Extended Kripke lemma and decidability for hypersequent substructural logics

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Abstract

We establish the decidability of every axiomatic extension of the commutative Full Lambek calculus with contraction $\mathrm{FL_{ec}}$ that has a cut-free hypersequent calculus. The axioms include familiar properties such as linearity (fuzzy logics) and the substructural versions of bounded width and weak excluded middle. Kripke famously proved the decidability of $\mathrm{FL_{ec}}$ by combining structural proof theory and combinatorics. This work significantly extends both ingredients: height-preserving admissibility of contraction by internalising a fixed amount of contraction (a Curry's lemma for hypersequent calculi) and an extended Kripke lemma for hypersequents that relies on the componentwise partial order on n-tuples being an ω^2-well-quasi-order.

CCS Concepts: • **Theory of computation → Proof theory**; *Automated reasoning*; Linear logic; Complexity theory and logic; Turing machines; • **Mathematics of computing** → Combinatoric problems; Combinatorial algorithms.

Keywords: decidability, substructural logics, proof theory

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1 Introduction

A substructural logic is a logic that lacks some of the structural rules of classical and intuitionistic logic such as weakening, contraction, exchange or associativity. As a consequence, a substructural logic will distinguish logical connectives that are conflated in the classical and intuitionistic setting. While the logical connectives of classical (intuitionistic) logic model "truth" (resp. "constructibility"), the logical connectives of substructural logics model a much wider range of notions by varying the structural rules that are retained and adding further axioms in this setting. For example, linear logic and its many variants (computational and resource awareness), Lambek calculus (syntax and syntactic types of natural language, context-free grammars, linguistics), relevant logics (refined accounts of the implication connective to avoid the paradoxes of material implication), fuzzy logics (degrees of truth, fuzzy systems modelling), bunched implication logics (software program verification and systems modelling). Along with modal logics, substructural logics play a significant role in applied and theoretical computer science.

Throughout, we identify a logic with the set of its theorems. A logic is decidable if there is an algorithm to decide if a given formula is a theorem of the logic or not. Such is the interest in decidability that this question is likely to have been considered for every logic in the literature! In this work we obtain decidability for the commutative Full Lambek calculus with contraction[1] $\mathrm{FL_{ec}}$ extended with any finite choice from infinitely many properties, including many familiar ones such as linearity (every extension of $\mathrm{FL_{ec}}$ + linearity is a fuzzy logic [11]), and the substructural versions of bounded width (ubiquitous) and weak excluded middle (preservation property of rough sets). A striking feature is the expansive breadth of logics that are covered and the fact that almost all of the decidability results are new. To achieve this, we overcome two major challenges: the presence of the contraction rule (c) in the absence of the weakening rule (w), and taming the structural language of the hypersequent calculus.

To explain the nature of these challenges, let us first consider how we might establish the decidability of intuitionistic propositional logic. Its sequent (proof) calculus is obtained by adding the structural rule of weakening (w) to $\mathrm{FL_{ec}}$ (i.e. $\mathrm{FL_{ecw}}$). Due to Gentzen's cut-elimination theorem, it has the *subformula property* (every formula in a proof is a subformula of the end formula). Additionally, repetitions of a formula in the antecedent of a sequent are inter-derivable

[1] Also called: intuitionistic negation-free fragment of relevant logic R without distribution; the positive fragment of Lattice R (denoted $LR+$ [30]); intuitionistic multiplicative additive linear logic with contraction $IMALLC$.

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and hence need not be distinguished e.g. $A \Rightarrow B$ is derivable from $A, A \Rightarrow B$ by (c) and the other direction uses (w). (For this reason, a set is often used in this calculus as the data structure of the antecedent in place of a list.) These two observations imply that only finitely many different sequents need occur in the proof of a given formula up to inter-derivability. The length of each branch in the proof search tree can then be made finite and decidability follows. A more careful counting of branch length can even establish (optimal) PSPACE-complexity. In the absence of weakening, the number of repetitions matters, so this argument fails in $\mathrm{FL_{ec}}$ ($A, A \Rightarrow B$ is no longer derivable from $A \Rightarrow B$). Therefore the task was to find some other way of bounding the branch length in the proof search tree.

Enter Kripke [18] who in 1959 gave a famous decidability argument for $\mathrm{FL_{ec}}$ later described by Urquhart [30] as a "*tour de force* of combinatorial reasoning". Kripke established the finiteness of the proof search tree by combining (i) *Curry's lemma* from structural proof theory which shows how to internalise a fixed amount of contraction into the other rules to obtain height-preserving admissibility of contraction, and (ii) showing that if a branch of the proof search tree is infinite, then some sequent on the branch must be derivable by contraction from a later sequent and hence redundant. The latter is the celebrated *Kripke lemma* that Meyer (see [28]) subsequently identified as being equivalent to Dickson's lemma from number theory. From an order-theoretic perspective, it expresses that the usual componentwise partial order on n-tuples of natural numbers is a well-quasi-order (wqo).

Kripke's lemma is a 'sledge hammer' in that it says something about the property of *all* infinite sequences of sequents without any direct consideration of the rules of the calculus. Could we be missing a simpler decision argument? The tight upper and lower complexity bounds for $\mathrm{FL_{ec}}$ due to Urquhart [30] confirm that we are not: there is no primitive recursive algorithm; it is primitive recursive in the Ackermann function. Indeed, omit commutativity and the logic $\mathrm{FL_c}$ is undecidable [6]. Also undecidable is its predicate version $\forall\mathrm{FL_{ec}}$ [17]. Kripke's lemma was subsequently utilised to prove decidability for specific logics in the vicinity e.g. the pure implication fragments of E and R, their implication-negation fragments [5], and via translation, lattice R [22]. However, there is a fundamental proof theoretic limitation that prevents broader application: the sequent calculus formalism (which crucially underpins Kripke's lemma) is not expressive enough to give a proof calculus with the subformula property for most logics of interest.

This takes us into the realm of structural proof theory where the limitations of the sequent calculus have motivated the development of numerous new proof formalisms that extend the sequent calculus by adding new structure. These formalisms are then used to define proof calculi with the subformula property for the logics of interest. A notable result for substructural logics is the work of Ciabattoni *et al.* [7] who construct hypersequent calculi with the subformula property for infinitely many axiomatic extensions of $\mathrm{FL_e}$ (acyclic $\mathcal{P}_3'$ axioms in the *substructural hierarchy*) via the addition of *analytic structural rules*. These extensions comprise a significant portion of logics in the sense that no further axiomatic extensions can be obtained via analytic structural rule extension [8] and the hierarchy closes [16] (up to formula equivalence in $\mathrm{FL_e}$) at the next level $\mathcal{N}_3$. We can restate the central contribution of this paper as decidability of *every* logic that has an analytic structural rule extension of the hypersequent calculus for $\mathrm{FL_{ec}}$.[2]

There is a price to pay for using the hypersequent calculus. Its structural language is so expressive that it is challenging (compared to the sequent calculus) to use it to prove meta-logical results. As Metcalfe *et al.* [21] observe: "For hypersequent calculi, however, [establishing decidability] is further complicated by the presence of the external contraction rule (EC) that can duplicate whole sequents." The issue is the interaction of the structural rules with (EC), and it is independent of the complications due to (c) discussed above. For example, although MTL (= $\mathrm{FL_{ew}}$ + prelinearity) was proved decidable (see [10]) by specialised semantic arguments, attempts to find a syntactic proof have been obstructed by the complex interactions of (EC) with the structural rule (com) which expand the proof search tree unmanageably. The challenges posed by contraction and hypersequent calculi are also illustrated by the fact that recent decidability results in the vicinity make significant concessions: decidability and 2EXPTIME-complexity for analytic structural rule extensions of $\mathrm{FL_{ecm}}$ are obtained in [9]. Here m stands for the mingle axiom which is a weaker version of weakening, but enough to allow a set to be used for the antecedent. Meanwhile decidability and complexity results are presented in [29] for a large class of extensions of $\mathrm{FL_e}$ using semantic and syntactic means. However, the extensions are confined to those that have a cut-free sequent calculus, and the only extensions of $\mathrm{FL_{ec}}$ are by specific mingle-type axioms.

A hypersequent is a multiset of sequents so what we do—informally speaking—is extend Kripke's argument over sets of n-tuples of natural numbers. The argument is technically involved but the *extended Kripke lemma* that we obtain gives us a handle on *all* infinite sequences of hypersequents, and it is this that supports our general results. In Sec. 3 we extend Curry's lemma to all the hypersequent calculi introduced in [7]. Height-preserving admissibility of the contraction rules is a desirable property in many structural proof theoretic investigations so this is a valuable result in its own right. We have written this section independent of the remainder so that this generalisation can be lifted directly to other applications. Our extended Kripke lemma (Sec. 4) is deeper than

[2] The distributivity axiom of the relevant logic R is in $\mathcal{N}_3$ so it is not covered by our result. This is what we should expect given that R is undecidable.

the original in a formal sense as it relies on the stronger property that the usual componentwise partial order on n-tuples of natural numbers is an ω^2-wqo. Finally, to combine structural proof theory and order theory, a novel partial order on hypersequents is formulated (we subsequently identified this partial order as the Smyth ordering from domain theory) and the infinite Ramsey theorem is deployed (something that is not required in Kripke's argument). This work reiterates Meyer's insights on Kripke's lemma and on the potential for logical problems to be recast in a way that takes advantage of the mathematical literature.

2 Preliminaries

Let $\mathbb{N}$ denote the set $\{0, 1, 2, \ldots\}$ of natural numbers. For $m > 0$, let $\mathbb{N}^m$ denote the set $\{(a_1, \ldots, a_m) | a_i \in \mathbb{N}\}$ of m-tuples of natural numbers.

Logical formulas are given by the following grammar. Var denotes a countably infinite set of propositional variables.

$$F := p \in \mathsf{Var} | \top | \bot | 1 | 0 | F \wedge F | F \vee F | F \cdot F | F \to F$$

The connective $\cdot$ is called *fusion*. It is also called multiplicative conjunction to contrast it with the additive conjunction $\wedge$. The meaning of the connectives in the context of substructural logics is conveyed well through its algebraic semantics of residuated lattices [14]: $\wedge, \vee$ are the familiar lattice connectives; $\top$ and $\bot$ are respectively the top and bottom element of the lattice; $\cdot$ is a monoidal operation with unit 1, and 0 is an arbitrary element of the lattice; the implication connective $\to$ is residuated with $\cdot$.

