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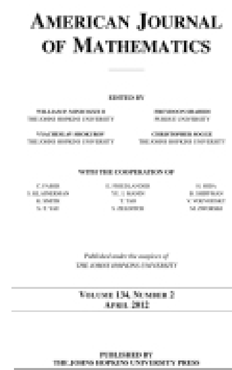
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EFFECTIVE BOUNDS FOR THE NUMBER OF TRANSCENDENTAL POINTS ON SUBVARIETIES OF SEMI-ABELIAN VARIETIES

By EHUD HRUSHOVSKI and ANAND PILLAY

Abstract. Let A be a semi-abelian variety, and X a subvariety of A, both defined over a number field. Assume that X does not contain $X_1 + X_2$ for any positive-dimensional subvarieties X_1, X_2 of A. Let Γ be a subgroup of $A(\mathbf{C})$ of finite rational rank. We give doubly exponential bounds for the size of $(X \cap \Gamma) \backslash X(\bar{\mathbf{Q}})$. Among the ingredients is a uniform bound, doubly exponential in the data, on finite sets which are quantifier-free definable in differentially closed fields. We also give uniform bounds on $X \cap \Gamma$ in the case where X contains no translate of any semi-abelian subvariety of A and Γ is a subgroup of $A(\mathbf{C})$ of finite rational rank which has trivial intersection with $A(\bar{\mathbf{Q}})$. (Here A is assumed to be defined over a number field, but X need not be.)

1. Introduction. Let A be a semi-abelian variety, X a 1-dimensional subvariety of A which is not a translate of a semi-abelian subvariety of A, and Γ a finite rational rank subgroup of A (namely $dim_{\mathbf{Q}}\Gamma \otimes \mathbf{Q}$ is finite). Assume the characteristic to be 0. It follows from [11] for example, that $X \cap \Gamma$ is finite. The issue is to obtain uniform and good bounds for the size of $X \cap \Gamma$, as a function of the data. It seems that very little is known about this in full generality. In [5], a slightly different question was posed, in the special case where A is the algebraic torus $(\mathbf{C}^\star)^2$, and both A and X are defined over a number field. The question (asked by Bombieri) was to find uniform bounds for the number of *generic* points of $X \cap \Gamma$, namely those points which are not rational over the algebraic closure of $\mathbf{Q}$. Buium obtained uniform bounds (of an iterated exponential nature) in the data ($deg(X)$, $rank(\Gamma)$). Buium used differential algebraic methods. The aim in the present paper is to improve the bounds to being doubly exponential in the relevant data, as well as to treat the more general case of semi-abelian varieties. We are also able to slightly weaken the hypothesis on X. The methods are also differential-algebraic, closely following Buium, but there are additional model-theoretic ingredients (coming from the theory of commutative groups of finite Morley rank), and there is also one more key observation regarding the differential-algebraic set-up which yields the doubly exponential bounds.

Before stating the results, let us discuss the nature of the data which for us will be quite naive. We work for now over the complex numbers. Let A be a semi-abelian variety of dimension n, and X a subvariety of dimension m, both defined over $\bar{\mathbf{Q}}$. Denote $\bar{\mathbf{Q}}$ by k. Assume A to be given as a locally closed

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subset of some projective space $\mathbf{P}$. Let $U_1, \dots, U_t$ be the intersections with A of the affine charts on $\mathbf{P}$. So the U_i form a covering of A by affine opens, and each U_i is defined over k. By definition $deg(U_i)$ is the degree of its Zariski closure in $\mathbf{P}$. Let $deg(A)$ denote $max\{deg(U_i)\colon\ i = 1, \dots, t\}$ and let $deg(X)$ be $max\{deg(X \cap U_i)\colon\ i = 1, \dots, t\}$. We assume we are also given a basis $\omega_1, \dots, \omega_n$ defined over k for the space of invariant differential forms on A, and that $N = max\{degree(\omega_i|U_j)\colon\ i = 1, \dots, n, j = 1, \dots, t\}$, where the degree of the restriction of the form to U_j means degree as a polynomial in $x_1, \dots, dx_1, \dots$.

