AN UPPER BOUND FOR THE NUMBER OF INTERSECTIONS BETWEEN A TRAJECTORY OF A POLYNOMIAL VECTOR FIELD AND AN ALGEBRAIC HYPERSURFACE IN THE n-SPACE

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Dedicated to Vladimir Igorevich Arnol'd on his 60th birthday.

Abstract. We give an explicit upper bound for the number of isolated intersections between an integral curve of a polynomial vector field in $\mathbb{R}^n$ and an algebraic hypersurface. The answer is polynomial in the height (the magnitude of coefficients) of the equation and the size of the curve in the space-time, with the exponent depending only on the degree and the dimension.

The problem turns out to be closely related to finding an explicit upper bound for the length of ascending chains of polynomial ideals spanned by consecutive derivatives.

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1. Entrée

The main problem that will be addressed (and solved) in this paper, concerns oscillatory properties of functions defined by polynomial ordinary differential equations. Geometrically the question is about the number of isolated intersections between an integral curve of a polynomial vector field and an algebraic hypersurface in the Euclidean n-space.

Despite the fact that, to the best of our knowledge, this problem was first discussed on Arnold's seminar in Moscow in the seventies, only a limited progress in this direction has been achieved so far, and very recently (in March 1997) the problem was once again formulated by R. Narasimhan in his lecture in the Fields institute. The most advanced contribution to this area, an upper bound for the multiplicity of contact between an integral curve and an algebraic hypersurface, is due to A. Gabriélov. It is discussed below in §1.3 and Appendix A.

This paper was preceded by an extended abstract [23] in which the main ideas of the construction have been already exposed together with motivations. The introductory section §1 begins with an informal discussion of the problem and some related results. Then we mention several obvious factors that may affect the number of intersections between a curve and a hypersurface: this gives (see §1.6) a minimal list of parameters on which the answer must depend. Then a precise formulation of the main theorem follows. Since the explicit dependence of the bound on the parameters looks somewhat unexpectedly, we explain why this is a meaningful (and nontrivial) result. The introductory section ends with formulations of several related results (bounds for complex intersections, equivalent problem with discrete time variable etc).

1.1. **Meandering of analytic spatial curves: the problem.** Consider a polynomial vector field in the Euclidean space $\mathbb{R}^n$ and let $\Gamma \subset \mathbb{R}^n$ be a compact connected piece of a phase trajectory of this field. Since Γ is a real analytic curve, for any affine hyperplane $\Pi \subset \mathbb{R}^n$ the following alternative holds: either $\Gamma \subset \Pi$, or the number of intersections $\#\Gamma \cap \Pi$ is finite and all of them are isolated on Γ. The problem is to place an explicit upper bound on the number of *isolated* intersections between Γ and any affine hyperplane. The supremum of the number of isolated intersections is a characteristic of the curve Γ, naturally measuring its meandering in the ambient space.

We require that the upper bound should be given in terms of the vector field (and eventually some simple geometric characteristics of the curve, e.g. its size), but *not* in terms of the explicit expression for Γ, that is, without solving the associated system of polynomial ordinary differential equations. The expected answer is that if the degrees of all polynomials are bounded from above and the absolute values of their coefficients are known, then one can place an explicit upper bound on the meandering of any finite piece of integral trajectory. Notice that the upper bound given in such terms, should necessarily be uniform over all integral trajectories belonging to any compact subset of the phase space.

The above formulation is not yet the most general one. We will also allow the vector field to be nonautonomous (explicitly time-dependent while still polynomial in the time variable). Besides, our results stated below imply a constructive explicit upper bound for the number of *complex isolated* intersections of the complexified curve $\Gamma^{\mathbb{C}}$ and any affine hyperplane in $\mathbb{C}^n$. Finally, one can count intersections with algebraic hypersurfaces of any known degree rather than with hyperplanes only. It is this strongest form in which the problem, later referred to as the *meandering problem*, will be solved.

1.2. **Analytic context: zeros of functions defined by systems of ordinary differential equations.** Consider several analytic functions in n variables, and let Z be their common zero locus in $\mathbb{R}^n$ or $\mathbb{C}^n$. The following problem occurs very often in theoretical considerations and applications. Assuming that Z is finite, one has to find an upper bound for $|Z|$, the number of points in Z.

The easiest case is that of algebraic varieties: if all functions were polynomials, then $|Z|$ does not exceed the nth power of the maximum of degrees of the equations (by Bézout theorem).

An important transcendental case that was recently extensively studied, is that of *Pfaffian functions* [14]. Namely, assume that together with polynomials one allows also solutions of Pfaffian equations with polynomial coefficients. Then one can place an upper bound on $|Z|$ in terms of the maximum of degrees of all polynomial data. To a certain extent these results can be generalized for counting complex points of Z.

The natural question arises: can one allow, among the functions determining the locus Z, also functions defined by ordinary differential equations and systems of such equations? This analytic formulation naturally reduces to the geometric meandering problem described above.

1.3. **Risler problem and Gabriélov theorem.** The question closely related to the original problem (an infinitesimal counterpart of the latter) is about the maximal *order of contact* between an integral trajectory Γ and a polynomial hypersurface. This question was raised by J.-J. Risler in [24] and therefore will be referred to as the *Risler problem*.

Let d be the degree of the vector field, i.e. the maximum of the degrees of the polynomials occurring in the right hand side of the corresponding differential equation, and n the dimension of the space. The following result was proved by Andrei Gabriélov (first with J.-M. Lion and R. Moussu in [7] for $n = 2$ and later in [5] for any n).

Theorem (A. Gabriélov [5]). *The order of* isolated *contact between* any *integral trajectory of this vector field and* any *hyperplane, does not exceed* $(d+1)^{2^n}$, *and a similar tower of exponents of height* 2 *gives an upper bound for the order of contact with polynomial hypersurfaces.*

We give a slightly different proof of this result in Appendix A. Actually, in a very recent preprint (March 1997) Gabriélov improved the double exponential in n upper bound on the order of contact, to be single exponential. However, this elegant improved result is based on a different set of arguments and we will not discuss it here.

1.4. **Meandering via integral curvatures: a geometric digression.** Assume for an instance that the parametrization of the curve were explicitly known. The result below gives an upper bound for the number of isolated intersections with hyperplanes in terms of certain integrals along the curve.

Recall that for any sufficiently smooth and sufficiently generic curve in $\mathbb{R}^n$ one can define the *osculating frame* (the Frenet frame), and a collection of $n-1$ curvature functions [21]. Integrating their absolute values against arclength yields *integral Frenet curvatures* of the curve. The *integral inflection* of the curve is by definition the number of points at which the osculating frame degenerates.

The principal result of [21] is an upper bound for the number of isolated intersections between a smooth curve and any affine hyperplane, in terms of a weighted sum of the integral curvatures and integral inflection.

The main reason why this result cannot be applied to the original problem, is very simple: to compute (or even estimate) the integral curvatures is in general impossible without solving explicitly the equations (i.e. integrating the vector field). Moreover, even for integrable systems it seems unlikely that there exist upper bounds for curvatures that would be uniform over all integral trajectories in the unit box, since the curvatures in general depend discontinuously on the initial conditions.

1.5. **Linear high order differential equations.** One rather specific though very important particular case when an explicit solution is available, is that of curves defined by nth order linear ordinary differential equations with variable coefficients. The underlying analytic assertion follows.

Assume that a C^n-smooth function $f(t)$ of the real argument $t \in I = [0, l]$ satisfies on this segment some linear homogeneous equation of order n with the variable coefficients that are real, continuous and bounded on the segment by some constant $A < \infty$.

Then one can place an explicit upper bound for the number of isolated zeros of f on I (counted with multiplicities, as always in this paper) in terms of A, l and n: *solutions of a linear equation with bounded coefficients cannot be strongly oscillating.*

This claim can be made precise in several ways, and generalized for complex analytic functions and equations. The exact formulations will be given below in section §2.2. Now we only want to stress the difference with the Pfaffian theory: *the magnitude of coefficients of the determining equation enters explicitly into upper bounds for the number of zeros.* This observation holds also for all nonlinear equations and systems.

The "main idea" of our approach to the meandering problem is to reduce the given system of polynomial (nonlinear) ordinary differential equations to one *quasi-linear* equation of a very high order, and use the results mentioned just above.

1.6. **On what the answer must depend.** The simplest example of linear vector fields with constant coefficients, say, the system in $\mathbb{R}^3$, given by the equations

$$\dot{x}_1 = \omega x_2, \ \dot{x}_2 = -\omega x_1, \ \dot{x}_3 = x_3,$$

shows that the *magnitude of coefficients of the polynomials defining the vector field, must enter explicitly into the bounds for the meandering of integral trajectories* (consider intersection of integral curves with the hyperplane $x_1 = 0$ for large $\omega \gg 1$). This same particular case also suggests that the number of isolated intersections of a trajectory Γ with hyperplanes must depend on the "size" of Γ, both with respect to the ambient space $\mathbb{R}^n$ and to the natural parameter ("time") on Γ.

Besides the magnitude-type data that should enter explicitly into the majorant, there are obviously relevant integer parameters of the problem, namely, the dimension of the phase space and the degrees of the equations determining the vector field (and, eventually, the hypersurface if the latter is not an affine hyperplane).

1.7. **Formulation of the main result in $\mathbb{R}^n$.** Consider the system of polynomial ordinary differential equations of degree d in n variables with a polynomial right hand side $v = (v_1, \dots, v_n)$:

$$\dot{x} = v(t,x) \iff \frac{d}{dt}x_j = \sum_{|\alpha|+k=0}^{d} v_{jk\alpha} t^k x^\alpha, \qquad j = 1, \dots, n \tag{1.1}$$

(the standard multiindex notation is assumed). Suppose that the *height* (see §2.1) of the polynomials $v_j(t,x) \in \mathbb{R}[t,x]$ is explicitly bounded from above, i.e. all the coefficients $v_{jk\alpha} \in \mathbb{R}$ of in (1.1) satisfy the inequality $|v_{jk\alpha}| \leqslant R$ for some known $R < \infty$.

Consider an arbitrary integral trajectory Γ of the system (1.1) entirely belonging to the centered box $\mathbb{B}_R = \{|x_j| < R,\ |t| < R\} \subset \mathbb{R}^{n+1}$ of size R in the space-time, i.e. a solution $t \longmapsto (x_1(t), \dots, x_n(t))$ defined on some interval $t \in [t_0, t_1] \subseteq [-R, R]$ and satisfying the inequalities $|x_j(t)| < R$ on it.

Finally, let $\Pi \subset \mathbb{R}^{n+1}$ be an algebraic hypersurface determined by an equation $\{p(t,x) = 0\}$ in the space-time, where $p \in \mathbb{R}[t,x]$ is a polynomial of degree $\leqslant d$:

$$\Pi = \{p(t,x) = 0\}, \qquad p(t,x) = \sum_{k+|\alpha|=0}^{d} p_{k\alpha} t^k x^\alpha. \tag{1.2}$$

Since the polynomial p is defined modulo a nonzero constant factpr, without loss of generality we may always assume that the height of p is also bounded by the same R, i.e. all coefficients $p_{k\alpha}$ satisfy the inequality $|p_{k\alpha}| \leqslant R$. Note that R is a common bound for the coefficients and for the "size" of Γ, whereas d is a common bound for the degrees of the vector field and the hypersurface. This is a convenient way to reduce the number of parameters pertinent to the problem, and not an additional restriction.

Theorem 1. *For any integral trajectory Γ of a polynomial vector field of degree d and height $\leqslant R$ in $\mathbb{R}^n$ and any algebraic hypersurface of the same degree and height, the number of* isolated *intersections between Γ and Π, counted with multiplicities inside the box $\mathbb{B}_R$, admits an upper bound of the form*

$$\#(\Gamma \cap \Pi \cap \mathbb{B}_R) \leqslant (2+R)^B, \qquad B = B(n,d) \in \mathbb{N}, \tag{1.3}$$

where $B = B(n,d)$ is a primitive recursive function of the integer arguments n and d. As $n, d \to \infty$, the function B grows no rapidly than the tower of four stories:

$$B(n,d) \leqslant \exp\exp\exp\exp(3n \ln d + O(1)). \tag{1.4}$$

1.8. **Discussion of the result: explicitness and computability.** The formulation given above, is in fact a very strong assertion on the computability of the upper bound for $\#\Gamma \cap \Pi$, rather than the explicit expression (though all O-terms can be easily made absolutely explicit). Indeed, the assertion of Theorem 1 implies that:

1. for any specific R and fixed d and n the number of intersections is bounded from above (the *existential finiteness theorem*); moreover, that
2. for any fixed d, n the number of intersections may grow no faster than polynomially in R as $R \to \infty$ (*existential polynomiality*) so that the estimate (1.3) holds for *some* finite growth exponent B; moreover, that
3. the growth exponent $B(n,d)$ can be explicitly computed from the integer data d, n (*theoretical computability*); moreover, that
4. $B(n,d)$ is a *primitive recursive* function (see below for the explanations) growing no faster than the tower (1.4).

More comments on why the assertion of this theorem appears in the above form, can be found in §1.10.

We do not make any claim as for the accuracy (or rather the degree of exaggeration) of the bound (1.4). Still the tower of exponentials in (1.4) looks so terrifying that one has to explain why this bound is meaningful at all. It turns out that another function of n, d, growing incomparably faster than the tower (1.4), naturally arises in connection with this problem, and the two sections §4 and §5 are devoted to proving that this monster in fact does not occur as an upper bound in the meandering problem.

1.9. **Ackermann generalized exponential.** Recall that a *primitive recursive function* of one or several integer arguments is a function that can be derived from constants and coordinate functions by a finite number of juxtapositions, substitutions and *primitive recursions*. While the former operations are obvious, the primitive recursion allows one to derive from some already constructed (primitive recursive) functions $F(x, y, z)$ and $g(y)$ the new function $h(x, y)$, using the rule

$$h(x+1, y) = F(x, y, h(x, y)), \qquad x = 1, 2, \dots, \; h(1, y) = g(y) \tag{1.5}$$

(here x is the recursion index and the variables y are to be considered as parameters). All elementary functions (polynomials, exponentials, towers of exponentials of a fixed height etc.) are primitive recursive. However, there exist an example showing that *not all algorithmically computable functions are primitive recursive*. This example is known as the *Ackermann generalized exponential*.

