see that, say, a 3-frame $\mathfrak{F}$ for $[\mathbf{K}, \mathbf{K}, \mathbf{K}]$ satisfies property Φ_{123} iff the following modal formula $\boldsymbol{cub}_{123}$ is valid in $\mathfrak{F}$ (cf. [39, 3.2.67] and [52]):

$$\begin{aligned}\boldsymbol{cub}_{123} \;=\; & [\Diamond_1(\Box_2 p_{12} \wedge \Box_3 p_{13}) \;\wedge\; \Diamond_2(\Box_1 p_{21} \wedge \Box_3 p_{23}) \;\wedge\; \Diamond_3(\Box_1 p_{31} \wedge \Box_2 p_{32}) \\ & \wedge\; \Box_1\Box_2(p_{12} \wedge p_{21} \rightarrow \Box_3 q_3) \;\wedge\; \Box_1\Box_3(p_{13} \wedge p_{31} \rightarrow \Box_2 q_2) \\ & \wedge\; \Box_2\Box_3(p_{23} \wedge p_{32} \rightarrow \Box_1 q_1)] \;\longrightarrow\; \Diamond_1\Diamond_2\Diamond_3(q_1 \wedge q_2 \wedge q_3)\,.\end{aligned}$$

Thus $\boldsymbol{cub}_{123}$ belongs to $\mathbf{K}^3$. On the other hand, Fig. 3 shows a 23-element frame for $[\mathbf{K}, \mathbf{K}, \mathbf{K}]$ (that is, a 3-frame satisfying $\boldsymbol{com}_{ij}$ and $\boldsymbol{chr}_{ij}$ for $i, j = 1, 2, 3$, $i \neq j$) that refutes $\boldsymbol{cub}_{123}$ (see again [39, 3.2.67]). So $[\mathbf{K}, \mathbf{K}, \mathbf{K}]$ and $\mathbf{K}^3$ are different.

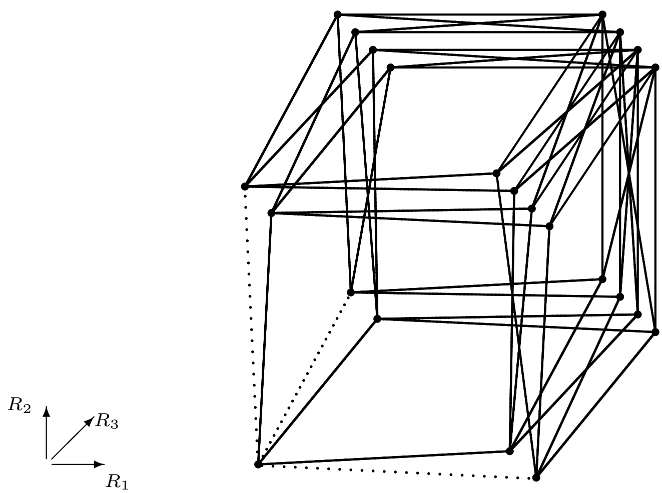


Figure 3. A frame for $[\mathbf{K}, \mathbf{K}, \mathbf{K}]$ that refutes $\boldsymbol{cub}_{123}$.

Moreover, in many cases the addition of cubifying properties does not help either. As is shown by Johnson [44] in the algebraic setting of diagonal-free cylindric algebras, $\mathbf{S5}^n$ is not finitely axiomatisable whenever $n \geq 3$. Generalisations of the cubifying properties are used in [52] to show that $\mathbf{K}^n$ is not finitely axiomatisable either for $n \geq 3$. Moreover, the following general result of [42] shows how hopeless the situation really is:

THEOREM 25. *Let $n \geq 3$ and let L be any n-modal logic such that $\mathbf{K}^n \subseteq L \subseteq \mathbf{S5}^n$. Then L is not finitely axiomatisable. Moreover, it is undecidable whether a finite n-frame is a frame for L.*

On the other hand, if frames for the component logics do not allow branching (like in the functional frames for **Alt**), then counterexamples like the above one do not work, and in fact the cubifying properties follow from the Church–Rosser properties. The following result of [24] says that any tuple of **Alt** logics is product-matching. It can be proven in a way similar to the proof of Theorem 21 above.

THEOREM 26. *For any natural number $n > 1$,* $\mathbf{Alt}^n = \overbrace{[\mathbf{Alt}, \ldots, \mathbf{Alt}]}^{n}$.

There are several interesting open questions concerning the axiomatisation of higher (≥ 3) dimensional product logics. For instance, it is not known whether logics like $\mathbf{K}^n$ or $\mathbf{S5}^n$ are axiomatisable using finitely many propositional variables, or whether $\mathbf{S5}^n$ is finitely axiomatisable over $\mathbf{K}^n$. Though logics like $\mathbf{K}^n$ or $\mathbf{S5}^n$ are known to be recursively enumerable by Theorem 17, no intuitive 'concrete' axiomatisation is known for most of them.

3.4 Decision problems and complexity of products

There are three basic approaches to establishing decidability of modal logics:

(1) Given such a logic L, one can try to prove that L has the finite model property (fmp). Even without a recursive bound on the size of the models, this can yield decidability if L is recursively enumerable, and the class of finite frames for L is recursively enumerable as well (up to isomorphism, of course). This is the case for instance if L is finitely axiomatisable.

(2) Even if a logic L does not enjoy the fmp, then one can try to show that it is characterised by some class of perhaps infinite models having a certain 'regular structure,' say, constructed from repeating finite pieces, so called 'blocks' or 'mosaics.'

(3) The third approach is to try to reduce the decision problem for L to another problem that is already known to be decidable (say, to the decision problem for another modal logic, or a suitable monadic second-order theory, or some problem about tree automata).

All three approaches have been successfully applied to uni- and multimodal logics; see e.g., [22, 12, 90]. As products of modal logics are special multimodal logics, in principle the same approaches can be applied to them as well.

As concerns (1), there is an even more tempting way. One can try to show the finite model property w.r.t. the 'intended' models, that is, those that are based on product frames. (It is important to stress that in general there are frames for product logics which are *not* product frames.)

DEFINITION 27. A modal logic L has the *product fmp* if L is characterised by the class of its finite product frames.

Note that by Proposition 14, for every product frame $\mathfrak{F} = \mathfrak{F}_1 \times \cdots \times \mathfrak{F}_n$ and product logic $L = L_1 \times \cdots \times L_n$,

$$\mathfrak{F} \models L \quad \text{iff} \quad \mathfrak{F}_i \models L_i, \text{ for all } 1 \leq i \leq n.$$

Obviously, the product fmp implies the fmp. As we shall see below, the converse does not necessarily hold.

We can enumerate the formulas that are not in a product logic L (and thereby obtain a decision algorithm for L whenever L is recursively enumerable) if

- L has the product fmp, and
- finite product frames for L are recursively enumerable (up to isomorphism).

The latter property clearly holds if L is a product of finitely axiomatisable Kripke complete logics such as **K**, **K4**, **K4.3**, **S5**, etc., so this approach looks very promising. Unfortunately, it is easy to see that most products of well-known unimodal logics lack the product fmp. Here is an example for a simple bimodal formula that 'forces' infinite product frames even for logics like $\mathbf{K4} \times \mathbf{K}$ or $\mathbf{K4} \times \mathbf{S5}$:

$$\Box_1^+ \Diamond_2 p \wedge \Box_1^+ \Box_2 (p \to \Diamond_1 \Box_1^+ \neg p) \tag{4}$$

(here $\Box_1^+ \psi$ abbreviates $\psi \wedge \Box_1 \psi$). However, as we shall see below, two-dimensional product logics with at least one $\mathbf{S5}_n$- or $\mathbf{K}_n$-component can have the (usual, 'abstract') fmp.

Products with '$\mathbf{S5}_n$- and $\mathbf{K}_n$-like' logics are usually decidable

Filtration.

Originating in the 1940s, the *filtration method* is one of the oldest and most well-known techniques for finite model property proofs in modal logic. Here we discuss how it can be used to show the fmp of two-dimensional product logics where one component is a special kind of Horn axiomatisable logic and the other is $\mathbf{S5}_n$ or $\mathbf{K}_n$.

A *QTC-logic* is a modal logic axiomatised by a finite set of formulas where each axiom is either variable-free or of the form

$$\Box_i p \to \Box_i^j p \quad (j \geq 0) \qquad \text{or} \qquad \Diamond_k \Box_k p \to p.$$

The following theorem is due to Shehtman [76]:

THEOREM 28. *Let L_1 be a QTC-logic and L_2 be either $\mathbf{S5}_n$ or $\mathbf{K}_n$. Then $L_1 \times L_2$ has the fmp.*

As it is easy to see that every QTC-logic is Horn axiomatisable, by Theorem 21 we obtain:

THEOREM 29. *Let L_1 be a QTC-logic and L_2 be either $\mathbf{S5}_n$ or $\mathbf{K}_n$. Then $L_1 \times L_2$ is decidable.*

Proof. We illustrate the proof of Theorem 28 by showing that $\mathbf{K4} \times \mathbf{K}$ has the fmp. Suppose $\varphi \notin \mathbf{K4} \times \mathbf{K}$ for some $\mathcal{ML}_2$-formula φ. We will construct a model refuting φ that is based on a finite frame for $[\mathbf{K4}, \mathbf{K}]$. As $[\mathbf{K4}, \mathbf{K}] = \mathbf{K4} \times \mathbf{K}$ by Theorem 21, this would suffice.

As is well-known, every rooted Kripke frame is a p-morphic image of an intransitive tree. Therefore, by Propositions 9 and 12, we may assume that there exists a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ refuting φ and based on the product $\mathfrak{F} = \langle W, \bar{R}_1, \bar{R}_2 \rangle$ of a transitive frame and an intransitive tree of depth $md(\varphi)$. Thus, $\langle W, \bar{R}_1 \rangle$ is transitive, $\langle W, \bar{R}_2 \rangle$ is the disjoint union of intransitive trees of depth $md(\varphi)$, and $\bar{R}_1$ and $\bar{R}_2$ have the commutativity and Church–Rosser properties. For each $x \in W$, let $\mathit{tree}(x) = \langle W_x, \bar{R}_{2,x} \rangle$ denote the intransitive tree x belongs to.

We define an equivalence relation $\sim$ on W. For all $x, y \in W$, let $x \sim y$ iff there exists a relation $E \subseteq W_x \times W_y$ satisfying the following properties:

- xEy
- for every $u \in W_x$ there is $v \in W_y$ such that uEv,
- for every $u \in W_y$ there is $v \in W_x$ such that uEv,
- for all $u \in W_x$, $v, z \in W_y$,
  - if uEv and $v\bar{R}_{2,y}z$ then there is $z' \in W_x$ such that $u\bar{R}_{2,x}z'$ and $z'Ez$,
  - if uEv and $z\bar{R}_{2,y}v$ then there is $z' \in W_x$ such that $z\bar{R}_{2,x}u$ and $z'Ez$,
- for all $u \in W_y$, $v, z \in W_x$,
  - if uEv and $v\bar{R}_{2,x}z$ then there is $z' \in W_y$ such that $u\bar{R}_{2,y}z'$ and $z'Ez$,
  - if uEv and $z\bar{R}_{2,x}v$ then there is $z' \in W_y$ such that $z\bar{R}_{2,y}u$ and $z'Ez$,

- for all $u \in W_x$, $v \in W_y$, and propositional variables $p \in sub\,\varphi$, $u \in \mathfrak{V}(p)$ iff $v \in \mathfrak{V}(p)$.

(In other words, E should be a bisimulation between $\langle W_x, \bar{R}_{2,x}, \bar{R}_{2,x}^{-1}\rangle$ and $\langle W_y, \bar{R}_{2,y}, \bar{R}_{2,y}^{-1}\rangle$ w.r.t. $sub\,\varphi$ that connects x and y.)

Now we define a new model $\mathfrak{M}^{\sim} = \langle \mathfrak{F}^{\sim}, \mathfrak{V}^{\sim}\rangle$ based on $\mathfrak{F}^{\sim} = \left\langle W^{\sim}, \bar{R}_1^{\sim}, \bar{R}_2^{\sim}\right\rangle$ as follows:

- $W^{\sim} = \{[x] \mid x \in W\}$, where $[x]$ denotes the $\sim$-equivalence class of x;
- for all $x, y \in W$,

$$[x]\bar{R}_2^{\sim}[y] \qquad \text{iff} \qquad \exists x'\exists y'\ (x' \sim x,\ y' \sim y \text{ and } x'\bar{R}_2 y');$$

- $\bar{R}_1^{\sim}$ is the transitive closure of the relation $\bar{R}_1^{\bullet}$ defined by taking, for all $x, y \in W$,

$$[x]\bar{R}_1^{\bullet}[y] \qquad \text{iff} \qquad \exists x'\exists y'\ (x' \sim x,\ y' \sim y \text{ and } x'\bar{R}_1 y');$$

- $\mathfrak{V}^{\sim}(p) = \{[x] \mid x \in \mathfrak{V}(p)\}$, for all $p \in sub\,\varphi$, and $\mathfrak{V}^{\sim}(q) = \emptyset$, for all other propositional variables q.

We claim that

$$\mathfrak{M}^{\sim} \text{ refutes } \varphi, \text{ and} \tag{5}$$

$$\left\langle W^{\sim}, \bar{R}_1^{\sim}, \bar{R}_2^{\sim}\right\rangle \text{ is a finite frame for } [\mathbf{K4}, \mathbf{K}]. \tag{6}$$

Claim (5) follows from the fact that $\mathfrak{M}^{\sim}$ is a *filtration* of $\mathfrak{M}$ in the sense that, for all $x, y \in W$, $i = 1, 2$, the following two conditions hold:

- if $x\bar{R}_i y$ then $[x]\bar{R}_i^{\sim}[y]$,
- if $[x]\bar{R}_i^{\sim}[y]$ then $(\mathfrak{M}, y) \models \psi$ whenever $\Box_i\psi \in sub\,\varphi$ and $(\mathfrak{M}, x) \models \Box_i\psi$.

($\bar{R}_2^{\sim}$ and $\bar{R}_1^{\sim}$ are known as the *least filtration* and the *Lemmon* (or *least transitive*) *filtration*, respectively; see e.g., [12, 31].) By induction on the construction of ψ, the reader can readily check that for every $\psi \in sub\,\varphi$ and every $x \in W$,

$$(\mathfrak{M}, x) \models \psi \qquad \text{iff} \qquad (\mathfrak{M}^{\sim}, [x]) \models \psi,$$

which yields (5).

To prove (6), observe first that $\bar{R}_1^{\sim}$ is transitive by definition. Using the definition of $\sim$, it is straightforward to show that $\sim$ commutes with $\bar{R}_2$. Then this fact can be used to show that $\bar{R}_1^{\sim}$ and $\bar{R}_2^{\sim}$ commute and have the Church–Rosser property.

Finally, we show that $W^{\sim}$ is finite. Observe first that since bisimilar paths are of equal length, if $x \sim y$ then both the depth and the co-depth of x and y (in the trees *tree*(x) and *tree*(y), respectively) are the same. Moreover, for all x, y, z, if $[x]\bar{R}_2^{\sim}[y]$, $[x]\bar{R}_2^{\sim}[z]$ and $y \not\sim z$ then the submodels generated by $[y]$ and $[z]$ in $\mathfrak{M}^{\sim}$ are not isomorphic (as far as propositional variables occurring in φ are concerned). So we have

$$|W^{\sim}| \ \leq \ \sum_{k=0}^{md(\varphi)} n_k(\varphi), \tag{7}$$

where $n_0(\varphi) = 2^{|sub\,\varphi|}$ and $n_{k+1}(\varphi) = 2^{|sub\,\varphi|} \cdot 2^{n_k(\varphi)}$. ❑

Observe that the bound in (7) is non-elementary in the size of φ. In fact, it is not known whether there exists an elementary decision algorithm for $\mathbf{K}\times\mathbf{K}$ or $\mathbf{K4}\times\mathbf{K}$. Note however that products of $\mathbf{K}$ with 'richer' dynamic and temporal logics, such as $\mathbf{PDL}\times\mathbf{K}$ and $\mathbf{PTL}\times\mathbf{K}$ are known to be non-elementary; see [23]. The same applies to products with $\mathbf{S5}_n$ whenever $n \geq 2$.

On the other hand, if one component-logic is not $\mathbf{K}$ but (unimodal) $\mathbf{S5}$, then one can do better. As is shown by Gabbay and Shehtman [24], in these cases the equivalence relation $\sim$ on worlds becomes more easily 'characterisable'. Namely, for each world x in W, let

$$\Sigma(x) = \{\psi \in sub\,\varphi \mid (\mathfrak{M}, x) \models \psi\},$$

and for $x, y \in W$, put

$$x \sim y \qquad \text{iff} \qquad \Sigma(x) = \Sigma(y) \ \text{ and } \ \{\Sigma(z) \mid x\bar{R}_2 z\} = \{\Sigma(z) \mid y\bar{R}_2 z\}.$$

As now each world $[x]$ in $\mathfrak{M}^{\sim}$ is uniquely determined by the pair $\big\langle\Sigma(x), \{\Sigma(z) \mid x\bar{R}_2 z\}\big\rangle$ of sets, we have the better, double-exponential bound

$$|W^{\sim}| \;\leq\; 2^{|sub\,\varphi|}\cdot 2^{2^{|sub\,\varphi|}}$$

on the size of the filtrated model. So the filtration method yields a coN2EXPTIME decision algorithm for products of QTC-logics with $\mathbf{S5}$.

Quasimodels.

If L is Kripke complete but not a QTC-logic then $L\times\mathbf{K}_n$ and $L\times\mathbf{S5}_n$ are out of the scope of Theorem 28. Yet, many of these products can be shown to be decidable by the *quasimodel method.* This method was first developed in the series of papers [84, 85, 86, 88] on description logics with various modal and temporal operators, and then extended to products in [83, 23] and to fragments of first-order modal and temporal logics in [87, 43, 89].

The idea is to finitise the '$\mathbf{K}_n$- (or $\mathbf{S5}_n$-)bit' of the models first, then build some kind of structure that manages to keep enough information about its 'two-dimensional' character on the one hand, and can be used to prove decidability (even if it is not necessarily finite) on the other.

