DETERMINISTIC AND NON AMBIGUOUS RATIONAL ω-LANGUAGES

André ARNOLD
Université de Poitiers

INTRODUCTION

Since not every rational ω-language can be recognized by a deterministic automaton, the class DRAT of deterministic rational ω-languages is strictly included in the class RAT of all rational ω-languages.

Then one can ask for a characteristic property of DRAT w.r.t. RAT, i.e. a property which is satisfied by a language of RAT if and only if this language is in DRAT.

There exists such a characteristic property, due to Landweber [5], which is a topological property : a language in RAT is in DRAT if and only if it is a countable intersection of open sets, for some natural topology on the set of all infinite words.

Indeed, using topological properties to characterize sets of infinite words (more precisely their position in the Borel hierarchy) turns out to be a very natural and very useful method [2,5,6,7] . Thus the interest of the Landweber's theorem is twofold : besides its specific interest as a characterization of DRAT, it is a good introduction to the use of topological methods in infinite words language theory.

Another technical tool is needed to prove this theorem : the so-called Müller (deterministic) automata which are nothing but deterministic Büchi automata with a different criterion for recognition. These automata recognize the closure $(\mathrm{DRAT})^B$ of DRAT, which is equal to RAT by Mac Naughton's theorem.

To end this paper we consider the notion of non-ambiguity, which is, as usual in language theory, a weaker notion than determinism : an automaton (in the Büchi sense) recognizes a non-ambiguous language if every word of the language has only one successful computation. We prove that every rational language is non ambiguous.

This paper contains four parts. The first one is devoted to Muller's automata. The second one introduces the topology on the set of infinite words and some of its properties. In the third one we prove the Landweber's theorem and in the last one we prove that rational ω-languages are non ambiguous.

Note : This paper is assumed to follow the D. Perrin's one, thus not all basic definitions are reminded.

I . DETERMINISTIC MULLER AUTOMATA

A deterministic Müller automaton is a deterministic Büchi automaton where the recognition criterion has been modified : it still depends on the set of states which are visited infinitely often, but, here, this set has to belong to some specified family of set of states.

Formally a deterministic Müller automaton is a tuple $\mathcal{A}_{\mathcal{C}} = \langle A, Q, q_*, \delta, \mathcal{C} \rangle$ where

A is an alphabet,

Q is a finite set of states,

$q_* \in Q$ is the initial state,

δ : QxA --> Q is the transition function,

and $\mathcal{C} \subset \mathcal{P}(Q)$ is a family of subsets of Q.

If $\mathcal{C}'$ is another subset of $\mathcal{P}(Q)$ then $\mathcal{A}_{\mathcal{C}'}$ will denote the automaton $\langle A, Q, q_*, \delta, \mathcal{C}' \rangle$.

Indeed one can consider that $\mathcal{A}_{\mathcal{C}}$ consist of two parts : the underlying labeled graph $\langle A, Q, q_*, \delta \rangle$ which is the same as for deterministic (total) Büchi automata and the recognizing family $\mathcal{C}$ of sets of states.

With every infinite word $u_i \in A^{\omega}$ is associated the unique sequence $(q_i)_{i \in N}$ of states, called the computation of u in [the underlying graph of] $\mathcal{A}_{\mathcal{C}}$, inductively defined by $q_0 = q_*$, $q_{i+1} = \delta(q_i, u(i+1))$ where u(i) is the i^{th} letter of u.

We also define $In(u) = \{q \in Q / \exists^{\infty} i : q_i = q\}$ where $(q_i)_i$ is the computation of u ; and the language recognized by $\mathcal{A}_{\mathcal{C}}$ is $L(\mathcal{A}_{\mathcal{C}}) = \{u \in A^{\omega} / I_n(u) \in \mathcal{C}\}$. In other words, u is recognized if the set of states which are visited infinitely often while $\mathcal{A}_{\mathcal{C}}$ reads u is in $\mathcal{C}$.

Let us denote by MRAT the class of languages recognized by deterministic Müller automata. We prove that MRAT is the boolean closure $(DRAT)^B$ of the class DRAT of languages recognized by deterministic Büchi automata.

Theorem I.1

$$MRAT = (DRAT)^B$$

Proof

1. We prove that MRAT is a boolean algebra containing DRAT, hence we get $(DRAT)^B \subset$ MRAT.

(i) MRAT is a boolean algebra.

