THE GENERALIZED GAMMA FUNCTIONS.

By Emil L. Post.

Introduction.

The difference equation

$$\phi(z + 1) = f(z)\phi(z) \tag{1}$$

has been studied directly* and indirectly† through

$$\phi(z + 1) - \phi(z) = \psi(z)$$

in the cases where $f(z)$ is a meromorphic function. In the present paper a solution of (1) is obtained under an entirely different assumption with regard to $f(z)$. One class of functions satisfying this condition is of the form

$$g(z)h(z)e^{\psi(z)}$$

where $g(z)$ is regular at infinity, and $h(z)$ and $\psi(z)$ are any algebraic functions, Abelian integrals, or finite combinations of these. It is also attempted to bring out the similarity of the solution and its properties to those of the ordinary Gamma function.

In part I a solution of (1) is obtained as an infinite product. An asymptotic expression is obtained for it, as well as an infinite integral. In part II a number of relations are obtained of which the generalization of the multiplication theorem of the ordinary Gamma function is characteristic.

Part I: Fundamental Expansions.

1. Construction of the Gaussian Form of the Generalized Gamma Functions. Let $f(z)$ satisfy the following two conditions: (a) that $\log f(z)$ be analytic in a sector enclosing the positive end of the real axis; (b) that for some value of r, a positive real value of ϵ may be found such that

$$\lim_{p \to +\infty} p^{1+\epsilon} \frac{d^{r+1}}{dz^{r+1}} \log f(z + p) = 0$$

uniformly over any finite region of the z plane. Under these conditions a

* Mellin, Acta Math., vol. 8 (1886), pp. 37–80; Barnes, Proc. London Math. Soc., ser. 2, vol. 2 (1905), pp. 438–469.

† Guichard, Ann. de L'Ecole Norm., ser. 3, vol. 4 (1887), pp. 361–380; Appell, Journ. de Math., ser. 4, vol. 7 (1891), pp. 157–219; Hurwitz, Acta Math., vol. 20 (1896), pp. 285–312.

solution of

$$\phi(z+1) = f(z)\phi(z), \tag{1}$$

denoted by $\Gamma_{f(u)}(z)$, will be obtained which is entirely analogous to the ordinary Gamma function.

In analogy with

$$\Gamma(z) = \operatorname*{Lim}_{p\to\infty} \frac{1\cdot 2 \cdots (p-1)}{z(z+1)\cdots(z+p-1)} p^z$$

set

$$\Gamma_{f(u)}(z) = \operatorname*{Lim}_{p\to\infty} \frac{f(1)f(2)\cdots f(p-1)}{f(z)f(z+1)\cdots f(z+p-1)} [f(p)]^z F(p, z) \tag{2}$$

where $F(p, z)$ is to be determined so that (2) converges, and satisfies the fundamental difference equation. For the latter condition

$$\operatorname*{Lim}_{p\to\infty} \frac{F(p, z+1)/F(p, z)}{f(z+p)/f(p)} = 1 \tag{3}$$

Taking into account (a) and (b), we may have condition (3) fulfilled by setting

$$\log F(p, z) = \frac{d}{dp}\log f(p)\frac{\phi_2(z)}{2!} + \cdots + \frac{d^r}{dp^r}\log f(p)\frac{\phi_{r+1}(z)}{(r+1)!} \tag{4}$$

where $\phi^2(z), \cdots \phi_{r+1}(z)$ are the Bernouillian polynomials.* Later we shall show that the same value of $F(p, z)$ insures the convergence of (2).

Since $\phi_n(1) = 0$,

$$\Gamma_{f(u)}(1) = 1$$

so that where q is a positive integer

$$\Gamma_{f(u)}(q) = f(1)f(2)\cdots f(q-1) = \underline{|f(q-1)}$$

in an evident notation.

The general solution of (1) imay be written

$$\phi(z) = \Gamma_{f(u)}(z)P(z) \tag{5}$$

where $P(z)$ is any periodic function of period unity. Clearly $\Gamma_{f(u)}(z)$ is the only solution of (1) such that

$$\operatorname*{Lim}_{p\to\infty} \frac{\phi(z+p)}{\underline{|f(p-1)}[f(p)]^z F(p, z)} = 1 \tag{6}$$

Equation (6) in connection with (1) may therefore be taken as defining $\Gamma_{f(u)}(z)$.

* Whittaker and Watson, Modern Analysis, Second edition, pp. 126, 127.

2. **Eulerian Form and Convergence.** Equation 2, §1 may be rewritten as follows:

$$\Gamma_{f(u)}(z) = \frac{[f(1)]^z F(1, z)}{f(z)} \prod_{p=1}^{\infty} \frac{f(p)}{f(z+p)} \left[\frac{f(p+1)}{f(p)}\right]^z \frac{F(p+1, z)}{F(p, z)} \qquad (1)$$

$$= \frac{[f(1)]^z F(1, z)}{f(z)} \prod_{p=1}^{\infty} Q(p).$$

Since

$$\frac{\phi_2(z)}{(s-1)!\,2!} + \frac{\phi_3(z)}{(s-2)!\,3!} + \cdots + \frac{\phi_s(z)}{s!} = \frac{z^s - z}{s!},$$

we have

$$\log F(p+1, z) = \log F(p, z) + \sum_{s=1}^{r} \frac{z^s - z}{s!} \frac{d^s \log f(p)}{dp^s}$$

$$+ \sum_{s=1}^{r-1} \lambda_s \frac{d^{r+1} \log f(p+\theta_s)}{dp^{r+1}} \frac{\phi_{s+1}(z)}{(r-s+1)!\,(s+1)!}$$