With the above notation and data we prove:

THEOREM 1.1. *Suppose A (semiabelian variety) and X (subvariety of A) are defined over $\bar{\mathbf{Q}}$, and that for no positive-dimensional subvarieties X_1 and X_2 of A is $X_1 + X_2$ contained in X. Let Γ be a subgroup of A of rational rank r. Then $|(X \cap \Gamma)\setminus X(\bar{\mathbf{Q}})| \leq t(deg(X)^{r2^{mr}}(N^n)^{2^{mr}-1})$.*

In the special case studied by Buium [5], we have:

COROLLARY 1.2. *Let $f(x, y)$ be an irreducible polynomial of degree d over $\bar{\mathbf{Q}}$, whose zero-locus in $\mathbf{C}^\star \times \mathbf{C}^\star$ is not a translate of a one-dimensional torus. Let Γ be a multiplicative subgroup of $\mathbf{C}^\star$ of rational rank r. Then $|\{(a, b) \in \Gamma \times \Gamma\colon\ f(a, b) = 0, a \notin \bar{\mathbf{Q}}\}| \leq d^{r2^r}((r + 1)^2)^{2^r-1}$.*

The theorem is proved by adjoining a derivation D to $\mathbf{C}$ such that $\bar{\mathbf{Q}} = k$ is the field of constants, finding a finite-dimensional differential algebraic subgroup G of A which contains Γ, and then proving the result for G in place of Γ. G will be a much larger group, which without loss of generality contains $A(k)$. In any case, proving the theorem for G in place of Γ has two aspects: (i) the qualitative result that $(X \cap G)\setminus X(k)$ is finite, (ii) a general result estimating the sizes of finite sets definable in differentially closed fields.

The methods here are of course related to those in [7], [2] on the Mordell-Lang conjecture for function fields. In those papers, however, the *nonisotrivial case* was the object of attention, and the model-theoretic results used in [7] were quite deep. In the situation of the present paper (transcendental points on subvarieties defined over k), the model-theoretic aspect is considerably easier, depending on the relatively elementary theory of socles.

The assumption on X in Proposition 2.1—that it does not contain a sum of positive-dimensional varieties—is rather unpleasant, but as the referee pointed out, to drop this restriction would require knowing the bounds over number fields.

It should be said that the main theorem is also valid in the nonisotrivial case (with a modification of the bounds). This uses some a priori bounds obtained by Buium [3] for the degrees of certain algebraic subgroups of extensions of an abelian variety by a vector group, together with rather deeper model-theoretic results as well as the results of this paper on cardinalities of finite sets definable in differentially closed fields. We have concentrated here on the isotrivial context (as in Theorem 1.1) because of the relatively elementary nature of the model-theoretic ingredients, as well as the rather explicit differential equations involved and the resulting bounds.

For differential algebra, the reader is referred to [4], [1], and to [9] for the model theory of differential fields. The non logician reader may find the survey [14] useful. The reader is referred to [15] and [13] for the relevant background on groups of finite Morley rank and stability theory For differentially algebraic groups (i.e. groups definable in differentially closed fields), the notions "finite-dimensional" (as in [1]) and "finite Morley rank" are synonymous. Our varieties are reduced and possibly reducible. By $X \cap Y$ we mean the set-theoretic intersection in some universal domain.

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2. A finiteness theorem in a differential-algebraic setting. In this section K will be a differentially closed field, with derivation D, and k denotes the field of constants of K. We consider K (and sometimes k) as a universal domain for algebraic geometry. The reader should see that there is no harm in assuming $(K, +, \cdot, D)$ to be ω-saturated. Our aim is to prove the following:

PROPOSITION 2.1. *Let A be a semi-abelian variety and X a (closed) subvariety of A, both defined over k. Assume that there do not exist nontrivial subvarieties X_1, X_2 of A such that $X_1 + X_2 \subseteq X$. Let G be a finite-dimensional differential algebraic subgroup of $A(K)$. Then $(X \cap G)\backslash X(k)$ is finite.*

Remark 2.2. First note that if X is a curve in A which is not a translate of an algebraic subgroup of A, then X satisfies the hypothesis of Proposition 2.1. Secondly, the requirement that K be differentially closed is irrelevant. One only requires (K, D) to be a differential field. We can (and do) ask whether Proposition 2.1 is still true under the weaker hypothesis that X contains no translate of a nontrivial algebraic subgroup of A. But see Section 6 for some positive information in this case.

The proof will depend on some facts about finite-dimensional differential algebraic groups, some general model theory of finite Morley rank groups, as well as some elementary facts about commutative algebraic groups.

FACT 2.3. *Let A be a commutative d-dimensional algebraic group defined over k. Then there is a differential algebraic surjective homomorphism μ: $A \to G_a^d$ whose kernel is $A(k)$. (μ is just Kolchin's logarithmic derivative, which will appear again and be defined formally in Section 4.)*

The following is due to Buium [1]. A proof also appears in the second author's article in [9]. $Tor(A)$ denotes the torsion subgroup of A.