. It is clear that $A^{\omega} - L(\mathcal{A}_{\mathcal{C}}) = L(\mathcal{A}_{\mathcal{P}(Q) - \mathcal{C}})$, hence MRAT is closed under complementation.

. If $L_1 = L(\mathcal{A}'_{\mathcal{C}'})$ and $L_2 = L(\mathcal{A}''_{\mathcal{C}''})$, we define $\mathcal{A}_{\mathcal{C}} = \langle A, Q, q_*, \delta, \mathcal{C} \rangle$ with $Q = Q' \times Q''$; $q_* = \langle q'_*, q''_* \rangle$; $\delta(\langle q', q'' \rangle, a) = \langle \delta'(q', a), \delta''(q'', a) \rangle$ and $\mathcal{C} = \{T \in Q \times Q'' / \pi'(T) \in \mathcal{C}'$ or $\pi''(T) \in \mathcal{C}''$ where π' (resp. π'') is the projection of $Q' \times Q''$

Then it is easy to check that $L(\mathcal{A}_{\mathcal{C}}) = L_1 \cup L_2$; for $(\langle q'_i, q''_i\rangle)_i$ is the computation of u in $\mathcal{A}$ iff $(q'_i)_i$ is the computation of u in $\mathcal{A}'$ and $(q''_i)_i$ is the computation of u in $\mathcal{A}''$; moreover if T(resp. T', T'') is In(u) with respect to $\mathcal{A}$ (resp. $\mathcal{A}'$, $\mathcal{A}''$) we have $\pi'(T) = T'$ and $\pi''(T) = T''$.

Hence MRAT is also closed under union, and it is a boolean algebra.

(ii) Every language recognized by the deterministic Büchi automaton $\mathcal{A}_F = \langle A, Q, q_*, \delta, F\rangle$, with $F \subset Q$ is $L_B(\mathcal{A}_F) = \{u \in A^\omega / In(u) \cap F \neq \emptyset\}$. Therefore it is recognized by the Müller automaton $\mathcal{A}_{\mathcal{C}}$ where $\mathcal{C} = \{T \subset Q / T \cap F \neq \emptyset\}$. It follows that DRAT $\subset$ MRAT.

2. It remains to prove that MRAT $\subset (\text{DRAT})^B$.

Let $\mathcal{A}_{\mathcal{C}}$ with $\mathcal{C} = \{T_1, \ldots, T_n\}$. It is obvious from the definition of $L(\mathcal{A}_{\mathcal{C}})$ that $L(\mathcal{A}_{\mathcal{C}}) = L(\mathcal{A}_{\{T_1\}}) \cup \cdots \cup L(\mathcal{A}_{\{T_n\}})$. Thus it suffices to prove : $L(\mathcal{A}_{\{T\}}) \in (\text{DRAT})^B$ where $T \subset Q$.

But $In(u) = T$ iff $T \subset In(u)$ and $In(u) \cap \overline{T} = \emptyset$ iff $\forall q \in T$, $In(u) \cap \{q\} \neq \emptyset$ and $In(u) \cap (Q-T) = \emptyset$ iff $\forall q \in T$, $u \in L_B(\mathcal{A}_{\{q\}})$ and $u \notin L_B(\mathcal{A}_{Q-T})$.

Hence $L(\mathcal{A}_{\{T\}}) = \bigcap_{q \in T} L_B(\mathcal{A}_{\{q\}}) - L_B(\mathcal{A}_{Q-T})$ which belongs to $(\text{DRAT})^B$.

Because of this theorem, the Mac Naughton's theorem can be stated in two equivalent forms

RAT = $(\text{DRAT})^B$ or

RAT = MRAT

i.e. every rational ω-language is a combination of deterministic ones or can be recognized by a deterministic Müller automaton.

II. TOPOLOGY ON A^ω

Let us define the following distance on A^ω :

$$d(u,v) = \begin{cases} 0 & \text{if } u=v, \\ 2^{-\inf\{i / u[i] \neq v[i]\}} & \text{if } u \neq v. \end{cases}$$

where u[i] is the word consisting of the first i letters of u.

