Robust Positivity Problems for low-order Linear Recurrence Sequences

The frontiers of decidability for explicitly given neighbourhoods

Mihir Vahanwala ⊠®

MPI-SWS, Saarland Informatics Campus, Saarbrücken, Germany

— Abstract

Linear Recurrence Sequences (LRS) are a fundamental mathematical primitive for a plethora of applications such as the verification of probabilistic systems, model checking, computational biology, and economics. Positivity (are all terms of the given LRS at least 0?) and Ultimate Positivity (are all but finitely many terms of the given LRS at least 0?) are important open number-theoretic decision problems. Recently, the robust versions of these problems, that ask whether the LRS is (Ultimately) Positive despite small perturbations to its initialisation, have gained attention as a means to model the imprecision that arises in practical settings. In this paper, we consider Robust Positivity and Ultimate Positivity problems where the neighbourhood of the initialisation, specified in a natural and general format, is also part of the input. We contribute by proving sharp decidability results: decision procedures at orders our techniques can't handle would entail significant number-theoretic breakthroughs.

2012 ACM Subject Classification Theory of computation; Logic and verification

Keywords and phrases Dynamical Systems, Verification, Robustness, Linear Recurrence Sequences, Positivity, Ultimate Positivity

Digital Object Identifier 10.4230/LIPIcs...

1 Introduction

A real Linear Recurrence Sequence (LRS) of order κ is an infinite sequence of real numbers $\langle u_0, u_1, u_2, \ldots \rangle$ having the following property: there exist κ real constants $a_0, \ldots, a_{\kappa-1}$, with $a_0 \neq 0$ such that for all $n \geq 0$:

$$u_{n+\kappa} = a_{\kappa-1}u_{n+\kappa-1} + \dots a_0u_n \tag{1}$$

An LRS is uniquely specified given the constants $a_0, \ldots, a_{\kappa-1}$, and the initial terms $u_0, \ldots, u_{\kappa-1}$. The best-known example is the Fibonacci sequence $(0, 1, 1, 2, 3, 5, 8, \ldots)$, satisfying the recurrence relation $u_{n+2} = u_{n+1} + u_n$: it is named after Leonardo of Pisa, who used it to model the population growth of rabbits. LRS have been extensively studied, and found several mathematical and scientific applications since. The monograph of Everest *et al.* [15] is a comprehensive treatise on the mathematical aspects of Recurrence Sequences.

Important number-theoretic decision problems for Linear Recurrence Sequences include Positivity (is $u_n \geq 0$ for all n?), Ultimate Positivity (is $u_n \geq 0$ for all but finitely many n?) and the Skolem Problem (is $u_n = 0$ for some n?). These problems have applications in software verification, probabilistic model checking, discrete dynamic systems, theoretical biology, and economics. Decidability has been open for decades: Ouaknine and Worrell [23] showed Positivity and Ultimate Positivity are decidable up to order 5 but number-theoretically hard at order 6, whereas Mignotte et al. [19] and Vereshchagin [27] independently proved the

XX:2 Robust Positivity for low-order LRS

Skolem Problem to be decidable up to order 4. These results were proven given the *rational* $a_0, \ldots, a_{\kappa-1}, u_0, \ldots u_{\kappa-1}$ as input, but can be generalised to real algebraic input as well. In this paper, we focus on Positivity and Ultimate Positivity for real algebraic sequences.

In contrast, the *uninitialised* variants of these problems are far more tractable. [10, 26] consider whether *every* possible initialisation keeps the sequence Positive, and decide so in PTIME. More recently, this result has been extended to processes with choices [3]. We argue that practical applications need a middle ground: recurrence relations that arise in practice need to be contextualised by actual instances of sequences; however, considering *precise* initialisations does not account for inherently imprecise real world measurements, and the requirement of safety margins. We thus study robust variants: given a recurrence and an initialisation, do all initialisations in a neighbourhood satisfy (Ultimate) Positivity?

Related Work

In this paper, we focus on the neighbourhood-of-initialisation notion of robustness, which was first introduced in [1], and more comprehensively treated in [2]. There are, however, different approaches to tackle imprecision: [21] considers a model of computation that can take arbitrary real numbers as input. Works with a more control-theoretic flavour include [4], which allows for rounding at every step before applying the recurrence; in the same vein, [13] allows for ε -disturbances at every step. Our notion of robustness has been considered in [1, 2, 13], however, these works primarily concern themselves with simply deciding whether there *exists* a neighbourhood around the given point that satisfies Positivity. Although they do identify that robust problems are hard when the neighbourhood is given as input, in the absence of decidability results, their hardness results are not sharp.

Our contribution

We address the gap in the robustness state-of-the-art by exploring the frontiers of decidability when the neighbourhood is given as input. In §2 we define the robustness problems we consider: we use the Mahalanobis distance to specify neighbourhoods. Our parameter is the positive definite matrix \mathbf{S} , and the neighbourhood of \mathbf{c} it specifies is the set of all points $\mathbf{c}' \in \mathbb{R}^{\kappa}$ such that $(\mathbf{c}' - \mathbf{c})^T \mathbf{S}(\mathbf{c}' - \mathbf{c}) \leq 1$. The size of neighbourhoods is usually parametrised by an ε : in our case, we can account for it by simply scaling \mathbf{S} . In the statistical context, \mathbf{S} is the inverse of a covariance matrix; and thus, our formulation is a rather natural way of capturing noise and measurement errors in the input. Our novelty, to the best of our knowledge, lies in identifying a general and practical way of explicitly specifying neighbourhoods, and establishing the first decidability results in such a setting, albeit at low orders. On the Diophantine approximation front, we prove hardness results at lower orders than ever before. Our contributions are summarised in the table below.

Problem: S-Robust	Decidability	Hardness
Positivity	§4, Up to order 4	§6, Diophantine hard at order 5
Uniform Ultimate Positivity	§4, Up to order 4	§6, Lagrange hard at order 5
Non-uniform Ultimate Positivity	§5, Up to order 4	[23, 2], Lagrange hard at order 6

Table 1 Problems are as formally defined in §2. The distinction between uniform and non-uniform refers to whether the threshold index for certifying Ultimate Positivity must be common for the entire neighbourhood. Hardness notions are defined in §3.

Notation and Prerequisites

For the purposes of discussing robustness, we shall use \mathcal{B} to denote the unit ball in \mathbb{R}^{κ} , centred at the origin. Similarly, we use $\mathcal{B}_{\mathbf{S}}$ to denote the set of \mathbf{d} such that $\mathbf{d}^T \mathbf{S} \mathbf{d} \leq 1$. For real column vectors \mathbf{x}, \mathbf{y} , we use $\langle \mathbf{x}, \mathbf{y} \rangle$ to denote the inner product $\mathbf{x}^T \mathbf{y} = \mathbf{y}^T \mathbf{x}$. $||\mathbf{x}||$ denotes the standard L2 norm $\sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$

Throughout this paper, \mathbb{N} , \mathbb{Z} , \mathbb{Q} , \mathbb{R} , and \mathbb{C} respectively denote the natural numbers, integers, rationals, reals, and complex numbers. $\alpha \in \mathbb{C}$ is said to be algebraic if it is a root of a polynomial with integer coefficients. Algebraic numbers form an algebraically closed field, denoted by $\overline{\mathbb{Q}}$. We denote the field of real algebraic numbers by \mathbb{A} .

In this paper, we assume our input consists of real algebraic numbers. Appendix A contains a brief initiation to this number field. The key takeaways are that the usual arithmetic can be carried out with perfect precision, and that the First Order Theory of the Reals $\langle \mathbb{R}; +, \cdot, \geq, 0, 1 \rangle$ is a decidable logical system powerful enough to fit our purposes.

▶ Theorem 1 (Renegar [25]). Let $M \in \mathbb{N}$ be fixed. Let $\chi(\mathbf{x})$ be a First Order Logic formula interpreted over the reals with fewer than M variables in total. There exists a procedure that returns an equivalent quantifier-free formula $\psi(\mathbf{x})$ in disjunctive normal form. This procedure runs in time polynomial in the size of the representation of χ .

2 Linear Recurrences, Robustness, and convenient bases

We start approaching robustness by decoupling the elements of an LRS: namely, the recurrence relation, and the initialisation.

