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Derivations and automorphisms on non-commutative power series

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ABSTRACT

We define a class of derivations on the algebra of the non-commutative power series in two variables and study the structure of the Lie algebra generated by them. After detecting the corresponding automorphisms under the exponential map explicitly, we clarify the structure of the automorphism group by means of the Hausdorff group associated to the Lie algebra.

1. Introduction

A fundamental and useful structure associated to any algebra is the interplay between the derivation Lie algebra and the automorphism group via the exponential/logarithm maps. In this paper, we will highlight this bijective correspondence between the Lie algebra of nilpotent derivations and the group of unipotent algebra automorphisms on the non-commutative algebra $\hat{R} = k\langle\langle a, b\rangle\rangle$ of the formal power series in two variables over a field k of characteristic zero. The main problem on which we focus is to give a concrete description of this correspondence.

However it is out of reach to give the complete answer to the problem at present, but in this paper we will show the various interesting aspects of this problem and give a partial answer to the problem.

The main result of this paper is as follows. We consider the series of derivations $D_n := D_n(\lambda; \alpha, \beta, \gamma, \delta)$ on $\hat{R} = k\langle\langle a, b\rangle\rangle$ for $n \geq 1$ defined by

$$D_n(a) = \lambda a^{n+1}, \qquad D_n(b) = \alpha a^{n+1} + \beta a^n b + \gamma b a^n + \delta b a^{n-1} b$$

where $\alpha, \beta, \gamma, \delta$ and λ are fixed elements in k. Since $\hat{R}$ is generated freely by two generators, a derivation of $\hat{R}$ is uniquely characterized by the images on the generators. (We will explain the detailed setup in Section 2.) The purpose of the article is to explicitly compute the exponentials of the derivations $\sum_n c_n D_n$, where $c_1, c_2, \ldots$ are arbitrary elements of k. The explicit expressions are given in Sections 5 and 6. This computation is related to the study of multiple zeta values. We will explain our motivation below in the last part of this section. The derivations $\{D_n\}$ are not commutative with each other for fixed parameters $\alpha, \beta, \gamma, \delta$ and λ in general. We study the structure of the Lie algebra generated by these derivations in Section 4. The defining equations of the Lie algebra is the same as that of the classical Witt algebra in the conformal field theory. In Section 7, we also clarify the structure of the automorphism group generated by their exponentials by using the general machinery of the Hausdorff series and discuss the relation with the derivation Lie algebra via exp/log. In Section 8, we point out that several elements defined in this paper satisfy the cocycle condition for a non-abelian group cohomology.

A special case of derivations defined above had appeared in [8,7] motivated by the study of multiple zeta values. In the remainder of this section, we explain more about our background problem.

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The multiple zeta value (MZV) of weight n is a real number defined by

$$\zeta(n_1,\ldots,n_r) := \sum_{m_1>\cdots>m_r>0} \frac{1}{m_1^{n_1}\cdots m_r^{n_r}} \tag{1}$$

where n_l are positive integers satisfying $n_1 > 1$ and $\sum n_l = n$. These numbers appear in various fields of mathematics and are related to the geometry of the variety $\mathbf{P}^1 \setminus \{0, 1, \infty\}$. It is known that there exist $\mathbf{Q}$-linear relations among MZVs of the same weight (e.g. $\zeta(3) = \zeta(2, 1)$). One of the crucial problems is to find all relations among them and to clarify the structure of the $\mathbf{Q}$-algebra generated by them.

In [8], Kaneko, Zagier and the author proved some linear relations among MZVs, the so-called *derivation relations*. They defined the derivation ∂_n on $\mathbf{Q}\langle\langle x, y\rangle\rangle$ by $\partial_n(x) = -\partial_n(y) := x(x+y)^{n-1}y$. Then the result says that for any word w in letters x, y, the element $\partial_n(xwy)$ gives a linear relation of MZVs after applying the evaluation map Z which maps $x^{n_1-1}y\cdots x^{n_r-1}y$ to $\zeta(n_1,\ldots,n_r)$ as it is shown there that $Z(\partial_n(xwy)) = 0$. For example, $\partial_1(xy) = xy^2 - x^2y$ corresponds to $\zeta(2, 1) - \zeta(3) = 0$. Note that $\partial_n = D_n(0; 0, 0, 1, -1)$ holds after changing of variables by $a = x + y$ and $b = x$. The key point of the proof of the derivation relations and also of their results is the computation of the exponentials of derivations. For example, they computed the exponential of $\partial := \sum_n \frac{1}{n}\partial_n$ as

$$e^{\partial}(x) = x(1-y)^{-1}, \qquad e^{\partial}(y) = (1 - x(1-y)^{-1})^{-1}y.$$

They proved the derivation relations and other properties by looking at the corresponding automorphisms rather than derivations themselves. Because of this phenomenon, the author wants to detect the exponentials of derivation of general forms.

The algebra $\mathbf{Q}\langle\langle x, y\rangle\rangle$ of two variables naturally appears in the theory of MZVs as the completed tensor algebra of the space of holomorphic 1-forms on $\mathbf{P}^1 \setminus \{0, 1, \infty\}$, or as the completed group ring of the fundamental group of $\mathbf{P}^1 \setminus \{0, 1, \infty\}$. (Both of which are isomorphic not only as algebras but as Hopf algebras discussed by Quillen in the appendix of [10].) It is natural to ask the problem for n-variables case ($n \geq 3$). The problem will certainly arise in the next step and must be related to a geometry of $\mathbf{P}^1 \setminus (\{0, \infty\} \cup \mu_N)$, where μ_N is the group of the Nth roots of unity, and the theory of *multiple L-values* in [4–6].

