NORTH-HOLLAND

Compact Graphs and Equitable Partitions

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ABSTRACT

Let G be a graph with adjacency matrix A, and let Γ be the set of all permutation matrices which commute with A. We call G *compact* if every doubly stochastic matrix which commutes with A is a convex combination of matrices from Γ. We characterize the graphs for which $S(A) = \{I\}$ and show that the automorphism group of a compact regular graph is generously transitive, i.e., given any two vertices, there is an automorphism which interchanges them. We also describe a polynomial time algorithm for determining whether a regular graph on a prime number of vertices is compact. © Elsevier Science Inc., 1997

1. EQUITABLE PARTITIONS AND DOUBLY STOCHASTIC MATRICES

A matrix is *doubly stochastic* if it is nonnegative and each of its rows and each of its columns sums to one. If A is the adjacency matrix of the graph G, we define $S(A)$ to be the set of all doubly stochastic matrices which commute with A. We note that $S(A)$ is a convex polytope, since it consists all matrices X such that

$$XA = AX, \qquad X\mathbf{1} = X^T\mathbf{1} = \mathbf{1}, \qquad X \geqslant 0.$$

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Each automorphism of G determines a permutation matrix which commutes with A; denote the set of these matrices by Γ. Then Γ is a matrix group isomorphic to the automorphism group of G, and each matrix in Γ is an extreme point of $S(A)$. We call G *compact* if all extreme points of $S(A)$ lie in Γ. The basic theory of compact graphs has been developed by Tinhofer, who has proved, amongst other things, that trees and cycles are compact [9, Theorems 2, 3] and that the disjoint union of isomorphic copies of a compact graph is compact [10, Theorem 6]. For related results, see [3].

Clearly the identity matrix I is contained in $S(A)$; the main result of this section is a characterization of the graphs for which $S(A) = I$. Our characterization makes use of equitable partitions, which we now discuss. (For more background, see Chapter 5 of [5].) Let G be a graph with n vertices, and let π be a partition of $V(G)$, with cells $C_1, \dots, C_r$. We call π *equitable* if, for any ordered pair of cells (C_i, C_j), the number of vertices in C_j adjacent to a fixed vertex in C_i only depends on i and j. We denote the number of cells in π by $|\pi|$. A partition is *discrete* if each cell is a singleton. The orbits of any group of automorphisms of G always form an equitable partition; we call such partitions *orbit partitions*. A partition π can be represented by what we call its *normalized characteristic matrix* $P(\pi)$, defined as follows. Suppose that $\pi = (C_1, \dots, C_m)$ and $c_i := |C_i|$. Then $P(\pi)$ is the $n \times m$ matrix with ith column equal to $c_i^{-1/2}$ times the characteristic vector of C_i, viewed as a subset of $V(G)$. Note that the columns of P are pairwise orthogonal unit vectors in $\mathbb{R}^n$.

LEMMA 1.1. *Let A be the adjacency matrix of G, and let π be a partition of $V(G)$ with normalized characteristic matrix P. Then π is equitable if and only if A and PP^T commute.*

Proof. From [6, Theorem 2.1] we know that π is equitable if and only if there is an $m \times m$ matrix B such that

$$AP = PB, \tag{1.1}$$

where $P = P(\pi)$. If π is equitable, then (1.1) yields that

$$B = P^T A P,$$

whence B is symmetric. Using (1.1) again, we see that

$$APP^T = PBP^T$$

and therefore APP^T is symmetric. Since A and PP^T are both symmetric, it follows that A and PP^T commute.

For the converse we note that π is equitable if and only if each cell induces a regular subgraph of G and the edges joining any two distinct cells form a semiregular bipartite graph. It is easy to verify that this holds if and only if $APP^T = PP^TA$. ■

If π is a partition with normalized characteristic matrix P, then PP^T is doubly stochastic; we denote the latter matrix by X_π. Given this, we have the following reformulation of Lemma 1.1.

COROLLARY 1.2. *Let π be a partition of the vertices of $V(G)$ with normalised characteristic matrix P. Then π is equitable if and only if $X_\pi \in S(A)$.*

As an immediate consequence we have:

COROLLARY 1.3. *If G is compact, then every equitable partition is an orbit partition.*

The *distance partition* with respect to a vertex v in G is the partition whose ith cell is the set of vertices in G at distance i from v, for each i. From the definition of distance-regular graphs (see, e.g., [2]) it follows that in a distance-regular graph the distance partition with respect to any vertex is equitable. From the previous corollary we deduce that the distance partition with respect to a vertex v is the partition formed by the orbits of the stabiliser of v in the automorphism group, and from this we obtain the following:

COROLLARY 1.4. *If G is compact and distance-regular, then it is distance-transitive.*

If $n \geqslant 7$, then the line graph of the complete graph K_n is distance-transitive, but not compact. To see this, choose a subgraph G of K_n isomorphic to $C_3 \cup C_{n-3}$. Let π be the partition of $L(K_n)$ with two cells, one consisting of the vertices corresponding to the edges of G, and the other formed by the remaining vertices. Then it is easy to verify that π is equitable, but it is not an orbit partition (since G is not vertex-transitive).

