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INVERSE LIMITS OF COMPACT SPACES

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This paper gives conditions under which the inverse limit of a system of compact (but non-Hausdorff) spaces will be non-empty, or compact, or hereditarily compact. The main result (Theorems 3 and 5) is that, if the spaces are compact, T_0 and non-empty and the maps are closed and continuous, then the inverse limit is compact and non-empty (and, trivially, T_0). Simple examples are given to show that the results are reasonably sharp.

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inverse system	hereditarily compact
inverse sequence	countably directed
inverse limit	compact

1. Introduction

The inverse limits of systems of *hereditarily* compact spaces (and continuous maps) are sometimes of interest to algebraists and algebraic geometers; it is desirable to know conditions under which the inverse limit will be (a) non-empty, or (b) compact, or (c) hereditarily compact. Since the spaces here will usually not be Hausdorff, the traditional method (as in [1, p. 217]) does not apply (see Example 5 at the end of the paper). In studying such systems one finds that the hereditariness of the compactness is usually superfluous in the hypotheses of the theorems, and unobtainable in their conclusions. Thus we shall study, more generally, inverse limits of compact (non-Hausdorff) spaces. On the other hand, it seems to be necessary to require more than mere continuity of the bonding maps; we shall often assume them to be closed (which, of course, holds automatically in the Hausdorff case). We shall also often assume that the spaces are T_0. However, all such assumptions will be stated explicitly.

Notation. In this paper, a space is "compact" if it is "quasicompact" in the sense of Bourbaki: that is, every open cover has a finite subcover. A *proper* map $f : X \to Y$ is one that is closed and continuous and such that each $f^{-1}(y)$, $y \in Y$, is compact in the present sense. It follows in the usual way that each set $f^{-1}(K)$, where K is a compact subset of Y, is compact.

An *inverse system* $\mathscr{X}=\{(X_\lambda, f_{\lambda\mu}): \Lambda\}$ of spaces is a collection of topological spaces X_λ, indexed by a directed set $(\Lambda, <)$, with continuous maps $f_{\lambda\mu}: X_\lambda \to X_\mu$ defined whenever $\lambda, \mu \in \Lambda$ and $\lambda > \mu$, in such a way that (for $\lambda > \mu > \nu$) $f_{\mu\nu} \circ f_{\lambda\mu} = f_{\lambda\nu}$. The limit, $\varprojlim \mathscr{X}$, is the subspace of $\prod\{X_\lambda: \lambda \in \Lambda\}$ consisting of all points x such that $f_{\lambda\mu}(x_\lambda) = x_\mu$ $(\lambda, \mu \in \Lambda, \lambda > \mu)$. In the special case of an inverse *sequence*, we use the notation $\mathscr{X}=\{(X_n, f_n): \mathbb{N}\}$, where $\mathbb{N}$ is the set of positive integers (in its usual ordering) and $f_n: X_{n+1} \to X_n$ is a continuous map; it is understood that (for $m > n$) $f_{mn} = f_n \circ f_{n+1} \circ \cdots \circ f_{m-1}$.

We shall say that an inverse system $\mathscr{Y}$ is a *subsystem* of $\mathscr{X}=\{(X_\lambda, f_{\lambda\mu}): \Lambda\}$ providing $\mathscr{Y}=\{(Y_\lambda, g_{\lambda\mu}): \Lambda\}$ is indexed by the same directed set $(\Lambda, <)$, and each Y_λ is a subspace of X_λ, and $g_{\lambda\mu} = f_{\lambda\mu} \mid Y_\lambda$ $(\lambda, \mu \in \Lambda, \lambda > \mu)$; note that this implies $f_{\lambda\mu}(Y_\lambda) \subset Y_\mu$. If each Y_λ is a *closed* subspace of X_λ $(\lambda \in \Lambda)$ then we call $\mathscr{Y}$ a *closed subsystem* of $\mathscr{X}$. It is easy to see that, when $\mathscr{Y}$ is a closed subsystem, each $g_{\lambda\mu}$ is closed if $f_{\lambda\mu}$ is closed, and proper if $f_{\lambda\mu}$ is proper.

I am grateful to Roy Douglas for calling my attention to questions of this nature.

2. Reduction to the surjective case

We begin by showing that, when the bonding maps are closed, the situation can be reduced to the case in which they are also surjective. We recall that they are always assumed to be continuous.