2. Setup

Let k be a field of characteristic zero. Let $R = k\langle a, b\rangle$ be the non-commutative polynomial algebra over k generated by two elements a, b of degree one, and $\mathfrak{m}$ the augmentation ideal of R. The completion $\hat{R}$ of R in terms of the filtration-topology induced from the sequence of ideals: $\mathfrak{m} \supset \mathfrak{m}^2 \supset \cdots$ can be identified with the algebra $k\langle\langle a, b\rangle\rangle$ of the non-commutative formal power series in two variables. For any ideal $\mathfrak{i}$ of R, $\hat{\mathfrak{i}}$ denotes the topological closure of $\mathfrak{i}$ in $\hat{R}$. For instance $(\mathfrak{m}^n)\hat{}$ is the ideal of $\hat{R}$ consists of the power series whose smallest degree of terms is larger than or equal to n. There are two exact sequences $0 \longrightarrow \hat{\mathfrak{m}} \longrightarrow \hat{R} \longrightarrow k \longrightarrow 0$ and $1 \longrightarrow 1 + \hat{\mathfrak{m}} \longrightarrow (\hat{R})^{\times} \longrightarrow k^{\times} \longrightarrow 1$. Furthermore $\hat{\mathfrak{m}}$ is a Lie algebra and $1 + \hat{\mathfrak{m}}$ is a group which are related bijectively via exp/log.

By definition, a derivation D on $\hat{R}$ over k is a k-linear endomorphism on $\hat{R}$ satisfies the Leibniz rule: $D(xy) = D(x)y + xD(y)$ for any x, y in $\hat{R}$. By $\mathrm{Der}(\hat{R})$ we denote the Lie algebra over k consisting of all derivations on $\hat{R}$ over k under the commutator bracket operation: $[D, D'] := DD' - D'D$. On the other hand, an automorphism Δ on $\hat{R}$ is a k-linear automorphism on $\hat{R}$ satisfying $\Delta(xy) = \Delta(x)\Delta(y)$ for any x, y in $\hat{R}$. $\mathrm{Aut}(\hat{R})$ denotes the group of all algebra automorphisms on $\hat{R}$ over k.

Let $\mathrm{Der}^+(\hat{R})$ be the Lie subalgebra consisting of derivations D which increase the degree: $D(\hat{\mathfrak{m}}) \subset (\mathfrak{m}^2)\hat{}$, where $(\mathfrak{m}^2)\hat{}$ is the closure of $\mathfrak{m}^2$ in $\hat{R}$. Let $\mathrm{Aut}^1(\hat{R})$ be the subgroup consisting of automorphisms Δ such that $(\mathrm{Id} - \Delta)(\hat{\mathfrak{m}}) \subset (\mathfrak{m}^2)\hat{}$. Such derivations and automorphisms induce the trivial derivation and the identity map, respectively on the associated graded algebra $\mathrm{gr}(\hat{R}) = \bigoplus(\mathfrak{m}^n)\hat{}/(\mathfrak{m}^{n+1})\hat{}$ of $\hat{R}$.

Since D and $\mathrm{Id} - \Delta$ have the degree-increasing properties, the exponential map: $\exp D = e^D = \sum_{m\geq 0} D^m/m!$ for $D \in \mathrm{Der}^+(\hat{R})$ and the logarithm map: $\log \Delta = -\sum_{m\geq 1}(\mathrm{Id} - \Delta)^m/m$ for $\Delta \in \mathrm{Aut}^1(\hat{R})$ are well-defined and give a one-to-one correspondence between $\mathrm{Der}^+(\hat{R})$ and $\mathrm{Aut}^1(\hat{R})$. Also the assumption on the characteristic of k is used to define the maps.

As mentioned above, derivations (resp. automorphisms) on $\hat{R} = k\langle\langle a, b\rangle\rangle$ are uniquely determined by the images on the generators. Since $\hat{R}$ is generated (topologically) freely by a and b, to define an element D in $\mathrm{Der}^+(\hat{R})$ (resp. Δ in $\mathrm{Aut}^1(\hat{R})$), one can choose any elements in $(\mathfrak{m}^2)\hat{}$ as values of D (resp. $\mathrm{Id} - \Delta$) on generators.

The problem what we like to focus here is to give the complete description of the correspondence between $\mathrm{Der}^+(\hat{R})$ and $\mathrm{Aut}^1(\hat{R})$ via the exponential/logarithm maps. For example, if one defines the derivation $D \in \mathrm{Der}^+(\hat{R})$ by $D(a) = a^2$ and $D(b) = ab - 3ba + b^2$, then what are the images on generators of $\exp D \in \mathrm{Aut}^1(\hat{R})$? The answers are given explicitly by

$$(\exp D)(a) = (1-a)^{-1}a, \qquad (\exp D)(b) = (1-a)^{-1}b(1-a-3b)^{-1}(1-a)^2,$$

where $(1-w)^{-1}$ means $1 + w + w^2 + \cdots$ for any $w \in \hat{\mathfrak{m}}$. This example may show the difficulty of the computation in general. Theorem 2 (or Corollary 3) in Section 7 implies the example above as a special case.

3. Inner derivations and automorphisms

In this section, we recall the basic properties of inner derivations and inner automorphisms of $\hat{R}$.

Any element $F \in \hat{R}$ defines a derivation $\operatorname{ad} F$ on $\hat{R}$ over k called an inner derivation by $(\operatorname{ad} F)(G) := FG - GF$ for $G \in \hat{R}$. $\operatorname{InnDer}(\hat{R})$ denotes the Lie subalgebra generated by inner derivations. We put

$$\operatorname{InnDer}^+(\hat{R}) := \operatorname{InnDer}(\hat{R}) \cap \operatorname{Der}^+(\hat{R}).$$

The $\operatorname{ad}F$ belongs to $\operatorname{InnDer}^+(\hat{R})$ if and only if $F \in \hat{\mathfrak{m}}$. The map $\operatorname{ad} : \hat{\mathfrak{m}} \longrightarrow \operatorname{InnDer}^+(\hat{R})$, $F \mapsto \operatorname{ad} F$ gives a Lie algebra isomorphism over k.

On the one hand, any invertible element $H \in \hat{R}$ defines the automorphism $\operatorname{Ad} H$ on $\hat{R}$ over k called the inner automorphism by $(\operatorname{Ad} H)(G) := HGH^{-1}$. We denote the subgroup generated by inner automorphisms by $\operatorname{InnAut}(\hat{R})$ and set

$$\operatorname{InnAut}^1(\hat{R}) := \operatorname{InnAut}(\hat{R}) \cap \operatorname{Aut}^1(\hat{R}).$$

In the case, the exp/log maps give a bijective correspondence between $\operatorname{InnDer}^+(\hat{R})$ and $\operatorname{InnAut}^1(\hat{R})$ explicitly by

$$\exp(\operatorname{ad} F) = \operatorname{Ad}(\exp F).$$

4. Derivations

In this section, we define a specific class of derivations of $\hat{R} = k\langle\langle a, b\rangle\rangle$ and study the structure of the Lie algebra generated by them.