We fix a Kripke complete modal logic L and an $\mathcal{ML}_2$-formula φ, and define the notion of an $L\times\mathbf{K}$-*quasimodel for* φ as follows. By a *type* for φ we mean any subset $\boldsymbol{t}$ of $sub\,\varphi$ which is *Boolean-saturated* (in the sense that, for instance,

- $\psi\wedge\chi\in\boldsymbol{t}$ iff $\psi\in\boldsymbol{t}$ and $\chi\in\boldsymbol{t}$, for every $\psi\wedge\chi\in sub\,\varphi$,
- $\neg\psi\in\boldsymbol{t}$ iff $\psi\notin\boldsymbol{t}$, for every $\neg\psi\in sub\,\varphi$,

and so on for the other Boolean connectives). A *quasistate candidate* for φ is a pair $\langle\langle T,<\rangle,\boldsymbol{t}\rangle$, where $\langle T,<\rangle$ is a finite intransitive tree of depth $\leq md(\varphi)$ and $\boldsymbol{t}$ a labeling function associating with each $x\in T$ a type $\boldsymbol{t}(x)$ for φ. (So we can think of a quasistate candidate as a tree of types.) Two quasistate candidates $\langle\langle T,<\rangle,\boldsymbol{t}\rangle$ and $\langle\langle T',<'\rangle,\boldsymbol{t}'\rangle$ are called *isomorphic* if there is an isomorphism f between the trees $\langle T,<\rangle$ and $\langle T',<'\rangle$ such that $\boldsymbol{t}(x)=\boldsymbol{t}'(f(x))$, for all $x\in T$. A quasistate candidate $\langle\langle T,<\rangle,\boldsymbol{t}\rangle$ is called a *quasistate* for φ if the following conditions hold:

(qm1) ($\Diamond_2$-*saturation*) For all $x \in T$ and $\Diamond_2\psi \in sub\,\varphi$,

$$\Diamond_2\psi \in \boldsymbol{t}(x) \quad \text{iff} \quad \exists y \in T\ \big(x < y\ \wedge\ \psi \in \boldsymbol{t}(y)\big).$$

(qm1′) (*smallness*) For all $x, x_1, x_2 \in T$ such that $x < x_1$, $x < x_2$ and $x_1 \neq x_2$, the structures $\langle\langle T^{x_1}, <^{x_1}\rangle, \boldsymbol{t}^{x_1}\rangle$ and $\langle\langle T^{x_2}, <^{x_2}\rangle, \boldsymbol{t}^{x_2}\rangle$ are not isomorphic,

where $\langle T^{x_i}, <^{x_i}\rangle$ is the subtree of $\langle T, <\rangle$ generated by x_i, and $\boldsymbol{t}^{x_i}$ is the restriction of $\boldsymbol{t}$ to T^{x_i}, $i = 1, 2$. Clearly,

$$b(\varphi) = \sum_{k=0}^{md(\varphi)} n_k(\varphi) \tag{8}$$

is an upper bound for the number of different quasistates for φ (cf. (7)). The number of points in any quasistate for φ is bounded by

$$n_0(\varphi) + \sum_{k=1}^{md(\varphi)} \prod_{j=1}^{k} n_{md(\varphi)-j}(\varphi) \leq b(\varphi)^{md(\varphi)}.$$

In what follows, we assume that nonisomorphic quasistates are disjoint and that isomorphic quasistates actually coincide.

A *basic structure* of depth m for φ is a pair $\langle \mathfrak{F}, \boldsymbol{q}\rangle$ such that $\mathfrak{F} = \langle W, R\rangle$ is a frame for L and $\boldsymbol{q}$ a function associating with each $w \in W$ a quasistate $\boldsymbol{q}(w) = \langle\langle T_w, <_w\rangle, \boldsymbol{t}_w\rangle$ for φ such that the depth of each $\langle T_w, <_w\rangle$ is m.

Let $\langle \mathfrak{F}, \boldsymbol{q}\rangle$ be a basic structure for φ of depth m and let $k \leq m$. A *k-run through* $\langle \mathfrak{F}, \boldsymbol{q}\rangle$ is a function r giving for each $w \in W$ a point $r(w) \in T_w$ of depth k. (That is, a run 'goes along' the frame $\mathfrak{F}$ and chooses a (location of a) type of the same depth from each type-tree $\langle T_w, <_w\rangle$.) Given a set $\mathfrak{R}$ of runs, we denote by $\mathfrak{R}_k$ the set of all k-runs from $\mathfrak{R}$. Clearly, if $\mathfrak{R}_0$ is not empty, then it is a singleton set, with its only member r_0 being the run through the roots of the quasistates.

A run r is called *coherent* if

$$\forall w \in W\, \forall \Diamond_1\psi \in sub\,\varphi\ \Big(\exists v \in W\ \big(wRv\ \wedge\ \psi \in \boldsymbol{t}_v(r(v))\big)\ \rightarrow\ \Diamond_1\psi \in \boldsymbol{t}_w(r(w))\Big),$$

and *saturated* if

$$\forall w \in W\, \forall \Diamond_1\psi \in sub\,\varphi\ \Big(\Diamond_1\psi \in \boldsymbol{t}_w(r(w))\ \rightarrow\ \exists v \in W\ \big(wRv\ \wedge\ \psi \in \boldsymbol{t}_v(r(v))\big)\Big).$$

Finally, we say that a quadruple $\mathfrak{Q} = \langle \mathfrak{F}, \boldsymbol{q}, \mathfrak{R}, \lhd\rangle$ is an $L \times \mathbf{K}$-*quasimodel for* φ (*based on* $\mathfrak{F}$) if $\mathfrak{F}$ is a frame for L, $\langle \mathfrak{F}, \boldsymbol{q}\rangle$ is a basic structure for φ of depth $m \leq md(\varphi)$ such that

(qm2) $\exists w_0 \in W\ \varphi \in \boldsymbol{t}_{w_0}(x_0)$, where x_0 is the root of $\langle T_{w_0}, <_{w_0}\rangle$,

$\mathfrak{R}$ is a set of coherent and saturated runs through $\langle \mathfrak{F}, \boldsymbol{q}\rangle$, and $\lhd$ is a binary relation on $\mathfrak{R}$ satisfying the following conditions:

(qm3) for all $r, r' \in \mathfrak{R}$, if $r \lhd r'$ then $r(w) <_w r'(w)$ for all $w \in W$;

(qm4) $\mathfrak{R}_0 \neq \emptyset$, and for all $k < m$, $r \in \mathfrak{R}_k$, $w \in W$ and $x \in T_w$, if $r(w) <_w x$ then there is $r' \in \mathfrak{R}_{k+1}$ such that $r'(w) = x$ and $r \lhd r'$.

Now, having the notion of a quasimodel been defined, what we need is the 'quasimodel truth-lemma:'

LEMMA 30. *Given a Kripke complete modal logic L, an $\mathcal{ML}_2$-formula φ is satisfiable in a product frame $\mathfrak{F} \times \mathfrak{G}$ for $L \times \mathbf{K}$ iff there is an $L \times \mathbf{K}$-quasimodel for φ based on $\mathfrak{F}$.*

Proof. ($\Leftarrow$) Suppose $\langle \mathfrak{F}, \boldsymbol{q}, \mathfrak{R}, \lhd \rangle$ is an $L \times \mathbf{K}$-quasimodel for φ. Take the product frame $\mathfrak{F} \times \langle \mathfrak{R}, \lhd \rangle$ and define a valuation $\mathfrak{V}$ in it as follows:

$$\mathfrak{V}(p) = \{\langle w, r \rangle \mid p \in \boldsymbol{t}_w(r(w))\}$$

for every propositional variable p. Let $\mathfrak{M} = \langle \mathfrak{F} \times \langle \mathfrak{R}, \lhd \rangle, \mathfrak{V} \rangle$. One can show by an easy induction on the construction of $\psi \in sub\,\varphi$ that for every $\langle w, r \rangle$ in $\mathfrak{M}$ we have

$$(\mathfrak{M}, \langle w, r \rangle) \models \psi \quad \text{iff} \quad \psi \in \boldsymbol{t}_w(r(w)).$$

In view of **(qm2)** and $\mathfrak{R}_0 \neq \emptyset$ (which we have by **(qm4)**), it follows that φ is satisfied in $\mathfrak{M}$.

($\Rightarrow$) Suppose that φ is satisfied in a model $\mathfrak{M}$ based on the product $\mathfrak{F} \times \mathfrak{G}$ of frames $\mathfrak{F} = \langle W, R \rangle$ and $\mathfrak{G} = \langle \Delta, < \rangle$. By Proposition 9 (i), we may assume that $\mathfrak{G}$ is an intransitive tree of depth $m \leq md(\varphi)$ and $(\mathfrak{M}, \langle w_0, x_0 \rangle) \models \varphi$ for some $w_0 \in W$, with x_0 being the root of $\mathfrak{G}$. With every pair $\langle w, x \rangle \in W \times \Delta$ we associate the type

$$\boldsymbol{t}(w, x) = \{\psi \in sub\,\varphi \mid (\mathfrak{M}, \langle w, x \rangle) \models \psi\}.$$

Now we have to construct a quasistate $\langle \langle T_w, <_w \rangle, \boldsymbol{t}_w \rangle$ for each $w \in W$. The obvious choice of $T_w = \Delta$, $<_w = <$ and $\boldsymbol{t}_w(x) = \boldsymbol{t}(w, x)$ does not work, because Δ can be infinite. So let us make it finite in such a way that the resulting structure still satisfies **(qm1)** and also complies with the smallness condition **(qm1′)**. Fix a $w \in W$ and define a binary relation $\sim_w$ on Δ as follows. If $x, y \in \Delta$ are of co-depth 0 (i.e., they are leaves of $\mathfrak{G}$) then

$$x \sim_w y \quad \text{iff} \quad \boldsymbol{t}(w, x) = \boldsymbol{t}(w, y).$$

For $x, y \in \Delta$ of co-depth k ($0 < k \leq md(\varphi)$), let

$$\begin{array}{lll} x \sim_w y \quad \text{iff} \quad \boldsymbol{t}(w, x) = \boldsymbol{t}(w, y) & \text{and} & \forall z \in \Delta \left(x < z \ \rightarrow \ \exists z' \in \Delta \ (y < z' \wedge z \sim_w z') \right) \\ & \text{and} & \forall z \in \Delta \left(y < z \ \rightarrow \ \exists z' \in \Delta \ (x < z' \wedge z \sim_w z') \right). \end{array}$$

Clearly $\sim_w$ is an equivalence relation on Δ. Denote by $[x]_w$ the $\sim_w$-equivalence class of x and put

$$\begin{aligned} &\Delta_w = \{[x]_w \mid x \in \Delta\}, \\ &[x]_w R_w [y]_w \quad \text{iff} \quad \exists y' \in [y]_w \ x < y', \\ &\boldsymbol{l}_w([x]_w) = \boldsymbol{t}(w, x). \end{aligned}$$

The structure $\langle \langle \Delta_w, R_w \rangle, \boldsymbol{l}_w \rangle$ is almost a quasistate, just $\langle \Delta_w, R_w \rangle$ is not necessarily a tree. The tree $\langle T_w, <_w \rangle$ we need can be obtained by unraveling $\langle \Delta_w, R_w \rangle$:

$$\begin{array}{lll} T_w = \{\langle [x_0]_w, \dots, [x_k]_w \rangle \mid k \leq m, \ [x_0]_w R_w [x_1]_w R_w \dots R_w [x_k]_w\}, & & \\ u <_w v \quad \text{iff} \quad u = \langle [x_0]_w, \dots, [x_k]_w \rangle, & v = \langle [x_0]_w, \dots, [x_k]_w, [x_{k+1}]_w \rangle & \\ \qquad\qquad\quad \text{and} \ \ [x_k]_w R_w [x_{k+1}]_w. & & \end{array}$$

Finally, let $\boldsymbol{t}_w(\langle [x_0]_w, \ldots, [x_k]_w \rangle) = \boldsymbol{l}_w([x_k]_w) = \boldsymbol{t}(w, x_k)$. It is not hard to see that, for any $w \in W$, $\langle \langle T_w, <_w \rangle, \boldsymbol{t}_w \rangle$ is a quasistate for φ. Moreover, by taking $\boldsymbol{q}(w) = \langle \langle T_w, <_w \rangle, \boldsymbol{t}_w \rangle$ for each $w \in W$, we obtain a basic structure $\langle \mathfrak{F}, \boldsymbol{q} \rangle$ for φ satisfying **(qm2)**.

It remains to define appropriate runs through $\langle \mathfrak{F}, \boldsymbol{q} \rangle$. To this end, for each $k \leq m$ and each sequence $\langle x_0, \ldots, x_k \rangle$ of points in Δ such that $x_0 < \cdots < x_k$, take the map

$$r : w \mapsto \langle [x_0]_w, \ldots, [x_k]_w \rangle ,$$

and let $\mathfrak{R}$ be the set of all such maps. For $r, r' \in \mathfrak{R}$, let $r \lhd r'$ iff $r(w) <_w r'(w)$ for all $w \in W$. It is straightforward to check that $\langle \mathfrak{F}, \boldsymbol{q}, \mathfrak{R}, \lhd \rangle$ is an $L \times \mathbf{K}$-quasimodel for φ. ❑

Note that $L \times \mathbf{S5}$-quasimodels are considerably simpler than $L \times \mathbf{K}$-quasimodels: instead of trees of types it is enough to consider *sets* of types only as quasistates. Similarly to the filtration case, this results in better upper bounds on the size of the constructed structures. On the other hand, $L \times \mathbf{K}_n$- and $L \times \mathbf{S5}_n$-quasimodels (for $n \geq 2$) are similar to the above complex ones, and one even has to take into account the several different accessibility relations when defining quasistates.

Although quasistates in quasimodels are always finite, quasimodels themselves are usually infinite (since the frame $\mathfrak{F}$ can be infinite). Depending on the component logics in question, there can be several ways of using them to prove decidability of products:

- In the simplest cases one can manage to find a finite quasimodel for φ and then to construct a finite product model out of it, thereby showing that the logic has the product fmp. This can be done in the case of $\mathbf{K} \times \mathbf{K}$. Moreover, for $\mathbf{S5} \times \mathbf{S5}$ and $\mathbf{K} \times \mathbf{S5}$ the resulting product model is of exponential size (see Chapter 3 of this handbook), so these logics are decidable in coNEXPTIME.

- In some cases, it can be shown that there is a quasimodel for φ iff there exists a *finite* set $\mathcal{S}$ of *finite* 'partial' quasimodels (called *blocks* or *mosaics*) satisfying some effectively checkable conditions and that the cardinality of $\mathcal{S}$ as well as the size of each block in it do not exceed a number effectively computable from φ. The 'effectively checkable conditions' are supposed to guarantee that blocks can be used as 'small mosaic pieces' to construct the quasimodel we need.

- In some cases, the statement that a quasimodel exists can be translated into monadic second-order logic or reduced to other known decidable problems.

Here we illustrate the second and the third techniques by showing—in two different ways—that $\mathbf{K4.3} \times \mathbf{K}$ is decidable. Note that the formula (4) shows that this logic lacks the product fmp, and it is not known whether it has the fmp.

Quasimodels and mosaics.

Throughout, we fix an $\mathcal{ML}_2$-formula φ. A *block for* φ is a quadruple

$$\mathfrak{B}^{uv} = \langle \mathfrak{F}^{uv}, \boldsymbol{q}^{uv}, \mathfrak{R}^{uv}, \lhd^{uv} \rangle$$

such that

- $\mathfrak{F}^{uv} = \langle \{u, v\}, < \rangle$ is a 2-element strict linear order with $u < v$,

- $\langle \mathfrak{F}^{uv}, \boldsymbol{q}^{uv} \rangle$ is a basic structure for φ of depth m, for some $m \leq md(\varphi)$,

- $\mathfrak{R}^{uv}$ is a set of runs through $\langle \mathfrak{F}^{uv}, \boldsymbol{q}^{uv} \rangle$ such that, for all $r \in \mathfrak{R}^{uv}$ and $\Diamond_1\psi \in sub\,\varphi$,

$$\text{if } \psi \in \boldsymbol{t}_v(r(v)) \text{ or } \Diamond_1\psi \in \boldsymbol{t}_v(r(v)) \text{ then } \Diamond_1\psi \in \boldsymbol{t}_u(r(u)),$$

- $\lhd^{uv}$ is a binary relation on $\mathfrak{R}^{uv}$ satisfying **(qm3)** and **(qm4)**.

We remind the reader that quasistates occurring in such a block are denoted by

$$\boldsymbol{q}^{uv}(u) = \langle \langle T_u, <_u \rangle, \boldsymbol{t}_u \rangle \quad \text{and} \quad \boldsymbol{q}(v)^{uv} = \langle \langle T_v, <_v \rangle, \boldsymbol{t}_v \rangle .$$

Observe that a block is almost a **K4.3** $\times$ **K**-quasimodel. The only thing missing is that its runs are (though coherent) not necessarily saturated. That is why we need an appropriate collection of blocks: By sticking them properly together, we can 'fix the defects' and converge to a real quasimodel.

To this end, we call a set $\mathcal{S}$ of blocks for φ *satisfying* if the following properties hold:

(ssb1) all blocks in $\mathcal{S}$ are of the same depth m, for some $m \leq md(\varphi)$;

(ssb2) $\mathcal{S}$ contains a block satisfying **(qm2)**;

(ssb3) for every $\mathfrak{B}^{uv}$ in $\mathcal{S}$, if $\Diamond_1\psi \in \boldsymbol{t}_v(r(v))$ for some run $r \in \mathfrak{R}^{uv}$ then there exist a block $\mathfrak{B}^{vw}$ in $\mathcal{S}$ and a sequence $\langle x_s \in T_w \mid s \in \mathfrak{R}^{uv} \rangle$ of points in T_w such that

- $\boldsymbol{q}^{uv}(v) = \boldsymbol{q}^{vw}(v)$,
- for every $s \in \mathfrak{R}^{uv}$, the function p defined by $p(v) = s(v)$, $p(w) = x_s$ is a run in $\mathfrak{R}^{vw}$,
- for all $s, s' \in \mathfrak{R}^{uv}$, if $s \lhd^{uv} s'$ then $x_s <_w x_{s'}$,
- $\psi \in \boldsymbol{t}_w(x_r)$;

(ssb4) for every block $\mathfrak{B}^{uv}$ in $\mathcal{S}$, if $\Diamond_1\psi \in \boldsymbol{t}_u(r(u))$, $\psi \notin \boldsymbol{t}_v(r(v))$ and $\Diamond_1\psi \notin \boldsymbol{t}_v(r(v))$ for some run $r \in \mathfrak{R}^{uv}$ then there are blocks $\mathfrak{B}^{uw}$ and $\mathfrak{B}^{wv}$ in $\mathcal{S}$ and a sequence $\langle x_s \in T_w \mid s \in \mathfrak{R}^{uv} \rangle$ of points in T_w such that

- $\boldsymbol{q}^{uv}(u) = \boldsymbol{q}^{uw}(u)$, $\boldsymbol{q}^{uw}(w) = \boldsymbol{q}^{wv}(w)$, $\boldsymbol{q}^{wv}(v) = \boldsymbol{q}^{uv}(v)$,
- for every $s \in \mathfrak{R}^{uv}$, the function p' defined by $p'(u) = s(u)$, $p'(w) = x_s$ is a run in $\mathfrak{R}^{uw}$, and the function p'' defined by $p''(w) = x_s$, $p''(v) = s(v)$ is a run in $\mathfrak{R}^{wv}$,
- for all $s, s' \in \mathfrak{R}^{uv}$, if $s \lhd^{uv} s'$ then $x_s <_w x_{s'}$,
- $\psi \in \boldsymbol{t}_w(x_r)$.

