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D-Finite Power Series

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finitely generated as a differential module

polynomials with complex coefficients

1. INTRODUCTION

A power series $f(x) = \sum a_i x^i$ is called *D*-finite if all the derivatives of f span a finite dimensional vector space over $\mathbb{C}(x)$, the field of rational functions in x. This is equivalent to saying that f satisfies a linear, homogeneous, differential equation with coefficients from $\mathbb{C}[x]$. A sequence (a_i) is called *P*-recursive if it satisfies a recursion of the form $p_d(i)\, a_i + p_{d-1}(i)\, a_{i-1} + \cdots + p_0(i)\, a_{i-d} = 0$ where the $p_j(i)$ are polynomials. The theory of *D*-finite power series (in one variable) and *P*-recursive sequences is presented in [SR]. The connection between the two concepts is that $\sum a_i x^i$ is *D*-finite if and only if (a_i) is *P*-recursive. Stanley 80

The concept of *D*-finiteness generalizes immediately to power series in several variables—see Definition 2.1 below. In Section 2 we present a number of results about *D*-finite power series in several variables. The generalization of *P*-recursiveness to higher dimension is not so obvious. A definition is given in [ZD]. However this definition suffers from a number of defects—for example, a sequence (a_{ij}) can satisfy this definition without $\sum a_{ij} x^i y^j$ being *D*-finite (see [GI] and Remark 3.5 below). In Section 3 we give a generalization of *P*-recursiveness to higher dimensions which does not have this defect. We also give a number of properties of *P*-recursive sequences in Section 3.

In Section 4 we prove a Hartogs'-type theorem for *D*-finite analytic functions $f(x, y)$—if the restriction of f to each line segment is *D*-finite as a function of one variable, then f is *D*-finite as a function of two variables. In Section 5 we prove that if the infinite matrix $(a_{ij})_{i,j \in \mathbb{N}}$ has the properties that (i) each row contains only finitely many nonzero entries and (ii) for every *P*-recursive sequence (b_j) the matrix product $(a_{ij})(b_j) = (\sum_j a_{ij} b_j)$ is *P*-recursive, then (a_{ij}) is *P*-recursive.

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2. D-Finite Power Series

In this section we shall define and derive some properties of D-finite power series in several variables. For results on D-finite power series in one variable see [SR].

DEFINITION 2.1. Let $f(x_1, ..., x_n) = \sum a(i_1, ..., i_n) x_1^{i_1} \cdots x_n^{i_n} \in \mathbb{C}[[x_1, ..., x_n]]$ be a formal power series in the variables $x = (x_1, ..., x_n)$ over $\mathbb{C}$. f is called D-finite if all the derivatives $(\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n} f$, for $\alpha_1, ..., \alpha_n \geqslant 0$, lie in a finite dimensional vector space over $\mathbb{C}(x_1, ..., x_n)$, the field of rational functions in $x_1, ..., x_n$.

PROPOSITION 2.2. *f is D-finite if and only if f satisfies a system of linear partial differential equations, one for each $i = 1, ..., n$, of the form* i.e., each projection is D-finite

$$\left\{A_{in_i}(x)\left(\frac{\partial}{\partial x_i}\right)^{n_i} + A_{in_i-1}(x)\left(\frac{\partial}{\partial x_i}\right)^{n_i-1} + \cdots + A_{i0}(x)\right\} f = 0, \tag{2.1}$$

the same x_i in every equation

where the $A_{ij}(x) \in \mathbb{C}[x]$. and $A_{i0}, \dots, A_{in_i}$ are not all 0

Proof. That the definition implies the existence of such a system is immediate. Conversely, given such a system we can solve for all the derivatives of f as linear combination of the $(\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n} f$, with $0 \leqslant \alpha_i < n_i$, with coefficients from $\mathbb{C}(x)$. ∎ $\Rightarrow$ the dimension is $\leq n_1 \cdots n_n$

We shall write an equation of the form (2.1) as $P_i(x_1, ..., x_n, \partial/\partial x_i) f = 0$, where P_i denotes the partial differential operator in $\partial/\partial x_i$ with coefficients from $\mathbb{C}[x]$. Similarly if $D_1, ..., D_k$ are derivatives (for example $\partial/\partial x_i$) and $z_1, ..., z_l$ are some of the x_i, we shall denote a linear differential operator in $D_1, ..., D_k$ with coefficients from $\mathbb{C}[z_1, ..., z_l]$ by $P(z_1, ..., z_l, D_1, ..., D_k)$.

The following proposition is proved as in the one variable case (cf. [SR]).

PROPOSITION 2.3. (i) *The D-finite power series form a subalgebra of* $\mathbb{C}[[x_1, ..., x_n]]$.

(ii) *If f is algebraic then f is D-finite.*

(iii) *If $f(x_1, ..., x_n)$ is D-finite, the $g_i(y)$ are algebraic ($y = (y_1, ..., y_m)$) and the substitution $f(g_1(y), ..., g_n(y))$ makes sense (i.e., the $g_i(0)$ lie in the polydisc of convergence of f) then $f(g_1(y), ..., g_n(y))$ is D-finite. In particular if f converges in the polydisc Δ then f is D-finite (as a function of one variable) on every line segment in Δ.*

If we partition $\{i, ..., n\} = \{i_1, ..., i_k\} \cup \{i'_1, ..., i'_l\}$ then we can write $f \in \mathbb{C}[[x]]$ uniquely as $f = \sum_{j_1, ..., j_k} f_{j_1, ..., j_k}^{\{i'_1, ..., i'_l\}}(x_{i'_1}, ..., x_{i'_l}) x_{i_1}^{j_1} \cdots x_{i_k}^{j_k}$ with the $f_{j_1, ..., j_k}^{\{i'_1, ..., i'_l\}} \in \mathbb{C}[[x_{i'_1}, ..., x_{i'_l}]]$. We call the $f_{j_1, ..., j_k}^{\{i'_1, ..., i'_l\}}$ for $l \geqslant 1$ *sections* of f.

In particular for the partitioning $\{1,\dots,n\}=\{2,\dots,n\}\cup\{1\}$: $P_i(x_1, D_2, \dots, D_n) f = 0$

LEMMA 2.4. *Let $\{1, ..., n\} = \{i_1, ..., i_k\} \cup \{i'_1, ..., i'_l\}$ be a partition and let $f(x_1, ..., x_n)$ be D-finite. Let $D_j = x_j(\partial/\partial x_j)$. Then for each $i \in \{i'_1, ..., i'_l\}$ f satisfies a differential equation of the form*

$$P_i\left(x_{i'_1}, ..., x_{i'_l}, D_{i_1}, ..., D_{i_k}, \frac{\partial}{\partial x_i}\right) f = 0, \tag{2.2}$$

and also one of the form

$$P'_i(x_{i'_1}, ..., x_{i'_l}, D_{i_1}, ..., D_{i_k}, D_i) f = 0. \tag{2.2$'$}$$

Proof. The proof is the same as that of Lemma 3 of [LL]. ∎

Since the labelling of the variables is not important, for notational convenience we shall only consider sections of the form $f^{1,...,r}_{i_{r+1},...,i_n}$ in the following.

PROPOSITION 2.5. *Let $f(x_1, ..., x_n) = \sum a(i_1, ..., i_n) x_1^{i_1} \cdots x_n^{i_n}$ be D-finite. Then*

(i) *All the sections $f^{1,...,r}_{i_{r+1},...,i_n}(x_1, ..., x_r) = \sum_{i_1,...,i_r} a(i_1,...,i_n) x_1^{i_1} \cdots x_r^{i_r}$ are D-finite.*

(ii) *There are parametrized systems of differential operators $Q_i(x_1, ..., x_r, t_{r+1}, ..., t_n, \partial/\partial x_i)$ for $i \leq r$, such that for each $i_{r+1},...,i_n$*

$$Q_i\left(x_1, ..., x, i_{r+1}, ..., i_n, \frac{\partial}{\partial x_i}\right) f^{1,...,r}_{i_{r+1},...,i_n}(x_1, ..., x_r) = 0.$$

(iii) *If all the sums $\sum_{i_{r+1},...,i_n} a(i_1, ..., i_n)$ converge (for example, if all are finite sums) then $\sum_{i_1,...,i_r} (\sum_{i_{r+1}\cdots i_n} a(i_1, ..., i_n)) x_1^{i_1} \cdots x_r^{i_r}$ is also D-finite.*

Proof. It is sufficient to consider the case $r = n-1$. $f_0^{1,...,n-1}(x_1, ..., x_{n-1}) = f(x_1, ..., x_{n-1}, 0)$ satisfies the equations $P_i(x_1, ..., x_{n-1}, 0, \partial/\partial x_i) f_0^{1,...,n-1} = 0$. By dividing by a suitable power of x_n we may assume that $P_i(x_1, ..., x_{n-1}, 0, \partial/\partial x_i) \neq 0$. If we know that the $f_j^{1,...,n-1}$ are D-finite for $j < k$ then $g(x_1, ..., x_n) = x_n^{-k}\{f - \sum_{j=0}^{k-1} f_j^{1,...,n-1} x_n^j\}$ is D-finite and hence so is $f_k^{1,...,n-1} = g(x_1, ..., x_{n-1}, 0)$.

induction

necessary so that g is a power series, instead of just a LAURENT series.