We identify a logic with the set of formulas that are theorems. The *axiomatic extension* $L + \mathcal{F}$ of the logic L by a finite set $\mathcal{F}$ of formulas is defined in the usual way as the closure of $L \cup \mathcal{F}$ under the axioms and rules of its Hilbert calculus.

A *sequent* is a tuple written $X \Rightarrow \Pi$ where X (the *antecedent*) is a multiset of formulas and Π (the *succedent*) is a multiset that is either empty or consists of a single formula.

A *hypersequent* is a (possibly empty) multiset of sequents and it is written explicitly as follows.

$$X_1 \Rightarrow \Pi_1 | \ldots | X_n \Rightarrow \Pi_n \tag{1}$$

Each $X_i \Rightarrow \Pi_i$ is called a *component* of the hypersequent. Of course, each of these components is a sequent.

The **hypersequent calculus** $\mathrm{HFL_{ec}}$ is given in Fig. 1. Note that the cut-rule is eliminable from this calculus so it is *not* included in the calculus.

As an aid to the reader unfamiliar with proof theory, we provide a brief exposition of the terminology below:

A "hypersequent calculus" is a type of formal proof system (introduced independently in [3, 23, 25]) that can be used to generate proofs ('derivations') of an infinite set of hypersequents. It is an extension of the "sequent calculus" formal proof system introduced by Gentzen [15] in 1935.

A hypersequent calculus consists of a finite set of *hypersequent rule schemas*. Each rule schema has a *conclusion* hypersequent and some number of *premise* hypersequent(s). A rule schema with no premises is called an *initial hypersequent*.

The conclusion and every premises of the rule schema has the form $s_1 | \ldots | s_n$ where each s_i is either (i) a hypersequent-variable (denoted by H), (ii) a sequent-variable (written here as $X \Rightarrow \Pi$), (iii) $\vec{\Gamma} \Rightarrow \Pi$, or (iv) $\vec{\Gamma} \Rightarrow$. In the last two cases, $\vec{\Gamma}$ is a finite list comprising of multiset-variables (denoted using X, Y, Z), formulas built from formula-variables (A, B) and propositional-variables (p, q, r), and Π is a succedent-multiset-variable. Each s_i is not a sequent, but it is still referred to as a *component*. This overloading of the terminology with (1) is standard practice. The variables in the rule schema are collectively referred to as *schematic variables*. A rule schema that contains neither formula-variables nor propositional-variables is called a *structural rule schema*.

The *extension of the hypersequent calculus* $\mathcal{H}$ by the finite set $\mathcal{R}$ of rule schemas is the hypersequent calculus $\mathcal{H} \cup \mathcal{R}$ (following standard convention, we write $\mathcal{H} + \mathcal{R}$).

A *rule instance* is obtained from a rule schema by uniformly instantiating the schematic variables with concrete objects of the corresponding type. I.e. a hypersequent-variable with a hypersequent, a sequent-variable with a sequent, a multiset-variable with a multiset, and so on. In particular, a succedent-multiset-variable is instantiated by a multiset that is either empty or consists of a single formula.

It is worth noting that the standard practice in the field is to not make a rigorous distinction between rule schema and rule instance as it can be determined easily from the context.

Any component in the premise or conclusion of a rule schema that is not a hypersequent-variable, as well as the corresponding component in a rule instance, is called an *active component*.

A *derivation* of the hypersequent h in the hypersequent calculus $\mathcal{H}$ is defined in the usual way as a finite tree whose nodes are labelled with hypersequents such that the root is labelled with h, the leaves are labelled with instances of initial hypersequents, and the labels of an interior node and its child node(s) are the conclusion and premise(s) of an instance of some rule schema in $\mathcal{H}$.

A *branch* in a derivation is a path from the root to a leaf. The *height* of a derivation is the number of nodes on the longest branch.

Example 2.1. Consider the rule schema named ($\wedge$R).

$$\frac{H|X \Rightarrow A \qquad H|X \Rightarrow B}{H|X \Rightarrow A \wedge B}\ (\wedge\mathrm{R})$$

It has two premises $H|X \Rightarrow A$ (the active component is $X \Rightarrow A$) and $H|X \Rightarrow B$ (active component $X \Rightarrow B$), and its conclusion is $H|X \Rightarrow A \wedge B$ (active component $X \Rightarrow A \wedge B$).

A rule instance of this schema is obtained by uniformly instantiating H by a hypersequent, X by a multiset of formulas, and the formula-variables A and B by formulas (the instantiation of $A \wedge B$ is fully determined by the latter). Here are some rule instances.

$$\frac{\Rightarrow p \qquad \Rightarrow q}{\Rightarrow p \wedge q} \qquad\qquad \frac{r, r \Rightarrow p \wedge q \qquad r, r \Rightarrow q}{r, r \Rightarrow (p \wedge q) \wedge q}$$

$$\frac{\Rightarrow s|q \to p \Rightarrow p|r \Rightarrow p \wedge q \qquad \Rightarrow s|q \to p \Rightarrow p|r \Rightarrow q}{\Rightarrow s|q \to p \Rightarrow p|r \Rightarrow (p \wedge q) \wedge q}$$

The active components in the rule instances above appear as the rightmost component of each hypersequent.

Example 2.2. Consider the rule schema named (com).

$$\frac{H|X_1, Y_1 \Rightarrow \Pi_1 \qquad H|X_2, Y_2 \Rightarrow \Pi_2}{H|X_1, Y_2 \Rightarrow \Pi_1|X_2, Y_1 \Rightarrow \Pi_2} \text{ (com)}$$

The conclusion has two active components $X_1, Y_2 \Rightarrow \Pi_1$ and $X_2, Y_1 \Rightarrow \Pi_2$. Meanwhile each premise has a single active component. Consider the following instantiations:

$$\begin{array}{llll} X_1 \mapsto \{r, r\} & Y_1 \mapsto \{p\} & \Pi_1 \mapsto \{p\} & \\ X_2 \mapsto \emptyset & Y_2 \mapsto \{r, q\} & \Pi_2 \mapsto \emptyset & H \mapsto p \Rightarrow |q \Rightarrow \end{array}$$

Here is the corresponding rule instance. Following a slight (and standard) abuse of notation, when writing a rule instance, the curly brackets { } denoting the multiset are dropped, and the comma between multiset-variables in the rule schema is interpreted as multiset union.

$$\frac{p \Rightarrow |q \Rightarrow |r, r, p \Rightarrow p \qquad p \Rightarrow |q \Rightarrow |r, q \Rightarrow \emptyset}{p \Rightarrow |q \Rightarrow |r, r, r, q \Rightarrow p|p \Rightarrow \emptyset}$$

Notation. We use lower case g, h to denote hypersequents to make it easy to distinguish them from the hypersequent-variable H. For an index set $J = \{r_1, \ldots, r_n\}$, let $H \mid X_j \Rightarrow \Pi_j (j \in J)$ denote the hypersequent

$$H \mid X_{r_1} \Rightarrow \Pi_{r_1} \mid \ldots \mid X_{r_n} \Rightarrow \Pi_{r_n}$$

Let A^n denote $A, \ldots, A$ (n copies). We say that X contains at least n copies of A (or A occurs at least n times in X) if there exists X' such that X is the multiset union of $\{A^n\}$ and X'. If additionally X' does not contain A (denoted $A \notin X'$) then "at least" can be replaced by "exactly".

Hypersequent calculi for substructural logics. Ciabattoni *et al.* [7, 8] establish equivalence between a large set of axiomatic extensions of $\mathrm{FL_e}$ and the analytic structural rule extensions of $\mathrm{HFL_e}$ (i.e. $\mathrm{HFL_{ec}}$ minus the contraction rule (c)). The following reformulates [7], [8, Def. 4.16].

Definition 2.3. An *analytic structural rule schema* has the following form where each $\vec{\Gamma}_{ik}$ and $\vec{\Delta}_l$ is a list of multiset-variables, and satisfies the conditions listed below.

$$\frac{\{H \mid Y_i, \vec{\Gamma}_{ik} \Rightarrow \Pi_i\}_{i \in I, k \in K_i} \qquad \{H \mid \vec{\Delta}_l \Rightarrow\}_{l \in L}}{H \mid Y_i, X_{i1}, \ldots, X_{i\alpha_i} \Rightarrow \Pi_i (i \in I) \mid Z_{j1}, \ldots, Z_{j\beta_j} \Rightarrow (j \in J)} \tag{2}$$

- **(linear conclusion)** The multiset-variables that occur in the conclusion occur exactly once.
- **(separation)** No multiset-variable occurs in an antecedent and in a succedent.
- **(coupling)** For every component in the conclusion with succedent Π, there is a multiset-variable Y in its antecedent such that the pair (Y, Π) always occur together in a premise, and Y occurs exactly once there.
- **(strong subformula property)** Each multiset-variable in the premise occurs in the conclusion.

Let $\mathcal{P}_0 = \mathcal{N}_0$ be the set Var of propositional variables. The *substructural hierarchy* [7] is defined as follows.

$$\begin{aligned} \mathcal{P}_{n+1} &:= 1 \mid \bot \mid \mathcal{N}_n \mid \mathcal{P}_{n+1} \vee \mathcal{P}_{n+} \mid \mathcal{P}_{n+1} \cdot \mathcal{P}_{n+1} \\ \mathcal{N}_{n+1} &:= 0 \mid \top \mid \mathcal{P}_n \mid \mathcal{N}_{n+1} \wedge \mathcal{N}_{n+} \mid \mathcal{P}_{n+1} \to \mathcal{N}_{n+1} \end{aligned}$$

$$\mathcal{P}_3' := 1 \mid \bot \mid \mathcal{N}_2 \wedge 1 \mid \mathcal{P}_3' \vee \mathcal{P}_3' \mid \mathcal{P}_3' \cdot \mathcal{P}_3'$$

The constructors in the definition of $\mathcal{P}_{i+1}$ (i.e. $\vee, \cdot$) are those whose rules in $\mathrm{HFL_e}$ are invertible in the antecedent; the constructors in the definition of $\mathcal{N}_{i+1}$ (i.e. $\wedge, \to$) are those whose rules in $\mathrm{HFL_e}$ are invertible in the succedent. Observe that $U_i \subset V_{i+1}$ ($U, V \in \{\mathcal{P}, \mathcal{N}\}$) and $\mathcal{P}_3' \subset \mathcal{P}_3$.