- ▶ Definition 2 (Linear Recurrence Relation (LRR)). A real algebraic LRR $\mathbf a$ of order κ is a $\kappa+1$ -ary relation, specified by κ numbers, $a_0,\ldots,a_{\kappa-1}\in\mathbb A$, with $a_0\neq 0$. $\mathbf a(Y_0,Y_1,\ldots,Y_\kappa)$ is interpreted as $Y_\kappa=\sum_{j=0}^{\kappa-1}a_jY_j$
- ▶ **Definition 3** (Characteristic polynomial of an LRR). Given an LRR **a**, its characteristic polynomial is $X^{\kappa} \sum_{j=0}^{\kappa-1} a_j X^j$
- ▶ **Definition 4** (Linear Recurrence Sequence (LRS)). A real algebraic LRS \mathbf{u} of order κ is an infinite sequence $\langle u_n \rangle_{n=0}^{\infty}$, given by a real algebraic order κ LRR \mathbf{a} and the initialisation $\mathbf{c} = (u_0, u_1, \dots, u_{\kappa-1}) \in \mathbb{A}^{\kappa}$. For all $n \in \mathbb{N}$, $\mathbf{a}(u_n, u_{n+1}, \dots, u_{n+\kappa})$ holds.

One can also encode the recurrence **a** as a $\kappa \times \kappa$ companion matrix **A**, and interpret the initialisation **c** as a vector. Then, u_n is given by the first coordinate of $\mathbf{A}^n \mathbf{c}$.

We now list decision problems: recall that in this paper, we consider that our input consists of real algebraic numbers.

- ▶ **Problem 1** (Positivity). An LRS $\langle u_n \rangle_{n=0}^{\infty}$ is given as (\mathbf{a}, \mathbf{c}) . Decide whether, for all $n \in \mathbb{N}$, $u_n \geq 0$.
- ▶ Problem 2 (Ultimate Positivity). Given an LRS (\mathbf{a}, \mathbf{c}) , decide whether there exists an N such that for all $n \geq N$, $u_n \geq 0$.

A Positive LRS is necessarily Ultimately Positive. In the general case, [23] shows both Positivity and Ultimate Positivity to be decidable up to order 5, while demonstrating number-theoretic hardness in the sense Definition 10 at order 6. On restricting our attention to simple LRS (the characteristic polynomial has no repeated root), Positivity is decidable up

to order nine [22], while Ultimate Positivity is decidable [24]. In this text, however, we shall focus on defining and tackling robust versions of these problems.

- ▶ **Problem 3** (S-Robust Positivity). Given an LRS (\mathbf{a}, \mathbf{c}) , and positive definite S, decide whether for all \mathbf{c}' such that $(\mathbf{c}' \mathbf{c})^T \mathbf{S} (\mathbf{c}' \mathbf{c}) \le 1$, the LRS $(\mathbf{a}, \mathbf{c}')$ is positive.
- ▶ Problem 4 (S-Robust Uniform Ultimate Positivity). Given an LRS (\mathbf{a}, \mathbf{c}), and positive definite S, decide whether there exists an N such that for all \mathbf{c}' with $(\mathbf{c}' \mathbf{c})^T \mathbf{S} (\mathbf{c}' \mathbf{c}) \leq 1$, the LRS (\mathbf{a}, \mathbf{c}') is positive from the Nth term onwards.

We can alternate quantifiers, and query a weaker notion of Robust Ultimate Positivity:

▶ Problem 5 (S-Robust Non-uniform Ultimate Positivity). Given an LRS (\mathbf{a}, \mathbf{c}) , and positive definite S, decide whether for all \mathbf{c}' with $(\mathbf{c}' - \mathbf{c})^T \mathbf{S} (\mathbf{c}' - \mathbf{c}) \leq 1$, there exists an N such that the LRS $(\mathbf{a}, \mathbf{c}')$ is positive from the Nth term onwards.

The attentive reader might have already noticed that we depart from convention and specify neighbourhoods as *closed* balls. Although [2] does not solve the problems we consider in this paper, it makes crucial observations about the geometry: for Problems 3 and 4, there is no difference between open and closed balls. On the other hand, Problem 5 becomes considerably easier with open balls, and its decidability assuming open balls is tackled in [2] itself.

For any order κ LRS $\langle \mathbf{a}, \mathbf{c} \rangle$, the n^{th} term u_n has the following closed form, consisting of κ monomials:

$$u_n = \sum_{j} \sum_{\ell=0}^{m_j - 1} p_{j\ell} \gamma_j^n n^{\ell} \tag{2}$$

here γ_j is a root of the characteristic polynomial with multiplicity m_j . We note that each $p_{j\ell}$ is linear in \mathbf{c} . This can be seen in two ways: (a) express the companion matrix \mathbf{a} in Jordan Normal form, and use it to compute $\mathbf{A}^n\mathbf{c}$ explicitly; (b) observe that sequences satisfying the linear recurrence \mathbf{a} form a vector space, and note (by taking derivatives of the characteristic polynomial) that the sequences $\langle \gamma_j^n \rangle_n, \langle n \gamma_j^n \rangle_n, \ldots, \langle n^{m_j-1} \gamma_j^n \rangle_n$ all satisfy the recurrence.

The above form 2 is called the **exponential polynomial** solution of **u**. Roots of the characteristic polynomial with the maximum modulus are called **dominant**. We note that we are working with a real algebraic LRS: thus, all γ are algebraic, and if complex, occur in conjugate pairs. Moreover, the corresponding complex coefficients $p_{j\ell}$ must also occur in conjugate pairs to make u_n real. Thus, we can also express u_n as:

$$u_n = \left(\sum_{j=1}^{k_1} \sum_{\ell=0}^{m_j - 1} z_{j\ell} \rho_j^n n^{\ell}\right) + \left(\sum_{j=k_1 + 1}^{k_2} (x_{j\ell} \cos n\theta_j + y_{j\ell} \sin n\theta_j) \rho_j^n n^{\ell}\right)$$
(3)

where each $\rho_j \in \mathbb{R}$. Note that γ_j , if not real, is written as $\rho_j(\cos\theta_j + i\sin\theta_j)$. We call the above the **real exponential polynomial** solution of the LRS **u**.

Let $\mathbf{p} \in \mathbb{A}^{\kappa} \subset \mathbb{R}^{\kappa}$ be the vector of coefficients in equation 3. We can thus write

$$u_n = \langle \mathbf{p}, \mathbf{q_n} \rangle = \begin{bmatrix} \rho_1^n & n\rho_1^n & \dots & n^{m_{k_1}-1}\rho_{k_1}^n & \rho_{k_1+1}^n \cos n\theta_{k_1+1} & \rho_{k_1+1}^n \sin n\theta_{k_1+1} & \dots \end{bmatrix} \mathbf{p}$$
 (4)

From the discussion surrounding equation 2, it is clear that we can easily compute an invertible linear map V such that Vp = c. Thus, we can equivalently consider our robust problems in the coefficient space, replacing S with $M = V^T S V$, which is also positive definite. We can also decompose M as BB^T : indeed, M, being symmetric positive definite, is unitarily diagonalisable with positive eigenvalues, so there is an algebraic unitary matrix U and algebraic diagonal matrix D such that $M = UD^2U^T$.

Consider Problems 3 and 4. They boil down to asking whether for all \mathbf{p}' with $(\mathbf{p}' - \mathbf{p})^T \mathbf{B} \mathbf{B}^T (\mathbf{p}' - \mathbf{p}) = 1$, for all (but finitely many n) $\mathbf{q_n}^T \mathbf{p}' \geq 0$. \mathbf{p}' , in turn, can be expressed as $\mathbf{p} + \mathbf{d}$. Let $\mathbf{f} = \mathbf{B}^T \mathbf{d}$. Thus, it is equivalent to ask whether

$$\langle \mathbf{p}, \mathbf{q_n} \rangle \ge \max_{\mathbf{f} \in \mathcal{B}} \mathbf{q_n}^T \left(\mathbf{B}^{-1} \right)^T \mathbf{f} = ||\mathbf{B}^{-1} \mathbf{q_n}|| = \sqrt{\langle \mathbf{b_1}, \mathbf{q_n} \rangle^2 + \dots + \langle \mathbf{b_{\kappa}}, \mathbf{q_n} \rangle^2}$$
 (5)

▶ Theorem 5 (First Main Decidability Result). Problems 3 and 4 are decidable up to order 4.