On the one hand, it is straightforward to see that one can effectively check whether there exists a satisfying set of blocks for φ. On the other hand, it is well-known that every rooted frame for **K4.3** is a p-morphic image of a sufficiently large strict linear order, so by Propositions 9 and 12, **K4.3** $\times$ **K** is determined by product frames whose first component is a strict linear order. As satisfiability in a single element strict linear order is trivially decidable, to establish the decidability of **K4.3** $\times$ **K**, it is enough to prove the following 'block lemma:'

LEMMA 31. *There is a* $\mathbf{K4.3} \times \mathbf{K}$*-quasimodel for* φ *based on a strict linear order with* ≥ 2 *elements iff there is a satisfying set of blocks for* φ*.*

Proof. The construction of a satisfying set from a quasimodel is easy. Suppose that $\mathfrak{Q} = \langle \mathfrak{F}, \boldsymbol{q}, \mathfrak{R}, \lhd \rangle$ is a quasimodel for φ, with $\mathfrak{F} = \langle W, R \rangle$ being a strict linear order with ≥ 2 elements. For all $u, v \in W$ such that uRv, define the *restriction* $\mathfrak{Q}^{uv}$ of $\mathfrak{Q}$ to the 2-element strict linear order on $\{u, v\}$ in the natural way. It is straightforward to check that these $\mathfrak{Q}^{uv}$ are blocks and that the collection $\mathcal{S}$ of them is a satisfying set.

Now we show how a quasimodel for φ can be constructed from a satisfying set $\mathcal{S}$ of blocks for φ. Starting from a block satisfying **(qm2)**, we will build a series of larger and larger quasimodel-like structures having not necessarily saturated runs. The 'defects' of these runs are 'corrected' one by one in such a way that the sequence of structures 'converges' to a quasimodel.

To begin with, we call a quadruple $\mathfrak{Q} = \langle \mathfrak{F}, \boldsymbol{q}, \mathfrak{R}, \lhd \rangle$ a *weak quasimodel for* φ if the following conditions hold:

(wq1) $\mathfrak{F} = \langle W, R \rangle$ is a finite strict linear order, $W = \{w_0, w_1, \dots, w_m\}$ for some $m > 0$, $w_0 R w_1 R \dots R w_m$, and $\langle \mathfrak{F}, \boldsymbol{q} \rangle$ is a basic structure for φ satisfying **(qm2)**;

(wq2) $\mathfrak{R}$ is a set of runs through $\langle \mathfrak{F}, \boldsymbol{q} \rangle$ such that for all $i < j \leq m$, $r \in \mathfrak{R}$ and $\Diamond_1 \psi \in sub\, \varphi$,

$$\text{if } \psi \in \boldsymbol{t}_{w_j}(r(w_j)) \text{ or } \Diamond_1 \psi \in \boldsymbol{t}_{w_j}(r(w_j)) \text{ then } \Diamond_1 \psi \in \boldsymbol{t}_{w_i}(r(w_i)),$$

(wq2′) $\lhd$ is a binary relation on $\mathfrak{R}$ satisfying **(qm4)** and such that, for all $r, s \in \mathfrak{R}$,

$$r \lhd s \qquad \text{iff} \qquad r(w_i) <_{w_i} s(w_i) \ \text{ for all } i \leq m,$$

(wq3) for every $i < m$, the restriction of $\mathfrak{Q}$ to the two-element strict linear order on $\{w_i, w_{i+1}\}$ is a block in $\mathcal{S}$.

(Note that property **(wq2′)** is a bit stronger than **(qm3)**.) Now take a triple $\langle i, r, \Diamond_1 \psi \rangle$ such that $i \leq m$, $r \in \mathfrak{R}$ and $\Diamond_1 \psi \in sub\, \varphi$. Such a triple is called a *defect in* $\mathfrak{Q}$ if $\Diamond_1 \psi \in \boldsymbol{t}_{w_i}(r(w_i))$ and for all j such that $i < j \leq m$, $\psi \notin \boldsymbol{t}_{w_j}(r(w_j))$ and $\Diamond_1 \psi \notin \boldsymbol{t}_{w_j}(r(w_j))$. If $i = m$ then such a defect is called an *end-defect*, otherwise it is a *middle-defect*.

We construct a sequence $\langle \mathfrak{Q}_n \mid n < \omega \rangle$ of weak quasimodels which 'converges' to a real quasimodel for φ. Take a block $\mathfrak{Q}_0 = \langle \mathfrak{F}_0, \boldsymbol{q}_0, \mathfrak{R}_0, \lhd_0 \rangle$ in $\mathcal{S}$ satisfying **(qm2)**. Clearly, it is a weak quasimodel for φ as well. Suppose now that we have already constructed $\mathfrak{Q}_n = \langle \mathfrak{F}_n, \boldsymbol{q}_n, \mathfrak{R}_n, \lhd_n \rangle$ such that $\mathfrak{F}_n = \langle W_n, R_n \rangle$, $W_n = \{w_0, w_1, \dots, w_m\}$ and $w_0 R_n w_1 R_n \dots R_n w_m$. If the set D_n of all defects in $\mathfrak{Q}_n$ is empty then we are done: $\mathfrak{Q}_n$ is obviously a quasimodel for φ. Otherwise, we take some $d = \langle i, r, \Diamond_1 \psi \rangle$ from D_n.

Case 1: d is a middle-defect, that is, $i < m$. By **(wq3)**, the restriction $\mathfrak{Q}^{w_i w_{i+1}}$ of $\mathfrak{Q}_n$ to the two-element strict linear order on $\{w_i, w_{i+1}\}$ is a block in $\mathcal{S}$. Choose two blocks $\mathfrak{B}^{w_i w}$ and $\mathfrak{B}^{w w_{i+1}}$ according to **(ssb4)** (with $u = w_i$ and $v = w_{i+1}$). We may assume that $w \notin W_n$. Define a basic structure $\langle \mathfrak{F}_n^d, \boldsymbol{q}_n^d \rangle$ by taking

$$\begin{aligned}
&W_n^d = W_n \cup \{w\},\\
&R_n^d = R_n \cup \{\langle w_j, w \rangle \mid j \leq i,\ w_j \in W_n\} \cup \{\langle w, w_j \rangle \mid i < j \leq m,\ w_j \in W_n\},\\
&\mathfrak{F}_n^d = \langle W_n^d, R_n^d \rangle,\\
&\boldsymbol{q}_n^d(v) = \begin{cases} \boldsymbol{q}^{w_i w}(v) = \boldsymbol{q}^{w w_{i+1}}(v), & \text{if } v = w,\\ \boldsymbol{q}_n(v), & \text{if } v \in W_n. \end{cases}
\end{aligned}$$

For all runs $s, p \in \mathfrak{R}_n$, $s' \in \mathfrak{R}^{w_i w}$, $s'' \in \mathfrak{R}^{w w_{i+1}}$, such that $s(w_i) = s'(w_i)$, $s'(w) = s''(w)$, $s''(w_{i+1}) = p(w_{i+1})$, define the function $s \cup s' \cup s'' \cup p$ on W_n^d by taking, for all $v \in W_n^d$,

$$(s \cup s' \cup s'' \cup p)(v) = \begin{cases} s(v), & \text{if } v = w_j,\ j \leq i, \\ s'(v) = s''(v), & \text{if } v = w, \\ p(v), & \text{if } v = w_j,\ i < j \leq m. \end{cases}$$

Let $\mathfrak{R}_n^d$ be the set of all such functions. Elements in $\mathfrak{R}_n^d$ of the form $s \cup s' \cup s'' \cup s$, for some $s \in \mathfrak{R}_n$, are called *extensions of* s. We call an extension $s \cup s' \cup s'' \cup s$ *good*, if $s'(w) = s''(w) = x_s$; cf. **(ssb4)**. Observe that every $s \in \mathfrak{R}_n$ has a unique good extension in $\mathfrak{R}_n^d$.

For all $s, s' \in \mathfrak{R}_n^d$, define

$$s \lhd_n^d s' \qquad \text{iff} \qquad s(v) <_v s'(v) \ \text{ for all } v \in W_n^d.$$

In other words, we 'glue together' the blocks $\mathfrak{B}^{w_i w}$ and $\mathfrak{B}^{w w_{i+1}}$ at w, and then 'insert' the resulting piece into $\mathfrak{Q}_n$ between w_i and w_{i+1}. It can be readily checked that $\mathfrak{Q}_n^d = \langle \mathfrak{F}_n^d, \boldsymbol{q}_n^d, \mathfrak{R}_n^d, \lhd_n^d \rangle$ is a weak quasimodel. Moreover, the defect d in $\mathfrak{Q}_n^d$ is 'cured' in the sense that (by **(ssb4)**) the good extension r^+ of r is such that $\psi \in \boldsymbol{t}_w(r^+(w))$.

Case 2: d is an end-defect. This case is analogous to Case 1, but we have to use **(ssb3)** instead of **(ssb4)** for 'gluing together' $\mathfrak{Q}_n$ and a block $\mathfrak{B}^{w_m w}$ at w_m.

Next we turn the remaining defects in $\mathfrak{Q}_n$ to a subset D_n^d of the set of defects in $\mathfrak{Q}_n^d$ as follows. Suppose $\langle j, s, \Diamond_1 \chi \rangle$ is a defect in D_n different from d. Let s^+ be the good extension of s and let $k = j$ if $j \leq i$ and $k = j + 1$ otherwise. If $\langle k, s^+, \Diamond_1 \chi \rangle$ is a defect in $\mathfrak{Q}_n^d$ then we put it into D_n^d. Clearly, $|D_n^d| \leq |D_n| - 1$. If $D_n^d \neq \emptyset$ then we take a defect $d' \in D_n^d$, construct $\mathfrak{Q}_n^{dd'}$, and so on. When all the finitely many defects in D_n are cured, we obtain a weak quasimodel $\mathfrak{Q}_{n+1}$. Note that every run $r_n \in \mathfrak{R}_n$ has a unique extension $r_{n+1} \in \mathfrak{R}_{n+1}$ obtained by taking at every step the good extension of the previous run. We call this r_{n+1} the *good extension of* r_n *in* $\mathfrak{Q}_{n+1}$.

The limit quasimodel is defined by taking $\mathfrak{F} = \langle W, R \rangle$, where $W = \bigcup_{n<\omega} W_n$, $R = \bigcup_{n<\omega} R_n$ and $\boldsymbol{q} = \bigcup_{n<\omega} \boldsymbol{q}_n$. Then clearly $\mathfrak{F}$ is a strict linear order and $\langle \mathfrak{F}, \boldsymbol{q} \rangle$ is a basic structure for φ.

For every $i < \omega$ and every sequence of runs $\langle r_n \in \mathfrak{R}_n \mid n \geq i \rangle$ such that r_{n+1} is the good extension of r_n in $\mathfrak{Q}_{n+1}$ for all $n \geq i$, take $r = \bigcup\{r_n \mid n \geq i\}$. Let $\mathfrak{R}$ be the set of such runs. For $r = \bigcup\{r_n \mid n \geq i\}$ and $r' = \bigcup\{r'_n \mid n \geq j\}$ in $\mathfrak{R}$, define

$$r \lhd r' \qquad \text{iff} \qquad r_n \lhd_n r'_n, \ \text{ for all } n \geq \max(i, j).$$

We show that $\mathfrak{R}$ and $\lhd$ satisfy **(qm3)** and **(qm4)**. Indeed, suppose that r and r' are of the above form and $r \lhd r'$. Take a $w \in W$. There is an $n \geq \max(i, j)$ such that $w \in W_n$. Then $r(w) = r_n(w)$, $r'(w) = r'_n(w)$ and $r_n \lhd_n r'_n$, which implies $r(w) <_w r'(w)$ by **(wq2′)**. For **(qm4)**, suppose that $r = \bigcup\{r_n \mid n \geq i\}$ and $r(w) <_w x$ for some $x \in T_w$. Then there is an $n \geq i$ such that $w \in W_n$, and so $r(w) = r_n(w)$. Since $\mathfrak{Q}_n$ satisfies **(qm4)**, there is an $s_n \in \mathfrak{R}_n$ such that $s_n(w) = x$ and $r_n \lhd_n s_n$. Let $s = \bigcup\{s_m \mid m \geq n\}$, where s_{m+1} is the good extension of s_m in $\mathfrak{Q}_{m+1}$ for all $m \geq n$. Then $s(w) = s_n(w) = x$, and it is not hard to see that, by **(ssb3)**, **(ssb4)** and **(wq2′)**, $r_m \lhd_m s_m$ hold for all $m \geq n$, from which $r \lhd s$.

Finally, we show that all the runs in $\mathfrak{R}$ are coherent and saturated. Indeed, suppose that $r = \bigcup\{r_n \mid n \geq i\}$ and $\Diamond_1 \psi \in \boldsymbol{t}_w(r(w))$ for some $w \in W$. Then there is an $n \geq i$ such that $w \in W_n$, and so $r(w) = r_n(w)$. If $\langle w, r_n, \Diamond_1\psi\rangle$ is not a defect in $\mathfrak{Q}_n$ then there is a $v \in W_n$ such that wRv, $r_n(v) = r(v)$ and $\psi \in \boldsymbol{t}_v(r_n(v))$. And if $\langle w, r_n, \Diamond_1\psi\rangle$ is a defect in $\mathfrak{Q}_n$ then it is cured in its good extension r_{n+1} in $\mathfrak{Q}_{n+1}$: there is $v \in W_{n+1}$ such that wRv, $r_{n+1}(v) = r(v)$ and $\psi \in \boldsymbol{t}_v(r_{n+1}(v))$. Conversely, assume that $\psi \in \boldsymbol{t}_w(r(w))$ and let vRw. Then there is an $n \geq i$ such that $v, w \in W_n$. Thus $r(w) = r_n(w)$, $r(v) = r_n(v)$ and vR_nw, and so $\Diamond_1\psi \in \boldsymbol{t}_v(r(v))$ follows by **(wq2)**.

Therefore, $\mathfrak{Q} = \langle \mathfrak{F}, \boldsymbol{q}, \mathfrak{R}, \lhd\rangle$ is a **K4.3** $\times$ **K**-quasimodel for φ, as required. ❑

Observe that the decision procedure given above is again non-elementary. In fact, no elementary decision procedure is known for **K4.3** $\times$ **K**. However, as quasistates in **K4.3** $\times$ **S5**-quasimodels are of double-exponential size, one can obtain a 2EXPTIME decision algorithm for **K4.3**$\times$**S5** as follows. Take the set of all blocks for φ (a straightforward computation shows that the cardinality of this set is also at most double-exponential in the size of φ). Eliminate iteratively those blocks for which there are no 'noneliminated' blocks satisfying **(ssb3)** and **(ssb4)**. This elimination procedure stops after at most double-exponentially many steps. Now it is not hard to show that φ is satisfiable iff the set $\mathcal{S}$ of remaining blocks contains a block satisfying **(qm2)**.

Quasimodels and reductions to monadic second-order theories.

Here we give a second proof for the decidability of **K4.3** $\times$ **K** by showing that one can translate the statement "there exists a **K4.3** $\times$ **K**-quasimodel for φ based on some strict linear order $\mathfrak{F}$" into monadic second-order logic.

Fix some $\mathcal{ML}_2$-formula φ. For every $m \leq md(\varphi)$, below we will define a monadic second order formula qm^m_φ (in the language having a binary predicate constant $<$) in such a way that the following holds:

LEMMA 32. *For any strict linear order $\mathfrak{F}$, $\mathfrak{F} \models \mathsf{qm}^m_\varphi$ for some $m \leq md(\varphi)$ iff there exists a* **K4.3** $\times$ **K**-*quasimodel for φ based on $\mathfrak{F}$.*

Though the monadic second-order theory of all strict linear orders is undecidable, we can still use this lemma to deduce decidability of **K4.3** $\times$ **K** as follows. It is not hard to see, using Theorem 16, that it is enough to consider quasimodels that are based on *countable* strict linear orders. Now for every monadic second-order formula ψ and a monadic predicate variable P not occurring in ψ, define the relativisation ψ^P of ψ to P inductively by taking $\psi^P = \psi$ for atomic ψ, $(\neg\psi)^P = \neg\psi^P$, $(\psi_1 \wedge \psi_2)^P = \psi_1^P \wedge \psi_2^P$, $(\forall x\psi)^P = \forall x(P(x) \to \psi^P)$, and $(\forall Q\psi)^P = \forall Q\psi^P$. Obviously, for any sentence ψ and any strict linear order $\mathfrak{F}$, we have $\mathfrak{F} \models \exists P(\exists x P(x) \wedge \psi^P)$ iff $\mathfrak{F}' \models \psi$ for some (nonempty) suborder $\mathfrak{F}'$ of $\mathfrak{F}$—the intended interpretation of P is the domain of $\mathfrak{F}'$. As is well-known, any countable strict linear order is a suborder of $\langle \mathbb{Q}, <\rangle$. Let λ be the first-order sentence defining the class of all strict linear orders. Then qm^m_φ (assumed not to involve P) is satisfiable in some countable strict linear order $\mathfrak{F}$ iff the monadic second-order formula

$$\exists P \left(\exists x P(x) \wedge (\lambda \wedge \mathsf{qm}^m_\varphi)^P\right)$$

holds in $\langle \mathbb{Q}, <\rangle$. As the monadic second-order theory of $\langle \mathbb{Q}, <\rangle$ is known to be decidable (see [67]), this proves the decidability of **K4.3** $\times$ **K**.

In order to define the necessary monadic second order formulas qm^m_φ for each $m \leq md(\varphi)$, we require a number of auxiliary formulas. Denote by Σ_m the set of all quasistates for φ of depth m. Given a quasistate $\boldsymbol{q} = \langle \langle T_{\boldsymbol{q}}, <_{\boldsymbol{q}}\rangle, \boldsymbol{t}_{\boldsymbol{q}}\rangle$ from Σ_m and a point a in $T_{\boldsymbol{q}}$, we denote the depth of a by $d_{\boldsymbol{q}}(a)$.