Definition 1. For integer $n \geq 1$ and for any elements $\alpha, \beta, \gamma, \delta$ and λ in k, define the sequence of derivations $D_n := D_n(\lambda; \alpha, \beta, \gamma, \delta) \in \operatorname{Der}^+(\hat{R})$ by

$$D_n(a) = \lambda a^{n+1}, \qquad D_n(b) = \alpha a^{n+1} + \beta a^n b + \gamma b a^n + \delta b a^{n-1} b.$$

We fix the parameters $\alpha, \beta, \gamma, \delta$ and λ throughout the paper. The derivations in the case $\lambda = 0$ are discussed in [7,8]. Let $\mathcal{L} = \mathcal{L}(\lambda; \alpha, \beta, \gamma, \delta)$ be the subspace of $\operatorname{Der}^+(\hat{R})$ (topologically) generated by all D_n for $n \geq 1$ for fixed $\alpha, \beta, \gamma, \delta$ and λ. When $\lambda = 0$, as proved in [7,8], each derivation commutes each other: $[D_m, D_n] = 0$ for any $m, n \geq 1$. In general case, we meet the following beautiful relations.

Proposition 1. *For any $m, n \geq 1$ and for fixed $\alpha, \beta, \gamma, \delta$ and λ in k, we have*

$$[D_m, D_n] = \lambda(n - m)D_{m+n}. \tag{2}$$

Consequently $\mathcal{L}$ is a Lie subalgebra of $\operatorname{Der}^+(\hat{R})$.

Proof. Since both sides of (2) are derivations on $\hat{R}$, it is enough to check that the values on generators are the same. By straightforward computation we have

$$\begin{aligned}
D_m D_n(a) &= \lambda^2(n+1)a^{m+n+1}, \\
D_m D_n(b) &= \alpha(\lambda+\beta+\gamma)a^{m+n+1} + (\beta^2+\alpha\delta)a^{m+n}b + (\gamma^2+\alpha\delta)ba^{m+n} + \delta(\beta+\gamma-\lambda)ba^{m+n-1}b \\
&\quad + \beta\gamma(a^n b a^m + a^m b a^n) + \beta\delta(a^n b a^{m-1} b + a^m b a^{n-1} b) \\
&\quad + \gamma\delta(a^{n-1} b a^m + a^{m-1} b a^n) + \delta^2(ba^{m-1}ba^{n-1}b + ba^{n-1}ba^{m-1}b) \\
&\quad + n\lambda(\alpha a^{m+n+1} + \beta a^{m+n} b + \gamma b a^{m+n} + \delta b a^{m+n-1} b).
\end{aligned}$$

From this, we can easily check the claim. □

When $\lambda = 1$, Eq. (2) is the same as the defining relations of the Witt algebra in the conformal theory. The following is a simple reason why they are the same: the Witt algebra is generated by the differential operators $z^n \frac{d}{dz}$ as a Lie algebra for $n \in \mathbf{Z}$. The relations

$$\left[z^m \frac{d}{dz}, z^n \frac{d}{dz}\right] = (n - m)z^{m+n} \frac{d}{dz}$$

hold for $m, n \in \mathbf{Z}$ and give the defining relations of the algebra. (e.g. [11]) If we restrict the domain $\hat{R} = k\langle\langle a, b\rangle\rangle$ of D_n to $k[[a]]$, the subalgebra of $\hat{R}$ generated topologically by a, we notice that $D_n = a^{n+1} \frac{d}{da}$ in the case of $\lambda = 1$.

Proposition 2. *For any $n \geq 0$ and for fixed $\alpha, \beta, \gamma, \delta$ and λ in k, we have*

$$\lambda^n n!\, D_{n+2} = (\operatorname{ad}^n D_1)(D_2) := [\underbrace{D_1, [D_1, \ldots [D_1}_{n}, D_2] \cdots]]. \tag{3}$$

In particular $\mathcal{L}$ is generated (topologically) by D_1 and D_2 as a Lie algebra.

Proof. It is proved easily by Proposition 2 and by induction. □

To consider the derivation which expressed as a linear combination of derivations D_n, we use the following notations defined in [8] and extend it to our case.

Let $k[[X]]$ be the commutative algebra of the formal power series in an indeterminate X over field k. Let $L = Xk[[X]]$ be the subspace consisting of a power series of constant term zero, and $G = 1 + Xk[[X]]$ be a set of power series of constant term one. The exponential/logarithm maps give a bijection between L and G.

Definition 2. For any $f(X) = \sum_{n\geq 1} c_n X^n \in L$, define the derivation $D_f := D_f(\lambda; \alpha, \beta, \gamma, \delta) \in \mathcal{L} \subset \mathrm{Der}^+(\hat{R})$ by $D_f(\lambda; \alpha, \beta, \gamma, \delta) := \sum_{n\geq 1} c_n D_n(\lambda; \alpha, \beta, \gamma, \delta)$. The action of D_f on generators are given by

$$D_f(a) = \lambda f(a)a, \qquad D_f(b) = \alpha f(a)a + \beta f(a)b + \gamma b f(a) + \delta b \frac{f(a)}{a} b.$$

The map $D : f \mapsto D_f$ gives a linear isomorphism from L and $\mathcal{L}$. In the following propositions, we introduce a Lie algebra structure on $L = Xk[[X]]$ which makes D a Lie isomorphism.

Proposition 3. *For $f \in L$, we put $f^\bullet = X\frac{df}{dX} \in L$. Then for fixed $\lambda \in k$ and any $f, g \in L$, the bracket defined by $[f, g]_\bullet := \lambda(fg^\bullet - f^\bullet g)$ gives a Lie algebra structure on L.*

Proof. The bracket is obviously bilinear and anti-symmetric. For $f, g, h \in L$,

$$\begin{aligned}
[[f, g]_\bullet, h]_\bullet &= \lambda[fg^\bullet - f^\bullet g, h]_\bullet \\
&= \lambda^2\{(fg^\bullet - f^\bullet g)h^\bullet - (fg^\bullet - f^\bullet g)^\bullet h\} \\
&= \lambda^2\{(fg^\bullet - f^\bullet g)h^\bullet - (f^\bullet g^\bullet + fg^{\bullet\bullet} - f^{\bullet\bullet} g - f^\bullet g^\bullet)h\} \\
&= \lambda^2(fg^\bullet h^\bullet - f^\bullet g h^\bullet - fg^{\bullet\bullet} h + f^{\bullet\bullet} gh).
\end{aligned}$$

By this form, one can check the Jacobi identity:

$$[[f, g]_\bullet, h]_\bullet + [[g, h]_\bullet, f]_\bullet + [[h, f]_\bullet, g]_\bullet = 0.$$

This completes the proof. □

Remark. Proposition 3 can be generalized as follows: if d is a derivation on a commutative algebra A, then the bracket defined by $[f, g] := fd(g) - d(f)g$ for $f, g \in A$ gives a Lie algebra structure on A. Furthermore d becomes a derivation of the Lie algebra A: $d([f, g]) = [d(f), g] + [f, d(g)]$.

