

# On the complexity of symmetric vs. functional PCSPs\*

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## Abstract

The complexity of the *promise constraint satisfaction problem*  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  is largely unknown, even for symmetric  $\mathbf{A}$  and  $\mathbf{B}$ , except for the case when  $\mathbf{A}$  and  $\mathbf{B}$  are Boolean.

First, we establish a dichotomy for  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  where  $\mathbf{A}, \mathbf{B}$  are symmetric,  $\mathbf{B}$  is *functional* (i.e. any  $r - 1$  elements of an  $r$ -ary tuple uniquely determines the last one), and  $(\mathbf{A}, \mathbf{B})$  satisfies technical conditions we introduce called *dependency* and *additivity*. This result implies a dichotomy for  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  with  $\mathbf{A}, \mathbf{B}$  symmetric and  $\mathbf{B}$  functional if (i)  $\mathbf{A}$  is Boolean, or (ii)  $\mathbf{A}$  is a hypergraph of a small uniformity, or (iii)  $\mathbf{A}$  has a relation  $R^{\mathbf{A}}$  of arity at least 3 such that the hypergraph diameter of  $(A, R^{\mathbf{A}})$  is at most 1.

Second, we show that for  $\text{PCSP}(\mathbf{A}, \mathbf{B})$ , where  $\mathbf{A}$  and  $\mathbf{B}$  contain a single relation,  $\mathbf{A}$  satisfies a technical condition called *balancedness*, and  $\mathbf{B}$  is arbitrary, the combined *basic linear programming* relaxation (BLP) and the *affine integer programming* relaxation (AIP) is no more powerful than the (in general strictly weaker) AIP relaxation. Balanced  $\mathbf{A}$  include symmetric  $\mathbf{A}$  or, more generally,  $\mathbf{A}$  preserved by a transitive permutation group.

## 1 Introduction

*Promise constraint satisfaction problems* (PCSPs) are a generalisation of constraint satisfaction problems (CSPs) that allow for capturing many more computational problems [4, 8, 6].

A canonical example of a CSP is the 3-colouring problem: Given a graph  $\mathbf{G}$ , is it 3-colourable? This can be cast as a CSP. Let  $\mathbf{K}_k$  denote a clique on  $k$  vertices. Then  $\text{CSP}(\mathbf{K}_3)$ , the constraint satisfaction problem with the template  $\mathbf{K}_3$ , is the following computational problem (in the decision version): Given a graph  $\mathbf{G}$ , say YES if there is a homomorphism from  $\mathbf{G}$  to  $\mathbf{K}_3$  (indicated by  $\mathbf{G} \rightarrow \mathbf{K}_3$ ) and say NO otherwise (indicated by  $\mathbf{G} \not\rightarrow \mathbf{K}_3$ ). Here a graph homomorphism is an edge preserving map [23]. As graph homomorphisms from  $\mathbf{G}$  to  $\mathbf{K}_3$  are 3-colourings of  $\mathbf{G}$ ,  $\text{CSP}(\mathbf{K}_3)$  is the 3-colouring problem.

Another example of a CSP is 1-in-3-SAT: Given a positive 3-CNF formula, is there an assignment that satisfies exactly one literal in each clause? This is  $\text{CSP}(\mathbf{1-in-3})$ , where

$$\mathbf{1-in-3} = (\{0, 1\}; \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}).$$

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Yet another example is NAE-3-SAT: Given a positive 3-CNF formula, is there an assignment that satisfies one or two literals in each clause? This is  $\text{CSP}(\mathbf{NAE})$ , where

$$\mathbf{NAE} = (\{(0, 1); \{0, 1\}^3 \setminus \{(0, 0, 0), (1, 1, 1)\}\}.$$

A canonical example of a PCSP is the approximate graph colouring problem [21]: Fix  $k \leq \ell$ . Given a graph  $\mathbf{G}$ , determine whether  $\mathbf{G}$  is  $k$ -colourable or not even  $\ell$ -colourable. (The case  $k = \ell$  is the same as  $k$ -colouring.) This is the same as the PCSP over cliques; i.e.,  $\text{PCSP}(\mathbf{K}_k, \mathbf{K}_\ell)$  is the following computational problem (in the decision version): Given a graph  $\mathbf{G}$ , say YES if  $\mathbf{G} \rightarrow \mathbf{K}_k$  and say NO if  $\mathbf{G} \not\rightarrow \mathbf{K}_\ell$ . In the search version, one is given a  $k$ -colourable graph  $\mathbf{G}$  and the task is to find an  $\ell$ -colouring of  $\mathbf{G}$  (which necessarily exists by the promise and the fact that  $k \leq \ell$ ).

Another example of a PCSP is  $\text{PCSP}(\mathbf{1-in-3}, \mathbf{NAE})$ , identified in [8]: Given a satisfiable instance  $\mathbf{X}$  of  $\text{CSP}(\mathbf{1-in-3})$ , can one find a solution if  $\mathbf{X}$  is seen as an instance of  $\text{CSP}(\mathbf{NAE})$ ? I.e., can one find a solution that satisfies one or two literals in each clause given the promise that a solution that satisfies exactly one literal in each clause exists? Although both  $\text{CSP}(\mathbf{1-in-3})$  and  $\text{CSP}(\mathbf{NAE})$  are NP-complete, Brakensiek and Guruswami showed in [8] that  $\text{PCSP}(\mathbf{1-in-3}, \mathbf{NAE})$  is solvable in polynomial time and in particular it is solved by the so-called affine integer programming relaxation (AIP), whose power was characterised in [6].

More generally, one fixes two relational structures  $\mathbf{A}$  and  $\mathbf{B}$  with  $\mathbf{A} \rightarrow \mathbf{B}$ . The  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  is then, in the decision version, the computational problem of distinguishing (input) relational structures  $\mathbf{X}$  with  $\mathbf{X} \rightarrow \mathbf{A}$  from those with  $\mathbf{X} \not\rightarrow \mathbf{B}$ . In the search version,  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  is the problem of finding a homomorphism from an input structure  $\mathbf{X}$  to  $\mathbf{B}$  given that one is promised that  $\mathbf{X} \rightarrow \mathbf{A}$ . One can think of  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  as an approximation version of  $\text{CSP}(\mathbf{A})$  on satisfiable instances. Another way is to think of  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  as  $\text{CSP}(\mathbf{B})$  with restricted inputs. We refer the reader to [25] for a very recent survey on PCSPs.

For CSPs, a dichotomy conjecture of Feder and Vardi [19] was resolved independently by Bulatov [15] and Zhuk [31] via the so-called algebraic approach [24, 14]: For every fixed finite  $\mathbf{A}$ ,  $\text{CSP}(\mathbf{A})$  is either solvable in polynomial time or  $\text{CSP}(\mathbf{A})$  is NP-complete.

For PCSPs, even the case of graphs and structures on Boolean domains is widely open; these two were established for CSPs a long time ago [23, 29] and constituted important evidence for conjecturing a dichotomy. Following the important work of Barto et al. [6] on extending the algebraic framework from the realms of CSPs to the world of PCSPs, there have been several recent works on complexity classifications of fragments of PCSPs [20, 22, 8, 12, 5, 2, 9, 13, 27, 26], hardness conditions [6, 12, 7, 30], and power of algorithms [6, 10, 3, 17]. Nevertheless, a classification of more concrete fragments of PCSPs is needed for making progress with the general theory, such as finding hardness and tractability criteria, as well as with resolving longstanding open questions, such as approximate graph colouring.

Brakensiek and Guruswami classified  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  for all Boolean symmetric structures  $\mathbf{A}$  and  $\mathbf{B}$  with disequalities [8]. Ficak, Kozik, Olšák, and Stankiewicz generalised this result by classifying  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  for all Boolean symmetric structures  $\mathbf{A}$  and  $\mathbf{B}$  [20].

Barto, Battistelli, and Berg [5] studied symmetric PCSPs on non-Boolean domains and in particular PCSPs of the form  $\text{PCSP}(\mathbf{1-in-3}, \mathbf{B})$ , where  $\mathbf{B}$  contains a single ternary relation over the domain  $\{0, 1, \dots, d-1\}$ . For  $d = 2$ , a complete classification  $\text{PCSP}(\mathbf{1-in-3}, \mathbf{B})$  is known [8, 20]. For  $d = 3$ , Barto et al. [5] managed to classify all but one structure  $\mathbf{B}$ . The remaining open case of “linearly ordered colouring” inspired further investigation in [27]. For  $d = 4$ , Barto et al. [5] obtained partial results. In particular, for certain structures

**B** they managed to rule out the applicability of the BLP + AIP algorithm from [10]. The significance of BLP + AIP here is that it is the strongest known algorithm for PCSPs for which a characterisation of its power is known both in terms of a minion and also in terms of polymorphism identities. This suggests that those cases are NP-hard (or new algorithmic techniques are needed).

**Contributions.** We continue the work from [8, 20] and [5] and focus on promise constraint satisfaction problems of the form  $\text{PCSP}(\mathbf{A}, \mathbf{B})$ , where  $\mathbf{A}$  is symmetric and  $\mathbf{B}$  is over an arbitrary finite domain.<sup>1</sup> Since the template  $\mathbf{A}$  is symmetric, we can assume without loss of generality that  $\mathbf{B}$  is symmetric, as observed in [5] and in [13].<sup>2</sup>

As our main result, we establish the following result. A structure  $\mathbf{B}$  is called *functional* if, for any relation  $R^{\mathbf{B}}$  in  $\mathbf{B}$  of, say, arity  $r$ , and any tuple  $x \in R^{\mathbf{B}}$ , any  $r - 1$  elements of  $x$  determine the last element. In detail,  $(x_1, \dots, x_{r-1}, y), (x_1, \dots, x_{r-1}, z) \in R^{\mathbf{B}}$  implies  $y = z$ , and similarly for the other  $r - 1$  positions.<sup>3</sup> The notions of dependency and additivity will be defined in Section 3: for the moment, consider them simply as technical conditions on  $(\mathbf{A}, \mathbf{B})$ . Finite tractability is defined in Section 2.

**Theorem 1.** *Let  $\mathbf{A}$  be a symmetric structure and  $\mathbf{B}$  be a functional structure such that  $\mathbf{A} \rightarrow \mathbf{B}$ . Assume that  $(\mathbf{A}, \mathbf{B})$  is dependent and additive. Then, either  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  is solvable in polynomial time by AIP and is finitely tractable, or  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  is NP-hard.*

Our main motivation for studying PCSPs with functional structures is the fact that more complexity classifications of PCSP fragments are needed to make progress with the general theory of PCSPs. Furthermore, functional PCSPs generalise linear equations, an important and fundamental class of CSPs. Finally, topological methods [26] seem inapplicable to resolving the computational complexity of functional PCSPs and thus other methods are required.

Theorem 1 has the following three corollaries. The first corollary applies to structures  $\mathbf{A}$  that are Boolean.

**Corollary 2.** *Let  $\mathbf{A}$  be a Boolean symmetric structure and  $\mathbf{B}$  be a functional structure such that  $\mathbf{A} \rightarrow \mathbf{B}$ . Then, either  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  is solvable in polynomial time by AIP and is finitely tractable, or  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  is NP-hard.*

The second corollary applies to structures  $\mathbf{A}$  consisting of a single relation of a small arity.

**Corollary 3.** *Let  $\mathbf{A}$  be a symmetric structure and  $\mathbf{B}$  be a functional structure such that  $\mathbf{A} \rightarrow \mathbf{B}$ . Assume that both  $\mathbf{A}$  and  $\mathbf{B}$  have exactly one relation of arity at most 4. Then, either  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  is solvable in polynomial time, or  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  is NP-hard.*

An example of a Boolean symmetric structure  $\mathbf{A}$  is **1-in-3**, and more generally **q-in-r**.<sup>4</sup> The structure **1-in-3** is not only Boolean (and thus is captured by Corollary 2) but also consists of a single relation of arity 3 (and thus is also captured by Corollary 3). In fact,

<sup>1</sup>All structures in this article can be assumed to be finite unless they are explicitly stated to be infinite.

<sup>2</sup>In detail, for any symmetric  $\mathbf{A}$  and (not necessarily symmetric)  $\mathbf{B}$  with  $\mathbf{A} \rightarrow \mathbf{B}$ , there is a symmetric  $\mathbf{B}'$  with  $\mathbf{A} \rightarrow \mathbf{B}'$  such that  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  and  $\text{PCSP}(\mathbf{A}, \mathbf{B}')$  are polynomial-time equivalent [5, 13]. This  $\mathbf{B}'$  is the largest symmetric substructure of  $\mathbf{B}$ . Observe that a functional structure has functional substructures, so if  $\mathbf{B}$  is functional then  $\mathbf{B}'$  remains functional.

<sup>3</sup>Note that for symmetric  $\mathbf{B}$  the requirement “for the other  $r - 1$  positions” is satisfied automatically.

<sup>4</sup>**q-in-r** is the structure on  $\{0, 1\}$  with a single (symmetric) relation of arity  $r$  containing all  $r$ -tuples with precisely  $q$  1s (and  $r - q$  0s).

it satisfies another property, leading to the following generalisation of **1-in-3**, which applies to structures  $\mathbf{A}$  with a certain connectivity property. We will need some notation. Define  $\text{dist}_R(x, y)$  as being the distance between  $x$  and  $y$  when viewed as vertices in a hypergraph whose edge relation is  $R$ . Thus, for instance,  $\text{dist}_R(x, x) = 0$ , and if  $x$  and  $y$  are in different connected components then  $\text{dist}_R(x, y) = \infty$ . Define  $\text{diam}(V, E) = \max_{u, v \in V} \text{dist}_E(u, v)$ .

**Corollary 4.** *Let  $\mathbf{A}$  be a symmetric structure and  $\mathbf{B}$  be a functional structure such that  $\mathbf{A} \rightarrow \mathbf{B}$ . Assume that  $\mathbf{A}$  has a relation  $R^{\mathbf{A}}$  of arity at least 3 for which  $\text{diam}(A, R^{\mathbf{A}}) \leq 1$ . Then, either  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  is solvable in polynomial time by AIP and is finitely tractable, or  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  is NP-hard.*

A hypergraph is called *linear* if no two distinct edges intersect in more than one vertex. We remark that any *linear*  $r$ -uniform hypergraph can be seen as a functional symmetric relational structure with one  $r$ -ary relation.

Several researchers have informally conjectured that  $\text{PCSP}(\mathbf{1-in-3}, \mathbf{B})$  admits a dichotomy. The authors, as well as other researchers, believe that in fact not only is there a dichotomy but also all tractable cases are solvable by AIP (cf. also Remark 39).

**Conjecture 5.** *For every structure  $\mathbf{B}$ , either  $\text{PCSP}(\mathbf{1-in-3}, \mathbf{B})$  is solvable in polynomial time by AIP, or  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  is NP-hard.*

Theorem 1 establishes the special case of Conjecture 5 for functional  $\mathbf{B}$ . We make further progress towards Conjecture 5 by proving that for any structure  $\mathbf{A}$  with a single (not necessarily Boolean) symmetric relation, and any (not necessarily functional) structure  $\mathbf{B}$  for which  $\mathbf{A} \rightarrow \mathbf{B}$ ,  $\text{BLP} + \text{AIP}$  from [10] is no more powerful for  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  than AIP from [8], although in general  $\text{BLP} + \text{AIP}$  is strictly stronger than AIP [10], already for (non-promise) CSPs with *two* Boolean symmetric relations, cf. Remark 57. In fact, we establish a more general result. We say that a relation  $R$  is *balanced* if there exists a matrix  $M$  whose columns are tuples of  $R$ , where each tuple of  $R$  appears as a column (possibly a multiple times), and where the rows of  $M$  are permutations of each other. The matrix  $M$  below shows that the Boolean **1-in-3** relation is balanced:

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

**Theorem 6.** *Let  $\mathbf{A}$  be any structure with a single relation. If the relation in  $\mathbf{A}$  is balanced then, for any  $\mathbf{B}$  such that  $\mathbf{A} \rightarrow \mathbf{B}$ ,  $\text{BLP} + \text{AIP}$  solves  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  if and only if AIP solves it.*

If the (only) relation in  $\mathbf{A}$  is *binary* (i.e., a digraph), the condition of balancedness has a natural combinatorial interpretation: A binary relation is balanced if and only if it is the disjoint union of strongly connected components (cf. Appendix A).

Theorem 6 implies the following corollary. We say that a relation of arity  $r$  is *preserved* by a group of permutations of degree  $r$  if and only if permuting any tuple of the relation according to any permutation of the group gives another tuple of the relation.

**Corollary 7.** *Suppose that  $G$  is a transitive group of permutations. Further, suppose that  $\mathbf{A}$  is a relational structure with one relation that is preserved by  $G$ . Then, for any  $\mathbf{A} \rightarrow \mathbf{B}$ ,  $\text{BLP} + \text{AIP}$  solves  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  if and only if AIP does.*

While Corollary 7 is more elegant than Theorem 6, it applies to fewer structures. Indeed, we will show in Remark 60 that there exist balanced relations that are not preserved by any transitive group. Examples of relations that are preserved by some transitive group of permutations  $G$  include symmetric relations (where  $G$  is the symmetric group) or cyclic relations (where  $G$  contains all cyclic shifts of appropriate degree).

