INFINITE GAMES OF PERFECT INFORMATION

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§ 1. INTRODUCTION

It is well-known that finite, two-person, zero-sum games with perfect information are strictly determined [1]. There have been attempts to remove from this result each of these restrictions. It is the first of these with which we will be concerned, i.e., we consider infinite games.

In a paper by Gale and Stewart [2], zero-sum, two-person, infinite games with perfect information are defined. Familiarity with this paper will be assumed.

The notation of this paper will lean heavily on the above paper, but some additions and modifications will be made. In referring to the game Γ we will mean the $(x_0, X_I, X_{II}, X, f, S, S_I, S_{II})$ of Gale and Stewart, where each element is understood as given in the game. We will also write $\Gamma(S_I = A)$ to stand for the game $(x_0, X_I, X_{II}, X_m, f, S, A, A^c)$ where $S = A \cup A^c$. We use here and elsewhere in the paper the superscript c to denote complement. We assume $f^{-1}(x)$ is always a finite set.

In this paper we extend the results of Gale and Stewart [2] and Wolfe [3], answer Questions 1 and 2 of [2] (assuming the Continuum Hypothesis), and finally, characterize the winning sets of a game suggested to me by Professor L. E. Dubins.

I would like to express my appreciation to Professor David Blackwell, who introduced me to the problem and who made several valuable suggestions, to Professor L. E. Dubins, who suggested the game in Section 4 and discussed this problem with me, and finally to Professor Jan Mycielski, who gave several alternative proofs which were shorter and clearer, and suggested the extensions of Section 2, which was more general than my original form.

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§ 2. THE DETERMINATENESS IN THE CASE $G_{\delta\sigma} \cup F_{\sigma\delta}$.

Gale and Stewart [2] have shown that if the winning sets (S_I or S_{II}) are either open or closed sets, then the game is determined. Wolfe has extended this result to G_δ and F_σ sets. In this section, this is still further extended to $G_{\delta\sigma}$ and $F_{\sigma\delta}$ sets.

DEFINITION 2.1. Player I(II) has a <u>loss</u> at $x \in X$ in the game Γ if and only if player I(II) has a <u>loss</u> for the subgame Γ_x.

DEFINITION 2.2. x is a <u>position</u> of s if and only if $s(i) = x$ for some i (or alternately, s is said to <u>pass through</u> x).

DEFINITION 2.3. A subgame Γ_C of Γ is I-imposed if and only if for all $x \in X_{II}^C$, $f^{C^{-1}}(x) = f^{-1}(x)$; (a similar definition is given for <u>II-imposed</u>) (see [2], Def. 8, Th. 4, 5 and Notation). Intuitively, this states that a I-imposed subgame is a subgame that may "be imposed" by player I, i.e., one in which player II has no restriction of his alternatives.

DEFINITION 2.4. For $A \subseteq X$, $O(A) = \bigcup_{x \in A} u(x)$ (see [2], p. 252, Notation).

LEMMA 2.1. If player II does not have a loss at $\bar{x} \in X_I$ then player II does not have a loss for all $x \in f^{-1}(\bar{x})$.

PROOF. The proof is trivial. If player II had a loss at $y \in f^{-1}(\bar{x})$ then at $\bar{x}$ player I could choose y so player II would then have had to have had a loss at $\bar{x}$ as well, contrary to the hypothesis.

LEMMA 2.2. If player II does not have a loss at $\bar{x} \in X_{II}$, then player II does not have a loss for some $x \in f^{-1}(\bar{x})$.

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PROOF. Trivial.

LEMMA 2.3. If Γ is not a loss for player II, $P \subseteq S_I$, P is an open set, then there exists a II-imposed subgame Γ_C of Γ such that Γ_C is not a loss for player II and $C \cap P = \emptyset$.

PROOF. Let C be all the plays in S that have the following property: if x is any position of s (for $s \in C$), then Γ_x is not a loss for player II. Since C is clearly a closed set, Γ_C is a subgame, and, in particular, Γ_C is a II-imposed subgame by Lemmas 2.1 and 2.2.

If player I had a winning strategy σ in Γ_C he would have had one in Γ as well, for if player I plays consistently with σ as long as he can in Γ_C (i.e., as long as player II does not reach a lost position) and conforms to a winning strategy σ_y when player II reaches his first lost position, y, player I will clearly win. Hence Γ_C is not a loss for player II.

Also $C \cap P = \emptyset$ for if $s \in P$ there is some position of s, x, where $\mathcal{U}(x) \in P$ and hence player II has a loss at x and $s \notin C$, and Lemma 2.3 follows.

