

Rank-Width Is Less Than or Equal to Branch-Width

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Abstract: We prove that the rank-width of the incidence graph of a graph G is either equal to or exactly one less than the branch-width of G , unless the maximum degree of G is 0 or 1. This implies that rank-width of a graph is less than or equal to branch-width of the graph unless the branch-width is 0. Moreover, this inequality is tight. © 2007 Wiley Periodicals, Inc. *J Graph Theory* 57: 239–244, 2008

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1. INTRODUCTION

In this article, graphs have no loops and no parallel edges. The *incidence graph* $I(G)$ of a graph $G = (V, E)$ is a graph on vertices in $V \cup E$ such that $x, y \in V \cup E$ are adjacent in $I(G)$ if one of x, y is a vertex of G , the other is an edge of G , and x is incident with y in G . In other words, $I(G)$ is the graph obtained from G by subdividing every edge exactly once.

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We prove that the rank-width of a graph G is less than or equal to the branch-width of G , unless the branch-width of G is 0. Definitions of branch-width and rank-width are in Section 2. To show this, we prove a stronger theorem stating that the rank-width of $I(G)$ is equal to or exactly one less than the branch-width of G , or the branch-width of G is 0. Another corollary of this theorem is that the rank-width of the line graph of a graph is less than or equal to the branch-width of the graph.

There are related works on this topic. In this article, we do not define clique-width [2] but rank-width is related to clique-width. Oum and Seymour [8] showed that $\text{rw}(G) \leq \text{cw}(G) \leq 2^{\text{rw}(G)+1} - 1$, where $\text{cw}(G)$, $\text{rw}(G)$ denote clique-width and rank-width, respectively. Let $\text{tw}(G)$ be the tree-width of G . Courcelle and Olariu [2] showed that clique-width is at most $2^{\text{tw}(G)+1} + 1$ and later Corneil and Rotices [1] proved that clique-width is at most $3 \cdot 2^{\text{tw}(G)-1}$. The previous results on clique-width imply that rank-width is smaller than or equal to $3 \cdot 2^{\text{tw}(G)-1}$. Kanté [4] showed that the rank-width is at most $4\text{tw}(G) + 2$. In this article, we prove that rank-width is smaller than or equal to $\text{tw}(G) + 1$.

2. PRELIMINARIES

Branch-width [9] and rank-width [8] of graphs are defined in a similar way. We will describe more general branch-width of symmetric submodular functions, and then define branch-width of graphs and rank-width of graphs in terms of branch-width of symmetric submodular functions. For a finite set V , let 2^V be the set of subsets of V . Let \mathbb{Z} be the set of integers. A function $f : 2^V \rightarrow \mathbb{Z}$ is *symmetric* if $f(X) = f(V \setminus X)$ for all $X \subseteq V$ and *submodular* if $f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$ for all $X, Y \subseteq V$. A tree is *subcubic* if every vertex has degree 1 or 3. A *branch-decomposition* of a symmetric submodular function $f : 2^V \rightarrow \mathbb{Z}$ is a pair (T, τ) of a subcubic tree T and a bijection $\tau : V \rightarrow \{t : t \text{ is a leaf of } T\}$. The *width* of an edge $e \in E(T)$ in a branch-decomposition (T, τ) is defined as $f(X_e)$ where (X_e, Y_e) is a partition of V from a partition of leaves of T induced by deleting e from T . The *width* of a branch-decomposition (T, τ) is the maximum width of all edges of T . The *branch-width* of f , denoted by $\text{bw}(f)$, is the minimum width of all branch-decompositions of f . (If $|V| < 2$, then f has no branch-decomposition. In this case, we assume that $\text{bw}(f) = f(\emptyset)$.)

Please be warned that in the above definition V can be any finite set, not just the set of vertices of graphs. We define branch-width of a graph $G = (V, E)$ as branch-width of a certain symmetric submodular function η_G on the set E of edges as follows. For a subset X of E , let $\text{mid}(X)$ be the set of vertices that are incident with both an edge in X and another edge in $E \setminus X$. Let $\eta_G(X) = |\text{mid}(X)|$. Then $\eta : 2^E \rightarrow \mathbb{Z}$ is a symmetric submodular function and so the branch-width of η_G is well defined. The *branch-width* $\text{bw}(G)$ of a graph G is defined as the branch-width of η_G .

The rank-width is defined by the *cut-rank* function $\rho_G : 2^V \rightarrow \mathbb{Z}$ of a graph $G = (V, E)$. For a subset X of V , consider a 0-1 matrix M_X over the binary field

$\text{GF}(2)$, in which the number of rows is $|X|$ (so rows are indexed by X), the number of columns is $|V \setminus X|$, (so columns are indexed by $V \setminus X$), and the entry is 1 if and only if the corresponding vertices of the row and the column are adjacent. Let $\rho_G(X) = \text{rank}(M_X)$ where rank is the matrix rank function. Then ρ_G is symmetric and submodular [8]. The *rank-width* $\text{rw}(G)$ of a graph G is defined as the branch-width of ρ_G .