The Ackermann generalized exponential $A(k, x, y)$ is an integer function of *three* integer arguments, that for small $k = 1, 2, 3, 4$ has the form

$$\begin{aligned} A(1, x, y) &= x + 1 + \cdots + 1 = x + y, \\ A(2, x, y) &= x + x + \cdots + x = xy, \\ A(3, x, y) &= xx \cdots x = x^y, \end{aligned} \qquad A(4, x, y) = \underbrace{x^{x^{\cdot^{\cdot^{\cdot^{x}}}}}}_{y \text{ times}} \quad \text{etc.}$$

It would be natural to refer to the function $A(k, \cdot, \cdot)$ as the *Ackermann exponential of level* k: the Ackermann exponential of level $k+1$ is obtained by iterations of the Ackermann exponential of level k, more precisely, by application of the latter *to itself*,

$$A(z+1, x, y+1) = A(z, x, A(z+1, x, y)) \tag{1.6}$$

coupled with the "boundary conditions" specifying for the first two levels $A(0, x, y) = y + 1$, $A(1, x, 0) = x$, $A(2, x, 0) = 0$, and then $A(3, x, 0) = A(4, x, 0) = \cdots = 1$. Notice that the rule (1.6) is not of the form (1.5), though for any *fixed* value of the first argument (the fixed level) the function $A(k, \cdot, \cdot)$ *is* primitive recursive in x, y.

One may indeed *prove* that the Ackermann exponential is not a primitive function of three integer variables, in particular, that it grows *faster than any primitive recursive function*.

In a surprising way this example rather naturally appears in connection with our problem, see Proposition 3 and Moreno theorem, and a large part of the proof of Theorem 1 is devoted to showing that the upper bound for the exponent B is "*infinitely better*" than the Ackermann exponential. Indeed, the achieved result (independently on the number of stories in the towers) should be considered as an impressive improvement upon the Ackermann-type bound: instead of Ackermann exponential of level n (the number of variables), we obtained the bound majorized by the Ackermann exponential of level $\leqslant 4$.

1.10. **Remarks about the demonstration of the main theorem: the structure of the paper.** The four levels of finiteness/computability/explicitness, as described in §1.8, are reached consecutively at various stages of the demonstration.

The existential finiteness level 1 (that can also be obtained from the most general properties of analytic functions [6]), is achieved in §2.3 as a corollary to the ascending chain property in the ring of functions analytic in the closed box. The corresponding result (Proposition 1 and Corollary 2) can be proved for trajectories of analytic vector fields.

For a (fixed) polynomial vector field one can prove that the upper bound for the number of isolated intersections in the box $\mathbb{B}_R$ grows polynomially with R as $R \to \infty$, but the exponent of this growth remains unknown at that moment.

Both these results hold for a single vector field (analytic or polynomial). To extend them for a parametric class of vector fields (e.g. all polynomial equations of a given degree d, parameterized by the collection of their coefficients $v_{jk\alpha}$), we use in §3.1 a simple but powerful trick. Namely, the parameters are considered as additional state variables governed by the trivial equations $\dot v_{jk\alpha} = 0$. This explains why the same upper bound R was imposed on both the coefficients and the geometric dimensions of the trajectory: after the extension the difference between the two disappears, and the bounds that were uniform over all trajectories of a single vector field, become uniform over the class of vector fields. In particular, we reach the second "level of computability" on that stage, showing that the dependence on *all real parameters* (except for the dimension and degree of the initial vector field) is polynomial (Corollary 3).

The price that one has to pay for this apparently innocent trick, is the dramatic increase in the dimension of the problem (the most sensitive parameter). Yet some complementary benefits, the most substantial of them the integrality of all input data, overweight this drawback. Based on this integrality, we reduce in §3 the original problem on meandering of phase curves, to a purely algebraic problem on the lengths of ascending chains of polynomial ideals.

In this new context one could simply apply the general results of Seidenberg [26, 27] on lengths of ascending chains of ideals in polynomial rings and prove the assertion of the main theorem on the level 3 (theoretical computability). If interrupted here, our arguments would prove that the function $B(n,d)$ in (1.3) is majorized by the Ackermann exponential, as this was shown in [20].

The "boundary layer" section §4 explains the origin of the Ackermann-type upper bounds and gives an insight on how the "moderate" bound (1.4) can be obtained for the function $B(n,d)$. The considerations of this section serve as a paradigm for a more technical exposition in §5. Yet the main result of §4 is an upper bound for the discrete Risler problem (see §1.12), a discrete analog of the problem on the maximal order of contact.

The remaining part §5 is purely algebraic and aimed at reducing the Ackermann-type upper bound for B to the much slowlier growing tower (1.4). In this section, independent from the previous exposition, we establish an upper bound for the length of ascending chains generated by consecutive derivations in polynomial rings.

All computations are collected in one section in the Appendix, in order to improve the readability of the text. Since it is not clear how inaccurate the estimate of

Theorem 1 is, the reader can skip the details of the computation. Moreover, since our arguments, even performed with the uttermost accuracy, cannot reduce the upper bound to anything better than a tower of exponents, we did not hesitate to choose simplier though more relaxed inequalities each time when making the computations. In particular, many times with a mixed feeling of relax and horror we observed how a tower of exponentials absorbs inferior but cumbersome terms.

1.11. **Complex intersections.** The method of the proof of Theorem 1 works also in the complex settings, given a proper modification of the theorem on oscillation of linear equations. Such theorems are also available [15, 13] and imply the similar upper bound for the number of isolated zeros of a polynomial $p(t,x) \in \mathbb{C}[t,x]$ restricted on the holomorphic integral curve $\varGamma^{\mathbb{C}}$. One has to exercise a special care concerning domains of definitions of $\varGamma$.

Notice that solutions (integral curves) of polynomial systems can blow up in a finite time and hence may exhibit the so called movable singularities (ramifications at infinity). If we take the integral curve $\{t \longmapsto x(t)\} \in \mathbb{C}^{n+1}$ passing through a certain initial point (t_0, x_0) in the box (*polydisk*) $\mathbb{B}_R$, then the set of t in the R-disk $\{|t| < R\} \subset \mathbb{C}$, for which the curve remains in $\mathbb{B}_R$, may be not simply connected, therefore one has to specify the choice of the branches.

In order to avoid these complications, we formulate the complex theorem in the "dual form", namely, for every initial point $(t_0, x_0) \in \mathbb{B}_R \subset \mathbb{C}^{n+1}$ we will explicitly specify the size of a small disk in the t-plane, in which the curve has no more than the given number (large than n, in general) of intersections with a polynomial hypersurface.

Theorem 2. *For any n and d one can specify the integer number $\ell = \ell(n,d)$ and a positive radius $\rho = \rho(R,n,d)$ in such a way that the integral curve through any point $(t_0, x_0) \in \mathbb{B}_R$ extends analytically on the disk $D_\rho = \{|t - t_0| < \rho\}$ and, restricted on this disk, can have no more than ℓ isolated intersections with any polynomial hypersurface of degree d in $\mathbb{C}^{n+1}$.*

The radius ρ depends polynomially on R, $\rho > (2+R)^{-B(n,d)}$, the functions $B(n,d)$ and $\ell(n,d)$ admit primitive recursive majorants growing no faster than (1.4) *and $d^{n^{O(n^2)}}$ respectively.*

1.12. **The discrete Risler problem and its ramifications.** A polynomial vector field is a dynamical system in continuous time, whose discrete time analog is a polynomial map (endomorpshim or automorphism of $\mathbb{R}^n$ or $\mathbb{C}^n$). The following is an obvious reformulation of the original problem on intersections between integral curves (orbits of the vector field) and polynomial hypersurfaces. Recall that an *orbit* of P is the sequence of points $\{x_i\}_{i=0}^{\infty} \subset \mathbb{R}^n$ obtaineb by iterations of the map:

$$x_{i+1} = P(x_i), \qquad i = 0, 1, \dots. \tag{1.7}$$

Let $P \colon \mathbb{R}^n \longmapsto \mathbb{R}^n$ be a polynomial map of degree d and $\Pi \subset \mathbb{R}^n$ an algebraic hypersurface defined by the equation $\{p(x) = 0\}$, $p \in \mathbb{R}[x], \deg p \leqslant d$. The problem is to find an upper bound (in terms of d and n) for the maximal number of consecutive zeros in the infinite numeric sequence $\{p(x_i) : \ i = 0, 1, \dots\}$, on the assumption that not all members of this sequence are zeros. Geometrically this problem concerns with the number of intersections between an orbit of the dynamical system and the hypersurface Π. This problem is the discrete analog of the Risler problem on the maximal order of contact, and the result is largely parallel to the Gabriélov theorem.

Suppose that the map P preserves dimensions of (semialgebraic) subsets, i.e. the dimension of $P(V)$ is always equal to the dimension of V.

Theorem 3. *Any orbit $\{x_i\}_{i=0}^{\infty} \subset \mathbb{R}^n$ of a dimension preserving dynamical system (1.7) of degree $\leqslant d$ that belongs to a polynomial hypersurface $\Pi = \{p = 0\}$ of degree $\leqslant d$ for $i = 0, 1, \dots, \ell = \ell(n, d)$, where*

$$\ell(n,d) \leqslant \underbrace{M^{M^{\cdot^{\cdot^{\cdot^{M}}}}}}_{n \text{ times}}, \qquad M = 1 + d^n,$$

necessarily remains on Π forever.

Theorem 3 is equivalent to another result (Theorem 5 from §4) giving an upper bound for the length of descending chains of algebraic varieties, as shown in §4.5.

Remark . A particular case of the discrete Risler problem was studied in [3], with P being a linear map and the surface $\Pi = \{p = 0\}$ being an algebraic sphere. However, this case differs radically from the general one, because *the degrees of the polynomials $p_k(x) = p(P(\cdots(P(x))\cdots))$* ($k$ times) *remain bounded as $k \to \infty$*, and hence the length of the corresponding chain of ideals can be bounded using linear algebraic tools only.

Besides the above mentioned reasons, the discrete Risler problem sometimes appears naturally in an attempt to solve the (original) Risler problem. Consider a *linear nonautonomous polynomial system* of differential equations having the form

$$\dot{x}(t) = A(t)x(t), \qquad A(t) = \sum_{j=0}^{d} A_j t^j, \;\; A_j \in \operatorname{Mat}_{n\times n}(\mathbb{R}), \;\; x \in \mathbb{R}^n.$$

Let $\Pi = \{p = 0\} \subset \mathbb{R}^n$ be a linear hyperplane. Any solution of this system can be expanded in a convergent Taylor series

$$x(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!}\, x_k, \qquad x_k \in \mathbb{R}^n, \;\; k = 0, 1, \dots.$$

Substituting this expansion into the equation, we obtain the recurrent formulas for the vector coefficients $\{x_k\}$: for each $k = d, d+1, \dots$ we have

$$x_{k+1} = A_0 x_k + kA_1 x_{k-1} + k(k-1)A_2 x_{k-2} + \cdots + k(k-1)\cdots(k-d+1)A_d x_{k-d},$$

that can be easily reduced to a first order linear difference scheme in $\mathbb{R}^{nd}$ with coefficients *polynomially depending* on the number k of the step. After the obvious suspension by the map $t \longmapsto t+1$ we see that the Risler problem on how many Taylor coefficients of the function $p(x(t))$ can vanish, gets reduced to the question on how many points of the orbit of the suspended polynomial map can belong to the (properly suspended) hyperplane.

This construction does not work in the nonlinear case. However, the linear systems constitute a very important class. In particular, one of the long standing open problems, that on the number of critical periods in a Newtonian system, can be reduced to a linear polynomial system [10, 25].

2. First Scene: Analysis

2.1. **Notations, language.** We introduce several notations and conventions. Recall that we denoted by $\mathbb{B}_R$ the centered closed box of size R in $\mathbb{R}^n$ or the centered polydisk in $\mathbb{C}^n$ (depending on context) defined by the inequalities $|x_j| \leqslant R$, $j = 1, \dots, n$.

Height of a polynomial. The *height* of a polynomial $p = \sum_{|\alpha| \leqslant d} p_\alpha x^\alpha$ in n variables with real or complex coefficients is the maximum of absolute values of the coefficients: $\mathcal{H}(p) = \max_\alpha |p_\alpha|$. Knowing the height and the degree is sufficient to place an upper bound for $|p(x)|$ on any box $\mathbb{B}_R$: estimating the number of monomials by $(d+1)^n$ and each monomial by $(1+R)^d$, we arrive to the inequality $\max_{x \in \mathbb{B}_R} |p(x)| \leqslant \mathcal{H}(p) \cdot (d+1)^n (1+R)^d$.

Parametrization of integral curves. We shall always assume that integral curves of differential equations (autonomous or not) are parameterized by the natural "time". It is very convenient to treat non-autonomous equations as autonomous, by adding the time variable t as one of the phase coordinates. On the other hand, for a system that was autonomous from the very beginning, a geometrically small piece of trajectory near a singualr point may be parameterized by a very long time interval. In order to reduce all restrictions to a common denominator, we shall always assume that the vector field is autonomous but *among the polynomial equations determining the vector field, one always has the equation* $\dot{x}_{n+1} = 1$, and this variable plays the role of time. Then the problem of counting *real* intersections will be always relative to a spatial box $\mathbb{B}_R$ of size R.

There is still a minor ambiguity in this formulation, since it may well happen that a trajectory $\Gamma\colon t \longmapsto x(t)$ leaves the box but eventually may return. To avoid this ambiguity, we will agree that all estimates apply only to *connected components* of integral curves, entirely belonging to $\mathbb{B}_R$.

This convention, while working well in the real context, may cause problems when dealing with complex zeros. Therefore the formulation of Theorem 2 is syntactically different from that of Theorem 1.

2.2. **Disconjugacy and oscillatory character of ordinary linear differential equations.** Consider a function satisfying a linear nth order differential equation with variable coefficients on a real or complex domain. There exists a direct connection between the magnitude of the coefficients of this equation and the number of isolated zeros of the function in this domain. This general principle admits several precise formulations, probably, the easiest of them being the following non-oscillation condition. Recall that a linear equation of order ℓ is called *non-oscillating*, or *disconjugate* on an interval or in a domain, if any nonzero solution has at most $\ell - 1$ isolated zeros on the interval (resp., in the domain), counted with their multiplicities. Note that one can always construct a solution with a root of multiplicity $\ell - 1$ at any point, so this bound cannot be improved.

