Weighted one-deterministic-counter automata

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- Abstract -

We introduce weighted one-deterministic-counter automata (ODCA). These are weighted one-counter automata (OCA) with the property of counter-determinacy, meaning that all paths labelled by a given word starting from the initial configuration have the same counter-effect. Weighted ODCAs are a strict extension of weighted visibly OCAs, which are weighted OCAs where the input alphabet determines the actions on the counter.

We present a novel problem called the co-VS (complement to a vector space) reachability problem for weighted odcas over fields, which seeks to determine if there exists a run from a given configuration of a weighted odca to another configuration whose weight vector lies outside a given vector space. We establish two significant properties of witnesses for co-VS reachability: they satisfy a pseudo-pumping lemma, and the lexicographically minimal witness has a special form. It follows that the co-VS reachability problem is in P.

These reachability problems help us to show that the equivalence problem of weighted ODCAs over fields is in P by adapting the equivalence proof of deterministic real-time OCAS [3] by Böhm et al. This is a step towards resolving the open question of the equivalence problem of weighted OCAS. Furthermore, we demonstrate that the regularity problem, the problem of checking whether an input weighted ODCA over a field is equivalent to some weighted automaton, is in P. Finally, we show that the covering and coverable equivalence problems for uninitialised weighted ODCAs are decidable in polynomial time. We also consider boolean ODCAs and show that the equivalence problem for (non-deterministic) boolean ODCAs is in PSPACE, whereas it is undecidable for (non-deterministic) boolean OCAs.

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1 Introduction

This paper investigates a restriction on weighted one-counter automata (OCA). Like weighted finite automata, weighted OCAs recognise functions - every word over a finite alphabet is mapped to a weight. We say that a weighted OCA has *counter-determinacy* (see Definition 11) if "all paths labelled by a given word, starting from the initial configuration, have the same counter-effect". Weighted one-deterministic-counter automata (ODCA) is a syntactic model equivalent to weighted OCA with counter-determinacy (see Definition 12). It consists of,

1. Counter: A counter that stays non-negative and allows zero tests.

- 2. Counter structure: A finite state deterministic machine where the transitions depend only on its current state, the input letter, and whether the counter is zero. The counter structure can increment/decrement the counter by one.
- 3. Finite state machine: A finite state weighted automaton whose transitions depend on its current state, the input letter, and whether the counter value is zero. This machine cannot modify the counter.

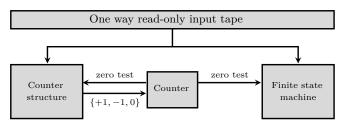


Figure 1 One-deterministic-counter automata

The counter structure and the finite state machine run synchronously on any word. The finite state machine computes the weight associated with the word. Our first observation is:

▶ **Theorem 2.** There is a polynomial time translation from a weighted OCA with counter-determinacy to a weighted ODCA and vice versa.

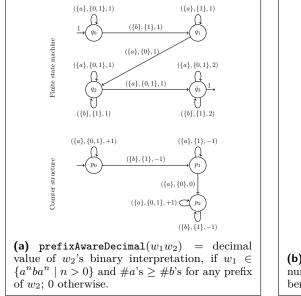
The proof is given in Appendix A. In the following example, the functions prefixAwareDecimal and equalPrefixPower are recognised by weighted OCA with counter-determinacy.

- ▶ **Example 2.** The functions are defined over the alphabet $\Sigma = \{a, b\}$. The transition weights of the ODCAs are from the field of rational numbers.
- (a) The function $\operatorname{prefixAwareDecimal}: \Sigma^* \to \mathbb{N}$ is defined as follows: $\operatorname{prefixAwareDecimal}(w) = \operatorname{decimal}(w_2)$ if $w = w_1w_2$, $w_1 \in \{a^nba^n \mid n > 0\}$, and the number of a's \geq number of b's for any prefix of w_2 , and 0 otherwise. Here, $\operatorname{decimal}(w_2)$ represents the decimal equivalent of w_2 when interpreted as a binary number, where 'a' is treated as a one and 'b' as a zero.
- (b) The function equalPrefixPower: $\Sigma^* \to \mathbb{N}$ is defined as follows: for all $w \in \Sigma^*$, equalPrefixPower $(w) = 2^k$ where k is the number of proper prefixes of w with equal number of a's and b's.

The weighted odcas recognising these functions are given in Figure 2. In the figure, if a transition from p_i to p_j of the counter structure is labelled (A, R, D) and $(a, r, d) \in A \times R \times D$, then there is a transition from p_i to p_j on reading the symbol a with counter action d. If a transition from q_i to q_j of the finite state machine is labelled (A, R, s) and $(a, r, s) \in A \times R \times \{s\}$, then there is a transition from q_i to q_j on reading the symbol a with weight s. In both cases, the current counter value should be 0 if r = 0 and greater than 0 if r = 1. For the finite state machine, the initial (resp. output) weight is marked using an inward (resp. outward) arrow. The weight of a path is the product of transition weights along that path. The accepting weight of a word is the sum of weights of all the paths from an initial state to an output state labelled by that word.

1.1 Comparisons with other models

Visibly pushdown automata (VPDA) were introduced by Alur and Madhusudan in 2004 [2]. They have received much attention as they are a strict subclass of pushdown automata suitable



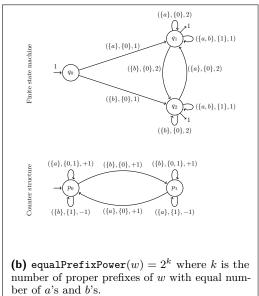


Figure 2 The figure shows weighted ODCAs recognising the functions given in Example 2.

for program analysis. VPDAs enjoy tractable decidable properties, which are undecidable in the general case. The visibly restriction, in essence, is that the stack operations are *input-driven*, i.e., only depends on the letter read. Weighted VPDA is a natural extension to the weighted setting. Counter-determinacy can be seen as a relaxation in the visibly constraint on OCAs, as the counter actions are no longer input-driven but are deterministic. The fact that weighted ODCAs are strictly more expressive than weighted visibly OCA can be noted from the fact that the functions in Example 2 are not recognised by a weighted visibly OCA.

Nowotka et al. [17] introduced height-deterministic pushdown automata, where the input string determines the stack height. Weighted ODCAs can be seen as weighted height-deterministic pushdown automata over a single stack alphabet and a bottom-of-stack symbol.

The reader might feel that a weighted ODCA is equivalent to a cartesian product of a deterministic OCA and a weighted finite automata. However, one can note that the functions prefixAwareDecimal and equalPrefixPower in Example 2 are not definable by the cartesian product of deterministic OCA and a weighted automaton. The reason is that the weighted automaton cannot "see" the counter values, so its power is restricted.

1.2 Motivation

Probabilistic pushdown automata (PPDA) have been studied for the analysis of stochastic programs with recursion [15, 18]. They are equivalent to recursive Markov chains [8, 14]. PPDAs are also a generalisation of stochastic context-free grammars [1] used in natural language processing and many variants of one-dimensional random walks [7].

The decidability of equivalence of probabilistic pushdown automata is a long-standing open problem [10]. The problem is inter-reducible to multiplicity equivalence of context-free grammars. In fact, the decidability is only known for some special subclasses of PPDA. It is known that the equivalence problem for PPDA is in PSPACE if the alphabet contains only one letter and is at least as hard as polynomial identity testing [10]. There is a randomised

1:4 Weighted one-deterministic-counter automata

polynomial time algorithm that determines the non-equivalence of two visibly PPDA over the alphabet triple ($\Sigma_{call}, \Sigma_{ret}, \Sigma_{int}$) where both machines perform push, pop, and no-action on the stack over the symbols in $\Sigma_{call}, \Sigma_{ret}$, and Σ_{int} respectively [12]. There is a polynomial-time reduction from polynomial identity testing to this problem. Hence it is highly unlikely that the problem is in P.

Since the equivalence problem for PPDA is unknown, the natural question to ask is the equivalence problem for probabilistic one-counter automata. However, this problem is also unresolved. In this paper, we identify a subclass of probabilistic OCAS (probabilistic ODCAS are also a superclass of visibly probabilistic OCAS) for which the equivalence problem is decidable. In particular, we show that the problem is in P. Note that our results are slightly more general since we consider weighted ODCAS where weights are from a field.

1.3 Our contributions on weighted ODCA (weights from a field)

The paper's primary focus is on the equivalence problem for weighted ODCAs where the weights are from a field.

We first introduce a novel reachability problem on weighted ODCA, called the complement to vector space (co-VS) reachability problem. The co-VS reachability problem (see Section 3) takes a weighted ODCA, an initial configuration, a vector space, a final counter state, and a final counter value as input. It asks, starting from the initial configuration, whether it is possible to reach a configuration with the final counter state, final counter value, and weight distribution over the states that is not in the vector space.

Let us call a word a *witness* if the run of the word 'reaches' a configuration desired by the reachability problem. We identify two interesting properties of witnesses.

- 1. pseudo-pumping lemma (Lemma 17): If the run of a witness encounters a 'large' counter value, then it can be pumped-down (resp. pumped-up) to get a run where the maximum counter value encountered is smaller (resp. larger). However, the lemma is distinct from a traditional pumping lemma, where the same subword can be pumped-down (or pumped-up) multiple times while maintaining reachability. In the case of a weighted ODCA, we only claim that a subword can be pumped, but the same subword may not be repeatedly pumped. It follows from the pseudo-pumping lemma that the co-VS reachability problem is in P (Theorem 22).
- 2. special-word lemma (Lemma 24): The lexicographically smallest witness is of the form $uy_1^{r_1}vy_2^{r_2}w$ where u, v, w, y_1 and y_2 are 'small' words and $r_1, r_2 \in \mathbb{N}$. The length of the word uy_1vy_2w is bounded by a polynomial in the number of states of the ODCA, whereas r_1 and r_2 also depend on the counter values of the initial and final configurations.

Comparing the above properties with that of deterministic one-counter automata will be interesting. In a deterministic OCA, the reachability problem is equivalent to asking whether there is a path to a final state (rather than a weight distribution over states) and a counter value from an initial state and counter value. Let z be an arbitrary 'long' witness. Consider the run on z of the deterministic OCA. By the Pigeon-hole principle (see Valiant and Paterson [23]), there will be words u, y_1, v, y_2 , and w such that $z = uy_1vy_2w$, and y_1 (and similarly y_2) starts and ends in the same state and the effect of y_1 on the counter is minus of the effect of y_2 on the counter. In short, y_1 and y_2 form loops with inverse counter-effects and can be pumped simultaneously. Therefore, for all $r \in \mathbb{N}$, the word $uy_1^rvy_2^rw$ is a witness. One can view this as a pumping lemma for deterministic OCA. Such a property does not hold in the case of weighted ODCA. The presence of weights at each state makes the problem inherently complex, necessitating a more sophisticated approach.

The proofs of Lemma 17 and Lemma 24 use linear algebra and combinatorics on words and are distinct from those employed for deterministic OCA. We also introduce a similar problem called co-VS coverability (see Section 3). The two properties of the witness and co-VS coverability are crucial along with the ideas developed by Böhm et al. [3, 4, 6] and Valiant and Paterson [23] in solving the equivalence problem.

▶ **Theorem 3.** There is a polynomial time algorithm that decides if two weighted ODCAs (weights from a field) are equivalent and outputs a word that distinguishes them otherwise.

Consider two non-equivalent weighted ODCAs. Let z be a minimal word that distinguishes them. We show that the counter values in the run of z are bounded by a polynomial in the number of states. A polynomial-time machine can then simulate the run of both machines up to this counter value to check for distinguishing words. It is sufficient to show that the counter values can be bounded in the run of the minimal distinguishing word. Let \mathcal{A} be the disjoint union of the two machines. It is easy to show the existence of a vector space \mathcal{V} such that the ODCAs are non-equivalent if there is a reachable configuration (in \mathcal{A}) that is not in \mathcal{V} . But \mathcal{A} contains two counters; hence, we cannot directly apply the properties of reachability witnesses. However, there are subruns in the run of z similar to runs of an ODCA. The pseudo-pumping lemma holds in these subruns, and therefore the counter values are bounded during these subruns of z (if the counter values are not bounded, then one can pump-down and generate a shorter witness violating the minimality of z). We use the special-word lemma to show the existence of such subruns. Section 4 provides a full proof.

Next, we consider the regularity problem - the problem of deciding whether a weighted ODCA is equivalent to some weighted automaton. The proof technique is adapted from the ideas developed by Böhm et al. [6] in the context of real-time OCA. The crucial idea in proving regularity is to check for the existence of infinitely many equivalence classes. The pseudo-pumping lemma (particularly pumping-up) is used in proving this. A detailed proof can be found in Section 5.

▶ **Theorem 4.** The regularity problem of weighted ODCA (weights from a field) is in P.

Finally, we look at uninitialised ODCAs - an ODCA without initial finite state distribution and initial counter state. We show that the "equivalence" problem for unitialised ODCAs are in polynomial time. Given two uninitialised ODCAs \mathcal{A}_1 and \mathcal{A}_2 , we say \mathcal{A}_2 covers \mathcal{A}_1 if for all initial configurations of \mathcal{A}_1 there exists an initial configuration of \mathcal{A}_2 such that they are equivalent. The coverable equivalence problem asks whether \mathcal{A}_1 covers \mathcal{A}_2 and \mathcal{A}_2 covers \mathcal{A}_1 .

▶ **Theorem 5.** Covering and coverable equivalence problems of uninitialised weighted odds are in P.

The proof relies on the algorithm to check the equivalence of two weighted odcas. A detailed proof can be found in Section 6.

1.4 Related work

Extensive studies have been conducted on weighted automata with weights from semirings. Tzeng [22] (also see Schützenberger [19]) gave a polynomial time algorithm to decide the equivalence of two probabilistic automata. The result has been extended to weighted automata with weights over a field. On the other hand, the problem is undecidable if the weights are over the semiring $(\mathbb{N}, \min, +)$ [13]. Unlike the extensive literature on weighted automata, the study on weighted versions of pushdown or one-counter machines is limited [9, 11, 15]. The undecidability of several interesting problems creates a major bottleneck.

Moving on to the non-weighted models, the equivalence problem for non-deterministic pushdown automata is known to be undecidable. On the other hand, from the seminal result by Sénizergues [20], we know that the equivalence problem for deterministic pushdown automata is decidable. The lower bound, though, is primitive recursive [21]. The language equivalence of synchronised real-time height-deterministic pushdown automata is in EXPTIME [17]. The equivalence problem for deterministic one-counter automata (with and without ϵ transitions), similar to that of deterministic finite automata, is NL-complete [5].

2 Preliminaries

2.1 Basic notations

In this paper, we fix an alphabet Σ . We use Σ^* to denote the set of finite length words over Σ . Given a word $w \in \Sigma^*$, we use |w| to denote the length of the word w. For any set S, we use |S| to denote the number of elements in S. We use the notation [i,j] to denote the interval $\{i,i+1,\ldots,j\}$. We say that a word $u=a_1\cdots a_k$ is a subword of a word w, if $w=u_0a_1u_1a_2\cdots a_ku_k$, where $a_i\in\Sigma$, $u_j\in\Sigma^*$ for all $i\in[1,k]$ and $j\in[0,k]$. We call u a proper subword of w if $u\neq w$. We say that a word w is a prefix of a word w if there exists $v\in\Sigma^*$ such that w=uv. Given a word $w=a_0\cdots a_n$, we write $w[i\cdots j]$ to denote the factor $a_i\cdots a_j$. For a $d\in\mathbb{N}$, the sign of d (denoted by sign(d)) is defined as sign(d)=0 if d=0 and is 1 otherwise. For all $l\in\mathbb{N}$, we use $\Sigma^{\leq l}$ (resp. Σ^l) to denote the set of words over Σ having length less than or equal to l (resp. exactly equal to l).

2.2 Linear algebra

In this paper, we use $\mathbf{x}, \mathbf{y}, \mathbf{z}$ to denote vectors over a field \mathcal{F} , s, t, r to denote elements in a field \mathcal{F} and $\mathbb{A}, \mathbb{B}, \mathbb{M}$ to denote matrices over a field \mathcal{F} . We use \mathcal{U}, \mathcal{V} to denote vector spaces. We recall the following facts.

- ▶ **Lemma 6.** The following are true for a field \mathcal{F} .
- 1. For any set X of n vectors in \mathcal{F}^m with n > m, there exists a vector $\mathbf{x} \in X$ that is a linear combination of the other vectors in X.
- 2. Given a set B of n vectors in \mathcal{F}^m and a vector $\mathbf{x} \in \mathcal{F}^m$, we can check if \mathbf{x} is a linear combination of vectors in B in time polynomial in m and n.

The following properties of vector spaces are important.

- ▶ **Lemma 7.** Let V be a vector space, $k \in \mathbb{N}$ and for all $r \in [0, k]$ $\mathbf{z}_r \in \mathcal{F}^k$ and $\mathbb{M}_r \in \mathcal{F}^{k \times k}$. Then, there exists an $i \in [1, k]$ such that the following conditions are true:
- 1. \mathbf{z}_i is a linear combination of $\mathbf{z}_0, \dots \mathbf{z}_{i-1}$, and
- **2.** if $\mathbf{z}_i \mathbb{M}_i \notin \mathcal{V}$, then there exists j < i such that $\mathbf{z}_j \mathbb{M}_i \notin \mathcal{V}$.

Proof. Let $k \in \mathbb{N}, r \in [0, k], \mathbf{z}_r \in \mathcal{F}^k, \mathbb{M}_r \in \mathcal{F}^{k \times k}$ be matrices and \mathcal{V} be a vector space.

- 1. Consider the set $\{\mathbf{z}_0, \mathbf{z}_1, \dots, \mathbf{z}_k\}$ of k+1 vectors of dimension k. It follows from Lemma 6 that there are at most k independent vectors of dimension k, and hence not all elements of the set can be independent.
- **2.** Let $i \in [1, k]$ be such that \mathbf{z}_i is a linear combination of $\mathbf{z}_0, \dots \mathbf{z}_{i-1}$ and $\mathbf{z}_i \mathbb{M}_i \notin \mathcal{V}$. Let us assume for contradiction that $\mathbf{z}_j \mathbb{M}_i \in \mathcal{V}$ for all $j \in [0, i-1]$. Since \mathbf{z}_i is a linear combination on $\mathbf{z}_0, \dots \mathbf{z}_{i-1}$, there exists $s_0, \dots s_{i-1} \in \mathcal{F}$ such that

$$\mathbf{z}_i = s_0 \cdot \mathbf{z}_0 + s_1 \cdot \mathbf{z}_1 + \dots + s_{i-1} \cdot \mathbf{z}_{i-1}$$

Since $\mathbf{z}_i \mathbb{M}_i = \sum_{j=0}^{i-1} s_j \cdot \mathbf{z}_j \mathbb{M}_i$ and \mathcal{V} is closed under linear combinations, we get that $\mathbf{z}_i \mathbb{M}_i \in \mathcal{V}$ contradicting our initial assumption.

- ▶ **Lemma 8.** Let V be a vector space, $k \in \mathbb{N}$ and for all $r \in [0, k^2]$ $\mathbb{A}_r, \mathbb{M}_r, \mathbb{B}_r \in \mathcal{F}^{k \times k}$. Then, there exists an $i \in [1, k^2]$ such that for all $\mathbf{x} \in \mathcal{F}^k$ the following conditions are true:
- 1. \mathbb{M}_i is a linear combination of $\mathbb{M}_0, \dots, \mathbb{M}_{i-1}$, and
- 2. if $\mathbf{x} \mathbb{A}_i \mathbb{M}_i \mathbb{B}_i \notin \mathcal{V}$, then there exists a j < i such that $\mathbf{x} \mathbb{A}_i \mathbb{M}_j \mathbb{B}_i \notin \mathcal{V}$.

Proof. Let $\mathbb{A}_r, \mathbb{M}_r, \mathbb{B}_r \in \mathcal{F}^{k \times k}$ for $r \in [0, k^2]$, be matrices over \mathcal{F} and \mathcal{V} a vector space.

- 1. Consider the set $\{\mathbb{M}_0, \mathbb{M}_1, \dots, \mathbb{M}_{k^2}\}$ of $k^2 + 1$ matrices of dimension k^2 . It follows from Lemma 6 that there are at most k^2 independent vectors of dimension k^2 , and hence not all elements of this set can be independent.
- **2.** Let $i \in [1, k^2]$ be such that \mathbb{M}_i is a linear combination of $\mathbb{M}_0, \dots, \mathbb{M}_{i-1}$ and $\mathbf{x} \mathbb{A}_i \mathbb{M}_i \mathbb{B}_i \notin \mathcal{V}$. Since \mathbb{M}_i is dependent on $\mathbb{M}_0, \dots, \mathbb{M}_{i-1}$, we prove that there exists j < i such that $\mathbf{x} \mathbb{A}_i \mathbb{M}_j \mathbb{B}_i \notin \mathcal{V}$. Let us assume for contradiction that this is not the case. Since \mathbb{M}_i is a linear combination on $\mathbb{M}_0, \dots \mathbb{M}_{i-1}$, there exists $s_0, \dots s_{i-1} \in \mathcal{F}$ such that

$$\mathbb{M}_i = s_0 \cdot \mathbb{M}_0 + s_1 \cdot \mathbb{M}_1 + \dots + s_{i-1} \cdot \mathbb{M}_{i-1}$$

Since $\mathbf{x}\mathbb{A}_i\mathbb{M}_j\mathbb{B}_i \in \mathcal{V}$ for all $j \in [0, i-1]$ we get that $\mathbf{x}\mathbb{A}_i\mathbb{M}_i\mathbb{B}_i = \sum_{j=0}^{i-1} s_j \cdot \mathbf{x}\mathbb{A}_i\mathbb{M}_j\mathbb{B}_i \in \mathcal{V}$, which is a contradiction.