## 2 Preliminaries

We let  $[r] = \{1, \dots, r\}$ . We denote by  $2^S$  the powerset of  $S$ .

**Structures and PCSPs** Promise CSPs have been introduced in [4] and [8]. We follow the notation and terminology of [6].

A (relational) *structure* is a tuple  $\mathbf{A} = (A; R_1^{\mathbf{A}}, \dots, R_t^{\mathbf{A}})$ , where  $R_i^{\mathbf{A}} \subseteq A^{\text{ar}(R_i)}$  is a relation of arity  $\text{ar}(R_i)$  on a set  $A$ , called the *domain*. A structure  $\mathbf{A}$  is called *Boolean* if  $A = \{0, 1\}$  and is called *symmetric* if  $R_i^{\mathbf{A}}$  is a symmetric relation for each  $i \in [t]$ ; i.e, if  $(x_1, \dots, x_{\text{ar}(R_i)}) \in R_i^{\mathbf{A}}$  then for every permutation  $\pi$  on  $[\text{ar}(R_i)]$  we have  $(x_{\pi(1)}, \dots, x_{\pi(\text{ar}(R_i))}) \in R_i^{\mathbf{A}}$ . A structure  $\mathbf{A}$  is called *functional* if  $(x_1, \dots, x_{\text{ar}(R_i)-1}, y) \in R_i^{\mathbf{A}}$  and  $(x_1, \dots, x_{\text{ar}(R_i)-1}, z) \in R_i^{\mathbf{A}}$  implies  $y = z$  for any  $i \in [t]$ , and that the same hold for all other  $r - 1$  positions in the tuple. For any  $r$ -ary functional relation  $R \subseteq A^r$ , we define a partial map also called  $R$  from  $A^{r-1}$  to  $A$  in the following way: for any  $x_1, \dots, x_{r-1} \in A$ ,  $R(x_1, \dots, x_{r-1})$  is the unique value  $y$  such that  $(x_1, \dots, x_{r-1}, y) \in R$ , if it exists;  $R(x_1, \dots, x_{r-1})$  is undefined if no such value exists.

For two structures  $\mathbf{A} = (A; R_1^{\mathbf{A}}, \dots, R_t^{\mathbf{A}})$  and  $\mathbf{B} = (B; R_1^{\mathbf{B}}, \dots, R_t^{\mathbf{B}})$  with  $t$  relations with the same arities, a *homomorphism from  $\mathbf{A}$  to  $\mathbf{B}$*  is a function  $h : A \rightarrow B$  such that, for any  $i \in [t]$ , for each  $x = (x_1, \dots, x_{\text{ar}(R_i)}) \in R_i^{\mathbf{A}}$ , we have  $h(x) = (h(x_1), \dots, h(x_{\text{ar}(R_i)})) \in R_i^{\mathbf{B}}$ . We denote the existence of a homomorphism from  $\mathbf{A}$  to  $\mathbf{B}$  by  $\mathbf{A} \rightarrow \mathbf{B}$ .

Let  $\mathbf{A}$  and  $\mathbf{B}$  be two structures with  $\mathbf{A} \rightarrow \mathbf{B}$ ; we call  $(\mathbf{A}, \mathbf{B})$  a (PCSP) *template*. In the *search* version of the *promise constraint satisfaction problem* (PCSP) with the template  $(\mathbf{A}, \mathbf{B})$ , denoted by  $\text{PCSP}(\mathbf{A}, \mathbf{B})$ , the task is: Given a structure  $\mathbf{X}$  with the promise that  $\mathbf{X} \rightarrow \mathbf{A}$ , find a homomorphism from  $\mathbf{X}$  to  $\mathbf{B}$  (which necessarily exists as homomorphisms compose). In the *decision* version of  $\text{PCSP}(\mathbf{A}, \mathbf{B})$ , the task is: Given a structure  $\mathbf{X}$ , output YES if  $\mathbf{X} \rightarrow \mathbf{A}$ , and output NO if  $\mathbf{X} \not\rightarrow \mathbf{B}$ . The decision version trivially reduces to the search version. We will use the decision version in this paper.

We will be interested in the complexity of  $\text{PCSP}(\mathbf{A}, \mathbf{B})$ , in particular for symmetric  $\mathbf{A}$  and functional  $\mathbf{B}$ . (As discussed in Section 1, the symmetricity of  $\mathbf{A}$  means that we can without loss of generality assume symmetricity of  $\mathbf{B}$ .)

**Operations and polymorphisms** A function  $h : A^n \rightarrow B$  is called an *operation* of arity  $n$ . A  $(2n + 1)$ -ary operation  $f : A^{2n+1} \rightarrow B$  is called *2-block-symmetric* if  $f(a_1, \dots, a_{2n+1}) = f(a_{\pi(1)}, \dots, a_{\pi(2n+1)})$  for every  $a_1, \dots, a_{2n+1} \in A$  and every permutation  $\pi$  on  $[2n + 1]$  that preserves parity; i.e,  $\pi$  maps odd values to odd values and even values to even values.

A  $(2n + 1)$ -ary operation  $f : A^{2n+1} \rightarrow B$  is called *alternating* if it is 2-block-symmetric, and furthermore  $f(a_1, \dots, a_{2n-1}, a, a) = f(a_1, \dots, a_{2n-1}, a', a')$  for every  $a_1, \dots, a_{2n-1}, a, a' \in A$ .

Consider structures  $\mathbf{A}, \mathbf{B}$  with  $t$  relations with the same arities. We call  $h : A^n \rightarrow B$  a *polymorphism* of  $(\mathbf{A}, \mathbf{B})$  if the following holds for any relation  $R = R_i$ ,  $i \in [t]$ , or arity  $r = \text{ar}(R)$ . For any  $x^1, \dots, x^r \in A^n$ , where  $x^i = (x_1^i, \dots, x_n^i)$ , with  $(x_1^i, \dots, x_n^i) \in R^{\mathbf{A}}$  for every  $1 \leq i \leq n$ , we have  $(h(x^1), \dots, h(x^r)) \in R^{\mathbf{B}}$ . One can visualise this as an  $(r \times n)$  matrix

whose rows are the tuples  $x^1, \dots, x^r$ . The requirement is that if every column of the matrix is in  $R^{\mathbf{A}}$  then the application of  $h$  on the rows of the matrix results in a tuple from  $R^{\mathbf{B}}$ . We denote by  $\text{Pol}^{(n)}(\mathbf{A}, \mathbf{B})$  the set of  $n$ -ary polymorphisms of  $(\mathbf{A}, \mathbf{B})$  and by  $\text{Pol}(\mathbf{A}, \mathbf{B})$  the set of all polymorphisms of  $(\mathbf{A}, \mathbf{B})$ .

**Relaxations** There are two standard polynomial-time solvable relaxations for PCSPs, the *basic linear programming* relaxation (BLP) and the *affine integer programming* relaxation (AIP) [8]. The AIP solves most tractable PCSPs studies in this paper, with the exception of cases covered in Corollary 3 (cf. also Remark 39). There is also a combination of the two, called BLP + AIP [10], that is provably stronger than both BLP and AIP. We will show that for certain PCSPs, this is not the case (cf. Theorem 6). The precise definitions of the relaxations are not important for us as we will only need the notion of solvability of PCSPs by these relaxations and characterisations of the power of the relaxations; we refer the reader to [8, 6, 10] for details. Let  $\mathbf{X}$  be an instance of  $\text{PCSP}(\mathbf{A}, \mathbf{B})$ . It follows from the definitions of the relaxations that if  $\mathbf{X} \rightarrow \mathbf{A}$  then both AIP and BLP + AIP accept [8, 6]. We say that AIP (BLP + AIP, respectively) *solves*  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  if, for every  $\mathbf{X}$  with  $\mathbf{X} \not\rightarrow \mathbf{B}$ , AIP (BLP + AIP, respectively) rejects.

The power of AIP and BLP + AIP for PCSPs is characterised by the following results.

**Theorem 8** ([6]).  *$\text{PCSP}(\mathbf{A}, \mathbf{B})$  is solved by AIP if and only if  $\text{Pol}(\mathbf{A}, \mathbf{B})$  contains alternating operations of all odd arities.*

**Theorem 9** ([10]).  *$\text{PCSP}(\mathbf{A}, \mathbf{B})$  is solved BLP + AIP if and only if  $\text{Pol}(\mathbf{A}, \mathbf{B})$  contains 2-block-symmetric operations of all odd arities.*

We now define the notion of *finite tractability* [6, 2]. We say that  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  is finitely tractable if  $\mathbf{A} \rightarrow \mathbf{E} \rightarrow \mathbf{B}$  for some finite structure  $\mathbf{E}$  and  $\text{CSP}(\mathbf{E})$  is tractable. For a group  $G$ , we use the standard notation  $H \triangleleft G$  to indicate that  $H$  is a normal subgroup of  $G$ .

**Lemma 10.** *Suppose  $\mathbf{A} \rightarrow \mathbf{E} \rightarrow \mathbf{B}$ , where  $E = G$  for some Abelian group  $(G, +)$ , and each relation of  $\mathbf{E}$  is either of the form (i)  $c + H$  for some  $r \in \mathbb{N}$ ,  $c \in G^r$  and  $H \triangleleft G^r$ , or (ii) empty. Then,  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  is solvable in polynomial time by AIP and is finitely tractable.*

*Proof.* The following alternating operation is a polymorphism of  $\mathbf{E}$

$$f(x_1, y_1, \dots, y_k, x_{k+1}) = \sum_{i=1}^{k+1} x_i - \sum_{i=1}^k y_i.$$

Consider a relation  $R^{\mathbf{E}}$  of  $\mathbf{E}$ , of the form  $c + H$ . Consider a matrix of inputs whose columns are  $x_1, y_1, \dots, y_k, x_{k+1} \in R^{\mathbf{E}}$ . In other words,  $x_i \in c + H$  and  $y_i \in c + H$  for each  $x_i, y_i$ . Note that the column that results from applying  $f$  to the rows of this matrix is just

$$x_1 - y_1 + \dots - y_k + x_{k+1} \in (c + H) - (c + H) + \dots - (c + H) + (c + H) \subseteq c + H$$

Thus  $f$  is an alternating polymorphism of  $\mathbf{E}$ . It follows that  $\text{CSP}(\mathbf{E})$  is solved by AIP, from whence it follows that  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  is finitely tractable and solved by AIP.  $\square$



**Minions** We will use the theory of minions from [6]. Let  $\mathcal{M}$  be a set, where each element  $f \in \mathcal{M}$  is assigned an arity  $\text{ar}(f)$ . We write  $\mathcal{M}^{(n)} = \{f \in \mathcal{M} \mid \text{ar}(f) = n\}$ . Further, let  $\mathcal{M}$  be endowed with, for each  $\pi : [n] \rightarrow [m]$ , a (so-called minor) map  $f \mapsto f^\pi : \mathcal{M}^{(n)} \rightarrow \mathcal{M}^{(m)}$  such that, for  $\pi : [m] \rightarrow [k]$  and  $\sigma : [n] \rightarrow [m]$ , and any  $f \in \mathcal{M}^{(n)}$  we have  $(f^\pi)^\sigma = f^{\sigma \circ \pi}$ , and  $f^{\text{id}} = f$ . Then,  $\mathcal{M}$  is called a *minion*.<sup>5</sup> We often write  $f \xrightarrow{\pi} g$  instead of  $g = f^\pi$ .

Consider two minions  $\mathcal{M}, \mathcal{N}$ ; a *minion homomorphism* is a map  $\xi : \mathcal{M} \rightarrow \mathcal{N}$  such that, for any  $f \in \mathcal{M}^{(n)}$  and  $\pi : [n] \rightarrow [m]$ , we have that  $\xi(f)^\pi = \xi(f^\pi)$ .<sup>6</sup> If such a minion homomorphism exists, we write  $\mathcal{M} \rightarrow \mathcal{N}$ .

Given an  $n$ -ary operation  $f : A^n \rightarrow B$  and a map  $\pi : [n] \rightarrow [m]$ , an  $m$ -ary operation  $g : A^m \rightarrow B$  is called a *minor* of  $f$  given by the map  $\pi$  if

$$g(x_1, \dots, x_m) = f(x_{\pi(1)}, \dots, x_{\pi(n)}).$$

The polymorphisms  $\text{Pol}(\mathbf{A}, \mathbf{B})$  thus form a minion, where  $f^\pi$  is given by the minor of  $f$  at  $\pi$ .

The main hardness theorem that we will use is the following.<sup>7</sup>

**Theorem 11** ([6]). *Fix constants  $m$  and  $C$ . Take any template  $(\mathbf{A}, \mathbf{B})$  such that  $\text{Pol}(\mathbf{A}, \mathbf{B}) = \bigcup_{i=1}^m \mathcal{M}_i$  is the union of  $m$  parts. Suppose that for each  $i \in [m]$  there exists a map  $I_i$  that takes  $f \in \mathcal{M}_i$  to a subset of  $[\text{ar}(f)]$  of size at most  $C$  such that the following holds: for any  $f, g \in \mathcal{M}_i$  such that  $g = f^\pi$  we have that  $I_i(g) \cap \pi(I_i(f)) \neq \emptyset$ . Then,  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  is NP-hard.*

### 3 Additivity and dependency

In this section we will define two new concepts for a template  $(\mathbf{A}, \mathbf{B})$ , additivity and dependency. Let  $f \in \text{Pol}(\mathbf{A}, \mathbf{B})$  be a polymorphism of  $(\mathbf{A}, \mathbf{B})$ . Intuitively, additivity constrains the value of  $f$  evaluated at two elements and dependency ensures that  $f$  is determined by such evaluations. The end goal of this section is to prove the following two theorems.

**Theorem 12.** *Suppose  $\mathbf{A}$  has a relation  $R^{\mathbf{A}}$  of arity at least 3 for which  $\text{diam}(A, R^{\mathbf{A}}) \leq 1$ . Then, for any functional  $\mathbf{B}$  such that  $\mathbf{A} \rightarrow \mathbf{B}$ ,  $(\mathbf{A}, \mathbf{B})$  is additive and dependent.*

**Theorem 13.** *Suppose  $\mathbf{A}$  has a relation  $R^{\mathbf{A}}$  of arity 3 or 4 that is connected when viewed as the edge relation of a hypergraph on vertices  $A$ . Then, for any functional  $\mathbf{B}$  such that  $\mathbf{A} \rightarrow \mathbf{B}$ ,  $(\mathbf{A}, \mathbf{B})$  is additive and dependent.*

#### 3.1 Additivity

Fix a polymorphism  $f \in \text{Pol}^{(n)}(\mathbf{A}, \mathbf{B})$ . For any  $(i, j) \in A^2$  (including the case where  $i = j$ ), we can define a function  $f_{ij}$  derived from  $f$  in the following way:  $f_{ij} : 2^{[n]} \rightarrow B$  is a function where

$$f_{ij}(S) = f(x_1, \dots, x_n)$$

where  $x_k = j$  if  $k \in S$  and  $x_k = i$  otherwise. In other words,  $f_{ij}(S)$  is  $f$  evaluated at the characteristic vector of  $S$ , where  $j$  indicates membership in  $S$  and  $i$  indicates non-membership.

We define  $f^p : 2^{[n]} \rightarrow B^{A^2}$  to be the function

$$f^p(S)(i, j) = f_{ij}(S).$$

<sup>5</sup>A minion is a functor from the skeleton of the category of finite sets to the category of sets.

<sup>6</sup>Minion homomorphisms are natural transformations.

<sup>7</sup>In [6], this is Theorem 5.21 together with Lemma 5.11, as detailed after the proof of Theorem 5.21 therein.

We will be interested in templates  $(\mathbf{A}, \mathbf{B})$  that have the following property.

**Definition 14.** Fix a template  $(\mathbf{A}, \mathbf{B})$  with  $\mathbf{A}$  symmetric and  $\mathbf{B}$  functional. We say that  $(\mathbf{A}, \mathbf{B})$  is *additive* if there exists an operator  $+: B^{A^2} \times B^{A^2} \rightarrow B^{A^2}$  such that, for any  $f \in \text{Pol}^{(n)}(\mathbf{A}, \mathbf{B})$  and disjoint  $S, T \subseteq [n]$  we have

$$f^p(S) + f^p(T) = f^p(S \cup T).$$

**Lemma 15.** If  $(\mathbf{A}, \mathbf{B})$  is additive, there exists an operator  $-: B^{A^2} \times B^{A^2} \rightarrow B^{A^2}$  such that, for any  $f \in \text{Pol}^{(n)}(\mathbf{A}, \mathbf{B})$  and  $S \subseteq T \subseteq [n]$ , we have

$$f^p(T) - f^p(S) = f^p(T \setminus S).$$

*Proof.* While  $f^p(S)$  has been written as a function from  $A^2$  to  $B$  above, we can also see it as an  $|A| \times |A|$  matrix of elements of  $B$ . Thus we can take the transpose of this matrix, denoted by the superscript  $T$  below. Observe that  $(f^p(S))^T(i, j) = f_{ji}(S) = f_{ij}(\bar{S}) = f^p(\bar{S})(i, j)$ , where  $\bar{S}$  denotes the complement of  $S$ . In other words,  $(f^p(S))^T = f^p(\bar{S})$ .

