EFFECTIVE MODEL COMPLETENESS OF THE THEORY OF RESTRICTED PFAFFIAN FUNCTIONS

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First order theory of reals admits quantifier elimination (Tarski-Seidenberg theorem).

Geometric language:

Definition 1. A set $X \subset \mathbb{R}^n$ is called *semialgebraic* if X is representable in a form $X = \{\mathbf{x} \in \mathbb{R}^n | F(\mathbf{x})\}$, where $F(\mathbf{x})$ is a quantifier-free (Boolean) formula with atoms of the kind $f > 0$, where $f \in \mathbb{R}[\mathbf{x}]$.

Quantifier elimination $\Leftrightarrow$

Theorem 2. *A projection of a semialgebraic set $X \subset \mathbb{R}^n$ on any coordinate subspace of $\mathbb{R}^n$ is a semialgebraic set.*

We wish to expand the language by analytic functions different from polynomials, in particular, by elementary transcendental functions.

First important difference: *open domains*

Common domain for all functions that occur in the theory.

Definition 3. A set $X \subset \mathbb{R}^n$ is called *semianalytic* if X is representable in a form $X = \{\mathbf{x} \in \mathbb{R}^n | F(\mathbf{x})\}$, where $F(\mathbf{x})$ is a quantifier-free (Boolean) formula with atoms of the kind $f > 0$, where fs are real analytic functions defined in a common domain $G \subset \mathbb{R}^n$.

Second important difference:

A projection of a semianalytic set may not be semianalytic.

Example (Osgood, 1916).

$$Y := \{(x, y, z) \in \mathbb{R}^3 |\, \exists u \in [0, 1]$$

$$(y = xu \wedge z = xe^u)\}.$$

Set Y is two-dimensional and any real analytic function vanishing on Y in the neighbourhood of the origin is $\equiv 0$.

Corollary 4. *Quantifier elimination is not possible in a theory involving e^u.*

This motivates

Definition 5. A set $X \subset \mathbb{R}^n$ is called *subanalytic* in an open domain $G \subset \mathbb{R}^n$ if it is an image of a semianalytic set under a projection into a subspace.

We will consider only

"Restricted" case:

Definition 6. A semianalytic set X is *restricted* in the domain G if its topological closure lies in G.

Definition 7. Consider the closed cube $[-1,1]^{m+n}$ in an open domain $G \subset \mathbb{R}^{m+n}$ and the projection map $\pi : \mathbb{R}^{m+n} \to \mathbb{R}^n$. A subset $Y \subset [-1,1]^n$ is called *restricted subanalytic* if $Y = \pi(X)$ for a restricted semianalytic set $X \subset [-1,1]^{m+n}$.

Obvious: Finite unions and intersections of arbitrary subanalytic sets are subanalytic.

Hard (Gabrielov, Wilkie): For a wide class of *restricted* subanalytic sets, the complement of a set is also subanalytic from this class.

$\Rightarrow$ This class is a Boolean algebra.

Logic: The first order theory of the reals expanded by real analytic functions from a "wide class" is *model complete*:

projection $\Leftrightarrow$ $\exists$

complement $\Leftrightarrow$ $\neg$

$\forall$ $\Leftrightarrow$ $\neg\exists\neg$

Any first order formula in a prenex form is equivalent to a formula in a prenex form with only $\exists$*.*

Important particular case:
Pfaffian functions

- Natural notion of the *format* of a formula describing the subanalytic set.

- Explicit upper bound on the format of a formula describing the complement via the format of the original set.

- An algorithm (with *oracle*) for computing the complement.
  (*Oracle:* Decides whether or not a system of analytic equations and inequalities is consistent.)

The rest of this talk will be just a more detailed explanation of these items.

Pfaffian functions (Khovanskii, 1970s) are analytic functions satisfying triangular systems of first order partial differential equations with polynomial coefficients.

