On the Decidability of Reachability in Continuous Time Linear Time-Invariant Systems

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─── **Abstract** ───

We consider the decidability of state-to-state reachability in linear time-invariant control systems over continuous time. We analyse this problem with respect to the allowable control sets, which are assumed to be the image under a linear map of the unit hypercube. This naturally models bounded (sometimes called saturated) controls. Decidability of the version of the reachability problem in which control sets are affine subspaces of $\mathbb{R}^n$ is a fundamental result in control theory. Our first result is decidability in two dimensions ($n = 2$) if the matrix A satisfies some spectral conditions, and conditional decidablility in general. If the transformation matrix A is diagonal with rational entries (or rational multiples of the same algebraic number) then the reachability problem is decidable. If the transformation matrix A only has real eigenvalues, the reachability problem is conditionally decidable. The time-bounded reachability problem is conditionally decidable, and unconditionally decidable in two dimensions. Some of our decidability results are conditional in that they rely on the decidability of certain mathematical theories, namely the theory of the reals with exponential ($\mathfrak{R}_{\exp}$) and with bounded sine ($\mathfrak{R}_{\exp,\sin}$). We also obtain a hardness result for a mild generalization of the problem where the target is simple set (hypercube of dimension $n - 1$ or hyperplane) instead of a point, and the control set is a convex bounded polytope. In this case, we show that the problem is at least as hard as the *Continuous Positivity problem* or the *Nontangential Continuous Positivity problem*.

1 Introduction

This paper is concerned with *linear time-invariant (LTI) systems*. LTI systems are one of the most basic and fundamental models in control theory and have applications in circuit design, signal processing, and image processing, among many other areas. LTI systems have both discrete-time and continuous-time variants; here we are concerned solely with the continuous-time version.

A (continuous-time) LTI system in dimension n is specified by a transition matrix $A \in \mathbb{Q}^{n \times n}$, a control matrix $B \in \mathbb{Q}^{m \times n}$ and a set of controls $U \subseteq \mathbb{R}^m$. The evolution of the system is described by the differential equation $x'(t) = Ax(t) + Bu(t)$ where $u : \mathbb{R} \to U$ is a measurable function. Here we think of u as an input (or control) applied to the system. Note that the number of inputs is independent of the dimension: it is possible to have only one input ($m = 1$) in dimension n, or many inputs in small dimension ($m > n$).

Given such an LTI system, we say that state $x_0 \in \mathbb{R}^n$ can *reach* state $y \in \mathbb{R}^n$ if there exists $T \geqslant 0$ and a control* $u : [0, T] \to U$ such that the unique solution to the differential system $x'(t) = Ax(t) + Bu(t)$ for $t \in (0, T)$ with initial condition $x(0) = x_0$ satisfies $x(T) = y$. Similarly, given $t \geqslant 0$, We say that state x_0 can reach state y *in time at most* t if it can reach y at a time $T \leqslant t$. The problem of computing the set of all states reachable from a given state has been an active topic of research for several decades. Almost exclusively, the emphasis is typically on efficient and scalable methods to over- and under-approximate the reachable set [8, 15, 14, 25, 41]. Furthermore, this problem has plenty of practical applications and is a fundamental basic block for the analysis of more complicated models, such as hybrid systems [12, 1, 43]. By contrast, there are relatively few results concerning the *decidability* of the reachable set—the focus of the present paper.

We consider the *LTI Reachability problem*: given an LTI system, and target state y, decide whether **0** can reach y. Specifically, we primarily focus on the case where the inputs are *saturated*, that is $U = [-1, 1]^m$. Equivalently, one can think of BU as being a zonotope [34]. This naturally leads us to study the impact of the control matrix (B) on the LTI Reachability problem. We will also consider the *Bounded Time LTI Reachability problem* where we are given an upper bound on the time allowed to reach the target: given an LTI system, and target state y and a time bound T, decide whether **0** can reach y in time at most T. Finally, we consider the *LTI Set Reachability problem* where the target y becomes a set and we ask whether there exists a reachable point within this set.

Close variants of the LTI Reachability problem include the Controllability problem (set of points that can reach **0**). It is also possible to consider the set of points reachable from a given source x_0. The former problem is equivalent to the Reachability problem in backward time, the latter is equivalent to the Reachability problem albeit with a modification of the control matrix and set.

Somewhat confusingly, the literature of control theory often focuses on more "universal" versions of the above problem with almost the same names: *null reachability* (can one reach all states from the origin?) and *null controllability* (can one reach the origin from all states?) [5]. However these "universal" reachability problems are very different from the point-to-point version that we study. In particular, both null reachability and null controllable are decidable in polynomial time using linear algebra.

One of the first results about (continuous-time) LTI Reachability problems is that of Kalman [24] where the control sets are linear subspaces of $\mathbb{R}^n$. An important particular case is that of the *Orbit problem*: given a matrix A, an initial state x_0 and a state y, decide whether y is in the *orbit of x under A*, *i.e.* whether $y = e^{At}x_0$ for some $t \geqslant 0$. This corresponds to the case when the control is a singleton (or equivalently with non-zero $x(0)$ and zero control set). This problem which was shown to be decidable in polynomial time [18, 9]. These results yield (polynomial-time) decidability when the control sets are affine subspaces of $\mathbb{R}^n$. An exact description of the null controllable regions for general linear systems with saturating actuators was obtained [22], however this formula does not immediately yield an algorithm (see Section 3).

In this paper, we study the decidability of several special instances of the LTI Reachability problem with saturated inputs. Specifically, we show several condition decidability results:

- In two dimensions ($n = 2$), the reachability problem is conditionally ($\mathfrak{R}_{\mathrm{exp,sin}}$) decidable.
- If A has real spectrum then the reachability problem conditionally ($\mathfrak{R}_{\mathrm{exp}}$) decidable.
- The time-bounded reachability problem is conditionally decidable ($\mathfrak{R}_{\mathrm{exp,sin}}$).

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* From now on, all controls are necessarily measurable functions, we omit it most of the time.

Some of our decidability results are conditional in that they rely on the decidability of certain mathematical theories, namely the theory of the reals with exponential ($\mathfrak{R}_{exp}$) and with bounded sine ($\mathfrak{R}_{exp,sin}$). Both theories are known to be decidable assuming Schanuel's conjecture [32], a major conjecture in transcendental number theory that is widely believed to be true. We also manage to find to some class of LTI with decidable reachability problem:

- In two dimensions, when the A has real spectrum and there is only one input ($m = 1$).
- When A is diagonalizable with rational eigenvalues (or rational multiples of the same algebraic number).
- When A is real diagonal, there is only one input and it has at most two nonzero entries.
- When A only has one eigenvalue, which is real, and there is only one input.

While those subclasses look ad-hoc, they all correspond to specific forms of the boundary of the reachable set and the study of the transcendental points on this boundary. In particular, some of those cases require some nontrivial theorems in the field of transcendental number theory (Gelfond–Schneider, Lindermann-Weierstrass). See Section 3 for more details.

We also obtain a hardness result for a mild generalization of the problem where the target is simple set (compact convex of dimension $n - 1$ or hyperplane) instead of a point, and the control set is either $U = \{u\}$ or $U = [-1, 1]$. In this case, we show that the problem is at least as hard as the *Continuous Skolem problem* or the *Nontangential Continuous Skolem problem* which asks whether the first component $x_1(t)$ of the solution to a linear differential equation $x'(t) = Ax(t)$ has a zero (resp. nontangeantial zero). Showing decidability of any of these problems would entail a major new effectiveness result in Diophantine approximation, which suggests that the problem is very challenging.

Related Work. It well-known that besides linear systems, most control problems are undecidable [6, 7]. For example, point-to-point reachability is undecidable for *piecewise linear systems* [2, 5, 26], for *saturated linear systems* [39] and point-to-set reachability is undecidable for polynomial systems, a consequence of [17]. However to the best of our knowledge, there are no (un)decidability results within the class of LTI systems, except when the control sets are affine subspaces. An exact description of the null controllable regions for general linear systems with saturating actuators was obtained [22], but it does not immediately translate into a decidability result. On the other hand, the reachability problem is well-known to be challenging in practice, even for LTI systems. There is a vast literature on efficient and scalable methods to over- and under-approximate the reachable set [8, 15, 14, 25, 41]. However those methods, by construction, cannot lead to any decidability results on their own. In fact, one can observe that a corollary of these methods is that the "only" difficult part of the problem, in terms of decidability, is the boundary of the reachable set.

A range of different control problems for discrete- and continuous-time LTI systems under constraints on the set of controls have been studied in the literature [13, 11, 40, 41, 38, 21, 22, 44, 16, 37, 20, 14, 23, 45]. Kalman showed that when the control set is $U = \mathbb{R}^m$, the system is globally null-controllable if and only if (A, B) is controllable [24]. Lee and Markus considered U such that $\mathbf{0} \in U \subset \mathbb{R}^m$ and showed that if (A, B) is controllable and all eigenvalues have negative real parts, then the system is globally null-controllable [30]. Sontag considered the problem of asymptotic null-controllability which asks if there is a control that reaches the origin in the limit [40]. Summers discussed about over estimation of the reachable set (from origin) by n-dimensional ellipsoids when $U = [-1, 1]^m$ [41]. Schmitendorf considered time varying matrices $A(t)$ and $B(t)$ and gives a characterisation for a given point to be controllable when U is compact [37], however this does not immediately yield an algorithm (see Section 3). Lafferriere considered a different reachability problem where the inputs are expressible in the first-order theory of the reals with some unknown coefficients

[27, 28], but our problem is of a very different nature.

2 Examples

The idea of using an external input to manipulate the state of some system to achieve a certain goal is fundamental and everywhere in our lives. In order to give a better idea of the problem we are trying to solve, we will informally explain the theory and its goals via some examples.
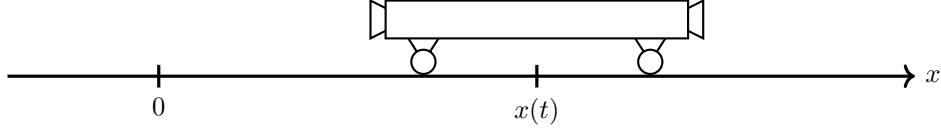
■ **Figure 1** Toy example of control in 1 dimension: a car with two boosters

Consider the toy example in Figure 1 (taken from Chapter 1 of [33]). A car has two boosters, one at the front and one at the back. At time $t = 0$, it starts at some position $x_0 \in \mathbb{R}$ on the real line, with velocity v_0. The objective is to reach the origin and stay there indefinitely, that is to reach origin with a speed of 0. The external input in the above problem is the effects of the boosters that affect the acceleration directly, thereby affecting the velocity and the position. We model the state of the system by its position and velocity $S(t) = (x(t), v(t)) \in \mathbb{R}^2$. Assuming the front and rear boosters are similar and give a max acceleration of M units, we can model the dynamics by

$$x'(t) = v(t), \qquad v'(t) = u(t)$$

where $u(t) \in [-M, M]$: the acceleration is positive when the rear booster is on, and negative when the positive booster is on. We call $U = [-M, M]$ the control set. Combining both equations and writing it in matrix form gives us

$$S(0) = \begin{bmatrix} x_0 \\ v_0 \end{bmatrix}, \qquad S'(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} S(t) + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} u(t) = AS(t) + Bu(t).$$

Here, the problem is to find or "synthesize" a control u such that we reach $(0,0)$ from the initial point. Note that in real life, we cannot change the control (booster output) arbitrarily fast, *i.e.* not all functions u are to be considered. In this work, we neglect this aspect and allow any function $u : \mathbb{R} \to U$ that is *measurable*, which is essentially the minimum mathematical condition for the problem to make sense. Observe that already in this toy example, it is natural to consider a bound on the acceleration: the control set is therefore bounded.