(ii) follows immediately from Lemma 2.4. If $P(x_1, ..., x_{n-1}, x_n(\partial/\partial x_n), \partial/\partial x_i) f = 0$ then $P(x_1, ..., x_{n-1}, i_n, \partial/\partial x_i) f_{i_n}^{1,...,n-1}(x_1, ..., x_{n-1}) = 0$.

(iii) can be seen by observing that if all the sums $\sum_{i_n} a(i_1, ..., i_n)$ converge then $\sum_{i_1\cdots i_{n-1}} (\sum_{i_n} a(i_1, ..., i_n)) x_1^{i_1}, ..., x_{n-1}^{i_{n-1}} = f(x_1, ..., x_{n-1}, 1)$ and we need only ensure that $P_i(x_1, ..., x_{n-1}, 1, \partial/\partial x_i) \neq 0$. ∎

DEFINITION 2.6. If $f(x) = \sum a(i_1, ..., i_n) x_1^{i_1} \cdots x_n^{i_n}$ and $g(x) = \sum b(i_1, ..., i_n)$

works because no negative powers of x_n in f

$x_1^{i_1} \cdots x_n^{i_n}$, then the *Hadamard product* $f * g(x) = \sum a(i_1, ..., i_n) b(i_1, ..., i_n) x_1^{i_1} \cdots x_n^{i_n}$. The primitive diagonal $I_{12} f(x_1, x_3, ..., x_n) = \sum a(i_1, i_1, i_3, ..., i_n) x_1^{i_1} x_3^{i_3} \cdots x_n^{i_n}$; the other I_{ij} we defined similarly. A *diagonal* is any composition of the I_{ij}'s. For example the *complete diagonal* is $I(f) = I_{12} I_{23} \cdots I_{n-1} f = \sum a(i_1, ..., i_1) x_1^{i_1}$. For results on diagonals of algebraic power series see [DL] and the references therein.

Note that

$$f * g = I_{1n+1} \cdots I_{n2n} f(x_1, ..., x_n)\, g(x_{n+1}, ..., x_{2n})$$

and

$$I_{12} f = \left\{ f * \left(\frac{1}{1 - x_1 x_2} \frac{1}{1 - x_3} \cdots \frac{1}{1 - x_n} \right) \right\} (x_1, 1, x_3, ..., x_n).$$

Also

$$\begin{aligned} &\int_0^{x_n} f(x_1, ..., x_{n-1}, t)\, dt \\ &\quad = \sum f_j^{1, ..., n-1}(x_1, ..., x_{n-1}) \frac{1}{j+1} x_n^{j+1} \\ &\quad = -(x_n f) * \left(\log(1 - x_n) \frac{1}{1 - x_1} \cdots \frac{1}{1 - x_{n-1}} \right). \end{aligned}$$

In [LL] we showed that every diagonal of a D-finite power series is D-finite. Hence we have the

THEOREM 2.7. *If $f(x_1, ..., x_n)$ and $g(x_1, ..., x_n)$ are D-finite then*

(i) *every diagonal of f is D-finite,*

(ii) *the Hadamard product $f * g$ is D-finite, and*

(iii) *$\int_0^{x_n} f(x_1, ..., x_{n-1}, t)\, dt$ is D-finite.*

Much is known about the algebraic structure of the noncommutative ring $A_n = \mathbb{C}[x_1, ..., x_n, \partial/\partial x_1, ..., \partial/\partial x_n]$ of linear differential operators with polynomial coefficients (with $(\partial/\partial x_i)\, x_i = x_i (\partial/\partial x_i) + 1$). See, for example, [BJ]. We shall need the following later.

LEMMA 2.8 [BJ, Prop. 8.4, p. 26]. *The ring A_n has no nonzero zero divisors. Further, given two nonzero differential operators P and Q there are nonzero differential operators R and S such that $RP = SQ$.* left Ore property

3. *P*-RECURSIVE SEQUENCES

In this section we consider sequences (i.e., functions $\mathbb{N}^n \to \mathbb{C}$). We define a notion of *P*-recursiveness for such sequences, which generalizes the case of $n=1$ (cf. [SR]). The main motivation of our definition is to guarantee that a sequence $a(i_1, ..., i_n)$ is *P*-recursive if and only if $\sum a(i_1, ..., i_n)\, x_1^{i_1} \dots x_n^{i_n}$ is *D*-finite (Theorem 3.7). The definition is a little cumbersome (since it has been designed to make the proof of Theorem 3.7 work), but it is often easier to show that a sequence is *P*-recursive than to check that the corresponding power series is *D*-finite.

DEFINITION 3.1. The *sections* of a sequence $a(i_1, ..., i_n)$ are the sequences of dimension $<n$ obtained from $a(i_1, ..., i_n)$ by holding one or more of the variables i_j fixed. A *k-section* of $a(i_1, ..., i_n)$ is a section of $a(i_1, ..., i_n)$ obtained by holding some of the i_j fixed at values $<k$. For example the sequence $b(i_1, i_3) = a(i_1, 7, i_3)$ is an 8-section of $a(i_1, i_2, i_3)$.

DEFINITION 3.2. We call a sequence $a(i_1, ..., i_n)$ *P-recursive* if there is a $k \in \mathbb{N}$ such that

(i) for each $j = 1, ..., n$ and each $v = (v_1, ..., v_n) \in \{0, ..., k\}^n$ there is a polynomial $p_v^{(j)}$ (with at least one $p_v^{(j)}(i) \neq 0$ for each j) such that

polynomial of one variable i_j

$$\sum_v p_v^{(j)}(i_j)\, a(i_1 - v_1, ..., i_n - v_n) = 0 \tag{3.1}$$

for all $i_1, ..., i_n \geq k$ (the sum is over $\{0, ..., k\}^n$), and

(ii) if $n > 1$ then all the k-sections of $a(i_1, ..., i_n)$ are *P*-recursive.

Remark 3.3. Part (ii) of the definition represents the information about $a(i_1, ..., i_n)$ that is lost in converting differential equations of the form (2.1) into recursion relations of the form (3.1). In the one-dimensional case this loss is negligible, but in the higher dimensional case it is not. The following examples show that part (ii) of the definition is needed.

EXAMPLES 3.4. (i) The sequence $a(i_1, i_2) = (i_1 + i_2)!$ is *P*-recursive. It satisfies the recursion $a(i_1, i_2 - 1) - a(i_1 - 1, i_2) = 0$. Since the coefficients are constant we may use the same recursion for $j = 1, 2$—i.e., we may take $k=1$ and $p_{(0,1)}^{(j)} = -p_{(1,0)}^{(j)} = 1$, $p_{(0,0)}^{(j)} = p_{(1,1)}^{(j)} = 0$ for $j = 1, 2$. Fixing $i_1 = 0$ gives the section $a(0, i_2) = i_2!$ which is certainly *P*-recursive. Similarly $a(i_1, 0)$ is *P*-recursive. Note that $(i_1 + i_2)!^\alpha$ satisfies the above recursion for any α. However the section $(i_1 + 0)!^\alpha = i_1!^\alpha$ is *P*-recursive if and only if $\alpha \in \mathbb{Z}$. This proof that, for example, $(i_1 + i_2)!^3$, is *P*-recursive seems easier than showing directly that $\sum (i_1 + i_2)!^3\, x_1^{i_1} x_2^{i_2}$ is *D*-finite.