Set  $x - y = (x^T + y)^T$ . Then for  $S \subseteq T \subseteq [n]$ ,

$$f^p(T) - f^p(S) = (f^p(T)^T + f^p(S))^T = f^p(\overline{\bar{T} \cup S}) = f^p(T \setminus S). \quad \square$$

**Lemma 16.** Suppose  $(\mathbf{A}, \mathbf{B})$  is additive. Fix a polymorphism  $f \in \text{Pol}^{(n)}(\mathbf{A}, \mathbf{B})$ . Consider any family of disjoint sets  $\mathcal{A} \subseteq 2^{[n]}$ , containing at least  $|B|^{|A|^2}$  sets. Then some nonempty subset  $\mathcal{B} \subseteq \mathcal{A}$  exists such that  $f^p(\bigcup \mathcal{B}) = f^p(\emptyset)$ .

The approach used to prove this is analogous to the following well known exercise (first set out by Vázsonyi and Sved, according to Erdős [1]): Prove that any sequence of  $n$  integers has a subsequence whose sum is divisible by  $n$ .

*Proof.* Note that  $\mathcal{A}$  contains at least  $|B|^{|A|^2} \geq |\text{range}(f^p)|$  different sets. Let  $A_1, \dots, A_{|\text{range}(f^p)|}$  be some of these sets. Define  $B_i = \bigcup_{j \leq i} A_j$  for  $0 \leq i \leq |\text{range}(f^p)|$ ; note that  $B_0 = \emptyset$ . By the pigeonhole principle there exists  $0 \leq i < j \leq |\text{range}(f^p)|$  such that  $f^p(B_i) = f^p(B_j)$ . Then, using Lemma 15,  $f^p(B_j \setminus B_i) = f^p(B_j) - f^p(B_i) = f^p(B_i) - f^p(B_i) = f^p(B_i \setminus B_i) = f^p(\emptyset)$ . Thus  $\mathcal{B} = \{A_{i+1}, \dots, A_j\}$  is the required family of sets.  $\square$

**Lemma 17.** Suppose  $(\mathbf{A}, \mathbf{B})$  is additive. Fix a polymorphism  $f \in \text{Pol}^{(n)}(\mathbf{A}, \mathbf{B})$ . Consider any  $S \subseteq [n]$ . There exists  $T \subseteq S$  of size at most  $|B|^{|A|^2}$  such that  $f^p(S) = f^p(T)$ .

*Proof.* Suppose this is not the case, and suppose that  $S$  is a minimal counterexample (with respect to inclusion) to this claim. Clearly  $|S| > |B|^{|A|^2}$ , or else taking  $T = S$  shows that  $S$  is no counterexample at all. Thus, apply Lemma 16 to the family  $\{\{x\} \mid x \in S\}$  to find that some nonempty subset  $U \subseteq S$  exists such that  $f^p(U) = f^p(\emptyset)$ . But now, take  $S' = S \setminus U \subseteq S$ , and note that  $f^p(S') = f^p(S \setminus U) = f^p(S) - f^p(U) = f^p(S) - f^p(\emptyset) = f^p(S \setminus \emptyset) = f^p(S)$ . By the minimality of  $S$ ,  $S'$  has a subset  $T$  of size at most  $|B|^{|A|^2}$  such that  $f^p(T) = f^p(S') = f^p(S)$ , which contradicts the fact that  $S$  is a counterexample.  $\square$



### 3.2 Dependency

Fix again a polymorphism  $f \in \text{Pol}^{(n)}(\mathbf{A}, \mathbf{B})$ .

**Definition 18.** We call  $(\mathbf{A}, \mathbf{B})$  *dependent* if there exists a map  $h$  such that, for every  $f \in \text{Pol}^{(n)}(\mathbf{A}, \mathbf{B})$ , we have that

$$f(x_1, \dots, x_n) = h(f(\alpha(x_1), \dots, \alpha(x_n)) \mid \alpha : A \rightarrow S, S \in A^{\leq 2}),$$

where  $A^{\leq 2}$  is the set of nonempty subsets of  $A$  of size at most 2.

Any polymorphism  $f \in \text{Pol}^{(n)}(\mathbf{A}, \mathbf{B})$  is just a function  $f : A^n \rightarrow B$ . Without loss of generality, identify  $A$  with the set  $[a]$  from now on. Note that a tuple from  $A^n = [a]^n$  can be seen as a partition of  $[n]$  into  $a$  parts  $S_1, \dots, S_a$ :  $S_i$  is the set of coordinates in the tuple set to  $i$ . We will thus denote by  $a^{[n]}$  both the set tuples and the set of such partitions. Thus we can, for example, evaluate  $f$  at a partition  $S_1, \dots, S_a$  of  $[n]$  and get  $f(S_1, \dots, S_a)$ .

**Definition 19.** For any polymorphism  $f \in \text{Pol}^{(n)}(\mathbf{A}, \mathbf{B})$ , we define  $f^* : a^{[n]} \rightarrow (B^{A^2})^a$  by

$$f^*(S_1, \dots, S_a) = (f^p(S_1), \dots, f^p(S_a)).$$

The usefulness of all the concepts introduced so far comes from the following fact.

**Lemma 20.** *Suppose  $(\mathbf{A}, \mathbf{B})$  is dependent and additive. Then there exists a function  $h : (B^{A^2})^a \rightarrow B$  such that for any polymorphism  $f \in \text{Pol}^{(n)}(\mathbf{A}, \mathbf{B})$ , we have  $f = h \circ f^*$ .*

In the proof we will freely mix the two notations for functions of form  $[a]^n \rightarrow B$ . If the input to such functions is written with capital letters it uses the “families of disjoint sets” notation, and if it is written with lower case letters it uses the tuple notation.

*Proof.* We claim that there exists a map  $h'$  that does not depend on  $f$  such that  $h'(f^*(x))$  contains all the values  $f(\alpha(x))$  for  $\alpha : A \rightarrow S$  and  $S \in A^{\leq 2}$ . If this is the case, then by the dependency of  $(\mathbf{A}, \mathbf{B})$  there exists a map  $h''$  such that  $f = h'' \circ h' \circ f^*$ , which implies our conclusion.

Showing that such an  $h'$  exists is the same as showing that we can deduce  $f(\alpha(x))$  for  $\alpha : A \rightarrow S, S \in A^{\leq 2}$  from just the values of  $f^*(x)$ , in a manner independent of  $f$ . First, if  $S = \{s\}$  then we want to find  $f(s, \dots, s)$ . But this value is just  $f_{ss}(X)$  for any  $X$ , and thus is included in every  $f^p(S_i)$  that is within  $f^*(x)$ .

Now suppose  $S = \{b, c\}$ , and that  $A = B \cup C$  such that  $\alpha$  maps  $B$  to  $b$  and  $C$  to  $c$ . In this case we observe that  $\alpha(x)$  contains a  $c$  at all places where  $x$  contained an element in  $C$ , and contains a  $b$  elsewhere. Suppose  $S_1, \dots, S_a$  are sets such that  $S_i$  contains all the indices where  $x$  is equal to  $i$ . Thus,  $f(\alpha(x)) = f_{bc}(\bigcup_{i \in C} S_i)$ . This value is an element of the tuple  $f^p(\bigcup_{i \in C} S_i)$ , so it is sufficient to show that we can deduce this latter value from  $f^*(x) = f^*(S_1, \dots, S_a)$ . But

$$f^p\left(\bigcup_{i \in C} S_i\right) = \sum_{i \in C} f^p(S_i).$$

All of the elements within this sum are contained within  $f^*(S_1, \dots, S_a) = (f^p(S_1), \dots, f^p(S_a))$ . Thus we can deduce the value of  $f(\alpha(x))$  uniquely from  $f^*(x)$ , without reference to  $f$ , as required.  $\square$

### 3.3 Formal system

We will now describe a system of formal proofs that will help us reason about additivity and dependency. First we define the semantics of this system.

**Definition 21.** Suppose  $S \subseteq A^n$  is a set of  $n$ -ary tuples, and  $t \in A^n$  is an  $n$ -tuple. We write  $S \models_{\mathbf{A}, \mathbf{B}} t$  (omitting  $\mathbf{A}, \mathbf{B}$  if there is no chance for confusion) if there exists a function  $h : B^S \rightarrow B$  (where we interpret  $B^S$  as being a tuple indexed by  $S$ ), such that, for any polymorphism  $f \in \text{Pol}^{(n)}(\mathbf{A}, \mathbf{B})$ , we have that

$$f(t) = h(f(u) \mid u \in S).$$

In other words, we write  $S \models t$  if the value of  $f(t)$  is uniquely determined by  $f(u)$  for  $u \in S$ , in a manner independent of  $f$ .

**Definition 22.** For any  $\mathbf{A}$ , we write  $\Gamma_{\mathbf{A}} = \{(r, s, s) \mid s, r \in A\} \cup \{(s, r, s) \mid s, r \in A\}$  for the set of triples from  $A^3$  with the last two elements equal or with the first and last elements equal. Write  $\Delta_{\mathbf{A}}^n = \{(x_1, \dots, x_n) \mid x_1, \dots, x_n \in A, |\{x_1, \dots, x_n\}| \leq 2\}$  i.e.  $\Delta_{\mathbf{A}}^n$  contains all the  $n$ -ary tuples that contain at most two distinct elements.

**Lemma 23.**  $(\mathbf{A}, \mathbf{B})$  is additive if for every  $p, q \in A$  we have

$$\Gamma_{\mathbf{A}} \models_{\mathbf{A}, \mathbf{B}} (p, p, q).$$

*Proof.* For  $p, q \in A$ , suppose  $h_{qp}$  is the function that witnesses  $\Gamma_{\mathbf{A}} \models_{\mathbf{A}, \mathbf{B}} (p, p, q)$ . Define

$$(x + y)(q, p) = h_{qp}(t_{xyz} \mid (x, y, z) \in \Gamma_{\mathbf{A}}),$$

where  $t_{sss} = x(s, s)$ ,  $t_{rss} = x(s, r)$  and  $t_{srs} = y(s, r)$ , for  $s, r \in A$ . (This defines  $t_{xyz}$  for all  $(x, y, z) \in \Gamma_{\mathbf{A}}$ .)

We claim that this shows the additivity of  $(\mathbf{A}, \mathbf{B})$ . Consider any polymorphism  $f \in \text{Pol}^{(n)}(\mathbf{A}, \mathbf{B})$  and disjoint  $S, T \subseteq [n]$ . Write  $\pi : [n] \rightarrow [3]$  for the function that takes  $S$  to 1,  $T$  to 2, and  $[n] \setminus (S \cup T)$  to 3. By the definition of  $h_{qp}$  and that of taking minors, we have

$$(f^p(S \cup T))(q, p) = f_{qp}(S \cup T) = f^\pi(p, p, q) = h_{qp}(f^\pi(x, y, z) \mid (x, y, z) \in \Gamma_{\mathbf{A}}).$$

Now, let  $t_{xyz} = f^\pi(x, y, z)$ . Observe that  $t_{sss} = f^\pi(s, s, s) = f^p(S)(ss)$ . Also,  $t_{rss} = f^\pi(r, s, s) = f^p(S)(s, r)$ , and  $t_{srs} = f^\pi(s, r, s) = f^p(T)(s, r)$ . Thus we deduce that

$$(f^p(S \cup T))(q, p) = h_{qp}(t_{xyz} \mid (x, y, z) \in \Gamma_{\mathbf{A}}) = (f^p(S) + f^p(T))(q, p).$$

Thus by Definition 14,

$$f^p(S \cup T) = f^p(S) + f^p(T),$$

as required. □

**Lemma 24.** Assuming  $A = [a]$ ,  $(\mathbf{A}, \mathbf{B})$  is dependent if we have

$$\Delta_{\mathbf{A}}^a \models_{\mathbf{A}, \mathbf{B}} (1, \dots, a).$$

*Proof.* Suppose  $h$  witnesses that  $\Delta_{\mathbf{A}}^a \models_{\mathbf{A}, \mathbf{B}} (1, \dots, a)$ . Note that every tuple in  $\Delta_{\mathbf{A}}^a$  can be seen as a function from  $[a]$  to some  $S \subseteq A$  for  $0 < |S| \leq 2$ . In other words  $h$  takes a tuple of elements from  $B$ , indexed by functions  $\alpha : A \rightarrow S, S \in A^{\leq 2}$ . In other words, we see that for any  $a$ -ary polymorphism  $f$ ,

$$f(1, \dots, a) = h(f(\alpha(1), \dots, \alpha(a)) \mid \alpha : A \rightarrow S, S \in A^{\leq 2}).$$

The type of this function (not coincidentally) is exactly the same as the function that ought to witness the dependency of  $(\mathbf{A}, \mathbf{B})$ ; and indeed, we claim that it does in fact witness this.

In other words, we must show that, for any  $f \in \text{Pol}^{(n)}(\mathbf{A}, \mathbf{B})$  and  $x_1, \dots, x_n \in A$  we have that

$$f(x_1, \dots, x_n) = h(f(\alpha(x_1), \dots, \alpha(x_n)) \mid A \rightarrow S, S \in A^{\leq 2}).$$

Observe that for any  $x_1, \dots, x_n$ , there exists a function  $\pi : [n] \rightarrow [a]$  such that  $f(x_1, \dots, x_n) = f^\pi(1, \dots, a)$ ; namely  $\pi(i) = x_i$ . Furthermore  $f^\pi$  is an  $a$ -ary polymorphism; thus

$$\begin{aligned} f(x_1, \dots, x_n) &= f^\pi(1, \dots, a) = h(f^\pi(\alpha(1), \dots, \alpha(a)) \mid \alpha : A \rightarrow S, S \in A^{\leq 2}) \\ &= h(f(\alpha(x_1), \dots, \alpha(x_n)) \mid A \rightarrow S, S \in A^{\leq 2}), \end{aligned}$$

as required.  $\square$

We now move on to a syntactic description of the formal proof system.

**Definition 25.** Fix some  $n \in \mathbb{N}$ . Fix also some relation  $R^{\mathbf{A}}$  of  $\mathbf{A}$ . For sets of  $n$ -ary tuples  $S \subseteq A^n$  and  $t \in A^n$ , we define  $S \vdash_{\mathbf{A}, R^{\mathbf{A}}} t$  as the minimal relation that satisfies the following.

1. If  $S \subseteq T$  and  $S \vdash t$  then  $T \vdash t$ .
2.  $t \vdash t$ .
3. If  $S \vdash t$  and  $t, T \vdash t'$  then  $S, T \vdash t'$ .
4. If there exists a matrix with  $n$  columns and  $r$  rows, whose rows are  $t_1, \dots, t_r$ , and whose columns are tuples of  $R^{\mathbf{A}}$ , then  $t_2, \dots, t_r \vdash t_1$ .

We omit  $\mathbf{A}, R^{\mathbf{A}}$  if they are obvious from context.

**Remark 26.** Suppose  $R^{\mathbf{A}}$  has arity  $r$ . Any judgement of the form  $S \vdash_{\mathbf{A}, R^{\mathbf{A}}} t$  must have a finite proof using the rules above by minimality. From this proof, we can create a *proof-tree* where the vertices are  $n$ -ary tuples of  $A$ , the root is  $t$ , the leaves belong to  $S$ , and every non-leaf has  $r - 1$  children such that if  $t_1$  is the non-leaf and  $t_2, \dots, t_r$  are its children, then the matrix whose rows are  $t_1, \dots, t_r$  has as its columns only tuples of  $R^{\mathbf{A}}$ . To create this tree, proceed inductively on the proof, from the conclusion backwards. The first and second rule do not modify the proof tree. The third rule corresponds to recursively constructing a subtree. The final rule is the only one that adds new vertices to the proof tree.

**Lemma 27.** If  $S \vdash_{\mathbf{A}, R^{\mathbf{A}}} t$  then  $S \models_{\mathbf{A}, \mathbf{B}} t$  for any symmetric  $\mathbf{A}$  and functional  $\mathbf{B}$ .

*Proof.* By minimality, it is sufficient to show that  $\models_{\mathbf{A}, \mathbf{B}}$  satisfies all the rules satisfied by  $\vdash_{\mathbf{A}, R^{\mathbf{A}}}$ , which we do now rule-by-rule. For the following assume always that all tuples are of arity  $n$ .