FACT 2.4. *Let A be a commutative algebraic group over k, and G a Zariski-dense differential algebraic subgroup. Then G contains $Tor(A)$.*

Before the next fact, we recall some model-theoretic notions. Suppose M is a (saturated) model of an ω-stable theory. Let G be a group definable in M. Following [7], G is said to be *semi-pluriminimal* if there is a finite number $X_1, \ldots, X_n$ of strongly minimal definable sets in M, such that G is contained in $acl(X_1, \ldots, X_n)$, after possibly adding a finite set of parameters, equivalently if G is contained in $acl(X)$ for some M-definable set X of Morley rank 1. ($acl(-)$ denotes model theoretic algebraic closure.) Suppose G is a definable group, defined over a finite set A of parameters. We say that G is *rigid*, if every connected definable subgroup of G is defined over $acl(A)$. If X is a definable subset of G of Morley multiplicity 1 the model-theoretic stabilizer of X in G, $Stab_G(X)$ is by definition the set of $a \in G$ such that $a \cdot X \cap X$ has Morley rank the same as that of X. A key tool in the present paper is Proposition 4.3 of [7] which yields the following:

FACT 2.5. *Suppose G is a connected commutative group of finite Morley rank, and H is a rigid connected definable subgroup of G which is semi-pluriminimal, and maximal such in G. Suppose X is a definable subset of G which has Morley multiplicity* 1 *and with $Stab_G(X)$ finite. Then, up to a set of smaller Morley rank, some translate of X is contained in H.*

We now return to the differential field $(K, +, \cdot, D)$, and for the remainder of this section "definable" will refer to definability in this structure. The next lemma generalizes an observation made in [10] and uses the same methods.

LEMMA 2.6. *Let A be a semi-abelian variety defined over k. Then $A(k)$ is the unique maximal connected definable subgroup of A of finite Morley rank which is semi-pluriminimal.*

Proof. It is enough to prove maximality, as if G_1, G_2 are semi-pluriminimal definable subgroups of A of finite Morley rank, then so is $G_1 + G_2$. Suppose that G is a definable subgroup of A which has finite Morley rank and contains $A(k)$. By Fact 2.3, $G/A(k)$ is definably isomorphic to a finite-dimensional vector space over k. Thus G is "analyzable" in k. In other words, G is a finite Morley rank group definable in $(K, +, \cdot, D)$ such that every infinite definable set in G is nonorthogonal to the definable set k. So if G is semi-pluriminimal, then actually G is contained in $acl(k)$ after naming some parameters. Thus there is some finite subgroup N of G such that G/N is definably isomorphic via some definable function f to a group definable in k, namely to a commutative algebraic group H in the field of constants k. N is defined over k, whereby A/N is also defined over k. Replacing A by A/N we may assume that N is trivial. We claim that H has a nontrivial unipotent part. For let E be the image of $A(k)$ under f. E is then a connected algebraic subgroup of H, and H/E is definably isomorphic in $(K, +, \cdot, D)$ to $G/A(k)$ and thus to a finite-dimensional vector space over k. Thus

H/E is a vector group. On the other hand, the structure theory for commutative algebraic groups tells us that H has a unique maximal connected linear algebraic subgroup L and the quotient is an abelian variety, and moreover L is a product of an algebraic torus and a vector group. As some quotient of H is a vector group, L must have a nontrivial vector group L_1 as a direct summand. Let G_1 be the preimage of L_1 under f. Then G_1 is a nontrivial torsion-free, connected definable subgroup of A. The Zariski closure of G_1 in A is also a semi-abelian variety so has nontrivial torsion. This contradicts Fact 2.4.

COROLLARY 2.7. *Let A be a semiabelian variety defined over k, G a finite Morley rank subgroup of A containing $A(k)$, and Y a definable subset of G which has Morley degree* 1 *and finite stabilizer (in the model-theoretic sense). Then, up to a set of smaller Morley rank, Y is contained in a translate of $A(k)$.*

Proof. Semi-abelian varieties are rigid (as definable groups in algebraically closed fields). Thus $A(k)$ is rigid, as any definable subgroup is an algebraic subgroup (in the sense of the algebraically closed field k). The result follows from Fact 2.5 and Lemma 2.6.

Proof of Proposition 2.1. First we may assume that G contains $A(k)$ (replace G by $G+A(k)$ which still has finite Morley rank). Suppose by way of contradiction that $(X \cap G)\setminus A(k)$ is infinite. Let Y be a definable subset of $(X \cap G)\setminus A(k)$ of Morley multiplicity 1.

Case (i). *$Stab_G(Y)$ is infinite*. Let k_1 be some differential field over which Y is defined. Let $H < G$ be $Stab_G(Y)$. H is also defined over k_1 and there will be generic $a \in H$ and generic $b \in Y$, independent over k_1 (in the sense of differential fields) such that $a+b \in Y$. But then $a+b \in X$, and note that a, b will be independent over k_1 in the sense of algebraic geometry also. Let X_1 be the locus of a over k_1 and X_2 the locus of b over k_1. Then X_1, X_2 are positive-dimensional subvarieties of A and clearly $X_1 + X_2 \subseteq X$, contradicting the hypothesis on X.