$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \cdot \left\{ {n-1 \atop k} \right\}$ not P-recursive!

(ii) Consider the recursion $b(i_1, i_2) - b(i_1 - 1, i_2 - 1) = 0$ (again we take the same recursion for $j = 1, 2$). This determines $b(i_1, i_2) = b(i_1 - i_2, 0)$ if $i_1 \geqslant i_2$ and $b(i_1, i_2) = b(0, i_2 - i_1)$ if $i_1 < i_2$. To know what the $b(i_1, i_2)$ are we still need to know the sections $b(i_1, 0)$ and $b(0, i_2)$. $b(i_1, i_2) = 1/(i_1 - i_2)!$ satisfies this recursion. The sections $b(i_1, 0) = 1/i_1!$ and $b(0, i_2) = 0$ for $i_2 > 0$, $b(0, 0) = 1$ are certainly P-recursive. Observe that if $b(i_1, i_2)$ satisfies a recursion of the form

$$\sum_{j=0}^{d} p_j(i_1, i_2)\, b(i_1 + j, i_2) = 0$$

with $p_d(i_1, i_2) \not\equiv 0$ then we must have for $m > d$ that $p_d(m - d, m) = 0$ since $b(m, m) = 1$ and $b(l, m) = 0$ for $l < m$. In particular it is not possible for $p_d(i_1, i_2)$ to be a function of just i_1.

(iii) Some very simple sequences are not P-recursive. While $1/(i_1 + i_2)$ is, $c(i_1, i_2) = 1/(i_1^2 + i_2)$ is not. For suppose $\sum_v p_v(i_1)(1/((i_1 - v_1)^2 + i_2)) = 0$. Choose $\bar{v}$ so that $p_{\bar{v}} \not\equiv 0$ and choose $0 \ll i_1 \ll i_2$ so that $\rho = (i_1 - \overline{v_1})^2 + i_2$ is prime, $p_{\bar{v}}(i_1) \neq 0$, $\rho > p_{\bar{v}}(i_1)$ and if $v \neq \bar{v}$ then $(i_1 - v_1)^2 + i_2$ is not divisible by ρ. It is then clear that this equation is impossible.

Remark 3.5. In [ZD] it was proposed to call a sequence $a(i_1, ..., i_n)$ (multi-) P-recursive if it satisfies a family of recursions (one for each $j = 1, ..., n$) of the form

too general

$$\sum_{l=0}^{r_j} p_l^{(j)}(i_1, ..., i_n)\, a(i_1, ..., i_{j-1}, i_j + l, i_{j+1}, ..., i_n) = 0, \tag{3.2}$$

where the $p_l^{(j)}$ are polynomials (and for each j at least one is $\not\equiv 0$). As was pointed out in [GI] this would lead one to take the sequence $a(i_1, i_2) = 0$ if $i_2 \neq i_1^2$ and $a(i, i^2) = 1$ as P-recursive. But neither $f(x_1, x_2) = \sum x_1^i x_2^{i^2}$ nor $f(1, x_2)$ is D-finite. This sequence satisfies $(i_2 - i_1^2)\, a(i_1, i_2) = 0$. This definition also suffers from the lack of an analogue of part (ii) of Definition 3.1. To see this note that the recursion $i_1 i_2 a(i_1, i_2) = 0$ has the solutions $a(i_1, i_2) = 0$ if $i_1 i_2 \neq 0$ and $a(i_1, 0)$, $a(0, i_2)$ arbitrary. We will show below (Theorem 3.8(vii)) that every P-recursive sequence does satisfy a system of recursions of the form (3.2). The above example shows that the converse is false. We do however have

PROPOSITION 3.6. *Let the sequence $a(i_1, ..., i_n)$ satisfy a system of recursions, one for each $j = 1, ..., n$, of the form*

$$p_0^{(j)}(i_j)\, a(i_1, ..., i_n) + \sum_{l=1}^{r_j} p_l^{(j)}(i_1, ..., i_n)\, a(i_1, ..., i_{j-1}, i_j - l, i_{j+1}, ..., i_n) = 0, \tag{3.3}$$

where the $P_0^{(j)}$ *are nonzero polynomials of one variable. Then the sequence* $a(i_1, ..., i_n)$ *is P-recursive.*

Proof. We prove this by induction on n. If we hold one of the i_j's fixed then the corresponding section satisfies a system of $n-1$ recursions of the form (3.6) and hence is P-recursive by induction. It remains to show that the sequence $a(i_1, ..., i_n)$ satisfies a family of recursions of the form (3.1) Let $r = \max\{r_j : j = 1, ..., n\}$ and let $l \in \mathbb{N}$ be larger than all the zeros of all the $p_0^{(j)}$'s. Let $M = \max\{\text{degree } p_l^{(j)} : j = 1, ..., n;\ l = 0, ..., r_j\}$ and let $K \in \mathbb{N}$ (to be specified later). Let $Q(i_1, ..., i_n) = \prod_j \prod_{m=0}^{K} p_0^{(j)}(i_j - m)$. Then using the recurrences (3.3) we can write each $Q(i_1, ..., i_n)\, a(i_1 - v_1, ..., i_n - v_n)$ for $0 \leqslant v_i \leqslant K$ as a linear combination of the $a(i_1 - K + \mu_1, ..., i_n - K + \mu_n)$ for $0 \leqslant \mu_i < r$, with coefficients which are polynomials in $i_1, ..., i_n$ of degrees $\leqslant M(K+1)n$. All such form a vector space over $\mathbb{C}(i_j)$ of dimension $r^n\binom{M(K+1)n+n-1}{n-1} < c(K+1)^{n-1}$ for some c. Here we are regarding the coefficients as polynomials in the i_m, for $m \neq j$, with coefficients from $\mathbb{C}(i_j)$. On the other hand the number of v's of the above type is $(K+1)^n$. For K large enough we certainly have $(K+1)^n > c(K+1)^{n-1}$. Hence there are polynomials $p_v^{(j)}(i_j)$, not all zero, such that

$$Q(i_1, ..., i_n) \sum_{0 \leqslant v_i \leqslant K} p_v^{(j)}(i_j)\, a(i_1 - v_1, ..., i_n - v_n) = 0.$$

If in addition the i_m are all $> K + l$ then $Q(i_1, ..., i_n) \neq 0$ and so

$$\sum_{0 \leqslant v_i \leqslant K} p_v^{(j)}(i_j)\, a(i_1 - v_1, ..., i_n - v_n) = 0$$

for all $i_1, ..., i_n \geqslant k = K + l$. ∎

EXAMPLES. (i) Proposition 3.6 and the recursions

$$(i_1 + 1)\, a(i_1 + 1, i_2) = (i_1 + i_2 + 1)\, a(i_1, i_2)$$

$$(i_2 + 1)\, a(i_1, i_2 + 1) = (i_1 + i_2 + 1)\, a(i_1, i_2)$$

show that $a(i_1, i_2) = \binom{i_1 + i_2}{i_1}$ is P-recursive. On the other hand to see that $\binom{i_1}{i_2}$ is P-recursive it is easier to use Definition 3.1 and the recursion $\binom{i_1}{i_2} + \binom{i_1}{i_2+1} = \binom{i_2+1}{i_2+1}$.

(ii) The observation at the end of Example 3.4(ii) shows that the converse of Proposition 3.6 is false.

THEOREM 3.7. *The sequence* $a(i_1, ..., i_n)$ *is P-recursive if and only if the power series* $f(x_1, ..., x_n) = \sum a(i_1, ..., i_n)\, x_1^{i_1} \cdots x_n^{i_n}$ *is D-finite.*

Proof. If $f(x_1, ..., x_n)$ is D-finite, from the equations $P_i(x_1, ..., x_n, \partial/\partial x_i) f = 0$ we get recursions of the form (3.1) by equating the coefficient of $x_1^{i_1} \cdots x_n^{i_n}$ to zero for $i_1, ..., i_n$ large enough. That the sequence $a(i_1, ..., i_n)$ satisfies part (ii) of Definition 3.1 follows from Proposition 2.5(i) and induction.