Proposition 4. *The map $D : L \longrightarrow \mathcal{L}$ is a Lie isomorphism, i.e., for $f, g \in L$ we have*

$$[D_f, D_g] = D_{[f,g]_\bullet}.$$

Proof. For elements $f(X) = \sum c_m X^m$, $g(X) = \sum d_n X^n$ in L, by (2) we have

$$[D_f, D_g] = \sum c_m d_n [D_m, D_n] = \lambda \sum (n - m) c_m d_n D_{m+n} = D_{[f,g]_\bullet},$$

where the last equality is because $[f, g]_\bullet = \lambda \sum_{m,n\geq 1} (n - m) c_m d_n X^{m+n}$. □

The $\mathcal{G}$ denotes the image of $\mathcal{L}$ under the exponential map: $\mathcal{G} := \exp \mathcal{L} \subset \mathrm{Aut}^1(\hat{R})$. We will discuss on a group structure of $\mathcal{G}$ in Section 7. We are interested in the relation between $\mathcal{L}$ and $\mathcal{G}$ under the exponential/logarithm maps. Next we will detect the values of e^{D_f} on generators explicitly.

5. Image of generator *a*

5.1. Explicit expression

The derivation D_f defined in previous section can be restricted to the subalgebra $k[[a]]$ of $\hat{R}$ generated topologically by a. It is also denoted by the same letter D_f. Next we refer a formula discovered by Comtet in 1973 in [3], which gives an explicit expression of D_f^n (nth composition). As a corollary we are able to write down the element $e^{D_f}(a)$.

Theorem 1 (*Comtet*)**.** *Let n be a positive integer and $g = g(a)$ a differentiable function on a. Then nth composition of the derivation $g(a)\frac{d}{da}$ is expressed by*

$$\left(g(a)\frac{d}{da}\right)^n = \sum_{l=1}^{n} T_{n,l} \left(\frac{d}{da}\right)^l$$

where for $1 \le l \le n$

$$T_{n,l} = \sum_{\substack{k_1+\cdots+k_{n-1}=n-l\ (\forall k_\bullet \ge 0)\\ k_1+\cdots+k_i \le i\ (1\le \forall i<n)}} \frac{g(a)}{l!} \prod_{j=1}^{n-1} (j+1-k_1-\cdots-k_j) \frac{g_{k_j}(a)}{k_j!} \tag{4}$$

where $g_{k_j}(a)$ *is the* k_j*th derivative function of* $g(a)$.

The following are examples in the case $n = 2, 3, 4$, where we write $g = g(a)$ for simplicity.

$$\left(g\tfrac{d}{da}\right)^2 = gg_1\left(\tfrac{d}{da}\right) + g^2\left(\tfrac{d}{da}\right)^2,$$

$$\left(g\tfrac{d}{da}\right)^3 = (gg_1^2 + g^2g_2)\left(\tfrac{d}{da}\right) + 3g^2g_1\left(\tfrac{d}{da}\right)^2 + g^3\left(\tfrac{d}{da}\right)^3,$$

$$\left(g\tfrac{d}{da}\right)^4 = (gg_1^3 + 4g^2g_1g_2 + g^3g_3)\left(\tfrac{d}{da}\right) + (7g^2g_1^2 + 4g^3g_2)\left(\tfrac{d}{da}\right)^2 + 6g^3g_1\left(\tfrac{d}{da}\right)^3 + g^4\left(\tfrac{d}{da}\right)^4.$$

The numbers appearing in $\{T_{n,l}\}$ as coefficients can be found in [13]; [A1 39 605] and are related to the number of *rooted trees*. They are also related to the *theory of species* in combinatorics. See [1,9] for the development in this direction.

It is also shown in [3] that $T_{n,l}(a)$ equals to the coefficient of $t_1 \cdots t_n$ in the Taylor expansion (at the origin) of

$$\frac{1}{l!}(t_1+\cdots+t_n)^l g(a) \prod_{j=1}^{n-1} g(a+t_1+\cdots+t_j).$$

Corollary 1. *For* $n \ge 1$ *and* $f \in L$, *we have* $D_f^n(a) = T_n(\lambda f)$ *where*

$$T_n(f) := \sum_{\substack{k_1+\cdots+k_{n-1}=n-1\ (\forall k_\bullet \ge 0)\\ k_1+\cdots+k_i \le i\ (1\le \forall i \le n-1)}} f(a)a \prod_{j=1}^{n-1} (j+1-k_1-\cdots-k_j) \frac{(f(a)a)_{k_j}}{k_j!}.$$

Proof. It holds $D_f = \lambda f(a)a\, d/da$ as derivations on $k[[a]]$, since both of them send the generator a to $\lambda f(a)a$. Hence we can apply Theorem 1 to compute D_f^n by putting $g(a) = \lambda f(a)a$. □

For $h \in G$, we define an element $u_h(a) \in 1 + ak[[a]]$ by $u_h(a)a = \sum_{n\ge 0} T_n(\lambda f)/n!$ for $f = \log h$ with $T_0(f) := a$.

Corollary 2. *We have*

$$e^{D_f}(a) = u_h(a)a. \tag{5}$$

We may not have simple expression of $u_h(a)$ in terms of elementary functions in general. See Example 2 in next subsection for such an example. In Sections 7 and 8 we will see a cocycle property of this element.

5.2. Analytic expression

In this subsection, we show an alternative way to express $u_h(a)$ by analytic methods. We assume that $k = \mathbf{R}$, the field of real numbers, and $\lambda \ne 0$ throughout this subsection.