Introduce a unary predicate variable $P_{\boldsymbol{q}}$ for each $\boldsymbol{q} \in \Sigma_m$ and a unary predicate variable R^k_ψ for each $\psi \in sub\,\varphi$ and each $k \leq m$. Given a type $\boldsymbol{t}$ for φ and $k \leq m$, let

$$\chi_{\boldsymbol{t}}(\overline{R^k}(x)) = \bigwedge_{\psi \in \boldsymbol{t}} R^k_\psi(x) \wedge \bigwedge_{\substack{\psi \notin \boldsymbol{t} \\ \psi \in sub\,\varphi}} \neg R^k_\psi(x),$$

saying that the type $\boldsymbol{t}$ at point x of depth k is defined with the help of

$$\overline{R^k}(x) = \left\langle R^k_\psi(x) \mid \psi \in sub\,\varphi \right\rangle .$$

For each $k \leq m$, let

$$\begin{aligned}\mathsf{run}_0(\overline{P}, \overline{R^k}) \;=\; & \forall x \bigwedge_{\boldsymbol{q} \in \Sigma_m} \big(P_{\boldsymbol{q}}(x) \rightarrow \\ & \bigvee_{\substack{a \in T_q \\ d_q(a)=k}} \chi_{\boldsymbol{t}_q(a)}(\overline{R^k}(x))\big) \;\wedge\; \forall x \bigwedge_{\Diamond_1\psi \in sub\,\varphi} \big[R^k_{\Diamond_1\psi}(x) \leftrightarrow \exists y\big(x < y \wedge R^k_\psi(y)\big)\big].\end{aligned}$$

This is intended to say that $\overline{R^k}$ defines a coherent and saturated k-run through a sequence of quasistates defined with the help of $\overline{P} = \langle P_{\boldsymbol{q}} \mid \boldsymbol{q} \in \Sigma_m \rangle$.

However, we have to refine this definition in order to ensure that condition **(qm4)** holds. To this end, we define, by 'backwards' induction on k, another formula $\mathsf{run}(\overline{P}, \overline{R^k})$ as follows. If $k = m$ (that is, we are at the 'leaf-level') then take $\mathsf{run}(\overline{P}, \overline{R^m}) = \mathsf{run}_0(\overline{P}, \overline{R^m})$.

Suppose, inductively, that for $k \leq m$ we have already defined $\mathsf{run}(\overline{P}, \overline{R^k})$. Then let $\mathsf{run}(\overline{P}, \overline{R^{k-1}})$ be the following formula:

$$\begin{aligned}&\mathsf{run}_0(\overline{P}, \overline{R^{k-1}}) \wedge \\ &\forall x \bigwedge_{\boldsymbol{q} \in \Sigma_m} \bigwedge_{\substack{a \in T_q \\ d_q(a)=k-1}} \Big[P_{\boldsymbol{q}}(x) \wedge \chi_{\boldsymbol{t}_q(a)}(\overline{R^{k-1}}(x)) \rightarrow \bigwedge_{\substack{b \in T_q \\ a <_q b}} \exists_{\psi \in sub\,\varphi} R^k_\psi \Big(\mathsf{run}(\overline{P}, \overline{R^k}) \wedge \chi_{\boldsymbol{t}_q(b)}(\overline{R^k}(x)) \wedge \\ &\qquad \forall z \bigwedge_{\boldsymbol{s} \in \Sigma_m} \bigwedge_{\substack{c \in T_s \\ d_s(c)=k-1}} \big(P_{\boldsymbol{s}}(z) \wedge \chi_{\boldsymbol{t}_s(c)}(\overline{R^{k-1}}(z)) \rightarrow \bigvee_{\substack{d \in T_s \\ c <_s d}} \chi_{\boldsymbol{t}_s(d)}(\overline{R^k}(z))\big)\Big)\Big].\end{aligned}$$

Finally, we define a monadic second-order sentence qm^m_φ by taking

$$\begin{aligned}\mathsf{qm}^m_\varphi = \exists_{\boldsymbol{q} \in \Sigma_m} P_{\boldsymbol{q}} \Big[& \forall x \bigvee_{\boldsymbol{q} \in \Sigma_m} \big(P_{\boldsymbol{q}}(x) \wedge \bigwedge_{\substack{\boldsymbol{q}' \in \Sigma_m \\ \boldsymbol{q} \neq \boldsymbol{q}'}} \neg P_{\boldsymbol{q}'}(x)\big) \;\wedge \\ & \bigvee_{\substack{\boldsymbol{s} \in \Sigma_m,\, a \in T_s \\ d_s(a)=0 \\ \varphi \in \boldsymbol{t}_s(a)}} \exists x \Big(P_{\boldsymbol{s}}(x) \wedge \exists_{\psi \in sub\,\varphi} R^0_\psi \,\big(\mathsf{run}(\overline{P}, \overline{R^0}) \wedge \chi_{\boldsymbol{t}_s(a)}(\overline{R^0}(x))\big)\Big)\Big].\end{aligned}$$

Evaluated in a strict linear order $\mathfrak{F} = \langle W, < \rangle$, the first line of qm^m_φ says that the sets $P_{\boldsymbol{q}} \subseteq W$ ($\boldsymbol{q} \in \Sigma_m$) form a partition of W. By defining the map $\boldsymbol{q} : W \rightarrow \Sigma_m$ as

$$\boldsymbol{q}(w) = \boldsymbol{q} \quad \text{iff} \quad w \in P_{\boldsymbol{q}}$$

and a relation $\lhd$ on the runs by taking $r \lhd r'$ iff r is defined by $\overline{R^{k-1}}$ and r' is defined by $\overline{R^k}$ for some $k \leq m$, we obtain a quasimodel $\mathfrak{Q} = \langle \mathfrak{F}, \boldsymbol{q}, \mathfrak{R}, \lhd \rangle$ for φ: the second line of qm_φ^m states condition **(qm2)**; conditions **(qm3)** and **(qm4)** are satisfied by the definitions of $\lhd$ and the formulas $\mathsf{run}(\overline{P}, \overline{R^k})$, respectively.

Lower complexity bounds.

The following general result was obtained by Marx [59]. It is proved by reducing the NEXPTIME-complete "$n \times n$ bounded tiling problem" to the satisfiability problem of the logics in question, see Chapter 3 of this handbook:

THEOREM 33. *Let L be a Kripke complete bimodal logic between* $\mathbf{K} \times \mathbf{K}$ *and* $\mathbf{S5} \times \mathbf{S5}$. *Then L is* CONEXPTIME-*hard.*

For products of 'linear' logics with **S5** (such as $\mathsf{Log}\{\langle \mathbb{N}, < \rangle\} \times \mathbf{S5}$, $\mathsf{Log}\{\langle \mathbb{Q}, < \rangle\} \times \mathbf{S5}$, $\mathbf{K4.3} \times \mathbf{S5}$) one can obtain an EXPSPACE lower bound by reducing the "2^n corridor tiling problem" to their satisfiability problem, see [23, Theorem 6.64].

Products of 'transitive' modal logics are usually undecidable

None of the techniques for proving decidability discussed above work if we consider two-dimensional products where both component logics are determined by *transitive* frames of unbounded 'cluster-depth' (such as $\mathbf{K4} \times \mathbf{K4}$). As we shall see below, these product logics are in fact undecidable, and often lack the 'abstract' fmp.

Given a transitive frame $\mathfrak{F} = \langle W, R \rangle$, a point $x \in W$ is said to be of *cluster-depth* $n < \omega$ in $\mathfrak{F}$ if there is a path $x = x_0 R x_1 R \ldots R x_n$ of points from distinct clusters in $\mathfrak{F}$ (that is, $x_{i+1} R x_i$ does not hold for any $i < n$) and there is no such path of greater length. If for every $n < \omega$ there is a path of n points from distinct clusters starting from x, then we say that x is of *infinite cluster-depth*, or x is of *cluster-depth* ∞. The *cluster-depth of* $\mathfrak{F}$ is defined to be the supremum of the cluster-depths of its points (with $n < \infty$ for all $n < \omega$). For instance, $\mathfrak{F}$ is of infinite cluster-depth if it contains points of arbitrary finite cluster-depth. By the *cluster-depth* of a bimodal frame $\langle W, R_1, R_2 \rangle$ with transitive R_1, R_2 we understand the minimal cluster-depth of $\langle W, R_1 \rangle$ and $\langle W, R_2 \rangle$.

We remind the reader that a frame $\langle W, R \rangle$ is called *Noetherian* if there is no infinite strictly ascending chain $x_0 R x_1 R x_2 R \ldots$ of points from W (i.e., no R-chain such that $x_i \neq x_{i+1}$, for all $i < \omega$).

THEOREM 34. (i) [28] *Let L_1 and L_2 be Kripke complete unimodal logics containing* **K4** *and such that both L_1 and L_2 have among their frames a rooted Noetherian linear order with an infinite descending chain of distinct points. Then all bimodal logics L in the interval*

$$[L_1, L_2] \;\subseteq\; L \;\subseteq\; L_1 \times L_2$$

lack the fmp.

(ii) [26] *If L is any Kripke complete bimodal logic containing* $\mathbf{K4} \times \mathbf{K4}$ *and having product frames of arbitrarily large finite or infinite cluster-depth, then L is undecidable.*

Note that as (by Theorem 21) $\mathbf{K4} \times \mathbf{K4} = [\mathbf{K4}, \mathbf{K4}]$, it is a simple and natural example for a finitely axiomatisable but undecidable modal logic.

Below we discuss the main points of the proof of Theorem 34. For more details, consult [26].

Lack of finite model property.

We will define a bimodal formula 'forcing' infinite $[\mathbf{K4}, \mathbf{K4}]$-frames. We want to 'get rid of' the clusters first: take two fresh propositional variables h and v, and define new modal operators by setting, for every bimodal formula ψ,

$$\begin{aligned}
\bar{\Diamond}_1 \psi &= \big[h \to \Diamond_1\big(\neg h \wedge (\psi \vee \Diamond_1 \psi)\big)\big] \wedge \big[\neg h \to \Diamond_1\big(h \wedge (\psi \vee \Diamond_1 \psi)\big)\big],\\
\bar{\Diamond}_2 \psi &= \big[v \to \Diamond_2\big(\neg v \wedge (\psi \vee \Diamond_2 \psi)\big)\big] \wedge \big[\neg v \to \Diamond_2\big(v \wedge (\psi \vee \Diamond_2 \psi)\big)\big],\\
\bar{\Box}_1 \psi &= \neg \bar{\Diamond}_1 \neg \psi, \quad \text{and} \quad \bar{\Box}_2 \psi = \neg \bar{\Diamond}_2 \neg \psi.
\end{aligned}$$

(Similar operators were used by Spaan [77] and by Reynolds and Zakharyaschev [70].)

Now define φ_∞ to be the conjunction of the following formulas:

$$\Box_1 \Box_2 \big((h \vee \Diamond_2 h \to \Box_2 h) \wedge (\neg h \vee \Diamond_2 \neg h \to \Box_2 \neg h)\big), \tag{9}$$
$$\Box_1 \Box_2 \big((v \vee \Diamond_1 v \to \Box_1 v) \wedge (\neg v \vee \Diamond_1 \neg v \to \Box_1 \neg v)\big), \tag{10}$$
$$\bar{\Diamond}_2 \bar{\Diamond}_1 (\bar{\Box}_2 \bot \wedge \bar{\Box}_1 \bot), \tag{11}$$
$$\bar{\Box}_1 \bar{\Box}_2 (\bar{\Box}_2 \bot \wedge \bar{\Box}_1 \bot \to d), \tag{12}$$
$$\bar{\Box}_2 \bar{\Diamond}_1 (\neg d \wedge \bar{\Box}_1 d), \tag{13}$$
$$\bar{\Box}_1 \bar{\Diamond}_2 (d \wedge \bar{\Box}_2 \neg d), \tag{14}$$
$$\bar{\Box}_1 \bar{\Box}_2 (d \to \bar{\Box}_1 \bar{\Diamond}_2 d), \tag{15}$$
$$\bar{\Box}_1 \bar{\Box}_2 (\neg d \to \bar{\Box}_2 \bar{\Diamond}_1 \neg d). \tag{16}$$

On the one hand, it is easy to see that φ is satisfiable in a product of two rooted Noetherian linear orders each of which contains an infinite descending chain of distinct points, see Fig. 4. Note that such a frame is *infinite.*

On the other hand, we show that φ_∞ cannot be satisfied in a *finite* frame for $[\mathbf{K4}, \mathbf{K4}]$. The idea behind the proof is that, though the points 'generated by' φ_∞ do not necessarily form a nice 'backward looking $\omega \times \omega$-grid' like on Fig. 4, yet each of them can be 'characterised' by a unique pair $\langle n, m \rangle$ of natural numbers.

To this end, suppose that φ_∞ is satisfied at the root r of a model $\mathfrak{M}$ based on a (not necessarily product) frame $\mathfrak{F} = \langle W, R_1, R_2 \rangle$ for $[\mathbf{K4}, \mathbf{K4}]$. Then both R_1 and R_2 are transitive, they commute and satisfy the Church–Rosser property.

We define new ($\mathfrak{M}$-dependent) binary relations $\bar{R}_1$ and $\bar{R}_2$ on W by taking, for all $x, y \in W$,

$$\begin{aligned}
x\bar{R}_1 y \quad &\text{iff} \quad \exists z \in W \ \big[xR_1 z \text{ and } \big((\mathfrak{M}, x) \models h \iff (\mathfrak{M}, z) \models \neg h\big)\\
&\qquad \text{and (either } z = y \text{ or } zR_1 y)\big],\\
x\bar{R}_2 y \quad &\text{iff} \quad \exists z \in W \ \big[xR_2 z \text{ and } \big((\mathfrak{M}, x) \models v \iff (\mathfrak{M}, z) \models \neg v\big)\\
&\qquad \text{and (either } z = y \text{ or } zR_2 y)\big].
\end{aligned}$$

In other words, $x\bar{R}_1 y$ iff $xR_1 y$ and either x, y are of different 'horizontal colours' in the sense that h is true in precisely one of them, or x, y are of the same h-colour (i.e., $x \models h$ iff $y \models h$), but there is a point z of different h-colour such that $xR_1 zR_1 y$. Clearly, we always have $\bar{R}_i \subseteq R_i$ ($i = 1, 2$). It is not hard to see that, by (9)–(10), $\big\langle W, \bar{R}_1, \bar{R}_2 \big\rangle$ is a

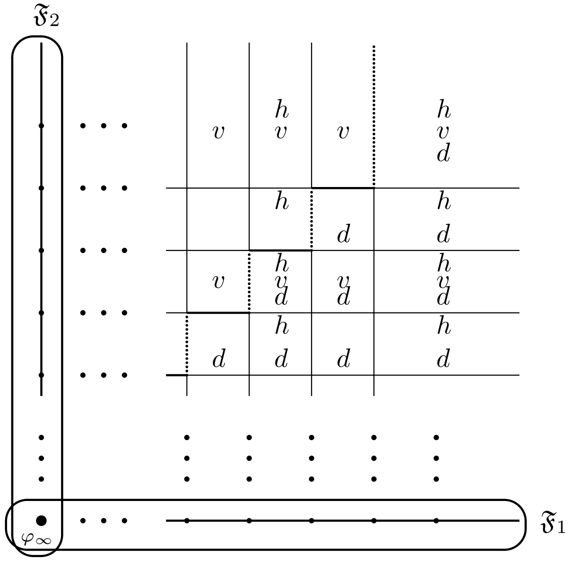


Figure 4. Satisfying φ_∞ in an infinite product frame.

(not necessarily rooted) frame for $[\mathbf{K4}, \mathbf{K4}]$, that is,

$$\begin{aligned}
&\text{both } \bar{R}_1 \text{ and } \bar{R}_2 \text{ are transitive,} && \text{(tran)}\\
&\bar{R}_1 \text{ and } \bar{R}_2 \text{ commute, and} && \text{(com)}\\
&\bar{R}_1 \text{ and } \bar{R}_2 \text{ are Church–Rosser.} && \text{(chro)}
\end{aligned}$$

Moreover, for all $x \in W$,

$$\begin{array}{lcl}
(\mathfrak{M}, x) \models \bar{\Diamond}_1 \psi & \text{iff} & \exists y \in W \ (x\bar{R}_1 y \text{ and } (\mathfrak{M}, y) \models \psi),\\
(\mathfrak{M}, x) \models \bar{\Diamond}_2 \psi & \text{iff} & \exists y \in W \ (x\bar{R}_2 y \text{ and } (\mathfrak{M}, y) \models \psi).
\end{array}$$

We define inductively four infinite sequences

$$x_0, x_1, x_2, \dots, \quad y_0, y_1, y_2, \dots, \quad u_0, u_1, u_2, \dots \quad \text{and} \quad v_0, v_1, v_2, \dots$$

of points from W such that, for every $i < \omega$,

(gen1) $(\mathfrak{M}, x_i) \models d \wedge \bar{\Box}_2 \neg d$,

(gen2) $(\mathfrak{M}, y_i) \models \neg d \wedge \bar{\Box}_1 d$,

(gen3) $r\bar{R}_2 u_i$, $u_i \bar{R}_1 x_i$ and $u_i \bar{R}_1 y_i$,

(gen4) if $i > 0$ then $r\bar{R}_1 v_i$, $v_i \bar{R}_2 x_i$ and $v_i \bar{R}_2 y_{i-1}$,

see Fig. 5. (We do not claim at this point that, say, all the x_i are distinct.)

To begin with, by (11), there are u_0, x_0 such that $r\bar{R}_2 u_0 \bar{R}_1 x_0$ and

$$(\mathfrak{M}, x_0) \models \bar{\Box}_1 \bot \wedge \bar{\Box}_2 \bot. \tag{17}$$

By (12), $(\mathfrak{M}, x_0) \models d$. By (13), there is y_0 such that $u_0 \bar{R}_1 y_0$ and $(\mathfrak{M}, y_0) \models \neg d \wedge \bar{\Box}_1 d$. So **(gen1)**–**(gen3)** hold for $i = 0$.

