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Effective analytic functions

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Abstract

One approach for computations with special functions in computer algebra is the systematic use of analytic functions whenever possible. This naturally leads to problems of how to answer questions about analytic functions in a fully effective way. Such questions comprise the determination of the radius of convergence or the evaluation of the analytic continuation of the function at the endpoint of a broken line path. In this paper, we propose a first definition for the notion of an effective analytic function and we show how to effectively solve several types of differential equations in this context. We will limit ourselves to functions in one variable.

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1. Introduction

An important problem in computer algebra is how to compute with special functions which are more complicated than polynomials. A systematic approach to this problem is to recognize that most interesting special functions are analytic, so they are completely determined by their power series expansions at a point.

Of course, the concept of "special function" is a bit vague. One may for instance study expressions which are built up from a finite number of classical special functions, like exp, log or erf. But one may also study larger classes of "special functions", such as the

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solutions to systems of algebraic differential equations with constant coefficients. Let us assume that we have fixed a class $\mathcal{F}$ of such special functions or expressions for the rest of this introduction.

In order to develop a satisfactory computational theory for the functions or expressions in $\mathcal{F}$, one may distinguish the following main subproblems:

- How do we test whether $f \in \mathcal{F}$ locally represents the zero power series?
- How do we evaluate $f \in \mathcal{F}$ safely and efficiently at any point where f may be defined?

The first subproblem is called the zero-test problem and it has been studied before in several works (Risch, 1975; Denef and Lipshitz, 1984, 1989; Khovanskii, 1991; Shackell, 1989, 1993; Péladan-Germa, 1997; Shackell and van der Hoeven, 2001; van der Hoeven, 2002a). For large classes of special functions, it turns out that the zero-test problem for power series can be reduced to the zero-test problem for constants (see also van der Hoeven (2001b) for a discussion of this latter problem).

In this paper, we will focus on the second subproblem, while leaving aside the efficiency considerations and restricting our attention to functions in one variable. By "safe evaluation" of $f \in \mathcal{F}$ at z, we mean that we want an algorithm which computes an approximation $\tilde{f} \in (\mathbb{Z} + \mathrm{i}\mathbb{Z})2^{\mathbb{Z}}$ for $f(z)$ with $|\tilde{f} - f(z)| < \varepsilon$ for any $\varepsilon \in 2^{\mathbb{Z}}$. If such an algorithm exists, then we say that $f(z)$ is an effective complex number. By "any point where the expression may be defined", we mean that we do not merely plan to study f near the point where a power series expansion was given, but that we also want to study all possible analytic continuations of f.

In other words, the aim of this paper is to develop an "effective complex analysis" for computations with analytic functions in a class $\mathcal{F}$ which we wish to be as large as possible. Such computations mainly consist of safe evaluations, bound computations for convergence radii and absolute values of functions on closed disks, and analytic continuation. Part of the philosophy behind this paper also occurs in Bishop and Bridges (1985) and Chudnovsky and Chudnovsky (1990), but without the joint emphasis on effectiveness at all stages and usefulness from the implementation point of view. In previous papers (van der Hoeven, 1999, 2001a), we have also studied in detail the fast and safe evaluation of holonomic functions. When studying solutions to non-linear differential equations, one must carefully avoid undecidable problems:

Theorem 1 (*Denef and Lipshitz, 1989*). *Given a power series $f = \sum f_n z^n$ with rational coefficients, which is the unique solution of an algebraic differential equation*

$$P(z, f, \ldots, f^{(l)}) = 0,$$

with rational coefficients and rational initial conditions, one cannot in general decide whether the radius of convergence $\rho(f)$ of f is <1 or $\geqslant 1$.

What we will show in this paper is that whenever we *know* that an analytic function $f = \sum f_n z^n$ as in the above theorem may be continued analytically along an "effective broken line path" γ, then the value $f(\gamma)$ of f at the endpoint of γ is effective (Theorem 3). We will also show that we may "effectively solve" any monic linear differential equation, without introducing singularities which were not already present in the coefficients (Theorem 2).

In order to prove these results, we will carefully introduce the concept of an "effective analytic function" in Section 2. The idea is that such a function f is given locally at the origin and that we require algorithms for

- computing coefficients of the power series expansion;
- computing a lower bound ρ for the radius of convergence;
- computing an upper bound for $|f|$ on any closed disk of radius $<\rho$;
- analytic continuation.

But we will *also* require additional conditions, which will ensure that the computed bounds are good enough from a more global point of view. In Section 3, we will show that all analytic functions, which are constructed from the effective complex numbers and the identity function using $+, -, \cdot, /, \frac{\mathrm{d}}{\mathrm{d}z}, \int$, exp and log, are effective. In Section 4, we will study the resolution of differential equations.

It is convenient to specify the actual algorithms for computations with effective analytic functions in an object oriented language with abstract data types (like C++). Effective types and functions will be indicated through the use of a $\mathsf{sanserif}$ font. We will not detail memory management issues and assume that the garbage collector takes care of this. The COLUMBUS program is a concrete implementation of some of the ideas in this paper (van der Hoeven, 2000–2002), although this program works with double precision instead of effective complex numbers.

2. Effective analytic functions

2.1. Effective numbers and power series

A complex number $z \in \mathbb{C}$ is said to be *effective* if there exists an algorithm which takes a positive number $0 < \varepsilon \in 2^{\mathbb{Z}}$ on input and which returns an approximation $\tilde{z} \in (\mathbb{Z} + \mathrm{i}\mathbb{Z})2^{\mathbb{Z}}$ for z with $|\tilde{z} - z| < \varepsilon$. We will also denote the approximation by $\tilde{z} = z_{<\varepsilon}$. In practice, we will represent z by an abstract data structure $\mathsf{Complex}$ with a method $\mathsf{approximate}$ which implements the above approximation algorithm. The *asymptotic complexity* of an effective complex number $z \in \mathsf{Complex}$ is the asymptotic complexity of its approximation algorithm.

We have a natural subtype $\mathsf{Real} \subseteq \mathsf{Complex}$ of effective real numbers. Recall, however, that there is no algorithm to decide whether an effective complex number is real. In the following, we will use the notation $\mathbb{R}^{>} = \{x \in \mathbb{R} : x > 0\}$ and $\mathbb{R}^{\geqslant} = \{x \in \mathbb{R} : x \geqslant 0\}$.

Let R be a *weakly effective ring*, in the sense that all elements of R can be represented by explicit data structures and that we have algorithms for the ring operations $0, 1, +, -$ and $\cdot$. If we also have an effective zero-test, then we say that R is an *effective ring*.

An *effective series* over R is a series $f \in \mathsf{R}[[z]]$, such that there exists an algorithm for computing the n-th coefficient of f. Effective series over R are represented by instances of the abstract data type $\mathsf{Series}(\mathsf{R})$, which has a method $\mathsf{expand} : \mathbb{N} \to \mathsf{R}[z]$, which computes the truncation $f_0 + \cdots + f_{n-1}z^{n-1} \in \mathsf{R}[z]$ of f at order n as a function of $n \in \mathbb{N}$. The *asymptotic complexity* of f is the asymptotic complexity of this expansion algorithm. In particular, we have an algorithm to compute the n-th coefficient f_n of an effective series. A survey of efficient methods for computing with effective series can be found in van der Hoeven (2002b).

