CANTOR ALGEBRAS WITH SINGLE GENERATOR I

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INTRODUCTION

Cantor functions $\varphi_1(x), \ldots, \varphi_n(x)$, $\omega(x_1, \ldots, x_n)$, which can be realized by numbering n-tuples $(x_1, \ldots, x_n)$ of positive integers are related to each other by the following identities (see [1], pp. 63-66):

$$\left.\begin{aligned} \varphi_i(\omega(x_1, \ldots, x_n)) = x_i \qquad (i = 1, \ldots, n), \\ \omega(\varphi_1(x), \ldots, \varphi_n(x)) = x. \end{aligned}\right\} \tag{1}$$

Therefore, an algebra $\langle A, \varphi_1, \ldots, \varphi_n, \omega\rangle$ of the type $\langle 1, \ldots, 1, n\rangle$ $(n \geqslant 2)$, given on any nonempty set A and satisfying the identities (1) is called a Cantor algebra. It is agreed that the type of Cantor algebra be denoted by $\langle\langle n\rangle\rangle$. The manifold of all Cantor algebras $\langle A, \varphi_1, \ldots, \varphi_n, \omega\rangle$ of the given type $\langle\langle n\rangle\rangle$ is denoted by $\mathfrak{A}_{1,n}$.

It was shown in [2] that for any positive integer $n \geqslant 2$ the manifold $\mathfrak{A}_{1,n}$ is minimal. In other words, all nonsingle-element (and hence infinite) Cantor algebras of any specified type $\langle\langle n\rangle\rangle$ $(n \geqslant 2)$ are equationally equivalent, that is, they have the same system of identities which are formal corollaries of the identities (1). It was shown by Akataev in [3] that contrary to the above the manifold $\mathfrak{A}_{1,n}$ $(n \geqslant 2)$ has a continuum of quasibasic manifolds. To investigate further the manifolds $\mathfrak{A}_{1,n}$ $(n \geqslant 2)$ it is essential to consider the construction of single-generator Cantor algebras.

In the present work, it is shown by us that for each $n \geqslant 2$ there exists a continuum of nonisomorphic single-generator Cantor algebras $\langle A, \varphi_1, \ldots, \varphi_n, \omega\rangle$, with no proper subalgebras. Hence it follows that, in particular, for any $n \geqslant 2$ there does not exist a universal countable Cantor algebra of the type $\langle\langle n\rangle\rangle$, that is, an algebra such that any countable Cantor algebra $\langle A, \varphi_1, \ldots, \varphi_n, \omega\rangle$ is isomorphic to a subalgebra of it.

It is also shown that any countable Cantor algebra can be embedded in a Cantor algebra with a single generator.

It is now noted for future reference that any ω-homomorphism (as well as any $\langle \varphi_1, \ldots, \varphi_n\rangle$-homomorphism) of the Cantor algebra $\langle A, \varphi_1, \ldots, \varphi_n, \omega\rangle$ is also a homomorphism of that algebra, that is, it retains all other fundamental operations of the algebra under consideration. For example, if σ is a ω-homomorphism and $x \in A$ then by setting $\sigma(\varphi_i(x)) = y_i$ one obtains

$$\omega(y_1, \ldots, y_n) = \sigma(\omega(\varphi_1(x), \ldots, \varphi_n(x)) = \sigma(x),$$

and hence $\varphi_i(\sigma(x)) = y_i = \sigma(\varphi_i(x)) \quad (i = 1, \ldots, n)$.

1. Construction of Continuum of Pairwise Nonisomorphic Cantor Algebras with Single Generator

Let $\mathbf{A} = \langle A, \varphi_1, \ldots, \varphi_n, \omega\rangle$ be any partial algebra of the type $\langle 1, \ldots, 1, n\rangle$ $(n \geqslant 2)$, which satisfies the following three conditions:

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(i) the unary operations $\varphi_1, \dots, \varphi_n$ are defined in the set A ;

(ii) if for some elements $a_1, \dots, a_n$ of A there is defined in A an element $\omega(a_1, \dots, a_n)$, then

$$\varphi_i(\omega(a_1, \dots, a_n)) = a_i \quad (i = 1, \dots, n);$$

(iii) for any element $a \in A$ the element $\omega(\varphi_1(a), \dots, \varphi_n(a))$ is defined in A and is equal to a .

We set

$$A_0 = A, \; A_{k+1} = A_k + \omega(A_k), \qquad F(A) = \bigcup_{k=0}^{\infty} A_k ,$$

where $\omega(A_k)$ denotes the set of all terms of the form $\omega(b_1, \dots, b_n)\,(b_1, \dots, b_n \in A_k)$ not defined in A_k. By definition, each element b of the set $F(A) \setminus A$ has a unique description of the form

$$b = \omega(b_1, \dots, b_n)\ (b_1, \dots, b_n \in F(A)) \cdot$$

By setting

$$\varphi_i(b) = b_i \qquad (i = 1, \dots, n),$$

one obviously obtains the algebra

$$\boldsymbol{F}(\mathbf{A}) = \langle F(A), \varphi_1, \dots, \varphi_n, \omega \rangle ,$$

which belongs to the manifold $\mathfrak{A}_{1,n}$. We shall call it the free closure of the partial algebra $\mathbf{A}$.

