Petri Nets, Commutative Context-Free Grammars, and Basic Parallel Processes

Javier Esparza
Institut für Informatik
Technische Universität München*

1 Introduction

The reachability problem plays a central rôle in Petri net theory, and has been studied in numerous papers (see [5] for a comprehensive list of references). In the first part of this paper we study it for the nets in which every transition needs exactly one token to occur. Following [8], we call them communication-free nets, because no cooperation between places is needed in order to fire a transition; every transition is activated by one single token, and the tokens may flow freely through the net independently of each other. We obtain a structural characterisation of the set of reachable markings of communication-free nets, and use it to prove that the reachability problem for this class is NP-complete. Another consequence of the characterisation is that the set of reachable markings of communication-free nets is effectively semilinear (this same result has been proved for many other net classes, see again [5] for a survey).

In the second part of the paper we apply the results of the first part to two different problems in the areas of formal languages and process algebras.

The first problem concerns commutative grammars. Huynh proved in [12] the NP-completeness of the uniform word problem for commutative context-free grammars. The proof is rather involved (8 journal pages). It is easy to see that this problem coincides with the reachability problem for communication-free nets. Therefore, our results lead immediately to a new and considerably shorter proof of Huynh's result. In passing, we also derive a new proof of Parikh's theorem [6].

The second problem concerns the decidability of process equivalences for infinite-state systems (see [9, 5] for a survey of results in this area). Strong bisimulation equivalence [14] has been shown to be decidable for the processes of Basic Process Algebra (BPA) [2], and the Basic Parallel Processes (BPP) [3], a natural subset of Milner's CCS. Since weak bisimulation is more useful than strong bisimulation for verification problems, it is natural to ask about the decidability of weak bisimulation. Using our results, we prove that weak bisimulation is at least semidecidable for BPPs.

$\star$ Part of this work was done while the author was at the Laboratory for Foundations of Computer Science, University of Edinburgh.

2 Petri nets and labelled Petri nets

For the purposes of this paper it is convenient to describe Petri nets using some notations on monoids. Given a finite alphabet $V = \{v_1, \ldots, v_n\}$, the symbols $V^\star$, $V^\oplus$ denote the free monoid and free commutative monoid generated by V, respectively. Given a word w of $V^\star$ or $V^\oplus$, and an element v of V, $w(v)$ denotes the number of times that v appears in w. A word w of $V^\oplus$ will be represented in two different ways:

- as a multiset of elements of V (for instance, $\{v_1, v_1, v_2\}$ is the word containing two copies of v_1 and one of v_2);
- as the vector $(w(v_1), \ldots, w(v_n))$.

The context will indicate which representation is being used at each moment. The *Parikh mapping* $\mathcal{P}: V^\star \to V^\oplus$ is defined by $\mathcal{P}(w) = (w(v_1), \ldots, w(v_n))$. Given $u, v \in V^\oplus$, $u + v$ denotes the concatenation of u and v, which corresponds to addition of multisets or sum of vectors.

A *net* is a triple $N = (S, T, W)$, where S is a set of *places*, T is a set of *transitions*, and $W: (S \times T) \cup (T \times S) \to I\!N$ is a *weight function*. Graphically, places are represented by circles, and transitions by boxes. If $W(x, y) > 0$, then there is an arc from x to y labeled by $W(x, y)$ (when $W(x, y) = 1$, the arc is not labeled for clarity). We denote by ${}^\bullet x$ the set $\{y \mid W(y, x) > 0\}$ and by $x^\bullet$ the set $\{y \mid W(x, y) > 0\}$. For a set X, ${}^\bullet X$ ($X^\bullet$) denotes the union of ${}^\bullet x$ ($x^\bullet$) for every element x of X. An element of $S^\oplus$ is called a *marking* of N. A marking M is graphically represented by putting $M(s)$ *tokens* (black dots) in each place s. A *Petri net* is a pair (N, M_0), where N is a net and M_0 is a marking of N called the *initial marking*. A marking M of a net $N = (S, T, W)$ *enables* a transition t if $M(s) \geq W(s, t)$ for every place $s \in {}^\bullet t$. If t is enabled at M, then it can *occur*, and its occurrence leads to the marking M', given by $M'(s) = M(s) + W(t, s) - W(s, t)$ for every place s. This is denoted by $M \xrightarrow{t} M'$. Given $\sigma = t_1 t_2 \ldots t_n$, $M \xrightarrow{\sigma} M'$ denotes that there exist markings $M_1, M_2, \ldots, M_{n-1}$ such that $M \xrightarrow{t_1} M_1 \xrightarrow{t_2} M_2 \ldots M_{n-1} \xrightarrow{t_n} M'$ and we say that M' is *reachable* from M. We denote by $M \xrightarrow{\sigma}$ (read "σ is enabled at M") the fact that $M \xrightarrow{\sigma} M'$ for some marking M'. The *reachability problem* for a class of Petri nets consists of deciding, given a Petri net (N, M_0) of the class and a marking M of N, if M is reachable form M_0.