When f is an effective series in $\mathsf{Series}(\mathsf{Complex})$, then we denote by $\bar{f} \in \mathbb{C}[[z]]$ the actual series which is represented by f. If $\bar{f}$ is the germ of analytic function, then we will denote by $\rho(\bar{f})$ the radius of convergence of $\bar{f}$ and by $|\bar{f}|_r$ the maximum of $|\bar{f}|$ on the closed disk $\bar{B}_r = \{z \in \mathbb{C} : |z| \leqslant r\}$ of radius r, for each $r < \rho(\bar{f})$.

2.2. Effective germs

An *effective germ* of an analytic function f at 0 is an abstract data structure Germ which inherits from $\mathsf{Series}(\mathsf{Complex})$ and with the following additional methods:

- A method $\mathsf{radius} : () \to \mathsf{Real} \cup \{+\infty\}$ which returns a lower bound $\rho(f) > 0$ for $\rho(\bar{f})$. This bound is called the *effective radius of convergence* of f.
- A method $\mathsf{norm} : \mathsf{Real} \to \mathsf{Real}$ which, given $0 < r < \rho(f)$, returns an upper bound $|f|_r$ for $|\bar{f}|_r$. We assume that $|f|_r$ is increasing in r.

Remark 1. For the sake of simplicity, we have reused the notation $\rho(f)$ and $|f|_r$ in order to denote the applications of the methods radius and norm to f. Clearly, one should carefully distinguish between $\rho(f)$ resp. $|f|_r$ and $\rho(\bar{f})$ resp. $|\bar{f}|_r$. The n-th coefficient of f will always be denoted by $f_n = \bar{f}_n$.

Given an effective germ f, we may implement a method $\mathsf{evaluate}$, which takes an effective complex number $z \in \mathsf{Complex}$ with $|z| < \rho(f)$ on input, and which computes its effective value $f(z) \in \mathsf{Complex}$ at z. For this, we have to show how to compute arbitrarily good approximations for $f(z)$:

Algorithm $\mathsf{evaluate\text{-}approx}(f, z, \varepsilon)$

Input: $f \in \mathsf{Germ}$ and $z \in \mathsf{Complex}$ with $|z| < \rho(f)$, and $\varepsilon \in 2^{\mathbb{Z}}$
Output: an approximation $\tilde{f}$ for $f(z)$ with $|\tilde{f} - f(z)| < \varepsilon$

Step 1. [Compute expansion order]
Let $r = (\rho(f) + |z|)/2$ and $M = |f|_r$
Let $n \in \mathbb{N}$ be smallest such that

$$\frac{Mr}{r - |z|}\left(\frac{|z|}{r}\right)^n < \frac{\varepsilon}{2}$$

Step 2. [Approximate the series expansion]
Compute $\hat{f} = f_0 + f_1 z + \cdots + f_{n-1} z^{n-1} \in \mathsf{Complex}$
Return $\hat{f}_{<\varepsilon/2}$

The correctness of this algorithm follows from Cauchy's formula:

$$\begin{aligned}
|\hat{f} - f(z)| &= \left|\frac{z^n}{2\pi \mathrm{i}} \oint_{|w|=r} \frac{f(w)}{(w-z)w^n} \mathrm{d}w\right| \\
&\leqslant \frac{|z|^n}{2\pi} \left|\oint_{|w|=r} \frac{M}{(r-|z|)r^n} \mathrm{d}w\right| \\
&= \frac{Mr}{r-|z|}\left(\frac{|z|}{r}\right)^n \\
&< \frac{\varepsilon}{2}.
\end{aligned}$$

2.3. Effective paths

Any $(l+1)$-tuple $(z_0, \ldots, z_l)$ of complex numbers determines a unique *affine broken line path* or *affine path* in $\mathbb{C}$, which is denoted by $z_0 \to z_1 \to \cdots \to z_{l-1} \to z_l$. If $z_0 = 0$, then we call $\gamma = z_0 \to z_1 \to \cdots \to z_{l-1} \to z_l$ a *broken line path* or a *path* of *length* $l_\gamma = l$. We denote by $\mathbb{P}$ the space of all paths and by $\mathbb{P}^{\text{aff}}$ the space of all affine paths. The trivial path of length 0 is denoted by $\bullet$. An affine path $z_0 \to \cdots \to z_l$ is said to be effective if $z_0, \ldots, z_l \in \mathsf{Complex}$. We denote by $\mathsf{AffinePath}$ the type of all effective affine paths and by Path the type of all effective paths.

Another notation for the path $0 = z_0 \to z_1 \to \cdots \to z_{l-1} \to z_l$ is $[z_1 - z_0, \ldots, z_l - z_{l-1}]$; the first notation is called the *usual notation*; the second one is called the *incremental notation*. Let $\gamma = [\delta_1, \ldots, \delta_l] = 0 \to z_1 \to \cdots \to z_{l-1} \to z_l$ and $\gamma' = [\delta'_1, \ldots, \delta'_{l'}] = 0 \to z'_1 \to \cdots \to z'_{l'-1} \to z'_{l'}$ be two paths in $\mathbb{P}$. Then we define their *concatenation* $\gamma + \gamma'$ by

$$\begin{aligned}\gamma + \gamma' &= [\delta_1, \ldots, \delta_l, \delta'_1, \ldots, \delta'_{l'}] \\ &= 0 \to z_1 \to \cdots \to z_l \to z_l + z'_1 \to \cdots \to z_l + z'_{l'}.\end{aligned}$$

We also define the *reversion* $-\gamma$ of γ by

$$\begin{aligned}-\gamma &= [-\delta_l, \ldots, -\delta_1] \\ &= 0 \to z_{l-1} - z_l \to \cdots \to z_1 - z_l \to -z_l.\end{aligned}$$

The *norm* of γ is defined by

$$|\gamma| = |\delta_1| + \cdots + |\delta_l|.$$

Notice that $|-\gamma| = |\gamma|$ and $|\gamma + \gamma'| = |\gamma| + |\gamma'|$.

We say that γ is a *truncation* of γ' if $l \leqslant l'$ and $\delta_i = \delta'_i$ for $i = 1, \ldots, l$. In this case, we write $\gamma \trianglelefteq \gamma'$ and $\gamma' - \gamma = [\delta'_{l+1}, \ldots, \delta'_{l'}]$, so that $(\gamma' - \gamma) + \gamma = \gamma'$ (however, we do not always have $\gamma' - \gamma = \gamma' + (-\gamma)$). The *longest common truncation* of two general paths γ and γ' always exists and we denote it by $\gamma \triangle \gamma'$. If we restrict our attention to paths $z_1 \to \cdots \to z_l$ such that $z_1, \ldots, z_l$ are in a subfield of $\mathsf{Complex}$ with an effective zero-test (like $\mathbb{Q}[i]$), then $\trianglelefteq$ and $\triangle$ are clearly effective.