More precisely:

Definition 8. A *Pfaffian chain* of the order $r \geq 0$ and degree $\alpha \geq 1$ in an open domain $G \subset \mathbb{R}^n$ is a sequence of analytic functions $f_1, \ldots, f_r$ in G satisfying differential equations

$$\frac{df_j}{dx_i}(\mathbf{x}) = g_{ij}(\mathbf{x}, f_1(\mathbf{x}), \ldots, f_j(\mathbf{x}))$$

for $1 \leq j \leq r$, $1 \leq i \leq n$. Here $g_{ij}(\mathbf{x}, y_1, \ldots, y_j)$ are polynomials in $\mathbf{x} = (x_1, \ldots, x_n)$, $y_1, \ldots, y_j$ of degrees not exceeding α. A function $f(\mathbf{x}) = P(\mathbf{x}, f_1(\mathbf{x}), \ldots, f_r(\mathbf{x}))$, where $P(\mathbf{x}, y_1, \ldots, y_r)$ is a polynomial of a degree not exceeding $\beta \geq 1$, is called a *Pfaffian function* of order r and degree (α, β).

Examples.

(a) Pfaffian functions of order 0 and degree $(1, \beta)$ are polynomials of degrees not exceeding β.

(b) $f(x) = e^{ax}$ is a Pfaffian function of order 1 and degree $(1, 1)$ in $G = \mathbb{R}$ because $df(x)/dx = af(x)$.

(c) $f(x) = 1/x$ is Pfaffian of order 1 and degree $(2, 1)$ in $\{x \in \mathbb{R} | \, x \neq 0\}$ because $df(x)/dx = -f^2(x)$.

(d) $f(x) = \ln(|x|)$ is Pfaffian of order 2 and degree $(2, 1)$ in $\{x \in \mathbb{R} | \, x \neq 0\}$ because $df(x)/dx = g(x)$ and $dg(x)/dx = -g^2(x)$, where $g(x) = 1/x$.

(e) Fewnomials.

Exercise. Show that $f(x) = \cos(x)$ is Pfaffian of order 2 and degree $(2,1)$ in $\bigcap_{k\in\mathbb{Z}}\{x \in \mathbb{R} |\, x \neq (2k+1)\pi\}$.

We will see that $\cos(x)$ is *not* Pfaffian in the whole $\mathbb{R}$.
We'll assume that G is "simple", like $\mathbb{R}^n$, $\{\mathbf{x}|\, \|\mathbf{x}\|^2 < 1\}$, or $(-1,1)^n$.

Theorem 9. (Khovanskii) *Consider*

$$f_1 = \cdots = f_n = 0,$$

where f_i*,* $1 \le i \le n$ *are Pfaffian functions in a domain* $G \subset \mathbb{R}^n$*, having a common Pfaffian chain of order* r *and degrees* (α, β_i) *respectively. Then the number of non-degenerate solutions of this system does not exceed*

$$2^{r(r-1)/2}\beta_1 \cdots \beta_n \cdot$$

$$\cdot(\min\{n,r\}\alpha + \beta_1 + \cdots + \beta_n - n + 1)^r.$$

Semi- and subanalytic sets defined by formulae with Pfaffian functions are called *semi- and sub-Pfaffian* sets respectively.

Aim: to give an "effective" proof of the *complement theorem* for restricted sub-Pfaffian sets $\Leftrightarrow$ effective model completeness of the theory of restricted Pfaffian functions.

Gabrielov (1968, 1996): geometric proof
Wilkie (1995, 1999): model-theoretic proof
Gabrielov-Vorobjov (2001): effective proof
Pericleous-Vorobjov (2003): alternative effective proof

The complement theorem immediately follows from the existence of a *cylindrical cell decomposition (CCD)* of the ambient space compatible with a given subanalytic set.

CCD compatible with X is a partition of the space into disjoint simple subanalytic subsets, called *cells*, such that for any cell C either $C \subset X$ or $C \cap X = \emptyset$.

Definition 10. *Cylindrical cell* is defined by induction as follows.