Lemma 2.1. *Let $I = [t_0, t_1] \subset \mathbb{R}$ be a real interval of length $r = t_1 - t_0$ and $f(t)$ a C^n-smooth function satisfying a linear equation*

$$y^{(\ell)} + a_1(t)\, y^{(\ell-1)} + \cdots + a_{\ell-2}(t)\, y'' + a_{\ell-1}(t) y' + a_\ell(t) y = 0 \tag{2.1}$$

with real bounded coefficients: $|a_k(t)| < c_k < \infty$ *for all* $t \in I$.

If $c_\ell r^\ell + c_{\ell-1} r^{\ell-1} + \cdots + c_1 r < 1$, then f may have no more than $\ell - 1$ real isolated zeros on I, counted with multiplicities.

Proof. Assume that f has at least ℓ isolated root on I. Then by the Rolle theorem, each derivative $f', f'', \dots, f^{(\ell-1)}$ must have at least one root on the interval I.

If $f^{(\ell)} \equiv 0$, then f is a polynomial of degree $\leqslant \ell - 1$ and the claim is obviously true. Otherwise without loss of generality we may assume that $|f^{(\ell)}| \leqslant 1$ on I, and the equality $|f^{(\ell)}(t_*)| = 1$ holds at some point t_*. Then by the mean value theorem, we have the estimate $|f^{(k)}(t)| \leqslant |f^{(k)}(a)| + \int_a^t |f^{(k+1)}(\tau)|\,d\tau$, where a is any point in I. Choosing a being one of the roots of $f^{(k)}$ (at least one such root must exist), we arrive to the inequality $\max_I |f^{(k)}(t)| \leqslant r \max_I |f^{(k+1)}(t)|$ for all $k = \ell-1, \ell-2, \dots, 2, 1$, which implies that $\max_I |f^{(k)}(t)| \leqslant r^{\ell-k}$. After substitution to the equation (2.1) we arrive to the contradiction: at the point t_* the leading term of the equation overweights all the rest, making the equality impossible. $\square$

Breaking an arbitrary finite real interval on sufficiently short subintervals satisfying the above lemma, we obtain the following corollary.

Corollary 1. *If all coefficients of the equation* (2.1) *are bounded by the common constant C on I, then any nontrivial solution cannot have more than $\ell(\ell r C + 1)$ isolated zeros there.* $\square$

One can improve substantially the dependence of this upper bound on n by using a more sophisticate way of restoring the given function from its known nth derivative and the position of n roots of the function *itself*, as in [15]. The construction therein allows also to establish a sufficient condition also for disconjugacy in some complex domains (say, open disks), which in turn implies an upper bound for the number of isolated complex zeros in terms of the magnitude of the coefficients of the equation.

Lemma 2.2 (W. J. Kim [15])**.** *A linear equation* (2.1) *with coefficients analytic in some convex domain $\mathcal{D}$ of diameter r in the complex plane, is disconjugate in this domain provided that*

$$c_\ell \,\frac{r^\ell}{\ell!} + \dots + c_1 \,\frac{r}{1!} \leqslant 1, \qquad c_k = \sup_{k \in D} |a_k(t)|. \tag{2.2}$$

Note that this result is stronger than Lemma 2.1 for equations with real analytic coefficients.

Another approach, established for real analytic functions in [13] and for general analytic case in [22]. The estimates established in these papers are sometimes better, sometimes worse than those the bounds for the number of zeros, implied by Lemma 2.2. However, all these inequalities yield asymptotically equivalent answers to our original problem, therefore we use the simplest of them.

2.3. **Derivation of the quasilinear equation.** The above claims concerning high order linear differential equations, do not admit straightforward generalizations for curves defined by *systems* of (first order) differential equations. However, one can prove similar results by deriving *quasilinear* equations. Our first statement is almost obvious, and follows directly from the most general principles of analytic geometry. However, we give a detailed proof, since the same construction will be later supplied with additional details and used in the proof of our main result. Thus the second proof below is a core of the analytic section of the paper.

Let $v = v(x) = (v_1(x), \dots, v_n(x))$ be an autonomous vector field (real) analytic in the (closed) box $\mathbb{B}_R = \{|x_j| \leqslant R\}$ in the phase space, and let $\Pi \subset \mathbb{R}^n$ be an analytic hypersurface given by the equation $\{p(x) = 0\}$, where p is also analytic in $\mathbb{B}_R$.

Proposition 1. *The number of isolated intersections between an integral trajectory Γ of the vector field v and the hypersurface Π in the box $\mathbb{B}_R$ is uniformly bounded over all integral trajectories of time length not exceeding $2R$.*

First proof. Consider the flow map $(t,x) \longmapsto \phi(t,x)$ which is real analytic on some box of the form $(-\varepsilon,\varepsilon) \times \mathbb{B}_R$. The number of isolated roots of the equation $p(\phi(t,x)) = 0$ is bounded uniformly in x by the general Gabriélov principle [6]. $\square$

Second proof. Let A_R be the ring of functions analytic on the (closed) box $\mathbb{B}_R$. Denote by L the Lie derivation of the ring A_R along the field v, and consider the ascending chain of ideals spanned by the first derivatives of the original (seed) function $p = p_0 \in A_R$:

$$\begin{gathered}(p_0) \subset (p_0, p_1) \subset \cdots \subset (p_0, \dots, p_k) \subset \cdots, \\ p_0 = p, \quad p_{k+1} = Lp_k,\ k = 1, 2, \dots, \quad L = \sum_{j=1}^{n} v_j \partial_{x_j}.\end{gathered} \tag{2.3}$$

Since the polydisk $\mathbb{B}_R$ is semianalytic and compact, the ring A_R is Noetherian [4, 28], and hence the chain (2.3) of ideals in this ring must stabilize. Thus at a certain moment $\ell < \infty$ we shall have the identity

$$p_\ell = \sum_{k=0}^{\ell-1} h_k p_k, \qquad h_k \in A_R. \tag{2.4}$$

Restricting the identity (2.4) on an arbitrary trajectory $\Gamma\colon t \longmapsto x(t)$ of the field v, we see that the function $y(t) = p_0(x(t))$ satisfies the linear ordinary differential equation (depending, in general, on the choice of the curve Γ),

$$y^{(\ell)} = \sum_{k=0}^{\ell-1} a_k(t) y^{(k)}, \qquad a_k(t) = h_k(x(t)) \iff a_k = h_k|_\Gamma. \tag{2.5}$$

Since the norm $C = \max_{k=0,\dots,\ell-1} \max_{x \in \mathbb{B}_R} |h_k(x)| < \infty$ is finite, we have an upper bound for the coefficients of this equation, which by Corollary 1 implies that all solutions of it have a bounded number of isolated roots. As this *bound* is independent of the choice of Γ *inside the box* $\mathbb{B}_R$, our claim is proved. $\square$

This result concerns intersections with a fixed hypersurface only. However, a minor modification allows to extend the above finiteness for parametric families of analytic hypersurfaces, in particular, for all affine hyperplanes.

Corollary 2. *Let the field v and the function p depend analytically on additional variables $z = (z_1, \dots, z_m)$ also satisfying the inequalities $|z_j| \leqslant R$. Then the number of intersections between any integral curve Γ and any hypersurface $\Pi_z = \{p(x,z) = 0\}$ in the box $\mathbb{B}_R$ is uniformly bounded over all curves, vector fields and hypersurfaces.*

Proof. The first proof needs almost no modifications to incorporate this case as well. In the second proof one has to consider z_j as new dependent variables governed by the trivial equations $\dot z_j = 0$, and count intersections with the "universal" hypersurface $\{(x,z) \colon p(x,z) = 0\}$ in the extended phase space. $\square$

Assuming that the vector field v is analytic in the whole space $\mathbb{R}^n$, one can prove the existence of a finite bound for the number of intersections in any finite size box $\mathbb{B}_R \subset \mathbb{R}^n$, yet the bound may grow arbitrarily rapidly as R grows. In the polynomial case one may guarantee that this growth is at most polynomial.

2.4. **Polynomial growth.** Assume now that v is a polynomial vector field and Π a polynomial hypersurface. Then the number of isolated intersections in the box $\mathbb{B}_R$ may grow at most polynomially as $R \to \infty$.

Lemma 2.3. *Let $v = (v_1, \dots, v_n)$ be a polynomial vector field, $v_j \in \mathbb{R}[x]$, $j = 1, \dots, n$, and $\Pi = \{p = 0\} \subset \mathbb{R}^n$ an algebraic hypersurface, $p \in \mathbb{R}[x]$.*

Then the number of interscetions between Π and orbits of v in the box $\mathbb{B}_R$ may grow at most polynomially as $R \to \infty$:

$$\#(\Gamma \cap \Pi \cap \mathbb{B}_R) \leqslant (2+R)^B, \qquad B = B(v, \Pi) < \infty. \tag{2.6}$$

Proof. We will adjust the second proof of the Proposition above. Consider once again the chain (2.3) of ideals: since all data were polynomial, all ideals in this chain belong to the ring $\mathbb{R}[x]$. Since this ring is also Noetherian (this assertion is much simpler than the Noetherian property of the ring A_R above), the ascending chain of ideals (2.3) must stabilize at a certain moment ℓ, which means that the same identity (2.4) but now with polynomial coefficients $h_k \in \mathbb{R}[x]$ holds between the first iterations $p_0, \dots, p_\ell$. Let D be the maximum of the degrees of h_k and H the maximum of their heights:

$$D = \max_{k=0,\dots,\ell-1} \deg h_k, \qquad H = \max_{k=0,\dots,\ell-1} \mathcal{H}(h_k). \tag{2.7}$$

Note that both constants are finite and depend only on v and Π, and not on the choice of Γ or the size R of the box.

Knowing H and D is sufficient to place an upper bound for the modulus $|h_k|$ on any box $\mathbb{B}_R$:

$$\forall k = 0, \dots, \ell, \ \forall x \in \mathbb{B}_R \qquad |h_k(x)| < C = H(D+1)^n \cdot (1+R)^D. \tag{2.8}$$

In the same way as before, the restriction $p_0|_\Gamma$ satisfies the same linear ordinary equation (2.1), with the coefficients bounded by the expression (2.8), polynomial in R. Substituting this inequality for C into the bound provided by Corollary 1, we arrive to the claim. □

In the same way as before, one can extend this result for intersections with algebraic hypersurfaces of some fixed degree: it is sufficient to consider the coefficients of the equations defining these surfaces, as additional variables governed by trivial equation. This observation immediately proves the following claim (in more details discussed in the next section).

Corollary 3. *Consider all polynomial vector fields in $\mathbb{R}^n$ of some degree d and all algebraic hypersurfaces of degree $\leqslant d$. The number of isolated intersections between any integral trajectory and any hypersurface in the box of size R grows at most polynomially with the height of the vector field and the size of the box. The exponent of growth depends on d and n only.* □

Remark . The number of intersections may indeed grow polynomially (an not just linearly) in R, as the simplest planar example $\dot z = iz|z|^q$ shows: the larger q, the more rapidly grows the number of intersections, say, with the real axis.

3. Interlude: the importance of being integral

The result of Lemma 2.3 provides an upper bound uniform over all (boxed) trajectories of a *fixed* differential equation. The Corollary 3 shows how one can derive from it the bound that would be uniform over all fields of bounded degree and height.

We discuss the "universalization" trick here in more details and show how the problem of estimating the exponent naturally reduces to bounding *only* the lengths of ascending chains of ideals.

3.1. **Universal equation with integral coefficients.** Recall that we consider now only the autonomous differential equations with one of the variables being governed by the equation $\dot{x}_n = 1$. Declare the coefficients $v_{jk\alpha}$ in (1.1) and $p_{k\alpha}$ in (1.2) to be the new dependent (state) variables governed by the trivial equations

$$\frac{d}{dt}v_{jk\alpha} = 0, \quad \frac{d}{dt}p_{k\alpha} = 0, \qquad j = 1, \dots, n, \quad 0 \leqslant k + |\alpha| \leqslant d. \tag{3.1}$$

Then (1.1) becomes a system of autonomous polynomial differential equations with coefficients taking only the values 0 and 1. This system (one for any choice of n and d) will be later referred to as the *universal system* (or universal vector field) and denoted by $U_{n,d}$. The same applies also to the equation defining the hypersurface Π: after the extension it becomes a universal surface $\Pi_{n,d}$ in the phase space of the universal vector field $U_{n,d}$.

3.2. **Case of binary coefficients: formulation of the result.** Any specified choice of the coefficients in the original system (1.1) and the hypersurface (1.2) is tantamount to specifying the additional state variables along the trajectory Γ. The assumption that the magnitude of the coefficients is bounded by R, transforms into the assumption that the corresponding integral curve is again a subset of the box with side $2R$ in the phase space of the system (and this is the additional reason why the same bound was imposed as a restriction on the coefficients and the geometric dimensions).

The result we need (Theorem 1) is a corollary to the following Theorem. Consider an arbitrary polynomial vector field v in $\mathbb{R}^n$ of degree $\leqslant d$ with zero-one coefficients as above (and hence of height 1) and an arbitrary polynomial hypersurface also with zero-one coefficients and of degree $\leqslant d$. As usual, we assume that one of the variables is governed by the equation $\dot{x}_n = 1$.

Theorem 4. *The number of isolated intersections between any integral curve $\Gamma \subset \mathbb{R}^n$ of the zero-one polynomial vector field of degree $\leqslant d$ and any zero-one polynomial hypersurface of degree $\leqslant d$, counted with multiplicities in the box $\mathbb{B}_R$, satisfies the estimate* (1.3)*, in which $B = B(n, d)$ is a primitive recursive function satisfying the estimate*

$$B(n, d) \leqslant \exp\exp\exp(n^3 + d + O(1)). \tag{3.2}$$

Reduction of Theorem 1 to Theorem 4. We apply Theorem 4 to the universal vector field $U_{n,d}$ and the universal hypersurface $\Pi_{n,d}$ (notice that the meaning of the variable n is changed).

