Theoretical Computer Science 3 (1977) 85–104.

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SOME DECISION PROBLEMS RELATED TO THE REACHABILITY PROBLEM FOR PETRI NETS*

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Communicated by M. Nivat
Received October 1975
Revised February 1976

In this paper, we show that (1) the question to decide whether a given Petri net is consistent, M_0-reversible or live is reduced to the reachability problem in a unified manner, (2) the reachability problem for Petri nets is equivalent to the equality problem and the inclusion problem for the sets of all firing sequences of two Petri nets, (3) the equality problem for the sets of firing sequences of two Petri nets with only two unbounded places under homomorphism is undecidable, (4) the coverability and reachability problems are undecidable for generalized Petri nets in which a distinguished transition has priority over the other transitions, and (5) the reachability problem is undecidable for generalized Petri nets in which some transitions can reset a certain place to zero marking.

1. Introduction

Vector addition systems were first introduced by Karp and Miller [7], and subsequently studied by Rabin, Nash [13], Van Leeuwen [14], Hack [3–5], Keller [10] and others in varying context. Petri net [6] is a formal model suitable for representing concurrent processes that use the synchronization primitive PV due to Dijkstra [2] or one of the generalizations of PV (see [11]). Vector addition systems can be represented as Petri nets [3].

The reachability problem for Petri nets and vector addition systems is an open problem. But it is known that this problem is equivalent to the arcone reachability problem [14], the zero reachability problem [13], the liveness problem [3, 4, 8], the deadlock-freeness problem [8] for Petri nets and the emptiness problem for context-free matrix grammers [14].

For a Petri net N and a set A of markings, let $L(N, A)$ denote the set of firing sequences from the initial marking of N to markings in A. In this paper, the following results are shown in Theorems 1 through 5.

* This paper is based on the authors' papers [1, 8, 9] in the References. The research of the second author was supported in part by National Science Foundation Grant No. GJ-43362 and in part by the Army Research Office.

(1) The question to decide whether a given Petri net is consistent, M_0-reversible or live is reduced to the reachability problem in a unified manner.

(2) The reachability problem for Petri nets is equivalent to the equality problem for the sets of all firing sequences of two Petri nets and also to the question to decide for any given Petri nets N_1 and N_2 and any given arcones A_1 and A_2 whether or not $L(N_1, A_1) \subseteq L(N_2, A_2)$.

(3) Given Petri nets with only two unbounded places N_1, N_2, arcones A_1, A_2 and a mapping Φ from the set of transitions of N_2 to that of N_1, it is undecidable in general whether or not $L(N_1, A_1) = \Phi(L(N_2, A_2))$. Furthermore, the result above holds for Petri nets N_1 and N_2, arcones A_1 and A_2 and mapping Φ which have the restrictions stated in Remark 1.

(4) The coverability and reachability problems are undecidable for generalized Petri nets which have only two unbounded places and a distinguished transition with priority over the other transitions.

(5) The reachability problem is undecidable for generalized Petri nets in which some transitions can reset one of two distinguished places to zero marking.

M. Hack and the authors have been working on closely related subjects independently. It was first proved by one of the authors in [8] that the equality problem for the sets of all firing sequences of two Petri nets is equivalent to the reachability problem, and its generalization (the first part of Theorem 2) was presented by the authors in [9]. Hack also proved in [5] that the equality problem for sets of firing sequences is reducible to the reachability problem. Theorem 3 was presented by the authors via simulation of a two counter automaton in [9]. Hack also proved via Hilbert's tenth problem in [5] that the equality problem for sets of firing sequences under homomorphism is undecidable. The proof presented here yields stronger results than Hack's proof. The main part of the proof of Theorem 4 was published by the authors in [1]. Hack also proved in [5] that the coverability, boundedness and reachability problems are undecidable in Petri nets subject to the priority firing rule via simulation of counter automaton. Theorem 4 shows that the undecidability holds for Petri nets in which a distinguished transition has priority over the other transitions.

2. Definitions

In this section necessary definitions on Petri nets are listed [1, 3, 4, 8, 9].

Definition 1. A *Petri net* $N = \langle \Pi, \Sigma, E, M_0 \rangle$ consists of the following:

(1) a finite set of places, $\Pi = \{r_1, r_2, \ldots, r_{|\Pi|}\}$,

(2) a finite set of transitions, $\Sigma = \{t_1, t_2, \ldots, t_{|\Sigma|}\}$ disjoint from Π,

(3) a finite set of arcs, E, where an arc goes from a place to a transition or from a transition to a place,

(4) an initial marking, $M_0 : \Pi \to \mathbb{N}$, where $\mathbb{N}$ is the set of nonnegative integers.

If there is an arc from place r to transition t (or from transition t to place r), then r is called an input place (or output place) of the transition t.

Definition 2. A *marking* M is a mapping from Π to $\mathbb{N}$ and represented by a $|\Pi|$-dimensional vector $(s_1, s_2, \ldots, s_i, \ldots, s_{|\Pi|})$, where $|\Pi|$ is the number of places and s_i denotes $M(r_i)$.

Definition 3. A Petri net is represented graphically as follows:

(1) places are represented by circles, ○,

(2) transitions are represented by bars, | ,

(3) arcs are represented by arrows, →,

(4) a marking M is represented by drawing $M(r_i)$ tokens into place r_i, and tokens are represented by dots.

Definition 4. Σ^* denotes the *set of all strings* composed of symbols of Σ, including the empty string λ whose length is zero.

Definition 5. Let $F(r, t)$ and $B(t, r)$ denote the number of those arcs which go from place r to transition t and that of those arcs which go from transition t to place r, respectively. A transition t is said to be firable at marking M, if and only if $M(r) \geqslant F(r, t)$ for every input place r. A firing of transition t is accomplished by removing $F(r, t)$ tokens from every input place r of the transition t and adding $B(t, r)$ tokens to every output place r of the transition t. Let M' denote the resulting marking. Then we write that $M \xrightarrow{t} M'$.

Definition 6. A *firing sequence* from marking M to marking M' is a string σt defined recursively as follows:

$$M \xrightarrow{\sigma t} M' \stackrel{\Delta}{=} \exists M'' \in \mathbb{N}^{|\Pi|} : M \xrightarrow{\sigma} M'' \ \& \ M'' \xrightarrow{t} M',$$

where $\sigma \in \Sigma^*$, $t \in \Sigma$, $\mathbb{N}^{|\Pi|}$ is the set of $|\Pi|$-dimensional vectors over $\mathbb{N}$. For the empty string λ, we define $M \xrightarrow{\lambda} M$.

Definition 7. A *marking* M' is said to be reachable from marking M if there exists a firing sequence σ such that $M \xrightarrow{\sigma} M'$.

Definition 8. Given a Petri net $N = \langle \Pi, \Sigma, E, M_0 \rangle$ with initial marking M_0, we define *reachability set* $R_N(M_0)$ as follows:

$$R_N(M_0) = \{M \in \mathbb{N}^{|\Pi|} \mid \exists \sigma \in \Sigma^*, M_0 \xrightarrow{\sigma} M\}.$$

Definition 9. Given a Petri net $N = \langle \Pi, \Sigma, E, M_0 \rangle$ with initial marking M_0, we define a set $L_0(N)$ of *firing sequences* as follows:

$$L_0(N)=\{\sigma\in\Sigma^* \mid \exists M\in \mathbf{N}^{|\Pi|},\ M_0\xrightarrow{\sigma} M\}.$$

Definition 10. For a marking M and a subset Π' of Π, we define *arcone* $A(M,\Pi')$ as follows:

$$A(M,\Pi')=\{M' \mid M'(r)=M(r) \text{ for } r\in\Pi' \text{ and } M'(r)\geq M(r) \text{ for } r\in\Pi-\Pi'\}.$$

If Π' is empty, then $A(M,\Pi')$ is said to be principal.