LEMMA 2.4. If Γ is not a loss for player II, $H \subseteq S_I$, H is a G_δ set, then there is a II-imposed subgame Γ_C of Γ such that $C \cap H = \emptyset$ and Γ_C is not a loss for player II.

PROOF. Let B be the set of all $x \in X$ such that in Γ_x (see [2], p. 254, Notation) there is a II-imposed subgame $[\Gamma_x]_{D_x}$ which is not a loss for player II and such that $D_x \cap H = \emptyset$. Let $H = \bigcap_{i=0}^{\infty} O(C_i)$ where $C_i \subseteq X$ and $X = C_0 \supseteq C_1 \supseteq C_2 \cdots$.

We first prove the following sequence of statements:

(A_i) If $x \in B^c \cap C_i$, then $\Gamma'_x = \Gamma_x(S'_I = S_I \cup O\{B^c \cap C_{i+1}\})$

is a loss for player II.

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If this were not true there would exist by Lemma 2.3 a II-imposed subgame $[\Gamma'_x]_E$ of Γ'_x which is no loss for player II and such that $E \cap O(B^c \cap C_{i+1}) = \emptyset$. We may further modify E so that the first time a position y in $B \cap C_{i+1}$ is reached, player II plays in the II-imposed subgame D_y (the existence of which follows from y being in B) which is no loss for him, and where $D_y \cap H = \emptyset$.

Clearly, the resulting play will either not pass through a point of C_{i+1} (and therefore not be in H) <u>or</u> it will pass through a point of $B \cap C_{i+1}$ and also not be in H. Since this modified II-imposed subgame is not a loss for player II, $x \in B$ contrary to the hypothesis, and (A_i) follows.

Suppose that the initial position x_0 of the game belongs to B^c, i.e., $x_0 \in B^c \cap C_0$ contrary to the conclusion of Lemma 2.4. Then by (A_0) player I has a winning strategy in $\Gamma'(S'_I = S_I \cup O\{B^c \cap C_1\})$. Player I can use that strategy until a position in $B_c \cap C_1$ is reached. He then can shift to the winning strategy assured by (A_1) in $\Gamma''(S'_I = S_I \cup O\{B^c \cap C_2\})$ until (and if) a position in $B^c \cap C_2$ is reached, when he shifts to the strategy guaranteed by (A_2) etc. The resulting play must pass through x_0, $x_1, x_2, \ldots$ where $x_i \in C_i$ and hence $s \in \bigcap_{i=0}^{\infty} O(C_i) = H \subseteq S_I$ or $s \notin O(C_i)$ for some i and so $s \in S_I$. So player I has a win contrary to the hypothesis. Hence $x_0 \in B$ and Lemma 2.4 follows.

THEOREM 2.1. If $S_I \in G_{\delta\sigma} \cup F_{\sigma\delta}$, then Γ is determined.

PROOF. Since the two players play essentially dual roles in this game, we may assume $S_I \in G_{\delta\sigma}$. Let $S_I = \bigcup_{i=1}^{\infty} H_i$ where $H_i \in G_\delta$.

Assume Γ is not a loss for player II. Then by Lemma 2.2 there is a II-imposed subgame Γ_{D_1} of Γ such that $D_1 \cap H_1 = \emptyset$ and Γ_{D_1} is not a loss for player II.

In general, if Γ_{D_i} is a II-imposed subgame of Γ such that $D_i \cap H_i = \emptyset$ and Γ_{D_i} is not a loss for player II, there is another II-imposed subgame of this subgame (and hence of the original game), $\Gamma_{D_{i+1}}$, such that $D_{i+1} \cap H_{i+1} = \emptyset$, and $\Gamma_{D_{i+1}}$ is no loss for player II. Now $D = \bigcap_{i=1}^{\infty} D_i$ is a non-null closed set, and determines a II-imposed subgame Γ_D where $D \cap S_I = \emptyset$, so player II has a win. Q.E.D.

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§ 3. EXAMPLES OF NON-DETERMINED GAMES

In the Gale-Stewart paper [2], the definition of absolutely-determined games is made and certain questions are raised concerning it. In particular, the question is asked if the property of absolute determinacy is invariant under unions and intersections of the winning sets. (If this were true for countably many unions and intersections of winning sets, this would imply games that had Borel sets as winning sets would be absolutely determined since it is shown that closed and open winning sets form absolutely determined games.) Unfortunately, this is not the case; in the next section this will be shown and an answer to another question raised by Gale and Stewart will be given.

