LINEAR EQUATIONS IN NON-COMMUTATIVE FIELDS.*

To Professor JAMES PIERPONT on his 65-th Anniversary.

By Oystein Ore.

The problem of solving linear equations with coefficients in a non-commutative field (division-algebra) has recently been studied by A. R. Richardson,[1] Heyting[2] and Study.[3] The paper by Study is mainly confined to the case where the coefficients are contained in a quaternion field. In order to obtain the solutions of a system of simultaneous linear equations, certain expressions are introduced by Heyting and Richardson, which present numerous analogies to the determinants in the commutative case. Their usefulness for the solution of equations is however inconveniently limited by the fact, that they are not defined for all values of the coefficients and certain restrictions must be placed on the elements involved. This is particularly striking for the "designants" of Heyting which only exist if certain "principal minors" do not vanish. In his last paper A. R. Richardson obtains a general definition by means of recursion-formulas; there exists however in this formula a definite lack of symmetry depending on vanishing or non-vanishing of the coefficients, and this fact, it seems to me, makes the definition unsatisfactory.

Another type of expressions with various invariant properties has been investigated by MacDuffee.[4] These expressions do not seem to have any connection with the elimination in linear systems and are therefore outside the scope of this article.

I have proposed in this paper to determine the rings (algebras) in which elimination between linear systems can be performed. By adhering strictly to the elimination properties of determinants, a new definition of deter-

* Received December 8, 1930.

[1] A. R. Richardson, Hypercomplex determinants. Messenger of Math. 55 (1926), pp. 145–152. A. R. Richardson, Simultaneous linear equations over a division algebra. London Math. Soc. 28 (1928), pp. 395–420.

[2] A. Heyting, Die Theorie der linearen Gleichungen in einer Zahlenspezies mit nichtkommutativer Multiplikation. Math. Annalen 98 (1927), pp. 465–490.

[3] E. Study, Zur Theorie der linearen Gleichungen. Acta Mathematica 42 (1918), pp. 1–61. Certain older considerations on the same problem can be found by Caley, Philosophical Magazine 26 (1845), pp. 141–145, C. J. Joly in the second edition of Hamilton, Elements of quaternions.

[4] C. C. MacDuffee, Invariantive characterizations of linear algebras with the associative law not assumed. Transactions Am. Math. Soc. 23 (1922), pp. 135–150.

minants in a non-commutative field is introduced, which is not open to the criticisms made above. By special choices of the elements in this determinants it reduces to the expressions of Heyting or Richardson. By means of this definition a farreaching analogy to the commutative case is obtained. I shall only prove the most important properties of these determinants and give the principal results on the solution of equations. A series of further results can then be immediately implied from the commutative case. The rank of a system can be introduced and all the ordinary results on linear dependency can be derived.

If one had been primarily interested in the dependency of linear expressions, Toeplitz's[5] method for solving equations without determinants could have been generalized. Another method is given by Noether;[6] in this way one obtains existence theorems for solutions, but not a simple procedure to determine them.

In § 1 and § 2 I discuss the properties of rings in which the elimination can be performed; these rings must satisfy a certain axiom M_{V} and this is, as I show, equivalent to the fact, that the ring can be completed to a non-commutative field ("Quotientenkörper") by the introduction of formal quotients of elements in the ring. In the commutative case all domains of integrity (rings without divisors of zero) have a uniquely defined quotient-field, which is the least field containing the ring. For the non-commutative case v. d. Waerden[7] has recently indicated this problem as unsolved. The result mentioned above gives all rings for which a quotient-field can exist. For rings without quotient-field it might however, as I show by an example, be possible to construct by a different process a field that contains the given ring.

1. **The axioms.** We shall in the following consider a system S with *more than one element,* for which the following axioms are supposed to hold:

Equality. The equality of two elements a and b in S is defined by the following properties:

G_{I}. *Determination.* For two elements either $a = b$ or $a \neq b$.

G_{II}. *Reflexitivity.* $a = a$.

G_{III}. *Symmetry.* From $a = b$ follows $b = a$.

G_{IV}. *Transitivity.* From $a = b$, $b = c$ follows $a = c$.

Addition. For two arbitrary elements a and b a sum $a+b$ exists, having the properties:

[5] O. Toeplitz, Über die Auflösung unendlich vieler linearen Gleichungen mit unendlich vielen Unbekannten. Rendiconti Palermo 28 (1909), pp. 88–96. Compare also H. Hasse, Algebra, vol. 1, Berlin 1926. O. Haupt, Einführung in die Algebra, vol. 1, Leipzig 1929.