Interpretation of parameters as state variables increases largely the dimension of the phase space of the problem and (inessentially) its degree. For our purposes it will be sufficient to estimate the number of monomials of degree $\leqslant d$ in $n + 1$ variables by $(d+1)^{n+1}$, and coefficients occurring in different equations should be counted separately, so that the "new" dimension will be $\leqslant (n+1)+(n+1)(d+1)^{n+1}$. The "new" degree will be $d+1$ (all "coefficients" $v_{jk\alpha}, p_{k\alpha}$ have degree 1), and substituting these expressions in the estimate (3.2), we obtain (after returning to the initial notation) the estimate (1.4).

The rest of this section contains the proof of Theorem 4 modulo the principal result on the lengths of ascending chains of ideals, Theorem 6, proved in §5. The computations establishing the asymptotics (3.2), are given in the Appendix C.

3.3. **The advantage of integrality.** Note that, given any particular combination of n and d, there is a unique universal vector field $U_{n,d}$ and therefore the unique derivation $L = L_{n,d}$ generating the chain (2.3) from the universal polynomial p. Thus the stabilization length ℓ, the maximal degree D and the maximal height H are also uniquely defined by these data (i.e. n and d only). In this section we show that in fact the length is all what one needs to place a primitive recursive upper bound on H and D and hence prove Theorem 4 by using the explicit information in (2.8).

Lemma 3.1. *Let L be the derivation of the polynomial ring $\mathbb{R}[x_1,\dots,x_n]$ with polynomial coefficients of height 1 and degree d, and p_0 a polynomial of height 1 and degree d in this ring. Consider the chain of ideals* (2.3) *and let ℓ be the length of this chain.*

Then one can find a combination (2.4) *in such a way that all the polynomials h_k, $k=0,1,\dots,\ell-1$ would satisfy the inequalities:*

$$\begin{aligned} &\deg h_k \leqslant D = D(n,d,\ell) \leqslant (\ell d)^{2^n}, \\ &\mathcal{H}(h_k) \leqslant H = H(n,d,\ell), \end{aligned} \qquad \forall k = 0,1,\dots,\ell-1, \tag{3.3}$$

where $D(n,d,\ell)$ and $H(n,d,\ell)$ are primitive recursive (*in fact, elementary*) *functions of their arguments.*

Proof. One can easily see that the degrees of the polynomials p_k grow linearly with k, i.e. $\deg p_k \leqslant (k+1)(d-1)+1 \leqslant \ell d$ for all $k=0,1,2,\dots,\ell-1$. Knowing ℓ, this is already sufficient to determine the degrees of the multipliers h_k: by the famous inequality due to Greta Hermann [12] shown to be essentially sharp by Mayr and Meyer [19],

$$\deg h_k \leqslant \deg p_\ell + 2(2\max_k \deg p_k)^{2^{n-1}} \leqslant (\ell d)^{2^n} \qquad \text{for all } k. \tag{3.4}$$

To place an upper bound for the height of the multipliers h_k, we use the integrality of all polynomials p_k. Since the upper bound for $\deg h_k$ is already known (denote it by D), the polynomials themselves can be written as $h_k = \sum_{|\alpha|\leqslant D} h_{k\alpha}x^\alpha$. The identity (2.4) becomes then a linear non-homogeneous system of algebraic equations with respect to the collection $\{h_{k\alpha}\colon 0\leqslant k\leqslant \ell-1,\ |\alpha|\leqslant D\}$ of $N\leqslant \ell(D+1)^{n+1}$ unknowns. We can write this system symbolically as $Ah=b$, where A is a huge (though not exceeding $N\times N$) rectangular matrix and b an N-dimensional column vector.

This system is known to admit at least one solution, therefore one can use the Cramer rule to produce it. According to this rule, the solution can be expressed as a ratio of two nonzero minors of the extended matrix $(A\,|\,b)$. Now note, that all entries of this extended matrix are integer numbers explicitly bounded from above.

This bound is a primitive recursive function of n, d and ℓ, as one can easily see after writing the recurrent equation

$$\mathcal{H}(p_{k+1}) \leqslant \deg p_k \times n(\deg p_k + 1)^n (d+1)^n \times \mathcal{H}(p_k) \tag{3.5}$$

for all $k = 0, \dots, \ell - 1$. The first multiplier comes from computing the partial derivative $\partial_{x_j} p_k$, the second term majorizes the number of monomials when reducing similar terms in the products $v_j \partial_{x_j} p_k$ and adding them together, and the last multiplier is equal to the height of p_k, since the height of all v_j is 1.

The upper bound for $\mathcal{H}(p_k)$ provides an upper bound for the numerator of the ratio in the Cramer rule, as the size of the corresponding minor is at most N, and all entries are already bounded. On the other hand, the denominator of this ratio is a nonzero *integer* number, hence at least 1 in absolute value.

Altogheter, these considerations give us a possibility of constructing the decomposition (2.4) together with an explicit bound for $H = \max_{k=0,\dots,\ell-1} \mathcal{H}(h_k)$ in terms of ℓ. □

Remark . This proof uses explicitly for the first and the last time the integrality of the coefficients. Without this integrality one would be unable to place any upper bound on the heights of the multipliers h_k.

3.4. **Proof of Theorem 4 modulo Theorem 6.** By Lemma 3.1 the proof of Theorem 4 is reduced to the following problem: *place an upper bound for the length of ascending chain of ideals generated by consecutive derivatives* (2.3) *in the ring of polynomials in n variables*. This question is addressed in the subsequent sections, where it is proved that the length of the chain (2.3) generated by consecutive derivatives, is majorized by a primitive recursive function $\ell = \ell(n, d)$ of n and d (Theorem 6).

The inequalities from Lemma 3.1 show that in this case the degrees and the heights of the polynomials h_k occurring in the identity (2.4), also admit primitive recursive majorants.

Restricting the identity (2.4) on any integral curve of the universal system $U_{n,d}$ belonging to the box $\mathbb{B}_R$ yields an upper bound for the absolute values of this restriction in the form $(2+R)^{\widetilde{B}(n,d)}$, where $\widetilde{B}$ is some primitive recursive majorant.

Applying Corollary 1 to the corresponding differential equation (2.5), we conclude with the required upper bound for the number of intersections in the universal system, that would be polynomial in R and primitive recursive in n and d.

The computations proving that the growth of the exponent B is as asserted, are straightforward but cumbersome, and are collected in Appendix C. □

3.5. **Proof of Theorem 2.** To prove this result, it is sufficient to establish a similar "disconjugacy" property for complex solutions of the universal equation $U_{n,d}$. In the same way as in the real settings, we derive a polynomial identity (2.4) with bounded degrees and heights of all terms.

Let a be the reference point in the box $\mathbb{B}_R$ of the phase space of the universal system (this point is obtained by the obvious manipulations from the reference point (x_0, t_0) of the original polynomial system). The complex solution Γ passing through the reference point can blow up in a finite time (exhibit singularities), but the distance to these singularities can be easily estimated from below: along trajectories of the universal field $U_{n,d}$ the polar radius r grows at most as the solution of the equation $\dot r = C(n, d) r^{d+1}$, where C is a simple expression (the number of terms, since each term comes with the coefficient 1). Thus the trajectory never leaves the box $\mathbb{B}_{2R}$ for all $|t| < \rho_0 = (C'(n,d)R^d)^{-1}$ (assuming that $t = 0$ corresponds to the reference point, as the system is autonomous).

Restricting the identity (2.4) on the part of $\Gamma \subset \mathbb{B}_{2R}$ parameterized by this small disk $\{|t| < \rho_0\}$, we obtain a linear differential equation with bounded coefficients (in the same way, as before—all those arguments were independent on the ground field). It remains only to choose $\rho \ll \rho_1$ in such a way that the inequality of Lemma 2.2 be satisfied. Then the equation (2.1) restricted on this small disk, is disconjugate and hence has no more than ℓ roots. Note that ℓ is now very large, and not equal to the ambient dimension. The computations in this case remain the same as in the real case. $\square$

4. Second Scene: Geometry

In this section we consider the geometric version of the problem on the length of chains, namely, that concerning *descending chains of algebraic varieties*. This question arises not only in connection with the discrete Risler problem §1.12, but also clarifies the constructions used in the analysis of a more elaborate case, that of ascending chains of ideals in §5.

4.1. **Descending chains of varieties: the settings.** Let $p_0, p_1, \dots, p_k, \dots$ be a sequence of polynomials in $\mathbb{C}[x] = \mathbb{C}[x_1, \dots, x_n]$, and let $X_k = \{x \in \mathbb{C}^n : p_0(x) = \cdots = p_k(x) = 0\} \subset \mathbb{C}^n$, $k = 0, 1, 2, \dots$ be the corresponding zero loci. By construction, they constitute the *descending chain of algebraic varieties*:

$$\mathbb{C}^n \supset X_0 \supset X_1 \supset \cdots \supset X_k \supset \cdots, \qquad X_k = \bigcap_{j=0}^{k} \{p_j = 0\}. \tag{4.1}$$

Since the ring $\mathbb{C}[x]$ is Noetherian, the chain must stabilize (which means that $X_\ell = X_{\ell+1} = \cdots = X_{\ell+k} = \cdots$ for some $\ell < \infty$). The problem is to determine the moment ℓ.

We shall solve this problem in two different settings, namely:

1. assuming nothing about the polynomials p_k except for the upper bounds on their degrees, we suggest a simple algorithm for estimation of the length of a *strictly ascending* part of the chain. However, for the case when the degrees are growing with k, this algorithm in general yields an upper bound that is not primitive, but rather a general recursive function that grows like the Ackermann exponential;
2. assuming that the polynomials p_k are produced by iterations of a ring homomorphism, we establish a primitive recursive upper bound for the length of the descending chain (i.e. until the moment of complete stabilization).

In both cases the solution is obtained by monitoring the number and dimensions of *irreducible components* of the varieties in the chain.

4.2. **Irreducible decomposition of algebraic varieties: a reminder.** Recall that an algebraic variety $X \subset \mathbb{C}^n$ is irreducible if it cannot be represented as a nontrivial union of two proper algebraic subvarieties. Any algebraic variety X in $\mathbb{C}^n$ admits a *unique* representation as an irredundant union of irreducible algebraic varieties, $X = X_1 \cup \cdots \cup X_s$ (the irredundance means that neither X_j belongs to the union of the remaining components). Thus it makes sense to speak about the number of irreducible components of a variety.

An irreducible variety X has one and the same *dimension* at any smooth point. Moreover, the smooth points of X constitute a connected stratum of its Whitney stratification, and any proper subvariety of an irreducible variety X has the dimension *strictly inferior* to $\dim X$.

The number of irreducible components of an algebraic variety defined as the zero locus of some number of polynomials in $\mathbb{C}^n$, can be estimated in terms of the degrees of the polynomials. The most known case is that covered by the *Bézout theorem*: if n polynomial equations of degrees $d_1, \dots, d_n$ in n variables define a zero-dimensional variety (a finite number of isolated points), then this number is *equal* to the product $d_1 \cdots d_n$ provided that the points are counted with their *multiplicities in* $\mathbb{C}P^n$. The difficult part of this result is to prove that the equality holds. A more simple inequality-type result can be used to estimate the number of components of all dimensions.

Proposition 2. *Assume that an algebraic variety in $\mathbb{C}^n$ is defined by any number of polynomial equations of degree $\leqslant d$, and has an irreducible decomposition with $m_0 \geqslant 0$ isolated points, $m_1 \geqslant 0$ one-dimensional varieties, ..., $m_{n-1} \geqslant 0$ irreducible $(n-1)$-dimensional components.*

Then $m_{n-1} + m_{n-2} + \cdots + m_{n-k} \leqslant d^k$ for all $k = 1, \dots, n$. In particular, the number of isolated points of any such variety, regardless of its dimension and the number of determining equations, does not exceed d^n.

Note that the *number* of equations can be arbitrary: only their degrees (and the ambient dimension) do matter. The proof of Proposition 2 will be given in the Appendix §B.1. This Proposition is almost identical to the result by J. Heintz [11], published considerably earlier.

4.3. **Descending chains of algebraic varieties without additional assumptions and appearance of the Ackermann-type bounds.** Consider the descending chain of varieties (4.1) under the following assumptions:

1. the chain is strictly descending, i.e. $X_k \supsetneq X_{k+1}$ for all $k = 0, 1, \dots, \ell$;
2. the degrees of the polynomials grow at most exponentially: $d_k = \deg p_k \leqslant d^k$.

Proposition 3. *The length of any descending chain of algebraic varieties, satisfying the above two assumptions, is bounded by a general recursive function.*

Proof. Together with the chain of varieties (4.1), consider the associated sequence of integral vectors $w_k \in \mathbb{Z}^n_+$,

$$w_k = (w_k^{n-1}, w_k^{n-2}, \dots, w_k^1, w_k^0), \qquad k = 0, 1, \dots, \ell,$$
$$0 \leqslant w_k^r = \text{the number of } r\text{-dimensional irreducible components in } X_k.$$

Then one can easily see that *the sequence $\{w_k\}$ is strictly lexicographically decreasing*. This means that for every k we have for some r between $n-1$ and 0 the relations $w_{k+1}^{n-1} = w_k^{n-1}$, $w_{k+1}^{n-2} = w_k^{n-2}$, ..., $w_{k+1}^{r+1} = w_k^{r+1}$, and $w_{k+1}^r < w_k^r$. In the geometric terms, addition of each new polynomial equation $\{p_{k+1} = 0\}$ to the previous ones preserves all irreducible components of dimensions $> r$ and destroys some r-dimensional components. (The equality $w_{k+1} = w_k$ is impossible, since it would imply the equality $X_{k+1} = X_k$).

What happens with the irreducible components of dimensions r and below, is not determined by the monotonicity alone, since an r-dimensional component after intersection with the hyperpsurface $\{p_{k+1} = 0\}$ can give rise to a large number of irreducible varieties of inferior dimension(s). Yet if the degrees of the equations grow with k in some controllable fashion, say, as d^k, then by Proposition 2 (the Bézout inequality), we have for all k

$$|w_k| = w_k^{n-1} + \cdots + w_k^0 \leqslant (d^k)^n = M^k, \qquad M = d^n. \tag{4.2}$$

In general, the length of any chain of words w_k that are lexicographically decreasing, must be finite (an analog of the Noetherian property), but no bound can be given. Under the additional constraint $|w_k| \leqslant M^k$ (or any other explicit control of the growth of $|w_k|$), this length can be estimated by a solution of a recurrent equation.