Conversely suppose that $a(i_1, ..., i_n)$ is P-recursive and hence satisfies recursions of the form (3.1). For notational convenience we shall only consider the case $j = n$. Then

$$\sum_{k \leqslant i_1, ..., i_n} \sum_{0 \leqslant v_1, ..., v_n \leqslant k} p_v^{(n)}(i_n)\, a(i_1 - v_1, ..., i_n - v_n)\, x_1^{i_1} \cdots x_n^{i_n} = 0. \tag{3.4}$$

This is a linear combination (over $\mathbb{C}$) of terms of the form

$$\begin{aligned}
&\sum_{k \leqslant i_1, ..., i_n} i_n^{\alpha} a(i_1 - v_1, ..., i_n - v_n)\, x_1^{i_1} \cdots x_n^{i_n} \\
&\quad = \sum_{k \leqslant i_1, ..., i_n} a(i_1 - v_1, ..., i_n - v_n) \left(x_n \frac{\partial}{\partial x_n} \right)^{\alpha} x_1^{i_1} \cdots x_n^{i_n} \\
&\quad = \left(x_n \frac{\partial}{\partial x_n} \right)^{\alpha} x_1^{v_1} \cdots x_n^{v_n} \sum_{k \leqslant i_1, ..., i_n} a(i_1 - v_1, ..., i_n - v_n)\, x_1^{i_1 - v_1} \cdots x_n^{i_n - v_n} \\
&\quad = \left(x_n \frac{\partial}{\partial x_n} \right)^{\alpha} x_1^{v_1} \cdots x_n^{v_n} \sum_{\substack{k - v_j < i_j \\ j = 1, ..., n}} a(i_1, ..., i_n)\, x_1^{i_1} \cdots x_n^{i_n} \\
&\quad = \left(x_n \frac{\partial}{x_n} \right)^{\alpha} x_1^{v_1} \cdots x_n^{v_n} (f(x) - g(x)),
\end{aligned}$$

where $g(x)$ is a $\mathbb{C}[x]$-linear combination of k-sections of f. By induction $g(x)$ is D-finite and hence (3.4) can be written as

$$Q\left(x_1, ..., x_n, \frac{\partial}{\partial x_n} \right) f = h(x),$$

where $h(x)$ is D-finite. By considering the largest possible α and the largest possible $v_n, v_{n-1}, ...$ one sees that Q is not zero. Let $R(x_1, ..., x_n, \partial/\partial x_n) \neq 0$ be such that $Rh = 0$. Then (by Lemma 2.8) we can take $P(x_1, ..., x_n, \partial/\partial x_n) = RQ \neq 0$. ▌

In analogy to Section 2 we define the diagonals of a sequence $a(i_1, ..., i_n)$. For example, $I_{12} a(i_1, ..., i_n) = a(i_1, i_1, i_3, ..., i_n)$. The convolution of two sequences is defined by $a \odot b(i_1, ..., i_n) = \sum_{l_j + m_j = i_j, j = 1, ..., n} a(l_1, ..., l_n)\, b(m_1, ..., m_n)$. The generating function of $a \odot b$ is just the product of he generating functions of a and b. We now draw some consequences from Theorem 3.7 and the results of Section 2.

THEOREM 3.8. (i) *The P-recursive sequences (of dimension n) form an algebra over* $\mathbb{C}[i_1, ..., i_n]$.

(ii) *The convolution of two P-recursive sequences is P-recursive.*

(iii) *Any diagonal of a P-recursive sequence is P-recursive.*

(iv) *All sections of a P-recursive sequence are P-recursive.*

(v) *Let* $C \subseteq \mathbb{N}^n$ *be defined by a finite set of inequalities of the form* $\sum a_j i_j \geq b$ *where the* a_j *and* $b \in \mathbb{Q}$. *Let* $b(i_1, ..., i_n) = a(i_1, ..., i_n)$ *if* $(i_1, ..., i_n) \in C$ *and* $b(i_1, ..., i_n) = 0$ *otherwise. If* $a(i_1, ..., i_n)$ *is P-recursive then so is* $b(i_1, ..., i_n)$.

(vi) *If* $a(i_1, ..., i_n)$ *is P-recursive and* $\sum_{i_n} a(i_1, ..., i_n)$ *converges for every* $i_1, ..., i_{n-1}$ *then* $b(i_1, ..., i_{n-1}) = \sum_{i_n} a(i_1, ..., i_n)$ *is P-recursive.*

(vii) *If* $a(i_1, ..., i_n)$ *is P-recursive then it satisfies a system of recursions of the form* (3.2).

(viii) *If* $a(i_1, i_2)$ *is P-recursive and for every* i_1 *there are only finitely many* i_2 *with* $a(i_2, i_2) \neq 0$ *then for every P-recursive* $b(i_2)$ *the matrix product* $(a(i_1, i_2))(b(i_2)) = (\sum_{i_2} a(i_1, i_2)\, b(i_2))$ *is P-recursive.*

Proof. (i) follows from Proposition 2.3(i), (ii) and Theorem 2.7(ii). (Note that if f is D-finite and Q is a differential operator then Qf is D-finite.) (ii) follows from Proposition 2.3(i). (iii) follows from Theorem 2.7(i). (iv) follows from Proposition 2.5(i). (v) uses Remark 4 of [LL]. (vi) follows from Proposition 2.5(iii). (vii) follows from Lemma 2.4 using the equation of the form $P'(x_i, D_1, ..., D_n) f = 0$. (viii) uses parts (i) and (vi). ∎

EXAMPLE 3.9. In order to see that $\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+2k}{k}^3$ is P-recursive we can argue as follows: $a(n, k) = \binom{n}{k}$ satisfies $a(n, k) + a(n, k+1) - a(n+1, k+1) = 0$ and $b(n, k) = \binom{n+2k}{k}$ satisfies $b(n, k) + b(n-2, k+1) - b(n-1, k+1) = 0$. It is trivial that the sections of a and b are P-recursive. Then use Theorem 3.8(i) and (vi). The argument could also be used to find a recursion for this sequence.

3.10. Computing the $a(i_1, ..., i_n)$ from the recursions. It is not immediately obvious how to use Definition 3.2 to compute the $a(i_1, ..., i_n)$ from finitely many of them. If $a(i_1, ..., i_n)$ satisfies recursions of the form (3.3) or of the form (3.2) with the $p_{r_j}^{(j)}(i_1, ..., i_n)$ each having only finitely many zeros then it is obvious how to do this and in fact one can do it quite efficiently. We next discuss the general case, and draw some consequences about the rate of growth of the $a(i_1, ..., i_n)$.

Let $a(i_1, ..., i_n)$ be P-recursive, and suppose it satisfies recursions of the form (3.1). For each $j = 1, ..., n$ let $\mathbb{N}^n \supseteq X_j = \{(k - j_1, ..., k - j_n)$: $p_{(j_1, ..., j_n)}(t) \not\equiv 0\}$. Choose a "weight" $(w_1, ..., w_n) \in \mathbb{Q}^n$, $w_i > 0$ for all i, and

$\sum w_i = 1$, such that for each j there is a unique $(\mu_1^{(j)}, ..., \mu_n^{(j)}) \in X_j$ such that the weight $\sum_i \mu_i^{(j)} w_i$ is a maximum. (There certainly is such.) Define the weight of a point $(i_1, ..., i_n)$ to be $\sum_j i_j w_j$. Let $p_m^{(j)}(i_j) = p_{k-\mu_1^{(j)}, ..., k-\mu_n^{(j)}}(i_j)$. We shall call the $p_m^{(j)}(i_j)$ the leading coefficients of the recursions (3.1). Let $S = \{(i_1, ..., i_n): p_m^{(j)}(i_j) = 0 \text{ for } j = 1, ..., n\}$. Then S is a finite set in $\mathbb{N}^n$. Notice that there is a $c \in \mathbb{R}$ such that for each $w \in \mathbb{N}$ there are at most cw^n points in $\mathbb{N}^n$ with weight $\leqslant w$ and also that if $(i_1, ..., i_n) \notin S$ and $i_1, ..., i_n \geqslant k$, then at least one of the recursions (3.1) can be used to express $a(i_1, ..., i_n)$ as a linear combination of the $a(i'_1, ..., i'_n)$ where the $(i'_1, ..., i'_n)$ have lower weights than $(i_1, ..., i_n)$. In fact if $p_m^{(j)}(i_j) \neq 0$ then

$$a(i_1, ..., i_n) = -(p_m^{(j)}(i_j))^{-1} \sum_{\nu \neq k-\mu^{(j)}} p_\nu^{(j)}(i_1 - \nu_1, ..., i_n - \nu_n) a(i_1 - \nu_1, ..., i_n - \nu_n).$$

Hence each $a(i_1, ..., i_n)$ can be expressed as a linear combination of the $a(i'_1, ..., i'_n)$ for $(i'_1, ..., i'_n) \in S$ and the $a(i''_1, ..., i''_n)$ where some $i''_j < k$ and all the $i''_j < d(i_1 + \cdots + i_n)$. We can take $d = \max w_j^{-1}$. We may assume, by induction, that we know how to compute these $a(i''_1, ..., i''_n)$, since they come from k-sections of a. The same induction gives us the following proposition which generalizes Theorem 16 on p. 200 and Theorem 18 on p. 206 of [MK] to higher dimensions.