Let $f \in L = Xk[[X]]$. Assume that $f(a) \in a\mathbf{R}[[a]]$ is a power series with positive convergent radius. Hence $f(a)$ and $g(a) := \lambda f(a)a$ define real analytic functions near the origin $a = 0$. We can take $\varepsilon > 0$ such that $1/g(a)$ is positive or negative on the interval $I = (0, \varepsilon)$, which depends on the signature of the coefficient of $g(a)$ of minimum degree. By the fundamental theorem of ordinary differential equation and the continuity of $1/g(a)$ on I, there exists a unique (up to constant) differentiable function $t = t(a)$ on I satisfying the differential equation $dt/da = 1/g(a)$, that is, $t(a) = \int g(a)^{-1} da$. One can choose any integral constant and fix it. $t(a)$ is a monotone (increasing or decreasing) function on I valued onto $t(I)$, since dt/da is positive or negative on I. Hence we have the composite-inverse function $y = y(t)$ of t on $t \in t(I)$, which is also differentiable by virtue of the inverse function theorem. In summary, we have two differentiable functions $t : I \longrightarrow t(I)$ and $y : t(I) \longrightarrow I$ such that $t(y(t)) = t, y(t(a)) = a$.

Proposition 5. *For* $f \in L$, *let* D_f *be the derivation on* $\mathbf{R}[[a]]$ *defined by* $D_f(a) = \lambda f(a)a$ *with* $\lambda \ne 0$. *Assume that* $f(a) \in a\mathbf{R}[[a]]$ *has a positive convergent radius. Then we have*

$$e^{sD_f}(a) = y(s + t(a)) \tag{6}$$

for $a \in I$ *and any* $s \in \mathbf{R}$ *such that* $s + t(a) \in t(I)$.

Proof. When parameter s runs over small neighborhood of $s = 0$, one can check that LHS of (6) makes a one-parameter subgroup whose tangent vector at $s = 0$ is $D_f(a)$. Indeed clearly $e^{sD_f}(a) = e^{s\lambda f(a)a}$ is differentiable in s and satisfies

$$\frac{d}{ds}e^{sD_f}(a)|_{s=0} = D_f(a), \qquad e^{sD_f}(e^{s'D_f}(a)) = e^{(s+s')D_f}(a).$$

We will show RHS also satisfies the same properties:

$$\frac{d}{ds}y(s+t(a)) = D_f(a), \qquad y(s+t(y(s'+t(a)))) = y(s+s'+t(a)).$$

For the former equation, by using $dy/dt = g(y)$ and the chain rule,

$$\frac{d}{ds}y(s+t(a))|_{s=0} = \frac{dy}{dt}(s+t(a))\frac{d}{ds}(s+t(a))|_{s=0} = g(y(s+t(a)))|_{s=0} = g(y(t(a))) = g(a) = D_f(a).$$

The latter equation is clear. From the uniqueness of one-parameter subgroups, we conclude the proof. $\square$

Example 1. If $f(X) = X^m \in L$ $(m \geq 1)$, we have $D_f = D_m$ and $D_f(a) = \lambda a^{m+1} =: g(a)$. In this case, a solution $t(a)$ to the differential equation $dt/da = a^{-m-1}/\lambda$ is $t(a) = -a^{-m}/\lambda m$. The inverse function is obtained by $y(t) = (-\lambda m t)^{-1/m}$. From Proposition 5, we have $e^{sD_f}(a) = y(s+t(a)) = (1 - s\lambda m a^m)^{-1/m}a$. This equation makes sense as the formal power series and holds for any base field k. Therefore by Corollary 2, we have $u_h(a) = (1-\lambda m a^m)^{-1/m}$ for $h = \exp(X^m)$ and for any $\lambda \in k$.

Example 2. If $f(X) = X/(1-X) = X+X^2+\cdots \in L$, we have $D_f = D_1+D_2+\cdots$ and $D_f(a) = \lambda a^2/(1-a) =: g(a)$. In this case, a solution $t(a)$ to the differential equation $dt/da = (1-a)/(\lambda a^2)$ is $t(a) = (-a^{-1} - \log(a))/\lambda$. This function is monotone on $I = (0, 1)$. Denote the inverse function by $y(t)$. (We do not know the simple description of $y(t)$ in terms of elementary functions.) From Proposition 5, we have $e^{sD_f}(a) = y(s+t(a)) = y(s+(-a^{-1}-\log(a))/\lambda) = a+s\lambda a^2+s\lambda(1+s\lambda)a^3+s\lambda(1+\frac{5}{2}s\lambda+(s\lambda)^2)a^4+\cdots$. By Corollary 2, $u_h(a) = y(1+(-a^{-1}-\log(a))/\lambda)/a = 1+\lambda a+(\lambda+\lambda^2)a^2+\lambda(1+\frac{5}{2}\lambda+\lambda^2)a^3+\cdots$ for $h = \exp(X/(1-X))$.

6. Image of generator *b*

We return to the general situation. Let k be any field of characteristic zero. Recall that $\mathcal{L} = \{D_f | f \in L\}$ and $\mathcal{G} := \exp \mathcal{L}$. In this section, $\exp D_f$ is detected explicitly.

Definition 3. For $h \in G = 1 + Xk[[X]]$ and for $\alpha, \beta, \gamma, \delta$ and $\lambda \in k$, we define an automorphism $\Delta_h = \Delta_h(\lambda; \alpha, \beta, \gamma, \delta) \in \mathrm{Aut}^1(\hat{R})$ by the following action on generators:

$$\Delta_h(a) = u_h(a)a, \qquad \Delta_h(b) = A_h B_h^{-1}$$

where $u_h(a)$ is defined in Corollary 2 in Section 5.1 and

$$A_h := v_h(a)^{\beta+\varepsilon}\left[b + \frac{v_h(a)^{-\omega}-1}{-\omega}(\alpha a - \varepsilon b)\right]$$

$$= v_h(a)^{\beta}\left[(v_h(a)^{\varepsilon} - v_h(a)^{\varepsilon'})\alpha a - (\varepsilon' v_h(a)^{\varepsilon} - \varepsilon v_h(a)^{\varepsilon'})b\right]\Big/\omega \tag{7}$$

$$B_h := v_h(a)^{-(\gamma+\varepsilon)}\left[1 + \frac{v_h(a)^{\omega}-1}{\omega a}(\varepsilon a - \delta b)\right]$$

$$= v_h(a)^{\beta-\lambda}\left[(\varepsilon v_h(a)^{\varepsilon} - \varepsilon' v_h(a)^{\varepsilon'}) - \frac{v_h(a)^{\varepsilon} - v_h(a)^{\varepsilon'}}{a}\delta b\right]\Big/\omega \tag{8}$$

where $v_h(a) := u_h(a)^{1/\lambda} = \exp(\lambda^{-1}\log(u_h(a))) \in 1 + ak[[a]]$ (resp. $v_h(a) := h(a)$) for $\lambda \neq 0$ (resp. $\lambda = 0$). The ε and ε' are roots of the equation $T^2 + (\beta+\gamma-\lambda)T + \alpha\delta = 0$ and $\omega = \varepsilon - \varepsilon'$.