Theorem 1. *Let $\mathscr{X}=\{(X_\lambda, f_{\lambda\mu}): \Lambda\}$ be an inverse system of non-empty compact spaces X_λ and closed maps $f_{\lambda\mu}$ $(\lambda, \mu \in \Lambda, \lambda > \mu)$. Then there is a closed subsystem $\mathscr{Y}=\{(Y_\lambda, g_{\lambda\mu}): \Lambda\}$ of $\mathscr{X}$, consisting of nonempty compact spaces Y_λ and closed surjective maps $g_{\lambda\mu}$ $(\lambda, \mu \in \Lambda, \lambda > \mu)$ such that $\varprojlim \mathscr{Y} = \varprojlim \mathscr{X}$.*

Remark. When $\mathscr{Y}$ is a subsystem of $\mathscr{X}$, the equality of $\varprojlim \mathscr{Y}$ and $\varprojlim \mathscr{X}$ as *sets* implies their equality as *spaces*, because $\prod\{Y_\lambda: \lambda \in \Lambda\}$ will be a subspace of $\prod\{X_\lambda: \lambda \in \Lambda\}$.

Proof. Let Φ denote the family of all closed subsystems $\mathscr{Y}$ of $\mathscr{X}$ such that

(a) the spaces Y_λ $(\lambda \in \Lambda)$ are all non-empty,

(b) $\varprojlim \mathscr{Y} = \varprojlim \mathscr{X}$ (for instance, $\mathscr{Y}=\mathscr{X}$).

Partially order Φ by defining $\mathscr{Z} \leq \mathscr{Y}$ (where $\mathscr{Y}, \mathscr{Z} \in \Phi$) to mean that $\mathscr{Z}$ is a subsystem (necessarily closed) of $\mathscr{Y}$. To show that Zorn's Lemma applies, suppose Ψ is a linearly ordered subfamily of Φ; we show Ψ has a lower bound in Φ. For each $\lambda \in \Lambda$, put $L_\lambda = \bigcap\{Y_\lambda: \mathscr{Y} \in \Psi\}$; this is a closed subset of X_λ, and it is nonempty because X_λ is compact and the sets Y_λ, $\mathscr{Y} \in \Psi$, are closed in X_λ and have the finite intersection property. If $\lambda, \mu \in \Lambda$ and $\lambda > \mu$, then

$$f_{\lambda\mu}(L_\lambda) \subset \bigcap\{f_{\lambda\mu}(Y_\lambda): \mathscr{Y} \in \Psi\} \subset \bigcap\{Y_\mu: \mathscr{Y} \in \Psi\} = L_\mu.$$

Thus, on defining $l_{\lambda\mu} = f_{\lambda\mu} | L_\lambda$, we have that $\mathscr{L} = \{(L_\lambda, l_{\lambda\mu}) : \Lambda\}$ is a closed subsystem of $\mathscr{X}$. Trivially $\varprojlim \mathscr{L} \subset \varprojlim \mathscr{X}$; conversely, given $x \in \varprojlim \mathscr{X}$, we have $x_\lambda \in Y_\lambda$ for each $\lambda \in \Lambda$ and each $\mathscr{Y} \in \Psi$ (because $x \in \varprojlim \mathscr{Y}$), so that $x_\lambda \in L_\lambda$. Since $l_{\lambda\mu}(x_\lambda) = f_{\lambda\mu}(x_\lambda) = x_\mu$ whenever $\lambda > \mu$, we have $x \in \varprojlim \mathscr{L}$. Thus $\mathscr{L} \in \Phi$; and clearly $\mathscr{L} \leqslant \mathscr{Y}$ for all $\mathscr{Y} \in \Psi$.

So, by Zorn's Lemma, there is a *minimal* $\mathscr{Y} \in \Phi$. To complete the proof, we have only to show that the corresponding maps $g_{\lambda\mu} : Y_\lambda \to Y_\mu$ $(\lambda > \mu)$ are surjective.

Define (for all $\lambda \in \Lambda$) $Z_\lambda = \bigcap \{g_{\nu\lambda}(Y_\nu) : \nu > \lambda\}$; this is a closed subset of Y_λ, because the maps $g_{\nu\lambda}$ are closed. One readily verifies that, if $\nu > \mu > \lambda$, then $g_{\nu\lambda}(Y_\nu) \subset g_{\mu\lambda}(Y_\mu)$, so the sets $g_{\nu\lambda}(Y_\nu)$ for $\nu > \lambda$ form a decreasing directed system of nonempty closed subsets of the compact space Y_ν. Hence their intersection, Z_λ, is also non-empty – and, of course, compact.

