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Coherence completions of categories

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Abstract

This is the first of a series of papers on coherence completions of categories. Here we show that there is a close connection between Girard's coherence spaces and free bicomplete categories. We introduce a new construction for creating models of linear logic, the coherence completion of a category. By presenting coherence completions as categories enriched over the category of pointed sets and the category of coherence spaces, the free structures on coherence completions are obtained in a very natural way. We show that if $\mathbf{C}$ is monoidal closed or $\star$-autonomous then so is its coherence completion. We also prove that if $\mathbf{C}$ is a model of linear logic then so is its coherence completion. A key idea of the paper which is introduced into linear logic is the notion of softness. We hope that this idea could be of use in solving the full completeness for larger fragments of linear logic. © 1999 Elsevier Science B.V. All rights reserved.

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0. Introduction

Linear logic arose from the semantic study of the structure of the proofs in intuitionistic logic. In [11] linear logic first appeared as a kind of linear algebra built on *coherence spaces*. Coherence spaces are certain simplified *Scott domains* [33, 34] which have good properties with respect to *stability* [4]. These spaces were first intended as a denotational semantics for intuitionistic logic, but further analysis revealed that the semantics of the connectives of intuitionistic logic could be decomposed into more primitive structure. This led to the creation of linear logic. Recently, Lafont and Streicher [25], Ehrhard [8], Lamarche [26] and Girard [14, 15] have developed semantics of linear logic by generalizing coherence spaces. These approaches have proven fruitful in providing new models of linear logic.

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In this paper we present *coherence completions* of categories which, in a certain sense, generalize Girard's coherence spaces. We show that there is a close connection between coherence completions and *free bicomplete categories* [21].

The present work is based on the categorical notion of *softness* as it applied to the category *Coh* of coherence spaces and *linear morphisms* (see Section 3.3). The concept of softness appeared in the study of free bicomplete categories [21, 22] as a categorical generalization of *Whitman's condition* on lattice theory [9, 36]. Softness is a structural rule concerning morphisms from limits into colimits. In the case of lattice theory, a lattice L is said to be *soft* if it satisfies Whitman's condition, that is, for any a, b, c and $d \in L$, $a \wedge b \leqslant c \vee d$ can hold in L only in one of the four trivial ways:

$$a \leqslant c \vee d, \text{ or } b \leqslant c \vee d, \ a \wedge b \leqslant c, \text{ or } a \wedge b \leqslant d.$$

In *Coh* softness has the following consequence: for any coherence spaces I, J, K and L, each linear morphism from the *conjunction* (*binary product*) of I and J into the *disjunction* (*binary coproduct*) of K and L factors through either a product projection or a coproduct injection. We show that the full subcategory of *Coh* whose objects are generated from the singleton space under products and coproducts is exactly the free bicomplete category of the singleton under the zero object and products and coproducts (see Section 4.3).

The first step we take here is to extend coherence spaces to $\mathbf{C}$-*coherence spaces* for an arbitrary category $\mathbf{C}$, called the coherence completion $Coh(\mathbf{C})$ of $\mathbf{C}$. *Coh* is now viewed as $Coh(\{*\})$, the coherence completion of the singleton. The construction of $Coh(\mathbf{C})$ is based on *Coh*: each $\mathbf{C}$-coherence space consists of a coherence space and a family of objects in $\mathbf{C}$; each $\mathbf{C}$-*linear morphism* is a linear morphism together with a family of arrows in $\mathbf{C}$. This construction is analogous to constructions of *free completions* of categories [10, 18, 29], where the category of sets played a similar role as that of *Coh*.

The main techniques we used here are *categorically enriched structures*. More precisely, we view $Coh(\mathbf{C})$ as a category enriched over the category $\mathbf{Set}_*$ of pointed sets as well as enriched over *Coh*. With the approach of $Coh(\mathbf{C})$ enriched over $\mathbf{Set}_*$, the full subcategory of $Coh(\mathbf{C})$ whose objects are *contractible* $\mathbf{C}$-*coherence spaces* is exactly the free bicomplete category of $\mathbf{C}$ under the zero object and products and coproducts. In view of $Coh(\mathbf{C})$ enriched over *Coh*, we present the coherence completions as a monad on the category of categories and functors, and show the pointwise nature of connectives of linear logic in the coherence completions.

A key idea explored in this paper is the *enriched softness* between products and coproducts. This important feature is directly related to the associativity of composition, which has been investigated by a number of researchers working in the semantics of computation (see [1, 5, 32]). One of relevant works we would like to emphasize is Abramsky's *interaction categories* which were proposed in [2] as a new paradigm for the semantics of functional and concurrent computation. The major difference between [2] and the present work is that interaction categories admit (possibly weak) *biproducts*.

Other enriched softness studied in the paper are those between the *external tensor products* and the *external par* in the coherence completions. We investigate the *mixed associativities* (called *weak distributivities* in [6, 19, 27]) between those operations and show additional properties between those operations, which make further connections to Cockett and Seely's work on *weakly distributive categories* [6] and Hyland and de Paiva's work on *tensor-par logic* [19].

The paper is organized as follows. In Section 1 we first define the coherence completion of a category and then study their enriched structures. In Section 2 we present the coherence completions as a monad on the category of categories and functors and study the external tensor product and its dual in the coherence completions. Using the pointwise nature of connectives of linear logic in the coherence completions, we show that if **C** is *monoidal closed*, or $\star$-*autonomous* then so is the coherence completion of **C**. Furthermore, if **C** is a model of linear logic then so is its coherence completion. Section 3 studies various forms of softness in coherence completions. We first show that the coherence completions as enriched structures (over category of pointed sets and the category of coherence spaces) have softness between products and coproducts and then study the softness between the external tensor products and their duals. In Section 4 we introduce contractible **C**-coherence spaces and show that the category of contractible **C**-coherence spaces is exactly the free bicomplete category of **C** under the zero object and products and coproducts. We also prove that *non-contractible* **C**-*coherence spaces* can be constructed from contractible **C**-coherence spaces under limits, or dually under colimits.

1. C-valued coherence spaces

In this section, we define the coherence completion of a category and describe its enriched structures. After giving a brief review of Girard's coherence spaces, we extend coherence spaces to **C**-valued coherence spaces with an arbitrary category **C**, called the coherence completion of **C**. The construction of the coherence completion of a category is quite similar to those of free completions of a category under products and coproducts: its objects are just families of objects of **C** associated with coherence spaces and, its arrows are families of arrows of **C** associated with linear morphisms between coherence spaces. The enriched structures of the coherence completion described in this section are crucial for the present work. Indeed, as we will see, the later discussions are mainly based on those enriched structures, namely the enriched structures over the category of pointed sets and the category of coherence spaces.

Recall from [11, 35] that a *web* is a pair $web(A) = (|A|, \sim_A)$ where $|A|$ is a set and $\sim_A$ is a symmetric and reflexive relation on $|A|$. A subset a of $|A|$ is said to be a *coherence subset* if for all $x, y \in a$, $x \sim_A y$. The set of coherence subsets of $web(A)$

ordered under inclusion

$$A = \{a \subset A \mid \forall x, y \in a \ (x \sim_A y)\}$$

is called a *coherence space* (or called a *Girard domain*, see [8, 32, 35]). We write A_{fin} for the collection of finite coherence subsets of A, and denote the relation $\sim_A$ on $|A|$ mostly by $\sim$, when that is convenient.

The elements of $|A|$ are called *atoms* (or *tokens*) of the coherence space A. The atoms of a coherence space represent atomic bits of information; a coherence set is a consistent piece of information. Coherence of atoms means that the atoms may be regarded as bits of information concerning the same object. The order of information is reflected by inclusion: $a \subset b$ means that b presents more information than a.

For any coherence space A, we have basic properties as follows:

(i) A contains all singletons $\{x\}$, for $x \in |A|$;
(ii) if $a \in A$ and $b \subset a$ then $b \in A$;
(iii) $X \subset A$, $\forall x, y \in X (x \cup y \in A) \Rightarrow \bigcup X \in A$;
(iv) $\emptyset \in A$;
(v) if $Y \subset A$ is directed with respect to inclusion, then $\bigcup Y \in A$.

When $X \subset P(|A|)$ satisfies (i)–(iii), we can define a reflexive and symmetric relation $\sim$ on $|A|$:

$$x \sim y \Leftrightarrow \{x, y\} \in X.$$

It is clear that $X = Coh(A)$.

Definition 1.1 (*Girard* [11]). (i) Let A and B be coherence spaces. The linear implication $web(A) \multimap web(B)$ from $web(A)$ into $web(B)$ is defined by a $web(|A| \times |B|, \sim_{\multimap})$ such that

$$(x, y) \sim_{\multimap} (x', y') \Leftrightarrow x \sim_A x' \Rightarrow (y \sim_B y' \text{ and } (y = y' \Rightarrow x = x')).$$

The coherence space corresponding to this *web* is called the linear implication of A and B, and is written as $A \multimap B$.

(ii) A linear morphism from A into B is defined by a coherence subset of $web(A) \multimap web(B)$, i.e., a set $X \subset |A| \times |B|$ such that

(a) if $(x, y), (x', y') \in X$, then $x \sim_A x' \Rightarrow y \sim_B y'$; and
(b) if $(x, y), (x', y) \in X$, then $x \sim_A x' \Rightarrow x = x'$.

For any coherence spaces A, B and C, the set $id_A = \{(x,x) | x \in |A|\}$ is a linear morphism on A, i.e., it is the identity on A. Also, for any linear morphisms $f : A \to B$ and $g : B \to C$, the composite $g \circ f$ is defined by

$$g \circ f = \{(x,z) | \exists y \in |B| ((x, y) \in f \text{ and } (y,z) \in g)\},$$

which is just the composition of graphs (relations). Notice that if $(x,z) \in g \circ f$ then there is a unique y such that $(x,y) \in f$ and $(y,z) \in g$. Indeed, suppose that there is another y' such that $(x,y') \in f$ and $(y',z) \in g$. That $y \sim_B y'$ follows from (ii)(a) of Definition 1.1 as $(x,y) \in f$ and $(x,y') \in f$. But $(y,z) \in g$ and $(y',z) \in g$, $y = y'$ therefore follows from (ii)(b) of Definition 1.1. Also, it is easy to verify that $g \circ f$ is a linear morphism from A into C and the composition is associative. We therefore form a category *Coh* whose objects and arrows are coherence spaces and linear morphisms, respectively.

Remark 1.2. (1) The above linear morphisms between coherence spaces A and B are described as coherence subsets of $web(A) \multimap web(B)$. There is an alternative definition of linear morphisms related to the linear maps. We summarize as follows (see [12] for more details). Consider coherence spaces as posets, an order-preserving map $F : A \to B$ is said to be *continuous* if for X directed with respect to $\subset$ in A,

$$F(\bigcup X) = \bigcup \{F(b) \mid b \in X\}.$$

F is said to be *stable* if F is continuous and satisfies

$$a \cup b \in A \;\Rightarrow\; F(a \cap b) = F(a) \cap F(b) \quad \text{(stability)}.$$

F is said to be a *linear map* if F is stable and satisfies

$$X \subset A, \text{ and for all } b, c \in X, \quad b \cup c \in A \;\Rightarrow\; F(\bigcup X) = \bigcup \{F(b) \mid b \in X\}.$$

There is a one to one correspondence between linear maps from A to B and coherence subsets of $web(A) \multimap web(B)$:

(a) For any linear map F from A to B, we associate its *trace*

$$tr(F) = \{(a,b) \mid b \in F(\{a\})\},$$

which is a coherence subset of $web(A) \multimap web(B)$.