1. Suppose  $S \subseteq T$  and  $S \models t$ . Thus there exists a function  $h : A^S \rightarrow A$  such that, for any polymorphism  $f \in \text{Pol}^{(n)}(\mathbf{A}, \mathbf{B})$ , we have

$$f(t) = h(f(u) \mid u \in S).$$

Now, we define the function  $h' : A^T \rightarrow A$  as follows:

$$h'(x_u \mid u \in T) = h(x_u \mid u \in S).$$

In other words,  $h'$  ignores all inputs in  $T \setminus S$  and otherwise acts like  $h$ . We claim that  $h'$  witnesses that  $T \models t$ : for any polymorphism  $f \in \text{Pol}^{(n)}(\mathbf{A}, \mathbf{B})$ , we have

$$f(t) = h(f(u) \mid u \in S) = h'(f(u) \mid u \in T).$$

2. For any tuple  $t$ , we have  $t \models t$ ; indeed, the identity function  $\text{id}_B : B \rightarrow B$  witnesses this fact.
3. Suppose  $S \models t$  and  $t, T \models u$ , as witnessed by  $h : B^S \rightarrow B$  and  $h' : B^{t+T} \rightarrow B$ . If  $t \in T$  then  $T \models u$  and we have by the first rule that  $S, T \models u$  as required; thus suppose  $t \notin T$ . We will thus interpret  $h'$  as having signature  $h' : B \rightarrow B^T \rightarrow B$ . With this interpretation, by definition, for any polymorphism  $f \in \text{Pol}^{(n)}(\mathbf{A}, \mathbf{B})$ , we have

$$\begin{aligned} f(t) &= h(f(v) \mid v \in S) \\ f(u) &= h'(f(t))(f(v) \mid v \in T). \end{aligned}$$

Now define  $h'' : B^{S \cup T} \rightarrow B$  in the following way:

$$h''(x_v \mid v \in S \cup T) = h'(h(x_v \mid v \in S))(x_v \mid v \in T).$$

We find that, for any  $f \in \text{Pol}^{(n)}(\mathbf{A}, \mathbf{B})$ ,

$$f(u) = h'(f(t))(f(v) \mid v \in T) = h'(h(f(v) \mid v \in S))(f(v) \mid v \in T) = h''(f(v) \mid v \in S \cup T).$$

Thus we find that  $S, T \models u$ , as required.

4. Suppose  $R^{\mathbf{A}}$  has arity  $r$ , and suppose there exists a matrix with  $n$  columns and  $r$  rows, whose rows are  $t_1, \dots, t_r$ , and whose columns are tuples of  $R^{\mathbf{A}}$ . Let  $h : B^{\{t_2, \dots, t_r\}} \rightarrow B$  be defined as follows, interpreting an element of  $B^{\{t_2, \dots, t_r\}}$  as an  $(r-1)$ -ary tuple in the natural way:

$$h(x_2, \dots, x_r) = R^{\mathbf{B}}(x_2, \dots, x_r).$$

Now, we claim that  $h$  witnesses that  $t_2, \dots, t_r \models t_1$ . Indeed, for any  $n$ -ary polymorphism  $f \in \text{Pol}^{(n)}(\mathbf{A}, \mathbf{B})$ , we have that  $(f(t_1), \dots, f(t_r)) \in R^{\mathbf{B}}$ , and thus

$$f(t_1) = R^{\mathbf{B}}(f(t_2), \dots, f(t_r)) = h(f(t_2), \dots, f(t_r)) = h(f(u) \mid u \in \{t_2, \dots, t_r\}).$$

Thus we conclude that  $\models_{\mathbf{A}, \mathbf{B}}$  satisfies the rules of  $\vdash_{\mathbf{A}, R^{\mathbf{A}}}$ , which implies our conclusion.  $\square$

**Lemma 28.** *Suppose that  $S \vdash_{\mathbf{A}, R^{\mathbf{A}}} t$  and  $x \in \text{supp}(R^{\mathbf{A}})$ . Then  $S \times \text{supp}(R^{\mathbf{A}}) \vdash_{\mathbf{A}, R^{\mathbf{A}}} (t, x)$ .<sup>8</sup>*

<sup>8</sup>Here, if  $t = (t_1, \dots, t_r)$  then  $(t, x) = (t_1, \dots, t_r, x)$ , and likewise  $S \times \text{supp}(R^{\mathbf{A}}) = \{(s_1, \dots, s_r, x) \mid (s_1, \dots, s_r) \in S, x \in \text{supp}(R^{\mathbf{A}})\}$ .

*Proof.* We show this by induction on the formal proof that proves that  $S \vdash_{\mathbf{A}, R^{\mathbf{A}}} t$ . Based on the last step in the proof used to prove this fact, we have the following cases.

1. Suppose  $S' \subseteq S$  and  $S' \vdash t$ . By the inductive hypothesis, we know that  $S' \times \text{supp}(R^{\mathbf{A}}) \vdash (t, x)$ . As  $S' \times \text{supp}(R^{\mathbf{A}}) \subseteq S \times \text{supp}(R^{\mathbf{A}})$ , then  $S \times \text{supp}(R^{\mathbf{A}}) \vdash (t, x)$  as required.
2. Suppose  $S = \{t\}$ . In this case the result is immediate, as  $S \times \text{supp}(R^{\mathbf{A}}) \supseteq \{(t, u)\} \vdash (t, u)$ .
3. Suppose  $S = X \cup Y$ ,  $X \vdash s$  and  $s, Y \vdash t$ . By the inductive hypothesis, for  $y \in \text{supp}(R^{\mathbf{A}})$ ,  $X \times \text{supp}(R^{\mathbf{A}}) \vdash (s, y)$  and  $\{(s, y) \mid y \in \text{supp}(R^{\mathbf{A}})\}, Y \times \text{supp}(R^{\mathbf{A}}) \vdash (t, x)$ . We can therefore deduce, by this rule applied  $|\text{supp}(R^{\mathbf{A}})|$  times, that  $S \times \text{supp}(R^{\mathbf{A}}) = (X \cup Y) \times \text{supp}(R^{\mathbf{A}}) \vdash (t, x)$ .
4. Suppose that  $S = \{t_2, \dots, t_r\}$  and there exists a matrix whose rows are  $t_1, \dots, t_r$  and all of whose columns are elements of  $R^{\mathbf{A}}$ . Suppose  $(x, x_2, \dots, x_r) \in R^{\mathbf{A}}$ . Then by applying this rule we find that  $(t_2, x), \dots, (t_r, x_r) \vdash (t, x)$ , which implies that  $S \times \text{supp}(R^{\mathbf{A}}) \vdash (t, x)$ .

Thus our conclusion follows.  $\square$

Any tuple  $x \in A^n$  can be seen as a function from  $[n]$  to  $A$ . Thus for any function  $\pi : [m] \rightarrow [n]$ , we let  $x \circ \pi \in A^m$  be the tuple which, at position  $i$ , has value  $x_{\pi(i)}$ . For a set of tuples  $S \in A^n$ , we let  $S \circ \pi = \{x \circ \pi \mid x \in S\}$ .

**Lemma 29.** *Suppose  $S \subseteq A^n$ ,  $t \in A^n$  and  $\pi : [m] \rightarrow [n]$ . If  $S \vdash_{\mathbf{A}, R^{\mathbf{A}}} t$  then  $S \circ \pi \vdash_{\mathbf{A}, R^{\mathbf{A}}} t \circ \pi$ .*

*Proof.* First, consider any matrix  $M$  whose rows are  $r_1, \dots, r_n$  and whose columns are  $c_1, \dots, c_m$ . If we create a matrix  $M'$  whose rows are  $r_1 \circ \sigma, \dots, r_n \circ \sigma$  for some  $\sigma : [k] \rightarrow [n]$ , we observe that the columns of this new matrix are  $c_{\sigma(1)}, \dots, c_{\sigma(k)}$ . Thus, the columns of  $M'$  are a subset of the columns of  $M$ .

Consider the proof tree that proves that  $S \vdash_{\mathbf{A}, R^{\mathbf{A}}} t$ . If a vertex is labeled by a tuple  $u$ , transform it into a vertex labeled by  $u \circ \pi$ . By the previous observation, this remains a valid proof tree, whose leaves belong to  $S \circ \pi$  and whose root is  $t \circ \pi$ . Thus  $S \circ \pi \vdash_{\mathbf{A}, R^{\mathbf{A}}} t \circ \pi$ .  $\square$

### 3.4 Super-connectedness

Rather than dealing directly with additivity and dependency, we will instead deal with a notion that implies them.

**Definition 30.** We say that  $\mathbf{A}$  is *super-connected* if  $\mathbf{A}$  has a relation  $R^{\mathbf{A}}$  such that, for every  $x, y, z \in A$  (perhaps even with  $|\{x, y, z\}| < 3$ ), we have

$$\Gamma_{\mathbf{A}} \vdash_{\mathbf{A}, R^{\mathbf{A}}} (x, y, z).$$

Super-connectedness is a very useful concept: it implies both additivity and dependency.

**Lemma 31.** *If  $\mathbf{A}$  is super-connected then  $(\mathbf{A}, \mathbf{B})$  is additive for functional  $\mathbf{B}$  where  $\mathbf{A} \rightarrow \mathbf{B}$ .*

*Proof.* An immediate consequence of Lemma 23 and Lemma 27.  $\square$

**Lemma 32.** *If  $\mathbf{A}$  is super-connected then  $(\mathbf{A}, \mathbf{B})$  is dependent for functional  $\mathbf{B}$  where  $\mathbf{A} \rightarrow \mathbf{B}$ .*

We first prove two simple propositions. Assume for them that  $\mathbf{A}$  is super-connected, witnessed by relation  $R^{\mathbf{A}}$ , and that  $|A| > 1$ .

**Proposition 33.**  $\text{supp}(R^{\mathbf{A}}) = A$ .

*Proof.* Suppose for contradiction that this is not the case. Let  $x \in A \setminus \text{supp}(R^{\mathbf{A}})$ , and take  $y \in A$  such that  $x \neq y$ . Now,

$$\Gamma_{\mathbf{A}} \vdash_{\mathbf{A}, R^{\mathbf{A}}} (x, x, y).$$

Consider any proof tree concluding in this judgement. The root vertex cannot have any children, since no matrix containing  $(x, x, y)$  can have its columns belong to  $R^{\mathbf{A}}$ , as  $x \notin \text{supp}(R^{\mathbf{A}})$ . Thus  $(x, x, y)$  is a leaf vertex in the tree, and  $(x, x, y) \in \Gamma_{\mathbf{A}}$ . This is not possible, as  $x \neq y$ .  $\square$

**Proposition 34.**  $\Delta_{\mathbf{A}}^n \vdash (x_1, \dots, x_n)$ .

*Proof.* We prove this fact by induction. We do induction over the lexicographical ordering on  $\{(n, d) \in \mathbb{N}^2 \mid n \geq d\}$ . For any  $(n, d) \in \mathbb{N}^2$  with  $n \geq d$  we prove that  $\Delta_{\mathbf{A}}^n \vdash (x_1, \dots, x_n)$  whenever  $|\{x_1, \dots, x_n\}| \leq d$ . As our base case, note that the result is immediate when  $n \leq 2$ , as  $(x_1, \dots, x_n) \in \Delta_{\mathbf{A}}^n$  in this case. Thus suppose  $n \geq 3$ .

First suppose that  $x_1, \dots, x_n$  contains a duplicated pair of values, say  $x_{n-1} = x_n$ . Now, define  $\pi : [n-1] \rightarrow [n]$ , where  $\pi(i) = i$ . By the inductive hypothesis, we know that  $\Delta_{\mathbf{A}}^{n-1} \vdash (x_1, \dots, x_{n-1})$ . Thus, by Lemma 29,  $\Delta_{\mathbf{A}}^{n-1} \circ \pi \vdash (x_1, \dots, x_{n-1}) \circ \pi = (x_1, \dots, x_{n-1}, x_{n-1}) = (x_1, \dots, x_n)$ . Furthermore, every tuple of  $\Delta_{\mathbf{A}}^{n-1} \circ \pi$  contains at most 2 values, so it belongs to  $\Delta_{\mathbf{A}}^n$ . Thus in this case  $\Delta_{\mathbf{A}}^n \vdash (x_1, \dots, x_n)$ .

Now suppose that  $x_1, \dots, x_n$  are all distinct. Since  $\mathbf{A}$  is super-connected, we find that  $\Gamma_{\mathbf{A}} \vdash (x_1, x_2, x_3)$ . Thus by Lemma 28, we find that  $\Gamma_{\mathbf{A}} \times \text{supp}(R^{\mathbf{A}})^{n-3} \vdash (x_1, \dots, x_n)$ . By Proposition 33,  $\text{supp}(R^{\mathbf{A}}) = A$ , so  $\Gamma_{\mathbf{A}} \times A^{n-3} \vdash (x_1, \dots, x_n)$ . But, every tuple  $t \in \Gamma_{\mathbf{A}} \times A^{n-3}$  contains at most  $n-1$  distinct values, so we can apply the inductive hypothesis to them, and find that  $\Delta_{\mathbf{A}}^n \vdash t$ . Thus  $\Delta_{\mathbf{A}}^n \vdash (x_1, \dots, x_n)$ , as required. This completes the proof.  $\square$

*Proof of Lemma 32.*  $\Delta_{\mathbf{A}}^a \vdash (1, \dots, a)$  by Proposition 34. Thus, by Lemma 27 we have that  $\Delta_{\mathbf{A}}^a \models_{\mathbf{A}, \mathbf{B}} (1, \dots, a)$ , and therefore dependency follows by Lemma 24.  $\square$

**Lemma 35.** *Suppose  $\mathbf{A}$  has a relation  $R^{\mathbf{A}}$  of arity at least 3 for which  $\text{diam}(A, R^{\mathbf{A}}) \leq 1$ . Then  $\mathbf{A}$  is super-connected.*

*Proof.* Fix  $x, y, z \in A$ ; we must show that  $\Gamma_{\mathbf{A}} \vdash (x, y, z)$ . We have several cases.

**$x = z$  or  $y = z$ .** In this case  $(x, y, z) \in \Gamma_{\mathbf{A}}$ , so there is nothing left to be proved.

**$x = y \neq z$ .** In this case  $\text{dist}_{R^{\mathbf{A}}}(x, z) = 1$ , so there exists an edge  $(x, z, a_3, \dots, a_r) \in R^{\mathbf{A}}$ . Consider the matrices

$$M = \begin{pmatrix} x & x & z \\ z & a_3 & x \\ a_3 & z & a_3 \\ a_4 & a_4 & a_4 \\ \vdots & \vdots & \vdots \\ a_r & a_r & a_r \end{pmatrix}, \quad N = \begin{pmatrix} z & a_3 & x \\ x & z & z \\ a_3 & x & a_3 \\ a_4 & a_4 & a_4 \\ \vdots & \vdots & \vdots \\ a_r & a_r & a_r \end{pmatrix}.$$



Thus we deduce that  $(z, a_3, x), \Gamma_{\mathbf{A}} \vdash (x, x, z)$  and  $\Gamma_{\mathbf{A}} \vdash (z, a_3, x)$ , and thus  $\Gamma_{\mathbf{A}} \vdash (x, x, z) = (x, y, z)$ .

$x \neq y, y \neq z, x \neq z$ . In this case  $\text{dist}_{R^{\mathbf{A}}}(x, z) = \text{dist}_{R^{\mathbf{A}}}(x, y) = 1$ , so there must exist edges  $(x, z, a_3, \dots, a_r), (x, y, b_3, \dots, b_r) \in R^{\mathbf{A}}$ . Consider the following matrix

$$\begin{pmatrix} x & y & z \\ z & x & x \\ a_3 & b_3 & a_3 \\ \vdots & \vdots & \vdots \\ a_r & b_r & a_r \end{pmatrix}.$$

Thus we find that  $(z, x, x), (a_3, b_3, a_3), \dots, (a_r, b_r, a_r) \vdash (x, y, z)$ , whence  $\Gamma_{\mathbf{A}} \vdash (x, y, z)$ .

Thus our conclusion follows in all cases.  $\square$

**Lemma 36.** *If  $\mathbf{A}$  has a connected relation  $R^{\mathbf{A}}$  of arity 3 then  $\mathbf{A}$  is super-connected.*

*Proof.* We see  $(A, R^{\mathbf{A}})$  as a connected 3-uniform hypergraph. For  $x, y \in A$  define  $\text{dist}(x, y)$  as the distance between  $x$  and  $y$  in this hypergraph. We show that  $\Gamma_{\mathbf{A}} \vdash_{\mathbf{A}, R^{\mathbf{A}}} (x, y, z)$  for all  $x, y, z \in A$  by induction on  $\min(\text{dist}(x, z), \text{dist}(y, z))$ . There are multiple cases.

**dist  $(x, z) = 0$  or dist  $(y, z) = 0$ .** In this case  $(x, y, z) \in \{(x, y, x), (x, y, y)\} \subseteq \Gamma_{\mathbf{A}}$ , so there is nothing left to prove.