The elements ε, ε' and ω are in a quadratic extension of k in general, but one can see $A_h \in \hat{R}$ and $B_h \in (\hat{R})^\times$ (the unit group), since each expression (7) and (8) is symmetric for ε and ε'. The following is the main theorem and will be proved in Section 7.

Theorem 2. *For any $f \in L$, we set $h = \exp f \in G$. Then we have*

$$\Delta_h = \exp D_f,$$

i.e., the following diagram commutes:

$$\begin{array}{ccc} L & \xrightarrow{\exp} & G \\ {\scriptstyle D}\downarrow & & \downarrow{\scriptstyle \Delta} \\ \mathcal{L} & \xrightarrow{\exp} & \mathcal{G} \end{array} \tag{9}$$

In particular we conclude $\mathcal{G} := \exp \mathcal{L} = \Delta(G) := \{\Delta_h \in \mathrm{Aut}^1(\hat{R}) \mid h \in G\}$.

Corollary 3. *For* $m \geq 1$ *and any* $s \in k$, $\exp(sD_m) \in \mathrm{Aut}^1(\hat{R})$ *is determined by*

$$\begin{aligned} e^{sD_m}(a) &= (1 - s\lambda m a^m)^{-1/m} a, \\ e^{sD_m}(b) &= (1 - s\lambda m a^m)^{-\frac{\beta+\varepsilon}{\lambda m}} \left[b + \frac{(1 - s\lambda m a^m)^{\frac{\omega}{\lambda m}} - 1}{-\omega} (\alpha a - \varepsilon b) \right] \\ &\quad \times \left[1 + \frac{(1 - s\lambda m a^m)^{\frac{-\omega}{\lambda m}} - 1}{\omega a} (\varepsilon a - \delta b) \right]^{-1} (1 - s\lambda m a^m)^{-\frac{\gamma+\varepsilon}{\lambda m}}, \end{aligned}$$

where $\varepsilon, \varepsilon'$, *and* ω *are defined in Definition 3 and* $(1 - s\lambda m a^m)^{-\frac{1}{\lambda m}} = e^{sa^m}$ *if* $\lambda = 0$.

Proof. This corollary is a special case of the theorem for $f(X) = sX^m$. In this case we can use $u_h(a) = (1 - s\lambda m a^m)^{-1/m}$ from Example 2. □

7. Automorphisms

In this section we introduce a non-trivial group structure on $G = 1 + Xk[[X]]$ which makes Δ $(h \mapsto \Delta_h)$ a group homomorphism. First we will define two *Hausdorff groups* associated to Lie algebras $(L, [\ ,\ ]_\bullet)$ and $(\mathcal{L}, [\ ,\ ])$ defined in Section 4 respectively. One can refer [2] for the general theory on Hausdorff groups.

The Baker–Campbell–Hausdorff series $H(x, y)$ is a power series in two indeterminates x, y defined by

$$H(x, y) = \log(e^x e^y) = x + y + \frac{1}{2}[x, y] + \frac{1}{12}([[x, y], y] - [[x, y], x]) + \cdots .$$

Each homogeneous term of $H(x, y)$ is a $\mathbf{Q}$-linear combination of commutator brackets. It is easy to show the formal group law:

$$\begin{aligned} &H(H(x, y), z) = H(x, H(y, z)), \\ &H(x, 0) = H(0, x) = x, \qquad H(x, -x) = H(-x, x) = 0 \end{aligned}$$

for indeterminates x, y, z. Using this series, define a product $H_\bullet$ on L by

$$H_\bullet(f, g) := f + g + \frac{1}{2}[f, g]_\bullet + \cdots \in L$$

for $f, g \in L$. Note that this product is well-defined and makes $(L, H_\bullet)$ a group. Similarly, define the product H on $\mathcal{L}$ by

$$H(D_f, D_g) := D_f + D_g + \frac{1}{2}[D_f, D_g] + \cdots \in \mathcal{L}.$$

Then $(\mathcal{L}, H)$ has a group structure.

Proposition 6. *For* $g, h \in G$, *the product* $*$ *defined by*

$$g * h := \exp H_\bullet(\log g, \log h)$$

gives a group structure on G.

It is clear that every map in the diagram is group homomorphisms:

$$\begin{array}{ccc} (L, H_\bullet) & \xrightarrow{\exp} & (G, *) \\ {\scriptstyle D}\downarrow & & \\ (\mathcal{L}, H) & \xrightarrow{\exp} & (\mathcal{G}, \cdot) \end{array} \tag{10}$$

$$D_{H_\bullet(f,g)} = H(D_f, D_g), \qquad e^{H(D_f, D_g)} = e^{D_f} e^{D_g} \tag{11}$$

for $f, g \in L$. The following theorem asserts Δ supplies the missing map in the diagram.

Theorem 3. *The map $\Delta : G \longrightarrow \mathrm{Aut}^1(\hat{R})$ $(h \mapsto \Delta_h)$ is a group homomorphism: for any $h, h' \in G$ we have*

$$\Delta_{h*h'} = \Delta_h \Delta_{h'}. \tag{12}$$

By combining Theorem 2, we obtain $\Delta : (G, *) \longrightarrow (\mathcal{G}, \cdot)$ is a group isomorphism. Theorem 3 is proved after the lemmas.

Lemma 1. *For any $h, h' \in G$, we have*

$$u_{h*h'}(a) = u_{h'}(u_h(a)a)u_h(a) \tag{13}$$

where $u_h(a)$ is defined in Corollary 2.

