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LINEAR RECURRENT SEQUENCES AND POWERS OF A SQUARE MATRIX

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Abstract

In this paper, we establish a formula expressing explicitly the general term of a linear recurrent sequence, allowing us to generalize the original result of J. McLaughlin [7] concerning powers of a matrix of size 2, to the case of a square matrix of size $m \geq 2$. Identities concerning Fibonacci and Stirling numbers and various combinatorial relations are derived.

1. Introduction

The main theorem of J. McLaughlin [7] states the following:

Theorem 1. *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be a square matrix of order two, let* $T = a + d$ *be its trace, and let* $D = ad - bc$ *be its determinant. Let*

$$y_n = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} T^{n-2i} (-D)^i . \tag{1}$$

Then, for $n \geq 1$,

$$A^n = \begin{pmatrix} y_n - d y_{n-1} & b y_{n-1} \\ c y_{n-1} & y_n - a y_{n-1} \end{pmatrix} . \tag{2}$$

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We remark that in this theorem, $(y_n)_{n\geq -1}$ is a linear recurrent sequence that satisfies

$$\begin{cases} y_{-1} = 0, \\ y_0 = 1, \\ y_n = Ty_{n-1} - Dy_{n-2} \text{ for all integer } n \geq 1. \end{cases} \tag{3}$$

By setting $A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $A_1 = \begin{pmatrix} -d & b \\ c & -a \end{pmatrix}$, relation (2) of Theorem 1 may be written as follows:

$$A^n = y_n A_0 + y_{n-1} A_1 \text{ for all integers } n \geq 0, \tag{4}$$

with $A_0 = I_2$ and $A_1 = A - TI_2$ (where I_2 is the unit matrix).

In Section 3, we extend this result (relation (4)) to any matrix $A \in \mathcal{M}_m(\mathcal{A})$ of order $m \geq 2$, $\mathcal{A}$ being a unitary commutative ring.

We prove the following result:

Let $A \in M_m(\mathcal{A})$ and let $P(X) = X^m - a_1 X^{m-1} - \cdots - a_{m-1}X - a_m$ be the characteristic polynomial of A. Let $A_0, A_1, \ldots, A_{m-1}$ be matrices of $M_m(\mathcal{A})$ defined by

$$A_k = -\sum_{i=0}^{k} a_i A^{k-i}, \quad \text{for } \; 0 \leq k \leq m-1, \text{ with } a_0 = -1,$$

and let $(y_n)_{n>-m}$ be the sequence of elements of $\mathcal{A}$ satisfying

$$y_n = \sum_{k_1+2k_2+\cdots+mk_m=n} \binom{k_1+k_2+\cdots+k_m}{k_1, k_2, \ldots, k_m} a_1^{k_1} a_2^{k_2} \ldots a_m^{k_m}, \quad \text{for } \; n > -m.$$

Then, for all integers $n \geq 0$, $A^n = y_n A_0 + y_{n-1} A_1 + \cdots + y_{n-m+1} A_{m-1}$.

The proof of this result is based on Theorem 2 given in Section 2.

In this section, we generalize Theorem 2, which permits us to express the general term u_n of a recurrent linear sequence satisfying the relation

$$u_n = a_1 u_{n-1} + a_2 u_{n-2} + \cdots + a_m u_{n-m} \quad \text{for all } n \geq 1,$$

in terms of the coefficients $a_1, a_2, \ldots, a_m$ and $u_0, u_{-1}, u_{-2}, \ldots, u_{-(m-1)}$. Applications to Fibonacci, generalized Fibonacci and "multibonacci" sequences are also given.

Finally, in Section 4, further combinatorial identities are derived, including identities concerning the Stirling numbers of the first and second kind.

As an illustration, we give a nice duality between the two following relations (Corollaries 5 and 7):

$$\sum_{k_1+2k_2+\cdots+mk_m=n} \binom{k_1+\cdots+k_m}{k_1,\ldots,k_m} (-1)^{n-(\sum_{i=1}^m k_i)} \binom{m}{m-1}^{k_1} \cdots \binom{m}{0}^{k_m} = \binom{n+m-1}{m-1},$$

$$\sum_{k_1+2k_2+\cdots+mk_m=n} \binom{k_1+\cdots+k_m}{k_1,\ldots,k_m} (-1)^{n-(\sum_{i=1}^m k_i)} \begin{bmatrix} m \\ m-1 \end{bmatrix}^{k_1} \cdots \begin{bmatrix} m \\ 0 \end{bmatrix}^{k_m} = \begin{Bmatrix} n+m-1 \\ m-1 \end{Bmatrix},$$

where $\binom{k_1+\cdots+k_m}{k_1,\ldots,k_m}$ is the multinomial coefficient (Section 2), and $\begin{bmatrix} n \\ k \end{bmatrix}$ and $\begin{Bmatrix} n \\ k \end{Bmatrix}$ are, respectively, the Stirling numbers of the first and second kind as defined in [5].

2. Explicit Expression of the General Term of a Recurrent Linear Sequence

In this section, we let $m \geq 2$ be an integer, $\mathcal{A}$ a unitary commutative ring, $a_1, a_2, \ldots, a_m$, $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$ elements of $\mathcal{A}$, and $(u_n)_{n>-m}$ a sequence of elements of $\mathcal{A}$ defined by

$$\begin{cases} u_{-j} = \alpha_j \quad \text{for } 0 \leq j \leq m-1, \\ u_n = a_1 u_{n-1} + a_2 u_{n-2} + \cdots + a_m u_{n-m} \text{ for } n \geq 1. \end{cases} \tag{5}$$

The aim of this section is to give an explicit expression of u_n in terms of n, $a_1, a_2, \ldots, a_m$, $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$ (Theorem 3).