DEFINITION 3.1. $\underline{P_\sigma}$ is the set of all plays that may occur if player I uses strategy $\underline{\sigma}$.

DEFINITION 3.2. $\underline{P^\tau}$ is the set of all plays that may occur if player II uses the strategy $\underline{\tau}$.

DEFINITION 3.3. A vertex $x \in X$ is <u>a</u> <u>win</u> (<u>loss</u>) <u>for</u> <u>player</u> <u>I</u> (<u>II</u>) if and only if the subgame Γ_x is <u>a</u> <u>win</u> (<u>loss</u>) <u>for</u> <u>player</u> <u>I</u> (<u>II</u>).

DEFINITION 3.4. A vertex $x \in X$ is <u>undetermined</u> if and only if it is not a win for either player I or II.

DEFINITION 3.5. A play $s \in S$ is <u>plausible</u> if and only if one of these three statements is true:

(i) $\Gamma_{s(i)}$ is a win for player I for all i,

(ii) $\Gamma_{s(i)}$ is a win for player II for all i,

(iii) $\Gamma_{s(i)}$ is undetermined for all i.

REMARK. The set of all plausible plays, $\tilde{P}$, is a closed set.

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DEFINITION 3.6. $\Gamma_{\widetilde{P}}$ the subgame determined by the closed set $\widetilde{P}$, will be called the <u>plausible subgame</u> of Γ.

DEFINITION 3.7. A game is <u>absolutely undetermined</u> if and only if x is undetermined for all $x \in X$.

DEFINITION 3.8. <u>Γ^*</u> will denote the following game (mentioned in Gale-Stewart):

X_I = {binary sequences with an even number of bits}

X_{II} = {binary sequences with an odd number of bits}

S = {set of infinite binary sequences}

$f(x)$ = carries finite binary sequences into the same finite binary sequence with the last bit removed. (A bit is a binary digit, i.e., 0 or 1.)

REMARK. We note the following obvious facts:

If $x \in X_I$: x a loss for player I implies for all $y \in f^{-1}(x)$, y a loss for player I; x undetermined implies there is no $y \in f^{-1}(x)$ which is a win for player I <u>and</u> there exists some $y \in f^{-1}(x)$ which is not a loss for player I; x a win for player I implies there exists a $y \in f^{-1}(x)$ which is also a win for player I; σ a winning strategy for player I, $s \in P_\sigma$, implies $s(i)$ is a win for player I for all i.

DEFINITION 3.9. $s \in S$ is <u>consistent</u> with $\sigma(\tau)$ if and only if $s \in P_\sigma$ $(s \in P^\tau)$.

DEFINITION 3.10. $x \in X$ is <u>consistent</u> with $s \in S$ if and only if $s(i) = x$ for some i.

DEFINITION 3.11. $x \in X$ is <u>consistent</u> with $A \subseteq S$ if and only if x is consistent with s for some $s \in A$.

DEFINITION 3.12. $\sigma(\tau)$ is consistent with Γ_C if and only if for all $x \in X_I$ $(x \in X_{II})$ such that x is

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consistent with C, $\sigma(x)$ ($\tau(x)$) is also consistent with C.

DEFINITION 3.13. A <u>mesh</u> of a strategy σ is a subset Q of P_σ such that there is a $1-1$ correspondence between each x consistent with P_σ and some s, consistent with x, in Q. $s(\sigma, x, Q)$ will denote that $s \in Q$ is associated with x.

DEFINITION 3.14. A game is <u>semi-absolutely determined</u> if and only if every subgame is determined.

DEFINITION 3.15. The <u>negative game</u> of a game is the identical game with S_I, S_{II} replacing S_{II}, S_I, respectively.

DEFINITION 3.16. A set $M \subseteq S$ has a <u>countable</u> $\sigma(\tau)$ <u>span</u> if and only if there exists a finite or countable number of σ's (τ's) such that

$$M \subseteq \bigcup_{i=1}^{\infty} P_{\sigma_i} \quad \left(M \subseteq \bigcup_{i=1}^{\infty} P^{\tau_i}\right).$$

LEMMA 3.1. If (1) $\Gamma = \Gamma^*$, (2) $P_\sigma \subseteq \left(\bigcup_{i=1}^{\infty} s_i\right) \cup \left(\bigcup_{i=1}^{\infty} P_{\sigma_i} \cap C_i\right)$, and (3) Q_i = (a fixed mesh of σ_i) $\cap\, C_i$, then there exists $\underline{s}$ such that $\underline{s} \in P_\sigma \cap Q_i$ for some i.