Now suppose that, for some $n < \omega$, x_i and y_i with **(gen1)**–**(gen4)** have already been defined for all $i \leq n$. By **(gen3)** for $i = n$ and by (com), there is v_{n+1} such that $r \bar{R}_1 v_{n+1} \bar{R}_2 y_n$. So by (14), there is x_{n+1} such that $(\mathfrak{M}, x_{n+1}) \models d \wedge \bar{\Box}_2 \neg d$ and $v_{n+1} \bar{R}_2 x_{n+1}$. Now again by (com), there is u_{n+1} such that $r \bar{R}_2 u_{n+1} \bar{R}_1 x_{n+1}$. So, by (13), there is y_{n+1} such that $u_{n+1} \bar{R}_1 y_{n+1}$ and $(\mathfrak{M}, y_{n+1}) \models \neg d \wedge \bar{\Box}_1 d$, as required (see Fig. 5).

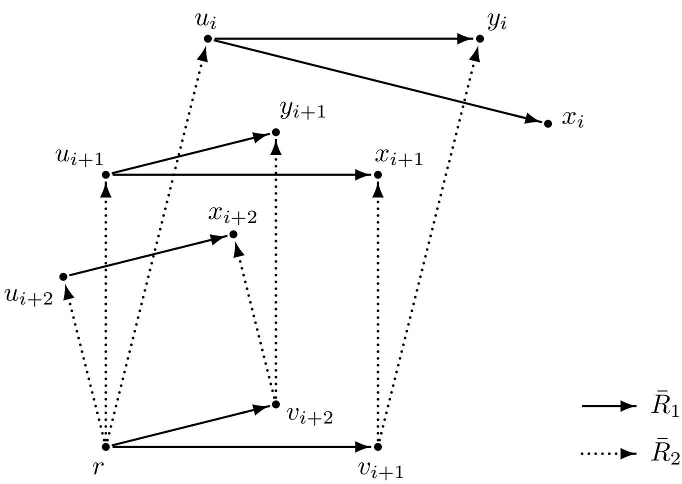


Figure 5. Generating the points x_i, y_i, u_i and v_i.

The following lemma is our basic tool in showing that all the x_n are different:

LEMMA 35. *For all* $i, n < \omega$,

(i) $(\mathfrak{M}, x_i) \models \bar{\Diamond}_1^n \top \leftrightarrow \bar{\Diamond}_2^n \top$,

(ii) $(\mathfrak{M}, y_i) \models \bar{\Diamond}_1^{n+1} \top \leftrightarrow \bar{\Diamond}_2^n \top$.

Proof. First, it is a straightforward consequence of (12), (16) and (com) that

$$\bar{\Box}_1 \bar{\Box}_2 (\neg d \rightarrow \bar{\Diamond}_1 \top) \tag{18}$$

holds in $\mathfrak{M}$. Further, it is not hard to show by induction on n that for all $n < \omega$,

$$\bar{\Box}_1 \bar{\Box}_2 (d \rightarrow \bar{\Box}_1^n \bar{\Diamond}_2^n d), \tag{19}$$

$$\bar{\Box}_1 \bar{\Box}_2 (\neg d \rightarrow \bar{\Box}_2^n \bar{\Diamond}_1^n \neg d). \tag{20}$$

are also true in $\mathfrak{M}$. Now to prove (i), suppose first that we have $(\mathfrak{M}, x_i) \models \bar{\Diamond}_1^n \top$. Then there is a point z such that $x_i \bar{R}_1^n z$. By **(gen1)**, $(\mathfrak{M}, x_i) \models d$. So, $(\mathfrak{M}, z) \models \bar{\Diamond}_2^n d$, by (19). Using (com), we find a point v such that $x_i \bar{R}_2^n v$ and $v \bar{R}_1^n u$, so $(\mathfrak{M}, x_i) \models \bar{\Diamond}_2^n \top$ follows. Conversely, suppose $(\mathfrak{M}, x_i) \models \bar{\Diamond}_2^n \top$, that is, there are points $z_1, \ldots, z_n$ such that $x_i\bar{R}_2z_1\bar{R}_2\ldots\bar{R}_2z_n$. By **(gen1)**, $(\mathfrak{M},x_i)\models\bar{\Box}_2\neg d$, and so $(\mathfrak{M},z_1)\models\neg d$. Therefore, by (20) and (18), we have $(\mathfrak{M},z_n)\models\bar{\Diamond}_1^n\top$, and then obtain $(\mathfrak{M},x_i)\models\bar{\Diamond}_1^n\top$ using (com).

To show (ii), assume first that we have $(\mathfrak{M},y_i)\models\bar{\Diamond}_2^n\top$. Then there is a point z such that $y_i\bar{R}_2^n z$. By **(gen2)**, $(\mathfrak{M},y_i)\models\neg d$. So, by (20), $(\mathfrak{M},z)\models\bar{\Diamond}_1^n\neg d$, and by (18), $(\mathfrak{M},z)\models\bar{\Diamond}_1^{n+1}\top$. Now $(\mathfrak{M},y_i)\models\bar{\Diamond}_1^{n+1}\top$ follows by (com). Conversely, suppose $(\mathfrak{M},y_i)\models\bar{\Diamond}_1^{n+1}\top$, that is, there are points $z_1,\ldots,z_n,z_{n+1}$ such that

$$y_i\bar{R}_1z_1\bar{R}_1\ldots\bar{R}_1z_n\bar{R}_1z_{n+1}.$$

By **(gen2)**, $(\mathfrak{M},y_i)\models\bar{\Box}_1 d$, and so $(\mathfrak{M},z_1)\models d$. Therefore, by (19), $(\mathfrak{M},z_{n+1})\models\bar{\Diamond}_2^n\top$. Finally, using (com) we obtain $(\mathfrak{M},y_i)\models\bar{\Diamond}_2^n\top$. ❑

Now we can show that all the x_n are distinct as follows. For every formula ψ and $\Diamond\in\{\bar{\Diamond}_1,\bar{\Diamond}_2\}$, we introduce

$$\Diamond^{=\mathbf{n}}\psi \;=\; \Diamond^n\psi\wedge\Box^{n+1}\neg\psi,$$

meaning 'see ψ in n steps but not in $n+1$ steps.' Define the horizontal and vertical ranks $hr(x)$ and $vr(x)$ of a point x (in model $\mathfrak{M}$) by taking

$$hr(x) \;=\; \begin{cases} n, & \text{if } n<\omega \text{ and } (\mathfrak{M},x)\models\bar{\Diamond}_1^{=\mathbf{n}}\top,\\ \infty, & \text{otherwise,}\end{cases}$$

$$vr(x) \;=\; \begin{cases} n, & \text{if } n<\omega \text{ and } (\mathfrak{M},x)\models\bar{\Diamond}_2^{=\mathbf{n}}\top,\\ \infty, & \text{otherwise.}\end{cases}$$

The reader can readily check, using (com) and (chro), that if $x\bar{R}_1y$ then $vr(x)=vr(y)$, and if $x\bar{R}_2y$ then $hr(x)=hr(y)$.

We claim that, for all $n<\omega$,

$$vr(u_n) \;=\; n, \tag{21}$$

$$hr(v_n) \;=\; n, \tag{22}$$

$$hr(x_n) \;=\; vr(x_n)=n. \tag{23}$$

First we prove (21) by induction on n. For $n=0$, it follows from the definition of x_0 (see (17)) and **(gen3)**. Suppose that (21) holds for some $n<\omega$. Then

$$\begin{aligned} vr(u_{n+1}) &\overset{\textbf{(gen3)}}{=} vr(x_{n+1}) \overset{\text{L.35(i)}}{=} hr(x_{n+1})\\ &\overset{\textbf{(gen4)}}{=} hr(y_n) \overset{\text{L.35(ii)}}{=} vr(y_n)+1 \overset{\textbf{(gen3)}}{=} vr(u_n)+1 \overset{\text{(IH)}}{=} n+1. \end{aligned}$$

Now (22) and (23) follow from (21) and

$$hr(v_n) \overset{\textbf{(gen4)}}{=} hr(x_n) \overset{\text{L.35(i)}}{=} vr(x_n) \overset{\textbf{(gen3)}}{=} vr(u_n).$$

Undecidability.

We discuss first how the 'diagonal points' x_n (with finite rank $hr(x_n)=vr(x_n)=n$) can be used not only to show the lack of fmp, but also to encode arbitrarily large finite parts of the '$\omega\times\omega$-grid' in frames for $[\mathbf{K4},\mathbf{K4}]$. The enumeration of the points of $\omega\times\omega$

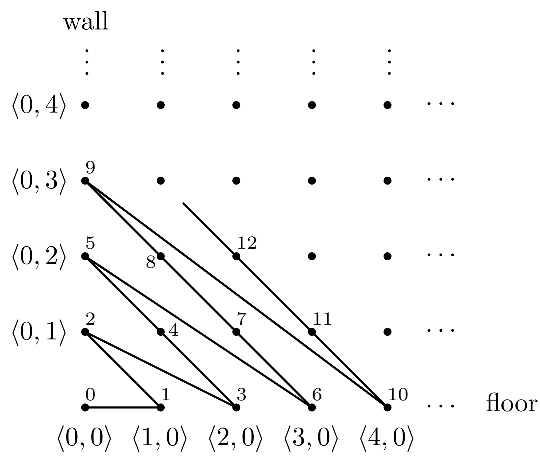


Figure 6. The enumeration *pair*.

we use below has been introduced in several papers dealing with undecidable multimodal logics; see, e.g., [36, 60, 70]. (Note that in all these cases either the language had *next-time operators* or all the frames were *linear*, neither is the case now.)

Let $pair : \omega \to \omega \times \omega$ be the function defined recursively by taking:

- $pair(0) = \langle 0, 0\rangle$,
- if $pair(n) = \langle 0, j\rangle$ then $pair(n+1) = \langle j+1, 0\rangle$,
- otherwise, if $pair(n) = \langle i+1, j\rangle$ then $pair(n+1) = \langle i, j+1\rangle$;

see Fig. 6. It is easy to see that *pair* is one-one and onto. Let $\sharp : \omega \times \omega \to \omega$ denote the inverse of the function *pair*. If $pair(n)$ is not on the wall (that is, the first coordinate of $pair(n)$ is different from 0) then define $left_n$ to be the $\sharp$ of the left neighbour of $pair(n)$. The reader can readily check the following important properties of these functions, for all $n > 0$:

(t1) If neither $pair(n)$ nor $pair(n-1)$ are on the wall then $left_n = left_{n-1} + 1$.

(t2) If $n > 1$ and $pair(n)$ is not on the wall, but $pair(n-1)$ is on the wall, then $n > 2$, $pair(n-2)$ is not on the wall, and $left_n = left_{n-2} + 1$.

(t3) $pair(n)$ is on the wall iff $pair(left_{n-1})$ is on the wall.

(t4) Either $pair(n)$ or $pair(n-1)$ is not on the wall.

We will require the following propositional variables:

- grid (marking the points of the grid),
- left (a pointer from n to $left_n$ when $pair(n)$ is not on the wall),
- wall (marking the wall, i.e., the pairs of the form $\langle 0, n\rangle$).

Let φ_{grid} be the conjunction of (9), (10) and the following formulas:

$$\bar{\Box}_1\,\bar{\Box}_2\,\big(\bar{\Box}_1\,\bot \to (\mathsf{grid} \leftrightarrow \bar{\Box}_2\,\bot)\big),$$
$$\bar{\Box}_1\,\bar{\Box}_2\,(\bar{\Box}_1\,\bot \wedge \mathsf{grid} \to \mathsf{wall}),$$
$$\bar{\Box}_1\,\bar{\Box}_2\,(\mathsf{wall} \to \mathsf{grid}),$$
$$\bar{\Box}_1\,\bar{\Box}_2\,\big(\bar{\Diamond}_1\,\mathsf{wall} \to \bar{\Box}_1\,(\mathsf{grid} \to \mathsf{wall})\big),$$
$$\bar{\Box}_1\,\bar{\Box}_2\,\big(\bar{\Diamond}_1\,\top \to (\mathsf{grid} \leftrightarrow \bar{\Diamond}_2^{=\mathbf{1}}\,\bar{\Diamond}_1^{=\mathbf{1}}\,\mathsf{grid})\big),$$
$$\bar{\Box}_1\,\bar{\Box}_2\,\big(\mathsf{grid} \wedge \bar{\Diamond}_1\,\top \to \big(\mathsf{wall} \leftrightarrow \bar{\Diamond}_2\,(\bar{\Diamond}_1^{=\mathbf{1}}\,\mathsf{left} \wedge \bar{\Diamond}_1\,\mathsf{wall})\big)\big),$$
$$\bar{\Box}_1\,\bar{\Box}_2\,\Big[\mathsf{left} \leftrightarrow \Big((\bar{\Diamond}_1^{=\mathbf{1}}\,\top \wedge \bar{\Box}_2\,\bot) \vee \big(\bar{\Diamond}_2\,(\bar{\Diamond}_1^{=\mathbf{2}}\,\mathsf{left} \wedge \bar{\Diamond}_1\,\mathsf{wall}) \wedge \bar{\Diamond}_2^{=\mathbf{1}}\,\bar{\Diamond}_1^{=\mathbf{2}}\,\mathsf{left}\big)$$
$$\vee\ \big(\bar{\Diamond}_2\,(\bar{\Diamond}_1^{=\mathbf{1}}\,\mathsf{left} \wedge \neg\bar{\Diamond}_1\,\mathsf{wall}) \wedge \bar{\Diamond}_2^{=\mathbf{1}}\,\bar{\Diamond}_1^{=\mathbf{1}}\,\mathsf{left}\big)\Big)\Big].$$

The following lemma, showing that φ_{grid} 'forces' the $\omega \times \omega$-grid onto 'diagonal points of finite rank', is proved in [26]:

LEMMA 36. *Suppose that $\mathfrak{M}$ is a model based on a rooted frame $\mathfrak{F} = \langle W, R_1, R_2\rangle$ for* $[\mathbf{K4}, \mathbf{K4}]$. *If $(\mathfrak{M}, r) \models \varphi_{grid}$ then the following hold, for all $n, m < \omega$ and all $x \in W$ such that $hr(x) = n$ and $vr(x) = m$:*

(i) $(\mathfrak{M}, x) \models \mathsf{grid}$ *iff* $n = m$,

(ii) $(\mathfrak{M}, x) \models \bar{\Diamond}_1^{=\mathbf{1}}\,\mathsf{left}$ *iff* $n > 0$, *$pair(n-1)$ is not on the wall and $m = left_{n-1}$,*

(iii) $(\mathfrak{M}, x) \models \mathsf{wall}$ *iff* $n = m$ *and $pair(n)$ is on the wall,*

(iv) $(\mathfrak{M}, x) \models \mathsf{left}$ *iff* *$pair(n)$ is not on the wall and $m = left_n$.*

Various undecidable problems can be 'represented' on the $\omega \times \omega$-grid, say, versions of the halting problems for Turing machines, register machines, etc., Post's correspondence problem, as well as infinite tiling problems.

Here we show as an example for reducing an undecidable tiling problem to the satisfiability problem for logics that (i) contain $[\mathbf{K4}, \mathbf{K4}]$ and (ii) have among their frames a product of two rooted Noetherian linear orders each of which contains an infinite descending chain of distinct points. (For other similar logics slight modifications of the proof might be necessary, see [26] for a general argument.)

A *tile type* is a 4-tuple of colours

$$t = \langle left(t), right(t), up(t), down(t)\rangle\,.$$

For a *finite* set T of tile types and a subset $X \subseteq \omega \times \omega$, we say that T *tiles* X if there exists a function (called a *tiling*) $\boldsymbol{\tau}$ from X to T such that, for all $\langle i, j\rangle \in X$,

- if $\langle i, j+1\rangle \in X$ then $up(\boldsymbol{\tau}(i,j)) = down(\boldsymbol{\tau}(i, j+1))$ and
- if $\langle i+1, j\rangle \in X$ then $right(\boldsymbol{\tau}(i,j)) = left(\boldsymbol{\tau}(i+1, j))$.

The following '*$\omega \times \omega$-tiling problem*' is undecidable (see [81, 9]): given a finite set T of tile types, decide whether T can tile $\omega \times \omega$.

Given a finite set T of tile types, we introduce a propositional variable t, for every $t \in T$. Let φ_T be the conjunction of the following formulas:

$$\bar{\Box}_1 \bar{\Box}_2 \,(\mathsf{grid} \leftrightarrow \bigvee_{t \in T} t),$$

$$\bar{\Box}_1 \bar{\Box}_2 \bigwedge_{t \neq t' \in T} \neg(t \wedge t'),$$

$$\bar{\Box}_1 \bar{\Box}_2 \bigwedge_{\substack{t,t' \in T \\ up(t') \neq down(t)}} \big(t \to \bar{\Box}_2 \,(\bar{\Diamond}_1^{=1} \,\mathsf{left} \to \neg \bar{\Diamond}_1 t')\big),$$

$$\bar{\Box}_1 \bar{\Box}_2 \bigwedge_{\substack{t,t' \in T \\ right(t') \neq left(t)}} \big(t \to \bar{\Box}_2 \,(\mathsf{left} \to \neg \bar{\Diamond}_1 t')\big).$$

For every $n < \omega$, let

$$plane_n = \{\langle i, j\rangle \mid \sharp(i,j) \leq n\}.$$

If formulas (9) and (10) are satisfied in a model $\mathfrak{M}$ based on a frame for $[\mathbf{K4}, \mathbf{K4}]$, then for all numbers $a, b < \omega$ and $x \in W$ with $hr(x) = a$ and $vr(x) = b$, there exists what we call a *perfect* $a \times b$*-rectangle starting at* x, that is, there are points $x_{i,j}$ (for $i \leq a$, $j \leq b$) such that

- $x = x_{a,b}$,
- $hr(x_{i,j}) = i$ and $vr(x_{i,j}) = j$,
- $x_{i,j} \bar{R}_1 x_{k,j}$ for $i > k$, and $x_{i,j} \bar{R}_2 x_{i,k}$ for $j > k$.

(Indeed, given x, take an a-long $\bar{R}_1$-path and a b-long $\bar{R}_2$-path starting from x, and then 'close them' under the Church-Rosser property.)

Now a straightforward induction on n shows the following:

LEMMA 37. *Let* $\mathfrak{M}$ *be a model that is based on a frame for* $[\mathbf{K4}, \mathbf{K4}]$ *with root* r *and suppose that* $(\mathfrak{M}, r) \models \varphi_{grid} \wedge \varphi_T$*. Then, for every* $n < \omega$*, every* $x \in W$ *such that* $hr(x) = vr(x) = n$*, and every perfect* $n \times n$*-rectangle* $x_{i,j}$ $(i \leq n,\ j \leq n)$ *starting at* x*, the function* $\tau : plane_n \to T$ *defined by*

$$\tau(i,j) = t \qquad \textit{iff} \qquad (\mathfrak{M}, x_{\sharp(i,j),\sharp(i,j)}) \models t$$

is a tiling of $plane_n$.