[6] E. Noether, Hyperkomplexe Größen und Darstellungstheorie, Math. Zeitschr. 30 (1929), pp. 641–692.

[7] B. L. v. d. Waerden, Moderne Algebra, vol. 1, § 12. Berlin 1930.

A_{I}. *Uniqueness.* $a+b$ is a uniquely defined element of S.
A_{II}. *Equality.* From $a = b$ and $a_1 = b_1$ follows $a+a_1 = b+b_1$.
A_{III}. *Associative Law.* $a+(b+c) = (a+b)+c$.
A_{IV}. *Zero-element.* There exists an element 0 for which $0+a = a+0 = a$.
A_{V}. *Commutative Law.* $a+b = b+a$.
A_{VI}. *Subtraction.* To every element a exists another $-a$ such that $a+(-a) = 0$.

It is well known, that from these axioms follows, that 0 and $-a$ are uniquely defined.

Multiplication.

M_{I}. *Uniqueness.* $a \cdot b$ is a uniquely defined element in S.
M_{II}. *Equality.* From $a = b$ and $a_1 = b_1$ follows $aa_1 = bb_1$.
M_{III}. *Associative Law.* $a(bc) = (ab)c$.
M_{IV}. *Distributive Law.*

a) $(b+c)a = ba+ca$,
b) $a(b+c) = ab+ac$,
c) both a) and b).

Systems S satisfying these axioms are called (non-commutative) *rings* (or *algebras*). We shall in the following consider systems of linear equations with coefficients which are elements of such a ring. In order to perform an elimination to obtain a solution of a linear system, it seems necessary that the coefficients should satisfy the axioms mentioned. (Axiom M_{IV} possibly only in part.) The main operation for the usual elimination is however to multiply one equation by a factor and another equation by another factor to make the coefficients of one of the unknowns equal in the two equations. We must therefore also demand: ORE Condition

M_{V}. *Existence of common multiplum.* When $a \neq 0$, $b \neq 0$ are two arbitrary elements of S, then it is always possible to determine two other elements $m \neq 0$, $n \neq 0$ such that

Common right multiples

$$an = bm. \tag{1}$$

For all rings satisfying these axioms the elimination-process can be carried out, and certain necessary conditions for the solvability can be established. In order to obtain necessary and sufficient conditions it is, as in the commutative rings, necessary to suppose that the ring does not contain any *divisors of zero*, i. e. from $ab = 0$ follows $a = 0$ or $b = 0$ or both. This is equivalent to: cancellation law

M_{VI}. *Converse equality axiom.* From $ab = ac$ or $ba = ca$, $a \neq 0$ follows $b = c$.

A ring satisfying the axioms M_{V} and M_{VI} shall be called a *regular* ring. It is not necessary that a regular ring contains a *unit-element*, i. e. that

M_{VII}. *Unit element.* There exists an element 1, for which $1 \cdot a = a \cdot 1 = a$ for an arbitrary element a in S.

is satisfied. If for a regular ring both M_{VII} and the axiom

M_{VIII}. *Division.* For every $a \neq 0$ exists an element a^{-1} such that $a \cdot a^{-1} = 1$.

skew field

are satisfied, then the domain is called a (non-commutative) *field* (or *division-algebra*).

Let us from now on suppose that the set S considered is a regular ring. From M_{V} follows by induction, that if $a_1 \neq 0 \cdots a_m \neq 0$, then such numbers $n_1 \neq 0$, $n_2 \neq 0 \cdots n_m \neq 0$ can be determined, that

$$a_1 n_1 = a_2 n_2 = \cdots = a_m n_m. \tag{2}$$

Furthermore if at the same time

$$an = bm, \qquad a n_1 = b m_1, \qquad n_1 \neq 0, \qquad m_1 \neq 0, \tag{3}$$

then r and s can be so determined, that

$$nr = n_1 s, \tag{4}$$

from which one easily obtains

$$mr = m_1 s. \tag{5}$$

2. Quotient fields. We shall now prove the following theorem:

THEOREM 1. *All regular rings can be considered as subrings* (more exactly: *are isomorphic to a subring*) *of a non-commutative field.*

Let a and $b \neq 0$ be two arbitrary elements in S. We then introduce the symbol $\left(\frac{a}{b}\right) = (a \cdot b^{-1})$ and for these symbols (fractions) such rules of operation shall be defined, that their totality forms a field K.