It is clear that the open ball $B(u, 2^{-n})$ of center u and radius 2^{-n} is the set $u[n]A^\omega$. Conversely, if $u \in A^*$, then uA^ω is an open ball of radius $2^{-|u|}$ (and any element of uA^ω is the center of this ball : that is an interesting property of the above defined distance).

It follows that the open sets for the topology induced by this distance are the sets LA^ω for $L \subset A^*$.

It is also easy to see that the topological closure of a set $L \subset A^\omega$ is $\mathfrak{Cl}(L) = \{u \in A^\omega / FG(u) \subset FG(L)\}$ where $FG(u) = \{u[u]/n \in \mathbb{N}\}$ and $FG(L) = \bigcup_{u \in L} FG(u)$. Indeed $u \in \mathfrak{Cl}(L)$ iff $\forall n, \exists v \in L : d(u,v) < 2^n$ iff $\forall n, \exists v \in L : u[n] = v[n]$.

In this topology we have an interesting characterization of G_δ-sets, i.e. sets which are countable intersections of open sets. This characterization is due to Landweber [5] and uses the notion of Eilenberg's limit of a language [4] .

Let $K \subset A^*$; we denote by $\mathrm{Elim}(K)$ the set $\{u \in A^\omega / \exists^\infty n : u[n] \in K\}$. Obviously $\mathcal{Cl}(L) = \mathrm{Elim}(FG(L))$. Therefore every closed set is an Eilenberg's limit. Indeed every G_δ-set is an Eilenberg's limit and the converse is true, as proved by Landweber.

Lemma II.1 [5]

A set $L \subset A^\omega$ *is a* G_δ *-set iff it is the Eilenberg's limit of some set* $M \subset A^*$.

Proof

1. Let us assume that $L = \mathrm{Elim}(M)$ and let us prove that $L = \bigcap_n M_n A^\omega$, where $M_n = M \cap A^n A^* = \{u \in M \mid |u| \geq n\} \subset A^*$. Since $M_n A^\omega$ are open sets, it will yield L is a G_δ-set.

(i) if $u \in \mathrm{Elim}(M)$, we have
$\exists^\infty n : u[n] \in M$, thus
$\exists^\infty n : u[n] \in M_n$, thus
$\exists^\infty n : u \in M_n A^\omega$
and, since $(M_n)_n$ is a decreasing sequence for inclusion, we get $u \in \bigcap_n M_n A^\omega$.

(ii) if $u \in \bigcap_n M_n A^\omega$, we have
. $u \in M_1 A^\omega \Rightarrow \exists n_1 \geq 1 : u[n_1] \in M$;
. $u \in M_{n_1+1} A^\omega \Rightarrow \exists n_2 > n_1 : u[n_2] \in M$;
. $u \in M_{n_2+1} A^\omega \Rightarrow \exists n_3 > n_2 : u[n_3] \in M$;
.....
and so on, we get $\exists^\infty n : u[n] \in M$.

2. Let us assume that $L = \bigcap_n O_n$ where O_n is an open set, i.e. $O_n = M_n A^\omega$. We can assume w.l.o.g. that the sequence $(M_n)_n$ is decreasing : indeed let $M'_n = \bigcap_{1\leq i\leq n} M_i A^*$; $(M'_n)_n$ is obviously decreasing and we have $M'_n A^\omega = (\bigcap_{1\leq i\leq n} M_i A^*)A^\omega = \bigcap_{1\leq i\leq n} M_i A^\omega = \bigcap_{1\leq i\leq n} O_i$ hence $\bigcap_n M'_n A^\omega = \bigcap_n(\bigcap_{1\leq i\leq n} O_i) = \bigcap_n O_n = L$.

Now let us denote by K_i the set $\mathrm{Pref}(M_i)A^i$ where $\mathrm{Pref}(M_i) = M_i - M_i A^+$. We prove that $L = \mathrm{Elim}(\bigcup_i K_i)$.

(i) $u \in \bigcap_n M_n A^\omega$ implies $\forall i, \exists j : u[j] \in M_i$. By definition of $\mathrm{Pref}(M_i)$ we get $\exists j' \leq j : u[j'] \in \mathrm{Pref}(M_i)$, and by definition of K_i, $u[j'+i] \in K_i$. It follows that $\forall i, \exists j' : u[i+j'] \in K_i$ and $u \in \mathrm{Elim}(\bigcup_i K_i)$.