Definition 11. Given a Petri net $N=\langle\Pi,\Sigma,E,M_0\rangle$, we define a *set* $L(N,A)$ *of firing sequences* from the initial marking M_0 to markings in a set A as follows:

$$L(N,A)=\{\sigma\in\Sigma^* \mid \exists M\in A,\ M_0\xrightarrow{\sigma} M\}.$$

Definition 12. *Zero marking* M_Z is the marking such that $M_Z(r)=0$ for every place r.

Definition 13. A place r is said to be bounded in Petri net $N=\langle\Pi,\Sigma,E,M_0\rangle$ if and only if there exists a nonnegative integer b such that $M(r)\leq b$ for every marking M in $R_N(M_0)$.

3. Consistency, M_0-reversibility and liveness

The following theorem is already known [13, 14].

Theorem 0. *The following decision problems are equivalent to each other.*

(1) *The reachability problem: for a given marking M, whether $M\in R_N(M_0)$ or not?*

(2) *The zero reachability problem: whether $M_Z\in R_N(M_0)$ or not?*

(3) *The arcone reachability problem: for a given arcone A, whether $A\cap R_N(M_0)\neq\emptyset$ or not?*

For a set $W\subseteq\mathbf{N}^{|\Pi|}$, let $m(W)$ denote the smallest subset of W such that for any M in W there is M' in $m(W)$ with $M'\leq M$. By König's theorem [3], $m(W)$ is finite.

Lemma 1. *For a Petri net N with initial marking $M^{(0)}$ and a principal arcone A, $m(R_N(M^{(0)})\cap A)$ can be obtained effectively if the reachability problem is decidable.*

Proof. Assume that the reachability problem is decidable. Then, the arcone reachability problem is also decidable. In this proof, the ith component of a marking M will be denoted by M_i. Let $i_1, i_2, \ldots, i_n$ be different positive integers less than $|\Pi|+1$ and $k_1, k_2, \ldots, k_n$ be nonnegative integers. Let

$$\begin{aligned}&R(i_1,k_1;i_2,k_2;\ldots;i_n,k_n)\\&\quad=\{M \mid M_{i_1}=k_1, M_{i_2}=k_2,\ldots,M_{i_n}=k_n, M\in R_N(M^{(0)})\cap A\}.\end{aligned}$$

For $n = 0$, let $R(\lambda) = R_N(M^{(0)}) \cap A$. By the assumption, it is decidable whether $R(i_1, k_1; i_2, k_2; \ldots; i_n, k_n)$ is empty or not. We shall prove the following proposition by the induction on $|\Pi| - n$.

(P) *Suppose that* $R(i_1, k_1; i_2, k_2; \ldots; i_n, k_n)$ *is not empty.*

(P1) *A marking in* $R(i_1, k_1; i_2, k_2; \ldots; i_n, k_n)$ *can be found effectively.*

(P2) $m(R(i_1, k_1; i_2, k_2; \ldots; i_n, k_n))$ *can be found effectively.*

Proof. The proposition is trivially true for $n = |\Pi|$. Suppose that the proposition is true for $n = l + 1$ and consider the case of $n = l$. For simplicity, let $I_l = \{i_1, i_2, \ldots, i_l\}$, let $R(i_1, k_1; i_2, k_2; \ldots; i_l, k_l)$ be denoted by R_l and for $i \in \{1, 2, \ldots, |\Pi|\} - I_l$ and a nonnegative integer k, let $R_l(i, k)$ denote $R(i_1, k_1; \ldots; i_l, k_l; i, k)$. For the proof of (P1), choose a positive integer i which is less than $|\Pi| + 1$ and different from $i_1, i_2, \ldots, i_l$. Decide whether $R_l(i, 0)$ is empty. If so, decide whether $R_l(i, 1)$ is empty. Repeat the process until we find a nonnegative integer k for which $R_l(i, k)$ is not empty. By the induction hypothesis, a marking M in $R_l(i, k) \subseteq R_l$ can be found. For the proof of (P2), note that $m(R_l) = m(\{M\} \cup \{M' \mid M' \not\geq M, M' \in R_l\})$ and that

$$\{M' \mid M' \not\geq M, M' \in R_l\} = \bigcup_{i \in \{1,2,\ldots,|\Pi|\} - I_l} \; \bigcup_{0 \leq k < M_i} R_l(i, k).$$

By the induction hypothesis, $m(R_l(i, k))$ can be found effectively. Since[1]

$$m(R_l) = m\left(\{M\} \cup \bigcup_{i \in \{1,2,\ldots,|\Pi|\} - I_l} \; \bigcup_{0 \leq k < M_i} m(R_l(i, k))\right),$$

$m(R_l)$ can be found effectively. □

Lemma 2. *Let* $N = \langle \Pi, \Sigma, E, M_0 \rangle$ *be a Petri net. Assume that a predicate* $Q(M)$ *defined on the markings in a principal arcone* A *on* Π *satisfies the following conditions.*

(1) *If* $Q(M')$ *is true and* $M \geq M'$, *then* $Q(M)$ *is true.*

(2) *If the reachability problem is decidable, then it is decidable whether* $Q(M)$ *is true or not for any given marking* M *in* A.

Then, given a Petri net N *with an initial marking* M_0, *it is decidable whether* $Q(M)$ *is true or not for every marking* M *in* $R_N(M_0) \cap A$ *if the reachability problem is decidable.*

Proof. By condition (1), $Q(M)$ is true for every marking in $R_N(M_0) \cap A$ if and only if $Q(M)$ is true for every marking in $m(R_N(M_0) \cap A)$. By Lemma 1, we can

[1] We use the fact that $m(X \cup Y) = m(m(X) \cup m(Y))$. Since $X \supseteq m(X)$ and $Y \supseteq m(Y)$, $m(X \cup Y) \supseteq m(m(X) \cup m(Y))$. Conversely, for any M in $m(X \cup Y)$, if we assume that M belongs to X, then there is no M' in Y such that $M > M'$. Thus M belongs to $m(X)$ and there is no M' in $m(Y)$ such that $M > M'$. Hence M belongs to $m(m(X) \cup m(Y))$.

obtain $m(R_N(M_0) \cap A)$ effectively. Since $m(R_N(M_0) \cap A)$ is finite, it is decidable whether $Q(M)$ is true for every marking M in $m(R_N(M_0) \cap A)$ by condition (2). □

As examples of the above lemma, we shall consider the following properties.

Definition 14. A Petri net N is said to be *live* if for any firing sequence $\alpha \in L_0(N)$ and any transition t, there is a firing sequence βt such that $\alpha\beta t \in L_0(N)$.

Definition 15. A Petri net N is said to be (strong) consistent if for any marking $M \in R_N(M_0)$ there is a firing sequence which fires each transition at least once and returns to the original marking M.

The notion of consistency defined above is stronger than a definition which has been in use. As used here, (strong) consistency implies the (weak) consistency at every reachable marking.

Definition 16. A Petri net N is said to be *M_0-reversible* if the initial marking M_0 is reachable from any marking M in $R_N(M_0)$.

We define the following predicates associated with Petri net $N = \langle \Pi, \Sigma, E, M_0 \rangle$ with respect to the properties above.

(1) For a marking M on Π, $Q_L^N(M)$ is defined to be true if and only if for any transition $t \in \Sigma$ there is a firing sequence α such that $M \xrightarrow{\alpha} M_1 \xrightarrow{t} M_2$ in N. Petri net N is live if and only if $Q_L^N(M)$ is true for all markings in $R_N(M_0)$. Predicate Q_L^N satisfies the conditions (1) and (2) in Lemma 2. If $M \geq M'$ and $Q_L^N(M')$ is true, that is, for any transition $t \in \Sigma$ there is a firing sequence α such that $M' \xrightarrow{\alpha} M_1 \xrightarrow{t} M_2$ in N, then it holds that $M = M' + (M - M') \xrightarrow{\alpha} M_1 + (M - M') \xrightarrow{t} M_2 + (M - M')$ in N, that is, $Q_L^N(M)$ is true. Thus condition (1) holds. By definition, $Q_L^N(M)$ is true if and only if $\{M' \mid \forall r \in \Pi,\ M'(r) \geq F(r, t)\} \cap R_N(M) \neq \emptyset$ for every $t \in \Sigma$. Since $\{M' \mid \forall r \in \Pi,\ M'(r) \geq F(r, t)\}$ is arcone, the latter problem is decidable if the reachability problem is decidable. Thus condition (2) holds.