We will need a definition of matroid branch-width. A *matroid* is a pair $M = (E, r)$ of a finite set E and a rank function $r : 2^E \rightarrow \mathbb{Z}$ satisfying the following axioms: $r(\emptyset) = 0$, $r(X) \leq |X|$ for all $X \subseteq E$, $r(X) \leq r(Y)$ if $X \subseteq Y$, and r is submodular. The *connectivity function* of a matroid $M = (E, r)$ is $\lambda_M(X) = r(X) + r(E \setminus X) - r(E) + 1$. It is easy to see that λ_M is symmetric and submodular. The *branch-width* $\text{bw}(M)$ of a matroid M is defined as the branch-width of λ_M .

Given a matrix A over $\text{GF}(2)$ whose columns are indexed by E , let $r_A(X) = \text{rank} A_X$ where A_X is the submatrix of A obtained by removing columns not in X . Then r_A satisfies the matroid rank axiom and therefore $M = (E, r_A)$ is a matroid. A matroid that has such a representation is called a *binary matroid*. Since the elementary row operations do not change r_A , every binary matroid has the *standard representation* A in which A_B is an identity matrix for some $B \subseteq E$. The *fundamental graph* $F(M)$ of a binary matroid with respect to the above standard representation is a bipartite graph on vertices E with a bipartition $(B, V \setminus B)$ such that $x \in B$ and $y \in V \setminus B$ are adjacent if and only if the row having 1 in the column vector of x in A has 1 in the column vector of y in A . Oum [6] showed the following.

Lemma 1 (Oum [6]). *The branch-width of a binary matroid M is exactly one more than the rank-width of its fundamental graph.*

The *cycle matroid* $M(G)$ of a graph $G = (V, E)$ is a binary matroid having the following standard representation: let B be an edge set of the spanning forest F of G . Let A be a 0-1 matrix $(a_{ij})_{i \in B, j \in V}$ such that $a_{ij} = 1$ if and only if $i = j \in B$ or the fundamental circuit of $j \notin B$ with respect to F contains i . Then A is a (standard) representation of $M(G)$. It is well known that $\lambda_{M(G)}(X) \leq \eta_G(X)$ for all nonempty $X \subset E(G)$. This implies that the branch-width of $M(G)$ is at most the branch-width of G if G has at least two edges. The following theorem was shown by Hicks and McMurray [3] and Mazoit and Thomassé [5], independently.

Theorem 2 (Hicks and McMurray [3]; Mazoit and Thomassé [5]). *The branch-width of a 2-connected graph G is equal to the branch-width of the cycle matroid $M(G)$.*

3. MAIN THEOREM

Now let us prove the main theorem.

Theorem 3. *For a graph G , $\text{rw}(I(G))$ is equal to $\text{bw}(G) - 1$ or $\text{bw}(G)$ unless the maximum degree of G is 0 or 1. (If the maximum degree of G is 0 or 1, then $\text{rw}(I(G)) \leq 1$ and $\text{bw}(G) = 0$.)*

Proof. This proof will work even if G has parallel edges. (If G has loops, then $I(G)$ has parallel edges, but $\text{rw}(I(G))$ is defined only if $I(G)$ has no parallel edges and no loops.) Without loss of generality, we may assume that G is connected and has at least two vertices. If $|V(G)| = 2$ then $\text{rw}(I(G)) = 1$ and $\text{bw}(G) \leq |V(G)| = 2$.

Now we assume that $|V(G)| > 2$, $|E(G)| > 1$, and $\text{bw}(G) \geq 1$ and so G admits rank-decompositions and branch-decompositions. Let us construct a graph \hat{G} by adding a new vertex v to G that is adjacent to all vertices of G . Then \hat{G} is 2-connected. Let F be a spanning tree of \hat{G} consisting of all edges incident with v and $B = E(F)$. It is easy to see that the fundamental graph of $M(\hat{G})$ with respect to B is $I(G)$. Therefore by Lemma 1, $\text{rw}(I(G)) = \text{bw}(M(\hat{G})) - 1$. By Theorem 2, $\text{bw}(M(\hat{G})) = \text{bw}(\hat{G})$.

Now it is enough to show that $\text{bw}(\hat{G}) = \text{bw}(G)$ or $\text{bw}(G) + 1$ when G is connected and $|V(G)| > 2$. (This is false when G has a single edge; $\text{bw}(G) = 0$ but $\text{bw}(\hat{G}) = 2$.) Since G is a minor of \hat{G} , $\text{bw}(G) \leq \text{bw}(\hat{G})$. Let (T, τ) be a branch-decomposition of G of width $\text{bw}(G)$. For every vertex w of G , pick an edge f_w of G incident with w . Then let e_w be the unique edge of T incident to the leaf $\tau(f_w)$. We subdivide e_w and attach a new leaf corresponding to the edge vw of \hat{G} . (If an edge xy of G is chosen twice and exactly one of its ends, say x , has degree 1, then we apply the above operation for vy first and then apply for vx . This is to avoid having $\{xy, vy\}$ in one side of the branch-decomposition because $\eta_{\hat{G}}(\{xy, vy\}) = 3$.) It is easy to see that the obtained branch-decomposition of \hat{G} has width at most $\text{bw}(G) + 1$. (Notice that $\text{bw}(G) \geq 1$ and therefore $\text{bw}(G) + 1 \geq 2$.) Consequently $\text{bw}(\hat{G}) \leq \text{bw}(G) + 1$. This proves the theorem. ■

If we use an easy inequality $\text{bw}(M(G)) \leq \text{bw}(G)$ instead of Theorem 2, we can still prove that $\text{rw}(I(G)) \leq \text{bw}(G)$ unless the maximum degree of G is 0 or 1. Actually this will be enough to prove the following two corollaries of Theorem 3.