1. Cylindrical 0-cell in $\mathbb{R}^n$ is an isolated point. Cylindrical 1-cell in $\mathbb{R}$ is an open interval $(a, b) \subset \mathbb{R}$.

2. For $n \geq 2$ and $0 \leq k < n$ a cylindrical $(k+1)$-cell B in $\mathbb{R}^n$ is either a graph of a continuous bounded function $f : C \to \mathbb{R}$, where C is a cylindrical a cylindrical $(k+1)$-cell in $\mathbb{R}^{n-1}$, or else a set of the form

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n |\, (x_1, \ldots, x_{n-1}) \in C$$

and $f(x_1, \ldots, x_{n-1}) < x_n < g(x_1, \ldots, x_{n-1})\}$, where C is a cylindrical k-cell in $\mathbb{R}^{n-1}$, and $f, g : C \to \mathbb{R}$ are continuous bounded functions such that

$$f(x_1, \ldots, x_{n-1}) < g(x_1, \ldots, x_{n-1})$$

for all points $(x_1, \ldots, x_{n-1}) \in C$.

Definition 11. *Cylindrical cell decomposition* $\mathcal{D}$ of a subset $A \subset \mathbb{R}^n$ is defined by induction as follows.

1. If $n = 1$, then $\mathcal{D}$ is a finite family of pairwise disjoint cylindrical cells (i.e., isolated points and intervals) whose union is A.

2. If $n \geq 2$, then $\mathcal{D}$ is a finite family of pairwise disjoint cylindrical cells in $\mathbb{R}^n$ whose union is A and there is a cylindrical cell decomposition $\mathcal{D}'$ of $\pi(A)$ such that $\pi(C)$ is its cell for each $C \in \mathcal{D}$, where $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is the projection map onto the coordinate subspace of $x_1, \ldots, x_{n-1}$. We say that $\mathcal{D}'$ is *induced* by $\mathcal{D}$.

Definition 12. Let $B \subset A \subset \mathbb{R}^n$ and $\mathcal{D}$ be a CCD of A. Then $\mathcal{D}$ is *compatible* with B if for any $C \in \mathcal{D}$ we have either $C \subset B$ or $C \cap B = \emptyset$ (i.e., some subset $\mathcal{D}' \subset \mathcal{D}$ is a CCD of B).

MAIN RESULT

Given:

A semi-Pfaffian set

$$X := \bigcup_{1 \le i \le M} \{\mathbf{x} \in \mathbb{R}^{m+n} | \, f_{i1} = \cdots = f_{iI_i} = 0,$$

$$g_{i1} > 0, \ldots, g_{iJ_i} > 0\} \subset (-1,1)^{m+n},$$

where f_{ij}, g_{ij} are Pfaffian functions with a common Pfaffian chain in an open domain $G \subset \mathbb{R}^{m+n}$ and $[-1,1]^{m+n} \subset G$.

The projection map $\pi: \mathbb{R}^{m+n} \to \mathbb{R}^n$.

$Y := \pi(X)$.

Then:

There is an algorithm (with oracle) producing a cylindrical cell decomposition $\mathcal{D}$ of $(-1,1)^n$ compatible with Y (modulo a linear coordinate change).

Output:

Each cell in $\mathcal{D}$ is described as a projection of a semi-Pfaffian set in DNF:

$$\pi' \left(\bigcup_{1 \le i \le M} \bigcap_{1 \le j \le M_i} \{h_{ij} *_{ij} 0\} \right),$$

where h_{ij} are Pfaffian functions in $n' \geq m + n$ variables, $\pi' : \mathbb{R}^{n'} \to \mathbb{R}^n$ is the projection map, $*_{ij} \in \{=, >\}$, and M, M_i $(i = 1, \dots, M)$ are certain integers.

Complexity:

Let there be N Pfaffian functions in the input formula, having order r and degrees (α, β). Let $\dim(Y) = d$.

Then

- The number of cells in the CCD $\mathcal{D}$ is
$$N^{(r+m+2n)^{2d}} (\alpha + \beta)^{r^{O(d(m+dn))}}.$$

- Integers n', M, M_i do not exceed the same bound.