$|\lambda_s| \leqq 1; \quad 0 < \theta_s < 1,$

by using Darboux's* form of the remainder in Taylor's series for complex variables in connection with (4 §1). Hence substituting in (1) and reducing we obtain

$$\log Q(p) = \sum_{s=1}^{r-1} \lambda_s \frac{d^{r+1} \log f(p+\theta_s)}{dp^{r+1}} \frac{\phi_{s+1}(z)}{(r-s+1)!\,(s+1)!}$$

$$+ \frac{z\lambda_r}{(r+1)!} \frac{d^{r+1}}{dp^{r+1}} \log f(p+\theta_r) - \frac{z^{r+1}\lambda_{r+1}}{(r+1)!} \frac{d^{r+1}}{dz^{r+1}} \log f(p+\theta_{r+1} z).$$

By means of condition (b) we see that from some value of (p) on, the terms of $\Sigma \log Q(p)$ are less in absolute value than those of the convergent series $\Sigma(1/(p^{1+\epsilon}))$. Hence (1) is absolutely convergent for all finite values of z which are not zeros of some $f(z+p)$. Condition (b) likewise proves (1) to be uniformly convergent over any finite region of the z plane which excludes these points. $\Gamma_{f(u)}(z)$ is therefore an analytic function, except for isolated points, in the entire sector at least over which $f(z)$ is analytic.

3. **Weierstrassian Form, and Derivatives.** When $r \leqq 1$,† equations (2 § 1) and (1 § 2) become

$$\Gamma_{f(u)}(z) = \operatorname*{Lim}_{p\to\infty} \frac{f(1)f(2)\cdots f(p-1)}{f(z)f(z+1)\cdots f(z+p-1)} [f(p)]^z, \qquad (1)$$

and

$$\Gamma_{f(u)}(z) = \frac{[f(1)]^z}{f(z)} \prod_{p=1}^{\infty} \left\{ \frac{f(p)}{f(z+p)} \left[\frac{f(p+1)}{f(p)}\right]^z \right\}. \qquad (2)$$

Since (1) is uniformly convergent, we may differentiate logarithmically so

* Journ. de Math., series 3, vol. 2 (1876), p. 291.

† Whenever r can be taken ≤ 1. This is not in general true.

that

$$\frac{\Gamma'_{f(u)}(z)}{\Gamma_{f(u)}(z)} = -\gamma_{f(u)} - \left[\left(\frac{f'(z)}{f(z)} - \frac{f'(1)}{f(1)}\right) + \left(\frac{f'(z+1)}{f(z+1)} - \frac{f'(2)}{f(2)}\right) + \cdots\right],$$

where

$$\gamma_{f(u)} = \lim_{p\to\infty}\left[\frac{f'(1)}{f(1)} + \frac{f'(2)}{f(2)} + \cdots + \frac{f'(p)}{f(p)} - \log f(p)\right] \tag{3}$$

convergence of $\gamma_{f(u)}$ being easily established. Clearly

$$\Gamma'_{f(u)}(1) = -\gamma_{f(u)}.$$

The analogy with $\Gamma(z)$ is further brought out by transforming (1) into

$$\frac{1}{\Gamma_{f(u)}(z)} = f(z)e^{\gamma_{f(u)}z}\prod_{p=1}^{\infty}\left\{\frac{f(p+z)}{f(p)}e^{-\frac{f'(p)}{f(p)}z}\right\}, \tag{4}$$

the generalization of the Weierstrassian form of $\Gamma(z)$.

From (3) we find

$$-\frac{d^2}{dz^2}\log\Gamma_{f(u)}(z) = \frac{d^2}{dz^2}\log f(z) + \frac{d^2}{dz^2}\log f(z+1) + \cdots.$$

More generally for r unrestricted, we have

$$-\frac{d^{r+1}}{dz^{r+1}}\log\Gamma_{f(u)}(z) = \frac{d^{r+1}}{dz^{r+1}}\log f(z) + \frac{d^{r+1}}{dz^{r+1}}\log f(z+1) + \cdots. \tag{5}$$

4. Asymptotic Expansions. In the notation we have adopted

$$\log \underline{|f(p)} = \log f(1) + \log f(2) + \cdots + \log f(p).$$

If $\phi(z)$ is analytic for $R(z) > a$, $-1 \leqq I(z) \leqq 1$, then

$$\phi(1) + \phi(2) + \cdots + \phi(p) = C + \int_a^p \phi(t)dt + \tfrac{1}{2}\phi(p) + \frac{B_1}{2!}\phi'(p)$$
$$- \frac{B_2}{4!}\phi'''(p) + \cdots + (-)^q\frac{B_q}{(2q)!}\phi^{(2q-1)}(p) + R_p,$$

where R_p can be put in either of the forms

$$\sum_{t=1}^{2q} A_t \sum_{s=p}^{\infty}\phi^{(2q+1)}(s + {}_t\theta_{s-p}), \qquad \sum_{t=1}^{2q+1} A_t \sum_{s=q}^{\infty}\phi^{(2q+2)}(s + {}_t\theta_{s-p}),$$

provided these forms converge. In the case where $\phi(z) = \log f(z)$, condition (a) insures the fulfilment of the condition of this formula. Let that form of the remainder be chosen which makes the index of differentiation $r + 1$. Then by condition (b) a value of p may be chosen such that for all greater values

$$| R_p | < \frac{1}{p^{1+\epsilon}} + \frac{1}{(p+1)^{1+\epsilon}} + \cdots < \frac{1}{\epsilon(p-1)^{\epsilon}},$$

Hence $\lim_{p\to\infty} R_p = 0$. Letting $C = \log G_{f(u)}$, we have

$$\log f(p) \sim \log G_{f(u)} + \int_a^p \log f(t)dt + \tfrac{1}{2}\log f(p) + \frac{B_1}{2!}\frac{d}{dp}\log f(p) - \cdots + (-)^{q-1}\frac{B_q}{(2q)!}\frac{d^{2q-1}\log f(p)}{dp^{2q-1}}. \tag{1}$$

In general q may be taken to be larger than the value above adopted, since condition (b) will usually be satisfied for larger values of r than the one used in the infinite product.