For future reference one notes that for any element f of the algebra $\boldsymbol{F}(\mathbf{A})$ a positive integer $k \geqslant 1$ obviously exists such that

$$\varphi_i^{k}(f) \in A \qquad (i = 1, \dots, n).$$

Let $\mathbf{Y}_{\mathfrak{n}} = \langle Y, \varphi_1, \dots, \varphi_n \rangle$ be a free algebra in the class of all algebras with unary operations $\varphi_1, \dots, \varphi_n$, freely generated by the set X , $|X| = \mathfrak{n}$. By setting $\omega(\varphi_1(y), \dots, \varphi_n(y)) = y$ for any element $y \in Y$, one obtains the partial algebra

$$\mathbf{Y} = \langle Y, \varphi_1, \dots, \varphi_n, \omega \rangle ,$$

which satisfies all the conditions (i), (ii), (iii) since the symbol $\omega(y_1, \dots, y_n)$ is defined in Y only at the points given by

$$(\varphi_1(y), \dots, \varphi_n(y)) \quad (y \in Y).$$

According to [4] the free closure $\boldsymbol{F}_{\mathfrak{n}} = \boldsymbol{F}(\mathbf{Y})$ of a partial algebra $\mathbf{Y}$ is a free algebra of rank $\mathfrak{n}$ in the manifold $\mathfrak{A}_{1,n}$, freely generated by the set X . This can easily be verified directly.

For the above introduced algebras,

$$\mathbf{Y}_{\mathfrak{n}} = \langle Y, \varphi_1, \dots, \varphi_n \rangle \quad \text{и} \quad \boldsymbol{F}_{\mathfrak{n}} = \langle F(Y), \varphi_1, \dots, \varphi_n, \omega \rangle;$$

this notation is retained throughout.

THEOREM 1. For any positive integer $n \geqslant 2$ there exists a continuum of pairwise nonisomorphic Cantor algebras of the type «n» with no proper subalgebras.

Proof. The positive integer $n \geqslant 2$ is kept constant and any group $\mathcal{G}$ with n generators $a_1, \dots, a_n$ is considered. By setting for each element g of $\mathcal{G}$

$$\varphi_i(g) = a_i g \qquad (i = 1, \dots, n),$$

the unary algebra $\langle \mathcal{G}, \varphi_1, \ldots, \varphi_n \rangle$ is obtained. Since in the latter the equality $\varphi_i(g) = \varphi_i(h)$ implies $g = h$, therefore by defining

$$\omega(g_1, \ldots, g_n) = g \iff (\exists g)(g_1 = \varphi_1(g) \,\&\, \ldots \,\&\, g_n = \varphi_n(g)), \tag{2}$$

where $g, g_1, \ldots, g_n \in \mathcal{G}$, a partial algebra $\langle \mathcal{G}, \varphi_1, \ldots, \varphi_n, \omega \rangle$ is obtained with the properties (i), (ii), (iii). Let $F(\mathcal{G}; a_1, \ldots, a_n)$ be the free closure in the previously specified sense of the partial algebra $\langle \mathcal{G}, \varphi_1, \ldots, \varphi_n, \omega \rangle$.

<u>LEMMA 1.</u> If $a_1, \ldots, a_n$ are periodic generators of the group $\mathcal{G}$ then the Cantor algebra $F(\mathcal{G}; a_1, \ldots, a_n)$ is generated by any of its elements and has thus no proper subalgebras.

Indeed, in this case any element $g \in \mathcal{G}$ can be written as a product of positive powers of the generators $a_1, \ldots, a_n$. The abbreviated notation $\Phi(a_1, \ldots, a_n)$ is used for it. Then the element g can be expressed in terms of unity 1 of the group $\mathcal{G}$ with the aid of the operations $\varphi_1, \ldots, \varphi_n$ in the form

$$g = \Phi(\varphi_1, \ldots, \varphi_n)(1).$$

Since the set $\mathcal{G}$ generates the algebra $F(\mathcal{G}; a_1, \ldots, a_n)$ this algebra can therefore be generated by the element 1.

Let f be any element of the algebra $F(\mathcal{G}; a_1, \ldots, a_n)$. Then for some positive power φ_1^{K} of the operator φ_1 the element $\varphi_1^{K}(f) \in \mathcal{G}$, i.e.,

$$\varphi_1^{K}(f) = \varphi_{i_1}^{K_1} \ldots \varphi_{i_s}^{K_s}(1) \quad (i_1, \ldots, i_s \in \{1, \ldots, n\}).$$

By assumption, the operations $\varphi_1, \ldots, \varphi_n$ in $\mathcal{G}$ can be inverted. Therefore, the unit 1 can be expressed in terms of f only by using the operations $\varphi_1, \ldots, \varphi_n$. Therefore the element f also generates the algebra $F(\mathcal{G}; a_1, \ldots, a_n)$.