(2) For a marking M on Π, $Q_C^N(M)$ is defined to be true if and only if there is a firing sequence α such that (1) $M \xrightarrow{\alpha} M$ in N, and (2) α contains each transition at least once. Petri net N is consistent if and only if $Q_C^N(M)$ is true for all markings in $R_N(M_0)$. It is clear that predicate Q_C^N satisfies the condition (1). For each transition $t \in \Sigma$ in N, we add an output place r_t of t to a copy of N in order to count the number of firings of transition t. Let $N' = \langle \Pi', \Sigma, E', M_0' \rangle$ be the resulting Petri net, where $\Pi' = \Pi \cup \{r_t \mid t \in \Sigma\}$. Given a marking M on Π, let M_0' be the marking such that $M_0'(r) = M(r)$ for every place r in Π and $M_0'(r_t) = 0$ for every place r_t with $t \in \Sigma$. Then $Q_C^N(M)$ is true if and only if a marking M' such that $M'(r) = M(r)$ for every place r in Π and $M'(r_t) \geq 1$ for every place r_t with $t \in \Sigma$ is reachable from M_0' in N'. Hence predicate Q_C^N satisfies the condition (2).

(3) For transition $t \in \Sigma$, let $A_t = \{M \mid t \text{ is firable at } M\}$. For a marking M in A_t, $Q_t^N(M)$ is defined to be true if and only if there is a firing sequence α such that $M \xrightarrow{t} M' \xrightarrow{\alpha} M$ in N. Then Petri net N is M_0-reversible if and only if $Q_t^N(M)$ is true for all transitions t and all markings M in $R_N(M_0) \cap A_t$. It is clear that predicate Q_t^N satisfies the condition (1). $Q_t^N(M)$ is true if and only if M is reachable from M' in N. Hence predicate Q_t^N satisfies the condition (2).

The following theorem summarizes the above discussion.

Theorem 1. *If the reachability problem is decidable, then the following problems are also decidable.*

(1) *To decide whether a Petri net is live.*
(2) *To decide whether a Petri net is consistent.*
(3) *To decide whether a Petri net is M_0-reversible.*

The proof above is based on the authors' previous paper [9]. The first part (1) of Theorem 1 and its converse are known [4, 8]. M. Hack proved it in a somewhat different way.

4. Equality problem for the set of firing sequences

Lemma 3. *If it is decidable whether $L_0(N_1) = L_0(N_2)$ for any given Petri nets N_1 and N_2, then the reachability problem is also decidable.*

Proof. From a given Petri net $N = \langle \Pi, \Sigma, E, M_0 \rangle$, we construct two Petri nets N_1 and N_2 in such a way that $L_0(N_1) \neq L_0(N_2)$ if and only if $M_Z \in R_N(M_0)$. Let $F(r, t)$ and $B(t, r)$ denote the number of arcs from place r to transition t and that of arcs from transition t to place r in N, respectively. First, we construct N_1 with initial marking M_{10} by adding place C with $M_{10}(C) = 1$, transition t_0 and one arc from C to t_0 to a copy of N as shown in Fig. 1. The initial marking M_{10} of N_1 is the initial marking M_0 of N augmented with the initial marking of place C. Next, we construct

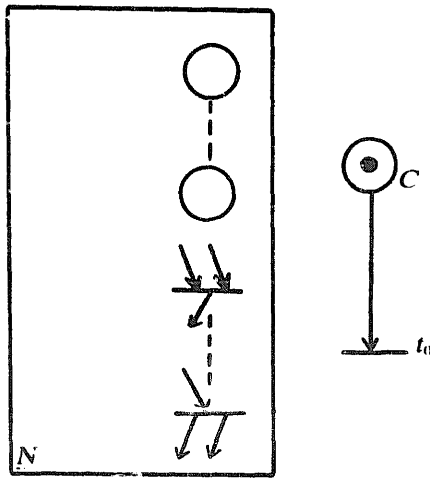


Fig. 1. Petri net N_1.

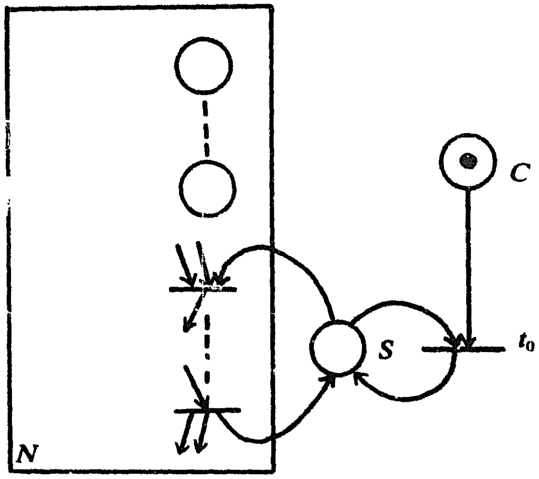


Fig. 2. Petri net N_2.

N_2 with initial marking M_{20} as shown in Fig. 2, by adding place S and several arcs to a copy of N_1 as follows:

(1) a place S for which

$$M_{20}(S) = \sum_{r \in \Pi} M_0(r),$$

(2) an arc from S to t_0,

(3) an arc from t_0 to S, and

(4) for each transition t in a copy of N,

$$\sum_{r \in \Pi} B(t, r) - \sum_{r \in \Pi} F(r, t)$$

arcs from t to S if

$$\sum_{r \in \Pi} B(t, r) \geqslant \sum_{r \in \Pi} F(r, t)$$

and

$$\sum_{r \in \Pi} F(r, t) - \sum_{r \in \Pi} B(t, r)$$

arcs from S to t if

$$\sum_{r \in \Pi} B(t, r) < \sum_{r \in \Pi} F(r, t).$$

The initial marking M_{20} of N_2 is the initial marking M_0 of N augmented with the initial markings of two places C and S.

The following properties hold from the construction of N_1 and N_2.

(P1) $L_0(N_1) \supseteq L_0(N_2)$.

(P2) *Suppose that $M_{10} \xrightarrow{\alpha} M_1$ in N_1 and $M_{20} \xrightarrow{\alpha} M_2$ in N_2. Then the following hold.*

(1) *$M_1(r) = M_2(r)$ for every place r other than S.*

(2) *$M_2(S) = \Sigma_{r\in\Pi} M_2(r)$.*

(3) *If transition t other than t_0 is firable at M_1 in N_1, then $M_1(r) \geq F(r,t)$ for every input place r of t and therefore*

$$M_2(S) = \sum_{r\in\Pi} M_2(r) = \sum_{r\in\Pi} M_1(r) \geq \sum_{r\in\Pi} F(r,t).$$

Since a number of arcs from place S to transition t is not greater than $\Sigma_{r\in\Pi} F(r,t)$, t is firable at M in N_2.