We shall now show that if for p we substitute z in the above formula, we obtain an asymptotic expansion for $\log \Gamma_{f(u)}(z+1)$, or

$$\log \Gamma_{f(u)}(z) \sim \log G_{f(u)} + \int_a^z \log f(t)dt - \tfrac{1}{2}\log f(z) + \frac{B_1}{2!}\frac{d\log f(z)}{dz} - \cdots + \frac{(-)^{q-1}B_q}{(2q)!}\frac{d^{2q-1}\log f(z)}{dz^{2q-1}}. \tag{2}$$

Denote the right hand member by S_z. Using the recurrence formulæ for the Bernoullian numbers, we easily find, provided z is within the analytic sector,

$$S_{z+1} = S_z + \log f(z) + P_z,$$

where P_z can be written in either of the forms

$$\sum_{t=1}^{q+1} A_t \frac{d^{2q+1}}{dz^{2q+1}}\log f(z+\theta_t), \qquad \sum_{t=1}^{q+2} A_t \frac{d^{2q+2}}{dz^{2q+2}}\log f(z+\theta_t).$$

Hence

$$S_{z+p} - S_z = \log [f(z)f(z+1)\cdots f(z+p-1)] + \sum_{y=z}^{z+p-1} P_y. \tag{3}$$

Substituting this result, and (1) above in the infinite product for $\Gamma_{f(u)}(z)$, we have

$$\log \Gamma_{f(u)}(z) = \lim_{p\to\infty}\Big[S_p + S_z - S_{z+p} + z\log f(p) + \log F(p,z) + R_p - \sum_{y=z}^{z+p-1} P_y \Big].$$

Now

$$S_{z+p} - S_p = \int_p^{z+p} \log f(t)dt - \tfrac{1}{2}\log\frac{f(z+p)}{f(p)} + \cdots$$

$$+\frac{(-)^{q-1}B_q}{(2q)!}\frac{d^{2q-1}}{dp^{2q-1}}\log\frac{f(z+p)}{f(p)}$$

$$= z\log f(p) + \frac{d}{dp}\log f(p)\left[\frac{z^2}{2!}-\frac{z}{2}\right] + \frac{d^2}{dp^2}\log f(p)\left[\frac{z^3}{3!}-\frac{z^2}{2\cdot 2!}\right.$$

$$\left.+\frac{B_1 z}{2!}\right] + \cdots + \frac{d^r}{dp^r}\log f(p)\left[\frac{z^{r+1}}{(r+1)!}-\frac{z^r}{2r!}+\frac{z^{r-1}B_1}{2!\,(r-1)!}\right.$$

$$\left.-\frac{z^{r-3}B_2}{4!\,(r-3)!}+\cdots\right] + Q_p$$

$$= z\log f(p) + \log F(p, z) + Q_p,$$

where

$$Q_p = \frac{z^{r+2}\lambda_1}{(r+2)!}\frac{d^{r+1}}{dp^{r+1}}\log f(p+\theta_1 z) - \frac{1}{2}\frac{z^{r+1}\lambda_2}{(r+1)!}\frac{d^{r+1}}{dp^{r+1}}\log f(p+\theta_2 z)$$

$$+\frac{B_1}{2!}\frac{z^r\lambda_3}{r!}\frac{d^{r+1}}{dp^{r+1}}\log f(p+\theta_3 z) - \cdots,$$

the last term ending in z^2 or z^3. Hence

$$\lim_{p\to\infty} Q_p = 0; \qquad \lim_{p\to\infty} R_p = 0; \qquad \lim_{p\to\infty}\sum_{y=z}^{z+p-1} P_y = \sum_{s=0}^{\infty} P_{z+s},$$

and

$$\log \Gamma_{f(u)}(z) = S_z - \sum_{s=0}^{\infty} P_{z+s},$$

S_z will therefore be the asymptotic expansion of $\log \Gamma_{f(u)}(z)$, provided z approaches infinity in such a way that $\sum_{s=0}^{\infty} P_{z+s} \to 0$. This will clearly be the case if $z \to \infty$ along a line parallel to or on the real axis in the positive direction. Under this condition (2) holds. We cannot infer more from the conditions we have assumed. If however condition (b) holds when $p \to \infty$ along any line,* (2) will hold for $z \to \infty$ along any line not parallel to or on the real axis in the negative direction.

5. **An Integral for $\Gamma_{f(u)}(z)$.** If x_1 and x_2 are integers, and $\phi(\xi)$ is a function which is analytic and bounded for all values of ξ such that

$$x_1 \leqq R(\xi) \leqq x_2,$$

then†

$$\tfrac{1}{2}\phi(x_1) + \phi(x_1+1) + \phi(x_1+2) + \cdots + \phi(x_2-1) + \tfrac{1}{2}\phi(x_2)$$

$$= \int_{x_1}^{x_2}\phi(\xi)d\xi + \frac{1}{i}\int_0^{\infty}\frac{\phi(x_2+iy)-\phi(x_1+iy)-\phi(x_2-iy)+\phi(x_1-iy)}{e^{2\pi y}-1}dy.$$

* This extended condition is satisfied by the class of functions suggested in the introduction.
† Whittaker and Watson, p. 145, Ex. 7.