Our next observation is that every matrix in $S(A)$ determines a nontrivial equitable partition of G. To prove this we need one property of doubly stochastic matrices. Suppose X is a doubly stochastic matrix. Define $D(X)$ to be the directed graph with the rows of X as its vertices, and ij entry equal to one if and only if $(X)_{ij} \neq 0$.

THEOREM 1.5. *If $X \in S(A)$, then the partition whose cells are the strong components of $D(X)$ is equitable.*

Proof. We show first that any weak component of X is a strong component. Assume that C is a subset of $V(D)$ such that there is no arc (u, v) with $u \in C$ and $v \notin C$. Then the sum of the entries of X in the rows corresponding to C is $|C|$, whence the sum of the entries in the submatrix of X with rows and columns indexed by C is again $|C|$. But this implies that if $v \notin C$ and $u \in C$ then $(X)_{vu} = 0$, and therefore there are no arcs in D from a vertex not in C to a vertex in C. It follows that if X is doubly stochastic, then we may write it in block-diagonal form as

$$X = \begin{pmatrix} X_1 & & \\ & \ddots & \\ & & X_r \end{pmatrix}$$

where $X_1, \dots, X_r$ are doubly stochastic matrices and $D(X_1), \dots, D(X_r)$ are strongly connected.

Since $D(X_i)$ is strongly connected, 1 is a simple eigenvalue of it, whence we see that 1 has geometric and algebraic multiplicity r as an eigenvalue of X. Let U denote the right eigenspace of X associated to 1. Then U consists of the vectors which are constant on the components of $D(X)$, and therefore the matrix representing orthogonal projection onto it has block-diagonal form:

$$\begin{pmatrix} m_1^{-1} J_{m1} & & \\ & \ddots & \\ & & m_r^{-1} J_{m_r} \end{pmatrix} \tag{1.2}$$

If $u \in U$ then $u^T X = u^T$. Hence if $y \in U^{\perp}$ and $u \in U$ then $u^T X y = u^T y = 0$, whence we see that $U^{\perp}$ is invariant under X.

If $p(T) := \det(tI - X)/(t - 1)^r$ and $y \in U$, then $p(X)y = p(1)y$. By the Cayley-Hamilton theorem, $p(X)(X - I)^r = 0$, and if $y \in U^{\perp}$ then

$$0 = p(X)(X - I)^r y = (X - I)^r p(X) y.$$

But $p(X)y \in U^{\perp}$, and the nullspace of $(X - I)^r$ is U; consequently $p(X)y$ must be zero. If E is the matrix $p(1)^{-1}p(X)$, it follows that E is diagonalizable and that its eigenvalues are 0 and 1. Hence $E^2 = E$.

If u and v belong to U, then $(Xu, v) = (u, v) = (u, Xv)$. Using this, it follows easily that $p(X)$ is symmetric, and hence E is a projection. Since E has rank r, it must be equal to the matrix in (1.2), and consequently it can be written as PP^T, where P is the normalized characteristic matrix of the partition whose cells are the components of X. Since E commutes with A, it follows that π is equitable.

COROLLARY 1.6. *We have $S(A) = \{I\}$ if and only if G has no nontrivial equitable partitions.*

From [4], for example, we know that the coarsest equitable partition of a graph can be found in polynomial time.

2. COMPACT REGULAR GRAPHS

Tinhofer [10; Section 4] observes, and it also follows from our Corollary 1.3, that a compact regular graph must be vertex transitive. In fact a somewhat stronger statement can be proved. The *rank* of transitive permutation group is defined to be the number of orbits of the stabilizer of a point. A permutation group on a set X is *generously transitive* if, given any two points, there is a permutation which interchanges them. (So the dihedral group acting on n points is generously transitive, and a regular permutation group is generously transitive if and only if it is an elementary abelian 2-group.)

THEOREM 2.1. *Let G be a regular graph with exactly r distinct eigenvalues. If G is compact, then* Aut(G) *is a generously transitive permutation group with rank r.*

Proof. If G is compact and regular, then it is vertex-transitive. Hence its components are all isomorphic, and can easily be seen to be compact. It follows that we may assume without loss that G is connected. Let Γ be the set of all permutation matrices which commute with A, and let $\mathscr{C}$ be the convex hull of Γ. We aim to compare the dimensions of $S(A)$ and $\mathscr{C}$.

Let m_i be the multiplicity of the ith eigenvalue of G. The space $C(A)$ of matrices which commute with A has dimension

$$\sum_{m=1}^{r} m_i^2.$$

As G is connected, J is a polynomial in A, and therefore it commutes with any matrix in $C(A)$. Accordingly all matrices in $C(A)$ have constant row and column sums. Consequently the dimension of $S(A)$ is equal to the dimension of the span of the nonnegative elements of $C(A)$. If $M \in C(A)$, then for all sufficiently small values of ϵ,

$$J + \epsilon M \in C(A).$$

This implies that $S(A)$ and $C(A)$ have the same (linear) dimension.