In the presence of weakening, every formula in $\mathcal{P}_3$ can be effectively transformed into an equivalent analytic structural rule schema. In its absence, this holds for formulas in $\mathcal{P}_3'$ that are *acyclic* (i.e. those that do not lead the transformation into cycles, see [8, Def. 4.11] for further details). This motivates the following definition.

Definition 2.4 (amenable)**.** A set $\mathcal{F}$ of formulas is *amenable* if (i) $\mathcal{F} \subseteq \mathcal{P}_3$ and left weakening $p \cdot q \to p \in \mathcal{F}$, or (ii) $\mathcal{F} \subseteq \mathcal{P}_3'$ consists of acyclic formulas.

We say that $\mathcal{H}$ is a *hypersequent calculus for the logic L* if for every formula B: $B \in L$ iff $\mathcal{H}$ derives $\Rightarrow B$.

Theorem 2.5 ([7, 8])**.** *(i) From every finite set $\mathcal{F}$ of amenable formulas, a finite set $R_{\mathcal{F}}$ of analytic structural rule schemas is computable such that $HFL_e + R_{\mathcal{F}}$ is a calculus for $FL_e + \mathcal{F}$ with cut-elimination and the subformula property.*
(ii) Every analytic structural rule extension $HFL_e + R$ is a calculus for an amenable axiomatic extension of FL_e.

Some amenable formulas and the analytic structural rules computed from them are shown in Fig. 2

Remark 1. *In addition to the above syntactic characterisation of analytic structural rule extensions of HFL_e via amenable formulas, there is also a semantic characterisation in terms of closure under hyper-MacNeille algebraic completions (see [8]).*

3 Internalising a limited contraction: from $\mathbf{HFL_{ec}} + R$ to $\mathbf{HFL_{ec}^{\rightsquigarrow}} + R^{\rightsquigarrow}$

A rule schema is *hp-admissible* (height-preserving admissible) if whenever the premises of a rule instance are derivable, there is a derivation with no greater height of its conclusion.

$$\frac{}{H|p \Rightarrow p} \qquad \frac{}{H|\bot, X \Rightarrow \Pi} \qquad \frac{}{H|X \Rightarrow \top} \qquad \frac{}{H|0 \Rightarrow} \qquad \frac{}{H| \Rightarrow 1} \qquad \frac{H|X \Rightarrow \Pi}{H|1, X \Rightarrow \Pi} \qquad \frac{H|X \Rightarrow}{H|X \Rightarrow 0}$$

$$\frac{H|X, A, B \Rightarrow \Pi}{H|X, A \cdot B \Rightarrow \Pi}\ (\cdot\mathrm{L}) \qquad \frac{H|X \Rightarrow A \qquad H|Y \Rightarrow B}{H|X, Y \Rightarrow A \cdot B}\ (\cdot\mathrm{R}) \qquad \frac{H|X, A \Rightarrow \Pi \qquad H|X, B \Rightarrow \Pi}{H|X, A \vee B \Rightarrow \Pi}\ (\vee\mathrm{L}) \qquad \frac{H|X \Rightarrow A}{H|X \Rightarrow A \vee B}\ (\vee\mathrm{R})$$

$$\frac{H|X, A \Rightarrow \Pi}{H|X, A \wedge B \Rightarrow \Pi}\ (\wedge\mathrm{L}) \qquad \frac{H|X \Rightarrow A \qquad H|X \Rightarrow B}{H|X \Rightarrow A \wedge B}\ (\wedge\mathrm{R}) \qquad \frac{H|X \Rightarrow A \qquad H|Y, B \Rightarrow \Pi}{H|X, Y, A \to B \Rightarrow \Pi}\ (\to\mathrm{L}) \qquad \frac{H|X, A \Rightarrow B}{H|X \Rightarrow A \to B}\ (\to\mathrm{R})$$

$$\frac{H|X \Rightarrow \Pi|X \Rightarrow \Pi}{H|X \Rightarrow \Pi}\ (\mathrm{EC}) \qquad \frac{H}{H|X \Rightarrow \Pi}\ (\mathrm{EW}) \qquad \frac{H|X, A, A \Rightarrow \Pi}{H|X, A \Rightarrow \Pi}\ (\mathrm{c})$$

Figure 1. The (cut-free) hypersequent calculus $\mathrm{HFL_{ec}}$ for $\mathrm{FL_{ec}}$

In this section we will construct a new hypersequent calculus $\mathrm{HFL}_{\mathrm{ec}}^{\rightsquigarrow} + R^{\rightsquigarrow}$ from an analytic structural rule extension $\mathrm{HFL_{ec}} + R$ that derives exactly the same hypersequents but where (EW), (c) and (EC) are hp-admissible. Hp-admissibility of (EW) even holds in $\mathrm{HFL_{ec}} + R$, so the challenge is (c) and (EC). The idea is to show that these rules can be permuted with every rule instance above it. The insight of Curry's lemma was that such a permutation is achievable if the rules internalise a fixed amount of contraction. Here we present the extension of Curry's lemma to the hypersequent calculus.

3.1 Formula multiplicity, component multiplicity

Consider how we might permute an instance of (c) with a rule instance above it.

If the duplicate formulas to be contracted are in the instantiation of the same schematic variable in the conclusion of the rule instance, then contraction is applied in every premise where this variable occurs and permutation is achieved. The obstacle therefore is when each of the duplicate formulas belongs to the instantiation of a *different* schematic variable. The solution is to internalise a fixed amount of contraction into the rule to handle this situation. The more variables that are present in the antecedent of an active component in the conclusion, the more contractions we need to internalise until we get to the stage where there are enough duplicate formulas in the conclusion to ensure that the instantiation of some variable *must* contains multiple copies. This motivates the definition of formula multiplicity (the terminology is due to [13], where an alternative definition is given).

Definition 3.1. The *formula multiplicity* of a rule schema is the maximum of the numbers of elements in the antecedents of each of the active components in its conclusion.

To permute (EC) with any rule instance above it, we require a similar definition, this time at the level of components rather than elements in the antecedent of a component.

Definition 3.2. The *component multiplicity* of a rule schema is the number of components in its conclusion (including the hypersequent-variable).

Let us compute some of the values by way of example.

The initial hypersequent with conclusion $H|\bot, X \Rightarrow \Pi$ has two components (hence the component multiplicity is 2) and the antecedent of the active component is $\bot, X$. There are 2 elements in this list so the formula multiplicity is 2.

The initial hypersequent with conclusion $H| \Rightarrow 1$ has two components (hence the component multiplicity is 2) and the antecedent of the active component is empty. Hence the formula multiplicity is 0.

The conclusion $H|X, Y, A \to B \Rightarrow \Pi$ of ($\to$L) has two components (hence the component multiplicity is 2) and the antecedent of the active component is $X, Y, A \to B$. There are 3 elements in this list so the formula multiplicity is 3.

The conclusion $H|X, A \Rightarrow \Pi$ of (c) has two components and the antecedent of the active component is X, A. There are two elements in this list so the formula multiplicity is 2.

The general form of an analytic structural rule schema is given in (2). In particular, its conclusion is:

$$H \mid Y_i, X_{i1}, \ldots, X_{i\alpha_i} \Rightarrow \Pi_i (i \in I) \mid Z_{j1}, \ldots, Z_{j\beta_j} \Rightarrow (j \in J)$$

By inspection, its component multiplicity is $1 + |I| + |J|$, and its formula multiplicity is

$$\max\Big(\max(\{\alpha_i | i \in I\}) + 1\,,\ \max(\{\beta_j | j \in J\})\Big)$$

Some analytic structural rule schemas and their formula and component multiplicities are given in Fig. 2.

Definition 3.3. Let $\mathrm{HFL_{ec}} + R$ be any analytic structural rule extension of $\mathrm{HFL_{ec}}$.

The *formula multiplicity of* $\mathit{HFL_{ec}} + R$ is the maximum of the formula multiplicities of its rule schemas.

The *component multiplicity of* $\mathit{HFL_{ec}} + R$ is the maximum of the component multiplicities of its rule schemas.

A quick inspection of Fig. 1 reveals that the formula multiplicity of $\mathrm{HFL_{ec}}$ is 3 and its component multiplicity is 2.

3.2 (EW), (c), (EC) are hp-admissible in $\mathbf{HFL}_{\mathbf{ec}}^{\rightsquigarrow} + R^{\rightsquigarrow}$

Define these binary relations on hypersequents h and h_1.

- $h \rightsquigarrow^{k}_{(\mathrm{c})} h_1$ iff h_1 can be obtained from h by applying some number of (c) such that every formula is up to k occurrences fewer in a component in h_1 than in the corresponding component in h.

$$\frac{H|X_1, Y_1 \Rightarrow \Pi_1 \qquad H|X_2, Y_2 \Rightarrow \Pi_2}{H|X_1, Y_2 \Rightarrow \Pi_1 | X_2, Y_1 \Rightarrow \Pi_2}\ (\mathrm{com})$$
$$(p \to q)_{\wedge 1} \vee (q \to p)_{\wedge 1}$$
$$(2, 3)$$

$$\frac{H|X, Y \Rightarrow}{H|X \Rightarrow |Y \Rightarrow}\ (\mathrm{wem})$$
$$(p \to 0)_{\wedge 1} \vee ((p \to 0) \to 0)_{\wedge 1}$$
$$(1, 3)$$

$$\frac{H|X_i, X_j \Rightarrow \Pi_i (0 \le i, j \le k; i \ne j)}{H|X_0 \Rightarrow \Pi_0 | \ldots | X_k \Rightarrow \Pi_k}\ (Bwk)$$
$$\vee_{i=0}^{k} (p_i \to (\vee_{j \ne i} p_j))_{\wedge 1}$$
$$(1, k+2)$$

$$\frac{H|X_i, X_j \Rightarrow \Pi_i (0 \le i \le k-1; i+1 \le j \le k)}{H|X_0 \Rightarrow \Pi_0 | \ldots | X_{k-1} \Rightarrow \Pi_{k-1} | X_k \Rightarrow}\ (Bck)$$
$$(p_0)_{\wedge 1} \vee (p_0 \to p_1)_{\wedge 1} \vee \ldots \vee ((p_0 \wedge \ldots \wedge p_{k-1}) \to p_k)_{\wedge 1}$$
$$(1, k+2)$$

$$\frac{H|Y, X_1 \Rightarrow \Pi \qquad H|Y, X_2 \Rightarrow \Pi}{H|Y, X_1, X_2 \Rightarrow \Pi}\ \text{mingle}$$
$$p \cdot p \to p$$
$$(3, 2)$$

$$\frac{\{H|Y, X_{i_1}, \ldots, X_{i_m} \Rightarrow \Pi \ s.t.\ \{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\}\}}{H|Y, X_1, \ldots, X_n \Rightarrow \Pi}\ \text{knot}_m^n$$
$$p^n \to p^m\ (n, m \ge 0)$$
$$(n+1, 2)$$

Figure 2. Some analytic structural rule schemas, each computed from the amenable formula that appears below it. Below that is its (formula multiplicity, component multiplicity). Note: $(A)_{\wedge 1}$ is notation for $(A \wedge 1)$.