In Section 5 we shall consider labelled Petri nets. A *labelled net* is a fourtuple (S, T, W, l), where (S, T, W) is a net and $l: T \to Act$ is a labelling function on a set Act of actions. Markings, enabledness and ocurrence of transitions are defined as for unlabelled nets. $M \xrightarrow{a} M'$ denotes that $M \xrightarrow{t} M'$ for some transition t such that $l(t) = a$.

3 Communication-free Petri nets

aka BPP
Commutative CFG

Definition 1. Communication-free Petri Nets
A net $N = (S, T, W)$ is *communication-free* if $|{}^\bullet t| = 1$ for every $t \in T$, and $W(s,t) \leq 1$ for every $s \in S$ and every $t \in T$. A Petri net (N, M_0) is communication-free if N is communication-free.

We characterise structurally the set $\{\mathcal{P}(\sigma) \mid M_0 \xrightarrow{\sigma}\}$ of a communication-free Petri net (N, M_0). We need the notions of siphon and subnet generated by a set of transitions.

A subset of places R of a net is a *siphon* if ${}^\bullet R \subseteq R^\bullet$. A nonempty siphon is said to be *proper*. A siphon is *unmarked* at a marking M if none of its places is marked at M. It follows immediately from the definition of the occurrence rule for Petri nets that if a siphon is unmarked at a marking, then it remains unmarked, i.e., it cannot become marked by the occurrence of transitions.

Given a net $N = (S, T, W)$ ans a subset of transitions U, the *subnet of N generated by U* is $({}^\bullet U \cup U^\bullet, U, W_U)$, where W_U is the restriction of W to the pairs (x, y) such that x or y is a transition of U. Given $X \in T^\oplus$, the net N_X is defined as the subnet of N generated by the set of transitions that appear in X.

We can now state the characterisation of the set $\{\mathcal{P}(\sigma) \mid M_0 \xrightarrow{\sigma}\}$.

Theorem 2.
Let (N, M_0) be a communication-free Petri net, where $N = (S, T, W)$, and let $X \in T^\oplus$. There exists a sequence $\sigma \in T^*$ such that $M_0 \xrightarrow{\sigma}$ and $\mathcal{P}(\sigma) = X$ iff

(a) $M_0(s) + \sum_{t \in T} (W(t,s) - W(s,t)) \cdot X(t) \geq 0$ *for every $s \in S$, and*

(b) every proper siphon of N_X is marked at M_0.

The proof is omitted for lack of space. It would take about two pages of this volume. We illustrate the result by means of an example. Figure 1 shows a communication-free net. There is no occurrence sequence with Parikh mapping $X = (1, 0, 0, 1)$, because for the place s_5 we have

$$M_0(s_5) + \sum_{t \in T} (W(t, s_5) - W(s_5, t)) \cdot X(t) = -X(t_4) = -1$$

and therefore (a) does not hold. There is no occurrence sequence with Parikh mapping $X = (0, 1, 1, 0)$ either. In this case, (a) holds, but not (b): the net N_X contains the transitions t_2, t_3, and the places s_2, s_3, s_4, s_5, and so the set $\{s_2, s_5\}$ is a proper siphon of N_X unmarked at M_0. Finally, the reader may check that $(1, 1, 0, 1)$ satisfies both (a) and (b), and in fact $t_1\, t_2\, t_4$ is an occurrence sequence having this Parikh mapping.

