Countable homogeneous and partially homogeneous ordered structures

J K Truss

University of Leeds

Abstract

I survey classification results for countable homogeneous or 'partially homogeneous' ordered structures. This includes some account of Schmerl's classification of the countable homogeneous partial orders, outlining an extension of this to the coloured case, and also treating results on linear orders, and their generalizations, trees and cycle-free partial orders.

2000 Mathematics Subject Classification 06F99
keywords: partial ordering, tree, homogeneous

1 Introduction

In this paper I shall describe a number of classifications of countable homogeneous or partially homogeneous structures, concentrating on partial and linear orders and coloured versions of these. The starting points are therefore [22] and [8]. By saying that a structure is 'homogeneous' we mean that any isomorphism between finite substructures extends to an automorphism. (In some papers this is called 'ultrahomogeneous' to distinguish it from other uses of the word 'homogeneous'.) We may regard it as an interesting exercise to try to classify at least some of the structures in a certain class, in the hope that this will enable us to understand the whole class better; in addition, since homogeneity is a restriction on the automorphism group, such classifications can furnish interesting examples of permutation groups having a rich structure, which may in some cases enable us to 'recognize' the structure from its group.

It was Schmerl who succeeded in [22] in giving a complete list of all the countable homogeneous partial orders. This list is relatively restricted. In particular it only contains countably many structures, and the corresponding automorphism groups have been studied in a number of papers, with most work being done on the 'generic' partial order. Simplicity results were for instance presented in [12]. If one wishes to consider larger classes, various weakenings of the notion of homogeneity are available. The first is to consider 'partial homogeneity'. We say, for instance, that a structure is *k-homogeneous* if any isomorphism between substructures of size k extends to an automorphism. We also say that it is *k-transitive* if *some* isomorphism between isomorphic substructures of size k (not necessarily the originally given one) extends to an automorphism. The case of 1-transitive linear orders was studied by Morel [18], and in his thesis [8], Droste obtained results about k-homogeneous partial orders for various k. He concentrated on the case of trees, the most important classification being of all the countable 2-transitive trees. This was extended to 'weakly 2-transitive trees' in [10], and to 1-transitive trees in [6]. Certain non-trees, called 'cycle-free partial orders' were classified in [7, 25, 27].

Another direction is to consider 'coloured' versions of the original structures. A *coloured partial order* is a structure of the form $(P, <, F)$ where $(P, <)$ is a partial order and F is a function from P onto a set C, thought of as the 'colour set'. Here isomorphisms are required to preserve colours as well as the order, and we immediately have notions of homogeneity and partial homogeneity for such structures. Colours actually arise naturally anyway, even in the monochromatic context, since any structure can automatically be made 1-homogeneous, say, by giving different colours to each of the orbits of its points under the action of its automorphism group. In addition, colours arose naturally in [10] and [27] as certain cuts in chains, which meant that it was necessary to classify particular coloured chains, which arose in the general description of these other more involved structures. This led directly to the work done by Campero-Arena [2], generalizing Morel's work on 1-transitive chains to the coloured case. For finite sets of colours this amounted to a modification of Morel's work, but allowing infinitely many colours greatly added to the complexity of the problem. The results for finite colour sets were related to work of Rosenstein [20]. Instead of homogeneity or partial homogeneity as a hypothesis he considered $\aleph_0$-categoricity, and gave a complete classification of all $\aleph_0$-categorical linear orders.

In yet another direction, we may consider structures which are not even partial orders at all, namely graphs or directed graphs, and I only mention these in passing to compare with the partial orders results. In fact classifications in these cases were one of the main initial motivations for a study of various homogeneous orders. Lachlan and Woodrow [16] extended Gardiner's classification [11] of finite homogeneous graphs to the countable case, and in later work, Lachlan [17] and Cherlin [5] classified the countable homogenous tournaments, and the countable homogeneous digraphs respectively. We only mention the following points about these results. Gardiner's list, and the extension by Lachlan and Woodrow, each comprises just countably many structures (and there are only five countable homogeneous tournaments). However, Cherlin's classification contains $2^{\aleph_0}$ structures (the majority constructed by Henson in [15]). Before Cherlin's proof, it was thought that because there were known to be uncountably many examples, that in itself would rule out any 'classification'.

Despite this, Cherlin's list is reasonably explicit, and most of the uncountably many structures are described in terms of the choice of a set of integers (a 'real number'). In the classifications I describe in this paper, there are just countably many in these cases: homogeneous partial orders, $\aleph_0$-categorical linear orders, 2-transitive trees, there are $2^{\aleph_0}$ for homogeneous coloured partial orders, 1-transitive coloured linear orders when the colour set is infinite, weakly 2-transitive trees, and 1-transitive trees. There is a third possibility, namely $\aleph_1$ (which of course may be different from $2^{\aleph_0}$) which applies to these cases: 1-transitive linear orders, 1-transitive coloured linear orders where the colour set is finite, and $3-CS$-transitive cycle-free partial orders.

What we ultimately seek in each case is a 'classifier'. In the easiest and simplest cases this may just be a natural number, or pair of natural numbers, as for instance in the classification of finite fields (we may just use the cardinality of the field, or else the pair consisting of characteristic and dimension). The idea however is that we should be able to read off from a classifier what the structure is, and that it should give us some insight into how a general structure in the class being classified is built up, or what it looks like, and therefore it is often preferable for the classifier itself to carry some structure. Now Rosenstein described the $\aleph_0$-categorical countable linear orders by saying that they are formed from singletons by closing up under two operations, sums (concatenations) and 'shuffle' (see later for the details). If we want to describe these more explicitly, then a natural method is to represent them by a (finite) tree describing which operations were used, and at what stage. This is precisely the route taken in [3], and these 'coding trees' provide quite a neat and visual method for describing the countable 1-transitive coloured linear orders with finite colour set. In the extension to infinitely many colours, it was found that the use of coding trees, previously merely cosmetic and organizational, is really essential, since the class of structures being classified is so much more involved. In the other instances mentioned above the classifiers comprise, for homogeneous partial orders and for 2-transitive trees, at most two cardinality parameters from 1 to $\aleph_0$; for homogeneous directed graphs, a real (subset of ω) together with cardinalities in some cases; likewise for countable weakly 2-transitive trees; for countable 1-transitive linear orders, essentially just a countable ordinal; and for homogeneous countable coloured partial orders, what Torrezão calls a 'skeleton'. This is really just an analogue of 'coding tree' tailored for the type of structure under consideration.

Now describing the countable homogeneous structures of a particular relational similarity type is equivalent to determining all the amalgamation classes of finite structures in that class, as was shown by Fraïssé. Specifically, if $\mathcal{A}$ is a countable homogeneous structure in a countable relational language, then the *age* of $\mathcal{A}$, which is defined to be the class of all structures isomorphic to finite substructures of $\mathcal{A}$, has the following properties: it is closed under isomorphisms and substructures, it has at most countably many members up to isomorphism, and it has the joint embedding and amalgamation properties. Conversely, for any class of finite structures fulfilling these properties (an *amalgamation class*), there is a countable homogeneous structure unique up to isomorphism, whose age is the class that we started with. Often the homogeneous structure arising from a given amalgamation class in this way is referred to as the corresponding *Fraïssé generic* or *Fraïssé limit*. So classifying a class of countable homogeneous structures is equivalent to classifying the corresponding amalgamation classes, and we shall pass freely between the two as the occasion arises.

The structures $\mathbb{Q}_C$ and P_C

A Fraïssé generic structure which will feature frequently in the paper is the 'C-coloured version of the rationals' $\mathbb{Q}_C$ where C is a finite non-empty or countable set. This has domain (isomorphic to) $\mathbb{Q}$, and is 'interdense', meaning that between any two points, there are points of all possible colours. It exists and is unique up to isomorphism, and it arises as the Fraïssé limit of the class of all finite C-coloured linear orders, so is homogeneous. In a similar way we may see that there is a countable generic coloured partial order P_C, which is the Fraïssé limit of the class of all finite C-coloured partial orders. It has to be verified that this is indeed an amalgamation class, which is not completely obvious and requires a little work.

The paper is organized as follows. In section 2 we describe Schmerl's classification of the countable homogeneous partial orders, and its modification by Torrezão in the coloured case. In section 3 we move on to consider 1-transitive linear orders as classified by Morel, and the corresponding coloured versions, also touching on $\aleph_0$-categorical linear orders. Then in section 4 we look at trees, considering three main possibilities, the 2-transitive, weakly 2-transitive, and 1-transitive cases. Finally in section 5 we outline Warren's work on cycle-free partial orders.

We conclude the introduction by listing a few further definitions and notations. We usually take a partial order as being given by a 'strict' relation, that is $<$ rather than $\leq$, though occasionally we may use $\leq$, and $\parallel$ stands for the relation of incomparability. Set notation such as $A < B$ or $A \parallel B$ means that every element of A is less than (incomparable with respectively) every member of B. A *chain* is a linearly ordered set (or a subset of a partially ordered set which is linearly ordered by the induced relation). An *antichain* is a partial order in which every two distinct elements are incomparable. A 3-element partial order $\{x, y, z\}$ with $x > y, z$ and $y \parallel z$ is a Λ-*shape* (with colouring unspecified) and a 3-element partial order $\{x, y, z\}$ with $x < y, z$ and $y \parallel z$ is a V-*shape*. We write $\mathcal{P}_|$ for a 2-element chain, $\mathcal{P}_-$ for a 2-element antichain, $\mathcal{P}_{|.}$ for a 3-element partial order containing a 2-element chain both of whose elements are incomparable with the third point, $\mathcal{P}_\Lambda$ for a Λ-shape, and $\mathcal{P}_{\mathrm{V}}$ for a V-shape. As usual we write $[x, y] = \{z : x \leq z \leq y\}$ and $(x, y) = \{z : x < z < y\}$, (which are not necessarily linear orders).

2 Homogeneous partial orders and the coloured versions

Schmerl's classification of the countable homogeneous partial orders

In this section I outline Schmerl's classification, and the way it is modified in the coloured case. He showed that every countable homogeneous partial order is of one of the following types:

a dense *chain of antichains*, which is a partial order $(P, <)$ obtained from a chain $(X, <)$ by replacing each point x by an antichain A_x of some fixed size $\leq \aleph_0$, and decreeing that if $x < y$ then all points of A_x lie below all points of A_y, with no other relations,

an *antichain of chains*, which is a partial order which can be written as the disjoint union of chains, so that points in distinct chains are incomparable, and which are all singletons, or all isomorphic to $\mathbb{Q}$,

and the (Fraïssé) *generic partial order* $(P, <)$.

There are one or two 'degenerate' cases. A single chain may be viewed either as a chain of antichains or an antichain of chains, and we take it to be the former, if non-trivial, though a singleton is viewed as an antichain. In other words, by an *antichain of (non-trivial) chains* we understand that there are at least two chains involved (though an antichain may have size 1). Most of the time we can treat antichains and antichains of chains as parallel cases, but on one or two occasions their behaviours differ, in which case we refer explicitly to an 'antichain of non-trivial chains'. The same conventions will be followed in the coloured case.

The fact that each of these structures is homogeneous is easy to verify. Conversely one supposes that $(P, <)$ is a countable homogeneous partial order, and shows that it must be of one of the above types. This is done by considering which of $\mathcal{P}_|$, $\mathcal{P}_-$, $\mathcal{P}_{|\cdot}$, $\mathcal{P}_\Lambda$,$\mathcal{P}_\mathrm{V}$ are embeddable in P.

Since the first stage of the classification of countable homogeneous coloured partial orders (the 'interdense' case) is more-or-less the same as Schmerl's, we give the outline for these, and the monochromatic classification will be an immediate consequence. The definition is that $(X, <, F)$ is *interdense* if for any $x < y$ in X and $c \in C$, there is z such that $x < z < y$ and $F(z) = c$. This notion is therefore dependent on precisely which colour set C we are taking, and as usual to make sense of the notion, it is assumed that C is the set of all colours actually appearing.

Most of the interesting new structures in the coloured case arise when interdensity fails. After outlining the interdense classification, we show how a general coloured homogeneous partial order may be written as a union of interdense components. The major part of the classification consists of describing how the components can fit together, which we sketch at the end of this section. Notice that if we think of trying to cut a general coloured partial order into interdense pieces, then each piece will only be interdense with respect to *its* *own* colour set.

In terms of P_C and $\mathbb{Q}_C$, given in the introduction, we can now list the following partial orders, which are all easily seen to be countable homogeneous interdensely coloured partial orders, and will form the members of our classification in the interdense case.

Any antichain at all, even coloured by many colours, is interdense, (since $x < y$ is always false), and it is also clearly homogeneous, since it is essentially a (multi-)set. (In practice we prefer in this case to reduce to the monochromatic subsets as 'components'.)

Each $\mathbb{Q}_C$ is interdense and homogeneous, and any non-trivial countable homogeneous interdense chain is of this form.

Building on these two cases, we have an *antichain of chains*, which is a union of a finite or countable set of copies of some $\mathbb{Q}_C$, with elements in distinct copies incomparable, and we also have a *chain of antichains*, which is obtained from some $\mathbb{Q}_C$ by replacing all points coloured by the same colour by a finite or countable coloured antichain, where points of the same colour must be replaced by isomorphic antichains, and the colour sets of antichains replacing differently coloured points of $\mathbb{Q}_C$ must be disjoint. The ordering is given by $x < y$ if for some $q < r$ in $\mathbb{Q}_C$, x and y lie in the antichains replacing q and r respectively.

Finally we have the generics P_C.

Following Schmerl's method, we may characterize these by which of $\mathcal{P}_{|}$, $\mathcal{P}_{-}$, $\mathcal{P}_{|\cdot}$, $\mathcal{P}_{\Lambda}$,$\mathcal{P}_{\mathrm{V}}$ embed (for appropriate colours on their points). We remark that any (coloured) partial order not embedding $\mathcal{P}_{|}$ is an antichain, and any partial order not embedding $\mathcal{P}_{-}$ (under any colouring) is a chain.