We shall show that, if $\lambda > \mu$, then $g_{\lambda\mu}(Z_\lambda) \subset Z_\mu$. In fact,

$$g_{\lambda\mu}(Z_\lambda) = g_{\lambda\mu}(\bigcap \{g_{\rho\lambda}(Y_\rho) : \rho > \lambda\}) \subset \bigcap \{g_{\lambda\mu}(g_{\rho\lambda}(Y_\rho)) : \rho > \lambda\}$$
$$= \bigcap \{g_{\rho\mu}(Y_\rho) : \rho > \lambda\},$$

and we show that this $\subset$ (in fact $=$) $\bigcap \{g_{\rho\mu}(Y_\rho) : \rho > \mu\} = Z_\mu$. Suppose $y_\mu \in \bigcap \{g_{\rho\mu}(Y_\rho) : \rho > \lambda\}$; it is enough to prove $y_\mu \in g_{\nu\mu}(Y_\nu)$ for every $\nu > \mu$. Given such a ν, take $\rho \in \Lambda$ such that $\rho > \lambda$ and $\rho > \nu$. Then $y_\mu \in g_{\rho\mu}(Y_\rho)$, so take $y_\rho \in Y_\rho$ such that $g_{\rho\mu}(y_\rho) = y_\mu$, and define $y_\nu = g_{\rho\nu}(y_\rho)$. We have $y_\nu \in Y_\nu$ and $g_{\nu\mu}(y_\nu) = g_{\nu\mu}(g_{\rho\nu} y_\rho) = g_{\rho\nu}(y_\rho) = y_\mu$, proving $y_\mu \in g_{\nu\mu}(Y_\nu)$ as required.

Hence, on defining $h_{\lambda\mu} = g_{\lambda\mu} | Z_\lambda$, we have that $\mathscr{Z} = \{(Z_\lambda, h_{\lambda\mu}) : \Lambda\}$ is a closed subsystem of $\mathscr{Y}$ (and hence of $\mathscr{X}$) in which the sets Z_λ are non-empty. Furthermore, we have trivially that $\varprojlim \mathscr{Z} \subset \varprojlim \mathscr{Y} = \varprojlim \mathscr{X}$; on the other hand, given $y \in \varprojlim \mathscr{Y}$, we have (for each $\lambda \in \Lambda$ and all $\nu > \lambda$) $y_\lambda = g_{\nu\lambda}(y_\nu)$, and therefore $y_\lambda \in \bigcap \{g_{\nu\lambda}(Y_\nu) : \nu > \lambda\} = Z_\lambda$. Since $h_{\nu\lambda}(y_\lambda) = g_{\nu\lambda}(y_\lambda) = y_\nu$ $(\nu > \lambda)$ we have $y \in \varprojlim \mathscr{Z}$. Thus $\varprojlim \mathscr{Z} = \varprojlim \mathscr{X}$.

By minimality of $\mathscr{Y}$ we must have $\mathscr{Z} = \mathscr{Y}$; that is, $Z_\lambda = Y_\lambda$ for all $\lambda \in \Lambda$. Thus $Y_\lambda \subset g_{\nu\lambda}(Y_\nu)$ for all $\nu > \lambda$, showing that each $g_{\nu\lambda}$ is surjective; and the proof is complete.

Remark. In the special case in which all the maps $f_{\lambda\mu}$ are *proper*, Theorem 1 has a simpler proof; it is enough to define $Y_\lambda = \bigcap \{f_{\nu\lambda}(X_\nu) : \nu > \lambda\}$ (and, of course, $g_{\lambda\mu} = f_{\lambda\mu} | Y_\lambda$). The verification that this works is straight-forward, but depends on the following well-known property:

(1) Suppose $\{S_\nu : \nu \in \Lambda\}$ is a decreasing directed family of closed subsets of a space X, and let $f : X \to Y$ be a continuous map such that each $f^{-1}(y)$, $y \in Y$, is compact. Then $f(\bigcap_\nu S_\nu) = \bigcap_\nu f(S_\nu)$.