Equality. Let $\left(\frac{a}{b}\right)$ and $\left(\frac{a_1}{b_1}\right)$ be two arbitrary fractions. According to M_{V} the elements $\beta \neq 0$, $\beta_1 \neq 0$ can be determined so that

$$b\beta_1 = b_1 \beta \tag{6}$$

and we say

$$\left(\frac{a}{b}\right) = \left(\frac{a_1}{b_1}\right) \tag{7}$$

when

$$a\beta_1 = a_1 \beta. \tag{8}$$

From the last remarks in § 1 it follows immediately that the equality of the two fractions (7) does not depend on the particular choice of the elements β and β_1 in (6). The axiom G_{I} is therefore satisfied. G_{II} and G_{III} are obviously satisfied. To derive G_{IV} let

$$\left(\frac{a}{b}\right) = \left(\frac{a_1}{b_1}\right), \qquad \left(\frac{a_1}{b_1}\right) = \left(\frac{a_2}{b_2}\right),$$

where

(9) $$b\beta_1 = b_1\beta, \qquad a\beta_1 = a_1\beta;$$

(10) $$b_1\beta_2 = b_2\beta_1', \qquad a_1\beta_2 = a_2\beta_1'.$$

We choose r and s such that

$$\beta r = \beta_2 s.$$

From (9) and (10) then it follows that

(11) $$b(\beta_1 r) = b_2(\beta_1' s)$$

and from the second equations of (9) and (10) in the same way that

(12) $$a(\beta_1 r) = a_2(\beta_1' s)$$

and (11) and (12) are equivalent to $\left(\frac{a}{b}\right) = \left(\frac{a_1}{b_1}\right)$. We note

(13) $$\left(\frac{ac}{bc}\right) = \left(\frac{a}{b}\right)$$

for all $c \neq 0$.

Addition. We define

(14) $$\left(\frac{a}{b}\right) + \left(\frac{a_1}{b_1}\right) = \left(\frac{a\beta_1 + a_1\beta}{b\beta_1}\right) = \left(\frac{a\beta_1 + a_1\beta}{b_1\beta}\right)$$

where β and β_1 satisfy (6). One easily sees that the sum (14) does not depend on the particular choice of β and β_1 and A_{I} is consequently satisfied. In order to prove A_{II}, we suppose

$$\left(\frac{a}{b}\right) = \left(\frac{a'}{b'}\right), \qquad \left(\frac{a_1}{b_1}\right) = \left(\frac{a_1'}{b_1'}\right)$$

i. e.:

(15) $$b\beta' = b'\beta, \qquad a\beta' = a'\beta$$

and correspondingly for the second fraction. Then

(16) $$\left(\frac{a}{b}\right) + \left(\frac{a_1}{b_1}\right) = \left(\frac{a\lambda + a_1\mu}{b_1\mu}\right), \qquad \left(\frac{a'}{b'}\right) + \left(\frac{a_1}{b_1}\right) = \left(\frac{a'\lambda' + a_1\mu'}{b_1\mu'}\right),$$

where

(17) $$b\lambda = b_1\mu, \qquad b'\lambda' = b_1\mu'.$$

To compare the two fractions (16) we determine ϱ and σ by the condition

(18) $$\mu\sigma = \mu'\varrho$$

and the two fractions are equal if

$$(a\lambda + a_1\mu)\sigma = (a'\lambda' + a_1\mu')\varrho$$

or according to (18)

$$a\lambda\sigma = a'\lambda'\varrho.$$

This is a simple consequence of (15), (17) and (18) and it is therefore

$$\left(\frac{a}{b}\right) + \left(\frac{a_1}{b_1}\right) = \left(\frac{a'}{b'}\right) + \left(\frac{a_1'}{b_1'}\right).$$

In the same way it follows that

$$\left(\frac{a'}{b'}\right) + \left(\frac{a_1}{b_1}\right) = \left(\frac{a'}{b'}\right) + \left(\frac{a_1'}{b_1'}\right)$$

and A_{II} must hold.