(4) *If transition t_0 is firable at M_1 in N_1, then t_0 is firable at M_2 in N_2 if and only if $\Sigma_{r\in\Pi} M_2(r) \neq 0$.*

(P3) *For any firing sequence α where t_0 does not occur, we have that*

$$\alpha \in L_0(N) \Leftrightarrow \alpha \in L_0(N_1) \Leftrightarrow \alpha \in L_0(N_2)$$

and that if $M_0 \xrightarrow{\alpha} M$ in N, $M_{10} \xrightarrow{\alpha} M_1$ in N_1 and $M_{20} \xrightarrow{\alpha} M_2$ in N_2, then $M(r) = M_1(r) = M_2(r)$ for every place r other than S and C. This follows from (P2) *and an induction on the length of α.*

(P4) *If $M_Z \notin R_N(M_0)$, then $L_0(N_1) = L_0(N_2)$.*

Proof. Let γ be in $L_0(N_1)$. If γ does not contain transition t_0, then $\gamma \in L_0(N_2)$ by (P3). Otherwise, let $\gamma = \alpha t_0 \beta$, where α does not contain t_0. Since transition t_0 can fire at most once, β does not contain t_0. By (P3), $\alpha \in L_0(N) \cap L_0(N_2)$. Let $M_0 \xrightarrow{\alpha} M$ in N, $M_{10} \xrightarrow{\alpha} M_1$ in N_1 and $M_{20} \xrightarrow{\alpha} M_2$ in N_2. Since $M \neq M_Z$,

$$M_2(S) = \sum_{r\in\Pi} M_2(r) = \sum_{r\in\Pi} M_1(r) = \sum_{r\in\Pi} M(r) > 0$$

by (P2) and (P3). Hence, t_0 is firable at M_2 in N_2 by (P2) through (P4). By applying (P2) repeatedly, we have that $\gamma \in L_0(N_2)$. That is, $L_0(N_1) \subseteq L_0(N_2)$. Thus (P4) follows from (P1).

(P5) *If $M_Z \in R_N(M_0)$, then $L_0(N_1) \neq L_0(N_2)$.*

Proof. Let $M_0 \xrightarrow{\alpha} M_Z$ in N. By (P3), $\alpha \in L_0(N_1) \cap L_0(N_2)$, $M_{10} \xrightarrow{\alpha} M_1$ in N_1 and $M_{20} \xrightarrow{\alpha} M_2$ in N_2, where $M_1(r) = M_2(r) = 0$ for every place r other than C including S. Since $M_2(S) = 0$, transition t_0 is not firable at M_2 in N_2 but is firable at M_1 in N_1. □

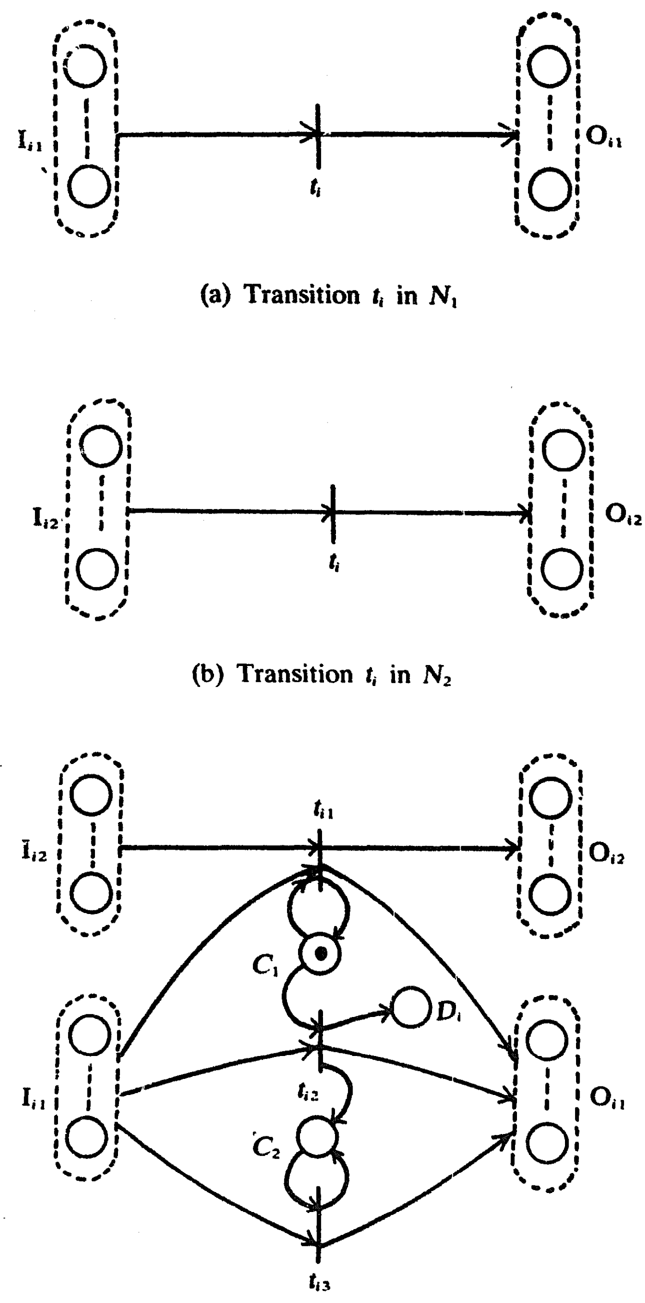
Lemma 4. *If the reachability problem is decidable, then it is also decidable whether $L(N_1, A_1) \subseteq L(N_2, A_2)$ for any given arcones A_1 and A_2 and any given Petri nets N_1 and N_2 which have the same set of transitions.*

Proof. From two given Petri nets $N_1 = \langle \Pi_1, \Sigma, E_1, M_{10}\rangle$ and $N_2 = \langle \Pi_2, \Sigma, E_2, M_{20}\rangle$, we construct a new Petri net $N = \langle \Pi, \Sigma', E, M_0\rangle$. We may assume that $\Pi_1 \cap \Pi_2 = \emptyset$.

Let $\Sigma = \{t_1, t_2, \ldots, t_{|\Sigma|}\}$ and let $F(r, t)$, $F_1(r, t)$ and $F_2(r, t)$ (or $B(t, r)$, $B_1(t, r)$ and $B_2(t, r)$) denote the number of arcs from place r (or transition t) to transition t (or place r) in N, N_1 and N_2, respectively. Let I_{i1} and O_{i1} (or I_{i2} and O_{i2}) be the set of the input places and that of the output places of transition t_i in N_1 (or N_2), respectively. Petri net N consists of the following places and transitions.

(1) $\Pi = \Pi_1 \cup \Pi_2 \cup \{C_1, C_2\} \cup \{D_i\ (1 \leqslant i \leqslant |\Sigma|)\}$, where C_1, C_2 and D_i with $1 \leqslant i \leqslant |\Sigma|$ are not in $\Pi_1 \cup \Pi_2$.

(2) For each transition $t_i \in \Sigma$ $(1 \leqslant i \leqslant |\Sigma|)$, N has three transitions t_{i1}, t_{i2} and t_{i3} as shown in Fig. 3, where



(a) Transition t_i in N_1

(b) Transition t_i in N_2

(c) Transitions t_{i1}, t_{i2} and t_{i3} in N

Fig. 3. Construction of N for transition t_i.

(1) the input places (or the output places) of t_{i1} are C_1 and the places in $I_{i1} \cup I_{i2}$ (or C_1 and the places in $O_{i1} \cup O_{i2}$),

(2) the input places (or the output places) of t_{i2} are C_1 and the places in I_{i1} (or C_2, D_i and the places in O_{i1}),

(3) the input places (or the output places) of t_{i3} are C_2 and the places in I_{i1} (or C_2 and the places in O_{i1}), and

(4) $F(r, t_{i1}) = F(r, t_{i2}) = F(r, t_{i3}) = F_1(r, t_i)$ for $r \in \Pi_1$, $F(r, t_{i1}) = F_2(r, t_i)$ for $r \in \Pi_2$, $B(t_{i1}, r) = B(t_{i2}, r) = B(t_{i3}, r) = B_1(t_i, r)$ for $r \in \Pi_1$, $B(t_{i1}, r) = B_2(t_i, r)$ for $r \in \Pi_2$, and $F(C_1, t_{i1}) = F(C_1, t_{i2}) = F(C_2, t_{i3}) = B(t_{i1}, C_1) = B(t_{i2}, C_2) = B(t_{i3}, C_2) = B(t_{i2}, D_i) = 1$.

The initial marking M_0 of N is the marking such that $M_0(r) = M_i(r)$ for $r \in \Pi_i$ with $1 \leqslant i \leqslant 2$, $M_0(C_1) = 1$, $M_0(C_2) = 0$ and $M_0(D_i) = 0$ for $1 \leqslant i \leqslant |\Sigma|$.