However, (1) becomes false if the assumption that f has compact point-inverses is dropped (even if f is now assumed closed); thus the more elaborate argument adopted for Theorem 1 seems to be necessary.

Incidentally, it is easy to see that the assumption, in Theorem 1, that the maps $f_{\lambda\mu}$ are closed, cannot be omitted: see for instance Example 2 below.

3. Non-emptiness of the limit

There is a trivial situation in which the limit is nonempty without any topological conditions being imposed: this is the case of an inverse *sequence* $\mathscr{X} = \{(X_n, f_n): \mathbb{N}\}$ with all the sets X_n nonempty and all the maps f_n surjective. For we choose $x_1 \in X_1$ and, recursively, $x_{n+1} \in (f_n)^{-1}(x_n)$, to produce $x \in \varprojlim \mathscr{X}$. This observation, combined with Theorem 1, gives:

Theorem 2. *If $\mathscr{X} = \{(X_n, f_n): \mathbb{N}\}$ is an inverse sequence of nonempty compact spaces and closed maps, then $\varprojlim \mathscr{X} \neq \emptyset$.*

However, when the index set $\mathbb{N}$ is replaced by a more complicated directed set Λ – for instance, the countable ordinals – even quite stringent conditions on the maps $f_{\lambda\mu}$ do not by themselves ensure that $\mathscr{X} = \{(X_\lambda, f_{\lambda\mu}): \Lambda\}$ have non-empty limit, as the following example shows.

Example 1. Let Λ consist of the countable ordinals, in their usual order (in fact, an arbitrary directed system with no greatest element and no cofinal subsequence will do), and take an inverse system of nonempty sets X_λ and surjective maps $f_{\lambda\mu}$ $(\lambda, \mu \in \Lambda, \lambda > \mu)$ having empty inverse limit. (See [2] or [3].) Give each X_λ the trivial ("indiscrete") topology; this is (hereditarily) compact, and each $f_{\lambda\mu}$ will be proper, as well as surjective. Yet $\varprojlim \mathscr{X} = \emptyset$.

Thus a theorem asserting that $\varprojlim \mathscr{X} \neq \emptyset$ in reasonable generality must require that the X_λ's satisfy some separation axioms. We shall see (in Theorem 3 below) that the T_0 axiom is enough, provided the maps $f_{\lambda\mu}$ are closed. The proof uses the following simple lemma.

Lemma 1. *Let X be a compact T_0 space. Then each closed nonempty subset F of X contains a closed point (that is, a point x such that $\{x\}$ is closed).*

Proof. Using the compactness we get a minimal non-empty closed subset F_0 of F; and, using the T_0 property, we see that F_0 cannot have more than one point.

Theorem 3. *Let $\mathscr{X} = \{(X_\lambda, f_{\lambda\mu}): \Lambda\}$ be an inverse system of compact non-empty T_0 spaces and closed maps. Then $\varprojlim \mathscr{X} \neq \emptyset$; in fact, there exists $x \in \varprojlim \mathscr{X}$ such that x_λ is a closed point of X_λ for all $\lambda \in \Lambda$.*

Proof. Consider the family Θ of all closed subsystems $\mathscr{Y}$ of $\mathscr{X}$ such that all the spaces Y_λ $(\lambda \in \Lambda)$ are nonempty; partially order Θ by the subsystem relation. By an argument similar to that in the proof of Theorem 1 we see that Θ has a minimal member, say $\mathscr{Z} = \{(Z_\lambda, h_{\lambda\mu}): \Lambda\}$. By applying Theorem 1 to $\mathscr{Z}$ we see that $\mathscr{Z}$ has a closed subsystem $\mathscr{Y} \in \Theta$ for which the maps are surjective; but from the minimality of $\mathscr{Z}$ it follows that $\mathscr{Z} = \mathscr{Y}$. Thus each $h_{\lambda\mu}$ $(\lambda > \mu)$ is a surjection.