- $h \rightsquigarrow^{k}_{(\mathrm{EC})} h_1$ iff h_1 can be obtained from h by applying some number of (EC) such that every component is up to k occurrences fewer in h_1 than in h.
- $h \rightsquigarrow^{k}_{l} h_1$ iff there exists h' such that $h \rightsquigarrow^{k}_{(c)} h'$ and $h' \rightsquigarrow^{l}_{(EC)} h_1$.

Example 3.4. Consider the following hypersequents.

$$\begin{aligned}
h_0 &:= \quad p^3, q^2 \Rightarrow r \,|\, p^2, q \Rightarrow r \,|\, p, q \Rightarrow r \,|\, \Rightarrow s \\
h_1 &:= \quad p^2, q^2 \Rightarrow r \,|\, p^2, q \Rightarrow r \,|\, p, q \Rightarrow r \,|\, \Rightarrow s \\
h_2 &:= \quad p, q \Rightarrow r \,|\, p, q \Rightarrow r \,|\, p, q \Rightarrow r \,|\, \Rightarrow s \\
h_3 &:= \quad p, q \Rightarrow r \,|\, \Rightarrow s \\
h_4 &:= \quad \Rightarrow s \\
h_5 &:= \quad p^3 \Rightarrow r \,|\, p^2, q \Rightarrow r \,|\, p, q \Rightarrow r \,|\, \Rightarrow s
\end{aligned}$$

Then $h_0 \rightsquigarrow^{2}_{(\mathrm{c})} h_1$. Also: $h_0 \rightsquigarrow^{2}_{(\mathrm{c})} h_2$ and $h_2 \rightsquigarrow^{2}_{(\mathrm{EC})} h_3$ and hence $h_0 \rightsquigarrow^{2}_{2} h_3$. However it is *not* the case that $h_3 \rightsquigarrow^{2}_{(\mathrm{EC})} h_4$ since h_4 *cannot* be obtained from h_3 by (EC). Nor is it that $h_2 \rightsquigarrow^{1}_{(\mathrm{EC})} h_3$ since h_3 has two fewer $p, q \Rightarrow r$ than h_2 . Nor is it that $h_0 \rightsquigarrow^{2}_{(\mathrm{c})} h_5$ since h_5 *cannot* be obtained from h_1 by (c).

Example 3.5. The analytic structural rule extension $\mathrm{HFL_{ec}}$+(com)+(Bw4) has formula multiplicity 3 and component multiplicity 6 (refer Figs. 1,2). By Thm. 2.5 it is a calculus for the logic $\mathrm{FL_{ec}} + (p \to q)_{\wedge 1} \vee (q \to p)_{\wedge 1} + \vee_{i=0}^{4} (p_i \to (\vee_{j \ne i} p_j))_{\wedge 1}$.

For the remainder of this section: fix an arbitrary analytic structural rule extension $\mathrm{HFL_{ec}} + R$. Denote its formula multiplicity by fm. Denote its component multiplicity by cm.

The hypersequent calculus $\mathrm{HFL}^{\rightsquigarrow}_{\mathrm{ec}} + R^{\rightsquigarrow}$ is obtained from the calculus $\mathrm{HFL_{ec}} + R$ by replacing each rule schema r (below left) in the latter with the rule $\mathrm{r}^{\rightsquigarrow}$ (below right) that internalises a fixed amount of contraction in the conclusion.

$$\frac{h_1 \quad \ldots \quad h_n}{h_0}\ r \qquad \frac{h_1 \quad \ldots \quad h_n}{g}\ r^{\rightsquigarrow} \text{ with } h_0 \rightsquigarrow^{\mathrm{fm}-1}_{\mathrm{cm}-1} g$$

For each instance of r there corresponds a finite number of instances of $r^{\rightsquigarrow}$. A *base instance* of $r^{\rightsquigarrow}$ is a particular instance such that the conclusion is h_0. In other words, the conclusion is the hypersequent that would have been obtained by applying r.

Lemma 3.6. *$HFL_{ec} + R$ and $HFL^{\rightsquigarrow}_{ec} + R^{\rightsquigarrow}$ derive the identical set of hypersequents.*

Proof. Every instance of a rule schema $r^{\rightsquigarrow} \in \mathrm{HFL}^{\rightsquigarrow}_{\mathrm{ec}} + R^{\rightsquigarrow}$ can be simulated in $\mathrm{HFL_{ec}} + R$ by a rule instance of r followed by some applications of (c) and (EC). Meanwhile every instance of a rule schema $r \in \mathrm{HFL_{ec}} + R$ can be simulated in $\mathrm{HFL}^{\rightsquigarrow}_{\mathrm{ec}} + R^{\rightsquigarrow}$ since the latter is a base instance of $r^{\rightsquigarrow}$. □

Since $(\mathrm{EW}^{\rightsquigarrow})$, $(\mathrm{c}^{\rightsquigarrow})$ and $(\mathrm{EC}^{\rightsquigarrow})$ are in $\mathrm{HFL}^{\rightsquigarrow}_{\mathrm{ec}} + R^{\rightsquigarrow}$, the admissibility of (EW), (c), and (EC) is trivial. The significance of the following lemma is that this admissibility is height-preserving.

Lemma 3.7. *(EW), (c), (EC) are hp-admissible in $HFL^{\rightsquigarrow}_{ec} + R^{\rightsquigarrow}$.*

Proof. We prove hp-admissibility of each rule schema in turn.

(EW): given a derivation δ in $\mathrm{HFL}^{\rightsquigarrow}_{\mathrm{ec}} + R^{\rightsquigarrow}$ of h, let us obtain a derivation of $h|X \Rightarrow \Pi$ of no greater height. Proof by induction on the height of δ.

Base case: if h is an instance of an initial hypersequent, then so is $h|X \Rightarrow \Pi$.

Inductive case: let $\mathrm{r}^{\rightsquigarrow}$ be the last rule instance in δ. Let us illustrate the argument when $\mathrm{r}^{\rightsquigarrow}$ has two premises $g|g_1$ and $g|g_2$ (the general case of ≥ 1 premise is analogous) and conclusion $g|g_0$, and g is the hypersequent that instantiates the hypersequent-variable of the rule (every rule schema in $\mathrm{HFL}^{\rightsquigarrow}_{\mathrm{ec}} + R^{\rightsquigarrow}$ has exactly one hypersequent-variable). By the induction hypothesis (IH) we obtain derivations of height lesser than δ of $g|X \Rightarrow \Pi|g_1$ and $g|X \Rightarrow \Pi|g_2$. Applying $\mathrm{r}^{\rightsquigarrow}$

to these hypersequents with the multiset-variable now instantiated by $g|X \Rightarrow \Pi$, we obtain $g|X \Rightarrow \Pi|g_0$. This is a derivation of $h|X \Rightarrow \Pi$ with height no greater than δ.

(c): given a derivation δ of $h|A, A, X \Rightarrow \Pi$ let us obtain a derivation of $h|A, X \Rightarrow \Pi$ of the same height. Induction on the height of δ.

Base case: if $h|A, A, X \Rightarrow \Pi$ is an instance of an initial hypersequent, then so is $h|A, X \Rightarrow \Pi$.

Inductive case. Let $r^{\rightsquigarrow}$ be the last rule in δ. Then there must be a base instance g_0 of $r^{\rightsquigarrow}$ and some g_1 such that

$$g_0 \rightsquigarrow^{\mathsf{fm}-1}_{(c)} g_1 \rightsquigarrow^{\mathsf{cm}-1}_{(EC)} h|A, A, X \Rightarrow \Pi \tag{3}$$

Since X contains exactly k copies of A for some $k \geq 0$, write X as A^k, X^- such that $A \notin X^-$. Since h contains exactly l identical components of $A, A, X \Rightarrow \Pi$ (i.e. $A^{k+2}, X^- \Rightarrow \Pi$) for some $l \geq 0$, partition it into the portion h^- that does not contain $A^{k+2}, X^- \Rightarrow \Pi$ and the remainder. So

$$h|A, A, X \Rightarrow \Pi :=$$
$$\underbrace{h^-|\overbrace{A^{k+2}, X^- \Rightarrow \Pi|\ldots|A^{k+2}, X^- \Rightarrow \Pi}^{l \text{ components}}}_{h}|A^{k+2}, X^- \Rightarrow \Pi$$

Since $g_1 \rightsquigarrow^{\mathsf{cm}-1}_{(EC)} h|A^{k+2}, X^- \Rightarrow \Pi$ by (3), partition g_1 into the portion g_1' that externally contracts to h^- (i.e. $g_1' \rightsquigarrow^{\mathsf{cm}-1}_{(EC)} h^-$), and the portion that externally contracts to the remainder. Specifically, there exists α with $0 \leq \alpha \leq \mathsf{cm} - 1$ such that g_1 has the following form. Here α is the number of times fewer that $A^{k+2}, X^- \Rightarrow \Pi$ appears in $h|A, A, X \Rightarrow \Pi$ than in g_1 (due to $\rightsquigarrow^{\mathsf{cm}-1}_{(EC)}$).

