# A Modal Logic for Chopping Intervals

YDE VENEMA, *Faculteit Wiskunde en Informatica, Universiteit van Amsterdam, Plantage Muidergracht* 24, 1018 *TV Amsterdam*

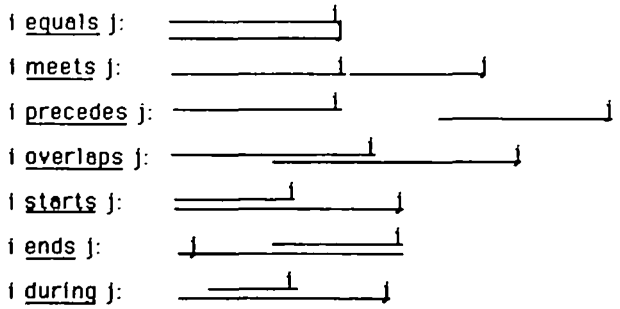
## 1. Introduction: representation of time in intervals

When developing a logical formalism to represent time, several choices are to be made in the field of language, ontology and semantics. E.g. concerning the language an important question is whether to use classical logic or some kind of a modal formalism; the ontology can be such that the time*points* are the basic entities, or one may start with *periods* of time. Two choices, four possible outcomes: of these, the modal logic of periods seems to have received the least attention [3, 8, 9, 19, 20, 7, 24]. We hope to help fill this gap with this article, which is devoted to a modal system of intervals. We will use the term 'interval' for an uninterrupted stretch of time informally visualized by a horizontal line segment:

$i$.

Although we only study linear time here, on the whole we want to be as general as possible. We make no further assumptions about the underlying nature of time, such as denseness or discreteness, but we show some of these properties to be expressible in the modal language, and very easily so. We do not impose any restructions on the semantics, like homogeneity: an atomic formula need not be true on all subintervals of $i$ if it is true on $i$ itself, or locality: if two intervals start simultaneously, they need not make the same atomic formulas true.

Except for [19], all modal systems of intervals developed up till now have *unary* modal operators associated with *binary* accessibility relations of intervals. It is well known [1, 3] that in linear time structures there are thirteen such relations. In the figure below, we visualize seven of them; the other relations are the converses of the ones we show (here **equals** is its own converse).

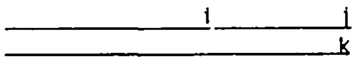



*J. Logic Computat.*, Vol. 1 No. 4, pp. 453–476, 1991

In [9] and [3], the systems defined have operators $F$, $P$ and $\Diamond$, corresponding to respectively the following relations: **precedes**$^\vee$, **precedes,** and the union of **during**$^\vee$, **starts**$^\vee$ and **ends**$^\vee$. In [7] one has operators corresponding to the **meets, starts** and **ends** relations and their converses, e.g. $\langle B\rangle\phi$ holding at an interval $i$ if it has an interval $j$ such that $j$ starts $i$ and $\phi$ holds at $j$. In the same article one can find some results concerning the complexity of the validity problem for formulas of this language with respect to several classes of frames. In [24] sound and complete axiom systems are given for the system of [7].

Here we present a system with *binary* modal operators connected with a *ternary* accessibility relation $A$. This relation $A$ may be defined as follows:

$$Aijk \equiv i \ \textbf{starts}\ k \wedge i \ \textbf{meets}\ j \wedge j \ \textbf{ends}\ k,$$




The introduced system CDT is called a logic for *chopping* intervals because it has an operator $C$, the interpretation of $\phi C\psi$ being formally

$$M, k \models \phi C\psi \text{ if there are } i, j \text{ with } Aijk \text{ and } M, i \models \phi \text{ and } M, j \models \psi,$$

and informally: $\phi C\psi$ holds at an interval iff we can chop it into a $\phi$-part and a $\psi$-part.

We would like to thank one of the anonymous referees for bringing the article [19] to out attention. Where the CDT-operators can be seen as totally describing the situation when there is one extra point given besides the beginning and endpoint of the current interval, Nishimura gives similar operators for every number of extra points. So his system, having infinitely many operators, of increasing arity, can be seen as a generalization of ours.

The CHOP-operator seems to be a very natural one; one may see it, just like von Wrights ANDNEXT-operator [3], as a formalization of the temporal connective in sentences like

He came home **and** went to bed.

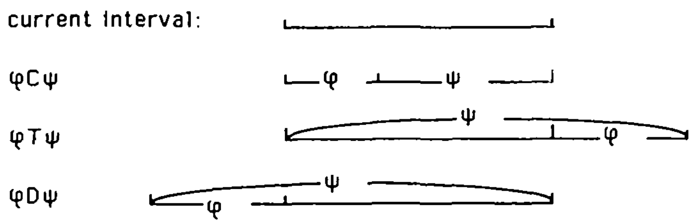
Besides this relevance for natural language processing (and artificial intelligence), the CHOP-operator also has computational and mathematical interest: it has a part to play in the branch of computer science where temporal logic is used to prove program correctness. Interpreting formulas not in intervals but in paths, i.e. sequences of computation states, $\phi C\psi$ holds in sequences which are the *concatenation* of two sequences in which $\phi$ resp. $\psi$ holds. Some results on this approach can be found in [21]. Finally, the modal logic studied here has a close link with the mathematical theory of *Relation Algebras* [11, 12], which form an algebraic treatment of the logic of binary relations, in the same way as Boolean algebras form an algebraic approach towards propositional logic. Here the link is formed by the observation, that if one views an interval as the ordered pair consisting of its beginning and endpoint, a set of intervals can be seen as a binary relation (on the set of timepoints). The details of this connection, which inspired our completeness results of sections 4 and 5, are given in [25].

The paper is organized as follows: first we give the necessary definitions concerning syntax and semantics; in sections 3 and 4 we give some results concerning expressiveness and correspondence; in section 5 we briefly consider sound and complete axiom systems, and the last section treats a complete natural deduction system for CDT.

## 2. CDT, a modal logic for chopping intervals

### 2.1. Syntax

Besides the usual Boolean connectives, the system CDT has three binary modal operators, $C$, $D$ and $T$, and a propositional constant, $\pi$. The intuitive picture is given by the following figure:

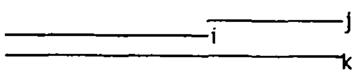



i.e. $\phi C\psi$ holds at an interval if it can be chopped into two pieces, at the first of which $\phi$ is to hold, and at the second $\Psi$. The constant $\pi$ will hold for 'point-intervals', i.e. intervals of zero duration. As abbreviations we use **tt** (**ff**) for an arbitrary tautology (falsity), e.g. $\mathbf{tt} \equiv \pi \vee \neg\pi$, $\mathbf{ff} \equiv \pi \wedge \neg\pi$.

### 2.2. Semantics

We have a choice between an ontology in which intervals are the primary objects and one in which they are defined as a sets of points in an underlying linear order. we present both approaches, the most general one first:

DEFINITION 2.2.1.
An *i-frame* is a triple $J = (I, A, P)$ with $A \subseteq {}^3I$ and $P \subseteq I$. Intuitively, $Aijk$ represents the following situation:



i.e. $k$ is the 'sum' of two adjacent intervals $i$ and $j$. An *i-model* is a pair $M, = (J, V)$ with $J$ an $i$-frame and $V$ a *valuation*, i.e. a map assigning subsets of $I$ to atomic propositions. We can define a *truth relation* $\models$ for $i$-models in the following way:

$$M\, i \models q \text{ if } i \in V(q)$$

$$M, i \models \pi \text{ if } i \in P$$

$$M, i \models \phi \wedge \psi \text{ if } M, i \models \phi \text{ and } M, i \models \psi,$$

$$M, i \models \phi C \psi \text{ if there are } j, k \text{ in } I \text{ with } Ajki \text{ and } j \models \phi \text{ and } k \models \psi,$$

$$M, i \models \phi T \psi \text{ if there are } j, k \text{ in } I \text{ with } Aijk \text{ and } j \models \phi \text{ and } k \models \psi,$$

$$M, i \models \phi D \psi \text{ if there are } j, k \text{ in } I \text{ with } Ajik \text{ and } j \models \phi \text{ and } k \models \psi.$$

A *frame* is a pair $F = (T, <)$ with $<$ a linear ordering on $T$. For a frame $F$ we define the *set of F-intervals* $INT(F) = \{[s, t] \mid s, t \in F, s \leq t\}$ and the *i-frame on F*, $I(F) = (\mathrm{INT}(F), A, P)$ with

$$([s, t], [u, v], [w, x] \in A \text{ if } s = w,\ t = u \text{ and } v = x,$$

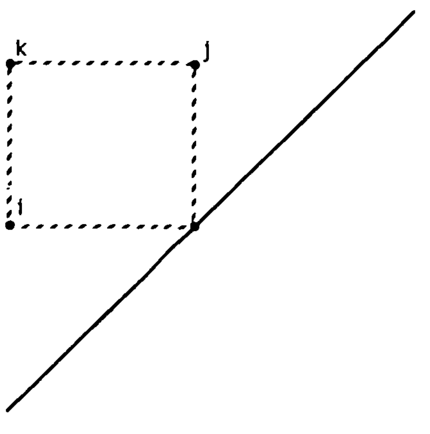
$$[s, t] \in P \text{ iff } s = t.$$

A *model* is an $i$-model $(J, V)$ with $J$ based on a frame $F$. In such a case we denote the model by $(F, V)$. Note that for models we have $F, V, [s, t] \models \phi C \psi$ iff there is a $u$ with $s \leq u \leq t$, $[s, u] \models \phi$ and $[u, t] \models \psi$.