Now, using Lemma 37, it is straightforward to show that if $\varphi_\infty \wedge \varphi_{grid} \wedge \varphi_T$ is satisfiable in a frame for $[\mathbf{K4}, \mathbf{K4}]$ then T tiles $plane_n$, for all $n < \omega$. A standard compactness argument (or König's lemma) shows that if a given finite set T of tile types tiles $plane_n$ for every $n < \omega$, then it actually tiles the whole $\omega \times \omega$-grid. On the other hand, it is easy to see that if T tiles $\omega \times \omega$ then $\varphi_\infty \wedge \varphi_{grid} \wedge \varphi_T$ is satisfiable in a product of two rooted Noetherian linear orders each of which contains an infinite descending chain of distinct points.

With the help of some additional 'machinery', one can even reduce 'stronger' undecidable statements (like recurrent Turing machine and tiling problems) to the satisfiability problem for certain products of 'transitive' logics. For instance, the following is shown in [26]:

THEOREM 38. *Let L_1 be any logic from the list*

$$\mathbf{K4},\ \mathbf{K4.1},\ \mathbf{K4.2},\ \mathbf{K4.3},\ \mathbf{S4},\ \mathbf{S4.1},\ \mathbf{S4.2},\ \mathbf{S4.3},\\ \mathbf{GL},\ \mathbf{GL.3},\ \mathbf{Grz},\ \mathbf{Grz.3},\ \mathsf{Log}\{\langle\omega,<\rangle\},\ \mathsf{Log}\{\langle\omega,\leq\rangle\},$$

and L_2 be any of

$$\mathsf{Log}\{\langle\omega,<\rangle\},\ \mathsf{Log}\{\langle\omega,\leq\rangle\},\ \mathbf{GL.3},\ \mathbf{Grz.3}.$$

Then any Kripke complete bimodal logic L in the interval

$$[L_1, L_2]\ \subseteq\ L\ \subseteq\ L_1 \times L_2$$

is Π^1_1-hard.

We also obtain the following interesting corollary. As the commutator of two recursively axiomatisable logics is recursively axiomatisable by definition, Theorem 38 yields a number of *Kripke incomplete* commutators of Kripke complete and finitely axiomatisable logics:

COROLLARY 39. *Let L_1 and L_2 be like in Theorem 38. Then the commutator $[L_1, L_2]$ is Kripke incomplete.*

Higher dimensional decidable and undecidable products

Products of more than two modal logics are often undecidable and lack the fmp. Let us first discuss some exceptions.

It is not hard to see that any product $L_1 \times \cdots \times L_n$ of **Alt** and **K** logics has the *finite depth property*, that is, it is determined by some class of frames of finite depth. Indeed, suppose $\varphi \notin L_1 \times \cdots \times L_n$ for some $\mathcal{ML}_n$-formula φ. Then there are rooted frames $\mathfrak{F}_i$, $i = 1, \ldots, n$, such that $\mathfrak{F}_i \models L_i$ and φ is refuted at the root of $\mathfrak{F}_1 \times \cdots \times \mathfrak{F}_n$. By a standard unravelling argument, for each $i = 1, \ldots, n$, there is an intransitive tree $\mathfrak{T}_i$ and a p-morphism h_i from $\mathfrak{T}_i$ onto $\mathfrak{F}_i$. So we always have $\mathfrak{T}_i \models L_i$. Note that if $L_i = \mathbf{Alt}$ then the unravelling $\mathfrak{T}_i$ of $\mathfrak{F}_i$ is just a chain of irreflexive points. It is straightforward to check that the function h defined by

$$h(x_1, \ldots, x_n) = \langle h_1(x_1), \ldots, h_n(x_n)\rangle$$

is a p-morphism from $\mathfrak{T}_1 \times \cdots \times \mathfrak{T}_n$ onto $\mathfrak{F}_1 \times \cdots \times \mathfrak{F}_n$ (cf. Proposition 9). Now we prune all the trees $\mathfrak{T}_i$ down to the modal depth $md(\varphi)$ of φ. Clearly, the resulting product frame $\mathfrak{T}_1^- \times \cdots \times \mathfrak{T}_n^-$ is of depth $n \cdot md(\varphi)$ and it is a frame for $L_1 \times \cdots \times L_n$. A straightforward induction on the stucture of φ shows that it refutes φ at its root.

As one can prove by a standard filtration argument that if an n-modal logic L has the finite depth property, then it has the fmp as well, we obtain the following theorem of Gabbay and Shehtman [24]:

THEOREM 40. *Any product of* **Alt** *and* **K** *logics has the fmp. In particular,* $\mathbf{K}^n$ *and* $\mathbf{Alt}^n$ *have the fmp, for any natural number $n \geq 2$.*

As a consequence of this theorem and Theorem 26 we have:

THEOREM 41. $\mathbf{Alt}^n$ *is decidable, for any natural number* $n \geq 2$.

In fact, it can be shown that $\mathbf{Alt}^n$ has the polynomial *product* fmp and it is coNP-complete.

On the other hand, the logic $\mathbf{K}^n$ is not so simple. Though it has the fmp, one cannot use it for a decision algorithm, as $\mathbf{K}^n$ is not only not finitely axiomatisable, but it is undecidable whether a finite n-frame is a frame for it (cf. Theorem 25). In fact, the following general result is shown in [42]:

THEOREM 42. *Let* $n \geq 3$ *and let* L *be any* n*-modal logic such that* $\mathbf{K}^n \subseteq L \subseteq \mathbf{S5}^n$. *Then* L *is undecidable and lacks the product fmp.*

The proof of this theorem (and that of Theorem 25) uses a reduction of a deep result of Hirsch and Hodkinson [40] saying that representability is undecidable for finite relation algebras.

Note that, unlike $\mathbf{K}^n$, logics like $\mathbf{K4}^n$ and $\mathbf{S5}^n$ do not even have the fmp (for $\mathbf{K4}^n$ this follows from Theorem 34, and for $\mathbf{S5}^n$ this was shown in [53]). The undecidability of $\mathbf{S5}^n$ was first shown by Maddux [58] in the algebraic framework of diagonal-free cylindric algebras. He used a reduction of the word problem of semigroups to prove the following general result:

THEOREM 43. *Any* n*-modal logic* L *in the interval*

$$[\mathbf{S5}, \mathbf{S5}, \ldots, \mathbf{S5}] \subseteq L \subseteq \mathbf{S5}^n$$

is undecidable whenever $n \geq 3$.

Another proof via the connection with first-order logic (see Section 3.2) that uses a reduction of the $\omega \times \omega$ tiling problem can be found in [23] (see also [47] for possible generalisations).

4 BETWEEN FUSIONS AND PRODUCTS

A natural idea for reducing the strong interaction between modal operators of product logics is to consider logics determined by (not necessarily generated) *subframes* of product frames. Worlds are still tuples, the relations still act coordinate-wise, but not all tuples of the Cartesian product are present, and so the commutativity and Church–Rosser properties do not necessarily hold. This kind of restriction on the 'domains' of modal operators is similar to 'relativisations' of the quantifiers in first-order logic and algebraic logic, where it indeed results in improving the bad algorithmic behaviour, cf. [63, 61].

This idea gives rise to the following combinations of logics. First, we choose a class of 'desirable' subframes of product frames. This can be any class: the class of all such subframes, the so-called 'locally cubic' frames, frames that 'expand' along one of the coordinates (see below for precise definitions), a class of frames satisfying some (modal or first-order) formulas, etc. Having chosen such a class $\mathcal{K}$, we then take the logic determined by those subframes of the appropriate product frames that belong to $\mathcal{K}$. Thus, each choice of $\mathcal{K}$ defines a new combination operator on logics:

DEFINITION 44. Let $n > 1$ be a natural number and $\mathcal{K}$ a class of subframes of n-ary product frames. Given Kripke complete (uni)modal logics $L_1, \ldots, L_n$ formulated in the language having $\Box_i$ ($i = 1, \ldots, n$), the *$\mathcal{K}$-relativised product* $(L_1 \times \cdots \times L_n)^{\mathcal{K}}$ of $L_1, \ldots, L_n$

is defined by taking

$$(L_1 \times \cdots \times L_n)^{\mathcal{K}} = \mathsf{Log}\{\mathfrak{G} \in \mathcal{K} \mid \mathfrak{G} \subseteq \mathfrak{F}_1 \times \cdots \times \mathfrak{F}_n \text{ for some } \mathfrak{F}_i \in \mathsf{Fr}L_i,\ i = 1, \ldots, n\}.$$

Observe that if we choose $\mathcal{K}$ to be the class of all product frames $\mathfrak{F}_1 \times \cdots \times \mathfrak{F}_n$ such that $\mathfrak{F}_i \in \mathsf{Fr}L_i$, then the $\mathcal{K}$-relativised product of the L_i is just their usual product.

We discuss here in detail two kinds of 'relativised product' operators: arbitrary and expanding relativisations.

Arbitrary relativisations.

We begin by considering the combination operator determined by the class SF_n of *all* subframes of n-ary product frames. SF_n-relativised products of logics will be called *arbitrarily relativised products*. Since SF_n contains frames that do not satisfy commutativity and/or Church–Rosser properties,

$$(L_1 \times \cdots \times L_n)^{\mathsf{SF}_n} \ \subsetneq\ L_1 \times \cdots \times L_n.$$

On the other hand, unlike product logics, arbitrarily relativised products do not necessarily contain the fusion of their components. For example, consider the minimal deontic logic $\mathbf{D}$, which is known to be determined by the class of *serial* frames. The formula $\Diamond_2\top$ clearly belongs to $\mathbf{K} \otimes \mathbf{D}$, but is refuted in any *finite* subframe of, say, $\langle \omega, < \rangle \times \langle \omega, < \rangle$, and so $\Diamond_2\top \notin (\mathbf{K} \times \mathbf{D})^{\mathsf{SF}_2}$.

However, for a large class of natural logics, arbitrarily relativised products do contain the fusions. A Kripke complete modal logic L is called a *subframe logic* if the class of Kripke frames for L is closed under taking (not necessarily generated) subframes (see Chapter 7 of this handbook). Typical examples of subframe logics are modal logics whose classes of Kripke frames are definable by universal first-order formulas, such as $\mathbf{K}$, $\mathbf{Alt}$, $\mathbf{T}$, $\mathbf{K4}$, $\mathbf{S4}$, $\mathbf{S5}$, $\mathbf{K5}$, $\mathbf{K45}$, $\mathbf{S4.3}$, and $\mathbf{K4.3}$. Note, however, that subframe logics like $\mathbf{GL}$, $\mathbf{GL.3}$, $\mathbf{Grz}$ are not first-order definable. It is not hard to see the following:

PROPOSITION 45. *If $L_1, \ldots, L_n$ are subframe logics, then*

$$L_1 \otimes \cdots \otimes L_n \ \subseteq\ (L_1 \times \cdots \times L_n)^{\mathsf{SF}_n}.$$

As the following result of [54] shows, for many standard subframe logics the converse inclusion holds as well. Thus in several cases 'arbitrary relativisation' can be regarded as a 'many-dimensional' semantical characterisation of fusions of these logics.

THEOREM 46. *Let $L_i \in \{\mathbf{K}, \mathbf{T}, \mathbf{K4}, \mathbf{S4}, \mathbf{S5}, \mathbf{S4.3}\}$, for $i = 1, \ldots, n$. Then*

$$(L_1 \times \cdots \times L_n)^{\mathsf{SF}_n} = L_1 \otimes \cdots \otimes L_n.$$

The proof is based on the following lemma that can be proved by constructing the necessary p-morphism in a step-by-step manner, see [54]:

LEMMA 47. *Suppose that $L_i \in \{\mathbf{K}, \mathbf{T}, \mathbf{K4}, \mathbf{S4}, \mathbf{S5}, \mathbf{S4.3}\}$, $i = 1, \ldots, n$, and let $\mathfrak{G} = \langle W, S_1, \ldots, S_n \rangle$ be a countable rooted n-frame such that $\langle W, S_i \rangle \models L_i$ for all $i = 1, \ldots, n$. Then $\mathfrak{G}$ is a p-morphic image of a subframe of some product frame for $L_1 \times \cdots \times L_n$.*

It is not clear how far Theorem 46 can be generalised. It definitely does not hold for all subframe logics, not even for those of them that are characterised by universally definable classes of frames. Take, for instance, the logic **K5** that is characterised by the class of Euclidean frames, i.e., frames $\langle W, R\rangle$ satisfying the universal (Horn) sentence

$$\forall x \forall y \forall u \left(R(u,x) \wedge R(u,y) \to R(x,y)\right).$$

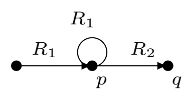
In particular, frames for **K5** have the property

$$\forall x \forall u \left(R(u,x) \to R(x,x)\right).$$

Now consider the formula

$$\varphi \;=\; \Diamond_1\big(p \wedge \Diamond_2(q \wedge \neg p)\big) \wedge \Box_1\Box_2(q \to \neg\Diamond_1 q).$$

It is clearly satisfiable in the following frame for $\mathbf{K5} \otimes \mathbf{K}$:



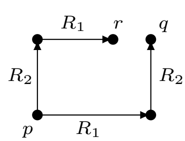
On the other hand, it is not hard to see that φ is not satisfiable in any subframe of a product frame for $\mathbf{K5} \times \mathbf{K}$. Therefore,

$$\mathbf{K5} \otimes \mathbf{K} \;\subsetneq\; (\mathbf{K5} \times \mathbf{K})^{\mathsf{SF}_2} \;\subsetneq\; \mathbf{K5} \times \mathbf{K}.$$

Other kinds of logics for which Theorem 46 does not hold are those having frames with a finite bound on their branching like **Alt**. The formula

$$\psi \;=\; p \wedge \Diamond_1\big(\neg p \wedge \Diamond_2 q\big) \wedge \Diamond_2\big(\neg p \wedge \Diamond_1 r\big) \wedge \Box_1\Box_2(q \to \neg r)$$

is clearly satisfiable in the $\mathbf{Alt} \otimes \mathbf{Alt}$-frame



On the other hand, it should be clear that ψ is not satisfiable in any subframe of a frame for $\mathbf{Alt} \times \mathbf{Alt}$. Thus,

$$\mathbf{Alt} \otimes \mathbf{Alt} \;\subsetneq\; (\mathbf{Alt} \times \mathbf{Alt})^{\mathsf{SF}_2} \;\subsetneq\; \mathbf{Alt} \times \mathbf{Alt}.$$

Expanding relativisations.

First-order modal and intuitionistic logics motivate our next combination operator. (To keep the notation simple, we concentrate on the $n = 2$ case only.)

DEFINITION 48. A 2-frame $\mathfrak{G} = \langle W, S_1, S_2\rangle$ is called an *expanding relativised product frame* if there exist frames $\mathfrak{F}_1 = \langle U_1, R_1\rangle$ and $\mathfrak{F}_2 = \langle U_2, R_2\rangle$ such that

- $\mathfrak{G}$ is a subframe of $\mathfrak{F}_1 \times \mathfrak{F}_2$, and
- for all $\langle w_1, w_2\rangle \in W$ and $u \in U_1$, if $w_1 R_1 u$ then $\langle u, w_2\rangle \in W$.

Define EX to be the class of all expanding relativised product frames. Given Kripke complete unimodal logics L_1 and L_2, the logic $(L_1 \times L_2)^{\mathsf{EX}}$ is called the *expanding relativised product* of L_1 and L_2.

Similarly to Proposition 45, if both L_1 and L_2 are subframe logics then $(L_1 \times L_2)^{\mathsf{EX}}$ is a (conservative) extension of both L_1 and L_2. Moreover, it is easy to see that every expanding relativised product frame satisfies the *left* commutativity and Church–Rosser properties (but not necessarily right commutativity). Now define the *e-commutator*

$$[L_1, L_2]^{\mathsf{EX}}$$

of L_1 *and* L_2 as the smallest bimodal logic containing L_1, L_2 and the axioms $\boldsymbol{com}^{\boldsymbol{l}}_{12}$ and $\boldsymbol{chr}_{12}$. Then clearly we have

$$[L_1, L_2]^{\mathsf{EX}} \;\subseteq\; (L_1 \times L_2)^{\mathsf{EX}}.$$

Similarly to Theorem 21, it can be shown that for some cases the e-commutator and the expanding relativised product coincide:

THEOREM 49. *Suppose L_1 and L_2 are Kripke complete unimodal logics such that L_1 is one of* **K**, **T**, **K4**, **S4**, **S5** *and L_2 is Horn axiomatisable. Then*

$$(L_1 \times L_2)^{\mathsf{EX}} \;=\; [L_1, L_2]^{\mathsf{EX}}.$$

No other general axiomatisation result for expanding relativised products is known.

As concerns decision problems, it is not hard to see that expanding relativised products are reducible to products. Indeed, let φ be an $\mathcal{ML}_2$-formula and e a propositional variable which does not occur in φ. Define by induction on the construction of φ an $\mathcal{ML}_2$-formula φ^e as follows:

$$\begin{array}{rcll} p^e & = & p \quad (p \text{ a propositional variable}), & \\ (\psi \wedge \chi)^e & = & \psi^e \wedge \chi^e, & (\neg\psi)^e \;=\; \neg\psi^e, \\ (\Box_1\psi)^e & = & \Box_1\psi^e, & (\Box_2\psi)^e \;=\; \Box_2(e \to \psi^e). \end{array}$$

By a structural induction on φ, one can easily prove the following:

PROPOSITION 50. *For all Kripke complete unimodal logics L_1 and L_2 and all $\mathcal{ML}_2$-formulas φ,*

$$\varphi \in (L_1 \times L_2)^{\mathsf{EX}} \quad \textit{iff} \quad \Big(e \wedge \Box_1^{\leq md(\varphi)}\Box_2^{\leq md(\varphi)}(e \to \Box_1 e)\Big) \to \varphi^e \in L_1 \times L_2.$$

As a consequence of this and the results in Section 3.4, we obtain that expanding relativised products are usually decidable if one of their components is an **S5**- or **K**-like logic.

On the other hand, as we saw in Section 3.4, products of 'transitive' logics with frames of arbitrarily large finite or infinite cluster-depth are undecidable. The following result of [27] shows that expanding relativised product logics with components having transitive frames of arbitrarily large finite cluster-depths can be *decidable*:

THEOREM 51. *If L_1, $L_2 \in \{$***GL**, **Grz**, **GL.3**, **Grz.3***$\}$ then $(L_1 \times L_2)^{\mathsf{EX}}$ is decidable.*

Here we discuss the main points of the proof for the case of $(\mathbf{GL}\times\mathbf{GL})^{\mathsf{EX}}$ only. For the other cases, as well as for more general results, consult [27].