(ii) if $u \in \mathrm{Elim}(\bigcup_i K_i)$ then there exists a strictly increasing sequence $(i_n)_n$ of integers and a sequence $(j_n)_n$ such that $u[i_n] \in K_{j_n}$. By definition of K_j we get $u[i_n] \in K_{j_n}$ implies $i_n \geq j_n$ and $u[i_n - j_n] \in \mathrm{Pref}(M_{j_n}) \subset M_{j_n}$. Therefore $u \in M_{j_n} A^\omega$.

If the sequence $(j_n)_n$ is unbounded, since $(M_n)_n$ is decreasing we will get $u \in \bigcap_n M_{j_n} A^\omega = \bigcap_n M_n A^\omega$ and the proof is complete. Let us prove that $(j_n)_n$ is always unbounded : if not we get $n \neq n'$ with $j_n = j_{n'}$, hence $u[i_n - j_n]$ and $u[i_{n'} - j_{n'}]$ both belong to Pref (M_{j_n}) ; this implies $i_n - j_n = i_{n'} - j_{n'}$, hence $i_n = i_{n'}$ which is impossible. □

III. A CHARACTERIZATION OF DETERMINISTIC LANGUAGES

In this section we want to prove the Landweber's theorem [5] which states that $L \in$ RAT is in DRAT iff it is a G_δ-set. In order to do that we need some preliminary definitions and lemmata.

Let $\mathcal{A}_\mathcal{C} = \langle A, Q, q_*, \delta, \mathcal{C} \rangle$ be a Müller automaton.

If u is a word of A^* and q a state of Q we denote by $P^-(q,u)$ the set $\{\delta^*(q,u[i]) / o < i < |u|\}$ and $P^+(q,u)$ the set $\{\delta^*(q,u[i]) / o \leq i \leq |u|\}$ where δ^* is the obvious extension of δ to $Q \times A^*$.

In other words if $q_o, q_1, \ldots, q_{|u|}$ is the computation of u from $q = q_o$, then

$P^+(q,u)$ is $\{q_o, q_1, \ldots, q_{|u|}\}$ and

$P^-(q,u)$ is $\{q_1, \ldots, q_{|u|-1}\}$.

Now let $\mathcal{H}_q$ be the set $\{P^+(q,u) / u \in A^+, \delta^*(q,u) = q\} \subset \mathcal{P}(Q)$. Obviously, $\mathcal{H}_q$ is closed under union.

We say that $\mathcal{A}_\mathcal{C}$ is "cycle-closed" if $\forall q \in Q$, $\forall T, T' \in \mathcal{H}(q)$, if $T \in \mathcal{C}$ then $T \cup T' \in \mathcal{C}$.

<u>Lemma III.1</u> *If $L(\mathcal{A}_\mathcal{C})$ is a G_δ-set, then $\mathcal{A}_\mathcal{C}$ is cycle-closed.*

<u>Proof</u> Let $T \in \mathcal{H}q \cap \mathcal{C}$ and $T' \in \mathcal{H}q$. There exist three words $x, u, v \in A^+$ such that $\delta^*(q_*, x) = q$,

$\delta^*(q,u) = \delta^*(q,v) = q$,

$T = P^+(q,u)$, $T' = P^+(q,v)$.

It is clear that if $w \in x(u^*v)^*u^\omega$ then $In(w) = P^+(q,u) = T \in \mathcal{C}$, hence $w \in L(\mathcal{A}_\mathcal{C})$, hence $x(u^*v)^*u^\omega \subset L(\mathcal{A}_\mathcal{C})$.

By lemma II.1 we have $L(\mathcal{A}_\mathcal{C}) = \mathrm{Elim}(K)$ for some $K \subset A^*$. Therefore, if $w \in x(u^*v)^*$, then $w\,u^\omega \in \mathrm{Elim}(K)$ and it exists $n > o$, $m \geq o$ such that

. $|w| < m < |w| + n\,|u| = |w\,u^n|$,

. $w\,u^n[m] \in K$.