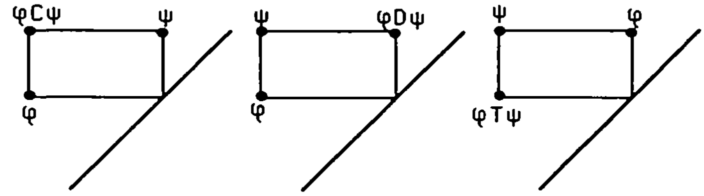
For both kind of models, the notions of *satisfiability* and *validity* are defined in the usual way. An $i$-frame on a linear ordering is called *two-dimensional* if it is isomorphic to an $i$-frame on a linear ordering; the class of two-dimensional frames is denoted by $F_2$.

### 2.3. Geometrical representation

If we represent intervals $[s, t]$ of $\mathrm{INT}(F)$ as points $(s, t)$ in the 'North-western halfplane' of $F^2$, we get a 'rectangular' interpretation for $(i, j, k) \in A : (i, j, k)$ forms an $A$-triple iff we can make a a rectangle in $F^{2NW}$ having as



its vertices: $i$, $k$, $j$ and the intersection of the horizontal line through $i$ and the vertical one through $j$, viz. the above figure. The set $P$ of point intervals is visualized as the diagonal $\{(s, t) \mid s = t\}$; Points $(s, t)$ with $s < t$ represent *stretched* intervals. For the interpretation of the binary modal operators, we obtain:




### 2.4. Compass-operators

The system HS defined by Halpern and Shoham in [7] can be seen as a very natural subsystem of CDT: define the following modal operators as abbreviations in CDT:

$\sigma$ for $\neg\pi$,

$\Diamond_{\rightarrow}\phi$ for $\sigma C\phi$, $\qquad$ $\Diamond_{\downarrow}\phi$ for $\phi C\sigma$

$\Diamond_{\leftarrow}\phi$ for $\sigma D\phi$, $\qquad$ $\Diamond_{\uparrow}\phi$ for $\sigma T\phi$,

$\Diamond_{\leftrightarrow}\phi$ for $(\Diamond_{\rightarrow}\phi \vee \phi \vee \Diamond_{\leftarrow}\phi)$, $\qquad$ $\Diamond_{\updownarrow}\phi$ for $(\Diamond_{\downarrow}\phi \vee \phi\Diamond_{\uparrow}\phi)$,

$\Box_{\uparrow}\phi$ for $\neg\Diamond_{\uparrow}\neg\phi$, etc.

In the light of the previous paragraph it will be clear how we can give these operators a 'compass interpretation' in point-based frames, e.g. $\Diamond_{\downarrow}\phi$ holds at a two-dimensional point $X$ iff there is a $\phi$-point right *south* of $X$. Note however, that our set of possible worlds still consists of the *North-western halfplane*. This will mean, for example, that the formula $\pi$ is equivalent to $\Box_{\downarrow}\mathbf{ff}$.

In terms of intervals, the meaning of the compass operators is given by the following scheme:

$\Diamond_{\downarrow}\phi$ holds at an interval if it has a *starting* interval where $\phi$ holds.

$\Diamond_{\rightarrow}\phi$ holds at an interval if it has an *ending* interval where $\phi$ holds.

$\Diamond_{\rightarrow}\phi$ holds at an interval if it *starts a* $\phi$-interval.

$\Diamond_{\leftarrow}\phi$ holds at an interval if it *ends* a $\phi$-interval.

Here the relations 'starts' and 'ends' are supposed to be irreflexive: an interval does not start or end itself. We use the name *HS* for the sublogic of CDT in which we only have the compass operators. The system HS is in a certain sense complete with respect to the binary relations of intervals: for each of the thirteen accessibility relations given in the introduction, we can define in HS the operator connected with it. For example, $i \models \langle\rangle\langle\rangle\phi$ if there is an interval $j$ **during** $i$ at which $\phi$ holds.

On the other hand, HS is less expressive than CDT itself: in [24] it is proved that we cannot even define the simple CDT-formula $qCq$ in HS.

## 3. CDT and classical logic of time points

When a new modal logic is defined, it is natural to apply the familiar theory of modalities to this logic and to ask old questions about the new subject. We will do this here, but for lack of surprising results we will only touch a small number of topics, and only lightly.

To start with, we can look at the capacity of CDT to characterize classes of frames and develop its *correspondence theory* [4]. Analogous to the choice in semantics for CDT, we have a choice in the first-order language which is to be the target of the correspondence translation: we may choose an 'interval language' with a triadic predicate symbol $A$ and a monadic $P$, or a 'point language' with only a dyadic symbol $<$. This section is devoted to the second option: we will compare the expressive strength of CDT with that of the first-order language $L_P$ having identity and a binary relation symbol $<$. Let $L_{PQ}$ be the language $L_P$ extended with binary relation symbols $Q_0, Q_1, \ldots$

First, we can easily give a straightforward translation $\tau$ of CDT-formulas into the set of $L_{PQ}$-formulas having two free variables, in such a way that $\phi$ and $\phi^\tau$ are locally equivalent on the model level: Let $M = (F, V)$ be a model on the $p$-frame $F$. We may consider $M$ as a structure for $L_{PQ}$ (in the usual classical sense) by interpreting every $Q_i$ by $V(q_i)$. We then have, for any pair $(s_1, s_2)$ with $s_1 \leq s_2$:

$$M \models \phi[s_1, s_2] \Leftrightarrow M \models_c \phi^\tau(x_1, x_2)[x_i \mapsto s_i].$$

Here $\models_c$ denotes the classical truth relation.

Equivalence on the frame level may then be obtained by a universal quantification over all relation symbols $Q_1, \ldots, Q_k$ appearing in $\phi^\tau$, global equivalence by a first order universal closure:

$$F \models \phi \Leftrightarrow F \models_c \forall x_1 \forall x_2 \forall Q_1 \ldots \forall Q_k \phi^\tau(x_1, x_2).$$

Fortunately, we need not always go as far as second-order $L_P$-logic: some important first order definable properties of linear orders can be characterized by a simple CDT-formula.

DEFINITION 3.1
Call a frame *dense* if it satisfies $\forall st(s < t \rightarrow \exists u(s < u < t))$, *discrete* if $\forall st(s < t \rightarrow [\exists u(s < u \wedge \neg\exists v(s < v < u)) \wedge \exists u(u < t \wedge \neg\exists v(u < v < t))])$ holds in $F$.

Let **length 1** be the CDT-formula $\sigma \wedge \neg(\sigma C \sigma)$, then **length 1** holds at intervals with no proper subintervals. Furthermore, define the following CDT-formulas:

$$\text{DENSE} \equiv \sigma \rightarrow (\sigma C \sigma)$$
$$\text{DISCRETE} \equiv [(\textbf{length 1 C tt}) \wedge (\textbf{tt C length 1})].$$

PROPOSITION 3.2
Let $F$ be a $p$-frame.
(1) $F \models \text{DENSE} \Leftrightarrow F$ is dense.
(2) $F \models \text{DISCRETE} \Leftrightarrow F$ is discrete.

PROOF. Straightforward. □

Note that the CDT-formulas characterizing density and discreteness only use atomic constants $\sigma$, **tt**. This will turn out to be of great use in Section 5.

In CDT we can also characterize the classes of, for example, the Dedekind-complete frames or the 'iso-choppable' ones (i.e. $F$ consisting of a frame $F'$ with an isomorphic copy of $F'$ glued behind it), but as this has already been shown in [7], resp. [24], for the subsystem HS of CDT, we omit it here.

Now let us return to the model level. Using a trick of Gabbay [6], we might easily show (cf. also the proof of lemma 2.7 in [24]) that the translation $\tau$ may be defined so that $\phi^{\tau}$ uses only *three* variables $x_1, x_2, x_3$ (possibly bound by different quantifiers at different occurrences). For notational simplicity, we drop subscripts of the predicate symbols and modal atomic propositions, and let $Q$ range over the $Q_i$'s.

DEFINITION 3.3
The set $L_3$ of $L$-formulas in at most 3 variables is inductively defined as the smallest set of $L$-formulas containing the atomic formulas $x_i < x_j$, $Qx_ix_j$ and $x_i = x_j$ (with $i, j \in \{1, 2, 3\}$), which is closed under Boolean formula-building, and under the quantifications $\exists x_1$, $\exists x_2$ and $\exists x_3$. $L_3(x_i, x_j)$ denotes the set of $L_3$-formulas having (at most) $x_i$ and $x_j$ as their free variables.