Case (ii). *$Stab_G(Y)$ is finite*. By Corollary 2.7, Y is contained in a translate of $A(k)$, up to a set of smaller Morley rank. Without loss, Y is of the form $Z+b$, for some infinite $Z \subset A(k)$. As $Y \cap A(k) = \emptyset$, $b \notin A(k)$. Let k_0 be a small subfield of k such that $tp(b/k)$ is definable over k_0, Z is defined over k_0 (all in the sense of differential fields), and both A, X are defined over k_0. Let a be a generic point of Z over k_0. Then $a+b \in Y$, and a is independent from b over k_0 in the sense of differential fields. So again a is independent from b over k_0 in the sense of algebraic geometry. As $a, b \notin acl(k_0)$, the loci X_1, X_2 of a, b respectively over k_0 are both infinite. We have again $X_1 + X_2 \subseteq X$, contradicting the hypothesis.

3. Finite sets definable in differentially closed fields. The aim in this section is to find good estimates (or bounds) for the size of a finite definable set

X in a differentially closed field, as a function of data entering into the formula defining X. Of course if $X = \{a_1, \ldots, a_n\}$ then the formula $x = a_1 \vee \cdots \vee x = a_n$ will define X and we can read off the number n from the formula. But this is cheating. In general X will be given to us simply as a Boolean combination of differential algebraic sets, and it is from the shape of this data that we wish to compute the size of X. For the model-theorist, what we are doing here is proving an effective "non-fcp" for differentially closed fields, with good bounds. It is well known that the theory of differentially closed fields has the non-fcp, namely that for any formula $\phi(x, y)$ in the language of differential fields (where x, y denote tuples of variables), there is a natural number N such that in any differentially closed field $(K, +, \cdot, D)$, if b is any tuple from K such that the set defined by $\phi(x, b)$ in K is finite, then it has size at most N. (See Dave Marker's article of the model theory of differential fields in [9] where a proof due to van den Dries is given.)

As in the previous section we will be working in a differentially closed field $(K, +, \cdot, D)$ which we can assume to be saturated. In particular K is an algebraically closed field of characteristic 0. Although in the application, the finite sets we are interested in will be subsets of semi-abelian varieties, it will be convenient here (and sufficient) to consider sets in affine spaces. However, some basic results in intersection theory (Bezout's Theorem) will be used, which apply to projective varieties. So we will consider affine n-space $\mathbf{A}^n$ over K as a Zariski open subset of $\mathbf{P}^n$ in the usual way. By the degree of an irreducible Zariski closed subset of $\mathbf{A}^n$ we mean the degree of its Zariski closure in $\mathbf{P}^n$. The degree of a possibly reducible closed subvariety of $\mathbf{P}^n$ is by definition the sum of the degrees of its irreducible components.

By quantifier elimination in differentially closed fields, any subset of K^n definable in $(K, +, \cdot, D)$ will be a finite union of sets of the following kind: $\{x \in K^n\colon (x, Dx, \ldots, D^l(x)) \in S \setminus T\}$ where S, T are Zariski closed subsets of $K^{n(l+1)}$. We will prove:

PROPOSITION 3.1. *Let X be a closed subvariety of K^n, with $dim(X) = m$, and let S, T be closed subvarieties of $K^{n(l+1)}$. Let $Z = \{x \in X\colon (x, Dx, \ldots, D^l(x)) \in S \setminus T\}$. Assume Z is finite. Then $card(Z) \leq deg(X)^{l2^{ml}} deg(S)^{2^{ml}-1}$.*

Remark 3.2. Note that T does not appear in the conclusion, which is not so surprising. Also the proof of Proposition 3.2 will yield the same bounds for the degree of the Zariski closure of Z, whether Z is finite or not.

There will be three ingredients in the proof of the proposition. The first is Bezout's Theorem from Fulton [6].

FACT 3.3. *Let X, Y be closed subvarieties of some $\mathbf{P}^n$, and let Z be the set-theoretic intersection of X and Y. Then $deg(Z) \leq deg(X)deg(Y)$.*

The second concerns the canonical prolongations related to those studied by Buium.

Definition 3.4. Let X be an irreducible subvariety of K^n. Let $l \geq 1$. Then the *lth* prolongation $B_l(X)$ of X is the Zariski closure (in $K^{n(l+1)}$) of $\{(x, Dx, \ldots, D^l(x))\colon x \in X\}$.