Lemma 2.1 *Let $\mathcal{P}$ be a countable homogeneous coloured partial order. If $\mathcal{P}_{|\cdot}$ and $\mathcal{P}_{\Lambda}$ embed in $\mathcal{P}$ for every possible colouring of their points, then $\mathcal{P}$ is isomorphic to the generic coloured partial order. Similarly if we are given that $\mathcal{P}_{|\cdot}$ and $\mathcal{P}_{\mathrm{V}}$ embed in $\mathcal{P}$ for every possible colouring of their points.*

Proof: Schmerl shows in [22] that if (the monochromatic versions of) $\mathcal{P}_{|\cdot}$ and $\mathcal{P}_{\Lambda}$ both embed in a countable homogeneous partial order, then it is isomorphic to the (monochromatic) generic. This proof is readily adapted to the situation described in the lemma. $\Box$

Lemma 2.2 *Let $\mathcal{P}$ be a countable homogeneous coloured partial order. If $\mathcal{P}_{|\cdot}$ embeds in $\mathcal{P}$ for all possible colourings of its points and $\mathcal{P}_{\Lambda}$ embeds in $\mathcal{P}$ for some colouring of its points then $\mathcal{P}_{\Lambda}$ embeds in $\mathcal{P}$ for all possible colourings of its points, with a similar statement for $\mathcal{P}_{|\cdot}$ and $\mathcal{P}_{\mathrm{V}}$. In either case, $\mathcal{P}$ is then isomorphic to the generic coloured partial order.*

Proof: Starting from an instance of $\mathcal{P}_{\Lambda}$, we may add points above and below using the hypothesis on copies of $\mathcal{P}_{|\cdot}$ to find another instance of $\mathcal{P}_{\Lambda}$ correctly coloured.

The proof for $\mathcal{P}_{\mathrm{V}}$ is the dual, and the final remark follows from Lemma 2.1. $\square$

In the interdense case we have the following rather stronger implication.

Lemma 2.3 *Let $\mathcal{P}$ be a homogeneous interdensely coloured partial order. Then $\mathcal{P}_{\Lambda}$ embeds in $\mathcal{P}$ for some colouring of its points if and only if $\mathcal{P}_{\mathrm{V}}$ embeds in $\mathcal{P}$ for some colouring of its points.*

Proof: We just do one direction, and suppose for a contradiction that $\mathcal{P}_{\mathrm{V}}$ does not embed for any colouring of its points but that $\mathcal{P}_{\Lambda}$ does. Let $x, y < z$, with $x \parallel y$ be a Λ-shape in $\mathcal{P}$. One uses homogeneity and interdensity to find $t, u < x$, $v, w < y$ such that $t \parallel u$, $v \parallel w$ and $F(u) = F(v)$. Since $\mathcal{P}_{\mathrm{V}}$ does not embed, $t, u \parallel y$ and $v, w \parallel x$. By homogeneity there is an automorphism g fixing t and w and taking u to v. Since $\mathcal{P}_{\mathrm{V}}$ does not embed, x and gx are comparable. If $gx \leq x$ then $v = gu < gx \leq x$, contrary to $v \parallel x$. Hence $x < gx$. Since $gu = v < y$, $u < g^{-1}y, x$ so again as $\mathcal{P}_{\mathrm{V}}$ does not embed, x and $g^{-1}y$ are comparable. We cannot have $g^{-1}y \leq x$, as this would give $w = g^{-1}w < g^{-1}y \leq x$, contradiction, and so $x < g^{-1}y$. Hence $x < gx < y$, which again is impossible, completing the proof. $\square$

Lemma 2.4 *Let $\mathcal{P}$ be a countable homogeneous interdensely coloured partial order with colour set C. If $\mathcal{P}_{\Lambda}$ does not embed for any colouring of its points then $\mathcal{P}$ is an antichain, or an antichain of chains isomorphic to $\mathbb{Q}_C$.*

Note that by the previous lemma, we could replace $\mathcal{P}_{\Lambda}$ by $\mathcal{P}_{\mathrm{V}}$ in this.

Proof: Since $\mathcal{P}_{\Lambda}$ does not embed, nor does $\mathcal{P}_{\mathrm{V}}$ by Lemma 2.3, so $\mathcal{P}$ is the disjoint union of its maximal chains. Since the colours are interdense, each chain must be a singleton, or isomorphic to $\mathbb{Q}_C$. $\square$

Lemma 2.5 *Let $\mathcal{P}$ be a countable homogeneous coloured partial order with interdense colours, which does not embed $\mathcal{P}_{|.}$ for any colouring of its points and which is not an antichain. Then $\mathcal{P}$ is a chain of antichains. More specifically, C may be written as the disjoint union of sets $D_{c'}$ for $c' \in C'$, and $\mathcal{P}$ is obtained from $\mathbb{Q}_{C'}$ by replacing each element having colour $c' \in C'$ by an antichain coloured by $D_{c'}$, such that if y, z have the same colour then $A_y \cong A_z$, and if y, z have different colours, then A_y and A_z have disjoint colour sets, and for $p_1 \in A_y$ and $p_2 \in A_z$, $p_1 < p_2 \Leftrightarrow y < z$ in $\mathbb{Q}_{C'}$.*

For a full description of a chain of antichains, it is not enough to give the colour sets $D_{c'}$; we also have to specify for each c', how many elements of the antichain are coloured by each element of $D_{c'}$. The fact that this is the same for each point corresponding to c' is ensured by requiring that they are isomorphic. Once this choice has been made, then this description does uniquely determine

$\mathcal{P}$ up to isomorphism. We call this partition $\{D_{c'} : c' \in C'\}$ of the colour set of a chain of antichains $\mathcal{P}$, its *colour-structure partition.*

Proof: 'Incomparability or equality' is an equivalence relation $\sim$ on $\mathcal{P}$ (since $\mathcal{P}_{|\cdot}$ does not embed), so that $\mathcal{P}$ can be partitioned into maximal antichains. By homogeneity, any two maximal antichains sharing a colour are isomorphic. Let Y be the set of these antichains and for each $y \in Y$ let D_y equal the set of colours in C occurring in that antichain. Let $C' = \{D_y : y \in Y\}$ and we view C' as a set of colours, and colour the points of Y accordingly. We order Y by letting $y_1 < y_2$ if some member of y_1 is below some member of y_2. It follows by homogeneity that this is equivalent to saying that every member of y_1 is below every member of y_2, and $<$ is clearly a partial order of Y. It is linear, since if $y_1, y_2 \in Y$ are incomparable, then they would have to be contained in the same antichain, hence equal. It is then easy to check that $Y \cong \mathbb{Q}_{\mathbb{C}'}$ and that $\mathcal{P}$ is obtained from Y in the way described. The other option, that Y is a singleton, is ruled out by the hypothesis that $\mathcal{P}$ is not an antichain. $\Box$

Lemma 2.6 *Any countable homogeneous interdensely coloured partial order $\mathcal{P}$ containing a Λ-shape, and also points x, y, z, t such that $x < y$, $t < z$, $x \parallel t, z$ and $y \parallel t, z$, is isomorphic to the generic coloured partial order.*

Proof: By interdensity, any colour in the colour set C occurs in (x, y) and also in (t, z). Since $x \parallel t, z$ and $y \parallel t, z$, $\mathcal{P}_{|\cdot}$ embeds for any colouring of its points. But we are assuming that $\mathcal{P}_\Lambda$ embeds for some colouring of its points, so by Lemma 2.2, it embeds for any colouring of its points and hence, by Lemma 2.1, $\mathcal{P}$ is the generic partial order. $\Box$

Lemma 2.7 *Let $\mathcal{P}$ be a countable homogeneous interdensely coloured partial order. If $\mathcal{P}_\Lambda$ (or $\mathcal{P}_V$) and $\mathcal{P}_{|\cdot}$ embed in $\mathcal{P}$ for some colouring of their points then $\mathcal{P}$ is isomorphic to the generic coloured partial order.*

Proof: By the previous lemma, it is sufficient to show that the partial order $Q = \{x_1, x_2, x_3, x_4\}$ where $x_1 < x_2$, $x_3 < x_4$, $x_1 \parallel x_3, x_4$ and $x_2 \parallel x_3, x_4$, also embeds in $\mathcal{P}$ for some colouring of its points. Suppose otherwise for a contradiction. As $\mathcal{P}_{|\cdot}$ embeds, there are x, y, z with $x < y$ and $x, y \parallel z$, and clearly $s \parallel z$ for all $s \in [x, y]$. ¿From the fact that Q does not embed we deduce that any $t > z$ is strictly above every element of $[x, y)$ (and dually, any $t < z$ is strictly below every element of $(x, y]$).

Now, by interdensity, there is $z' \in (x, y)$ such that $F(z') = F(z)$. By homogeneity, there are x' and y' such that $x' < z < y'$ with $F(x') = F(x)$ and $F(y') = F(y)$. By the above remark, $y' > [x, y)$ and $x' < (x, y]$. Also, $z \parallel z'$. By homogeneity, there is an automorphism f of $\mathcal{P}$ that interchanges z and z'. By interdensity, there is $y'' \in (z', y)$ such that $F(y'') = F(y') = F(y)$. Since $z' < y''$, then $z < fy''$ and therefore, by our previous remark, $y'' < fy''$. But also $z = f^{-1}z' < f^{-1}y''$ and therefore, $y'' < f^{-1}y''$, a contradiction. $\Box$

We can now prove the result for the interdense case.

Theorem 2.8 *Any countable homogeneous interdensely coloured partial order $\mathcal{P}$ is isomorphic to one of the following:*

an antichain,

an antichain of chains each isomorphic to $\mathbb{Q}_C$,

a chain of antichains obtained from $Q_{C'}$ by replacing each point by a coloured antichain, so that points coloured the same are replaced by isomorphic antichains, and the colour sets of antichains replacing differently coloured points are disjoint,

the C-coloured generic.

Furthermore, each of the coloured partial orders described is homogeneous.

Proof: If $\mathcal{P}_|$ does not embed, then $\mathcal{P}$ is an antichain, and if $\mathcal{P}_-$ does not embed then it is a chain, so as remarked above is isomorphic to $\mathbb{Q}_C$ (or a singleton). All these cases are homogeneous. Now suppose that both $\mathcal{P}_|$ and $\mathcal{P}_-$ embed (for some colouring of their points). It follows easily that $\mathcal{P}_{|\cdot}$ or $\mathcal{P}_\Lambda$ or both embed for some colouring of their points. Recall also that $\mathcal{P}_\Lambda$ embeds if and only if $\mathcal{P}_V$ embeds.

If $\mathcal{P}_{|\cdot}$ and $\mathcal{P}_\Lambda$ both embed for some colouring of their points then, by Lemma 2.7, $\mathcal{P}$ is the generic coloured partial order.

If $\mathcal{P}_{|\cdot}$ embeds for some colouring of its points but $\mathcal{P}_\Lambda$ does not embed for any colouring of its points, then, by Lemma 2.4, $\mathcal{P}$ is an antichain of chains. If $\mathcal{P}_\Lambda$ embeds for some colouring of its points but $\mathcal{P}_{|\cdot}$ does not embed for any colouring of its points, then, by Lemma 2.5, $\mathcal{P}$ is a chain of antichains. $\Box$

The following lemma will be quite useful in various places.

Lemma 2.9 *Let $\mathcal{P}$ be a homogeneous coloured partial order with colour set C, and let C' be a non-empty subset of C. Then the restriction $\mathcal{P}'$ of $\mathcal{P}$ to C' is also homogeneous. Furthermore, if $\mathcal{P}$ is the Fraïssé limit of an amalgamation class $\mathcal{K}$, then $\mathcal{P}'$ is the Fraïssé limit of the family $\mathcal{K}'$ of restrictions of members of $\mathcal{K}$ to C'.*

Proof: Any finite partial automorphism of $\mathcal{P}'$ is also a partial automorphism of $\mathcal{P}$, so extends to an automorphism of $\mathcal{P}$ whose restriction to C' is the desired automorphism of $\mathcal{P}'$. The final statement follows from Fraïssé's theorem, since the age of $\mathcal{P}'$ is clearly the family of restrictions to C' of members of the age of $\mathcal{P}$. $\Box$

Now moving on to the general case in which interdensity is not assumed, we introduce an equivalence relation $\sim$ whose equivalence classes are the maximal interdensely coloured substructures of a given homogeneous coloured partial order $\mathcal{P}$ with colour set C (except that antichain components will be monochromatic). Let $\approx$ on C and $\sim$ on $\mathcal{P}$ be given by $c_0 \approx c_1$ if there are x, y and z such that $x \leq y \leq z$ and $F(x) = F(z) = c_0$, $F(y) = c_1$, and $x \sim y$ if $F(x) \approx F(y)$. (This is an adaptation of [3] Lemma 3.2 to the partially ordered case.) We refer to the $\sim$-classes as *components* of $\mathcal{P}$.

Lemma 2.10 *If $\mathcal{P} = (P, <, F)$ is a homogeneous coloured partial order, then $\approx$ and $\sim$ are equivalence relations on C and $\mathcal{P}$ respectively. Each component is convex, homogeneous and interdensely coloured (with respect to its colour set), and the colour sets of distinct components are disjoint.*

Proof: The *definition* of $\approx$ is asymmetric, but by homogeneity it easily follows that actually it is symmetric and transitive.

It is immediate that $\sim$ is an equivalence relation on $\mathcal{P}$. Convexity of each component follows from the definition of $\approx$. The fact that each component is itself homogeneous follows from Lemma 2.9.

To show that each component X has interdense colours for its colour set C', let $x < y$ in X, and $c \in C'$. Let $F(x) = c_1$ and $F(y) = c_2$. We find points $u < v < w$ coloured c_1, c, c_2 respectively, and by taking (u, w) to (x, y) using homogeneity we find a point (the image of v) between x and y coloured c as required. The existence of u, v, w follows from $c_1 \approx c_2 \approx c$ and homogeneity (splitting into the cases $c_1 \neq c$, $c_1 = c$). $\square$

Since each component is homogeneous, it is isomorphic to one of the structures described in Theorem 2.8, namely, a chain of antichains, an antichain of chains or the generic. To completely describe the general case, we need to see how the different components can be 'fitted together'. In the following lemmas we outline what the different cases are, and how they can be handled. In treating a general countable coloured homogeneous partial order, it suffices to consider just the cases where there are finitely many components. This is because of the following 'compactness' type result.

Lemma 2.11 *Let $\mathcal{P}$ be a countable coloured partial order which is expressible as a union of a family $\mathcal{F}$ of subsets which are coloured by pairwise disjoint colour sets and are interdensely coloured. Then $\mathcal{P}$ is homogeneous if and only if the union of every finite subfamily of $\mathcal{F}$ is homogeneous.*

We would like to say that $\mathcal{P}$ is homogeneous if and only if every finite union of components is homogeneous, but we cannot because the proof of Lemma 2.10 requires homogeneity, so we have to refer to the components 'indirectly'.