Let us apply this property to $w_o = x$; we get n_o and m_o such that $x\,u^{n_o}[m_o] \in K$ and $|x| < m_o < |x\,u^{n_o}|$. Let us apply it again to $w_1 = x\,u^{n_o}\,v$; we get n_1 and m_1 such that $x\,u^{n_o}\,v\,u^{n_1}[m_1] \in K$ and $m_o < m_1 < |x\,u^{n_o} v\,u^{n_1}|$; and so on we get a sequence $m_o < m_1 < m_2 < \ldots$ and a sequence $n_o, n_1, \ldots, n_i, \ldots$ such that $n_i \geq 1$, $m_i < |x\,u^{n_o}\,v\,u^{u_1}\,v \ldots v\,u^{n_i}|$ and $x\,u^{n_o}\,v\,u^{n_1}\,v \ldots v\,u^{n_i}[m_i] \in K$. It follows that the infinite word $w = x\,u^{n_o}\,v\,u^{n_1}\,v \ldots v\,u^{n_i}\,v \ldots$ has an infinite number of left factors in K ; then $w \in \mathrm{Elim}(K) = L(\mathcal{A}_\mathcal{C})$.

But it is clear that $In(w) = P(q,u) \cup P(q,v) = T \cup T'$. Since $w \in L(\mathcal{A}_{\mathcal{C}})$, $T \cup T' \in \mathcal{C}$.

Now we prove a technical lemma :

<u>Lemma III.2</u> *Let* $v = u_0u_1u_2\ldots$ *such that :* $u_i \in A^+$, *and for some state* q *of* $\mathcal{A}_{\mathcal{C}}$
$$\delta^*(q_*,u_0) = q = \delta^*(q,u_{i+1})$$

Then $\forall i,\ \exists j \geq i,\ \exists k \geq 0 : In(v) = P^+(q,u_j\, u_{j+1}\ldots u_{j+k})$.

<u>Proof</u> Every state not in $In(v)$ can appear only a finite number of times in the computation of u. Thus there exists an integer k_0 such that $\forall_{k\geq k_0}$, $\delta^*(q_*,v[k]) \notin In(v)$. It follows that $\forall j \geq \max(i,k_0)$, $P^+(q,v_j) \subset In(v)$, and $P^+(q,u_j\, u_{j+1}\ldots u_{j+k}) = P^+(q,u_j) \cup P^+(q,u_{j+1})\ldots \cup P^+(q,u_{j+k}) \subset In(v)$.

Now, every state of $In(v)$ appears infinitely often in the computation of u. Hence $\forall q' \in In(v)$, $\forall_j$, there exists $k \geq 0$ such that $q' \in P^+(q,u_j\, u_{j+1}\ldots u_{j+k})$.
Thus for k large enough we get $P(q,u_j\, u_{j+1}\ldots u_{j+k}) = In(v)$.

Now if $\mathcal{A}_{\mathcal{C}}$ is cycle-closed we can construct a deterministic Büchi automaton which recognizes $L(\mathcal{A}_{\mathcal{C}})$.

<u>Construction</u> : Let $\mathcal{A}_{\mathcal{C}} = \langle A,Q,q_*,\delta,\mathcal{C}\rangle$ be a deterministic Müller automaton. For every state s of Q we construct the following deterministic Büchi automaton :
$\mathcal{M}_s = \langle A,Q \cup Q\times \mathcal{P}(Q),q_*,\delta',\{\langle s,\emptyset\rangle\}\rangle$ where

$$\cdot\ \delta'(q,a) = \begin{cases} q' \text{ if } q' = \delta(q,a) \text{ and } q' \neq s \\ \langle s,\emptyset\rangle \text{ if } \delta(q,a)=s \end{cases}$$

$$\cdot\ \delta'(\langle q,P\rangle,a) = \langle \delta(q,a),P'\rangle$$

$$\text{where } P' = \begin{cases} \emptyset \text{ if } \delta(q,a)=s \text{ and } \\ \qquad P \cup \{s\} \in \mathcal{C} \\ P \cup \{\delta(q,a)\} \text{ otherwise.} \end{cases}$$

Now it is easy to construct a deterministic Büchi automaton $\mathcal{B}$ such that $L_B(\mathcal{B}) = \bigcup_{s\in Q} L_B(\mathcal{M}_s)$. This construction is quite similar to the construction of the "union" of Müller automata in part 1(i) of the proof of Theorem I.1.

<u>Lemma III.3</u> *If* $\mathcal{A}_{\mathcal{C}}$ *is cycle-closed then* $L(\mathcal{A}_{\mathcal{C}}) = \bigcup_{s\in Q} L_B(\mathcal{M}_s)$.