(b) For any coherence subset X of $web(A) \multimap web(B)$, we associate a linear map from A to B

$$F_X(a) = \{y \mid \exists x \in a((x,y) \in X)\}.$$

(2) The category *Coh* is a category enriched over the category $\mathbf{Set}_*$ of *pointed sets*. In fact, for coherence spaces A and B, the empty set $\emptyset$ is a coherence subset of $web(A) \multimap web(B)$, i.e., it is a linear morphism from A into B. The hom-set between A and B can therefore be viewed as a pointed set $(Coh(A,B), \emptyset)$.

(3) Since the objects and the arrows of *Coh* are entirely given by the corresponding webs and coherence subsets, in the remaining part of the paper all properties of *Coh* and the coherence completion of a category will be treated in terms of the corresponding webs and coherence subsets.

For any category $\mathbf{C}$, we define the *coherence completion* $Coh(\mathbf{C})$ of $\mathbf{C}$, i.e., the category of $\mathbf{C}$-*valued coherence spaces* and $\mathbf{C}$-*linear morphisms*, as follows.

(i) Objects of $Coh(\mathbf{C})$ are determined by $\mathbf{C}$-*webs*: for any coherence space I with $web(I)=(|I|,\sim)$, we form objects

$$A_I=(|I|,\sim,\{A_i\}_{i\in|I|}),$$

which are called $\mathbf{C}$-webs; here A_i are arbitrary objects of $\mathbf{C}$. The $\mathbf{C}$-(*valued*) *coherence subsets* of A_I and $\mathbf{C}$-(*valued*) *coherence spaces* associated with A_I are defined in the same way as those of *Coh*. When $|I|=\emptyset$, we have the empty web $(\emptyset,\sim,\emptyset)$, which is the zero object.

(ii) Arrows of $Coh(\mathbf{C})$: for $\mathbf{C}$-webs $(|I|,\sim,\{A_i\}_{i\in|I|})$ and $(|J|,\sim,\{B_j\}_{j\in|J|})$, a $\mathbf{C}$-*linear morphism* from $(|I|,\sim,\{A_i\}_{i\in I})$ to $(|J|,\sim,\{B_j\}_{j\in|J|})$ is defined by a pair

$$f=(t,\{f_{i,j}\}_{(i,j)\in t}):A_I\to B_J,$$

where $t:I\to J$ is a linear morphism of *Coh* and $f_{i,j}:A_i\to B_j$ are arrows of $\mathbf{C}$, for all $(i,j)\in t$. When $t=\emptyset$, we have $(\emptyset,\emptyset)$, which is the zero arrow from A_I to B_J.

Let $f=(t,\{f_{i,j}\}_{(i,j)\in t}):A_I\to B_J$ and $g=(s,\{g_{j,k}\}_{(j,k)\in s}):B_J\to C_K$ be $\mathbf{C}$-linear morphisms. The compositions $g\circ f$ is defined to be the pair

$$g\circ f=(s\circ t,\{g_{j,k}\circ f_{i,j}\})$$

where the family $\{g_{j,k}\circ f_{i,j}\}$ is the collection of all compositions $g_{j,k}\circ f_{i,j}$ with $(i,j)\in t$ and $(j,k)\in s$.

Proposition 1.3. *For any category* $\mathbf{C}$, $Coh(\mathbf{C})$ *forms a category. Moreover*, *Coh is isomorphic to* $Coh(\{*\})$; *here* $\{*\}$ *is the category with one object* $*$ *and one* (*identity*) *arrow.*

Proof. Let $A_I=(|I|,\sim,\{A_i\}_{i\in|I|})$, $B_J=(|J|,\sim,\{B_j\}_{j\in|J|})$ and $C_K=(|K|,\sim\{C_k\}_{k\in|K|})$ be $\mathbf{C}$-webs. It is clear that

$$id_{A_I}=(id_I,(id_{A_i})_{i\in I}):A_I\to A_I$$

is a $\mathbf{C}$-linear morphism, i.e., it is the identity on A_I. Let $f=(t,\{f_{i,j}\}_{(i,j)\in t}):A_I\to B_J$ and $g=(s,\{g_{j,k}\}_{(j,k)\in s}):B_J\to C_K$ be $\mathbf{C}$-linear morphisms. That $g\circ f$ is a $\mathbf{C}$-linear morphism from A_I into C_K follows from $s\circ t$ being a linear morphism from I into J in *Coh*. In order to verify the associativity of composition in $Coh(\mathbf{C})$, let $h=(w,\{h_{k,l}\}_{(k,l)\in w}):C_K\to D_L$ be a $\mathbf{C}$-linear morphism. If $(i,l)\in w\circ(s\circ t)$ then there is a unique $k\in K$ such that $(i,k)\in s\circ t$ and $(k,l)\in w$. From $(i,k)\in s\circ t$, there is a unique $j\in J$ such that $(i,j)\in t$ and $(j,k)\in s$. The uniqueness of k and j implies that $(i,l)\in w\circ(s\circ t)$ iff $(i,l)\in(w\circ s)\circ t$. We obtain that $h\circ(g\circ f)=(h\circ g)\circ f$ from the identity

$$h_{k,l}\circ(g_{j,k}\circ f_{i,j})=(h_{k,l}\circ g_{j,k})\circ f_{i,j}$$

of $\mathbf{C}$.

This shows that $Coh(\mathbf{C})$ forms a category.

The isomorphism between Coh and $Coh(\{*\})$ is quite clear, as the objects and arrows of $Coh(\{*\})$ are only dependent on those of Coh. $\square$

Proposition 1.4. *$Coh(\mathbf{C})$ has products, coproducts and a zero object.*

Proof. We derive products, coproducts and the zero object of $Coh(\mathbf{C})$ from those of Coh as follows.

Firstly, the zero object of Coh is the coherence space 0 with the empty web. Consequently, the zero object of $Coh(\mathbf{C})$ is given by the empty web $(\emptyset, \sim, \emptyset)$.

Consider an arbitrary family of coherence spaces $(X_i)_{i\in I}$, the product of (X_i) in Coh is determined by the web

$$\prod_{i\in I} web(X_i) = \left(\bigcup_{i\in I} |X_i|, \sim\right);$$

here $\bigcup_{i\in I} |X_i|$ is the disjoint union of all $|X_i|$, and we represent it as $\bigcup(\{i\}\times|X_i|)$. The relation $\sim$ on $\bigcup_{i\in I}|X_i|$ is defined by

(i) $(i,x)\sim(i,x')$ iff $x \sim_{X_i} x'$ for $i\in I$; and

(ii) $(i,x)\sim(j,y)$ for $i\neq j$, $x\in|X_i|$ and $y\in|X_j|$.

The projection $p_{X_i}: \prod X_i \to X_i$ is given by the set

$$\{((i,x),x) \mid x\in|X_i|\}.$$

For $\mathbf{C}$-coherence spaces $A_{X_i} = (|X_i|, \sim_{X_i}, \{A_x\}_{x\in|X_i|})$ with $i\in I$, we can see that the product of all A_{X_i} is given by the $\mathbf{C}$-web:

$$\left(\bigcup_{i\in I}|X_i|, \sim, \bigcup_{i\in I}\{A_x\}_{x\in|X_i|}\right).$$

And the projection

$$p_{A_{X_i}} = (p_{X_i}, \{id_{A_x}\}_{x\in|X_i|}) : \prod A_{X_i} \to X_i;$$

here p_{X_i} is the projection of the product of X_i and id_{A_x} is the identity on A_x for each $x\in|X_i|$.

The coproduct of coherence spaces X_i is determined by the web

$$\sqcap_{i\in I} X_i = (\bigcup |X_i|, \sim_{\cup});$$

here $\sim_{\cup}$ is the relation on $\bigcup|X_i|$ with the above (i) and

(ii)$'$ $(i,x) \not\sim_{\cup} (j,y)$ for $i\neq j$, $x\in|X_i|$ and $y\in|X_j|$.

The injection $q_{X_i} : X_i \to \sqcup X_i$ is the set

$$q_{X_i} = \{(x,(i,x)) \mid x \in X_i\}$$

for all $i \in I$. Hence the coproduct of **C**-webs A_{X_i} is given by the **C**-web:

$$\left(\bigcup_{i \in I} |X_i|, \sim_{\cup}, \bigcup_{i \in I} \{A_x\}_{x \in |X_i|} \right)$$

with the injections $q_{A_{X_i}} = (q_{X_i}, \{id_{A_x}\}_{x \in |X_i|})$ for all $i \in I$. $\square$

We now discuss the *enriched structures* of *Coh*(**C**). Let us first review some basic concepts to be needed later.

Let $\mathscr{V}$ be a category. $\mathscr{V}$ is *monoidal* if there is a functor $(-) \otimes (-) : \mathscr{V} \times \mathscr{V} \to \mathscr{V}$, an object T (called the unit) and isomorphisms

$$c_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$

$$r_A : A \otimes T \to A \quad \text{and} \quad l_A : T \otimes A \to A$$

defined and natural for all objects A, B and C of $\mathscr{V}$. These are subject to coherence conditions that essentially means that no non-trivial automorphisms can be constructed using only these arrows. See [24] for more details on this and later coherences.

The monoidal category is *symmetric* if there are isomorphisms

$$s_{A,B} : A \otimes B \to B \otimes A$$

that are subject to similar naturality and coherence conditions. The monoidal category is *closed* if there is a bifunctor $(-) \multimap (-) : \mathscr{V}^{op} \times \mathscr{V} \to \mathscr{V}$ such that for each object A of $\mathscr{V}$, the functor $A \multimap (-)$ is right adjoint to $A \otimes (-)$. This means that for any objects B and C of $\mathscr{V}$, we have

$$\mathscr{V}(A \otimes B, C) \cong \mathscr{V}(B, A \multimap C)$$

and these isomorphisms are natural in B and C.

Proposition 1.5 (Girard [12]). *Coh is a symmetric monoidal closed category.*

Proof. For coherence spaces I and J, the *tensor product* $I \otimes J$ is the coherence space with the web $(|I| \times |J|, \sim_{\otimes})$; where the relation $\sim_{\otimes}$ on $|I| \times |J|$ is defined by

$$(i,j) \sim_{\otimes} (i',j') \Leftrightarrow i \sim_I i' \quad \text{and} \quad j \sim_J j'.$$

It is clear that there is an isomorphism between $I \otimes J$ and $J \otimes I$. The *unit* T is given by the coherence space $1 = (\{*\}, \sim)$ with the singleton web. The bifunctor $(-) \multimap (-)$ is defined by the linear implication $I \multimap J$ for all I and J of *Coh*, and up to canonical isomorphisms,

$$(I \otimes J) \multimap K = I \multimap (J \multimap K). \quad \square$$

For a monoidal category $\mathscr{V}$ with the unit T, recall from [24] that a category $\mathbf{A}$ *enriched over* $\mathscr{V}$ (also called $\mathscr{V}$*-category*) is defined by a map

$$[-,-] : Ob(\mathbf{A}) \times Ob(\mathbf{A}) \to Ob(\mathscr{V})$$

together with associative morphisms and unit morphisms

$$c_{A,B,C} : [A,B] \otimes [B,C] \to [A,C]$$

$$u_A : T \to [A,A]$$

for all $A,B,C \in \mathbf{A}$. Here $c_{A,B,C}$ and u_A satisfy the associativity axiom and the unit axiom, respectively.

Our first observation is the fact that $Coh(\mathbf{C})$ is a category enriched over $\mathbf{Set}_*$. Indeed, for any A_I and B_J of $Coh(\mathbf{C})$, $[A_I, B_J]$ is defined by the pair of the set $Hom(A_I, B_J)$ and the zero morphism from A_I to B_J, c_{A_I,B_J,C_K} is just the usual composition with $\otimes$ being the binary product in $\mathbf{Set}_*$, and u_{A_I} is defined by the morphism which maps the singleton $*$ into the identity on A_I.