Proof: By Lemma 2.9, if $\mathcal{P}$ is homogeneous, the union of any finite subfamily of $\mathcal{F}$ is homogeneous.

Conversely, we use a back-and-forth argument. Let p be a finite partial automorphism of $\mathcal{P}$ and $x \in \mathcal{P}$. We shall show that p can be extended to a finite partial automorphism q having x in its domain. Since p is finite, there is a union $\mathcal{Q}$ of a finite subfamily of $\mathcal{F}$ such that p is a partial automorphism of $\mathcal{Q}$. Since $\mathcal{Q}$ is homogeneous, we can extend p to an automorphism f of $\mathcal{Q}$, and then $q = p \cup \{(x, f(x))\}$ is the desired finite partial automorphism of $\mathcal{P}$. This shows that any finite partial automorphism can be extended to include any specified element of $\mathcal{P}$ in its domain. A similar argument applies to the range. Since $\mathcal{P}$ is assumed countable, it follows by back and forth that $\mathcal{P}$ is homogeneous. $\square$

¿From now on we may therefore if we wish restrict to just finitely many components. Of course as the number of components increases towards infinity, we shall have $2^{\aleph_0}$ possibilities, but they are essentially 'controlled' by the finite component substructures.

The key steps are working out how two components can be related, and similarly for three components in two possible patterns, a 'V-shape', or a chain of length 3 ('Λ-shapes' are just the dual to V-shapes). Once we know this, the possibilities for the overall structure can be expressed in terms of the components, with natural restrictions on the relations between them.

If P_1 and P_2 are components, let us write $P_1 \prec P_2$ to mean that some element of P_1 lies below some element of P_2. It follows easily from homogeneity, and the fact that the components are convex, that this is a partial order. What will be needed for the classification is to know first of all which kind of component we have at each point, and what the relations are between different components. Let us say that P_1 is *completely below* P_2 if for all $x \in P_1$ and $y \in P_2$, $x < y$, and we write $P_1 < P_2$ (which accords with the earlier notation). This can happen for all possible types of component for P_1 and P_2 (since there is essentially 'no interaction' between the structures of P_1 and P_2). We also say that P_1 is *partially below* P_2 if there are $x_1, x_2 \in P_1$ and $y_1, y_2 \in P_2$ such that $x_1 < y_1$ and $x_2 \parallel y_2$. This can happen in some but not all cases.

Lemma 2.12 *(***The 2-chain lemma***) Let $\mathcal{P}$ be a homogeneous coloured partial order having two components P_1 and P_2 such that $P_1 \prec P_2$. Then one of the following must hold. Furthermore, each of the possibilities listed can occur, and is uniquely determined up to isomorphism, given the appropriate cardinalities and colour sets arising:*

(i) P_1 and P_2 are of any possible type and $P_1 < P_2$,

(ii) P_1 and P_2 are both chains of antichains, and P_1 is partially below P_2 (written $<_{cc}$),

(iii) P_1 and P_2 are both antichains, or antichains of chains, and there is a 1–1 correspondence between the sets of constituent chains such that $x < y$ if and only if x and y lie in chains which correspond; we write $P_1 <_{pm} P_2$ (for 'perfect matching'),

(iv) P_1 and P_2 are both antichains, or antichains of chains, and there is a 1–1 correspondence between the sets of constituent chains such that $x < y$ if and only if x and y lie in chains which do not correspond; we write $P_1 <_{cpm} P_2$ (for 'complement of perfect matching'),

(v) P_1 and P_2 are both antichains, or antichains of chains, and for any x_1, x_2 in the same chain of P_1 and y_1, y_2 in the same chain of P_2, $x_1 < y_1 \Leftrightarrow x_2 < y_2$, and for any finite disjoint unions U and V of chains of P_1 there is $y \in P_2$ such that y is above all members of U and not above any member of V, and for any finite disjoint unions U and V of chains of P_2 there is $x \in P_1$ such that x is below all members of U and not below any member of V; we write $P_1 <_g P_2$ (for 'generic'),

(vi) P_1 is an infinite antichain and P_2 is generic, and P_1 is partially below P_2 (written $<_{ag}$), or the same thing with P_1 and P_2 interchanged (written $<_{ga}$),

(vii) P_1 and P_2 are both generic, and P_1 is partially below P_2 (written $<_{gg}$).

Proof: (ii) We remark that in this case we can give an explicit description of the structure of $P_1 \cup P_2$. Let $\{D_i(c) : c \in C_i'\}$ for $i = 1, 2$ be the colour structure partition for P_i (together with 'multiplicities', which we do not indicate explicitly). We obtain $P_1 \cup P_2$ from $\mathbb{Q}_{C_1' \cup C_2'}$ by replacing each point coloured $c \in C_i'$ by an antichain of points coloured by the members of $D_i(c)$, letting P_i be the set of points arising from C_i', and decreeing that $x < y$ provided that x and y are ordered this way in $\mathbb{Q}_{C_1' \cup C_2'}$, and either they both lie in the same one of P_1, P_2, or else $x \in P_1$ and $y \in P_2$. The fact that this is homogeneous follows quite easily from its bi-definability with the chain of antichains derived from $\mathbb{Q}_{C_1' \cup C_2'}$ and the colour structure partition $\{D_1(c) : c \in C_1'\} \cup \{D_2(c) : c \in C_2'\}$.

(iii), (iv), and (v) arise from the classification of countable homogeneous bipartite graphs, which is mentioned in [14] for instance. These are of five possible kinds, empty (which doesn't arise here, since we assume $P_1 \prec P_2$), complete (that is $P_1 < P_2$), perfect matching, complement of a perfect matching, and 'generic' (where P_1 and P_2 must both have $\aleph_0$ constituent chains). The main additional point is that for homogeneity, if X_1 and X_2 are maximal chains of P_1 and P_2 respectively, then either $X_1 < X_2$ or $X_1 \parallel X_2$, which relies on there being at least two constituent chains for each (except where one or both are antichains, in which case it is clear anyway).

For (vi) and (vii) the relevant structures are most easily constructed by Fraïssé amalgamation. For (vii) the class $\mathcal{K}$ is taken to be the family of all finite partial orders of the form $X_1 \cup X_2$ where the colours of X_1 and X_2 are in $F(P_1)$, $F(P_2)$ respectively, and no point of X_1 is above any point of X_2, and for (vi) it is the same except that X_1 also has to be an antichain. The fact that these are amalgamation classes, establishing existence, is easy. The main labour is to demonstrate uniqueness, which is quite involved, particularly for (vii).

The other thing needed to complete the proof is to show that for all combinations $P_1 \prec P_2$ not otherwise mentioned, we must have $P_1 < P_2$, for instance if P_1 is a chain of antichains, and P_2 is generic. This is not hard, but varying details are needed in each of the cases. $\Box$

The following lemmas are useful when considering more than two components.

Lemma 2.13 *Let $\mathcal{P}$ and $\mathcal{Q}$ be homogeneous coloured partial orders with disjoint colour sets. Suppose that $\mathcal{P} \cup \mathcal{Q}$ is a partial ordering extending each of $\mathcal{P}$ and $\mathcal{Q}$, and such that for each component X of $\mathcal{P}$ and Y of $\mathcal{Q}$, $X < Y$, $Y < X$, or $X \parallel Y$. Then $\mathcal{P} \cup \mathcal{Q}$ is also homogeneous.*

Proof: Let $p : A \to B$ be a finite partial automorphism of $\mathcal{P} \cup \mathcal{Q}$, and let p_1 and p_2 be its restrictions to $\mathcal{P}$ and $\mathcal{Q}$ respectively. As each of $\mathcal{P}$ and $\mathcal{Q}$ is homogeneous, these may be extended to automorphisms f_1 and f_2 of $\mathcal{P}$, $\mathcal{Q}$. Then $f = f_1 \cup f_2$ is an automorphism of $\mathcal{P} \cup \mathcal{Q}$ extending p. This is because f_1 must preserve components X of $\mathcal{P}$ and f_2 must preserve components Y of $\mathcal{Q}$, and by hypothesis, all points of X are related in the same way to the points of Y. □

For countable homogeneous coloured partial orders having three components P_1, P_2, P_3 forming a V-shape with $P_2 \parallel P_3$, since each component can be a chain of antichains, an antichain of chains, or generic, there appear to be 27 cases to consider. In view of the evident symmetry between P_2 and P_3, this at once reduces to 18. Several more are eliminated by Lemma 2.13, since by the 2-chain lemma 2.12 the majority of 2-component relations must be complete. In all cases, the next lemma gives the uniqueness of the V-shape structures given the 2-component relations.

Lemma 2.14 *Let $\mathcal{P}$ and $\mathcal{Q}$ be countable homogeneous coloured partial orders having three components P_1, P_2 and P_3 forming a* V*-shape so that $P_1 \prec P_2, P_3$ and $P_2 \parallel P_3$, and similarly for $\mathcal{Q}$. If the union of any two components of $\mathcal{P}$ is isomorphic to the union of the corresponding two components of $\mathcal{Q}$, then $\mathcal{P}$ and $\mathcal{Q}$ are isomorphic.*

Proof: Since $\mathcal{P}$ and $\mathcal{Q}$ are countable and homogeneous, it suffices to show that they have the same age. Let $X_1 \cup X_2 \cup X_3$ be a finite substructure of $\mathcal{P}$ with $X_i \subseteq P_i$. Then by hypothesis, $X_1 \cup X_2$ embeds in $Q_1 \cup Q_2$ and $X_1 \cup X_3$ embeds in $Q_1 \cup Q_3$. By homogeneity of $\mathcal{Q}$ we may assume that the two embeddings agree on X_1. Since $P_2 \parallel P_3$ and $Q_2 \parallel Q_3$, the union of the two embeddings is an embedding of $X_1 \cup X_2 \cup X_3$ into $\mathcal{Q}$. Thus the age of $\mathcal{P}$ is contained in the age of $\mathcal{Q}$. The same argument applies in reverse, so they have the same ages, as required. □

Lemma 2.15 *(***The** V**-shape lemma***) Let $\mathcal{P}$ be a countable homogeneous coloured partial order having three components P_1, P_2 and P_3, where $P_1 \prec P_2, P_3$ and $P_2 \parallel P_3$. Then one of the following must hold. Furthermore, each of the possibilities listed can occur, and is uniquely determined up to isomorphism, given the appropriate cardinalities and colour sets arising:*

(i) $P_1 < P_2$, and P_1, P_3 follow one of the cases listed in the 2-chain lemma 2.12, and similarly with P_2 and P_3 interchanged,

(ii) P_1, P_2, and P_3 are all chains of antichains, and $P_1 <_{cc} P_2$ and $P_1 <_{cc} P_3$.

(iii) P_1, P_2 and P_3 are all antichains of chains and $P_1 <_g P_2$ and $P_1 <_g P_3$,

(iv) P_1 is an antichain, P_2 is an antichain of chains and P_3 is a generic, and $P_1 <_g P_3$ and $P_1 <_{ag} P_3$ (or the same with P_2 and P_3 interchanged),

(v) P_1 is an antichain, P_2 and P_3 are generics, and $P_1 <_{ag} P_2$ and $P_1 <_{ag} P_3$,

(vi) P_1, P_2 and P_3 are generics and $P_1 <_{gg} P_2$ and $P_1 <_{gg} P_3$,

(vii) P_1 and P_2 are generics, P_3 is an antichain and $P_1 <_{gg} P_2$ and $P_1 <_{ga} P_3$ (or the same with P_2 and P_3 interchanged),

(viii) P_1 is generic, P_2 and P_3 are antichains, and $P_1 <_{ga} P_2$ and $P_1 <_{ga} P_3$.

The next remark is straightforward but useful, and in the 3-chain lemma enables us to discount the cases where either lower or upper link is complete.

Lemma 2.16 *Let $\mathcal{P}$ be a countable homogeneous coloured partial order having three components P_1, P_2 and P_3 such that $P_1 \prec P_2 \prec P_3$. If $P_1 < P_2$ or $P_2 < P_3$ then $P_1 < P_3$. Furthermore, if P_1 and $P_2 \cup P_3$ (similarly for $P_1 \cup P_2$ and P_3) are known to be countable homogeneous coloured partial orders with P_1 interdense, P_2 and P_3 components of $P_2 \cup P_3$ with $P_1 < P_2 \prec P_3$, (or $P_1 \prec P_2 < P_3$ in the other case), then $P_1 \cup P_2 \cup P_3$ is homogeneous and uniquely determined up to isomorphism.*

Lemma 2.17 *(***The 3-chain lemma***): Let $\mathcal{P}$ be a homogeneous coloured partial order consisting of three components P_1, P_2 and P_3, such that $P_1 \prec P_2 \prec P_3$. Then the relation between P_1 and P_3 is the transitive closure of the other two relations, and the relation between the three components is one of the following:*

(i) $P_1 < P_2$ and $P_1 < P_3$ and the relation between P_2 and P_3 is any of the ones allowed by the 2-chain lemma 2.12,

(ii) $P_1 < P_3$ and $P_2 < P_3$ and the relation between P_1 and P_2 is any of the ones allowed by the 2-chain lemma 2.12,

(iii) P_1, P_2 and P_3 are all chains of antichains, $P_1 <_{cc} P_2 <_{cc} P_3$ and $P_1 <_{cc} P_3$,

(iv) P_1, P_2 and P_3 are all antichains of chains, $P_1 <_{pm} P_2$ and the relation between P_2 and P_3 is one of $<_{pm}$, $<_{cpm}$, $<_g$, in which case the relation between P_1 and P_3 is the same as that between P_2 and P_3,

(v) P_1, P_2 and P_3 are all antichains of chains, $P_2 <_{pm} P_3$ and the relation between P_1 and P_2 is one of $<_{pm}$, $<_{cpm}$, $<_g$, which is the same as the relation between P_1 and P_3,

(vi) P_1, P_2 and P_3 are all antichains of chains, $P_1 <_g P_2$ and $P_2 <_g P_3$, and the relation between P_1 and P_3 is $<_g$, $<_{cpm}$ or $<$,

(vii) P_1, P_2 and P_3 are all generics, $P_1 <_{gg} P_2 <_{gg} P_3$, and $P_1 <_{gg} P_3$ or $P_1 < P_3$,

(viii) P_1 and P_3 are antichains, P_2 is a generic, $P_1 <_{ag} P_2 <_{ga} P_3$, and the relation between P_1 and P_3 is $<_g$, $<_{cpm}$ or $<$,

(ix) P_1 and P_2 are antichains, P_3 is a generic, $P_1 <_{pm} P_2$ or $P_1 <_g P_2$, and $P_1, P_2 <_{ag} P_3$,

(x) P_1 is a generic, P_2 and P_3 are antichains, $P_1 <_{ga} P_2, P_3$, and $P_2 <_{pm} P_3$ or $P_2 <_g P_3$,

(xi) P_1 is an antichain of chains, P_2 is an antichain, P_3 is a generic, $P_1 <_g P_2 <_{ag} P_3$, and $P_1 < P_3$,

(xii) P_1 is a generic, P_2 is an antichain, P_3 is an antichain of chains, $P_1 <_{ga} P_2 <_g P_3$, and $P_1 < P_3$,

(xiii) P_1 and P_2 are generics, P_3 is an antichain, $P_1 <_{gg} P_2 <_{ga} P_3$, and $P_1 < P_3$ or $P_1 <_{ga} P_3$,

(xiv) P_2 and P_3 are generics, P_1 is an antichain, $P_1 <_{ag} P_2 <_{gg} P_3$, and $P_1 < P_3$ or $P_1 <_{ag} P_3$.