Let the initial conditions (a) and (b) imposed on $f(z)$ be extended to the following:

(a') $\log f(z)$ is analytic to the right of a line at distance a from the axis of imaginaries;

(b') for some value of r and ϵ, $\epsilon > 0$,

$$\operatorname*{Lim}_{z\to\infty} z^{1+\epsilon}\frac{d^{r+1}}{dz^{r+1}}\log f(z) = 0,$$

where $z \to \infty$ along any line included in the analytic region of (a').

We shall then have, if $R(z) > a$, $x_1 = 0$, $x_2 \to \infty$,

$$-\frac{d^{r+1}}{dz^{r+1}}\log\Gamma_{f(u)}(z) = \frac{d^{r+1}}{dz^{r+1}}\log f(z) + \frac{d^{r+1}}{dz^{r+1}}\log f(z+1) + \cdots$$

$$= \frac{1}{2}\frac{d^{r+1}}{dz^{r+1}}\log f(z) + \int_0^\infty \frac{d^{r+1}}{d\xi^{r+1}}\log f(z+\xi)d\xi$$

$$-\frac{1}{i}\int_0^\infty \frac{\frac{d^{r+1}}{dz^{r+1}}\log f(z+iy) - \frac{d^{r+1}}{dz^{r+1}}\log f(z-iy)}{e^{2\pi y}-1}dy + \operatorname*{Lim}_{x_2\to\infty} Rx_2,$$

where

$$Rx_2 = \frac{1}{i}\int_0^\infty \frac{\frac{d^{r+1}}{dz^{r+1}}\log f(z+x_2+iy) - \frac{d^{r+1}}{dz^{r+1}}\log f(z+x_2-iy)}{e^{2\pi y}-1}dy.$$

From (b') we easily find

$$\operatorname*{Lim}_{x_2\to\infty} x_2^{\epsilon} Rx_2 = 0, \quad i.\ e., \operatorname*{Lim}_{x_2\to\infty} Rx_2 = 0.$$

Also

$$\int_0^\infty \frac{d^{r+1}}{d\xi^{r+1}}\log f(z+\xi)d\xi = -\frac{d^r\log f(z)}{dz^r},$$

so that on integration

$$\log\Gamma_{f(u)}(z) = c_0 + c_1 z + \cdots + c_r z^r - \frac{1}{2}\log f(z)$$

$$+ \int_a^z \log f(z)dz + \frac{1}{i}\int_0^\infty \frac{\log f(z+iy) - \log f(z-iy)}{e^{2\pi y}-1}dy.$$

If we let $z \to \infty$, on expanding the latter integral and comparing with the asymptotic expansion, we find

$$c_0 = \log G_{f(u)}; \quad c_1 = c_2 = \cdots c_r = 0,$$

and

$$\log\Gamma_{f(u)}(z) = \log G_{f(u)} - \frac{1}{2}\log f(z) + \int_0^z \log f(z)dz$$

$$+ 2\int_0^\infty \frac{\log f(z+iy) - \log f(z-iy)}{2i}\frac{dy}{e^{2\pi y}-1}.$$

As a matter of notation, let

$$\frac{\phi(z+iy)-\phi(z-iy)}{2i}=\sin_{\phi(u)}(z,y),$$

$$\frac{\phi(z+iy)+\phi(z-iy)}{2}=\cos_{\phi(u)}(z,y).$$

Then

$$\log \Gamma_{f(u)}(z)=\log G_{f(u)}-\frac{1}{2}\log f(z)+\int_a^z \log f(z)dz+2\int_0^\infty \frac{\sin_{\log f(u)}(z,y)}{e^{2\pi y}-1}dy \quad (1)$$

and by differentiation

$$\frac{\Gamma_{f(u)}'(z)}{\Gamma_{f(u)}(z)}=\log f(z)-\frac{1}{2}\frac{f'(z)}{f(z)}+2\int_0^\infty \frac{\cos_{f'(u)/f(u)}(z,y)}{e^{2\pi y}-1}dy. \quad (2)$$

PART II. TRANSFORMATIONS.

6. **Integral for the Asymptotic Constant.** In the present section and in the one following we shall obtain results which are very useful in establishing particular relations between the generalized Gamma functions.

Let*

$$u=\int_z^{z+1}\log \Gamma_{f(u)}(t)dt;$$

then

$$\frac{du}{dz}=\log \Gamma_{f(u)}(z+1)-\log \Gamma_{f(u)}(z)=\log f(z),$$

and

$$u=\int_a^z \log f(t)dt+C.$$

Since

$$\log \Gamma_{f(u)}(z)\sim \log G_{f(u)}+\int_a^z \log f(t)dt-\frac{1}{2}\log f(z)$$
$$+\sum_{s=1}^{q}\frac{(-)^{s-1}B_s}{(2s)!}\frac{d^{2s-1}\log f(z)}{dz^{2s-1}},$$

it easily follows that

$$\int_z^{z+1}\log \Gamma_{f(u)}(t)dt\sim \log G_{f(u)}+\int_z^{z+1}\left[\int_a^t \log f(u)du\right]dt$$
$$-\frac{1}{2}\int_z^{z+1}\log f(t)dt+\sum_{s=1}^{q}\frac{(-)^{s-1}B_s}{(2s)!}\frac{d^{2s-2}}{dz^{2s-2}}\log\frac{f(z+1)}{f(z)}.$$

But, using the Euler-Maclaurin Sum formula,† we get

* For the analogue of the ordinary Gamma function see Whittaker and Watson, p. 255, Ex. 21.
† Whittaker and Watson, p. 128.