Let us define the sequence $(y_n)_{n\in\mathbb{Z}}$ of elements of $\mathcal{A}$, with the convention that an empty sum is zero, by

$$y_n = \sum_{k_1+2k_2+\cdots+mk_m=n} \binom{k_1+k_2+\cdots+k_m}{k_1,k_2,\ldots,k_m} a_1^{k_1} a_2^{k_2} \ldots a_m^{k_m}, \quad \text{for } n \in \mathbb{Z}, \tag{6}$$

the summation being taken over all m-tuples $(k_1, k_2, \ldots, k_m)$ of integers $k_j \geq 0$ satisfying $k_1 + 2k_2 + \cdots + mk_m = n$. With the previous convention, we have $y_n = 0$ for $n < 0$. The multinomial coefficient that appears in the summation is defined for integers $k_1, k_2, \ldots, k_m \geq 0$, by

$$\binom{k_1+k_2+\cdots+k_m}{k_1,k_2,\ldots,k_m} = \frac{(k_1+k_2+\cdots+k_m)!}{k_1!k_2!\cdots k_m!},$$

and can always be written as a product of binomial coefficients

$$\binom{k_1+k_2+\cdots+k_m}{k_1,k_2,\ldots,k_m} = \binom{k_1+k_2+\cdots+k_m}{k_1+k_2+\cdots+k_{m-1}} \cdots \binom{k_1+k_2+k_3}{k_1+k_2} \binom{k_1+k_2}{k_1}.$$

Let us adopt the following convention. For $k_1 + k_2 + \cdots + k_m \geq 1$, we put

$$\binom{k_1+k_2+\cdots+(k_j-1)+\cdots+k_m}{k_1,k_2,\ldots,k_j-1,\ldots,k_m} = 0 \quad \text{when } k_j = 0,$$

for any $j \in \{1, 2, \ldots, m\}$. We can now state the following lemma [p. 80 (Vol. 1), 4].

Lemma 1. *Let $k_j \geq 0$ be integers for $j \in \{1, 2, \ldots, m\}$, such that $k_1 + \cdots + k_m \geq 1$. Then*

$$\binom{k_1+k_2+\cdots+k_m}{k_1,k_2,\ldots,k_m} = \sum_{j=1}^{m} \binom{k_1+\cdots+(k_j-1)+\cdots+k_m}{k_1,\ldots,k_j-1,\ldots,k_m}.$$

This lemma permits us to easily prove the following result.

Lemma 2. *The sequence* $(y_n)_{n\in\mathbb{Z}}$ *, defined by relation* (6) *, satisfies the recurrence relation*

$$y_n = a_1 y_{n-1} + a_2 y_{n-2} + \cdots + a_m y_{n-m} \quad \text{for } n \geq 1$$

with $y_0 = 1$ *and* $y_n = 0$ *for* $n < 0$.

Proof. First notice that, for $n \geq 1$, for all $j \in \{1, 2, \ldots, m\}$ we have

$$a_j y_{n-j} = \sum_{k_1+2k_2+\cdots+mk_m=n} \binom{k_1+\cdots+(k_j-1)+\cdots+k_m}{k_1,\ldots,k_j-1,\ldots,k_m} a_1^{k_1}\ldots a_j^{k_j}\ldots a_m^{k_m}.$$

Applying Lemma 1, we obtain $\sum\limits_{j=1}^{m} a_j y_{n-j} = y_n$. The relations $y_0 = 1$ and $y_n = 0$ for $n < 0$ follow immediately. $\square$

We can now state the following result.

Theorem 2. *Let* $(u_n)_{n>-m}$ *the sequence of elements of* $\mathcal{A}$ *defined by*

$$\begin{cases} u_{-j} = 0 & \text{for } 1 \leq j \leq m-1, \\ u_0 = 1, \\ u_n = a_1 u_{n-1} + a_2 u_{n-2} + \cdots + a_m u_{n-m} & \text{for } n \geq 1. \end{cases}$$

Then for all integers $n > -m$,

$$u_n = \sum_{k_1+2k_2+\cdots+mk_m=n} \binom{k_1+k_2+\cdots+k_m}{k_1,k_2,\ldots,k_m} a_1^{k_1} a_2^{k_2}\ldots a_m^{k_m}.$$

Corollary 1. *Let* $q \geq 1$ *be an integer,* $a, b \in \mathcal{A}$*, and let* $(v_n)_{n\geq -q}$ *be the sequence of elements of* $\mathcal{A}$ *defined by*

$$\begin{cases} v_{-j} = 0 & \text{for } 1 \leq j \leq q, \\ v_0 = 1, \\ v_{n+1} = av_n + bv_{n-q} & \text{for } n \geq 0. \end{cases} \tag{7}$$

Then, for all $n \geq -q$,

$$v_n = \sum_{k=0}^{\left\lfloor \frac{n}{q+1} \right\rfloor} \binom{n-kq}{k} a^{n-k(q+1)} b^k, \tag{8}$$

and, for all $n \geq 0$,

$$v_{n+1} + bv_{n-q} = 2v_{n+1} - av_n = \sum_{k=0}^{\left\lfloor \frac{n+1}{q+1} \right\rfloor} \frac{n+1-k(q-1)}{n+1-kq} \binom{n+1-kq}{k} a^{n+1-k(q+1)} b^k. \tag{9}$$

Proof. We deduce relation (8) directly from Theorem 2, with $m = q+1$, $a_1 = a$, $a_m = b$ and $a_k = 0$ for $1 < k < m$. From (8), we deduce (9) as follows:

$$\begin{aligned}
v_{n+1} + bv_{n-q} &= \sum_{k=0}^{\lfloor \frac{n+1}{q+1} \rfloor} \binom{n+1-kq}{k} a^{n+1-k(q+1)} b^k + \sum_{k=0}^{\lfloor \frac{n-q}{q+1} \rfloor} \binom{n-(k+1)q}{k} a^{n-q-(q+1)k} b^{k+1} \\
&= \sum_{k=0}^{\lfloor \frac{n+1}{q+1} \rfloor} \binom{n+1-kq}{k} a^{n+1-k(q+1)} b^k + \sum_{k=1}^{\lfloor \frac{n+1}{q+1} \rfloor} \binom{n-kq}{k-1} a^{n+1-k(q+1)} b^k \\
&= \sum_{k=0}^{\lfloor \frac{n+1}{q+1} \rfloor} \binom{n+1-kq}{k} a^{n+1-k(q+1)} b^k + \sum_{k=0}^{\lfloor \frac{n+1}{q+1} \rfloor} \frac{k}{n+1-kq} \binom{n+1-kq}{k} a^{n+1-k(q+1)} b^k \\
&= \sum_{k=0}^{\lfloor \frac{n+1}{q+1} \rfloor} \frac{n+1-k(q-1)}{n+1-kq} \binom{n+1-kq}{k} a^{n+1-k(q+1)} b^k.
\end{aligned}$$

□

We now give some applications of the above corollary.