To tackle Problem 5, we consider the following rearrangement of equation 4:

$$u_n/n^d \rho^n = \begin{bmatrix} \mathbf{q}_{dom}^T(n) & \mathbf{q}_{res}^T(n) \end{bmatrix} \begin{bmatrix} \mathbf{p}_{dom} \\ \mathbf{p}_{res} \end{bmatrix}$$
 (6)

We split the inner product into a dominant (red) and a residual (black) part. In the limit, the latter becomes negligible relative to the former. We have normalised the inner product, so the dominant part is $\Theta(1)$. Let $\mu = \liminf_{n \in \mathbb{N}} \langle \mathbf{q}_{dom}(n), \mathbf{p}_{dom} \rangle$. If $\mu < 0$, then Ultimate Positivity is impossible. If $\mu > 0$, then we can provide an N such that $u_n > 0$ beyond the N^{th} term. Thus, the core idea is to ensure that the given neighbourhood lies entirely in the region where $\mu \geq 0$. Although Ultimate Positivity is guaranteed for the points where the inequality is strict, the neighbourhood may intersect the critical region where $\mu = 0$. If there are no non-dominant terms, this is irrelevant; but otherwise decidability hinges on whether we can handle the intersection.

▶ **Theorem 6** (Second Decidability Result). Problem 5 is decidable up to order 4.

3 Diophantine Approximation

Diophantine Approximation is a vast and active number-theoretic field of research concerned, among other things, with the approximation of reals by rational numbers. In this section, we follow Lagarias and Shallit's terminology [16] and briefly introduce classes of constants whose computation is an open problem. In what follows, [x] denotes the shortest distance from x to an integer; while $[x]_b$ denotes the shortest distance from x to an integer multiple of x. $[x]_b = b[x/b]$

- ▶ **Definition 7** (Diophantine Approximation Type). The homogenous Diophantine approximation type L(t) is defined to be $\inf_{n \in \mathbb{N} \setminus 0} n[nt]$. The inhomogeneous Diophantine approximation type L(t,s) is defined to be $\inf_{n \in \mathbb{N} \setminus 0} n[nt-s]$, $s \notin \mathbb{Z} + t\mathbb{Z}$.
- ▶ Definition 8 (Lagrange constant). The homogenous Lagrange constant $L_{\infty}(t)$ is defined to be $\liminf_{n\in\mathbb{N}} n[nt]$. The inhomogeneous Lagrange constant $L_{\infty}(t,s)$ is defined to be $\liminf_{n\in\mathbb{N}} n[nt-s], s \notin \mathbb{Z} + t\mathbb{Z}$.

It is known (Dirichlet, Minkowski [20]) that these constants lie between 0 and 1. As an immediate corollary, one observes that

$$\exists c. \ \forall t, s. \ \exists^{\infty} n. \ 1 - \cos(nt - s) \le \frac{1}{2} \left[nt - s \right]_{2\pi}^{2} \le \frac{c}{n^{2}}$$
 (7)

We record a useful number-theoretic fact: its proof relies on the Ostrowski numeration system [9, 7], and is deferred to Appendix B.

▶ **Lemma 9.** For every irrational number x, strictly decreasing real positive function ψ , and interval $\mathcal{I} = [a,b] \subset [0,1]$, $a \neq b$, there exists $y \in \mathcal{I}$ such that $[nx-y] < \psi(n)$ for infinitely many odd, and infinitely many even n.

Following [23], we note that the Diophantine approximation type and Lagrange constant of most transcendental numbers are unknown, and define

$$\mathcal{A} = \{ p + qi \in \mathbb{C} \mid p, q \in \mathbb{A}, p^2 + q^2 = 1, \forall n. \ (p + qi)^n \neq 1 \}$$
 (8)

i.e., the set of points on the unit circle of $\mathbb C$ with rational real and imaginary parts, excluding 1,-1,i and -i. The set $\mathcal A$ consists of algebraic numbers, none of which are roots of unity. In particular, writing $p+qi=e^{i2\pi\theta}$, we have that $\theta\notin\mathbb Q$. We denote:

$$\mathcal{T} = \left\{ \theta \in (-1/2, 1/2] \mid e^{2\pi i \theta} \in \mathcal{A} \right\} \tag{9}$$

The set \mathcal{T} is dense in $\left(-\frac{1}{2}, \frac{1}{2}\right]$. In general, we don't have a method to compute $L(\theta)$ or $L_{\infty}(\theta)$ for $\theta \in \mathcal{T}$, or approximate them with arbitrary precision.

- ▶ **Definition 10** (Number-theoretic hardness). Let \mathcal{T} be as above. A decision problem is said to be \mathcal{T} -Diophantine hard (resp. \mathcal{T} -Lagrange hard), if its decidability entails that given any $t \in \mathcal{T}$ and $\varepsilon > 0$, one can compute ℓ such that $|\ell L(t)| < \varepsilon$ (resp. $|\ell L_{\infty}(t)| < \varepsilon$).
- ▶ **Theorem 11** (Main Hardness Result). Problem 3 (resp. Problem 4) is \mathcal{T} -Diophantine hard (resp. \mathcal{T} -Lagrange hard) at order 5.
- [2] notes that in view of the Lagrange hardness (Definition 10) of Ultimate Positivity at order 6 [23], Problem 5 is also Lagrange hard at order 6: one can simply construct a neighbourhood that lies entirely in the region $\mu > 0$ (i.e. neighbourhoods for which Ultimate Positivity can be *certified*), except for a hard instance of Ultimate Positivity on its surface. In this way, Ultimate Positivity is guaranteed for all but the single critical point on the boundary.
- ▶ **Theorem 12.** Problem 5 is \mathcal{T} -Lagrange hard at order 6.

4 Uniform Robustness: Decidability at order four

In this section, we prove Theorem 5. The techniques naturally apply to lower orders, and we omit their explicit treatment. Recall that we must show how to check the validity of inequality 5 for all (but finitely many) n:

$$\langle \mathbf{p}, \mathbf{q_n} \rangle \geq \max_{\mathbf{f} \in \mathcal{B}} \mathbf{q_n}^T \left(\mathbf{B}^{-1} \right)^T \mathbf{f} = ||\mathbf{B}^{-1} \mathbf{q_n}|| = \sqrt{\langle \mathbf{b_1}, \mathbf{q_n} \rangle^2 + \dots + \langle \mathbf{b_{\kappa}}, \mathbf{q_n} \rangle^2}$$

Our main case distinction is on the number of pairs of complex conjugates among the roots. In each case, the plan is to first decide whether there exists a threshold index N beyond which the LHS is always positive, and if so, compute it. This can always be done for order four LRS [23]. We then square throughout, and transfer all terms to the LHS, thus obtaining a new instance of (Ultimate) Positivity. This plan is trivial to execute when all characteristic roots are real.

Thanks to the following old result [6, Thm. 2], the answer to Ultimate Positivity is trivially NO if there are two pairs of complex conjugates.

▶ Proposition 13. If the characteristic polynomial has no real dominant root of maximum multiplicity, then in any full-dimensional neighbourhood of initialisations, there exists an initialisation, such that the sequence has infinitely many positive terms, and infinitely many negative terms.

Thus, we shall concern ourselves with the case there is one pair of complex conjugate roots. Without loss of generality, we also assume that the real dominant root is 1. We rely on specific applications of the following general results to analyse the trigonometric terms that arise.

▶ **Theorem 14** (Masser [17]). Let $e^{i\theta_1}, ..., e^{i\theta_k}$ be complex algebraic numbers of unit modulus. Consider the free abelian group L defined by $L = \{(\lambda_1, ..., \lambda_k) \in \mathbb{Z}^k : e^{i(\lambda_1\theta_1 + ... + \lambda_k\theta_k)} = 1\}$. The group L has a finite generator set $\{\mathbf{l_1}, ..., \mathbf{l_p}\} \subset \mathbb{Z}^k$ with $p \leq k$. The generator set can be computed in time polynomial and each entry in the generator set is polynomially bounded in the sizes of the representations of $e^{i\theta_1}, ..., e^{i\theta_k}$.