$$g_1 := g_1'|\overbrace{A^{k+2}, X^- \Rightarrow \Pi|\ldots|A^{k+2}, X^- \Rightarrow \Pi}^{l+1+\alpha \text{ components}}$$

For each of the $l+1+\alpha$ components of $A^{k+2}, X^- \Rightarrow \Pi$ above, the corresponding component in g_0 will have up to $\mathsf{fm} - 1$ more occurrences of A (due to $\rightsquigarrow^{\mathsf{fm}-1}_{(c)}$). Therefore for each i ($1 \leq i \leq l + 1 + \alpha$) there exists β_i with $0 \leq \beta_i \leq \mathsf{fm} - 1$, and also a multiset X_i where the number of occurrences of each formula is no less and at most $\mathsf{fm} - 1$ greater than in X. Let g_0' be such that $g_0' \rightsquigarrow^{\mathsf{fm}-1}_{(c)} g_1'$. Then

$$g_0 := g_0'|\overbrace{A^{k+2+\beta_1}, X_1 \Rightarrow \Pi|\ldots|A^{k+2+\beta_{l+1+\alpha}}, X_{l+1+\alpha} \Rightarrow \Pi}^{l+1+\alpha \text{ components}}$$

Without loss of generality, in the list $\beta_{l+1}, \ldots, \beta_{l+1+\alpha}$, we can assume that there is some initial segment $\beta_{l+1}, \ldots, \beta_{l+N}$ (possibly empty) such that each value is $\mathsf{fm} - 1$ and that the rest take values strictly less than $\mathsf{fm} - 1$. I.e.

$$g_0'|\overbrace{A^{k+2}, X_1 \Rightarrow \Pi|\ldots|A^{k+2}, X_l \Rightarrow \Pi}^{l \text{ components}}|$$
$$A^{k+2+\mathsf{fm}-1}, X_{l+1} \Rightarrow \Pi|\ldots|A^{k+2+\mathsf{fm}-1}, X_{l+N} \Rightarrow \Pi|$$
$$A^{k+2+\beta_{l+N+1}}, X_{l+N+1} \Rightarrow \Pi|\ldots|A^{k+2+\beta_{l+1+\alpha}}, X_{l+1+\alpha} \Rightarrow \Pi$$

For every i such that $l + 1 \leq i \leq l + N$, either the component $A^{k+2+\mathsf{fm}-1}, X_i \Rightarrow \Pi$ is in the instantiation of the hypersequent-variable, or at least 2 of the copies of A occur in the instantiation of some multiset-variable (since the antecedent of every active component in the conclusion of the rule schema has at most fm elements by the definition of formula multiplicity). In either case apply the IH to those instantiations in the premise(s)[3] to make A, A into A. Then apply $r^{\rightsquigarrow}$ to get the following base instance. Note that $A, A \mapsto A$ does not change anything else because the hypersequent variable (see (2)) and each multiset-variable (by linear conclusion Def. 2.3) occur exactly once in the conclusion.

$$g_0'|\overbrace{A^{k+2}, X_1 \Rightarrow \Pi|\ldots|A^{k+2}, X_l \Rightarrow \Pi}^{l \text{ components}}|$$
$$\overbrace{A^{k+1+\mathsf{fm}-1}, X_{l+1} \Rightarrow \Pi|\ldots|A^{k+1+\mathsf{fm}-1}, X_{l+N} \Rightarrow \Pi}^{A,\, A \mapsto A \text{ via IH on premises}}|$$
$$A^{k+2+\beta_{l+N+1}}, X_{l+N+1} \Rightarrow \Pi|\ldots|A^{k+2+\beta_{l+1+\alpha}}, X_{l+1+\alpha} \Rightarrow \Pi$$

Since $\beta_i < \mathsf{fm}-1$ for every i such that $l+N+1 \leq i \leq l+1+\alpha$, the above is related under $\rightsquigarrow^{\mathsf{fm}-1}_{(c)}$ to

$$g_1'|\overbrace{A^{k+2}, X^- \Rightarrow \Pi|\ldots|A^{k+2}, X^- \Rightarrow \Pi}^{l \text{ components}}|$$
$$\overbrace{A^{k+1}, X^- \Rightarrow \Pi|\ldots|A^{k+1}, X^- \Rightarrow \Pi}^{1+\alpha \text{ components}}$$

Since $0 \leq \alpha \leq \mathsf{cm} - 1$ and $g_1' \rightsquigarrow^{\mathsf{cm}-1}_{(EC)} h^-$, the above is related under $\rightsquigarrow^{\mathsf{cm}-1}_{(EC)}$ to

$$h^-|\overbrace{A^{k+2}, X^- \Rightarrow \Pi|\ldots|A^{k+2}, X^- \Rightarrow \Pi}^{l \text{ components}}|A^{k+1}, X^- \Rightarrow \Pi$$

Since A^{k+1}, X^- is A, X, this is exactly $h|A, X \Rightarrow \Pi$.

(EC): given a derivation δ of $h|s|s$ let us obtain a derivation of $h|s$ of the same height. Induction on the height of δ.

Base case: if $h|s|s$ is an instance of an initial hypersequent, then so is $h|s$.

Inductive case. Let $r^{\rightsquigarrow}$ be the last rule in δ. Then there must be a base instance g_0 of $r^{\rightsquigarrow}$ and some g_1 such that

$$g_0 \rightsquigarrow^{\mathsf{fm}-1}_{(c)} g_1 \rightsquigarrow^{\mathsf{cm}-1}_{(EC)} h|s|s \tag{4}$$

[3]The multiset-variable is not required to occur in the premises (e.g. consider if r was the standard weakening rule). In that case, no IH on the premises is required; simply use A instead of A, A in that variable's instantiation.

Since h contains exactly l ($l \geq 0$) identical components of s for some $l \geq 0$, write $h|s|s$ as follows such that $s \notin h^-$.

$$h|s|s := h^- | \overbrace{s| \dots |s}^{l} |s|s$$

Since $g_1 \rightsquigarrow^{\mathrm{cm}-1}_{\mathrm{(EC)}} h|s|s$ by (4), we can partition g_1 into the portion g_1' that externally contracts to h^- (i.e. $g_1' \rightsquigarrow^{\mathrm{cm}-1}_{\mathrm{(EC)}} h^-$), and the portion that externally contracts to the remainder. Specifically, there exists α with $0 \leq \alpha \leq \mathrm{cm}-1$ such that g_1 has the following form. Here α is the number of times fewer that s appears in $h|s|s$ than g_1 due to $\rightsquigarrow^{\mathrm{cm}-1}_{\mathrm{(EC)}}$.

$$g_1 := g_1' | \overbrace{s| \dots |s}^{l+2+\alpha}$$

If $\alpha < \mathrm{cm}-1$ then $g_1 \rightsquigarrow^{\mathrm{cm}-1}_{\mathrm{(EC)}} h|s$, so we have $g_0 \rightsquigarrow^{\mathrm{fm}-1}_{\mathrm{cm}-1} h|s$ as required. So assume that $\alpha = \mathrm{cm}-1$. In that case g_0 has the following form for some g_0' with $g_0' \rightsquigarrow^{\mathrm{fm}-1}_{\mathrm{(c)}} g_1'$, and sequents s_i ($1 \leq i \leq l+1+\mathrm{cm}$) such that $s_i \rightsquigarrow^{\mathrm{fm}-1}_{\mathrm{(c)}} s$.

$$g_0 := g_0' | \overbrace{s_1 | \dots |s_{l+\mathrm{cm}-1}|s_{l+\mathrm{cm}}|s_{l+1+\mathrm{cm}}}^{l+1+\mathrm{cm}}$$

Since the number of components in the conclusion of a rule schema is at most cm, and since one of these is the hypersequent-variable, the number of active components is at most $\mathrm{cm}-1$. Therefore we may assume without loss of generality that the last two components $s_{l+\mathrm{cm}}$ and $s_{l+1+\mathrm{cm}}$ are in the instantiation of the hypersequent-variable. In each premise, make use of hp-admissibility of (c) (proved above) to replace $s_{l+\mathrm{cm}}|s_{l+1+\mathrm{cm}}$ with $s|s$, and then replace $s|s$ with s using the IH. Reapplying $\mathrm{r}^{\rightsquigarrow}$ we can obtain the base instance

$$g_0' | \overbrace{s_1 | \dots |s_{l+\mathrm{cm}-1}}^{l+\mathrm{cm}-1} |s$$

Because we had $g_0' \rightsquigarrow^{\mathrm{fm}-1}_{\mathrm{(c)}} g_1'$ and each $s_i \rightsquigarrow^{\mathrm{fm}-1}_{\mathrm{(c)}} s$, the above is related under $\rightsquigarrow^{\mathrm{fm}-1}_{\mathrm{(c)}}$ to

$$g_1' | \overbrace{s| \dots |s}^{l+\mathrm{cm}-1} |s$$

Because we had $g_1' \rightsquigarrow^{\mathrm{cm}-1}_{\mathrm{(EC)}} h^-$ it follows that the above is related under $\rightsquigarrow^{\mathrm{cm}-1}_{\mathrm{(EC)}}$ to

$$h^- | \overbrace{s| \dots |s}^{l} |s$$

This is exactly $h|s$. □

Remark 2. *From the above lemma it follows using everywhere minimal derivations (see [19]) that (EW), (c), and (EC) are hp-admissible also in* $HFL_{ec}^{\rightsquigarrow} + R^{\rightsquigarrow} \setminus \{(EW^{\rightsquigarrow}), (c^{\rightsquigarrow}), (EC^{\rightsquigarrow})\}$.

4 The extended Kripke lemma

We write (a_i) to denote an infinite sequence with i^{th} element a_i, more standardly denoted $(a_i)_{i \in \mathbb{N}}$. For strictly increasing functions r and s on $\mathbb{N}$: $(a_{r(i)})$ is a subsequence of (a_i) (denoted $(a_{r(i)}) \sqsubseteq (a_i)$); also we drop parentheses and write $(a_{rs(i)})$ to mean the subsequence $(a_{r(s(i))})$ of $(a_{r(i)})$.

