On Borel Inseparability of Game Tree Languages

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GAMES 2008

Separability

In contexts where we study hierarchies
(Descriptive Set Theory, Automata Theory, Logic, Complexity Theory)
we ask about:

$$\begin{tikzcd}[column sep=small, row sep=small]
\vdots & \vdots & \vdots \\
\Sigma^0_n \arrow[dr, dash] & & \Pi^0_n \arrow[dl, dash] \\
& \Delta^0_n & \\
\vdots & \vdots & \vdots \\
\Sigma^0_2 \arrow[dr, dash] & & \Pi^0_2 \arrow[dl, dash] \\
& \Delta^0_2 & \\
\Sigma^0_1 \arrow[ur, dash] \arrow[dr, dash] & & \Pi^0_1 \arrow[ul, dash] \arrow[dl, dash] \\
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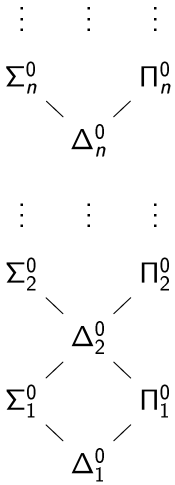

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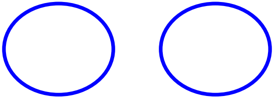

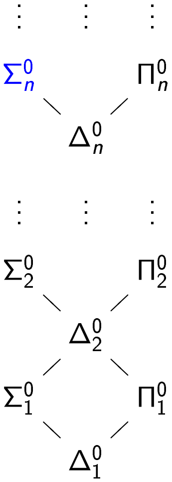

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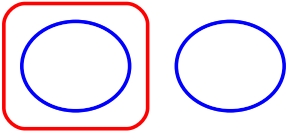
  - can they be separated by a set from the lower level?



$$\begin{array}{ccc}
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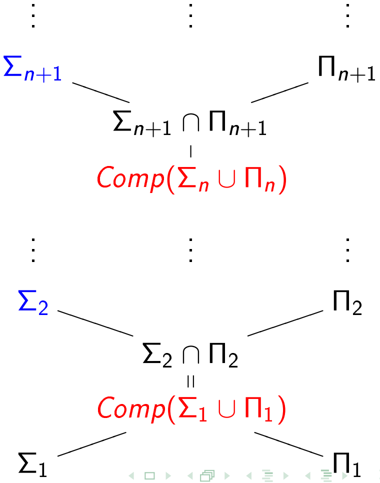
Nondeterministic Index Hierarchy

Santocanale and Arnold studied separation property in the context of μ-terms and parity automata.



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separation fails
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$$|$$

$$Comp(\Sigma_n \cup \Pi_n)$$

$$\vdots \qquad \vdots \qquad \vdots$$

?

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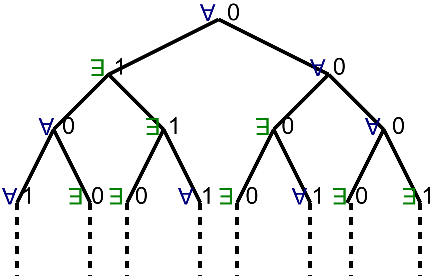
Inseparable Pair

- The inseparable pair:
  - $W_{(0,1)}$ — one of the game tree languages that witness strictness of alternating index hierarchy:



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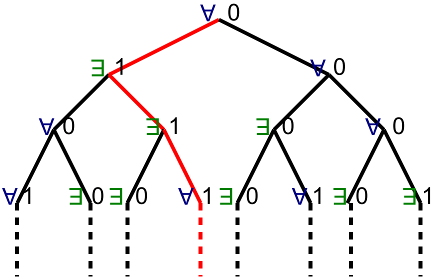

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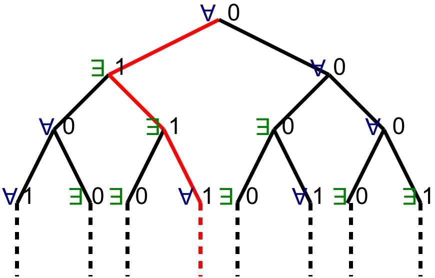