Given $e^{2i\theta}$, if an n such that $e^{2in\theta} = 1$ exists (θ is a rational multiple of 2π), this n is bounded, and can be efficiently found by enumeration. Similarly, one can check whether $\alpha^n = \beta$ (is $n\theta - \varphi$ ever an integer multiple of 2π ?). Suppose θ is not a rational multiple of 2π . Then the following theorem establishes that $\{n\theta \text{ modulo } 2\pi\}_{n\in\mathbb{N}}$ is dense in $[0, 2\pi]$. Let h be a trigonometric function of t with period 2π . Another immediate consequence is that

$$\liminf_{n} h(n\theta) = \min_{t \in [0,2\pi]} h(t) \tag{10}$$

- ▶ **Theorem 15** (Kronecker [8]). Let $\theta_1, ..., \theta_k, \phi_1, ..., \phi_k \in [0, 2\pi)$. Then the following are equivalent:
- 1. For every tuple $(\lambda_1,...\lambda_k)$ of integers with $\lambda_1\theta_1 + \cdots + \lambda_k\theta_k \in 2\pi\mathbb{Z}$, we have $\lambda_1\phi_1 + \cdots + \lambda_k\phi_k \in 2\pi\mathbb{Z}$.
- **2.** For any $\epsilon > 0$, there exists an arbitrarily large n such that for all $1 \leq j \leq k$ we have $|n\theta_j \phi_j| \leq \epsilon$.

Considering the state-of-the-art, the following lemma is rather immediate.

- ▶ Lemma 16 (Decidability for Simple LRS). Problems 3 and 4 are decidable for simple LRS of order four.
- **Proof.** Let the distinct characteristic roots be $1, \alpha, \gamma, \bar{\gamma}$. Indeed, any inner product $\langle \mathbf{v}, \mathbf{q_n} \rangle$ may also expressed as $f_1 + f_2 \alpha^n + f_3 \gamma^n + \bar{f_3} \bar{\gamma}^n$. On squaring throughout after the initial Positivity check, and transferring all terms to the LHS, we get a Positivity (Problem 1) instance for a new simple LRS, this time of order at most 10. This time, the characteristic roots are $1, \alpha^2, \gamma^2, \bar{\gamma}^2, \alpha, \gamma, \bar{\gamma}, \alpha\gamma, \alpha\bar{\gamma}, \gamma\bar{\gamma}$: these are precisely the bases of the exponents in $(f_1 + f_2 \alpha^n + f_3 \gamma^n + \bar{f_3} \bar{\gamma}^n)^2$.

We assume $\alpha \neq -1$: if it were, the resulting LRS could be decomposed into two LRS of lower order, and both Ultimate Positivity [24] and Positivity [22] for simple LRS is known to be decidable for order up to nine. For the same reason, we can also assume $\gamma/\bar{\gamma}$ is not a root of unity: this can be efficiently detected, see Theorem 14. Thus, depending on whether $|\gamma| < 1$ or = 1, the resulting LRS either has only 1 as a dominant root, and nine non-dominant roots, or has five dominant roots, $1, \gamma, \bar{\gamma}, \gamma^2, \bar{\gamma}^2$ ($\gamma\bar{\gamma} = 1$) and four non-dominant roots. The former case is trivial, while the latter is handled by [22].

The only remaining possibility, therefore, is that the characteristic roots are $1, 1, \gamma, \bar{\gamma}$. Let $0 < |\gamma| = \rho \le 1$, and we again assume that $\gamma/\bar{\gamma}$ is not a root of unity, for reasons described above. On squaring after the initial check, our LRS is of the form

$$z_2 n^2 + z_1 n + z_0$$

$$+ x_2 n \rho^n \cos n\theta + y_2 n \rho^n \sin n\theta$$

$$+ x_1 \rho^n \cos n\theta + y_1 \rho^n \sin n\theta$$

$$+ x_0 \rho^{2n} \cos 2n\theta + y_0 \rho^{2n} \sin 2n\theta + w \rho^{2n} \ge 0$$

If $\rho < 1$, the above can trivially be resolved with growth arguments, or is an easy order 3 LRS. Thus, we assume $\rho = 1$. If $z_2 \neq 0$, then decidability is trivial; hence we assume $z_2 = 0$. In this case, our inequality can be arranged as

$$n(z_1 + x_2 \cos n\theta + y_2 \sin n\theta) + (z_0 + x_1 \cos n\theta + y_1 \sin n\theta + x_0 \cos 2n\theta + y_0 \sin 2n\theta) \ge 0$$
 (11)

Decidability is most clearly seen through a slight shift in perspective: for any x, y, ϕ , there exist x', y' such that $x \cos \alpha + y \sin \alpha = x' \cos(\alpha - \phi) + y' \sin(\alpha - \phi)$ is an identity in α . Thus, note that for a convenient choice of φ , inequality 11 can easily be rewritten as (we use t_n as shorthand for $n\theta - \varphi$, and choose φ such that $x'_2 \leq 0$)

$$n(z_1 + x_2' \cos t_n) + (z_0 + x_1' \cos t_n + y_1' \sin t_n + x_0' \cos 2t_n + y_0' \sin 2t_n) \ge 0$$
(12)

Observe that there is nothing special about the particular representation of inequality 11. The common proposition that both 11 and 12 convey is that

$$\left(\mathbf{e_1}^T \mathbf{A}^n \mathbf{c}\right)^2 - \left(\max_{\mathbf{d} \in \mathcal{B}_{\mathbf{S}}} \mathbf{e_1}^T \mathbf{A}^n \mathbf{d}\right)^2 \ge 0$$

On retracing our steps back to the discussion surrounding equation 4, we could well have chosen our basis of solutions to be $\mathbf{x'_n} \begin{bmatrix} n & 1 & \cos(n\theta - \varphi) & \sin(n\theta - \varphi) \end{bmatrix}$. This would have given us different $\mathbf{V'}, \mathbf{M'}$, and hence $\mathbf{B'}$ in equation 5, but importantly, it would generate inequality 12 for the same original input.

Since we assume θ is not a rational multiple of 2π , we can argue by Theorem 15 (Kronecker) that $\{(n\theta - \varphi) \text{ modulo } 2\pi\}_{n \in \mathbb{N}}$ is dense in $[0, 2\pi]$. We define

$$f(t) = z_1 + x_2' \cos t \tag{13}$$

$$g(t) = z_0 + x_1' \cos t + y_1' \sin t + x_0' \cos 2t + y_0' \sin 2t \tag{14}$$

Since we chose $x_2' \leq 0$, it is clear that f attains its minima at 0. If $z + x_2' < 0$, then the inequality will be violated for infinitely many n for which $[n\theta - \varphi]_{2\pi}$ is close enough to 0; on the other hand, if $z + x_2' > 0$, then it is guaranteed to hold beyond a computable threshold index N.

Thus, we assume, $z_1 + x_2' = 0$, i.e. the minimal f(0) = 0. If g(0) > 0, we are done: we can easily get a positive lower bound on f for values where g < 0, and get an N beyond which the validity of inequality 12 is guaranteed. On the other hand, if g < 0, we argue that inequality 12 is violated for infinitely many n. Recall inequality 7. It tells us that there are infinitely many n for which $f(n\theta - \varphi) < c/n^2$. Thus, the terms in red are infinitely often lower bounded by c/n, while the terms in black, for the same n, would be close to a negative constant. Thus, we can return NO for robust Ultimate Positivity.

The final case that remains is g(0) = 0. We argue that remarkably, it does not arise at all!

▶ **Lemma 17.** The scenario where $z_2 = 0$, $z_1 + x_2' = 0$, and $z_0 + x_1' + x_0' = 0$ is impossible.

Proof. Suppose that $\mathbf{b_1}^T, \dots, \mathbf{b_4}^T$ are the rows of the matrix $(\mathbf{B}')^{-1}$, and $\mathbf{u_1}, \dots, \mathbf{u_4}$ are the columns. Our inequality is:

$$(p_1 n + p_2 + p_3 \cos(n\theta - \varphi) + p_4 \sin(n\theta - \varphi))^2 - (\dots + (b_{i1} n + b_{i2} + b_{i3} \cos(n\theta - \varphi) + b_{i4} \sin(n\theta - \varphi))^2 + \dots) \ge 0$$

In the table, we explicitly give each coefficient of inequality 12.