Remark 3.5. (1) By Kolchin's irreducibility theorem [8], $B_l(X)$ is also irreducible.

(2) Assume that X is defined over k_0 where k_0 is a (small) differential subfield of K. Then $B_l(X)$ is also defined over k_0.

(3) Let X be defined over k_0 as in (2). Let $P_1, \ldots, P_r$ be polynomials in $k_0[X_1, \ldots, X_n]$ which generate $I(X)$. Let $\tau(X)$ be $\{(a, v) \in K^{2n}\colon a \in X, \sum_{i=1}^n \partial_i P_j(a) v_i + P_j^D(a) = 0, j = 1, \ldots, r\}$ (where ∂_i denotes the partial derivative with respect to X_i and P^D denotes the polynomial obtained from P by applying D to the coefficients). Then $B_1(X)$ is contained in $\tau(X)$, and is equal to it if X is smooth.

LEMMA 3.6. *(With the notation and assumptions of Definition 3.4.)*
(1) $dim(B_l(X)) = (l+1)dim(X)$.
(2) $deg(B_l(X) \leq deg(X)^{l+1}$.

Proof. (1) By Remark 3.5 (3). (2) One can reduce the situation to the case where X is a hypersurface, defined by irreducible $f(x_1, \ldots, x_n) = 0$ of degree d (so $d = deg(X)$). Applying the derivation D to f, we obtain polynomials $f, Df, \ldots, D^l f$, where $D^r f$ is in the variables $D^j(x_i)$ for $j = 0, \ldots, r$, $i = 1, \ldots n$. Each $D^r(f)$ has degree at most d. The $D^r(f)$ vanish on $B_l(X)$, and define a variety Y of dimension at most $(l+1)dim(X)$. By (1) $B_l(X)$ is an irreducible component of Y. It follows from Bezout's theorem that $deg(B_l(X) \leq d^{l+1}$ as required.

The third ingredient is a characteristic property of differentially closed fields, pointed out in [12], but originating in an earlier draft of the present paper:

FACT 3.7. *Suppose W is an irreducible subvariety of $B_1(X)$ which projects generically onto X. Then for any nonempty Zariski open subset U of W there is $a \in X$ such that $(a, Da) \in U$.*

Proof of Proposition 3.1. The first step will be to reduce the problem to the situation where $l = 1$. Note that if Proposition 3.1 has been proved for irreducible X of dimension $\leq m$, then it holds for arbitrary X of dimension m (as $a^c + b^d \leq (a+b)^{max(c,d)}$). Thus we may assume X to be irreducible. Let $Y = B_{l-1}(X)$. Let $S_1 = \{(x_0, \ldots, x_{l-1}, y_0, \ldots, y_{l-1}) \in K^{2nl}\colon (x_0, \ldots, x_{l-1}, y_{l-1}) \in S, y_0 = x_1, \ldots, y_{l-2} = x_{l-1}\}$, and similarly for T_1. Let $Z_1 = \{z \in Y\colon (z, Dz) \in S_1 \setminus T_1\}$. Clearly Z_1 is in bijection with Z. Also by Lemma 3.6, $dim(Y) = ml$, $deg(Y) \leq m^l$, and by Fact 3.3 applied repeatedly to the intersection of S with various hyperplanes, $deg(S_1) \leq deg(S)$. So to prove the proposition, it is enough to prove that $card(Z_1) \leq deg(Y)^{2^{dim(Y)}} deg(S_1)^{2^{dim(Y)}-1}$. Thus it is enough to prove the proposition for the case $l = 1$.

We now proceed to do this, by induction on $dim(X)$. Again we may assume X to be irreducible. If $dim(X) = 0$ then X is a point, and Z has cardinality at most 1, so there is nothing to prove. So we may assume $dim(X) = m > 0$. Now consider $B_1(X) \cap S$. If this set is contained in T then Z is empty and we are finished. So we may assume $(B_1(X) \cap S)\backslash T \neq \emptyset$. Let $Y_1, \ldots, Y_r$ be the irreducible components of $B_1 \cap S$ which are not contained in T. Suppose that some Y_i projects generically onto X. So $Y_i\backslash T$ is a Zariski open subset of Y_i, and by Fact 3.7, there will be infinitely many $a \in X$ such that $(x, Dx) \in Y_i\backslash T$, so Z will be infinite, contradicting our hypothesis. Thus each Y_i projects densely into a proper subvariety X_i of X. Let X' be the union of the X_i. So $dim(X') \leq m - 1$, and $Z = \{a \in X'\colon (a, Da) \in S\backslash T\}$. Now the Zariski closure of each Y_i in $\mathbf{P}^{2n}$ will be an irreducible component of the Zariski closure in $\mathbf{P}^{2n}$ of $B_1(X) \cap S$. So by Fact 3.3 and Lemma 3.6(2), $deg(Y_1 \cup \cdots \cup Y_r) \leq deg(B_1(X))deg(S) \leq deg(X)^2 deg(S)$. Thus $deg(X') \leq deg(X)^2 deg(S)$. (*).