We easily derive the following two results:

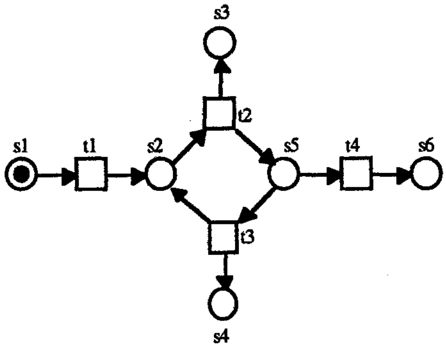


Fig. 1 Illustration of Theorem 2

Theorem 3.
The reachability problem for communication-free Petri nets is NP-complete.

SAT

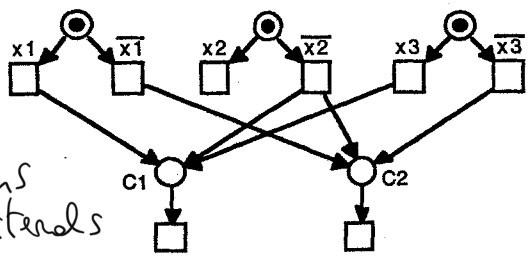
Proof: NP-hardness follows from a straightforward reduction from the satisfiability problem for boolean formulae in conjunctive normal form. Given such a formula F, we construct a communication-free Petri net (N, M_0) and a marking M such that F is satisfiable iff M is reachable from M_0. Figure 2 shows the communication-free Petri net corresponding to the formula $C_1 \wedge C_2$, where

$$C_1 = x_1 \vee \overline{x_2} \vee x_3 \quad C_2 = \overline{x_1} \vee \overline{x_2} \vee \overline{x_3}$$

To prove membership in NP, use the following nondeterministic algorithm,

□: transitions
○: places

C_1 contains as many tokens as satisfied literals



For CFG it is NP-complete if a word can be derived containing at least one occurrence of each C_i's.

Fig. 2 Communication-free Petri net for the formula of the text. The formula is satisfiable iff the marking $\{C_1, C_2\}$ is reachable

(Even for DFA!)

whose correctness follows immediately from Theorem 2. Given a communication-free Petri net (N, M_0) and a marking M, guess a set U of transitions of N. Check in polynomial time whether every proper siphon of N_U is marked at M_0. For that, let N'_U be the net obtained from N_U by removing all places marked by M_0, and use the following greedy algorithm, due to Starke [15], to compute the largest siphon R of N'_U, (clearly, every proper siphon of N_U is marked at M_0 iff $R = \emptyset$):

Input: The net $N'_U = (S, U, W)$.
Output: $R \subseteq S$, the largest siphon of N'_U.
Initialization: $R = S$.

begin
  while there exists $s \in R$ and $t \in s^\bullet$ such that $t \notin {}^\bullet R$ **do**
    $R := R - \{s\}$
  endwhile
end

Now, construct the system of linear equations containing for each place s the equation

$$M(s) = M_0(s) + \sum_{t \in T} (W(t, s) - W(s, t)) \cdot X(t)$$

Guess in polynomial time the smallest solution $X \geq 0$ of the system which satisfies $X(t) = 0$ for every $t \notin U$ (the smallest solution has polynomial size in the input due to the result of von zur Gathen and Sieveking [11]). ■

A subset $\mathcal{X}$ of a free commutative monoid $V^{\oplus}$ is *linear* if there exist elements $v, v_1, \ldots, v_n \in V^{\oplus}$ such that

$$\mathcal{X} = \{v + a_1 v_1 + \ldots + a_n v_n \mid a_1, \ldots, a_n \geq 0\}$$

$\mathcal{X}$ is *semilinear* if it is a finite union of linear sets. We have:

Theorem 4.
The set of reachable markings of a communication-free Petri net is effectively semilinear.

Proof: Let (S, T, W, M_0) be a communication-free Petri net. It is easy to see that the set of reachable markings of (N, M_0) is

$$\{M_0 + C \cdot \mathcal{P}(\sigma) \mid M_0 \xrightarrow{\sigma}\}$$

where C is the matrix given by $C(s, t) = W(s, t) - W(t, s)$. By Theorem 2, this set is the union of the sets of solutions of a finite number of systems of linear equations. Since the set of solutions of a system is semilinear, the result follows. ■