In the example of the car, we viewed the input as a something under our control that we used to achieve some objective. A dual view is to consider certain safety problems and check whether the input, now controlled by an adversary, can be used to steer the system to a bad state. Consider the system in Figure 2, a spring-mass-damper system with an external force acting on it. This typically models a vehicle's suspension. For example, consider a bike travelling on a road and encountering a speed breaker or hump. We are interested in the vertical movement of the tires, after they cross the hump. Here, the tire acts as the mass, the damper and spring form the bike suspension, which provides shock absorption and the recoil force upon hitting the ground, the road is modelled by the external force u. A bad
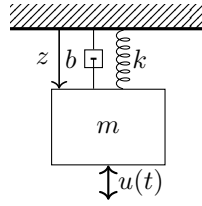
Figure 2 A spring-mass-damper system: a mass m is attached to the ceiling by a spring of stiffness k and a damper of factor b. The mass is excited by a force $u(t)$ viewed as the input of the system.

state is one when the tire's vertical movement is higher than certain admissible value which we want to avoid (for it could damage or even break the suspension). Here the problem is to decide whether it is possible via some external input to reach a bad state. In a similar fashion to the previous example, we could model the above system by an LTI system with a bounded control set.

3 Challenges

A major challenge in solving the continuous-time reachability problem is the fact that there is no simple formula, or more exactly, no formula that is immediately computable. Given a LTI system $x'(t) = Ax(t) + Bu(t)$, there is a general expression for $x(t)$ given u that involves an integral and exponential of matrix (see Section 4):

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}Bu(s)\,\mathrm{d}s. \tag{1}$$

Therefore, the reachability problem is equivalent to checking whether there exists a u such that (1) is equal to the target y. Unfortunately, it seems impossible to obtain any more actionable *symbolic* formula without knowing more about the shape of u. In the particular case where $x(0) = \mathbf{0}$ and $u : \mathbb{R} \to U$ for some convex set U, it is possible say more.

The simplest case is when $U = \mathbb{R}^m$: the reachable set can be shown to be a linear subspace, the image of the so-called controllability matrix $\begin{bmatrix} B & AB & \cdots A^{n-1}B \end{bmatrix}$ and therefore the reachability problems reduces to an orbit problem (if the image of the controllability matrix is not the whole space, what happens on the remaining space is exactly an orbit problem).

A more interesting case, and the subject of this paper, is when U is a compact convex polytope and in particular a hypercube: $U = [-1, 1]^m$. This is known as the *saturated input* case. It is not hard to see that when $x(0) = \mathbf{0}$ and U is convex, the reachable set $\mathcal{R}$ is strictly convex. Unfortunately, the set $\mathcal{R}$ can still be very complicated and checking whether a single point lies inside it turns out be a challenging problem. We are aware of two distinct but similar results in this direction. Schmitendorf et al [37] have some general conditions under which a control can steer a point to the origin. In the particular case at hand, they turn out to be equivalent to another formulation by [22] where the boundary of the reachable region is described by the set of

$$\int_0^{-\infty} e^{At}b\,\mathrm{sgn}(c^T e^{At}b)\,\mathrm{d}t$$

where $c \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. The main challenge is that evaluating this integral is potentially a hard problem. In particular, if A has a complex eigenvalue whose argument is not rational multiple of π, then the sign of $c^T e^{At}b$ will follow a completely irregular pattern. In fact,
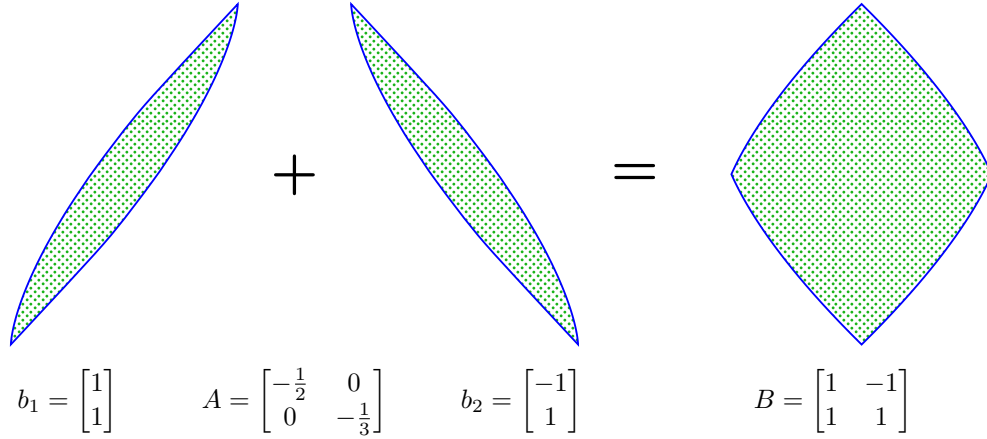
■ **Figure 3** Example of a simple LTI system: the reachable set is depicted in three cases. The two pictures on the left correspond to the case where of one control ($B = b_1$ and b_2 respectively). The picture on the right corresponds to the case where $B = [b_1, b_2]$: the reachable set is then the Minkowski sum of the two reachable sets.

the a priori simpler problem of deciding whether $c^T e^{At} b$ will change sign *at all* is exactly the continuous Skolem problem. This problem is open and has been shown to be related to difficult number theoretical questions (see Section 4.3). Note, however, that computing this integral, or rather deciding if this integral is less than some prescribed number, does not necessarily require solving the continuous Skolem problem. In fact, a solution to Skolem would not help per se (there could be infinitely many changes, whose values are not even algebraic), and conversely, computing this integral does not necessarily help deciding the existence of a sign change.

At the geometrical level, the difficulty of the problem is apparent when looking at pictures of reachable set. Already in dimension 2, the boundary of the reachable set when $U = [-1, 1]$ and A is stable can be complicated. As illustrated on Figure 3, it consists of two smooth curves joining at singular points. Furthermore, the smooth curves are not necessarily expressible with polynomial equations and may involve exponentials, such as $\{(x, y) \in \mathbb{R}^2 : y = xe^x\}$. In fact, in the particular case illustrated on the left of Figure 3, the boundary is exactly (see Appendix J for the details) the set

$$\partial\mathcal{R}(A, b_1) = \left\{ \pm \begin{bmatrix} 2 - 4\alpha^{-3} \\ 3 - 6\alpha^{-2} \end{bmatrix} : \alpha \in [1, \infty) \right\} \cup \left\{ \pm \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}.$$

In the case where the eigenvalues of the matrix are real, the formulas will only involve real exponentials and the boundary can be expressed in the theory of reals with exponential ($\mathfrak{R}_{\exp}$). This theory is known to be decidable subject to Schanuel's conjecture (see Section 4.5). Unfortunately, the formulas may further involve sine and cosine when A has complex eigenvalues, and the theory of reals with sine and cosine is undecidable. This suggests that the reachability problem is hard, but surprisingly, we have only been able to show some hardness results in the case of set reachability.

4   Preliminaries

In this document, $\|x\|$ denotes the usual Euclidean norm of vectors $x \in \mathbb{C}^n$ and $\|A\|$ the induced norm on matrices $A \in \mathbb{C}^{n \times n}$. Recall that any induced norm is consistent ($\|Ax\| \leqslant \|A\|\|x\|$) and therefore submultiplicative ($\|AB\| \leqslant \|A\|\|B\|$). Given a matrix $A \in \mathbb{C}^{n \times n}$, e^A denotes the matrix exponential of A. In particular, we have that $\|e^A\| \leqslant e^{\|A\|}$. We denote the boundary of set S by ∂S and its closure by $\overline{S}$. We denote the tranpose of a vector or matrix A by A^T and its spectrum by $\sigma(A)$.

4.1   Jordan decomposition and matrix exponential

Given a square matrix A of order n with rational entries, one can find matrices P and Λ (possibly with complex algebraic entries) such that $A = P\Lambda P^{-1}$. Here Λ is a block diagonal matrix $\mathrm{diag}(J_1, J_2, \ldots, J_m)$ where the J_i are matrices of a special form (given below) known as the Jordan blocks A particular application of Jordan decomposition is to compute the exponential of a matrix. From the above definition, it is clear that if $A = P\Lambda P^{-1}$, then $e^{At} = Pe^{\Lambda t}P^{-1}$. If $\Lambda = \mathrm{diag}(J_1, J_2, \ldots, J_m)$, then it is not hard to see that $e^{\Lambda t} = \mathrm{diag}(e^{J_1 t}, e^{J_2 t}, \ldots, e^{J_m t})$. A closed form expression for a jordan block $e^{J_i t}$ is given by

$$
e^{J_i t} = e^{\lambda_i t}
\begin{bmatrix}
1 & t & \frac{t^2}{2} & \cdots & \frac{t^{k-1}}{(k-1)!} \\
0 & 1 & t & \cdots & \frac{t^{k-2}}{(k-2)!} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & t \\
0 & 0 & \cdots & 0 & 1
\end{bmatrix}
\quad \text{when} \quad
J_i =
\begin{bmatrix}
\lambda_i & 1 & 0 & \cdots & 0 \\
0 & \lambda_i & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_i & 1 \\
0 & 0 & \cdots & 0 & \lambda_i
\end{bmatrix}
$$

where λ_i is an eigenvalues of A and k is the size of the Jordan block. A consequence of this normal form is the *real Jordan normal form*: if A is real then its Jordan form can be nonreal. However, one can allow more general blocks to recover a real representation: a real Jordan block is either a complex Jordan block with a real λ_i, or a block matrix of the form

$$
J_i' =
\begin{bmatrix}
C_i & I_2 & 0 & \cdots & 0 \\
0 & C_i & I_2 & \cdots 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & C_i & I_2 \\
0 & 0 & \cdots & 0 & C_i
\end{bmatrix}
\quad \text{where} \quad
C_i =
\begin{bmatrix}
a_i & -b_i \\
b_i & a_i
\end{bmatrix}
$$

where $\lambda_i = a_i + ib_i$. In this case, one can ensure that the transformation matrix P is also real. This form is particularly useful for simple blocks since the exponential of C_i is a scaling-and-rotate matrix:

$$
e^{C_i t} = e^{a_i t}
\begin{bmatrix}
\cos(b_i t) & \sin(b_i t) \\
-\sin(b_i t) & \cos(b_i t)
\end{bmatrix}.
$$

4.2   Control Theory

The following definition is standard in the literature of control theory.