▶ Lemma 9. Let $k \in \mathbb{N}$, $\mathbb{A} \in \mathcal{F}^{k \times k}$ and $\mathcal{V} \subseteq \mathcal{F}^k$ be a vector space. Then the following set is a vector space,

$$\mathcal{U} = \{ \mathbf{y} \in \mathcal{F}^k \mid \mathbf{y} \mathbb{A} \in \mathcal{V} \}.$$

Proof. To prove that \mathcal{U} is a vector space, it suffices to show that it is closed under vector addition and scalar multiplication. First, we prove that \mathcal{U} is closed under vector addition. Let $\mathbf{z}_1, \mathbf{z}_2 \in \mathcal{U}$ be two vectors, since $\mathbf{z}_1 \mathbb{A}, \mathbf{z}_2 \mathbb{A} \in \mathcal{V}$, $(\mathbf{z}_1 + \mathbf{z}_2) \mathbb{A} = \mathbf{z}_1 \mathbb{A} + \mathbf{z}_2 \mathbb{A} \in \mathcal{V}$. Therefore, $\mathbf{z}_1 + \mathbf{z}_2 \in \mathcal{U}$. Now we prove that \mathcal{U} is closed under scalar multiplication. For any vector $\mathbf{z}_1 \in \mathcal{U}$, we know that $\mathbf{z}_1 \mathbb{A} \in \mathcal{V}$. Since \mathcal{V} is a vector space, for any scalar $r \in \mathcal{F}$, $(r \cdot \mathbf{z}_1) \mathbb{A} \in \mathcal{V}$, and therefore $r \cdot \mathbf{z}_1 \in \mathcal{U}$. This concludes the proof.

In particular, the above lemma holds for the vector space $\{\mathbf{0} \in \mathcal{F}^k\}$.

2.3 Weighted one-deterministic-counter automata

In this section, we define weighted ODCA, where the weights are from a semiring. However, our results require that the weights come from some field \mathcal{F} except for Section 7, where the weights are from the boolean semiring. First, we define a weighted one-counter automata.

▶ **Definition 10.** A weighted one-counter automata $\mathcal{A} = (Q, \lambda, \delta_0, \delta_1, \eta)$, is defined over an alphabet Σ where, Q is a non-empty finite set of states, $\lambda \in \mathcal{F}^{|Q|}$ is the initial distribution where the i^{th} component of λ indicates the initial weight on state $q_i \in Q$, $\delta_0 : Q \times \Sigma \times Q \times \{0, +1\} \to \mathcal{F}$ and $\delta_1 : Q \times \Sigma \times Q \times \{-1, 0, +1\} \to \mathcal{F}$ are the transition functions, and $\eta \in \mathcal{F}^{|Q|}$ is the final distribution, where the i^{th} component of η indicates the output weight on state $q_i \in Q$.

Note that the counter values do not go below zero. Let $p, q \in Q, a \in \Sigma, n \in \mathbb{N}, e \in \{-1, 0, +1\}$, and $s \in \mathcal{F}$. We say $(q, n) \hookrightarrow^{a|s} (p, n + e)$ if $\delta_{sign(n)}(q, a, p, e) = s$. Let $w = a_1 a_2 \cdots a_t \in \Sigma^*$ for some $t \in \mathbb{N}$. For a $q_0 \in Q$ and $n_0 \in \mathbb{N}$, we say $(q_0, n_0) \hookrightarrow^{w|s} (q_t, n_t)$ if for all $i \in [1, t]$, there are $q_i \in Q, n_i \in \mathbb{N}, s_i \in \mathcal{F}$ such that $(q_{i-1}, n_{i-1}) \hookrightarrow^{a_i|s_i} (q_i, n_i)$ and $s = \prod_{i=1}^t s_i$.

▶ **Definition 11.** A weighted OCA with counter-determinacy is a weighted one-counter automata $\mathcal{A} = (Q, \lambda, \delta_0, \delta_1, \eta)$ with the following restriction: if $\lambda[i]$ and $\lambda[j]$ are non-zero for some $i, j \in [1, |Q|]$, then for all $w \in \Sigma^*$, if $(q_i, 0) \hookrightarrow^{w|s_1} (p_1, n_1)$ and $(q_j, 0) \hookrightarrow^{w|s_2} (p_2, n_2)$ for some $p_1, p_2 \in Q$, $n_1, n_2 \in \mathbb{N}$ and $s_1, s_2 \in \mathcal{F}$, then $n_1 = n_2$.

The configuration of a weighted OCA with counter-determinacy is therefore of the form (\mathbf{x}, n) where $\mathbf{x} \in \mathcal{F}^{|Q|}$ such that for $i \in [0, |Q|], \mathbf{x}[i]$ denotes the weight with which the machine is in state q_i and $n \in \mathbb{N}$ denotes the current counter value. We present a definition for weighted ODCA, which is an equivalent syntactic model.

- ▶ **Definition 12.** A weighted ODCA, $\mathcal{A} = ((C, \delta_0, \delta_1, p_0), (Q, \lambda, \Delta, \eta))$ is defined over an alphabet Σ where,
- $(C, \delta_0, \delta_1, p_0)$ represents the counter structure and $(Q, \lambda, \Delta, \eta)$ represents the finite state machine.
- C is a non-empty finite set of counter states.
- $\delta_0: C \times \Sigma \to C \times \{0, +1\}, \ \delta_1: C \times \Sigma \to C \times \{-1, 0, +1\} \ are \ counter \ transitions.$
- $p_0 \in C$ is the start state for the counter structure.
- Q is a non-empty finite set of states of the finite state machine.
- $\lambda \in \mathcal{F}^{|Q|}$ is the initial distribution where the i^{th} component of λ indicates the initial weight on state $q_i \in Q$.
- $\Delta: \Sigma \times \{0,1\} \to \mathcal{F}^{|Q| \times |Q|}$ gives the transition matrix. For all $a \in \Sigma$ and $d \in \{0,1\}$, the component in the i^{th} row and j^{th} column of $\Delta(a,d)$ denotes the weight on the transition from state $q_i \in Q$ to state $q_j \in Q$ on reading symbol a from counter value n with sign(n) = d.
- $\mathbf{\eta} \in \mathcal{F}^{|Q|}$ is the final distribution, where the i^{th} component of $\mathbf{\eta}$ indicates the output weight on state $q_i \in Q$.

Note that δ_0 and δ_1 are deterministic transition functions. The counter structure and the finite state machine run synchronously on any given word. A configuration c of an ODCA is of the form $(\mathbf{x}_{\mathsf{c}}, p_{\mathsf{c}}, n_{\mathsf{c}}) \in \mathcal{F}^{|Q|} \times C \times \mathbb{N}$. We use the notation WEIGHT-VECTOR(c) to denote \mathbf{x}_{c} , COUNTER-STATE(c) to denote p_{c} , and COUNTER-VALUE(c) to denote n_{c} . The initial configuration is $(\lambda, p_0, 0)$. A transition is a tuple $\tau = (\iota, d, a, \mathsf{ce}, \mathbb{A}, \theta)$ where $\iota, \theta \in C$ are counter states, $d \in \{0, 1\}$ denotes whether the counter value is zero or not, $a \in \Sigma$, $\mathsf{ce} \in \{-1, 0, 1\}$ is the counter-effect, $\mathbb{A} \in \mathcal{F}^{|Q| \times |Q|}$ such that $\Delta(a, d) = \mathbb{A}$, and $\delta_d(\iota, a) = (\theta, \mathsf{ce})$. Given a transition $\tau = (\iota, d, a, \mathsf{ce}, \mathbb{A}, \theta)$ and a configuration $\mathsf{c} = (\mathbf{x}, n, p)$, we denote the application of τ to c as $\tau(\mathsf{c}) = (\mathbf{x}\mathbb{A}, \theta, n + \mathsf{ce})$ if $p = \iota$ and d = sign(n); $\tau(\mathsf{c})$ is undefined otherwise. Note that the counter values are always non-negative.

Consider a sequence of transitions $T = \tau_0 \cdots \tau_\ell$ where $\tau_i = (\iota_i, d_i, a_i, \mathsf{ce}_i, \mathbb{A}_i, \theta_i)$ for all $i \in [0, \ell]$. We denote by $\mathsf{word}(T) = a_0 \cdots a_\ell$ the word labelling it, $\mathsf{we}(T) = \mathbb{A}_0 \cdots \mathbb{A}_\ell$ its weight-effect matrix, and $\mathsf{ce}(T) = \mathsf{ce}_0 + \cdots + \mathsf{ce}_\ell$ its counter-effect. For all $0 \le i < j \le \ell$, we use $T_{i \cdots j}$ to denote the sequence of transitions $\tau_i \cdots \tau_j$ and |T| to denote its length $\ell + 1$. We call T floating if for all $i \in [0, \ell - 1]$, $d_i = 1$ and non-floating otherwise. We denote $\min_{\mathsf{ce}}(T) = \min_i(\mathsf{ce}(\tau_0 \cdots \tau_i))$ the minimal effect of its prefixes and $\max_{\mathsf{ce}}(T) = \max_i(\mathsf{ce}(\tau_0 \cdots \tau_i))$ is the maximal effect of its prefixes.

A run π is an alternating sequence of configurations and transitions denoted as $\pi = c_0 \tau_0 c_1 \cdots \tau_{\ell-1} c_\ell$ such that for every i, $c_{i+1} = \tau_i(c_i)$. The word labelling, length, weight-effect, and counter-effect of the run are those of its underlying sequence of transitions. Given a sequence of transitions $T = \tau_0 \cdots \tau_{\ell-1}$ and a configuration c, we denote by T(c) the run (if it is defined) $c_0 \tau_0 c_1 \cdots \tau_{\ell-1} c_\ell$ where $c_0 = c$.

Let c be a configuration with COUNTER-VALUE(c) = n for some $n \in \mathbb{N}$. Observe that, for a valid floating sequence of transitions, T(c) is defined if and only if $n > -\min_{ce}(T)$, and for a valid non-floating sequence of transitions, T(c) is defined if and only if $n = -\min_{ce}(T)$ and for every i, $d_{\tau_i} = 0$ if and only if $ce(\tau_0 \cdots \tau_{i-1}) = \min_{ce}(T)$. In particular, observe that if a valid floating sequence of transition T is applicable to a configuration $c = (\mathbf{x}, p, n)$, then for every $n' \geq n$ and vector $\mathbf{x}' \in \mathcal{F}^{|Q|}$, it is applicable to (\mathbf{x}', p, n') .

For any word w, there is at most one run labelled by w starting from a given configuration c_0 . We denote this run $\pi(w, c_0)$. A run $\pi(w, c_0) = c_0 \tau_0 c_1 \cdots \tau_{\ell-1} c_\ell$ is also represented as $c_0 \xrightarrow{w} c_\ell$. We say $c_0 \to^* c_\ell$ if there is some word w such that $c_0 \xrightarrow{w} c_\ell$. For a weighted ODCA \mathcal{A} , the accepting weight of w is denoted by $f_{\mathcal{A}}(w, c) = \lambda we(\pi(w, c)) \eta^{\top}$, where c is the initial configuration of \mathcal{A} . Two weighted ODCAS \mathcal{A} and \mathcal{B} are equivalent if for all $w \in \Sigma^*$, $f_{\mathcal{A}}(w, c) = f_{\mathcal{B}}(w, d)$ where c and d are the initial configurations of \mathcal{A} and \mathcal{B} respectively. Let c and d be configurations of ODCAS \mathcal{A} and \mathcal{B} respectively. We say that $c \equiv_l d$ if and only if for all $w \in \Sigma^{\leq l}$, $f_{\mathcal{A}}(w, c) = f_{\mathcal{B}}(w, d)$ otherwise $c \not\equiv_l d$. We use the notation $f_{\mathcal{A}}(w)$ to denote $f_{\mathcal{A}}(w, (\lambda, p_0, 0))$.

We say that two configurations c and d are equivalent if and only if $c \equiv_l d$ for all $l \in \mathbb{N}$ and we denote this by $c \equiv d$. If the weighted ODCA is defined over the boolean semiring, then we call it a non-deterministic/deterministic ODCA. The class of weighted ODCAs includes deterministic OCA, visibly weighted OCA, and deterministic weighted OCA.

An uninitialised weighted ODCA \mathcal{A} is a weighted ODCA without an initial counter state and initial distribution. Formally, $\mathcal{A} = ((C, \delta_0, \delta_1), (Q, \Delta, \eta))$. Given an uninitialised weighted ODCA \mathcal{A} and an initial configuration $\mathbf{c}_0 = (\mathbf{x}, p, 0)$, we define the weighted ODCA $\mathcal{A}\langle\mathbf{c}_0\rangle = ((C, \delta_0, \delta_1, p), (Q, \mathbf{x}, \Delta, \eta))$.

In Theorem 2 (proof in Appendix A), we show that weighted OCA with counter-determinacy is equivalent to weighted ODCA.

Consider the weighted ODCA \mathcal{C} recognising the function prefixAwareDecimal given in Figure 2a. Here, $\lambda = [1,0,0,0]$ and $\eta = [0,0,0,1]$. The configuration $c_0 = ([1,0,0,0],p_0,0)$ is the initial configuration of this machine. Let w = abaaab. The run of this machine on the word w can be written as:

$$\pi(w, \mathbf{c}_0) = ([1, 0, 0, 0], p_0, 0) \xrightarrow{a} ([1, 0, 0, 0], p_0, 1) \xrightarrow{b} ([0, 1, 0, 0], p_1, 0) \xrightarrow{a} ([0, 0, 1, 0], p_2, 0) \xrightarrow{a} ([0, 0, 1, 1], p_2, 1) \xrightarrow{a} ([0, 0, 1, 2], p_2, 2) \xrightarrow{b} ([0, 0, 1, 6], p_2, 1).$$

The counter-effect of this run is $ce(\pi(w, c_0)) = 1$ and the weight-effect matrix is given by

$$\mathrm{we}(\pi(w,\mathbf{c}_0)) = \Delta(a,0) \ \Delta(b,1) \ \Delta(a,0) \ \Delta(a,1) \ \Delta(a,1) \ \Delta(b,1) = \begin{bmatrix} 0 & 0 & 1 & 6 \\ 0 & 0 & 1 & 14 \\ 0 & 0 & 1 & 46 \\ 0 & 0 & 0 & 64 \end{bmatrix}.$$

The accepting weight of the word w is $f_{\mathcal{C}}(w, c_0) = \lambda we(\pi(w, c_0)) \eta^{\top} = 6$.

Weighted automata (WA) is a restricted form of an ODCA where the counter value is fixed at zero. The above notions of transitions, runs, acceptance, etc. are used for WA also. We also use the classical notion and represent weighted automata as $\mathcal{A} = (Q, \lambda, \Delta, \eta)$, without counter states. Given a weighted ODCA \mathcal{A} over the alphabet Σ and a field \mathcal{F} , we define its M-unfolding weighted automaton \mathcal{A}^M as a finite state weighted automaton that recognises the same function as \mathcal{A} for all runs where the counter value does not exceed M. A formal definition is given in Definition 13.

- ▶ Definition 13 (M-unfolding weighted automata). Let $\mathcal{A} = ((C, \delta_0, \delta_1, p_0), (Q, \lambda, \Delta, \eta))$ be a weighted odca over the alphabet Σ and a field \mathcal{F} . For a given $M \in \mathbb{N}$, we define an M-unfolding weighted automata \mathcal{A}^M of \mathcal{A} as follows, $\mathcal{A}^M = (C', \delta', p'_0; Q', \lambda', \Delta', \eta'_F)$ where,
- $C' = C \times [0, M]$ is the finite set of counter states.
- $\delta': C' \times \Sigma \to C'$ is the deterministic counter transition. Let $p, q \in C, m \in \mathbb{N}$, $a \in \Sigma$ and $d \in \{-1, 0, +1\}$. $\delta'((p, m), a) = (q, m + d)$, if $\delta_{sign(m)}(p, a) = (q, d)$.
- $p'_0 = (p_0, 0)$ is the initial counter state.
- $Q' = Q \times [0, M]$ is the finite set of states.
- $\lambda' \in \mathcal{F}^{|Q'|}$ is the initial distribution.

$$\lambda'[i] = \begin{cases} \lambda[i], & \text{if } i < |Q| \\ 0, & \text{otherwise} \end{cases}$$

 $\Delta': \Sigma \to \mathcal{F}^{|Q|' \times |Q'|}$ gives the transition matrix. For $i, j \in |Q'|$ and $a \in \Sigma$,

$$\Delta'(a)[i][j] = \begin{cases} \Delta(a,0)[i][j], & \text{if } i,j < |Q| \\ \Delta(a,1)[i \bmod |Q|][j \bmod |Q|], & \text{if } \frac{i}{|Q|} = \frac{j}{|Q|} \\ 0, & \text{otherwise} \end{cases}$$

 $\eta_F' \in \mathcal{F}^{|Q'|}$ is the final distribution.

$$\eta_F'[i] = \eta[i \bmod |Q|]$$

3 Reachability problems of weighted ODCA

In this section, we introduce the co-VS reachability and co-VS coverability problems for weighted odcas over a field \mathcal{F} . We fix a weighted odca $\mathcal{A} = ((C, \delta_0, \delta_1, p_0), (Q, \lambda, \Delta, \eta))$. We use $\mathcal{V} \subseteq \mathcal{F}^{|Q|}$ to denote a vector space and $\overline{\mathcal{V}}$ its complement. Let $S \subseteq C$ be a subset of the set of counter states, $X \subseteq \mathbb{N}$ a set of counter values, and $w \in \Sigma^*$. The notation $\mathbf{c} \xrightarrow{w} \overline{\mathcal{V}} \times S \times X$ denotes the run $\mathbf{c} \xrightarrow{w} \mathbf{d}$ where $\mathbf{d} \in \overline{\mathcal{V}} \times S \times X$ if it exists. We use $\mathbf{c} \xrightarrow{*} \overline{\mathcal{V}} \times S \times X$ to denote that there exists a word $u \in \Sigma^*$ such that $\mathbf{c} \xrightarrow{u} \overline{\mathcal{V}} \times S \times X$.

CO-VS REACHABILITY PROBLEM

INPUT: a weighted ODCA \mathcal{A} , an initial configuration c, a vector space \mathcal{V} , a set of counter states S, and a counter value m.

OUTPUT: Yes, if there exists a run $c \xrightarrow{*} \overline{\mathcal{V}} \times S \times \{m\}$ in \mathcal{A} . No, otherwise.

CO-VS COVERABILITY PROBLEM

INPUT: a weighted odca A, an initial configuration c, a vector space V, and a set of counter states S.

OUTPUT: Yes, if there exists a run $c \xrightarrow{*} \overline{\mathcal{V}} \times S \times \mathbb{N}$ in \mathcal{A} . No, otherwise.

Unlike the co-VS reachability problem, the final configuration's counter value is not considered part of the input for co-VS coverability problem. We assume that the vector space $\mathcal{V} \subseteq \mathcal{F}^{|Q|}$ is provided by giving a suitable basis. We call $z \in \Sigma^*$ a witness of $(\mathbf{c}, \overline{\mathcal{V}}, S, X)$ if $\mathbf{c} \xrightarrow{z} \overline{\mathcal{V}} \times S \times X$. Furthermore, z is called a minimal witness for $(\mathbf{c}, \overline{\mathcal{V}}, S, X)$ if for all $u \in \Sigma^*$ with $\mathbf{c} \xrightarrow{u} \overline{\mathcal{V}} \times S \times X$, $|u| \geq |z|$.

First, we look at the particular case of co-VS reachability problem for weighted automata. Note that for weighted automata, the counter value is always zero. Given a weighted automata \mathcal{B} , with k states, an initial configuration $\bar{\mathbf{c}}$, a vector space $\mathcal{U} \subseteq \mathcal{F}^k$ and a set of counter states S, the co-VS reachability problem asks whether there exists a run $\bar{\mathbf{c}} \stackrel{*}{\to} \overline{\mathcal{U}} \times S \times \{0\}$.

▶ **Theorem 14.** There is a polynomial time algorithm that decides the co-VS reachability problem for weighted automata and outputs a minimal reachability witness if it exists.

Proof. The idea of equivalence checking of weighted automata goes back to the seminal paper by Schützenberger [19]. Tzeng [22] provided a polynomial time algorithm for the equivalence of two probabilistic automata by reducing the problem to the co-VS reachability problem where $\mathcal{V} = \{\mathbf{0}\}$. The same algorithm can be modified to solve the general co-VS reachability problem.

In the upcoming subsection, we give some interesting properties of minimal witnesses. In Section 3.2, we provide a pseudo-pumping lemma which helps us show that co-VS reachability and co-VS coverability are in P if the counter values are given in unary notation. Finally, in Section 3.3, we demonstrate that the lexicographically minimal witness has a canonical form. In the following subsections, $\mathcal V$ denote a vector space, $\mathbf c$ a configuration, S a subset of counter states, and S is the set of counter states, and S is the set of states of the finite state machine.

3.1 Minimal witness and its properties

The following observation helps in breaking down the reachability problem into sub-problems. If $z \in \Sigma^*$ is a minimal witness for $(c, \overline{\mathcal{V}}, S, X)$, then for every z_1, z_2 such that $z = z_1 z_2$, there is a computable vector space \mathcal{U} such that z_1 is a minimal witness for $(c, \overline{\mathcal{U}}, \{p\}, \{n\})$ where p is the counter state and p is the counter value reached after reading z_1 from c.

▶ Observation 15. Consider arbitrary $z, z_1, z_2 \in \Sigma^*$ such that $z = z_1 z_2$. Let $\mathbf{d} = (\mathbf{x}_d, p_d, n_d)$ and $\mathbf{e} = (\mathbf{x}_e, p_e, n_e)$ be configurations such that $\mathbf{c} \xrightarrow{z_1} \mathbf{d} \xrightarrow{z_2} \mathbf{e}$ and $\mathbb{A} \in \mathcal{F}^{|Q| \times |Q|}$ be such that $\mathbf{x}_d \mathbb{A} = \mathbf{x}_e$. If z is a minimal witness for $(\mathbf{c}, \overline{\mathcal{V}}, S, X)$, then z_1 is a minimal witness for $(\mathbf{c}, \overline{\mathcal{U}}, \{p_d\}, \{n_d\})$, where $\mathcal{U} = \{\mathbf{y} \in \mathcal{F}^{|Q|} \mid \mathbf{y} \mathbb{A} \in \mathcal{V}\}$.