(xv) P_1 and P_3 are generics, P_2 is an antichain, $P_1 <_{ga} P_2 <_{ag} P_3$, and $P_1 < P_3$ or $P_1 <_{gg} P_3$.

Guided by these preliminary lemmas, we can now describe how the final classification of all countable homogeneous coloured partial orders with finitely many components is achieved.

Skeletons

In this case the classifiers are referred to as 'skeletons'. We start by saying what the skeleton of a homogeneous coloured partial order $\mathcal{P} = (\mathcal{P}, <, \mathcal{F})$ is. What then will remain is to characterize abstractly which structures can arise as skeletons, and to show that $\mathcal{P}$ is uniquely determined by its skeleton.

Definition 2.18 *The* skeleton of $\mathcal{P}$ *is the coloured partial order $\mathcal{Q}$ with labels on its elements and pairs of comparable elements given by:*

the domain of $\mathcal{Q}$ is the set of components of $\mathcal{P}$, partially ordered by $\prec$,

each element X of $\mathcal{Q}$ is labelled CA, A, AC, or Ge according as it is a (non-trivial) chain of antichains, an antichain, an antichain of at least two non-trivial chains, or generic, by the set of colours occurring in X, by its colour structure partition if it is a non-trivial chain of antichains, and by the number of constituent chains if X is an antichain or an antichain of chains,

pairs (X, Y) of elements of $\mathcal{Q}$ such that $X \prec Y$ are labelled C if $X < Y$, PM, CPM, or G if both X and Y are antichains or antichains of chains, and the relation between them is a perfect matching, its complement, or generic respectively, CC if both X and Y are chains of antichains, AG if X is an antichain and Y generic, GA the dual of this, and GG if both are generic, and in all of the last four cases, X is partially below Y.

It is clear then from the previous lemmas that any countable homogeneous coloured partial order has a skeleton which obeys all these conditions.

We now turn to the 'converse' idea. We have defined skeleton *of* $\mathcal{P}$, and we can now define 'skeleton' abstractly. Although by Lemma 2.11 we could restrict to finite skeletons, in fact the definition is given for the countable case.

Definition 2.19 *A* skeleton *is a finite or countable partial order $\mathcal{Q}$, with labels on the vertices and pairs of comparable vertices fulfilling the following conditions:*

(i) each vertex x is labelled by CA, A, AC, or Ge, and by a set C_x (of colours), so that $x \neq y \Rightarrow C_x \cap C_y = \emptyset$,

(ii) if x is labelled CA then it is also labelled by a colour structure partition $\{D_{c'} : c' \in C'\}$ where $\bigcup C' = C_x$,

(iii) if x is labelled A or AC, then it also has a label $N(x) \in \{1, 2, \ldots, \aleph_0\}$, and if A, then $|C_x| = 1$ (antichain components are monochromatic), where if the label is AC, then $N(x) \neq 1$,

(iv) (conditions required by the 2-chain lemma) pairs of comparable elements $x_1 < x_2$ are labelled by one of C, CC, PM, CPM, G, AG, GA and GG (and we write $x_1 X x_2$ in place of $(x_1, x_2) \in X$ for such labels), where if the label is CC, then x_1 and x_2 are both labelled CA, if PM, CPM, G then they are labelled A or AC, if AG then they are labelled A and Ge, if GA then they are labelled Ge and A, and if GG then they are both labelled Ge,

(v) (cardinality restrictions) if $x_1 PM x_2$ or $x_1 CPM x_2$ then $N(x_1) = N(x_2)$, if $x_1 G x_2$ then $N(x_1) = N(x_2) = \aleph_0$, if $x_1 AG x_2$ then $N(x_1) = \aleph_0$, and if $x_1 GA x_2$ then $N(x_2) = \aleph_0$,

(vi) (conditions arising from the V-shape lemma) if $x_1 < x_2, x_3$ and $x_2 \parallel x_3$, then one of the following must apply: $x_1 C x_2$; $x_1 C x_3$; $x_1 CC x_2$ and $x_1 CC x_3$; $x_1 G x_2$ and $x_1 G x_3$; $x_1 G x_2$ and $x_1 AG x_3$; $x_1 AG x_2$ and $x_1 G x_3$; $x_1 AG x_2$ and $x_1 AG x_3$; $x_1 GG x_2$ and $x_1 GG x_3$; $x_1 GG x_2$ and $x_1 GA x_3$; $x_1 GA x_2$ and $x_1 GG x_3$; $x_1 GA x_2$ and $x_1 GA x_3$;

(vii) (conditions arising from the Λ-shape lemma) the duals of (vi)

(viii) (conditions arising from the 3-chain lemma) if $x_1 < x_2 < x_3$, then one of the following must apply: $x_1 C x_2$ and $x_1 C x_3$; $x_1 C x_3$ and $x_2 C x_3$; $x_1 CC x_2 CC x_3$ and $x_1 CC x_3$; $x_1 PM x_2 R x_3$ and $x_1 R x_3$ where R is PM, CPM, or G; $x_1 R x_2 PM x_3$ and $x_1 R x_3$ where R is PM, CPM, or G; $x_1 G x_2 G x_3$ and $x_1 R x_3$ where R is G, CPM, or C; $x_1 GG x_2 GG x_3$ and $x_1 GG x_3$ or $x_1 C x_3$; $x_1 AG x_2 GA x_3$ and $x_1 R x_3$ where R is G, CPM, or G; $x_1 PM x_2$ and $x_1, x_2 AG x_3$; $x_1 G x_2$ and $x_1, x_2 AG x_3$; $x_1 GA x_2 PM x_3$ and $x_1 GA x_3$; $x_1 GA x_2, x_3$ and $x_2 G x_3$; $x_1 G x_2 AG x_3$ and $x_1 C x_3$; $x_1 GA x_2 G x_3$ and $x_1 C x_3$; $x_1 GG x_2 GA x_3$ and $x_1 GA x_3$ or $x_1 C x_3$; $x_1 AG x_2 GG x_3$ and $x_1 AG x_3$ or $x_1 C x_3$; $x_1 GA x_2 AG x_3$ and $x_1 GG x_3$ or $x_1 C x_3$.

Theorem 2.20 *For any skeleton $\mathcal{Q}$ there is a countable homogeneous coloured partial order $\mathcal{P}$ having $\mathcal{Q}$ as its skeleton.*

Proof: The structure $\mathcal{P}$ is built by Fraïssé amalgamation from a class $\mathcal{K}$ of finite structures. The definition of $\mathcal{K}$ is obtained by 'interpreting' the instructions enshrined in $\mathcal{Q}$. The ideas are clear, though there are some annoying technical problems caused by the requirement to ensure that $\mathcal{K}$ is closed under formation of substructures. This means for instance, that with regard to perfect matchings, we have to allow substructures which may not actually *be* perfect matchings, but which can still be extended to them. We omit the details. $\square$

3 Linear orders and and coloured linear orders

In the remainder of the paper, I shall consider partially homogeneous structures, beginning with linear orders. The particular interest here is that this provides a natural class of structures, namely the countable 1-transitive linear orders, with precisely $\aleph_1$ members (whether or not the continuum hypothesis is true). This might be thought to have some bearing on Vaught's conjecture, but this is certainly not the case as it stands, since the notion of '1-transitivity' is not first order axiomatizable. Conceivably, a clever coding procedure could be used to derive an axiomatizable class from it, but at present, we regard the class of interest just in its own right. First we explain why the only interesting value of k to take here in k-homogeneous is 1.

Lemma 3.1 *: Any 2-homogeneous chain is homogeneous (hence k-homogeneous for each k). Similarly for coloured chains.*

Proof: Given an isomorphism from $a_1 < a_2 < \ldots < a_k$ to $b_1 < b_2 < \ldots < b_k$, by 2-homogeneity there are automorphisms taking $(-\infty, a_1], [a_1, a_2], \ldots, [a_{k-1}, a_k]$, $[a_k, \infty)$ to $(-\infty, b_1], [b_1, b_2], \ldots, [b_{k-1}, b_k], [b_k, \infty)$ respectively, and we can patch them to give a single automorphism taking a_i to b_i for each i. □

We remark that we could just have said that any countable non-trivial 2-homogeneous chain is isomorphic to $\mathbb{Q}$. However, this would not carry across quite so simply to the coloured case, where there are slightly more complicated examples which can arise, namely finite or countable unions of convex subsets, each of which is isomorphic to some $\mathbb{Q}_n$ or a singleton, where the pieces are coloured by pairwise disjoint colour sets.

The most obvious examples of countable 1-transitive chains are $(\mathbb{Z}, \leq)$ and $(\mathbb{Q}, \leq)$. The key difference between these two is therefore that $\mathbb{Z}$ is only 1-transitive, whereas $\mathbb{Q}$ is (fully) homogeneous. We now give Morel's result, which lists all the countable 1-transitive linear orders. The lexicographic product of chains $(X, \leq)$ and $(Y, \leq)$, 'X copies of Y', written $X.Y$, is the cartesian product ordered by $(x_1, y_1) < (x_2, y_2)$ if $x_1 < x_2$ or $(x_1 = x_2 \wedge y_1 < y_2)$, and we then define $\mathbb{Z}^\alpha$, where α is an ordinal, by transfinite induction thus:

$\mathbb{Z}^0 = \{0\}$, $\mathbb{Z}^{\alpha+1} = \mathbb{Z}.\mathbb{Z}^\alpha$, and, viewing $\mathbb{Z}^\alpha$ as a subset of $\mathbb{Z}^{\alpha+1}$ via $\{(0, z) : z \in \mathbb{Z}^\alpha\}$, we let $\mathbb{Z}^\lambda = \bigcup_{\alpha<\lambda} \mathbb{Z}^\alpha$, if λ is a limit ordinal.

Morel's classification

Theorem 3.2 *: (i) If $(X, \leq)$ and $(Y, \leq)$ are 1-transitive linear orders, then so is $(X.Y, \leq)$,*

(ii) if $\{X_\alpha : \alpha \in A\}$ are 1-transitive chains and $\leq$ is a linear ordering of A such that $\alpha_1 \leq \alpha_2 \Rightarrow X_{\alpha_1}$ is a convex subset of X_{α_2}, then $\bigcup_{\alpha \in A} X_\alpha$ is a 1-transitive chain,

(iii) for every ordinal α, $\mathbb{Z}^\alpha$ and $\mathbb{Q}.\mathbb{Z}^\alpha$ are 1-transitive chains,

(iv) every countable 1-transitive chain is isomorphic to $\mathbb{Z}^\alpha$ or $\mathbb{Q}.\mathbb{Z}^\alpha$ for some countable ordinal α.

Proof: (i) In $X.Y$ to pass from any point to another, we first use the 1-transitivity of X to move it to the correct copy of Y. Then we use the 1-transitivity of Y to move it to the precisely correct point in this copy of Y (fixing all the other copies).

(ii) If $x, y \in \bigcup_{\alpha \in A} X_\alpha$ are given, there is $\alpha \in A$ such that $x, y \in X_\alpha$. Now move x to y in X_α, fixing all points outside X_α (possible by convexity).

(iii) This follows from (i) and (ii), since $\mathbb{Z}^\alpha$ is convex in $\mathbb{Z}^\beta$ whenever $\alpha \leq \beta$.

(iv) (Sketch) If $(X, \leq)$ is densely ordered, then 1-transitivity implies that it has no endpoints, therefore $(X, \leq) \cong (\mathbb{Q}, \leq)$. If it is not dense (and $|X| > 1$), then each point lies in a copy of $\mathbb{Z}$. Amalgamate them and repeat, transfinitely if necessary. Eventually we reach 1 or $\mathbb{Q}$, and these give the two cases $\mathbb{Z}^\alpha$ and $\mathbb{Q}.\mathbb{Z}^\alpha$. $\square$

For uncountable chains, things are much more complicated, and we do not investigate them here. See [13] for instance.

We now move on to consider the case of coloured chains, as given in [3, 4]. Now for (monochromatic) linear orders, the main building blocks were $\mathbb{Z}$ and $\mathbb{Q}$. In the coloured case, we also need to consider coloured versions $\mathbb{Q}_C$ of the rationals, which were described in the introduction. There is a subdivision into two cases, those in which the colour set is finite, and those in which it is (countably) infinite. The former, treated in [3], is relatively straightforward, amounting to a modification of Morel's list. In the latter some much more complicated ideas are involved. This is mirrored in the fact that for the former there are just $\aleph_1$ examples, in the latter $2^{\aleph_0}$, the $\aleph_1$ arising just because of the part played by Morel's structures.