<u>LEMMA 2.</u> If A and B are groups with periodic generators $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$, respectively then the Cantor algebras $F(A; a_1, \ldots, a_n)$ and $F(B; b_1, \ldots, b_n)$ are isomorphic if and only if there exists an isomorphic mapping σ of the group A into the group B so that

$$\sigma(a_1) = b_1, \ldots, \sigma(a_n) = b_n. \tag{3}$$

In fact, let us suppose that there exists an isomorphic mapping $\sigma: A \to B$, such that the relations (3) are satisfied. This is an ismorphism of the partial algebra $\langle A, \omega \rangle$ onto the partial algebra $\langle B, \omega \rangle$. Indeed, if $x_1, \ldots, x_n \in A$, then in view of (2) there exists an element $\omega(x_1, \ldots, x_n)$ in A if and only if there exists in B an element $\omega(\sigma(x_1), \ldots, \sigma(x_n))$, and the equality takes place:

$$\sigma(\omega(x_1, \ldots, x_n)) = \omega(\sigma(x_1), \ldots, \sigma(x_n)).$$

For example, if in A there exists an element $x_0 = \omega(x_1, \ldots, x_n)$, then $x_i = a_i x_0$ $(i = 1, \ldots, n)$. Let $y_i = \sigma(x_i)$ $(i = 0, 1, \ldots, n)$. Since $y_i = b_i y_0$ $(i = 1, \ldots, n)$, therefore in B an element $\omega(y_1, \ldots, y_n)$ is defined and it is equal to y_0.

If the mapping σ has already been extended to the isomorphism of the partial algebra $\langle A_K, \omega \rangle$ onto the partial algebra $\langle B_K, \omega \rangle$ and also if $f \in A_{K+1} \setminus A_K$, then f has a unique description,

$$f = \omega(x_1, \ldots, x_n)(x_1, \ldots, x_n \in A_K),$$

We set

$$\sigma(f) = \omega(\sigma(x_1), \ldots, \sigma(x_n)).$$

Obviously, σ is an isomorphism of the partial algebra $\langle A_{\kappa+1}, \omega \rangle$ onto the partial algebra $\langle B_{\kappa+1}, \omega \rangle$.

The union σ of thus constructed mappings $A_\kappa \longrightarrow B_\kappa$ is a one-to-one mapping of the algebra $\mathbf{F}(A; a_1, \ldots, a_n)$ onto the algebra $\mathbf{F}(B; b_1, \ldots, b_n)$, which preserves the operation and also satisfies the relation (3). In view of our remark in the introduction, σ is the sought isomorphism of the Cantor algebras under consideration.

Conversely, let an isomorphism ρ be given of the algebra $\mathbf{F}(A; a_1, \ldots, a_n)$ onto the algebra $\mathbf{F}(B; b_1, \ldots, b_n)$. We set $\tau = \rho(1)$. In view of the way the algebra $\mathbf{F}(B; b_1, \ldots, b_n)$ was constructed one can find a positive integer $\kappa > 0$, such that $\varphi_1^\kappa(\tau) = b \in B$. Hence

$$\rho(a_1^\kappa) = \rho(\varphi_1^\kappa(1)) = \varphi_1^\kappa(\rho(1)) = b.$$

By selecting the number $m \geq 0$ such that $a_1^{\kappa+m} = 1$ one obtains

$$\tau = \rho(1) = \rho(\varphi_1^m(a_1^\kappa)) = \varphi_1^m(b) = b_1^m b \in B.$$

If $x = \varphi_{i_1}^{\kappa_1} \ldots \varphi_{i_s}^{\kappa_s}(1)$ is an arbitrary element of A then also

$$\rho(x) = \varphi_{i_1}^{\kappa_1} \ldots \varphi_{i_s}^{\kappa_s}(\tau) \in B.$$

For the inverse mapping ρ^{-1} one obtains $\rho^{-1}(y) \in A$ for any element $y \in B$. Thus, the restriction $\rho|A$ is a one-to-one mapping of A onto B.

To complete the proof of Lemma 2 it only remains to verify that the mapping $\sigma : A \longrightarrow B$, defined by the formula

$$\sigma(x) = \rho(x)\, \tau^{-1} \qquad (x \in A),$$

is a group homomorphism transforming a_i into b_i $(i = 1, \ldots, n)$.

Let $x = \Phi(\varphi_1, \ldots, \varphi_n)(1)$, $y = \Psi(\varphi_1, \ldots, \varphi_n)(1)$ be arbitrary elements of A described in the form of terms of 1 with the aid of the operations $\varphi_1, \ldots, \varphi_n$. Then

$$xy = \Phi(\varphi_1, \ldots, \varphi_n)(\Psi(\varphi_1, \ldots, \varphi_n)(1)),$$

hence

$$\sigma(xy) = \rho(xy)\tau^{-1} = \Phi(b_1, \ldots, b_n)\Psi(b_1, \ldots, b_n) = \sigma(x)\sigma(y).$$

One also has

$$\sigma(a_i) = \rho(\varphi_i(1))\tau^{-1} = (b_i \tau)\tau^{-1} = b_i \qquad (i = 1, \ldots, n).$$

The lemma has thus been proved.