Proof. Let $z \in \Sigma^*$ be a minimal reachability witness for $(\mathbf{c}, \overline{\mathcal{V}}, S, X)$, $\mathbf{d} = (\mathbf{x}_{\mathsf{d}}, p_{\mathsf{d}}, n_{\mathsf{d}})$ and $\mathbf{e} = (\mathbf{x}_{\mathsf{e}}, p_{\mathsf{e}}, n_{\mathsf{e}})$ be configurations such that $\mathbf{c} \stackrel{z_1}{\longrightarrow} \mathbf{d} \stackrel{z_2}{\longrightarrow} \mathbf{e}$ where $z_1, z_2 \in \Sigma^*$ with $z = z_1 z_2$ and $\mathbb{A} \in \mathcal{F}^{|Q| \times |Q|}$ be such that $\mathbf{x}_{\mathsf{d}} \mathbb{A} = \mathbf{x}_{\mathsf{e}}$. Let $\mathcal{U} = \{\mathbf{y} \in \mathcal{F}^{|Q|} \mid \mathbf{y} \mathbb{A} \in \mathcal{V}\}$. Assume for contradiction that there exists $z_1' \in \Sigma^*$ smaller than z_1 and $\mathbf{c} \stackrel{z_1'}{\longrightarrow} \mathbf{f}$ for some configuration $\mathbf{f} \in \overline{\mathcal{U}} \times \{p_{\mathsf{d}}\} \times \{n_{\mathsf{d}}\}$. Note that for all $\mathbf{y} \in \overline{\mathcal{U}}$, the vector $\mathbf{y} \mathbb{A} \in \overline{\mathcal{V}}$. Since the configurations \mathbf{f} and \mathbf{d} have the same counter state and counter value, $\mathbf{c} \stackrel{z_1'}{\longrightarrow} \mathbf{f} \stackrel{z_2}{\longrightarrow} \overline{\mathcal{V}} \times \{p_{\mathsf{e}}\} \times \{n_{\mathsf{e}}\}$ is a run and the word $z_1'z_2$ contradicts the minimality of z.

We aim to show that the length of a minimal witness for $(c, \overline{\mathcal{V}}, S, X)$ is polynomially bounded. The following lemma shows that if the counter values are polynomially bounded during the run of a minimal witness, then its length is also polynomially bounded.

▶ **Lemma 16.** Let $z \in \Sigma^*$ be a minimal witness for $(c, \overline{\mathcal{V}}, S, X)$. If the number of distinct counter values encountered during the run $c \xrightarrow{z} \overline{\mathcal{V}} \times S \times X$ is t, then $|z| \leq t \cdot K$.

Proof. Let $\mathbf{c} = \mathbf{c}_1$ and $T(\mathbf{c}_1) = \mathbf{c}_1 \tau_1 \mathbf{c}_2 \cdots \tau_{h-1} \mathbf{c}_h$ be the run on word z from \mathbf{c}_1 and T the corresponding sequence of transitions. Let t be the number of distinct counter values encountered during this run. Now assume for contradiction that $h > |Q| \cdot |C| \cdot t$, then by Pigeon-hole principle, there are |Q| + 1 many configurations $\mathbf{c}_{i_0}, \mathbf{c}_{i_1}, \ldots, \mathbf{c}_{i_{|Q|}}$ with the same counter state and counter value during this run. Given a configuration \mathbf{c} , let $\mathbf{x}_{\mathbf{c}}$ denote WEIGHT-VECTOR(\mathbf{c}). Let \mathbb{A}_j denote the matrix such that $\mathbf{x}_{\mathbf{c}_{i_j}} \mathbb{A}_j = \mathbf{x}_{\mathbf{c}_h}$ for all $j \in [0, |Q|]$. From Lemma 8 we get that there exists $r \leq |Q|$, and $t \in [0, r-1]$ such that

 $\mathbf{x}_{\mathsf{c}_{i_t}} \mathbb{A}_r \in \overline{\mathcal{V}}$. Consider the sequence of transitions $T' = \tau_1...i_t \tau_r...\ell_{-1}$ and $v = \mathsf{word}(T')$. The run $\pi(v, \mathsf{c}_1) = T'(\mathsf{c}_1)$ is a run since configurations c_t and c_r have the same counter state and counter value. This is a shorter run than $\pi(z, \mathsf{c}_1)$ and $\mathsf{c}_1 \xrightarrow{v} \overline{\mathcal{V}} \times S \times X$. This is a contradiction since z was assumed to be minimal.

It now suffices to show that the counter values encountered during the run of a minimal witness are polynomially bounded.

3.2 Pseudo-pumping lemma

The pseudo-pumping lemma is a valuable tool in our analysis, allowing us to pump up or down a sufficiently long word while maintaining the reachability conditions.

- ▶ Lemma 17 (pseudo-pumping lemma). Let $m, R \in \mathbb{N}$, be such that COUNTER-VALUE(\mathbf{c}) = m and $z \in \Sigma^*$ be such that $\mathbf{c} \xrightarrow{z} \overline{\mathcal{V}} \times S \times \{m\}$ is a floating run, and the maximum counter value encountered during this run is m + R. If $R > K^2$, then there exists $z_{sub}, z_{sup} \in \Sigma^*$ such that the following hold:
- 1. there exist $x, y, u, v, w \in \Sigma^*$ such that $z = xyuvw, z_{sub} = xuw$, $c \xrightarrow{z_{sub}} \overline{\mathcal{V}} \times S \times \{m\}$ is a floating run, and the counter values encountered during this run are less than m + R, and
- 2. there exist $x, y, u, v, w \in \Sigma^*$ such that $z = xyuvw, z_{sup} = xy^2uv^2w$, $c \xrightarrow{z_{sup}} \overline{\mathcal{V}} \times S \times \{m\}$ is a floating run, and the maximum counter value encountered in this run exceeds m + R.

Proof. Let $z \in \Sigma^*$ be a witness for $(c, \overline{\mathcal{V}}, S, \{m\})$ and $e \in \overline{\mathcal{V}} \times S \times \{m\}$ be such that $c \xrightarrow{z} e$ is a floating run, and the maximum counter value encountered in this run be m+R where $R > K^2$. Let COUNTER-VALUE(c) = m. There exist $z_1, z_2 \in \Sigma^*$ and configuration f such that $z = z_1 z_2$ and $c \xrightarrow{z_1} f \xrightarrow{z_2} e$, where COUNTER-VALUE(f) = m+R (see Figure 3).

Let $c_1 = c$ and $\pi = c_1 \tau_1 c_2 \cdots \tau_{\ell-1} c_\ell$ denote the run on word z from the configuration c_1 and $T = \tau_1 \tau_2 \cdots \tau_{\ell-1}$ the sequence of transitions of π . For any $i \in [0, R]$, we denote by l_i and d_i the indices such that a configuration with counter value m+i is encountered for the last (resp. first) time before (resp. after) reaching counter value m+R in π . That is, COUNTER-VALUE(c_{l_i}) = COUNTER-VALUE(c_{d_i}) = m+i, and for any j where $l_i < j < d_i$, COUNTER-VALUE(c_j) > m+i. To simplify the notation, we denote by $c_i = c_{l_i}$ and $c_i' = c_{d_i}$.

Consider the pairs of configurations $(\mathbf{g}_1,\mathbf{g}_1'), (\mathbf{g}_2,\mathbf{g}_2'),\dots, (\mathbf{g}_R,\mathbf{g}_R')$. Since $R>(|Q|\cdot |C|)^2$, by the Pigeonhole principle, there exist two counter states p,q, and a set of indices $X\subseteq [0,R]$ where $|X|=|Q|^2+1$ such that for all $h\in X$, Counter-state $(\mathbf{g}_h)=p$ and Counter-state $(\mathbf{g}_h')=q$. For all $j\in X$, let $u_j,v_j,w_j\in \Sigma^*$ be such that $\mathbf{c}_1\xrightarrow{u_j}\mathbf{g}_j\xrightarrow{v_j}\mathbf{g}_j'\xrightarrow{w_j}\mathbf{e}$. We use the following shorthand for any configuration $\mathbf{g}\colon \mathbf{x}_{\mathbf{g}}=\mathbf{w}\text{Eight-vector}(\mathbf{g})$. For all $j\in X$, let matrix \mathbb{A}_j and \mathbb{B}_j be such that $\mathbf{x}_{\mathbf{g}_j'}=\mathbf{x}_{\mathbf{g}_j}\mathbb{A}_j$ and $\mathbf{x}_{\mathbf{e}}=\mathbf{x}_{\mathbf{g}_j'}\mathbb{B}_j$. Since $\mathbf{x}_{\mathbf{e}}\in \overline{\mathcal{V}}$, for all $j\in X$, $\mathbf{x}_{\mathbf{g}_j}\mathbb{A}_j\mathbb{B}_j\in \overline{\mathcal{V}}$. Let $r=|Q|^2+1$, and $i_1< i_2<\dots< i_r$ be the indices in X. We prove the two cases separately.

1. Consider the sequence of matrices: $\mathbb{A}_{i_r}, \mathbb{A}_{i_{r-1}}, \dots, \mathbb{A}_{i_1}$. Since there can be at most $|Q|^2$ independent matrices, there exists $k \in [1,r]$ such that \mathbb{A}_{i_k} is a linear combination of $\mathbb{A}_{i_r}, \dots, \mathbb{A}_{i_{k+1}}$. Hence, there exists $h \in \{i_r, \dots, i_{k+1}\}$ such that $\mathbf{x}_{\mathbf{g}_{i_k}} \mathbb{A}_h \mathbb{B}_{i_k} \in \overline{\mathcal{V}}$. Let $z_{sub} = u_{i_k} v_h w_{i_k}$. It is easy to observe that z_{sub} is a subword of z as mentioned in the lemma. To conclude the proof, it now suffices to show that z_{sub} is a witness for $(\mathbf{c}, \overline{\mathcal{V}}, S, \{m\})$ and the counter values encountered during the run $\mathbf{c} \xrightarrow{z_{sub}} \mathbf{h}$ are less than m+R. Consider the floating run $\mathbf{g}_h \xrightarrow{v_h} \mathbf{g}_h'$. From the choice of \mathbf{g}_h and \mathbf{g}_h' we know that COUNTER-VALUE $(\mathbf{g}_h) = \text{COUNTER-VALUE}(\mathbf{g}_h') = m+h$ and for all j where $l_h < j < d_h$, COUNTER-VALUE $(\mathbf{c}_j) > m+h$. Since COUNTER-STATE $(\mathbf{g}_h) = \text{COUNTER-STATE}(\mathbf{g}_{i_k}), \pi(v_h, \mathbf{g}_{i_k})$ is also a floating run

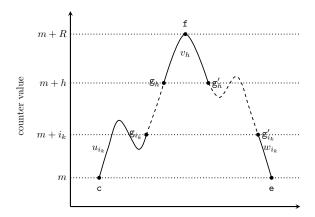


Figure 3 The figure shows the floating run from a configuration c with COUNTER-VALUE(c) = m to a configuration $\mathbf{e} = (\mathbf{x}, p, m)$ such that $\mathbf{x} \in \overline{\mathcal{U}}$. Configurations \mathbf{g}_{i_k} and \mathbf{g}_h (resp. \mathbf{g}'_{i_k} and \mathbf{g}'_h) are where the counter values $m+i_k$ and m+h are encountered for the last (resp. first) time before (resp. after) reaching m+R. Also, COUNTER-STATE(\mathbf{g}_{i_k}) = COUNTER-STATE(\mathbf{g}_h) and COUNTER-STATE(\mathbf{g}'_h) = COUNTER-STATE(\mathbf{g}'_{i_k}). The dashed line denotes the part of the run that can be removed to get a shorter witness for $(\mathbf{c}, \overline{\mathcal{U}}, \{p\}, \{n\})$.

 $\mathbf{g}_{i_k} \xrightarrow{v_h} \mathbf{d}$ such that COUNTER-STATE(\mathbf{g}'_h) = COUNTER-STATE(\mathbf{d}), COUNTER-VALUE(\mathbf{g}_{i_k}) = COUNTER-VALUE(\mathbf{d}) = $m+i_k < m+h$, and the minimum and maximum counter values encountered in the run is $m+i_k$ and $m+R-(h-i_k)$ respectively (see Figure 3). Furthermore, $\mathbf{x}_d = \mathbf{x}_{\mathbf{g}_{i_k}} \mathbb{A}_h$. Since COUNTER-STATE(\mathbf{g}'_{i_k}) = COUNTER-STATE(\mathbf{g}'_{i_k}) = COUNTER-STATE(\mathbf{g}'_{i_k}) = COUNTER-VALUE(\mathbf{d}). Therefore, $\pi(w_{i_k}, \mathbf{d})$ is the run $\mathbf{d} \xrightarrow{w_{i_k}} \mathbf{h}$ where $\mathbf{x}_h = \mathbf{x}_{\mathbf{d}} \mathbb{B}_{i_k}$ and hence $\mathbf{x}_h = \mathbf{x}_{\mathbf{g}_{i_k}} \mathbb{A}_h \mathbb{B}_{i_k} \in \overline{\mathcal{V}}$. This concludes that z_{sub} is a witness for ($\mathbf{c}, \overline{\mathcal{V}}, S, \{m\}$) and satisfies the properties mentioned in the lemma.

Consider the sequence of matrices: $\mathbb{A}_{i_1}, \mathbb{A}_{i_2}, \dots, \mathbb{A}_{i_r}$. Since there can be at most $|Q|^2$ independent matrices, there exists $k \in [1,r]$ such that \mathbb{A}_{i_k} is a linear combination of $\mathbb{A}_{i_1}, \dots, \mathbb{A}_{i_{k-1}}$. Hence, there exists an $h \in \{i_1, \dots, i_{k-1}\}$ such that $\mathbf{x}_{\mathbf{g}_{i_k}} \mathbb{A}_h \mathbb{B}_{i_k} \in \overline{\mathcal{V}}$. Let $z_{sup} = u_{i_k} v_h w_{i_k}$. It is easy to observe that z_{sup} is a superword of z as mentioned in the lemma. To conclude the proof, it now suffices to show that z_{sup} is a witness for $(c, \overline{\mathcal{V}}, S, \{m\})$ and the counter values encountered during the run $c \xrightarrow{z_{sup}} h$ is greater than m+R. Consider the floating run $g_h \xrightarrow{v_h} g'_h$. From the choice of g_h and g'_h we know that COUNTER-VALUE(g_h) = COUNTER-VALUE(g'_h) = m + h and for all j where $l_h < j < d_h$, COUNTER-VALUE(c_j) > m+h. Since COUNTER-STATE(\mathbf{g}_h) = COUNTER-STATE(\mathbf{g}_{i_k}), $\pi(\mathbf{g}_{i_k}, v_h)$ is also a floating run $\mathsf{g}_{i_k} \xrightarrow{v_h} \mathsf{d}$ such that Counter-State $(\mathsf{g}'_h) = \text{Counter-State}(\mathsf{d}), \text{Counter-Value}(\mathsf{g}_{i_k}) =$ COUNTER-VALUE(d) = $m + i_k > m + h$, and the minimum and maximum counter values encountered in the run is $m + i_k$ and $m + R + (i_k - h)$ respectively. Furthermore, $\mathbf{x}_d = \mathbf{x}_{\mathsf{g}_{i_k}} \mathbb{A}_h. \text{ Since Counter-state}(\mathsf{g}'_{i_k}) = \text{counter-state}(\mathsf{g}'_h), \text{ counter-state}(\mathsf{g}'_{i_k}) =$ COUNTER-STATE(d). Moreover, since COUNTER-VALUE(g'_{i_k}) = COUNTER-VALUE(g_{i_k}), we have Counter-value(\mathbf{g}'_{i_k}) = Counter-value(\mathbf{d}). Therefore $\pi(\mathbf{d}, w_{i_k})$ is the run $\mathbf{d} \xrightarrow{w_{i_k}} \mathbf{h}$ where $\mathbf{x}_h = \mathbf{x}_d \mathbb{B}_{i_k}$ and hence $\mathbf{x}_h = \mathbf{x}_{g_{i_k}} \mathbb{A}_h \mathbb{B}_{i_k} \in \overline{\mathcal{V}}$. This concludes that z_{sup} is a witness for $(c, \overline{V}, S, \{m\})$ and satisfies the properties mentioned in the lemma.

It is important to note that we do not end up in the same configuration while pumping up/down, but we ensure that we reach a configuration with the same counter state, counter

value, and whose weight vector is in the complement of the given vector space.

Now, we prove that for any run (it need not necessarily be a floating run) of a minimal reachability witness z for $(c, \overline{\mathcal{V}}, S, \{m\})$, the maximum counter value encountered during the run $c \stackrel{z}{\to} \overline{\mathcal{V}} \times S \times \{m\}$ is bounded by a polynomial in the number of states of the machine, and the initial and final counter values. This can be achieved by iteratively applying Lemma 17 on the run of the minimal witness (refer Figure 5) and using Observation 15 and Lemma 16.

- ▶ Corollary 18. If $z \in \Sigma^*$ is a minimal witness for $(c, \overline{V}, S, \{m\})$, then
- 1. the maximum counter value encountered during the run $c \xrightarrow{z} \overline{\mathcal{V}} \times S \times \{m\}$ is less than $max(COUNTER-VALUE(c), m) + K^2$, and
- 2. $|z| \le K^3 + max(COUNTER-VALUE(c), m) \cdot K$.

Proof. Let $z \in \Sigma^*$ be a minimal reachability witness for $(c, \overline{V}, S, \{m\})$, where c is a configuration with counter value n.

1. Consider the run of word z from c. Let $\mathbf{d} \in \overline{\mathcal{V}} \times S \times \{m\}$ such that $\mathbf{c} \stackrel{z}{\to} \mathbf{d}$. Assume for contradiction that the maximum counter value encountered during the run $\mathbf{c} \stackrel{z}{\to} \mathbf{d}$ is greater than $\max(n,m) + (|Q| \cdot |C|)^2$. Let $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_t$ be all the configurations in this run such that their counter values are zero. There exists words $u_1, u_2, \cdots, u_{t+1} \in \Sigma^*$ such that $z = u_1 u_2 \cdots u_{t+1}$ and

$$\mathsf{c} \xrightarrow{u_1} \mathsf{e}_1 \xrightarrow{u_2} \mathsf{e}_2 \xrightarrow{u_3} \cdots \xrightarrow{u_t} \mathsf{e}_t \xrightarrow{u_{t+1}} \mathsf{d}$$

Note that $c \xrightarrow{u_1} e_1$, $e_t \xrightarrow{u_{t+1}} d$ and $e_i \xrightarrow{u_{i+1}} e_{i+1}$ for all $i \in [1, t-1]$ are floating runs (refer Figure 5).

We show that the counter values are bounded during these floating runs. First, we consider the floating run $c \xrightarrow{u_1} e_1$. Given a configuration c, we use \mathbf{x}_c to denote WEIGHT-VECTOR(c). Let $\mathbb{A} \in \mathcal{F}^{|Q| \times |Q|}$ be such that $\mathbf{x}_d = \mathbf{x}_{e_1} \mathbb{A}$. From Lemma 9 we know that the set $\mathcal{U} = \{\mathbf{y} \in \mathcal{F}^{|Q|} \mid \mathbf{y} \mathbb{A} \in \mathcal{V}\}$ is a vector space and hence the vector $\mathbf{x}_{e_1} \in \overline{\mathcal{U}}$. From Observation 15, we know that u_1 is a minimal reachability witness for $(c, \overline{\mathcal{U}}, \{p_{e_1}\}, \{0\})$ and therefore by Lemma 17 we know that the maximum counter value encountered during the run $\pi(u_1, c)$ is less than $n + (|Q| \cdot |C|)^2$.

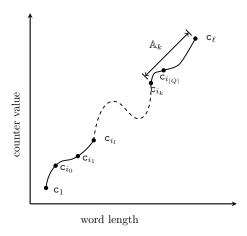
Similarly for the floating run $\mathbf{e}_t \xrightarrow{u_{t+1}} \mathbf{d}$, the maximum counter value is bounded by $m + (|Q| \cdot |C|)^2$. Now consider the floating runs $\mathbf{e}_i \xrightarrow{u_{i+1}} \mathbf{e}_{i+1}$ for all $i \in [1, t-1]$. Again by applying Lemma 17 we get that the maximum counter value encountered during these sub-runs is less than $(|Q| \cdot |C|)^2$. Therefore, the maximum counter value encountered during the run $\mathbf{c} \xrightarrow{z} \overline{\mathcal{V}} \times S \times \{m\}$ is less than $\max(n, m) + (|Q| \cdot |C|)^2$.

2. From the previous point, we know that the maximum counter value encountered during the run $c \stackrel{z}{\to} \overline{\mathcal{V}} \times S \times \{m\}$ is less than $max(n,m) + (|Q| \cdot |C|)^2$. Therefore, there are at most $max(n,m) + (|Q| \cdot |C|)^2$ many distinct counter values encountered during this run. Now from Lemma 16 we get that $|z| \leq (|Q| \cdot |C|) \cdot (max(n,m) + (|Q| \cdot |C|)^2)$.

The following lemma helps us show that the length of a minimal witness for co-VS coverability is polynomially bounded in the number of states.

▶ Lemma 19 (cut lemma). Let $z \in \Sigma^*$ be a witness for $(\mathbf{c}, \overline{\mathcal{V}}, S, \mathbb{N})$, where \mathbf{c} is a configuration with COUNTER-VALUE(\mathbf{c}) = n for some $n \in \mathbb{N}$, and $\mathbf{c} \stackrel{z}{\to} \overline{\mathcal{V}} \times S \times \{m\}$ is a floating run for some $m \in \mathbb{N}$. If m - n > K, then there exists $z_{sub} \in \Sigma^*$ such that z_{sub} is a subword of z, $\mathbf{c} \stackrel{z_{sub}}{\to} \overline{\mathcal{V}} \times S \times \{m'\}$ is a floating run and m' - n < m - n.