Application 1. Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence

$$\begin{cases} F_0 = 0, \\ F_1 = 1, \\ F_{n+1} = F_n + F_{n-1} \quad \text{for } n \geq 1. \end{cases}$$

Then, by setting $\varphi_n = F_{n+1}$ for $n \geq -1$, we see that $(\varphi_n)_{n\geq -1}$ is also defined by

$$\begin{cases} \varphi_{-1} = 0, \\ \varphi_0 = 1, \\ \varphi_n = \varphi_{n-1} + \varphi_{n-2} \quad \text{for } n \geq 1. \end{cases}$$

The application of Corollary 1 gives us that

$$\varphi_n = F_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}, \quad \text{for } n \geq -1,$$

the relation given in [pp. 18-20, 12], and announced in [9]. Also

$$F_n + F_{n+2} = \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \frac{n+1}{n+1-k} \binom{n+1-k}{k}, \quad \text{for } n \geq 0,$$

and we find the relations given in Problem 6.98 of [10], which state that

$$F_{2n-1} + F_{2n+1} = \sum_{k=0}^{n} \frac{2n}{2n-k} \binom{2n-k}{k}$$

and

$$F_{2n} + F_{2n+2} = \sum_{k=0}^{n} \frac{2n+1}{2n+1-k} \binom{2n+1-k}{k}.$$

Application 2. For $q \geq 1$, let $\left(G_n^{(q)}\right)_{n\geq 0}$ be the generalized Fibonacci sequence as cited in [8] , and let $\left(H_n^{(q)}\right)_{n\geq 0}$ be a sequence of numbers defined as follows:

$$\begin{cases} G_0^{(q)} = G_1^{(q)} = \cdots = G_q^{(q)} = 1, \\ G_{n+1}^{(q)} = G_n^{(q)} + G_{n-q}^{(q)} \quad \text{for } n \geq q, \end{cases} \quad \text{and} \quad \begin{cases} H_0^{(q)} = H_1^{(q)} = \cdots = H_q^{(q)} = 1, \\ H_{n+1}^{(q)} = H_n^{(q)} - H_{n-q}^{(q)} \text{ for } n \geq q. \end{cases}$$

We can extend easily the above sequences to $(G_n)_{n\geq -q}$ and $(H_n)_{n\geq -q}$ by

$$\begin{cases} G_{-j}^{(q)} = 0 \ \text{ for } 1 \leq j \leq q, \\ G_0^{(q)} = 1, \\ G_{n+1}^{(q)} = G_n^{(q)} + G_{n-q}^{(q)} \quad \text{for } n \geq 0, \end{cases} \quad \text{and} \quad \begin{cases} H_{-j}^{(q)} = 0 \quad \text{for } 1 \leq j \leq q, \\ H_0^{(q)} = 1, \\ H_{n+1}^{(q)} = H_n^{(q)} - H_{n-q}^{(q)} \quad \text{for } n \geq 0. \end{cases}$$

The application of Corollary 1 gives us, for $n \geq -q$, the relations

$$G_n^{(q)} = \sum_{k=0}^{\lfloor \frac{n}{q+1} \rfloor} \binom{n-kq}{k}, \qquad \text{and} \qquad H_n^{(q)} = \sum_{k=0}^{\lfloor \frac{n}{q+1} \rfloor} (-1)^k \binom{n-kq}{k}.$$

Notice that $G_n^{(1)} = F_{n+1} = \varphi_n$, and $H_n^{(2)}$ is the integer function studied by L. Bernstein [1] , who showed that the only zeros of $H_n^{(2)}$ are at $n = 3$ and $n = 12$. This result was treated also by L. Carlitz [2, 3] and recently by J. McLaughlin and B. Sury [8].

The following theorem gives us an explicit formulation for u_n in terms of n, a_1, $a_2, \ldots$, a_m, α_0, $\alpha_1, \ldots, \alpha_{m-1}$, and thus generalizes Theorem 2.

Theorem 3. *Let $(u_n)_{n>-m}$ be a sequence of elements of $\mathcal{A}$ defined by*

$$\begin{cases} u_{-j} = \alpha_j \quad \text{for } 0 \leq j \leq m-1, \\ u_n = a_1 u_{n-1} + a_2 u_{n-2} + \cdots + a_m u_{n-m} \quad \text{for } n \geq 1. \end{cases} \tag{10}$$

Let $(\lambda_j)_{0\leq j\leq m-1}$ and $(y_n)_{n>-m}$ be the sequences of elements of $\mathcal{A}$ defined by

$$\lambda_j = -\sum_{k=j}^{m-1} a_{k-j}\alpha_k, \quad \text{for } 0 \leq j \leq m-1, \ \text{with } a_0 = -1, \tag{11}$$

and

$$y_n = \sum_{k_1+2k_2+\cdots+mk_m=n} \binom{k_1+k_2+\cdots+k_m}{k_1, k_2, \ldots, k_m} a_1^{k_1} a_2^{k_2} \ldots a_m^{k_m}, \quad \text{for } n > -m.$$

Then for all integer $n > -m$, we have $u_n = \lambda_0 y_n + \lambda_1 y_{n+1} + \cdots + \lambda_{m-1} y_{n+m-1}$.