Term	Coefficient	Explicitly
n^2	$z_2 = 0$	$p_1^2 - \langle \mathbf{u_1}, \mathbf{u_1} angle$
n	z_1	$2p_1p_2 - 2\langle \mathbf{u_1}, \mathbf{u_2} \rangle$
$n\cos(n\theta-\varphi)$	x_2'	$2p_1p_3 - 2\langle \mathbf{u_1}, \mathbf{u_3} \rangle$
$n\sin(n\theta - \varphi)$	$y_2' = 0$	$2p_1p_4 - 2\langle \mathbf{u_1}, \mathbf{u_4} \rangle$
1	z_0	$p_2^2 + \frac{1}{2}(p_3^2 + p_4^2) - \langle \mathbf{u_2}, \mathbf{u_2} \rangle - \frac{1}{2}(\langle \mathbf{u_3}, \mathbf{u_3} \rangle + \langle \mathbf{u_4}, \mathbf{u_4} \rangle)$
$\cos(n\theta - \varphi)$	x_1'	$2p_2p_3-2\langle \mathbf{u_2},\mathbf{u_3}\rangle$
$\sin(n\theta - \varphi)$	y_1'	$2p_2p_4 - 2\langle \mathbf{u_2}, \mathbf{u_4} \rangle$
$\cos(2n\theta - 2\varphi)$	x'_0	$\frac{1}{2}(p_3^2-p_4^2)-\frac{1}{2}(\langle \mathbf{u_3},\mathbf{u_3}\rangle-\langle \mathbf{u_4},\mathbf{u_4}\rangle)$
$\sin(2n\theta - 2\varphi)$	y_0'	$2p_3p_4 - 2\langle \mathbf{u_3}, \mathbf{u_4} \rangle$

Suppose, for the sake of contradiction, the scenario actually occurs. We then respectively have

$$p_1^2 = \langle \mathbf{u_1}, \mathbf{u_1} \rangle$$

$$p_1(p_2 + p_3) = \langle \mathbf{u_1}, \mathbf{u_2} + \mathbf{u_3} \rangle$$

$$(p_2 + p_3)^2 = \langle \mathbf{u_2} + \mathbf{u_3}, \mathbf{u_2} + \mathbf{u_3} \rangle$$

This implies that $|\langle \mathbf{u_1}, \mathbf{u_2} + \mathbf{u_3} \rangle| = ||\mathbf{u_1}|| \cdot ||\mathbf{u_2} + \mathbf{u_3}||$, i.e. $\mathbf{u_1}$ is a scaled multiple of $\mathbf{u_2} + \mathbf{u_3}$. This contradicts the fact that the columns of the invertible $(\mathbf{B}')^{-1}$ are linearly independent, and we're done.

5 Non-uniform Robustness: Decidability at order four

In this section, we prove Theorem 6. As before, the techniques naturally apply to lower orders, and we omit their explicit treatment. Recall equation 6 and the surrounding discussion:

$$u_n/n^d \rho^n = \begin{bmatrix} \mathbf{q}_{dom}^T(n) & \mathbf{q}_{res}^T(n) \end{bmatrix} \begin{bmatrix} \mathbf{p}_{dom} \\ \mathbf{p}_{res} \end{bmatrix}$$
 (15)

The crucial first task is to check whether for all points \mathbf{p}' in the neighbourhood, $\liminf_{n\in\mathbb{N}}\langle\mathbf{q}_{dom}(n),\mathbf{p'}_{dom}\rangle=\mu(\mathbf{p'})\geq0$. Although Ultimate Positivity is guaranteed for the points where the inequality is strict, the neighbourhood may intersect the critical region where $\mu=0$. If there are no non-dominant terms, this is irrelevant; but otherwise decidability hinges on whether we can handle the intersection.



Figure 1 Visual intuition

Recall Proposition 13: for Ultimate Positivity, there must be a real positive dominant root. We make cases, based on the presence of a pair of complex conjugates among the dominant roots. If the dominant terms are all real, then there are at most two of them, and $\langle \mathbf{q}_{dom}(n), \mathbf{p'}_{dom} \rangle \geq 0$ is the intersection of at most two halfspaces (of the form $z - |\mathbf{w}| \geq 0$). The neighbourhood must lie entirely above the separating hyperplanes (ball atop hyperplane in Figure 1). The points where the neighbourhood is tangent to the planes have algebraic coordinates. The polynomial equations for the coordinates come from the facts that: (i) the point lies on the plane, (ii) the point lies on the surface of the neighbourhood, (iii) the gradient of $(\mathbf{p'} - \mathbf{p})^T \mathbf{M} (\mathbf{p'} - \mathbf{p})$ is along the normal to the plane. Alternately, one could simply use the First Order Theory of the Reals, as discussed below. Solving for critical points of tangency gives us low-order, decidable instances of Ultimate Positivity.

Otherwise, the dominant terms contain a pair of complex conjugates. The case where the remaining root is -1 (dominant) is similar, and simpler. We thus assume $\langle \mathbf{q}_{dom}(n), \mathbf{p'}_{dom} \rangle = z + x \cos n\theta + y \sin n\theta$. We assume θ is not a rational multiple of 2π (can be detected, Theorem 14): otherwise, the region where $\mu(\mathbf{p'}) \geq 0$ is again a finite union of halfspaces, which can be handled as before. Recall equation 10. We get that $\mu(\mathbf{p'}) = z - \sqrt{x^2 + y^2}$. As shown in Figure 1, the critical region $\mu(\mathbf{p'}) = 0$ is a cone. Consider the following first order formula, which conveys that the dominant contribution for all points in the neighbourhood is guaranteed to be non-negative.

$$\chi_0 := \forall \mathbf{p}'. \ (\mathbf{p}' - \mathbf{p})^T \mathbf{M} (\mathbf{p}' - \mathbf{p}) \le 1 \Rightarrow \mu(\mathbf{p}') \ge 0$$
(16)

We can use Theorem 1 (Renegar) to evaluate the truth of this sentence in the First Order Theory of the Reals. If -1 is a root, we adjust the expression for μ accordingly. Here, evaluating the truth of the formula suffices to decide Ultimate Positivity, because there are no non-dominant terms.

Thus, we concentrate on the case where there is one non-dominant term, i.e. $\langle \mathbf{q_n}, \mathbf{p'} \rangle = z + x \cos n\theta + y \sin n\theta + w\alpha^n$. As discussed, we need to analyse the points where the (surface of the) neighbourhood intersects the region $\mu(\mathbf{p'}) = 0$, and where the non-dominant terms can make a negative contribution. Notice that the cone $z - \sqrt{x^2 + y^2} = 0$ is carved out by infinitely many hyperplanes of the form $z + x \cos \phi + y \sin \phi = 0$. Consider the following first order formula with free variable c

$$\chi_1(c) := \exists s \exists \mathbf{p}'. \ (\mathbf{p}' - \mathbf{p})^T \mathbf{M} (\mathbf{p}' - \mathbf{p}) = 1 \land z + cx + sy = 0 \land c^2 + s^2 = 1 \land w \sim 0$$
 (17)

In the above \sim is \neq if the non-dominant root $\alpha < 0$, and is < otherwise. (α is necessarily nonzero!) The free variable c stands for $\cos \phi$, and denotes the hyperplane where the neighbourhood, which lies within the cone, possibly touches the surface of the cone. Moreover, by the inequalities on w, we also enforce that the non-dominant terms make a negative contribution at the point of contact. We can use Theorem 1 to get an equivalent quantifier free formula: this comprises purely of polynomial (in-)equalities in the free variable c. The set of c, and hence $\cos \phi$, satisfying these, consists of finitely many intervals. Of course, Ultimate Positivity is guaranteed when this set is empty: the non-dominant terms are never adversarial when the dominant contribution is dangerously small.

We first dispose of the case where all intervals consist of single points. Consider an interval $\{c_0\}$ consisting of a single point. This is illustrated by the case of the ball touching the cone in Figure 1. Since we know that for the corresponding witness $s_0, z_0, x_0, y_0, z_0 \ge \sqrt{x_0^2 + y_0^2}$, it must be the case that $x_0 = -z_0c_0$, $y_0 = -z_0s_0$. Since c_0, s_0 are algebraic; regardless of z_0, w_0 , this point of tangency is an order 4 decidable instance of Ultimate Positivity: in fact, it is a YES instance, due to the techniques of [24].

If, however, the set of c satisfying χ_1 consists of intervals that have more than one point, then the techniques of [24] to decide Ultimate Positivity for a single point with algebraic coordinates are no longer accessible. This situation is illustrated by the case of the ball nestled in cone in Figure 1. Let $[\phi_1, \phi_2]$ be an interval of ϕ such that: a) all values of c between $\cos \phi_1$ and $\cos \phi_2$ satisfy χ_1 , b) The corresponding witnesses z are at most z_0 , and c) The corresponding witnesses w have magnitude at least some fixed w_0 . Then, we must have for each ϕ (and corresponding $z(c), x = -cz, y = -z\sin\phi, w$)) in this interval, the following inequality is violated only finitely often:

$$z - z\cos(n\theta - \phi) + w\alpha^n \ge 0 \tag{18}$$

We consider an even weaker inequality, which, in this context, we argue is bound to be violated infinitely often:

$$z_0[n\theta - \phi]_{2\pi}^2 \ge 2w_0\alpha^n \tag{19}$$

The argument hinges on Lemma 9, which we restate:

▶ **Lemma 18.** For every irrational number x, strictly decreasing real positive function ψ , and interval $\mathcal{I} = [a,b] \subset [0,1], \ a \neq b$, there exists $y \in \mathcal{I}$ such that $[nx-y] < \psi(n)$ for infinitely many odd, and infinitely many even n.