As $dim(X') \leq m - 1$ we may apply the induction hypothesis to conclude that $card(Z) \leq deg(X')^{2^{m-1}} deg(S)^{2^{(m-1)}-1}$. So by (*), $card(Z) \leq (deg(X)^2 deg(S))^{2^{m-1}}$ $deg(S)^{2^{m-1}-1} = deg(X)^{2^m} deg(S)^{2^m - 1}$, proving the proposition.

4. Finding the differential algebraic group.

Here A will be a commutative algebraic group over a differentially closed field K. We assume that A is defined over the field of constants k of K. Let Γ be a finite rational rank subgroup of A. We will find a finite-dimensional differential algebraic subgroup G of A which contains Γ. Also, assuming we are given a basis $\omega_1, \ldots, \omega_n$ for the invariant differential forms on A, we will give explicit equations defining G. We will recall Kolchin's "logarithmic derivative," a differential algebraic homomorphism from A into $T_e(A)$, the tangent space to A at the identity. For now we just assume that A is given to us as an abstract commutative algebraic group defined over k, namely we are given a covering of A by affine open subsets defined over k. We begin by stating a few easy facts. Let $T(A)$ be the tangent bundle of A, also a commutative group defined over k. For a point $a \in A(K)$, $D(a)$ is well defined as a point of $T_a(A)(K)$, and the differential algebraic section taking a to $(a, D(a))$ is a homomorphism which we will also call D from $A(K)$ into $T(A)(K)$. On the other hand the 0-section yields a canonical trivialization $T(A) = A \times T_e(A)$ of $T(A)$, and let π be the projection of $T(A)$ onto $T_e(A)$. ($\pi(a, v)$ is just the image of (a, v) under the differential of multiplication by a^{-1}.)

Definition 4.1. By $lD\colon A(K) \to T_e(A)(K)$ we mean $\pi \circ D$, the composition of D with π.

Remark 4.2. (i) lD is clearly a differential algebraic homomorphism from A into (in fact onto) $T_e(A)$ whose kernel is $A(k)$.

(ii) Suppose that $e_1, \ldots, e_n$ form a basis defined over k for $T_e(A)$, and suppose that $\omega_1, \ldots, \omega_n$ is the dual basis (also defined over k) for the space of invariant differential forms of A. Suppose moreover that in some affine open neighborhood U of $a \in A(K)$, ω_i is given by $f_{i1}dg_{i1} + \cdots + f_{in}dg_{in}$. Then, with respect to the above basis for $T_e(A)$, $lD(a) = (\sum_j f_{1j}(a)D(g_{1j}(a)), \ldots, \sum_j f_{nj}D(g_{nj}(a)))$.

We want to find explicit equations (at least locally) for some finite Morley rank definable group G which contains Γ, assuming we are given invariant forms as in Remark 4.2 (ii).

LEMMA 4.3. *Let $V < K^n$ be an r-dimensional vector space over k. Then V is contained in a finite-dimensional subgroup W which is the common zero-set of n linear homogeneous differential polynomials, each of order at most r in a single indeterminate.*

Proof. Let $V_1 < K$ be the projection of V on the first coordinate. V_1 is a vector space over k of dimension $s \leq r$. Let $a_1, \ldots, a_s$ be a basis for V_1 over k. The matrix $(D^j(a_i))$ $(j = 0, \ldots, s, i = 1, \ldots, s)$ defines a linear mapping from K^{s+1} to K^s, so must have a nontrivial zero $(b_0, \ldots, b_s)$. Thus the function $f_1(x, D(x), \ldots, D^s(x)) = b_0x + b_1D(x) + \cdots b_sD^s(x)$ is nonzero and vanishes on V_1. Do the same thing for each projection V_i of V on K to find f_i. V is contained in the common zero set W of all the f_i, and the latter is clearly finite-dimensional over k.