In N, transition t_{i3} $(1 \leqslant i \leqslant |\Sigma|)$ cannot fire when place C_1 has a token, and after one of the transitions t_{j2} $(1 \leqslant j \leqslant |\Sigma|)$ has fired, transition t_{i1} and t_{i2} $(1 \leqslant i \leqslant |\Sigma|)$ cannot fire and only transition t_{i3} $(1 \leqslant i \leqslant |\Sigma|)$ can fire. Output place D_j of t_{j2} has a token if and only if transition t_{j2} has fired. Let Ψ_1 and Ψ_3 be mappings such that $\Psi_1(t_i) = t_{i1}$ and $\Psi_3(t_i) = t_{i3}$. For a subset Π' of Π and a marking M on Π, let $M[\Pi']$ denote the projection of M on Π'.

By the construction of N, the following properties hold.

(P1) *Let $M_0 \xrightarrow{\gamma} M$ in N. If $M(D_j) = 1$, then γ is represented as $\Psi_1(\alpha)t_{j2}\Psi_3(\beta)$, where α and β are in Σ^*. If $M(C_1) = 1$, then there is $\alpha \in \Sigma^*$ such that $\gamma = \Psi_1(\alpha)$.*

(P2) *$\alpha \in L_0(N_1) \cap L_0(N_2)$ if and only if $\Psi_1(\alpha) \in L_0(N)$. If $M_0 \xrightarrow{\Psi_1(\alpha)} M$ in N and $M_{i0} \xrightarrow{\alpha} M_i$ in N_i for $i = 1, 2$, then $M[\Pi_i] = M_i$ for $i = 1, 2$, and $M(C_2) = M(D_i) = 0$ for $1 \leqslant i \leqslant |\Sigma|$.*

(P3) *$\alpha t_j \beta \in L_0(N_1)$ and $\alpha \in L_0(N_2)$ if and only if $\Psi_1(\alpha)t_{j2}\Psi_3(\beta) \in L_0(N)$. If $M_{10} \xrightarrow{\alpha t_j \beta} M_1$ in N_1, $M_{20} \xrightarrow{\alpha} M_2$ in N_2 and $M_0 \xrightarrow{\Psi_1(\alpha)} M' \xrightarrow{t_{j2}\Psi_3(\beta)} M$ in N, then $M[\Pi_1] = M_1$, $M[\Pi_2] = M'[\Pi_2] = M_2$, $M(C_1) = 0$, $M(C_2) = 1$, $M(D_j) = 1$ and $M(D_i) = 0$ for $i \neq j$.*

It is clear that $L(N_1, A_1) \subseteq L(N_2, A_2)$ if and only if (I) there is no firing sequence α such that $\alpha \in L(N_1, A_1)$, $\alpha \in L_0(N_2)$ and $\alpha \notin L(N_2, A_2)$, and (II) there is no firing sequence α such that $\alpha \in L(N_1, A_1)$ and $\alpha \notin L_0(N_2)$.

First, we will consider condition (I). Let A denote the set of markings M on Π such that $M[\Pi_1] \in A_1$, $M[\Pi_2] \in \bar{A}_2$ (the complement of A_2, which is a union of a finite number of arcones [14]), $M(C_1) = 1$, $M(C_2) = 0$ and $M(D_i) = 0$ for $1 \leqslant i \leqslant |\Sigma|$. A is a union of a finite number of arcones. By (P1) and (P2), condition (I) is true if and only if no markings in A are reachable from M_0 in N. If the reachability problem is decidable, the latter problem is decidable. Therefore it is decidable whether condition (I) is true.

Next, we will show that condition (II) is true if and only if no markings which belong to a union A_0 of a finite number of arcones are reachable in N. Since the latter problem is decidable if the reachability problem is decidable, it is decidable whether condition (II) is true.

We construct A_0 as follows. Let A_{0j} be a union of $|\Pi_2|+|\Sigma|$ dimensional arcones such that

$$A_{0j} = \bigcup_{\{r \mid F_2(r,t_j)\neq 0,\, r\in \Pi_2\}} \bigcup_{0\le l< F_2(r,t_j)} \left\{ M \left| \begin{array}{l} M \text{ is a marking on } \Pi_3 \\ \text{such that } M(r)=l, \\ M(D_j)=1,\ M(D_i)=0\ (i\neq j) \end{array} \right.\right\}.$$

Let A_0 be the set of markings M on Π such that $M[\Pi_1]\in A_1$, $M[\Pi_3]\in \bigcup_{1\le j\le|\Sigma|} A_{0j}$, $M(C_1)=0$ and $M(C_2)=1$, where $\Pi_3=\Pi_2\cup\{D_i(1\le i\le|\Sigma|)\}$. Then, A_0 is a union of a finite number of arcones.

If condition (II) is false, that is, there is a firing sequence γ such that $\gamma\in L(N_1, A_1)$ and $\gamma\notin L_0(N_2)$, then there exists t_j such that $\gamma=\alpha t_j\beta$, $\alpha\in L_0(N_2)$ and $\alpha t_j\notin L_0(N_2)$. By property (P3), $\Psi_1(\alpha)t_{j2}\Psi_3(\beta)\in L_0(N)$. Let $M_{10}\xrightarrow{\alpha t_j\beta} M_1$ in N_1, $M_{20}\xrightarrow{\alpha} M_2$ in N_2 and $M_0\xrightarrow{\Psi_1(\alpha)t_{j2}\Psi_3(\beta)} M$ in N. Since $\alpha t_j\notin L_0(N_2)$, there is a place r in Π_2 such that $M_2(r)<F_2(r,t_j)$. Hence it follows from (P3) that $M[\Pi_1]\in A_1$, $M[\Pi_3]\in A_{0j}$, $M(C_1)=0$ and $M(C_2)=1$. That is, $M\in A_0$.

If there is a firing sequence γ in $L(N, A_0)$, then γ can be represented as $\Psi_1(\alpha)t_{j2}\Psi_3(\beta)$ by (P1). By (P3), $\alpha t_j\beta\in L(N_1, A_1)$ and $\alpha\in L_0(N_2)$. Let $M_0\xrightarrow{\Psi_1(\alpha)} M'\xrightarrow{t_{j2}\Psi_3(\beta)} M$ in N and $M_{20}\xrightarrow{\alpha} M_2$ in N_2. Since $M[\Pi_3]\in\bigcup_j A_{0j}$ and $M(D_j)=1$, there exists a place r in Π_2 such that $M(r)=l<F(r,t_j)$. By (P3), $M_2=M[\Pi_2]$. Thus transition t_j cannot fire at marking M_2, that is, $\alpha t_j\beta\notin L_0(N_2)$. Therefore condition (II) is false. □

From Lemmas 3 and 4, we have the following theorem.

Theorem 2. *The following decision problems are equivalent to the reachability problem.*

(1) *To decide for any given Petri nets N_1 and N_2 and any given arcones A_1 and A_2 whether $L(N_1, A_1)\subseteq L(N_2, A_2)$.*

(2) *To decide for any given Petri nets N_1 and N_2 whether $L_0(N_1)=L_0(N_2)$.*

5. Undecidability of the equality problem under homomorphism

To show undecidability of some decision problems, we consider a deterministic two counter automaton (abbreviated as 2ca) with no input tape which starts from an initial state s_0 and halts when and only when it goes to a final state s_f. Let c_l $(l=1,2)$ denote a counter and the initial contents of c_1 and c_2 are assumed to be zero. The operations of a 2ca are of the following types.

(I) If the current state is s, then add one to counter c_l and go to state s'. This operation will be denoted by (s, s', l).

(II) If the current state is s and counter c_l is not equal to zero, then subtract one from counter c_l and go to state s'. If the current state is s and counter c_l is equal to zero, then go to state s''. This operation will be denoted by (s, s', s'', l).