Remark. Note that (11) is equivalent to

$$\begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{m-2} \\ \lambda_{m-1} \end{pmatrix} = \begin{pmatrix} 1 & -a_1 & -a_2 & \cdots & -a_{m-2} & -a_{m-1} \\ 0 & 1 & -a_1 & -a_2 & \cdots & -a_{m-2} \\ 0 & 0 & 1 & -a_1 & \cdots & -a_{m-3} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & -a_1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{m-2} \\ \alpha_{m-1} \end{pmatrix}.$$

or

$$[\lambda_0, \lambda_1, \ldots, \lambda_{m-1}]^t = C\,[\alpha_0, \alpha_1, \ldots, \alpha_{m-1}]^t, \tag{12}$$

where

$$C = (c_{ij})_{1 \le i,j \le m}, \quad \text{with } c_{ij} = \begin{cases} 0 & \text{if } i > j, \\ 1 & \text{if } i = j, \\ -a_{j-i} & \text{if } i < j. \end{cases}$$

We deduce also from relations (10) and (12) the matrix equality

$$\begin{pmatrix} u_n \\ u_{n+1} \\ \vdots \\ u_{n+m-2} \\ u_{n+m-1} \end{pmatrix} = \begin{pmatrix} y_n & y_{n+1} & \cdots & y_{n+m-2} & y_{n+m-1} \\ y_{n+1} & y_{n+2} & \cdots & y_{n+m-1} & y_{n+m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ y_{n+m-2} & y_{n+m-1} & \cdots & y_{n+2m-4} & y_{n+2m-3} \\ y_{n+m-1} & y_{n+m} & \cdots & y_{n+2m-3} & y_{n+2m-2} \end{pmatrix} \times$$

$$\times \begin{pmatrix} 1 & -a_1 & -a_2 & \cdots & -a_{m-1} \\ 0 & 1 & -a_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -a_2 \\ 0 & & 0 & 1 & -a_1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{m-2} \\ \alpha_{m-1} \end{pmatrix}$$

Proof. Let $\mathcal{S}$ be the $\mathcal{A}$-module of sequences $(v_n)_{n>-m}$ satisfying the recurrence relation

$$v_n - a_1 v_{n-1} - a_2 v_{n-2} - \cdots - a_m v_{n-m} = 0 \quad \text{for all } n \ge 1.$$

Let us consider the family of sequences $\left(v_n^{(k)}\right)_{n>-m}$, for $0 \le k \le m-1$, defined by

$$\begin{cases} v_n^{(0)} &= y_n, \\ v_n^{(1)} &= y_{n+1} - a_1 y_n, \\ v_n^{(2)} &= y_{n+2} - a_1 y_{n+1} - a_2 y_n, \\ &\vdots \\ v_n^{(m-1)} &= y_{n+m-1} - a_1 y_{n+m-2} - a_2 y_{n+m-3} - \cdots - a_{m-1} y_n, \end{cases}$$

i.e.,

$$\begin{aligned} v_n^{(k)} &= y_{n+k} - (a_1 y_{n+k-1} + a_2 y_{n+k-2} + \cdots + a_k y_n) \\ &= \sum_{i=0}^{k} -a_i y_{n+k-i}, \qquad \text{with } a_0 = -1. \end{aligned} \tag{13}$$

By Theorem 2, we have $(y_n)_{n>-m} \in \mathcal{S}$. Consequently, $(y_{n+q})_{n>-m} \in \mathcal{S}$ for $q \geq 0$, and finally we deduce that

$$\left(v_n^{(k)}\right)_{n>-m} \in \mathcal{S}, \quad \text{for } 0 \leq k \leq m-1. \tag{14}$$

It is easy to observe that we also have, for $j, k \in \{0, 1, 2, \ldots, m-1\}$,

$$v_{-j}^{(k)} = \delta_{kj} = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases} \tag{15}$$

In fact, with $a_0 = -1$, we can write, for $j, k \in \{0, 1, \ldots, m-1\}$,$v_{-j}^{(k)} = -\sum_{i=0}^{k} a_i y_{-j+k-i}$.

If $k < j$, then $-j + k - i < 0$ for $1 \leq i \leq k$ and $v_{-j}^{(k)} = 0$ (because $y_q = 0$ for $q < 0$). If $k = j$, then $v_{-j}^{(k)} = v_{-j}^{(j)} = -\sum\limits_{i=0}^{j} a_i y_{-i} = -a_0 y_0 = 1$. If $k > j$, then for $r = k - j < 0$, we have $r \geq 1$ and

$$\begin{aligned} v_{-j}^{(k)} &= y_r - (a_1 y_{r-1} + a_2 y_{r-2} + \cdots + a_k y_{r-k}), \quad \text{and by using } r - k = -j \leq 0, \\ &= y_r - (a_1 y_{r-1} + a_2 y_{r-2} + \cdots + a_k y_{r-k} + \cdots + a_m y_{r-m}) \\ &= 0 \qquad \text{because } (y_n)_{n>-m} \in \mathcal{S} \text{ and } r \geq 1. \end{aligned}$$

Relations (14) and (15) give easily, that for all $n > -m$,$u_n = \sum\limits_{k=0}^{m-1} \alpha_k v_n^{(k)}$ and, with (13),

$$\begin{aligned} u_n &= \sum_{k=0}^{m-1} \alpha_k \left(\sum_{i=0}^{k} -a_i y_{n+k-i} \right) \\ &= \sum_{k=0}^{m-1} \sum_{j=0}^{k} -a_{k-j} \alpha_k y_{n+j} \\ &= \sum_{j=0}^{m-1} \left(\sum_{k=j}^{m-1} -a_{k-j} \alpha_k \right) y_{n+j} \\ &= \sum_{j=0}^{m-1} \lambda_j y_{n+j}, \end{aligned}$$

where λ_j is as defined in (11). $\square$

Theorem 3 allows us to find various formulae for the Fibonacci numbers.

Corollary 2. *For all integers $n \geq 1$,*

$$F_n = \frac{1}{2^{n+2}} \sum_{k=1}^{n} \frac{n+1-k}{k} \binom{k}{n+1-2k} 8^k.$$

Remark. In this summation, we may, in fact, restrict the sum to those integers i ranging between $\frac{n+1}{3}$ and $\frac{n+1}{2}$, the binomial coefficient of the formula being zero for the other integers.