A *subdivision* of a path $\gamma = [\delta_1, \ldots, \delta_l]$ is a path of the form

$$\gamma' = [\lambda_{1,1}\delta_1, \ldots, \lambda_{1,k_1}\delta_1, \ldots, \lambda_{l,1}\delta_l, \ldots, \lambda_{l,k_l}\delta_l],$$

where $\lambda_{i,j} \in (0, 1)$ with $\lambda_{i,1} + \cdots + \lambda_{i,k_i} = 1$ for all i. If γ' is a subdivision of γ, then we write $\gamma \sqsubseteq \gamma'$. Given γ, γ' and γ'' with $\gamma \sqsubseteq \gamma'$ and $\gamma \sqsubseteq \gamma''$, there exists a shortest path γ''' with $\gamma' \sqsubseteq \gamma'''$ and $\gamma'' \sqsubseteq \gamma'''$. We call γ''' the *shortest common subdivision* of γ' and γ'' and we denote it by $\gamma''' = \gamma' \sqcup \gamma''$.

Given an analytic function f at the origin, which can be continued analytically along a path $\gamma \in \mathbb{P}$, we will denote by $f(\gamma)$ the value of f at the endpoint of the path and by $f_{+\gamma}$ the analytic function at zero, such that $f_{+\gamma}(\varepsilon) = f(\gamma + \varepsilon) := f(\gamma + [\varepsilon])$ for all sufficiently small ε. If $\gamma = [\delta_1, \ldots, \delta_l]$, then we will also write $f_{+\gamma} = f_{+\delta_1,\ldots,+\delta_l}$. The *domain* $\operatorname{Dom} f$ of f is the set of all paths γ along which f can be continued analytically.

2.4. Effective analytic functions

A *quasi-effective analytic function* is an instance f of the abstract data type AnFunc, which inherits from Germ, and with the following additional method:

- continue : Complex → AnFunc computes f_{+z} for all z with $|z| < \rho(f)$.

We will denote by $\bar{f}$ the analytic function which is represented by f. The *domain* $\operatorname{Dom} f \subseteq \operatorname{Dom} \bar{f}$ of f is the set of all paths $[\delta_1, \ldots, \delta_l] \in$ Path with

$$|\delta_i| < \rho(f_{+\delta_1,\ldots,+\delta_{i-1}})$$

for all $i \in \{1, \ldots, n\}$. The functions evaluate and continue may be extended to the class Path, for all paths which are in the domain of f.

Remark 2. Notice that, similarly to in Remark 1, we have reused the notation f_{+z} in order to denote the application of the method $f \to$ continue to z. So f_{+z} is again a quasi-effective analytic function, which should be distinguished from $\bar{f}_{+z} = \overline{f_{+z}}$. Given $\gamma = [\delta_1, \ldots, \delta_l]$, we will also write $f(\gamma) = f_{+\delta_1,\ldots,+\delta_{l-1}}(\delta_l)$ and $f_{+\gamma} = f_{+\delta_1,\ldots,+\delta_l}$.

Let f and g be two quasi-effective analytic functions. We will write $f = g$ as soon as $\bar{f} = \bar{g}$. We say that f and g are *strongly equal*, and we write $f \equiv g$, if $\operatorname{Dom} f = \operatorname{Dom} g$ and $\rho(f_{+\gamma}) = \rho(g_{+\gamma})$ and $|f_{+\gamma}|_r = |g_{+\gamma}|_r$ for all $\gamma \in \operatorname{Dom} f$ and $r < \rho(f_{+\gamma})$. We say that g is *better* than f, if $\operatorname{Dom} f \subseteq \operatorname{Dom} g$, and $\rho(f_{+\gamma}) \leqslant \rho(g_{+\gamma})$ and $|f_{+\gamma}|_r \geqslant |g_{+\gamma}|_r$ for all $\gamma \in \operatorname{Dom} f$ and $r < \rho(f_{+\gamma})$. We say that a quasi-effective analytic function f satisfies the *homotopy condition* if:

EA1. If $\gamma, \gamma + [\delta_1], \gamma + [\delta_1, \delta_2], \gamma + [\delta_1 + \delta_2] \in \operatorname{Dom} f$, then $f_{+\gamma,+(\delta_1+\delta_2)} \equiv f_{+\gamma,+\delta_1,+\delta_2}$.

Here we understand that $f_{+\gamma,+(\delta_1+\delta_2)} \overset{\text{def}}{=} f_{+\gamma}$ if $\delta_1 + \delta_2 = 0$.

Let f be a quasi-effective analytic function and consider the functions $\delta \mapsto \rho(f_{+\delta})$ and $(\delta, r) \mapsto |f_{+\delta}|_r$. We say that f satisfies the *local continuity condition* if there exist continuous functions

$$R : B_{\rho(f)} = \{z \in \mathbb{C} : |z| < \rho(f)\} \rightarrow \mathbb{R}^{>};$$
$$N : \{(\delta, r) \in B_{\rho(f)} \times \mathbb{R}^{\geqslant} : r < R(\delta)\} \rightarrow \mathbb{R}^{>},$$

such that $\rho(f_{+\delta}) = R(\delta)$ and $|f_{+\delta}|_r = N(\delta, r)$ for all $\delta \in B_{\rho(f)} \cap$ Complex and $r \in$ Real with $0 \leqslant r < \rho(f_{+\delta})$. We say that f satisfies the *continuity condition* if:

EA2. $f_{+\gamma}$ satisfies the local continuity condition for each $\gamma \in \operatorname{Dom} f$.

We say that f is an *effective analytic function* if it satisfies both the homotopy condition and the continuity condition. In what follows, we will always assume that instances of the type AnFunc satisfy the conditions **EA1** and **EA2**.

Remark 3. In fact, there are several alternatives for the definition of effective analytic functions, by changing the homotopy and continuity conditions. The future will teach us which conditions are best suited for complex computations. Nevertheless, there is no doubt that the "spirit" of the definition should be preserved: quasi-effectiveness plus additional conditions which will allow us to prove global properties.

2.5. Analytic continuation along subdivided paths

The *extended domain* $\mathrm{Dom}^\sharp f$ of an effective analytic function f is the set of all paths γ such that there exists a subdivision γ' of γ with $\gamma' \in \mathrm{Dom}\, f$. For a tuple $f = (f_1, \ldots, f_n)$ of effective analytic functions, we also define $\mathrm{Dom}\, f = \mathrm{Dom}\, f_1 \cap \cdots \cap \mathrm{Dom}\, f_n$ and similarly for $\mathrm{Dom}^\sharp f$ and $\mathrm{Dom}\, \bar{f}$. We say that an effective analytic function f (or a tuple of such functions) is *faithful*, if $\mathrm{Dom}^\sharp f = \mathrm{Dom}^\sharp \bar{f} \cap \mathsf{Path}$. We say that a subset P of Path is *effective* if there exists an algorithm for deciding whether a given effective path γ belongs to Path.