The second enriched structure of $Coh(\mathbf{C})$ we want to stress is given in the following.

Proposition 1.6. *For a category* $\mathbf{C}$, $Coh(\mathbf{C})$ *is a category enriched over Coh.*

Proof. For $\mathbf{C}$-coherence spaces $A_I = (|I|, \sim_I, \{A_i\}_{i \in |I|})$ and $B_J = (|J|, \sim_J, \{B_j\}_{j \in |J|})$, we define $[A_I, B_J]$ as a coherence space determined by the following web $(|[A_I, B_J]|, \sim)$. $|[A_I, B_J]|$ is the set of all triples (i,j,f), for all $i \in |I|, j \in |J|$ and $f : A_i \to B_j$ of $\mathbf{C}$. The relation $\sim$ is defined by that for any (i,j,f) and (i',j',g) of $|[A_I, B_J]|$

$$(i,j,f) \sim (i',j',g) \quad \text{iff} \quad i \sim_I i' \;\Rightarrow\; (j \sim_J j' \text{ and } (j = j' \;\Rightarrow\; i = i' \text{ and } f = g)).$$

For any A_I, B_J and $C_K = (|K|, \sim_K, \{C_k\}_{k \in |K|})$ of $Coh(\mathbf{C})$, the associativity morphism

$$c_{A_I,B_J,C_K} : [A_I, B_J] \otimes [B_J, C_K] \to [A_I, C_K]$$

is defined by

$$((i,j,f),(j,k,g)) \mapsto (i,k,g \circ f).$$

That the map c_{A_I,B_J,C_K} is a linear morphism follows from the fact that the associativity morphism

$$c_{I,\ J,\ K} : [I,J] \otimes [J,K] \to [I,K]$$

$$((i,j),(j,k)) \mapsto (i,k)$$

is linear. More precisely, the associativity morphism $c_{I,J,K}$ corresponds by adjunction with the composition of the linear morphisms

$$\begin{array}{c}
[I,J]\otimes[J,k]\otimes I \\
\Big\downarrow{\cong} \\
[I,J]\otimes I\otimes[J,\ K] \\
\Big\downarrow{ev_{I,J}\otimes id_{[J,K]}} \\
J\otimes[J,K] \\
\Big\downarrow{\cong} \\
[J,K]\otimes J \\
\Big\downarrow{ev_{J,K}} \\
K
\end{array}$$

where the evaluation morphism

$$ev_{I,J}:[I,J]\otimes I\to J$$

$$((i,j),i)\mapsto j$$

corresponds by the adjunction with the identity on $[I,J]$.

The unit morphism $u_{A_I}:1\to[A_I,A_I]$ is defined to be the set of all pairs $(*,(i,i,id_{A_i}))$ with $i\in|I|$, which of course is a linear morphism.

Since *Coh* is a symmetric monoidal closed category, *Coh* is enriched over *Coh*. The associativity axiom on c_{A_I,B_J,C_K} and the unit axiom on u_{A_I} are derived from those on $c_{I,J,K}$ and u_I, respectively.

2. The connectives of linear logic in *Coh*(C)

The goal of this section is to study the *connectives* of linear logic in coherence completion. After presenting the coherence completions of categories as a monad on the category of categories and functors, we introduce the *external tensor product* and its dual on coherence completions of categories. These *external operations* provide the pointwise nature of connectives of linear logic in the coherence completion. The main result proved here is that if $\mathbf{C}$ is monoidal closed, or $\star$-autonomous then so is $Coh(\mathbf{C})$; moreover, if $\mathbf{C}$ is a model of linear logic then so is $Coh(\mathbf{C})$.

Let *Cat* be the category of categories and functors. We start to describe a *monad structure*

$$Coh(-): Cat \to Cat.$$

For a functor $F: \mathbf{B} \to \mathbf{C}$, $Coh(-)$ takes F into the functor $Coh(F): Coh(\mathbf{B}) \to Coh(\mathbf{C})$ which is defined by

$$Coh(F)(|I|, \sim, \{A_i\}_{\in I}) = (|I|, \sim, \{F(A_i)\}_{i \in I})$$

$$Coh(F)((t, \{f_{i,j}\}_{(i,j) \in t}) = (t, \{F(f_{i,j})\}_{(i,j) \in t})$$

for any arrow $(t, \{f_{i,J}\}_{(i,j) \in t})$ of $Coh(\mathbf{B})$. The functorialities of $Coh(F)$ and $Coh(-)$ can be checked easily.

The *unit* $i: id_{Cat} \to Coh(-)$ is defined as follows. For a category $\mathbf{B}$, the component

$$i_{\mathbf{B}}: \mathbf{B} \to Coh(\mathbf{B})$$

$$B \mapsto (\{*\}, \{B\})$$

is the functor so that $i_{\mathbf{B}}(f) = (id_{\{*\}}, \{f\})$ for any arrow $f: A \to B$ of $\mathbf{B}$. Let $F: \mathbf{B} \to \mathbf{C}$ be a functor between categories $\mathbf{B}$ and $\mathbf{C}$, then $Coh(F) \circ i_{\mathbf{B}} = i_{\mathbf{C}} \circ F$. This says that i is a natural transformation between id_{Cat} and $Coh(-)$.

To describe the *multiplication* $U: Coh(-)^2 \to Coh(-)$, let $\mathbf{C}$ be a category, we define the component

$$U_{\mathbf{C}}: Coh^2(\mathbf{C}) \to Coh(\mathbf{C})$$

as follows. For any object

$$B_I = (|I|, \sim_I, \{B_i\}_{i \in |I|})$$

of $Coh^2(\mathbf{C})$, then each $B_i = (|M_i|, \sim_{M_i}, \{B_{i_m}\}_{i_m \in |M_i|})$ is a $\mathbf{C}$-web. We define a $\mathbf{C}$-web

$$U_{\mathbf{C}}(B_I) = (|M|, \sim_M, \{B_{i_m}\}_{i_m} \in |M|)$$

so that $|M| = \bigcup |M_i|$, the disjoint union of all $|M_i|$ and, the relation $\sim_M$ on $|M|$ is defined by that for any $i_m \in |M_i|$ and $i'_{m'} \in |M_{i'}|$,

(a) $i \neq i' \;\Rightarrow\; (i_m \sim_M i'_{m'} \text{ iff } i \sim_I i')$;

(b) $i = i' \;\Rightarrow\; (i_m \sim_M i_{m'} \text{ iff } i_m \sim_{M_i} i_{m'})$.

For any arrow

$$f = (t, \{f_{i,j}\}_{(i,j) \in t}): B_I \to C_J$$

of $Coh^2(\mathbf{B})$ with $C_J = (|J|, \sim_J, \{C_j\}_{j \in |J|})$, then each $C_j = (|N_j|, \sim_{N_j}, \{C_{j_n}\}_{j_n \in |N_j|})$ is a $\mathbf{C}$-web, and for $(i,j) \in t$, $f_{i,j} : B_i \to C_j$ is given by a pair $(s_{i,j}, \{g_{i_m,j_n}\}_{(i_m,j_n) \in s_{i,j}})$ where each $s_{i,j} : M_i \to N_j$ is a linear morphism and each $g_{i_m,j_n} : B_{i_m} \to C_{j_n}$ is an arrow of $\mathbf{C}$. We define $U_{\mathbf{C}}(f) : U_{\mathbf{C}}(B_I) \to U_{\mathbf{C}}(C_J)$ to be a pair

$$(s, \{g_{i_m,j_n}\}_{(i_m,j_n) \in s})$$

so that s is the disjoint union of $s_{i,j}$ for all $(i,j) \in t$. That s is a linear morphism from M into N is shown as follows.

For any (i_m, j_n) and $(i'_{m'}, j'_{n'})$ of s, if $i_m \sim_M i'_{m'}$, there are only two possibilities:

Case 1: $i = i'$ and $i_m \sim_{M_i} i_{m'}$. Since $s_{i,j}$ is a linear morphism, Case 1 implies that $j = j'$ and $j_n \sim_{N_j} j_{n'}$. So $j_n \sim_N j_{n'}$.

Case 2: $i \neq i'$ and $i \sim_I i'$. Since (i,j) and (i',j') are in t, Case 2 implies that $j \neq j'$ and $j \sim_J j'$. Consequently, $j_n \neq j'_{n'}$ and $j_n \sim_N j'_{n'}$.

The functoriality of $U_{\mathbf{C}}$ is quite clear. For any functor $F : \mathbf{C} \to \mathbf{D}$, the following diagram

$$\begin{tikzcd} Coh^2(\mathbf{C}) \arrow[r, "U_{\mathbf{C}}"] \arrow[d, "Coh^2(F)"'] & Coh(\mathbf{C}) \arrow[d, "Coh(F)"] \\ Coh^2(\mathbf{D}) \arrow[r, "U_{\mathbf{D}}"] & Coh(\mathbf{D}) \end{tikzcd}$$

commutes. Indeed, for $f : B_I \to C_J$ of described above, we can see that

$$U_{\mathbf{D}} \circ Coh^2(F)(B_I) = Coh(F) \circ U_{\mathbf{C}}(B_I) = (|M|, \sim_M, \{F(B_{i_m})\}_{i_m \in |M_i|}),$$

$$U_{\mathbf{D}} \circ Coh^2(F)(f) = Coh(F) \circ U_{\mathbf{C}}(f) = (s, \{F(g_{i_m,j_n})\}_{i_m,j_n) \in s}).$$

This shows that $U : Coh(-)^2 \to Coh(-)$ is a natural transformation.

Proposition 2.1. *The triple* $(Coh(-), i, U)$ *is a monad on Cat, that is, we have*

(i) (*Associative law*) *The following diagram*:

$$\begin{tikzcd} Coh(-)^3 \arrow[r, "Coh(-) \circ U"] \arrow[d, "U \circ Coh(-)"'] & Coh(-)^2 \arrow[d, "U"] \\ Coh(-)^2 \arrow[r, "U"] & Coh(-) \end{tikzcd}$$

commutes; *here* $Coh(-) \circ U : Coh(-)^3 \to Coh(-)^2$ *denotes the natural transformation with components* $(Coh(-) \circ U)_{\mathbf{C}} = Coh(U_{\mathbf{C}})$, *while* $Coh(-) \circ U$ *has components* $(U \circ Coh(-))_{\mathbf{C}} = U_{Coh(\mathbf{C})}$ *for any category* $\mathbf{C}$.

(ii) (*Unit law*) *The following diagram*

$$\begin{tikzcd}
Coh(-) \arrow[r] \arrow[d, equal] & Coh(-)^2 \arrow[r] \arrow[d, "U"] & Coh(-) \arrow[d, equal] \\
Coh(-) \arrow[r, equal] & Coh(-) \arrow[r, equal] & Coh(-)
\end{tikzcd}$$

commutes; *where the top arrows of the squares are the natural transformations* $i \circ Coh(-)$ *and* $Coh(-) \circ i$ *with components* $i_{Coh(\mathbf{C})}$ *and* $Coh(i_{\mathbf{C}})$ *respectively*, *for any category* $\mathbf{C}$.