The associative law follows directly by assuming that the three fractions have the same denominator, which can always be obtained according to (2). A_{IV} follows from

$$\left(\frac{a}{b}\right) + \left(\frac{0}{c}\right) = \left(\frac{a}{b}\right), \qquad \left(\frac{0}{c}\right) = 0.$$

A_{V} is obvious, and A_{VI} is a consequence of

$$\left(\frac{a}{b}\right) + \left(\frac{-a}{b}\right) = 0.$$

Multiplication. We define

$$\left(\frac{a}{b}\right) \cdot \left(\frac{a_1}{b_1}\right) = \left(\frac{a\alpha_1}{b_1\beta}\right)$$

where

$$b\alpha_1 = a_1\beta, \ \beta \neq 0.$$

One easily proves that the product does not depend on the particular choice of α_1 and β, and M_{I} is fulfilled. M_{II} and M_{III} are derived by simple calculations. M_{IV} will hold to the extent which it holds in S. M_{V} follows easily and the unit-element is $\left(\frac{a}{a}\right) = 1$, which is independent of the choice of a according to (13). Finally M_{VIII} is satisfied since

$$\left(\frac{a}{b}\right) \cdot \left(\frac{b}{a}\right) = 1.$$

It is therefore proved, that the totality K of fractions $\left(\frac{a}{b}\right)$ forms a field; the regular ring S is, as one easily sees, isomorphic to the ring of all elements in K of the form $\left(\frac{ac}{c}\right)$. As a corollary it follows, that every regular ring is a subring of a ring with unit elements.[8]

One can also easily prove the following theorem, which is, to a certain extent the converse of Theorem 1:

THEOREM II. *Let a and $b \neq 0$ run through all elements of a ring S without divisors of zero. If then the formal solutions of all equations*

$$xb = a \tag{19}$$

form a field, the ring S must be a regular ring.

If one supposes that all solutions $\left(\frac{a}{b}\right)$ of (19) form a field K, then as formerly the ring S must be isomorphic to the ring of elements of the form $\left(\frac{ac}{c}\right)$ in K. Furthermore in this field there must exist a b^{-1} and the equation

$$bx = a \tag{20}$$

must have a solution $x = \left(\frac{r}{s}\right)$ which is also contained in K. Then obviously

$$br = as$$

follows.

One can replace (19) by (20) in Theorem II or even in a slightly more general way by

$$axb = c.$$

A proper quotient-field can therefore only exist for regular rings. This result does however not exclude the possibility of rings, which are not regular, from still being subrings of fields; it might even be possible, as in the commutative case, for all rings without divisors of zero to be subrings in fields. A general construction of this kind seems to be difficult to define. later disproved

If for a ring S the axiom M_V is not satisfied, there must exist at least two elements A and B such that a relation

$$Aa + Bb = 0$$

[8] Every ring without divisors of zero is a subring of a ring with unit element; this follows immediately by considering the ring of elements $\left(\frac{ac}{c}\right)$ where $\left(\frac{ac}{c}\right) = \left(\frac{ad}{d}\right)$, $\left(\frac{ac}{c}\right)\left(\frac{bd}{d}\right) = \left(\frac{abc}{c}\right)$ etc. $\left(\frac{c}{c}\right)$ is then the unit element.

can only hold if $a = b = 0$. It would be natural to characterize such rings by the maximum number N of elements $A_1 \cdots A_N$ in the ring such that a relation

$$A_1 a_1 + \cdots + A_N a_N = 0$$

could only hold for $a_1 = \cdots = a_N = 0$, where the a_i are elements of S. The number N, which is finite or infinite, might suitably be called the *order of irregularity*. For an arbitrary element b in S the elements $b, A_1 \cdots A_N$ would be dependent with respect to S and a $c \neq 0$ could be found, such that

$$b c = A_1 a_1 + \cdots + A_N a_N.$$

Various interesting types of irregular rings exist, but I shall refrain from any further studies. Only one example will be given to show the existence of irregular rings and prove that even such rings can be contained in fields.

We consider all polynomials of the form

$$A(x) = a_1 x + \cdots + a_n x^n$$

where the coefficients are arbitrary complex numbers. Equality, addition and subtraction are defined the ordinary way. Multiplication is defined as composition

$$A(x) \times B(x) = A(B(x)).$$

In this way a non-commutative ring has been defined, in which M_{I}, M_{II}, M_{III} are satisfied; x is the unit-element. The distributive law holds for right-hand multiplication, but not for left-hand. From some interesting investigations by Ritt[9] on the composition of polynomials it follows easily that this ring is irregular. This ring is however obviously contained in the field of all algebraic functions that vanish for $x = 0$, when multiplication is defined as composition.