To prove Theorem 1 it only remains to mention that according to the results of Neiman ([5], p. 245) and Levin [6] the set of all pairwise nonisomorphic groups with two periodic generators is of continuum power.

From Theorem 1 there follows directly

COROLLARY. For any positive integer $n \geq 2$ no universal countable Cantor algebra exists of the type «n».

2. Embedding Theorem

THEOREM 2. Any countable Cantor algebra $\langle A, \varphi_1, \ldots, \varphi_n, \omega \rangle$ $(n \geq 2)$ can be embedded in a Cantor algebra with single generator.

Proof. By virtue of the isomorphism theorem (see [7], p. 65) it is sufficient to show as in other similar cases that a free Cantor algebra $\boldsymbol{F}_1$ of rank 1 (of the specified type $\langle\!\langle n \rangle\!\rangle$) contains a subalgebra $\boldsymbol{F}_\infty$ such that 1) $\boldsymbol{F}_\infty$ is a free Cantor algebra of infinite rank and 2) for any congruence $\boldsymbol{F}_\infty$ in η there exists a congruence ϑ in $\boldsymbol{F}_1$, satisfying the relation

$$\vartheta \cap (\boldsymbol{F}_\infty \times \boldsymbol{F}_\infty) = \eta .$$

Before proving the above proposition we shall consider a congruence ρ of an arbitrary algebraic system $\mathbf{A} = \langle A, \Omega \rangle$ generated by a specified relation $\rho_0 \subseteq A \times A$. By analogy with the subgroups (see [8], p. 18) it admits the following description.

We associate with each element a of the set the unknown x_a. To the set of the unknowns $\{x_a \mid a \in A\}$ another unknown x is adjoined and a term $T(x, x_{a_1}, \ldots, x_{a_m})$ is considered of the signature Ω of the unknowns x, $x_{a_1}, \ldots, x_{a_m}$, which contains only one appearance of the unknown x. A derivative operation in $\mathbf{A}$, defined by the formula

$$E(x) = T(x, a_1, \ldots, a_m),$$

is called an elementary translation of the system $\mathbf{A}$ (see [9]).

Let

$$\rho_1 = \rho_0 \cup \rho_0^{-1} \cup \iota_A ,$$

where ι_A is the equality relation in A. For $a, b \in A$ we set $a \rho_2 b$ if and only if there exist elements $u, v \in A$ and an elementary translation $E(x)$ of the system $\mathbf{A}$, such that

$$a = E(u), \quad b = E(v) \quad \text{and} \quad (u, v) \in \rho_1 .$$

It is easily verified that the relation

$$\rho = \rho_2 \cup \rho_2^2 \cup \rho_2^3 \cup \ldots$$

is the least congruence in $\mathbf{A}$, which contains the given relation ρ_0, that is, ρ is a congruence generated by the relation ρ_0.

LEMMA 3. Let $\mathbf{A}$ be an arbitrary unary algebra and $\boldsymbol{B}$ a subalgebra of the latter. Then for any congruence $\boldsymbol{B}$ in β there exists a congruence $\alpha = \alpha(\beta)$ in $\mathbf{A}$, such that

$$\alpha \cap (B \times B) = \beta .$$

Indeed, let α be a congruence in $\mathbf{A}$, which is generated by the relation β. Then

$$\alpha = \beta_2 \cup \beta_2^2 \cup \beta_2^3 \cup \ldots, \tag{4}$$

where β_2 is the set of pairs of the form $(E(u), E(v))$, in which $E(x)$ is any elementary translation in $\mathbf{A}$, such that

$$(u, v) \in \beta_1 = \beta \cup \iota_A .$$

Obviously, $\beta \subseteq \alpha \cap (B \times B)$.

Conversely, let $(a, b) \in \alpha \cap (B \times B)$. Then $a, b \in B$ and $a \alpha b$. Consequently, there exist elements $c_1, \ldots, c_k$ in A, such that

$$a \beta_2 c_1 \beta_2 c_2 \ldots c_k \beta_2 b .$$

It will be shown that $c_1, \ldots, c_k \in B$.

Let $a=E(u)$, $c_1=E(v)$, where $(u,v)\in\beta\cup\iota_A$. If $(u,v)\in\beta$, then $v\in B$. Since the fundamental operations in $\mathbf{A}$ are unary and since $\mathbf{B}$ is a subalgebra in $\mathbf{A}$, it is also obtained that $c_1=E(v)\in B$. If, however, $u=v\in A$, then again $c_1=a\in B$. Similarly, $c_2,\dots,c_k$ belong to B. In view of the proper implication

$$x\beta_2 y \ \&\ x,y\in B \Longrightarrow x\beta y,$$

one has $a\beta b$ and therefore $\alpha\cap(B\times B)\subseteq\beta$. Consequently, $\beta=\alpha\cap(B\times B)$. Lemma 3 has been proved.

Consider now the algebras $\mathbf{Y}_n=\langle Y,\varphi_1,\dots,\varphi_n\rangle$, $\mathbf{F}_n=\langle F(Y),\varphi_1,\dots,\varphi_n,\omega\rangle$, defined in Section 1.