Now, if $\alpha < 0$, we use Lemma 18 on the irrational $\theta/2\pi$, and the decreasing $\sqrt{\frac{w_0|\alpha|^n}{2\pi^2z_0}}$ to argue that there exists a ϕ in the desired interval, such that the weaker inequality will be violated for infinitely many n of the the appropriate parity. Thus, we can return NO if we are in the case where the set of c satisfying χ_1 (equation 17) consists of intervals that contain more than a single point.

6 Uniform Robustness: Hardness at order five

We shall prove Theorem 11 in this section. That is, given $\theta \in \mathcal{T}$ as defined in equation 9, we shall give rational \mathbf{a}, \mathbf{c} such that varying \mathbf{S} while invoking \mathbf{S} -Robust Positivity decision procedures will enable us to approximate L(t) and $L_{\infty}(t)$ to arbitrary precision.

6.1 The hard sequence

We assume θ is specified by $c \in \overline{\mathbb{Q}}$, 0 < |p| < 1, such that $\theta = \frac{\arccos c}{2\pi}$. Our LRR **a** is such that the roots of the characteristic polynomial are $1, 1, 1, e^{2\pi i\theta}, e^{-2\pi i\theta}$, i.e. the characteristic polynomial is $(X-1)^3(X^2-2cX+1)$.

Here, $u_n = \begin{bmatrix} n^2 & n & 1 & \cos 2\pi n\theta & \sin 2\pi n\theta \end{bmatrix} \mathbf{p}$, and we choose the input $\mathbf{S} = \frac{1}{r^2} (\mathbf{V}^{-1})^T \mathbf{V}^{-1}$ such that its translation \mathbf{M} to the solution space is $1/r^2$ times the identity matrix.

We note that since both c, s from the root c + si are algebraic, the linear map \mathbf{V}^{-1} from the space of initialisations to the space of real exponential polynomial coefficients is also algebraic. In the solution space, we consider a ball of radius r around $(r, 0, 1 + \frac{r}{2}, -1, 0)$, i.e. by equation 5, we ask whether for all (but finitely many) n

$$rn^{2} + \frac{r}{2} + 1 - \cos 2\pi n\theta \ge r\sqrt{n^{4} + n^{2} + 2} \Leftrightarrow \frac{r}{2} + 1 - \cos 2\pi n\theta \ge r\left(\frac{n^{2} + 2}{n^{2} + \sqrt{n^{4} + n^{2} + 2}}\right)$$
$$\Leftrightarrow 1 - \cos 2\pi n\theta \ge \frac{r}{2}\left(\frac{n^{2} + 4 - \sqrt{n^{4} + n^{2} + 2}}{n^{2} + \sqrt{n^{4} + n^{2} + 2}}\right)$$

Simplifying to a slightly more indicative form, we ask whether for all (but finitely many) n

$$1 - \cos 2\pi n\theta \ge \frac{r}{2} \left(\frac{7n^2 + 14}{(n^2 + \sqrt{n^4 + n^2 + 2})(n^2 + 4 + \sqrt{n^4 + n^2 + 2})} \right) = r \cdot Q(n)$$
 (20)

6.2 Numerical analysis

Inequality 20 is pivotal to our reduction. We note that in the limit, the ratio of Q(n) to $7/8n^2$ tends to 1 from below. On the other hand, for small values of $[2\pi n\theta]_{2\pi}$, we shall approximate $1 - \cos 2\pi n\theta$ by $\frac{[2\pi n\theta]_{2\pi}^2}{2}$, which itself is tightly lower bounded by a constant multiple of $L(\theta)/n^2$. Thus, the universal validity of inequality 20 hinges on how r relates to L. We capture the crucial interdependence in the following technical lemma.

▶ **Lemma 19.** Let r < 100. For every $\varepsilon > 0$, we can compute N such that for all $n \ge N$,

1.
$$Q(n) > \frac{7(1-\varepsilon)^2}{8n^2}$$

2.
$$1 - \cos x < \frac{7r}{8n^2} < \frac{700}{8N^2} \Rightarrow 1 - \cos x \ge (1 - \varepsilon)^2 \frac{x^2}{2}$$

6.3 Computing the Lagrange constant

We use the purported decidability of S-Robust Uniform Ultimate Positivity (does inequality 20 hold for all but finitely many n?) to approximate $L_{\infty}(\theta)$ to arbitrary precision.

Case YES:

Suppose indeed, for all but finitely many $n, 1 - \cos 2\pi n\theta \ge r \cdot Q(n)$

 $1-\cos 2\pi n\theta$ is always upper bounded by $\frac{[2\pi n\theta]_{2\pi}^2}{2}$. In this case, we use Part 1 of Lemma 19 to argue that for every ε , there exists an N such that for all $n \geq N$,

$$\frac{[2\pi n\theta]_{2\pi}^2}{2} > \frac{7r(1-\varepsilon)^2}{8n^2} \Leftrightarrow n[n\theta] > (1-\varepsilon)\frac{\sqrt{7r}}{4\pi}$$

This allows us to conclude that $L_{\infty}(\theta) \geq \frac{\sqrt{7r}}{4\pi}$.

Case NO:

Suppose there exist infinitely many n such that $1 - \cos 2\pi n\theta < r \cdot Q(n)$.

 $r \cdot Q(n)$ is always upper bounded by $7r/8n^2$. In this case, we use the second part of Lemma 19 on the infinitely many times $1 - \cos 2\pi n\theta$ is small enough. Here, for any ε , we will have infinitely many n such that

$$(1-\varepsilon)^2 \frac{[2\pi n\theta]_{2\pi}^2}{2} < \frac{7r}{8n^2} \Leftrightarrow n[n\theta] < \left(\frac{1}{1-\varepsilon}\right) \frac{\sqrt{7r}}{4\pi}$$

Thus, we can conclude that $L_{\infty}(\theta) \leq \frac{\sqrt{7r}}{4\pi}$.

6.4 Computing the Diophantine approximation type

Here, we choose a precision ε , and compute the N in Lemma 19. We use this N to get an appropriate instance of S-Robust Positivity for the hard LRS.

Our goal is to be able to reason as follows, similar to the previous subsection,

YES
$$\Rightarrow \inf_{n>N} n[n\theta] \ge (1-\varepsilon) \frac{\sqrt{7r}}{4\pi}$$

and

$$\mathrm{NO}\Rightarrow\inf_{n\geq N}n[n\theta]\leq \left(\frac{1}{1-\varepsilon}\right)\frac{\sqrt{7r}}{4\pi}$$

For the finite prefix 1 to N-1, we can compute $n[n\theta]$ to arbitrary precision and compare with the purported lower/upper bounds for $L(\theta) = \inf_{n \in \mathbb{N}} n[n\theta]$.

Our query, therefore is

$$\forall n \geq N. \ \mathbf{x_n}^T \mathbf{p} \geq r \sqrt{\langle \mathbf{x_n}, \mathbf{x_n} \rangle}$$

which, on taking $\mathbf{S} = r^2(\mathbf{V}^{-1})^T \mathbf{V}^{-1}$, is equivalent to

$$\forall n \geq 0. \ \mathbf{A}^n(\mathbf{A}^N \mathbf{c}) \geq \max_{\mathbf{d} \in \mathcal{B}_{\mathbf{c}}} \mathbf{A}^n(\mathbf{A}^N \mathbf{d})$$

Let $\mathbf{c}' = \mathbf{A}^N \mathbf{c}$, $\mathbf{S}' = (\mathbf{A}^{-N})^T \mathbf{S} \mathbf{A}^{-N}$. Thus, our instance of robust Positivity is:

$$\forall n \geq 0. \ \mathbf{A}^n \mathbf{c}' \geq \max_{\mathbf{d}' \in \mathcal{B}_{\mathbf{S}'}} \mathbf{A}^n \mathbf{d}'$$

Finally, we note that by replacing the coefficients $(r, 0, 1 + \frac{r}{2}, -1, 0)$ with $(r, 0, 1 + \frac{r}{2}, \cos 2\pi\varphi, \sin 2\pi\varphi)$, the same reduction allows us to compute **inhomogeneous** Langrange constants and Diophantine approximation types.