GROWTH RATE

PROPOSITION 3.11. *Let $a(i_1, ..., i_n)$ be P-recursive. There are $c, \gamma \in \mathbb{N}$ such that for all $i_1, ..., i_n$ we have $|a(i_1, ..., i_n)| \leqslant c((i_1 + \cdots + i_n)!)^\gamma$. If in addition all the $a(i_1, ..., i_n) \in \mathbb{Q}$ there are $0 < c', \gamma'$ such that if $a(i_1, ..., i_n) \neq 0$ then $|a(i_1, ..., i_n)| > c'((i_1 + \cdots + i_n)!)^{-\gamma'}$.*

4. A HARTOGS'-TYPE THEOREM FOR *D*-FINITE FUNCTIONS

In this section we shall prove the following converse of Proposition 2.3(iii).

THEOREM 4.1. *If $f(x, y)$ is analytic on the polydisc $\Delta = \{(x, y): |x|, |y| < 1\}$ and is D-finite on every line segment in this polydisc (as a function of one variable) then f is D-finite as a function of two variables.*

Remarks 4.2. (i) The similar result is true in higher dimension, with essentially the same proof.

(ii) A power series $f(x_1, ..., x_n)$ is called differentially algebraic if all the derivatives of f lie in a field of finite transcendences degree over $\mathbb{C}(x_1, ..., x_n)$. In [OA] Ostrowski showed that if $f(x, y)$ is analytic for $|x|$, $|y| < 1$ and for all a, b with $|a|, |b| < 1$ the functions of one variable, $f(a, y)$

and $f(x, b)$, are differentially algebraic then f is differentially algebraic as a function of two variables. The analogous result is not true for *D*-finite analytic functions as is shown by the example

$$f(x, y) = (1+x)^y = 1 + yx + \frac{y(y-1)}{2!}x^2 + \frac{y(y-1)(y-2)}{3!}x^3 + \cdots.$$

For every a with $|a| < 1$, $f(a, y)$ is *D*-finite and $f(x, b)$ is *D*-finite for every b, but $f(x, y)$ is not *D*-finite, since $f(x, x)$ is not. (We leave the verification of this as an exercise.) Note that $g(x, y) = f(x, y/x)$ is also a power series and $g(x, bx) = f(x, b)$. Hence $g(x, y)$ is *D*-finite on every line through $(0, 0)$, but it is not *D*-finite as a function of two variables.

(iii) Over any countable subfield K of $\mathbb{C}$ there is an analytic function $f(x, y) = \sum a_{ij} x^i y^j$ with the $a(i, j) \in K$ such that, on every line defined over K, $f(x, y)$ is a polynomial, but f is not *D*-finite.

(iv) There exist C^∞-functions which are *D*-finite on every line (in fact on every algebraic curve) but are not *D*-finite.

In this section we will write sequences as a_{ij} instead of $a(i, j)$ and similarly for power series.

LEMMA 4.3. *If $f(x, y) = \sum a_{ij} x^i y^j$ is not D-finite and if $K \subset \mathbb{C}$ is any countable field, there is a set $X \subset \Delta$, of the first category, such that for every $(a, b) \in \Delta - X$ the vector space dimension of the ring $K(a, b)[((\partial^{i+j} f)/(\partial x^i \, \partial y^j))(a, b)\colon i, j \in \mathbb{N}]$ over $K(a, b)$ is infinite.*

Proof. Suppose that $f(x, y) = \sum a_{ij} x^i y^j$ is not *D*-finite. Interchanging x and y if necessary, we may assume that the $(\partial^i f/\partial x^i)(x, y)$, $i \in \mathbb{N}$, are linearly independent over $K(x, y)$. Hence any linear relation among the $(\partial^i f/\partial x^i)(x, y)$ over $K(x, y)$ holds only on a nowhere dense subset of Δ. Since K is countable there are only countably many such possible relations and the lemma follows. ∎

LEMMA 4.4. *If the power series $f(t) = \sum c_i t^i$ is D-finite then it satisfies an equation of the form $P(t, \partial/\partial t) f = 0$ where $P(t, \partial/\partial t) \in K[t, \partial/\partial t]$ and $K = \mathbb{Q}(c_i\colon i \in \mathbb{N})$. Conversely if f satisfies an* nth order equation *$\{A_n(t)(\partial/\partial t)^n + \cdots + A_0(t)\} f = 0$ where all the $A_i(t) \in K[t]$ and $A_n(0) \neq 0$, then for all i $(\partial^i f/\partial t^i)(0) \in K[f(0), (\partial f/\partial t)(0), ..., ((\partial^{n-1} f)/(\partial t^{n-1}))(0)]$.*

Proof. Suppose f satisfies $Q(t, \partial/\partial t) f = 0$ where $Q \in \mathbb{C}(t)[\partial/\partial t]$ has leading coefficient 1. If $Q \notin K(t)[\partial/\partial t]$ then there is an automorphism σ of $\mathbb{C}$ over K which moves some coefficient of Q. Let Q^σ result from Q by applying σ to all the coefficients of Q. Then $Q^\sigma f = 0$. Hence $(Q - Q^\sigma) f = 0$

and this is a lower order equation. Hence if we choose P of lowest possible order, $P=A_n(t)(\partial/\partial t)^n+\cdots+A_0(t)$, then all the $A_i(t)/A_n(t)\in K(t)$ and, hence, clearing denominators we may take $P\in K[t, \partial/\partial t]$. The converse direction is immediate. ▮

Let f be analytic on Δ and suppose that f is D-finite on every line segment in Δ. For $(a, b)\in\Delta$ let $f=\sum a_{ij}(a, b)(x-a)^i(y-b)^j$ be the Taylor series of f at (a, b). Let K_{ab} be the field generated over $\mathbb{Q}$ by a, b and the $a_{ij}(a, b)$. Choose $\alpha\in\mathbb{C}$ algebraically independent over K_{ab}. Set $x-a=t$, $y-b=\alpha t$. Then

$$g(t)=\sum a_{ij}(a, b)\, t^i(\alpha t)^j=\sum_n\left(\sum_{i+j=n} a_{ij}(a, b)\,\alpha^j\right)t^n,$$

is D-finite. Hence by Lemma 4.4 g satisfies an equation $P(t, \partial/\partial t)\,g=0$ where $P\in K_{ab}(\alpha)[t, \partial/\partial t]$. Clearing denominators we may assume $P\in K_{ab}[\alpha][t, \partial/\partial t]$. Now we may regard α as a variable, z say, and we have

$$P\left(z, t, \frac{\partial}{\partial t}\right)\sum a_{ij}(a, b)\, t^i(zt)^j=0.$$

Hence for all $\beta\in\mathbb{C}$ we have $P(\beta, t, \partial/\partial t)\sum a_{ij}(a, b)\, t^i(\beta t)^j=0$. We may further assume for all $\beta\in\mathbb{C}$ that $P(\beta, t, \partial/\partial t)\not\equiv 0$. Hence to each point $(a, b)\in\Delta$ we have associated a differential operator $P_{ab}(z, t, \partial/\partial t)$ such that for all $\beta\in\mathbb{C}$, $P_{ab}(\beta, t, \partial/\partial t)\sum a_{ij}(a, b)\, t^i(\beta t)^j=0$, i.e., f restricted to the line through (a, b) in the direction $(1, \beta)$ satisfies the nontrivial equation

$$P_{ab}\left(\beta, t, \frac{\partial}{\partial t}\right)f(a+t, b+\beta t)=0.$$

Let P_{ab} have order n_{ab} and leading coefficient $c_{ab}(z, t)\in K_{ab}[z, t]$. For some $n_0\in\mathbb{N}$ the set $Y_{n_0}=\{(a, b)\in\Delta\colon n_{ab}=n_0\}$ is of the second category.