Proof. Note that $F_n = 2F_{n-2} + F_{n-3}$ for $n \geq 3$. Denoting by $(y_n)_{n \geq -2}$ the sequence defined by

$$\begin{cases} y_{-2} = y_{-1} = 0, \\ y_0 = 1, \\ y_n = 2y_{n-2} + y_{n-3} & \text{for } n \geq 1, \end{cases}$$

we see that, for $n \geq 1, F_n = y_{n-1} + y_{n-2}$. Theorem 3 allows us to state that, for $n \geq -2$,

$$y_n = \sum_{2k+3l=n} \binom{k+l}{l} 2^k = \sum_{0 \leq t \leq n} \binom{t}{n-2t} 2^{3t-n},$$

and Corollary 2 follows by simple calculations. □

The following result can also be easily deduced from Theorem 3.

Corollary 3. *For all integers $m \geq 1$, $F_{2m-1} = \sum_{k=0}^{\lfloor m/3 \rfloor} \frac{m+k}{m-k} \binom{m-k}{2k} 2^{m-1-3k}$.*

Now, let us consider for $q > 1$, the "multibonacci" sequence $\left(\phi_n^{(q)}\right)_{n > -q}$ defined by

$$\begin{cases} \phi_{-(q-1)}^{(q)} = \cdots = \phi_{-2}^{(q)} = \phi_{-1}^{(q)} = 0, \\ \phi_0^{(q)} = 1, \\ \phi_n^{(q)} = \phi_{n-1}^{(q)} + \phi_{n-2}^{(q)} + \cdots + \phi_{n-q}^{(q)} & \text{for } n \geq 1, \end{cases} \tag{16}$$

where $\phi_n^{(2)} = \varphi_n = F_{n+1} = G_n^{(1)}$. Theorem 3 also implies that, for all $n \geq 0$,

$$\phi_n^{(q)} = \sum_{k_1 + 2k_2 + \cdots + qk_q = n} \binom{k_1 + k_2 + \cdots + k_q}{k_1, k_2, \ldots, k_q}.$$

Thus, for $q = 3$, we obtain

$$\phi_n^{(3)} = \sum_{i+2j+3k=n} \binom{i+j+k}{i, j, k} = \sum_{2i+3j \leq n} \binom{n-i-2j}{i+j} \binom{i+j}{i}.$$

We deduce from (16) that $\phi_{n+1}^{(q)} = 2\phi_n^{(q)} - \phi_{n-q}^{(q)}$, for $n \geq 1$. Let us consider $(\psi_n)_{n\geq -q}$, the sequence defined by $\psi_n := \phi_n^{(q)}$, for $n > -q$ and $\psi_{-q} = 1$, which satisfies the recurrence relation$\psi_{n+1} = 2\psi_n - \psi_{n-q}$, for $n \geq 0$. After applying Theorem 3, we find that, for $n \geq 0$,

$$\begin{aligned} \psi_n \left(= \phi_n^{(q)}\right) &= \lambda_0 z_n + \lambda_1 z_{n+1} + \cdots + \lambda_q z_{n+q} \\ &= z_n - 2z_{n+q-1} + z_{n+q}, \end{aligned}$$

with $z_n = \sum_{k=0}^{\lfloor \frac{n}{q+1} \rfloor} \binom{n-kq}{k} 2^{n-k(q+1)} (-1)^k$, for $n \geq -q$. We know, via Theorem 3 and Corollary 1, that the sequence $(z_n)_{n\geq -q}$ satisfies the recurrence relation $z_{n+1} - 2z_n + z_{n-q} = 0$, for $n \geq 0$. This gives, for $n \geq 0$,

$$\begin{aligned} \phi_n^{(q)} &= z_n - z_{n-1} + (z_{n+q} - 2z_{n+q-1} + z_{n-1}) \\ &= z_n - z_{n-1}. \end{aligned}$$

Applying relation (9) in Corollary 1, we can write, for $n \geq 1$,

$$\phi_n^{(q)} = \sum_{k=0}^{\lfloor \frac{n}{q+1} \rfloor} \frac{n - k(q-1)}{n-kq} \binom{n-kq}{k} 2^{n-1-k(q+1)} (-1)^k.$$

Thus,

$$\sum_{k_1+2k_2+\cdots+qk_q=n} \binom{k_1+\cdots+k_q}{k_1,\ldots,k_q} = \sum_{k=0}^{\lfloor \frac{n}{q+1} \rfloor} \frac{n-k(q-1)}{n-kq} \binom{n-kq}{k} 2^{n-1-k(q+1)} (-1)^k. \quad (17)$$

For $q = 3$, we obtain

$$\begin{aligned} \sum_{i+2j+3k=n} \binom{i+j+k}{i,j,k} &= \sum_{2i+3j\leq n} \binom{n-i-2j}{i+j} \binom{i+j}{i} \\ &= \sum_{k=0}^{\lfloor \frac{n}{4} \rfloor} \frac{n-2k}{n-3k} (-1)^k \binom{n-3k}{k} 2^{n-1-4k}. \end{aligned}$$

3. Powers of a Square Matrix of Order m

We start this section with the main result of this paper.