4 Context-free and commutative context-free grammars

A *context-free grammar* is a 4-tuple $G = (Non, Ter, A, P)$, where *Non* and *Ter* are disjoint sets, called the sets of nonterminals and terminals, respectively, A is an element of *Non* called the *axiom*, and P is a finite subset of $Non \times (Non \cup Ter)^*$, called the set of *productions*. The *language* $L(G)$ of a context-free grammar G is defined as usual. A *commutative context-free grammar* (or ccf-grammar for short) is a 4-tuple $G^c = (Non, Ter, A, P^c)$, where *Non*, *Ter*, and A are as above, and $P^c \subseteq Non \times (Non \cup Ter)^{\oplus}$. That is, free monoids are replaced by free commutative monoids. Given two commutative words $\alpha, \beta \in (Non \cup Ter)^{\oplus}$, α *directly generates* β, written $\alpha \longrightarrow \beta$, if $\alpha = \alpha_1 + \gamma$, $\beta = \alpha_1 + \delta$, and $(\gamma, \delta) \in P^c$. α *generates* β if $\alpha \xrightarrow{*} \beta$, where $\xrightarrow{*}$ denotes the reflexive and transitive closure of $\longrightarrow$.

The following ccf-grammar with $Non = \{A, B\}$ and $Ter = \{a, b, c\}$ generates the language $\{\{a, b^{2n}, c^n\} \mid n \geq 0\}$.

$$A \rightarrow \{a\} \quad A \rightarrow \{b, b, A, B\} \quad B \rightarrow \{c\}$$

We define a mapping which assigns to a ccf-grammar $G^c = (Non, Ter, A, P^c)$ a Petri net (S, T, W, M_0). The Petri net of Figure 3 is the one assigned to the grammar above. The reader can possibly guess the definition of the Petri net from

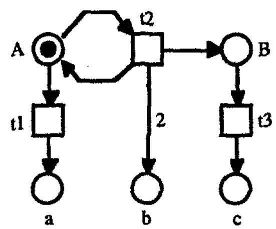


Fig. 3 A Petri net

this example: S is the set $Non \cup Ter$, i.e., there is a place for each terminal and for each nonterminal; T is the set P^c, i.e., there is a transition for each production. Given a place s and a transition $t = (X, w)$, the weight $W(s, t)$ is 1 if $s = X$ and 0 otherwise, whereas the weight $W(t, s)$ is the number of times that s appears in the commutative word w. Finally, M_0 is the marking that puts one token on the axiom A and no tokens on the rest. It follows directly from this description that commutative context-free grammars are assigned communication-free Petri nets.

Every word w of $(Non \cup Ter)^{\oplus}$ is a marking of the net (S, T, W). The application of a production corresponds to the occurrence of a transition. In particular, the relation $\xrightarrow{*}$ on words of $(Non \cup Ter)^{\oplus}$ corresponds to the reachability relation on markings.

The uniform word problem for commutative context-free grammars. The uniform word problem for commutative grammars is the problem of deciding, given a grammar $G^c = (Non, Ter, A, P^c)$ and a commutative word w of terminals, if $A \xrightarrow{*} w$. It follows immediately from our description of the net assigned to G^c that $A \xrightarrow{*} w$ iff w is a reachable marking of the Petri net translation of G^c. So the uniform word problem for ccf-grammars can be reduced in linear time to the reachability problem for communication-free Petri nets, and vice versa. Therefore, we have a new proof for the following result of [12]:

Theorem 5. *[12]*
The uniform word problem for commutative context-free grammars is NP-complete. ∎

The connection between commutative context-free grammars and Petri nets was pointed out in [12], but not used. The proof of membership in NP of [12] takes 8 journal pages, and is rather involved. Our proof is shorter[2], and it uses only standard techniques of net theory.

Parikh's Theorem. As a second consequence of Theorem 2, we obtain a new proof of Parikh's Theorem.

Theorem 6.
Let $G = (Non, Ter, A, P)$ be a context-free grammar. The set $\{\mathcal{P}(w) \in Ter^{\oplus} \mid w \in L(G)\}$ is semilinear.

Proof: Let $G^c = (Non, Ter, A, P^c)$ be the commutative grammar obtained after replacing the productions of G by their commutative versions. We have:

- if $w \in L(G)$, then $\mathcal{P}(w) \in L(G^c)$;
- if $w \in L(G^c)$, then there exists $w' \in L(G)$ such that $w = \mathcal{P}(w')$.