▶ **Definition 1** (Controllable). *A pair of matrices (A, B) is called controllable if the rank of $[B, AB, \ldots, A^{n-1}B]$ is n, where A is an $n \times n$ matrix.*

Consider two vectors c and b in $\mathbb{R}^n$ and define a function $f_{c,A,b}(t) = c^T e^{At} b$. Then it follows from the Jordan decomposition that $f_{c,A,b}(t) = \sum_{j=1}^m P_j(t) e^{\theta_j t}$ where each θ_j is an eigenvalue of A. The following properties of f are well-known (see e.g. [19]).

▶ **Lemma 2.** *Let $f_{c,A,b}(t)$ be the function defined above. Then*
- *if $f_{c,A,b} \neq 0$, then the number of zeros of $f_{c,A,b}$ in any bounded interval is finite,*
- *(A, b) is controllable $\iff$ for all c, $f_{c,A,b} \neq 0$,*
- *if the eigenvalues of A are real and b, c are nonzero then $f_{c,A,b}$ has at most $n-1$ zeros.*

Given a matrix A and a control set U, define the null-controllable set $\mathcal{C}$ and the reachable set $\mathcal{R}$ as

$$\mathcal{C}(A, B, U) = \bigcup_{T \geqslant 0} \left\{ -\int_0^T e^{-At} B u(t) \, \mathrm{d}t \,\middle|\, u : [0, T] \to U \text{ measurable} \right\}, \tag{2}$$

$$\mathcal{R}(A, B, U) = \bigcup_{T \geqslant 0} \left\{ \int_0^T e^{At} B u(t) \, \mathrm{d}t \,\middle|\, u : [0, T] \to U \text{ measurable} \right\}. \tag{3}$$

It follows immediately from those definitions that $\mathcal{R}(A, B, U) = \mathcal{C}(-A, B, -U)$ and $U = -U$ for a hypercube (or any symmetric set). It is customary in the literature to express results about the null-controllable sets. However, since we are interested in reachability questions, we find it more convenient to state all results using the reachable set.

Define a matrix A to be *stable* if all its eigenvalues have negative real part, *antistable* if all its eigenvalues have positive real part, *weakly-stable* (also called *semi-stable* in [22]) if all its eigenvalues have nonpositive real parts and *weakly-antistable* if all its eigenvalues have nonnegative real parts. Clearly A is stable (resp. weakly-stable) if and only if $-A$ is antistable (resp. weakly-antistable).

In some cases, it is well-known that it is possible to decompose the system into its stable and weakly-antistable parts (or dually into its antistable and semistable parts). In particular, this is possible when the control set is a hypercube:

▶ **Proposition 3** ([22]). *Let (A, B) be controllable, $U = [-1, 1]^m$.*
- *If A is weakly-antistable, then $\mathcal{R} = \mathbb{R}^n$*
- *If A is stable, then $\mathcal{R}$ is a bounded convex open set containing the origin*
- *If $A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ where $A_1 \in \mathbb{R}^{n_1 \times n_1}$ stable and $A_2 \in \mathbb{R}^{n_2 \times n_2}$ weakly-antistable and B is partitioned as $\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ accordingly, then $\mathcal{R} = \mathcal{R}(A_1, B_1, U) \times \mathbb{R}^{n_2}$.*

This fact, together with Proposition 3, suggests that we need to study the reachable region of stable systems. Decompose B as $[b_1, \ldots, b_m]$ and assume that $U = [-1, 1]^m$, then it is not hard to check that

$$\mathcal{R}(A, B, [-1, 1]^m) = \mathcal{R}(A, b_1, [-1, 1]) + \cdots + \mathcal{R}(A, b_m, [-1, 1]) \tag{4}$$

is a Minkowski sum of reachable regions in which $m = 1$, *i.e.* B is a column vector. In this case, one can obtain an explicit description of $\mathcal{R}$. Note however that (4) does not, by itself, allows for a reduction to this simpler case: even if he have an algorithm to decide membership in each smaller control set, deciding membership in the Minkowski sum is nontrivial.

▶ **Theorem 4** ([22]). *Let A be stable, $b \in \mathbb{R}^{n \times 1}$ such that (A, b) is controllable and $U = [-1, 1]$. Then $\mathcal{R}(A, b, U)$ is an open convex set containing $\mathbf{0}$ and its boundary is given by*

$$\partial \mathcal{R}(A, b, U) = \left\{ \int_0^\infty e^{At} b \operatorname{sgn}(c^T e^{At} b) \, \mathrm{d}t : c \in \mathbb{R}^n \setminus \{\mathbf{0}\} \right\}$$

which is a strictly convex set.

4.3 Continuous Skolem Problem

The *Continuous Skolem problem* is a fundamental decision problem concerning the reachability of linear continuous-time dynamical system [4]. Given an initial point and a system of linear differential equation, the problem asks whether the orbit ever intersects a given hyperplane. More precisely, given a matrix $A \in \mathbb{R}^{n \times n}$, two vectors $c, b \in \mathbb{R}^n$, with rational (or algebraic coefficients), the question is whether there exists $t \geqslant 0$ such that $c^T e^{At} b = 0$. One can also consider the bounded time version of this problem, where one asks about the existence of a zero at time $t \leqslant T$ for some prescribed rational number T. While similar in spirit to the Orbit problem (does the orbit reach a given point?), it is of a very different nature. In fact, decidability of this problem is still open, even when restricting to the case of a bounded time interval [10]. The Continuous Skolem problem admits several reformulations, notably whether a linear differential equation or an exponential polynomial admits a zero [4]

The Continuous Skolem problem can be seen as the continuous analog of the *Skolem problem*, which asks whether a linear recurrent sequence has a zero. The Skolem problem is a famously open problem in number theory and computer science, which is know to be decidable up to dimension 4 and not known to be either decidable or undecidable starting from dimension 5. We refer the reader to [35] for a recent survey on the Skolem problem.

The Continuous Skolem problem is only known to be decidable in very specific cases: in low dimension, with a dominant real eigenvalue or a particular spectrum [4, 10]. Recent developments suggest that the Continuous Skolem problem is a very challenging problem. Indeed, decidability of the problem in the case of two (or more) rationally linearly independent frequencies would imply a new effectiveness result in Diophantine approximation that seem far off at the moment [10]. Even decidability in the bounded case is nontrivial because of *tangential zeros*, and has only been shown recently subject to Schanuel's Conjecture, a unifying conjecture in transcendental number theory. While Schanuel's Conjecture is widely believed to be true, its far reaching consequences suggest that that any proof is a long way off. For instance, it easily implies that $\pi + e$ is transcendental, but meanwhile the much weaker fact that $\pi + e$ is irrational is still unknown! It should be noted however that from a TCS perspective, the Continuous Skolem problem is only known to be at least NP-hard [4].

We now introduce the *Continuous Nontangential Skolem problem*, a variant of this problem where only zero-crossings are considered. Given a matrix $A \in \mathbb{R}^{n \times n}$, two vectors $c, b \in \mathbb{R}^n$, with rational (or algebraic coefficients), the question is whether there exists $t \geqslant 0$ such that $f(t) = 0$ and $f'(t) \neq 0$, where $f(t) = c^T e^{At} b = 0$. We call such a time t a *nontangential zero*, as opposed to *tangential zero* that would satisfy $f(t) = f'(t) = 0$. Clearly any nontangential zero is a zero but some system admit tangential zeros.

We believe that this problem is essentially as hard as the Continuous Skolem problem. Indeed, one of the reasons why the Continuous Skolem problem is believed to be hard is a Diophantine hardness proof [10]. In short, this reduction shows that decidability would entail some major new effectiveness result in Diophantine approximation, namely computability of the Diophantine-approximation types of all real algebraic numbers. But one can observe that

the reduction of [10] only relies on nontangential zeros, hence decidability of the Nontangential Skolem problem would also entail those Diophantine effectiveness results.

We note that there is a subtlety in the definition of the Nontangential Skolem problem: one needs to decide the existence of nontangeantial zeros but it is entirely possible that it also has some tangential zeros. Hence, even over a bounded interval, it is not clear that the problem is decidable. For instance, a Newton-based method would not be able to distinguish between a tangential or a nontangential zero using a finite number of iterations. The problem easily becomes decidable, *over a bounded interval* under the promise that there are no tangential zeros. We also believe that a variant of the decidability argument in [10] would show that the problem is decidable over bounded interval, assuming Schanuel's conjecture. As we have seen before, the problem is hard for Diophantine-approximation types over unbounded intervals.

4.4 First-order theory of the reals

A *sentence* in the first-order theory of the reals is (although one can allow more general expression that interleave quantifiers and connectives) an expression of the form $\phi = Q_1 x_1 \cdots Q_n x_n.\psi(x_1, \ldots, x_n)$ where each $Q_1, \ldots, Q_n$ is one of the quantifiers $\exists$ or $\forall$, and ψ is a Boolean combinations (built from connectives $\wedge$, $\vee$ and $\neg$) of *atomic predicates* of the form $P(x) \sim 0$ where P is polynomial with integer coefficients and $\sim$ is one of the relations $<, \leqslant, =, >, \geqslant, \neq$. A theory is said to be decidable if there is an algorithm that, given a sentence, can determine if it is true or false. A famous result by Tarksi is that first-order theory of reals admits quantifier elimination and is decidable. In this paper, we denote by $\mathfrak{R}_0$ this theory, formally this is the first-order theory of the structure $(\mathbb{R}, 0, 1, <, +, \cdot)$.

▶ **Theorem 5** ([42]). *The first-order theory $\mathfrak{R}_0$ of reals is decidable.*

We note that although the theory only allows integer coefficients, one can easily introduce algebraic coefficients by creating new variables and express that they are the roots on some polynomial. See also [36, 3] for more efficient decision procedures for the first-order theory of reals.

4.5 Transcendental number theory

A complex number is said to be *algebraic* if it is the root of a nonzero polynomial with integer coefficients. We denote by $\overline{\mathbb{Q}}$ the field of all algebraic numbers. A non-algebraic number is called *transcendental*. We will use that all field operations on algebraic numbers (including comparisons) are effective, see e.g. [3]. We will use transcendence theory in our proofs, essentially to argue that some equalities between two numbers are impossible. A classical results concerns powers of algebraic numbers.

▶ **Theorem 6** (Gelfond–Schneider). *If a and b are algebraic numbers with $a \neq 0, 1$ and b irrational, then any value[†] of a^b is transcendental.*

An important generalization of this result is the Lindemann-Weierstrass Theorem. In particular, we will use the following reformulation by Baker:

▶ **Theorem 7** (Lindemann-Weierstrass, Baker's reformulation). *If $\alpha_1, \ldots, \alpha_k$ are distinct algebraic numbers, then $e^{\alpha_1}, \ldots, e^{\alpha_k}$ are linearly independent over the algebraic numbers.*

[†] In general, a^b is defined by $e^{b \log a}$ and can have several values depending on the branch of the logarithm.