Choose points $(a_i, b_i)\in Y_{n_0}$, $i\in\mathbb{N}$ such that no three of them lie on the same line. Let $K_i=K_{a_ib_i}$ and let K be the compositum of the K_i's. We say that $P_i=P_{a_ib_i}$ is singular at point $(a, b)\in\Delta$ if $c(((b-b_i)/(a-a_i)), a-a_i)=0$, or if $a=a_i$. Each P_i is only singular at a set of the first category in Δ.

Now suppose that f is D-finite on every line segment in Δ, but is not D-finite, and choose $(a, b)\in\Delta$ such that

(i) the vector space dimension of $K(a, b)[((\partial^{i+j})/(\partial x^i\,\partial y^j))\, f(a, b)\colon i, j\in\mathbb{N}]$ over $K(a, b)$ is infinite.

(ii) (a, b) is not a singular point of any of the P_i's.

This is possible by Lemma 4.3 and the above discussion. By Lemma 4.3, all the derivatives of f at (a, b) in the direction $(1, ((b-b_i)/(a-a_i)))$ lie in the vector space

$$V = \left\{ \sum_{i,j<n_0} \gamma_{ij} \frac{\partial^{i+j} f}{\partial x^i\, \partial y^j}(a, b) \colon \gamma_{ij} \in K(a, b) \right\}.$$

Hence all the derivatives of f at (a, b) lie in V. To see this note that the nth derivative of f at (a, b) in the direction $(1, \alpha_i)$ is $\sum_{j=0}^{n} \alpha_i^j (\partial/\partial x)^{n-j} (\partial/\partial y^j) f(a, b) \in V$ and that if $\alpha_i = ((b-b_i)/(a-a_i))$, $i = 1, ..., n$ then $0 \neq \det(\alpha_i^j) \in K(a, b)$. This, however, contradicts our choice of (a, b) and completes the proof of Theorem 4.1.

SECTION 5

In this section we prove the converse of Theorem 3.8(viii).

THEOREM 5.1. *Let $a(i, j)$ satisfy*

for each i only finitely many of the $a(i, j)$ are nonzero, (5.1)

and suppose that for every P-recursive $b(j)$ the matrix product $(a(i, j))(b(j)) = (\sum_j a(i, j) b(j))$ is P-recursive. Then $a(i, j)$ is P-recursive.

We shall find it easier to work with the generating function $f(x, y) = \sum a(i, j)\, x^i\, y^j$. We assume that the sequence $a(i, j)$ satisfies the hypotheses of Theorem 5.1.

LEMMA 5.2. *$f(x, y)$ satisfies a nontrivial equation of the form*

$$P\left(x, y, \frac{\partial}{\partial x}\right) f = 0. \tag{5.2}$$

Proof. Let K be the field generated by the $a(i, j)$ and let α be transcendental over K. Then $b(j) = \alpha^j$ is certainly P-recursive so $h(x) = \sum a(i, j)\, \alpha^j x^i$ is D-finite. By Lemma 4.4 there is a $\bar{P}(x, \partial/\partial x) \in K(\alpha)[x, \partial/\partial x]$ satisfying $\bar{P}(x, \partial/\partial x) h = 0$. Clearing denominators we may assume $\bar{P} \in K[\alpha, x, \partial/\partial x]$, say $\bar{P} = P(x, \alpha, \partial/\partial x)$. Replacing α in the equation $P(x, \alpha, \partial/\partial x) h(x) = 0$ by y we get $P(x, y, \partial/\partial x) f(x, y) = 0$. ∎

LEMMA 5.3. *f satisfies an equation of the form*

$$Q\left(x, y\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right) f = 0. \tag{5.3}$$

Proof. Let α and β be algebraically independent over K and let $b(j) = (j!/\alpha(1-\alpha)\cdots(j-\alpha))\,\beta^j$. Then $b(j)$ is certainly P-recursive. Hence $l(x) = \sum a(i,j)(j!/\alpha\cdots(j-\alpha))\,\beta^j x^i$ is D-finite. As in the proof of Lemma 5.2 $l(x)$ satisfies an equation of the form

$$R\left(x, \alpha, \beta, \frac{\partial}{\partial x}\right) l(x) = 0$$

with $R(x, \alpha, \beta, \partial/\partial x) \in K[x, \alpha, \beta][\partial/\partial x]$. For any $c \in \mathbb{C}$, $cl(x)$ also satisfies this equation. Let $n \in \mathbb{N}$ and set $c = n - \alpha$. Then

$$R\left(x, \alpha, \beta, \frac{\partial}{\partial x}\right)(n-\alpha)\sum_{i,j} a(i,j)\frac{j!}{\alpha\cdots(j-\alpha)}\beta^j x^i = 0$$

so

$$\begin{aligned} &R\left(x, \alpha, \beta, \frac{\partial}{\partial x}\right)(n-\alpha)\sum_{\substack{i\\ j<n}} \frac{j!}{\alpha\cdots(j-\alpha)} a(i,j)\,\beta^j x^i \\ &\quad + R\left(x, \alpha, \beta, \frac{\partial}{\partial x}\right)\frac{1}{\alpha\cdots(n-1-\alpha)} \\ &\quad \times \sum_{\substack{i\\ j\geqslant n}} \frac{j!}{(n+1-\alpha)\cdots(j-\alpha)} a(i,j)\,\beta^j x^i = 0. \end{aligned}$$

This equation holds in $K(\alpha, \beta)[[x]]$. Since α and β are algebraically independent over K we may treat them as variables. In this equation we may set $\alpha = n$ (since no denominator is divisible by $n - \alpha$) to get

$$R\left(x, n, \beta, \frac{\partial}{\partial x}\right)\frac{1}{n(1-n)\cdots(-1)}\sum_{\substack{i\\ j\geqslant n}} \frac{j!}{1.2.\cdots(j-n)} a(i,j)\,\beta^j x^i = 0.$$

So, taking out the common factor of $\beta^n/n(1-n)\cdots(-1)$, we have

$$R\left(x, n, \beta, \frac{\partial}{\partial x}\right)\sum_{\substack{i\\ j\geqslant n}} j(j-1)\cdots(j-n+1)a(i,j)\,\beta^{j-n}x^i = 0.$$

Now set $\beta = 0$ to get

$$R\left(x, n, 0, \frac{\partial}{\partial x}\right)\sum_i a(i,n)\,x^i = 0. \tag{5.4}$$

(By extracting a suitable power of β we may assume that $R(x, \alpha, 0, \partial/\partial x) \not\equiv 0$. Hence $R(x, n, 0, \partial/\partial x) \in \mathbb{C}[x, n, \partial/\partial x]$ is not identically zero. It may be zero for a finite number of values of n). Now let $D_y = y(\partial/\partial y)$. Then $D_y y^n = ny^n$ and from (4) we have $R(x, n, 0, \partial/\partial x) \sum_i a(i, n) x^i y^n = 0$ and hence

$$R\left(x, D_y, 0, \frac{\partial}{\partial x}\right) \sum_i a(i, n) x^i y^n = 0$$

and finally

$$R\left(x, D_y, 0, \frac{\partial}{\partial x}\right) \sum_{i,j} a(i, j) x^i y^j = 0.$$

Hence if we take $Q(x, D_y, \partial/\partial x) = R(x, D_y, 0, \partial/\partial x)$ the lemma is proved. ∎

LEMMA 5.4. *Let f be as above. Then f satisfies two nontrivial equations of the form*

$$\begin{aligned} R\left(x, y, \frac{\partial}{\partial x}\right) T\left(x, \frac{\partial}{\partial x}\right) f &= 0 \\ S\left(x, y, \frac{\partial}{\partial y}\right) T\left(x, \frac{\partial}{\partial x}\right) f &= 0. \end{aligned} \tag{5.5}$$