$$\int_z^{z+1}\left[\int_a^t \log f(u)du\right]dt \sim \int_a^z \log f(t)dt$$
$$+\frac{1}{2}\int_z^{z+1}\log f(t)dt + \sum_{s=1}^{q}\frac{(-)^s B_s}{(2s)!}\frac{d^{2s-2}}{dz^{2s-2}}\log\frac{f(z+1)}{f(z)},$$

so that

$$\int_z^{z+1}\log \Gamma_{f(u)}(t)dt \sim \int_a^z \log f(t)dt + \log G_{f(u)}.$$

Comparing with the above, we see that

$$C = \log G_{f(u)}$$

and

$$\int_z^{z+1}\log \Gamma_{f(u)}(t)dt = \int_a^z \log f(t)dt + \log G_{f(u}. \tag{1}$$

Letting $z = a$, we obtain

$$\log G_{f(u)} = \int_a^{a+1}\log \Gamma_{f(u)}(t)dt, \tag{2}$$

an analytical expression for the constant $G_{f(u)}$.

Since $G_{f(u)}$ depends on the value of a chosen, we shall write it ${}_aG_{f(u)}$. Then

$$\log_{a_1} G_{f(u)} = \log_{a_2} G_{f(u)} + \int_{a_2}^{a_1}\log f(t)dt. \tag{3}$$

7. **The Asymptotic Test.** We have seen that $\log \Gamma_{f(u)}(z)$ has the same asymptotic expansion as $\log \underline{|f(p-1)}$. Since any other solution than $\Gamma_{f(u)}(z)$ of

$$\phi(z+1) = f(z)\phi(z) \tag{1}$$

must be in the form

$$\phi(z) = \Gamma_{f(u)}(z)P(z), \tag{2}$$

where $P(z)$ is periodic of period unity, it is evident that $\Gamma_{f(u)}(z)$ is the only solution of (1) possessing this property. The following is a more useful expression of the above principle.

From

$$-\frac{d^{r+1}}{dz^{r+1}}\log \Gamma_{f(u)}(z) = \frac{d^{r+1}}{dz^{r+1}}\log f(z) + \frac{d^{r+1}}{dz^{r+1}}\log f(z+1) + \cdots,$$

we have

$$\operatorname*{Lim}_{z\to+\infty}\frac{d^{r+1}}{dz^{r+1}}\log \Gamma_{f(u)}(z) = 0. \tag{3}$$

Let $\psi(z)$ be some other solution of (1) possessing property (3). Then if $z = p + x$, where p is an integer, we must have for all values of x

$$\operatorname*{Lim}_{p\to\infty}\frac{d^{r+1}}{dx^{r+1}}\log \psi(x+p) = \operatorname*{Lim}_{p\to\infty}\left[\frac{d^{r+1}}{dx^{r+1}}\log \Gamma_{f(u)}(x+p) + \frac{d^{r+1}}{dx^{r+1}}\log P(x)\right] = 0,$$

so that

$$\frac{d^{r+1}}{dx^{r+1}} \log P(x) = 0.$$

Since $P(z)$ is periodic we can only have

$$P(z) = ce^{2\pi ipz},$$

so that

$$\psi(z) = ce^{2\pi ipz}\Gamma_{f(u)}(z), \tag{4}$$

where p is an integer. If furthermore $\psi(z)$ have an asymptotic expansion of the form

$$b + a_0 \int_a^z \log f(t)dt + a_1 \log f(z) + \cdots + a_{2q} \frac{d^{2q-1}}{dz^{2q-1}} \log f(z),$$

where the a's are independent of $f(z)$, we find by letting $f(z) = e^{z^{r-\epsilon}}$ and using (4) that

$$a_0 = 1, \quad a_1 = -\tfrac{1}{2}, \qquad a_{2s+1} = 0, \qquad a_{2s} = \frac{B_s}{(2s)!}(-)^{s-1},$$

and

$$\psi(z) = c\Gamma_{f(u)}(z). \tag{5}$$

We shall refer to condition (3) and the one just given as the asymptotic test. Hence if two solutions $\psi_1(z)$ and $\psi_2(z)$ of (1) satisfy the asymptotic test,

$$\psi_1(z) = c\psi_2(z).$$

The same is clearly true of solutions of

$$\phi(z + n) = f(z)\phi(z),$$

where n is real and positive, since this equation can be transformed into

$$\psi(z + 1) = f(nz)\psi(z).$$

8. **Elementary Transformations.** From the Gaussian form of $\Gamma_{f(u)}(z)$ we see that

$$\Gamma_{[f_1(u)]^m[f_2(u)]^n}(z) = [\Gamma_{f_1(u)}(z)]^m[\Gamma_{f_2(u)}(z)]^n. \tag{1}$$

In particular

$$\Gamma_c(z) = c^{z-1}; \qquad \Gamma_u(z) = \Gamma(z); \qquad \Gamma_{e^{(u^r)}}(z) = e^{[\phi_{r+1}(z)]/(r+1)}.*$$

Again, both $\Gamma_{f(u+b)}(z)$ and $\Gamma_{f(u)}(z + b)$ are solutions of

$$\phi(z + 1) = f(z + b)\phi(z).$$

They clearly have the same asymptotic expansions, except for a constant

* By means of these results and (2), we easily evaluate $\Gamma_{f_1(u)e^{f_2(u)}}(z)$ where $f_1(u)$ is a rational and $f_2(u)$ an integral algebraic function.