**dist  $(x, z) = \text{dist}(y, z) = 1$ .** In this case, there exists edges  $(x, a, z), (y, b, z)$ . Now consider the following matrices:

$$M = \begin{pmatrix} x & y & z \\ z & z & x \\ a & b & a \end{pmatrix}, \quad N = \begin{pmatrix} z & z & x \\ a & x & z \\ x & a & a \end{pmatrix}, \quad P = \begin{pmatrix} a & x & z \\ x & z & x \\ z & a & a \end{pmatrix}.$$

Thus we conclude that  $(a, b, a), (z, z, x) \vdash (x, y, z)$ ,  $(a, x, z), (x, a, a) \vdash (z, z, x)$  and also  $(x, z, x), (z, a, a) \vdash (a, x, z)$ , from where the conclusion follows.

**dist  $(x, z) > 1$  or dist  $(y, z) > 1$ .** Without loss of generality suppose  $\text{dist}(x, z) > 1$ . Thus there exist edges  $(x, a, x'), (z', b, z), (y, c, d)$  such that  $\text{dist}(x', z') = \text{dist}(x, z) - 2$ , and therefore  $\text{dist}(a, z'), \text{dist}(b, x) < \text{dist}(x, z)$ . Consider the matrix

$$\begin{pmatrix} x & y & z \\ a & c & z' \\ x' & d & b \end{pmatrix}.$$

Thus we deduce that  $(a, c, z'), (x', d, b) \vdash (x, y, z)$ ; furthermore the inductive hypothesis can be applied to the tuples to the left of  $\vdash$ , so this completes the proof in this case.

Thus we find by induction that  $\mathbf{A}$  is super-connected.  $\square$

**Lemma 37.** *If  $\mathbf{A}$  has a connected relation  $R^{\mathbf{A}}$  of arity 4 then  $\mathbf{A}$  is super-connected.*

*Proof.* We see  $(A, R^A)$  as a connected 4-uniform hypergraph. For any  $x, y \in A$ , define  $\text{dist}(x, y)$  as the distance between  $x$  and  $y$  in this hypergraph. We need to show that, for any  $x, y, z \in A$ , we have that  $\Gamma_A \vdash_{A, R^A} (x, y, z)$ . We show this by induction on  $\text{minmax}(\text{dist}(x, z), \text{dist}(y, z))$  (where  $\text{minmax}(a, b) = (\min(a, b), \max(a, b))$ ), ordered by the lexicographical ordering. Note that  $\Gamma_A$  is symmetric under swapping its first two elements, as is our inductive variant, and thus our proposition is thus symmetric in this way in general; thus suppose without loss of generality that  $\text{dist}(x, z) \leq \text{dist}(y, z)$ , and thus that  $\text{minmax}(\text{dist}(x, z), \text{dist}(y, z)) = (\text{dist}(x, z), \text{dist}(y, z))$ .

**$\text{dist}(x, z) = 0$ .** In this case  $(x, y, z) = (z, y, z) \in \Gamma_A$ , so nothing needs to be proved.

**$\text{dist}(x, z) = \text{dist}(y, z) = 1$ .** In this case, there exist  $a, b, c, d$  such that  $(x, a, b, z), (y, c, d, z) \in R^A$ . (It may be possible that  $a, b, c, d, x, y, z$  are not all distinct, but this does not change the correctness of the proof that follows.) Now consider the matrices

$$M = \begin{pmatrix} x & y & z \\ a & c & a \\ b & d & b \\ z & z & x \end{pmatrix}, \quad N = \begin{pmatrix} z & z & x \\ b & a & z \\ a & x & b \\ x & b & z \end{pmatrix}$$

Thus  $(a, c, a), (b, d, b), (z, z, x) \vdash (x, y, z)$  i.e.  $(z, z, x), \Gamma_A \vdash (x, y, z)$ . Now, by the proof of the previous lemma since they belong to the tuple  $(x, a, b, z)$  of arity at least 3, we find that  $\Gamma_A \vdash (z, z, x)$ . Combining these facts we get our conclusion.

**$\text{dist}(y, z) > \text{dist}(x, z) = 1$ .** In this case, there exist edges  $(x, a, b, z)$  and  $(y, c, d, y')$  such that  $\text{dist}(y', z) = \text{dist}(y, z) - 1$ . Furthermore, there exists an edge  $(y', f, g, y')$  such that

$$\text{dist}(f, z), \text{dist}(g, z), \text{dist}(y', z) \leq \text{dist}(y', z).$$

(This second edge is why we must assume  $\text{dist}(y, z) > 1$ , since this fails for  $\text{dist}(y, z) = 1$ !) Now consider the following two matrices

$$M = \begin{pmatrix} x & y & z \\ a & c & a \\ b & d & b \\ z & y' & x \end{pmatrix}, \quad N = \begin{pmatrix} z & y' & x \\ a & f & a \\ b & g & b \\ x & y' & z \end{pmatrix}.$$

These matrices show  $(x, y', z), (b, g, b), (a, f, a), (b, d, b), (a, c, a) \vdash (x, y, z)$ . We can apply the inductive hypothesis to all the terms on the left of  $\vdash$  so we are done in this case.

**$\text{dist}(y, z) \geq \text{dist}(x, z) > 1$ .** In this case there exist edges  $(x, a, b, x')$ ,  $(z, c, d, z')$ ,  $(y, e, f, y')$  and  $(y', g, h, y'')$  such that  $\text{dist}(a, z') < \text{dist}(x, z)$ ,  $\text{dist}(x', d) < \text{dist}(x, z)$ ,  $\text{dist}(y'', z) < \text{dist}(y, z)$ . (Namely,  $(x, a, b, x')$  is the edge taken from  $x$  towards  $z$ , with  $x'$  being closer to  $z$  than  $x$ ;  $(z, c, d, z')$  is the edge taken from  $z$  to  $x$ , with  $z'$  being closer to  $x$  than  $z$ ; and  $(y, e, f, y'), (y', g, h, y'')$  are the edges taken from  $y$  to  $z$ , with  $y'$  and  $y''$  being closer and closer to  $z$  than  $y$ .) Now consider

$$M = \begin{pmatrix} x & y & z \\ a & e & z' \\ x' & f & d \\ b & y' & c \end{pmatrix}, \quad N = \begin{pmatrix} b & y' & c \\ a & g & z' \\ x & y'' & z \\ x' & h & d \end{pmatrix}$$

Due to these matrices,  $(a, e, z'), (x', f, d), (a, g, z'), (x, y'', z), (x', y, d) \vdash (x, y, z)$ ; by the distance properties mentioned above, we can apply the inductive hypothesis to all of the tuples on the left of  $\vdash$ , so  $\Gamma_{\mathbf{A}} \vdash (x, y, z)$ .

Thus, by induction,  $\Gamma_{\mathbf{A}} \vdash (x, y, z)$  for all  $x, y, z \in A$ , which implies our conclusion.  $\square$

With all of this, we are finally ready to prove Theorem 12 and Theorem 13.

*Proof of Theorem 12 and Theorem 13.* The structures from Theorem 12 are super-connected by Lemma 35. The structures from Theorem 13 are super-connected by Lemma 36 and Lemma 37. This is sufficient for additivity by Lemma 31 and for dependency by Lemma 32.  $\square$

## 4 Dichotomy

In this section we will prove our main result.

**Theorem 1.** *Let  $\mathbf{A}$  be a symmetric structure and  $\mathbf{B}$  be a functional structure such that  $\mathbf{A} \rightarrow \mathbf{B}$ . Assume that  $(\mathbf{A}, \mathbf{B})$  is dependent and additive. Then, either  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  is solvable in polynomial time by AIP and is finitely tractable, or  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  is NP-hard.*

Before proving Theorem 1, we will prove three interesting corollaries.

**Corollary 2.** *Let  $\mathbf{A}$  be a Boolean symmetric structure and  $\mathbf{B}$  be a functional structure such that  $\mathbf{A} \rightarrow \mathbf{B}$ . Then, either  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  is solvable in polynomial time by AIP and is finitely tractable, or  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  is NP-hard.*

For the proof of Corollary 2, we will need a simple lemma.

**Lemma 38.** *Suppose  $\mathbf{A}$  is Boolean,  $\mathbf{B}$  is functional, and every relation of  $\mathbf{A}$  is either binary or contains only constant tuples. Then,  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  is solvable in polynomial time by AIP and is finitely tractable.*

*Proof.* Consider any  $h : \mathbf{A} \rightarrow \mathbf{B}$ . Suppose  $h(0) = h(1)$ . Then every relation in  $\mathbf{B}$  contains a constant tuple of the form  $(h(0), \dots, h(0))$ ; in this case,  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  is trivially solved by AIP and is finitely tractable. Thus suppose  $h(0) \neq h(1)$ . Any empty relation in  $\mathbf{A}$  can be removed (together with the corresponding relation in  $\mathbf{B}$ ) as it does not affect  $\text{Pol}(\mathbf{A}, \mathbf{B})$  and the complexity of  $\text{PCSP}(\mathbf{A}, \mathbf{B})$ .

Since  $\mathbf{B}$  is functional, the binary relations of  $\mathbf{A}$  that do not contain only constant tuples must be the binary disequality. To see why, consider any relation  $R^{\mathbf{A}}$  in  $\mathbf{A}$  that contains the tuple  $(0, 1)$ .  $R^{\mathbf{A}}$  cannot contain  $(0, 0)$  or  $(1, 1)$ , since the corresponding relation  $R^{\mathbf{B}}$  in  $\mathbf{B}$  contains  $(h(0), h(1))$  and if it contained  $(h(0), h(0))$  or  $(h(1), h(1))$  it would not be functional. Since  $\mathbf{A}$  is symmetric,  $R^{\mathbf{A}}$  also contains the tuple  $(1, 0)$ . Thus  $R^{\mathbf{A}} = \{(0, 1), (1, 0)\}$  is the disequality relation. It follows that every relation in  $\mathbf{A}$  is either a binary disequality, or consists only of constant tuples. In this case,  $\text{CSP}(\mathbf{A})$  is solved by AIP and thus  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  is solved by AIP and is finitely tractable.  $\square$

*Proof of Corollary 2.* If  $\mathbf{A}$  contains only binary relations or relations that contain only constant tuples, the conclusion follows by Lemma 38. Otherwise,  $\mathbf{A}$  has a relation  $R^{\mathbf{A}}$  of arity at least 3 with a non-constant tuple i.e. for which  $\text{diam}(A, R^{\mathbf{A}}) = 1$ , and the conclusion follows from Theorem 1 together with super-connectivity of such structures: super-connectedness implies additivity (cf. Lemma 31) and dependency (cf. Lemma 32).  $\square$

**Corollary 3.** *Let  $\mathbf{A}$  be a symmetric structure and  $\mathbf{B}$  be a functional structure such that  $\mathbf{A} \rightarrow \mathbf{B}$ . Assume that both  $\mathbf{A}$  and  $\mathbf{B}$  have exactly one relation of arity at most 4. Then, either  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  is solvable in polynomial time, or  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  is NP-hard.*

*Proof.* First, observe that the conclusion holds trivially if  $R^{\mathbf{B}}$  has arity 1 as in this case  $\text{CSP}(\mathbf{B})$  is trivially tractable. Second, suppose  $R^{\mathbf{B}}$  has arity 2. In this case we claim that  $\text{CSP}(\mathbf{B})$  is solvable in polynomial time. Indeed, an easy modification of the algorithm for 2-colouring will solve this problem. Thus  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  is tractable in this case.

Now, suppose  $R^{\mathbf{B}}$  has arity 3 or 4. The conclusion holds for connected  $\mathbf{A}$  by the super-connectedness of structures  $\mathbf{A}$  that have a relation of arity 3 or 4 that is connected. Thus suppose that  $\mathbf{A}$  is not connected, and that its connected components are  $\mathbf{A}_1, \dots, \mathbf{A}_k$  i.e.  $\mathbf{A} = \mathbf{A}_1 + \dots + \mathbf{A}_k$ . If  $\text{PCSP}(\mathbf{A}_i, \mathbf{B})$  is NP-hard for some  $i \in [k]$  then  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  will also be NP-hard (since there is a trivial reduction from  $\text{PCSP}(\mathbf{A}_i, \mathbf{B})$  to  $\text{PCSP}(\mathbf{A}, \mathbf{B})$ , as  $\mathbf{A}_i \rightarrow \mathbf{A}$ ). Thus suppose  $\text{PCSP}(\mathbf{A}_i, \mathbf{B})$  is solvable in polynomial time for all  $i \in [k]$ . Consider any input hypergraph  $\mathbf{X} = \mathbf{X}_1 + \dots + \mathbf{X}_n$ , where  $\mathbf{X}_1, \dots, \mathbf{X}_n$  are connected. Since the homomorphic image of a connected hypergraph is connected, if  $\mathbf{X}_i \rightarrow \mathbf{A}$  then  $\mathbf{X}_i \rightarrow \mathbf{A}_j$  for some  $j \in [k]$ . Thus, for the decision version of  $\text{PCSP}(\mathbf{A}, \mathbf{B})$ , it is sufficient to see if, for all  $i \in [n]$  there exists some  $j \in [k]$  such that  $\mathbf{X}_i$  is a YES-instance of  $\text{PCSP}(\mathbf{A}_j, \mathbf{B})$ . If so then  $\mathbf{X}$  is a YES-instance overall. For the search version something similar happens: see if we can produce a homomorphism  $\mathbf{X}_i \rightarrow \mathbf{B}$  for each  $\mathbf{X}_i$  by running the algorithm for  $\text{PCSP}(\mathbf{A}_j, \mathbf{B})$  for each  $\mathbf{A}_j$ , and combine these homomorphisms to find a homomorphism  $\mathbf{A} \rightarrow \mathbf{B}$ .  $\square$

**Remark 39.** Unlike in all other results in this paper, solvability by AIP is not proved in Corollary 3, only polynomial-time solvability. As far as we know, it well may be that the PCSPs considered in Corollary 3 are not solved by AIP (or even BLP + AIP). The way that tractability is deduced in the proof of Corollary 3 is as follows: If  $\mathbf{A}_1, \dots, \mathbf{A}_k$  are connected and  $\text{PCSP}(\mathbf{A}_i, \mathbf{B})$  is tractable, then  $\text{PCSP}(\mathbf{A}_1 + \dots + \mathbf{A}_k, \mathbf{B})$  is tractable. We give a concrete example of a template with a single symmetric relation of arity 6 that shows that this reduction *does not* preserve solvability by AIP, or even by BLP + AIP. Note however that Corollary 3 only applies to relations of arity at most 4, so the corollary might still be strengthened to show solvability by AIP.

Let  $\mathbb{Z}' = \{x' \mid x \in \mathbb{Z}\}$  be a disjoint copy of  $\mathbb{Z}$ . Then, let  $\mathbf{B} = \mathbf{A}_1 + \mathbf{A}_2$ , where

$$\begin{aligned} \mathbf{A}_1 &= (\{0, 1\}; \{(x_1, \dots, x_6) \mid x_1 + \dots + x_6 \equiv 1 \pmod{2}\}), \\ \mathbf{A}_2 &= (\{0', 1', 2'\}; \{(x_1, \dots, x_6) \mid x_1 + \dots + x_6 \equiv 2' \pmod{3'}\}). \end{aligned}$$

Note that  $\mathbf{A}_1, \mathbf{A}_2, \mathbf{B}$  are all functional and symmetric. Furthermore,  $\text{PCSP}(\mathbf{A}_1, \mathbf{B})$  and  $\text{PCSP}(\mathbf{A}_2, \mathbf{B})$  are solved by AIP by Theorem 8: a  $(2k + 1)$ -ary alternating polymorphism is

$$(x_1, y_1, \dots, x_k, y_k, x_{k+1}) \mapsto x_1 - y_1 + \dots + x_k - y_k + x_{k+1} \pmod{m},$$

where  $m = 2$  for  $\mathbf{A}_1$  and  $m = 3'$  for  $\mathbf{A}_2$ . We will now show that BLP + AIP fails to solve  $\text{PCSP}(\mathbf{B}, \mathbf{B}) = \text{CSP}(\mathbf{B})$ ; to see why, suppose that it does solve it. Thus, by Theorem 9,  $\text{Pol}(\mathbf{B}, \mathbf{B})$  should contain a 2-block symmetric polymorphism of arity 5, say  $f \in \text{Pol}^{(5)}(\mathbf{B}, \mathbf{B})$ . For the ease of notation, suppose that the first block of symmetry of  $f$  contains the first 3 inputs, and the second block of symmetry contains the last 2 inputs (rather than the blocks

being based on parity). Now consider the following matrix

$$\left( \begin{array}{ccc|cc} 1' & 0' & 0' & 1 & 0 \\ 0' & 1' & 0' & 0 & 1 \\ 0' & 0' & 1' & 1 & 0 \\ 1' & 0' & 0' & 0 & 1 \\ 0' & 1' & 0' & 1 & 0 \\ 0' & 0' & 1' & 0 & 1 \end{array} \right)$$

Every column of this matrix is an element of  $R^{\mathbf{B}}$ ; thus  $f$  applied to every row gives a tuple of  $R^{\mathbf{B}}$ . But, due to block-symmetry, the image of every row through  $f$  is the same! This contradicts the lack of constant tuples in  $\mathbf{B}$ .