Now we consider the dimension of the space spanned by Γ. If ρ denotes the permutation representation of Γ on the vertices of G, then there are irreducible representations Ψ_i and nonnegative integers c_i such that

$$\rho = \sum_{i=1}^{s} c_i \psi_i.$$

(If the c_i are all equal to one, ρ is said to be *multiplicity-free.*) From Theorem II.1 in [7] it follows that the space spanned by $\rho(\Gamma)$ has dimension

$$\sum_{i=1}^{s} \psi_i(e)^2,$$

where e denotes the identity of Γ.

Next we relate the two pieces of information we have gained. Each eigenspace of A is Γ-invariant, and ρ is the direct sum of the representations of Γ on the distinct eigenspaces of A. This implies that the dimension of the span of Γ is bounded above by the dimension of $S(A)$, with equality if and only if $r = s$ and $m_i = \psi_i(e)$ for $i = 1, \dots, r$ (perhaps after some reordering). Further, since

$$n = \sum m_i = \sum c_i \psi_i,$$

we see that, if equality holds, then $c_i = 1$ for all i, and ρ is multiplicity-free.

By a result of P. Cameron (see [2, Proposition 2.9.2]) a multiplicity-free permutation group is generously transitive if and only all irreducible constituents of its permutation character are real. Hence the theorem follows. ■

It follows from [6, Theorem 4.8] that a vertex-transitive graph on n vertices has at most $3n/4$ distinct eigenvalues when $n > 2$. As a transitive permutation group on n points is regular if and only if its rank is n, the automorphism group of a compact graph X with more than two vertices cannot act regularly on $V(X)$. If G is the path on five vertices, then the space of matrices which commute with A and J has dimension three, being spanned by J and the projections onto the eigenspaces of A with eigenvalues

1 and -1. However, G is compact (by [9, Theorem 3]) and $|\Gamma| = 2$, so $S(A)$ has dimension two. This shows that if G is not regular, then the dimensions of $S(A)$ and $C(A)$ may differ.

Theorem 2.1 implies that a compact connected regular graph G is the union of some classes in a symmetric association scheme on the same set of vertices.

The proof of Theorem 2.1 raises the problem of deciding when the intersection of the span of Γ with $S(A)$ is equal to the convex hull of Γ. Equality must hold for compact graphs, of course. Schreck and Tinhofer [8] show that a transitive graph on p points (p prime) which is neither complete nor empty can be compact if and only if its automorphism is dihedral of order $2p$. Their proof shows that if the automorphism group is larger than this, then the intersection of $S(A)$ with the real span of Γ strictly contains $\mathscr{C}$.

Using Schreck and Tinhofer's result, we can decide in polynomial time whether a regular graph on a prime number of vertices is compact. For this we need the following result.

LEMMA 2.2. *Let G be a connected regular graph on a prime number of vertices. If G has an eigenvalue with multiplicity at least three and is not a complete graph, it is not compact.*

Assume $p = |V(G)|$, and let k denote the valency of G. If G is not vertex-transitive, it is not compact. If G is vertex-transitive, then the Sylow p-subgroup of $\mathrm{Aut}(G)$ acts transitively on $V(G)$, and therefore G is a circulant.

Let θ be a primitive pth root of unity, and let V be the Van der Monde matrix with ij entry equal to $\theta^{(i-1)(j-1)}$. Then the columns of V form a set of n pairwise orthogonal eigenvectors for $A = A(G)$. (Although V will have complex entries in general, the eigenvalues corresponding to these eigenvectors will all be real.) Let V_i denote the ith column of V. The vectors $V_2, \dots, V_p$ are algebraically conjugate over the rationals. Now one eigenspace of A is spanned by V_1, and each of the remaining eigenspaces is spanned by some subset of the vectors $V_2, \dots, V_p$. It follows that these eigenspaces are also algebraically conjugate, and so they all have the same dimension. Therefore all eigenvalues of G not equal to k have the same multiplicity, m say.

Now, from the proof the previous theorem, the dimension of $S(A)$ is

$$1 + m(p-1).$$

Let Γ be the set of all permutation matrices which commute with A. If the dimension of the span of Γ is less than $1 + m(p-1)$, then G is not compact. If $\mathrm{Aut}(G)$ is dihedral of order $2p$, then G is compact, whence the dimension of $S(A)$ and that of the span of Γ both equal $2p - 1$. However, $m \geqslant 3$, and thus either Aut(G) is not dihedral, or the dimension of the span of Γ is smaller than the dimension of $S(A)$. In either case G is not compact. ∎

So suppose that G is a regular graph on p vertices. We may compute the characteristic polynomial of $\varphi(G, x)$ of G. The greatest common divisor of $\varphi(G, x)$ and its second derivative is the constant polynomial if and only if all eigenvalues of G have multiplicity at most two. However, if all eigenvalues of G have multiplicity at most two, then we can compute generators for, and the order of, Aut(G) in polynomial time. (See [1, theorem 4.1].) Using the generators, we can determine whether Aut(G) is vertex-transitive. If it is not, then G is not compact. If Aut(G) is vertex-transitive, then it is a subgroup of the 1-dimensional affine group over $GF(p)$, and hence it is dihedral if and only if $|\mathrm{Aut}(G)| = 2p$. This completes our argument.

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