So, every CDT-formula has an equivalent (on the model level) in $L_3(x_i, x_j)$. Now an interesting feature of CDT is that the converse holds as well: we can show that every $L_3(x_1, x_2)$-formula has an equivalent CDT-formula over the class of linear orders. There is a small technical problem, however: in a CDT-model, the truth of formulas is only evaluated at pairs $(s_1, s_2)$ of points for which $s_1 \leq s_2$, while in classical logic we do not have such a constraint. We solve this problem by giving *two* translations $\phi^+$, $\phi^-$ of a classical formula $\phi$, and by introducing adapted CDT-models:

DEFINITION 3.4
Let $M = (F, *)$ be a structure for $L_P$. Introduce, for every predicate symbol $Q$, atomic CDT-formulas $q^+$, $q^-$. Let $M^0$ be the CDT-model with

$$V(q^+) = \{(s, t\} \mid s \leq t \text{ and } (s, t) \in Q^*\}$$
$$V(q^-) = \{(s, t) \mid s \leq t \text{ and } (t, s) \in Q^*\}.$$

Furthermore, we need a handier characterization of $L_3$-formulas. In the sequel $i$, $j$ and $k$ will be such that $\{i, j, k\} \subseteq \{1, 2, 3\}$.

DEFINITION 3.5
$L'(x_i, x_j)$ is inductively defined as follows:

(1) $Qx_ix_i$, $x_i = x_i$, $x_i < x_i$, $Qx_ix_j$, $x_i = x_j$ and $x_i < x_j$ are in $L'(x_i, x_j)$ and in $L'(x_j, x_i)$.
(2) If $\phi$ and $\psi$ are in $L'(x_i, x_j)$, then so are $\neg\phi$ and $\phi \wedge \psi$.
(3) If $\psi \in L'(x_i, x_k)$ and $\xi \in L'(x_k, x_j)$, then $\exists x_k(\psi \wedge \xi)$ is in $L'(x_i, x_j)$.

PROPOSITION 3.6
Every formula in $L_3(x_1, x_2)$ has an equivalent in $L'(x_1, x_2)$.

PROOF. By a straightforward induction on the complexity of $L_3$-formulas we prove the following strengthening of the above proposition:

> Every $L_3$-formula $\phi$ is equivalent to a Boolean combination $\phi'$ of $L'$-formulas, where $\phi'$ has the same free variables as $\phi$.

The only interesting case of the proof is the quantifier step: let $\phi$ be of the form $\exists x_1\ \psi(x_1, x_2, x_3)$, then by Induction Hypothesis $\phi$ is equivalent to $\exists x_1\ \psi'(x_1, x_2, x_3)$, where we may assume that $\psi'$ is a disjunction of a conjunction of $L'$-formulas.

Distribute the $\exists x_1$ over the disjuncts. Let $\exists x_1 \xi(x_1, x_2, x_3)$ be one of the new disjuncts, then $\xi$ can be written as $\xi \equiv \xi_{12} \wedge \xi_{13} \wedge \xi_{23}$, with every $\xi_{ij} \in L'(x_i, x_j)$. Clearly then $\exists x_1 \xi(x_1, x_2, x_3)$ is equivalent to $\xi_{23} \wedge \exists x_1(\xi_{12} \wedge \xi_{13})$, which is a Boolean combination of two $L'(x_2, x_3)$-formulas.

We leave it to the reader to verify that the Boolean combination of $L'$-formulas has the proper free variables. □

We have now come to the main result of this section; first consider

DEFINITION 3.7
By induction on the complexity of $L'$-formulas we define translations ${}^{ij}$:

$$(x_i = x_i)^{ij} \quad = \quad \mathbf{tt}$$
$$(x_i = x_j)^{ij} \quad = \quad \pi$$

$$
\begin{array}{lcl}
(x_j = x_j)^{ij} & = & \mathbf{tt} \\
(x_i < x_i)^{ij} & = & \mathbf{ff} \\
(x_i < x_j)^{ij} & = & \sigma \quad (= \neg\pi) \\
(x_j < x_i)^{ij} & = & \mathbf{ff} \\
(Qx_ix_i)^{ij} & = & \Diamond(\pi \wedge q^+) \\
(Qx_ix_j)^{ij} & = & q^+ \\
(Qx_jx_i)^{ii} & = & q^- \\
(Qx_jx_j)^{ii} & = & \Diamond(\pi \wedge q^+) \\
(\neg\phi)^{ij} & = & \neg\phi^{ij} \\
(\phi \wedge \psi)^{ij} & = & \phi^{ij} \wedge \psi^{ij}
\end{array}
$$

$$(\exists x_k(\psi(x_i, x_k) \wedge \xi(x_k, x_j)))^{ij} = \psi^{ki} D \xi^{kj} \vee \psi^{ik} C \xi^{kj} \vee \xi^{jk} T \psi^{ik}$$

Lemma 3.8. For all $L'(x_i, x_j)$-formulas

$$M \vDash_c \phi(x_i, x_j)[x_i \mapsto s_i, x_j \mapsto s_j] \Leftrightarrow M^0 \vDash \phi^{ij}[s_i, s_j] \text{ if } s_i \leq s_j. \qquad (*)$$

Proof. The proof is of course by induction on the complexity of $L'$-formulas. For the atomic case, the validity of (*) can easily be verified; the Boolean induction step does not cause any problems, so let us suppose $\phi(x_i, x_j) \equiv \exists x_k(\psi(x_i, x_k) \wedge \xi(x_k, x_j))$, and $s_i \leq s_j$.

Then $M \vDash_c \phi(x_i, x_j)[x_i \mapsto s_i, x_j \mapsto s_j]$ iff there is an $s_k$ in $M$ with

$$M \vDash_c \psi(x_i, x_k)[x_i \mapsto s_i, x_k \mapsto s_k] \text{ and } M \vDash_c \xi(x_k, x_j)[x_k \mapsto s_k, x_j \mapsto s_j] \qquad (**)$$

As $M$ is linear, there are only three possibilities for the position of $s_k$ with respect to $s_i$ and $s_j$: either $s_k \leq s_i \leq s_j$ or $s_i \leq s_k \leq s_j$ or $s_i \leq s_j \leq s_k$. Together with the induction hypothesis this gives the equivalence of (**) with

$$
\begin{array}{rlcl}
 & (M^0 \vDash \psi^{ki}[s_k, s_i] & \text{and} & M^0 \vDash \xi^{kj}[s_k, s_j]) \\
\text{or} & (M^0 \vDash \psi^{ik}(s_i, s_k] & \text{and} & M^0 \vDash \xi^{kj}[s_k, s_j]) \\
\text{or} & (M \vDash \xi^{jk}[s_j, s_k] & \text{and} & M^0 \vDash \psi[s_i, s_k]).
\end{array}
$$

So by the truth definition of the CDT-operators, indeed $M \vDash_c \phi(x_i, x_j)[x_i \mapsto s_i, x_j \mapsto s_j]$ iff $M^0 \vDash \phi^{ij}[s_i, s_j]$. □

Now let $\phi(x_1, x_2)$ be an $L_3$-formula. By 3.6, $\phi$ has an equivalent $\phi' \in L'(x_1, x_2)$. Let $\phi^+$ be the formula $(\phi')^{12}$, $\phi^-$ the formula $(\phi')^{21}$. Lemma 9 then gives

Theorem 3.9
For every $L_3$-formula $\phi$:

$$
\begin{array}{rl}
M \vDash \phi[x_1 \mapsto s_1, x_2 \mapsto s_2] \Leftrightarrow & M^0 \vDash \phi^+[s_1, s_2] \text{ if } s_1 \leq s_2 \\
 & M^0 \vDash \phi^-[s_2, s_1] \text{ if } s_2 \leq s_1.
\end{array}
$$

Thus we have established the equivalence of CDT to the three-variable fragment of $L_{PQ}$ over the class of linear models. Now suppose we impose the condition of *locality* upon our semantics:

$$\text{for atomic } p,\ [s, t] \in V(p) \text{ iff } [s, s] \in V(p).$$

Then CDT collapses to $\mathrm{CDT}_{\mathrm{loc}}$, a sort of extended temporal logic of points. Formulas of CDT will then correspond to classical formulas in an extension $L_{\mathrm{PM}}$ of $L_{\mathrm{P}}$ with *monadic* relation symbols $Q_1, Q_2, \ldots$ Now as the three-variable fragment of $L_{\mathrm{PM}}$ is as expressive as $L_{\mathrm{PM}}$ itself over the class of linear orders ([6, 10]), Theorem 3.9 gives a functional completeness result of $\mathrm{CDT}_{\mathrm{loc}}$ with respect to the class of linear orders.

## 4. CDT and classical logic of intervals

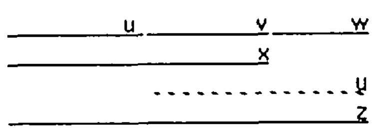
In this section we treat the 'other' correspondence theory of CDT, viz. the one in which we compare CDT with the first order interval language $L_{\mathrm{I}}$ having a triadic relation symbol A and a monadic P. We will not give the formal definition of the straightforward translation and only treat the frame level. First we discuss some formulas which will come back as axioms for the class of $p$-frames later on. The fact that the formulas involved have first order correspondents over the class of $i$-frames is a direct consequence of their Sahlqvist form [4, 23].