PROOF. Assume the contrary; i.e., $P_\sigma \cap Q_i = \emptyset$ (for all i). Construct $\underline{s}$ as follows:

<u>Step 1</u>: Let $\underline{s}(0) = x_0$; let $n_1 = \min\limits_k$ such that $\sigma(x_0) = \sigma_k(x_0)$; let $s^1 = s(\sigma_{n_1}, x_0, Q_{n_1})$.

By hypothesis, $s^1 \notin P_\sigma$; therefore there exists $x \in X_I$ consistent with s^1 and such that $\sigma(x) \neq \sigma_{n_1}(x)$; define $\underline{s}(i) = s^1(i)$ for all i such that $i \leq i_1$ where $s^1(i_1) = x$; take $\underline{s}(i_1 + 1) = \sigma(x_1)$, $s_1(i_1 + 2) \neq \underline{s}(i_1 + 2) = x_1$.

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Step m: Assume $\underline{s}(i)$ has been defined for all $i \leq i_{m-1} + 2$; define $n_m = \min_k \{k \mid x_{m-1}$ is consistent with σ_k and $\sigma_k(x_{k-1})\}$. (If there is not such a k, assumption (2) is false, since it cannot happen that

$$P_\sigma \cap u(x) \subseteq \bigcup_{i=1}^{\infty} s_i$$

for any x consistent with σ.)

Since $s^m = s(\sigma_{n_m}, x_{k-1}, Q_{n_m}) \notin p_\sigma$, by hypothesis, there exists $x \in X_I$ consistent with s^m such that $\sigma(x) \neq \sigma_{n_m}(x)$. Define $\underline{s}(i) = s^m(i)$ for all $1 \leq i_m$ where $s^m(i_m) = x$; take $\underline{s}(i_m + 1) = \sigma(x)$ and $x_m = \underline{s}(i_m + 2) \neq s_m(i_m + 2)$.

The resulting play $\underline{s}$ is inconsistent with each s_i, σ_i, which contradicts hypothesis (2). Q.E.D.

LEMMA 3.2. If Γ is any Gale-Stewart game where S is compact and if there exists a countable set $\bigcup_{i=1}^{\infty} s_i \subseteq S_{II}$ such that

$$P_\sigma \cap \left[\bigcup_{i=1}^{\infty} s_i\right] \neq \emptyset$$

for all strategies σ for player I, then player II has a win.

PROOF. Player I has a win for neither the above game nor the game Γ' where $S_{II} = \bigcup_{i=1}^{\infty} s_i$. But the above S_{II} set is a F_σ set and therefore Γ' is strictly determined and player II must have a win for the game Γ', and a fortiori, for the original game Γ. Q.E.D.

LEMMA 3.3. If x_0 is an undetermined vertex of Γ, then $\Gamma_{\tilde{p}}$, the plausible subgame of Γ, is absolutely undetermined.

PROOF. Assume the contrary, i.e., that there exists $\sigma_{\tilde{p}}$, a winning strategy for $\Gamma_{\tilde{p}}$. For each $x \in X_I$ where player I has a win, we associate σ_x : a winning strategy for player I in Γ_x.

For each $x \in X_I$, x an undetermined vertex, x consistent with

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$\sigma_{\tilde{P}}$, let $\sigma(x) = \sigma_{\tilde{P}}(x)$.

For each $x \in X_I$, x a winning vertex for player I, we will take $\sigma(x) = \sigma_z(x)$ where $z = f^n(x)$ and $n = \max_n \{m \mid \Gamma_{(f^m(x))}$ is a win for player I$\}$.

For all other $x \in X_I$, define $\sigma(x)$ arbitrarily.

Let $s \in P_\sigma$: If (1) $s \in \tilde{P}$ then $\sigma = \sigma_{\tilde{P}}$ and $s \in S_I$; (2) $s \notin \tilde{P}$, take first i such that $x = s(i)$ is determined. x is in X_{II} since $\sigma_{\tilde{P}}$ carries $\{x \mid x$ consistent with $\tilde{P}\}$ into itself. Also, it is clear that x must be a win for player I. Therefore, $P_\sigma \subseteq S_I$, σ is a winning strategy, which contradicts the hypothesis. Q.E.D.

LEMMA 3.4. (a) If $S_I \subseteq \bigcup_{i=1}^{\infty} P_{\sigma_i}$, then the game Γ is determined.