Fixing $\nu \in \Lambda$ for the present, we show that Z_ν must be a singleton. By Lemma 1, Z_ν contains a closed point x_ν. For each $\lambda \in \Lambda$ take $\rho \in \Lambda$ so that $\rho >$ both λ and ν, and define $Y_{\rho\lambda} = h_{\rho\lambda}((h_{\rho\nu})^{-1}(x_\nu))$. We show that $Y_{\rho\lambda}$ is independent of the choice of ρ (subject to $\rho > \lambda, \nu$). It is enough to show that $Y_{\rho\lambda} = Y_{\sigma\lambda}$ if $\sigma > \rho > \lambda, \nu$. Suppose $y_\lambda \in Y_{\rho\lambda}$: then $y_\lambda = h_{\rho\lambda}(y_\rho)$ for some $y_\rho \in Y_\rho$ such that $h_{\rho\nu}(y_\rho) = x_\nu$. Since $h_{\sigma\rho}$ is surjective, choose $y_\sigma \in Y_\sigma$ such that $h_{\sigma\rho}(y_\sigma) = y_\rho$. Then

$$h_{\sigma\lambda}(y_\sigma) = h_{\rho\lambda}(h_{\sigma\rho}(y_\sigma)) = h_{\rho\lambda}(y_\rho) = y_\lambda,$$

and similarly $h_{\sigma\nu}(y_\sigma) = h_{\rho\nu}(y_\rho) = x_\nu$. This shows that $y_\lambda \in Y_{\sigma\lambda}$. Thus $Y_{\rho\lambda} \subset Y_{\sigma\lambda}$. The proof of the reverse inclusion is similar but simpler.

Define Y_λ to be the common value of the sets $Y_{\rho\lambda}$ ($\rho > \lambda, \nu$). Clearly Y_λ is a closed, non-empty subset of Z_λ. Further, if $\lambda > \mu$, a routine verification shows that $h_{\lambda\mu}(Y_\lambda) \subset Y_\mu$. Thus, on defining $g_{\lambda\mu} = h_{\lambda\mu} \mid Y_\lambda$ ($\lambda > \mu$), we have a closed subsystem $\mathscr{Y} = \{(y_\lambda, g_{\lambda\mu}) : \Lambda\}$ of $\mathscr{Z}$. Clearly $\mathscr{Y} \in \Theta$; but $\mathscr{Z}$ is a minimal element of Θ, and therefore $\mathscr{Y} = \mathscr{Z}$. In particular,

$$Z_\nu = Y_\nu = h_{\sigma\nu}((h_{\rho\nu})^{-1}(x_\nu)) = \{x_\nu\}.$$

Since the above applies to all $\nu \in \Lambda$, every Z_ν must be a singleton $\{x_\nu\}$. But then the point $x = \{x_\nu : \nu \in \Lambda\}$ is in $\varprojlim \mathscr{X}$, and the theorem is proved.

Remark. As the proof shows, the T_0 axiom could be replaced by the weaker assumption (genuinely weaker, even for compact spaces):

If F is a closed set with more than one point, then there exists a closed non-empty subset of F, different from F.

This weaker assumption can also be used instead of T_0 throughout what follows.

In Theorems 2 and 3, the assumption that the maps $f_{\lambda\mu}$ are *closed* cannot be omitted, even if the spaces X_λ are T_1 and hereditarily compact, as the following example shows.

Example 2. For each $n \in \mathbb{N} = \{1, 2, \dots\}$ take $X_n = \mathbb{N}$ with the cofinite topology; this is T_1 and hereditarily compact. Define $f_n : X_{n+1} \to X_n$ by $f_n(x_{n+1}) = x_{n+1} + 1$ ($x_{n+1} \in X_{n+1}$). This is continuous but not closed. And here clearly $\varprojlim \mathscr{X} = \emptyset$.

However, at least for some Λ's, the assumption that the maps $f_{\lambda\mu}$ are *closed* can be replaced by the assumption that they are *onto*:

Theorem 4. *Let $\mathscr{X} = \{(X_\lambda, f_{\lambda\mu}) : \Lambda\}$ be an inverse system of nonempty compact T_0 spaces and continuous maps, and suppose that each $f_{\lambda\mu}$ is a surjection (for all $\lambda, \mu \in \Lambda$ with $\lambda > \mu$) and that Λ is linearly ordered. Then $\varprojlim \mathscr{X} \neq \emptyset$; in fact, there exists $x \in \varprojlim \mathscr{X}$ such that (for all $\lambda \in \Lambda$) x_λ is a closed point of X_λ.*