Lemma 4.1 (Kripke lemma [18])**.** *For every sequence (s_i) of sequents built using a finite set Ω of formulas, there exists i, j such that $i < j$, and s_i can be obtained from s_j by repeated applications of (c).*

Meyer observed (see [28]) that the above is equivalent to Dickson's lemma from number theory. Order-theoretically it expresses that the standard componentwise ordering on $\mathbb{N}^m$ is a wqo.

The main result in this section is the following.

Lemma 4.2 (extended Kripke lemma)**.** *For every sequence (h_i) of hypersequents built using a finite set Ω of formulas, there exists $i < j$ such that h_i can be obtained from h_j by repeated applications of (c), (EC) and (EW).*

4.1 Some order theoretic observations

A partially ordered set $(S, \leq)$ is a set S with a partial order $\leq$. Let $x < y$ (also $y > x$) denote $x \leq y$ and $x \neq y$. A sequence (x_i) from S is a *descending chain* wrt $\leq$ if $x_{i+1} < x_i$ for every $i \in \mathbb{N}$; it is an *antichain* wrt $\leq$ if for all $i, j \in \mathbb{N}$, $x_i \leq x_j$ implies $i = j$; it is an *ascending chain* wrt $\leq$ if $x_{i+1} > x_i$ for every $i \in \mathbb{N}$. The elements x and y are *incomparable* if neither $x \leq y$ nor $y \leq x$. Thus an antichain (x_i) is a sequence such that x_i and x_j ($i \neq j$) are incomparable.

The following is standard (see e.g. [1]).

Lemma 4.3. *Let $(A, \leq)$ be a partially ordered set, and let (x_i) be a sequence from A. Then there is a subsequence $(x_{r(i)})$ such that one of the following holds wrt $\leq$:*

1. *$(x_{r(i)})$ is a descending chain*
2. *$(x_{r(i)})$ is an antichain*
3. *$(x_{r(i)})$ is an ascending chain*
4. *$(x_{r(i)})$ is constant i.e. $x_{r_i} = x_{r_j}$ for all $i, j \in \mathbb{N}$*

For a partially ordered set $(S, \leq)$, a subset $U \subseteq S$ is *upward closed* if $x \in U$ and $x \leq y$ implies $y \in U$. For a subset $X \subseteq S$, the *upward closed set generated by* X is defined $\uparrow X := \{y \in S | \exists x. x \in X \text{ and } x \leq y\}$. The set $U(S, \leq)$ of upward closed subsets of S is partially ordered by set inclusion $\subseteq$.

We use the symbol $\leq$ specifically to denote the standard componentwise partial order on $\mathbb{N}^m$ defined: $(a_1, \dots, a_m) \leq (b_1, \dots, b_m)$ iff $a_i \leq b_i$ for every i ($1 \leq i \leq m$).

Theorem 4.4. *Let $m > 0$.*

(i) $(\mathbb{N}^m, \leq)$ *has no descending chain and no antichain.*
(ii) $(U(\mathbb{N}^m, \leq), \subseteq)$ *has no ascending chain and no antichain.*

Proof. (i) This result is known as Dickson's lemma [1, 12].

(ii) See [1, 2, 20]. This proof relies on the fact that $(\mathbb{N}^m, \leq)$ is a ω^2-wqo. In particular, it does not hold for every wqo (Rado [26] has given a counterexample). ◻

Let $\mathcal{P}(\mathbb{N}^m)$ denote the powerset of $\mathbb{N}^m$.

For $X, Y \in \mathcal{P}(\mathbb{N}^m)$, let us define $\preccurlyeq^1_s$ as follows. It is easily seen that this relation is reflexive and transitive.

$$X \preccurlyeq^1_s Y \text{ iff } \forall y \in Y \exists x \in X (x \leq y)$$

However it is not antisymmetric since e.g. when $m = 2$: $\{(1,2)\} \preccurlyeq^1_s \{(1,2),(10,20)\} \preccurlyeq^1_s \{(1,2)\}$. To obtain antisymmetry (and hence a partial order), we will need to treat such elements as equivalent. We therefore define the following equivalence relation:

$$X \sim_s Y \text{ iff } X \preccurlyeq^1_s Y \text{ and } Y \preccurlyeq^1_s X$$

Then $\sim_s$ partitions $\mathcal{P}(\mathbb{N}^m)$ into the set $\mathcal{P}(\mathbb{N}^m)/{\sim_s}$ of equivalence classes. As usual, let $[X]$ denote the equivalence class containing X. For $X, Y \in \mathcal{P}(\mathbb{N}^m)$, define

$$[X] \preccurlyeq_s [Y] \text{ iff } X \preccurlyeq^1_s Y$$

It is easy to check that the definition of $\preccurlyeq_s$ does not depend on the representative elements that are chosen and hence is well-defined: let $X \sim_s X'$ and $Y \sim_s Y'$, and suppose that $[X] \preccurlyeq_s [Y]$. Then $X' \preccurlyeq^1_s X \preccurlyeq^1_s Y \preccurlyeq^1_s Y'$ so $[X'] \preccurlyeq_s [Y']$ as required.

Clearly $\preccurlyeq_s$ is reflexive, transitive and antisymmetric so $\preccurlyeq_s$ is a partial order on $\mathcal{P}(\mathbb{N}^m)/{\sim_s}$.

The definition of $\preccurlyeq^1_s$ is motivated by the fact that it parallels the operations of (EW), (c) and (EC) on hypersequents. This will become clear in the next subsection. In turns out that $\preccurlyeq^1_s$ is known as the *minoring ordering* (see [4]), and is also called the *Smyth ordering* in the power domain literature.

Lemma 4.5. *For $X, Y \in \mathcal{P}(\mathbb{N}^m)$: (i) $[X] \prec_s [Y]$ iff $\uparrow X \supset \uparrow Y$. (ii) If X and Y are incomparable wrt $\preccurlyeq_s$, then $\uparrow X$ and $\uparrow Y$ are incomparable wrt $\subseteq$.*

Proof. (i) Assume $[X] \prec_s [Y]$ (so $[X] \preccurlyeq_s [Y]$ and $[X] \neq [Y]$). For every $u \in \uparrow Y$ there exists $y \in Y$ such that $y \leq u$. Since $[X] \preccurlyeq_s [Y]$, there exists $x \in X$ such that $x \leq y$ and hence $x \leq u$. Thus $u \in \uparrow X$ so $\uparrow X \supseteq \uparrow Y$. Let us show that the inequality is strict. Since $[X] \neq [Y]$ then $X \not\succcurlyeq_s Y$ and hence $\forall x \in X \exists y \in y (y \leq x)$ does *not* hold. So there exists $x_0 \in X$ such that $y \not\leq x_0$ for every $y \in Y$. Thus $x_0 \notin \uparrow Y$, so $\uparrow X \supset \uparrow Y$.

Assume $\uparrow X \supset \uparrow Y$. For arbitrary $y \in Y$: $y \in \uparrow X$ so there exists $x \in X$ such that $x \leq y$ and hence $[X] \preccurlyeq_s [Y]$. Let us show that the inequality is strict. Let $u \in \uparrow X \setminus \uparrow Y$. Then there exists $x \in X$ such that $x \leq u$. If $[X] = [Y]$ then $X \sim_s Y$ and hence there would exist $y \in Y$ such that $y \leq x$. But then $y \leq u$ and hence $u \in \uparrow Y$ and this is a contradiction. So $[X] \prec_s [Y]$.

(ii) Assume that X and Y are incomparable. $\uparrow X \subset \uparrow Y$ is not possible because this would imply $[X] \succ_s [Y]$ by (i), and hence $X \succ^1_s Y$. Similarly $\uparrow Y \subset \uparrow X$ is not possible. Finally, suppose $\uparrow X = \uparrow Y$. Then $x \in X$ implies $x \in \uparrow Y$ and hence there exists $y \in Y$ such that $y \leq x$. Similarly $y \in Y$ implies $y \in \uparrow X$ and hence there exists $x \in X$ such that $x \leq y$. This implies $X \sim_s Y$, contradicting the incomparability of X and Y. We conclude that $\uparrow X$ and $\uparrow Y$ are incomparable. ◻

Lemma 4.6. *$(\mathcal{P}(\mathbb{N}^m)/{\sim_s}, \preccurlyeq_s)$ has no descending chain and no antichain.*

Proof. If (X_i) is a descending chain on $(\mathcal{P}(\mathbb{N})/{\sim_s}, \preccurlyeq_s)$, then $(\uparrow X_i)$ is an ascending chain on $(U(\mathbb{N}^m, \leq), \subseteq)$ by Lem. 4.5(i). If (X_i) is an antichain chain on $(\mathcal{P}(\mathbb{N})/{\sim_s}, \preccurlyeq_s)$, then $(\uparrow X_i)$ is an antichain on $(U(\mathbb{N}^m, \leq), \subseteq)$ by Lem. 4.5(ii). Each of these consequences contradicts Thm. 4.4(ii). ◻

4.2 Relating to hypersequents

Let set(M) denote the set of elements that occur a positive number of times in multiset M. E.g. if M is multiset $\{p, p, q, r\}$ then set$(M) = \{p, q, r\}$.

Define the following restriction (EW^r) of (EW).

$$\frac{h}{h|X \Rightarrow A}\ (\text{EW}^r) \quad \begin{array}{l} h \text{ contains a component } Y \Rightarrow A \\ \text{such that set}(X) = \text{set}(Y)\end{array}$$

Example 4.7. An instance of (EW^r) is given below.

$$\frac{p^5, q^3 \Rightarrow r}{p^5, q^3 \Rightarrow r | p^2, q^4 \Rightarrow r}\ (\text{EW}^r)$$

However e.g. $p^5, q^3 \Rightarrow r | q^4 \Rightarrow r$ is not obtainable from $p^5, q^3 \Rightarrow r$ because $\text{set}(\{q^4\}) = q \neq \{p, q\} = \text{set}(\{p^5, q^3\})$.

Let Ω be a set of formulas, and let $\mathcal{H}_\Omega$ denote the set of hypersequents built using formulas from Ω.

For $h_1, h_2 \in \mathcal{H}_\Omega$, define $h_1 \sim_h h_2$ iff each hypersequent is obtainable from the other using repeated applications of (c), (EC) and (EW^r).

Clearly $\sim_h$ is an equivalence relation. Therefore it partitions $\mathcal{H}_\Omega$ into the set $\mathcal{H}_\Omega/{\sim_h}$ of equivalence classes. Let $[h]$ denote the equivalence class containing h.