Theorem 4. *Let $A \in \mathcal{M}_m(\mathcal{A})$ and let $P(X) = X^m - a_1X^{m-1} - \cdots - a_{m-1}X - a_m$ be the characteristic polynomial of A. Let $A_0, A_1, \ldots, A_{m-1}$ be matrices of $\mathcal{M}_m(\mathcal{A})$ defined by*

$$A_k = -\sum_{i=0}^{k} a_i A^{k-i}, \quad \text{for } 0 \leq k \leq m-1, \text{ with } a_0 = -1, \quad (18)$$

and let $(y_n)_{n>-m}$ *be the sequence of elements of* A *satisfying*

$$y_n = \sum_{k_1+2k_2+\cdots+mk_m=n} \binom{k_1+k_2+\cdots+k_m}{k_1,k_2,\ldots,k_m} a_1^{k_1}a_2^{k_2}\ldots a_m^{k_m}, \quad \text{for } n>-m.$$

Then, for all integers $n \geq 0$,

$$A^n = y_n A_0 + y_{n-1}A_1 + \cdots + y_{n-m+1}A_{m-1} \tag{19}$$

Proof. Define, for $0 \leq k \leq m$,

$$P_k(X) = -\sum_{i=0}^{k} a_i X^{k-i}, \quad \text{with } a_0 = -1. \tag{20}$$

For $0 \leq k \leq m-1$, we have

$$XP_k(X) = P_{k+1}(X) + a_{k+1}. \tag{21}$$

Relation (20) shows that the degree of P_k is k for $0 \leq k \leq m$. It implies that $(P_0, P_1, \ldots, P_{m-1})$ is a basis of the free $\mathcal{A}$-module $\mathcal{A}_{m-1}[X]$ consisting of polynomials of $\mathcal{A}[X]$ of degree $\leq m-1$.

For all $n \geq 0$, the remainder R_n of the euclidean division of X^n by P_m is written as a linear combination of polynomials $P_0, P_1, \ldots, P_{m-1}$. Then, for all $n \geq 0$, there exists a unique family $(\alpha_{n,k})_{0\leq k\leq m-1}$ such that

$$R_n = \sum_{k=0}^{m-1} \alpha_{n,k} P_k. \tag{22}$$

For $0 \leq n \leq m-1$, we have $R_n(X) = X^n$, where X^n is a linear combination of $P_0, \ldots, P_{m-1}$, and

$$\begin{cases} \alpha_{0,0} = 1 \\ \alpha_{n,k} = 0 \quad \text{for } n < k \leq m-1. \end{cases} \tag{23}$$

Relations (21) and (22) imply that

$$\begin{aligned} XR_n(X) &= \sum_{k=0}^{m-1} \alpha_{n,k}\left(P_{k+1}(X) + a_{k+1}\right) \\ &= \sum_{k=0}^{m-1} a_{k+1}\alpha_{n,k} + \sum_{k=1}^{m-1} \alpha_{n,k-1}P_k(X) + \alpha_{n,m-1}P_m(X). \end{aligned}$$

As a consequence, the polynomial $R_{n+1}(X) - XR_n(X) - \alpha_{n,m-1}P_m(X)$, of degree $\leq m-1$, is divisible by $P_m(X)$, which is of degree m. This polynomial is thus the zero polynomial as well as its components in the basis $(P_0, P_1, \ldots, P_{m-1})$. This provides us with the following relations:

$$\alpha_{n+1,0} = \sum_{k=0}^{m-1} a_{k+1}\alpha_{n,k}, \quad \text{for } n \geq 0, \tag{24}$$

$$\alpha_{n+1,k} = \alpha_{n,k-1} \quad \text{for } 1 \leq k \leq m-1. \tag{25}$$

Let us set, for all integers $n \in \mathbb{Z}$,

$$z_n = \begin{cases} \alpha_{n,0} & \text{for } n \geq 0, \\ 0 & \text{for } n < 0. \end{cases} \tag{26}$$

One checks easily that for all integers $n \geq 0$ and for $0 \leq k \leq m-1$,

$$\alpha_{n,k} = z_{n-k}. \tag{27}$$

Indeed, if $n \geq k$ this relation follows from (25) and (26), and if $0 \leq n < k \leq m-1$ it follows from (23) and (26). From (25), (26) and (27), we find that

$$\begin{cases} z_n = 0 \quad \text{for } n < 0, \\ z_0 = 1, \\ z_n = a_1 z_{n-1} + a_2 z_{n-2} + \cdots + a_{m-1} z_{n-m-1} + a_m z_{n-m} \;\; \text{for } n \geq 1. \end{cases}$$

Theorem 2 implies that $z_n = y_n$, for all $n \in \mathbb{Z}$, where

$$y_n = \sum_{k_1+2k_2+\cdots+mk_m=n} \binom{k_1+k_2+\cdots+k_m}{k_1, k_2, \ldots, k_m} a_1^{k_1} a_2^{k_2} \ldots a_m^{k_m}.$$

This last fact, together with the Cayley-Hamilton Theorem, (22), (20), (18) and (27) now give that

$$A^n = R_n(A) = \sum_{i=0}^{m-1} \alpha_{n,i} P_i(A) = \sum_{i=0}^{m-1} z_{n-i} A_i = \sum_{i=0}^{m-1} y_{n-i} A_i.$$

which completes the proof of (19). □

4. Further Combinatorial Identities

Some nice combinatorial identities can be derived from Theorem 4 by considering various particular matrices with simple forms.

Corollary 4. *Let n be a positive integer. Then*

$$\sum_{k_1+2k_2+\cdots+mk_m=n} \binom{k_1+\cdots+k_m}{k_1,\ldots,k_m} (-1)^{n-(k_1+\cdots+k_m)} \binom{m}{m-1}^{k_1} \cdots \binom{m}{0}^{k_m} = \binom{n+m-1}{m-1},$$

where $\binom{n+m-1}{m-1}$ *is the number of m- combinations with repetition of finite set* $\{1,\ldots,n\}$, *and the summation is taken over all m-tuples* $(k_1,k_2,\ldots,k_m)$ *of integers* $k_j \geq 0$ *satisfying the relation* $k_1+2k_2+\cdots+mk_m=n$.

Proof. Let J_m be the $m\times m$ Jordan matrix,

$$J_m=\begin{pmatrix} 1 & 1 & 0 & \cdots & 0\\ 0 & 1 & 1 & \ddots & \vdots\\ 0 & 0 & \ddots & \ddots & 0\\ \vdots & \vdots & \ddots & 1 & 1\\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

The characteristic polynomial of J_m is $(X-1)^m$. We also have $J_m^n=\left(\binom{n}{j-i}\right)_{1\leq i,j\leq m}$.