Proof. For any category $\mathbf{C}$, each object $C_{K,I_k,M_{i,k}}$ of $Coh^3(\mathbf{C})$ can be represented as

$$(|K|, \sim_K, \{(|I_k|. \sim_{I_k}, \{(|M_{k,i}|, \sim_{M_{k,i}}, \{C_{k,i,m}\}_{m\in|M_{k,i}|})\}_{i\in|I_k|})\}_{k\in|K|})$$

where $K, I_k, M_{i,k} \in Coh$ and, $C_{k,i,m} \in \mathbf{C}$. A straightforward calculation shows that

$$\begin{aligned}
&U_{\mathbf{C}}(Coh(U_{\mathbf{C}})(C_{K,I_k,M_{k,i}})) \\
&= U_{\mathbf{C}}\left(\left(K, \left\{\left(\bigcup_{i\in|I_k|}|M_{k,i}|, \sim_{I_M}, \{C_{k,i,m}\}_{(i,m)\in|I_k|\times|M_{k,i}|}\right)\right\}_{k\in|K|}\right)\right) \\
&= \left(\bigcup_{(k,i)\in|K|\times|I_k|}|M_{k,i}|, \sim_{K_{I_M}}, \{C_{k,i,m}\}_{(k,i,m)\in|K|\times|I_k|\times|M_{k,i}|}\right).
\end{aligned}$$

Here $(i,m) \sim_{I_M} (i',m')$ is defined by

$$(i \neq i' \Rightarrow i \sim_{I_k} i') \quad \text{and} \quad (i = i' \Rightarrow m \sim_{M_{k,i}} m')$$

and $(k,i,m) \sim_{K_{I_M}} (k',i',m')$ is defined by

$$(k \neq k' \Rightarrow k \sim_K k') \quad \text{and} \quad (k = k' \Rightarrow (i,m) \sim_{I_M} (i',m')).$$

On the other hand, we can see that

$$\begin{aligned}
&U_{Coh(\mathbf{B})}(U_{\mathbf{B}}(C_{K,I_k,M_{k,i}})) \\
&= U_{\mathbf{B}}\left(\left(\bigcup_{k\in|K|}|I_k|, \sim_{K_I}, \{(|M_{k,i}|, \sim_{M_{k,i}}, \{C_{k,i,m}\}_{m\in|M_{k,i}|})\}_{(k,i)\in|K|\times|I_k|}\right)\right) \\
&= \left(\bigcup_{(k,i)\in|K|\times|I_k|}|M_{k,i}|, \sim_{M_{K_I}}, \{C_{k,i,m}\}_{(k,i,m)\in|K|\times|I_k|\times|M_{k,i}|}\right).
\end{aligned}$$

Here $(k,i) \sim_{K_I} (k',i')$ is defined by

$$(k \neq k' \Rightarrow k \sim_K k') \quad \text{and} \quad (k = k' \Rightarrow i \sim_{I_k} i')$$

and $(k,i,m) \sim_{M_{K_I}} (k',i',m')$ is defined by

$$((k,i) \neq (k',i') \Rightarrow (k,i) \sim_{K_I} (k',i')) \quad \text{and} \quad ((k,i) = (k',i') \Rightarrow m \sim_{M_{k,i}} m').$$

We conclude that both $(k,i,m) \sim_{K_{I_M}} (k',i',m')$ and $(k,i,m) \sim_{M_{K_I}} (k',i',m')$ are defined by

(a) $k \neq k' \Rightarrow k \sim_K k'$;

(b) $k = k' \,\&\, i \neq i' \Rightarrow i \sim_{I_k} i'$;

(c) $k = k' \,\&\, i = i' \Rightarrow m \sim_{M_{k,i}} m'$.

This shows that the images of the functors $U_{\mathbf{C}} \circ Coh(U_{\mathbf{C}})$ and $U_{\mathbf{C}} \circ U_{Coh(\mathbf{C})}$ at objects of $Coh^3(\mathbf{C})$ coincide. A similar argumentation shows that these functors take an arrow of $Coh^3(\mathbf{C})$ into the same arrow of $Coh(\mathbf{C})$.

(ii) follows from the identities

$$U_{\mathbf{C}}(i_{Coh(\mathbf{C})}((t,f))) = U_{\mathbf{C}}((id_*, \{(t,f)\})) = (t,f)$$

and

$$U_{\mathbf{C}}(Coh(i_{\mathbf{C}})((t,f)) = U_{\mathbf{C}}((t, \{i_{\mathbf{C}}(f_{i,j})\}_{(i,j)\in t})) = (t,f)$$

for any $\mathbf{C}$-linear morphism $(t,f): A_I \to B_J$ with $f = \{f_{i,j}\}_{(i,j)\in t}$. $\square$

Proposition 2.2. *For any functor $F: \mathbf{B} \to \mathbf{C}$, the functor $Coh(F): Coh(\mathbf{B}) \to Coh(\mathbf{C})$ induced by F preserves products and coproducts.*

Proof. The structures of the product and the coproduct of a family $\{A_{I_n}\}_{i\in N}$ of $Coh(\mathbf{B})$ are entirely determined by the structures of I_n. $\square$

In order to describe the monoidal structure on the free coherence completion, for categories $\mathbf{B}$ and $\mathbf{C}$, we define the *external tensor product* $\otimes^{\mathrm{e}}$ and the dual external tensor product (or, called *external par*) $\odot^{\mathrm{e}}$ on $Coh(\mathbf{B}) \times Coh(\mathbf{C})$ as follows. Let

$$\otimes^{\mathrm{e}}, \odot^{\mathrm{e}} : Coh(\mathbf{C}) \times Coh(\mathbf{D}) \to Coh(\mathbf{C} \times \mathbf{D})$$

be the functors so that for $\mathbf{C}$-coherence spaces $A_I = (|I|, \sim_I, \{A_i\}_{i\in|I|})$ and $B_J = (|J|, \sim_J, \{B_j\}_{j\in|J|})$, $A_I \otimes^{\mathrm{e}} B_J$ is the $(\mathbf{C} \times \mathbf{D})$-coherence space

$$(|I| \times |J|, \sim_{I\otimes J}, \{(A_i, B_j)\}_{(i,j)\in|I|\times|J|}),$$

where $I \otimes J$ is defined in the Section 1. And $A_I \odot^{\mathrm{e}} B_J$ is the $(\mathbf{C} \times \mathbf{D})$-coherence space

$$(|I| \times |J|, \sim_{I\odot J}, \{A_i, B_j\}_{(i,j)\in(|I|\times|J|)}),$$

where $I \odot J$ is the *par* of I and J in Coh, that is, $I \odot J$ is the coherence space with the web $(|I| \times |J|, \sim_{\odot})$ so that

$$(i,j) \sim_{\odot} (i',j') \quad \Leftrightarrow \quad (i \sim_I i') \text{ or } (j \sim_J j').$$

It is clear that if $\mathbf{C}=\mathbf{D}=\{*\}$ then the external tensor product and the external par are exactly the tensor and the par of *Coh*.

Let $\circ : Coh(\mathbf{C}^{op}) \to Coh(\mathbf{C})^{op}$ be the functor such that for $A_I \in Coh(\mathbf{C})$, $\circ(A_I)$ is $A_{\neg(I)}$ with $\neg(I)=I \multimap 1$, denoted by A_I° and, for $f=(s,\{f_{i,j}\}) : A_I \to B_J$, $\circ(f)$ is $(\neg(s),\{f_{j,i}\})$, denoted by f°.

Proposition 2.3. *The above functor* $\circ$ *is an isomorphism. Moreover, for* $A_I \in Coh(\mathbf{C})$ *and* $B_J \in Coh(\mathbf{D})$, *up to isomorphisms, we have*

$$(A_I \odot^{\mathrm{e}} B_J)^\circ = A_I^\circ \otimes^{\mathrm{e}} B_J^\circ.$$

Proof. The proof follows from the duality between the tensor and par in *Coh*. □

Proposition 2.4. *Let* $\mathbf{B}$, $\mathbf{C}$ *and* $\mathbf{D}$ *be three categories. For* $B_I \in Coh(\mathbf{B})$, $C_J \in Coh(\mathbf{C})$ *and* $D_K \in Coh(\mathbf{D})$, *up to isomorphisms, we have the following associativities*:

$$B_I \otimes^{\mathrm{e}} (C_J \otimes^{\mathrm{e}} D_K) = (B_I \otimes^{\mathrm{e}} C_J) \otimes^{\mathrm{e}} D_K,$$

$$B_J \odot^{\mathrm{e}} (C_J \odot^{\mathrm{e}} D_K) = (B_I \odot^{\mathrm{e}} C_J) \odot^{\mathrm{e}} D_K.$$

Proof. An easy calculation. □

Let $\mathbf{C}=\{*\}$. For $A_I \in Coh(\mathbf{D})$, consider the functor

$$(-) \otimes A_I : Coh \to Coh(\mathbf{D})$$

$$J \mapsto (|J| \times |I|, \sim_{\otimes}, \{A_{j,i}\}_{(j,i) \in |J| \times |I|});$$

here $A_{j,i}=A_i$ for all $j \in J$. We have the bijections

$$\frac{J \otimes A_I \to B_K}{J \to [A_I, B_K]}$$

which is natural in J and B_K. We also have a functor

$$(-)^\circ \odot A_I : Coh^{op} \to Coh(\mathbf{D})$$

$$J \mapsto (|J| \times |I|, \sim_{\odot}, \{A_{j,i}\}_{(j,i) \in |J| \times |I|});$$

with the bijections

$$\begin{array}{c} B_K \to J^\circ \odot A_I \\ \hline J \otimes A_I^\circ \to B_K^\circ \\ \hline J \to [A_I^\circ, B_K^\circ] \\ \hline J \to [B_K, A_I] \end{array}$$

which is natural in J and B_K. This shows the following to be true.

Proposition 2.5. *For* $A_I \in Coh(\mathbf{D})$,
(i) *the functor* $(-) \otimes A_I$ *is left adjoint to the functor*

$$[A_I, (-)] : Coh(\mathbf{D}) \to Coh;$$

(ii) *the functor* $(-)^\circ \odot A_I$ *is right adjoint to the functor*

$$[(-), A_I] : Coh(\mathbf{D})^{op} \to Coh.$$

Let $\mathbf{C}$ be a monoidal closed category with respect to the bifunctors $(-)\star(-) : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ and $(-)/(-) : \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{C}$. We define the *tensor product*

$$(-) \otimes (-) : Coh(\mathbf{C}) \otimes Coh(\mathbf{C}) \to Coh(\mathbf{C})$$

on $Coh(\mathbf{C})$ to be the composition of the functors

$$\begin{tikzcd} Coh(\mathbf{C}) \times Coh(\mathbf{C}) \arrow[d, "(-) \otimes^e (-)"'] \\ Coh(\mathbf{C} \times \mathbf{C}) \arrow[d, "Coh((-)\star(-))"'] \\ Coh(\mathbf{C}) \end{tikzcd}$$

And the *function space* $(-) \multimap (-)$ on $Coh(\mathbf{C})$ is defined by the composition

$$\begin{tikzcd} Coh(\mathbf{C})^{op} \times Coh(\mathbf{C}) \arrow[d, "{[(-),(-)]}"'] \\ Coh(\mathbf{C} \times \mathbf{C}) \arrow[d, "Coh((-)/(-))"'] \\ Coh(\mathbf{C}) \end{tikzcd}$$

The above tensor product and the function space define a monoidal closed structure on $Coh(\mathbf{C})$. We have the following proposition.

Proposition 2.6. *If* $\mathbf{C}$ *is monoidal closed, or* $\star$*-autonomous, then so is* $Coh(\mathbf{C})$.