To derive this equation, notice that any alphabetic (lexicographically decreasing) sequence $\{w_k\}$ can be subdivided into "dictionary sections" each section embracing the words from the sequence, starting with the same letter (the corresponding varieties all having the same number of upper-dimensional components). Denote by $L(n,s,k)$ the maximal length of an alphabetic sequence that begins with a "word" of size $s = |w_0|$, in dimension n (equal to the length of the words) and containing no more than k "dictionary sections" (on the tacit assumption that the growth is as in (4.2)). Then the maximal length of the sequence that begins with the word of size s is $L(n,s,s)$ (all s components may be initially upper-dimensional, hence the number of "dictionary sections" can be as large as s). Moreover, the length of the "dictionary section" that begins at some step q, cannot be larger than $\leqslant L(n-1, M^q, M^q)$, since the remaining $n-1$ letters of all words in this section can form any alphabetic sequence, and M^q is (the majorant for) the size of the initial word in this sequence. Thus we have the recurrent rule

$$L(n,s,k+1) \leqslant L(n,s,k) + L(n-1, M^q, M^q), \qquad q = L(n,s,k), \tag{4.3}$$

which describes completely the function L. This rule has the same nature as the rule defining the Ackermann exponential, since the right hand side of (4.3) contains the function L applied to "itself". □

It is clear how the longest possible chain can be obtained. The polynomial p_{k+1} should be chosen in such a way that the corresponding word w_{k+1} be different from w_k in the lowest possible dimension $r = r_k$ and the impact on the components of dimension $j-1$ should be maximal compatible with the degrees. More formally, we assume that:

1. $w^r_{k+1} < w^r_k$ only if $w^{r-1}_k = \cdots = w^0_k = 0$, and in this case
2. $w^r_{k+1} = w^r_k - 1$ and w^{r-1}_k is maximal compatible with the Bézout theorem.

According to this scenario, first all zero dimensional components (isolated points) should be removed. Then, when there are no such points, one of the 1-dimensional components (curves) will be "chopped" into a number of points; this number is rapidly growing with the number of the step on which this "chopping" occurs, as the degree of the corresponding polynomial p_k grows with k and the Bézout inequlities are essentially sharp. Then once again the points will be removed one by one and when the turn of the next curve comes, it gives rise to a much bigger number of points. After the initial supply of curves is exhausted, one of the 2-dimensional components will be chopped into a (very large already) number of curves, and the process repeats again on the new scale.

Clearly the length of the chain constructed in accordance with these rules, grows tremendously with the dimension of the original variety, and the resulting upper bound will be of the same type as the Ackermann generalized exponential. Of course, one should verify that the above rules can be indeed realized (in particular, that only one component is split into the maximal Bézout-compatible number of pieces while the rest are preserved after adding only one polynomial). We will not prove this circumstance (the corresponding examples can be easily produced starting from the example given in [20]), referring instead to the Moreno theorem below and hoping that the above considerations clarified sufficiently the reason why the Ackermann exponential appears in connection with the problem on ascending/descending chains.

4.4. **Iterations of polynomial maps.** The situation changes completely if the polynomial equations defining the descending chain of varieties (4.1) are produced by iterations of a homomorphism. Recall that each polynomial map $P\colon \mathbb{C}^n \to \mathbb{C}^n$ defines the ring homomorphism $P^*\colon \mathbb{C}[x] \to \mathbb{C}[x]$, $(P^*p)(x) = p(P(x))$, and vice versa.

In this section we analyze the chains of varieties generated using iterates of homomorphisms and prove that such chains must be strictly descending and, moreover, under an additional genericity-type assumption their lengths admit much better estimates than in the general case considered earlier.

Definition 1. A polynomial map $P\colon \mathbb{C}^n \to \mathbb{C}^n$ is called *dimension-preserving*, if the image of any semialgebraic k-dimensional variety is again k-dimensional.

If X and Y are two semialgebraic varieties with $P(X) \subset Y$, then the assumption that P is dimension-preserving, guarantees that $\dim X \leqslant \dim Y$.

Theorem 5. *Let $P\colon \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map of degree d, $p_0 \in \mathbb{C}[x] = \mathbb{C}[x_1,\dots,x_n]$ a polynomial of degree $\leqslant d$ and the sequence of polynomials $p_k \in \mathbb{C}[x]$ is defined using the rule*

$$p_{k+1} = P^*p_k \iff p_{k+1}(x) = p_k(P(x)), \qquad k = 0,1,2,\dots. \tag{4.4}$$

Then the descending chain of algebraic varieties built as in (4.1) *is strictly descending: $X_\ell = X_{\ell+1} \implies X_{\ell+k} = X_{\ell+k+1}$ for all k.*

Under the additional assumption that P is dimension preserving, the length of this chain is bounded by a primitive recursive function of n and d. For n and d large, this function grows no rapidly than the tower of n stories,

$$\ell(n,d) \leqslant \underbrace{M^{M^{\cdot^{\cdot^{\cdot^{M}}}}}}_{n \text{ times}}, \qquad M = 1 + d^n. \tag{4.5}$$

4.5. **Proof of Theorem 5: the paradigm.** We give a detailed proof of this theorem, since it will later be used as a prototype for a more technical proof of Theorem 6 in §5. It will be clear from the proof that the main reason why the Ackermann function appears as an upper bound in the general problem on descending chains of varieties, is the non-monotonicity of the dimensions of the deleted pieces. As soon as the dimensions of the deleted pieces need to be monotonous, the upper bound immediately becomes primitive recursive.

Dynamical interpretation of the problem. Consider the dynamical system in $\mathbb{C}^n$ defined by iterations of the polynomial map $P\colon x \longmapsto P(x)$. Note that all points have well-defined orbits $\{P^{[k]}(x)\}_{k=0}^{\infty}$. Let us start with the following observation: the locus X_k is the set of initial conditions $x \in \mathbb{C}^n$ such that the first k points of their respective orbits belong to the algebraic surface $X_0 = \{p_0 = 0\} \subset \mathbb{C}^n$. This is a trivial reformulation of the rules (4.1), (4.4).

Strict decrease. The difference $X_k \smallsetminus X_{k+1}$ consists of points that remain on X_0 during the first k steps of their life, and leave it on the $(k+1)$st step. The P-image of any such point will remain on X_0 for $k-1$ more steps and then leave it. This proves the inclusion

$$P(X_k \smallsetminus X_{k+1}) \subseteq X_{k-1} \smallsetminus X_k, \qquad k = 1,2,\dots. \tag{4.6}$$

Since the polynomial map is everywhere defined, the P-image of a nonvoid set is again nonvoid, which implies that a *nonempty difference* at a step k cannot be preceded by an empty one. In other words, the equality $X_k = X_{k+1}$ ensures that the rest of the chain immediately stabilizes.

Monotonicity of dimensions. If P is a dimension-preserving polynomial map, then from (4.6) one can immediately see that *the dimensions of the differences* $X_k \smallsetminus X_{k+1}$ *are monotonously non-increasing.* This observation implies that the chain (4.1) can be subdivided by some moments $k_{n-1} \leqslant k_{n-2} \leqslant \cdots \leqslant k_1 \leqslant k_0$ into n segments of finite length,

$$\begin{aligned} &\underbrace{X_0 \supset X_1 \supset \cdots \supset X_{k_{n-1}}}_{\dim X_k \smallsetminus X_{k+1} = n-1} \supset \underbrace{X_{k_{n-1}+1} \supset \cdots \supset X_{k_{n-2}}}_{\dim X_k \smallsetminus X_{k+1} = n-2} \supset \cdots \\ &\qquad \cdots \supset \underbrace{X_{k_s+1} \supset \cdots \supset X_{k_{s-1}}}_{\dim X_k \smallsetminus X_{k+1} = s} \supset \cdots \supset \underbrace{X_{k_1+1} \supset \cdots \supset X_{k_0}}_{\dim X_k \smallsetminus X_{k+1} = 0} \end{aligned} \tag{4.7}$$

such that along the sth (from the right) segment the differences $X_k \smallsetminus X_{k+1}$, $k_s+1 \leqslant k \leqslant k_{s-1}$, are exactly s-dimensional semialgebraic varieties (some segments can be eventually empty).

The length of each such segment does not exceed the number of s-dimensional irreducible components in the starting set X_{k_s+1} of this segment, since this number must strictly decrease on each step inside the segment. Indeed, inside the segment all components of dimension $> s$ must be preserved, otherwise the difference will be more-than-s-dimensional. On the other hand, if all s-dimensional components are preserved on some step, this means that the difference $X_k \smallsetminus X_{k+1}$ is at most $(s-1)$-dimensional, and one starts the next segment.

Computation. It remains only to observe that the degrees $\deg p_k$ grow exponentially, $\deg p_k \leqslant d^{k+1}$, whereas the number of irreducible components can be estimated by Proposition 2 by the nth power of the maximal degree $(\deg p_k)^n = (d^n)^{k_s+1}$. Hence for the lengths $k_s - k_{s-1}$ we have the recurrent inequality, $k_{s-1} - k_s \leqslant (d^n)^{k_s+1}$ for downward going values of $s = n-1, \dots, 1, 0$ and the initial condition $k_n = d$ (the initial polynomial p_0 of degree d may have at most d factors). The solution of this recurrent inequality is majorized by the solution of a more simple one $k_{s-1} + 1 \leqslant M^{k_s+1}$, $M = d^n + 1$, that gives the tower of height n for $k_0 + 1$ with M on each level: thus for the length of the descending chain we obtain the required estimate (4.5). $\square$

Discussion. Roughly, the upper bound for the length of the descending chain (4.1) is given by the Ackermann function of level 4, and this is related to the exponential growth of degrees of p_k: if the degrees were growing linearly with k, then the result would be given by the Ackermann function of rank 3, or, more exactly, by a double exponential in n estimate, as in [5].

Note that within the hierarchy of Ackermann exponentials of different levels, the towers of exponents of any *fixed* height (2, 3 or 1000) all occupy an intermediate place between the Ackermann functions of levels 3 and 4, “closer” to the former one. Thus the difference between the tower of height 4 that occurs in Theorem 1 and that of height 2 typical for the (continuous) Risler problem, is “negligeable”.

4.6. **Digression: chains of ideals generated by iterated ring homomorphisms and Yomdin theorem.** Together with decreasing chains of varieties, one can consider the ascending chains of ideals generated by iterations of ring homomorphism, as in Theorem 5.

It turns out that, using the tools developed in the next section, we can sometimes place a primitive recursive bound on the length of such chains: if P^* is the ring homomorphism and $I_k = (p_0, \dots, p_k)$ the corresponding ideals, then in the same way as in Lemma 5.4 one can easily show that

$$P^*(I_{k-1} : I_k) \subset I_k : I_{k+1}$$

for all k. This immediately implies that the chain $\{I_k\}_{k=0}^{\infty}$ is *strictly* ascending, and under some rather mild additional assumptions the dimensions $\dim(I_{k-1} : I_k)$ are non-increasing. This is already sufficient to reproduce all proofs from §5 for such chain.

Starting from the moment of stabilization, one can produce upper bounds for the norm p_k of the form $|p_k(x)| \leqslant C\rho^k \cdot \max_{j=0,1,\dots,\ell-1} |p_j(x)|$ for some explicit C and ρ. Interpreting $p_k(x)$ as the Taylor coefficients of a certain analytic function (see §1.12) and using the results by M. Briskin and Y. Yomdin [2, 29], one can explicitly construct (at least in some particular but important cases) certain effective bounds for the number of zeros of this function.

5. Third Scene: Algebra

The geometric results obtained in the previous section, have their counterparts in the theory of polynomial rings. Our goal is to establish here an analog of Theorem 5 for ascending chains of ideals. From now on we will denote our principal ring $\mathbb{C}[x] = \mathbb{C}[x_1, \dots, x_n]$ by $\mathfrak{R}$.

Remark . Throughout this section we will always assume that the ground field is $\mathbb{C}$. This will not restrict the generality, since in the real case the polynomials p_k spanning the chain will be real, and taking the real part of the identity (2.4) would yield the similar representation with real multipliers $h_k \in \mathbb{R}[x]$.

Throughout this section, we will consider ascending chains of ideals of the form

$$\begin{aligned} &I_0 \subset I_1 \subset \cdots \subset I_k \subset \cdots \subset \mathfrak{R} = \mathbb{C}[x_1, \dots, x_n], \\ &I_{k+1} = I_k + (p_{k+1}), \qquad k = 0, 1, 2, \dots, \quad p_k \in \mathfrak{R}, \end{aligned} \tag{5.1}$$

under one of the two following sets of assumptions:

1. the polynomials p_k are arbitrary, and the only information available is an upper bound for their degrees: $\deg p_k \leqslant \phi(k)$, where ϕ is a certain explicit function (most often $\phi(k) = dk$ with some $d \in \mathbb{N}$);
2. the polynomials p_k are consecutive derivatives of the initial polynomial p_0 (the seed), as in (2.3):

$$p_{k+1} = Lp_k, \qquad L\colon \mathfrak{R} \to \mathfrak{R} \text{ a derivation of degree } d. \tag{5.2}$$

We start with the algebraic analog of Proposition 3: the constructive part of this result is due to A. Seidenberg, and the "negative" result showing that in general the bound on the length of ascending chains grows as the Ackermann exponential, belongs to G. Moreno.

5.1. Theorems by Seidenberg and Moreno.

The general question on how to estimate the length of *strictly* ascending chains of polynomial ideals, was studied by A. Seidenberg. He established an algorithm that inputs an integer-valued function $\phi(\cdot)$ (the bound for $\deg p_k$) and the number of variables n and produces the maximal possible length of an ascending chain of ideals. The following result is an algebraic counterpart of the geometric claim already established in Proposition 3.

Theorem (A. Seidenberg [27]). *Suppose that an ascending chain of ideals* (5.1) *in the ring of polynomials in n variables is obtained by adding polynomial generators p_k of degrees growing as a computable function of k.*

Then the length of this chain is an algorithmically computable (*recursive, but not necessarily primitive recursive*) *function of n.*

In particular, if the degrees of the polynomials grow linearly ($\deg p_k \leqslant dk$ for some d) or exponentially ($\deg p_k \leqslant d^k$), the length of the chain is an algorithmically computable function of n and d.