To analyze a general countable 1-transitive coloured linear order coloured by a set C, we need the following constructions. The first is derived from $\mathbb{Q}_n$ where $2 \leq n \leq \aleph_0$ ($\aleph_0$ only needed when C is infinite). If Y_i for $i < n$ are coloured linear orders, then $\mathbb{Q}_n(Y_0, \ldots, Y_{n-1})$ denotes the result of substituting a copy of Y_i for each point of $\mathbb{Q}_n$ coloured i, called a $\mathbb{Q}_n$*-combination*, or in [20] 'shuffle'. (Here, if n is infinite, we just mean $\mathbb{Q}_n(Y_0, \ldots)$, and in future we shall take this kind of minor modification in the notation for granted.) The next is *concatenation*, in general over a countable linear ordering $(\gamma, <)$ (though in practice it suffices to concatenate two sets at a time). If $\{Y_x : x \in \gamma\}$ are coloured linear orders, then the concatenation is obtained from the linear ordering γ by replacing each x by Y_x, ensuring that these sets are disjoint, ordering each Y_x as before, and inducing the ordering between different Y_xs from that of γ. The third is that if Z is a (monochromatic) linear order, and Y is a coloured linear order, then $Z.Y$ is the *lexicographic product*, with colours given by the second co-ordinates. There are two other constructions which arise when C is infinite, one is lim, which corresponds to taking the union of an infinite nested family, the other, which is more obscure, is select$_n$, again for $2 \leq n \leq \aleph_0$, which makes a 'selection' among possible points.

Coding trees

To keep track of what is happening, in [2] the notion of 'coding tree' was introduced. For the finite colour set case this is not really necessary, and indeed it is not mentioned at all in Rosenstein's related work on $\aleph_0$-categorical structures, but for C infinite some device like this seems essential. It is similar to the idea of a 'skeleton' used in section 2. For technical reasons, we require coding trees to be Dedekind–MacNeille complete (a notion explained in the next section). For finite colour sets, the coding trees will be be finite, hence automatically Dedekind–MacNeille complete. For infinite coding trees, Dedekind–MacNeille completeness corresponds to the behaviour of certain subsets of the colour set which we refer to as 'clumps' (meaning that they colour a convex subset of the linear order). The definition then is that a *coding tree* is a Dedekind–MacNeille labelled tree with a root (at the top, since coding tees grow 'downwards', see the next paragraph), at most $\aleph_0$ leaves, every vertex has to be a leaf or above a leaf, all cones at ramification points have greatest elements (see the next section for the definition of these notions), at most countably many vertices have only one child, and the labels are among $\mathbb{Q}_n$, a countable linear order $(\gamma, <)$, $select_n$ for some n, $1 < n \leq \aleph_0$, a countable 1-transitive linear order Z, lim, 1; and these must obey various conditions expressing the intended interpretation, for instance vertices labelled $\mathbb{Q}_n$ or $select_n$ have n children, and a bijection between n and these children is specified, vertices labelled γ have children in bijective correspondence with γ, vertices labelled lim have no children but just one cone below, vertices labelled 1 are leaves and they also have (distinct) colours attached, together with some other conditions.

We remark that 'trees' will be used in more than one sense in this paper. To help distinguish them from the trees whose classification we discuss in the next section, and which have no direct connection with coding trees, we envisage coding trees as growing downwards, and the trees being classified as growing upwards.

Theorem 3.3 *: (i) Any coding tree encodes some countable 1-transitive coloured linear order.*

(ii) The countable 1-transitive coloured linear order encoded by any coding tree is unique up to isomorphism.

(iii) Any countable 1-transitive coloured linear order is encoded by some coding tree.

Proof: We remark that it isn't even clear in general what is meant by saying that a coding tree 'encodes' a coloured linear order. For finite coding trees there is no problem, since they are so explicit, and merely a 'book-keeping device'. For instance, in Figure 1 I show the coding tree for the coloured order $\mathbb{Q}_3(\mathbb{Z}^2.(\mathbb{Q}(green)^\wedge\mathbb{Q}(yellow)), \mathbb{Q}_2(red, blue), \mathbb{Z}.\mathbb{Q}(white))$ where $^\wedge$ stands for

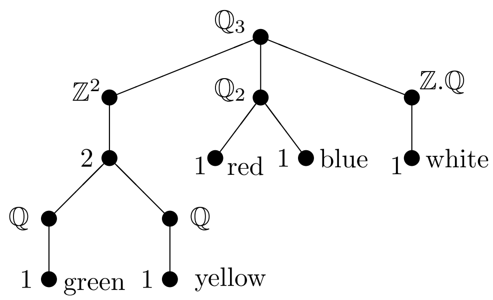

Figure 1: example of a coding tree

'concatenation'. For infinite coding trees, which may not be well-founded or conversely well-founded, it is not so clear. What we do is to introduce an auxiliary notion, called 'expanded coding tree', which is the same as the coding tree, but with each non-leaf vertex having below it the whole tree represented at that point, not just the code. Thus for instance, at a vertex labelled $\mathbb{Q}_n$, in the coding tree there are just n children, but in the corresponding expanded coding tree, there are infinitely many, indexed by the members of $\mathbb{Q}_n$, and each child in the coding tree corresponds to all the children in the expanded coding tree having its colour. For concatenation and lim there is no difference between coding tree and the expanded version. For lexicographic product, in the expanded coding tree, instead of 1 child, we have children indexed by the label Z at that point. The most subtle point is the handling of $select_n$. Here in the coding tree there are n children of the vertex x (indicating the possibilities for what comes below), but in the expanded coding tree there is just one (the choice actually made). So that all possibilities are chosen somewhere, though in different places, we also have to say that in the expanded coding tree, for every vertex y above x, all possible choices are made at some point below y.

This outlines the definition of 'expanded coding tree'. There is then a fairly easy notion of this being *associated with* a coding tree, and we say that a coding tree *encodes* a coloured linear order if it is isomorphic to the coloured linear ordering of the leaves of some expanded coding tree associated with it.

The proofs of the three parts are then accomplished in outline as follows. For (i) we derive an expanded coding tree associated with the given coding tree by taking suitable functions on the branches having 'finite support' (to avoid increasing the cardinality too much, and even to be able to verify the most basic properties). Then (ii) may be proved using a careful application of back-and-forth. Finally for (iii) one starts with a given countable coloured linear ordering, and seeks to recognize inside it the ingredients constituting an expanded coding tree, which can then be 'collapsed' to form a coding tree. As mentioned before, a key technique here is look at 'clumps', which are sets of colours colouring convex subsets, and the coding tree is built up from a chosen maximal tree of clumps. □

We also now mention the related class of $\aleph_0$-categorical linear, or coloured, linear orders. These were treated by Rosenstein [20], and are rather like the finite colour set 1-transitive coloured linear orders with modifications. The differences are these. In the first place, there is no longer any requirement of 1-transitivity, so that in one sense there are *more* examples. This is replaced instead by deducing from the Ryll–Nardzewski Theorem that there are only finitely many orbits under the action of the automorphism group. On the other hand, we no longer have any of the examples in which $\mathbb{Z}$ appears, since this would give infinitely many 2-types (pairs at greater and greater distances), contrary to the Ryll–Nardzewski Theorem, so in another sense we also have *fewer* examples. This means that when taking lexicographic products, we can only take them over $\mathbb{Q}$ and not over any non-trivial $\mathbb{Z}^\alpha$, and to ease notation, we may regard this as a $\mathbb{Q}_n$-combination with $n = 1$. The conclusion is thus that a countable linear ordering is $\aleph_0$-categorical if and only if it can be built up in finitely many steps from singletons by taking concatenations and $\mathbb{Q}_n$-combinations where $1 \leq n < \aleph_0$. The proof of this result can be carried out by methods which are quite similar to that of the classification of countable 1-transitive coloured linear orders with a finite colour set. Coding trees may be used as a method for describing the stages in the construction.

4 Trees

Trees form an important example intermediate between chains and cycle-free partial orders. In fact, these last were proposed as a generalization of trees by Rubin [21]. In his thesis [8], Droste initiated the study of sufficiently transitive countable trees. The idea is that trees can grow upwards, or downwards—the theory is identical; here we suppose that they grow upwards. A *tree* then is a partially ordered set $(T, \leq)$ in which every two elements have a common lower bound, and for every $x \in T$, $\{y \in T : y \leq x\}$ is linearly ordered. We generally assume that T is not itself linearly ordered, in which case it is called *proper*, thus there are at least two incomparable elements. Trees are also called 'semilinear orders' (that is, linear in one direction).

A key feature required to give a description of a tree is how it 'ramifies'. A *ramification point* is a point of the Dedekind–MacNeille completion which is the infimum of two incomparable points. Merely requiring 1-transitivity of a tree, the weakest sensible hypothesis, guarantees that either all ramification points lie in T, and we then say that it has *positive type*, or none do, in which case it has *negative type*. In order to make sense of this, and because it is also used even to give the *definition* of a cycle-free partial order, we now introduce the Dedekind–MacNeille completion X^D of an arbitrary partially ordered set $(X, <)$.

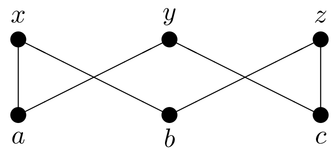

Figure 2: a 6-crown

Intuitively, this is obtained by adjoining all the Dedekind cuts of maximal chains, and the upper and lower bounds which 'ought' to be present. To make this more precise, we require the following definitions.

Definition 4.1 *$I \subseteq X$ is an* ideal *if $x \leq y \in I \Rightarrow x \in I$. For $A \subseteq X$, let $A^+ = \{x \in X : (\forall a \in A) a \leq x\}$ and $A^- = \{x \in X : (\forall a \in A) x \leq a\}$.*

Then A^- is always an ideal. Ideals of the form $\{a\}^- = \{x \in X : x \leq a\}$ are called principal. *We say that $(X, \leq)$ is* Dedekind–MacNeille complete *(D–M complete) if every non-empty subset I of X, bounded above, and such that $I = I^{+-}$, is a principal ideal.*

Examples: In the 6-crown shown in Figure 2, the only suitable sets I are $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b, x\}$, $\{a, c, y\}$, and $\{b, c, z\}$. Here $\{a, b, x\} = \{x\}^-$, $\{a, c, y\} = \{y\}^-$, and $\{b, c, z\} = \{z\}^-$, so these ideals are all principal, and this partially ordered set (clearly not a tree) is D-M-complete.

However the partial order in Figure 3(a) is not D-M-complete, since $I = \{a, b\}$ satisfies $I^{+-} = I$: $I^+ = \{x, y\}$ and $I^{+-} = \{a, b\}$, and I is not principal. The Dedekind–MacNeille completion here is shown in Figure 3(b).

Generally, for any partial order $(X, \leq)$ we can form a minimal D-M-complete partially ordered set $(X^D, \leq)$ extending X, called its *D-M-completion.* In fact we can take the elements of X^D to be the non-empty bounded above ideals I satisfying $I^{+-} = I$, partially ordered by $\subseteq$, and with X identified with a subset of X^D via principal ideals.

Lemma 4.2 *: If $(X, \leq)$ is D-M-complete, then*

(i) every maximal chain C of X is Dedekind-complete (in the usual sense),

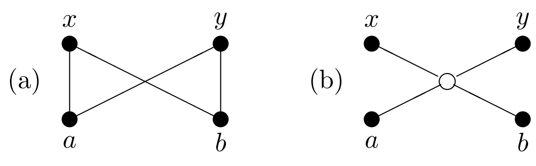

Figure 3: (a) a '4-crown', which is not D-M-complete, and (b) its completion

(ii) if $x, y \in X$ have a common upper bound (common lower bound) then they have a least upper bound $\sup(x, y)$ *(greatest lower bound* $\inf(x, y)$ *respectively).*

Proof: (i) Let $A \subseteq C$ be non-empty bounded above, and let $I = A^{+-}$. Then $A \subseteq I$ and I is bounded above and satisfies $I^{+-} = I$, thus $I = \{a\}^-$ for some a. This is the least upper bound of A in C.

(ii) If $I = \{x, y\}^{+-}$, then $I = \{a\}^-$ for some a, and $a = \sup(x, y)$. If x, y have a common lower bound, let $I = \{x, y\}^{-+-}$. Then if $I = \{b\}^-$, $b = \inf(x, y)$. $\square$

If $c = \sup(a, b)$, where a and b are incomparable, then c is called a *lower ramification point*. Similarly, if $c = \inf(a, b)$ where a and b are incomparable, then c is called an *upper ramification point*.

In a tree we have upper ramification points, but not lower ones. In general we may have both.

Definition 4.3 *For every partially ordered set $(X, \leq)$ with Dedekind–MacNeille completion X^D, let* $\uparrow\mathrm{Ram}(X)$*,* $\downarrow\mathrm{Ram}(X)$ *be the sets of upper and lower ramification points of X^D, and* $\mathrm{Ram}(X) = \uparrow\mathrm{Ram}(X) \cup \downarrow\mathrm{Ram}(X)$*. We also write X^+ for $X \cup \mathrm{Ram}(X)$ (not now with the same meaning as before).*

We define $\sim$ on $\{y \in T : x < y\}$, where x is a fixed element of T^+, and T is a tree, as follows: let $y_1 \sim y_2$ if $\exists y(x < y \leq y_1, y_2)$. This is clearly reflexive and symmetric. and to see that it is transitive, let $y_1 \sim y_2 \sim y_3$ and $x < y \leq y_1, y_2$ and $x < y' \leq y_2, y_3$. Since $y, y' \leq y_2$ and T is a tree, $y \leq y'$ or $y' \leq y$, suppose the former. Then $x < y \leq y_1, y_3$, so $y_1 \sim y_3$.

The $\sim$-classes are called the (upper) cones *at x. Thus $x \in \mathrm{Ram}(T) \Leftrightarrow$ there are at least 2 cones, and the number of cones is called the* ramification order *of x, r.o.(x).*

In set theory, 'tree' is usually used to mean that for every $x \in T$, $\{y \in T : y \leq x\}$ is *well*-ordered rather than just linearly ordered. In this case there is a unique minimal element, called the *root*, and we have a notion of 'level' in the tree, enumerated by ordinals. To distinguish these from 'trees' in our sense, we call them *well-founded* trees. They will play a (small) part in what follows. (There is another notion of 'well-founded tree' in descriptive set theory, see [19], namely trees whose dual is well-founded.)

The classification of proper countable k-transitive trees, $k \geq 2$.