PROPOSITION 4.1

$$J \models (\phi C \psi) C \xi \rightarrow \phi C(\psi C \xi) \qquad \text{(M1)}$$

$$\Leftrightarrow J \models \forall uvwz[\exists x(Auvx \wedge Axwz) \rightarrow \exists y(Auyz \wedge Avwy)] \qquad \text{(C1)}$$

PROOF. First, the following picture should make clear what is going on:




Suppose $J \models (C1)$ and there is a valuation $V$ and an interval $z$ with $z \models (\phi C \psi) C \xi$. Then there are $u$, $v$, $w$ and $x$ with $Axwz$ and $w \models \xi$, $Auvx$, $u \models \phi$ and $v \models \psi$. By (C1) there is a $y$ with $Auyz \wedge Avwy$. This means $y \models \psi C \xi$, and then $z \models \phi C(\psi C \xi)$.

For the other direction, suppose $J \not\models (C1)$. Then there are $u$, $v$, $w$, $z$ and $x$ with $Auvx$ and $Axwz$, but for no $y$ we have $Auyz$ and $Avwy$. If $V$ is a valuation with $V(p) = \{u\}$, $V(q) = \{v\}$ and $V(r) = \{w\}$, then it is straightforward to show that $J, V, z \models (pCq)Cr \wedge \neg(pC(qCr))$. □

In the same way one can show that (C1) is characterized by the following

CDT-formulas as well:

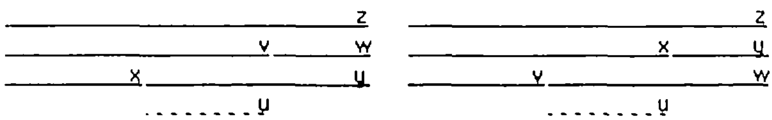
$$\phi T(\psi T\xi) \rightarrow (\phi T\psi)T\xi \qquad \text{(M1')}$$
$$\psi D(\phi T\xi) \rightarrow \phi T(\psi D\xi) \qquad \text{(M1'')}$$

Now suppose $J \models (\sigma C\sigma) \rightarrow \sigma$. Then (C1) implies transitivity of the starting relation of intervals: if $u$ starts $x$ and $x$ starts $z$ then $u$ starts $z$. This transitivity is reflected in (M1) if we take both $\phi$ and $\psi$ to be $\sigma$. (1) then reads as $\Diamond\Diamond\xi \rightarrow (\sigma C\sigma)T\xi$, and as $\sigma C\sigma$ implies $\sigma$, we get $\Diamond\Diamond\xi \rightarrow \Diamond\xi$, a familiar correspondent of transitivity.

PROPOSITION 4.2

$$J \models \phi T(\psi C\xi) \rightarrow [\psi C(\phi T\xi)V(\xi T\phi)T\psi] \qquad \text{(M2)}$$
$$\Leftrightarrow J \models \forall vwxy[\exists z(Avwz \wedge Axyz)$$
$$\rightarrow \exists u[(Axuv \wedge Auwy)V(Avux \wedge Auyw)]] \qquad \text{(C2)}$$

PROOF. The proof runs in the same line as the previous one. The following picture may be of use; here $v$ should be seen as the current interval and $\phi$, $\psi$ and $\xi$ as holding in $w$, $x$ and $y$.



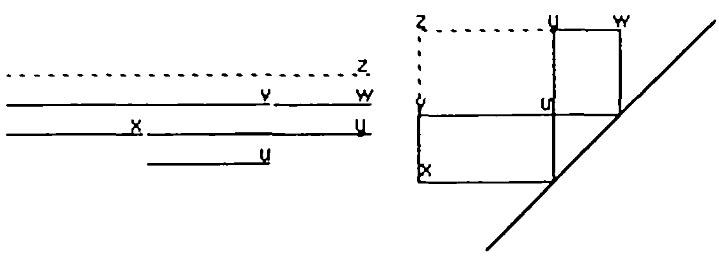

Note that in (C2) some kind of 'internal linearity' is involved: if both meeting points, of $v$ and $w$ and of $x$ and $y$, are in the interval $z$, then one of them is before the other (or they are identical).

In some respect the following pair of formulae can be seen as the converses of the previous ones:

PROPOSITION 4.3

$$J \models \psi C(\phi T\xi) \rightarrow \phi T(\psi C\xi) \qquad \text{(M3)}$$
$$\Leftrightarrow J \models \forall vwxy[\exists u(Axuv \wedge Auwy) \rightarrow \exists z(Axyz \wedge Avwz)] \qquad \text{(C3)}$$

PROOF. Again we only give a picture, and here its two-dimensional counterpart too:



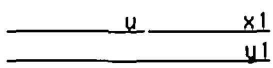
Again, $v$ is the current interval and in the proof one should consider a valuation $V$ with $V(\phi) = \{w\}$, $V(\psi) = \{x\}$ and $V(\xi) = \{y\}$. □

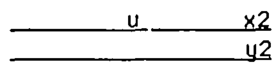
Note that (C3) implies what is called in [24] 'north-western directedness': if $v$ and $y$ are respectively west and north of $u$, then there is a $z$ north of $v$ and west of $y$. One can easily show that this property is characterized by the HS-formula $\langle E\rangle\langle N\rangle\xi \rightarrow \langle N\rangle\langle E\rangle\xi$, the same one we obtain by taking both $\phi$ and $\psi$ to be $\sigma$ in (M3).

PROPOSITION 4.4

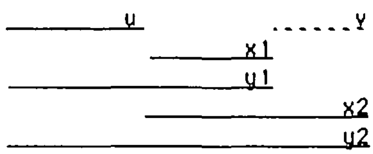
$$J \models (\phi T\psi)C\xi \rightarrow [(\xi D\phi)T\psi \vee \psi C(\phi D\xi) \qquad \text{(M4)}$$
$$\Leftrightarrow J \models \forall x1x2y1y2[\exists u(Aux1y1 \wedge Aux2y2) \rightarrow$$
$$\exists v[(Ax1vx2 \wedge Ay1vy2) \vee (Ax2vx1 \wedge Ay2vy1)]] \qquad \text{(C4)}$$

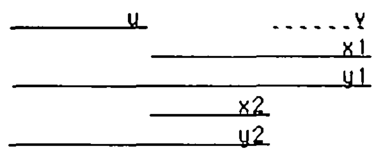
PROOF. (C4) expresses that if we have both



and



then there is an interval $v$ connecting the $x1$, $y1$ and the $x2$, $y2$-endpoints; either



or



If we keep this in mind, the proof is straightforward: take $y1$ as the current interval and let $x1$ be the $\xi$-interval, $x2$ the $\phi$-one and $y2$ the $\psi$-one. □

Here we may consider (C4) as a sort of generalization of 'external linearity': if $x1$ and $x2$ have the same beginning point then one of them starts the other, or they are identical.

We conclude this part by mentioning two correspondences involving the $\pi$-constant: Proposition 15 says that a stretched interval cannot start a point, (16) is about chopping off the starting-point interval, leaving the current interval undamaged and (17) expresses the fact that two intervals $u$ and $w$ have the same starting-point interval if $u$ starts $w$.

PROPOSITION 4.5

$$J \models \pi \rightarrow \neg(\neg\pi \mathbf{tt}) \qquad \text{(M5)}$$
$$J \models \forall w(Pw \rightarrow \forall uv(Auvw \rightarrow Pu). \qquad \text{(C5)}$$

Proposition 4.6

$$J \models \pi C\phi \leftrightarrow \phi \tag{M6}$$

$$\Leftrightarrow J \models \forall vw(\exists u(Auvw \wedge Pu) \leftrightarrow v = w) \tag{C6}$$

Proposition 4.7

$$J \models [(\pi \wedge \phi)C\mathbf{tt} \wedge ((\pi \wedge \psi)C\mathbf{tt})C\mathbf{tt}] \rightarrow (\pi \wedge \psi)C\mathbf{tt} \tag{M7}$$

$$J \models \forall uvwxypq[Auvw \wedge Apux \wedge Aqwy \wedge Pp \wedge Pq \rightarrow p = q] \tag{C6}$$

The above results show that CDT can express many essential properties of two-dimensional frames. However, CDT cannot *characterize* this class $K_2$: to show this, we apply a version of the well known theorem of preservation of modal validity under zigzagmorphisms:

Definition 4.8

Let $J = (I, A, P)$ and $J' = (I', A', P')$ be two $i$-frames. A function $f: I \mapsto I'$ is a *zigzagmorphism from $J$ onto $J'$ if*

(1) $f$ is surjective
(2) $f$ is a homomorphism
(3) $P'fu \Rightarrow Pu$
(4) Assume $A'u'v'w'$.