Let (N, M_0) be the Petri net assigned to G^c. We have $\mathcal{P}(w) \in L(G^c)$ iff $\mathcal{P}(w)$, seen as a marking of N, is reachable from M_0. Therefore, $\mathcal{P}(L(G))$ is the set of reachable markings of (N, M_0) that only put tokens in the places corresponding to terminals. This latter condition can be expressed using linear equations. The result follows now from Theorem 4. ∎

We do not claim this proof to be simpler than Parikh's proof, which is rather straightforward, and takes little more than 3 pages in [6]. However, it shows still another connection between Petri nets and formal language theory.

5 Weak bisimilarity in Basic Parallel Processes

We show that communication-free Petri nets are also strongly related to Basic Parallel Processes (BPPs), a subset of CCS.

Basic Parallel Process expressions are generated by the following grammar:

$$\begin{array}{rll} E ::= & \mathbf{0} & \text{(inaction)} \\ \mid & X & \text{(process variable)} \\ \mid & aE & \text{(action prefix)} \\ \mid & E + E & \text{(choice)} \\ \mid & E \parallel E & \text{(merge)} \end{array}$$

where a belongs either to a set of *atomic actions* Act or is the *silent action* τ. A BPP is a family of recursive equations $\mathcal{E} = \{X_i \stackrel{\text{def}}{=} E_i \mid 1 \le i \le n\}$, where the

2 The comparison of the lengths is fair, because both proofs use the result of von zur Gathen and Sieveking.

X_i are distinct and the E_i are BPP expressions at most containing the variables $\{X_1, \ldots, X_n\}$. We further assume that every variable occurrence in the E_i is *guarded*, that is, appears within the scope of an action prefix. The variable X_1 is singled out as the *leading variable* and $X_1 \stackrel{\text{def}}{=} E_1$ is the *leading equation.*

Any BPP determines a labelled transition system, whose transition relations $\xrightarrow{a}$ are the least relations satisfying the following rules:

$$aE \xrightarrow{a} E \qquad \frac{E \xrightarrow{a} E'}{E + F \xrightarrow{a} E'} \qquad \frac{E \xrightarrow{a} E'}{E \parallel F \xrightarrow{a} E' \parallel F'}$$

$$\frac{E \xrightarrow{a} E'}{X \xrightarrow{a} E'} (X \stackrel{\text{def}}{=} E) \qquad \frac{F \xrightarrow{a} F'}{E + F \xrightarrow{a} F'} \qquad \frac{F \xrightarrow{a} F'}{E \parallel F \xrightarrow{a} E \parallel F'}$$

The states of the transition system are the BPP expressions E that satisfy $X_1 \xrightarrow{w} E$ for some string w of actions.

Let $\mathcal{E}$ and $\mathcal{F}$ be two BPPs with disjoint sets of variables. Consider the labelled transition system obtained by putting the transition systems of $\mathcal{E}$ and $\mathcal{F}$ side by side. A binary relation $\mathcal{R}$ between the states of this labelled transition system is a weak bisimulation if whenever $E\mathcal{R}F$ then, for each $a \in Act$,

- if $E \xrightarrow{a} E'$ then $F \stackrel{a}{\Longrightarrow} F'$ for some F' with $E'\mathcal{R}F'$;
- if $F \xrightarrow{a} F'$ then $E \stackrel{a}{\Longrightarrow} E'$ for some E' with $F'\mathcal{R}E'$;

and, moreover

- if $E \xrightarrow{\tau} E'$ then $F \stackrel{\epsilon}{\Longrightarrow} F'$ for some F' with $E'\mathcal{R}F'$;
- if $F \xrightarrow{\tau} F'$ then $E \stackrel{\epsilon}{\Longrightarrow} E'$ for some E' with $F'\mathcal{R}E'$,

where $\stackrel{a}{\Longrightarrow} = (\xrightarrow{\tau})^* \xrightarrow{a} (\xrightarrow{\tau})^*$, and $\stackrel{\epsilon}{\Longrightarrow} = (\xrightarrow{\tau})^*$.

$\mathcal{E}$ and $\mathcal{F}$ are weakly bisimilar if their leading variables are related by some weak bisimulation.