Applying Theorem 4 with $A=J_m$, and considering the $(1,m)$ entries of both sides of (19), we obtain the relation $\binom{n}{m-1}=y_{n-(m-1)}$, which leads to the result. □

From Corollary 4, we obtain the following combinatorial identities, for $m=2,3,4$.

$$\sum_{2i\leq n}(-1)^i\binom{n-i}{i}2^{n-2i}=n+1,$$

$$\sum_{2i+3j\leq n}(-1)^i\binom{n-i-2j}{i+j}\binom{i+j}{i}3^{n-i-3j}=\binom{n+1}{2}$$

$$\sum_{2i+3j+4k\leq n}(-1)^{i+k}\binom{n-i-2j-3k}{i+j+k}\binom{i+j+k}{i+j}\binom{i+j}{i}2^{2n-3i-4j-8k}3^i=\binom{n+3}{3}$$

Like J. Mc Laughlin and B. Sury [8, Corollary 6], we can also derive Corollary 4 from the following result.

Corollary 5. ([8, Theorem 1]) *Let* $x_1,x_2,\ldots,x_m$ *be elements of the unitary commutative ring* $\mathcal{A}$ *with* $\mathrm{s}_k=\sum_{1\leq i_1<i_2<\cdots<i_k\leq m}x_{i_1}x_{i_2}\cdots x_{i_k}$, *for* $1\leq k\leq m$. *Then, for each positive integer n,*

$$\sum_{k_1+k_2+\cdots+k_m=n}x_1^{k_1}\ldots x_m^{k_m}=\sum_{k_1+2k_2+\cdots+mk_m=n}\binom{k_1+\cdots+k_m}{k_1,\ldots,k_m}(-1)^{n-k_1-\cdots-k_m}\mathrm{s}_1^{k_1}\ldots\mathrm{s}_m^{k_m},$$

with the summations being taken over all m-tuples $(k_1,k_2,\ldots,k_m)$ *of integers* $k_j\geq 0$ *satisfying the relations* $k_1+k_2+\cdots+k_m=n$ *for the left-hand side and* $k_1+2k_2+\cdots+mk_m=n$ *for the right-hand side.*

Proof. Let us give a proof of this result by using Theorem 2. For $n > -m$ and $1 \leq l \leq m$, consider $q_n^{(l)} := \sum_{k_1+k_2+\cdots+k_l=n} x_1^{k_1} x_2^{k_2} \ldots x_l^{k_l}$, and $q_n := q_n^{(m)}$. Let $E : \beta_n \longmapsto \beta_{n+1}$ be the shift operator which acts on any sequence $(\beta_n)_n$, and, for $1 \leq l \leq m$, let Q_l be the operator given by $Q_l := (E - x_1)(E - x_2) \cdots (E - x_l)$. Notice that $Q_m = E^m - \mathrm{s}_1 E^{m-1} + \cdots + (-1)^m \mathrm{s}_m$. Then, for $n \geq 1$ and $2 \leq l \leq m$, we have $(E - x_l) \cdot q_{n-l}^{(l)} = q_{n-(l-1)}^{(l-1)}$. Therefore,

$$Q_m \left(q_{n-m}^{(m)} \right) = Q_{m-1} \left(q_{n-(m-1)}^{(m-1)} \right) = \cdots = Q_1 \left(q_{n-1}^{(1)} \right) = 0.$$

Thus,

$$q_n = \mathrm{s}_1 q_{n-1} - \mathrm{s}_2 q_{n-2} + \cdots + (-1)^{m-1} \mathrm{s}_m q_{n-m}, \quad \text{for } n \geq 1. \tag{28}$$

By the definition of q_n, we also have

$$q_n = 0 \text{ for } n < 0 \text{ and } q_0 = 1. \tag{29}$$

Applying Theorem 2 to the sequence $(q_n)_{n>-m}$, we obtain the result. □

Corollary 4 follows immediately by setting $x_i = 1$ for all i in the previous corollary.

The following theorem is an extension of a result of J. Mc Laughlin and B. Sury [8, Theorem 3].

Theorem 5. *Let K be a field of characteristic zero, and let $x_1, x_2, \ldots, x_m$ be independent variables. Then, in $K(x_1, x_2, \ldots, x_m)$,*

$$\sum_{k_1+\cdots+k_m=n} x_1^{k_1} x_2^{k_2} \ldots x_m^{k_m} = \sum_{i=1}^{m} \frac{x_i^{n+m-1}}{\prod_{j \neq i} (x_i - x_j)}.$$

Proof. For $\gamma_1, \gamma_2, \ldots, \gamma_m \in K(x_1, x_2, \ldots, x_m)$, let $V(\gamma_1, \gamma_2, \ldots, \gamma_m)$ denote the Vandermonde determinant defined by $V(\gamma_1, \gamma_2, \ldots, \gamma_m) = \det \left(\gamma_i^{j-1} \right)_{1 \leq i,j \leq m}$. It is well known that $V(\gamma_1, \gamma_2, \ldots, \gamma_m) = \prod_{1 \leq i < j \leq m} (\gamma_j - \gamma_i)$. By (28), the sequence $(q_n)_{n>-m}$ defined by

$$q_n = \sum_{k_1+k_2+\cdots+k_m=n} x_1^{k_1} x_2^{k_2} \ldots x_m^{k_m}$$

is a recurrent sequence with characteristic polynomial

$$X^m - \mathrm{s}_1 X^{m-1} + \cdots + (-1)^m \mathrm{s}_m = (X - x_1)(X - x_2) \cdots (X - x_m).$$