A congruence ϑ of the algebra $\mathbf{Y}_n$ is said to be a Cantor congruence if for any elements u, v of Y the proper implication holds,

$$(\varphi_1(u)\vartheta\varphi_1(v))\ \&\ \dots\ \&\ (\varphi_n(u)\vartheta\varphi_n(v))\longrightarrow u\vartheta v.$$

LEMMA 4. For any congruence ϑ of the algebra $\mathbf{F}_n$ the restriction

$$\vartheta|_Y=\vartheta\cap(Y\times Y)$$

is a Cantor congruence in $\mathbf{Y}_n$. Conversely, for any Cantor congruence ϑ_0 of the algebra $\mathbf{Y}_n$ there exists a unique congruence ϑ of the algebra $\mathbf{F}_n$, which satisfies the relation

$$\vartheta\cap(Y\times Y)=\vartheta_0.$$

Indeed, the first assertion of Lemma 4 follows directly from the fundamental identity $\omega(\varphi_1(x),\dots,\varphi_n(x))=x$, which is basic for any Cantor algebra of the type $\langle\langle n\rangle\rangle$.

Conversely, let us suppose that a Cantor congruence ϑ_0 of the algebra $\mathbf{Y}_n$ is given and let σ_0 : $\mathbf{Y}_n\longrightarrow\mathbf{Y}_n/\vartheta_0$ be the natural homomorphism. Now in the set $A=Y/\vartheta_0$ the partial operation is defined

$$\omega(a_1,\dots,a_n)=a\Longleftrightarrow a_1=\varphi_1(a)\ \&\ \dots\ \&\ a_n=\varphi_n(a),$$

where a, $a_1,\dots,a_n$ are elements of A. Since the congruence ϑ_0 is a Cantor congruence, the element $[a]$, if it does exist, is determined uniquely by the given elements $a_1,\dots,a_n$. One obtains the partial algebra $\mathbf{A}=\langle A,\varphi_1,\dots,\varphi_n,\omega\rangle$, with the properties (i), (ii), (iii) (see Section 1). Let $F(\mathbf{A})$ be the free closure for $\mathbf{A}$. It will be shown that the mapping σ_0 can be extended to the homomorphism of the algebra $\mathbf{F}_n=\langle F(Y),\varphi_1,\dots,\varphi_n,\omega\rangle$ into the algebra

$$\mathbf{F}(\mathbf{A})=\langle F(A),\varphi_1,\dots,\varphi_n,\omega\rangle.$$

The term $\omega(y_1,\dots,y_n)(y_1,\dots,y_n\in Y)$ is defined in Y only if an element $y\in Y$ exists such that $y_1=\varphi_1(y),\dots,y_n=\varphi_n(y)$. Under these conditions $\omega(y_1,\dots,y_n)=y$, and therefore

$$\sigma_0(\omega(y_1,\dots,y_n))=\sigma_0(y)=\omega(\varphi_1(\sigma_0(y)),\dots,\varphi_n(\sigma_0(y)))=$$
$$\omega(\sigma_0(y_1),\dots,\sigma_0(y_n)).$$

Thus σ_0 is a homomorphism of the partial algebra $\langle Y,\omega\rangle$ into the algebra $\langle F(A),\omega\rangle$. Suppose that it has already been proved that one can extend the mapping σ_0 to the homomorphism σ of the partial algebra $\langle Y_k,\omega\rangle$ into the algebra $\langle F(A),\omega\rangle$. If $y_1,\dots,y_n\in Y_k$ and if the term $\omega(y_1,\dots,y_n)$ is defined in Y_k, then one sets

$$\sigma(\omega(y_1,\dots,y_n))=\omega(\sigma(y_1),\dots,\sigma(y_n)).$$

In this manner a homomorphism σ is constructed of the partial algebra $\langle Y_{\kappa+1}, \omega\rangle$ into the algebra $\langle F(A), \omega\rangle$ which is an extension of σ_0. Since $F(Y) = \bigcup Y_\kappa$, there exists a homomorphism σ of the algebra $\langle F(Y), \omega\rangle$ into the algebra $\langle F(A), \omega\rangle$ extending the given mapping σ_0. Bearing in mind the remark made in the introduction one can say that σ is a homomorphism of the algebra $\mathbf{F}_{\#}$ into the algebra $\mathbf{F(A)}$.

Let ϑ be a nuclear congruence in $\mathbf{F}_{\#}$ for σ. Then $\vartheta|_Y = \vartheta_0$, since σ is an extension of σ_0.

Let η be any congruence of the algebra $\mathbf{F}_{\#}$, such that $\eta|_Y = \vartheta_0$. It will be shown that $\eta = \vartheta$.