Proof. We may assume that Λ is well-ordered (by replacing it by a suitable cofinal subset), say consisting of the ordinals $\lambda < \alpha$. Using Lemma 1, pick a closed point x_0 of X_0, then a closed point x_1 of the nonempty closed subset $(f_{10})^{-1}(x_0)$ of X_1; and so on, transfinitely. The step at a limit ordinal λ is dealt with as follows. Suppose a closed point $x_\mu \in X_\mu$ has been chosen for all $\mu < \lambda$, in such a way that whenever $\nu < \mu < \lambda$ we have $f_{\mu\nu}(x_\mu) = x_\nu$. Put $F_\mu = (f_{\lambda\mu})^{-1}(x_\mu)$ $(\mu < \lambda)$; this is a nonempty closed subset of X_λ. It is easily verified that whenever $\nu < \mu < \lambda$ we have $F_\nu \supset F_\mu$; hence, from the compactness of X_λ, we have that $\bigcap\{F_\mu: \mu < \lambda\}$ is a closed nonempty subset of X_λ. By Lemma 1, pick a closed point $x_\lambda \in \bigcap\{F_\mu: \mu < \lambda\}$, and the inductive step has been achieved.

The point $x = \{x_\lambda : \lambda < \alpha\}$ thus constructed is in $\varprojlim \mathscr{X}$, proving the theorem.

It would be interesting to know whether Theorem 4 would remain true without the assumption that Λ is linearly ordered.

4. Compactness of the limit

Theorem 5. *Let $\mathscr{X} = \{(X_\lambda, f_{\lambda\mu}): \Lambda\}$ be an inverse system of compact T_0 spaces and continuous closed maps. Then $\varprojlim \mathscr{X}$ is compact (and T_0).*

Proof. Put $L = \varprojlim \mathscr{X}$, $X = \prod\{X_\lambda : \lambda \in \Lambda\}$; thus L is a subspace of X. Let $\xi = \{\xi^\alpha : \alpha \in A\}$ be an ultranet ("universal net") on L; we must prove $\xi \to z$ for some $z \in L$.

For each $\lambda \in \Lambda$, let π_λ denote the projection map $X \to X_\lambda$. Then $\pi_\lambda \circ \xi$ is an ultranet on X_λ; because X_λ is compact, $\pi_\lambda \circ \xi$ converges. Put Z_λ = the set of *all* points of X_λ to which $\pi_\lambda \circ \xi$ converges; thus $Z_\lambda \neq \emptyset$, and (because Z_λ is also the set of all cluster points of $\pi_\lambda \circ \xi$) Z_λ is closed in X_λ.

Note that, if $\lambda > \mu$, then $f_{\lambda\mu}(Z_\lambda) \subset Z_\mu$. For, given $z_\lambda \in Z_\lambda$, we have $\pi_\lambda \circ \xi \to z_\lambda$, and therefore (because $f_{\lambda\mu}$ is continuous) $f_{\lambda\mu} \circ \pi_\lambda \circ \xi \to f_{\lambda\mu}(z_\lambda)$. But $f_{\lambda\mu} \circ \pi_\lambda = \pi_\mu$ for points of L, so we have $\pi_\mu \circ \xi \to f_{\lambda\mu}(z_\lambda)$; that is, $f_{\lambda\mu}(z_\lambda) \in Z_\mu$, as required.

Writing $h_{\lambda\mu} = f_{\lambda\mu} | Z_\lambda$, we therefore have that $\mathscr{Z} = \{(Z_\lambda, h_{\lambda\mu}): \Lambda\}$ is a closed subsystem of $\mathscr{X}$. The maps $h_{\lambda\mu}$ $(\lambda > \mu)$ are closed and continuous, and the spaces Z_λ are non-empty, compact and T_0. By Theorem 3, $\varprojlim \mathscr{Z} \neq \emptyset$. Take a point $z \in \varprojlim \mathscr{Z}$; then clearly $z \in \varprojlim \mathscr{X} = L$, and we have that (for each $\lambda \in \Lambda$) $\pi_\lambda(z) \in Z_\lambda$ and therefore $\pi_\lambda \circ \xi \to \pi_\lambda(z)$. Thus $\xi \to z$, as required.