(b) If $S_I \subseteq \bigcup_{i=1}^{\infty} P^{\tau_i}$, then the game Γ is determined.

PROOF of (a). Assume the contrary. Then Γ is undetermined and so is $\Gamma_{\tilde{P}}$; in the game $\Gamma_{\tilde{P}}$, $(S_I) \subseteq \bigcup_{i=1}^{\infty} P_{\sigma_{\tilde{P}_i}}$. Now, given $x \in (X_{II})_{\tilde{P}}$ there exists $j = \min_i$ such that x is consistent with $\sigma_{\tilde{P}_i}$. Since Γ_x is undetermined for all $x \in X$, there exists $\tau_{\tilde{P}_j}$ such that $< \sigma_{\tilde{P}_j}, \tau_{\tilde{P}_j} > \in S_{II}$. Let $\tau_{\tilde{P}}(x) = \tau_{\tilde{P}_j}(x)$. It is clear that for $s \in P^{\tau_{\tilde{P}}}$, <u>either</u> $s \notin \bigcup_{i=1}^{\infty} P_{\sigma_{\tilde{P}_j}}$, and therefore $s \in S_{II}$, <u>or</u> there exists $\min j$ such that $s \in P_{\sigma_{\tilde{P}_j}}$, and hence $s = < \sigma_{\tilde{P}_j}, \tau_{\tilde{P}_j} > \in S_{II}$. Hence player II has a win which contradicts the hypothesis by Lemma 3.3.

PROOF of (b). Assume the contrary, i.e., that Γ, and therefore $\Gamma_{\tilde{P}}$, is undetermined. Also, it is clear that $(S_I \cap P \subseteq \bigcup_{i=1}^{\infty} P_{\sigma_{\tilde{P}_i}}$. With each $x \in X_{II}$ where player II has two choices, take $j = \min_i$ where x is consistent with τ_i, and take $\tau(x) \neq \tau_i(x)$. Since $(\Gamma_{\tilde{P}})_x$ is undetermined for all x it follows that for all $s \in P$, there exist an infinite number of $x \in X_{II}$ such that player II has two choices. Hence, $s \in S_{II}$, which contradicts the hypothesis.

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REMARK. It is clear that the roles of σ and τ may be interchanged and a similar theorem obtained.

LEMMA 3.5. P_σ is a closed set.

PROOF. $P_\sigma = S - \bigcup_{x \in A} u(x)$; $A = \{x \mid x$ not consistent with $\sigma\}$.

LEMMA 3.6. If Γ^* is a win for player I, $S_I = \bigcup_{x \in A} u(x)$, then there exists a finite subset of A, B where $S_I = \bigcup_{x \in B} u(x)$ still is a win for player I.

PROOF. This follows directly from: $u(x)$ is open (a base set), S is compact, P_σ is a closed subset of a compact set, and the Heine-Borel theorem.

Answer to Question 2 of Gale and Stewart.

We will consider three games of the form Γ^*, differing only in the sets S_I, as follows:

(1) Γ' where S_I' is the winning set for player I,

(2) Γ'' where S_I'' is the winning set for player I,

(3) Γ where $S_I' \cap S_I''$ is the winning set for player I,

We will define S_I', S_I'', so that Γ' and Γ'' are absolutely determined games but Γ is undetermined, thus answering the question raised by Gale and Stewart (by symmetry the same could be obtained for union).

It was shown by Gale and Stewart that the set of strategies for player I or II is of the power of the continuum. We note that the cardinal number of the set of all closed sets is also of the power of the continuum since the space is Hausdorff, has a countable basis, and has an infinite number of points.

The first thing we do is associate the σ's, the τ's, and the C's (the closed sets) with the set of ordinal numbers less than Ω (the first uncountable ordinal). We then proceed by induction over these ordinals, executing the following steps:

Step 1:

(i) With σ_1, the first strategy for player I, take an arbitrary

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strategy for player II, τ_{σ_1}, and set $< \sigma_1, \tau_{\sigma_1} > = s_{1a}$ into S'_{II}.

(ii) With τ_1, take $< \sigma_{\tau_1}, \tau_1 > = s_{1b}$ $(s_{1b} \neq s_{1a})$ and assign s_{1b} to $S'_I \cap S''_I$.

(iii) Consider C_1, the first closed set, and the associated subgame Γ_{C_1}. (In (iii) to (vi) we will consider all notation as pertaining to the subgame Γ_{C_1}.)