Proof. From Lemmas 5.2 and 5.3 f satisfies equations of the form (5.2) and (5.3) above. Let

$$Q\left(x, D_y, \frac{\partial}{\partial x}\right) = Q_0\left(x, \frac{\partial}{\partial x}\right) D_y^k + Q_1\left(x, \frac{\partial}{\partial x}\right) D_y^{k-1} + \cdots + Q_k\left(x, \frac{\partial}{\partial x}\right).$$

Applying D_y^l to equation (5.3) we get

$$\left\{Q_0\left(x, \frac{\partial}{\partial x}\right) D_y^{k+l} + Q_1\left(x, \frac{\partial}{\partial x}\right) D_y^{k+l-1} + \cdots + Q_k\left(x, \frac{\partial}{\partial x}\right) D_y^l\right\} f = 0$$

and so

$$Q_0 D_y^{k+l} f = -Q_1 D_y^{k+l-1} f - \cdots - Q_k D_y^l f.$$

By Lemma 2.8 there are nonzero $A(x, \partial/\partial x)$, $B(x, \partial/\partial x)$ such that $AQ_1 = BQ_0$. Then

$$\begin{aligned} AQ_0 D_y^{k+l} f &= -BQ_0 D_y^{k+l-1} f - AQ_2 D_y^{k+l-2} f - \cdots - AQ_k D_y^l f \\ &= -B\{-Q_1 D_y^{k+l-2} f - \cdots - Q_k D_y^{l-1} f\} \\ &\quad - AQ_2 D_y^{k+l-2} f - \cdots - AQ_k D_y^l f \\ &= C\left(x, \frac{\partial}{\partial x}\right) D_y^{k+l-2} f + D\left(x, \frac{\partial}{\partial x}\right) D_y^{k+l-3} f + \cdots. \end{aligned}$$

Iterating, we get that for each l there are nonzero $\tilde{Q}_l(x, \partial/\partial x)$, $\tilde{Q}_{l_i}(x, \partial/\partial x)$ $i = 0, ..., k-1$ such that $\tilde{Q}_l D_y^{k+l} f = \sum_{i<k} \tilde{Q}_{l_i} D_y^i f$. Again by Lemma 2.8, for any $L \in \mathbb{N}$ (to be chosen later) there are nonzero $Q(x, \partial/\partial x)$ and $Q_{l_i}(x, \partial/\partial x)$ such that for each $l \leqslant L$

$$QD_y^{k+l} f = \sum_{i<k} Q_{l_i} D_y^i f. \tag{5.6}$$

From equation (5.2) we have (if n is the order of $P(x, y, \partial/\partial x)$) that for each j

$$\left(\frac{\partial}{\partial x}\right)^j f = \sum_{m<n} r_{jm}(x, y) \left(\frac{\partial}{\partial x}\right)^m f,$$

where the $r_{jm}(x, y) \in \mathbb{C}(x, y)$. Applying this to Eq. (5.6) we have that for each $l \leqslant L$

$$\begin{aligned} QD_y^{k+l} f &= \sum_{\substack{i<k \\ m<n}} D_y^i s_{im}(x, y) \left(\frac{\partial}{\partial x}\right)^m f \\ &= \sum_{\substack{i<k \\ m<n}} t_{im}(x, y)\, D_y^i \left(\frac{\partial}{\partial x}\right)^m f \end{aligned}$$

for some $s_{im}(x, y)$, $t_{im}(x, y) \in \mathbb{C}(x, y)$. Now let $L = kn$. Then we see that the $D_y^{k+l} Q(x, \partial/\partial x) f$ for $0 \leqslant l \leqslant L$ are linearly dependent over $\mathbb{C}(x, y)$. Hence there are $u_l(x, y) \in \mathbb{C}(x, y)$ not all zero such that

$$\sum_{l=0}^{L} u_l(x, y)\, D_y^{k+l} Qf = 0.$$

Hence we may take $T(x, \partial/\partial x) = Q(x, \partial/\partial x)$ and $S(x, y, \partial/\partial y) = \sum_{l=0}^{L} u_l(x, y)\, D_y^{k+l}$ to get

$$S\left(x, y, \frac{\partial}{\partial y}\right) T\left(x, \frac{\partial}{\partial x}\right) f = 0.$$

From Eq. (5.2) we have that the $(\partial/\partial x)^i f$ are linearly dependent over

$\mathbb{C}(x, y)$. Hence also the $(\partial/\partial x)^i T(x, \partial/\partial x) f$ are. Hence there is a nonzero $R(x, y, \partial/\partial x)$ such that

$$R\left(x, y, \frac{\partial}{\partial x}\right) T\left(x, \frac{\partial}{\partial x}\right) f = 0. \quad \blacksquare$$

LEMMA 5.5. *There are $c, \gamma \in \mathbb{N}$ such that $|a(i, j)| \leqslant c(i!)^\gamma$.*

Proof. Let $f(x, y) = \sum f_i^1(y) x^i$. Then by (5.1) all the f_i^1 are polynomials and from (5.2) they satisfy a recursion of the form

$$P_k(n, y) f_{n+k}^1(y) = \sum_{j=0}^{k-1} p_j(n, y) f_{n+j}^1(y). \tag{5.7}$$

It follows that there is an integer s such that $f_n^1(y)$ is a polynomial of degree $\leqslant (n+1)s$, for all n. Hence

$$a(i, j) = 0 \qquad \text{for} \quad j > (i+1)s. \tag{5.8}$$

From (5.2) it follows that each of the section of $a(i, j)$, for j fixed, is P-recursive. The recursion for the $a(i, j)$ corresponding to (5.2) together with condition (5.8) and the recursions for these sections now determine all the $a(i, j)$ in terms of finitely many of them, and the Lemma follows exactly as in the proof of Proposition 3.11. $\blacksquare$

Since both of the sequences $(i!)^\gamma$ and $(i!)^{-\gamma}$ are P-recursive, replacing $a(i, j)$ by $a(i, j)(i!)^{-\gamma}$ we may assume that $f(x, y)$ is analytic in a neighborhood of $(0, 0)$. We assume this from now on.

LEMMA 5.6. *For $\varepsilon > 0$ but small enough there are D-finite analytic functions $f_i(x)$ and $A(x, y)$ and analytic functions $g_i(y)$ such that*

$$f(x, y) = \sum_{i=0}^{n-1} f_i(x)\, g_i(y) + A(x, y)$$

and the f_i are analytic for $0 < x < \varepsilon$, the g_i for $|y| < \varepsilon$, and A for $0 < x < \varepsilon$ and $|y| < \varepsilon$. If $x = 0$ is an ordinary point of $T(x, \partial/\partial x)$ then we can replace $0 < x < \varepsilon$ by $|x| < \varepsilon$.

Proof. From Eq. (5.5) above $T(x, \partial/\partial x) f = B(x, y)$ is D-finite and analytic near $(0, 0)$. Choose (a, b) near $(0, 0)$ such that a is an ordinary point of the equation $T(x, \partial/\partial x) F = 0$ and (a, b) lies in the circle of convergence of $f(x, y)$. Let the $f_i(x)$ be n independent analytic solutions near $x = a$ of this equation (n is the order of T). We next must find a particular solution $A(x, y) = \sum u_i(x, y) f_i(x)$ of the equation $T(x, \partial/\partial x)F = B(x, y)$. Solving this by variation of parameters we get the equations

$$\begin{bmatrix} f_0 & \cdots & f_{n-1} \\ f'_0 & \cdots & f'_{n-1} \\ \cdots & \cdots & \cdots \\ f_0^{(n-1)} & \cdots & f_{n-1}^{(n-1)} \end{bmatrix} \begin{bmatrix} u'_1 \\ u'_2 \\ \cdots \\ u'_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ B(x, y) \end{bmatrix}.$$

Let $W(x)$ be the determinant of the coefficient matrix (i.e., the Wronskian). Then

$$u'_i = \frac{C_i(x)\, B(x, y)}{W(x)},$$

where the $C_i(x)$ are sums of products of the $f_l^{(j)}(x)$. Hence each $C_i(x)\, B(x, y)$ is D-finite. Let $T(x, \partial/\partial x) = a_0(x)(\partial/\partial x)^n + a_1(x)(\partial/\partial x)^{n-1} + \cdots$. Then W satisfies the equation

$$W' = aW,$$

where $a = -a_1(x)/a_0(x)$. Hence $(1/W)' = -(1/W^2)\, W' = -aW/W^2 = -a/W$ and we see that $1/W$ is also D-finite. Hence the $u'_i(x, y)$ are all D-finite, and the D-finiteness of the $u_i(x, y)$ follows from Theorem 2.7(iii). $A(x, y)$ is analytic for $|y| < \varepsilon$, $0 < x < \varepsilon$ and hence the $g_i(y)$ are analytic for $|y| < \varepsilon$.