We will need a definition of a vertex-minor. The *local complementation* at a vertex v of a graph G is an operation to obtain the graph $G * v$ on the vertices of G such that two distinct vertices x, y in $G * v$ are adjacent if and only if either (i) both x and y are neighbors of v and they are nonadjacent in G , or (ii) at least one of x or y is nonadjacent to v and x, y are adjacent in G . It is shown in [6] that local complementations preserve the cut-rank functions and rank-width. A *vertex-minor* of a graph is a graph obtainable by successive local complementations and vertex deletions.

Lemma 4 (Oum [6]). *If H is a vertex-minor of G , then $\text{rw}(H) \leq \text{rw}(G)$.*

Corollary 5. *For a graph G , $\text{rw}(G) \leq \max(\text{bw}(G), 1)$.*

Proof. A graph G is a vertex-minor of $I(G)$, because we get G by applying local complementations to vertices of $I(G)$ corresponding to edges of G and deleting those vertices. Then by Lemma 4, $\text{rw}(G) \leq \text{rw}(I(G))$. By Theorem 3, either $\text{rw}(I(G)) \leq \text{bw}(G)$, or $\text{rw}(I(G)) \leq 1$ and $\text{bw}(G) = 0$. ■

The above corollary implies that $\text{rw}(G) \leq \text{tw}(G) + 1$, because $\text{bw}(G) \leq \text{tw}(G) + 1$ [9].

Corollary 6. *Let G be a graph. The rank-width of the line graph $L(G)$ of G is at most the branch-width of G .*

Proof. If no two edges of G are adjacent, then $L(G)$ has no edges and therefore $\text{rw}(L(G)) = \text{bw}(G) = 0$. We may now assume that $\text{bw}(G) > 0$. It is enough to show that $L(G)$ is a vertex-minor of $I(G)$. We apply local complementations to vertices in $I(G)$ corresponding to vertices of G and then remove those vertices. The remaining graph is a line graph of G . ■

We remark that with different methods, Oum [7] showed that the rank-width of $L(G)$ is exactly one of $\text{bw}(G) - 2$, $\text{bw}(G) - 1$, or $\text{bw}(G)$ if G is 2-connected.

Let us now show that the inequalities in Theorem 3 and Corollaries 5 and 6 are tight. Robertson and Seymour [9] showed that $\text{bw}(K_n) = \lceil 2n/3 \rceil$ for $n \geq 3$. It is straightforward to prove that $\text{bw}(I(K_n)) = \text{bw}(K_n) = \lceil 2n/3 \rceil$ for $n \geq 3$. (In fact, $\text{bw}(G) = \text{bw}(I(G))$ for any graphs G with branch-width at least 2.) We know from the proof of Theorem 3 that $\text{rw}(I(K_n)) = \text{bw}(M(\hat{K}_n)) - 1 = \text{bw}(M(K_{n+1})) - 1 = \text{bw}(K_{n+1}) - 1 = \lceil 2(n+1)/3 \rceil - 1 = \lceil (2n-1)/3 \rceil$. So if $n \geq 3$ and $n \equiv 0, 1 \pmod{3}$, then $\text{rw}(I(K_n)) = \text{bw}(I(K_n))$. This proves that Theorem 3 and Corollary 5 are tight.

It remains to prove that Corollary 6 is tight. Let $P(G)$ be a graph obtained from G by attaching a pendant vertex to each vertex of G . (So, $|V(P(G))| = 2|V(G)|$, G is an induced subgraph of $P(G)$, and all vertices of $P(G)$ not in the subgraph G have degree 1 and have distinct neighbors in the subgraph G .) Oum [6] showed that $\text{rw}(P(G)) = \text{rw}(G)$ if G has at least one edge. We observe that if we apply local complementations to vertices of $L(P(K_n))$ corresponding to edges of $P(G)$ incident with pendant vertices, then we obtain $I(K_n)$. Therefore, $\text{rw}(L(P(K_n))) = \text{rw}(I(K_n))$. It is routine to prove that $\text{bw}(P(K_n)) = \text{bw}(K_n)$ if $n \geq 3$. So if $n \geq 3$ and $n \equiv 0, 1 \pmod{3}$, then $\text{rw}(L(P(K_n))) = \text{bw}(P(K_n))$. Therefore, Corollary 6 is tight.

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