3. Equations with 2 unknowns. Let

$$\begin{aligned} x_1 a_{11} + x_2 a_{12} &= b_1, \\ x_1 a_{21} + x_2 a_{22} &= b_2 \end{aligned}$$

be two linear equations with coefficients in a regular ring S. We can then always find two elements A_{12} and A_{22} in S, such that

$$a_{12} A_{22} = a_{22} A_{12}. \tag{22}$$

[9] J. F. Ritt, Prime and composite polynomials. Trans. Am. Math. Soc. 23 (1922), pp. 51–66.

We make the assumption in the following, that by all solutions of equations of the form (22) $A_{22} \neq 0$, $A_{12} \neq 0$ when $a_{12} \neq 0$, $a_{22} \neq 0$. If $a_{12} = 0$, $a_{22} \neq 0$, then $A_{12} = 0$, $A_{22} \neq 0$, correspondingly $A_{12} \neq 0$, $A_{22} = 0$ if $a_{12} \neq 0$, $a_{22} = 0$; finally $A_{12} = A_{22} = 0$ if $a_{12} = a_{22} = 0$.

From (21) we then obtain

$$x_1 \left|\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix}\right\| = \left|\begin{matrix} b_1 & a_{12} \\ b_2 & a_{22} \end{matrix}\right\| \tag{23}$$

where

$$\Delta_{12}^{(12)} = \left|\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix}\right\| = a_{11} A_{22} - a_{21} A_{12}, \quad \left|\begin{matrix} b_1 & a_{12} \\ b_2 & a_{22} \end{matrix}\right\| = b_1 A_{22} - b_2 A_{12} \tag{24}$$

are called *right-hand determinants* of second order. By a different choice of the A_{12} and A_{22} in (22) we obtain different determinants, but by r. h. multiplication with elements in S they can all be obtained one from another. If we introduce the quotientfield K corresponding to S, all the different expressions (24) for the determinant can be obtained by multiplying one of them r. h. by an element $k \neq 0$ in K.

A determinant is therefore non-zero or zero. In most problems it is necessary only to decide, whether a determinant vanishes or not. We shall say that two determinants are *equivalent* (denoted by $\sim$) if they both vanish or do not vanish.

Obviously one obtains, by using the same values for A_{12} and A_{22} as in (22)

$$\Delta_{21}^{(12)} = \left|\begin{matrix} a_{12} & a_{11} \\ a_{22} & a_{21} \end{matrix}\right\| = a_{12} A_{21} - a_{22} A_{11} \sim \Delta_{12}^{(12)}$$

or by a different choice of A's

$$\Delta_{21}^{(12)} = -\Delta_{12}^{(12)} k, \quad k \neq 0, \quad \text{i. e. } \Delta_{21}^{(12)} \sim \Delta_{12}^{(12)}.$$

It can also be shown, that two columns can be interchanged

$$\Delta_{12}^{(21)} = \left|\begin{matrix} a_{12} & a_{11} \\ a_{22} & a_{21} \end{matrix}\right\| = a_{12} A_{21} - a_{22} A_{11} \sim \Delta_{12}^{(12)}. \tag{25}$$

It is sufficient to prove, that from $\Delta_{12}^{(12)} = 0$ follows $\Delta_{12}^{(21)} = 0$. Let us first suppose $a_{12} \neq 0$, $a_{11} \neq 0$, i. e. $A_{12} \neq 0$, $A_{11} \neq 0$. Then from (22) one obtains $a_{22} = a_{12} k$, and from $\Delta_{12}^{(12)} = 0$ in the same way $a_{21} = a_{11} k$. Substituting this in

$$a_{11} A_{21} = a_{21} A_{11}$$

it follows that $A_{21} = k A_{11}$ and (25) must also vanish. The same holds, as can easily be seen, when $a_{12} = 0$ or $a_{11} = 0$.

One therefore has the equivalences

$$\Delta_{12}^{(12)} \sim \Delta_{12}^{(21)} \sim \Delta_{21}^{(12)} \sim \Delta_{21}^{(21)}. \tag{26}$$

Through elementary calculations one also shows

$$\left|\begin{matrix} k\,a_{11}, & a_{12} \\ k\,a_{21}, & a_{22} \end{matrix}\right\| \sim \left|\begin{matrix} a_{11}, & k\,a_{12} \\ a_{21}, & k\,a_{22} \end{matrix}\right\| \sim \Delta_{12}^{(12)}, \quad k \neq 0. \tag{27}$$

$$\left|\begin{matrix} a_{11}k, & a_{12}k \\ a_{21}, & a_{22} \end{matrix}\right\| \sim \left|\begin{matrix} a_{11}, & a_{12} \\ a_{21}k, & a_{22}k \end{matrix}\right\| \sim \Delta_{12}^{(12)}, \quad k \neq 0. \tag{28}$$