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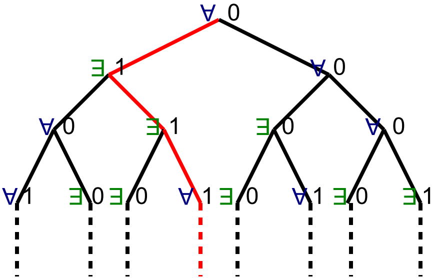

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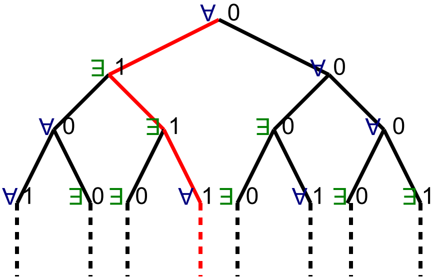

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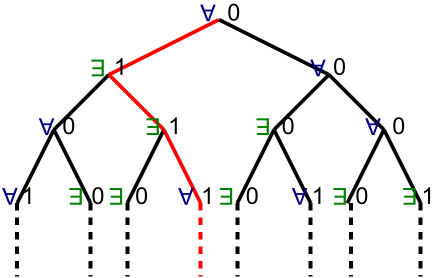

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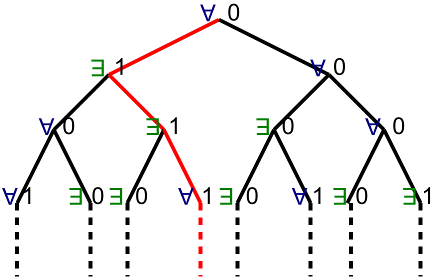
- We use this to prove something stronger than needed:



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First Separation Property *fails for co-Büchi class.*

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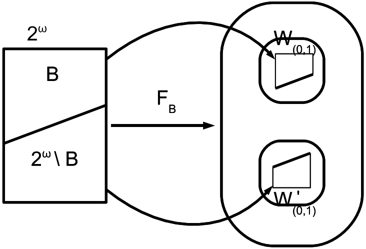
Lemma

Let $B \subseteq 2^\omega$ be an arbitrary Borel set. There exists a continuous function $F_B : 2^\omega \to T_{\{\exists,\forall\}\times\{0,1\}}$ such, that:

$$\begin{aligned} {F_B}^{-1}(W_{(0,1)}) &= B \\ {F_B}^{-1}(W'_{(0,1)}) &= 2^\omega \setminus B \end{aligned}$$



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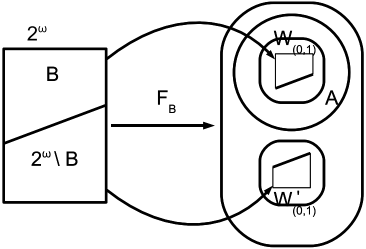

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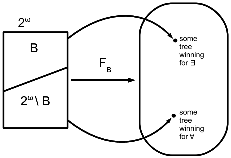

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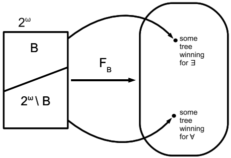

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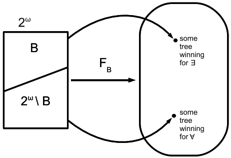

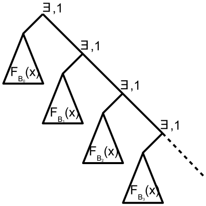

Topological View

- $W_{(0,1)}$ and $W'_{(0,1)}$ are Π^1_1-complete (coanalytic complete) sets
- Borel inseparable Π^1_1 pairs known so far were all similar to the classical one:

$$\begin{array}{rcl} WF & = & \{t \in T_{\{0,1\}} : \text{every path has only finitely many 1's}\} \\ UB & = & \{t \in T_{\{0,1\}} : \text{exactly one path has infinite number of 1's}\} \end{array}$$

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- Büchi sets are all Σ^1_1
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- But we can ask:

  Does Büchi class have First Separation Property?

  (It would complement another fact by Santocanale and Arnold)