We ask whether there exists a strategy for player I, σ_1^*, such that $P_{\sigma_1^*}$ has no points thus far assigned to S'_{II}, and that for each x consistent with σ_1^* there exists an infinite number of points in $\mathcal{u}(x) \cap P_{\sigma_1^*}$ not yet assigned to S''_1. If the answer is yes to the above question, choose one such σ_1^*, assign $P_{\sigma_1^*}$ to S'_I and assign Q_i to S''_{II} where Q_i is some mesh of σ_1^*; then go to (iv). If the answer is no, leave out (iii) and go directly to (iv).

(iv) If there exists σ_1^{**} such that no point of $P_{\sigma_1^{**}}$ has yet been assigned to S''_{II}, assign $P_{\sigma_1^{**}}$ to S''_I and go to (v); if not, go to (v) directly.

(v) If there exists τ_1^* such that no point of $P^{\tau_1^*}$ has yet been assigned to S'_{II}, assign $P^{\tau_1^*}$ to S'_I and go on to (vi); if not, go to (vi) directly.

(vi) If there exists τ_1^{**} such that no point of $P^{\tau_1^{**}}$ has yet been assigned to S''_{II}, assign $P^{\tau_1^{**}}$ and go to Step α (i) directly.

Step α (the inductive step):

(i) If $P_{\sigma_\alpha} \leq \left[\bigcup_{K<\alpha} s_{Kb}\right] \cup \left[\bigcup_{K<\alpha} < \sigma_\alpha, \tau_K^* >\right] \cup \left[\bigcup_{K<\alpha} (P_{\sigma_K} \cap C_1)\right]$ take $s_{\alpha a} = < \sigma_\alpha, \tau_{\alpha_a} >$ which is not in the above union and assign it to S'_{II}. If the inclusion does hold, we note that the above union consists of a countable number of points and $\bigcup_{K<\alpha} (P_{\sigma_K^*} \cap C_1)$.

By Lemma 3.1 there exists a point in Q_j for some j, which also lies in P_{σ_α}; hence there exists $s \in P_{\sigma_\alpha}$ already assigned to S''_{II}.

(ii) At each step there are only a countable number of points assigned to $S'_{II} \cup S''_{II}$, so we may assign $s_{ab} = < \sigma_{\tau_\alpha}, \tau_\alpha >$ to $S'_I \cap S''_I$.

(iii), (iv), (v), and (vi) are conditional, so we may do these without diffifulty, substituting the subscript α for the subscript 1 (used in Step 1), where appropriate.

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It is clear from the previous construction that the game Γ, with winning set for player I, $S_I = S'_I \cap S''_I$, is undetermined. It remains only to show the absolute determinacy of the two games Γ' and Γ''.

Consider C_α: Consider the negative of the subgame Γ'_{C_α}, $\overline{\Gamma}$. By construction, there existed at the α^{th} step a τ^*_α such that $P^{\tau^*_\alpha}$ was assigned to S'_I, or, since there had been only a countable number of points (s_{1a}) assigned to S'_{II}, by Lemma 3.2 there must have been a sufficient number of points assigned to S'_{II} to insure a win for player I. In any case the negative subgame is determined.

We may show in precisely the same way that the negative of subgame Γ'' is also semi-absolutely determined.

Again consider C_α. Assume Γ'_{C_α} is an undetermined subgame. Let $C_\beta = P$ and $\Gamma_{C_\beta} = \Gamma_P$, the plausible subgame of Γ'_{C_β}. Now, at the β^{th} step there must have existed σ^*_β, such that no element of $P_{\sigma^*_\beta}$ had yet been assigned to S'_{II}, and with each x consistent with σ^*_β and C_β, there existed an infinite number of plays (also consistent with σ^*_β and C_β) not yet assigned to S''_I. This follows from Lemmas 3.3 and 3.4 (since $[\Gamma_P]_x$ is undetermined for all x), and hence, σ^*_β is a win for Γ_{C_β}. But this contradicts Lemma 3.3, which states the plausible subgame of an undetermined game is undetermined. So Γ' is semi-absolutely determined.

By a method analogous to the proof of the negative determinacy of Γ', we may show that the game Γ'' is also semi-absolutely determined. Therefore, both games, Γ' and Γ'', are absolutely determined, and the proof is complete.

REMARK. Those points not assigned to any set (or assigned to a set in the game $\Gamma'(\Gamma'')$, but not assigned in the game $\Gamma''(\Gamma')$) may be assigned arbitrarily without affecting the construction.