LEMMA 4.4. *Assume the subgroup Γ of $A(K)$ has rational rank r.Then there is a finite-dimensional differential algebraic subgroup G of $A(K)$ such that*

(i) *G contains Γ.*

(ii) *Suppose that $\omega_1, \ldots, \omega_n$ are a basis defined over k for the invariant differential forms on A as in Remark 4.2 (ii), and that U is an affine open subset of A (defined over k) on which each ω_i has total degree at most N (as a polynomial function over k in $x_1, \ldots, x_m, dx_1, \ldots, dx_m$). Then there is a variety S of degree at most N^n such that $G \cap U = \{a \in U\colon (a, D(a), \ldots, D^r(a)) \in S\}$.*

Proof. Let $e_1, \ldots, e_n$ be a basis of $T_e(A)$ defined over k as in Remark 4.2 (ii), and so we identify $T_e(A)(K)$ with K^n. Let Λ be the image of A under lD. So Λ has rank at most r, whereby the tensor product of Λ with k is a k-vector space V of dimension at most r. Let W be as given by Lemma 4.3, and let G be $(lD)^{-1}(W)$. Note that G contains $A(k)$ and that $G/A(k)$ is isomorphic (differential algebraically) to W. As W is finite-dimensional, and $A(k)$ is finite-dimensional, so is G. Now let U be as in (ii) and suppose $\omega_i = g_i(x_1, \ldots, x_m, dx_1, \ldots, dx_m)$ on U, where g_i is a polynomial over k, and each g_i has degree at most N. So for $a \in U(K)$, $lD(a) = (g_1(a, D(a)), \ldots, g_n(a, D(a))$ (where $a = (a_1, \ldots, a_m)$). Thus, with notation as in the proof of Lemma 4.3, given $x = (x_1, \ldots, x_m) \in U(K)$, $x \in G$ iff for $i = 1, \ldots, n$, we have that $f_i(g_i(x, D(x)), D(g_i(x, D(x))), \ldots, D^r(g_i(x, D(x)))) = 0$. Note that the last polynomial has degree at most N as a polynomial h_i in $x_1, \ldots, x_m, D(x_1), \ldots, D(x_m), \ldots, D^r(x_1), \ldots, D^r(x_m)$. Consider now m-tuples of indeterminates $x, x^1, \ldots, x^r$ and let $S = \{(x, x^1, \ldots, x^r)\colon\ h_i(x, x^1, \ldots, x^r) = 0, i = 1, \ldots, n\}$. By Bezout's theorem $deg(S) \leq N^n$, and clearly $G \cap U = \{x \in U\colon\ (x, D(x), \ldots, D^r(x)) \in S\}$.

5. Proof of Theorem 1.1. We are given a semiabelian variety A and a subvariety X of A, both defined over $\overline{\mathbf{Q}} = k$. Recall the data $dim(A) = n$, $dim(X) = m$, $deg(X)$, N, and r = rational rank of Γ, defined in Section 1 before the statement of Theorem 1.1 (where $deg(X)$ and N, depend on a given presentation of A as a locally closed subset of some projective space, as well as on the invariant differential forms ω_i). We assume that X is not of the form $X_1 + X_2$ for X_1, X_2 positive-dimensional subvarieties of A. We want to prove that the cardinality of $\{a \in X \cap \Gamma \backslash A(k)\}$ is at most $t(deg(X)^{r2^{mr}}(N^n)^{2^{mr}-1})$.

Adjoin a derivation D of $\mathbf{C}$ such that k is the field of constants and $(\mathbf{C}, D)$ is differentially closed. Choose a finite-dimensional differential algebraic subgroup G of $A(\mathbf{C})$ containing Γ as in Lemma 4.4. It is enough to obtain the bounds above for G in place of Γ. By Proposition 2.1, $X \cap G \backslash X(k)$ is finite. Fix one of the sets U in the given open affine covering of A. So $Z = (X \cap U) \cap (G \cap U) \backslash (X \cap U)(k)$ is finite. By Lemma 4.4, $Z = \{x \in (X \cap U)(\mathbf{C})\colon\ (x, D(x), \ldots, D^r(X)) \in S \backslash T\}$ where $deg(S)$ is at most N^n and where $T = \{(x, x^1, \ldots, x^r)\colon\ x^1 = 0\}$. By Proposition 3.1, the cardinality of Z is at most $deg(X \cap U)^{r2^{mr}}(N^n)^{2^{mr}-1}$. Counting the points over all the affine open sets $U_1, \ldots, U_t$ gives the result.

Let us consider the bounds in the case considered by Buium [5], namely where A is $\mathbf{C}^* \times \mathbf{C}^*$, $f(x, y)$is an irreducible polynomial of degree d over $\overline{\mathbf{Q}}$, $X = \{(a, b) \in \mathbf{C}^* \times \mathbf{C}^*\colon\ f(a, b) = 0\}$ is not a translate of a subtorus, and $\Gamma = \Gamma_1 \times \Gamma_1$ where Γ_1 is a subgroup of $\mathbf{C}^*$ of rational rank r.