Our results in some cases depend on on Schanuel's conjecture, a unifying conjecture in transcendental number theory [29] that generalises many of the classical results in the field (including Theorems 6 and 7). The conjecture states that if $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$ are linearly independent over $\mathbb{Q}$ then some k-element subset of $\{\alpha_1, \ldots, \alpha_k, e^{\alpha_1}, \ldots, e^{\alpha_k}\}$ is algebraically independent.

Assuming Schanuel's Conjecture, MacIntyre and Wilkie [31] have shown decidability of the first-order theory of the expansion of the real field with the exponentiation function and the sin and cos functions restricted to bounded intervals.

▶ **Theorem 8** (Wilkie and MacIntyre). *If Schanuel's conjecture is true, then, for each $n \in \mathbb{N}$, the first-order theory of the structure $(\mathbb{R}, 0, 1, <, +, \cdot, \exp, \cos \upharpoonright_{[0,n]}, \sin \upharpoonright_{[0,n]})$ is decidable.*

In the rest of the paper, we denote by $\mathfrak{R}_{\exp}$ the first-order theory of the reals with the exponential, and $\mathfrak{R}_{\exp,\sin}$ the first-order theory of the reals with the exponential and the sin and cos functions restricted to a bounded interval.

5 Decidability

The goal of this section is to study the decidability of the LTI Reachability problem in various special cases. We will always restrict ourselves to the case where the control set is a hypercube $U = [-1, 1]^m$. Surprisingly, and despite the explicit description given by Theorem 4, this problem remains challenging (see Section 6 for some hardness results). A first observation, already made in Section 4.2 is that we can simplify the problem when the input lies in a hypercube.

▶ **Proposition 9** (Appendix A). *For any $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$, if A is stable then there exists computable matrices[‡] $C_1, \ldots, C_m, P_1, \ldots, P_m$ such that*

$$\mathcal{R}(A, B, [-1, 1]^m) = \sum_{i=1}^{m} P_i \mathcal{R}(C_i, b_i, [-1, 1])$$

where the b_i are the columns of B, $\sigma(C_i) \subseteq \sigma(A)$ and (C_i, b_i) is controllable for all i. In particular, if for every i, the membership in $\mathcal{R}(C_i, b_i, [-1, 1])$ or in $\partial \mathcal{R}(C_i, b_i, [-1, 1])$ is expressible in a theory $\mathfrak{R}$ that contains $\mathfrak{R}_0$, then membership in $\mathcal{R}(A, B, [-1, 1]^m)$ is expressible in that theory $\mathfrak{R}$.

A second observation is that, for the purpose of decidability, we can focus on the border of the reachable set. Indeed, we can compute arbitrary good approximations of the border and hence semi-decide the problem if the target is not on the boundary. This is useful in combination of a decision procedure for the algebraic points on the border: if we can decide if an algebraic point is on the border, then we can make sure that semi-decision procedure will finish.

▶ **Proposition 10** (Appendix B). *There is an algorithm that, given $A \in \mathbb{R}^{n \times n}$ anti-stable and $B \in \mathbb{R}^{n \times k}$ with rational coefficients[§] and $p \in \mathbb{N}$, computes two convex polytopes P_- and P_+ such that $P_- \subseteq \partial \mathcal{R}(A, b, [-1, 1]) \subseteq P_+$ and the Hausdorff distance[¶] between P_- and P_+ is less than 2^{-p}.*

[‡] Note that the C_i can be of lower dimension that n and the P_i are not necessarily square.

[§] In fact, this is still true for algebraic and even computable coefficients. A real is computable if one can produce arbitrary precise rational approximations of it.

[¶] Recall that the Hausdorff distance, which measures how far two sets are from each other, between two sets X and Y is defined by $d(X, Y) = \max \left(\sup_{x \in X} \inf_{y \in Y} \|x - y\|, \sup_{y \in Y} \inf_{x \in X} \|x - y\| \right)$.

We start with the simplest case where A is already diagonal. In fact, this seemingly easy case is already difficult and we only manage to solve it unconditionally in some cases.

▶ **Proposition 11** (Appendix C)**.** *The LTI Reachability problem is decidable when $U = [-1,1]^m$ and one of the following conditions holds:*

- *A is real diagonal, B is a column (i.e. $m = 1$) and it has at most 2 nonzero entries,*
- *A is real diagonalizable and its eigenvalues are rational, or a rational multiple of the same algebraic number,*
- *A only has one eigenvalue, which is real, and B is a column (i.e. $m = 1$).*

The main obstacle to generalizing this result is that the Boolean formulas involved in description of the boundary become too complicated, either involved three distinct exponentials or a combination of exponentials and polynomials. In fact, deciding if an exponential zero has zero is exactly the Continuous Skolem problem, and is not known to be decidable, even for real eigenvalues. We can recover decidability if we assume that the first-order theory of the reals with exponential is decidable. This is known to be true if Schanuel's conjecture hold, see Theorem 8.

▶ **Proposition 12** (Appendix D)**.** *The LTI Reachability problem when $U = [-1,1]^m$ and A has real eigenvalues reduces to deciding $\mathfrak{R}_{\exp}$. In particular, it is decidable if Schanuel's conjecture is true.*

One cannot easily generalise the previous result to any matrix because complex eigenvalues involve expression with exp and sin over unbounded domains. It is well-know that the first-order theory of reals with unbounded sin is undecidable (by embedding of Peano arithmetic). This explains why very few results are known about the Continuous Skolem problem in the unbounded case, even assuming Schanuel's conjecture. Nevertheless, one can show that in dimension two, only bounded sine and cosine are necessary to solve the problem.

▶ **Proposition 13** (Appendix E)**.** *The LTI Reachability problem in dimension 2 when $U = [-1,1]^m$ reduces to deciding $\mathfrak{R}_{\exp,\sin}$. In particular, it is decidable if Schanuel's conjecture is true.*

In fact, dimension 2 is special enough that we can show unconditional decidability of the reachability problem if A has real eigenvalues and B is a column (i.e. there is only one input).

▶ **Proposition 14** (Appendix F)**.** *The LTI Reachability problem in dimension 2 when $U = [-1,1]$, B is a column and A has real eigenvalues is decidable.*

Finally, another way to avoid the use of unbounded sine and cosine is to consider the Bounded Time LTI Reachability problem, which is also very natural in control theory.

▶ **Proposition 15** (Appendix G)**.** *The Bounded Time LTI Reachability problem when $U = [-1,1]^m$ reduces to deciding $\mathfrak{R}_{\exp,\sin}$. In particular, it is decidable if Schanuel's conjecture is true.*

6   Hardness

We saw in the previous section that the LTI Reachability problem seems very challenging, requiring powerful tools like the first-order theory of the reals with exponential and Schanuel's conjecture. In this section, we give some evidence that the problem is indeed difficult. Our first observation is that, in some sense, the LTI Set Reachability problem trivially contains the Skolem problem when the input set is $\{0\}$, in other word, when there is no input.

▶ **Theorem 16** (Appendix H). *The Continuous Skolem problem reduces to the LTI Set Reachability problem with input set $U = \{u\}$, where $u = (1, \ldots, 1)$, the matrix A is stable and the target set is a compact convex set of dimension $n - 1$.*

However, we are not really satisfied with this hardness result. Indeed, the problem is fundamentally different when U is a singleton. To see that, observe that if $U = \{u\}$, then the reachable set is just the orbit of u under $x' = Ax$. In particular, we can see that if A is stable then the orbit is a *closed set* minus an algebraic point (0). In particular, we can trivially decide whether an algebraic point is **0** or not, so deciding reachability is really about **deciding membership in a closed set**. Now compare that with the situation when $U = [-1, 1]^n$: by Proposition 3, when A is stable, the reachable set is **open**. This topological difference can lead to some difficulty because deciding membership in the boundary may involve some difficult transcendence results. For this reason, it is important to study hardness when U is not a singleton.

We show that problem remains hard when $U = [-1, 1]$, by reducing to the Continuous Nontangential Skolem problem. Recall that this problem, asks whether an exponential polynomial (or equivalently a linear differential equation) has a zero-crossing (nontangential zero). We argued in Section 4.3 that this problem is essentially as hard as the Skolem problem.

▶ **Theorem 17** (Appendix I). *The Continuous Nontangential Skolem problem reduces to the LTI Set Reachability problem with a single saturated input, i.e. $x' = Ax + bu$ with $b \in \mathbb{R}^n$ and $u(t) \in [-1, 1]$, and the target set can be chosen to be either a hyperplane, or a convex compact set of dimension $n - 1$.*

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A   Proof of Proposition 9

Following the observations of Section 4.2 and in particular (4), we have that $\mathcal{R}(A, B, [-1, 1]^m) = \sum_{i=1}^m \mathcal{R}(A, b_i, [-1, 1])$. We now focus on the case where $B = b$ is a column vector. If (A, b) is controllable then there is nothing to do. Otherwise let $V = \mathrm{span}(b, Ab, \ldots, A^{n-1}b)$, and assume that $k := \dim V < n$. Then $b \in V$ and $AV \subseteq V$, so by a change of basis P sending V to $\mathbb{R}^k$ we have

$$P^{-1}AP = \begin{bmatrix} A_V & 0 \\ 0 & * \end{bmatrix}, \qquad P^{-1}b = \begin{bmatrix} b_V \\ 0 \end{bmatrix}.$$

Then $\mathrm{span}(b_V, A_V b_V, \ldots, A_V^{k-1}b_V) = \mathbb{R}^k$ by construction, therefore (A_V, b_V) is controllable. Furthermore, it is clear that the spectrum of A_V is included in that of A. On the other hand, for any input u,

$$\int_0^\infty e^{At}bu(t)\,\mathrm{d}t = P\int_0^\infty \begin{bmatrix} e^{A_V t} & * \\ 0 & * \end{bmatrix}\begin{bmatrix} b_V \\ 0 \end{bmatrix} u(t)\,\mathrm{d}t = P\begin{bmatrix} \int_0^\infty e^{A_V t}b_V u(t)\,\mathrm{d}t \\ 0 \end{bmatrix}.$$

It follows that

$$\mathcal{R}(A, b, [-1, 1]) = P J_k \mathcal{R}(A_V, b_V, [-1, 1]) \quad \text{where} \quad J_k = \begin{bmatrix} I_k \\ 0 \end{bmatrix} \in \mathbb{R}^{n \times k}.$$

Going back to the general case, we now have that

$$\mathcal{R}(A, B, [-1, 1]^m) = \sum_{i=1}^m P_i \mathcal{R}(C_i, b_i, [-1, 1]).$$

Assume there are formulas $\Phi_1, \ldots, \Phi_m$ in some theory $\mathfrak{R}$ to expressible membership in $\mathcal{R}(C_i, b_i, [-1, 1])$. Then we can express membership of some $y \in \mathbb{R}^n$ in $\mathcal{R}(A, B, [-1, 1]^m)$ by the formula

$$\Phi(y) := \exists z_1, \ldots, \exists z_k. \, \Phi_1(z_1) \wedge \cdots \wedge \Phi_m(z_m) \wedge y = P_1 z_1 + \cdots + P_m z_m.$$

Clearly if the theory $\mathfrak{R}$ contains $\mathfrak{R}_0$ then Φ is in $\mathfrak{R}$. If instead of a formula Φ_i for $\mathcal{R}(C_i, b_i, [-1, 1])$, we have a formula for its boundary $\partial \mathcal{R}(C_i, b_i, [-1, 1])$, then we note that $\mathcal{R}(C_i, b_i, [-1, 1])$ is open convex by Proposition 3, hence we can write a formula for $\mathcal{R}(C_i, b_i, [-1, 1])$ from Φ_i, as shown below.