Proof

1. If $v \in L_B(\mathcal{A}_S)$, then $v = u_o\, u_1 \ldots u_n \ldots$ with $u_i \in A^+$ such that

- $\delta'^*(q_*, u_o) = \langle s, \emptyset \rangle$ and $P^-(q_*, u_o) \not\ni \langle s, \emptyset \rangle$,
- $\delta'^*(\langle s, \emptyset \rangle, u_{i+1}) = \langle s, \emptyset \rangle$ and $P^-(\langle s, \emptyset \rangle, u_{i+1}) \not\ni \langle s, \emptyset \rangle$.

By definition of $\mathcal{A}_S$, we have $\delta^*(q_*, u_o) = s$, $\delta^*(s, u_{i+1}) = s$ and $P^+(s, u_{i+1}) \in \mathcal{C}$.

By lemma III.2 we have $In(v) = P^+(s, u_{i+1} \ldots u_{i+k}) = P^+(s, u_{i+1}) \cup P^+(s, u_{i+2}) \ldots \cup P^+(s, u_{i+k})$ for some i and k. Since $\mathcal{A}_{\mathcal{C}}$ is cycle-closed, we get, since $P^+(s, u_{i+1}) \in \mathcal{C}$, $In(v) \in \mathcal{C}$, hence $v \in L(\mathcal{A}_{\mathcal{C}})$.

2. If $v \in L(\mathcal{A}_{\mathcal{C}})$, then $In(v) \in \mathcal{C}$ and $v = u_o u_1 \ldots u_n \ldots$ such that $u_i \in A^+$ and for some $s \in Q$, $\delta^*(q_*, u_o) = s = \delta^*(s, u_{i+1})$, and $s \notin P^-(q_*, u_o)$, $s \notin P^-(s, u_{i+1})$.

Thus, in $\mathcal{A}_S$, we have

- $\delta'^*(q_*, u_o) = \langle s, P_o \rangle$,
- $\delta'^*(\langle s, P_i \rangle, u_{i+1}) \doteq \langle s, P_i \rangle$,

with $P_o = \emptyset$.

Let us prove that if $P_i = \emptyset$, then $P_j = \emptyset$ for some $j > i$; it will follow $w \in L_B(\mathcal{A}_S)$.

By lemma III.2, $In(v) = P^+(s, u_j \ldots u_{j+k})$ for some $j > i$. Then since $In(v) \in \mathcal{C}$ and since $\mathcal{A}_{\mathcal{C}}$ is cycle-closed, $P^+(s, u_{i+1} \ldots u_{j+k}) \in \mathcal{C}$ and the set $\{j > i / P^+(s, u_{i+1} \ldots u_j) \in \mathcal{C}\}$ is not empty. If j is the least element in this set, obviously we have $\delta'^*(\langle s, \emptyset \rangle, u_{i+1} \ldots u_j) = \langle s, \emptyset \rangle$. Hence, if $P_i = \emptyset$, $P_j = \emptyset$.

Now we are ready to prove the main theorem.

Theorem III.1 [5]

Let L *be a language of* RAT. L *is in* DRAT *iff* L *is a* G_δ*-set.*

Proof

1. Let us assume that L is in DRAT ; it is recognized by a deterministic Büchi automaton $\mathcal{B} = \langle A, Q, q_*, \delta, F \rangle$. Let us denote by $L_*(\mathcal{B})$ the set of finite words u such that $\delta^*(q_*, u) \in F$. Because $\mathcal{B}$ is deterministic, it is easy to see that $L_B(\mathcal{B}) = Elim(L_*(\mathcal{B}))$ and by lemma II.1, $L_B(\mathcal{B})$ is a G_δ-set.

2. Let $L(\mathcal{A}_{\mathcal{C}})$ be a G_δ-set. By lemma III.1 $\mathcal{A}_{\mathcal{C}}$ is cycle-closed and by lemma III.3, $L(\mathcal{A}_{\mathcal{C}})$ is in DRAT.