If $f(u) = u'$ then there are $v$, $w$ with $Auvw$, $f(v) = v'$ and $f(w) = w'$.
If $f(v) = v'$ then there are $u$, $w$ with $Auvw$, $f(u) = u'$ and $f(w) = w'$.
If $f(w) = w'$ then there are $u$, $v$ with $Auvw$, $f(u) = u'$ and $f(v) = v'$.

Proposition 4.9

If $f$ is a zigzagmorphism from $F$ onto $F'$ then for all CDT-formulas:

$$F \models \phi \Rightarrow F' \models \phi.$$

The *proof* of this proposition is like in ordinary modal logic [4]. The following theorem expresses the fact that there is no (set of) CDT-formula(e) characterizing the two-dimensional frames.

Theorem 4.10

Let $\Phi$ be a set of CDT-formulas. If $J \models \Phi$, for all two-dimensional frames $J$, then there is a $J'$ not in $F_2$ with $J' \models \Phi$.

Proof. Consider the $i$-frames $F = I(\mathbb{Q}, <)$ and $F' = (I, A, P)$ with $I = \{q \in \mathbb{Q} \mid q \geq 0\}$, $A = \{\langle p, q, r\rangle \mid p + q = r\}$ and $P = \{0\}$. Let $f$ be the function from INT$(\mathbb{Q}, \leq)$ onto $I$ mapping intervals on their length, i.e. $f([p, q]) = q - p$. It is straightforward to verify that $f$ is a zigzagmorphism, so if $\Phi$ is a set of CDT-formulas valid on $F_2$, by the previous proposition we have $F' \models \phi$ for all $\phi$ in $\Phi$. Obviously, $F'$ is not two-dimensional; e.g. it does not satisfy the property C8 below, as every 'interval' in $F'$ has 0 as its starting point and 0 as its endpoint. □

In the first-order language with predicates $A$ and $P$ we don't have any problems in defining two-dimensional frames. We might do this by defining the **meets** relation $|(u \mid v \equiv \exists w A u v w)$ and then proceeding like Allen and Hayes [2] (cf. [13], Ch. 5), where points are defined as equivalence classes of meeting pairs of intervals. In this way we wouldn't even need the $P$-predicate (except for some defining formulas, as our $i$-frames need an interpretation for $P$); we choose to give a proof in our own language here.

DEFINITION 4.11
Consider the following formulas:

$$\forall puvq(Pp \wedge Pq \wedge Apuu \wedge Apvv \wedge Auqu \wedge Avqv \rightarrow u = v) \tag{C8}$$

$$\forall pq(Pp \wedge Pq \rightarrow [p = q \textbf{ or } \exists u(\neg Pu \wedge Apuu \wedge Auqu) \textbf{ or } \exists u(\neg Pu \wedge Aquu \wedge Aupu)]) \tag{C9}$$

where **or** is the exclusive or, i.e. $\phi \textbf{ or } \psi \equiv (\phi \vee \psi) \wedge \neg(\phi \wedge \psi)$.
Let TWO-DIM be the conjunction of (C1)–(C9) and their mirror images, where the mirror image of a formula $\phi$ is obtained by replacing all occurrences of a predicate $Av_1v_2v_3$ in $\phi$ by $Av_2v_1v_3$.

THEOREM 4.12
Let $J = (I, A, P)$ be an $i$-frame. Then

$$J \models \text{TWO-DIM} \Leftrightarrow J \text{ is two-dimensional.}$$

PROOF. We only prove the direction from left to right: suppose POINT-BASED holds in $J$; we will show that $J$ is isomorphic to a point frame based on the set of point-intervals $P$.

(1) First we need the fact that $A$ is functional in all its arguments, i.e.

$$\forall uvww'(Auvw \wedge Auvw' \rightarrow w = w')$$
$$\forall uu'vw(Auvw \wedge Au'vw \rightarrow u = u').$$

For, suppose $Auvw \wedge Auvw'$; by (C6) there are $p$, $q$, $p'$, $q'$, $p''$, $q''$, in $P$ with $Apuu$, $Ap'ww$, $Ap''w'w'$, $Awq'w$ and $Aw'q''w'$; by (C7) $p' = p = p''$ and $q' = q = q''$, so by (C8), $w = w'$.

We can now define the linear ordering on $P$: set, for $p, q \in P$, $p < q$ if there is a $u \notin P$ with $Apuu$ and $Auqu$.

We prove the following:

(2) $\forall pq(p < q \textbf{ or } p = q \textbf{ or } q < p)$: this is immediately by (C9)
(3) $<$ is transitive: suppose $p < q < r$ as $Apuu$, $Auqu$, $Aqvv$ and $Avrv$.

By (C3) there is a $w$ such that $Auvw$, by (C4) $w \notin P$. By (C1) there is a $w'$ with $Apw'w$ and $Auvw'$, (1) gives $w = w'$ whence $Apww$. In the same manner we prove $Awrw$, so $p < r$.

Now let $u \in I$; by (C6) and (C7) we may define unique intervals $Lu$ and $Ru$ such that

$$ALuuu,\ AuRuu,\ Lu \in P,\ Ru \in P.$$

Of course, $Lu$ and $Ru$ are to be seen as the beginning- and endpoint of $u$. It will be clear that $u \in P \Rightarrow Lu \leq Ru$, $u \in P \Rightarrow Lu = Ru$. Now we are ready to show that the map $f : I \mapsto I(P, <)$, given by

$$f(u) = (Lu, Ru) \text{ is the desired isomorphism.}$$

(5) $f$ is a bijection: surjectivity is immediate by definition of $<$; for injectivity, suppose $f(u) = f(v)$. Then $Lu = Lv$ and $Ru = Rv$, so by (C8), $u = v$.
(6) $f$ preserves $A$ and $P$. For $P : u \in P \Rightarrow Lu = Ru \Rightarrow f(u) \in P'$. For $A$: suppose $Auvw$; we must show $Lu = Lw$, $Rv = Rw$ and $Ru = Lv$, of which we only prove the latter: by (C2) there is an $x$ with $Auxu$ and $Axvv$. By (C7) we have $x = Ru$ and $x = Lv$, so $Ru = Lv$.
(7) $f$ anti-preserves $A$ and $P$. For $P$: $u \notin P \Rightarrow Lu < Ru \Rightarrow Lu \neq Ru$ by (2). For $A$: suppose $A'f(u)f(v)f(w) \Rightarrow$ by (C3) there is a $z$ with $Auvz$; by (C7), $Lu = Lz$ and $Rv = Rz$. This implies $f(z) = f(w)$, so by injectivity of $f$ we get $z = w$, which implies $Auvw$. $\Box$

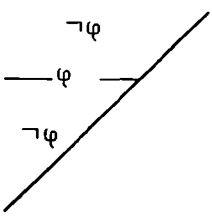
## 5. Axiomatic completeness for CDT

In this section and the next we turn to the matter of recursively enumerating the CDT-formulas valid in several classes of frames. As the classes of frames involved are all first-order definable (viz. the linear, the dense and the discrete flows of time), it is quite easy to show that the set of valid CDT-formulas is recursively enumerable: the easiest proof uses the embedding of CDT-formulas into first order logic which we gave above. The problem is to give *explicit* derivation systems. In this section we take an axiomatic approach, in the next we give a natural deduction system.

The axiom system of this section has a derivation rule (CR) for which we need the following definition: let

$$\mathrm{HOR}(\phi) \equiv \boxminus\phi \wedge \Box\boxminus\neg\phi \wedge \Box\boxminus\neg\phi,$$

then in a two-dimensional model, $M, [s, t] \models \mathrm{HOR}(\phi)$ iff $\phi$ holds at all points on the horizontal line through $(s, t)$, viz.



DEFINITION 5.1

The axiomatic system $ACDT$ consists of

(1) (all substitution instances of) the following *axioms* and their mirror images:

A all propositional tautologies

B $(\phi \vee \psi)C\chi \leftrightarrow \phi C\chi \vee \psi C\chi$

$(\phi \vee \psi)T\chi \leftrightarrow \phi T\chi \vee \psi T\chi$

$\phi T(\psi \vee \chi) \leftrightarrow \phi T\psi \vee \phi T\chi$

C 1 $\neg(\phi T\psi)C\phi \rightarrow \neg\psi$

2 $\neg(\phi T\psi)D\psi \rightarrow \neg\phi$

3 $\phi T\neg(\psi C\phi) \rightarrow \neg\psi$

D $\sigma C\mathbf{tt} \leftrightarrow \sigma$

E 1 $\pi C\phi \leftrightarrow \phi$

2 $\pi T\phi \leftrightarrow \phi$

3 $\phi T\pi \leftrightarrow \phi$

F $[(\pi \wedge \phi)C\mathbf{tt} \wedge ((\pi \wedge \psi)C\mathbf{tt})C\mathbf{tt}] \rightarrow (\pi \wedge \psi)C\mathbf{tt}$

G 1 $\phi C(\psi C\psi) \leftrightarrow (\phi C\psi)C\chi$

2 $\phi T(\psi T\chi) \leftrightarrow (\psi C(\phi T\chi) \vee (\chi T\phi)T\psi$

3 $\psi C(\phi T\xi) \rightarrow \phi T(\psi C\xi)$

4 $(\phi T\psi)C\xi \rightarrow [(\xi D\phi)T\psi \vee \psi C(\phi D\xi)$

(2) the following *derivation* rules:

MP Modus Ponens:

if $\phi$ and $\psi$ are theses, then so is $\psi$.