Proof. Put $f = \log h$, $g = \log h' \in L$. From (11), one has

$$e^{D_f} e^{D_g} = e^{D_{H_\bullet(f,g)}}. \tag{14}$$

Let us compute the images of a of both sides of (14). By Corollary 2 and Proposition 6 we have

$$e^{D_f}(e^{D_g}(a)) = e^{D_f}(u_{h'}(a)a) = u_{h'}(e^{D_f}(a))e^{D_f}(a) = u_{h'}(u_h(a)a)u_h(a)a,$$
$$e^{D_{H_\bullet(f,g)}}(a) = u_{e^{H_\bullet(f,g)}}(a)a = u_{h*h'}(a)a.$$

Comparing both, we have (13). $\square$

Lemma 2. *For any $h, h' \in G$, we obtain*

$$\Delta_h(A_{h'})B_h = A_{h*h'}, \qquad \Delta_h(B_{h'})B_h = B_{h*h'} \tag{15}$$

where A_h, B_h are the elements defined in Definition 3 in Section 6.

Proof. For simplicity we write v_h (resp. u_h) instead of $v_h(a)$ (resp. $u_h(a)$). Hence we can write

$$A_h = v_h^\beta\big[(v_h^\varepsilon - v_h^{\varepsilon'})\alpha a - (\varepsilon' v_h^\varepsilon - \varepsilon v_h^{\varepsilon'})b\big]\big/\omega,$$
$$B_h = v_h^{\beta-\lambda}\Big[(\varepsilon v_h^\varepsilon - \varepsilon' v_h^{\varepsilon'}) - \frac{v_h^\varepsilon - v_h^{\varepsilon'}}{a}\delta b\Big]\Big/\omega$$

and $\Delta_h(a) = u_h a$, $\Delta_h(b) = A_h B_h^{-1}$. First we show that $F := \Delta_h(v_{h'}) = v_{h*h'}v_h^{-1}$. By Lemma 1, one has $\Delta_h(u_{h'}) = u_{h*h'}u_h^{-1}$. When $\lambda \neq 0$ it implies

$$F = \Delta_h(v_{h'}) = \Delta_h(u_{h'}^{1/\lambda}) = \Delta_h(u_{h'})^{1/\lambda} = (u_{h*h'}u_h^{-1})^{1/\lambda} = v_{h*h'}v_h^{-1}.$$

If $\lambda = 0$, since Δ_h is the identity map on $k[[a]]$ and $h(a) * h'(a) = h(a)h'(a)$, we have

$$F = \Delta_h(v_{h'}) = \Delta_h(h'(a)) = h'(a) = h(a)h'(a)h(a)^{-1} = v_{h*h'}v_h^{-1}.$$

Using this we have

$$\begin{aligned}
\Delta_h(A_{h'})B_h &= \Delta_h\left(v_{h'}^\beta\big[(v_{h'}^\varepsilon - v_{h'}^{\varepsilon'})\alpha a - (\varepsilon' v_{h'}^\varepsilon - \varepsilon v_{h'}^{\varepsilon'})b\big]\right)B_h\Big/\omega \\
&= F^\beta\big[(F^\varepsilon - F^{\varepsilon'})\alpha\Delta_h(a)B_h - (\varepsilon' F^\varepsilon - \varepsilon F^{\varepsilon'})A_h\big]\Big/\omega \\
&= v_{h*h'}^\beta\Big[(F^\varepsilon - F^{\varepsilon'})\alpha u_h a v_h^{-\lambda}\Big\{(\varepsilon v_h^\varepsilon - \varepsilon' v_h^{\varepsilon'}) - \frac{v_h^\varepsilon - v_h^{\varepsilon'}}{a}\delta b\Big\} \\
&\qquad - (\varepsilon' F^\varepsilon - \varepsilon F^{\varepsilon'})\big\{(v_h^\varepsilon - v_h^{\varepsilon'})\alpha a - (\varepsilon' v_h^\varepsilon - \varepsilon v_h^{\varepsilon'})b\big\}\Big]\Big/\omega^2 \\
&= v_{h*h'}^\beta\Big[\big\{(F^\varepsilon - F^{\varepsilon'})(\varepsilon v_h^\varepsilon - \varepsilon' v_h^{\varepsilon'}) - (\varepsilon' F^\varepsilon - \varepsilon F^{\varepsilon'})(v_h^\varepsilon - v_h^{\varepsilon'})\big\}\alpha a \\
&\qquad - \big\{(F^\varepsilon - F^{\varepsilon'})(v_h^\varepsilon - v_h^{\varepsilon'})\alpha\delta - (\varepsilon' F^\varepsilon - \varepsilon F^{\varepsilon'})(\varepsilon' v_h^\varepsilon - \varepsilon v_h^{\varepsilon'})\big\}b\Big]\Big/\omega^2 \\
&= v_{h*h'}^\beta\big[(v_{h*h'}^\varepsilon - v_{h*h'}^{\varepsilon'})\alpha a - (\varepsilon' v_{h*h'}^\varepsilon - \varepsilon v_{h*h'}^{\varepsilon'})b\big]\big/\omega = A_{h*h'}.
\end{aligned}$$

Similarly we have

$$\begin{aligned}
\Delta_h(B_{h'})B_h &= \Delta_h\left(v_{h'}^{\beta-\lambda}\left[(\varepsilon v_{h'}^\varepsilon - \varepsilon' v_{h'}^{\varepsilon'}) - \frac{v_{h'}^\varepsilon - v_{h'}^{\varepsilon'}}{a}\delta b\right]\right)B_h\Big/\omega \\
&= F^{\beta-\lambda}\left[(\varepsilon F^\varepsilon - \varepsilon' F^{\varepsilon'})B_h - \frac{F^\varepsilon - F^{\varepsilon'}}{u_h a}\delta A_h\right]\Big/\omega
\end{aligned}$$