Kolchin's logarithmic derivative on $\mathbf{C}^*$ is just $lD(x) = D(x)/x$. $lD(\Gamma_1)$ is contained in a vector space V_1 over k which is the zero set of $g(y, D(y), \ldots, D^r(y))$ where g is some linear polynomial. Let $G_1 = (lD)^{-1}(V)$. Then G_1 is a differential algebraic subgroup of $\mathbf{C}^*$ defined by $g((D(x)/x, D(D(x)/x), \ldots, D^r(D(x)/x)) = 0$. The latter can be written in the form $h(x, D(x), \ldots, D^r(x)) = 0$ where h is a polynomial in $x, D(x), \ldots, D^r(x)$ of degree $r+1$. Let $G = G_1 \times G_1$. Then G is $\{(x_1, x_2)\colon\ (x_1, x_2, D(x_1), D(x_2) \ldots, D^r(x_1), D^r(x_2))) \in S\}$ for some variety S such that $deg(S)$ is at most $(r+1)^2$. By Proposition 2.1 and Proposition 3.1, the cardinality of $X \cap G \backslash X(k)$ is at most $d^{r2^r}((r+1)^2)^{2^r-1}$.

6. Additional remarks. In this section we point out what the methods (essentially Corollary 2.7) yield under the weaker assumption on the subvariety X of A that X contains no translate of any positive-dimensional algebraic subgroup of A and where now X need not be defined over a number field.

LEMMA 6.1. *Let $(K, +, \cdot, D)$ be a differentially closed field with field of constants k. Let A be a semiabelian variety defined over k. Let X be a subvariety of A also defined over K containing no translate of any semiabelian subvariety of A. Let G be a definable (in $(K, +, \cdot, D)$) subgroup of $A(K)$ which has finite Morley rank and* *contains $A(k)$. Then $X \cap G$ intersects only finitely many translates of $A(k)$. Moreover there is a uniform bound, depending on the formulas defining A, X and G, on this finite number.*

Proof. To prove the finiteness statement we will prove by induction on the Morley rank ($RM(Z)$) of Z that

(*) for any definable subset Z of $X \cap G$, Z is contained in finitely many translates of $A(k)$.

If $RM(Z) = 0$ then Z is finite and there is nothing to prove. Suppose $RM(Z) = n$. Writing Z as a finite disjoint union of definable sets of Morley rank n and Morley multiplicity 1, we may assume that the Morley multiplicity of Z is 1. Let Y be the Zariski closure of Z in A, let $Y_1, \dots, Y_r$ the the irreducible components of Y and let $Z_i = Y_i \cap G$. So Z is the union of the Z_i's. If $RM(Z_i) < n$ induction yields that Z_i is contained in a finite union of translates of $A(k)$. Suppose $RM(Z_i) = n$. Then $mult(Z_i) = 1$ too. Note that Z_i is Zariski-dense in Y_i. Our assumptions on X imply that the algebraic-geometric stabilizer of Y_i in A is finite. It follows easily that $Stab_G(Z_i)$ is also finite. By Corollary 2.7, there is a definable subset Z_i' of Z_i such that $RM(Z_i \backslash Z_i') < n$ and Z_i' is contained in a single translate of $A(k)$. Now we can apply induction to $Z_i \backslash Z_i'$, to conclude that Z_i is contained in finitely many translates of $A(k)$. Thus (*) holds for Z.

Clearly (*) (for the case $Z = X \cap G$) gives the lemma.

The uniform bound comes from the non-fcp for differentially closed fields, mentioned at the beginning of Section 3.

Remark 6.2. With a little more work the uniform bound in the above lemma can be chosen to be doubly exponential in the data.

COROLLARY 6.3. *Let A be a semi-abelian variety defined over $\bar{\mathbf{Q}}$ and X a subvariety of A (not necessarily defined over $\bar{\mathbf{Q}}$), such that X contains no translates of abelian subvarieties of A. Let Γ be a finitely generated subgroup of A such that $\Gamma \cap A(\bar{\mathbf{Q}}) = \{0\}$. Then $X \cap \Gamma$ is finite and there is moreover a uniform bound on its size, depending on the same data as in Theorem 1.1.*

Proof. As in the proof of Theorem 1.1, adjoin a derivation D to $\mathbf{C}$ such that $(\mathbf{C}, D)$ is differentially closed with field of constants $k = \bar{\mathbf{Q}}$, and find a finite Morley rank definable group G containing Γ as well as $A(k)$. The assumptions on Γ imply that Γ intersects each translate of $A(k)$ in at most one point. The bound on the number of translates of $A(k)$ intersecting $X \cap G$ given by Lemma 6.1 thus yields a bound for the size of $X \cap \Gamma$.

Remark 6.4. As in Remark 6.2 we can obtain doubly exponential bounds for the size of $X \cap \Gamma$ in Corollary 6.3.

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