7 Extensions and Perspective

We note that our techniques for **S**-Robust Non-uniform Ultimate Positivity hinge on the First Order Theory of the Reals. Observe that this was rather agnostic to the exact shape of the neighbourhood: we can easily extend the same techniques to arbitrary semi-algebraic neighbourhoods. The uniform variants, on the other hand, can easily be seen to reduce to Positivity and Ultimate Positivity when the neighbourhoods are polyhedra. Let $\mathcal P$ be a polyhedron that contains the origin, and let $\mathbf d_1,\ldots,\mathbf d_k$ be its corners. Our critical inequality 5 in this case is:

$$\langle \mathbf{p}, \mathbf{q_n} \rangle \ge \max_{\mathbf{d} \in \mathcal{P}} \langle \mathbf{d}, \mathbf{q_n} \rangle$$
 (21)

XX:14 Robust Positivity for low-order LRS

The inner product on the right is always maximised at a *corner*. Thus, it is equivalent to ask whether for all $j \in \{1, ..., k\}$,

$$\langle \mathbf{p} - \mathbf{d_i}, \mathbf{q_n} \rangle \ge 0$$
 (22)

These are precisely k many instances of (Ultimate) Positivity, which are decidable up to order five, and number-theoretically hard at order six [23].

As outlined at the outset, we contributed towards a sharp and comprehensive picture of what is *decidable* about Robust Positivity Problems for real algebraic Linear Recurrence Sequences. We find it remarkable that number-theoretic analyses involving Diophantine approximation, which usually show up in the context of hardness, also play a significant role in our *decidability* proofs! However, a rather conspicuous gap in our picture is the status of S-Robust Non-uniform Ultimate Positivity at order 5: this seems to require even more delicate analysis.

An obvious, but possibly tedious future direction would be to tie up the book-keeping loose ends, and meticulously account for the complexity of our techniques. We chose to work with algebraic numbers; in settings involving rational numbers where scaling to integers and accessing an PosSLP oracle is viable, the complexity usually lies in PSPACE. However, this might blow up significantly in the absence of efficient positivity testing for a different class of arithmetic circuit.

At a higher level, we note that we chose our norm to be based on the standard matrix inner product. It is interesting to investigate what kinds of decidability and hardness results hold for neighbourhoods specified using different norms. Perhaps, results could be universal across a wider class of norms, and there could be a profound underlying linear-algebraic reason whose discovery would be mathematically significant.

In the grand Formal Methods scheme, the study of Hyper-properties [11] is an exciting natural way robustness problems for Linear Dynamical Systems could fit in. Hyper-logics reason about sets of traces of an infinite time system, rather than a single trace. They gained importance as a means to verify security in view of attacks like Meltdown and Spectre. A quintessential hyper-property, for instance, would specify a reasonable notion of indistinguishability of traces. In that regard, our notions of S-Robust Positivity and S-Robust Uniform Ultimate Positivity bear striking resemblance. Exploring deeper connections is a fascinating future research avenue.

References

1 S. Akshay, Hugo Bazille, Blaise Genest, and Mihir Vahanwala. On robustness for the skolem and positivity problems. In *STACS 2022*. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2022. URL: https://drops.dagstuhl.de/opus/volltexte/2022/15815/, doi:10.4230/LIPICS.STACS.2022.5.

- 2 S. Akshay, Hugo Bazille, Blaise Genest, and Mihir Vahanwala. On robustness for the skolem, positivity and ultimate positivity problems, 2022. arXiv:2211.02365.
- 3 S. Akshay, Blaise Genest, and Nikhil Vyas. Distribution-based objectives for Markov Decision Processes. In 33rf Symposium on Logic in Computer Science (LICS 2018), volume IEEE, pages 36–45, 2018.
- 4 Christel Baier, Florian Funke, Simon Jantsch, Toghrul Karimov, Engel Lefaucheux, Joël Ouaknine, Amaury Pouly, David Purser, and Markus A. Whiteland. Reachability in dynamical systems with rounding. In Nitin Saxena and Sunil Simon, editors, 40th IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science, FSTTCS 2020, volume 182 of LIPIcs, pages 36:1–36:17, 2020.
- 5 S. Basu, R. Pollack, and M. F. Roy. Algorithms in Real Algebraic Geometry. Springer, 2nd edition, 2006.
- 6 J. P. Bell and S. Gerhold. On the positivity set of a linear recurrence. Israel Jour. Math, 57, 2007.
- 7 Valérie Berthé and Jungwon Lee. Dynamics of ostrowski skew-product: I. limit laws and hausdorff dimensions, 2022. arXiv:2108.06780.
- 8 N. Bourbaki. Elements of Mathematics: General Topology (Part 2). Addison-Wesley, 1966.
- 9 Avraham Bourla. The ostrowski expansions revealed, 2016. arXiv:1605.07992.
- 10 Mark Braverman. Termination of integer linear programs. In *International Conference on Computer Aided Verification*, pages 372–385. Springer, 2006.
- 11 Michael R. Clarkson and Fred B. Schneider. Hyperproperties. In 2008 21st IEEE Computer Security Foundations Symposium, pages 51–65, 2008. doi:10.1109/CSF.2008.7.
- 12 H. Cohen. A Course in Computational Algebraic Number Theory. Springer-Verlag, 1993.
- Julian D'Costa, Toghrul Karimov, Rupak Majumdar, Joël Ouaknine, Mahmoud Salamati, Sadegh Soudjani, and James Worrell. The pseudo-Skolem problem is decidable. In Filippo Bonchi and Simon J. Puglisi, editors, 46th International Symposium on Mathematical Foundations of Computer Science, MFCS 2021, volume 202 of LIPIcs, pages 34:1–34:21. Schloss Dagstuhl Leibniz-Zentrum für Informatik, 2021.
- Manuel Eberl and René Thiemann. Factorization of polynomials with algebraic coefficients. *Archive of Formal Proofs*, November 2021. https://isa-afp.org/entries/Factor_Algebraic_Polynomial.html, Formal proof development.
- 15 Graham Everest, Alfred J. van der Poorten, Igor E. Shparlinski, and Thomas Ward. Recurrence Sequences. In Mathematical surveys and monographs, 2003.
- J. C. Lagarias and J. O. Shallit. Linear fractional transformations of continued fractions with bounded partial quotients. Journal de théorie des nombres de Bordeaux, 9:267–279, 1997.
- 17 David W. Masser. Linear relations on algebraic groups. In New Advances in Transcendence Theory. Cambridge University Press, 1988.
- 18 M. Mignotte. Some useful bounds. In Computer Algebra, 1982.

XX:16 Robust Positivity for low-order LRS

- Maurice Mignotte, Tarlok Nath Shorey, and Robert Tijdeman. The distance between terms of an algebraic recurrence sequence. *Journal für die reine und angewandte Mathematik*, 349:63–76, 1984.
- 20 H. Minkowski. Diophantische Approximationen: Eine Einfuhrung in Die Zahlentheorie. Chelsea Scientific Books. Chelsea, 1957. URL: https://books.google.de/books?id=m2JbAAAACAAJ.
- 21 Eike Neumann. Decision problems for linear recurrences involving arbitrary real numbers. Logical Methods in Computer Science, 17(3), 2021.
- 22 Joël Ouaknine and James Worrell. On the positivity problem for simple linear recurrence sequences. In *International Colloquium on Automata, Languages, and Programming*, pages 318–329. Springer, 2014.
- 23 Joël Ouaknine and James Worrell. Positivity problems for low-order linear recurrence sequences. In Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2014, Portland, Oregon, USA, January 5-7, 2014, pages 366-379. SIAM, 2014.
- 24 Joël Ouaknine and James Worrell. Ultimate positivity is decidable for simple linear recurrence sequences. In *International Colloquium on Automata*, *Languages*, and *Programming*, pages 330–341. Springer, 2014.
- James Renegar. On the Computational Complexity and Geometry of the First-Order Theory of the Reals, Part I: Introduction. Preliminaries. The Geometry of Semi-Algebraic Sets. The Decision Problem for the Existential Theory of the Reals. J. Symb. Comput., 13:255–300, 1992.
- 26 Ashish Tiwari. Termination of linear programs. In *Computer-Aided Verification, CAV*, volume 3114 of *LNCS*, pages 70–82. Springer, July 2004.
- Nikolai Vereshchagin. The problem of appearance of a zero in a linear recurrence sequence. Mat. Zametki, 38(2):609–615, 1985.