Let $C \subseteq \mathbb{R}^n$ be an open bounded convex set, if we have a formula Φ to express membership in ∂C in some theory $\mathfrak{R}$ that subsumes $\mathfrak{R}_0$, then we can write a formula in $\mathfrak{R}$ to express membership in C. Indeed, by Krein–Milman theorem, the closure $\overline{C}$ of C is the convex hull of its extreme points, but the extreme points of $\overline{C}$ are on the boundary ∂C. Hence $\overline{C}$ is the convex hull of ∂C. It follows by Carathéodory's theorem that any point in $\overline{C}$ is the convex combination of at most $n + 1$ in ∂C. Hence we can write a formula ψ to decide membership in $\overline{C}$. But since C is open, we know that $C = \overline{C} \setminus \partial C$, hence we can write a formula for C.

B Proof of Proposition 10

We first observe that we can reduce to the case where $B = b$ is a column vector and (A, b) is controllable. Indeed, apply Proposition 9 to get computable $C_1, \ldots, C_k$ and $P_1, \ldots, P_k$ such that

$$\mathcal{R}(A, B, [-1, 1]^m) = \sum_{i=1}^m P_i \mathcal{R}(C_i, b_i, [-1, 1]).$$

Now assume that we have some convex under/over-approximation Q_i^-, Q_i^+ of $\mathcal{R}(C_i, b_i, [-1, 1])$ for all i. Then $\sum_{i=1}^m P_i Q_i^{\pm}$ is a convex under/over-approximation by the property of the Minkowski sum of convex sets, furthermore this sum can be computed effectively. Note that in this reduction, the matrix A has not changed, hence it is still stable.

We now focus on the case where $B = b$ is a column vector such that (A, b) is controllable. We can apply Theorem 4 to get that $\mathcal{C}(A, b, U)$ is an open convex set containing $\mathbf{0}$ and

$$\partial \mathcal{R}(A, b, U) = \left\{ \int_0^\infty e^{At} b \operatorname{sgn}(c^T e^{At} b) \, dt : c \in \mathbb{R}^n \setminus \{\mathbf{0}\} \right\}.$$

Observe that only the direction of c matters so we can restrict the set to the *compact* subset of c such that $\|c\| = 1$. Let $f_c(t) = e^{At} b \operatorname{sgn}(c^T e^{At} b)$ and observe that since A is stable, $f_c(t) \to 0$ as $t \to -\infty$. In fact, one can compute constants D and $\alpha > 0$ such that $|f_c(t)| \leqslant D e^{\alpha t}$ for all $t \leqslant 0$. Let T to be fixed later, then

$$\left| \int_{-\infty}^0 f_c(t) \, dt - \int_{-T}^0 f_c(t) \, dt \right| \leqslant \int_{-\infty}^T D e^{\alpha t} \, dt = D \alpha^{-1} e^{-\alpha T}.$$

Furthermore, one can approximate $\int_{-T}^{0} f_c(t)\,\mathrm{d}t$ with arbitrary precision given T and c, by using the fact that $c^T e^{At} b$ has isolated zeros. Furthermore, $c \mapsto \int_{-T}^{0} f_c(t)\,\mathrm{d}t$ is continuous since the zero-crossings of $c^T e^{At} B$ move continuously with c and the (discontinous) tangential zeros that can appear do not change the integral. It follows that on the compact set $\{c : \|c\| = 1\}$, it has bounded variations, with a computable bound. Putting everything together, this allows us to sample the border with sufficiently many points as to obtain an underapproximation and overapproximation of the border, in the form of a convex set.

C Proof of Proposition 11

We start by observing that in the second case, we can reduce to the case where A is diagonal. Indeed, since A is real diagonalizable, we can write $A = P^{-1}MP$ where P is real and M is diagonal. Note that M satisfies all the assumptions since it contains only the eigenvalues of M. Furthermore, the reachable set is easily observed to be $\mathcal{R}(A, B, U) = P^{-1}\mathcal{R}(M, PB, U)$ hence it is equivalent to decide if $Py \in \mathcal{R}(M, PB, U)$.

Assume that A is diagonal (this covers the first two cases of the theorem with the above remark). Write $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ with $\lambda_i \geq \lambda_{i+1}$ without loss of generality. Decompose A into $A = \mathrm{diag}(A_1, A_2)$ where A_1 contains the nonnegative λ_i and A_2 the negative ones. Then A_1 is weakly antistable and A_2 is stable. Decompose B into B_1 and B_2 accordingly. Then by Proposition 3, $\mathcal{R}(A, B, U) = \mathbb{R}^{n_1} \times \mathcal{R}(A_2, B_2, U)$. Then by Proposition 9, we have that $\mathcal{R}(A_2, B_2, [-1, 1]^m) = \sum_{i=1}^{m} P_i \mathcal{R}(C_i, b_i, [-1, 1])$ where the b_i are the columns of B_2 and (C_i, b_i) is controllable for all i.

We now assume that A is diagonal with positive eigenvalues, $B = b$ is a column vector, (A, b) is controllable and $U = [-1, 1]$. Write $A = \mathrm{diag}(\mu_1, \ldots, \mu_k)$ where the μ_i are negative. Then by Theorem 4, we have that

$$\partial \mathcal{R}(A, b, [-1, 1]) = \{\beta_c : c \in \mathbb{R}^n \setminus \{\mathbf{0}\}\}, \qquad \beta_c := \int_0^\infty e^{At} b\, \mathrm{sgn}(c^T e^{At} b)\,\mathrm{d}t$$

And observe that

$$f_c(t) := c^T e^{At} b = c^T \mathrm{diag}(e^{\mu_1 t}, \ldots, e^{\mu_k t}) b = \sum_{i=1}^{k} c_i e^{\mu_i t} b_i.$$

If all entries of A are a rational multiple of the same algebraic number: write[||] $\mu_i = p_i \alpha$ where $p_i \in \mathbb{Z}$ and $\alpha \in \overline{\mathbb{Q}}$. Then

$$f_c(t) = \sum_{i=1}^{k} c_i b_i \left(e^{\alpha t}\right)^{p_i} = Q(c, e^{\alpha t})$$

where Q is a polynomial with algebraic coefficients. Let d be the degree of $Q(c, \cdot)$ (that does not depend on c but only on the μ_i), then $Q(c, \cdot)$ has at most d nontangential[**] zeros, call them $z_1 < z_2 < \cdots < z_k$. Each gives rise to some unique t_i satisfying $z_i = e^{\alpha t_i}$. It follows

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[||] We can put the least common denominator in α, hence they become integer multiples.

[**] We are only interested in zero-crossings, since tangential zeros do not change the integral. In doing so, we also get for free that all nontangential zeros have multiplicity 1, hence they are all distincts.

that, up to a sign,

$$\pm\beta_c = \int_0^{t_1} e^{At} b \, dt - \int_{t_1}^{t_2} e^{At} b \, dt + \cdots + (-1)^k \int_{t_k}^{\infty} e^{At} b \, dt$$

$$= A^{-1}\left((e^{At_1} - I) - (e^{At_2} - e^{At_1}) + \cdots - (-1)^k e^{At_k}\right) b$$

$$= A^{-1}\left(2\sum_{i=1}^{k}(-1)^{i-1}e^{At_i} - I\right) b.$$

In particular, since A is diagonal, the jth component of β_c is

$$\beta_{c,j} = \pm\frac{1}{\mu_j}\left(2\sum_{i=1}^{k}(-1)^{i-1}e^{-\mu_j t_i} - 1\right) b_j$$

$$= \pm\frac{1}{\mu_j}\left(2\sum_{i=1}^{k}(-1)^{i-1}\left(e^{-\alpha t_i} - 1\right)^{p_i}\right) b_j$$

$$= \pm\frac{1}{\mu_j}\left(2\sum_{i=1}^{k}(-1)^{i-1}z_i^{p_i} - 1\right) b_j$$

$$= R_{k,j}(z_1, \ldots, z_k)$$

where $R_{k,j}$ is a polynomial with algebraic coefficients that does not depend on c. Note that the sign can be determined easily: it is the sign of $f_c(0) = \sum_{i=1}^{k} b_i c_i$. We can now write a formula in the first-order theory of the reals to express that a target y is on the border[††]:

$$\Phi(y) := \exists c.c \neq 0 \bigwedge \bigvee_{k=0}^{d} \Phi_k(y, c) \qquad\qquad \text{on a border in direction } c$$

$$\Phi_k(y, c) := \exists z_1, \ldots, z_k.\Psi_k(c, z) \bigwedge \Psi'_k(c, z) \bigwedge \bigwedge_{j=1}^{n} (y_j = R_{k,j}(z)) \quad \text{match target}$$

$$\Psi_k(c, z) := \bigwedge_{i=1}^{k} (Q(c, z_i) = 0 \wedge Q'(c, z_i) \neq 0) \bigwedge 0 < z_1 < \cdots < z_k \quad z \text{ are zeros of } Q(c, \cdot)$$

$$\Psi'_k(c, z) := \forall u.(u > 0 \wedge Q(c, u) = 0 \wedge Q'(c, u) \neq 0) \Rightarrow \bigvee_{i=1}^{k} u = z_i \quad z \text{ are the only zeros.}$$

We have shown that membership in the border is expressible is $\mathfrak{R}_0$, which shows that membership in the entire reachable set is expressible in $\mathfrak{R}_0$ by Proposition 9, and hence decidable.