Proof. We recall from [3] that a category $\mathbf{C}$ is $\star$-*autonomous* if $\mathbf{C}$ is a closed symmetric monoidal category with a *negation* $\neg$ given by a *dualizing object*. For a coherence space I, remember that $\neg I$ is given by $I \multimap 1$ where 1 is the coherence space with singleton web. If $\mathbf{C}$ has a negation $\neg : \mathbf{C}^{op} \to \mathbf{C}$ given by a dualizing object, we then define the negation on $Coh(\mathbf{C})$ by

$$\neg(|I|, \sim, \{A_i\}_{i\in I}) = (|I|, \sim_{\neg I}, \{\neg A_i\}_{i\in I}).$$

Let A_I, B_J and C_K be objects of $Coh(\mathbf{C})$. The tensor product $A_I \otimes B_J$ and the linear implication $A_I \multimap B_J$ can be expilictly written as

$$A_I \otimes B_J = (|I| \times |J|, \sim_{\otimes}, \{A_i \star B_j\}_{(i,j)\in(|I|\times|J|)});$$

$$A_I \multimap B_J = (|I| \times |J|, \sim_{\multimap}, \{A_i/B_j\}_{(i,j)\in(|I|\times|J|)}).$$

Up to canonical isomorphisms, the identity

$$(A_I \otimes B_J) \multimap C_K = A_I \multimap (B_J \multimap C_K)$$

follows immediately from the identities

$$(A_i \star B_j)/C_k = A_i/(B_j/C_k),$$

$$(I \otimes J) \multimap K = I \multimap (J \multimap K).$$

Finally, let U be the unit for $\star$. Then the unit for the tensor in $Coh(\mathbf{C})$ is given by

$$1_U = (|1|, \sim_1, \{U\}).$$

A category is said to be *linear* if it is $\star$-autonomous with binary products (dually with binary coproducts). Combining Propositions 1.4 and 2.6, we have

Corollary 2.7. *If* $\mathbf{C}$ *is* $\star$-*autonomous then* $Coh(\mathbf{C})$ *is linear.*

We continue to discuss the connectives of classical linear logic in $Coh(\mathbf{C})$.

The *conjunction* $\sqcap$ and *disjunction* $\sqcup$ are just the binary product and coproduct discussed in Proposition 1.4. We have

Proposition 2.8. *Let* $\mathbf{C}$ *have a negation* $\neg : \mathbf{C}^{op} \to \mathbf{C}$. *Up to isomorphisms, we have*
(i) *for any* A_I *of* $Coh(\mathbf{C})$, $\neg(\neg(A_I)) = A_I$.
(ii) *For* A_I *and* B_J *of* $Coh(\mathbf{C})$,

$$A_I \sqcup B_J = \neg(\neg(A_I) \sqcap \neg(B_J)).$$

Proof. The proof is trivial. $\square$

In order to construct the *exponentials* on $Coh(\mathbf{C})$, recall that for any coherence space I, the exponential $!I$ can be defined by the following two approaches: the first approach is to define $|!I|$ to be the set of finite coherent subsets of I (see [11]); another approach is to define $|!I|$ to be the set of *coherent multisets* of atoms of $|I|$, i.e., the atoms of $|!I|$ are all the formal expressions $[x_1,\ldots,x_n]$ with $x=\{x_1,\ldots,x_n\}$ coherent subsets of I (see [13]). In those two approaches $\sim_{!I}$ is defined by

$$a \sim_{!I} b \quad \text{iff } a\cup b\in I_{\text{fin}}$$

$$[x_1,\ldots,x_n] \sim_{!I} [y_1,\ldots,y_m] \quad \text{iff } x\cup y\in I_{\text{fin}},$$

respectively. It is well-known that both approaches define *comonad structures* $(!,\varepsilon,\delta)$ such that for each $I\in Coh$, $!I$ is a $\otimes$-comonoid with the additional property that $!(I\sqcap J)=!I\otimes !J$.

Let us assume that $\mathbf{C}$ is a linear category equipped with a comonad $(!_{\mathbf{C}},\varepsilon_{\mathbf{C}},\delta_{\mathbf{C}})$ so that for each $C\in\mathbf{C}$, $!_{\mathbf{C}}C$ is a $\otimes$-comonoid with the additional property that $!_{\mathbf{C}}(C\sqcap D)=!_{\mathbf{C}}C\otimes !_{\mathbf{C}}D$. For any C-coherence space A_I, we define $!A_I$ to be

$$\left(|!I|,\sim_{!I},\left\{\bigotimes_{i\in a}!A_i\right\}_{a\in|!I|}\right).$$

Here $!I$ is one of the two ways mentioned above.

Theorem 2.9. *For the above* $\mathbf{C}$, *we have a comonad* $(!,\varepsilon,\delta)$ *on* $Coh(\mathbf{C})$ *such that for any* $\mathbf{C}$*-coherence spaces* A_I *and* B_J, $!A_I$ *is a* $\otimes$*-comonoid with*

$$!(A_I\sqcap B_J)=!A_I\otimes !B_J.$$

Proof. We only consider the first approach on $!I$ mentioned above as there is no essential difference between both approaches.

We start to define a functor $!:Coh(\mathbf{C})\to Coh(\mathbf{C})$. Recall that the functor $!:Coh\to Coh$ takes a linear morphism $t:I\to J$ into a linear morphism $!t:!I\to !J$ such that for any $a=\{x_1,\ldots,x_n\}\in|!I|$ and $b=\{y_1,\ldots,y_m\}\in|!J|$, $(a,b)\in !t$ iff $n=m$ and $y_u\in t(x_u)$ for each $u\leqslant n$. Let $f:(|I|,\sim,\{A_i\}_{i\in|I|})\to(|J|,\sim,\{B_j\}_{j\in|J|})$ be a $\mathbf{C}$-linear morphism of $Coh(\mathbf{C})$ determined by t and a family $f_{i,v}:A_i\to B_v$ for all i and $(i,v)\in t$. We define $!f:!A_I\to !B_J$ as follows. For the above $(a,b)\in !t$, note that $!_{\mathbf{C}}A_1\otimes\cdots\otimes !_{\mathbf{C}}A_n=!_{\mathbf{C}}(A_1\times\cdots\times A_n)$. Let $\prod f_{u,u}:A_1\times\cdots\times A_n\to B_1\times\cdots\times B_n$ be the unique arrow of $\mathbf{C}$ induced by all $f_{u,u}:A_u\to B_u$ for $u\leqslant n$. We then define $!_{\mathbf{C}}f_{(a,b)}:!_{\mathbf{C}}A_1\otimes\cdots\otimes !_{\mathbf{C}}A_n\to !_{\mathbf{C}}B_1\otimes\cdots\otimes !_{\mathbf{C}}B_n$ as the arrow $!_{\mathbf{C}}(\prod f_{u,u})$ of $\mathbf{C}$. The functoriality of $!$ on $Coh(\mathbf{C})$ follows directly from the functoriality of $!$ on Coh and $!_{\mathbf{C}}$ on $\mathbf{C}$.

To define the counit ε and the comultiplication δ, remember that for any coherence space I, $\varepsilon_I:!I\to I$ is given by the set $\{(\{i\},i)\,|\,i\in I\}$. We let $\varepsilon_{A_I}:!A_I\to A_I$ be the $\mathbf{C}$-linear morphism which is given by ε_I and the family of ε_{A_i} for all $i\in I$. Also, we let $\delta_{A_I}:!A_I\to !!A_I$ be the $\mathbf{C}$-linear morphism which is given by δ_I and the family of δ_{A_i} for all $i\in I$; here $\delta_I:!I\to !!I$ is the set

$$\left\{(a,\{a_1,\ldots,a_n\})\,|\,a,a_1,\ldots,a_n\in|!I| \text{ and } \bigcup_{i\leqslant n}a_i=a\right\}.$$

By the pointwise nature of $(!, \varepsilon, \delta)$ on $Coh(\mathbf{C})$, that $(!, \varepsilon, \delta)$ is a comonad on $Coh(\mathbf{C})$ follows easily from the comonad structures ! on Coh and $!_{\mathbf{C}}$ on $\mathbf{C}$.

Proving the rest of Proposition 2.9 is a matter of straightforward calculation. □

Theorem 2.10. *If the co-Kleisli category $K(!_{\mathbf{C}})$ is cartesian closed then so is the co-Kleisli category $K(!)$ of the comonad ! on $Coh(\mathbf{C})$.*

Proof. Recall from [28] that the objects of $K(!)$ are those of $Coh(\mathbf{C})$, whose morphisms are given by

$$Hom_{K(!)}(A_I, B_J) = Hom_{Coh(\mathbf{C})}(!A_I, B_J),$$

i.e., a morphism $A_I \to B_J$ of $K(!)$ is a linear morphism $!A_I \to B_J$. The products of $K(!)$ are just those of $Coh(\mathbf{C})$.

For cartesian closedness, let A_I, B_J and C_K be $\mathbf{C}$-webs. Since $\otimes$ and $\multimap$ on $Coh(\mathbf{C})$ are defined pointwise, up to isomorphisms, the following identities

$$\begin{aligned} &!\,(A_I \textstyle\prod B_J) \multimap C_K \\ &\quad = (!A_I \otimes !B_J) \multimap C_K \\ &\quad = !A_I \multimap (!B_J \multimap C_K) \end{aligned}$$

follow directly from those in Coh and $\mathbf{C}$. We therefore have the appropriate bijection

$$\frac{(A_I \prod B_J) \to C_K}{A_I \to (B_J \to C_K)}$$

in $K(!)$. □

3. Softness in *Coh*(C)

The main interest of this section is to study various forms of *softness* in $Coh(\mathbf{C})$. As we have already seen that $Coh(\mathbf{C})$ is a category enriched over both $\mathbf{Set}_*$ and Coh, the first result we prove here is that $Coh(\mathbf{C})$ has softness between products and coproducts in both enriched structures. We also show that there is softness between the external tensor product and its dual. This softness makes further connections to Cockett and Seely's work on *weakly distributive categories* [5] and Hyland and de Paiva on *tensor-par logic* [19].

We start with the notion of its softness which was introduced in [21]. Softness is a categorical generalization of *Whitman's condition* on free lattices. As an important part of Whitman's solution to the word problem for free lattices (see [8, 23, 36]), Whitman showed that for any poset P, the free lattice $F(P)$ of P satisfies the following property (known as Whitman's condition) : for any a, b, c and d in $F(P)$,

$$a \wedge b \leqslant c \vee d \;\Leftrightarrow\; a \leqslant c \vee d, \text{ or } b \leqslant c \vee d, \text{ or } a \wedge b \leqslant c, \text{ or } a \wedge b \leqslant d.$$

We say that a lattice is *soft* if it satisfies Whitman's condition.

Definition 3.1 (*Joyal* [21]). (i) Let $\mathbf{B}$ be a category with limits and colimits. $\mathbf{B}$ is said to be soft (or, has softness between limits and colimits) if for any pair of diagrams $D: I \to \mathbf{B}$ and $E: J \to \mathbf{B}$ the commutative square of canonical maps

$$\begin{tikzcd}
\operatorname{colim} \mathbf{B}(D, E) \arrow[r] \arrow[d] & \operatorname{colim} \mathbf{B}(D, \operatorname{colim} E) \arrow[d] \\
\operatorname{colim} \mathbf{B}(\lim D, E) \arrow[r] & \mathbf{B}(\lim D, \operatorname{colim} E)
\end{tikzcd}$$

is a pushout in $\mathbf{Set}$.

(ii) An object A of $\mathbf{B}$ is σ-atomic if the functor $\mathbf{B}(A, -): \mathbf{B} \to \mathbf{Set}$ preserves colimits. A is π-atomic if it is σ-atomic in $\mathbf{B}^{op}$. A is atomic if it is both σ- and π-atomic.

Remark 3.2. (i) For an arbitrary class $\mathscr{F}$ of limits and class $\mathscr{G}$ of colimits, the softness between $\mathscr{F}$ and $\mathscr{G}$ and atomic objects in a category $\mathbf{B}$ can be defined by restricting the limit diagrams and colimits in $\mathscr{F}$ and $\mathscr{G}$, respectively.

(ii) Let $\mathbf{B}$ be a category having softness between $\mathscr{F}$ and $\mathscr{G}$. The pushout diagram of 3.1 implies that any arrow from a $\lim D$ of $\mathscr{F}$ to a $\operatorname{colim} E$ of $\mathscr{G}$ factors through either a limit projection or a colimit injection.