The non-solvability of  $\text{PCSP}(\mathbf{B}, \mathbf{B})$  by AIP is relevant in view of Conjecture 5 — it shows that the “AIP being a universal algorithm for  $\text{PCSP}(\mathbf{1-in-3}, -)$ ” part of Conjecture 5 cannot be extended to *arbitrary* symmetric templates. We believe that it might hold true for any *connected*  $\mathbf{A}$  with one symmetric relation.

**Corollary 4.** *Let  $\mathbf{A}$  be a symmetric structure and  $\mathbf{B}$  be a functional structure such that  $\mathbf{A} \rightarrow \mathbf{B}$ . Assume that  $\mathbf{A}$  has a relation  $R^{\mathbf{A}}$  of arity at least 3 for which  $\text{diam}(A, R^{\mathbf{A}}) \leq 1$ . Then, either  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  is solvable in polynomial time by AIP and is finitely tractable, or  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  is NP-hard.*

*Proof.* Such structures are super-connected by Lemma 35, which implies additivity and dependency of  $(\mathbf{A}, \mathbf{B})$  by Lemma 31 and Lemma 32, respectively.  $\square$

We will now move on to a proof of Theorem 1. Suppose  $(\mathbf{A}, \mathbf{B})$  is dependent and additive. Suppose generally that  $A = [a]$ .

**Definition 40.** Consider a polymorphism  $f \in \text{Pol}^{(n)}(\mathbf{A}, \mathbf{B})$ . We call it *k-degenerate* if there exist  $x_1, \dots, x_k \in \text{range}(f^p)$  such that for any  $S_1, \dots, S_k \subseteq [n]$  for which  $f^p(S_i) = x_i$  we have that not all  $S_i$  are disjoint. Note that no polymorphism can be 1-degenerate as a single set is a disjoint family.

For any polymorphism  $f \in \text{Pol}^{(n)}(\mathbf{A}, \mathbf{B})$ , we call a set  $S \subseteq [n]$  a *hard set* if, for any  $T \supseteq S$ , we have  $f^p(T) \neq f^p(\emptyset)$ .<sup>9</sup>

We will prove Theorem 1 using the following two cases. For the following, define  $N_d = \max(1 + |B|^{a^2} a^{2r_{\max}}, 3)$  and  $N_h = |B|^{a^2}$ , where  $r_{\max}$  is the maximum arity of any relation in  $\mathbf{A}$ .

**Theorem 41.** *If  $\text{Pol}(\mathbf{A}, \mathbf{B})$  contains a polymorphism that is not  $k$ -degenerate, for any  $k \leq N_d$ , and that has no hard sets of size at most  $N_h$ , then  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  is solved by AIP and is finitely tractable.*

**Theorem 42.** *If every polymorphism within  $\text{Pol}(\mathbf{A}, \mathbf{B})$  is  $k$ -degenerate for some  $k \leq N_d$ , or has a hard set of size at most  $N_h$ , then  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  is NP-hard.*

These two theorems will be proved in their own sections later.

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<sup>9</sup>These two notions are similar to those of *unbounded antichains* and *fixing sets* in [20]. The notion of hard-set is similar to the notion of an  $f(\emptyset)$ -avoiding set from [18, 30].

*Proof of Theorem 1.* A result of Theorem 41 and Theorem 42.  $\square$

**Remark 43.** The following turns out to be an equivalent characterisation of the solvability of  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  for symmetric  $\mathbf{A}$  and functional  $\mathbf{B}$ . Suppose  $\mathbf{A}$  has domain  $A$  and relations  $R_1^{\mathbf{A}}, \dots, R_k^{\mathbf{A}}$ . For any positive integer  $m$ , define a structure  $\mathbf{A}_m$  in the following way. The domain of  $\mathbf{A}_m$  is the free  $\mathbb{Z}_m$ -module of functions  $\mathbb{Z}_m^A$ . We will write elements of this module as formal sums of the form  $\sum_{a \in A} x_i \bar{a}$ , where  $x_i \in \mathbb{Z}_m$  and  $\{\bar{a} \mid a \in A\}$  is a basis for  $\mathbb{Z}_m^A$ . Extend the map  $x \mapsto \bar{x}$  to tuples of functions, in the following way:  $\overline{(x_1, \dots, x_n)} = (\bar{x}_1, \dots, \bar{x}_n)$ . To define the relation  $R_k^{\mathbf{A}_m}$ , consider the set of tuples of functions  $S = \{\bar{t} \mid t \in R_k^{\mathbf{A}}\}$ . Tuples of functions also form a free  $\mathbb{Z}_m$ -module; thus take  $R_k^{\mathbf{A}_m}$  to be the minimal affine space containing  $S$ . Equivalently,  $R_k^{\mathbf{A}_m}$  is the set of tuples that are equivalent to some  $t \in R_k^{\mathbf{A}}$  modulo  $\{t - t' \mid t, t' \in R_k^{\mathbf{A}}\}$ , if the set of tuple of functions from  $\mathbb{Z}_m^A$  is seen as merely an Abelian group.

With the relational structure  $\mathbf{A}_m$  thus defined, and noting that  $\mathbf{A} \rightarrow \mathbf{A}_m$  always via the homomorphism  $x \mapsto \bar{x}$ , we note that our result is equivalent to the following: for symmetric  $\mathbf{A}$  and functional  $\mathbf{B}$  where  $(\mathbf{A}, \mathbf{B})$  is additive and dependent,  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  is solvable in polynomial time if and only if  $\mathbf{A} \rightarrow \mathbf{A}_m \rightarrow \mathbf{B}$  for some positive integer  $m$ . That this condition is sufficient is clear, as  $\text{CSP}(\mathbf{A}_m)$  is solved by AIP as per Lemma 10. Necessity follow from our proof of Theorem 41 and Theorem 42.

We note also that  $\mathbf{A}_m$  can also be described as a free structure [6]. Namely, define  $\mathcal{Z}_m$  to be a minion such that

$$\mathcal{Z}_m^{(n)} = \{(x_1, \dots, x_n) \mid 0 \leq x_1, \dots, x_n < m, \sum_{i=1}^n x_i \equiv 1 \pmod{m}\},$$

and let minoring be defined as follows. For  $x = (x_1, \dots, x_n) \in \mathcal{Z}_m^{(n)}$ , and for  $\pi : [n] \rightarrow [n']$ , define  $y = x^\pi$  by  $y_i = \sum_{\pi(j)=i} x_j \pmod{m}$ . With this in mind, it can be seen that  $\mathbf{A}_m = \mathbf{F}_{\mathcal{Z}_m}(\mathbf{A})$ .

#### 4.1 Proof of Theorem 41

In this section we assume that  $\text{Pol}^{(n)}(\mathbf{A}, \mathbf{B})$  has a polymorphism  $f$  of arity  $n$  that is not  $k$ -degenerate for  $k$  at most  $N_d$ , and has no hard sets of size at most  $N_h$ . Given this, we will prove that  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  is solved by AIP and is finitely tractable.

**Definition 44.** Define  $0 = f^p(\emptyset)$  and  $1 = f^p([n])$ .

**Lemma 45.**  $(\text{range}(f^p), +, 0)$  forms a group.<sup>10</sup>

*Proof.* We prove this in a few parts.

**Closure, well-definedness.** Consider  $x, y \in \text{range}(f^p)$ . As  $f$  is not 2-degenerate, there exist disjoint  $S, T$  such that  $f^p(S) = x, f^p(T) = y$ . Thus  $x + y = f^p(S) + f^p(T) = f^p(S \cup T) \in \text{range}(f^p)$ , so  $+$  is closed and well-defined.

**Associativity.** Consider any  $x, y, z \in \text{range}(f^p)$ . Since  $f$  is not 3-degenerate, there exist disjoint  $S, T, U \subseteq [n]$  such that  $f^p(S) = x, f^p(T) = y, f^p(U) = z$ . Thus,

$$x + (y + z) = x + f(T \cup U) = f(S \cup (T \cup U)) = f((S \cup T) \cup U) = f(S \cup T) + z = (x + y) + z.$$

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<sup>10</sup>This group happens to be Abelian, but this is not needed for the proof.



**Identity element.** Consider any  $x \in \text{range}(f^p)$ . Suppose  $f^p(S) = x$  for some  $S \subseteq [n]$ . Thus,  $x + 0 = f^p(S \cup \emptyset) = f^p(S) = x$ .

**Inverses.** Consider any  $x \in \text{range}(f^p)$ . Suppose that  $f(S) = x$ ; by Lemma 17, some  $T \subseteq S$  exists with size at most  $|B|^{a^2}$  such that  $f^p(T) = f^p(S) = x$ . Since  $f$  has no hard sets of size at most  $N_h = |B|^{a^2}$ ,  $T$  is not a hard set, and thus some  $U \supseteq T$  exists such that  $f^p(U) = 0$ . Thus  $x + f^p(U \setminus T) = f^p(T) + f^p(U \setminus T) = f^p(U) = 0$ , so  $x$  has an inverse.

Thus we conclude that  $(\text{range}(f^p), +, 0)$  is a group.  $\square$

**Definition 46.** Let  $G$  be the Abelian subgroup of  $(\text{range}(f^p), +, 0)$  generated by  $1 = f^p([n])$ . Let  $m$  be the order of 1. Thus  $G \cong \mathbb{Z}_m$ . Note that  $m \leq |\text{range}(f^p)| \leq |B|^{a^2}$ . We will identify  $\mathbb{Z}_m$  with  $G$  (e.g. we allow ourselves to write  $1 + 1 = 2$ , provided  $m \geq 3$ , where  $1, 2 \in \text{range}(f^p)$ ).

Define the Abelian group  $(H, +) = G^a$ . We will identify  $H$  with  $\mathbb{Z}_m^a$ . We will also define 0 to be the 0 element in  $H$  as well as  $G$ .

For any  $i \in [a]$ , define  $\bar{i} \in H$  as the unit vector that has a 1 at position  $i$ . For some tuple  $(x_1, \dots, x_s) \in [a]^r$ , define  $\overline{(x_1, \dots, x_s)} = (\overline{x_1}, \dots, \overline{x_s}) \in H^r$ . Define 0 to be the zero vector in  $H^r$  as well.<sup>11</sup>

For any relation  $R^{\mathbf{A}}$  of  $\mathbf{A}$  of arity  $r$ , define  $M(R^{\mathbf{A}})$  to be the subgroup of  $H^r$  generated by  $\bar{p} - \bar{q}$  for  $p, q \in R^{\mathbf{A}}$ . Since  $H^r$  is Abelian,  $M(R^{\mathbf{A}})$  is a normal subgroup.

**Lemma 47.** Fix some relation  $R^{\mathbf{A}}$  of  $\mathbf{A}$ ; suppose it has arity  $r$ . Let  $t$  be some tuple of  $R^{\mathbf{A}}$ . Define  $M = M(R^{\mathbf{A}})$ . Consider any  $(a_1, \dots, a_s) \in H^r$  such that  $(a_1, \dots, a_s) \equiv \bar{t} \pmod{M}$ . There exists a matrix  $(x_{ij})$  with  $N \leq N_d$  columns and  $r$  rows, where  $N \equiv 1 \pmod{m}$ , with elements in  $[a]$ , such that each column is a tuple of  $R^{\mathbf{A}}$ , and, for each row  $i$ , we have

$$\sum_{j=1}^N \overline{x_{ij}} = a_i.$$

*Proof.* Note that every element in  $H^r$  has order that divides  $m$  (since  $H^r \cong (G^a)^r \cong (\mathbb{Z}_m^a)^r$ ). Thus, since  $(a_1, \dots, a_s) \equiv \bar{t} \pmod{M}$ , and since  $M$  is generated by  $\bar{p} - \bar{q}$  for  $p, q \in R^{\mathbf{A}}$ , it follows that there exist coefficients  $c_{pq} \in \{0, \dots, m-1\}$  for  $p, q \in R^{\mathbf{A}}$  such that

$$(a_1, \dots, a_s) = \bar{t} + \sum_{p,q \in R^{\mathbf{A}}} c_{pq}(\bar{p} - \bar{q}) = \bar{t} + \sum_{p,q \in R^{\mathbf{A}}} c_{pq}\bar{p} + (m - c_{pq})\bar{q}. \quad (1)$$

This indicates the matrix we will use: let  $(x_{ij})$  be a matrix whose first column is  $t$ , and, for each  $p, q \in R^{\mathbf{A}}$ , has  $c_{pq}$  columns equal to  $p$  and  $m - c_{pq}$  columns equal to  $q$ . Clearly we use  $N \leq 1 + m|R^{\mathbf{A}}|^2 = 1 + |B|^{a^2} a^{2r_{\max}} \leq N_d$  columns, and  $N \equiv 1 \pmod{m}$ . To see why  $\sum_{j=1}^N \overline{x_{ij}} = a_i$  for each  $i$ , note that this condition is equivalent to  $(a_1, \dots, a_s) = \sum_{j=1}^N \overline{c_j}$ , where  $c_1, \dots, c_N$  are the columns of the matrix. But this is precisely Equation (1). Thus we have created the required matrix.  $\square$

<sup>11</sup>We can see the elements of  $H$  as frequency vectors modulo  $m$ . Indeed, for  $x_1, \dots, x_n \in [a]$ ,  $\overline{x_1} + \dots + \overline{x_n}$  counts the number of appearances of  $1, 2, \dots, a$  modulo  $m$  among  $x_1, \dots, x_n$ . In line with this, the elements of  $H^r$  can be seen as tuples of  $r$  frequency vectors. Under this view, for  $t_1, \dots, t_n \in [a]^r$ , the sum  $\overline{t_1} + \dots + \overline{t_n}$  is a tuple of  $r$  frequency vectors, where the  $i$ -th frequency vector counts the frequencies of the elements of  $[a]$  among the  $i$ -th elements of the tuples  $t_1, \dots, t_n$ , modulo  $m$ .

**Lemma 48.** Suppose  $f = h \circ f^*$ . For every  $N \leq N_d$  such that  $N \equiv 1 \pmod m$ , the function  $h_N : A^N \rightarrow B$  defined by

$$h_N(x_1, \dots, x_N) = h\left(\sum_{i=1}^N \overline{x_i}\right)$$

is a polymorphism of  $(\mathbf{A}, \mathbf{B})$ .

*Proof.* By assumption,  $f$  is not  $N$ -degenerate. Thus there exist disjoint subsets  $S_1, \dots, S_N$  of  $[n]$  where  $f^p(S_1) = \dots = f^p(S_N) = f^p([n]) = 1$ . Let  $T = [n] \setminus (S_1 \cup \dots \cup S_N)$ . Note that  $S_1, \dots, S_N, T$  form a partition of  $[n]$ . Furthermore,

$$1 = f^p([n]) = f^p(S_1) + \dots + f^p(S_N) + f^p(T) = N + f^p(T) = 1 + f^p(T).$$

The last equation holds as  $N \equiv 1 \pmod m$ , and addition is done in  $G \cong \mathbb{Z}_m$ . Thus  $f^p(T) = 0$ .

Let  $\pi : [n] \rightarrow [N+1]$  be a function that takes  $x \in S_i$  to  $i$  and  $x \in T$  to  $N+1$ . Consider the polymorphism  $f^\pi$ . Since  $f = h \circ f^*$ , by the definition of  $f^*$  we can see that

$$f^\pi(U_1, \dots, U_a) = h(f^p(\pi^{-1}(U_1)), \dots, f^p(\pi^{-1}(U_a))).$$

Now consider  $f^p(\pi^{-1}(U))$ . Note that  $\pi^{-1}(U) = T_U \cup \bigcup_{i \in U \cap [N]} S_i$ , where  $T_U = U$  if  $N+1 \in U$ , and  $T_U = \emptyset$  otherwise. Thus

$$f^p(\pi^{-1}(U)) = f^p\left(T_U \cup \bigcup_{i \in U \cap [N]} S_i\right) = f^p(T_U) + \sum_{i \in U \cap [N]} f^p(S_i) = 0 + \sum_{i \in U \cap [N]} 1 = |U \cap [N]| \pmod m,$$

where  $|U \cap [N]| \pmod m$  is taken as an element of  $\mathbb{Z}_m \cong G$ . In other words,

$$f^\pi(U_1, \dots, U_a) = h(|U_1 \cap [N]| \pmod m, \dots, |U_a \cap [N]| \pmod m).$$

Suppose now that  $U_1, \dots, U_a$  are the set family representation of the input vector  $x_1, \dots, x_{N+1}$  (i.e.  $x_i = j$  if and only if  $i \in U_j$ ) and consider the sum  $\sum_{i=1}^N \overline{x_i}$ . The  $j$ -th coordinate of this sum is the number of  $j$ 's that appear in  $x_1, \dots, x_N$ , modulo  $m$ , i.e.  $|U_j \cap [N]| \pmod m$ . Thus we see that the equation above is equivalent to

$$f^\pi(x_1, \dots, x_{N+1}) = h\left(\sum_{i=1}^N \overline{x_i}\right).$$

This polymorphism ignores  $x_{N+1}$ , so we find that the function

$$(x_1, \dots, x_N) \mapsto h\left(\sum_{i=1}^N \overline{x_i}\right)$$

is also a polymorphism. But this is just  $h_N$ , which is thus a polymorphism as required.  $\square$

We can now prove the main theorem in this subsection.