If b has at most two nonzero entries: let $i \neq j$ be those two entries, then

$$f_c(t) = 0 \Leftrightarrow c_i e^{\mu_i t} b_i + c_j e^{\mu_j t} b_j = 0$$

$$\Leftrightarrow 1 + \frac{c_j b_j}{c_i b_i} e^{(\mu_i - \mu_j)t} = 0$$

$$\Leftrightarrow t = t_1 := \frac{1}{\mu_i - \mu_j} \ln \frac{-c_i b_i}{c_j b_j}.$$

We assume that we are not in the previous case, so in particular μ_i and μ_j must be distincts and $\mathbb{Q}$-linearly independent. If $c_i = 0$ then f_c has no zero unless $c_j = 0$, in which case it is

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[††] We write Q' for $\frac{\partial Q(c,z)}{\partial z}$ which is also a polynomial.

constant equal to zero and $\beta_c = 0$. When $c_i = 0$ and $c_j \neq 0$, the sign of f_c is constant. If $c_i b_i = -c_j b_j$ then the only zero of f_c is at $t = 0$ and the sign is then constant once again. In all case where the sign is constant on $(0, \infty)$, we have that

$$\beta_c = \pm \int_0^\infty e^{At} b \, \mathrm{d}t = -A^{-1} b$$

which is algebraic and hence can easily be checked against the target. In all other cases, we have

$$\beta_c = \int_0^{t_1} e^{At} b \, \mathrm{d}t - \int_{t_1}^\infty e^{At} b \, \mathrm{d}t = A^{-1}(2e^{At_1} - I_n)b = 2A^{-1}e^{At_1}b - A^{-1}b.$$

Recall that b has two nonzero entries and A is diagonal, hence $A^{-1}e^{At_1}b$ also has two nonzero entries: $\mu_i^{-1}e^{\mu_i t_1}b_i$ and $\mu_j^{-1}e^{\mu_j t_1}b_j$ respectively. We now argue that one or both of those values are transcendental which prevents the target from being on the border. Observe that $\mu_i^{-1}e^{\mu_i t_1}b_i$ is algebraic if and only if $e^{\mu_i t_1}$ is algebraic. But

$$e^{\mu_i t_1} = e^{\frac{\mu_i}{\mu_i - \mu_j} \ln \frac{-c_i b_i}{c_j b_j}} = \left(\frac{-c_i b_i}{c_j b_j}\right)^{\frac{\mu_i}{\mu_i - \mu_j}}$$

which is transcendental by Theorem 6 if $\frac{\mu_i}{\mu_i - \mu_j}$ is irrational, since we assumed that $\frac{-c_i b_i}{c_j b_j}$ is not 0 or 1. Therefore, β_c is algebraic only when $\frac{\mu_i}{\mu_i - \mu_j}, \frac{\mu_j}{\mu_i - \mu_j} \in \mathbb{Q}$. This would imply that $\frac{\mu_i}{\mu_j} \in \mathbb{Q}$, a contradiction since we assume that they are $\mathbb{Q}$-linearly independent. In summary, the only two possible algebraic points on the border are 0 and $A^{-1}b$ and it is easy to check (i) if they are indeed on the border, (ii) if they are equal to the target. Clearly one can write a formula in $\mathfrak{R}_0$ to decide if this is the case.

At this point, we can conclude for the general case because we assume that B only consist of one column. Note that we would not be able to conclude if B had several columns because we can only write a formula for $\partial \mathcal{R}(A_i, b_i, [-1,]1) \cap \overline{\mathbb{Q}}^n$. Indeed, if we have two convex sets C and D, then $\partial(C + D) \subseteq \partial C + \partial D$ but in general we do not have $\partial(C + D) \cap \overline{\mathbb{Q}}^n \subseteq (\partial C \cap \overline{\mathbb{Q}}^n) + (\partial D \cap \overline{\mathbb{Q}}^n)$. A simple counter-example is $C = [0, \pi]$ and $D = [0, 4 - \pi]$: then $C + D = [0, 4]$, $\partial(C+D) \cap \overline{\mathbb{Q}} = \{0, 4\}$ but $\partial C \cap \overline{\mathbb{Q}} = \{0\}$ and $\partial D \cap \overline{\mathbb{Q}} = \{0\}$. It is unclear whether such a counter-example can be built with actual reachable sets however.

If A only has one eigenvalue which is real: by using Proposition 9 as before, it suffice to show that membership is expressible in $\mathfrak{R}_0$ for some controllable pairs (A_i, b_i). The crucial point here is that the spectrum of A_i is included in that of A. Since A has a unique eigenvalue λ, A_i also has a unique eigenvalue λ. If $\lambda > 0$ the the reachable set is everything by Proposition 3, which is trivial to express in $\mathfrak{R}_0$.

We now assume that $\lambda < 0$, $B = b$ is a column vector, (A, b) is controllable and $U = [-1, 1]$. Then by Theorem 4, we have that

$$\partial \mathcal{R}(A, b, [-1, 1]) = \{\beta_c : c \in \mathbb{R}^n \setminus \{\mathbf{0}\}\}, \qquad \beta_c := \int_0^\infty e^{At} b \, \mathrm{sgn}(c^T e^{At} b) \, \mathrm{d}t$$

But since A has a unique real eigenvalue λ, $e^{At} = e^{\lambda t}P(t)$ where $P(t)$ is a matrix where each entry is a polynomial in t with real algebraic coefficients. It follows that $c^T e^{At} b = e^{\lambda t}Q(c, t)$ where Q is a polynomial with algebraic coefficients and $Q(c, \cdot)$ has at most d nontangeantial zeros (tangeantial zeros do not change the integral) which are distincts, and where d is

independent of c. We can write a formula in $\mathfrak{R}_0$ to express those zeros $t_1 < t_2 < \cdots < t_k$. Furthermore, note that by integration by part, we have

$$\int_u^v e^{At} b \, \mathrm{d}t = \int_u^v e^{\lambda t} P(t) \, \mathrm{d}t = \left[e^{\lambda t} R(t) \right]_u^v$$

where R is some polynomial matrix with algebraic coefficients. It follows that, up to a sign,

$$\begin{aligned}
\pm \beta_c &= \int_0^{t_1} e^{At} b \, \mathrm{d}t - \int_{t_1}^{t_2} e^{At} b \, \mathrm{d}t + \cdots + (-1)^k \int_{t_k}^{\infty} e^{At} b \, \mathrm{d}t \\
&= \left[e^{\lambda t} R(t) \right]_0^{t_1} b - \left[e^{\lambda t} R(t) \right]_{t_1}^{t_2} b + \cdots + (-1)^k \left[e^{\lambda t} R(t) \right]_{t_k}^{\infty} b \\
&= \left(2 \sum_{i=1}^k (-1)^{i-1} e^{\lambda t_i} R(t_i) - R(0) \right) b.
\end{aligned}$$

In particular, the jth component of β_c is of the form

$$\beta_{c,j} = S_{j,0}(b) + \sum_{i=1}^k e^{\lambda t_i} S_{j,i}(t_1, \ldots, t_k, b)$$

where the S_i are polynomials with algebraic coefficients. But now recall that the t_i are algebraic since they are the roots of $Q(c, \cdot)$ and they are distinct because they are in fact the nontangeantial zeros. Furthermore, the target β has algebraic coordinates. Hence, by Theorem 7 (take $\alpha_i = \lambda t_i \in \overline{\mathbb{Q}}$ and add $\alpha_0 = 0$), the only way this can happen is if $S_{j,i}(t_1, \ldots, t_k, b) = 0$ for $1 \leqslant i \leqslant k$ and $\beta_{c,j} = S_{j,0}(b)$. Crucially, those conditions do no involve any expoenntials so we can express all those conditions in $\mathfrak{R}_0$. Here again, we can only conclude when B is a column, because we only have a formula for the algebraic point on the border of each controllable system.

D   Proof of Proposition 12

By putting A is Jordan Normal Form, write $A = Q^{-1} M Q$ where Q is invertible and M is made of Jordan blocks. Since A has real spectrum, Q and M are real matrices and $\mathcal{R}(A, B, U) = Q^{-1} \mathcal{R}(M, QB, U)$ so we can now assume that A only consists of Jordan blocks since $y \in \mathcal{R}(A, B, U)$ if and only if $Qy \in \mathcal{R}(M, QB, U)$.

Assume that $A = \mathrm{diag}(J_1, \ldots, J_k)$ consist of Jordan blocks. Without loss of generality, we can assume that the blocks are ordered by increasing eigenvalue. Hence we can write $A = \mathrm{diag}(A_1, A_2)$ where A_1 contains the nonnegative λ_i and A_2 the negative ones. Then A_1 is weakly-antistable and A_2 is stable. Decompose B into B_1 and B_2 accordingly. Then by Proposition 3, $\mathcal{R}(A, B, U) = \mathbb{R}^{n_1} \times \mathcal{R}(A_2, B_2, U)$. We can then apply Proposition 9 to decompose $\mathcal{R}(A_2, B_2, U)$ into smaller controllable problems (C_i, b_i), where each C_i also has a real spectrum. It then suffices to show that membership in $\partial \mathcal{R}(C_i, b_i, [-1, 1])$ is expressible in $\mathfrak{R}_{\exp}$ for each subproblem to conclude by Proposition 9.

Assume that A only has negative eigenvalues, $B = b$ is a column vector, (A, b) is controllable and $U = [-1, 1]$. Then by Theorem 4 we have that

$$\partial \mathcal{R}(A, b, [-1, 1]) = \{\beta_c : c \in \mathbb{R}^n \setminus \{\mathbf{0}\}\}, \qquad \beta_c := \int_0^{\infty} e^{-At} b \, \mathrm{sgn}(c^T e^{-At} b) \, \mathrm{d}t$$

But observe that $f_c(t) := c^T e^{-At} b$ is an exponential polynomial in t. Furthermore, it has at most $n - 1$ zeros by Lemma 2 since A has real eigenvalues and b, c are nonzero. Let

$0 \leqslant k \leqslant n-1$ and assume that f_c has k nontangeantial zeros (the tangeantial zeros will play no role) $0 < t_1 < \cdots < t_k$. Then, depending on the sign of $f_c(0)$, and noting that the sign of f_c changes at each t_i, we have that

$$\beta_c = \pm \left(\int_0^{t_1} e^{-At}b\,\mathrm{d}t + \sum_{i=1}^{k-1} \int_{t_i}^{t_{i+1}} (-1)^i\,\mathrm{d}t + (-1)^k \int_{t_k}^\infty e^{-At}b\,\mathrm{d}t \right)$$

$$= \pm A^{-1} \left(I_n + 2\sum_{i=1}^k (-1)^i e^{At_i} \right) b$$

$$= \pm R_k(t_1, \ldots, t_k)$$

which is also a (vector of) exponential polynomials. We can now write a formula in $\mathfrak{R}_{\exp}$ to express that a target y is on the border:

$$\Phi(y) := \exists c.c \neq 0 \bigwedge \bigvee_{k=0}^d \Phi_k(y,c) \qquad\qquad \text{on a border in direction } c$$

$$\Phi_k(y,c) := \exists t_1, \ldots, t_k.\Psi_k(c,z) \bigwedge \Psi_k'(c,z) \bigwedge \bigwedge_{j=1}^n (y_j = R_{k,j}(t)) \quad \text{match target}$$

$$\Psi_k(c,z) := \bigwedge_{i=1}^k (f_c(t_i) = 0 \wedge f_c'(t_i) \neq 0) \bigwedge 0 < t_1 < \cdots < t_k \qquad z \text{ are zeros of } f_c$$

$$\Psi_k'(c,z) := \forall u.(u > 0 \wedge f_c(u) = 0 \wedge f_c'(u) \neq 0) \Rightarrow \bigvee_{i=1}^k u = z_i \qquad z \text{ are the only zeros.}$$

We note that those are indeed formulas in $\mathfrak{R}_{\exp}$ because f_c, f_c' and e^{At} are exponential polynomials in t and c and they can be computed by putting A in Jordan normal form, and all those exponential polynomials have algebraic coefficients since b and A have algebraic coefficients. This shows that deciding the border reduces to deciding a formula in $\mathfrak{R}_{\exp}$.

