Combinatory Logic Synthesis and Alternation

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Combinatory Logic Synthesis

- Using combinatory logic (CL) as a foundation for synthesis by type inhabitation
- Components from a repository
- Component-based Synthesis is (re-)emerging research topic
- Combinatory logic
  - Schönfinkel and Curry (1920's)
  - Variable-free
  - Hilbert-style

Alternation

- Alternating Turing Machine (ATM)
- Chandra, Kozen and Stockmeyer (1981)
- Existential / universal states
- Acceptance rules adapted

Simply Typed λ-calculus ($\lambda_{\to}$)

$$e ::= x \mid \lambda x.e \mid (e\ e')$$
$$\tau ::= a \mid \tau \to \tau'$$
$$\Gamma = \{x : \tau, \ldots\}$$

$$\frac{}{\Gamma, x : \tau \vdash x : \tau}(\mathsf{var}) \qquad \frac{\Gamma \vdash e : \tau \to \tau' \quad \Gamma \vdash e' : \tau}{\Gamma \vdash (e\ e') : \tau'}(\to\mathsf{E})$$

$$\frac{\Gamma, x : \sigma \vdash e : \tau}{\Gamma \vdash \lambda x.e : \sigma \to \tau}(\to\mathsf{I}) \qquad \frac{\Gamma \vdash e : \tau \quad \tau \leq \tau'}{\Gamma \vdash e : \tau'}(\leq)$$

Type inhabitation

Does there exist λ-term e with $\Gamma \vdash e : \tau$?

Curry-Howard Isomorphism

Propositions-as-types correspondence

$$\begin{aligned} \lambda_{\to} &\leftrightarrow \text{minimal inutitionistic propositional logic} \\ \text{Types} &\leftrightarrow \text{propositions} \\ \lambda_{\to}\ \text{terms} &\leftrightarrow \text{proofs} \\ \text{Inhabitation} &\leftrightarrow \text{provability} \end{aligned}$$

Statman's theorem

Provability in intuitionistic logic is PSPACE-complete.

Fixed-base CL (Hilbert-style System)

$$
\begin{aligned}
S &: (\sigma \to \tau \to \rho) \to (\sigma \to \tau) \to \sigma \to \rho \\
K &: \sigma \to \tau \to \sigma \\
I &: \tau \to \tau
\end{aligned}
$$

$$
\frac{[S \text{ is substitution}]}{\Gamma, x : \tau \vdash x : S(\tau)}(\text{var}) \qquad \frac{\Gamma \vdash e : \tau \to \tau' \quad \Gamma \vdash e' : \tau}{\Gamma \vdash (e\ e') : \tau'}(\to\text{E})
$$

Relativized Inhabitation

Set of given combinators is *not fixed* but given as part of the input.

Given Γ and τ does there exist CL-term e with $\Gamma \vdash e : \tau$?

Related Results

Relativized inhabitation in CL for simple types is undecidable.
(Linial-Post)

Example Repository

$$
\begin{aligned}
\Gamma = \{ & \\
& Tr : () \to D((\mathtt{R}, \mathtt{R}), \mathtt{R}, \mathtt{R}), \\
& pos : D((\mathtt{R}, \mathtt{R}), \mathtt{R}, \mathtt{R}) \to ((\mathtt{R}, \mathtt{R}), \mathtt{R}), \\
& cdn : ((\mathtt{R}, \mathtt{R}), \mathtt{R}) \to (\mathtt{R}, \mathtt{R}), \\
& fst : (\mathtt{R}, \mathtt{R}) \to \mathtt{R}, \\
& snd : (\mathtt{R}, \mathtt{R}) \to \mathtt{R}, \\
& \vdots \\
& \}
\end{aligned}
$$

Example Repository with Semantic Specification

$$
\begin{aligned}
\Gamma = \{ & \\
& Tr : () \to D((\mathtt{R}, \mathtt{R}) \cap \mathtt{Cart}, \mathtt{R} \cap \mathtt{GPST}, \mathtt{R} \cap \mathtt{Cel}), \\
& pos : D((\mathtt{R}, \mathtt{R}) \cap \epsilon, \mathtt{R} \cap \mu, \mathtt{R}) \to ((\mathtt{R}, \mathtt{R}) \cap \epsilon, \mathtt{R} \cap \mu) \cap \mathtt{Pos}, \\
& cdn : ((\mathtt{R}, \mathtt{R}) \cap \epsilon, \mathtt{R}) \cap \mathtt{Pos} \to (\mathtt{R}, \mathtt{R}) \cap \epsilon, \\
& fst : ((\mathtt{R}, \mathtt{R}) \cap \mathtt{Coord} \to \mathtt{R}) \cap (\mathtt{Cart} \to \mathtt{Cx}) \cap (\mathtt{Polar} \to \mathtt{Radius}), \\
& snd : ((\mathtt{R}, \mathtt{R}) \cap \mathtt{Coord} \to \mathtt{R}) \cap (\mathtt{Cart} \to \mathtt{Cy}) \cap (\mathtt{Polar} \to \mathtt{Angle}), \\
& \vdots \\
& \}
\end{aligned}
$$

$\Gamma \vdash ? : \mathtt{R} \cap \mathtt{Angle}$ has inhabitant $? = snd(cdn(pos\ Tr\ ()))$

Relation of Inhabitation in $\lambda_{\rightarrow}$ and Alternation

Solving $\Gamma \vdash ? : \tau$

To answer $\Gamma \vdash ? : \tau$ apply one of the following tactics:

- for $\tau = \tau_1 \rightarrow \tau_2$, ask $\Gamma \cup \{\tau_1\} \vdash ? : \tau_2$
- for $\tau = a$, choose $x \in \Gamma$ with $x : \sigma_1 \rightarrow \ldots \rightarrow \sigma_n \rightarrow a$ then ask $\Gamma \vdash ? : \sigma_i$ for all $1 \leq i \leq n$. Success if $n = 0$.