Remark. One might expect that, analogously, by similar reasoning but with Theorem 4 replacing Theorem 3 one could prove: "if $\mathscr{X} = \{(X_\lambda, f_{\lambda\mu}): \Lambda\}$ is an inverse system of compact T_0 spaces and surjective maps, and if Λ is linearly ordered, then $\varprojlim \mathscr{X}$ is compact". This, however, is false, as the following example shows. (The difficulty is that the maps $h_{\lambda\mu}$, in this analogue of the proof of Theorem 5, need not be surjective.)

Example 3. There exists an inverse sequence of hereditarily compact T_1 spaces and continuous bijective maps having non-compact limit.

For $n \in \mathbb{N}$ $(=\{1, 2, \ldots\})$ define $X_n = \mathbb{N}$ with the topology specified as follows. Each of the points $1, 2, \ldots, n$ is isolated; and, for each $m > n$, a neighbourhood base at m consists of the cofinite subsets of $\mathbb{N}$ that contain m. This is easily seen to give a hereditarily compact T_1 topology. Define $f_n : X_{n+1} \to X_n$ to be the identity map. This is continuous, because the only point at which the topologies of X_n and X_{n+1} differ is $n+1$, which is isolated in X_{n+1}. The inverse limit L is the subspace of $\mathbb{N}^{\aleph_0}$ consisting of the points all of whose coordinates are equal. This is an infinite discrete set because, given $p = (m, m, \ldots, m, \ldots)$ of L, take $n > m$; then $(\pi_n)^{-1}(m)$ is a neighbourhood of p in $\mathbb{N}^{\aleph_0}$ that contains no other point of L. Thus L is not compact.

However, we do have a special result for inverse *sequences*:

Theorem 6. *Let $\mathscr{X} = \{(X_n, f_n): \mathbb{N}\}$ be an inverse sequence of compact spaces and closed maps. Then $\varprojlim \mathscr{X}$ is compact.*

(Note that no separation axioms are assumed).

Proof. Let $L = \varprojlim \mathscr{X} \subset X = \prod \{X_n : n \in \mathbb{N}\}$. As in the proof of Theorem 5, let ξ be an ultranet on L and (for each $n \in \mathbb{N}$) let Z_n be the set of all points of X_n to which $\pi_n \circ \xi$ converges. We have as before that $\mathscr{Z} = \{(Z_n, h_n): \mathbb{N}\}$, where $h_n = f_n | Z_{n+1}$, is an inverse sequence of nonempty compact spaces and closed maps. So $\varprojlim \mathscr{Z} \neq \emptyset$, by Theorem 2; and this shows that ξ converges to some $z \in L$, proving that L is compact.

5. Hereditary compactness of the limit

The inverse limit of even a *sequence* of hereditarily compact T_1 spaces, with very "nice" maps, need not itself be hereditarily compact; this is shown by the following example.

Example 4. Let $Y_1, Y_2, \ldots$ be arbitrary non-trivial hereditarily compact spaces, and let $X_n = Y_1 \times Y_2 \times \cdots \times Y_n$ $(n = 1, 2, \ldots)$. Let $f_n : X_{n+1} \to X_n$ be the projection map. Each X_n is hereditarily compact [4, p. 911] and each f_n is an open surjection and also a proper map. (Only the fact that f_n is closed requires verification; and this is straightforward, even in the absence of separation axioms.) It is easy to see that $\varprojlim \mathscr{X}$ here is homeomorphic to $\prod \{Y_n : n \in \mathbb{N}\}$; and this is *not* hereditarily compact [4, p. 912].

Thus it is natural to require, in the following theorem, that the directed indexing set Λ be very different from a sequence. Call a directed system $(\Lambda, <)$ "*countably directed*" if every *countable* subset of Λ has an upper bound in Λ. (For instance, the set of all countable ordinals is countably directed.)

Theorem 7. *Let $\mathscr{X} = \{(X_\lambda, f_{\lambda\mu}): \Lambda\}$ be an inverse system of hereditarily compact spaces (and continuous maps), and suppose that Λ is countably directed. Then $\varprojlim \mathscr{X}$ is hereditarily compact.*

The proof depends on the following lemma.

Lemma 2. *Given a set L, and a family $\mathcal{T}$ of topologies on L such that*
(1) *$\mathcal{T}$ is countably directed by inclusion,*
(2) *each $T \in \mathcal{T}$ is hereditarily compact;*
then the topology T^* generated by $\bigcup \mathcal{T}$ is hereditarily compact.