Proof. Let $z \in \Sigma^*$ be a witness for $(c, \overline{\mathcal{V}}, S, \mathbb{N})$ and $c \xrightarrow{z} \overline{\mathcal{V}} \times S \times \{m\}$ is a floating run. Let n be the counter value of configuration c and $m > n + |Q| \cdot |C|$. Let $c_1 = c$ and



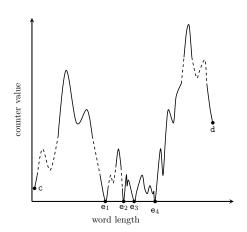


Figure 4 The figure shows a run from configuration c_1 to $c_\ell = (\mathbf{x}_{c_\ell}, p_{c_\ell}, n_{c_\ell})$ such that $\mathbf{x}_{c_\ell} \in \overline{\mathcal{V}}$. The configurations c_{i_l} and c_{i_k} are where the counter values $n_{c_{i_l}}$ and $n_{c_{i_k}}$ are encountered for the last time. Also the configurations c_{i_l} and c_{i_k} have the same counter state. The dashed lines denote a part of that run that can be removed to get a shorter witness for $(c, \overline{\mathcal{V}}, \{p_{c_\ell}\}, \mathbb{N})$.

Figure 5 The figure shows a run from configuration c to $\mathbf{d} = (\mathbf{x}_d, p_d, n_d)$ such that $\mathbf{x}_d \in \overline{\mathcal{V}}$. Configurations $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ are where the counter value zero is encountered during the run. The dashed lines denote the parts that can be removed to obtain a shorter witness for $(\mathbf{c}, \overline{\mathcal{V}}, \{p_d\}, \{n_d\})$.

 $\pi(z, \mathbf{c}_1) = \mathbf{c}_1 \tau_1 \mathbf{c}_2 \cdots \tau_{\ell-1} \mathbf{c}_\ell$ be such that configuration \mathbf{c}_ℓ has counter value m. Consider the sequence of transitions $T = \tau_0 \tau_1 \cdots \tau_{\ell-1}$ in $\pi(z, \mathbf{c}_1)$.

Since there are only |C| counter states, by the Pigeon-hole principle, there exists a strictly increasing sequence $I=0< i_0< i_1< \cdots < i_{|Q|}\leq \ell$ such that for all $j,j'\in I$ (Condition 1) COUNTER-STATE(\mathbf{c}_j) = COUNTER-STATE($\mathbf{c}_{j'}$) and (Condition 2) if j< j' then COUNTER-VALUE(\mathbf{c}_j) < COUNTER-VALUE(\mathbf{c}_j) and for all $d\in [j+1,j'-1]$, COUNTER-VALUE(\mathbf{c}_j) < COUNTER-VALUE(\mathbf{c}_j). Given a configuration \mathbf{c} , let $\mathbf{x}_{\mathbf{c}}$ denote WEIGHT-VECTOR(\mathbf{c}). Consider the set of configurations $\mathbf{c}_{i_0}, \mathbf{c}_{i_1}, \ldots, \mathbf{c}_{i_{|Q|}}$ (see Figure 4). For any $j\in [0,|Q|]$, let \mathbb{A}_j denote the matrix such that $\mathbf{x}_{\mathbf{c}_{i_j}}\mathbb{A}_j=\mathbf{x}_{\mathbf{c}_\ell}$. Since $\mathbf{x}_{\mathbf{c}_{i_d}}\mathbb{A}_d\in\overline{\mathcal{V}}$ for all $d\in [0,|Q|]$, from Lemma 7 we get that there exists $l,k\in [0,|Q|]$ with l< k such that $\mathbf{x}_{\mathbf{c}_{i_l}}\mathbb{A}_k\in\overline{\mathcal{V}}$. Consider a configuration $\mathbf{e}=(\mathbf{x},p,n)$. If $\pi(u,\mathbf{e})$ is a floating run with $\min_{\mathbf{c}\in(\pi(u,\mathbf{e}))>0}$, then for all $m\in\mathbb{N}$ and $\mathbf{y}\in\mathcal{F}^{|Q|}$, $\pi(u,(\mathbf{y},p,m))$ is a run. Consider the sequence of transitions $T'=\tau_{i_k\cdots\ell-1}$ and let $u=\mathbf{word}(T')$. Because of Condition 2, $\min_{\mathbf{c}\in(\pi(u,\mathbf{c}_{i_k}))>0}$. Therefore the run $T''(\mathbf{c}_1)$ where $T''=\tau_{1\cdots i_l-1}\tau_{i_k\cdots\ell-1}$ is a run shorter than $\pi(z,\mathbf{c}_1)$ with smaller counter effect.

Note that if the run of a minimal coverability witness z for $(c, \overline{\mathcal{V}}, S, \mathbb{N})$ is a floating run, then the number of distinct counter values encountered during the run $c \stackrel{z}{\to} \overline{\mathcal{V}} \times S \times \mathbb{N}$ is polynomially bounded in the number of states of the machine. Now we show that for any run (need not be floating) of a minimal coverability witness z for $(c, \overline{\mathcal{V}}, S, \mathbb{N})$, the maximum counter value encountered during the run $c \stackrel{z}{\to} \overline{\mathcal{V}} \times S \times \mathbb{N}$ is bounded by a polynomial in the number of states of the machine and the initial counter value.

▶ Corollary 20. If $z \in \Sigma^*$ is a minimal witness for $(c, \overline{\mathcal{V}}, S, \mathbb{N})$, where c is a configuration with counter value n, then the maximum counter value encountered during the run $c \stackrel{z}{\to} \overline{\mathcal{V}} \times S \times \mathbb{N}$ is less than $max(n, |Q| \cdot |C|) + (|Q| \cdot |C|)^2$.

Proof. Let $z \in \Sigma^*$ be a minimal witness for $(c, \overline{V}, S, \mathbb{N})$, where c is a configuration with

counter value n. Consider the run of word z from \mathbf{c} . Let $\mathbf{d} \in \overline{\mathcal{V}} \times S \times \mathbb{N}$ such that $\mathbf{c} \xrightarrow{z} \mathbf{d}$. If $\mathbf{c} \xrightarrow{z} \mathbf{d}$ is a floating run, then by Lemma 19 the maximum counter value encountered during this run will be less than $n + |Q| \cdot |C|$. Now if $\mathbf{c} \xrightarrow{z} \mathbf{d}$ is not a floating run, then there exists $u_1, u_2 \in \Sigma^*$ such that $z = u_1 u_2$ and $\mathbf{c} \xrightarrow{u_1} \mathbf{e} \xrightarrow{u_2} \mathbf{d}$ where, counter value of configuration \mathbf{e} is zero and $\mathbf{e} \xrightarrow{u_2} \mathbf{d}$ is a floating run.

Given a configuration c, let \mathbf{x}_c denote WEIGHT-VECTOR(c). Let $\mathbb{A} \in \mathcal{F}^{|Q| \times |Q|}$ be such that $\mathbf{x}_d = \mathbf{x}_e \mathbb{A}$. From Lemma 9, we know that the the set $\mathcal{U} = \{\mathbf{y} \in \mathcal{F}^{|Q|} \mid \mathbf{y} \mathbb{A} \in \mathcal{V}\}$ is a vector space and hence the vector $\mathbf{x}_e \in \overline{\mathcal{U}}$. Note that for all $\mathbf{y} \in \overline{\mathcal{U}}$, the vector $\mathbf{y} \mathbb{A} \in \overline{\mathcal{V}}$. From Observation 15, we know that u_1 is a minimal reachability witness for $(c, \overline{\mathcal{U}}, \{p_e\}, \{0\})$, where p_e is the counter state of configuration e, and therefore by Corollary 18, we know that the maximum counter value encountered during the run $\pi(u_1, c)$ is less than $n + (|Q| \cdot |C|)^2$. Now since $e \xrightarrow{u_2} d$ is a floating run and u_2 is the minimal such word, from Lemma 19, we get that the counter value of configuration d is less than or equal to $|Q| \cdot |C|$, and by Lemma 17, we know that the maximum counter value encountered during this run is less than $|Q| \cdot |C| + (|Q| \cdot |C|)^2$. Therefore, we get that the maximum counter value encountered during the run $c \xrightarrow{z} d$ is less than $\max(n, |Q| \cdot |C|) + (|Q| \cdot |C|)^2$.

Our next objective is to show that the counter values are polynomially bounded during the run of a minimal coverability witness. The problem is similar to co-VS reachability, except that now we are not given a final counter value. A crucial ingredient in proving this is Lemma 19 which will help us in proving that if the run of a minimal coverability witness z for $(c, \overline{\mathcal{V}}, S, \mathbb{N})$ is a floating run, then the number of distinct counter values encountered during the run $c \stackrel{z}{\to} \overline{\mathcal{V}} \times S \times \mathbb{N}$ is polynomially bounded in the number of states of the machine and the initial counter value. Using this and the ideas presented earlier for co-VS reachability, we can prove the existence of a polynomial length witness for the co-VS coverability problem.

▶ Corollary 21. Let c be a configuration with counter value n. If z is a minimal witness for $(c, \overline{V}, S, \mathbb{N})$ then $|z| \leq (|Q| \cdot |C|) \cdot (max(n, (|Q| \cdot |C|)) + (|Q| \cdot |C|)^2)$.

Proof. Let $z \in \Sigma^*$ be a minimal reachability witness for $(c, \overline{\mathcal{V}}, S, \mathbb{N})$. From Corollary 20, we know that the maximum counter value encountered during the run $c \xrightarrow{z} \overline{\mathcal{V}} \times S \times \mathbb{N}$ is less than $\max(n, (|Q| \cdot |C|)) + (|Q| \cdot |C|)^2$. Therefore, there are at most $\max(n, (|Q| \cdot |C|)) + (|Q| \cdot |C|)^2$ many distinct counter values encountered during this run. Now from Lemma 16 we get that $|z| \leq (|Q| \cdot |C|) \cdot (\max(n, (|Q| \cdot |C|)) + (|Q| \cdot |C|)^2)$.

Now, we prove that the co-VS reachability and co-VS coverability problems of weighted ODCA are in P by demonstrating a small model property. We have already established using Lemma 17, Corollary 18, and Lemma 19 that the maximum and minimum counter values encountered during the run of the minimal witness do not exceed some polynomial bound. This, in turn, implies a polynomial bound on the length of the witness by Lemma 16. As a result, we get the following theorem.

▶ **Theorem 22.** The co-VS reachability and co-VS coverability problems for weighted odd can be decided in polynomial time when the counter values are given in unary notation.

Proof. Assume we are given a weighted ODCA $\mathcal{A} = ((C, \delta_0, \delta_1, p_0), (Q, \lambda, \Delta, \eta))$, initial configuration $\mathbf{c} = (\mathbf{x}, p, n)$, vector space \mathcal{V} , set of counter states S and counter value m as inputs for the co-VS reachability problem. For solving this reachability problem, we first consider the $\max(n, m) + (|Q| \cdot |C|)^2$ -unfolding weighted automata $\mathcal{A}^{\max(n, m) + (|Q| \cdot |C|)^2} = (C', \delta', p'_0; Q', \lambda', \Delta', \eta'_F)$ of \mathcal{A} as described in Definition 13. From Corollary 18, we know that the maximum counter value encountered during the run of the minimal reachability

witness z for $(c, \overline{\mathcal{V}}, S, \{m\})$ is less than $max(n, m) + (|Q| \cdot |C|)^2$. We define a vector space $\mathcal{U} \subseteq \mathcal{F}^{|Q'|}$ as follows: A vector $\mathbf{z} \in \mathcal{F}^{|Q'|}$ is in \mathcal{U} if there exists $\mathbf{y} \in \mathcal{V}$ such that for all $i \in [0, |Q| - 1]$, $\mathbf{z}[|Q| \cdot m + i] = \mathbf{y}[i]$ and for all $m' \neq m$ and $i \in [0, |Q| - 1]$, $\mathbf{z}[|Q| \cdot m' + i] = 0$. Given a configuration $\mathbf{c} = (\mathbf{x}, p, n)$ of a weighted ODCA, we define the vector $\mathbf{z}_{\mathbf{c}} \in \mathcal{F}^{|Q'|}$.

$$\mathbf{z}_{c}[i] = \begin{cases} \mathbf{x}[i \bmod |Q|], & \text{if } \frac{i}{|Q|} = n\\ 0, & \text{otherwise} \end{cases}$$

Now, consider the configuration $\bar{\mathsf{c}} = (\mathbf{z}_{\mathsf{c}}, (p, n))$ of $\mathcal{A}^{\max(n, m) + (|Q| \cdot |C|)^2}$ and check whether $\bar{\mathsf{c}} \stackrel{*}{\to} \overline{\mathcal{U}} \times S \times \{0\}$. This is a co-VS reachability problem of weighted automata. Using Theorem 14, this can be solved in polynomial time.

For solving co-VS coverability problem when a weighted ODCA \mathcal{A} with an initial configuration $\mathbf{c}=(\mathbf{z},p,n)$, a vector space \mathcal{V} and a set of counter states S are given as inputs, we consider the $\max(n,(|Q|\cdot|C|))+(|Q|\cdot|C|)^2$ -unfolding weighted automata $\mathcal{A}^{\max(n,(|Q|\cdot|C|))+(|Q|\cdot|C|)^2}=(C',\delta',p'_0;\ Q',\lambda',\Delta',\eta'_F)$ of \mathcal{A} . From Corollary 20, we know that the maximum counter value encountered during the run of a minimal reachability witness z for $(\mathbf{c},\overline{\mathcal{V}},S,\mathbb{N})$ is less than $\max(n,(|Q|\cdot|C|))+(|Q|\cdot|C|)^2$. We define a vector space $\mathcal{U}\subseteq\mathcal{F}^{|Q'|}$ as follows: A vector $\mathbf{x}\in\mathcal{F}^{|Q'|}$ is in \mathcal{U} if there exists $\mathbf{y}\in\mathcal{V}$ and $m\in\mathbb{N}$ such that for all $i\in[0,|Q|-1]$, $\mathbf{x}[|Q|\cdot m+i]=\mathbf{y}[i]$ and for all $m'\neq m$ and $i\in[0,|Q|-1]$, $\mathbf{x}[|Q|\cdot m'+i]=0$. Given a configuration $\mathbf{c}=(\mathbf{x},p,n)$ of a weighted ODCA, we define the vector $\mathbf{z}_{\mathbf{c}}\in\mathcal{F}^{|Q'|}$.

$$\mathbf{z}_{c}[i] = \begin{cases} \mathbf{x}[i \bmod |Q|], & \text{if } \frac{i}{|Q|} = n\\ 0, & \text{otherwise} \end{cases}$$

Now, consider the configuration $\bar{\mathbf{c}} = (\mathbf{z}_{\mathbf{c}}, (p, n))$ of $\mathcal{A}^{max(n, (|Q| \cdot |C|)) + (|Q| \cdot |C|)^2}$ and check whether $\bar{\mathbf{c}} \stackrel{*}{\to} \overline{\mathcal{U}} \times S \times \{0\}$. This is a co-VS reachability problem of a weighted automaton. From Theorem 14, we know that this can be solved in polynomial time.

3.3 Lexicographically minimal witness

This section will show that the lexicographically minimal witness has a distinct structure. We assume a total order on the symbols in Σ . Given two words $u,v\in\Sigma^*$, we say that u precedes v in the lexicographical ordering if |u|<|v| or if |u|=|v| and there exists an $i\in[0,|u|-1]$ such that u[0,i-1]=v[0,i-1] and u[i] precedes v[i] in the total ordering assumed on Σ . A word $z\in\Sigma^*$ is called the lexicographically minimal witness for $(c,\overline{\mathcal{V}},S,\{m\})$, if $c\xrightarrow{z}\overline{\mathcal{V}}\times S\times X$ and for all $u\in\Sigma^*$ with $c\xrightarrow{u}\overline{\mathcal{V}}\times S\times X$, z precedes u in the lexicographical ordering. We show that the lexicographically minimal witness z for $(c,\overline{\mathcal{V}},S,\{m\})$ has a canonical form. First, we prove this for floating runs.

- ▶ **Lemma 23.** There exist $poly_1 : \mathbb{N} \to \mathbb{N}$, and $poly_2 : \mathbb{N}^2 \to \mathbb{N}$ such that, if $z \in \Sigma^*$ is the lexicographically minimal witness for $(\mathbf{c}, \overline{\mathcal{V}}, S, \{m\})$ and $\mathbf{c} \xrightarrow{z} \overline{\mathcal{V}} \times S \times \{m\}$ is a floating run, then there exist $u, y, w \in \Sigma^*$ and $r \in \mathbb{N}$ such that $z = uy^r w$ and the following are true:
- 1. $|uyw| \leq poly_1(K)$, and
- 2. $r \leq poly_2(K, |COUNTER-VALUE(c) m|)$.

Proof. Let z be the lexicographically minimal witness for $(c, \overline{\mathcal{V}}, S, \{m\})$, and $g \in \overline{\mathcal{V}} \times S \times \{m\}$ such that $c \xrightarrow{z} g$ is a floating run. Let n be such that COUNTER-VALUE(c) = n. We consider the case n > m. The case where m > n is analogous. Let t = n - m.

▶ Claim 1.
$$|z| \le 2K^3 + t \cdot K$$
.

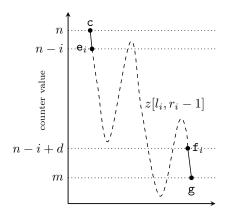


Figure 6 The figure shows the floating run from a configuration c (where COUNTER-VALUE(c) = n) to a configuration $\mathbf{g} = (\mathbf{x}, p, m)$ such that $\mathbf{x} \in \overline{\mathcal{V}}$. The configurations \mathbf{e}_i and \mathbf{f}_i are where the counter values n-i and n-i+d are encountered for the first (resp. last) time during this run. The dashed line is the part of the run due to factor $z[l_i, r_i - 1]$ and has a counter effect d.

<u>Proof:</u> From Point 1 of Lemma 17, it follows that the maximum counter value during the run $c \xrightarrow{z} g$ is less than $n + K^2$. By a symmetric argument, it follows that the minimum counter value during the run is greater than $m - K^2$. Hence, there are at most $t + 2K^2$ distinct counter values during the run. From Lemma 16 it follows that $|z| \le 2K^3 + t \cdot K$. $\blacktriangleleft_{Claim:1}$

If $t \leq K^2$, then from Claim 1, we get that $|z| \leq 3K^3$, and the lemma is trivially true. Let us assume $t > K^2$ and let $d = K^2 - t$. Let $\mathbf{c}_1 = \mathbf{c}$ and $\pi(z, \mathbf{c}_1) = \mathbf{c}_1 \tau_1 \mathbf{c}_2 \cdots \tau_{\ell-1} \mathbf{c}_\ell$ denote the run on word z from \mathbf{c} . For any $i \in [0, K^2]$, we denote by l_i the index such that the counter value n-i is encountered for the first time, and r_i the index such that the counter value n-i+d is encountered for the last time in $\pi(z, \mathbf{c}_1)$ (see Figure 6). Let $X = \{(l_i, r_i)\}_{i \in [0, K^2]}$ be the set of these pairs of indices, and let $W = \{z[l, r-1] \mid (l, r) \in X\}$ be the set of corresponding factors. Note that $|X| > K^2$. We argue that these factors $z[l_i, r_i - 1]$ for $i \in [0, K^2]$ need not all be distinct.

▶ Claim 2. $|W| \le K^2$.

<u>Proof:</u> Assume for contradiction that $|W| > (|Q| \cdot |C|)^2$. Since the number of counter states is |C|, by Pigeon-hole principle there exists $Y \subseteq X$ with $|Y| = |Q|^2 + 1$ such that for all $(l,r), (l',r') \in Y$, configurations \mathbf{c}_l and $\mathbf{c}_{l'}$ have the same counter state, configurations \mathbf{c}_r and $\mathbf{c}_{r'}$ have the same counter state, and $z[l,r-1] \neq z[l',r'-1]$. We say (l,r) < (l',r') if z[l,r-1] precedes z[l',r'-1] in the lexicographical order. Therefore, the elements in Y have an ordering as follows: $(l_0,r_0) < (l_1,r_1) < \cdots < (l_{|Q|^2},r_{|Q|^2})$. For any configuration \mathbf{h} , let $\mathbf{x}_\mathbf{h} = \mathbf{w} \in \mathbb{C}[\mathbf{h}]$. For all $i \in [0,|Q|^2]$, let $u_i = z[1,l_i-1], x_i = z[l_i,r_i-1], w_i = z[r_i,\ell-1]$, configurations \mathbf{e}_i , \mathbf{f}_i be such that $\mathbf{c} \xrightarrow{u_i} \mathbf{e}_i \xrightarrow{w_i} \mathbf{f}_i \xrightarrow{w_i} \mathbf{g}$ and matrices $\mathbb{A}_i, \mathbb{M}_i, \mathbb{B}_i$ be such that $\mathbf{x}_{\mathbf{e}_i} = \mathbf{x}_{\mathbf{c}} \mathbb{A}_i$, $\mathbf{x}_{\mathbf{f}_i} = \mathbf{x}_{\mathbf{e}_i} \mathbb{M}_i$, $\mathbf{x}_{\mathbf{g}} = \mathbf{x}_{\mathbf{f}_i} \mathbb{B}_i$.