Alternation

- Nondeterminism: choose $x \in \Gamma$
- Alternation: all $\Gamma \vdash ? : \sigma_i$ must be solved in parallel

Finite Combinatory Logic FCL($\cap, \leq$)

$$\tau ::= a \mid \tau \to \tau \mid \tau \cap \tau$$

$$\frac{}{\Gamma, x : \tau \vdash x : \tau}(\text{var}) \qquad \frac{\Gamma \vdash e : \tau \to \tau' \quad \Gamma \vdash e' : \tau}{\Gamma \vdash (e\ e') : \tau'}(\to\text{E})$$

$$\frac{\Gamma \vdash e : \tau_1 \quad \Gamma \vdash e : \tau_2}{\Gamma \vdash e : \tau_1 \cap \tau_2}(\cap\text{I}) \qquad \frac{\Gamma \vdash e : \tau \quad \tau \leq \tau'}{\Gamma \vdash e : \tau'}(\leq)$$

RU, TLCA 2011

	Complexity	lower-bound	upper-bound
FCL	PTIME		
FCL($\cap$)	EXPTIME	bottom-up TA	top-down ATA
FCL($\cap, \leq$)	EXPTIME	bottom-up TA	ATM

Bounded Combinatory Logic $\mathrm{BCL}_k(\cap, \leq)$

$$\tau ::= a \mid \alpha \mid \tau \to \tau \mid \tau \cap \tau$$

$$\frac{[S : \mathbb{V} \to \mathbb{T}_k]}{\Gamma, x : \tau \vdash x : S(\tau)}(\text{var}) \qquad \frac{\Gamma \vdash e : \tau \to \tau' \quad \Gamma \vdash e' : \tau}{\Gamma \vdash (e\ e') : \tau'}(\to\text{E})$$

$$\frac{\Gamma \vdash e : \tau_1 \quad \Gamma \vdash e : \tau_2}{\Gamma \vdash e : \tau_1 \cap \tau_2}(\cap\text{I}) \qquad \frac{\Gamma \vdash e : \tau \quad \tau \leq \tau'}{\Gamma \vdash e : \tau'}(\leq)$$

DMRU, CSL 2012

	Complexity	lower-bound	upper-bound
$\mathrm{BCL}_k(\leq)$	EXPTIME	ATM	ATM
$\mathrm{BCL}_k(\cap, \leq)$	2+k-EXPTIME	ATM	ATM

Complexity of Inhabitation Problems

System	Complexity	Authors
FCL($\leq$)	PTIME	RU, TLCA 2011
FCL($\cap, \leq$)	EXPTIME	RU, TLCA 2011
$\mathsf{BCL}_k(\leq)$	EXPTIME	DMRU, CSL 2012
$\mathsf{BCL}_k(\cap,\leq)$	(k+2)-EXPTIME	DMRU, CSL 2012
$\mathrm{CL}(\mathbf{SK}) \to$	PSPACE	S, 1979
$\lambda(- \cap I)$	EXPSPACE	RU, KF 2012
$\lambda^{r2}\cap$	EXPSPACE	U, TLCA 2009
$\lambda\cap$	∞	U, 1999
$\mathrm{CL}(\cap)$	∞	DH, 1992 + U, 1999

Open Research Questions

- Relation between LTL-style and combinatory synthesis
- Generally: relation to MSO synthesis

ATM for Inhabitation in $\mathrm{BCL}_k(\cap, \leq)$

Input : Γ, τ

1 // loop
2 $\quad$ CHOOSE $(x : \sigma) \in \Gamma$;
3 $\quad$ $\sigma' := \bigcap\{S(\sigma) \mid S : \mathit{Var}(\Gamma, \tau) \to \mathbb{T}_k(\Gamma, \tau)\}$;
4 $\quad$ CHOOSE $n \in \{0, \ldots, \|\sigma'\|\}$;
5 $\quad$ CHOOSE $P \subseteq \mathbb{P}_n(\sigma')$;
6 $\quad$ IF $(\bigcap_{\pi \in P} \mathit{tgt}_n(\pi) \leq \tau)$ THEN
7 $\quad\quad$ IF $(n = 0)$ THEN ACCEPT;
8 $\quad\quad$ ELSE
9 $\quad\quad\quad$ FORALL$(i = 1 \ldots n)$
10 $\quad\quad\quad\quad$ $\tau := \bigcap_{\pi \in P} \mathit{arg}_i(\pi)$;
11 $\quad\quad\quad$ GOTO LINE 2;
12 $\quad$ ELSE REJECT;