Glivenko's theorem

Valéry Glivenko was one of several Soviet logicians who, like Kolmogorov, worked on the intersection of algebraic logic and probability theory. There are a few fundamental theorems bearing his name. This note is about his embedding of classical logic in intuitionistic logic.

Specifically, Glivenko proved that

1. THEOREM (1929): If $\vdash_{\mathrm{CPC}} \phi$, then $\vdash_{\mathrm{IPC}} \neg\neg\phi$.

Because the converse of this theorem is obviously also true, Glivenko's theorem assures us that IPC can be used to discover the classical validities. For this reason, it is sometimes suggested that IPC should be thought of as a "stronger" logic than CPC: With it one can see everything that one can see with CPC, and in addition one can make many distinctions that CPC "overlooks" (Gödel sometimes suggested this attitude).

Glivenko's theorem was improved upon in a couple of ways in the decade following its publication, in the "negative translations" of Gentzen and Gödel. But Glivenko's original result remains of interest because (1) it is readily applied in many contexts, (2) it was a breakthrough at the time, introducing completely new concepts, and (3) it can be recast in order to relate logics other than CPC and IPC, such as logics intermediate between these and predicate calculi.

Glivenko's original proof is a fairly strightforward induction on the complexity of the formula ϕ, and this is the approach found in many texts. A more modern presentation uses semantic methods, proving the theorem by relating truth-functional validities and Kripke frames. The following proof seems to be in some ways simpler either of those. (Throughout this note we think of a natural deduction presentation of IPC.)

We observe some preliminary facts.

2. FACT: $\vdash_{\mathrm{IPC}} \neg\neg(\phi \vee \neg\phi)$

3. FACT: $\neg\neg\phi \vdash_{\mathrm{IPC}+p\vee\neg p} \phi$

4. FACT: $\neg\neg\neg\phi \vdash_{\mathrm{IPC}} \neg\phi$

For the proof of the main theorem, suppose $\vdash_{\mathrm{CPC}} \phi$. Then by **3** we have $\vdash_{\mathrm{IPC}+p\vee\neg p} \phi$, hence a proof in IPC+$p \vee \neg p$:

$\psi_1 \vee \neg\psi_1$	axiom
$\ldots$	
$\psi_n \vee \neg\psi_n$	axiom
.	
.	
.	
ϕ	

From this proof one can construct a pure IPC proof:

$\neg\neg(\psi_1 \vee \neg\psi_1)$ fact 2
$\neg\neg(\psi_2 \vee \neg\psi_2)$ fact 2
$\ldots$
$\neg\neg(\psi_{n-1} \vee \neg\psi_{n-1})$ fact 2
$\neg\neg(\psi_n \vee \neg\psi_n)$ fact 2

   $\neg\phi$

     $\psi_1 \vee \neg\psi_1$

       $\psi_2 \vee \neg\psi_2$
       $\ldots$

         $\psi_{n-1} \vee \neg\psi_{n-1}$

           $\psi_n \vee \neg\psi_n$

             .
             .
             .
             ϕ
             $\bot$ $\neg$ -elim
           $\neg(\psi_n \vee \neg\psi_n)$ $\neg$ -int
           $\bot$ $\neg$ -elim

         $\ldots$
         $\bot$ $\neg$ -elim
       $\neg(\psi_2 \vee \neg\psi_2)$ $\neg$ -int
       $\bot$ $\neg$ -elim
     $\neg(\psi_1 \vee \neg\psi_1)$ $\neg$ -int
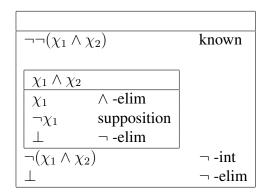 $\bot$ $\neg$ -elim
   $\neg\neg\phi$ $\neg$ -int

That completes the proof of the main theorem. We observe three corollaries:

5. COROLLARY: If $\vdash_{\text{CPC}} \neg\phi$, then $\vdash_{\text{IPC}} \neg\phi$.

   This follows immediately from the main theorem and fact **4**.

6. COROLLARY: CPC is inconsistent only if IPC is.

   For if CPC is inconsistent, then there is a formula ϕ such that $\vdash_{\text{CPC}} \phi$ and $\vdash_{\text{CPC}} \neg\phi$. But then $\vdash_{\text{IPC}} \neg\neg\phi$ and $\vdash_{\text{CPC}} \neg\phi$.