We remind the reader that $\mathsf{Fr}\,\mathbf{GL}$ consists of all the irreflexive, transitive and Noetherian frames. Recall that the *depth* $d^{\mathfrak{F}}(x)$ of a point x in an irreflexive tree $\mathfrak{F}=\langle W,R\rangle$ is defined to be the R-distance of x from the root. More precisely, the depth of the root is 0, and the depth of immediate R-successors of a point of depth n is $n+1$. If for no $n<\omega$ the point x is of depth n, then we say that x is of *infinite depth.*

The first important step in the proof is to show that $(\mathbf{GL}\times\mathbf{GL})^{\mathsf{EX}}$ has the 'expanding' version of the product fmp:

LEMMA 52. *Given some $\mathcal{ML}_2$-formula φ, if φ is satisfiable in a frame for $(\mathbf{GL}\times\mathbf{GL})^{\mathsf{EX}}$ then φ is satisfiable in an expanding relativised product frame $\mathfrak{H}$ that is a subframe of a product of two finite trees $\mathfrak{G}_1$ and $\mathfrak{G}_2$. Moreover, $\mathfrak{G}_1=\langle U_1,S_1\rangle$ can be chosen such that, for every $x\in U_1$,*

- $|\{y \mid \langle x,y\rangle \text{ in } \mathfrak{H}\}| \leq \big(|sub\,\varphi|+1\big)!^{\,d^{\mathfrak{G}_1}(x)+1}$*, and*
- *x has at most $|sub\,\varphi|\cdot\big(|sub\,\varphi|+1\big)!^{\,d^{\mathfrak{G}_1}(x)+1}$ immediate S_1-successors.*

Proof. By a standard unravelling argument one can show that every rooted frame for $\mathbf{GL}$ is a p-morphic image of a Noetherian tree-like frame. Moreover, similarly to Proposition 9, one can show that $(\mathbf{GL}\times\mathbf{GL})^{\mathsf{EX}}$ is determined by expanding relativised product frames that are subframes of products of two Noetherian tree-like frames. So we may assume that our formula φ is satisfied at the root $\langle r_1,r_2\rangle$ of some model $\mathfrak{M}=\langle\mathfrak{H},\mathfrak{V}\rangle$ that is based on an expanding relativised subframe $\mathfrak{F}=\langle W,R_1',R_2'\rangle$ of the product of two (possibly infinite) Noetherian tree-like frames $\mathfrak{F}_1=\langle W_1,R_1\rangle$ and $\mathfrak{F}_2=\langle W_2,R_2\rangle$.

For $i=1,2$, call a point x in W_i *R_i-maximal in a set* $X\subseteq W_i$, if $x\in X$ and there is no $x'\in X$ with xR_ix'. Now we take the closure U of the set $\{\langle r_1,r_2\rangle\}$ under the the following three rules:

- **$\Diamond_1$-rule:** if $\langle x,y\rangle\in U$, $(\mathfrak{M},\langle x,y\rangle)\models\Diamond_1\psi$, for some $\Diamond_1\psi\in sub\,\varphi$, and there is no $\langle x',y\rangle\in U$ such that xR_1x' and $(\mathfrak{M},\langle x',y\rangle)\models\psi$, then choose a point $x'\in W_1$ that is R_1-maximal in the set $\{z \mid xR_1z,\ \langle z,y\rangle\in W \text{ and } (\mathfrak{M},\langle z,y\rangle)\models\psi\}$ (such a point exists because $\mathfrak{F}_1$ is Noetherian), and set $U:=U\cup\{\langle x',y\rangle\}$.
- **$\Diamond_2$-rule:** if $\langle x,y\rangle\in U$, $(\mathfrak{M},\langle x,y\rangle)\models\Diamond_2\psi$, for some $\Diamond_2\psi\in sub\,\varphi$, and there is no $\langle x,y'\rangle\in U$ such that yR_2y' and $(\mathfrak{M},\langle x,y'\rangle)\models\psi$, then choose a point $y'\in W_2$ that is R_2-maximal in the set $\{z \mid yR_2z,\ \langle x,z\rangle\in W \text{ and } (\mathfrak{M},\langle x,z\rangle)\models\psi\}$ (such a point exists because $\mathfrak{F}_2$ is Noetherian), and set $U:=U\cup\{\langle x,y'\rangle\}$.
- **Square-rule:** if $\langle x,y\rangle\in U$, xR_1x' and $\langle x',y\rangle\notin U$, then set $U:=U\cup\{\langle x',y\rangle\}$.

Now let $S_i'=R_i'\cap(U\times U)$ $(i=1,2)$ and $\mathfrak{H}=\langle U,S_1',S_2'\rangle$. Take also $\mathfrak{G}_1=\langle U_1,S_1\rangle$ and $\mathfrak{G}_2=\langle U_2,S_2\rangle$, where $U_1=\{x\in W_1 \mid \langle x,r_2\rangle\in U\}$, $U_2=\{y\in W_2 \mid \exists x\in U_1\ \langle x,y\rangle\in U\}$, and $S_i=R_i\cap(U_i\times U_i)$ $(i=1,2)$. Then clearly, $\mathfrak{H}$ is an expanding relativised subframe of the product of Noetherian tree-like frames $\mathfrak{G}_1$ and $\mathfrak{G}_2$.

We show that $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are in fact finite trees with the required bounds. First, we claim that

$$\text{if } x \text{ is of finite depth in } \mathfrak{G}_1, \text{ then } |\{y \mid \langle x,y\rangle\in U\}| \;\leq\; \big(|sub\,\varphi|+1\big)!^{\,d^{\mathfrak{G}_1}(x)+1}. \tag{24}$$

Indeed, we can proceed by induction on n. If $n = 0$, then by applying the $\Diamond_2$-rule to the root $\langle r_1, r_2\rangle$ of $\mathfrak{H}$, we can obtain $\leq |sub\,\varphi|$ immediate S_2'-successors of the form $\langle r_1, y\rangle$. In view of maximality, at each of these points the number of formulas of the form $\Diamond_2\psi \in sub\,\varphi$ to which the $\Diamond_2$-rule still applies is $\leq |sub\,\varphi| - 1$. We proceed with the same kind of argument and finally get

$$|\{y \mid \langle x, y\rangle \in U\}| \quad \leq \quad 1 + |sub\,\varphi| + |sub\,\varphi| \cdot (|sub\,\varphi| - 1) + \cdots + |sub\,\varphi|\,! \;\leq\; (|sub\,\varphi| + 1)\,!\,.$$

The induction step for y of depth $n+1$ is considered analogously. The only difference is that instead of one 'starting' point we should start applying the $\Diamond_2$-rule to all points of the form $\langle x, y\rangle$ such that $\langle z, y\rangle \in U$ for the unique point z with $d(z) = n$ and zS_1y, that is to $|\{y \mid \langle z, y\rangle \in U\}| \leq \big(|sub\,\varphi| + 1\big)!^{\,n+1}$ many points.

Next, we claim that every point x of finite depth in $\mathfrak{G}_1$ has $\leq |sub\,\varphi| \cdot \big(|sub\,\varphi| + 1\big)!^{\,d(x)+1}$ immediate S_1-successors. Indeed, it follows from (24) and the fact that the $\Diamond_1$-rule can be applied at most $|sub\,\varphi|$ times to a point $\langle x, y\rangle$.

Finally, we claim that every point in $\mathfrak{G}_1$ is of finite depth, that is, $\mathfrak{G}_1$ is a tree. Indeed, since $\mathfrak{G}_1$ is Noetherian, we cannot have infinite ascending chains of distinct points in it. Suppose $\mathfrak{G}_1$ still contains a point x of infinite depth. This means that there is an infinite descending chain $\dots S_1 x_2 S_1 x_1 S_1 x$. Let z be an S_1-maximal point of finite depth such that zS_1x. By (24), $|\{y \mid \langle z, y\rangle \in U\}|$ is finite. Therefore, we may apply the $\Diamond_1$-rule to points of the form $\langle z, y\rangle$ finitely many times only, and so there exists an immediate S_1-successor z' of z located properly between z and x. But then $d(z') = d(z) + 1$, and so the depth of z' is finite, which is a contradiction.

Therefore, $\mathfrak{G}_1$ is a Noetherian tree with finite branching. Therefore, by König's lemma, it must be finite. So $\mathfrak{G}_2$ is finite as well. This completes the proof of Lemma 52. ❑

We are now in a position to prove that $(\mathbf{GL} \times \mathbf{GL})^{\mathsf{EX}}$ is decidable. It is to be noted that the 'expanding product fmp' does not give decidability automatically because (i) Lemma 52 does not provide us with an effective upper bound for the size of a model refuting a given formula, nor (ii) do we know that $(\mathbf{GL} \times \mathbf{GL})^{\mathsf{EX}}$ is finitely axiomatisable.

Instead, we will use a version of Kruskal's tree theorem [50]. Given a finite set Σ, a *labelled* Σ*-tree* is a pair $\mathfrak{T} = \langle\langle T, <\rangle, l\rangle$, where $\langle T, <\rangle$ is a (transitive, irreflexive) tree and l is a function from T to Σ. Given two finite labelled Σ-trees $\mathfrak{T}_i = \langle\langle T_i, <_i\rangle, l_i\rangle$, $i = 1, 2$, we say that $\mathfrak{T}_1$ is *embeddable* into $\mathfrak{T}_2$ if there exists an injective map $\iota : T_1 \to T_2$ such that, for all $u, v \in T_1$,

- $u <_1 v$ iff $\iota(u) <_2 \iota(v)$,
- $l_2(\iota(u)) = l_1(u)$.

Now Kruskal's tree theorem says that for every infinite sequence $\mathfrak{T}_1, \mathfrak{T}_2, \dots$ of finite labelled Σ-trees, there exist $i < j < \omega$ such that $\mathfrak{T}_i$ is embeddable into $\mathfrak{T}_j$.

In order to use this theorem, we again turn our models to *quasimodels*. The quasimodels used here are similar to the $L \times \mathbf{K}$-quasimodels of Section 3.4, but they do differ from them in two important aspects: (i) quasistates are now not intransitive, but *transitive* and *irreflexive* trees; (ii) runs are not total, but only *partial* functions over the underlying frame.

To be precise, given an $\mathcal{ML}_2$-formula φ, a *quasistate* for φ is a finite labelled (transitive, irreflexive) tree $\langle\langle T, <\rangle, \boldsymbol{t}\rangle$ where the label $\boldsymbol{t}(x)$ of each $x \in T$ is a type for φ, and $\langle\langle T, <\rangle, \boldsymbol{t}\rangle$ satisfies the $\Diamond_2$-saturation condition **(qm1)** of Section 3.4.

A *basic structure* for φ is a pair $\langle \mathfrak{F}, \boldsymbol{q}\rangle$ such that $\mathfrak{F} = \langle W, R\rangle$ is a finite (transitive, irreflexive) tree and $\boldsymbol{q}$ a function associating with each $w \in W$ a quasistate $\boldsymbol{q}(w) = \langle\langle T_w, <_w\rangle, \boldsymbol{t}_w\rangle$ for φ. We call such a basic structure *small* if, for all $x, y \in W$,

(sm1) $|T_x| \le \big(|sub\,\varphi| + 1\big)!^{\,d^{\mathfrak{F}}(x)+1}$,

(sm2) x has at most $|sub\,\varphi| \cdot \big(|sub\,\varphi| + 1\big)!^{\,d^{\mathfrak{F}}(x)+1}$ immediate R-successors in $\mathfrak{F}$, and

(sm3) if xRy and $x \ne y$ then $\boldsymbol{q}(x)$ is not embeddable into $\boldsymbol{q}(y)$.

For every $n < \omega$, let Q_n be the set of all small basic structures $\langle \mathfrak{F}, \boldsymbol{q}\rangle$ such that $\mathfrak{F}$ is a finite (transitive, irreflexive) tree of depth n.

LEMMA 53. *There is an $n < \omega$ such that $Q_n = \emptyset$, and so the set of small basic structures for φ is finite and can be constructed effectively from φ.*

Proof. Suppose otherwise. Define a relation E on the set Q of all small basic structures as follows. For $\mathfrak{Q} = \langle \mathfrak{F}, \boldsymbol{q}\rangle$, $\mathfrak{Q}' = \langle \mathfrak{F}', \boldsymbol{q}'\rangle$ in Q, set $\mathfrak{Q} E \mathfrak{Q}'$ iff $\mathfrak{F}$ is an 'initial subtree' of $\mathfrak{F}'$ and $\boldsymbol{q}$ coincides with $\boldsymbol{q}'$ on the points of $\mathfrak{F}$. Clearly, for every $\mathfrak{Q}' \in Q_{n+1}$, there is some $\mathfrak{Q} \in Q_n$ such that $\mathfrak{Q} E \mathfrak{Q}'$. Therefore, by König's infinity lemma, there is an infinite E-chain $\mathfrak{Q}_0 E \mathfrak{Q}_1 E \dots E \mathfrak{Q}_n E \dots$ in Q such that $\mathfrak{Q}_n \in Q_n$ for $n < \omega$. Since $\mathfrak{Q}_{n+1}$ is always an extension of $\mathfrak{Q}_n$, their union $\mathfrak{Q} = \bigcup_{n<\omega} \mathfrak{Q}_n$ is also a basic structure. Let $\mathfrak{Q} = \langle \mathfrak{F}, \boldsymbol{q}\rangle$ and $\mathfrak{F} = \langle W, R\rangle$. Then $\mathfrak{F}$ is an infinite tree with finite branching. By König's lemma, it must have an infinite branch $x_0 R x_1 R \dots$. Then, by Kruskal's theorem, there exist $i < j < \omega$ such that $\boldsymbol{q}(x_i)$ is embeddable into $\boldsymbol{q}(x_j)$. But x_i and x_j already belonged to the underlying tree of $\mathfrak{Q}_j$, contrary to $\mathfrak{Q}_j$ being in Q_j. ❑

A *run through* a basic structure $\langle \mathfrak{F}, \boldsymbol{q}\rangle$ is a partial function r from W giving for each $w \in \operatorname{dom} r$ a point $r(w) \in T_w$ such that, for all $x \in W$, if $x \in \operatorname{dom} r$ and xRy then $y \in \operatorname{dom} r$. Coherent and saturated runs are defined as in Section 3.4. Finally, we call a triple $\langle \mathfrak{F}, \boldsymbol{q}, \mathcal{R}\rangle$ a $(\mathbf{GL} \times \mathbf{GL})^{\mathsf{EX}}$-*quasimodel* (for φ) if $\langle \mathfrak{F}, \boldsymbol{q}\rangle$ is a basic structure and $\mathfrak{R}$ is a set of coherent and saturated runs through it, satisfying the following conditions (cf. **(qm2)**–**(qm4)** of Section 3.4):

(eqm2) $\varphi \in \boldsymbol{t}_{w_0}(x_0)$, where w_0 and x_0 are the roots of $\mathfrak{F}$ and $\langle T_{w_0}, <_{w_0}\rangle$, respectively;

(eqm3) for all $r, r' \in \mathcal{R}$ and for all $x, y \in \operatorname{dom} r \cap \operatorname{dom} r'$, $w_{r(x)} <_x w_{r'(x)}$ iff $w_{r(y)} <_y w_{r'(y)}$;

(eqm4) for all $x \in W$ and $w \in T_x$ there is $r \in \mathcal{R}$ such that $r(x) = w$.

We call a quasimodel *small* if the underlying basic structure is small.

LEMMA 54. *φ is satisfiable in a frame for $(\mathbf{GL}\times\mathbf{GL})^{\mathsf{EX}}$ iff there is a small $(\mathbf{GL}\times\mathbf{GL})^{\mathsf{EX}}$-quasimodel for φ.*

Proof. Turning a quasimodel to a 'real' model is easy, so let us concentrate on the opposite direction. We may assume that φ is satisfied in a model $\mathfrak{M} = \langle \mathfrak{H}, \mathfrak{V}\rangle$ based on an expanding relativised subframe $\mathfrak{H}$ of a product $\mathfrak{G}_1 \times \mathfrak{G}_2$ satisfying the conditions of Lemma 52. We can turn $\mathfrak{M}$ to a $(\mathbf{GL}\times\mathbf{GL})^{\mathsf{EX}}$-quasimodel $\langle \mathfrak{G}_1, \boldsymbol{q}, \mathcal{R}\rangle$ as follows. Suppose that $\mathfrak{G}_i = \langle U_i, S_i\rangle$, for $i = 1, 2$. For every $x \in U_1$, let $\boldsymbol{q}(x) = \langle\langle T_x, <_x\rangle, \boldsymbol{t}_x\rangle$, where

$$T_x = \{y \in U_2 \mid \langle x, y\rangle \text{ in } \mathfrak{H}\}, \qquad <_x = S_2 \cap (T_x \times T_x),$$
$$\boldsymbol{t}_x(y) = \{\psi \in sub\,\varphi \mid (\mathfrak{M}, \langle x, y\rangle) \models \psi\}.$$

Finally, for every $y \in U_2$ define a run r_y through $\langle \mathfrak{G}_1, \boldsymbol{q} \rangle$ by taking

$$\operatorname{dom} r_y = \{x \in U_1 \mid \langle x, y \rangle \text{ in } \mathfrak{H}\}$$

and then $r_y(x) = y$, for every $x \in \operatorname{dom} r_y$. Put $\mathcal{R} = \{r_y \mid y \in U_2\}$. It is straightforward to check that $\langle \mathfrak{G}_1, \boldsymbol{q}, \mathcal{R} \rangle$ is indeed a $(\mathbf{GL} \times \mathbf{GL})^{\mathsf{EX}}$-quasimodel for φ. Moreover, by the assumption on $\mathfrak{M}$, the basic structure $\langle \mathfrak{G}_1, \boldsymbol{q} \rangle$ is finite. To show that we can turn it to a basic structure satisfying **(sm3)**, suppose that there are $x, y \in U_1$ such that xS_1y and $\boldsymbol{q}(x)$ is embeddable into $\boldsymbol{q}(y)$ by an embedding ι. Then we replace in $\mathfrak{G}_1$ the subtree generated by x with the subtree generated by y, thus obtaining some tree $\mathfrak{G}' = \langle U', S' \rangle$. Let $\boldsymbol{q}'$ be the restriction of $\boldsymbol{q}$ to U'. We define new runs through $\langle \mathfrak{G}', \boldsymbol{q}' \rangle$ by taking, for all $r, r' \in \mathcal{R}$ such that $x \in \operatorname{dom} r$, $y \in \operatorname{dom} r'$, $\iota(r(x)) = r'(y)$, and for all $z \in U'$, $z \in \operatorname{dom} r$,

$$(r + r')(z) \;=\; \begin{cases} r(z), & \text{if } zS_1x, \\ r'(z), & \text{if } z = y \text{ or } yS_1z. \end{cases}$$

Let $\mathcal{R}'$ be the collection of these new runs together with those runs from $\mathcal{R}$ that 'start at' a point z with yS_1z. It is straightforward to check that $\langle \mathfrak{G}', \boldsymbol{q}', \mathcal{R}' \rangle$ is a $(\mathbf{GL} \times \mathbf{GL})^{\mathsf{EX}}$-quasimodel for φ. Since $\mathfrak{G}_1$ is finite, after finitely many repetitions of this procedure the underlying basic structure will satisfy **(sm3)**. To comply with the cardinality conditions **(sm1)** and **(sm2)**, we can use the construction from the proof of Lemma 52. Then, again we can get rid of the embeddable pairs as above, and so on. As at each step the underlying tree can get only smaller, we end up with a small $(\mathbf{GL} \times \mathbf{GL})^{\mathsf{EX}}$-quasimodel for φ. ❑

Now we can describe the decision algorithm for $(\mathbf{GL} \times \mathbf{GL})^{\mathsf{EX}}$ as follows. Given a formula φ, by Lemma 53, we can effectively construct the set of all small basic structures for φ. Then for each such small basic structure, we check whether it is a $(\mathbf{GL} \times \mathbf{GL})^{\mathsf{EX}}$-quasimodel for φ. By Lemma 54, this way we find a quasimodel for φ iff φ is satisfiable in a frame for $(\mathbf{GL} \times \mathbf{GL})^{\mathsf{EX}}$.