Corollary III.1 5

$L(\mathcal{A}_{\mathcal{C}})$ *is in* DRAT *iff* $\mathcal{A}_{\mathcal{C}}$ *is cycle-closed*

Proof By lemma III.3 if $\mathcal{A}_{\mathcal{T}}$ is cycle-closed then $L(\mathcal{A}_{\mathcal{T}})$ is in DRAT. If $L(\mathcal{A}_{\mathcal{T}})$ is in DRAT then by theorem III.1, $L(\mathcal{A}_{\mathcal{T}})$ is a G_δ-set and by lemma III.1, $\mathcal{A}_{\mathcal{T}}$ is cycle-closed. □

It follows that "cycle-closedness" is a characteristic property of these Müller automata which recognize deterministic languages. Moreover it is clear that "cycle-closedness" is a decidable property.

IV - NON-AMBIGUITY

We say that a non deterministic Büchi automaton is non-ambiguous if every word recognized by this automaton has only one successful computation, and a rational ω-language is non-ambiguous if it is recognized by a non-ambiguous Büchi automaton. Here we prove that

Theorem IV-1 1

Any rational ω-language is non-ambiguous.

Let $L(\mathcal{A}_{\mathcal{T}})$ be a deterministic Müller automaton. It is clear that $L(\mathcal{A}_{\mathcal{T}})$ is the disjoint union of $L(\mathcal{A}_{\{T\}})$ for $T \in \mathcal{T}$: if $v \in L(\mathcal{A}_{\{T\}}) \cap L(\mathcal{A}_{\{T'\}})$ then $In(v) = T = T'$.

Obviously a disjoint union of non-ambiguous languages is non-ambiguous since the disjoint union of non-ambiguous Büchi automata $\mathcal{B}_i$ is a non-ambiguous Büchi automaton which recognizes the union of $L_B(\mathcal{B}_i)$.

Given a Müller automaton $\mathcal{A}_{\{T\}} = \langle A, Q, q_*, \delta, \{T\}\rangle$ we construct the following Büchi automaton

$\mathcal{B} = \langle A, Q', Q'_*, \delta', F\rangle$ where

$Q' = Q \cup T \times \mathcal{P}(T)$;

$$Q'_* = \begin{cases} \{q_*\} \text{ if } q^* \notin T, \\ \{q_*, \langle q_*, \{q_*\}\rangle\} \text{ if } q^* \in T \text{ ;} \end{cases}$$

$$\delta'(q,a) = \begin{cases} \{q', \langle q', \{q'\}\rangle\} \text{ if } q \notin T \text{ and } q' \in T, \\ q' \text{ otherwise ;} \end{cases}$$

where $q' = \delta(q,a)$

$$\delta'(\langle q,P\rangle, a) = \begin{cases} \langle q', \{q'\}\rangle \quad \text{if } P=T \\ \langle q', P \cup \{q'\}\rangle \quad \text{if } P \neq T \end{cases}$$

where $q' = \delta(q,a)$

and $F = \{\langle q,T\rangle / q \in T\}$.

The intuitive idea behind this construction is the following : a computation of u in $\mathcal{B}$ proceeds like that

- Compute in the automaton $\mathcal{A}$ (this is done by transitions $q \xrightarrow{a} \delta(q,a)$)

- At some point, guess the automaton $\mathcal{A}$ has entered a state of Q- T for the last time in the computation in $\mathcal{A}$ ($q \xrightarrow{a} \langle q',\{q'\}\rangle$)

- Continue the computation in $\mathcal{A}$, but remembering the set of visited states ; when this set is T, start a new record.

Now we prove that $L(\mathcal{A}_{\{T\}}) = L_B(\mathcal{B})$.

Let $w \in L(\mathcal{A}_{\{T\}})$; then there exists a unique factorisation $u_o u_1 \ldots u_n \ldots$ of v which satisfies

(i) $\forall i \geq 1, q_i \in T$, where $q_o = q_*$ and $q_{i+1} = \delta^*(q_i, u_i)$

(ii) . $u_o = \varepsilon \implies q_* \in T$,

. $u_o = u'_o\ a \implies \delta^*(q_*, u'_o) \notin T$,

(iii) $u_{i+1} = u'_{i+1}\ a$,

$\{\delta^*(q_{i+1}, u'_{i+1}[j]) / o \leq j \leq |u'_{i+1}|\} = T$,

$\{\delta^*(q_{i+1}, u'_{i+1}[j]) / o \leq j < |u'_{i+1}|\} \neq T$.