Lemma 4.4 *: If $(T, \leq)$ is proper countable k-transitive tree where $k \geq 2$, then, either all its maximal chains have length 2, and T has the form shown in Figure 4, or $k \leq 3$, and all its maximal chains are densely ordered without endpoints.*

Proof: First suppose that there is a chain $x_0 < x_1 < x_2 < \ldots < x_k$ of length $k + 1$. Then p defined by $p(x_i) = x_{i+1}$ for $0 \leq i < k$ is an isomorphism of k-element substructures, so by k-transitivity there is an automorphism f which

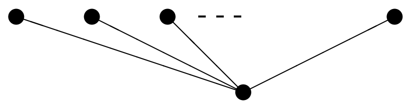

Figure 4: a 2-transitive tree with just 2 levels

takes $\{x_0, \ldots, x_{k-1}\}$ to $\{x_1, \ldots, x_k\}$, and which must take x_i to x_{i+1}. Then $\{f^n x_0 : n \in \mathbb{Z}\}$ is an infinite chain. Let C be a maximal chain containing this set. Repeating the argument (with different choices of the x_i) we see that C has no endpoints. To see that C is densely ordered, let $x_0 < x_1$ be any two points of C. As C has no endpoints, we can choose $x_k > x_{k-1} > \ldots > x_2 > x_1$. This time let $p(x_0) = x_0, p(x_i) = x_{i-1}$ for $2 \leq i \leq k$, and let f be an automorphism which extends p. Since $x_0 < x_1 < x_2$, $f(x_0) < f(x_1) < f(x_2)$, so $x_0 < f(x_1) < x_1$.

Now let C' be any other maximal chain. Since T is a tree, $C \cap C' \neq \emptyset$. Let $x \in C \cap C'$. Then $\{y \in C : y \leq x\}$ is infinite. But this is contained in C', so C' is also infinite, Therefore, by the preceding argument, C' is densely ordered without endpoints.

As T is proper, there are incomparable x and y. Since T is a tree, there is $z_k \leq x, y$. Since all the maximal chains have no endpoints, there are $z_1 < z_2 < \ldots < z_k$. Consider p given by $p(z_i) = z_{i+1}$, for $i < k$, $p(z_k) = x$ (or y), and extending to an automorphism, we see that there are incomparable $t, u > x$ and incomparable $v, w > y$, and repeating the argument, incomparable $r, s > t$. This is illustrated in Figure 5. Now $\{r, s, u, y\}$ and $\{t, u, v, w\}$ are 4-element antichains, so $\{r, s, u, y, z_1, \ldots, z_{k-4}\} \cong \{t, u, v, w, z_1, \ldots, z_{k-4}\}$. However, no automorphism can take $\{r, s, u, y\}$ to $\{t, u, v, w\}$, since the infima of each two of r, s, u, y are comparable, but of t, u and v, w are not. We deduce that $k \leq 3$.

If all chains of T have length $\leq k$, then T is a well-founded rooted tree, with at most k levels. Since T is infinite, the tree shown in Figure 4 embeds, with root a and upper level infinite. If a is not the root of T, then there

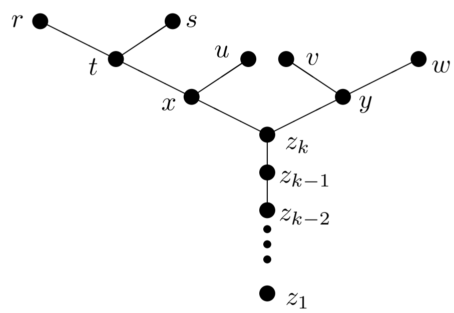

Figure 5: the argument in Lemma 4.4

is $b < a$, and some automorphism takes $\{x_1, \ldots, x_{k-1}, a\}$ to $\{x_1, \ldots, x_{k-1}, b\}$ where $x_1, \ldots, x_{k-1}$ are $k-1$ of the points on the upper level. But this takes a down, which is impossible, since the chains are finite. Similarly, each point above a is maximal, since otherwise we could find an automorphism moving it up. We deduce that all the maximal chains have length 2, and T has the claimed form. $\square$

Lemma 4.5 *: Let $(T, \leq)$ be a proper infinite k-transitive tree with $k \geq 2$, and with infinite chains. Then all the ramification points of T have the same ramification order, and either all or none of the ramification points are in T.*

Proof: Let x, y be ramification points. Then there are $a, b > x$ $c, d > y$, in T, such that $x = \inf(a, b)$ and $y = \inf(c, d)$. Choose $z_1 < z_2 < \ldots < z_{k-2} < x$ and $t_1 < t_2 < \ldots < t_{k-2} < y$ in T. Then $\{z_1, z_2, \ldots, z_{k-2}, a, b\} \cong \{t_1, t_2, \ldots, t_{k-2}, c, d\}$, so there is an automorphism f which takes the first set to the second. This takes $\{a, b\}$ to $\{c, d\}$, hence x to y. We deduce that x and y have equal ramification order.

In addition, $x \in T \Leftrightarrow y \in T$, so either $\mathrm{Ram}(T) \subseteq T$ or $T \cap \mathrm{Ram}(T) = \emptyset$. $\square$

We say that T has *positive type* if $T^+ = T$, (that is $\mathrm{Ram}(T) \subseteq T$), and *negative type* otherwise.

Theorem 4.6 *(Droste): For every cardinal κ, $2 \leq \kappa \leq \aleph_0$, there are countable 2-homogeneous trees T_κ and T'_κ such that all points of T_κ, T'_κ have ramification order κ, T_κ of positive type, T'_κ of negative type. Moreover, any proper countable infinite k-transitive tree for $k \geq 2$ is isomorphic to a tree as in Figure 4 or to some T_κ or T'_κ.*

Proof (Sketch): Intuitively, T_κ is constructed by starting with one maximal chain ordered like $\mathbb{Q}$ and allowing all its points to ramify to $\kappa - 1$ other branches, and then repeating infinitely many times on the new branches formed, and taking the union. Formally, the elements of T_κ may be represented by finite sequences of rationals. For T'_κ we modify this by first choosing a countable dense set of irrationals at which the ramification is performed.

The fact that T_κ and T'_κ are 2-homogeneous relies on the constructions' following the same 'recipe' at each stage. Since the only 2-element substructures are a chain and an antichain, it suffices to consider just these cases. Given any two 2-element chains we choose maximal chains added during the construction containing them. Since $\mathbb{Q}$ is 2-homogeneous, we take the first to the second so that the first 2-element chain goes to the second. Now this isomorphism can be extended since the construction starting from any two of these maximal chains is the same. For 2-element antichains the same argument is used, but now using two pairs of maximal chains.

Given any countable k-transitive proper tree with infinite chains, by Lemma 4.4 we know what κ has to be, and also whether it has positive or negative type. This enables us to show that there is an isomorphism to T_κ (T'_κ) by using back-and-forth, in the style of the previous paragraph, where we map the whole of a maximal chain at a time. □

Weak 2-transitivity

A key point about the above classification, apart from the fact that it is rather explicit, is that it comprises just countably many structures. It was discovered by Droste, Holland, and Macpherson [10], that an apparently minor change in the definition, from 2-transitivity, to 'weak 2-transitivity', increases the supply of examples to uncountably many, even though they are still fairly easy to describe. By saying that a partial order is *weakly 2-transitive* is meant that its automorphism group acts transitively on the set of 2-element chains (whereas 2-transitivity requires it also to be transitive on the family of 2-element antichains).

I now describe the general form that a weakly 2-transitive tree can take. First observe that once again, (except in the trivial case of just 2 levels), all maximal chains are ordered like $\mathbb{Q}$. This follows as for the 2-transitive case, since only transitivity on 2-element chains was used for this. We can also deduce by back-and-forth that the automorphism group acts transitively on the set of all maximal chains. It is no longer the case however that the group must act transitively on the ramification points. Letting T be such a tree, we adjoin its ramification points to form T^+, and let C be any maximal chain (by the above remarks, it doesn't matter which) and let C^+ be the union of C and all members of T^+ lying below a member of C.

Now one possibility which can arise is that a cone at some ramification point a has a least element. If so, the ramification point is called *special*, and as one easily shows that no two members of $T^+ - T$ are consecutive, each such least element of a cone at a lies in T. Furthermore, special ramification points clearly form a single orbit under the action of $\mathrm{Aut}(T)$. A notion of *type* is then given for such a tree (which I do not define here precisely), which provides the following information:

(i) whether T has positive or negative type, and in the former case, what the ramification order is at points of T,

(ii) whether there are any special ramification points a, and if so, how many cones there are at a with or without minimal members,

(iii) all other ramification orders.

The main result is then the following.

Theorem 4.7 *: Any two countable weakly 2-transitive trees having the same type are isomorphic. Furthermore, any type is the type of some countable weakly 2-transitive tree.*

We remark that the second sentence is carried out by starting with a suitable coloured version of $\mathbb{Q}$. If there are special ramification points, then one colour is reserved to stand for pairs of points consisting of a special ramification point and the point of T above it; if there are none, then this colour is just reserved for points of T. The other colours correspond to all other ramification orders specified by the type. Now branches are added in stages. At each stage we add the correct number of branches above each ramification point; for the 'typical' point this is just as for the 2-transitive case. At the lower point of a pair, we have to add the correct number of immediate successors (lying in T), and the correct number without successors. Then one iterates. It follows from this characterization that there are $2^{\aleph_0}$ pairwise non-isomorphic countable weakly 2-transitive trees.

The 1-transitive case

Finally in this section we consider a further relaxation of transitivity to just 1-transitive, a case which was tackled in [6]. Here there are three things which apply even in the weakly 2-transitive case which we can no longer assert. These are that all maximal chains are ordered like $\mathbb{Q}$, are 2-homogeneous linear orders, and are all isomorphic. The argument that they are all isomorphic to $\mathbb{Q}$ clearly requires the stronger hypothesis, so we expect examples now for instance having maximal chains ordered like $\mathbb{Z}$. We might also expect 2-homogeneity of maximal chains to be replaced by 1-transitivity, but even this is false in general, and we can construct examples for instance having maximal chains ordered like $\omega.\mathbb{Z}$ (ω copies of $\mathbb{Z}$). Third, it need not be the case that all maximal chains are isomorphic.

Given these setbacks, it might seem that the quest for any meaningful 'classification' would be hopeless in this generality. However Chicot [6] successfully carried this out, though a purist might object that the characterization given doesn't really provide a classification in the strict sense of the word (the classifiers are 'too complicated'). As in the case of countable 1-transitive coloured linear orders, the characterization does though provide a great deal of information about what the possibilities are, and details on how they are constructed, directly generalizing the weakly 2-transitive case.

I now outline the main steps in Chicot's analysis (though the full details are far too complicated to describe in a brief survey). First, some definitions.

Definition 4.8 *A linear order* $(X, <)$ *is* lower 1-transitive *if for every* $a, b \in X$, $(-\infty, a] \cong (-\infty, b]$.

This is a weakening of 1-transitivity. The most obvious example of a lower 1-transitive but not 1-transitive linear order is ω^*, that is, ω backwards. The point is that any branch of a 1-transitive tree is lower 1-transitive, though it need not be 1-transitive, so the first step in classifying all the countable 1-transitive trees is to find all the countable lower 1-transitive linear orders. Even this, it turns out, is a substantial extension of Morel's list. The corresponding natural notion in place of 'isomorphic' is this:

Definition 4.9 $(X, <)$ *and* $(Y, <)$ *are* lower isomorphic *if for some* $x \in X$ *and*

$y \in Y$, $(-\infty, x] \cong (-\infty, y]$ (where these intervals are taken in X, Y respectively), and we write $X \cong_l Y$.

For example $\omega^* \cong_l \mathbb{Z}$.

Lemma 4.10 *: If $(T, \leq)$ is a 1-transitive tree, then all its branches are lower 1-transitive linear orders, and they are all lower isomorphic.*

To describe the countable lower 1-transitive linear orders, we again use coding trees. We shall have additional constructions, and corresponding labels of points in the coding trees. We have already mentioned ω^*, and $\dot{\mathbb{Q}}$ and $\dot{\mathbb{Q}}_n$ are used for $\mathbb{Q}$ and $\mathbb{Q}_n$ respectively with an extra point added on the right. The reason for the presence of $\mathbb{Q}_n$ and $\dot{\mathbb{Q}}_n$ is that if $Y_0, Y_1, \ldots, Y_n$ are lower 1-transitive, and all lower isomorphic, then $\mathbb{Q}_n(Y_0, Y_1, \ldots, Y_{n-1})$ and $\dot{\mathbb{Q}}_n(Y_0, Y_1, \ldots, Y_n)$ are also lower 1-transitive, and this represents a typical way of constructing many new examples of lower 1-transitive linear orders. For instance, $\mathbb{Q}_2(\omega^*, \mathbb{Z})$ is a case in point.

Now the coding trees used here differ somewhat from those used earlier, in that they are required to be *levelled*, which means that they can be partitioned into maximal antichains, so that the partition is linearly ordered compatibly with the tree ordering, meaning that for levels L_1 and L_2, $L_1 < L_2$ if and only if there are some $x_i \in L_i$ such that $x_1 < x_2$ in the tree, if and only if for every $x_1 \in L_1$ ($x_2 \in L_2$) there is $x_2 \in L_2$ ($x_1 \in L_1$ respectively) such that $x_1 < x_2$. In what follows we shall use the word 'branch' for a maximal chain of a coding tree of points greater than or equal to some leaf (or for any maximal chain of the trees being classified).

Definition 4.11 *A* coding tree *here is a Dedekind–MacNeille complete levelled tree $(T, \leq)$ with greatest element (root) having countably many leaves such that every element is greater than or equal to some leaf, and the vertices are labelled by $\mathbb{Z}$, ω^*, $\mathbb{Q}$, $\dot{\mathbb{Q}}$, $\mathbb{Q}_n$, $\dot{\mathbb{Q}}_n$ (for $2 \leq n \leq \aleph_0$), 1, or lim, and*

(i) x is labelled 1 if and only if it is a leaf,

(ii) two vertices on the same level have equal, or lower-isomorphic, labels,

(iii) the number of children of a vertex is 1, 2, n, $n+1$ if it is labelled $\mathbb{Z}$ or $\mathbb{Q}$, ω^* or $\dot{\mathbb{Q}}$, or $\mathbb{Q}_n$ or $\dot{\mathbb{Q}}_n$ respectively, and if it is labelled lim, then it has no children and just one cone below it, and for ω^*, $\dot{\mathbb{Q}}$, and $\dot{\mathbb{Q}}_n$ one child is designated as the 'right child',

(iv) at each level, the 'left forests' from that level are isomorphic as levelled labelled forests, and

(v) the (labelled levelled) trees with roots at distinct left children of any parent vertex are not isomorphic.