*Proof of Theorem 41.* We will show that  $(\mathbf{A}, \mathbf{B})$  admits a homomorphic sandwich  $\mathbf{A} \rightarrow \mathbf{E} \rightarrow \mathbf{B}$ , where  $\mathbf{E}$  is a relational structure whose domain is  $H$ , and where each relation will be of the form (i)  $c + M$  for some  $c \in H^r$  and  $M \triangleleft H^r$ , or (ii) empty. By Lemma 10 this implies

our desired conclusion. The homomorphism  $\mathbf{A} \rightarrow \mathbf{E}$  will be given by the map  $g(x) = \bar{x}$ . The homomorphism  $\mathbf{E} \rightarrow \mathbf{B}$  will be given by any function  $h$  for which  $f = h \circ g$ . (Recall that such a function exists by Lemma 20.) We will construct  $\mathbf{E}$  relation by relation, showing along the way that  $g$  and  $h$  are in fact homomorphisms.

Consider some relation  $R^{\mathbf{A}}$  of  $\mathbf{A}$ , of arity  $r$ , that corresponds to a relation  $R^{\mathbf{B}}$  of  $\mathbf{B}$ , and  $R^{\mathbf{E}}$  in  $\mathbf{E}$ . If  $R^{\mathbf{A}}$  is empty then we can simply set  $R^{\mathbf{E}}$  to be empty, and then  $g$  and  $h$  map tuples of  $R^{\mathbf{A}}$  to tuples of  $R^{\mathbf{E}}$ , and then to tuples of  $R^{\mathbf{B}}$  vacuously. Thus, suppose  $t = (t_1, \dots, t_s)$  is some tuple of  $R^{\mathbf{A}}$ , and let  $M = M(R^{\mathbf{A}})$ . Then we set  $R^{\mathbf{E}} = \bar{t} + M$ ; in other words, a tuple  $x \in H^r$  will belong to this relation if and only if  $x \equiv \bar{t} \pmod{M}$ .

We first show that  $g$  maps  $R^{\mathbf{A}}$  to  $R^{\mathbf{E}} = \bar{t} + M$ . Indeed, consider any tuple  $x \in R^{\mathbf{A}}$ . We know that  $g(x) = \bar{x}$  by definition. Thus,  $g(x) = \bar{x} = \bar{t} + (\bar{x} - \bar{t}) \in \bar{t} + M$ . Thus  $g$  maps  $R^{\mathbf{A}}$  to  $\bar{t} + M$ .

We now show that  $h$  maps  $R^{\mathbf{E}} = \bar{t} + M$  to  $R^{\mathbf{B}}$ . Consider any tuple  $(a_1, \dots, a_s) \in \bar{t} + M$ . By Lemma 47 there exists some matrix  $X = (x_{ij})$  with  $N \leq N_d$  columns and  $r$  rows, where  $N \equiv 1 \pmod{m}$ , such that each column is an element of  $R^{\mathbf{A}}$ , and for each  $i \in [r]$  we have

$$\sum_{j=1}^N \overline{x_{ij}} = a_i.$$

Furthermore, by Lemma 48, the function  $h_N : A^N \rightarrow B$  given by

$$h_N(x_1, \dots, x_N) = h\left(\sum_{i=1}^N \overline{x_i}\right)$$

is a polymorphism. Note now that

$$\begin{aligned} (h(a_1), \dots, h(a_s)) &= \left( h\left(\sum_{j=1}^N \overline{x_{1j}}\right), \dots, h\left(\sum_{j=1}^N \overline{x_{sj}}\right) \right) \\ &= (h_N(x_{11}, \dots, x_{1N}), \dots, h_N(x_{s1}, \dots, x_{sN})) \in R^{\mathbf{B}}. \end{aligned}$$

The last inclusion holds since the tuple in question is the result of applying  $h_N$ , a polymorphism of  $(\mathbf{A}, \mathbf{B})$ , to the rows of a matrix whose columns are elements in  $R^{\mathbf{A}}$ .

Thus we note that  $\mathbf{A} \rightarrow \mathbf{E} \rightarrow \mathbf{B}$  for some structure  $\mathbf{E}$  that satisfies the conditions in Lemma 10. In conclusion,  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  is solved by AIP and is finitely tractable.  $\square$

## 4.2 Proof of Theorem 42

In this section we will prove that  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  is NP-hard if each polymorphism  $f \in \text{Pol}(\mathbf{A}, \mathbf{B})$  is  $k$ -degenerate for some  $k$  at most  $N_d$ , or has a hard set of size at most  $N_h$ .

**Lemma 49.** *If  $f \in \text{Pol}(\mathbf{A}, \mathbf{B})$  then  $f$  cannot have more than  $|B|^{a^2}$  disjoint hard sets.*

*Proof.* Equivalently we show that any family  $\mathcal{G}$  of more than  $|B|^{a^2}$  disjoint sets contains a non-hard set. Apply Lemma 16 to  $\mathcal{G}$  to find a nonempty subfamily  $\{G_1, \dots\}$  such that  $f^p(\bigcup_i G_i) = f^p(\emptyset)$ . Thus  $G_i \in \mathcal{G}$  is not a hard set.  $\square$

**Lemma 50.** *Suppose  $f \in \text{Pol}^{(n)}(\mathbf{A}, \mathbf{B})$  and  $\pi : [n] \rightarrow [m]$ . Then  $(f^\pi)^p = f^p \circ \pi^{-1}$ .*

*Proof.* Note that  $f_{ij}(S) = f(T_1, \dots, T_a)$  where  $T_j = S$ ,  $T_i = [n] \setminus S$ , and all the other inputs are  $\emptyset$ . Now,  $(f^\pi)_{ij}(S) = f^\pi(T_1, \dots, T_a) = f(\pi^{-1}(T_1), \dots, \pi^{-1}(T_a)) = f^p(\pi^{-1}(S))$ , and so  $(f^\pi)_{ij} = f_{ij} \circ \pi^{-1}$ . Our conclusion follows by applying this fact for each  $i, j \in A$ .  $\square$

**Lemma 51.** *Suppose  $f \in \text{Pol}^{(n)}(\mathbf{A}, \mathbf{B})$  and  $\pi : [n] \rightarrow [m]$ . If  $S$  is a hard set of  $f$  then  $\pi(S)$  is a hard set of  $f^\pi$ .*

*Proof.* We prove this by contrapositive. Suppose  $\pi(S)$  is not a hard set of  $f^\pi$ . Then some  $T \supseteq \pi(S)$  exists such that  $(f^\pi)^p(T) = (f^\pi)^p(\emptyset)$ . So  $(f^p)(\pi^{-1}(T)) = (f^\pi)^p(T) = (f^\pi)^p(\emptyset) = (f^p)(\pi^{-1}(\emptyset)) = f^p(\emptyset)$ . Thus  $f^p(\pi^{-1}(T)) = f^p(\emptyset)$ , and  $S$  is not a hard set, as  $S \subseteq \pi^{-1}(T)$ .  $\square$

Let  $\mathcal{M}_h$  denote the subset of  $\text{Pol}(\mathbf{A}, \mathbf{B})$  whose polymorphisms have hard sets of size at most  $N_h$ . Let  $\mathcal{M}_{x_1, \dots, x_k}$  denote the subset of  $\text{Pol}(\mathbf{A}, \mathbf{B})$  whose polymorphisms are  $k$ -degenerate, yet not  $(k-1)$ -degenerate, where  $x_1, \dots, x_k \in B^{A^2}$  are witnesses to this degeneracy. By assumption, and as no polymorphism is 1-degenerate,

$$\text{Pol}(\mathbf{A}, \mathbf{B}) = \mathcal{M}_h \cup \bigcup_{k=2}^{N_d} \bigcup_{x_1 \in \text{range}(f^p)} \dots \bigcup_{x_k \in \text{range}(f^p)} \mathcal{M}_{x_1, \dots, x_k}. \quad (2)$$

**Lemma 52.** *There exists some assignment  $I$  that takes  $f \in \mathcal{M}_h^{(n)}$  to a subset of  $[n]$  of size at most  $|B|^{2a^2}$  such that, whenever  $g \in \mathcal{M}_h^{(m)}$  and  $g = f^\pi$  for some  $\pi : [n] \rightarrow [m]$ , we have that  $\pi(I(f)) \cap I(g) \neq \emptyset$ .*

*Proof.* To construct  $I(f)$ , let  $S_1, \dots$  be a maximal sequence of disjoint hard sets of  $f$  of size at most  $|B|^{a^2}$ , constructed greedily, and then set  $I(f)$  to be the union of these sets. Since there can be at most  $|B|^{a^2}$  disjoint hard sets by Lemma 49, it follows that  $|I(f)| \leq |B|^{2a^2}$ .

Consider now any  $f, g \in \mathcal{M}_h$  such that  $g = f^\pi$ . Note that  $I(f)$  contains within it a hard set  $S$  of size at most  $|B|^{a^2}$ . Thus  $\pi(I(f)) \supseteq \pi(S)$ , which is a hard set of size at most  $|B|^{a^2}$  by Lemma 51, and thus must intersect  $I(g)$  by maximality. It follows that  $\pi(I(f)) \cap I(g) \neq \emptyset$ .  $\square$

**Lemma 53.** *For  $k \geq 2, x_1, \dots, x_k \in \text{range}(f^p)$ , there exists some assignment  $I$  that takes  $f \in \mathcal{M}_{x_1, \dots, x_k}^{(n)}$  to a subset of  $[n]$  of size at most  $k|B|^{a^2}$  such that, whenever  $g \in \mathcal{M}_{x_1, \dots, x_k}^{(m)}$  and  $g = f^\pi$  for some  $\pi : [n] \rightarrow [m]$  we have that  $\pi(I(f)) \cap I(g) \neq \emptyset$ .*

*Proof.* To construct  $I(f)$ , take  $S_1, \dots, S_{k-1}$  to be disjoint sets such that  $f^p(S_i) = x_i$ , and take  $T$  to be any set such that  $f(T) = x_k$ . Such sets exist since  $f$  is not  $(k-1)$ -degenerate, and we can take all of these sets to be of size at most  $|B|^{a^2}$ , by replacing them with the subsets given by Lemma 17. Let  $I(f)$  be the union of  $S_1, \dots, S_{k-1}, T$ . Note that  $|I(f)| \leq k|B|^{a^2}$ .

Consider now any  $f, g \in \mathcal{M}_{x_1, \dots, x_k}$  such that  $g = f^\pi$ . Note that  $I(f)$  contains within it disjoint sets  $S_1, \dots, S_{k-1}$  such that  $f^p(S_i) = x_i$ , and  $I(g)$  contains within it a set  $T$  such that  $g^p(T) = x_k$ . Now,  $f^p(\pi^{-1}(T)) = (f^\pi)^p(T) = g^p(T) = x_k$ , and thus by the  $k$ -degeneracy of  $f$  and the disjointness of  $S_1, \dots, S_{k-1}$  it follows that  $\pi^{-1}(T)$  and  $S_1, \dots, S_{k-1}$  must intersect. It follows that  $\pi(I(f)) \cap I(g) \neq \emptyset$ , as required.  $\square$

*Proof of Theorem 42.* We see in (2) that  $\text{Pol}(\mathbf{A}, \mathbf{B})$  is the union of  $m = 1 + \sum_{k=2}^{N_d} (|B|^{a^2})^k$  sets, each of which has an assignment  $I$  that satisfies the condition of Theorem 11 if we take  $C = \max(N_d|B|^{a^2}, |B|^{2a^2})$ . Thus  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  is NP-hard.  $\square$

## 5 BLP+AIP = AIP when A has one balanced relation

In this section we prove Theorem 6 and Corollary 7. Recall that we call a relation  $R$  balanced if there exists a matrix  $M$  whose columns are tuples of  $R$ , where each tuple of  $R$  appears as a column (possibly a multiple times), and where the rows of  $M$  are permutations of each other.

**Theorem 6.** *Let  $\mathbf{A}$  be any structure with a single relation. If the relation in  $\mathbf{A}$  is balanced then, for any  $\mathbf{B}$  such that  $\mathbf{A} \rightarrow \mathbf{B}$ , BLP + AIP solves PCSP( $\mathbf{A}, \mathbf{B}$ ) if and only if AIP solves it.*

Suppose that  $A = [a]$ , and the relation of  $\mathbf{A}$  is  $R = R^{\mathbf{A}}$ . Furthermore suppose that each element in  $[a]$  appears in  $R$  (otherwise these elements can just be eliminated from  $A$ ). Suppose  $A \neq \emptyset, R \neq \emptyset$  (otherwise the conclusion is trivially true). We name the columns of the matrix that witness the balancedness of  $R$  as  $t_1, \dots, t_N \in R$ .

For any  $i \in [a]$ , let  $\bar{i}$  be a unit vector in  $\mathbb{Z}^a$ ; i.e., it has a 1 at position  $i$ . For any tuple  $(a_1, \dots, a_r) \in A^r$ , let  $(\overline{a_1}, \dots, \overline{a_r}) = (\bar{a}_1, \dots, \bar{a}_r) \in (\mathbb{Z}^a)^r$ . Let  $\overline{R} = \{\bar{t} \mid t \in R\} \subseteq (\mathbb{Z}^a)^r$ . (We see the elements of  $\mathbb{Z}^a$  as frequency vectors, and the elements of  $(\mathbb{Z}^a)^r$  as tuples of frequency vectors.) Endow all of these with additive structure. For any  $k \in \mathbb{Z}$ , we denote by  $k\overline{R}$  the set of sums of  $k$  vectors from  $\overline{R}$ .

**Lemma 54.**  $(k+1)\overline{R} - k\overline{R} + k \sum_i \bar{t}_i \subseteq (kN+1)\overline{R}$ .

*Proof.* If  $x \in (k+1)\overline{R} - k\overline{R} + k \sum_i \bar{t}_i$ , it can be written as a sum of  $k+1$  vectors from  $\overline{R}$ , minus  $k$  vectors from  $\overline{R}$ , plus  $k$  copies of each vector  $\bar{t}_i$ . Since each tuple of  $R$  appears among  $t_1, \dots, t_N$ , the last  $kN$  vectors in the sum above include at least  $k$  copies of each vector in  $\overline{R}$ . By removing the  $k$  subtracted vectors from the  $k$  copies of each vector from  $\overline{R}$ , we find that  $x$  can be written as a sum of  $k+1 - k + kN = kN+1$  vectors from  $\overline{R}$ , i.e.  $x \in (kN+1)\overline{R}$ .  $\square$

For any  $k \in \mathbb{Z}$ , we define  $S_k \subseteq \mathbb{Z}^a$  to be the set of sequences of integers that sum up to  $k$ , with non-negative coordinates.

**Lemma 55.** *If  $(\mathbf{A}, \mathbf{B})$  has a 2-block-symmetric polymorphism  $f$  of arity  $2k+1$  then there exists a function  $g : S_k \times S_{k+1} \rightarrow B$  such that  $(g(x_1, y_1), \dots, g(x_r, y_r)) \in R^{\mathbf{B}}$  for all  $(x_1, \dots, x_r) \in k\overline{R}, (y_1, \dots, y_r) \in (k+1)\overline{R}$ .*

*Proof.* To compute  $g(x, y)$ , create two sequences of elements in  $[a]$ , of lengths  $k$  and  $k+1$ , whose frequencies correspond to  $x$  and  $y$  respectively (i.e. the sequence for  $x = (x_1, \dots, x_a)$  has  $x_i$  appearances of  $i$ ), and interleave these to create a sequence  $a_1, \dots, a_{2k+1}$ . Then  $g(x, y) = f(a_1, \dots, a_{2k+1})$ .

To see why this function satisfies the required condition, suppose  $(x_1, \dots, x_r) \in k\overline{R}$  and  $(y_1, \dots, y_r) \in (k+1)\overline{R}$ . Thus we can, by definition, create matrices  $M$  and  $N$ , with  $k$  and  $k+1$  columns respectively, and  $r$  rows, where each column is an element of  $R$ , and each row  $i$  has frequencies corresponding to  $x_i$  and  $y_i$  respectively. Interleave the columns of these matrices to create a matrix  $A$ . Apply  $f$  to the rows of  $A$ . We find that the image of row  $i$  of  $A$  is  $g(x_i, y_i)$  by the symmetry of  $f$ ; furthermore, the images of the rows of  $A$  must form a tuple of  $R^{\mathbf{B}}$ , since  $f$  is a polymorphism. This is the desired conclusion.  $\square$

**Lemma 56.** *Assume there exists a function  $f : (S_{k+1} - S_k) \rightarrow B$  such that  $(f(x_1), \dots, f(x_r)) \in R^{\mathbf{B}}$  for any  $x_1, \dots, x_r \in S_{k+1} - S_k$  with  $(x_1, \dots, x_r) \in (k+1)\overline{R} - k\overline{R}$ . Then, PCSP( $\mathbf{A}, \mathbf{B}$ ) has an alternating polymorphism of arity  $2k+1$ .*

*Proof.* If such a function exists, then

$$g(x_1, \dots, x_{2k+1}) = f(\overline{x_1} + \overline{x_3} + \dots + \overline{x_{2k+1}} - \overline{x_2} - \overline{x_4} - \dots - \overline{x_{2k}})$$

is the required polymorphism.  $\square$

*Proof of Theorem 6.* By Theorem 8, AIP solves PCSP( $\mathbf{A}, \mathbf{B}$ ) if and only if  $\text{Pol}(\mathbf{A}, \mathbf{B})$  contains alternating operations of all odd arities. By Theorem 9, BLP + AIP solves PCSP( $\mathbf{A}, \mathbf{B}$ ) if and only if  $\text{Pol}(\mathbf{A}, \mathbf{B})$  contains 2-block-symmetric operations of all odd arities. As any alternating operation is 2-block-symmetric, it follows that any PCSP solved by AIP is also solved by BLP + AIP.<sup>12</sup> It suffices to show that 2-block-symmetric operations in  $\text{Pol}(\mathbf{A}, \mathbf{B})$  imply alternating operations.