G Generalization:

if $\phi$ is a thesis, then so are $\neg(\neg\phi C\psi)$, $\neg(\neg\phi T\psi)$, $\neg(\psi T\neg\phi)$ and their mirror images.

CR Consistency Rule:

if $\mathrm{HOR}(q) \rightarrow \phi$ is a thesis, with $q$ an atomic constant not occurring in $\phi$, then $\phi$ is a thesis as well.

## 5.2. The meaning of the axioms

The B-axioms, expressing distributivity, are needed for any binary existential modal operator; the C-axioms correspond, in a certain sense, to the axioms $\phi \rightarrow GP\phi$ and $\phi \rightarrow HF\phi$ in ordinary temporal logic. There these axioms are needed to ensure that the accessibility relation associated with the $F$-operator is the converse of the one associated with the $P$-operator. In the case of CDT-logic something similar is going on: as we have three binary modal operators, in general the frames for the logic will have three ternary accessibility relations $R_c$, $R_d$ and $R_t$. Just like in the $F$, $P$-case described above, the E-axioms are expressing that $R_c$ $R_d$ and $R_t$ are no more than 'directions' of one ternary relation $A$.

The meaning of the other axioms is explained in the previous section.

We can now establish our completeness result. As the completeness proof is quite complex from a technical point of view, yet a straightforward generalization of the proof given in [24] for the subsystem HS of CDT, we confine ourselves to a very rough sketch:

THEOREM 5.2.1 (Soundness and Completeness)
A CDT-formula is a thesis iff it is valid on the class of $p$-frames:

$$\vdash \phi \Leftrightarrow \models \phi.$$

PROOF. The soundness part of the proof is straightforward.

In the completeness proof, we build, for an arbitrary maximal ACDT-consistent set $\Sigma$ of formulas, a model for $\Sigma$. This model $M_\Sigma = (T, V)$ is built up in stages, in each of which we deal with a finite approximation of $M_\Sigma$. Such an approximation has the form of a pair $\lambda_n = (T_n, \Lambda_n)$ where $T_n = \{s_0, s_1, \ldots, s_n\}$ is a finite subset of $T$ and $\Lambda_n$ is a function mapping every $T_n$-interval $[s_i, s_j]$ onto a maximal consistent set. $\Lambda_n([s_i, s_j])$ is to be seen as the set of formulas holding at $[s_i, s_j]$ in the final model $M_\Sigma$; $\Lambda_1$ maps $[s_0, s_1]$ to $\Sigma(*)$.

Every $\Lambda_n$ must satisfy conditions like

$$\pi \in \Lambda_n([s_i, s_j]) \text{ iff } s_i = s_j$$

and

$$\text{If } \phi \in \Lambda_n([s_i, s_j]) \text{ and } \psi \in \Lambda_n([s_j, s_k]), \text{ then } \phi C \psi \in \Lambda_n([s_i, s_k]).$$

The ACDT-axioms ensure that we may always choose maximal consistent sets satisfying these conditions.

As an approximation has finitely many intervals, in general it will have *defects* like

$$\text{there is a } \phi C \psi \in \Lambda_n([s_i, s_k]) \text{ yet no } s_j \in T_n \text{ with } \phi \in \Lambda_n([s_i, s_j])$$
$$\text{and } \psi \in \Lambda_n([s_j, s_k]).$$

The construction of the $\Lambda_n$'s will be such that every defect will eventually be repaired, e.g. for the example: there will be a $m > n$ with an $s_j$ in $T_m$ such that $\phi \in \Lambda_m([s_i, s_j])$ and $\psi \in \Lambda_m([s_j, s_k])$.

The Consistency Rule is needed to implement these repairments.

The above sketched procedure will yield a chain of approximations of which the union $(\bigcup_{n\in\omega} T_n, \bigcup_{n\in\omega} \Lambda_n)$ does not have any defects. Then, defining $M_\Sigma$ by $T = \bigcup_{n\in\omega} T_n$ and $V(q) = \{[s_i, s_j] \mid q \in \Lambda_n([s_i, s_j])$, we can prove the *truth lemma*

$$\phi \in \Lambda_n([s_i, s_j]) \Leftrightarrow M_\Sigma \models \phi[s_i, s_j].$$

So by (*), $M_\Sigma$ is a model for $\Sigma$. $\square$

We can now proceed to define sound and complete axiom systems for the classes of the dense and the discrete frames (cf. definition 2):

DEFINITION 5.2.2

(1) $\mathrm{ACDT}_{de}$ is the axiom system ACDT where we have added the formula DENSE as an axiom.
(2) $\mathrm{ACDT}_{di}$ is the axiom system ACDT where we have added the formula DISCRETE as an axiom.
(3) $\mathrm{ACDT}_{Q}$ is the axiom system ACDT where we have added the formulas DENSE, $\Diamond$**tt** and $\Diamond$**tt** as axioms.

THEOREM 5.2.3

(1) $\mathrm{ACDT}_{de} \vdash \phi$ iff $\phi$ is valid on the class of dense $p$-frames
(2) $\mathrm{ACDT}_{di} \vdash \phi$ iff $\phi$ is valid on the class of discrete $p$-frames
(3) $\mathrm{ACDT}_{Q} \vdash \phi$ iff $F$ is valid on $(\mathbb{Q}, <)$.

PROOF. The proofs for these propositions are more or less the same: Soundness is straightforward; for completeness, we simply copy the completeness proof for ACDT. Then, arriving at the truth lemma, we observe that the formula DENSE (resp. DISCRETE, resp. DENSE $\wedge \Diamond$**tt** $\wedge \Diamond$**tt**) is an element of every MCS, so DENSE (resp. ...) is valid on the model we constructed, and as this formula only uses the propositional constants $\pi$ and **tt**, it is valid on the underlying frame as well. It is then straightforward (cf. proposition 3.2.) to verify that this frame is dense (resp. discrete, resp. isomorphic to the ordering of the rationals). □

## 5.3. Undecidability

For several classes $K$ of linear orders, Halpern and Shoham showed in [7] that the problem whether an HS-formula is valid on every $K$-frame, is (highly) undecidable. This means that the above given recursive enumerations are about the best results we can get for CDT, or for any subsystem of CDT containing HS. Such a fact cannot come as a big surprise, because any two-dimensional modal logic is a (considerable) fragment of second-order logic with infinitely many binary predicates, whereas the first-order logic with one binary predicate is already known to be undecidable.

Still, there might be useful fragments of the language which are decidable: e.g. the Sahlqvist forms ([4]) have first-order equivalents, so by decidability of the theory of linear orders, decidability for the Sahlqvist fragment follows immediately. It might also be interesting to try to give a sound and complete axiom system for $\mathrm{CDT}_{loc}$ (cf. the last remark of section 3), because $\mathrm{CDT}_{loc}$ is expressively complete with respect to universal *monadic* second-order logic, and so we know the set of locally valid CDT-formulas to be decidable, for e.g. $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, all linear orders, etc.

### 5.4. Some connections

In the introduction we mentioned the paper [19] by Nishimura with which we only got acquainted after finishing this paper. Nishimura defines an axiom system $I$ for his logic. It is very interesting to see that his system and the way he proceeds to prove its completeness are very similar to an approach taken in the theory of Relation Algebras, where the aim is to give an axiomatization of the class of Representable Relation Algebras ([14, 16]). The resemblance between these approaches lies in the fact that both axiom systems have infinitely many axioms, each of which more or less describes, in an exhaustive manner, all possible positions of $n$ points. In our ACDT we can deal with finitely many axioms because of having the consistency rule. Later, we hope to be able to give a more elaborate comparison of Nishimura's work and our results.

## 6. A natural deduction system

### 6.1. Introduction

As was mentioned in the introduction, there is a close link between the algebraic theory of binary relations and the logic CDT. Here we will use a result by William Wadge, who gave in [26] a sound and complete natural deduction system for the identities holding in RRA, the variety of so-called *Representable* Relation Algebras: as usual in algebraic logic, representation theorems for algegras correspond to completeness theorems in logics. As in Wadge's approach, the idea of our proof is to treat a CDT-formula as a sort of compound relation symbol in a classical formalism, and then to give an enumeration of the formulas in this language which are valid on the class of linear orders. Following Wadge, we only take special care of the fact that our semantics presupposes the frames to be linear orderings. This explains the presence of the symbols '$S$' ($\approx<$) and '$R$' ($\approx\leq$) in the target set of predicate formulas, and the need to give deduction rules for these symbols.