We know that for all $k \in [0, |Q|^2]$, $\mathbf{x}_c \mathbb{A}_k \mathbb{M}_k \mathbb{B}_k \in \overline{\mathcal{V}}$. Consider the sequence of matrices $\mathbb{M}_0, \mathbb{M}_1, \cdots, \mathbb{M}_{|Q|^2}$. Since there can be at most $|Q|^2$ independent matrices, we get that there exists $i \in [0, |Q|^2]$ such that \mathbb{M}_i is a linear combination of $\mathbb{M}_0, \ldots, \mathbb{M}_{i-1}$. Hence, we get that there exists a j where j < i such that $\mathbf{x}_c \mathbb{A}_i \mathbb{M}_j \mathbb{B}_i \in \overline{\mathcal{V}}$. Since $x_j = z[l_j, r_j - 1]$ precedes $x_i = z[l_i, r_i - 1]$, the word $u_i x_j w_i$ precedes z in the lexicographical ordering. Therefore the run $\pi(u_i x_j w_i, \mathbf{c})$ contradicts the lexicographical minimality of z.

Since $|W| \leq K^2$ and $|X| > K^2$, there exists $i, j \in [0, K^2]$, with i < j and $x \in \Sigma^*$ such that $(l_i, r_i) \in X$, $(l_j, r_j) \in X$ and $x = z[l_i, r_i - 1] = z[l_j, r_j - 1]$ (see Figure 7). Let $u_1, w_1, u_2, w_2 \in \Sigma^*$ such that $z = u_1 x w_1 = u_2 x w_2$. Since $u_1 \neq u_2$, either u_1 is a prefix of u_2 or u_2 a prefix of u_1 . Without loss of generality, let us assume u_1 is a prefix of u_2 . Therefore, there exists $v \in \Sigma^*$ such that $u_2 = u_1 v$. Let \mathbf{e} be a configuration such that $\mathbf{c} \xrightarrow{u_1} \mathbf{e}$.

▶ Claim 3. $|u_1|, |v|, |w_1| \le 3K^3$.

<u>Proof:</u> Consider the set X. For any $i, j \in [0, K^2]$, the difference between the counter values of configurations c_{l_i} and c_{l_j} and the difference between the counter values of the configurations c_{r_j} and c_{r_i} is at most $K^2 + 1$. Therefore the counter-effect of u_2 , w_2 , and v can be at most K^2 . Since $\pi(v, \mathbf{e})$ is a floating run from Claim 1, we get that $|v| \leq 3K^3$. By similar arguments, the counter-effect of u_1 and w_1 can be at most K^2 , and again by Claim 1, we get that their lengths are at most $3K^3$.

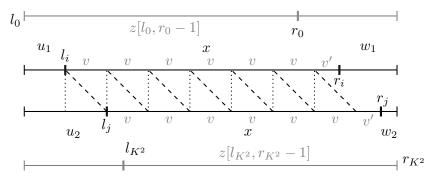


Figure 7 The figure shows the factorisation of a word $z = u_1xw_1 = u_2xw_2$, where $x = z[l_i, r_i - 1] = z[l_j, r_j - 1]$, and $u_1 \neq u_2$. The factor v is a prefix of x such that $u_2 = u_1v$. The word z can be written as $u_1v^iv'w_2$ for some $i \in \mathbb{N}$ and v' prefix of v. For $k \in [0, K^2]$, l_k is the index such that the counter value n - k is encountered for the first time and r_k the index such that the counter value n - k + d is encountered for the last time during the run $\mathbf{c} \xrightarrow{z} \mathbf{g}$.

▶ Claim 4. There exist $v' \in \Sigma^*$ and $r \in [0, K^3 + t \cdot K]$ such that $x = v^r v'$ with $|v'| \le |v|$.

<u>Proof:</u> Let $r \in \mathbb{N}$ be the largest number such that x is of the form v^rv' for some $v' \in \Sigma^*$ (see Figure 7). We know that $z = u_2xw_2$ and $u_2 = u_1v$. Therefore, $z = u_1vxw_2 = u_1v^rv'vw_2 = u_1v^rvv'w_2$. Furthermore, $z = u_1xw_1 = u_1v^rv'w_1$. Now since $u_1v^rvv'w_2 = u_1v^rv'w_1$, we get that $vv'w_2 = v'w_1$. Hence, if $|v'| \geq |v|$, then v is a prefix of v'. This is a contradiction since v was chosen to be the largest number such that v is of the form v^rv' .

To show the bound on the value r, we observe the following. We know that the counter effect of the run $\pi(x, \mathbf{e})$ is d. Therefore from Claim 1, we get that $|x| \leq 2K^3 + |d| \cdot K$. Hence, $r \leq 2K^3 + |d| \cdot K$.

From Claim 4 and Claim 3, we get that $|u_1vv'w_1| \leq 12K^3$ and $z = u_1v^rv'w_1$ for some $r \in [0, 2K^3 + |d| \cdot K)$].

We now establish that the lexicographically minimal witness z (whose run need not be floating) for a co-VS reachability problem has the form $uy_1^{r_1}vy_2^{r_2}w$. Here, lengths of the words u, y_1, y_2, v , and w are polynomially bounded in the number of states, and r_1 and r_2 are polynomial values dependent on the number of states and the input counter values.

- ▶ Lemma 24 (special-word lemma). If $z \in \Sigma^*$ is the lexicographically minimal witness for $(c, \overline{\mathcal{V}}, S, \{m\})$, then there exists $u, y_1, v_1, v_2, v_3, y_2, w \in \Sigma^*$ and $r_1, r_2 \in \mathbb{N}$ such that $z = uy_1^{r_1}vy_2^{r_2}w$ and the following are true:
- 1. $|uy_1vy_2w|$ is polynomially bounded in the number of states of the machine.
- 2. r_1 and r_2 are polynomially bounded in the number of states of the machine, m, and COUNTER-VALUE(c).

Proof. Let $z \in \Sigma^*$ be the lexicographically minimal reachability witness for $(c, \overline{\mathcal{V}}, S, \{m\})$, where c is a configuration with counter value n. Consider the run of word z from c. Let $d \in \overline{\mathcal{V}} \times S \times \{m\}$ such that $c \stackrel{z}{\to} d$. Let $c = c_1$ and $T(c_1) = c_1 \tau_1 c_2 \cdots \tau_{\ell-1} c_\ell$ denote the run on word z from the configuration c_1 and T the corresponding sequence of transitions. Let e_1 be the first configuration with counter value zero and e_2 be the last configuration with counter value zero during this run. Let $z_1, z_2, z_3 \in \Sigma^*$ be such that $c \stackrel{z_1}{\longrightarrow} e_1 \stackrel{z_2}{\longrightarrow} e_2 \stackrel{z_3}{\longrightarrow} c_\ell$ and $z = z_1 z_2 z_3$. Observe that $c \stackrel{z_1}{\longrightarrow} e_1$ and $e_2 \stackrel{z_3}{\longrightarrow} c_\ell$ are floating runs.

From Lemma 23, we know that there exists $u_1, u_3, v_1, v_3, y_1, y_3 \in \Sigma^*$ and $r_1, r_3 \in \mathbb{N}$ such that $z_1 = u_1 y_1^{r_1} v_1$, $z_3 = u_3 y_3^{r_3} v_3$, $|u_1|, |u_3| \leq 3 \cdot |Q|^3 \cdot |C|^4$, $|v_1|, |v_3| \leq 6 \cdot |Q|^3 \cdot |C|^4$, $|y_1|, |y_3| \leq 3 \cdot |Q|^3 \cdot |C|^4$, $r_1 \in [0, n \cdot |Q| \cdot |C| + (|Q| \cdot |C|)^3]$ and $r_3 \in [0, m \cdot |Q| \cdot |C| + (|Q| \cdot |C|)^3]$.

4 Equivalence of weighted ODCA

In this section, we present a polynomial time algorithm to decide the equivalence of two weighted odds whose weights come from a fixed field. The techniques developed in the previous section in conjunction with those presented in Valiant and Paterson [23], and Böhm et al. [3] for deterministic real-time odd give us the algorithm. The idea here is to prove that the maximum counter value encountered during the run of a minimal witness is polynomially bounded. We use this to reduce the equivalence problem to that of weighted automata.

In the remainder of this section, we fix two non-equivalent weighted ODCAs A_1 and A_2 over an alphabet Σ and a field \mathcal{F} . For $i \in \{1, 2\}$,

$$\mathcal{A}_i = (C_i, \delta_{0_i}, \delta_{1_i}, p_{0_i}; Q_i, \boldsymbol{\lambda}_i, \Delta_i, \boldsymbol{\eta}_i).$$

Without loss of generality assume $K = |C_1| = |Q_1| = |C_2| = |Q_2|$. We will reason on the synchronised runs on pairs of configurations. Given two weighted ODCAS, \mathcal{A}_1 and \mathcal{A}_2 and $i \in \mathbb{N}$, we denote a configuration pair as $\mathbf{h}_i = \langle \mathbf{c}_i, \mathbf{d}_i \rangle$ where \mathbf{c}_i is a configuration of \mathcal{A}_1 and \mathbf{d}_i is a configuration of \mathcal{A}_2 . We similarly consider transition pairs of \mathcal{A}_1 and \mathcal{A}_2 , and consider synchronised runs as the application of a sequence of transition pairs to a configuration pair. We fix a minimal word z (also called witness) that distinguishes \mathcal{A}_1 and \mathcal{A}_2 and $\ell = |z|$. Henceforth we will denote by

$$\Pi = \mathtt{h}_0 \tau_0 \mathtt{h}_1 \cdots \tau_{\ell-1} \mathtt{h}_\ell$$

the synchronisation of runs over z in \mathcal{A}_1 and \mathcal{A}_2 from their initial configurations, where \mathbf{h}_i are pairs of configurations and τ_i are pairs of transitions. We denote by $T = \tau_0 \cdots \tau_{\ell-1}$ the sequence of transition pairs of this run pair. The main idea to prove Theorem 3 is to show that the length of z is polynomially bounded in the size of the two weighted ODCAS.

▶ Lemma 25. There is a polynomial $\operatorname{poly}_0: \mathbb{N} \to \mathbb{N}$ such that if two weighted ODCAs \mathcal{A}_1 and \mathcal{A}_2 are not equivalent, then there exists a witness z such that the counter values encountered during Π are less than $\operatorname{poly}_0(\mathsf{K})$.

We use Lemma 25 to show that the length of a minimal witness z is bounded by a polynomial $\operatorname{poly}_1(\mathsf{K}) = 2\mathsf{K}^5\operatorname{poly}_0(\mathsf{K})$.

▶ Lemma 26. There is a polynomial $\operatorname{poly}_1: \mathbb{N} \to \mathbb{N}$ such that if two weighted ODCAs \mathcal{A}_1 and \mathcal{A}_2 are not equivalent, then there exists a witness z such that |z| is less than or equal to $\operatorname{poly}_1(\mathsf{K})$.

Proof. Assume for contradiction that the length of a minimal witness z is greater than $\operatorname{poly}_1(\mathsf{K})$. From Lemma 25, we know that the counter values encountered during the run Π in less than $\operatorname{poly}_0(\mathsf{K})$. Since $|z| > \operatorname{poly}_1(\mathsf{K})$, by the Pigeonhole principle, we get that there exist indices $0 \le i_0 < i_2 < \dots < i_{2\mathsf{K}} \le \ell$ such that for all configuration pairs $\mathbf{h}_{i_j}, j \in [1, 2\mathsf{K}]$, \mathbf{c}_{i_j} and $\mathbf{c}_{i_{j-1}}$ have the same counter value and counter state and \mathbf{d}_{i_j} and $\mathbf{d}_{i_{j-1}}$ have the same counter state and counter value.

For all $j \in [0, 2K]$ we define the vector $\mathbf{x}_j \in \mathcal{F}^{2K}$ such that $\mathbf{x}_j[r] = \mathbf{x}_{c_{i_j}}[r]$, if r < K and $\mathbf{x}_{d_{i_j}}[r-K]$, otherwise. We also define the vector $\boldsymbol{\eta} \in \mathcal{F}^{2K}$ such that $\boldsymbol{\eta}[r] = \boldsymbol{\eta}_1[r]$, if r < K and $\boldsymbol{\eta}_2[r-K]$, otherwise. For all $j \in [0, 2K]$, let \mathbb{A}_j denote the matrix such that $\mathbf{x}_j\mathbb{A}_j = \mathbf{x}_\ell$. Since z is a minimal witness, we know that for all $j \in [0, 2K]$, $\mathbf{x}_j\mathbb{A}_j\boldsymbol{\eta}^\top \neq 0$. From Lemma 8, we get that there exists $r, r' \in [0, 2K]$, with r' < r such that $\mathbf{x}_{r'}\mathbb{A}_r\boldsymbol{\eta}^\top \neq 0$. The sequence of transitions $\tau_{i_r+1}\cdots\tau_\ell$ can be taken from $\mathbf{h}_{i_r'}$ since the counter values and counter states are the same for both configurations. Consider the sequence of transitions $T' = \tau_0 \cdots \tau_{i_r'} \tau_{i_r+1} \cdots \tau_\ell$ and let $w = \mathbf{word}(T')$. The word w is a shorter witness than z and contradicts its minimality.

Thus we can reduce the equivalence problem of weighted ODCA over fields to that of weighted automata over fields (which is in P [22]) by "simulating" the runs of weighted ODCAS \mathcal{A}_1 and \mathcal{A}_2 up to length $\operatorname{poly}_1(\mathsf{K})$ by two weighted automata. The naive algorithm will only give us a PSPACE procedure, but there is a polynomial time procedure to do this, and the proof is given below.

Proof of Theorem 3. We consider the two weighted ODCAS \mathcal{A}_1 and \mathcal{A}_2 . From Lemma 26, we know that the length of the minimal witness z is less than $\operatorname{poly}_1(\mathsf{K})$. Let $M = \operatorname{poly}_1(\mathsf{K})$. We construct the M-unfolding weighted automata \mathcal{A}_1^M and \mathcal{A}_2^M as described in Definition 13. It follows that, \mathcal{A}_1 is non-equivalent to \mathcal{A}_2 if and only if there exists a word $w \in \Sigma^{\leq M}$ such that $f_{\mathcal{A}_1^M}(w) \neq f_{\mathcal{A}_2^M}(w)$. Tzeng [22, Lemma 3.4] gives a polynomial time algorithm to output a minimal word that distinguishes two probabilistic automata. We conclude the proof by noting that the algorithm can be extended to the case of weighted automata.

The rest of this section is dedicated to proving Lemma 25. We adapt techniques developed by Böhm et al. [3] for OCAs. We start by labelling some configuration pairs as background points (see Figure 8). Consider the case where there is no background point in Π . By reducing the problem to co-VS reachability/coverability we show that the counter values in Π are polynomially bounded. Now consider the case where there is a background point \mathbf{h}_j in Π . We show that the counter values encountered during the run of Π till \mathbf{h}_j is polynomially bounded. This is shown by Lemma 30 and Lemma 36. We conclude by arguing that the length of the run from \mathbf{h}_j is polynomially bounded.

$$\underbrace{\mathbf{h}_{0}\tau_{0}\mathbf{h}_{1}\tau_{1}\mathbf{h}_{2}\cdots\mathbf{h}_{j-1}\tau_{j-1}}_{\text{configuration pair in background space}}\underbrace{\mathbf{h}_{j} = \langle \mathbf{c}_{j}, \mathbf{d}_{j} \rangle}_{\text{configuration pair in background space}}\underbrace{\tau_{j}\cdots\tau_{\ell-1}\mathbf{h}_{\ell}}_{\text{configuration pair in background space}}$$

Following Böhm et al. [3], we define a partition of the set of configuration pairs to facilitate this.

4.1 Configuration Space

Each pair of configuration $h = \langle c, d \rangle$ is mapped to a point in the space $\mathbb{N} \times \mathbb{N} \times (C_1 \times C_2) \times \mathcal{F}^{\mathsf{K}} \times \mathcal{F}^{\mathsf{K}}$, henceforth referred to as the *configuration space*. Here, the first two dimensions represent the two counter values, the third dimension $C_1 \times C_2$ corresponds to the pair of counter states, and the remaining dimensions represent the weight vector. The projection of the configuration space onto the first two dimensions is depicted in Figure 8. We partition the configuration space into three: initial space, belt space, and background space. The size of the initial space and, thickness and number of belts will be polynomially bounded in K. These partitions are indexed on two carefully chosen polynomials $\operatorname{poly}_2(\mathsf{K}) = 516\mathsf{K}^{21}$ and $\operatorname{poly}_3(\mathsf{K}) = 42\mathsf{K}^{14}$, , so that all belts are disjoint outside the initial space. The precise polynomials are required in the proofs of Lemma 27 and Lemma 33. We use some properties of these partitions to show that the length of a minimal witness is bounded. Given a configuration c, we use n_c to denote COUNTER-VALUE(c).

- initial space: All configuration pairs $\langle c, d \rangle$ such that $n_c, n_d < \text{poly}_2(K)$.
- belt space: Let $\alpha, \beta \in [1, 3\mathsf{K}^7]$ be co-prime. A belt of slope $\frac{\alpha}{\beta}$ consists of those configuration pairs $\langle \mathsf{c}, \mathsf{d} \rangle$ outside the initial space that satisfies $|\alpha.n_\mathsf{c} \beta.n_\mathsf{d}| \leq \mathrm{poly}_3(\mathsf{K})$. The belt space contains all configuration pairs $\langle \mathsf{c}, \mathsf{d} \rangle$ that is inside belts with slope $\frac{\alpha}{\beta}$.
- background space: All remaining configuration pairs.

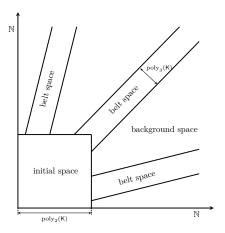


Figure 8 Projection of configuration space

The proof of the following lemma is similar to that of the non-weighted case presented in [3].

▶ **Lemma 27.** If $\langle c, d \rangle$ and $\langle e, f \rangle$ are configuration pairs inside two distinct belts and lie outside the initial space, then there is no $a \in \Sigma$ such that $\langle c, d \rangle \xrightarrow{a} \langle e, f \rangle$.

Proof. Recall $\operatorname{poly}_2(\mathsf{K}) = 516\mathsf{K}^{21}$ and $\operatorname{poly}_3(\mathsf{K}) = 42\mathsf{K}^{14}$. Let B and B' be two distinct belts with μ being the slope of the belt B and μ' the slope of the belt B'. Hence $\mu \neq \mu'$. Without loss of generality, let us assume that $\mu' > \mu$. It suffices to show that for all $x > \operatorname{poly}_2(\mathsf{K})$, we have

$$\mu x + \text{poly}_3(\mathsf{K}) + 1 < \mu' x - \text{poly}_3(\mathsf{K}) - 1.$$

We know that $\mu' - \mu \ge \frac{1}{3\mathsf{K}^7}$ and $x > 516\mathsf{K}^{21}$.

Therefore,
$$\frac{516\mathsf{K}^{21}}{6\mathsf{K}^7} < (\mu' - \mu) \cdot x$$
.

$$\begin{split} &\Longrightarrow \mu x + \frac{86\mathsf{K}^{14}}{2} < \mu' x - \frac{86\mathsf{K}^{14}}{2} \\ &\Longrightarrow \mu x + 42\mathsf{K}^{14} + \mathsf{K}^{14} < \mu' x - 42\mathsf{K}^{14} - \mathsf{K}^{14} \\ &\Longrightarrow \mu x + 42\mathsf{K}^{14} + 1 < \mu' x - 42\mathsf{K}^{14} - 1 \end{split}$$

Lemma 27 ensures that the belts are disjoint outside the initial space and that no run can go from one belt to another without passing through the initial space or background space. To prove Lemma 25, there are two cases to consider: either there is no background space point in Π , or there is a background space point in Π .

4.2 Case 1: When there is no background space point in Π .

Since there is no background space point in Π , all the points in Π are either in the initial or belt space. By definition, the counter values of configuration pairs inside the initial space are bounded by $\operatorname{poly}_2(\mathsf{K})$. Now, we look at the sub-run of Π inside the belt space. If a sub-run of Π enters and exits a belt from the initial space or if Π ends inside a belt, then we show that the counter values encountered during that belt visit are polynomially bounded. This is shown by reducing to co-VS reachability of an ODCA. For this proof, it is crucial that the belts are disjoint.

Let $\Pi_b = \mathbf{h}_i \tau_i \mathbf{h}_{i+1} \cdots \tau_{j-1} \mathbf{h}_j$ be a sub-run of the run of z inside a belt with slope $\frac{\alpha}{\beta}$. Similar to the technique mentioned in [5], each configuration pair $\mathbf{h}_r = ((\mathbf{x}_{\mathsf{c}_r}, p_{\mathsf{c}_r}, n_{\mathsf{c}_r}), (\mathbf{x}_{\mathsf{d}_r}, p_{\mathsf{d}_r}, n_{\mathsf{d}_r}))$, where $r \in [i, j]$ can alternatively be presented as $((\mathbf{x}_{\mathsf{c}_r}, \mathbf{x}_{\mathsf{d}_r}), p_{\mathsf{c}_r}, p_{\mathsf{d}_r}, l_r)$ where l_r denotes a line with slope $\frac{\alpha}{\beta}$ inside the given belt that contains the point $(n_{\mathsf{c}_r}, n_{\mathsf{d}_r})$. Let L be the set of all lines with slope $\frac{\alpha}{\beta}$ inside the given belt. Note that $|L| = \text{poly}_3(\mathsf{K})$. The run Π_b is similar to the run of a weighted ODCA \mathcal{D} that has the tuple $(p_{\mathsf{c}_r}, p_{\mathsf{d}_r}, l_r)$ as the state of the finite state machine and $\mathbf{x}_r \in \mathcal{F}^{2\mathsf{K}}$ as its weight vector where $\mathbf{x}_r[i] = \mathbf{x}_{\mathsf{c}_r}[i]$, if $i < \mathsf{K}$ and $\mathbf{x}_r[i] = \mathbf{x}_{\mathsf{d}_r}[i - \mathsf{K}]$, otherwise. A formal definition of the ODCA \mathcal{D} is given in Definition 28.