(iii) As we showed before, $Coh(\mathbf{C})$ is a category enriched over $\mathbf{Set}_*$ and Coh. For a category $\mathbf{B}$ enriched over $\mathbf{Set}_*$, the canonical square of 2.1 should be viewed as a diagram in $\mathbf{Set}_*$ as the forgetful functor $\mathbf{Set}_* \to \mathbf{Set}$ does not preserve colimit diagrams. We say that $\mathbf{B}$ has *softness* between $\mathscr{F}$ and $\mathscr{G}$ if the canonical square of Definition 3.1 is a pushout in $\mathbf{Set}_*$. Also, to say atomic objects of $\mathbf{B}$, we mean that the representable functors $\mathbf{B}(B, -)$ and $\mathbf{B}(-, B)$ into $\mathbf{Set}_*$ preserve colimits and limits in $\mathscr{G}$ and $\mathscr{F}$, respectively. For a category $\mathbf{B}$ enriched over Coh, we then say that $\mathbf{B}$ has softness between $\mathscr{F}$ and $\mathscr{G}$ if the canonical square of 3.1 is a pushout in Coh.

In $Coh(\mathbf{C})$, we will denote 0 for the empty $\mathbf{C}$-coherence space, i.e., $0 = (0, \emptyset)$ which is determined by the empty web 0 of Coh. For any $C \in \mathbf{C}$, we use the same C for the singleton $\mathbf{C}$-coherence space, i.e., $C = (web(\{*\}), \{C\})$; here $web(\{*\})$ is the singleton web of Coh.

For A_I, B_J of $Coh(\mathbf{C})$, $[A_I, B_J]$ of the following theorem will denote for the pointed set $(Coh(\mathbf{C})(A_I, B_J), 0_{A_I, B_J})$ where $0_{A_I, B_J}$ is the zero arrow from A_I to B_J.

Theorem 3.3. *Let $\mathbf{C}$ be a category. Consider $Coh(\mathbf{C})$ as a category enriched over $\mathbf{Set}_*$, then*

(i) *0 and C are the only atoms in $Coh(\mathbf{C})$ under products and coproducts.*

(ii) *$Coh(\mathbf{C})$ has softness between products and coproducts in the following sense: for any families $\{A_{I_m}\}$ and $\{B_{J_n}\}$ of $Coh(\mathbf{C})$, with $A = \prod A_{I_m}$ and $B = \coprod B_{J_n}$, the* *commutative square of the canonical morphisms*

$$\begin{array}{ccc}
\coprod_{m,n}[A_{I_m}, B_{J_n}] & \longrightarrow & \coprod_m [A_{I_m}, B] \\
\Big\downarrow & & \Big\downarrow \\
\coprod_n [A, B_{J_n}] & \longrightarrow & [A, B]
\end{array}$$

is pushout in $\mathbf{Set}_*$.

Proof. That 0 is an atom is trivial, as the hom-set from (or, into) 0 into (or, from) any A_I has unique morphism, i.e., the zero morphism. We now show that C is an atom. For the σ-atomness of C, we only need to consider the hom-set from C into any non-empty coproduct because of the atomness of 0. Also, only the non-zero morphisms play a role in the canonical diagram of Definition 3.1. Let $\{A_{I_m}\}_{m\in M}$ be a family of $\mathbf{C}$-coherence spaces. For a $\mathbf{C}$-linear morphism

$$f = (t, \{f_{*,u}\}_{u\in\amalg I_m}) : C \to \coprod A_{I_m},$$

if $(*,a),(*,b)\in t$ then $a\sim b$. Hence there is a unique $m\in M$ such that $\{a,b\}\in Coh(I_m)$. Consequently, C factors uniquely through the injection $q_m : A_{I_m} \to \coprod A_{I_m}$. Dually, we can show that C is π-atomic.

To prove the uniqueness part of (i). Let $A_I \neq 0$ and C, for all $C\in\mathbf{C}$. There are at least two elements $x, y\in|I|$. (a) If $x\sim_I y$, consider a $\mathbf{C}$-linear morphism

$$f = (t, \{f_{i,j}\}) : A_I \to C_x \amalg C_y$$

$$t = \{(x,x),(y,y)\}, \quad f_{x,x} = id_{C_x} \text{ and } f_{y,y} = id_{C_y}.$$

f cannot factor through coproduct injections of $C_x \amalg C_y$, i.e., A_I is not σ-atomic.

(b) If $x \not\sim_I y$, we can dually show that A_I is not π-atomic. This proves (i).

For (ii), let

$$f = (t, \{f_{i,j}\}) : \prod_{m\in M} A_{I_m} \to \coprod_{n\in N} B_{J_n}$$

be a non-zero arrow. There are only three possibilities:

Case 1: There are $m\neq m'$, $x\in|I_m|$ and $x'\in|I_{m'}|$ such that $(x,y),(x',y')\in t$. In $\prod I_m$ we have that $x\sim x'$, hence $y\sim y'$ is true in $\coprod J_n$. This implies that there is a unique n such that $y,y'\in|J_n|$. Consider arbitrary $(z,z')\in t$. (a) If $z\in|I_m|$ then $z\sim y$. So $z'\sim y'$ is true in $\coprod J_n$. Consequently, $z'\in|J_n|$. (b) If $z\notin|I_m|$ then $z\sim x$. So $z'\sim x'$ is true in $\coprod J_n$. Again, we have $z'\in|J_n|$. This shows that f uniquely factors through the injection $B_{J_n}\to\coprod B_{J_n}$.

Case 2: There is unique m such that if $(x,x')\in t$ then $x\in|I_m|$, and there are $n\neq n'$, $y\in J_n$ and $y'\in J_{n'}$ such that $(x,y),(x',y')\in t$.

Case 2 implies that f factors uniquely through the projection $\prod A_{I_m} \to A_{I_m}$.

Case 3: f factors through some B_{J_n} and A_m.

By Cases 1 and 2, m and n must be unique. Thus we have completed the proof of (ii). □

Proposition 3.4. *Consider* $Coh(\mathbf{C})$ *to be a category enriched over* $\mathbf{Set}_*$, *we have*

(i) *The only* π*-atoms of* $Coh(\mathbf{C})$ *are coproducts of families of objects of the forms* C.

(ii) *The only* σ*-atoms of* $Coh(\mathbf{C})$ *are products of families of objects of the forms* C.

Proof. The proof is similar to that of Theorem 3.3. □

Remark 3.5. (i) In the literature, for instance, in [14, 35], 0 is said to be an atomic coherence space. We emphasize this in order to understand better the free structure on *Coh* in Section 4.

(ii) Recently Ehrhard [8] refined coherence spaces into *hypercoherences* with applications to the question of sequentiality. A hypercoherence X is a pair $(|X|, \Gamma(X))$ where $\Gamma(X)$ is a set of finite subsets of $|X|$ that includes all the singletons. A linear morphism $X \to Y$ with $Y = (|Y|, \Gamma(X))$ is a subset of $|X| \times |Y|$ so that for any finite subset w of f with the projections w_X and w_Y,

$$w_X \in \Gamma(X) \Rightarrow (w_Y \in \Gamma(Y) \;\&\; (\#w_Y = 1 \Rightarrow \#w_X = 1)).$$

We notice that there is softness between products and coproducts in the category of hypercoherences and linear morphisms. More detail discussions on this aspect will appear in a forthcoming paper.

(iii) The singleton $\mathbf{C}$-coherence space is not π- or σ-atomic in $Coh(\mathbf{C})$ under limits and colimits. Indeed, for $Coh(\mathbf{C}) = Coh$, Let $a = \{x\}$, $b = \{y\}$ and $c = \{z\}$ be the singleton webs of Coh. Consider the commutative square of linear morphisms

$$\begin{tikzcd}
a\Pi(b \amalg c) \arrow[r, "f"] & a\Pi b\Pi c \\
a \amalg b \amalg c \arrow[u, "g'"] \arrow[r, "f'"] & a \amalg (b\Pi c) \arrow[u, "g"']
\end{tikzcd}$$

Here

$$f = \{(x,x),(y,y),(z,z)\}, \quad g = \{(x,x),(y,y),(z,z)\}$$

$$f' = \{(x,x),(y,y),(z,z)\} \quad \text{and} \quad g' = \{(x,x),(y,y),(z,z)\}.$$

It is easy to verify that (f', g') is the pullback of f and g in Coh. But the linear morphism

$$h : a \amalg b \amalg c \to 1$$

$$h = \{(a,*),(b,*),(c,*)\}$$

cannot factor through any canonical projection in the pullback diagram.

(vi) There is no softness in *Coh* under arbitrary limits and colimits. Indeed, consider the above pullback diagram, and

$$h : a \amalg b \amalg c \to b \amalg c$$

$$h = \{(x, y), (y, y), (z, z)\}.$$

h is a linear morphism from the pullback diagram (i.e., $a \amalg b \amalg c$ as the pullback of the mentioned diagram) into the coproduct $b \amalg c$. But h cannot factor through any canonical projection or any coproduct injection.

In the remaining part of this section we view $Coh(\mathbf{C})$ as a category enriched over *Coh*. For C-valued coherence spaces A_I and B_J, $[A_I, B_J]$ is the coherence space described in the proof of Proposition 1.6.

Let $(A_{I_m})_{m \in M}$ and $(B_{J_n})_{n \in N}$ be the families of objects of $Coh(\mathbf{C})$, $A = \prod_m A_{I_m}$ and $B = \coprod_n B_{J_n}$. Consider the canonical diagram of Definition 3.1

$$\begin{array}{ccc} \coprod_{m,n}[A_{I_m}, B_{J_n}] & \xrightarrow{f} & \coprod_n [A, B_{J_n}] \\ \Big\downarrow{\scriptstyle g} & & \Big\downarrow{\scriptstyle g'} \\ \coprod_m [A_{I_m}, B] & \xrightarrow{f'} & [A, B] \end{array}$$

where f, g, f' and g' are linear morphisms defined by

$$f : (i, j, f) \mapsto (i, j, f),$$

$$g : (i, j, f) \mapsto (i, j, f),$$

$$f' : (i, j, f) \mapsto (i, j, f),$$

$$g' : (i, j, f) \mapsto (i, j, f)$$

for all $(i, j, f) \in |[A_{I_m}, B_{J_n}]|$.