The unicity of this factorisation of v comes from the fact that if $v \in L(\mathcal{A}_{\{T\}})$, its computation has a last state not in T, which determines u'_o, or all its states are in T (in this case, $q_* \in T$) and then $u_o = \varepsilon$; then the u_{i+1}'s are uniquely determined by the condition (iii).

Then we obviously have, in $\mathcal{B}$,

. if $u_o = u'_o\ a$, $q_* \xrightarrow{u'_o} (q_*, u'_o) \xrightarrow{a} \langle q_1, \{q_1\}\rangle$;

. if $u_o = \varepsilon$, $q_1 = q_* \in T$;

. $\langle q_{i+1}, \{q_{i+1}\}\rangle \xrightarrow{u'_{i+1}} \langle \delta^*(q_{i+1}, u'_{i+1}), T\rangle$

$\xrightarrow{a} \langle q_{i+2}, \{q_{i+2}\}\rangle$.

So we get a computation in $\mathcal{B}$ which goes infinitely often in $\{\langle q,T\rangle / q \in T\}$ and $v \in L_B(\mathcal{B})$.

Conversely if $v \in L_B(\mathcal{B})$, then v can be written $u_o\ u'_1\ a_1\ u'_2\ a_2 \ldots$ and there exists a sequence $q_o, q_1, q'_2, q_2, q'_3, q_3 \ldots$ such that

. $u_o = \varepsilon$ and $q_1 = q_*$, or

$u_o = u'_o\ a_o$ and $q_* \xrightarrow{u'_o} q_o \xrightarrow{a_o} \langle q_1, \{q_1\}\rangle$ with $q_o \notin T$;

. $\langle q_i, \{q_i\}\rangle \xrightarrow{u'_i} \langle q'_{i+1}, T\rangle \xrightarrow{a_i} \langle q_{i+1}, \{q_{i+1}\}\rangle$;

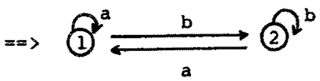
. the computation $\langle q_i, \{q_i\}\rangle \xrightarrow{u'_i} \langle q'_{i+1}, T\rangle$ does not cross any other state $\langle q, T\rangle$.

Then it is easy to see that the factorisation $u_o\ u_1\ u_2 \ldots u_n \ldots$ of v defined by $u_{i+1} = u'_{i+1}\ a_i$ satisfies (i)-(ii)-(iii) above. It follows that $v \in L(\mathcal{A}_{\{T\}})$ and therefore $L_B(\mathcal{B}) = L(\mathcal{A}_{\{T\}})$.

The non-ambiguity of $\mathcal{B}$ comes from the fact that with two different computations of v in $\mathcal{B}$, two different factorizations of v satisfying (i)-(ii)-(iii) are associated, which is impossible.

An example

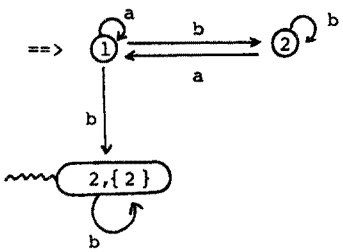
Let us consider the Müller automaton



with $\mathcal{C} = \{\{2\}\}$

which recognizes $\{a,b\}^* b^\omega$.

The above construction gives the Büchi automaton



It is clear that the only way to recognize a word in $\{a,b\}^* b^\omega$ is to go from 1 to 2,{2} reading the first b which is not followed by an a.

REFERENCES

1 **A. Arnold.** Rational ω-languages are non-ambiguous (Note). Theoret. Comput. Sci. 26 (1983) 221-223.

2 **A. Arnold.** Topological characterizations of infinite behaviours of transition systems. in "Automata, Languages and Programming", 10^{th} Colloquium, Barcelona, 1983 (J. Diaz, ed), LNCS 154 (1983) 28-38.

3 **S. Eilenberg.** Automata, Languages and Machines. Vol. A, Academic Press, New-York (1974).

4 **K. Kuratowski.** Topology I. Academic Press, New-York (1966).

5 **L.H. Landweber.** Decision probelms for ω-automata. Math. System Theor. 3 (1969) 376-384.

6 **L. Staiger.** Finite-State ω-languages. J. Comput. System Sci. 27 (1983) 434-448.

7 **L. Staiger, K. Wagner.** Automaton theoretische und automatonfreie charakterisierungen topologischer Klassen regulärer Folgenmangen. EIK 10 (1974) 379-392.