- Order of h_{ij} is r, degrees are
$$(\alpha + \beta)^{r^{O(d(m+dn))}}.$$

- The complexity of the algorithm is
$$N^{(r+m+n)^{O(d)}} (\alpha + \beta)^{(r+m+n)^{O(d(m+dn))}}.$$

Relaxing and simplifying:

All parameters of the output and the complexity are bounded from above by

$$(N(\alpha+\beta))^{(r+m+n)^{O(n^3)}}.$$

Corollary 13. *The complement* $\widetilde{Y} := (-1,1)^n \setminus Y$ *is a sub-Pfaffian set. There is an algorithm (with oracle) for computing* $\widetilde{Y}$ *having the same complexity as above. The complement* $\widetilde{Y}$ *is represented by the algorithm as a union of some cells of the CCD* $\mathcal{D}$.

How the CCD algorithm works.

Subroutines:

- Computing frontier and closure of a semi-Pfaffian set X.
  The *closure* of X in G is

  $$\bar{X} := \{\mathbf{x} \in G |\, \forall \varepsilon > 0\, \exists \mathbf{y} \in X\ (|\mathbf{x} - \mathbf{y}| < \varepsilon)\}.$$

  The *frontier* of X in G is

  $$\partial X := \bar{X} \setminus X.$$

  Both $\bar{X}$ and ∂X are *semi-Pfaffian*.

- Computing smooth (weak) stratification of a semi-Pfaffian set X.
  Partition of X into a disjoint union of non-singular, not necessarily connected, possibly empty, semi-Pfaffian sets called *strata*.

EXAMPLE:

$$X = \{(\mathbf{y}, \mathbf{x}) = (x_1, x_2, x_3) |$$
$$f := x_1^2 + x_2^2 + x_3^2 - 1/2 = 0\},$$

$$Y = \{\mathbf{y} = (x_1, x_2) |\, x_1^2 + x_2^2 \leq 1/2\}.$$

$n = d = 2$, $m = 1$.

Two recursive procedures: *down* and *up*.

DOWN:

FIRST STEP:

X is non-singular (else we would use a subroutine to stratify X).

$$X' := \{(x_1, x_2, x_3) \in X | \, \partial f/\partial x_3 \neq 0\}$$

set of all regular values of the restriction of $\pi : (x_1, x_2, x_3) \mapsto (x_1, x_2)$ on X.

$$X_2 := \{(x_1, x_2, x_3) \in X | \, \partial f/\partial x_3 = 0\} =$$

$$= \{(x_1, x_2, x_3) \in X | \, x_3 = 0, x_1^2 + x_2^2 - 1/2 = 0\}$$

set of all critical values of the restriction of $\pi : (x_1, x_2, x_3) \mapsto (x_1, x_2)$ on X.

$Y_2 := \pi(X_2)$, $d_2 := \dim(Y_2) = 1$

END OF THE FIRST STEP OF *DOWN*.

SECOND STEP:

X_2, Y_2 play the role of X, Y respectively.

All points of X_2 are regular for $\pi|_{X_2}$.
A new feature: $d_2 < n$.
Projection map $\rho_2 :\ (x_1, x_2) \mapsto x_1$.

$S_2 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 |\, x_1^2 = 1/2, x_2 = x_3 = 0\}$
the set of all critical points of $\rho_2 \pi|_{X_2}$,

$Z_2 := \rho_2 \pi(X_3)$.

Let $X_3 := S_2$, $Y_3 := Y_2 \cap \rho_2^{-1}(Z_2) = \pi(X_3)$,
$d_3 := \dim(Y_3) = 0$.

END OF THE SECOND STEP OF *DOWN*.

LAST (DEGENERATE) STEP:

All points of X_3 are regular for $\pi|_{X_3}$.
Similar to the second step, $d_3 < n$.
Projection map $\rho_3 :\ (x_1, x_2) \mapsto 0$.