<u>Answer to Question 1 of Gale and Stewart</u>:

Again we use the game Γ^*. We first well-order the σ's, τ's, and the closed sets C, putting them, as we did in the first construction, into $1-1$ correspondence with the set of ordinal numbers less than Ω, the first uncountable ordinal number. We then construct a game whose negative is

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semi-absolutely determined, but which itself is undetermined.

Step 1:

(i) With σ_1 we associate the play $s_1' = < \sigma_1, \tau_{\sigma_1} >$ and assign it to S_{II}.

(ii) With τ_1 we associate the play $s_1'' = < \sigma_{\tau_1}, \tau_1 >$ and assign it to S_I.

(iii) Consider C_1. If there exists a σ_1' consistent with C_1 such that no point of P'_{σ_1} has yet been assigned to S_I, choose one such σ_1', and assign P'_{σ_1} to S_{II} and go on to Step α; if there does not exist such a σ_1', go to Step α directly.

Step α:

(i) We note that any countable step there has been only a countable number of points assigned to S_I; therefore, there must exist $s_\alpha' = < \sigma_\alpha, \tau_{\sigma_\alpha} >$ not yet assigned to S_I (since P_{σ_α} is uncountable). Assign s_α' to S_{II}.

(ii) We note first there have been only a countable number of points consistent with τ_α, assigned thus far to S_{II}:

$$\left[\bigcup_{K<\alpha} < \sigma_K', \tau_\alpha >\right] \cup \left[\bigcup_{K<\alpha} s_K'\right] .$$

Since there are an uncountable number of points in P^{τ_α}, it follows that there must exist $s_\alpha'' = < \sigma_{\tau_\alpha}, \tau_\alpha >$ not yet assigned to S_{II}; assign s_α'' to S_I.

(iii) This step (where α replaces 1 in the appropriate places of Step I (iii)) is conditional so there is no difficulty here.

We assign all points in S not assigned by the induction, arbitrarily, and note the game is undetermined. We show the negative game is semi-absolutely determined as follows:

Let C_α be an arbitrarily closed set. At Step α (iii), either we assigned P'_{σ_α} to S_{II}, assuring a win for player I, or (since only a countable number of points, $\bigcup_{K<\alpha} s_K''$, have thus far been assigned to S_I) the hypotheses of Lemma 3.2 are met, and the subgame is determined. Therefore, the negative of the original game is semi-absolutely determined and the construction is complete.

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§ 4. THE UNSYMMETRIC GAME

Consider the following game, suggested by L. E. Dubins:

Let S be the set of all infinite binary sequences, and divide S into two complementary sets S_I and S_{II}, the winning sets for players I and II, respectively, as in the earlier sections. Also, as before, X is the set of all finite binary sequences, and $u(x) = \{s \in S \mid s(i) = x$ for some $i\}$. (Here $s(i)$ denotes the i^{th} initial segment of the play s.) P^{τ} will again designate the set of plays that may occur if player II uses strategy τ, play being in the sense of sequence s, without regard to history.

Player I plays first and chooses any binary sequence of length n_1, where n_1 is a finite integer chosen by player I. Player II may then choose a 0 or a 1 to be the next bit (binary digit) of the sequence. Player I may now choose a finite sequence (say of length n_2) of bits and add it on to the $n_1 + 1$ bits already formed. The players alternate in this manner, and the final infinite sequence will lie in either set S_I or set S_{II}, in which case player I or II will win, respectively.

In the earlier games considered, if we were given a sequence, the way in which it was formed (i.e., which player chose which bits) was determined. Now a sequence may be formed in many ways, and, in general, a strategy may dictate at a position x different choices depending on how x was formed.

DEFINITION 4.1. A <u>history</u> $h = (x ; f)$ is an ordered pair pair consisting of x, a finite sequence of bits, and f, the way in which x was formed (x is said to be formed by f).

DEFINITION 4.2. A strategy τ is said to be <u>independent</u> at x if $\tau(x ; f)$ is invariant with respect to f, i.e., $\tau(x ; f_1) = \tau(x ; f_2)$ for all f_1, f_2 that form x. The sequence x_1 followed by x_2 will be denoted by (x_1, x_2). If x_1 is a finite sequence formed by f_1 followed by x_2 formed by f_2, we may write the total history $h = (x_1, x_2 ; f_1, f_2)$. (We assume the move of

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one of the players actually starts at the beginning of x_2.)

DEFINITION 4.3. A strategy, τ, is said to be a positional strategy if τ is independent at x for all x in X.