A BPP is in *normal form* if every expression E_i on the right hand side of an equation is of the form $a_1\alpha_1 + \ldots + a_n\alpha_n$, where for each i the expression α_i is a merge of variables. It is proved in [1] that every BBP is weakly (even strongly) bisimilar to a BPP in normal form, which can be effectively (and efficiently) constructed (the proof is very similar to that for Greibach normal form and context-free grammars). Therefore, the problem of deciding strong or weak bisimilarity for BPPs can be reduced to the same problem for BPPs in normal form.

Every BPP in normal form can be translated into a labelled communication-free Petri net. The translation is graphically illustrated by means of an example in Figure 4. The net has a place for each variable X_i. For each subexpression $a_j\alpha_j$ in the defining equation of X_i, a transition is added having the place X_i in its preset, and the variables that appear in α_j in its postset. If a variable appears n times in α_j, then the arc leading to it is given the weight n. The transition is labelled by a_j.

It follows easily from the rules of the operational semantics that every state E of a BPP $\{X_i \stackrel{\text{def}}{=} E_i \mid 1 \leq i \leq n\}$ in normal form can be written as

$$X_1^{i_1} \parallel X_2^{i_2} \parallel \ldots \parallel X_n^{i_n}$$

X=a(X‖Y)+b(Y‖Y)
Y=b(X‖Y)

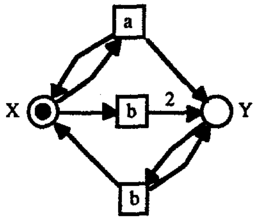


Fig. 4 A BPP and its corresponding Petri net

where $X^i = \underbrace{X \parallel \ldots \parallel X}_{i}$. Such a state corresponds to the marking $X_1^{i_1} X_2^{i_2} \cdots X_n^{i_n}$ of the labelled Petri net assigned to $\mathcal{E}$, and vice versa. Moreover, if M_E and $M_{E'}$ are the markings corresponding to the states E and E', then $E \xrightarrow{a} E'$ iff $M_E \xrightarrow{a} M_{E'}$. So the transition system of a BPP in normal form and the reachability graph of its labelled Petri net are isomorphic. It follows that two BPPs are weakly bisimilar iff their corresponding labelled communication-free Petri nets are weakly bisimilar, where weak bisimilarity for labelled Petri nets is defined as for BPPs, just replacing the states of the labelled transition systems by markings, and the relations $\xrightarrow{a}$ between BPP expressions by the corresponding reachability relations between markings.

5.1 Weak bisimilarity is semidecidable for BPPs

We give a positive test of weak bisimilarity for labelled communication-free Petri nets. We first observe that it suffices to consider instances of the problem in which the two Petri nets have the same underlying nets, i.e., they differ only in the initial markings. To prove it, given two labelled communication-free Petri nets (N_1, M_{01}) and (N_2, M_{02}), we construct another two, (N, M'_{01}) and (N, M'_{02}) as follows. N is the result of putting N_1 and N_2 side by side. M'_{01} coincides with M_{01} on N_1, and puts no tokens on N_2. M'_{02} coincides with M_{02} on N_2, and puts no tokens on N_1. Clearly, (N, M'_{01}) and (N, M'_{02}) are weakly bisimiliar iff (N_1, M_{01}) and (N_2, M_{02}) are weakly bisimilar.

In the sequel we fix a unique labelled communication-free net $N = (S, T, W, l)$, and study the weak bisimilarity of two initial markings M_{01} and M_{02} of N.

It follows immediately from the definitions that the union of two bisimulations is a bisimulation. Therefore, there exists a unique maximal weak bisimulation between the markings of N. Clearly, (N, M_{01}) and (N, M_{02}) are weakly bisimilar iff the pair (M_{01}, M_{02}) belongs to the maximal weak bisimulation. We study the properties of the maximal weak bisimulation. Given a commutative monoid $A^{\oplus}$, an equivalence relation $\mathcal{R} \subseteq A^{\oplus} \times A^{\oplus}$ is a *congruence* iff for every $x, y, z \in A^{\oplus}$ $(x, y) \in \mathcal{R}$ implies $(x + z, y + z) \in R$. We have:

Lemma 7.
The maximal weak bisimulation between markings of a labelled communication-free net is a congruence.