From this result and the arguments given in §3, it already follows that the exponent $B(n, d)$ in Theorem 1 is an algorithmically computable function of n and d. The problem is to describe this function explicitly and/or estimate its growth.

In an attempt to compute the length explicitly in the simplest case when the degrees grow linearly with k (the question raised by Teo Mora in 1991), G. Moreno discovered that the upper bound cannot be given by a primitive recursive function:

> Trying to bound ... the maximal degree attained, one could expect a doubly exponential growth, the traditional "resonable" complexity in Computer Algebra. Nothing of this kind: it grows so fast that it has "no complexity" ([20, p. 30]).

More precisely, in [20] G. Moreno considers the chain of homogeneous polynomials $p_d, p_{d+1}, \dots, p_k, \dots, p_\ell$, of degrees growing at a "minimal" nontrivial rate $\deg p_k = k$, starting with a polynomial p_d of degree d, such that p_k is not in the ideal generated by the preceding polynomials. Then the length ℓ of this chain, as a function of the number of variables, overtakes any explicit expression (and even any primitive recursive function). The following result is obtained in [20] (see Proposition II.1.1).

Theorem (G. Moreno [20])**.** *Let p_k be homogeneous polynomials in n variables of degree $d+k$ for some d. Then the length of the chain* (5.1) *does not exceed $A(n,2,d+2)-4$, where A is the Ackermann generalized exponential* (*see* §*1.9*)*, and this estimate is sharp.*

The reasons for the occurrence of the Ackermann exponential as an upper bound, are roughly the same as in §4 when studying descending chains of varieties.

Thus without additional assumptions on the ascending chain, it is impossible to majorize its length "in a closed form". The rest of this section is devoted to showing that the chains generated by consecutive derivatives, must stabilize inifinitely faster than in the general case.

5.2. **Formulation of the main result.** The main positive result of this section follows. Consider the chain (5.1) generated by the rule (5.2) and assume for simplicity that both the seed polynomial p_0 and the derivation L have the same degree d.

Theorem 6. *The length of any ascending chain of polynomials generated by iterated derivatives along a polynomial vector field as in* (5.1)–(5.2)*, is bounded by a primitive recursive function $\ell = \ell(n,d)$ of n* (*the number of variables*) *and d* (*the degree of the derivation L and the seed polynomial p_0*)*.*

As n and d are large, this function grows polynomially in d and "slightly more than doubly" exponential in n:

$$\ell = \ell(n,d) \leqslant d^{n^{O(n^2)}}. \tag{5.3}$$

Note that this bound, though awful from any computational point of view, is still very modest if measured against the scale of Ackermann exponentials of various levels, as described in §1.9.

The proof of this result is largely parallel to Theorem 5 and consists in monitoring components of the primary decomposition of the ideals constituting the chain. The source of additional difficulties is twofold: first, in the algebraic context one has to care about *multiplicities* of the components and, second and more troublesome, the construction should be modified to avoid explicit and implicit using of the *uniqueness* of the primary decomposition that is known to fail in general (in particular, this circumstance prevents one from speaking about the number of primary components).

5.3. **Prerequisites from the Commutative Algebra.**

Dimension. Any algebraic subvariety in $\mathbb{C}^n$ is a stratified set [18] that has a certain dimension. If $I \subset \mathfrak{R}$ is an ideal and $X = V(I) = \{x \in \mathbb{C}^n : p(x) = 0\ \forall p \in I\}$ its *zero locus*, then we put $\dim I$ be the (complex) dimension of its zero locus. This number (between 0 and n) can be given a purely algebraic description, known as *Krull dimension*.

Primary decomposition and its uniqueness. One of the basic results of commutative algebra, known as the *Lasker–Noether theorem* [30, Ch. IV, §4], asserts that any polynomial ideal $I \subset \mathfrak{R}$ can be represented as a finite intersection of *primary* ideals, $I = Q_1 \cap \cdots \cap Q_s$. Recall that an ideal Q is primary, if $pq \in Q$ and $p \notin Q$ implies that $q^r \in Q$ for some natural exponent r. The radical $\sqrt{Q} = \{q \in \mathfrak{R} \colon q^r \in Q\}$ is a prime ideal called the *associated prime*, and by the Nullstellensatz it consists of all polynomials vanishing on the zero locus $V(Q) \subset \mathbb{C}^n$.

The primary decomposition in general is *not* unique, even if we assume that it is *irredundant*. However, in an irredundant primary decomposition the primary components whose associated primes are minimal (i.e. contain no prime ideals associated with other components), are *uniquely* defined [30, Theorem 8, p. 211]. In particular, the *leading term*

$$\operatorname{l.t.}(I) = \bigcap_j \{Q_j \colon \dim Q_j = \dim I\}, \tag{5.4}$$

the intersection of all upper-dimensional primary components, is uniquely defined, since the corresponing prime ideals are minimal for dimensionality reasons. As an application of the uniqueness part we have the following simple fact on monotonicity of the leading terms (the proof is given in Appendix B.2).

Lemma 5.1. *Suppose that $J \subset J'$ are two polynomial ideals of equal dimensions with the leading terms*

$$\operatorname{l.t.}(J) = Q_1 \cap \cdots \cap Q_s, \quad \operatorname{l.t.}(J') = Q'_1 \cap \cdots \cap Q'_{s'} \tag{5.5}$$

(*as usual, the decomposition is assumed to be irredundant*).

Then each component Q'_j contains one of the components Q_i.

Multiplicity. The notion of multiplicity of an ideal is rather subtle. However, for our purposes it would be sufficient to use it only in a restricted environment, where the following construction works.

Let $I \subset \mathfrak{R}$ be an ideal and assume that $0 \in \mathbb{C}^n$ is an *isolated point* of its locus $V(I)$. Denote by $\mathfrak{m} = (x_1, \dots, x_n) \subset \mathfrak{R}$ the maximal ideal of the ring and let $\mathfrak{R}_\mathfrak{m}$ be the corresponding localization (the ring of rational fractions whose denominators do not vanish at the origin). Then I is *cofinite* at the origin, which means that the dimension of the quotient ring $\mathfrak{R}_\mathfrak{m}/I \cdot \mathfrak{R}_\mathfrak{m}$ over $\mathbb{C}$ is finite [1], i.e. $\mu_0(I) = \dim_\mathbb{C} \mathfrak{R}_\mathfrak{m}/I \cdot \mathfrak{R}_\mathfrak{m} < \infty$. The number $\mu_0(I)$ is called the *multiplicity* of I at the origin $0 \in \mathbb{C}^n$. In the similar way one may define the multiplicity $\mu_a(I)$ of any ideal I at any isolated point $a \in V(I)$ of its zero locus.

Definition 2. Let a be a regular (smooth) point of the zero locus of a polynomial ideal $I \subset \mathfrak{R}$ of some dimension r between 0 and n. Let Π be an affine subspace in $\mathbb{C}^n$ of codimension r, transversal to $V(I)$ at a, and L the corresponding ideal generated by r affine forms. Then the ideal $I + L$ is zero-dimensional at a and hence cofinite.

The *multiplicity* $\mu_a(I)$ of I at a is the multiplicity of $I + L$ at a (the complex dimension of the corresponding quotient algebra in the local ring). The multiplicity

$\mu(I)$ is the generic value of $\mu_a(I)$ (the minimum over all smooth points $a \in V(I)$), and this definition will be applied to primary ideals only.

The multiplicity of an ideal in $\mathfrak{R} = \mathbb{C}[x_1, \dots, x_n]$ generated by polynomials of degree $\leqslant d$, can be easily estimated from above: by virtue of the Bézout theorem, the multiplicity of such ideal never exceeds d^n.

The following simple property of multiplicities is proved in the Appendix, see §B.3. This property allows to control the length of ascending chains of primary ideals with the same associated prime: the multiplicities should strictly decrease along such a chain.

Lemma 5.2. *If $I \subset J$ are two non-equal primary ideals in $\mathfrak{R}$ with the same associated prime, then $\mu(I) > \mu(J)$.*

5.4. **Chains of ideals of constant uniform dimension.** The above two lemmas already imply an upper bound for the length of an ascending chain under rather specific restrictions. However, we will use this particular case as a building block for the general construction.

Let $\mathcal{P} = \{P_1, P_2, \dots, P_\nu\}$ be a finite collection of pairwise different prime ideals, $P_i \subset \mathfrak{R}$, of the same dimension m. Consider a *strictly* ascending (finite) chain of ideals

$$J_0 \subsetneq J_1 \subsetneq J_2 \subsetneq \cdots \subsetneq J_{\ell-1} \subsetneq J_\ell \tag{5.6}$$

under the additional assumption that *each ideal from this chain is an intersection of primary ideals with associated primes only from the predefined collection $\mathcal{P}$.* Denoting by Q_{kj} the primary component of J_k with the associated prime P_j (if there is no such component, we introduce a fictitious term $Q_{kj} = (1)$ to make our considerations uniform), we can write

$$J_k = Q_{k1} \cap \cdots \cap Q_{k\nu},$$
$$Q_{kj} \text{ is either } (1) \text{ or primary with } \sqrt{Q_{kj}} = P_j.$$

By the uniqueness part of the Noether–Lasker theorem, all ideals Q_{kj}, whether fictitious or not, are uniquely determined.

Lemma 5.3. *The length ℓ of the strictly ascending chain* (5.6) *does not exceed the number of primary components of J_0, counted with their multiplicities*:

$$\ell \leqslant \sum_{j=1}^{\nu} \mu(Q_{0j}).$$

Proof. This is an obvious corollary to Lemma 5.1 and Lemma 5.2. Indeed, by Lemma 5.1, the monotonicity of the chain (5.6) imples the monotonicity of all the chains

$$\begin{matrix} Q_{01} \subset Q_{11} \subset \cdots \subset Q_{\ell 1}, \\ Q_{02} \subset Q_{12} \subset \cdots \subset Q_{\ell 2}, \\ \vdots \quad\ \vdots \qquad\qquad \vdots \\ Q_{0\nu} \subset Q_{1\nu} \subset \cdots \subset Q_{\ell\nu}. \end{matrix}$$

Moreover, in every column k at least one inclusion must be strict (otherwise $J_k = J_{k+1}$). Since all ideals in each line are primary with the same associated prime (unless they become trivial), the assertion of Lemma 5.2 applied to each line, shows that the numbers $\mu(Q_{kj})$ are nonincreasing (by definition we put $\mu((1)) = 0$), and their sum $\sum_{j=1}^{\nu} \mu(Q_{kj})$ is *strictly* decreasing as k ranges from 1 to ℓ. Since all multiplicities are nonnegative, this last observation proves the claim. $\square$

5.5. **Convexity of chains generated by consecutive derivatives.** We prove now an extremely simple but important property of chains generated by consecutive derivations. Recall that the *colon ratio* of any two ideals $I, J \subset \mathfrak{R}$ is the ideal $I : J = \{q \in \mathfrak{R} : qJ \subset I\}$.

Definition 3. An ascending chain of ideals (5.1) is called *convex* if

$$I_{k-1} : I_k \subset I_k : I_{k+1} \qquad \forall k = 1, 2, \dots. \tag{5.7}$$

Lemma 5.4 (main). *Any chain of polynomial ideals $\{I_k\}$ generated by adding consecutive drivatives as in (5.1)–(5.2), is convex.*

Proof. Obviously, $I_k : I_{k+1} = I_k : (p_{k+1})$. If $q \in I_{k-1} : (p_k)$, then $qp_k = \sum_{i=0}^{k-1} h_i p_i$. Applying L, we conclude:

$$\begin{aligned} qp_{k+1} &= q \cdot Lp_k = L(qp_k) - Lq \cdot p_k \\ &= \sum_{i=0}^{k-1} Lh_i \cdot p_i + \sum_{i=1}^{k} h_{i-1} p_i + Lq \cdot p_k = \sum_{i=0}^{k} \tilde{h}_i p_i \in I_k, \end{aligned}$$

which implies that $q \in I_k : (p_{k+1})$. $\square$

Remark . If the property of a chain being ascending can be referred to as its monotonicity, then it would be only natural to refer to the property (5.7) as the *convexity* of chains. This remark justifies the terminology introduced above.

The convexity just established, immediately implies a number of very strong assertions concerning the chain.

Corollary 4. *A convex ascending chain is* strictly *ascending. In other words, if the chain* (5.1) *possesses the property* (5.7) *and $I_\ell = I_{\ell+1}$ for some ℓ, then $I_\ell = I_{\ell+1} = I_{\ell+2} = \cdots = \bigcup_{j=0}^{\infty} I_j$.*

Proof. If $I_\ell = I_{\ell+1}$, then $I_\ell : I_{\ell+1} = (1)$ (the unit ideal, i.e. the whole ring $\mathfrak{R}$), so by (5.7) all other colon ratios $I_{\ell+s} : I_{\ell+s+1}$ for any natural $s > 1$ are also trivial as they must contain the ideal (1). $\square$

Remark . The strict monotonicity of chains ideals is a very characteristic feature of the problem on counting intersections. There are many classical problems (probably, the most famous of them is the center problem) that also can be reduced to estimating lengths of ascending chains of polynomial ideals. However, in these problems the chains may include coinciding ideals (meaning non-strict ascent). This circumstance makes these problems much harder, in the contrast to the case studied in this paper.

Since ascending colon ratios must have non-decreasing dimensions, we have immediately the following fact.

Corollary 5. *The dimensions* $\dim(I_k : I_{k+1})$ *constitute a non-increasing sequence.* $\square$

The main result of this section, Theorem 6, is a direct corollary to Lemma 5.4 and the following result.

Theorem 7. *The length of any convex chain obtained by adding polynomials of degrees satisfying the inequalities* $\deg p_k \leqslant dk$, *is bounded by a primitive recursive function* $\ell = \ell(n,d)$. *This function grows as* (5.3) *as* n,d *grow.*

The proof of this theorem occupies the rest of the section: we first consider chains with additional restrictions that will be dropped in the sequel.