Suppose that it has already been shown that $\eta|_{Y_\kappa} = \vartheta|_{Y_\kappa}$. Consider now the set $Y_{\kappa+1} = Y_\kappa + \omega(Y_\kappa)$. If $f \vartheta g$ $(f, g \in Y_{\kappa+1})$, then

$$\varphi_i(f)\, \vartheta\, \varphi_i(g) \quad (i = 1, \ldots, n).$$

Since each element of $\omega(Y_\kappa)$ can be represented as $\omega(y_1, \ldots, y_n)$, where $y_1, \ldots, y_n \in Y_\kappa$, therefore $\varphi_i(f)$ and $\varphi_i(g)$ belong to the set Y_κ and thus

$$\varphi_i(f)\, \eta\, \varphi_i(g) \quad (i = 1, \ldots, n).$$

Hence one obtains

$$\omega(\varphi_1(f), \ldots, \varphi_n(f))\, \eta\, \omega(\varphi_1(g), \ldots, \varphi_n(g)),$$

that is $f \eta g$. Thus, $\vartheta|_{Y_{\kappa+1}} \subseteq \eta|_{Y_{\kappa+1}}$. In view of symmetry the converse inclusion also holds and hence, $\eta|_{Y_{\kappa+1}} = \vartheta|_{Y_{\kappa+1}}$. Consequently, $\eta = \vartheta$, Lemma 4 has been proved.

Consider now the free Cantor algebra $\mathbf{F}_1 = \langle F(Y), \varphi_1, \ldots, \varphi_n, \omega\rangle$ (of the given type $\langle\langle n\rangle\rangle$) freely generated by the element x. The absolutely free unary algebra $\mathbf{Y}_1 = \langle Y, \varphi_1, \ldots, \varphi_n\rangle$, freely generated by the element x contains a free subalgebra $\mathbf{H} = \langle H, \varphi_1, \ldots, \varphi_n\rangle$ of infinite rank. One selects as $\mathbf{H}$ the subalgebra in $\mathbf{Y}_1$, generated by the elements

$$h_1 = \varphi_2\varphi_1(x),\ h_2 = \varphi_2\varphi_1^2(x),\ h_3 = \varphi_2\varphi_1^3(x), \ldots.$$

Since $\Phi_i(h_i) = \Phi_j(h_j)$ of two $\langle\varphi_1, \ldots, \varphi_n\rangle$-terms in $\mathbf{H}$ is only possible if $i = j$, therefore the elements $h_1, h_2, \ldots$ freely generate H (see [7], p. 313). Let $\mathbf{F}_\infty = \langle F_\infty, \varphi_1, \ldots, \varphi_n, \omega\rangle$ be the subalgebra of the Cantor algebra $\mathbf{F}_1$, generated by the elements $h_1, h_2, \ldots$. The following proposition will be proved.

<u>LEMMA 5.</u> $F_\infty \cap Y = H$.

It is obvious that $F_\infty \cap Y \supseteq H$. Let $f \in (F_\infty \cap Y)$. Since it is an element of Y, f has a unique description $f = \Phi(x)$ in the form of $\langle\varphi_1, \ldots, \varphi_n\rangle$-term of x. On the other hand, f can be expressed as an $\langle\varphi_1, \ldots, \varphi_n, \omega\rangle$-term of finite number of generators $h_1, \ldots, h_m$ of the algebra $\mathbf{F}_\infty$. It is assumed that this term is already reduced, that is, that it contains no subterms of the form $\omega(\varphi_1(t), \ldots, \varphi_n(t))$, $\varphi_i(\omega(u_1, \ldots, u_n))$. If it does not contain ω, then it is a $\langle\varphi_1, \ldots, \varphi_n\rangle$-term $f = \Psi(h_i)$ and therefore, $f \in H$. Let $f = \omega(v_1, \ldots, v_n)$, where $v_1, \ldots, v_n$ are $\langle\varphi_1, \ldots, \varphi_n, \omega\rangle$-terms with a smaller number of appearances of the operator ω. One can find a positive integer $\kappa \geq 1$, such that

$$\varphi_i^\kappa(f) \in H \quad (i = 1, \ldots, n).$$

Consequently, the elements $\varphi_i^\kappa(f)$ can be represented by $\langle\varphi_1, \ldots, \varphi_n\rangle$-terms of $h_1, h_2, \ldots$:

$$\varphi_i^\kappa(f) = \Phi_i(h_{j(i)}) \quad (i = 1, \ldots, n).$$

One obtains

$$\varphi_i^{\kappa}(\Phi(x)) = \Phi_i(\varphi_2 \varphi_1^{j(i)}(x)) \qquad (i = 1, \ldots, n).$$

Hence, in view of the fact that the algebra $\langle Y, \varphi_1, \ldots, \varphi_n \rangle$, $j(1) = j(2) = \ldots = j(n) = j$ is absolutely free and from the equalities

$$\varphi_i^{\kappa}(\Phi(x)) = \Phi_i(\varphi_2 \varphi_1^{j}(x)) \qquad (i = 1, \ldots, n;\ n \geq 2)$$

it follows that the element $f = \Phi(x)$ is equal to a $\langle \varphi_1, \ldots, \varphi_n \rangle$-term of h_j, that is, $f \in H$. Lemma 5 has been proved.