- ▶ Definition 28. Let $\mathcal{A}_i = ((C_i, \delta_{0_i}, \delta_{1_i}, p_{0_i}), (Q_i, \lambda_i, \Delta_i, \eta_i))$ for $i \in \{1, 2\}$, be the two odcas given. Let L be the set of all lines with slope $\frac{\alpha}{\beta}$ inside the given belt. We define the odca $\mathcal{D} = ((C, \delta_0, \delta_1, p_0), (Q, \lambda, \Delta, \eta))$, where the initial state p_0 and the initial distribution λ are arbitrarily chosen.
- $C = C_1 \times C_2 \times L$ is a non-empty finite set of states.
- $\delta_1: C \times \Sigma \to C \times \{-1,0,+1\}$ is the deterministic counter transition. Let $p_1,q_1 \in C_1, p_2, q_2 \in C_2$, $a \in \Sigma$ and $d_1, d_2 \in \{-1,0,+1\}$. Let $l_1, l_2 \in L$ and $m_1, m_2 \in \mathbb{N}$, such that the point (m_1, m_2) lies on the line l_1 . $\delta_1((p_1, p_2, l_1), a) = ((q_1, q_2, l_2), d_1)$, if $\delta_{1_1}(p_1, a) = (q_1, d_1)$ and $\delta_{1_2}(p_2, a) = (q_2, d_2)$ and the point $(m_1 + d_1, m_2 + d_2)$ lies on the line l_2 . It is undefined otherwise. The function $\delta_0: C \times \Sigma \to C \times \{0, +1\}$ is arbitrarily chosen.
- $Q = Q_1 \cup Q_2$ is a non-empty finite set of states of the finite state machine.
- $\Delta: \Sigma \times \{0,1\} \to \mathcal{F}^{2\mathsf{K} \times 2\mathsf{K}}$ gives the transition matrix for all $a \in \Sigma$ and $d \in \{0,1\}$. For $l \in L, m \in \mathbb{N}$ and $a \in \Sigma$,

$$\Delta(l,a)[i][j] = \begin{cases} \Delta_1(a,1)[i][j], & \text{if } i,j < \mathsf{K} \\ \Delta_2(a,1)[i-\mathsf{K}][j-\mathsf{K}], & \text{if } i,j > \mathsf{K} \\ 0, & \text{otherwise} \end{cases}$$

 $\eta \in \mathcal{F}^{2K}$ is the final distribution.

$$oldsymbol{\eta}[i] = egin{cases} oldsymbol{\eta}_1[i], & \textit{if } i < \mathsf{K} \\ oldsymbol{\eta}_2[i - \mathsf{K}], & \textit{otherwise} \end{cases}$$

The sub-run Π_b can now be seen as a floating run of a weighted ODCA \mathcal{D} . If the run Π ends inside a belt, then $\Pi_b = \mathbf{h}_i \tau_i \cdots \tau_{\ell-1} \mathbf{h}_{\ell}$. In this case, we show that the difference between the counter values of the first and last configuration pairs is smaller than a polynomial in K.

▶ Lemma 29. There is a polynomial $poly : \mathbb{N} \to \mathbb{N}$, such that if $\Pi_b = h_i \tau_i \cdots \tau_{\ell-1} h_\ell$ lies inside a belt where $h_r = \langle \mathbf{c}_r, \mathbf{d}_r \rangle$, for $r \in [i, \ell]$, then $|\text{COUNTER-VALUE}(\mathbf{c}_\ell) - \text{COUNTER-VALUE}(\mathbf{c}_\ell)| \leq poly(\mathsf{K})$ and $|\text{COUNTER-VALUE}(\mathbf{d}_\ell) - \text{COUNTER-VALUE}(\mathbf{d}_i)| \leq poly(\mathsf{K})$.

Proof. Let $\Pi_b = \mathbf{h}_i \tau_i \mathbf{h}_{i+1} \cdots \tau_{\ell-1} \mathbf{h}_{\ell}$ be a sub-run of the run of a minimal witness inside a belt and ends in the belt, where $\mathbf{h}_r = \langle (\mathbf{x}_{\mathsf{c}_r}, p_{\mathsf{c}_r}, n_{\mathsf{c}_r}), (\mathbf{x}_{\mathsf{d}_r}, p_{\mathsf{d}_r}, n_{\mathsf{d}_r}) \rangle$, for $r \in [i, \ell]$. As mentioned in Definition 28, we consider this as the run of the weighted ODCA \mathcal{D} . Since it is the run of a witness, $\mathbf{x}_j \boldsymbol{\eta}^\top \neq 0$. Consider the vector space $\mathcal{U} = \{\mathbf{y} \in \mathcal{F}^{2\mathsf{K}} \mid \mathbf{y} \boldsymbol{\eta}^\top = 0\}$. Our problem now reduces to the co-VS coverability problem in machine \mathcal{D} and asks whether $(\mathbf{x}_i, (p_{\mathsf{c}_i}, p_{\mathsf{d}_i}, l_i), n_{\mathsf{c}_i}) \stackrel{*}{\to} \overline{\mathcal{U}} \times \{(p_{\mathsf{c}_\ell}, p_{\mathsf{d}_\ell}, l_\ell)\} \times \mathbb{N}$. From Corollary 18, we know that the length of a minimal reachability witness for $((\mathbf{x}_i, (p_{\mathsf{c}_i}, p_{\mathsf{d}_i}, l_i), n_{\mathsf{c}_i}), \overline{\mathcal{U}}, (p_{\mathsf{c}_\ell}, p_{\mathsf{d}_\ell}, l_\ell), \mathbb{N})$ is polynomially bounded in n_{c_i} and K. Hence proved.

In the following lemma, we show that if $\Pi_b = \mathbf{h}_i \tau_i \mathbf{h}_{i+1} \cdots \tau_{j-1} \mathbf{h}_j$ is a sub-run of Π inside a belt and either COUNTER-VALUE(\mathbf{c}_i) = COUNTER-VALUE(\mathbf{c}_j) or COUNTER-VALUE(\mathbf{d}_i) = COUNTER-VALUE(\mathbf{d}_j), where $\mathbf{h}_r = \langle \mathbf{c}_r, \mathbf{d}_r \rangle$, for $r \in [i, j]$, then the counter values in Π_b cannot increase more than a polynomial in K from COUNTER-VALUE(\mathbf{c}_i) and COUNTER-VALUE(\mathbf{d}_i).

▶ Lemma 30. There is a polynomial $poly : \mathbb{N} \to \mathbb{N}$ such that, if $\Pi_b = h_i \tau_i h_{i+1} \cdots \tau_{j-1} h_j$ is a run inside a belt with COUNTER-VALUE(\mathbf{c}_i) = COUNTER-VALUE(\mathbf{c}_j) or COUNTER-VALUE(\mathbf{d}_i) = COUNTER-VALUE(\mathbf{d}_j), where $h_r = \langle \mathbf{c}_r, \mathbf{d}_r \rangle$, for $r \in [i, j]$, then the counter effect of any sub-run of Π_b is less than or equal to poly(K).

Proof. Let $\Pi_b = \mathbf{h}_i \tau_i \mathbf{h}_{i+1} \cdots \tau_{j-1} \mathbf{h}_j$ be a sub-run of the run of a minimal witness inside a belt such that $n_{\mathsf{c}_i} = n_{\mathsf{c}_j}$, where $\mathbf{h}_r = \langle (\mathbf{x}_{\mathsf{c}_r}, p_{\mathsf{c}_r}, n_{\mathsf{c}_r}), (\mathbf{x}_{\mathsf{d}_r}, p_{\mathsf{d}_r}, n_{\mathsf{d}_r}) \rangle$, for $r \in [i, j]$. We consider this as the run of the weighted ODCA \mathcal{D} as mentioned in Definition 28. Since it is the run of a witness, we know that there exists $\mathbb{A} \in \mathcal{F}^{2\mathsf{K} \times 2\mathsf{K}}$ such that $\mathbf{x}_j \mathbb{A} \boldsymbol{\eta}^\top \neq 0$. Consider the vector space $\mathcal{U} = \{\mathbf{y} \in \mathcal{F}^{2\mathsf{K}} \mid \mathbf{y} \mathbb{A} \boldsymbol{\eta}^\top = 0\}$.

Our problem now reduces to the co-VS reachability problem in machine \mathcal{D} and asks whether $(\mathbf{x}_i, (p_{\mathsf{c}_i}, p_{\mathsf{d}_i}, l_i), n_{\mathsf{c}_i}) \stackrel{*}{\to} \overline{\mathcal{U}} \times \{(p_{\mathsf{c}_j}, p_{\mathsf{d}_j}, l_j)\} \times \{n_{\mathsf{c}_i}\}$. From Corollary 18, the length of a minimal reachability witness for $((\mathbf{x}_i, (p_{\mathsf{c}_i}, p_{\mathsf{d}_i}, l_i), n_{\mathsf{c}_i}), \overline{\mathcal{U}}, (p_{\mathsf{c}_j}, p_{\mathsf{d}_j}, l_j), \{n_{\mathsf{c}_i}\})$ is bounded by a polynomial in n_{c_i} and K. Hence proved.

Hence, we get that the pair of runs of the minimal witness cannot reach counter values higher than some polynomial bound if it does not enter the background space. Now we look at the case where the run enters the background space.

4.3 Case 2: When there is a background space point in Π .

We now consider the case where the witness ultimately enters the background space. Using co-VS reachability, we prove that the counter values encountered during Π till the first background space point are polynomially bounded. We also show that the length of the remaining run is polynomially bounded in the number of states of the machines.

To that end, we need the notion of underlying uninitialised weighted automaton. Roughly speaking, an underlying uninitialised weighted automaton of an ODCA \mathcal{A} is the uninitialised weighted automaton $U(\mathcal{A})$ that is syntactically equivalent to \mathcal{A} without zero tests. In other words, the transition function of $U(\mathcal{A})$ will be determined by the transition functions of \mathcal{A} for positive counter values. Floating runs of \mathcal{A} are isomorphic to runs of this weighted automaton $U(\mathcal{A})$.

▶ **Definition 31.** For $l \in \{1,2\}$, the underlying uninitialised weighted automaton of \mathcal{A} is the uninitialised weighted automaton $U(\mathcal{A}_l) = (Q'_l, \Delta'_l, \eta'_l)$, where $Q'_l = C_l \times Q_l$ and $\eta'_l \in \mathcal{F}^{\mathsf{K}^2}$ is the final distribution. For $i < \mathsf{K}^2, \eta'_l[i] = \eta_l[i \bmod \mathsf{K}]$. The transition matrix is given by $\Delta'_l : \Sigma \to \mathcal{F}^{\mathsf{K}^2 \times \mathsf{K}^2}$. Let $a \in \Sigma$, $d \in \{-1, 0, +1\}$, $i, j < \mathsf{K}^2$,

$$\Delta_l'(a)[i][j] = \begin{cases} \Delta_l(p_{\frac{i}{\mathsf{K}}}, a, 1)[i \bmod \mathsf{K}][j \bmod \mathsf{K}], & \text{if } \delta_{l_1}(p_{\frac{i}{\mathsf{K}}}, a) = (p_{\frac{j}{\mathsf{K}}}, d) \\ 0 & \text{otherwise} \end{cases}$$

A configuration c of a weighted ODCA \mathcal{A} is said to be k-equivalent to a configuration \bar{c} of an uninitialised weighted automata \mathcal{B} , denoted $c \sim_k \bar{c}$, if for all $w \in \Sigma^{\leq k}$, $f_{\mathcal{A}}(w, c) = f_{\mathcal{B}}(w, \bar{c})$. We say that c is not k-equivalent to \bar{c} otherwise and denote this as $c \nsim_k \bar{c}$.

As we need to test the equivalence of configurations from \mathcal{A}_1 and \mathcal{A}_2 , we consider the uninitialised weighted automata \mathcal{B} , which is a disjoint union of $\mathrm{U}(\mathcal{A}_1)$ and $\mathrm{U}(\mathcal{A}_2)$. This gives us a single automaton with which we can compare their configurations. Let $i \in \{1,2\}$ and c be a configuration of \mathcal{A}_i . For all $p \in C_i$ and $m < |\mathcal{B}|$, we define the sets $\mathcal{W}_i^{p,m}$. The set $\mathcal{W}_i^{p,m}$ contains vectors $\mathbf{x} \in \mathcal{F}^{\mathsf{K}}$ such that the configuration (\mathbf{x}, p, m) is $|\mathcal{B}|$ -equivalent to some configuration of \mathcal{B} . The set $\overline{\mathcal{W}}_i^{p,m}$ is the set $\mathcal{F}^{\mathsf{K}} \setminus \mathcal{W}_i^{p,m}$. Formally,

$$\mathcal{W}_i^{p,m} = \{\mathbf{x} \in \mathcal{F}^{\mathsf{K}} | \exists \bar{\mathbf{c}} \in \mathcal{F}^{|\mathcal{B}|}, \mathbf{c} = (\mathbf{x}, p, m) \sim_{|\mathcal{B}|} \bar{\mathbf{c}} \}$$

▶ **Lemma 32.** For any $i \in \{1, 2\}$, $p \in C_i$ and $m < |\mathcal{B}|$, the set $\mathcal{W}_i^{p,m}$ is a vector space.

Proof. To prove this, it suffices to show that it is closed under vector addition and scalar multiplication. We fix a set $\mathcal{W}_i^{p,m}$. First, we prove that it is closed under scalar multiplication. For any vector $\mathbf{z}_1 \in \mathcal{W}_i^{p,m}$, we know that there exists a configuration $\mathbf{c} = (\mathbf{z}_1, p, m)$ and $\bar{\mathbf{c}} \in \mathcal{F}^{|\mathcal{B}|}$ such that $\mathbf{c} \sim_{|\mathcal{B}|} \bar{\mathbf{c}}$. Now, for any scalar $r \in \mathcal{F}$, the configuration $(r.\mathbf{z}_1, p, m) \sim_{|\mathcal{B}|} \mathbf{z}_1 \cdot \bar{\mathbf{c}}$. Therefore $r \cdot \mathbf{z}_1 \in \mathcal{W}_i^{p,m}$. Now, we show that it is closed under vector addition. Let $\mathbf{z}_1, \mathbf{z}_2 \in \mathcal{W}_i^{p,m}$ be two vectors. Therefore, there exists configurations $\mathbf{c}_1 = (\mathbf{z}_1, p, m)$, $\mathbf{c}_2 = (\mathbf{z}_2, p, m)$, $\bar{\mathbf{c}}_1 \in \mathcal{F}^{|\mathcal{B}|}$ and $\bar{\mathbf{c}}_2 \in \mathcal{F}^{|\mathcal{B}|}$, such that $\mathbf{c}_1 \sim_{|\mathcal{B}|} \bar{\mathbf{c}}_1$ and $\mathbf{c}_2 \sim_{|\mathcal{B}|} \bar{\mathbf{c}}_2$. Consider the configuration $\mathbf{c}_3 = (\mathbf{z}_1 + \mathbf{z}_2, p, m)$, $\mathbf{c}_3 \sim_{|\mathcal{B}|} \bar{\mathbf{c}}_1 + \bar{\mathbf{c}}_2$. Therefore, $\mathbf{z}_1 + \mathbf{z}_2 \in \mathcal{W}_i^{p,m}$.

The distance of a configuration c of \mathcal{A}_i (denoted as $\operatorname{dist}_{\mathcal{A}_i}(c)$) is the length of a minimal word that takes you from c to a configuration (\mathbf{x}, p, m) for some $m < |\mathcal{B}|$ and $p \in C_i$ such that $\mathbf{x} \in \overline{\mathcal{W}}_i^{p,m}$. We define $\operatorname{dist}_{\mathcal{A}_i}(c)$ as:

$$\min\{|w| \mid \mathbf{c} \xrightarrow{w} (\mathbf{x}, p, m) \exists p \in C_i, m < |\mathcal{B}|, \mathbf{x} \in \overline{\mathcal{W}}_i^{p, m}\}$$

The notion of distance play a key role in determining which parts of the run of a witness can be pumped out if it is not minimal. Given two configurations c, d of \mathcal{A}_1 and \mathcal{A}_2 respectively, if $\operatorname{dist}_{\mathcal{A}_1}(c) \neq \operatorname{dist}_{\mathcal{A}_2}(d)$, then $c \not\equiv d$. By special word lemma (Lemma 23), the lexicographically minimal reachability witness has a special form. This is used to show that if a configuration c of an ODCA \mathcal{A} has finite distance, then $\operatorname{dist}_{\mathcal{A}}(c) = \frac{a}{b}\operatorname{COUNTER-VALUE}(c) + t$, where $a, b, t \in \mathbb{N}$ and are polynomially bounded in $|\mathcal{A}|$. This helps us in proving that configuration pairs outside the initial space having equal distance lie inside a belt.

▶ **Lemma 33.** Let $c = (\mathbf{x}_c, p_c, n_c)$ be a configuration of weighted ODCA \mathcal{A} . If $\operatorname{dist}_{\mathcal{A}}(c) < \infty$ then, $\operatorname{dist}_{\mathcal{A}}(c) = \frac{a}{h}n_c + t$ where $a, b \in [0, 3\mathsf{K}^7]$ and $|t| < 42\mathsf{K}^{14}$.

Proof. Without loss of generality, let us consider the weighted ODCA \mathcal{A}_1 and a configuration $\mathbf{c}=(\mathbf{x}_{\mathtt{c}},p_{\mathtt{c}},n_{\mathtt{c}})$ of \mathcal{A}_1 . Let us assume that $\mathrm{dist}_{\mathcal{A}_1}(\mathbf{c})<\infty$. This means that $\mathbf{c}\to^*\mathrm{d}$, for some configuration $\mathbf{d}=(\mathbf{x}_{\mathtt{d}},p,m)$ with $\mathbf{x}_{\mathtt{d}}\in\overline{\mathcal{W}}_1^{p,m}$ for some $p\in C_1$ and $m<2\mathsf{K}^2$. Since COUNTER-VALUE(\mathbf{d}) = m, by Lemma 23, we know that there is a word $u=u_1u_2^ru_3$ (with $r\geq 0$) such that that $\mathbf{c}\stackrel{u}{\to}\mathrm{d}$ where $|u|=\mathrm{dist}_{\mathcal{A}_1}(\mathbf{c}), |u_1u_3|\leq 9\mathsf{K}^7, |u_2|\leq 3\mathsf{K}^7$ and u_2 has a negative counter effect ℓ . Let g be the combined counter effect of u_1,u_3 and $\alpha=\frac{|u_2|}{\ell}$. Since $|u_1u_3|\leq 9\mathsf{K}^7$, we have $|g|\leq 9\mathsf{K}^7$.

$$\operatorname{dist}_{\mathcal{A}_1}(\mathsf{c}) = \frac{n_\mathsf{c} - m - g}{\ell} |u_2| + |u_1 u_3|$$
$$= \alpha n_\mathsf{c} - \underbrace{\alpha(m+g) + |u_1 u_3|}_{t}$$

Since $1 \le \alpha \le 3K^7$ it follows that $-42K^{14} < t < 42K^{14}$. Hence proved.

Therefore, the background space points either have unequal or infinite distances.

▶ **Lemma 34.** For any configuration pair $\langle c, d \rangle$, in the background space, either $\operatorname{dist}_{\mathcal{A}_1}(c) \neq \operatorname{dist}_{\mathcal{A}_2}(d)$ or $\operatorname{dist}_{\mathcal{A}_1}(c) = \operatorname{dist}_{\mathcal{A}_2}(d) = \infty$.

Proof. Assume for contradiction that there is a configuration pair $\langle c, d \rangle$, in the background space such that $\operatorname{dist}_{\mathcal{A}_1}(c) = \operatorname{dist}_{\mathcal{A}_2}(d) < \infty$. Let $n_c = \operatorname{COUNTER-VALUE}(c)$ and $n_d = \operatorname{COUNTER-VALUE}(d)$. Since $\operatorname{dist}_{\mathcal{A}_1}(c) = \operatorname{dist}_{\mathcal{A}_2}(d)$. From Lemma 33, there exists $a_1, b_1, a_2, b_2 \in [0, 3\mathsf{K}^7]$ and $d_1, d_2 < 42\mathsf{K}^{14}$ such that

$$\frac{a_1}{b_1}n_{\mathtt{c}} + d_1 = \mathrm{dist}_{\mathcal{A}_1}(\mathtt{c}) = \mathrm{dist}_{\mathcal{A}_1}(\mathtt{d}) = \frac{a_2}{b_2}n_{\mathtt{d}} + d_2$$

Therefore $\left|\frac{a_1}{b_1}n_c - \frac{a_2}{b_2}n_d\right| \le |d_2 - d_1| < 42\mathsf{K}^{14}$. This satisfies the belt condition and is a configuration pair in the belt space. This contradicts our initial assumptions.

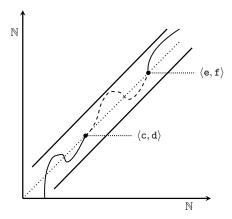
Similar to that in [3], we can show that the length of the run Π in the background space is polynomially bounded in K, and the counter values of the first background point in Π .

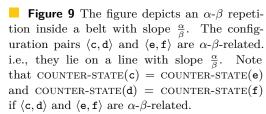
▶ **Lemma 35.** If $h_j = \langle c_j, d_j \rangle$ is the first configuration pair in the background space during Π , then $\ell - j$ is bounded by a polynomial in COUNTER-VALUE(c_j), COUNTER-VALUE(d_j), and K.

Proof. Let $h_j = \langle c_j, d_j \rangle$ be the first configuration pair in the background space during the run Π , then from Lemma 34, either $\operatorname{dist}_{\mathcal{A}_1}(c_j) = \operatorname{dist}_{\mathcal{A}_2}(d_j) = \infty$ or $\operatorname{dist}_{\mathcal{A}_1}(c_j) \neq \operatorname{dist}_{\mathcal{A}_2}(d_j)$. We separately consider the two cases.