The set S_3 of all critical points of $\rho_3\pi|_{X_3}$ is empty, thus $Z_3 := \rho_3\pi(S_3) = \emptyset$.

DOWN IS COMPLETED.

UP:

Recursion.
On each step consider the pair Y_i, Z_i starting from the largest i, in our case, Y_3, Z_3.

FIRST STEP:

Since Y_3 consists of just two points, $(1/\sqrt{2}, 0)$ and $(-1/\sqrt{2}, 0)$, the CCD $\mathcal{D}_3$ of $(-1,1)^2$ compatible with Y_3 is trivial.

SECOND STEP:

Consider Y_2, Z_2.
The CCD $\mathcal{D}_3$ induces the CCD $\mathcal{D}'_3$ of $(-1,1)$ into five cells compatible with Z_2:

$C_1 := \{x_1 |\, -1 < x_1 < -1/\sqrt{2}\}$,
$C_2 := \{x_1 |\, x_1 = -1/\sqrt{2}\}$,
$C_3 := \{x_1 |\, -1/\sqrt{2} < x_1 < 1/\sqrt{2}\}$,
$C_4 := \{x_1 |\, x_1 = 1/\sqrt{2}\}$,
$C_5 := \{x_1 |\, 1/\sqrt{2} < x_1 < 1\}$.

By the choice of Z_2, for any $z \in C_3$ the cardinality of $\rho_2^{-1}(z) \cap Y_2$ is constant (=2).

Moreover, the following cells form CCD of $\rho_2^{-1}(C_3) \cap (-1,1)^2$ compatible with $\rho_2^{-1}(C_3) \cap Y$:

- $\{(x_1, x_2) \in \rho_2^{-1}(C_3) \cap (-1,1)^2 | \ \exists (y_1, y_2) \in Y_2 \ \exists (y'_1, y'_2) \in Y_2 (y_1 = y'_1, y_2 < y'_2 < x_2)\}$

- $\{(x_1, x_2) \in \rho_2^{-1}(C_3) \cap (-1,1)^2 | \ \exists (y_1, y_2) \in Y_2 \ \exists (y'_1, y'_2) \in Y_2 (y_1 = y'_1, y_2 < y'_2 = x_2)\}$

- $\{(x_1, x_2) \in \rho_2^{-1}(C_3) \cap (-1,1)^2 | \ \exists (y_1, y_2) \in Y_2 \ \exists (y'_1, y'_2) \in Y_2 (y_1 = y'_1, y_2 < x_2 < y'_2)\}$

- $\{(x_1, x_2) \in \rho_2^{-1}(C_3) \cap (-1,1)^2 | \ \exists (y_1, y_2) \in Y_2 \ \exists (y'_1, y'_2) \in Y_2 (y_1 = y'_1, y_2 = x_2 < y'_2)\}$

- $\{(x_1, x_2) \in \rho_2^{-1}(C_3) \cap (-1,1)^2 | \ \exists (y_1, y_2) \in Y_2 \ \exists (y'_1, y'_2) \in Y_2 (y_1 = y'_1, x_2 < y_2 < y'_2)\}$.

Similar CCD of $\rho_2^{-1}(C_i) \cap (-1,1)^2$ can be constructed for all other cells C_i.

Combining all CCD for $\rho_2^{-1}(C_i) \cap (-1,1)^2$ with $\mathcal{D}_3$, we get a CCD of $(-1,1)^2$ compatible with Y.

END OF *UP*

O-minimal structures involving Pfaffian functions

Charbonnel, Wilkie: "closure at infinity" operation.
Main theorem: sets constructed from semi-Pfaffian sets by a finite sequence of projections on subspaces and closures at infinity form an o-minimal structure.

Gabrielov: "relative closure" operation for a one-parameter family of semi-Pfaffian sets and "limit set" (a finite union of relative closures of semi-Pfaffian families).
Main theorem: limit sets form an o-minimal structure.

Problem: find upper bounds on the formats of the results of Boolean and projection operations in these o-minimal structures.

References

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