factor, so that

$$\Gamma_{f(u+b)}(z) = c\Gamma_{f(u)}(z+b).$$

To determine c, let $z = 1$. Since $\Gamma_{f(u+b)}(1) = 1$,

$$c = \frac{1}{\Gamma_{f(u)}(1+b)},$$

and

$$\Gamma_{f(u+b)}(z) = \frac{\Gamma_{f(u)}(z+b)}{\Gamma_{f(u)}(1+b)}. \tag{2}$$

By means of the integral of §6 these results may be directly applied to ${}_aG_{f(u)}$. We thus find

$${}_aG_{[f_1(u)]^m[f_2(u)]^n} = [{}_aG_{f_1(u)}]^m[{}_aG_{f_2(u)}]^n,$$

$${}_oG_c = c^{-1/2}, \qquad {}_oG_{\epsilon(u^r)} = 1,$$

$${}_aG_{f(u+b)} = \frac{{}_{a+b}G_{f(u)}}{\Gamma_{f(u)}(1+b)}, \tag{3}$$

9. **Infinite Products in Terms of Generalized Gamma Functions.** Consider first

$$\prod_{p=0}^{\infty}\frac{f(a_1+p)f(a_2+p)\cdots f(a_k+p)}{f(b_1+p)f(b_2+p)\cdots f(b_l+p)} = \prod_{p=0}^{\infty}F(p). \tag{1$'$}$$

A sufficient condition for convergence is that for some positive value of ϵ

$$\operatorname*{Lim}_{p\to\infty} p^{1+\epsilon}\log F(p) = 0. \tag{2$'$}$$

But

$$\log F(p) = \log f(a_1+p) + \cdots - \log f(b_1+p) - \cdots$$

$$= (k-l)\log f(p) + \frac{\Sigma a - \Sigma b}{1}\frac{d}{dp}\log f(p) + \cdots$$

$$+ \frac{\Sigma a^r - \Sigma b^r}{r!}\frac{d^r}{dp^r}\log f(p) + R_p,$$

where

$$\operatorname*{Lim}_{p\to\infty} p^{1+\epsilon}R_p = 0, \qquad \operatorname*{Lim}_{p\to\infty} p^{1+\epsilon}\frac{d^r}{dp^r}\log f(p) \neq 0.$$

We must therefore have

$$k = l, \qquad \Sigma a = \Sigma b, \cdots \Sigma a^r = \Sigma b^r, \tag{3$'$}$$

if (2)′ is to be fulfilled. Now

$$\prod_{p=0}^{\infty}\frac{f(a_1+p)\cdots f(a_k+p)}{f(b_1+p)\cdots f(b_k+p)} = \frac{\Gamma_{f(u)}(b_1)\cdots\Gamma_{f(u)}(b_k)}{\Gamma_{f(u)}(a_1)\cdots\Gamma_{f(u)}(a_k)}S_\infty,$$

where

$$S_p = \frac{[f(p)]^{a_1}F(p, a_1) \cdots [f(p)]^{a_k}F(p, a_k)}{[f(p)]^{b_1}F(p, b_1) \cdots [f(p)]^{b_k}F(p, b_k)}.$$

Using (3)′, we see that $S_p = 1$, so that

$$\prod_{p=0}^{\infty}\frac{f(a_1 + p)f(a_2 + p) \cdots f(a_k + p)}{f(b_1 + p)f(b_2 + p) \cdots f(b_k + p)} = \frac{\Gamma_{f(u)}(b_1) \cdots \Gamma_{f(u)}(b_k)}{\Gamma_{f(u)}(a_1) \cdots \Gamma_{f(u)}(a_k)}. \tag{1}$$

Consider now more generally

$$\prod_{p=0}^{\infty}\frac{f_1(a_1 + p) \cdots f_k(a_k + p)}{\phi_1(b_1 + p) \cdots \phi_l(b_l + p)} = \prod_{p=0}^{\infty} F(p), \tag{4'}$$

where (2)′ is again a sufficient condition. We may write

$$\prod_{p=0}^{\infty} F(p) = \lim_{q\to\infty}\prod_{p=1}^{q} F(p - 1) = \lim_{q\to\infty} \underline{|F(q - 1)}$$

$$= {}_aG_{F(u-1)}e^{\int_a^\infty \log F(u - 1)du} \tag{5'}$$

by (1 §4). Furthermore, provided the functions used exist, from §8 it follows that

$${}_aG_{F(u-1)} = \frac{{}_aG_{f_1(a_1-1+u)} \cdots {}_aG_{f_k(a_k-1+u)}}{{}_aG_{\phi_1(b_1 1-1+u)} \cdots {}_aG_{\phi_l(b_l-1+u)}}$$

$$= \frac{{}_{a+a_1-1}G_{f_1(u)} \cdots {}_{a+a_k-1}G_{f_k(u)}}{{}_{a+b_1-1}G_{\phi_1(u)} \cdots {}_{a+b_l-1}G_{\phi_l(u)}}\,\frac{\Gamma_{\phi_1(u)}(b_1) \cdots \Gamma_{\phi_l(u)}(b_l)}{\Gamma_{f_1(u)}(a_1) \cdots \Gamma_{f_k(u)}(a_k)},$$

so that finally

$$\prod_{p=0}^{\infty}\frac{f_1(a_1 + p) \cdots f_k(a_k + p)}{\phi_1(b_1 + p) \cdots \phi_l(b_l + p)} = A\,\frac{{}_aG_{f_1(u)} \cdots {}_aG_{f_k(u)}}{{}_aG_{\phi_1(u)} \cdots {}_aG_{\phi_l(u)}}\,\frac{\Gamma_{\phi_1(u)}(b_1) \cdots \Gamma_{\phi_l(u)}(b_l)}{\Gamma_{f_1(u)}(a_1) \cdots \Gamma_{f_k(u)}(a_k)},$$

where

$$\log A = \int_a^\infty \log F(u - 1)du + \sum_{\lambda=1}^{k}\int_a^{a+a_\lambda-1}\log f_\lambda(u)du - \sum_{\mu=1}^{l}\int_a^{a+b_\mu-1}\log \phi_\mu(u)du. \tag{2}$$