E   Proof of Proposition 13

If A has real eigenvalues, then the decidability reduces to $\mathfrak{R}_{\exp}$ by Proposition 12, which is decidable in $\mathfrak{R}_{\exp,\sin}$. Otherwise, since A is real, it must have two conjugate complex but nonreal eigenvalues. Hence, we can put A in *real* Jordan Form: $A = P^{-1}JP$ where P is real and

$$J = \begin{bmatrix} \lambda & \theta \\ -\theta & \lambda \end{bmatrix},$$

where $\lambda, \theta \in \mathbb{R}$. It follows that $\mathcal{R}(A,B,U) = P^{-1}\mathcal{R}(J,PB,U)$ so we now focus on this particular case. Assume that A is a real Jordan block of the form above. Then

$$e^{Jt} = e^{\lambda t} \begin{bmatrix} \cos(\theta t) & \sin(\theta t) \\ -\sin(\theta t) & \cos(\theta t) \end{bmatrix}.$$

Furthermore, we can decompose B into columns vectors $b_1, \ldots, b_m$ so that

$$\mathcal{R}(A,B,[-1,1]^m) = \mathcal{R}(A,b_1,[-1,1]) + \cdots + \mathcal{R}(A,b_m,[-1,1])$$

so we now assume that $B = b$ is a column vector. We further reduce to the case where b has norm 1 by noticing that $\mathcal{R}(A,b_1,[-1,1]) = \|b\|\mathcal{R}(A,b/\|b\|,[-1,1])$. Now (A,b) is controllable

(unless $b = 0$ which is trivial) since A rotates (and rescale) b by an angle θ and $\theta \neq 0 \pmod{\pi}$ (indeed, θ is nonzero and algebraic). Hence we can apply Proposition 3 and Theorem 4 to get that either $\mathcal{R}(A, b, [-1, 1]) = \mathbb{R}^2$ if $\lambda > 0$, or

$$\partial \mathcal{R}(A, B, [-1, 1]^m) = \{\beta_c : c \in \mathbb{R}^2 \setminus \{0\}\}, \qquad \beta_c := \int_0^\infty e^{At} b \, \mathrm{sgn}(c^T e^{At} b) \, \mathrm{d}t.$$

We now focus on this case since the other one is trivial. Since the dimension is two, we can write $c_\phi := \begin{bmatrix} \cos(\phi) & \sin(\phi) \end{bmatrix}^T$ for $\phi \in [0, 2\pi)$ and $b = \begin{bmatrix} \cos \beta & \sin \beta \end{bmatrix}^T$. Then,

$$\begin{bmatrix} \cos(\theta t) & \sin(\theta t) \\ -\sin(\theta t) & \cos(\theta t) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \cos(\theta t + \beta) \\ \sin(\theta t + \beta) \end{bmatrix}.$$

and

$$\mathrm{sgn}(c^T e^{At} b) = \mathrm{sgn}\left(\begin{bmatrix} \cos(\phi) & \sin(\phi) \end{bmatrix} \begin{bmatrix} \cos(\theta t + \beta) \\ \sin(\theta t + \beta) \end{bmatrix} \right) = \mathrm{sgn}(\cos(\theta t + \beta + \phi)).$$

Hence,

$$\beta_\phi := \beta_{c_\phi} = \int_0^\infty e^{\lambda t} \begin{bmatrix} \cos(\theta t + \beta) \\ \sin(\theta t + \beta) \end{bmatrix} \mathrm{sgn}(\cos(\theta t + \beta + \phi)) \, \mathrm{d}t = e^{i\beta} \int_0^t e^{(\lambda + i\theta)t} \mathrm{sgn}(\cos((\theta t + \beta + \phi)) \, \mathrm{d}t$$

when viewed as a complex number (to simplify computations). Let t_ϕ denote the smallest $t \geqslant 0$ such that $\cos(\theta t + \beta + \phi) = 0$ and $\varepsilon_\phi = \mathrm{sgn}(\cos(\beta + \phi))$, except when $t_\phi = 0$ in which case we let $\varepsilon_\phi = -1$. Then

$$\beta_\phi = e^{i\beta} \varepsilon_\phi \left(\int_0^{t_\phi} e^{(\lambda + i\theta)t} \, \mathrm{d}t - \sum_{k=0}^\infty \int_{t_\phi + k\frac{\pi}{\theta}}^{t_\phi + (k+1)\frac{\pi}{\theta}} (-1)^k e^{(\lambda + i\theta)t} \, \mathrm{d}t \right)$$

$$= \frac{e^{i\beta} \varepsilon_\phi}{\lambda + i\theta} \left(e^{(\lambda + i\theta)t_\phi} - 1 - \sum_{k=0}^\infty (-1)^k \left(e^{(\lambda + i\theta)(t_\phi + (k+1)\frac{\pi}{\theta})} - e^{(\lambda + i\theta)(t_\phi + (k+1)\frac{\pi}{\theta})} \right) \right)$$

$$= \frac{e^{i\beta} \varepsilon_\phi}{\lambda + i\theta} \left(-1 + 2 \sum_{k=0}^\infty (-1)^k e^{(\lambda + i\theta)(t_\phi + k\frac{\pi}{\theta})} \right)$$

$$= \frac{e^{i\beta} \varepsilon_\phi}{\lambda + i\theta} \left(-1 + 2 e^{(\lambda + i\theta)t_\phi} \sum_{k=0}^\infty (-1)^k e^{k(\lambda + i\theta)\frac{\pi}{\theta}} \right)$$

$$= \frac{e^{i\beta} \varepsilon_\phi}{\lambda + i\theta} \left(-1 + 2 e^{(\lambda + i\theta)t_\phi} \frac{1}{1 + e^{(\lambda + i\theta)\frac{\pi}{\theta}}} \right)$$

$$= \frac{e^{i\beta} \varepsilon_\phi}{\lambda + i\theta} \left(-1 + 2 e^{(\lambda + i\theta)t_\phi} \frac{1}{1 - e^{\frac{\lambda \pi}{\theta}}} \right).$$

One can then obtain an expression for each coordinate of β_ϕ viewed as a 2D vector. In particular, this expression is expressible in $\mathfrak{R}_{\exp,\sin}$ where sin is taken over some bounded interval. Indeed, λ, β and θ are algebraic, t_ϕ can be express in with an equation involving a bounded sin since we have the trivial bound $t_\phi \leqslant \frac{2\pi}{\theta}$. We can then express $e^{(\lambda + i\theta)t_\phi}$ using a combination of exponential and bounded sin, again noting that $\theta t_\phi \leqslant 2\pi$. We can also express π using an equation on bounded sin ($\sin \pi = 0 \wedge (\forall y. \, 0 < y < \pi \implies \sin y \neq 0)$) hence we can define $e^{\frac{\lambda \pi}{\theta}}$. It follows that we can express β_ϕ and hence the boundary in $\mathfrak{R}_{\exp,\sin}$. We then conclude using Proposition 9.

F   Proof of Proposition 14

If A is diagonalizable, and since it has real eigenvalues, we can write $A = P^{-1}DP$ where P is real and D is real diagonal. Then $\mathcal{R}(A, B, U) = P^{-1}\mathcal{R}(D, PB, U)$ so we only need to decide if $Py \in \mathcal{R}(D, PB, U)$. But in dimension 2 all columns of PB necessarily have at most two nonzero entries so we can conclude using Proposition 11. Otherwise, A only has one eigenvalue, which is real, so we conclude with Proposition 11.

G   Proof of Proposition 15

H   Proof of Theorem 16

Let c, A, b be an instance of the Continuous Skolen problem. Let $f(t) = c^T e^{At}b$, the problem asks whether f has any zero at $t \geqslant 0$. Without loss of generality, we can assume that $c^T b \geqslant 0$ by changing b into $-b$. Now let $u = (1, \ldots, 1)$, $B \in \mathbb{Q}^{n \times n}$ be such that $Bu = b$ and $U = \{u\}$, and observe that by definition,

$$
\mathcal{R}(A, AB, U) = \left\{ \int_0^T e^{At} ABu \, \mathrm{d}t : T \geqslant 0 \right\} = \left\{ (e^{At} - I_n)b : t \geqslant 0 \right\}.
$$

But notice that for any $t \geqslant 0$,

$$
c^T (e^{At} - I_n)b = c^T e^{At}b - c^T b = f(t) - c^T b.
$$

Hence if we define the set $Y = \{x \in \mathbb{R}^n : c^T x \leqslant -c^T b\}$, which is a hyperplane, then Y is reachable if and only if there exists $t \in \mathbb{R}^n$ such that $f(t) \leqslant 0$. But since $f(0) = c^T b \geqslant 0$ by assumption, this last condition is equivalent to the existence of a zero by continuity. This shows that Skolem instance (c, A, b) is positive if and only the Set Reachability instance (A, AB, U, Y) is positive.

Note that we can easily modify the instance to further strengthen the result like in the proof of Theorem 17. More precisely, we can ensure that the LTI instance is stable and that the set Y is compact convex.

I   Proof of Theorem 17

Let c, A, b be an instance of the Continuous Nontangential Skolem problem. Let $f_c(t) = c^T e^{At}b$, the problem asks whether f_c has any zero-crossing at $t \geqslant 0$. Note that for any $\alpha > 0$, $f_c(t)$ is zero-crossing if and only if $e^{-\alpha t}f_c(t)$ is zero-crossing. Furthermore, $e^{-\alpha t}f_c(t)$ is still an exponential polynomial, so without loss of generality we can assume that all eigenvalues of A have negative real parts, by taking α sufficiently large. In other words, we can assume that $-A$ is anti-stable. In particular, A must be invertible.

We now show that we can assume that (A, b) is controllable. Let $V = \mathrm{span}[b, Ab, \ldots, A^{n-1}b]$ where n is the dimension of A, and assume that $\dim V < n$. Then $b \in V$ and $AV \subseteq V$ by Cayley–Hamilton theorem, so by an orthogonal change of basis P,

$$
P^{-1}c = \begin{bmatrix} c_V \\ * \end{bmatrix}, \qquad P^{-1}AP = \begin{bmatrix} A_V & * \\ 0 & * \end{bmatrix}, \qquad P^{-1}b = \begin{bmatrix} b_V \\ 0 \end{bmatrix}.
$$

It then follows that $c^T e^{At}b = c_V e^{A_V t}b_V$, but note that $\mathrm{span}[b_V, A_V b_V, \ldots, A_V^{k-1}b_V] = V$ by construction, therefore (A_V, b_V) is controllable.