Now let us choose constants $\lambda, \mu \in (0, 1) \cap \mathsf{Real}$ with $\lambda < \mu$ and consider an effective analytic function f. Then the following algorithm may be used in order to evaluate f at any path $\gamma \in \mathrm{Dom}^\sharp f$:

Algorithm evaluate-subdiv(f, γ)

Input: $f \in \mathsf{AnFunc}$ and $\gamma \in \mathsf{Path}$, such that $\gamma \in \mathrm{Dom}^\sharp f$
Output: $f(\gamma)$

Step 1. [Handle trivial cases]
If $\gamma = \bullet$, then return $f(0)$
Write $\gamma = [\delta] + \gamma'$
Compute an approximation $\tilde{\epsilon} \in \mathbb{Z}2^{\mathbb{Z}}$ of $\epsilon = \delta - \frac{1+\mu}{2}\rho(f)$ with $|\tilde{\epsilon} - \epsilon| < \frac{1-\mu}{2}\rho(f)$.
If $\tilde{\epsilon} < 0$ then return evaluate - subdiv$(f_{+\delta}, \gamma')$
Step 2. [Subdivide path]
Let $\delta' = \lambda|\rho(f)|\frac{\delta}{|\delta|}$
Return evaluate-subdiv$(f_{+\delta'}, [\delta - \delta'] + \gamma')$

We notice that $\tilde{\epsilon} < 0$ implies $\delta < \rho(f)$ and $\tilde{\epsilon} \geqslant 0$ implies $\delta > \mu\rho(f)$. The correctness proof of this algorithm relies on three lemmas:

Lemma 1. *Let* $\gamma_1, \gamma_2 \in \mathrm{Dom}\, f$ *with* $\gamma_1 \sqsubseteq \gamma_2$. *Then* $f_{+\gamma_1} \equiv f_{+\gamma_2}$.

Proof. Let us proof the lemma by induction over the difference d of the lengths of γ_2 and γ_1. If $d = 0$, then $\gamma_1 = \gamma_2$ and we are done. Otherwise, let γ'' be longest, such that there exist paths γ_1' and γ_2' and numbers δ_1, δ_2 and δ_3 with $\gamma_1 = \gamma_1' + [\delta_1] + \gamma''$ and $\gamma_2 = \gamma_2' + [\delta_2, \delta_3] + \gamma''$. If $d > 1$, then the induction hypothesis implies

$$f_{+\gamma_2} \equiv f_{+\gamma_2' + [\delta_2 + \delta_3] + \gamma''} \equiv f_{+\gamma_1}.$$

Otherwise, we have $\gamma_1' = \gamma_2'$ and $\delta_1 = \delta_2 + \delta_3$. Consequently, the homotopy condition implies that $f_{+\gamma_1' + [\delta_1]} \equiv f_{+\gamma_2' + [\delta_2, \delta_3]}$, whence $f_{+\gamma_1} \equiv f_{+\gamma_2}$. $\square$

Remark 4. In fact, the above lemma even holds for homotopic paths $\gamma_1, \gamma_2 \in \mathrm{Dom}\, f$, but we will not need this in what follows.

Lemma 2. *If* $\gamma, \gamma_1, \gamma_2 \in \mathrm{Dom}\, f$ *are such that* $\gamma \sqsubseteq \gamma_1$ *and* $\gamma \sqsubseteq \gamma_2$, *then* $\gamma_1 \sqcup \gamma_2 \in \mathrm{Dom}\, f$.

Proof. Let $\gamma_1 \sqcup \gamma_2 = [\delta'_1, \ldots, \delta'_l]$. Let us prove by induction over i that $\gamma'_i = [\delta'_1, \ldots, \delta'_i] \in \operatorname{Dom} f$. If $i = 0$, then we have nothing to do. Assume now that we have proved the assertion for a given $i < l$ and let us prove it for $i + 1$. Since $\gamma_1 \sqcup \gamma_2$ is the shortest common subdivision of γ_1 and γ_2, there exist a $k \in \{1, 2\}$ and a path $\gamma'' \lhd \gamma_k$ such that $\gamma'' \sqsubseteq \gamma'_i$. By Lemma 1, we have $\rho(f_{+\gamma'_i}) = \rho(f_{+\gamma''})$. Therefore, $|\delta'_{i+1}| \leqslant |\delta''| < \rho(f_{+\gamma'_i})$, where δ'' is such that $\gamma'' + [\delta''] \trianglelefteqslant \gamma_k$. This shows that $\gamma'_{i+1} \in \operatorname{Dom} f$, as desired. □

Lemma 3. *Let $\gamma \in \operatorname{Dom} f$. Then there exists a $\sigma > 0$ such that for any $\gamma' \in \operatorname{Dom} f$, with $\gamma' \trianglelefteqslant \gamma''$ for some $\gamma'' \sqsupseteq \gamma$, we have $\rho(f_{+\gamma'}) \geqslant \sigma$.*

Proof. Write $\gamma = [\delta_1, \ldots, \delta_l]$ and let $i \in \{1, \ldots, l\}$. The continuity condition implies that the function $\alpha \in [0, \delta_i] \cap \mathsf{Complex} \mapsto \rho(f_{+\delta_1,\ldots,+\delta_{i-1},+\alpha})$ is the restriction of a continuous function R on the compact set $[0, \delta_i]$. Consequently, there exists a lower bound $\sigma_i > 0$ for R on $[0, \delta_i]$. On the other hand, any $\gamma' \in \operatorname{Dom} f$, with $\gamma' \trianglelefteqslant \gamma''$ for some $\gamma'' \sqsupseteq \gamma$, is a subdivision of a path of the form $[\delta_1, \ldots, \delta_{i-1}, \alpha]$ with $\alpha \in [0, \delta_i]$. Taking $\sigma = \min\{\sigma_1, \ldots, \sigma_l\}$, we conclude by Lemma 1. □

Proof (**Proof of the Algorithm*). The algorithm is clearly correct if it terminates. Assume that the algorithm does not terminate for $\gamma = [\delta_1, \ldots, \delta_l]$ and let $\gamma' = [\delta'_1, \ldots, \delta'_{l'}]$ be a subdivision of γ in $\operatorname{Dom} f$. Let $\delta''_1, \delta''_2, \ldots$ be the sequence of increments such that $\mathsf{evaluate\text{-}subdiv}$ is called successively for f, $f_{+\delta''_1}$, $f_{+\delta''_1,+\delta''_2}$, …. Let $\sigma > 0$ be the constant we find when applying Lemma 3 to γ'. Since $\delta''_1 + \delta''_2 + \cdots \leqslant |\gamma|$, there exists an i with $|\delta''_{i+1}| < \lambda\sigma$.