This polynomial has m distincts roots $x_1, x_2, \ldots, x_m$. We deduce that there exist m elements $A_i = A_i(x_1, x_2, \ldots, x_m) \in K(x_1, x_2, \ldots, x_m)$, $1 \leq i \leq m$, such that

$$q_n = \sum_{i=1}^{m} A_i x_i^n \quad \text{for } n \geq -m.$$

The initial conditions given by (28) lead to the Cramer system $\sum_{i=1}^{m} \frac{A_i}{x_i^j} = \delta_{j,0}$ for $0 \leq j \leq m-1$. The resolution of this system gives

$$A_i = \frac{V\left(\frac{1}{x_1}, \ldots, \frac{1}{x_{i-1}}, 0, \frac{1}{x_{i+1}}, \ldots, \frac{1}{x_m}\right)}{V\left(\frac{1}{x_1}, \ldots, \frac{1}{x_{i-1}}, \frac{1}{x_i}, \frac{1}{x_{i+1}}, \ldots, \frac{1}{x_m}\right)} = \frac{x_i^{m-1}}{\prod_{j\neq i}(x_i - x_j)}, \quad \text{for } 1 \leq i \leq m.$$

This completes the proof. □

Let us now give an application to Stirling numbers. For $n \geq 0$, with the notations of [5], the Stirling numbers of the first kind $\left[\begin{array}{c} n \\ k \end{array}\right]$, and the Stirling numbers of the second kind $\left\{\begin{array}{c} n \\ k \end{array}\right\}$, can be defined by the equations

$$X^{\overline{n}} = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right] X^k, \tag{30}$$

with $X^{\overline{n}} := \begin{cases} 1 & \text{if } n = 0, \\ X(X+1)(X+2)\cdots(X+n-1) & \text{if } n \geq 1, \end{cases}$ and

$$X^n = \sum_{k=0}^{n} \left\{\begin{array}{c} n \\ k \end{array}\right\} X^{\underline{k}}, \tag{31}$$

with $X^{\underline{k}} := \begin{cases} 1 & \text{if } k = 0, \\ X(X-1)(X-2)\cdots(X-k+1) & \text{if } k \geq 1. \end{cases}$

It is well known [p. 38 (Vol. 2), 4] that $\left\{\begin{array}{c} n \\ k \end{array}\right\} = \frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i} i^n$. From Theorem 5 and Corollary 5, we deduce

$$\sum_{k_1+2k_2+\cdots+mk_m=n} \binom{k_1+\cdots+k_m}{k_1,\ldots,k_m} (-1)^{n-k_1-\cdots-k_m} \mathrm{s}_1^{k_1}\ldots\mathrm{s}_m^{k_m} = \sum_{i=1}^{m} \frac{x_i^{n+m-1}}{\prod_{j\neq i}(x_i-x_j)}. \tag{32}$$

If we take $x_k = -(k-1)$ for $1 \leq k \leq m$, we have $\mathrm{s}_k = (-1)^k \left[\begin{array}{c} m \\ m-k \end{array}\right]$, for $1 \leq k \leq m$, and relation (32) gives the following result

Corollary 6. *For all positive integers m and n,*

$$\sum_{k_1+2k_2+\cdots+mk_m=n} \binom{k_1+\cdots+k_m}{k_1,\ldots,k_m} (-1)^{n-(k_1+\cdots+k_m)} \left[\begin{array}{c} m \\ m-1 \end{array}\right]^{k_1} \cdots \left[\begin{array}{c} m \\ 0 \end{array}\right]^{k_m} = \left\{\begin{array}{c} n+m-1 \\ m-1 \end{array}\right\}.$$

Note that Theorem 5 gives the following relation, which is stated in [p. 42, (Vol. 2), 4].

For all positive integers m and n, $\left\{ \begin{matrix} n+m-1 \\ m-1 \end{matrix} \right\} = \sum_{k_1+\cdots+k_{m-1}=n} 1^{k_1} 2^{k_2} \ldots (m-1)^{k_{m-1}}$.

For $m = 3, 4$ and 5, we obtain

$$\sum_{2i\le n} (-1)^i \binom{n-i}{i} 2^i 3^{n-2i} = 2^{n+1} - 1,$$

$$\sum_{2i+3j\le n} (-1)^i \binom{n-i-2j}{i+j} \binom{i+j}{i} 6^{n-2i-2j} 11^i = \frac{3^{n+2} - 2^{n+3} + 1}{2},$$

$$\sum_{2i+3j+4k\le n} (-1)^{i+k} \binom{n-i-2j-3k}{i+j+k} \binom{i+j+k}{i+j} \binom{i+j}{i} 2^{n-2i-2j-k} 3^k 5^{n-i-j-4k} 7^i = \frac{4^{n+3} - 3^{n+4} + 3 \cdot 2^{n+3} - 1}{6}.$$

Corollary 7. *Let n be a positive integer and let x and y be indeterminates. Then*

$$\sum_{2i+3j+4k\le n} (-1)^{i+k} \binom{n-i-2j-3k}{i+j+k} \binom{i+j+k}{i+j} \binom{i+j}{i} (2x+2y)^{n-2i-2j-4k} \times$$
$$\times (xy)^{j+2k} \left(x^2 + 4xy + y^2\right)^i = \frac{(n+1)\left(x^{n+3} - y^{n+3}\right) - (n+3)\, xy \left(x^{n+1} - y^{n+1}\right)}{(x-y)^3}.$$

Corollary 8. *Let n a positive integer and x, y be indeterminates. Then*

$$\sum_{2i+3j+4k\le n} (-1)^{i+k} \binom{n-i-2j-3k}{i+j+k} \binom{i+j+k}{i+j} \binom{i+j}{i} \times$$
$$\times (3x+y)^{n-2i-3j-4k} x^{i+2j+3k} y^k (3x+3y)^i (x+3y)^j$$
$$= \frac{(n+1)(n+2)\left(x^{n+3} - y^{n+3}\right) - (n+3)\, y \left[n\left(x^{n+2} - y^{n+2}\right) + (n+2)\, x^{n+1} (x-y)\right]}{2\,(x-y)^3}.$$

Proof of Corollaries 7 and 8. It suffices to put, in Corollary 4, $x_1 = x_2 = x$ and $x_3 = x_4 = y$ to obtain Corollary 7; $x_1 = x_2 = x_3 = x$ and $x_4 = y$ to obtain Corollary 8. □

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