It is noted that by virtue of Lemma 5 and in view of the results in [4] (see Theorem 2 and its Corollary 1) F_∞ is a free Cantor algebra of infinite rank. Indeed, as shown in [4], F_∞ can be freely generated by the basis of absolutely free algebra $\langle F_\infty \cap H;\ \varphi_1, \ldots, \varphi_n \rangle$, that is, by the elements $h_1, h_2, \ldots$. To be able, however, to avail oneself of Lemma 4 one must make sure that F_∞ is a free closure for H in the sense of Section 1. Since F_∞ and $F(H)$ are both generated by the elements $h_1, h_2, \ldots$, it is sufficient to show that $F(H)$ is a subalgebra of the algebra F_1. This in turn can be done by verifying that $\langle F(H), \omega \rangle$ is a subalgebra of the algebra $\langle F(Y), \omega \rangle$.

Let

$$Y_0 = Y,\ Y_{\kappa+1} = Y_\kappa + \omega(Y_\kappa); \quad H_0 = H,\ H_{\kappa+1} = H_\kappa + \omega(H_\kappa).$$

Then $F(Y) = \bigcup Y_\kappa$, $F(H) = \bigcup H_\kappa$. The operation ω in $F(Y)$ and $F(H)$ is denoted respectively by $\omega_{F(Y)}$ and $\omega_{F(H)}$. An element $\omega_{F(H)}(u_1, \ldots, u_n)$ is defined in H for $u_1, \ldots, u_n$ of H if and only if there exists an element $u \in H$, such that $u_1 = \varphi_1(u), \ldots, u_n = \varphi_n(u)$. Therefore the partial operation $\omega_{F(Y)}$ in Y generates a partial operation $\omega_{F(H)}$ in H. However, if for the elements $u_1, \ldots, u_n$ of H the element $\omega_{F(H)}(u_1, \ldots, u_n)$ is not defined in H, then by Lemma 5 the element $\omega_{F(Y)}(u_1, \ldots, u_n)$ is not defined in Y either, both these elements being identical with a term $\omega(u_1, \ldots, u_n)$ of $\omega(Y)$. Thus, $H \subseteq Y$, $\omega(H) \subseteq \omega(Y)$, and

$$\omega_{F(Y)}(u_1, \ldots, u_n) = \omega_{F(H)}(u_1, \ldots, u_n) \tag{5}$$

for all elements $u_1, \ldots, u_n$ of H. Suppose that it has already been shown that $H_\kappa \subseteq Y_\kappa$, $\omega(H_\kappa) \subseteq \omega(Y_\kappa)$ and that the equality (5) holds for all $u_1, \ldots, u_n$ of H_κ.

Let $u_1, \ldots, u_n \in H_{\kappa+1}$. If the element $\omega_{F(H)}(u_1, \ldots, u_n)$ is defined in $H_{\kappa+1}$, then $u_1, \ldots, u_n \in H_\kappa$ and the equality (5) is valid by our inductive hypothesis. Let now the element $\omega_{F(H)}(u_1, \ldots, u_n)$ be not defined in $H_{\kappa+1}$. It will be shown that the element $\omega_{F(Y)}(u_1, \ldots, u_n)$ is not defined in $Y_{\kappa+1}$ and therefore both these elements are identical with a term $\omega(u_1, \ldots, u_n)$ of $\omega(Y_{\kappa+1})$. Indeed, let us assume that the element $\omega_{F(Y)}(u_1, \ldots, u_n)$ is defined in $Y_{\kappa+1}$. Then $u_1, \ldots, u_n \in Y_\kappa$. From the inclusion $\omega(H_\kappa) \subseteq \omega(Y_\kappa)$ one obtains $Y_\kappa \cap \omega(H_\kappa) = \emptyset$ and hence

$$Y_\kappa \cap H_{\kappa+1} = (Y_\kappa \cap H_\kappa) + (Y_\kappa \cap \omega(H_\kappa)) = H_\kappa.$$

Consequently, $u_1, \ldots, u_n \in H_\kappa$, and the element $\omega_{F(H)}(u_1, \ldots, u_n)$ is defined in $H_{\kappa+1}$ which is contrary to our assumption. Thus, $H_{\kappa+1} \subseteq Y_{\kappa+1}$, $\omega(H_{\kappa+1}) \subseteq \omega(Y_{\kappa+1})$, and the equality (5) is valid for all elements $u_1, \ldots, u_n$ of $H_{\kappa+1}$. This proves that $F_\infty = F(H)$.

Let η be any congruence of the algebra F_∞. The restriction

$$\beta = \eta \cap (H \times H) \tag{6}$$

is a Cantor congruence in $\boldsymbol{H}=\langle H, \varphi_1, \dots, \varphi_n\rangle$. By Lemma 3 the congruence α of the algebra $\mathbf{Y}_1 = \langle Y, \varphi_1, \dots, \varphi_n\rangle$, generated by the relation β, satisfies the relation

$$\beta = \alpha \cap (H \times H).$$

It will be shown that with such a selection of $\boldsymbol{H}$ the congruence α of the algebra $\langle Y, \varphi_1, \dots, \varphi_n\rangle$ is again a Cantor congruence.