Case-1, $\operatorname{dist}_{\mathcal{A}_1}(\mathsf{c}_j) = \operatorname{dist}_{\mathcal{A}_2}(\mathsf{d}_j) = \infty$: then we prove that the remaining length of the witness from $\langle \mathsf{c}_j, \mathsf{d}_j \rangle$ is bounded by $2\mathsf{K}^2$. Assume for contradiction that this is not the case and $\mathsf{c}_j \sim_{2\mathsf{K}^2} \mathsf{d}_j$ but $\mathsf{c}_j \not\equiv \mathsf{d}_j$. Let $v \in \Sigma^{>2\mathsf{K}^2}$ be the word which distinguishes c and d . Therefore, there exists a prefix of $v, u \in \Sigma^{|v|-2\mathsf{K}^2}$, and $i = \ell - 2\mathsf{K}^2$ such that $\langle \mathsf{c}_j, \mathsf{d}_j \rangle \xrightarrow{u} \langle \mathsf{c}_i, \mathsf{d}_i \rangle$ and $\mathsf{c}_i \not\equiv_{2\mathsf{K}^2} \mathsf{d}_i$.

Since v is a minimal witness $\mathbf{c}_i \equiv_{2\mathsf{K}^2-1} \mathbf{d}_i$ and $\mathbf{c}_i \not\equiv_{2\mathsf{K}^2} \mathbf{d}_i$. Since $\mathrm{dist}_{\mathcal{A}_1}(\mathbf{c}_j) = \mathrm{dist}_{\mathcal{A}_2}(\mathbf{d}_j) = \infty$, there exists configurations $\bar{\mathbf{c}}_i$ and $\bar{\mathbf{d}}_i$ in the underlying automaton \mathcal{B} such that $\mathbf{c}_i \sim_{2\mathsf{K}^2} \bar{\mathbf{c}}_i$ and $\mathbf{d}_i \sim_{2\mathsf{K}^2} \bar{\mathbf{d}}_i$. Since $\mathbf{c}_i \equiv_{2\mathsf{K}^2-1} \mathbf{d}_i$, it follows that $\bar{\mathbf{c}}_i \sim_{2\mathsf{K}^2-1} \bar{\mathbf{d}}_i$. From the equivalence result of weighted automata, we know that if two configurations of a weighted automata with k





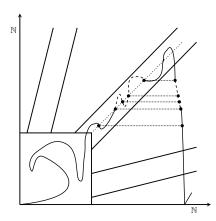


Figure 10 The figure shows the run of a word that enters the background space from the belt such that the counter values of the first configuration pair in the background space exceed a polynomial bound. Some α - β repetitions inside the belt can be removed to show the existence of a shorter witness.

states are non-equivalent, then there is a word of length less than k which distinguishes them. Therefore, this is sufficient to prove that the underlying weighted automata with $\bar{\mathbf{c}}_i$ and $\bar{\mathbf{d}}_i$ as initial distributions are equivalent, and thus $\bar{\mathbf{c}}_i \sim_{2\mathsf{K}^2} \bar{\mathbf{d}}_i$. This allows us to deduce that $\mathbf{c}_i \equiv_{2\mathsf{K}^2} \mathbf{d}_i$, which is a contradiction. Therefore, the remaining length of the witness from $\langle \mathbf{c}_j, \mathbf{d}_j \rangle$ is bounded by $2\mathsf{K}^2$.

Case-2, $\operatorname{dist}_{\mathcal{A}_1}(\mathsf{c}_j) \neq \operatorname{dist}_{\mathcal{A}_2}(\mathsf{d}_j)$: Without loss of generality, we suppose $\operatorname{dist}_{\mathcal{A}_1}(\mathsf{c}_j) > \operatorname{dist}_{\mathcal{A}_2}(\mathsf{d}_j)$. By definition of $\operatorname{dist}_{\mathcal{A}_2}$, there exists $u \in \Sigma^{\operatorname{dist}_{\mathcal{A}_2}(\mathsf{d}_j)}$, i > j and a configuration $\bar{\mathsf{c}}$ of the underlying automaton \mathcal{B} such that $\mathsf{c}_j \stackrel{u}{\to} \mathsf{c}_i$, $\mathsf{d}_j \stackrel{u}{\to} \mathsf{d}_i$, $\mathsf{c}_i \sim_{2\mathsf{K}^2} \bar{\mathsf{c}}_i$ and $\mathsf{d}_i \not\sim_{2\mathsf{K}^2} \bar{\mathsf{c}}_i$. Therefore $\mathsf{c}_i \not\equiv_{2\mathsf{K}^2} \mathsf{d}_i$. By definition, there exists $v \in \Sigma^{\leq 2\mathsf{K}^2}$ such that $f_{\mathcal{A}_1}(v,\mathsf{c}_i) \not= f_{\mathcal{A}_2}(v,\mathsf{d}_i)$ and hence $f_{\mathcal{A}_1}(uv,\mathsf{c}_j) \not= f_{\mathcal{A}_2}(uv,\mathsf{d}_j)$. As $uv \in \Sigma^{\operatorname{dist}_{\mathcal{A}_2}(\mathsf{d}_j) + 2\mathsf{K}^2}$, we get that $\mathsf{c}_j \not\equiv_{\operatorname{dist}_{\mathcal{A}_2}(\mathsf{d}_j) + 2\mathsf{K}^2}$ d_j . Therefore, there is $w \in \Sigma^{\leq \min\{\operatorname{dist}_{\mathcal{A}_1}(\mathsf{c}_j), \operatorname{dist}_{\mathcal{A}_2}(\mathsf{d}_j)\} + 2\mathsf{K}^2}$ that distinguishes c_j and d_j .

Let $\alpha, \beta \in [1, 3\mathsf{K}^7]$ be co-prime. We say configuration pairs $\langle (\mathbf{x}_\mathsf{c}, p_\mathsf{c}, n_\mathsf{c}), (\mathbf{x}_\mathsf{d}, p_\mathsf{d}, n_\mathsf{d}) \rangle$ and $\langle (\mathbf{x}_\mathsf{e}, p_\mathsf{e}, n_\mathsf{e}), (\mathbf{x}_\mathsf{f}, p_\mathsf{f}, n_\mathsf{f}) \rangle$ are α - β related if $p_\mathsf{c} = p_\mathsf{e}$, $p_\mathsf{d} = p_\mathsf{f}$ and $\alpha \cdot n_\mathsf{c} - \beta \cdot n_\mathsf{d} = \alpha \cdot n_\mathsf{e} - \beta \cdot n_\mathsf{f}$. Roughly speaking, two configuration pairs are α - β related if they have the same state pairs and lie on a line with slope $\frac{\alpha}{\beta}$. An α - β repetition is a run $\bar{\pi}_1 = c_i \tau_i c_{i+1} \tau_{i+1} \cdots \tau_{j-1} c_j$ that lies inside a belt with slope $\frac{\alpha}{\beta}$ such that c_i and c_j are α - β related. The following lemma bounds the counter values of the first configuration pair in the background space, if it exists, during the run Π .

▶ **Lemma 36.** If $h_j = \langle c_j, d_j \rangle$ is the first background point in Π then, COUNTER-VALUE (c_j) and COUNTER-VALUE (d_i) are less than $\mathsf{K}^5 \cdot 42\mathsf{K}^{14}$.

Proof. Let $\mathbf{h}_j = \langle \mathbf{c}_j, \mathbf{d}_j \rangle$ be the first point in the background space during the run Π . Given any configuration \mathbf{c} , let $n_{\mathbf{c}}$ denote COUNTER-VALUE(\mathbf{c}). Assume for contradiction that $n_{\mathbf{c}_j}$ is greater than $\mathsf{K}^5 \cdot 42\mathsf{K}^{14}$. Let $\Pi = \mathbf{h}_0\tau_0 \cdots \mathbf{h}_{j-1}\tau_{j-1}\mathbf{h}_j \cdots \mathbf{h}_\ell$ be a run of a minimal witness. Since \mathbf{h}_j is the first point in the background space in this run and $n_{\mathbf{c}_j} > \mathsf{K}^5 \cdot 42\mathsf{K}^{14}$, there exists 0 < i < j such that the sub-run $\Pi_b = \mathbf{h}_i\tau_i\mathbf{h}_{i+1}\cdots\tau_{j-2}\mathbf{h}_{j-1}$ lies inside a belt B with slope $\frac{\alpha}{\beta}$ for some $\alpha, \beta \in [1, 3\mathsf{K}^7]$. Since we are looking at the run of a minimal witness, from Lemma 34 either $\mathbf{c}_j \not\equiv_{\mathsf{Z}\mathsf{K}^2} \mathbf{d}_j$ or $\mathrm{dist}(\mathbf{c}_j) \not\equiv \mathrm{dist}(\mathbf{d}_j)$. We separately consider the two cases.

Case-1: $\operatorname{dist}_{\mathcal{A}_1}(\mathsf{c}_j) \neq \operatorname{dist}_{\mathcal{A}_2}(\mathsf{d}_j)$: Without loss of generality, let us assume $\operatorname{dist}_{\mathcal{A}_1}(\mathsf{c}_j) < \operatorname{dist}_{\mathcal{A}_2}(\mathsf{d}_j)$. Therefore there exists $t \in \mathbb{N}$ with $j < t \leq \ell$ and configuration pair h_t such that $\mathsf{h}_t = \langle (\mathbf{x}_{\mathsf{c}_t}, p, m), (\mathbf{x}_{\mathsf{d}_t}, p_{\mathsf{d}_t}, n_{\mathsf{d}_t}) \rangle$, $m < 2\mathsf{K}^2$ and $\mathbf{x}_{\mathsf{c}_t} \in \overline{\mathcal{W}}_1^{p,m}$. We show that we can pump some portion out from Π_b to reach a configuration in the background space with unequal distance and smaller counter values.

Since $n_{c_j} > \mathsf{K}^5 \cdot 42\mathsf{K}^{14}$, by Pigeonhole principle, there exists indices $i_0 < i_1 < i_2 < \cdots, i_{\mathsf{K}^2} < i_0' < i_1' < i_2', \cdots, < i_{\mathsf{K}^2}'$ such that for all $r \in [1, \mathsf{K}^2]$, (1) $\mathsf{h}_{i_{r-1}}$ and h_{i_r} are $\alpha \cdot \beta$ related and lie in belt B, (2) $n_{\mathsf{c}_{i_{r-1}}} < n_{\mathsf{c}_{i_r}} = n_{\mathsf{c}_{i_r'}}$, (3) $p_{\mathsf{c}_{i_r'}} = p_{\mathsf{c}_{i_{r-1}'}}$, (4) for all $t \in \mathbb{N}$ with $i_r < t < j$, $n_{\mathsf{c}_t} > n_{\mathsf{c}_{i_r}}$, and (5) for all $t \in \mathbb{N}$ with $j < t < i_r'$, $n_{\mathsf{c}_t} < n_{\mathsf{c}_{i_t'}}$.

Given any configuration \mathbf{c} let $\mathbf{x}_{\mathbf{c}}$ denote WEIGHT-VECTOR(\mathbf{c}). For $r \in [0, \mathsf{K}^2]$ let $\mathbb{A}_r \in \mathcal{F}^{\mathsf{K} \times \mathsf{K}}$ denote the matrix such that $\mathbf{x}_{\mathsf{c}_{i_r}} \mathbb{A}_r = \mathbf{x}_{\mathsf{c}_{i'_r}}$ and $\mathbb{B}_r \in \mathcal{F}^{\mathsf{K} \times \mathsf{K}}$ denote the matrix such that $\mathbf{x}_{\mathsf{c}_{i'_r}} \mathbb{B}_r = \mathbf{x}_{\mathsf{c}_t} \in \overline{\mathcal{W}}_1^{p,m}$. Therefore for all $r \in [0, \mathsf{K}^2]$, we have $\mathbf{x}_{\mathsf{c}_{i_r}} \mathbb{A}_r \mathbb{B}_r \in \overline{\mathcal{W}}_1^{p,m}$. From Lemma 8, we have that there exists $s, r \in [0, \mathsf{K}^2]$ with s < r such that $\mathbf{x}_{\mathsf{c}_{i_s}} \mathbb{A}_r \mathbb{B}_s \in \overline{\mathcal{W}}_1^{p,m}$. Consider the sequence of transitions $T' = \tau_0, \cdots, \tau_{i_s-1}\tau_{i_r}, \cdots, \tau_{j-1}$ and let $w = \mathsf{word}(T')$. Let $\mathbf{h}_{j'}$ be the configuration such that $\mathbf{h}_0 \xrightarrow{w} \mathbf{h}_{j'}$. Since we have removed an α - β repetitions inside the belt, the configuration pair $\mathbf{h}_{j'}$ is a point in the background space (see Figure 10). Moreover, $n_{\mathsf{c}_{j'}} < n_{\mathsf{c}_j}$ and $\mathsf{dist}_{\mathcal{A}_1}(\mathsf{c}_{j'}) < \infty$. Since it is a point in the background space, from Lemma 34, we get that $\mathsf{dist}_{\mathcal{A}_1}(\mathsf{c}_{j'}) \neq \mathsf{dist}_{\mathcal{A}_2}(\mathsf{d}_{j'})$. Therefore, there is a shorter path to a configuration in background space with smaller counter values and unequal distance. This is a contradiction.

Case-2: $c_j \not\equiv_{2K^2} d_j$: Consider the sub-run Π_b . Since it is a run inside a belt, we can consider this as the run of the weighted ODCA \mathcal{D} . Since $n_{c_j} > \mathsf{K}^4 \cdot 42\mathsf{K}^{14}$, by Pigeon-hole principle, there exists indices $i_0, i_1, i_2, \cdots, i_{\mathsf{K}^2}$ such that for all $r \in [1, \mathsf{K}^2]$, $\mathbf{h}_{i_{r-1}}$ and \mathbf{h}_{i_r} are α - β related with $n_{c_{i_{r-1}}} < n_{c_{i_r}}$ and for all $t \in \mathbb{N}$ with $i_r < t < j$, $n_{c_t} > n_{c_{i_r}}$.

Since it is the run of a minimal witness, we know that there exists $\mathbb{A} \in \mathcal{F}^{2\mathsf{K} \times 2\mathsf{K}}$ such that $\mathbf{x}_{j-1}\mathbb{A}\eta_F^{\top} \neq 0$. Consider the vector space $\mathcal{U} = \{\mathbf{y} \in \mathcal{F}^{2\mathsf{K}} \mid \mathbf{y}\mathbb{A}\eta_F^{\top} = 0\}$. For $r \in [0, \mathsf{K}^2]$, let \mathbb{A}_r denote the matrices such that $\mathbf{x}_{i_r}\mathbb{A}_r = \mathbf{x}_j \in \mathcal{U}$. Since $\mathbf{x}_{i_r}\mathbb{A}_r \in \overline{\mathcal{U}}$ for all $r \in [0, \mathsf{K}^2]$, from Lemma 8, we get that there exists $r' \in [0, r-1]$ such that $\mathbf{x}_{c_{i'_r}}\mathbb{A}_r \in \overline{\mathcal{V}}$. The sequence of transitions $\tau_{i_r+1}\cdots\tau_\ell$ can be taken from $\mathbf{h}_{i'_r}$ since the counter values always stay positive. Consider the sequence of transitions $T' = \tau_0 \cdots \tau_{i'_r}\tau_{i_r+1}\cdots\tau_\ell$ and let $w = \mathsf{word}(T')$. The word w is a shorter witness than z and contradicts its minimality.

Finally, we prove that the counter values encountered during the run Π are polynomially bounded in K using above lemmas.

Proof of Lemma 25. Consider the run Π . From Lemma 29, Lemma 30 and Lemma 36, we get that the counter values of configuration pairs inside the belt space during this run in polynomially bounded in K. Therefore, if it exists, the first background point in Π has polynomially bounded counter values. From Lemma 35, the length of Π after the first background point is polynomially bounded in K. Since the counter values of configuration pairs inside the initial space are also bounded by a polynomial in K, the maximum counter value in Π is polynomially bounded in K.

5 Regularity of ODCA is in P

We say that a weighted odca \mathcal{A} is regular if there is a weighted automaton \mathcal{B} that is equivalent to it. We look at the regularity problem - the problem of deciding whether a

weighted ODCA is regular. We fix a weighted ODCA $\mathcal{A} = ((C, \delta_0, \delta_1, p_0), (Q, \lambda, \Delta, \eta))$ and use N to denote $|C| \cdot |Q|$.

The proof technique is adapted from the ideas developed by Böhm et al. [6] in the context of real-time OCA. The crucial idea in proving regularity is to check for the existence of infinitely many equivalence classes. The proof relies on the notion of distance of configurations. Distance of a configuration is the length of a minimal word to be read to reach a configuration that does not have an N equivalent configuration in the underlying automata. The challenge is to find infinitely many configurations reachable from the initial configuration, so that no two of them have same distance. Our main contribution is in designing a "pumping" like argument to show this (Lemma 17, Point 2).

Recall the definition of U(A) from Definition 31. We use c to denote some configuration of A and \bar{c} to denote some configuration of U(A). For a $p \in C$ and $m \in \mathbb{N}$, we define

$$\mathcal{W}^{p,m} = \{ \mathbf{x} \in \mathcal{F}^{|Q|} | \exists \bar{\mathbf{c}} \in \mathcal{F}^{\mathsf{N}}, \mathbf{c} = (\mathbf{x}, p, m) \sim_{\mathsf{N}} \bar{\mathbf{c}} \} .$$

The set $\overline{\mathcal{W}}^{p,m}$ is $\mathcal{F}^{|Q|} \setminus \mathcal{W}^{p,m}$. The distance of a configuration c (denoted by dist(c)) is

$$\min\{|w| \mid c \xrightarrow{w} (\mathbf{x}, p, m) \exists p \in C, m < \mathsf{N}, \text{ and } \mathbf{x} \in \overline{\mathcal{W}}^{p, m}\}$$
.

The following lemma shows when \mathcal{A} is not regular. Given any configuration \mathbf{c} , we use $\mathbf{x}_{\mathbf{c}}$ to denote WEIGHT-VECTOR(\mathbf{c}), $p_{\mathbf{c}}$ to denote COUNTER-STATE(\mathbf{c}) and $n_{\mathbf{c}}$ to denote COUNTER-VALUE(\mathbf{c}).

- ▶ **Lemma 37.** Let c be an initial configuration of an ODCA A. Then the following are equivalent.
- 1. A is not regular.
- 2. for all $t \in \mathbb{N}$, there exists configurations \mathbf{d} , \mathbf{e} s.t. $n_{\mathbf{e}} < \mathbf{N}, \mathbf{c} \overset{*}{\to} \mathbf{d} \overset{*}{\to} \mathbf{e}$, $\mathbf{x}_{\mathbf{e}} \in \overline{\mathcal{W}}^{p_{\mathbf{e}}, n_{\mathbf{e}}}$ and $t < \operatorname{dist}(\mathbf{d}) < \infty$.
- 3. there exists configurations \mathbf{d} , \mathbf{e} and a run $\mathbf{c} \stackrel{*}{\to} \mathbf{d} \stackrel{*}{\to} \mathbf{e}$ s.t. $N^2 + N \leq n_d \leq 2N^2 + N$, $\mathbf{x}_e \in \overline{\mathcal{W}}^{p_e, n_e}$ with $n_e < N$.
- **Proof.** $3 \to 2$: Consider an arbitrary $q \in C$, $m < \mathbb{N}$ and vector space $\mathcal{V} = \mathcal{W}^{q,m}$. Let us assume for contradiction the complement of Point 2. That is, there exists a $t \in \mathbb{N}$ such that for all configurations \mathbf{d}' where $\mathbf{c} \stackrel{*}{\to} \mathbf{d}' \stackrel{*}{\to} \overline{\mathcal{V}} \times \{q\} \times \{m\}$, $\mathrm{dist}(\mathbf{d}') \leq t$. Note that for all \mathbf{d}' where $n_{\mathbf{d}'} > \mathbb{N}$, $\mathrm{dist}(\mathbf{d}') \geq n_{\mathbf{d}'} \mathbb{N}$. Hence there exists an $M \in \mathbb{N}$ such that for all \mathbf{d}' where $\mathbf{c} \stackrel{*}{\to} \mathbf{d}' \stackrel{*}{\to} \overline{\mathcal{V}} \times \{q\} \times \{m\}$, $n_{\mathbf{d}'} \leq M$.

Consider a configuration d where $n_d > N^2 + N$ and a run $c \stackrel{*}{\to} d \stackrel{*}{\to} \overline{\mathcal{V}} \times \{q\} \times \{m\}$. Point 3 shows the existence of such a run. From Lemma 17, Point 2, we get that there exists a d' such that $c \stackrel{*}{\to} d' \stackrel{*}{\to} \overline{\mathcal{V}} \times \{q\} \times \{m\}$ and $n_{d'} > n_d$, which is a contradiction.

 $2 \to 1$: Assume for contradiction that for all $t \in \mathbb{N}$, there exists configurations d, e such that $c \stackrel{*}{\to} d \stackrel{*}{\to} e$, $\mathbf{x}_e \in \overline{\mathcal{W}}^{p_e,n_e}$, $n_e < \mathsf{N}$ and $t < \mathrm{dist}(d) < \infty$ but \mathcal{A} is regular. Let \mathcal{B} be the weighted automaton equivalent to \mathcal{A} . We use $|\mathcal{B}|$ to represent the number of states of \mathcal{B} .