Fix some natural number  $k$ ; we will now show that there exists an alternating operation in  $\text{Pol}(\mathbf{A}, \mathbf{B})$  of arity  $2k + 1$ . Since  $\text{Pol}(\mathbf{A}, \mathbf{B})$  contains a 2-block-symmetric operation of arity  $2kN + 1$ , let  $f : S_{kN} \times S_{kN+1} \rightarrow B$  be the function given by Lemma 55. We will construct the function required by Lemma 56 in order to prove the existence of an alternating polymorphism.

Consider the vector  $v = \sum_i \bar{t}_i \in (\mathbb{Z}^a)^r$ . We claim that  $v$  is a constant vector. To see why this is the case, observe that one way to compute  $\sum_i \bar{t}_i$  is to make  $t_1, \dots, t_N$  into the columns of a matrix, and then to compute the frequencies of each element of  $[a]$  in each row. Element  $i$  of  $v$  is a tuple, where element  $j$  is the number of appearances of  $j$  in row  $i$  in this matrix. But, since  $t_1, \dots, t_N$  witness the balancedness of  $R$ , these frequencies are equal for each row. Thus  $v$  is indeed a constant vector; suppose that  $v = (c, \dots, c)$  for some  $c \in S_N$ . Note that each element in  $[a]$  appears in some tuple of  $R$  by assumption, and each tuple of  $R$  appears in the sum  $\sum_i \bar{t}_i$ . Thus each coordinate in  $c \in \mathbb{Z}^a$  is at least 1.

The function we are interested in is  $g : (S_{k+1} - S_k) \rightarrow B$ , where  $g(x) = f(kc, x + kc)$ . First note that these inputs are legal inputs for the function  $f$ . To see why, note first that  $c \in S_N$  and thus  $kc \in S_{kN}$ . Second, consider  $x + kc$ . As  $x \in S_{k+1} - S_k$ , the elements in  $x$  sum up to 1. Since the elements in  $kc$  sum up to  $kN$ , it follows that the elements in  $x + kc$  sum up to  $1 + kN$  as required. Furthermore, all the elements of  $x + kc$  are non-negative: each element of  $x$  is at least  $-k$ , whereas each element of  $c$  is at least 1, and thus each element of  $kc$  is at least  $k$ . Thus  $x + kc \in S_{kN+1}$ .

Why does  $g$  satisfy the conditions from Lemma 56? Consider any  $x_1, \dots, x_r \in S_{k+1} - S_k$  such that  $(x_1, \dots, x_r) \in (k+1)\bar{R} - k\bar{R}$ . Note that

$$(kc, \dots, kc) = k(c, \dots, c) = k \sum_i \bar{t}_i \in kN\bar{R},$$

$$(x_1 + kc, \dots, x_r + kc) = (x_1, \dots, x_r) + k(c, \dots, c) \in (k+1)\bar{R} - k\bar{R} + k \sum_i \bar{t}_i \subseteq (kN+1)\bar{R},$$

due to Lemma 54. Thus, since  $f$  satisfies the conditions in Lemma 55,

$$(g(x_1), \dots, g(x_r)) = (f(kc, x_1 + kc), \dots, f(kc, x_r + kc)) \in R^B.$$

Thus  $(g(x_1), \dots, g(x_r)) \in R^B$ , as required.  $\square$

Theorem 6 does not generalise to structures with multiple relations (even just two), as the following examples show.

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<sup>12</sup>This also directly follows from the definitions of the AIP and BLP + AIP algorithms [10].



**Remark 57.** Consider a Boolean symmetric template  $\mathbf{A}$  that has two balanced (and in fact even symmetric) relations, namely  $R^{\mathbf{A}} = \{(0)\}$  and  $Q^{\mathbf{A}} = \{(0,1), (1,0), (1,1)\}$ , which are unary and binary, respectively. Then  $\text{CSP}(\mathbf{A})$  is solved by BLP + AIP, and indeed by BLP, since the symmetric operation  $\max(x_1, \dots, x_n)$  is a polymorphism for any  $n$  [6], but *not* by AIP. This is because  $\mathbf{A}$  fails to have any alternating non-unary polymorphisms, even of arity 3: suppose  $f(x, y, z)$  is such a polymorphism. Then  $f(1, 1, 0) = f(0, 0, 0) = f(0, 1, 1)$  as  $f$  is alternating; and  $f(0, 0, 0) = 0$  due to  $R^{\mathbf{A}}$ . However, due to  $Q^{\mathbf{A}}$ ,  $f(1, 1, 0)$  and  $f(0, 1, 1)$  cannot both be 0. This contradiction implies our conclusion.

One cannot simply remove the balancedness condition from Theorem 6, as the following example shows.

**Remark 58.** Let  $\mathbf{A}$  be a Boolean template with relation  $S^{\mathbf{A}} = \{(0, 0, 1), (0, 1, 0), (0, 1, 1)\}$ . Note that  $S^{\mathbf{A}}$  is *not* balanced. Then  $\text{CSP}(\mathbf{A})$  is solved by BLP + AIP, and indeed by BLP, since the symmetric operation  $\max(x_1, \dots, x_n)$  is a polymorphism for any  $n$  [6], but not by AIP.  $\mathbf{A}$  fails to have any alternating polymorphism, even of arity 3, for exactly the same reason as the problem from Remark 57. (The identities that would result from  $R^{\mathbf{A}}$  in that example now result from the first element in each tuple in  $S^{\mathbf{A}}$ , and the identities that would result from  $Q^{\mathbf{A}}$  in that example now result from the last two elements in each tuple in  $S^{\mathbf{A}}$ .)

On the other hand, there are templates that are unbalanced for which AIP and BLP + AIP have equivalent power, as the following example shows.

**Remark 59.** Consider a Boolean template  $\mathbf{A}$  that has one relation  $P^{\mathbf{A}} = \{(0, 1)\}$ . Then  $\text{CSP}(\mathbf{A})$  is solved by AIP and by BLP + AIP, since the alternating operation  $x_1 + \dots + x_n \bmod 2$  is a polymorphism of  $\mathbf{A}$  for every odd  $n$ . This is in spite of the fact that  $P^{\mathbf{A}}$  is unbalanced.

We now prove Corollary 7.

**Corollary 7.** *Suppose that  $G$  is a transitive group of permutations. Further, suppose that  $\mathbf{A}$  is a relational structure with one relation that is preserved by  $G$ . Then, for any  $\mathbf{A} \rightarrow \mathbf{B}$ , BLP + AIP solves  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  if and only if AIP does.*

*Proof.* Let  $R$  be the relation of  $\mathbf{A}$ , of arity  $r$ . It is sufficient to show that  $R$  is balanced. Let  $M$  be a matrix whose columns are the tuples of  $R$ . Suppose that the rows of  $M$  are  $r_1, \dots, r_n$ . We show that row  $i$  is a permutation of row  $j$ , for arbitrary  $i, j \in [r]$ .

Represent the elements of  $G$  as permutation matrices. Let  $\pi \in G$  be a permutation (matrix) that sends  $i$  to  $j$  (it exists by transitivity). Consider  $\pi M$ . Note that no two columns of  $\pi M$  can be equal, since then two columns of  $\pi^{-1}\pi M = M$  would be equal, which is false. Furthermore each column of  $\pi M$  is a tuple of  $R$ , and thus a column of  $M$ , since  $R$  is preserved by  $\pi$ . Thus we see that  $\pi M$  can be seen as  $M$  but with its columns permuted. In other words, for some permutation matrix  $\sigma$ , we have  $\pi M = M\sigma^T$ .

Now, let us look at row  $j$  in  $\pi M = M\sigma^T$ . In  $\pi M$  this is  $r_i$  (since  $\pi$  sends  $i$  to  $j$ ). In  $M\sigma^T$  this is  $r_j\sigma^T$ . Thus  $r_i = r_j\sigma^T$ , i.e. row  $i$  of  $M$  and row  $j$  of  $M$  are permutations of each other. We conclude that  $R$  is balanced, as required.  $\square$

Corollary 7 applies to fewer structures than Theorem 6, as shown in the next example.

**Remark 60.** Consider any digraph  $\mathbf{A}$  with edge relation  $E^{\mathbf{A}}$  that is strongly connected but not symmetric. Then  $E^{\mathbf{A}}$  is balanced (cf. Appendix A). On the other hand, the unique transitive permutation group with degree 2 (i.e. the group containing the identity permutation and the permutation swapping two elements) does not preserve  $E^{\mathbf{A}}$ .

## 6 Conclusion

Our first result classifies certain  $\text{PCSP}(\mathbf{A}, \mathbf{B})$ , where  $\mathbf{A}$  and  $\mathbf{B}$  are symmetric and  $\mathbf{B}$  is functional, into being either NP-hard or solvable in polynomial time. This is the first step towards the following more general problem.

**Problem 61.** Classify the complexity of  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  for functional  $\mathbf{B}$ .

Looking more specifically at the case  $\text{PCSP}(\mathbf{1-in-3}, \mathbf{B})$ , we note that our proof of Theorem 1 implies that, for functional  $\mathbf{B}$ , we have that  $\text{PCSP}(\mathbf{1-in-3}, \mathbf{B})$  is tractable if and only if  $\mathbf{Eqn}_{m,1} \rightarrow \mathbf{B}$  for some  $m \leq |B|$ , where  $\mathbf{Eqn}_{m,1}$  is a relational structure over  $\{0, \dots, m-1\}$  with one ternary relation defined by  $x + y + z \equiv 1 \pmod m$ . By using the Chinese remainder theorem,  $\mathbf{Eqn}_{m,1} = \mathbf{Eqn}_{3^p,1} \times \mathbf{Eqn}_{q,1}$ , where  $q$  is coprime to 3. Since this latter template contains a constant tuple (namely  $(x, x, x)$  where  $x$  is the inverse of 3 modulo  $q$ ), we find that, for functional  $\mathbf{B}$ ,  $\text{PCSP}(\mathbf{1-in-3}, \mathbf{B})$  is tractable if and only if  $\mathbf{Eqn}_{3^p,1} \rightarrow \mathbf{B}$ .

Looking at non-functional templates  $\text{PCSP}(\mathbf{1-in-3}, \mathbf{B})$  that are tractable, all the examples the authors are aware of are either tractable for the same reason as a functional template is (i.e.  $\mathbf{Eqn}_{3^p,1} \rightarrow \mathbf{B}$ ), or because they include the not-all-equal predicate (i.e.  $\mathbf{NAE} \rightarrow \mathbf{B}$ ). Thus, we pose the following problem.

**Problem 62.** Is  $\text{PCSP}(\mathbf{1-in-3}, \mathbf{B})$  tractable if and only if  $\mathbf{Eqn}_{3^p,1} \times \mathbf{NAE} \rightarrow \mathbf{B}$  for some  $p$ ?

Problem 62 has a link with the problem of determining the complexity of  $\text{PCSP}(\mathbf{1-in-3}, \mathbf{C}_k^+)$ , where  $\mathbf{C}_k^+$  is a ternary symmetric template on domain  $[k]$  which contains tuples of the form  $(1, 1, 2), \dots, (k-1, k-1, k), (k, k, 1)$ , as well as all tuples of three distinct elements (rainbow tuples). Such templates are called *cyclic*, with the cycle being  $1 \rightarrow \dots \rightarrow k \rightarrow 1$ .

The link is the following:  $\mathbf{Eqn}_{3^p,1} \times \mathbf{NAE}$  is a template containing one cycle of length  $2 \times 3^p$ , together with certain rainbow tuples — in other words,  $\mathbf{Eqn}_{3^p,1} \times \mathbf{NAE} \rightarrow \mathbf{C}_{2 \times 3^p}^+$ . Likewise,  $\mathbf{Eqn}_{3^p,1}$  has a cycle of length  $3^p$  and some rainbow tuples, i.e.  $\mathbf{Eqn}_{3^p,1} \rightarrow \mathbf{C}_{3^p}^+$ . That  $\text{PCSP}(\mathbf{1-in-3}, \mathbf{C}_k^+)$  is tractable whenever  $k = 3^p$  or  $k = 2 \times 3^p$  was first observed in [11]. If Problem 62 were answered in the affirmative then we would have that  $\text{PCSP}(\mathbf{1-in-3}, \mathbf{C}_k^+)$  is tractable if and only if  $k = 3^p$  or  $k = 2 \times 3^p$ . In particular, this would mean that  $\text{PCSP}(\mathbf{1-in-3}, \mathbf{C}_4^+)$  is NP-hard, as conjectured in [5].<sup>13</sup>

Answering Problem 62 in the affirmative would resolve Conjecture 5, i.e.,  $\text{PCSP}(\mathbf{1-in-3}, \mathbf{B})$  would be tractable (via AIP) if and only if  $\mathbf{Eqn}_{3^p,1} \times \mathbf{NAE} \rightarrow \mathbf{B}$ . Perhaps determining whether this equivalence is true might be easier than resolving Conjecture 5; thus we pose the following problem.

**Problem 63.** Is  $\text{PCSP}(\mathbf{1-in-3}, \mathbf{B})$  solved by AIP if and only if  $\mathbf{Eqn}_{3^p,1} \times \mathbf{NAE} \rightarrow \mathbf{B}$  for some  $p$ ?

There already exists such a characterisation for the power of AIP using an infinite structure [6]. In particular, if we let  $\mathbf{Z}$  be an infinite structure whose domain is  $\mathbb{Z}$ , and with a tuple  $(x, y, z)$  in the relation if and only if  $x + y + z = 1$ , then  $\text{PCSP}(\mathbf{1-in-3}, \mathbf{B})$  is solved by AIP if and only if  $\mathbf{Z} \rightarrow \mathbf{B}$ . We are interested in a finite template of this kind.

Turning from problems to algorithms, our second result shows us that, for certain problems of the form  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  where  $\mathbf{B}$  need not be functional, and  $\mathbf{A}, \mathbf{B}$  have one relation, AIP

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<sup>13</sup>Our structure  $\mathbf{C}_4^+$  is called  $\check{\mathbf{C}}^+$  in [5].

and BLP + AIP have the same power. A natural question is for which other templates is it true?

**Problem 64.** For which templates  $(\mathbf{A}, \mathbf{B})$  do AIP and BLP + AIP have the same power?

Equivalently [6, 10], for which templates  $(\mathbf{A}, \mathbf{B})$  does the existence of 2-block symmetric operations of all odd arities in  $\text{Pol}(\mathbf{A}, \mathbf{B})$  imply the existence of alternating operations of all odd arities in  $\text{Pol}(\mathbf{A}, \mathbf{B})$ ?

We remark that the recent work [16] does not answer any problem from this section, and the results from [16] are consistent with positive answers to Problems 62 and 63.

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## A Deferred proofs

**Lemma 65.** *A digraph  $\mathbf{A}$  is balanced if and only if it is a disjoint union of strongly connected components.*

*Proof.* If the relation of  $\mathbf{A}$  is balanced, then there exists a collection of tours of  $\mathbf{A}$  containing every edge at least once. These exist since the multi-digraph formed by the columns of the matrix witnessing the balancedness of the relation of  $\mathbf{A}$  is Eulerian. Thus, considering the tour that contains some edge  $(u, v)$ , we see that there exists a walk from  $v$  to  $u$ . It follows that every weakly connected component of  $\mathbf{A}$  is strongly connected, or equivalently  $\mathbf{A}$  is the disjoint union of strongly connected components.

If  $\mathbf{A}$  is the disjoint union of strongly connected components, then we create a matrix that witnesses the balancedness of the relation of  $\mathbf{A}$ . Consider any edge  $(u, v)$  of  $\mathbf{A}$ . Take the cycle formed by  $(u, v)$  together with the path from  $v$  to  $u$ . Add all of these edges to the matrix as columns. The resulting matrix contains each edge at least once; furthermore every vertex appears the same number of times in the first and the second row (since the edges form cycles). Thus  $\mathbf{A}$  is balanced.  $\square$