A total state of a 2ca K is represented as $[s, x_1, x_2]$, where s is the current state of K and x_1 and x_2 denote the current contents of counters c_1 and c_2 of K, respectively. The initial total state is $[s_0, 0, 0]$. If $[s', x_1', x_2']$ is the total state of K where K starting from a total state $[s, x_1, x_2]$ reaches after a sequence α of operations, then we write $[s, x_1, x_2] \xrightarrow{\alpha} [s', x_1', x_2']$.

It is known to be undecidable in general whether a 2ca K halts or not [12].

Theorem 3. *Given Petri nets with only two unbounded places $N_1 = \langle \Pi_1, \Sigma_1, E_1, M_{10} \rangle$, $N_2 = \langle \Pi_2, \Sigma_2, E_2, M_{20} \rangle$, arcones A_1, A_2 and a mapping Φ from Σ_2 to Σ_1, it is undecidable in general whether $L(N_1, A_1) = \Phi(L(N_2, A_2))$ or not, where $\Phi(L(N_2, A_2)) = \{\Phi(\alpha) \mid \alpha \in L(N_2, A_2)\}$.*

Proof. For any 2ca K, we construct two Petri nets $N_1 = \langle \Pi_1, \Sigma_1, E_1, M_{10} \rangle$ and $N_2 = \langle \Pi_2, \Sigma_2, E_2, M_{20} \rangle$ in such a way that $L(N_1, A_1) \neq \Phi(L(N_2, A_2))$ if and only if 2ca K halts. N_1 and N_2 consist of the following places and transitions.

(1) For each state s in 2ca K, both N_1 and N_2 have a place designated by S.

(2) N_1 and N_2 have two places C_1 and C_2, where C_1 and C_2 correspond to counters c_1 and c_2 in 2ca K, respectively.

(3) N_2 has a place D.

(4) For each operation $\tau = (s, s', l)$ of type I in 2ca K, N_1 and N_2 have a transition $t(\tau)$ whose input place is S and whose output places are S' and C_l as shown in Fig. 4.

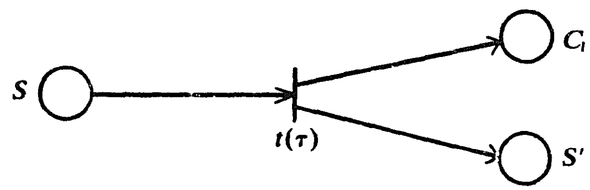


Fig. 4. Simulation of operation τ of type I in Petri nets N_1 and N_2.

(5) For each operation $\tau = (s, s', s'', l)$ of type II in 2ca K, N_1 has two transitions $t_{NZ}(\tau)$ and $t_Z(\tau)$ as shown in Fig. 5, where the input places (or the output places) of $t_{NZ}(\tau)$ are S and C_l (or S') and the input place (or the output place) of $t_Z(\tau)$ is S (or

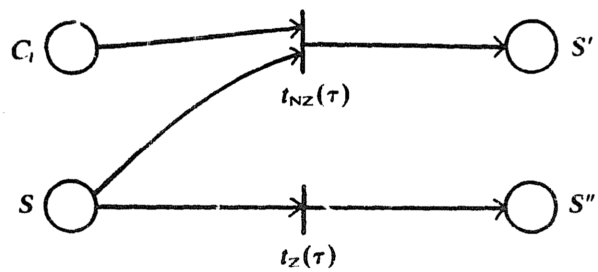


Fig. 5. Simulation of operation τ of type II in Petri net N_1.

S''), and N_2 has three transitions $t_{NZ}(\tau)$, $t_Z(\tau)$ and $t'_Z(\tau)$ as shown in Fig. 6, where the input places (or the output places) of $t_{NZ}(\tau)$ are S and C_l (or S'), the input place (or the output place) of $t_Z(\tau)$ is S (or S''), and the input places (or the output places) of $t'_Z(\tau)$ are S, C_l and D (or S'' and C_l).

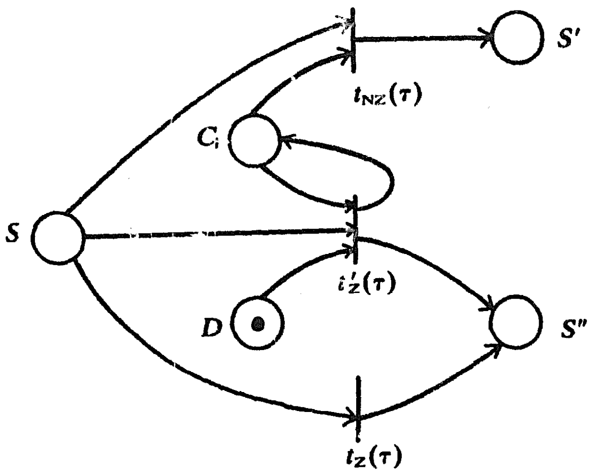


Fig. 6. Simulation of operation τ of type II in Petri net N_2.

The initial marking M_{10} of N_1 is the marking such that $M_{10}(S_0) = 1$ and $M_{10}(r) = 0$ for every place r other than S_0. The initial marking M_{20} of N_2 is the marking such that $M_{20}(S_0) = M_{20}(D) = 1$ and $M_{20}(r) = 0$ for every place r other than S_0 and D. Let A_1 and A_2 be arcones $\{M \mid M(S_f) = 1\}$ and $\{M \mid M(S_f) = 1,\ M(D) = 0\}$, respectively. Let Φ be a mapping such that $\Phi(t(\tau)) = t(\tau)$ for every operation τ of type I, $\Phi(t_{NZ}(\tau)) = t_{NZ}(\tau)$, and $\Phi(t_Z(\tau)) = \Phi(t'_Z(\tau)) = t_Z(\tau)$ for every operation τ of type II.

For a finite sequence α of operations from the initial total state in 2ca K, define $f(\alpha) \in \Sigma_1^*$ recursively as follows.

(1) If $\alpha = \lambda$, then $f(\alpha) = \lambda$.

(2) If $\alpha = \alpha_1\tau$ and τ is an operation of type I, then $f(\alpha) = f(\alpha_1)t(\tau)$.

(3) Suppose that $\alpha = \alpha_1\tau$ and $\tau = (s, s', s'', l)$ is of type II and $[s_0, 0, 0] \xrightarrow{\alpha_1} [s, x_1, x_2]$. If $x_l > 0$, then $f(\alpha) = f(\alpha_1)t_{NZ}(\tau)$. Otherwise, $f(\alpha) = f(\alpha_1)t_Z(\tau)$.

From the construction of N_1 and N_2, we have the following (P1) through (P6).

(P1) *For a finite sequence α of operations in K such that $[s_0, 0, 0] \xrightarrow{\alpha} [s, x_1, x_2]$, firing sequence $f(\alpha)$ belongs to $L_0(N_1)$. Let $M_{10} \xrightarrow{f(\alpha)} M$. Then $M(S) = 1$, $M(C_l) = x_l$ $(l = 1,2)$ and $M(r) = 0$ for every place r other than S, C_1 and C_2. If $s = s_f$, that is, 2ca K halts, then $f(\alpha) \in L(N_1, A_1)$.*

(P2) *Suppose that $\beta \in L_0(N_1)$ and any firing of transition of type $t_Z((s, s', s'', l))$ in β occurs at a marking whose C_l component is zero. Then, there is a sequence α of* *operations from the initial total state in K such that $f(\alpha)=\beta$, and β is said to simulate* 2ca *K correctly.*

(P3) *Let γ be a firing sequence in $L_0(N_2)$ such that $M_{20} \xrightarrow{\gamma} M_2$ in N_2. Then $\Phi(\gamma) \in L_0(N_1)$. Let $M_{10} \xrightarrow{\Phi(\gamma)} M_1$ in N_1. Then we have that for all places r other than D,*

$$M_1(r) = M_2(r). \tag{$*$}$$

Hence, $L(N_1, A_1) \supseteq \Phi(L(N_2, A_2))$.