We can now assume that $-A$ is anti-stable and (A, b) controllable. Without loss of generality, we can assume that $f_c(0) = c^T b \geqslant 0$, by considering $-c$ instead of c if this is not the case. Also recall that $\mathcal{R} := \mathcal{R}(A, b, [-1, 1]) = \mathcal{C}(-A, b, [-1, 1])$. Therefore, by Theorem 4, $\mathcal{R}(A, b, U)$ is an open convex set containing 0 and its boundary is given by

$$\partial \mathcal{R}(A, b, U) = \{\beta_v : v \in \mathbb{R}^n \setminus \{\mathbf{0}\}\}, \qquad \beta_v := \int_0^\infty e^{At} b \operatorname{sgn}(v^T e^{At} b) \, dt$$

which is a strictly convex set. We start by a few observations/claim:

- $x \in \overline{\mathcal{R}}$ if and only if $x = \int_0^\infty e^{At} b u(t) \, dt$ for some $u : \mathbb{R} \to [-1, 1]$,
- β_c is the unique extermal point in direction c: $\overline{\mathcal{R}} \cap (\beta_c + H_c) = \{\beta_c\}$ where $H_c = \{x \in \mathbb{R}^n : c^T x = 0\}$ is the hyperplane of normal c,
- $-A^{-1} b \in \overline{\mathcal{R}}$,
- $-A^{-1} b = \beta_c$ if and only if f_c has no zero-crossings (nontangential zeros),
- $-A^{-1} b \neq \beta_{-c}$,
- $-A^{-1} b = \beta_c$ if and only if $(-A^{-1} b + H_c) \cap \mathcal{R} = \varnothing$

We now go though those claims one by one: the first one is direct consequence of (1) with the change of variable $\xi = t - s$. The second claim is essentially already proven in [22] but we give the gist of the proof. Pick $x \in \overline{\mathcal{R}}$, then by the first claim there exists $u : \mathbb{R} \to [-1, 1]$ such that $x = \int_0^\infty e^{At} b u(t) \, dt$. Then check that, since $|u(t)| \leqslant 1$,

$$c^T x = \int_0^\infty c^T e^{At} b u(t) \, dt \leqslant \int_0^\infty |c^T e^{At} b| \, d = \int_0^\infty c^T e^{At} b \operatorname{sgn}(c^T e^{At} b) \, dt = c^T \beta_c.$$

Therefore, β_c is a maximizer in direction c. But by strict convexity of $\overline{\mathcal{R}}$, there can only be a unique one, for otherwise a whole line segment would be in $\partial \mathcal{R}$ which would be a contradiction. To show the third claim, simply observe that since $e^{At} \to 0$ as $t \to \infty$, we have, by the first claim,

$$-A^{-1} b = \int_0^\infty e^{At} b \, dt = \int_0^\infty e^{At} b u(t) \, dt \in \overline{\mathcal{R}} \qquad \text{where } u(t) = 1.$$

The fourth claim follows from the computation above and the remark that

$$c^T(\beta_c + A^{-1} b) = \int_0^\infty c^T e^{At} b \big(\operatorname{sgn}(c^T e^{At} b) - 1 \big) \, dt = \int_0^\infty |f_c(t)| - f_c(t) \, dt.$$

Indeed, there are two cases (since we assumed that $f_c(0) \geqslant 0$):

- Either $f_c(t) \geqslant 0$ for all $t \geqslant 0$, then f_c has no zero-crossings and $c^T(\beta_c + A^{-1}) b = 0$. But $-A^{-1} b \in \mathcal{R}$ and β_c is the maximizer in direction c, so by strict convexity, $\beta_c = -A^{-1} b$.
- Either f_c has at least one zero-crossing, then by continuity there exists an open interval (a, b) such that $|f_c(t)| - f_c(t) > 0$. But since $|f_c(t)| - f_c(t) \geqslant 0$ for all t, it follows that $c^T(\beta_c + A^{-1} b) > 0$ hence $\beta_c \neq -A^{-1} b$.

The fifth claim follows from a similar argument since $\beta_{-c} = -\beta_c$ and

$$c^T(\beta_c - A^{-1} b) = \int_0^\infty c^T e^{At} b \big(\operatorname{sgn}(c^T e^{At} b) + 1 \big) \, dt = \int_0^\infty |f_c(t)| + f_c(t) \, dt.$$

But now assume, for contradiction, that $\beta_{-c} = -A^{-1} b$. Then $c^T(\beta_c - A^{-1} b) = 0$ so $f_c(t) \leqslant 0$ for all t, hence f_c has no zero-crossings. But then $-A^{-1} b = \beta_c$ by the previous fact. This implies that $c^T(\beta_c + A^{-1} b)$ and $f_c(t) \geqslant 0$ by the computation of the fourth fact. Therefore $f_c \equiv 0$ and (A, b) is not controllable by Lemma 2, a contradiction.

Finally, to show the sixth claim, observe that if $-A^{-1}b \in \mathcal{R}$ then $-A^{-1}b \neq \beta_c$ since $\beta_c \in \partial\mathcal{R}$ and $\mathcal{R}$ is open so the result is true. Otherwise, $-A^{-1}b \in \partial\mathcal{R}$ by a previous fact. Hence there are two cases: if $-A^{-1}b = \beta_c$ then the intersection is empty by the second fact. Otherwise $-A^{-1}b \neq \beta_c$, but $-A^{-1}b \neq \beta_{-c}$ by the fifth fact so it must be the case that $-A^{-1}b + H_c$ interesects $\mathcal{R}$. To show this formally, observe that by strict convexity $(\beta_{-c}, \beta_c) \subseteq \mathcal{R}$. Also, since β_c is the maximizer in direction c, $-A^{-1}b \neq \beta_c$ implies that $c^T x < c^T \beta_c$ where $x = -A^{-1}b$. A similar reasoning for β_{-c} shows that $c^T\beta_{-c} < c^T x < c^T\beta_c$. Define $y = \beta_{-c} + \alpha(\beta_c - \beta_{-c})$ where $\alpha = \frac{c^T x - c^T \beta_{-c}}{c^T\beta_c - c^T\beta_{-c}}$. Then $\alpha \in (0,1)$ by the previous inequality, hence $y \in (\beta_{-c}, \beta_c) \subseteq \mathcal{R}$. But by construction $c^T y = c^T\beta_{-c} + \alpha c^T(\beta_c - \beta_{-c}) = c^T x$, therefore $y \in x + H_c$.

We can now describe an algorithm that solves this instance of Continuous Nontangential Skolem problem by reducing to the LTI Set Reachability problem. Consider the LTI with matrix A, control matrix b, input set $[-1,1]$ and target set $Y = \{x \in \mathbb{R}^n : c^T x = -c^T A^{-1}b\}$. Note that Y is a hyperplane with algebraic coefficients, and check that $Y = -A^{-1}b + H_c$. Let $\mathcal{R}$ be the reachable set of this LTI, then it follows by the third and sixth facts above that $Y \cap \mathcal{R} = \varnothing$ if and only if $-A^{-1}b = \beta_c$ if and only if f_c has no zero-crossings. Hence Y is reachable if and only if the Continuous Nontangential Skolem problem instance has a zero-crossing.

The set Y above is convex but not compact, but since we have chosen A to be stable, the reachable set is bounded by Proposition 3 and it is easy to compute a bound M such that $\mathcal{R} \subseteq [-M, M]^n$. Then we can define $\hat{Y} = Y \cap [-M, M]^n$ which is now compacy convex, and clearly $\mathcal{R} \cap Y = \varnothing$ if and ony if $\mathcal{R} \cap \hat{Y} = \varnothing$.

J Details on Figure 3

Since all eigenvalues A of negative, it is stable. Furthermore, one checks that b_1 and Ab_1 are linearly independent, hence (A, b_1) is controllable. We can apply Theorem 4 to get the description of the boundary:

$$\partial\mathcal{R}(A, b_1) = \left\{ \int_0^\infty e^{At}b\,\mathrm{sgn}(c^T e^{At}b)\,\mathrm{d}t : c \in \mathbb{R}^2 \setminus \{\mathbf{0}\} \right\}$$

$$= \left\{ \int_0^\infty \begin{bmatrix} e^{-\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{3}} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathrm{sgn}\left(\begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} e^{-\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{3}} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \mathrm{d}t : c_1, c_2 \in \mathbb{R} \setminus \{0\} \right\}$$

$$= \left\{ \int_0^\infty \begin{bmatrix} e^{-\frac{t}{2}} \\ e^{-\frac{t}{3}} \end{bmatrix} \mathrm{sgn}\left(c_1 e^{-\frac{t}{2}} + c_2 e^{-\frac{t}{3}} \right) \mathrm{d}t : c_1, c_2 \in \mathbb{R} \setminus \{0\} \right\}.$$

Given c_1, c_2 nonzero, observe that

$$c_1 e^{-\frac{t}{2}} + c_2 e^{-\frac{t}{3}} = 0 \iff 1 + \frac{c_2}{c_1} e^{\frac{t}{6}} = 0 \iff t = 6\ln\frac{-c_1}{c_2}.$$

Since only the ratio c_1/c_2 is important and c_2 must be nonzero, we can parametrize it as $c_2 = 1$ and $c_1 = -\alpha$, where $\alpha \in (1, +\infty)$, so that the only possible solution becomes $t = t_1 := -6\ln\tan\alpha \geqslant 0$. Now there are two cases:

- if $c_1 c_2 \geqslant 0$, then there is no solution and the integral becomes

$$\int_0^\infty \begin{bmatrix} e^{-\frac{t}{2}} \\ e^{-\frac{t}{3}} \end{bmatrix} \mathrm{sgn}(c_1)\,\mathrm{d}t = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \mathrm{sgn}(c_1);$$

- if $c_1 c_2 < 0$, then we can split the integral into two parts (the sign must chance in this case):

$$\int_0^{t_1} \begin{bmatrix} e^{-\frac{t}{2}} \\ e^{-\frac{t}{3}} \end{bmatrix} \operatorname{sgn}(c_1) \, \mathrm{d}t - \int_{t_1}^{\infty} \begin{bmatrix} e^{-\frac{t}{2}} \\ e^{-\frac{t}{3}} \end{bmatrix} \operatorname{sgn}(c_1) \, \mathrm{d}t = \begin{bmatrix} 2 - 4e^{-t_1/2} \\ 3 - 6e^{-t_1/3} \end{bmatrix} \operatorname{sgn}(c_1)$$

$$= \begin{bmatrix} 2 - 4e^{-3\alpha} \\ 3 - 6e^{-2\alpha} \end{bmatrix} \operatorname{sgn}(c_1)$$

$$= \begin{bmatrix} 2 - 4\alpha^{-3} \\ 3 - 6\alpha^{-2} \end{bmatrix} \operatorname{sgn}(c_1).$$

Hence the boundary of the reachable set is

$$\partial \mathcal{R}(A, b_1) = \left\{ \pm \begin{bmatrix} 2 - 4\alpha^{-3} \\ 3 - 6\alpha^{-2} \end{bmatrix} : \alpha \in [1, \infty) \right\} \cup \left\{ \pm \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}.$$