To make sense of the final clauses we say that any child which is not 'right' is *left* (for instance, *all* children of $\mathbb{Q}_n$ vertices) and the *left forest* of a vertex is the forest consisting of its left children and their descendants. For $\mathbb{Z}$, ω^*, $\mathbb{Q}$, and $\dot{\mathbb{Q}}$ the left forest is a single tree, but for $\mathbb{Q}_n$ and $\dot{\mathbb{Q}}_n$ it is a union of n trees. Saying that two forests are isomorphic means that their trees can be put into 1–1 correspondence in such a way that corresponding trees are isomorphic.

As for 1-transitive coloured linear orders we have a notion of *expanded coding tree* which is formed from a coding tree by interpreting the labels. If E is an expanded coding tree associated with a coding tree T, then the set of leaves of E under the natural left-right order is the linear ordering *encoded* by T.

Theorem 4.12 *: Any coding tree encodes some countable lower 1-transitive linear order, which is unique up to isomorphism. Conversely, any countable lower 1-transitive linear order is encoded by some coding tree.*

The next stage in this particular problem is to analyze certain countable lower 1-transitive coloured linear orders, since by analogy with the case of weakly 2-transitive trees these will describe the possible structures of branches of a 1-transitive tree with information about ramification included. It is (fortunately) not necessary to classify *all* such, as for instance, the points corresponding to the points of the structure, which will be assigned one fixed colour $\bar{c}$, must occur densely. The precise definitions here are motivated by the case of weakly 2-transitive trees, but the extra requirements in the 1-transitive case are considerably more involved. In essence however there is a definition of *colour coding tree* and with respect to this, an analogue of Theorem 4.12 for the coloured countable lower 1-transitive linear orders needed as branches for the description of all the countable 1-transitive trees.

Moving towards a description in outline of all the countable 1-transitive trees T, let us denote by Υ the class of branches of T^+ viewed as coloured chains, up to isomorphism. We remark that members of Υ will have no maximal elements, and they will form a subset of a lower isomorphism class.

We next need to introduce the notion of 'cone type' of a ramification point. Now in the weakly 2-transitive case we had the notion of a *special* ramification point, and we had to count the numbers of cones at such a point with or without least members. This time, we need to do a similar thing, but corresonding to all possible levels of the coding trees of members of Υ (and a key point is that since all members of Υ are lower isomorphic, their colour coding trees have the same sets of levels, and are closely linked, so may be considered simultaneously). This gives rise to a pair of sequences indexed by the levels of the coding tree of an element of Υ telling us how many special and normal cones there are in the corresponding quotient of that lower 1-transitive coloured linear order at a particular ramification point a, and this is called the *cone type* of a.

We say that a countable tree $(T, <)$ is *structured* if it is proper, and there is a colouring F of T^+ such that the set Υ of branches of T^+ is (up to isomorphism) a subset of some colour lower isomorphism class of lower 1-transitive coloured linear orders without maxima, and having one colour $\bar{c}$ dense in the others (which precisely colours the elements of T), any two elements having the same colour have the equal cone types, every final segment of a member of Υ occurs infinitely often above every point of T, and whenever $x < y$ are consecutive points of a member of Υ and $F(x) = \bar{c}$ then also $F(y) = \bar{c}$.

Finally we have the notion of the *type* $t(T)$ of a tree $(T, <)$, which comprises the set Υ of isomorphism types of branches of T^+, the colour set of T^+ (which may be viewed as the set of orbits of T^+ under the action of $\mathrm{Aut}(T)$), and the family of cone types of ramification points. Conversely one may describe in abstract terms what a 'type' is, based on all the restrictions that one can show are necessary for it to arise in this way. Given all this, the main result is then as follows.

Theorem 4.13 *: Any countable tree is structured if and only if it is 1-transitive. Furthermore, any countable 1-transitive tree has a type, two countable 1-transitive trees are isomorphic if and only if they have the same type, and any type is the type of some countable 1-transitive tree.*

5 Cycle-free partial orders

M. Rubin [21] carried out an exhaustive study of trees (not just countable, and not just 2-transitive ones), and proposed that his work be generalized to what he called 'cycle-free' partial order, CFPOs. He had a clear intuition as to what this should mean, but there were some problems in formulating the correct definition. Roughly speaking, we may say that the idea is that whereas in a tree one can only branch while passing upwards, in a CFPO one can branch going up, or down, and repeatedly, provided one never returns to the starting point.

Peter Neumann [1] proposed the following 'desiderata' for such a notion:

(1) subsets of CFPOs should be cycle-free,

(2) trees and their duals should be cycle-free,

(3) the diagram representing a cycle-free partial order should be 'treelike' (in the graph-theoretical sense),

(4) cycle-freeness should be first order expressible,

(5) the Dedekind–MacNeille completion of a cycle-free partial order should be cycle-free.

He included (4) and (5) because the definition of a CFPO given by Richard Warren [27] went via the Dedekind–MacNeille completion, and was ostensibly not first order expressible. The reason why Warren felt it necessary to go by way of X^D is explained by considering (1).

According to Rubin's ideas, the partial ordering in Figure 3(b) definitely should be a CFPO, and when we began considering the notion, we thought that the one in Figure 3(a) should not be, since one can traverse $a - x - b - y - a$ and obtain a cycle. This however already contradicts (1), since the one in Figure 3(a), which (we thought) *has* a cycle, is clearly a subset of that in Figure 3(b), which does *not*. In the light of this example, Warren felt that, to assess whether

Figure 6: a substructure of a CFPO need not be a CFPO

X was cycle-free, one should really look at X^D, and then (guided by (3) and the graph-theoretical idea) see whether there were still any 'genuine' cycles. His definition was this: $(X, \leq)$ is a cycle-free partial order (CFPO) if between any two points u and v of X, there is *in* X^D a unique path from u to v. So, in our example, we get over the fact that in X there are distinct paths from a to b, via x or via y, by noting that in X^D these become $a\ t\ c\ t\ b$, $a\ t\ d\ t\ b$ where $t = \sup(a, b)$, neither of which is a 'path', as it 'retraces its steps'. To formalize this, we say that a *connecting set* in a partial order from x to y is a finite sequence $x = x_0, x_1, x_2, \ldots, x_n = y$ such that, for each i, x_i and x_{i+1} are comparable. A *path* is a set of the form $\bigcup_{0 \leq i < n} C_i$ for some connecting set $\{x_i : 0 \leq i \leq n\}$ where C_i is a maximal chain of $[x_i, x_{i+1}] = \{y : x_i \leq y \leq x_{i+1}\}$ if $x_i \leq x_{i+1}$ ($= \{y : x_{i+1} \leq y \leq x_i\}$ if $x_{i+1} \leq x_i$ for simplicity), and such that the C_i only overlap at the endpoints.

Now this definition captured some of what was desired, and it was possible to develop the desired theory. However, (4) was still unlikely. In fact, the definition as it stood could not be first order, and even (1) would be false. That is because Warren's definition entailed 'connectedness'—between any two points there is a connecting set. So for instance the partial order shown in Figure 6(a) would be a CFPO, but its substructure in Figure 6(b) would not. In addition, by general model-theoretic considerations, one easily sees that the infinite 'alternating chain' ALT shown in Figure 7, which is connected, and is surely a CFPO, is elementarily equivalent to two incomparable copies of ALT, which is not. To have any chance of axiomatizing the class of CFPOs therefore, we have to relax the connectedness stipulation, and instead say that a CFPO is a partial order $(X, <)$ in which for any $x, y \in X$ there is *at most* one path from x to y. This is equivalent to saying that all its connected components are CFPOs in Warren's sense.

To achieve an axiomatization as given in [24] we require the special partial order shown in Figure 8, a *diamond* and $2n$-*crowns* for $n \geq 3$ (a 6-crown is shown in Figure 2).

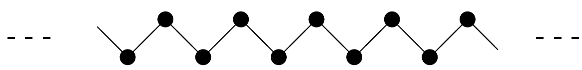

Figure 7: the infinite 'alternating chain' ALT

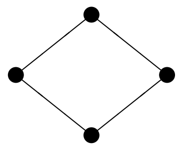

Figure 8: a diamond

Theorem 5.1 *: $(X, \leq)$ is a CFPO (not necessarily connected) if and only if no diamond or $2n$-crown for $n \geq 3$ embeds in X^+.*

To get an axiomatization for CFPOs from this theorem, one has to eliminate the reference to X^+. Since X^+ is interpretable in X, that is, we may express all properties of X^+ just by talking about X, this can be done. In fact there is an apparently slightly weaker equivalent (X^+ embeds no diamond, and X embeds no $2n$-crown for $n \geq 3$).

Theorem 5.2 *: (i) The class of (not necessarily connected) CFPOs is axiomatizable by universal axioms, but is not finitely axiomatizable.*
(ii) The class of connected CFPOs is not axiomatizable.

An important and related concept is that of 'being in between'. Any (now connected) CFPO has a natural betweenness relation $B(x; y, z) : x$ is between y and z if x lies on the unique path from y to z. This fulfils various axioms studied by Adeleke and Neumann [1], and is occasionally employed in our work.

The CFPOs studied by Warren were just those which are countable, proper (now meaning 'not a tree or its dual'), and k-CS-transitive for some $k \geq 3$. In non-trivial cases we cannot expect any interest in restricting to k-transitive CFPOs, since we could then not even embed the alternating chain with 5 points shown in Figure 6(a), since $\{a, c\} \cong \{a, e\}$, but as a is at different 'distances' 2 and 4 from c and e, there can be no automorphism taking $\{a, c\}$ to $\{a, e\}$. Instead we go for 'k-connected-set-transitive', meaning that for any two isomorphic connected k-element substructures there is an automorphism taking the first to the second. This rules out the above counter-example, as the substructures used are not connected. The classification falls into the following cases, for a CFPO M:
(1) Sporadic [27]: ALT embeds in M, but all chains of M^+ are finite,
(2) Skeletal [27]: all the maximal chains of M have length 2, but M^+ has infinite chains (from which it follows that ALT embeds),
(3) Infinite chain [7]: M has an infinite chain,
(4) ALT does not embed [25].

The most surprising feature of this classification one may say is the 'skeletal' case. The word is intended to suggest that the structure M is a mere 'skeleton', consisting of just upper and lower points (maximal chains of length 2); however, on passing to M^+, where all the ramification points are included, suddenly infinite chains appear; it is as if flesh has been put on the bones. This situation contrasts starkly with that for trees, where the finite chain case is essentially trivial. So we have a much richer situation, with many more beautiful and interesting structures, and having complicated automorphism groups.

Lemma 5.3 *: Let M be an infinite, proper, k-CS-transitive CFPO, for some $k \geq 2$, all of whose chains are finite. Then all maximal chains of M have length 2.*

If $k = 2$, the proof is as for trees. If $k > 2$ one has to argue that, given a chain $\{a, b, c\}$ of length 3, one can find $k - 2$ points 'at the side' so that there is a non-trivial order-preserving map on a connected set of size k fixing each of these $k - 2$ points, and whose domain and range each intersect $\{a, b, c\}$ in a set of size two. This ensures that one can map strictly upwards within a finite chain, leading to a contradiction.

We now describe in outline the first three cases (1), (2), (3).

Sporadics: These are CFPOs M in which all the chains of M^+ have finite length, in which case it turns out that this length is ≤ 4, but which nevertheless embeds ALT. There are four kinds:

$M = \mathcal{M}_{\kappa_1\kappa_2}$ Each minimal point is an upward ramification point of order κ_1, and each maximal point is a downward ramification point of order κ_2. See Figure 9.

$M = \mathcal{N}_{\kappa_1\kappa_2}$ This is a 'decorated' version of ALT. See Figure 10.

$M = \mathcal{P}_{\kappa_1\kappa_2}$ Each minimal point is an upward ramification point of order 2. The maximal points do not ramify. The middle points ramify upwards with order κ_1, and downwards with order κ_2. See Figure 11.

In addition there is $M = \mathcal{P}'_{\kappa_1\kappa_2}$, the dual of $\mathcal{P}_{\kappa_1\kappa_2}$, that is $\mathcal{P}_{\kappa_1\kappa_2}$ turned upside down. Notice that in these diagrams, we fully shade in points of M.

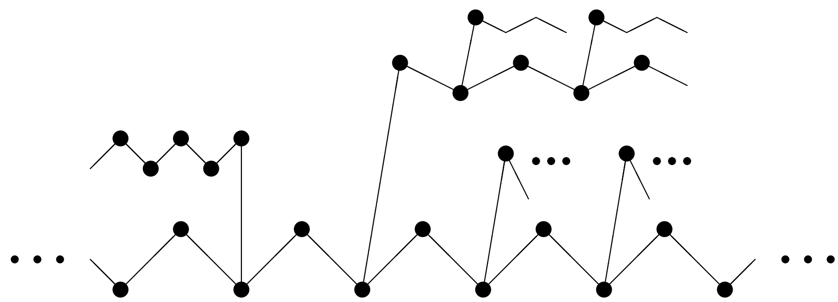

Figure 9: $\mathcal{M}_{3,2}$; maximal chains of M^+ have length 2

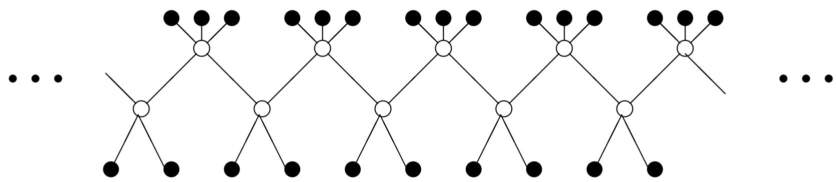

Figure 10: $\mathcal{N}_{\kappa_1,\kappa_2}$; a 'decorated' version of ALT

Points of $M^+ - M$ are not shaded in. However, all the points of M^+ are *definable* from points of M.