$$\left|\begin{matrix} a_{11} + k\,a_{12}, & a_{12} \\ a_{21} + k\,a_{22}, & a_{22} \end{matrix}\right\| \sim \left|\begin{matrix} a_{11}, & a_{12} + k\,a_{11} \\ a_{21}, & a_{22} + k\,a_{21} \end{matrix}\right\| \sim \Delta_{12}^{(12)}. \tag{29}$$

$$\left|\begin{matrix} a_{11} + a_{21}k, & a_{12} + a_{22}k \\ a_{21}, & a_{22} \end{matrix}\right\| \sim \left|\begin{matrix} a_{11}, & a_{12} \\ a_{21} + a_{11}k, & a_{22} + a_{12}k \end{matrix}\right\| \sim \Delta_{12}^{(12)}. \tag{30}$$

From (21) one obtains analogously to (23)

$$x_2\,\Delta_{12}^{(21)} = \left|\begin{matrix} b_1 & a_{11} \\ b_2 & a_{21} \end{matrix}\right\|. \tag{31}$$

Two determinants with the same second column are called *proportional*, if we agree, that in the calculation of the determinants the same set of A's shall be used in both cases. The determinants (23) are proportional; for two proportional determinants the quotient $\Delta' \Delta^{-1}$ always has the same value.

The relations (23) and (31) show that *the necessary and sufficient condition that the system* (21) *have a unique solution in the quotient-field K, is that the determinant $\Delta_{12}^{(12)}$ of* (21) *does not vanish. The solutions can then be found as quotients of proportional determinants.*

When the determinant vanishes, there must exist a linear relation

$$L_1(x_1, x_2)\,A_{22} - L_2(x_1, x_2)\,A_{12} = 0$$

where L_1 and L_2 denote the left-hand sides of the equations (21). In order that a solution then exist it is necessary that the same relation hold for b_1 and b_2 and the second determinant (24) therefore also vanishes. This can be expressed by the fact, that all second-order determinants in the matrix

$$\left|\begin{matrix} a_{11}, & a_{12}, & b_1 \\ a_{21}, & a_{22}, & b_2 \end{matrix}\right\|$$

vanish. In this case one of the equations (21) is a consequence of the other. The analogy to the commutative case is obvious.

Corresponding considerations give the same result for the solution of a left-hand system

$$(32) \qquad \begin{aligned} c_{11}x_1 + c_{12}x_2 &= d_1, \\ c_{21}x_1 + c_{22}x_2 &= d_2. \end{aligned}$$

The l. h. determinant is

$$\left\| \begin{matrix} c_{11}, & c_{12} \\ c_{21}, & c_{22} \end{matrix} \right| = C_{22}c_{11} - C_{12}c_{21},$$

where

$$C_{22}c_{12} = C_{12}c_{22}.$$

The notion of equivalence and proportionality are introduced in an analogous way, and corresponding conditions are obtained for the solvability of a system (32). The complete system of equivalences (27)-(30) also holds for l. h. determinants, when only l. h. and r. h. multiplication with k are interchanged. Finally one should observe the reciprocity between r. h. and l. h. determinants

$$(33) \qquad \left\| \begin{matrix} a_{11}, & a_{21} \\ a_{12}, & a_{22} \end{matrix} \right| \sim \left| \begin{matrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{matrix} \right\|$$

which can easily be derived.

4. **The general case.** After having studied the case $n = 2$ both Heyting and Richardson define the general determinant by induction formulas. We shall also use an inductive definition, which is however selected on the principle of preserving as directly as possible the elimination properties of the determinant Δ.

Let

$$(34) \qquad \sum_{i=1}^{n} x_i a_{ji} = b_j \qquad (j = 1, 2, \cdots, n)$$

be a system of linear equations. We then define the right-hand determinant by

$$(35) \qquad |a_{ji}\| = a_{11}A_1^{(1)} + a_{21}A_1^{(2)} + \cdots + a_{n1}A_1^{(n)},$$

where the $A_1^{(j)}$ are a set of solutions of the homogeneous l. h. system

$$(36) \qquad \begin{aligned} a_{12}A_1^{(1)} + a_{22}A_1^{(2)} + \cdots + a_{n2}A_1^{(n)} &= 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{1n}A_1^{(1)} + a_{2n}A_1^{(2)} + \cdots + a_{nn}A_1^{(n)} &= 0. \end{aligned}$$

Let us now assume that the properties of l. h. and r. h. linear systems with $n-1$ unknowns have already been obtained as in the case $n = 2$.