Let $t_1, t_2, \ldots t_{|\mathcal{B}|+1} \in \mathbb{N}$ such that for $i \in [1, |\mathcal{B}|]$, $t_i < t_{i+1}$, and d_{t_i} be such that $c \stackrel{*}{\to} d_{t_i} \stackrel{*}{\to} (\mathbf{x}_i, p_{\mathbf{e}}, n_{\mathbf{e}})$, $\mathbf{x}_i \in \overline{\mathcal{W}}^{p_{\mathbf{e}}, n_{\mathbf{e}}}$ and $t_i < \operatorname{dist}(d_{t_i}) < t_{i+1} < \infty$. Clearly $d_{t_i} \not\equiv d_{t_j}$ for $i \neq j$ and hence corresponds to two different states of \mathcal{B} . Since we can find more than $|\mathcal{B}|$ pairwise non-equivalent configurations, it contradicts the assumption that \mathcal{B} is equivalent to \mathcal{A}

 $1 \to 3$: We prove the contrapositive of the statement. Let us assume that there is no configurations d, e and a run c $\stackrel{*}{\to}$ d $\stackrel{*}{\to}$ e such that $N^2 + N \le n_d \le 2N^2 + N$, $\mathbf{x_e} \in \overline{\mathcal{W}}^{p_e, n_e}$ with $n_e < N$. This implies that there does not exists a configuration d' such that $n_{d'} > 2N^2$,

 $c \stackrel{*}{\to} d' \stackrel{*}{\to} (\mathbf{y}, p_e, n_e)$ for some $\mathbf{y} \in \overline{\mathcal{W}}^{p_e, n_e}$. Assume for contradiction that there is such a run, then there exists a portion in this run that can be "pumped down" to get a run $c \stackrel{*}{\to} d'' \stackrel{*}{\to} (\mathbf{y}', p_e, n_e)$ for some configuration d'' such that $N^2 + N \le n_{d''} \le 2N^2 + N$ and $\mathbf{y}' \in \overline{\mathcal{W}}^{p_e, n_e}$. This is a contradiction. Therefore all runs starting from configuration with counter value greater than or equal to $N^2 + N$ "looks" similar to a run on a weighted automaton.

▶ **Theorem 38.** There is a polynomial time algorithm to decide whether a weighted odca is equivalent to some weighted automata.

Proof. Let \mathcal{A} be a weighted odca. Lemma 37 shows that if \mathcal{A} is not regular, then there are words $u,v\in\Sigma^*$ and configurations d,e such that there is a run of the form $c\xrightarrow{u}d\xrightarrow{v}e$ such that $N^2+N\leq \text{COUNTER-VALUE}(d)\leq 2N^2+N$, weight-vector(e) $\in\overline{\mathcal{W}}^{\text{COUNTER-STATE}(e),\text{COUNTER-VALUE}(e)}$ with Counter-value(e) < N. The existence of such words u and v can be decided in polynomial time since the minimal length of such a path if it exists, is polynomially bounded in the number of states of the weighted odca by Corollary 18. This concludes the proof.

6 Covering

Let \mathcal{A}_1 and \mathcal{A}_2 be two uninitialised weighted odcas. We say \mathcal{A}_2 covers \mathcal{A}_1 if for all initial configurations \mathbf{c}_0 of \mathcal{A}_1 there exists an initial configuration \mathbf{d}_0 of \mathcal{A}_2 such that $\mathcal{A}_1\langle\mathbf{c}_0\rangle$ and $\mathcal{A}_2\langle\mathbf{d}_0\rangle$ are equivalent. We say \mathcal{A}_1 and \mathcal{A}_2 are coverable equivalent if \mathcal{A}_1 covers \mathcal{A}_2 , and \mathcal{A}_2 covers \mathcal{A}_1 . We show that the covering and coverable equivalence problems for uninitialised weighted odcas are decidable in polynomial time. The proof relies on the algorithm to check the equivalence of two weighted odcas.

▶ **Theorem 39.** Covering and coverable equivalence problems of uninitialised weighted ODCAs are in P.

Proof. We fix two uninitialised weighted odcas $A_1 = (C_1, \delta_1; Q_1, \Delta_1, \eta_1)$ and $A_2 = (C_2, \delta_2; Q_2, \Delta_2, \eta_2)$ for this section. Without loss of generality, assume $K = |C_1| = |Q_1| = |C_2| = |Q_2|$. For $i \in [1, K]$ we define the vector $\mathbf{e}_i \in \mathcal{F}^K$ as follows:

$$\mathbf{e}_{i}[j] = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

For $j \in [1, K]$, $q \in C_1$, we use $\mathbf{h}_{j,q}$ to denote the configuration $(\mathbf{e}_j, q, 0)$ of \mathcal{A}_1 and for $i \in [1, K]$, $p \in C_2$, we use $\mathbf{g}_{i,p}$ to denote the configuration $(\mathbf{e}_i, p, 0)$ of \mathcal{A}_2 .

▶ Claim 1. There is a polynomial time algorithm to decide whether A_2 covers $A_1\langle h_{j,q}\rangle$ for any $j \in [1, K]$ and $q \in C_1$.

<u>Proof:</u> First, we check, in polynomial time (equivalence with a zero machine), whether $\mathcal{A}_1\langle \mathbf{h}_{j,q}\rangle$ accepts all words with weight $0 \in \mathcal{F}$. If that is the case, then $\mathcal{A}_1\langle \mathbf{h}_{j,q}\rangle$ and $\mathcal{A}_2\langle \mathbf{g}_0\rangle$ are equivalent for the configuration $\mathbf{g}_0 = (\{0\}^K, p, 0)$, for any $p \in C_2$. Otherwise, there is some word w_0 accepted by $\mathcal{A}_1\langle \mathbf{h}_{j,q}\rangle$ with non-zero weight s (returned by the previous equivalence check). Without loss of generality, we consider the smallest one, whose size is polynomial in K.

We pick a $p \in C_2$ and check whether there exists an initial distribution from counter state p that makes the two machines equivalent. Assume that such an initial distribution

exists and for all $i \in [1, K]$, let α_i denote the initial weight on state $q_i \in Q_2$. We use α to denote the resultant initial distribution. We initialise an empty set B to store a system of linear equations.

The following steps will be repeated at most K times to check the existence of an initial distribution with initial state $p \in C_2$. Let w be the counter-example returned by the equivalence query in the previous step. For all $i \in [1, K]$, we compute $f_{\mathcal{A}_2 \langle \mathsf{g}_{i,p} \rangle}(w)$. We add the linear equation $\sum_{i=1}^K \alpha_i \cdot f_{\mathcal{A}_2 \langle \mathsf{g}_{i,p} \rangle}(w) = f_{\mathcal{A}_1 \langle \mathsf{h}_{j,q} \rangle}(w)$ to B and compute values for α_i , $i \in [1, K]$, such that it satisfies the system of linear equations in B. We check whether $\mathcal{A}_1 \langle \mathsf{h}_{j,q} \rangle$ and $\mathcal{A}_2 \langle (\alpha, p, 0) \rangle$ are equivalent or not. If they are not equivalent, we get a new counter example that distinguishes them. Now we repeat the procedure to compute a new initial distribution.

Note that the above procedure is executed at most K times to find an initial distribution if it exists. This is because we can find only K many linearly independent linear equations in K variables. Suppose the above procedure fails to find an initial distribution for which the machines are equivalent. In that case, there is an initial distribution of \mathcal{A}_1 with initial counter state q, for which \mathcal{A}_2 with initial counter state p does not have an equivalent initial distribution. We now pick a different counter state of C_2 and repeat the process until we find a $p \in C_2$ for which the algorithm finds an equivalent initial distribution. If for all $p \in C_2$, the algorithm returns false, then \mathcal{A}_2 does not cover $\mathcal{A}_1\langle \mathbf{h}_{j,q}\rangle$.

First, we show the existence of a polynomial time procedure to check whether \mathcal{A}_2 covers \mathcal{A}_1 . For all $j \in [1, \mathsf{K}]$, we check whether there exists an initial state $p \in C_2$ such that \mathcal{A}_2 with initial counter state p has an initial distribution that makes it equivalent to $\mathcal{A}_1 \langle \mathbf{h}_{j,q} \rangle$ using Claim 1. If we fail to find such a state in C_2 then we return false. We repeat this procedure for all $q \in C_1$. If for all $q \in C_1$ there exists a $p \in C_2$ such that \mathcal{A}_2 with initial counter state p has an initial distribution that makes it equivalent to $\mathcal{A}_1 \langle \mathbf{h}_{j,q} \rangle$ for all $j \in [1, K]$, then we say that \mathcal{A}_2 covers \mathcal{A}_1 otherwise we say that \mathcal{A}_2 does not cover \mathcal{A}_1 . Let us see why this is true. For simplifying the arguments we fix a $q \in C_1$. Assume that for all $j \in [1, K]$, there exits $p \in C_2$ such that $\mathcal{A}_1 \langle \mathbf{h}_{j,q} \rangle$ is equivalent to the configuration $(\mathbf{x}_{j,q}, p, 0)$ for some $\mathbf{x}_{j,q} \in \mathcal{F}^{K}$. Now, any initial configuration $(\lambda, q, 0)$ of \mathcal{A}_1 is equivalent to the configuration $(\sum_{j=1}^{K} \lambda[j] \cdot \mathbf{x}_{j,q}, p, 0)$ of \mathcal{A}_2 .

The coverable equivalence problem can now be solved by checking whether A_1 covers A_2 and A_2 covers A_1 , which can be done in time polynomial in K.

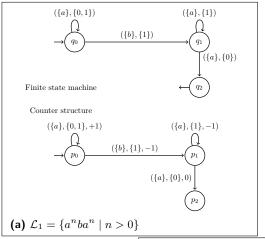
7 Non-deterministic ODCA

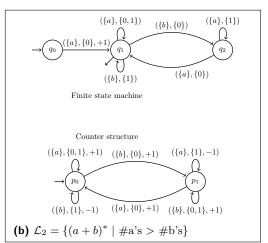
In this section, we consider the counter-determinacy restriction over weightless OCAs (equivalently, with weights from the boolean semiring). These results do not follow from previous sections, as booleans are not a field.

- ▶ **Example 39.** The following languages are defined over the alphabet $\Sigma = \{a, b\}$ and are recognised by non-deterministic OCA with counter-determinacy.
- (a) The language $\mathcal{L}_1 = \{a^n b a^n \mid n > 0\}.$
- (b) The language $\mathcal{L}_2 = \{(a+b)^* \mid \text{number of a's is greater than number of b's}\}.$
- (c) The language $\mathcal{L}_3 = \{a^n(b+c)^m b(b+c)^k \mid m, n \in \mathbb{N} \text{ and } m > n\}.$

Note that none of the above languages are definable by visibly pushdown automata. The ODCAs corresponding to these languages are given in Figure 11.

We observe that the relationship between non-deterministic and deterministic ODCAs is similar to that between non-deterministic and deterministic finite automata. By definition,





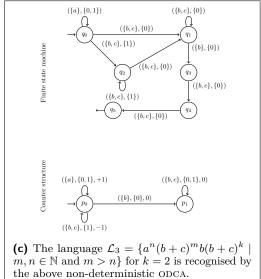


Figure 11 The figure gives odcas corresponding to examples (a), (b), and (c) given in Example 39. Let $A \subseteq \Sigma$, $R \subseteq \{0,1\}$ and $D \in \{-1,0,+1\}$ are non-empty sets. For $i,j \in \mathbb{N}$, if a transition from q_i to q_j is labelled (A,R) and $(a,r) \in A \times R$, then there is a transition from q_i to q_j on reading the symbol a. The current counter value should be 0 if r=0 and greater than 0 if r=1. Similarly, if a transition from p_i to p_j is labelled (A,R,D) and $(a,r,d) \in A \times R \times \{D\}$, then there is a transition from p_i to p_j on reading the symbol a with counter action d. The current counter value should be 0 if r=0 and greater than 0 if r=1.

deterministic ODCAs have at most one unique path for any fixed word. Therefore, they are deterministic OCAs with counter-determinacy. It is also easy to observe that deterministic OCAs are deterministic ODCAs. It follows that deterministic ODCAs and deterministic OCAs are expressively equivalent. Similar to non-deterministic finite automata, we observe that non-deterministic ODCAs can be determinised by a subset construction of the states of the finite state machine. However, this results in an exponential blow-up. In Example 39, the deterministic ODCA that recognises the language \mathcal{L}_3 has to check whether every b encountered after reading the word $a^n(b+c)^{n+1}$ is at the k^{th} position from the end. This will require at least $\mathcal{O}(2^k)$ states. On the other hand, there is a non-deterministic ODCA with $\mathcal{O}(k)$ states recognising the same language. Similar to finite automata, non-deterministic ODCAs are a "succinct" way to represent deterministic OCAs.

A deterministic/non-deterministic ODCA \mathcal{A} is an ODCA over the boolean semiring $\mathcal{S} = (\{0,1\},\vee,\wedge)$. The language recognised by \mathcal{A} is given by $\mathcal{L}(\mathcal{A}) = \{w \mid f_{\mathcal{A}}(w) = 1\}$. We say an ODCA $\mathcal{A} = ((C,\delta_0,\delta_1,p_0),(Q,\boldsymbol{\lambda},\Delta,\boldsymbol{\eta}))$ is deterministic if for every transition sequence $T = \tau_0 \cdots \tau_{\ell-1}$, the vector $\boldsymbol{\lambda} \mathbf{we}(T)$ contains exactly one 1 and non-deterministic otherwise.

For every language recognised by a non-deterministic ODCA, there is a deterministic ODCA of at most exponential size that recognises it. The idea is a simple subset construction (see Theorem 40).

▶ Theorem 40 (Determinisation). Given a non-deterministic ODCA, a polynomial space machine can output an equivalent deterministic ODCA of exponential size.

Proof. Let $\mathcal{A} = ((C, \delta_0, \delta_1, p_0), (Q, \lambda, \Delta, \eta))$ be a non-deterministic ODCA. Given a vector $\mathbf{x} \in \mathcal{S}^k$ for some $k \in \mathbb{N}$, we define the function IsDet: $\mathcal{S}^k \to \{true, false\}$ as follows:

IsDet(
$$\mathbf{x}$$
) =
$$\begin{cases} \text{true, if } \exists i < k \text{ s.t } \mathbf{x}[i] = 1 \text{ and } \forall j \neq i, \mathbf{x}[i] = 0 \\ \text{false, otherwise.} \end{cases}$$

Given a transition matrix \mathbb{A} corresponding to the states Q, we define its determinisation $\det(\mathbb{A})$ as follows. There are rows and columns corresponding to each set in 2^Q . For any $q_i \in Q$, let $\mathcal{M}(q_i, \mathbb{A}) = \{q_j \mid \mathbb{A}[i][j] = 1\}$ be the set of all states in the row of q_i whose entries are 1. With the notation that $\det(\mathbb{A})[s][s']$ corresponds to the entry of the cell corresponding to the sets $s, s' \in 2^Q$, we let $\det(\mathbb{A})[s][s'] = 1$ if and only if $s' = \bigcup_{q \in s} \mathcal{M}(q_i, \mathbb{A})$. We claim that $\mathcal{A}_{\det} = ((C, \delta_0, \delta_1, p_0), (Q, \lambda, \Delta', \eta'))$, with η' such that for any $S \in 2^Q, \eta'[S] = \bigvee_{s \in S} \eta[s]$ and for all $a \in \Sigma$ and $d \in \{0, 1\}$, $\Delta'(a, d) = \det(\Delta(a, d))$ is such that it is deterministic and $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}_{\det})$.

For this, for any sequence of operations $T = \tau_0 \cdots \tau_{\ell-1}$, let $\mathbf{v}_T, \mathbf{v}_T'$ be the vectors corresponding to $\lambda \mathbf{we}(T)$ in \mathcal{A} and \mathcal{A}_{det} respectively. Then we have $\text{IsDet}(\mathbf{v}_T') = 1$ and for any $S \in 2^Q$, $\mathbf{v}_T'[S] = 1$ if and only if for all $q_i \in S$, $\mathbf{v}_T[i] = 1$.

The above result and the fact that equivalence of deterministic ODCA is in NL gives us the upper bound in the following theorem. The lower bound follows from that of NFAs [16].

▶ **Theorem 41.** The equivalence problem for non-deterministic ODCA is PSPACE-complete.

The equivalence of non-deterministic OCA is undecidable [23]. Our theorem shows that undecidability is due to non-determinism in the component that modifies the counter.

8 Conclusion

We introduced a new model called one-deterministic-counter automata. The model "separates" the machine into two components, (1) counter structure - that can modify the counter, and

(2) finite state machine - that can access the counter. This separation of the "writing" and "reading" part gives some natural advantages to the model. We show that the equivalence problem for weighted ODCA is in P if the weights are from a field while that of non-deterministic ODCA is in PSPACE. Note that the equivalence problems on weighted automata (where weights are from a field) and non-deterministic finite automata are in P and PSPACE respectively. On the other hand, the equivalence problem for non-deterministic OCA is undecidable and that of weighted OCA (weights from a field) is not-known. It will be interesting to look at other models where we can separate the "writing" and the "reading" parts. For example, a natural extension is to consider *stack-deterministic* pushdown automata - where a deterministic machine updates the stack. We also leave open the question of learning of weighted ODCAs.

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A Section 1. Introduction

▶ Theorem 2. There is a polynomial time translation from a weighted OCA with counter-determinacy to a weighted ODCA and vice versa.

Proof. First, we show that we can construct an equivalent weighted ODCA from a given weighted OCA with counter-determinacy in polynomial time. Let $\mathcal{A} = (Q, \lambda, \delta_0, \delta_1, \eta)$ be a weighted OCA with counter-determinacy. For this purpose, we define a function $color: [1, |Q|] \to [1, |Q|]$ as follows.

 $color(i) = min\{j \mid \forall w \in \Sigma^*, n \in \mathbb{N}, \text{ counter-effect of } w \text{ from } (q_i, n) \text{ and } (q_j, n) \text{ are equal}\}$

Given a weighted ODCA with an initial configuration $(\lambda,0)$, we can find this coloring function in polynomial time. First, we look at the smallest $i \in [1,|Q|]$ such that $\lambda[i] \neq 0$. For all $j \in [0,|Q|]$, where $\lambda[j] \neq 0$, we say color(j) = i. Now, we look at the configuration reachable from $(\lambda,0)$ in one step. Let (\mathbf{x},c) for some $c \in \mathbb{N}$ be this configuration. If there exists a $j \in [1,|Q|]$ with $\mathbf{x}[j] \neq 0$ and color(j) = i for some $i \in [1,|Q|]$, then for all $k \in [0,|Q|]$, where $\mathbf{x}[k] \neq 0$, we say color(k) = i. If for all $j \in [1,|Q|]$ with $\mathbf{x}[j] \neq 0$, color(j) is not defined, then we look at the smallest $i \in [1,|Q|]$ such that $\mathbf{x}[i] \neq 0$. For all $j \in [0,|Q|]$, where $\mathbf{x}[j] \neq 0$, we say color(j) = i. This process is repeated until color(i) is defined for all $i \in [1,|Q|]$. This terminates after polynomial steps as all reachable states from the initial configuration can be reached by reading a polynomial length word. Let (\mathbf{x},n) be a configuration reachable from the initial configuration of a weighted OCA \mathcal{A} with counter-determinacy.

▶ Claim 1. If $\mathbf{x}[i] \neq 0$ and $\mathbf{x}[j] \neq 0$, then color(i) = color(j).

<u>Proof:</u> This follows from the notion of counter-determinacy. If $\mathbf{x}[i] \neq 0$ and $\mathbf{x}[j] \neq 0$ and $\operatorname{color}(i) \neq \operatorname{color}(j)$, there there exists a $w \in \Sigma^*$ such that $(q_i, n) \hookrightarrow^{w|s_1} (p_1, n_1)$ and $(q_j, n) \hookrightarrow^{w|s_2} (p_2, n_2)$ for some $p_1, p_2 \in Q, s_1, s_2 \in \mathcal{F}$ and $n_1, n_2 \in \mathbb{N}$ such that $n_1 \neq n_2$. This contradicts the fact that the machine is counter-deterministic.

Therefore, for all $w \in \Sigma^*$, if $(q_i, n) \hookrightarrow^{w|s_1} (p_1, n_1)$ and $(q_j, n) \hookrightarrow^{w|s_2} (p_2, n_1)$ for some $q, p_1, p_2 \in Q, n, n_1 \in \mathbb{N}$ and $s_1, s_2 \in \mathcal{F}$, then $color(p_1) = color(p_2)$. The colors are analogous to the counter-states in the syntactic definition. The transition from one color to another depends solely on the current input symbol and whether the current counter value is zero or non-zero. Hence, in a weighted ODCA the counter transitions are determined by a deterministic one-counter automata where the states represent these colors and transitions represent the transition from one color to another. Now, we formally define the equivalent weighted ODCA $((C, \delta'_0, \delta'_1, p_0), (Q, \lambda, \Delta, \eta))$ as follows:

- lacksquare $C = \{j \mid color(i) = j, i \in [1, |Q|]\}$ is the set of counter states.
- $\delta'_0: C \times \Sigma \times \{0\} \to C \times \{0, +1\}$ and $\delta'_1: C \times \Sigma \times \{1\} \to C \times \{-1, 0, +1\}$ are the deterministic counter transitions. For all $q \in |Q|, a \in \Sigma$ we define $\delta'_1(q, a)$ and $\delta'_0(q, a)$ as: $\delta'_1(q, a) = (p, d)$ if $(q, 1) \stackrel{a|s}{\longleftrightarrow} (p, 1 + d)$ and $\delta'_0(q, a) = (p, d)$ if $(q, 0) \stackrel{a|s}{\longleftrightarrow} (p, d)$ for some $p \in Q, s \in \mathcal{F}$, and $d \in \{-1, 0, +1\}$.
- Let $i \in [1, |Q|]$ such that $\lambda[i] \neq 0$. $p_0 = j$, is the start state for counter transition, where j = color(i).
- $\Delta: \Sigma \times \{0,1\} \to \mathcal{F}^{|Q| \times |Q|}$ gives the transition matrix for all $a \in \Sigma$ and $d \in \{0,1\}$. For $i,j \in [1,|Q|], \ \Delta(a,d)[i][j] = s$, if $\delta_d(q_i,a,q_j,e) = s$ for $q_i,q_j \in Q, s \in \mathcal{F}$ and $e \in \{-1,0,+1\}$.

Hence, we can construct an equivalent weighted ODCA from a given weighted OCA with counter-determinacy in polynomial time. Proving the converse is straightforward.