Now let $\gamma'' = [\delta''_1, \ldots, \delta''_i]$ and let j be such that

$$|[\delta'_1, \ldots, \delta'_j]| < |\gamma''|$$

and

$$|[\delta'_1, \ldots, \delta'_{j+1}]| \geqslant |\gamma''|.$$

Then

$$\gamma''' = [\delta'_1, \ldots, \delta'_j, \delta'_{j+1} + \delta''_1 + \cdots + \delta''_i - \delta'_1 - \cdots - \delta'_j] \in \operatorname{Dom} f.$$

By Lemma 2, it follows that $\gamma'' \sqcup \gamma''' \in \operatorname{Dom} f$. Hence $\rho(f_{+\gamma''}) = \rho(f_{+(\gamma'' \sqcup \gamma''')}) \geqslant \sigma$. This yields the desired contradiction, since $|\delta''_{i+1}| = \lambda\rho(f_{+\gamma''}) \geqslant \lambda\sigma$.

3. Operations on effective analytic functions

In this section we will show how to effectively construct elementary analytic functions from the constants in $\mathsf{Complex}$ and the identity function z, using the operations $+, -, \cdot, /, \frac{\mathrm{d}}{\mathrm{d}z}, \int$, exp and log. In our specifications of the corresponding concrete data types which inherit from AnFunc, we will omit the algorithms for computing the coefficients of the series expansions, and refer the reader to van der Hoeven (2002b) for a detailed treatment of this matter.

3.1. Basic effective analytic functions

Constant effective analytic functions are implemented by the following concrete type ConstantAnFunc which derives from AnFunc (this is reflected through the $\triangleright$ symbol below):

Class ConstantAnFunc $\triangleright$ AnFunc

- $z \in$ Complex
- new : $\tilde{z} \in$ Complex $\mapsto z := \tilde{z}$
- radius : $() \mapsto \infty$
- norm : $r \mapsto |z|$
- continue : $\delta \mapsto$ ConstantAnFunc(z)

The method new is the constructor for ConstantAnFunc. In the method continue it is shown how to call the constructor. In a similar way, the following data type implements the identity function:

Class IdentityAnFunc $\triangleright$ AnFunc

- $z \in$ Complex
- new : $() \mapsto z := 0$
- new : $\tilde{z} \in$ Complex $\mapsto z := \tilde{z}$
- radius : $() \mapsto \infty$
- norm : $r \mapsto |z| + r$
- continue : $\delta \mapsto$ IdentityAnFunc$(z + \delta)$

The default constructor with zero arguments returns the identity function centered at 0. The other constructor with one argument z returns the identity function centered at z.

The conditions **EA1** and **EA2** are trivially satisfied by the constant functions and the identity function. They all have domain Path.

3.2. The ring operations

The addition of effective analytic functions is implemented as follows:

Class SumAnFunc $\triangleright$ AnFunc

- $f, g \in$ AnFunc
- new : $(\tilde{f} \in$ AnFunc$, \tilde{g} \in$ AnFunc$) \mapsto f := \tilde{f}; g := \tilde{g}$
- radius : $() \mapsto \min(\rho(f), \rho(g))$
- norm : $r \mapsto |f|_r + |g|_r$
- continue : $\delta \mapsto$ SumAnFunc$(f_{+\delta}, g_{+\delta})$

We clearly have $\mathrm{Dom}\,(f + g) = \mathrm{Dom}\, f \cap \mathrm{Dom}\, g$ and $(f + g)_\gamma \equiv f_\gamma + g_\gamma$ for all paths in $\mathrm{Dom}\,(f + g)$. Consequently, condition **EA1** is satisfied by $f + g$. Since min is a continuous function, condition **EA2** is also satisfied.

Subtraction is implemented in the same way as addition: only the series computation changes. Multiplication is implemented as follows:

Class ProductAnFunc $\triangleright$ AnFunc

- $f, g \in$ AnFunc
- new : $(\tilde{f} \in$ AnFunc$, \tilde{g} \in$ AnFunc$) \mapsto f := \tilde{f}; g := \tilde{g}$
- radius : $() \mapsto \min(\rho(f), \rho(g))$
- norm : $r \mapsto |f|_r |g|_r$
- continue : $\delta \mapsto$ ProductAnFunc$(f_{+\delta}, g_{+\delta})$

We again have $\operatorname{Dom}(fg) = \operatorname{Dom} f \cap \operatorname{Dom} g$ and the conditions **EA1** and **EA2** are verified in a similar way to in the case of addition.

3.3. *Differentiation and integration*

In order to differentiate an effective analytic function f, we have to be able to bound f' on each disk D_r with $r < \rho(f)$. Fixing a number $\lambda \in (0, 1) \cap$ Real, this can be done as follows:

Class DerAnFunc $\triangleright$ AnFunc

- $f \in$ AnFunc
- new : $\tilde{f} \in$ AnFunc $\mapsto f := \tilde{f}$
- radius : $() \mapsto \rho(f)$
- norm : $r \mapsto \frac{s}{(s-r)^2} |f|_s$, where $s := \rho(f) + \lambda(\rho(f) - r)$
- continue : $\delta \mapsto$ DerAnFunc$(f_{+\delta})$

Let us show that the bound for the norm is indeed correct. Given the bound $|f|_s$ for s on $\bar{D}_s$, Cauchy's formula implies that $|f_n| \leqslant |f|_s / s^n$ for all n. Consequently, for all $z \in \bar{B}_r$:

$$|f'(z)| = \left| \sum_{n=0}^{\infty} (n+1) f_{n+1} z^n \right| \leqslant \sum_{n=0}^{\infty} (n+1) |f|_s \frac{r^n}{s^{n+1}} = \frac{s}{(s-r)^2} |f|_s.$$

We have $\operatorname{Dom} f' = \operatorname{Dom} f$ and the fact that λ is a constant, which is fixed once and for all, ensures that condition **EA2** is again satisfied by f'. The actual choice of λ is a compromise between keeping $|f|_s$ as small as possible while keeping $s - r$ as large as possible.

Bounding the value of an integral on a disk is simpler, using the formula

$$\left| \int_0^{\delta} f(z) \right| \leqslant |\delta| \max_{z \in [0,d]} |f(z)|.$$

For the analytic continuation of integrals, we have to keep track of the integration constant, which can be determined using the evaluation algorithm from Section 2.2. In the algorithm below, this integration constant corresponds to c.

Class IntAnFunc $\triangleright$ AnFunc

- $f \in$ AnFunc
- $c \in$ Complex
- new : $\tilde{f} \mapsto f := \tilde{f}, c := 0$

- new : $(\tilde{f} \in \mathsf{AnFunc}, \tilde{c} \in \mathsf{Complex}) \mapsto f := \tilde{f}, c := \tilde{c}$
- radius : $() \mapsto \rho(f)$
- norm : $r \mapsto |c| + r|f|_r$
- continue : $\delta \mapsto \mathsf{IntAnFunc}(f_{+\delta}, \mathsf{this}(\delta))$

The domain of $\int_0^z f(t)\mathrm{d}t$ is the same as the domain of f.