By the choice of the solution

$$(37) \qquad A_1^{(j)} \qquad (j = 1, 2, \cdots, n)$$

of (36) we define $A_1^{(j)} = 0$ if all the l. h. determinants of order $n-1$ in the matrix

$$(38) \qquad \left\| \begin{matrix} a_{12}, & a_{22}, & \cdots & a_{n2} \\ . & . & . & . \\ a_{1n}, & a_{2n}, & \cdots & a_{nn} \end{matrix} \right|$$

vanish. If one of the determinants in (38) is not equivalent to zero, the system (36) must have a fundamental solution (37) such that the most general solution can be obtained from it by r. h. multiplication of an arbitrary constant. If one supposes for example that the first determinant in (38) does not vanish, an arbitrary value $A_1^{(n)} \neq 0$ can be assigned to the last unknown in (36) and the others are then determined by equations of the type

$$(39) \qquad \| \Delta_j | \, A_1^{(j)} = \| \Delta_j' | \, A_1^{(n)}$$

where $\|\Delta_j| \neq 0$ and $\|\Delta_j'|$ are $(n-1)$-order proportional determinants from (38).

We multiply the equations (34) respectively by $A_1^{(1)}, A_1^{(2)} \cdots$ and add; considering (36) one obtains

$$(40) \qquad x_1 \cdot |a_{ij}\| = \left| \begin{matrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ . & . & . & . \\ b_n & a_{n2} & \cdots & a_{nn} \end{matrix} \right\| = b_1 A_1^{(1)} + \cdots + b_n A_1^{(n)}.$$

The right side determinant in (40) differs from $|a_{ij}\|$ only by the elements in the first column, and the same $A_1^{(j)}$ have been used in its calculation. Such determinants are called *proportional*. The equation (40) gives, when $|a_{ij}\| \neq 0$, a single value for x_1.

We shall now prove, chiefly by induction, a few of the properties of n-th order determinants. From the definition follows immediately:

An equivalent determinant is obtained when two rows are interchanged.

Such a change will only invert the order of a_{j1} and a_{i1} in (35), and then the corresponding $A_1^{(j)}$ and $A_1^{(i)}$ will also be interchanged.

A little more difficult to prove is:

When two columns are interchanged one obtains an equivalent determinant.

When the first column remains unchanged, the equations (36) will only change in order and their solutions are the same. Let us then suppose that the first column is interchanged with the second, and let us assume $|a_{ji}\| \neq 0$. We shall then prove that the new determinant

$$(41) \qquad \Delta' = a_{12} B_2^{(1)} + a_{22} B_2^{(2)} + \cdots + a_{n2} B_2^{(n)}$$

does not vanish. In (41) the $B_2^{(j)}$ must satisfy the same equations (36) as the $A_1^{(j)}$ except that the first equation must be replaced by

$$a_{11} B_2^{(1)} + a_{21} B_2^{(2)} + \cdots + a_{n1} B_2^{(n)} = 0. \tag{42}$$

If now $\Delta' = 0$, it follows from (41) that the $B_2^{(j)}$ would satisfy exactly the same $n-1$ equations as the $A_1^{(j)}$ and one must have $B_2^{(j)} = A_1^{(j)} k$. From (42) then follows $|a_{ji}|| = 0$ contrary to the supposition.

When all elements in a column in a r. h. determinant are multiplied l. h. with $k \neq 0$ one obtains an equivalent determinant.

Since the columns can be interchanged one can always suppose, that the first column is multiplied l. h. by k. From the definition (35) the theorem then follows immediately.

The following theorems are easily proved:

If one multiplies all elements of a row in a r. h. determinant r. h. by $k \neq 0$, one obtains an equivalent determinant.

When a row is added to another row or a column to a column, the resulting determinant is equivalent to the original.

Analogous theorems hold for l. h. determinants.

5. Let us now first consider the homogeneous system of equations

$$\sum_{i=1}^{n} x_i a_{ji} = 0 \tag{43}$$

where $|a_{ji}|| \neq 0$. As in (40) we then obtain relations

$$x_j \cdot |\Delta_j|| = 0 \tag{44}$$

where $|\Delta_j||$ is obtained from $|a_{ji}||$ by interchanging columns. All these determinants are therefore $\neq 0$ and we therefore have:

If the determinant of the system (43) *does not vanish, the only solution is $x_j = 0$ $(j = 1, 2, \cdots, n)$.*

When the determinant vanishes, there exists a linear relation between the l. h. sides and one of the equations is redundant. If all $(n-1)$-order determinants in the corresponding matrix vanish, at least two of the equations are a consequence of the others etc. The notion of *rank* of a system can therefore be introduced, and all theorems on the solution of homogeneous systems can be derived as in the commutative case.