(P4) *Any firing sequence γ in $L(N_2, A_2)$ must contain a transition of type $t'_Z((s, s', s'', l))$ because D has a token unless a transition of type $t'_Z(\tau)$ fires. Since transition $t'_Z((s, s', s'', l))$ can fire only at a marking whose C_l component is positive, it follows from* (∗), (P1) *and* (P2) *that $\Phi(\gamma)$ does not simulate* 2ca *K correctly.*

(P5) *Suppose that $\beta \in L(N_1, A_1)$ does not simulate* 2ca *K correctly, that is, β contains a transition of type $t_Z((s, s', s'', l))$ whose firing occurs at a marking M_1 such that $M_1(C_l) > 0$. Let γ be the sequence derived from β by substituting $t'_Z((s, s', s'', l))$ for an occurrence of $t_Z((s, s', s'', l))$ which fires at a marking whose C_l component is positive. Then $\beta = \Phi(\gamma)$ and $\gamma \in L(N_2, A_2)$.*

(P6) *Suppose that* 2ca *K halts. Let α be the sequence of operations from the initial total state to a final total state in K. By* (P1), *$f(\alpha) \in L(N_1, A_1)$. By* (P4), *$f(\alpha) \notin \Phi(L(N_2, A_2))$. Suppose that* 2ca *K does not halt. Then, any firing sequence β in $L(N_1, A_1)$ does not simulate* 2ca *K correctly, and therefore $\beta \in \Phi(L(N_2, A_2))$ by* (P5). *Hence, $L(N_1, A_1) = \Phi(L(N_2, A_2))$ by* (P3). *Thus $L(N_1, A_1) \neq \Phi(L(N_2, A_2))$ if and only if* 2ca *K halts.* □

Remark 1. Petri nets N_1 and N_2 and arcones A_1 and A_2 in the proof above have the following properties:

(1) If $\Phi(t_1) = \Phi(t_2)$, then $B_2(t_1, r) - F_2(r, t_1) = B_2(t_2, r) - F_2(r, t_2)$ for every unbounded place r, that is, the net effect for each unbounded place by transition t_1 is the same as that by transition t_2.

(2) Arcones A_1 and A_2 are defined by specifying the number of tokens of some bounded places only.

6. Generalized Petri nets

The coverability problem is to decide whether there is a marking M in $R_N(M_0)$ such that $M \geqslant M_f$ for a given marking M_f. This is known to be decidable in general [3, 7]. But the coverability problem for Petri nets in which a specified transition has priority over the other transitions will be shown to be undecidable.

Definition 17. Transition t_0 is said to have *priority over transition t* if t is not allowed to fire when t_0 is firable.

Theorem 4. *The coverability and reachability problems are undecidable for generalized Petri nets which have only two unbounded places and distinguished transition with priority over the other transitions.*

Proof. For any 2ca K, we construct a Petri net $N = \langle \Pi, \Sigma, E, M_0 \rangle$ and show that for a marking M_f there is a reachable marking M such that $M \geqslant M_f$ if and only if 2ca K halts. Petri net N consists of the following places and transitions.

(1) For each state s in 2ca K, N has a place designated by S.

(2) N has three places D, C_1 and C_2. C_1 and C_2 correspond to counters c_1 and c_2 in 2ca K, respectively.

(3) For each operation $\tau = (s, s', l)$ of type I in 2ca K, N has a transition $t(\tau)$ whose input place is S and whose output places are S' and C_l as shown in Fig. 7.

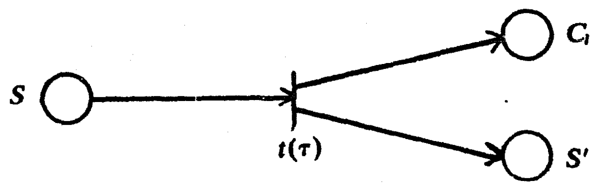


Fig. 7. Simulation of operation τ of type I.

(4) For each operation $\tau = (s, s', s'', l)$ of type II in 2ca K, N has three transitions $t_{NZ}(\tau)$, $t_Z(\tau)$ and $t_Z'(\tau)$ and a place $G(\tau)$ as shown in Fig. 8, where the input places (or the output places) of $t_{NZ}(\tau)$ are S and C_l (or S'), the input places (or the output places) of $t_Z(\tau)$ are S (or D, $G(\tau)$ and C_{3-l}), and the input places (or the output places) of $t_Z'(\tau)$ are D, $G(\tau)$ and C_{3-l} (or S'').

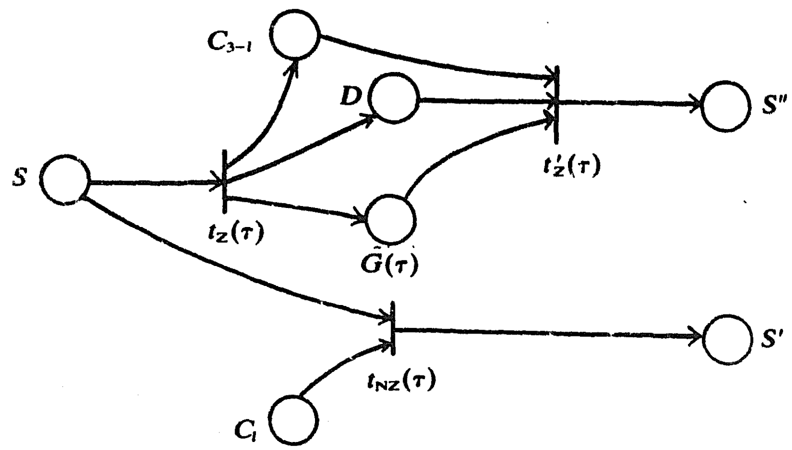


Fig. 8. Simulation of operation τ of type II.

(5) N has a transition t_0 whose input places are D, C_1 and C_2 as shown in Fig. 9, and which has priority over the other transitions.

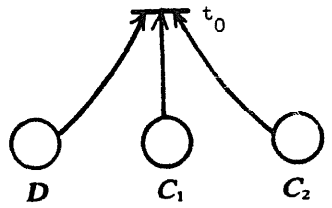


Fig. 9. Transition t_0 with priority.

The initial marking M_0 is the marking such that $M_0(S_0)=1$ and $M_0(r)=0$ for every place r other than S_0. Let M_f denote the marking such that $M_f(S_f)=1$ and $M_f(r)=0$ for every place r other than S_f. It follows from the construction of N that any marking M reachable from the initial marking M_0 satisfies the following equation:

$$\sum_{r\in\Pi-\{C_1,C_2,D\}} M(r)=1.$$

Marking M such that $M(S)=1$ is said to correspond to state s in 2ca K. Then the initial marking M_0 corresponds to the intial state s_0 and markings M such that $M\geqslant M_f$ correspond to the final state s_f.

For a finite sequence α of operations from the initial total state in K, define $f(\alpha)\in\Sigma^*$ recursively as follows.

(1) If $\alpha=\lambda$, then $f(\alpha)=\lambda$.

(2) If $\alpha=\alpha_1\tau$ and τ is an operation of type I, then $f(\alpha)=f(\alpha_1)t(\tau)$.

(3) Suppose that $\alpha=\alpha_1\tau$ and $\tau=(s,s',s'',l)$ is of type II and $[s_0,0,0]\xrightarrow{\alpha_1}[s,x_1,x_2]$. If $x_l>0$, then $f(\alpha)=f(\alpha_1)t_{\mathrm{NZ}}(\tau)$. Otherwise, $f(\alpha)=f(\alpha_1)t_{\mathrm{Z}}(\tau)t'_{\mathrm{Z}}(\tau)$.

From the construction of N, we have the following properties.