Proof: It is easy to see that the maximal weak bisimulation is an equivalence relation. It remains to prove that if M_1 is weakly bisimilar to M_2, then M_1+M is weakly bisimilar to M_2+M for every marking M. Let $\mathcal{R}$ be a weak bisimulation containing (M_1, M_2). The relation $\mathcal{R}' = \{(M_1' + M, M_2' + M) \mid (M_1', M_2') \in \mathcal{R}, M \in S^{\oplus}\}$ contains $(M_1 + M, M_2 + M)$ for every marking M. We show that $\mathcal{R}'$ is a weak bisimulation.

Let $(M_1' + M, M_2' + M)$ be a pair of $\mathcal{R}'$. If $M_1' + M \xrightarrow{a} M''$, then, since a transition needs only one token to occur, we have two possible cases:

(1) $M_1' \xrightarrow{a} M_1''$ and $M'' = M_1'' + M$.
Since $(M_1', M_2') \in \mathcal{R}$, there exists M_2'' such that $M_2' \overset{a}{\Longrightarrow} M_2''$ and $(M_1'', M_2'') \in R$. Then $M_2' + M \overset{a}{\Longrightarrow} M_2'' + M$ and $(M_1'' + M, M_2'' + M) \in \mathcal{R}'$.

(2) $M \xrightarrow{a} M'$ and $M'' = M_1' + M'$.
Then $M_2' + M \overset{a}{\Longrightarrow} M_2' + M'$ and $(M_1' + M', M_2' + M') \in \mathcal{R}'$.

The other direction and the case $M_1' + M \xrightarrow{\tau} M''$ are proved similarly. ■

We can now apply (following [13]) the following result of [4],

Theorem 8. *[4]*
Every congruence on a commutative monoid $A^{\oplus}$ is a semilinear subset of $A^{\oplus} \times A^{\oplus}$. ■

By this theorem, the maximal weak bisimulation between markings of a comunication-free net is a semilinear subset of $S^{\oplus} \times S^{\oplus}$, where S is the set of places of the net.

The set of semilinear relations on a commutative monoid is recursively enumerable. Let $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 \ldots$ be an effective enumeration of this set. We have to check if some $\mathcal{S}_i$ is a weak bisimulation containing (M_1, M_2).

Whether $(M_1, M_2) \in \mathcal{S}_i$ or not can be decided by solving a set of linear diophantine equations, which can be done in nondeterministic polynomial time.

In order to decide whether $\mathcal{S}_i$ is a weak bisimulation we need some well-known results about semilinear sets and Presburger arithmetic. Presburger arithmetic is the first order theory of addition. More precisely, formulae of Presburger arithmetic are built out of variables, quantifiers, and the symbols $0, \leq, +$. Formulae are interpreted on the natural numbers, and the symbols are interpreted as the number 0, the natural total order on $\mathbb{N}$, and addition.

A subset A of $\mathbb{N}^n$ is *expressible* in Presburger arithmetic if there exists a Presburger formula $\mathrm{A}(x_1, \ldots, x_k)$ with free variables $x_1, \ldots, x_n$ such that for every $(n_1, \ldots, n_k) \in \mathbb{N}^n$, the closed formula $\mathrm{A}(n_1, \ldots, n_k)$ is true if and only if $(n_1, \ldots, n_k) \in A$. We say that $\mathrm{A}(x_1, \ldots, x_k)$ *expresses* A.

Ginsburg and Spanier obtain in [7] the following result:

Theorem 9. *[7]*
A subset $A \subseteq \mathbb{N}^n$ is semilinear iff it is expressible in Presburger arithmetic. Moreover, the transformations between semilinear sets and formulae are effective. ■

Presburger's classical theorem can be seen as a consequence of this result:

Theorem 10.
It is decidable if a formula of Presburger arithmetic is true. ■

With the help of these theorems and Theorem 2 we can finally prove:

Theorem 11.
Let $N = (S, T, W, l)$ be a labelled communication-free net, and let $\mathcal{R} \subseteq S^{\oplus} \times S^{\oplus}$ be a semilinear relation. It is decidable if $\mathcal{R}$ is a weak bisimulation.

Proof: The following relations on the markings of N are effectively semilinear:

(1) $\{(M, M') \mid M \xrightarrow{a} M'\}$.
The definition of $M \xrightarrow{a} M'$ can be easily encoded in Presburger arithmetic.