3.4. Inversion

In order to compute the inverse $1/f$ of an effective analytic function f with $f(0) \neq 0$, we should in particular show how to compute a lower bound for the norm of the smallest zero. Moreover, this computation should be continuous as a function of the path. Again, let $\lambda \in (0, 1) \cap \mathsf{Real}$ be a parameter and consider

Class InvAnFunc $\triangleright$ AnFunc

- $f \in \mathsf{AnFunc}$
- new : $\tilde{f} \in \mathsf{AnFunc} \mapsto f := \tilde{f}$
- radius : $() \mapsto \min\left(\lambda\rho(f), \frac{|f(0)|}{|f'|_{\lambda\rho(f)}}\right)$
- norm : $r \mapsto \frac{1}{|f(0)|-|f'|_r r}$
- continue : $\delta \mapsto \mathsf{InvAnFunc}(f_{+\delta})$

Again, the choice of λ is a compromise between keeping $\lambda\rho(f)$ reasonably large, while keeping the bound $|f'|_{\lambda\rho(f)}$ as small as possible. We have

$$\mathrm{Dom}^\sharp \frac{1}{f} = \{\gamma \in \mathrm{Dom}^\sharp f \,|\, f(\gamma) \neq 0\}.$$

Notice that we cannot necessarily test whether $f(\gamma) = 0$. Consequently, $\mathrm{Dom}^\sharp \frac{1}{f}$ is not necessarily effective.

Remark 5. Instead of fixing a $\lambda \in (0, 1) \cap \mathsf{Real}$, it is also possible to compute λ such that

$$\lambda\rho(f) = \frac{|f(0)|}{|f'|_{\lambda\rho(f)}},$$

using a fast algorithm for finding zeros, like the secant method.

3.5. Exponentiation and logarithm

The logarithm of an effective analytic function f can be computed using the formula

$$\log f = \log f(0) + \int_0^z \frac{f'(t)}{f(t)}\mathrm{d}t.$$

As to exponentiation, we use the following method:

Class ExpAnFunc $\triangleright$ AnFunc

- $f \in \mathsf{AnFunc}$
- new : $\tilde{f} \in \mathsf{AnFunc} \mapsto f := \tilde{f}$

- radius : $() \mapsto \rho(f)$
- norm : $r \mapsto \exp |f|_r$
- continue : $\delta \mapsto \mathsf{ExpAnFunc}(f_{+\delta})$

We have $\mathrm{Dom}^\sharp (\log f) = \{\gamma \in \mathrm{Dom}^\sharp f \,|\, f(\gamma) \neq 0\}$ and $\mathrm{Dom}\, \mathrm{e}^f = \mathrm{Dom}\, f$.

4. Solving differential equations

In this section, we will show how to effectively solve linear and algebraic differential equations. As in the previous section, we will omit the algorithms for computing the series expansions and refer the reader to van der Hoeven (2002b). We will use the classical majorant technique from Cauchy and Kovalevskaya in order to compute effective bounds.

Given two power series $f, g \in \mathbb{C}[[z_1, \ldots, z_n]]$, we say that f is *majored* by g (see also van der Hoeven (2003)), and we write $f \trianglelefteq g$, if $g \in \mathbb{R}^{\geqslant}[[z_1, \ldots, z_n]]$ and $|f_{k_1,\ldots,k_n}| \leqslant g_{k_1,\ldots,k_n}$ for all $k_1, \ldots, k_n \in \mathbb{N}$. If $n = 1$ and $f \in \mathsf{AnFunc}$, then we write $f \trianglelefteq g$ if $\bar{f} \trianglelefteq g$. Given $\alpha > 0$, we also write $\mathbb{b}_\alpha = (1 - \alpha z)^{-1}$.

4.1. Linear differential equations

Let $L_0, \ldots, L_{l-1}$ be effective analytic functions and consider the equation

$$f^{(l)} = L_{l-1} f^{(l-1)} + \cdots + L_0 f \tag{1}$$

with initial conditions $f^{(0)}(0) = \nu_0, \ldots, f^{(l-1)}(0) = \nu_{l-1}$ in $\mathsf{Complex}$. We will show that the unique solution to this equation can again be represented by an effective analytic function f, with $\mathrm{Dom}\, f = \mathrm{Dom}\, L_0 \cap \cdots \cap \mathrm{Dom}\, L_{l-1}$. Notice that any linear differential equation of the form $L_l f^{(l)} + \cdots + L_0 f = g$ with $L_l(0) \neq 0$ can be reduced to the above form (using division by L_l, differentiation, and linearity if $g(0) = 0$).

We first notice that the coefficients of f may be computed recursively using the equation

$$f = \left(\nu_0 + \int\right) \cdots \left(\nu_{l-1} + \int\right) \left(L_{l-1} f^{(l-1)} + \cdots + L_0 f\right). \tag{2}$$

Assume that $M \in \mathbb{R}^{\geqslant}$ and $\alpha \in \mathbb{R}^{>}$ are such that $L_i \trianglelefteq M \mathbb{b}_\alpha$ for all i. Then the equation

$$\hat{f} = \left(\hat{\nu}_0 + \int\right) \cdots \left(\hat{\nu}_{l-1} + \int\right) \left(M \mathbb{b}_\alpha \hat{f}^{(l-1)} + \cdots + M \mathbb{b}_\alpha \hat{f}\right) + R, \tag{3}$$

is a majorant equation (von Kowalevsky, 1875; Cartan, 1961) of (2) for any choices of $\hat{\nu}_0, \ldots, \hat{\nu}_{l-1} \in \mathbb{R}^{\geqslant}$ and $R \in \mathbb{R}^{\geqslant}[[z]]$ such that $|\nu_0| \leqslant \hat{\nu}_0, \ldots, |\nu_{l-1}| \leqslant \hat{\nu}_{l-1}$. Let

$$h = \mathbb{b}_\alpha^{(M+1)/\alpha}.$$

We take $\hat{\nu}_i = C h^{(i)}(0)$ for all $i \in \{0, \ldots, l-1\}$, where

$$C = \max\{|\nu_0|, \ldots, |\nu_{l-1}|\} \geqslant \max \left\{ \frac{|\nu_0|}{h^{(0)}(0)}, \ldots, \frac{|\nu_{l-1}|}{h^{(l-1)}(0)} \right\}.$$

Now we observe that

$$\begin{aligned}
&M\mathbb{b}_\alpha h^{(l-1)} + \cdots + M\mathbb{b}_\alpha h \\
&\trianglelefteq M\left[(M+1)\cdots(M+(l-2)\alpha+1)\mathbb{b}_\alpha^{(M+l\alpha+1)/\alpha} + \cdots + \mathbb{b}_\alpha^{(M+\alpha+1)/\alpha}\right] \\
&\trianglelefteq (M+1)\cdots(M+(l-1)\alpha+1)\mathbb{b}_\alpha^{(M+l\alpha+1)/\alpha} \\
&= h^{(l)}.
\end{aligned}$$

Therefore, we may take

$$\begin{aligned}
R = &\left(\hat{\nu}_0 + \int\right)\cdots\left(\hat{\nu}_{l-1} + \int\right)\left(M\mathbb{b}_\alpha Ch^{(l-1)} + \cdots + M\mathbb{b}_\alpha Ch\right) \\
&- Ch \in \mathbb{R}^{\geqslant}[[z]].
\end{aligned}$$

This choice ensures that (3) has the particularly simple solution Ch. The majorant technique now implies that $f \trianglelefteq Ch$.