Example 4.8. Observe that $p, q \Rightarrow r$ is equivalent under $\sim_h$ to $p, q \Rightarrow r | p^2, q^3 \Rightarrow r$ since

$$\frac{p, q \Rightarrow r}{p, q \Rightarrow r | p^2, q^3 \Rightarrow r}\ (\text{EW}^r) \qquad \frac{p, q \Rightarrow r | p^2, q^3 \Rightarrow r}{p, q \Rightarrow r}\ (\text{c}),(\text{EC})$$

For $[h_1], [h_2] \in \mathcal{H}_\Omega/{\sim_h}$, define $[h_1] \preccurlyeq_h [h_2]$ iff h_1 can be obtained from h_2 by repeated applications of (c), (EC) and (EW^r).

Let us establish that the definition of $\preccurlyeq_h$ does not depend on the representative elements that are chosen and hence is well-defined. Let $h_1 \sim_h h'_1$ and $h_2 \sim_h h'_2$, and suppose that $[h_1] \preccurlyeq_h [h_2]$. Then h'_1 is obtainable from h_1 using (c), (EC), (EW^r), similarly the latter from h_2, and similarly the latter from h'_2. Thus $[h'_1] \preccurlyeq_h [h'_2]$.

Clearly $\preccurlyeq_h$ is reflexive, transitive, and antisymmetric, so it is a partial order on $\mathcal{H}_\Omega/{\sim_h}$.

Definition 4.9 ((S, Π)-component, hypersequent, sequence)**.** Let S be a finite set of formulas, and let Π be either empty or consist of a single formula.

A (S, Π)-*component* has the form $X \Rightarrow \Pi$ with $\mathrm{set}(X) = S$.

If every component in a hypersequent is a (S, Π)-component, call this a (S, Π)-*hypersequent*.

A sequence of (S, Π)-hypersequents is called a (S, Π)-*sequence*.

For a non-empty finite set S, fix an enumeration $A_1, \ldots, A_{|S|}$. Let $X_1 \Rightarrow \Pi | \ldots | X_N \Rightarrow \Pi$ be any (S, Π)-hypersequent h. Then define $h^{\dashv} \in \mathcal{P}(\mathbb{N}^{|S|})$:

$$\bigcup_{1 \le j \le N} \{(a_1, \ldots, a_{|S|}) | A_i \text{ occurs exactly } a_i \text{ times in } X_j\}$$

Example 4.10. Let h be the $(\{p, q, p \to q\}, r)$-hypersequent below left. Let us enumerate $\{p, q, p \to q\}$ as $p, q, p \to q$. Then $h^{\dashv} \in \mathcal{P}(\mathbb{N}^3)$ is given below right.

$$p, p, p \to q \Rightarrow r | p, q, p \to q \Rightarrow r \qquad \{ (2, 0, 1) \, , \, (1, 1, 1) \}$$

Observe that $(g|h)^{\dashv} = g^{\dashv} \cup h^{\dashv}$.

Lemma 4.11. *For a non-empty finite set S of formulas, and (S, Π)-hypersequents g, h: $[h] \preccurlyeq_h [g]$ iff $[h^{\dashv}] \preccurlyeq_s [g^{\dashv}]$.*

Proof. By inspection, the relation $\preccurlyeq_s^1$ satisfies the following properties for arbitrary $X_1, X_2, Y_1, Y_2 \in \mathcal{P}(\mathbb{N}^m)$.

(i) $X_1 \preccurlyeq_s^1 Y_1$ and $X_2 \preccurlyeq_s^1 Y_2$ implies $X_1 \cup X_2 \preccurlyeq_s^1 Y_1 \cup Y_2$.
(ii) $X_1 \preccurlyeq_s^1 Y_1$ implies $X_1 \cup X_2 \preccurlyeq_s Y_1$.

In the following rule instances, suppose that the premise (and hence the conclusion) is a (S, Π)-hypersequent.

$$\frac{h|X, A, A \Rightarrow \Pi}{h|X, A \Rightarrow \Pi} \qquad \frac{h|X \Rightarrow \Pi | X \Rightarrow \Pi}{h|X \Rightarrow \Pi} \qquad \frac{h}{h|X \Rightarrow \Pi} \, (\mathrm{EW}^r)$$

We observe—using (i) and (ii)—that in every instance of one of the above rule schemas, conclusion$^{\dashv} \preccurlyeq_s^1$ premise$^{\dashv}$. I.e.

$$\begin{aligned}(h|X, A \Rightarrow \Pi)^{\dashv} &= h^{\dashv} \cup (X, A \Rightarrow \Pi)^{\dashv} \preccurlyeq_s^1 h^{\dashv} \cup (X, A, A \Rightarrow \Pi)^{\dashv} \\ &= (h|X, A, A \Rightarrow \Pi)^{\dashv}\end{aligned}$$

Hence $[(h|X, A \Rightarrow \Pi)^{\dashv}] \preccurlyeq_s [(h|X, A, A \Rightarrow \Pi)^{\dashv}]$.

$$\begin{aligned}(h|X \Rightarrow \Pi)^{\dashv} &= h^{\dashv} \cup (X \Rightarrow \Pi)^{\dashv} = h^{\dashv} \cup (X \Rightarrow \Pi)^{\dashv} \cup (X \Rightarrow \Pi)^{\dashv} \\ &= (h|X \Rightarrow \Pi | X \Rightarrow \Pi)^{\dashv}\end{aligned}$$

Hence $[(h|X \Rightarrow \Pi)^{\dashv}] \preccurlyeq_s [(h|X \Rightarrow \Pi | X \Rightarrow \Pi)^{\dashv}]$.

Also $(h|X \Rightarrow \Pi)^{\dashv} = h^{\dashv} \cup (X \Rightarrow \Pi)^{\dashv} \preccurlyeq_s^1 h^{\dashv}$ and hence we have $[(h|X \Rightarrow \Pi)^{\dashv}] \preccurlyeq_s [h^{\dashv}]$.

Suppose that $[h] \preccurlyeq_h [g]$. By definition, there is some finite sequence $g = g_0, \ldots, g_N = h$ such that g_{i+1} is obtained from g_i by (c), (EC) or (EW^r). By the observations above, $g_{i+1}^{\dashv} \preccurlyeq_s^1 g_i^{\dashv}$. By transitivity of $\preccurlyeq_s^1$ it follow that $h^{\dashv} \preccurlyeq_s^1 g^{\dashv}$, and hence $[h^{\dashv}] \preccurlyeq_s [g^{\dashv}]$.

Now suppose that $[h^{\dashv}] \preccurlyeq_s [g^{\dashv}]$. By definition $h^{\dashv} \preccurlyeq_s^1 g^{\dashv}$ and thus $\forall y \in g^{\dashv} \exists x \in h^{\dashv} (x \le y)$. Since g and h are both (S, Π)-hypersequents, every component in g can be made identical to some component in h using (c). Let g_1 be the hypersequent obtained from g in this way. Note that a sequent may occur as a component more times in g_1 than in h. Obtain g_2 from g_1 by using (EC) to remove excess copies so this is no longer the case. Finally, observe that h may contain some (S, Π)-components that do not occur in g_2. Obtain g_3 from g_2 by applying (EW^r) to insert these missing components. It follows that g_3 is identical to h, and thus $[h] \preccurlyeq_h [g]$. ◻

From the above, also $[h] \prec_h [g]$ iff $[h^{\dashv}] \prec_s [g^{\dashv}]$.

Corollary 4.12. *If (h_i) is a (S, Π)-sequence with S finite, $([h_i])$ is neither a descending chain nor an antichain wrt $\preccurlyeq_h$.*

Proof. Immediate if $S = \emptyset$ as $([h_i])$ would be constant. Suppose that $S \neq \emptyset$. If $([h_i])$ is a descending chain, then $[h_i] \succ_h [h_{i+1}]$ for every i. Lem. 4.11 implies $[h_i^{\dashv}] \succ_s [h_{i+1}^{\dashv}]$, so $([h_i^{\dashv}])$ would be a descending chain on $(\mathcal{P}(\mathbb{N}^{|S|})/{\sim_s}, \preccurlyeq_s)$. If $([h_i])$ is an antichain, for every i, j neither $[h_i] \preccurlyeq_h [h_j]$ nor $[h_j] \preccurlyeq_h [h_i]$. From Lem. 4.11: neither $[h_i^{\dashv}] \preccurlyeq_h [h_j^{\dashv}]$ nor $[h_j^{\dashv}] \preccurlyeq_h [h_i^{\dashv}]$, so $([h_i^{\dashv}])$ would be an antichain on $(\mathcal{P}(\mathbb{N}^{|S|})/{\sim_s}, \preccurlyeq_s)$. Each of these consequences contradicts Lem. 4.6. ◻

4.3 Proof of the extended Kripke lemma

We will use the infinite Ramsey theorem (see [1, 24, 27]).

Theorem 4.13 (Ramsey [27])**.** *For every function c mapping each 2-element subset of $\mathbb{N}$ to an element of a finite set, there exists an infinite set $Y \subseteq \mathbb{N}$ such that the restriction of c to the 2-element subsets of Y is a constant.*

The (S, Π)-*reduct* of a hypersequent h—denoted $h(S, \Pi)$—is obtained by deleting every non-(S, Π)-component in h. Clearly $h(S, \Pi)$ is a (S, Π)-hypersequent. Also, $(h_i(S, \Pi))$ is a (S, Π)-sequence for every sequence (h_i) of hypersequents.

Proof of Lemma 4.2. Let (h_i) be a sequence of hypersequents from $\mathcal{H}_\Omega$ for a finite set Ω of formulas. Since there are only finitely many choices for $S \subseteq \Omega$ and Π (empty or an element of Ω), it follows that there is a subsequence $(h_{q(i)}) \sqsubseteq (h_i)$ such that the *signature of* $h_{q(i)}$—define this as the set $\{(S, \Pi) | h_{q(i)} \text{ has a } (S, \Pi)\text{-component}\}$—is the same set $T = \{(S_1, \Pi_1), \ldots, (S_N, \Pi_N)\}$ for every i. If T is empty then $(h_{q(i)})$ is a sequence of empty hypersequents so the result follows trivially. Therefore take T as non-empty.