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7. COROLLARY: If ϕ is any formula and $\sigma(\phi)$ is created by substituting $p \supset p$ for some of the sentence letters in ϕ and substituing $p \wedge \neg p$ for the others, then $\vdash_{\text{IPC}} \phi$ if $\vdash_{\text{CPC}} \phi$.
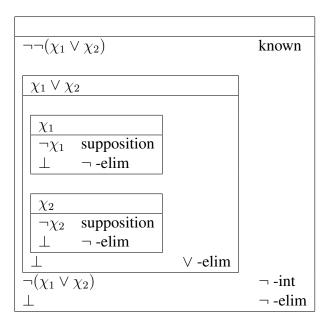
To prove this it is convenient to observe first that if the immediate subformulas χ_i of a formula ψ are each such that either $\vdash_{\text{IPC}} \chi_i$ or $\vdash_{\text{IPC}} \neg\chi_i$, then $\vdash_{\text{IPC}} \psi$ if $\vdash_{\text{CPC}} \psi$ and $\vdash_{\text{IPC}} \neg\psi$ if $\vdash_{\text{CPC}} \neg\psi$: Suppose $\vdash_{\text{CPC}} \neg\psi$. Then $\vdash_{\text{IPC}} \neg\psi$ by corollary **5**. Suppose on the other hand that $\vdash_{\text{CPC}} \psi$, and consider whether ψ is of the form $\chi_1 \wedge \chi_2$, $\chi_1 \vee \chi_2$, $\chi_1 \supset \chi_2$, or $\neg\chi$.

a (ψ is of the form $\chi_1 \wedge \chi_2$.) If $\vdash_{\text{IPC}} \neg\chi_1$, then because by the main theorem, $\vdash_{\text{IPC}} \neg\neg\psi$, there is an IPC proof:
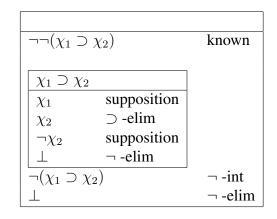
$\neg\neg(\chi_1 \wedge \chi_2)$	known
$\quad$ $\chi_1 \wedge \chi_2$	
$\quad$ χ_1 $\qquad$ $\wedge$ -elim	
$\quad$ $\neg\chi_1$ $\qquad$ supposition	
$\quad$ $\bot$ $\qquad$ $\neg$ -elim	
$\neg(\chi_1 \wedge \chi_2)$	$\neg$ -int
$\bot$	$\neg$ -elim

Therefore, $\vdash_{\text{IPC}} \chi_1$. By the same reasoning, $\vdash_{\text{IPC}} \chi_1$. Hence $\vdash_{\text{IPC}} \psi$.

b (ψ is of the form $\chi_1 \vee \chi_2$.) If $\vdash_{\text{IPC}} \neg\chi_1$ and $\vdash_{\text{IPC}} \neg\chi_2$, then because by the main theorem, $\vdash_{\text{IPC}} \neg\neg\psi$, there is an IPC proof:

$\neg\neg(\chi_1 \vee \chi_2)$	known
$\quad$ $\chi_1 \vee \chi_2$	
$\quad\quad$ χ_1	
$\quad\quad$ $\neg\chi_1$ $\quad$ supposition	
$\quad\quad$ $\bot$ $\qquad$ $\neg$ -elim	
$\quad\quad$ χ_2	
$\quad\quad$ $\neg\chi_2$ $\quad$ supposition	
$\quad\quad$ $\bot$ $\qquad$ $\neg$ -elim	
$\quad$ $\bot$ $\qquad\qquad$ $\vee$ -elim	
$\neg(\chi_1 \vee \chi_2)$	$\neg$ -int
$\bot$	$\neg$ -elim

Therefore, either $\vdash_{\text{IPC}} \chi_1$ or $\vdash_{\text{IPC}} \chi_2$, and either way $\vdash_{\text{IPC}} \psi$.

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c (ψ is of the form $\chi_1 \supset \chi_2$.) If $\vdash_{\text{IPC}} \chi_1$ and $\vdash_{\text{IPC}} \neg\chi_2$, then because by the main theorem, $\vdash_{\text{IPC}} \neg\neg\psi$, there is an IPC proof:

$\neg\neg(\chi_1 \supset \chi_2)$	known
$\boxed{\begin{array}{ll} \chi_1 \supset \chi_2 \\ \chi_1 & \text{supposition} \\ \chi_2 & \supset\text{-elim} \\ \neg\chi_2 & \text{supposition} \\ \bot & \neg\text{-elim} \end{array}}$	
$\neg(\chi_1 \supset \chi_2)$	$\neg$-int
$\bot$	$\neg$-elim

Therefore, either $\vdash_{\text{IPC}} \neg\chi_1$ or $\vdash_{\text{IPC}} \chi_2$, and either way $\vdash_{\text{IPC}} \psi$ (**verify this!**).

d (ψ is of the form $\neg\chi$.) Then by corollary **5**, $\vdash_{\text{IPC}} \psi$.

Consider now the formula $\sigma(\phi)$. Because it contains only finitely many connectives, one need only iterate the result just demonstrated to establish that its immediate subformulas are each such that either $\vdash_{\text{IPC}} \chi_i$ or $\vdash_{\text{IPC}} \neg\chi_i$ ($p \supset p$ and $p \wedge \neg p$ were chosen for σ because each is either provable or refutable in IPC, and obviously any "truth functional" compound of formulas provable or refutable in IPC will itself be either provable or refutable in CPC, hence also in IPC, ...). Therefore if $\vdash_{\text{CPC}} \sigma(\phi)$, then $\vdash_{\text{IPC}} \sigma(\phi)$.