Theorem 3.6. *The above canonical diagram is a pushout in Coh, i.e.,* $Coh(\mathbf{C})$ *has the enriched softness over Coh between products and coproducts.*

Proof. For a coherence space K, let $s : \coprod_m [A_{I_m}, B] \to K$ and $t : \coprod_n [A, B_{J_n}] \to K$ be linear morphisms such that $s \circ f = t \circ g$. Then we can see that $s(\{(i, j, f)\}) = t(\{(i, j, f)\})$ for all $(i, j, f) \in |\coprod_m [A_{I_m}, B]| (= |\coprod_n [A, B_{J_n}]| = |[A, B]|)$. Let $v : [A, B] \to K$ with $v(\{(i, j, f)\}) = s(\{(i, j, f)\})$. It is clear that $s = v \circ f'$ and $t = v \circ g'$. We need to check whether v is a linear morphism. But this follows from the facts that $(i, j, f) \sim (i', j', g)$ holds in $[A, B]$ iff it holds in $\coprod_m [A_{I_m}, B]$ or $\coprod_n [A, B_{J_n}]$, and $(i, j, f) \sim (i', j', g)$ holds in $\coprod_{m,n}[A_{I_m}, B_{J_n}]$ iff it holds $\coprod_m [A_{I_m}, B]$ and $\coprod_n [A, B_{J_n}]$. Also, the uniqueness of v can be easily detected. $\square$

We now turn to the softness between the external tensor porduct $\otimes$ and its dual $\odot$ described in Section 2. We first look at the *mixed associativities* (also called *weak distributivities* in [6, 19]) between $\otimes$ and $\odot$. For $B_I \in Coh(\mathbf{B})$, $C_J \in Coh(\mathbf{C})$ and $D_K \in Coh(\mathbf{D})$, the mixed associativities are defined by the canonical arrows

$$\begin{aligned}
\varepsilon_{B_I, C_J, D_K} &: (B_I \odot C_J) \otimes D_K \to B_I \odot (C_J \otimes D_K) \\
&\quad ((b_i, c_j), d_k) \mapsto (b_i, (c_j, d_k)) \\
\delta_{B_I, C_J, D_K} &: B_I \otimes (C_J \odot D_K) \to (B_I \otimes C_J) \odot D_K \\
&\quad (b_i, (c_j, d_k)) \mapsto ((b_i, c_j), d_k).
\end{aligned}$$

Let A, B, C and D be arbitary categories. For $A_I \in Coh(\mathbf{A})$, $B_J \in Coh(\mathbf{B})$, $C_K \in Coh(\mathbf{C})$ and $D_L \in Coh(\mathbf{D})$, consider the canonical diagram

$$\begin{tikzcd}
A_I \otimes (B_J \odot C_K) \otimes D_L \arrow[r, "f"] \arrow[d, "g"'] & A_I \otimes (B_J \odot (C_J \otimes D_L)) \arrow[d, "g'"] \\
((A_I \otimes B_J) \odot C_K) \otimes D_L \arrow[r, "f'"] & (A_I \otimes B_J) \odot (C_K \otimes D_L)
\end{tikzcd}$$

with

$$f = id_{A_I} \otimes \varepsilon_{B_J, C_K, D_L},$$

$$g = \delta_{A_I, B_J, C_K} \otimes id_{D_L},$$

$$f' = \varepsilon_{A_I \otimes B_J, C_K, D_L},$$

$$g' = \delta_{A_I, B_J, C_J \otimes D_L}.$$

The commutativity of this diagram is one of the axioms of *weakly distributive categories* [6].

Proposition 3.7. *The above canonical diagram is a pushout in* $Coh(\mathbf{A} \times \mathbf{B} \times \mathbf{C} \times \mathbf{D})$.

Proof. The proof is similar to one of Theorem 3.6. We outline the main points as follows. For any $a, a' \in A_I$, $b, b' \in B_J$, $c, c' \in C_K$ and $d, d' \in D_L$, we can show that

$$((a,b),(c,d)) \sim ((a',b'),(c',d'))$$

holds in $(A_I \otimes B_J) \odot (C_K \otimes D_L)$ iff

$$(a,(b,(c,,d))) \sim (a',(b',(c',d')))$$

holds in $A_I \otimes (B_J \odot (C_K \otimes D_L))$ or

$$(((a,b),c),d) \sim (((a',b'),c'),d')$$

holds in $((A_I \otimes B_J) \odot C_K) \otimes D_L$. Also,

$$(a,(b,c),d) \sim (a',(b',c'),d')$$

holds in $A_I \otimes (B_J \odot C_K) \otimes D_L$ iff

$$(a,(b,(c,d))) \sim (a',(b',(c',d')))$$

holds in $A_I \otimes (B_J \odot (C_K \otimes D_L))$ and

$$(((a,b),c),d) \sim (((a',b'),c'),d')$$

holds in $((A_I \otimes B_J) \odot C_K) \otimes D_L$. $\square$

Let $(S_n)_{n\in N}$ and $(T_m)_{m\in M}$ be the families of coherence spaces, and let (A_{I_n}) and (B_{J_m}) be the families of **D**-coherence spaces,

$$A = \prod_{n\in N} S_n^{\circ} \odot A_{I_n} \quad \text{and} \quad B = \coprod_{m\in M} T_m \otimes B_{J_m}.$$

Consider the canonical diagram

$$\begin{tikzcd}
\coprod_{n,m} S_n \otimes [A_{I_n}, B_{J_m}] \otimes T_m \arrow[r, "f"] \arrow[d, "g"] & \coprod_n S_n \otimes [A_{I_n}, B] \arrow[d, "g'"] \\
\coprod_m [A, B_{J_m}] \otimes T_m \arrow[r, "f'"] & [A,B]
\end{tikzcd}$$

where

$$f : (s,(i,j,f),t) \mapsto (s,(i,(t,j),f))$$

$$g : (s,(i,j,f),t) \mapsto (((s,i),j,f),t)$$

$$f' : (((s,i),j),f),t) \mapsto ((s,i),(t,j),f)$$

$$g' : (s,(i,(t,j),f)) \mapsto ((s,i),(t,j),f)$$

for all $s\in|S_n|$, $t\in|T_m|$ and $(i,j,f_{i,j}) \in |[A_{I_n}, B_{J_m}]|$.

Theorem 3.8. *The above canonical diagram is a pushout in Coh.*

Proof. The proof is a combination of Theorem 3.6 and Proposition 3.7. Indeed, for any $s,s' \in |S_n|$, $t,t' \in |T_m|$, and $(i,j,f),(i',j',g) \in |[A_{I_n}, B_{J_m}]|$, we can show that

$$((s,i),(t,j),f) \sim ((s',i'),(t',j'),g)$$

holds in $[A,B]$ iff

$$(s,(i,(t,j),f)) \sim (s',(i',(t',j'),g))$$

holds in $\coprod_n S_n \otimes [A_{I_n},B]$ or

$$((s,i),j,f),t) \sim ((s',i'),j',g),t')$$

holds in $\coprod_m [A,B_{J_m}] \otimes T_m$. Also,

$$(s,(i,j,f),t) \sim (s',(i',j',g),t')$$

holds in $\coprod_{m,n} S_n \otimes [A_{I_n},B_{J_m}] \otimes T_m$ iff

$$(s,(i,(t,j),f)) \sim (s',(i',(t',j'),g))$$

holds in $\coprod_n S_n \otimes [A_{I_n},B]$ and

$$((s,i),j,f),t) \sim ((s',i'),j',g),t')$$

holds in $\coprod_m [A,B_{J_m}] \otimes T_m$. $\square$

Remark 3.9. (i) In Proposition 3.7, let $\mathbf{A}=\mathbf{B}=\mathbf{C}=\mathbf{D}=\{*\}$, and let $\mathbf{D}=\{*\}$ and $\#M=\#N=1$ in Theorem 3.8, it is easy to see that the pushouts of Proposition 3.7 and Theorem 3.8 coincide.

(ii) In Theorem 3.8, let both S_n and T_m be the coherence space with singleton web, then the pushouts of Theorems 3.6 and 3.8 coincide.

(iii) Softness of Theorem 3.8 plays an important role in the free bicompletions of enriched categories (see [22])

4. Contractible C-coherence spaces

This section studies the free structure in $Coh(\mathbf{C})$. We will show that the *free bicomplete category* $CCoh(\mathbf{C})$ of $\mathbf{C}$ under the zero object, products and coproducts is a full subcategory of $Coh(\mathbf{C})$ which is closed under products and coproducts. Moreover, all $\mathbf{C}$-coherence spaces can be constructed from objects of $CCoh(\mathbf{C})$ under limits (dually, under colimits).

Definition 4.1 (*Joyal* [22]). For a category $\mathbf{C}$, the free bicompletion of $\mathbf{C}$ under limits and colimits is a pair $(i,\Lambda(\mathbf{C}))$ where $\Lambda(\mathbf{C})$ has limits and colimits and $i:\mathbf{C}\to\Lambda(\mathbf{C})$ is a functor such that

(i) (Existence) for any functor $F:\mathbf{C}\to\mathbf{B}$ with $\mathbf{B}$ having limits and colimits there is a functor $F':\Lambda(\mathbf{C})\to\mathbf{B}$ preserving limits and colimits such that $F=F'\circ i$;

(ii) (Uniqueness) if $F',F'':\Lambda(\mathbf{C})\to\mathbf{B}$ are functors which satisfy (i) then there is a unique isomorphism $u:F'\to F''$ such that $u\circ i=id_F$.

The following characterization theorem of $\Lambda(\mathbf{C})$ is established in [22].

Theorem 4.2. *For a category* $\mathbf{C}$, *the free bicompletion* $i:\mathbf{C}\to\Lambda(\mathbf{C})$ *has the following properites*:
(i) $\Lambda(\mathbf{C})$ *is soft.*
(ii) $i(A)$ *is atomic for any* $A\in\mathbf{C}$.
(iii) *The functor* i *is full and faithful.*
(iv) $\Lambda(\mathbf{C})$ *is generated from* $i(C)$ *under limits and colimits.*
Moreover, these properties characterize the pair $(i,\Lambda(\mathbf{C}))$ *up to an equivalance of categories.*

Let $\mathbf{C}$ be a category. A $\mathbf{C}$-coherence space (or web) is said to be *contractible* if it is generated from objects of the form $(\{*\},\{C\})$ under products and coproducts; here C is in $\mathbf{C}$. We denote by $CCoh(\mathbf{C})$ the full subcategory of $Coh(\mathbf{C})$ whose objects are contractible $\mathbf{C}$-coherence spaces, and $CCoh$ for $CCoh(\{*\})$.

Let

$$i_{\mathbf{C}}:\mathbf{C}\to CCoh(\mathbf{C})$$

$$C\mapsto(\{*\},\{C\})$$

be the inclusion. We have

Theorem 4.3. $CCoh(\mathbf{C})$ *is the free bicompletion of* $\mathbf{C}$ *under the zero object and non-empty products and coproducts. More precisely, the inclusion* $i_{\mathbf{C}}:\mathbf{C}\to CCoh(\mathbf{C})$ *satisfies*
(i) (*Existence*) *for any functor* $F:\mathbf{C}\to\mathbf{B}$ *with* B *having products, coproducts and the zero object, there is a functor* $F':CCoh(\mathbf{C})\to\mathbf{B}$ *preserving products and coproducts such that* $F=F'\circ i_{\mathbf{C}}$;
(ii) (*Uniqueness*) *if* $F',F'':CCoh\to\mathbf{B}$ *are functors which satisfy* (i) *then there is a unique isomorphism* $u:F'\to F''$ *such that* $u\circ i=id_F$.

Proof. To prove Theorem 4.3, we first establish some connections between $CCoh(\mathbf{C})$ and the *free coproduct-completion* of $\mathbf{C}$. Recall that the free coproduct-completion of $\mathbf{C}$ can be described as the category of families in $\mathbf{C}$, $Fam(\mathbf{C})$, whose objects are pairs $(I,\{A_i\}_{i\in I})$ where I is a set (we can view it as the coproduct of singleton web $\{i\}$), and whose morphisms $f:(I,\{A_i\})\to(J,\{B_j\})$ are pairs $(t,\{f_i\}_{i\in I})$ where $t:I\to J$ is a function (we can view it as the linear morphism between I and J) and $f_i:A_i\to B_{t(i)}$ for all $i\in I$. The composition of f and $g=(s,\{g_j\}_{j\in J}):(J,\{B_j\})\to(K,\{C_k\})$ is defined by $(s\circ t,\{g_{t(i)}\circ f_i\})$, i.e., the composition of linear morphisms in $Coh(\mathbf{C})$. The main properties of $Fam(\mathbf{C})$ are that: (a) $(\{*\},\{C\})$ is σ-atomic of $Fam(\mathbf{C})$ for $C\in\mathbf{C}$; (b) each object $(I,\{A_i\})$ of $Fam(\mathbf{C})$ is the coproduct of atoms $(\{*\},\{A_i\})$. By using (a) and (b), we can derive the freeness of $Fam(\mathbf{C})$ (e.g., see [17]).