Proof of Lemma 2. By [4, p. 901] it is enough to prove that, for each *countable* subspace A of L, the subspace topology induced on A by T^* is compact. Enumerate A as $\{a_n : n \in \mathbb{N}\}$, and suppose $\mathcal{U}^*$ is a T^*-open cover of A; we must produce a finite sub-cover. Choose for each $n \in \mathbb{N}$ a set $U_n \in \mathcal{U}^*$ that contains a_n. Since $\bigcup \mathcal{T}$ is easily seen to be a base for T^*, we can choose (for each n) a topology $T_n \in \mathcal{T}$, and a set $V_n \in T_n$, such that $a_n \in V_n \subset U_n$. The countable subset $\{T_1, T_2, \ldots\}$ of $\mathcal{T}$ has an upper bound, say T', in $\mathcal{T}$; and $\{V_1, V_2, \ldots\}$ is a T'-open cover of A. Since T' is hereditarily compact, we have $A \subset V_1 \cup V_2 \cup \cdots \cup V_m$ for some (finite) m; and then $A \subset U_1 \cup \cdots \cup U_m$, as required.

To prove Theorem 7, let X_λ have topology $\mathcal{U}_\lambda$ ($\lambda \in \Lambda$), and put $X = \prod\{X_\lambda : \lambda \in \Lambda\}$, $L = \varprojlim \mathcal{X}$ (a subspace of X). For each $\lambda \in \Lambda$ let $\mathcal{V}_\lambda = \{(\pi_\lambda)^{-1}(U_\lambda) : U_\lambda \in \mathcal{U}_\lambda\}$, where $\pi_\lambda : X \to X_\lambda$ is the projection map; then $\mathcal{V}_\lambda$ is a hereditarily compact topology on X. Hence the subspace topology T_λ induced on L by $\mathcal{V}_\lambda$ is a hereditarily compact topology on L.

If $\lambda > \mu$ (and $\lambda, \mu \in \Lambda$), then $T_\mu \subset T_\lambda$. For a typical set in T_μ is of the form $L \cap V_\mu$, where $V_\mu = (\pi_\mu)^{-1}(U_\mu)$ and $U_\mu \in \mathcal{U}_\mu$. Define $U_\lambda = (f_{\lambda\mu})^{-1}(U_\mu)$; then $U_\lambda \in \mathcal{U}_\lambda$ and therefore $L \cap V_\lambda \in T_\lambda$, where $V_\lambda = (\pi_\lambda)^{-1}(U_\lambda)$. But it is not hard to check that $L \cap V_\mu = L \cap V_\lambda$, proving $T_\mu \subset T_\lambda$.

Thus the family $\mathcal{T} = \{T_\lambda : \lambda \in \Lambda\}$ of topologies on L is countably directed by inclusion. By Lemma 2, the topology T^* (on L) generated by $\bigcup \mathcal{T}$ is hereditarily compact. And it is not hard to see that T^* is just the subspace topology on L induced by the product topology on X – that is, the standard topology of the inverse limit. Thus $\varprojlim \mathcal{X}$ is hereditarily compact, as asserted by the theorem.

We conclude with an example to show that the usual method of dealing with inverse limits of compact T_2 spaces – that is, exhibiting $\varprojlim \mathcal{X}$ as the intersection of a system (with the finite intersection property) of closed subsets of $\prod\{X_\lambda : \lambda \in \Lambda\}$ – fails even for hereditarily compact T_1 spaces; $\varprojlim \mathcal{X}$ can be very far from closed in $\prod\{X_\lambda : \lambda \in \Lambda\}$. This fact is already implicit in Example 3 above, but is shown even more clearly in the following example.

Example 5. Take $\Lambda = \mathbb{N}$, and for each $n \in \mathbb{N}$ let X_n be $\mathbb{N}$ (or any other fixed infinite set) in the cofinite topology. Let $f_n : X_{n+1} \to X_n$ be the identity map, for all $n \in \mathbb{N}$. Then $\mathcal{X} = \{(X_n, f_n) : \mathbb{N}\}$ is an inverse sequence of nonempty hereditarily compact T_1 spaces, with each f_n a homeomorphism. Here $\varprojlim \mathcal{X}$ is, of course, the "diagonal", consisting of all points of $X = \prod \{X_n : n \in \mathbb{N}\}$ all of whose coordinates are equal. And this is dense in X; in particular it is not closed in X.

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