(P1) *For a finite sequence α of operations in K such that $[s_0,0,0]\xrightarrow{\alpha}[s,x_1,x_2]$, firing sequence $f(\alpha)$ belongs to $L_0(N)$. Let $M_0\xrightarrow{f(\alpha)}M$. Then $M(S)=1$, $M(C_l)=x_l$ $(l=1,2)$ and $M(r)=0$ for every place r other than S, C_1 and C_2. If $s=s_f$, then $M\geqslant M_f$. That is, if K halts, then there is a marking M reachable from M_0 such that $M\geqslant M_f$.*

(P2) *Let $\beta_1 t$ be a firing sequence in $L_0(N)$ such that $t=t_{\mathrm{Z}}(\tau)$, $\tau=(s,s',s'',l)$ and $M_0\xrightarrow{\beta_1}M_1\xrightarrow{t}M$. Then it follows from Fig. 8 that $M(G(\tau))=M(D)=1$. $M(C_{3-l})\geqslant 1$ and $M(r)=0$ for every place r other than $G(\tau)$, D, C_1 and C_2. If $M_1(C_l)=0$, that is, N simulates operation τ of K correctly, then transition t_0 cannot fire and only $t'_{\mathrm{Z}}(\tau)$ is firable at marking M. If $M_1(C_l)\geqslant 1$, that is, N simulates operation τ of K incorrectly, then t_0 can fire at M and no other transitions can fire since t_0 has priority over other transitions. If t_0 has fired, then the resulting marking M' has zero component for every place other than $G(\tau)$, C_1 and C_2. Hence, $M'\not\geqslant M_f$ and no transitions can fire at M'.*

(P3) *Suppose that $M_0\xrightarrow{\beta}M$ and $M\geqslant M_f$. Then, it follows from* (P2) *that* (1) *β does not contain t_0, and* (2) *any firing of a transition of type $t_{\mathrm{Z}}((s,s',s'',l))$ in β occurs at a* *marking whose C_l component is zero and is followed immediately by a firing of transition $t'_Z((s, s', s'', l))$. Hence, β can be decomposed as $\beta = \beta_1 \beta_2 \cdots \beta_m$, where β_i is either $t(\tau_i)$, or $t_{NZ}(\tau_i)$ or $t_Z(\tau_i)t'_Z(\tau_i)$ for $1 \leq i \leq m$. Let $\alpha = \tau_1 \tau_2 \cdots \tau_m$. Then, by the definitions of N and f we have that $[s_0, 0, 0] \xrightarrow{\alpha} [s_f, x_1, x_2]$ where x_1 and x_2 are nonnegative integers. That is, K halts.*

Consequently, we have that:

$$\exists M \in R_N(M_0): \quad M \geq M_f \iff 2\text{ca } K \text{ halts.}$$

By simple modifications of the arguments above, we can prove that the reachability problem is also undecidable [1]. □

We shall introduce a new kind of arc called *reset arc* and denote them by "⊖>". A reset arc e goes out of a transition and enters to a place which is denoted by $r(e)$. A firing of a transition with reset arcs $e_1, e_2, \ldots$ is defined in the same way as that of a transition without reset arcs except that all tokens in places $r(e_1), r(e_2), \ldots$ are removed.

Theorem 5. *The reachability problem is undecidable for generalized Petri nets in which some transitions can reset one of two distinguished places to zero marking.*

Proof. In this proof, assume that two counters are equal to zero when 2ca K halts. For any such 2ca K, we construct a Petri net $N = \langle \Pi, \Sigma, E, M_0 \rangle$ consisting of the following places and transitions.

(1) For each state s in 2ca K, N has a place designated by S.

(2) N has four places C_1, C_2, C'_1 and C'_2. Places C_l and C'_l correspond to counter c_l in 2ca K $(l = 1, 2)$.

(3) For each operation $\tau = (s, s', l)$ of type I in 2ca K, N has a transition $t(\tau)$ whose input place is S and whose output places are S', C_l and C'_l as shown in Fig. 10.

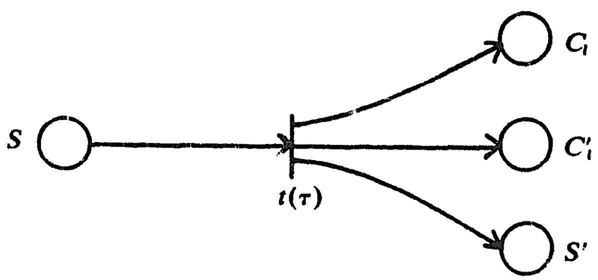


Fig. 10. Simulation of operation of type I.

(4) For each operation $\tau = (s, s', s'', l)$ of type II in 2ca K, N has two transitions $t_{NZ}(\tau)$ and $t_Z(\tau)$ as shown in Fig. 11, where the input places (or the output places) of

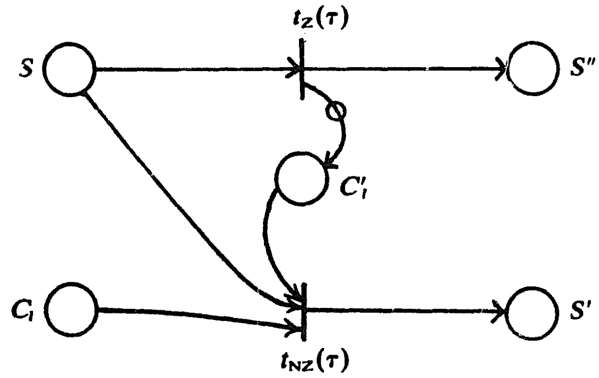


Fig. 11. Simulation of operation of type II.

$t_{NZ}(\tau)$ are S, C_l and C'_l (or S'), the input place (or the output place) of $t_Z(\tau)$ is S (or S''), and $t_Z(\tau)$ resets C'_l.

The initial marking M_0 is the marking such that $M_0(S_0)=1$ and $M_0(r)=0$ for every place r other than S_0. Let M_f denote the marking such that $M_f(S_f)=1$ and $M_f(r)=0$ for every place r other than S_f. Let f be a mapping defined in the proof of Theorem 4. Let $F(r,t)$ (or $B(t,r)$) denote the number of arcs from place r (or transition t) to transition t (or place r) in N.

From the construction of N, we have the following properties.

(P1) *Suppose that $\beta \in L_0(N)$ and any firing of transition of type $t_Z((s,s',s'',l))$ in β occurs at a marking whose C_l component is zero. Then, there is a sequence of operations from the initial total state in K such that $f(\alpha)=\beta$, and β is said to simulate* 2ca *K correctly.*

(P2) *A number of arcs between place C_l and any transition t is equal to a number of arcs between place C'_l and t except for reset arcs. Therefore it holds that $M(C_l) \geq M(C'_l)$ for any marking M in $R_N(M_0)$. Since there is no transition t such that $B(t,C'_l)-F(C'_l,t) > B(t,C_l)-F(C_l,t)$, it holds that $M'(C_l) > M'(C'_l)$ for any marking M' reachable from marking M such that $M(C_l) > M(C'_l)$.*

(P3) *Consider two markings M and M' such that $M \xrightarrow{t_Z(\tau)} M'$. If $M(C_l)=0$, then $M'(C_l)=M'(C'_l)=0$. If $M(C_l)>0$, then $M'(C_l)>M'(C'_l)=0$. Therefore by* (P2), *$M(C_l)=M(C'_l)$ $(l=1,2)$ for any marking M such that $M_0 \xrightarrow{\beta} M$ if and only if β simulates a sequence of operations of* 2ca *K correctly.*

(P4) *We will show that $M_f \in R_N(M_0)$ if and only if* 2ca *K halts. If $M_f \in R_N(M_0)$, then $M_f(C_l)=M_f(C'_l)=0$ for $l=1,2$ by the definition. By* (P3), *a firing sequence β such that $M_0 \xrightarrow{\beta} M_f$ simulates a sequence of operations from the initial total state $[s_0,0,0]$ to the final total state $[s_f,0,0]$ in K correctly. Therefore* 2ca *K halts. Conversely, if* 2ca *K halts, then there is a firing sequence $f(\alpha)$ which simulates the sequence α of operations from $[s_0,0,0]$ to $[s_f,0,0]$ in K correctly, and therefore $M_0 \xrightarrow{f(\alpha)} M_f$.* □

Acknowledgments

We are grateful to Professors W.W. Peterson, N. Tokura and K. Taniguchi for their useful discussions and to the referee for his helpful suggestions.

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