The proof of Theorem 4.3 is a modification of the proof of the freeness on $Fam(\mathbf{C})$ by using matrices of morphisms instead of families of morphisms. Let $A_I=(|I|,\sim,$

$\{A_i\}_{i\in|I|})$ and $B_J=(|J|,\sim,\{B_j\}_{j\in|J|})$ be contractible **C**-webs, and let

$$f=(t,\{f_{i,j}\}_{(i,j)\in t}):A_I\to B_J$$

be a **C**-linear morphism. We define a functor $F' : CCoh(\mathbf{C})\to\mathbf{B}$ as follows. Notice that for a contractible **C**-coherence space A_I, A_I is entirely determined by the structure of I and the family of A_i. Since the structure of I is organized by a number of products alternating with a number of coproducts, we formally let

$$F'(A_I)=(|I|,\sim,\{F(A_i)\}_{i\in|I|})$$

which means that $F'(A_I)$ is the object of **B** obtained from the family of $F(A_i)$ by taking those of products and coproducts of I. For $f=(t,\{f_{i,j}\}_{(i,j)\in t})$, we represent f as a matrix $(f'_{i,j})_{|I|\times|J|}$ in the following way. $f'_{i,j}=f_{i,j}:A_i\to B_j$ if $(i,j)\in t$; otherwise, let $f'_{i,j}$ be the zero morphism from A_i into B_j. The composition of two matrices f and

$$g=(g_{j,k}):\ B_J\to C_K$$

is defined by the matrix $(g_{j,k}\circ f_{i,j})$ (the reader can easily see that the above representation of $CCoh(\mathbf{C})$ is isomorphic to $CCoh(\mathbf{C})$). Using the universal properties of products and coproducts, we then define

$$F'(f):F'(A_I)\to F'(\mathbf{B}_J)$$

by the morphism determined by the matrix $(F'(f'_{i,j}))$. Here $F'(f'_{i,j})=F(f_{i,j})$ if $(i,j)\in t$; otherwise, $F'(f'_{i,j})$ is the zero morphism from $F(A_i)$ into $F(B_j)$. The functoriality of F' can be easily verified: the composition of $F'(f)$ and $F'(g)$ is determined by the composition of the matrices $(F(f'_{i,j}))$ and $(F'(g'_{j,k}))$ which, of course, implies that $F'(g\circ f)=F'(g)\circ F'(f)$. That $F'(id_{A_I})=id_{F'(A_I)}$ follows from the fact that the representing matrix of $F'(id_{A_I})$ has identities in its diagonal entries, and whose other entries are the zero morphisms. Also, the definition of F' implies that $F=F'\circ i_{\mathbf{C}}$.

To see that F' preserves products and coproducts, let $\{A_{I_n}\}_{n\in N}$ be a family of **C**-webs $A_{I_n}=(|I_n|,\sim,\{A_{i_n}\}_{i_n\in|I_n|})$, and let A_I be the product of $\{A_{I_n}\}$. That $F'(A_I)=\prod F'(A_{I_n})$ follows from the structure of I as $I=\prod I_n$. Similarly, we can show that F' preserves coproducts.

The proof of the uniqueness of F' is straightforward. □

For categories **B** and **C**, let F: $\mathbf{C}\to\mathbf{B}$ be a functor. $Coh(F)$ of Proposition 2.2 restricted on $CCoh(\mathbf{C})$ induces a functor

$$CCoh(F):CCoh(\mathbf{C})\to CCoh(\mathbf{B})$$

which preserves products and coproducts such that $i_{\mathbf{B}}\circ F=CCoh(F)\circ i_{\mathbf{C}}$. Here $i_{\mathbf{B}}:\mathbf{B}\to CCoh(\mathbf{B})$ and $i_{\mathbf{C}}:\mathbf{C}\to CCoh(\mathbf{C})$ are inclusions.

Corollary 4.4. *Up to isomorphism, $CCoh(F)$ is the unique extension of F which preserves products and coproducts.*

Proof. Let $G = i_{\mathbf{B}} \circ F$. Apply Theorem 4.3 to G, then $G' = CCoh(F)$ as required. $\square$

The following characterization theorem ensures that a category is the free bicompletion of a category under the zero object and non-empty products and coproducts, i.e., first adding the zero object into it, then taking free bicompletion of the new category under non-empty products and coproducts.

Theorem 4.5. *For any category* $\mathbf{C}$, *the free bicompletion* $F : \mathbf{C} \to \Lambda(\mathbf{C})$ *of* C *under the zero object and non-empty products and coproducts has the following properties*:

(i) $\Lambda(\mathbf{C})$ *has the zero object*, *products*, *coproducts*, *and softness between products and coproducts* (*enriched over* $\mathbf{Set}_*$).

(ii) *The zero object and* $F(C)$ *are atomic*, *for* $C \in \mathbf{C}$.

(iii) *For any objects* A *and* B *of* $\mathbf{C}$, *if* $g : F(A) \to F(B)$ *is non-zero morphism of* $\mathbf{B}$ *then there is a unique morphism* f: $A \to B$ *of* $\mathbf{C}$ *such that* $F(f) = g$.

(iv) $\Lambda(\mathbf{C})$ *is generated from* $F(\mathbf{C})$ *under products and coproducts.*

Moreover, (i) *to* (iv) *characterize the pair* $(F, \Lambda(\mathbf{C}))$ *up to an equivalence of categories.*

Remark 4.6. (i), (ii), (iv) of Theorem 4.5 are quite natural from the discussions of previous sections. (iii) of Theorem 4.5 means that for the free bicompletion $\mathbf{C}_0$ of $\mathbf{C}$ under the zero object, the unique extension $F_0 : \mathbf{C}_0 \to \mathbf{B}$ of F along with the inclusion i_0: $\mathbf{C} \to \mathbf{C}_0$ is full and faithful. Readers can easily see the analogy between Theorems 4.2 and 4.5.

Proof of Theorem 4.5. From Theorem 3.3, $\Lambda(\mathbf{C})$ satisfies properties (i)–(iv). Assume that $\mathbf{B}$ is a category satisfying (i)–(iv), by Theorem 4.3, we have a unique extension F' of F: $\mathbf{C} \to \mathbf{B}$

$$F' : CCoh(\mathbf{C}) \to \mathbf{B}$$

$$A_I = (|I|, \sim, \{A_i\}_{i \in |I|}) \mapsto F'(A_I)$$

where $F'(A_I)$ is obtained from the family of A_i by taking the products alternating with the coproducts of I in $\mathbf{B}$. Notice that F' preserves products and coproducts. That F' is surjective on objects follows from (iv), i.e., each object of $\mathbf{B}$ is obtained from objects of $F(\mathbf{C})$ under products and coproducts. To show that F' is full and faithful, let $A_I = ((|I|, \sim, \{A_i\})$ and $B_J = (|J|, \sim, \{B_j\})$ be objects of $CCoh(\mathbf{C})$, and let $g : F'(A_I) \to F'(B_J)$ be a morphism in $\mathbf{B}$. From the atomness of objects $F(A_i)$ and $F(B_j)$, g is entirely determined by a family of arrows $g_{i,j} : F(A_i) \to F(B_j)$ for all $i \in |I|$ and $j \in |J|$. We construct a $\mathbf{C}$-linear morphism $(t, \{f_{i,j}\}) : A_I \to A_J$ as follows. For any $(i,j) \in |I| \times |J|$, if $g_{i,j}$ is not a zero morphism, we let $(i,j) \in t$ and let $f_{i,j} : A_i \to B_j$ be the unique morphism such that $F(f_{i,j}) = g_{i,j}$ (by (iii)); otherwise, let $(i,j) \notin t$. We therefore have $F'((t, \{f_{i,j}\})) = g$. This shows the fullness of F'. The faithfulness of F' follows from the uniqueness of $f_{i,j}$. $\square$

Example 4.7. Not every **C**-coherence space is contractible. For instance, take $\mathbf{C}=\{*\}$, the following coherence space I_4 is the simplest *non-contractible* one: $|I_4|=\{a,b,c,d\}$, the relation on $|I_4|$ is defined by $a\sim b$, $b\sim c$, and $c\sim d$. In fact, for any coherence space I, if $|I|\leqslant 3$ then I is contractible. If $|I|=4$ and I is non-contractible, then, up to isomorphism, I is the same as I_4. Consequently, for an arbitary category **C**, $A_{I_4}=(|I_4|,\sim,\{A_1,A_2,A_3,A_4\})$ is non-contractible in $Coh(\mathbf{C})$, and A_I is contractible for $|I|\leqslant 3$.

Proposition 4.8. *Let* **C** *be a category. Then every non-contractible* **C**-*coherence space is a limit of a diagram whose objects are contractible. Dually, every non-contractible* **C**-*coherence space is a colimit of a diagram whose objects are contractible.*

Proof. We first consider $Coh(\mathbf{C})=Coh$. Notice that for any coherence space I, if there are only two elements $i,\ j\in|I|$ such that $i\not\sim j$, then $web(I)$ is the product of $\{i\}\coprod\{j\}$ and all other singleton webs $\{k\}$ with $k\in|I|$, i.e., I is contractible. Assume that I is non-contractible coherence space. We therefore have at least two pairs (i,j) and (i',j') such that $i\not\sim j$ and $i'\not\sim j'$. For any pair (i,j) with $i\not\sim j$, we denote $I_{i,j}$ for the contractible coherence space which is the product of $\{i\}\coprod\{j\}$ and all other singleton webs $\{k\}$ for $k\in|I|$, i.e.,

$$I_{i,j}=\prod_{k\neq i,j}\{k\}\prod(\{i\}\coprod\{j\}).$$

Let $J=\prod_{i\in|I|}\{i\}$. We have linear morphisms

$$t_{i,j}:I_{i,j}\to J,$$

$$t_{i,j}=\{(i,i)\}_{i\in|I|}.$$

It is easy to see that the family of linear morphisms

$$t'_{i,j}:I\to I_{i,j}$$

$$t'_{i,j}=\{(i,i)\}_{i\in|I|}$$

forms the wide-pullback of the family $t_{i,j}$ in Coh.

Let $A_I=(|I|,\sim,\{A_i\}_{i\in|I|})$ be a non-contractible **C**-coherence space. Note that a **C**-coherence space A_I is non-contractible iff I is non-contractible. Let $A_{I_{i,j}}=(|I_{i,j}|,\sim_{I_{i,j}},\{A_i\}_{i\in|I|})$ and $A_J=(|J|,\sim_J,\{A_i\}_{i\in|I|})$. Consider **C**-linear morphisms

$$f_{i,j}=(t_{i,j},\{id_{A_i}\}_{i\in|I|}):A_{I_{i,j}}\to A_J,$$

then the family of **C**-linear morphisms

$$f'_{i,j}=(t'_{i,j},\{id_{A_i}\}_{i\in|I|}):A_I\to A_{I_{i,j}}$$

is the wide-pullback of the family of $f_{i,j}$ in $Coh(\mathbf{C})$. $\square$

The following example shows that there are contractible coherence spaces whose tensor product are not contractible:

Example 4.9. Let $B = a \prod (b \coprod c)$ and its copy B' $= a' \prod (b' \coprod c')$, C is the tensor product B and B'. The corresponding web of C has a subweb $(|A|, \sim)$; here $|A| = \{(a, b'), (b, a'), (c, a'), (c, c')\}$ with $\sim$:

$$(b, a') \sim (a, b') \sim (c, a') \sim (c, c').$$

C is non-contractible since $(|A|, \sim)$ is isomorphic to I_4.

Remark 4.10. In a forthcoming paper we will explore game semantics aspects of $Coh(\mathbf{C})$ and, show how to construct a model of linear logic by expanding $CCoh(\mathbf{C})$.

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