5.6. **Length of convex chains with strictly ascending leading terms.** Consider the ascending chain (5.1) with an additional assumption that the colon ratios have the same dimension as the ideals I_k themselves. We show then that the length of this chain is completely determined by the leading term of the first ideal in the chain.

Lemma 5.5. *Consider a finite strictly ascending chain of ideals has the form* (5.1). *Assume that all colon ratios have the same dimension as the starting ideal of this chain*:

$$\dim I_0 = \dim I_0 : I_1 = \cdots = \dim I_k : I_{k+1} = \cdots = \dim I_{\ell-1} : I_\ell. \tag{5.8}$$

Then the length ℓ *of this chain does not exceed the number of primary components of the leading term* $J_0 = \mathrm{l.\,t.}\,(I_0)$ *of the starting ideal* (*counted with multiplicities*).

Proof. This is a simple corollary to Lemma 5.3. Indeed, consider the chain of leading terms $\{J_k\}_{k=0}^{\ell}$, $J_k = \mathrm{l.\,t.}\,(I_k)$. Denote by m the common dimension of all colon ratios.

1. The chain $\{J_k\}$ is ascending by Lemma 5.1: all upper-dimensional primary components of I_{k+1} can be obtained by enlarging primary components of J_k (sometimes making them trivial).

2. Moreover, the chain $\{J_k\}$ is in fact *strictly* ascending and this is where we use the assumption on dimensions. Indeed, if $J_k = J_{k+1}$ for some k, then

$$p_{k+1} \in I_k + (p_{k+1}) = I_{k+1} \subset \mathrm{l.\,t.}\,(I_{k+1}) = J_{k+1} = J_k = \mathrm{l.\,t.}\,(I_k),$$

which means that p_{k+1} in fact belongs to all upper-dimensional primary components of I_k. Writing $I_k = J_k \cap R_k$, where R_k is the intersection of all primary components of dimension strictly inferior to m, we conclude that

$$I_k : I_{k+1} = I_k : p_{k+1} = (J_k : p_{k+1}) \cap (R_k : p_{k+1}) = (1) \cap (R_k : p_{k+1}) \supset R_k,$$

therefore $\dim I_k : I_{k+1} \leqslant \dim R_k < m$. This contradicts the second assumption.

3. It remains only to verify that the collection of prime ideals associated with all primary components of the ideals J_k, is non-expanding (in particular, contained in that of J_0). We note that the upper-dimensional associated primes can be detected as ideals of upper-dimensional irreducible components of the loci $X_k = V(K_k) \subset \mathbb{C}^n$. The chain of algebraic varieties X_k is descending, $X_k \supset X_{k+1}$.

Each irreducible upper-dimensional component of X_k either belongs to X_{k+1}, if p_{k+1} vanishes identically on this component, or becomes an algebraic variety of dimension strictly inferior to m after intersection with the hypersurface $\{p_{k+1} = 0\} \subset \mathbb{C}^n$. Thus the collections of irreducible upper-dimensional components of the varieties X_k are non-expanding as k grows from 0 to ℓ, and the same holds for the collections of prime ideals associated with the leading terms J_k.

4. All assumptions of Lemma 5.3 are thus verified for the chain $\{J_k\}$. Applying this lemma, one produces the required estimate. $\square$

5.7. **Revealed growth.** The next step is to get rid of the assumption that the dimension of the colon ratios coincides with that of the starting ideal in the chain in Lemma 5.5. Using the lemma below, one can transform any chain with a constant dimension of colon ratios, into another chain with strictly increasing leading terms.

Lemma 5.6. *Assume that the dimension of the colon ratios $I_k : I_{k+1}$ remains the same along the strcitly ascending chain* (5.1):

$$\dim I_0 : I_1 = \cdots = \dim I_{\ell-1} : I_\ell = m. \tag{5.9}$$

Consider any primary decomposition of the starting ideal I_0 written in the form $I_0 = I_0' \cap S$, where I_0' is the intersection of all primary components of dimension m and below, and S is the intersection of all primary components of I_0 of dimensions $\geqslant m+1$.

Then the chain built from I_0' using the same rule

$$I_{k+1}' = I_k' + (p_{k+1}), \qquad k = 0, 1, \dots, \ell-1, \tag{5.10}$$

will be strictly ascending, and $I_k = I_k' \cap S$.

Proof. It is sufficient to prove the identity $I_k = I_k' \cap S$, since the equality $I_k' = I_{k+1}'$ would imply $I_k = I_{k+1}$ that would contradict the strict monotonicity of the initial chain.

1. By induction we show that $p_{k+1} \in S$ for all $k = 0, 1, \dots, \ell-1$. Indeed, if this is not true for some k, then p_{k+1} would not belong to at least one primary component of S of dimension $m+1$ or more. But then the colon ratio will be at least $(m+1)$-dimensional, contrary to our assumption. Indeed, for a P-primary ideal $Q \subset \mathfrak{R}$ and any $p \in \mathfrak{R}$ we have the following alternative:

$$Q : (p) = \begin{cases} (1), & \text{if } p \in Q, \\ P\text{-primary}, & \text{if } p \in P \smallsetminus Q, \\ Q, & \text{if } p \notin P. \end{cases}$$

In all nontrivial cases the dimension is preserved.

2. The proof of the identity also goes by induction: $I_0 = I_0' \cap S$ by construction of I_0' and S. The justify induction step we use the modular law [30]: since $p_{k+1} \in S$ by the previous argument, we have $(p_{k+1}) \cap S = (p_{k+1})$ and hence

$$I_{k+1}' \cap S = (I_k' + (p_{k+1})) \cap S = I_k' \cap S + (p_{k+1}) = I_{k+1}.$$

This completes the proof of of Lemma 5.6. □

Corollary 6. *In a strictly ascending chain of ideals* (5.1) *the length of any segment corresponding to a constant dimension m of colon ratios, does not exceed the "number" of m-dimensional primary components of an arbitrary primatry decomposition of the starting ideal in this segment.*

The word "number" has to be used in the quotation marks since it depends, in general, on the choice of the primary decomposition. Needless to say, the components are to be counted with their multiplicities, which (according to our definitions) means exactly the following: one has to construct an arbitrary primary decomposition, and then collect all terms of dimension m and below. For the resulting ideal the upper-dimensional primary components are already well-defined and hence can be counted with their multiplicities. It is this number which bounds the length of the corresponding segment. The assertion of the Corollary follows from Lemma 5.5

and Lemma 5.6: one has to construct the auxiliary chain with strictly ascending leading terms as in Lemma 5.6 and then estimate its length by Lemma 5.5.

5.8. **Effective commutative algebra.** We had already come across some results from the realm of effective commutative algebra. In §3.3, assuming that a polynomial p_ℓ belongs to the ideal $(p_0, \dots, p_{\ell-1}) \subset \mathfrak{R}$, we implicitly used an algorithm to construct the representation $p_\ell = \sum_{k=0}^{\ell-1} h_k p_k$; the complexity estimate for this algorithm implies certain upper bounds for the degrees $\deg h_k$ [12, 19].

This is but one of many results in the same spirit. We agree that to construct (or define) an ideal in the polynomial ring $\mathfrak{R}$ means to construct (or specify) a set of generators of this ideal. Then many operations on ideals become algorithmically implementable. In particular, given polynomials generating some input ideals I and J, one can explicitly do the following [26]:

- construct the intersection $I : J$ and the colon ratio $A : B$;
- decide whether a given polynomial belongs to I and if so, construct an explicit expansion for the former in the generators of the latter;
- construct *some* primary decomposition of I and the intersection of all primary components of upper and lower dimensions,
- and many other (but not all) algebraic operations.

The algorithms performing the above mentioned operations, are discussed and perfected in a number of works. However, we are interested here not in the manipulations themselves, but rather in the upper bounds for the degrees of the generators of polynomial ideals. The result that will be used further, describes the algorithmical complexity of the primary decomposition.

Theorem ([12, 26, 27, 8, 9, 16, 17]). *If a polynomial ideal $I \in \mathfrak{R} = \mathbb{C}[x_1, \dots, x_n]$ is generated by polynomials of degree $\leqslant d$, then one can effectively construct polynomial bases of all ideals in the primary decomposition of I. The number of primary components and the degrees of polynomials in the bases can be majorized by primitive recusrive functions of n and d, and each of these functions grows no more rapidly than $d^{n^{O(n)}}$ as $n \to \infty$.*

This theorem in fact constitutes a synopsis of several references, see [12, 26, 27] for the primitive recursivity of the bound, [8, 9] for explicit estimates and [16, 17] for more precise bounds. Combination of this result with our definition of multiplicity and Bézout inequalities implies the following.

Corollary 7. *For an ideal generated by polynomials of degree d in $\mathbb{C}[x_1, \dots, x_n]$, the "number" of all primary components, counted with their multiplicities, is majorized by a primitive recursive function of d and n, and grows no more rapidly than $d^{n^{O(n)}}$ as $n \to \infty$.* □

The meaning of the "number" of irreducible components is the same as above.

5.9. **Proof of Theorem 7.** According to Corollary 5, any (finite strictly ascending) convex chain (5.1) can be subdivided into no more than n finite strictly ascending segments in such a way that along each segment the dimension of the colon ratios is the same and equal to $n - s$, where $s = 1, 2, \dots, n$ is the number of the segment:

$$\underbrace{I_0 \subset I_1 \subset \cdots \subset I_{k_1}}_{\dim I_k : I_{k+1} = n-1} \subset \underbrace{I_{k_1+1} \subset \cdots \subset I_{k_2}}_{\dim I_k : I_{k+1} = n-2} \subset \cdots$$
$$\cdots \subset \underbrace{I_{k_s+1} \subset \cdots \subset I_{k_{s+1}}}_{\dim I_k : I_{k+1} = n-s+1} \subset \cdots \subset \underbrace{I_{k_{n-1}+1} \subset \cdots \subset I_{k_n}}_{\dim I_k : I_{k+1} = 0} \tag{5.11}$$

By Corollary 6, the length of each such segment does not exceed the "number" of primary m-dimensional components. Denote the primitive recursive bound for the number of components occurring in Corollary 7, by $c(d, n)$. As it follows from Lemma 6, the length of each segment in the subdivision (5.11) can be explicitly bounded:

$$k_0 = 1, \quad k_{j+1} - k_j \leqslant c(dk_j, n) = (dk_j)^{n^{O(n)}}. \tag{5.12}$$

This recurrent identity immediately proves that the length of the chain $\ell = \ell(n, d) = k_n$ is a primitive recursive function of d and n and grows as asserted in the theorem.

Indeed, the growth of the sequence $\{k_j\}$ as in (5.12) is overtaken (for large values of d and n) by the growth of the linear difference equation $l_{j+1} = Ml_j^\alpha$ with $M = d^{n^{O(n)}}$, $\alpha = n^{O(n)}$, $l_0 = 1$, whose solutions can be found and estimated explicitly: since one can always assume $\alpha > 2$, we have $\log l_j \leqslant 2 \log M \cdot \alpha^j$ so that $l_n \leqslant M^{2\alpha^n} \sim M^{n^{O(n^2)}} \sim d^{n^{O(n^2)}}$. $\square$

Appendix A. Gabriélov theorem

Consider the problem analyzed by Gabriélov [5], that occupies in some sense the intermediate place between algebraic and geometric versions of the problem on ascending/descending chains.

Namely, assume that $\{I_k\}$ is the ascending chain of ideals (5.1) generated by consecutive derivations (5.2). Then one can associate with this chain of ideals the chain of their respective zero loci $\{X_k\}$, $X_k = V(I_k)$. Despite the fact that the chain of ideals must be *strictly* ascending by Corollary 4, the descent of the chain of varieties *should not necessarily* be strict. The easiest example is the sequence of derivatives of the polynomial x^μ in one variable, corresponding to $n = 1$ and $L = \partial/\partial x$. The chain of the respective zero loci drops from $\{0\}$ to $\varnothing$ after μ stable steps.

However, one may estimate the length of the chain $\{X_k\}$ along which the total stabilization must occur. The considerations below provide the arguments sufficient for proving the result described in §1.3. We start with the folowing trivial observation.

Let $L\colon \mathfrak{R} \to \mathfrak{R}$ be a Lie derivation, v the corresponding polynomial vector field in $\mathbb{C}^n$ and Y a submanifold in $\mathbb{C}^n$ with the coordinate ring $\mathfrak{R}' = \mathfrak{R}/I(Y)$. Denote by $\pi : \mathfrak{R} \to \mathfrak{R}'$ the canonical projection. Then L "covers" a well-defined derivation L' of the ring $\mathfrak{R}'$ if Y is invariant by the flow of the field v. This claim admits a local reformulation with all rings being the rings of germs.

Now we proceed with the proof of Gabriélov theorem. Denote by X_∞ the stable limit, the intersection of all varieties X_k, and consider the decreasing chain of semialgebraic varieties $X'_k = X_k \smallsetminus X_\infty$. The chain $\{X'_k\}$ eventually vanishes. Suppose that on some step $k = k_s$ the variety X'_{k_s} is s-dimensional (recall that each semialgebraic variety has dimension), and denote by X the set of points at which the field v is transversal to X'_{k_s}.

The complement $X'_{k_s} \smallsetminus X$ is less-than-s-dimensional. Indeed, since the tangency condition is algebraic, its violation on a relatively open set would mean that this set is locally invariant by the flow of v and hence belongs to X_∞. But this contradicts the definition of X'_{k_s} as a part of the complement to X_∞.

We show that after some number m of steps, any point a on X will *not* belong to X'_{k_s+m}, and hence the latter semialgebraic variety should be less-than-s-dimensional. The number m can be explcitly majorized.

Since the integral curve of v through a is transversal to X at a, we can construct the germ of a codimension s analytic surface (Y, a) in $\mathbb{C}^n$ such that $X \cap Y = \{a\}$ and Y be invariant by the flow of v. Let $\mathfrak{R}'$ be the local ring of germs on (Y, a). The ascending chain of ideals $\{I_k\}$ restricted on Y, yields an ascending chain of ideals $\{I'_k\} \subset \mathfrak{R}'$, $k = k_s, k_s + 1, \dots$, in the local ring, generated by adding consecutive derivatives $\pi L^k p_0 = (L')^k \pi p_0$ (by virtue of the above observation). Therefore this chain is convex and hence, by Corollary 4, *strictly* ascending.

