Semirings and their Applications

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This book is an updated and considerably expended
version of the book "The Theory of Semirings,
with Applications to Mathematics and Theoretical science",
published by Longinov in 1992.

In memory of my mother,
Prof. Naomi Golan

יקרה היא מפנינים וכל חפציה לא ישוו בה

CONTENTS

PREFACE

There is no branch of mathematics, however abstract, which may not some day be applied to phenomena of the real world.

—Nikolai Ivanovich Lobatchevsky

This book is an extensively-revised and expanded version of "**The Theory of Semirings, with Applications in Mathematics and Theoretical Computer Science**" [Golan, 1992], first published by Longman. When that book went out of print, it became clear — in light of the significant advances in semiring theory over the past years and its new important applications in such areas as idempotent analysis and the theory of discrete-event dynamical systems — that a second edition incorporating minor changes would not be sufficient and that a major revision of the book was in order. Therefore, though the structure of the first edition was preserved, the text was extensively rewritten and substantially expanded.

In particular, references to many interesting and applications of semiring theory, developed in the past few years, had to be added. Unfortunately, I find that it is best not to go into these applications in detail, for that would entail long digressions into various domains of pure and applied mathematics which would only detract from the unity of the volume and increase its length considerably. However, I have tried to provide an extensive collection of examples to arouse the reader's interest in applications, as well as sufficient citations to allow the interested reader to locate them. For the reader's convenience, an index to these citations is given at the end of the book.

Thanks are due to the many people who, in the past six years, have offered suggestions and criticisms of the preceding volume. Foremost among them is Dr. Susan LaGrassa, who was kind enough to send me a detailed list of errors — typographical and mathematical — which she found in it. I have tried to correct them all. During the 1997/8 academic year I conducted a seminar on semirings while on sabbatical at the University of Idaho. Many thanks are due to the participants of that seminar, and in particular to Prof. Erol Barbut and Prof. Willy Brandal, for their incisive comments. Prof. Dan Butnariu of the University of Haifa was also

very instrumental in introducing me to applications of semirings in the theory of fuzzy sets, as where departmental guests Prof. Ivan Chajda, Prof. E. P. Klement and Prof. Radko Mesiar, while Dr. Larry Manevitz of the Department of Computer Science at the University of Haifa was always ready to help me understand applications in artificial intelligence and other areas of computer science. I also owe a large debt to my two former Ph.D. students, Dr. Wang Huaxiong and Dr. Wu Fuming, who listened patiently to my various p-baked ideas as they formed and contributed many original insights on semiring theory, which have been incorporated in this edition.

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Semirings abound in the mathematical world around us. Indeed, the first mathematical structure we encounter – the set of natural numbers – is a semiring. Other semirings arise naturally in such diverse areas of mathematics as combinatorics, functional analysis, topology, graph theory, Euclidean geometry, probability theory, commutative and noncommutative ring theory, optimization theory, discrete event dynamical systems, automata theory, formal language theory and the mathematical modeling of quantum physics and parallel computation systems. From an algebraic point of view, semirings provide the most natural common generalization of the theories of rings and bounded distributive lattices, and the techniques used in analysing them are taken from both areas.

Historically, semirings first appear implicitly in [Dedekind, 1894] and later in [Macaulay, 1916], [Krull, 1924], [Noether,1927], and [Lorenzen, 1939] in connection with the study of ideals of a ring. They also appear in [Hilbert, 1899] and [Huntington, 1902] in connection with the axiomatization of the natural numbers and nonnegative rational numbers. Semirings per se were first considered explicitly in [Vandiver, 1934], also in connection with the axiomatization of the arithmetic of the natural numbers. His approach was later developed in a series of expository articles culminating in [Vandiver & Weaver, 1956]. Over the years, semirings have been studied by various researchers either in their own right, in an attempt to broaden techniques coming from semigroup theory or ring theory, or in connection with applications. However, despite such categorical pronouncements as "... the above shows that *the ring is not the fundamental system for associative algebra of double composition*" (italics in the original) found in [Vandiver, 1939], semirings never became popular and the interest in them among algebraists gradually petered out, although it never died completely. The only attempt to present the algebraic theory of semirings as an integral part of modern algebra seems to be in [Rédei, 1967] and [Almeida Costa, 1974]. Nonetheless, semirings – and semimodules over them – have become an important tool in applied mathematics and theoretical computer science and appear, under various names, with consistent and increasing frequency in the literature of those subjects. Were there more communication between theoretical algebraists and these utilizers of algebra, it is likely that the former would find in the work of the latter sufficiently many "naturally arising" problems to revive and revitalize research in semiring theory in its own right, while the latter would find at their disposal a supply of theoretical results which they can use.

Since the results on semirings are scattered through the mathematical literature and are for most part inaccessible, they are not easily available to those who have to use them. A further problem is that the terminology used by different authors is not standard many authors use the term "semiring" for what we call here a "hemiring" and vice versa. Others, translating directly from the German, use the term "halfring". Some do not require that a semiring have a multiplicative identity or even an additive zero. On the other hand, some insist that multiplication, as well as addition, be commutative. In [Gondran & Minoux, 1984], the term "dioïd" (i.e., double monoid) is used in place of "semiring" and the term "semiring" is used in a stronger sense, while [Shier, 1973] prefers the term "binoid" for a commutative semiring in which addition is idempotent; others use "dioïd" for that purpose. A categorical definition of "semiring" (namely as a semiadditive category having one object) is given in [Manes, 1976]. To add to the confusion, some sources, e.g. [Sturm, 1986], use the term "semiring" to mean something else entirely. The reader must therefore be extremely wary.

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The notation used throughout the book will be explained as it is introduced. In addition, we will use the following standard notation:

$\mathbb{B} = \{0, 1\}$,
$\mathbb{P}$ = the set of all positive integers,
$\mathbb{N}$ = the set of all nonnegative integers,
$\mathbb{Z}$ = the set of all integers,
$\mathbb{Q}$ = the set of all rational numbers,
$\mathbb{Q}^+$ = the set of all nonnegative rational numbers
$\mathbb{I}$ = the unit interval on the real line,
$\mathbb{R}$ = the set of all real numbers,
$\mathbb{R}^+$ = the set of all nonnegative real numbers,
$\mathbb{C}$ = the set of all complex numbers.

If n is a positive integer then:

$\mathcal{S}_n$ is the group of all permutations of $\{1, \ldots, n\}$,
$\mathcal{A}_n$ is the group of all even permutations of $\{1, \ldots, n\}$.

If A and B are sets then:

$sub(A)$ is the family of all subsets of A,
$fsub(A)$ is the family of all finite subsets of A,
B^A is the set of all functions from A to B.