Observe that this decision procedure does not give an even primitive recursive complexity bound. In fact, using a reduction of the reachability problem for *lossy channel systems* (known to be non-primitive recursive by Schnoebelen [73]), it is shown in [27] that there is no primitive recursive decision algorithm for $(\mathbf{GL} \times \mathbf{GL})^{\mathsf{EX}}$.

Other expanding relativised products can even be more complex. The results in [46] suggest that logics like $(\mathsf{Log}\{\langle \mathbb{N}, < \rangle\} \times \mathsf{Log}\{\langle \mathbb{N}, < \rangle\})^{\mathsf{EX}}$ or $(\mathsf{Log}\{\langle \mathbb{N}, < \rangle\} \times \mathbf{S4})^{\mathsf{EX}}$ are undecidable. However, nothing is known about the decision problem of products 'expanding along' branching transitive frames with infinite ascending chains, such as $(\mathbf{K4} \times \mathbf{K4})^{\mathsf{EX}}$ or $(\mathbf{S4} \times \mathbf{S4.3})^{\mathsf{EX}}$. Note that these logics do not have the 'expanding product fmp'.

Another open direction of research is to study the decision problem for the finitely axiomatisable logics obtained by adding either only one of the commutativity axioms or the Church–Rosser axioms to decidable fusions.

5 OTHER COMBINATIONS

Of course, even within the constraints of the combination principles **(C1)**–**(C3)** formulated in the introduction, there are infinitely many ways of combining modal logics. Though much research have been done on multimodal logics, very little of it can be regarded as systematic investigation into properties of some combination method. Moreover, the 'global analysis'—as explained in Chapter 7 of this handbook for the unimodal case—of multimodal logics is still in its infancy. (In fact, most of the investigations into fusions and products can be considered as the first detailed case studies.) The translation of [49] of multimodal logics into unimodal ones is not helpful either in the combination context, as it makes the information about the 'components' virtually disappear.

Releasing (parts of) the criteria **(C2)** and **(C3)** allows us to treat more 'multi-aspect' approaches to modal logic as combinations. The possibilities are again endless, below we discuss a rather ad hoc list of examples. Many more ideas that are relevant to combining modal logics can be found in the 'combining systems' literature, see e.g. the series 'Frontiers of Combining Systems (FroCoS)' [8, 14, 45, 2].

Interaction operators. Interaction between the components can be handled not only by adding interaction axioms, but also by enriching the language with 'dimension-connecting' connectives.

Perhaps the simplest and most natural operations of this sort are the *diagonal constants* d_{ij}. Given two natural numbers i and j with $1 \leq i, j \leq n$, the truth-relation for the constant d_{ij} in models over (subframes of) n-ary product frames is defined as follows:

$$(\mathfrak{M}, \langle u_1, \ldots, u_n \rangle) \models \mathsf{d}_{ij} \quad \text{iff} \quad u_i = u_j.$$

The set of n-tuples satisfying d_{ij} is usually called the (i,j)-*diagonal element*. The main reason for introducing such constants has been to give a 'modal treatment' of equality of classical first-order logic, see Section 3.2 above. Modal algebras for the product logic $\mathbf{S5}^n$ extended with diagonal elements are called *representable cylindric algebras* and are extensively studied in the algebraic logic literature; see e.g. [39, 41] and the references therein.

Another natural way of connecting dimensions is via so-called 'jump' modalities. Given a function $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ (such a map can be called a *jump*), define the truth-relation for the unary modal operator s_π in models over (subframes of) product frames as follows:

$$(\mathfrak{M}, \langle u_1, \ldots, u_n \rangle) \models \mathsf{s}_\pi \varphi \quad \text{iff} \quad (\mathfrak{M}, \langle u_{\pi(1)}, \ldots, u_{\pi(n)} \rangle) \models \varphi.$$

These modal operators are often called (*generalised*) *substitutions*, since by taking

$$P(x_{\pi(1)-1}, \ldots, x_{\pi(n)-1})^\bullet = \mathsf{s}_\pi P(x_0, \ldots, x_{n-1}) \qquad (P \text{ an atomic formula})$$

one can extend the translation $^\bullet$ of Section 3.2 above from formulas with a fixed order of the variables to arbitrary first-order formulas. Note that in cubic universal product $\mathbf{S5}^n$-frames certain substitutions are expressible with the help of the boxes and the diagonal constants [39]. Various versions of modal algebras corresponding to products of $\mathbf{S5}$ logics with substitutions and with or without diagonal constants (e.g., *polyadic* and *substitution algebras*) are introduced by Halmos [33, 34] and Pinter [64, 65] and have been studied in the algebraic logic literature ever since.

Valuation restrictions. One may try to loosen the strong interaction between the components of product logics by imposing restrictions on possible valuations in models over (subframes of) product frames. In general, the resulting formalisms will not be closed under the rule of Substitution, and so do not satisfy **(C2)**.

DEFINITION 55. Let $\mathfrak{M}_1 = \langle \mathfrak{F}_1, \mathfrak{V}_1 \rangle$ and $\mathfrak{M}_2 = \langle \mathfrak{F}_2, \mathfrak{V}_2 \rangle$ be Kripke models that are based on respective frames $\mathfrak{F}_1 = \langle W_1, R_1 \rangle$ and $\mathfrak{F}_2 = \langle W_2, R_2 \rangle$, and let $\Phi(x_1, x_2)$ be a formula in the first-order language $\mathcal{L}$ having two unary predicate symbols V_1, V_2 and two binary predicate symbols. Then a model

$$\mathfrak{M}^{\Phi} = \langle \mathfrak{F}_1 \times \mathfrak{F}_2, \mathfrak{V}^{\Phi} \rangle$$

is said to be a Φ-*flat product of* $\mathfrak{M}_1$ *and* $\mathfrak{M}_2$ if, for all propositional variables p and all $u_1 \in W_1$, $u_2 \in W_2$,

$$\langle u_1, u_2 \rangle \in \mathfrak{V}^{\Phi}(p) \quad \text{iff} \quad I_p \models \Phi[u_1, u_2],$$

where I_p is the first-order $\mathcal{L}$-structure $I_p = \langle W_1 \cup W_2, \mathfrak{V}_1(p), \mathfrak{V}_2(p), R_1, R_2 \rangle$. The valuations in flat-product models are called *flat valuations*.

If Φ is a Boolean combination of $V_1(x_1)$ and $V_2(x_2)$ then we say that $\mathfrak{M}^{\Phi}$ is a *Boolean-flat model*.

Boolean-flat models are introduced and studied in [21, 37]. Various special cases of flat valuations are discussed for many-dimensional temporal logics in [21, 22] and for temporal arrow logics in [61]. Satisfiability in Boolean-flat models can be reduced to satisfiability in the component models, as the following 'flat product decomposition theorem' of Gabbay and Shehtman [25, 23] shows:

THEOREM 56. *Let $\mathfrak{M}^{\Phi}$ be a Boolean-flat product of models $\mathfrak{M}_1$ and $\mathfrak{M}_2$. Then for every $\mathcal{ML}_2$-formula φ, there are a finite set I_φ and unimodal formulas $\varphi_i^{\mathbf{1}}$ (with $\Box_1$) and $\varphi_i^{\mathbf{2}}$ (with $\Box_2$), $i \in I_\varphi$, such that, for all worlds $\langle u_1, u_2 \rangle$ in $\mathfrak{M}^{\Phi}$,*

$$(\mathfrak{M}^{\Phi}, \langle u_1, u_2 \rangle) \models \varphi \quad \textit{iff} \quad \exists i \in I_\varphi \big((\mathfrak{M}_1, u_1) \models \varphi_i^{\mathbf{1}} \textit{ and } (\mathfrak{M}_2, u_2) \models \varphi_i^{\mathbf{2}} \big).$$

Modalising one logic with another. Another possibility is to take some combination method satisfying **(C1)**–**(C3)** and then consider a *fragment* of the full modal language only. The general methodology of 'temporalising' a logic, introduced by Finger and Gabbay [17], results in such a combination when applied to two modal logics:

DEFINITION 57. The set of *modalised formulas* is the smallest set Γ of $\mathcal{ML}_2$-formulas such that:

- if φ is an $\mathcal{ML}_1$-formula then $\varphi \in \Gamma$,
- Γ is closed under Boolean combinations,
- if $\varphi \in \Gamma$ then $\Diamond_2 \varphi \in \Gamma$ and $\Box_2 \varphi \in \Gamma$.

We will evaluate modalised formulas in *modalised models*. These are structures of the form $\mathfrak{M} = \langle \mathfrak{F}, f \rangle$, where $\mathfrak{F} = \langle W, R \rangle$ is a frame and f is a function mapping each $w \in W$ to a pair $f(w) = \langle \mathfrak{M}_w, x_w \rangle$ with $\mathfrak{M}_w$ being a Kripke model and x_w a world in it. The *truth-relation* '$\mathfrak{M}, w \models \varphi$' for modalised formulas φ and worlds w in $\mathfrak{F}$ is defined inductively as follows:

- $\mathfrak{M}, w \models \psi$ iff $(\mathfrak{M}_w, x_w) \models \psi$, whenever ψ is an $\mathcal{ML}_1$-formula,
- $\mathfrak{M}, w \models \neg\psi$ iff $\mathfrak{M}, w \not\models \psi$,
- $\mathfrak{M}, w \models \psi \wedge \chi$ iff $\mathfrak{M}, w \models \psi$ and $\mathfrak{M}, w \models \chi$,

- $\mathfrak{M}, w \models \Diamond_2\psi$ iff there is $v \in W$ such that wRv and $\mathfrak{M}, v \models \psi$.

We say that φ *is true in* $\mathfrak{M}$ if $\mathfrak{M}, w \models \varphi$ for all $w \in W$.

Now let L_1 and L_2 be two Kripke complete unimodal logics formulated in the language $\mathcal{ML}_1$ in such a way that they have different modal operators (say, $\Diamond_1, \Box_1$ and $\Diamond_2, \Box_2$, respectively). The *modalisation of* L_1 *with* L_2 is the set $L_2[L_1]$ of modalised formulas that are true in all modalised models $\mathfrak{M} = \langle \mathfrak{F}, f \rangle$ where

- $\mathfrak{F}$ is a frame for L_2, and
- for all w in $\mathfrak{F}$, the underlying frame of $\mathfrak{M}_w$ is a frame for L_1.

It is not hard to see that $L_2[L_1]$ is a decidable subset of all $\mathcal{ML}_2$-formulas, whenever L_1 and L_2 are both decidable. In fact, this is a consequence of Theorem 5, as $L_2[L_1]$ is a fragment of the fusion of L_1 and L_2 in the sense that $L_2[L_1]$ is the set of all modalised formulas in $L_1 \otimes L_2$ (cf. the proof of Theorem 3).

$\mathcal{E}$-connections. This combination method is introduced by Kutz *et al.* [56, 55] in the more general setting of 'abstract description systems.' When applied to modal logics, this method takes disjoint Kripke models for each component and connects their domains via 'link relations.' These 'connections' then also appear explicitly in the language.

DEFINITION 58. Suppose that we have n 'copies' of the unimodal language $\mathcal{ML}_1$ in such a way that their sets of propositional variables are disjoint (say, $p_0^i, p_1^i, \dots$ for the ith copy) and their modal operators are disjoint as well (say, $\Box_i$ and $\Diamond_i$ for the ith copy). Let J be a non-empty set and take an $n-1$-ary new connective ${}^i\langle E_j \rangle$, for each $j \in J$, $i = 1, \dots, n$. Then the *n-ary $\mathcal{E}$-connection language* $\mathcal{EL}_n^J$ is defined as follows. *$\mathcal{EL}_n^J$-formulas* are partitioned into n sets, each one containing the so-called *i-formulas* for some $i = 1, \dots, n$. For all $i = 1, \dots, n$, the sets of i-formulas are defined by simultaneous induction:

- the propositional variables $p_0^i, p_1^i, \dots$ are i-formulas,
- the set of i-formulas is closed under the Boolean connectives and the modal operators $\Box_i$ and $\Diamond_i$,
- if φ_k is a k-formula, for each $k = 1, \dots, i-1, i+1, \dots, n$, then
$$ {}^i\langle E_j \rangle \, (\varphi_1, \dots, \varphi_{i-1}, \varphi_{i+1}, \dots, \varphi_n) $$
is an i-formula, for every $j \in J$.

$\mathcal{EL}_n^J$-formulas are evaluated in *$\mathcal{EL}_n^J$-models.* These are structures of the form
$$ \mathfrak{M} = \left\langle \mathfrak{M}_1, \dots, \mathfrak{M}_n, E_j^{\mathfrak{M}} \right\rangle_{j \in J}, $$
where the $\mathfrak{M}_i = \langle \langle W_i, R_i \rangle, \mathfrak{V}_i \rangle$ are (unimodal) Kripke models and $E_j^{\mathfrak{M}} \subseteq W_1 \times \dots \times W_n$, for each $j \in J$. The *truth-relation* '$\mathfrak{M}, w \models \varphi$' for i-formulas φ and worlds w in $\mathfrak{M}_i$ is defined inductively as follows, simultaneously for $i = 1, \dots, n$:

- $\mathfrak{M}, w \models p_k^i$ iff $w \in \mathfrak{V}_i(p_k^i)$,
- $\mathfrak{M}, w \models \neg\psi$ iff $\mathfrak{M}, w \not\models \psi$,

- $\mathfrak{M}, w \models \psi \wedge \chi$ iff $\mathfrak{M}, w \models \psi$ and $\mathfrak{M}, w \models \chi$,
- $\mathfrak{M}, w \models \Diamond_i \psi$ iff there is $v \in W_i$ such that wR_iv and $\mathfrak{M}, v \models \psi$,
- $\mathfrak{M}, w \models {}^i\langle E_j \rangle (\psi_1, \ldots, \psi_{i-1}, \psi_{i+1}, \ldots, \psi_n)$ iff for all $k = 1, \ldots, i-1, i+1, \ldots, n$ there are $v_k \in W_k$ such that $\mathfrak{M}, v_k \models \psi_k$ and $\langle v_1, \ldots, v_{i-1}, w, v_{i+1}, \ldots, v_n \rangle \in E_j^{\mathfrak{M}}$.

We say that an i-formula φ *is true in* $\mathfrak{M}$ if $\mathfrak{M}, w \models \varphi$ for all $w \in W_i$.

Now let $L_1, \ldots, L_n$ be unimodal logics formulated in n 'almost disjoint' copies of $\mathcal{ML}_1$ as described above, and let J be a non-empty set. The *basic $\mathcal{E}$-connection*

$$\mathcal{EC}^J(L_1, \ldots, L_n)$$

of $L_1, \ldots, L_n$ is the set of all $\mathcal{EL}_n^J$*-formulas* that are true in all $\mathcal{EL}_n^J$-models $\mathfrak{M} = \langle \mathfrak{M}_1, \ldots, \mathfrak{M}_n, E_j^{\mathfrak{M}} \rangle_{j \in J}$ where $\mathfrak{M}_i$ is a model for L_i, for $i = 1, \ldots, n$.

The following theorem on the transfer of decidability is proved in [55] in the more general setting of 'abstract desription systems:'

THEOREM 59. *If $L_1, \ldots, L_n$ are all decidable unimodal modal logics, then the basic $\mathcal{E}$-connection $\mathcal{EC}^J(L_1, \ldots, L_n)$ is decidable.*

Intuively, the decision procedure for, say, $\mathcal{EC}^{\{0\}}(L_1, L_2)$ works as follows. As usual, we can consider satisfiability instead of validity. To check whether there exists a model $\mathfrak{M} = \langle \mathfrak{M}_1, \mathfrak{M}_2, E_0 \rangle$ and a world w in $\mathfrak{M}_i$ such that $\mathfrak{M}, w \models \varphi$ for a given i-formula φ, the algorithm non-deterministically 'guesses'

- the 1-types that are realised in $\mathfrak{M}_1$ and the 2-types that are realised in $\mathfrak{M}_2$ (where a k*-type* is a set of k-formulas that are true at a world of $\mathfrak{M}_k$), and
- a binary relation e between the guessed sets of 1-types and 2-types.

Then it checks whether the guessed sets satisfy a collection of 'integrity conditions.' This check involves satisfiability tests of certain sets of k-formulas constructed from φ, for $k = 1, 2$—here we exploit that L_1 and L_2 are decidable. If the integrity conditions are satisfied, then it is possible to construct a model satisfying φ using models of the constructed sets of k-formulas. If the integrity conditions are not satisfied then φ is not satisfiable. This algorithm also provides an upper bound for the satisfiability problem for $\mathcal{EC}^J(L_1, \ldots, L_n)$: the time complexity is non-deterministic and one exponent higher than the maximal time complexity of the component procedures.

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