Let $y_1, y_2 \in Y$ and $\varphi_i(y_1)\alpha\varphi_i(y_2)$ $(i=1,\dots,n)$. Since the congruence α is generated by the relation β therefore the relation (4) is valid. Therefore, there exist elements $z_{ij} \in Y$ $(i=1,\dots,n;\ j=1,\dots,\kappa)$, such that

$$\varphi_i(y_1)\beta_2 z_{i1}\beta_2 \ \cdots\ \beta_2 z_{i\kappa}\beta_2\varphi_i(y_2). \tag{7}$$

It is required to show that $y_1\alpha y_2$. The proof is by induction with respect to the nonnegative integer κ.

It is first noted that for any elements z_1 and z_2 of the algebra $\mathbf{Y}_1$ the proper implication holds,

$$z_1\beta_2 z_2 \Longrightarrow (z_1=z_2)\vee(z_1, z_2 \in H \ \&\ z_1\beta z_2). \tag{8}$$

Indeed, let $z_1\beta_2 z_2$. Then there exists an elementary translation $E(t)$ of the algebra $\mathbf{Y}_1$ and elements u, v of this algebra so that

$$z_1 = E(u), \quad z_2 = E(v) \quad \text{and} \quad (u,v)\in\beta_1=\beta\cup\iota_Y$$

Thus, if $u=v$, then $z_1=z_2$. However, if $(u,v)\in\beta$, then $u,v\in H$; hence since the fundamental operations of the algebra $\mathbf{Y}_1$ are unary, the elements $z_1=E(u)$ and $z_2=E(v)$ also belong to the subalgebra $\boldsymbol{H}$ and $z_1\beta z_2$.

Now, let $\kappa=0$. Then for each $i=1,\dots,n$, in view of (7) and (8), one has

$$\varphi_i(y_1)=\varphi_i(y_2) \qquad \text{or} \qquad \varphi_i(y_1), \varphi_i(y_2)\in H \text{ and } \varphi_i(y_1)\beta\varphi_i(y_2).$$

If even for one value of i the equality $\varphi_i(y_1)=\varphi_i(y_2)$ is valid then in view of the fact that the algebra $\mathbf{Y}_1$ is absolutely free one obtains $y_1=y_2$ and hence $y_1\alpha y_2$. Let for each $i=1,\dots,n$ the elements $\varphi_i(y_1)$ and $\varphi_i(y_2)$ be in H and $\varphi_i(y_1)\beta\varphi_i(y_2)$. Then by Lemma 5 one has

$$y_1 = \omega(\varphi_1(y_1), \dots, \varphi_n(y_1)) \in (F_\infty\cap Y) = H.$$

Similarly, $y_2\in H$. Moreover, since the congruence β in H is a Cantor congruence one obtains $y_1\beta y_2$ and thus certainly $y_1\alpha y_2$.

Let us assume that the proposition $y_1\alpha y_2$ has already been proved for the case $\kappa-1$ and let the relations (7) be true for all $i=1,\dots,n$. We shall make use of the remark (8). If $z_{i\kappa}=\varphi_i(y_2)$ for each $i=1,\dots,n$, then

$$\varphi_i(y_1)\beta_2 z_{i1}\beta_2\ \cdots\ \beta_2 z_{i,\kappa-1}\beta_2\varphi_i(y_2) \quad (i=1,\dots,n) \tag{9}$$

and by inductive hypothesis $y_1\alpha y_2$. Let the elements $z_{i\kappa}$, $\varphi_i(y_2)$ belong to H and $z_{i\kappa}\beta\varphi_i(y_2)$ for at least one value of $i=1,\dots,n$. Then for each i one obtains in accordance with the formula (8) $z_{i,\kappa-1}\beta z_{i\kappa}$ hence $z_{i,\kappa-1}\beta\varphi_i(y_2)$ and, of course, $z_{i,\kappa-1}\beta_2\varphi_i(y_2)$. Thus the relations (9) are again true and again by inductive hypothesis $y_1\alpha y_2$. It has been shown that α in $\mathbf{Y}_1$ is a Cantor congruence.

The proof of Theorem 2 is now completed in the following manner. By Lemma 4 there exists a congruence ϑ of the algebra $\boldsymbol{F}_1$ which satisfies the relation

$$\vartheta\cap(Y\times Y)=\alpha$$

The intersection

$$\gamma = \vartheta \cap (F(H) \times F(H))$$

is a congruence of the algebra F_∞ for which

$$\gamma \cap (H \times H) = \vartheta \cap (H \times H) = \alpha \cap (H \times H) = \beta. \tag{10}$$

Since $F_\infty = F(H)$, therefore by Lemma 4 a unique congruence exists in F_∞ which generates the Cantor congruence β in H. Thus, from (6) and (10) one obtains that $\gamma = \eta$, that is, $\vartheta \cap (F_\infty \times F_\infty) = \eta$. Theorem 2 has been proved.

A simple proposition similar to Theorem 2 follows from the above considerations:

COROLLARY. For $n \geqslant 2$ any countable unary algebra $\langle A, \varphi_1, \ldots, \varphi_n \rangle$ can be embedded in a unary algebra of the same type with single generator.

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