$$= F^{\beta-\lambda}\left[(\varepsilon F^{\varepsilon}-\varepsilon' F^{\varepsilon'})v_h^{\beta-\lambda}\left\{(\varepsilon v_h^{\varepsilon}-\varepsilon' v_h^{\varepsilon'})-\frac{v_h^{\varepsilon}-v_h^{\varepsilon'}}{a}\delta b\right\}\right.$$
$$\left.-\frac{F^{\varepsilon}-F^{\varepsilon'}}{u_h a}\delta v_h^{\beta}\left\{(v_h^{\varepsilon}-v_h^{\varepsilon'})\alpha a-(\varepsilon' v_h^{\varepsilon}-\varepsilon v_h^{\varepsilon'})b\right\}\right]\Big/\omega^2$$
$$= v_{h*h'}^{\beta-\lambda}\left[\left\{(\varepsilon F^{\varepsilon}-\varepsilon' F^{\varepsilon'})(\varepsilon v_h^{\varepsilon}-\varepsilon' v_h^{\varepsilon'})-\frac{F^{\varepsilon}-F^{\varepsilon'}}{a}(v_h^{\varepsilon}-v_h^{\varepsilon'})\alpha\delta a\right\}\right.$$
$$\left.-\left\{(\varepsilon F^{\varepsilon}-\varepsilon' F^{\varepsilon'})\frac{v_h^{\varepsilon}-v_h^{\varepsilon'}}{a}-\frac{F^{\varepsilon}-F^{\varepsilon'}}{a}(\varepsilon' v_h^{\varepsilon}-\varepsilon v_h^{\varepsilon'})\right\}\delta b\right]\Big/\omega^2$$
$$= v_{h*h'}^{\beta-\lambda}\left[\left(\varepsilon v_{h*h'}^{\varepsilon}-\varepsilon' v_{h*h'}^{\varepsilon'}\right)-\frac{v_{h*h'}^{\varepsilon}-v_{h*h'}^{\varepsilon'}}{a}\delta b\right]\Big/\omega = B_{h*h'}.$$

Thus we conclude Lemma 2. □

Proof of Theorem 3. One can show Theorem 3 by checking the values on generators of both sides of (12). The image of a has checked in Corollary 2 in Section 5.1. For b, by Lemma 2

$$\Delta_h(\Delta_{h'}(b)) = \Delta_h(A_{h'}B_{h'}^{-1}) = \Delta_h(A_{h'})\Delta_h(B_{h'})^{-1} = (\Delta_h(A_{h'})B_h)(\Delta_h(B_{h'})B_h)^{-1} = A_{h*h'}B_{h*h'}^{-1} = \Delta_{h*h'}(b).$$

These show Theorem 3. □

Proof of Theorem 2. It is enough to show that

(i) $\frac{d}{ds}\Delta_{h^s}|_{s=0} = D_f$ for $h = e^f$,
(ii) $\Delta_{h^{s+s'}} = \Delta_{h^s}\Delta_{h^{s'}}$

for any parameters $s, s' \in k$ and $h \in G$, because $\exp(sD_f)$ satisfies the same conditions and these conditions uniquely determine the corresponding automorphism: $\Delta_{h^s} = \exp(sD_f)$. These conditions say Δ_{h^s} is a one-parameter subgroup and the tangent vector at $s = 0$ is D_f.

For (i), we have $\frac{d}{ds}u_{h^s}(a)|_{s=0} = \lambda f(a)$, because of $e^{sD_f}(a) = u_{h^s}(a)a$ in Corollary 2. Therefore $\frac{d}{ds}\Delta_{h^s}(a)|_{s=0} = \frac{d}{ds}u_{h^s}(a)a|_{s=0} = \lambda f(a)a = D_f(a)$. Similarly we can check $\frac{d}{ds}v_{h^s}(a)|_{s=0} = f(a)$. Using this and (7) and (8), one has

$$\frac{d}{ds}A_{h^s}|_{s=0} = (\beta+\varepsilon)f(a)b + f(a)(\alpha a-\varepsilon b) = \alpha f(a)a + \beta f(a)b,$$
$$\frac{d}{ds}B_{h^s}^{-1}|_{s=0} = -\frac{f(a)}{a}(\varepsilon a-\delta b) + (\gamma+\varepsilon)f(a) = \gamma f(a) + \delta\frac{f(a)}{a}b.$$

These imply $\frac{d}{ds}u_{h^s}(b)|_{s=0} = D_f(b)$. Thus (i) is proved. (ii) is a direct consequence of (12) in Theorem 3, which is obtained by putting $h = h^s$ and $h' = h^{s'}$, since $h^s * h^{s'} = h^{s+s'}$. □

8. Non-abelian cohomology

In this last section we point out that several elements defined in this paper satisfy the 1-cocycle condition for group cohomology.

We recall the definition of (non-abelian) group cohomology. The basic notions and properties can be found in [12]. Let G and N be arbitrary groups for a moment. Suppose G acts on N from left as group automorphisms: $G \times N \longrightarrow N, (h, n) \mapsto hn$. The map $w : G \longrightarrow N$ satisfying $w(hh') = (hw(h'))w(h)$ for any $h, h' \in G$ is called a 1-cocycle and the set of 1-cocycles is denoted by $Z^1(G, N)$. If N is abelian, $Z^1(G, N)$ has a natural structure of abelian group. Two cocycles w, w' are called cohomologous, $v \sim v'$, if there exists $n \in N$ such that $w(h) = (hn)w'(h)n^{-1}$ for all $h \in G$. The 1st cohomology $H^1(G, N)$ of G with coefficient N is defined as the quotient set $Z^1(G, N)/\sim$.

Let us return to our situation. Recall that $G = 1 + Xk[[X]]$ forms a group under the operation $*$ defined in Proposition 6. By means of Theorem 3 G acts on $\hat{R}$ (from left) by Δ_h as algebra automorphisms. Especially G acts on $(\hat{R})^{\times}$ (the unit group of $\hat{R}$) as group automorphisms. Hence we take $G = (1 + Xk[[X]], *)$ and $N = (\hat{R})^{\times}$. In this case

$$Z^1(G, N) = \{w : G \longrightarrow N \mid w_{h*h'} = \Delta_h(w_{h'})w_h\}.$$

In Lemma 2 and in the proof, we have proved the following.

$$u_{h*h'} = \Delta_h(u_{h'})u_h, \qquad v_{h*h'} = \Delta_h(v_{h'})v_h, \quad B_{h*h'} = \Delta_h(B_{h'})B_h.$$

These show the following proposition.

Proposition 7. *The maps u, v and B are 1-cocycles: $u, v, B \in Z^1(G, (\hat{R})^{\times})$.*

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