From the algorithmic point of view, let $\rho = \min\{\rho(L_0), \ldots, \rho(L_{l-1})\}$ and assume that we want to compute a bound for $|\bar{f}|$ on $\bar{B}_r$ for some $r < \rho$. Let $\lambda \in (0,1) \cap \mathsf{Real}$ be a fixed constant. Then we may apply the above computation for $\alpha = s^{-1}$ with $s = r + (\rho - r)\lambda$ and $M = \max\{|L_0|_s, \ldots, |L_{l-1}|_s\}$. From the majoration $f \trianglelefteq Ch$, we deduce in particular that

$$|f|_r \leqslant C\left(\frac{s}{s-r}\right)^{(M+1)s}.$$

This leads to the following effective solution of (1):

Class $\mathsf{LDEAnFunc} \rhd \mathsf{AnFunc}$

- $L = (L_0, \ldots, L_{l-1}) \in \mathsf{Array}(\mathsf{AnFunc})$
- $\nu = (\nu_0, \ldots, \nu_{l-1}) \in \mathsf{Array}(\mathsf{Complex})$
- $\mathsf{new} : (\tilde{L} \in \mathsf{Array}(\mathsf{AnFunc}), \tilde{\nu} \in \mathsf{Array}(\mathsf{Complex})) \mapsto L := \tilde{L}, \nu := \tilde{\nu}$
- $\mathsf{radius} : () \mapsto \min(\rho(L_0), \ldots, \rho(L_{l-1}))$
- $\mathsf{continue} : \delta \mapsto \mathsf{LDEAnFunc}((L_{0,+\delta}, \ldots, L_{l-1,+\delta}), (\mathsf{this}(\delta), \ldots, \mathsf{this}^{(l-1)}(\delta))$

Like in C++, the keyword this stands for the current instance of the data type, which is implicit to the method. The norm method is given by

Method $\mathsf{LDEAnFunc} \to \mathsf{norm}(r)$
$s := r + \lambda(\rho(\mathsf{this}) - r)$
$C := \max\{|\nu_0|, \ldots, |\nu_{l-1}|\}$
$M := \max\{|L_0|_s, \ldots, |L_{l-1}|_s\}$
Return $C\left(\frac{s}{s-r}\right)^{(M+1)s}$

We have proved the following theorem:

Theorem 2. *Let $L_0, \ldots, L_{l-1}$ be effective analytic functions and let $\nu_0, \ldots, \nu_{l-1} \in$ Complex. Then there exists a faithful effective analytic solution f to (1) with* $\mathrm{Dom}\, f = \mathrm{Dom}\,(L_1, \ldots, L_l)$. *In particular, if* $\mathrm{Dom}\,(L_1, \ldots, L_l)$ *is effective, then so is* $\mathrm{Dom}\, f$.

4.2. Systems of algebraic differential equations

Let us now consider a system of algebraic differential equations

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{pmatrix} f_1 \\ \vdots \\ f_l \end{pmatrix} = \begin{pmatrix} P_1(f_1, \ldots, f_l) \\ \vdots \\ P_l(f_1, \ldots, f_l) \end{pmatrix} \tag{4}$$

with initial conditions $f_1(0) = \nu_1, \ldots, f_l(0) = \nu_l$ in $\mathsf{Complex}$, where $P_1, \ldots, P_l$ are polynomials in l variables with coefficients in $\mathsf{Complex}$. Modulo the change of variables

$$\begin{aligned} f_i &= \nu_i + \tilde{f}_i \\ P_i(f_1, \ldots, f_l) &= \tilde{P}_i(\tilde{f}_1, \ldots, \tilde{f}_l) \end{aligned}$$

we obtain a new system

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{pmatrix} \tilde{f}_1 \\ \vdots \\ \tilde{f}_l \end{pmatrix} = \begin{pmatrix} \tilde{P}_1(\tilde{f}_1, \ldots, \tilde{f}_l) \\ \vdots \\ \tilde{P}_l(\tilde{f}_1, \ldots, \tilde{f}_l) \end{pmatrix}, \tag{5}$$

with initial conditions $\tilde{f}_1(0) = \cdots = \tilde{f}_l(0) = 0$.

Let $M > 0$ and $\alpha > 0$ be such that

$$\tilde{P}_i(z_1, \ldots, z_l) \trianglelefteqslant M \mathbb{b}_\alpha(z_1 + \cdots + z_l)$$

for all i. Then the system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{pmatrix} \hat{f}_1 \\ \vdots \\ \hat{f}_l \end{pmatrix} = \begin{pmatrix} \mathbb{b}_\alpha(\hat{f}_1 + \cdots + \hat{f}_l) \\ \vdots \\ \mathbb{b}_\alpha(\hat{f}_1 + \cdots + \hat{f}_l) \end{pmatrix} \tag{6}$$

with initial conditions $\hat{f}_1(0) = \cdots = \hat{f}_l(0) = 0$ is a majorant system of (5). The unique solution of this system therefore satisfies $\tilde{f}_i \trianglelefteqslant \hat{f}_i$ for all i. Now the $\hat{f}_i$ really all satisfy the same first order equation

$$\hat{f}_i' = M \mathbb{b}_{l\alpha}(\hat{f}_i)$$

with the same initial condition $\hat{f}_i(0) = 0$. The unique solution of this equation is

$$\hat{f}_1 = \cdots = \hat{f}_l = \frac{1 - \sqrt{1 - 2l\alpha M z}}{l\alpha},$$

which is a power series with radius of convergence

$$\rho = \frac{1}{2l\alpha M}$$

and positive coefficients, so that $|\hat{f}_i(z)| \leqslant \hat{f}_i(r)$ for any $r < \rho$.

As to the implementation, we may fix $\alpha = 1$. We will denote the transformed polynomials $\tilde{P}_i$ by $\tilde{P}_i = P_{i,+\nu}$. We will also write $\|P\|_1$ for the smallest number

$M \in \mathbb{R}^{\geqslant}$ with $P(z_1, \ldots, z_l) \trianglelefteq M \mathbb{b}_\alpha(z_1 + \cdots + z_l)$. The implementation uses a class $\mathsf{ADESystem}$ with information about the entire system of equations and solutions and a class $\mathsf{ADEAnFunc}$ for each of the actual solutions f_i.