(2) $\{(M, M') \mid M \stackrel{a}{\Longrightarrow} M'\}$.
$M \stackrel{a}{\Longrightarrow} M'$ iff there exists $X \in T^{\oplus}$ such that conditions (a) and (b) of Theorem 2 hold, and moreover
(c) for every $t \in T$ such that $l(t) \notin \{a, \tau\}$, $X(t) = 0$, and
(d) $\sum_{t \in l^{-1}(a)} X(t) = 1$.

The sets satisfying one of the conditions (a)–(d) are all effectively semilinear. By Theorem 9, its intersection is also effectively semilinear.

By Theorem 9 there exist formulae of Presburger arithmetic expressing the relations (1) and (2) above and the relation $\mathcal{R}$. Using them, it is easy to encode the definition of weak bisimulation as another formula of Presburger arithmetic. The result follows from Theorem 10. ■

The algorithm to decide if a semilinear relation is a weak bisimulation has a high complexity. This is not very surprising, because the algorithms given so far to decide strong bisimulations for BPPs are non-elementary [3]. There exist so far no lower bounds for this problem. However, Hirshfeld, Jerrum and Moller have recently shown in [10] that deciding strong bisimulation is polynomial for the class of *normed* BPPs, in which a process may always terminate.

6 Conclusions

We have solved the reachability problem for communication-free Petri nets using very well-known techniques of net theory. We have then shown that this solution has several applications to context-free and commutative context-free grammars, and Basic Parallel Processes. More precisely, and in order of increasing interest, we have obtained a new proof of Parikh's theorem, a simpler proof of the NP-completeness of the uniform word problem for commutative context-free grammars, and the first positive result about the decidability of weak bisimulation for Basic Parallel Processes.

Acknowledgments

Many thanks to Soren Christensen and Yoram Hirshfeld for helpful discussions. Special thanks are also due to Richard Mayr and an anonymous referee for patiently correcting many typos in an earlier version, and for very useful suggestions.

References

1. S. Christensen. Decidability and Decomposition in Process Algebras. Ph.D. Thesis, University of Edinburgh, CST-105-93, (1993).
2. S. Christensen, H. Hüttel, and C. Stirling. Bisimulation equivalence is decidable for all context-free processes. Proceedings of CONCUR '92, LNCS 630, pp. 138–147 (1992).
3. S. Christensen, Y. Hirshfeld and F. Moller. Bisimulation equivalence is decidable for basic parallel processes. CONCUR '93, LNCS 715, 143–157 (1993).
4. S. Eilenberg and M.P. Schützenberger. Rational sets in commutative monoids. Journal of Algebra 13, 173–191 (1969).
5. J. Esparza and M. Nielsen. Decidability Issues for Petri Nets – a Survey. EATCS Bulletin 52 (1994). Also: Journal of Information Processing and Cybernetics.
6. S. Ginsburg. The Mathematical Theory of Context-free Languages. McGraw-Hill (1966).
7. S. Ginsburg and E.H. Spanier. Semigroups, Presburger formulas and languages. Pacific Journal of Mathematics 16, pp. 285–296 (1966).
8. Y. Hirshfeld. Petri Nets and the Equivalence Problem. CSL '93, LNCS 832 pp. 165–174 (1994).
9. Y. Hirshfeld and Faron Moller. Deciding Equivalences in Simple Process Algebras. In: "Modal Logic and Process Algebra: Proceedings of a 3-day Workshop on Bisimulation", CSLI Press (1995).
10. Y. Hirshfeld, M. Jerrum and F. Moller. A polynomial algorithm for deciding bisimulation of normed basic parallel processes. LFCS report 94-288, Edinburgh University (1994). To appear in Journal of Mathematical Structures in Computer Science.
11. J.E. Hopcroft and J.D. Ullman. Introduction to Automata Theory, Languages and Computation. Addison-Wesley (1979).
12. D.T. Huynh. Commutative Grammars: The Complexity of Uniform Word Problems. Information and Control 57, 21–39 (1983).
13. P. Jančar. Decidability Questions for Bisimilarity of Petri Nets and Some Related Problems. STACS '94, LNCS 775, pp. 581–592 (1994). To appear in Theoretical Computer Science.
14. R. Milner. Communication and Concurrency. Prentice-Hall (1989).
15. P.H. Starke. Analyse von Petri-Netz Modellen. Teubner (1990).