Let us suppose that $([h_{r(i)}]) \sqsubseteq ([h_{q(i)}])$ is a descending chain and obtain a contradiction. Then for every $i, j \in \mathbb{N}$ with $i < j$: $[h_{r(i)}] \succ_h [h_{r(j)}]$. Hence $[h_{r(i)}] \not\preccurlyeq_h [h_{r(j)}]$. Since $h_{r(i)}$ and $h_{r(j)}$ have the same signature, any missing components of $h_{r(i)}$ can be supplied by (EW^r). Therefore there must exist some (S^{ij}, Π^{ij})-component ($(S^{ij}, \Pi^{ij}) \in T$) in $h_{r(j)}$ that cannot be contracted to any component in $h_{r(i)}$. Assign $(S^{ij}, \Pi^{ij}) \in T$ to the 2-element subset $\{i, j\} \subset \mathbb{N}$. By Ramsey's theorem there is an infinite $Y \subseteq \mathbb{N}$ such that every 2-element subset of Y is assigned the same element $(S^*, \Pi^*) \in T$. Write $Y = \{s(0), s(1), s(2), \ldots\}$ with $s(0) < s(1) < s(2) < \ldots$. Then $[h_{rs(i)}(S^*, \Pi^*)] \succ_h [h_{rs(i+1)}(S^*, \Pi^*)]$, and thus we have that $([h_{rs(i)}(S^*, \Pi^*)])$ is a descending chain, contradicting Cor. 4.12.

Let us suppose that $([h_{r(i)}]) \sqsubseteq ([h_{q(i)}])$ is an antichain and obtain a contradiction. Then for every $i, j \in \mathbb{N}$ with $i < j$: $[h_{r(i)}] \not\preccurlyeq_h [h_{r(j)}]$ and $[h_{r(j)}] \not\preccurlyeq_h [h_{r(i)}]$. From the former, because $h_{r(i)}$ and $h_{r(j)}$ have the same signature, there exists some (S^{ij}, Π^{ij})-component ($(S^{ij}, \Pi^{ij}) \in T$) in $h_{r(j)}$ that cannot be contracted to any component in $h_{r(i)}$. Assign $(S^{ij}, \Pi^{ij}) \in T$ to the 2-element subset $\{i, j\} \subset \mathbb{N}$. Applying Ramsey's theorem there is an infinite set $Y \subseteq \mathbb{N}$ such that every 2-element subset of Y is assigned the same element (S^*, Π^*). Write $Y = \{s(0), s(1), s(2), \ldots\}$ with $s(0) < s(1) < s(2) < \ldots$. Then $[h_{rs(i)}(S^*, \Pi^*)] \not\preccurlyeq_h [h_{rs(j)}(S^*, \Pi^*)]$ for every $i < j$. We continue:

For every $i, j \in \mathbb{N}$ with $i < j$, define $c(\{i, j\}) = 1$ if $[h_{rs(i)}(S^*, \Pi^*)] \succ_h [h_{rs(j)}(S^*, \Pi^*)]$ and 0 otherwise. Once again by Ramsey's theorem, there is an infinite set $Z \subseteq \mathbb{N}$ such such that every 2-element subset of Z is assigned either 1 or 0. Write $Z = \{t(0), t(1), t(2), \ldots\}$ with $t(0) < t(1) < t(2) < \ldots$. If it is 1 then $([h_{rst(i)}(S^*, \Pi^*)])$ is a descending chain so by Cor. 4.12 we obtain a descending chain in $(\mathcal{P}(\mathbb{N}^{|S|}), \preccurlyeq_s)$ contradicting Thm. 4.6. So it must be 0. Then for every $i, j \in \mathbb{N}$ such that $i < j$: $[h_{rst(i)}(S^*, \Pi^*)] \not\preccurlyeq_h [h_{rst(j)}(S^*, \Pi^*)]$ (previous paragraph) and $[h_{rst(i)}(S^*, \Pi^*)] \not\succ_h [h_{rst(j)}(S^*, \Pi^*)]$. It follows that $([h_{rst(i)}(S^*, \Pi^*)])$ is an antichain, contradicting Cor. 4.12.

Since no subsequence of $([h_{q(i)}])$ is a decreasing chain or an antichain, it must have a constant or ascending subseqence $([h_{u(i)}])$ (Lem. 4.3). Then $[h_{u(1)}] \preccurlyeq_h [h_{u(2)}]$, so $h_{u(1)}$ can be obtained from $h_{u(2)}$ using (c), (EC) and (EW^r). ◻

5 Decidability via backward proof search

We are ready to establish the decidability of every amenable axiomatic extension of $\mathrm{FL_{ec}}$. Equivalently, these are the logics that can be formalised by extending the hypersequent calculus $\mathrm{HFL_{ec}}$ with analytic structural rules.

Theorem 5.1 (Main theorem)**.** *Every axiomatic extension of $\mathrm{FL_{ec}}$ by amenable formulas is decidable.*

Proof. Let L be an arbitrary amenable axiomatic extension of $\mathrm{FL_{ec}}$. By Thm. 2.5(i) we can compute a finite set R of analytic structural rules such that for every formula F: $F \in L$ iff $\Rightarrow F$ is derivable in $\mathrm{HFL_{ec}} + R$. From Lem. 3.6: $F \in L$ iff $\Rightarrow F$ is derivable in $\mathrm{HFL}_{\mathrm{ec}}^{\rightsquigarrow} + R^{\rightsquigarrow}$. Therefore in the following we solely consider derivations in $\mathrm{HFL}_{\mathrm{ec}}^{\rightsquigarrow} + R^{\rightsquigarrow}$.

A derivation has minimal height if there is no derivation of the same hypersequent with lesser height. A derivation is *everywhere minimal* [19] if every subderivation of it has minimal height. A derivation is *irredundant* if no hypersequent in the derivation can be obtained from a hypersquent above it by repeated applications from (c), (EC), (EW).

Observation 1. Every derivable hypersequent has an everywhere minimal derivation. We proceed following [19]. Given a derivation d of h, obtain a derivation $\min(d)$ of h of minimal height by exhaustively searching for derivations ('backward proof search') with height at most that of d. This is a finite search because there are only finitely many possible rule instances in $\mathrm{HFL}_{\mathrm{ec}}^{\rightsquigarrow} + R^{\rightsquigarrow}$ with a given hypersequent conclusion (this was why we internalised a fixed amount of contraction!). Replace each immediate subderivation d' of $\min(d)$ with $\min(d')$. In the derivation so obtained, replace each immediate subderivation d'' of those $\min(d')$ with $\min(d'')$. Proceed in this way until the initial hypersequents are reached. The result is an everywhere minimal derivation of h.

Observation 2. An everywhere minimal derivation is irredundant. Indeed, if d were a everywhere minimal derivation that is not irredundant, then it would contain a hypersequent h_1 and a hypersequent h_2 above it from which h_1 could be obtained by repeated applications from (c), (EC), (EW). Then the subderivation of h_1 would not have minimal height since a derivation of lesser height is obtainable from h_2 by hp-admissibility of the latter rules (Lem. 3.7).

We have shown that if $\Rightarrow F$ is derivable, it has an irredundant derivation. The point of irredundancy (unlike everywhere minimality) is that we can enforce it during backward proof search. Perform *irredundant backward proof search* on $\Rightarrow F$ by constructing a proof search tree as follows: place $\Rightarrow F$ at the root. Repeatedly, for each hypersequent in the tree, place all the premises of all possible rule instances as its children, omitting those rule instances that would introduce a premise that violates irredundancy of the tree. The proof search tree is finitely branching since there are only finitely many possible rule instances that apply to a given hypersequent conclusion. Moreover every branch has finite length in the limit since, by Lem. 4.2, there is no infinite sequence (h_i) of hypersequents built using subformulas of F that can satisfy irredundancy (i.e. no element is obtainable from a later element by (c), (EC), (EW)). The proof search tree is therefore finite by König's lemma and thus the construction will terminate. Finally, check if the proof search tree contains a derivation as a subtree. If it does, then $\Rightarrow F$ is derivable, otherwise it is not. ◻

6 Future work

Complexity bounds for the amenable extensions of $\mathrm{FL_{ec}}$ are a challenging question in their own right, as indicated by Urquhart's [30] complexity arguments for $\mathrm{FL_{ec}}$.

It is likely that an upper bound could be obtained here along the lines of [30]: the idea is to show that arbitrarily long necessarily finite non-constant non-ascending sequences can be replaced by ones with bounded length ('controlled bad sequences') and use an upper bound for controlled bad sequences. Controlled bad sequences can be used because the amount of contraction internalised in each rule is fixed and hence the number of copies of a formula in the antecedent can only increase by a fixed amount from conclusion to premise. Balasubramanian [4] gives lower and upper bounds for controlled bad sequences over the Smyth (minoring) wqo of finite sets of $\mathbb{N}^m$ that exceed Urquhart's tight 'primitive recursive in the Ackermann function' bound for $\mathrm{FL_{ec}}$.

The question of lower bounds is of particular interest: is there an amenable extensions of $\mathrm{FL_{ec}}$ that has a lower bound exceeding the complexity of $\mathrm{FL_{ec}}$? Given the reliance on the hypersequent calculus, and hence the Smyth wqo, it seems plausible that such an extension exists.

The decidability problem for arbitrary analytic structural rule extensions of $\mathrm{HFL_{ew}}$ is open. In the specific case of monoidal t-norm logic MTL ($\mathrm{FL_{ew}} + (p \to q) \lor (q \to p)$), decidability is shown by a semantic argument establishing the finite embeddability property—attributed to Ono, see [10] for a proof—and it is not known how this could be argued syntactically i.e. from its hypersequent calculus $\mathrm{HFL_{ew}}$ + (*com*). In the proof of Thm. 5.1, the possibility of e.g. $A^2 \Rightarrow B$ appearing above $A \Rightarrow B$ was excluded from the irredundant backward proof search, justified by the hp-admissibility of (c). However, if (c) is replaced by (w) then it is not clear how to exclude from proof search an infinite branch containing $A \Rightarrow B, A^2 \Rightarrow B, A^3 \Rightarrow B, \ldots$, even if weakening is hp-admissible. It seems therefore that a new idea is required.

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