We now show that in the above defined game, player I has a win if and only if S_I contains a perfect set and player II has a win if and only if S_I has at most a countably infinite number of points. This characterizes the winning sets for both players, and from the existence of sets which have a non-countable number of points and contain no perfect sets one infers the existence of undetermined games.

THEOREM 4.1. Player I has a win if and only if S_I contains a perfect set.

PROOF. Let S_I contain a perfect set P.

Let $M = \{x \mid \mathcal{U}(x, 1) \cap P$ and $\mathcal{U}(x, 0) \cap P$ are both perfect$\}$.

We note $\mathcal{U}(x) \cap P$ is null or perfect for any x in X since $\mathcal{U}(x)$ is both open and closed. Also M is not null, for if we take 2 points in P and take x to be the initial portion they have in common, we find this x is in M. Therefore, if player I chooses an x in M initially, $\mathcal{U}(x, y) \cap P$ is perfect (where y is 0 or 1 depending upon player II's choice), and as before player I may choose another x in M, etc. Continuing thus, we obtain a point in closure $(P) = P$, and therefore in S_I.

On the other hand, if player I has a winning strategy σ, it is trivial to verify that P_σ (Def. 3.1) is perfect.

LEMMA 4.1. Let τ_1 be any strategy for player II and let x_1 be a fixed finite sequence. Then player II has a strategy τ_2 such that $P^{\tau_1} \supseteq P^{\tau_2}$ (Def. 3.2), where τ_2 is independent at x. We will call τ_2 a refinement of τ_1 at x_1.

PROOF. Assume there are f_1, f_2 such that $\tau_1(x_1 ; f_1) \neq \tau_1(x_1 ; f_2)$.

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(If this is not the case, τ_1 is its own refinement with respect to x_1.) Pick a fixed f_1 which form x_1.

Define τ_2 as follows:

$$\tau_2(x_1 ; f) = \tau_1(x_1 ; f_1),$$

$$\tau_2(x_1, x_2 ; f, f_2) = \tau_1(x_1, x_2 ; f_1, f_2),$$

if player II chose the first bit of x_2;

otherwise, define τ_2 as

$$\tau_2(x ; f) = \tau_1(x ; f).$$

In effect, τ_2 dictates the same plays as τ_1, <u>except</u> when a history arises where the initial sequence is x_1 and player II made the following move, in which case player II "forgets" how x_1 was actually formed and plays as though the initial sequence x_1 was formed by f_1. It is easy to see that τ_2 is a new strategy meeting the requirements of the lemma.

LEMMA 4.2. If player II has a winning strategy, then he has a winning positional strategy.

PROOF. Assume τ_0 is a winning historical strategy and let x_1, x_2, ... be the positions at which τ is not independent and such that if $i > j$ then the length of $x_i \geq$ the length of x_j. Let τ_i be defined inductively as a refinement of τ_{i-1} at x_i. If we define $\tau(x) = \tau_i(x ; f)$, where i is so large that length $x_i >$ length x, then $P^{\tau} \subseteq P^{\tau_0}$ (by construction) and $P^{\tau_0} \subseteq S_{II}$ (by hypothesis), so $P^{\tau} \subseteq S_{II}$, and τ is a winning positional strategy for player II.

THEOREM 4.2. Player II has a win if and only if S_I is at most countably infinite.

PROOF. If S_I is countably infinite, i.e., $S_I = \{s_1, s_2, \ldots\}$, player II may make at his k^{th} turn to move, a choice that precludes the

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formation of s_k, which assures the resulting play being in S_{II}.

If player II has a winning historical strategy he has a winning positional strategy, say τ, by Lemma 4.2.

Each s in S_I has the property that for some N (dependent on s), $i \geq N$ implies $\tau(s(i)) \neq s(i+1)$. Define $N(s)$ to be the least such N. (If this were not the case there would exist an s in S and an infinite number of n_i, $i = 1, 2, 3, \ldots$, such that $\tau(s(n_i)) = s(n_i + 1)$ for all i. If player I chooses his move so that it terminates in one such $s(n_i)$, the resulting play would be s in S_I, which contradicts the hypothesis that τ is a winning strategy for player II.)

The set $B_K = \{s \mid s \text{ in } S_I \text{ and } N(s) \leq K\}$ is finite since for any x of length K there exists at most one s in B_K whose initial sequence is x (for $i > K$, $s(i) \neq \tau[s(i-1)]$, and there exist at most 2^K different sequences of length K. $S_I \subseteq \bigcup_{K=1}^{\infty} B_K$, so S_I is at most countable and Theorem 4.2 follows.

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