Skeletals: These are the countable k-CS-transitive CFPOs M for $k = 3$ or 4 in which all maximal chains of M have length 2, but the maximal chains of M^+ are infinite. For these, as for the sporadics, to specify the whole of M^+ it suffices to give one maximal chain of M^+, (with top and bottom points in M) and instructions about which points ramify up and down and with what ramification order. In these cases, for upward ramification points x and y there is an automorphism taking x to y (similarly for the downward ramification points), since we may find $a, b, c, d, e, f \in M$ such that $x = \inf(b, c)$, $y = \inf(e, f)$, $a < b, c$, and $d < e, f$. In fact this shows that $\mathcal{P}_{\kappa_1\kappa_2}$ is *not* 3-CS-transitive (though it may be k-CS-transitive for larger k). The maximal chains of M^+ are best thought of as *coloured chains*, that tell us what kinds of points there are. The automorphism group will then act transitively on the points of any fixed colour.

Case 1: $\uparrow\mathrm{Ram}(M) = \downarrow\mathrm{Ram}(M)$ (see Definition 4.3): $\mathcal{A}^Z_{\kappa\lambda}$. Here Z is an infinite 1-transitive countable linear order (from Morel's list, Theorem 3.2). The order-type of a maximal chain of M^+ is $1 + Z + 1$; all points of $\mathrm{Ram}(M)$ ramify both upwards and downwards, with orders κ, λ respectively.

Otherwise $\uparrow\mathrm{Ram}(M) \cap \downarrow\mathrm{Ram}(M) = \emptyset$.

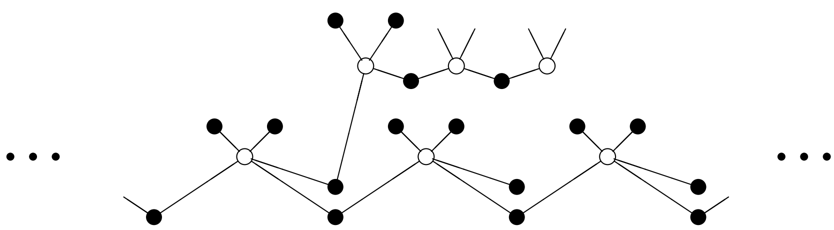

Figure 11: $\mathcal{P}_{\kappa_1,\kappa_2}$

Case 2: Ram(M) densely ordered: $\mathcal{B}_{\kappa\lambda}$. Here the upward and downward ramification points are interdensely ordered.

Case 3: Fully covered CFPOs; every upper cone has a least element, and every lower cone has a greatest element. Here the maximal chain consists of $\mathbb{Q}$ 'pairs' (together with the top and bottom elements): $\mathcal{C}_{\kappa\lambda}$.

Case 4: Partially covered CFPOs: Every lower cone has a greatest element, but $\uparrow$r.o.$(M) = 2$ and just one of the two upper cones has a least element (and the dual of this case). The maximal chains are listed:
$\mathcal{D}_\lambda$ $\mathbb{Q}_2(2)$ $\quad \mathcal{E}_\lambda$ $\mathbb{Q}.(\mathbb{Q}.2+1)$ $\quad \mathcal{F}^Z_\lambda$ $Z.(\mathbb{Q}.2+1)$, $Z \not\cong \mathbb{Q}$.

Case 5: Non-covered CFPOs: $\uparrow$r.o.$(M) =\downarrow$r.o.$(M) = 2$, and at each upward ramification point just one cone has a least element and similarly for downward ramification points: $\mathcal{G}^Z$, $\mathcal{H}^Z$, $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$ (with some duals).

Infinite chain CFPOs

Now we move on to the classification of infinite chain CFPOs. These structures are superficially 'less surprising' in the sense that they are ones one might have suspected existed, whereas the finite chain ones one might have doubted. Their classification is rather more complicated, though it turns out that there is a close connection, which can be made explicit, between this and the finite chain case.

As indicated in the finite chain case, the whole behaviour of any one of the CFPOs M in our list can be described by a maximal chain of M^+ with sufficient information about the ramification points.

Lemma 5.4 *: Suppose that M is a proper k-CS-transitive CFPO, $k \geq 2$, with an infinite chain. Then*

(i) for every $x \in M$ there are infinite chains A_x, B_x in M such that $A_x < x < B_x$,

(ii) M is dense and 1-transitive,

(iii) for every $x \in M$ there are infinite antichains X_x, Y_x in M such that $X_x < x < Y_x$,

(iv) $\uparrow$Ram(M), $\downarrow$Ram(M) are dense in M.

Proof: As for proofs about trees, the important thing is to establish this for one maximal chain. Let C be an infinite maximal chain of M, and let $x \in C$. Then either $\{y \in C : x < y\}$ or $\{y \in C : x > y\}$ is infinite. Suppose the former, and let $x < x_1 < x_2 < \ldots < x_k$ in C. Then $p(x) = x_1$, $p(x_i) = x_{i+1}$ for $1 \leq i < k$ is an isomorphism of k-element substructures, so there is an automorphism f extending it. Then $B_x = \{f^n(x) : n > 0\}$ is an infinite chain above x, and $A_x = \{f^{-n}(x) : n > 0\}$ is an infinite chain below x.

(ii) If $x < y$, by (i) we may choose $x_k > x_{k-1} > \ldots > x_2 > y$. Then the isomorphism $p(x) = x$, $p(x_i) = x_{i-1}$ for $2 < i \leq k$, $p(x_2) = y$ extends to an automorphism, and the image of y lies between x and y. This gives density. For

1-transitivity, let $x, y \in M$ be given. Choose $x_k > x_{k-1} > \ldots > x_2 > x$ and $y_k > y_{k-1} > \ldots > y_2 > y$, and take $\{x, x_2, \ldots, x_k\}$ to $\{y, y_2, \ldots, y_k\}$.

(iii) Since M is proper, M contains a V-shape. By (ii), every point has two incomparable points above it. By (i), given x, there are x_i with $x < x_0 < x_1 < x_2 < \ldots$. Since every point has two incomparable points above it, there is y_i such that $x_i < y_i$ and y_i is incomparable with x_{i+1}. Then $Y_x = \{y_i : i \in \omega\}$ is an infinite antichain above x. Similarly for X_x.

(iv) Since there is a V-shape $a < b, c$ say in M, $a \leq \inf(b, c)$ which is an upward ramification point. Let $x < y$ be arbitrary in M. Pick $b_k > b_{k-1} > \ldots > b_3 > b > a$, and $y_k > y_{k-1} > \ldots > y_3 > y > x$, and take $\{a, b, b_3, \ldots, b_k\}$ to $\{x, y, y_3, \ldots, y_k\}$. Then a goes to x, and b to y, so the image of $\inf(b, c)$ lies between x and y (possibly equal to x—though, since we can easily show by 3-CS-transitivity that $M \cap \mathrm{Ram}(M) = \emptyset$, this cannot actually happen). Similarly for $\downarrow\mathrm{Ram}(M)$. $\Box$

The classification is given in terms of a maximal chain C of M^+ (it is easy to see that, for every two maximal chains, there is an automorphism taking the first to the second).

Case 1: M is dense in M^+.

If $\uparrow\mathrm{Ram}(M) = \downarrow\mathrm{Ram}(M)$: $\mathfrak{A}_{\kappa\lambda}$, M and $\mathrm{Ram}(M)$ interdense on C.

If $\uparrow\mathrm{Ram}(M) \cap \downarrow\mathrm{Ram}(M) = \emptyset$: $\mathfrak{B}_{\kappa\lambda}$, M, $\uparrow\mathrm{Ram}(M)$, $\downarrow\mathrm{Ram}(M)$ all interdense.

Now suppose M is not dense in M^+. Define $\sim$ on $C \cap \mathrm{Ram}(M)$ by: $x \sim y$ for $x \leq y$ if $(x, y) \cap M = \emptyset$. Since M is not dense in M^+, some $\sim$-classes are non-trivial.

Case 2: M is not dense in M^+ and $\uparrow\mathrm{Ram}(M) = \downarrow\mathrm{Ram}(M)$. Then each $\sim$-class is a 1-transitive linear order Z (in Morel's list), and they are all isomorphic (hence non-trivial). These and the points of $M \cap C$ are interdense: $\mathfrak{C}^Z_{\kappa\lambda}$.

Case 3: M not dense in M^+, $\uparrow\mathrm{Ram}(M) \cap \downarrow\mathrm{Ram}(M) = \emptyset$, and all $\sim$-classes are finite. A *pair* consists of $x < y$ where $x \in \uparrow\mathrm{Ram}(M)$, $y \in \downarrow\mathrm{Ram}(M)$, and there is nothing in between. An upward ramification point x which equals $\inf\{y > x : y \in \mathrm{Ram}(M) \cap C\}$ is called an *upper limit ramification point.*

$\mathfrak{D}_{\kappa\lambda}$: only pairs occur as $\sim$-classes.

$\mathfrak{E}_{\lambda}$: only pairs and singleton upward ramification points occur.

$\mathfrak{F}$: pairs and singleton upward and downward ramification points all occur.

Case 4: M not dense in M^+, $\uparrow\mathrm{Ram}(M) \cap \downarrow\mathrm{Ram}(M) = \emptyset$, and all $\sim$-classes are infinite.

Then it turns out that all $\sim$-classes are isomorphic.

There are many more cases, depending on whether or not each $\sim$-class has a pair, upward or downward limit ramification points, etc.

In summary, the structure of all the CFPOs in this part of the classification is specified by a maximal chain C of M^+, viewed as 'coloured' by finitely many (at most 4) colours which will tell us

(i) if the point lies in M or $\mathrm{Ram}(M)$,

(ii) if in $\mathrm{Ram}(M)$, what 'kind' of ramification point it is, upward/downward, ramification order, upper or lower limit, member of a pair, etc. One can show that $\mathrm{Aut}(M)$ acts transitively on each colour (using k-CS-transitivity on M), so what is involved is classifying certain 1-transitive *coloured* linear orders. The cases of the subdivision correspond to different instances of this classification. So this has strong links with the material presented in section 3.

Simplicity

Finally we consider some simplicity questions about the automorphism groups of these structures [9]. For chains, Higman showed that $\mathrm{Aut}(\mathbb{Q}, \leq)$ and $\mathrm{Aut}(\mathbb{R}, \leq)$ both have exactly three proper non-trivial normal subgroups, so they are not far from being simple. By contrast, Droste, Holland and Macpherson showed that for T a countable 2-transitive tree with infinite chains, $\mathrm{Aut}(T)$ has the greatest possible number, $2^{2^{\aleph_0}}$, of normal subgroups. In other words, these groups, unlike $\mathrm{Aut}(\mathbb{Q}, \leq)$, are very far from being simple.

It is something of a surprise, therefore, that many of the automorphism groups of the CFPOs in our list are simple groups. This adds to quite a long list of groups of infinite 'homogeneous' structures that are known to have simple automorphism groups. Moreover, there are neat characterizations of which ones are or are not simple, and for which a 'bounded number of conjugates' suffices.

I shall outline the proof of the simplicity of the automorphism groups of one or two of the CFPOs in the classification. In some cases we are able to give the proof in the following strong form: if g and h are non-identity elements of G, then h may be expressed as the product of 12 conjugates of g or g^{-1}. ¿From this one deduces simplicity of G thus. Let N be a non-trivial normal subgroup. Then there is some $g \neq 1$ in N. Let h be any other non-identity element of G. Since h may be written as the product of 12 conjugates of g, g^{-1}, and N is normal, also $h \in N$. It follows that $N = G$.

In other cases, there is no 'bounded' number of conjugates, such as 12. In other words, for given $g, h \neq 1$, there is always a number n such that h is equal to a product of n conjugates of g, g^{-1}, but this n may depend on g and h. This is no disadvantage from the point of view of establishing simplicity, but it does mean that we may not be able to express simplicity by a first order formula. In our case, the point is that there is quite an attractive demarcation between the cases non-simple, simple with ≤ 12 conjugates, and simple with unboundedly many conjugates required, that corresponds to combinatorial properties of the CFPOs in question.

First, as an easy case, let us consider $\mathcal{M}_{\kappa\lambda}$ with $\kappa, \lambda \geq 3$ (see Figure 9), and we show that $G = \mathrm{Aut}(\mathcal{M}_{\kappa\lambda})$ is simple. (Here there is a bound on the number of conjugates.) Here, all maximal chains even of $\mathcal{M}^+_{\kappa\lambda}$ have length 2, all the minimal points have upward ramification order κ, and all maximal points have downward ramification order λ. Suppose that $1 < N \lhd G$, and let $g \in N - \{1\}$. Then g moves some point. If g fixes all minimal points, then it must move some maximal point a to b, say. But there is a minimal point c below a not below b, and this would also have to be moved. Hence g moves a minimal point, a say, to ga. Let the path from a to ga be $a = x_0 < x_1 > x_2 < \ldots > x_{2n} = ga$, and let $h \in G$ fix all the points y such that the path from a to y passes through x_1, and permute the other points above a in any given way (there are at least 2 such as $\kappa \geq 3$). Then $hx_i = x_i$ for each i, and $h(ga) = ga$ and $h(gx_1) = gx_1$. Hence $hg^{-1}h^{-1}g$ fixes $\{a, x_1\}$ and permutes the points above a in the same way that h does. As $hg^{-1}h^{-1}g \in N$, N contains an element fixing a and permuting all points above a except x_1 in any given manner. By varying these, and also the point above a that is fixed, we can show that N contains the stabilizer of a. The proof is concluded by showing that any element of G is a product of two elements in stabilizers of points.

For the more complicated cases—skeletal and infinite chain case, we need to isolate a particular class Σ of elements of the group. If M is skeletal and $\mathrm{Aut}(M)$ acts 'sufficiently transitively' on some maximal chain of M^+, we let $g \in \Sigma$ provided that for some $a < b$ in M, $ga = a$ and $gb = b$ and for every $x \in (a, b)^{M^D}$, $x < gx$, with a similar condition in the infinite chain case. Then under the transitivity condition, one can show that any two members of Σ are conjugate. Next one shows that if $1 \neq N \lhd G$, then N contains a member of Σ, hence it contains all of Σ, and finally that Σ generates G.

Examples: $\mathcal{A}^{\mathbb{Q}}_{\kappa\lambda}$ has a simple automorphism group, with a bounded number of conjugates; $\mathcal{C}_{\kappa\lambda}$ has a simple automorphism group, but with no bound.

$\mathcal{A}^{\mathbb{Z}}_{\kappa\lambda}$ does not have simple automorphism group. Here, there is a notion of 'level' on the points of M^+, and the permutations which preserve the levels form a proper non-trivial normal subgroup.

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Authors' address
Department of Pure Mathematics, University of Leeds,
Leeds LS2 9JT, UK, (e-mail pmtjkt@leeds.ac.uk)