10. **The Multiplication Theorem.** Both $\Gamma_{f(u/n)}(nz)$ and

$$\Gamma_{f(u)}(z)\Gamma_{f(u)}\left(z + \frac{1}{n}\right) \cdots \Gamma_{f(u)}\left(z + \frac{n - 1}{n}\right)$$

are solutions of

$$\phi(z + 1) = f(z)f\left(z + \frac{1}{n}\right) \cdots f\left(z + \frac{n - 1}{n}\right)\phi(z).$$

Furthermore it is easily shown that they both satisfy the asymptotic test since $\Gamma_{f(u)}(z)$ does. Hence

$$\Gamma_{f(u)}(z)\Gamma_{f(u)}\left(z+\frac{1}{n}\right)\cdots\Gamma_{f(u)}\left(z+\frac{n-1}{n}\right)=\psi(n)\Gamma_{f(u/n)}(nz).$$

To determine $\psi(n)$, we have

$$\log {}_{na}G_{f(u/u)} = \int_{na}^{na+1}\log\Gamma_{f(u/u)}(z)dz = n\int_{a}^{a+(1/n)}\log\Gamma_{f(u/u)}(nz)dz$$

$$= n\int_{a}^{a+(1/n)}\left[\log\Gamma_{f(u)}(z)+\cdots+\log\Gamma_{f(u)}\left(z+\frac{n-1}{n}\right)-\log\psi(n)\right]dz$$

$$= n\left[\int_{a}^{a+(1/n)}\log\Gamma_{f(u)}(z)dz+\int_{a+(1/n)}^{a+(2/n)}\log\Gamma_{f(u)}(z)dz+\cdots\right.$$

$$\left.+\int_{a+[(n-1)/n]}^{a+1}\log\Gamma_{f(u)}(z)dz\right]-\log\psi(n)$$

$$= n\int_{a}^{a+1}\log\Gamma_{f(u)}(z)dz-\log\psi(n),$$

so that

$$\psi(n) = \frac{[{}_aG_{f(u)}]^n}{{}_{na}G_{f(u/u)}},$$

and

$$\Gamma_{f(u)}(z)\Gamma_{f(u)}\left(z+\frac{1}{n}\right)\cdots\Gamma_{f(u)}\left(z+\frac{n-1}{n}\right)=\frac{[{}_aG_{f(u)}]^n}{{}_{na}G_{f(u/u)}}\Gamma_{f(u/n)}(nz) \qquad (1)$$

When $f(u) = u$, by using the results of §8, the ordinary multiplication theorem results.

Let $z = 1/n$, and we obtain

$$\Gamma_{f(u)}\left(\frac{1}{n}\right)\Gamma_{f(u)}\left(\frac{2}{n}\right)\cdots\Gamma_{f(u)}\left(\frac{n-1}{n}\right)=\frac{[{}_aG_{f(u)}]^n}{{}_{na}G_{f(u/n)}}. \qquad (2)$$

If in this $n = 2$, we obtain

$$\Gamma_{f(u)}\left(\frac{1}{2}\right)=\frac{[{}_aG_{f(u)}]^2}{{}_{2a}G_{f(u/2)}}, \qquad (3)$$

the analogue of $\Gamma(1/2) = \sqrt{\pi}$.

11. **An Integration Theorem Generalized.** When n is a positive integer

$$\int_{a}^{a+1}\log f(z)dz = \int_{a}^{a+(1/n)}\log f(z)dz+\int_{a+(1/n)}^{a+(2/n)}\log f(z)dz+\cdots$$

$$+\int_{a+[(n-1)/n]}^{a+1}\log f(z)dz.$$

By means of the Generalized Gamma functions the corresponding relation may be obtained for any positive real value of n;*

$$\int_a^{a+(1/n)} \log f(z)dz + \cdots + \int_{a+(n-1/n)}^{a+1} \log f(z)dz$$

$$= \int_a^{a+(1/n)} \log \left[f(z) f\left(z + \frac{1}{n}\right) \cdots f\left(z + \frac{n-1}{n}\right) \right] dz$$

$$= \int_a^{a+(1/n)} \log \left[f(z) \Gamma_{f[z+(u/n)]}(n) \right] dz.$$

Let now n have any positive real value. Then

$$\int_a^{a+(1/n)} \log [f(z)\Gamma_{f[z+(u/n)]}(n)]dz = \int_a^{a+(1/n)} \log \frac{\Gamma_{f[u/n]}(n+nz)}{\Gamma_{f[u/n]}(nz)} dz\dagger$$

$$= \frac{1}{n}\left[\int_{na+n}^{na+n+1} \log \Gamma_{f[u/n]}(z)dz - \int_{na}^{na+1} \log \Gamma_{f[u/n]}(z)dz \right],$$

so that, using §6, we find the relation desired

$$\int_a^{a+(1/n)} \log [f(z)\Gamma_{f[z+(u/n)]}(n)]dz = \int_a^{a+1} \log f(z)dz. \tag{1}$$

12. **The Multiplication Theorem Generalized.** If condition (b) be extended so that for some value of r

$$\lim_{p\to\infty} p^{2+\epsilon} \frac{d^{r+2} \log f(z+p)}{dz^{r+2}} = 0,$$

the multiplication theorem may be generalized to admit all positive real values of n.‡ We have for positive integral values of n

$$\Gamma_{f(u)}(z)\Gamma_{f(u)}\left(z+\frac{1}{n}\right) \cdots \Gamma_{f(u)}\left(z+\frac{n-1}{n}\right) = \Gamma_{f(u)}(z)\Gamma_{\Gamma_{f(u)}(z+(v/n))}(n).$$