If $x = 0$ is an ordinary point of the equation $T(x, \partial/\partial x)F = 0$ then the f_i are analytic near $x = 0$ and $A(x, y)$ is analytic near $(0, 0)$. By taking a different basis for the solution space of $T(x, \partial/\partial x)F = 0$ we may assume that $(\partial^j f/\partial x^j)(0) = \delta_{ij}$ for $1 \leqslant i, j \leqslant n$. Then

$$\frac{\partial^j f}{\partial x^j}(0, y) = g_j(y) + A(0, y).$$

Since each $(\partial^j f/\partial x^j)(0, y)$ is a polynomial and $A(x, y)$ is D-finite so are the $g_j(y)$ and hence so is $f(x, y)$. If $x = 0$ is not an ordinary point of $T(x, \partial/\partial x)F = 0$, then the $f_i(x)$ are analytic for $0 < x < \varepsilon$, the $g_i(y)$ are analytic for $|y| < \varepsilon$, $A(x, y)$ is analytic for $0 < x < \varepsilon$ and $|y| < \varepsilon$, and $f(x, y)$ is analytic for $|x|, |y| < \varepsilon$. Since $A(x, y)$ is D-finite there is an

$$H\left(x, y, \frac{\partial}{\partial y}\right) = H_0\left(y, \frac{\partial}{\partial y}\right) + xH_1\left(y, \frac{\partial}{\partial y}\right) + \cdots$$

such that $HA=0$. We may assume $H_0 \not\equiv 0$. Then

$$Hf = \sum_{i=1}^{n} f_i(x)\, Hg_i(y). \tag{5.9}$$

We may assume that the set $\{1, H_0 g_1, ..., H_0 g_n\}$ is linearly independent for if $\sum \alpha_i H_0 g_i(y) + \beta^1 = 0$ with not all the $\alpha_i = 0$ then $(\partial/\partial y)\, H_0(y, \partial/\partial y) \sum \alpha_i g_i(y) = 0$ and hence $\sum \alpha_i g_i(y)$ is D-finite. Suppose $\alpha_1 = 1$. Then

$$f(x, y) = f_1(x) \sum \alpha_i g_i(y) + \sum_{i=2}^{n} (f_i - \alpha_i f_1)\, g_i(y) + A(x, y)$$

and $f_1(x) \sum \alpha_i g_i(y)$ is D-finite so we can absorb it into the particular solution $A(x, y)$, and use the $f_i - \alpha_i f_1$ in place of the f_i for $i = 2, ..., n$. Equation (5.9) is of the form

$$k(x, y) = \sum_{i=1}^{n} f_i(x)\, h_i(x, y), \tag{5.10}$$

where $k(x, y)$ and the $h_i(x, y)$ are analytic for $|x|$, $|y| < \varepsilon$, the $f_i(x)$ are analytic for $0 < x < \varepsilon$, and the set $\{1, h_1(0, y), ..., h_n(0, y)\}$ is linearly independent over $\mathbb{C}$. We shall show that the $f_i(x)$ are actually analytic at $x = 0$. For suppose that $f_i(x), ..., f_l(x)$ are not, and $f_{l+1}(x), ..., f_n(x)$ are. Then we may absorb the sum $\sum_{i=l+1}^{n} f_i(x)\, h_i(x, y)$ into $k(x, y)$ to get an equation of the form (5.10) with none of the $f_i(x)$ analytic at $x = 0$. Choose b with $|b| < \varepsilon$ such that the $h_i(0, b) \neq 0$ for $i = 1, ..., l$. (This is possible since the $h_i(0, y)$ are nonconstant because the set $\{1, h_i(0, y), ..., h_n(0, y)\}$ is independent.) Then

$$k(x, b) = \sum_{i=1}^{l} f_i(x)\, h_i(x, b)$$

and $f_1(x) = k(x, b)/h_1(x, b) - \sum_{i=2}^{l} f_i(x)(h_i(x, b)/h_1(x, b))$. Since $h_1(0, b) \neq 0$, $1/h_1(x, b)$ is analytic for x near zero. Substituting this into (5.10) gives

$$k(x, y) = \frac{k(x, b)}{h_1(x, b)}\, h_1(x, y) + \sum_{i=2}^{l} f_i(x) \left\{ h_i(x, y) - \frac{h_i(x, b)}{h_1(x, b)}\, h_1(x, y) \right\}.$$

Let $\bar{k}(x, y) = k(x, y) - (k(x, b)/h_1(x, b))\, h_1(x, y)$ and let $\bar{h}_i(x, y) = h_i(x, y) - (h_i(x, b)/h_1(x, b))\, h_1(x, y)$ so that this equation becomes

$$\bar{k}(x, y) = \sum_{i-2}^{l} f_i(x)\, \bar{h}_i(x, y).$$

Note that $\bar{k}$ and the $\bar{h}_i$ are analytic for (x, y) near $(0, 0)$ and that the set $\{1, \bar{h}_2(0, y), ..., \bar{h}_l(0, y)\}$ is again independent and we have an equation of the same form with one fewer $f_i(x)$ in it. Hence by induction all the $f_i(x)$ are analytic at $x=0$ (the case of just one $f_i(x)$ is trivial). Hence in equation (5.9) all the functions Hf, $f_i(x)$, and $Hg_i(y)$ are analytic near $(0, 0)$. By replacing the f_i by linear combinations of the f_i we may assume that $f_1(x) = a_j x^j +$ higher order terms with $a_j \neq 0$ and that the $f_i(x)$ for $i > 1$ vanish to order greater than j at $x=0$. Then

$$\left(\frac{\partial}{\partial x}\right)^j Hf(0, y) = a_j H_0 g_1(y)$$

and so $H_0 g_1(y)$ is a polynomial in y. Hence $g_1(y)$ is D-finite and we could have absorbed $f_1(x)\, g_1(y)$ into the D-finite particular solution $A(x, y)$. Hence by induction all the $g_i(y)$ are D-finite and so is $f(x, y)$. This completes the proof of Theorem 5.1. ∎

Remark 5.7. (i) Theorem 5.1 is true over every uncountable field of characteristic zero. If one restricts oneself to a countable subfield $L \subset \mathbb{C}$, there is a sequence $a(i, j)$ satisfying (5.1) and such that for every P-recursive $b(j)$ from L all but finitely many of the $\sum_j a(i, j) b(j)$ are zero, but $a(i, j)$ is not P-recursive. This is reminiscent of the theorem of [PR].

(ii) Conditions (5.1) and (5.2) are not enough to ensure that f is D-finite (or in fact even differentially algebraic). This is shown by the following example. Let $f(x, y) = \int_0^{1+x} t^y e^{-t}\, dt$. Then $\Gamma(y+1) = \lim_{x \to \infty} f(x, y)$. If $f(x, y)$ were D-finite then so would $\lim_{x \to \infty} f(x, y)$ be. (Divide the equation $Q(x, y, \partial/\partial y) f(x, y) = 0$ by the highest power of x occurring in Q before taking the limit.) Now $\partial f/\partial x = (1+x)^y e^{-(1+x)}$, $\partial^2 f/\partial x^2 = y(1+x)^{y-1} - (1+x)^y e^{-(1+x)} = \{y/(1+x) - 1\}(\partial f/\partial x)$ and hence f satisfies an equation of the above form. Write $f(x, y) = \sum A_n(y)\, x^n$. Then the A_n satisfy

$$(n+1)nA_{n+1} + n(n-1)\, A_n = (y-1)nA_n - (n-1)\, A_{n-1}.$$

So $A_0 = 0$, $A_1 = 1/e$, $A_2 = (1/e)(y-1)$ and in general $A_n(y)$ is a polynomial in y of degree $n-1$. Hence $f(x, y)$ satisfies (5.1).

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