The difference between the type of ascent of the chains $\{I_k\}$ and $\{I'_k\}$ is twofold:

1. all ideals of the latter chain are cofinite, so the numbers $\mu_k = \dim_{\mathbb{C}} \mathfrak{R}'/\mathfrak{R}' \cdot I'_k$ are finite, and
2. the chain $\{I'_k\}$ must terminate by the trivial ideal $I'_{k_s+m} = (1) \in \mathfrak{R}'$, since $a \notin X_\infty$.

Now it is obvious that the codimensions μ_k of the cofinite ideals must be *strictly* decreasing: otherwise we would have the equality $I'_k = I'_{k+1} \neq (1)$. However, this is impossible since by Corollary 4 this would mean that I_k stabilize on a non-trivial ideal. Therefore after no more than $m = \mu_{k_s}$ steps the chain of local ideals must stabilize on the trivial ideal $(1) = \mathfrak{R}'$.

To place an upper bound for the number of steps m after which the variety X'_{k_s+m} must become less-than-s-dimensional, we need to estimate the multiplicity μ_{k_s}. Note that this multiplicity does not depend on the choice of the transversal section Y as soon as the latter remains transversal to X at a. Choosing Y being an affine subspace in $\mathbb{C}^n$, we can majorize the multiplicity of the ideal (by virtue of Bézout theorem) by the nth power of the maximal degree of the polynomials generating the ideal, that is, by $(dk_s)^n$.

Putting everything together, we have the recurrent inequalities $k_{s-1} - k_s \leqslant (dk_s)^n$ that finally imply the double exponential in n and polynomial in d upper bound for the length of the chain $\{X'_k\}$ (compare with the end of §5.9). □

Appendix B. Demonstration of auxiliary results

B.1. **Bézout inequalities.** Recall that the *degree* of an irreducible k-dimensional variety $V \subset \mathbb{C}^n$ is the number of intersections of V and a generic affine subspace $L \subset \mathbb{C}^n$, *transversal* to V. If V is of uniform dimension k, then $\deg V$ is the sum of degrees of its irreducible components (the previous definition also applies, though). Finally, for an arbitrary algebraic V we define $\deg V$ as the sum of degrees of all its equidimensional parts. The following proprerties of degree can be easily established, see [18]:

1. if $V \subsetneq W$ are both of uniform dimension k, then $\deg V < \deg W$;
2. if M is an affine subspace, then $\deg(V \cap M) \leqslant \deg V$;
3. $\deg(V \times W) = (\deg V)(\deg W)$;
4. $\deg(V \cap W) = \deg\big((V \times W) \cap \Delta\big) \leqslant (\deg V)(\deg W)$, where $\Delta \subset \mathbb{C}^n \times \mathbb{C}^n$ is the diagonal (by the previous assertions).
5. $\deg V > 0$ if $V \neq \varnothing$, hence
6. the number of irreducible components of V (of all dimensions) does not exceed $\deg V$.

Having recalled all this, we proceed with the proof of Proposition 2.

Proof. We start with the space $\mathbb{C}^n$ and add polynomial equations one by one, controlling the number and degrees of irreducible components on each step. The result is better understood on the "genealogic tree" which has the root at $\mathbb{C}^n$ on level 0, and on the level k are listed all components of dimension $n - k$.

The set $\mathbb{C}^n$ after intersection with the hypersurface $\{p_1 = 0\}$ is the union of irreducible $(n-1)$-dimensional components, the sum of their degrees $\sum_i d_i$ being $\leqslant d$, unless $p_1 = 0$. These components are portrayed by points on the first level of the tree, and we call them descendants of the root, marking the edges leading from the root to each vertex by p_1 (or the first nonzero p_j, if $p_1 = 0$).

After adding the second equation (suppose that it is $\{p_2 = 0\}$), for each irreducible component we have two possibilities: either it belongs to the hypersurface $\{p_2 = 0\}$, and then the corresponding vertex remains unchanged, or it gives rise to no more than $d_i \times d$ descendants of dimension $n - 2$, portrayed by vertices on the second level. We draw then the edges connecting this ancestor with its descendants, marking them by p_2, and do this for all vertices on the level 1, that survive taking intersection with the hypersurface $\{p_2 = 0\}$.

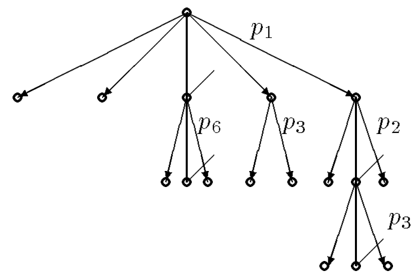

FIGURE 1. In this picture the polynomial p_3 produces new ancestors from two components of different dimensions; the polynomials p_4 and p_5 have made no impact. The two remaining components of dimension $n-1$ belong to the common level surface $\{p_1 = \cdots = p_6 = 0\}$.

On the next step we take the polynomial p_3 and repeat this procedure with *all terminal vertices* (leaves) of the tree constructed before, regardless of their dimension. Again, some vertices spawn a number of descendants, while some remain infertile. But in any case the number of immediate descendants from each ancestor can be at most d times the degree of this ancestor. We draw then the corresponding edges, marking them by p_3 and continue the process of adding new equations in their natural order.

As a result we obtain a tree of at most n levels, each edge being marked by one of p_k. Now by induction one easily proves that the sum of degrees of all components on the level k or above does not exceed d^k. Indeed, if d_i are degrees of all vertices on some level k, then $\sum_i d_i \leqslant d^k$, and each gives rise to no more than d descendants, irregardless of the moment of fertilization (since all polynomials p_k have degrees bounded by d). The number of vertices on the $(k+1)$st level will be therefore $\leqslant \sum_i d \times d_i \leqslant d^{k+1}$.

It only remains to note that all irreducible components of V occur (eventually, with repetitions) as leaves of this tree after exhausting the list of polynomials, and the level of each occurrence is equal to the codimension of the corresponding component. $\square$

B.2. **Proof of Lemma 5.1.** We start with the obvious identity $J = J \cap J'$ and consider the decomposition of the leading terms:

$$Q_1 \cap \cdots \cap Q_s = Q_1 \cap \cdots \cap Q_s \cap Q_1' \cap \cdots \cap Q_{s'}'. \tag{B.1}$$

The decomposition in the right hand side is not irredundant. However, all prime ideals associated with the primary terms Q_j', must be among the primes associated with Q_j. Indeed, this follows from the simple fact that all m-dimensional irreducible components of the variety $X' = V(J') \subset X = V(J)$ should be among those of X.

Rearranging if necessary the components of $\operatorname{l.t.}(J')$, we can assume that Q_j' and Q_j have the same associated prime for all $j = 1, \ldots, s'$ and $s' \leqslant s$. After collecting "similar terms" in the right hand side of (B.1), we observe that it becomes

$$(Q_1 \cap Q_1') \cap \cdots \cap (Q_{s'} \cap Q_{s'}') \cap Q_{s'+1} \cap \cdots \cap Q_s.$$

From the uniqueness theorem it follows that $Q_j = Q_j \cap Q_j'$ for all $j = 1, \dots, s'$, which implies that $Q_j \subset Q_j'$ for all such j. □

B.3. **Proof of Lemma 5.2.** Denote by I_a (resp., J_a) the localizations of the two ideals (i.e. their images in the local ring $\mathfrak{R}_a$). The proof consists of two steps: first we show that if for two primary ideals with the same associated prime the equality $I_a = J_a$ holds after localization at almost all points, then in fact $I = J$, and the second observation is that if for $I \subset J$ the equality $\mu_a(I) = \mu_a(J)$ holds for almost all points, then $I_a = J_a$ for almost all points also.

1. If $p_1, \dots, p_s$ are generators of I, and q is an arbitrary polynomial in J, then the condition $I_a = J_a$ implies that $q = \sum r_j p_j$, where r_j are rational fractions with the denominators not vanishing at a, hence (by getting rid of the denominators) we arrive to the representation $hq = \sum h_j p_j$, where $h \in \mathfrak{R}$ is a polynomial not vanishing at a, and h_j are polynomials as well. Consider the colon ideal $I : q$. The above conclusion means that for almost all $a \in X$ the colon ideal $I : q$ contains a polynomial with $h(a) \neq 0$. For obvious reasons, for $a \notin X$ this is valid as well. Since I is primary, then by [30, Ch. III, §9, Theorem 14] the ideal $I : q$, if not trivial, is also primary with the same associated prime. But from the above assertion it follows that the zero locus of $I : q$ is strictly contained in X, so the only possibility left is that $I : q = \mathfrak{R}$, i.e. $q \in I$. Since $q \in J$ was chosen arbitrary, this proves the first assertion (note that we used only the fact that I is primary; the bigger ideal J could in fact be arbitrary).

2. Let $a \in X$ be a smooth point, $\mathfrak{R}_a$ and I_a being the corresponding localizations. Since the situation is local, without loss of generality we may assume that X is a coordinate subspace. Choose T being the complementary coordinate subspace and denote by L the corresponding ideal. Let (x, ε) be the associated local coordinates, so that $X = \{x = 0\}$, and $T = \{\varepsilon = 0\}$. The point a is therefore at the origin $(0, 0)$.

If the germs $f_1, \dots, f_\mu \in \mathfrak{R}_{(0,0)}$ generate the local algebra of the cofinite ideal $(I+L)_{(0,0)}$, then any germ $q(x)$ from the restriction of $I_{(0,0)}$ on T can be represented as

$$q(x) = \sum_{j=1}^{\mu} c_j\, f_j(x, 0) + \sum_{i=1}^{s} h_i(x) p_i(x, 0), \tag{B.2}$$

where $p_i = p_i(x, \varepsilon)$ are generators of the ideal I, and $p_i(x, 0)$ are their respective restrictions on the transversal L. If this representation is minimal (i.e. the number of germs f_j cannot be reduced), then μ is the multiplicity of I at the point $a = (0, 0)$.

By the Preparation theorem in the Thom–Martinet version [18, Chapter I, §3], the representation (B.2) can be "extended" for all small nonzero ε: any element $q \in \mathfrak{R}_{(0,0)}$ admits a representation

$$q(x, \varepsilon) = \sum_{j=1}^{\mu} c_j(\varepsilon)\, f_j(x, \varepsilon) + \sum_{i=1}^{s} h_i(x, \varepsilon) p_i(x, \varepsilon). \tag{B.3}$$

Now assume that the localization $J_{(0,0)}$ is strictly bigger than $I_{(0,0)}$ and take an element $q \in J_{(0,0)} \smallsetminus I_{(0,0)}$. By (B.3), it can be expanded after a certain choice of the coefficients $c_j(\varepsilon)$.

The situation when all $c_j(\varepsilon)$ are identical zeros, is impossible, since this would mean that $q \in I_{(0,0)}$ contrary to our assumptions. Therefore for almost all values of ε the equality

$$\sum_{j=1}^{\mu} c_j(\varepsilon) f_j(x,\varepsilon) = q + \sum_{i=1}^{s} h_i p_i \in J_a$$

means a nontrivial linear dependence between the generators $f_j(\cdot,\varepsilon)$ of the corresponding local algebra $\mathfrak{R}_{(0,\varepsilon)}/J_{(0,\varepsilon)} \cdot \mathfrak{R}_{(0,\varepsilon)}$, which fact implies that its dimension is strictly smaller than μ.

Thus we proved that $\mu(I) = \mu(J)$ implies that for almost all points a the localizations I_a and J_a coincide, hence by part 1, $I = J$. □

Appendix C. Computations

Recall that we deal with a Lie derivative L corresponding to the universal polynomial vector field of height 1 and degree d in n independent variables. Iterations of L generate a sequence of of polynomials $p_{k+1} = Lp_k$, starting from some seed polynomial of degree d and height 1, and we are interested in estimating the degrees and heights of all terms in the identity

$$p_\ell = \sum_{k=0}^{\ell-1} h_k p_k, \qquad \ell = d^{n^{O(n^2)}}. \tag{C.1}$$

We shall repeatedly use the symbol ℓ in the "absorbing" sense (similar to O of the classical analysis), so that $d\ell = \ell$, $\ell^n = \ell$ etc.

1. The degrees of the polynomials p_k grow linearly with k: $\deg p_k \leqslant kd + O(1)$.

2. The heights of the polynomials p_k are controlled by the recurrent inequalities (3.5). From these inequalities it easily follows that

$$\mathcal{H}(p_{k+1}) \leqslant \mathcal{H}(p_k) \cdot (d\ell) n (d\ell+1)^n (d+1)^n \asymp \mathcal{H}(p_k) \cdot \ell, \tag{C.2}$$

whence for all $k = 0, 1, \dots, \ell$ we have the inequality

$$\mathcal{H}(p_k) \leqslant \ell^\ell \asymp d^\ell = d^{d^{n^{O(n^2)}}}. \tag{C.3}$$

3. The degrees of the polynomials h_k do not exceed $(\ell d)^{2^n} \asymp \ell$ by (3.4).

4. The size of the matrix of the linear system described in the proof of Lemma 3.1, is therefore bounded by the expression $\ell^{n+1} \asymp \ell$. This matrix and the column in the right hand side are filled by integer numbers not exceeding d^ℓ. By the Cramer rule, each component of the solution can be found as a ratio of two minors of this matrix. But any such minor does not exceed the sum of $\ell!$ terms, each being att most $(d^\ell)^\ell$. Since the denominator of the ratio is at least 1 (being a nonzero integer), we have the upper bound for the height of all h_k:

$$\mathcal{H}(h_k) \leqslant \ell!(d^\ell)^\ell \asymp d^\ell. \tag{C.4}$$

5. A polynomial of degree ℓ and height d^ℓ restricted on the box $\mathbb{B}_R$ in $\mathbb{R}^n$, does not exceed $\ell^n d^\ell (1+R)^\ell \asymp (2+R)^{d^\ell}$. This expression is also the bound for the coefficients of the quasilinear equation. By Corollary 1 the number of real zeros of any solution of this equation does not exceed the upper bound asymptotically equivalent to $(2+R)^{d^\ell}$, which is the ultimate answer that appears in Theorem 4, since d^ℓ is asymptotically overtaken by the tower of three exponents $\exp\exp\exp(n^3 + d + O(1))$. □

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