By the condition above imposed we have

$$\lim_{p\to\infty} p^{1+\epsilon} \frac{d^{r+2}}{d\zeta^{r+2}} \log \Gamma_{f(u)}\left(z+\frac{\zeta+p}{n}\right) = 0.$$

It therefore follows easily that $\Gamma_{\Gamma_{f(u)}(z+(v/n))}(n)$ exists when n is any real and positive number. Furthermore, with the aid of (2 §8)

$$\Gamma_{f(u)}(z)\Gamma_{\Gamma_{f(u)}(z+(v/n))}(n)$$

* The theorem depends only on the existence of the functions involved. That n be real and positive is sufficient for this purpose, under conditions (a) and (b), but not always necessary. For the class of functions suggested in the introduction the theorem holds for all values of n.

† By (2 § 8) and (1 § 1).

‡ The note to §11 applies here too except that n may not be a negative real number.

is seen to satisfy the equation

$$\phi\left(z+\frac{1}{n}\right)=f(z)\phi(z).$$

Since after transformation by (2 §8) it may be shown to satisfy the asymptotic test, we obtain

$$\Gamma_{f(u)}(z)\Gamma_{\Gamma_{f(u)}(z+(v/n))}(n) = \psi(n)\Gamma_{f(u/n)}(nz).$$

The previous section enables us to determine $\psi(n)$ as in the simpler case.

$$\begin{aligned}\log {}_{na}G_{f(u/n)} &= n\int_a^{a+(1/n)}\log \Gamma_{f(u/n)}(nz)dz \\ &= n\int_a^{a+(1/n)}[\log \Gamma_{f(u)}(z)\Gamma_{\Gamma_{f(u)}(z+(v/n))}(n)]dz - \log\psi(n) \\ &= n\int_a^{a+1}\log \Gamma_{f(u)}(z)dz - \log\psi(n).\end{aligned}$$

Hence, as before

$$\psi(n) = \frac{[{}_aG_{f(u)}]^n}{{}_{na}G_{f(u/n)}},$$

and

$$\Gamma_{f(u)}(z)\Gamma_{\Gamma_{f(u)}(z+(v/n))}(n) = \frac{[{}_aG_{f(u)}]^n}{{}_{na}G_{f(u/n)}}\Gamma_{f(u/n)}(nz) \tag{1}$$

Letting $z = 1/n$, and using (2 §8), we obtain

$$\Gamma_{\Gamma_{f(u)}(v/n)}(n) = \frac{[{}_aG_{f(u)}]^n}{{}_{na}G_{f(u/n)}}. \tag{2}$$

13. **The Associated Periodic Functions.** Let $f(z)$ be such that both $\Gamma_{f(u)}(z)$ and $\Gamma_{f(-u)}(z)$ exist, and let

$$\Gamma_{f(u)}(z)\Gamma_{f(-u)}(1-z) = F(z),$$

Then

$$F(z+1) = F(z).$$

We shall denote this periodic function by $P_{f(u)}(z)$, *i. e.*,

$$\Gamma_{f(u)}(z)\Gamma_{f(-u)}(1-z) = P_{f(u)}(z). \tag{1}$$

Let ${}_aG_{f(u)}\,{}_aG_{f(-u)} = {}_a\pi_{f(u)}$. Then using the integral of §6, we easily obtain

$$\log {}_a\pi_{f(u)} = \int_a^{a+1}\log P_{f(u)}(z)dz. \tag{2}$$

Clearly

$$P_{f(u)}(0) = \frac{1}{f(0)}, \qquad \text{or} \qquad \operatorname*{Lim}_{z\to 0} P_{f(u)}(z)f(z) = 1,$$

$$P_{f(u)}(\tfrac{1}{2}) = \Gamma_{f(u)}(\tfrac{1}{2})\Gamma_{f(-u)}(\tfrac{1}{2}) = \frac{[{}_a\pi_{f(u)}]^2}{2a\pi_{f(u/2)}}. \tag{3}$$

Again

$$P_{f(u)}(z) = P_{f(-u)}(-z), \tag{4}$$

$$P_{f(u+b)}(z) = \frac{1}{f(b)}\frac{P_{f(u)}(z+b)}{P_{f(u)}(1+b)}, \tag{5}$$

The multiplication theorem gives

$$\Gamma_{f(u)}(z)\Gamma_{f(u)}\left(z+\frac{1}{n}\right)\cdots\Gamma_{f(u)}\left(z+\frac{n-1}{n}\right) = \frac{[{}_aG_{f(u)}]^n}{{}_{na}G_{f(u/n)}}\Gamma_{f(u/n)}(nz),$$

$$\Gamma_{f(-u)}\left(\frac{1}{n}-z\right)\Gamma_{f(-u)}\left(\frac{2}{n}-z\right)\cdots\Gamma_{f(-u)}(1-z) = \frac{[{}_{-a}G_{f(-u)}]^n}{{}_{-na}G_{f(-u/n)}}\Gamma_{f(-u/n)}(1-nz).$$

Inverting and multiplying,* we finally obtain

$$P_{f(u)}(z)P_{f(u)}\left(z+\frac{1}{n}\right)\cdots P_{f(u)}\left(z+\frac{n-1}{n}\right) = \frac{[{}_a\pi_{f(u)}]^n}{{}_{na}\pi_{f(u/n)}}P_{f(u/n)}(nz). \tag{6}$$

COLUMBIA UNIVERSITY.

* It will be noticed that whereas in the case of the ordinary Γ function it is usual to obtain the multiplication theorem from that of the sine function, the reverse method is used here.