DEFINITION 6.1.1
The set $AT$ of *atomic terms* consists of all CDT-formulas and the symbols $S$, $R$ and $\Omega$. The set $T$ of *terms* is defined as the closure of $AT$ under Boolean operators. A *($V$-)formula* is an expression of the form $x\alpha y$, where $x$ and $y$ are variables and $\alpha$ is a term. The *negation* $\neg F$ of a formula $F \equiv x\alpha y$ is defined as $\neg F \equiv x\neg\alpha y$. For a set $\Gamma$ of $V$-formulas we denote the set of variables occurring in $\Gamma$ by $Var(\Gamma)$.

Note that with this definition, the set of CDT-formulas is a (proper) subset of the set of terms. Note too that a V-formula $x_1(q_1Cq_2)x_2$ can be seen as an abbreviation of $\exists x_3(x_1q_1x_3 \wedge x_3q_2x_2)$, which is more or less the $L_{PQ}$-correspondent (cf. Section 2) of the CDT-formula $q_1Cq_2$. In fact, the whole deductive system defined below may be seen as a calculus directed towards recursively enumerating the set of CDT-correspondents valid on the class of linear orders.

Definition 6.1.2
A ($V$-)*model* is a pair $M = (D, <, b)$, where $(D, <)$ is a linear frame and $b$ is a function mapping variables on elements of $D$ and terms on subsets of $\mathrm{INT}(D)$. Furthermore, $b$ is required to satisfy the following properties:

(1) $b(\pi) = \{(s, s) \mid s \in D\}$, $b(S) = <$, $b(R) = \mathrm{INT}(D)$, $b(\Omega) = \varnothing$.
(2) $b(\alpha \wedge \beta) = b(\alpha) \cap b(\beta)$, etc.
(3) $b(\alpha C \beta) = \{(s, t) \in \mathrm{INT}(D) \mid \exists u \in D[(s, u) \in b(\alpha) \wedge (u, t) \in b(\beta)]\}$,
$b(\alpha T \beta) = \{(s, t) \in \mathrm{INT}(D) \mid \exists u \in D[(s, u) \in b(\beta) \wedge (t, u) \in b(\alpha)]\}$,
$b(\alpha D \beta) = \{(s, t) \in \mathrm{INT}(D) \mid \exists u \in D[(u, s) \in b(\alpha) \wedge (u, t) \in b(\beta)]\}$.

Definition 6.1.3
Let $M$ be a model, $F$ a formula, say $x\alpha y$, and $\Gamma$ a set of formulas. *$F$ is true in $M$, $M$ satisfies $F$,* or *$M$ is a model for $F$,* notation $M \models F$, if $(b(x), b(y))$ is in $b(\alpha)$. *$\Gamma$ is true in $M$* if all its formulas are, *$F(\Gamma)$ is valid,* notation $\models F$ ($\models \Gamma$), if it is true in all models. *$F$ follows semantically from $\Gamma$,* or *$\Gamma$ implies $F$* if $F$ is true in all the models for $\Gamma$.

Lemma 6.1.4
For CDT-formulas $\phi$, $\phi$ is valid on the class of linear models iff $x(\neg RV\phi)y$ is valid.

Proof. By a straightforward induction on the complexity of $\phi$. □

Thus we may obtain a recursive enumeration of all valid CDT-formulas by doing so for the valid entailments $\models F$.

Definition 6.1.5
We give a *natural deduction system* for V-formulas by defining a notion '$\vdash$' of *deducibility* between sets of formulas and formulas. Formally, $\vdash$ is defined as the smallest relation for which $\{\phi\} \vdash \phi$ holds and which is closed under the following *rules of inference*:

In these rules $x, y$ and $z$ are arbitrary variables, $\phi$ is an arbitrary CDT-formula, $\alpha$ and $\beta$ are arbitrary terms, $F$ and $G$ are arbitrary formulas and $\Gamma$ is an arbitrary set of formulas, with the exception that the rules $C^-$, $D^-$, and $T^-$ and may only be applied when $z \notin \mathrm{Var}(\Gamma)$.

$$(F^+)\ \frac{\Gamma \vdash F}{\Gamma, G \vdash F} \qquad\qquad (F^-)\ \frac{\Gamma \vdash G \quad \Gamma, G \vdash F}{\Gamma \vdash F}$$

$$(\wedge^+)\ \frac{x\alpha y \quad x\beta y}{x\alpha \wedge \beta y} \qquad\qquad (\wedge^-)\ \frac{x\alpha \wedge \beta y}{x\alpha y} \quad \frac{x\alpha \wedge \beta y}{x\beta y}$$

$$(\vee^+)\ \frac{x\alpha y}{x\alpha y \vee \beta y}\quad \frac{x\beta y}{x\alpha \vee \beta y} \qquad (\wedge^-)\ \frac{\Gamma, x\alpha y \vdash F \quad \Gamma, x\beta y \vdash F \quad \Gamma \vdash x\alpha \vee \beta y}{\Gamma \vdash F}$$

$$(\neg^+)\ \frac{\Gamma, x\alpha y \vdash z\Omega z}{\Gamma \vdash x\neg\alpha y} \qquad (\neg^-)\ \frac{\Gamma, x\alpha y \vdash F \quad \Gamma, x\neg\alpha y \vdash F}{\Gamma \vdash F}$$

$$(\Omega^+)\ \frac{x\alpha y \quad x\neg\alpha y}{z\Omega z} \qquad (\Omega^-)\ \frac{x\Omega x}{F}$$

$$(C^+)\ \frac{x\alpha z,\ z\beta y}{x\alpha C\beta y} \qquad (C^-)\ \frac{\Gamma, x\alpha z, z\beta y \vdash F \quad \Gamma \vdash x\alpha C\beta y}{\Gamma \vdash F}$$

$$(T^+)\ \frac{xRy,\ x\alpha z,\ y\beta z}{x\alpha T\beta y} \qquad (T^-)\ \frac{\Gamma, xRy, x\alpha z, x\beta z \vdash F \quad \Gamma \vdash x\alpha T\beta y}{\Gamma \vdash F}$$

$$(D^+)\ \frac{xRy,\ z\alpha x,\ z\beta y}{x\alpha D\beta y} \qquad (D^-)\ \frac{\Gamma, xRy, z\alpha x, z\beta y \vdash F \quad \Gamma \vdash x\alpha D\beta y}{\Gamma \vdash F}$$

$$(R^+)\ \frac{xS \vee \pi y}{xRy} \qquad (R^-)\ \frac{xRy}{xS \vee \pi y}$$

$$(S_1)\ \frac{xSy \quad ySz}{xSz} \qquad (\pi_1)\ \frac{}{x\pi x}$$

$$(S_2)\ \frac{x\neg Sy \quad x\neg\pi y}{ySx} \qquad (\pi_2)\ \frac{x\pi y \quad y\pi z}{x\pi z}$$

$$(S_3)\ \frac{xSy \quad ySx}{x\Omega x} \qquad (\pi_3)\ \frac{x\pi y}{y\pi x}$$

$$(S_4)\ \frac{x\phi y}{xRy} \qquad (\pi_4)\ \frac{x\alpha y \quad y\pi z}{x\alpha z}$$

Note that, although in the above formulation we only have three (meta)variables $x$, $y$ and $z$, proofs in general may need more variables. We conjecture that the situation is just like the one for Relation Algebras: let $\Gamma \vdash_n F$ denote 'there is a derivation of $F$ from $\Gamma$ which uses at most $n$ variables'. Maddux showed in [15] that the sets $V_n$ of formulas given by $V_n = \{F \mid \vdash_n F\}$ form a *strictly* increasing sequence.

DEFINITION 6.1.6

A *theory* is a set formulas which is closed under $\vdash$, i.e. $\Gamma \vdash F$ implies $F \in \Gamma$. A theory $\Gamma$ is *consistent* if for no $x$, $x\Omega x$ is deducible from $\Gamma$, it is *complete* if for every CDT-term $\alpha$ and variables $x, y$, one of $x\alpha y$, $y\alpha x$, $x\neg\alpha y$ or $y\neg\alpha x$ belongs to $\Gamma$.

A theory $\Gamma'$ is a *saturation* of $\Gamma$ if it contains $\Gamma$ and satisfies the following condition for the operator $C$:

whenever $x\alpha C\beta y$ is in $\Gamma$, there is a $z$ with $x\alpha z$ and $z\alpha y$ in $\Gamma'$

and the analogons for the operators $D$ and $T$. (Note that in such a case $\Gamma'$ might need more variables than $\Gamma$.)

A theory $\Gamma$ is *saturated* if it is a saturation of itself.

THEOREM 6.1.7 (Soundness)
If $\Gamma \vdash F$ then $\Gamma \models F$.

PROOF. Straightforward. □

We now proceed to prove completeness, i.e. we want to show that $\Gamma \models \phi$ implies $\Gamma \vdash \phi$. As usual this is done by contraposition: given a $\Gamma$ and an $F$ for which $\Gamma \nvdash F$, we will construct a model for $\Gamma$ in which $F$ is not true. It easily follows from the inference rules for negation that this can be done by showing that every consistent theory has a model. We need the following lemmas:

LEMMA 6.1.3
Every consistent theory has a complete extension in the same variables.