We shall now generalize the reciprocity relation (33) to determinants of order n; to prove

$$\| a_{ij} | \sim | a_{ji} \| \qquad (i, j = 1, 2, \cdots n) \tag{45}$$

it is sufficient to prove, that when one of these determinants does not vanish, the other must also be different from zero. Let us suppose

$|a_{ji}\| \neq 0$. Then according to definition not all determinants of order $n-1$ in (38) can vanish; by a rearrangement of rows and columns we can suppose

$$\| a_{ij} | \neq 0 \qquad (i, j = 2, 3, \cdots, n).$$

If we assume that (45) has been proved for all determinants of order $\leqq n-1$, we conclude

$$\| a_{ij} | \sim |a_{ji}\| \neq 0 \qquad (i, j = 2, 3, \cdots, n). \tag{46}$$

The determinant on the left in (45) has the expression

$$\| a_{ij} | = B_1^{(1)} a_{11} + B_2^{(1)} a_{12} + \cdots + B_n^{(1)} a_{1n} \tag{47}$$

where the $B_i^{(1)}$ must satisfy the equations

$$\sum_{i=1}^{n} B_i^{(1)} a_{ji} = 0 \qquad (j = 2, 3, \cdots, n). \tag{48}$$

From (46) follows, that at least one of the $B_i^{(1)}$ does not vanish. On the other hand, if (47) vanishes, the relations (48) must be satisfied even for $j=1$, i. e. the $B_i^{(1)}$ must be a solution of (43) and since the determinant of this system does not vanish, we must have $B_i^{(1)} = 0$ $(i = 1, 2, \cdots, n)$ contrary to the supposition.

Let us now finally consider the complete system of equations (34) and suppose $| a_{ji} \| \neq 0$. Then one obtains analogously to (40)

$$x_i \cdot | \Delta_i \| = | \Delta_i' \| \tag{49}$$

where $| \Delta_i \| \neq 0$, since it is obtained from $| a_{ji} \|$ by interchanging columns. The relations (49) prove, that if the determinant of the system (34) is different from zero, there can be only a unique solution given by (49). In order to prove the existence of this soltution, it is necessary to prove, that the x_i defined by (49) actually satisfy (34).

Let us put

$$\begin{aligned} L_j(x) &= x_1 a_{j1} + x_2 a_{j2} + \cdots + x_n a_{jn} - b_j, \\ M_j(x) &= x_j | \Delta_j \| - | \Delta_j' \|. \end{aligned} \qquad (j = 1, 2, \cdots, n).$$

From the way the relation (49) has been derived, it follows that

$$\sum_{j=1}^{n} L_j(x) A_i^{(j)} = M_i(x) \qquad (i = 1, 2, \cdots, n). \tag{50}$$

To show that the x_i defined by (49) satisfy (34), one must show that conversely the $L_j(x)$ can be homogeneously expressed by the $M_j(x)$. This is according to (50) always possible, if the *complementary determinant* $|A_i^{(j)}||$ to $|a_{ji}||$ does not vanish. It is however simple to prove that

$$|\Delta_i^{(j)}|| \sim ||A_j^{(i)}| \sim |a_{ji}||.$$

One observes easily that the complementary determinant to $|A_i^{(j)}||$ must be equivalent to $|a_{ji}||$ itself.

We have therefore proved:

A linear system (34) *has one and only one solution if its determinant does not vanish.*

Furthermore follows directly:

The necessary and sufficient condition, that a system (34) *have a solution is that any identical relation between the left-hand side linear forms must also be satisfied by the constants* b_i.

It is only necessary to indicate that a series of the properties of determinants in the commutative case can be extended by means of the principles used in this outline. Among the theorems of interest I shall only mention, that by a linear substitution one obtains also in the non-commutative case a *composed* determinant corresponding to the product of two determinants in the commutative case; it can be shown that this composed determinant is equivalent to the product of the two constituents.

It is perfectly obvious from the preceding results how matrices and rank of matrices can be introduced, and that the corresponding results from the commutative case can be derived in an analogous way. I shall therefore not go into the details of this theory.

YALE UNIVERSITY.