A Appendix: Prerequisites

A.1 Algebraic Numbers: Arithmetic

For an algebraic number α , its defining polynomial p_{α} is the unique polynomial in $\mathbb{Z}[X]$ of least degree such that the GCD of its coefficients is 1 and α is one of its roots. Given a polynomial $p \in \mathbb{Z}[X]$, we denote the length of its representation by $\operatorname{size}(p)$; its height, denoted by H(p), is the maximum absolute value of the coefficients of p; d(p) denotes the degree of p. The height $H(\alpha)$ and degree $d(\alpha)$ of α are defined to be the height and degree of p_{α} .

For any $p \in \mathbb{Z}[X]$, the distance between distinct roots is effectively lower bounded in terms of its degree and height [18]. This bound allows one to represent an algebraic number α as a 4-tuple (p, a, b, r) where p is the defining polynomial, and a + bi is a rational approximation of sufficient precision $r \in \mathbb{Q}$. We use size α to denote the size of this representation, i.e., number of bits needed to write down this 4-tuple.

Given a polynomial $p \in \mathbb{Z}[X]$, one can compute its roots in polynomial time [5]. Recently, implementations of algorithms to factor polynomials in $\overline{\mathbb{Q}}[X]$ have been verified [14]. Given α , β two algebraic numbers, one can always compute the representations of $\alpha + \beta$, $\alpha\beta$, $\frac{1}{\alpha}$, $\Re(\alpha)$, $\Re(\alpha)$, $|\alpha|$, and decide $\alpha = \beta$, $\alpha > \beta$ in polynomial time wrt the size of their representations. [5, 12].

A.2 First Order Theory of the Reals

This logical theory reasons about the universe of real numbers, and is denoted $\langle \mathbb{R}; +, \cdot, \geq, 0, 1 \rangle$. That is, variables take real values; terms can be added and multiplied, we have the comparison predicate, and direct access to the constants 0 and 1. Thus, our propositional atoms are inequalities involving polynomials with integer coefficients. With existential quantifiers and polynomials, we can thus express algebraic constants too. Formally, we have access to only the existential quantifier, negation, and disjunction; however, this can express the universal quantifier and all other Boolean connectives as well.

Variables are either quantified or free. Remarkably, the First Order Theory of the Reals admits quantifier elimination: for any formula $\chi(\mathbf{x})$, whose free variables are \mathbf{x} , there exists an **equivalent** formula $\psi(\mathbf{x})$ that does not contain any quantified variables. The following result is relevant to us.

▶ Theorem 20 (Renegar [25]). Let $M \in \mathbb{N}$ be fixed. Let $\chi(\mathbf{x})$ be a formula with fewer than M variables in total. There exists a procedure that returns an equivalent quantifier-free formula $\psi(\mathbf{x})$ in disjunctive normal form. This procedure runs in time polynomial in the size of the representation of χ .

B Appendix: Ostrowski Numeration System

In this appendix, we prove Lemma 9. We state number-theoretic properties of the continued fraction representation and Ostrowski Numeration System without proof. We refer the reader to [9] for a more detailed exposition, and we closely follow the discussion surrounding [7, Propositions 1.1, 2.1] in our own proof. We first prove a slightly simpler statement.

▶ Lemma 21. For every irrational number x, strictly decreasing real positive function ψ , and interval $\mathcal{I} = [\alpha, \beta] \subset [0, 1], \ \alpha \neq \beta$, there exists $y \in \mathcal{I}$ such that $[nx - y] < \psi(n)$ for

infinitely many n.

Proof. Without loss of generality, we can assume that $x \in (0,1)$. Consider the continued fraction representation of x: $[0; a_1, a_2, a_3, \dots]$

$$x = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\ddots}}}}$$

where $a_1, a_2, a_3, \dots \in \mathbb{N}$. Let the rational approximation of x obtained by truncating the expansion at the k^{th} level be $\frac{p_k}{q_k}$, i.e. $\frac{p_1}{q_1} = \frac{1}{a_1}$, and so on. Let $\theta_k = q_k x - p_k$. We have that $|\theta_k| = (-1)^k \theta_k$. It is well known that $|\theta_k| < 1/q_k$. We define $q_{-1} = p_0 := 0$, and $p_{-1} = q_0 := 1$, so that for $k \ge 1$, the following recurrences hold:

$$p_k = a_k p_{k-1} + p_{k-2}, \ q_k = a_k q_{k-1} + q_{k-2}$$

We thus have that $q_k \ge \left(\frac{1+\sqrt{5}}{2}\right)^k = \phi^k$.

▶ Proposition 22. Let irrational x and its continued fraction representation $[0; a_1, a_2, a_3, \ldots]$ be as above. The infinite series

$$\sum_{i=1}^{\infty} a_i |\theta_{i-1}|$$

converges.

▶ Proposition 23 (Ostrowski Numeration System). Every real number $y \in [0,1)$ can be written uniquely in the form

$$y = \sum_{i=1}^{\infty} b_i |\theta_{i-1}| = \sum_{i=1}^{\infty} (-1)^{i-1} b_i \theta_{i-1}$$

where $b_i \in \mathbb{N}$ $b_i \leq a_i$ for all $i \geq 1$. If for some i, $a_i = b_i$, then $b_{i+1} = 0$. $a_i \neq b_i$ for infinitely many odd, and infinitely many even indices i.

We prove Lemma 21 by using the free choice of b_i in this system to construct appropriate y. We first handle the issue of placing y in the correct interval $[\alpha, \beta]$. Let $\beta - \alpha = \delta$. We use Proposition 22 to argue that there exists a suffix of the infinite series, such that changing the suffix does not change the real number it represents by more than $\delta/2$. Then, we can simply fix the corresponding prefix of $(\alpha + \beta)/2$ to be the prefix of y.

Once this prefix is locked in, our strategy is to set b_i to 0 in even positions, and 1 in some odd positions, to ensure that for sufficiently large k, $n_k = \sum_{i=1}^k b_i (-1)^{i-1} q_{i-1}$ is positive, and increasing in k.

Now, notice that since b_i , p_i are all integers, for any y,

$$[n_k x - y] = \left[\sum_{i=1}^k b_i (-1)^{i-1} q_{i-1} x - \sum_{i=1}^k b_i (-1)^{i-1} p_{i-1} - y \right]$$

$$= \left[\sum_{i=1}^k b_i (-1)^{i-1} \theta_{i-1} - y \right]$$

$$= \left[-\sum_{i=k+1}^\infty b_i (-1)^{i-1} \theta_{i-1} \right] = \sum_{i=k+1}^\infty b_i |\theta_{i-1}|$$

$$< \sum_{i=k+1}^\infty b_i \frac{1}{q_{i-1}} \le \sum_{i=k+1}^\infty b_i \frac{1}{\phi^{i-1}} \le \frac{c}{\phi^k}$$

Note that the last constant c can be set independently of the choice of which b_i are 1, and which are 0: it comes from the convergence of the geometric sum. We now make the choice of where to set $b_i = 1$. To conclude the proof, we shall show that given a decreasing function ψ , we can ensure that for infinitely many distinct n_k ,

$$[n_k x - y] < \frac{c}{\phi^k} \le \psi(n_k) = \psi\left(\sum_{i=1}^k b_i (-1)^{i-1} q_{i-1}\right)$$

.

The first inequality is guaranteed. Suppose the second inequality does not hold. Then, from i = k onwards, we keep assigning $b_i := 0$. This holds n_k constant as k increases, but decreases $\frac{c}{\phi^k}$. Eventually, the second inequality will indeed hold. After this point, for the next odd i, we can set b_i to 1, and get a new n_k . We continue this ad infinitum, and we are done.

Now, to get infinitely many even n, apply Lemma 21 with 2x, [a,b], $\psi_0(n) = \psi(2n)$. For some choice of y, there will be infinitely many n such that $[2nx-y] < \psi_0(n) = \psi(2n)$. To get infinitely many odd n, we can take a subset of the interval, and shift it by x. Take $\psi_1(n) = \psi(2n+1)$. For some choice of y-x, there will be infinitely many n such that $[n(2x) - (y-x)] = [(2n+1)x-y] < \psi_1(n) = \psi(2n+1)$.