PROOF. By the standard Lindenbaum construction. □

LEMMA 6.1.9
Every consistent theory has a consistent saturation.

PROOF. Suppose $\Gamma$ is a consistent theory, $x\alpha C\beta y \in \Gamma$ but there is no $z$ in $\Gamma$ with $x\alpha z$ and $z\beta y$ in $\Gamma$. Let $z$ be a variable not occurring in $\Gamma$. We claim that the set $\Gamma \cup \{x\alpha z, z\beta y\}$ is consistent. For, suppose this is not the case, then $\Gamma, x\alpha z, z\beta y \vdash a\Omega a$. As $x\alpha C\beta y \in \Gamma$ we have $\Gamma \vdash x\alpha C\beta y$. Applying the rule $C^-$ this gives $\Gamma \vdash a\Omega a$, contradicting the consistency of $\Gamma$.

So, using a similar procedure for every quintuple like the above $(x, y, C, \alpha, \beta)$, we can find extensions $\Gamma_1 \subseteq \Gamma_2 \subseteq \Gamma_3 \ldots$ of $\Gamma$ such that the union of these extensions will be a consistent saturation of $\Gamma$. (By introducing a *new* variable $z_i$ for every $\Gamma_i$ we can ensure that no added formulae in different $\Gamma_i$'s will conflict.) □

LEMMA 6.1.10
Every consistent theory has a complete, saturated extension.

PROOF. Using the previous two lemmas, we can construct for any consistent theory $\Gamma$, a sequence of consistent theories $\Gamma = \Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \Gamma_3 \ldots$ satisfying

(1) every $\Gamma_{2i+1}$ is a complete extension of $\Gamma_{2i}$,
(2) every $\Gamma_{2i+2}$ is a saturation of $\Gamma_{2i+1}$.

The union of these theories will be the complete, consistent and saturated extension of $\Gamma$ we were looking for. □

THEOREM 6.1.11 (Completeness)
Every consistent theory has a model.

PROOF. Let $\Gamma$ be a consistent theory. By the previous lemma we may assume for this proof that $\Gamma$ is complete and saturated.

Define the following relation $\equiv$ on $\mathrm{Var}(\Gamma)$: $x \equiv y$ if $x\pi y$ is in $\Gamma$. Using the rules $(\pi_1)$, $(\pi_2)$ and $(\pi_3)$ one can easily verify that $\equiv$ is an equivalence relation. Let $[x]$ be the equivalence class of the variable $x$ and define $D = \mathrm{Var}(\Gamma)/\!\equiv\, = \{[x] \mid x \in \mathrm{Var}(\Gamma)\}$.

We want $D$ to be the domain of our model; we set $b(x) = [x]$ for variables $x$, $b(\alpha) = \{([x], [y]) \mid x\alpha y \in \Gamma\}$. (Rule $\pi_4$ ensures that this is a correct definition). The ordering $<$ is defined as $b(S)$. It is easily proved that $<$ is a strict linear ordering, the rules $S1$, $S2$ and $S3$ giving that $<$ is resp. transitive, linear and asymmetric. That $b(\alpha) \subseteq\, \leq$ for any $\alpha$, follows from $S4$. Condition (1) of Definition 6.12 is satisfied by definition of $D$ and $b$, and by the rules $R^+$ and $R^-$, (2) is obtained by using the inference rules for the Boolean connectives. Of condition (3) we only prove the part for $C$:

$$b(\alpha C\beta) = \{(s, t) \in \mathrm{INT}(D) \mid \exists u \in D[(s, u) \in b(\alpha) \wedge (u, t) \in b(\beta)]\}.$$

This statement is equivalent to the following ones:

$$([x], [y]) \in b(\alpha C\beta) \text{ iff there is a } z \text{ with } ([x], [z]) \in b(\alpha) \text{ and } ([z], [y]) \in b(\beta)$$
$$x\alpha C\beta y \in \Gamma \text{ iff there is a } z \text{ with } x\alpha z \text{ and } z\beta y \text{ in } \Gamma.$$

Now of the above equivalence the direction from right to left is immediate, as by definition theories are closed under deduction and $x\alpha z,\ z\beta y \vdash x\alpha C\beta y$ by rule $C^+$. The other direction is just the statement that $\Gamma$ is saturated.

It then only remains to be shown that $M = (D, <, b)$ is a model for $\Gamma$. But by definition of $D$, $<$ and $b$ it is almost immediate that $M \models F$ for all formulas $F$ in $\Gamma$. □

We finish this paper with a short comparison of the two deductive systems we have developed: we feel that in the axiomatic system, it will be very hard to find actual derivations of theorems in which the Consistency Rule is needed, whereas in the natural deduction system derivations are usually very intuitive. On the other hand, the disadvantage of the latter method is that it expands the formalism by adding point variables, thus violating the paradigm of the modal *variable-free* approach.

## References


[1] J. F. Allen (1983). Maintaining knowledge about temporal intervals, *Artificial Intelligence*, **26**, 832–43.

[2] J. F. Allen and P. J. Hayes (1987). Short time periods, *Proceedings of the 10th International Joint Conference on Artificial Intelligence*, Milano, Italy. Morgan-Kauffman.

[3] J. van Benthem (1983). *The Logic of Time*, Reidel, Dordrecht.

[4] J. van Benthem (1983). *Modal Logic and Classical Logic*, Indices, Napoli.

[5] D. M. Gabbay (1981). An irreflexivity lemma with applications to axiomatizations on linear frames. In U. Monnich (ed.), *Aspects of Philosophical Logic*, 67–89. Reidel, Dordrecht.
[6] D. M. Gabbay (1981). Expressive functional completeness in tense logic. In U. Monnich (ed.), *Aspects of Philosophical Logic*, 27–42. Reidel, Dordrecht.
[7] J. Y. Halpern and Y. Shoham (1986). A propositional modal logic of time intervals. In *Proceedings of the First IEEE Symposium on Logic in Computer Science*, Cambridge, Mass. 279–92. Computer Society Press, Washington.
[8] J. Halpern, Z. Manna and B. Moszkowski (1983). A hardware semantics based on temporal intervals. *Proceedings 10th Conference on Automata, Languages and Programming*, Barcelona, 278–91.
[9] I. L. Humberstone (1979). Interval semantics for time logic: some remarks, *Journal of Philosophical Logic*, **VIII**, 171–96.
[10] N. Immerman and K. Dexter (1987). Definability with bounded number of bound variables. In *Proceedings of Symposium on Logic in Computer Science*, Ithaka, New York, 236–44. Computer Society Press, Washington.
[11] B. Jonsson (1982). Varieties of relation algebras, *Algebra Universalis*, **15**, 273–98.
[12] B. Jonsson and A. Tarski (1952). Boolean algebras with operators, *American Journal of Mathematics*, **73**, 891–939; **74**, 127–62.
[13] P. Ladkin (1987). The Logic of Time Representation. Dissertation, Kestral Institute, Palo Alto.
[14] R. Mackenzie (1991). A general method for constructing elementary axioms for classes of representable structures. Manuscript.
[15] R. D. Maddux (1989). Relation algebras of every dimension. Manuscript, Dept. of Mathematics, Iowa State University.
[16] J. D. Monk (1964). Nonfinitizability of classes of representable cylindric algebras, *Michigan Mathematics Journal*, **11**, 207–210.
[17] U. Monnich (ed.) (1989). *Aspects of Philosophical Logic*. Reidel, Dordrecht.
[18] B. Moszkowski and Z. Manna (1984). Reasoning in interval temporal logic. In E. Clarke and D. Kozen (eds), *Logics of Programs*, 371–84. Springer Lecture Notes in Computer Science Vol. 164.
[19] H. Nishimura (1980). *Interval Logic With Applications to Study of Tense and Aspect in English*. Publication of the Research Institute for Mathematical Sciences, Kyoto University, **16**, 417–59.
[20] P. Roper (1980). Intervals and tenses, *Journal of Philosophical Logic*, **9**, 451–69.
[21] R. Rosner and A. Pnueli (1986). A choppy logic. In *Proceedings of the First IEEE Symposium on Logic in Computer Science*, Cambridge, Mass. 306–313. Computer Society Press, Washington.
[22] H. Sahlqvist (1975). Completeness and correspondence in first and second-order semantics for modal logic. In S. Kanger (ed.), *Proceedings of the Third Scandinavian Logic Symposium*. North-Holland, Amsterdam.
[23] G. Sambin and V. Vaccaro (1989). A new proof of Sahlqvist's theorem on modal definability and completeness. *Journal of Symbolic Logic*, **54**, 992–9.
[24] Y. Venema (to appear). Expressiveness and completeness of an interval tense logic, *Notre Dame Journal of Formal Logic*.
[25] Y. Venema (1989). *Two-dimensional Modal Logics for Relation Algebras and Temporal Logic of Intervals*. ITLI Prepublication Series LP-89-03, University of Amsterdam.
[26] W. W. Wadge (1989). *A Complete Natural Deduction System for the Relation Calculus*. University of Warwick, Theory of Computation Report No. 5.

*Received 4 April 1990*