Class $\mathsf{ADESystem}$

- $P = (P_1, \ldots, P_l) \in \mathsf{Array}(\mathsf{Polynomial}(\mathsf{Complex}, l))$
- $\nu = (\nu_1, \ldots, \nu_l) \in \mathsf{Array}(\mathsf{Complex})$
- $M \in \mathsf{Real}$
- $\mathsf{new} : (\tilde{P}, \tilde{\nu}) \mapsto P := \tilde{P}, \nu := \tilde{\nu}, M := \max(\|P_{1,+\nu}\|_1, \ldots, \|P_{l,+\nu}\|_1)$
- $\mathsf{component} : 1 \leqslant i \leqslant l \mapsto \mathsf{ADEAnFunc}(\mathsf{this}, i)$
- $\mathsf{continue} : \delta \mapsto \mathsf{ADESystem}(P, (\mathsf{component}(1)(\delta), \ldots, \mathsf{component}(l)(\delta))$

Class $\mathsf{ADEAnFunc} \rhd \mathsf{AnFunc}$

- $\Sigma \in \mathsf{ADESystem}$
- $i \in \{1, \ldots, l\}$
- $\mathsf{new} : (\tilde{\Sigma}, \tilde{\imath}) \mapsto \Sigma := \tilde{\Sigma}, i := \tilde{\imath}$
- $\mathsf{radius} : () \mapsto 1/(2l(\Sigma \to M))$
- $\mathsf{norm} : r \mapsto (1 - \sqrt{1 - 2lr(\Sigma \to M)})/l$
- $\mathsf{continue} : \delta \mapsto \Sigma \to \mathsf{continue}(\delta) \to \mathsf{component}(i)$

In contrast with the linear case, the domain of the solution $(f_1, \ldots, f_l)$ to (4) is not necessarily effective. Nevertheless, the solution is faithful:

Theorem 3. *Let $P_1, \ldots, P_l$ be polynomials with coefficients in* $\mathsf{Complex}$ *and let $\nu_1, \ldots, \nu_l \in$* $\mathsf{Complex}$. *Then the system* (4) *admits a faithful effective analytic solution $(f_1, \ldots, f_l)$.*

Proof. Let $\gamma = [\delta_1, \ldots, \delta_l] \in \operatorname{Dom} \overline{(f_1, \ldots, f_l)} \cap \mathsf{Path}$. Let us prove by induction over $i = \{0, \ldots, l\}$ that $[\delta_1, \ldots, \delta_i] \in \operatorname{Dom}^\sharp (f_1, \ldots, f_l)$. For $i = 0$ we have nothing to prove, so assume that $i > 0$. For all $t \in [0, 1]$, let

$$\begin{aligned} \gamma_{;t} &= [\delta_1, \ldots, \delta_{i-1}, t\delta_i] \\ \nu_{;t} &= (\overline{f_1}(\gamma_{;t}), \ldots, \overline{f_l}(\gamma_{;t})) \\ M_{;t} &= \max(\|P_{1,+\nu_{;t}}\|_1, \ldots, \|P_{l,+\nu_{;t}}\|_1). \end{aligned}$$

Then $M_{;t}$ is a continuous function, which admits a global maximum $M_{;[0,1]}$ on $[0, 1]$. Now let $\gamma' \sqsupseteq [\delta_1, \ldots, \delta_{i-1}]$ be such that $\gamma' \in \operatorname{Dom} f$ and let $\delta' = (\delta_i - \delta_{i-1})/n$ be such that $n \in \mathbb{N}$ and $|\delta'| < 1/(2lM_{;[0,1]})$. Then we have $\gamma'' = \gamma' + [\delta', \overset{n\times}{\ldots}, \delta'] \in \operatorname{Dom} f$ and $\gamma'' \sqsupseteq [\delta_1, \ldots, \delta_i]$. This proves that $[\delta_1, \ldots, \delta_i] \in \operatorname{Dom}^\sharp (f_1, \ldots, f_l)$ and we conclude by induction. $\square$

Remark 6. In principle, it is possible to replace the algebraic differential equations by more general non-linear differential equations, by taking convergent power series for the P_i. However, this would require the generalization of the theory in this paper to analytic functions in several variables (interesting exceptions are power series which are polynomial in all but one variable, or entire functions). One would also need to handle the transformations $P_i \mapsto P_{i,+\nu}$ with additional care; these transformations really correspond to the analytic continuation of the P_i.

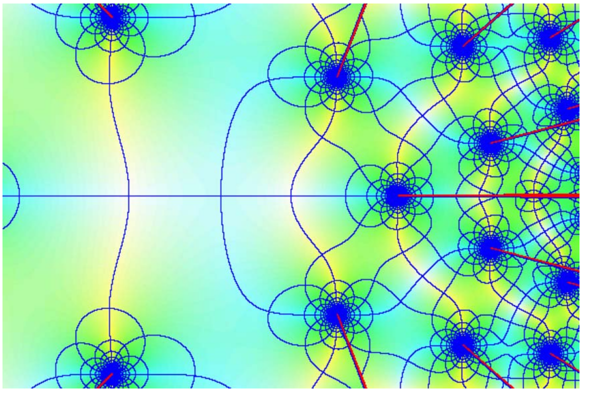

Fig. 1. Plot of a solution to $f'' = 1 + 4f^3 + f'$ with the COLUMBUS program.

5. Conclusion and final remarks

Using a careful definition of effective analytic functions, we have shown how to answer many numerical problems about analytic solutions to differential equations. In order to generalize the present theory to analytic functions in several variables or more general analytic functions, like solutions to convolution equations, we probably have to weaken the conditions **EA1** and **EA2**. Nevertheless, it is plausible that further research will lead to a more suitable definition which preserves the spirit of the present one.

The COLUMBUS program implements the approach of the present paper in the weaker setting of double precision complex numbers instead of effective complex numbers. We plan to describe this program in more detail in a forthcoming paper and in particular the "radar algorithm" which is used to graphically represent analytic functions (see Fig. 1). It would be nice to adapt or reimplement the COLUMBUS program so as to permit computations with effective complex numbers when desired.

The implementation of the COLUMBUS program has also been instructive for understanding the complexity of our algorithms. For instance, the lower bound for the smallest zero in our algorithm for inversion can be extremely bad, as in the example

$$f = \frac{1}{1 - \exp(n - z)}.$$

The problem here is that the effective radius of convergence is of the order of 1, while the real radius is n. Consequently, the analytic continuation from 0 to $n - 1$ will take $O(n)$ steps instead of only 1. In the COLUMBUS program, this problem has been solved by using a numerical algorithm in order to determine the radius of convergence instead of the theoretically correct one. In some cases, this leads to an exponential speed-up. Theoretically correct approaches for solving the problem are computing the smallest zero of the denominator in an appropriate radius or using transseries-like expansions. We plan to explain these approaches in more detail in a forthcoming paper.

Another trick which can be used in concrete implementations is overriding the default evaluation method from Section 2.5 with a more efficient one when possible, or implementing methods for the evaluation or computation of higher derivatives.

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